Stability of Couette flow for 2D Boussinesq system in a uniform magnetic field

Dongfen Bian ∗ Shouyi Dai † Jingjing Mao ‡

Abstract

In this paper, we consider the Boussinesq equations with magnetohydrodynamics convection in the domain $T \times \mathbb{R}$ and establishes the nonlinear stability of the Couette flow $(u_{sh} = (y, 0), b_{sh} = (1, 0), p_{sh} = 0, \theta_{sh} = 0)$. The novelty in this paper is that we design a new Fourier multiplier operator by using the properties of the enhanced dissipation to overcome the difficult term $\partial_{xy}(-\Delta)^{-1}j$ in the linearized and nonlinear system. Then, we prove the asymptotic stability for the linearized system. Finally, we establish the nonlinear stability for the full system by bootstrap principle.

AMS Subject Classification (2020): 35Q35; 76D03

Key Words: Boussinesq-MHD system; Couette flow; stability

1 Introduction and Main Results

In this paper, we consider the following 2D incompressible Boussinesq equations for magnetohydrodynamics (MHD) convection [11, 27]

\[
\begin{cases}
    u_t + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla p - \nu \Delta u = \theta e_2, \\
    b_t + (u \cdot \nabla)b - (b \cdot \nabla)u - \mu \Delta b = 0, \\
    \theta_t + (u \cdot \nabla)\theta - \eta \Delta \theta = 0, \\
    \nabla \cdot u = \nabla \cdot b = 0.
\end{cases}
\]

(1.1)

The unknowns are the velocity field $u = (u^1, u^2)$, the magnetic field $b = (b^1, b^2)$, the temperature $\theta$ and the scalar pressure $p$. In addition, we denote here by $\nu$ the fluid viscosity, $\mu$ the magnetic diffusivity, $\eta$ the thermal diffusivity and $e_2 = (0, 1)$. The spatial domain $\Omega$ here is taken to be $\Omega = T \times \mathbb{R}$ with $T = [0, 2\pi]$ being the periodic box and $\mathbb{R}$ being the whole line.

Physically, the first equation of (1.1) represents the conservation law of the momentum with the effect of the buoyancy $\theta e_2$. The second equation of (1.1) shows that the electromagnetic field is governed by the Maxwell equation. The third equation is a balance of the

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∗School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China. Email: biandongfen@bit.edu.cn/dongfen_bian@brown.edu.
†School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China. Email: daishou@outlook.com.
‡School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China. Email: mao.jingjing@outlook.com.
temperature convection and diffusion. For more physics and numerical simulations, the interested readers may refer to [11, 22] and the references therein. Two fundamental problems, the global regularity problem and the stability problem, have been among the main driving forces in advancing the mathematical theory on the Boussinesq-MHD system. Significant progress has been made on the global regularity of the nonlinear Boussinesq-MHD system [7, 8, 9, 10, 23, 24, 25, 37, 38]. The goal of this paper is the nonlinear stability around the Couette flow \((u_{sh} = (y, 0), b_{sh} = (1, 0), p_{sh} = 0, \theta_{sh} = 0)\).

When the fluid is not affected by the temperature, that is, \(\theta \equiv 0\), then the equations (1.1) become the incompressible MHD system and govern the dynamics of the velocity and the magnetic field in electrically conducting fluids such as plasmas and reflect the basic physics conservation laws. Also the issue of stability on the MHD equations has been extensively studied. Liss considered the sobolev stability threshold of 3D Couette flow in a uniform magnetic field [25]. In [21], the author obtained the stability and large-time behavior of perturbations near a stationary solution of the 2D resistive MHD equation. Further background and motivation for the MHD system may be found in [18, 32] and references therein. Two fundamental problems, temperature convection and diffusion. For more physics and numerical simulations, the interested readers may refer to [11, 22] and the references therein. Two fundamental problems, the global regularity problem and the stability problem, have been among the main driving forces in advancing the mathematical theory on the Boussinesq-MHD system. Significant progress has been made on the global regularity of the nonlinear Boussinesq-MHD system [7, 8, 9, 10, 23, 24, 25, 37, 38]. The goal of this paper is the nonlinear stability around the Couette flow \((u_{sh} = (y, 0), b_{sh} = (1, 0), p_{sh} = 0, \theta_{sh} = 0)\).

When the fluid is not affected by the Lorentz force, that is, \(b \equiv 0\), then the equations (1.1) become the classical Boussinesq system. The Boussinesq system reflects the basic physics laws obeyed by buoyancy-driven fluids. It is one of the most frequently used model for atmospheric and oceanographic flows and serves as the centerpiece in the study of the Rayleigh convection [15, 19, 27, 29]. Important progress has been made on the stability and large-time behavior of Couette flow for the 2D Boussinesq system [16]. Motivated by this work, we consider the Boussinesq system in the presence of magnetic field.

Note that when the fluid is not affected by the temperature and the Lorentz force, that is, \(\theta \equiv 0\) and \(b \equiv 0\), then the equations (1.1) become the Navier-Stokes equations, and become the Euler equations if without viscosity. For the Couette flow of Euler and Navier-Stokes equations, there are many interesting results [1, 2, 5, 6, 13, 14, 30, 31, 35]. Very recently, Deng, Wu and Zhang prove the nonlinear stability of Couette flow for the 2D Boussinesq system [16]. Motivated by this work, we consider the Boussinesq system in the presence of magnetic field.

In this paper we mainly focus on the nonlinear stability for the full system (1.1) and consider the Couette flow

\[
\tilde{u} = (\tilde{u}_1, \tilde{u}_2) = (u_1 - y, u_2), \quad \tilde{b} = (\tilde{b}_1, \tilde{b}_2) = (b_1 - 1, b_2), \quad \tilde{p} = p, \quad \tilde{\theta} = \theta,
\]

and satisfy the following system

\[
\begin{cases}
\tilde{u}_1 + (\tilde{u} \cdot \nabla) \tilde{u}_1 + y \partial_y \tilde{u}_1 + \tilde{u}_2 - (\tilde{b} \cdot \nabla) \tilde{b}_1 - \partial_x \tilde{b}_1 + \partial_x \tilde{p} - \nu \Delta \tilde{u}_1 = 0, \\
\tilde{u}_2 + (\tilde{u} \cdot \nabla) \tilde{u}_2 + y \partial_y \tilde{u}_2 - (\tilde{b} \cdot \nabla) \tilde{b}_2 - \partial_x \tilde{b}_2 + \partial_y \tilde{p} - \nu \Delta \tilde{u}_2 = 0, \\
\tilde{b}_1 + (\tilde{u} \cdot \nabla) \tilde{b}_1 + y \partial_y \tilde{b}_1 - (\tilde{b} \cdot \nabla) \tilde{u}_1 - \partial_x \tilde{u}_1 - \tilde{b}_1 - \mu \Delta \tilde{b}_1 = 0, \\
\tilde{b}_2 + (\tilde{u} \cdot \nabla) \tilde{b}_2 + y \partial_y \tilde{b}_2 - (\tilde{b} \cdot \nabla) \tilde{u}_2 - \partial_x \tilde{u}_2 - \mu \Delta \tilde{b}_2 = 0, \\
\tilde{\theta}_1 + (\tilde{u} \cdot \nabla) \tilde{\theta} + y \partial_y \tilde{\theta} - \eta \Delta \tilde{\theta} = 0, \\
\nabla \cdot \tilde{u} = \nabla \cdot \tilde{b} = 0.
\end{cases}
\]

The corresponding perturbed vorticity and current density near the steady vorticity \(w_{sh} = -1\) and the steady current density \(j_{sh} = 0\) take the form of

\[
\tilde{w} = \partial_x \tilde{u}_2 - \partial_y \tilde{u}_1, \quad \tilde{j} = \partial_x \tilde{b}_2 - \partial_y \tilde{b}_1.
\]
and verify, together with $\tilde{\theta}$, the following system

$$
\begin{align*}
\begin{cases}
\partial_t \tilde{w} + (\tilde{u} \cdot \nabla)\tilde{w} - (\tilde{b} \cdot \nabla)\tilde{w} + y\partial_x \tilde{w} - \partial_x \tilde{j} - \nu \Delta \tilde{w} = \partial_x \tilde{\theta}, \\
\partial_t \tilde{j} + (\tilde{u} \cdot \nabla)\tilde{j} - (\tilde{b} \cdot \nabla)\tilde{w} + y\partial_x \tilde{j} - \partial_x \tilde{w} - \mu \Delta \tilde{j} - 2\partial_x \tilde{b}^1 - Q(\nabla \tilde{u}, \nabla \tilde{b}) = 0, \\
\partial_t \tilde{\theta} + (\tilde{u} \cdot \nabla)\tilde{\theta} + y\partial_x \tilde{\theta} - \eta \Delta \tilde{\theta} = 0, \\
\tilde{u} = -\nabla \perp (-\Delta)^{-1} \tilde{w}, \\
\tilde{b} = -\nabla \perp (-\Delta)^{-1} \tilde{j},
\end{cases}
\end{align*}
\tag{1.2}
$$

where $Q(\nabla \tilde{u}, \nabla \tilde{b}) = 2\partial_x \tilde{b}^1(\partial_x \tilde{u}^2 + \partial_y \tilde{u}^1) - 2\partial_x \tilde{u}^1(\partial_x \tilde{b}^2 + \partial_y \tilde{b}^1)$.

For notational convenience, we shall write $w$ for $\tilde{w}$, $\theta$ for $\tilde{\theta}$, $j$ for $\tilde{j}$, $u$ for $\tilde{u}$ and $b$ for $\tilde{b}$ from now on. We rewrite the nonlinear system (1.2) as follows

$$
\begin{align*}
\begin{cases}
\partial_t w + (u \cdot \nabla)w - (b \cdot \nabla)j + y\partial_x w - \partial_x j - \nu \Delta w = \partial_x \theta, \\
\partial_t j + (u \cdot \nabla)j - (b \cdot \nabla)w + y\partial_x j - \partial_x w - \mu \Delta j - 2\partial_x b^1 - Q(u, b) = 0, \\
\partial_t \theta + (u \cdot \nabla)\theta + y\partial_x \theta - \eta \Delta \theta = 0, \\
u = -\nabla \perp (-\Delta)^{-1} w, \\
b = -\nabla \perp (-\Delta)^{-1} j,
\end{cases}
\end{align*}
\tag{1.3}
$$

where $Q(u, b) = 2\partial_x b^1(\partial_x u^2 + \partial_y u^1) - 2\partial_x u^1(\partial_x b^2 + \partial_y b^1)$.

The linearization of (1.3) takes the form of

$$
\begin{align*}
\begin{cases}
\partial_t w + y\partial_x w - \partial_x j - \nu \Delta w - \partial_x \theta = 0, \\
\partial_t j + y\partial_x j - \partial_x w - \mu \Delta j - 2\partial_x b^1 = 0, \\
\partial_t \theta + y\partial_x \theta - \eta \Delta \theta = 0, \\
u = -\nabla \perp (-\Delta)^{-1} w, \\
b = -\nabla \perp (-\Delta)^{-1} j, \\
w|_{t=0} = w(0), \quad j|_{t=0} = j(0), \quad \theta|_{t=0} = \theta(0).
\end{cases}
\end{align*}
\tag{1.4}
$$

Now we state our main results. To make the statement precise, we define, for $f = f(x, y)$ with $(x, y) \in T \times \mathbb{R}$ and $k \in \mathbb{Z}$,

$$
f_k(y) := \frac{1}{2\pi} \int_T f(x, y)e^{-ikx} \, dx.
$$

Then, we write $f_\varphi(x, y) = f(x, y) - f_0(y)$. In addition, for two functions $f$ and $g$ and a norm $\| \cdot \|_X$ we write

$$
\|(f, g)\|_X = \sqrt{\|f\|_X^2 + \|g\|_X^2}.
$$

For the linearized system (1.4), we can get the following linear stability.

**Theorem 1.1.** Let $0 < \nu = \mu \leq \eta \leq 1$ and $(w, j, \theta)$ be the solution to the system (1.4) with initial data $(w(0), j(0), \theta(0))$. Then there exist two positive constants $c > 0$ and $C > 0$ such that for any $k \in \mathbb{Z}, t > 0$,

$$
\|\theta_k(t)\|_{L^2_y} \leq C\|\theta_k(0)\|_{L^2_y}e^{-c\eta \frac{1}{2}k^2t},
$$

3
\[ \| (w_k(t), j_k(t)) \|_{L^2_\delta} \leq C \left( \nu^{-2} \| (w_k(0), j_k(0)) \|_{L^2_\delta} + \nu^{-6} (\nu^{-1} |k|)^{\frac{2}{3}} \| \theta_k(0) \|_{L^2_\delta} \right) e^{-c\nu^{\frac{2}{3}} |k|^2 t}. \] (1.5)

Moreover, for \( N > 0 \), there exist \( c_N > 0 \) and \( C_N > 0 \) such that for any \( k \in \mathbb{Z}, t > 0 \),
\[ \| D^N_y \theta_k(t) \|_{L^2_\delta} \leq C N e^{-c_N \nu^{\frac{2}{3}} |k|^2 t} \left( \| D^N_y \theta_k(0) \|_{L^2_\delta} + (\nu^{-1} |k|)^{\frac{N}{2}} \| \theta_k(0) \|_{L^2_\delta} \right), \] (1.6)
\[ \| (D^N_y w_k(t), D^N_y j_k(t)) \|_{L^2_\delta} \leq C N e^{-c_N \nu^{\frac{2}{3}} |k|^2 t} \left( \| (D^N_y w_k(0), D^N_y j_k(0)) \|_{L^2_\delta} + \nu^{-6} (\nu^{-1} |k|)^{\frac{N}{2}} \| D^N_y \theta_k(0) \|_{L^2_\delta} \right) \]
\[ + \nu^{-6N} (\nu^{-1} |k|)^{\frac{N}{2}} (\nu^{-2} \| (w_k(0), j_k(0)) \|_{L^2_\delta} + \nu^{-6} (\nu^{-1} |k|)^{\frac{N}{2}} \| \theta_k(0) \|_{L^2_\delta}). \] (1.7)

The linear stability result in Theorem 1.1 can be converted into the estimate in physical space by introducing the time-dependent operator, for \( t \geq 0 \),
\[ \Lambda^b_k = (1 - \partial_x^2 - (\partial_y + t \partial_x)^2)^{\frac{1}{2}}, \]
or, in terms of its symbol, \( \Lambda^b_k(k, \xi) = (1 + k^2 + (\xi + tk)^2)^{\frac{1}{2}} \). And for any \( b \in \mathbb{R} \), it is easy to check that \( \Lambda^b_k \) commutes with \( \partial_t + y \partial_x \). The estimate in physical space is stated in the following theorem.

**Theorem 1.2.** Let \((w, j, \theta)\) be the solution to (1.4) with initial data \((w(0), j(0), \theta(0))\), and \(0 < \nu = \mu \leq \eta < 1\). There exists a constant \( C > 0 \) such that for \( b \in \mathbb{R} \),
\[ \| (\Lambda^b_k w, \Lambda^b_k j) \|_{L^\infty_t(L^2)} + \nu^{\frac{1}{2}} \| (\nabla \Lambda^b_k w, \nabla \Lambda^b_k j) \|_{L^2_t(L^2)} + \nu^{\frac{1}{2}} \| (\partial_x \Lambda^b_k w, |D_x|^{\frac{1}{2}} \Lambda^b_k j) \|_{L^2_t(L^2)} \]
\[ + \nu^{-4} (\nu \eta)^{-\frac{1}{2}} \left( \| \partial_x \Lambda^{\frac{1}{2}} \theta \|_{L^2_t(L^2)} + \eta \nu^{\frac{1}{2}} \| \nabla |D_x|^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \theta \|_{L^2_t(L^2)} + \eta \nu^{\frac{1}{2}} \| |D_x|^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \theta \|_{L^2_t(L^2)} \right) \]
\[ \leq C \left( \nu^{-2} \| (w(0), j(0)) \|_{H^b} + \nu^{-4} (\nu \eta)^{-\frac{1}{2}} \| |D_x|^{\frac{1}{2}} \theta(0) \|_{H^\beta} \right). \]

The nonlinear stability of the full system (1.3) is stated as follows.

**Theorem 1.3.** Assume \( \nu = \mu = \eta, 0 < \nu \leq 1, b > 1, \beta \geq \frac{11}{3}, \delta \geq \beta + \frac{13}{3}, \alpha \geq \delta - \beta + \frac{14}{3} \) and assume that the initial data \((w(0), j(0), \theta(0))\) satisfies
\[ \| \theta(0) \|_{H^b} \leq \varepsilon \nu^\alpha, \quad \| (w(0), j(0)) \|_{H^b} \leq \varepsilon \nu^\beta, \quad \| |D_x|^{\frac{1}{2}} \theta(0) \|_{H^\beta} \leq \varepsilon \nu^\delta, \]
for some sufficiently small \( \varepsilon > 0 \). Then the solution \((w, j, \theta)\) to the system (1.3) satisfies
\[ \| \Lambda^b_k \theta \|_{L^\infty_t(L^2)} + \nu^{\frac{1}{2}} \| \nabla \Lambda^b_k \theta \|_{L^2_t(L^2)} + \nu^{\frac{1}{2}} \| |D_x|^{\frac{1}{2}} \Lambda^b_k j \|_{L^2_t(L^2)} + \| (-\Delta)^{-\frac{1}{2}} \Lambda^b_k \theta \|_{L^2_t(L^2)} \leq C \varepsilon \nu^\alpha, \]
\[ \| (\Lambda^b_k w, \Lambda^b_k j) \|_{L^\infty_t(L^2)} + \nu^{\frac{1}{2}} \| (\nabla \Lambda^b_k w, \nabla \Lambda^b_k j) \|_{L^2_t(L^2)} + \nu^{\frac{1}{2}} \| (|D_x|^{\frac{1}{2}} \Lambda^b_k w, |D_x|^{\frac{1}{2}} \Lambda^b_k j) \|_{L^2_t(L^2)} \]
\[ + \| (-\Delta)^{-\frac{1}{2}} \Lambda^b_w, (-\Delta)^{-\frac{1}{2}} \Lambda^b_j \|_{L^2_t(L^2)} \leq C \varepsilon \nu^\beta, \]
and
\[ \| |D_x|^{\frac{1}{2}} \Lambda^b_k \theta \|_{L^\infty_t(L^2)} + \nu^{\frac{1}{2}} \| |D_x|^{\frac{1}{2}} \Lambda^b_k \theta \|_{L^2_t(L^2)} + \nu^{\frac{1}{2}} \| |D_x|^{\frac{1}{2}} \Lambda^b_k \theta \|_{L^2_t(L^2)} \]
\[ + \| (-\Delta)^{-\frac{1}{2}} |D_x|^{\frac{1}{2}} \Lambda^b_k \theta \|_{L^2_t(L^2)} \leq C \varepsilon \nu^\delta, \]
where \( C \) is a positive constant.
The stability problem of the full system (1.3) is not trivial and more difficult than that in [16]. Including the similar difficulty in [16], there is an extra difficult term \( \partial_{xy}(-\Delta)^{-1}j \). More precisely, due to the presence of the buoyancy forcing term, the Sobolev norms or even the \( L^2 \)-norm of the velocity field could grow in time if the three linear terms \( y \partial_x w \), \( y \partial_x j \) and \( y \partial_x \theta \) were not included in (1.3). Fortunately, the enhanced dissipation makes the stability of the Couette flow for the system (1.3) possible. Following the idea of [16], some operators are designed to extract the properties of enhanced dissipation. For the stability, we define a function \( \phi_k \) for \( k \neq 0 \) as follows,

\[
\phi_k(\xi) = \begin{cases} 
\frac{6(k^2 + \xi_0^2)}{k^3 - 2\zeta_0^2} - (2 + \pi), & \xi > 0, \\
\frac{6(k^2 + \xi_0^2)}{k^2 + \xi_0^2} - (2 + \pi), & \xi \in [-\xi_0, 0], \\
(4 - \pi)e^{4\xi_0(\xi + \xi_0)}, & \xi \in (-\infty, -\xi_0).
\end{cases}
\]

Here \( \xi_0 \) is a real positive solution of the equation \( \nu \xi_0(k^2 + \xi_0^2) = 96|k| \). It is easy to check that \( \phi_k \in C^1(\mathbb{R}) \). Then we add \( \phi_k(\text{sgn}(k)\xi) \) for \( k \neq 0 \) to the symbol function of the operator constructed in [16] to overcome the term \( \partial_{xy}(-\Delta)^{-1}j \) and establish the new operator \( \mathcal{M} \) for which symbol function satisfies the following properties,

\[
2\nu(\xi^2 + k^2)\mathcal{M}(k, \xi) + k\partial_\xi \mathcal{M}(k, \xi) \geq \nu(\xi^2 + k^2) + \frac{1}{4}\nu \frac{\zeta_0}{|k|} + \frac{1}{\xi^2 + k^2},
\]

\[
2\nu(\xi^2 + k^2)\mathcal{M}(k, \xi) + k\partial_\xi \mathcal{M}(k, \xi) + \mathcal{M}(k, \xi) \frac{4k\xi}{k^2 + \xi^2} \geq \nu(\xi^2 + k^2) + \frac{1}{4}\nu \frac{\zeta_0}{|k|} + \frac{1}{\xi^2 + k^2}.
\]

The upper bound of the new operator \( \mathcal{M} \) is depending on diffusivity. This is why we need bigger \( \alpha, \beta, \delta \) compared with that in [16] in Theorem 1.3.

2 The linear stability

This section is devoted to the proofs of the linear stability stated in Theorem 1.1 and Theorem 1.2. In order to prove the desired linear stability results, we construct a new Fourier multiplier operator to overcome the difficult term \( \partial_{xy}(-\Delta)^{-1}j \) and the details are in the following subsections.

2.1 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1.

Proof. By projecting the equations in (1.4) onto each frequency, we obtain the system in the \( y \)-variable only,

\[
\begin{cases}
\partial_tw_k + ikyw_k - ikj_k + \nu(D_y^2 + k^2)w_k = ik\theta_k, \\
\partial_tj_k + ikyj_k - ikw_k - 2ikb_j^1 + \mu(D_y^2 + k^2)j_k = 0, \\
\partial_t\theta_k + ikyb\theta_k + \eta(D_y^2 + k^2)\theta_k = 0, \\
w_k|_{t=0} = w_k(0), \quad j_k|_{t=0} = \theta_k(0), \quad \theta_k|_{t=0} = \theta_k(0).
\end{cases}
\]
By taking the $L^2_y$-inner product with $\theta_k$ in the third equation of (2.1), we get

$$
\langle \partial_t \theta_k, \theta_k \rangle_{L^2_y} + \langle \theta_k, \partial_t \theta_k \rangle_{L^2_y} = \frac{d}{dt} \langle \theta_k, \theta_k \rangle_{L^2_y} = \frac{d}{dt} \| \theta_k \|^2_{L^2_y},
$$

$$
\langle iky \theta_k, \theta_k \rangle_{L^2_y} + \langle \theta_k, iky \theta_k \rangle_{L^2_y} = \langle iky \theta_k, \theta_k \rangle_{L^2_y} + \langle -iky \theta_k, \theta_k \rangle_{L^2_y} = 0,
$$

$$
\langle \eta(D_y^2 + k^2) \theta_k, \theta_k \rangle_{L^2_y} + \langle \theta_k, \eta(D_y^2 + k^2) \theta_k \rangle_{L^2_y} = 2\eta \| D_y \theta_k \|^2_{L^2_y} + 2\eta k^2 \| \theta_k \|^2_{L^2_y},
$$

which implies that

$$
\frac{1}{2} \frac{d}{dt} \| \theta_k \|^2_{L^2_y} + \eta \| D_y \theta_k \|^2_{L^2_y} + \eta k^2 \| \theta_k \|^2_{L^2_y} = 0. \tag{2.2}
$$

To further the estimates, we apply the Fourier multiplier operator defined in [16]. If $k \in \mathbb{Z}$ and $k \neq 0$, the multiplier $M_k$ is given by

$$
M_k \theta_k := \varphi(\eta^{\frac{1}{2}} |k|^{-\frac{1}{4}} \text{sgn}(k) D_y) \theta_k,
$$

where $\varphi$ is a real-valued, non-decreasing function, and $\varphi \in C^\infty(\mathbb{R})$ satisfies $0 \leq \varphi(x) \leq 1$, $0 \leq \varphi'(x) \leq \frac{1}{4}$ for all $x \in \mathbb{R}$, and $\varphi'(x) = \frac{1}{4}$ for $x \in [-1, 1]$. Clearly, $M_k$ is a self-adjoint and non-negative Fourier multiplier operator. We take the $L^2_y$-inner product of the third equation in (2.1) with $M_k \theta_k$. The following basic identities hold:

$$
2\text{Re} \langle \partial_t \theta_k, M_k \theta_k \rangle_{L^2_y} = \frac{d}{dt} \langle M_k \theta_k, \theta_k \rangle_{L^2_y},
$$

$$
2\text{Re} \langle \eta(D_y^2 + k^2) \theta_k, M_k \theta_k \rangle_{L^2_y} = \langle \eta(D_y^2 + k^2) \theta_k, M_k \theta_k \rangle_{L^2_y} + \langle M_k \theta_k, \eta(D_y^2 + k^2) \theta_k \rangle_{L^2_y}
= \langle \eta(D_y^2 + k^2)M_k \theta_k, \theta_k \rangle_{L^2_y} + \langle \eta(D_y^2 + k^2)M_k \theta_k, \theta_k \rangle_{L^2_y}
= \langle 2\eta(D_y^2 + k^2)M_k \theta_k, \theta_k \rangle_{L^2_y},
$$

$$
2\text{Re} \langle iky \theta_k, M_k \theta_k \rangle_{L^2_y} = \langle [M_k, iky] \theta_k, \theta_k \rangle_{L^2_y},
$$

where in the last equation we have used the fact that $M_k$ is self-adjoint and $iky$ is skew-adjoint.

Here the bracket in $[M_k, iky]$ denotes the standard commutator. Noticing that

$$
\langle [M_k, iky] f, g \rangle_{L^2_y} = \langle M_k f, iky g \rangle_{L^2_y} + \langle iky f, M_k g \rangle_{L^2_y}
= \langle -iky (M_k f), g \rangle_{L^2_y} + \langle M_k (iky f), g \rangle_{L^2_y}
= \langle k \partial_k M_k \tilde{f} + kM_k \partial_k \tilde{f}, g \rangle_{L^2_y} + \langle -kM_k \partial_k \tilde{f}, \tilde{g} \rangle_{L^2_y}
= \langle \langle k \partial_k M_k \rangle (D_y) f, g \rangle_{L^2_y}
= \langle \eta^\frac{1}{2} |k|^{-\frac{1}{4}} \varphi(\eta^\frac{1}{2} |k|^{-\frac{1}{4}} \text{sgn}(k) D_y) f, g \rangle_{L^2_y},
$$

we obtain

$$
\frac{d}{dt} \langle M_k \theta_k, \theta_k \rangle_{L^2_y} + \langle 2\eta(D_y^2 + k^2) \varphi(\eta^\frac{1}{2} |k|^{-\frac{1}{4}} \text{sgn}(k) D_y) \theta_k, \theta_k \rangle_{L^2_y}
+ \langle \eta^\frac{1}{2} |k|^\frac{1}{2} \varphi(\eta^\frac{1}{2} |k|^{-\frac{1}{4}} \text{sgn}(k) D_y) \theta_k, \theta_k \rangle_{L^2_y} = 0.
$$

Together with (2.2), this gives

$$
\frac{d}{dt} \left( \langle M_k \theta_k, \theta_k \rangle_{L^2_y} + \| \theta_k \|^2_{L^2_y} \right) + \left( 2\eta(D_y^2 + k^2) \left( \varphi(\eta^\frac{1}{2} |k|^{-\frac{1}{4}} \text{sgn}(k) D_y) + 1 \right) \right) \langle \theta_k, \theta_k \rangle_{L^2_y} = 0. \tag{2.3}
$$
By the choice of the function $\varphi$, there holds
\[ \eta(t^2 + k^2)(1 + 2\varphi(\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi)) + \eta^\frac{3}{2}|k|\overset{\circ}{\xi} \varphi'(\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi) \geq \frac{1}{4}\eta^\frac{3}{2}|k|\overset{\circ}{t}, \]
for all $k \in \mathbb{Z}$, $k \neq 0, \eta > 0, \xi \in \mathbb{R}$. In fact, when $|\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi| \leq 1$, we have
\[ \varphi'(\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi) = \frac{1}{4}, \]
and the above inequality clearly holds. When $|\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi| > 1$, we have
\[ \frac{1}{4}\eta^\frac{3}{2}|k|\overset{\circ}{t} \leq \frac{1}{4}\eta^\frac{3}{2}|k|\overset{\circ}{t}|\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi|^2 \leq \frac{1}{4}\eta^2. \]
According to
\[
\begin{aligned}
&\left\langle \left(2\eta(D_y^2 + k^2)(\varphi(\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)D_y) + 1) + \eta^\frac{3}{2}|k|\overset{\circ}{t}\varphi'(\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)D_y)\right)\theta_k, \theta_k\right\rangle_{L^2_{\xi}}
= \left\langle \left(\eta(t^2 + k^2)(2\varphi(\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi) + 1) + \eta^\frac{3}{2}|k|\overset{\circ}{t}\varphi'(\eta^\frac{3}{2}|k|^{-\frac{1}{2}}\text{sgn}(k)\xi)\right)\hat{\theta}_k, \hat{\theta}_k\right\rangle_{L^2_{\xi}}
+ \left\langle \eta(t^2 + k^2)\hat{\theta}_k, \hat{\theta}_k\right\rangle_{L^2_{\xi}}
\geq \left\langle \frac{1}{4}\eta^\frac{3}{2}|k|\overset{\circ}{t}\theta_k, \theta_k\right\rangle_{L^2_{\xi}} + \left\langle \eta(t^2 + k^2)\hat{\theta}_k, \hat{\theta}_k\right\rangle_{L^2_{\xi}}
= \frac{1}{4}\eta^\frac{3}{2}|k|\overset{\circ}{t}\left\|\theta_k\right\|_{L^2_{\xi}}^2 + \eta\left\|D_y\theta_k\right\|_{L^2_{\xi}}^2 + \eta k^2\left\|\theta_k\right\|_{L^2_{\xi}}^2,
\end{aligned}
\]

Together with (2.3), we get
\[ \frac{d}{dt}\left\langle \langle M_k\theta_k, \theta_k\rangle_{L^2_{\xi}} + \left\|\theta_k\right\|_{L^2_{\xi}}^2\right\rangle + \frac{1}{4}\eta^\frac{3}{2}|k|\overset{\circ}{t}\left\|\theta_k\right\|_{L^2_{\xi}}^2 + \eta\left\|D_y\theta_k\right\|_{L^2_{\xi}}^2 + \eta k^2\left\|\theta_k\right\|_{L^2_{\xi}}^2 \leq 0. \]

We see
\[
\begin{aligned}
&\frac{d}{dt}\left(\left\|\sqrt{M_k + 1}\theta_k(t)\right\|_{L^2_{\xi}}^2 e^{\frac{3}{2}\eta^\frac{3}{2}|k|\overset{\circ}{t}}\right)
= e^{\frac{3}{2}\eta^\frac{3}{2}|k|\overset{\circ}{t}}\left(\frac{d}{dt}\left(\left\|\sqrt{M_k + 1}\theta_k(t)\right\|_{L^2_{\xi}}^2\right) + \frac{1}{8}\eta^\frac{3}{2}|k|\overset{\circ}{t}\left\|\sqrt{M_k + 1}\theta_k(t)\right\|_{L^2_{\xi}}^2\right)
\leq e^{\frac{3}{2}\eta^\frac{3}{2}|k|\overset{\circ}{t}}\left(\frac{d}{dt}\left(\langle M_k\theta_k, \theta_k\rangle_{L^2_{\xi}} + \left\|\theta_k\right\|_{L^2_{\xi}}^2\right) + \frac{1}{4}\eta^\frac{3}{2}|k|\overset{\circ}{t}\left\|\theta_k\right\|_{L^2_{\xi}}^2\right)
\leq 0.
\end{aligned}
\]
 Integrating (2.4) in $t$ and using the properties of $M_k$, we obtain
\[ \left\|\theta_k(t)\right\|_{L^2_{\xi}} \leq \sqrt{2}\left\|\theta_k(0)\right\|_{L^2_{\xi}} e^{-\frac{3}{2}\eta^\frac{3}{2}|k|\overset{\circ}{t}}. \]

When $k = 0$, for (2.2), we get $\frac{1}{2}\frac{d}{dt}\left\|\theta_0(t)\right\|_{L^2_{\xi}}^2 \leq 0$, which implies that $\left\|\theta_0(t)\right\|_{L^2_{\xi}} \leq \left\|\theta_0(0)\right\|_{L^2_{\xi}}$. Differentiating the third equation in (2.1) with respect to $y$ leads to
\[ \partial_tD^N_y\theta_k + ikyD^N_y\theta_k + kND^{N-1}_y\theta_k + \eta(D^2_y + k^2)D^N_y\theta_k = 0. \]
Taking the $L^2_y$-inner product with $(1 + M_k)D^N_y \theta_k$ gives
\[
\frac{d}{dt} \left( (M_k D^N_y \theta_k, D^N_y \theta_k)_{L^2_y} + \|D^N_y \theta_k\|_{L^2_y}^2 \right) + \frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} \|D^N_y \theta_k\|_{L^2_y}^2
+ \eta \|D^{N+1} \theta_k\|_{L^2_y}^2 + \eta k^2 \|D^N \theta_k\|_{L^2_y}^2
\leq -2 \Re(k N D^N_y \theta_k, (1 + M_k)D^N_y \theta_k)_{L^2_y}
\leq \frac{1}{8} \frac{1}{3} |k|^\frac{4}{3} \|D^N_y \theta_k\|_{L^2_y}^2 + 32 N^2 \eta \frac{1}{3} |k|^\frac{4}{3} \|D^{N-1} \theta_k\|_{L^2_y}^2,
\]
therefore,
\[
\frac{d}{dt} \left( (M_k D^N_y \theta_k, D^N_y \theta_k)_{L^2_y} + \|D^N_y \theta_k\|_{L^2_y}^2 \right) + \frac{1}{8} \frac{1}{3} |k|^\frac{4}{3} \|D^N_y \theta_k\|_{L^2_y}^2
+ \eta \|D^{N+1} \theta_k\|_{L^2_y}^2 + \eta k^2 \|D^N \theta_k\|_{L^2_y}^2
\leq 32 N^2 \eta \frac{1}{3} |k|^\frac{4}{3} \|D^{N-1} \theta_k\|_{L^2_y}^2.
\]

Similarly, we get
\[
\frac{d}{dt} \left( \| \sqrt{M_k} + 1 \xi^N \theta_k(t) \|_{L^2_y}^2 e^{\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} \right) \leq e^{\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} 32 N^2 \eta \frac{1}{3} |k|^\frac{4}{3} \|D^{N-1} \theta_k\|_{L^2_y}^2,
\]
which leads to
\[
\|D^N_y \theta_k(t)\|_{L^2_y}^2 \leq e^{\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} \left( 2 \|D^N_y \theta_k(0)\|_{L^2_y}^2 + 32 N^2 \int_0^t \eta \frac{1}{3} |k|^\frac{4}{3} \|D^{N-1} \theta_k(s)\|_{L^2_y}^2 e^{\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} s} ds \right).
\]

Now we prove the inequality \((L.6)\) by induction. For $N = 1$,
\[
\|D_y \theta_k(t)\|_{L^2_y}^2 \leq e^{-\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} \left( 2 \|D_y \theta_k(0)\|_{L^2_y}^2 + 32 \int_0^t \eta \frac{1}{3} |k|^\frac{4}{3} \|\theta_k(s)\|_{L^2_y}^2 e^{\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} s} ds \right)
\leq e^{-\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} \left( 2 \|D_y \theta_k(0)\|_{L^2_y}^2 + 64 \eta \frac{1}{3} |k|^\frac{4}{3} \|\theta_k(0)\|_{L^2_y}^2 \int_0^t \eta \frac{1}{3} |k|^\frac{4}{3} s ds \right)
\leq e^{-\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} \left( 2 \|D_y \theta_k(0)\|_{L^2_y}^2 + 1024 \eta \frac{1}{3} |k|^\frac{4}{3} \|\theta_k(0)\|_{L^2_y}^2 \right),
\]
which leads to
\[
\|D_y \theta_k(t)\|_{L^2_y} \leq C_1 e^{-\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} \left( \|D_y \theta_k(0)\|_{L^2_y} + (\eta^{-1} |k|) \frac{3}{4} \|\theta_k(0)\|_{L^2_y} \right).
\]
Assume that for $n \leq N$, there exist two positive constants $C_n > 0$ and $c_n > 0$, such that
\[
\|D^n_y \theta_k(t)\|_{L^2_y} \leq C_n e^{-c_n \eta \frac{1}{3} |k|^\frac{4}{3} t} \left( \|D^n_y \theta_k(0)\|_{L^2_y} + (\eta^{-1} |k|) \frac{3}{4} \|\theta_k(0)\|_{L^2_y} \right).
\]
Then for $n = N + 1$, we have
\[
\|D^{N+1}_y \theta_k(t)\|_{L^2_y}^2 \leq e^{-\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} t} \left( 2 \|D^{N+1}_y \theta_k(0)\|_{L^2_y}^2
+ 32 (N + 1)^2 \int_0^t \eta \frac{1}{3} |k|^\frac{4}{3} \|D^N \theta_k(s)\|_{L^2_y}^2 e^{\frac{1}{4} \eta \frac{1}{3} |k|^\frac{4}{3} s} ds \right).
\]
When $\frac{1}{16} - 2c_N < 0$, we have
\[
\begin{aligned}
\int_0^t \eta^{-\frac{1}{3}}|k|^{\frac{2}{3}} \|D_y^N \theta_k(s)\|_{L_y^2}^2 e^{\frac{1}{16} \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} s} ds \\
\leq C_N^2 \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} \left( \|D_y^N \theta_k(0)\|_{L_y^2} + (\eta^{-1} |k|)^{\frac{N}{3}} \|\theta_k(0)\|_{L_y^2} \right)^2 \int_0^t e^{(\frac{1}{16} - 2c_N) \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} s} ds \\
\leq \frac{2C_N^2}{2c_N - \frac{1}{16}} (\eta^{-1} |k|)^{\frac{2}{3}} \left( \|D_y^N \theta_k(0)\|_{L_y^2}^2 + (\eta^{-1} |k|)^{\frac{2N}{3}} \|\theta_k(0)\|_{L_y^2}^2 \right).
\end{aligned}
\]
Because
\[
(\eta^{-1} |k|)^{\frac{2}{3}} \|D_y^N \theta_k(0)\|_{L_y^2}^2 \leq (\eta^{-1} |k|)^{\frac{2}{3}} \|\theta_k(0)\|_{L_y^2}^2 \frac{2N}{N + 1 \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} (\|D_y^N \theta(0)\|_{L_y^2}^2) + \frac{N}{N + 1} \|D_y^N \theta(0)\|_{L_y^2}^2,
\]
we have
\[
\|D_y^{N+1} \theta_k(t)\|_{L_y^2} \leq C_{N+1} e^{-\frac{1}{16} \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \left( \|D_y^{N+1} \theta_k(0)\|_{L_y^2} + (\eta^{-1} |k|)^{\frac{N+1}{3}} \|\theta_k(0)\|_{L_y^2} \right). \tag{2.6}
\]
When $0 \leq \frac{1}{16} - 2c_N < \alpha_N < \frac{1}{16}$, we get
\[
\begin{aligned}
\int_0^t \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} \|D_y^N \theta_k(s)\|_{L_y^2}^2 e^{\frac{1}{16} \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} s} ds \\
\leq C_N^2 \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} \left( \|D_y^N \theta_k(0)\|_{L_y^2} + (\eta^{-1} |k|)^{\frac{N}{3}} \|\theta_k(0)\|_{L_y^2} \right)^2 \int_0^t e^{\alpha_N \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} s} ds \\
\leq 2C_N^2 \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} \left( \|D_y^N \theta_k(0)\|_{L_y^2}^2 + (\eta^{-1} |k|)^{\frac{2N}{3}} \|\theta_k(0)\|_{L_y^2}^2 \right) \frac{e^{\alpha_N \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t}}{\alpha_N \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \\
\leq C_{N+1} \left( (\eta^{-1} |k|)^{\frac{2}{3}} \|D_y^N \theta_k(0)\|_{L_y^2}^2 + (\eta^{-1} |k|)^{\frac{2N+2}{3}} \|\theta_k(0)\|_{L_y^2}^2 \right) e^{\alpha_N \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \\
\leq C_{N+1} \|D_y^{N+1} \theta_k(0)\|_{L_y^2}^2 + (\eta^{-1} |k|)^{\frac{2N+2}{3}} \|\theta_k(0)\|_{L_y^2}^2 \right) e^{\alpha_N \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t},
\end{aligned}
\]
where $\alpha_N$ is a positive constant and $C_{N+1}$ is a generic positive constant which may change from line to line, and we will often do so without any remark. Hence,
\[
\begin{aligned}
\|D_y^{N+1} \theta_k(t)\|_{L_y^2}^2 \leq e^{-\frac{1}{16} \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \left( 2 \|D_y^{N+1} \theta_k(0)\|_{L_y^2}^2 \\
+ 32(N + 1)^2 \int_0^t \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} \|D_y^N \theta_k(s)\|_{L_y^2}^2 e^{\frac{1}{16} \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} s} ds \right) \\
\leq 2e^{-\frac{1}{16} \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \|D_y^{N+1} \theta_k(0)\|_{L_y^2}^2 + 32(N + 1)^2 C_{N+1} \left( \|D_y^{N+1} \theta_k(0)\|_{L_y^2}^2 \\
+ (\eta^{-1} |k|)^{\frac{2N+2}{3}} \|\theta_k(0)\|_{L_y^2}^2 \right) e^{(\alpha_N - \frac{1}{16}) \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \\
\leq C_{N+1} e^{(\alpha_N - \frac{1}{16}) \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \left( \|D_y^{N+1} \theta_k(0)\|_{L_y^2}^2 + (\eta^{-1} |k|)^{\frac{2N+2}{3}} \|\theta_k(0)\|_{L_y^2}^2 \right). \tag{2.6}
\end{aligned}
\]
By the estimate (2.6) and letting $c_{N+1} = \frac{1}{32} - \frac{1}{2} \alpha_N$, we can show
\[
\|D_y^{N+1} \theta_k(t)\|_{L_y^2} \leq C_{N+1} e^{-c_{N+1} \eta^{-\frac{1}{3}} |k|^{\frac{2}{3}} t} \left( \|D_y^{N+1} \theta_k(0)\|_{L_y^2} + (\eta^{-1} |k|)^{\frac{N+1}{3}} \|\theta_k(0)\|_{L_y^2} \right).
Now, we prove the inequality (1.5). For \( w_k \) and \( j_k \), we have
\[
\frac{d}{dt} \| w_k \|^2_{L_y^2} + (2 \nu (D_y^2 + k^2) w_k, w_k)_{L_y^2} = 2 \text{Re} \langle ik \theta_k, w_k \rangle_{L_y^2} + 2 \text{Re} \langle ik j_k, w_k \rangle_{L_y^2},
\]
\[
\frac{d}{dt} \| j_k \|^2_{L_y^2} + (2 \mu (D_y^2 + k^2) j_k, j_k)_{L_y^2} = 2 \text{Re} \langle ik w_k, j_k \rangle_{L_y^2} + 2 \text{Re} \langle 2ik \partial_y (k^2 + D_y^2)^{-1} j_k, j_k \rangle_{L_y^2}.
\]
And we define function \( \phi_k \) for \( k \neq 0 \) as follows,
\[
\phi_k(\xi) = \begin{cases} 
\frac{6(k^2 + \xi_0^2)^2}{k^2 + \xi^2} - (2 + \pi), & \xi > 0, \\
\frac{6(k^2 + \xi_0^2)^2}{k^2 + \xi^2} - (2 + \pi), & \xi \in [-\xi_0, 0], \\
(4 - \pi) e^{(4-\pi)(k^2 + \xi_0^2)/(k^2 + \xi^2)}, & \xi \in (-\infty, -\xi_0),
\end{cases}
\]
where \( \xi_0 \) is a real positive solution of the equation \( \nu \xi_0 (k^2 + \xi_0^2) = 96|k| \). It is easy to check that \( \phi_k(\xi) \in C^1(\mathbb{R}) \) for \( k \neq 0 \). Therefore, we get \( \xi_0 \leq 96\nu^{-1}|k|^{-1} \leq 96\nu^{-1} \), which implies that \( 0 < \phi_k(\xi) \leq C \nu^{-4} \) and \( 0 \leq \phi'_k(\xi) \leq C \frac{\nu^{-3}}{|k|}, \xi \in \mathbb{R}, \) where \( C \) is a constant positive. And we define
\[
M' = \varphi(\nu^{\frac{1}{4}} |k|^{-\frac{3}{4}} \text{sgn}(k) D_y) + \phi_k(\text{sgn}(k) D_y), \quad k \neq 0, \quad M' = 0.
\]
Then we can make a conclusion
\[
\nu (\xi^2 + k^2) (1 + 2M'_k(\xi)) + k \partial_x M'_k(\xi) \geq \frac{1}{4} \nu^\frac{3}{2} |k|^2,
\]
\[
\nu (\xi^2 + k^2) (1 + 2M'_k(\xi)) + k \partial_x M'_k(\xi) + (1 + M'_k(\xi)) \frac{4k \xi}{k^2 + \xi^2} \geq \frac{1}{4} \nu^\frac{3}{2} |k|^2,
\]
with \( k \in \mathbb{Z} \) and \( k \neq 0 \). Multiplying the \( w_k \) equation by \( (1 + M'_k) w_k \) and the \( j_k \) equation by \( (1 + M'_k) j_k \), we can get
\[
\frac{d}{dt} (\langle 1 + M'_k \rangle w_k, w_k \rangle_{L_y^2} + \nu \| D_y w_k \|^2_{L_y^2} + \nu k^2 \| w_k \|^2_{L_y^2} + \frac{1}{4} \nu^\frac{3}{2} |k|^\frac{3}{2} \| w_k \|^2_{L_y^2})
\]
\[
+ \frac{d}{dt} (\langle 1 + M'_k \rangle j_k, j_k \rangle_{L_y^2} + \nu \| D_y j_k \|^2_{L_y^2} + \nu k^2 \| j_k \|^2_{L_y^2} + \frac{1}{4} \nu^\frac{3}{2} |k|^\frac{3}{2} \| j_k \|^2_{L_y^2})
\]
\[
\leq 2 \text{Re} \langle ik \theta_k, (1 + M'_k) w_k \rangle_{L_y^2}.
\]
Applying Young’s inequality to the right-hand side and removing some terms on the left-hand side yields
\[
\frac{d}{dt} \left( \langle 1 + M'_k \rangle w_k, w_k \rangle_{L_y^2} + \langle 1 + M'_k \rangle j_k, j_k \rangle_{L_y^2} \right) + \frac{1}{8} \nu^\frac{3}{2} |k|^\frac{3}{2} \| (w_k, j_k) \|^2_{L_y^2}
\]
\[
\leq C \nu^{-\frac{3}{8}} |k|^\frac{3}{4} \| \theta_k \|^2_{L_y^2},
\]
which gives that
\[
\frac{d}{dt} \left( \| \sqrt{1 + M'_k} w_k \|^2_{L_y^2} + \| \sqrt{1 + M'_k} j_k \|^2_{L_y^2} \right) e^{\frac{1}{8} \nu^{\frac{3}{2}} |k|^\frac{3}{2} t}
\]
\[
\leq e^{\frac{1}{8} \nu^{\frac{3}{2}} |k|^\frac{3}{2} t} \left( \frac{d}{dt} \left( \| \sqrt{1 + M'_k} w_k \|^2_{L_y^2} + \| \sqrt{1 + M'_k} j_k \|^2_{L_y^2} \right) + \frac{1}{8} \nu^\frac{3}{2} |k|^\frac{3}{2} \| (w_k, j_k) \|^2_{L_y^2} \right)
\]
\[
\leq C e^{\frac{1}{8} \nu^{\frac{3}{2}} |k|^\frac{3}{2} t} \nu^{-\frac{3}{8}} |k|^\frac{3}{4} \| \theta_k \|^2_{L_y^2}.
\]
Integrating in $t$ and using the inequality (2.5), we obtain
\[
\| (w_k(t), j_k(t)) \|_{L^2_y}^2 \leq C^2 \left( \nu^{-4} \| (w_k(0), j_k(0)) \|_{L^2_y}^2 + \nu^{-12} (\nu^{-1} |k|) \frac{4}{3} \| \theta_k(0) \|_{L^2_y}^2 \right) e^{-2c \frac{3}{4} |k| \frac{4}{3} t}.
\]

Differentiating the $w_k$ equation and the $j_k$ equation in (2.1) with respect to $y$, respectively, leads to
\[
\begin{align*}
\partial_t D_y^N w_k + i k y D_y^N w_k + k N D_y^N - 1 w_k + \nu (D_y^2 + k^2) D_y^N w_k &= i k D_y^N \theta_k + i k D_y^N j_k, \\
\partial_t D_y^N j_k + i k y D_y^N j_k + k N D_y^N - 1 j_k + \nu (D_y^2 + k^2) D_y^N j_k - 2 i k D_y^N b_k &= i k D_y^N w_k.
\end{align*}
\]

Taking the $L^2_y$-inner product with $(1 + M_k') D_y^N w_k$ and $(1 + M_k') D_y^N j_k$, we get
\[
\begin{align*}
\frac{d}{dt} \left( \langle (1 + M_k') D_y^N w_k, D_y^N w_k \rangle_{L^2_y} + \langle (1 + M_k') D_y^N j_k, D_y^N j_k \rangle_{L^2_y} \right) &+ \frac{1}{4} \nu \frac{4}{3} |k| \frac{4}{3} \| (D_y^N w_k, D_y^N j_k) \|_{L^2_y}^2 \\
&\leq -2 \text{Re} \langle k N D_y^N - 1 w_k, (1 + M_k') D_y^N w_k \rangle_{L^2_y} - 2 \text{Re} \langle k N D_y^N - 1 j_k, (1 + M_k') D_y^N j_k \rangle_{L^2_y} \\
&\quad + 2 \text{Re} \langle i k D_y^N \theta_k, (1 + M_k') D_y^N w_k \rangle_{L^2_y} \\
&\leq \frac{1}{16} \nu \frac{4}{3} |k| \frac{4}{3} \| (D_y^N w_k, D_y^N j_k) \|_{L^2_y}^2 + C N^2 \nu^{-\frac{4}{3} - 8} |k| \frac{4}{3} \| (D_y^N - 1 w_k, D_y^N - 1 j_k) \|_{L^2_y}^2 \\
&\quad + \frac{1}{16} \nu \frac{4}{3} |k| \frac{4}{3} \| D_y^N w_k \|_{L^2_y}^2 + C \nu^{-\frac{4}{3} - 8} |k| \frac{4}{3} \| D_y^N \theta_k \|_{L^2_y}^2,
\end{align*}
\]

which gives that
\[
\begin{align*}
\frac{d}{dt} \left( \langle (1 + M_k') D_y^N w_k, D_y^N w_k \rangle_{L^2_y} + \langle (1 + M_k') D_y^N j_k, D_y^N j_k \rangle_{L^2_y} \right) &+ \frac{1}{8} \nu \frac{4}{3} |k| \frac{4}{3} \| (D_y^N w_k, D_y^N j_k) \|_{L^2_y}^2 \\
&\leq C N^2 \nu^{-\frac{4}{3} - 8} |k| \frac{4}{3} \| (D_y^N - 1 w_k, D_y^N - 1 j_k) \|_{L^2_y}^2 + C \nu^{-\frac{4}{3} - 8} |k| \frac{4}{3} \| D_y^N \theta_k \|_{L^2_y}^2.
\end{align*}
\]

Then we have
\[
\begin{align*}
\| D_y^N w_k(t) \|_{L^2_y}^2 + \| D_y^N j_k(t) \|_{L^2_y}^2 &\leq C \nu^{-4} \| (D_y^N w_k(0), D_y^N j_k(0)) \|_{L^2_y}^2 e^{-\frac{1}{8 (1 + M_k')} \nu \frac{4}{3} |k| \frac{4}{3} t} \\
&\quad + \int_0^t C N^2 \nu^{-\frac{4}{3} - 8} |k| \frac{4}{3} \| (D_y^N - 1 w_k(s), D_y^N - 1 j_k(s)) \|_{L^2_y}^2 e^{-\frac{1}{8 (1 + M_k')} \nu \frac{4}{3} |k| \frac{4}{3} (t-s)} ds \\
&\quad + \int_0^t C \nu^{-\frac{4}{3} - 8} |k| \frac{4}{3} \| D_y^N \theta_k(s) \|_{L^2_y}^2 e^{-\frac{1}{8 (1 + M_k')} \nu \frac{4}{3} |k| \frac{4}{3} (t-s)} ds.
\end{align*}
\]
Next, we prove the inequality (1.7) by induction. For $N = 1$,

$$\|D_y w_k(t)\|_{L_y^2}^2 + \|D_y j_k(t)\|_{L_y^2}^2 \leq \frac{1}{s(1+M_B)} e^{-\frac{1}{s(1+M_B)} \|k\|_y^2 t}$$

$$\leq C_1 \nu^{-4}(\|D_y w_k(0), D_y j_k(0)\|_{L_y^2}^2 e^{-c_1 \nu \frac{1}{3} |k|_y^2 t}$$

$$\leq C_1 \nu^{-4}(\|D_y w_k(0), D_y j_k(0)\|_{L_y^2}^2 e^{-c_1 \nu \frac{1}{3} |k|_y^2 t}$$

$$\leq C_1 \nu^{-4}(\|D_y w_k(0), D_y j_k(0)\|_{L_y^2}^2 e^{-c_1 \nu \frac{1}{3} |k|_y^2 t}$$

where $C_1 > 0$ and $c_1 > 0$. That is,

$$\|D_y w_k(t)\|_{L_y^2} + \|D_y j_k(t)\|_{L_y^2} \leq C_1 \nu^{-4}(\|D_y w_k(0), D_y j_k(0)\|_{L_y^2}^2 e^{-c_1 \nu \frac{1}{3} |k|_y^2 t}$$

Now assume that for $n \leq N$, there exist two positive constants $C_n$ and $c_n$ such that

$$\|D_y^n w_k(t)\|_{L_y^2}^2 + \|D_y^n j_k(t)\|_{L_y^2}^2 \leq C_n \nu^{-4}(\|D_y^n w_k(0), D_y^n j_k(0)\|_{L_y^2}^2 e^{-c_n \nu \frac{1}{3} |k|_y^2 t}$$

$$\leq C_n \nu^{-4}(\|D_y^n w_k(0), D_y^n j_k(0)\|_{L_y^2}^2 e^{-c_n \nu \frac{1}{3} |k|_y^2 t}$$

Then for $n = N + 1$, we have

$$\|D_y^{N+1} w_k(t)\|_{L_y^2}^2 + \|D_y^{N+1} j_k(t)\|_{L_y^2}^2 \leq C \nu^{-4}(\|D_y^{N+1} w_k(0), D_y^{N+1} j_k(0)\|_{L_y^2}^2 e^{-c_n \nu \frac{1}{3} |k|_y^2 t}$$

$$\leq C \nu^{-4}(\|D_y^{N+1} w_k(0), D_y^{N+1} j_k(0)\|_{L_y^2}^2 e^{-c_n \nu \frac{1}{3} |k|_y^2 t}$$

$$\leq C \nu^{-4}(\|D_y^{N+1} w_k(0), D_y^{N+1} j_k(0)\|_{L_y^2}^2 e^{-c_n \nu \frac{1}{3} |k|_y^2 t}$$

$1 + I_2$.  

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From the induction assumption, we have

\[
I_1 = \int_0^t C(N + 1)^2 \nu^{-\frac{1}{3}} \rho\left|D_y^{N} w_k(s), D_y^{N} j_k(s)\right|^2 L_y^2 e^{-\frac{1}{8(1 + A_k^2)} \nu^\frac{2}{3} \rho(t-s)} ds
\]

\[
\leq C_{N+1}^2 e^{-2\beta_N \nu \frac{12}{13} \rho |k|^2 t \nu^{-1} |k|^2} \nu^{-12} \left( \nu^{-2} \left|D_y^{N} w_k(0), D_y^{N} j_k(0)\right|^2 L_y^2 + \nu^{-12} (\nu^{-1} |k|^2) \left|D_y^{N} \theta_k(0)\right|^2 L_y^2 \right)
\]

\[
+ C_{N+1}^2 e^{-2\beta_N \nu \frac{12}{13} \rho |k|^2 t \nu^{-1} |k|^2} \nu^{-12(\nu^{-1} |k|^2)} 2(\nu^{-1} |k|^2) \nu^{-12(\nu^{-1} |k|^2)} 2(\nu^{-1} |k|^2) \|
\]

\[
\leq C_{N+1}^2 e^{-2\beta_N \nu \frac{12}{13} \rho |k|^2 t \nu^{-1} |k|^2} \nu^{-12} \left( \nu^{-2} \left|D_y^{N} w_k(0), D_y^{N} j_k(0)\right|^2 L_y^2 + \nu^{-12} (\nu^{-1} |k|^2) \left|D_y^{N} \theta_k(0)\right|^2 L_y^2 \right),
\]

with $\beta_N > 0$. Note that

\[
\nu^{-2} |(\nu^{-1} |k|^2) \left|D_y^{N} w_k(0)\right|^2 L_y^2 \leq \nu^{-12} (\nu^{-1} |k|^2) \left|D_y^{N} w_k(0)\right|^2 L_y^2
\]

\[
\nu^{-12} (\nu^{-1} |k|^2) \left|D_y^{N} j_k(0)\right|^2 L_y^2 \leq \nu^{-12} (\nu^{-1} |k|^2) \left|D_y^{N} j_k(0)\right|^2 L_y^2
\]

\[
\nu^{-12} (\nu^{-1} |k|^2) \left|D_y^{N} \theta_k(0)\right|^2 L_y^2 \leq \nu^{-12} (\nu^{-1} |k|^2) \left|D_y^{N} \theta_k(0)\right|^2 L_y^2
\]

For $I_2$, we have

\[
I_2 = \int_0^t C\nu^{-\frac{1}{3}} \rho |k|^2 \left|D_y^{N+1} \theta_k(s)\right|^2 L_y^2 e^{-\frac{1}{8(1 + A_k^2)} \nu^\frac{2}{3} \rho(t-s)} ds
\]

\[
\leq C_{N+1}^2 e^{-2\gamma_N \nu \frac{12}{13} \rho |k|^2 t \nu^{-1} |k|^2} \left( \nu^2 \left|D_y^{N+1} \theta_k(0)\right|^2 L_y^2 + \nu^{-6} (\nu^{-1} |k|^2) \left|D_y^{N+1} \theta_k(0)\right|^2 L_y^2 \right),
\]

where the positive constants $\beta_N$ and $\gamma_N$ are dependent of $N$. Thus we have

\[
\left|D_y^{N+1} w_k(t), D_y^{N+1} j_k(t)\right| L_y^2
\]

\[
\leq C_{N+1} e^{-c_N + \nu \frac{12}{13} \rho |k|^2 t} \left( \nu^{-2} \left|D_y^{N+1} w_k(0), D_y^{N+1} j_k(0)\right|^2 L_y^2 + \nu^{-6} (\nu^{-1} |k|^2) \left|D_y^{N+1} \theta_k(0)\right|^2 L_y^2 \right),
\]

\[
+ \nu^{-6+\nu^{-1} |k|^2} \left( \nu^{-2} \left|D_y^{N+1} \theta_k(0)\right|^2 L_y^2 + \nu^{-6} (\nu^{-1} |k|^2) \left|D_y^{N+1} \theta_k(0)\right|^2 L_y^2 \right),
\]

with $c_{N+1} = \min\{\beta_N, \gamma_N\}.$

This completes the proof. \hfill \Box
2.2 Proof of Theorem 1.2

In this subsection, we prove Theorem 1.2, which is a consequence of Theorem 1.1.

Proof. For any $b \in \mathbb{R}$, we apply $\Lambda_k^b$ to the equations in (2.1) to obtain
\[
\begin{aligned}
\partial_t \Lambda_k^b w_k + iKy \Lambda_k^b w_k + \nu(D_y^2 + k^2)\Lambda_k^b w_k &= ik\Lambda_k^b \theta_k + ik\Lambda_k^b \theta_k, \\
\partial_t \Lambda_k^b j_k + iKy \Lambda_k^b j_k + \nu(D_y^2 + k^2)\Lambda_k^b j_k - 2iK\Lambda_k^b b_k &= ik\Lambda_k^b w_k.
\end{aligned}
\]

We multiply the above equations by $(1 + M_k^b)\Lambda_k^b w_k$ and $(1 + M_k^b)\Lambda_k^b j_k$, respectively, and integrate over $\mathbb{R}$. Then using the properties of $M_k$, we can show that
\[
\begin{aligned}
\frac{d}{dt}(1 + M_k^b)\Lambda_k^b w_k, \Lambda_k^b w_k)_{L^2_y} + \nu\|D_y \Lambda_k^b w_k\|_{L^2_y}^2 + \nu k^2\|\Lambda_k^b w_k\|_{L^2_y}^2 + \frac{1}{8} \nu \frac{1}{k} \|\Lambda_k^b w_k\|_{L^2_y}^2 \\
+ \frac{d}{dt}(1 + M_k^b)\Lambda_k^b j_k, \Lambda_k^b j_k)_{L^2_y} + \nu\|D_y \Lambda_k^b j_k\|_{L^2_y}^2 + \nu k^2\|\Lambda_k^b j_k\|_{L^2_y}^2 + \frac{1}{8} \nu \frac{1}{k} \|\Lambda_k^b j_k\|_{L^2_y}^2 \\
\leq C\nu^{-\frac{1}{3} - \delta} ||\Lambda_k^b \theta_k||_{L^2_y}^2.
\end{aligned}
\]

Summing over $k$ and integrating in $t$ yields
\[
\begin{aligned}
\|\Lambda_k^b w(t)\|_{L^2_y}^2 - \sum \langle (1 + M_k^b)\Lambda_k^b w_k, \Lambda_k^b w_k \rangle_{L^2_y} + \nu \int_0^t \sum \|D_y \Lambda_k^b w_k(t)\|_{L^2_y}^2 dt \\
+ \frac{d}{dt}(1 + M_k^b)\Lambda_k^b w_k, \Lambda_k^b w_k)_{L^2_y} + \nu\|D_y \Lambda_k^b w_k\|_{L^2_y}^2 + \nu k^2\|\Lambda_k^b w_k\|_{L^2_y}^2 + \frac{1}{8} \nu \frac{1}{k} \|\Lambda_k^b w_k\|_{L^2_y}^2 \\
+ \frac{d}{dt}(1 + M_k^b)\Lambda_k^b j_k, \Lambda_k^b j_k)_{L^2_y} + \nu\|D_y \Lambda_k^b j_k\|_{L^2_y}^2 + \nu k^2\|\Lambda_k^b j_k\|_{L^2_y}^2 + \frac{1}{8} \nu \frac{1}{k} \|\Lambda_k^b j_k\|_{L^2_y}^2 \\
\leq C\nu^{-\frac{1}{3} - \delta} \int_0^t \sum \|\Lambda_k^b\theta_k\|_{L^2_y}^2 dt.
\end{aligned}
\]

Thus we have
\[
\|\Lambda_k^b w(t)\|_{L^\infty_y(L^2_x)} + \|\Lambda_k^b j(t)\|_{L^\infty_x(L^2_y)} + \nu \hat{\psi}(\|\nabla \Lambda_k^b w\|_{L^2_y(L^2_x)} + \|\nabla \Lambda_k^b j\|_{L^2_y(L^2_x)}) \\
+ \nu \hat{\psi}(\|D_x \frac{1}{2} \Lambda_k^b w\|_{L^2_y(L^2_x)} + \|D_x \frac{1}{2} \Lambda_k^b j\|_{L^2_y(L^2_x)}) \\
\leq C(\nu^{-\frac{1}{3} - \delta} \|D_x \frac{1}{2} \psi \sigma(0)\|_{H^b} + \nu^{-2}\|w(0)\|_{H^b} + \nu^{-2}\|j(0)\|_{H^b}).
\]

Similarly for $|D_x|\frac{1}{2} \Lambda_k^b \theta$, we have
\[
\|\Lambda_k^b |D_x|\frac{1}{2} \psi \theta(t)\|_{L^\infty_y(L^2_x)} + \eta \hat{\psi}(\|\nabla |D_x|\frac{1}{2} \psi \theta\|_{L^2_y(L^2_x)} + \|\nabla |D_x|\frac{1}{2} \psi \theta\|_{L^2_y(L^2_x)}) \\
+ \|D_x|\frac{1}{2} \psi \theta\|_{L^2_y(L^2_x)}\|D_x|\frac{1}{2} \psi \theta\|_{L^2_y(L^2_x)} \\
\leq C(\nu^{-\frac{1}{3} - \delta} \|D_x \frac{1}{2} \psi \theta(0)\|_{H^b} + \nu^{-2}\|w(0)\|_{H^b} + \nu^{-2}\|j(0)\|_{H^b}).
\]

According to (2.7) and (2.8), we get
\[
\begin{aligned}
\|\Lambda_k^b w(t)\|_{L^\infty_y(L^2_x)} + \nu \hat{\psi}(\|\nabla \Lambda_k^b w\|_{L^2_y(L^2_x)} + \|\nabla \Lambda_k^b j\|_{L^2_y(L^2_x)}) \\
+ \|\Lambda_k^b j(t)\|_{L^\infty_y(L^2_x)} + \nu \hat{\psi}(\|\nabla \Lambda_k^b j\|_{L^2_y(L^2_x)} + \|\nabla \Lambda_k^b j\|_{L^2_y(L^2_x)}) \\
\leq C(\nu^{-\frac{1}{3} - \delta} \|D_x \frac{1}{2} \psi \theta(0)\|_{H^b} + \nu^{-2}\|w(0)\|_{H^b} + \nu^{-2}\|j(0)\|_{H^b} + \nu^{-4}(\nu, \eta)^{-\frac{1}{2}} \|D_x \frac{1}{2} \psi \theta(0)\|_{H^b}).
\end{aligned}
\]
which concludes the proof.

\[\square\]

## 3 The nonlinear stability

For the proof of the nonlinear stability, we recall the time-dependent elliptic operator which is defined in introduction. For \( t \geq 0 \),

\[ \Lambda^2_t = 1 - \partial_x^2 - (\partial_y + t\partial_x)^2, \]

or, in terms of its symbol, \( \Lambda^2_t(k, \xi) = 1 + k^2 + (\xi + tk)^2 \). And for any \( b \in \mathbb{R} \), \( \Lambda^b_t(k, \xi) = (1 + k^2 + (\xi + tk)^2)^{\frac{b}{2}} \) and the operator \( \Lambda^b_t \) satisfies the following basic properties, which can be found in \([16]\).

**Lemma 3.1.** The operator \( \Lambda^b_t \) satisfies the following properties

1. For any \( b \in \mathbb{R} \), \( \Lambda^b_t \) commutes with \( \partial_t + y\partial_x \), namely
   \[ \Lambda^b_t(\partial_t + y\partial_x) = (\partial_t + y\partial_x)\Lambda^b_t. \]

2. For any \( b > 0 \),
   \[ \|\Lambda^b_t(fg)\|_{L^2} \leq \|f\|_{L^\infty} \|\Lambda^b_t g\|_{L^2} + \|g\|_{L^\infty} \|\Lambda^b_t f\|_{L^2}. \]
   Moreover, for \( b > 1 \), we have
   \[ \|f(t)\|_{L^\infty(\Omega)} \leq C\|\hat{f}(t)\|_{L^1} \leq C\|\Lambda^b_t f(t)\|_{L^2(\Omega)}. \]

   And consequently,
   \[ \|\Lambda^b_t(fg)\|_{L^2} \leq C\|\Lambda^b_t f\|_{L^2} \|\Lambda^b_t g\|_{L^2}. \]

The framework is the bootstrap argument \([34]\), which is stated as follows.

**Lemma 3.2** (Abstract bootstrap principle). Let \( I \) be a time interval, and for each \( t \in I \) suppose we have two statements, that is, “hypothesis” \( H(t) \) and “conclusion” \( C(t) \). Suppose we can verify the following four assertions:

1. (Hypothesis implies conclusion) If \( H(t) \) is true for some time \( t \in I \), then \( C(t) \) is also true for that time \( t \).

2. (Conclusion is stronger than hypothesis) If \( C(t) \) is true for some time \( t \in I \), then \( H(t') \) is true for all \( t' \in I \) in a neighbourhood of \( t \).

3. (Conclusion is closed) If \( t_1, t_2, \ldots \) is a sequence of times in \( I \) which converges to another time \( t \in I \), and \( C(t_n) \) is true for all \( t_n \), then \( C(t) \) is true.

4. (Base case) \( H(t) \) is true for at least one time \( t \in I \).

Then \( C(t) \) is true for all \( t \in I \).
In order to use abstract bootstrap principle to prove Theorem 1.3, we define $I = [0, \infty]$, the statement of $H(T)$ as an estimation less than or equal to $C\varepsilon \nu^\alpha$ and the statement of $C(T)$ as the same estimation less than or equal to $\frac{1}{2}C\varepsilon \nu^\alpha$. According to non-decreasing estimation for time $t$ it is easy to check the assertion 3. And when $H(t)$ is true on $t = 0$, the remaining task is to prove $H(T) \Rightarrow C(T)$ based on the priori bounds. Then we prove that $C(t)$ is true for all $t \in [0, \infty]$.

Let $\nu = \eta = \mu$ in (1.3). Invoking the properties of $\Lambda^b_1$ in Lemma 3.1, we have

$$
\begin{cases}
\partial_t \Lambda^b_k w + y \partial_y \Lambda^b_k w - \nu \Delta \Lambda^b_k w + \Lambda^b_k ((u \cdot \nabla) w) - \Lambda^b_k ((b \cdot \nabla) j) = \partial_x \Lambda^b_k j + \partial_y \Lambda^b_k \theta, \\
\partial_t \Lambda^b_k j + y \partial_y \Lambda^b_k j - \nu \Delta \Lambda^b_k j + \Lambda^b_k ((u \cdot \nabla) j) = 0, \\
\partial_t \Lambda^b_k \theta + y \partial_y \Lambda^b_k \theta - \nu \Delta \Lambda^b_k \theta + \Lambda^b_k ((u \cdot \nabla) \theta) = 0, \\
u = -\nabla^\perp (-\Delta)^{-1} w, \\
b = -\nabla^\perp (-\Delta)^{-1} j,
\end{cases}
$$

where $Q(\nabla u, \nabla b) = 2\partial_y b (\partial_x u^2 + \partial_y u^1) - 2\partial_x u (\partial_x b^2 + \partial_y b^1)$.

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3** Let $\varphi$ be a real-valued, non-decreasing function, and $\varphi \in C^{\infty}(\mathbb{R})$ satisfies $0 \leq \varphi(x) \leq 1$, $0 \leq \varphi'(x) \leq \frac{1}{2}$ for all $x \in \mathbb{R}$ and $\varphi'(x) = \frac{1}{2}$ for $x \in [-1, 1]$. For $k \neq 0$, we choose the function $\phi_k$ as follows,

$$
\phi_k(\xi) = \begin{cases} 
\frac{6(k^2 + \xi_0^2)^2}{k^4} - (2 + \pi), & \xi > 0, \\
\frac{6(k^2 + \xi_0^2)^2}{k^4} - (2 + \pi), & \xi \in [-\xi_0, 0], \\
(4 - \pi)e^{(4 - \pi)(k^2 + \xi_0^2)}, & \xi \in (-\infty, -\xi_0),
\end{cases}
$$

where $\xi_0$ is a real positive solution of the equation $\nu \xi_0(k^2 + \xi_0^2) = 96|k|$. $\phi_k(\xi) \in C^1(\mathbb{R})$ for $k \neq 0$, and $0 < \phi_k(\xi) \leq C
\nu^{-4}$, $0 \leq \phi_k'(\xi) \leq C\nu^{-3}$, $\xi \in \mathbb{R}$ where $C$ is a positive constant. The Fourier multiplier operator $\mathcal{M}$ employed here is defined as $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + 1$ with $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ given by

$$
\begin{align*}
\mathcal{M}_1(k, \xi) &= \varphi(\nu^\frac{1}{4}|k|^{-\frac{1}{2}} \text{sgn}(k)\xi), \quad k \neq 0, \\
\mathcal{M}_2(k, \xi) &= \phi_k(\text{sgn}(k)\xi), \quad k \neq 0, \\
\mathcal{M}_3(k, \xi) &= \frac{1}{k^2} \left( \arctan \frac{\xi}{k} + \frac{\pi}{2} \right), \quad k \neq 0, \\
\mathcal{M}_1(0, \xi) &= \mathcal{M}_2(0, \xi) = \mathcal{M}_3(0, \xi) = 0.
\end{align*}
$$

Then $\mathcal{M}$ is a self-adjoint Fourier multiplier and verifies that

$$
1 \leq \mathcal{M} \leq (C + 6)\nu^{-4}.
$$

Finally we recall the projectors onto the horizontal zeroth mode and the non-zeroth modes,

$$
\begin{align*}
f_0(y) &:= (P_0 f)(y) = \frac{1}{2\pi} \int_T f(x, y) dx, \\
f_\neq(x, y) &:= (P_\neq f)(x, y) = f(x, y) - (P_0 f)(x, y).
\end{align*}
$$

(3.2)
Then we have
\[(f_\varphi)_k(y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) e^{-ikx} \, dx = f_k(y), \text{ for } k \neq 0, \quad (f_\varphi)_0(y) = 0.\]

Multiplying the equations (3.1) by $\mathcal{M} \lambda^b_k w$, $\mathcal{M} \lambda^b_j$ and $\mathcal{M} \lambda^b_\theta$, respectively, and integrating over $\mathbb{T} \times \mathbb{R}$, we can prove that
\[
\frac{d}{dt} \|\sqrt{\mathcal{M}} \lambda^b_k w\|^2_{L^2} + 2\nu \|\nabla \sqrt{\mathcal{M}} \lambda^b_k w\|^2_{L^2} + \langle (k \partial_\xi \mathcal{M})(D) \lambda^b_k, \lambda^b_k \rangle_{L^2} = -2\langle \lambda^b_k (u \cdot \nabla \theta), \mathcal{M} \lambda^b_k \theta \rangle_{L^2},
\]
\[
\frac{d}{dt} \|\sqrt{\mathcal{M}} \lambda^b_j \|^2_{L^2} + 2\nu \|\nabla \sqrt{\mathcal{M}} \lambda^b_j \|^2_{L^2} + \langle (k \partial_\xi \mathcal{M})(D) \lambda^b_j, \lambda^b_j \rangle_{L^2} = -2\langle \lambda^b_j (u \cdot \nabla w), \mathcal{M} \lambda^b_j \rangle_{L^2} + 2\langle \lambda^b_j \cdot \nabla \lambda^b_j, \mathcal{M} \lambda^b_j \rangle_{L^2},
\]
\[
\frac{d}{dt} \|\sqrt{\mathcal{M}} |D_x|^{\frac{3}{2}} \lambda^b_\theta \|^2_{L^2} + 2\nu \|\nabla \sqrt{\mathcal{M}} |D_x|^{\frac{3}{2}} \lambda^b_\theta \|^2_{L^2} + \langle (|D_x|^{\frac{3}{2}} k \partial_\xi \mathcal{M})(D) \lambda^b_\theta, \lambda^b_\theta \rangle_{L^2} = -2\langle \lambda^b_\theta (u \cdot \nabla \theta), |D_x|^{\frac{3}{2}} \mathcal{M} \lambda^b_\theta \rangle_{L^2}.
\]

According to the definition of $\mathcal{M}_2$, we can get
\[
k \partial_\xi \mathcal{M}_2(k, \xi) + (2 + \pi + \mathcal{M}_2(k, \xi)) \frac{4k \xi}{k^2 + \xi^2} \geq 0, \text{ for } \text{sgn}(k) \xi \in (-\xi_0, 0),
\]
\[
\frac{1}{4} \nu \xi^2 + (2 + \pi + \mathcal{M}_2(k, \xi)) \frac{4k \xi}{k^2 + \xi^2} \geq 0, \text{ for } \text{sgn}(k) \xi \in (-\infty, -\xi_0].
\]

By $k \partial_\xi \mathcal{M}_2(k, \xi) \geq 0$ for all $\xi \in \mathbb{R}$, we have
\[
\frac{1}{4} \nu \xi^2 + k \partial_\xi \mathcal{M}_2(k, \xi) + (1 + \mathcal{M}_1(k, \xi) + \mathcal{M}_2(k, \xi) + \mathcal{M}_3(k, \xi)) \frac{4k \xi}{k^2 + \xi^2} \geq 0.
\]
Hence, we can get
\[
2\nu (\xi^2 + k^2) M(k, \xi) + k \partial_\xi M(k, \xi) \geq \nu (\xi^2 + k^2) + \frac{1}{4} \nu \xi^2 + \frac{1}{4} \xi^2 + \frac{1}{k^2},
\]
\[
2\nu (\xi^2 + k^2) M(k, \xi) + k \partial_\xi M(k, \xi) + M(k, \xi) \frac{4k \xi}{k^2 + \xi^2} \geq \nu (\xi^2 + k^2) + \frac{1}{4} \nu \xi^2 + \frac{1}{4} \xi^2 + \frac{1}{k^2}.\]
Therefore,
\[2\nu\|\nabla\sqrt{M}f\|_{L^2}^2 + \langle (k\partial_x M)(D)f, f\rangle_{L^2} \geq \nu\|\nabla f\|_{L^2}^2 + \frac{1}{4}\nu^\frac{3}{2}\||D_x|^\frac{3}{2}f\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}f\|_{L^2}^2,\]

\[2\nu\|\nabla\sqrt{M}f\|_{L^2}^2 + \langle (k\partial_x M)(D)f, f\rangle_{L^2} - 2\langle 2\partial_{xy}(-\Delta)^{-1}f, M(D)f\rangle_{L^2} \geq \nu\|\nabla f\|_{L^2}^2 + \frac{1}{4}\nu^\frac{3}{2}\||D_x|^\frac{3}{2}f\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}f\|_{L^2}^2,\]

where \(f\) is defined in (3.2). The formulas (3.3), (3.4) and (3.5) become

\[
\frac{d}{dt}\||\Lambda^b\theta\|_{L^2}^2 + \nu\|\nabla\Lambda^b\theta\|_{L^2}^2 + \frac{1}{4}\nu^\frac{3}{2}\||\Lambda^b\theta\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}\Lambda^b\theta\|_{L^2}^2
\leq -2\langle \Lambda^b(u \cdot \nabla \theta), M\Lambda^b\theta\rangle_{L^2},
\]

(3.7)

\[
\frac{d}{dt}\||\Lambda^b w\|_{L^2}^2 + \nu\|\nabla\Lambda^b w\|_{L^2}^2 + \frac{1}{4}\nu^\frac{3}{2}\||\Lambda^b w\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}\Lambda^b w\|_{L^2}^2
\leq -2\langle \Lambda^b(u \cdot \nabla w), M\Lambda^b w\rangle_{L^2} + 2\langle \Lambda^b(b \cdot \nabla j), M\Lambda^b w\rangle_{L^2} + 2\langle \partial_x\Lambda^b j, M\Lambda^b w\rangle_{L^2} + 2\langle \partial_x\Lambda^b\theta, M\Lambda^b w\rangle_{L^2}
\]

(3.8)

\[
\frac{d}{dt}\||\Lambda^b j\|_{L^2}^2 + \nu\|\nabla\Lambda^b j\|_{L^2}^2 + \frac{1}{4}\nu^\frac{3}{2}\||\Lambda^b j\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}\Lambda^b j\|_{L^2}^2
\leq -2\langle \Lambda^b(u \cdot \nabla j), M\Lambda^b j\rangle_{L^2} + 2\langle \Lambda^b(b \cdot \nabla w), M\Lambda^b j\rangle_{L^2} + 2\langle \partial_x\Lambda^b j, M\Lambda^b j\rangle_{L^2} + 2\langle \Lambda^b w, M\Lambda^b j\rangle_{L^2}
\]

(3.9)

And the formula (3.6) becomes

\[
\frac{d}{dt}\||D_x|^\frac{3}{2}\Lambda^b\theta\|_{L^2}^2 + \nu\||D_x|^\frac{3}{2}\Lambda^b\theta\|_{L^2}^2 + \frac{1}{4}\nu^\frac{3}{2}\||D_x|^\frac{3}{2}\Lambda^b\theta\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}|D_x|^\frac{3}{2}\Lambda^b\theta\|_{L^2}^2
\leq -2\langle \Lambda^b(u \cdot \nabla \theta), |D_x|^\frac{3}{2}M\Lambda^b\theta\rangle_{L^2},
\]

(3.10)

Since \(u\) and \(b\) are given by \(w\) and \(j\) via the Biot-Savart law,

\[
u = -\nabla^\perp(-\Delta)^{-1}w = \begin{pmatrix} \partial_y(-\Delta)^{-1}w \\ -\partial_x(-\Delta)^{-1}w \end{pmatrix},
\]

\[
u = -\nabla^\perp(-\Delta)^{-1}j = \begin{pmatrix} \partial_y(-\Delta)^{-1}j \\ -\partial_x(-\Delta)^{-1}j \end{pmatrix},
\]

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we can decompose \( u \) into two parts according to (5.2),

\[
\begin{align*}
    u_0 &= \mathbb{P}_0 u = \mathbb{P}_0 \left( \frac{\partial_y (-\Delta)^{-1} w}{-\partial_x (-\Delta)^{-1} w} \right) \\
    &= \begin{pmatrix}
        \frac{1}{2\pi} \int f(x,y) dx \\
        \frac{1}{2\pi} \int g(x,y) dy
    \end{pmatrix}.
\end{align*}
\]

Let \( w(x,y) = -\Delta f(x,y) = -\partial_x^2 f(x,y) - \partial_y^2 f(x,y) \), then

\[
    w_0 = \frac{1}{2\pi} \int f(x,y) + \partial_y^2 f(x,y) dx = \frac{1}{2\pi} \int \partial_y f(x,y) dx = -\partial_y^2 f_0.
\]

Therefore,

\[
    u_0 = \mathbb{P}_0 u = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}, \quad \text{with } u_0 = \partial_y (-\partial_y^2)^{-1} w_0,
\]

similarly for \( b_0 \),

\[
    b_0 = \mathbb{P}_0 b = \begin{pmatrix} b_0 \\ 0 \end{pmatrix}, \quad \text{with } b_0 = \partial_y (-\partial_y^2)^{-1} j_0.
\]

Note that

\[
    u_\neq = \mathbb{P}_\neq u = u - u_0 = -\nabla^L (-\Delta)^{-1} w_\neq,
\]

\[
    j_\neq = \mathbb{P}_\neq j = j - j_0 = -\nabla^L (-\Delta)^{-1} j_\neq.
\]

Therefore, we can write

\[
    I_1 = (\Lambda^b_t (u \cdot \nabla \theta), M \Lambda^b_t \theta)_{L^2} = I_{11} + I_{12},
\]

with

\[
    I_{11} = (\Lambda^b_t (u_\neq \cdot \nabla \theta), M \Lambda^b_t \theta)_{L^2}, \quad I_{12} = (\Lambda^b_t (u_0 \cdot \nabla \theta), M \Lambda^b_t \theta)_{L^2}.
\]

Using the boundedness of \( M \) and Lemma 8.1, we have for \( b > 1 \),

\[
    |I_{11}| \leq C \nu^{-4} \| \Lambda^b_t (u_\neq \cdot \nabla \theta) \|_{L^2} \| \Lambda^b_t \theta \|_{L^2}
\]

\[
\leq C \nu^{-4} \| \Lambda^b_t u_\neq \|_{L^2} \| \nabla \Lambda^b_t \theta \|_{L^2} \| \Lambda^b_t \theta \|_{L^2}
\]

\[
\leq C \nu^{-4} \| \nabla^L (-\Delta)^{-1} \Lambda^b_t w_\neq \|_{L^2} \| \nabla \Lambda^b_t \theta \|_{L^2} \| \Lambda^b_t \theta \|_{L^2}
\]

\[
\leq C \nu^{-4} \| (-\Delta)^{-\frac{1}{2}} \Lambda^b_t w_\neq \|_{L^2} \| \nabla \Lambda^b_t \theta \|_{L^2} \| \Lambda^b_t \theta \|_{L^2}.
\]

We write \( M^b_t = \sqrt{M^b_t} \) or

\[
    M^b_t (k, \xi) := \sqrt{M(k, \xi)} (1 + k^2 + (\xi + kt)^2)^{\frac{b}{2}}.
\]

Using the explicit expression of \( M^b_t \), we deduce that

\[
| \partial_\xi M^b_t (k, \xi) | \leq C (\nu^{\frac{1}{3}} |k|^{-\frac{1}{3}} + \frac{\nu^{-3}}{|k|})(1 + k^2 + (\xi + kt)^2)^{\frac{b}{2}}.
\]
Since $\theta_0$ is independent of $x$, we have

$$I_{12} = \langle \Lambda_t^k(u_0 \cdot \nabla \theta), \mathcal{M} \Lambda_t^k \theta \rangle_{L^2} = \langle \Lambda_t^k(u_0 \partial_x \theta \neq 0), \mathcal{M} \Lambda_t^k \theta \rangle_{L^2}.$$

Due to the cancellations

$$\langle \mathcal{M}_t^k(u_0 \partial_x \theta \neq 0), \mathcal{M}_t^k \theta_0 \rangle_{L^2} = \langle \mathcal{M}_t^k(u_0 \theta \neq 0), \partial_x \mathcal{M}_t^k \theta_0 \rangle_{L^2} = 0,$$

$$\langle u_0 \partial_x (\mathcal{M}_t^k \theta \neq 0), \mathcal{M}_t^k \theta \rangle_{L^2} = 0,$$

we have

$$I_{12} = \langle \mathcal{M}_t^k(u_0 \partial_x \theta \neq 0), \mathcal{M}_t^k \theta \rangle_{L^2} = \langle \mathcal{M}_t^k(u_0 \partial_x \theta \neq 0) - u_0 \partial_x (\mathcal{M}_t^k \theta \neq 0), \mathcal{M}_t^k \theta \rangle_{L^2}.$$

By Taylor’s formula,

$$\left| \mathcal{M}_t^k(k, \xi) - \mathcal{M}_t^k(k, \xi - z) \right| \leq \int^1_0 |\partial_x \mathcal{M}_t^k(k, \xi - sz)| |z| ds.$$

Therefore by (3.11) and $\max_{s \in [0,1]} |\xi - sz + kt| \leq 2|z| + |\xi + kt - z|$, we get

$$\left| \mathcal{M}_t^k(k, \xi) - \mathcal{M}_t^k(k, \xi - z) \right| \leq \int^1_0 |\partial_x \mathcal{M}_t^k(k, \xi - sz)| |z| ds \leq \int^1_0 C(\nu^{\frac{1}{2}}|k|^{-\frac{1}{2}} + \nu^{-3}|k|)(1 + k^2 + (\xi - sz + kt)^2)^{\frac{1}{2}} |z| ds$$

$$\leq \int^1_0 C(\nu^{\frac{1}{2}}|k|^{-\frac{1}{2}} + \nu^{-3}|k|)(1 + k^2 + z^2 + (\xi + kt - z)^2)^{\frac{1}{2}} |z| ds$$

$$\leq C(\nu^{\frac{1}{2}}|k|^{-\frac{1}{2}} + \nu^{-3}|k|) \left( (1 + k^2 + (\xi - sz)^2)^{\frac{1}{2}} + (1 + z^2)^{\frac{1}{2}} \right) |z|.$$

So $I_{12}$ can be estimated as

$$|I_{12}| \leq \sum_{k \neq 0} C(\nu^{\frac{1}{2}}|k|^{\frac{1}{2}} + \nu^{-3}) \int \int (\Lambda^k_t(k, \xi - z) + \Lambda^k_t(0, z))$$

$$\times |\hat{w}(0, z)||\hat{\theta}_\neq(k, \xi) - \mathcal{M}_t^k(k, \xi)|d\xi dz \leq C\nu^{-\frac{3}{2}}\|\hat{w}_0\|_{L^1} ||D_x|^{\frac{1}{2}} \Lambda^k_t \theta_\neq \|^2_{L^2} + C\nu^{-\frac{3}{2}}\|\Lambda^k_t w_0\|_{L^2} ||D_x|^{\frac{1}{2}} \theta_\neq \|^2_{L^1} ||D_x|^{\frac{1}{2}} \Lambda^k_t \theta_\neq \|^2_{L^2}$$

$$+ C\nu^{-\frac{5}{2}}||\hat{w}_0||_{L^1} ||\Lambda^k_t \theta_\neq \|^2_{L^2} + C\nu^{-\frac{5}{2}}||\Lambda^k_t w_0||_{L^2} ||\theta_\neq \|^2_{L^1} ||\Lambda^k_t \theta_\neq \|^2_{L^2}$$

$$\leq C\nu^{-\frac{3}{2}}\|\Lambda^k_t w_0\|_{L^2} ||D_x|^{\frac{1}{2}} \Lambda^k_t \theta_\neq \|^2_{L^2} + C\nu^{-\frac{5}{2}}\|\Lambda^k_t w_0\|_{L^2} ||\Lambda^k_t \theta_\neq \|^2_{L^2}.$$
Similarly as

\[ |I_1| \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} \theta\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} + C\nu^{-\frac{5}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} \theta\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} + C\nu^{-\frac{5}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \] (3.12)

Consequently,

\[ |I_1| \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} \theta\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} + C\nu^{-\frac{5}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} \theta\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} + C\nu^{-\frac{5}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \] (3.13)

Similarly as \( I_1 \), we can estimate \( I_2 \) and \( I_6 \) as follows

\[ |I_2| \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} w\|_{L^2} \|\Lambda_{b}^{\nu} w\|_{L^2} + C\nu^{-\frac{3}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} w\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} w\|_{L^2} \|\Lambda_{b}^{\nu} w\|_{L^2} + C\nu^{-\frac{3}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} w\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \] (3.14)

\[ |I_6| \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} j\|_{L^2} \|\Lambda_{b}^{\nu} j\|_{L^2} + C\nu^{-\frac{3}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} j\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} \|\nabla \Lambda_{b}^{\nu} j\|_{L^2} \|\Lambda_{b}^{\nu} j\|_{L^2} + C\nu^{-\frac{3}{2}} \|\Lambda_{b}^{\nu} w_0\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} j\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} w_0\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \] (3.15)

Due to the cancellations

\[ \langle M_{b}^{\nu}(b_0 \partial_x w \neq \cdot), M_{b}^{\nu}(j \neq \cdot) \rangle_{L^2} = 0, \]
\[ \langle M_{b}^{\nu}(b_0 \partial_x j \neq \cdot), M_{b}^{\nu}(w \neq \cdot) \rangle_{L^2} = 0, \]
\[ \langle b_0 \partial_x (M_{b}^{\nu} w \neq \cdot), M_{b}^{\nu}(j \neq \cdot) \rangle_{L^2} + \langle b_0 \partial_x (M_{b}^{\nu} j \neq \cdot), M_{b}^{\nu}(w \neq \cdot) \rangle_{L^2} = 0, \]

similarly as \( I_1 \), we can estimate \( I_3 + I_7 \) as follows

\[ |I_3 + I_7| \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} j \|_{L^2} \|\nabla \Lambda_{b}^{\nu} j\|_{L^2} \|\Lambda_{b}^{\nu} w\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} \|\Lambda_{b}^{\nu} j\|_{L^2} \|\Lambda_{b}^{\nu} w\|_{L^2} \]
\[ + C\nu^{-\frac{3}{2}} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} j\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} w\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} \|\Lambda_{b}^{\nu} j\|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} w\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} j\|_{L^2} \]
\[ \leq C\nu^{-4} \|(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} j \|_{L^2} \|\nabla \Lambda_{b}^{\nu} j\|_{L^2} \|\Lambda_{b}^{\nu} w\|_{L^2} \] (3.15)

\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} ||(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} j \|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ + C\nu^{-\frac{3}{2}} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} j\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} w\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} ||(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} j \|_{L^2} \|\Lambda_{b}^{\nu} \theta\|_{L^2} \]
\[ + C\nu^{-5} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} ||(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} w \|_{L^2} ||(-\Delta)^{-\frac{3}{2}} \Lambda_{b}^{\nu} j \|_{L^2} \]
\[ + C\nu^{-\frac{3}{2}} \|\Lambda_{b}^{\nu} j_{0}\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} w\|_{L^2} ||D_x|^\frac{3}{2} \Lambda_{b}^{\nu} j\|_{L^2} \].
Using the self-adjointness of $\mathcal{M}$ and the skew-adjointness of $\partial_x$, we have

\[ I_5 + I_8 = \langle \partial_x \Lambda^b_{ij}, \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} + \langle \partial_x \Lambda^b_{ij}, \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} = 0. \]  

(3.16)

Using the upper bound of $\mathcal{M}$, we have

\[ |I_4| = |\langle \partial_x \Lambda^b_i, \mathcal{M} \Lambda^b_i \rangle_{L^2}| \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_i \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_i w||_{L^2}. \]  

(3.17)

For $I_9$, we have

\[ I_9 = \langle \Lambda^b_i Q, \mathcal{M} \Lambda^b_j \rangle_{L^2} = \langle \Lambda^b_i (2\partial_x b^1(\partial_x u^2 + \partial_y u^1) - 2\partial_x u^1(\partial_x b^2 + \partial_y b^1)), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} \]

\[ = \langle \Lambda^b_i (2\partial_x b^1(\partial_x u^2 - \partial_y b^1) - 2\partial_x u^1(2\partial_x b^2 - j)), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2}. \]

For $I_{9_1}$,

\[ I_{9_1} \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2} \]

\[ \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2} \]

\[ \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}. \]

For $I_{9_2}$,

\[ I_{9_2} = \langle \Lambda^b_i (w_0 \partial_x b^1), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} + \langle \Lambda^b_i (w_0 \partial_x b^1), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} \]

\[ = \langle \Lambda^b_i (w_0 \partial_x b^1), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} \]

\[ \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2} \]

\[ \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}. \]

For $I_{9_3}$,

\[ I_{9_3} \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2} \]

\[ \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2} \]

\[ \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}. \]

For $I_{9_4}$,

\[ I_{9_4} = \langle \Lambda^b_i (j_0 \partial_x u^1), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} + \langle \Lambda^b_i (j_0 \partial_x u^1), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} \]

\[ = \langle \Lambda^b_i (j_0 \partial_x u^1), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} + \langle \Lambda^b_i (j_0 \partial_x u^1), \mathcal{M} \Lambda^b_{ij} \rangle_{L^2} \]

\[ \leq C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2} \]

\[ + C \nu^{-4} \|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}|||D_x|^\frac{2}{3} \Lambda^b_{ij} \theta||_{L^2}. \]
Hence, we have

\[ |I_0| \leq C\nu^{-4} \| (\Delta)^{-\frac{1}{4}} \Lambda_t^b j \|_{L^2}^2 \| \nabla \Lambda_t^b j \|_{L^2}^2 \| (\Delta)^{-\frac{1}{2}} \Lambda_t^n w \|_{L^2}^2 \| \nabla \Lambda_t^n w \|_{L^2} \| \Lambda_t^b j \|_{L^2}^2 \]

(3.18)

We decompose \( I_{10} \) as \( I_{101} + I_{102} \) with

\[ I_{101} = \langle \Lambda_t^b (u_\cdot \cdot \cdot \nabla \theta), |D_x|^2 \nabla \Lambda_t^n \theta \rangle_{L^2}, \quad I_{102} = \langle \Lambda_t^b (u_0 \cdot \nabla \theta), |D_x|^2 \nabla \Lambda_t^n \theta \rangle_{L^2}. \]

For \( I_{101} \), we have

\[ |I_{101}| \leq C\nu^{-4} \| D_x |\frac{1}{2} \Lambda_t^b (u_\cdot \cdot \cdot \nabla \theta) \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \]
\[ \leq C\nu^{-4} \| \Lambda_t^b u_\cdot \cdot \cdot \nabla \theta \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \nabla \theta \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \]
\[ + C\nu^{-4} \| D_x |\frac{1}{2} \Lambda_t^b u_\cdot \cdot \cdot \nabla \theta \|_{L^2} \| \Lambda_t^b \nabla \theta \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \]
\[ \leq C\nu^{-4} \| (\Delta)^{-\frac{1}{2}} \Lambda_t^n w \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \]
\[ + C\nu^{-4} \| D_x |\frac{1}{2} \Lambda_t^b w \|_{L^2} \| \Lambda_t^b \nabla \theta \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^n \theta \|_{L^2}. \]

The estimates for \( I_{102} \) are the same as those for \( I_{12} \),

\[ |I_{102}| \leq C\nu^{-\frac{3}{2}} \| \Lambda_t^b w_0 \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2}^2 \]
\[ + C\nu^{-5} \| \Lambda_t^b w_0 \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \]

Therefore, we deduce that

\[ |I_{10}| \leq C\nu^{-4} \| (\Delta)^{-\frac{1}{2}} \Lambda_t^n w \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \]
\[ + C\nu^{-5} \| \Lambda_t^b w_0 \|_{L^2} \| (\Delta)^{-\frac{1}{2}} |D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2} \| D_x |\frac{1}{2} \nabla \Lambda_t^n \theta \|_{L^2} \]
\[ + C\nu^{-4} \| D_x |\frac{1}{2} \Lambda_t^b w \|_{L^2} \| \Lambda_t^b \nabla \theta \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^n \theta \|_{L^2} + C\nu^{-\frac{3}{2}} \| \Lambda_t^b w_0 \|_{L^2} \| D_x |\frac{1}{2} \Lambda_t^n \theta \|_{L^2} \]

(3.19)

Inserting the upper bounds (3.12)–(3.19) into the estimates (3.7), (3.8), (3.9) and (3.10), and integrating in time, we obtain

\[ \| \Lambda_t^b \|_{L^\infty_t(L^2)}^2 + \nu \| \nabla \Lambda_t^n \theta \|_{L^2_t(L^2)}^2 + \frac{1}{4} \nu^\frac{1}{2} \| D_x |\frac{1}{2} \Lambda_t^b \theta \|_{L^2_t(L^2)}^2 \]
\[ \leq 2 \| \Lambda_t^b \theta(0) \|_{L^2}^2 + C_1 \nu^{-4} \| (\Delta)^{-\frac{1}{2}} \Lambda_t^n w \|_{L^2_t(L^2)} \| \nabla \Lambda_t^n \theta \|_{L^2_t(L^2)} \| \Lambda_t^n \theta \|_{L^\infty_t(L^2)} \]
\[ + C_1 \nu^{-\frac{1}{2}} \| \Lambda_t^n w \|_{L^\infty_t(L^2)} \| D_x |\frac{1}{2} \Lambda_t^n \theta \|_{L^2_t(L^2)}^2 \]
\[ + C_1 \nu^{-5} \| \Lambda_t^n w \|_{L^\infty_t(L^2)} \| (\Delta)^{-\frac{1}{2}} \Lambda_t^n w \|_{L^2_t(L^2)} \| \nabla \Lambda_t^n \theta \|_{L^2_t(L^2)}, \]

(3.20)
\[ \| \Lambda_t^b w \|^2_{L^\infty_t(L^2)} + \nu \| \nabla \Lambda_t^b w \|^2_{L_t^2(L^2)} + \frac{1}{8} \nu^{\frac{3}{4}} \| D_x \frac{1}{2} \Lambda_t^b w \|^2_{L_t^2(L^2)} + \| (\Delta)^{-\frac{1}{2}} \Lambda_t^b w \|^2_{L_t^2(L^2)} \\
+ \| \Lambda_t^b j \|^2_{L^\infty_t(L^2)} + \nu \| \nabla \Lambda_t^b j \|^2_{L_t^2(L^2)} + \frac{1}{8} \nu^{\frac{3}{4}} \| D_x \frac{1}{2} \Lambda_t^b j \|^2_{L_t^2(L^2)} + \| (\Delta)^{-\frac{1}{2}} \Lambda_t^b j \|^2_{L_t^2(L^2)} \leq 2 \| \Lambda_0^b w(0) \|^2_{L^2_t(L^2)} + 2 \| \Lambda_0^b j(0) \|^2_{L^2_t(L^2)} + C_2 \nu^{\frac{5}{4}} \| D_x \frac{1}{2} \Lambda_t^b \theta \|^2_{L_t^2(L^2)} \\
+ C_2 \nu^{-\frac{3}{5}} \| \Lambda_t^b w \|^2_{L^2_t(L^2)} \| D_x \frac{1}{2} \Lambda_t^b \theta \|_{L^2_t(L^2)} + C_2 \nu^{-\frac{3}{5}} \| \Lambda_t^b w \|_{L^\infty_t(L^2)} \| D_x \frac{1}{2} \Lambda_t^b \theta \|_{L^2_t(L^2)} \\
+ C_2 \nu^{-\frac{3}{5}} \| \Lambda_t^b w \|_{L^\infty_t(L^2)} \| (\Delta)^{-\frac{1}{2}} \Lambda_t^b \theta \|_{L^2_t(L^2)} \| D_x \frac{1}{2} \Lambda_t^b \theta \|_{L^2_t(L^2)} \\
+ C_2 \nu^{-\frac{3}{5}} \| \Lambda_t^b j \|_{L^\infty_t(L^2)} \| (\Delta)^{-\frac{1}{2}} \Lambda_t^b \theta \|_{L^2_t(L^2)} \| D_x \frac{1}{2} \Lambda_t^b \theta \|_{L^2_t(L^2)} \]
To apply the bootstrap argument, we make the ansatz that, for $T \leq \infty$, the solution of (3.1) obeys

$$\|\Lambda^b\theta\|_{L_t^\infty([0,T])(L^2)} + \nu \frac{1}{2} \|\nabla \Lambda^b\theta\|_{L_t^2([0,T])(L^2)} + \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^2([0,T])(L^2)}$$

$$+ \|(-\Delta)^{-\frac{1}{2}} \Lambda^b\theta\|_{L_t^2([0,T])(L^2)} \leq C\varepsilon \nu^\alpha, \tag{3.24}$$

$$\|\Lambda^b w\|_{L_t^\infty([0,T])(L^2)} + \nu \frac{1}{2} \|\nabla \Lambda^b w\|_{L_t^2([0,T])(L^2)} + \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b w\|_{L_t^2([0,T])(L^2)}$$

$$+ \|\Lambda^b j\|_{L_t^\infty([0,T])(L^2)} + \nu \frac{1}{2} \|\nabla \Lambda^b j\|_{L_t^2([0,T])(L^2)} + \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b j\|_{L_t^2([0,T])(L^2)}$$

$$+ \|(-\Delta)^{-\frac{1}{2}} \Lambda^b w\|_{L_t^2([0,T])(L^2)} + \|(-\Delta)^{-\frac{1}{2}} \Lambda^b j\|_{L_t^2([0,T])(L^2)} \leq C\varepsilon \nu^\beta, \tag{3.25}$$

$$\|D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^\infty([0,T])(L^2)} + \nu \frac{1}{2} \|\nabla D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^2([0,T])(L^2)} + \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^2([0,T])(L^2)}$$

$$+ \|(-\Delta)^{-\frac{1}{2}} D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^2([0,T])(L^2)} \leq \tilde{C} \varepsilon \nu^\delta. \tag{3.26}$$

We then show that (3.24), (3.25), and (3.26) actually hold with $C$ replaced by $C/2$ and $\tilde{C}$ by $\tilde{C}/2$. In fact, if we insert the initial condition and the ansatz in priori, we find

$$\|\Lambda^b\theta\|_{L_t^2([0,T])(L^2)}^2 + \nu \|\nabla \Lambda^b\theta\|_{L_t^2([0,T])(L^2)}^2 + \frac{1}{4} \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^2([0,T])(L^2)}^2$$

$$+ \|(-\Delta)^{-\frac{1}{2}} \Lambda^b\theta\|_{L_t^2([0,T])(L^2)}^2 \leq 2\varepsilon^2 \nu^\alpha + C_1 C^3 \varepsilon^3 (\nu^{2\alpha + \beta - \frac{3}{2}} + \nu^{2\alpha + \beta - 2} + \nu^{2\alpha + \beta - \frac{3}{4}}), \tag{3.28}$$

$$\|\Lambda^b w\|_{L_t^\infty([0,T])(L^2)} + \nu \|\nabla \Lambda^b w\|_{L_t^\infty([0,T])(L^2)} + \frac{1}{8} \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b w\|_{L_t^\infty([0,T])(L^2)}^2$$

$$+ \|\Lambda^b j\|_{L_t^\infty([0,T])(L^2)} + \nu \|\nabla \Lambda^b j\|_{L_t^\infty([0,T])(L^2)} + \frac{1}{8} \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b j\|_{L_t^\infty([0,T])(L^2)}^2$$

$$+ \|(-\Delta)^{-\frac{1}{2}} \Lambda^b w\|_{L_t^\infty([0,T])(L^2)} + \|(-\Delta)^{-\frac{1}{2}} \Lambda^b j\|_{L_t^\infty([0,T])(L^2)} \leq 2\varepsilon^2 \nu^{2\beta + \frac{1}{2}} + 2\varepsilon^2 \nu^{2\beta + \frac{1}{2}} + C_2 C^2 \varepsilon^2 \nu^{2\beta - \frac{3}{2}} + C_2 C^3 \varepsilon^3 \nu^{3\beta - 2} + C_2 C^2 \varepsilon^2 \nu^{3\beta - 2} + C_2 C^3 \varepsilon^3 \nu^{3\beta - \frac{1}{2}}$$

$$+ C_2 C^3 \varepsilon^3 \nu^{3\beta - 2} + 2C_2 C^3 \varepsilon^3 \nu^{\beta - \frac{1}{2}} + 2C_2 C^3 \varepsilon^3 \nu^{\beta - 1} + 2C_2 C^3 \varepsilon^3 \nu^{\beta - \frac{1}{2}} + C_2 C^3 \varepsilon^3 \nu^{\beta - 2} + C_2 C^3 \varepsilon^3 \nu^{\beta - \frac{1}{4}}$$

$$+ C_2 C^3 \varepsilon^3 \nu^{\beta - \frac{1}{2}} + C_2 C^3 \varepsilon^3 \nu^{\beta - 2} + C_2 C^3 \varepsilon^3 \nu^{\beta - \frac{1}{4}} \leq 4\varepsilon^2 \nu^{2\beta + \frac{1}{2}} + C_2 C^2 \varepsilon^2 \nu^{2\beta - \frac{3}{2}} + C_2 C^3 \varepsilon^3 (6\nu^{3\beta - \frac{3}{2}} + 4\nu^{3\beta - 2} + 4\nu^{3\beta - \frac{1}{4}}), \tag{3.29}$$

$$\|D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^\infty([0,T])(L^2)} + \nu \|\nabla D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^\infty([0,T])(L^2)} + \frac{1}{4} \nu \frac{1}{2} \|D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^\infty([0,T])(L^2)}^2$$

$$+ \|(-\Delta)^{-\frac{1}{2}} D_x |\frac{3}{2} \Lambda^b\theta\|_{L_t^\infty([0,T])(L^2)} \leq 2\varepsilon^2 \nu^{2\beta} + C_3 C\tilde{C} \varepsilon^3 \nu^{2\beta + \beta - \frac{3}{2}} + C_3 C^2 \tilde{C} \varepsilon^3 \nu^{\alpha + \beta - \frac{1}{4}}$$

$$+ C_3 C\tilde{C} \varepsilon^3 \nu^{2\beta + \beta - 2} + C_3 C\tilde{C} \varepsilon^3 \nu^{2\beta + \beta - \frac{1}{4}} \leq 2\varepsilon^2 \nu^{2\beta} + C_3 C\tilde{C} \varepsilon^3 (\tilde{C} \nu^{2\beta + \beta - \frac{3}{2}} + C\varepsilon^3 \nu^{\alpha + \beta - \frac{1}{4}} + \tilde{C} \nu^{2\beta + \beta - 2} + \tilde{C} \nu^{2\beta + \beta - \frac{1}{4}}), \tag{3.30}$$
which implies that
\[
\|\Lambda^t_\theta\|_{L^\infty_x((0,T)|L^2)} + \nu^{\frac{1}{2}}\|\nabla \Lambda^t_\theta\|_{L^\infty_x((0,T)|L^2)} + \nu^{\frac{1}{2}}\|D_x\|^{\frac{3}{2}}\Lambda^t_\theta\|_{L^s_x((0,T)|L^2)} \\
+ \|(-\Delta)^{-\frac{1}{2}}\Lambda^t_\theta\|_{L^\delta_x((0,T)|L^2)} \leq 4\varepsilon \nu^\alpha + 3C_1\varepsilon^\frac{\beta}{2}(\nu^{\alpha+\frac{1}{2}\beta-\frac{\alpha}{4}} + \nu^{\alpha+\frac{1}{4}\beta-\frac{\alpha}{8}} + \nu^{\frac{1}{2}\beta-\frac{1}{4}}),
\]
\[
\|\Lambda^t_j\|_{L^\infty_x((0,T)|L^2)} + \nu^{\frac{1}{2}}\|\nabla \Lambda^t_j\|_{L^\infty_x((0,T)|L^2)} + \nu^{\frac{1}{2}}\|D_x\|^{\frac{3}{2}}\Lambda^t_j\|_{L^s_x((0,T)|L^2)} \\
+ \|(-\Delta)^{-\frac{1}{2}}\Lambda^t_j\|_{L^\delta_x((0,T)|L^2)} \leq 10\varepsilon \nu^\beta + 5C_2\varepsilon^\frac{\beta}{2}(\nu^{\frac{1}{2}\beta-\frac{\alpha}{4}} + \nu^{\frac{1}{4}\beta-\frac{\alpha}{8}} + \nu^{\frac{1}{4}\beta-\frac{1}{4}}),
\]
\[
\|D_x\|^{\frac{3}{2}}\Lambda^t_\theta\|_{L^\infty_x((0,T)|L^2)} + \nu^{\frac{1}{2}}\|D_x\|^{\frac{3}{2}}\Lambda^t_\theta\|_{L^s_x((0,T)|L^2)} + \nu^{\frac{1}{2}}\|D_x\|^{\frac{3}{2}}\Lambda^t_\theta\|_{L^s_x((0,T)|L^2)} \\
+ \|(-\Delta)^{-\frac{1}{2}}\Lambda^t_\theta\|_{L^\delta_x((0,T)|L^2)} \leq 4\varepsilon \nu^\delta + 3C_3\varepsilon^\frac{\beta}{2}(\nu^{\frac{1}{2}\beta-\frac{\alpha}{4}} + \nu^{\frac{1}{4}\beta-\frac{\alpha}{8}} + \nu^{\frac{1}{4}\beta-\frac{1}{4}}).
\]
If we invoke (3.23) and choose
\[
\tilde{C} \geq 32, C \geq 80, C \geq 40C_2^\frac{1}{2}\tilde{C}, \varepsilon \leq \min\{\frac{1}{288^2CC_2}, \frac{\tilde{C}}{24^2C^2C_3}, \frac{1}{72^2CC_3}, \frac{1}{36^2C_1C}\},
\]
then the inequalities hold with $C$ replaced by $C/2$ and $\tilde{C}$ replaced by $\tilde{C}/2$. Hence, the ansatz is true for all $T \geq 0$. We redefine $C$ as the max{$C, \tilde{C}$} and this completes the proof.

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