Vibration–Based Damage Diagnosis in a Laboratory Cable–Stayed Bridge Model via an RCP–ARX Model Based Method

P.G. Michaelides, P.G. Apostolellis, S.D. Fassois

Laboratory for Stochastic Mechanical Systems and Automation (SMSA), Department of Mechanical and Aeronautical Engineering, University of Patras, GR 265 00 Patras, Greece
URL: http://www.smsa.upatras.gr
E-mail: mixail@mech.upatras.gr; fassois@mech.upatras.gr

Abstract. Vibration–based damage detection and identification in a laboratory cable–stay bridge model is addressed under inherent, environmental, and experimental uncertainties. The problem is challenging as conventional stochastic methods face difficulties due to uncertainty underestimation. A novel method is formulated based on identified Random Coefficient Pooled ARX (RCP–ARX) representations of the dynamics and statistical hypothesis testing. The method benefits from the ability of RCP models in properly capturing uncertainty. Its effectiveness is demonstrated via a high number of experiments under a variety of damage scenarios.

1. Introduction
The interest in the ability to monitor a structure and detect damage at an early stage is pervasive throughout the mechanical, aerospace and civil engineering communities. In fact, the problems of early detection and identification of damage are of paramount importance, as prompt detection may lead to better dynamic performance, increased safety and proper maintenance [1]. Vibration–based statistical time series methods for damage detection and assessment are among the most accurate and effective “global” methods [2–5]. They offer a number of potential advantages, such as no requirement for visual inspection, automation capability, “global” coverage, and the ability to work at a system level. They utilize (i) random excitation and/or response signals (time series), (ii) statistical model building, and (iii) statistical decision making for inferring the health state of a structure. As with all vibration based methods, the fundamental principle upon which they are founded is that small changes (damage) in a structure cause discrepancies in its vibration response, which may be detected and associated with a specific cause (damage type).

Nevertheless, discrepancies in structural response may be also induced by physical uncertainties – caused by various reasons such as varying structural configurations, loading, boundary, and environmental conditions, unaccounted excitations, variations in the geometrical and material characteristics, data acquisition and signal noise, and so on – which are inherent in structures and propagate to the dynamic response characteristics. These uncertainties may affect damage diagnosis, as it may be difficult to distinguish the effects of actual damage from those of uncertainty. To a certain
degree, the problem may be handled by “properly adjusting” the risk level $\alpha$ (type I error or false alarm probability) – for instance see [4]. Nevertheless, proper accounting of uncertainties is far more appropriate and desirable.

Unfortunately, conventional stochastic models – such as AutoRegressive with eXogenous excitation (ARX) or other similar models [6] – which are capable of describing uncertainties in the excitation and/or response signals are characterized by deterministic parameters and their identification is based on a single data record. Thus they are incapable of representing the full spectrum of uncertainties – in this context see [7]. It is thus obvious that alternative models are necessary for this purpose. A logical possibility is to use Random Coefficient Pooled (RCP) stochastic models, as postulated in a recent study by two of the authors [8]. RCP models resemble the form of their conventional counterparts, yet they are different in that they are capable of representing a structure under (an infinite number of) “experimental conditions”. In each “experimental condition”, and for each “structural health state” (healthy, damage A, damage B, and so on), the RCP model parameters are random variables which vary from one single experiment to another (under the same “experimental conditions” and “structural health state”). The identification of RCP models is thus based on multiple data records, each one obtained under a specific “experimental condition.”

The aim of the present study is the formulation of a damage diagnosis (detection and identification) method based on RCP–ARX models that properly represent uncertainties. The method is capable of offering an effective solution to the damage diagnosis problem under uncertainty and is also capable of operating even under a single pair of signal (like force excitation and vibration response) measurements. The rest of the paper is organized as follows: The experimental set-up and procedures are described in section 2, while the damage diagnosis methodology is presented in section 3. Experimental damage detection and identification results are presented in section 4, and the conclusions are summarized in section 5.

2. The Experimental Set-Up

2.1. The Test Rig

The laboratory cable–stayed bridge model and the test rig are shown in Figure 1. The bridge deck is represented by an $1470 \times 190 \times 2\,mm$ aluminium plate suspended via 20 cables attached to the central steel pylon and clamped at each edge of the deck. Seven $200 \times 120 \times 5\,mm$ steel plates are attached to the
Table 1. Damage types (scenarios) and experimental details.

| Structural Damage Types (scenarios) | Description | Number of Experiments |
|-------------------------------------|-------------|-----------------------|
| Healthy                             | —           | Baseline phase 50     |
| Damage type A                        | relaxing cable | 50                  |
| Damage type B                        | relaxing cable | 50                  |
| Damage type C                        | relaxing cable | 50                  |
| Damage type D                        | relaxing cable | 50                  |
| Damage type E                        | loosening deck bolt | 50                  |
| Damage type F                        | loosening tower bolt | 50                  |

Deck: 1470 × 190 × 2 mm aluminum plate, reinforced with 7 steel plates (200 × 120 × 5 mm)
Actuator: MB Dynamics Modal 50A electromechanical shaker
Sensors: PCB 740B02 dynamic strain gauge, PCB 288D01 impedance head
Data acquisition device: Spectral Dynamics Siglab 20-42
Excitation signal amplifier: PCB 482A20
Response signal amplifier: PCB 481
Analysis bandwidth: 0 – 100 Hz
Sampling frequency: $f_s = 256$ Hz
Signal length: $N = 4,000$ samples ($\approx 15.6$ s)

2.2. The Damage Types
Six damage types (scenarios) are considered (Table 1 and Figure 1). Each one of the first four (damage types A–D) correspond to loosening a single suspension cable. The fifth (damage type E) corresponds to loosening of one of the four bolts on one edge of the deck, while the sixth (damage type F) corresponds to loosening one of the four bolts of the central pylon’s base.

2.3. The Experiments
Damage detection and identification is based on a pair of forced excitation and vibration response signals. In each test case the excitation is a realization of a random, zero–mean, stationary Gaussian force with bandwidth $0 – 100$ Hz. A number of experiments are carried out, initially with the healthy structure and subsequently with the structure under each one of the aforementioned damage types. The acquired signals are digitized with sampling frequency of $f_s = 256$ Hz (analysis bandwidth $0 – 100$ Hz), with each resulting signal being $N=4,000$ samples ($\approx 15.6$ s) long.

3. The Random Coefficient Pooled ARX model based method
The Random Coefficient Pooled ARX (RCP–ARX) model based method consists of two phases: (a) The baseline phase, which includes modeling of the structure under each “structural health state” via a stochastic RCP–ARX model for each state, and (b) the inspection phase, which is performed periodically or on demand during the structure’s service cycle and includes the functions of damage detection and identification (localization).
3.1. Baseline Phase
As already explained, an RCP–ARX model is capable of representing the uncertain structural dynamics under a number (possibly infinite) of “experimental conditions” for a given “structural health state.” To comply with standard terminology used for similar (particularly pooled linear regression) models employed in econometrics and often referred to as Swamy’s random coefficient models [9–11], each “experimental condition” is also referred to as cross-section.

An RCP–ARX model is of the following form:

\[ y_k[t] + \sum_{i=1}^{na} a_{ik} y_k[t - i] = \sum_{i=0}^{nb} b_{ik} x_k[t - i] + e_k[t] \]

\[ e_k[t] \sim \text{iid} \mathcal{N}(0, \sigma_k^2), \quad E\{e_k[t] \cdot e_l[t - \tau]\} = \begin{cases} \sigma_k^2 \cdot \delta[\tau], & k = l, \\ 0, & k \neq l \end{cases}, \quad k, l \in [1, M] \]

\[ \theta_k = \begin{bmatrix} a_{1k}, a_{2k}, \ldots, a_{(na)k}; b_{0k}, b_{1k}, \ldots, b_{(nb)k} \end{bmatrix}^T \sim \text{iid} \mathcal{N}(\theta_0, \Gamma), \quad k \in [1, M] \]

\[ \theta_k \text{ and } e_l[t] \text{ mutually independent } \forall k, l, t \]

In the above expressions \( t \) designates normalized discrete time \((t = 1, 2, \ldots, N)\), \( k \) the “experimental condition” \((k \in [1, M])\), \( x_k[t] \), \( y_k[t] \), \( e_k[t] \), the measurable excitation, vibration response, innovations signals, respectively, and \( \sigma_k^2 \) the random innovations variance – all corresponding to the \( k \)-th “experimental condition.” The indices \( na \) and \( nb \) designate the AutoRegressive (AR) and eXogenous (X) model orders, respectively. iid stands for identically independently distributed, \( \mathcal{N}(\cdot, \cdot) \) for the normal distribution with the indicated mean and covariance, \( \delta[\tau] \) for the Kronecker delta \((1 \text{ for } \tau = 0, 0 \text{ otherwise})\). The \( a_{ik} \)'s, \( b_{ik} \)'s are the random AR, X model parameters, respectively, corresponding to the \( k \)-th “experimental condition.” These are stacked together to form the random parameter vector \( \theta_k \). Notice that all \( \theta_k \)'s are iid, following a common multivariate normal distribution with mean \( \theta_0 \) and covariance \( \Gamma \). Therefore it is appropriate to just use a random vector \( \theta \sim \mathcal{N}(\theta_0, \Gamma) \) to designate any one or collectively all of them. Finally notice that according to the last of the above expressions, all random parameter vectors are mutually independent with all innovations signals.

3.1.1. Model Identification. The estimation of an RCP–ARX model, corresponding to a single “structural health state,” is based on \( M \) data records, each one corresponding to a single “experimental condition” (cross-section). Each such data record is \( N \) samples long.

A simple identification approach is presently used, according to which the AR, X model orders are first selected based upon conventional ARX\((n, n)\) identification for increasing \( n \) until model adequacy is achieved according to a number of criteria, such as the BIC, modal stabilization diagrams, and so on [6].

Once proper model orders have been selected, RCP–ARX model parameter estimation, that is mainly estimation of the parameter vector common mean \( \theta_0 \) and covariance \( \Gamma \), is accomplished by a simple Mean Group (MG) approach [12] as follows: A conventional ARX model is estimated via an Ordinary Least Squares (OLS) estimator for each data record \( k = 1, \ldots, M \) (corresponding to each “experimental condition”). Using these OLS estimates \( \hat{\theta}_k^{\text{OLS}} \) for \( k = 1, \ldots, M \), the following estimators are postulated:

\[ \hat{\theta}_0 = \frac{1}{M} \sum_{k=1}^{M} \hat{\theta}_k^{\text{OLS}} \]

\[ \hat{\Gamma}^E = \frac{1}{M-1} \sum_{k=1}^{M} \left( \hat{\theta}_k^{\text{OLS}} - \hat{\theta}_0 \right) \left( \hat{\theta}_k^{\text{OLS}} - \hat{\theta}_0 \right)^T \]
with the superscript “E” designating the empirical covariance estimator and distinguishing it from potential alternatives (see next subsection). Since the estimators \( \hat{\theta}_k \)'s are also random vectors, each characterized by its own covariance \( \hat{P}_{\theta_k} \) [13, pp. 212-213], the above covariance estimator may (if necessary) be modified as follows in order to have this “extra” uncertainty removed [9, 10]:

\[
\hat{\Gamma} = \hat{\Gamma}^E - \frac{1}{M} \sum_{k=1}^{M} \hat{P}_{\theta_k}^{OLs}
\]  

(7)

It should be noted that the above, “corrected,” covariance estimator is particularly useful when the focus is on the precise estimation of the “inherent” structural uncertainties, excluding the estimator uncertainty stemming from the noise in the vibration signals. In a damage diagnosis context, though, where it is desirable to have all kinds of uncertainties incorporated into the covariance, the above “correction” is not needed.

An important issue emerges in estimating \( \Gamma \) in situations where the parameter vector dimensionality (presently \( p = na + nb + 1 \)) is not sufficiently smaller (by at least one order of magnitude) from the number of data records (“experimental conditions”) \( M \). In such cases the empirical unbiased covariance estimate is non–positive definite and ill–conditioned [14], and thus non–invertible (invertibility is required in the sequel – see section 3.2). This “large \( p \) – small \( M \)” situation is quite common in a structural dynamics context, as the structural models are typically of quite high order while only few data records are available. To overcome this difficulty, the empirical covariance estimator \( \hat{\Gamma}^E \) in eq. (7) is replaced by an alternative estimator, referred to as the shrinkage based covariance estimator \( \hat{\Gamma}^S \), which is always well conditioned and optimum in the mean square error sense [14–16].

### 3.1.2. The Shrinkage Based Covariance Estimator [14–16]

Let \( \hat{\Gamma}^E \) be the empirical unbiased covariance matrix estimate of the \( p \times 1 \) random vector \( \theta_k \) and \( M \) the number of available data records. Let \( T \) be a positive definite, well conditioned, arbitrary target covariance matrix. Then, the shrinkage based estimate of the covariance is of the form:

\[
\hat{\Gamma}^S = \lambda T + (1 - \lambda)\hat{\Gamma}^E
\]  

(8)

with \( \lambda \in [0,1] \) denoting the shrinkage intensity. Clearly, the shrinkage based estimator is a weighted average of the empirical unbiased covariance estimator \( \hat{\Gamma}^E \) and the target covariance \( T \). Six commonly used covariance targets are shown in Table 2.
The squared error between the shrinkage estimator and the true covariance may be expressed as a quadratic function of $\lambda$ [14, eq.(5)] and be thus analytically minimized. The resulting optimal shrinkage intensity $\lambda^*$ is [14, 15]:

$$\lambda^* = \frac{\sum_{i=1}^{p} \sum_{j=1}^{p} \{\text{Var}(\hat{E}_{ij}) - \text{Cov}(t_{ij}, \hat{E}_{ij})\}}{\sum_{i=1}^{p} \sum_{j=1}^{p} E_{ij}(t_{ij} - \hat{E}_{ij})^2}$$  \hspace{1cm} (9)

where $\hat{E}_{ij}$, $t_{ij}$ designate the $(i,j)$-th elements of $\hat{\Gamma}$ and $T$, respectively. In estimating $\lambda^*$ sample values of the above indicated theoretical quantities are used [14].

As may be observed from eq. (9), the smaller the variance of the empirical estimate’s elements $\hat{E}_{ij}$, the smaller the optimum shrinkage intensity $\lambda^*$. In other words, as the number of data records $M$ increases, providing a more accurate empirical covariance estimate, the effect of the arbitrary target diminishes. Furthermore, the shrinkage intensity decreases as the mean squared difference between the target and the empirical covariance elements (denominator of eq. (9)) increases, which means that a highly misspecified target will not practically affect the result. Finally, when the target and the empirical covariance are positively correlated (second term of the numerator of eq. (9)), which means that they are both inferred from the same information (data), the shrinkage intensity decreases, neglecting the redundant information.

### 3.2. Inspection Phase

In the inspection phase a fresh pair of excitation – response signals is acquired from the structure being in unknown “health state”, and the objective is to infer that unknown state. Towards this end, a conventional ARX model, of the same orders as that of the baseline RCP–ARX model, is estimated using an OLS estimator. This model provides a representation of the structure in its current (unknown) state. The idea then is to examine – through a proper statistical hypothesis test and at a selected risk level – whether the parameter vector $\hat{\theta}$ of the estimated current ARX model is a sample of the random vector $\theta (\theta \sim \mathcal{N}(\theta_0, \Gamma))$ corresponding to the RCP–ARX model of the healthy (for the detection problem) structure. If this hypothesis can be accepted as true, then the current state of the structure is accepted as being healthy (null $H_0$ hypothesis). If not, then it is accepted as being damaged (alternative $H_1$ hypothesis).

Under the null $H_0$ (healthy) hypothesis, the statistic $\chi^2_\theta$, below, follows $\chi^2$ distribution with $p (= \text{dim} (\theta))$ degrees of freedom (sum of squares of independent standardized Gaussian variables [13, p. 558]):

$$\chi^2_\theta = (\hat{\theta} - \theta_0)^T\Gamma^{-1}(\hat{\theta} - \theta_0) \sim \chi^2(p)$$  \hspace{1cm} (10)

As the true mean $\theta_0$ and covariance $\Gamma$ are unavailable, their corresponding estimates (sample quantities) are used by assuming their negligible variability. This leads to the following test constructed at the $\alpha$ (type I or false alarm) risk level:

$$\chi^2_\theta \leq \chi^2_{1-\alpha}(p) \quad \Rightarrow \quad H_0 \text{ is accepted} \quad \text{(healthy structure)}$$

$$\text{Else} \quad \Rightarrow \quad H_1 \text{ is accepted} \quad \text{(damaged structure)}$$  \hspace{1cm} (11)

with $\chi^2_{1-\alpha}(p)$ designating the chi–square distribution’s $1 - \alpha$ critical point. It is noted that in case the empirical covariance estimate in eq. (6) is ill conditioned, the shrinkage based estimator is used.

Once the presence of damage is detected, identification (presently localization) may be based on similar hypothesis testing procedures, in each one of which the estimate $\hat{\theta}$ is examined as to whether or not it is a sample of the random vector $\theta$ corresponding to the RCP–ARX model of each damage type (presently damage types A, . . . , F).

It is also mentioned that it is clearly not necessary – or even best – to employ the complete ARX parameter vector $\theta$ in the diagnosis phase. Either a smaller portion (dimensionality reduction), or a
Figure 2. Order selection for RCP–ARX$(n, n)$ identification under the healthy state: (a) RSS/SSS versus model order, (b) BIC versus model order, (c) SPP versus model order, (d) frequency stabilization diagram plus average (over 50 healthy data records) non–parametric FRF magnitude estimate, and (e) average (over 50 healthy data records) non–parametric coherence estimate. The selected order of $n = 51$ is indicated by a red line or arrows.

proper transformation of it, or a combination of the two may be alternatively employed. In the present study only a portion of the actual parameter vector is used. The individual elements (parameters) retained are those that may be confidently considered as being non–zero, in the sense that their $99.7\%$ confidence intervals (mean $\pm 3\sigma$) do not include zero. Of course, various other possibilities and transformations (such as Principal Component Analysis [17, p. 459]) may be also considered.

4. Experimental Damage Diagnosis Results

4.1. Baseline Phase: structural identification under various health states

RCP–ARX identification of the healthy structural dynamics is based on $M = 50$ data records, each consisting of $N = 4000$ sample ($\approx 15.6$ s) long scalar excitation and vibration response signals. The modelling strategy consists of the successive fitting of RCP–ARX$(n, n)$ models until an adequate model is selected. Model order selection, which is crucial for successful identification, is based on a combination of tools: monitoring the Residual Sum of Squares over the Signal Sum of Squares (RSS/SSS) criterion (Figure 2a), the Bayesian Information Criterion (BIC) (Figure 2b) [6], and use of stabilization diagrams (Figure 2d) which depict the estimated modal parameters (usually frequencies) as a function of increasing model order [6]. The number of signal samples per estimated parameters (Samples Per Parameter – SPP) is also monitored in order to maintain a sufficiently high value that warrants statistical reliability (Figure 2c). The order selection procedure is conducted for the baseline healthy case only, and the selected orders are subsequently used in all other “structural health states.” In each estimation procedure the parameter covariance $\Gamma$ is estimated via the shrinkage estimator using target 6 (see Table 2), as the empirical covariance estimate generally is ill–conditioned.

Based on the results of Fig. 2, a model order $n = 51$ is selected, as the BIC presents a global
minimum for it, the SPP is quite adequate, and frequency stabilization has been achieved for all major natural frequencies. An average (over the 50 healthy data records) non–parametric FRF magnitude estimate (based on Welch spectral estimates – MATLAB function tfestimate.m) is also depicted on the frequency stabilization diagram, whereas a corresponding average coherence estimate (MATLAB function mscohere.m) is shown on the next subplot. This last curve indicates a good linear relationship for frequencies above 22 Hz, but not for lower ones.

The RCP–ARX model based structural dynamics uncertainty is, for the healthy structural state and in the form of FRF magnitude mean ± 3σ bounds, indicated in Figure 3. On the same plot, 150 conventional ARX based (OLS estimation; MATLAB function arx.m) FRF magnitude curves are also shown for the same (healthy) structural state. As may be readily observed, the RCP–ARX model based uncertainty bounds nicely correspond to the zone defined by the 150 conventional ARX model based estimates, indicating a precise capture of the actual uncertainty. It is worth noting that uncertainty propagation from the model parameters to the FRF magnitude is based on linearization (via Taylor series expansion) of the functional relation between the model parameters and the FRF magnitude [8].

Maintaining the healthy RCP–ARX model orders, the estimation procedure is repeated for each one of the rest “structural health states”, hence an RCP–ARX model is obtained for each one of the damage types A, B, . . . F.

4.2. Inspection Phase
Only those RCP–ARX parameters whose mean ± 3σ confidence interval does not include zero are retained and used in the sequel. Thus the reduced parameter vector employed is of dimensionality $p = 57$ (instead of $p = 103$).

Damage detection results for 100 healthy (black circles) and 300 damage test cases (various colors and symbols depending on the exact damage type) are presented in Figure 4. The statistical limit is, at the $\alpha = 10^{-3}$ risk (false alarm) level, depicted as a red horizontal dashed line. A test statistic below
Figure 4. RCP–ARX damage detection results for all 400 test cases at the $\alpha = 10^{-3}$ risk level. A damage is detected if the test statistic (various markers depending on the actual damage scenario) exceeds the critical limit (---).

Figure 5. RCP–ARX damage identification results for all 300 test cases at the $\alpha = 10^{-3}$ risk level. In each subplot the hypothesis of a single damage type is examined. The actual damage is indicated above the plot. A hypothesis is accepted if the test statistic (various markers depending on the actual damage scenario) does not exceed the critical limit (---).
the statistical limit indicates healthy structural state, whereas a statistic above it indicates the presence of damage. It is evident that all healthy and damage cases are properly detected as such (no false alarms or missed damage cases).

Once a damage is detected, the current conventional ARX model is subsequently “compared” to each one of the baseline RCP–ARX models corresponding to each damage type (scenario). Damage identification (localization) results are presented in Figure 5. In each case, the current type is identified as coinciding with that being considered if the corresponding statistic does not exceed the statistical limit at the $\alpha = 10^{-3}$ risk level (red dashed line). Evidently, the correct damage type is identified in all test cases, indicating the effectiveness of the method.

5. Conclusions
A Random Coefficient Pooled ARX (RCP–ARX) model based damage detection and identification method has been formulated and applied to a laboratory cable–stayed bridge model. RCP–ARX models have been shown to adequately capture uncertainty and offer a good basis for developing proper damage diagnosis strategies. As the parameter covariance matrix is typically estimated based on a limited number of data records, a shrinkage type covariance estimator has been effectively employed. The damage detection and identification (localization) results have indicated the method’s effectiveness with a high number of test cases under various damage types. The overall performance has been excellent, characterized by zero false alarm, missed damage, and wrong damage localization rates.

References
[1] S.W. Doebling, C.R. Farrar, M.B. Prime and D.W. Shevitz 1996 Report LA-13070-MS (Los Alamos National Laboratory, USA).
[2] S.D. Fassois and J.S. Sakellariou 2007 Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 365 p. 411.
[3] M. Basseville, M. Abdelghani and A. Benveniste 2000 Automatica 36 p. 101.
[4] F.P. Kopsaftopoulos and S.D. Fassois 2010 Mechanical Systems & Signal Processing 24 p. 1977.
[5] S.D. Fassois and J.S. Sakellariou 2009 Statistical time series methods for structural health monitoring (Encyclopaedia of Structural Health Monitoring pp. 443–472) ed C. Boller, F.K. Chang, Y. Fujino (John Wiley & Sons Ltd.)
[6] S.D. Fassois 2001 Parametric identification of vibrating structures (Encyclopaedia of Vibration pp. 673–685) ed S. Braun, D. Ewins, S Rao (Academic Press).
[7] P.G. Michaelides and S.D. Fassois 2008 Proc. Int. Conf. on Noise and Vibration Engineering – ISMA (Leuven) p. 3927
[8] P.G. Michaelides and S.D. Fassois 2010 Proc. Int. Conf. on Uncertainty in Structural Dynamics – USD (Leuven) p. 5305.
[9] P.A.V.B. Swamy 1970 Econometrica 38 p. 311.
[10] N. Beck and J. Katz 2001 Statistica Neerlandica 55 p. 111.
[11] N. Beck and J. Katz 2007 Political Analysis 15 p. 182.
[12] M. Pesaran and R. Smith 1995 Journal of Econometrics 68 p. 79.
[13] L. Ljung 1999 System Identification: Theory for the User, 2nd ed. (Prentice Hall)
[14] J. Schäfer, and K. Strimmer 2005 Statistical Applications in Genetics and Molecular Biology 4(1) article 32.
[15] O. Ledoit and M. Wolf 2003 Journal of Empirical Finance 10 p. 603.
[16] O. Ledoit and M. Wolf 2004 Journal of Multivariate Analysis 88 p. 365.
[17] T.W. Anderson 2003 An Introduction to Multivariate Statistical Analysis, 3rd ed. John Wiley