Eigenvalue gaps for the Cauchy process and a Poincaré inequality

Rodrigo Bañuelos*  
Department of Mathematics  
Purdue University  
West Lafayette, IN 47906  
bannelos@math.purdue.edu

Tadeusz Kulczycki†  
Institute of Mathematics  
Wrocław University of Technology  
50-370 Wrocław, Poland  
tkulczyc@im.pwr.wroc.pl

Abstract

A connection between the semigroup of the Cauchy process killed upon exiting a domain $D$ and a mixed boundary value problem for the Laplacian in one dimension higher known as the mixed Steklov problem, was established in [6]. From this, a variational characterization for the eigenvalues $\lambda_n$, $n \geq 1$, of the Cauchy process in $D$ was obtained. In this paper we obtain a variational characterization of the difference between $\lambda_n$ and $\lambda_1$. We study bounded convex domains which are symmetric with respect to one of the coordinate axis and obtain lower bound estimates for $\lambda^* - \lambda_1$ where $\lambda^*$ is the eigenvalue corresponding to the “first” antisymmetric eigenfunction for $D$. The proof is based on a variational characterization of $\lambda^* - \lambda_1$ and on a weighted Poincaré-type inequality. The Poincaré inequality is valid for all $\alpha$ symmetric stable processes, $0 < \alpha \leq 2$, and any other process obtained from Brownian motion by subordination. We also prove upper bound estimates for the spectral gap $\lambda_2 - \lambda_1$ in bounded convex domains.

Contents

§1. Introduction  
§2. Variational formulas  
§3. Weighted Poincaré inequalities  
§4. Proof of Theorems 1.1 and 1.2  
§5. Concluding Remarks

*Supported in part by NSF Grant # 9700585-DMS  
†Supported by KBN grant 2 P03A 041 22 and RTN Harmonic Analysis and Related Problems, contract HPRN-CT-2001-00273-HARP
1 Introduction

The spectral gap estimates for eigenvalues of the Laplacian with Dirichlet boundary conditions, henceforth referred to as the Dirichlet Laplacian, have attracted considerable attention for many years [2], [3], [10], [29], [38], [37], [41]. The Dirichlet Laplacian is the infinitesimal generator of the semigroup of Brownian motion killed upon leaving a domain. Therefore questions concerning eigenvalues of this operator can be studied both by analytic and probabilistic methods. The question of precise lower bounds for the spectral gap for the Dirichlet Laplacian (the difference between the first two eigenvalues) was raised by M. van den Berg [10] (see also Yau [39], problem #44) and was motivated by problems in mathematical physics related to the behavior of free Boson gases. The conjecture, which remains open, asserts that for any convex bounded domain $D$ of diameter $d_D$, the spectral gap is bounded below by $3\pi^2/d_D^2$. (See [5], [8], [20] where some special cases of the conjecture are proved and [21], [10] for more general “partition function” inequalities.) The spectral gap has also been studied for the Laplacian with Neumann boundary conditions and for Schrödinger operators, [34], [38], [2], [37]. From the probabilistic point of view, the spectral gap for the Dirichlet Laplacian determines the rate to equilibrium for the Brownian motion conditioned to remain forever in $D$, the Doob $h$-process corresponding to the ground state eigenfunction.

The natural question arises as to whether these results can be extended to other non-local, pseudo-differential operators. The class of such operators which are most closely related to the Laplacian $\Delta$ from the point of view of Brownian motion are $-(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$. These are the infinitesimal generators of the symmetric $\alpha$-stable processes. These processes do not have continuous paths which is related to non-locality of $-(-\Delta)^{\alpha/2}$. As in the case of Brownian motion, we can consider the semigroup of these processes killed upon exiting domains and we can consider the eigenvalues of such semigroup. Here again, the spectral gap determines the asymptotic exponential rate of convergence to equilibrium for the process conditioned to remain forever in the domain. Instead of speaking of the eigenvalue gap for the operator $-(-\Delta)^{\alpha/2}$ we will very often refer to it as the eigenvalue gap for the corresponding process.

The purpose of this paper is to obtain eigenvalue gap estimates for the Cauchy process, the symmetric $\alpha$-stable process for $\alpha = 1$. This is done using the connection (established in [6]) between the eigenvalue problem for the Cauchy process and the mixed Steklov problem. Both, the methods and the results, are new. The results raise natural questions concerning spectral gaps for other symmetric $\alpha$-stable processes and for more general Markov processes. We believe that as with the results in [6] which have motivated subsequent work by others, see [22], [23], [18], the current results will also be of interest. Let $X_t$ be a symmetric $\alpha$-stable
process in \( \mathbb{R}^d \), \( \alpha \in (0, 2] \). This is a process with independent and stationary increments and characteristic function \( E^0 e^{i\xi X_t} = e^{-t|\xi|^\alpha} \), \( \xi \in \mathbb{R}^d \), \( t > 0 \). \( E_x \), \( P_x \) denote the expectation and probability of this process starting at \( x \), respectively. By \( p^{(\alpha)}(t, x, y) = p_t^{(\alpha)}(x - y) \) we will denote the transition density of this process. That is,
\[
P_x(X_t \in B) = \int_B p^{(\alpha)}(t, x, y), dy.
\]
When \( \alpha = 2 \) the process \( X_t \) is just the Brownian motion in \( \mathbb{R}^d \) but running at twice the speed. That is, if \( \alpha = 2 \) then
\[
p^{(2)}(t, x, y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}, \quad t > 0, \; x, y \in \mathbb{R}^d.
\]
When \( \alpha = 1 \), the process \( X_t \) is the Cauchy process in \( \mathbb{R}^d \) whose transition densities are given by
\[
p^{(1)}(t, x, y) = \frac{c_d t}{(t^2 + |x-y|^2)^{(d+1)/2}}, \quad t > 0, \; x, y \in \mathbb{R}^d,
\]
where
\[
c_d = \Gamma((d + 1)/2)/\pi^{(d+1)/2}.
\]

Our main interest in this paper are the eigenvalues of the semigroup of the process \( X_t \) killed upon leaving a domain. Let \( D \subset \mathbb{R}^d \) be a bounded connected domain and \( \tau_D = \inf\{ t \geq 0 : X_t \notin D \} \) be the first exit time of \( D \). By \( \{P^D_t\}_{t \geq 0} \) we denote the semigroup on \( L^2(D) \) of \( X_t \) killed upon exiting \( D \). That is,
\[
P^D_t f(x) = E_x(f(X_t), \tau_D > t), \quad x \in D, \; t > 0, \; f \in L^2(D).
\]
The semigroup has transition densities \( p^D(t, x, y) \) satisfying
\[
P^D_t f(x) = \int_D p^D(t, x, y) f(y) dy.
\]
The kernel \( p^D(t, x, y) \) is strictly positive symmetric and
\[
p^D(t, x, y) \leq p^{(\alpha)}(t, x, y) \leq c_{\alpha,d} t^{-d/\alpha}, \quad x, y \in D, \; t > 0.
\]
The fact that \( D \) is bounded implies that for any \( t > 0 \) the operator \( P^D_t \) maps \( L^2(D) \) into \( L^\infty(D) \). From the general theory of semigroups \(^{14}\) it follows that there exists an orthonormal basis of eigenfunctions \( \{\varphi_n\}_{n=1}^\infty \) for \( L^2(D) \) and corresponding eigenvalues \( \{\lambda_n\}_{n=1}^\infty \) satisfying
\[
0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots
\]
with \( \lambda_n \to \infty \) as \( n \to \infty \). That is, the pair \( \{ \varphi_n, \lambda_n \} \) satisfies

\[
P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x), \quad x \in D, \quad t > 0.
\]

The eigenfunctions \( \varphi_n \) are continuous and bounded on \( D \). In addition, \( \lambda_1 \) is simple and the corresponding eigenfunction \( \varphi_1 \), often called the ground state eigenfunction, is strictly positive on \( D \). For more general properties of the semigroups \( \{ P_t^D \}_{t \geq 0} \), see [26], [12], [16].

It is well known (see [4], [16], [17], [28]) that if \( D \) is a bounded connected Lipschitz domain and \( \alpha = 2 \), or that if \( D \) is a bounded connected domain for \( 0 < \alpha < 2 \), then \( \{ P_t^D \}_{t \geq 0} \) is intrinsically ultracontractive. This implies, among many other things, that

\[
\lim_{t \to \infty} \frac{e^{\lambda_1 t} P(t, x, y)}{\varphi_1(x) \varphi_1(y)} = 1,
\]

uniformly in both variables \( x, y \in D \). In addition, the rate of convergence is given by the spectral gap \( \lambda_2 - \lambda_1 \). That is, for any \( t \geq 1 \) we have

\[
e^{- (\lambda_2 - \lambda_1) t} \leq \sup_{x, y \in D} \left| \frac{e^{\lambda_1 t} P(t, x, y)}{\varphi_1(x) \varphi_1(y)} - 1 \right| \leq C(D, \alpha) e^{- (\lambda_2 - \lambda_1) t}.
\]

The proof of this for \( \alpha = 2 \) may be found in [38]. The proof in our setting is exactly the same.

In the Brownian motion case the properties of eigenfunctions and eigenvalues have been extensively studied for many years, both analytically and probabilistically. It is well known that geometric information on \( D \), such as convexity, symmetry, volume growth, smoothness of its boundary, etc., provides information not only on the ground state eigenfunction \( \varphi_1 \) and the ground state eigenvalue \( \lambda_1 \), but also on the spectral gap \( \lambda_2 - \lambda_1 \), and on the geometry of the nodal domains of \( \varphi_2 \).

In the case of stable processes of index \( 0 < \alpha < 2 \), very little is known. (We refer the reader to [6] where some of the known results are reviewed and for a discussion of the many open questions.) Except for the one-dimensional case ([6], [18]) we are not at present able to estimate from below the spectral gap \( \lambda_2 - \lambda_1 \) or obtain much useful geometric information on the eigenfunction corresponding to \( \lambda_2 \). In this paper we will instead study domains with one axis of symmetry and obtain estimates for \( \lambda_* - \lambda_1 \) where \( \lambda_* \) is the eigenvalue corresponding to the “first” antisymmetric eigenfunction for \( D \). In the Brownian motion case \( \lambda_* = \lambda_2 \) in many important cases (we will discuss this later in the sequel). Therefore estimates on \( \lambda_* - \lambda_1 \) are very closely related to estimates on \( \lambda_2 - \lambda_1 \). It is natural to conjecture that \( \lambda_* - \lambda_1 = \lambda_2 - \lambda_1 \) for the Cauchy process and for other symmetric \( \alpha \)-stable processes in various symmetric domains but this remains open.
For each $x = (x_1, x_2, \ldots, x_d)$ we put $\hat{x} = (-x_1, x_2, \ldots, x_d)$. For any domain $D \subset \mathbb{R}^d$, we set $D_+ = \{x \in D : x_1 > 0\}$ and $D_- = \{x \in D : x_1 < 0\}$. We say that $D$ is symmetric relative to the $x_1$-axis if $\hat{x} \in D$ whenever $x \in D$. Recall that the inradius $r_D$ of $D$ is the radius of the largest ball contained in $D$.

In Theorem 4.3 of [6] we proved that if $D \subset \mathbb{R}^d$ is a connected, bounded Lipschitz domain which is symmetric relative to the $x_1$–axis, then there exists an eigenfunction $\varphi_*$ for the Cauchy process with corresponding eigenvalue $\lambda_*$ which is antisymmetric relative to the $x_1$–axis ($\varphi_*(x) = -\varphi_*(\hat{x}), x \in D$) and (up to a sign) $\varphi_*(x) > 0$ for $x \in D_+$ and $\varphi_*(x) < 0$ for $x \in D_-$. Moreover, if $\varphi$ is any eigenfunction with eigenvalue $\lambda$ such that $\varphi$ is antisymmetric relative to the $x_1$–axis and $\varphi$ is different from $\varphi_*$ ($\varphi \notin \text{Span}\{\varphi_*\}$), then $\lambda_* < \lambda$. In other words, $\varphi_*$ has the smallest eigenvalue amongst all eigenfunctions which are antisymmetric relative to $x_1$–axis.

The main result of this paper is the following theorem.

**Theorem 1.1.** Let $D \subset \mathbb{R}^d$ be a bounded convex Lipschitz domain which is symmetric relative to the $x_1$–axis and $\{P_t^D\}_{t \geq 0}$ be the semigroup of the Cauchy process killed upon exiting $D$. Let $\lambda_*$ be the eigenvalue for $\{P_t^D\}_{t \geq 0}$ corresponding to the unique eigenfunction $\varphi_*$ which is antisymmetric relative to the $x_1$–axis and strictly positive on $D_+$ and strictly negative on $D_-$. Let $L = \sup\{x_1 : x = (x_1, \ldots, x_d) \in D\}$ and assume that the inradius $r_D$ of $D$ is equal to 1. Then we have

$$
\min \left( \frac{C_d}{L^2}, \frac{C_d'}{r_D} \right) \leq \lambda_* - \lambda_1
$$

where

$$C_d = \frac{\pi^2(d+1)}{2\pi d(d+2) + 4(d+1)}$$

and $C_d' = 4C_d/\pi^2$.

The eigenvalues $\lambda_n$ satisfies the scaling property $\lambda_n(kD) = \lambda_n(D)/k$, $k > 0$. This leads to the following easy conclusion.

**Corollary 1.1.** Let $D \subset \mathbb{R}^d$ satisfy the same assumption as in Theorem 1.1 except that now the inradius $r_D$ is arbitrary. Then we have

$$
\min \left( \frac{C_d r_D}{L^2}, \frac{C_d'}{r_D} \right) \leq \lambda_* - \lambda_1
$$

where $C_d, C_d'$ are the same as in Theorem 1.1. In particular, for a disk $D = B(0, r) \subset \mathbb{R}^2, r > 0$ we have

$$\frac{1}{6r} \leq \lambda_* - \lambda_1.$$
In terms of an upper estimate for the gap, we have the following

**Theorem 1.2.** Let $D \subset \mathbb{R}^d$ be a bounded convex domain of inradius $r_D$ and let $\lambda_1$, $\lambda_2$ eigenvalues for the semigroup of the Cauchy process killed upon exiting $D$. Then

\begin{equation}
\lambda_2 - \lambda_1 \leq \frac{\sqrt{\mu_2} - \frac{1}{2}\sqrt{\mu_1}}{r_D}
\end{equation}

where $\mu_1$ and $\mu_2$ are, respectively, the first and second eigenvalues for the Dirichlet Laplacian for the unit ball, $B(0,1)$, in $\mathbb{R}^d$. In fact, $\mu_2 = j_{d/2,1}^2$ and $\mu_1 = j_{d/2-1,1}^2$, where $j_{p,k}$ denotes the $k$th positive zero of the Bessel function $J_p(x)$.

The constants $C'_d$, $C''_d$ in Theorem 1.1 are of course not optimal. An easy calculation shows that $C_1 \approx 0.735$, ($C_1 > 7/10$), $C'_1 \approx 0.297$ ($C'_1 > 1/4$), $C_2 \approx 0.475$, ($C_2 > 4/10$), $C'_2 \approx 0.192$, ($C'_2 > 1/6$), $C_3 \approx 0.358$, ($C_3 > 1/3$), $C'_3 \approx 0.145$, ($C'_3 > 1/7$). In particular, for rectangles $R = [-L,L] \times [-1,1]$, where $L \geq 1$, we have

$$\min \left( \frac{2}{5L^2}, \frac{1}{6} \right) < \lambda_\ast - \lambda_1$$

As we shall see below, Theorem 1.2 holds in greater generality and it also raises interesting questions concerning sharp upper bounds; see Conjecture 4.4 below.

In the case of Brownian motion under the same assumptions on $D$ there is also an antisymmetric eigenfunction $\varphi_\ast$. In fact, for Brownian motion $\varphi_\ast$ restricted to $D_+$ is the first eigenfunction for $D_+$ and hence $\lambda_\ast(D) = \lambda_1(D_+)$. This fact has been used by several authors to study the van den Berg conjecture mentioned above ([8], [20]). For the Cauchy process $\lambda_1(D_+) \neq \lambda_\ast(D)$ (in fact $\lambda_1(D_+) < \lambda_\ast(D)$) and $\varphi_\ast$ restricted to $D_+$ is not the first eigenfunction for $D_+$. Such effect is due to the discontinuity of the paths of the Cauchy process. This is the reason for introducing the special eigenvalue $\lambda_\ast$ instead of studying $\lambda_1(D_+)$ as in the case of Brownian motion.

In the case of Brownian motion for a bounded domain $D$ the Courant-Hilbert nodal domain theorem asserts that the second eigenfunction $\varphi_2$ has exactly 2 nodal domains. That is, $D$ is divided into 2 connected subdomains $D_+$ and $D_-$ such that $\varphi_2 > 0$ on $D_+$ and $\varphi_2 < 0$ on $D_-$. If in addition $D \subset \mathbb{R}^2$ is convex, the nodal line $N = \{x \in D : \varphi_2(x) = 0\}$ touches the boundary at exactly 2 points ([30], [1]). Moreover, when $D \subset \mathbb{R}^2$ is convex and double symmetric, that is, $D$ is symmetric relative to both coordinate axes, there exists an eigenfunction corresponding to $\lambda_2$ with nodal line lying on one of the coordinate axes (see L. E. Payne [33]). In other words $\varphi_2 = \varphi_\ast$ or $\varphi_2$ is an antisymmetric eigenfunction defined analogously as $\varphi_\ast$ but with respect to the $x_2$–axis. Therefore in the case of Brownian motion, when
$D \subset \mathbb{R}^2$ is a convex double symmetric domain, estimates for $\lambda_2 - \lambda_1$ gives estimates for $\lambda_2 - \lambda_1$. Unfortunately, in the case of the Cauchy process we do not know anything about the location of the nodal line for the second eigenfunctions even in the simplest possible planar regions such as a disk or a rectangle. Nevertheless, it seems that the following conjecture should be true.

**Conjecture 1.1.** Let $D \subset \mathbb{R}^2$ be a convex domain which is symmetric relative to both coordinate axis. Let $\lambda_n, \varphi_n$ be the eigenvalues and eigenfunctions for the Cauchy process in $D$. Then there exists an eigenfunction corresponding to $\lambda_2$ such that its nodal line lies on one of the coordinate axis.

If this conjecture were true then the estimates for $\lambda_n - \lambda_1$ would give estimates for $\lambda_2 - \lambda_1$. We are not able to prove the Conjecture 1.1 partly because we do not know whether the Courant-Hilbert nodal domain theorem holds for the Cauchy process. It may be possible to gain some information on this conjecture by analyzing $\partial \varphi_1(x)/\partial x_i$ as in [33] but so far this remains open. In the simplest geometric situation of $D = (-1,1)$ we know the “shape” of the second eigenfunction, that $\lambda_2$ has multiplicity 1 and that $\lambda_2 = \lambda_\ast$, ([6], Theorem 5.3). However, even in this simple geometric setting the situation is fairly nontrivial.

The paper is organized as follows. In §2, we recall the connection between eigenvalues and eigenfunctions for the Cauchy process and the Steklov problem, ([6]). Using this we derive a variational formulas for $\lambda_\ast - \lambda_1$ and $\lambda_n - \lambda_1$. Such variational formulas are of independent interest. Also, in §2 we present some auxiliary lemmas which allow us to replace the Steklov eigenfunction $u_1(x,t)$ in the variational formula by the simpler expression $e^{-\lambda_1 t} \varphi_1(x)$.

In §3, we prove the weighted Poincaré–type inequality for the first eigenfunction $\varphi_1$. The Poincaré inequality has been used in the Brownian motion case in [37] and [38] to estimate $\lambda_2 - \lambda_1$. In that case the Poincaré inequality depends on the fact that for convex domains the first eigenfunction $\varphi_1$ is log–concave. For the Cauchy process this remains unknown. (For some geometric properties related to concavity for the eigenfunction in rectangles, see [9].) Nevertheless, by subordination we can show that $\varphi_1$ is the limit of integrals of log-concave functions and this allows us to obtain the appropriate inequality. We will show this Poincaré inequality not only for the Cauchy process but for all symmetric $\alpha$-stable processes $0 < \alpha < 2$.

In §4 we prove Theorems 1.1 and 1.2. The lower bound (Theorem 1.1) will follow from the variational formula and the Poincaré inequality. The upper bound is an easy observation that follows from a deep result of Ashbaugh and Benguria, [3], and a recent result of Chen and Song [18]. In §5 we present some open questions and possible extensions of our results.
2 Variational formulas

Unless otherwise explicitly mentioned, we assume throughout this section that \( \alpha = 1 \). We briefly recall the connection between our eigenvalue problem (1.3) and the mixed Steklov problem discussed in [6]. Let \( D \) be a bounded Lipschitz domain (see [6] for the precise definition of Lipschitz domain). For \( f \in L^1(\mathbb{R}^d) \) we set

\[
P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy
\]

where \( p(t, x, y) \) is given by (1.2). For \( f \in L^2(D) \) we extend it to all of \( \mathbb{R}^d \) by putting \( f(x) = 0 \) for \( x \in D^c \). Since \( D \) is bounded we see that such functions are also in \( L^1(\mathbb{R}^d) \). Thus \( P_t f(x) \) is well defined for \( f \in L^2(D) \) by our bound on \( p(t, x, y) \) and in particular it is well defined for any eigenfunction \( \varphi_n \) of our eigenvalue problem (1.3) extended to be zero outside of \( D \). For any \( n \in \mathbb{N}, x \in \mathbb{R}^d \) and \( t > 0 \) we put

\[
(2.1) \quad u_n(x, t) = P_t \varphi_n(x) \quad \text{and} \quad u_n(x, 0) = \varphi_n(x).
\]

This defines a function in

\[
H = \{(x, t) : x \in \mathbb{R}^d, t \geq 0 \}.
\]

For bounded Lipschitz domains, \( \varphi_n \) is continuous on all of \( \mathbb{R}^d \) (see [6], inequality (3.2)), so that \( u_n \) is continuous on all of \( H \). We will denote by \( H_+ \) the interior of the set \( H \). That is, \( H_+ = \{(x, t) : x \in \mathbb{R}^d, t > 0 \} \). Let

\[
\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial t^2}
\]

denote the Laplace operator in \( H_+ \).

We have (Theorem 1.1 [6])

\[
(2.2) \quad \Delta u_n(x, t) = 0; \quad (x, t) \in H_+,
\]

\[
(2.3) \quad \frac{\partial u_n}{\partial t}(x, 0) = -\lambda_n u_n(x, 0); \quad x \in D,
\]

\[
(2.4) \quad u_n(x, 0) = 0; \quad x \in D^c.
\]

The problem (2.2)–(2.3) is called a mixed Steklov problem. The functions \( u_n \) are called Steklov eigenfunctions. On bounded domains this problem has been extensively studied (see for example, [27], [25]). The transformation of our eigenvalue problem (1.3) for the Cauchy process to (2.2)–(2.4) enables us to derive a variational formula for \( \lambda_n \). This was done in [6], Theorem 3.8.
In this paper we will prove variational formulas for eigenvalue gaps $\lambda_n - \lambda_1$ and for $\lambda_* - \lambda_1$. For $D \subset \mathbb{R}^d$ we set

$$H_D = H_+ \cup \{(x, 0) \in H : x \in D\}.$$ 

For $\varepsilon > 0$ we set

$$H_\varepsilon = \{(x, t) \in H : t > \varepsilon\}.$$ 

By $\nabla$ we denote the “full” gradient in $H$. That is,

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}\right).$$

For brevity, $D_1, \ldots, D_d$ will denote $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$ and $D_{d+1}$ will denote $\frac{\partial}{\partial t}$. Similarly, $D_1^2, \ldots, D_d^2$ will denote $\frac{\partial^2}{\partial x_1^2}, \ldots, \frac{\partial^2}{\partial x_d^2}$ and $D_{d+1}^2$ will denote $\frac{\partial^2}{\partial t^2}$. Coordinate axes in $H$ will be denoted by $0x_1, \ldots, 0x_d, 0x_{d+1}$ and $0x_{d+1}$ denotes the $0t$ axis.

**Definition 2.1.** We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **piecewise $C^1$** on $\mathbb{R}$ if the following conditions (i) and (ii) are satisfied:

(i) There exist a set $A \subset \mathbb{R}$ consisting of at most finitely many points ($A$ may be empty) such that for any $x \in \mathbb{R} \setminus A$ the derivative $f'(x)$ exists, is finite and continuous at $x$.

(ii) $f'$ is bounded on $\mathbb{R} \setminus A$.

If we assume that $f : \mathbb{R} \to \mathbb{R}$ is piecewise $C^1$ on $\mathbb{R}$ and $f$ is continuous on $\mathbb{R}$ then $f$ has the following basic property. For any $a, b \in \mathbb{R}$ we have

$$\int_a^b f'(x) \, dx = f(b) - f(a).$$

We shall need the definition of the class of $C^1$ functions on $H_\varepsilon$.

**Definition 2.2.** Let $\varepsilon > 0$ and $f : H_\varepsilon \to \mathbb{R}$. We say that $f$ is **piecewise $C^1$ on $H_\varepsilon$** if the following conditions (i) and (ii) are satisfied for each $i = 1, \ldots, d, d+1$.

(i) For any line $l \subset H_\varepsilon$ parallel to $0x_i$ when $i = 1, \ldots, d$ or half-line $l \subset H_\varepsilon$ parallel to $0x_i$ when $i = d+1$ there exists a subset $A(l, i) \subset l$ ($A(l, i)$ depends on $l$ and $i$) consisting of at most finitely many points ($A(l, i)$ may be empty) such that for any $(x, t) \in l \setminus A(l, i)$ the derivative $D_i f(x, t)$ exists, is finite and is continuous at $(x, t)$ as a function on $l$.

(ii) There exists a constant $c(\varepsilon, i)$ such that for any $(x, t) \in H_\varepsilon$ which does not belong to any $A(l, i)$ we have $|D_i f(x, t)| \leq c(\varepsilon, i)$.
In the variational formulas for $\lambda_n - \lambda_1$ the functions $u_n/u_1$ will play a crucial role. We know that $\varphi_1 > 0$ on $D$ so $u_1 > 0$ on $H_D$, which implies that $u_n/u_1$ is well defined on $H_D$. Since for any $n \in \mathbb{N}$, $u_n$ is continuous on $H_D$, $u_n/u_1$ is also continuous on $H_D$. Intrinsic ultracontractivity for the semigroup $\{P_t\}_{t \geq 0}$ proved in [28] implies that that for any $n \in \mathbb{N}$ there exists a constant $c(D,n)$ such that for any for any $x \in D$ we have $\varphi_n(x) \leq c(D,n)\varphi_1(x)$. It follows from this that $u_n/u_1$ is bounded on $H_D$. We also have

$$
(2.5) \quad D_i \left( \frac{u_n}{u_1} \right)(x,t) = \frac{(D_i u_n(x,t))u_1(x,t) - (D_i u_1(x,t))u_n(x,t)}{u_1^2(x,t)}
$$

and

$$
(2.6) \quad u_1(x,t) = \int_D \frac{c dt}{(t^2 + |x-y|^2)^{(d+1)/2}} \varphi_1(y) dy.
$$

Fix $\varepsilon > 0$. Note that there exists a constant $c(D,\varepsilon)$ such that for any $(x,t) \in H_\varepsilon$ and $y \in D$ we have $t^2 + |x-y|^2 \leq c(D,\varepsilon)(t^2 + |x|^2)$. It follows that there is a constant $c(D,\varepsilon)$ such that for any $(x,t) \in H_\varepsilon$ we have $u_1(x,t) \geq c(D,\varepsilon)t(t^2 + |x|^2)^{-(d+1)/2}$. Lemma 3.3(e) in [6] states that there exists a constant $c(D,n,\varepsilon)$ such that for any $n \in \mathbb{N}$ and $(x,t) \in H_\varepsilon$ we have $|\nabla u_n(x,t)| \leq c(D,n,\varepsilon)(t^2 + |x|^2)^{-(d+1)/2}$. Therefore, we see from (2.5) that for any $i = 1, \ldots, d+1$ and $n \in \mathbb{N}$, $n \geq 2$ the derivative $D_i(u_n/u_1)$ is bounded on $H_\varepsilon$. In fact, there exists a constant $c = c(D,n,\varepsilon)$ such that $\nabla(u_n/u_1)(x,t) \leq c/t$ for any $(x,t) \in H_\varepsilon$.

We will now introduce the classes of functions $\mathcal{G}(D)$ and $\mathcal{G}_n(D)$ which we shall use in the variational characterization of $\lambda_n - \lambda_1$. (Note that the set $\mathcal{G}(D)$ is a linear space.)

**Definition 2.3.** Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We define $\mathcal{G}(D)$ to be the set of all functions $u : H_D \rightarrow \mathbb{R}$ satisfying the following conditions:

(i) $u$ is continuous and bounded on $H_D$.

(ii) For any $\varepsilon > 0$ $u$ is piecewise $C^1$ on $H_\varepsilon$.

(iii)

$$
\int_H |\nabla u(x,t)|^2 u_1^2(x,t) dx dt < \infty.
$$

When $D \subset \mathbb{R}^d$ is fixed and $u : H_D \rightarrow \mathbb{R}$, we simply set $\tilde{u}(x) = u(x,0)$, $x \in D$ and

$$
||\tilde{u}||_2 = \left( \int_D \tilde{u}^2(x) dx \right)^{1/2}.
$$
Definition 2.4. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For $n \geq 2$, set

$$G_n(D) = \{ u \in G(D) : \tilde{u} \varphi_1 \perp \varphi_1, \ldots, \varphi_{n-1}; \|\tilde{u}\varphi_1\| = 1 \}.$$ 

We will often simply write $G(D)$ for $G$ and $G_n(D)$ for $G_n$ when there is no danger of confusion.

Theorem 2.1. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then for any $n \geq 2$ we have

$$\lambda_n - \lambda_1 = \inf_{u \in G_n} \int_H |\nabla u(x,t)|^2 u_1^2(x,t) \, dx \, dt.$$ 

Moreover, the function $u_n/u_1 \in G_n$ and the infimum is achieved on this function. That is,

$$\lambda_n - \lambda_1 = \int_H \left| \nabla \left( \frac{u_n}{u_1} \right) (x,t) \right|^2 u_1^2(x,t) \, dx \, dt.$$ 

Definition 2.5. Let $D \subset \mathbb{R}^d$ be a connected bounded Lipschitz domain which is symmetric relative to the $x_1$-axis. We set

$$G_*(D) = \{ u \in G(D) : \tilde{u}$ is antisymmetric relative to $x_1$-axis and $\|\tilde{u}\varphi_1\| = 1 \}.$$ 

As above, we will often write $G_*(D)$ for $G_*$. Put $u_*(x,t) = P_t \varphi_*(x)$, $(x,t) \in H_+$, $u_*(x,0) = \varphi_*(x)$, $x \in \mathbb{R}^d$ as in formula (2.1).

Theorem 2.2. Let $D \subset \mathbb{R}^d$ be a connected bounded Lipschitz domain which is symmetric relative to the $x_1$-axis. We have

$$\lambda_* - \lambda_1 = \inf_{u \in G_*} \int_H |\nabla u(x,t)|^2 u_1^2(x,t) \, dx \, dt.$$ 

Moreover, the function $u_*/u_1 \in G_*$ and the infimum is achieved on this function. That is

$$\lambda_* - \lambda_1 = \int_H \left| \nabla \left( \frac{u_*}{u_1} \right) (x,t) \right|^2 u_1^2(x,t) \, dx \, dt.$$ 

The proofs of these results will be very similar to the proofs of the variational formulas for $\lambda_n$ and $\lambda_*$ proved in [6] (see the proofs of Propositions 3.4, 3.5, 3.6, 3.7, Theorem 3.8 and Proposition 4.8 in [6]). As in [6], we first need some auxiliary propositions.
Proposition 2.1. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and assume that $u : H_D \to \mathbb{R}$ satisfies conditions (i) and (ii) from Definition 2.3. Then for $\varepsilon > 0$ and $n \geq 2$,

\[
\int_{H_\varepsilon} \nabla u(x,t) \nabla \left( \frac{u_n}{u_1} \right)(x,t) u_1^2(x,t) \, dx \, dt = -\int_{\mathbb{R}^d} u(x,\varepsilon) u_1^2 \frac{\partial}{\partial t} \left( \frac{u_n}{u_1} \right)(x,\varepsilon) \, dx.
\]  

(2.7)

Both integrals are absolutely convergent.

Proof. First note that if $f : \mathbb{R} \to \mathbb{R}$ is piecewise $C^1$ on $\mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is $C^2$ on $\mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ is $C^1$ on $\mathbb{R}$, then a simple integration by parts gives

\[
\int_a^b f'g' h = [fg']_a^b - \int_a^b fg'' h - \int_a^b fg' h',
\]  

(2.8)

for any $a, b \in \mathbb{R}$, $a < b$. To prove (2.7) we need a multidimensional version of (2.8).

For this we need some more notation. For any $\varepsilon > 0$ and $a > \varepsilon$ let

\[
\Omega = \Omega(a,\varepsilon) = (-a,a) \times \ldots \times (-a,a) \times (\varepsilon,a).
\]  

d times

Of course, $\Omega \subset H_+$. Let $f : H_+ \to \mathbb{R}$, $g : H_+ \to \mathbb{R}$ and $h : H_+ \to \mathbb{R}$. Assume that for any $\varepsilon > 0$ is piecewise $C^1$ on $H_\varepsilon$, $g$ is $C^2$ on $H_\varepsilon$ and $h$ is $C^1$ on $H_\varepsilon$. Then (2.8) implies that for any $\varepsilon > 0$ and any $a > \varepsilon$ we have

\[
\int_\Omega (\nabla f)(\nabla g)h = \int_{\partial \Omega} f(D_\nu g)h - \int_\Omega f(\Delta g)h - \int_{\Omega} f(\nabla g)(\nabla h),
\]  

(2.9)

where $D_\nu$ is the outer normal derivative on $\partial \Omega$.

The identity (2.9) is a well known version of the Green formula, see [24], page 280, formula 5. But here, because of a very simple shape of $\Omega$ this formula follows directly from (2.8).

Let us fix $\varepsilon > 0$, $a > \varepsilon$ and apply (2.9) to $f = u$, $g = u_n/u_1$, $h = u_1^2$. We have

\[
\int_\Omega (\nabla u)(\nabla \left( \frac{u_n}{u_1} \right)) u_1^2 = \int_{\partial \Omega} u \left( D_\nu \left( \frac{u_n}{u_1} \right) \right) u_1^2
\]  

- \int_\Omega u \Delta \left( \frac{u_n}{u_1} \right) u_1^2 - \int_{\Omega} u \left( \nabla \left( \frac{u_n}{u_1} \right) \right) (\nabla (u_1^2)) = I - II - III.
\]  

(2.10)

We first calculate the integrals II and III. Recall that for any $i = 1, \ldots, d, d + 1$ we have

\[
D_i \left( \frac{u_n}{u_1} \right) = \frac{(D_i u_n) u_1 - (D_i u_1) u_n}{u_1^2}.
\]  

11
Simple calculations gives

\[
D_i^2 \left( \frac{u_n}{u_1} \right) = \frac{(D_i^2 u_n) u_1 - (D_i^2 u_1) u_n - 2(D_i u_1)(D_i u_n)}{u_1^2} + \frac{2(D_i u_1)^2 u_n}{u_1^2}.
\]

It follows that

\[
II = \int_\Omega uu_1 \sum_{i=1}^{d+1} D_i^2 \left( \frac{u_n}{u_1} \right) \]

(2.11)

\[
= \int_\Omega uu_1 \sum_{i=1}^{d+1} D_i^2 u_n - \int_\Omega uu_n \sum_{i=1}^{d+1} D_i^2 u_1 - 2 \int_\Omega u \sum_{i=1}^{d+1} (D_i u_1)(D_i u_n) \]

\[
+ 2 \int_\Omega u \frac{u_n}{u_1} \sum_{i=1}^{d+1} (D_i u_1)^2.
\]

Since the functions \(u_n\) are all harmonic in \(H_+\), \(\Delta u_n = 0\) and it follows that the first two integrals in (2.11) are zero.

Similarly,

\[
III = \int_\Omega u \sum_{i=1}^{d+1} \left( D_i \left( \frac{u_n}{u_1} \right) \right) D_i \left( u_1^2 \right) \]

\[
= 2 \int_\Omega u \sum_{i=1}^{d+1} (D_i u_1)(D_i u_n) - 2 \int_\Omega u \frac{u_n}{u_1} \sum_{i=1}^{d+1} (D_i u_1)^2.
\]

Comparing the expressions for \(II\) and \(III\) we obtain that \(II + III = 0\). By (2.10) we get

\[
(2.12) \quad \int_\Omega (\nabla u) \left( \nabla \left( \frac{u_n}{u_1} \right) \right) u_1^2 = \int_{\partial_\Omega} uu_1 \left( D_{\nu} \left( \frac{u_n}{u_1} \right) \right).
\]

Next we estimate \(u_1^2(x, t)\). For \((x, t) \in \overline{H_\varepsilon}\) (the closure of \(H_\varepsilon\)) we have

\[
(2.13) \quad u_1(x, t) = \int_D \frac{c_d}{(t^2 + |x - y|^2)^{(d+1)/2}} \varphi_1(y) dy \leq c(D, \varepsilon)(t^2 + |x|^2)^{-d/2}.
\]

Hence \(u_1^2(x, t) \leq c(D, \varepsilon)(t^2 + |x|^2)^{-d}\). Note also that \(u\) satisfies condition (i) from Definition 2.3 so \(u\) is bounded on \(\overline{H_\varepsilon}\). By the remarks before Definition 2.3 \(\nabla(u_n/u_1)\) is bounded on \(\overline{H_\varepsilon}\) so that \(D_{\nu}(u_n/u_1)\) is bounded on \(\partial \Omega = \partial(\Omega(a, \varepsilon))\), independently on \(a\).

The boundary of \(\Omega\) consists of \(2(d+1)\) faces. We denote by \((\partial \Omega)_1\) the face which is a subset of \(\partial H_\varepsilon\). For any \((x, t) \in \partial \Omega \setminus (\partial \Omega)_1\) we have \(|x|^2 + t^2 \geq a^2\) so
for such \((x, t)\) we have \(u_1^2(x, t) \leq c_1(D, \varepsilon)a^{-2d}\). The measure of \(\partial \Omega\) is bounded by \(c(d)a^d\). It follows that

\[
\int_{\partial\Omega \setminus \partial\Omega_1} uu_1^2 D_\nu \left( \frac{u_n}{u_1} \right) \leq c(D, \varepsilon)a^{-d},
\]

so when \(\varepsilon > 0\) is fixed and \(a \to \infty\) this integral tends to 0. Note that for \((x, t) \in (\partial \Omega)_1\) we have \(D_\nu u_1 = -\frac{\partial}{\partial t} u_1\). It follows that

\[
\lim_{a \to \infty} \int_{\partial \Omega} uu_1^2 D_\nu \left( \frac{u_n}{u_1} \right) u_1 = -\int_{\partial H_\varepsilon} uu_1^2 \frac{\partial}{\partial t} \left( \frac{u_n}{u_1} \right).
\]

The last integral is absolutely convergent by (2.13). When \(\varepsilon > 0\) is fixed and \(a \to \infty\) the set \(\Omega\) tends to \(H_\varepsilon\). Therefore the left hand side of (2.12) tends to

\[
\int_{H_\varepsilon} (\nabla u) \left( \nabla \left( \frac{u_n}{u_1} \right) \right) u_1^2,
\]

when \(a \to \infty\). When \(d \geq 2\) this integral is absolutely convergent by (2.13) and by the fact that \(\nabla u\) and \(\nabla (u_n/u_1)\) are bounded on \(H_\varepsilon\). When \(d = 1\) the last integral is absolutely convergent by (2.13), the fact that \(\nabla u\) is bounded on \(H_\varepsilon\) and the fact that \(\nabla (u_n/u_1)(x, t) \leq c/t\) for \(c = c(D, n, \varepsilon)\) and any \((x, t) \in H_\varepsilon\).

**Proposition 2.2.** Let \(D \subset \mathbb{R}^d\) be a bounded Lipschitz domain and assume that \(u : H_D \to \mathbb{R}\) satisfies conditions (i) and (ii) from Definition 2.3. Then for \(n \geq 2\) we have

\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} u(x, \varepsilon)u_1^2(x, \varepsilon) \frac{\partial}{\partial t} \left( \frac{u_n}{u_1} \right)(x, \varepsilon) dx = -(\lambda_n - \lambda_1) \int_D \varphi_n(x)\varphi_1(x) u(x, 0) dx.
\]

**Proof.** Let \(r_n\) be defined as in Proposition 3.1 in [6]. By Proposition 3.2 (iii) in [6] we get

\[
u_1^2(x, \varepsilon) \frac{\partial}{\partial t} \left( \frac{u_n}{u_1} \right)(x, \varepsilon) = \frac{\partial u_n}{\partial t}(x, \varepsilon)u_1(x, \varepsilon) - \frac{\partial u_1}{\partial t}(x, \varepsilon)u_n(x, \varepsilon)
\]

\[
= (-\lambda_n u_n(x, \varepsilon) + P\varepsilon r_n(x))u_1(x, \varepsilon) - (-\lambda_1 u_1(x, \varepsilon) + P\varepsilon r_1(x))u_n(x, \varepsilon).
\]

Since \(u\) is bounded we obtain

\[
\left| \int_{\mathbb{R}^d} u(x, \varepsilon)P\varepsilon r_n(x)u_1(x, \varepsilon) dx \right| \leq \|u\|_{\infty} \int_{\mathbb{R}^d} |P\varepsilon r_n(x)| u_1(x, \varepsilon) dx.
\]
The last integral tends to 0 as $\varepsilon$ tends to $0^+$ by Proposition 3.5 (formula (3.14)) in [6]. Exactly in the same way
\[
\left| \int_{\mathbb{R}^d} u(x, \varepsilon) P_\varepsilon r_1(x) u_n(x, \varepsilon) \, dx \right|
\]
tends to 0 as $\varepsilon$ tends to $0^+$.

The only thing which remains is to verify that
\[
(2.14) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} u_n(x, \varepsilon) u_1(x, \varepsilon) u(x, \varepsilon) \, dx = \int_D \varphi_n(x) \varphi_1(x) u(x, 0) \, dx.
\]
Note that $u$ is bounded and $\lim_{\varepsilon \to 0^+} u_n(x, \varepsilon) = \varphi_n(x), \, x \in \mathbb{R}^d$ (recall that $\varphi_n(x) = 0$ for $x \in D^c$). By definition of $u_n$, for any $x \in \mathbb{R}^d$ and $\varepsilon \in (0, 1)$ we have
\[
|u_n(x, \varepsilon)| = |P_\varepsilon \varphi_n(x)| \leq c(D)||\varphi_n||_\infty (1 + \delta_D(x))^{-d-1},
\]
where $\delta_D(x) = \text{dist}(x, \partial D)$. Now (2.14) follows by the bounded convergence theorem.

**Proposition 2.3.** Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then for $n \geq 2$ we have
\[
\int_H \left| \nabla \left( \frac{u_n}{u_1} \right)(x, t) \right|^2 u_1^2(x, t) \, dx \, dt = \lambda_n - \lambda_1.
\]
In particular, we conclude that $u_n / u_1$ satisfies condition (iii) of Definition 2.3 and hence $u_n / u_1 \in \mathcal{G}$.

**Proof.** Since $u_n / u_1$ satisfies conditions (i) and (ii) from Definition 2.3 we can apply Propositions 2.1 and 2.2. This gives
\[
\int_H \left| \nabla \left( \frac{u_n}{u_1} \right)(x, t) \right|^2 u_1^2(x, t) \, dx \, dt = \lim_{\varepsilon \to 0^+} \int_{H\varepsilon} \left| \nabla \left( \frac{u_n}{u_1} \right)(x, t) \right|^2 u_1^2(x, t) \, dx \, dt
\]
\[
= - \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} \frac{u_n(x, \varepsilon)}{u_1(x, \varepsilon)} \frac{u_1^2(x, \varepsilon)}{u_1^2(x, \varepsilon)} \frac{\partial}{\partial t} \left( \frac{u_n}{u_1} \right)(x, \varepsilon) \, dx
\]
\[
= (\lambda_n - \lambda_1) \int_D \varphi_n(x) \varphi_1(x) \frac{u_n(x, 0)}{u_1(x, 0)} \, dx = \lambda_n - \lambda_1.
\]

**Proposition 2.4.** Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $u \in \mathcal{G}$. Then for $n \geq 2$
\[
\int_H \nabla u(x, t) \nabla \left( \frac{u_n}{u_1} \right)(x, t) u_1^2(x, t) \, dx \, dt = (\lambda_n - \lambda_1) \int_D \varphi_n(x) \varphi_1(x) u(x, 0) \, dx.
\]
Both integrals are absolutely convergent.
Proof. Since $u$ and $u_n/u_1$ satisfy condition (iii) of Definition 2.3 we have
\[
\lim_{\varepsilon \to 0^+} \int_{H_\varepsilon} \nabla u(x,t) \nabla \left( \frac{u_n}{u_1} \right) (x,t) u_1^2(x,t) \, dx \, dt = \int_{H} \nabla u(x,t) \nabla \left( \frac{u_n}{u_1} \right) (x,t) u_1^2(x,t) \, dx \, dt
\]
and the integral on the right hand side is absolutely convergent. The proposition follows from Propositions 2.1 and 2.2.

To simplify notation set
\[
Q(u,v) = \int_{H} \nabla u(x,t) \nabla v(x,t) u_1^2(x,t) \, dx \, dt.
\]
Note that for any $u, v \in \mathcal{G}$ the expression $Q(u, v)$ is well defined and finite.

Proof of Theorem 2.1. We must show that $\lambda_n - \lambda_1 = \inf_{u \in \mathcal{G}_n} Q(u,u)$. Of course, $u_n/u_1 \in \mathcal{G}_n$ and by Proposition 2.3,
\[
\inf_{u \in \mathcal{G}_n} Q(u,u) \leq Q(u_n/u_1, u_n/u_1) = \lambda_n - \lambda_1.
\]

It remains to show that
\[
\inf_{u \in \mathcal{G}_n} Q(u,u) \geq \lambda_n - \lambda_1.
\]
Fix $u \in \mathcal{G}_n$. For $(x,t) \in H_D$ set
\[
v_k(x,t) = (\sum_{m=1}^{k} c_m u_m(x,t)) / u_1(x,t),
\]
where $c_m = \int_D \tilde{u}(x) \varphi_1(x) \varphi_m(x) \, dx$. Since $u \in \mathcal{G}_n$, $n \geq 2$ we know that $\tilde{u} \perp \varphi_1^2$ so $c_1 = 0$. It follows that $v_k \in \mathcal{G}$ because $u_m/u_1 \in \mathcal{G}$ ($m \geq 2$) and $\mathcal{G}$ is a linear space.

We have
\[
(2.15) \quad Q(u,u) = Q(v_k, v_k) + Q(u - v_k, u - v_k) + 2Q(u - v_k, v_k)
\]
and
\[
(2.16) \quad Q(u - v_k, v_k) = \sum_{m=1}^{k} c_m Q(u, u_m/u_1) - \sum_{m=1}^{k} c_m Q(v_k, u_m/u_1).
\]
By Proposition 2.4 the right hand side equals
\[
\sum_{m=1}^{k} c_m(\lambda_m - \lambda_1) \left( \int_D \varphi_m(x) \varphi_1(x) u(x, 0) \, dx - \int_D \varphi_m(x) \varphi_1(x) v_k(x, 0) \, dx \right).
\]
But \( \int_D \varphi_m(x) \varphi_1(x) u(x, 0) \, dx = c_m \) and for \( m = 1, \ldots, k \)
\[
\int_D \varphi_m(x) \varphi_1(x) v_k(x, 0) \, dx = \sum_{l=1}^{k} \int_D \varphi_m(x) \varphi_1(x) \varphi_l(x) / \varphi_1(x) \, dx = c_m.
\]
Thus the expression in (2.16) must be zero. We have also shown that \( Q(v_k, v_k) = \sum_{m=1}^{k} c_m^2(\lambda_m - \lambda_1) \). Since \( u \in \mathcal{G}_n \) we have \( \|\tilde{u} \varphi_1\|_2 = 1 \), \( \tilde{u} \varphi_1 \perp \varphi_1, \ldots, \varphi_{n-1} \) so \( c_1 = \ldots = c_{n-1} = 0 \). Hence \( \sum_{m=n}^{\infty} c_m^2 = 1 \). Therefore for \( k \geq n \) we get by (2.15)
\[
Q(u, u) \geq Q(v_k, v_k) = \sum_{m=n}^{k} c_m^2(\lambda_m - \lambda_1) \geq (\lambda_n - \lambda_1) \sum_{m=n}^{k} c_m^2.
\]
Since \( k \geq n \) is arbitrary, we conclude that \( Q(u, u) \geq \lambda_n - \lambda_1 \).

Proof of Theorem 2.2. We must show that \( \lambda_* - \lambda_1 = \inf_u Q(u, u) \). Assume that \( \lambda_* \) has multiplicity \( m \geq 1 \) and that it is one of the eigenvalues \( \lambda_k = \ldots = \lambda_{k+m-1} \), for some \( k \geq 2 \). We may assume that \( u_* = u_k \). Note also that \( u_*/u_1 \in \mathcal{G}_* \). By Proposition 2.3 we get
\[
\lambda_* - \lambda_1 = \lambda_k - \lambda_1 = Q(u_k/u_1, u_k/u_1) = Q(u_*/u_1, u_*/u_1) \geq \inf_{u \in \mathcal{G}_*} Q(u, u).
\]
It remains to show that
\[
\inf_{u \in \mathcal{G}_*} Q(u, u) \geq \lambda_* - \lambda_1.
\]
We have \( \lambda_* - \lambda_1 = \lambda_k - \lambda_1 = \inf_{u \in \mathcal{G}_k} Q(u, u) \), where \( \mathcal{G}_k = \{ u \in \mathcal{G} : \tilde{u} \varphi_1 \perp \varphi_1, \ldots, \varphi_{k-1}; \|\tilde{u} \varphi_1\|_2 = 1 \} \). By the proof of Proposition 4.8 in [6], \( \varphi_1, \ldots, \varphi_{k-1} \) are all symmetric relative to \( x_1 \)-axis. It follows that \( \mathcal{G}_* \subset \mathcal{G}_k \) and hence,
\[
\lambda_* - \lambda_1 = \inf_{u \in \mathcal{G}_k} Q(u, u) \leq \inf_{u \in \mathcal{G}_*} Q(u, u).
\]

We end this section with two lemmas which allow us to replace the Steklov eigenfunction \( u_1(x, t) \) in the variational formula by the simpler expression \( e^{-\lambda_1 t} \varphi_1(x) \).

Lemma 2.1. For any \( x \in D \) and \( t > 0 \) we have
\[
\inf_{u \in \mathcal{G}_*} Q(u, u) \geq \inf_{u \in \mathcal{G}_*} Q(u, u).
\]
Proof. By Proposition 3.2 (iii) in [6] we have
\[
\frac{\partial u_1}{\partial t}(x, t) = -\lambda_1 u_1(x, t) + P_t r_1(x),
\]
for \(x \in \mathbb{R}^d, t > 0\). Moreover we have \(r_1(x) \geq 0\) for all \(x \in \mathbb{R}^d\). This follows from Proposition 3.1 [6] and the fact that \(\varphi_1 \geq 0\) on \(\mathbb{R}^d\). Put \(f(x, t) = e^{\lambda_1 t} u_1(x, t), (x, t) \in H\). We have
\[
\frac{\partial f}{\partial t}(x, t) = e^{\lambda_1 t} P_t r_1(x) \geq 0,
\]
for \(x \in \mathbb{R}^d, t > 0\). Hence, for each fixed \(x \in D\) the function \(f(x, t)\) is nondecreasing as a function of \(t\). Therefore for \(x \in D\) we have \(f(x, t) \geq f(x, 0) = \varphi_1(x)\).

The following lemma is an immediate conclusion of Theorem 2.2 and Lemma 2.1.

Lemma 2.2. We have
\[
\lambda_* - \lambda_1 \geq \int_0^\infty \int_D \left| \nabla \left( \frac{u_*}{u_1} \right) (x, t) \right|^2 \varphi_1^2(x) e^{-2\lambda_1 t} dx dt.
\]

3 Weighted Poincaré inequalities

Let us recall that the positive function \(g\) defined on the interval \((-l, l)\) is log–concave if the function \(\log(g)\) is concave in \((-l, l)\). That is, for all \(x, y \in (-l, l)\) and \(0 \leq \lambda \leq 1\),
\[
\log(g(\lambda x + (1 - \lambda)y)) \geq \lambda \log(g(x)) + (1 - \lambda) \log(g(y))
\]
or equivalently,
\[
g(\lambda x + (1 - \lambda)y) \geq \log(g(x))^\lambda \log(g(y))^{1-\lambda}.
\]

If \(g\) is a positive function defined on a convex domain \(D \subset \mathbb{R}^d\), then \(g\) is said to be log–concave on \(D\) if it is log–concave on every segment contained in \(D\). The celebrated theorem of Brascamp and Lieb [14] asserts that in the case of Brownian motion, \(\varphi_1\) is log–concave if \(D\) is convex. In fact, their result is more general than that and it is one of this more general versions that we shall use below. We state it here in the form that we need. Let us recall that in the introduction we have defined (see [14])
\[
p_t^{(2)}(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.
\]
This is just the Gaussian density in $\mathbb{R}^d$. This is the density for Brownian motion running at twice the usual speed. By $B_t$ we denote the standard Brownian motion in $\mathbb{R}^d$. That is, in our notation we have $P_x(B_{2t} \in A) = \int_A p_t^{(2)}(x - y) \, dy$, $x \in \mathbb{R}^d$, $t > 0$, $A \subset \mathbb{R}^d$.

**Proposition 3.1.** (Brascamp–Lieb [17]) Let $D \subset \mathbb{R}^d$ be a bounded convex domain and for $n \in \mathbb{N}$, let $t_1, t_2, \ldots, t_n$ be real numbers in $(0, \infty)$. For $x \in D$ define the function

$$G_n(x; t_1, \ldots, t_n) = \int_D \cdots \int_D \prod_{i=1}^n p_t^{(2)}(x_{i-1} - x_i) \, dx_1 \ldots dx_n,$$

where $x_0 = x$. As a function of $x$, $G_n(x; t_1, t_2, \ldots, t_n)$ is log–concave in $D$.

Note that

$$G_n(x; t_1, \ldots, t_n) = P_x\{B_{2t_1} \in D, B_{2(t_1+t_2)} \in D, \ldots, B_{2(t_1+t_2+\cdots+t_n)} \in D\}.$$

Our desired Poincaré inequality will follow from this proposition, subordination and inequalities already known for log–concave functions. First, we recall the latter.

**Proposition 3.2.** (Payne–Weinberger [34], Smits [38]). Let $l > 0$, $g : (-l, l) \to \mathbb{R}$ be positive and log-concave. Let $f : (-l, l) \to \mathbb{R}$ be piecewise $C^1$ and satisfying

$$\int_{-l}^l f(x)g(x) \, dx = 0.$$

Then

$$\int_{-l}^l (f'(x))^2 g(x) \, dx \geq \frac{\pi^2}{4l^2} \int_{-l}^l f^2(x)g(x) \, dx.$$

As an easy consequence of this proposition we obtain the following result.

**Corollary 3.1.** Let $l > 0$, $g : (-l, l) \to \mathbb{R}$ be positive, log-concave, and satisfying $g(-x) = g(x)$, $x \in (-l, l)$. That is $g$ is symmetric. Let $f : (-l, l) \to \mathbb{R}$ be piecewise $C^1$ and satisfying $f(-x) = -f(x)$, $x \in (-l, l)$. That is $f$ is antisymmetric. Then

$$\int_{-l}^l (f'(x))^2 g(x) \, dx \geq \frac{\pi^2}{4l^2} \int_{-l}^l f^2(x)g(x) \, dx.$$

From now on we assume that $D$ satisfies the assumptions of Theorem 1.1, $L = \sup\{x_1 : x = (x_1, \ldots, x_d) \in D\}$. As an easy conclusion of the above corollary we get the following proposition.

18
Proposition 3.3. (Smits[38]) Let \( g : D \to \mathbb{R} \) be positive, log-concave, and satisfying \( g(\tilde{x}) = g(x), x \in D \). That is, \( g \) is symmetric relative to \( x_1 \)-axis. Let \( f : D \to \mathbb{R}, f \in C^\infty(D) \) and satisfy \( f(\tilde{x}) = -f(x), x \in D \). That is, \( f \) is antisymmetric relative to \( x_1 \)-axis. Then

\[
\int_D \left\| \frac{\partial f}{\partial x_1}(x) \right\|^2 g(x) \, dx \geq \frac{\pi^2}{4L^2} \int_D f^2(x) g(x) \, dx.
\]

These type of inequalities are commonly known as Poincaré inequalities (see Payne-Weinberger [34]).

Although we are not able to prove that the first eigenfunction \( \varphi_1 \) for the Cauchy process for the domain \( D \) is log-concave, we will be able to show that the assertion of the previous proposition holds for \( g = \varphi_2^2 \) using Proposition 3.1 and subordination. That is, we have

**Theorem 3.1.** Let \( f : D \to \mathbb{R}, f \in C^\infty(D) \) and satisfying \( f(\tilde{x}) = -f(x), x \in D \). That is, \( f \) is antisymmetric relative to \( x_1 \)-axis. Then

\[
\int_D \left| \frac{\partial f}{\partial x_1}(x) \right|^2 \varphi_1^2(x) \, dx \geq \frac{\pi^2}{4L^2} \int_D f^2(x) \varphi_1^2(x) \, dx
\]

where \( \varphi_1 \) is the first eigenfunction for the symmetric stable process of index \( 0 < \alpha < 2 \).

**Proof.** Let us recall that for \( 0 < \alpha < 2 \) the symmetric stable process \( X_t \) in \( \mathbb{R}^d \) has the representation

\[
X_t = B_{2\sigma_t},
\]

where \( \sigma_t \) is a stable subordinator of index \( \alpha/2 \) independent of \( B_t \) (see [11]). Thus

\[
p_t^{(\alpha)}(x - y) = \int_0^\infty p_s^{(2)}(x - y) g_{\alpha/2}(t, s) \, ds,
\]

where \( g_{\alpha/2}(t, s) \) is the transition density of \( \sigma_t \).

Let \( x \in D, n \in \mathbb{N}, 0 < t_1 < t_2 < \ldots, t_n \) and set \( x_0 = x \) and \( t_0 = 0 \). Using the Markov property for the stable process \( X_t \), the subordination formula (3.5),

19
Fubini’s theorem, in this order, we obtain,

\( F_n(x; t_1, \ldots, t_n) = P_x\{X_{t_1} \in D, \ldots, X_{t_n} \in D\} \)

\[ = \int_D \cdots \int_D \prod_{i=1}^{n} p^{(a)}_{t_i-t_{i-1}}(x_{i-1} - x_i) \, dx_1 \cdots dx_n \]

\[ = \int_0^\infty \cdots \int_0^\infty \left( \int_D \cdots \int_D \prod_{i=1}^{n} p^{(2)}_{s_i}(x_{i-1} - x_i) \, dx_1 \cdots dx_n \right) \]

\[ \times \prod_{i=1}^{n} g_{\alpha/2}(t_i - t_{i-1}, s_i) \, ds_1 \cdots ds_n \]

\[ = \int_0^\infty \cdots \int_0^\infty G_n(x; s_1, \ldots, s_n) \prod_{i=1}^{n} g_{\alpha/2}(t_i - t_{i-1}, s_i) \, ds_1 \cdots ds_n, \]

where \( G_n \) is defined as in Proposition 3.1.

Let us note that the product of log-concave functions is log-concave. Using this, Proposition 3.1 and Proposition 3.3, for each sequence of positive numbers \( s_1, s_2, \ldots, s_n \) and \( \tilde{s}_1, \tilde{s}_n, \ldots, \tilde{s}_m, n, m \in \mathbb{N} \) we have (with \( f \) as in the statement of the theorem),

\[ \int_D \left| \frac{\partial f}{\partial x_1}(x) \right|^2 G_n(x; s_1, \ldots, s_n)G_m(x; \tilde{s}_1, \ldots, \tilde{s}_m) \, dx \geq \frac{\pi^2}{4L^2} \int_D f^2(x)G_n(x; s_1, \ldots, s_n)G_m(x; \tilde{s}_1, \ldots, \tilde{s}_m) \, dx. \]

Integrating this inequality with respect to \( s_1 \ldots s_n \) and \( \tilde{s}_1, \ldots, \tilde{s}_m \) we obtain by (3.6) – (3.9),

\[ \int_D \left| \frac{\partial f}{\partial x_1}(x) \right|^2 F_n(x; t_1, \ldots, t_n)F_m(x; \tilde{t}_1, \ldots, \tilde{t}_m) \, dx \geq \frac{\pi^2}{4L^2} \int_D f^2(x)F_n(x; t_1, \ldots, t_n)F_m(x; \tilde{t}_1, \ldots, \tilde{t}_m) \, dx, \]

for \( 0 < t_1 < t_2 < \cdots < t_n \) and \( 0 < \tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_m \).

Now, let \( \tau_D = \inf \{ t \geq 0 : X_t \notin D \} \). Since \( D \) is bounded and has a Lipschitz boundary Lemma 6 from [13] gives that \( P_x(X(\tau_D) \in \partial D) = 0 \), for any \( x \in D \). Using this and the right continuity of the sample paths we obtain that for any
\[ x \in D \]

\[ P_x \{ \tau_D > t \} = P_x \{ X_s \in D, \forall 0 \leq s \leq t \} = \lim_{n \to \infty} P_x \{ X_{\frac{t}{n}} \in D, i = 1, \ldots, n \} = \lim_{n \to \infty} F_n \left( x; \frac{t}{n}, \frac{2t}{n}, \ldots, \frac{(n-1)t}{n}, t \right). \]

Fix \( t > 0 \), let \( n, m \in \mathbb{N} \), \( t_i = it/n, i = 1, \ldots, n \) and \( \tilde{t}_i = it/m, i = 1, \ldots, m \). Letting \( n \) and \( m \) go to \( \infty \), it follows from (3.10) and (3.11) that

\[ \int_D \left| \frac{\partial f}{\partial x_1}(x) \right|^2 (P_x \{ \tau_D > t \})^2 \, dx \geq \frac{\pi^2}{4L^2} \int_D f^2(x)(P_x \{ \tau_D > t \})^2 \, dx \]

for all \( t > 0 \).

From the “intrinsic ultracontractive” properties of the semigroup for stable processes in general bounded domains (see [16], [17], [28]), it follows that for any symmetric stable process

\[ \lim_{t \to \infty} e^{\lambda t} P_x \{ \tau_D > t \} = \varphi_1(x) \]

and this convergence is uniform for \( x \in D \). The inequality (3.3) follows from (3.12) and (3.13) and the theorem is proved.

We call the inequality (3.3) a “weighted Poincaré-type inequality for stable processes.” It is interesting to note that the eigenfunction \( \varphi_1 \) in (3.3) can be replaced by various other similarly generated functions from \( P_x \{ \tau_D > t \} \). For example, we may replace \( \varphi_1 \) by \( E_x \tau_D^p \) or by \( (E_x \tau_D)^p \), for any \( 0 < p < \infty \). In addition, the theorem holds for any process obtained from Brownian motion by subordination such as the relativistic process studied in [35].

4 Proof of Theorems 1.1 and 1.2

Unless otherwise explicitly mentioned, we assume throughout this section that \( \alpha = 1 \) and that \( D \) satisfies the assumptions of Theorem 1.1. We shall now apply the results of the previous section and our variational characterization for \( \lambda_s - \lambda_1 \) to prove Theorem 1.1.

As an immediate conclusion of Theorem 3.1 we get the following proposition.
Proposition 4.1. Let $u : D \times (0, \infty) \to \mathbb{R}$ be such that for any $t \in (0, \infty)$ the function $u(\cdot, t) \in C^\infty(D)$. Assume also that $u(\hat{x}, t) = -u(x, t)$ for any $x \in D$ and $t \in (0, \infty)$. Then for any $t \in (0, \infty)$ we have

$$
\int_D \left| \frac{\partial u}{\partial x_1}(x, t) \right|^2 \varphi_1^2(x) dx \geq \frac{\pi^2}{4L^2} \int_D u^2(x, t) \varphi_1^2(x) dx.
$$

Recall that $\nabla$ is the “full” gradient in $H$, that is,

$$\nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right).$$

Observe that the function $u(x, t) = u_*(x, t)/u_1(x, t)$ satisfies the assumptions of Proposition 4.1. Therefore

$$\int_0^\infty \int_D \left| \nabla \left( \frac{u_*}{u_1} \right) (x, t) \right|^2 \varphi_1^2(x) e^{-2\lambda_1 t} dx dt \geq \min \left( \frac{\pi^2}{4L^2}, 1 \right) \int_0^\infty \int_D \left( \frac{u_*^2(x, t)}{u_1^2(x, t)} + \left| \frac{\partial}{\partial t} \left( \frac{u_*}{u_1} \right) (x, t) \right|^2 \right) \varphi_1^2(x) e^{-2\lambda_1 t} dx dt.
$$

Lemma 4.1. Let $f : [0, \infty) \to \mathbb{R}$ be a bounded continuous function such that its first derivative $f'$ exists and is bounded on $[0, \infty)$. Then for any $c > 0$ we have

$$I(f) = \int_0^\infty \left( f^2(t) + (f'(t))^2 \right) e^{-ct} dt \geq \frac{f^2(0)}{c + 1}.
$$

Proof. We have

$$I(f) \geq \int_0^\infty (-2f(t)f'(t)) e^{-ct} dt = f^2(0) - c \int_0^\infty f^2(t)e^{-ct} dt.
$$

It follows that

$$(c + 1)I(f) \geq c \int_0^\infty f^2(t)e^{-ct} dt + I(f) \geq f^2(0).$$

We do not know whether the inequality (4.2) is optimal. Note only that if we put $f(t) \equiv f(0)$, $t \geq 0$, then $I(f) = f^2(0)/c$. 

22
Proof of Theorem 1.1. By Lemma 2.2, (4.1) and Lemma 4.1, we obtain

\[ \lambda_\ast - \lambda_1 \geq \int_0^\infty \int_D \left| \nabla \left( \frac{u_\ast(x,t)}{u_1(x,t)} \right) \right|^2 \varphi_1^2(x) e^{-2\lambda_1 t} \, dx \, dt \]

\[ \geq \min \left( \frac{\pi^2}{4L^2}, 1 \right) \int_D \int_0^\infty \left( \frac{u_\ast^2(x,t)}{u_1^2(x,t)} + \left| \frac{\partial}{\partial t} \left( \frac{u_\ast}{u_1} \right)(x,t) \right| \right)^2 e^{-2\lambda_1 t} \, dt \varphi_1^2(x) \, dx \]

\[ \geq \frac{1}{2\lambda_1 + 1} \min \left( \frac{\pi^2}{4L^2}, 1 \right) \int_D \frac{u_\ast^2(x,0)}{u_1^2(x,0)} \varphi_1^2(x) \, dx \]

\[ = \frac{1}{2\lambda_1 + 1} \min \left( \frac{\pi^2}{4L^2}, 1 \right) \int_D \varphi_2^2(x) \, dx = \frac{1}{2\lambda_1 + 1} \min \left( \frac{\pi^2}{4L^2}, 1 \right), \]

using the fact that

\[ \int_D \varphi_2^2(x) \, dx = 1. \]

Rewriting this we find that

\[ (4.3) \quad \lambda_\ast - \lambda_1 \geq \min \left( \frac{\pi^2}{4(2\lambda_1 + 1)L^2}, \frac{1}{2\lambda_1 + 1} \right). \]

Since the inradius of D is equal to 1 we have \( \lambda_1 \leq \lambda_1(B(0,1)). \) By Corollary 2.2 [6] we have \( \lambda_1(B(0,1)) \leq C(d), \) where

\[ C(d) = \frac{\pi d(d + 2)}{4(d + 1)}. \]

It follows that \( \lambda_1 \leq C(d). \) This and (4.3) conclude the proof of Theorem 1.1.

Proof of Theorem 1.2. We first recall the following deep result of Ashbaugh and Benguria, the so called “Payne–Pólya–Weinberger conjecture” proved in [3]. For any bounded connected domain \( D \subset \mathbb{R}^d, \) we denote by \( \mu_2(D) \) and \( \mu_1(D) \) the second and first eigenvalues of the Dirichlet Laplacian in \( D, \) respectively. (Of course, \( \mu_2(D) \) and \( \mu_1(D) \) are the second and first eigenvalues for the semigroup of Brownian motion killed upon exiting \( D. \)) Let \( B \) be any ball in \( \mathbb{R}^d. \) The Payne–Pólya–Weinberger conjecture proved in [3] asserts that

\[ (4.4) \quad \frac{\mu_2(D)}{\mu_1(D)} \leq \frac{\mu_2(B)}{\mu_1(B)} = \frac{j_{d/2,1}^2}{j_{d/2-1,1}^2}. \]

Furthermore, equality holds if and only if \( D \) is a ball (we will not use this fact here). To avoid confusion let us also denote by \( \lambda_1(D) \) and \( \lambda_2(D) \) the first and
second eigenvalues for the semigroup of the Cauchy process killed upon exiting $D$. It follows by the upper bound in [6] and the lower bound in [18] that for $i = 1, 2$, and for convex domains $D$,

\begin{equation}
\frac{1}{2} \sqrt{\mu_i(D)} \leq \lambda_i(D) \leq \sqrt{\mu_i(D)}.
\end{equation}

From this,

\begin{equation}
\lambda_2(D) - \lambda_1(D) \leq \sqrt{\mu_2(D)} - \frac{1}{2} \sqrt{\mu_1(D)}.
\end{equation}

However, by (4.4),

\begin{equation}
\frac{\sqrt{\mu_2(D)}}{\frac{1}{2} \sqrt{\mu_1(D)}} - 1 \leq \frac{\sqrt{\mu_2(B)}}{\frac{1}{2} \sqrt{\mu_1(B)}} - 1,
\end{equation}

where here we choose $B$ to be the largest ball contained in the domain $D$. This inequality can be written as

\begin{equation}
\frac{\sqrt{\mu_2(D)} - \frac{1}{2} \sqrt{\mu_1(D)}}{\frac{1}{2} \sqrt{\mu_1(D)}} \leq \frac{\sqrt{\mu_2(B)} - \frac{1}{2} \sqrt{\mu_1(B)}}{\frac{1}{2} \sqrt{\mu_1(B)}},
\end{equation}

which leads to

\begin{equation}
\sqrt{\mu_2(D)} - \frac{1}{2} \sqrt{\mu_1(D)} \leq \left( \frac{\sqrt{\mu_2(B)} - \frac{1}{2} \sqrt{\mu_1(B)}}{\frac{1}{2} \sqrt{\mu_1(B)}} \right) \left( \frac{\sqrt{\mu_1(D)}}{\sqrt{\mu_1(B)}} \right)
\leq \sqrt{\mu_2(B)} - \frac{1}{2} \sqrt{\mu_1(B)},
\end{equation}

where we used the fact that $\sqrt{\mu_1(D)} \leq \sqrt{\mu_1(B)}$, by domain monotonicity of the first eigenvalue. By scaling, $\mu_2(B) = \frac{j_{\alpha/2}^{\alpha/2}}{r_B^{2\alpha}}$ and $\mu_1(B) = \frac{j_{\alpha/2}^{\alpha/2}}{r_B^{2\alpha}}$, which proves the desired inequality.

Let $\lambda_i(D)$ be the eigenvalues for the semigroup of the symmetric $\alpha$-stable process killed on exiting a bounded convex domain $D$. Using the more general inequality

\begin{equation}
\frac{1}{2} (\mu_i(D))^{\alpha/2} \leq \lambda_i(D) \leq (\mu_i(D))^{\alpha/2},
\end{equation}

valid for any $0 < \alpha < 2$, see [23], [18] and the argument above we have the following generalization of Theorem 1.2.
Theorem 4.1. Let $D \subset \mathbb{R}^d$ be a bounded convex domain of inradius $r_D$. Let $\lambda_1$ and $\lambda_2$ be the first and second eigenvalues for the semigroup of the symmetric $\alpha$-stable process $0 < \alpha < 2$ killed upon exiting $D$. Then

$$\lambda_2 - \lambda_1 \leq \frac{j_{d/2,1}^\alpha - \frac{1}{2}j_{d/2-1,1}^\alpha}{r_D^\alpha}.$$  

In the case of Brownian motion the above argument gives that for any bounded domain $D$ of inradius $r_D$,

$$\mu_2(D) - \mu_1(D) \leq \mu_2(B) - \mu_1(B) = \frac{j_{d/2,1}^2 - j_{d/2-1,1}^2}{r_D^2},$$

with equality if and only if $D$ is a ball. We believe the following conjecture should be true.

Conjecture 4.1. Let $D \subset \mathbb{R}^d$ be a bounded domain and let $\lambda_2(D)$ and $\lambda_1(D)$ be the second and first eigenvalues for the semigroup of the symmetric $\alpha$-stable process $0 < \alpha < 2$ killed upon exiting $D$. Then

(i) (The $\alpha$–stable version of the Payne–Pólya–Weinberger Conjecture):

$$\frac{\lambda_2(D)}{\lambda_1(D)} \leq \frac{\lambda_2(B)}{\lambda_1(B)}$$

with equality if and only if $D$ is a ball. In particular,

$$\lambda_2(D) - \lambda_1(D) \leq \lambda_2(B) - \lambda_1(B)$$

with equality if and only if $D$ is a ball.

(ii) If $D$ has inradius $r_D$, then

$$\lambda_2(D) - \lambda_1(D) \leq \frac{j_{d/2,1}^\alpha - \frac{1}{2}j_{d/2-1,1}^\alpha}{r_D^\alpha}.$$  

We refer the reader to [7] and [31] where many of the classical isoperimetric–type inequalities which hold for Brownian motion are shown to also hold for symmetric stable processes.

As for a conjecture concerning a sharp lower bound we have the following (see also Remark 5.1 below).

Conjecture 4.2. Let $D \subset \mathbb{R}^2$ be a bounded convex domain which is symmetric relative to both coordinate axes. Let $R$ be the smallest oriented (sides parallel to the coordinate axes) rectangle containing $D$. For any $0 < \alpha < 2$,

$$\lambda_2(R) - \lambda_1(R) \leq \lambda_2(D) - \lambda_1(D).$$

For Brownian motion ($\alpha = 2$) this is proved in [5], [8], [20].
5 Concluding Remarks

We end this paper with several remarks and questions which naturally arise from our results.

**Remark 5.1.** For planar domains $D$ with the symmetry assumptions of Theorem 1.1 and for Brownian motion, it follows from [5], [8], [20] that $\lambda_* - \lambda_1 \geq \frac{3\pi^2}{(4L^2)}$, and for arbitrary convex domains of diameter $d$, $\lambda_2 - \lambda_1 > \frac{\pi^2}{d^2}$, [38], [41]. We may ask whether our estimates for the Cauchy process, $\lambda_* - \lambda_1$ is optimal in terms of the order of $L$. Let us consider the rectangle $R = [-L, L] \times [-1, 1]$ where $L \geq 1$. In a forthcoming paper we will show that

$$(5.1) \quad \frac{c \ln(L + 1)}{L^2} \leq \lambda_* - \lambda_1 \leq \frac{C \ln(L + 1)}{L^2}$$

for two absolute positive constants $c, C$. For this case, the methods in this paper give only $1/L^2$ due to the fact that we integrate the expression in the variational formula over $D \times [0, \infty)$ (see Lemma 2.2) and the extra term $\ln(L + 1)$ comes from integration over all of $H$.

When $D \subset \mathbb{R}^2$ is a dump–bell shaped domain (say two disjoint unit balls joined by a sufficiently thin corridor) which is symmetric according to the $x_1$-axis, one can show that

$$(5.2) \quad \lambda_* - \lambda_1 \leq \frac{C}{L^3},$$

where $C > 0$ does not depend on $D$ and $L >> 1$. (Since trivially $\lambda_2 - \lambda_1 \leq \lambda_* - \lambda_1$, the upper bound estimate for $\lambda_* - \lambda_1$ also gives the same estimate for the spectral gap $\lambda_2 - \lambda_1$.) Thus the lower bound result of this paper is not true for arbitrary non-convex domain. It may also be that we have here a different situation than in the case of Brownian motion case where the spectral gap $\lambda_2 - \lambda_1$ tends to zero as the corridor becomes thinner and thinner and the domain becomes two disjoint balls. It is probably the case that the spectral gap $\lambda_2 - \lambda_1$ (for the Cauchy process) of this dump-bell tends to the spectral gap of the set which consists of two disjoint balls, and the spectral gap for such a set is strictly positive.

The existence and properties of $\lambda_*$ and $\varphi_*$ (Theorem 4.3 [6]) were formulated and proved for connected, bounded and symmetric Lipschitz domains. In fact these assumptions were needed only for technical reasons and the existence and other basic properties are true without the assumptions of connectedness and Lipschitz boundary. This leads to the following question. Assume $D \subset \mathbb{R}^2$ has diameter $d_D$, inradius $r_D$ and is symmetric relative to the $x_1$–axis. What is the best lower bound estimate for $\lambda_* - \lambda_1$ in terms of $d_D$ and $r_D$ (regardless of connectedness or...
convexity of $D$? Of course, the same question may be asked for the spectral gap $\lambda_2 - \lambda_1$. These questions are non-trivial even in the one-dimensional case when $D$ consists of finite number of disjoint intervals.

**Remark 5.2.** It may be possible to apply the techniques used in this paper and in [6] to study eigenvalues and eigenfunctions for other processes. Of course, the most obvious extensions would be to other symmetric stable processes. It would also be of interest to extend these results to the relativistic process ([15], [35]) with characteristic function $E_0 e^{i \xi X_t} = e^{-t(\sqrt{m^2 + |\xi|^2} - m)}$, $t > 0$, $\xi \in \mathbb{R}^d$, $m > 0$. The infinitesimal generator of this process is the so-called relativistic Hamiltonian $-\sqrt{-\Delta + m^2} + m$. As with the Cauchy process one can build a "relativistic" Steklov problem of the type

$$\begin{align*}
\Delta u_n(x, t) + 2m \frac{\partial u_n}{\partial t}(x, t) &= 0; \\
\frac{\partial u_n}{\partial t}(x, 0) &= -\lambda_n u_n(x, 0); \\
u_n(x, 0) &= 0;
\end{align*}$$

for $(x, t) \in H_+$, $x \in D$, $x \in D^c$.

Using the identity

$$e^{2mt} \left( \Delta + 2m \frac{\partial}{\partial t} \right) = e^{2mt} \left( \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \right) + \frac{\partial}{\partial t} \left( e^{2mt} \frac{\partial}{\partial t} \right)$$

one can show that the eigenvalues of the relativistic process are given by the variational formula

$$\lambda_n = \inf_{u \in \tilde{F}_n} \int_H |\nabla u(x, t)|^2 e^{2mt} dx dt,$$

for an appropriately chosen class of functions $\tilde{F}_n$. Thus the eigenvalue problem for the relativistic process is similar to that of the Cauchy process. Nevertheless, extending the results which we now have for the Cauchy process remains mostly open (although some results follow from the recent paper [18], see Example 6.2).

**Remark 5.3.** As mentioned in the introduction, the spectral gap $\lambda_2 - \lambda_1$ measures the rate at which the Cauchy process conditioned to remain forever in the domain $D$ tends to equilibrium. That is, for any $\varepsilon > 0$, we define (as in [33]) the time to equilibrium $T_\varepsilon$ by

$$T_\varepsilon = \inf \{ t > 0 : \sup_{x,y \in D} \left| \frac{e^{\lambda_1 t} p_D(t, x, y)}{\varphi_1(x) \varphi_1(y)} - 1 \right| \leq \varepsilon \}.$$
It follows from (1.4) that

$$\frac{1}{\lambda_2 - \lambda_1} \log \frac{1}{\varepsilon} \leq T_{\varepsilon} \leq C_1 + \frac{1}{\lambda_2 - \lambda_1} \log \frac{1}{\varepsilon}.$$  

While a probabilistic interpretation of $\lambda_\ast - \lambda_1$ is not as “clean” and useful as the one above, we do have the following.  Recall that $D_+ = \{x \in D : x_1 > 0\}$ and $D_- = \{x \in D : x_1 < 0\}$.  Then for any $x \in D_+$

$$-(\lambda_\ast - \lambda_1) = \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{P^x(X_t \in D_+, \tau_D > t) - P^x(X_t \in D_-, \tau_D > t)}{P^x(\tau_D > t)} \right).$$

This follows from the proof of Theorem 4.3 in [6], the definition of $\tilde{p}_D(t, x, y)$ (see Lemma 4.5 in [6]) and the general theory of semigroups.

References

[1] G. Alessandrini, *Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains*, Comment. Math. Helv. 69(1) (1994), 142–154.

[2] M. S. Ashbaugh and R. Benguria, *Optimal lower bounds for eigenvalue gaps for Schrödinger operators with symmetric single-well potentials and related results*, Maximum principles and eigenvalue problems in partial differential equations, Longman, White Plains, NY, 1988.

[3] M. S. Ashbaugh and R. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. Math. 135(2) (1992), 601-628.

[4] R. Bañuelos, *Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators*, J. Funct. Anal. 100 (1991), 181-206.

[5] R. Bañuelos and P. Kröger, *Gradient estimates for the ground state Schrödinger eigenfunction and applications*, Comm. Math. Physics, 224 (2002), 454–550.

[6] R. Bañuelos, T. Kulczycki, *The Cauchy process and the Steklov problem*, J. Funct. Anal. 211 (2004), 355–423.

[7] R. Bañuelos, R. Latała, P. J. Méndez-Hernández, *A Brascamp-Lieb-Luttinger-type inequality and applications to symmetric stable processes*, Proc. Amer. Math. Soc. 129(10) (2001), 2997–3008 (electronic).
[8] R. Bañuelos, P. J. Méndez-Hernández, *Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps*, J. Funct. Anal. 176(2) (2000), 368–399.

[9] R. Bañuelos, T. Kulczycki, P. J. Méndez-Hernández, *On the shape of the ground state eigenfunction for stable processes*, preprint.

[10] M. van den Berg, *On condensation in the free--boson gas and the spectrum of the Laplacian*, J. Statist. Phys., 31 (1983), 623–637.

[11] R.M. Blumenthal and R.K. Getoor, *Markov Processes and Potential Theory*, Springer, New York, 1968.

[12] R.M. Blumenthal and R.K. Getoor, *The asymptotic distribution of the eigenvalues for a class of Markov operators*, Pacific J. Math. 9 (1959), 399–408.

[13] K. Bogdan *The boundary Harnack principle for the fractional Laplacian*, Studia Math. 123(1) (1997), 43-80.

[14] H.L. Brascamp and E.H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis, 22(1976), 366–389

[15] R. Carmona, W. C. Masters and B. Simon *Relativistic Schrödinger operators: asymptotic behavior of eigenfunctions*, J. Funct. Anal. 91 (1990), 117-142.

[16] Z.Q. Chen and R. Song, *Intrinsic ultracontractivity and conditional gauge for symmetric stable processes*, J. Funct. Anal. 150(1) (1997), 204–239.

[17] Z.Q. Chen and R. Song, *Intrinsic ultracontractivity, conditional lifetimes and conditional gauge for symmetric stable processes on rough domains*, Illinois J. Math. 44(1) (2000), 138–160.

[18] Z.Q. Chen and R. Song, *Two sided eigenvalue estimates for subordinate Brownian motion in bounded domains*, preprint

[19] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[20] B. Davis, *On the spectral gap for fixed membranes*, Ark. Mat. 39(1) (2001), 65–74.

[21] B. Davis and M. Hosseini, *On ratio inequalities for heat content*, J. London Math. Soc. 69 (2004), 97–106.
[22] R. D. DeBlassie, *Higher order PDEs and symmetric stable processes* Probab. Theory Related Fields (2004).

[23] R. D. DeBlassie and P. J. Méndez-Hernández, *α–continuity properties of symmetric α–stable process*, Preprint.

[24] R. Courant D. Hilbert, *Methods of mathematical physics*, Vol I, New York, Wiley (1989).

[25] B. Dittmar, A. Y. Solynin, *The mixed Steklov eigenvalue problem and new extremal properties of the Grötzsch ring.* (Russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 270 (2000), Issled. po Linein. Oper. i Teor. Funkts. 28, 51–79, 365.

[26] R. K. Getoor, *Markov operators and their associated semi-groups*, Pacific J. Math. 9 (1959) 449–472.

[27] J. Hersch, L. E. Payne, *Extremal principles and isoperimetric inequalities for some mixed problems of Stekloff’s type*, Z. Angew. Math. Phys. 19 (1968), 802-817.

[28] T. Kulczycki, *Intrinsic ultracontractivity for symmetric stable processes*, Bull. Polish Acad. Sci. Math. 46(3) (1998), 325–334.

[29] J. Ling, *A lower bound for the gap between the first two eigenvalues of Schrödinger operators on convex domains in $S^n$ or $R^n$*, Michigan Math. J. 40(2) (1993), 259–270.

[30] A.D. Melas, *On the nodal line of the second eigenfunction of the Laplacian in $R^2$*, J. Differential Geom, 35 (1992), 255–266.

[31] P. J. Méndez-Hernández, *Brascamp-Lieb-Luttinger Inequalities for Convex Domains of Finite Inradius*, Duke Math. J., 13 (2002), 93–131

[32] L. E. Payne, *Isoperimetric inequalities and their applications*, SIAM Rev. 9 (1967), 453–488.

[33] L. E. Payne, *On two conjectures in the fixed membrane eigenvalue problem*, Z. Angew. Math. Phys. 24 (1973), 721-729.

[34] L. E. Payne and H. F. Weinberg, *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal. 5 (1960), 286-292.

[35] M. Ryznar, *Estimates of Green functions for relativistic α–stable processes*, Potential Analysis, 17 (2002), 1–23.
[36] L. Saloff–Coste, *Precise estimates on the rate at which certain diffusions tend to equilibrium*, Math. Z., 217 (1994), 641–677.

[37] I. M. Singer, B. Wong, S.-T. Yau, S. S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12(2) (1985), 319–333.

[38] R. G. Smits, *Spectral gaps and rates to equilibrium for diffusions in convex domains* Michigan Math. J. 43(1) (1996), 141-157.

[39] S.-T. Yau, *Open problems in geometry*, in “Chern—a great geometer of the twentieth century,” Edited by S.-T. Yau, Internat. Press, Hong Kong, 1992.

[40] D. You, *Sharp inequalities for ratios and partition functions of Schrödinger operators*, Potential Anal. 18 (2003), 219–250.

[41] Q. Yu and J. Q. Zhong, *Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator*, Trans. Amer. Math. Soc. 294 (1986), 341-349.