Discrete Conics

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Abstract

In this paper, we introduce discrete conics, polygonal analogues of conics. We show that discrete conics satisfy a number of nice properties analogous to those of conics, and arise naturally from several constructions, including the discrete negative pedal construction and an action of a group acting on a focus-sharing pencil of conics.

1 Introduction

In [1], we showed that a certain family of Simson polygons, polygons which admit a point whose projections into the sides of the polygon are collinear, can be fruitfully viewed as discrete analogs of the parabola. In this paper, we find a family of polygons which are discrete analogs of a general conic.

Intuitively, a regular polygon is the best candidate for being called a "discrete circle" - it has the maximal amount of symmetry as well as other geometric properties that are similar to those of a circle. In the same spirit, we define discrete analogues of conics as follows (see Figure 1):

Definition 1.1 (Discrete Conic) Let $C$ be a conic with focus $F$. A discrete conic is a polygon $V_1V_2\cdots V_n$ such that for every $i \in \{1, 2, \ldots, n\}$ and some fixed $\theta \in \mathbb{R}$, $\angle V_i F V_{i+1} = \theta$.

We will show later that these discrete conics are projective images of regular polygons. One of our goals in this paper will be to show how these discrete conics arise naturally in two constructions: via a group acting on

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1When $C$ is a hyperbola, parabola or a degenerate conic, we allow the rays used in measuring an angle to wrap around the plane.
Figure 1: A discrete conic.

a pencil of focus-sharing conics and through the discretized negative pedal construction. In the last section, we will prove a number of nice properties of discrete conics.

2 Properties of Discrete Conics

We now list some of the nice properties of discrete conics. Proofs can be found in section 5.

Theorem 1 A discrete conic is projectively equivalent to a regular polygon.

Recall that a Poncelet Polygon is a polygon which circumscribes a conic and is inscribed in a conic.

Theorem 2 A discrete conic is a Poncelet Polygon and the circumscribing and inscribed conics share a focus.

Let us fix notation. The discrete conic $D = V_1V_2\cdots V_n$ lies on conic $C$ with focus $F$. The vertices of $D$ satisfy $\angle V_i F V_{i+1} = \theta$. We set $S_i = V_i V_{i+1}$.
The sides $S_i$ are tangent to a conic with focus $F$ which we will denote by $C_L$. Let $F'$ be its other focus. We define $F$ and $F'$ to be the *focii of the discrete conic $D$*. Finally, we set $M_i = V_iV_{i+1} \cap C_L$ to be a tangency point of the inner conic.

**Theorem 3** Let $D = V_1 \cdots V_n$ be a discrete conic with $n$ sides. If $n$ is even then the diagonals $V_jV_{j+\frac{n}{2}}$ of $D$ are coincident at $F$. If $n$ is odd then chords $V_jM_{j+\frac{n-1}{2}}$ are coincident at $F$.

An analog of the reflective property of a conic exists in the case when $D$ is closed. Unlike the previous results, this is not a projective property and does not follow from Theorem 1. We will symbolically represent a path taken by a ray via a sequence of alternating positions and arrows. For instance, if $r$ is a ray which starts at $F'$, reflects off a side $S$ and then passes through $F$, then we will say that $r$ traverses the path $F' \rightarrow S \rightarrow F$.

![Figure 2: Optic (reflective) property for even- and odd-sided discrete conics.](image.png)

**Theorem 4** *(Reflective Property)* Let $r$ be a ray which traverses the path $F' \rightarrow S_j \rightarrow F$. If the discrete conic $D$ is an even-sided polygon then $r$ traverses

$$F' \rightarrow S_j \rightarrow F \rightarrow S_{j+\frac{n}{2}} \rightarrow F'.$$
If \( n \) is odd-sided then \( r \) traverses the path

\[ F' \rightarrow S_j \rightarrow F \rightarrow V_{j+n-1}. \]

As a particular example of Theorem 4, consider the case when \( C \) is a circle. Then \( D \) is a regular polygon. In this case, the Theorem states that if the polygon is even-sided and regular then a ray through the center which reflects back to the center must consequently reflects back to the center again. In the case that the regular polygon is odd-sided then a ray passing through the center that reflects back to the center must subsequently pass through one of the vertices.

We also have an analog of the Isogonal Property of a conic. Recall that if tangents at \( X \) and \( Y \) to a conic intersect at \( Z \) and \( F, F' \) are the focii of the conic, then \( \angle FZX = \angle F'ZY \) [3]. Analogously,

**Theorem 5 (Isogonal Property)** If \( Z \) is a point lying on the intersection of sides \( S_i \) and \( S_j \) of two sides of the discrete conic \( D \) then \( FZ \) bisects angles \( \angle M_i FM_j, \angle V_i FV_{j+1} \) and \( \angle V_{i+1} FV_j \). Moreover, \( FZ \) and \( F'Z \) are isogonal with respect to \( S_i \) and \( S_j \).

![Figure 3: Isogonal properties of discrete conics.](image)

We can describe a discrete conic by the conic it is on (parameter \( t \)), the angle \( \theta \) of the relation \( \theta = \angle V_i FV_{i+1} \) and the phase angle \( \phi \mod \theta \). Thus we
Figure 4: Three members of an infinite family of discrete conics. Given a discrete conic $D_i$ on conic $C_i$, the subsequent discrete conic $D_{i+1}$ has vertices which are the intersections of tangents of $C_i$ at the vertices of $D_i$. All conics share a single focus.

will write $D_{t,\theta,\phi}$. The group operations that can be applied to this discrete conic correspond to multiples of angle $\theta$. In particular, $G_{k\theta}$ sends $D_{t,\theta,\phi}$ to

$$G_{k\theta}(D_{t,\theta,\phi}) = D_{t \sec^2\left(\frac{\theta}{2}\right),\theta,\phi+\frac{\theta}{2}}.$$ 

Using this fact, we will discuss the relation between discrete conics and the Poncelet Grid Theorem. R. Schwartz’s Poncelet Grid Theorem states that if $P = P_1P_2\cdots P_n$ is a Poncelet polygon then the intersection points of lines $L_i = P_iP_{i+1}$ and $L_{i+k} = P_{i+k}P_{i+k+1}$ lie on a conic $c'$ and form a polygon $P'$ projectively equivalent to $P$ (see [4] and [5]). More can be said when $P$ is a discrete conic.

**Theorem 6** If $P$ is a discrete conic lying on conic $c$ then $P' = G_{k\theta}(H_\theta(P))$. In particular, $P'$ is a discrete conic having the same angle parameter $\theta$ and lying on a a conic which shares a focus with $c$.

A consequence of what we have shown is that there exists an infinite family of discrete conics, each discrete conic lying on a conic of this pencil,
Figure 5: Theorem 7 for a hexagon. We set \( K_i = V_i V_{i+1} \cap V_{i+n} V_{i+n+1} \) for \( i = 1, 2, ..., 2n - 1 \) to be the intersections of opposite sides. These are collinear and their line is orthogonal to \( FF' \). Moreover, these form a discrete conic, so that the angles as seen from \( F \) are equal.

and with tangents to the corresponding conic at the vertices of the discrete conic forming the sides of the subsequent discrete conic (see Figure 4).

We conclude with an interesting result, reminiscent of Pascal’s famous hexagon theorem.

**Theorem 7** Let \( D = V_1 V_2 \cdots V_{2n} \) be an even-sided discrete conic with foci \( F, F' \). The intersections of opposite sides lie on a line orthogonal to \( FF' \) and form another discrete conic. In the special case when \( C \) is a parabola, this line is the directrix of the parabola.

3 A Group Acting on a Pencil of Focus-Sharing Conics

Let \( C \) be a circle with center \( O \). Let \( X \) and \( Y \) be variable points on \( C \). Define \( G_\theta(C) \) to be the locus of points \( Z \) such that \( Z \) is the intersection of tangent to \( C \) at \( X \) and \( Y \) for \( X \) and \( Y \) satisfying \( \angle XOY = \theta \). We define \( H_\theta(C) \) to be the locus enveloped by lines \( z = XY \) for \( X \) and \( Y \) satisfying \( \angle XOY = \theta \). We restrict \( \theta \in [0, \pi) \).
Lemma 3.1 Consider a family of concentric plane-foliating circles $OF$. The operations $G$ and $H$ define an abelian group acting on $OF$ with $G_{\theta} = H_{\theta}^{-1}$ and $G_{\theta} \circ G_{\phi} = G_{2 \cos^{-1}(\cos(\frac{\theta}{2}) \cos(\frac{\phi}{2}))}$.

Proof. Let $R$ be the radius of circle $C$ with center $O$. From right triangle $\triangle OXZ$, we see that the radius of $G_{\theta}(C)$ is $R \sec(\frac{\theta}{2})$. On the other hand, let $M$ be the midpoint of $XY$. From right triangle $\triangle OMX$, the radius $OM$ of $H_{\theta}(C)$ is equal to $R \cos(\frac{\theta}{2})$. It follows that $G_{\theta} = H_{\theta}^{-1}$ and $G_{\theta} \circ G_{\phi} = G_{2 \cos^{-1}(\cos(\frac{\theta}{2}) \cos(\frac{\phi}{2}))}$. The identity is just $G_{0}$. \(\square\)

The map $\theta \rightarrow \cos(\frac{\theta}{2})$ sending angle $\theta \in [0, \pi)$ to the number $\cos(\frac{\theta}{2})$ shows that the group is isomorphic to the positive real numbers under multiplication.

We now extend this group operation to conics. Consider a conic with focus $F$. Define $G'_{\theta}(C)$ to be the locus of points $Z$ such that $Z$ is the intersection of tangents at $X$ and $Y$ satisfying $\angle XFY = \theta$. We define $H'_{\theta}(C)$ to be the locus enveloped by lines $z$ such $z = XY$ for $X$ and $Y$ satisfying $\angle XFY = \theta$.

Theorem 8 Let $C$ be a conic with focus $F$. Then $G'_{\theta}(C)$ and $H'_{\theta}(C)$ are conics with focus $F$. 
For the proof, we will use a technique suggested by Arseniy Akopyan, namely reciprocation (also known as polar duality). For a detailed discussion and proofs we refer the reader to [2]. Consider a circle with center $O$. Each point $P$ other than $O$ determines a corresponding line $p$, the polar of $P$, which is the line perpendicular to $OP$ and passing through the inverse of $P$. Conversely, given a line $p$, we call the point which is the inverse of the foot of the perpendicular from $O$ to $p$ its pole. The transformation taking a set of points to their polar lines or a set of lines to their poles is called reciprocation. We do not distinguish the points of a curve from the envelope of tangents to the curve.

A fact we will employ in the proof is that a conic is the reciprocal of a circle and vice versa. Moreover, point $O$ is the focus of the conic. To see why this is true, consider the reciprocal of the tangent lines to the conic. To reciprocate a line, we first find the foot of the perpendicular from $O$. Since the pedal of a conic with respect to its focus is a circle, the locus of feet of perpendiculars is the pedal circle. We then invert these to obtain the points of the reciprocal of the conic. Hence these form a circle, and in particular are the inverse image of the pedal circle.

**Proof of Theorem 8.** Let $x$ and $y$ be the tangent lines at $X$ and $Y$, respectively. We apply a duality transformation $d$ about a circle centered at $F$. The transformation takes the conic $C$ to a circle $d(C)$ upon which points $d(x)$ and $d(y)$ lie. It is not hard to see that we may choose the circle of inversion so that $F$ is inside of $d(C)$. Point $Z$ maps to the line $d(Z)$ passing through $d(x)$ and $d(y)$. The images $d(X)$ and $d(Y)$ are the tangents to $d(C)$ at $d(x)$ and $d(y)$. Let $X'$ and $Y'$ be the feet of the perpendiculars $Fd(X)$.
Figure 8: The operations $G'_\theta(C)$ and $H'_\theta(C)$ produce conics which share focus $F$. \\
and $Fd(Y)$. Since $\angle XFY = \theta$, we have $\angle X'FY' = \theta$. Let $\bar{c}$ be the center of $d(C)$. Then

$$\angle d(x)\bar{c}d(y) = \pi - \angle d(x)d(z)d(y) = \pi - \angle X'd(z)Y' = \angle X'FY' = \theta$$

Therefore the envelope of $d(Z)$ is a concentric circle. Applying $d$ to this concentric circle shows that $G'_\theta(C)$ is a conic with focus $F$. Similar reasoning shows that $H'_\theta(C)$ is also a conic with focus $F$. \(\square\)

**Corollary 3.2** Let $C$ be a conic with focus $F$, and $d$ be the duality transform about $F$. The following diagrams commute for every $\theta$:

\[
\begin{array}{c}
C \xrightarrow{d} d(C) \\
\downarrow H'_\theta \\
H'_\theta(C) \xrightarrow{d} d(H'_\theta(C)) = G'_\theta(d(C))
\end{array}

\[
\begin{array}{c}
C \xrightarrow{d} d(C) \\
\downarrow G'_\theta \\
G'_\theta(C) \xrightarrow{d} d(G'_\theta(C)) = H'_\theta(d(C))
\end{array}
\]
### Proof
The commutativity of the diagrams follows from the proof of Theorem 8. □

Using the result of Corollary 3.2, we can show that \( G' \) and \( H' \) form a group isomorphic to that of \( G \) and \( H \).

**Theorem 9** The operations \( G' \) and \( H' \) define an abelian group with \( G'_{\theta} = H'^{-1}_{\theta} \) and \( G'_{\theta} \circ G'_{\phi} = G'_{2\cos^{-1}(\cos(\frac{\theta}{2}) \cos(\frac{\phi}{2}))} \).

**Proof.** By Corollary 3.2, we have \( H'_{\theta} = d \circ H_{\theta} \circ d^{-1} \) and \( G'_{\theta} = d \circ G_{\theta} \circ d^{-1} \). Therefore \( G'_{\theta} \circ H'_{\theta} = d \circ H_{\theta} \circ d^{-1} \circ d \circ G_{\theta} \circ d^{-1} = I \) and similarly \( H'_{\theta} \circ G'_{\theta} = I \). We also have \( G'_{\theta} \circ G'_{\phi} = d \circ G_{\theta} \circ d^{-1} \circ d \circ G_{\phi} \circ d^{-1} = d \circ G_{\theta} \circ G_{\phi} \circ d^{-1} \). Finally, \( G'_{\theta} \circ G'_{0} = d \circ G_{\theta} \circ G_{0} \circ d^{-1} = d \circ G_{\theta} \circ d^{-1} = G'_{\theta} \).

Consider the map \( \Theta : G(CF) \rightarrow G(OF) \) from the group acting on conics to the group acting on the circles given by

\[
\Theta(G'_{\theta}) = d^{-1} \circ G'_{\theta} \circ d = H_{\theta}.
\]

This map is clearly a bijection. Moreover,

\[
\Theta(G'_{\theta} \circ G'_{\phi}) = d^{-1} \circ G'_{\theta} \circ G'_{\phi} \circ d = \Theta(G'_{\theta}) \circ \Theta(G'_{\phi}).
\]

Therefore \( G(CF) \) is isomorphic to \( G(OF) \). □

Since \( G' \) and \( G \) coincide for circles, we will from now on drop the primes.

We fix one of the conics of the family \( CF \) to be \((p + x)^2 + y^2 = (1 + px)^2\). Such a conic has eccentricity \( p \), though the case \( p = 1 \) is degenerate, being a line rather than a parabola. A calculation shows that the conics of the family have equation

\[
(p + x)^2 + y^2 = (1 + px)^2 t
\]

for \( t \in [0, \infty) \) as the parameter. We denote this pencil of conics by \( CF \).

**Proposition 3.3** The pencil of conics \( CF \) shares focus \( F = (-p, 0) \) and foliates \( \mathbb{R}^2 \setminus \{(x, y) : x = -\frac{1}{p}\} \).
Proof. It is clear from the duality transformation that the conics share focus \((-p, 0)\). Two conics with differing values of \(t\) are clearly disjoint. Let \((x, y) \in \mathbb{R}^2 \setminus \{(x, y) : x = -\frac{1}{p}\}\). If \(x \neq -\frac{1}{p}\) then the ratio

\[
\frac{(p + x)^2 + y^2}{(1 + px)^2}
\]

is defined. Since it is nonnegative, there is some \(t\) which equals to this ratio. If \(x = -\frac{1}{p}\), then the equation

\[
(p - \frac{1}{p})^2 + y^2 = 0.
\]

has no solution since \(p \neq \pm 1\). \(\Box\)

Figure 9: A portion of the pencil of conics \(CF\) when \(p = \frac{3}{4}\) and when \(p = \sqrt{2}\).

Note that in the special case that \(p = 0\) we obtain a family of concentric circles.

It is also important to note that the \emph{discrete conics are precisely those polygons for which there is an action of the group}. This action is obtained by taking \(X = V_i\) and \(Y = V_{i+k}\) to be vertices of the discrete conic. Then \(\angle XFY = k\theta\). This is unique to discrete conics. Indeed, if \(\tilde{D}\) is a polygon lying on a conic \(C\) of the family \(CF\) which is not a discrete conic, then at least one pair of consecutive vertices do not form the same angle with respect to the focus as the other vertices. Therefore the image of \(\tilde{D}\), call it \(G(\tilde{D})\),
will not lie on a single conic of $CF$. Consequently, the operation $G^2(\tilde{D})$ will not be well-defined.

4 The Discrete Negative Pedal Construction

For a given curve $C$ in the plane and a given fixed point $P$, called the pedal point, the pedal curve of $C$ is the locus of points $X$ such that $PX$ is perpendicular to a tangent to the curve passing through $X$.

The negative pedal curve is the inverse of the pedal curve. More precisely, the negative pedal curve of $C$ with respect to $P$ is the envelope of lines $XP$ for $X$ lying on the given curve. The negative pedal curve of a pedal curve with the same pedal point is the original curve.

For a conic $C$, if the pedal point is the focus $F$, then the pedal curve is a circle (a line in the case of the parabola). Conversely, given a circle $o$, if $P$ does not lie on $o$, then the negative pedal is a conic with $P$ as a focus.

In [1], we used a discretized version of the negative pedal construction to construct discrete analogs of parabolas. Our construction for the general conic is analogous except our construction before prioritized distance over angle. For a general conic, no such choice is possible. We recall the previous construction. Let $P$ be a point, $L$ a line and $X_1, X_2, \ldots, X_n$ points on $L$ with $|X_iX_{i+1}| = \Delta$ for $i = 1, 2, \ldots, n-1$. Let $L_i$ be the line orthogonal to $PX_i$ and passing through $X_i$, $i = 1, 2, \ldots, n-1$. The vertices of the discrete parabola are $V_i = L_i \cap L_{i+1}$, $i = 1, 2, \ldots, n-1$. (see Figure 10).

A few remarks are in order. The cases $p = \pm 1$ are degenerate for our construction, so we do not consider them. We also assume that the pedal point $P$ is on the $x$-axis because the general case can be obtained via rotation.

We will show that the $V_i$’s lie on a conic independent of $\phi$. We will also show that if $P' = (-p, 0)$, then $\angle V_iP'V_{i+1} = \theta$ - a surprising fact considering that no mention of $P'$ is made in the construction.
Figure 10: Construction of a discrete conic $V_0, V_1, V_2, \ldots V_n$ via the discrete negative pedal construction.

In the limit $\theta \to 0$, we obtain the negative pedal construction with respect to a circle. Thus the negative pedal is the envelope of a conic $C_L$. We call this conic the limiting conic. The sides of the discrete conic $V_0V_1 \cdots V_{n-1}$ are tangent to $C_L$, since they are a subset of the enveloping lines.

The following easy lemma will allow us to understand $C_L$ better.

**Lemma 4.1** Let $X$ and $Y$ be points on a circle centered at $O$. Let $B$ be line $XY$, $L_X$ and $L_Y$ be the lines perpendicular to $B$ and passing through $X$ and $Y$, respectively. Consider an arbitrary line $L$ passing through $O$. Set $P_1 = L \cap L_X$ and $P_2 = L \cap L_Y$. Then the distance between $O$ and $P_1$ is the same as the distance between $O$ and $P_2$.

**Proof.** The result clearly holds when $L \parallel B$. It is easy to see that rotating $L$ preserves equality. $\Box$

**Theorem 10** The focii of the limiting conic $C_L$ are $P = (p, 0)$ and $P' = (-p, 0)$. The equation of $C_L$ is $x^2 + \frac{y^2}{1-p^2} = 1$.

**Proof.** We argue that the negative pedal of $P'$ with respect to the unit circle is also $C_L$ so that $P'$ must be the other focus of $C_L$. We do this by showing that the set of lines $L_i$ in the limit $\theta \to 0$ produced from $P$ is the
same as those produced from $P'$. This would imply that the negative pedal from $P'$ has the same envelope of lines as that of $P$, so that its negative pedal is $C_L$.

Let $X \in S^1$. Let $B$ be the line orthogonal to $PX$ and passing through $X$. If $B$ is tangent to $S^1$ then $X$ is on the $x$-axis, so it is also part of the envelope corresponding to $P'$. Otherwise, $B$ intersects $S^1$ in one more point - call it $Y$. Applying Lemma 4.1 we conclude that $B$ is also part of the envelope corresponding to $P'$, so that $P$ and $P'$ produce the same envelope. It follows that the focii of $C_L$ are $P$ and $P'$.

This implies that $C_L$ is of the form

$$\frac{x^2}{\lambda} + \frac{y^2}{\lambda - p^2} = 1.$$ 

Since the line $x = 1$ is part of the envelope, it must be tangent to $C_L$. Therefore $C_L$ passes through $(1, 0)$. It follows that $\lambda = 1$. □

The following two Lemmas will culminate in the result that the $V_i$'s lie on a conic independent of $\phi$. The first may be viewed as a simple result in mathematical billiards. The second follows from the first, and is a possibly new property of the pedal circle with respect to its conic.

**Lemma 4.2** Let $ABCD$ be a rectangle with center $O$. Take points $E \in AB$ and $F \in CD$ such that $E, F, O$ are collinear. Let $G \in BC$ and $H \in DA$ satisfy $\angle EGB = \angle FGC$ and $\angle EHA = \angle FHD$. Then
1. $E, G, F, H$ is a 4-periodic billiard trajectory.
2. $\angle EGB = \angle FGC = \angle EHA = \angle FHD$ and $\angle AEH = \angle BEG = \angle CFG = \angle DFH$.
3. $G, H, O$ are collinear.
4. $EGFH$ is a parallelogram with sides parallel to the diagonals of $ABCD$.

**Proof.** The equality $\angle EGB = \angle FGC = \angle EHA = \angle FHD$ and (3) follow by odd symmetry about $O$. (1) and (2) then follow by a simple angle-count.

To prove (4), observe that (1) implies that $EFGH$ is a parallelogram. Let $\theta = \angle BEG$, $x = BE$ and $y = BG$. Triangle $\triangle EBG$ implies that

$$\tan \theta = \frac{y}{x}.$$ 

Similarly, triangle $\triangle EAH$ implies that

$$\tan \theta = \frac{|BC| - y}{|AB| - x},$$

so that $\triangle EBG$ is similar to $\triangle ABC$. □

**Lemma 4.3** Let $C$ be a conic with foci $F$ and $F'$, let $o$ be the pedal circle with respect to $F$ and call its center $O$. Consider any point $X$ on $o$. Set $L$ to be the line through $X$ which is orthogonal to $FX$ and let its intersection with $o$ be $Y$ and its intersection with $C$ be $Z$. Then $F'Z \parallel OX$ and $FZ \parallel OY$. 
Proof. We extend \( XF \) to intersect \( o \) at \( X' \) and \( YF' \) to intersect \( o \) at \( Y' \). Then \( XYX'Y' \) is a rectangle with center \( o \). Call the intersection of \( X'Y' \) with \( CZ' \). We apply Lemma 4.2 with \( F \) and \( F' \) being \( E \) and \( F \) from the Lemma and \( Z \) and \( Z' \) being \( G \) and \( H \).

We are now ready to consider the construction at the beginning of the section. Let \( M_j = C_L \cap L_j \) for each \( j \) and set \( M = M_1, M_2 \cdots M_n \). A consequence of Lemma 4.3 is the following result, which shows that \( D = V_1, V_2 \ldots \) and \( M \) are discrete conics.

**Theorem 11** The intersections \( V_j = L_j \cap L_{j+1} \) lie on a conic independent of \( \phi \) and having focus \( P' \) and form a discrete conic of angle \( \theta \). Moreover, \( D = G_{\phi}(M) \).

Proof. We argue that \( \angle M_j P'M_{j+1} = \theta \). Indeed, line \( P'M_j \) is parallel to \( OX_j \) by Lemma 4.3. Since \( \angle X_j OX_{j+1} = \theta \), the result follows.

Recall that if tangents at \( X \) and \( Y \) to a conic intersect at \( Z \) and \( F \) is a focus of the conic then \( ZF \) bisects angle \( \angle XFY \). It follows that \( \angle V_j P'M_j = \frac{\theta}{2} \) and \( \angle V_j P'V_{j+1} = \theta \). Using Theorem 8 with this last fact, we conclude that the \( V_j \)'s lie on a conic independent of \( \phi \) which shares focus \( P' \) with \( C_L \). \( \square \)
Figure 14: Two different values of $p$. By Theorem 11, the $V_i$'s lie on a conic independent of the phase of $X_0, X_1, ...$ and form a discrete conic.

Let $C$ have equation $x^2 + \frac{y^2}{1+p^2} = 1$. A calculation shows that for a discrete conic with parameter $\theta$ and phase angle $\phi$ has vertices with coordinates

$$V_j = \left( \frac{p - \cos((j-1)\theta + \phi)}{p \cos((j-1)\theta + \phi) - 1}, \frac{(p^2 - 1) \sin((j-1)\theta + \phi)}{p \cos((j-1)\theta + \phi) - 1} \right).$$

5 Proofs of Theorems from Section 1

**Proof of Theorem 1.** Let $D = V_1 V_2 \cdots V_n$ and let $T_i$ denote the tangent at $V_i$ to the conic $C$. Applying the duality transformation $d$ as in the proof of Theorem 8, we see that each $d(V_i)$ is a tangent line to circle $d(C)$, and $d(T_i)$ becomes the tangency point of $d(V_i)$ on $d(C)$. Let $p_i$ denote the projection of $F$ into $d(V_i)$. Since $\angle V_i F V_{i+1} = \theta$ for each $i$, we have $\angle p_i F p_{i+1} = \theta$. This implies that the angle formed by consecutive tangents is $\pi - \theta$, and in particular, is always equal. This shows that the vertices of $D$ map to the sides of a regular polygon. The sides of $D$ map to the vertices of this regular polygon. $\square$

**Proof of Theorem 2.** This follows from Theorem 1 but we can provide an additional proof. Given a discrete conic $D$ on a conic $C$ with angle parameter $\theta$, the discrete conic $D' = H_\theta(D)$ lies on a conic $C'$ sharing a focus with $C$ and
satisfying \(D = G_\theta(D')\). This means that the vertices of \(D\) are intersections of tangents to \(C'\) at consecutive pairs of vertices of \(D'\). In particular, the sides of \(D\) are tangent to \(C\). \(\square\)

**Proof of Theorem 3.** This too follows from Theorem 1 but we provide another proof. Assume first that \(n\) is even. Using the fact that \(\angle V_j F V_{j+1} = \theta\) we see that \(\angle V_j F V_{j+\frac{n}{2}} = \frac{\theta}{2}\). Since \(n = \frac{2\pi}{\theta}\), we conclude that \(V_j, F\) and \(V_{j+\frac{n}{2}}\) are collinear.

In the odd case, we have \(\angle V_i F M_i = \frac{\theta}{2}\) and \(\angle M_i F M_{i+1} = \theta\). Therefore \(\angle V_i F M_{i+\frac{n-1}{2}} = \frac{\theta}{2} + \frac{n-1}{2}\theta = \frac{n}{2}\theta\), so that \(V_j, F\) and \(M_{j+\frac{n-1}{2}}\) are collinear. \(\square\)

**Proof of Theorem 4.** Assume first that the number of sides \(n\) of \(D\) is even. Since \(r\) traverses \(F' \rightarrow S_j \rightarrow F\), and \(S_j\) is a line tangent to \(C_L\), the reflection of \(r\) in \(S_j\) passes through \(M_j\) and \(F\). Theorem 3 implies that \(M_j, F\) and \(M_{j+\frac{n}{2}}\) are collinear, since \(M_1 M_2 \cdots M_n\) is also a discrete conic with parameter \(\theta\) and focus \(F\). Therefore \(r\) must subsequently reflect in side \(S_{j+\frac{n}{2}}\) at \(M_{j+\frac{n}{2}}\) and pass through \(F'\).

Now assume that \(n\) is odd. Then Theorem 3 implies that \(r\) passes through \(V_{j-\frac{n-1}{2}} = V_{j+\frac{n-1}{2}}\). \(\square\)

**Proof of Theorem 5.** The lines \(S_i\) and \(S_j\) are tangent to \(C_L\) at \(M_i\) and \(M_j\). Since \(F\) is a focus of \(C_L\) and \(Z\) is the intersection of the tangents, \(FZ\) bisects angle \(\angle M_i F M_j\). Since \(\angle M_i F V_i = \angle M_j F V_{j+1}\), we see that \(FZ\) bisects angle \(\angle V_i F V_{j+1}\). Similar reasoning shows that \(FZ\) bisects angle \(\angle V_{i+1} F V_j\).

Finally, since \(S_i\) and \(S_j\) are tangent to \(C_L\), the isogonal property of a conic implies that \(FZ\) and \(F'Z\) are isogonal with respect to \(S_i\) and \(S_j\). \(\square\)

**Proof of Theorem 6.** Analogous to the proof of Theorem 2. \(\square\)

**Proof of Theorem 7.** The intersections of opposite lines are the vertices of the discrete conic \(G_{n\theta}(H_\theta(D))\), implying that they are a discrete conic. In particular, the intersection of sides \(S_i S_{i+n}\) maps to the line \(d(S_i)d(S_{i+n})\) under duality. Since \(d(S_1)d(S_2) \cdots d(S_{2n})\) form a regular polygon, the dual of the intersection of opposite sides are the \(2n\) diagonals of the regular polygon (each diagonal is counted twice). These pass through the center \(O\) of the circumscribing circle of this regular polygon. Note that \(F'\) is the image of the center of \(O\). Applying duality once more, we see that these diagonals map to points lying on the dual of \(O\), which is a line orthogonal to \(FF'\). \(\square\)
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