Asymptotics of the persistence exponent of integrated FBM and Riemann-Liouville process

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July 3, 2020

Abstract

We consider the persistence probability for the integrated fractional Brownian motion and the Riemann-Liouville process with parameter $H$, respectively. For the integrated fractional Brownian motion, we discuss a conjecture of Molchan and Khokhlov and determine the asymptotic behaviour of the persistence exponent when $H \to 0$ and $H \to 1$, which is in accordance with the conjecture. For the Riemann-Liouville process, we find the asymptotic behaviour of the persistence exponent for $H \to 0$.

2010 Mathematics Subject Classification: 60G15; 60G22

Keywords: Gaussian process; integrated fractional Brownian motion; persistence; one-sided exit problem; Riemann-Liouville process; stationary process; zero crossing

1 Introduction and main results

The area of persistence probabilities deals with properties of stochastic processes when they have long excursions. The simplest question is the persistence probability itself: For a self-similar process $(X_t)_{t \geq 0}$ one expects that

$$\mathbb{P}(X_t < 1 \forall t \in [0, T]) = T^{-\theta + o(1)}, \quad T \to \infty,$$

(1)

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for some constant $\theta = \theta(X) \in (0, \infty)$, called persistence exponent, which is to be determined.

This type of problem originates in the theoretical physics literature where the persistence exponent serves as a simple measure of how fast a complicated physical system returns from a disordered initial condition to its stationary state. The question has received quite some attention in recent years for various types of processes. We refer to [11] for an overview of the theoretical physics point of view and to [3] for a survey of the mathematics literature.

The present paper deals with the persistence exponents of two related processes, namely the integrated fractional Brownian motion $I^H$ and the Riemann-Liouville process $R^H$, which we will define now.

For $H \in (0, 1)$, let $B^H$ be a standard fractional Brownian motion (FBM), that is, a centered Gaussian process with covariance
\[
\mathbb{E}[B^H_t B^H_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad t, s \geq 0.
\]
The persistence exponent of FBM is known to be $\theta(B^H) = 1 - H$, see [17] ([3], [5], [6] for refinements). In this paper, we deal with the integrated version of $B^H$, which we call $I^H$, i.e.
\[
I^H_t := \int_0^t B^H_s \, ds, \quad t \geq 0.
\]
The persistence exponent $\theta_I(H) := \theta(I^H)$ exists due to the fact that $I^H$ has non-negative correlations. However, its value is unknown unless $H = 1/2$: $B^{1/2}$ is a usual Brownian motion and $I^{1/2}$ is thus integrated Brownian motion, and it could be shown using Markov techniques that $\theta_I(1/2) = 1/4$ (cf. [13], [27], and [14]).

For the general case, Molchan and Khokhlov stated the following conjecture [22]:
\[
\theta_I(H) = H(1 - H).
\]
This conjecture is very surprising because of its symmetry, as it is clear that $B^H$ (and thus $I^H$) are very different processes for $H < 1/2$ and $H > 1/2$, respectively. Further, in the sequence of papers [22], [18], [19], [21] the following properties of $\theta_I(H)$ could be established: $\frac{1}{2} \min(H, 1-H) \leq \theta_I(H) \leq \min(H, 1-H)$ for all $H \in (0, 1)$; $\theta_I(1-H) \leq \theta_I(H)$ for $H < 1/2$; and $\theta_I(H) \leq \max(1/4, \sqrt{(1-H^2)/12})$ for all $H \in (0, 1)$.

The present paper determines the asymptotic behaviour of $\theta_I(H)$ for $H \to 0$ and $H \to 1$, respectively. This is our first main result. Here and elsewhere, $f(x) \sim g(x)$ stands for $\lim f(x)/g(x) = 1$. 2
Theorem 1. The function \( H \mapsto \theta_1(H) \) is continuous on \((0, 1)\). Further, one has \( \theta_1(H) \sim H \) for \( H \to 0 \) and \( \theta_1(H) \sim 1 - H \) for \( H \to 1 \).

The second result of this paper deals with Riemann-Liouville processes. For \( H > 0 \), define

\[
R_t^H := \int_0^t (t - s)^{H - \frac{1}{2}} dB_s, \quad t \geq 0,
\]

to be the Riemann-Liouville fractional integral of a Brownian motion \((B_t)_{t \geq 0}\).

Riemann-Liouville processes are closely related to fractional Brownian motion \(B^H\) via the Mandelbrot-van Ness integral representation:

\[
\sigma_H B_t^H = \int_0^t (t - s)^{H - \frac{1}{2}} dB_s + \int_{-\infty}^0 (t - s)^{H - \frac{1}{2}} - (-s)^{H - \frac{1}{2}} dB_s =: R_t^H + M_t^H,
\]

where \( \sigma_H^2 := (\Gamma(H + 1/2))^2 \Gamma(2 - 2H) \cos(\pi H)/(\pi H(1 - 2H)) \), see e.g. [16, Section 2]. Clearly, \( R_t^H \) and \( M_t^H \) are independent processes. For \( H = 1/2 \) we have \( M_t^{1/2} \equiv 0 \) so that \( R_t^{1/2} \) is a usual Brownian motion (as can also be seen from the defining integral representation (2)). Furthermore, a simple Fubini argument shows that for integer \( n \geq 0 \) we have

\[
R_t^{n+1/2} = n! \int_0^t \ldots \int_0^{s_{n-1}} B_{s_n} ds_n \ldots ds_1 = n! (I^n B)_t, \quad t \geq 0,
\]

where \( (If)_t := \int_0^t f(s) ds \) is the simple integration operator. So, the Riemann-Liouville process is simply a (fractionally) integrated Brownian motion.

The persistence exponent of the Riemann-Liouville process, \( \theta_R(H) := \theta(R^H) \) exists due to the fact that the process has non-negative correlations. In [4], it was shown that \( \theta_R(H) \) is non-increasing and that it is the same exponent as for a number of further processes.

For \( H \to \infty \), the correlation function of the Lamperti transform of \( R^H \) (see below for precise definitions) converges to the correlation function \( \tau \mapsto 1/\cosh(\tau/2) \), the corresponding process having persistence exponent \( \theta_R(\infty) \in (0, \infty) \). Now, a continuity theorem for persistence exponents (see [10, Theorem 1.6], [11, Lemma 3.1], or [7, Lemma 3.6]; summarized suitably for our purposes in Lemma 3 below) shows that \( \theta_R(H) \to \theta_R(\infty) \) for \( H \to \infty \). In [23], the exponent is derived to be \( \theta_R(\infty) = 3/16 \).

This paper is concerned with the behaviour of \( \theta_R(H) \) for \( H \to 0 \). We show that \( \theta_R(H) \) tends to infinity for \( H \to 0 \) and that the asymptotic behaviour is in the range \( H^{-1} \) to \( H^{-2} \). This is our second main result.

Theorem 2. The function \( H \mapsto \theta_R(H) \) is continuous on \((0, \infty)\). Further, \( \lim \sup_{H \to 0} \theta_R(H) H^2 < \infty \) and \( \lim \inf_{H \to 0} \theta_R(H) H > 0 \).
Figure 1 illustrates the behaviour of the persistence exponents of Riemann-Liouville process, (integrated) fractional Brownian, and (integrated) Brownian motion.

The study of the persistence probabilities of FBM, iFBM, and related processes has received considerable attention in theoretical physics and mathematics. For instance, see [22] and [20] where a relation between the Hausdorff dimension of Lagrangian regular points for the inviscid Burgers equation with FBM initial velocity and the persistence probabilities of integrated FBM is established; the interest for it arises from [25] and [27].

The outline of the rest of this paper is as follows. In the next subsection, we transform the problem for $I^H$ and $R^H$ into a persistence problem for stationary Gaussian processes (GSP) and sketch the proof technique. Section 2 contains the proofs related to Theorem 1 while Section 3 is devoted to the proofs related to Theorem 2.

**Transformation to GSP; ideas of the proofs**

The first step in our proofs is to transform the involved self-similar processes – $I^H$ and $R^H$ are $(H + 1)$-self-similar and $H$-self-similar, respectively – into stationary Gaussian processes via an exponential time change, also called Lamperti transform.
More generally, for an $H$-self-similar process $(X_t)_{t \geq 0}$, we consider its Lamperti transform $Z_t := e^{-H \tau} X_{e^\tau}$, $\tau \in \mathbb{R}$. It can often be shown that (1) turns into
\[ \mathbb{P}(Z_\tau < 0 \ \forall \tau \in [0, T]) = e^{-T(\theta + o(1))}, \quad T \to \infty, \] (3)
where $\theta = \theta(X)$ is as in (1).

Consequently, we consider the Lamperti transform of $I^H$ defined by
\[ U^H_\tau := \sqrt{2(1 + H)} e^{-(1+H)\tau} I^H_{e^\tau}, \quad \tau \in \mathbb{R}, \]
where we also renormalized in order to have a unit variance process.

Similarly, we consider the normalized Lamperti transform of $R^H$ defined by
\[ V^H_\tau := \sqrt{2H} e^{-\tau H} R^H_{e^\tau}, \quad \tau \in \mathbb{R}. \]

The basic idea of our proofs is as follows. The first step is to show that indeed (1) for $I^H$ ($R^H$, respectively) is the same as (3) for $U^H$ ($V^H$, respectively). This is a standard argument where we follow [4, Proposition 1.6] or [18, Theorem 1].

The second step to prove Theorem 1 (Theorem 2 is proved similarly, but is significantly more technical) is to consider the GSP $(U^H_{\tau/H})_{\tau \in \mathbb{R}}$ for $H \to 0$ and the GSP $(U^H_{\tau/(1-H)})_{\tau \in \mathbb{R}}$ for $H \to 1$, respectively: Their persistence exponents are given by $\theta_1(H)/H$ and $\theta_1(H)/(1 - H)$, respectively, as a quick computation shows. We will show that in both of these cases, the respective correlation function of that stationary Gaussian process tends to the correlation function $\tau \mapsto e^{-\tau}$, which is the correlation function of an Ornstein-Uhlenbeck process, which has persistence exponent 1. Then, one uses the following lemma, which is Lemma 3.6 in [7] together with Remark 3.8 in [7] and Theorem 1.6 in [10] as well as Lemma 3.10 in [7], to conclude the convergence of the persistence exponents $\theta_1(H)/H \to 1$ for $H \to 0$ and, respectively, $\theta_1(H)/(1 - H) \to 1$ for $H \to 1$.

**Lemma 3.** For $k \in \mathbb{N}$, let $(Z^{(k)}_\tau)_{\tau \geq 0}$ be a stationary, centered Gaussian process with non-negative correlation function $A_k(\tau)$, $\tau \geq 0$, satisfying $A_k(0) = 1$. Suppose that $A_k(\tau) \to A(\tau)$ for $k \to \infty$ and all $\tau \geq 0$, where $A: [0, \infty) \to [0, 1]$ is the correlation function of a centered, stationary Gaussian process $(Z_\tau)_{\tau \geq 0}$.
(a) If $Z^{(k)}$ and $Z$ have continuous sample paths and the conditions

\[
\lim_{L \to \infty} \limsup_{k \to \infty} \sum_{\tau=L}^{\infty} A_k \left( \frac{\tau}{L} \right) = 0 \text{ for every } \ell \in \mathbb{N}, \tag{4}
\]

\[
\limsup_{\varepsilon \downarrow 0} |\log \varepsilon|^\eta \sup_{k \in \mathbb{N}, \tau \in [0, \varepsilon]} (1 - A_k(\tau)) < \infty \text{ for some } \eta > 1, \text{ and} \tag{5}
\]

\[
\limsup_{\tau \to \infty} \frac{\log A(\tau)}{\log \tau} < -1 \tag{6}
\]

are fulfilled, then

\[
\lim_{k,T \to \infty} \frac{1}{T} \log P(Z^{(k)}_\tau < 0 \forall \tau \in [0, T]) = \lim_{T \to \infty} \frac{1}{T} \log P(Z_\tau < 0 \forall \tau \in [0, T]). \tag{7}
\]

(b) If $A(\tau) = 0$ for all $\tau > 0$ and (4) is fulfilled, then

\[- \lim_{k,T \to \infty} \frac{1}{T} \log P(Z^{(k)}_\tau < 0 \forall \tau \in [0, T]) = \infty.
\]

The lemma says that if the correlation functions of the processes $Z^{(k)}$ converge pointwise to the correlation function of the process $Z$ and the technical conditions (4)–(6) are satisfied, then the persistence exponents of the processes $Z^{(k)}$ converge to the persistence exponent of the process $Z$. Here, the existence of the persistence exponents, i.e. the existence of the (negative) limits in (7), follows from non-negative correlations, Slepian’s lemma, and subadditivity.

## 2 Proofs for the case of integrated FBM

In this section, we prove Theorem 1. We start with a lemma giving important properties of the correlation function of $U^H$. In particular, we see the mentioned convergence of the correlation functions of $(U^H_{\tau/H})_{\tau \in \mathbb{R}}$ for $H \to 0$ and of $(U^H_{\tau/(1-H)})_{\tau \in \mathbb{R}}$ for $H \to 1$, respectively.

**Lemma 4.** The correlation function $\rho_H(\tau)$, $\tau \geq 0$, of the process $U^H$ is given by

\[
\rho_H(\tau) = \mathbb{E}[U^H_0 U^H_\tau] = \frac{(1 + H) \left( e^{-H \tau} + e^{H \tau} \right)}{1 + 2H} + \frac{\left( e^{\tau/2} - e^{-\tau/2} \right)^2 (1+H)^2 - e^{(1+H)\tau} - e^{-(1+H)\tau}}{2 (1 + 2H)}. \tag{8}
\]
Further, $\rho_H$ is non-increasing and $\lim_{H \to 0} \rho_H \left( \frac{\tau}{H} \right) = \lim_{H \to 1} \rho_H \left( \frac{\tau}{1-H} \right) = e^{-\tau}$ for all $\tau \geq 0$.

Proof. For the representation in (8) and the monotonicity, see [18, Lemma 2]. For the convergence, note that

$$
\rho_H \left( \frac{\tau}{H} \right) = \frac{(1 + H)(e^\tau + e^{-\tau})}{1 + 2H} + \frac{\left( e^{\frac{\tau}{H}} - e^{-\frac{\tau}{H}} \right)^2}{2 (1 + 2H)}
$$

$$
= \frac{(1 + H)(e^\tau + e^{-\tau})}{1 + 2H} + \frac{\sum_{k=0}^{\infty} (-1)^k (2 + 2H)e^{\frac{\tau}{1+H-k}} e^{-\frac{1+H}{H} \tau} - e^{-\frac{1+H}{H} \tau}}{2 (1 + 2H)}, \quad \tau \geq 0,
$$

by the generalized binomial theorem. Note that in this case, the binomial theorem also holds for $\tau = 0$ due to the fact that $2 + 2H > 0$, see e.g. [1]. As $H < 1$, we have for all $k \geq 2$

$$
\left| \binom{2 + 2H}{k} \right| = \frac{2H + 2}{1} \cdot \frac{2H + 1}{2} \cdot \frac{2H + 4 - k}{2} \cdot \frac{2H + 3 - k}{k - 1} \cdot \frac{2H + 1 - k}{k} \leq 4 \cdot \frac{3}{2} \cdot 1 \cdot 1 = 6.
$$

Therefore, we can majorize $(-1)^k (2 + 2H)e^{\frac{\tau}{1+H-k}}$, $k \geq 2$, by the summable sequence $6 e^{\tau(2-k)}$, $k \geq 2$, to conclude with the dominated convergence theorem that for all $\tau \geq 0$

$$
\lim_{H \to 0} \rho_H \left( \frac{\tau}{H} \right) = e^{-\tau} + \frac{\sum_{k=2}^{\infty} (-1)^k \lim_{H \to 0} \left( 2 + 2H \right) e^{\frac{k-1+H}{H} \tau} - \lim_{H \to 0} e^{-\frac{1+H}{H} \tau}}{2} = e^{-\tau}.
$$

Analogously to (9), one derives for $\tau \geq 0$

$$
\rho_H \left( \frac{\tau}{1-H} \right) = \frac{(1 + H)(e^{\frac{\tau}{1-H} \tau})}{1 + 2H} + \frac{\sum_{k=2}^{\infty} (-1)^k (2 + 2H)e^{\frac{\tau}{1-H} (1+H-k)} e^{-\frac{1+H}{H} \tau}}{2 (1 + 2H)}
$$

$$
= \frac{1 + H}{1 + 2H} e^{\frac{\tau}{1-H} \tau} + \frac{1 + H}{2} e^{-\tau} + \frac{\sum_{k=3}^{\infty} (-1)^k (2 + 2H)e^{\frac{\tau}{1-H} (1+H-k)} - e^{-\frac{1+H}{H} \tau}}{2 (1 + 2H)}.
$$

(11)
By (10), we can majorize \((-1)^k\binom{2+2H}{k} e^{\frac{-\tau}{1-H} (1+H-k)}, k \geq 3\), by the summable sequence \(6 e^{\tau (3-k)}, k \geq 3\). Thus, the dominated convergence theorem yields

\[
\lim_{H \to 1} \rho_H \left( \frac{\tau}{1-H} \right) = \frac{2}{3} \lim_{H \to 1} e^{-\frac{\tau}{1-H} \tau} + e^{-\tau} + \sum_{k=3}^{\infty} \left( -1 \right)^k \lim_{H \to 1} \left( \binom{2+2H}{k} e^{-\frac{k+1+H}{1-H} \tau} - \lim_{H \to 1} e^{-\frac{k+1+H}{1-H} \tau} \right) = \frac{1}{6} \lim_{H \to 1} \left( \binom{2+2H}{k} \right) e^{-\frac{k+1+H}{1-H} \tau} = e^{-\tau},
\]

for all \(\tau \geq 0\), which finishes the proof.

\[\Box\]

**Proof of Theorem 1.** Due to subadditivity, Slepian’s lemma and the fact that \(U^H\) is a stationary, centered Gaussian process with non-negative correlations, the persistence exponent

\[
-\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}(U^H_T < 0 \forall \tau \in [0, T]) \tag{12}
\]

exists. Further, (12) equals \(\theta_I(H)\). Indeed, note that

\[
\mathbb{P}(U^H_T < 0 \forall \tau \in [0, T]) = \mathbb{P}(I^H_T < 0 \forall t \in [1, eT])
\]

and that \(\mathbb{P}(I^H_T < 0 \forall t \in [1, T])\) has the same polynomial rate for \(T \to \infty\) as \(\mathbb{P}(I^H_T < 1 \forall t \in [0, T])\), see [18, Theorem 1].

The case \(H \to 0\). Observe that

\[
\frac{\theta_I(H)}{H} = -\frac{1}{H} \lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}(U^H_T < 0 \forall \tau \in [0, T])
= -\frac{1}{H} \lim_{T \to \infty} \frac{1}{T/H} \log \mathbb{P}(U^H_{T/H} < 0 \forall \tau \in [0, T/H])
= -\frac{1}{H} \lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}(U^H_{T/H} < 0 \forall \tau \in [0, T]) \tag{13}
\]

By Lemma 4, the correlation function \(\tau \mapsto \rho_H \left( \frac{\tau}{H} \right)\) of \((U^H_{T/H})_{\tau \in \mathbb{R}}\) converges pointwise for \(H \to 0\) to \(\tau \mapsto e^{-\tau}\). This is the correlation function of the Ornstein-Uhlenbeck process, which has persistence exponent 1, see [26].

So, as soon as we have proven that also the persistence exponents converge, the desired convergence \(\theta_I(H)/H \to 1\) for \(H \to 0\) follows. In order to achieve this, we want to apply Lemma 3(a), i.e. we check the conditions (4)–(6) for the process \((U^H_{T/H})_{\tau \in \mathbb{R}}\) with correlation function \(\tau \mapsto \rho_H \left( \frac{\tau}{H} \right)\). Obviously, (6) is fulfilled for the limiting correlation function \(\tau \mapsto e^{-\tau}\).
For checking (4), note that for $H \in (0, 1/2]$ and every $k \geq 4$, one has
\[(−1)^k \binom{2 + 2H}{k} = -\frac{2H + 2}{1} \cdot \frac{2H + 1}{2} \cdot \frac{2}{3} \cdot \frac{1 − 2H}{4} \cdot \frac{2 − 2H}{5} \cdot \cdots \cdot \frac{k − 3 − 2H}{k} \leq 0 \quad (14)\]
and $(-1)^k \binom{2 + 2H}{k} = -\frac{2H + 2}{1} \cdot \frac{2H + 1}{2} \cdot \frac{2H}{3} < 0$ for $k = 3$. Thus, by using the representation in (9) and estimating all non-positive terms by $0$, we have
\[
\rho_H \left( \frac{\tau}{H} \right) \leq \frac{1 + H}{1 + 2H} e^{-\tau} + \frac{1}{2(1 + 2H)} \cdot \frac{2H + 2}{1} \cdot \frac{2H + 1}{2} e^{\tau(1 + H - 2)} + 0
\]
for $H \in (0, 1/2]$, and (4) follows.
For checking (5), note that due to the monotonicity of $\rho_H$ and (9), one gets
\[
\sup_{\tau \in [0, \epsilon]} \left( 1 - \rho_H \left( \frac{\tau}{H} \right) \right) = \sup_{\tau \in [0, \epsilon]} \left( 1 - \rho_H \left( \frac{0}{H} \right) - \rho_H \left( \frac{\epsilon}{H} \right) \right)
\]
\[
\leq \frac{1 + H}{1 + 2H} (1 - e^{-\epsilon}) + \frac{1}{2(1 + 2H)} \sum_{k=2}^{\infty} (-1)^k \binom{2 + 2H}{k} \left( 1 - e^{\tau(1 + H - k)} \right) - \left( 1 - e^{-1/H \epsilon} \right)
\]
\[
\leq (1 - e^{-\epsilon}) + \left( \frac{1 + H}{2} - \frac{1}{2(1 + 2H)} \right) (1 - e^{-\epsilon})
\]
\[
\leq \epsilon + \left( \frac{1 + H}{2} - \frac{1}{2(1 + 2H)} \right) \epsilon = \epsilon + \frac{3 + 2H}{2(1 + 2H)} \epsilon \leq \epsilon + \frac{\epsilon}{2} = 3 \epsilon, \quad (15)
\]
for all $\epsilon > 0$ and $H \in (0, 1/2]$, where we used again (14) to estimate non-positive terms by $0$ and the fact that $1 - e^{-x} \leq x$. This shows (5) for every $\eta > 1$.

The case $H \to 1$. Similarly to (13), one has
\[
\frac{\theta(t)}{1 - H} = -\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P} \left( \frac{U_t^H}{1 - H} < 0 \forall \tau \in [0, T] \right)
\]
and by Lemma [4], the correlation function $\tau \mapsto \rho_H \left( \frac{\tau}{1 - H} \right)$ of $\left( U^H_{\tau/(1 - H)} \right)_{\tau \in \mathbb{R}}$ converges pointwise for $H \to 1$ to $\tau \mapsto e^{-\tau}$. Again, this is the correlation function of the Ornstein-Uhlenbeck process with persistence exponent $1$. Applying Lemma [3](a) for the process $\left( U^H_{\tau/(1 - H)} \right)_{\tau \in \mathbb{R}}$ finishes the proof of
the asymptotics, subject to checking the technical conditions. We have al-
ready seen that (6) is fulfilled, as the limiting correlation function is the same
as in the $H \to 0$ case. Next, we check condition (4): Considering (14) for
$H \in [1/2, 1)$, one sees that $(-1)^k \binom{2+2H}{k} < 0$ for $k = 3$ and $(-1)^k \binom{2+2H}{k} \geq 0$
for $k \geq 4$. So, using the representation in (11) and estimating again the
negative terms by 0, one gets

$$
\rho_H \left( \frac{\tau}{1-H} \right) = \frac{1 + H}{1 + 2H} e^{-\frac{H}{1-H} \tau} + \frac{1 + H}{2} e^{-\tau} + \frac{\sum_{k=3}^{\infty} (-1)^k \binom{2+2H}{k} e^{\tau(1+H-k)} - e^{-\frac{H}{1-H} \tau}}{2(1+2H)}
\leq \frac{1 + H}{1 + 2H} e^{-\frac{H}{1-H} \tau} + \frac{1 + H}{2} e^{-\tau} + \frac{\sum_{k=4}^{\infty} (-1)^k \binom{2+2H}{k} e^{\tau(1+H-k)}}{2(1+2H)}
\leq \left( 2 + \frac{\sum_{k=4}^{\infty} (-1)^k \binom{2+2H}{k}}{2(1+2H)} \right) e^{-\tau}
$$

for $H \in [1/2, 1)$. By the generalized binomial theorem, it holds

$$
\sum_{k=4}^{\infty} (-1)^k \binom{2+2H}{k} = \sum_{k=0}^{\infty} (-1)^k \binom{2+2H}{k} - \sum_{k=0}^{3} (-1)^k \binom{2+2H}{k}
= (1 - 1)^{2+2H} - 1 + (2 + 2H) - \frac{(2 + 2H)(1 + 2H)}{2}
+ \frac{(2 + 2H)(1 + 2H) \cdot 2H}{3!}
\leq 1 + 2H + \frac{(2 + 2H)(1 + 2H)H}{3},
$$

and thus, we can estimate

$$
\rho_H \left( \frac{\tau}{1-H} \right) \leq \left( 2 + \frac{1}{2} + \frac{H(1 + H)}{3} \right) e^{-\tau} \leq 4 e^{-\tau},
$$

and (4) follows. Further, using (11) again, condition (5) is shown similarly.
to \text{[15]}, as in this case

\begin{equation*}
\sup_{\tau \in [0, \varepsilon]} \left( 1 - \rho_H \left( \frac{\tau}{1 - H} \right) \right) = \rho_H \left( \frac{0}{1 - H} \right) - \rho_H \left( \frac{\varepsilon}{1 - H} \right) = \frac{1 + H}{1 + 2H} \left( 1 - e^{-\frac{H\varepsilon}{1 - H}} \right) + \frac{1 + H}{2} (1 - e^{-\varepsilon}) + \sum_{k=3}^{\infty} (-1)^k \left( \frac{2 + 2H}{k} \right) \left( 1 - e^{\frac{k\varepsilon}{1 + H - k}} \right) - \left( 1 - e^{-\frac{1 + H}{1 - H} \varepsilon} \right)
\end{equation*}

\begin{align*}
&= \frac{1 + H}{2} (1 - e^{-\varepsilon}) + \frac{1 + H}{1 + 2H} \left( 1 - e^{-\frac{H\varepsilon}{1 - H}} \right) - \frac{H(1 + H)}{3} \left( 1 - e^{-\frac{3H\varepsilon}{1 - 3H}} \right) \\
&\quad + \frac{H(1 + H)(2H - 1)}{12} \left( 1 - e^{-\frac{(2H - 1)\varepsilon}{1 - (2H - 1)}} \right) - \frac{1}{2(1 + 2H)} \left( 1 - e^{-\frac{(2H - 1)\varepsilon}{1 - (2H - 1)}} \right) \\
&\quad + \sum_{k=5}^{\infty} (-1)^k \left( \frac{2 + 2H}{k} \right) \left( k - 1 - H \right) \varepsilon \frac{1}{1 - H},
\end{align*}

where we used again $1 - e^{-x} \leq x$. Note that

\begin{equation*}
\frac{1 + H}{1 + 2H} - \frac{H(1 + H)}{3} = \frac{(3 + 2H)(1 + H)(1 - H)}{3(1 + 2H)},
\end{equation*}

that

\begin{equation*}
\frac{H(1 + H)(2H - 1)}{12} \left( 1 - e^{-\frac{(2H - 1)\varepsilon}{1 - (2H - 1)}} \right) - \frac{1}{2(1 + 2H)} \left( 1 - e^{-\frac{(2H - 1)\varepsilon}{1 - (2H - 1)}} \right) < 0,
\end{equation*}

for $H \in (0, 1)$ and $\varepsilon > 0$; and that, for $k \geq 5$,

\begin{equation*}
(-1)^k \left( \frac{2 + 2H}{k} \right) \left( k - 1 - H \right) \frac{1}{1 - H} = \frac{2H + 2}{k - 2} \cdot \frac{(k - 1 - H)(2H + 1)}{k - 1} \cdot \frac{2H}{k} \cdot \frac{2H - 1}{1} \cdot \frac{2 - 2H}{2(1 - H)} \cdot \frac{3 - 2H}{3} \cdots \frac{k - 3 - 2H}{k - 3} \leq \frac{4}{k - 2} \cdot \frac{2}{k} \cdot 1 \cdots 1 = \frac{12}{k(k - 2)}.
\end{equation*}
which is summable in $k$. Putting these facts together, we get, for every $\eta > 1$,

$$\limsup_{\varepsilon \to 0} |\log \varepsilon|^\eta \sup_{H \in [1/2, 1], \tau \in [0, \varepsilon]} \left( 1 - \rho_H \left( \frac{\tau}{1 - H} \right) \right)$$

$$\leq \limsup_{\varepsilon \to 0} |\log \varepsilon|^\eta \sup_{H \in [1/2, 1]} \left( 1 + \frac{(3 + 2H)(1 + H)}{3(1 + 2H)} + \sum_{k=5}^{\infty} \frac{12}{k(k - 2)} \right)$$

$$= 0 < \infty,$$

showing (5).

Finally, the continuity of $\theta_I$ follows by the continuity of $H \mapsto \rho_H(\tau)$ and Lemma 3(a), as one sees easily that the conditions (4)–(6) are satisfied for the sequence $\tau \mapsto \rho_H(\tau)$, $H \in [H_0 - \delta, H_0 + \delta]$, with fixed $H_0 \in (0, 1)$, small $\delta > 0$ and $H \to H_0$.

\[ \square \]

3 Proofs for the case of Riemann-Liouville processes

In this section, we prove Theorem 2. For this purpose, we will need the following two lemmas about the correlation function of $V^H$.

**Lemma 5.** The correlation function $r_H(\tau)$, $\tau \geq 0$, of the process $V^H$ is given by

$$r_H(\tau) = \mathbb{E}[V_0^H V_{\tau}^H] = \frac{4H}{1 + 2H} e^{-\frac{\tau}{2}} \cdot 2F_1(1, 1/2 - H, 3/2 + H; e^{-\tau}), \quad (16)$$

where $2F_1$ denotes Gauss’ hypergeometric function. In particular, $r_H$ is non-increasing.

Further, for $\tau \geq 0$, we have

$$e^{-\tau/2} - r_H(\tau)$$

$$= \frac{4H \Gamma(2H) \Gamma(\frac{3}{2} + H)}{(1 + 2H) \Gamma(\frac{1}{2} - H) \Gamma(1 + 2H)} e^{-\frac{\tau}{2}} (1 - e^{-\tau}) \sum_{\ell=0}^{\infty} e^{-\ell\tau} \frac{\Gamma(\ell + \frac{3}{2} - H)}{\Gamma(\ell + \frac{3}{2} + H)}. \quad (17)$$

**Proof.** For the equality in (16), see e.g. [16, Equation (12)]. For (17), first note that $r_H(0) = 1$ and thus

$$\frac{1 + 2H}{4H} e^{\tau/2} (e^{-\tau/2} - r_H(\tau))$$

$$= 2F_1(1, 1/2 - H, 3/2 + H; 1) - 2F_1(1, 1/2 - H, 3/2 + H; e^{-\tau}). \quad (18)$$
By Euler’s integral representation of the hypergeometric function, we have

\[ 2 F_1(1, 1/2 - H, 3/2 + H; e^{-\tau}) = \frac{\Gamma\left(\frac{3}{2} + H\right)}{\Gamma\left(\frac{1}{2} - H\right) \Gamma(1 + 2H)} \int_0^1 t^{-(H+1/2)}(1 - t)^{2H} \left(1 - e^{-\tau t}\right)^{-1} dt. \]  

(19)

Inserting this into (18), we get

\[ \frac{(1 + 2H) \Gamma\left(\frac{1}{2} - H\right) \Gamma(1 + 2H)}{4H \Gamma\left(\frac{3}{2} + H\right)} e^{-\tau/2} \left(e^{-\tau/2} - r_H(\tau)\right) \]

\[ = \int_0^1 t^{-(H+1/2)}(1 - t)^{2H} \left[(1 - t)^{-1} - (1 - e^{-\tau t})^{-1}\right] dt \]

\[ = \int_0^1 t^{-(H+1/2)}(1 - t)^{2H-1} \cdot \left[1 - \frac{1 - t}{1 - e^{-\tau t}}\right] \frac{1}{1 - e^{-\tau t}} dt \]

\[ = (1 - e^{-\tau}) \int_0^1 t^{1-(H+1/2)}(1 - t)^{2H-1} \cdot \sum_{\ell=0}^\infty e^{-\ell\tau} \ell \cdot \frac{1}{1 - e^{-\tau t}} dt \]

\[ = (1 - e^{-\tau}) \sum_{\ell=0}^\infty e^{-\ell\tau} \int_0^1 t^{\ell+3/2-H-1}(1 - t)^{2H-1} dt \]

\[ = (1 - e^{-\tau}) \sum_{\ell=0}^\infty e^{-\ell\tau} \frac{\Gamma(\ell + 3/2 - H) \Gamma(2H)}{\Gamma(\ell + 3/2 + H)}, \]

which shows (17). Finally, the monotonicity of \(r_H\) follows by the monotonicity of \(\tau \mapsto e^{-\tau/2}\) and the monotonicity of the hypergeometric function, see (19).

We now analyse the behaviour of the rescaled correlation \(r_H(\tau/\gamma)\) with \(\gamma = \gamma_H \to \infty\) with \(H \to 0\).

Lemma 6. Let \(\gamma = \gamma_H\) be a function tending to infinity with \(H \to 0\). If \(\gamma^{-2H} \to c\) for \(H \to 0\) and some \(c \in [0, 1]\), then \(r_H(\tau/\gamma) \to 1 - c\) for \(H \to 0\) and all \(\tau > 0\).

Proof. By (17), we have

\[ e^{-\frac{\tau}{\gamma}} - r_H\left(\frac{\tau}{\gamma}\right) \sim \frac{\tau}{\gamma} \cdot \sum_{\ell=0}^\infty e^{-\ell\tau/\gamma} \frac{\Gamma(\ell + 3/2 - H) \Gamma(2H)}{\Gamma(\ell + 3/2 + H)}, \quad H \to 0, \]

(20)
as $\gamma^{-1} \to 0$ with $H \to 0$ and $\Gamma(2H)\Gamma(1-2H) = \pi/\sin(2\pi H) \sim (2H)^{-1}$ for $H \to 0$. To estimate the fraction $\frac{\Gamma(\ell+\frac{3}{2}-H)}{\Gamma(\ell+\frac{3}{2}+H)}$, recall Stirling’s formula for the gamma function, [28, Section XII.3.3], yielding

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} e^{\mu(x)}, \quad 0 < \mu(x) < \frac{1}{12x}, \quad (21)$$

for $x > 0$ and thus

$$\frac{\Gamma(\ell+\frac{3}{2}-H)}{\Gamma(\ell+\frac{3}{2}+H)} = \left(\ell + \frac{3}{2} - H\right)^{\ell+\frac{1}{2}-H} \left(\ell + \frac{3}{2} + H\right)^{-\ell-\frac{1}{2}+H} e^{2H e^{\mu(\ell+\frac{3}{2}-H) - \mu(\ell+\frac{3}{2}+H)}} \quad (22)$$

for $H \in (0, 1/2]$ and $\ell \in \mathbb{N}_0$. Note that due to the range of $\mu$ given in (21), the last factor in (22) converges to 1 uniformly in $H$ for $\ell \to \infty$. Further,

$$\left(\ell + \frac{3}{2} - H\right)^{\ell+\frac{1}{2}-H} \left(\ell + \frac{3}{2} + H\right)^{-\ell-\frac{1}{2}+H} e^{2H} \quad (22)$$

for $\ell \to \infty$, as

$$e^{2H} \left(\frac{\ell + \frac{3}{2} - H}{\ell + \frac{3}{2} + H}\right)^{\ell+\frac{1}{2}} \left(\frac{\ell + \frac{3}{2} + H}{\ell + \frac{3}{2}}\right)^{-\ell-\frac{1}{2}} = e^{2H} \left(1 + \frac{1 - H}{\ell + \frac{3}{2}}\right)^{\ell+\frac{1}{2}} \left(1 + \frac{1 + H}{\ell + \frac{3}{2}}\right)^{-\ell-\frac{1}{2}} \to e^{2H} \cdot e^{1-H} \cdot e^{-1-H} = 1$$

and again, the convergence is uniform in $H \in (0, 1/2]$. Thus, for every $\varepsilon > 0$, there exists $\ell_0 \in \mathbb{N}_0$ independent of $H$ s.t.

$$(1 - \varepsilon) \ell^{-2H} \leq \frac{\Gamma(\ell+\frac{3}{2}-H)}{\Gamma(\ell+\frac{3}{2}+H)} \leq (1 + \varepsilon) \ell^{-2H}, \quad \ell \geq \ell_0, \; H \in (0, 1/2]. \quad (23)$$

Now, let us first show $\lim_{H \to 0}\left(e^{-r(\tau/2\gamma)} - r_H(\tau/\gamma)\right) \geq c$. Fix $\varepsilon \in (0, 1)$.  


Considering the lower bound of (23) in (20) leads to

\[
\lim_{H \to 0} \left( e^{-\frac{\tau}{\gamma}} - r_H \left( \frac{\tau}{\gamma} \right) \right) = \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \sum_{\ell = 0}^{\infty} e^{-\ell \tau / \gamma} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)}
\]
\[
\geq \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \sum_{\ell = 0}^{\infty} e^{-\ell \tau / \gamma} \ell^{-2H}
\]
\[
\geq (1 - \varepsilon) \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \int_{\ell \tau / \gamma}^{\infty} e^{-y / \gamma} y^{-2H} dy
\]
\[
= (1 - \varepsilon) \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \int_{\ell \tau / \gamma}^{\infty} e^{-y / \gamma} dy \cdot \lim_{H \to 0} \frac{(\tau / \gamma)^{2H}}{\Gamma \left( \ell + \frac{3}{2} - H \right)} = (1 - \varepsilon) c.
\]

Letting \(\varepsilon \to 0\) finishes the proof of the lower bound.

Finally, we show \(\lim_{H \to 0} (e^{-\tau / (2\gamma)} - r_H(\tau / \gamma)) \leq c\). Analogously to the lower bound case, the upper bound of (23) together with (20) yields

\[
\lim_{H \to 0} \left( e^{-\frac{\tau}{\gamma}} - r_H \left( \frac{\tau}{\gamma} \right) \right) = \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \sum_{\ell = 0}^{\infty} e^{-\ell \tau / \gamma} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)}
\]
\[
\leq \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \sum_{\ell = 0}^{\ell_0 - 1} e^{-\ell \tau / \gamma} \ell^{-2H} + (1 + \varepsilon) \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \sum_{\ell = \ell_0}^{\infty} e^{-\ell \tau / \gamma} \ell^{-2H}
\]
\[
\leq 0 \cdot \ell_0 + (1 + \varepsilon) \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \int_{(\ell_0 - 1)\tau / \gamma}^{\infty} e^{-y / \gamma} y^{-2H} dy
\]
\[
= (1 + \varepsilon) \lim_{H \to 0} \frac{\Gamma \left( \ell + \frac{3}{2} - H \right)}{\Gamma \left( \ell + \frac{3}{2} + H \right)} \int_{(\ell_0 - 1)\tau / \gamma}^{\infty} e^{-y / \gamma} dy \cdot \lim_{H \to 0} \frac{(\tau / \gamma)^{2H}}{\Gamma \left( \ell + \frac{3}{2} - H \right)} = (1 + \varepsilon) c,
\]
which shows the assertion by letting \(\varepsilon \to 0\).

\(\Box\)

**Proof of Theorem 2**. Similarly to the proof of Theorem 1 due to non-negative correlations, the persistence exponent

\[\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}(V^H_\tau < 0 \forall \tau \in [0, T])\]
exists and equals $\theta_R(H)$: Again, one may apply [18 Theorem 1] to see that $\mathbb{P}(R^H_t < 0 \forall t \in [1, T])$ and $\mathbb{P}(R^H_t < 1 \forall t \in [0, T])$ have the same polynomial rate for $T \to \infty$. Note that, according to [1 Corollary 3.1], a proper choice for $\phi_T$ in [18 Theorem 1] is given by $\phi_T := \phi \equiv 0$ on $[0, 1/2)$ and

$$\varphi(t) := c \int_{1/2}^{t} (t-s)^{H-\frac{1}{2}} s^{-\eta} ds, \quad t \geq \frac{1}{2},$$

for $\eta \in (1/2, 1/2 + H)$ and $c$ large enough s.t. $\varphi(t) \geq 1$ for all $t \geq 1$. Such $c$ exists due to the fact that $\phi$ is continuous on $[1, \infty)$ and that

$$c^{-1} \varphi(t) = t^{H+\frac{1}{2}-\eta} \int_{1/2}^{t} \left(1 - \frac{s}{t}\right)^{H-\frac{1}{2}} \left(\frac{s}{t}\right)^{-\eta} ds = t^{H+\frac{1}{2}-\eta} \int_{1/(2t)}^{1} (1-u)^{H-\frac{1}{2}} u^{-\eta} du \sim t^{H+\frac{1}{2}-\eta} \int_{0}^{1} (1-u)^{H-\frac{1}{2}} u^{-\eta} du \to \infty, \quad t \to \infty.$$

Now, similarly to (13), for every function $\gamma = \gamma_H$, we have

$$\frac{\theta_R(H)}{\gamma} = -\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}(V^H_{\tau/\gamma} < 0 \forall \tau \in [0, T]).$$

We will show $\theta_R(H)/\gamma \to 0$ for any function $\gamma = \gamma_H$ with $\gamma \gg H^{-2}$ as well as $\theta_R(H)/\gamma \to \infty$ for any function $\gamma = \gamma_H$ with $\gamma \ll H^{-1}$, where $f(x) \ll g(x)$ means $\lim f(x)/g(x) = 0$. This proves the assertion.

The case $\gamma \gg H^{-2}$. We proceed similarly to the proof of [7 Lemma 3.2], define

$$Y^H_T := \int_{0}^{T} V^H_{\tau/\gamma} d\tau, \quad \sigma^2_T := \mathbb{V}Y^H_T$$

and fix $\delta > 0$. Then

$$\mathbb{P}(V^H_{\tau/\gamma} < 0 \forall \tau \in [0, T]) \geq \mathbb{E} \left[ \mathbb{P} \left( \sup_{\tau \in [0, T]} V^H_{\tau/\gamma} < 0, \ Y^H_T < -\delta \sigma_T \sqrt{T} \right| Y^H_T \right]$$

$$= \mathbb{E} \left[ \mathbb{P} \left( \sup_{\tau \in [0, T]} V^H_{\tau/\gamma} < 0 \right| Y^H_T < -\delta \sigma_T \sqrt{T} \right] \mathbb{I}_{\{Y^H_T < -\delta \sigma_T \sqrt{T}\}}. \quad (25)$$

As $V^H$ and thus also $Y^H_T$ are Gaussian processes, by [12 Proposition 3.13], $(V^H_{\tau/\gamma})_{\tau \in [0, T]}$ given $Y^H_T = y$ is a Gaussian process with mean function

$$\mu(\tau) \cdot y := \sigma^2_T \int_{0}^{\tau} r_H \left(\frac{\tau - t}{\gamma}\right) dt \cdot y = \sigma^2_T \left( \int_{0}^{\tau} r_H \left(\frac{t}{\gamma}\right) dt + \int_{\tau}^{T-\tau} r_H \left(\frac{t}{\gamma}\right) dt \right) \cdot y, \ \tau \in [0, T],$$

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and covariance function
\[ r_H\left(\frac{\tau_1 - \tau_2}{\gamma}\right) - \sigma_t^{-2} \int_0^T \int_0^T \int_0^T \int_0^T r_H\left(\frac{\tau_1 - t_1}{\gamma}\right) r_H\left(\frac{\tau_2 - t_2}{\gamma}\right) \text{d}t_1 \text{d}t_2, \quad \tau_1, \tau_2 \in [0, T]. \] (26)

We have
\[ \sigma_t^2 = \int_0^T \int_0^T r_H\left(\frac{\tau_1 - \tau_2}{\gamma}\right) \text{d}\tau_1 \text{d}\tau_2 = 2 \int_0^T \int_0^T \int_0^T \int_0^T r_H\left(\frac{t - t}{\gamma}\right) \text{d}t \text{d}t \leq 2 T \int_0^\infty \int_0^\infty r_H\left(\frac{t}{\gamma}\right) \text{d}t, \]

using the non-negativity of \( r_H \), and, by (16) and the series representation of the hypergeometric function,
\[ \int_0^\infty \int_0^\infty r_H\left(\frac{t}{\gamma}\right) \text{d}t = \frac{4H \Gamma(\frac{3}{2} + H)}{(1 + 2H) \Gamma(\frac{1}{2} + H)} \sum_{k=0}^\infty \frac{\Gamma\left(\frac{1}{2} - H + k\right)}{\Gamma\left(\frac{1}{2} + H + k\right)} \int_0^\infty e^{-\tau(1 + \frac{1}{2})} \text{d}\tau \]
\[ \sim 2 \gamma H \sum_{k=0}^\infty \left(k + \frac{1}{2}\right)^{-1} = c \cdot \gamma H, \quad H \to 0. \] (27)

Thus, there exists \( H_0 > 0 \) s.t. \( c/2 \cdot \gamma H \leq \int_0^\infty \int_0^\infty r_H\left(\frac{t}{\gamma}\right) \text{d}t \leq 2 c \cdot \gamma H \) for all \( H \in (0, H_0] \). Now, we fix \( H \in (0, H_0] \). Then \( \sigma_t^2 \leq 4cT \gamma H \) and
\[ \mu(\tau) \geq \sigma_t^{-2} \int_0^{T/2} r_H\left(\frac{t}{\gamma}\right) \text{d}t \geq \frac{1}{2} \sigma_t^{-2} \int_0^\infty r_H\left(\frac{t}{\gamma}\right) \text{d}t \geq \frac{c}{4} \sigma_t^{-2} \gamma H \]
for \( \tau \in [0, T] \) and \( T \) large enough, as \( \int_0^\infty \int_0^\infty r_H\left(\frac{t}{\gamma}\right) \text{d}t < \infty \).

This gives, on the set \( \{ Y_H^T < -\delta \sigma_T \sqrt{T} \} \) and for \( T \) large enough,
\[ -\mu(\tau) Y_H^T \geq \frac{\delta c}{4} \sigma_T^{-1} \sqrt{T} \gamma H \geq \frac{\delta \sqrt{c}}{8} \sqrt{\gamma H} =: \delta c' \sqrt{\gamma H}. \]

Thus, again on the set \( \{ Y_H^T < -\delta \sigma_T \sqrt{T} \} \),
\[ \mathbb{P}\left( \sup_{\tau \in [0, T]} V_{\tau\gamma}^H < 0 \mid Y_T^H \right) \geq \mathbb{P}\left( \sup_{\tau \in [0, T]} \left| V_{\tau\gamma}^H - \mu(\tau) Y_T^H \right| < \delta c' \sqrt{\gamma H} \mid Y_T^H \right) \]
\[ \geq \prod_{k=1}^{[T]} \mathbb{P}\left( \sup_{\tau \in [k-1, k]} \left| V_{\tau\gamma}^H - \mu(\tau) Y_T^H \right| < \delta c' \sqrt{\gamma H} \mid Y_T^H \right), \] (28)
using the Gaussian correlation inequality, \([24]\), in the second step. Considering \((20)\), we have, for \(\tau_1, \tau_2 \in [0, T]\),

\[
\mathbb{V}[V_{\tau_1/\gamma}^H - V_{\tau_2/\gamma}^H \mid Y_T^H] = \mathbb{V}[V_{\tau_1/\gamma}^H \mid Y_T^H] + \mathbb{V}[V_{\tau_2/\gamma}^H \mid Y_T^H] - 2 \text{cov} \left( V_{\tau_1/\gamma}^H, V_{\tau_2/\gamma}^H \mid Y_T^H \right)
\]

\[
= 2 - 2 r_H \left( \frac{\tau_1 - \tau_2}{\gamma} \right) - \sigma_T^{-2} \left( \left( \int_0^T r_H \left( \frac{\tau_1 - t}{\gamma} \right) \, dt \right)^2 
\right.
\]

\[
+ \left. \left( \int_0^T r_H \left( \frac{\tau_2 - t}{\gamma} \right) \, dt \right)^2 - 2 \int_0^T \int_0^T r_H \left( \frac{\tau_1 - t_1}{\gamma} \right) r_H \left( \frac{\tau_2 - t_2}{\gamma} \right) \, dt_1 \, dt_2 \right)
\]

\[
= \mathbb{V}[V_{\tau_1/\gamma}^H - V_{\tau_2/\gamma}^H] - \sigma_T^2 \left( \int_0^T r_H \left( \frac{\tau_1 - t}{\gamma} \right) \, dt - \int_0^T r_H \left( \frac{\tau_2 - t}{\gamma} \right) \, dt \right)^2
\]

\[
\leq \mathbb{V}[V_{\tau_1/\gamma}^H - V_{\tau_2/\gamma}^H].
\]

Therefore, by the Sudakov-Fernique inequality, \([2\text{, Theorem } 2.9]\), and the fact that \((V_{\tau/\gamma}^H)\) is a.s. continuous and therefore a.s. bounded on the intervals \([k-1, k]\), one estimates

\[
\mathbb{E} \left[ \sup_{\tau \in [k-1, k]} \left( V_{\tau/\gamma}^H - \mu(\tau) Y_T^H \right) \mid Y_T^H \right] \leq \mathbb{E} \sup_{\tau \in [k-1, k]} V_{\tau/\gamma}^H = \mathbb{E} \sup_{\tau \in [0, 1]} V_{\tau/\gamma}^H
\]

\[
= \mathbb{E} \sup_{\tau \in [0, 1/\gamma]} V_{\tau}^H \leq \mathbb{E} \sup_{\tau \in [0, 1]} V_{\tau}^H =: \alpha(H),
\]

using the stationarity in the second step, for all \(k \in \{1, \ldots, \lfloor T \rfloor\}\), and \(\gamma \geq 1\) by making \(H_0\) smaller if necessary, as \(\gamma \to \infty\) with \(H \to 0\).

Assume for a moment that \(\alpha(H) \ll \sqrt{\gamma H}\). Then, by Borell’s inequality, \([2\text{, Theorem } 2.1]\), and again a.s. boundedness, this implies

\[
\mathbb{P} \left( \sup_{\tau \in [k-1, k]} \left( V_{\tau/\gamma}^H - \mu(\tau) Y_T^H \right) > \delta c' \sqrt{\gamma H} \mid Y_T^H \right)
\]

\[
\leq 2 \mathbb{P} \left( \sup_{\tau \in [k-1, k]} \left( V_{\tau/\gamma}^H - \mu(\tau) Y_T^H \right) > \delta c' \sqrt{\gamma H} \mid Y_T^H \right)
\]

\[
\leq 2 e^{-\frac{1}{2}(\delta c' \sqrt{\gamma H - E[\sup_{\tau \in [k-1, k]} (V_{\tau/\gamma}^H - \mu(\tau) Y_T^H)]Y_T^H])^2} \leq 2 e^{-\frac{1}{2}(\delta c' \sqrt{\gamma H - \alpha(H)})^2}, \quad (29)
\]

where we used \(\sup_{\tau \in [k-1, k]} \mathbb{V}[V_{\tau/\gamma}^H \mid Y_T^H] \leq \sup_{\tau \in [k-1, k]} \mathbb{V}[V_{\tau/\gamma}^H] = 1\) due to \(r_H\) being non-negative and where we possibly have to make \(H_0\) even smaller s.t. \(\alpha(H) < \delta c' \sqrt{\gamma H}\).

Now, let us verify \(\alpha(H) \ll \sqrt{\gamma H}\). As \(\sup_{\tau \in [0, 1]} \mathbb{V}[V_{\tau}^H] = 1\), Dudley’s Theorem,
\[ \alpha(H) \leq 4\sqrt{2} \int_0^{1/2} \sqrt{\log N([0,1], d_{V^H}, \varepsilon)} \, d\varepsilon, \]  

(30)

where \( N([0,1], d_{V^H}, \varepsilon) \) is the minimal number of points in an \( \varepsilon \)-net for \([0,1]\) w.r.t. the intrinsic pseudometric \( d_{V^H}(\tau_1, \tau_2) := \sqrt{\mathbb{E}[(V^H_{\tau_1} - V^H_{\tau_2})^2]}, \ \tau_1, \tau_2 \in [0,1], \) of \( V^H \). By (17), the fact that \( 1 - e^{-x} \leq x \) and the steps in the derivation of (21), we have

\[ d_{V^H}(\tau_1, \tau_2)^2 = 2(1 - r_H(\tau)) = 2 \left( 1 - e^{-\frac{\log |\tau_1 - \tau_2|}{2\gamma H}} + e^{-\frac{\log |\tau_1 - \tau_2|}{2\gamma H}} - r_H(\tau) \right) \]

\[ \leq |\tau_1 - \tau_2| + 2(2\ell_0 |\tau_1 - \tau_2| + 2|\tau_1 - \tau_2|^{2H}) \leq 5\ell_0 |\tau_1 - \tau_2|^{2H} \]

for some \( \ell_0 \in \mathbb{N} \), all \( \tau_1, \tau_2 \in [0,1] \) and \( H \) small enough, where we used the fact that \( x \leq x^{2H} \) for \( x \in [0,1] \) and \( 2H < 1 \) in the last inequality. Thus, we have \( d_{V^H}(\tau_1, \tau_2) \leq c|\tau_1 - \tau_2|H \), giving the estimate

\[ N([0,1], d_{V^H}, \varepsilon) \leq \frac{|[0,1]|}{c^{-1/2H} \varepsilon^{1/2H}} = c^{1/H} \varepsilon^{-1/H}, \]

as \( |\tau_1 - \tau_2| \leq c^{-1/H} \varepsilon^{1/H} \) then implies \( d_{V^H}(\tau_1, \tau_2) \leq \varepsilon \). Together with (30), this leads to

\[ \alpha(H) \leq 4\sqrt{2} \int_0^{1/2} \sqrt{H^{-1} (\log c - \log \varepsilon)} \, d\varepsilon \]

\[ = \sqrt{H^{-1}} \cdot 4\sqrt{2} \int_2^{\infty} x^{-2} \sqrt{\log c + \log x} \, dx \ll \sqrt{\gamma H} \]

(31)

for \( H \to 0 \), as \( \gamma \gg H^{-2} \). Combining this with (28) and (25), we get

\[ \mathbb{P}\left(V^H_{\tau,\gamma} < 0 \ \forall \tau \in [0,T]\right) \geq \left( 1 - 2e^{-\frac{\log(\sqrt{\gamma H} - \alpha(H))}{2}} \right)^{|T|} \mathbb{P}\left(\frac{Y^H}{\sigma_T} < -\delta T\right) \]

for all \( T \) large enough. Taking the logarithm, dividing by \( T \) and letting \( T \to \infty \) gives

\[ -\frac{\theta_R(H)}{\gamma} \geq \log\left( 1 - 2e^{-\frac{\log(\sqrt{\gamma H} - \alpha(H))}{2}} \right) - \frac{\delta^2}{2}, \]

where we used that \( Y^H_T/\sigma_T \) is \( \mathcal{N}(0,1) \)-distributed. By (31), letting first \( H \to 0 \) and then \( \delta \to 0 \) gives \( \lim_{H \to 0} \theta_R(H)/\gamma = 0 \).

The case \( \gamma \to \infty, \gamma \ll H^{-1} \). As \( \lim_{H \to 0} \gamma^{-2H} \geq \lim_{H \to 0} H^{2H} = 1 \) and \( \lim_{H \to 0} \gamma^{-2H} \leq \lim_{H \to 0} 1^{-2H} = 1 \), Lemma 9 yields \( r_H(\tau/\gamma) \to 0 \) for
$H \to 0$ and all $\tau > 0$. To conclude the assertion $\theta_R(H)/\gamma \to \infty$, we want to apply Lemma 3(b) and thus have to check (4) for the correlation function $\tau \mapsto r_H(\tau/\gamma)$. Indeed, one has, for every $\ell, L \in \mathbb{N},$

$$
\limsup_{H \to 0} \sum_{\tau=L}^{\infty} r_H \left( \frac{\tau}{\ell \gamma} \right) \leq \limsup_{H \to 0} \int_{L-1}^{\infty} r_H \left( \frac{\tau}{\ell \gamma} \right) d\tau \leq \limsup_{H \to 0} \int_{0}^{\infty} r_H \left( \frac{\tau}{\ell \gamma} \right) d\tau
$$

where we used monotonicity and non-negativity of $r_H$ as well as (27).

Finally, similarly to the proof of Theorem 1, the continuity of $\theta_R$ follows by the continuity of $H \mapsto r_H(\tau)$ and Lemma 3(a), as the sequence $\tau \mapsto r_H(\tau)$, $H \in [H_0 - \delta, H_0 + \delta]$, with fixed $H_0 \in (0, \infty)$, small $\delta > 0$ and $H \to H_0$ fulfills the conditions (4)–(6). One checks easily (1) and (6), while for checking (5), note that

$$
1 - r_H(\varepsilon) = 1 - e^{-\frac{\varepsilon}{2}} + e^{-\frac{\varepsilon}{2}} - r_H(\tau) \leq \frac{\varepsilon}{2} + c_{H_0} \left( \varepsilon + \varepsilon^{2(H_0 - \delta)} \right)
$$

for suitable $c_{H_0}$ and small $\varepsilon$ by doing the steps as in the derivation of (23). \hfill $\Box$

Acknowledgement. This work was supported by Deutsche Forschungsgemeinschaft (DFG grant AU370/5).

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