Remarks on the Donaldson metric

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Abstract

The Donaldson metric is a metric on the space of symplectic two-forms in a fixed cohomology class. It was introduced in [2]. We compute the associated Levi-Civita connection, describe its geodesics and compute the formula for the covariant Hessian of an energy functional on the space of symplectic structures in a fixed cohomology class, introduced by S. Donaldson in [1].

Let $M$ be a closed oriented Riemannian four-manifold. Denote by $g$ the Riemannian metric on $M$, denote by $d\text{vol} \in \Omega^4(M)$ the volume form of $g$, and let $*: \Omega^k(M) \to \Omega^{4-k}(M)$ be the Hodge $*$-operator associated to the metric and orientation. Fix a cohomology class $a \in H^2(M; \mathbb{R})$ such that $a^2 > 0$ and consider the space

$$\mathcal{S}_a := \{ \rho \in \Omega^2(M) \mid d\rho = 0, \rho \wedge \rho > 0, [\rho] = a \}$$

of symplectic forms on $M$ representing the class $a$. This is an infinite-dimensional manifold and the tangent space of $\mathcal{S}_a$ at any element $\rho \in \mathcal{S}_a$ is the space of exact 2-forms on $M$. The next proposition is proved in [2]. It summarizes the properties of a family of Riemannian metrics $g^\rho$ on $M$, one for each nondegenerate 2-form $\rho$ (and for each fixed background metric $g$). For each nondegenerate 2-form define the function $u$ by the equation

$$2ud\text{vol} = \rho \wedge \rho$$

(1)

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Proposition 1. (Symplectic Forms and Riemannian Metrics).
Fix a nondegenerate 2-form \( \rho \in \Omega^2(M) \) such that \( \rho \wedge \rho > 0 \) and define the function \( u : M \to (0, \infty) \) by \( u = \frac{\rho \wedge \star(\rho \wedge \lambda)}{u} \).

(i) The volume form of \( g^\rho \) agrees with the volume form of \( g \).

(ii) The Hodge \( \star \)-operator \( \star^\rho : \Omega^1(M) \to \Omega^3(M) \) associated to \( g^\rho \) is given by
\[
\star^\rho \lambda = \rho \wedge \star(\rho \wedge \lambda)
\]
for \( \lambda \in \Omega^1(M) \) and by \( \star^\rho \iota(X)\rho = -\rho \wedge g(X, \cdot) \) for \( X \in \text{Vect}(M) \).

(iii) The Hodge \( \star \)-operator \( \star^\rho : \Omega^2(M) \to \Omega^2(M) \) associated to \( g^\rho \) is given by
\[
\star^\rho \omega = R^\rho \star R^\rho \omega, \quad R^\rho \omega := \omega - \frac{\omega \wedge \rho}{d\text{vol}_\rho} \rho,
\]
for \( \omega \in \Omega^2(M) \). The linear map \( R^\rho : \Omega^2(M) \to \Omega^2(M) \) is an involution that preserves the exterior product, acts as the identity on the orthogonal complement of \( \rho \) with respect to the exterior product, and sends \( \rho \) to \( -\rho \).

(iv) Let \( \omega \in \Omega^2(M) \) be a nondegenerate 2-form and let \( J : TM \to TM \) be an almost complex structure such that \( g = \omega(\cdot, J\cdot) \). Define the almost complex structure \( J^\rho \) by \( \rho(J^\rho \cdot, \cdot) := \rho(\cdot, J\cdot) \) and define the 2-form \( \omega^\rho \in \Omega^2(M) \) by \( \omega^\rho := R^\rho \omega \). Then \( g^\rho = \omega^\rho(\cdot, J^\rho \cdot) \) and so \( \omega^\rho \) is self-dual with respect to \( g^\rho \).

The following is the central object of this paper.

Definition 2. Each nondegenerate 2-form \( \rho \in \Omega^2(M) \) with \( \rho^2 > 0 \) determines an inner product \( \langle \cdot, \cdot \rangle_\rho \) on the space of exact 2-forms defined by
\[
\langle \hat{\rho}_1, \hat{\rho}_2 \rangle_\rho := \int_M \hat{\lambda}_1 \wedge \star^\rho \hat{\lambda}_2, \quad d\hat{\lambda}_i = \hat{\rho}_i, \quad \star^\rho \hat{\lambda}_i \in \text{im}d.
\]
These inner products determine a Riemannian metric on the infinite-dimensional manifold \( \mathcal{S}_a \) called the Donaldson metric.

Definition 3. A vector field \( X_{\hat{\rho}} \) is associated to an exact 2-form \( \hat{\rho} \in T_{\hat{\rho}} \mathcal{S}_a \) if it is the unique vector field satisfying
\[
-d\iota(X_{\hat{\rho}})\rho = \hat{\rho}, \quad \star^\rho \iota(X_{\hat{\rho}})\rho \in \text{im}d.
\]
Every metric has an associated Levi-Civita connection. This is the unique torsion free and Riemannian connection with respect to the metric. The formula and computation is the content of the following theorem.

**Theorem 4 (Levi Civita Connection).** Let $\rho_t : \mathbb{R} \to \mathcal{S}_a$ be a smooth path of symplectic forms with $\rho := \rho_0$ and $\hat{\rho} := \partial_t|_{t=0} \rho_t$. Let $X$ be the associated vector field of $\hat{\rho}$. Let $Y_t : \mathbb{R} \to \text{Vect}(M)$ be a smooth path of vector fields such that $\ast^\rho(Y_t) \rho_t$ is exact and define

$$\hat{\sigma}_t := -d_t(Y_t) \rho_t, \quad \hat{\tau} := \hat{\sigma}_0, \quad Y := Y_0.$$ 

The unique Levi-Civita connection associated to the Donaldson metric is given by

$$\nabla^g_{\hat{\rho}} \hat{\sigma} = \frac{d}{dt} \bigg|_{t=0} \hat{\sigma}_t + \frac{1}{2} d_t(Y) \hat{\rho} + \frac{1}{2} d_t(X) \hat{\sigma} - \frac{1}{2} d_t (\nabla_X Y + \nabla_Y X) \rho. \quad (6)$$

Here $\nabla_X Y$ denotes the covariant derivative of the Levi-Civita connection of the metric $g$ for two vector fields $X$ and $Y$.

**Proof.** The Levi-Civita connection of the Donaldson metric is the unique connection that is torsion free and Riemannian with respect to the Donaldson metric. Since the Christoffel symbol given by (6) is symmetric in $\hat{\rho}$ and $\hat{\sigma}$, the torsion of the connection $\nabla^g$ vanishes. It remains to show that it is Riemannian. Let $Z_t$ be a smooth path of vector fields such that $\ast^\rho(Y_t) \rho_t$ is exact. Denote $\hat{\tau}_t := -d_t(Z_t) \rho_t, \hat{\tau} := \hat{\tau}_0$. We claim that

$$\left. \frac{d}{dt} \right|_{t=0} \langle \hat{\sigma}_t, \hat{\tau}_t \rangle_{\rho_t} = \langle \nabla^g_{\hat{\rho}} \hat{\sigma}, \hat{\tau} \rangle_{\rho} + \langle \hat{\sigma}, \nabla^g_{\hat{\rho}} \hat{\tau} \rangle_{\rho}.$$ 

By the definition of the Donaldson metric and since $\ast^\rho(Y) \rho = -\rho \wedge \iota(X)g$ by Proposition 3.1 ii) in [2],

$$\langle \hat{\sigma}_t, \hat{\tau}_t \rangle_{\rho_t} = \int_M (\iota(Y_t) \rho_t) \wedge \ast^\rho(Y_t) \rho_t = -\int_M (\iota(Y_t) \rho_t) \wedge \rho_t \wedge \iota(Z_t)g$$

and

$$\left. \frac{d}{dt} \right|_{t=0} \langle \hat{\sigma}_t, \hat{\tau} \rangle_{\rho} = -\int_M (\iota(\hat{Y}) \rho) \wedge \rho \wedge \iota(Z)g - \int_M (\iota(Y) \hat{\rho}) \wedge \rho \wedge \iota(Z)g$$

and

$$\left. \frac{d}{dt} \right|_{t=0} \langle \hat{\sigma}, \hat{\tau}_t \rangle_{\rho} = -\int_M (\iota(\hat{Z}) \rho) \wedge \rho \wedge \iota(Y)g - \int_M (\iota(Z) \hat{\rho}) \wedge \rho \wedge \iota(Y)g.$$
where \( \vec{Y} := \frac{d}{dt}igg|_{t=0} Y_t, \vec{Z} := \frac{d}{dt}igg|_{t=0} Z_t \). Hence,

\[
\frac{d}{dt} \bigg|_{t=0} \langle \bar{\sigma}_t, \bar{\tau}_t \rangle_{\rho} = \langle \frac{d}{dt} \bigg|_{t=0} \bar{\sigma}_t, \bar{\tau}_t \rangle_{\rho} - \int_M (\iota(Y)\rho) \wedge \bar{\rho} \wedge \iota(Z)g - \int_M (\iota(Y)\rho) \wedge \rho \wedge \iota(\vec{Z})g
\]

\[
= \langle \frac{d}{dt} \bigg|_{t=0} \bar{\sigma}_t, \bar{\tau}_t \rangle_{\rho} - \int_M (\iota(Y)\rho) \wedge \bar{\rho} \wedge \iota(Z)g - \int_M (\iota(\vec{Z})\rho) \wedge \rho \wedge \iota(Y)g
\]

\[
= \langle \frac{d}{dt} \bigg|_{t=0} \bar{\sigma}_t, \bar{\tau}_t \rangle_{\rho} + \langle \bar{\sigma}, \frac{d}{dt} \bigg|_{t=0} \bar{\tau}_t \rangle_{\rho} - \int_M (\iota(Y)\rho) \wedge \bar{\rho} \wedge \iota(Z)g + \int_M (\iota(Z)\rho) \wedge \rho \wedge \iota(Y)g.
\]

Define the Christoffel symbols \( \tilde{\Gamma}_{\tilde{\alpha}\tilde{\beta}} \) by

\[
2\tilde{\Gamma}_{\tilde{\alpha}\tilde{\beta}} := (\iota(Y)\rho) \wedge (d_t(X)\rho) \wedge \iota(Z)g - (\iota(Z)d_t(X)\rho) \wedge \rho \wedge \iota(Y)g
\]

\[
+ (\iota(Y)\rho) \wedge (d_t(Z)\rho) \wedge \iota(X)g - (\iota(X)d_t(Z)\rho) \wedge \rho \wedge \iota(Y)g
\]

\[
- (\iota(Z)\rho) \wedge (d_t(Y)\rho) \wedge \iota(X)g + (\iota(X)d_t(Y)\rho) \wedge \rho \wedge \iota(Z)g.
\]

Then

\[
\tilde{\Gamma}_{\tilde{\alpha}\tilde{\beta}} + \tilde{\Gamma}_{\tilde{\beta}\tilde{\alpha}} = (\iota(Y)\rho) \wedge d_t(X)\rho \wedge \iota(Z)g - (\iota(Z)d_t(X)\rho) \wedge \rho \wedge \iota(Y)g
\]

\[
= (-\iota(Y)\rho) \wedge \bar{\rho} \wedge \iota(Z)g + (\iota(Z)\rho) \wedge \rho \wedge \iota(Y)g.
\]

Hence, it remains to show that

\[
\int_M \Gamma_{\tilde{\alpha}\tilde{\beta}} = \langle \bar{\sigma}, \frac{1}{2}d_t(Z)\rho + \frac{1}{2}d_t(X)\bar{\rho} - \frac{1}{2}d_t(\nabla Z X + \nabla_X Z)\rho \rangle_{\rho}.
\]

Let

\[
A := (-\iota(Z)d_t(X)\rho) \wedge \rho \wedge \iota(Y)g - (\iota(X)d_t(Z)\rho) \wedge \rho \wedge \iota(Y)g
\]

\[
B := 2\tilde{\Gamma}_{\tilde{\alpha}\tilde{\beta}} - A
\]

\[
= (\iota(Y)\rho) \wedge (d_t(X)\rho) \wedge \iota(Z)g + (\iota(Y)\rho) \wedge (d_t(Z)\rho) \wedge \iota(X)g
\]

\[
- (\iota(Z)\rho) \wedge (d_t(Y)\rho) \wedge \iota(X)g + (\iota(X)d_t(Y)\rho) \wedge \rho \wedge \iota(Z)g.
\]

Since \(*\rho \iota(X)\rho = -\rho \wedge \iota(X)g\) for any vector field \(X\),

\[
\int_M A = -\int_M \iota(Z)d_t(X)\rho \wedge \rho \wedge \iota(Y)g - \int_M \iota(X)d_t(Z)\rho \wedge \rho \wedge \iota(Y)g
\]

\[
= \langle \bar{\sigma}, d_t(Z)\rho + d_t(X)\bar{\rho} \rangle_{\rho}.
\]
Since
\[ Zg(X, Y) + Xg(Y, Z) - Yg(X, Z) = g(\nabla_Z X + \nabla_X Z, Y) + g([Y, Z], X) + g([Y, X], Z) \]
(see our sign convention for the Lie bracket), we have
\[ \langle \tilde{\sigma}, -d\iota(\nabla_Z X + \nabla_X Z)\rangle_{\rho} = \int_M g(Y, \nabla_Z X + \nabla_X Z) d\text{vol}_{\rho} \]
\[ = \int_M (Zg(X, Y) + Xg(Y, Z) - Yg(X, Z)) - g([Y, Z], X) - g([Y, X], Z)) d\text{vol}_{\rho} \]
Here we used that \( \langle -d\iota(X)\rho, -d\iota(Y)\rho \rangle_{\rho} = \int_M g(X, Y) d\text{vol}_{\rho} \) for any vector field \( X \) and a vector field \( Y \) such that \( \ast^\rho d\iota(Y)\rho \) is exact. Then
\[ - \int_M g([Y, Z], X) + g([Y, X], Z) d\text{vol}_{\rho} \]
\[ = -\langle d\iota([Y, Z])\rho, d\iota(X)\rho \rangle_{\rho} - \langle d\iota([Y, X])\rho, d\iota(Z)\rho \rangle_{\rho} \]
\[ = -\langle \mathcal{L}_{[Y, Z]}\rho, d\iota(X)\rho \rangle_{\rho} - \langle \mathcal{L}_{[Y, X]}\rho, d\iota(Z)\rho \rangle_{\rho} \]
\[ = \langle [\mathcal{L}_Y, \mathcal{L}_Z]\rho, d\iota(X)\rho \rangle_{\rho} + \langle [\mathcal{L}_Y, \mathcal{L}_X]\rho, d\iota(Z)\rho \rangle_{\rho} \]
\[ = - \int_M (\iota(Y) d\iota(Z)\rho) \wedge \rho \wedge \iota(X) g + \int_M (\iota(Z) d\iota(Y)\rho) \wedge \rho \wedge \iota(X) g \]
\[ - \int_M (\iota(Y) d\iota(X)\rho) \wedge \rho \wedge \iota(Z) g + \int_M (\iota(X) d\iota(Y)\rho) \wedge \rho \wedge \iota(Z) g. \]
Here we used the identity \( \mathcal{L}_{[X, Y]} = -[\mathcal{L}_X, \mathcal{L}_Y] \) for all vector fields \( X, Y \) in the third equality. Using the Leibniz rule for the interior product for the first three terms yields,
\[ - \int_M g([Y, Z], X) + g([Y, X], Z) d\text{vol}_{\rho} \]
\[ = \int_M (d\iota(Z)\rho) \wedge (\iota(Y)\rho) \wedge \iota(X) g + \int_M g(X, Y)(d\iota(Z)\rho) \wedge \rho \]
\[ - \int_M (d\iota(Z)\rho) \wedge (\iota(Z)\rho) \wedge \iota(X) g - \int_M g(X, Z)(d\iota(Y)\rho) \wedge \rho \]
\[ + \int_M (d\iota(X)\rho) \wedge (\iota(Y)\rho) \wedge \iota(Z) g + \int_M g(Y, Z)(d\iota(X)\rho) \wedge \rho \]
\[ + \int_M (\iota(X) d\iota(Y)\rho) \wedge \rho \wedge \iota(Z) g. \]
Since
\[ \int_M Xg(Y, Z)d\text{vol}_\rho = \int_M (\iota(X)d\text{g}(Y, Z))d\text{vol}_\rho \]
\[ = \int_M (d\text{g}(Y, Z))\iota(X)\rho \wedge \rho \]
\[ = -\int_M g(Y, Z)\iota(X)\rho \wedge \rho \]
for all vector fields $X, Y, Z$ we find
\[ \langle \hat{\sigma}, -d\iota(\nabla_X Y + \nabla_X Z)\rho \rangle \rho \]
\[ = \int_M (d\iota(Z)\rho) \wedge (\iota(Y)\rho) \wedge \iota(X)g - \int_M (d\iota(Y)\rho) \wedge (\iota(Z)\rho) \wedge \iota(X)g \]
\[ + \int_M (d\iota(X)\rho) \wedge (\iota(Y)\rho) \wedge \iota(Z)g + \int_M (\iota(X)d\iota(Y)\rho) \wedge \rho \wedge \iota(Z)g \]
\[ = \int_M B. \]
Hence
\[ \langle \hat{\sigma}, \frac{1}{2}d\iota(\nabla_X Y + \nabla_X Z)\rho \rangle \rho \]
\[ = \frac{1}{2} \int_M A + B = \Gamma_{\hat{\sigma}\hat{\sigma}\hat{\rho}}. \]
This proves the claim and the theorem. \qed

The following is an immediate corollary.

Corollary 5 (Geodesic Equation). The geodesic equation on the space $\mathcal{S}_a$ with respect to the Donaldson metric is
\[ \frac{d^2}{dt^2}\rho_t = d\iota(X_t)d\iota(X_t)\rho_t + d\iota(\nabla_{X_t}X_t)\rho_t, \] (7)
where $X_t$ is the associated vector field of $\partial_t\rho_t$.

The next lemma gives an alternative formula for the covariant derivative.

Lemma 6. Let $\rho_t : \mathbb{R} \to \mathcal{S}_a$ be a smooth path of symplectic forms with $\rho := \rho_0$ and $\hat{\rho} := \partial_t|_{t=0}\rho_t$. Let $X$ be the associated vector field of $\hat{\rho}$. Let
$Y_t : \mathbb{R} \to \text{Vect}(M)$ be a smooth path of vector fields such that $*^\rho_t(Y_t)\rho_t$ is exact and define

$$\hat{\tau}_t := -dt(Y_t)\rho_t, \quad \hat{\sigma} := \hat{\sigma}_0, \quad Y := Y_0.$$ 

Then

$$\nabla^\rho_{\hat{\rho}} \hat{\sigma} = -dt(\hat{Y} + \nabla_X Y)\rho, \quad \hat{Y} := \frac{d}{dt}\bigg|_{t=0} Y_t.$$ 

**Proof.** We have

$$\nabla^\rho_{\hat{\rho}} \hat{\sigma} = -\frac{d}{dt}\bigg|_{t=0} dt(Y_t)\rho_t + \frac{1}{2}\frac{d}{dt}(Y)\hat{\rho} + \frac{1}{2}\frac{d}{dt}(X)\hat{\sigma} - \frac{1}{2}\frac{d}{dt}(\nabla_Y X + \nabla_X Y)\rho$$

$$= -dt(\hat{Y})\rho + \frac{1}{2}dt(Y)\hat{\rho} + \frac{1}{2}dt(X)\hat{\sigma} - \frac{1}{2}dt(\nabla_Y X + \nabla_X Y)\rho.$$

Using the identity $\mathcal{L}_{[X,Y]} = -[\mathcal{L}_X, \mathcal{L}_Y]$ and Cartan’s formula for the Lie derivative we compute

$$-2dt(Y)\hat{\rho} + dt(Y)\hat{\rho} + dt(X)\hat{\sigma} - dt(\nabla_Y X + \nabla_X Y)\rho$$

$$= -dt(Y)\hat{\rho} + dt(X)\hat{\sigma} - dt(\nabla_Y X + \nabla_X Y)\rho$$

$$= \mathcal{L}_Y \mathcal{L}_X \rho - \mathcal{L}_X \mathcal{L}_Y \rho - dt(\nabla_Y X + \nabla_X Y)\rho$$

$$= -\mathcal{L}_{[Y,X]} \rho - dt(\nabla_Y X + \nabla_X Y)\rho$$

$$= -2dt(\nabla_X Y)\rho.$$

Hence,

$$\nabla^\rho_{\hat{\rho}} \hat{\sigma} = -dt(\hat{Y} + \nabla_X Y)\rho.$$ 

This proves the lemma.

S. Donaldson introduced the following energy functional on the space of symplectic structures in a fixed cohomology class in [1],

$$F : \mathcal{S}_a \to \mathbb{R}, \quad F(\rho) := \int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2} \text{dvol}.$$ 

The functional and the corresponding negative gradient flow with respect to the Donaldson metric are further studied in [2] and [3]. It is shown in [2] that the gradient of $F$ with respect to the Donaldson metric is the operator

$$\text{grad} F : \mathcal{S}_a \to T_p \mathcal{S}_a$$

$$\rho \mapsto -d *^\rho d\Theta^\rho,$$
where

\[ \Theta^\rho := \star \rho u - \frac{1}{2} \left| \frac{\rho}{u} \right|^2 \rho. \]

We compute its associated vector field.

**Lemma 7.** The associated vector field \( X_{\text{grad}^\rho} \) of \( \text{grad}^\rho(\rho) \) is given by the two equivalent equations

\[ \star^\rho d\Theta^\rho = \iota_{X_{\text{grad}^\rho}} \rho \iff d\Theta^\rho = \rho \wedge \iota_{X_{\text{grad}^\rho}} g. \tag{8} \]

In the hyperKähler case,

\[ X_{\text{grad}^\rho} = -\sum_{i=1}^3 J_i X_{K_i}, \]

where \( K_i := \frac{\omega_i \wedge \rho}{\text{dvol}_\rho} \) and \( X_{K_i} \) is the Hamiltonian vector field of \( K_i \) with respect to the symplectic structure \( \rho \).

**Proof.** It is immediate that a vector field \( X_{\text{grad}^\rho} \) defined by the first equation of (8) satisfies the two conditions (2) for \( \hat{\rho} = \text{grad}^\rho(\rho) = -d \star^\rho d\Theta^\rho \). That the second equation is equivalent to the first follows from the identity \( \star^\rho \iota_X \rho = -\rho \wedge \iota_X g \) proved in [2]. In the hyperKähler case it is shown in [2] that \( d\Theta^\rho = \star^\rho \sum_i \rho(J_i X_{K_i}, \cdot) \). Hence it follows from the first equation in (8) that \( \rho(X_{\text{grad}^\rho}, \cdot) = -\rho(\sum_i J_i X_{K_i}, \cdot) \). This proves the lemma.

The Hessian operator of the energy functional \( E \) is the operator \( \mathcal{H} : T_\rho \mathcal{J}_a \rightarrow T_\rho \mathcal{J}_a \) defined by

\[ \mathcal{H}_\rho \hat{\rho} := \nabla^\rho_{\text{grad}^\rho}(\rho). \]

Associated to this operator is the Hessian quadratic form \( \mathcal{H}_\rho : T_\rho \mathcal{J}_a \rightarrow \mathbb{R} \) given by

\[ \mathcal{H}_\rho(\hat{\rho}) := \langle \mathcal{H}_\rho \hat{\rho}, \hat{\rho} \rangle. \]

Since \( \nabla^\rho \) is the Levi-Civita connection of the Donaldson metric, the Hessian quadratic form equals \( \frac{d^2}{dt^2} \bigg|_{t=0} E(\rho_t) \) for a curve \( \mathbb{R} \rightarrow \mathcal{J}_a : t \rightarrow \rho_t \) satisfying \( \rho_0 = \rho, \left. \frac{d}{dt} \right|_{t=0} = \hat{\rho} \) and \( \left. \frac{d^2}{dt^2} \right|_{t=0} \rho_t = 0. \)
Theorem 8 (Covariant Hessian). Let \( \rho \in \mathcal{S}_a \). Then the following holds.

(i) The Hessian operator of the energy functional \( E: \mathcal{S}_a \to \mathbb{R} \) is the linear operator

\[
\mathcal{H}_\rho \hat{\rho} = -d \star^\rho d \hat{\Theta} + d \star^\rho (\hat{\rho} \wedge \iota (X_{\text{grad}E}) g) - d \iota (\nabla_X X_{\text{grad}E}) \rho, 
\]

where \( \hat{\Theta} := \hat{\rho} + \rho \wedge \iota (X_{\text{grad}E}) g \)

\[
\hat{\Theta} := \frac{\hat{\rho} + \rho \wedge \iota (X_{\text{grad}E}) g}{u} - \left| \frac{\rho}{u} \right|^2 \hat{\rho}, \nabla \text{ denotes the Levi-Civita connection of the metric } g \text{ and } X, X_{\text{grad}E} \text{ are the associated vector fields to } \hat{\rho} \text{ respectively } \text{grad}E(\rho).
\]

(ii) The Hessian of \( E \) is the quadratic form

\[
\mathcal{H}_\rho (\hat{\rho}) := \int_M \hat{\rho} \wedge \hat{\rho} + \int_M (\iota (X) \hat{\rho} - \iota (\nabla_X X) \rho) \wedge \star^\rho (X_{\text{grad}E}) \rho.
\]

(iii) In the hyperKähler case the Hessian of \( E \) is given by

\[
\mathcal{H}_\rho (\hat{\rho}) = \int_M \sum_i \left( \hat{H}_i d\text{vol}_\rho + \omega_i (X, \nabla_X K, X) \right) d\text{vol}_\rho,
\]

where \( \hat{H}_i := \frac{(d\iota (X_i ) \omega_i ) \wedge \rho}{d\text{vol}_\rho} \) and \( K_i := \frac{\omega_i \wedge \rho}{d\text{vol}_\rho} \).

Proof. We prove (i). Let \( X \) and \( X_{\text{grad}E} \) be the associated vector fields of \( \hat{\rho} \) and \( \text{grad}E \). Let \( \rho_t: \mathbb{R} \to \mathcal{S}_a \) be a path of symplectic forms such that

\[
\frac{d}{dt} \bigg|_{t=0} \rho_t = \hat{\rho}.
\]

By Lemma 6

\[
\nabla_{\hat{\rho} \text{grad}E}(\rho) = -\iota (\hat{X}_{\text{grad}E} + \nabla_X X_{\text{grad}E}) \rho
\]

where \( \hat{X}_{\text{grad}E} = \frac{d}{dt} \bigg|_{t=0} X_{\text{grad}E} \) and \( \Theta := \frac{d\Theta}{d\text{vol}_\rho} \). By Lemma 7 we have \( d\Theta^\rho = \rho \wedge \iota (X_{\text{grad}E}) g \) and hence

\[
\frac{d}{dt} \bigg|_{t=0} \Theta^{\rho_t} = \rho \wedge \iota (X_{\text{grad}E}) g + \rho \wedge \iota (\hat{X}_{\text{grad}E}) g,
\]

where \( \hat{\Theta} := \frac{d}{dt} \bigg|_{t=0} \hat{\Theta}^{\rho_t} \). It follows that

\[
\iota (\hat{X}_{\text{grad}E}) \rho = \star^\rho \left( \rho \wedge \iota (\hat{X}_{\text{grad}E}) g \right) = \star^\rho d \hat{\Theta} - \star^\rho (\hat{\rho} \wedge \iota (X_{\text{grad}E}) g).
\]

Hence,

\[
\nabla_{\hat{\rho} \text{grad}E}(\rho) = -d \star^\rho d \hat{\Theta} + d \star^\rho (\hat{\rho} \wedge \iota (X_{\text{grad}E}) g) - d \iota (\nabla_X X_{\text{grad}E}) \rho.
\]

That \( \hat{\Theta} = \frac{\hat{\rho} + \rho \wedge \iota (X_{\text{grad}E}) g}{u} - \left| \frac{\rho}{u} \right|^2 \hat{\rho} \) is proved in [2]. This proves (i).
We prove (ii). By part (i)

\[ \mathcal{H}_\rho(\rho) = \langle H_\rho(\rho), \rho \rangle_\rho \]

\[ = \langle -d* d \left( \frac{\rho + *\rho}{u} - |\rho|^2 \rho \right), \rho \rangle_\rho \]

\[ + \langle d* (\rho \wedge \iota(X_{\text{grad}}) g), \rho \rangle_\rho + \langle -d\iota(\nabla X X_{\text{grad}}) \rho, \rho \rangle_\rho \]

\[ =: A + B + C. \]

By the definition of the Donaldson metric

\[ A = \int_M \left( * d \left( \frac{\rho + *\rho}{u} - |\rho|^2 \rho \right) \right) \wedge * \iota(X) \rho \]

\[ = \int_M \left( \frac{\rho + *\rho}{u} - |\rho|^2 \rho \right) \wedge (-d\iota(X) \rho) \]

\[ = \int_M \left( \frac{\rho + *\rho}{u} - |\rho|^2 \rho \right) \wedge \rho. \]

Likewise,

\[ B = -\int_M \rho \wedge (\iota(X_{\text{grad}}) g) \wedge \iota(X) \rho \]

\[ = -\int_M (\iota(X) \rho) \wedge (\iota(X_{\text{grad}}) g) \wedge \rho - \int_M g(X_{\text{grad}}, X) \rho \wedge \rho \]

\[ = \int_M (\iota(X) \rho) \wedge * (\iota(X_{\text{grad}}) \rho) - \int_M g(X_{\text{grad}}, X) \rho \wedge \rho. \]

Since \( \langle -d\iota(X) \rho, -d\iota(Y) \rho \rangle_\rho = \int_M (\iota(X) \rho) \wedge * \iota(Y) \rho = \int_M g(X, Y) d\text{vol}_\rho \) for \( X \) associated to \( \rho \) and \( Y \) an arbitrary vector field we have

\[ C = \int_M g(\nabla X X_{\text{grad}}), X) d\text{vol}_\rho \]

\[ = \int_M (\iota(X)dg(X_{\text{grad}}, X) - g(X_{\text{grad}}, \nabla X X)) d\text{vol}_\rho \]

\[ = \int_M dg(X_{\text{grad}}, X) \wedge (\iota(X) \rho) \wedge \rho - \int_M g(X_{\text{grad}}, \nabla X X) d\text{vol}_\rho \]

\[ = \int_M g(X_{\text{grad}}, X) \rho \wedge \rho - \int_M \iota(\nabla X X) \rho \wedge * \iota(X_{\text{grad}}) \rho. \]
Hence,
\[ A + B + C = \int_M \left( \hat{\rho} + \frac{\rho^+}{u} - \left| \frac{\rho^+}{u} \right|^2 \right) \wedge \hat{\rho} \]
\[ + \int_M (\iota(X)\hat{\rho} - \iota(\nabla_X X)\rho) \wedge \iota(\nabla_{X_{\text{grad}}} \rho). \]

This proves (iii).

We prove (iii). Assume the hyperKähler case. The following identities are proved in [2],
\[ \text{grad} \varepsilon(\rho) = d \sum_i dK_i \circ J_i^\rho \]
\[ \hat{\Theta} = \sum_i \hat{K}_i^2 \omega_i^\rho - \frac{1}{2} \sum_i \hat{K}_i^2 \hat{\rho} \]
\[ \int M \hat{\Theta} \wedge \hat{\rho} = \int M \sum_i \left( \hat{K}_i^2 \text{dvol}_\rho - \frac{1}{2} \hat{K}_i^2 \hat{\rho} \wedge \hat{\rho} \right), \]
where \( \hat{K}_i = \frac{\omega_i \wedge \hat{\rho}}{\text{dvol}_\rho} \), \( \rho(J_i^\rho \cdot \cdot) := \rho(\cdot, J_i \cdot) \) and \( \omega_i^\rho = \omega_i - K_i \rho \). From (i) we have
\[ \mathcal{H}_\rho(\hat{\rho}) = \int M \hat{\Theta} \wedge \hat{\rho} - \int M \hat{\rho} \wedge \iota(\text{grad} \varepsilon) g \wedge \iota(X) \rho \]
\[ + \int M g (\nabla_X X_{\text{grad}}, X) \text{dvol}_\rho \]
\[ =: \int M \hat{\Theta} \wedge \hat{\rho} + D + E. \]

By Lemma 7 we have \( X_{\text{grad} \varepsilon} = - \sum_i J_i X_{K_i} \). Therefore
\[ D = - \int M \hat{\rho} \wedge \iota(\text{grad} \varepsilon) g \wedge \iota(X) \rho = \int M \sum_i \iota(X_{K_i}) \omega_i \wedge \iota(X) \rho \wedge \hat{\rho} \]
and
\[ E = \int M g (\nabla_X X_{\text{grad} \varepsilon}, X) \text{dvol}_\rho = \int M \sum_i \omega_i (X, \nabla_X X_{K_i}) \text{dvol}_\rho. \]
It now follows from Lemma 4.3 in [2] that
\[
\mathcal{H}_\rho(\hat{\rho}) = \int_M \sum_i \left( \hat{K}_i^2 \mathrm{dvol}_\rho - \frac{1}{2} K_i^2 \hat{\rho} \right) + D + E
\]
\[
= \int_M \sum_i \left( \hat{H}_i^2 \mathrm{dvol}_\rho + \omega_i \left( X, \nabla_{X_i} X \right) \right) \mathrm{dvol}_\rho.
\]
This proves (iii) and the theorem. \(\Box\)

**Remark 9.** The Hessian operator \(\mathcal{H} : T_\rho \mathcal{X} \to T_\rho \mathcal{X}\) given by (9) is a non-local differential operator of degree two. It is non-local because of the last term \(d\mathbf{v} (\nabla_X X_{\text{grad} \rho})\), which involves solving the equation

\[-d\mathbf{v}(X) \rho = \hat{\rho}, \quad \ast\rho(X) \rho \in \text{im} d\]

for the associated vector field \(X\) of \(\hat{\rho}\). Its leading term

\[-d \ast \rho d \hat{\Theta} = -d \ast \rho d \left( \frac{\hat{\rho} + \ast \rho \hat{\rho}}{u} - \left| \frac{\rho}{u} \right|^2 \hat{\rho} \right)\]

\[= -2d \ast \rho \frac{d \hat{\rho}}{u} + d \ast \rho \left( \frac{du}{u^2} \wedge (\hat{\rho} + \ast \rho \hat{\rho}) \right) + d \ast \rho \left( d \left| \frac{\rho}{u} \right|^2 \wedge \hat{\rho} \right)\]

\[= \left( d \ast \rho \frac{1}{u} d + d \ast \rho \frac{1}{u} d \ast \rho \right) \hat{\rho} + d \ast \rho \left( \frac{du}{u^2} \wedge (\hat{\rho} + \ast \rho \hat{\rho}) \right)\]

\[+ d \ast \rho \left( d \left| \frac{\rho}{u} \right|^2 \wedge \hat{\rho} \right)\]

is an elliptic differential operator.

**References**

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