EVOLUTES OF PLANE CURVES AND NULL CURVES IN
MINKOWSKI 3-SPACE

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Abstract. We use the isotropic projection of Laguerre geometry in order

to establish a correspondence between plane curves and null curves in the

Minkowski 3-space. We describe the geometry of null curves (Cartan frame,

pseudo-arc parameter, pseudo-torsion, pairs of associated curves) in terms of

the curvature of the corresponding plane curves. This leads to an alternative

description of all plane curves which are Laguerre congruent to a given one.

1. Introduction

Two dimensional Laguerre geometry is the geometry of oriented contact be-

tween circles in the Euclidean plane \( E^2 \) (points of \( E^2 \) are viewed as circles with

radius zero). An efficient and intuitive model of Laguerre geometry is the so called

Minkowski model \([3, 8]\). This model establishes a 1-1 correspondence, called the

isotropic projection \([3, 8]\), between points in the Minkowski 3-space \( E^3_1 \) and oriented

circles in the Euclidean plane. The group of Laguerre transformations is identified

with the subgroup of the group of affine transformations of \( E^3_1 \) which is generated

by linear (Lorentzian) isometries, translations and dilations of \( E^3_1 \).

A differentiable one-parameter family of circles in the Euclidean plane corre-

sponds, via isotropic projection, to a differentiable curve in the Minkowski 3-space.

In particular, the family of osculating circles to a given plane curve \( \gamma \) corresponds

to a certain curve \( \varepsilon \) in \( E^3_1 \), which we will call the \( L \)-evolute of \( \gamma \). The \( L \)-evolute of a

plane curve is a null curve and, conversely, any null curve is the \( L \)-evolute of some

plane curve (see Section 5).

The importance of light like (degenerate) submanifolds in relativity theory has

been emphasized by many mathematical and physical researchers (see the mono-

graphs \( [5, 6] \)). Several authors have also investigated the particular case of null

curves in the Minkowski 3-space (see the surveys \( [13, 14] \)). In the present paper we

describe the geometry of null curves (Cartan frame, pseudo-arc parameter, pseudo-
torsion) in terms of the curvature of the corresponding plane curves. This leads

us to the notion of potential function (see Definition 5.8). A potential function

together with an initial condition completely determines the null curve and the

underlying plane curve through the formulae of Theorem 5.10. As a consequence,

we obtain an alternative description of all plane curves which are Laguerre con-
gruent to a given curve \( \gamma \) as follows (see Remark 5.12 for details): starting with

the plane curve \( \gamma \), compute the pseudo-torsion \( \tau \) of its \( L \)-evolute; up to scale, the

pseudo-torsion is invariant under Laguerre transformations of \( \gamma \) and the potential

functions associated to null curves with pseudo-torsion \( \tau \) are precisely the solutions

of a certain second order differential equation; use formulae of Theorem 5.10 to
recover from these solutions the corresponding plane curves; in this way, we obtain all the plane curves which are Laguerre congruent with \( \gamma \).

In Section 6, we describe, in terms of their potential functions, some classes of associated null curves: Bertrand pairs [1, 13], null curves with common binormal lines [12], and binormal-directional curves [4]. We shall also prove the following two interesting results on null curves: 1) a null helix parameterized by pseudo-arc admits a null curve parameterized by pseudo-arc with common binormal lines at corresponding points if, and only if, it has pseudo-torsion \( \tau = 0 \) (see Corollary 6.6); 2) given a null curve \( \varepsilon \) parameterized by pseudo-arc with constant pseudo-torsion \( \bar{\tau} = 0 \) and a 1-1 correspondence between points of the two curves \( \varepsilon \) and \( \bar{\varepsilon} \) such that, at corresponding points, the tangent lines are parallel (see Theorem 6.10).

2. Curves in the Euclidean plane

We start by fixing some notation and by recalling standard facts concerning curves in the Euclidean plane.

Let \( I \) be an open interval of \( \mathbb{R} \) and \( \gamma : I \to \mathbb{E}^2 \) be a regular plane curve, that is, \( \gamma \) is differentiable sufficiently many times and \( \dot{\gamma}(t) \neq 0 \) for all \( t \in I \), where \( \mathbb{E}^2 = (\mathbb{R}^2, \cdot) \) and \( \cdot \) stands for the standard Euclidean inner product on \( \mathbb{R}^2 \). Consider the unit tangent vector \( \mathbf{t} = \dot{\gamma}/|\dot{\gamma}(t)| \) and the unit normal vector \( \mathbf{n} = J(t) \) of \( \gamma \), where \( J \) is the anti-clockwise rotation by \( \pi/2 \). The pair \((\mathbf{t}, \mathbf{n})\) satisfies the Frenet equations \( \mathbf{i} = |\dot{\gamma}| \mathbf{n} \mathbf{u} \) and \( \mathbf{n} = -|\dot{\gamma}| \mathbf{k} \mathbf{t} \), where \( k = \text{det}(\dot{\gamma}, \ddot{\gamma})/|\dot{\gamma}|^3 \) is the curvature of \( \gamma \). We denote by \( l_{\gamma, t_0}(t) \) the arc length at \( t \) of \( \gamma \) with respect to the starting point \( t_0 \in I \).

Assume that the curvature \( k \) is a nonvanishing function on \( I \). Let \( u(t) = 1/k(t) \) be the (signed) radius of curvature at \( t \). The evolute of \( \gamma \) is the curve \( \gamma_e : I \to \mathbb{R}^2 \) defined by \( \gamma_e(t) = \gamma(t) + u(t) \mathbf{n}(t) \), which is a regular curve if, additionally, \( \dot{u} \) is a nonvanishing function on \( I \). In this case, the curvature \( k_e \) of \( \gamma_e \) is given by

\[
 k_e = \frac{\text{det}(\dot{\gamma}_e, \ddot{\gamma}_e)}{|\dot{\gamma}_e|^3} = \frac{|\dot{\gamma}|}{u\dot{u}}. \tag{2.1}
\]

We also have

\[
l_{\gamma, t_0}(t) = \int_{t_0}^t |\dot{\gamma}_e(s)|ds = \pm \int_{t_0}^t \dot{u}(s)ds = \pm(u(t) - u(t_0)),
\]

which means that \( u \) is an arc length parameter of the evolute.

If \( t \) is an arc length parameter of \( \gamma \), the curve \( \gamma_t(t) = \gamma(t) - t\mathbf{t}(t) \) is an involute of \( \gamma \). It is well known that the evolute of the involute \( \gamma_t \) is precisely \( \gamma \) and that, for any other choice of arc length parameter, the corresponding involute of \( \gamma \) is parallel to \( \gamma \).

Finally, recall that, given a smooth function \( k : I \to \mathbb{R} \), there exists a plane curve \( \gamma : I \to \mathbb{R}^2 \) parameterized by arc length whose curvature is \( k \). Moreover, \( \gamma \) is unique up to rigid motion and is given by \( \gamma(t) = (\int \cos \theta(t)dt, \int \sin \theta(t)dt) \) where the turning angle \( \theta(t) \) is given by \( \theta(t) = \int k(t)dt \).

3. Laguerre geometry and the Minkowski 3-space

Consider on \( \mathbb{R}^3 \) the Lorentzian inner product defined by \( \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = u_1 v_1 + u_2 v_2 - u_3 v_3 \), for \( \bar{\mathbf{u}} = (u_1, u_2, u_3) \) and \( \bar{\mathbf{v}} = (v_1, v_2, v_3) \). The Minkowski 3-space is the metric space
$E^3_1 = (\mathbb{R}^3, \cdot)$. If $\vec{u} \in E^3_1$ is a spacelike vector, which means that $\vec{u} \cdot \vec{u} > 0$, we denote $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$. The light cone $C_P$ with vertex at $P \in E^3_1$ is the quadric

$$C_P = \{ X \in E^3_1 | (X - P) \cdot (X - P) = 0 \}.$$

We have a 1-1 correspondence, called the isotropic projection, between points in $E^3_1$ and oriented circles in the Euclidean plane, which is defined as follows. Given $P = (p_1, p_2, p_3)$ in $E^3_1$, consider the light cone $C_P$. The intersection of $C_P$ with the Euclidean plane $E^2 \cong \{(u_1, u_2, u_3) \in E^3_1 | u_3 = 0\}$ is a circle centered at $(p_1, p_2)$ with radius $|p_3|$. The orientation of this circle is anti-clockwise if $p_3 > 0$ and clockwise if $p_3 < 0$. Points in $E^2$ are regarded as circles of zero radius and correspond to points $P = (p_1, p_2, p_3)$ in $E^3_1$ with $p_3 = 0$. Making use of this correspondence, Laguerre transformations are precisely those affine transformations $L : E^3_1 \rightarrow E^3_1$ of the form $L(\vec{u}) = \lambda \vec{u} + \vec{t}$, where $\lambda \in \mathbb{R} \setminus \{0\}$, $\vec{t} \in E^2_1$ and $A \in O_1(3)$ is an orthogonal transformation of $E^3_1$. For details see [3].

4. NULL CURVES IN MINKOWSKI 3-SPACE

Next we recall some standard facts concerning null curves in the Minkowski 3-space. For details, we refer the reader to [13, 14].

A regular curve $\varepsilon : I \rightarrow E^3_1$ with parameter $t$ is called a null curve if $\dot{\varepsilon}$ is lightlike, that is $\dot{\varepsilon} \cdot \dot{\varepsilon} = 0$. Differentiating this, we obtain $\ddot{\varepsilon} \cdot \dot{\varepsilon} = 0$, which means that $\ddot{\varepsilon}$ lies in $\text{Span}\{\dot{\varepsilon}\}^\perp$. We will assume throughout this paper that $\dot{\varepsilon}(t)$ and $\ddot{\varepsilon}(t)$ are linearly independent for all $t \in I$ (in particular, $\varepsilon$ can not be a straight line). Then we have $\text{Span}\{\dot{\varepsilon}\}^\perp = \text{Span}\{\ddot{\varepsilon}, \varepsilon\}$ and $\ddot{\varepsilon}$ is spacelike. If $t = \phi(s)$ is a solution of the differential equation

$$\frac{d\phi}{ds} = \pm \frac{1}{|\varepsilon \circ \phi|},$$

then $\nu = \varepsilon \circ \phi$ satisfies $|\ddot{\varepsilon}| = 1$. In this case, we say that $s$ is a pseudo-arc parameter $[4, 13, 14]$ of $\varepsilon$.

Suppose now that the null curve $\varepsilon : I \rightarrow E^3_1$ is parameterized by pseudo-arc parameter $s$, that is $|\dot{\varepsilon}| = 1$. The tangent vector is $T = \dot{\varepsilon}$ and the (unit) normal vector is $N = \ddot{\varepsilon}$. Define also the binormal vector as the unique lightlike vector $B$ orthogonal to $N$ satisfying $T \cdot B = -1$. The Cartan frame $\{T, B, N\}$ of $\varepsilon$ satisfies the following Frenet equations:

$$\begin{pmatrix} T \\ B \\ N \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \tau \\ \tau & 1 & 0 \end{pmatrix} \begin{pmatrix} T \\ B \\ N \end{pmatrix},$$

where $\tau : I \rightarrow \mathbb{R}$ is called the pseudo-torsion of $\varepsilon$. It follows from (4.2) that

$$\dddot{T} - 2\tau \dot{T} - \dot{\tau} T = 0.$$  

(4.3)

On the other hand, since $\varepsilon$ is a null curve parameterized by pseudo-arc length, we have $T \cdot T = 0$ and $T \cdot \dot{T} = 1$. Hence the components $f_i$ of $T = (f_1, f_2, f_3)$ satisfy

$$\ddot{f}_i - 2\tau \dot{f}_i - \dot{\tau} f_i = 0, \quad f_1^2 + f_2^2 - f_3^2 = 0, \quad \ddot{f}_i^2 + \dot{f}_2^2 - \dot{f}_3^2 = 1.$$  

(4.4)
Conversely, given a function $\tau : I \to \mathbb{R}$, $s_0 \in I$ and a fixed basis $\{T_0, B_0, N_0\}$ of $E^3_1$ satisfying
\[ T_0 \cdot T_0 = B_0 \cdot B_0 = 0, \quad N_0 \cdot N_0 = 1, \quad T_0 \cdot N_0 = B_0 \cdot N_0 = 0, \quad T_0 \cdot B_0 = -1, \]
consider the linear map which is represented, with respect to this basis, by the matrix $A_\tau$. This linear map lies in the Lie algebra $o_1(3)$, which means that we can integrate in order to get a map $F : I \to O_1(3)$, with $F(s_0) = Id$, such that the frame $\{T, B, N\} = \{FT_0, FB_0, FN_0\}$ satisfies (4.2). Hence $\varepsilon(s) = \int T(s) ds$ defines a regular null curve parameterized by pseudo-arc length with pseudo-torsion $\tau$, and $\varepsilon$ is unique up to Lorentz isometry.

The pseudo-arc parameter and pseudo-torsion are preserved under Lorentz isometries. Regarding dilations, we have the following.

**Proposition 4.1.** If $\varepsilon$ is a null curve with pseudo-arc parameter $s$ and pseudo-torsion $\tau$, and $\lambda \neq 0$ is a real number, then $\bar{s} = \sqrt{|\lambda|} s$ is a pseudo-arc parameter of the null curve $\bar{\varepsilon} = \lambda \varepsilon$, which has pseudo-torsion
\[ \bar{\tau}(\bar{s}) = \frac{1}{|\lambda|} \tau(\bar{s}/\sqrt{|\lambda|}). \tag{4.5} \]

**Proof.** By applying twice the chain rule, we see that the tangent vector $T = \frac{d}{ds}$ and the normal vector $N = \frac{d^2 \varepsilon}{ds^2}$ of $\varepsilon$ are related with the tangent vector $T$ and the normal vector $N$ of $\varepsilon$ by
\[ T(\bar{s}) = \frac{\lambda}{\sqrt{|\lambda|}} T(\bar{s}/\sqrt{|\lambda|}), \quad N(\bar{s}) = \text{sign}(\lambda) N(\bar{s}/\sqrt{|\lambda|}). \]

This shows that $\bar{s}$ is a pseudo-arc parameter. On the other hand, since the pseudotorsion $\bar{\tau}$ is the component of $N$ along $T$, formula (4.5) can now be easily verified. \hfill \Box

5. The L-evolute of a plane curve

A differentiable one-parameter family of circles in the Euclidean plane corresponds, via isotropic projection, to a differentiable curve in the Minkowski 3-space $E^3_1$. In particular, the family of osculating circles to a given plane curve $\gamma$ corresponds to a certain curve in $E^3_1$, which we will call the $L$-evolute of $\gamma$. In the present section, we show that, the $L$-evolute of a plane curve is a null curve and that, conversely, any null curve is the $L$-evolute of some plane curve if it has non-vanishing third coordinate. We also describe the geometry of null curves in terms of the curvature of the corresponding plane curves.

**Definition 5.1.** Consider a regular curve $\gamma : I \to E^2$, with curvature $k$ and parameter $t$. Assume that $k$ and its derivative $\dot{k}$ are nonvanishing functions on $I$. Let $\gamma_\varepsilon$ be its evolute and $u = 1/k$ its (signed) radius of curvature. The $L$-evolute of $\gamma$ is the curve $\varepsilon$ in the Minkowski 3-space $E^3_1$ defined by $\varepsilon = (\gamma_\varepsilon, u)$; that is, for each $t$, $\varepsilon(t) \in E^3_1$ corresponds to the osculating circle of $\gamma$ at $t$ under the isotropic projection.

Observe that whereas the evolute of a curve is independent of the parameterization, the $L$-evolute depends on the orientation: as a matter of fact, if $\gamma_R$ is a orientation reversing reparameterization of $\gamma$, then the trace of the $L$-evolute of $\gamma_R$ is that of $(\gamma_\varepsilon, -u)$. 
Proposition 5.2. Given a regular plane curve \( \gamma : I \to \mathbb{E}^2 \) in the conditions of Definition 5.1, its L-evolute \( \varepsilon : I \to \mathbb{E}^3_1 \) is a null curve in \( \mathbb{E}^3_1 \). Conversely, any null curve \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) : I \to \mathbb{E}^3_1 \) is the L-evolute of some plane curve \( \gamma : I \to \mathbb{E}^2 \) if \( \varepsilon_3 \) is nonvanishing on \( I \).

Proof. Taking into account the Frenet equations for the regular plane curve \( \gamma \), we have
\[
\dot{\varepsilon} = (\dot{\gamma}_\varepsilon, \dot{u}) = (\dot{\gamma} + \dot{u} \mathbf{n} + u \mathbf{t}, \dot{u}) = \dot{u}(\mathbf{n}, 1).
\]
Hence \( \dot{\varepsilon} \cdot \dot{\varepsilon} = 0 \), since \((\mathbf{n}, 1)\) is a lightlike vector. Moreover, \( \varepsilon \) is regular, since \( \dot{u} = -\dot{k}/k^2 \neq 0 \) on \( I \). Hence \( \varepsilon \) is a null curve.

Conversely, let \( \varepsilon \) be a null curve in \( \mathbb{E}^3_1 \) with parameter \( t \). We must have \( \dot{\varepsilon}_3(t) \neq 0 \) for all \( t \); as a matter of fact, if \( \dot{\varepsilon}_3(t_0) = 0 \) for some \( t_0 \), then \( \dot{\varepsilon}_1(t_0)^2 + \dot{\varepsilon}_2(t_0)^2 = 0 \), since \( \varepsilon \) is null; hence \( \dot{\varepsilon}(t_0) = 0 \) and \( \varepsilon \) is not regular, which is a contradiction. This means that we can reparameterize \( \varepsilon \) with the parameter \( u = \varepsilon_3(t) \) and assume that \( \varepsilon \) is of the form \( \varepsilon(u) = (\varepsilon_1(u), \varepsilon_2(u), u) \). By hypothesis, \( u \neq 0 \). Consider the plane curve \( \gamma \) defined by \( \gamma(u) = \gamma_\varepsilon(u) - u^{-\frac{4}{3}} \varepsilon_3(u) \), where \( \gamma_\varepsilon = (\varepsilon_1, \varepsilon_2) \). Observe that \( \gamma_\varepsilon \) is a unit speed curve: since \( \dot{\varepsilon}(u) = (\dot{\varepsilon}_1(u), \dot{\varepsilon}_2(u), 1) \) and \( \dot{\varepsilon} \cdot \dot{\varepsilon} = 0 \) = 0, we get \( |\gamma_\varepsilon| = 1 \). Hence \( \gamma \) is an involute of \( \gamma_\varepsilon \); and, consequently, \( \gamma_\varepsilon \) is the evolute of \( \gamma \). In particular, the curvature \( k \) of \( \gamma \) satisfies \( k = \pm 1/u \). If \( k = 1/u \), then \( \varepsilon \) is the L-evolute of \( \gamma \). If \( k = -1/u \), then \( \varepsilon \) is the L-evolute of a orientation reversing reparameterization \( \tilde{\gamma} \) of \( \gamma \).

To make it clear, throughout the rest of this paper, we will assume that

- a) all plane curves are regular, with nonvanishing \( k \) and \( \dot{k} \);
- b) all null curves, and the L-evolutes in particular, are such that \( \dot{\varepsilon} \) and \( \ddot{\varepsilon} \) are everywhere linearly independent – in particular, we are excluding null straight lines in \( \mathbb{E}^3_1 \) and we can always reparameterize by pseudo-arc.

Proposition 5.3. Let \( t \) be an arc length parameter of \( \gamma \) and \( \varepsilon \) the L-evolute of \( \gamma \). Let \( s \) be a pseudo-arc parameter of \( \varepsilon \), where \( t = \phi(s) \) and \( \phi \) is a solution of (4.1). Then
\[
\dot{\phi}(s) = \pm \sqrt{|\frac{u(\phi(s))}{\dot{u}(\phi(s))}|},
\]
where \( u = 1/k \) is the radius of curvature of \( \gamma \).

Proof. By hypothesis, the null curve \( v = \varepsilon \circ \phi \) satisfies \( |\dot{v}| = 1 \). Taking into account the Frenet equations for plane curves and the chain rule, we have
\[
\dot{v}(s) = \dot{\phi}(s)\dot{\varepsilon}(\phi(s)) = \dot{\phi}(s)\dot{u}(\phi(s))(\mathbf{n}(\phi(s)), 1)
\]
and
\[
\dot{v}(s) = \{\dot{\phi}(s)\dot{u}(\phi(s)) + \dot{\phi}(s)^2\ddot{u}(\phi(s))\}(\mathbf{n}(\phi(s)), 1)
\]
\[
- \dot{\phi}(s)^2k(\phi(s))\dot{u}(\phi(s))(\mathbf{t}(\phi(s)), 0)
\]
Since \((\mathbf{n}, 1)\) is a lightlike vector orthogonal to \((\mathbf{t}, 0)\) it follows that
\[
1 = |\dot{v}(s)| = \dot{\phi}(s)^4(k(\phi(s))\dot{u}(\phi(s)))^2,
\]
from which we deduce (5.1).
Definition 5.4. Two plane curves $\gamma$ and $\tilde{\gamma}$, with $L$-evolutes $\varepsilon$ and $\tilde{\varepsilon}$, respectively, are said to be *Laguerre congruent* if the corresponding families of osculating circles are related by a Laguerre transformation, that is, if (up to reparameterization) $\tilde{\varepsilon} = \lambda \varepsilon + \tilde{t}$ for some $\lambda \neq 0$, $A \in O_1(3)$, and $\tilde{t} \in \mathbb{E}^2$.

The identification $\mathbb{E}^2 \cong \{(u_1, u_2, u_3) \in \mathbb{E}_1^3 \mid u_3 = 0\}$ induces a natural embedding of the group $\text{Iso}^+(2) = \mathbb{R}^2 \times SO(2)$ of all rigid motions of $\mathbb{E}^2$ in the group $\mathcal{L}$ of Laguerre transformations. The subgroup of $\mathcal{L}$ generated by $\text{Iso}^+(2)$ together with the translation group of $\mathbb{E}_1^3$ will be denoted by $\mathcal{L}_I$. We point out that if $L \in \mathcal{L}_I$ corresponds to a translation along the timelike axis $\varepsilon_3 = (0, 0, 1)$, that is, $L(\tilde{u}) = \tilde{u} + \alpha \varepsilon_3$ for some real number $\alpha$, then the projections of $\varepsilon$ and $\tilde{\varepsilon} = L(\varepsilon)$ into the Euclidean plane $\mathbb{E}^2$ coincide, which implies that $\gamma$ and $\tilde{\gamma}$ are involutes of the same curve and, consequently, they are parallel: $\tilde{\gamma} = \gamma + \alpha n$ where $n$ is the unit normal vector of $\tilde{\gamma}$.

We also have the following.

Theorem 5.5. Two plane curves $\gamma$ and $\tilde{\gamma}$ are Laguerre congruent if, and only if, the pseudo-torsions $\tau$ and $\tilde{\tau}$ of $\varepsilon$ and $\tilde{\varepsilon}$, respectively, are related by (5.3) for some $\lambda \neq 0$.

Proof. Taking into account that the pseudo-torsion and the pseudo-arc parameter are invariant under Lorentz isometries and that, for dilations, (4.5) holds, the assertion follows from the fact that the pseudo-torsion determines the null curve up to Lorentz isometry, as observed in Section 2. $\square$

5.1. The Tait theorem for osculating circles of a plane curve. The correspondence between null curves and curves in the Euclidean plane allows one to relate an old theorem by P.G. Tait on the osculating circles of a plane curve and the following property for null curves observed by L.K. Graves.

Proposition 5.6 (Graves, [10]). A null curve $\varepsilon$ starting at $P$ lies in the inside of the light cone $C_P$.

If $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is a null curve with $\varepsilon(t_0) = P$, then either $\varepsilon(t)$ lies in the inside the upper part of the light cone $C_P$ for all $t > t_0$ or $\varepsilon(t)$ lies in the inside of the lower part of the light cone $C_P$ for all $t > t_0$. Consequently, in both cases, the circle associated to $P = \varepsilon(t_0)$ under the isotropic projection does not intersect the circle associated to $\varepsilon(t)$ for all $t > 0$ (see Figure 1). This implies the following theorem.

Proposition 5.7 (Tait, [16]). The osculating circles of a curve with monotonic positive curvature are pairwise disjoint and nested.

For some variations on the P.G. Tait result, see [9].

5.2. The potential function. Let $\gamma$ be a regular curve in $\mathbb{E}^2$ with arc length parameter $t$ and (signed) radius of curvature $u$. Observe that the sign of $uu$ is changed if the orientation of $\gamma$ is reversed.

Definition 5.8. Take an arc length parameter $t$ of $\gamma$ such that $uu > 0$. In (4.1), choose

$$\text{sign}(\dot{\phi}) = \text{sign}(\dot{u}) = \text{sign}(u)$$

and let $s$ be the corresponding pseudo-arc parameter of the $L$-evolute of $\gamma$. The potential function of $\gamma$ is the (positive) function

$$f(s) = \dot{\phi}(s)\dot{u}(\phi(s)) = \sqrt{u(\phi(s))u(u(s))}.$$  (5.5)
Proposition 5.9. If $f$ is the potential function of $\gamma$, the pseudo-torsion $\tau$ of $\varepsilon$ is given by

$$\tau = \frac{1}{f} \ddot{f} - \frac{1}{2f^2} (\dot{f}^2 + 1).$$

(5.6)

Proof. In view of (5.2), (5.3) and (5.4), the tangent and normal vectors of $u = \varepsilon \circ \phi$ are given by

$$T(s) = f(s)(n(\phi(s)), 1), \quad N = \dot{f}(s)(n(\phi(s)), 1) - (t(\phi(s)), 0).$$

(5.7)

From this one can check that the binormal vector is given by

$$B(s) = \begin{pmatrix} -\dot{f}(s) t(\phi(s)) + \frac{1}{2f(s)} (\dot{f}(s)^2 - 1) n(\phi(s)) \end{pmatrix}.$$  \(\dot{f}(s) + 1 \end{pmatrix}.

(5.8)

On the other hand, differentiating $N$, we get

$$\dot{N}(s) = \begin{pmatrix} \{\dot{f}(s) - k(\phi(s)) \phi(s)\} n(\phi(s)) - f(s) k(\phi(s)) \phi(s) t(\phi(s)), \dot{f}(s) \end{pmatrix}.$$  \(\dot{f}(s) + 1 \end{pmatrix}.

Observe also that

$$k(\phi(s)) \phi(s) = \frac{\dot{\phi}(s)}{u(\phi(s))} = \frac{\dot{\phi}(s) \dot{u}(\phi(s))}{u(\phi(s)) \dot{u}(\phi(s))} = \frac{1}{f(s)}.$$  \(\dot{f}(s) + 1 \end{pmatrix}.$$

Hence

$$\dot{N}(s) = \begin{pmatrix} \{\dot{f}(s) - \frac{1}{f(s)} \} n(\phi(s)) - \frac{\dot{f}(s)}{f(s)} t(\phi(s)), \dot{f}(s) \end{pmatrix}.$$  \(\dot{f}(s) + 1 \end{pmatrix}.$$

Since $\tau = -\dot{N} \cdot B$, we deduce (5.6). \(\Box\)

Up to $L_I$-congruence, the curve $\gamma$ and its $L$-evolute can be recovered from its potential function $f$ as follows.

Theorem 5.10. Let $f : (\alpha, \beta) \to \mathbb{R}$ be a positive and differentiable function on the open interval $(\alpha, \beta)$. Take $s_0 \in (\alpha, \beta)$ and a constant $b_0$ such that $\int_{s_0}^s f(v) dv + b_0$ is nonvanishing on $(s_0, \beta)$. Set $\theta(s) = \int_{s_0}^s \frac{1}{f(s)} dv$. Then the null curve $\varepsilon : (s_0, \beta) \to E^3$ given by

$$\varepsilon(s) = \int_{s_0}^s \begin{pmatrix} \cos \theta(v), \sin \theta(v), 1 \end{pmatrix} f(v) dv + (0, 0, b_0)$$

(5.9)
has pseudo-arc parameter $s$ and $\varepsilon$ is the $L$-evolute of some regular plane curve $\gamma$ with potential function $f : (s_0, \beta) \to \mathbb{R}$. Up to a rigid motion in $\mathbb{E}^2$, this plane curve is given by

$$
\gamma(t) = \int_{s_0}^{\phi^{-1}(t)} \left( \cos \theta(v), \sin \theta(v) \right) \dot{\phi}(v) dv,
$$

(5.10)

where the arc length parameter $t$ of $\gamma$ satisfies $t = \phi(s)$ for some strictly monotone function $\phi : (s_0, \beta) \to \mathbb{R}$ with derivative

$$
\dot{\phi}(s) = \frac{\int_{s_0}^{s} f(v) dv + b_0}{f(s)}.
$$

(5.11)

Moreover, if $\bar{\gamma}$ is another regular plane curve with potential function $f : (s_0, \beta) \to \mathbb{R}$, then $\bar{\gamma}$ coincides with $\gamma$ up to rigid motion, for some constant $b$. Consequently, any two plane curves with the same potential function are $\mathcal{L}_1$-congruent. Conversely, if $\gamma$ and $\bar{\gamma}$ are $\mathcal{L}_1$-congruent, then they have the same potential function.

**Proof.** Differentiating (5.9) we get

$$
\dot{\varepsilon}(s) = f(s)(\cos \theta(s), \sin \theta(s), 1).
$$

From this we see that $\dot{\varepsilon} \cdot \dot{\varepsilon} = 0$, that is, $\varepsilon$ is a null curve. Differentiating again, we obtain

$$
\ddot{\varepsilon}(s) = (-\sin \theta(s), \cos \theta(s), 0) + \dot{f}(s)(\cos \theta(s), \sin \theta(s), 1).
$$

Hence $\ddot{\varepsilon} \cdot \ddot{\varepsilon} = 1$, which means that $s$ is a pseudo-arc parameter of $\varepsilon$.

By hypothesis, $\varepsilon_3(s) = \int_{s_0}^{s} f(v) dv + b_0$ is nonvanishing on $(s_0, \beta)$. Hence, by Proposition 5.2 $\varepsilon$ is the $L$-evolute of some plane curve. The radius of curvature $u$ of this plane curve at $s$ is precisely $\varepsilon_3(s)$. On the other hand, a simple computation shows that $t$ is an arc length parameter for the curve $\gamma$ defined by (5.10) and that the radius of curvature of $\gamma$ at $s$ is $\varepsilon_3(s)$. Hence, the fundamental theorem of plane curves assures that, up to rigid motion in $\mathbb{E}^2$, the plane curve whose $L$-evolute is $\varepsilon$ coincides with $\gamma$.

Now, take any curve $\bar{\gamma}$ with potential function $f : (s_0, \beta) \to \mathbb{R}$. Let $\bar{t}$ and $\bar{u}$ be an arc length parameter and the radius of curvature, respectively, of $\bar{\gamma}$, so that, by definition of potential function,

$$
f(s) = \bar{\phi}(s)\bar{u}(\bar{\phi}(s)) = \sqrt{\bar{u}(\bar{\phi}(s))\bar{u}(\bar{\phi}(s))},
$$

with $\bar{t} = \bar{\phi}(s)$. According to our choices in the definition of potential function, we have

$$
\frac{d\bar{t}}{ds} = \epsilon \sqrt{\bar{u}/\dot{\bar{u}}},
$$

where $\epsilon := \text{sign}(\dot{\bar{\phi}}) = \text{sign}(\bar{u}) = \text{sign}(\bar{u})$. From the first equation we see that $f = \bar{u}\dot{u}/ds$ and multiplying both it follows that $\frac{d\bar{t}}{ds} = \bar{u}/f$. Hence

$$
\bar{u}(\bar{\phi}(s)) = \int_{s_0}^{s} f(v) dv + \bar{u}(\bar{\phi}(s_0)), \quad \frac{d\bar{t}}{ds} = \frac{\int_{s_0}^{s} f(v) dv + \bar{u}(\bar{\phi}(s_0))}{f(s)}.
$$

(5.12)

Taking $b_0 := \bar{u}(\bar{\phi}(s_0))$ we see from (5.12) that $\gamma$ and $\bar{\gamma}$ have the same curvature function and the same arc length parameter $t = \bar{t}$, which means that $\gamma$ and $\bar{\gamma}$ are related by a rigid motion. In particular, the $L$-evolute $\varepsilon$ of $\bar{\gamma}$ is also given by (5.9) up to rigid motion acting on the first two coordinates. We can see this constructively as follows.
Let \( \vec{e} = (\vec{e}_1, \vec{e}_2, \vec{e}_3) \) be the L-evolute of \( \vec{\gamma} \). By definition of L-evolute, \( \vec{e}_3(s) = \vec{u}(\vec{\phi}(s)) \). On the other hand, we know that the evolute \( \gamma_\varepsilon = (\vec{e}_1, \vec{e}_2) \) of \( \vec{\gamma} \) has arc length \( \bar{u} \) and curvature \( k_\varepsilon = 1/\bar{u} \). Since

\[
\int k_\varepsilon d\bar{u} = \int \frac{1}{\bar{u} \bar{u}} d\bar{s} ds = \int \frac{1}{f} ds,
\]

setting \( \theta(s) = \int_{s_0}^{s} \frac{1}{f} dv \), the curve \( \gamma_\varepsilon \) is given, up to rigid motion, by

\[
\gamma_\varepsilon(\bar{u}(\vec{\phi}(s))) = \int_{s_0}^{s} (\cos \theta(v), \sin \theta(v)) f(v) dv,
\]

by the fundamental theorem of plane curves. Consequently, the L-evolute \( \vec{e} \) is given, up to rigid motion acting on the first two coordinates, by

\[
\vec{e}(s) = \int_{s_0}^{s} (\cos \theta(v), \sin \theta(v), 1) f(v) dv + (0, 0, \bar{u}(\vec{\phi}(s_0))).
\]

Finally, if \( \gamma \) and \( \vec{\gamma} \) are \( L_t \) congruent, then their L-evolutes \( \vec{e} \) and \( \varepsilon \) satisfy \( \vec{e}(s) = A\varepsilon(s) + \vec{b} \), with common pseudo-arc parameter \( s \), where \( A \) is a rigid motion acting on the first two coordinates and \( \vec{b} = (0, 0, b_0) \in \mathbb{E}_1^3 \). Hence, the corresponding curvature radius satisfy \( \bar{u}(\vec{\phi}(s)) = u(\phi(s)) + b \). Consequently, \( f(s) = d\bar{u}/ds = du/ds = f(s) \), and we are done.

\( \square \)

**Remark 5.11.** These results provide a scheme to integrate equations (4.4). Given a function \( \tau(s) \), if \( f(s) \) is a solution to the differential equation (5.10), then the null curve (5.9) has pseudo-torsion \( \tau \) and pseudo-arc parameter \( s \). This means that the components of the tangent vector

\[
\mathbf{T} = \left( \cos \left( \int \frac{1}{f} ds \right) f, \sin \left( \int \frac{1}{f} ds \right) f, f \right)
\]

of \( \varepsilon \) satisfy (4.4). Moreover, all solutions of (5.10) for a given \( \tau(s) \) arise in this way.

**Remark 5.12.** We have also obtained a description of all plane curves which are Laguerre congruent to a given curve \( \gamma \). As a matter of fact, starting with \( \gamma \), compute its potential function and the pseudo-torsion \( \tau \) of its L-evolute making use of (5.5) and (5.6); in view of Theorem 5.5 find the general solution of the equation

\[
\frac{1}{|\lambda|} \tau(s/|\lambda|) = \frac{1}{f(s)} \dot{f}(s) - \frac{1}{2f^2(s)} (\dot{f}^2(s) + 1)
\]

for each \( \lambda \neq 0 \); since this is a second order differential equation, the two initial conditions together with the parameter \( \lambda \) determine a three-parameter family of potential functions; for any such function \( f \), formulas (5.10) and (5.11) define a curve in the plane which is Laguerre equivalent to \( \gamma \). Conversely, any curve which is Laguerre congruent to \( \gamma \) arises in this way, up to rigid motion.

**Example 1.** Equation (5.6) is equivalent to \( 2\tau f^2 = 2\dot{f} \dot{f} + (\dot{f}^2 + 1) \). Differentiating this, we obtain the third order linear ordinary differential equation

\[
\ddot{f} - 2\tau \dot{f} - f = 0.
\]

For \( \tau = -\frac{5}{2x^2} \), the general solution of (5.14) is

\[
f(s) = as + bs \sin(2\ln s) + cs \cos(2\ln s),
\]
and a straightforward computation shows that the solutions of (5.6), with \( \tau = -\frac{5}{2x^2} \), are precisely those functions (5.14) satisfying \( b^2 + c^2 - a^2 = -\frac{1}{2} \). In particular, for \( c = b = 0 \) and \( a = \frac{1}{2} \) we get the solution \( f(s) = \frac{s}{2} \). In view of Theorem 5.10, the arc length \( t \) of the plane curve \( \gamma \) associated to this potential function is given by \( t = \frac{s^2}{4} \), and we have

\[
\gamma(t) = \frac{t}{2} (\sin(4t) + \cos(4t), \sin(4t) - \cos(4t)).
\]  

(5.16)

Up to Euclidean motion, \( \gamma \) is the logarithmic spiral \( \theta \mapsto e^{\theta}(\cos \theta, \sin \theta) \) reparameterized by arc length \( t \). The L-evolute of \( \gamma \) is given by

\[
\varepsilon(t(s)) = \frac{s^2}{8} (\sin(2\ln s) + \cos(2\ln s), \sin(2\ln s) - \cos(2\ln s), 2).
\]

This null curve is an example of a *Cartan slant helix* in \( \mathbb{E}_3^1 \). A Cartan slant helix in \( \mathbb{E}_3^1 \) is a null curve parameterized by pseudo-arc whose normal vector makes a constant angle with a fixed direction. Accordingly to the classification established in [4], Cartan slant helices are precisely those null curves whose pseudo-torsions are of the form \( \pm \frac{1}{(c_s + b_s)} \), where \( c \neq 0 \) and \( b \) are constants.

**Example 2.** Let us consider the *Cornu’s Spiral*

\[
\gamma(t) = \left( \int_0^t \cos\left(\frac{u^2}{2}\right)du, \int_0^t \sin\left(\frac{u^2}{2}\right)dv \right).
\]

This is a plane curve with arc length \( t \) and radius of curvature \( u = 1/t \). From (5.1), (5.5) and (5.6) we can see that the L-evolute of \( \gamma \) has pseudo-arc \( s = 2\sqrt{t} \), for \( t > 0 \), the potential function is \( f(\gamma(s)) = 8/s^3 \) and the pseudo-torsion is \( \tau = 15/2s^2 - s^6/128 \). For this pseudo-torsion \( \tau \), the general solution of (5.14) is

\[
f(s) = \frac{a + b \sin \left(\frac{s^4}{32}\right) + c \cos \left(\frac{s^4}{32}\right)}{s^3},
\]

and the solutions of (5.6) are precisely those functions \( f(s) \) satisfying \( a^2 - b^2 - c^2 = 64 \).

**Example 3.** The potential functions associated to the pseudo-torsion \( \tau = -\frac{3}{8s^2} - \frac{1}{2s} \) are of the form

\[
f(s) = a\sqrt{s} + b\sin(2\sqrt{s})\sqrt{s} + c\cos(2\sqrt{s})\sqrt{s},
\]

with \( a^2 - b^2 - c^2 = 1 \). For \( a = 1 \) and \( b = c = 0 \), formula (5.10), with \( s_0 = 0 \) and \( b_0 = 0 \), gives \( \gamma(t(s)) = \frac{3}{2}(x(t(s)), y(t(s))) \), where

\[
x(t(s)) = \frac{1}{2}\sqrt{s}(2s - 3)\sin(2\sqrt{s}) + \frac{3}{4}(2s - 1)\cos(2\sqrt{s}) + \frac{3}{4}
\]

\[
y(t(s)) = \frac{3}{4}(2s - 1)\sin(2\sqrt{s}) - \frac{1}{2}\sqrt{s}(2s - 3)\cos(2\sqrt{s}).
\]

In the Figure 2, the curve \( \gamma \) is represented on the left, for \( 0 < s < 500 \); on the right, one can see the plane curve \( \tilde{\gamma} \) associated to the potential function \( f_{\tilde{\gamma}}(s) = \sqrt{2s} + \sin(2\sqrt{s})\sqrt{s} \) (which corresponds to the choice \( a = \sqrt{2}, b = 1 \) and \( c = 0 \)), obtained by numerical integration of (5.10), with \( s_0 = 0 \) and \( b_0 = 0 \), also for \( 0 < s < 500 \).
5.3. Pseudo-torsion and Schwarzian derivatives. Very recently, Z. Olszak [15] observed that the pseudo-torsion of a null curve can be described as follows.

**Theorem 5.13.** [15] If \( \varepsilon : I \to E^3_1 \) is a null curve with pseudo-arc parameter \( s \), then

\[
\varepsilon(s) = \varepsilon(s_0) + \frac{1}{2} \int_{s_0}^{s} \frac{1}{g(v)} \left( 2g(v), g(v)^2 - 1, g(v)^2 + 1 \right) dv,
\]

where \( s, s_0 \in I \), for some non-zero function \( g \) with nonvanishing derivative \( \dot{g} \) on \( I \).

The pseudo-torsion \( \tau \) of \( \varepsilon \) is precisely the Schwarzian derivative of \( g \):

\[
\tau = S(g) = \frac{\ddot{g}}{g} - \frac{3}{2} \left( \frac{\ddot{g}}{g} \right)^2.
\]

Observe that (5.17) can be obtained, up to Euclidean isometry in the first two coordinates, from (5.9) by using the Weierstrass substitution

\[
g(s) = \pm \tan(\theta(s)/2)
\]

where \( \theta(s) \) is the turning angle of the corresponding plane curve \( \gamma \) at \( s \).

5.4. Potential function of the evolute.

**Theorem 5.14.** Let \( \gamma \) be a plane curve parameterized by arc length \( t \) and \( \gamma_e \) be its evolute. The potential function \( f \) and the pseudo-arc parameter \( s \) associated to \( \gamma \) are related with the potential function \( f_e \) and the pseudo-arc parameter \( s_e \) associated to the evolute \( \gamma_e \) by

\[
f_e^2(s_e(s)) = 2f^2(s) \left| \frac{df}{ds}(s) \right|
\]

and \( s_e = \beta(s) \) with \( \beta(s) = \int_{s_0}^{s} \sqrt{2\left| \frac{df}{ds}(v) \right|} dv + s_e(s_0) \). Consequently, the L-evolute of \( \gamma_e \) is given by

\[
\varepsilon_{\gamma_e}(\beta(s)) = 2 \int_{s_0}^{s} f(v) \left| \frac{df}{ds}(v) \right| \left( \cos \theta(v), \sin \theta(v), 1 \right) dv,
\]

up to congruence in \( L_1 \), where \( \theta(s) = \int_{s_0}^{s} \frac{1}{f(v)} dv \).
Proof. Recall that the arc length parameter of $\gamma_\varepsilon$ is precisely the radius of curvature $u$ of $\gamma$. By (2.1) and taking into account the definition of potential function, the radius of curvature of $\gamma_\varepsilon$ is given by $u_\varepsilon = u \dot{u} = f^2$. Hence, since $f = \frac{df}{ds}$,

$$f_\varepsilon^2 = \left| u_\varepsilon \frac{du_\varepsilon}{du} \right| = \left| f^2 \frac{du_\varepsilon}{ds} \frac{ds}{du} \right| = 2 f_\varepsilon^2 \left| \frac{df}{ds} \right|.$$

On the other hand, taking into account (5.1) and the previous observations, we also have

$$\frac{ds_\varepsilon}{ds} = \frac{ds}{ds_\varepsilon} \frac{du}{du_\varepsilon} = \sqrt{\left| \frac{1}{u_\varepsilon} \frac{du_\varepsilon}{du} \right|} \left| f = \frac{f}{f} = \sqrt{2 \left| \frac{df}{ds} \right|} \right|,$$

hence $s_\varepsilon = \beta(s)$ with $\beta(s) = \int_0^s \sqrt{2 \left| \frac{df}{ds}(v) \right|} dv + s_\varepsilon(s_0)$.

Formula (5.20) follows straightforwardly by applying (5.9) to $f_\varepsilon$ and by taking the change of parameter $s_\varepsilon = \beta(s)$. \hfill \Box

**Corollary 5.15.** The pseudo-torsion $\tau_\varepsilon$ of $\varepsilon_{\gamma_\varepsilon}$, the L-evolute of $\gamma_\varepsilon$, is given by

$$\tau_\varepsilon(s_\varepsilon = \beta(s)) = \frac{\tau(s) - S(\beta(s))}{2 \left| \frac{df}{ds} \right|},$$

where $S(\beta(s))$ is the Schwarzian derivative of $\beta(s)$.

Proof. The turning angles of $\gamma$ and its evolute $\gamma_\varepsilon$ differ by $\pi/2$, which implies that the corresponding functions $g$ and $g_\varepsilon$ given by (5.13) satisfy

$$g_\varepsilon \circ \beta(s) = \pm \tan(\theta(s)/2 + \pi/4) = \pm \frac{\tan(\theta(s)/2) + 1}{1 - \tan(\theta(s)/2)} = h \circ g(s)$$

for some fractional linear transformation $h$. In particular, $S(g) = S(g_\varepsilon \circ \beta)$. Hence, Theorem 5.13 together with the chain rule for the Schwarzian derivative yield

$$\tau(s) = S(g) = S(g_\varepsilon \circ \beta) = (S(g_\varepsilon) \circ \beta) \beta^2 + S(\beta) = (\tau_\varepsilon \circ \beta) \beta^2 + S(\beta).$$

Since $\beta^2 = 2 \left| \frac{df}{ds} \right|$, we are done. \hfill \Box

**Example 4.** If the pseudo-arc of $\gamma_\varepsilon$ coincides with that of $\gamma$, that is $s_\varepsilon(s) = s$, then we see from (5.19) that $f_\varepsilon = f$ and $\frac{df_\varepsilon}{du} = \pm \frac{1}{4}$. Hence $f(s) = \pm \frac{1}{4} s + c$ for some constant $c$.\footnote{This is due to the fact that $\beta(s)$ is a logarithmic spiral, which is a special case of a logarithmic spiral.} and (5.21) yields $\tau_\varepsilon(s) = \tau(s)$ for $f(s) = \frac{1}{4} s$. The plane curve $\gamma$ is the logarithmic spiral (5.16). More generally, each potential function $f(s) = \pm \frac{1}{4} s + c$ corresponds to a plane curve whose L-evolute is a Cartan slant helix with pseudotorsion $\tau = -\frac{5}{2(5s+2c)^2}$.

**Example 5.** Let $\gamma$ be the plane curve associated to the potential function $f(s) = \sqrt{s}$ (see Example 3). Then its evolute has potential function $f_\varepsilon(\beta(s)) = s^{1/4}$ and $s_\varepsilon = \beta(s)$ satisfies $\hat{\beta}(s) = s^{-1/4}$. Integrating this we obtain $f_\varepsilon(s_\varepsilon) = \left(\frac{3}{4} s_\varepsilon\right)^{1/3}$. The pseudo-torsion of the extended evolute $\varepsilon_{\gamma_\varepsilon}$ is then given by $\tau_\varepsilon(s_\varepsilon) = -\frac{5}{24} \left(3s_\varepsilon/4\right)^{-2} - \frac{1}{2} \left(3s_\varepsilon/4\right)^{-2/3}$ and we have

$$\varepsilon_{\gamma_\varepsilon}(s) = \left(\sqrt{s} \sin(2 \sqrt{s}) + \frac{1}{2} \cos(2 \sqrt{s}), \frac{1}{2} \sin(2 \sqrt{s}) - \sqrt{s} \cos(2 \sqrt{s}), s\right).$$
5.5. **Null helices and the corresponding potential functions.** A null curve $\bar{\epsilon}$ parameterized by the pseudo-arc parameter is called a *null helix* if its pseudo-torsion $\tau$ is constant \[7\,\[13\,\[14\]. Null helices admit the following classification.

**Proposition 5.16.** \[7\] A null helix with pseudo-torsion $\tau$ and parameterized by the pseudo-arc parameter $s$ is congruent to one of the following:

1. if $\tau < 0$, $\bar{\epsilon}_1(s) = \frac{1}{2|\tau|} \left( \cos(\sqrt{2|\tau|} s), \sin(\sqrt{2|\tau|} s), \sqrt{2|\tau|} s \right)$;
2. if $\tau = 0$, $\bar{\epsilon}_2(s) = \left( \frac{1}{\tau}, \frac{1}{\tau}, \frac{\tau}{4}, \frac{\tau}{4} \right)$;
3. if $\tau > 0$, $\bar{\epsilon}_3(s) = \frac{1}{\sqrt{2\tau}} (\sqrt{2\tau} s, \cosh(\sqrt{2\tau} s), \sinh(\sqrt{2\tau} s))$.

Next we describe the corresponding potential functions.

**Theorem 5.17.** The potential functions of plane curves whose $L$-evolutes have constant pseudo-torsion $\tau$ are precisely the following:

1. if $\tau < 0$, then $f_1(s) = a \cos(\sqrt{2|\tau|} s) + b \sin(\sqrt{2|\tau|} s) + c$, with $2|\tau|(a^2 + b^2) + 1 = 2|\tau|c^2$;
2. if $\tau = 0$, then $f_2(s) = a^2 s^2 + bs + c$, with $4ac = 1 + b^2$;
3. if $\tau > 0$, then $f_3(s) = ae^{\sqrt{2\tau}s} + be^{-\sqrt{2\tau}s} + c$, with $2\tau c^2 + 1 = 8\tau ab$.

**Proof.** For $\tau = 0$, the general solution of (5.14) is $f(s) = as^2 + bs + c$; and any such function is a solution of (5.10) with $\tau = 0$ if, and only if, $4ac = 1 + b^2$. The remaining cases are deduced similarly.

The curve $\bar{\epsilon}_1$ corresponds to the potential function $f_1$ with $a = b = 0$ and $c = 1/\sqrt{2|\tau|}$; $\bar{\epsilon}_2$ corresponds to $f_2$ with $a = 3/4$, $b = 0$ and $c = 1/3$; $\bar{\epsilon}_3$ corresponds to $f_3$ with $a = b = 1/(2\sqrt{2\tau})$ and $c = 0$.

**Example 6.** Take the potential function $f(s) = 1/\sqrt{2|\tau|}$, which corresponds to the null helix $\bar{\epsilon}_1$. The corresponding plane curve $\gamma$ is the involute of a circle. Explicitly, by (5.11), we have $\frac{ds}{dt} = s$, where $t$ is the arc length parameter of $\gamma$, hence we have $s = \sqrt{2}t$ for $t > 0$; from (5.10), we conclude that $\gamma(t) = \frac{1}{\sqrt{2|\tau|}} (x(t), y(t))$, with

\[
\begin{align*}
x(t) &= 2\sqrt{|\tau|} t \sin(2\sqrt{|\tau|} t) + \cos(2\sqrt{|\tau|} t) \\
y(t) &= \sin(2\sqrt{|\tau|} t) - 2\sqrt{|\tau|} t \cos(2\sqrt{|\tau|} t).
\end{align*}
\]

6. **Associated curves**

In this section we describe, in terms of their potential function, some classes of associated null curves: Bertrand pairs \[1\,\[13\,\[14\], null curves with common binormal direction \[12\], and binormal-directional curves \[4\].

6.1. **Bertrand pairs.** Motivated by the definition of Bertrand curve in the Euclidean space, null Bertrand curves are defined as follows.

**Definition 6.1.** Let $\bar{\epsilon} : I \to \mathbf{E}^3$ be a null curve parameterized by pseudo-arc $s$. The curve $\bar{\epsilon}$ is a *null Bertrand curve* if there exists a null curve $\bar{\epsilon} : I \to \mathbf{E}^3$ and a one-to-one differentiable correspondence $\beta : I \to I$ such that, for each $s \in I$, the principal normal lines of $\bar{\epsilon}$ and $\bar{\epsilon}$ at $s$ and $\beta(s)$ are equal. In this case, $\bar{\epsilon}$ is called a *null Bertrand mate* of $\bar{\epsilon}$ and $(\bar{\epsilon}, \bar{\epsilon})$ is a *null Bertrand pair.*
Theorem 6.2. Let \( \varepsilon : I \to \mathbb{E}^3_1 \) be a null curve parameterized by pseudo-arc \( s \). The null curve \( \varepsilon \) is a null Bertrand curve if, and only if, it has nonzero constant pseudo-torsion \( \tau \). In this case, if \( \bar{\varepsilon} : \bar{I} \to \mathbb{E}^3_1 \) is a null Bertrand mate of \( \varepsilon \), with one-to-one correspondence \( \beta : I \to \bar{I} \), then \( \bar{s} := \beta(s) \) is a pseudo-arc parameter of \( \bar{\varepsilon} \), the null curve \( \bar{\varepsilon} \) has the same pseudo-torsion \( \tau \), and \( \beta \) satisfies \( \dot{\beta}(s) = \pm 1 \). In particular, if \( \varepsilon \) has nonzero constant pseudo-torsion \( \tau \), then

\[
\bar{\varepsilon}(s) = \varepsilon(s) - \frac{1}{\tau} N(s)
\]

(6.1)

defined on \( I \) is a null Bertrand mate of \( \varepsilon \).

Corollary 6.3. Let \((\varepsilon, \bar{\varepsilon})\) be a null Bertrand pair satisfying (6.1). Let \( f \) and \( \bar{f} \) be the corresponding potential functions. Then

\[
\bar{f}(s) = f(s) - \frac{1}{\tau} \frac{d^2 f}{ds^2}(s).
\]

(6.2)

Proof. Differentiating (6.1), it follows that \( \bar{T}(s) = T(s) - \frac{1}{\tau} \frac{dT}{ds}(s) \). In view of (6.13), and equating the third components, we conclude (6.2).

Corollary 6.4. Let \( \gamma \) and \( \bar{\gamma} \) be two plane curves in \( \mathbb{E}^2 \) and let \( \varepsilon \) and \( \bar{\varepsilon} \) be the corresponding L-evolutes. Assume that \((\varepsilon, \bar{\varepsilon})\) is a null Bertrand pair with constant pseudo-torsion \( \tau \) satisfying (6.1). Then \( \gamma \) and \( \bar{\gamma} \) are congruent in \( L_1 \) if, and only if the potential function \( f \) of \( \gamma \) is \( f(s) = \frac{1}{\sqrt{2|\tau|}} \) (see example 6).

Proof. Since \( \varepsilon \) and \( \bar{\varepsilon} \) have the same pseudo-arc \( s \), \( \gamma \) and \( \bar{\gamma} \) are congruent in \( L_1 \) if, and only if, \( \bar{f} = f \). If \( \tau < 0 \), we have, from (6.2),

\[
\bar{f}(s) = -a \cos(\sqrt{2|\tau|} s) - b \sin(\sqrt{2|\tau|} s) + c,
\]

with \( 2|\tau|(a^2+b^2)+1 = 2|\tau|c^2 \). Hence \( \bar{f} = f \) if, and only if, \( a = b = 0 \) and \( c = \frac{1}{\sqrt{2|\tau|}} \).

Similarly, if \( \tau > 0 \), it is easy to check that we can not have \( \bar{f} = f \). \( \square \)

6.2. Null curves with common binormal lines.

Theorem 6.5. Let \( \varepsilon : I \to \mathbb{E}^3_1 \) be a null curve parameterized by pseudo-arc parameter \( s \), with pseudo-torsion \( \tau \). Then the following assertions are equivalent:

a) there exists a null curve \( \bar{\varepsilon} : \bar{I} \to \mathbb{E}^3_1 \) and a one-to-one correspondence \( \beta : I \to \bar{I} \) such that, for each \( s \in I \), the binormal lines of \( \varepsilon \) and \( \bar{\varepsilon} \) are equal at \( s \) and \( \beta(s) \), and \( \bar{s} := \beta(s) \) is a pseudo-arc parameter of \( \bar{\varepsilon} \);

b) \( \tau \) satisfies

\[
a_0^2 v(s)^4 = \pm (1 + v(s) \tau(s)),
\]

(6.3)

for some nonzero constant \( a_0 \), where \( v(s) \) satisfies

\[
\frac{1}{v(s)} = -\frac{1}{2} \int \tau^2 ds.
\]

(6.4)

Moreover, in this case, the pseudo-torsion of \( \bar{\varepsilon} \) satisfies \( \bar{\tau}(\bar{s} = \beta(s)) = \pm \tau(s) \).

Proof. Assume that \( \bar{s} = \beta(s) \) is a pseudo-arc parameter of \( \bar{\varepsilon} \) and that \( \varepsilon \) and \( \bar{\varepsilon} \) have common binormal lines at corresponding points, that is

\[
\mathbf{B}(\beta(s)) = a(s) \mathbf{B}(s),
\]

and

\[
\bar{\varepsilon}(\beta(s)) = \varepsilon(s) + v(s) \mathbf{B}(s),
\]

(see example 6).
for some function \(v(s) \neq 0\). Differentiating this with respect to \(s\), we get
\[
\dot{\beta}\mathbf{T} = \mathbf{T} + \dot{v}\mathbf{B} + v\tau\mathbf{N}. \tag{6.5}
\]
Taking the inner product of both terms of this equation with \(\mathbf{B}\), we obtain
\[
\dot{\beta} = a. \tag{6.6}
\]
Since \(\mathbf{T}\) is a lightlike vector, that is \(\mathbf{T} \cdot \mathbf{T} = 0\), it also follows from (6.5) that
\[
-2\dot{v} + v^2\tau^2 = 0, \quad \tag{6.7}
\]
which means that (6.4) holds.

Differentiating (6.5) with respect to \(s\), and taking (6.6) into account, we have
\[
a^3\mathbf{N} = (a + av\tau + 2av\tau - \dot{a}_v)\mathbf{N} + (\dot{a}v + av\tau - \dot{a}_v)\mathbf{B} + (av\tau^2 - \dot{a})\mathbf{T}. \tag{6.8}
\]
Since \(\mathbf{N} \cdot \mathbf{B} = 0\), the component of \(\mathbf{N}\) along \(\mathbf{T}\) must vanish, that is
\[
\dot{a} = av\tau^2. \tag{6.9}
\]
Hence, taking (6.7) and (6.9) into account, we can rewrite (6.8) as
\[
a^3\mathbf{N} = a(1 + v\tau)\mathbf{N} + av\tau(1 + \tilde{\tau}v)\mathbf{B}. \tag{6.10}
\]
Now, since \(\mathbf{N} \cdot \mathbf{N} = 1\), from this we get
\[
a^4 = (1 + v\tau)^2. \tag{6.10}
\]
Hence
\[
\mathbf{N} = \pm(\mathbf{N} + v\tau\mathbf{B}).
\]

By differentiation of \(B(\beta(s)) = a(s)B(s)\) with respect to \(s\), it follows that
\[
a\dot{\mathbf{B}} = a\mathbf{B} + a\tau\mathbf{N} = a\tau(\mathbf{N} + v\tau\mathbf{B}) = \pm a\tau\mathbf{N}.
\]
Since the pseudo-torsion \(\tilde{\tau}\) is defined by \(\mathbf{B} = \tau\mathbf{N}\), we conclude that \(\tilde{\tau}(\beta(s)) = \pm\tau(s)\).

Observe also that (6.7) and (6.9) imply that \(\dot{a}/a = 2\dot{v}/v\), and consequently \(a = a_0v^2\) for some constant \(a_0\). Inserting this in (6.10), we obtain (6.9).

Conversely, given a null curve \(\varepsilon\) with pseudo-arc parameter \(s\) and pseudo-torsion \(\tau\), take a function \(v(s)\) satisfying (6.3) and (6.4) for some nonconstant \(a_0\). Consequently, \(v(s)\) also satisfies (6.7). For \(a = a_0v^2\), it is clear that the equalities (6.9) and (6.10) hold. Define \(\zeta(s) = \varepsilon(s) + v(s)B(s)\). Differentiating twice with respect to \(s\), we get
\[
\dot{\zeta}(s) = \mathbf{T} + \dot{v}\mathbf{B} + v\tau\mathbf{N}, \quad \ddot{\zeta}(s) = (1 + 2\dot{v}\tau + v\tau)\mathbf{N} + (\ddot{v} + v\tau)\mathbf{B} + v\tau^2\mathbf{T}.
\]
From this, straightforward computations show that \(\zeta\) is a null curve and that \(\vert\dot{\zeta}\vert^2 = a^4\). This means, taking (4.11) into account, that the parameter defined by \(\bar{s} = \beta(s)\), with \(\beta\) satisfying \(\dot{\beta} = a\), is a pseudo-arc parameter of \(\zeta\). Consider the corresponding reparameterization \(\bar{\varepsilon} = \zeta \circ \beta^{-1}\). We can check as above that the tangent vector \(\mathbf{T}\) and the principal normal vector \(\mathbf{N}\) of \(\bar{\varepsilon}\) satisfy
\[
\mathbf{T} = \frac{1}{a}(\mathbf{T} + \dot{v}\mathbf{B} + v\tau\mathbf{N}), \quad \mathbf{N} = \pm(\mathbf{N} + v\tau\mathbf{B}). \tag{6.11}
\]
Since, by definition, \(\mathbf{B}\) is the unique null vector such that \(\mathbf{B} \cdot \mathbf{T} = -1\) and \(\mathbf{B} \cdot \mathbf{N} = 0\), we conclude that \(\mathbf{B}(\beta(s)) = a(s)\mathbf{B}(s)\). \(\square\)
Given a curve $\gamma : I \to \mathbb{E}^3$ in the Euclidean space $\mathbb{E}^3$, parameterized by arc length $t$, if there exists another curve $\bar{\gamma} : \bar{I} \to \mathbb{E}^3$ and a one-to-one correspondence $\beta : I \to \bar{I}$ such that, for each $t \in I$, the binormal lines of $\gamma$ and $\bar{\gamma}$ are equal at $t$ and $\bar{t} := \beta(t)$ is an arc length parameter of $\bar{\gamma}$, then both curves are plane curves, that is, their torsions vanish identically (see [11], page 161). For null helices in the Minkowski space we have a similar result.

**Corollary 6.6.** A null helix $\varepsilon$, parameterized by pseudo-arc $s$, with constant pseudo-torsion $\tau$, admits a null curve $\bar{\varepsilon}$, parameterized by pseudo-arc $\bar{s} = \beta(s)$, with common binormal lines at corresponding points (that is, $\bar{B}(\beta(s)) = a(s)B(s)$) if, and only if, $\tau = 0$.

**Proof.** If $\tau$ is constant, then, any function $v(s)$ defined by (6.4) must be of the form

$$v(s) = \frac{1}{v_0 - \frac{1}{2} \tau^2 s},$$

for some integration constant $v_0$, and (6.3) implies that

$$\frac{\dot{a}^2}{(v_0 - \frac{1}{2} \tau^2 s)^4} = \pm 1,$$

which holds if, and only if, $\tau = 0$ and $a_0 = \pm v_0^2$. □

Since the potential function $\check{f}$ of $\check{\varepsilon}$ is the third component of $\check{T}$, we see from (5.7), (5.8) and (6.11) that

$$\check{f}(\check{s} = \beta(s)) = \frac{1}{a_0 v^2(s)} \left\{ f(s) + \frac{v^2(s) \tau^2(s)}{4f(s)} (f^2(s) + 1) + v(s) \tau(s) \dot{f}(s) \right\}.$$

**Remark 6.7.** In [12], the authors investigated pairs of null curves possessing common binormal lines. We point out that Theorem 6.5 does not contradict the main result in [12] since in that paper the parameters are not necessarily the pseudo-arc parameters. If one considers that the binormals coincide at corresponding points, $B(\beta(s)) = a(s)B(s)$, and that $s$ and $\check{s} = \beta(s)$ are precisely the pseudo-arc parameters of $\varepsilon$ and $\check{\varepsilon}$, then the additional condition (6.3) is necessary.

### 6.3. W-directional curves.

**Definition 6.8.** Let $\varepsilon$ be a null curve in $\mathbb{E}_1^3$ parameterized by the pseudo-arc parameter $s$ and $W$ a null vector field along $\varepsilon$. A null curve $\check{\varepsilon}$ parameterized by pseudo-arc $\check{s} = \beta(s)$ is a $W$-directional curve of $\varepsilon$ if the tangent vector $\check{T}$ coincides with the vector $W$ at corresponding points: $\check{T}(\beta(s)) = W(s)$.

Let us consider first the binormal-directional curve of $\varepsilon$, that is, take $W = B$, where $B$ is the binormal vector field of $\varepsilon$.

**Theorem 6.9.** [4] Let $\varepsilon$ be a null curve in $\mathbb{E}_1^3$ parameterized by the pseudo-arc parameter $s$ with non-zero pseudo-torsion $\tau$. Let $\check{\varepsilon}$ be its null binormal-directional curve with pseudo-torsion $\check{\tau}$ and pseudo-arc parameter $\check{s} = \beta(s)$. Then, $\frac{d\check{s}}{ds} = \pm \tau(s)$ and

$$\check{\tau}(\beta(s)) = \frac{1}{\tau(s)}.$$
Since $\bar{T}(\beta(s)) = B(s)$, from (5.8) we see that the potential function of $\bar{\varepsilon}$ is given by
\[ \bar{f}(\bar{s} = \beta(s)) = \frac{1}{2f(s)}(\hat{f}^2(s) + 1), \]
with $\frac{d\bar{s}}{ds} = \pm \tau(s)$.

**Example 7.** Consider the potential function $f(s) = s/2$. As shown in Example 1, this is the potential function associated to the $L$-evolute $\varepsilon$ of the logarithmic spiral \((5.16)\), which has pseudo-torsion $\tau(s) = -\frac{5}{2}s^2$. The potential function associated to the binormal-directional curve $\bar{\varepsilon}$ of $\varepsilon$ is then given by $\bar{f}(\beta(s)) = \frac{5}{4}s$. Since $\frac{d\bar{s}}{ds} = \pm \frac{5}{2}s^2$, we can take $\bar{s} = \frac{5}{2}s$. Hence $\bar{f}(\bar{s}) = s/2$, that is $\bar{\varepsilon}$ is the $L$-evolute of a logarithmic spiral congruent to \((5.16)\) in $L^I$.

We finish this paper with the observation that, given a null curve $\varepsilon$ parameterized by pseudo-arc, there exists a null helix $\bar{\varepsilon}$, parameterized by pseudo-arc, with pseudo-torsion $\bar{\tau} = 0$ and a 1-1 correspondence between points of the two curves $\varepsilon$ and $\bar{\varepsilon}$ such that, at corresponding points, the tangent lines are parallel.

**Theorem 6.10.** Let $\varepsilon$ be a null curve in $E^3_1$ parameterized by the pseudo-arc parameter $s$ with nonzero pseudo-torsion $\tau$. Let $\lambda$ be a function whose Schwarzian derivative satisfies $S(\lambda) = -\tau/2$. Define $\bar{s} = \lambda(s)$ and $W(\bar{s} = \lambda(s)) := \hat{\lambda}(s)T(s)$, where $T$ is the tangent vector of $\varepsilon$. Then the $W$-directional curve $\bar{\varepsilon}$ is a null helix with pseudo-parameter $\bar{s}$ and pseudo-torsion $\bar{\tau} = 0$.

**Proof.** Let $\bar{\varepsilon}$ be the $W$-directional curve (unique up to translation), that is $\hat{\bar{\varepsilon}}(\bar{s}) = W(\bar{s})$. Since
\[ \frac{dW}{d\bar{s}}(\lambda(s)) = \frac{\hat{\lambda}(s)}{\lambda(s)}T(s) + \hat{T}(s), \]
and $s$ is a pseudo-arc parameter of $\varepsilon$, meaning that $\hat{T} \cdot \hat{T} = 1$, we see that we also have $W \cdot W = 1$. Hence $\bar{s}$ is a pseudo-arc parameter of $\varepsilon$. Since $T$ is a solution of \((4.3)\), a straightforward computation shows that $W$ satisfies
\[ \frac{d^3W}{d\bar{s}^3} = 0, \]
which means that $\bar{\varepsilon}$ is a null helix with pseudo-torsion $\bar{\tau} = 0$. As a matter of fact, this is a particular case of the reduction procedure of a third order linear differential equation detailed in Cartan’s book [2] (page 48). \(\square\)

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