Lambert’s theorem through an affine lens

Alain Albouy
IMCCE, UMR 8028,
77, avenue Denfert-Rochereau
F-75014 Paris
Alain.Albouy@obspm.fr

Abstract. We give two new proofs of Lambert’s theorem on the elapsed time along a Keplerian arc. The first one, in Hamilton’s style, uses a variational principle and seems to be minimal in the sense that we doubt that a shorter argument may exist. The second one, in Lambert’s style, is constructive and elementary. It starts with the remark that two Keplerian arcs related by Lambert’s theorem correspond with each other through an affine map. We also give some related statements on Keplerian arcs and conic sections. We review an impressive list of published proofs of Lambert’s theorem, which appears as an unachieved quest for a simpler argument.

1. The statement. Lambert’s theorem may be stated as follows. Call a Keplerian arc, or an arc of Keplerian orbit, a solution \( [t_A, t_B] \rightarrow \mathbb{R}^d, t \mapsto q \), of Newton’s differential system \( \ddot{q} = -q/r^3 \). Here \( r = \|q\| \) is the Euclidean norm of the position vector \( q \in \mathbb{R}^d \). The origin of \( \mathbb{R}^d \) is called the fixed center and denoted by \( O \). Call \( A \in \mathbb{R}^d \) the initial position at an initial time \( t_A \in \mathbb{R} \), and \( B \in \mathbb{R}^d \) the final position, at a final time \( t_B > t_A \). The dimension \( d \) is a nonzero natural number. The case \( d = 1 \) is related to special motions of the system in an arbitrary dimension: rectilinear arcs are Keplerian arcs that lie on a straight line passing through \( O \).

We accept by extension \( O = A \) or \( O = B \), meaning a collision in the limit, which is only possible if the orbit is rectilinear. We also accept “continued” rectilinear orbits, where \( q \) bounces at \( O \) with infinite velocity, keeping its energy \( H = \|\dot{q}\|^2/2 - 1/r \) constant. This continuation is classical and natural: one may simply think of a Keplerian orbit which is nearly rectilinear (see figure 2).

Theorem 1 (Lambert). Let \( K \) be the space of Keplerian arcs in a Euclidean vector space. Let \( O \) be the fixed center and the origin of the vector space. To any arc in \( K \) are associated \( A \) and \( B \), the ends of the arc, and \( H \), the energy of a point moving along the arc. Consider a nonempty level set \( \mathcal{L} \subset K \) of the triple \( (\|AB\|, \|OA\| + \|OB\|, H) \). In any connected component of \( \mathcal{L} \),

(i) the elapsed time \( \Delta t = t_B - t_A \) is constant,
(ii) there exists a rectilinear arc.

Remark 1. The classical way of stating Lambert’s theorem is: \( \Delta t \) is a function of \( H, \|AB\| \) and \( \|OA\| + \|OB\| \). However, the “function” is ramified and multivalued for two reasons. Firstly, in the case of an ellipse, \( q \) can go from \( A \) to \( B \) clockwise or counterclockwise, and can make several turns, which gives various
arcs with same $H$ and different $\Delta t$. Secondly, if A, B and O are all distinct and if $H > H_{\text{min}}$, with

\[
H_{\text{min}} = -\frac{2}{\|AB\| + \|OA\| + \|OB\|},
\]

(1)

there are two Keplerian conic sections with energy $H$ passing through A and B. These multi-valuations produce connected components of $\mathcal{L}$ with different $\Delta t$.

**Remark 2.** The second conclusion (ii) is usually absent from the statements classically called Lambert’s theorem. But the classical authors always explain, after crediting to Lambert a statement similar to (i), how to use it to simplify the computation of $\Delta t$, by reducing the general case to the rectilinear case. The conclusion (ii) is used but its proof is often neglected.

**Remark 3.** The theorem is still true if we consider classes of arcs instead of arcs, by identifying arcs which are deduced one from another by a time translation or by a Euclidean isometry. We will assume these identifications when specifying the dimensions of subspaces of arcs.

### 2. A minimal proof of conclusion (i).

The main idea of this proof is extracted from the proof of Lambert’s theorem that Hamilton published in [1]. But our computations are shorter. Denote by $v = \dot{q}$ the velocity and consider the action integral

\[
w = \int_{t_A}^{t_B} \|v\|^2 dt,
\]

(2)

which is stationary on any Keplerian arc, among the paths with arbitrary values $t_A$ and $t_B$, but with same ends A and B and same energy $H$.

**Proposition 1.** On any variation of a Keplerian arc among the Keplerian arcs with same ($\|AB\|, \|OA\| + \|OB\|, H$) the variation $\delta w$ of the action integral vanishes.

**Proof.** As $H$ is fixed, the variation of the integral is expressed through the vectorial variations $\delta A$ and $\delta B$ of the ends and the velocities $v_A$ and $v_B$ at these ends:

\[
\delta w = \langle \delta B, v_B \rangle - \langle \delta A, v_A \rangle.
\]

(3)

By using the rotational invariance we may start with a Keplerian arc in a plane $Oxy$ with ends at equal ordinates $y_A = y_B$, and consider only variations in the same plane, having ends with the same property. As $\|AB\|$ is fixed, the segment AB is only translated, and $\delta A = \delta B$. We get

\[
\delta w = \langle \delta A, v_B - v_A \rangle.
\]

(4)

**Lemma 1.** Let $q \in \mathbb{R}^d$ be the position vector, $v = \dot{q}$ the velocity vector, $\varepsilon = q/\|q\|$ be the radial unit vector. Consider two positions A and B on a same Keplerian orbit. The vectors $v_B - v_A$ and $\varepsilon_A + \varepsilon_B$ are linearly dependent.
Proof. Recall the expression of the eccentricity vector $E$, which is constant in a Keplerian motion:

$$E = \varepsilon + v \mathbin{\langle} C,$$

(5)

where $C = q \wedge v$ is a bivector, $\mathbin{\langle}$ is the contracted product (see remark 4). As we have

$$0 = \varepsilon_A - \varepsilon_B + (v_A - v_B) \mathbin{\langle} C,$$

(6)

the direction of $v_A - v_B$ is orthogonal to the direction of $\varepsilon_A - \varepsilon_B$, i.e., is the direction of $\varepsilon_A + \varepsilon_B$.

Now it is enough to prove that if $\|OA\| + \|OB\|$ is fixed we have

$$\langle \delta A, \varepsilon_A + \varepsilon_B \rangle = 0.$$

(7)

The vector $\delta A$ is indeed the variation of the vector $OA$ and can be seen as minus the variation of $O$ relatively to $A$. This variation $\delta O$ is orthogonal to the gradient $-\varepsilon_A - \varepsilon_B$ of the function $\|OA\| + \|OB\|$, considered as a function of $O$, which proves (7).

Proof of conclusion (i) of the theorem. Proposition 1 states that $w$ is constant on any connected component of a level set $L$. Hamilton’s theory ([1], §2) shows that in a variation of an integral (2) among stationary paths with given $A$ and $B$, $\Delta t = \delta w / \delta H$. So $\Delta t$ has the same property as $w$.

Remark 4. The vectorial notation in (5) is valid in any dimension. It is used in this context in Cushman-Duistermaat [1]. Vectors and covectors are identified through the Euclidean form. We have $v \mathbin{\langle} (q \wedge v) = \langle v, q \rangle v - \langle v, v \rangle q$. One may as well use another notation. One may prefer to focus on the 3-dimensional case and use the cross product, as in Gibbs [1], §61:

$$E = \varepsilon - v \times C, \quad C = q \times v.$$

Actually the 2-dimensional case expressed in a frame $Oxy$ is enough for our purpose, since all the vectors are in the orbital plane. The formulas are:

$$E = (\alpha, \beta), \quad \alpha = \frac{x}{r} - \dot{y} C, \quad \beta = \frac{y}{r} + \dot{x} C, \quad C = x\dot{y} - y\dot{x}.$$

(8)

Remark 5. In the last step of the proof of Proposition 1, the point $O$ moves on an ellipse with foci $A$ and $B$. When $A \neq B$, it may move continuously till it reaches the line $AB$. There, a Keplerian arc with ends $A$ and $B$ should be rectilinear, since a non-rectilinear Keplerian orbit passes through the same point when it crosses again a ray from $O$. This argument would prove conclusion (ii) if we could prove the existence of an arc for all the intermediate positions of $O$. In remark 1, we claimed the existence of orbits under condition (1), but we did not prove anything. The existence proof is not difficult, but it fits better with the constructive proof of Lambert’s theorem that we will present now. The arguments are more elementary and barely use differential calculus. We construct the arcs and reach conclusions (i) and (ii) directly.
3. A short constructive proof. Theorem 1 easily reduces to the bidimensional case \( d = 2 \). We will prove it on \( \mathbb{R}^2 = \text{Oxy} \).

In this proof the case \( \|AB\| = 0 \) requires a separate study. The velocity vectors \( v_A \) and \( v_B \) at times \( t_A \) and \( t_B \) have same norm, which is given by \( H \). If \( v_A = v_B \), the orbit is periodic and the period \( T \) is obtained through Kepler’s third law:

\[
T = 2\pi(-2H)^{-3/2}.
\]

The connected components of \( \mathcal{L} \) are numbered by the number \( n \) of turns. On each component, \( \Delta t = nT \) is constant. So (i) is proved, and we simply push the eccentricity to 1 to get (ii). There is another case, \( v_A = -v_B \neq 0 \), where the orbit is rectilinear and goes back to the initial point after bouncing inward or culminating outward. In the space of Keplerian arcs, this second case defines isolated arcs: \( \Delta t \) is tautologically constant.

We will assume \( A \neq B \) in the rest of the proof. A curve in \( \mathbb{R}^2 \) is an irreducible conic section with a focus at \( O \) if and only if its equation is of the form:

\[
x^2 + y^2 - (\alpha x + \beta y + \gamma)^2 = 0, \quad \text{with} \quad (\alpha, \beta) \in \mathbb{R}^2, \quad \gamma \in [0, +\infty[.
\]

By irreducible we mean that the conic section is not a pair of lines. The eccentricity is \( \sqrt{\alpha^2 + \beta^2} \), the semi-parameter is \( \gamma \). Setting \( r = \sqrt{x^2 + y^2} \geq 0 \), we get this other equation

\[
r = \alpha x + \beta y + \gamma,
\]

which only includes one branch of the conic section. This branch is the whole conic section, except in the case of the hyperbola, where it is the branch whose convex hull contains \( O \).

We introduce an angular parameter \( \phi \) and a number \( f \). We write the identity

\[
(x \cos \phi + f \sin \phi)^2 + (x \sin \phi - f \cos \phi)^2 = x^2 + y^2 \quad \text{in the form}
\]

\[
(x \sin \phi - f \cos \phi)^2 + y^2 = x^2 + y^2 - (x \cos \phi + f \sin \phi)^2.
\]

Lemma 2. For any \( \phi \in [0, \pi[ \), any \( \mu \in \mathbb{R} \) and any \( \nu \in [0, +\infty[ \), the image of the branch \( C \) with equation \( r = \mu y + \nu \) by the affine map \( (x_1, y_1) \mapsto (x_2, y_2) \), where \( x_1 = x_2 \sin \phi - \nu y_2 \mu \cos \phi - \nu \cos \phi \), \( y_1 = y_2 \), has a focus at \( O \).

Proof. Replace \((x_1, y_1)\) by its expression in \((x_2, y_2)\) in the equation \( x_1^2 + y_1^2 - (\mu y_1 + \nu)^2 = 0 \), then set \( f = \mu y_2 + \nu \) and use (11). The equation of the image is \( r = x \cos \phi + y \mu \sin \phi + \nu \sin \phi \). \( \square \)

The principal axis of \( C \) is vertical. If \( \phi = \pi/2 \) the affine map is the identity. If not, the map is not a transvection since \( x_2 \) has coefficient \( \sin \phi \neq 1 \). If we rescale by compounding with \((x_2, y_2) \mapsto (x_3, y_3)\), \( x_2 = x_3 / \sin \phi \), \( y_2 = y_3 / \sin \phi \), it will still not be a transvection, now because of the coefficient \( 1 / \sin \phi \neq 1 \) of \( y_3 \). But it will be the transformation we need.

Lemma 3. For any \( \phi \in [0, \pi[ \), any \( \mu \in \mathbb{R} \) and any \( \nu \in [0, +\infty[ \), the image of the branch \( C \) with equation \( r = \mu y + \nu \) by the affine map \( (x_1, y_1) \mapsto (x_3, y_3) \), where

\[
x_1 = x_3 - y_3 \mu \cot \phi - \nu \cos \phi, \quad y_1 = \frac{y_3}{\sin \phi},
\]

has a focus at \( O \). Its equation is \( r = x \cos \phi + y \mu \sin \phi + \nu \sin^2 \phi \).
Lemma 4. The affine map (12) sends any horizontal chord $AB$ of the branch $C$ on a horizontal chord of same length $\|AB\|$ and same $\|OA\| + \|OB\|$.

Proof. The preservation of the horizontal length $\|AB\|$ is simply due to the unit coefficient of $x_3$ in formula (12). Now for $A$ and $B$ on the image branch, we have $r_A = \alpha x_A + \beta y + \gamma$ and $r_B = \alpha x_B + \beta y + \gamma$ where $r_A = \|OA\|$, $\alpha = \cos \phi$, etc. To get $r_A + r_B$ we should compute $x_A + x_B$. The midpoint of a horizontal chord of $C$ is on the line $x_1 = 0$, which is sent on the line $x_3 = y_3 \mu \cot \phi + \nu \cos \phi$. So

$$\frac{r_A + r_B}{2} = (\cos \phi)(\mu y \cot \phi + \nu \cos \phi) + y \mu \sin \phi + \nu \sin^2 \phi = \frac{\mu y}{\sin \phi} + \nu.$$

But $y / \sin \phi = y_3 / \sin \phi = y_1$. Thus $r_A + r_B$ does not depend on $\phi$ and remains unchanged.

Lemma 5. For any $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that $\gamma > 0$, there are exactly two Keplerian orbits describing the branch with equation (10). Their constants of areas are $C = \sqrt{\gamma}$ and $C = -\sqrt{\gamma}$ and their energy is $H = (\alpha^2 + \beta^2 - 1)/(2 \gamma)$.

Proof. That $C = \pm \sqrt{\gamma}$ is essentially Proposition XIV of Newton’s Principia. The other statements are not more difficult and not less classical.

From now on, we consider abusively that the affine map (12) sends a Keplerian orbit on a Keplerian orbit and a Keplerian arc on a Keplerian arc. The map indeed sends an unparametrized arc on an unparametrized arc. The orientation is induced by the affine map, but the time parametrization is not. Instead, we consider on the image branch a time parametrization compatible with Newton’s system.

Lemma 6. Under the affine map (12), the constant of areas $C$ is shrunk as the areas, i.e., is multiplied by $\sin \phi$, while the energy $H$ is unchanged.
Proof. The affine map sends a conic section with semi-parameter $\nu$ to a conic section with semi-parameter $\nu \sin^2 \phi$. According to Lemma 5, $C$ is multiplied by $\sin \phi$. The Jacobian of the map is also $\sin \phi$, as the ordinates are multiplied by $\sin \phi$ and the horizontal distances are preserved. The factor $\alpha^2 + \beta^2 - 1$ in the formula for $H$ was $\mu^2 - 1$ and becomes $\cos^2 \phi + \mu^2 \sin^2 \phi - 1 = (\mu^2 - 1) \sin^2 \phi$. 

Lemma 7. The elapsed time $\Delta t = t_B - t_A$ on a Keplerian arc with endpoints A and B such that $y_A = y_B$ is unchanged by the affine map (12).

Proof. The period $T = 2\pi(-2H)^{-3/2}$ of an elliptic orbit is unchanged, since the energy $H$ is unchanged. This reduces the study to arcs making less than one turn around O. The elapsed time is twice the area swept divided by the constant of areas. According to Lemma 6, it is enough to prove that the area swept is shrunk as all the areas, i.e., multiplied by $\sin \phi$. But this area is made of a segment of conic section $S$ and of the triangle $T = OAB$ (see figure 1). The segment $S$ is sent on the corresponding segment, whose area is shrunk as any area. The triangle $T$ is sent on a triangle formed by the image of the chord AB and the image of O, which is on the $x$-axis. Its area is shrunk as any area. This is not the triangle we need in order to compute the area swept from O. But the triangle we need, with vertex at O, has same area.

Lemma 8. To any branch of conic section $B$ with equation $r = \alpha x + \beta y + \gamma$, with $\gamma > 0$, which possesses a horizontal chord, is associated uniquely a triple $(\phi, \mu, \nu) \in [0, \pi] \times \mathbb{R} \times [0, +\infty]$ such that $B$ is the image by the affine map (12) of the vertical branch $C$ with equation $r = \mu y + \nu$.

Proof. There are two points A and B of same ordinate $y_A = y_B$ on the branch $B$. Thus $r_A = \alpha x_A + \beta y_A + \gamma$, $r_B = \alpha x_B + \beta y_B + \gamma$ give $r_A - r_B = \alpha (x_A - x_B)$. The triangular inequality gives $\alpha^2 < 1$. We set $\phi = \arccos \alpha$, $\mu = \beta / \sin \phi$, $\nu = \gamma / \sin^2 \phi$.

Let us conclude the proof of Theorem 1. We are in the case $||AB|| > 0$. We consider only Keplerian arcs with ends A and B such that $y_A = y_B$, $x_B < x_A$. The other arcs are connected to these ones by a rotation. First notice that the rectilinear arcs satisfy $y = 0$ for all time, and either $x_B < x_A < 0$ or $0 < x_B < x_A$. In both orders, when we fix $||AB||$, $||OA|| + ||OB||$ and $H$, we fix $x_A$, $x_B$ and the orbit, and we are left with arcs which are isolated in the set of rectilinear arcs. The only choices are discrete choices, namely, the number of bounces at collision and the number of outward culminations.

If there were a connected family of non-rectilinear arcs belonging to some $\mathcal{L}$, with non-constant $\Delta t$, we could embed each of them in a family with free parameter $\phi$ according to Lemma 8, which would stay in $\mathcal{L}$ according to Lemmas 4 and 6. We would pass to the limit as $\phi \to 0$ while keeping the same $\Delta t$ according to Lemma 7, and obtain a connected family of rectilinear arcs with non-constant $\Delta t$, which we just proved to be impossible.

4. Further results. Our statement of Lambert’s theorem considers the connected components in the space of Keplerian arcs of a nonempty level set $\mathcal{L}$ of the triple $(||AB||, ||OA|| + ||OB||, H)$. We will describe these connected components.
Definition 1. A Lambert family of planar Keplerian arcs consists of
(i) an arc $A$ carried by a vertical branch $C$ with equation $r = \mu y + \nu$ with $\nu > 0$, with two distinct endpoints at the same ordinate $y_1$,
(ii) all the images of $A$ by the affine maps (12), for all $\phi \in [0, \pi[$,
(iii) all the reflected arcs with respect to the horizontal axis, which indeed correspond to the $\phi \in ]-\pi, 0[$,
(iv) both limiting rectilinear Keplerian arcs as $\phi \to 0$ and as $\phi \to \pi$.

A rotated Lambert family consists of all these arcs rotated in all ways around $O$.

Remark 6. An arc $A$ and its image by reflection generate the same Lambert family. Their coordinates are $(\mu, \nu, y_1)$ and $(-\mu, \nu, -y_1)$. An arc and its image by a rotation by $\pi$ generate two distinct Lambert families, but the same rotated Lambert family.

A description of a Lambert family. We will use a short notation
\begin{equation}
2\bar{r} = \|OA\| + \|OB\|, \quad 2m = \|AB\|. \tag{13}
\end{equation}
Since we fix $\bar{r}$, the center $O$ describes relatively to the chord $AB$ an ellipse with foci $A$ and $B$. As indeed $O$ is fixed, we may think of a Lambert family of horizontal chords as follows. The midpoint $M$ of the chord $AB$ describes an ellipse centered at $O$, with horizontal major axis, semi-major axis $\bar{r}$, eccentricity $m/\bar{r}$ and, consequently, semi-minor axis $\sqrt{\bar{r}^2 - m^2}$. Then $y_1^2 = \bar{r}^2 - m^2$, since $y_1$ is the ordinate of the chord when $\phi = \pi/2$. Formula (12) gives $M = (\bar{r} \cos \phi, y_1 \sin \phi)$. Lemma 3 gives a similar expression $(\cos \phi, \mu \sin \phi)$ for the eccentricity vector. The second focus $F$ of the Keplerian conic section is consequently at $-H^{-1}(\cos \phi, \mu \sin \phi)$. The angle $\phi$ may be determined through the classical relation $\|OA\| - \|OB\| = \pm 2m \cos \phi$, where the sign is a plus if $x_A < x_B$.

Figure 2. Ellipses with a chord, of a Lambert family, $\phi = \pi/4, \pi/6, \pi/12, 0$.

Definition 2. A Keplerian arc making less than one turn is said to be
– direct, or $D_O$, if its convex hull does not contain $O$; indirect, or $I_O$, if its convex hull contains $O$.

– direct with respect to the second focus $F$, or $D_F$, if its convex hull does not contain $F$; indirect with respect to $F$, or $I_F$, if its convex hull contains $F$.

**Remark 7.** The hyperbolic and parabolic arcs are always $D_F$. The choices in the definition classify as well the rectilinear arcs, where in the negative energy case, the second focus is the culmination point. A “turn” should be understood as a return to the initial condition. See figure 2 where in each ellipse the lower arc is $I_O D_F$ and the upper arc is $D_O I_F$.

**Lemma 9.** If the generator $A$ is $D_O$ (respectively $I_O, D_F, I_F$) all the arcs of the Lambert family are $D_O$ (respectively $I_O, D_F, I_F$).

**Proof.** There is indeed another classification, which amounts to the same thing for hyperbolic and parabolic arcs, but which is strictly finer for most elliptic arcs. It consists in ordering the ordinate $y_1 \sin \phi$ of the chord, assumed to be horizontal, with respect to the ordinates of both foci, namely, 0 and $-H^{-1} \mu \sin \phi$. We can distinguish $D_O D_F$ arcs which are beyond the second focus $F$ from $D_O D_F$ arcs which are beyond the first focus $O$. The common factor $\sin \phi$ shows that this finer classification remains unchanged along a Lambert family, as soon as $\sin \phi \neq 0$. Thus the Lemma is proved for the non-rectilinear arcs. The rectilinear arcs are defined by passing to the limit. The definitions in term of convex hull remain consistent in the limit. \(\square\)

**Lemma 10.** Let $A \neq B$ and $O \notin [A, B]$. There are exactly one $D_O D_F$ arc and one $I_O D_F$ arc with ends $A$ and $B$ and given energy $H > H_{\min}$, where $H_{\min}$ is expressed by formula (1).

![Figure 3. Another family, with same chords and same energy as in figure 2.](image)

**Proof.** The case where $O$, $A$ and $B$ are collinear but $O$ is not between $A$ and $B$ is easy. The arcs are rectilinear. The $I_O$ arc bounces at $O$, while the $D_O$ arc does not. Both arcs are $D_F$, i.e., do not culminate. The condition $H > H_{\min}$ gives a major axis $2a = -1/H$ greater than $\|OA\|$ and $\|OB\|$. 

\(8\)
Lemmas 8 and 9 reduce the other cases to the case of a vertical branch with a horizontal chord at ordinate $y_1 = \sqrt{\bar{r}^2 - m^2} \geq 0$, where $\bar{r}$ and $m$ are given by (13). The case $y_1 = 0$ is excluded as $O$ is not in the chord $[A, B]$. The relations $\bar{r} = \mu y_1 + \nu$ and $2H \nu = \mu^2 - 1$ give $\mu$ through the second degree equation $2H(\bar{r} - \mu y_1) = \mu^2 - 1$. The reduced discriminant is

$$H^2 y_1^2 + 1 + 2H \bar{r} = (H \bar{r} + 1)^2 - m^2 H^2 = (H(\bar{r} + m) + 1)(H(\bar{r} - m) + 1).$$

If $H > H_{\text{min}}$ both factors are positive and there are two roots $\mu_1$ and $\mu_2$ with $\mu_1 \leq -H y_1 \leq \mu_2$. Clearly $\mu = \pm 1$ is root only if $H = 0$, which proves among other things that $\nu > 0$ for both roots $\mu_1$ and $\mu_2$. Since $\mu_2 > 0$, the lower arc of the branch associated to $\mu_2$ always exists and is $IO$. As $-H y_1 \leq \mu_2$ the second focus is above the chord and this arc is $DF$. Since $\mu_1 \leq -H y_1$ the upper arc of the branch associated to $\mu_1$ always exists and is $DO$ and $DF$. So we have the existence of both arcs. As for the uniqueness, there are no other arcs in the case $H \geq 0$, and there are two other arcs if $H < 0$. But these two arcs are $IF$ as complements of $DF$ arcs.

**Corollary.** Under the same hypotheses and furthermore $H < 0$, there are exactly four arcs making less than one turn, one $DODF$, one $IODF$, one $DOIF$, one $IOIF$. If $H = H_{\text{min}}$ there are two arcs of the same ellipse with $F$ on the chord. The upper is $DOIF$, the lower $IOIF$.

**Remark 8.** If $O \in [A, B]$ the arcs of Lemma 10 become symmetric one each other with respect to the chord and they are both $IOIF$.

**Corollary.** Any Keplerian arc with distinct ends of same ordinate belongs to a unique Lambert family.

**Proof.** The non-rectilinear case is Lemma 8 and remark 6. A rectilinear arc is characterized by $\bar{r}$, $m$ and $H$, a choice $DO$ or $IO$, and if $H < 0$, a choice $DF$ or $IF$ and a number of turns. It first defines a Lambert family of horizontal chords, and then the unique arc on each chord with same $H$ and same discrete choices. This forms a Lambert family of arcs.

**Corollary.** If $\|AB\|$ is fixed to a nonzero value a connected component of $L$ in Theorem 1 is a rotated Lambert family.

5. Geometrical analogues. Some geometrical statements are closely related to Lambert’s theorem. Even if they can be expressed in many simple ways, they never appear as well-known. We will give three propositions. We begin with a lemma which appears in Terquem [1] as Theorem V.

**Lemma 11.** In an ellipse

- an arbitrary chord passing through a focus,
- the parallel chord passing through the center and
- the major axis

have their three lengths in geometric progression.
Proof. We take the direction of the chord as the $x$-axis and a focus as the origin. We compute the horizontal semi-chord at ordinate $y$ as $\sqrt{\delta/(1-\alpha^2)}$ where $\delta = \alpha^2(\beta y + \gamma)^2 - (1-\alpha^2)(y^2 - (\beta y + \gamma)^2) = (\alpha^2 - 1)y^2 + (\beta y + \gamma)^2$ is the reduced discriminant of equation (9) seen as a trinomial in $x$. At $y = 0$ the semi-chord is $\gamma/(1-\alpha^2)$. At $y = \beta a$, where $a$ is the semi-major axis, which satisfies $\gamma = a(1 - \alpha^2 - \beta^2)$, $\delta = (\alpha^2 - 1)a^2\beta^2 + a^2(1-\alpha^2)^2 = a(1-\alpha^2)\gamma$. Consequently $\sqrt{\delta/(1-\alpha^2)}$ is the geometric mean of $a$ and $\gamma/(1-\alpha^2)$.

Proposition 2. Consider in a Euclidean plane an ellipse and a chord. Apply an affine map. Any two of these three properties imply the remaining one:

- a parallel chord passing through a focus is sent on a chord passing through a focus,
- the length of the given chord is preserved,
- the semi-major axis is preserved.

Proof. Call the three lengths in Lemma 11, corresponding to the direction of the given chord, $f, g, h$ before applying the map and $f', g', h'$ after. An affine map sends all the parallel chords to parallel chords, multiplying their length by a common factor $\lambda$. As the center of the ellipse is sent to the center of the image, we have $g' = \lambda g$. Observe now that in the family of parallel chords the length starts from zero, increases until it reaches a maximum and then decreases to zero. Thus a “parallel chord passing through a focus” is also a “parallel chord of same length as a parallel chord passing through a focus”: the first condition in the statement is $\lambda f = f'$. The second is $g = g'$, i.e., $\lambda = 1$, the third is $h' = h$. An easy analysis shows that if two conditions are satisfied, the geometric progression implies the remaining one.

Lemma 3 provides affine maps satisfying the three properties in Proposition 2, if the term $-\nu \cos \phi$ is removed expression (12). But Lemma 3 works as well for parabolas and hyperbolas. Let us extend Proposition 2 accordingly.

Proposition 3. The image of a conic section with semi-parameter $\gamma > 0$, with a focus on a line $D$, by an affine map with Jacobian determinant $J$, which fixes all the points of $D$, has semi-parameter $J^2\gamma$ if and only if it has a focus on $D$.

In the hypothesis and in the conclusion we should consider that a parabola has a focus at infinity, which is a point on the line at infinity. This focus is on $D$ if and only if the axis of the parabola is parallel to $D$. The proof of this proposition is a case by case study, which we leave to the reader.

In the case of an ellipse or a hyperbola, the image has semi-parameter $J^2\gamma$ if and only if the energy $H = -1/(2a)$ is preserved. To prove this we may use the expression $\pi a^{3/2} \sqrt{\gamma}$ of the area of an ellipse, and the expression $|a|^{3/2} \sqrt{\gamma}$ of the area of a triangle delimited by two asymptotes of a hyperbola and a tangent.

Proposition 4. Consider an ellipse drawn in an affine plane. Consider two Euclidean forms making this plane Euclidean in two different ways, each defining a pair of foci, each defining a semi-major axis of the ellipse. Both semi-major axes are equal if and only if both Euclidean forms induce equal units of length.
on a chord passing through two foci, one of each pair.

**Proof.** This is obtained from Proposition 2. Consider that the first Euclidean form defines the Euclidean structure of the plane, and that the second is the pull-back of the first by an affine map. By the well-known theory of the Gram matrix, there exists an affine map with such a pull-back. Consider the chord passing through two foci as the given chord: the first hypothesis of Proposition 2 is satisfied. The second and the third hypotheses are then equivalent. □

6. A plethora of demonstrations. Here is a timeline of Lambert’s theorem.  

**1687.** Newton considers the problem of the determination of the orbit of a comet from three observations, in Proposition XLI, Book III, of his *Principia*. Lagrange [3, 6] will show later how two of Newton’s lemmas give a proof of Lambert’s theorem in the parabolic case.

**1743.** Euler considers the same problem as Newton, about comets on parabolic orbits, and concludes §XIII of [1] by the formula:

\[ 6(t_B - t_A) = (\|OA\| + \|OB\| + \|AB\|)^{3/2} - (\|OA\| + \|OB\| - \|AB\|)^{3/2}. \] (14)

The time \( t \) from the collision to the position \( x \) in a rectilinear Keplerian motion with zero energy satisfies \( 6t = (2x)^{3/2} \). Euler’s formula is this expression together with the reduction to the rectilinear case proposed by Theorem 1. Euler’s choice of sign happens to correspond to a direct arc. His proof, which we call \( P_1 \), is based on a simplification that appears when dividing the area swept by the square root of the semi-parameter. In §XIV, Euler gives another proof, of same nature, of the same formula. In §XV, he considers the “more difficult” elliptic case and gets a formula in terms of the eccentricity and the three distances which is not as elegant.

**1744.** At the opportunity of the observation of another comet, Euler reconsiders the determination of nearly parabolic orbits in his book [2]. He presents the computations differently and does not mention equation (14). He compares his new numerical results with what he got in 1743.

**1761.** In a letter in february (see Bopp [1]), Lambert announces to Euler his discovery of formula (14).

**1761.** Lambert publishes his fundamental book [1] where he presents formula (14) at §63, giving a proof of style \( P_1 \), and later the elliptic case of Theorem 1. His main step is the construction, from a general ellipse with a chord, of another ellipse with a chord perpendicular to the principal axis. Both ellipses belong to the same rotated Lambert family (our Definition 1). The second corresponds to our vertical branch \( C \). Our figure 1 should be compared to Lambert’s figure 21, our Lemma 6, to his §173, our Lemma 4, to his §177, our Lemma 7, to his §178. Lambert expresses, through the rectilinear motion, the elliptic \( \Delta t \) as an integral and as a series at §210, which he uses at §211 to obtain again formula (14) as a limiting case. He mentions the hyperbolic case at §213, but only about the rectilinear motion. We will number this proof \( P_2 \). Lambert refers to Euler’s
book of 1744 in his introduction, but not to Euler’s article of 1743.

1761. In March Lambert sends his book to Euler who answers “Votre théorème pour exprimer l’aire d’un secteur parabolique est excellent, j’en puis bien voir la vérité, mais par de tels détours, que je ne serois jamais arrivé, si je ne l’avoyssu d’avance; je attend donc avec impatience de voir l’analyse qui y a conduit sans détour” in a first letter, “la belle démonstration de l’aire du secteur parabolique, dont Vous m’avez communiqué l’expression m’a causé un très sensible plaisir; mais je fus bien plus surpris d’en voir l’application aux secteurs elliptiques [...] je reconnais aisément que les méthodes, que j’avoysses proposées autrefois, peuvent être très considérablement perfectionnées” in a second, “Vos remarques sur la reduction du mouvement curviligne des corps celestes la chute rectiligne sont très sublimes, et nous découvrent en effet des prometes qui sans cette reduction paroissent tout faire indechiffrables” in a third.

1773. Lagrange [2], §XI, deduces Lambert’s theorem while discussing Euler’s two fixed centers problem and analyzing the limiting case where one of the centers has zero mass and is on the orbit. We call this proof $P_3$. See 1780, 1815, Jacobi 1866. We will briefly discuss in §7.3 a related work by Lagrange published in the same volume of Miscellanea Taurinensia.

1780. In memory of his friend and colleague who died in 1777, Lagrange publishes a series of memoirs. In [3], he writes “C’est ce que M. Lambert a fait depuis dans son beau Traité De orbitis Cometarum, où il est parvenu un des Théorèmes les plus élégants et les plus utiles qui aient été trouvés jusqu’ici sur ce sujet, et qui a en même temps l’avantage de s’appliquer aussi aux orbites elliptiques” and “Théorème qui, par sa simplicité et par sa généralité, doit être regardé comme une des plus ingénieuses découvertes qui aient été faites dans la Théorie du système du monde”. He also analyses Euler’s book of 1744, and several published consequences of Lambert’s theorem.

1780. Lagrange [4] presents three other proofs of Lambert’s theorem, introducing then in §1 by “mais ce théorème mérite particulièrement l’attention des Géomètres par lui-même, et parce qu’il paraît difficile d’y parvenir par le calcul; en sorte qu’on pourrait le mettre dans le petit nombre de ceux pour lesquels l’Analyse géométrique semble avoir de l’avantage sur l’Analyse algébrique.” He rejects his proof $P_3$ as too indirect and complicated, but proposes a similar proof $P_5$ which does not refer explicitly to the two fixed centers problem (see §14). The first proof in [4], which we call $P_4$, uses the eccentric anomaly. Note that the difference $u_B - u_A$ of the final and initial eccentric anomalies is obviously an invariant of our map (12), and that $(u_A + u_B)/2$ is the eccentric anomaly of the highest or lowest point of the ellipse. The second proof, which we call $P_5$, is concluded in §7. It starts with the expression of the elliptic $\Delta t$ by a quadrature of a function of the distance $r$, and concludes by using general methods rather than formulas for the Keplerian motion. In the proof $P_6$, the concluding identity of $P_5$ is presented as a particular case of more general identities (see our §7.3).

1784. Lexell [1], in a volume announcing the death of his master Euler, discusses the proofs by Lambert and Lagrange, extends them to the case of hyper-
bolic motions, and discusses reality conditions in Lagrange’s identities. He also proposes some reciprocal statements.

1797. Olbers publishes a method of orbit determination in his book [1], with many references, including to formula (14), to Lambert’s works and its continuations. He also discusses a method published by Laplace in 1780.

1798. Laplace publishes his *Mécanique céleste* [1] with, at §27, a proof of Lambert’s theorem, similar to $P_4$, and concludes with three formulas, the first for the elliptic case, calling attention to the choices of arcs, the second for the parabolic case, being formula (14) where the choices of signs are characterized, the third for the hyperbolic case. He republishes his orbit determination method, which does not use Lambert’s theorem.

1809. Gauss publishes his *Theoria motus* [1], a book on orbit determination. In §106 he gives the correct attribution of (14): “This formula appears to have been first discovered, for the parabola, by the illustrious Euler, (Miscell. Berolin, T. VII. p. 20,) who nevertheless subsequently neglected it, and did not extend it to the ellipse and hyperbola: they are mistaken, therefore, who attribute the formula to the illustrious Lambert, although the merit cannot be denied this geometer, of having independently obtained this expression when buried in oblivion, and of having extended it to the remaining conic sections. Although this subject is treated by several geometers, still the careful reader will acknowledge that the following explanation is not superfluous. We begin with the elliptic motion.” Gauss gives a proof of Lambert’s theorem of style $P_4$. He insists on a remaining ambiguity of sign, which he explains by the existence of two ellipses: the second focus is constructed as the intersection of two circles, giving two possible positions. In §108 he discusses the limiting process to get formula (14) from the elliptic case, but decides to give a proof of style $P_1$, discussing the signs. In §109 he gives a proof and formulas for the hyperbolic case. He advertises the same series as Lambert, which are valid for the three conic sections, as being better for practical use than a method he had devised previously.

1815. Lagrange, in the second, posthumous, edition [6] of his *Mécanique analytique*, section VII, §25, gives a proof of style $P_1$ of formula (14), cites Euler’s article in §26, and presents a method for orbit determination which uses (14). In §84 he presents briefly his proof $P_3$. He shows how his final integral formula applies to the three types of conic sections, and advertises the same series as Lambert and Gauss.

1820. Legendre [1], p. 7, gives a proof of (14). He recalls that he published the least-squares method before Gauss in *Theoria Motus*, in the same context.

1831. Encke [1] gives a proof of (14) in an article explaining Olbers’s method. He introduces it as follows: “Although this method was already carried to such a degree of perfection in the first memoir, that even the master-hand of the author of the *Theoria motus, &c.*, made no essential alteration in it, but only some abbreviations, [...] Lambert’s theorem is a main part of Olbers’s method. The manner of solving it given by Olbers admitting of some abbreviations, I
shall begin with explaining this little improvement."

1834. Hamilton studies in [1] the properties of what he calls the characteristic function, namely, the integral of $2T dt$, where $T$ is the kinetic energy. This is $w$ of formula (2) in the case of a point particle. In §15 he shows that $w$ on elliptic arcs depends on $||AB||$, $||OA|| + ||OB||$ and $H$. Together with the relation $\delta w/\delta H = t$, on which he insists in §2, this gives a new proof of Lambert’s theorem. The method to deal with $w$ has common features with $P_4$, see e.g. equation (108).

1837. Jacobi [1], §7, presents Hamilton’s formulas in another order. He uses Lambert’s theorem to deduce the trigonometrical expression of $w$ that Hamilton used to deduce Lambert’s theorem. He insists on the analogy of the expressions of $t$ and $w$ (see 1866, Tait). He deduces from the expression of $w$ elegant formulas for the initial and final velocity vectors as $v_A = k + \rho \varepsilon_A$ and $v_B = k - \rho \varepsilon_B$ respectively, where $k$ is a vector along the chord, $\varepsilon_A$ and $\varepsilon_B$ are unit radial vectors, and $\rho$ is a number. See 1866, 1888, 1961 and §7.1. He checks that $w$ satisfies the Hamilton-Jacobi equation.

1837. Chasles [1], IV, §37, opposes again, after Lagrange, analysis and geometry: “Le célébre Lambert, autre Leibnitz par l’universalité et la profondeur de ses connaissances, doit être placé au nombre des mathématiciens qui, dans un temps où les prodiges de l’analyse occupaient tous les esprits, ont conservé la connaissance et le got de la Géométrie et ont su en faire les plus savantes applications. [...] Ces considérations géométriques sont simples, et cependant elles ont suffi pour conduire Lambert au théorème le plus important de la théorie des comètes, dont les démonstrations qu’on en a données depuis par la voie du calcul ont exigé toutes les ressources de l’analyse la plus relevée.”

1847. Hamilton [2] states a “Theorem of hodographic isochronism: If two circular hodographs, having a common chord, which passes through or tends towards a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.” We can rephrase the statement in this way. If a circle $C$ cuts orthogonally two hodographs $H_1$ and $H_2$ of two Keplerian orbits of the same Lambert family, its center is on $Ox$. The arcs cut on $H_1$ and $H_2$ are described in the same time.

1862. Cayley [1] gives a description of Lambert’s original results and of the rotated Lambert family, and a computational proof of Lambert’s theorem similar to $P_4$, which is guided by Lambert’s constructions.

1866. In Jacobi’s famous book [2], which consists of lectures that he gave in Königsberg in the winter 1842–43, edited from notes by Borchardt, the whole lecture 25 presents a proof of Lambert’s theorem. Jacobi separates the Hamilton-Jacobi equation in elliptic coordinates, with a focus at O and another at the initial point A. He slightly changes the presentation of his formulas for the initial and final velocities (see 1837). He shows how the separation produces elliptic integrals if the foci are O and an arbitrary point, even if this arbitrary point is a second fixed center. Except for the introduction of Hamilton’s characteristic function, the proof follows $P_3$, by reversing the order of generality. Lagrange is not cited for his proofs of Lambert’s theorem, but only for his
related article [1] (see our §7.3).

1866. Sylvester [1] fixes the semi-major axis of ellipses to 1 and proves by direct computation of the Jacobian that $\|OA\| + \|OB\|$, $\|AB\|$ and $\Delta t$ are functionally dependent. This is a proof of Lambert’s theorem which he considers to be close to $P_4$. But he actually removed part of Lagrange’s computation, replacing it by the simpler-minded computation of the Jacobian. Then he takes the eccentricity $e$ as a parameter of what we call a Lambert family. He states that $\Delta t$ does not depend on $e$ and evaluates $\Delta t$ at $e = 1$ (see our remarks 2 and 5).

1866. Sylvester [2] presents his previous proof with these words: “Notwithstanding this plethora of demonstrations I venture to add a seventh, the simplest, briefest, and most natural of all”. He reacts to Lagrange’s and Chasles’s arguments about the advantage of geometry: “In the nature of things such advantage can never be otherwise than temporary. Geometry may sometimes appear to take the lead of analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on the way. The interval between the two is as wide as between empiricism and science, as between the understanding and the reason; or as between the finite and the infinite”. He proves the hyperbolic and parabolic cases of Lambert’s theorem as he had proved the elliptic case, and continues as described in the long title of his paper.

1866. Hamilton [3], article 419, proves his theorem of hodographic isochronism (see 1847), and deduces Lambert’s theorem from it. He then gives a proof, using variations, quaternions and hodographs, of a “new form of Lambert’s Theorem”: the principal function from $A$ to $B$ depends on $\|OA\| + \|OB\|$, $\|AB\|$, $t$ and the mass of the attracting body, while the characteristic function and the elapsed time depend on $\|OA\| + \|OB\|$, $\|AB\|$, $H$ and the mass.

1866. Tait [1], or [2] p. 163, interprets the analogy between time and characteristic function in Hamilton [1]: “while the time is proportional to the area described about one focus, the action is proportional to that described about the other.”

1869. Cayley [2] resolves the ambiguity of sign pointed out by Gauss with a geometrical criterion. One should ask if the line passing through $A$ and the second focus separates $O$ from $B$.

1878. Adams [1] publishes a proof of type $P_4$ in the elliptic and hyperbolic cases, and then in the parabolic case by passing to the limit. He presents the same formulas as Gauss in a more transparent way. He notices that three functions are expressed in term of two quantities only, $u_B - u_A$ and $e \cos((u_A + u_B)/2)$. This reminds Sylvester’s argument. This presentation is adopted in Dziobek [1] and Routh [1].

1884. Ioukovsky [1] proposes a proof based on the variation of the characteristic function $w$. He uses the analogy between $t$ and $w$ pointed out by Jacobi and Tait instead of using Hamilton’s relation $\delta w/\delta H = t$. He does not cite any author except Euler and Lambert. As the proof involves the second focus, it should be adapted to each type of conic section.
1884. Catalan [1] presents a proof of Lambert’s theorem of style \( P_4 \), where he interprets each step with a geometrical construction. He gives some related geometrical statements, one of them being a construction, from a general ellipse with a chord, of what we call the vertical branch \( C \) (see 1761, 1862), others being new.

1888. Dziobek’s book [1] gives a short proof using Adams’ argument and a proof inspired by Hamilton and Jacobi. He comments: “For a long time, the proposition was regarded as a curiosity. Its true source was shown by the investigations of Hamilton and Jacobi.” He advertises Jacobi’s expression of \( v_A \) and \( v_B \) and writes: “no one would have succeeded \textit{a priori} in getting the notable equations [of \( v_A \) and \( v_B \)] from those §1.” We will comment his words in §7.1.

1901. Bourget [1] complains that Jacobi [2] does not cite Lagrange’s proof \( P_6 \). He generalizes the main identity in \( P_6 \).

1941. Wintner’s book [1], §247-248, has an interesting presentation with sometimes imprecise attributions: ‘A proof of Lambert’s theorem can be obtained by an application of the theorem of Gauss-Bonnet on the surface of revolution \( S_h \) of §244. However, the proof is shorter if use is made of the “Beltrami-Hilbert integral” or the “isoenergetic action \( W \)” not via \( S_h \) but in a more direct manner, as follows.’ His historical note at p. 422 compares the lengths of various proofs.

1961. Godal [1] presents as Jacobi and Dziobek the initial and final velocity vectors as \( k + \rho e_A \) and \( k - \rho e_B \) respectively. He notices that \( \rho \parallel k \parallel \) depends only on A and B, not on the orbit. See §7.1 and remark 10.

1983. Souriau [1], p. 376, proposes a new proof of Lambert’s theorem, resulting of a collection of remarkable and elegant formulas about the Kepler problem.

2002. Marchal [1] presents several formulas and proofs of style \( P_4 \) which include, as Jacobi’s paper in 1837, formulas for the action. Remarkable inequalities are deduced and used to estimate the minimizers of the action in the \( n \)-body problem.

2016. A recent work as Linet & Teyssandier [1] shows that Lambert’s theorem may still be rediscovered by skillful calculators. They consider the gravitational influence of a spherically symmetric body on the propagation of light within the weak-field, linear approximation of general relativity. Their formula (39) is typically “Lambertian”.

7. Final comments. Many proofs were proposed after Lambert’s proof in 1761. Such a “plethora of demonstrations”, in Sylvester’s words, gives the impression of a chronic dissatisfaction. After Lambert’s publication, which was found obscure, most attempts were “analytical”. The geometrical arguments by Lambert remained essentially untouched, being only described in few words by Lagrange in 1780 and in a short note by Cayley in 1862. The fact that two unparametrized arcs belonging to the same Lambert family correspond with each other through an affine transformation of the plane has never been stated, as far as I know.
7.1. Comments on our minimal proof. Dziobek claims in 1888 that Hamilton and Jacobi found the “true source” of Lambert’s theorem. He is not convincing: of the two proofs in his book, the short one does not involve such “source”, while the long one does. If Hamilton himself was convinced he got the “true source” in 1834, he would not have published other proofs based on different ideas. What Hamilton shared indeed with his contemporaries is an obsession of Lambert’s theorem. Uncovering deep features of dynamics and geometry, namely, the properties of the characteristic function, the circular hodograph of the Keplerian motion and the quaternion algebra, he successively used them to produce a new demonstration. Jacobi does not appear to be convinced in 1837 that Hamilton got the “true source”, and the key of Jacobi’s second proof is the elliptic system of coordinates rather than the characteristic function.

The simplicity of our minimal proof supports Dziobek’s opinion about the “true source”, and at the same time contradicts his words “no one would have succeeded” (see 1888). Lemma 1 is remarkable. The direction of \( v_B - v_A \) does not depend on the choice of the conic section passing through A and B. If \( \varepsilon_A + \varepsilon_B \neq 0 \), there is a \( \rho \) such that \( v_A - v_B = \rho(\varepsilon_A + \varepsilon_B) \). We set \( k = v_A - \rho \varepsilon_A = v_B + \rho \varepsilon_B \) and get Jacobi’s expressions \( v_A = k + \rho \varepsilon_A, v_B = k - \rho \varepsilon_B \), where there just remains to express \( k \) and \( \rho \), if needed. Interestingly, Dziobek refers to his §1 as not giving this key lemma, but this first section of his excellent book does present the eccentricity vector at (17a), in a new and deep way, and does use it to compute velocities, in his proof of the circularity of the hodograph.

Remark 9. Hamilton obtained in 1834 from the expression of the characteristic function “the following curious, but not novel property, of the ellipse”, which is republished in 1866, just after the “new form of Lambert’s theorem”, as “this known theorem: that if two tangents \( (QP, QP') \) to a conic section be drawn from any common point \( Q \), they subtend equal angles at a focus \( O \), whatever the special form of the conic may be”. One should understand that the equal angles are QOP and QOP’. The same property appears in Ioukovsky, now as an argument used to prove Lambert’s theorem. According to Berger [1], 17.2.1.6, this property is one of Poncelet’s “small theorems”. Poncelet [1], p. 265, states this property and the fact that the external bisector of POP’ meets the chord on the directrix, but he gives credit to De Lahire and l’Hôpital. The earliest statement we know of the “curious property” belongs to de La Hire (see [1], book 8, Proposition 24, p. 190).

De La Hire’s property is related to Lemma 1. A simple proof is found by writing equation (10) in the same notation as (5), which gives \( r - \langle q, E \rangle = \gamma \). Any point \( Q \) of the tangent at \( q \) satisfies \( \langle OQ, \varepsilon - E \rangle = \gamma \). The intersection of the tangent lines \( \langle OQ, \varepsilon_A - E \rangle = \gamma \) and \( \langle OQ, \varepsilon_B - E \rangle = \gamma \) satisfies \( \langle OQ, \varepsilon_A - \varepsilon_B \rangle = 0 \). So OQ has for direction \( \varepsilon_A + \varepsilon_B \), the angle bisector of both positions seen from a focus. Lemma 1 can be restated as:

Lemma 12. Consider two positions A and B on a Keplerian orbit in a plane with origin the fixed center O. Let \( v_A \) and \( v_B \) be the velocity vectors at these
positions. The interior bisector line of the angle $AOB$ contains the vector $v_B - v_A$ and the intersection point of the tangents at $A$ and $B$.

**7.2. Comments on our constructive proof.** This proof improves Lambert’s original proof $P_2$ and Lagrange’s proof $P_4$, by pointing out the affine transformations and figure 1, and by getting the three kinds of conic sections in a single computation. Note that $P_4$ has longer computations than our proof, only for the elliptic case. We simply followed an opinion emitted by Gauss about the treatment of conic sections in celestial mechanics (see his book [1] §3, p. 3):

“Inquiries into the motions of the heavenly bodies, so far as they take place in conic sections, by no means demand a complete theory of this class of curves; but a single general equation rather, on which all others can be based, will answer our purpose. And it appears to be particularly advantageous to select that one to which, while investigating the curve described according to the law of attraction, we are conducted as a characteristic equation.”

After changing a sign convention, this is equation (10). Lagrange [5], §7, implicitly advertised this equation, by presenting an elegant and useful solution of the Kepler problem, which consists in deducing (10) from the expression (8) of the eccentricity vector.

We are indeed dealing with the famous focus-directrix description of a conic section. The directrix is the line $0 = \alpha x + \beta y + \gamma$. The right-hand side is the distance to the directrix multiplied by the eccentricity $\sqrt{\alpha^2 + \beta^2}$. The left-hand side is the distance to the focus $O$. Pappus proved that a curve described in this way is a conic section in his report about the *Surface-loci*, a lost book by Euclid (see Thomas [1], p. 493, Heath [1], p. 243, Chasles [1], p. 44).

Our fundamental identity (11) uses in a non-intuitive way the most typical operation of Algebra, the “al-jabr” operation, which consists in translating a term from the left hand side to the right hand side of an equation. We were not able to find a purely geometrical argument with comparable simplicity. In all other attempts, the parabolic case required a special treatment (see Proposition 3).

**7.3. Comments on another proof.** The elliptic coordinates $\sigma$ and $\tau$ of a point moving in a plane attracted by two fixed centers are two elliptic functions of a parameter, and the time is expressed as an elliptic integral in $\sigma$ minus an elliptic integral in $\tau$. As the analytic expression of the Keplerian motion does not involve elliptic functions, simplifications should occur when one of the two centers has a zero mass. Observing these simplifications, Lagrange obtained a proof of Lambert’s theorem which was published in 1773. Indeed, when one of the masses is zero, $\sigma$ and $\tau$ are expressed by the same elliptic function, and the time is expressed as an elliptic integral in $\sigma$ minus the same elliptic integral in $\tau$. A strange simplification occurs when subtracting. The formulas appear in Lagrange, as (M), (N) and (T) in [2], and in §10 of [4]. The deduction of (T) uses a method that Lagrange explains in [1], without explicit reference to the mechanical problem, but with a reference to previous works by Euler about elliptic integrals. Euler commented Lagrange’s works [1] and [2] in his last letter to Lagrange. These works became classical in the theory of elliptic functions,
about the addition theorem (see Euler [3] and the notes by the editors, and Houzel [1], p. 89). They concern the Keplerian motion expressed in elliptic coordinates rather than Lambert’s theorem. Jacobi uses them in his lecture 25 (see 1866). Sylvester [2] advertises Lagrange’s identities without mentioning Euler or Lexell. Bourget [1] cites works by Euler, Raffy, Fagnano, Graves and Chasles.

**Remark 10.** Lambert’s theorem became of wide interest in the context of the space exploration. Many works have been published. Some confusion of concepts and misattributions were unavoidable, which are now easy to correct. It would be safe to always distinguish Lambert’s theorem from what was called in this new context the “Lambert problem”. This new terminology should be used as we use names for “boundary problems” as the “Dirichlet problem” or the “Neumann problem”, i.e., a problem that is supposed to be solved numerically in each new case. The Lambert problem is: given A, B and the elapsed time \( \Delta t \), find a Keplerian arc going from A to B in the time \( \Delta t \). Of course we wait for a corresponding existence and uniqueness result, stated as clearly and generally as possible. Rather than referring to “Lambert’s formula” or “Lambert’s equation”, we should think of Lambert’s theorem as a reduction to the rectilinear case. We refer to the short presentation in our previous work [1], to which we add for easier understanding the following figure 4. The name “Lambert problem” was quite happily chosen. We can read in Bopp [1], p. 24, statements that Lambert sent to Euler about the possible sets of data that can be used to determine uniquely a Keplerian orbit: “J’ai oublié de tourner le problème §210 c’est que l’orbite se trouvera

1° par les 3 cotés \( FN, FM, NM \) et le temps \( T \) employé parcourir l’arc \( NM \).

2° par le rapport \( (FM : FN) \), l’angle \( NFM \), le temps \( T \), et le temps périodique. Si le diamètre du Soleil peut être mesuré assez exactement, deux observations suffisent pour déterminer l’orbite de la Terre par ce dernier théorème.”

![Figure 4. A sequence of arcs passing through two fixed positions.](image-url)
Acknowledgements. Thanks to Alain Chenciner for many precious suggestions, to Richard Montgomery for his encouraging help, to Pierre Teyssandier and Christian Velpry for criticizing my drafts.

References

J.C. Adams [1] On a simple proof of Lambert’s theorem, *Messenger of Mathematics*, 7 (1878), pp. 97–100

A. Albouy [1] Lectures on the two-body problem, *Classical and Celestial Mechanics. The Recife Lectures*. H. Cabral and F. Diacu ed., Princeton University Press, Princeton, 2002, pp. 63–116

M. Berger [1] *Géométrie*, Nathan, Paris, 1977, 1990

K. Bopp [1] Leonhard Eulers und Johann Heinrich Lamberts Briefwechsel, *Abhandlungen der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse*, 2 (1924), pp. 7–37

H. Bourget [1] Sur une formule de Lagrange et le théorème de Lambert, *Annales de la faculté des sciences de Toulouse*, 2e série, tome 3, no 1 (1901), pp. 69–75

E. Catalan [1] Note sur le théorème de Lambert, *Nouv. annal. de math.*, 3/3 (1884), pp. 506–513

A. Cayley [1] On Lambert’s Theorem for Elliptic Motion, *Monthly Notices of the Royal Astronomical Society*, 22 (1862), pp. 238–242; *Mathematical Papers*, 3, Cambridge, pp. 562–565

A. Cayley [2] Note on Lambert’s Theorem for Elliptic Motion, *M.N.R.A.S.*, 29 (1869), pp. 318–320; *Math. Papers*, 7, pp. 387–389

M. Chasles [1] *Aperu historique sur l'origine et le développpement des méthodes en géométrie, particulièrement de celles qui se rapportent la géométrie moderne*, Hayez, Bruxelles, 1837

R.H. Cushman, J.J. Duistermaat [1] A Characterization of the Ligon-Schaaf Regularization Map, *Communications on Pure and Applied Mathematics*, Vol. L (1997), pp. 773–787

O. Dziobek [1] *Mathematical Theories of Planetary Motions*, translation from the German edition (Leipzig, 1888), Register publishing Co. 1892; Dover 1962

J.F. Encke [1] On Olbers’s Method of determining the Orbits of Comets (translation of *Jahrbuch* for 1833, p. 264), *The London and Edinburgh philosophical magazine and journal of science*, third series, vol. 7 (1835), pp. 7–25, 123–132, 203–206, 280–288

L. Euler [1] Determinatio orbitae cometae qui mense Martio huius anni 1742 potissimum fuit observatus, *Miscellanea Berolinensia*, 7 (1743), pp. 1–90; *opera omnia*, II28, pp. 28–104

L. Euler [2] *Theoria motuum planetarum et cometarum*, Ambrosii Haude, Berlin, 1744

L. Euler [3] Euler Lagrange, St Petersbourg, 23 Mars 1775, *Leonhardi Euleri
K.F. Gauss [1] *Theoria Motus Corporum Coelestium in sectionibus conicis solem ambientium*, Perthes & Besser, Hamburg, 1809; translation by C.H. Davis, Little, Brown & Co., New York, 1857; Dover, New York, 1963

J.W. Gibbs [1] *Vector analysis*, Charles Scribner’s sons, New York, 1909; Dover, 1960

Th. Godal [1] Conditions of Compatibility of Terminal Positions and Velocities, *Xth International Astronautical Congress*, Springer-Verlag (1961), pp. 40–44

W.R. Hamilton [1] On a General Method in Dynamics by which the Study of the Motions of all free Systems of attracting or repelling Points is reduced to the Search and Differentiation of one central Relation, or characteristic Function, *Philosophical Transactions of the Royal Society*, Part II (1834), pp. 247–308; *Math. Papers*, vol. 2, pp. 103–161

W.R. Hamilton [2] On a Theorem of Hodographic Isochronism, *Proceedings of the Royal Irish Academy*, 3 (1845–47), p. 417; *Math. Papers*, vol. 2, p. 293 and Appendix p. 630

W.R. Hamilton [3] *Elements of Quaternions*, Longmans, Green, & Co., London, 1866; edition in two volumes, Longmans, Green, & Co., 1899, 1901, Chelsea, 1969

T. Heath [1] *A History of Greek mathematics, volume 1, From Thales to Euclid*, Oxford, 1921; Dover, 1981

C. Houzel [1] *La géométrie algébrique. Recherches historiques*. A. Blanchard, Paris, 2002

N. Ioukovsky [1] Sur une démonstration nouvelle du théorème de Lambert, *Nouvelles annales de mathématiques*, 3er série, tome 3 (1884), pp. 90–96

C.G.J. Jacobi [1] Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variablen auf die Integration eines einzigen Systemes gewöhnlicher Differentialgleichungen, *Journal für die reine und angewandte Mathematik*, Bd. 17 (1837), pp. 97–162; *Werke* Bd. 4, pp. 57–127; French translation: Sur la réduction de l’intégration des équations différentielles du premier ordre entre un nombre quelconque de variables l’intégration d’un seul système d’équations différentielles ordinaires. *Journal de mathématiques pures et appliquées*, (1) 3 (1838), pp. 60–96, 161–201

C.G.J. Jacobi [2] *Vorlesungen über Dynamik*, A. Clebsch, ed., Berlin, 1866; *Gesammelte Werke*, Supplementband, Berlin, 1884; english translation by K. Balagangadharan, *Jacobi’s Lectures on Dynamics: Second Edition*, Biswarup Banerjee, ed., New Delhi, Hindustan Book Agency, 2009.

J.L. Lagrange [1] Sur l’intégration de quelques équations différentielles dont les indéterminées sont séparées, mais dont chaque membre en particulier n’est point intégrable, *Miscellanea Taurinensia*, IV, 1766-69 (1773), pp. 98–125; *Œuvres II,
J.L. Lagrange [2] Recherches sur le mouvement d’un corps qui est attiré vers deux centres fixes. Premier mémoire, où l’on suppose que l’attraction est en raison inverse des carrés des distances. *Miscellanea Taurinensia*, IV, 1766-69 (1773), pp. 188–215; *Œuvres* II, pp. 67–94

J.L. Lagrange [3] Sur le problème de la détermination des orbites des Comètes d’après trois observations. Premier mémoire. *Nouveaux mémoires de l’Académie royale des sciences et belles-lettres*, 1778 (1780), pp. 111–123; *Œuvres* IV, pp. 439–451

J.L. Lagrange [4] Sur une manière particulière d’exprimer le temps dans les sections coniques, décrites par des forces tendantes au foyer et réciproquement proportionnelles aux carrés des distances, *Nouveaux mémoires de l’Académie royale des sciences et belles-lettres*, 1778 (1780), pp. 181–202; *Œuvres* IV, pp. 559–582

J.L. Lagrange [5] Théorie des variations séculaires des éléments des Planètes, première partie, *Nouveaux mémoires de l’Académie royale des sciences et belles-lettres*, 1781 (1783), pp. 199–276; *Œuvres* V, pp. 125–207

J.L. Lagrange [6] *Mécanique analytique* (tome second), Veuve Courcier, Paris, 1815; *Œuvres* XII

Ph. de La Hire [1], *Sectiones conicae in novem libros distributa*, Michallet, Paris, 1685

J.H. Lambert [1] *Insigniores Orbitae Cometarum Proprietates*, Augustae Vindelicolorum, Augsburg, 1761; German translation in: *Abhandlungen zur Bahnbestimmung der Cometen*, Deutsch herausgegeben und mit Anmerkungen versehen von J. Bauschinger, Ostwald’s Klassiker der exakten Wissenschaften, Verlag von Willen Engelmann, no 133, 1902

P.S. Laplace [1] *Traité de Mécanique Céleste*, livre II, 1798; *Œuvres*, tome 1

A.M. Legendre [1] *Nouvelles méthodes pour la détermination de l’orbite des comètes*, second supplément, Huzard-Courcier, Paris, 1820

A.L. Lexell [1] *Disquisitio de theoremate quodam singulari celeb. Lamberti, pro aestimandis temporibus, quibus arcus sectionum conicarum describuntur a corporibus, quae ad alterutrum focum attrahuntur viribus reciproce proportionalibus quadratis distantiarum* (1784), *Histoire de l’académie impériale des sciences*, pp. 233–236, *Nova acta Academiae scientiarum imperialis*, 1, pp. 140–183

B. Linet, P. Teyssandier [1] Time transfer functions in Schwarzschild-like metrics in the weak-field limit: A unified description of Shapiro and lensing effects, *Phys. Rev. D*, 93 (2016), 044028

C. Marchal [1] How the method of minimization of action avoids singularities, *Celestial Mech. Dynam. Astronom.*, 83 (2002), pp. 325–353

W. Olbers [1] *Abhandlung über die leichteste und bequemste Methode die Bahn*
J.V. Poncelet [1] Traité des propriétés projectives des figures; ouvrage utile ceux qui s’occupent des applications de la géométrie descriptive et d’opérations géométriques sur le terrain, Bachelier, Paris, 1822

E.J. Routh [1] A treatise on dynamics of a particle, Cambridge U. Press 1898; Dover, 1960

J.M. Souriau [1] Géométrie globale du problème deux corps, Proceedings of the IUTAM–ISIMM, Symposium on Modern Developments in Analytical Mechanics, Academy of Sciences of Turin, Turin (1983), pp. 369–418

J.J. Sylvester [1] On Lambert’s Theorem for Elliptic Motion, Monthly Notices of the Royal Astronomical Society, 26 (1866), pp. 27–29; Math. Papers, 2, Chelsea, pp. 496–497

J.J. Sylvester [2] Astronomical prolusions: Commencing with an instantaneous proof of Lambert’s and Euler’s Theorems, and modulating through a construction of the orbit of a heavenly body from two heliocentric distances, the subtended chord, and the periodic time, and the focal property of cartesian ovals, into a discussion of motion in a circle and its relation to planetary motion, Philosophical Magazine, 31 (1866), pp. 52–76; Math. Papers, 2, Chelsea, pp. 519–541

P.G. Tait [1] Note on the action in an elliptic orbit, Quarterly Journal of Pure and Applied Mathematics, 7 (1866), p. 45

P.G. Tait [2] On the application of Hamilton’s characteristic function to special cases of constraint, Trans. of the Royal Soc. of Edinburgh, 24 (1867), pp. 147–166; Scientific Papers 1, Cambridge, pp. 54–73

O. Terquem [1] Relations d’identité et équations fondamentales relatives aux courbes du second degré, Nouvelles annales de mathématiques, (1) 2 (1843), pp. 532–538

I. Thomas [1] Greek mathematical works, Thales to Euclid, Loeb classical library, London, 1939, 1980, 1991

A. Wintner [1] The Analytical Foundations of Celestial Mechanics, Princeton Math. Series 5, Princeton University Press, Princeton, 1941