Non-linear PI regulators in control problems for holonomic mechanical systems

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ABSTRACT
In this paper we propose a solution to the regulation problem of holonomic mechanical systems without velocity measurements by constructing nonlinear PI controllers. An approach based on the Lyapunov functional method is used to analyse the stability of the closed-loop system which has the form of Volterra integro-differential equation. The paper presents the quasi-invariance principle for Volterra nonlinear integral differential equations. Furthermore, we present sufficient conditions that one can use to conclude the asymptotic stability of the closed-loop system via using a Lyapunov functional with semi-definite time derivative. The simulation results of the two-link robot manipulator demonstrate the effectiveness of the proposed PI controller.

1. Introduction
Several control strategies that solve the regulation problem for robot manipulators using only position measurements have been presented in the literature, see for instance Berghuis and Nijmeijer (1993a, 1993b), Burkov (1995, 1998, 2009), Canudas de Wit and Fixot (1991), Loria, Kelly, Ortega, and Santibanez (1997), Nicosia and Tomei (1990) and Siciliano and Villani (1996).

One of the approaches to this problem consists in using a state-feedback controller with an observer which reconstructs the lacking velocity signals, see for instance Canudas de Wit and Fixot (1991) and Nicosia and Tomei (1990). As mentioned in Berghuis and Nijmeijer (1993a) the main drawback of such approach is the local stability conditions for the closed-loop system. A different approach to the regulation problem for robot manipulators without velocity measurements consists in using a dynamic position feedback controller via a first-order linear compensator by invoking LaSalle's invariance principle, see for instance Berghuis and Nijmeijer (1993a, 1993b), Burkov (1995, 1998, 2009), Loria et al. (1997) and Siciliano and Villani (1996). However, these results are applicable only for non-dissipative mechanical systems with the potential energy which doesn’t depend on time.

In fact the use of dynamic compensators is an adding integral terms in the regulator. Using the representation of integral terms in the structure of control signals as regulators with unlimited after-effect (Anan’evskii & Kolmanovskii, 1989; Andreev, 2009), the motion of mechanical systems with these types of regulators can be modeled by Volterra integro-differential equations (Volterra, 1959). Such equations arise in the mathematical modelling of viscoelastic materials (Dafermos & Nohel 1981; Mac 1977; Sergeev 2007b; Volterra, 1959), population dynamics (Britton 1990; Volterra 1959), age-dependent epidemic of a disease (El-Doma, 1987), nuclear reactor dynamics (Kappel & Di, 1972). The study of the qualitative theory of Volterra integro-differential equations including the stability problem attracts great attention of numerous researches, see for instance Burton (1983), Grimmer and Seifert (1975) and Sergeev (2007a, 2017) and their bibliographies. In the paper (Anan’evskii and Kolmanovskii 1989) the stabilizability of the rigid body systems was established by using PID controllers and Lyapunov functional method. Nevertheless, the stability analysis for Volterra nonlinear integro-differential equations still remains an open problem.

In this paper, we show that Volterra integro-differential equations can serve as a powerful tool for mathematical modelling the controlled mechanical systems without velocity measurement. We consider a large class of holonomic mechanical systems under the action of potential, dissipative and gyroscopic forces assuming that the potential energy can depend on time. We present a nonlinear PI controller which solves the regulation problem for such systems.

There are two contributions of this paper. In the first part of the paper new theorems of LaSalle’s type on the limit behaviour of the solutions, on asymptotic stability...
of the zero solution are proved for Volterra integro-differential equations. The second part of the paper contains the results of solving the regulation problem for a holonomic mechanical system based on the construction of nonlinear PI regulators.

2. The quasi-invariance principle for Volterra integro-differential equation

Consider a Volterra nonlinear integro-differential equation in the form

$$\dot{x} = f(t, x(t)) + \int_{0}^{t} g(t - s, x(t), x(s)) \, ds \quad (1)$$

where $x \in \mathbb{R}^{n}$ is the phase vector; $f$ and $g$ are the functions defined and continuous in the domains $R \times D$ ($D \subset \mathbb{R}^{n}$) and $R_{+} \times D \times D$ respectively.

**Assumption 2.1:** Assume that the function $f = f(t, x)$ satisfies the Lipschitz condition, i.e. for each compact set $K_{1} \subset D$ there exists $L_{1} = L_{1}(K_{1}) = \text{const} > 0$ such that the following inequality holds

$$\|f(t, x^{(2)}) - f(t, x^{(1)})\| \leq L_{1}\|x^{(2)} - x^{(1)}\| \quad (2)$$

**Assumption 2.2:** Assume that the function $g(t, x, y)$ satisfies the following conditions: for each compact set $K_{2} \subset D \times D$ the following inequalities hold

$$\|g(\tau, x, y)\| \leq g_{1}(\tau, K_{2}) \quad \forall (\tau, x, y) \in R_{+} \times K_{2},$$

$$\times \int_{0}^{\infty} g_{1}(\tau, K_{2})\, d\tau < +\infty \quad (3)$$

$$\|g(\tau, x^{(2)}, y^{(2)}) - g(\tau, x^{(1)}, y^{(1)})\| \leq L_{21}\|x^{(2)} - x^{(1)}\| + L_{22}\|y^{(2)} - y^{(1)}\| \quad (4)$$

where $L_{2j} = L_{2j}(K_{2})$ ($j = 1, 2$).

**Proposition 2.1:** Let Assumptions 2.1 and 2.2 hold. Then, for each initial point $x_{0} \in D$ there exists a unique solution $x = x(t, x_{0})$ ($x(0, x_{0}) = x_{0}$) of the Equation (1) defined on the maximal interval $[0, \alpha]$.

Consider a family of translates of the function $f = f(t, x)$ in some functional space $F = \{f : R \times D \to \mathbb{R}^{n}\}$ as

$$\{F^{t}(t, x) = f(t + \tau, x), \tau \in R_{+}\} \quad (5)$$

Using the precompactness property of the family (5), we can find the set of limiting functions (Artstein, 1977)

$$f^{*}(t, x) = \frac{d}{dt} \lim_{t_{k} \to +\infty} \int_{0}^{t} f(t_{k} + s, x) \, ds \quad (6)$$

For (1) define the family of limiting integro-differential equations as

$$\dot{x}(t) = f^{*}(t, x(t)) + \int_{-\infty}^{t} g(t - s, x(t), x(s)) \, ds \quad (7)$$

**Remark 2.1:** Equation (7) is the functional-differential equation with infinite delay. The domain of its definition is $R \times C$.

**Definition 2.1:** Let $x = x(t, x_{0})$ be some solution of (1) bounded by some compact set $K \subset D$ for all $t \geq 0$. Define a positive limit point $p \in D$ and the corresponding positive limit set $\omega^{+}$ as follows (Artstein, 1977)

$$p = \lim_{t_{k} \to +\infty} x(t_{k}, x_{0}),$$

$$\omega^{+} = \{p \in D : x(t_{k}, x_{0}) \to p, t_{k} \to +\infty\}$$

The following property of the set $\omega^{+}$ holds.

**Theorem 2.1:** Let $x = x(t, x_{0})$ be some solution of (1) bounded by some compact set $K \subset D$ for all $t \geq 0$. Then, for each limit point $p \in \omega^{+}$ there exists a solution $x = \varphi(t)$ ($t \in R$) of some Equation (7) such that $\varphi(0) = p$ and $\{\varphi(t), t \in R\} \subset \omega^{+}$.

**Proof:** Let $p \in \omega^{+}$ be a limit point defined by the sequence $t_{k} \to +\infty$, i.e.

$$\lim_{t_{k} \to +\infty} x(t_{k}) = p$$

From the sequence $\{t_{k}\}$ choose the subsequence $\{t_{k_{j}}\}$ such that the sequence $\{f(t_{k_{j}}, x(t_{k_{j}}))\}$ converges to the function $f^{*} : R \times D \to \mathbb{R}^{n}$ in the space $F$. For simplicity, denote by $\{t_{k}\}$ the sequence $\{t_{k_{j}}\}$.

The sequence of the functions $x^{(k)}(t) = x(t + t_{k} + 0)$ is uniformly bounded and equicontinuous on $t \in [-T_{1}; T_{1}]$, where $T_{1} = t_{1} - t_{0}$. Therefore, there exist the subsequence $\{x^{(k)}(t)\}$ and the function $x = x^{(k)}$, $-T_{1} \leq t \leq T_{1}$ such that $x^{(k)}(t) \to x^{(\infty)}(t)$ for $t \in [-T_{1}; T_{1}]$. From the sequence $\{x^{(k)}(t)\}$ choose the subsequence $\{x^{(k_{m})}(t)\}$ which converges uniformly on $t \in [-T_{2}; T_{2}]$, $T_{2} = t_{2} - t_{0}$. Continuing that process further one can find the subsequences $T_{m}^{\infty} \to +\infty$, $T_{m}^{\infty} \to +\infty$ and the function $x = \varphi(t)$ such that the sequence $x^{(m)}(t) = x(t + T_{m}^{\infty})$ converges to $x = \varphi(t)$ uniformly on $t \in [-T_{m}^{\infty}; T_{m}^{\infty}]$. Since the function $x(t)$ ($x(0) = x_{0}$) is a solution of (1), one can get...
Consider a family of translates \( W_1(t,x) = W_1(t + t,x), \quad t \in \mathbb{R}^+. \) We introduce the limiting functions of \( W_1 \) as follows

\[
W_1^*(t,x) = \frac{d}{dt} \lim_{t_k \to +\infty} \int_0^t W_1(t_k + s,x) \, ds \tag{12}
\]

The function \( W_1^*(t,x) \) is defined in the domain \( \mathbb{R} \times D \) for almost all \( t \in \mathbb{R} \).

**Theorem 2.2:** Assume that one can find a Lyapunov functional candidate \( V = V(t,x) \) bounded for all continuous function \( x : \mathbb{R}^+ \to D \) whose upper right-hand derivative satisfies the inequality (10). Then, for each bounded by some compact set \( K \subset D \) solution \( x = x(t) \) of Equation (1) the set \( \omega^+ \) consists of the solutions of Equation (1) which satisfy the following equalities

\[
W_1(t,x(t)) = 0, \quad W_2(t-s,x(t),x(s)) = 0, \quad s \leq t \tag{13}
\]

**Proof:** The function \( V(t,x(t)) \) satisfies the inequality (10) and is monotonically decreasing along the solution \( x = x(t,x_0) \) due to the condition (10). Therefore, the following holds

\[
\lim_{t \to +\infty} V(t,x(t)) = V^* \geq 0 \tag{14}
\]

From the inequality (9) for all \( T > 0 \) one can find that

\[
V(t+T,x(t+T)) - V(t,x(t)) \leq - \int_{t-T}^{t+T} W_1(t,x(t)) \, dt \tag{15}
\]

Let \( \omega^+ \) be a positive limit set and \( p \in \omega^+ \) be a positive limit point defined by the sequence \( t_k \to +\infty, x(t_k,x_0) \to p \). As in the proof of Theorem 2.1 one can find the solution \( x = \phi(t) \) of the Equation (7) which passes through the point \( p, \phi(0) = p \). Thereafter, for the sequences \( t_m^* \to +\infty \) and \( T_m^* \to +\infty \) constructed in Theorem 2.1 we have

\[
\begin{align*}
V(t,x^{(m)}(t)) &|_{t = t_m^*+T_m^*} - V(t,x^{(m)}(t)) |_{t = t_m^*-T_m^*} \\
& \leq - \int_{t_m^*-T_m^*}^{t_m^*+T_m^*} W_1(t,x^{(m)}(t)) \, dt \\
& \quad - \int_{t_m^*-T_m^*}^{t_m^*+T_m^*} \left( \int_0^r W_2(t-s,x^{(m)}(t), \phi(s)) \, ds \right) \, dt
\end{align*}
\]

Hence, passing to the limit in (16) for \( m \to +\infty \) and using (14) we obtain

\[
W_1(t,\phi(t)) = 0, \quad \int_{-\infty}^t W_2(t-s,\phi(t),\phi(s)) \, ds = 0 \tag{16}
\]

for all \( t \in \mathbb{R} \) and correspondingly \( W_2(t-s,\phi(t),\phi(s)) = 0, \quad s \leq t \).

Assume that for Equation (1) one can find a Lyapunov functional candidate as

\[
V(t,x) = V_1(t,x(t)) + \int_0^t V_2(t-s,x(t),x(s)) \, ds \tag{9}
\]

where \( V_1 \) and \( V_2 \) are some nonnegative scalar functions which are defined and continuous in the domains \( \mathbb{R} \times D \) and \( \mathbb{R}^+ \times D \times D \) respectively.

Assume also that the following estimate holds

\[
\dot{V}^*(t,x(t)) \leq - W(t,x(t)), \quad W(t,x) = W_1(t,x(t)) + \int_0^t W_2(t-s,x(t),x(s)) \, ds \tag{10}
\]

where \( W_1(t,x) \) and \( W_2(t,x,y) \) are some non-negative functions defined and continuous in the domains \( \mathbb{R}^+ \times D \) and \( \mathbb{R}^+ \times D \times D \). The functions \( W_1(t,x) \) and \( W_2(t,x,y) \) satisfy in these domains the conditions such as (2), (3) and (4).

Hence in particular, for a continuous function \( x : \mathbb{R} \to K \) (\( K \subset D \) is compact set) there exists an integral

\[
\int_{-\infty}^t W_2(t-s,x(t),x(s)) \, ds \tag{11}
\]

Consider a family of translates \( W^*_1(t,x) = W_1(t + t,x), \quad t \in \mathbb{R}^+ \). We introduce the limiting functions of \( W^*_1 \) as follows

\[
W^*_1(t,x) = \frac{d}{dt} \lim_{t_k \to +\infty} \int_0^t W_1(t_k + s,x) \, ds \tag{12}
\]
Remark 2.2: Theorem 2.2 presents an invariance principle for Volterra integro-differential Equations (1).

Assume that in Equation (1) the following holds $f(0) = 0$, $g(t, 0, 0) = 0$. Therefore, Equation (1) has a zero solution $x(t, 0) = 0$.

Using Theorem 2.2 one can easily obtain the following sufficient conditions for the asymptotic stability of the Equation (1).

Theorem 2.3: Assume that one can find a functional (9) with a function $V_1(t, x) \geq a(\|x\|)$ whose upper right-hand derivative satisfies the inequality (10). Assume also that there are no solutions $x = \phi(t)$ of the Equation (7) satisfying the following equalities

$$W_1(t, \phi(t)) = 0, \quad W_2(t - s, \phi(t), \phi(s)) = 0, \quad s \leq t, \quad \forall t \in \mathbb{R}$$

except for the zero solution $\phi(t) = 0$. Then, the zero solution $x = 0$ of the Equation (1) is asymptotically stable.

3. The regulation problem for a holonomic mechanical system on the base of PI regulator

Consider a controlled mechanical system with $n$ degrees of freedom described by Lagrange equations in the form

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial \dot{q}} + Q(q, \dot{q}) + U$$

where $q$ is the vector of generalized coordinates, $T = \dot{q}'A(q)\dot{q}/2$ is the kinetic energy of the system with inertial matrix $A(q)$, $\dot{q} = dq/dt$, $Q(q, \dot{q})$ is the vector of generalized dissipative and gyroscopic forces, $Q(q, 0) = 0, Q' \dot{q} \leq 0$, $\Pi = \Pi(t, q)$ is the potential energy, $U$ is the generalized control force, $(\cdot)'$ is the transpose operation. Suppose that included in (18) functions are defined and continuous for all $q \in \mathbb{R}^n$. Assume that restrictions on the control input $U$ are not imposed.

Represent the Equations (18) resolved with respect to $\ddot{q}$ in the form

$$\frac{dq}{dt} = \dot{q}$$

$$\frac{d\dot{q}}{dt} = A^{-1}(q) \left( C(q, \dot{q}) \dot{q} + Q(q, \dot{q}) - \frac{\partial \Pi}{\partial \dot{q}} + U \right)$$

The coefficients of the matrix of inertial forces $C = (c_{jk})$ are defined by the following equality

$$c_{jk} = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\partial a_{ik}}{\partial q_j} - \frac{\partial a_{ij}}{\partial q_k} \right) \dot{q}_j \dot{q}_k, \quad j, k = 1, \ldots, n$$

(20)

Consider the stabilization problem of programme position

$$\dot{q} = 0, \quad q = q^{(0)} = \text{const}$$

(21)

Show that the stabilization problem is solved by using a nonlinear PI regulator (see the block diagram in Figure 1) such as

$$U = \left. \frac{\partial \Pi_1(t, q)}{\partial q} \right|_{q=q^{(0)}} - \left. \frac{\partial \Pi_2(t, q)}{\partial q} \right|_{q=q^{(0)}} - \left( \frac{\partial f}{\partial q} \right)' \times \int_0^t P(t - \nu) \left( f(q(t)) - f(q(\nu)) \right) d\nu$$

(22)

Figure 1. Block diagram of nonlinear PI regulator (22).
where \( \Pi_u \in R \times R^n \rightarrow R \) is some continuously differentiable function, \( P : R^+ \rightarrow R^{n \times n} \) is some nonnegative matrix function with a derivative \( \partial P(s)/\partial s \) such that
\[
x^T \frac{\partial P(s)}{\partial s} x \leq -\alpha(s) \| x \|^2 \quad \forall s \geq 0
\] (23)

where \( \alpha(s) > 0 \forall s \geq 0, f : R \rightarrow R^n \) is some differentiable function which has finite number of the prototypes \( f(c) \) in any bounded domain \( \{ q \in R^n : \| q \| \leq \mu = \text{const} \} \). In other words, there exists finite number of solutions of the equation \( f(q) = c \). Let us make the change of variables such as \( x = q - q^{(0)}, y = q \). Then, the Equations (18) with the controller (22) can be written as
\[
\frac{dx(t)}{dt} = y(t)
\]
\[
\frac{dy(t)}{dt} = A_1^{-1}(x(t)) \left( C_1(x(t), y(t))y(t) + Q_1(x(t), y(t)) \right.
\]
\[
\left. - \frac{\partial \Pi_1(t, x(t))}{\partial x} - \frac{\partial \Pi_u(t, x(t))}{\partial x} - \left( \frac{\partial f_1(x(t))}{\partial x} \right)' \right)
\]
\[
\times \int_0^t P(t - \nu)(f_1(x(\nu)) - f_1(x(t))) d\nu \right)
\] (24)

where the subscript ‘1’ denotes the functions which are obtained from the corresponding functions included in (21) and (22) as a result of the aforementioned change of variables.

Consider the Lyapunov functional candidate as
\[
V = \frac{1}{2} y'(t)A_1(x(t), y(t))y(t) + \tilde{\Pi}_1(t, x(t))
\]
\[
+ \frac{1}{2} \int_0^t (\frac{\partial \Pi_1(t, x(t))}{\partial x} - \frac{\partial f_1(x(t))}{\partial x} \right)
\]
\[
\times (f_1(x(t)) - f_1(x(\nu))) d\nu
\]
where \( \tilde{\Pi}_1(t, x) = \Pi_1(t, x) + \Pi_u(t, x) \).

For the time derivative of the functional (25) due to the Equation (24) we obtain
\[
\dot{V} = \frac{1}{2} \int_0^t (f_1(x(t)) - f_1(x(\nu))) \frac{\partial P(t - \nu)}{\partial t} d\nu \leq 0
\] (25)

Accordingly to Theorems 2.2 and 2.3 one can simply obtain the following result.

**Theorem 3.1:** Let the controller (22) be such that the function \( \tilde{\Pi}_1(t, x) \) is definitely positive and \( \partial \tilde{\Pi}_1/\partial t \leq 0 \). Let also there exist \( \Delta = \text{const} \geq 0 \) such that \( \forall \varepsilon = \text{const} > 0 : \varepsilon < \Delta \) and \( \exists \delta = \delta(\varepsilon, \Delta) > 0 \) in the domain \( \varepsilon < \| x \| < \Delta \) the following inequality hold
\[
\left| \frac{\partial \tilde{\Pi}_1}{\partial x} \right| > \delta(\varepsilon, \Delta)
\]

Then, the controller (22) stabilizes the programme position (21).

**Remark 3.1:** If in Theorem 3.1 \( \Delta = +\infty \), then the controller (22) globally stabilizes the programme position (21).

**Remark 3.2:** Theorem 3.1 presents the basis for constructing nonlinear PI regulators for solution to the regulation problem for a large class of holonomic mechanical systems.

### 4. Example

Consider the regulation problem for a double-link planar robot manipulator (Fantoni & Lozano, 2002). The manipulator (see Figure 2) consists of two absolutely rigid links \( G_1, G_2 \) and two cylinder joints \( O_1, O_2 \). Both the links can move in the vertical plane only. The mass centre \( C_1 \) of the link \( G_1 \) is situated on the ray \( O_1 O_2 \). The position of the mass center \( C_2 \) of the link \( G_2 \) doesn’t coincide with the position of the joint \( O_2 \).

Let us introduce the following notation. \( q_1 \) is an angle between the horizon line and the ray \( O_1 C_1; q_2 \) is an angle between the lines \( O_1 O_2 \) and \( O_2 C_2 \); \( l_i \) is a length of the segment \( O_i C_i; l \) is a length of the segment \( O_1 O_2; m_i \) is a mass of the link \( G_i; l_i \) is an inertia moment of the link \( G_i \) about the axis of the joint \( O_i; g \) is the acceleration due to gravity.

![Figure 2. The two-link planar robot manipulator.](image-url)
The gravitational moments in the \( \mu \) where 
\[
M_1 = -\mu_1 \cos q_1 - \mu_2 \cos(q_1 + q_2), \\
M_2 = -\mu_2 \cos(q_1 + q_2)
\]

where \( \mu_1 = (m_1 l_1 + m_2 l_2) g \) and \( \mu_2 = m_2 l_2 g \).

Choose the reals \( \nu_1, p_i \) and \( \alpha_i \) \((i = 1, 2)\) such that the following inequalities hold
\[
\nu_1 > \mu_1 + \mu_2, \quad (\nu_1 - \mu_1 - \mu_2)(\nu_2 - \mu_2) > \mu_2^2, \\
p_i > 0, \quad \alpha_i > 0 \quad (i = 1, 2)
\]

Then, using Theorem 3.1, one can conclude that the control moments
\[
\begin{align*}
U_1 &= \mu_1 \cos q_1^{(0)} \cos(q_1(t) - q_1^{(0)}) \\
&\quad + \mu_2 \cos(q_1^{(0)} + q_2^{(0)}) \cos(q_1(t) + q_2(t) - q_1^{(0)} - q_2^{(0)}) \\
&\quad - \nu_1 \text{sat}(q_1(t) - q_1^{(0)}) - p_1 \cos q_1(t) \\
&\quad \times \int_0^t e^{-\alpha_1(t-s)} (\sin q_1(t) - \sin q_1(s)) \, ds \\
U_2 &= \mu_2 \cos(q_1^{(0)} + q_2^{(0)}) \cos(q_1(t) + q_2(t) - q_1^{(0)} - q_2^{(0)}) \\
&\quad - \nu_2 \text{sat}(q_2(t) - q_2^{(0)}) \\
&\quad - p_2 \cos q_2(t) \int_0^t e^{-\alpha_2(t-s)} (\sin q_2(t) - \sin q_2(s)) \, ds
\end{align*}
\]
globally stabilize the programme position
\[
\dot{q}_1 = \dot{q}_2 = 0, \quad q_1 = q_1^{(0)} = \text{const}, \quad q_2 = q_2^{(0)} = \text{const}
\]

In order to illustrate the performance of the controller (27), we performed simulations on Scilab 5.5.2.

The parameters of the manipulator are chosen as
\[
m_1 = 2 \text{ kg}, \quad m_2 = 0.5 \text{ kg}, \\
l = 0.4 \text{ m}, \quad l_1 = 0.25 \text{ m}, \quad l_2 = 0.3 \text{ m} \\
l_1 = 0.167 \text{ kg} \cdot \text{m}^2, \quad l_2 = 0.083 \text{ kg} \cdot \text{m}^2
\]

We have set the programme positions to \( q_1^{(0)} = \pi/2 \text{ rad} \) and \( q_2^{(0)} = -\pi/4 \text{ rad} \).

The parameters of the controller (27) were fixed in
\[
\nu_1 = 7.4, \quad \nu_2 = 1.5, \quad p_1 = p_2 = 10, \quad \alpha_1 = \alpha_2 = 3.
\]
The time interval was chosen as 20 sec.

The angular positions \( q_1(t), q_2(t) \) and programme position coordinates \( q_1^{(0)}, q_2^{(0)} \) are depicted in Figures 3 and 4 for an initial position \( q_1(0) = q_1^{(0)} + 2 \text{ rad}, q_2(0) = q_2^{(0)} - 3 \text{ rad}, q_1(0) = 0.5 \text{ rad/s} \) and \( q_2(0) = -0.5 \text{ rad/s} \).

**Figure 3.** First link angular position \( q_1(t) \) and programme position \( q_1^{(0)} \).
These graphs show that the controller (27) solves the global regulation problem for the manipulator. The link angular positions $q_1(t)$ and $q_2(t)$ converge to the corresponding reference positions.

The terms containing $\mu_1$ and $\mu_2$ in the controller (27) compensate the action of the gravitational moments in the programme state (28) of the manipulator. From a practical viewpoint, an advantage of the saturated controller (27) is that it satisfies the input constraints. It should be noted that our controller (27) is given in explicit form and has been designed more easily and simply than the bounded control schemes via dynamic output feedback proposed by Burkov (1995, 2009) and Loria et al. (1997). Moreover, the performance of the controller was illustrated via simulation on a two-link planar robot manipulator.

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**5. Conclusion**

The paper presents the solution to the regulation problem for holonomic mechanical systems without velocity measurements. Nonlinear PI controller has been proposed which solves this problem. The closed-loop system has been represented in the form of a nonlinear Volterra integro-differential equation. An asymptotic stability property of Volterra equation has been studied by constructing a Lyapunov functional with a semi-definite time derivative. The important advantages of the proposed controller are its explicit form and simple design. Through the new design, the proposed control scheme does not require the on-line integration of the differential equations. Moreover, the proposed control scheme allows the use of uncontrolled forces to solve the regulation problem. The performance of the controller was illustrated via simulation on a two-link planar robot manipulator.

**Nomenclature/Notation**

Let $\mathbb{R}^n$ be $n$-dimensional linear real space with the norm $\|x\|$. The symbol $R^+$ denotes the positive real semi-axis $R^+ = [0; +\infty)$. The symbol $R$ denotes the real axis $R = (-\infty; +\infty)$. $C$ is Banach space of continuous functions $\varphi : R \to \mathbb{R}$ with the norm $||\varphi|| = \sup_{s \in [a,b]} \|\varphi(s)\|$, $a, b \in \mathbb{R}$ and $a < b$. Denote by $a : R^+ \to R^+$ a function of Hahn type, i.e. $a(0) = 0$ and $a$ is continuous and strictly monotonically increasing. The symbol $E$ denotes the identity matrix $E \in \mathbb{R}^{n \times n}$. Denote by $(f * g)(t)$ a convolution of two
functions $f(t)$ and $g(t)$ defined for $t \geq 0$, i.e. $(f \ast g)(t) = \int_0^t f(\tau)g(t - \tau)\,d\tau$. Denote by $\text{sat}: \mathbb{R} \to \mathbb{R}$ the function defined as follows

$$\text{sat}(x) = \begin{cases} 
1, & \text{if } x \geq \pi/2 \\
\sin x, & \text{if } |x| < \pi/2 \\
-1, & \text{if } x \leq -\pi/2
\end{cases}$$

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