Determinism, independence and objectivity are inconsistent

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Hidden-variable models aim to reproduce the results of quantum theory and satisfy our classical intuition. Their refutation is based on deriving predictions that are different from those of quantum mechanics. Here instead we study the mutual consistency of apparently reasonable classical assumptions. We introduce a version of the delayed-choice experiment which combines determinism, independence of hidden variables on the conducted experiments, and wave-particle objectivity (an assertion that quantum systems at any moment are either particles or waves, but not both). These three ideas are incompatible with any theory, not only quantum mechanics.

Introduction. Most of the quantum formalism was in place by 1932 [1]. Since then quantum theory has been spectacularly successful across all the scales and systems it has been applied to. Yet many of its results contradict both common sense and classical physical intuition. Wave-particle duality, superposition, and entanglement are among these counterintuitive features [2,3] and the “strictly instrumentalist” [4] core of quantum theory abandons many familiar traits of classical physics. Hence the profound differences of opinion on the meaning of quantum theory, and the desire to explain, or even to remove its puzzling properties [2,5].

Hidden-variable (HV) theories endeavour to give a satisfactory representation of our intuition while reproducing the experimental predictions of quantum theory [2,6]. Imposing classical concepts (determinism, versions of locality, etc) on HV models constrains the resulting probability distributions. This may lead to “paradoxes”, i.e., an incompatibility of the allegedly reasonable assumptions with the predictions of quantum theory.

Quantum technologies [2, 3] enable the realization of now classic gedankenexperiments and the development of new tests to confront the predictions of HV theories with those of quantum mechanics. When the latter are experimentally confirmed, HV models fail the crucial test of adequacy and, unless some loophole for the experiment is found [9], should either be abandoned or amended to include deep, possibly unacceptable [3], conspiratorial correlations. The loopholes, in turn, may be countered by more sophisticated set-ups [10].

An implicit premise of these debates is that the classically reasonable assumptions form a world view that may be experimentally inadequate but nevertheless consistent. We question this tacit assumption and investigate the mutual consistency of classical requirements. Specifically, in the case of wave-particle duality and delayed-choice experiments [11,12] we inquire if it is possible to find any probability distribution that satisfies all the classical constraints. Here we answer this question in the negative: determinism, independence, and wave-particle objectivity are incompatible.

Setting. Wave-particle duality and the opposing idea of wave-particle objectivity are best illustrated by the classic Wheeler delayed-choice experiment (WDC) [11], Fig. 1a-b.

We briefly introduce it and then proceed to our generalized model.

Classical concepts — “wave” or “particle” — represent only one aspect of quantum objects. A single-photon interference (a definite wave-like behaviour), is produced by particle-like single-photon detections [11,13]. Hence we adapt operational definitions of “wave” and “particle” counting statistics as dependence (independence) on the phase shift in one of the arms of the Mach-Zehnder interferometer (MZI). These data are obtained when the MZI is closed (open), providing us with the operational definitions of the wave statistics

\[ e_w = \left( \cos^2 \frac{\varphi}{2}, \sin^2 \frac{\varphi}{2} \right), \] (1)

and, respectively, particle statistics

\[ e_p = \left( \frac{1}{2}, \frac{1}{2} \right), \] (2)

We assume that the beam-splitters are balanced (50/50) and polarization-insensitive.

The complementarity [2,3,14] of setups [13] of the MZI needed to observe the particle and the wave behaviours obscures a simultaneous presence of both properties and allows one to entertain an “objective” view [12,15] that at any moment of time a photon is either a particle or a wave. Randomly choosing whether or not to insert the second beamsplitter after the photon enters the interferometer (Fig. 1b) prevents a possible causal link between the experimental setup and the photon’s behaviour: the photon should not “know” beforehand if it has to behave like a particle or like a wave.

The delayed-choice experiment with quantum control (Fig. 1b) allows to first detect the photon and only later to find out what type of test was performed [12]. A variable bias \( \alpha \) is necessary for the theoretical discussion and allows the observation of a “morphing” of particle-like into wave-like statistics and vice-versa [13,15]. The quantum-controlled delayed-choice experiment has been recently implemented in several different systems [16,17].

In the quantum-mechanical description of the experiment, the joint state of the photon \( A \) and ancilla \( B \) just before the measurements is:

\[ |\psi\rangle = \cos \alpha |p\rangle |0\rangle + \sin \alpha |w\rangle |1\rangle, \] (3)
FIG. 1: The evolution of the delayed-choice experiment. (a) In Wheeler’s classic experiment, the second beam-splitter is inserted or removed after the photon is inside the interferometer. The detectors observe either an interference pattern depending on the phase ϕ (wave behaviour), or a flat (constant) distribution of hits (particle behaviour) [11]. A quantum random number generator (QRNG) determines whether BS₂ is inserted or not. (b) Quantum network. The beamsplitter is equivalent to a Hadamard gate [7]. The QRNG is an auxiliary quantum system initially prepared in the equal superposition state | + ⟩ = 1/√2 (|0⟩ + |1⟩) and then measured. In the delayed-choice experiment with a quantum control the Hadamard gate is controlled by the ancilla prepared in the state $e^{iϕ/2}$ and can be measured after the photon is detected by $D_b$ [12].

where the wavefunctions $| p ⟩ = e^{-iϕ/2}(|0⟩ + e^{iϕ}|1⟩)$ and $| w ⟩ = e^{iϕ/2}(\cos (ϕ/2)|0⟩ - i \sin (ϕ/2)|1⟩)$ result in particle and wave statistics, respectively [12, 13, 17]. We represent the counting statistics as a vector of relative frequencies and arrange the entries alphanumerically, $ab = 00, 01, 10, 11$. With this notation, the statistics predicted by quantum theory is:

$$q(a, b) = \left( \frac{1}{2} \cos^2 α, \sin^2 α \cos^2 \frac{ϕ}{2}, \frac{1}{2} \cos^2 α, \sin^2 α \sin^2 \frac{ϕ}{2} \right).$$

We now introduce an abstract setting (see Fig. 2) which separates the classical assumptions (leading to the delayed-choice “paradoxes”) from quantum mechanics. As before, the type of measurement on the system $A$ is determined by the setting of the ancilla $B$. A hidden variable theory assumes that the state of $A$ and $B$ is fully determined by $Λ$.

FIG. 2: Abstraction of the delayed-choice experiments. Two distinct statistics for the system $A$ are observed depending on the setting of the ancilla $B$. A hidden variable theory assumes that the state of $A$ and $B$ is fully determined by $Λ$.

We now introduce an abstract setting (see Fig. 2) which separates the classical assumptions (leading to the delayed-choice “paradoxes”) from quantum mechanics. As before, the type of measurement on the system $A$ is determined by the setting of the ancilla $B$ which is revealed by the outcome $b = 0, 1$ of the detector $D_b$. Two statistically distinguishable probability distributions are observed for the system $A$:

$$\bar{e}_p(a) ≡ e(a|b = 0) = (e_p, 1 - e_p),$$

$$\bar{e}_w(a) ≡ e(a|b = 1) = (e_w, 1 - e_w),$$

for some numbers $0 \leq e_p, e_w \leq 1$, where $\bar{e}_p(a)$ represents the “p-statistics” (analogously to the particle-like behaviour in the open MZI), and $\bar{e}_w(a)$ represents the “w-statistics” (analogously to the wave-like behaviour in the closed MZI). We still refer to the parameter that controls $B$ as $α$, and represent the statistics of $D_b$ as

$$e(b) = (x, 1 - x),$$

where $0 \leq x(α) \leq 1$. In the following we assume the standard rules for the marginal $p(i) = \sum_j p(i, j)$ and conditional probability distribution $p(i, j) = p(i|j)p(j) = p(j|i)p(i)$ (Bayes’ rule).

Thus, from the two conditional probability distributions [5] and the marginal statistics $e(b)$ we can reconstruct the joint distribution $e(a, b)$:

$$e(a, b) = (xe_p, (1 - x)e_w, x(1 - e_p), (1 - x)(1 - e_w)).$$

Any general probability distribution $e(a, b)$ can be represented by three independent parameters $x$, $e_p$, $e_w$. For the quantum delayed-choice experiment these parameters are defined by Eqs. (1), (3), and (4).

Hidden-variable models. A HV theory is encapsulated in two elements: a conditional probability distribution of the observable quantities given the value of HV $Λ$, $p(a, b, \ldots | Λ)$ and a probability distribution of $Λ$, $p(Λ)$. The observed probabilities are obtained by an appropriate integration or summation.

A HV theory is adequate if it reproduces the experimentally observed statistics. Here we investigate not if a proposed HV theory is adequate (in a world described by the quantum theory), but if any statistics with marginal distributions [5] and [7] can be based on it. Hence we require

$$e(a, b) = p(a, b) = \sum_Λ p(a, b|Λ) p(Λ),$$

without assuming anything about the parameters $x$, $e_p$, and $e_w$ apart from a generic dependence of $x$ on the settings $α$.

HV theories intend to complete or improve quantum mechanics by incorporating classical intuitions, and thus should satisfy additional properties. We consider the reification of the counting statistics of Eq. (5) summarized by:

(i) Objectivity. We objectify the statistics given by $\bar{e}_w$ and $\bar{e}_p$ as reflecting an intrinsic property of the system, like “wave” or “particle” in the WDC experiment, which is unchanged during its lifetime [12, 13]. This property is expressed by a binary function $λ = w, p$ of the HV $Λ$, $λ = λ(Λ)$. This is a property of an individual system, but could be causally influenced by changing the experimental settings. It is revealed in one setting of the apparatus (e.g., in the closed MZI) as

$$p(a|b = 1, λ = w) = \bar{e}_w(a),$$

and in another setting as

$$p(a|b = 0, λ = p) = \bar{e}_p(a).$$

(ii) Determinism: a knowledge of the hidden variables $Λ$ determines the individual outcomes of the detectors. This is a standard feature of HV models [11, 14]. Specifically, for our setup we demand its weak form [6]

$$p(a, b|Λ) = χ_{ab}(Λ),$$
where the indicator function $\chi = 1$, if $\Lambda$ belongs to some pre-determined set $\mathcal{L}$, and $\chi = 0$ otherwise.

(iii) **Independence:** the property of $\lambda$-independence [3, 5] assumes the nature of the system, as determined by the value of a hidden variable, does not depend on the experimental setting. In our context it means that choosing $\alpha$ [in the quantum delayed-choice experiment, the rotation $R(\alpha)$] does not affect $\Lambda$. In line with the standard HV practice we assume the setting $\alpha$ can be selected independently [3, 6].

**Solution to the constraints.** The unknown parameters at our disposal are the eight probabilities $p(a, b, \lambda)$. At this stage we do not enquire about the algorithm that generates them from the underlying HV $\Lambda$. The probabilities $p(a, b, \lambda)$ satisfy the normalization and adequacy constraints, Eq. (9). The adequacy conditions can be written as

$$e(a, b) = p(a, b) = p(a, b, p) + p(a, b, w). \quad (12)$$

In addition, (i) and the standard rules for the conditional probabilities, such as

$$p(a|b, \lambda) = \frac{p(a, b, \lambda)}{p(0, b, \lambda) + p(1, b, \lambda)} \quad (13)$$

imply two additional constraints,

$$p(0, 0, p)(1 - e_p) = p(1, 0, p)e_p, \quad (14)$$

$$p(0, 1, w)(1 - e_w) = p(1, 1, w)e_w. \quad (15)$$

The resulting linear system has a two-parameter family of solutions $p_2(a, b, \lambda)$. However, for all these solutions the resulting statistics is independent of $\lambda$,

$$p_2(a|b = 0, p) = p_2(a|b = 0, w) = \bar{e}_p(a), \quad (16)$$

$$p_2(a|b = 1, w) = p_2(a|b = 1, p) = \bar{e}_w(a), \quad (17)$$

i.e., the statistics of $D_a$ is determined solely by the state of the apparatus. Any such theory reintroduces w-p (wave-particle) duality and therefore defeats its own purpose.

We can construct a nontrivial HV theory using a special solution

$$p_s(b|\lambda) = \delta_{x_0}b_{\delta_0} + \delta_{x_1}b_{\delta_1} \equiv p_s(\lambda|b) \quad (18)$$

which introduces the $b$-$\lambda$ correlation. As a result,

$$p(b = 0, \lambda = w) = p(b = 1, \lambda = p) = 0, \quad (19)$$

implying

$$\sum_a p(a|1, p) = \sum_a p(a, 0, w) = 0. \quad (20)$$

Since the probabilities are positive numbers, the above four probabilities are all zero. The system appears overconstrained, but it still has a unique solution

$$p_s(a, b, \lambda) = e(a, b)p_s(b|\lambda). \quad (21)$$

In particular,

$$p_s(\lambda) = \sum_{a, b} p_s(a, b, \lambda) = (x, 1 - x). \quad (22)$$

**The contradiction.** The determinism criterion (ii) enables us to decompose the set $\mathcal{L} := \{\Lambda\}$ into four disjoint domains $\mathcal{L}_{ab}$, $a, b = 0, 1$, where $\mathcal{L}_{ab}$ is the subset of the HV $\Lambda$ resulting in the outcome $(a, b)$. From Eq. (18) it follows that

$$\lambda = p \quad \forall \Lambda \in \mathcal{L}_{00}, \quad \lambda = w \quad \forall \Lambda \in \mathcal{L}_{11}. \quad (23)$$

Choosing the rotation $\alpha$, and thus influencing the probability $p(b) = (x, 1 - x)$, independently from the preparation of the system and the ancilla [assumption (iii)] leads to a contradiction. Consider, e.g.,

$$p(0, 0) = \sum_{\Lambda \in \mathcal{L}_{00}} p(\Lambda), \quad (24)$$

From Eq. (20) we have

$$p(0, 0) = p(0, 0, p), \quad p(0, 0, w) = 0. \quad (25)$$

and

$$p(0, 0) + p(1, 0) = p(0, 0, p) + p(1, 0, p). \quad (26)$$

The probabilities $p(\Lambda)$ do not depend on $\alpha$ [the assumption of independence, (iii)], but the domains $\mathcal{L}_{ab}$ and thus the resulting probabilities $p(a, b)$ depend on the experimental parameters.

Consider two different values $\alpha_1$ and $\alpha_2$. Take a point $\Lambda$ which (due to the shifting boundaries) belongs to $\mathcal{L}_{00}(\alpha_1)$, but not to $\mathcal{L}_{00}(\alpha_2)$. If $\Lambda$ is now in either $\mathcal{L}_{01}(\alpha_2)$ or $\mathcal{L}_{11}(\alpha_2)$, we immediately have a contradiction, since $\lambda = \lambda(\Lambda)$ and $p(\Lambda)$ is independent of $\alpha$. Namely, from $\Lambda \in \mathcal{L}_{00}(\alpha_1)$ we obtain $\lambda = p$ (from Eq. (23)). On the other hand, if $\Lambda \in \mathcal{L}_{01}(\alpha_2)$ we have $\lambda = w$.

As the only remaining option the boundary can move between $\mathcal{L}_{00}$ and $\mathcal{L}_{10}$ (and, similarly, between $\mathcal{L}_{01}$ and $\mathcal{L}_{11}$), implying

$$\mathcal{L}_{00}(\alpha_1) \cup \mathcal{L}_{10}(\alpha_1) = \mathcal{L}_{00}(\alpha_2) \cup \mathcal{L}_{10}(\alpha_2). \quad (27)$$

Since $p(\Lambda)$ is independent of $\alpha$ we have

$$p(0, 0, p)(\alpha_1) + p(1, 0, p)(\alpha_1) = p(0, 0, p)(\alpha_2) + p(1, 0, p)(\alpha_2) \equiv p(b = 0) = \text{const}, \quad (28)$$

which makes the outcomes $b = 0, 1$ independent of the external parameters and thus contradicts Eq. (6).

**Discussion.** This result implies the following. The assumptions (i)-(iii) are consistent in classical physics, where all systems behave either as particles or as waves. However, if the system (e.g., a photon) demonstrates two types of statistics (particle or wave statistics in the WDC experiments) in two different experimental setups (MZI open or closed), then it is impossible to construct a causal deterministic theory which
promotes the two observed statistics to the status of objective properties of the system.

We stress that these statistics do not need to be derived from the quantum predictions (such as Eq. (4)) – all is required is that different setups yield different statistics. Consequently this result does not depend on comparison of the predictions of a candidate HV theory with quantum mechanics (compare with [18]).

It seems natural to drop the objectivity (i) from the list of classical desiderata. It is known that weak determinism and \( \lambda \)-independence (constraints (ii) and (iii)) are consistent with quantum mechanics [6]. Nevertheless, before an attempt to supplement quantum mechanics can start, one of its counterintuitive features (say, wave-particle duality) must be accepted. Whether this indicates a failure of the HV program or not, it is a matter of opinion. Our work establishes that there are situations where a set of plausible classical ideas is self-contradictory.

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