ARCS ON PUNCTURED DISKS INTERSECTING AT MOST TWICE WITH ENDPOINTS ON THE BOUNDARY

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Abstract

Let $D_n$ be the $n$-punctured disk. We prove that a family of essential simple arcs starting and ending at the boundary and pairwise intersecting at most twice is of size at most $\binom{n+1}{3}$. On the way, we also show that any nontrivial square complex homeomorphic to a disk whose hyperplanes are simple arcs intersecting at most twice must have a corner or a spur.

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1. Introduction

It is a classical theorem that maximal cliques in the curve graph $C^0(S)$ of a hyperbolic closed orientable surface $S$ are of size $\frac{3}{2}|\chi|$ (see [6], for example).

We define the augmented curve graph $C^k(S)$ as follows:

- The vertices of $C^k(S)$ are homotopy classes of essential simple closed curves on $S$.
- Two vertices are connected by an edge if they have representatives intersecting at most $k$ times.

The sizes of cliques of these graphs were explored in [1], [3], [2] and, more recently, in [4]. The combination of these papers show that $N_1(S)$, the maximal clique size in
\( C^1(S) \) satisfies:

\[
O(|\chi|^2) \leq N_1(S) \leq O\left(\frac{|\chi|^3}{(\log |\chi|)^2}\right)
\]

In the case of arcs, the augmented arc graph, \( A^k(S) \) is similarly defined for punctured surfaces. P. Przytycki \[2\] showed that this arc graph has maximal clique size growing asymptotically as \( \Theta(|\chi|^{k+1}) \). In particular, for \( k = 1 \) it has maximal clique size \( 2|\chi|(|\chi| + 1) \). When \( S \) is a punctured sphere, the maximal clique size of \( A^2(S) \) is known to be \( |\chi|(|\chi| + 1)(|\chi| + 2) \), see \[5\]. When the endpoints of arcs are fixed, then we get a subgraph of \( A^1(S) \), whose maximal clique size is calculated in \[2\] to be \( \binom{|\chi|+1}{2} \). The main result of this paper will be to compute the maximal clique size of the corresponding subgraph of \( A^2(S) \).

**Definition 1.1.** Let \( D_n \) be the \( n \)-punctured disk. An arc in \( D_n \) is an embedding \( \alpha : (0, 1) \to D_n \) converging to punctures or the boundary \( \partial D_n = S^1 \) at 0, 1. We often identify arcs with their images in \( D_n \). We write \( \alpha \sim \beta \) if \( \alpha \) is homotopic to \( \beta \) relative to \( \partial D_n \). In this article, we assume that all arcs are essential, meaning that they are not homotopic to a constant arc.

**Definition 1.2.** Two arcs \( \alpha, \beta \) are said to be in minimal position if they minimize \( |\alpha \cap \beta| \) in their respective homotopy classes relative to \( \partial D_n \).

**Definition 1.3.** A family of simple arcs \( A \) on \( D_n \) is called good if:

1. For any arc \( \alpha \in A \), the endpoints of \( \alpha \) lie in \( \partial D_n \).
2. For any arcs \( \alpha, \beta \in A \), \( \alpha \neq \beta \).
3. For any \( \alpha, \beta \in A \), \( |\alpha \cap \beta| \leq 2 \).
4. The arcs in \( A \) are in minimal position.

We are now ready to state the main result of this paper.

**Theorem 1.4.** The maximal cardinality of a good family of arcs on \( D_n \) is \( \binom{n+1}{3} \).

In order to prove this theorem, we will first obtain a result on square complexes, which is of independent interest:

**Definition 1.5.** Let \( X \) be a finite planar square complex homotopy equivalent to a disk.

1. A corner at \( v \) of \( X \) consists of two consecutive boundary edges intersecting at \( v \) which belong to a single square.
2. A spur (at \( v \)) of \( X \) is a 1-cell with a vertex \( v \) which is not contained in any other cell of \( X \).

**Theorem 1.6.** Let \( X \) be a finite planar square complex homotopy equivalent to a disk. Suppose that the hyperplanes in \( X \) are simple arcs, pairwise intersecting at most twice, and that \( X \) is not a single 0-cell. Then \( X \) has a corner or a spur.

If hyperplanes in \( X \) are simple arcs pairwise intersecting at most twice, then we say that they satisfy condition \((*)\).
2. Proof Plan

In this section, we provide an outline for the proof of Theorem 1.4.

Definition 2.1. Let \( \mathcal{A} \) be a good family of arcs. An arc \( \alpha \in \mathcal{A} \) is called isolated if \( \alpha \) is disjoint from any other arc \( \beta \in \mathcal{A} \), and splits \( D_n \) into two components, one of which contains a single puncture.

The following two propositions will be used to prove Theorem 1.4.

Proposition 2.2. In any maximal good family of arcs \( \mathcal{A} \) there exists an isolated arc \( \alpha \).

Proposition 2.3. Assume that \( \alpha_p \in \mathcal{A} \) is an isolated arc, and let \( p \) be the unique puncture in one of the connected components of \( D_n \setminus \alpha_p \). Let \( \mathcal{A}' \) be the set of homotopy classes of essential arcs obtained from \( \mathcal{A} \) by removing \( p \). Then

\[ |\mathcal{A}| - |\mathcal{A}'| \leq \binom{n}{2} \]

Succinctly, the deduction arguments of this paper are as follows:

\[ \text{Theorem 1.6} \Rightarrow \text{Proposition 2.2} \]
\[ \text{Proposition 2.3} + \text{Proposition 2.2} \Rightarrow \text{Theorem 1.4} \]

Proof of Theorem 1.4 from the propositions. The proof follows by induction. The base case, where \( n = 1 \) is obvious. By Proposition 2.2, there exists an isolated arc \( \alpha_p \). After removing \( p \) and identifying arcs that become homotopic to each other, we will get a new family of arcs \( \mathcal{A}' \) on \( D_{n-1} \). By induction, \( |\mathcal{A}'| \leq \binom{n}{3}, \) and thus, by Proposition 2.3

\[ |\mathcal{A}| \leq |\mathcal{A}'| + \binom{n}{2} \leq \binom{n+1}{3}, \]

which is what we want to show.

We now show how Proposition 2.2 follows from Theorem 1.6.

Definition 2.4. Let \( \mathcal{A} \) be a good family of arcs. After a small perturbation, we can assume that there are no triple intersections of arcs in \( \mathcal{A} \). Consider the square complex, \( X(\mathcal{A}) \), dual to \( \mathcal{A} \): each vertex in \( X(\mathcal{A}) \) corresponds to a connected component of \( D_n \setminus \cup \mathcal{A} \), and edges correspond to arc segments in the boundary of adjacent components.

Remark. If \( \mathcal{A} \) is any family of arcs starting and ending at \( \partial D_n \), then \( X(\mathcal{A}) \) is homotopy equivalent to a disk, using the retraction as in Figure 1.

Lemma 2.5.

(1) If \( X(\mathcal{A}) \) has a spur, then \( \mathcal{A} \) has an isolated arc or is not maximal.

(2) If \( X(\mathcal{A}) \) has a corner, then \( \mathcal{A} \) is not maximal.

Proof. (1) A spur corresponds to a region \( A \) that is adjacent to a single other region \( B \) along a single arc \( \alpha \). This means that \( \partial A \cap \partial B = \alpha \), and thus, \( \alpha \) is disjoint from any other arc in \( \mathcal{A} \). Since \( \alpha \) is essential, \( A \) contains at least one puncture. Since \( X(\mathcal{A}) \) has a spur terminating at \( A \), it follows that \( \mathcal{A} \setminus \alpha \) is
Figure 1. A retraction from the disk to $X(\mathcal{A})$ where $\mathcal{A}$ is a good family (in red) on $D_3$ (punctures in black).

Figure 2. The region $A$ cannot contain more than one puncture, or else the arc $\beta$ in the figure could be added to $A$.

Figure 3. The region $A_v$ defined from the corner at $v$.

contained in $B$. If $A$ has at least two punctures, this contradicts maximality of $\mathcal{A}$ (see Figure 2).

(2) Let $e_1, e_2$ be the edges forming a corner at a vertex $v$ in $X(\mathcal{A})$. Let $h_1, h_2$ be the hyperplanes determined by $e_1, e_2$ (see Definition 5.2). Consider the region in $D_n \setminus \cup \mathcal{A}$ corresponding to $v$, call it $A_v$ (see Figure 3).

If $A_v$ does not contain a puncture, then the arcs corresponding to $h_1, h_2$ will not be in minimal position. If $A_v$ does contain a puncture, then the arc $\partial A_v \setminus (\partial D_n)$ is disjoint from all other arcs in $\mathcal{A}$ after a homotopy, and can be added to the family. Thus, $\mathcal{A}$ is not maximal, contradiction.

Proof of Proposition 2.2 from Theorem 1.6. If $\mathcal{A}$ has a spur, then we are done by Lemma 2.5(1). Otherwise, $X(\mathcal{A})$ has a corner, which by Lemma 2.5(2) contradicts maximality.
3. HELPFUL LEMMAS

In this section, we present some helpful lemmas and definitions used throughout this paper to prove Proposition 2.3.

Definition 3.1. Here, we regard the punctures as marked points. Let $\alpha, \beta$ be arcs in $D_n$.

A bigon (resp. half-bigon, strip) between $\alpha$ and $\beta$ is a connected component $D$ of $D_n \setminus (\alpha \cup \beta)$ which is

1. homeomorphic to a disk,
2. $\alpha \cap \bar{D}, \beta \cap \bar{D}$ are nonempty and connected, and $D \cap \partial D_n$ is empty (resp. nonempty and connected, disconnected).

Definition 3.2. We say that two arcs $\alpha, \beta$ on $D_n$ are in minimal position if they minimize $|\alpha \cap \beta|$ in their respective homotopy classes.

Lemma 3.3. (The Bigon Criterion, Lemma 1.7 in [6]) Two arcs are in minimal position if and only if they intersect transversely, and there are no bigons or half-bigons between them.

Corollary 3.4. Let $\alpha$ be an arc with endpoints on $\partial D_n$ and $\beta$ an arc with endpoints outside of $\partial D_n$. If $\alpha$ and $\beta$ intersect once and transversely, then they are in minimal position.

Definition 3.5. We say that an arc $\alpha$ separates two punctures $q, r$ if $D_n \setminus \alpha$ is disconnected and $p$ and $r$ lie on different connected components of $D_n \setminus \alpha$.

Remark 3.6. An arc with endpoints on $\partial D_n$ is essential if and only if it separates some punctures.

4. EXAMPLES

We now give some explicit constructions for maximal good families of arcs, showing the lower bound in Theorem 1.4.

We will think of $D_n$ as $\mathbb{R}^2$, with punctures placed on the $x$-axis at $\frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}$. For simplicity, we will name the punctures by their $x$-coordinates. In this model, the arcs we are interested in are arcs in the plane starting and ending at $\infty$, where $\infty$ can be identified as the specified start and end puncture in the one-point compactification of $\mathbb{R}^2$. For any $0 \leq a < b < c \leq n$, let $\alpha_{abc}$ be the arc given by the graph of the function $(t-a)(t-b)(t-c)$.

Let $\mathcal{A}$ be the set of all arcs $\alpha_{abc}$. Obviously, $|\mathcal{A}| = \binom{n+1}{3}$.
Figure 5. Showing that if \(a = 0, b = a', c = b'\) and \(c' = n\) then \(\alpha_{abc} \not\sim \alpha_{a'b'c'}\).

**Lemma 4.1.** The arcs in \(A\) pairwise intersect at most twice.

**Proof.** If \((a, b, c) \neq (a', b', c')\), then the arc \(\alpha_{abc}\) is given by the graph of \(t^3 + p(t)\) and the arc \(\alpha_{a'b'c'}\) is given by the graph of \(t^3 + q(t)\) where \(p(t) \neq q(t)\) are two quadratic polynomials. The graphs of these polynomials will intersect at the solutions of \(q(t) - p(t) = 0\), of which there are at most 2, as \(q(t) - p(t) = 0\) is a quadratic equation. \(\square\)

**Lemma 4.2.** The arcs in \(A\) are essential and not homotopic to each other.

**Proof.** These arcs are essential by Remark 3.6 since they separate punctures. Namely, if \(0 \leq a < b < c \leq n + 1\), then \(\alpha_{abc}\) separates \(b-\frac{1}{2}\) and \(b+\frac{1}{2}\).

We now show that these arcs are not homotopic to each other.

**Step 1.** Let \(1 \leq i < n - 1\). Let \(\gamma_i\) be the horizontal segment between \(i-\frac{1}{2}\) and \(i+\frac{1}{2}\). Then for any \(\alpha \in A\), \(\alpha\) and \(\gamma_i\) are in minimal position, which follows from Corollary 3.4.

**Step 2.** We now show that if \(|\alpha_{abc} \cap \gamma_i| = |\alpha_{a'b'c'} \cap \gamma_i|\) for all \(i\) then one of the following holds:

1. \((a, b, c) = (a', b', c')\), or
2. \(a = 0, b = a', c = b'\) and \(c' = n\)

Note that for \(1 \leq i \leq n - 1\), \(|\alpha_{abc} \cap \gamma_i| = 1\) if and only if \(i \in \{a, b, c\}\). Thus, if \(0 < a < b < c < n\) then \(\alpha_{abc}\) is uniquely determined by these intersection numbers. Otherwise, it might happen that only two of these intersection numbers are nonzero, in which case, when condition (1) does not hold, then (2) must. Finally, if only one of the intersection numbers \(|\alpha_{abc} \cap \gamma_i|\) is nonzero, by the condition that \(a < b < c\), it follows that \(a = 0, c = n,\) and \(b = i\), uniquely determining \(\alpha_{abc}\).

**Step 3.** By Step 2, to show that \(\alpha_{abc} \not\sim \alpha_{a'b'c'}\), it suffices to check the case where \(a = 0, b = a', c = b'\) and \(c' = n\). In this case, there are two half-bigons and a bigon formed between \(\alpha_{abc}\) and \(\alpha_{a'b'c'}\), and each contains a puncture (see Figure 5), and hence by Lemma 3.3 they are not homotopic to each other. \(\square\)

5. Square Complexes and Theorem 1.6

**Definition 5.1.** Let \(X\) be a square complex. We say that two edges in a square are parallel if they are disjoint. We then define parallelism between any two edges in \(X\)
Figure 6. Folding the red square along $v$.

by taking the transitive closure. In other words, two edges $e, e'$ in $X$ are parallel if there exists a sequence of edges $e = e_1, \ldots, e_n = e'$ such that $e_i, e_{i+1}$ lie in the same square and are parallel for all $0 \leq i < n$.

**Definition 5.2.** A parallelism class of edges in $X$ is called a hyperplane. We say that an edge in $X$ is dual to a hyperplane $h$ if it is part of the parallelism class of $h$.

**Remark.** If $X$ is a planar square complex, we will also treat hyperplanes in $X$ as immersed curves or arcs starting and terminating in $\partial X$. If $h$ is a simple hyperplane, then its corresponding arc will bisect every square that contains an edge in $h$.

**Definition 5.3.** The union of the cells intersecting a hyperplane $h$ is called the carrier of $h$, and will be denoted by $N(h)$. Similarly, for $s \subset h$ a connected segment, we define the carrier $N(s)$ to be the union of all cells intersecting $s$.

**Definition 5.4.** Let $X$ be a square complex as in Theorem 1.6.

1. A square in $X$ is called a boundary square if it contains an edge $e$ which is also contained in $\partial X$.
2. We say that a hyperplane $h$ in $X$ is a boundary hyperplane if it bisects a boundary square.
3. Assume that $X$ is not separated by a single cell. A boundary (hyperplane) segment of a hyperplane $h$ is a maximal hyperplane segment of $h$ starting and ending in square centers with the additional requirement that every dual edge has a vertex in $\partial X$. For examples, see Figure 11.

**Definition 5.5.** Let $X$ be a square complex, let $S$ be a square in $X$, and let $v, x \in S$ be opposite vertices. Let $u, w$ be the other vertices of $S$. We can consider a new square complex, obtained from $X$ by removing $S$ and identifying $vu \sim vw$ and $xu \sim xw$.

We say that the new complex $X'$ is obtained from $X$ by folding $S$ along $v$.

The following technical lemma will be used in the proof of Theorem 1.6. We recommend to skip it on first reading. In the following, we denote by $|X|$ the number of squares in $X$. Recall that a square complex satisfies condition $(\ast)$ if its hyperplanes are simple arcs intersecting at most twice.

**Lemma 5.6.** Let $Y$ be a square complex with the following properties:

1. $Y$ is homeomorphic to a disk,
2. $Y$ satisfies condition $(\ast)$,
3. $Y$ has a hyperplane $h$ such that $N(h)$ consists of two boundary squares as in Figure 7,
4. $Y$ has no corner outside of $N(h)$,
Then there exists a square complex $Y'$ with $|Y'| < |Y|$ such that:

1. $Y'$ is homeomorphic to a disk,
2. $Y'$ satisfies condition $(\ast)$,
3. $Y'$ has a hyperplane, $h'$ such that $N(h')$ consists of two boundary squares as in Figure 7,
4. $Y'$ has no corner outside of $N(h')$,
5. The two hyperplanes intersecting $h'$ are disjoint.

Proof. Let $\alpha, \beta$ be two hyperplanes in $Y$ which form a bigon $B$ disjoint from the boundary squares of $Y$. We define $B$ as the planar square complex composed of all cells intersecting $B$ in $Y$. The square complex $B$ will be called minimal if there are no hyperplane bigons contained in it distinct from $B$. Suppose $B$ is minimal. Let $S_1, S_2$ be the squares of $B$ in which $\alpha$ and $\beta$ intersect. Let $v_1, v_2$ be vertices of $S_1, S_2$ such that neither the vertices $v_i$, nor their opposite vertices in $S_i$ lie in $B$. We define a new square complex, $Y^*$ by folding $S_1, S_2$ along $v_1, v_2$ respectively, as in Figure 8.

Claim. $Y^*$ is a square complex which satisfies conditions (1)-(4) in the statement of Lemma 5.6.

Proof. Conditions (1),(3),(4) are immediate, since the folding occurred on the interior of $Y$. It remains to show condition (2). Note that after folding, all hyperplane intersection numbers remain the same apart from the ones for $\alpha$ and $\beta$. After folding, the remaining hyperplane segments of $\alpha$ and $\beta$ glue together to form two new hyperplanes, $\alpha^*$ and $\beta^*$ as in Figure 8. $\alpha^*$ is simple, because any self-intersection would have to come from a third intersection between $\alpha$ and $\beta$. Additionally, note that $\alpha^*$ and $\beta^*$ are disjoint. We now need to show that $|\alpha^* \cap s^*| \leq 2$ for any hyperplane $s^*$ in $Y^*$ originating from a hyperplane $s$ in $Y$ (see Figure 8). Let $B^*$ be the square complex $B$ after folding $S_1, S_2$. If $s$ does not enter $B$ in $Y$, then, because $\alpha = \alpha^*$ on $Y \setminus B = Y^* \setminus B^*$, it follows that $|\alpha^* \cap s^*| \leq 2$. Since $B$ is minimal, it follows that if $s$ enters $B$ by intersecting $\alpha$, it must exit $B$ by intersecting $\beta$. After passing to $\alpha^*$ and $\beta^*$, we still get that $\alpha^*, \beta^*$ intersect $s^*$ once in $B^*$, and hence, by condition (2) of $Y$, we get that $Y^*$ satisfies condition (2). This proves the claim. \[ \square \]
Figure 9. An example of the above construction. (a) illustrates the complex $X$, (b) illustrates the smallest component $X_1$ of $X \setminus v$, and (c) illustrates the constructed complex $\hat{X}$.

We return to the proof of Lemma 5.6. Let $\alpha_0, \beta_0$ be the two hyperplanes intersecting $h$, and let $B_0$ be the bigon they form, and $B_0$ be the respective square complex. Since $Y$ is finite, there must exist a minimal bigon complex $B$ contained in $B_0$. By the claim, we get $Y^*$, which is a square complex with one fewer bigons contained in $B_0$. We can then continue to remove the bigons in $B_0$ until $B_0$ is minimal. When $B_0$ is minimal, we can remove it, giving our desired complex $Y'$, which will also satisfy condition (5) in addition to (1)-(4).

□

Definition 5.7. Let $X$ be a square complex homeomorphic to a disk, and let $h$ be a simple hyperplane in $X$. Consider the square complex $X'$ obtained from $X$ by removing $N(h)$, and gluing along opposite edges in $\partial N(h)$. We say that $X'$ is obtained from $X$ by collapsing along $h$.

Proof of Theorem 1.6. If $X$ contains a spur, then we are done. Otherwise, we collapse all hyperplanes containing 1-cells not contained in 2-cells to obtain a new complex $X'$. Obviously, a corner of $X'$ corresponds to a corner of $X$, and the hyperplanes of $X'$ satisfy condition $(*).

It remains to prove the theorem under the hypothesis that every 1-cell in $X$ is contained in a 2-cell. We do this by induction on the number of squares in $X$.

Claim. If $X$ contains a separating vertex, then it has a corner

Proof. Let $X_1, \ldots, X_n$ be the connected components of $X \setminus \{v\}$. By induction, there is a corner in every $X_i$. This corner is either a corner of $X$, and we are done, or a corner at $v$. Let $X_1$ be the component with the smallest number of squares, and let $S_v$ be the square in $X_1$ containing $v$. We define $\hat{X}$ as the double of $X_1 \setminus S_v$ along the two remaining edges $e_1, e_2$ from $S_v$. Let $\hat{e}_1, \hat{e}_2$ be the edges corresponding to $e_1, e_2$ in $\hat{X}$. Note that the number of squares in $\hat{X}$, satisfies $|\hat{X}| = 2|X_1| - 2 < |X|$. 

Remark. It cannot happen that a hyperplane in $\hat{X}$ passes through $\hat{e}_1$ and $\hat{e}_2$. This would correspond to a non-simple hyperplane in $X$. In particular, it follows that all hyperplanes in $\hat{X}$ are simple arcs.

We now justify why the hyperplanes in $\hat{X}$ satisfy condition $(*).$ Let $\hat{\alpha}, \hat{\beta}$ be hyperplanes in the double $\hat{X} = (X \setminus S_v) \sqcup (X \setminus S_v)/\sim$.

There are four cases we must check:
\[ \hat{\alpha}, \hat{\beta} \text{ lie in the first copy of } (X \setminus S_v). \text{ In this case, } \hat{X} \text{ satisfies condition } (\ast) \text{ because } X \text{ satisfies it.} \]

\[ \hat{\alpha}, \hat{\beta} \text{ lie in different copies of } (X \setminus S_v). \text{ In this case, } \hat{\alpha} \cap \hat{\beta} = \emptyset, \text{ so condition } (\ast) \text{ is automatically satisfied.} \]

\[ \hat{\alpha} \text{ lies in both copies, } \hat{\beta} \text{ lies in the first. In this case, by the remark, it follows without loss of generality that } \hat{\alpha} \text{ passes through } \hat{e}_1 \text{ and not } \hat{e}_2. \text{ Thus, in the copy of } X \setminus S_v \text{ containing } \hat{\beta}, \hat{\alpha} \text{ coincides with its corresponding hyperplane, } \alpha \text{ in } X \setminus S_v. \text{ Since } \hat{\beta} \text{ coincides with its corresponding hyperplane } \beta \text{ in } X \setminus S_v, \text{ by condition } (\ast) \text{ for } X, \text{ we get that } \hat{\alpha}, \hat{\beta} \text{ satisfy condition } (\ast). \]

\[ \hat{\alpha}, \hat{\beta} \text{ both lie in both copies of } (X \setminus S_v). \text{ This case can only happen if } \hat{\alpha}, \hat{\beta} \text{ are obtained as doubles of hyperplanes } \alpha, \beta \text{ passing through } e_1, e_2 \text{ respectively. If } \hat{\alpha} \neq \hat{\beta}, \text{ then } \alpha, \beta \text{ must cross in } S_v, \text{ meaning that in } (X \setminus S_v), \text{ we have } |\hat{\alpha} \cap \hat{\beta}| \leq 2 - 1 = 1 \text{ (see Figure 10). Thus, doubling these hyperplanes gives } |\hat{\alpha} \cap \hat{\beta}| \leq 1 + 1 = 2. \]

Thus, by induction, \( \hat{X} \) contains a corner, which gives rise to a corner in \( X \). This justifies that there is no separating vertex.

\[ \square \]

If \( X \) contains a separating square \( S \), then let \( X_1 \) be the smallest component of \( X \setminus S \). Let \( \hat{X} \) be the double of \( X_1 \) along \( X_1 \cap S_v \), which might consist of one, two, or three edges. The argument justifying condition \((\ast)\) is as above, and the doubling works in the same way. Hence we can assume that \( X \) does not contain separating squares. Note that likewise, there are no separating edges, as they would imply the existence of a corner or a separating square.

We will attempt to reproduce this doubling trick in the remaining case, where \( X \) does not contain a separating cell. We now assume by contradiction that \( X \) has no corners, and we will show that there exists a smaller complex \( Z \) satisfying the hypothesis with no corners.

Since \( X \) contains no separating cells, \( X \) is homeomorphic to a disk, and the union of the boundary segments is a circle (see Figure 11).

We choose an orientation on this circle which induces an orientation on each boundary segment.

This means that every boundary segment \( s \subset h \) induces a decomposition of \( h \) as an oriented arc \( h = s_- s s_+ \). Note that the orientation induced on \( h \) in the above way depends on the boundary segment \( s \) (see Figure 12).
Let $S$ denote the set of all boundary segments. For every $s \in S$, we denote $s_-, s_+$ as above. $X \setminus N(s_-)$ has two (possibly disconnected) sides. Let $\hat{X}$ be the side not containing $s$. Let $X_{s^-} = \hat{X} \cup N(s_-)$ as in Figure 13. We analogously define $X_{s^+}$.

Let $M = \max_{s \in S} \max\{|X_{s^-}|, |X_{s^+}|\}$. Without loss of generality, assume that $M = |X_{s^-}|$ for some $s \in S$. Set $X' = X_{s^-}$.

**Remark.** By choice of $s$, $|X'| \leq \frac{|X|}{2}$.

**Claim.** If $t$ is a boundary segment in $X'$, then it cannot happen that both $t_-$ and $t_+$ intersect $s_+$.

**Proof.** If $t_+$ and $t_-$ both intersect $s_+$, then by condition (*), it follows that neither of them may intersect $N(s_-)$. In particular, $t_- \cap N(s_-) = \emptyset$, so $X_{s_-} \subset X_{t_-}$ as in Figure 14. This containment is strict, since $N(t_-) \subset X_{t_-} \setminus N(s_-)$. This contradicts maximality of $|X_{s_-}|$. 
Figure 14. If $s$ is chosen such that $X_{s_-}$ is maximal, then it cannot happen that both $t_-$ and $t_+$ intersect $s_+$.

Figure 15. Finding a sequence of consecutive segments labelled $-,0,\ldots,0,+$.

By the claim, we can label each boundary segment $t \neq s_+$ in $X'$ by $+$, $-$, or $0$ depending on whether $t_+$, $t_-$, or neither intersect $s_+$. We order the boundary segments of $X'$ different from $s_+$ using the orientation induced from $\partial X$. Note that the first and last boundary segments in this order are labelled by $-$ and $+$ respectively. Let $t^1$ be the last boundary segment labelled by $-$. Then we take the consecutive boundary segments $t^2, t^3, \ldots, t^{k-1}$ as long as they are labelled by $0$. This gives a sequence of consecutive segments, $t^1, \ldots, t^k$ labelled $-,0,\ldots,0,+$ (see Figure 15).

We now examine $t^1_-$ and $t^k_+$ and treat three cases, each time constructing $Z$ differently.

(a) If $t^1_-$ and $t^k_+$ intersect each other before intersecting $s_+$, then we can look at the subcomplex $Y$ of $X'$ bounded by $N(t^1_-) \cup N(t^k_+) \cup \bigcup_{1 \leq i \leq k} N(t^i)$ as in Figure 16.

Consider the square $Q = N(t^1_-) \cap N(t^k_+)$, and let $\hat{Y} = Y \setminus Q$. Let $Z$ be the double of $\hat{Y}$ along the remaining edges of $Q$. We also know that $|Z| \leq 2|Y| - 2 \leq 2|X'| - 2 < |X|$.

(b) If $t^1_-$ and $t^k_+$ do not intersect each other before intersecting $s_+$, and $|t^1_- \cap t^k_| \leq 1$, as in Figure 17, then we define $Y$ as the complex bounded by $N(s_+) \cup N(t^1_-) \cup N(t^k_+) \cup \bigcup_{1 \leq i \leq k} N(t^i)$.

We keep the notation $s_+$ for the segment of $s_+$ between the intersections $s_+ \cap t^1_-$ and $s_+ \cap t^k_+$. 

□
Figure 16. Defining the complex \( Y \), as the union of all the squares intersecting the region \( \hat{Y} \) in the figure.

Figure 17. The second case, where the segments \( t_1^+ \) and \( t_k^- \) are disjoint or intersect once.

Figure 18. The construction of \( \hat{Y} \) by removing nonboundary hyperplanes does not change the homeomorphism type of the complex.

Let \( \hat{Y} \) be the complex obtained from \( Y \) by collapsing all non-boundary hyperplanes. This operation does not produce corners and keeps \( \hat{Y} \) homeomorphic to a disk (see Figure 18).

The boundary hyperplanes of \( Y \) are the same as those for \( \hat{Y} \), so if \( s_+, \hat{h}^1, \ldots, \hat{h}^k \) were the boundary hyperplanes of \( Y \), we denote the boundary hyperplanes of \( \hat{Y} \) by \( \hat{s}_+, \hat{h}^1, \ldots, \hat{h}^k \).

Doubling \( \hat{Y} \) along \( \hat{s}_+ \), and then collapsing \( N(\hat{s}_+) \) gives us \( Z \). As in the previous case, \( |Z| = 2|\hat{Y}| - 4 \leq 2|Y| - 4 < |X| \).

(c) If \( t_1^- \) and \( t_k^+ \) do not intersect each other before intersecting \( s_+ \), and \( t_1^+, t_k^- \) form a bigon, as in Figure 19, then we define \( Y, \hat{Y} \) exactly as above.

We verify that \( \hat{Y} \) satisfies the conditions of Lemma 5.6.

(1) Follows from Figure 18.
(2) Immediate,
(3) \( \hat{s}_+ \) is this hyperplane, with only \( \hat{h}_1, \hat{h}_k \) intersecting it,
Figure 19. The third case, where the segments $t_1^+$ and $t_k^-$ form a bigon.

(4) Since $X$ had no corner, $Y$ has exactly two corners which lie in $N(s_+)$. Since collapsing nonboundary hyperplanes does not introduce new corners, the same holds for $\hat{Y}$.

(5) The bigon formed by $h_1, h_k$ must not intersect the boundary squares of $X'$, or else we would get $t'_j$ labelled $+$ or $-$ for $1 < j < k$. This property then extends to $Y$ and $\hat{Y}$.

By Lemma 5.6, we can replace $\hat{Y}$ with a complex $Y'$ with no corners outside of $N(\hat{s}_+)$, and reduce to the second case, defining $Z$ as above.

In all three cases, we have $|Z| < |X|$, so by the induction hypothesis, $Z$ has a corner at a vertex $v$. Note that $v$ cannot belong to the edges along which we doubled. Consequently, it must belong to $Y \setminus Q$ in the first case or $\hat{Y} \setminus N(\hat{s}_+)$ in the second case. This is a contradiction. \hfill \Box

6. Strips, bigons, and Proposition 2.3

In this section, we will prove Proposition 2.3. Throughout this section, we assume that $A$ is a family of arcs as in Proposition 2.3. As a warm-up, we show:

Lemma 6.1. $\alpha_p$ is the unique arc becoming nonessential after removing $p$.

Proof. Let $\alpha \in A$ be distinct from $\alpha_p$. Since $\alpha$ and $\alpha_p$ are disjoint, they form a strip, where $p$ is outside of this strip. Note that since $\alpha \neq \alpha_p$, there must be some puncture $r$ in this strip. Since $\alpha$ is essential, there exists a puncture $q$ on the other side of this strip, as in Figure 26.

Figure 20. $\alpha$ remains essential after removing $p$. 

Figure 21. Two possible configurations for arcs becoming homotopic after removing $p$.

Figure 22. $\alpha$ and $\beta$ split $D_n$ into four quadrants.

Now, $\alpha$ separates $q$ and $r$, and thus remains essential after removing $p$. □

Lemma 6.2. If two arcs $\alpha, \beta \in \mathcal{A}$ become homotopic and stay essential after removing $p$, then before removing $p$ they must have been in one of the two configurations in Figure 21.

Proof. We prove this by considering cases.

(1) Firstly, if $\alpha$ and $\beta$ are disjoint, then they cut the disk $D_n$ into three connected components, one of which is a strip bounded by $\partial D_n$, $\alpha$ and $\beta$. If $\alpha$ and $\beta$ were not homotopic to each other before removing the puncture, then the puncture must have lied in this strip. Since $\alpha$ and $\beta$ are homotopic after removing $p$, it follows that $p$ is the only puncture in this strip. This gives Figure 21a.

(2) Secondly, we consider the case where $|\alpha \cap \beta| = 1$. In this case, $\alpha \cup \beta$ splits $D_n$ into four quadrants. Without loss of generality, let $p$ lie in the bottom right quadrant (IV) as in Figure 22.

By Lemma 3.3, every quadrant must have a puncture. However, if quadrant II has a puncture $q$, then quadrant III cannot have a puncture $r$, since $\beta$ would separate $r$ from $q$, whereas $\alpha$ would not. This would then contradict the fact that $\alpha$ and $\beta$ become homotopic after removing $p$.

(3) The last scenario is when $|\alpha \cap \beta| = 2$. In this case, $\alpha$ and $\beta$ lie as in Figure 23.

By Lemma 3.3, there must be a puncture in the bigon and the two half-bigons in Figure 23. Let $r \neq p$ be a puncture in one of the half-bigons.

If there was a puncture $q$ outside the bigons and half-bigons (without loss of generality, let it lie in the right region of Figure 23), then $\beta$ separates $r$ and...
Figure 23. Simple arcs intersecting twice lie in this configuration.

Figure 24. If \( \alpha, \beta \in A \), then the arc \( \gamma \) must be in \( A \).

\( q \), while \( \alpha \) does not, and this property remains after \( p \) is removed. Thus, \( \alpha \) and \( \beta \) do not become homotopic to each other.

This means that the punctures in the bigon and half-bigons are the only possible punctures in this configuration. If \( p \) lies in a bigon or half-bigon, then it must be the only puncture in this bigon or half-bigon, otherwise, after its removal, \( \alpha \) and \( \beta \) would still be in minimal position, and non-homotopic. If \( p \) lies in the bigon, then removing \( p \) would make \( \beta \) nullhomotopic by the above discussion, giving a contradiction.

Thus, \( p \) lies in one of the half-bigons, it is the only puncture in this half-bigon, and all other punctures are contained in the bigon or other half-bigon between \( \alpha \) and \( \beta \), giving the configuration in Figure 21a.

Lemma 6.2 yields the following immediate consequence:

**Corollary 6.3.** Let \( q \neq p \) be a puncture, and let \( \delta \neq \delta' \in A \) be arcs becoming homotopic after removing \( p \). Then \( |\delta \cap \delta'| = 0 \) if and only if exactly one of \( \delta \), \( \delta' \) separate \( p \) from \( q \). In particular, there are no triples of disjoint arcs becoming pairwise homotopic after removing \( p \).

**Lemma 6.4.** If \( A \) is maximal, then if there are two arcs \( \alpha, \beta \in A \) as in Figure 21b), then the arc \( \gamma \) in Figure 24 belongs to \( A \).

We think of \( \gamma \) as a subset of \( \alpha \cup \beta \), and set \( \gamma_\alpha = \gamma \cap \alpha \) and \( \gamma_\beta = \gamma \cap \beta \).

**Proof.** We must check that if \( \delta \) is an arc in \( A \), it cannot intersect \( \gamma \) more than twice. Note that by construction of \( \gamma \), if \( \delta \) intersects \( \gamma \) at any point, then it must either intersect \( \gamma_\alpha \) or \( \gamma_\beta \).
We assume that $|\delta \cap \gamma| \geq 3$. By the pigeonhole principle, without loss of generality, we can assume $|\delta \cap \gamma| = 2$. Let $\{x, y\}$ be the two intersection points, and consider the segment of $\delta$ between them, denoted $I_\delta$ as in Figure 25.

The arc $\beta$ splits $D_n$ into two disks, one of which contains $p$, and the other, denoted $D$, which does not. Suppose first that $I_\delta \subset D$. Then since all of the punctures in $D$ are enclosed by $\alpha$ (see Figure 24), it follows that the only way in which $I_\delta$ will not form an empty bigon with $\beta$ is if it intersects $\alpha$ twice in $D$ (see Figure 25(a) above). However, this means that $\delta$ intersects $\alpha$ twice outside of $\gamma$ and $|\gamma \cap \delta| = 0$. Thus, $|\gamma \cap \delta| = 2$, contradiction.

On the other hand, if $I_\delta \subset D_n \setminus D$, as in Figure 25(b), then consider the segment of $\delta$ from $x$ to $\partial D_n$ not containing $y$. Since $\delta$ and $\beta$ are in minimal position, this segment does not form an empty half-bigon with $\beta$. This means that it must enter the bigon between $\alpha$ and $\beta$. Thus it intersects $\alpha$ twice, so as before, $|\gamma \cap \delta| = 2$, contradiction. □

**Lemma 6.5.** If $\mathcal{A}$ is maximal, then any set $S$ of arcs in $\mathcal{A}$ which become pairwise homotopic and remain essential after removing $p$ is of size at most 3. If $|S| = 3$, then the arcs in $S$ are as in Figure 24, and if $|S| = 2$, then they are as in Figure 21a.

**Proof.** If $|S| \geq 3$, then by Corollary 6.3 and Lemma 6.2 it follows that there exist $\alpha, \beta \in S$ that are in configuration $b$) of Figure 21 and that the arc $\gamma$ described in this figure is also in $S$. We will show that no fourth arc $\delta$ can be added to $S$ in this configuration.

Using Figure 21b, we split $D_n$ into five regions, I-V:

**Claim.** $|\delta \cap \gamma| = 2$
Figure 27. The arc $\delta$ forms an empty bigon with $\beta$.

Proof. We justify the claim by contradiction. If $\delta \cap \gamma = \emptyset$, then by Corollary 6.3, $\delta$ lies in the complement of III and must intersect $\alpha$ and $\beta$ twice. Note that by parity of intersection numbers, $\delta$ must have both endpoints in the same region, and this region cannot be I, or else $\delta$ would span an empty half-bigon with $\alpha$ or $\beta$. Without loss of generality, let the endpoints of $\delta$ lie in II. In order for no half-bigons to be formed, if we follow $\delta$ from these endpoints, we must enter V, and from there we must go to IV (see Figure 27).

Then, having intersected $\alpha$ and $\beta$ twice, they must join and form an empty bigon with $\beta$ as in Figure 27 contradiction. This justifies the claim. \[\square\]

Now, since $|\delta \cap \gamma| = 2$, by Corollary 6.3, it follows that $\delta$ separates all the punctures in V from $p$. Since $\alpha, \beta$ also separate all of the punctures in V from $p$, we get by Corollary 6.3 that $|\delta \cap \alpha| = |\delta \cap \beta| = 0$, which is a contradiction, since $\gamma \subset \alpha \cup \beta$. \[\square\]

Let $S$ be the set of all strips between arcs in $\mathcal{A}$ which contain the single puncture $p$. Note that by Lemma 6.1, $\alpha_p$ cannot be an arc in such a strip.

Corollary 6.6. Let $\mathcal{A}$ be a maximal good family of arcs. Assume that $\alpha \in \mathcal{A}$ is an isolated arc, such that one of the components of $D_n \setminus \alpha$ contains a single puncture $p$. Let $\mathcal{A}'$ be the set of homotopy classes of essential arcs obtained from $\mathcal{A}$ by removing $p$. Then

$$|\mathcal{A}| - |\mathcal{A}'| \leq 1 + |S|$$

Proof. Consider the equivalence relation $\sim_p$ on $\mathcal{A}$ where $\alpha \sim_p \beta$ if $\alpha$ and $\beta$ are homotopic after removing $p$. By Lemma 6.2, Lemma 6.4, and Lemma 6.5, every equivalence class contains one, two, or three elements, forming zero, one, or two strips, respectively. Since $|\mathcal{A}'| = \{[\alpha] : \alpha \in \mathcal{A}\} \setminus \alpha_p$, we get the result (where the +1 comes from $\alpha_p$). \[\square\]

Lemma 6.7. $\alpha_p \subset \bigcap_{S \in S} S$, and its endpoints are on the same connected component of $\partial D_n \cap S$ for any strip $S \in S$.

Proof. Let $S \in S$ be a given strip between two arcs $\alpha, \beta \in \mathcal{A}$. Since $\alpha_p$ is disjoint from every $\alpha, \beta \in \mathcal{A}$, it follows that it must lie in one connected component of $D_n \setminus (\alpha \cup \beta)$.

If it does not lie in $S$, then without loss of generality, it must lie in the connected component bounded by $\alpha$ and $\partial D_n$. In this case, since $\beta$ is essential, there must be
a puncture $r$ in the connected component bounded by $\beta$ and $\partial D_n$, and $\alpha_p$ cannot separate $r$ and $p$, giving a contradiction.

If the endpoints of $\alpha_p$ lie in different connected components of $\partial D_n \cap S$ for some strip $S$, then $\alpha_p \sim \alpha$ or $\alpha_p \sim \beta$, since $p$ is the only puncture in $S$, contradiction. □

**Definition 6.8.** There exists an arc $\varepsilon$ from $p$ to $\partial D_n$ disjoint from $\alpha_p$. By Lemma 6.7, $\varepsilon$ lies in every $S \in S$. If $S \in S$ then there are two unique (up to homotopy) arcs in $S$ from $p$ to $\partial D_n \cap S$. The first arc is $\varepsilon$, and the second will be denoted by $\delta_S$ (see Figure 28).

We denote

$$\mathcal{G} = \{\delta_S\}_{S \in S} \cup \{\varepsilon\}$$

**Remark.** If $S \neq S'$ are two strips, then $\delta_S \neq \delta_{S'}$, since $\delta_S$ and $\varepsilon$ uniquely determine $S$.

**Lemma 6.9.** $\mathcal{G}$ is a family of arcs from $p$ to $\partial D_n$ which pairwise intersect at most once.

**Proof.** Firstly, we note that by construction, $\varepsilon$ is disjoint from $\delta_S$ for any $\delta_S \in \mathcal{C}$. Now, let $\delta_S \neq \delta_{S'}$ be two arcs in minimal position coming from strips $S \neq S'$. Let $\alpha, \beta \in \mathcal{A}$ be the sides of these strips as in Figure 29.

It is clear that $|\alpha \cap \beta| = |\delta_S \cap \delta_{S'}| + 1$, so we just need to show that $\alpha$ and $\beta$ are in minimal position. Let $x$ be the intersection point of $\alpha$ and $\beta$ near $p$ (see Figure 29).
If there is an empty bigon or half-bigon between $\alpha$ and $\beta$ whose boundary does not contain $x$, then this will also be a bigon or half-bigon will also be between $\delta S, \delta S'$ contradicting their minimal position. The half bigon containing $\varepsilon$ (whose boundary contains $x$) contains the puncture $p$, and is thus nonempty. Since $\delta S, \delta S', \varepsilon$ are not pairwise homotopic and $\delta S, \delta S'$ do not form an empty bigon, it follows that the other regions between $\alpha$ and $\beta$ whose boundaries contain $x$ are not empty bigons or half-bigons.

We are now ready to prove Proposition 2.3:

\textit{Proof.} By Corollary 6.6 the number of arcs which become pairwise homotopic after removing $p$ is equal to the number of strips formed by arcs in $A$ after removing $p$. By Lemma 6.9 and Theorem 1.7 in [2], $|G| \leq \binom{n}{2}$. Since $|S| = |G|-1$, the proposition follows. $\square$

7. Discussion and Corollaries

We present a few corollaries of Theorem 1.4 and Theorem 1.6

Corollary 7.1. Let $X$ be a planar square complex homotopy equivalent to a disk, whose hyperplanes satisfy condition $(\ast)$. Then if $X$ contains at least two $X$ has two corners and/or spurs.

\textit{Proof.} By Theorem 1.6 $X$ has a corner or a spur. If $X$ has only one corner or spur at $v$, then since $X$ contains at least two 1-cells, it follows that the double $Y = X \sqcup_v X$ will have no corners or spurs. Condition $(\ast)$ is immediately satisfied by $Y$, and the homotopy type of $Y$ is still that of a disk, contradicting Theorem 1.6. $\square$

This corollary is reminiscent of Greendlinger’s Lemma from small cancellation theory:

Theorem 7.2. Let $X$ be a $C(6)$-complex, and $D \to X$ a minimal disc diagram. Then one of the following holds:

- $D$ is a single cell,
- $D$ is a ladder, or
- $D$ has at least three spurs or shells of degree $\leq 3$.

The main difference is that in the above corollary, we only require knowledge of the hyperplane intersection data instead of negative curvature assumptions.

7.1. Generalizing Theorems 1.4 and 1.6

The proof method outlined in this paper can also be used to simplify the case for once-intersecting families:

Theorem 7.3 (Theorem 1.7 in [2]). The maximal cardinality of a good family of arcs pairwise intersecting at most once on $D_n$ is $\binom{n}{2}$.

Note that in general, it is not true that when allowing for $k$-intersections, the maximal families are of size $\binom{n+k-1}{k+1}$. For example, any maximal family of arcs on $D_3$ pairwise intersecting at most three times is of size at most 4.

As for Theorem 1.6 it is not true when allowing for triple intersections:
Figure 30. Theorem 1.6 is not true when we allow for three intersections.

Figure 31. Two good families on $D_4$. Both are maximal, as their size is $10 = \binom{5}{3}$, and are in minimal position.

7.2. Actions of $\text{Mod}(D_n)$. It is natural to ask whether the maximal families of size $\binom{n+1}{3}$ on $D_n$ are related by homeomorphisms in the mapping class group of $D_n$. Alternatively, does the mapping class group act transitively on maximal good families of arcs? The answer is no, as illustrated in Figure 31.

Claim. The good families in Figure 31 are not in the same orbit of $\text{Mod}(D_n)$.

Proof. We know that elements of the mapping class group preserve intersection numbers of arcs. We first note that by Lemma 3.3, both of these families are in minimal position. The left family in Figure 31 has an arc (the blue arc) which has precisely three intersections with other arcs in the family, whereas in the right family, no such arc exists. Thus, these families are not obtained from one another by a homeomorphism in $\text{Mod}(D_n)$. □

7.3. Further Questions.

Question. Is every maximal good family of arcs on $D_n$ of size $\binom{n+1}{3}$? This is true for the curve complex, but not much is known in the case of the arc complexes.

Question. Does the upper bound from Theorem 1.4 hold when we consider arcs from $\partial D_n$ to a specified puncture $p \neq \partial D_n$?

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