GENERIC CHARACTER SHEAVES ON
DISCONNECTED GROUPS AND CHARACTER VALUES

G. Lusztig

INTRODUCTION

The theory of character sheaves [L3] on a reductive group $G$ over an algebraically closed field and the theory of irreducible characters of $G$ over a finite field are two parallel theories; the first one is geometric (involving intersection cohomology complexes on $G$), the second one involves functions on the group of rational points of $G$. In the case where $G$ is connected, a bridge between the two theories was constructed in [L1] and strengthened in [L2], [S]. In this paper we begin the construction of the analogous bridge in the general case, extending the method of [L1]. Here we restrict ourselves to character sheaves which are ”generic” (in particular their support is a full connected component of $G$) and show how such character sheaves are related to characters of representations (see Theorem 1.2).

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1. Statement of the Theorem

1.1. Let $k$ be an algebraic closure of a finite field $\mathbb{F}_q$. Let $G$ be a reductive algebraic group over $k$ with identity component $G^0$ such that $G/G^0$ is cyclic, generated by a fixed connected component $D$. We assume that $G$ has a fixed $\mathbb{F}_q$-rational structure with Frobenius map $F : G \to G$ such that $F(D) = D$. Let $l$ be a prime number invertible in $k$; let $\mathbb{Q}_l$ be an algebraic closure of the $l$-adic numbers. All group representations are assumed to be finite dimensional over $\mathbb{Q}_l$. We say ”local system” instead of ”$\mathbb{Q}_l$-local system”.

Let $\mathcal{B}$ be the variety of Borel subgroups of $G^0$. Now $F : G \to G$ induces a morphism $\mathcal{B} \to \mathcal{B}$ denoted again by $F$. We fix $B^* \in \mathcal{B}$ and a maximal torus $T$ of $B^*$ such that $F(B^*) = B^*, F(T) = T$. Let $U^*$ be the unipotent radical of $B^*$. Let

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NB* (resp. NT) be the normalizer of B* (resp. T) in G. Let $\hat{T} = NT \cap NB^*$, a closed F-stable subgroup of G with identity component T. Let $\hat{T}_D = \hat{T} \cap D$.

Let $\mathcal{N} = NT \cap G^0$. Let $W = \mathcal{N}/T$ be the Weyl group. Let $\mathcal{D} : T \sim \homoth, D : W \sim \homoth W$ be the automorphisms induced by Ad$(d) : \mathcal{N} \rightarrow \mathcal{N}$ where $d$ is any element of $\hat{T}_D$. Now $F : \mathcal{N} \rightarrow \mathcal{N}$ induces an automorphism of $W$ denoted again by $F$. For $w \in W$ let $[w]$ be the inverse image of $w$ under the obvious map $\mathcal{N} \rightarrow W$ and let $w$ be the automorphism Ad$(x) : T \rightarrow T$ for any $x \in [w]$. For $w \in W$ let $\mathcal{O}_w$ be the $G^0$-orbit in $\mathcal{B} \times \mathcal{B}$ ($G^0$ acting by simultaneous conjugation on both factors) that contains $(B^*, xB^*x^{-1}d^{-1})$ for some $x \in [w]$. Define the "length function" $l : W \rightarrow \mathbb{N}$ by $l(w) = \dim \mathcal{O}_w - \dim \mathcal{B}$. For any $y \in G^0$ we define $k(y) \in \mathcal{N}$ by $y \in U^*k(y)U^*$. For $y \in G^0, \tau \in \hat{T}$ we have $k(\tau y \tau^{-1}) = \tau k(y)\tau^{-1}$ and $F(k(y)) = k(F(y))$. For $x \in G^0$ we define $F_x : G \rightarrow G$ by $F_x(g) = xF(g)x^{-1}$; this is the Frobenius map for an $\mathbb{F}_q$-rational structure on $G$. (Indeed if $y \in G^0$ is such that $x = y^{-1}F(y)$, then Ad$(y) : G \sim \homoth G$ carries $F_x$ to $F$.) If $w \in W$ satisfies $D(w) = w$ and $x \in [w]$ then $T, \hat{T}$ are $F_x$-stable; thus $F_x$ is the Frobenius map for an $\mathbb{F}_q$-rational structure on $\hat{T}$ whose group of rational points is $\hat{T}^F$. Since $\hat{T}_D^F$ is the set of rational points of $\hat{T}_D$ (a homogeneous $T$-space under left translation) for the rational structure defined by $F_x : \hat{T}_D \rightarrow \hat{T}_D$, we have $\hat{T}_D^F \neq \emptyset$.

Let $Z_\emptyset = \{(B_0, g) \in \mathcal{B} \times D; gB_0g^{-1} = B_0\}$. Let $d \in \hat{T}_D$. We set $\tilde{Z}_{\emptyset, d} = \{(h_0U^*, g) \in (G^0/U^*) \times D; h_0^{-1}gh_0d^{-1} \in B^*\}$.

Define $a_\emptyset : \tilde{Z}_{\emptyset, d} \rightarrow Z_\emptyset$ by $(h_0U^*, g) \mapsto (h_0B^*h_0^{-1}, g)$. Now $a_\emptyset$ is a principal $T$-bundle where $T$ acts (freely) on $Z_{\emptyset, d}$ by $t_0 : (h_0U^*, g) \mapsto (h_0t_0^{-1}, g)$. Define $p_\emptyset : Z_\emptyset \rightarrow D$ by $(B_0, g) \mapsto g$. We define $b_\emptyset : \tilde{Z}_{\emptyset, d} \rightarrow T$ by $(h_0U^*, g) \mapsto k(h_0^{-1}gh_0d^{-1})$. Note that $b_\emptyset$ commutes with the $T$-actions where $T$ acts on $T$ by

(a) $t_0 : t \mapsto t_0tD(t_0^{-1})$.

Let $\mathcal{L}$ be a local system of rank 1 on $T$ such that

(i) $\mathcal{L}^\otimes n \cong \mathbb{Q}$ for some $n \geq 1$ invertible in $k$;

(ii) $D^*\mathcal{L} \cong \mathcal{L}$;

From (i),(ii) we see (using [L3, 28.2(a)]) that $\mathcal{L}$ is equivariant for the $T$-action (a) on $T$. Hence $b_\emptyset^*\mathcal{L}$ is a $T$-equivariant local system on $\tilde{Z}_{\emptyset, d}$. Since $a_\emptyset$ is a principal $T$-bundle there is a well defined local system $\tilde{\mathcal{L}}_\emptyset$ on $Z_\emptyset$ such that $a_\emptyset^*\tilde{\mathcal{L}}_\emptyset = b_\emptyset^*\mathcal{L}$. Note that the isomorphism class of $\tilde{\mathcal{L}}_\emptyset$ is independent of the choice of $d$. Assume in addition that:

(iii) $\{w \in W; D(w) = w, w^*\mathcal{L} \cong \mathcal{L}\} = \{1\}$.

We show:

(b) $p_\emptyset!\tilde{\mathcal{L}}_\emptyset$ is an irreducible intersection cohomology complex on $D$.

We identify $Z_\emptyset$ with the variety $X = \{(g, xB^*) \in G \times G^0/B^*; x^{-1}gx \in NB^*\}$ (as in [L3, I, 5.4] with $P = B^*, L = T, S = T_D$) by $(g, xB^*) \mapsto (xB^*x^{-1}, g)$. Then $\tilde{\mathcal{L}}_\emptyset$ becomes the local system $\tilde{\mathcal{E}}$ on $X$ defined as in [L3, I, 5.6] in terms of the local system $\mathcal{E} = j^*\mathcal{L}$ on $T_D$ where $j : T_D \rightarrow T$ is $y \mapsto d^{-1}y$. (Note that $\mathcal{E}$ is equivariant
for the conjugation action of $T$ on $\hat{T}_D$.) In our case we have $\mathcal{E} = IC(X, \mathcal{E})$ since $X$ is smooth. Hence from [L3, I, 5.7] we see that $p_{\emptyset}!\mathcal{E}$ is an intersection cohomology complex on $D$ corresponding to a semisimple local system on an open dense subset of $D$ which, by the results in [L3, II, 7.10], is irreducible if and only if the following condition is satisfied: if $w \in W, x \in [w]$ satisfy $\text{Ad}(x)(\hat{T}_D) = \hat{T}_D$ and $\text{Ad}(x)^*\mathcal{E} \cong \mathcal{E}$, then $w = 1$. This is clearly equivalent to condition (iii). This proves (b).

From (b) and the definitions we see that $p_{\emptyset}!\hat{\mathcal{L}}_{\emptyset}[\dim D]$ is a character sheaf on $D$ in the sense of [L3, VI]. A character sheaf on $D$ of this form is said to be generic. We can state the following result.

**Theorem 1.2.** Let $A$ be a generic character sheaf on $D$ such that $F^*A \cong A$ where $F : D \to D$ is the restriction of $F : G \to G$. Let $\psi : F^*A \to A$ be an isomorphism. Define $\chi_\psi : D^F \to \mathcal{Q}_l$ by $g \mapsto \sum_{i \in \mathbf{Z}} (-1)^i \text{tr}(\psi, \mathcal{H}_g(A))$ where $\mathcal{H}_g$ is the $i$-th cohomology sheaf and $\mathcal{H}_g^i$ is its stalk at $g$. There exists a $G^F$-module $V$ and a scalar $\lambda \in \mathbf{Q}_l^*$ such that $\chi_\psi(g) = \lambda \text{tr}(g, V)$ for all $g \in D^F$.

The proof is given in §3. We now make some preliminary observations. In the setup of 1.1 we have $A = p_{\emptyset}!\mathcal{L}_\emptyset[\dim D]$ where $\mathcal{L}$ satisfies 1.1(i),(ii),(iii) and $F^*(p_{\emptyset}!\mathcal{L}_\emptyset) \cong p_{\emptyset}!\hat{\mathcal{L}}_{\emptyset}$. Hence we have $p_{\emptyset}!F^*\mathcal{L}_{\emptyset} \cong p_{\emptyset}!\hat{\mathcal{L}}_{\emptyset}$. By a computation in [L3, IV, 21.18] we deduce that there exists $w' \in W$ such that $D(w') = w'$, $w'^*F^*\mathcal{L} \cong \mathcal{L}$. Setting $w = F(w')$ we see that

(a) $D(w) = w$, $F^*w^*\mathcal{L} \cong \mathcal{L}$.

**1.3.** Let $w = (w_1, w_2, \ldots, w_r)$ be a sequence in $W$. Let $l_w = l(w_1) + l(w_2) + \cdots + l(w_r)$. Let

$$Z_w = \{(B_0, B_1, \ldots, B_r, g) \in B^{r+1} \times D; gB_0g^{-1} = B_r, (B_{i-1}, B_i) \in \mathcal{O}_{w_i}(i \in [1, r])\}.$$ 

This agrees with the definition in 1.1 when $r = 0$, that is $w = \emptyset$. Let $d \in \hat{T}_D$. We define $\hat{Z}_{w,d}$ as in 1.1 when $r = 0$ and by

$$\hat{Z}_{w,d} = \{(h_0 U^*, h_1 B^*, \ldots, h_{r-1} B^*, h_r U^*, g) \in$$

$$(G^0/U^*) \times (G^0/B^*) \times \cdots \times (G^0/B^*) \times (G^0/U^*) \times D;$$

$$k(h_{i-1}^{-1} h_i) \in [w_i](i \in [1, r]), h_r^{-1} g h_0 d^{-1} \in U^*\};$$

when $r \geq 1$. Define $a_w : \hat{Z}_{w,d} \to Z_w$ as in 1.1 when $r = 0$ and by

$$(h_0 U^*, h_1 B^*, \ldots, h_{r-1} B^*, h_r U^*, g) \mapsto$$

$$(h_0 B^* h_0^{-1}, h_1 B^* h_1^{-1}, \ldots, h_{r-1} B^* h_{r-1}, h_r B^* h_r^{-1}, g),$$

when $r \geq 1$. Note that $a_w$ is a principal $T$-bundle where $T$ acts (freely) on $\hat{Z}_{w,d}$ as in 1.1 when $r = 0$ and by

$$t_0 : (h_0 U^*, h_1 B^*, \ldots, h_{r-1} B^*, h_r U^*, g) \mapsto$$

$$(h_0^{-1} U^*, h_1 B^*, \ldots, h_{r-1} B^*, h_r d t_0^{-1} d^{-1} U^*, g).$$
when $r \geq 1$. Define $p_w : Z_w \to D$ by $(B_0, B_1, \ldots, B_r, g) \mapsto g$.

In the remainder of this subsection we assume that $w_1 w_2 \ldots w_r = 1$; this holds automatically when $r = 0$. We define $b_w : \hat{Z}_{w,d} \to T$ as in 1.1 when $r = 0$ and by

$$(h_0 U^*, h_1 B^*, \ldots, h_{r-1} B^*, h_r U^*, g) \mapsto k(h_0^{-1}h_1)k(h_1^{-1}h_2)\ldots k(h_{r-1}^{-1}h_r)$$

when $r \geq 1$. Note that $b_w$ commutes with the $T$-actions where $T$ acts on $T$ as in 1.1(a).

Let $\mathcal{L}$ be a local system of rank 1 on $T$ such that 1.1(i),(ii) hold. As in 1.1, $\mathcal{L}$ is equivariant for the $T$-action 1.1(a) on $T$. Hence $b_w^* \mathcal{L}$ is a $T$-equivariant local system on $\hat{Z}_{w,d}$. Since $a_w$ is a principal $T$-bundle there is a well defined local system $\hat{L}_w$ on $Z_w$ such that $a_w^* \hat{L}_w = b_w^* \mathcal{L}$.

**Lemma 1.4.** Assume that $w_1 w_2 \ldots w_r = 1$ and that $\mathcal{L}$ (as in 1.3) satisfies

(i) $\alpha^* \mathcal{L} \not\equiv \mathcal{Q}_l$ for any coroot $\alpha : k^* \to T$.

Then $p_{w!} \hat{L}_w[l_w]/(l_w/2) \cong p_{\tilde{w}}! \hat{L}_{\tilde{w}}[l_{\tilde{w}}](l_{\tilde{w}}/2)$. (Note that $l_w$ is even.)

Assume first that for some $i \in [1, r]$ we have $w_i = w_i'w_i''$ where $w_i', w_i''$ in $W$ satisfy $l(w_i'w_i'') = l(w_i') + l(w_i'')$. Let

$$w' = (w_1, w_2, \ldots, w_{i-1}, w_i', w_i'', w_{i+1}, \ldots, w_n).$$

The map $(B_0, B_1, \ldots, B_{r+1}, g) \mapsto (B_0, B_1, B_{i-1}, B_{i+1}, \ldots, B_{r+1}, g)$ defines an isomorphism $Z_{w'} \to Z_w$ compatible with the maps $p_{w'}, p_w$ and with the local systems $\hat{L}_{w'}$, $\hat{L}_w$. Since $l_{w'} = l_w$ we have

(a) $p_{w!} \hat{L}_w[l_w]/(l_w/2) \cong p_{w'}! \hat{L}_{w'}[l_{w'}](l_{w'}/2)$.

Using (a) repeatedly we can assume that $l(w_i) = 1$ for all $i \in [1, r]$. We will prove the result in this case by induction on $r$. Note that $r$ is even. When $r = 0$ the result is obvious. We now assume that $r \geq 2$. Since $w_1 w_2 \ldots w_r = 1$, we can find $j \in [1, r-1]$ such that $l(w_1 w_2 \ldots w_j) = j$, $l(w_1 w_2 \ldots w_{j+1}) = j - 1$. We can find a sequence $w' = (w_1', w_2', \ldots, w_r')$ in $W$ such that $l(w_i') = 1$ for all $i \in [1, r]$, $w_i' w_j' = w_1 w_2 \ldots w_j$, $w_j' = w_{j+1}'$, $w_i = w_i$ for $i \in [j+1, r]$. Let

$$u = (w_1 w_2 \ldots w_j, w_{j+1}, \ldots, w_r) = (w_1 w_2 \ldots w_j, w_{j+1}', \ldots, w_r').$$

Using (a) repeatedly we see that

$$p_{w!} \hat{L}_w[l_w]/(l_w/2) \cong p_{u!} \hat{L}_u[l_u]/(l_u/2) \cong p_{w'}! \hat{L}_{w'}[l_{w'}]/(l_{w'}/2).$$

Replacing $w$ by $w'$ we see that we may assume in addition that $w_j = w_{j+1}$ for some $j \in [1, r-1]$. We have a partition $Z_w = Z'_w \cup Z''_w$ where $Z'_w$ (resp. $Z''_w$) is defined by the condition $B_{j-1} = B_{j+1}$ (resp. $B_{j-1} \neq B_{j+1}$). Let $w' = (w_1, w, \ldots, w_{j-1}, w_{j+2}, \ldots, w_r)$, $w'' = (w_1, w, \ldots, w_{j-1}, w_{j+1}, \ldots, w_r)$. Define $c : Z'_w \to Z_{w'}$ by

$$(B_0, B_1, \ldots, B_r, g) \mapsto (B_0, B_1, \ldots, B_{j-1}, B_{j+2}, \ldots, B_r, g).$$
This is an affine line bundle and $\tilde{L}_w|_{Z_w'} = c^*\tilde{L}_w'$. Let $p'_w$ be the restriction of $p_w$ to $Z_w'$. We have $p'_w = p_w\cdot c$. Since the induction hypothesis applies to $w'$ we have

$$p'_w!(\tilde{L}_w|_{Z_w'})[l_w](l_w/2) = p_w'|_w e^*\tilde{L}_w'[l_w](l_w/2)$$

(b) $= p_w'|_w \tilde{L}_w'[2](1)|l_w|(l_w/2) = p_w'|_w \tilde{L}_w'[l_w](l_w'/2) = p_0! \tilde{L}_0$.

Define $e : Z_w'' \to Z_w''$ by

$$(B_0, B_1, \ldots, B_r, g) \mapsto (B_0, B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_r, g).$$

Let $p''_w$ be the restriction of $p_w$ to $Z_w''$. We have $p''_w = p''_w e$. We show that $p''_w!(\tilde{L}_w|_{Z_w''}) = 0$. It is enough to show that

$$p''_w!(e_1(\tilde{L}_w|_{Z_w''}) = 0.$$ 

Hence it is enough to show that $e_1(\tilde{L}_w|_{Z_w''}) = 0$. It is also enough to show that, if $E$ is a fibre of $e$, then $H^i_c(E, \tilde{L}_w|_E) = 0$ for any $i$. As in the proof of [L3, VI, 28.10] we may identify $E = k^*$ in such a way that $\tilde{L}_w|_E$ becomes $\tilde{\alpha}^*(L)$ for some coroot $\tilde{\alpha} : k^* \to T$. We then use that $H^i_c(k^*, \tilde{\alpha}^*L) = 0$ which follows from $\tilde{\alpha}^*L \not\in \mathbb{Q}_L$.

Using (c) and the exact triangle

$$(p''_w!(e_1(\tilde{L}_w|_{Z_w''}), p_w!\tilde{L}_w, p'_w!(\tilde{L}_w|_{Z_w'}))$$

we see that

$$p_w!\tilde{L}_w[L](l_w/2) = p'_w!(\tilde{L}_w|_{Z_w'})[l_w](l_w/2) = p_0! \tilde{L}_0$$

(the last equality follows from (b)). The lemma is proved.

**Lemma 1.5.** Assume that $\mathcal{L}$ (as in 1.3) satisfies 1.1(iii). Then $\mathcal{L}$ satisfies 1.4(i).

Let $R_L$ be the set of roots $\alpha : T \to k^*$ such that the corresponding coroot $\tilde{\alpha}$ satisfies $\tilde{\alpha}^*\mathcal{L} \cong \mathcal{Q}_L$. Let $W_L$ be the subgroup of $W$ generated by the reflections with respect to the various $\alpha \in R_L$. Since $D^*\mathcal{L} \cong \mathcal{L}$ we have $D(W_L) = W_L$. Assume that 1.4(i) does not hold. Then $R_L \neq \emptyset$ and $W_L \neq \{1\}$. By [DL, 5.17] the fixed point set of $D : W_L \to W_L$ is $\neq \{1\}$. Let $w \in W_L - \{1\}$ be such that $D(w) = w$. Since $w \in W_L$ we have $w^*\mathcal{L} \cong \mathcal{L}$ (see [L3, VI, 28.3(b)]). Thus 1.1(iii) does not hold. The lemma is proved.

2. Constructing representations of $G^F$

2.1. In this section we construct some representations of $G^F$ using the method of [DL]. See [M],[DM] for other results in this direction.

Let $\mathcal{L}$ be a local system of rank 1 on $T$ such that 1.1(i) holds. For any $t \in T$ let $\mathcal{L}_t$ be the stalk of $\mathcal{L}$ at $t$. Assume that we are given $w \in W$ and $x \in [w]$ such that
(i) $F_x^r \mathcal{L} \cong \mathcal{L}$

$(F_x : T \to T$ as in 1.1). Let $\phi : F_x^r \mathcal{L} \to \mathcal{L}$ be the unique isomorphism of local systems on $T$ which induces the identity map on $\mathcal{L}_1$. For $t \in T$, $\phi$ induces an isomorphism $\mathcal{L}_{F_x(t)} \cong \mathcal{L}_t$. When $t \in T^{F_x}$ this is an automorphism of the 1-dimensional vector space $\mathcal{L}_t$ given by multiplication by $\theta(t) \in \mathbb{Q}_t^*$. It is well known that $t \mapsto \theta(t)$ is a group homomorphism $T^{F_x} \to \mathbb{Q}_t^*$.

Following [DL] we define

$$Y = \{ hU^* \in G^0/U^*; h^{-1}F(h) \in U^*xU^* \}. $$

For $(g, t) \in G^{0F} \times T^{F_x}$ we define $e_{g,t} : Y \to Y$ by $hU^* \mapsto ght^{-1}U^*$. Note that $(g, t) \mapsto e_{g,t}$ is an action of $G^{0F} \times T^{F_x}$ on $Y$. Hence $G^{0F} \times T^{F_x}$ acts on $H_c^i(Y) := H_c^i(Y, \mathbb{Q})$ by $(g, t) \mapsto e_{g^{-1}, t^{-1}}^*$. We set

$$H_c^i(Y)_\theta = \{ \xi \in H_c^i(Y); e_{1,t-1}^* \xi = \theta(t)^{-1}\xi \text{ for all } t \in T^{F_x} \};$$

this is a $G^{0F} \times T^{F_x}$-stable subspace of $H_c^i(Y)$.

For $g \in G^{0F}$ we define $e_g : H_c^i(Y)_\theta \to H_c^i(Y)_\theta$ by $e_g(\xi) = e_{g^{-1}, 1}^*$. This makes $H_c^i(Y)_\theta$ into a $G^{0F}$-module.

We can find an integer $r \geq 1$ such that

$$F^r(x) = x, \quad xF(x) \ldots F^{r-1}(x) = 1.$$ 

Indeed we first find an integer $r_1 \geq 1$ such that $F^{r_1}(x) = x$ and then we find an integer $r_2 \geq 1$ such that $(xF(x) \ldots F^{r_2-1}(x))^{r_2} = 1$. Then $r = r_1r_2$ has the required properties. Then $hU^* \mapsto F^r(h)U^*$ is a well defined map $Y \to Y$ denoted again by $F^r$. Also,

$$F^r = F_x^r : G \to G.$$ 

(We have $F_x^r(g) = (xF(x) \ldots F^{r-1}(x))F^r(g)(xF(x) \ldots F^{r-1}(x))^{-1} = F^r(g)$.) Hence $F^r$ acts trivially on $T^{F_x}$. We see that $F^r : Y \to Y$ commutes with $e_{g,t} : Y \to Y$ for any $(g, t) \in G^{0F} \times T^{F_x}$. Hence $(F^r)^* : H_c^i(Y) \to H_c^i(Y)$ leaves stable the subspace $H_c^i(Y)_\theta$. Note that:

for any $i$, all eigenvalues of $(F^r)^* : H_c^i(Y) \to H_c^i(Y)$ are of the form root of 1 times $q^{nr/2}$ where $n \in \mathbb{Z}$.

(See [L1, 6.1(e)] and the references there.)

Replacing $r$ by an integer multiple we may therefore assume that $r$ satisfies in addition the following condition:

(a) for any $i$, all eigenvalues of $(F^r)^* : H_c^i(Y) \to H_c^i(Y)$ are of the form $q^{nr/2}$ where $n \in \mathbb{Z}$.

2.2. We preserve the setup of 2.1 and assume in addition that $\mathcal{L}$ satisfies 1.4(i).

Let $i_0 = 2 \dim U^* - l(w)$. Note that

(a) $H_c^i(Y)_\theta = 0$ for $i \neq i_0$; if $i = i_0$ then all eigenvalues of $(F^r)^* : H_c^i(Y)_\theta \to H_c^i(Y)_\theta$ are of the form $q^{nr/2}$.

For the first statement in (a) see [DL, 9.9] and the remarks in the proof of [L1, 8.15]. The second statement in (a) is deduced from 2.1(a) as in the proof of [L1, 6.6(c)].
2.3. We preserve the setup of 2.1 and assume in addition that \( \mathcal{L} \) satisfies 1.1(ii) and that \( w \in W \) satisfies \( \mathcal{D}(w) = w \). From the definitions we see that \( \mathcal{D} : T \to T \) commutes with \( F_x : T \to T \) hence \( \mathcal{D} \) restricts to an automorphism of \( T^{F_x} \) and that

(a) \( \theta(\mathcal{D}(t)) = \theta(t) \) for any \( t \in T^{F_x} \).

We show:

(b) there exists a homomorphism \( \tilde{\theta} : \tilde{T}^{F_x} \to \tilde{Q}_i^* \) such that \( \tilde{\theta}|_{T^{F_x}} = \theta \).

Let \( d \in \tilde{T}_D^{F_x} \). Let \( n = |G/G^0| = |\tilde{T}^{F_x}/T^{F_x}| \). Then \( t_0 := d^n \in T^{F_x} \). Let \( c \in \tilde{Q}_i^* \) be such that \( c^n = \theta(t_0) \). For any \( t \in T^{F_x} \) and \( j \in \mathbf{Z} \) we set \( \tilde{\theta}(djt) = c^j \theta(t) \).

This is well defined: if \( djt = d'j't' \) with \( j, j' \in \mathbf{Z} \) and \( t, t' \in T^{F_x} \) then \( j' = j + nj_0 \), \( j_0 \in \mathbf{Z} \) and \( t' = t_0^n t \) so that \( \theta(t') = c^{nj_0} \theta(t) \) and \( c^j \theta(t) = c^{j'} \theta(t') \). We show that if \( j, j' \in \mathbf{Z} \) and \( t, t' \in T^{F_x} \) then \( \tilde{\theta}(djt d'j't') = \tilde{\theta}(djt) \tilde{\theta}(d'j't') \) that is \( c^{j+j'} \theta(D^{j'}(t)t') = c^j \theta(t)c^{j'} \theta(t') \); this follows from (a). This proves (b).

Let \( \Gamma = \{(g, \tau) \in G^F \times \tilde{T}^{F_x}; g\tau^{-1} \in G^0\} \), a subgroup of \( G^F \times \tilde{T}^{F_x} \). For \( (g, \tau) \in \Gamma \) we define \( e_{g,\tau} : Y \to Y \) by \( hU^* \mapsto gh\tau^{-1}U^* \). To see that this is well defined we assume that \( h \in G^0 \) satisfies \( h^{-1}F(h) \in U^*xU^* \) and \( (g, \tau) \in \Gamma \); we compute

\[
(\tau h \tau^{-1})^{-1}F(\tau h \tau^{-1}) = \tau h^{-1}g^{-1}F(h)F(\tau^{-1}) = \tau h^{-1}F(h)F(\tau^{-1}) \in U^*xU^*F(\tau^{-1}) = U^*xU^*,
\]

since \( \tau x F(\tau^{-1}) = x \) (that is \( F_x(\tau) = \tau \)). Note that \( (g, \tau) \mapsto e_{g,\tau} \) is an action of \( \Gamma \) on \( Y \) (extending the action of \( G^0F \times T^{F_x} \)). Hence \( \Gamma \) acts on \( H^i_c(Y) \) by \( (g, \tau) \mapsto e_{g^{-1},\tau^{-1}}^* \). Note that \( H^i_c(Y)_\theta \) is a \( \Gamma \)-stable subspace of \( H^i_c(Y) \). This follows from the identity

\[
e_{g^{-1},\tau^{-1}} - e_{1,t^{-1}} = e_{1,t^{-1}}e_{g^{-1},\tau^{-1}} - e_{g^{-1},\tau^{-1}}
\]

for \( g \in G^F \), \( \tau \in \tilde{T}^{F_x} \), \( t \in T^{F_x} \) together with the identity \( \theta(t) = \theta(\tau^{-1}t\tau) \) which is a consequence of (a).

For \( g \in G^F \) we define \( e_g : H^i_c(Y)_\theta \to H^i_c(Y)_\theta \) by

\[
e_g(\xi) = \tilde{\theta}(\tau)e_{g^{-1},\tau^{-1}}^*\xi
\]

for any \( \xi \in H^i_c(Y)_\theta \) and any \( \tau \in \tilde{T}^{F_x} \) such that \( g\tau^{-1} \in G^0 \). Assume that \( \tau' \in \tilde{T}^{F_x} \) is another element such that \( g\tau'^{-1} \in G^0 \). Then \( \tau' = \tau t \) with \( t \in T^{F_x} \) and

\[
\tilde{\theta}(\tau')e_{g^{-1},\tau'^{-1}}^*\xi = \tilde{\theta}(\tau) \theta(t)e_{g^{-1},\tau^{-1}}e_{1,t^{-1}}^*\xi = \tilde{\theta}(\tau)e_{g^{-1},\tau^{-1}}^*\xi
\]

so that \( e_g \) is well defined. For \( g, g' \) in \( G^F \) we choose \( \tau, \tau' \) in \( \tilde{T}^{F_x} \) such that \( g\tau^{-1} \in G^0, g'\tau'^{-1} \in G^0 \); we have

\[
e_{gg'} \xi = \tilde{\theta}(\tau') \tilde{\theta}(\tau)e_{g^{-1},\tau^{-1}}e_{g'^{-1},\tau'^{-1}}e_{\tau\tau'^{-1}}^*\xi = \tilde{\theta}(\tau')e_{(gg')^{-1},(\tau\tau'^{-1})^{-1}}^*\xi = e_{gg'} \xi.
\]

We see that
\( g \mapsto \epsilon_g \) defines a \( GF \)-module structure on \( H_c^i(Y_\theta) \) extending the \( G^{0F} \)-module structure in 2.1.

(Note that this extension depends on the choice of \( \tilde{\theta} \).) We show:

(c) If \( (g, \tau) \in \Gamma \) then \( F^r e_{g,\tau} : Y \to Y \) is the Frobenius map of an \( F_q \)-rational structure on \( Y \).

Since \( e_{g,t} \) is a part of a \( \Gamma \)-action, it has finite order. Since \( F^r = F^r_x : G \to G \) (see 2.1), we see that \( F^r : Y \to Y \) commutes with \( e_{g,\tau} : Y \to Y \). Hence (c) holds.

2.4. We preserve the setup of 2.3 and assume in addition that \( L \) satisfies 1.3(i). Let \( i_0 = 2 \dim U^* - l(w) \). Using 2.2(a), 2.3(c) and Grothendieck’s trace formula we see that for \( (g, d) \in \Gamma \) we have

\[
(-1)^{i_0}(w) \tilde{\theta}(d) q^{i_0 r/2} \text{tr}(\epsilon_g, H^i_c(Y_\theta))
\]

\[
= \tilde{\theta}(d) \sum_i (-1)^i \text{tr}((F^r)^* \epsilon_g, H^i_c(Y_\theta)) = \sum_i (-1)^i \text{tr}((F^r)^* \epsilon_g, H^i_c(Y_\theta))
\]

\[
= \sum_i (-1)^i |T^F_x|^{-1} \sum_{t \in T^F_x} \text{tr}((F^r)^* \epsilon_g, H^i_c(Y_\theta)) \tilde{\theta}(t)
\]

\[
= |T^F_x|^{-1} \sum_{t \in T^F_x} \text{tr}((F^r)^* \epsilon_g, H^i_c(Y_\theta)) \tilde{\theta}(t)
\]

\[
= |T^F_x|^{-1} \sum_{t \in T^F_x} |Y^{F^r} \epsilon_{g-1,(dt)^{-1}}| \tilde{\theta}(t)
\]

\[
= |T^F_x|^{-1} \sum_{t \in T^F_x} |\{hU^* \in (G^0/U^*); h^{-1} F(h) \in U^* U^*, h^{-1} g^{-1} F^r(h) dt \in U^*\}| \tilde{\theta}(t).
\]

3. Proof of Theorem 1.2

3.1. Let \( A, \psi, \chi_{\psi} \) be as in 1.2. Let \( L, w \) be as in the end of 1.2. Let \( x \in [w] \). From 1.2(a) we see that 2.1(i) holds. Let \( r \geq 1 \) be as in 2.1. Let

\[
w = (w, F(w), \ldots, F^{r-1}(w)).
\]

By the choice of \( r \) we have \( wF(w) \ldots F^{r-1}(w) = 1 \). Define a morphism \( \tilde{F} : Z_w \to Z_w \) by

\[
\tilde{F}(B_0, B_1, \ldots, B_r, g) = (F(g^{-1} B_{r-1}), F(B_0), F(B_1), \ldots, F(B_{r-1}), F(g)).
\]

We show:

(a) Let \( g \in D^F \) and let \( \tilde{F}_g : p_w^{-1}(g) \to p_w^{-1}(g) \) be the restriction of \( \tilde{F} : Z_w \to Z_w \). Then \( \tilde{F}_g \) is the Frobenius map of an \( F_q \)-rational structure on \( p_w^{-1}(g) \).

It is enough to note that the map \( B_{r+1} \to B_{r+1} \) given by

\[
(B_0, B_1, \ldots, B_r) \mapsto (F(g^{-1} B_{r-1}), F(B_0), F(B_1), \ldots, F(B_{r-1}))
\]
is the composition of the map

\[ F' : (B_0, B_1, \ldots, B_r) \mapsto (F(B_0), F(B_1), \ldots, F(B_r)) \]

(the Frobenius map of an $F_q$-rational structure on $B^{r+1}$) with the automorphism

\[ \left( B_0, B_1, \ldots, B_r \right) \mapsto (g^{-1}B_{r-1}g, B_0, B_1, \ldots, B_{r-1}) \]

of $B^{r+1}$ which commutes with $F'$ and has finite order (since $g$ has finite order in $G$).

Let $d \in \hat{T}_D^F$. Define a morphism $\tilde{F}' : \hat{Z}_{w,d} \to \hat{Z}_{w,d}$ by

\[ \tilde{F}'(h_0U^*, h_1B^*, \ldots, h_{r-1}B^*, h_rU^*, g) = (h'_0U^*, h'_1B^*, \ldots, h'_{r-1}B^*, h'_rU^*, F(g)) \]

where

\[ h'_0 = F(g^{-1}h_{r-1}k(h^{-1}_{r-1}h_r)x^{-1}d), \quad h'_r = F(h_{r-1}k(h^{-1}_{r-1}h_r)x^{-1}), \]

\[ h'_i = F(h_{i-1}) \text{ for } i \in [1, r-1]. \]

This is well defined since

\[ (F(h_{r-1}k(h^{-1}_{r-1}h_r)x^{-1})^{-1}F(g)F(g^{-1}h_{r-1}k(h^{-1}_{r-1}h_r))x^{-1}dd^{-1} = 1. \]

We show that the $T$-action on $\hat{Z}_{w,d}$ (see 1.3) satisfies $\tilde{F}'(t_0\hat{x}) = F_x(t_0)\tilde{F}'(\hat{x})$ for $t_0 \in T, \hat{x} \in \hat{Z}_{w,d}$. Let $(h_i)$ be as above. We must show:

\[ F(g^{-1}h_{r-1}k(h^{-1}_{r-1}h_r)dt_0^{-1}d^{-1})x^{-1}d = F(g^{-1}h_{r-1}k(h^{-1}_{r-1}h_r))x^{-1}dxF(t_0^{-1})x^{-1}, \]

\[ F(h_{r-1}k(h^{-1}_{r-1}h_r)dt_0^{-1}d^{-1})x^{-1} = F(h_{r-1}k(h^{-1}_{r-1}h_r)x^{-1}dxF(t_0)^{-1}x^{-1}d^{-1}, \]

which follow from $F(d) = x^{-1}dx$. Note that

(b) $a_w \tilde{F}' = \tilde{F}a_w : \hat{Z}_{w,d} \to \hat{Z}_w$.

We show:

(c) $|a_w^{-1}(y)\tilde{F}'| = |T^F_x|$ for any $y \in Z_{w}^{\tilde{F}}$.

Since $a_w^{-1}(y)$ is a homogeneous $T$-space this follows from Lang’s theorem applied to $(T, F_x)$.

We have

(d) $p_w \tilde{F} = Fp_w : Z_w \to D$.

3.2. We show:

(a) $b_w \tilde{F}' = F_xb_w : \hat{Z}_{w,d} \to T$.

Let $(h_0, h_1, \ldots, h_r, g) \in (G^0)^{r+1} \times D$ be such that

\[ (h_0U^*, h_1B^*, \ldots, h_{r-1}B^*, h_rU^*, g) \in \hat{Z}_{w,d}. \]
Let \((h'_1, h'_2, \ldots, h'_r)\) be as in 3.1. We set
\[
\mu = k(h_0^{-1}h_1)k(h_1^{-1}h_2) \cdots k(h_{r-1}^{-1}h_r) \in T,
\]
\[
\mu' = k(h_0^{-1}h_1)k(h_1^{-1}h_2) \cdots k(h_{r-2}^{-1}h_{r-1}) \in B^*F^{r-1}(x)^{-1}B^*
\]
\[
\tilde{\mu} = k(h'_0^{-1}h'_1)k(h'_1^{-1}h'_2) \cdots k(h'_{r-1}^{-1}h'_r) \in T,
\]
so that \(\mu = \mu'k(h_{r-1}^{-1}h_r)\) and
\[
\tilde{\mu} = k(d^{-1}xF(k(h_{r-1}^{-1}h_r)^{-1}h_{r-1}^{-1}gh_0))
\times k(F(h_0^{-1}h_1) \cdots k(F(h_{r-3}^{-1}h_{r-2}))k(F(h_{r-2}^{-1}h_{r-1}k(h_{r-1}^{-1}h_r))x^{-1})
\]
\[
= d^{-1}xF(k(h_{r-1}^{-1}h_r)^{-1})F(d)k(F(d^{-1})F(h_{r-1}^{-1}gh_0))F(\mu')F(k(h_{r-1}^{-1}h_r))x^{-1}
\]
\[
= d^{-1}xF(d)F(\mu)x^{-1} = XF(\mu)x^{-1} = F_x(\mu),
\]
as required.

### 3.3
Let \(\phi : F_x^{*}\mathcal{L} \xrightarrow{\sim} \mathcal{L} , \theta : T^{F_x} \rightarrow \hat{Q}_l^{*}\) be as in 2.1. We shall denote by \(?\) the various isomorphisms induced by \(\phi\) such as:
(a) \(\hat{F}^{**}b_w^{*}\mathcal{L} \xrightarrow{\sim} b_w^{*}\mathcal{L}\) (see 3.2(a)),
(b) \(\hat{F}^{**}a_w^{*}\mathcal{L} \xrightarrow{\sim} a_w^{*}\mathcal{L}\) (coming from (a)),
(c) \(a_w^{*}\hat{F}^{*}\mathcal{L} \xrightarrow{\sim} a_w^{*}\mathcal{L}\) (coming from (b) and 3.1(b)),
(d) \(\hat{F}^{*}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\) (coming from (c)),
(e) \(p_w^{*}\hat{F}^{*}\mathcal{L} \xrightarrow{\sim} p_w^{*}\mathcal{L}\) (coming from (d)),
(f) \(F^{*}p_w^{*}\mathcal{L} \xrightarrow{\sim} p_w^{*}\mathcal{L}\) (coming from (e) and 3.1(d)),
(g) \(F^{*}(p_w^{*}\mathcal{L}[l_w]) \xrightarrow{\sim} p_w^{*}\mathcal{L}[l_w]\) (coming from (f)).

### 3.4
For any \(g \in D^F\) we compute
\[
\sum_i (-1)^i \text{tr}(\mathcal{H}_g(p_w^{*}\mathcal{L}_w)) = \sum_i (-1)^i \text{tr}(H^i_c(p_w^{-1}(g), \mathcal{L}_w))
\]
\[
= \sum_{y \in p_w^{-1}(g);
\hat{F}(y) = y} \text{tr}(?, \mathcal{L}_w)_{y}
\]
where \(\mathcal{H}_i\) is the \(i\)-th cohomology sheaf. (The last two sums are equal by the Grothendieck trace formula applied in the context of 3.1(a).) Using 3.1(c) we see that the last sum equals
\[
|T^{F_x}|^{-1} \sum_{\tilde{g} \in a_w^{-1}(p_w^{-1}(g))} \text{tr}(?, (a_w^{*}\mathcal{L}_w)_{\tilde{g}}) = |T^{F_x}|^{-1} \sum_{\tilde{g} \in a_w^{-1}(p_w^{-1}(g))} \text{tr}(?, (b_w^{*}\mathcal{L}_w)_{\tilde{g}})
\]
\[
= |T^{F_x}|^{-1} \sum_{\tilde{g} \in a_w^{-1}(p_w^{-1}(g))} \text{tr}(?, (\mathcal{L}_w)_{b_w(\tilde{g})}).
\]
Now $a_{w}^{-1}(p_{w}^{-1}(g))$ can be identified with the set of all

$$(h_{0}U^{*}, h_{1}B^{*}, \ldots, h_{r-1}B^{*}, h_{r}U^{*}) \in (G^{0}/U^{*}) \times (G^{0}/B^{*}) \times \ldots \times (G^{0}/B^{*}) \times (G^{0}/U^{*})$$

such that

(a) $k(h_{i-1}^{-1}h_{i}) \in F^{i-1}(x)T$ for $i \in [1, r]$, 
(b) $h_{r}^{-1}g_{0}d^{-1} \in U^{*}$,
(c) $h_{0}U^{*} = F(g^{-1}h_{r-1}k(h_{r-1}^{-1}h_{r}))x^{-1}dU^{*}$,
(d) $h_{i}B^{*} = F(h_{i-1})B^{*}$ for $i \in [1, r - 1]$.

(We then have automatically $h_{r}U^{*} = F(h_{r-1}k(h_{r-1}^{-1}h_{r})x^{-1}U^{*})$. If $h_{0}U^{*}$ is given, then (d) determines successively $h_{2}B^{*}, \ldots, h_{r-1}B^{*}$ in a unique way and (b) determines $h_{r}U^{*}$ in a unique way. We see that the equations (a)-(d) are equivalent to the following equations for $h_{0}U^{*}$:

$$h_{0}^{-1}F(h_{0}) \in B^{*}x^{B^{*}}, \quad F^{-1}(h_{0})^{-1}g_{0}d^{-1} \in B^{*}F^{-1}(x)B^{*},$$

$$F^{r}(h_{0})^{-1}g_{0}d^{-1}U^{*} = k(F^{r}(h_{0})^{-1}gF(h_{0})F(d^{-1}))x^{-1}U^{*}$$

(if $r \geq 2$) and

$$h_{0}^{-1}g_{0}d^{-1} \in B^{*}x^{B^{*}}, \quad F^{r}(h_{0})^{-1}g_{0}d^{-1}U^{*} = k(F(h_{0})^{-1}gF(h_{0})F(d^{-1}))x^{-1}U^{*}$$

(if $r = 1$). In both cases these equations are equivalent to

(e) $h_{0}^{-1}F(h_{0}) \in U^{*}txF(t)^{-1}U^{*}, \quad F^{r}(h_{0})^{-1}g_{0}d^{-1} \in F^{r}(t)U^{*}$

for some $t \in T$. We then have $F^{-1}(h_{0})^{-1}g_{0}d^{-1} \in U^{*}F^{-1}(t)F^{-1}(x)U^{*}$. For $h_{0}U^{*}, t$ as in (e) we compute

$$k(h_{0}^{-1}F(h_{0})), k(F(h_{0})^{-1}F^{2}(h_{0})), \ldots k(F^{-2}(h_{0})^{-1}F^{-1}(h_{0}))k(F^{-1}(h_{0})^{-1}g_{0}d^{-1})$$

$$= (txF(t)^{-1})(F(t)F(x)F^{2}(t^{-1})), \ldots (F^{-2}(t)F^{-1}(t)F^{-1}(t^{-1}))(F^{-1}(t)F^{r-1}(t))$$

$$= txF(x) \ldots F^{-1}(x) = t.$$

By 3.2(a) the result of the last computation is necessarily in $T^{F_{x}}$. Thus $F_{x}(t) = t$. Hence $F^{r}(t) = t$ and the equations (e) become

(f) $h_{0}^{-1}F(h_{0}) \in U^{*}x^{U^{*}}, \quad F^{r}(h_{0})^{-1}g_{0}d^{-1} \in T^{F_{x}}U^{*}$.

We see that

$$\sum_{i}(-1)^{i}\text{tr}(?, \mathcal{H}_{j}(\bar{p}_{w}^{\dagger} \hat{L}_{w})) = |T^{F_{x}}|^{-1} \sum_{t \in T^{F_{x}}} a_{t} = |T^{F_{x}}|^{-1} \sum_{t' \in T^{F_{x}}} a'_{t'},$$

where

$$a_{t} = |\{hU^{*} \in (G^{0}/U^{*}); h^{-1}F(h) \in U^{*}x^{U^{*}}, dh^{-1}g_{0}F^{r}(h)t \in U^{*}\}| \theta(t),$$
$a'_t = |\{hU^* \in (G^0/U^*); h^{-1}F(h) \in U^*xU^*, h^{-1}g^{-1}F'(h)dt' \in U^*\}|\theta(dt'd^{-1})$.

Comparing with the last formula in 2.4 and using $\theta(dt'd^{-1}) = \theta(t')$ for $t' \in T^{Fz}$ we obtain (with $i_0$ as in 2.4):

$$\sum_i (-1)^{i} tr(?, H^i_g(p_{w!}L_w)) = (-1)^{(w)}\tilde{\theta}(d)q^{i_0r/2} tr(\epsilon_g, H^{i_0}_c(Y)_{\theta}).$$

Let us choose an isomorphism $p_{w!}L_w[l_w] \cong p_{\emptyset!}L_{\emptyset}$. (This exists by 1.4; note that 1.4(i) holds by 1.5.) Via this isomorphism, the isomorphism 3.3(g) corresponds to an isomorphism $F^* (p_{\emptyset!}L_{\emptyset}) \to p_{\emptyset!}L_{\emptyset}$ that is to an isomorphism $\psi' : F^* A \cong A$ so that

$$\sum_i (-1)^{i} tr(?, H^i_g(p_{w!}L_w)) = \sum_i (-1)^{i} tr(\psi', H^i_g(A))$$

for any $g \in D^F$. (We use that $l_w$ is even.) Since $A$ is irreducible, we must have $\psi = \lambda'\psi'$ for some $\lambda' \in \mathbb{Q}_l^*$. It follows that

$$\sum_{i \in \mathbb{Z}} (-1)^{i} tr(\psi, H^i_g(A)) = \lambda'(-1)^{(w)}\tilde{\theta}(d)q^{i_0r/2} tr(\epsilon_g, H^{i_0}_c(Y)_{\theta})$$

for any $g \in D^F$. Thus Theorem 1.2 holds with $V$ being the $G^F$-module $H^{i_0}_c(Y)_{\theta}$, which is irreducible (even as a $G^{0F}$-module) if $G^0$ has connected centre, but is not necessarily irreducible in general.

**References**

[DL] P.Deligne and G.Lusztig, *Representations of reductive groups over finite fields*, Ann.Math. 103 (1976), 103-161.

[DM] F.Digne and J.Michel, *Groupes réductifs non connexes*, Ann.Sci. École Norm.Sup. 27 (1994), 345-406.

[L1] G.Lusztig, *Green functions and character sheaves*, Ann.Math. 131 (1990), 355-408.

[L2] G. Lusztig, *Remarks on computing irreducible characters*, J.Amer.Math.Soc. 5 (1992), 971-986.

[L3] G.Lusztig, *Character sheaves on disconnected groups, I*, Represent. Th. (electronic) 7 (2003), 374-403; II 8 (2004), 72-124; III 8 (2004), 125-144; IV 8 (2004), 145-178; Errata 8 (2004), 179-179; V 8 (2004), 346-376; VI 8 (2004), 377-413; VII 9 (2005), 209-266; VIII 10 (2006), 314-352; IX 10 (2006), 353-379.

[M] G.Malle, *Generalized Deligne-Lusztig characters*, J.Algebra 159 (1993), 64-97.

[S] T.Shoji, *Character sheaves and almost characters of reductive groups*, Adv.in Math. 111 (1995), 244-313; II 111 (1995), 314-354.

Department of Mathematics, M.I.T., Cambridge, MA 02139