RECURSION FORMULAS FOR $G_1$ AND $G_2$ HORN HYPERGEOMETRIC FUNCTIONS

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Abstract. The aim of this paper is to present various recursion formulas for Horn hypergeometric functions by the contiguous relations of hypergeometric series. These recursion formulas allow us to state the functions $G_1$ and $G_2$ Horn hypergeometric functions as a combination of themselves.

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1. INTRODUCTION

The first $G_1$ and $G_2$ Horn hypergeometric functions were defined by the series [3–5]

$$G_1 (\alpha, \beta, \beta'; x, y) = \sum_{m, p=0}^{\infty} (\alpha)_m (\beta)_p (\beta')_{m-p} \frac{x^m y^p}{m! p!}$$

$$|x| < r, \quad |y| < s, \quad r + s = 1$$

and

$$G_2 (\alpha, \alpha', \beta, \beta'; x, y) = \sum_{m, p=0}^{\infty} (\alpha)_m (\alpha')_p (\beta')_{m-p} \frac{x^m y^p}{m! p!}$$

$$|x| < 1, \quad |y| < 1,$$

respectively. Recently, Opps et al. [1] have obtained some recursion formulas for the function $F_2$ by the contiguous relation of the Gauss hypergeometric function $_2F_1$. In [6], Wang gave some recursion formulas for Appell hypergeometric functions. The aim of our present investigation is to construct various recursion formulas for each of Horn hypergeometric functions $G_1$ and $G_2$.

Recall that gamma function is defined in [2, 3] by

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} \, dt, \quad \text{Re}(n) > 0.$$
The Pochhammer symbol \((\lambda)_n\) is denoted by
\[
(\lambda)_n := \lambda (\lambda + 1) \cdots (\lambda + n - 1), \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \quad \text{and} \quad (\lambda)_0 := 1
\]
and its well known form is also given in [1] as
\[
(\lambda)_n = \frac{(-1)^n}{(1-\lambda)_n}, \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.1}
\]
It easily follows from (1.1) that
\[
(\lambda)_{m-n} = (\lambda)_m (\lambda + m)_{-n}, \tag{1.2}
\]
for \(m, n \in \mathbb{N}\).

2. Recursion Formulas of \(G_1\)

In this section, we give some recursion formulas for the function \(G_1\). We start the following theorem.

**Theorem 1.** Recursion formulas for the function \(G_1\) are as follows:

\[
G_1(\alpha + n, \beta, \beta'; x, y) = G_1(\alpha, \beta, \beta'; x, y) \tag{2.1}
\]

\[
+ (\beta)_{-1} \beta' x \sum_{k=1}^{n} G_1(\alpha + k, \beta - 1, \beta' + 1; x, y)
\]

\[
+ \beta (\beta')_{-1} y \sum_{k=1}^{n} G_1(\alpha + k, \beta, \beta' - 1; x, y)
\]

and

\[
G_1(\alpha - n, \beta, \beta'; x, y) = G_1(\alpha, \beta, \beta'; x, y) \tag{2.2}
\]

\[
- (\beta)_{-1} \beta' x \sum_{k=0}^{n-1} G_1(\alpha - k, \beta - 1, \beta' + 1; x, y)
\]

\[
- \beta (\beta')_{-1} y \sum_{k=0}^{n-1} G_1(\alpha - k, \beta, \beta' - 1; x, y)
\]

**Proof.** From the definition of the function \(G_1\) and transformation
\[
(\alpha + 1)_{m+p} = (\alpha)_{m+p} \left( 1 + \frac{m}{\alpha} + \frac{p}{\alpha} \right)
\]
we can get the following relation:

\[
G_1(\alpha + 1, \beta, \beta'; x, y) = G_1(\alpha, \beta, \beta'; x, y) \tag{2.3}
\]

\[
+ (\beta)_{-1} \beta' x G_1(\alpha + 1, \beta - 1, \beta' + 1; x, y)
\]

\[
+ \beta (\beta')_{-1} y G_1(\alpha + 1, \beta + 1, \beta' - 1; x, y).
\]
By applying this contiguous relation to function \( G_1 \) with the parameter \( \alpha + 2 \), we have

\[
G_1 (\alpha + 2, \beta, \beta'; x, y) = G_1 (\alpha + 1, \beta, \beta'; x, y) + (\beta)_{-1} \beta' x G_1 (\alpha + 2, \beta - 1, \beta' + 1; x, y)
+ \beta \left( \beta' \right)_{-1} y G_1 (\alpha + 2, \beta + 1, \beta' - 1; x, y)
= G_1 (\alpha, \beta, \beta'; x, y)
+ (\beta)_{-1} \beta' x \left[ G_1 (\alpha + 1, \beta - 1, \beta' + 1; x, y) + G_1 (\alpha + 2, \beta - 1, \beta' + 1; x, y) \right]
+ \beta \left( \beta' \right)_{-1} y \left[ G_1 (\alpha + 1, \beta + 1, \beta' - 1; x, y) + G_1 (\alpha + 2, \beta + 1, \beta' - 1; x, y) \right]
\]

\[
G_1 (\alpha + n, \beta, \beta'; x, y) = G_1 (\alpha, \beta, \beta'; x, y)
+ (\beta)_{-1} \beta' x \sum_{k=1}^{n} G_1 (\alpha + k, \beta - 1, \beta' + 1; x, y)
+ \beta \left( \beta' \right)_{-1} y \sum_{k=1}^{n} G_1 (\alpha + k, \beta + 1, \beta' - 1; x, y)
\]

If we compute the function \( G_1 \) with the parameter \( \alpha + n \) by relation \( (2.3) \) for \( n \) times, we find the formula given by \( (2.1) \). Replacing \( \alpha \) by \( \alpha - 1 \) in the contiguous relation \( (2.3) \), we get

\[
G_1 (\alpha - 1, \beta, \beta'; x, y) = G_1 (\alpha, \beta, \beta'; x, y) - (\beta)_{-1} \beta' x G_1 (\alpha, \beta - 1, \beta' + 1; x, y)
- \beta \left( \beta' \right)_{-1} y G_1 (\alpha, \beta + 1, \beta' - 1; x, y).
\]

If we apply this relation to the function \( G_1 \) with the parameter \( \alpha - n \) for \( n \) times, we obtain the recursion formula \( (2.2) \) similar to the proof of formula \( (2.1) \).

By the same contiguous relations \( (2.1) \) and \( (2.2) \) we can express the functions \( G_1 \) in the above theorem in other forms.

**Theorem 2.** The function \( G_1 \) satisfies the recursion formulas:

\[
G_1 (\alpha + n, \beta, \beta'; x, y) = \sum_{i=0}^{n} \sum_{k=0}^{n-i} \binom{n}{i} \binom{n-i}{k} (\beta)_{i-k} \left( \beta' \right)_{k-i} x^{k} y^{i} G_1 (\alpha + i + k, \beta + i - k, \beta' + k - i; x, y)
\]

\[
G_1 (\alpha - n, \beta, \beta'; x, y) = \sum_{i=0}^{n} \sum_{k=0}^{n-i} \binom{n}{i} \binom{n-i}{k} (\beta)_{i-k} \left( \beta' \right)_{k-i} x^{-k} y^{-i} G_1 (\alpha + i + k, \beta + i - k, \beta' + k - i; x, y)
\]
Proof. By the induction method, we only prove the recursion formula given by (2.4). For \( n = 1 \), formula (2.4) is satisfied. Assume that the result (2.4) is true for \( n = t \). Then, we need to show that the relation (2.4) is satisfied for \( n = t + 1 \). Setting \( n = t \) in (2.4), we have

\[
G_1(\alpha + t, \beta, \beta'; x, y) = \sum_{i=0}^{t} \sum_{k=0}^{t-i} \binom{t}{i} \binom{t-i}{k} (\beta)_{i-k} (\beta')_{k-i} x^k y^i G_1(\alpha + i + k, \beta + i - k, \beta' + k - i; x, y).
\]

Replacing \( \alpha \) by \( \alpha + 1 \) in the above relation, we obtain

\[
G_1(\alpha + t + 1, \beta, \beta'; x, y) = \sum_{i=0}^{t} \sum_{k=0}^{t-i} \binom{t}{i} \binom{t-i}{k} (\beta)_{i-k} (\beta')_{k-i} x^k y^i G_1(\alpha + i + k + 1, \beta + i - k, \beta' + k - i; x, y).
\]

In the above equality, we apply the contiguous relation (2.3) with the transformations \( \alpha \to \alpha + i + k, \beta \to \beta + i - k, \beta' \to \beta' + k - i \). Using the relations

\[
\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}
\]

and

\[
\binom{n}{k} = 0, \text{ when } k > n \text{ or } k < 0
\]

and with some simplifications, we deduce

\[
G_1(\alpha + t + 1, \beta, \beta'; x, y)
\]

\[
= \sum_{i=0}^{t} \sum_{k=0}^{t-i} \binom{t}{i} \binom{t-i}{k} (\beta)_{i-k} (\beta')_{k-i} x^k y^i G_1(\alpha + i + k, \beta + i - k, \beta' + k - i; x, y)
\]

\[
+ \sum_{i=0}^{t} \sum_{k=0}^{t-i} \binom{t}{i} \binom{t-i}{k} (\beta)_{i-k} (\beta')_{k-i} x^k y^i (\beta + i - k - 1)_{-1} (\beta' + k - i)_{-1}
\]

\[
G_1(\alpha + 1 + i + k, \beta - 1 + i - k, \beta' + 1 + k - i; x, y)
\]

\[
+ \sum_{i=0}^{t} \sum_{k=0}^{t-i} \binom{t}{i} \binom{t-i}{k} (\beta)_{i-k} (\beta')_{k-i} x^k y^{i+1} (\beta + i - k)_{1} (\beta' + k - i)_{-1}
\]

\[
G_1(\alpha + 1 + i + k, \beta + 1 + i - k, \beta' + 1 + k - i; x, y)
\]

\[
= \sum_{i=0}^{t+1} \sum_{k=0}^{t+1-i} \binom{t+1}{i} \binom{t+1-i}{k} (\beta)_{i-k} (\beta')_{k-i} x^k y^i
\]
where we replace \( k \) by \( k - 1 \) in the second summation term and \( i \) by \( i - 1 \) in the third summation term in the first equality. So, we obtain the recursion formula (2.4). In a similar manner, the relation (2.5) can be easily proved.

**Theorem 3.** For the function \( G_1 \), we have

\[
G_1 (\alpha, \beta + n, \beta'; x, y) = G_1 (\alpha, \beta, \beta'; x, y) + \alpha (\beta')_{-1} y \sum_{k=1}^{n} G_1 (\alpha + 1, \beta + k, \beta' - 1; x, y)
- \alpha \beta' x \sum_{k=1}^{n} \frac{1}{(\beta + k - 1)} G_1 (\alpha + 1, \beta + k - 2, \beta' + 1; x, y)
\]

and

\[
G_1 (\alpha, \beta - n, \beta'; x, y) = G_1 (\alpha, \beta, \beta'; x, y) - \alpha (\beta')_{-1} y \sum_{k=1}^{n} G_1 (\alpha + 1, \beta - k + 1, \beta' - 1; x, y)
- \alpha \beta' x \sum_{k=1}^{n} \frac{1}{(\beta - k)} G_1 (\alpha + 1, \beta - k - 1, \beta' + 1; x, y).
\]

**Proof.** Using the definition of the function \( G_1 \) and the equality

\[
(\beta + 1)_{p-m} = (\beta)_{p-m} \left( 1 + \frac{p}{\beta} - \frac{m}{\beta} \right)
\]

we can easily obtain the contiguous function

\[
G_1 (\alpha, \beta + 1, \beta'; x, y) = G_1 (\alpha, \beta, \beta'; x, y) + \alpha (\beta')_{-1} y G_1 (\alpha + 1, \beta + 1, \beta' - 1; x, y)
- \alpha \frac{(\beta)_{-1}}{\beta} \beta' x G_1 (\alpha + 1, \beta - 1, \beta' + 1; x, y).
\]

If we apply this contiguous relation for two times for the function \( G_1 \) with the parameter \( \beta + 2 \), we get

\[
G_1 (\alpha, \beta + 2, \beta'; x, y) = G_1 (\alpha, \beta + 1, \beta'; x, y) + \alpha (\beta')_{-1} y G_1 (\alpha + 1, \beta + 2, \beta' - 1; x, y)
- \alpha \frac{(\beta + 1)_{-1}}{(\beta + 1)} x G_1 (\alpha + 1, \beta, \beta' + 1; x, y)
= G_1 (\alpha, \beta, \beta'; x, y).
\]
By iterating this method on $G_1$ with $\beta + n$ for $n$ times, we find
\[
G_1 (\alpha, \beta + n, \beta'; x, y) = G_1 (\alpha, \beta + 1, \beta'; x, y)
\]
\[
+ \alpha (\beta')^{-1} \sum_{k=1}^{n} G_1 (\alpha + 1, \beta + k, \beta' - 1; x, y)
\]
\[
- \alpha \beta' \sum_{k=1}^{n} \frac{(\beta' + k - 1)_{-1}}{(\beta' + k - 1)} G_1 (\alpha + 1, \beta + k - 2, \beta' + 1; x, y).
\]

Replacing $\beta$ by $\beta - 1$ in contiguous relation (2.8), we obtain
\[
G_1 (\alpha, \beta - 1, \beta'; x, y) = G_1 (\alpha, \beta, \beta'; x, y) - \alpha (\beta')^{-1} \sum_{k=1}^{n} G_1 (\alpha + 1, \beta, \beta' - 1; x, y)
\]
\[
+ \alpha \frac{(\beta - 1)_{-1}}{(\beta - 1)} \beta' x G_1 (\alpha + 1, \beta - 2, \beta' + 1; x, y).
\]

If we apply this relation to the function $G_1$ with the parameter $\beta - n$ for $n$ times, we obtain the recursion formulas (2.7). □

**Theorem 4.** For the function $G_1$, the equalities
\[
G_1 (\alpha, \beta, \beta' + n; x, y) = G_1 (\alpha, \beta, \beta'; x, y)
\]
\[
+ \alpha (\beta')^{-1} \sum_{k=1}^{n} G_1 (\alpha + 1, \beta - 1, \beta' + k; x, y)
\]
\[
- \alpha \beta y \sum_{k=1}^{n} \frac{(\beta' + k - 1)_{-1}}{(\beta' + k - 1)} G_1 (\alpha + 1, \beta + 1, \beta' + k - 2; x, y)
\]

and
\[
G_1 (\alpha, \beta, \beta' - n; x, y) = G_1 (\alpha, \beta, \beta'; x, y)
\]
\[
- \alpha (\beta')^{-1} \sum_{k=1}^{n} G_1 (\alpha + 1, \beta - 1, \beta' - k + 1; x, y)
\]
\[
+ \alpha \beta y \sum_{k=1}^{n} \frac{(\beta' - k)_{-1}}{(\beta' - k)} G_1 (\alpha + 1, \beta + 1, \beta' - k - 1; x, y)
\]

hold.
Proof. If we use following equalities
\[
G_1 (\alpha, \beta, \beta' + 1; x, y) = G_1 (\alpha, \beta, \beta'; x, y) \\
+ \alpha (\beta)_{-1} x G_1 (\alpha + 1, \beta - 1, \beta' + 1; x, y) \\
- \alpha \beta (\beta')_{-1} y G_1 (\alpha + 1, \beta + 1, \beta' - 1; x, y)
\]
\[
G_1 (\alpha, \beta, \beta' - 1; x, y) = G_1 (\alpha, \beta, \beta'; x, y) \\
- \alpha (\beta)_{-1} x G_1 (\alpha + 1, \beta - 1, \beta'; x, y) \\
+ \alpha \beta (\beta' - 1)_{-1} y G_1 (\alpha + 1, \beta + 1, \beta' - 2; x, y)
\]
we obtain recursion formulas given by (2.9) and (2.10).

3. Recursion Formulas of \(G_2\)

In this section, we give some recursion formulas for the function \(G_2\). We first present the recursion formulas for the function \(G_2\) about the parameter \(\alpha\) and \(\alpha'\).

**Theorem 5.** The function \(G_2\) satisfies the recursion formulas:
\[
G_2 (\alpha + n, \alpha', \beta, \beta'; x, y) = G_2 (\alpha, \alpha', \beta, \beta'; x, y) \\
+ (\beta)_{-1} \beta' x \sum_{k=1}^{n} G_2 (\alpha + k, \alpha', \beta - 1, \beta' + 1; x, y)
\]
and
\[
G_2 (\alpha - n, \alpha', \beta, \beta'; x, y) = G_2 (\alpha, \alpha', \beta, \beta'; x, y) \\
- (\beta)_{-1} \beta' x \sum_{k=0}^{n-1} G_2 (\alpha - k, \alpha', \beta - 1, \beta' + 1; x, y).
\]

**Proof.** By the definition of the function \(G_2\), we get
\[
G_2 (\alpha + 1, \alpha', \beta, \beta'; x, y) = G_2 (\alpha, \alpha', \beta, \beta'; x, y) \\
+ (\beta)_{-1} \beta' x G_2 (\alpha + 1, \alpha', \beta - 1, \beta' + 1; x, y).
\]
Applying this relation \(n\) times recursively, as the same as we have done in the proof of Theorem 1, we immediately have complete the proof (3.1).

The function \(G_2\) in the above theorem can be expressed in other forms as follows:

**Theorem 6.** For the function \(G_2\), the equalities
\[
G_2 (\alpha + n, \alpha', \beta, \beta'; x, y)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} (\beta)_{-k} (\beta')_{k} x^k G_2(\alpha + k, \alpha', \beta - k, \beta' + k; x, y) \tag{3.3}
\]

and
\[
G_2(\alpha - n, \alpha', \beta, \beta'; x, y) = \sum_{k=0}^{n} \binom{n}{k} (\beta)_{-k} (\beta')_{k} (-x)^k G_2(\alpha, \alpha', \beta - k, \beta' + k; x, y), \tag{3.4}
\]

hold.

**Proof.** By the inductive method as we have done in the proof of Theorem 2, the proof can be easily seen. \(\square\)

**Theorem 7.** The function \(G_2\) satisfies the following recursion formulas
\[
G_2(\alpha, \alpha' + n, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) \tag{3.5}
+ \beta (\beta')_{-1} y \sum_{k=1}^{n} G_2(\alpha, \alpha' + k, \beta + 1, \beta' - 1; x, y)
= \sum_{k=0}^{n} \binom{n}{k} (\beta)_{k} (\beta')_{-k} x^k G_2(\alpha, \alpha' + k, \beta + k, \beta' - k; x, y)
\]

and
\[
G_2(\alpha, \alpha' - n, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) \tag{3.6}
- \beta (\beta')_{-1} y \sum_{k=0}^{n-1} G_2(\alpha, \alpha' - k, \beta + 1, \beta' - 1; x, y)
= \sum_{k=0}^{n} \binom{n}{k} (\beta)_{k} (\beta')_{-k} (-y)^k G_2(\alpha, \alpha' - k, \beta + k, \beta' - k; x, y)
\]

**Proof.** By the definition of the function \(G_2\), we get
\[
G_2(\alpha, \alpha' + 1, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y)
+ \beta (\beta')_{-1} y G_2(\alpha, \alpha' + 1, \beta + 1, \beta' - 1; x, y)
\]

and
\[
G_2(\alpha, \alpha' - 1, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y)
- \beta (\beta')_{-1} y G_2(\alpha, \alpha' - 1, \beta + 1, \beta' - 1; x, y).
\]

By applying this relation for \(n\) times, as the same as we have done in the proof of Theorem 1, we complete the proof. \(\square\)
Now, we present the recursion formulas of the function $G_2$ about the parameter $\beta$ and $\beta'$.

**Theorem 8.** Recursion formulas for the function $G_2$ are as follows

\[
G_2(\alpha, \alpha', \beta + n, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) + \alpha'(\beta')^{-1} y \sum_{k=1}^{n} G_2(\alpha, \alpha' + 1, \beta + k, \beta' - 1; x, y)
\]

\[-\alpha \beta' x \sum_{k=1}^{n} \frac{(\beta + k - 1)}{(\beta + k - 1)} G_2(\alpha + 1, \alpha', \beta + 2, \beta' + 1; x, y) \]

and

\[
G_2(\alpha, \alpha', \beta - n, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) - \alpha'(\beta')^{-1} y \sum_{k=1}^{n} G_2(\alpha, \alpha' + 1, \beta - k + 1, \beta' - 1; x, y)
\]

\[+ \alpha \beta' x \sum_{k=1}^{n} \frac{(-k)}{(-k)} G_2(\alpha + 1, \alpha', \beta - 1, \beta' + 1; x, y).\]

**Proof:** From the definition of the function $G_2$, we have the following relation:

\[
G_2(\alpha, \alpha', \beta + 1, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) + \alpha'(\beta')^{-1} y G_2(\alpha, \alpha' + 1, \beta + 1, \beta' - 1; x, y)
\]

\[-\alpha \beta' x \frac{(\beta - 1)}{\beta} G_2(\alpha + 1, \alpha', \beta - 1, \beta' + 1; x, y).\]

If we apply this relation for $n$ times, we can easily prove the theorem by the same method as we have done in the proof of Theorem 3. \qed

**Theorem 9.** Recursion formulas for the function $G_2$ are as follows

\[
G_2(\alpha, \alpha', \beta, \beta' + n; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) + \alpha(\beta')^{-1} x \sum_{k=1}^{n} G_2(\alpha + 1, \alpha', \beta - 1, \beta' + k; x, y)
\]

\[-\alpha' \beta y \sum_{k=1}^{n} \frac{(\beta' + k - 1)}{(\beta' + k - 1)} G_2(\alpha + 1, \alpha', \beta + 1, \beta' + k - 2; x, y)\]

and

\[
G_2(\alpha, \alpha', \beta, \beta' - n; x, y)
\]
\[
\begin{align*}
&= G_2(\alpha, \alpha', \beta, \beta'; x, y) - \alpha (\beta)_{-1} x \sum_{k=1}^{n} G_2(\alpha + 1, \alpha', \beta - 1, \beta' - k + 1; x, y) \\
&+ \alpha' \beta y \sum_{k=1}^{n} \frac{(\beta' - k)_{-1}}{(\beta' - k)} G_2(\alpha, \alpha' + 1, \beta + 1, \beta' - k - 1; x, y). 
\end{align*}
\]

**Proof.** From the definition of the function $G_2$, we have the following relation:

\[
G_2(\alpha, \alpha', \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) + \alpha (\beta)_{-1} x G_2(\alpha + 1, \alpha', \beta - 1, \beta + 1; x, y) \\
- \alpha' \beta y \frac{(\beta')_{-1}}{\beta'} G_2(\alpha, \alpha' + 1, \beta + 1, \beta' - 1; x, y). 
\]

If we apply this relation for $n$ times, we can easily prove the theorem by the same method as we have done in the proof of Theorem 3. \hfill \Box

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