BOUNDINESS OF MODULI OF VARIETIES OF GENERAL TYPE

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Abstract. We show that the family of semi log canonical pairs with ample log canonical class and with fixed volume is bounded.

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Date: December 4, 2014.

The first author was partially supported by DMS-1300750, DMS-1265285 and a grant from the Simons foundation, the second author was partially supported by NSF research grant no: 0701101, no: 1200656 and no: 1265263 and this research was partially funded by the Simons foundation and by the Mathematische Forschungsinstitut Oberwolfach and the third author was partially supported by “The Recruitment Program of Global Experts” grant from China. Part of this work was completed whilst the second and third authors were visiting the Freiburg Institute of Advanced Studies and they would like to thank Stefan Kebekus and the Institute for providing such a congenial place to work. We are grateful to János Kollár and Mihai Păun for many useful comments and suggestions.
1. Introduction

The aim of this paper is to show that the moduli functor of semi log canonical stable pairs is bounded:

**Theorem 1.1.** Fix an integer $n$, a positive rational number $d$ and a set $I \subset [0,1]$ which satisfies the DCC.

Then the set $\mathcal{F}_{slc}(n,d,I)$ of all log pairs $(X, \Delta)$ such that

1. $X$ is projective of dimension $n$,
2. $(X, \Delta)$ is semi log canonical,
3. the coefficients of $\Delta$ belong to $I$,
4. $K_X + \Delta$ is an ample $\mathbb{Q}$-divisor, and
5. $(K_X + \Delta)^n = d$,

is bounded.

In particular there is a finite set $I_0$ such that $\mathcal{F}_{slc}(n,d,I) = \mathcal{F}_{slc}(n,d,I_0)$.

The main new technical result we need to prove (1.1) is to show that abundance behaves well in families:

**Theorem 1.2.** Suppose that $(X, \Delta)$ is a log pair where the coefficients of $\Delta$ belong to $(0,1] \cap \mathbb{Q}$. Let $\pi : X \rightarrow U$ be a projective morphism to a smooth variety $U$. Suppose that $(X, \Delta)$ is log smooth over $U$.

If there is a closed point $0 \in U$ such that the fibre $(X_0, \Delta_0)$ has a good minimal model then $(X, \Delta)$ has a good minimal model over $U$ and every fibre has a good minimal model.

**Corollary 1.3.** Let $(X, \Delta)$ be a log pair where $\Delta$ is a $\mathbb{Q}$-divisor and let $X \rightarrow U$ be a projective morphism to a variety $U$.

Then the subset $U_0 \subset U$ of points $u \in U$ such that the fibre $(X_u, \Delta_u)$ is divisorially log terminal and has a good minimal model is constructible.

**Corollary 1.4.** Let $\pi : X \rightarrow U$ be a projective morphism to a smooth variety $U$ and let $(X, \Delta)$ be log smooth over $U$. Suppose that the coefficients of $\Delta$ belong to $(0,1] \cap \mathbb{Q}$.

If there is a closed point $0 \in U$ such that the fibre $(X_0, \Delta_0)$ has a good minimal model then the restriction morphism

$$\pi_* \mathcal{O}_X(m(K_X + \Delta)) \rightarrow H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u)))$$

is surjective for any $m \in \mathbb{N}$ such that $m\Delta$ is integral and for any closed point $u \in U$.

In particular if $\psi : X \rightarrow Z$ is the ample model of $(X, \Delta)$ then $\psi_u : X_u \rightarrow Z_u$ is the ample model of $(X_u, \Delta_u)$ for every closed point $u \in U$.

The moduli space of stable curves is one of the most intensively studied varieties. The moduli space of stable varieties of general type
is the higher dimensional analogue of the moduli space of curves. Unfortunately constructing this moduli space is more complicated than constructing the moduli space of curves. In particular it does not seem easy to use GIT to construct the moduli space in higher dimensions; for example see [28] for a precise example of how badly behaved the situation can be. Instead Kollár and Shepherd-Barron initiated a program to construct the moduli space in all dimensions in [25]. This program was carried out in large part by Alexeev for surfaces, [1] and [2].

We recall the definition of the moduli functor. For simplicity, in the definition of the functor, we restrict ourselves to the case with no boundary. We refer to the forthcoming book [17] for a detailed discussion of this subject and to [22] for a more concise survey.

**Definition 1.5** (Moduli of slc models, cf. [22, 29]). Let $H(m)$ be an integer valued function. The moduli functor of semi log canonical models with Hilbert function $H$ is

$$M_{H}^{slc}(S) = \begin{cases} \text{flat projective morphisms } X \to S, \text{ whose} \\
\text{fibres are slc models with ample canonical class} \\
\text{and Hilbert function } H(m), \omega_X \text{ is flat over } S \\
\text{and commutes with base change.} \end{cases}$$

In this paper we focus on the problem of showing that the moduli functor is bounded, so that if we fix the degree, we get a bounded family. The precise statement is given in (1.1). We now describe the proof of (1.1). We first explain how to reduce to (1.2).

For curves if one fixes the genus $g$ then the moduli space is irreducible. In particular stable curves are always limits of smooth curves. This fails in higher dimensions, so that there are components of the moduli space whose general point corresponds to a non-normal variety, or better, a semi log canonical variety.

Fortunately, cf. [21, 23, 24] and [23, 5.13], one can reduce boundedness of semi log canonical pairs to boundedness of log canonical pairs in a straightforward manner. If $(X, \Delta)$ is semi log canonical then let $n: Y \to X$ be the normalisation. $X$ has nodal singularities in codimension one, so that informally $X$ is obtained from $Y$ by identifying points of the double locus, the closure of the codimension one singular locus. More precisely, we may write

$$K_Y + \Gamma = n^{*}(K_X + \Delta),$$

where $\Gamma$ is the sum of the strict transform of $\Delta$ plus the double locus and $(Y, \Gamma)$ is log canonical. If $K_X + \Delta$ is ample then $(X, \Delta)$ is determined by $(Y, \Gamma)$ and the data of the involution $\tau: S \to S$ of the normalisation.
of the double locus. Note that the involution $\tau$ fixes the different, the divisor $\Theta$ defined by adjunction in the following formula:

$$(K_Y + \Gamma)|_S = K_S + \Theta.$$ 

Conversely, if $(Y, \Gamma)$ is log canonical, $K_Y + \Gamma$ is ample, $\tau$ is an involution of the normalisation $S$ of a divisor supported on $[\Gamma]$ which fixes the different, then we may construct a semi log canonical pair $(X, \Delta)$, whose normalisation is $(Y, \Gamma)$ and whose double locus is $S$.

Note that $\tau$ fixes the pullback $L$ of the very ample line bundle determined by a multiple of $K_X + \Delta$. The group of all automorphisms of $S$ which fixes $L$ is a linear algebraic group. It follows, by standard properties of the scheme Isom, that if $(Y, \Gamma)$ is bounded then $\tau$ is bounded.

Thus to prove (1.1) it suffices to prove the result, when $X$ is normal, that is, when $(X, \Delta)$ is log canonical, cf. (7.3). The first problem is that a priori $X$ might have arbitrarily many components. Note that if $X = C$ is a curve of genus $g$ then $K_X$ has degree $2g - 2$ and so $X$ has at most $2g - 2$ components. In higher dimensions the situation is more complicated since $K_X$ is not necessarily Cartier and so $d$ is not necessarily an integer.

Instead we use [12, 1.3], which was conjectured by Alexeev [1] and Kollár [19]:

**Theorem 1.6.** Fix a positive integer $n$ and a set $I \subset [0, 1]$ which satisfies the DCC. Let $\mathcal{D}$ be the set of log canonical pairs $(X, \Delta)$ such that the dimension of $X$ is $n$ and the coefficients of $\Delta$ belong to $I$.

Then the set

$$\{ \text{vol}(X, K_X + \Delta) | (X, \Delta) \in \mathcal{D} \},$$

also satisfies the DCC.

Since there are only finitely many ways to write $d$ as a sum of elements $d_1, d_2, \ldots, d_k$ taken from a set which satisfies the DCC, cf. (2.4.1), we are reduced to proving (1.1) when $X$ is normal and irreducible.

Let $\mathfrak{F} \subset \mathfrak{F}_{\text{slc}}(n, d, I)$ be the subset of all log canonical pairs $(X, \Delta)$ where $X$ is irreducible. Since the coefficients of $\Delta$ belong to a set which satisfies the DCC, [12, 1.3] implies that some fixed multiple of $K_X + \Delta$ defines a birational map to projective space. As the degree of $K_X + \Delta$ is bounded by assumption, $\mathfrak{F}$ is log birationally bounded, that is, there is a log pair $(Z, B)$ and a projective morphism $\pi: Z \to U$, such that given any $(X, \Delta) \in \mathfrak{F}$, we may find $u \in U$ such that $X$ is birational to
and the strict transform $\Phi$ of $\Delta$ plus the exceptional divisors are components of $B_u$.

[12, 1.6] proves that $\mathfrak{F}$ is a bounded family provided if in addition we assume that the total log discrepancy of $(X, \Delta)$ is bounded away from zero (meaning that the coefficients of $\Delta$ are bounded away from one as well as the log discrepancy is bounded away from zero). For applications to moduli this is far too strong; the double locus occurs with coefficient one.

Instead we proceed as follows. By standard arguments we may assume that $U$, smooth, the morphism $\pi$ is smooth and its restriction to any strata of $B$ is smooth, that is, $(Z, B)$ is log smooth over $U$. We first reduce to the case when $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$. We are looking for a higher model $Y \to Z$ such that $\text{vol}(Y, K_Y + \Gamma) = d$ where $\Gamma$ is the transform of $\Delta$ plus the exceptionals. At this point we use some of the ideas that go into the proof of [11, 1.9]. By deformation invariance of log plurigenera we may assume that $U$ is a point, (7.2).

In general $\text{vol}(Z_u, K_{Z_u} + \Phi) \geq \text{vol}(X, K_X + \Delta) = d$. Since the volume satisfies the DCC, (1.6), we may assume that the model $Z$ minimises the supremum of $\text{vol}(Z, K_Z + \Phi)$. In this case, by a standard diagonalisation argument, we are given a sequence of pairs $(X_i, \Delta_i) \in \mathfrak{F}$ and it suffices to find a higher model $Y \to Z$ where the limit has smaller volume. This follows using some results from [11], cf. (7.1).

So we may assume that $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$. Since $(X, \Delta)$ is log canonical and $K_X + \Delta$ is ample, we can recover $(X, \Delta)$ from $(Z_u, \Phi)$ as the log canonical model, cf. (2.2.2). Conversely if $u \in U$ is a point such that $(Z_u, \Phi)$ has a log canonical model, $f : Z_u \to X$, where

$$X = \text{Proj} R(Z_u, K_{Z_u} + \Phi) \quad \text{and} \quad \Delta = f_* \Phi,$$

the coefficients of $0 \leq \Phi \leq B_u$ belong to $I$ and $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$ then $(X, \Delta) \in \mathfrak{F}$.

It therefore suffices to prove that the set of fibres with a log canonical model is constructible. Note that $(X, \Delta)$ has a log canonical model if and only if the log canonical section ring

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, O_X(m(K_X + \Delta)))$$

is finitely generated. Conjecturally every fibre has a log canonical model. Once again the problem are the components of $\Delta$ with coefficient one. The main result of [6] implies that if there are no components of $\Delta$ with coefficient one, that is, $(X, \Delta)$ is kawamata log terminal, then the log canonical section ring is finitely generated.
In general, (2.9.1), the existence of the log canonical model $Z$ is equivalent to the existence of a good minimal model $f: X \rightarrow Y$, that is, a model $(Y, \Gamma)$ such that $K_Y + \Gamma$ is semi-ample. In this case the log canonical model is simply the model $Y \rightarrow Z$ such that $K_Y + \Gamma$ is the pullback of an ample divisor.

In fact we prove, (1.2), a much stronger result. We prove that if one fibre $(X_0, \Delta_0)$ has a good minimal model then every fibre has a good minimal model. By [14, 1.1] it suffices to prove that every fibre over an open subset has a good minimal model, equivalently, that the generic fibre has a good minimal model.

Let $\eta \in U$ be the generic point. We may assume that $U$ is affine. We prove the existence of a good minimal model for the pair $(X_\eta, \Delta_\eta)$ in two steps. We first show that $(X_\eta, \Delta_\eta)$ has a minimal model. For this we run the $(K_X + \Delta)$-MMP with scaling of an ample divisor. We know that if we run the $(K_{X_0} + \Delta_0)$-MMP with scaling of an ample divisor then this MMP terminates with a good minimal model. However, if we run the $(K_X + \Delta)$-MMP we might lose the property that $X_0$ is irreducible, the flipping locus might be an extra component of the new central fibre. Using [14, 2.10] and (5.3) we can reduce to the case when the diminished stable base locus of $K_{X_0} + \Delta_0$ does not contain any non-canonical centres. In this case we show, (3.1), that every step of the $(K_X + \Delta)$-MMP induces a $(K_{X_0} + \Delta_0)$-negative map. This generalises [11, 4.1], which assumes that $U$ is a curve and that $(X, \Delta)$ is terminal.

This MMP ends $f: X \rightarrow Y$ with a minimal model for the generic fibre, (3.2).

To finish off we need to show that the minimal model is a good minimal model. There are two cases. We may write $(X, \Delta = S + B)$, where $S = \lfloor \Delta \rfloor$.

In the first case, if $K_X + (1 - \epsilon)S + B$ is not pseudo-effective for any $\epsilon > 0$ then we may run $Y \rightarrow W$ the $(K_X + (1 - \epsilon)S + B)$-MMP until we reach a Mori fibre space (5.2) $W \rightarrow Z$. If $\epsilon > 0$ is sufficiently small, this MMP induces a $(K_{X_0} + \Delta_0)$-non-positive map, see (5.1). It follows that this MMP is $(K_X + \Delta)$-non-positive. We know that there is a component $D$ of $S$ whose image dominates the base $Z$ of the Mori fibre space. By induction the generic fibre of the image $E$ of $D$ in $Y$ is a good minimal model. The restriction $E \rightarrow F$ of the map $Y \rightarrow W$ need not be a birational contraction but we won’t lose semi-ampleness. The image of the divisor is pulled back from $Z$ and so $K_X + \Delta$ has a semi-ample model.

In the second case $K_X + (1 - \epsilon)S + B$ is pseudo-effective. As $K_X + (1 - \epsilon)S + B$ is kawamata log terminal, it follows by work of B. Berndtsson and M. Păun, (4.1), that the Kodaira dimension is invariant, see
As $K_X + (1 - \epsilon)S + B$ is pseudo-effective and $(X_0, \Delta_0)$ has a good minimal model, it follows that $K_{X_0} + \Delta_0$ is abundant, that is, the Kodaira dimension is the same as the numerical dimension. By deformation invariance of log plurigenera the generic fibre is abundant. As the restriction of $K_Y + \Gamma$ to every component of coefficient one is semi-ample, the restriction of $K_Y + \Gamma$ to the sum of the coefficient one part is semi-ample by (2.5.1) and we are done by (2.6.1).

2. Preliminaries

2.1. Notations and Conventions. We will follow the terminology from [24]. Let $f: X \rightarrow Y$ be a proper birational map of normal quasi-projective varieties and let $p: W \rightarrow X$ and $q: W \rightarrow Y$ be a common resolution of $f$. We say that $f$ is a birational contraction if every $p$-exceptional divisor is $q$-exceptional. If $D$ is an $\mathbb{R}$-Cartier divisor on $X$ such that $D' := f_*D$ is $\mathbb{R}$-Cartier then we say that $f$ is $D$-non-positive (resp. $D$-negative) if we have $p^*D = q^*D' + E$ where $E \geq 0$ and $E$ is $q$-exceptional (respectively $E$ is $q$-exceptional and the support of $E$ contains the strict transform of the $f$-exceptional divisors).

We say a proper morphism $\pi: X \rightarrow U$ is a contraction morphism if $\pi_*\mathcal{O}_X = \mathcal{O}_U$. Recall that for any divisor $D$ on $X$, the sheaf $\pi_*\mathcal{O}_X(D)$ is defined to be $\pi_*\mathcal{O}_X([D])$. If we are given a morphism $X \rightarrow U$, then we say that $(X, \Delta)$ is log smooth over $U$ if $(X, \Delta)$ has simple normal crossings and both $X$ and every stratum of $(X, \Delta)$ is smooth over $U$, where $D$ is the support of $\Delta$. If $\pi: X \rightarrow U$ and $Y \rightarrow U$ are projective morphisms, $f: X \rightarrow Y$ is a birational contraction over $U$ and $(X, \Delta)$ is a log canonical pair (respectively divisorially log terminal $\mathbb{Q}$-factorial pair) such that $f$ is $(K_X + \Delta)$-non-positive (respectively $(K_X + \Delta)$-negative) and $K_Y + \Gamma$ is nef over $U$ (respectively and $Y$ is $\mathbb{Q}$-factorial), then we say that $f: X \rightarrow Y$ is a weak log canonical model (respectively a minimal model) of $K_X + \Delta$ over $U$.

We say $K_Y + \Gamma$ is semi-ample over $U$ if there exists a surjective morphism $\psi: Y \rightarrow Z$ over $U$ such that $K_Y + \Gamma \sim_{\mathbb{R}} \psi^*A$ for some $\mathbb{R}$-divisor $A$ on $Y$ which is ample over $U$. Equivalently, when $K_Y + \Gamma$ is $\mathbb{Q}$-Cartier, $K_Y + \Gamma$ is semi-ample over $U$ if there exists an integer $m > 0$ such that $\mathcal{O}_Y(m(K_Y + \Gamma))$ is generated over $U$. Note that in this case

$$R(Y/U, K_Y + \Gamma) := \bigoplus_{m \geq 0} \pi_*\mathcal{O}_Y(m(K_Y + \Gamma))$$

is a finitely generated $\mathcal{O}_U$-algebra, and

$$Z = \text{Proj} \ R(Y/U, K_Y + \Gamma).$$
If $K_Y + \Gamma$ is semi-ample and big over $U$, then $Z$ is the log canonical model of $(X, \Delta)$ over $U$. A weak log canonical model $f : X \rightarrow Y$ is called a semi-ample model if $K_Y + \Gamma$ is semi-ample.

Let $D$ be an $\mathbb{R}$-Cartier divisor on a projective variety $X$. Let $C$ be a prime divisor. If $D$ is big then

$$\sigma_C(D) = \inf \{ \text{mult}_C(D') \mid D' \sim_{\mathbb{R}} D, D' \geq 0 \}.$$ 

Now let $A$ be any ample $\mathbb{Q}$-divisor. Following [27], let

$$\sigma_C(D) = \lim_{\epsilon \rightarrow 0} \sigma_C(D + \epsilon A).$$

Then $\sigma_C(D)$ exists and is independent of the choice of $A$. There are only finitely many prime divisors $C$ such that $\sigma_C(D) > 0$ and the $\mathbb{R}$-divisor $N_\sigma(X, D) = \sum_C \sigma_C(D) C$ is determined by the numerical equivalence class of $D$, cf. [6, 3.3.1] and [27] for more details.

Following [27] we define the numerical dimension

$$\kappa_\sigma(X, D) = \max_{H \in \text{Pic}(X)} \{ k \in \mathbb{N} \mid \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD + H))}{m^k} > 0 \}.$$ 

If $D$ is nef then this is the same as

$$\nu(X, D) = \max \{ k \in \mathbb{N} \mid H^{n-k} \cdot D^k > 0 \}$$

for any ample divisor $H$, see [27]. $D$ is called abundant if $\kappa_\sigma(X, D) = \kappa(X, D)$, that is, the numerical dimension is equal to the Iitaka dimension. If we drop the condition that $X$ is projective and instead we have a projective morphism $\pi : X \rightarrow U$, then an $\mathbb{R}$-Cartier divisor $D$ on $X$, is called abundant over $U$ if its restriction to the generic fibre is abundant.

If $(X, \Delta)$ is a log pair then a non canonical centre is the centre of a valuation of log discrepancy less than one.

### 2.2. The volume.

**Definition 2.2.1.** Let $X$ be a normal $n$-dimensional irreducible projective variety and let $D$ be an $\mathbb{R}$-divisor. The volume of $D$ is

$$\text{vol}(X, D) = \limsup_{m \rightarrow \infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$ 

Let $V \subset X$ be an irreducible subvariety of dimension $d$. Suppose that $D$ is $\mathbb{R}$-Cartier. The restricted volume of $D$ along $V$ is

$$\text{vol}(X|V, D) = \limsup_{m \rightarrow \infty} \frac{d!(\dim \text{Im}(H^0(X, \mathcal{O}_X(mD)) \rightarrow H^0(V, \mathcal{O}_V(mD))))}{m^d}.$$
Lemma 2.2.2. Let $f : X \to Z$ be a birational morphism between log canonical pairs $(X, \Delta)$ and $(Z, B)$. Suppose that $K_X + \Delta$ is big and that $(X, \Delta)$ has a log canonical model $g : X \to Y$.

If $f_*\Delta \leq B$ and $\text{vol}(X, K_X + \Delta) = \text{vol}(Z, K_Z + B)$ then the induced birational map $Z \to Y$ is the log canonical model of $(Z, B)$.

Proof. Let $\pi : W \to X$ be a log resolution of $(X, C + F)$, which also resolves the map $g$, where $C$ is the strict transform of $B$ and $F$ is the sum of the $f$-exceptional divisors. We may write $K_W + \Theta = \pi^*(K_X + \Delta) + E$, where $\Theta \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Theta = \Delta$ and $\pi_*E = 0$. Then the log canonical model of $(W, \Theta)$ is the same as the log canonical model of $(X, \Delta)$. Replacing $(X, \Delta)$ by $(W, \Theta)$ we may assume that $(X, C + F)$ is log smooth and $g : X \to Y$ is a morphism. Replacing $(Z, B)$ by the pair $(X, D = C + F)$, we may assume $Z = X$.

If $A = g_*(K_X + \Delta)$ and $H = g^*A$ then $A$ is ample and $K_X + \Delta - H \geq 0$. Let $L = D - \Delta \geq 0$, let $S$ be a component of $L$ with coefficient $a$ and let

$$v(t) = \text{vol}(X, H + tS).$$

Then $v(t)$ is a non-decreasing function of $t$ and

$$v(0) = \text{vol}(X, H) = \text{vol}(X, K_X + \Delta) = \text{vol}(X, K_X + D) \geq \text{vol}(X, H + L) \geq \text{vol}(X, H + aS) = v(a).$$

Thus $v(t)$ is constant over the range $[0, a]$. [26.4.25 (iii)] implies that

$$\frac{1}{n} \frac{dv}{dt} \bigg|_{t=0} = \text{vol}_{X|S}(H) \geq S \cdot H^{n-1} = g_*S \cdot A^{n-1}$$

so that $g_*S = 0$. But then every component of $L$ is exceptional for $g$ and $g$ is the log canonical model of $(X, D)$. \hfill \Box

2.3. Deformation Invariance.

Lemma 2.3.1. Let $\pi : X \to U$ be a projective morphism to a smooth variety $U$ and let $(X, \Delta)$ be a log smooth pair over $U$.

If the coefficients of $\Delta$ belong to $[0, 1]$ then

$$N_\sigma(X, K_X + \Delta)|_{X_u} = N_\sigma(X_u, K_{X_u} + \Delta_u)$$

for every $u \in U$. 9
Proof. Pick a relatively ample Cartier divisor $A$ such that $(X, \Delta + A)$ is log smooth over $U$. Fix $u \in U$. Then [11, 1.8.1] implies that

$$f_\ast \mathcal{O}_X(m(K_X + \Delta) + A) \to H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u) + A_u))$$

is surjective for all positive integers $m$ such that $m\Delta$ is integral. It follows that

$$N_\sigma(X, K_X + \Delta)|_{X_u} \leq N_\sigma(X_u, K_{X_u} + \Delta_u)$$

and the reverse inequality is clear. □

Lemma 2.3.2. Let $\pi : X \to U$ be a projective morphism to a smooth variety $U$ and let $(X, \Delta)$ be a log smooth pair over $U$. Let $0 \in U$ be a closed point, let

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)$$

and let $0 \leq \Theta \leq \Delta$ be the unique divisor so that $\Theta_0 = \Theta|_{X_0}$.

If the coefficients of $\Delta$ belong to $[0, 1]$ then

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta).$$

Proof. Fix a positive integer $k$ such that $k\Delta \geq \lceil \Delta \rceil$. Pick a relatively ample Cartier divisor $H$ such that $(X, \Delta + H)$ is log smooth over $U$ and $kK_X + H$ is big. Let $m > k$ be an integer. Consider the commutative diagram

$$\pi_\ast \mathcal{O}_X(m(K_X + \Theta) + H) \to \pi_\ast \mathcal{O}_X(m(K_X + \Delta) + H)$$

$$H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Theta_0) + H_0)) \to H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0) + H_0)).$$

The top row is an inclusion and the bottom row is an isomorphism by assumption. As

$$[m(K_X + \Delta)] + H = (m - k)(K_X + \Delta) + kK_X + H + (k\Delta - \{m\Delta\})$$

is big, the first column is surjective by [11, 1.8.1]. Nakayama’s Lemma implies that the top row is an isomorphism in a neighbourhood of $X_0$. It follows that

$$\Theta \geq \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta).$$

and the reverse inequality follows by (2.3.1). □

Lemma 2.3.3. Let $\pi : X \to U$ be a projective morphism to a smooth variety $U$ and let $(X, D)$ be log smooth over $U$, where the coefficients of $D$ are all one. Let $0 \in U$ be a closed point.

Then the restriction morphism

$$\pi_\ast \mathcal{O}_X(K_X + D) \to H^0(X_0, \mathcal{O}_{X_0}(K_{X_0} + D_0))$$

is surjective.
Proof. Since the result is local we may assume that $U$ is affine. Cutting by hyperplanes we may assume that $U$ is a curve. Thus we want to show that the restriction map

$$H^0(X, \mathcal{O}_X(K_X + X_0 + D)) \rightarrow H^0(X_0, \mathcal{O}_{X_0}(K_{X_0} + D_0))$$

is surjective. This is equivalent to showing that multiplication by a local parameter

$$H^1(X, \mathcal{O}_X(K_X + D)) \rightarrow H^1(X, \mathcal{O}_X(K_X + D + X_0))$$

is injective.

By assumption the image of every strata of $D$ is the whole of $U$ and $0 = (K_X + D) - (K_X + D)$ is semi-ample. Therefore a generalisation of Kollár’s injectivity theorem (see [18], [8, 6.3] and [4, 5.4]) implies that

$$H^1(X, \mathcal{O}_X(K_X + D)) \rightarrow H^1(X, \mathcal{O}_X(K_X + D + X_0))$$

is injective.

2.4. DCC sets.

Lemma 2.4.1. Let $I \subset \mathbb{R}$ be a set of positive real numbers which satisfies the DCC. Fix a constant $d$.

Then the set

$$T = \{(d_1, d_2, \ldots, d_k) | k \in \mathbb{N}, d_i \in I, \sum d_i = d\}$$

is finite.

Proof. As $I$ satisfies the DCC there is a real number $\delta > 0$ such that if $i \in I$ then $i \geq \delta$. Thus

$$k \leq \frac{d}{\delta}.$$ 

It is enough to show that given any infinite sequence $t_1, t_2, \ldots$ of elements of $T$ that we may find a constant subsequence. Possibly passing to a subsequence we may assume that the number of entries $k$ of each vector $t_i = (d_{i_1}, d_{i_2}, \ldots, d_{i_k})$ is constant. Since $I$ satisfies the DCC, possibly passing to a subsequence, we may assume that the entries are not decreasing. Since the sum is constant, it is clear that the entries are constant, so that $t_1, t_2, \ldots$ is a constant sequence. 

Lemma 2.4.2. Let $J$ be a finite set of real numbers at most one.

If

$$I = \{ a \in (0, 1] | a = 1 + \sum_{i \leq k} a_i - k, a_1, a_2, \ldots, a_k \in J \}.$$ 

then $I$ is finite.
Proof. If $a_k = 1$ then
\[ \sum_{i \leq k} a_i - k = \sum_{i \leq k-1} a_i - (k - 1). \]
Thus there is no harm in assuming that $1 \notin J$. If $a_k < 0$ then
\[ 1 + \sum_{i \leq k} a_i - k < 0. \]
Thus we may assume that $J \subset [0, 1)$.

Note that
\[ 1 + \sum_{i \leq k} a_i - k > 0 \quad \text{if and only if} \quad \sum_{i \leq k} (1 - a_i) < 1. \]
Since $J$ is finite we may find $\delta > 0$ such that if $a \in J$ then $1 - a \geq \delta$. This bounds $k$ and the result is clear. \[ \square \]

2.5. **Semi log canonical varieties.** We will need the definition of certain singularities of semi-normal pairs, [20, 7.2.1]. Let $X$ be a semi-normal variety which satisfies Serre’s condition $S_2$ and let $\Delta$ be an $\mathbb{R}$-divisor on $X$, such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $n: Y \to X$ be the normalisation of $X$ and write
\[ K_Y + \Gamma = n^*(K_X + \Delta), \]
where $\Gamma$ is the sum of the strict transform of $\Delta$ and the double locus. We say that $(X, \Delta)$ is **semi log canonical** if $(Y, \Gamma)$ is log canonical and $(X, \Delta)$ is **divisorially semi log terminal** if $(Y, \Gamma)$ is divisorially log terminal. Note that if $(X, \Delta)$ is divisorially log terminal and $S$ is the union of the components of $\lfloor \Delta \rfloor$, then $(S, \Theta)$ is divisorially semi log terminal where
\[ (K_X + \Delta)|_S = K_S + \Theta. \]

**Theorem 2.5.1.** Let $(X, \Delta)$ be a semi log canonical pair and let $n: Y \to X$ be the normalisation. By adjunction we may write
\[ K_Y + \Gamma = n^*(K_X + \Delta), \]
where $(Y, \Gamma)$ is log canonical.

If $X$ is projective and $\Delta$ is a $\mathbb{Q}$-divisor then $K_X + \Delta$ is semi-ample if and only if $K_Y + \Gamma$ is semi-ample.

**Proof.** See [9] or [13, 1.4]. \[ \square \]

Suppose that $(X, \Delta)$ is log canonical and $\pi: X \to U$ is a morphism of quasi-projective varieties. If $(X_0, \Delta_0)$ is the fibre over a closed point $0 \in U$ then note that
\[ (K_X + \Delta)|_{X_0} = K_{X_0} + \Delta_0. \]
2.6. **Base Point Free Theorem.** Recall the following generalization of Kawamata’s theorem:

**Theorem 2.6.1.** Let \((X, \Delta = S + B)\) be a divisorially log terminal pair, where \(S = \lfloor \Delta \rfloor\) and \(B\) is a \(\mathbb{Q}\)-divisor. Let \(H\) be a \(\mathbb{Q}\)-Cartier divisor on \(X\) and let \(X \to U\) be a proper surjective morphism of varieties.

If there is a constant \(a_0\) such that

1. \(H|_S\) is semi-ample over \(U\),
2. \(aH - (K_X + \Delta)\) is nef and abundant over \(U\), for all \(a > a_0\),

then \(H\) is semi-ample over \(U\).

**Proof.** See [15], [3], [8], [7], [9] and [14, 4.1].

2.7. **Minimal models.**

**Lemma 2.7.1.** Let \((X, \Delta)\) be a divisorially log terminal pair where \(X\) is \(\mathbb{Q}\)-factorial and projective. Assume that \(K_X + \Delta\) is pseudo-effective. Suppose that we run \(f: X \to Y\) the \((K_X + \Delta)\)-MMP with scaling of an ample divisor \(A\), so that \((Y, \Gamma + tB)\) is nef, where \(\Gamma = f^*\Delta\) and \(B = f^*A\).

1. If \(F\) is \(f\)-exceptional then \(F\) is a component of \(N_\sigma(X, K_X + \Delta)\).
2. If \(t > 0\) is sufficiently small then every component of \(N_\sigma(X, K_X + \Delta)\) is \(f\)-exceptional.
3. If \((X, \Delta)\) has a minimal model and \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier then \(N_\sigma(X, K_X + \Delta)\) is a \(\mathbb{Q}\)-divisor.

**Proof.** Let \(p: W \to X\) and \(q: W \to Y\) resolve \(f\). As \(f\) is a minimal model of \((X, tA + \Delta)\), for some some \(t \geq 0\), we may write

\[ p^*(K_X + tA + \Delta) = q^*(K_Y + tB + \Gamma) + E, \]

where \(E = E_t \geq 0\) is \(q\)-exceptional. As \(q^*(K_Y + tB + \Gamma)\) is nef, it follows that

\[ N_\sigma(X, K_X + tA + \Delta) = p_*E. \]

As \(A\) is ample, (1) holds. If \(t\) is sufficiently small then \(N_\sigma(X, K_X + tA + \Delta)\) and \(N_\sigma(X, K_X + \Delta)\) have the same support and so (2) holds.

If \((X, \Delta)\) has a minimal model then we may assume that \(t = 0\) and so

\[ N_\sigma(X, K_X + \Delta) = p_*E_0 \]

is a \(\mathbb{Q}\)-divisor. \(\square\)
Lemma 2.7.2. Let \((X, \Delta)\) be a divisorially log terminal pair where \(X\) is \(\mathbb{Q}\)-factorial and projective. Assume that \(K_X + \Delta\) is pseudo-effective.

If \(f : X \rightarrow Y\) is a birational contraction such that \(K_Y + \Gamma = f_*(K_X + \Delta)\) is nef and \(f\) only contracts components of \(N_\sigma(X, K_X + \Delta)\) then \(f\) is a minimal model of \((X, \Delta)\).

Proof. Let \(p : W \rightarrow X\) and \(q : W \rightarrow Y\) resolve \(f\). We may write
\[ p^*(K_X + \Delta) + E = q^*(K_Y + \Gamma) + F, \]
where \(E \geq 0\) and \(F \geq 0\) have no common components and both \(E\) and \(F\) are \(q\)-exceptional.

As \(K_Y + \Gamma\) is nef, the support of \(F\) and the support of \(N_\sigma(Y, q^*(K_Y + \Gamma) + F)\) coincide. On the other hand, every component of \(E\) is a component of \(N_\sigma(X, p^*(K_X + \Delta) + E)\). Thus \(E = 0\) and any divisor contracted by \(f\) is a component of \(F\). \(\square\)

2.8. Blowing up log pairs.

Lemma 2.8.1. Let \((X, \Delta)\) be a log smooth pair.
If \(|\Delta| = 0\) then there is a sequence \(\pi : Y \rightarrow X\) of smooth blow ups of the strata of \((X, \Delta)\) such that if we write
\[ K_Y + \Gamma = \pi^*(K_X + \Delta) + E, \]
where \(\Gamma \geq 0\) and \(E \geq 0\) have no common components, \(\pi_* \Gamma = \Delta\) and \(\pi_* E = 0\), then no two components of \(\Gamma\) intersect.

Proof. This is standard, see for example [10, 6.5]. \(\square\)

Lemma 2.8.2. Let \((X, \Delta)\) be a sub log canonical pair.
We may find a finite set \(I \subset (0, 1]\) such that if \(\pi : Y \rightarrow X\) is any birational morphism and we write
\[ K_Y + \Gamma = \pi^*(K_X + \Delta) \]
then the coefficients of \(\Gamma\) which are positive belong to \(I\).

Proof. Replacing \((X, \Delta)\) by a log resolution we may assume that \((X, \Delta)\) is log smooth. Let \(J\) be the set of coefficients of \(\Delta\) and let \(I\) be the set given by (2.4.2).

Suppose that \(\pi : Y \rightarrow X\) is a birational morphism. We may write
\[ K_Y + \Gamma = \pi^*(K_X + \Delta). \]
We claim that the coefficients of \(\Gamma\) which are positive belong to \(I\). Possibly blowing up more we may assume that \(\pi\) is a sequence of smooth blow ups. If \(Z \subset X\) is smooth of codimension \(k\) and \(a_1, a_2, \ldots, a_k\) are
the coefficients of the components of $\Delta$ containing $Z$ then the coefficient of the exceptional divisor is

$$a = 1 + \sum_{i \leq k} a_i - k.$$ 

If $a > 0$ then $a \in I$ and we are done by induction on the number of blow ups. □

**Lemma 2.8.3.** Let $(X, \Delta)$ be a log smooth pair where the coefficients of $\Delta$ belong to $(0, 1]$ and $X$ is projective.

If $(X, \Delta)$ has a weak log canonical model then there is a sequence $\pi: Y \rightarrow X$ of smooth blow ups of the strata of $\Delta$ such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$ and if we write

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma),$$

then $B_-(Y, K_Y + \Gamma')$ contains no strata of $\Gamma'$. If $\Delta$ is a $\mathbb{Q}$-divisor then $\Gamma'$ is a $\mathbb{Q}$-divisor.

**Proof.** Let $f: X \rightarrow W$ be a weak log canonical model of $(X, \Delta)$. Let $\Phi = f_*\Delta$. Let $I$ be the finite set whose existence is guaranteed by (2.8.2) applied to $(W, \Phi)$.

Suppose that $\pi: Y \rightarrow X$ is a sequence of smooth blow ups of the strata of $\Delta$. We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$.

Let $p: V \rightarrow Y$ and $q: V \rightarrow W$ resolve the induced birational map $Y \rightarrow W$, so that the strict transform of $\Phi$ and the exceptional locus of $q$ has global normal crossings. We may write

$$K_V + \Psi = q^*(K_W + \Phi) + F,$$

where $\Psi \geq 0$ and $F \geq 0$ have no common components, $q_*\Psi = \Phi$ and $q_*F = 0$. Note that the coefficients of $\Psi$ belong to $I$.

As $q^*(K_W + \Phi)$ is nef, $\Psi$ has no components in common with $N_\sigma(V, K_V + \Psi)$. Thus

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma) = p_*\Psi$$

so that the coefficients of $\Gamma'$ belong to $I$.

Suppose that $Z$ is a strata of $(X, \Delta)$ which is contained in $N_\sigma(X, K_X + \Delta)$. Let $\pi: Y \rightarrow X$ blow up $Z$ and let $E$ be the exceptional divisor. The coefficient of $E$ in $\Gamma$ is no more than the minimum coefficient of
any component of $\Delta$ containing $Z$. $E$ is a component of $\Gamma - \Gamma'$, so that the coefficient of $E$ in $\Gamma'$ is strictly less than the coefficient of any component of $\Delta$ containing $Z$. Since $I$ is a finite set and $(X, \Delta)$ has only finitely many strata, it is clear that after finitely many blow ups we must have $\Gamma = \Gamma'$.

Lemma 2.8.4. Let $(X, \Delta)$ be a log pair and let $\pi : X \rightarrow U$ be a morphism of quasi-projective varieties.

Then the subset $U_0 \subset U$ of points $u \in U$ such that the fibre $(X_u, \Delta_u)$ is divisorially log terminal is constructible. Further if $U_1 \subset U_0$ is open then $(X, \Delta)$ is divisorially log terminal over $U_1$.

Proof. It suffices to prove that if $U_0$ is dense then it contains an open subset.

Let $f : Y \rightarrow X$ be a log resolution. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components. Passing to an open subset of $U$ we may assume that $(Y, \Gamma)$ is log smooth over $U$. As $\Gamma_u$ is a boundary for a dense set of points $u \in U_0$, it follows that $\Gamma$ is a boundary.

Suppose that $F$ is an exceptional divisor of log discrepancy zero with respect to $(X, \Delta)$, that is, coefficient one in $\Gamma$. Let $Z = f(F)$ be the centre of $F$ in $X$. Note that $F_u$ has log discrepancy zero with respect to $(X_u, \Delta_u)$; for any $u \in U_0$. As $(X_u, \Delta_u)$ is divisorially log terminal, it follows that $(X_u, \Delta_u)$ is log smooth in a neighbourhood of the generic point of $Z_u$. But then $(X, \Delta)$ is log smooth in a neighbourhood of the generic point of $Z$ and so $(X, \Delta)$ is divisorially log terminal.

But then $(X_u, \Delta_u)$ is divisorially log terminal for some open subset of points $U_1 \subset U$. □

2.9. Good minimal models.

Lemma 2.9.1. Let $(X, \Delta)$ be a divisorially log terminal pair, where $X$ is projective and $\mathbb{Q}$-factorial.

If $(X, \Delta)$ has a weak log canonical model then the following are equivalent

1. every weak log canonical model of $(X, \Delta)$ is a semi-ample model,
2. $(X, \Delta)$ has a semi-ample model, and
3. $(X, \Delta)$ has a good minimal model.

Proof. (1) implies (2) is clear.

We show that (2) implies (3). Suppose that $g : X \rightarrow Z$ is a semi-ample model of $(X, \Delta)$. Let $p : W \rightarrow X$ be a log resolution of $(X, \Delta)$
which also $q: W \to Z$ resolves $g$. We may write

$$K_W + \Phi = p^*(K_X + \Delta) + E,$$

where $\Phi \geq 0$ and $E \geq 0$ have no common components, $p_*\Phi = \Delta$ and $p_*E = 0$. [4, 2.10] implies that $(X, \Delta)$ has a good minimal model if and only if $(W, \Phi)$ has a good minimal model.

Replacing $(X, \Delta)$ by $(W, \Phi)$ we may assume that $g$ is a morphism. We run $f: X \to Y$ the $(K_X + \Delta)$-MMP with scaling of an ample divisor over $Z$. Note that running the $(K_X + \Delta)$-MMP over $Z$ is the same as running the absolute $(K_X + \Delta + H)$-MMP, where $H$ is the pullback of a sufficiently ample divisor from $Z$. Note also that $N_\sigma(X, K_X + \Delta)$ and $N_\sigma(X, K_X + \Delta + H)$ have the same components. By (2) of (2.7.1) we may run the $(K_X + \Delta)$-MMP with scaling until $f$ contracts every component of $N_\sigma(X, K_X + \Delta)$. If $h: Y \to Z$ is the induced birational morphism then $h$ is small. As $h^*(K_Y + \Gamma) = g^*(K_X + \Delta)$ is semi-ample

$$K_Y + \Gamma = h^*h_*(K_Y + \Gamma),$$

is semi-ample and $f$ is a good minimal model. Thus (2) implies (3).

Suppose that $f: X \to Y$ is a minimal model and $g: X \to Z$ is a weak log canonical model. Let $p: W \to Y$ and $q: W \to Z$ be a common resolution over $X$, $r: W \to X$. Then we may write

$$p^*(K_Y + \Gamma) + E_1 = r^*(K_X + \Delta) = q^*(K_Z + \Phi) + E_2$$

where $\Gamma = f_*\Delta$, $\Phi = g_*\Delta$, $E_1 \geq 0$ is $p$-exceptional and $E_2 \geq 0$ is $q$-exceptional. As $f$ is a minimal model and $g$ is a weak log canonical model, every $f$-exceptional divisor is $g$-exceptional. Thus

$$p^*(K_Y + \Gamma) + E = q^*(K_Z + \Phi),$$

where $E = E_1 - E_2$ is $q$-exceptional. Negativity of contraction applied to $q$ implies that $E \geq 0$, so that $E \geq 0$ is $p$-exceptional. Negativity of contraction applied to $p$ implies that $E = 0$. But then $K_Y + \Gamma$ is semi-ample if and only if $K_Z + \Phi$ is semi-ample. Thus (3) implies (1). \[\square\]

**Lemma 2.9.2.** Let $(X, \Delta)$ be a divisorially log terminal pair, where $X$ is $\mathbb{Q}$-factorial and projective. Let $A$ be an ample divisor.

If $(X, \Delta)$ has a good minimal model then there is a constant $\epsilon > 0$ with the following properties:

1. If $g_t: X \to Z_t$ is the log canonical model of $(X, \Delta + tA)$ then $Z_t$ is independent of $t \in (0, \epsilon)$ and there is a morphism $Z_t \to Z_0$.
2. If $h: X \to Y$ is a weak log canonical model of $(X, \Delta + tA)$ for some $t \in [0, \epsilon)$ then $h$ is a semi-ample model of $(X, \Delta)$.  

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Proof. Suppose that we run $f_t : X \dasharrow W_t$ the $(K_X + \Delta)$-MMP with scaling of $A$. [14, 2.9] implies that this MMP terminates with a minimal model, so that we may find $\epsilon > 0$ such that $f = f_0 = f_t : X \dasharrow W = W_t$ is independent of $t \in [0, \epsilon)$. Let $\Phi = f_*\Delta$ and let $B = f_*A$. If $C \subset W$ is a curve then

$$(K_W + \Phi + tB) \cdot C = 0 \text{ whenever } (K_W + \Phi + sB) \cdot C = 0,$$

for all $t \in [0, \epsilon)$ and $s \in (0, \epsilon)$, since $K_W + \Phi + \lambda B$ is nef for all $\lambda \in (0, \epsilon)$. Let

$$Z_t = \text{Proj} R(X, K_X + \Delta + tA),$$

be the ample model. The induced contraction morphism $W \to Z_t$ contracts those curves $C$ such that $(K_W + \Phi + tB) \cdot C = 0$ so that $Z = Z_t$ is independent of $t \in (0, \epsilon)$ and there is a contraction morphism $Z_t \to Z_0$ This is (1).

Let $h : X \dasharrow Y$ be a weak log canonical model of $(X, \Delta + tA)$. Then $h$ is a semi-ample model of $(X, \Delta + tA)$ and there is an induced morphism $\psi : Y \to Z$.

Possibly replacing $\epsilon$ with a smaller number we may assume that $h$ contracts every component of $N_\sigma(X, K_X + \Delta)$ by (2.7.1). Note that if $P$ is a prime divisor which is not a component of $N_\sigma(X, K_X + \Delta)$ then $(K_X + \Delta + tA)|_P$ is big. Thus $h$ also contracts precisely the components of $N_\sigma(X, K_X + \Delta)$. It follows that $\psi$ is a small morphism.

If $\Gamma = h_*\Delta, B = h_*A, \Psi = \psi_*\Gamma$ and $C = \psi_*B$ then

$$K_Y + \Gamma + sB = \psi^*(K_Z + \Psi + sC),$$

for any $s$. By assumption $K_Z + \Psi + sC$ is ample for $s \in (0, \epsilon)$ and so $K_Y + \Gamma + sB$ is nef for $s \in (0, \epsilon)$. Thus $K_Y + \Gamma$ is nef and so $h$ is a semi-ample model of $(X, \Delta)$ by (2.9.1).

Lemma 2.9.3. Let $k$ be any field of characteristic zero. Let $(X, \Delta)$ be a divisorial log terminal pair, where $X$ is $\mathbb{Q}$-factorial and projective. Let $(\bar{X}, \bar{\Delta})$ be the corresponding pair over the algebraic closure $\bar{k}$ of $k$.

Then $(X, \Delta)$ has a good minimal model if and only if $(\bar{X}, \bar{\Delta})$ has a good minimal model.

Proof. If $W$ is a scheme over $k$ then $\bar{W}$ denotes the corresponding scheme over $\bar{k}$. One direction is clear; if $f : X \dasharrow Y$ is a good minimal model of $(X, \Delta)$ then $\bar{f} : \bar{X} \dasharrow \bar{Y}$ is a semi-ample model of $(\bar{X}, \bar{\Delta})$ and so $(\bar{X}, \bar{\Delta})$ has a good minimal model by (2.9.1).

Conversely suppose that $(\bar{X}, \bar{\Delta})$ has a good minimal model. Pick an ample divisor $A$ on $X$. We run $f : X \dasharrow Y$ the $(K_X + \Delta)$-MMP with scaling of $A$. Then $f$ is a weak log canonical model of $(X, \Delta + tA)$ and so $\bar{f} : \bar{X} \dasharrow \bar{Y}$ is a weak log canonical model of $(\bar{X}, \bar{\Delta} + t\bar{A})$. (2.9.2)
implies that we may find $\epsilon > 0$ such that $\tilde{f}$ is a semi-ample model of $(X, \Delta)$ for $t \in [0, \epsilon]$. If $\Gamma = f_\ast \Delta$ then $K_{\tilde{X}} + \Gamma$ is semi-ample so that $K_Y + \Gamma$ is semi-ample. But then $f$ is a good minimal model of $(X, \Delta)$. $\square$

3. The MMP in families I

**Lemma 3.1.** Let $(X, \Delta)$ be a divisorially log terminal pair and let $\pi: X \rightarrow U$ be a projective morphism, where $U$ is smooth, affine, of dimension $k$ and $X$ is $\mathbb{Q}$-factorial. Let $0 \in U$ be a closed point such that

1. there are $k$ divisors $D_1, D_2, \ldots, D_k$ containing $0$ such that if $H_i = \pi^\ast D_i$ and $H = H_1 + H_2 + \cdots + H_k$ is the sum then $(X, H + \Delta)$ is divisorially log terminal,
2. $X_0$ is reduced, $\dim X_0 = \dim X - \dim U$ and $\dim V_0 = \dim V - \dim U$, for all non canonical centres $V$ of $(X, \Delta)$, and
3. $\mathcal{B}_-(X_0, K_{X_0} + \Delta_0)$ contains no non canonical centres of $(X_0, \Delta_0)$. Let $f: X \rightarrow Y$ be a step of the $(K_X + \Delta)$-MMP. If $f$ is birational and $V$ is a non canonical centre of $(X, \Delta)$ then $V$ is not contained in the indeterminacy locus of $f$, $V_0$ is not contained in the indeterminacy locus of $f_0$ and the induced maps $\phi: V \rightarrow W$ and $\phi_0: V_0 \rightarrow W_0$ are birational, where $W = f(V)$. Let $\Gamma = f_\ast \Delta$. Further

1. if $G_i$ is the pullback of $D_i$ to $Y$ and $G = G_1 + G_2 + \cdots + G_k$ is the sum then $(Y, G + \Gamma)$ is divisorially log terminal,
2. $Y_0$ is reduced, $\dim Y_0 = \dim Y - \dim U$ and $\dim W_0 = \dim W - \dim U$, for all non canonical centres $W$ of $(Y, \Gamma)$, and
3. $\mathcal{B}_-(Y_0, K_{Y_0} + \Gamma_0)$ contains no non canonical centres of $(Y_0, \Gamma_0)$. If $V$ is a non kawamata log terminal centre, or $V = X$ then $\phi: V \rightarrow W$ and $\phi_0: V_0 \rightarrow W_0$ are birational contractions.

On the other hand, if $f$ is a Mori fibre space then $f_0$ is not birational.

**Proof.** Suppose that $f$ is birational.

As $f$ is a step of the $(K_X + \Delta)$-MMP and $H$ is pulled back from $U$, it follows that it is also a step of the $(K_X + H + \Delta)$-MMP, and so $(Y, G + \Gamma)$ is divisorially log terminal. As every component of $Y_0$ is a non kawamata log terminal centre of $(Y, G)$ and $X_0$ is irreducible, it follows that $Y_0$ is irreducible.

Let $V$ be a non canonical centre of $(X, \Delta)$. Then $V$ is a non canonical centre of $(X, H + \Delta)$. Let $g: X \rightarrow Z$ be the contraction of the extremal ray associated to $f$ (so that $f = g$ unless $f$ is a flip). Let $Q = g(V)$ and let $\psi: V \rightarrow Q$ be the induced morphism. As $V_0$ is a non canonical centre of $(X_0, \Delta_0)$ it is not contained in $\mathcal{B}_-(X_0, K_{X_0} + \Delta_0)$.
and so the induced morphism $\psi_0: V_0 \rightarrow Q_0$ is birational. As $Q$ is irreducible and dominates $U$, and the dimension of the fibres of $V \rightarrow Q$ are upper-semicontinuous, $\psi$ is birational. Thus $V$ does not belong to the indeterminacy locus of $f$, $V_0$ does not belong to the indeterminacy locus of $f_0$, and both $\phi: V \rightarrow W$ and $\phi_0: V_0 \rightarrow W_0$ are birational.

Now suppose that $V$ is a non kawamata log terminal centre or $V = X$. If $V$ is a non kawamata log terminal centre then $V$ is a non canonical centre and so $\phi: V \rightarrow W$ and $\phi_0: V_0 \rightarrow W_0$ are both birational. We can define divisors $\Sigma_0$ and $\Theta_0$ on $V_0$ and $W_0$ by adjunction:

$$(K_{X_0} + \Delta_0)|_{V_0} = K_{V_0} + \Sigma_0.$$  

and

$$(K_{Y_0} + \Gamma_0)|_{W_0} = K_{W_0} + \Theta_0.$$  

If $P$ is a divisor on $W_0$ and $f$ is not an isomorphism at the generic point of the centre $N$ of $P$ on $V_0$ then

$$a(P; V_0; \Sigma_0) < a(P; W_0; \Theta_0) \leq 1.$$  

Thus $N$ is a non-canonical centre of $(X, \Delta)$. Therefore $N$ is birational to $P$ so that $N$ is a divisor on $V_0$. Thus $\phi_0: V_0 \rightarrow W_0$ is a birational contraction. In particular $f_0: X_0 \rightarrow Y_0$ is a birational contraction and so (1–3) clearly hold. As $\phi_0: V_0 \rightarrow W_0$ is a birational contraction it follows that $\phi: V \rightarrow W$ is a birational contraction.

Suppose that $f$ is a Mori fibre space. As the dimension of the fibres of $f: X \rightarrow Y$ are upper-semicontinuous, $f_0$ is not birational. \hfill \Box

**Lemma 3.2.** Let $(X, \Delta)$ be a divisorially log terminal pair and let $\pi: X \rightarrow U$ be a projective morphism, where $U$ is smooth and affine and $X$ is $\mathbb{Q}$-factorial. Let $\eta \in U$ be the generic point and let $0 \in U$ be a closed point. Suppose that either

1. there are $k$ divisors $D_1, D_2, \ldots, D_k$ containing $0$ such that if $H_i = \pi^* D_i$ and $H = H_1 + H_2 + \cdots + H_k$ is the sum then $(X, H + \Delta)$ is divisorially log terminal,
2. $X_0$ is reduced, $\dim X_0 = \dim X - \dim U$ and $\dim V_0 = \dim V - \dim U$, for all non canonical centres $V$ of $(X, \Delta)$, and
3. $B_-(X_0, K_{X_0} + \Delta_0)$ contains no non canonical centres of $(X_0, \Delta_0)$.

or $(X, \Delta)$ is log smooth over $U$ and (3) holds.

If $(X_0, \Delta_0)$ has a good minimal model then we may run $f: X \rightarrow Y$ the $(K_X + \Delta)$-MMP until $f_0: X_0 \rightarrow Y_0$ is a minimal model of $(X_0, \Delta_0)$ and $f_0: X_0 \rightarrow Y_0$ is a semi-ample model of $(X_0, \Delta_0)$. If $D$ is a component of $|\Delta|$, $E$ is the image of $D$ and $\phi: D \rightarrow E$ is the restriction of $f$ to $D$ then the induced map $\phi_0: D_0 \rightarrow E_0$ is a semi-ample model of $(D_0, \Sigma_0)$, where $\Sigma_0$ is defined by adjunction

$$(K_{X_0} + \Delta_0)|_{D_0} = K_{D_0} + \Sigma_0.$$  

Further $B_-(X, K_X + \Delta)$ contains no non-canonical centres of $(X, \Delta)$.
Proof. Suppose that \((X, \Delta)\) is log smooth over \(U\). If \(D_1, D_2, \ldots, D_k\) are \(k\) general divisors containing 0 then \((X, H + \Delta)\) is log smooth, so that (1) and (2) hold. Thus we may assume (1–3) hold.

We run \(f: X \rightarrow Y\) the \((K_X + \Delta)\)-MMP with scaling of an ample divisor \(A\). Let \(\Gamma = f_*\Delta\) and \(B = f_*A\). By construction \(K_Y + tB + \Gamma\) is nef for some \(t > 0\). Since \(\pi: X \rightarrow U\) satisfies the hypotheses of (3.1), \(f_0: X_0 \rightarrow Y_0\) is a weak log canonical model of \((X_0, tA_0 + \Delta_0)\).

If \(K_X + \Delta\) is not pseudo-effective then this MMP ends with a Mori fibre space for some \(t > 0\) and so \(Y_0\) is covered by curves on which \(K_{Y_0} + tB_0 + \Gamma_0\) is negative by (3.1). This contradicts the fact that \(K_{X_0} + tA_0 + \Delta_0\) is big. Thus \(K_X + \Delta\) is pseudo-effective and given any \(\epsilon > 0\) we may run the MMP until \(t < \epsilon\).

Since \(K_{X_0} + \Delta_0\) has a good minimal model (2.9.2) implies that there is a constant \(\epsilon > 0\) such that if \(t \in (0, \epsilon)\) then any more steps of this MMP are an isomorphism in a neighbourhood of \(Y_0\). It follows that \(K_{Y_0} + tB_0 + \Gamma_0\) is nef for all \(t \in (0, \epsilon)\), so that \(K_{Y_0} + \Gamma_0\) is nef. Thus \(f_0: X_0 \rightarrow Y_0\) is a minimal model of \((X_0, \Delta_0)\).

Suppose that \(D\) is a component of \(\lfloor \Delta \rfloor\). (3.1) implies that the induced map \(\phi_0: D_0 \rightarrow E_0\) is a birational contraction so that \(\phi_0\) is a semi-ample model of \((D_0, \Sigma_0)\).

As
\[
(K_Y + \Gamma)|_{Y_0} = K_{Y_0} + \Gamma_0
\]
is nef, it follows that \(B_\cdot(X, K_X + \Delta)|_{X_0}\) is contained in the indeterminacy locus of \(f_0: X_0 \rightarrow Y_0\). Thus \(B_\cdot(X, K_X + \Delta)\) contains no non-canonical centres of \((X, \Delta)\). 

4. Invariance of plurigenera

We will need the following result of B. Berndtsson and M. Păun.

**Theorem 4.1.** Let \(f: X \rightarrow \mathbb{D}\) be a projective morphism to the unit disk \(\mathbb{D}\) and let \((X, \Delta)\) be a log pair.

If

1. \((X, \Delta)\) is log smooth over \(\mathbb{D}\) and \(|\Delta| = 0\),
2. the components of \(\Delta\) do not intersect,
3. \(K_X + \Delta\) is pseudo-effective, and
4. \(B_\cdot(X, K_X + \Delta)\) does not contain any components of \(\Delta_0\),

then
\[
H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \rightarrow H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))
\]
is surjective for any integer \(m\) such that \(m\Delta\) is integral.
Proof. We check that the hypotheses of [5, Theorem 0.2] are satisfied and we will use the notation established there.

We take $\alpha = 0$ and $p = m$ so that if $L = \mathcal{O}_X(m\Delta)$ then

$$p(\lfloor \Delta \rfloor + \alpha) = m\lfloor \Delta \rfloor \in c_1(L),$$

is automatic. $K_X + \Delta$ is pseudo-effective by assumption. As we are assuming (4), $\nu_{\min}\{K_X + \Delta\}, X_0) = 0$ and $\rho_{\min, \infty}^j = 0$. In particular $J = J'$ and $\Xi = 0$. As we are assuming that the components of $\Delta$ do not intersect the transversality hypothesis is automatically satisfied.

If $u \in H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$ is a non-zero section then we choose $h_0 = e^{-\varphi_0}$ such that $\varphi_0 \leq 0 = \varphi_\Xi$ and

$$\Theta h_0(K_{X_0} + \Delta_0) = \frac{1}{m}[Z_u].$$

Since $u$ has no poles and $\lfloor \Delta \rfloor = 0$, we have

$$\int_{X_0} e^{\varphi_0 - \frac{1}{m} \varphi m \Delta} < \infty.$$

Condition (⋆) is automatically satisfied, as $\rho_{\min, \infty}^j = 0$ and $J = J'$.

[5, Theorem 0.2] implies that we can extend $u$ to $U \in H^0(X, \mathcal{O}_X(m(K_X + \Delta))).$ 

Theorem 4.2. Let $\pi: X \to U$ be a projective morphism to a smooth variety $U$ and let $(X, \Delta)$ be a log smooth pair over $U$.

If $\lfloor \Delta \rfloor = 0$ then

$$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))),$$

is independent of the point $u \in U$, for all positive integers $m$.

In particular $\kappa(X_u, K_{X_u} + \Delta_u)$ is independent of $u \in U$ and

$$f_*\mathcal{O}_X(m(K_X + \Delta)) \to H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u)))$$

is surjective for all positive integers $m > 0$ and for all $u \in U$.

Proof. Fix a positive integer $m$. We may assume that $U$ is affine.

Replacing $\Delta$ by

$$\Delta_m = \lfloor m\Delta \rfloor$$

we may assume that $m\Delta$ is integral.

By [2.8.1] there is a composition of smooth blow ups of the strata of $\Delta$ such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$
where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_* \Gamma = \Delta$ and $\pi_* E = 0$, then no two components of $\Gamma$ intersect. Then $(Y, \Gamma)$ is log smooth over $U$, $m\Gamma$ is integral and $[\Gamma] = 0$.

As

$$h^0(Y_u, \mathcal{O}_{Y_u}(m(K_{Y_u} + \Gamma_u))) = h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))),$$

replacing $(X, \Delta)$ by $(Y, \Gamma)$ we may assume that no two components of $\Delta$ intersect.

We may assume that

$$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))) \neq 0,$$

for some $u \in U$. Let $F$ be the fixed divisor of the linear system $|m(K_{X_u} + \Delta_u)|$ and let

$$\Theta_u = \Delta_u - \Delta_u \wedge F/m.$$

There is a unique divisor $0 \leq \Theta \leq \Delta$ such that

$$\Theta|_{X_u} = \Theta_u.$$

Note that $m\Theta$ is integral,

$$f_\ast \mathcal{O}_X(m(K_X + \Theta)) \subset f_\ast \mathcal{O}_X(m(K_X + \Delta))$$

and

$$H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Theta_u))) = H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))).$$

Replacing $(X, \Delta)$ by $(X, \Theta)$ we may assume that no component of $\Delta_u$ is in the base locus of $|m(K_{X_u} + \Delta_u)|$. In particular $\mathcal{B}_-(X_u, K_{X_u} + \Delta_u)$ does not contain any components of $\Delta_u$. \[5.2\] implies that $\mathcal{B}_-(X, K_X + \Delta)$ does not contain any components of $\Delta_u$ and we may apply \[4.1\].

5. The MMP in families II

**Lemma 5.1.** Let $(X, \Delta)$ be a log canonical pair and let $(X, \Phi)$ be a divisorially log terminal pair, where $X$ is $\mathbb{Q}$-factorial of dimension $n$.

Let

$$\Delta(t) = (1 - t) \Delta + t \Phi.$$

Suppose that $X \to U$ is projective. Let $f \colon X \to Y$ be a step of the $(K_X + \Delta(t))$-MMP over $U$ and let $\Gamma = f_\ast \Delta$.

Suppose $0 \in U$ is a closed point such that $K_{X_0} + \Delta_0$ is nef and $(X_0, \Delta_0)$ is log canonical. Let $r$ be a positive integer, such that $r(K_{X_0} + \Delta_0)$ is Cartier.

If

$$0 < t \leq \frac{1}{1 + 2nr}$$

then

$$\left(\Delta(t), \frac{\partial}{\partial t}\right)$$

is $C^1$-smooth over $U$, $m\Gamma$ is integral and $[\Gamma] = 0$. Then

$$h^0(Y_u, \mathcal{O}_{Y_u}(m(K_{Y_u} + \Gamma_u))) = h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))),$$

replacing $(X, \Delta)$ by $(Y, \Gamma)$ we may assume that no two components of $\Delta$ intersect.
then \( f \) is \((K_X + \Delta)\)-trivial in a neighbourhood of \( X_0 \). In particular \((Y_0, \Gamma_0)\) is log canonical, \( K_{Y_0} + \Gamma_0 \) is nef, \( r(K_{Y_0} + \Gamma_0) \) is Cartier and \((Y, \Gamma)\) is log canonical in a neighbourhood of \( Y_0 \).

**Proof.** Let \( R \) be the extremal ray corresponding to \( f \).

If \( f \) is an isomorphism in a neighbourhood of \( X_0 \) there is nothing to prove and if \((K_X + \Delta) \cdot R = 0\), the result follows by [24, 3.17].

Otherwise, as \( K_{X_0} + \Delta_0 \) is nef, \((K_X + \Delta) \cdot R > 0\) and so \((K_X + \Phi) \cdot R < 0\). [16] implies that \( R \) is spanned by a rational curve \( C \) contained in \( X_0 \) such that 
\[
-(K_X + \Phi) \cdot C \leq 2n.
\]
As \( r(K_{X_0} + \Delta_0) \) is Cartier
\[
(K_X + \Delta) \cdot C = (K_{X_0} + \Delta_0) \cdot C \geq \frac{1}{r}.
\]
Thus
\[
0 > (K_X + \Delta(t)) \cdot C = (1 - t)(K_X + \Delta) \cdot C + t(K_X + \Phi) \cdot C \\
\geq \frac{(1 - t)}{r} - 2nt \\
= \frac{1}{r} - t\left(1 + 2nr\right) \\
\geq 0,
\]
a contradiction. \(\Box\)

**Lemma 5.2.** Let \((X, \Delta = S + B)\) be a divisorially log terminal pair, where \( S \leq \lfloor \Delta \rfloor \) and \( X \) is \(\mathbb{Q}\)-factorial. Let \( \pi : X \rightarrow U \) be a projective morphism, where \( U \) is smooth and affine. Let \( 0 \in U \) be a closed point, let \( n \) be the dimension of \( X \) and let \( r \) be a positive integer such that \( r(K_{X_0} + \Delta_0) \) is Cartier. Fix
\[
\epsilon < \frac{1}{2nr + 1}.
\]
If \((X_0, \Delta_0)\) is log canonical, \( K_{X_0} + \Delta_0 \) is nef but \( K_X + (1 - \epsilon)S + B \) is not pseudo-effective, then we may run \( f : X \dashrightarrow Y \) the \((K_X + (1 - \epsilon)S + B)\)-MMP over \( U \), the steps of which are all \((K_X + \Delta)\)-trivial in a neighbourhood of \( X_0 \), until we arrive at a Mori fibre space \( \psi : Y \rightarrow Z \) such that the strict transform of \( S \) dominates \( Z \) and \( K_Y + \Gamma \sim_{\mathbb{Q}} \psi^*L \), for some divisor \( L \) on \( Z \).

**Proof.** We run \( f : X \dashrightarrow Y \) the \((K_X + (1 - \epsilon)S + B)\)-MMP with scaling of an ample divisor over \( U \). [5.1] implies that every step of this MMP is \((K_X + \Delta)\)-trivial in a neighbourhood of \( X_0 \). As \( K_X + (1 - \epsilon)S + B \) is not
pseudo-effective this MMP ends with a Mori fibre space $\psi: Y \to Z$. As every step of this MMP is $(K_X + \Delta)$-trivial in a neighbourhood of $X_0$, it follows that the strict transform of $S$ dominates $Z$. □

**Lemma 5.3.** Let $(X, \Delta)$ be a divisorially log terminal pair, where $X$ is $\mathbb{Q}$-factorial and projective and $\Delta$ is a $\mathbb{Q}$-divisor. If $\Phi$ is a $\mathbb{Q}$-divisor such that

$$0 \leq \Delta - \Phi \leq N_\sigma(X, K_X + \Delta),$$

then $(X, \Phi)$ has a good minimal model if and only if $(X, \Delta)$ has a good minimal model.

**Proof.** Suppose that $f: X \to Y$ is a minimal model of $(X, \Delta)$. Let $\Gamma = f_*\Delta$. (2) of (2.7.1) implies that $f$ contracts every component of $N_\sigma(X, K_X + \Delta)$ so that

$$f_*(K_X + \Delta) = K_Y + \Gamma = f_*(K_X + \Phi).$$

Let $p: W \to X$ and $q: W \to Y$ resolve $f$. If we write

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E,$$

then $E \geq 0$ is $q$-exceptional and $p_*E = N_\sigma(X, K_X + \Delta)$. It follows that if we write

$$p^*(K_X + \Phi) = q^*(K_Y + \Gamma) + F,$$

then

$$F = E - p^*(\Delta - \Phi) \geq E - p^*(N_\sigma(X, K_X + \Delta)) = E - p^*p_*E,$$

is $p$-exceptional. Therefore $F \geq 0$ by negativity of contraction and so $f$ is a weak log canonical model of $(X, \Phi)$. If $f$ is a good minimal model of $(X, \Delta)$ then $f$ is a semi-ample model of $(X, \Phi)$ and so $(X, \Phi)$ has a good minimal model by (2.9.1).

Now suppose that $(X, \Phi)$ has a good minimal model. We may run the $(K_X + \Phi)$-MMP until we get a minimal model $f: X \to Y$ of $(X, \Phi)$. Let $Y \to Z$ be the ample model of $K_X + \Phi$.

If $t > 0$ is sufficiently small then $f$ is also a run of the $(K_X + \Delta_t)$-MMP, where

$$\Delta_t = \Phi + t(\Delta - \Phi).$$

Let $n$ be the dimension of $X$ and let $r$ be a positive integer such that $r(K_X + \Phi)$ is Cartier. If

$$0 < t < \frac{1}{1 + 2nr}$$

and we continue to run the $(K_X + \Delta_t)$-MMP with scaling of an ample divisor then (5.1) implies that every step of this MMP is $(K_X + \Phi)$-trivial, so that every step is over $Z$. After finitely many steps (2.7.1)
implies that we obtain a model \(g : X \to W\) which contracts the components of \(N_\sigma(X, K_X + \Delta_i)\). As the support of \(N_\sigma(X, K_X + \Delta)\) is the same as the support of \(N_\sigma(X, K_X + \Delta_i)\) and the support of \(\Delta - \Phi\) is contained in \(N_\sigma(X, K_X + \Delta)\) it follows that

\[
g_*(K_X + \Delta) = g_*(K_Y + \Phi).
\]

Thus \(g_*(K_X + \Delta)\) is semi-ample. On the other hand \(g\) only contracts divisors in \(N_\sigma(X, K_X + \Delta)\) so that (2.7.2) implies that \(g\) is a minimal model of \((X, \Delta)\). Thus \(g : X \to W\) is a good minimal model of \((X, \Delta)\).

\[\square\]

6. Abundance in families

**Lemma 6.1.** Suppose that \((X, \Delta)\) is a log pair where the coefficients of \(\Delta\) belong to \((0, 1] \cap \mathbb{Q}\). Let \(\pi : X \to U\) be a projective morphism to a smooth affine variety \(U\). Suppose that \((X, \Delta)\) is log smooth over \(U\).

If there is a closed point \(0 \in U\) such that the fibre \((X_0, \Delta_0)\) has a good minimal model then the generic fibre \((X_\eta, \Delta_\eta)\) has a good minimal model.

**Proof.** By [2.9.3] it is enough to prove that the geometric generic fibre has a good minimal model. Replacing \(U\) by a finite cover we may therefore assume that the strata of \(\Delta\) have irreducible fibres over \(U\).

Let \(f_0 : Y_0 \to X_0\) be the birational morphism given by (2.8.3). As \((X, \Delta)\) is log smooth over \(U\), the strata of \(\Delta\) have irreducible fibres over \(U\) and \(f_0\) blows up strata of \(\Delta_0\), we may extend \(f_0\) to a birational morphism \(f : Y \to X\) which is a composition of smooth blow ups of strata of \(\Delta\). We may write

\[
K_Y + \Gamma = f^*(K_X + \Delta) + E,
\]

where \(\Gamma \geq 0\) and \(E \geq 0\) have no common components, \(f_* \Gamma = \Delta\) and \(f_* E = 0\). \((Y, \Gamma)\) is log smooth and the fibres of the components of \(\Gamma\) are irreducible. [14] 2.10] implies that \((Y_0, \Gamma_0)\) has a good minimal model, as \((X_0, \Delta_0)\) has a good minimal model; similarly [14] 2.10] also implies that if \((Y_\eta, \Gamma_\eta)\) has a good minimal model then \((X_\eta, \Delta_\eta)\) has a good minimal model.

Replacing \((X, \Delta)\) by \((Y, \Gamma)\) we may assume that if \(\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)\) then \(B_-(X_0, K_{X_0} + \Theta_0)\) contains no strata of \(\Theta_0\). There is a unique divisor \(0 \leq \Theta \leq \Delta\) such that \(\Theta|_{X_0} = \Theta_0\). [2.3.2] implies that

\[
\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)\]
so that

$$\Delta - \Theta \leq N_\sigma(X, K_X + \Delta).$$

Hence by (5.3) and (2.9.3) it suffices to prove that \((X_\eta, \Theta_\eta)\) has a good minimal model. Replacing \((X, \Delta)\) by \((X, \Theta)\) we may assume that \(B_-(X_0, K_{X_0} + \Delta_0)\) contains no strata of \(\Delta_0\). (3.2) implies that we can run \(f: X \dasharrow Y\) the \((K_X + \Delta)\)-MMP over \(U\) to obtain a minimal model of the generic fibre. Let \(\Gamma = f_*\Delta\).

Pick a component \(D\) of \([\Delta]\). Let \(\phi: D \dasharrow E\) be the restriction of \(f\) to \(D\). (3.2) implies that \(\phi_0\) is a semi-ample model of \((D_0, (\Delta_0 - D_0)|_{D_0})\). (2.9.1) implies that \((D_0, (\Delta_0 - D_0)|_{D_0})\) has a good minimal model. By induction on the dimension \((D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})\) has a good minimal model. But then \(\phi_\eta: D_\eta \dasharrow E_\eta\) is a semi-ample model of \((D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})\).

Let \(S = [\Delta]\) and \(B = \{\Delta\} = \Delta - S\). Let \(T = f_*S\) and \(C = f_*B\). Suppose that \(K_{Y_0} + (1 - \epsilon)T_0 + C_0\) is not pseudo-effective for any \(\epsilon > 0\). Then \(K_{X_0} + (1 - \epsilon)S_0 + B_0\) is not pseudo-effective for any \(\epsilon > 0\). (1.2) implies that \(K_X + (1 - \epsilon)S + B\) is not pseudo-effective for any \(\epsilon > 0\). But then \(K_Y + (1 - \epsilon)T + C\) is not pseudo-effective for any \(\epsilon > 0\). (3.2) implies that we may run the \((K_Y + (1 - \epsilon)T + C)\)-MMP until we get to a Mori fibre space \(g: Y \dasharrow W, \psi: W \dasharrow V\) over \(U\). By assumption \(g_*(K_Y + \Gamma) \sim_\mathbb{Q} \psi^*L\) for some divisor \(L\).

Pick a component \(D\) of \(S\) whose image \(F\) in \(W\) dominates \(V\). Let \(E\) be the image of \(D\) in \(Y\). As we already observed, \(\phi_\eta: D_\eta \dasharrow E_\eta\) is a semi-ample model of \((D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})\). As the birational map \(g_0: Y_0 \dasharrow W_0\) is \((K_{Y_0} + \Gamma_0)\)-trivial, the birational map \(g_\eta: Y_\eta \dasharrow W_\eta\) is also \((K_{Y_\eta} + \Gamma_\eta)\)-trivial. Then \(L_\eta\) is semi-ample as \((\psi^*L)|_{F_\eta}\) is semi-ample. The composition \(X_\eta \dasharrow W_\eta\) is a semi-ample model of \((X_\eta, \Delta_\eta)\) and so \((X_\eta, \Delta_\eta)\) has a good minimal model by (2.9.1).

Otherwise, \(K_{Y_0} + (1 - \epsilon)T_0 + C_0\) is pseudo-effective for some \(\epsilon > 0\). If \(Y_0 \dasharrow Z_0\) is the log canonical model of \((Y_0, \Gamma_0)\) then \(T_0\) does not dominate \(Z_0\) and so if \(\epsilon\) is sufficiently small then \(K_{X_0} + (1 - \epsilon)S_0 + B_0\) has the same Kodaira dimension as \(K_{X_0} + \Delta_0\).

$$\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) \geq \kappa(X_\eta, K_{X_\eta} + (1 - \epsilon)S_0 + B_\eta)$$

$$= \kappa(X_0, K_{X_0} + (1 - \epsilon)S_0 + B_0)$$

$$= \kappa(X_0, K_{X_0} + \Delta_0)$$

$$= \kappa_\sigma(X_0, K_{X_0} + \Delta_0)$$

$$= \nu(Y_0, K_{Y_0} + \Gamma_0)$$

$$= \nu(Y_\eta, K_{Y_\eta} + \Gamma_\eta).$$
The first inequality holds as $S^0 \geq 0$, the second equality holds by \eqref{4.2} (note that $(X_0, (1-\epsilon)S_0+B_0)$ is Kawamata log terminal as $(X_0, \Delta_0)$ is divisorially log terminal) and the last equality holds as intersection numbers are deformation invariant.

We have already seen that if $E$ is a component of $T$ then $(K_Y+\Gamma)|_{E_0}$ is semi-ample. \eqref{2.5.1} implies that $(K_Y+\Gamma)|_{E_0}$ is semi-ample. Let $H = K_{Y_0} + \Gamma_0$. Then $H|_{T_0}$ is semi-ample and $aH - (K_{Y_0} + \Gamma_0)$ is nef and abundant for all $a > 1$. Thus $f_0: X_0 \dashrightarrow Y_0$ is a good minimal model by \eqref{2.6.1}.

\[\begin{proof}
\end{proof}\]

**Lemma 6.2.** Suppose that $(X, \Delta)$ is a log pair where the coefficients of $\Delta$ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi: X \longrightarrow U$ be a projective morphism to a smooth affine variety $U$. Suppose that $(X, \Delta)$ is log smooth over $U$.

If $(X, \Delta)$ has a good minimal model then every fibre $(X_u, \Delta_u)$ has a good minimal model.

\[\begin{proof}
\end{proof}\]
\( \Delta_u \), (3.1) implies that \( \pi_u : X_u \twoheadrightarrow Y_u \) is a semi-ample model of \((X_u, \Delta_u)\).

(2.9.1) implies that \((X_u, \Delta_u)\) has a good minimal model. \( \square \)

**Proof of (1.2).** By (6.1) the generic fibre \((X_\eta, \Delta_\eta)\) has a good minimal model. Hence we may find a good minimal model of \( \pi^{-1}(U_0) \) over an open subset \( U_0 \) of \( U \). As \((X, \Delta)\) is log smooth over \( U \), every strata of \( S = \lfloor \Delta \rfloor \) intersects \( \pi^{-1}(U_0) \). Therefore we may apply [14, 1.1] to conclude that \((X, \Delta)\) has a good minimal model over \( U \). (6.2) implies that every fibre has a good minimal model. \( \square \)

**Proof of (1.3).** It suffices to prove that if \( U_0 \) is dense then it contains an open subset. By (2.8.4) we may assume that \((X, \Delta)\) is divisorially log terminal and every fibre \((X_u, \Delta_u)\) is divisorially log terminal.

Let \( \pi : Y \longrightarrow X \) be a log resolution. We may write

\[
K_Y + \Gamma = \pi^*(K_X + \Delta) + E,
\]

where \( \Gamma \geq 0 \) and \( E \geq 0 \) have no common components. Passing to an open subset we may assume that \((Y, \Gamma)\) is log smooth over \( U \), so that

\[
K_{Y_u} + \Gamma_u = \pi^*(K_{X_u} + \Delta_u) + E_u,
\]

for all \( u \in U \). [14, 2.10] implies that if \((Y, \Gamma)\) has a good minimal model over \( U \) then \((X, \Delta)\) has a good minimal model over \( U \). Similarly [14, 2.10] implies that if \((X_u, \Delta_u)\) has a good minimal model then \((Y_u, \Gamma_u)\) has a good minimal model.

Replacing \((X, \Delta)\) by \((Y, \Gamma)\) we may assume that \((X, \Delta)\) is log smooth over \( U \). (1.2) implies that \( U_0 = U \). \( \square \)

**Lemma 6.3.** Let \( \pi : X \longrightarrow U \) be a projective morphism to a smooth variety \( U \) and let \((X, \Delta)\) be log smooth over \( U \). Suppose that the coefficients of \( \Delta \) belong to \((0, 1] \cap \mathbb{Q})

If there is a closed point \( 0 \in U \) such that the fibre \((X_0, \Delta_0)\) has a good minimal model then the restriction morphism

\[
\pi_* \mathcal{O}_X(m(K_X + \Delta)) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))
\]

is surjective for any \( m \in \mathbb{N} \) such that \( m\Delta \) is integral.

**Proof.** (2.3.3) implies that we may assume that \( m \geq 2 \). Replacing \( U \) by a finite cover we may assume that the strata of \( \Delta \) have irreducible fibres over \( U \). Since the result is local we may assume that \( U \) is affine and so we want to show that the restriction map

\[
H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))
\]

is surjective. Cutting by hyperplanes we may assume that \( U \) is a curve.

Let \( f_0 : Y_0 \longrightarrow X_0 \) be the birational morphism given by (2.8.3). As \((X, \Delta)\) is log smooth over \( U \), the strata of \( \Delta \) have irreducible fibres
over $U$ and $f_0$ blows up strata of $\Delta_0$, we may extend $f_0$ to a birational morphism $f: Y \rightarrow X$ which is a composition of smooth blow ups of strata of $\Delta$. We may write
\[ K_Y + \Gamma = f^*(K_X + \Delta) + E, \]
where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. $(Y, \Gamma)$ is log smooth and the fibres of the components of $\Gamma$ are irreducible. Note that $m\Gamma$ is integral and the natural maps induce isomorphisms
\[ H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \simeq H^0(Y, \mathcal{O}_Y(m(K_Y + \Gamma))) \]
and
\[ H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0))) \simeq H^0(Y_0, \mathcal{O}_{Y_0}(m(K_{Y_0} + \Gamma_0))) \]
Replacing $(X, \Delta)$ by $(Y, \Gamma)$ we may assume that if $\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)$, then $B_-(X_0, K_{X_0} + \Theta_0)$ contains no strata of $\Theta_0$. There is a unique divisor $0 \leq \Theta \leq \Delta$ such that $\Theta|_{X_0} = \Theta_0$. (2.3.2) implies that $\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$.

(6.1) implies that we may run the $(K_X + \Theta)$-MMP over $U$ until we get to a minimal model $f: X \rightarrow Y$. (3.1) implies that $f$ is an isomorphism in a neighbourhood of the generic point of every non kawamata log terminal centre of $(X, X_0 + \Theta)$. Let $V \subset X \times Y$ be the graph. Then $V \rightarrow X$ is an isomorphism in a neighbourhood of the generic point of each non kawamata log terminal centre of $(X, X_0 + \Theta)$. We may find a log resolution $W \rightarrow V$ of the strict transform of $\Theta$ and the exceptional divisor of $V \rightarrow Y$ which is an isomorphism in a neighbourhood of the generic point of each non kawamata log terminal centre of $(X, X_0 + \Theta)$. If $p: W \rightarrow X$ and $q: W \rightarrow Y$ are the induced morphisms then we may write
\[ K_W + \Phi + W_0 = p^*(K_X + X_0 + \Theta) + E, \]
where $W_0$ is the strict transform of $X_0$, $\Phi$ is the strict transform of $[\Theta]$ and $[E] \geq 0$ as $p$ is an isomorphism in a neighbourhood of the generic point of each non kawamata log terminal centre of $(X, X_0 + \Theta)$.

We may also write
\[ p^*((m - 1)(K_X + \Theta)) = q^*f_*((m - 1)(K_X + \Theta)) + F. \]
Possibly shrinking $U$, we may assume $X_0$ is $\mathbb{Q}$-linearly equivalent to zero. If we set
\[ A = p^*(m(K_X + \Theta)) + E - F, \quad L = [A] \quad \text{and} \quad C = \{-A\} \]
then
\[ L - W_0 = p^*(m(K_X + \Theta)) + E - F + C - W_0 \]
\[ = p^*(K_X + \Theta) + E + p^*((m - 1)(K_X + \Theta)) - F + C - W_0 \]
\[ \sim Q K_W + \Phi + C + q^*f_*((m - 1)(K_X + \Theta)). \]

\((W, \Phi + C)\) is log canonical, as \((W, \Phi + C)\) is log smooth and \(\Phi + C\) is a boundary. Since all non kawamata log terminal centres of \((W, \Phi + C)\) dominate \(U\), a generalisation of Kollár’s injectivity theorem (see [18], [8, 6.3] and [4, 5.4]) implies that multiplication by a local parameter
\[
H^1(W, \mathcal{O}_W(L - W_0)) \rightarrow H^1(W, \mathcal{O}_W(L))
\]
is an injective morphism and so the restriction morphism
\[
H^0(W, \mathcal{O}_W(L)) \rightarrow H^0(W_0, \mathcal{O}_{W_0}(L|_{W_0}))
\]
is surjective. Note that the support of \(L - [q^*f_*(m(K_X + \Theta))]\) does not contain \(W_0\) and
\[
L - [q^*f_*(m(K_X + \Theta))] = [A] - [q^*f_*(m(K_X + \Theta))]
\geq [A - q^*f_*(m(K_X + \Theta))]
\geq [E + \frac{1}{m - 1}F]
\geq 0.
\]

We also have
\[
|L| \subset |mp^*(K_X + \Delta) + \lceil E - F \rceil|
\subset |mp^*(K_X + \Delta) + \lceil E \rceil|
= |m(K_X + \Delta)|.
\]
Let \(q_0 : W_0 \rightarrow Y_0\) be the restriction of \(q\) to \(W_0\). We have
\[
|m(K_{X_0} + \Delta_0)| = |m(K_{X_0} + \Theta_0)|
= |m(K_{Y_0} + f_0, \Theta_0)|
= |q_0^*m(K_{Y_0} + f_0, \Theta_0)|
\subset |L|_{W_0}
= |L|_{W_0}
\subset |m(K_X + \Delta)|_{X_0}.
\]

**Proof of (1.4).** Immediate from (6.3) and (1.2).
Lemma 7.1. Let \( w \) be a positive real number and let \( I \subset [0, 1] \) be a set which satisfies the DCC. Fix a log smooth pair \((Z, B)\), where \( Z \) is a projective variety. Let \( \mathfrak{F} \) be the set of all log smooth pairs \((X, \Delta)\) such that \( \text{vol}(X, K_X + \Delta) = w \), the coefficients of \( \Delta \) belong to \( I \) and there is a sequence of smooth blow ups \( f : X \rightarrow Z \) of the strata of \( B \) such that \( f_* \Delta \leq B \).

Then there is a sequence of blow ups \( Y \rightarrow Z \) of the strata of \( B \) such that:

If \((X, \Delta) \in \mathfrak{F}\) then
\[
\text{vol}(Y, K_Y + \Gamma) = w
\]
where \( \Gamma \) is the sum of the strict transform of \( \Delta \) and the exceptional divisors of the induced birational map \( Y \rightarrow X \).

Proof. We may suppose that \( 1 \in I \) and that \( I \) is closed. Let
\[
v(Z, B) = \sup \{ \text{vol}(Z, K_Z + \Phi) \mid \Phi = f_* \Delta \text{ for some } (X, \Delta) \in \mathfrak{F} \}.
\]
Let \( \mathfrak{D} \) be the set of log smooth pairs \((X, \Delta)\) such that \( X \) is projective and the coefficients of \( \Delta \) belong to \( I \). If \((X, \Delta) \in \mathfrak{F}\) then \((Z, \Phi) \in \mathfrak{D}\) so that \( v(Z, B) \in \bar{V} \), where
\[
V = \{ \text{vol}(X, K_X + \Delta) \mid (X, \Delta) \in \mathfrak{D} \}.
\]
Suppose that \( g : Y \rightarrow Z \) blows up the strata of \( B \) and let \( C \) be the strict transform of \( B \) plus the \( g \)-exceptional divisors. If \((X, \Delta) \in \mathfrak{F}\) and \( \Gamma \) is the sum of the strict transform of \( \Delta \) and the exceptional divisors of \( Y \rightarrow X \) then
\[
\text{vol}(Y, K_Y + \Gamma) \leq \text{vol}(Z, K_Z + \Phi),
\]
and so \( v(Y, C) \leq v(Z, B) \). As \( V \) satisfies the DCC, possibly replacing \( Z \) by a higher model \( Y \) we may assume that \( v = v(Z, B) \) is minimal and it suffices to prove that \( v = w \).

Clearly \( v \geq w \). Pick \( g_j : Y_j \rightarrow Z \) which blow up the strata of \( B \) such that the natural birational map \( Y_j \rightarrow Y_k \) is a morphism whenever \( j \geq k \) and given any \( g : Y \rightarrow Z \) which blows up the strata of \( B \) we may find \( j \) such that the induced birational map \( h_j : Y_j \rightarrow Y \) is a morphism. As \( v(Y_j, C_j) = v \), where \( C_j \) is the strict transform of \( B \) plus the exceptionals, we may find pairs \((X^j_i, \Delta^j_i) \in \mathfrak{F}\) such that if \( \Gamma^j_i \) is the strict transform of \( \Delta^j_i \) plus the exceptional divisors then
\[
\lim_i \text{vol}(Y_j, K_{Y_j} + \Gamma^j_i) = v.
\]
Possibly passing to a subsequence we may assume that
\[ \text{vol}(Y_j, K_{Y_j} + \Gamma_i^j) > v - 1/j. \]

Let \((X_i, \Delta_i)\) be the diagonal sequence, so that \(X_i = X_i^i\) and \(\Delta_i = \Delta_i^i\). Suppose that \(g: Y \rightarrow Z\) blows up the strata of \(B\). Let \(\Gamma_i\) be the strict transform of \(\Delta_i\) plus the exceptional divisors. Pick \(j_0\) such that the induced birational map \(h_j: Y_j \rightarrow Y\) is a morphism for all \(j \geq j_0\). Then \(\Gamma_i = h_i^*\Gamma_i^i\) for all \(i \geq j_0\), so that
\[ \text{vol}(Y, K_Y + \Gamma) \geq \text{vol}(Y_i, K_{Y_i} + \Gamma_i^i) \geq v - 1/i. \]

Thus \(\lim_i \text{vol}(Y, K_Y + \Gamma_i) = v\).

As the set \(\{ \frac{r - 1}{r^i} | r \in \mathbb{N}, i \in I \}\) satisfies the DCC, by [12, (1.5)] we may find \(r \in \mathbb{N}\) such that \(K_{X_i} + \frac{r - 1}{r} \Delta_i\) is big, for all \(i\). In this case,
\[ K_{X_i} + a \Delta_i = (1 - \epsilon)(K_{X_i} + \Delta_i) + \epsilon \left( K_{X_i} + \frac{r - 1}{r} \Delta_i \right), \]
where \(a = 1 - \frac{\epsilon}{r}\) so that
\[ \text{vol}(Y, K_Y + a \Gamma) \geq \text{vol}(Y, (1 - \epsilon)(K_Y + \Gamma)) = (1 - \epsilon)^n v, \]
where \(\dim Z = n\), independently of the model \(Y\). If we fix \(\epsilon > 0\) then \((Z, (1 - \epsilon)\Phi)\) is kawamata log terminal. Hence (2.8.1) implies we may pick a birational morphism \(g: Y \rightarrow Z\) such that if we write
\[ K_Y + \Psi = g^*(K_Z + (1 - \epsilon)\Phi) + E \]
where \(\Psi \geq 0\) and \(E \geq 0\) have no common components, \(g_* \Psi = (1 - \epsilon)\Phi\) and \(g_* E = 0\), then no two components of \(\Psi\) intersect. In particular \((Y, \Psi)\) is terminal.

Let
\[ \Sigma = \Psi \wedge (1 - \epsilon)\Gamma. \]
As \(\Gamma\) is the limit of \(\Gamma_i\), given \(0 < \eta < 1\) it follows that we may find \(i\) such that
\[ (1 - \eta)\Sigma \leq \Psi \wedge (1 - \epsilon)\Gamma_i. \]
Let $\Sigma_i \leq \Delta_i$ be the strict transform of $(1-\eta)\Sigma$ on $X_i$. We have
\[
\text{vol}(Y, K_Y + (1-\eta)\Sigma) \leq \text{vol}(X_i, K_{X_i} + \Sigma_i) \\
\leq \text{vol}(X_i, K_{X_i} + \Delta_i) = w.
\]
where we used the fact that $(Y, \Sigma)$ is terminal, as $(Y, \Psi)$ is terminal, for the first inequality. Taking the limit as $\eta$ goes to zero, [12, (5.3.2)] implies that
\[
\text{vol}(Y, K_Y + (1-\epsilon)\Gamma) = \text{vol}(Y, K_Y + \Sigma) \leq w.
\]
But we already showed that
\[
\text{vol}(Y, K_Y + (1-\epsilon)\Gamma) \geq (1-\epsilon)^n w,
\]
independently of the model $Y$. Taking the limit as $\epsilon$ goes to zero, we must have $v = w$. \qed

**Lemma 7.2.** Let $w$ be a positive real number and let $I \subset [0, 1]$ be a set which satisfies the DCC. Let $\mathfrak{F}$ be a set of log canonical pairs $(X, \Delta)$ such that $X$ is projective, the coefficients of $\Delta$ belong to $I$ and $\text{vol}(X, K_X + \Delta) = w$.

Then there is a projective morphism $Z \rightarrow U$ and a log smooth pair $(Z, B)$ over $U$ such that if $(X, \Delta) \in \mathfrak{F}$ then there is a point $u \in U$ and a birational map $f_u: X \dashrightarrow Z_u$ such that
\[
\text{vol}(Z_u, K_{Z_u} + \Phi) = w
\]
where $\Phi \leq B_u$ is the sum of the strict transform of $\Delta$ and the exceptional divisors of $f_u^{-1}$.

**Proof.** We may assume that $1 \in I$.

By [12, 1.3] there is a constant $r$ such that if $(X, \Delta) \in \mathfrak{F}$ then $\phi_r(K_X+\Delta)$ is birational. (2.3.4) and (3.1) of [11] imply that the set $\mathfrak{F}$ is log birationally bounded.

Therefore we may find a projective morphism $\pi: Z \rightarrow U$ and a log pair $(Z, B)$ such that if $(X, \Delta) \in D$ then there is a point $u \in U$ and a birational map $f: X \dashrightarrow Z_u$ such that the support of the strict transform of $\Delta$ plus the $f^{-1}$-exceptional divisor is contained in the support of $B_u$. By standard arguments, see for example the proof of [11, 1.9], we may assume that $(Z, B)$ is log smooth over $U$ and the intersection of strata of $B$ with the fibres is irreducible.

Let $0$ be a closed point of $U$. Let $\mathfrak{F}_0 \subset \mathfrak{F}$ be the set of log canonical pairs $(X, \Delta)$ such that there is a birational morphism $f: X \rightarrow Z_0$ and $f_*\Delta \leq B_0$. By (7.1) there is a sequence of blow ups $g: Y_0 \rightarrow Z_0$ of the strata of $B_0$ such that if $(X, \Delta) \in \mathfrak{F}_0$ and $\Gamma$ is the strict transform
of \( \Delta \) plus the exceptional divisors then
\[
\text{vol}(Y_0, K_{Y_0} + \Gamma) = w.
\]
Let \( g : Y \to Z \) be the sequence of blow ups of the strata of \( B \) induced by \( g_0 \). Replacing \((Z,B)\) by \((Y,C)\), where \( C \) is the sum of the strict transform of \( B \) and the exceptional divisors of \( g \), we may assume that
\[
\text{vol}(Z_0, K_{Z_0} + \Psi_0) = w,
\]
where \( \Psi_0 = f_*\Delta \).

Suppose that \((X,\Delta)\) \( \in \mathcal{F} \). By a standard argument, see the proof of \([11,(1.9)]\), we may assume that \((X,\Delta)\) is log smooth and \( f : X \to Z \) blows up the strata of \( B \). Let \( h : W \to Z \) blow up the corresponding strata of \( B \) so that \( W_u = X \) and \( h_u = f \). Let \( \Theta \) be the divisor on \( W \) such that \( \Theta_u = \Delta \). Then
\[
\text{vol}(W_0, K_{W_0} + \Theta_0) = \text{vol}(X, K_X + \Delta) = w,
\]
by deformation invariance of plurigenera, \([1.2]\), so that \((W_0, \Theta_0) \in \mathcal{F}_0 \).

But then
\[
\text{vol}(Z_0, K_{Z_0} + \Phi_0) = w,
\]
where \( \Phi_0 = f_*\Theta_0 \). Let \( \Phi = h_*\Theta \). Then \( \Phi_u \) is the strict transform of \( \Delta \) plus the exceptional divisors and
\[
\text{vol}(Z_u, K_{Z_u} + \Phi_u) = w,
\]
by deformation invariance of plurigenera, \([1.2]\).

**Proposition 7.3.** Fix an integer \( n \), a constant \( d \) and a set \( I \subset [0,1] \) which satisfies the DCC.

Then the set \( \mathfrak{F}_{lc}(n,d,I) \) of all \((X,\Delta)\) such that

1. \( X \) is a union of projective varieties of dimension \( n \),
2. \((X,\Delta)\) is log canonical,
3. the coefficients of \( \Delta \) belong to \( I \),
4. \( K_X + \Delta \) is an ample \( \mathbb{Q} \)-divisor, and
5. \((K_X + \Delta)^n = d \),

is bounded.

**Proof.** If
\[
X = \coprod_{i=1}^k X_i,
\]
and \((X_i,\Delta_i)\) is the corresponding log canonical pair then \( K_{X_i} + \Delta_i \) is ample and if \( d_i = (K_{X_i} + \Delta_i)^n \) then \( d = \sum d_i \). \([2.4.1]\) and \([1.6]\) imply that there are only finitely many tuples \((d_1,d_2,\ldots,d_k)\).

Thus it is enough to show that the set \( \mathfrak{F} \) of irreducible pairs \((X,\Delta)\) satisfying (1–5) is bounded.
By $(7.2)$ there is a projective morphism $Z \to U$ and a log smooth pair $(Z, B)$ over $U$, such that if $(X, \Delta) \in \mathfrak{F}$ then there is a closed point $u \in U$ and a birational map $f_u : Z_u \to X$ such that
\[
\text{vol}(Z_u, K_{Z_u} + \Phi) = d,
\]
where $\Phi \leq B_u$ is the sum of the strict transform of $\Delta$ and the $f$-exceptional divisors. $(2.2.2)$ implies that $f_u$ is the log canonical model of $(Z_u, \Phi)$.

On the other hand, $(1.3)$ implies that if we replace $U$ by a finite disjoint union of locally closed subsets then we may assume that every fibre has of $\pi$ has a log canonical model. Replacing $(Z, B)$ by the log canonical model over $U$, the fibres of $\pi$ are the elements of $\mathfrak{F}$.

Proof of $(1.1)$. Let $\mathfrak{F}$ be the set of triples $(X, \Delta, \tau)$ where $(X, \Delta) \in \mathfrak{F}_{\text{lc}}(n, d, I)$ and $\tau : S \to S$ is an involution of thenormalisation of a divisor supported on $\lfloor \Delta \rfloor$, which fixes the different of $(K_X + \Delta)|_S$. Then $\tau$ fixes the ample divisor $H$, the pullback of $K_X + \Delta$ to $S$. Note that the set of all automorphisms which fix $H$ is an algebraic group and the set of all involutions fixing the different is a closed subset.

It is enough to prove that $\mathfrak{F}$ is bounded. $(7.3)$ implies that $\mathfrak{F}_{\text{lc}}(n, d, I)$ is bounded and the boundedness of $\tau$ is then automatic. □

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