The Inflationary Energy Scale in Braneworld Cosmology

Rachael M. Hawkins and James E. Lidsey
Astronomy Unit, School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, LONDON, E1 4NS, U.K.

Upper bounds on the energy scale at the end of inflation in the Randall–Sundrum type II braneworld scenario are derived. The analysis is made exact by introducing new parameters that represent extensions of the Hubble flow parameters. Only very weak assumptions about the form of the inflaton potential are made. In the high energy and slow roll regime the bounds depend on the amplitude of gravitational waves produced during inflation and become stronger as this amplitude increases.

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I. INTRODUCTION

Inflation is presently the most favoured model for describing the earliest stages of the universe’s history [1]. (For a review, see, e.g., Ref. [2]). As well as resolving the horizon, flatness and monopole problems of the standard, big bang cosmology, it provides a quantum mechanical origin for generating the primordial density perturbations that subsequently formed large-scale structure through gravitational instability [3]. The inflationary paradigm is strongly supported by recent observations from the Wilkinson Microwave Anisotropy Probe (WMAP) [4, 5, 6, 7, 8]. These indicate that the cosmic microwave background (CMB) power spectrum is consistent with the gaussian, adiabatic and nearly scale–invariant form expected from an inflationary epoch and that the universe is spatially flat to within the limits of observational accuracy [3]. Moreover, the observed anti–correlation on degree angular scales between the temperature and polarization E–mode maps of the CMB [7] provides strong evidence for the existence of correlations on length scales beyond the Hubble radius at the epoch of decoupling [3].

In the simplest class of inflationary models, the energy density of the universe is dominated by the potential energy of a single, scalar ‘inflaton’ field, \( \phi \), that slowly rolls down its self–interaction potential [1]. In light of the above developments, there is a pressing need to understand the origin of the inflaton field within the context of a unified field theory. Recent developments in our understanding of the non–perturbative properties of string/M–theory [10, 11] have led to the proposal that our observable, four–dimensional universe may be viewed as a domain wall embedded in a higher–dimensional ‘bulk’ space [12, 13]. The standard model gauge interactions are confined to the four–dimensional hypersurface, but gravitational interactions may propagate in the bulk dimensions. From a cosmological point of view, the Randall–Sundrum type II (RSII) scenario has generated considerable interest [15]. In the simplest version of this scenario, our observable universe is represented by a codimension one brane embedded in five–dimensional, Anti–de Sitter (AdS) space. Although the fifth dimension is infinite in extent, corrections to Newton gravity remain undetectable if the warping of the non–factorizable geometry is sufficiently strong [13]. (For a recent review of the cosmological implications of the RSII scenario, see, e.g., Ref. [14]).

In the case where a single inflaton field is confined to the brane, the effective Friedmann equation is derived by projecting the five–dimensional Einstein field equations onto a four–dimensional hypersurface sourced by the scalar field [16]. The Friedmann equation acquires a quadratic dependence on the energy density that becomes important at high energy scales [12, 16, 17, 18, 19]. Such a modification to the braneworld Friedmann equation alters the dynamics of the inflaton field, compared with standard cosmology, by providing an additional source of friction as the field rolls down its potential [20]. This has made the RSII braneworld scenario attractive to inflationary model builders as it increases the number of inflationary potentials that can be considered. In the steep inflationary scenario, for example, potentials that can not support inflationary expansion in the standard cosmology can provide a sufficient number of e–foldings of inflation in the braneworld scenario [21, 22, 23].

In this paper, we focus on the energy scale at the end of inflation in the RSII scenario. The inflationary energy scale is important for a number of reasons. Firstly, it is directly related to the reheating temperature of the universe immediately after inflation. If this temperature is too high, the overproduction of gravitinos and other moduli fields may violate constraints imposed at the epoch of nucleosynthesis [24]. Within the context of steep inflation, it is possible that reheating may proceed through gravitational particle production [25]. However, in this case the universe becomes dominated by gravitational waves before the epoch of nucleosynthesis if the energy scale at the end of inflation is lower than a critical value [26]. The inflationary energy scale is also important to quintessential inflationary models [22, 26, 27, 28, 29], where the inflaton field survives the reheating process to act today as the quintessence field. These models must satisfy the stringent coincidence constraint that the densities of dark energy and matter are comparable at the present epoch. Since the conditions at the end of inflation determine the subsequent evolution of the inflaton...
field, it is important to derive constraints on this earlier epoch. Moreover, one of the principle uncertainties in single field inflationary models at present is the number of e–foldings of inflationary expansion that elapsed between the epoch when observable scales first crossed the Hubble radius and the end of inflation [30, 31]. Since this quantity is related to the reheating process, and therefore the energy scale of inflation, constraints on such a scale can in principle lead to constraints on the number of e–foldings.

Unfortunately the process which ends inflation is not directly observable, although the possible overproduction and subsequent evaporation of primordial black holes (PBHs) place strong constraints on the processes immediately after inflation [32]. The earliest observational constraints available to us correspond to approximately 60 e–foldings before the end of inflation and arise from the CMB power spectrum at low multipoles. In particular, the amplitude of gravitational waves is directly related to the energy scale at this epoch [33]. In this paper, we relate the energy scale at the end of RSII inflation to the energy scale at the end of inflation [33]. In this paper, we relate the energy scale at the end of RSII inflation to the energy scale at this epoch [33]. In this paper, we relate the energy scale at the end of RSII inflation to the energy scale at this epoch [33].

The Friedmann equation for the RSII scenario is given by

$$H^2 = \frac{8\pi}{3m_4^2} \rho \left(1 + \frac{\rho}{2\lambda}\right),$$

where $H \equiv \dot{a}/a$ represents the Hubble parameter, a dot denotes differentiation with respect to time, $\rho$ is the energy density of matter confined on the brane, $m_4^2$ is the four–dimensional Planck mass, $\lambda$ is the brane tension and we have assumed that the four–dimensional cosmological constant is zero. The standard cosmic dynamics of Einstein gravity is formally recovered in the limit $\lambda \to \infty$, or equivalently, in the low–energy regime corresponding to $\rho \ll \lambda$.

We assume throughout that the energy–momentum of the brane matter is dominated by a single inflaton field such that $\rho = \phi^2/2 + V(\phi)$, where $V(\phi)$ represents the interaction potential. The equation of motion of the scalar field is then given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,$$

where the prime denotes differentiation with respect to the field. The dynamics of each particular inflationary model are determined by the Friedmann equation and the scalar field equation of motion once the functional form of the inflaton potential has been specified.

Algorithms for finding exact solutions to this system were developed in Ref. [38] by defining a new dimension–less parameter, $y$:

$$\rho \equiv \frac{2\lambda y^2}{1 - y^2},$$

in terms of the energy density, where the restriction $y^2 < 1$ ensures the weak energy condition is satisfied. Substituting Eq. [3] into Eq. [1] then yields the Hubble parameter as a function of $y$:

$$H(y) = \left(\frac{16\pi \lambda}{3m_4^2}\right)^{1/2} \frac{y}{1 - y^2}.$$

The Hubble parameter increases monotonically with increasing $y$ in the range of physical interest.

It is reasonable to assume that the scalar field monotonically rolls down its potential during the final stages of inflation ($\dot{\phi} \neq 0$). This implies that the inflaton field may be viewed as the dynamical variable and Eq. [2] may then be simplified to

$$\rho' = -3H\dot{\phi}.$$

Substituting Eq. [3] into Eq. [5] then yields

$$\dot{\phi} = -\left(\frac{\lambda m_4^2}{3\pi}\right)^{1/2} \frac{y'}{1 - y^2}$$

and it follows from Eqs. [4] and [6] that the scale factor, $a(\phi)$, satisfies [38]

$$y' a' = \frac{4\pi}{m_4^2} y a.$$

Integration of Eq. [7] yields the scale factor in terms of a single quadrature with respect to the inflaton field:

$$a(\phi) = \exp \left[-\frac{4\pi}{m_4^2} \int d\phi \frac{y}{y'}\right].$$
and the inflaton potential is determined by substituting Eqs. (4) and (6) into Eq. (3):

$$V(\phi) = \frac{2\lambda y^2}{1 - y^2} - \frac{\lambda m_y^2}{6\pi} \left( \frac{y'}{1 - y^2} \right)^2.$$  \hspace{1cm} (9)

Thus, in the braneworld scenario the function $y(\phi)$, as defined in Eq. (3), plays a similar role to that of the Hubble parameter, $H(\phi)$, of conventional scalar field cosmology. Employing the scalar field dynamics as the dynamical variable implies that the second–order system of equations (1) and (2) as defined such that the dynamics may be described by the ‘Hubble flow’ parameters, defined such that

$$\epsilon_H(\phi) \equiv -\frac{\dot{H}}{H^2} = \frac{m_y^2 H''}{4\pi H^2},$$  \hspace{1cm} (10)

$$\eta_H(\phi) \equiv \frac{m_y^2 H''}{4\pi H}$$  \hspace{1cm} (11)

and inflation proceeds for $\epsilon_H < 1$.

An important parameter pair in the RSII scenario are the braneworld slow–roll parameters. These are defined by $\epsilon_B \equiv -\ddot{H}/H^2$ and $\eta_B \equiv V''/(3H^2)$ and are given by [20]

$$\epsilon_B \approx \frac{m_y^2}{4\pi} \left( \frac{V'}{V} \right)^2 \left[ \frac{1 + V/\lambda}{(2 + V/\lambda)^2} \right],$$  \hspace{1cm} (12)

$$\eta_B \approx \frac{m_y^2}{8\pi} \left( \frac{V''}{V} \right) \left[ \frac{2\lambda}{2\lambda + V} \right]$$  \hspace{1cm} (13)

in the slow–roll limit, $\ddot{\phi} \ll V$ and $|\ddot{\phi}| \ll H|\dot{\phi}|$. Self–consistency of the slow–roll approximation requires that $\max\{\epsilon_B, |\eta_B|\} \ll 1$. These expressions are important when relating observable quantities such as the spectral indices and gravitational wave amplitudes directly to the inflaton potential and its first two derivatives, but only approximately describe the classical scalar field dynamics. Thus, in the low–energy limit, Eqs. (12) and (13) reduce to the potential slow–roll parameters, $\epsilon_V \approx m_y^2 V'^2/(16\pi V^2)$ and $\eta_V = m_y^2 V''/(8\pi V)$, respectively, and not the Hubble flow parameters (10) and (11).

In the previous Subsection it was shown that the variable $y(\phi)$ acts in an analogous way to the Hamilton–Jacobi function $H(\phi)$. Motivated by these considerations, we now introduce two new parameters, $\beta(y)$ and $\gamma(y)$, defined such that

$$\beta \equiv \frac{4\pi \phi^2}{m_y^2 H^2} = \frac{m_y^2}{4\pi} \frac{y^2}{y'}^2,$$  \hspace{1cm} (14)

$$\gamma \equiv \frac{m_y^2 y''}{4\pi y}$$  \hspace{1cm} (15)

$$\gamma = -\frac{m_y^2}{16\pi \sqrt{\beta}}$$  \hspace{1cm} (16)

Since $H \propto y$ in the low–energy regime ($\rho/2\lambda \rightarrow 0, y \rightarrow 0$), it follows immediately that Eqs. (14) and (15) reduce to Eqs. (10) and (11) in this limit. These parameters may therefore be viewed as generalizations to the RSII inflationary scenario of the Hubble flow parameters of standard inflation. We refer to the parameters (14) and (15) as the braneworld flow parameters. In principle an infinite hierarchy of such parameters involving higher derivatives of $y(\phi)$ may be defined along the lines of Ref. [39, 41]. We emphasize that no reference has yet been made to the slow–roll approximation. Indeed, it can be shown by combining Eqs. (4), (6) and (14) that

$$\beta = \left( \frac{1 - y^2}{1 + y^2} \right) \epsilon,$$  \hspace{1cm} (17)

where $\epsilon = -\dot{H}/H^2$.

The number of e–foldings of inflationary expansion, $N \equiv \ln(a_{end}/a_N) = \int_{t_{end}}^{t_N} dt H$, that occur when the scalar field rolls from some value, $\phi_N$, to the value, $\phi_{end}$, is given exactly in terms of a quadrature involving $\beta(\phi)$:

$$N = \frac{4\pi}{m_y^2} \int_{\phi_N}^{\phi_{end}} \frac{d\phi y}{y'} = \frac{4\pi}{m_y^2} \int_{\phi_N}^{\phi_{end}} \frac{d\phi}{\sqrt{\beta(\phi)}},$$  \hspace{1cm} (18)

where the subscript ‘$N$’ denotes the value $N$ e–foldings before the end of inflation and a subscript ‘end’ denotes values at the end of inflation, respectively. Moreover, rearranging Eq. (14), and integrating yields a second quadrature in terms of $\beta$:

$$y_{end}/y_N = \exp \left[ -\sqrt{\frac{4\pi}{m_y^2}} \int_{\phi_N}^{\phi_{end}} \frac{d\phi}{\sqrt{\beta(\phi)}} \right].$$  \hspace{1cm} (19)

The RSII braneworld cosmology may therefore be parametrized in terms of the three functions $y(\phi)$, $\beta(y(\phi))$ and $\gamma(y(\phi))$, which are analogous to the familiar functions arising in the Hamilton–Jacobi formalism of standard cosmological models: $H(\phi)$, $\epsilon_H[H(\phi)]$ and $\eta_H[H(\phi)]$ [36, 57, 88]. The flow parameters (14) and (15) play an important role in developing upper bounds on the inflationary energy scale as they provide a relationship between the variable $y$ and its derivatives without specifying the functional form of the inflaton potential or restricting the analysis to the slow–roll regime. If the functional form of $\beta(\phi)$ is known, Eqs. (18) and (19) may be solved to determine $y(\phi_{end})$. The magnitude of the Hubble parameter at this epoch then follows directly from Eq. (4). We proceed in the following Section to derive an upper limit on this parameter.
III. UPPER LIMITS ON THE BRANEWORLD INFLATION ENERGY SCALE

We begin by noting that Eq. (11) implies that $y$ is a monotonically decreasing function of time ($\dot{y} < 0$). It then follows from Eq. (9) that the maximal value of the energy scale at the end of inflation is deduced by maximizing the value of $y_{\text{end}}$. Since inflation ends when $\epsilon = -H/H^2 = 1$, we deduce from Eq. (17) that

$$\beta_{\text{end}} = 1 - \frac{y_{\text{end}}^2}{1 + y_{\text{end}}^2}$$  \hspace{1cm} (20)

and $y_{\text{end}}$ is therefore maximized by minimizing the value of $\beta_{\text{end}}$.

Similarly, Eq. (10) implies that the value of $y_{\text{end}}$ is maximized relative to its value $N$ e–folds before the end of inflation by minimizing the area under the curve $\sqrt{\beta(\phi)}$ in the range $\phi = (\phi_N, \phi_{\text{end}})$. In the low–energy limit, $\beta \to \epsilon_H$, and the conventional scenario is recovered. In this case, Liddle noted that the area under the curve $1/\sqrt{\epsilon_H(\phi)}$ is fixed by Eq. (18) once the number of e–folds is determined, since inflation ends precisely when $\epsilon_H = 1$. It then follows that the smallest decrease in the Hubble parameter over the last $N$ e–folds is achieved when $\epsilon_H$ is kept as small as possible. A similar conclusion holds for RSII inflation, although now it is $\beta$ that must be kept as small as possible.

To proceed further, it is necessary to specify the behaviour of $\beta(\phi)$ during the final stages of inflation. We consider two cases in what follows, generalizing the method of Ref. [31] to the braneworld scenario.

A. Case A: $\beta(\phi)$ increases monotonically

Liddle has argued convincingly that in the conventional scenario, $\epsilon$ should increase monotonically as the end of inflation approaches for a wide class of physically well–motivated models. This is also a very reasonable assumption to make in the braneworld scenario [50]. It is straightforward to verify that the quantity $(1 - y^2)/(1 + y^2)$ also increases monotonically with decreasing $y$ and so it follows from Eq. (17) that $\beta$ increases monotonically if $\epsilon$ increases monotonically. Thus, the maximal value of $y_{\text{end}}$ follows in the limit where $\beta$ remains constant during the last $N$ e–folds of inflation.

When $\beta$ is constant, the integral in Eq. (17) may be readily evaluated, yielding

$$y_{\text{end}}^{\text{max}} = y_N \exp \left[ -\frac{4\beta N}{m_N^2} \phi_{\text{end}} - \phi_N \right]$$  \hspace{1cm} (21)

and integrating Eq. (18) and substituting the result into Eq. (21) yields the maximum value for $y_{\text{end}}$ relative to $y_N$:

$$y_{\text{end}}^{\text{max}} = y_N \exp(-N\beta_N).$$  \hspace{1cm} (22)

The maximum value for the Hubble parameter then follows from Eq. (4):

$$H_{\text{end}}^{\text{max}} = \left( \frac{16\pi\lambda}{3m_N^2} \right)^{1/2} \frac{y_N e^{-N\beta_N}}{1 - y_N^2 e^{-2N\beta_N}}.$$  \hspace{1cm} (23)

It should be emphasized that the derivation of Eq. (23) has not invoked the slow–roll approximation. In order to evaluate the upper limit, however, the parameters $\{y_N, \beta_N, \lambda\}$ must be known. The direct dependence on the tension of the brane may be eliminated by expressing the upper limit (23) in terms of the value of the Hubble parameter at $N$ e–folds. The ratio is given by

$$\frac{H_{\text{end}}^{\text{max}}}{H_N} = \frac{y_{\text{end}}^{\text{max}}}{y_N} \left( \frac{1 - y_N^2}{1 - (y_{\text{end}}^{\text{max}})^2} \right).$$  \hspace{1cm} (24)

and substituting Eq. (22) results in the upper limit on the energy scale at the end of inflation:

$$\frac{H_{\text{end}}^{\text{max}}}{H_N} = e^{-N\beta_N} \frac{1 - y_N^2}{1 - y_N^2 e^{-2N\beta_N}}.$$  \hspace{1cm} (25)

In principle, $\beta_N$ and $y_N$ can be determined in terms of the potential and its derivatives.

B. Case B: $\beta(\phi)$ and $\beta'(\phi)$ increase monotonically

The second case assumes that both $\beta(\phi)$ and $\beta'(\phi)$ increase monotonically [37]. This results in a potentially stronger constraint than that of case A since the area under the curve $\sqrt{\beta(\phi)}$ in Eq. (19) is enhanced. Given these restrictions, the form of $\beta$ that maximizes the number of e–folds and the energy scale is linear:

$$\beta = B\phi,$$  \hspace{1cm} (26)

where, without loss of generality, we may assume that $B$ is a positive constant and $\phi > 0$. This implies that $\beta' > 0$ and hence $\beta > \gamma$. The total number of e–folds consistent with this behaviour follows from Eqs. (16) and (18):

$$N = \frac{\sqrt{\beta_{\text{end}} - \sqrt{\beta}}}{\sqrt{\beta(\beta - \gamma)}}$$  \hspace{1cm} (27)

and this implies that $\beta$ must be sufficiently close to $\gamma$ to ensure that enough inflation occurred.

An expression for $\phi_N$ can be found by rearranging Eq. (16) and noting that $B = \beta_{\text{end}}/\phi_{\text{end}}$:

$$\phi_N^2 = \frac{m_N^2}{16\pi} \frac{\beta_N}{(\beta_N - \gamma N)^2}. $$  \hspace{1cm} (28)

Likewise, an expression for $\phi_{\text{end}}$ follows from $\phi_{\text{end}} = \beta_{\text{end}}\phi/\beta_N$:

$$\phi_{\text{end}} = \sqrt{\frac{m_N^2}{16\pi} \frac{1}{\sqrt{\beta_N(\beta_N - \gamma N)}}}. $$  \hspace{1cm} (29)
Given Eq. (26), the integral in Eq. 18 can be evaluated and rearranged to yield

$$N(\beta_N - \gamma_N) + 1 \over \sqrt{\beta_{\text{end}}} = \frac{1}{\sqrt{\beta_N}}$$  \hspace{1cm} (30)

and the integral in Eq. 19 reduces to

$$y_{\text{end}} / y_N = \exp \left[ - \frac{16\pi B}{9m_4^2} \left( \phi_{\text{end}}^{3/2} - \phi_N^{3/2} \right) \right].$$  \hspace{1cm} (31)

Substituting Eqs. (28) and (29) into Eq. 31 implies that

$$y_{\text{end}} / y_N = \exp \left[ - \frac{1}{3} \frac{\beta_N}{\beta_N - \gamma_N} \left( \frac{\beta_{\text{end}}^{3/2}}{\beta_N^{3/2}} - 1 \right) \right]$$  \hspace{1cm} (32)

and it follows, after substituting Eq. 30 into Eq. 32, that

$$y_{\text{end}}^{\text{max}} = y_N \exp \left[ - \frac{\beta_N}{3(\beta_N - \gamma_N)} \left( 1 + N(\beta_N - \gamma_N)^3 - 1 \right) \right].$$  \hspace{1cm} (33)

Eq. 33 can be used to find the ratio of the Hubble parameters $H_{\text{end}}^{\text{max}}$ and $H_N$, as shown in Eq. 24, by substituting Eq. 33 where required. Case A can be recovered from Case B in the limit that $\beta_N \rightarrow \gamma_N$, i.e., $\beta' \rightarrow 0$.

A particular form for the function $y(\phi)$ motivated by particle physics considerations could now be chosen and an estimate for the energy scale at the end of inflation deduced. However, given the enhanced accuracy of recent CMB measurements, it is of more interest to consider what observations may reveal about the underlying inflationary model. We therefore explore this possibility in the following Section.

IV. OBSERVATIONAL CONSTRAINTS ON THE END OF INFLATION

In the previous Section upper limits on the energy scale at the end of RSII inflation were derived without limiting the analysis to the slow–roll regime. However, it is necessary to assume that the slow–roll approximation applied during the epoch when observable scales went beyond the Hubble radius, since the derivations of the perturbation spectra are only valid in this regime. In the slow–roll limit, Eq. 15 reduces to

$$y^2 = \frac{V}{V + 2\lambda}$$  \hspace{1cm} (34)

and Eqs. 12 and 17 then imply that

$$\epsilon_B = \left( 1 + \frac{V}{\lambda} \right) \beta.$$  \hspace{1cm} (35)

The amplitudes of the scalar and tensor perturbations are given by

$$A_S^2 = \frac{1}{25\pi^2} \frac{H^4}{\phi^2}$$  \hspace{1cm} (36)

$$A_T^2 = \frac{4}{25\pi m_4^2} H^2 F^2,$$  \hspace{1cm} (37)

respectively, where

$$\frac{1}{F^2} = \sqrt{1 + s^2 - s^2 \sinh^{-1} \left( \frac{1}{s} \right)}$$  \hspace{1cm} (38)

and

$$s \equiv \left( \frac{3H^2 m_4^2}{4\pi \lambda} \right)^{1/2}.$$  \hspace{1cm} (39)

The function $\beta$ may be viewed as a function of the inflaton potential.

The corresponding spectral indices (tilts) are given by

$$n_S - 1 = -6\epsilon_B + 2\eta_B$$  \hspace{1cm} (40)

$$r = -8\eta_T,$$  \hspace{1cm} (41)

where $r \equiv 16A_T^2 / A_S^2$. Eq. 11 represents the consistency equation for RSII inflation.

It follows from Eqs. 14, 36 and 37 that the ratio of tensor to scalar amplitudes can be expressed directly in terms of the first braneworld flow parameter:

$$\beta = \frac{r}{16F^2}.$$  \hspace{1cm} (42)

We now consider in turn the two cases discussed in Section 3.

A. Case A

Substituting Eqs. 31 and 12 into Eq. 26 allows the upper limit on the inflationary energy scale for case A to be expressed in terms of the relative amplitude of the gravitational wave perturbations

$$\frac{H_{\text{end}}^{\text{max}}}{H_N} = \exp \left( - \frac{r N}{16F^2} \right) \left|_{N} \right.$$  \hspace{1cm} (43)

where $x \equiv V/(2\lambda)$ and quantities on the right hand side of Eq. 15 are to be evaluated $N$ e–foldings before the end of inflation.

Thus far, only the slow–roll approximation has been invoked and the magnitude of the inflaton potential relative to that of the brane tension has not been specified. We now focus on the high–energy limit ($V \gg \lambda$, $x \gg 1$, $s \gg 1$), corresponding to the region of parameter space relevant to the steep inflationary scenario.
In this limit, the function \( F^2 \approx (27H^2m_\gamma^2/(16\pi\lambda))^{1/2} \approx 3x \approx 3s/2 \) and Eq. (42) reduces to

\[
\frac{H_{\text{end}}^{\text{max}}}{H_N} = \frac{e^{-m_\lambda/x}}{1 + x(1 - e^{-2m_\lambda/x})},
\]

where

\[
m_\lambda \equiv \frac{rN}{48}. \tag{45}
\]

In the high energy limit, the gravitational wave amplitude varies as \( A_T^2 \propto (\lambda/m_\gamma^4)(V/2\lambda)^3 \). Thus, a positive detection of the tensor perturbations would not by itself constrain the magnitude of the inflaton potential unless the brane tension has also been specified. However, at present the tension is viewed as a free parameter since a definitive particle physics model for inflation has yet to be developed. In view of this, and anticipating a future detection of the gravitational wave spectrum, we consider how the bound (44) varies with \( \{r, N\} \).

Since \( r \) is small, the high–energy limit of Eq. (44) may be approximated by expanding the exponential terms to first–order in a Taylor series:

\[
\frac{H_{\text{end}}^{\text{max}}}{H_N} = \frac{1 - rN/(48x_N)}{1 + rN/24}. \tag{46}
\]

We therefore arrive at the remarkably simple result that the maximal energy scale at the end of steep inflation can not exceed

\[
\frac{H_{\text{end}}^{\text{max}}}{H_N} = \frac{1}{1 + rN/24}. \tag{47}
\]

Thus, at sufficiently high energy scales, the upper limit is determined entirely by the relative gravitational wave amplitude for a given number of e–foldings. Increasing the gravitational wave amplitude strengthens the bound on the energy scale at the end of inflation. We consider this limit in more detail in Section 4.3.

### B. Case B

For this case, both braneworld flow parameters \( \{r, N\} \) need to be considered. To arrive at a bound given in terms of observable parameters, we restrict the analysis to the high energy limit \( V \gg 2\lambda, x \gg 1 \) and express Eq. (42) in terms of the inflaton potential and its derivatives. These quantities can then be related to the spectral index of the scalar perturbation spectrum and the gravitational wave amplitude. The scalar spectral index is deduced from Eqs. (12), (13) and (40):

\[
n_S - 1 = -\frac{\lambda m_\gamma^2}{2\pi V} \left[ \frac{V''}{V} - \frac{3V'^2}{V^2} \right] \tag{48}
\]

and the gravitational wave amplitude follows from Eq. (42):

\[
\beta = \frac{m_\gamma^2}{4\pi} \frac{\lambda^2 V'^2}{V^4} = \frac{r\lambda}{24V^2}. \tag{49}
\]

The relationship (16) between the braneworld flow parameters implies that the combination \( (\beta - \gamma) \) can be expressed in terms of \( \beta \) and its first derivative \( \beta' \). Hence, by employing Eq. (19), the bound (48) can be written as

\[
y_{\text{end}}^{\text{max}} = y_N \exp \left[ \frac{23V'^2}{8\kappa^2V^2 - 4\lambda^2V^2} \left( \frac{1}{2} \right) \right],
\]

where \( \kappa^2 = 8\pi/m_\gamma^2 \). Eq. (50) can then be expressed succinctly in terms of the scalar spectral index and gravitational wave amplitude by substituting Eqs. (48) and (49):

\[
y_{\text{end}}^{\text{max}} = y_N e^{-m_B/\kappa x N}, \tag{51}
\]

where the constant

\[
m_B \equiv -\frac{r}{72(n_S - 1) + 6x} \left[ 1 - \frac{N}{2} \left( n_S - 1 + \frac{r}{12} \right) \right] \tag{52}
\]

is determined by \( n_S \) and \( r \). We therefore deduce that

\[
\frac{H_{\text{end}}^{\text{max}}}{H_N} = \frac{e^{-m_B/\kappa x N}}{1 + x_N \left( 1 - e^{-2m_B/\kappa x N} \right)}. \tag{53}
\]

It should be emphasized that the bound (53) follows from the assumption that the braneworld flow parameter \( \{r, N\} \) and its first derivative are monotonically increasing functions. In this case, Eq. (16) implies that \( \beta > \gamma \) and this is equivalent to imposing the restriction \( 1 - n_S > r/12 \) at the level of approximation considered in this Subsection. The bound (53) is valid only in this region of parameter space.

To summarize, the bounds (44) and (53) on the steep inflationary energy scale may be written in a unified form, where each limit is determined by the numerical value of the parameters \( m_{A,B} \). At sufficiently high energy scales \( (x \gg 1) \), the relative bounds asymptote to the constant values:

\[
\frac{H_{\text{end}}^{\text{max}}}{H_N} = \frac{1}{1 + 2m_i}, \tag{54}
\]

where \( i = \{A, B\} \).

### C. Parameter Estimates

It is necessary to consider a couple of provisos when interpreting the data from recent measurements of the
FIG. 1: Illustrating Eq. (43) for case A studied in the text. The maximal value of the Hubble parameter at the end of braneworld inflation, \( \frac{H_{\text{end}}}{H_N} \), is shown on the vertical axis and the gravitational wave amplitude, \( r \), on the horizontal axis. The bound is parametrized relative to the value of the Hubble parameter when observable scales first crossed the Hubble radius during inflation and this is assumed to have occurred 60 e–foldings before the end of inflation. The solid line corresponds to \( \frac{V_N}{2\lambda} = 10 \) and the dashed line to \( \frac{V_N}{2\lambda} = 10^4 \). There is little difference despite the three orders of magnitude increase in \( \frac{V_N}{2\lambda} \).

FIG. 2: Illustrating the dependence of the relative bound (43) on the energy scale for fixed gravitational wave amplitudes: \( r = 0.1 \) (top plot), \( r = 0.3 \), \( r = 0.5 \), \( r = 0.7 \) and \( r = 0.9 \) (bottom plot). The variable \( s \) is defined in Eq. (39). Qualitatively similar behaviour arises for case B. In both cases, the ratio \( H_{\text{end}}^{\max}/H_{60} \) asymptotes to a constant value that is determined by the magnitude of the tensor perturbations.

FIG. 3: Illustrating the relative values of the parameters \( m_A \) and \( m_B \) defined in the text in Eqs. (45) and (52) for different spectral tilts and gravitational wave amplitudes.

To arrive at numerical estimates for the bounds, we must specify the number of e–foldings, \( N \), that corresponds to observable scales. There is some uncertainty in this value given its dependence on the reheating temperature. Recently, upper limits on \( N \) were estimated for standard inflation, where it was concluded that a reliable range is \( 50 < N < 60 \) in the absence of a specific inflationary model [30, 31]. In the RSII braneworld scenario, this may alter slightly if inflation ends in the high–energy regime, since the evolution of the scale factor immediately after inflation is different. Here we specify \( N = 60 \).

For case A, the asymptotic value (47) is only increased by a factor of \( \frac{12 + 30r}{12 + 25r} \) by choosing \( N = 50 \). This relaxes the upper bound by 3% for \( r = 0.1 \) and by 13% for \( r = 0.9 \).

The upper bound for case A is stronger for higher gravitational wave amplitudes. We therefore take the weakest upper limit of \( r \leq 0.9 \) quoted by WMAP from the combined data set [6]. Fig. 1 illustrates the bound as a function of gravitational wave amplitude for two different energy scales. The bound is fairly insensitive to the energy scale corresponding to observable scales. Fig. 2 illustrates the bound with varying energy scale for fixed values of \( r \) in the range \( r \in [0.1, 0.9] \). When \( r = 0.9 \), the energy scale at the end of inflation is constrained to be no more than 30% that of the observable scale.

A direct comparison between cases A and B in the high energy limit is made possible by comparing the relative magnitude of the asymptotic values (54) for given gravitational wave amplitudes. It follows from Eqs. (45) and
that

\[ \frac{m_B}{m_A} = -\frac{8}{N\{12(n_S - 1) + r\}} \left[ (1 - \frac{N}{2} \left( n_S - 1 + \frac{r}{12} \right)^3 - 1 \right] \]  

(55)

and Fig. 3 illustrates the dependence of this ratio on observable parameters. The two cases become degenerate, \( m_B \rightarrow m_A \), in the limit \( r \rightarrow 12(1 - n_S) \). The bound on the energy scale for case B is made stronger by increasing the difference between \( r \) and \( (1 - n_S) \) and the ratio between the two bounds is enhanced for smaller gravitational wave amplitudes. Moreover, the ratio increases as the scalar perturbation spectrum is tilted away from a Harrison–Zeldovich form. For example, taking \( n_S = 0.94 \) as the smallest value consistent with WMAP [6] implies that \( m_B/m_A = 3.4 \) for \( r = 0.1 \) and \( m_B/m_A = 2.2 \) for \( r = 0.35 \). This enhances the upper limit on the energy scale by 68% and 64%, respectively.

V. CONCLUSION

The energy scale at which inflation ends is not precisely known. Until the correct particle physics theory is identified and this scale determined, our understanding of the dynamics of the very early universe is incomplete. It is therefore important to constrain the energy scale at the end of inflation.

By introducing new ‘braneworld flow parameters’ that generalize the Hubble flow parameters of the standard scenario, we have derived an upper bound on the energy scale at the end of RSII inflation by invoking only very weak assumptions about the form of the inflaton potential, namely that the first braneworld flow parameter, \( \beta(\phi) \), is a monotonically increasing function during the final stages of inflation. This should be the case for a wide class of realistic potentials which are smooth, differentiable, monotonically decreasing functions. By further assuming that observable scales correspond to the high energy and slow–roll regimes relevant to the steep inflationary scenario [21], this upper bound was expressed in terms of observable parameters. A positive detection of the gravitational wave background would determine the energy scale when observable modes crossed the Hubble radius. In the absence of such a detection, we have expressed the bounds on the scale at the end of inflation relative to this scale.

At progressively higher energy scales, this relative bound is only very weakly dependent on the magnitude of the inflaton potential and asymptotes to a constant value. The asymptotic limit depends on the gravitational wave amplitude and the constraint becomes tighter as \( r \) increases. This is qualitatively similar behaviour to the standard scenario [27]. The amplitude of the scalar perturbations varies as \( A_N^2 \propto H_N^2/\beta_N \) and increasing \( \beta_N \) (and consequently \( r \)) for a given COBE normalization increases the energy scale \( N \) e–foldings before the end of inflation. On the other hand, it follows from Eq. (22) that this has the effect of reducing the maximal energy scale at the end of inflation. Hence, maximizing \( H_N \) does not maximize \( H_{end} \).

This has implications for models of steep inflation, where the logarithmic slope of the potential is large. Whilst this may be viewed as a positive feature, in that it leads to an amplitude of gravitational waves that is potentially observable [21], it results in a stronger upper limit on \( H_{end} \). Consequently, when reheating proceeds through gravitational particle production, the era immediately after inflation where the universe is dominated by the kinetic energy of the inflaton field is prolonged. This makes is harder to satisfy the bounds imposed by primordial nucleosynthesis on excessive gravitational wave contributions to the energy density [20, 29].

In this paper, we have considered the region of parameter space where observable scales correspond to the high-energy regime, i.e., where the quadratic contribution in the Friedmann equation (1) dominates the dynamics. However, it is possible that for some models, the last \( N \) e–foldings of inflation occur when this term is becoming sub–dominant. In the low–energy regime (\( \rho/2\Lambda \ll 1 \)), the limits of the standard scenario are recovered [35], since the braneworld parameters \( \{\beta, \gamma\} \rightarrow \{\epsilon_B, \eta_B\} \). It should be emphasized, however, that the bounds derived in Section 3, Eqs. (25) and (33), are valid over all energy scales. Moreover, it follows from the definitions (14) and (15) that the relation

\[ \gamma - \beta = \eta_B - 2\epsilon_B \]  

(56)
is also valid for all energy regimes when the field is slowly rolling. Since the second flow parameter [15] only appears in the bound (33) through the combination \( (\gamma - \beta) \), it follows that Eqs. (35) and (56) are sufficient to relate the braneworld flow parameters directly to the corresponding braneworld slow–roll parameters, [12] and [13], and hence the scalar spectral index and gravitational wave amplitude. In principle, therefore, the bound for case B derived in Section 4.2 could be extended to cover all energy scales.

Finally, the upper bounds on the energy scale for RSII inflation followed once a variable was identified that played the role of the Hubble flow parameter [10]. In principle, therefore, other scenarios where the Friedmann equation is modified may be developed along similar lines to those of the present work. One extension of the RSII scenario is to relax the assumption of a reflection symmetry in the bulk dimension. This introduces a further term in the Friedmann equation that scales as \( \rho^{-2} \) [13, 16]. The inclusion of a Gauss–Bonnet combination of curvature invariants in the bulk action also modifies the dynamics and at high energies the Friedmann equation asymptotically takes the form \( H^2 \propto \rho^{2/3} \) [17]. The implications of this modification for inflation were recently considered [18]. It would be interesting to develop a framework involving generalized flow parameters
for these scenarios and to investigate how these new features alter the bounds on the inflationary energy scale.

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[1] A. A. Starobinsky, Phys. Lett. 91B, 99 (1980); A. H. Guth, Phys. Rev. D 23, 347 (1981); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982); S. W. Hawking and I. G. Moss, Phys. Lett. 110B, 35 (1982); A. D. Linde, Phys. Lett. 108B, 389 (1982); A. D. Linde, Phys. Lett. 129B, 177 (1983).

[2] A. R. Liddle and D. H. Lyth, Cosmological Inflation and Large-Scale Structure (Cambridge University Press, Cambridge, 2000).

[3] V. Mukhanov and G. Chibisov, Pis’ma Zh. Eksp. Teor. Fiz. 33, 549 (1981) [JETP Lett. 33, 532 (1981), astro-ph/0303077]; A. H. Guth and S. Y. Pi, Phys. Rev. Lett. 49, 1110 (1982); S. W. Hawking, Phys. Lett. 115B, 295 (1982); A. D. Linde, Phys. Lett. 116B, 335 (1982); A. A. Starobinsky, Phys. Lett. 117B, 175 (1982); A. A. Starobinsky, Sov. Astron. Lett. 9, 302 (1983); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D 28, 679 (1983); D. H. Lyth, Phys. Rev. D 31, 1792 (1985).

[4] C. L. Bennett et al., astro-ph/0302207; G. Hinshaw et al., astro-ph/0302217.

[5] D. N. Spergel et al., astro-ph/0302209.

[6] H. V. Peiris et al., astro-ph/0302225.

[7] A. Kogut et al., astro-ph/0302213.

[8] E. Komatsu et al., astro-ph/0302223.

[9] W. Hu and M. White, Astrophys. J. 479, 568 (1997); D. N. Spergel and M. Zaldarriaga, Phys. Rev. Lett. 79, 2180 (1997).

[10] E. Witten, Nucl. Phys. B443, 85 (1995); P. Townsend, Phys. Lett. B 350, 184 (1995); J. Polchinski, Phys. Rev. Lett. 75, 4724 (1995). For reviews, see, e.g., J. Polchinski, String Theory (Cambridge University Press, Cambridge, 1998); C. V. Johnson, D–Branes (Cambridge University Press, Cambridge, 2003).

[11] P. Hoˇrava and E. Witten, Nucl. Phys. B460, 506 (1996); P. Hoˇrava and E. Witten, Nucl. Phys. B475, 94 (1996).

[12] K. Akama, Lect. Notes Phys. 176, 267 (1982), hep-th/0001117; V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. 125B, 136 (1983); M. Vissot, Phys. Lett. 159B, 22 (1985); N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B 429, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B 436, 257 (1998); A. Lukas, B. A. Ovrut, K. S. Stelle, and D. Waldram, Phys. Rev. D 59, 086001 (1999); A. Lukas, B. A. Ovrut, and D. Waldram, Phys. Rev. D 60, 086001 (1999); L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); H. S. Reall, Phys. Rev. D 59, 103506 (1999); M. Gogberashvili, Europhys. Lett. 49, 396 (2000); J. E. Lidsey, Class. Quantum Grav. 17, L39 (2000); J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D 64, 123522 (2001).

[13] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).

[14] J. E. Lidsey, astro-ph/0305228.

[15] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D 62, 024012 (2000).

[16] J. M. Cline, C. Grojean, and G. Servant, Phys. Rev. Lett. 83, 4245 (1999); C. Csáki, M. Graesser, C. Kolda, and J. Terning, Phys. Lett. B 462, 34 (1999).

[17] P. Binétruy, C. Deffayet, and D. Langlois, Nucl. Phys. B565, 269 (2000); P. Binétruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B 477, 285 (2000).

[18] E. E. Flanagan, S.-H. Tye, and I. Wasserman, Phys. Rev. D 62, 044039 (2000).

[19] P. Kraus, J. High Energy Phys. 12, 011 (1999); D. Ida, J. High Energy Phys. 09, 014 (2000); C. Barcelo and M. Visser, Phys. Lett. B 482, 183 (2000).

[20] R. Maartens, D. Wands, B. Bassett, and I. Heard, Phys. Rev. D 62, 041301 (2000).

[21] E. J. Copeland, A. R. Liddle, and J. E. Lidsey, Phys. Rev. D 64, 023509 (2001).

[22] G. Huey and J. E. Lidsey, Phys. Lett. B 514, 217 (2001).

[23] S. Sarkir, Rep. Prog. Phys. 59, 1493 (1996); D. H. Lyth, Phys. Lett. B 476, 356 (2000); R. Kallosh, L. Kofman, A. Linde, and A. Van Proeyen, Class. Quantum Grav. 17, 4269 (2000).

[24] L. H. Ford, Phys. Rev. D 35, 2955 (1987); L. P. Grishchuk and Y. V. Sidorov, Phys. Rev. D 42, 341 (1990); B. Spokoiny, Phys. Lett. B 315, 40 (1993).

[25] V. Sahni, M. Sami, and T. Souradeep, Phys. Rev. D 65, 023518 (2002).

[26] P. J. E. Peebles and A. Vilenkin, Phys. Rev. D 59, 063505 (1999).

[27] A. S. Majumdar, Phys. Rev. D 64, 083506 (2001); S. Mizuno and K. Maeda, Phys. Rev. D 64, 123521 (2001); K. E. Kunze and V. A. Vazquez-Mozo, Phys. Rev. D 65, 044002 (2002); M. Malquarti and A. R. Liddle, Phys. Rev. D 66, 023524 (2002); N. J. Nunes and E. J. Copeland, Phys. Rev. D 66, 043524 (2002); K. Dimopoulos and J. W. F. Valle, Astropart. Phys. 18, 287 (2002); S. Mizuno, K. Maeda, and K. Yamamoto, Phys. Rev. D 67, 023516 (2003).

[28] K. Dimopoulos, astro-ph/0212261.

[29] S. Dodelson and L. Hui, astro-ph/0305113.

[30] A. R. Liddle and S. M. Leach, astro-ph/0305263.
[32] B. J. Carr and J. E. Lidsey, Phys. Rev. D 48, 543 (1993); B. J. Carr, J. H. Gilbert, and J. E. Lidsey, Phys. Rev. D 50, 4853 (1994); A. M. Green and A. R. Liddle, Phys. Rev. D 56, 6166 (1997); R. Guedens, D. Clancy, and A. R. Liddle, Phys. Rev. D 66, 043513 (2002); R. Guedens, D. Clancy, and A. R. Liddle, Phys. Rev. D 66, 083509 (2002); D. Clancy, R. Guedens, and A. R. Liddle, astro-ph/0301568.

[33] A. A. Starobinsky, JETP Lett. 30, 682 (1979); L. F. Abbott and M. B. Wise, Nucl. Phys. B244, 541 (1984); A. A. Starobinsky, Pis’ma Astron. Zh. 11, 323 (1985) [Sov. Astron. Lett. 11, 133 (1985)].

[34] D. H. Lyth, Phys. Lett. B 246, 359 (1990).

[35] A. R. Liddle, Phys. Rev. D 49, 739 (1994).

[36] L. P. Grishchuk and Yu. V. Sidorav, in Fourth Seminar on Quantum Gravity, eds. M. A. Markov, V. A. Berezin, and A. G. Muslimov, Class. Quantum Grav. 7, 231 (1990); D. S. Salopek, J. R. Bond, and J. M. Bardeen, Phys. Rev. D 40, 1753 (1989); D. S. Salopek and J. R. Bond, Phys. Rev. D 42, 3930 (1990).

[37] J. E. Lidsey, Phys. Lett. B 273, 42 (1991).

[38] R. M. Hawkins and J. E. Lidsey, Phys. Rev. D 62, 041301 (2001).

[39] A. R. Liddle, P. Parsons, and J. D. Barrow, Phys. Rev. D 50, 7222 (1994).

[40] D. Langlois, R. Maartens, and D. Wands, Phys. Lett. B489, 259 (2000).

[41] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. W. Copeland, T. Barreiro, and M. Abney, Rev. Mod. Phys. 69, 373 (1997).

[42] C. Gordon and R. Maartens, Phys. Rev. D 63, 044022 (2001).

[43] D. Langlois, R. Maartens, M. Sasaki, and D. Wands, Phys. Rev. D 63, 084009 (2001).

[44] B. Leong, A. Challinor, R. Maartens, and A. Lasenby, Phys. Rev. D 66, 104010 (2002).

[45] N. Deruelle, Astrophys. Space Sci. 283, 619 (2003).

[46] H. Stoica, S. -H. Tye, and I. Wasserman, Phys. Lett. B 482, 205 (2000); H. Collins and B. Holdom, Phys. Rev. D 62, 105009 (2000); N. Deruelle and T. Dolezel, Phys. Rev. D 62, 103502 (2000); P. Bowcock, C. Charmousis, and R. Gregory, Class. Quantum Grav. 17, 4745 (2000); W. B. Perkins, Phys. Lett. B 504, 28 (2001).

[47] C. Charmousis and J. Dufaux, Class. Quantum Grav. 19, 4671 (2002); S. C. Davis, Phys. Rev. D 67, 024030 (2003); E. Gravanis and S. Willison, hep-th/0209076.

[48] J. E. Lidsey and N. Nunes, astro-ph/0303168.

[49] Since we are assuming throughout that the inflaton field is a monotonically varying function of time, we may specify $\dot{\phi} > 0$ without loss of generality. Eq. (6) then implies that $y' < 0$ and hence Eq. (14) implies that $\sqrt{\beta} = -\sqrt{m^2/4\pi y'/y}$.

[50] Indeed, within the context of the slow-roll approximation, it follows that if the logarithmic derivative of the potential increases as the field rolls down its potential (as is to be expected for a single field inflation model that is undergoing a smooth exit), the slow-roll parameter also increases monotonically with decreasing $V(\phi)$.

[51] We employ the normalization conventions of Ref. [12]. The ratio of the tensor to scalar amplitudes is related to the quantity, $r$, employed by Peiris et al. [6] by $r = 16A_T^2/A_S^2$. 