Gauge theory of the Maxwell and semi-simple extended (anti) de Sitter algebra

Salih Kibaroğlu and Oktay Cebeciöğlu

Department of Physics, Kocaeli University, 41380 Kocaeli, Turkey
(Dated: April 22, 2021)

In this paper, a semi-simple and Maxwell extension of the (anti) de Sitter algebra is constructed. Then, a gauge-invariant model has been presented by gauging the Maxwell semi-simple extension of the (anti) de Sitter algebra. We firstly construct a Stelle-West like model action for five-dimensional space-time in which the effects of spontaneous symmetry breaking have been taken into account. In doing so, we get an extended version of Einstein’s field equations. Next, we decompose the five-dimensional extended Lie algebra and establish a MacDowell-Mansouri like action that contains the Einstein-Hilbert term, the cosmological term as well as new terms coming from Maxwell extension in four-dimensional space-time where the torsion-free condition is assumed. Finally, we have shown that both models are equivalent for an appropriately chosen gauge condition.

PACS numbers: 02.20.Sv; 11.25.Hf; 11.15.-q; 04.50.Kd
Keywords: Lie algebra, gauge field theory, modified theory of gravity, cosmological constant

I. INTRODUCTION

The gauge theories provide a useful theoretical background to describe the fundamental interactions of particle physics. The three basic interactions, strong, weak and electromagnetic are described by Yang-Mills gauge theory, which is based on the Lie-groups $SU(3)$, $SU(2)$ and $U(1)$, respectively. These interactions form the basis of the Standard Model. The fourth interaction, gravitational interaction, characterized by Einstein’s general theory of relativity is not a genuine gauge theory in the sense of Yang-Mills type. On the other hand, it can be constructed as the Yang-Mills gauge theory by a symmetry-breaking mechanism. Before we start, it is important to point out the non-commutativity of translation generators, $[P^a, P^b] = \pm iM_{ab}$, for the de Sitter ($dS$) groups. Stelle and West demonstrated that the Einstein-Cartan theory is reproduced by spontaneously broken $dS$ group to the Lorentz group $[1, 2]$. This approach opened new doors to unify the fundamental interactions in the framework of gauge theory $[4]$.

The (anti) de Sitter ($A_dS$) gauge theory of gravity was first formulated in papers $[2, 5–7]$. There are two main advantages of ($A_dS$) gauge theory: The first one is to construct a consistent and renormalizable theory. It is well-known that the general theory of relativity is not a renormalizable theory. To overcome this problem, many theoretical models have been proposed. Among these theories, ($A_dS$) gauge theory emerges as a useful candidate to solve this issue because it can be constructed as a Yang-Mills gauge theory $[8]$. The second one is the explanation of the cosmological constant. The astronomical observations show that our universe is accelerating as in de Sitter spacetime. Therefore, one can say that ($A_dS$) gauge theory of gravity have an important potential to explain the accelerating expansion with a positive cosmological constant (for more detail see $[9]$).

Similar to the de Sitter group, there exists a group having non-commuting translational generators, called the Maxwell group. This group can be interpreted as a modification of the Poincaré group which describes the empty Minkowski space-time, and the translation generators are no longer abelian but satisfy $[P^a, P^b] = iz_{ab}$ $[10–15]$. Here, the antisymmetric central charges $z_{ab}$ represent additional degrees of freedom. In early studies, these extra degrees of freedom were associated with constant electromagnetic fields $[16, 17]$. Recently, the Maxwell group and its modifications have extensively studied to generalize Einstein’s theory of gravity and supergravity $[18–20, 22, 25, 26, 29, 30]$. Furthermore, gauge theory of gravity based on the Maxwell (super)algebras lead to the cosmological constant as in ($A_dS$)-like gravity theories $[18]$. This feature may take an essential role because the studies on the cosmological constant indicate that there should be a background field filling our space-time. In this new context, the additional degrees of freedom represent the uniform gauge field strengths in (super)space leading to uniform constant energy density $[14]$. Therefore, we can say that the Maxwell symmetries may provide a powerful geometrical framework for the cosmological constant and dark energy $[33, 34]$. So, it can be said that Maxwell symmetry contains a useful structure to generalize the general theory of relativity. Moreover, this symmetry is used in various areas such as describing planar dynamics of the Landau problem $[35]$, the higher spin fields $[36, 37]$, and it was also used in the string theory as an internal symmetry of the matter gauge fields $[38]$.
In the present paper, we consider a Maxwell generalization of \((A)\) \(dS\) gauge theory of gravity for two reasons. The first one is the compatibility with renormalizability requirements of \(dS\) gauge group. Namely, the semi-simple structure of \(dS\) gauge group makes it possible to form the \(dS\) gravity as a Yang-Mills gauge theory \([47]\). The second is the present accelerated expansion of our universe due to the basic properties of the de Sitter geometry and its close connection with the cosmological constant. The second is the present accelerated expansion of our universe due to the basic properties of the de Sitter geometry and its close connection with the cosmological constant and it is also well known that the de Sitter vacuum keeps the position of the best candidate for a dark matter. For these reasons, the extension of (anti) de Sitter algebra may provide new insights into the nature of these links.

The organization of this letter is as follows. In Sect. II, we give a short introduction to the Stelle-West model for gravity based on \((A)\) \(dS\) algebra. We then give a semi-simple extension of \((A)\) \(dS\) algebra by considering the Maxwell symmetry. Using the resulting algebra, we construct a gauge theory of gravity and after symmetry-breaking, we get an extended version of the Einstein field equations. In Sect. III, firstly, we will be interested in decomposing extended five-dimensional algebra with respect to one of its regular, four-dimensional subalgebras. A regular subalgebra here is given by de Sitter algebra. Secondly, we construct gauge theory of gravity based on this decomposed algebra and establish a MacDowell-Mansouri like action. Then, we are able to show that both actions, Stelle-West and MacDowell-Mansouri, coincide with each other in a certain gauge.

II. THE (ANTI) DE SITTER GRAVITY AND ITS MAXWELL EXTENSION

We introduce our discussion by reviewing the theory of spontaneously broken \((A)\) \(dS\) gravity given by the Stelle-West model \([1, 2, 40]\). This model provides a useful background for both \(dS\) and \(AdS\) groups. The (anti) de Sitter algebra in 4D is

\[
[\mathcal{M}_{AB}, \mathcal{M}_{CD}] = i (\eta_{AD}\mathcal{M}_{BC} + \eta_{BC}\mathcal{M}_{AD} - \eta_{AC}\mathcal{M}_{BD} - \eta_{BD}\mathcal{M}_{AC}),
\]

where \(\mathcal{M}_{AB}\) are the generators of the group and the metric tensor is chosen to be \(\eta_{AB} = \text{diag}(1, -1, -1, -1, -1)\). Here the capital Latin indices run \(A, B, \ldots = 0, 1, 2, 3, 5\).

To construct a gravitational theory based on the given symmetry group, we will give only a summary of \((A)\) \(dS\) gravity as presented by Stelle and West. The Stelle-West action is given by the help of (anti) de Sitter curvature \(\mathcal{F}^{AB}(x)\) and its connection \(\mathcal{A}^{AB}(x)\)

\[
S_{SW} = \sigma \int V^E \epsilon_{ABCD} \mathcal{F}^{AB} \wedge \mathcal{F}^{CD} + \alpha (e^2 - V^AV^A)
\]

where \(V^A\) is a non-dynamical five vector field has a positive constant magnitude \(V^A V^A = e^2\). We also note that \(V^A\) does not have any degrees of freedom, but it helps to construct the geometrical structure of the theory \([1]\). Moreover, \(\sigma\) is an arbitrary constant and \(\alpha\) represents the Lagrange multiplier \([39, 40]\). This action reduces to the Einstein-Cartan action under the following constraints,

\[
V^A = (0, 0, 0, 0, e)
\]

where \(e^a (x)\) corresponds to the vierbein field, \(D\) is the Lorentz covariant derivative and the small Latin indices take \(a, b, \ldots = 0, 1, 2, 3\). Moreover, \(l\) is related to the cosmological constant and defined by \(l = \sqrt{3/|\Lambda|}\). Indeed, using these constraints, the action Eq.(2) spontaneously broken down to the Einstein-Cartan action together with a topological term (more detail see \([40]\)),

\[
S_{SW} = -\frac{3}{4\kappa A} \int \epsilon_{abcd} R^{ab} \wedge R^{cd} - \frac{2}{3} \Lambda \epsilon_{abcd} \left(R^{ab} \wedge e^c \wedge e^d - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d\right)
\]

where the constant is chosen as \(\sigma e = -\frac{3}{4\kappa A}\) and \(\kappa = 8\pi G c^{-4}\) is Einstein’s gravitational constant. Here, the first term corresponds to the topological sector and the remaining terms are the Einstein-Hilbert action with the cosmological term. Furthermore, we want to note that a similar action to Eq.(2) can also be found in \([41]\) where the topological
theories of gravity were investigated in any dimensions. Besides, to construct an action of this type, it is beneficial to analyze gauged Wess-Zumino-Witten model and transgression field theory [42].

Now, we aim to extend \((A) dS\) gravity by taking into account the Maxwell symmetry. To do this we start with the generalized \((A) dS\) algebra,

\[
[Y_{AB}, Y_{CD}] = i (\eta_{AD} Y_{BC} + \eta_{BC} Y_{AD} - \eta_{AC} Y_{BD} - \eta_{BD} Y_{AC}),
\]

where \(Y_{AB}\) represents the generator of the corresponding algebra. Moreover, this generator can be decomposed into the following form,

\[
Y_{AB} = \frac{1}{2} (M_{AB} + Z_{AB}),
\]

where \(M_{AB}\) corresponds to a generalized \((A) dS\) group generator and \(Z_{AB}\) is a new additional antisymmetric generator responsible for the Maxwell symmetry. These generators obey the following Lie algebra,

\[
[M_{AB}, M_{CD}] = i (\eta_{AD} M_{BC} + \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}),
\]

\[
[M_{AB}, Z_{CD}] = i (\eta_{AD} Z_{BC} + \eta_{BC} Z_{AD} - \eta_{AC} Z_{BD} - \eta_{BD} Z_{AC}),
\]

\[
[Z_{AB}, Z_{CD}] = i (\eta_{AD} M_{BC} + \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}).
\]

The resulting algebra is the Maxwell extension of the \((A) dS\) algebra. To gauge this algebra, we define Lie algebra valued 1-form \(A(x)\),

\[
A(x) = A^A X_A = \frac{1}{2} \omega^{AB} M_{AB} - \frac{1}{2} B^{AB} Z_{AB},
\]

where \(X^A = (M_{AB}, Z_{AB})\) correspond to the generators of the algebra and the associated gauge fields \(A^A = (\omega^{AB}, B^{AB})\) can be described by 1-form fields \(\omega^{AB} = \omega^{AB}_\mu dx^\mu\) and \(B^{AB} = B^{AB}_\mu dx^\mu\), respectively. By the help of the corresponding Lie algebra valued zero-form gauge generator \(\zeta(x)\) which is defined as follows,

\[
\zeta(x) = \zeta^A X_A = -\frac{1}{2} \tau^{AB} (x) M_{AB} - \frac{1}{2} \phi^{AB} (x) Z_{AB},
\]

the transformation of connection one-form is given by

\[
\delta A = -d\zeta - i [A, \zeta],
\]

and hence we get variations of the gauge fields

\[
\delta \omega^{AB} = -d\tau^{AB} - \omega^{[A} \tau^{C|B]} - B^{[A} \phi^{C|B]},
\]

\[
\delta B^{AB} = -d\phi^{AB} - \omega^{[A} \phi^{C|B]} + \tau^{[A} B^{C|B]},
\]

where \(\tau^{AB} (x)\) and \(\phi^{AB} (x)\) are the parameters of the corresponding generators, respectively. The curvature two-forms are defined to be

\[
\mathcal{F}(x) = \mathcal{F}^A X_A = -\frac{1}{2} \mathcal{R}^{AB} M_{AB} - \frac{1}{2} \mathcal{F}^{AB} Z_{AB},
\]

where \(\mathcal{R}^{AB}\) and \(\mathcal{F}^{AB}\) represent the curvatures corresponding to the respective generators. To find the explicit forms of these curvatures, we use the following structure equation

\[
\mathcal{F} = dA + \frac{i}{2} [A, A],
\]
and taking account of the gauge fields in Eq. (9), the group curvatures are found to be,

\[ R^{AB} = d\omega^{AB} + \omega^A_C \wedge \omega^C_B + B^A_C \wedge B^C_B, \]
\[ = R^{AB} + B^A_C \wedge B^C_B, \]
\[ F^{AB} = dB^{AB} + \omega^{[A}_C \wedge B^{C]B}, \]

(15)

where \( R^{AB} = d\omega^{AB} + \omega^A_C \wedge \omega^C_B \) denotes the usual \((A)dS\) curvature 2-forms. The transformation properties of the 2-forms curvature under the infinitesimal gauge transformation can be derived from the following expression,

\[ \delta F = i [\xi, F], \]

(16)

and so one can obtain

\[ \delta R^{AB} = \tau^{[A}_C R^{C]B} + \phi^{[A}_C F^{C]B}, \]
\[ \delta F^{AB} = \phi^{[A}_C R^{C]B} + \tau^{[A}_C F^{C]B}. \]

(17)

Now, we are in a position to construct the gauge-invariant action under local \((A)dS\)-Maxwell transformations. First, define the

\[ J^{AB} = R^{AB} + F^{AB}, \]

(18)

with the following variation

\[ \delta J^{AB} = \tilde{\tau}^{[A}_C J^{C]B}, \]

(19)

where the shifted zero-form parameter is \( \tilde{\tau}^{AB} = \tau^{AB} + \phi^{AB} \). Moreover, it can be shown that the exterior covariant derivative of the shifted curvature goes to zero, i.e. \( D\tilde{J}^{AB} = 0 \), where the extended covariant derivative is given by

\[ D\Phi = [d + \tilde{\omega}]\Phi. \]

(20)

Here, \( \tilde{\omega}^{AB} = \omega^{AB} + B^{AB} \) is the shifted connection. In terms of this connection, the shifted curvature can be written as

\[ \tilde{J}^{AB} = d\tilde{\omega}^{AB} + \tilde{\omega}^A_C \wedge \tilde{\omega}^C_B. \]

(21)

Using the shifted curvature 2-form, we can write down the Stelle-West type action as follows\cite{1, 2, 40}

\[ S = -\frac{3}{4\kappa \Lambda c} \int V^E \epsilon_{ABCDE} \tilde{J}^{AB} \wedge \tilde{J}^{CD} + \alpha (c^2 - V_A V^A). \]

(22)

By varying the action Eq. (22) with respect to \( \tilde{\omega}^{AB} \), one obtains

\[ \epsilon_{ABCDE} D V^E \wedge \tilde{J}^{CD} + V^E \epsilon_{ABCDE} D \tilde{J}^{CD} = 0. \]

(23)

We already know that exterior covariant derivative of the shifted curvature 2-form is zero. So, the equations of the motion reduce to

\[ \epsilon_{ABCDE} D V^E \wedge \tilde{J}^{CD} = 0, \]

(24)

if we impose the constraints similar to Eq. (4),

\[ e^a = -l D e^a = -l \tilde{\omega}^a_4, \quad DV^4 = 0, \]

(25)

where \( l = \sqrt{3/|\Lambda|} \), then the action Eq. (22) now takes the following form,
\[ S_{SW} = -\frac{3}{4\kappa L} \int \epsilon_{abcd} R^{ab} \wedge R^{cd} - \frac{2}{3} \Lambda \left( \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d - \frac{4}{9} \epsilon^a \wedge e^b \wedge e^c \wedge e^d \right) + \epsilon_{abcd} \left( 2 R^{ab} \wedge B^c_e \wedge B^{ed} + DB^{ab} \wedge DB^{ed} - \frac{2A}{3} DB^{ab} \wedge e^c \wedge e^d \right) + \epsilon_{abcd} \left( 2 DB^{ab} \wedge B^c_e \wedge B^{ed} + B^c_e \wedge B^{eb} \wedge B^f_e \wedge B^{fd} - \frac{4A}{3} B^a_e \wedge B^{eb} \wedge e^c \wedge e^d \right). \] (26)

where \( R^{ab} (\omega) = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} \) corresponds to the Riemann curvature 2-form. Here, the first term being Euler topological invariant term does not contribute to the equation of motion. The second terms correspond to the Einstein-Hilbert action together with a cosmological term, and the remaining terms contain the fields \( B^{ab}(x) \) coupled to the spin connection and vierbein. So, we get the Maxwell extension of the Stelle-West action given in Eq. (5).

Moreover, Eq. (24) reduces to the well-known form,

\[ \epsilon_{abcd} J^{ab} \wedge e^c = 0. \] (27)

and this equation leads to an extended version of the Einstein equation in the coordinate basis as follows,

\[ J_{\mu\nu} - \frac{1}{2} g_{\mu\nu} J = 0. \] (28)

### III. DECOMPOSITION OF \((A)\) dS-MAXWELL ALGEBRA

In the previous section, we briefly reviewed the Stelle-West model of gravity and gave its extension by using the Maxwell symmetry in five-dimensions. We also showed that the action Eq. (22) reduces to the generalized Einstein-Cartan gravity by choosing special constraints in Eq. (23). In this section, we establish the gauge theory of gravity based on the Maxwell extended \((A)\) dS group in 4-dimensional space-time. To do this, we first decompose the extended \((A)\) dS algebra in Eq. (5) in terms of the following generators,

\[ M_{ab} = M_{a'b'}, \quad P_a = \sqrt{\frac{\lambda}{2}} (M_{5a} + Z_{5a}), \quad Z_{ab} = Z_{ab}, \] (29)

the Lie algebra of the corresponding group is found to be

\[ [M_{ab}, M_{cd}] = i \left( \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \right), \]
\[ [M_{ab}, Z_{cd}] = i \left( \eta_{ad} Z_{bc} + \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} \right), \]
\[ [Z_{ab}, Z_{cd}] = i \left( \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \right), \]
\[ [P_a, P_b] = i\lambda (M_{ab} + Z_{ab}), \]
\[ [M_{ab}, P_a] = i(\eta_{ad} P_a - \eta_{ad} P_b), \]
\[ [Z_{ab}, P_a] = i(\eta_{ad} P_a - \eta_{ad} P_b), \] (30)

where the generators \( X_A = \{ P_a, M_{ab}, Z_{ab} \} \) correspond to the translations, the Lorentz and the Maxwell transformations. Here, the constant \( \lambda \) has the unit of \( L^{-2} \), and it will be related to the cosmological constant and the metric tensor defined as \( \eta_{ab} = \text{diag}(+,-,-,-,-) \). The self-consistency of this algebra can be checked with the help of the Jacobi identities. The algebra given by Eq. (30) is the semisimple extension of AdS-Maxwell algebra presented in [20].

We will follow similar methods for establishing a gauge theory of gravity as in [23-40] by using differential forms. Let us first define the \((A)\) dS-Maxwell algebra-valued one-form \( A(x) = A^A X_A \) as follows,

\[ A(x) = e^a P_a - \frac{1}{2} \omega_{ab} M_{ab} - \frac{1}{2} B^{ab} Z_{ab}. \] (31)

where \( \mathcal{A}^A (x) = \{ e^a, \omega^{ab}, B^{ab} \} \) are the gauge fields which correspond to the generators of the symmetry group, respectively. Moreover, the unit dimension of the all gauge fields have zero other than \( [e^a] = L \). Here \( L \) is considered as the unit of length. The variation of the gauge field \( A(x) \) under a gauge transformation can be found by using Eq. (11) and the following \((A)\) dS-Maxwell algebra-valued zero-form gauge generator,
Thus, one can find the transformation law of gauge fields as follows,

\[ \delta e^a = -dy^a - \omega^a e^c - B^a e^c + \varphi^a e^c, \]

\[ \delta \omega^{ab} = -d\tau^{ab} - \omega^{[a} \tau^{[b]} - B^{[a} \varphi^{[b]} - \lambda e^{[a} y^{b]}, \]

\[ \delta B^{ab} = -d\phi^{ab} - \omega^{[a} \phi^{[b]} + \tau^{[a} B^{[b]} - \lambda e^{[a} y^{b]}, \]

where \( y^a(x), \tau^{ab}(x) \) and \( \phi^{ab}(x) \) are the parameters of the associated generators. The curvature two-forms of the associated gauge fields \( F(x) = F^A X_A \) are defined to be

\[ F(x) = F^a P_a - \frac{1}{2} R^{ab} M_{ab} - \frac{1}{2} F^{ab} Z_{ab}, \]

where \( F^a(x), R^{ab}(x) \) and \( F^{ab}(x) \) represent the curvatures which come from the associated generators. To find the explicit forms of these curvatures, we have used the structure equation Eq.(14) together with the gauge fields in Eq.(31). The group curvature 2-forms are,

\[ F^a = de^a + \omega^a \wedge e^c + B^a \wedge e^c, \]

\[ R^{ab} = R^{ab}(\omega) + B^a \wedge B^b + \lambda e^a \wedge e^b, \]

\[ F^{ab} = dB^{ab} + \omega^{[a} \wedge B^{[b]} + \lambda e^{[a} \wedge e^{b}, \]

where \( R^{ab}(\omega) \) denotes the usual Riemann 2-form tensor. The transformation properties of the curvature 2-forms under the infinitesimal gauge transformation can be found by using Eq.(16), Eq.(32) and Eq.(34),

\[ \delta F^a = -R^a \wedge e^c - \tau^a F^e - F^a \wedge e^c + \phi^a F^e, \]

\[ \delta R^{ab} = \tau^{[a} R^{c[b]} + \phi^{[a} F^{c[b]} + \lambda y^{[a} F^{b]} \]

\[ \delta F^{ab} = \phi^{[a} R^{c[b]} + \tau^{[a} F^{c[b]} + \lambda y^{[a} F^{b]} \]

If we define a shifted curvature as \( J^{ab} = R^{ab} + F^{ab} \), then it transforms as \( \delta J^{ab} = -J^{[a} (\tau^{c[b]} + \phi^{c[b]}). \) We finally write the MacDowell-Mansouri like action as

\[ S = \frac{1}{4 \kappa^2} \int \epsilon_{abcd} J^{ab} \wedge J^{cd} \]

\[ = \frac{1}{4 \kappa^2} \int \epsilon_{abcd} R^{ab} \wedge R^{cd} + 4\lambda \left( \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d + \lambda \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \right) \]

\[ + 2 \epsilon_{abcd} R^{ab} \wedge B^c \wedge B^d + \epsilon_{abcd} B^a \wedge B^c \wedge B^e \wedge B^d \]

\[ + 4 \lambda \epsilon_{abcd} B^a \wedge B^b \wedge e^c \wedge e^d + 2 \epsilon_{abcd} R^{ab} \wedge D B^{cd} \]

\[ + 2 \epsilon_{abcd} B^a \wedge B^b \wedge D B^{cd} + \epsilon_{abcd} D B^{ab} \wedge D B^{cd} + 4 \lambda \epsilon_{abcd} D B^{ab} \wedge e^c \wedge e^d \]

where \( \gamma \) is an arbitrary constant. Furthermore, if we choose \( \lambda = -\frac{1}{4} \) and \( \gamma = -\frac{1}{4} \) then the resulting action takes the same form as the action given in Eq.(22).

Now, we are in a position to consider the field equations of the theory and they can be derived from a variational action principle. The equations of motion can be found by the variation of the action in Eq.(37) with respect to the gauge fields \( \omega^{ab}(x), B^{ab}(x) \) and \( e^a(x), \) respectively,

\[ D J^{cd} + B^c_e \wedge J^{[c|d]} = 0, \]

\[ \epsilon_{abcd} e^b \wedge J^{cd} = 0, \]

here, we want to note that the variation with respect to \( \omega^{ab}(x) \) and \( B^{ab}(x) \) lead to the same equation in Eq.(38). Furthermore, one can show that all these equations of motion verify each other. Making use of the shifted curvature and passing from the tangent indices to world indices with the help of
\[ \epsilon_a^\mu \epsilon_b^\nu J^{ab} = \frac{1}{2} J^\mu_{\rho\sigma} dx^\rho \wedge dx^\sigma \]  

one writes the field Eq.(39) as

\[ J^\mu_\rho - \frac{1}{2} \delta^\mu_\rho J = 0. \]  

and expanding the shifted curvature we get,

\[ R^\mu_\rho (\omega) - \frac{1}{2} R (\omega) \delta^\mu_\rho + \Lambda \delta^\mu_\rho = T^\mu_\rho (B) \]  

where

\[ T^\mu_\rho (B) = - \left( \epsilon_a^\mu \epsilon_b^\nu D_{[\rho} B^{ab}_{\nu]} + \epsilon_a^\mu \epsilon_b^\nu B^a_{[\rho c} B^{b\nu]} + \frac{1}{2} \delta^\mu_\rho \left( \epsilon_a^c \epsilon_b^d D_{[\gamma} B^{ab}_{\nu]} + \epsilon_a^c \epsilon_b^d B^a_{[\gamma c} \wedge B^{b\nu]} \right) \]  

and it represents the tensorial contribution of the Maxwell symmetry. Therefore, we demonstrated that a new extended framework leads to the generalized Einstein field equation together with a cosmological term plus additional energy-momentum tensor as a function of the gauge field \( B^{ab} (x) \). Moreover, in the limit of \( B^{ab} (x) = 0 \), Eq.(42) reduces to the well-known Einstein's gravitational field equation including the cosmological constant.

IV. CONCLUSION

In the present paper, we interested in a gauge theory of gravity based on the Maxwell extension of \( (A) dS \) algebra. In our extension, the translation generator satisfies a new commutation relationship as \([P_a, P_b] = i (M_{ab} + Z_{ab}).\) From this type of extension, we obtained a generalized Lie algebra by unifying the (anti) de Sitter with the Maxwell algebra presented in Eq.(8). In this generalization, we preserved the semi-simple structure of de Sitter algebra. We then constructed the gauge theory for the resulting algebra and establishing a Stelle-West like action and we derived a generalization of Einstein’s field equations. Moreover, we obtained the semi-simple extension of \( (A) dS-Maxwell \) algebra in Eq.(30) for four-dimensional case by decomposing the algebra given in Eq.(8) under the chosen conditions in Eq.(29). After that, we took this algebra for the construction of gauge theory and established a MacDowell-Mansouri like action. As a result of these calculations, we obtained the same field equations as the previous one in a certain condition. These field equations contain a positive cosmological constant and additional terms related to the Maxwell symmetry in addition to Einstein’s field equations. So the resulting gravitational theory can be seen as a generalization of the results given in [20]. If we take the Maxwell gauge field as \( B^{ab} (x) = 0 \), then this gravitational model reduces to well-known de Sitter gravity.

We want to remark that it is possible to construct a Yang-Mills like action [43] based on these extended algebras given in Eq.(8) and Eq.(30) because of their semi-simple characteristics. We know that three of four fundamental interactions except gravity are Yang-Mills type of gauge theories. Therefore, the resulting algebras may provide a useful background to study the unification problem of fundamental interactions [48].

As we mentioned before the Maxwell extended algebras (similar to the de Sitter algebras) provide a useful background to investigate the cosmological constant problem. So the unification of de Sitter and Maxwell algebras may open a new way to analyze the cosmological constant problem. Furthermore, there may be occurred interesting results if we construct a cosmological interpretation of the Maxwell extended theories because, in addition to minimal de Sitter gravity theory, the Maxwell extended theories contain new energy-momentum tensors dependent on \( B^{ab} (x) \) fields. Thus, it is known that additional source term can be related to the dark energy [49, 50], the Maxwell fields \( B^{ab} (x) \) may play an important role to explain the dark energy problem.

ACKNOWLEDGMENTS

This study is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) Research project No. 118F364.

[1] K. S. Stelle, P. C. West, J. Phys. A: Math. Gen. 12, L205 (1979).
