Topical Results on Lattice Chiral Fermions in the CFA

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We report new results on the lattice regularization of the chiral Schwinger model and the chiral U(1) model in four dimensions in the CFA.

1. INTRODUCTION

The continuum fermion approach (CFA) to regularizing chiral fermions appears to be a promising method. For a summary of results obtained so far see \[1\]. The basic idea of the approach is the following. We start from a lattice with spacing \(a\). We call this the original lattice. On this lattice the simulations are done. Next we construct a finer lattice with lattice spacing \(a_f\), using a suitable interpolation of the gauge field \[2\]. On this lattice we formulate the fermions. The action for a single fermion of chirality \(\epsilon_\alpha\) is taken to be

\[
S_{\epsilon_\alpha} = \bar{\psi}D_{\epsilon_\alpha}\psi = \frac{1}{2}\bar{\psi}\{\gamma_5(D_{\epsilon_\alpha}^+ + D_{\epsilon_\alpha}^-) \psi
- \frac{r}{2}a_f D_{\epsilon_\alpha}^+ D_{\epsilon_\alpha}^- \psi\},
\]

where, restricting ourselves to gauge group U(1),

\[
(D_{\epsilon_\alpha}^\pm \psi)(n) = \pm \frac{1}{a_f}[[P_{-\epsilon_\alpha} + P_{\epsilon_\alpha}(U^f_{\pm\mu}(n))^\epsilon_{\alpha}] \times \psi(n \pm \hat{\mu}) - \psi(n)],
\]

with \(U^f_{\pm\mu}\) being the link variable on the fine lattice, \(\epsilon_\alpha\) the fractional charge, and \(P_{\epsilon_\alpha} = (1 + \epsilon_\alpha \gamma_5)/2\). The effective action is then computed from \(\hat{S}_{\epsilon_\alpha}\) in the limit \(a_f \to 0\), where \(a\) is kept fixed. Hence the name continuum fermions.

A particular feature of this action – or better of the Wilson term we have chosen – is that the Wilson-Dirac operator \(D_{\epsilon_\alpha}\) fulfills the Ginsparg-Wilson relation \[3,4\]:

\[
\gamma_5 D_{\epsilon_\alpha} + D_{\epsilon_\alpha} \gamma_5 = r a_f D_{\epsilon_\alpha} \gamma_5 D_{\epsilon_\alpha} + O(a_f^2).
\]

This is not a great surprise. The motivation behind our construction was to find an action which obeys the index theorem – what, in fact, it does as we shall see below. The Ginsparg-Wilson relation \(\hat{S}_{\epsilon_\alpha}\) does, however, not guarantee that the theory is invariant under chiral gauge transformations.

Indeed, the resulting effective action, \(W_{\epsilon_\alpha}\), is not gauge invariant, even in the limit \(a_f \to 0\). But there exists a local, purely bosonic counterterm \(C\), so that

\[
\hat{W}_{\epsilon_\alpha} = \lim_{a_f \to 0} W_{\epsilon_\alpha}^\Sigma, W_{\epsilon_\alpha}^\Sigma = W_{\epsilon_\alpha} + C
\]

is invariant under chiral gauge transformations. For the imaginary part this is only true in the anomaly-free model. The counterterm can be – and has been – computed in perturbation theory.

A further important feature of the action \(\hat{S}_{\epsilon_\alpha}\) is that it has a shift symmetry which makes sure that the ungauged fermion with chirality \(-\epsilon_\alpha\) decouples.

In the chiral Schwinger model we found for the topologically trivial sector of the theory

\[
\hat{W}_{\epsilon_\alpha} = \frac{1}{2}(W_V + W_0) + i \text{Im} W_{\epsilon_\alpha},
\]

where \(W_V\) and \(W_0\) are the effective actions of the corresponding vector model and the free theory,
respectively. The imaginary part of the effective action turned out to be given by the harmonic part of the gauge field, i.e. the toron field, alone, and it could be computed analytically from the gauge field we started with. Thus we arrived at an action which can be simulated relatively easily on the original lattice. We believe to find a similar result in more realistic models in four dimensions.

After so much of introduction, let us now come to our new results.

2. CHIRAL SCHWINGER MODEL

Our first results concern the chiral Schwinger model. Here we want to test whether the index theorem is fulfilled. This is a non-trivial task because any topologically non-trivial gauge field involves at least one singular plaquette. Some authors have argued that the CFA would fail this test. This would be true had we employed the standard gauged or ungauged Wilson terms. Furthermore, we will investigate whether our approach reproduces the correct anomaly.

In two dimensions the index theorem says that in a background gauge field configuration of topological charge $Q$ we should find exactly

$$n_{\epsilon_o} = |Q| \theta(\epsilon_o Q)$$

(6)

zero modes of chirality $\epsilon_o$. For a configuration of, e.g., charge $Q = +1$ this would mean $n_+ = 1$ and $n_- = 0$.

To leading order in $a_f$ the Wilson-Dirac operator can be written

$$D_{\epsilon_o} = D^{\epsilon_o} = a_f \frac{2}{\pi} D^{-\epsilon_o} D^{\epsilon_o} + O(a_f^2),$$

(7)

where $D^{\epsilon_o}$ is the average of forward and backward derivatives. Being a finite matrix, the Dirac operator $D^{\epsilon_o} \equiv -D^{-\epsilon_o \dagger}$ has the same number of zero modes as the corresponding vector operator, namely $n_{\epsilon_o} + n_{-\epsilon_o} = |Q|$, thus violating the index theorem. But the situation is different for the Wilson term $D^{-\epsilon_o} D^{\epsilon_o} \equiv (D^{-\epsilon_o} D^{\epsilon_o})^\dagger$. It has $n_{\epsilon_o}$ zero modes of chirality $\epsilon_o$ and none of chirality $-\epsilon_o$, exactly as required by the index theorem. For small, but finite $a_f$ the zero modes approach the value

$$\left( \frac{a_f}{a} \right)^2 \frac{|Q|}{L^2},$$

(8)

where $L$ is the size of the original lattice.

We consider two configurations of charge $Q$. We denote the link variables on the original lattice by $U_\mu(s) = \exp(i\theta_\mu(s))$, $-\pi < \theta_\mu(s) \leq \pi$, where $s_\mu$ are the lattice points on the original lattice. The first configuration is

$$\theta_1(s) = F s_2 - \bar{\theta}_1,$$

(9)

$$\theta_2(s) = \begin{cases} -\bar{\theta}_2, & s_2 = 1, \ldots, L - 1, \\ FL s_1 - \bar{\theta}_2, & s_2 = L, \end{cases}$$

where $F = 2\pi Q/L^2$, $\bar{\theta}_1 = \pi(L - 1)/L^2$ and $\bar{\theta}_2 = \pi/L^2$. The second configuration is

$$\theta_\mu(s) = 2\pi Q \epsilon_{\mu\nu} \partial_\nu G(s - \bar{s}), \bar{s} = (L/2, L),$$

(10)

where $G$ is the inverse lattice Laplacian. Both configurations have constant field strength $F$, zero toron field, and for $|Q| = 1$ they have one singular plaquette at $s = (L/2, L)$. The two configurations are related by a periodic (topologically trivial) gauge transformation. In the following we shall take $L = 6$, and we shall use the interpolation given in \[2\].

Both configurations give the same value for the effective action \[\delta W_{\epsilon_o} / \delta h(s)\]. As the anomaly-free model we took $\epsilon_o \epsilon_{\mu\nu} = -1, -1, -1, 1, 2$. This indicates that we have gauge invariance in the background of singular gauge fields as well. For $Q = 1$ we find exactly one zero mode with chirality $+$, and none with chirality $-$. For $Q = -2$ we find two zero modes with chirality $+$, and none with chirality $-$. For negative charges we find the same result but with $+$ and $-$ interchanged, in agreement with the index theorem. The values of the zero modes are in good agreement with the analytical result \[\delta W_{\epsilon_o} / \delta h(s)\].

A further requirement of the method is that it reproduces the correct anomaly. In the chiral Schwinger model the anomaly condition reads

$$\delta W_{\epsilon_o} / \delta h(s) = i\epsilon_o \epsilon_{\mu\nu} \frac{1}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}(s),$$

(11)

where $h(s)$ is a gauge transformation. We consider configuration \[\bar{F} = -1\] with $\bar{F} = -1$ and $\bar{s} = (L/2, L/2)$ now. The r.h.s. of \[\delta W_{\epsilon_o} / \delta h(s)\] is known analytically. The l.h.s. is computed for $a/a_f = 5$, i.e. on the $30^2$ lattice. In Fig. 1 we show the result for both sides separately for $\epsilon_o \epsilon_{\mu\nu} = -1$ and 2, as well
Figure 1. The anomaly as a function of \( s_1 \) for \( s_2 = 4 \). The l.h.s. (r.h.s.) of (11) is marked by ✷ (+) for \( \epsilon_\alpha \epsilon_\alpha = -1 \), and by ✶ (×) for \( \epsilon_\alpha \epsilon_\alpha = 2 \). The result for the anomaly-free model is denoted by •. As before, the coefficients, \( c_2, c_4 \) and \( c_\partial \), can be computed in lattice perturbation theory. The result so far is
\[
c_2 = 0.03717 a_f^{-2}, \quad c_4 = 0.00052. \tag{13}
\]

We cannot quote any number for \( c_\partial \) yet. Numerically we find that gauge invariance is, within our present accuracy, already restored by the counterterms \( C_2 \) and \( C_4 \), if we use the perturbative values [13] for the coefficients. This indicates that \( c_\partial \) is very small, or zero. As an upper bound we can quote \( c_\partial < 10^{-4} \).

What distinguishes the chiral theory from the corresponding vector theory is the imaginary part, \( \text{Im} W_{\epsilon_\alpha} \), of the effective action. In two dimensions this was entirely determined by the toron field contribution. In four dimensions, however, we find that it is zero for toron field configurations. This leaves the interesting possibility that the imaginary part of the effective action is generally zero in this model.

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