CAUSAL HOLOGRAPHY OF TRAVERSING FLOWS

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ABSTRACT. In this paper, we continue our study (see [K1]-[K3]) of the trajectory spaces of traversally generic (see [K2] for the definition) flows on smooth compact manifolds with boundary. Unlike flows generated by vector fields with singularities, the trajectory spaces \( \mathcal{T}(v) \) of traversally generic flows are not pathological: in fact, they are compact CW-complexes. These spaces \( \mathcal{T}(v) \) are homotopy equivalent to the manifold \( X \) on which the vector field \( v \) is defined. Moreover, despite being spaces with singularities, they retain some residual smooth structure of \( X \).

With the help of a traversally generic field \( v \) on \( X \), we divide its boundary \( \partial_1 X \) into two complementary compact manifolds, \( \partial_+^1 X(v) \) and \( \partial_-^1 X(v) \), which share a common boundary \( \partial_2 X(v) \). Then we introduce the causality map \( \mathcal{C}_v : \partial_+^1 X(v) \to \partial_-^1 X(v) \), a far relative of the Poincaré return map. Our main results, Theorem 4.1 and Theorem 4.2, claim that, for traversally generic flows, the knowledge of the causality map \( \mathcal{C}_v \) is sufficient for a reconstruction of the pair \( (X, \mathcal{F}(v)) \), up to a piecewise differential homeomorphism \( \Phi : X \to X \) (respectively, up to a diffeomorphism \( \Phi \)) which is the identity on the boundary \( \partial_1 X \). Here \( \mathcal{F}(v) \) denotes the oriented 1-dimensional foliation on \( X \), produced by the \( v \)-flow.

We call these results “holographic” since the \((n+1)\)-dimensional \( X \) and the un-parameterized dynamics of the flow on it are captured by a single correspondence \( \mathcal{C}_v \) between two \( n \)-dimensional screens, \( \partial_+^1 X(v) \) and \( \partial_-^1 X(v) \).

This holography of traversing flows has numerous applications to the dynamics of general flows. Some of them include results about the geodesic flows on manifolds \( M \) with boundary and the classical scattering problems. These results are valid for special class of Riemannian metrics on \( M \) which we call traversally generic metrics. For such metrics, we prove that the space of geodesic curves on \( M \) can be reconstructed from the geodesic scattering data (see Theorem 5.3).

1. Introduction

This paper is the fourth in a sequence which studies Morse theory and gradient-like flows on manifolds with boundary (see [K1], [K2], and [K3]).

Let \( X \) be a compact smooth manifold with boundary. We employ the semi-local models for traversally generic fields \( v \) on \( X \) (as described in [K1]—[K3]) to get a better understanding of the trajectory space \( \mathcal{T}(v) \) of the \( v \)-flow, in particular, of the surrogate smooth structure (see Definition 2.2) \( \mathcal{T}(v) \) inherits from \( X \).

In Theorem 2.2, we prove that \( \mathcal{T}(v) \) can be given the structure of Whitney stratified space (see Definition 2.3), where the stratification is labeled by the elements of the universal poset \( \Omega^* \) that has been introduced in [K3]. As a result, for a traversally generic field
\(v\), the trajectory space \(T(v)\) admits a triangulation, amenable to the flow-induced \(\Omega^*\)-stratification (see Corollary 2.4).

For traversally generic fields \(v\), the trajectory space \(T(v)\) is not only homotopy equivalent to \(X\) (Corollary 2.4), but it also shares with \(X\) all stable characteristic classes of the surrogate tangent bundle \(\tau(T(v))\).

In Origami Theorem 3.1, for any given \((n+1)\)-manifold \(X\), we construct an open set of traversally generic fields \(v\), for which the trajectory space \(T(v)\) is the \((n+1)\)-to-1 ramified image of the standard ball \(D^n\); as a result, the \(n\)-ball can be “folded” into the trajectory spaces of some massive set of traversally generic flows.

In the rest of this paper, we are preoccupied with the following general question:

“For a traversing field \(v\) on a compact manifold \(X\), what kind of residual structure on the boundary \(\partial X\) allows for a reconstruction of the pair \((X,v)\), up to a diffeomorphism, or up to a piecewise differential ("PD" for short) homeomorphism?”

If such a structure on the boundary is available, it deserves to be called holographic, since the information about the \((n+1)\)-dimensional dynamics is recorded on a \(n\)-dimensional record (or rather on the appropriately linked pair of such records).

With the dream of holography in mind, for a traversing field \(v\), we introduce the causality map

\[ C_v : \partial_1^+ X(v) \to \partial_1^- X(v) \]

that takes any point \(x\) on the boundary, where the field is directed inwards, to the “next” along the trajectory \(\gamma_x\) point \(C_v(x)\), where the field is directed outwards. In general, \(C_v\) is a discontinuous map, with a very particular types of discontinuity.

The causality map plays a role somewhat similar to the one played by the classical Poincaré return map: continuous dynamics are reduced to a single map (and its iterations) of a lower-dimensional slice.

Let \(v_1\) be a traversally generic field on a manifold \(X_1\), and let \(v_2\) be a traversally generic field on a manifold \(X_2\), the dimensions of both manifolds being equal. We denote by \(\mathcal{F}(v_i)\) the oriented 1-dimensional foliation of the manifold \(X_i\) produced by the vector field \(v_i\), \(i = 1, 2\).

Theorem 4.1 and Theorem 4.2—the main results of this paper—claim that any PD-homeomorphism (respectively, any diffeomorphism) \(\Phi^\partial : \partial_1 X_1 \to \partial_1 X_2\) which commutes with the causality maps \(C_{v_1}\) and \(C_{v_2}\), extends to a PD-homeomorphism (respectively, any diffeomorphism) \(\Phi : X_1 \to X_2\). Moreover, \(\Phi\) takes each \(v_1\)-trajectory to a \(v_2\)-trajectory, thus mapping the \(v_1\)-oriented 1-dimensional foliation \(\mathcal{F}(v_1)\) to the \(v_2\)-oriented foliation \(\mathcal{F}(v_1)\).

In other words, the causality map \(C_v\) allows for a reconstruction of the pair \((X, \mathcal{F}(v))\) up to a PD-homeomorphism (respectively, up to a diffeomorphism). So the smooth topology of \(X\) and the unparametrized \(v\)-flow dynamics are rigid for the given “boundary conditions” \(C_v : \partial_1^+ X(v) \to \partial_1^- X(v)\).
This topological rigidity has a number of implications for general dynamical systems (see Theorem 4.3, Corollary 4.2, and Corollary 4.3). We summarize them in the Holographic Causality Principle. Vaguely, it states that the causality relation on a generic event horizon in the space-time space of a given dynamic system determines the compact portion of the event space, bounded by the event horizon, and the evolution of the system in it, up to a diffeomorphism of that portion.

In Section 5, we apply the Holographic Causality Principle to geodesic flows on the spaces $SM$ of unit tangent vectors on compact Riemannian manifolds $M$ with boundary (see Theorem 5.2 and Corollary 5.4).

This application is intimately linked to the classical inverse scattering problems. Let us briefly explain what we mean here by the scattering data. Given a compact Riemannian manifold $M$ with boundary, for each geodesic curve $\gamma \subset M$ which enters $M$ through a point $m \in \partial M$ in the direction of a unitary tangent vector $u \in T_m(M)$, we register the exit point $m' \in \partial M$ and the exit direction, given by a unitary tangent vector $u' \in T_{m'}(M)$ at $m'$. The correspondence $\{(m, v) \Rightarrow (m', v')\}_{(m, v)}$ is what we call “the scattering data”.

We strive to restore $M$ and the metric $g$ on it from the geodesic scattering data (this resembles the problem of reconstructing the mass distribution from the gravitational lensing). Moving towards this goal, we introduce certain classes of metrics $g$ which we call geodesically traversally generic (see Definition 5.1 and Definition 5.2). We speculate that the space of such metrics $\mathcal{G}(M)$ is open and nonempty for any $M$, and prove that it is indeed open (see Theorem 5.1).

Furthermore, in Theorem 5.2 we prove that if a metric $g \in \mathcal{G}(M)$, then the geodesic flow $v^g$ on $SM$ is topologically rigid for the corresponding scattering data. For general pairs $(M, g)$, one should take these results with a grain of salt, since we do not know that $\mathcal{G}(M) \neq \emptyset$ for any given $M$. We do however exhibit a rich gallery $\mathcal{G}(M) \supset \mathcal{G}(M)$ of so-called gradient type metrics (see Definition 5.1), and we do have nontrivial examples of the desired metrics $g \in \mathcal{G}(M)$. For instance, if the boundary $\partial M$ is concave in $g \in \mathcal{G}(M)$, then $g$ is geodesically traversally generic.

On the other hand, for all metrics of the gradient type, the geodesic field $v^g$ on $SM$ allows for arbitrary accurate $C^\infty$-approximations by traversally generic fields $w$ on $SM$, for which the topological restoration of the $w$-induced dynamics on $SM$ from the “approximate scattering data”

$$C_w : \partial^+ SM(w) \to \partial^- SM(w)$$

is possible.

In Theorem 5.3 we prove that, for a geodesically traversally generic metric $g$ on $M$, the scattering data are sufficient for a reconstruction of the smooth topological type of the space of geodesics in $M$. In general, that space is not a smooth manifold, but for $g \in \mathcal{G}(M)$, it is a compact $CW$-complex, carrying some “surrogate smooth structure”. It is this structure that is captured by the scattering data.

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1. which are not necessarily of the gradient type
2. The difficulty is to find the approximating field $w$ in the form $v^\tilde{g}$ for some metric $\tilde{g}$ on $M$. 
In Theorem 5.4 we prove that, for \( g \in \mathcal{G}(M) \), the geodesic scattering data on \( \partial M \) allow for a reconstruction of the homology of \( M \); moreover, if the tangent bundle of \( M \) is trivial, the same data allow for a reconstruction of the stable smooth topological type of the manifold \( M \).

In Section 6, we introduce the concepts of proto-holographic and holographic structures on closed manifolds \( Y \) (see Definitions 6.1-6.3). These definitions postulate the existence of a special cover of \( Y \) by local charts which mimic the intersections of a \( \hat{v} \)-adjusted cylindrical neighborhoods of \( \hat{v} \)-trajectories in \( \hat{X} \supset X \) with the boundary \( \partial X = Y \). Here \( v \) is a traversally generic field on a manifold \( X \) such that \( \dim X = \dim Y + 1 \).

In Theorem 6.1, we show that any traversally generic field \( v \) on \( X \) generates a holographic structure \( \mathcal{H}(v) \) on \( \partial X \), which is cobordant to the empty set (see Definition 6.3). Conversely, any cobordant to the empty set holographic structure \( \mathcal{H} \) on \( \partial X \) is generated by a traversally generic field \( v \) on \( X \).

It turns out, that the language of holographic and cobordant to \( \emptyset \) structures \( \{ \mathcal{H} \} \) on \( \partial_1 X \) is equivalent to the language of causality maps \( \{ C_v : \partial_1^+ X(v) \rightarrow \partial_1^- X(v) \} \). Thus \( \mathcal{H}(v) \) allows for a reconstruction of the pair \( (X, F(v)) \), up to a diffeomorphism.

We complete the section with few remarks about the holographic structures on homotopy spheres and their relation with the underling smooth exotic spheres.

2. On the Trajectory Spaces for Traversally Generic Flows

Let \( X \) be a compact smooth \((n+1)\)-dimensional manifold with boundary. A vector field \( v \) is called traversing if each \( v \)-trajectory is either a closed interval with both ends residing in \( \partial X \), or a singleton also residing in \( \partial X \) (see [K1] for the details). In particular, a traversing field does not vanish in \( X \).

We denote by \( \mathcal{V}_{\text{trav}}(X) \) the space of traversing fields on \( X \).

For traversing fields \( v \), the trajectory space \( \mathcal{T}(v) \) is homology equivalent to \( X \) (Theorem 5.1, [K3]). Moreover, as we will explain soon, for traversing fields \( v \), the trajectory space \( \mathcal{T}(v) \) has an interesting feature: it comes equipped with a vector \( n \)-bundle \( \tau(\mathcal{T}(v)) \) which plays the role of “surrogate tangent bundle”.

In this paper, we consider an important subclass of traversing fields which we call traversally generic (see Definition 3.2 from [K2]). Such fields admit special flow-adjusted coordinate systems, in which the boundary is given by a quite special polynomial equations (see formula (2.4)) and the trajectories are parallel to one of the preferred coordinates’ axis (see [K2], Lemma 3.4).

We denote by \( \mathcal{V}^\dagger(X) \) the space of traversally generic fields on \( X \).

It turns out that \( \mathcal{V}^\dagger(X) \) is an open and dense subspace of \( \mathcal{V}_{\text{trav}}(X) \) (see [K2], Theorem 3.5).

For traversally generic fields \( v \), the trajectory space \( \mathcal{T}(v) \) is stratified by closed subspaces, labeled by the elements \( \omega \) of an universal poset \( \Omega^\bullet_{(n)} \) which depends only on \( \dim(X) = n + 1 \) (see [K3], Section 2, for the definition and properties of \( \Omega^\bullet_{(n)} \)). The elements \( \omega \in \Omega^\bullet_{(n)} \) correspond to combinatorial patterns that describe the way in which \( v \)-trajectories \( \gamma \subset X \)
intersect the boundary $\partial_1 X := \partial X$. Each intersection point $a \in \gamma \cap \partial_1 X$ acquires a well-defined multiplicity $m(a)$, a natural number that reflects the order of tangency of $\gamma$ to $\partial_1 X$ at $a$ (see [K1] for the expanded definition of $m(a)$). So $\gamma \cap \partial_1 X$ can be viewed as a divisor $D_\gamma$ on $\gamma$. Then $\omega$ is just the ordered sequence of multiplicities $\{m(a)\}_{a \in \gamma \cap \partial_1 X}$, the order being prescribed by $v$.

The support of the divisor $D_\gamma$ is either a singleton $a$, in which case $m(a) \equiv 0 \mod 2$, or the minimum and maximum points of $\sup D_\gamma$ have odd multiplicities, and the rest of the points have even multiplicities.

Let $m(\gamma) := \sum_{a \in \gamma \cap \partial_1 X} m(a)$ and $m'(\gamma) := \sum_{a \in \gamma \cap \partial_1 X} (m(a) - 1)$. Similarly, for $\omega := (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ we introduce the norm and the reduced norm of $\omega$ by the formulas:

$$|\omega| := \sum_i \omega_i \quad \text{and} \quad |\omega'| := \sum_i (\omega_i - 1).$$

There exists another universal poset $\Omega_{(d)}$ which is equally important to our program: its elements are real divisors of real polynomials of degree $d$. See [K3], Definition 2.1, for the description of the partial order $\succ$ in $\Omega_{(d)}$.

In fact, $(\Omega_{(2n+2)}, \succ)$ is instrumental in defining the partial order $\succ_{\bullet}$ in $(\Omega^\bullet_{(n)}, \succ_{\bullet})$ (see [K3], Definition 2.5). As sets, $\Omega^\bullet_{(n)} \subset \Omega_{(2n+2)}$: the elements of $\Omega^\bullet_{(n)}$ correspond to portions of the real polynomial divisors such that the corresponding polynomial is non-positive on the interval bounded by the maximal and the minimal elements from that portion’s support. However, the relation between the partial orders $\succ_{\bullet}$ and $\succ$ is subtle.

Let $\partial_j X := \partial_j X(v)$ denote the locus of points $a \in \partial_1 X$ such that the multiplicity of the $v$-trajectory $\gamma_a$ through $a$ at $a$ is greater than or equal to $j$. This locus has an alternative description with the help of an auxiliary function $z : \hat{X} \to \mathbb{R}$ which satisfies the following three properties:

$$z^{-1}(0) = \partial_1 X,$$

$$z^{-1}(\mathbb{R}) = X.$$

In terms of $z$, the locus $\partial_j X := \partial_j X(v)$ is defined by the equations:

$$\{z = 0, \ L_v z = 0, \ldots, \ L_v^{(j-1)} z = 0\},$$

where $L_v^{(k)}$ stands for the $k$-th iteration of the Lie derivative operator $L_v$ in the direction of $v$ (see [K2]).

The pure stratum $\partial_j X^0 \subset \partial_j X$ is defined by the additional constraint $L_v^{(j)} z \neq 0$. The locus $\partial_j X$ is the union of two loci: (1) $\partial_j^+ X$, defined by the constraint $L_v^{(j)} z \geq 0$, and (2)
\(\partial^-_j X\), defined by the constraint \(L^{(j)}_v z \leq 0\). The two loci, \(\partial^+_j X\) and \(\partial^-_j X\), share a common boundary \(\partial_{j+1} X\).

For the rest of the paper, we assume that the field \(v\) on \(X\) extends to a non-vanishing field \(\hat{v}\) on some open manifold \(\hat{X}\) which properly contains \(X\). We treat the extension \((\hat{X}, \hat{v})\) as a germ. One can think of \(\hat{X}\) as being obtained from \(X\) by attaching an external collar to \(X\) along \(\partial_1 X\). In fact, the treatment of \((X, v)\) will not depend on the extension germ \((\hat{X}, \hat{v})\), but many constructions are simplified by introducing an extension.

We denote by \(X(v, \omega)\) the union of \(v\)-trajectories whose divisors are of a given combinatorial type \(\omega \in \Omega^*\). Its closure \(\cup_{\omega' \leq \omega} X(v, \omega')\) is denoted by \(X(v, \omega_{\infty})\).

Each pure stratum \(\mathcal{T}(v, \omega) \subset \mathcal{T}(v)\) is an open smooth manifold and, as such, has a “conventional” tangent bundle. In particular, the pure strata of maximal dimension \(n\) have tangent bundles. It turns out that these “honest” tangent \(n\)-bundles extend across the singularities of the space \(\mathcal{T}(v)\) to form a \(n\)-bundle \(\tau(\mathcal{T}(v))\) over \(\mathcal{T}(v)\)!

However, at the singularities, no exponential map that takes a vector from \(\tau(\mathcal{T}(v))\) to a point in \(\mathcal{T}(v)\) is available—the bundle \(\tau(\mathcal{T}(v))\) does not reflect faithfully the local geometry of the trajectory space \(\mathcal{T}(v)\).

In order to define the dual of the bundle \(\tau(\mathcal{T}(v))\) intrinsically, we need to consider a surrogate of smooth structure on the singular space \(\mathcal{T}(v)\).

**Definition 2.1.** Let \(\Gamma : X \to \mathcal{T}(v)\) be the projection that takes each point \(x \in X\) to the trajectory \(\gamma_x \in \mathcal{T}(v)\) that contains \(x\). We say that a function \(h : \mathcal{T}(v) \to \mathbb{R}\) is smooth, if the composition \(h \circ \Gamma\) is smooth on \(X\). We denote by \(C^\infty(\mathcal{T}(v))\) the algebra of all smooth functions on the space \(\mathcal{T}(v)\).

**Definition 2.2.** Let \(v_1, v_2\) be two traversally generic fields on manifolds \(X_1, X_2\), respectively.

- A map \(\Phi : \mathcal{T}(v_1) \to \mathcal{T}(v_2)\) is called smooth, if for any function \(h\) from \(C^\infty(\mathcal{T}(v_2))\), its pull-back \(\Phi^*(h) \in C^\infty(\mathcal{T}(v_1))\).
- A bijective map \(\Phi : \mathcal{T}(v_1) \to \mathcal{T}(v_2)\) is called a diffeomorphism, is both \(\Phi\) and \(\Phi^{-1}\) are smooth.

For any traversing (non-vanishing) field \(v\), the algebra \(C^\infty(\mathcal{T}(v))\) of smooth functions on the trajectory space \(\mathcal{T}(v)\) can be identified with the subalgebra of \(C^\infty(X)\), formed by functions \(f : X \to \mathbb{R}\) with the property \(L_v(f) = df(v) = 0\), where \(L_v\) stands for the \(v\)-directional derivative. Such functions are constant along each trajectory \(\gamma \subset X\).

We denote by \(C^\infty_{\gamma}(\mathcal{T}(v))\) the algebra of germs of smooth functions from \(C^\infty(\mathcal{T}(v))\) at a given point \(\gamma \in \mathcal{T}(v)\), by \(m_\gamma(\mathcal{T}(v)) \triangleq C^\infty_{\gamma}(\mathcal{T}(v))\) the maximal ideal, formed by the the germs of functions that vanish at \(\gamma\), and by \(m^2_\gamma(\mathcal{T}(v))\) the square of the ideal \(m_\gamma(\mathcal{T}(v))\).

Then the quotients \(m_\gamma(\mathcal{T}(v))/m^2_\gamma(\mathcal{T}(v))\) are real \(n\)-dimensional vector spaces. Indeed, since the pull-back of smooth functions on \(\mathcal{T}(v)\) are the smooth functions on \(X\) that are constants along each trajectory \(\gamma\), the quotient \(m_\gamma(\mathcal{T}(v))/m^2_\gamma(\mathcal{T}(v))\) can be canonically

\[3\text{This property is independent of an extension } (\hat{X}, \hat{v}) \text{ of } (X, v).\]
identified with the quotient \( m_x(S)/m_x^2(S) \), where \( S \) is a germ of a smooth transversal section of the \( \hat{v} \)-flow at \( x := \gamma \cap S \), and \( m_x(S) \) denotes the maximal ideal in the algebra \( C^\infty(S) \), an ideal comprised of functions that vanish at \( x \). It is well-known that \( m_x(S)/m_x^2(S) \) can be canonically identified with the cotangent space \( T_x^*(S) \) via the correspondence \( f \mapsto df \), where the germ of \( f : S \to \mathbb{R} \) at \( x \) belongs to the ideal \( m_x(S) \). Therefore the spaces

\[
\tau^*_\gamma(T(v)) := m_\gamma(T(v))/m_\gamma^2(T(v))
\]

form a vector \( n \)-bundle \( \tau^*(T(v)) \) over \( T(v) \). Its pull-back \( \Gamma^*(\tau^*(T(v))) \) can be identified with the subbundle \( \tau^*(v) \) of the cotangent bundle \( T^*(X) \), formed by the “horizontal” 1-forms that vanish on \( v \) and have the property \( \mathcal{L}_v(\alpha) = 0 \). The identification is via the correspondence \( \Gamma^*(f) \Rightarrow df(\Gamma(f)) \), where \( f \in m_\gamma(T(v)) \).

Now we define \( \tau(T(v)) \) as the dual bundle of \( \tau^*(T(v)) \).

Let \((11) \in \Omega^*\gamma\{[n]\}\) denote the unique maximal element of the poset; it labels the trajectories that intersect the boundary \( \partial_1 X \) only at a pair of distinct points, where they are transversal to the boundary.

**Lemma 2.1.** For any traversing field \( v \), the tangent bundles to the components of the maximal stratum \( T(v,(11)) \) extend to a \( n \)-dimensional vector bundle \( \tau(T(v)) \) over the trajectory space \( T(v) \).

Moreover, for a traversally generic field \( v \) and each element \( \omega \in \Omega^*\gamma\{[n]\} \), the tangent bundle of the pure stratum \( T(v,\omega) \) embeds in \( \tau(T(v))|_{T(v,\omega)} \) as a subbundle with a canonically trivialized complement.

**Proof.** We already have observed that the pull-back \( \Gamma^*(\tau^*(T(v))) \) of the cotangent bundle \( \tau^*(T(v)) \) can be identified with the bundle \( \tau^*(v) \) of the flow-invariant 1-forms on \( X \) that vanish on \( v \).

The map \( \Gamma : X(v,(11)) \to T(v,(11)) \) is a fibration with a closed segment for the fiber. Therefore \( \Gamma \) admits a smooth section \( S_{(11)} \subset X(v,(11)) \) which is transversal to the \( v \)-trajectories. Consider a decomposition of the \((n + 1)\)-bundle \( T(X)|_{S_{(11)}} \) into the tangent \( n \)-bundle \( T(S_{(11)}) \) and a line bundle \( L \) tangent to the \( v \)-trajectories. With the help of this decomposition, the cotangent bundle \( T^*(S_{(11)}) \) can be identified with the restriction \( \tau^*(v)|_{S_{(11)}} \) of \( \tau^*(v) \) to \( S_{(11)} \). Using the isomorphism \( \tau^*(v)|_{S_{(11)}} \approx \Gamma^*(\tau^*(T(v)))|_{S_{(11)}} \), we identify the cotangent bundle \( T^*(S_{(11)}) \) with the bundle \( \tau^*(T(v))|_{S_{(11)}} \), a bundle that evidently is defined on the whole space \( T(v) \).

A similar conclusion holds for any traversally generic field \( \hat{v} \) and each \( \omega \in \Omega^*\gamma\{[n]\} \): by Lemma 3.4 from [K2], the map \( \Gamma : X(v,\omega) \to T(v,\omega) \) is a fibration with its base being an open smooth \( (n - |\omega'|) \)-manifold and with a closed segment for the fiber, the fiber being consistently oriented by \( v \). Therefore \( \Gamma \) admits a smooth section \( S_\omega \). The cotangent bundle \( \tau^*(S_\omega) \) can be identified with the cotangent bundle \( \tau^*(T(v,\omega))|_{T(v,\omega)} \), a bundle that embeds into the bundle \( \tau^*(T(v)) \).

\[\footnote{It is dual to \( \tau(T(v)) \) under construction.} \]

\[\footnote{Here perhaps a much weaker assumption about \( v \) will do.}\]
The only non-trivial statement of the lemma is the existence of a preferred trivialization in the quotient bundle $\tau(T(v))|_{T(v,\omega)}/\tau(T(v,\omega))$. It follows from the last claim of Theorem 2.1 below. Thus we get the bundle isomorphism
\[ \Psi : \tau(T(v,\omega)) \oplus \mathbb{R}^{2\omega'} \approx \tau(T(v))|_{T(v,\omega)} \]
, canonically defined by $v$.

\[ \square \]

**Corollary 2.1.** For a traversally generic field $v$ on $X$, the stable characteristic classes of the tangent bundles $\tau(T(v))$ and $\tau(X)$ coincide via the cohomological isomorphism induced by the projection $\Gamma : X \to T(v)$.

**Proof.** Note that $T(X) \approx \Gamma^*(\tau(T(v))) \oplus \mathbb{R}$. Therefore, the cohomological isomorphism induced by $\Gamma$ (see Theorem 5.1, [K3]) helps to identify the stable characteristic classes of $\tau(T(v))$ and $T(X)$.

\[ \square \]

For a traversally generic $v$, the space $T(v)$ comes equipped with two distinct *intrinsically-defined orientations* of its pure strata $\{T(v,\omega)\}_\omega$. These orientations depend only on $v$ and the preferred orientation of $X$.

**Theorem 2.1.** Let $X$ be an oriented compact $(n+1)$-manifold, and $v$ a traversally generic field. Then

- each component of any pure stratum $T(v,\omega)$, where $\omega \in \Omega^*_n$ and $|\omega'| > 0$, acquires two distinct orientations, called preferred and versal. Switching the orientation of $X$ affects both orientations of $T(v,\omega)$ by the same factor $(-1)^{\exp(\omega)}$.
- With the help of these two orientations, each component of $T(v,\omega)$ acquires one of the two polarities “$\oplus$” and “$\ominus$”. They do not depend on the orientation of $X$.
- Each manifold $X(v,\omega)$ comes equipped with a $v$-induced normal framing in $X$. Similarly, the normal $|\omega'|$-dimensional bundle $\nu(T(v,\omega)) := \tau(T(v))|_{T(v,\omega)}/\tau(T(v,\omega))$ acquires a $v$-induced preferred framing.

**Proof.** Let us extend the field $v$ on $X$ to a non-vanishing field $\hat{v}$ on $\hat{X} \supset X$.

Local transversal sections $S$ of the $\hat{v}$-flow have a well-defined orientation due to the global orientation of $X$ and the preferred orientation of the $v$-trajectories.

For a traversally generic $v$ on a $(n+1)$-dimensional $X$, the vicinity $U \subset \hat{X}$ of each $v$-trajectory $\gamma$ of the combinatorial type $\omega$ has a special coordinate system
\[ (u, x, y) : U \to \mathbb{R} \times \mathbb{R}^{2\omega'} \times \mathbb{R}^{n-|\omega'|}. \]

By Lemma 3.4 and formula (3.17) from [K2], in these coordinates, the boundary $\partial_1X$ is given by the polynomial equation:
\[ P(u, x) := \prod_i \left[ (u - \alpha_i)^{\omega_i} + \sum_{l=0}^{\omega_i-2} x_{il}(u - \alpha_i)^l \right] = 0 \]
of an even degree $|\omega|$ in $u$. Here $x := \{x_{i,l}\}_{i,l}$, and the numbers $\{\alpha_i\}_i$ are all distinct real roots of the polynomial $P(u,0)$, ordered so that $\alpha_i < \alpha_{i+1}$ for all $i$.

At the same time, $X$ is given by the polynomial inequality $\{P(u,x) \leq 0\}$. Each $v$-trajectory in $U$ is produced by freezing all the coordinates $x, y$, while letting $u$ to be free.

We order the coordinates $\{x_{i,l}\}_{i,l}$ lexicographically: first we order them by the increasing $i$'s; then, for a fixed $i$, the ordering among $\{x_{i,l}\}_l$ is defined by the increasing powers $l$ of the binomial $(u - \alpha_i)$ in the formula (2.4). This ordering of $\{x_{i,l}\}_{i,l}$, together with the orientation in $S$ (induced with the help of $v$ by the orientation of $X$) gives rise to an orientation of the $y$-coordinates. The $y$-coordinates correspond to the space, tangent to the pure stratum $T(v, \omega)$ at $\gamma$.

We still have to check that this ordering of $\{x_{i,l}\}_l$ is determined by the geometry of tangency and does not depend on a particular choice of the special coordinates $\{x_{i,l}\}_l$.

Consider a $v$-trajectory $\gamma$ of the combinatorial type $\omega$. Let $\gamma \cap \partial_1 X = \prod a_i$, a finite set of points. For each $a_i \in \partial_1 X^o$ (by definition, $j_i := \omega_i$), we divide the local coordinates $(u, x, y)$, participating in the formula (2.3), into new groups $(u, x^{(i)}, y^{(i)})$, where by definition $x^{(i)} := (x_{i,0, x_{i,1}, \ldots, x_{i,j_i-2}})$ and $y^{(i)}$ denotes the rest of $\{x_{i'}\}_{i'}$-coordinates $(i' \neq i)$ together with all the $y$-coordinates.

In the vicinity of each $a_i \in \partial_1 X^o$, we write down the auxiliary function $z$ from (2.3) in two ways:

$$\begin{align*}
(1) & \quad u^j + \sum_{l=0}^{j_i-2} \phi_l(x^{(i)}, y^{(i)}) u^l, \\
(2) & \quad (u^j + \sum_{l=0}^{j_i-2} x_{i,l} u^l) Q(u, x^{(i)}, y^{(i)}).
\end{align*}$$

Here $\phi_l(0,0) = 0$ for all $l$.

To simplify these notations, put $j := j_i$, $\tilde{x} := x^{(i)}$, $\tilde{y} := y^{(i)}$.

We aim to show that, at the origin $(\tilde{x}, \tilde{y}) = (0, 0)$, the following two $(j - 1)$-forms are equal:

$$d\phi_0 \wedge d\phi_1 \wedge \cdots \wedge d\phi_{j-2} \mid_{(0,0)} = dx_{i,0} \wedge dx_{i,1} \wedge \cdots \wedge dx_{i,j-2} \mid_{(0,0)}. \tag{2.5}$$

We regard these two forms as $\pm$ volume forms in the normal bundle $\nu(\partial_1 X(v), X)$ at the point $a_i$. We identify $\nu_{a_i}(\partial_1 X(v), X)$ with the $\tilde{x}$-space $\mathbb{R}^{j-1}$. Hence the two orientations of the $\tilde{y}$-space $\mathbb{R}^{n-j+1} \approx T_{a_i}(\partial_1 X(v))$, induced by the two coordinate systems, $\{\phi_l\}_l$ and $\{x_{i,l}\}_l$, in the vicinity of $a_i$, do also agree.

The argument validating (2.5) is similar to the one we have used in [K2], Lemma 3.3.

First note that $q := Q(u, 0, 0) \neq 0$ must be 1: just plug $\tilde{x} = 0, \tilde{y} = 0$ in the identity

$$u^j + \sum_{l=0}^{j-2} \phi_l(\tilde{x}, \tilde{y}) u^l = (u^j + \sum_{l=0}^{j-2} x_{i,l} u^l) Q(u, \tilde{x}, \tilde{y}). \tag{2.6}$$

Form the row-vector $a(u) := (u^{j-2}, \ldots, u, 1)$ and the column-vector $d\Phi := (d\phi_{j-2}, \ldots, d\phi_1, d\phi_0)^t$. 


of 1-forms. Then the functional identity (2.6) can be written as
\[ u^\partial + a \ast \Phi = (u^\partial + a \ast \tilde{x}) Q \]
where \( a \ast \) stands for the matrix multiplication. Note that its RHS belongs to the ideal \( \langle u^\partial, \tilde{x} \rangle \) of \( C^\infty(\mathbb{R}^{n+1}) \), generated by the functions \( u^\partial \) and \( x_{i,0}, \ldots, x_{i,j-2} \).

Consider the differential of this identity and take its portion that does not contain the \( du \) terms. Then, modulo the ideal \( \langle u^{j-1}, \tilde{x} \rangle \), generated by the functions \( u^{j-1} \) and \( x_{i,0}, \ldots, x_{i,j-2} \), that portion produces the identity
\[ a \ast d\Phi = Qa \ast d\tilde{x} \mod \langle u^{j-1}, \tilde{x} \rangle \]
Here \( \mod \langle u^{j-1}, \tilde{x} \rangle \) means that we ignore the differential 1-forms whose coefficients belong to the functional ideal \( \langle u^{j-1}, \tilde{x} \rangle \).

We apply partial derivatives \( \frac{\partial}{\partial u^k}, \ldots, \frac{\partial}{\partial u^m} \) to the identity above to get a new system of identities:
\[ \frac{\partial^k}{\partial u^k} (a) \ast d\Phi = \frac{\partial^k}{\partial u^k} (Qa) \ast d\tilde{x} \mod \langle u^{j-1-k}, \tilde{x} \rangle \]
where \( k = 0, 1, \ldots, j-2 \).

Now substitute \( u = 0 \) and use that \( q = 1 \) to get the following triangular system of identities, modulo the ideal \( \langle \tilde{x} \rangle \) generated by \( x_{i,0}, \ldots, x_{i,j-2} \):

\[
\begin{align*}
    d\phi_0 &= dx_{i,0} \mod \langle \tilde{x} \rangle \\
    d\phi_1 &= dx_{i,1} + b_{1,0} dx_{i,0} \mod \langle \tilde{x} \rangle \\
    d\phi_2 &= dx_{i,2} + b_{2,0} dx_{i,0} + b_{2,1} dx_{i,1} \mod \langle \tilde{x} \rangle \\
    &\vdots \\
    d\phi_{j-2} &= dx_{i,j-2} + b_{j-2,0} dx_{i,0} + b_{j-2,1} dx_{i,1} + \cdots + b_{j-2,j-3} dx_{i,j-3} \mod \langle \tilde{x} \rangle
\end{align*}
\]
Here \( \{b_{s,t}\} \) are some functional coefficients whose computation we leave to the reader. Now (2.5) follows by taking exterior products of the 1-forms on the RHS and LHS of the system above and letting \( \tilde{x} = 0 \).

Let \( \theta_i := dx_{i,0} \wedge \cdots \wedge dx_{i,j_i-2} \) and let \( \theta := \wedge i \theta_i \). Then \( du \wedge \theta \), together with the volume form in \( X \), define the volume form in the \( y \)-coordinates. Therefore the orientation of the space \( \tau_\gamma(\mathcal{T}(v, \omega)) \), tangent to the pure stratum \( \mathcal{T}(v, \omega) \) at its typical point \( \gamma \) (this space can be identified with the space spanned by the vectors \( \partial_{y_1}, \ldots, \partial_{y_{n-|\omega|}} \)), is determined intrinsically by the local geometry of the \( v \)-flow in the vicinity of \( \gamma \subset \hat{X} \). For time being, let us call this orientation of \( \tau_\gamma(\mathcal{T}(v, \omega)) \) versal.

On the other hand, each manifold \( \partial_j X \), \( j > 0 \), comes equipped with its own preferred orientation, which depends only on the stratification \( \{\partial_k^+ X(v)\} \) (and not on properties of the field \( v \) inside \( X \)) and on the preferred orientation of \( X \). Here is the recipe for its construction: the orientation of \( X \), with the help of the inward normals, induces a preferred orientation of \( \partial_1 X \), and thus of \( \partial_1^+ X \). In turn, the inward normals to \( \partial_2 X = \partial(\partial_1^+ X) \) in \( \partial_1^+ X \) produce a preferred orientation of \( \partial_2 X \), and thus of \( \partial_2^+ X \). And the process goes on: the preferred orientation of \( \partial_{j-1} X \), with the help of the inward normal to \( \partial_j X \) in \( \partial_{j-1}^+ X \), determines a preferred orientation of \( \partial_j X \), and hence of \( \partial_j^+ X \).
So, along each trajectory $\gamma$, every space $T_i$, tangent to $\partial_j X^\circ$ and transversal to $\gamma$ at the points $a_i \in \gamma \cap \partial_i X$, is preferably oriented. For a traversally generic $v$, the $\dot{v}$-flow propagates these spaces $T_i$‘s along $\gamma$ in such a way that they form complementary vector bundles over $\gamma$ (see [K2], Definition 3.2). We order them by the increasing values of $i$. This ordering, together with the preferred orientations of the $T_i$‘s (based on the orientations of $\partial_j X$), generates a new preferred orientation of the tangent space $\tau_\gamma(T(v, \omega))$. This preferred orientation may agree or disagree with the versal orientation of the same space, produced with the help of special coordinates in the vicinity of $\gamma$ (the versal orientation is based on the increasing powers of $(z - \alpha_i)$’s, a feature of the special coordinates). In the first case, we attach the polarity “$\oplus$” to $\gamma$, in the second case, the polarity of $\gamma$ is defined to be “$\ominus$”.

Therefore not only the components of pure strata $\mathcal{T}(v, \omega)$ are canonically oriented open manifolds, but they also come in two flavors: “$\oplus$” and “$\ominus$”!

The ordered collection of $|\omega|$ linearly independent and globally defined 1-forms as in formula (3.30), [K2] produces a framing of the quotient bundle

$$\nu^*(\mathcal{T}(v, \omega)) := \tau^*(\mathcal{T}(v))|_{\mathcal{T}(v, \omega)} / \tau^*(\mathcal{T}(v, \omega))$$

, the “normal cotangent bundle” of $\mathcal{T}(v, \omega)$ in $\mathcal{T}(v)$. Let us explain this observation.

For any $\gamma \in \mathcal{T}(v, \omega)$ and any two points $a, x \in \gamma$, denote by $\phi_{a,x}$ the germ (taken in the vicinity of $\gamma \subset \hat{X}$) of the unique $v$-flow-generated and $\gamma$-localized diffeomorphism that maps $x$ to $a$.

Fix an auxiliary function $z : \hat{X} \to \mathbb{R}$ as in (2.3). For each point $a_i \in \gamma \cap \partial_1 X$ of multiplicity $j_i > 1$, let us consider the 1-forms

$$\{dz, L_v(dz), L_v^2(dz), \ldots, L_v^{j_i - 2}(dz)\}$$

, taken at the point $a_i$ (that is, view them as elements of $T_{a_i}^*(X)$). Then, with the help of one-parameter family of diffeomorphisms $\{\phi_{a_i,x}\}_{x \in \gamma}$, we spread the forms

$$\{dz|_{a_i}, L_v(dz)|_{a_i}, L_v^2(dz)|_{a_i}, \ldots, L_v^{j_i - 2}(dz)|_{a_i}\}$$

along $\gamma$ to get $j_i - 1$ independent sections $\eta_{i,0}, \eta_{i,1}, \ldots, \eta_{i,j_i - 2}$ of $T^*(X)|_{\gamma}$. By their very construction, these sections are flow-invariant. Moreover, since at points of $\partial_2 X$ the field $v$ is tangent to $\partial_1 X = \{z = 0\}$, we get $dz(v)|_{\partial_2 X} = L_v(z) = 0$. Thus $\eta_{i,0}(v)|_{\gamma} = 0$ for all $i$.

Similarly, for each $a_i \in \partial_3 X$ (i.e., $j_i > 2$), the field $v$ is tangent to

$$\partial_2 X = \{z = 0, L_v(z) = 0\}.$$ 

Therefore, using the identity

$$L_v(dz) = v \, d(z) + d(v \, dz) = d(v \, dz)$$

, we get $L_v(dz)(v)|_{\partial_1 X} = 0$. As a result, $\eta_{i,1}(v)|_{\gamma} = 0$ for all $i$ with $j_i > 2$. Similar considerations show that for each $i$, all the sections $\{\eta_{i,k}\}_{k < j_i - 1}$, have the property $\eta_{i,k}(v)|_{\gamma} = 0$—they are horizontal 1-forms. Therefore they can be viewed as independent sections of the subbundle $\tau^*(v) \subset T^*(X)$. With the help of $(\Gamma^*)^{-1}$, these sections produce independent sections of the quotient bundle $\nu^*(\mathcal{T}(v, \omega))$. 


Now take all $|\omega|$ sections $\{\eta_{i,0}, \eta_{i,1}, \ldots, \eta_{i,-2}\}_i$ of $T^*(X)|_{\gamma}$, ordered in groups by the increasing values of $i$. For a traversally generic $v$, by Theorem 3.3 from [K2], these sections of $\tau^*(v) \subset T^*(X)|_{\gamma}$ are linearly independent.

As long as the combinatorial type $\omega$ of $\gamma$ is fixed, these sections depend smoothly on $\gamma$. Since their construction relies only on $\omega$, $z$, and $v$, they are globally well-defined independent sections of the conormal bundle $\nu^*(\mathcal{T}(v, \omega))$, an intrinsically defined trivialization of this bundle. Their duals define independent sections of the normal bundle $\nu(\mathcal{T}(v, \omega))$.

The preferred orientation of each $\partial_j X$, $j \geq 1$, depends only on $v|_{\partial_j X}$ and the orientation of $X$. In particular, the preferred orientation of $\partial_1 X$ depends on the orientation of $X$ only. As we flip the orientation of $X$, the preferred orientation of each $\partial_j X$ flips as well. Therefore, the preferred orientation of the tangent bundle $\tau(\mathcal{T}(v, \omega))$ changes, as a result of flipping the orientation of $X$, only when the cardinality of the intersection $\gamma \cap \partial_1 X$—the interger $|\sup(\omega)|$—is odd.

The versal orientation of $\mathcal{T}(v, \omega)$ behaves similarly under the change of an orientation of $X$. As a result, the polarity “$\oplus$” or “$\ominus$” of each component of $\mathcal{T}(v, \omega)$ is independent of the orientation of $X$.

\begin{corollary}
For a traversally generic field $v$, the points of minimal 0-dimensional strata $\mathcal{T}(v, \omega)$ come equipped with two sets of polarities: “$+$, $-$” and “$\oplus$, $\ominus$”.
\end{corollary}

\begin{proof}
When $\omega$ has the maximal possible reduced multiplicity $|\omega| = n$, we can compare the versal and preferred orientations at each point $\gamma$ of the zero-dimensional set $\mathcal{T}(v, \omega)$. When the two agree, we attach the polarity “$\oplus$” to $\gamma$; otherwise, its polarity is defined to be “$\ominus$”. Of course, the preferred orientation of $\nu(\gamma, X)$ can be compared with the preferred orientation of $\partial_1 X$ at the lowest point in $\gamma \cap \partial_1 X$. This comparison allows for another pair $(+, -)$ of polarities to be attached to $\gamma$.
\end{proof}

Now let us investigate how flipping the field $v \Rightarrow -v$ affects the preferred and versal orientations of strata in $\mathcal{T}(v)$. Recall that replacing $v$ with $-v$ affects the Morse stratification according to the formula: $\partial^\epsilon_j X(-v) = \partial_j^\epsilon X(v)$, where $\epsilon = +$ when $(n+1) - j \equiv 0 \pmod{2}$, and $\epsilon = -$ otherwise (see [K1]).

This reversal of field’s direction also changes the order in which the points of $\gamma \cap \partial_1 X$ occur along a typical $v$-trajectory $\gamma$. As a result, the combinatorial type $\omega$ of $\gamma$ changes to $\tilde{\omega}$, the mirror image of the histogram $\omega$. In other words, $\tilde{\omega}(i) := \omega(q-i)$, where $q := |\sup(\omega)|$.

Note that changing $v$ to $-v$ transforms $\{dz, L_v(dz), L^2_v(dz), \ldots\}$ into $\{dz, L_{-v}(dz), L^2_{-v}(dz), \ldots\}$.

Thus the flip $v \Rightarrow -v$ results in the change of the sign of the differential form $\theta_j := dz \wedge L_v(dz) \wedge L^2_v(dz) \wedge \cdots \wedge L^{j-2}_v(dz)$ by the factor $(-1)^{(j-2)/2}$. In addition, the flip results in changing the order of the points from $\gamma \cap \partial_1 X$ to the opposite.

Let $\omega = (\omega(1), \omega(2), \omega(3), \ldots) = (j_1, j_2, j_3, \ldots)$. Consider a division of the ordered set $1, \ldots, j_1 - 1; j_1, \ldots, j_1 + j_2 - 2; j_1 + j_2 - 1, \ldots, j_1 + j_2 + j_3 - 3; \ldots$
into successive groups of the lengths \(j_1-1, j_2-1, j_3-1, \ldots\). Let \(\sigma_\omega\) denote the permutation that changes the order of groups to the opposite, but keeps the relative positions of elements within each group. We denote by \(\text{sign}(\sigma)\) the parity of permutation \(\sigma\). So the flip \(v \mapsto -v\) affects the form \(\lambda_i \theta_{\omega(i)}\) by multiplying it with the factor \((-1)^{\omega(i)}\), where

\[
\epsilon(\omega) = \text{sign}(\sigma_\omega) + \sum_i \left[\frac{\omega(i) - 2}{2}\right]
\]

Therefore this formula describes the connection between the versal orientation of \(T(v, \omega)\) and the versal orientation of \(T(-v, \bar{\omega})\). Of course, the trajectory spaces \(T(v)\) and \(T(-v)\) are canonically homeomorphic.

Our next goal is to prove that the trajectory space \(T(v)\) of a traversally generic field \(v\) is a Whitney stratified space (see Definition 2.3). Prior to establishing this fact in Theorem 2.2 below, we need to prove a few lemmas.

Recall that a function \(f\) on a closed subset \(Y\) of a smooth manifold \(X\) is called smooth if it is the restriction of a smooth function, defined in an open neighborhood of \(Y\).

**Lemma 2.2.** Let \(v \neq 0\) be a gradient-like vector field on a compact manifold \(X\), and \(\Gamma : X \to T(v)\) the obvious map. Let \(F \subset T(v)\) be a closed subset and \(\psi : F \to \mathbb{R}\) a function such that \(\Gamma^*(\psi)\) is a smooth function on \(\Gamma^{-1}(F) \subset X\) with the property \(L_v(\Gamma^*(\psi)) = 0\) on \(\Gamma^{-1}(F)\).

Then \(\psi : F \to \mathbb{R}\) admits an extension \(\Psi : T(v) \to \mathbb{R}\) such that \(\Gamma^*(\Psi)\) is a smooth function on \(X\) with the property \(L_v(\Gamma^*(\Psi)) = 0\).

**Proof.** Let \(h : X \to \mathbb{R}\) be a smooth function such that \(dh(v) > 0\). Using \(h\), we can find a finite set \(\mathcal{S}\) of closed smooth transversal sections \(\{S_\alpha \subset h^{-1}(c_\alpha)\}_\alpha\) of the \(v\)-flow, such that each trajectory hits some section from the collection \(\mathcal{S}\). Moreover, we can assume that all the heights \(\{c_\alpha\}\) are distinct and separated by some \(\epsilon > 0\). The set \(\mathcal{S}\) can be given a structure of poset: \(\beta \succ \alpha\) if there exists an ascending \(v\)-trajectory \(\gamma\) that first pierces \(S_\alpha\) and then \(S_\beta\).

Now the proof is an induction by the heights \(\{c_\alpha\}\), guided by the partial order in \(\mathcal{S}\). For a given \(\alpha\), consider the set \(\mathcal{S}_{\succ \alpha} := \{\beta \succ \alpha\}\). Assume that the desired extension

\[
\tilde{\Psi}_{\succ \alpha} : h^{-1}([c_\alpha + \epsilon/2, +\infty)) \to \mathbb{R}
\]

of the function \(\Gamma|_{\Gamma^{-1}(F)} \circ \psi\), subject to the property \(L_v(\tilde{\Psi}_{\succ \alpha}) = 0\), already has been constructed.

Denote by \(X(v, A)\) the union of \(v\) trajectories through a closed subset \(A \subset X\).

Consider two sets: \(F_\alpha := S_\alpha \cap \Gamma^{-1}(F)\) and \(Q_\alpha := X(v, \bigsqcup_{\beta \succ \alpha} S_\beta) \cap S_\alpha\).

Since \(\tilde{\Psi}_{\succ \alpha}\) is constant along each trajectory and \(S_\alpha\) is smooth and transversal to the flow, \(\tilde{\Psi}_{\succ \alpha}\) produces a well-defined smooth function \(\tilde{\Psi}_{\succ \alpha} : Q_\alpha \to \mathbb{R}\). On the other hand, the function \(\tilde{\psi} := \psi \circ \Gamma : \Gamma^{-1}(F) \to \mathbb{R}\) is smooth and constant along trajectories by the
lemma hypothesis. In particular, it is a smooth function on the closed set $F_{\alpha}$. Moreover, since $\tilde{\Psi}_{\succ \alpha}$ is an extension of $\tilde{\psi}$ in

$$h^{-1}([c_{\alpha} + \epsilon/2, +\infty)) \subset X$$

, both functions $\hat{\Psi}_{\succ \alpha}$ and $\hat{\psi}$ agree on $F_{\alpha} \cap Q_{\alpha}$. Therefore we have produced a function $\Psi_{\alpha} : F_{\alpha} \cup Q_{\alpha} \to \mathbb{R}$ which extends to is smooth function $\tilde{\Psi}_{\alpha}$ on $S_{\alpha}$. In turn, $\Psi_{\alpha} : S_{\alpha} \to \mathbb{R}$ defines a smooth function $\Psi_{\alpha} : X(v, S_{\alpha}) \to \mathbb{R}$ which is constant on each trajectory through $S_{\alpha}$. By their construction, $\hat{\Psi}_{\alpha}$ and $\tilde{\Psi}_{\succ \alpha}$ agree on

$$X(v, S_{\alpha}) \cap h^{-1}([c_{\alpha} + \epsilon/2, +\infty)).$$

Together, they produce a smooth function in $h^{-1}([c_{\alpha}, +\infty))$ which is constant along the trajectories through $\prod_{\beta \succeq \alpha} S_{\beta}$ and extends $\hat{\psi}$. This completes the induction step. \qed

**Figure 1.** The upper four diagrams show the flow sections $S_i$ and the sets $X(v, S_i)$ for $i = 1, 2, 3, 4$. The lower four diagrams show how the domains of $\Gamma^*(\psi)$-extensions propagate, as they appear in the proof (to simplify the picture, the original set $\Gamma^{-1}(F)$ is not shown).
Definition 2.3. Let $Z$ be a closed subset of a smooth manifold $M$, which decomposes as $Z = \bigsqcup_{\alpha \in \mathcal{S}} Z_\alpha$, where $\mathcal{S}$ is a finite poset.

We say that $Z$ is a Whitney space if the following properties hold:

1. each $Z_\alpha$ locally is a smooth submanifold of $M$,
2. take any pair $Z_\alpha \subset Z_\beta$ of strata and consider a pair of sequences $\{x_i \in Z_\beta\}_i$, $\{y_i \in Z_\alpha\}_i$, both converging to the same point $y \in Z_\alpha$. In a local coordinate system on $M$, centered on $y$, form the secant lines $\{l_i := \langle x_i, y_i \rangle\}_i$ so that that $\{l_i\}_i$ converge to a limiting line $l$. Also consider a sequence of tangent spaces $\{T_{x_i}(Z_\beta)\}_i$, that converge to a limiting space $\tau \subset T_y(M)$. Then we require that $l \subset \tau$.

If $Z \subset M$ is a Whitney space, then one can prove that $T_y(Z_\alpha) \subset \tau$ (see [GM2]).

Now we are going to verify that the standard models of traversally generic flows lead to spaces of trajectories which are Whitney spaces.

Lemma 2.3. Let $\omega \in \Omega_{\{d\}}$, where $d' = |\omega'|$, and let $\omega_\epsilon \subseteq \{\tilde{\omega} \in \Omega_{\{d'\}} : \tilde{\omega} \geq \omega\}$. Consider the semi-algebraic set

$$Z_\omega := \{P_\omega(u, x) \leq 0, \|x\| \leq \epsilon\}$$

where $\epsilon > 0$ is sufficiently small and the polynomial $P_\omega$ of even degree $d = |\omega|$ is as in (24) (its real divisor has the combinatorial type $\omega$). Let $T_\omega$ denote the $(\omega_\epsilon)$-stratified trajectory space of the constant field $u := \partial_u$ in $Z_\omega$.

Then there exists an embedding $K_\omega : \mathcal{T}_\omega \rightarrow \mathbb{R}^{2|\omega'|}$, given by some smooth functions on $Z_\omega$ which are constant along every $\partial_u$-trajectory residing in $Z_\omega$.

Proof. Evidently, the $x$-coordinates $x : Z_\omega \rightarrow \mathbb{R}^{|\omega'|}$ provide us with a map $\chi : \mathcal{T}_\omega \rightarrow \mathbb{R}^{|\omega'|}$. The map $\chi$ is given by the algebraic functions which are constant on the $\partial_u$-trajectories in $Z_\omega$. Unfortunately, $\chi$ does not separate some trajectories; that is, $\chi$ is not an embedding (just a finitely ramified map). We will complement $x$ with another smooth map $\tilde{x} : Z_\omega \rightarrow \mathbb{R}^{|\omega'|}$, also constant on the trajectories in $Z_\omega$ and such that the pair of maps $(x, \tilde{x})$ will separate the points of $\mathcal{T}_\omega$.

To construct $\tilde{x}$, we will use some facts from [K3], Section 4. We view $x$ as a point in the space of coefficients $\mathbb{R}^{|\omega'|}$ for the family of $u$-polynomials $\{P(u, x)\}_x$. Recall that the ball $B_\epsilon := \{|x| \leq \epsilon\}$ has a cone structure, given (with the help of the Vieté map) by the the local linear contractions in $\mathbb{C}$ of each “nearby” complex divisor $D_C(P(\sim, x_\ast))$ on the complex divisor $D_C(P(\sim, 0))$. This contraction produces a smooth algebraic curve $A_{x_\ast} : [0, 1] \rightarrow B_\epsilon$ in the coefficient $x$-space (a generator of the “cone”) which connects the given point $x_\ast \in \mathbb{R}^{|\omega'|}$ to the origin $0$. In particular, the combinatorial type of the divisor $D_B(P(\sim, A_{x_\ast}(t)))$ is constant for all $t \in (0, 1]$ (that type belongs to the poset $\Omega_{\{d\}}$).

Let $S_{x_\ast}$ be the ruled $(u, t)$-parametric surface that projects along the $u$-direction onto the curve $A_{x_\ast}$. Consider the intersection $\Sigma_{x_\ast}$ of $S_{x_\ast}$ with the set $Z_\omega$ (see Fig. 2, the left diagram). As $x_\ast \in \partial B_\epsilon$ varies, the surfaces $\{\Sigma_{x_\ast}\}$ span $Z_\omega$ (the trajectory $\{x = 0\}$ serves as a binder of an open book whose pages are the $\Sigma_{x_\ast}$’s).
We will define a new projection $\tilde{x} : \Sigma x \to A_{x^*}$ as follows. Consider the $u$-directed line $L_x$ through $x$. For a typical point $x \in A_{x^*}$, let $\Pi_x := L_x \cap \Sigma x$. The set $\Pi_x$ is a disjointed union of closed intervals $\{I_i(x) = [\alpha_i(x), \bar{\alpha}_i(x)]\}_{i}$ (where $\alpha_i(x) < \bar{\alpha}_i(x)$) residing in the line $L_x$. We order them so that $I_1(x) < I_2(x) < \cdots < I_s(x)$ as sets.

Put

$$\Pi^\wedge_x := (L_x \setminus \Pi_x) \cap [\alpha_{\min}(x), \alpha_{\max}(x)].$$

Here $\alpha_{\min}(x), \alpha_{\max}(x)$ denote the minimal and the maximal real roots of the $u$-polynomial $P_\omega(u, x)$. Thus $\Pi^\wedge_x$ is a finite disjoint union of closed intervals $\{I^\wedge_i(x) = [\bar{\alpha}_i(x), \alpha_{i+1}(x)]\}_{i}$ residing in the line $L_x$. We also order them so that $I^\wedge_1(x) < I^\wedge_2(x) < \cdots < I^\wedge_{s-1}(x)$.

Let $\phi : [0, 1] \to \mathbb{R}_+$ be a smooth monotonically increasing function such that $0 < \phi(t) < t$ for all $t \in (0, 1]$ and whose infinite order jet at 0 coincides with the jet of the function $t$.

For each $i$, we map the point

$$(\alpha_i(A_{x^*}(t)), t) \in \partial^+_1 \Sigma x^* (\partial_u)$$

to the point

$$(\alpha_i(A_{x^*}(\phi(t))), \phi(t)) \in \partial^-_1 \Sigma x^* (\partial_u).$$

We denote by $\theta^i_{x^*}$ this correspondence. As a function in $(u, t)$, it is smooth.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The map $\tilde{x} : \Sigma x^* \to A_{x^*}$ over an arc $A_{x^*}$ (on the left) and the map $\tilde{x} : Z_\omega \to \mathbb{R}[\omega]^-$ (on the right).}
\end{figure}

Now we define $\tilde{x} : \Sigma x^* \to A_{x^*}$ by the following formulas (see Fig. 2):

$$\begin{align*}
\tilde{x} &:= x \text{ for all points in } I_1(x), \\
\tilde{x} &:= \theta_{x^*,1} \text{ for all points in } I_2(x), \\
\tilde{x} &:= \theta_{x^*,2} \circ \theta_{x^*,1} \text{ for all points in } I_3(x), \\
\tilde{x} &:= \theta_{x^*,3} \circ \theta_{x^*,2} \circ \theta_{x^*,1} \text{ for all points in } I_4(x), \\
\end{align*}$$

(2.8)
Since $0 < \phi(t) < t$ for all $t \in (0, 1]$, $\tilde{x}$ separates the trajectories that are not distinguished by $x$. Therefore the smooth map $J_\omega := (x, \tilde{x}(u, x)) : Z_\omega \to \mathbb{R}^{2|\omega|'}$, being constant on each trajectory in $Z_\omega$, gives rise to a a smooth (in the sense of Definition 2.1) embedding $K_\omega : T_\omega \to \mathbb{R}^{2|\omega|'}$. □

**Remark 2.1.** It seems that the desired embedding $K_\omega : T_\omega \to \mathbb{R}^{2|\omega|'}$ cannot be delivered by analytic functions. □

**Corollary 2.3.** The image $K_\omega(T_\omega) \subset \mathbb{R}^{2|\omega|'}$ is a Whitney $(\omega_{\preceq}, \succeq)$-stratified space.

**Proof.** It is useful to consult continuously with Fig. 3 that illustrates some key elements in the proof.

Let $\pi : \mathbb{R} \times \mathbb{R}^{2|\omega|'} \to \mathbb{R}^{2|\omega|'}$ denote the obvious projection. Put $K := K_\omega$.

Consider the map

$$
E := E_\omega : Z_\omega \to \mathbb{R} \times \mathbb{R}^{2|\omega|'}
$$

, given by the formula $E(u, x) := (u, J(u, x))$, where $J := J_\omega = (x, \tilde{x}(u, x))$, and the function $\tilde{x} : Z_\omega \to \mathbb{R}^{|\omega|'}$ is as in the proof of Lemma 2.1. The map $E$ is a regular embedding, given by smooth functions on $Z_\omega$. Consider the projection

$$
Q : \mathbb{R} \times \mathbb{R}^{2|\omega|'} \to \mathbb{R} \times \mathbb{R}^{|\omega|'}
$$

, given by the formula $Q(u, x, \tilde{x}) := (u, x)$. By the definition, $Q(E(Z_\omega)) = Z_\omega$.

Let $\mu \prec \nu$ be two elements in the poset $\omega_{\preceq}$ that labels the combinatorial types of $\partial_u$-trajectories from $Z_\omega$, and let $K_\mu, K_\nu$ be the two pure strata of the space $K(T_\omega) \subset \mathbb{R}^{2|\omega|'}$, indexed by $\mu, \nu$ (thus $K_\mu \subset K_\nu$). Consider a sequence of points $\{y_m \in K_\nu\}_m$ and a sequence

![Figure 3](image-url)
of points \( \{ z_m \in K_m \}_m \), both converging to a point \( z_\ast \in K_\mu \). We need to verify that, if the tangent spaces \( \{ T_{y_m} K_\nu \}_m \) converge in \( \mathbb{R}^{2|\omega|'} \) to an affine space \( T_\ast \) containing \( z_\ast \), and the sequence of lines \( \{ m \ni [z_m, y_m] \}_m \) converges to a line \( l_\ast \subset \mathbb{R}^{2|\omega|'} \), then \( l_\ast \subset T_\ast \).

Equivalently, we need to verify that if the spaces \( \{ T_m := \pi^{-1}(T_{y_m} K_\nu) \}_m \) converge in \( \mathbb{R} \times \mathbb{R}^{2|\omega|'} \) to an affine space \( T_\ast := \pi^{-1}(T_\ast) \supset \pi^{-1}(z_\ast) \), and the sequence of 2-planes \( \{ l_m := \pi^{-1}(l_m) \}_m \) converges to a plane \( L_\ast := \pi^{-1}(l_\ast) \subset \mathbb{R} \times \mathbb{R}^{2|\omega|'} \), then \( L_\ast \subset T_\ast \). Let us call this conjectured property “\( B \)”. 

Note that all the affine spaces \( T_m, T_\ast, L_m, \) and \( L_\ast \), are fibrations with the line fibers parallel to the direction of \( \mathbb{R} \) in \( \mathbb{R} \times \mathbb{R}^{2|\omega|'} \).

We can think of \( E(Z_\omega) \) as a graph of a smooth map \( \tilde{x} \) from \( Z_\omega \subset \mathbb{R} \times \mathbb{R}^{|\omega|'} \) to \( \mathbb{R}^{|\omega|'} \).

Since \( Q : E(Z_\omega) \to Z_\omega \) is a \( (\omega_\prec) \)-stratification-preserving diffeomorphism which respects the \( \partial_\alpha \)-induced 1-foliations \( \mathcal{F} \) on \( E(Z_\omega) \) and \( G \) on \( Z_\omega \), the tangent spaces to the \( \nu \)-indexed pure stratum in \( E(Z_\omega) \) are mapped by \( Q \) isomorphically onto the tangent space to the \( \nu \)-indexed pure stratum in \( Z_\omega \). So, with the help of the graph-manifold \( E(Z_\omega) \), any tangent space to the \( \nu \)-indexed pure stratum in \( Z_\omega \) determines the corresponding tangent space to the \( \nu \)-indexed pure stratum in \( E(Z_\omega) \).

Let \( \tilde{T}_\ast \) denote the tangent space to \( E(Z_\omega) \) at a generic point \( \tilde{z}_\ast \in \pi^{-1}(z_\ast) \), and let \( \tau_\ast \) denote the tangent space to \( Z_\omega \) at the point \( Q(z_\ast) \). By the very definitions of \( T_\ast \) and \( L_\ast \) as limit objects, and using that \( E(Z_\omega) \) is a smooth manifold \( \tilde{B} \) carrying the foliation \( \mathcal{F} \) (whose leaves are \( \nu \)-parallel lines in \( \mathbb{R} \times \mathbb{R}^{2|\omega|'} \)), we get that \( T_\ast \subset \tilde{T}_\ast \) and \( L_\ast \subset \tilde{L}_\ast \).

Since \( Q : E(Z_\omega) \to Z_\omega \) is a diffeomorphism, \( Q : \tilde{T}_\ast \to \tau_\ast \) is an isomorphism of vector spaces. Therefore there exist unique subspaces of \( \tilde{T}_\ast \) that are mapped by \( Q \) onto \( Q(T_\ast) \) or onto \( Q(L_\ast) \); these are exactly the spaces \( T_\ast \) and \( L_\ast \), respectively. Thus, \( Q(L_\ast) \subset Q(T_\ast) \) if and only if \( L_\ast \subset T_\ast \).

Therefore the property \( \tilde{B} \) is equivalent to the following property \( B \): “if the spaces \( \{ Q(T_m) \}_m \) converge in \( \mathbb{R} \times \mathbb{R}^{|\omega|'} \) to the affine space \( Q(T_\ast) \), and the sequence of planes \( \{ Q(l_m) \}_m \) converges to a plane \( Q(L_\ast) \subset \mathbb{R} \times \mathbb{R}^{|\omega|'} \), then \( Q(L_\ast) \subset Q(T_\ast) \)”.

Note that the composition \( Q \circ K : T_\omega \to \mathbb{R}^{|\omega|'} \) is delivered employing the map \( x : Z_\omega \to \mathbb{R}^{|\omega|'} \). The image \( Q(K(T_\omega)) = \mathbb{R}^{|\omega|'} \) is stratified by the collection of real discriminant varieties, indexed by various elements \( \rho \in \omega_\prec \) (\( \{K_3\} \), Theorem 4.1 and Theorem 4.2). In particular, these strata are semi-algebraic sets whose interiors are smooth cells. By the fundamental results of \( \text{Har1}, \text{Har2}, \) and \( \text{Hi} \), the semialnalic sets are Whitney stratified spaces. As a result, the \( (\omega_\prec) \)-stratified space \( Q(K(T_\omega)) \) is a Whitney space. Thus property \( B \) is valid, since all the affine spaces, relevant to \( B \), are fibrations with the line \( \pi \)-fibers over the corresponding spaces in \( \mathbb{R}^{|\omega|'} = Q(K(T_\omega)) \).

On the other hand, each element \( \rho \in \omega_\prec \) decomposes canonically into an ordered sequence \( \Xi(\rho) := \{ \rho_\kappa \}_\kappa \) of elements from the set \( \omega_\prec_\ast \) (see the combinatorial construction

\[ \text{with a piecewise smooth boundary} \]
in \([K3]\) that follows Example 5.1\(^8\). This decomposition \(\Xi(\rho)\) is such that, if the \(\partial_n\)-trajectory through a point \((u, x) \in \omega\) has a combinatorial type \(\rho\), then \(u\) determines the unique type \(\rho_x \in \omega_{\leq}\) to which the point \(Q^{-1}((u, x)) \cap E(\omega)\) belongs. In that sense the \(\omega_{\leq}\)-stratification in \(\omega\) is finer than the \(\omega_{\leq*,}\)-stratification in \(E(\omega)\).

Therefore, the validated \(B\)-property for the \((\omega_{\leq})\)-stratified space \(Q(K(\mathcal{T}_\omega))\) implies the \(B\)-property for the \((\omega_{\leq*})\)-stratified space \(E(\omega)\). As a result, \(K(\mathcal{T}_\omega)\) is a Whitney stratified space in \(\mathbb{R}^{2|\omega'|}\).

**Theorem 2.2.** For a traversally generic field \(v\) on a compact smooth \((n + 1)\)-dimensional manifold \(X\), the \(\Omega_{\omega}(n)\)-stratified trajectory space \(\mathcal{T}(v)\) can be given the structure of Whitney space (residing in an Euclidean space).

**Proof.** By Lemma 3.4 from \([K2]\), there exists a finite \(v\)-adjusted closed cover \(\mathcal{U} := \{U_r\}_r\) of \(X\), such that each \(U_r \subset X\) admits special coordinates \((u, x) := (u^{(r)}, x^{(r)}, y^{(r)})\) in which \(\partial_{x}X\) is given by the polynomial equation \(\{P_r(u, x) = 0\}\) as in \([2.3]\). Recall that the equation is determined by the combinatorial type \(\omega_r\) of the core trajectory \(\gamma_r \subset U_r\).

Let us denote by \(\mathcal{T}_r\) the space of trajectories of the \(\partial_n\)-flow in the domain

\[
U_r := \{P_r(u, x) \leq 0, \|x\| \leq \epsilon, \|y\| \leq \epsilon'\}.
\]

It is a compact subset of \(\mathcal{T}(v)\).

Consider the embeddings

\[
K_r : \mathcal{T}_r \rightarrow \mathbb{R}^{2|\omega_r'|} \times \mathbb{R}^{n - |\omega_r'|},
E_r : U_r \rightarrow \mathbb{R} \times \mathbb{R}^{2|\omega_r'|} \times \mathbb{R}^{n - |\omega_r'|}
\]

, given by the formulas

\[
K_r(\gamma_{(u^{(r)}, x^{(r)}, y^{(r)})}) := (x^{(r)}, \bar{x}^{(r)}(u^{(r)}, x^{(r)}), y^{(r)}),
E_r(u^{(r)}, x^{(r)}, y^{(r)}) := (u^{(r)}, x^{(r)}, \bar{x}^{(r)}(u^{(r)}, x^{(r)}), y^{(r)}).
\]

Here \(\gamma_{(u^{(r)}, x^{(r)}, y^{(r)})}\) denotes the \(\partial_n\)-trajectory in \(U_r\), passing through the point \((u^{(r)}, x^{(r)}, y^{(r)})\), and \(\bar{x}^{(r)}(u^{(r)}, x^{(r)})\) is a function as in Corollary \([2.3]\) (see Fig. 2).

Smooth functions on \(\psi : \mathcal{T}_r \rightarrow \mathbb{R}\) are exactly the smooth functions on \(U_r \cap X\) that are constant along the trajectories. By Lemma \([2.2]\) each \(\psi\) extends to a smooth function on \(X\) which is constant on each trajectory. We denote this extension \(\hat{\psi}\).

Therefore, employing the local embeddings \(K_r : \mathcal{T}_r \rightarrow \mathbb{R}^{2|\omega_r'|} \times \mathbb{R}^{n - |\omega_r'|}\), we extend them to some smooth maps \(\hat{K}_r : \mathcal{T}(v) \rightarrow \mathbb{R}^{2|\omega_r'|} \times \mathbb{R}^{n - |\omega_r'|}\). Together they produce a smooth embedding \(K : \mathcal{T}(v) \rightarrow \mathbb{R}^N\), where \(K := \prod_r \hat{K}_r\) and \(\mathbb{R}^N := \prod_r (\mathbb{R}^{2|\omega_r'|} \times \mathbb{R}^{n - |\omega_r'|})\).

Let \(G : X \rightarrow \mathbb{R}^N\) be the composition \(\Gamma \circ K\), where \(\Gamma : X \rightarrow \mathcal{T}(v)\) is the obvious map.

By Lemma 4.1 in \([K1]\), we can choose a function \(h : \hat{X} \rightarrow \mathbb{R}\) such that \(dh(v) > 0\) in \(\hat{X}\). With the help of \(h\), we get a map \(E : X \rightarrow \mathbb{R} \times \mathbb{R}^N\) given by the formula \(E(z) :=\)

\(^8\)For example, in Fig. 3, \(\rho = (1111) \in (121)_{\leq}\) decomposes into \(\rho_1 = (11) \in (121)_{\leq*}\), and \(\rho_2 = (11) \in (121)_{\leq*}\).
\((h(z), G(z))\). Since \(dh(v) > 0\) and the Jacobian of each map \(J_r := (x(r), \tilde{x}(r), y(r))\) is of the maximal rank \(n\) in \(U_r\), the map \(E\) is a regular smooth embedding.

Composing \(E\) with the obvious projection \(\pi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N\), we get a smooth (see Definition 2.2) embedding \(K : T(v) \to \mathbb{R}^N\).

Our goal is to show that \(K(T(v))\) is a Whitney stratified space in \(\mathbb{R}^N\). Since Definition 2.2 of Whitney space is local, it suffices to check its validity in each local chart \(T_r \subset T(v)\), that is, to verify that \(K(T_r) \subset \mathbb{R}^N\) is a Whitney space. Next arguments are very similar to the ones used in proving Corollary 2.3.

Consider the projection \(p_r : \mathbb{R}^N \to \mathbb{R}^{2|\omega|}{\times} \mathbb{R}^{n-|\omega|}\), produced by omitting the product \(\prod_{s \neq r} (\mathbb{R}^{|\omega_s|}{\times} \mathbb{R}^{n-|\omega_s|})\) from the product \(\prod_q (\mathbb{R}^{2|\omega_q|}{\times} \mathbb{R}^{n-|\omega_q|})\). Let

\[Q_r : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^{2|\omega|}{\times} \mathbb{R}^{n-|\omega|}\]
denote the projection \(id \times p_r\).

Note that the projection \(Q_r\) generates a diffeomorphism between the manifold \(E(U_r) \subset \mathbb{R} \times \mathbb{R}^N\) and the manifold \(E_r(U_r) \subset \mathbb{R} \times \mathbb{R}^{2|\omega|}{\times} \mathbb{R}^{n-|\omega|}\), a diffeomorphism that respects the oriented 1-foliations, induced by the \(v\)-flow on \(X\), as well as the \((\omega_r)_{\leq}\)-stratifications of \(E(U_r)\) and \(E_r(U_r)\) by combinatorial types of \(v\)-trajectories (or rather of the \(\pi\)-fibers).

We denote these foliations by \(F_r\) and \(G_r\), respectively.

Let \(\mu \prec \nu\) be two elements in the poset \((\omega_r)_{\leq}\), and \(\mathcal{K}_\mu, \mathcal{K}_\nu\) the two pure strata of \(K(T_r) \subset \mathbb{R}^N\), indexed by \(\mu, \nu\) (thus \(\mathcal{K}_\mu \subset \mathcal{K}_\nu\)). Consider a sequence of points \(\{y_m \in \mathcal{K}_\nu\}_m\) and a sequence of points \(\{z_m \in \mathcal{K}_\mu\}_m\), both converging to a point \(z_* \in \mathcal{K}_\mu\).

We need to verify that, if the tangent spaces \(\{T_{y_m,\mathcal{K}_\nu}\}_m\) converge in \(\mathbb{R}^N\) to an affine space \(T_*\) containing \(z_*\), and the sequence of lines \(\{l_m, z_m, y_m\}_m\) converges to a line \(l_* \subset \mathbb{R}^N\), then \(l_* \subset T_*\).

Equivalently, we need to verify that if the spaces \(\{T_m := \pi^{-1}(T_{y_m,\mathcal{K}_\nu})\}_m\) converge in \(\mathbb{R} \times \mathbb{R}^N\) to an affine space \(T_* := \pi^{-1}(T_*) \subset \mathbb{R} \times \mathbb{R}^N\), and the sequence of 2-planes \(\{l_m := \pi^{-1}(l_m)\}_m\) converges to a plane \(L_* := \pi^{-1}(l_*) \subset \mathbb{R} \times \mathbb{R}^N\), then \(L_* \subset T_*\). Let us call this conjectured property “\(\Lambda\)”.

Note that all the affine spaces \(T_m, T_*, L_m,\) and \(L_*\), are fibrations with the line fibers parallel to the direction of \(\mathbb{R}\) in \(\mathbb{R} \times \mathbb{R}^N\).

We can think of \(E(U_r)\) as a graph of a smooth map from \(E_r(U_r)\) to \(\prod_{s \neq r} (\mathbb{R}^{2|\omega_s|}{\times} \mathbb{R}^{n-|\omega_s|})\). Since \(Q_r : E(U_r) \to E_r(U_r)\) is a stratification-preserving diffeomorphism which respects the \(v\)-induced 1-foliations \(F_r\) and \(G_r\), the tangent spaces to the \(v\)-indexed pure stratum in \(E(U_r)\) are mapped isomorphically by \(Q_r\) onto the tangent space to the \(v\)-indexed pure stratum in \(E_r(U_r)\). So, with the help of the graph-manifold \(E(U_r)\), any tangent space to the \(v\)-indexed pure stratum in \(E_r(U_r)\) determines the corresponding tangent space to the \(v\)-indexed pure stratum in \(E(U_r)\).

Let \(\tilde{T}_*\) denote the tangent space to \(E(U_r)\) at a generic point \(\tilde{x}_* \in \pi^{-1}(z_*)\), and let \(T_*\) denote the tangent space to \(E_r(U_r)\) at the point \(Q_r(z_*)\). By the very definitions of \(T_*\) and \(L_*\), as limit objects and using that \(E(U_r)\) is a smooth manifold carrying the foliation \(F_r\) whose leaves are parallel lines in \(\mathbb{R} \times \mathbb{R}^N\), we get that \(T_* \subset \tilde{T}_*\) and \(L_* \subset \tilde{T}_*\).
Since \( Q_r : E(U_r) \to E_r(U_r) \) is a diffeomorphism, \( Q_r : \overline{\tau} \to \tau \) is an isomorphism of vector spaces. Therefore there exist unique subspaces of \( \overline{\tau} \) that are mapped by \( Q_r \) onto \( Q_r(T_\ast) \) or onto \( Q_r(L_\ast) \); these are exactly the spaces \( T_\ast \) and \( L_\ast \), respectively. Thus, \( Q_r(L_\ast) \subseteq Q_r(T_\ast) \) if and only if \( L_\ast \subseteq T_\ast \).

Hence the property \( \hat{A} \) is equivalent to the following property \( A \): if the spaces \( \{Q_r(T_m)\}_m \) converge in \( \mathbb{R} \times \mathbb{R}^{2|\omega_r|'} \times \mathbb{R}^{n-|\omega_r|'} \) to the affine space \( Q_r(T_\ast) \), and the sequence of planes \( \{Q_r(L_m)\}_m \) converges to a plane \( Q_r(L_\ast) \subseteq \mathbb{R} \times \mathbb{R}^{2|\omega_r|'} \), then \( Q_r(L_\ast) \subseteq Q_r(T_\ast) \).

By Corollary 2.3 \( K_r(T_r) \subseteq \mathbb{R}^{2|\omega_r|'} \times \mathbb{R}^{n-|\omega_r|'} \) is a Whitney space. Therefore, property \( A \) is valid. So the property \( \hat{A} \) has been validated as well: \( K(T(v)) \) is a Whitney stratified space in \( \mathbb{R}^N \).

\[ \square \]

**Remark 2.2.** It is desirable to find a more direct proof of Theorem 2.2 the proof that will validate Whitney’s property \( \hat{B} \) geometrically, without relying on the heavy general theorems claiming: “the semianalytic sets are Whitney spaces”. In fact, the discriminant varieties in \( \mathbb{R}^\text{coeff} \) that correspond to various combinatorial patterns \( \omega \) for real divisors of real \( d \)-polynomials, do have remarkable intersection patterns for their tangent spaces and cones (see [K4]). Perhaps, these properties of discriminant varieties should be in the basis of any “more geometrical” proof.

\[ \square \]

**Corollary 2.4.** Let \( X \) be an \((n+1)\)-dimensional compact smooth manifold, carrying a traversally generic field \( v \). Then the following claims are valid:

- The space of trajectories \( T(v) \) admits the structure of finite cell/simplicial complex.
- For each \( \omega \in \Omega(v, [n]) \), the stratum \( T(v, \omega_{\geq \ast}) \) is a codimension \( |\omega|' \) subcomplex of \( T(v) \).
- With respect to an appropriate cellular/simplicial structure in \( X \), the obvious map \( \Gamma : X \to T(v) \) is cellular/simplicial.
- As a result, \( \Gamma \) is a homotopy equivalence.

**Proof.** By Theorem 2.2 the trajectory space \( T(v) \) of a traversally generic flow admits a structure of a Whitney space, being embedded in some ambient Euclidean space.

The fundamental results of [Go], [Jo], and [Ve] claim that the Whitney spaces \( Y \) admit smooth triangulations \( \tau : T \to Y \), amenable to their stratifications. The adjective “smooth” here refers to the homeomorphism \( \tau \) being smooth on the interior of each simplex \( \Delta \subseteq T \) (remember the pure strata \( T(v, \omega) \) are smooth manifolds?). With respect to such triangulations, the strata are subcomplexes.

Therefore \( T(v) \) admits a finite triangulation so that each stratum \( T(v, \omega_{\geq \ast}) \) is a subcomplex.

For traversing fields \( v \), over each open simplex \( \Delta^\circ \subseteq T(v) \), the map \( \Gamma : X \to T(v) \) is a trivial fibration whose fibers are either closed segments, or singletons. Thus each set \( \Gamma^{-1}(\Delta^\circ) \) is homeomorphic either to the cylinder \( \Delta^\circ \times [0,1] \), or to \( \Delta^\circ \). This introduces a cellular structure on \( X \) so that \( \Gamma \) becomes a cellular map. With a bit more work, one can refine the cellular structures in \( X \) and \( T(v) \), so that \( \Gamma \) becomes a simplicial map.

Since, by Theorem 5.1 from [K1], \( \Gamma : X \to T(v) \) is a weak homotopy equivalence and both spaces are \( CW \)-complexes, we conclude that \( \Gamma \) is a homotopy equivalence.

\[ \square \]
Remark 2.3. Most probably, $\mathcal{T}(v)$ is a compact $CW$-complex for any traversing and boundary generic (see Definition 2.1 from [K1]) (and not necessary traversally generic!) field $v$. However, for such fields, we do not have algebraic models for their interactions with boundary in the vicinity of a typical trajectory. So we do not know how to extend the previous arguments to a larger class of fields. □

3. The Origami Theorem

We introduce a filtration $\{\mathcal{T}^+_{\{\text{max} \geq k\}}(v)\}_k$ of the trajectory space $\mathcal{T}(v)$ by closed subspaces (actually, by cellular subcomplexes). This stratification is cruder than the stratification $\{\mathcal{T}(v, \omega_{\geq \ast})\}_\omega$. By definition, a trajectory $\gamma \in \mathcal{T}^+_{\{\text{max} \geq k\}}(v)$ if $\gamma \cap \partial^+_k X$ contains at least one point $x$ of multiplicity greater than or equal to $k$. Moreover, we insist that this $x \in \partial^+_k X$ (and not in $\partial^+_k X^0$). In other words, $\mathcal{T}^+_{\{\text{max} \geq k\}}(v)$ is exactly the image of $\partial^+_k X$ in the trajectory space under the obvious map $\Gamma : X \to \mathcal{T}(v)$. In particular, $\mathcal{T}^+_{\{\text{max} \geq 1\}}(v) = \mathcal{T}(v)$.

As we will see next, Corollary 3.3 from [K1] and Theorems 3.4 and 3.5 from [K2] imply that there is an nonempty open subset $\mathcal{D}(X) \subset \mathcal{V}(X)$ such that, for each field $v \in \mathcal{D}(X)$, all the strata $\{\partial^+_j X\}_j$ are diffeomorphic to closed balls, except for $\partial^+_n X$ (which is a finite union of 1-balls) and for the finite set $\partial^+_n X$. So let us consider a model filtration

$$Z^0 \subset Z^1 \subset D^2 \subset D^3 \subset \ldots D^n-1 \subset D^n$$

of a closed ball $D^n$, such that:

1. each ball $D^j \subset \partial D^{j+1}$,
2. $Z^1$ is a disjoint union of finitely many arcs in $\partial D^2$,
3. $Z^0 \subset \partial Z^1$ is a finite set.

It turns out that, for $v \in \mathcal{D}(X)$, the trajectory space $\mathcal{T}(v)$ can be produced by an origami-like folding of the ball $D^n$ (see Fig. 4 for an example of an origami map on a 2-ball). The result below should be compared with a similar statement in Example 4.3, as well as with Theorem 2.5 from [K], the latter dealing with the flow-generated spines, not trajectory spaces.

**Theorem 3.1. (Trajectory spaces as the ball-based origami)**

- Any compact connected smooth $(n+1)$-manifold $X$ with boundary admits a traversally generic vector field $v$ such that its trajectory space $\mathcal{T}(v)$ is the image of a closed ball $D^n$ under a continuous cellular map $\Gamma : D^n \to \mathcal{T}(v)$ which is $(n+1)$-to-$1$ at most.
- For each ball $D^k$ from the filtration in (3.1), its $\Gamma$-image is the space $\mathcal{T}^+_{\{\text{max} \geq n+1-k\}}(v)$, and the restriction $\Gamma|_{D^k}$ is a $[\frac{n}{n-k}]$-to-$1$ map at most. For $n > 2$, the maps $\Gamma : Z^1 \to \mathcal{T}^+_{\{\text{max} \geq n\}}(v)$ and $\Gamma : Z^0 \to \mathcal{T}^+_{\{\text{max} \geq n+1\}}(v)$ are both bijective.
- The restrictions of $\Gamma$ to $\partial D^{k+1} \setminus D^k$ are $1$-to-$1$ maps for all $1 < k < n$, and so are the restrictions of $\Gamma$ to $\partial D^2 \setminus Z^1$ and $\partial Z^1 \setminus Z^0$. 


The fields \( v \), for which the above statements hold, form an open nonempty set in the space \( \mathcal{V}^\perp(X) \) of traversally generic fields on \( X \), and thus in the space \( \mathcal{V}_{\text{trav}}(X) \) of all traversing fields.

**Proof.** If \( \partial_1 X \) has several connected components, we pick one of them, say, \( \partial^*_1 X \). The union of the remaining boundary components is denoted by \( \partial^{**}_1 X \). We can construct a Morse function \( f : X \to \mathbb{R} \) so that it is locally constant on \( \partial^{**}_1 X \), and these constants are the local maxima of \( f \) in a collar \( U \) of \( \partial^{**}_1 X \) in \( X \). Then by finger moves (as in the proof of Lemma 3.3 from [K1]) we eliminate all critical points of \( f \) without changing \( f \) in \( U \). Pick a Riemannian metric on \( X \) and let \( v \) be the gradient field of \( f \). Evidently, \( \partial^{**}_1 X \subset \partial^+_1 X(v) \) and \( \partial^-_1 X(v) \subset \partial_1^+ X \).

By an argument as in Corollary 3.3 from [K1], in the vicinity of \( \partial^*_1 X \), we can deform the field \( v \) to a new \( f \)-gradient-like field so that all the manifolds \( \partial^+_1 X, \partial^+_2 X, \ldots, \partial^+_n X \), residing in the component \( \partial^*_1 X \), will be diffeomorphic to balls, and \( Z^1 := \partial^+_n X \) will consist of a number of arcs. The argument in Corollary 3.3 from [K1] constructs such a \( v \) to be boundary generic in the sense of Definition 2.1 from [K1]. Moreover, by Theorem 3.5 from [K2], we can further perturb \( v \) inside \( X \), without changing it on \( \partial_1 X \), so that the new perturbation will be a traversally generic field. Abusing notations, we continue to denote the new field by \( v \).

Since now, for \( k > 2 \), the locus \( \partial^+_n X(v) \) is diffeomorphic to a disk \( D^k \), for each point \( x \in D^k \), the \( v \)-trajectory \( \gamma_x \) has at least one tangency point of multiplicity \( n + 1 - k \) residing in \( \partial^+_1 X \), namely \( x \) itself. Thus, \( \Gamma \) maps \( D_k \) onto \( T^+_{\{ \text{max} \geq n + 1 - k \}}(v) \). Similarly, \( \Gamma : Z^1 \to T^+_{\{ \text{max} \geq n \}}(v), \Gamma : Z^0 \to T^+_{\{ \text{max} \geq n + 1 \}}(v) \) are surjective maps.
We notice that, due to the convexity of the flow in their neighborhoods, the points of \( \partial_j^- X \) are “protected” in the following sense: no \( v \)-trajectory can reach \( \partial_2^+ X(v) \), unless the trajectory is a singleton which belongs to \( \partial_2^- X(v) \) in the first place, no \( v \)-trajectory can reach \( \partial_3^- X(v) \), unless the trajectory is a singleton \( \partial_3^- X(v) \), and so on ... (see Fig. 1 and Fig. 5 from [K1]). In particular, no \( v \)-trajectory through a point of \( \partial_j^+ X(v) \) can reach \( \partial_2^- X(v) \), unless the trajectory is a singleton which belongs to \( \partial_2^- X(v) \) in the first place, no \( v \)-trajectory through a point of \( \partial_3^+ X(v) \) can reach \( \partial_3^- X(v) \), unless the trajectory is a singleton \( \partial_3^- X(v) \), and so on ... The claim also follows from Theorem 2.2 from [K2].

Therefore all the maps \( \{ \Gamma : \partial_j^- X(v) \setminus \partial_{j+1} X(v) \to \mathcal{T}(v) \}_j \) are 1-to-1. Thus the claim in the third bullet has been validated.

For a traversally generic \( v \), by Corollary 5.1 from [K3], the map \( \Gamma : \partial_1 X \to \mathcal{T}(v) \) is \((n+2)\)-to-1 at most. Since each trajectory, distinct from a singleton, must exit through \( \partial_1^- X \) at a point of an odd multiplicity, the same argument shows that \( \Gamma : \partial_j^+ X \to \mathcal{T}(v) \) is \((n+1)\)-to-1 at most. Because for \( v \in \mathcal{V}^j(X) \), the tangent spaces to \( \partial_j^+ X^\circ \) along each trajectory \( \gamma \) must form, with the help of the flow, a stable configuration in the germ of a \( n \)-section transversal to \( \gamma \) (see Definition 3.2 from [K2]). Thus the cardinality of \( \gamma \cap \partial_j^+ X^\circ \), \( j = n+1-k \), cannot exceed \( \lceil \frac{n}{j-k} \rceil = \lceil \frac{n}{n-1} \rceil \), provided \( k < n \). The statement in second bullet has been established.

By the second bullet of Theorem 3.4 from [K2], the smooth topological type of the stratification \( \{ \partial_j X(v) \}_j \) is stable under perturbations of \( v \) within the space \( \mathcal{V}^j(X) \) of boundary generic fields. The same argument shows that \( \{ \partial_j^+ X(v) \}_j \) is stable as well. Thus, for all fields \( v' \) sufficiently close to \( v \), the stratification \( \{ \partial_j^+ X(v') \}_j \) will remain as in [3.1]. By Theorem 3.5 from [K2], all sufficiently close fields to a traversally generic field will remain traversally generic, and by Corollary 3.3 from [K2], all sufficiently close fields to a traversally generic \( f \)-gradient-like field will remain traversally generic and \( f \)-gradient-like. By the argument above, this gives the desired control of the cardinality for the fibers of the maps \( \Gamma : \partial_j^+ X \to \mathcal{T}(v) \) and of the smooth topology of the stratification \( \{ \partial_j^+ X(v') \}_j \) within an open set of traversally generic fields.

\[\square\]

**Remark 3.1.** Recall that the trajectory space \( \mathcal{T}(v) \) in the Origami Theorem 3.1 is not only homotopy equivalent to the manifold \( X \) (see Theorem 5.1 from [K3]), but also shares all stable characteristic classes with it (Corollary 2.1). So all this information about \( X \) is hidden in a subtle way in the geometry of the folding map \( \Gamma : D^n \to \mathcal{T}(v) \).

The Origami Theorem 3.1 oddly resembles the Noether Normalization Lemma in the Commutative Algebra [No], although the direction of the ramified morphism has been reversed. In its algebro-geometrical formulation, the Normalization Lemma states that any affine variety is a branched covering over an affine space. In contrast, in our setting, many trajectory spaces \( \mathcal{T}(v) \)—rather complex objects—have a simple and universal ramified cover—the ball.
To explain this analogy, for a traversally generic field $v$, consider the Lie derivation $L_v$ of the algebra $C^\infty(X)$ of smooth functions on $X$. Its kernel $C^\infty(T(v))$ is a part of the long exact sequence:

$$0 \to C^\infty(T(v)) \to C^\infty(X) \xrightarrow{L_v} C^\infty(X) \to \ldots$$

Recall that $C^\infty(T(v))$, the algebra of smooth functions on the space of trajectories, can be identified with the algebra of all smooth functions on $X$ that are constant along each $v$-trajectory.

When a traversally generic $v$ is such that $\partial^+_1 X(v)$ is diffeomorphic to $D^n$, then employing Theorem 3.1 and with the help of the finitely ramified surjective map

$$\Gamma_\delta : D^n = \partial^+_1 X(v) \subset X \xrightarrow{\Gamma} T(v)$$

we get the induced monomorphism $\Gamma^*_\delta : C^\infty(T(v)) \to C^\infty(D^n)$ of algebras, where the target algebra $C^\infty(D^n)$ of smooth functions on the $n$-ball is universal for a given dimension $n$.

Any point-trajectory $\gamma \in T(v)$ gives rise to the maximal ideal $m_\gamma \lhd C^\infty(T(v))$, comprising smooth functions that vanish at $\gamma$. On the other hand, if $m \lhd C^\infty(T(v))$ is a maximal ideal and a function $h \in m$ does not vanish on the compact $T(v)$, then the function $1 = (\frac{1}{h}) \cdot h \in m$, so that $m = C^\infty(T(v))$. Thus every maximal ideal $m \lhd C^\infty(T(v))$, distinct from the algebra itself, is of the form $m_\gamma$.

Recall that the map $\Gamma_\delta$ is finitely ramified with fibers of cardinality $(n + 1)$ at most (see Corollary 5.1 from [K3]). Therefore, for any maximal ideal $m \lhd C^\infty(T(v))$, its image $\Gamma^*_\delta(m)$ is the intersection $\cap_i m_i$ of $(n + 1)$ maximal ideals $m_i \lhd C^\infty(D^n)$ at most.

One can think of smooth vector fields on $X$ as derivations of the algebra $C^\infty(X)$. Let $C^+(X) \subset C^\infty(X)$ denote the open cone, formed by all positive functions. The gradient-like fields $v$ correspond to derivations $L_v$ of $C^\infty(X)$ such that $L_v(f) \in C^+(X)$ for some $f \in C^\infty(X)$.

By Theorem 3.5 from [K2], the traversally generic fields form a nonempty open set $V^\parallel(X)$ in the space of all fields. By Theorem 3.4 from [K2], the topological type of the stratification $\partial^+_1 X(v)$ is stable for any boundary generic field $v \in V^\parallel(X) \supset V^\parallel(X)$; so if the stratification in (3.1) is available for some boundary generic $v$, it is also available for all nearby fields. Therefore, the considerations above imply:

**Corollary 3.1.** (The origami resolutions $C^\infty(D^n)$ for the kernels of special derivations of the algebra $C^\infty(X)$)

For any $(n + 1)$-dimensional smooth and compact manifold $X$ with boundary, there exists an open nonempty set $\text{Der}^\circ(X)$ of derivations $L : C^\infty(X) \to C^\infty(X)$ of the algebra $C^\infty(X)$, which is characterized by the following property: for any $L \in \text{Der}^\circ(X)$, there exists a monomorphism of algebras

$$\Gamma^*_\delta : \ker(L) \to C^\infty(D^n)$$

$^9C^\infty(T(v))$ is just a subalgebra of $C^\infty(X)$, not an ideal.
such that, for any maximal ideal \( m \triangleleft \ker(\mathcal{L}) \), the image \( \Gamma_v^*(m) \) is an intersection \( \cap_i m_i \) of \( n + 1 \) maximal ideals \( m_i \triangleleft C^\infty(D^n) \) at most.

To describe the image of the Lie derivative \( \mathcal{L}_v \) looks as a more subtle problem. In fact, the cokernel \( \text{coker}(\mathcal{L}_v) \) can be interpreted as a 1-dimensional cohomology of \( X \) with coefficients in the sheaf of germs of functions from the kernel \( \ker(\mathcal{L}_v) \). Let us clarify this observation.

As before, we extend the pair \((X, v)\) to an ambient pair \((\hat{X}, \hat{v})\), where \( X \) is properly contained in \( \hat{X} \), and \( \hat{v} \neq 0 \) is a gradient-like field.

Recall that a function is called smooth on a closed subset \( K \) of a smooth manifold, if it is smooth in an open neighborhood of \( K \).

Let us denote by \( \mathcal{B} \) the sheaf of \( C^\infty \)-functions on \( X \), by \( \mathcal{A} \) the sheaf on \( X \) generated by germs of functions from the kernel of \( \mathcal{L}_v \), and by \( \mathcal{C} \) the sheaf on \( X \), generated by the germs of functions of the form \( \mathcal{L}_v(h) \), where \( h \) is the germ of a smooth function on \( X \).

By the definitions, we get a short exact sequence of sheaves \( 0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{\mathcal{L}_v} \mathcal{C} \to 0 \) on \( X \).

We claim that, in fact, \( \mathcal{C} = \mathcal{B} \). Indeed, consider a germ of an arbitrary smooth function \( f \) at \( x \in X \). By definition of a smooth function on a manifold with boundary, the germ is represented by a smooth function \( \hat{f} : \hat{U}_x \to \mathbb{R} \), where \( \hat{U}_x \) is an open neighborhood of \( x \) in \( \hat{X} \). Take a smooth transversal section \( S \) of the \( \hat{v} \)-flow so that \( x \in \text{int}(S) \). We pick \( S \) in the shape of a closed \( n \)-disk so small that \( S \subset \hat{U}_x \). There exists a “short” closed tube \( \hat{V}_x \subset \hat{U}_x \) which is diffeomorphic to the cylinder \( S \times [-1,1] \) and consists of closed segments of \( \hat{v} \)-trajectories that pass through the points of \( S \). Under this diffeomorphism \( \phi : \hat{V}_x \to S \times [-1,1], \phi(S) \) is identified with \( S \times \{0\} \).

Next, consider the smooth function \( \hat{g}(x) := \int_{\hat{x}}^x \hat{f} \, d\mu_{\hat{v}} \) in \( \text{int}({\hat{V}_x}) \), where the integration is performed along the \( \hat{v} \)-trajectories \( \hat{\gamma} \subset \hat{V} \), starting from the point \( \hat{\gamma} \cap S \), and \( d\mu_{\hat{v}} \) denotes the measure on \( \hat{\gamma} \), induced by the metric in which \( \hat{v} \) is of the unit length. Evidently, \( \mathcal{L}_v(\hat{g}) = \hat{f} \). Thus, in \( \text{int}({\hat{V}_x}) \cap X \), \( f \) is the \( \mathcal{L}_v \)-image of the function of \( \hat{g} \). As a result, in \( \text{int}({\hat{V}_x}) \cap X \), \( f \) is the \( \mathcal{L}_v \)-image of the function of \( \hat{g}|_X \).

Therefore the short exact sequence of sheaves \( 0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{\mathcal{L}_v} \mathcal{C} \to 0 \) transforms into the short exact sequence \( 0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{\mathcal{L}_v} \mathcal{B} \to 0 \).

Since smooth functions extend from closed subsets of \( X \) into the entire \( X \) ([W], [SL]), the sheaf \( \mathcal{B} \) on \( X \) is soft.

Although, by Lemma 2.2, the push-forward \( \Gamma_! (A) \) is a soft sheaf on \( T(v) \), the sheaf \( \mathcal{A} \) on \( X \) may not be soft: just think of the case when \( \gamma \cap F \) has many connected components; then locally constant functions on \( \gamma \cap F \) are not necessarily constant on \( \gamma \)!

As a result of \( \mathcal{B} \) being soft, we get an exact sequence in cohomology:

\[
0 \to H^0(X; A) \to H^0(X; B) \xrightarrow{\mathcal{L}_v} H^0(X; B) \to H^1(X; A) \to 0
\]

Thus,

\[
\text{coker}(\mathcal{L}_v : C^\infty(X) \to C^\infty(X)) \approx H^1(X; A).
\]
Note that $H^i(X; \mathcal{A}) = 0$ for all $i > 1$ since $\mathcal{B}$ is soft and therefore acyclic in all positive dimensions. □

Next we introduce one construction (see Fig. 5) which will turn to be very useful throughout our investigations.

![Figure 5. The embedding $\alpha(f, v)$ of $X$ into the product $\mathcal{T}(v) \times \mathbb{R}$.](image)

**Lemma 3.1.** For any non-vanishing gradient-like field $v$ on $X$, there is an embedding $\alpha : X \subset \mathcal{T}(v) \times \mathbb{R}$. In fact, any pair $(f, v)$ such that $df(v) > 0$ generates such an embedding $\alpha = \alpha(f, v)$ in a canonical fashion.

For any smooth map $\beta : \mathcal{T}(v) \to \mathbb{R}^N$, the composite map

$$A(v, f) : X \xrightarrow{\alpha} \mathcal{T}(v) \times \mathbb{R} \xrightarrow{\beta \times id} \mathbb{R}^N \times \mathbb{R}$$

is smooth.

Any two embeddings $\alpha(f_1, v)$ and $\alpha(f_2, v)$ are isotopic through homeomorphisms, provided that $df_1(v) > 0, df_2(v) > 0$.

**Proof.** Since $f$ is strictly increasing along the $v$-trajectories, any point $x \in X$ is determined by the $v$-trajectory $\gamma_x$ through $x$ and the value $f(x)$. Therefore, $x$ is determined by the point $\gamma_x \times f(x) \in \mathcal{T}(v) \times \mathbb{R}$. By the definition of topology in $\mathcal{T}(v)$, the correspondence $\alpha(f, v) : x \to \gamma_x \times f(x)$ is a continuous map.

In fact, $\alpha(f, v)$ is a smooth map in the spirit of Definition 2.1, more accurately, for any map $\beta : \mathcal{T}(v) \to \mathbb{R}^N$, given by $N$ smooth functions on $\mathcal{T}(v)$, the composite map $A(v, f) : X \to \mathbb{R}^N \times \mathbb{R}$ is smooth. The verification of this fact is on the level of definitions.

For a fixed $v$, the condition $df(v) > 0$ defines an open convex cone $C(v)$ in the space $C^\infty(X)$. Thus, $f_1$ and $f_2$ can be linked by a path in $C(v)$, which results in $\alpha(f_1, v)$ and $\alpha(f_2, v)$ being homotopic through homeomorphisms.

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10 equivalently, traversing
Remark 3.2. By examining Fig. 5, we observe an interesting phenomenon: the embedding
\( \alpha : X \subset \mathcal{T}(v) \times \mathbb{R} \) does not extend to an embedding of a larger manifold \( \hat{X} \supset X \), where \( \hat{X} \setminus X \approx \partial_1 X \times [0, \epsilon) \). In other words, \( \alpha(\partial_1 X) \) has no outward “normal field” in the ambient \( \mathcal{T}(v) \times \mathbb{R} \); in that sense, \( \alpha(\partial_1 X) \) is rigid in \( \mathcal{T}(v) \times \mathbb{R} \).

\( \Box \)

Corollary 3.2. Let \( X^0 \) denote the interior of \( X \). For any non-vanishing gradient-like field \( v \) on \( X \), the embedding
\[ \alpha(f, v) : \partial_1 X \to (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(f, v)(X^0) \]
is a homology equivalence. As a result, the space \( (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(f, v)(X^0) \) is a Poincaré complex of the formal dimension \( \dim(X) - 1 \).

**Proof.** Put \( \alpha := \alpha(f, v) \). Let us compare the homology long exact sequences of two pairs:
\( X \supset \partial_1 X \) and \( \mathcal{T}(v) \times [0, 1] \supset (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(X^0) \). They are connected by the vertical homomorphisms that are induced by \( \alpha \). Using the excision property,
\[ \alpha_* : H_* (X, \partial_1 X) \to H_* (\mathcal{T}(v) \times [0, 1], (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(X^0)) \]
are isomorphisms. On the other hand, since by Theorem 5.1 from [K3], \( \Gamma : X \to \mathcal{T}(v) \) is a homology equivalence, \( \alpha_* : H_* (X) \to H_* (\mathcal{T}(v) \times [0, 1]) \) are isomorphisms. Therefore by the Five Lemma,
\[ \alpha_* : H_* (\partial_1 X) \to H_* ((\mathcal{T}(v) \times [0, 1]) \setminus \alpha(X^0)) \]
must be isomorphisms as well. Since \( \partial_1 X \) is a closed \( n \)-manifold, it is a Poincaré complex of formal dimension \( n \), and thus so is the space \( (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(f, v)(X^0) \).

4. The Causality-based Holography Theorems

Now we are in position to formulate one question which is central to our program:
“Is it possible to reconstruct the manifold \( X \) and the nonsingular gradient flow from some data available on the boundary \( \partial X \)?”

When such a reconstruction is possible (see Theorem 4.1 and Theorem 4.2), the corresponding proposition deserves the adjective “holographic” in its name.\(^{11}\)

Given a traversing field \( v \) on \( X \), consider the map \( C_v : \partial_1^+ X \to \partial_1^- X \) that takes any point \( x \in \partial_1^+ X \) to the next point \( y \) from the set \( \gamma_x \cap \partial_1 X \), the order on the trajectory \( \gamma_x \) being defined by \( v \). We call \( C_v \) the causality map of \( v \) (see Theorem 4.3 for a justification of the name).

Of course the gradient fields have no closed trajectories. Nevertheless, in the world of such fields on manifolds \( X \) with boundary, the causality map can be thought as a weak substitute for the Poincaré return map (see [Te] for the definition of the Poincaré return map). The dynamics of \( C_v \) under (finitely many) iterations reflects the concavity of \( X \) with respect to the \( v \)-flow. The “iterations” of \( C_v \) are only partially-defined maps. However, for geodesic flows on Riemannian manifolds with boundary, \( C_v \) can be transformed into, so

\(^{11}\)We own an apology to the fellow physicists: the name does not suggest a connection to the holography principles in the quantum field theory and the dual theories of gravitation.
called, billiard map (see Section 5, in particular, the discussion that follows Example 5.4 REF). For the billiard maps, arbitrary iterations are available.

We are painfully familiar with the discontinuous nature of $C_v$ (implicitly, this discontinuity dominates the investigations in [K], [K1], [K2], and [K3]). The bright spot is that $C_v$ is semicontinuous relative to a nonsingular function $f: X \to \mathbb{R}$ with the property $df(v) > 0$. This semicontinuity has the following definition: for any $x \in \partial^+_1 X$ and $\epsilon > 0$, there is a neighborhood $U_\epsilon(x) \subset \partial_1 X$ such that

$$f(C_v(y)) - f(y) \geq \epsilon \quad \text{and} \quad f(C_v(y)) - f(y) > f(C_v(x)) - f(x) - \epsilon \quad \text{for all } y \in U_\epsilon(x).$$

Note that $C_v(x) = x$ exactly when $x \in \partial^{-}_2 X^0 \cup \partial^{-}_3 X^0 \cup \cdots \cup \partial^{-}_{n+1} X$.

We can take alternative and more formal views of the map $C_v$.

Note that the traversing $v$-flow on $X$ defines a structure of a partially ordered set on $\partial X$: we write $x < x'$, where $x, x' \in \partial X$, if there is an ascending $v$-trajectory (not a singleton) that connects $x$ to $x'$. Let us denote by $\mathcal{C}^0(v)$ this poset $(\partial X, \prec)$. Evidently, $x \preceq x'$ if and only if $x'$ is an image of $x$ under a number of iterations of the causality map $C_v$, provided $v$ being boundary generic. Therefore, the poset $\mathcal{C}^0(v)$ allows for a reconstruction of the causality map $C_v$.

Equivalently, we can consider the points of $\partial X$ as elements of a category $\mathcal{Cat}^0(v)$. For any $x, x' \in \partial X$, we define the set of morphisms $\text{Mor}(x, x')$ to be empty if no ascending $v$-trajectory in $X$ connects $x$ to $x'$. Otherwise, $\text{Mor}(x, x')$ consists of a single element—the portion of an ascending trajectory that starts at $x$ and terminates at $x'$.

If $\text{Mor}(x, x') \neq \emptyset$ and $\text{Mor}(x', x'') \neq \emptyset$, then $\text{Mor}(x, x'') \neq \emptyset$. This defines the obvious composition map $\text{Mor}(x, x') \times \text{Mor}(x', x'') \to \text{Mor}(x, x'')$.

**Remark 4.1.** Note that Lemma 3.4 and formula (3.19) from [K2] provide, among other things, for local models of the causality maps $C_v$, generated by traversally generic fields $v$. In the special coordinates $(u, x, y)$, $C_v$ amounts to taking each root of the $u$-polynomial $P(u, x)$, residing in a maximal interval $I(x)$ where $P(u, x) \leq 0$, either to the next root residing in $I(x)$, or to itself (when $I(x)$ happens to be a singleton). By Theorem 2.2 from [K2], this is a map from the semi-algebraic set $\{P(u, x) = 0, \frac{\partial P}{\partial u}(u, x) \geq 0, \|x\| \leq \epsilon\}$ to the semi-algebraic set $\{P(u, x) = 0, \frac{\partial P}{\partial u}(u, x) \leq 0, \|x\| \leq \epsilon\}$. These observations will form a base of Definitions 6.1, 6.3 that introduce the notion of holographic structure on $\partial_1 X$.

For a traversing field $v$, the smooth functions on $X$ that are constant along each $v$-trajectory $\gamma$ give rise to smooth functions on $\partial_1 X$. Evidently, such functions are constant along each $C_v$-trajectory $\gamma^0 = \gamma \cap \partial_1 X$. Furthermore, any smooth function on $\partial_1 X$ which is constant on each finite set $\gamma^0$ gives rise to a unique continuous function on $X$, which is constant along each trajectory $\gamma$. However, such functions may not be automatically smooth on $X$!

For a traversing $v$, consider the algebra $\text{Ker}(\mathcal{L}_v) \approx C^\infty(T(v))$ of smooth functions on $X$ that are constants along each $v$-trajectory.
Question 4.1. For a traversally generic field \( v \) on \( X \), how to characterize the image (trace) of the algebra \( \ker(\mathcal{L}_v) \) in the algebra \( C^\infty(\partial_1 X) \) in terms of the causality map \( C_v \)?

Let us describe a good candidate for the trace of the algebra \( \ker(\mathcal{L}_v) \) in the algebra \( C^\infty(\partial_1 X) \). For a generic field \( v \), let \( n_j(v) \) be the ideal of smooth functions on \( \partial_1 X \) that vanish on the submanifold \( \partial_{j+1} X(v) \). We denote by \( \mathcal{L}_v^{(k)} \) the \( k \)-th iteration of the Lie derivative in the direction of the field \( v \). Let \( m_j(v) \) be the ideal of smooth functions \( \psi \) on \( \partial_1 X \) such that \( (\mathcal{L}_v^{(k)} \psi)|_{\partial_h X} = 0 \) for all \( k \leq j \). In other words, \( \psi \in m_j(v) \) if and only if \( \mathcal{L}_v^{(k)} \psi \in n_k(v) \) for all \( k \leq j \).

Let us denote by \( m_j(v)^{C_v} \) the subalgebra of functions from \( m_j(v) \) that are constants on each \( C_v \)-trajectory \( \gamma^\partial \subset \partial_1 X \).

Conjecture 4.1. Let \( v \) be a traversally generic field on a smooth compact \((n+1)\)-manifold \( X \). Consider the algebra \( \ker(\mathcal{L}_v) \approx C^\infty(\mathcal{T}(v)) \) of smooth functions on \( X \) that are constant along each \( v \) trajectory. Then the restriction of \( \ker(\mathcal{L}_v) \) to the boundary \( \partial_1 X \subset X \) coincides with the subalgebra \( m_n(v)^{C_v} \subset C^\infty(\partial_1 X) \).

It is easy to check that \( \ker(\mathcal{L}_v)|_{\partial_1 X} \subset m_n(v)^{C_v} \).

Now we move away from the category smooth maps towards the category of piecewise differentiable (“PD” for short) maps.

Definition 4.1. Let \( v \) be a traversing vector field on a compact smooth manifold \( X \).

We say that a triangulation \( T^\partial \) of \( \partial X \) is invariant under the causality map \( C_v : \partial_1^\partial X \to \partial_1 X \), if the interior of each simplex from \( T^\partial \) is mapped homeomorphically by \( C_v \) onto the interior of a simple.

Lemma 4.1. If \( v \) is a traversally generic field on a compact smooth manifold \( X \), then the boundary \( \partial_1 X \) admits a \( C_v \)-invariant smooth triangulation.

Proof. For boundary generic fields \( v \), the map \( \Gamma : \partial_1 X \to \mathcal{T}(v) \) is finitely ramified surjection. For a traversally generic \( v \), the lemma follows from Corollary 2.4 which claims the existence of a triangulation \( T \) of the trajectory space \( \mathcal{T}(v) \), consistent with its \( \Omega^\bullet \)-stratification. Any such triangulation \( T \) of \( \mathcal{T}(v) \), with the help of \( \Gamma^{-1} \), lifts to a triangulation \( T^0 \) of \( \partial_1 X \). Indeed, for each \( \omega \in \Omega^\bullet \), by Corollary 5.1 from \[ K3 \], the smooth submersion

\[ \Gamma : \Gamma^{-1}(\mathcal{T}(v, \omega)) \cap \partial_1 X \to \mathcal{T}(v, \omega) \]

is a trivial covering whose fiber is a finite set. By its very construction, the triangulation \( T^0 \) is \( C_v \)-invariant.

Remark 4.2. The existence of a triangulation on \( \partial_1 X \) by itself does not imply the existence of a triangulation on \( \mathcal{T}(v) \): there are smooth manifolds that can serve as finite covering spaces over topological manifold bases that do not admit any triangulation! For example, the standard sphere may cover a non-triangulable fake real projective space (see \[ CS \]).

\[ \text{\[12\]} \text{Remember, } C_v \text{ is typically a discontinuous map!} \]
This is why the efforts that were invested in proving the existence of a triangulation $T$, amenable to the stratification in $\mathcal{T}(v)$, are justified. See [Gor] for general treatment of triangulations on stratified spaces.

\textbf{Theorem 4.1. (The PD-Holography Theorem)}

Let $X_1, X_2$ be two smooth compact $(n + 1)$-manifolds with boundary, equipped with traversally generic fields $v_1, v_2$, respectively. Let $T_i^\partial$, $i = 1, 2$, be a smooth triangulation of $\partial X_i$, invariant under the causality map $C_v^{13} \text{ where } i = 1, 2$.

Then any PD-homeomorphism $\Phi^\partial : (\partial X_1, T_1^\partial) \to (\partial X_2, T_2^\partial)$, such that

$$\Phi^\partial \circ C_{v_1} = C_{v_2} \circ \Phi^\partial$$

extends to a PD-homeomorphism $\Phi : X_1 \to X_2$ which takes the unparametrized $v_1$-trajectories to the unparametrized $v_2$-trajectories so that the field-induced orientations of trajectories are preserved.

\textit{Proof.} Since each $v_i \neq 0$ is a gradient field, any trajectory reaches the boundary; so the obvious maps $T_i^\partial : \partial_1 X_i \to \mathcal{T}(v_i)$ are onto.

Evidently, the fiber of $\Gamma_i^\partial$ consists of the maximal chain of points $x_1 \sim x_2 \sim \cdots \sim x_q$ from $\partial_1 X_i$ such that $C_{v_i}(x_j) = x_{j+1}$ for all $j \in [1, q-1]$. By the definition of $C_{v_i}$, such chain is exactly the ordered locus $\gamma_{x_1} \cap \partial_1 X_i$. Therefore, using that $\Phi^\partial : \partial_1 X_1 \to \partial_1 X_2$ commutes with the causality maps $C_{v_i}$, we see that $\Phi^\partial$ gives rise to a well-defined continuous map $\Phi^\gamma : \mathcal{T}(v_1) \to \mathcal{T}(v_2)$ of the trajectory spaces.

In the next paragraph, in order to simplify the notations, let $v := v_1$ and $X := X_1$.

For each point $y \in \partial_1 X$, its multiplicity $m(y)$ with respect to a boundary generic flow $v$ can be detected by the unique stratum $\partial_1 X^o := \partial_1 X^o(v)$, $j = m(y)$, to which $y$ belongs. On the other hand, it also can be detected in terms of the causality map $C_v$ and its iterations. Let us justify this claim. Recall that, for boundary generic fields, Lemma 3.1 from [K2] provides us with the model for the divisors $\{D_i\}_\gamma$, localized to a sufficiently small neighborhood $U_y$ of $y$ (the set $\gamma \cap \partial_1 X \cap U_y$ is the support of $D_k$). We choose the neighborhood $U_y$ with some care: first we chose a small smooth transversal section $S \subset \bar{X}$ of the $\hat{v}$-flow, which contains $y$, then we consider the union $V_y$ of $\hat{v}$-trajectories through the points of $S$, and finally we let $U_y := V_y \cap X$.

Using these models, the maximal length of a chain

$$z_1 \sim z_2 = C_v(z_1) \sim z_3 = C_v(z_2) \sim \ldots$$

in any sufficiently small $v$-adjusted neighborhood $U_y \subset \partial_1 X$ of $y$ is $\lceil m(y)/2 \rceil$, where $\lceil \cdot \rceil$ denotes the integral part of a positive number (see [K3]). Indeed, if $m(y)$ is even, then the maximal number of roots of even multiplicity for a polynomial of degree $m(y)$ is $m(y)/2$, and by Lemma 3.1 from [K2], such polynomials $u^{m(y)} + \sum_{j=0}^{m(y)/2} x_j u^j$ are present in an arbitrary small neighborhood of the polynomial $u^{m(y)}$ in the coefficient space. When $m(y)$ is odd, then the maximal length of a chain $z_1 \sim z_2 \sim \ldots$ in the vicinity of $y$ in $\partial_1 X$ is

\textsuperscript{13}By Lemma 4.1 such triangulations do exist.
(m(y) − 1)/2 = [m(y)/2]. It corresponds either to the m(y)-polynomials with one simple root, followed by the maximal number of multiplicity 2 roots (when y is the first in its localized chain), or to the m(y)-polynomials with the maximal number of multiplicity 2 roots, followed by a simple root (when y is the last in its localized chain).

Evidently, the order in which the points γ ∩ ∂1X appear along each trajectory γ is also determined by C_v. Therefore the combinatorial type ω(γ) ∈ Ω^• (see [K3], Definitions 2.3 and 2.5, for the description of the universal poset Ω^•) of each v-trajectory γ ⊂ X can be recovered from the causality map C_v : ∂1^+X → ∂1^-X and its partially-defined iterations. Thus the information encoded in C_v is sufficient for a reconstruction of the Ω^•-stratified space T(v).

So, the PD-homeomorphism Φ^θ : ∂1X_1 → ∂1X_2 which commutes with the causality maps C_v1 and C_v2, must take any chain

z_1 \leadsto z_2 = C_v1(z_1) \leadsto z_3 = C_v1(z_2) \leadsto \ldots

in ∂1X_1 to a similar chain in ∂1X_2 with the same multiplicity pattern.

Since the triangulation T_1^θ is invariant under the causality map C_v1, it gives rise to a triangulation T_1 of the trajectory space T(v_1). This triangulation T_1 is consistent with its Ω•-stratification of T(v_1). Therefore, Φ^θ gives rise to the PD-homeomorphism

Φ^T : T(v_1) → T(v_2)

which preserves the Ω•-stratifications of the trajectory spaces.

Next, consider the embeddings

{α_i := α_i(f_1,v_i) : X_i ⊂ T(v_i) × R} i=1,2

as in Lemma 3.1 and Fig. 4.

We would like to lift the newly constructed PD-homeomorphism Φ^T : T(v_1) → T(v_2) to a PD-homeomorphism

Φ : T(v_1) × R → T(v_2) × R

that takes α_1(X_1) to α_2(X_2) and has the property Φ ∩ α_1|∂X_1 = α_2 ∩ Φ^θ.

As usually, we abuse the notations by denoting the v-trajectories γ ⊂ X and the points Γ(γ) ∈ T(v) in the trajectory space they represent by the same symbol “γ”.

Put

(4.2) Φ(γ,t) = (Φ^T(γ), Φ(γ)(t))

, where γ ∈ T(v_1), t ∈ R, and Φ(γ) : R → R is an orientation-preserving PL-homeomorphism to be constructed.

Since Φ^T preserves the Ω•-stratifications, the combinatorial types ω(γ) and ω(Φ^T(γ)) of the divisors D_γ and D_{Φ^T(γ)} are the same.

There is a natural choice for an orientation-preserving PL-homeomorphism φ of R that maps the ordered set D_(γ)^{(1)} := α_1(γ ∩ ∂1X_1) to the ordered set D_(γ)^{(2)} := α_2(Φ^T(γ) ∩ ∂1X_2).
Let $q := |\sup(\omega)|$. In fact, there is a unique increasing PL-function $\phi : \mathbb{R} \to \mathbb{R}$ that takes the $k$-th point $x_k \in \alpha_1(\gamma \cap \partial_1 X_1)$ to the $k$-th point $y_k \in \alpha_2(\Phi^T(\gamma) \cap \partial_1 X_2)$, $1 \leq k \leq q$, and such that $\phi$ is a linear polynomial in each of the intervals

$(-\infty, x_1], [x_1, x_2], \ldots, [x_{q-1}, x_q], [x_q, +\infty)$,

and has slope 1 in $(-\infty, x_1) \cup (x_q, +\infty)$. We will denote such $\phi$ by $\phi_{\vec{x}, \vec{y}}$ (see Fig. 6, diagram A).

**Figure 6.** The PL (diagram A) and smooth (diagram B) interpolating homeomorphisms $\phi_{\vec{x}, \vec{y}} : \mathbb{R} \to \mathbb{R}$.

Let $\text{Hom}^\text{PL}_+(\mathbb{R})$ be the group of orientation-preserving piecewise linear homeomorphisms of $\mathbb{R}$ which are smooth away from a finite set of points. Each element of the group is represented by a monotone PL-function. Therefore, $\text{Hom}^\text{PL}_+(\mathbb{R})$ forms a convex cone in the vector space of all PL-functions on $\mathbb{R}$.

Evidently, the construction $\phi_{\vec{x}, \vec{y}}$ gives rise to a continuous map

$\phi_d : (\text{Sym}^d \mathbb{R})^\circ \times (\text{Sym}^d \mathbb{R})^\circ \to \text{Hom}^\text{PL}_+(\mathbb{R})$.

Here $(\text{Sym}^d \mathbb{R})^\circ$ denotes the space of simple divisors in $\mathbb{R}$. We view

$\vec{x} := (x_1 < x_2 < \cdots < x_d) \text{ and } \vec{y} := (y_1 < y_2 < \cdots < y_d)$

as points of $(\text{Sym}^d \mathbb{R})^\circ$. Note that when $x_k$ merges with $x_{k+1}$ and $y_k$ merges with $y_{k+1}$ to form new elements $\vec{x}', \vec{y}' \in \text{Sym}^d \mathbb{R}$, then the function $\phi_{\vec{x}', \vec{y}'}$ converges in the sup-norm on compacts to the function $\phi_{\vec{x}, \vec{y}}$. Therefore $\phi_d$ extends continuously to the subset $(\text{Sym}^d \mathbb{R})^\circ \times \text{Sym}^d \mathbb{R}$ that is formed by pars $(\vec{x}, \vec{y})$ with the same combinatorics of $\vec{x}$ and $\vec{y}$, that is, with the same sequence of symbols "<" and "\leq" in their representation of the length $d$.

With $\phi_d : (\text{Sym}^d \mathbb{R})^\circ \times \text{Sym}^d \mathbb{R}$ being continuous and “well-behaved” under the merge operations of divisors in $\mathbb{R}$, the difficulty shifts towards the discontinuity under the insert operations on such divisors (see formulas (2.1)-(2.3) from [K3], where such operations are discussed in some detail), in other words, towards the problem of matching the
maps \( \{ \phi_d \} \) for different \( d \)'s of the same parity. Although, thanks to the contractibility of \( \text{Hom}^{\text{PL}}(\mathbb{R}) \), this problem is solvable, it requires some attention and time.

Now the inductive argument, aimed at proving the existence of the extension \( \tilde{\Phi} \) (as in (1.2)), proceeds as follows. We assume that for a finite closed sub-poset (an ideal) \( \Theta \subseteq \Omega^* \), we already managed to construct a PL-homeomorphism

\[
\tilde{\Phi} : X_1(v_1, \Theta) \to X_2(v_2, \Theta)
\]

as in (1.2) which covers the homeomorphism \( \Phi^T : T(v_1, \Theta) \to T(v_2, \Theta) \) and has the property

\[
\tilde{\Phi} \circ \alpha_1 |_{\partial X_1(v_1, \Theta)} = \alpha_2 \circ \Phi^\theta |_{\partial X_1(v_1, \Theta)}.
\]

Let \( \omega \in \Omega^* \setminus \Theta \) be an element such that \( \omega_{+, i} \), the set of elements smaller than \( \omega_i \), is contained in \( \Theta \). The induction step requires to extend the map \( \tilde{\Phi} \) over the set \( X_1(v_1, \omega \cup \Theta) \) so that the extension would coincide on \( X_1(v_1, \omega \cup \Theta) \cap \partial X_1 \) with the given homeomorphism \( \Phi^\theta \).

We pick triangulations of trajectory spaces \( T(v_1) \), which lift against \( \Gamma_i \) to the smooth triangulations \( T_i^\gamma \) of \( \partial X_i \) and are consistent with \( \Omega^* \)-stratifications in \( T(v_1) \) (see Lemma 4.1).

For \( \omega \in \Omega^* \), consider the closure \( \overline{T}(v_1, \omega) \) of the pure stratum \( T(v_1, \omega) \) in \( T(v_1) \) and the closed set

\[
\partial T(v_1, \omega) := T(v_1, \omega) \setminus \overline{T}(v_1, \omega).
\]

Put \( D_1^{(1)} := f_1(\gamma \cup \partial X_1) \subset \mathbb{R} \) and \( D_2^{(2)} := f_2(\Phi^T(\gamma) \cap \partial X_2) \subset \mathbb{R} \).

Consider the map \( \tilde{\Phi} : T(v_1) \times \mathbb{R} \to T(v_2) \times \mathbb{R} \) defined by the formula

\[
\tilde{\Phi}(\gamma, t) := (\Phi^T(\gamma), \phi(D_1^{(1)}, D_2^{(2)})(t))
\]

where \( t \in \mathbb{R} \) (here, we suppress the dependence of \( \phi \in \text{Hom}^{\text{PL}}(\mathbb{R}) \) on the cardinality \( d \) of the set \( D_1^{(1)} \).

Since \( \Phi^T \) is determined by \( \Phi^\theta \) as has been described above, we get the desired property \( \alpha_2 \circ \Phi^\theta |_{\partial X_1} = \tilde{\Phi} \circ \alpha_1 |_{\partial X_1} \). Unfortunately, the map \( \tilde{\Phi} \) is discontinuous!

Using the induction \( \Theta \Rightarrow \omega \cup \Theta \) we will gradually repair \( \tilde{\Phi}(\gamma, t) \) to insure its continuity. Of course, for \( \Theta \) comprising the minimal combinatorial types of trajectories from \( X_1 \), the formula above delivers a PL-homeomorphism

\[
\tilde{\Phi} : \alpha_1(X_1(v_1, \Theta)) \to \alpha_2(X_2(v_2, \Theta))
\]

such that \( \alpha_2 \circ \Phi^\theta = \tilde{\Phi} \circ \alpha_1 \) over \( \partial X_1 \cap X_1(v_1, \Theta) \). For such \( \gamma \)'s, there is no need to modify the homeomorphism \( \phi(D_1^{(1)}, D_2^{(2)}) \). So the basis for the induction has been established.

By the inductive assumption, suppose that the desired matching homeomorphism \( \tilde{\phi}(\gamma) \in \text{Hom}^{\text{PL}}(\mathbb{R}) \), such that

\[
\alpha_2 \circ \Phi(x) = \tilde{\Phi} \circ \alpha_1(x) \quad \text{and} \quad \tilde{\Phi}(\gamma, t) := (\Phi^T(\gamma_1), \phi(\gamma_2)(t))
\]
Let
\[ T(v_i, \omega) := T(v_i, \omega) \setminus U_\omega \]
, where \( U_\omega \) is a regular neighborhood of \( \partial T(v_i, \omega) := \partial T(v_i, \omega) \) in \( T(v_i, \omega) \) with the properties as in Lemma 4.2 below.

Over \( T(v_1, \omega) \), we keep the continuous family of PL-homeomorphisms \( \phi(D^{(1)}_\gamma, D^{(2)}_\gamma) \). In the neighborhood \( U_\omega \), for each point \( \gamma \in \partial U_\omega \), we linearly interpolate between the function \( \phi(D^{(1)}_\gamma, D^{(2)}_\gamma) \) and previously constructed function \( \tilde{\phi}(D^{(1)}_{\gamma'}, D^{(2)}_{\gamma'}) \), where \( \gamma' := p_\omega(\gamma) \) (see Lemma 4.2 for the properties of the retraction \( p_\omega : U_\omega \rightarrow T(v, \omega_v) \)). The interpolation uses the product structure in \( U_\omega \setminus \partial T(v_1, \omega) \), facilitated by the special vector field \( w \) as in Lemma 4.2. We connect a typical point \( \gamma \in \partial U_\omega \) with the unique point \( p_\omega(\gamma) \in T(v_1, \omega') \), where \( \omega' < \omega \), by an arc \( \{ \eta_\gamma(s) \} s \subset U_\omega \) that integrates the field \( w \). Here \( s \in [0, 1] \), \( \eta_\gamma(0) = \gamma \), and \( \eta_\gamma(1) = p_\omega(\gamma) \). We define the new \( s \)-family \( \phi_{\eta_\gamma}(s) \in Hom^{PL}_+(\mathbb{R}) \) by the formula
\[
(4.3) \quad \varphi_{\eta_\gamma}(s) := (1 - s) \cdot \phi(D^{(1)}_\gamma, D^{(2)}_\gamma) + s \cdot \tilde{\phi}(D^{(1)}_{p_\omega(\gamma)}, D^{(2)}_{p_\omega(\gamma)})
\]
which is continuous in \( \gamma \in \partial U_\omega \) and \( s \in [0, 1] \). Here we rely on the fact that the linear homotopy of monotonically increasing PL-functions on \( \mathbb{R} \) is a monotonically increasing PL-function.

Note that, for \( s \in (0, 1) \), the graph of \( \varphi_{\eta_\gamma}(s) \), acquires singular (i.e. non-smooth) points from both sets, \( D^{(1)}_\gamma \) and \( D^{(1)}_{p_\omega(\gamma)} \).

However, when \( s \in (0, 1) \), the map \( \varphi_{\eta_\gamma}(s) \) still does not take the set
\[ D^{(1)}_{\eta_\gamma}(s) := \alpha_1(\eta_\gamma(s) \cap \partial_1 X_1) \]
exactly onto the set
\[ D^{(2)}_{\Phi^T(\eta_\gamma(s))} := \alpha_2(\Phi^T(\eta_\gamma(s)) \cap \partial_1 X_2) \]
, although the orders in which the \((\alpha_2 \circ \Phi^T \circ \alpha_1^{-1})\)-corresponding elements of two sets, \( \varphi_{\eta_\gamma}(s)(D^{(1)}_{\eta_\gamma(s)}) \) and \( D^{(2)}_{\Phi^T(\eta_\gamma(s))} \), appear are similar (in fact, by choosing the neighborhood \( U_\omega \) small enough, we can insure that the distance between these two sets in \( \mathbb{R} \) is uniformly small for all \( w \)-trajectories \( \eta_\gamma \) in \( U_\omega \)).

To compensate for this failure, consider an additional correction:
\[
(4.4) \quad \xi_{\eta_\gamma}(s) := \phi(\varphi_{\eta_\gamma}(s)(D^{(1)}_{\eta_\gamma(s)})), \ D^{(2)}_{\Phi^T(\eta_\gamma(s))}) \in Hom^{PL}_+(\mathbb{R}),
\]
, where \( \phi \) denotes the canonical PL-homeomorphism of the line
\[ \Phi^T(\eta_\gamma(s)) \times \mathbb{R} \subset T(v_2) \times \mathbb{R} \]
, which takes the ordered set \( \varphi_{\eta_\gamma}(s)(D^{(1)}_{\eta_\gamma(s)}) \) to the ordered set \( D^{(2)}_{\Phi^T(\eta_\gamma(s))} \).

Now the composition
\[ \chi_{\eta_\gamma}(s) := \xi_{\eta_\gamma}(s) \circ \varphi_{\eta_\gamma}(s) \in Hom^{PL}_+(\mathbb{R}) \]
does the job: it takes $D^{(1)}_{\eta_r(s)}$ bijectively to $D^{(2)}_{\Phi^r(\eta_r(s))}$ and is continuous in $\gamma \in \partial U_\omega$ and $s \in [0, 1]$. Indeed, chasing the definitions, we get $\xi_{\eta_r(0)} = Id = \xi_{\eta_r(1)}$, so that

$$\chi_{\eta_r(0)} = \phi(D^{(1)}_\gamma, D^{(2)}_\gamma)\text{ and } \chi_{\eta_r(1)} = \tilde{\phi}(D^{(1)}_{p_\omega(\gamma)}, D^{(2)}_{p_\omega(\gamma)}).$$

As $s \to 1$, each point $x(s)$ from the set $\alpha_1(\eta_r(s) \cap \partial X_1)$ approaches a unique point $x(1)$ from the set $\alpha_1(p_\omega(\gamma) \cap \partial X_1)$ (cf. Lemma 2.5 from [K3]). At the same time, the $s$-family of intervals $\{\alpha_1(\eta_r(s))\}$ converges to a closed interval $\beta(\eta_r(1))$ bounded by a pair of points from the set $\alpha_1(p_\omega(\gamma))$. By the definition of $\varphi_{\eta_r(s)}$, we get $\lim_{s \to 1} \varphi_{\eta_r(s)} = \varphi_{\eta_r(1)}$.

Moreover, since $D^{(1)}_{\eta_r(s)} \subset \mathbb{R}$ and $D^{(2)}_{\Phi^r(\eta_r(s))} \subset \mathbb{R}$ depend continuously on $s \in [0, 1]$ and $\gamma \in \partial U_\omega$, the the restrictions of the functions $\xi_{\eta_r(s)}$ to the intervals $\alpha_1(\eta_r(s))$ converge to the restriction of $\xi_{\eta_r(1)} = Id$ to the interval $\beta(\eta_r(1))$. As a result,

$$\lim_{s \to 1} \chi_{\eta_r(s)}|_{\beta(\eta_r(1))} = \chi_{\eta_r(1)}|_{\beta(\eta_r(1))}.$$  

Because of the piecewise linear nature of the field $w$ in $U_\omega$ (and thus of the retraction $p_\omega : U_\omega \to \partial \mathcal{T}(v_1, \omega_\gamma)$), the family maps $\{\chi_{\eta_r(s)}\}_{\gamma \in \partial U_\omega, s \in [0, 1]}$ helps to extend the piecewise differential homeomorphism from

$$\tilde{\Phi} : \alpha_1(X_1(v_1, \omega_\gamma)) \to \alpha_2(X_2(v_2, \omega_\gamma))$$

to a piecewise differential homeomorphism

$$\Phi : \alpha_1(X_1(v_1, \omega_\gamma)) \to \alpha_2(X_2(v_2, \omega_\gamma))$$

, thus completing the induction step in our construction of the desired PD-homeomorphism

$$\Phi := \alpha_2^{-1} \circ \tilde{\Phi} \circ \alpha_1 : X_1 \to X_2.$$  

\[ \square \]

**Lemma 4.2.** For each $\omega \in \Omega^\bullet$ and a traversally generic field $v$ on $X$, there exits a closed neighborhood $U_\omega$ of the space $\mathcal{T}(v, \omega_\gamma)$ in the space $\mathcal{T}(v, \omega_\gamma)$ and a piecewise linear vector field $w$ in $U_\omega$, subject to the following properties:

(4.5)  

(1) $w|_{\mathcal{T}(v, \omega_\gamma)} = 0,$

(2) $w$ does not vanish in $U_\omega \setminus \mathcal{T}(v, \omega_\gamma)$,

(3) $w$ is inward transversal to the boundary $\partial U_\omega$,

(4) the projection $p_\omega : U_\omega \to \mathcal{T}(v, \omega_\gamma)$, defined by the $w$-trajectories, is a PL-map which helps to identify $U_\omega$ with the mapping cylinder of $p_\omega : \partial U_\omega \to \mathcal{T}(v, \omega_\gamma)$.

\[ \text{Proof.} \] Using Corollary 2.4, we take a triangulation $T$ of $\mathcal{T}(v)$ consistent with its $\Omega^\bullet$-stratification. Consider the set of all simplices $\Delta$ in the second barycentric subdivision of $\mathcal{T}(v, \omega_\gamma)$ that share points with $\partial \mathcal{T}(v, \omega)$. Each such simplex $\Delta$ is a join of a maximal simplex $\Delta' \subset \partial \mathcal{T}(v, \omega)$ and the maximal subsimplex $\Delta'' \subset \Delta$ that does not intersect

\[ \text{properties (2) and (3) imply that } U_\omega \setminus \mathcal{T}(v, \omega_\gamma) \text{ is PD-homeomorphic to the product } \partial U_\omega \times [0, 1], \text{ the product structure being defined by the } w\text{-trajectories.} \]
\(\partial T(v, \omega)\). For any line segment \([x, y] \subset \Delta\) which connects \(x \in \Delta'\) with \(y \in \Delta''\), consider a subsegment \([x, \tilde{y}] \subset [x, y]\) where \(\tilde{y}\) is the midpoint of \([x, y]\). Denote by \(\hat{\Delta} \subset \Delta\) the polyhedron formed by all such segments \([x, \tilde{y}]\).

Now define \(U_\omega\) as the union of all the polyhedra \(\hat{\Delta}\) as above. This choice of \(U_\omega\) implies that \(U_\omega \setminus \partial T(v, \omega) \approx \partial U_\omega \times [0, 1]\), where the boundary \(\partial U_\omega\) is a codimension one PL-submanifold of \(T(v, \omega)\). Indeed, for each simplex \(\Delta\) as above, consider a standard vector field \(w_\Delta\) consistent with the join structure \(\text{join}(\Delta', \Delta'')\) in \(\Delta\). It points from \(\Delta''\) towards \(\Delta'\), vanishes only on \(\Delta' \cup \Delta''\), and is transversal to the face \(\delta \hat{\Delta} := \hat{\Delta} \cap (\Delta \setminus \hat{\Delta}) \subset \partial U_\omega\). In the barycentric coordinates \(\beta_0, \beta_1\) on the segment \([x, y]\), the magnitude of \(w_\Delta\) is prescribed by the function \((1 - \beta_0)(1 - \beta_1)\).

This local construction of \(w_\Delta\) gives rise to a global PL-vector field \(w\) in \(U_\omega\). In particular, \(w\) is inward transversal to \(\partial U_\omega\), vanishes on \(\partial T(v, \omega)\), and the \(w\)-trajectories \(\{\eta_x(t)\}_x\) define a PL-retraction \(p_\omega : U_\omega \to \partial T(v, \omega)\) via a formula \(p_\omega(x) = \lim_{t \to \infty} \eta_x(t)\).

The properties of \(w\) helps to identify \(U_\omega\) with the mapping cylinder of

\[p_\omega : \partial U_\omega \to T(v, \omega_\infty).\]

\(\square\)

**Remark 4.3.** Recall that, by Whitehead’s Theorem \([\text{Wh}]\), any smooth manifold admits a unique PD-structure (consistent with its differentiable structure). Therefore, different \(C_v\)-invariant smooth triangulations \(\{T^0\}\) of \(\partial_1 X\) all are PD-equivalent, but perhaps not as \(C_v\)-invariant triangulations! In other words, a common refinement of two \(C_v\)-invariant differentiable triangulations of \(\partial_1 X\) may be not \(C_v\)-invariant.

We conjecture that, in fact, any two smooth \(C_v\)-invariant triangulations have a \(C_v\)-invariant smooth refinement. In other words, the trajectory space \(T(v)\) admits a unique PD-structure that is consistent (via the pull-back \(\Gamma^{-1}\)) with the preferred PD-structure on the smooth manifold \(\partial_1 X\).

\(\square\)

**Remark 4.4.** For traversally generic fields, not any homeomorphism \(\Phi^T : T(v_1) \to T(v_2)\), which preserves the \(\Omega^*\)-stratification of the trajectory spaces, lifts to a homeomorphism \(\Phi : X_1 \to X_2\) which maps the \(v_1\)-trajectories to the \(v_2\)-trajectories so that their field-induced orientations are preserved. Therefore, using the homeomorphisms \(\Phi^0 : \partial_1 X_1 \to \partial_1 X_2\) which commute with the causality maps is essential for the reconstruction of \(\Phi : X_1 \to X_2\!\).!

For example, consider a surface \(X\) with boundary which admits generic Morse data \((f, v)\) with a single \(v\)-trajectory \(\gamma\) of the combinatorial type \(\omega = (121)\). Then in the vicinity of \(\gamma\), the space \(T(v)\) is a graph shaped as the letter \(Y\). The function \(f\) breaks the six-fold symmetry of the graph \(T(v)\). With the help of \(f\), the three edges of the graph can be labeled by the three distinct elements \(\rho_1, \rho_2, \rho_3 \in Mor(\{(11), (121)\})\). By definition, \(\rho_1\) sends the first 1 in \((11)\) to the first 1 in \((121)\), and the second one to 2; \(\rho_2\) sends the first 1 in \((11)\) to 2, and the second one to the third 1 in \((121)\); finally, \(\rho_3\) sends the first 1 in \((11)\) to the first 1 in \((121)\), and the second 1 to the third 1 in \((121)\).

Consider the automorphism \(\Phi^T : T(v) \to T(v)\) which switches the two edges that are labeled by \(\rho_1\) and \(\rho_2\) and keeps the edge labeled by \(\rho_3\) fixed. Evidently, \(\Phi^T\) preserves the
\[ \Omega^\ast\text{-stratification of } T(v). \] We leave to the reader to verify that \( \Phi^T \) cannot be lifted to a homeomorphism \( \Phi : X \to X \) which preserves the \( v \)-induced orientations of the trajectories (take a glance at Fig. 5).

We are now moving toward proving Theorem 4.2, the smooth analogue of Theorem 4.1, but first we need to establish a lemma below.

Recall again that a function \( f \) on a closed subset \( Y \) of a smooth manifold \( X \) is called smooth if it is the restriction of a smooth function, defined in an open neighborhood of \( Y \).

Let \( v \) be a traversing field on a compact manifold \( X \), and \( A \supset B \) two closed subsets of \( \partial_1X \). We denote by \( X(v,A) \) and \( X(v,B) \) the unions of \( v \)-trajectories that pass through points of \( A \) and \( B \), respectively.

**Lemma 4.3.** Let \( v \) be a traversally generic field\(^ {15} \) on a compact manifold \( X \) and \( A \subset B \subset \partial_1X \) closed subsets. Consider a smooth function \( f : B \cup X(v,A) \to \mathbb{R} \) such that \( f(x) < f(x') \) for any two points \( x \neq x' \) on the same trajectory such that \( x' \) can be reached from \( x \) by moving along the trajectory in the direction of \( v \).

Then \( f \) extends to a smooth function \( F : X(v,B) \to \mathbb{R} \) such that \( L_v(F) > 0 \) on \( X(v,B) \).

**Proof.** The argument is an induction by the increasing combinatorial types (from the universal poset \( \Omega^\ast \)) of the \( v \)-trajectories that pass through points of the set \( B \setminus A \). With \( B \) being fixed, we intend to increase gradually “the size” of \( A \subset B \), until it coincides with \( B \).

Let \( \Theta \subset \Omega^\ast \) be a poset, formed by the combinatorial types of trajectories from \( X(v,B) \). Since \( B \) is closed, the poset \( \Theta \) is closed in \( \Omega^\ast \).

Consider trajectories through the points of \( B \setminus A \) and their combinatorial types, which reside in the finite poset \( \Theta \). Among these types, we pick a *minimal* element \( \omega \in \Theta \). Evidently, if no such \( \omega \) exists, then \( A = B \) and we are done.

Denote by \( Z_\omega \) the subset of \( X(v,B) \) that is formed by the trajectories of the type \( \omega \). Let \( Z^0_\omega := Z_\omega \cap \partial_1X \).

By the choice of \( \omega \), the trajectories that are limits of trajectories from \( Z_\omega \) are contained in \( X(v,A) \).

We are going to show that the given smooth function \( f : Z^0_\omega \cup X(v,A) \to \mathbb{R} \), with the properties as in the lemma, extends to smooth function \( F : Z_\omega \cup X(v,A) \to \mathbb{R} \) so that \( L_v(F) > 0 \). Indeed, for each trajectory \( \gamma \subset Z_\omega \), there is a smooth strictly monotone function \( F_\gamma : \gamma \to \mathbb{R} \) that takes the given increasing values of the function \( f| : \gamma \cap Z^0_\omega \to \mathbb{R} \) (see Fig. 5, diagram B). The family \( \mathcal{F}_\omega(\gamma,f) \) of such functions \( F_\gamma \) forms a convex set in the space \( C^\infty(\gamma) \) of all smooth functions on \( \gamma \). In particular, \( \mathcal{F}_\omega(\gamma,f) \) is a contractible space.

\(^{15}\) Probably, the lemma is valid for any field \( v \in \mathcal{V}_{\text{trav}}(X) \cap \mathcal{V}(X) \).
In fact, we get a fibration

$$\mathcal{F}_\omega(f) := \bigcup_{\gamma \in Z_\omega} \mathcal{F}_\omega(\gamma, f) \longrightarrow \mathcal{T}_\omega$$

, where $\mathcal{T}_\omega := \mathcal{T}(v, \omega)$ denotes the $\Gamma$-image of $Z_\omega$ in the trajectory space $\mathcal{T}(v)$. The local triviality of the fibration $\mathcal{F}_\omega(f) \to \mathcal{T}_\omega$ can be validated using the special coordinates $(u, x, y)$ as in (2.7) in the vicinity of a given trajectory $\gamma_0 = \{x = 0, y = 0\}$ of the combinatorial type $\omega$. In these special coordinates, any other trajectory of the type $\omega$ is given by the equations $\{x = 0, y = \text{const}\}$. In fact, we can employ only the coordinate $u$ to transfer smooth functions on $\gamma$’s to smooth functions on $\mathbb{R}$ and back. So, with the help of $u$, we establish a homeomorphism between the fibers $\mathcal{F}_\omega(\gamma, f)$ and $\mathcal{F}_\omega(\gamma_0, f)$, where the point $\gamma \in \mathcal{T}_\omega$ is in vicinity of $\gamma_0$. In fact, in the coordinates $(u, x, y)$, that homeomorphism is the identity.

We denote by $\mathcal{T}_\omega(A)$ the $\Gamma$-image of $Z_\omega \cap X(v, A)$ in the trajectory space $\mathcal{T}(v)$. It is important to notice that $\mathcal{T}_\omega(A)$ is a closed subset in $\mathcal{T}_\omega$, which is a smooth manifold. This follows from the fact that $X(v, B)$ is a closed subset of $X$ and $\omega$ is a minimal combinatorial type of trajectories from $X(v, B \setminus A)$.

For the trajectories $\gamma$ from the set $X(v, A) \cap Z_\omega$, the given function $f : B \cup X(v, A) \to \mathbb{R}$ delivers a continuous section $\sigma_A : \mathcal{T}_\omega(A) \to \mathcal{F}_\omega(f)$ of the fibration $\mathcal{F}_\omega(f)$. Since the fibers $\mathcal{F}_\omega(\gamma, f)$ of $\mathcal{F}_\omega(f) \to \mathcal{T}_\omega$ are contractible and $\mathcal{T}_\omega(A)$ is closed in $\mathcal{T}_\omega$, $\sigma_A$ extends to a continuous section $\sigma_B$ of the fibration $\mathcal{F}_\omega(f) \to \mathcal{T}_\omega$. This claim can be validated by the following standard partition of unity argument. The section $\sigma_A$ extends in an open neighborhood $U_A$ of $\mathcal{T}_\omega(A)$ in $\mathcal{T}_\omega$ so that $L_v(\sigma_A(\gamma)) > 0$ for all $\gamma \in U_A$. We denote this extension by $\tilde{\sigma}_A$. Consider a partition of unity $\{\phi_A, \phi_B\}$ subordinate to the open cover $\{U_A, \mathcal{T}_\omega \setminus \mathcal{T}_\omega(A)\}$ of $\mathcal{T}_\omega$.

Let $\sigma_B^2 : \mathcal{T}_\omega \setminus \mathcal{T}_\omega(A) \to \mathcal{F}_\omega(f)$ be a continuous section, produced by the construction as in Fig. 5, diagram B. This interpolating construction is based on a standard block-function $\varphi_{a,b} : [0, a] \to [0, b]$ that smoothly depends on the two non-negative parameters $a, b$. The infinite jet of $\varphi_{a,b}$ at 0 coincides with the jet of the function $x$, the infinite jet of $\varphi_{a,b}$ at $a$ coincides with the jet of the function $x + b$, and $\frac{d}{dx} \varphi_{a,b}(x) > 0$ in the interval $[0, a]$. Note that in Fig. 5, diagram B, we show the four-points interpolation $\phi_{x,y}$ that uses three block-functions of the type $\varphi_{a,b}$.

Now put $\sigma_B := \phi_A \cdot \bar{\sigma}_A + \phi_B \cdot \sigma_B^2$. The section $\sigma_B$ admits an approximation $\hat{\sigma}_B : \mathcal{T}_\omega \to \mathcal{F}_\omega(f)$ which depends smoothly on $\gamma_x \subset Z_\omega$ (or rather on $x$) and still has the property $L_v(F_{\gamma_x}) > 0$ on $Z_\omega$. With the help of $\hat{\sigma}_B$, we managed to produce a smooth function $F : Z_\omega \cup X(v, A) \to \mathbb{R}$, subject to the condition $L_v(F_{\gamma}) > 0$ on $Z_\omega \cup X(v, A)$.

Finally we form the closed set $A' := A \cup Z_\omega^2 \subset \partial_1 X$ and apply the previous arguments to the new pair $B \supset A'$. This completes the inductive step.

Eventually, via a sequence of the steps “$(A, B) \Rightarrow (A', B)$” we will arrive to the pair $(B, B)$. □

By letting $A := \emptyset$ and $B := \partial_1 X$ in Lemma 4.3 we get its instant implication:
Corollary 4.1. Let $v$ be a traversally generic field on a compact manifold $X$. Consider a smooth function $f : \partial_1 X \to \mathbb{R}$ such that $f(x) < f(x')$ for any two points $x \neq x'$ on the same trajectory such that $x'$ can be reached from $x$ by moving in the direction of $v$.

Then $f$ extends to a smooth function $F : X \to \mathbb{R}$ such that $\mathcal{L}_v(F) > 0$ on $X$. \hfill $\square$

We are now in position to prove one of the main results of this paper:

Theorem 4.2. (The Smooth Holography Theorem)

Let $X_1, X_2$ be two smooth compact $(n + 1)$-manifolds with boundary, equipped with traversally generic fields $v_1, v_2$, respectively.

Then any smooth diffeomorphism $\Phi^\partial : \partial_1 X_1 \to \partial_1 X_2$, such that $\Phi^\partial \circ C_{v_1} = C_{v_2} \circ \Phi^\partial$, extends to a diffeomorphism $\Phi : X_1 \to X_2$ which maps $v_1$-trajectories to $v_2$-trajectories so that the field-induced orientations of trajectories are preserved.

Proof. The argument is a modification of the Theorem 4.1 proof, but is somewhat simpler due to a better control provided by the diffeomorphism $\Phi^\partial$. As in that proof, the combinatorial type $\omega(\gamma)$ of each $v_i$-trajectory $\gamma \subset X_i$ can be recovered from the causality map $C_{v_i} : \partial_1^\gamma X_i \to \partial_1 X_i$, where $i = 1, 2$. As a result, the information encoded in $C_{v_i}$ is sufficient for a reconstruction of the $\Omega^\bullet$-stratified space $\mathcal{T}(v_i)$, the target space of the finitely ramified map $\Gamma_i : \partial_1^\gamma X_i \to \mathcal{T}(v_i)$, determined by $C_{v_i}$. Therefore, any diffeomorphism $\Phi^\partial : \partial_1 X_1 \to \partial_1 X_2$ which commutes with the causality maps gives rise to a homeomorphism $\Phi^\partial : \mathcal{T}(v_1) \to \mathcal{T}(v_2)$ which preserves the $\Omega^\bullet$-stratifications of the two spaces.

Since the stratifications $\{\partial_1 X_i(v_i)\}_j$ can be recovered from the causality maps $C_{v_i}$ and the diffeomorphism $\Phi^\partial$ commutes with the causality maps, we get that $\Phi^\partial(\partial_j X_1(v_i)) = \partial_j X_2(v_2)$ for all $j > 0$.

We are moving now towards lifting $\Phi^\partial$ to a desired diffeomorphism $\Phi : X_1 \to X_2$.

Since $v_2 \neq 0$ is a gradient-like field, there exists a smooth function $f_2 : X_2 \to \mathbb{R}$ such that $\mathcal{L}_{v_2}(f_2) > 0$ everywhere in $X_2$. We use $f_2$ to form the auxiliary function $f_1^\partial := (\Phi^\partial)^*(f_2) : \partial_1 X_1 \to \mathbb{R}$.

Consider a finite open and $v_1$-adjusted cover $\{U_\beta\}$ of $X_1$, where each $U_\beta$, with the help of special coordinates $X_\beta := (u_\beta, x_\beta, y_\beta)$, is diffeomorphic to the semi-algebraic set

$$Z_\beta := \{P_\beta(u, x_\beta) \leq 0, \|x_\beta\| \leq \epsilon, \|y_\beta\| \leq \epsilon'\}.$$

Here the polynomial $P_\beta$ is as in as in (2.4). This $v_1$-adjusted cover $\{U_\beta\}$ gives rise to a finite open cover $\{V_\beta\}$ of the space $\mathcal{T}(v_1)$.

Let $\{\psi_\beta\}$ be a partition of unity associated with $\{V_\beta\}$, such that each function $\phi_\beta := (\Gamma_1)^*(\psi_\beta) \in C^\infty(X_1)$. Such partition of unity can be constructed by employing smooth nonnegative functions $\chi_\beta(x_\beta, y_\beta)$ on the polydisk $\Pi_\beta := \{\|x_\beta\| \leq \epsilon, \|y_\beta\| \leq \epsilon'\}$ with the support in its interior: just consider the restriction of $\chi_\beta$ to $Z_\beta$ and pull it back to $U_\beta$.

\footnote{In contrast, in Theorem 4.1 \(\Phi^\partial\) is just a PD-map.}
With the help of coordinates $X_β$, we transfer the smooth function $f_1^Φ : \partial_1 X \cap U_β \to \mathbb{R}$ to a smooth function $q_β : Z_β \to \mathbb{R}$ with the property

$$q_β(u', x', y') < q_β(u'', x', y')$$

for all $x', y'$ and $u'' > u'$ so that $(u', x', y')$ and $(u'', x', y')$ belong to the same connected component of the set $Z_β \cap \{x = x', y = y'\}$.

By Lemma 4.3, there exists a smooth function $Q_β : Z_β \to \mathbb{R}$ that extends $q_β$ and has the property $L_u(Q_β) > 0$ in $Z_β$. Its pull-back $f_β : U_β \to \mathbb{R}$ under the coordinate map $X_β$ has the property $L_υ(f_β) > 0$ in $U_β$. Moreover, by the construction, $(f_β)|_\partial_1 X_1 = f_1^Φ$.

Since $L_υ(φ_β) = 0$ and $φ_β ≥ 0$, by the construction of $φ_β$, we get that $L_υ(φ_β : f_β) > 0$ in $\text{int}(U_β)$. Now we form the smooth function

$$f_1 := \sum_β φ_β \cdot f_β.$$

Since $\sum_β φ_β = 1$ and $(f_β)|_\partial_1 X_1 = f_1^Φ$, the function $f_1 : X_1 \to \mathbb{R}$ extends the function $f_1^Φ : \partial_1 X_1 \to \mathbb{R}$ and has the property $L_υ(f_1) > 0$ in $X_1$.

With the gradient-like function $f_1 : X_1 \to \mathbb{R}$ for $υ$ in place, we define the diffeomorphism $φ_1^{12}$ that takes a typical $υ$-trajectory $γ \subset X_1$ to the $υ$-trajectory $γ' \subset X_2$ that projects, with the help of $Γ_2$, to the point $Φ^T(γ) \in T(υ_2)$. It is given by the formula

$$φ_1^{12} := (f_2|_Φ^T(γ))^{-1} \circ (f_1|_γ).$$

In this formula, as before, we abuse notations: $γ$ stands for both a $υ$-trajectory in $X$ and for the corresponding point in the trajectory space $T(υ)$.

Now the desired 1-to-1 map $Φ : X_1 \to X_2$ is introduced by the formula $Φ(x) := x'$, where $x'$ belongs to the trajectory over the point $Φ^T(γ_x) \in T(υ_2)$ and is such that $φ_1^{12}(x) = x'$. By the very construction of the function $f_1 : X_1 \to \mathbb{R}$, we get $Φ|_\partial_1 X_1 = Φ^Φ$.

We claim that $Φ$ is a diffeomorphism. Indeed, the trajectory $γ_x$ and the point $φ_1^{12}(x) ∈ γ'$ depend smoothly on $x$. Similar arguments apply to the inverse map $Φ^{-1}$.

**Remark 4.5.** Let $υ$ be a traversally generic field on $X$. Among other things, Theorem 4.2 claims that any diffeomorphism of the boundary $\partial_1 X$, which commutes with the (partially defined) causality map $C_υ$, extends to a diffeomorphism of $X$! □

**Corollary 4.2.** Let $X_1, X_2$ be two smooth compact $(n + 1)$-manifolds with boundary, equipped with traversally generic fields $υ_1, υ_2$, respectively. Then any diffeomorphism $Φ^Φ : \partial_1 X_1 \to \partial_1 X_2$ such that

$$Φ^Φ \circ C_υ = C_υ \circ Φ^Φ$$

generates a stratification-preserving homeomorphism $Φ^T : T(υ_1) \to T(υ_2)$ of the corresponding $Ω^*$-stratified trajectory spaces. In turn, $Φ^T$ induces an isomorphism

$$(Φ^T)^* : C^∞(T(υ_2)) \to C^∞(T(υ_2))$$

of the algebras of smooth functions on the two trajectory spaces—the two spaces are “diffeomorphic”. 
Proof. By Theorem 4.2, there exists a diffeomorphism \( \Phi : X_1 \to X_2 \) which takes \( v_1 \)-trajectories to \( v_2 \)-trajectories and extends \( \Phi^\partial \). Therefore, \( \Phi \) maps every smooth function \( f : X_2 \to \mathbb{R} \) that is constant on each \( v_2 \)-trajectory to a smooth function \( f \circ \Phi : X_1 \to \mathbb{R} \) that is constant on each \( v_1 \)-trajectory.

Similar argument applies to the inverse diffeomorphism \( (\Phi)^{-1} : X_2 \to X_1 \). □

For a traversing field \( v \), we denote by \( F(v) \) the 1-dimensional oriented foliation determined by \( v \).

Theorem 4.2 has another “holographic” implication:

**Corollary 4.3.** For a traversally generic field \( v \), the smooth type of the pair \((X, F(v))\) can be recovered from each of the following structures on its boundary \( \partial_1 X \):

- the causality map \( C_v : \partial_1^+ X \to \partial_1^- X \),
- the poset \( C^\partial(v) \) (whose elements are the points of \( \partial_1 X \)),
- the category \( \text{Cat}^\partial(v) \) (determined by \( C^\partial(v) \)).

In particular, all the invariants of the smooth structure on \( X \) can be recovered from each of the three previous structures on \( \partial_1 X \).

**Proof.** Consider two manifolds \( X_1 \) and \( X_2 \) which carry traversally generic fields \( v_1 \) and \( v_2 \). Assume that the manifolds share a common boundary: this means that there is a diffeomorphism \( \Phi^\partial : \partial_1^+ X_1 \to \partial_1^- X_2 \). If the two fields induce isomorphic causality maps, then by definition the diffeomorphism \( \Phi^\partial : \partial_1^+ X_1 \to \partial_1^- X_2 \) commutes with the causality maps \( C_{v_1} \) and \( C_{v_2} \).

According to Theorem 4.2, the diffeomorphism \( \Phi^\partial : \partial_1^+ X_1 \to \partial_1^- X_2 \) extends to a diffeomorphism \( \Phi : X_1 \to X_2 \) so that the \( v_1 \)-trajectories are mapped to the \( v_2 \)-trajectories.

The equivalence of the three structures in the statement of the corollary has been established in the discussion that has followed formula (4.1). □

**Example 4.1.** The statement of Corollary 4.3 is not obvious even for the nonsingular gradient flows on 2-dimensional manifolds. Consider a compact surface \( X \) with a connected boundary \( \partial_1 X \approx S^1 \) and a traversally generic field \( v \) on \( X \). Then \( \partial_1^+ X \) is a disjoint union of \( q \) arcs in \( S^1 \). Then the set \( \partial_1^- X \) is a disjoint union of \( q \) arcs as well.

The causality map \( C_v : \partial_1^+ X \to \partial_1^- X \) can be represented by a graph \( G_v \subset \partial_1^+ X \times \partial_1^- X \), drawn in a set of \( q \times q \) of black unitary squares of the \( 2q \times 2q \) checker board, the sums of indexes of each square in the \( 2q \times 2q \) table being odd. The graph \( G_v \) has a finite number of discontinuity points with the well-defined left and right limits for each arc of \( G_v \). The interior of each arc of \( G_v \) is smooth. (It looks that the total variation of the function \( C_v \) in this model is \( q \).

According to Corollary 4.3 this graph \( G_v \) “knows” everything about the topology of \( X \) and the dynamics of the \( v \)-flow on it, up to a diffeomorphism of \( X \)! Even the claim about the topological type of \( X \) has some subtlety: according to the Morse formula for vector fields \([\text{Mo}]\), to calculate \( \chi(X) \), and thus to determine the topological type of \( X \), we need to

\[ 17 \text{such as the stable characteristic classes of the tangent bundle } \tau(X) \]
know not only \( \chi(\partial^+_2 X) = q \) (which we obviously do), but also the integer \( \chi(\partial^+_2 X) \), which can be extracted by iterating the map \( C_v \). This presumes that the polarity of each of the 2q points from \( \partial_2 X \) can be recovered from \( C_v \) or \( G_v \). We leave to the reader to discover the recipe.

**Example 4.2.** All the Gauge invariants of compact smooth 4-manifolds \( X \) with boundary can be recovered from the causality map \( C_v : \partial^+_2 X \to \partial^-_1 X \) of any traversally generic field \( v \) on \( X \). As a practical matter, this recovery must be very challenging...

**Example 4.3.** Take any compact smooth manifold \( X \) with a spherical boundary \( \partial_1 X = S^n \). By Theorem 3.1 from [K1] and Theorem 3.5 from [K2], there is an open set \( D(X) \) of traversally generic fields \( v \), such that \( \partial^+_1 X \) is a ball \( D^n_+ \subset S^n \). Then \( \partial^-_1 X \) is the complimentary ball \( D^n_- \). According to Corollary 4.3, for any \( v \in D(X) \), the smooth type of the manifold \( X \) is determined by the semi-continuous causality map \( C_v : D^n_+ \to D^n_- \), (equivalently, by its graph \( \Gamma(C_v) \subset D^n_+ \times D^n_- \)). Therefore we conclude:

*Compact smooth manifolds with spherical boundary can be faithfully represented by special semi-continuous maps of \( n \)-balls.*

Compare this description of \( X \) with the description of the trajectory space \( T(v) \), given by the Origami Theorem [3,1].

**Example 4.4.** Consider a flow of liquid trough a given volume \( X \) with a smooth boundary. We assume that the pressure gradient \( v \) does not vanish in \( X \). We think about \( \partial X \) as the hypersurface where a multitude of measuring devices are positioned. The basic assumption is that their presence and measuring activity does not alter the flow. Any particle which enters the volume is registered and its next appearance at a point of \( \partial X \) is registered as well. According to the Holography Theorem 4.2, these data allow for a reconstruction of the volume \( X \) and the dynamics of the flow in it up to a diffeomorphism of \( X \) which is identity on its boundary.

Now consider any time-dependent vector field \( u(t), t \in \mathbb{R} \), on a \( n \)-dimensional manifold \( Y \) without boundary. Then \( u(t) \) gives rise to a non-vanishing vector field \( v := (u(t), 1) \) on the manifold \( Y \times \mathbb{R} \). Note that \( v \) is a gradient-like field with respect to the function \( T(y,t) := t \) on \( Y \times \mathbb{R} \).

Let \( X \subset Y \times \mathbb{R} \) be a 0-dimensional compact smooth submanifold. Since the field \( v \) is of gradient type with respect to the function \( T \) and \( X \) is compact, any \( v \)-trajectory \( \gamma(t) \) that passes through a point of \( X \) is contained in \( X \) for a compact set of instances \( t \in \mathbb{R} \).

Assume that \( X \subset Y \times \mathbb{R} \) is such that \( v \) is traversally generic with respect to \( \partial X \). In view of Theorem 3.5 from [K2], this assumption can be satisfied by a small perturbation \( \tilde{v} \) of \( v \). In fact, such perturbation \( \tilde{v} \) can be of the form \((\hat{u}(t,y),1)\) since the property of a field to be traversally generic depends only on its direction, and not on its magnitude.

Let us call \( X \) the “event manifold” and its boundary \( \partial X \) the “event horizon”. Note that the event manifold is chosen as independent set of data, not directly related to the time-dependent dynamic system \( u(t) \) on the manifold \( Y \).
Thus \( u(t) \) defines the causality map \( C_v : \partial^+_v X(v) \to \partial^-_v X(v) \) which takes each “entrance” point \( x_0 = (t_0, y_0) \) on the event horizon \( \partial X \) to the closest along the \( v \)-trajectory trough \( x_0 \) “exit” point \( x_1 = (t_1, y_1) \) on \( \partial X \).

We can think of the event \( x_0 \) as the cause of the event \( x_1 \), so that \( C_v \) indeed becomes the causality map or the causality relation on the horizon \( \partial X \).

The Holography Theorem 4.2 and Corollary 4.3 have the following pivotal interpretation:

**Theorem 4.3. (The Holographic Causality Principle)**

Let \( u(t), t \in \mathbb{R} \), be a time-dependent smooth vector field on a \( n \)-dimensional manifold \( Y \) without boundary.

For any compact \((n + 1)\)-dimensional smooth event manifold \( X \subset Y \times \mathbb{R} \), such that the field \( v = (u, 1) \) is traversally generic on \( X \), the causality relation on the event horizon \( \partial X \) determines the pair \((X, C(v))\), up to a smooth diffeomorphism of \( X \) which is the identity on the event horizon.

**Remark 4.6.** We do not claim that the reconstruction of the event manifold \( X \) from the causality map allows for the reconstruction of its slicing by the fixed-time frames!

In turn, Theorem 4.3 has the following interpretation:

**Corollary 4.4. (The topological rigidity of continuation for ODE’s)**

Let \( Y \) be a smooth \( n \)-manifold with no boundary and \( X \subset Y \times \mathbb{R} \) a compact submanifold of dimension \( n + 1 \). Let \( u_1(t), u_2(t), t \in \mathbb{R} \), be two time-dependent smooth vector fields on \( Y \) such that \( u_1(y, t) = u_2(y, t) \) for all events \((y, t) \in (Y \times \mathbb{R}) \setminus X\). Assume that the fields \( v_1 := (u_1, 1) \) and \( v_2 := (u_2, 1) \) are traversally generic on \( X \) and that the two causality maps, \( C_{v_1} : \partial^+_v X \to \partial^-_v X \) and \( C_{v_2} : \partial^+_v X \to \partial^-_v X \), are identical.

Then the two dynamical systems, generated by \( v_1 \) and \( v_2 \) on \( Y \times \mathbb{R} \), are topologically equivalent via a \( C^\infty \)-diffeomorphism which is an identity on the event horizon.

In search for further applications of Holography Theorem 4.2, let us briefly visit the Classical Hamiltonian/Lagrangian Mechanics.

Consider a mechanical system whose dynamics, in the local coordinates \((q, p)\), is described by the system of ODE’s:

\[
\begin{align*}
\dot{q} &= \frac{\partial L}{\partial p} = \frac{\partial H}{\partial p} \\
\dot{p} &= \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}
\end{align*}
\]

(4.6)

Here \( q \) denotes some local coordinates on a configuration space \( M \), and \((q, p)\) are local coordinates in the phase space \( TM \)—the tangent space of the smooth manifold \( M \). The function \( H = H(q, p, t) \) is called the (time-dependent) Hamiltonian, and the function \( L = L(q, p, t) \) is called the (time-dependent) Lagrangian of the system.

Consider the globally-defined Poincaré-Cartan 1-form \( \alpha := pdq - H dt \) on the manifold \( TM \times \mathbb{R} \). Its differential \( d\alpha \) is a nonsingular 2-form on \( TM \times \mathbb{R} \) of the rank \( 2 \text{dim}(M) \). The kernel of \( d\alpha \) is generated by the vector field \( v = (\dot{q}, \dot{p}, 1) \), defined by (4.6).
The fundamental relation between all these objects is described by the formula
\[ v \| d\alpha = -dH. \]

By the Liouville Theorem, the symplectic 2-form \( \omega := d\alpha \mid_{TM \times \{t\}} = dp \wedge dq \) is invariant under the flow \( v = (\dot{q}, \dot{p}, 1) \), defined by (4.6).

Applying Theorem 4.2 to the Hamiltonian system (4.6), we get:

**Corollary 4.5.** Consider the Hamiltonian dynamical system (4.6). Assume that, for some \( c \in \mathbb{R} \),
- the set \( X := \{(q,p,t) \mid L(q,p,t) \leq c\} \) is compact in \( TM \times \mathbb{R} \),
- \( c \) is a regular value of \( L : TM \times \mathbb{R} \to \mathbb{R} \),
- the field \( v := (\dot{q}, \dot{p}, 1) = (\frac{\partial L}{\partial p}, \frac{\partial L}{\partial q}, 1) \) is traversally generic with respect to \( \partial X \).

Then the causality map \( C_v \) on the event horizon \( \partial X := \{(q,p,t) \mid L(q,p,t) = c\} \) allows for a reconstruction of the pair \( (X, \mathcal{F}(v)) \), up to a diffeomorphism of \( X \) which is the identity on \( \partial X \). \( \square \)

**Question 4.2.** The main unresolved issue here is: “How abundant are the Hamiltonian systems that are traversally generic with respect to a given event horizon?”

We know that any gradient-like field \( v \neq 0 \) can be approximated by a traversally generic field (Theorem 3.5 from [K2]). So the open question is whether such an approximation is possible within the universe of Hamiltonian fields. \( \square \)

### 5. The Geodesic Scattering and Holography

In this section, we will apply the Holographic Causality Principle to geodesic flows on the spaces \( SM \) of unit tangent vectors on compact Riemannian manifolds \( M \) with boundary.

Let \( M \) be a compact \( k \)-dimensional smooth Riemannian manifold with boundary, and \( g \) a smooth Riemannian metric on \( M \). Let \( SM \to M \) denote the spherical bundle formed by tangent unitary vectors. The bundle \( SM \) is a subbundle of the tangent bundle \( TM \).

For simplicity, the notation “\( SM \)” does not reflect the dependence of \( SM \subset TM \) on \( g \); of course, the smooth topological type of the fibration \( SM \to M \) does not depend on \( g \).

The metric \( g \) on \( M \) induces a geodesic flow \( \{\Psi^g_t : SM \to SM\}_{t \in \mathbb{R}} \). Let us recall its construction. Each tangent vector \( u \) at a point \( m \in M \setminus \partial M \) determines a unique geodesic curve \( \gamma_{(m,u)} \subset M \) throug \( m \) in the direction of \( u \). When \( m \in \partial M \), the geodesic curve \( \gamma_{(m,u)} \) is well-defined for unit vectors \( u \in T_mM \) that point inside of \( M \).

By definition, \( \Psi^g_t(m,u) \) is the point \( (m',u') \in SM \) such that the distance along \( \gamma_{(m,u)} \) from \( m' \in \gamma_{(m,u)} \) to \( m \) is \( t \), and \( u' \) is the tangent vector to \( \gamma_{(m,u)} \) at \( m' \).

We stress that \( \Psi^g_t(m,u) \) may not be well-defined for all \( t \in \mathbb{R} \) and all \( u \in TM \); some geodesic curves \( \gamma \) may reach the boundary \( \partial M \) in finite time, and some tangent vectors

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18 Note that is the gradient field \( (\frac{\partial L}{\partial q}, \frac{\partial L}{\partial p}, \frac{\partial L}{\partial t}) \) is transversal to the horizon \( \partial X \).
$u \in TM|_{\partial M}$ may point outside of $M$. However, such constraints are familiar to us: our entire enterprise deals with such boundary-induced complications.

In the local coordinates $(x^1, \ldots, x^k, p_1, \ldots p_k)$ on the cotangent space $T^*M$, the equations of the cogeodesic flow are:

$$\dot{x}^\alpha = \sum_\beta g^{\alpha\beta} p_\beta$$

(5.1)

$$\dot{p}_\alpha = -\frac{1}{2} \sum_{\beta, \gamma} \frac{\partial g^{\beta\gamma}(x)}{\partial x^\alpha} p_\beta p_\gamma$$

, where $g^{\alpha\beta}(x)$ is the inverse of the metric tensor $g_{\alpha\beta}$ (so that $\sum_\beta g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$).

This system can be rewritten in terms of the Hamiltonian function $H^g(x, p) := \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha\beta}(x) p_\alpha p_\beta$—the kinetic energy—in the familiar Hamiltonian form:

$$\dot{x}^\alpha = \frac{\partial H^g}{\partial p_\alpha}$$

$$\dot{p}_\alpha = -\frac{\partial H^g}{\partial x^\alpha}$$

(5.2)

The projections of the trajectories of (5.1) (or of (5.2)) on $M$ are the geodesic curves. Note the obvious similarity between (4.6) and (5.2).

Let $(M, g)$ be a smooth Riemannian manifold. The manifold $TM$ has a preferred metric $g_1$, produced by $g$. Using the $g$-induced connection

$$K_g : T(TM) \to TM$$

, we can split $T(TM)$ into two complementary subbundles, the horizontal $H(TM)$—the kernel of $K_g$— and the vertical $V(TM)$. The connection defines a bundle isomorphism

$$H(TM) \oplus V(TM) \approx TM \oplus TM$$

(see [Be] for the details). The derived metric $g_1$ on $T(TM)$ is introduced so that this isomorphism is an isometry between $(T(TM), g_1)$ and $(TM \oplus TM, g \oplus g)$.

The metric $g_1$ has a number of nice properties. For example, in the metric $g_1$, the fibers of the bundle $T(TM) \to TM$ are the geodesic submanifolds ([Be], Prop. 1.102).

Let $v^g \in T(TM)$ be the field on the manifold $TM$, tangent to the trajectories of the geodesic flow $\Psi^g$ on $TM$. Then the integral trajectories of $v^g$ are geodesic curves in $g_1$-metric on $T(TM)$ ([Be], Prop. 1.106).

The metric $g_1$ induces a metric on $SM \subset TM$ which we also denote “$g_1$”.

This construction $(M, g) \Rightarrow (SM, g_1)$ can be iterated so that we get a tower of compact Riemannian manifolds:

$$(M, g) \Rightarrow (SM, g_1) \Rightarrow (S(SM), g_2) \Rightarrow \ldots$$
, where \( g_2 := (g_1)_1 \), etc., and each manifold \( S(\ldots (SM) \ldots) \) being equipped with a geodesic flow.

As any vector field, the geodesic field \( v^g \) divides the boundary \( \partial(SM) \) into two portions: \( \partial^+(SM) \) where \( v^g \) points inside of \( SM \), and \( \partial^-(SM) \) where it points outside of \( SM \).

**Definition 5.1.** Let \( (M, g) \) be a compact Riemannian manifold with boundary.

We say that a metric \( g \) on \( M \) is of the gradient type if the vector field \( v^g \in T(SM) \) that governs the geodesic flow is gradient-like: i.e., there exists a smooth function \( F : SM \to \mathbb{R} \) such that \( df(v^g) > 0 \).

**Remark 5.1.** The property of a metric \( g \) being of the gradient type is an open property in the \( C^\infty \)-topology. Indeed, if \( dF(v^g) > 0 \) on \( SM \), then \( F : SM \to \mathbb{R} \) extends in a compact neighborhood \( U \) of \( SM \) in \( TM \setminus M \) to a smooth function \( \tilde{F} : U \to \mathbb{R} \) so that \( d\tilde{F}(v^g) > 0 \) in \( U \). In this neighborhood, \( d\tilde{F}(v^g')|_U > 0 \) for all metrics \( g' \), sufficiently close to \( g \). For such metrics \( g' \), the space of unit spheres \( S'M \subset TM \) is fiber-wise close to \( SM \subset TM \); in particular, \( S'M \subset U \). Recall that the geodesic field \( v^{g'} \) is tangent to \( S'M \), thus \( d\tilde{F}(v^{g'}) > 0 \) on \( S'M \).

**Definition 5.2.** Let \( (M, g) \) be a compact Riemannian manifold with boundary.

- We say that a metric \( g \) on \( M \) is geodesically boundary generic if the vector field \( v^g \in T(SM) \) is boundary generic (see Definition 2.1 from [K1]) with respect to the boundary \( \partial(SM) = SM|_{\partial M} \).
- We say that a metric \( g \) on \( M \) is geodesically traversally generic if the vector field \( v^g \in T(SM) \) is of the gradient type and is traversally generic (see Definition 3.2 from [K2]) with respect to the boundary \( \partial(SM) = SM|_{\partial M} \).

We denote the space of all metrics of gradient type on \( M \) by the symbol \( G(M) \), and the space of all geodesically traversally generic metrics on \( M \) by the symbol \( GG(M) \).

**Question 5.1.** How to formulate the property of \( v^g \) being traversally generic with respect to \( \partial(SM) \) in terms of the Jacobi fields on \( M \) and their interactions with \( \partial M \)?

**Remark 5.2.** Of course, not any metric \( g \) on \( M \) is of the gradient type. Evidently, metrics that have closed geodesics cannot be of the gradient type.

At the same time, there are plenty examples of gradient-type metrics. By Remark 5.1, any metric sufficiently close to the metric from these examples is again a metric of the gradient type.

In contrast, to manufacture a geodesically traversally generic metric is a more delicate task. In fact, we know only few examples, where gradient type metrics are proven to be of the traversally generic type: they are described in Corollary 5.3 (these examples have gradient-type metrics in which the boundary \( \partial M \) is strictly convex) and Lemma 5.4 (where the double-tangent geodesics to \( \partial M \) are forbidden). However, we suspect that traversally generic metrics are abundant (see Conjecture 5.1). In any case, by Theorem 5.1 below, the property of a metric \( g \) to be traversally generic is stable under small perturbations of \( g \).
Lemma 5.1. The metric $g$ on a compact manifold $M$ is of the gradient type if and only if, for any pair $(x, w) \in \text{int}(SM)$, the geodesic curve through $x \in M$ in the direction of the vector $w$ reaches the boundary $\partial M$.

Proof. If for any pair $(x, w) \in \text{int}(SM)$, the geodesic line in $M$ through $x$ in the direction of the vector $w$ reaches the boundary $\partial M$, then any trajectory of the geodesic flow in $SM$ must reach the boundary $\partial (SM) = SM|_{\partial M}$ in finite time. Therefore, the geodesic field $v^g$ is traversing. By Lemma 4.1 from [K1], any traversing field is of the gradient type.

On the other hand, if $g$ is of gradient type, then the field $v^g$ is traversing; so any $v^g$-trajectory in $SM$ reaches the boundary $\partial (SM)$. Therefore, its projection in $M$ must reach the boundary $\partial M$. □

Corollary 5.1. For any compact domain $M \subset \mathbb{R}^n$, the flat metric $g_E$ on $M$ is of the gradient type, and so are all the metrics $g$ that are sufficiently close to $g_E$.

Proof. Thanks to the compactness of $M$, any oriented line $l$ through any point in $\text{int}(M)$ hits the boundary $\partial M$ in both directions. By Lemma 5.1, the pair $(M, g_E)$ is of the gradient type.

By Theorem 5.1, any $g$, which is sufficiently close to $g_E$, is of the gradient type. □

Given a metric $g$ on a compact manifold $M$ with boundary, we consider two sets:

1. Black $\partial^\theta(g) \subset \partial^+ \text{(SM)}(v^g)$ and White $\partial^\theta(g) \subset \partial^- \text{(SM)}(v^g)$.

By definition, a point $(m, v) \in \text{Black}^\theta(g)$ if the geodesic curve, determined by $(m, v) \in \partial^+ \text{(SM)}(v^g)$, for all the positive times, is trapped in $\text{int}(M)$; similarly, $(m, v) \in \text{White}^\theta(g)$ if the geodesic curve, determined by the point $(m, -v)$, for all the positives times, is trapped in $\text{int}(M)$. We also consider a subset $\text{Black}(g) \subset \text{int}(SM)$ comprising points $(m, v)$ such that the geodesic line through $m$ in both directions, $v$ and $-v$, does not reach the boundary $\partial M$.

Corollary 5.2. A metric $g$ on $M$ is of the gradient type, if and only if, the three sets $\text{Black}^\theta(g), \text{White}^\theta(g)$, and $\text{Black}(g)$ are empty.

Proof. Examining the definitions of the three sets, the corollary follows instantly from Lemma 5.1 □

We have only some weak evidence for the validity of following conjecture; however, the world in which it is valid seems to be a pleasing place...

Conjecture 5.1. The set $\mathcal{GG}(M)$ of geodesically traversally generic metrics is open and dense in the space $\mathcal{G}(M)$ of the gradient-type metrics. □

The openness of $\mathcal{GG}(M)$ in $\mathcal{G}(M)$ follows from the theorem below.

Theorem 5.1. Let $M$ be a compact smooth manifold with boundary.

In the space $\mathcal{R}(M)$ of all Riemannian metrics on $M$, equipped with the $C^\infty$-topology, the geodesically traversally generic metrics form an open set $\mathcal{GG}(M)$. □
If a metric $g$ on $M$ is of the gradient type, then the geodesic field $v^g$ on $SM$ can be approximated arbitrarily well in the $C^\infty$-topology by a traversally generic field $\tilde{v} \in \mathcal{V}(SM)$.

The space $\mathcal{GG}(M)$ is invariant under the natural action of the diffeomorphism group $\text{Diff}(M)$ on $\mathcal{R}(M)$.

Proof. The construction of the geodesic flow $g \Rightarrow v^g$ defines a map $\mathcal{F} : \mathcal{R}(M) \to \mathcal{V}(SM)$, where $\mathcal{V}(SM)$ denotes the space of all vector fields on $SM$. By Theorem 3.5 and Corollary 3.3 from [K2], the subspace $\mathcal{V}(SM)$, formed by traversally generic (and thus gradient-like) vector fields is open in $\mathcal{V}(SM)$. Since the geodesic $\gamma$ through a point $m \in M$ in the direction of a given tangent vector $u \neq 0$ depends smoothly on metric $g$, we conclude that $\mathcal{F}$ is a continuous map. Therefore, $\mathcal{GG}(M) := \mathcal{F}^{-1}(\mathcal{V}(SM))$ is an open set in $\mathcal{R}(M)$.

By definition, for any $g \in \mathcal{G}(M)$, the geodesic field $v^g$ on $SM$ is of the gradient type (and thus traversing). Again, by Theorem 3.5 from [K2], $v^g$ can be approximated by a traversally generic field $\tilde{v} \in \mathcal{V}(SM)$. Note that the projections of $\tilde{v}$-trajectories under the map $SM \to M$ stay $C^\infty$-close to the geodesic lines in the original metric $g$.

Nevertheless, the question whether $\mathcal{GG}(M) \neq \emptyset$ for a given $M$ remains open!

Evidently, by the “naturality” of the geodesic flow, the space $\mathcal{GG}(M)$ is invariant under the natural action of the diffeomorphism group $\text{Diff}(M)$ on $\mathcal{R}(M)$. □

Example 5.1. Consider a flat metric $g$ on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and form a punctured torus $M^2$ by removing an open disk $D^2$ from $T^2$. If $D^2$ is convex in the fundamental square domain $Q^2 \subset \mathbb{R}^2$, then there exist closed geodesics (with a rational slope with respect to the lattice $\mathbb{Z}^2$) that miss $D^2$. For such $M^2$, the flat metric is not of the gradient type.

However, it is possible to position $D^2 \subset T^2$ so that its lift $\tilde{D}^2$ to $\mathbb{R}^2$ will have intersections with any line that passes through $Q^2$ (see Fig. 7). For such a choice of $(M^2, g)$, thanks to Lemma 5.1, the metric $g$ is of the gradient type.

Moreover, by Theorem 5.1, any metric $g'$ on $M^2$, sufficiently close to this flat metric $g$, is also of the gradient type. □

Figure 7. The flat metric on torus becomes of gradient type on the complement to a curvy disk (whose “center” is at the corners of the square fundamental domain).
Remark 5.3. Let $N$ be a codimension 0 compact submanifold of a compact Riemannian manifold $M$ such that $N \subset \text{int}(M)$. If a metric $g$ on the ambient $M$ is of the gradient type, then, by Lemma 5.1, its restriction $g|_N$ is of the gradient type on $N$.

Of course, if $g$ is geodesically traversally generic on a compact manifold $M$, it may not be geodesically traversally generic on $N$.

Perhaps, for a given metric $g$ on $N$ of gradient type, there is a small isotopy of $M$ in $N$ so that $g^{\theta}$ becomes boundary generic (in the sense of Definition 5.2) with respect to the deformed $M$? This conjecture—a relative of Conjecture 5.1—at the first glance seems to be quite challenging. \[ \Box \]

Example 5.2. Consider the hyperbolic space $\mathbb{H}^n$ with its virtual spherical boundary $\partial \mathbb{H}^n$ and hyperbolic metric $g$ (so $\mathbb{H}^n$ is modeled by the open unit ball in the Euclidean space $E^n$). Each geodesic line hits $\partial \mathbb{H}^n$ at a pair of points, where it is orthogonal (in the Euclidean metric) to $\partial \mathbb{H}^n$. For each oriented geodesic line $\gamma$ through a given point $x \in \mathbb{H}^n$, consider the distance $d_{\mathbb{H}}(x, w)$ between $x$ and the unique point $y \in \gamma \cap \partial \mathbb{H}^n$ that can be reached from $x$ by moving along $\gamma$ in the direction of $-w$, that is, $d_{\mathbb{H}}(x, w)$ is the length of the circular arc $(y, x) \subset \gamma$ in $E^n$. Evidently, $d_{\mathbb{H}}(x, w)$ is strictly increasing, as one moves along the oriented $\gamma$.

Let $M \subset \mathbb{H}^n$ be a compact codimension 0 submanifold, equipped with the induced hyperbolic metric. Then the field $g^{\theta}$ of the geodesic flow $\Psi_t$ on the space $SM$ is of the gradient type since $d_{\mathbb{H}}(x, w)$ is strictly increasing along the oriented trajectories of $g^{\theta}$.

Again, by Theorem 5.1, any metric $g'$ on $M$, sufficiently close to this hyperbolic metric $g$, is also of the gradient type. \[ \Box \]

We are guided by an important observation: if a metric $g$ is of the gradient type, then the causality map

$$C_{vs}: \partial_1^+(SM) \rightarrow \partial_1^-(SM)$$

is available! This map represents a metric-induced scattering: indeed, with the help of $C_{vs}$, each unit tangent vector $u \in T^+M|_{\partial M}$ is mapped (“scattered”) to a unit tangent vector $u' \in T^-M|_{\partial M}$. Here $T^\pm M|_{\partial M} \subset TM|_{\partial M}$ denote the sets of vectors along $\partial M$ that are tangent to $M$ and point inside/outside of $M$.

We will need traversally generic metrics $g$ in order to control the local structure of the causality map $C_{vs}: \partial_1^+(SM) \rightarrow \partial_1^-(SM)$.

For each tangent vector $w \in T_xM|_{\partial M}$, consider its orthogonal decomposition $w = n \oplus u$ with respect to the metric $g$, where $n$ is the exterior normal to $\partial M$ and $u \in T_x(\partial M)$. We denote by $\tau_g(x, w)$ the point $(x, -n \oplus u)$.

Lemma 5.2. The manifolds $\partial_1^+(SM)(g^{\theta})$ and $\partial_1^-(SM)(g^{\theta})$ are diffeomorphic via the orientation-reversing involution $\tau_g: \partial_1(SM) \rightarrow \partial_1(SM)$.

Proof. Examining the constructions of the strata $\partial_1^+(SM)(g^{\theta})$ and the definition of $\tau_g$, we see that $\tau_g$ maps $\partial_1^+(SM)(g^{\theta})$ to $\partial_1^-(SM)(g^{\theta})$ by an orientation-reversing diffeomorphism. \[ \Box \]
Theorem 5.2. (The inverse problem of geodesic scattering is topologically rigid for traversally generic metrics)

Assume that \( M \) admits a geodesically traversally generic Riemannian metric \( g \).

Then the scattering (causality) map \( C_{v^g} : \partial^+_1(SM) \to \partial^-_1(SM) \) allows for a reconstruction of the pair \((SM, F(v^g))\), up to a \( C^\infty \)-diffeomorphism of \( SM \) which is the identity on \( \partial_1(SM) \).

Proof. The proof follows instantly from the Holography Theorem 4.2 and Corollary 4.3 by unraveling Definition 5.2. □

Remark 5.4. If a pair \((M, g)\) is such that the geodesic field \( v^g \) is traversing on \( SM \), then the trajectory space \( T(v^g) \) is the space of unparametrized geodesics on \( M \). It is given the quotient topology via the obvious map \( \pi : SM \to T(v^g) \). Moreover, the space of geodesics \( T(v^g) \), although not a manifold in general, inherits a surrogate smooth structure from \( SM \) (as in Definitions 2.1 and 2.2)! □

In view of Remark 5.4, Theorem 5.2 has an instant but philosophically important implication:

Theorem 5.3. Assume that \( M \) admits a geodesically traversally generic Riemannian metric \( g \). Then the scattering map \( C_{v^g} : \partial^+_1(SM) \to \partial^-_1(SM) \) allows for a reconstruction of the smooth topological type of the space \( T(v^g) \) of unparametrized geodesics on \( M \).

Proof. Just apply Theorem 5.2 and Corollary 4.2 to the field \( v^g \). □

It looks that the claim of Theorem 5.3 could be valid for any gradient type and boundary generic metric \( g \) on \( M \).

Question 5.2. Given a boundary generic metric \( g \) on \( M \), how to describe the strata \( \{\partial^j_0 SM(v^g)\}_j \) of \( \partial(SM) \) in terms of the local Riemannian geometry \(^{19}\) of the pair \((M, \partial M)\)? □

The lemma below is just a small step towards answering Question 5.2.

Lemma 5.3. For a boundary generic metric \( g \) on \( M \), the stratum \( \partial^j_0 SM(v^g) \) consists of pairs \((x, v) \in SM|_{\partial M}\) such that the germ of the geodesic curve \( \gamma \subset M \) through \( x \) in the direction of \( v \) is tangent to \( \partial M \) with the multiplicity \( m(x) = j - 1 \). Moreover, the stratum \( \partial^j_0 SM(v^g) \) also has a similar description in terms of the germ of \( \gamma \subset M \) (its exact formulation is described in the proof).

Proof. Take a smooth function \( z : M \to \mathbb{R}_+ \) with the properties: (1) 0 is a regular value of \( z \), (2) \( z^{-1}(0) = \partial M \), and (3) \( z^{-1}((\sim \infty, 0]) = M \) (compare these properties with the auxiliary function \( z \) in (2.3) whose domain is \( X := SM \)). Let \( \tilde{z} : SM \to \mathbb{R}_+ \) be the composition of the projection \( \pi : SM \to M \) with \( z \). Evidently, \( \tilde{z} : SM \to \mathbb{R} \) has the same three properties with respect to \( \partial(SM) \) as \( z : M \to \mathbb{R} \) has with respect to \( \partial M \).

\(^{19}\)like the curvature tensor of \( M \) and the normal curvature of \( \partial M \)
For any pair \((x, v) \in \partial(SM)\), consider the germ of the geodesic line \(\gamma \subset M\) through \(x\) in the direction of \(v\) and its lift \(\tilde{\gamma} \subset SM\), the geodesic flow curve through \((x, v)\). Then \(\pi : \tilde{\gamma} \rightarrow \gamma\) locally is an orientation preserving diffeomorphism of curves. Therefore the \(k\)-jet \(\text{jet}^k(\tilde{z}|_{\tilde{\gamma}}) = 0\) if and only if \(k\)-jet \(\text{jet}^k(z|_{\gamma}) = 0\).

By Lemmas 3.1 and 3.3 from \([K2]\), \((x, v) \in \partial_j(SM)(v^g)\) if and only if \(\text{jet}^{j-1}(\tilde{z}|_{\tilde{\gamma}}) = 0\); similarly, \((x, v) \in \partial^+_j(SM)(v^g)\) if, in addition, \(\frac{\partial^j}{\partial t^j}(\tilde{z}|_{\tilde{\gamma}})(x, v) \geq 0\) (here \(\tilde{t}\) is the natural parameter along \(\tilde{\gamma}\)).

Thanks to the orientation-preserving diffeomorphism \(\pi : \tilde{\gamma} \rightarrow \gamma\), these properties of \(\tilde{z}|_{\tilde{\gamma}}\) are equivalent to the similar properties \(\text{jet}^{j-1}(z|_{\gamma}) = 0\) and \(\frac{\partial^j}{\partial t^j}(z|_{\gamma})(x) \geq 0\) of \(z|_{\gamma}\). The latter ones describe the multiplicity \(m(x) := j - 1\) of tangency of \(\gamma\) to \(\partial M\) at \(x\).

**Remark 5.5.** If \((M, g)\) is such that there exists a geodesic curve \(\gamma \subset M\) whose arc is contained in \(\partial M\), then the metric \(g\) is not geodesically boundary generic in the sense of Definition 5.2. So we stay away from the metrics \(g\) on \(M\) in which the boundary \(\partial M\) is geodesically closed.

The next proposition claims that some refined convexity/concavity properties of the boundary \(\partial M\) with respect to the metric \(g\) on \(M\) can be recovered from the scattering data.

**Corollary 5.3.** Assume that \(M\) admits a geodesically traversally generic Riemannian metric \(g\). Then the scattering map \(C_{\vartheta} : \partial^+_1(SM) \rightarrow \partial^-_1(SM)\) allows for a reconstruction of the loci \(\{\partial^+_j SM(v^g)\}_j\).

As a result, for each \(j > 0\), the locus \(\{(x, v) \in SM|_{\partial M}\}\), such that the germ of the geodesic curve \(\gamma \subset M\) through \(x\) in the direction of \(v\) is tangent to \(\partial M\) with the multiplicity \(j - 1\), can be reconstructed from the scattering map.

**Proof.** The claims are immediate implications of Lemma 5.2 and Theorem 5.2. □

Assume that a metric \(g\) on \(M\) is geodesically boundary generic (see Definition 5.2). For an oriented geodesic curve \(\gamma \subset M\), consider its intersections \(\{x_i\}_i\) with \(\partial M\). We denote by \(m_i\) the multiplicity of the intersection \(x_i\) (see the proof of Lemma 5.3). Let \(m(\gamma) := \sum_i m_i\) and \(m'(\gamma) := \sum_i (m_i - 1)\) (cf. (2.1) and (2.2)).

The following corollary describes the ways in which geodesic arcs can be inscribed in \(M\), provided that the metric on \(M\) is traversally generic.

**Corollary 5.4.** Assume that a \(n\)-dimensional compact smooth manifold \(M\) admits a traversally generic Riemannian metric \(g\). Then any geodesic curve in \(M\) has \(2n - 2\) simple points of tangency to the boundary \(\partial M\) at most.

In general, any geodesic curve \(\gamma \subset M\) interacts with the boundary \(\partial M\) so that:

\[m(\gamma) \leq 4n - 2\quad\text{and}\quad m'(\gamma) \leq 2n - 2.\]

**Proof.** Since, by Lemma 5.3, the multiplicity of tangency of a geodesic curve \(\gamma\) to \(\partial M\) at a point \(x\) and the multiplicity of tangency of its lift \(\tilde{\gamma} \subset SM\) to \(\partial(SM)\) at the point \((x, \tilde{\gamma}(x))\) are equal, the claim follows from the second bullet in Theorem 5.3 from \([K2]\). □
In one special case of \((M, g)\), the traversal genericity of the gradient flow on SM comes “for free” at the expense of a very restricted topology of \(M\):

**Corollary 5.5.** Let \(M\) be a compact Riemannian \(n\)-manifold with boundary. Assume that the boundary \(\partial M\) is strictly convex with respect to a metric \(g\) of the gradient type on \(M\).

Then geodesic field \(v^g\) on \(SM\) is traversally generic. Moreover, the manifold \(SM\) must be diffeomorphic to the product \(DT(\partial M) \times [0, 1]\), the corners in the product being rounded. Here \(DT(\partial M)\) denotes the tangent \((n-1)\)-disk bundle of \(\partial M\).

**Proof.** For such metric \(g\), any geodesic curve \(\gamma \subset M\), tangent to \(\partial M\), is a singleton. Therefore \(\partial_2 SM(v^g) = \partial_2 SM(v^g)\) — the field \(v^g\) on \(SM\) is convex. Under these assumptions, the geodesic field \(v^g\) on \(SM\) is traversally generic since no strata \(\partial_2 SM(v^g)\) interact, with the help of the geodesic flow, through the bulk \(SM\). Also, for a strictly convex \(\partial M\) and \(g\) of the gradient type,

\[
\partial_1^+ SM(v^g) = \{(x, v) \mid x \in \partial M, v \text{ points inside } M\}.
\]

Thus \(\partial_1^+ SM(v^g)\) fibers over \(\partial M\) with a fiber being the hemisphere \(D_{n-1}^+ \subset S^{n-1}\) of dimension \(n - 1 = \dim(\partial M)\). This fibration is isomorphic to the unit disk tangent bundle of \(\partial M\). Thus, by Lemma 4.2 from [K1], the manifold \(SM\) must be diffeomorphic to the product \(DT(\partial M) \times [0, 1]\), the corners in the products being rounded. As a result, \(SM\) fibers over \(\partial M\) with the fiber \(D^n\) and over \(M\) with the fiber \(S^{n-1}\). This property of \(SM\) puts severe restrictions on the topology of \(M\); in particular, the space \(SM\) must be homotopy equivalent to \(\partial M\). Note that \(M = D^n\) has the desired property: \(M \times S^{n-1} \approx \partial M \times D^n\).

**Corollary 5.6.** Let \(M \subset \mathbb{H}^n\) be a codimension 0 compact smooth submanifold of the hyperbolic space.

Assuming the validity of Conjecture 5.1, the hyperbolic metric \(g\) on \(M\) can be perturbed in the \(C^\infty\)-topology to a metric \(\tilde{g}\) so that the scattering map \(C_{v^g} : \partial_1^+ (SM) \rightarrow \partial_1^+ (SM)\) allows for a reconstruction of the pair \((SM, F(v^g))\), up to a diffeomorphism of \(SM\) which is the identity on \(\partial_1 (SM)\).

If \(M \subset \mathbb{H}^n\) is strictly convex in the hyperbolic metric, then the scattering map \(C_{v^g}\) allows for a reconstruction of the pair \((SM, F(v^g))\), up to a diffeomorphism of \(SM\) which is the identity on \(\partial_1 (SM)\). In such a case, \(SM\) is diffeomorphic to the product \(DT(\partial M) \times [0, 1]\).

**Proof.** In view of Example 5.2 REF, the hyperbolic metric on \(M\) is of the gradient type. By Corollary 5.4 the strict convexity of \(\partial M\) implies that \(\partial_2^+ (SM)\) are empty; so that \(v^g\) is automatically traversally generic and convex. By Lemma 4.2 from [K1], the convexity of \(v^g\) in relation to \(\partial (SM)\) implies that \(SM \approx \partial_1^+ (SM)\) is convex. This proves the second claim of the corollary.

With the help of Theorem 5.3, the validity of the first claim is conditioned by the validity of Conjecture 5.1.

---

注释 20: Thus \(SM\) fibers over \(\partial M\) with the fiber \(D^n\) and over \(M\) with the fiber \(S^{n-1}\).
Occasionally, we can conclude that the geodesic field $v^g$ on $SM$ is traversally generic even if $\partial M$ is not convex, but the geodesic curves in $M$ have some special properties in relation to $\partial M$.

**Lemma 5.4.** If a boundary geodesically generic and gradient-type metric $g$ on $M$ is such that no geodesic curve $\gamma \subset M$ is tangent to the boundary $\partial M$ either at two distinct points, or at a single point, but from distinct directions, then $v^g$ is a traversally generic field.

As a result, Theorem 5.2—the topological rigidity of the inverse geodesic scattering problem—is applicable to such $(M,g)$. 

**Proof.** For a boundary generic $g$, each locus $\partial_j(SM)(v^g)$, $j \geq 2$, is a smooth submanifold of $\partial(SM)$. If every $v^g$-trajectory $\tilde{\gamma}$ is tangent to the boundary $\partial(SM)$ at a single point $x \in \partial_2(SM)$ at most, then $x$ belongs to a single pure stratum $\partial_j(SM)(v^g)$, $j := j(x) \geq 2$. Moreover, the $v^g$-guided projection of $\partial_j(SM)(v^g)$ on a local transversal to $\tilde{\gamma}$ section $\Sigma$ of the geodesic flow is a regular bijection, and no other strata $\partial_k(SM)(v^g)$, $k \geq 2$, intersect $\tilde{\gamma}$. Evidently, such a field $v^g$ is traversally generic (see Definition 3.2 from [K2]). Thanks to Lemma 5.3, the double-tangent trajectories $\tilde{\gamma}$ of $v^g$ correspond to double-tangent to $\partial M$ geodesic curves $\gamma$. The exceptions are the geodesic loops $\gamma$ in $M$ that are tangent (from two distinct directions) to $\partial M$ at some point $m \in \partial M$. Therefore, in the absence of such $\gamma$’s, the geodesic flow is traversally generic. □

**Example 5.3.** Consider a shell $M$, produced by removing a convex domain from the interior of a convex domain in the Euclidean space $E$ (so topologically $M$ is an annulus). Then evidently any geodesic line $\gamma$ in $M$ is tangent to the boundary of the interior convex domain at a single point and is transversal to the boundary of the exterior convex domain. Any geodesic line $\gamma \subset E$ in is tangent to the boundary of the exterior convex domain at a singleton. For such a pair $(M,g_E)$, the geodesic field $v^g$ on $SM$ is traversally generic. Thus, by Lemma 5.4 Theorem 5.2 is applicable to $(M,g_E)$, as well as to any metric $g$ in $M$, sufficiently close to $g_E$. □

The causality map $C_{v^g} : \partial^+_1(SM) \to \partial^-_1(SM)$ can be composed with the involution $\tau_g : \partial_1(SM) \to \partial_1(SM)$ (see Lemma 5.2) to create the **causality return map**

$$B_{v^g} : \partial^+_1(SM) \to \partial^+_1(SM)$$

, a vague analogue of the Poincaré return map. Its iterations determine the dynamics of geodesic billiard on the curvy table $(M,g)$ with boundary $\partial M$. So $B_{v^g}$ also deserves the name of “Billiard Map”.

We will devote a different paper to investigations of the dynamics of billiard maps on curved billiards with geodesically traversally generic metrics.

**Remark 5.6.** Note that, for a metric $g$ of the gradient type, the closed piecewise geodesic trajectories $\Upsilon$ of the billiard game on $M$ are in 1-to-1-correspondence with the closed trajectories $\Upsilon^\partial$ of the billiard map $B_{v^g} : \partial^+_1(SM) \to \partial^+_1(SM)$. Each periodic point $z$ of the transformation $B_{v^g}$ gives rise to such trajectory $\Upsilon^\partial$; the period of $z$ with respect to $B_{v^g}$ is the number of reflections (including the tangential ones) of $\Upsilon$ from the boundary $\partial M$. □
Example 5.4. The Poncelet Closure Theorem \([\text{BKOR}]\) produces classical examples of periodic billiard trajectories. Let us remind its content.

Consider two plane conics (quadratic curves) \(Q_1, Q_2\). Suppose that there exists a \(k\)-sided polygon \(P_k\) inscribed in \(Q_1\) and circumscribed around \(Q_2\). Then the theorem claims that there are infinitely many such polygons. In fact, any point of \(Q_1\) can serve as a vertex of such polygon.

Let us interpret the Poncelet Closure Theorem in terms of billiards. Let \(Q_1\) be an ellipse, and \(Q_2\) another ellipse contained in the interior of \(Q_1\). Let \(M\) be the compact domain, bounded by \(Q_1 \cup Q_2\). Then the special \(k\)-gon \(P_k\) that is inscribed in \(Q_1\) and circumscribed around \(Q_2\) can be viewed as a closed billiard trajectory on the flat annulus \(M\). Along the trajectory, the points of tangent reflection, residing in \(Q_2\), alternate with points of transversal reflection, residing in \(Q_1\).

In terms of the billiard map \(B_{v^g} : \partial^+\Gamma(SM) \to \partial^+\Gamma(SM)\), the polygon \(P_k\) gives rise to a closed \(B_{v^g}\)-trajectory of the period \(2k\). Then the Poncelet Closure Theorem implies that if the billiard map has one such special trajectory of period \(2k\), then it has infinitely many points of the period \(2k\). Moreover, if one point \(m \in Q_1\) has a unit tangent vector \(v \in T_x(M)\) such that the pair \((m, v)\) gives rise to a special \(k\)-gon \(P_k\), then any other point \(m' \in Q_1\) has this property. In such case, the set of \(2k\)-periodic points in \(\partial^+\Gamma(SM)(v^g)\) of the billiard map \(B_{v^g}\) projects onto \(\partial M = Q_1 \cup Q_2\). Here the locus \(\partial^+\Gamma(SM)(v^g)\) is diffeomorphic to \((Q_1 \cup Q_2) \times [0, 1]\).

Example 5.4 REF motivates a generalization. Given a compact Riemannian manifold \((M, g)\), consider the group \(\mathcal{PI}(M, g)\) of diffeomorphisms \(\phi : (M, \partial M) \to (M, \partial M)\) that map (unparametrized) geodesics to geodesics and are isometries in the vicinity of \(\partial M\) in \(M\). Evidently, \(\mathcal{PI}(M, g)\) contains the group of isometries \(I(M, g)\).

Then the group \(\mathcal{PI}(M, g)\) acts on the set of billiard trajectories. This action lifts to a \(\mathcal{PI}(M, g)\)-action on the manifold \((SM, \partial(SM))\) via the formula

\[
(m, v) \to (\phi(m), D\phi_m(v)/\|D\phi_m(v)\|_g)
\]

, where \(\phi \in \mathcal{PI}(M, g)\).

Now assume that the metric \(g\) on \(M\) is of the gradient type. Since the billiard map \(B_{v^g}\) depends only on the unparametrized oriented geodesics in \(M\), the lifted \(\mathcal{PI}(M, g)\)-action on \(SM\) commutes with \(B_{v^g}\). Therefore, if \(\alpha \in \mathcal{PI}(M, g)\) and \(z = (m, v) \in \partial^+\Gamma(SM)(v^g)\) is a periodic point for the map \(B_{v^g}\) of a period \(k\), then \(\alpha(z)\) is also of the period \(k\). Assuming that one \(k\)-periodic point \(z\) exists, the set of all \(k\)-periodic points is at least as big as the size of the orbit \(\mathcal{PI}(M, g)z\).

The billiard map \(B_{v^g}\) has its piecewise continuous analogue

\[
\{B_{v^g}^t : SM \to SM\}_{t \in [0, +\infty)}.
\]

Let us describe a typical map \(B_{v^g}^T\), \(T \in [0, +\infty)\), informally. Take a point \((x, v) \in SM\). When \(\tilde{\gamma}(t) \in \text{int}(SM)\) for all \(t < T\), we follow the \(v^g\)-trajectory \(\tilde{\gamma}(t)\) through \((x, v)\) for \(T\) units of time. Otherwise, there is a the first moment \(t_1 \leq T\), such that \(\tilde{\gamma}(t_1) \in \partial^+\Gamma(SM)(v^g)\). In such a case, we apply the involution \(\tau_g\) (see Lemma 5.2) to get to the point \(\tau_g(\tilde{\gamma}(t_1)) \in \text{int}(SM)\) for all \(t > t_1\), and then follow the \(v^g\)-trajectory of the period \(\tau_g(\tilde{\gamma}(t_1))\) for \(T - t_1\) units of time.
\( \partial_1^+(SM)(v^g) \) in order to continue the \( v^g \)-directed journey for \( T - t_1 \) moments of time. Again, the instructions for the continuation of the journey bifurcate, with the point \((x, v)\) being replaced with the point \( \tau_g(\tilde{\gamma}(t_1)) \) and the time interval \( T - t_1 \). This may lead to the next moment in time \( t_2 \leq T - t_1 \) when the trajectory will reach again the locus \( \partial_1^+(SM)(v^g) \). We repeat the recipe until, after a finite or infinite sequence of “bouncing” instances \( t_1, t_2, \ldots, t_k, \ldots \) (where \( \sum_k t_k \leq T \)) we will exhaust \( T \) and will arrive at the location \( B_T^T(x, v) \in SM \).

Note that the projection of each \( B_t^g \)-trajectory by the map \( \pi : SM \to M \) is a continuous piecewise smooth trajectory on the billiard table \( M \).

**Theorem 5.2** has another immediate interpretation in terms of billiards:

**Corollary 5.7.** Assume that a compact manifold \( M \) with boundary admits a geodesically traversally generic Riemannian metric \( g \). Then the billiard map \( B_{v^g} : \partial_1^+(SM) \to \partial_1^+(SM) \) allows for a reconstruction of:

- the pair \( (SM, F(v^g)) \), up to a diffeomorphism of \( SM \) which is the identity on \( \partial_1(SM) \),
- the smooth topological type of the space \( T(v^g) \) of unparametrized geodesics on \( M \),
- the unparameterized trajectories of the piecewise continuous family
  \[ \{ B_t^g : SM \to SM \}_{t \in \mathbb{R}_+} \]
  of billiard maps, up to a diffeomorphism of \( SM \) which is the identity on \( \partial_1(SM) \).

\[ \square \]

One might hope that, for a typical traversally generic metric \( g \) on \( M \), the reconstruction \( C_{v^g} \Rightarrow (M, g) \) is possible (see [Cr], [Cr1], and [We] for special cases of such a reconstruction). At the moment, this is just a wishful thinking. In the absence of a faithful reconstruction of the geometry \( g \) of \( M \) from the scattering data \( C_{v^g} \), with a mindset of a humble topologist, we will settle for less:

**Theorem 5.4.** Assume that a compact connected \( n \)-manifold \( M \) with boundary admits a geodesically traversally generic Riemannian metric \( g \).

Then the scattering map \( C_{v^g} : \partial_1^+(SM) \to \partial_1^+(SM) \) allows for a reconstruction of the cohomology groups \( H^*(M; \mathbb{Z}) \), as well as for the reconstruction of the homotopy groups \( \pi_i(M) \) for all \( i < n \).

If, in addition, \( M \) has a trivial tangent bundle, then the stable smooth type of \( M \) is reconstructable from the scattering map.

**Proof.** Since, by Theorem 5.2, the topological type of \( SM \) can be reconstructed from the scattering map \( C_{v^g} \), so is the cohomology of \( SM \). The spectral sequence of the spherical fibration \( SM \to M \) is trivial since \( H^n(M; \mathbb{Z}) = 0 \) due to the property \( \partial M \neq \emptyset \). Thus \( H^*(M; \mathbb{Z}) \) can be recovered from \( H^*(SM; \mathbb{Z}) \).

The long exact homotopy sequence of the fibration \( SM \to M \) identifies \( \pi_i(M) \) with \( \pi_i(SM) \) for all \( i < n \).

For a trivial tangent bundle \( TM \), the “reconstructible” space \( SM \) is diffeomorphic to the product \( M \times S^{n-1} \). Thus the stable smooth type of \( M \) (the stabilization being understood as
CAUSAL HOLOGRAPHY OF TRAVERSING FLOWS

6. HOLOGRAPHIC STRUCTURES ON THE BOUNDARY

The reconstruction procedure \((\partial_1 X, C_v) \Rightarrow (X, \mathcal{F}(v))\) (whose existence is claimed by Corollary \([3]\)) presumes that the space \(X\) and a traversally generic field \(v\) on it, giving rise to the boundary-confined data \((\partial_1 X, C_v)\), do exist.

Thanks to Lemma 3.4 \([K2]\), we have “semi-local” models for the causality map \(C_v\) in the vicinity of each \(C_v\)-“trajectory”

\[
\gamma^\partial_x := (x, C_v(x), C_v(C_v(x)), \ldots, C_v^{(d)}(x)) \in (\partial_1 X)^{d+1}
\]

where all the iterative \(C_v\)-images of \(x\) are distinct and \(x \neq C_v(x')\) for all \(x' \neq x\) (see Remark 4.1).

Still, the question “Which pairs \((\partial_1 X, C_v)\) arise from traversally generic fields on some manifold \(X\)?” requires some attention.

Similarly, for a traversally generic \(v\) on a \((n+1)\)-dimensional \(X\), we can model the smooth types of the \(\Omega_{\bullet'}\langle n\rangle\)-stratified trajectory spaces \(T(v)\) locally (Theorem 5.3 from \([K3]\)).

Again, the question “Which \(\Omega_{\bullet'}\langle n\rangle\)-stratified cellular complexes \(T\) can be produced as trajectory spaces \(T(v)\) of traversally generic pairs \((X, v)\)?” remains open.

In the definitions below, we will develop a language that is appropriate for answering these questions.

First, we need to reinforce some old notations. For any \(\omega \in \Omega_{\bullet'}\langle n\rangle\), consider the product \(\mathbb{R} \times \mathbb{R}^{\omega'} \times \mathbb{R}^{n-\omega'} \approx \mathbb{R}^{n+1}\). We denote by \((u, x, y) := (u, \vec{x}, \vec{y})\) a typical point in the product. Form a cylinder

\[
\Pi_\omega(\epsilon, r) := \{(u, x, y) : \|\vec{x}\| < \epsilon, \|\vec{y}\| < r\} \subset \mathbb{R} \times \mathbb{R}^{\omega'} \times \mathbb{R}^{n-\omega'}
\]

whose base is an open polydisk. We denote by \(\pi\) the obvious projection of \(\mathbb{R} \times \mathbb{R}^{\omega'} \times \mathbb{R}^{n-\omega'}\) onto \(\mathbb{R}^{\omega'} \times \mathbb{R}^{n-\omega'} \approx \mathbb{R}^{n}\).

We also consider the very familiar polynomial

\[
P_\omega(u, \vec{x}) := \prod_{i \in \sup(\omega)} \left[ (u - \alpha_i)^{\omega(i)} + \sum_{k=0}^{\omega(i)-2} x_{i,k}(u - \alpha_i)^k \right]
\]

, where the variables \(\vec{x} := \{x_{i,k}\}\) are considered in the lexicographic order, and \(\alpha_i < \alpha_{i+1}\) for all \(i\). In what follows, we assume that \(\|\vec{x}\| < \epsilon\), where \(\epsilon > 0\) has been chosen so small that all the real roots of \(P_\omega(u, \vec{x})\) belong to a union of disjoint intervals in \(\mathbb{R}\) with the centers at the roots \(\{\alpha_i\}\). We use such \(\epsilon\) to pick an appropriate cylinder \(\Pi_\omega(\epsilon, r)\).

Let us consider the two types of semi-algebraic sets:

\[
Z_\omega := \{P_\omega(u, \vec{x}) \leq 0\} \cap \Pi_\omega(\epsilon, r) \subset \mathbb{R} \times \mathbb{R}^n
\]

\[
W_\omega := \{P_\omega(u, \vec{x}) = 0\} \cap \Pi_\omega(\epsilon, r) \subset \mathbb{R} \times \mathbb{R}^n.
\]
We assume that $\mathbb{R} \times \mathbb{R}^n$ is oriented; from now and on, this orientation is fixed. The nonsingular hypersurface $W_{\omega} \subset \mathbb{R} \times \mathbb{R}^n$ is oriented with the help of the normal gradient vector field $\nabla(P_{\omega}(u, x))$ and the preferred orientation of $\mathbb{R} \times \mathbb{R}^n$.

For each $x := (\bar{x}, \bar{y}) \in \mathbb{R}^n$, we consider the connected components $\{F_\rho(x)\}_\rho$ of the fiber $F(x)$ over $x$ of the projection $\pi : Z_\omega \to \mathbb{R}^n$ (the fiber does not depend on $\bar{y}$).

Next, we will be dealing with a family of elements $\{\omega_\beta \in \Omega_{\bullet(\{n\})}\}_\beta$ indexed by $\beta$, a family of polynomials $\{P_\beta(u_\beta, \bar{x}^\beta) := P_{\omega_\beta}(u_\beta, \bar{x}^\beta)\}_\beta$, and a family of appropriate cylinders $\{\Pi_\beta := \Pi_{\omega_\beta}(\epsilon_\beta, r_\beta)\}_\beta$, equipped with the projections $\pi_\beta : \Pi_\beta \to \mathbb{R}^n$, induced by the obvious projection $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

We will also employ the connected components $\{F_{\beta, \rho}(x)\}_\rho$ of the fiber $F_\beta(x)$ of the projection $\pi_\beta : Z_\beta := Z_{\omega_\beta} \to \mathbb{R}^n$, where $x \in \mathbb{R}^n$.

**Definition 6.1.** Let $Y$ be a closed smooth $n$-manifold. A smooth (oriented) proto-holographic atlas on $Y$ is an open cover $\{U_\beta\}_\beta$ of $Y$ together with the following structures:

- for each index $\beta$, an element $\omega_\beta \in \Omega_{\bullet(\{n\})}$ is chosen,
- for each $\beta$, there exists a homeomorphism $h_\beta$ from $U_\beta$ onto a union of some connected components of the nonsingular real semi-algebraic set
  $$W_\beta := \{P_\beta(u_\beta, \bar{x}^\beta) = 0\} \cap \Pi_\beta \subset Z_\beta \subset \mathbb{R} \times \mathbb{R}^n,$$
- for any pair of indices $\beta, \beta'$, there is a (orientation preserving) $C^\infty$-diffeomorphism
  $$h_{\beta, \beta'} : h_\beta(U_\beta \cap U_{\beta'}) \to h_{\beta'}(U_\beta \cap U_{\beta'})$$
  of the hypersurfaces in $\mathbb{R} \times \mathbb{R}^n$ such that
  $$h_{\beta'}|_{U_\beta \cap U_{\beta'}} = h_{\beta, \beta'} \circ h_\beta|_{U_\beta \cap U_{\beta'}}.$$
- for each $x \in \mathbb{R}^n$ and each connected component $F_\rho(x)$ of the fiber $F(x) := \pi_\beta^{-1}(x)$ of the map $\pi_\beta : Z_\beta \to \mathbb{R}^n$, the diffeomorphism $h_{\beta, \beta'}$ takes the finite set
  $$h_\beta(U_\beta \cap U_{\beta'}) \cap F_\rho(x) \to h_{\beta'}(U_\beta \cap U_{\beta'}) \cap F_{\rho'}(x')$$
  and preserves the natural orders of points in the two fibers. Here $x' \in \mathbb{R}^n$ and $F_{\rho'}(x')$ denotes some component of the fiber of $\pi_{\beta'} : Z_{\beta'} \to \mathbb{R}^n$ over $x'$.

We say that a proto-holographic atlas is holographic if all the maps $h_\beta : U_\beta \to W_\beta$ are onto (see Fig. 8). □

**Remark 6.1.** The last bullet of Definition 6.1 imposes conditions on $h_{\beta, \beta'}$ which are a relaxed version of the following better looking but more stringent conditions: “$h_{\beta, \beta'}$ takes each fiber of the obvious projection

$$\pi : h_\beta(U_\beta \cap U_{\beta'}) \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

to a fiber of

$$\pi : h_{\beta'}(U_\beta \cap U_{\beta'}) \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

$^{21}$Its elements $U_\beta$ are not necessarily connected sets.
and preserves the natural orders of points in the two fibers”. However, the geometry of traversing flows requires the relaxed version from Definition 6.1.

Example 6.1. Here is the most trivial example of a smooth proto-holographic atlas. Pick a smooth structure on $Y$ by specifying an open cover $\{U_\beta\}_\beta$, where each $U_\beta$ is homeomorphic to an open $n$-ball $D^n$, together with some homeomorphisms $\phi_\beta : U_\beta \to D^n \subset \mathbb{R}^n$, and the linking diffeomorphisms

$$\{\phi_{\beta,\beta'} : \phi_\beta(U_\beta \cap U_{\beta'}) \to \phi_{\beta'}(U_\beta \cap U_{\beta'})\}_{\beta,\beta'}$$

which satisfy the usual cocycle conditions. We associate the same element $\omega = (11)$ with each $U_\beta$.

Then

$$P_{\omega,\beta}(u_\beta, \vec{x}^\beta) = (u - \alpha_1)(u - \alpha_2)$$

and

$$\Pi_\beta = \{(u, \vec{y}) : \|\vec{y}\| < r\}$$

for all $\beta$. So $W_\beta$ is a union of two $n$-balls

$$D^\alpha_- = \{(u, \vec{y}) : u = \alpha_1, \|\vec{y}\| < r\} \quad \text{and} \quad D^\alpha_+ = \{(u, \vec{y}) : u = \alpha_2, \|\vec{y}\| < r\}$$

, while $Z_\beta$ is the cylinder $\{(u, \vec{y}) : \alpha_1 \leq u \leq \alpha_2, \|\vec{y}\| < r\}$. We identify $D^n$ with $D^n_-$ so that now $\phi_\beta = h_\beta$ takes $U_\beta$ homeomorphically onto $D^n_\beta$. The verification of all the properties, required for a proto-holographic atlas, is straightforward.
The next example is a variation on the same theme. This time, each element \( U_\beta \) of a proto-holographic atlas is homeomorphic to a disjoint union of two open \( n \)-balls \( D^n \). We associate the same element \( \omega = (11) \) with each \( U_\beta \), but now

\[
h_\beta : U_\beta \to W_\beta = D^n_+ \coprod D^n_-
\]

is surjective—the atlas is holographic. Note that, if there exists a smooth holographic structure on \( Y \) which is amenable to such a choice of charts \( \{h_\beta : U_\beta \to W_\beta = D^n_+ \coprod D^n_-\} \), then \( Y \) is a covering space over some manifold \( Y_0 \) with the fiber of cardinality 2. □

**Definition 6.2.**

- A proto-holographic atlas \( \{V_\sigma\}_\sigma \) on \( Y \) is called a refinement of a proto-holographic atlas \( \{U_\beta\}_\beta \) if each set \( V_\sigma \) is contained in some set \( U_\beta \), where \( \omega_\sigma \geq \omega_\beta \) in \( \Omega_{*[n]} \), and the map \( h_\sigma : V_\sigma \to W_\sigma \) is a restriction of \( h_\beta : U_\beta \to W_\beta \) to \( V_\sigma \); moreover, the map \( h_{\sigma,\sigma'} : h_\sigma(U_\beta \cap U_{\beta'}) \to h_{\sigma'}(U_\beta \cap U_{\beta'}) \) is obtained from the map \( h_{\beta,\beta'} : h_\beta(U_\beta \cap U_{\beta'}) \to h_{\beta'}(U_\beta \cap U_{\beta'}) \) by restricting it to the hypersurface \( h_\sigma(V_\sigma \cap V_{\sigma'}) \).

- Two smooth proto-holographic atlases on \( Y \) are called equivalent if they admit a common refinement.

- A proto-holographic structure on \( Y \) is the equivalence class of smooth proto-holographic atlases on \( Y \) □

Given a proto-holographic/holographic structure \( H \) on \( Y \), any diffeomorphism \( \Phi : Y \to Y \) induces the pull-back proto-holographic/holographic structure \( \Phi^*H \) on \( Y \). It is defined by the formulas:

\[
\Phi^*(U_\beta) := \Phi^{-1}(U_\beta), \quad \Phi^*h_\beta : \Phi^{-1}(U_\beta) \to U_\beta \xrightarrow{h_\beta} W_\beta, \quad \Phi^*h_{\beta,\beta'} = h_{\beta,\beta'}.
\]

Thus the group of smooth diffeomorphisms \( \text{Diff}(Y) \) acts on the set of all proto-holographic structures \( H(Y) \) on \( Y \). Consider \( \overline{H}(Y) \), the orbit set of this action. We have little to say about the modular space \( \overline{H}(Y) \), an interesting object to study...

**Remark 6.2.** The proto-holographic structures do not behave contravariantly under general smooth maps of manifolds. However, a weaker version of the concept does. We call it appropriately “weak proto-holographic structure”. In the second bullet of Definition 6.1, we will not require that \( h_\beta : U_\beta \to W_\beta \) is a homeomorphism onto a union of some components of the semi-algebraic set \( W_\beta \); instead, we assume that \( h_\beta : U_\beta \to W_\beta \) is just a continuous map, while keeping the rest of the bullets enforced and adding the cocycle condition

\[
h_{\beta',\beta''} \circ h_{\beta',\beta''} \circ h_{\beta,\beta'} |_{h_\beta(U_\beta \cap U_{\beta'} \cap U_{\beta''})} = \text{Id}
\]

for each triple of indices \( \beta, \beta', \beta'' \). □

Given a map \( \pi : A \to B \) of two sets and a subset \( C \subset A \), we denote by \( \pi^!(C) \) the set \( \pi^{-1}(\pi(C)) \subset A \). Evidently \( C \subset \pi^!(C) \). When \( \pi : A \to B \) is a map of topological spaces,
we denote by \( \pi^1(C) \) the union of connected components of \( \pi^1(C) \) that have nonempty intersections with \( C \).

**Definition 6.3.** Let \( Y \) be a closed smooth \( n \)-manifold. Consider the semi-algebraic set

\[ Z_\beta := \{ P_\beta(u_\beta, x^\beta) \leq 0 \} \cap \Pi_\beta \subset \mathbb{R} \times \mathbb{R}^n. \]

We say that a smooth holographic atlas \( \{ U_\beta, h_\beta \}_{\beta} \) on \( Y \) \(^{22}\) is (orientably) cobordant to the empty set \( \emptyset \) if:

- for any pair of indices \( \beta, \beta' \), the (orientation-preserving) \( C^\infty \)-diffeomorphism
  \[ h_{\beta, \beta'} : h_\beta(U_\beta \cap U_{\beta'}) \rightarrow h_{\beta'}(U_\beta \cap U_{\beta'}) \]
  of the \( (\nabla P_\beta \text{-oriented}) \) hypersurfaces in \( \mathbb{R} \times \mathbb{R}^n \) extends to a (orientation-preserving) \( C^\infty \)-diffeomorphism
  \[ H_{\beta, \beta'} : Z^\beta_{\beta'} \rightarrow Z^{\beta'}_{\beta'} \]
  of the domains
  \[ Z^\beta_{\beta'} := Z_\beta \cap \pi^1_\beta(h_\beta(U_\beta \cap U_{\beta'})) \text{ and } Z^{\beta'}_{\beta'} := Z_{\beta'} \cap \pi^1_{\beta'}(h_{\beta'}(U_\beta \cap U_{\beta'})) \]
  so that \( H_{\beta, \beta'} \) maps every connected component of each fiber of the projection
  \[ \pi_\beta : Z^\beta_{\beta'} \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \]
  to a connected component of a fiber of
  \[ \pi_{\beta'} : Z^{\beta'}_{\beta'} \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \]
  and preserves their orientations,

- for any triple of indices \( \beta, \beta', \beta'' \),
  \[ H_{\beta'', \beta'} \circ H_{\beta', \beta''} \circ H_{\beta, \beta'} |_{Z_\beta \cap \pi^1_\beta(U_\beta \cap U_{\beta'} \cap U_{\beta''})} = Id. \]

We say that a holographic structure \( \mathcal{H} \) on \( Y \) is (orientably) cobordant to the empty set \( \emptyset \), if \( \mathcal{H} \) is represented by a holographic atlas which is (orientably) cobordant to \( \emptyset \). \( \square \)

**Remark 6.3.** Note that the quotient \( \frac{P_{\beta'} \circ H_{\beta, \beta'}}{P_\beta} \) of the two functions from Definition 6.3 must be positive in \( Z_\beta \cap \pi^1_\beta(h_\beta(U_\beta \cap U_{\beta'})) \). \( \square \)

**Remark 6.4.** Given two proto-holographic/holographic atlases on closed smooth manifolds \( Y_1 \) and \( Y_2 \) of equal dimensions, one can form a proto-holographic/holographic atlas on \( Y_1 \coprod Y_2 \) by considering the obvious disjoint union of atlases. Similarly, given two holographic and cobordant to \( \emptyset \) atlases on \( Y_1 \) and \( Y_2 \), the obvious union of atlases is a holographic and cobordant to \( \emptyset \) atlas on \( Y_1 \coprod Y_2 \).

These observations could lead to the notion of cobordism between two proto-holographic atlases/structures \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) on \( Y_1 \) and \( Y_2 \), respectively: \( \mathcal{H}_1 \) could be defined to be cobordant to \( \mathcal{H}_2 \) if \( \mathcal{H}_1 \coprod -\mathcal{H}_2 \) is cobordant to \( \emptyset \). Alas, such cobordism is not an equivalence relation! \( \square \)

\(^{22}\)See Definition 6.1.
Theorem 6.1.

- Any traversally generic field \( v \) on an (oriented) manifold \( X \) gives rise to a smooth holographic (orientably) cobordant to \( \emptyset \) structure \( H(v) \) on \( Y = \partial_1 X \).
- Conversely, any smooth holographic (orientably) cobordant to \( \emptyset \) structure \( \mathcal{H} \) on \( Y \) arises from a traversally generic field \( v \) on an (oriented) manifold \( X \) whose boundary is diffeomorphic to \( Y \).

Proof. Consider a traversally generic field \( v \) on \( X \). By Lemma 3.4 from [K2], \( X \) has a \( v \)-adjusted cover \( \{V_\beta\}_\beta \) by connected sets \( V_\beta \) which admit special coordinates \( (u_\beta, \bar{x}^3, \bar{y}^3) \).

With each set \( V_\beta \) we associate the combinatorial type \( \omega_\beta \in \Omega_{\bnu(n)} \) of the core trajectory \( \gamma_\beta \subset V_\beta \) (given by the equations \( \bar{x}^3 = 0, \bar{y}^3 = 0 \)). In these coordinates, \( \partial_1 X \cap V_\beta \) is given by a polynomial equation \( P_\beta(u_\beta, \bar{x}^3) = 0 \) (as in the second bullet of Definition 6.1) and \( V_\beta \)—by the inequality \( P_\beta(u_\beta, \bar{x}^3) \leq 0 \). These two local descriptions are coupled with the \( P_\beta \)-correlated inequalities \( \|\bar{x}^3\| < \epsilon_\beta, \|\bar{y}^3\| < r_\beta \), designed to partition real roots of \( u_\beta \)-polynomials \( \{P_\beta(u_\beta, \bar{x}^3)\}_{\beta} \) into separate groups, the groups being labeled by the roots of \( P_\beta(u_\beta, \tilde{0}) \) (see Lemma 3.2 from [K2]).

Let

\[ H_\beta : V_\beta \to Z_\beta \subset \mathbb{R}^{n+1} \]

be the diffeomorphism, given by the local coordinates \( (u_\beta, \bar{x}^3, \bar{y}^3) \). We denote by \( h_\beta \) its restriction to the boundary portion \( U_\beta := V_\beta \cap \partial_1 X \).

By fixing the coordinates \( (\bar{x}^3, \bar{y}^3) \) and a marker \( u_\beta^* \) such that \( P_\beta(u_\beta^*, \bar{x}^3) \leq 0 \), we are getting the \( H_\beta \)-images of the \( v \)-trajectories \( \gamma \subset V_\beta \), residing in the set

\[ Z_\beta := \{P_\beta(u_\beta, \bar{x}^3) \leq 0, \|\bar{x}^3\| \leq \epsilon_\beta, \|\bar{y}^3\| \leq r_\beta\}. \]

We define a diffeomorphism

\[ H_{\beta, \beta'} : H_\beta(V_\beta \cap V_{\beta'}) \to H_{\beta'}(V_\beta \cap V_{\beta'}) \]

by the formula \( H_{\beta'} \circ H_{\beta}^{-1} |_{H_\beta(V_\beta \cap V_{\beta'})} \). Due to the nature of the local coordinates \( (u_\beta, \bar{x}^3, \bar{y}^3) \), \( H_{\beta, \beta'} \) takes each connected component of every fiber of the \( u \)-directed map

\[ \pi_\beta : Z_\beta = H_\beta(V_\beta) \to \mathbb{R}^n \]

to a connected component of a fiber of the \( u \)-directed map

\[ \pi_{\beta'} : Z_{\beta'} = H_{\beta'}(V_{\beta'}) \to \mathbb{R}^n. \]

It preserves the \( u \)-induced orientation of the \( \pi \)-fibers.

Note that

\[ \pi_{\beta'}^{-1}(h_\beta(U_\beta \cap U_{\beta'})) = H_\beta(V_\beta \cap V_{\beta'}) \quad \text{and} \quad \pi_{\beta'}(h_\beta(U_\beta \cap U_{\beta'})) = H_{\beta'}(V_\beta \cap V_{\beta'}). \]

Evidently, for any triple \( \beta, \beta', \beta'' \), we obtain the identity

\[ H_{\beta'', \beta} \circ H_{\beta', \beta''} \circ H_{\beta, \beta'} |_{Z_\beta \cap H_\beta(V_\beta \cap V_{\beta'} \cap V_{\beta''})} = Id. \]
Consider $U_\beta := V_\beta \cap \partial_1 X$, $h_\beta := H_\beta|_{U_\beta}$, and $h_{\beta,\beta'} := H_{\beta,\beta'}|_{h_\beta(U_\beta \cap U_{\beta'})}$. Thus $\{U_\beta, h_\beta, h_{\beta,\beta'}\}$ form a holographic atlas on $\partial_1 X$, which is cobordant to $\emptyset$. The validation of this claim is on the level of definitions.

Now we turn to the validation of the second bullet in Theorem 6.1. Consider a holographic atlas $\{U_\beta, h_\beta, h_{\beta,\beta'}\}_\beta$ on $Y$, cobordant to $\emptyset$ (as in Definition 6.3). In particular, we have the diffeomorphisms $H_{\beta,\beta'} : Z_{\beta'} := Z_\beta \cap \pi_1^\beta(h_\beta(U_\beta \cap U_{\beta'})) \to Z_{\beta'} := Z_\beta \cap \pi_1^\beta(h_{\beta'}(U_\beta \cap U_{\beta'}))$ to play with.

Consider the hypersurface $W_\beta := \{P_\beta(u_\beta, \vec{x}) = 0\} \cap \Pi_\beta \subset \mathbb{R} \times \mathbb{R}^n$.

Let $X$ denote a quotient of the space $\bigsqcup_\beta Z_\beta$ by the following equivalence relation: $z \in Z_\beta$ is equivalent to $z' \in Z_{\beta'}$ if $z \in Z_{\beta'}$ and $z' = H_{\beta,\beta'}(z) \in Z_{\beta'}$. The cocycle condition $H_{\beta',\beta''} \circ H_{\beta,\beta'} \circ H_{\beta',\beta''} \mid_{\pi_1^\beta(U_\beta \cap U_{\beta'} \cap U_{\beta''})} = Id$ implies that if $z \sim z'$ and $z' \sim z''$, then $z'' \sim z$.

We denote by $q : \bigsqcup_\beta Z_\beta \to X$ the quotient map which takes each point $z$ to its equivalence class $q(z)$. Evidently, for each index $\beta$, $q : Z_\beta \to q(Z_\beta)$ is a homeomorphism.

It is on the level of definitions to verify that $X$ is a smooth compact $(n + 1)$-manifold with the smooth structure being defined by the atlas

$$\{q(Z_\beta), q^{-1} : q(Z_\beta) \to \mathbb{R} \times \mathbb{R}^n\}_\beta$$

and with the boundary $\partial X = \bigcup_\beta q(W_\beta)$ being diffeomorphic to $Y$.

Moreover, $X$ admits a 1-dimensional foliation $\mathcal{F}^u$ with oriented leaves. Indeed, each chart $q(Z_\beta)$ comes equipped with a 1-dimensional foliation $\mathcal{F}_\beta$ generated as the $q$-image of the connected components of fibers of the projection $\pi_\beta : Z_\beta \to \mathbb{R}^n$. Since the gluing diffeomorphisms $\{H_{\beta,\beta'}\}$ take components of the fibers of $\pi : Z_{\beta'} \to \mathbb{R}^n$ to components of the fibers of $\pi : Z_{\beta'} \to \mathbb{R}^n$ and preserve their orientation, these local foliations $\mathcal{F}_\beta$ produce a foliation $\mathcal{F}^u$ on $X$ with oriented leaves. We can equip $X$ with a Riemannian metric $g$. Let $v$ be the unit vector field tangent to the leaves of $\mathcal{F}^u$ and coherent with their orientation. By the very construction of $\mathcal{F}^u$ from the traversally generic building blocks $\{Z_\beta\}_\beta$, the field $v$ is traversally generic.

Thus, via this construction, we have built the desired pair $(X, v)$ from a given holographic and cobordant to $\emptyset$ atlas.

**Corollary 6.1.** If a closed smooth manifold $Y$ admits a holographic (orientably) cobordant to $\emptyset$ structure, then $Y$ is a boundary of a (orientable) smooth manifold $X$. □

Any proto-holographic structure $\mathcal{H}$ on $Y$ generates some familiar “proto-objects”. Among them is the stratification $\{\partial_j Y\}_{1 \leq j \leq n}$ of $Y$ by submanifolds of codimension $j$, an analogue of the Morse stratification $\{\partial_j \overline{X}(v)\}_{1 \leq j \leq n}$ (see [MQ]). When $\mathcal{H}$ is holographic, we can construct the surrogate “trajectory space” $\mathcal{T}(\mathcal{H})$ and the partially-defined “causality map”
Theorem 4.1, as well as [K3]). Note that the odd and even multiplicities are distinguished by their local topology.

**Lemma 6.1.** Every smooth proto-holographic structure $\mathcal{H}$ on a closed $n$-manifold $Y$ gives rise to a stratification $\{\partial_j^+Y := \partial_j^+Y(\mathcal{H})\}_{0 \leq j \leq n}$ of $Y$ by submanifolds $\partial_j^+Y$, $\partial_j^-Y$ of codimension $j$ such that:

- $\partial_j^+Y \cup \partial_0^-Y = Y$,
- $\partial(\partial_j^+Y) = \partial(\partial_j^-Y) = \partial_{j+1}^+Y \cup \partial_{j+1}^-Y$ for all $j$.

Every smooth holographic structure $\mathcal{H}$ on $Y$ gives rise to:

- a causality map $C_\mathcal{H} : \partial_0^+Y \to \partial_0^-Y$ (possibly discontinuous) whose fixed-point set is $\partial_1^-Y \cup \partial_2^-Y \cup \cdots \cup \partial_n^-Y$,
- a $C_\mathcal{H}$-trajectory space $T(\mathcal{H})$, the target of a finitely ramified map $\Gamma_\mathcal{H} : Y \to T(\mathcal{H})$.

Locally, $T(\mathcal{H})$ is modeled after the standard cellular complex $T_\omega$ (as in Theorem 5.3 from [K3]), where $\omega \in \Omega^*_{\mathcal{H}}$.

**Proof.** Consider a proto-holographic atlas $\mathcal{H} = \{U_\beta, h_\beta, h_{\beta,\beta'}\}$ on $Y$. Recall that any point $z_\beta^* = (u_\beta^*, x_\beta^*, y_\beta^*) \in W_\beta \subset \partial Z_\beta$ has a multiplicity $j(z_\beta^*)$, defined as the multiplicity of the $u_\beta$-polynomial $P_\beta(u_\beta, x_\beta)$ at its root $u_\beta^*$. In fact, $j(z_\beta^*)$ can be detected just in terms of the maximal cardinality of fibers of the map $\pi : W_\beta \to \mathbb{R}^n$, the fibers which are contained in a connected component of a fiber of $\pi : Z_\beta \to \mathbb{R}^n$. Here we localize the fibers of $\pi : W_\beta \to \mathbb{R}^n$ to the vicinity of $z_\beta^*$ in the hypersurface $W_\beta$. Indeed, if $j(z_\beta^*)$ is an even number, then any small neighborhood of $z_\beta^*$ in $Z_\beta$ contains trajectories of the combinatorial type $(122\ldots 21)$, where the number of 2’s is $(j(z_\beta^*) - 2)/2$; if $j(z_\beta^*)$ is an odd number, then any small neighborhood of $z_\beta^*$ in $Z_\beta$ contains trajectories of the combinatorial type $(122\ldots 2)$, where the number of 2’s is $(j(z_\beta^*) - 1)/2$ (see the arguments in the beginning of the proof of Theorem 4.1 as well as [K3]). Note that the odd and even multiplicities are distinguished by their local topology.

Thus, for any $y \in U_\beta \subset Y$, the point $z_\beta := h_\beta(y)$ acquires multiplicity $j_\beta(z_\beta)$. By the property of $h_{\beta,\beta'}$, described in the fourth bullet of Definition 6.1, the fibers of $\pi_\beta : h_\beta(U_\beta \cap U_{\beta'}) \to \mathbb{R}^n$, which belong to the same connected component of the corresponding fiber $\pi : Z_\beta \to \mathbb{R}^n$, are mapped to the fibers of $\pi_{\beta'} : h_{\beta'}(U_\beta \cap U_{\beta'}) \to \mathbb{R}^n$, which also belong to a connected component of the corresponding fiber $\pi : Z_{\beta'} \to \mathbb{R}^n$. Therefore, employing this property of diffeomorphism $h_{\beta,\beta'}$ and interpreting the multiplicities in terms of $\pi$-fibers’ cardinalities, we get $j_{\beta'}(h_{\beta'}(y)) = j_\beta(h_\beta(y))$ when $y \in U_\beta \cap U_{\beta'}$.

Consider the stratification $\{\partial_j^\pm W_\beta := \partial_{j+1}^\pm Z_\beta\}_j$ of $W_\beta$, generated by the field $\partial_\beta := \frac{\partial}{\partial u_\beta}$. We have seen that a proto-holographic atlas $\mathcal{H}$ helps to define a stratification

$$\{\partial_j Y := \partial_j Y(\mathcal{H})\}_{1 \leq j \leq n}$$

of $Y$ by closed submanifolds: just put

$$\partial_j Y \cap U_\beta := h_\beta^{-1}(\partial_{j+1} Z_\beta \cap h_\beta(U_\beta)).$$
The stratification \( \{ \partial_j Y(\mathcal{H}) \}_{1 \leq j \leq n} \) can be refined to a stratification
\[
\{ \partial_j^\pm Y := \partial_j Y^{\pm}(\mathcal{H}) \}_{1 \leq j \leq n}
\]
by compact submanifolds such that \( \partial_j Y = \partial_j^+ Y \cup \partial_j^- Y \) and \( \partial_j Y = \partial(\partial_j^+ Y) = \partial(\partial_j^- Y) \) for all \( j \). Let us describe its construction.

By Theorem 2.2 from [K2], the stratification \( \{ \partial_j^\pm W_\beta := \partial_j^\pm Z_\beta \}_{j} \) of \( W_\beta \), generated by the field \( \partial_\beta := \frac{\partial}{\partial \omega_\beta} \), is characterized as follows: \((u_\beta, \bar{x}^\beta, \bar{y}^\beta) \in \partial_j^+ Z_\beta \) if \( \frac{\partial^k}{\partial \omega_\beta^k} P_\beta(u_\beta, \bar{x}^\beta) = 0 \) for all \( k \leq j \), and \( \frac{\partial^{j+1}}{\partial \omega^{j+1}} P_\beta(u_\beta, \bar{x}^\beta) < 0 \).

In fact, the sign of the derivative \( \frac{\partial^{j+1}}{\partial \omega^{j+1}} P_\beta(u_\beta, \bar{x}^\beta) \) at a root \( u_\beta \) of multiplicity \( j \) can be determined by the simple combinatorial rule: consider the real divisor \( P_\beta(\bar{x}^\beta) \) of the \( u_\beta \)-polynomial \( P_\beta \) and decompose its support \( \text{sup}(P_\beta(\bar{x}^\beta)) \) into atoms and strings as in [K2], Fig. 1. Then any root of even multiplicity, which is an atom, is assigned the polarity “−”, any root of odd multiplicity which is inside of a string is assigned the polarity “+”, any root of odd multiplicity, forming the lower end of a string, is assigned the polarity “−”, while any root of odd multiplicity, forming the upper end of a string, is assigned the polarity “+”.

Therefore, the polarity \( \pm \) of a point \( z \in \partial_j W_\beta \) is determined by the ordered sequence of intersection points of the \( \partial_\beta \)-trajectory \( \gamma_z \subset Z_\beta \), through \( z = (u_\beta, \bar{x}^\beta, \bar{y}^\beta) \) with the hypersurface \( W_\beta \), each intersection point \( z \) being considered with its multiplicity \( j_\beta(z) \). As the combinatorial rule above implies, the polarities attached to each intersection \( z \in \gamma_z \cap W_\beta \) depend only on the multiplicities and the order of points from \( \gamma_z \cap W_\beta \) along \( \gamma_z \). Again, by the fourth bullet of Definition [6.1], this kind of data is preserved under the smooth change of coordinates \( h_{\beta,\beta'} : h_{\beta}(U_\beta \cap U_{\beta'}) \rightarrow h_{\beta'}(U_\beta \cap U_{\beta'}) \). Therefore, the local formulas \( \partial_j^\pm Y \cap U_\beta := h_{\beta}^{-1}(\partial_j^\pm Z_\beta \cap h_{\beta}(U_\beta)) \) produce the desired globally-defined stratifications.

For a given holographic atlas \( \mathcal{H} \), let us turn to the construction of the causality map \( C_\mathcal{H} : \partial_0^+ Y \rightarrow \partial_0^+ Y \). Take a point \( y \in U_\beta \cap \partial_0^+ Y \) to the point \( h_\beta(y) \in h_\beta(U_\beta) = W_\beta \). Then, in the fiber \( F := F_{h_\beta(y)} \) of the map \( \pi : h_{\beta}(U_\beta) \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), which contains the point \( h_\beta(y) \), take \( h_\beta(y) \) to the point \( z \) with the minimal first coordinate \( u_\beta(z) > u_\beta(h_\beta(y)) \) and such that \( P_\beta < 0 \) in the open interval \( (h_\beta(y), z) \subset \mathbb{R} \times \mathbb{R}^n \); if no such \( z \) exists, put \( z = h_\beta(y) \). Let \( C_\mathcal{H}(y) := h_{\beta}^{-1}(z) \). Since \( \mathcal{H} \) is a holographic structure, the map \( C_\mathcal{H} \) is well-defined for all points in \( \partial_0^+ Y \cap U_\beta \).

Note that the model causality map \( C_{\partial_\beta} : W_\beta^+ \rightarrow W_\beta^- \) has \( \partial_\beta^- W_\beta \cup \cdots \cup \partial_n^- W_{\beta} := \partial_\beta^- Z_{\beta} \cup \cdots \cup \partial_n^- Z_{\beta} \) for its fixed point set.

This local construction of \( C_\mathcal{H}(y) \) does not depend on the choice of a chart \( U_\beta \) which contains \( y \). Indeed, if \( y \in U_\beta \cap U_{\beta'} \), then, according to the fourth bullet in Definition [6.1], the connecting diffeomorphism \( h_{\beta,\beta'} \) maps each portion \( F \cap \gamma_{z_{\beta}} \) of a fiber \( F \) onto a portion

\[ \text{these signs are opposite to the signs of the lowest order non-trivial derivatives } \frac{\partial^{j+1}}{\partial \omega^{j+1}} P_\beta(u_\beta, \bar{x}^\beta). \]

\[ \text{These trajectories } \{ \gamma_z \} \text{ produce the strings and atoms in sup}(D_{P_\beta}(\bar{x}^\beta)). \]
$F' \cap \gamma_{z_{\beta'}}$ of the corresponding fiber $F'$. Here $z_{\beta} := h_{\beta}(y)$, and $\gamma_{z_{\beta}} \subset Z_{\beta}$ denotes the $\partial_{\beta}$-trajectory through $z_{\beta}$. Similarly, $F'$ denotes the fiber of $\pi : h_{\beta'}(U_{\beta} \cap U_{\beta'}) \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ that contains $z_{\beta'} := h_{\beta'}(y)$, and $\gamma_{z_{\beta'}} \subset Z_{\beta'}$ the corresponding $\partial_{\beta'}$-trajectory. Moreover, $h_{\beta,\beta'}$ respects the $u_{\beta}$-induced order of points in $F \cap \gamma_{z_{\beta}}$ and the $u_{\beta'}$-induced order of points in $F' \cap \gamma_{z_{\beta'}}$. Since the two locally-defined causality maps,

$$C_{H,\beta} : Y^+ \cap U_{\beta} \to Y^- \cap U_{\beta} \text{ and } C_{H,\beta'} : Y^+ \cap U_{\beta'} \to Y^- \cap U_{\beta'}$$

, operate within the sets $h_{\beta}^{-1}(F \cap \gamma_{z_{\beta}})$ and $h_{\beta'}^{-1}(F' \cap \gamma_{z_{\beta'}})$, respectively, by employing same combinatorial rules which depend only on the intrinsically-defined polarities of the corresponding points, by the third bullet in Definition [6.1] we get that $C_{H,\beta}(y) = C_{H,\beta'}(y)$.

With the given holographic atlas $\mathcal{H}$ on $Y$ in place, we declare two points $y_1, y_2 \in Y$ to be equivalent if there exists $y_0 \in Y$ such that both $y_1$ and $y_2$ can be obtained as images of $y_0$ under iterations of the causality map $C_H$. We denote by $\mathcal{T}(\mathcal{H})$ the quotient space of $Y$ by this equivalence relation and call it the trajectory space of $\mathcal{H}$. The obvious finitely-ramified map $\Gamma_H : Y \to \mathcal{T}(\mathcal{H})$ helps to define the quotient topology in the trajectory space $\mathcal{T}(\mathcal{H})$.

Note that, for each $y \in U_{\beta}$, the vicinity of its $C_H$-trajectory in $\mathcal{T}(\mathcal{H})$ is modeled after the standard cell complex $T_{\omega}$ as in Theorem 5.3 from [K3], where $\omega \in (\omega_{\beta})_{\subseteq} \subset \Omega^{\bullet}_{\{n\}}$. Indeed, the obvious projections $W_{\beta} \subset Z_{\beta} \to T_{\omega_{\beta}}$, with the help of $h_{\beta}^{-1}$, provide for the local models of $\Gamma_H$.

**Corollary 6.2.** For any traversally generic field $v$ on $X$, the $\Omega^{\bullet}_{\{n\}}$-stratified topological type of the trajectory space $\mathcal{T}(v)$ depends only on the holographic structure $\mathcal{H}(v)$

that $v$ generates on the boundary $\partial_1 X$.

**Proof.** By Theorem [6.1] the fields $v_i$ ($i = 1, 2$), give rise to holographic structures $\mathcal{H}(v_i)$. By Lemma [6.1] $\mathcal{H}(v_i)$ canonically produce causality maps $C_{\mathcal{H}(v_i)}$ (which happen to coincide with the causality maps $C_{v_i}$). By the hypotheses, $\mathcal{H}(v_1) = \mathcal{H}(v_2)$, which implies the equality $C_{\mathcal{H}(v_1)} = C_{\mathcal{H}(v_2)}$. Since the spaces $\mathcal{T}(v_i)$ depend only on $C_{\mathcal{H}(v_1)} = C_{v_1}$, we conclude that $\mathcal{T}(v_1)$ and $\mathcal{T}(v_2)$ are homeomorphic as $\Omega^{\bullet}_{\{n\}}$-stratified spaces.

**Remark 6.5.** In principle, in accordance with Theorem [6.1] a given holographic cobordant to $\emptyset$ structure $\mathcal{H}$ on $Y$, may have many realizations $\{\mathcal{H}(v)\}$ by traversally generic fields $v$ on a variety of $X$’s whose boundary is $Y$. However, Holography Theorem [4.2] claims that the smooth type of $X$, together with the $v$-induced oriented foliation $\mathcal{F}(v)$, is determined by $\mathcal{H}$ via the causality map $C_{\mathcal{H}} = C_{v}$.

**Definition 6.4.** Let $\mathcal{H}$ be a smooth holographic structure on $Y$ and $\mathcal{T}(\mathcal{H})$ its trajectory space.

A continuous function $h : \mathcal{T}(\mathcal{H}) \to \mathbb{R}$ is called holographically smooth, if the composite function $Y \xrightarrow{\gamma_H} \mathcal{T}(\mathcal{H}) \xrightarrow{h} \mathbb{R}$ is smooth on $Y$.

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25 which is cobordant to $\emptyset$
We denote by $C^\infty(\mathcal{T}(\mathcal{H}))$ the algebra of holographically smooth functions on $\mathcal{T}(\mathcal{H})$. Via the induced monomorphism $\Gamma_{\mathcal{H}}^*$, the algebra $C^\infty(\mathcal{T}(\mathcal{H}))$ maps onto a subalgebra $C^\infty_{\mathcal{H}}(Y)$ of $C^\infty(Y)$. □

One should compare Definition 6.4 of the algebra $C^\infty_{\mathcal{H}}(Y)$ with Conjecture 4.1.

**Corollary 6.3.** Let $\mathcal{H}$ be a holographic structure on a closed smooth manifold $Y$.

Then $\mathcal{H}$ determines the algebra $C^\infty(\mathcal{T}(\mathcal{H}))$ of holographically smooth functions on the trajectory space $\mathcal{T}(\mathcal{H})$ as a subalgebra $C^\infty_{\mathcal{H}}(Y)$ of $C^\infty(Y)$.

Moreover, any diffeomorphism $\Phi : Y \to Y$ induces an algebra isomorphism $\Phi^* : C^\infty_{\mathcal{H}}(Y) \to C^\infty_{\Phi^*\mathcal{H}}(Y)$, where $\Phi^*\mathcal{H}$ denotes the $\Phi$-induced holographic structure.

**Proof.** By Lemma 6.1, the obvious maps $\Gamma_{\mathcal{H}} : Y \to \mathcal{T}(\mathcal{H})$ and $\Gamma_{\Phi^*\mathcal{H}} : Y \to \mathcal{T}(\Phi^*\mathcal{H})$ are defined, via the proto-causality maps $C_{\mathcal{H}}$ and $C_{\Phi^*\mathcal{H}}$, in terms of the structures $\mathcal{H}$ and $\Phi^*\mathcal{H}$, respectively.

It is on the level of definitions to check that $\Phi \circ \Gamma_{\Phi^*\mathcal{H}} = \Gamma_{\mathcal{H}}$. Therefore, using that $\Phi$ is a diffeomorphism, we get that $\Phi^* : C^\infty_{\mathcal{H}}(Y) \to C^\infty_{\Phi^*\mathcal{H}}(Y)$ is an isomorphism of algebras. □

**Question 6.1.** Does the pair of algebras $C^\infty_{\mathcal{H}}(Y) \subset C^\infty(Y)$ determine the holographic structure $\mathcal{H}$ on $Y$? □

Now let us combine Holography Theorem 4.2 with Theorem 6.1 into a single proposition.

**Theorem 6.2. (The Smooth Holographic Principle)**

For a traversally generic field $v$ on a (oriented) manifold $X$, the smooth type of $(X, F(v))$ is determined by the $v$-induced (orientably) cobordant to $\emptyset$ holographic structure $\mathcal{H}(v)$ on $\partial_1 X$.

Conversely, any (orientably) cobordant to $\emptyset$ holographic structure $\mathcal{H}$ on $\partial_1 X$ is of the form $\mathcal{H}(v)$ for a traversally generic field $v$ on some (orientable) manifold $X$. Moreover, the smooth topological type of the pair $(X, F(v))$ is determined by $\mathcal{H}$.

**Proof.** By Theorem 6.1 a traversally generic field $v$ on $X$ gives rise to a holographic structure $\mathcal{H}(v)$ on $\partial_1 X$, $\mathcal{H}(v)$ being cobordant to $\emptyset$.

On the other hand, by the same Theorem 6.1, any given holographic structure $\mathcal{H}$ on $\partial_1 X$, which is cobordant to $\emptyset$, produces a traversally generic field $v$ on some $X$.

By Lemma 6.1 the causality map $C_v$ can be described intrinsically in terms of $\mathcal{H}$ as the map $C_{\mathcal{H}} : \partial_1^+ X(\mathcal{H}) \to \partial_1^- X(\mathcal{H})$. By Holography Theorem 4.2, $C_v$ allows for a reconstruction of the smooth type of the pair $(X, F(v))$. Therefore, $C_{\mathcal{H}}$ does the same job. □

Let us recall one old and appealing result of Wall [Wa], the classification of smooth highly connected even-dimensional manifolds:

\[26\text{Indeed, the map } Y \to \mathcal{T}(\mathcal{H}) \text{ is onto.}\]
**Theorem 6.3. (Wall)** For \( n \geq 3 \) the diffeomorphism class of \((n-1)\)-connected \(2n\)-manifold \(X\) with boundary a homotopy sphere are in natural bijections with the isomorphism classes of \(\mathbb{Z}\)-valued non-degenerate \((-1)^n\)- symmetric bilinear forms with a quadratic refinement in \(\pi_n(BSO(n))\).

**Remark 6.6.** Recall that, if a homotopy sphere \(\Sigma^{2n-1}\), \(n \geq 3\), bounds a framed smooth \((2n)\)-manifold \(Y\), then by surgery, \(\Sigma^{2n-1}\) bounds a framed \((n-1)\)-connected manifold \(X\) (cf. [Kos], Theorem 2.2 on page 201). Therefore, all such spheres satisfy the hypotheses of Theorem 6.3.\footnote{\(\pi_n(BSO(n)) \approx \mathbb{Z}\) if \(n \equiv 0(2)\), and \(\pi_n(BSO(n)) \approx \mathbb{Z}_2\) if \(n \equiv 1(2)\).} \(\square\)

The smooth structures on homotopy spheres have been a favorite subject in topology for a number of years. To support the tradition, we will make few superficial observations about holographic structures on the homotopy spheres and their relations to the underlying smooth structures.

**Corollary 6.4.** Let \(\mathcal{H}\) denote the set of holographic orientably cobordant to \(\emptyset\) structures on homotopy \((2n-1)\)-spheres.

For \(n \geq 3\), the set \(\mathcal{H}\) maps to the set of isomorphism classes of \(\mathbb{Z}\)-valued non-degenerate over the rationals \((-1)^n\)- symmetric bilinear forms on free finite-dimensional \(\mathbb{Z}\)-modules. Any such bilinear form \(\Phi\), which is non-degenerate over the integers \(\mathbb{Z}\) and admits a quadratic refinement \(\mu\) as in (6.1), is in the image of such map.

**Proof.** Let \(\mathcal{H}\) be a holographic orientably cobordant to \(\emptyset\) structure on a homotopy sphere \(\Sigma^{2n-1}\). By Theorem 6.2, such \(\mathcal{H}\) is of the form \(\mathcal{H}(v)\) for some traversally generic \(v\) on an oriented manifold \(X\) whose boundary is \(\Sigma^{2n-1}\). Moreover, the topological type of \(X\) is determined by \(\mathcal{H} = \mathcal{H}(v)\).

Let

\[
\Phi_{\mathcal{H}} : \left( H_n(X;\mathbb{Z})/\Tor \right) \otimes \left( H_n(X;\mathbb{Z})/\Tor \right) \to \mathbb{Z}
\]

be the bilinear \((-1)^n\)-symmetric intersection form on \(X\). Since \(\partial X\) is a homotopy sphere, the form \(\Phi_{\mathcal{H}}\) is non-degenerate over the rational field \(\mathbb{Q}\) (if \(H_{n-1}(X;\mathbb{Z})\) is torsion-free, then \(\Phi_{\mathcal{H}}\) is non-degenerated over \(\mathbb{Z}\)). Thus we have mapped any holographic orientably cobordant to \(\emptyset\) structure \(\mathcal{H}\) on a homotopy \((2n-1)\)-sphere to the bilinear form \(\Phi_{\mathcal{H}}\).

Let us show that any bilinear \((-1)^n\)-symmetric non-degenerate over \(\mathbb{Z}\) form \(\Phi : H \otimes H \to \mathbb{Z}\) on a finitely-dimensional free \(\mathbb{Z}\)-module \(H\) can be produced this way, provided that \(\Phi\) is enhanced by a quadratic form \(\mu : H \to \mathbb{Z}/(1 - (-1)^n)\). Recall that the relations between \(\Phi\) and \(\mu\) are described by the formulas:

\[
(6.1) \quad \Phi(a, a) = \mu(a) + (-1)^n \mu(a), \quad \Phi(a, b) = \mu(a + b) - \mu(a) - \mu(b) \mod (1 - (-1)^n)
\]

for any \(a, b \in H\). Note that the RHS of the formula for \(\Phi(a, a)\) is well-defined as an element of \(\mathbb{Z}\).

By Theorem 6.3, any such pair \((\Phi, \mu)\) is realized on some smooth orientable \((n-1)\)-connected \(2n\)-manifold \(X\) which bounds a homotopy sphere \(\Sigma^{2n-1}\). The quadratic form
Lemma 6.2. Let $\mathcal{H}(v_1)$ be a holographic orientably cobordant to $\emptyset$ structure on a homotopy $(2n - 1)$-sphere $\Sigma_i$, $i = 1, 2$.

Then the bilinear $\mathbb{Z}$-form $\Phi_{\mathcal{H}(v)}$ that, according to Corollary 6.4, corresponds to the holographic structure $\mathcal{H}(v) : = \mathcal{H}(v_1) \# \mathcal{H}(v_2)$ on the homotopy sphere $\Sigma : = \Sigma_1 \# \Sigma_2$ is the direct sum $\Phi_{\mathcal{H}(v_1)} \oplus \Phi_{\mathcal{H}(v_2)}$ of the forms $\Phi_{\mathcal{H}(v_1)}$ and $\Phi_{\mathcal{H}(v_2)}$. □

Proof. Employing the construction from the proof of Corollary 6.4, the validation of the lemma is at the level of definitions. □

The reader should be aware: now we are entering the zone of speculations...

We leave to the reader to discover a generalization of the boundary connected sum construction $\mathcal{H}(v) : = \mathcal{H}(v_1) \# \mathcal{H}(v_2)$, a generalization that will produce a holographic structure $\mathcal{H} : = \mathcal{H}_1 \# \mathcal{H}_2$ on $\Sigma : = \Sigma_1 \# \Sigma_2$, utilizing two given holographic structures: $\mathcal{H}_1$ on $\Sigma_1$ and $\mathcal{H}_2$ on $\Sigma_2$. Each of these two structures $\mathcal{H}_i$ ($i = 1, 2$) is enhanced by a choice of a pair of “base” points $s_i^+, s_i^- \in \Sigma_i$. Both points are contained in some element $U_{s_i}^{\text{base}}$ of the combinatorial type (11) from the atlas $\mathcal{H}_i$; their images, $h_i^{\text{base}}(s_i^+)$ and $h_i^{\text{base}}(s_i^-)$, under the atlas homeomorphism $h_i^{\text{base}} : U_i^{\text{base}} \to \mathbb{R}^n \times \mathbb{R}$, project to the same point in $\mathbb{R}^n$.

Note that the trajectory space $\mathcal{T}(\mathcal{H}_1 \# \mathcal{H}_2)$ is obtained from the spaces $\mathcal{T}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2)$ by gluing them along some $n$-disks $D^n_1$ and $D^n_2$, the disk $D^n_i$ resides in the maximal (i.e., indexed by $\omega = (11)$) stratum of $\mathcal{T}(\mathcal{H}_i)$.
The operation
\[
(6.2) \quad (\mathcal{H}_1; s_1^+, s_1^-, (\mathcal{H}_2; s_2^+, s_2^-) \Rightarrow (\mathcal{H}_1 \#_\partial \mathcal{H}_2; s_1^+, s_2^-)
\]
converts the set of all based holographic structures on smooth homotopy $n$-spheres into a semigroup (monoid) $\mathcal{H}_\Sigma_n$. The associativity of the operation in (6.2) is evident.

The order of two holographic structures $\mathcal{H}_1$ and $\mathcal{H}_2$ in the operation “$\#_\partial$” does matter: we think of $\Sigma_2$ as residing “above” $\Sigma_1$, at least in the vicinity of points $s_1^- \in \Sigma_1$ and $s_2^+ \in \Sigma_2$ where the “virtual” 1-handle $H$, responsible for the operation, is attached.

It seems that the holographic structure $\mathcal{H}_1 \#_\partial \mathcal{H}_2$ indeed may depend on the choice of base pairs $(s_1^+, s_1^-)$ and $(s_2^+, s_2^-)$ and may change at least when they have been chosen in different connected components of the pure strata of the combinatorial type (11) in $\Sigma_1$ or $\Sigma_2$.

Employing the connected sums of based holographic structures on homotopy $n$-spheres, we can form words, representing elements of $\mathcal{H}_\Sigma_n$, in an alphabet to be discovered and understood...

We denote by $\mathcal{HS}_n$ the sub-semigroup of based holographic structures on the standard $n$-sphere $S^n$. We also will employ a smaller semigroup $\mathcal{HB}_{n+1}$, based on holographic structures $\mathcal{H}(v)$ on $S^n$ that are induced by traversally generic fields $v$ on the standard ball $B^{n+1}$.

Unfortunately, we do not know much about “the place and size” of $\mathcal{HB}_{n+1}$ inside of $\mathcal{HS}_n$.

Note that both $\mathcal{HS}_n$ and $\mathcal{HB}_{n+1}$ act, via the connected sum operation $\#_\partial$, on $\mathcal{H}_\Sigma_n$ from the left and from the right.

Here is the informal question that motivates us:

What part of a given holographic structure on a closed manifold can be localized to the “spherical nuggets” representing an element of $\mathcal{HS}_n$ or even of $\mathcal{HB}_{n+1}$?

Specifically, if a based holographic structure $\mathcal{H}$ on some homotopy $n$-sphere $\Sigma$ is a connected sum of based holographic structures $\mathcal{H}_i$ on homotopy $n$-spheres $\Sigma_i$, then we would like to be able to “drop” the summands-words that correspond to the elements of $\mathcal{HB}_{n+1}$. When playing a cruder game, we would like to be able to “drop” the summands-words that correspond to the elements of $\mathcal{HS}_n$.

**Definition 6.5.** We introduce two sets of relations among the elements of the semigroup $\mathcal{H}_\Sigma_n$ by declaring:

1. either all the elements of $\mathcal{HS}_n$ as being trivial, or
2. all the elements of $\mathcal{HB}_{n+1}$ as being trivial.

We say that two based holographic structures on homotopy $n$-spheres are stably equivalent, if they become equal modulo the relations of the first type.

We denote by $\mathcal{H}_\Sigma_n^{st}$ the set of based stably equivalent holographic structures on the homotopy $n$-spheres.

Similarly, we say that two based holographic structures on homotopy $n$-spheres are strongly stably equivalent, if they become equal modulo the relations of the second type.
We denote by $\mathcal{H}\Sigma_n^\text{st}$ the set of based strongly stably equivalent holographic structures on the homotopy $n$-spheres.

The stable and strongly stable equivalence classes of based holographic structures on homotopy spheres are amenable to the connected sum operation. Therefore, these equivalences give rise to the obvious epimorphisms of semigroups:

\[
\mathcal{H}\Sigma_n \to \mathcal{H}\Sigma_n^\text{st} \to \mathcal{H}\Sigma_n
\]

Let $\Theta_n$ be the commutative semigroup of smooth homotopy $n$-spheres with the addition induced by the connected sum operation. For $n \neq 4$, thanks to the $h$-cobordism theorem (see [Mi], [P1], and [P2]), $\Theta_n$ is an abelian group. The role of opposite element $-\Sigma^n$ is played by the homotopy sphere $\Sigma^n$, taken with the opposite orientation.

Lemma 6.3. For $n \neq 4$, under the connected sum operation, the set $\mathcal{H}\Sigma_n^\text{st}$ is a group.

Proof. Given a based holographic structure $(\mathcal{H}; s^+, s^-)$ on $\Sigma^n$, represented by an atlas $A := \{ h_\beta : U_\beta \to \mathbb{R}^n \times \mathbb{R} \}$, consider the based holographic structure $(\sim \mathcal{H}; s^+, s^-)$ represented by the same atlas $\sim A$ on $\Sigma^n$ with the orientation of $\Sigma^n$ being reversed. The holographic structure $\mathcal{H}\#_0 \sim \mathcal{H}$ is defined on the homotopy sphere $\Sigma^n \#_0 - \Sigma^n$. By the $h$-cobordism theorem, $\Sigma^n \#_0 (-\Sigma^n) \approx S^n$ for any $n \neq 4$. Thus the holographic structure $\mathcal{H}\#_0 \sim \mathcal{H}$ is defined on $S^n$, and thus is trivial in $\mathcal{H}\Sigma_n^\text{st}$.

Corollary 6.5. For $n \neq 4$, the forgetful map $\mathcal{K}^\text{st}_n : \mathcal{H}\Sigma_n^\text{st} \to \Theta_n$ is an epimorphism of groups, while the forgetful map $\mathcal{K}_n : \mathcal{H}\Sigma_n \to \Theta_n$ is an epimorphism of semi-groups.

Proof. Homotopy spheres are stably parallelizable manifolds (see [Kos], Corollary 8.6). So, their Pontrjagin and Stiefel-Whitney characteristic classes vanish. Therefore any homotopy sphere $\Sigma^n$ bounds an oriented manifold $X$ (see [CF]).

By Theorem 3.5 from [K2], $X$ admits a traversally generic field $v$ which generates a holographic orientably cobordant to $\emptyset$ structure $\mathcal{H}(v)$ on $\Sigma^n$. As a result, any homotopy sphere admits a holographic cobordant to $\emptyset$ structure. Thus the forgetful map $\mathcal{K}^\text{st}_n$ is surjective. By Lemma 6.2, it is a homomorphism of groups.

Thanks to the pioneering works of Kervaire & Milnor [KM], Browder [Br], and recently, of Hill, Ravenel, & Hopkins [HIR], the groups $\Theta_n$, $n \neq 4$, are well-understood\(^{28}\) in terms of the stable homotopy groups of spheres $\pi_n^\text{st}$ and of the homomorphism

\[
J : \pi_n(SO(N)) \to \pi_{n+N}(S^N)
\]

, where $N \gg n$.

So, for $n \neq 4$, the group of stably equivalent holographic structures on homotopy $n$-spheres is at least as rich as its abelian image $\Theta_n$.

Against the background of our investigations of traversing flows, the kernels of the forgetful maps $\mathcal{K}^\text{st}_n : \mathcal{H}\Sigma_n^\text{st} \to \Theta_n$ and $\mathcal{K}_n : \mathcal{H}\Sigma_n \to \Theta_n$ are the most interesting objects to study.

\(^{28}\)Presently, the only remaining “problematic” dimensions are $n = 4$ and $n = 126$.\]
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