Robust T-S Fuzzy Control of Electrostimulation for Paraplegic Patients considering Norm-Bounded Uncertainties

Márcio Roberto Covacic, Marcelo Carvalho Minhoto Teixeira, Aparecido Augusto de Carvalho, Rodrigo Cardim, Edvaldo Assunção, Marcelo Augusto Assunção Sanches, Henrique Shuiti Fujimoto, Maxwell Simões Mineo, Anderson Ross Biazeto, and Ruberlei Gaino

1Department of Electrical Engineering, State University of Londrina, Rodovia Celso Garcia Cid (PR 445), Km 380, Londrina 86057-970, Paraná, Brazil
2Department of Electrical Engineering, São Paulo State University (UNESP), 1370 Prof. José Carlos Rossi Avenue, Ilha Solteira 15385-000, Brazil

Correspondence should be addressed to Márcio Roberto Covacic; marciocovacic@uel.br

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This manuscript presents a Takagi–Sugeno fuzzy control for a mathematical model of the knee position of paraplegic patients using functional electrical stimulation (FES). Each local model of the fuzzy system is represented considering norm-bounded uncertainties. After obtaining the model of FES with norm-bounded uncertainties, the fuzzy control strategy is designed through the solution of linear matrix inequalities (LMIs) using the conditions available in the literature, which consider these norm-bounded uncertainties. The strategy considers decay rate and constraints on the input signal. The model is simulated in the Matlab environment using the numerical parameters measured by experimental tests from a paraplegic patient.

1. Introduction

Several researchers have used functional electrical stimulation (FES) to restore some motion activities of people with injured spinal cord [1]. However, FES is not yet a regular clinical method because the amount of effort involved in using actual stimulation systems still outweighs the functional benefits they provide. One serious problem of using FES is that artificially activated muscles fatigue at a faster rate than those activated by the natural physiological processes. Due to this problem, a considerable effort has been directed toward developing FES systems based on closed-loop control. The movement is measured in real time with several types of sensors, and the stimulation pattern is modulated accordingly [1]. The dynamics of the lower limb is represented by a nonlinear second-order model, which considers the gravitational and inertial characteristics of the anatomical segment as well as the damping and stiffness properties of the knee joint.

In this paper, we present a Takagi–Sugeno nonlinear system with the aim of controlling the position of the leg of a paraplegic patient. The controller was designed in order to change the angle of the knee joint from $\theta^\circ$ to $30^\circ$ when electrical stimulation is applied in the quadriceps muscle.

The authors considered the leg mathematical model proposed by Ferrarin and Pedotti [1], with the parameter values given in [2, 3]. The parameters $B$ (viscous coefficient), $J$ (inertial moment), $\tau$ (time constant), and $G$ (static gain) of the shank-foot complex model have the nominal values given in [2, 3], but with a 20% tolerance range around these nominal values, that is, these values are in the range between 80% and 120% of their nominal values. The minimum and maximum values of the nonlinear term $f_{24}(x_1)$ are computed considering the angle variation from $0^\circ$ to $60^\circ$, that
is, $-30^\circ \leq x_i \leq 30^\circ$. The range of values of $B$, $J$, $\tau$, and $G$ are considered as norm-bounded uncertainties, whose analysis requires a lower number of linear matrix inequalities (LMIs), compared to polytopic uncertainty analysis, obtaining a lower computational cost. For the case studied in this manuscript, with two local models and four uncertain parameters, the control design methods that consider polytopic uncertainty analysis require the solution of a set of 49 LMIs, while, for the norm-bounded uncertainty analysis, only 4 LMIs are required. In this paper, the proposal for the knee position control design of paraplegic patients with functional electrical stimulation (FES) considers that the parameters of the mathematical model of the system are uncertain, whose uncertainties are bounded in norm. To the authors' knowledge, the Takagi–Sugeno (T-S) fuzzy control considering norm-bounded uncertainties, applied to the knee joint movement of the paraplegic patient, was not published yet.

The simplest design technique to obtain a design model for nonlinear plants is its linearization at an interest point. However, this linearized design described is not adequate when the system operates far from the operation point. A possible solution for this problem is the nonlinear plant representation by T-S fuzzy models, whose idea consists on the description of the nonlinear system as a combination of a certain number of local linear models. So, the global model is obtained by the fuzzy combination of these local linear models.

The T-S fuzzy methodology can also be applied if the nonlinear model contains polytopic uncertainties, as in [4]. The representation of the uncertainties considers that the uncertain system structure varies according to a convex combination of some vertices that limit the polytope, where each vertex is described as a fuzzy combination of local linear models. Fuzzy control theory is useful because the fuzzy systems can approximately represent real systems with a precision that can be specified by the designer. Furthermore, there are several types of models, suitable for different applications, since linguistic models for modeling a given system, even the T-S models, whose structure is suitable for control applications. The systematic procedure to design fuzzy control systems involves the fuzzy model construction for nonlinear systems [5].

The parallel distributed compensation (PDC) [6] in fuzzy regulator design can be used to stabilize nonlinear systems described by fuzzy models. The idea is to design a compensator for each fuzzy rule. For each rule, there exists an associated controller. The resulting global fuzzy regulator, which is nonlinear in general, is a fuzzy combination of each individual linear regulator. The PDC offers a procedure to design a regulator for each T-S fuzzy model, where each control rule is designed from the correspondent plant T-S model rule.

An important fact that motivates the use of the representation of a broad class of nonlinear plants by Takagi–Sugeno fuzzy models for designing suitable controllers is that they usually allow a design procedure, with PDC approach, based on LMIs. LMI-based designs can offer conditions for the stability of the equilibrium point, and also, it is possible to specify other performance indexes, such as decay rate, constraints of the input and/or output, and the minimization of the $\mathcal{H}_\infty$ cost, even for plants with uncertain parameters. Furthermore, the procedure for finding a feasible solution, when there exists one, becomes a convex optimization problem, and there exist easy methods such as polynomial time algorithms for solving this class of problem. With the aforementioned relevant facts, many researchers are using Takagi–Sugeno fuzzy models for obtaining adequate controllers for solving complex nonlinear control problems.

Muscle is a highly complex nonlinear system [7], capable of producing the same output for a variety of inputs. A property exploited by the physiologically activated muscle is its effort to minimize fatigue [8]. Considering that when the quadriceps is electrically stimulated, its response is nonlinear, we used T-S fuzzy models in order to design a controller for the knee angle variation.

Ferrarin and Pedotti [1] showed that, for the conditions considered in their experiments, a simple one-pole transfer function was able to model the relationship between stimulus pulse width and active muscle torque. The nonlinear term $f_{21}(x_1)$ is analyzed in a T-S fuzzy representation. So, the mathematical model of the functional electrical stimulation of the knee angle of the paraplegic patient is represented as a T-S fuzzy combination of two local models, considering the minimum and the maximum values of the nonlinear term $f_{21}(x_1)$.

In [9], the nonlinear term $f_{21}(x_1)$ is considered as the uncertain parameter, and the parameters $B$, $J$, $\tau$, and $G$ of the shank-foot complex model have definite values. However, these parameters assume different values each day, depending on the health conditions of the patient at each moment. Fatigue can also change these values. So, these parameters have unknown values, but these can be considered in determinate ranges, bounded by the minimum and the maximum values of each parameter [10].

According to Santos et al. and Gaino [9, 11], the application of FES on the quadriceps muscle, more particularly on the motor neurons of a person, causes an involuntary contraction of the muscle of the leg, that is, causes an action potential (AP). FES is also known by neuromuscular electrical stimulation (NMES) [11].

In order to obtain the muscle contraction, the amplitude (or intensity) and duration of the electrical stimulus must be inside specific bounds. Then, the AP is generated and propagates in both directions of the nerve fiber [9]. Complex mechanisms of electrochemical stimuli occur in the neuromuscular structure causing the process of excitation-contraction coupling responsible for the movement of the leg [11]. The modulation of the force, by the number of muscle fibers recruited, and the speed of fiber recruitment depend on several parameters. Some of these parameters include the proximity of the nerve fiber and the electrode, the electrode diameter, and the variation of the number of active states of the fibers by the variation
of the amplitude or pulse duration [9, 11]. As can be seen in Figure 1, the degree of muscle activation (\(\alpha\)) is a nonlinear function that depends on the duration of the stimulus \(d\).

By the knowledge of the authors, scientific studies on the application of T-S controllers to control the leg position of paraplegic patients are interesting, relevant, and challenging research topics. Recently, few articles have been published in this area, as, for instance, [12, 13]. In [11] and subsequent articles [2, 3, 14–17], published by our research group, some theories have been employed to control knee joint movement using neuromuscular electrical stimulation. In [18], an article was published describing control application in paraplegic patients.

The T-S fuzzy strategy for control of the knee joint angle of a paraplegic position using LMIs and considering polytopic uncertainties was studied in [19], but other papers involving T-S fuzzy control with polytopic uncertainties were not found in the literature. The T-S fuzzy control using LMIs and considering norm-bounded uncertainties was not found in the literature. In [20], LMI stabilization conditions considering polytopic and norm-bounded uncertainties were presented for fractional-order systems, but the method does not consider fuzzy models.

The list of the variables used in this manuscript is presented in Appendix.

2. General Takagi–Sugeno Fuzzy Representation

Certain classes of nonlinear systems can be exactly represented with T-S fuzzy models, using the method described in [5]. According to this construction method, the local models are obtained in function of the operation region.

For \(i = 1, 2, \ldots, q\), the \(i\)-th rule of the continuous-time T-S fuzzy model is described as

\[
\text{rule } i: \text{ if } z_1(t) \in M_i^1 \text{ and } \ldots z_p(t) \in M_i^p, \\
\text{ then } \begin{cases} \\
\dot{x}(t) = A_i x(t) + B_i u(t), \\
y(t) = C x(t). \end{cases}
\]

In the fuzzy model (1), \(M_i^j, j = 1, 2, \ldots, p\), is the fuzzy set of rule \(i\), and \(z_1(t), \ldots, z_p(t)\) are the premise variables. Let \(\mu_i^j(z_j(t))\) be the membership function of the fuzzy set \(M_i^j\), and define

\[
w_i^j(z(t)) = \prod_{j=1}^{p} \mu_i^j(z_j(t)), \\
z(t) = [z_1, \ldots, z_p]^T.
\]

Since \(\mu_i^j(z_j(t)) \leq 0\), one has, for \(i = 1, 2, \ldots, q\),

\[
w_i^j(z(t)) \geq 0, \quad i = 1, \ldots, q.
\]

The resulting fuzzy model is the weighted mean of the local models, given by

\[
\dot{x}(t) = \frac{\sum_{i=1}^{q} w_i^j(z(t))(A_i x(t) + B_i u(t))}{\sum_{i=1}^{q} w_i^j(z(t))} \\
= \sum_{i=1}^{q} \alpha_i(z(t))(A_i x(t) + B_i u(t)) = A(\alpha)(z(t)) + B(\alpha)u(t)
\]

\[
y(t) = C(x(t)),
\]
where

\[ \alpha = [\alpha_1, \ldots, \alpha_q]^T, \]

\[ \alpha_i(z(t)) = \frac{u_i(z(t))}{\sum_{i=1}^{q} u_i(z(t))} \]

\[ \alpha_i(z(t)) \geq 0, \quad i = 1, \ldots, q, \sum_{i=1}^{q} \alpha_i(z(t)) = 1. \]  \hspace{1cm} (5)

2.1. Regulators with Takagi–Sugeno Fuzzy Models. The PDC [6] offers a procedure to design a regulator for the T-S fuzzy model, where each control rule is designed from the corresponding plant T-S model rule. The designed fuzzy regulator shares the local controllers, each one given by

\[ \text{rule } i: \quad \text{if } z_1(t) \text{ is } \mathcal{M}_1^i \text{ and} \ldots \text{ and } z_p(t) \text{ is } \mathcal{M}_p^i, \]

then \( u(t) = -K_i x(t). \) \hspace{1cm} (6)

The fuzzy global regulator is given by

\[ u(t) = -\sum_{i=1}^{q} \alpha_i(z(t))K_i x(t) = -K(\alpha)x(t), \]

where \( \alpha = [\alpha_1, \ldots, \alpha_q]^T. \)

So, the closed-loop system is composed as follows:

\[ \dot{x}(t) = \sum_{i=1}^{q} \sum_{j=1}^{q} \alpha_i(z(t))\alpha_j(z(t))(A_i - B_j K_j)x(t). \]  \hspace{1cm} (8)

3. Norm-Bounded Uncertainties

According to [21], a control system with norm-bounded uncertainties is described by

\[ \dot{x}(t) = (A_n + \delta A)x(t) + (B_n + \delta B)u(t), \]

\[ y(t) = Cx(t), \]

with \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, \) and \( y(t) \in \mathbb{R}^p. \) Matrices \( A_n, B_n, \) and \( C_n \) are composed by the nominal values of the uncertain parameters.

The uncertain matrices \( \delta A \) and \( \delta B \) can be represented as

\[ \delta A = L \Delta R_A, \]

\[ \delta B = L \Delta R_B, \]  \hspace{1cm} (10)

where

\[ L = \begin{bmatrix} \ell_{11} & \ell_{12} & \cdots & \ell_{1v} \\ \ell_{21} & \ell_{22} & \cdots & \ell_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nv} \end{bmatrix}, \]

\[ \Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_v\}, \quad \|\Delta\| = \max|\delta_i| \leq 1, \]

\[ R_A = \begin{bmatrix} r_{a11} & r_{a12} & \cdots & r_{a1n} \\ r_{a21} & r_{a22} & \cdots & r_{a2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{am1} & r_{am2} & \cdots & r_{amn} \end{bmatrix}, \]

\[ R_B = \begin{bmatrix} r_{b11} & r_{b12} & \cdots & r_{b1m} \\ r_{b21} & r_{b22} & \cdots & r_{b2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{bmn1} & r_{bmn2} & \cdots & r_{bmnm} \end{bmatrix}. \]

So, matrix \( \delta A \) is given by

\[ \delta A = L \Delta R_A = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{mn1} & \phi_{mn2} & \cdots & \phi_{mnn} \end{bmatrix}, \]  \hspace{1cm} (12)

with

\[ \phi_{ij} = \sum_{k=1}^{v} \ell_{ik} \delta_k r_{akj}, \]  \hspace{1cm} (13)

for \( i, j = 1, 2, \ldots, n. \) Then, considering (13), one obtains \( n^2v \) equations that are given by

\[ \ell_{ik} r_{akj} = \frac{\partial \phi_{ij}}{\partial \delta_k}, \]  \hspace{1cm} (14)

for \( i, j = 1, 2, \ldots, n \) and \( k = 1, 2, \ldots, v. \)

Similarly, matrix \( \delta B \) is given by

\[ \delta B = L \Delta R_B = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{mn1} & \beta_{mn2} & \cdots & \beta_{mnm} \end{bmatrix}, \]  \hspace{1cm} (15)

where

\[ \beta_{ij} = \sum_{k=1}^{v} \ell_{ik} \delta_k r_{bkj}, \]  \hspace{1cm} (16)

for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m. \) Then, considering (16), one obtains \( nmv \) equations that are given by
\[ \ell_{ikr_{ikj}} = \frac{\partial \theta_{ij}}{\partial k} \]

for \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m, \) and \( k = 1, 2, \ldots, v. \)

### 4. Knee Joint Model

The mathematical model of the leg employed in this manuscript was proposed in [1]. This model relates the applied pulse width to the torque generated on the knee joint. In the modeling [1], the leg was considered as an open kinematic system composed of two stiff segments: the thigh and the shin-foot complex, as shown in Figure 2. From [1], the equilibrium equation around the knee joint is

\[ J \ddot{\theta} = -m gl \sin(\theta) - M_s - B \dot{\theta} + M_a, \]

where

- \( J \) is the inertial moment of the shin-foot complex
- \( \theta \) is the knee angle between the shin and the thigh on the sagittal plane
- \( \theta = \theta_v + 90^\circ \)
- \( \dot{\theta} \) is the knee angular velocity
- \( m \) is the mass of the shin-foot complex
- \( g \) is the gravitational acceleration
- \( l \) is the distance between the knee and the shin-foot complex mass center
- \( B \) is the viscous friction coefficient
- \( M_s \) is the torque due to the stiffness component
- \( M_a \) is the knee active torque generated by electrical stimulation

The stiffness moment is defined as

\[ M_s = \lambda e^{-\theta} (\theta - \omega), \]

where \( \lambda \) and \( E \) are the coefficients of the exponential terms and \( \omega \) is the elastic rest angle of the knee. In [1], it was observed that the torque applied to the muscle (\( M_a \)) and the electrical stimulation pulse width (\( P_u \)) can be adequately related by the transfer function described in (20), where \( G \) and \( \tau \) are positive constants obtained from parametric identification, when a stimulus is applied on the quadriceps and angular variation is read, as mentioned in [1]:

\[ H(s) = \frac{M_a(s)}{P_u(s)} = \frac{G}{1 + s \tau} \]

So, from [1], the knee joint mathematical model is given by

\[
\begin{align*}
\dot{x}(t) &= A(\alpha)x(t) + B(\alpha)u(t), \\
y(t) &= Cx(t),
\end{align*}
\]

with

\[ x(t) = \begin{bmatrix} \Delta \theta_1(t) \\ \Delta \theta_2(t) \\ \Delta M_a(t) \end{bmatrix}, \quad u(t) = P_N(t), \quad y(t) = \Delta \theta_v(t), \]

where \( \theta_{\text{v0}}, \dot{\theta}_{\text{v0}}, \) and \( M_{\text{a0}} \) are the knee angle, the knee angular velocity (\( \dot{\theta}_{\text{v0}} = 0 \)), and the active torque at the operation point. The input \( P_N(t) \) is the pulse width of the electrical stimulation.

\[
A(\alpha) = \begin{bmatrix} 0 & 1 & 0 \\ \tilde{f}_{21}(x_1) & B & 1 \\ 0 & 0 & \frac{1}{\tau} \end{bmatrix},
\]

\[
B(\alpha) = \begin{bmatrix} 0 \\ 0 \\ G \end{bmatrix} \frac{1}{\tau}, \quad C = [1 \ 0 \ 0].
\]

The nonlinear term \( \tilde{f}_{21}(x_1) \) is given by

\[
\tilde{f}_{21}(x_1) = \frac{1}{f_{x_1}} mg l \sin(x_1(t) + \theta_{v0}) - \frac{1}{f_{x_1}} \lambda \exp\left( -E \left( x_1(t) + \theta_{v0} + \frac{\pi}{2} \right) \right) \times \left( x_1(t) + \theta_{v0} + \frac{\pi}{2} - \omega \right) + M_{a0},
\]

where
\[ M_{ab} = mgl \sin(\theta_{a0}) + \lambda \exp\left(-E\left(\theta_{a0} + \frac{\pi}{2}\right)\right)(\theta_{a0} + \frac{\pi}{2} - \omega). \] (25)

4.1. Local Fuzzy Models of the Knee Joint considering Norm-Bounded Uncertainties. At the system described by (21), the uncertain parameter \( \bar{f}_{21}(x_1) \) given in (24) belongs to the interval \( \bar{f}_{21\text{min}} \leq \bar{f}_{21}(x_1) \leq \bar{f}_{21\text{max}} \). Then, the number of rules is \( q = 2 \), that is, the T-S fuzzy system has two local models. So, appropriate functions \( \alpha_1 \) and \( \alpha_2 \) are defined as

\[
\alpha_1 = \frac{\bar{f}_{21}(x_1) - \bar{f}_{21\text{max}}}{\bar{f}_{21\text{min}} - \bar{f}_{21\text{max}}} \\
\alpha_2 = \frac{\bar{f}_{21}(x_1) - \bar{f}_{21\text{min}}}{\bar{f}_{21\text{max}} - \bar{f}_{21\text{min}}} = 1 - \alpha_1.
\] (26)

The nonlinear function \( \bar{f}_{21}(x_1) \) is exactly represented by the convex combination of these two local models: \( \bar{f}_{21}(x_1) = \alpha_1 \bar{f}_{21\text{min}} + \alpha_2 \bar{f}_{21\text{max}} \) [2, 3, 5, 11, 13]. Note that, from (26), \( \alpha_1 \geq 0, \alpha_2 \geq 0, \) and \( \alpha_1 + \alpha_2 = 1. \)

So, considering two local models, matrices \( A(a) \) and \( B(a) \) are

\[
A(a) = \alpha_1 (A_{1n} + \delta A_1) + \alpha_2 (A_{2n} + \delta A_2),
\]

\[
B(a) = \alpha_1 (B_{1n} + \delta B_1) + \alpha_2 (B_{2n} + \delta B_2),
\]

\[
\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1,
\]

\[
A_{1n} + \delta A_1 = \begin{bmatrix} 0 & 1 & 0 \\ \bar{f}_{21\text{min}} & -\frac{B}{\bar{f}} & \frac{1}{\bar{f}} & 0 & 0 & -\frac{1}{\bar{f}} \end{bmatrix},
\]

\[
A_{2n} + \delta A_2 = \begin{bmatrix} 0 & 1 & 0 \\ \bar{f}_{21\text{max}} & -\frac{B}{\bar{f}} & \frac{1}{\bar{f}} & 0 & 0 & -\frac{1}{\bar{f}} \end{bmatrix},
\]

\[
B_{1n} + \delta B_1 = B_{2n} + \delta B_2 = \begin{bmatrix} 0 \\ 0 \\ G \\ \tau \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},
\]

with \( \bar{f}_{21}(x_1) \) described in (24) and (25), which belongs to the interval

\[
\bar{f}_{21\text{min}} \leq \bar{f}_{21} \leq \bar{f}_{21\text{max}}.
\] (28)

Functions \( \alpha_1 \) and \( \alpha_2 \) are given by (17), and the matrices that describe the local models are

\[
A_{1n} + \delta A_1 = \begin{bmatrix} 0 & 1 & 0 \\ \bar{f}_{21\text{min}} & a_{122} & a_{123} \\ 0 & 0 & a_{133} \end{bmatrix},
\]

\[
A_{2n} + \delta A_2 = \begin{bmatrix} \bar{f}_{21\text{max}} & a_{222} & a_{223} \\ 0 & 0 & a_{233} \end{bmatrix},
\]

\[
B_{1n} + \delta B_1 = \begin{bmatrix} 0 \\ 0 \\ b_{131} \end{bmatrix},
\]

\[
B_{2n} + \delta B_2 = \begin{bmatrix} 0 \\ 0 \\ b_{231} \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},
\]

where

\[
a_{122} = a_{222} = \frac{B}{\bar{f}},
\]

\[
a_{123} = a_{233} = \frac{1}{\bar{f}},
\]

\[
a_{133} = a_{233} = \frac{1}{\tau},
\]

\[
b_{131} = b_{231} = \frac{G}{\tau},
\]

with

\[
0 < B_{\text{min}} \leq B \leq B_{\text{max}},
\]

\[
0 < \bar{f}_{\text{min}} \leq \bar{f} \leq \bar{f}_{\text{max}},
\]

\[
0 < \tau_{\text{min}} \leq \tau \leq \tau_{\text{max}},
\]

\[
0 < G_{\text{min}} \leq G \leq G_{\text{max}}.
\] (30)

So,

\[
A_{1n} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{f}_{21\text{min}} & a_{122n} & a_{123n} \\ 0 & 0 & a_{133n} \end{bmatrix},
\]

\[
A_{2n} = \begin{bmatrix} \bar{f}_{21\text{max}} & a_{222n} & a_{223n} \\ 0 & 0 & a_{233n} \end{bmatrix},
\]

\[
B_{1n} = \begin{bmatrix} 0 \\ 0 \\ b_{131n} \end{bmatrix},
\]

\[
B_{2n} = \begin{bmatrix} 0 \\ 0 \\ b_{231n} \end{bmatrix}.
\] (31)
knowing that

\[
\begin{align*}
    a_{122\min} & \leq a_{122} \leq a_{122\max}, \\
    a_{123\min} & \leq a_{123} \leq a_{123\max}, \\
    a_{133\min} & \leq a_{133} \leq a_{133\max}, \\
    b_{131\min} & \leq b_{131} \leq b_{131\max}, \\
    a_{222\min} & \leq a_{222} \leq a_{222\max}, \\
    a_{223\min} & \leq a_{223} \leq a_{223\max}, \\
    a_{233\min} & \leq a_{233} \leq a_{233\max}, \\
    b_{231\min} & \leq b_{231} \leq b_{231\max}.
\end{align*}
\]

So,

\[
\begin{align*}
    \delta a_{122} &= \frac{a_{122\max} - a_{122\min}}{2} \delta_{11}, \quad -1 \leq \delta_{11} \leq 1, \\
    \delta a_{123} &= \frac{a_{123\max} - a_{123\min}}{2} \delta_{12}, \quad -1 \leq \delta_{12} \leq 1, \\
    \delta a_{133} &= \frac{a_{133\max} - a_{133\min}}{2} \delta_{13}, \quad -1 \leq \delta_{13} \leq 1, \\
    \delta b_{131} &= \frac{b_{131\max} - b_{131\min}}{2} \delta_{14}, \quad -1 \leq \delta_{14} \leq 1, \\
    \delta a_{222} &= \frac{a_{222\max} - a_{222\min}}{2} \delta_{21}, \quad -1 \leq \delta_{21} \leq 1, \\
    \delta a_{223} &= \frac{a_{223\max} - a_{223\min}}{2} \delta_{22}, \quad -1 \leq \delta_{22} \leq 1, \\
    \delta a_{233} &= \frac{a_{233\max} - a_{233\min}}{2} \delta_{23}, \quad -1 \leq \delta_{23} \leq 1, \\
    \delta b_{231} &= \frac{b_{231\max} - b_{231\min}}{2} \delta_{24}, \quad -1 \leq \delta_{24} \leq 1.
\end{align*}
\]

Replacing (35)–(38) in \( \delta A_1 \) and \( \delta B_1 \), one has

\[
\begin{align*}
    \delta A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho_{122} & \rho_{122} \delta_{11} \\ 0 & 0 & \rho_{133} \delta_{13} \end{bmatrix}, \\
    \delta B_1 &= \begin{bmatrix} 0 \\ 0 \\ \rho_{134} \delta_{14} \end{bmatrix},
\end{align*}
\]

where

\[
\begin{align*}
    \rho_{122} &= \frac{a_{122\max} - a_{122\min}}{2}, \\
    \rho_{123} &= \frac{a_{123\max} - a_{123\min}}{2}, \\
    \rho_{133} &= \frac{a_{133\max} - a_{133\min}}{2}.
\end{align*}
\]
\[
\rho_{134} = \frac{b_{131 \max} - b_{131 \min}}{2}
\] 

(48)

So, matrices \(\delta A_1\) and \(\delta B_1\) are decomposed according to (10), where

Following (13), the elements of \(\delta A_1\) are functions of \(\delta_{11}, \delta_{12}, \delta_{13}\), and \(\delta_{14}\), given by

\[
\phi_{ij} = \ell_{11i} \delta_{11} r_{a1j} + \ell_{12i} \delta_{12} r_{a12j} + \ell_{13i} \delta_{13} r_{a13j} + \ell_{14i} \delta_{14} r_{a14j},
\]

(49)

for \(i, j = 1, 2, 3\), where \(\phi_{ij}\) are the elements of matrix \(\delta A_1\) in (43).

From (16), the elements of \(\delta B_1\) are also the function of \(\delta_{11}, \delta_{12}, \delta_{13}\), and \(\delta_{14}\), given by

\[
\begin{align*}
\beta_{11i} &= \ell_{11i} \delta_{11} r_{b11} + \ell_{12i} \delta_{12} r_{b12} + \ell_{13i} \delta_{13} r_{b13} + \ell_{14i} \delta_{14} r_{b14}, \\
\beta_{12i} &= \ell_{12i} \delta_{12} r_{b12} + \ell_{12r} \delta_{12} r_{b12} + \ell_{13i} \delta_{13} r_{b13} + \ell_{14i} \delta_{14} r_{b14}, \\
\beta_{13i} &= \ell_{13i} \delta_{13} r_{b13} + \ell_{12r} \delta_{13} r_{b13} + \ell_{13i} \delta_{13} r_{b13} + \ell_{14i} \delta_{14} r_{b14}, \\
\beta_{14i} &= \ell_{14i} \delta_{14} r_{b14} + \ell_{12r} \delta_{14} r_{b14} + \ell_{13r} \delta_{14} r_{b14} + \ell_{14i} \delta_{14} r_{b14},
\end{align*}
\]

(50)

for \(i = 1, 2, 3, j = 1\), where \(\beta_{11i}\) are the elements of matrix \(\delta B_1\) in (44).

According to (14) and (17), the elements of \(L_1, R_{A1}\), and \(R_{B1}\) are solutions of the following equations:

\[
\begin{align*}
\ell_{11r} r_{a11} &= 0, & \ell_{11r} r_{a12} &= 0, & \ell_{11r} r_{a13} &= 0, & \ell_{11r} r_{a14} &= 0, \\
\ell_{11r} r_{a12} &= 0, & \ell_{11r} r_{a12} &= 0, & \ell_{11r} r_{a13} &= 0, & \ell_{11r} r_{a14} &= 0, \\
\ell_{11r} r_{a13} &= 0, & \ell_{11r} r_{a12} &= 0, & \ell_{11r} r_{a13} &= 0, & \ell_{11r} r_{a14} &= 0, \\
\ell_{11r} r_{b11} &= 0, & \ell_{11r} r_{b12} &= 0, & \ell_{11r} r_{b13} &= 0, & \ell_{11r} r_{b14} &= 0, \\
\ell_{11r} r_{b12} &= 0, & \ell_{11r} r_{b12} &= 0, & \ell_{11r} r_{b13} &= 0, & \ell_{11r} r_{b14} &= 0, \\
\ell_{11r} r_{b13} &= 0, & \ell_{11r} r_{b12} &= 0, & \ell_{11r} r_{b13} &= 0, & \ell_{11r} r_{b14} &= 0, \\
\ell_{11r} r_{b14} &= 0, & \ell_{11r} r_{b12} &= 0, & \ell_{11r} r_{b13} &= 0, & \ell_{11r} r_{b14} &= 0,
\end{align*}
\]

(51)

(52)
where $\rho_{122}$, $\rho_{123}$, $\rho_{133}$, and $\rho_{134}$ are given in (45)–(48).

So, the solution of the set of equations (52) is

$$L_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\rho_{122}/r_{a112} & \rho_{123}/r_{a123} & 0 & 0 \\
0 & 0 & \rho_{133}/r_{a133} & \rho_{134}/r_{a131}
\end{bmatrix},$$

$$R_{A1} = \begin{bmatrix}
0 & r_{a112} & 0 \\
0 & 0 & r_{a123} \\
0 & 0 & r_{a133} \\
0 & 0 & 0
\end{bmatrix},$$

$$R_{B1} = \begin{bmatrix}
0 \\
0 \\
0 \\
r_{a131}
\end{bmatrix},$$

For simplicity, one can choose $r_{a112} = r_{a123} = r_{a133} = r_{a131} = 1$. Therefore, matrices $L_1$, $R_{A1}$, and $R_{B1}$ are

$$L_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \rho_{123} & 0 \\
0 & \rho_{133} & \rho_{134}
\end{bmatrix},$$

$$R_{A1} = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix},$$

$$R_{B1} = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},$$

with $\rho_{122}$, $\rho_{123}$, $\rho_{133}$, and $\rho_{134}$ given in (45)–(48).

Similarly, replacing (39)–(42) in $\delta A_2$ and $\delta B_2$, one has

$$\delta A_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & \rho_{222}/\delta_{21} & \rho_{223}/\delta_{22} \\
0 & 0 & \rho_{233}/\delta_{23}
\end{bmatrix},$$

$$\delta B_2 = \begin{bmatrix}
0 & 0 \\
0 & \rho_{234}/\delta_{24}
\end{bmatrix},$$

where

$$\rho_{222} = \frac{\alpha_{222\max} - \alpha_{222\min}}{2},$$

$$\rho_{223} = \frac{\alpha_{223\max} - \alpha_{223\min}}{2},$$

$$\rho_{233} = \frac{\alpha_{233\max} - \alpha_{233\min}}{2},$$

$$\rho_{234} = \frac{\alpha_{234\max} - \alpha_{234\min}}{2}.$$
The values of these parameters were experimentally measured, for a 45-year-old paraplegic patient [2, 3]. However, several factors, such as changes of temperature, fatigue, and spasm, cause physiological changes on musculature that must be considered on process control. Furthermore, for other people, the physiological characteristics may be completely different since these characteristics depend on several factors, for instance, the age, weight, physical activities, and health conditions. So, calibration is needed before the beginning of the tests. The adjustments to perform a desired movement are made after the identification of each patient at a specific current and frequency.

The nonlinear term $f_{21}(x_1)$ is described in (24) and (25). Considering the parameters of Table 1, knowing that $g = 9.8 \text{ m/s}^2$ and taking the operation point angle as $\theta_{o0} = \pi/6 \text{ rad}$, these values are replaced in (25), obtaining $M_{a0} = 3.8028 \text{ N-m}$. For this operation point, the term $f_{21}(x_1)$ belongs to the interval $f_{21 \text{ min}} \leq f_{21}(x_1) \leq f_{21 \text{ max}}$, where

$$\begin{align*}
\bar{f}_{21 \text{ min}} &= -29.0297, \\
\bar{f}_{21 \text{ max}} &= -24.4981. 
\end{align*}$$

The new input of the system, $P_N$, is defined from the system input, $P_u$, and is known as the unreferenced pulse width [11, 22–24]. It is given by

$$P_N = P_u - \frac{M_{a0}}{G}. \hspace{1cm} (64)$$

Since the input is the pulse width which is applied on the skin of the patient, its value must be positive, that is, $P_N > 0$. So,

$$P_N > -\frac{M_{a0}}{G}. \hspace{1cm} (65)$$

\textbf{5.1. Norm-Bounded Uncertainties.} From the parameters presented in Table 1, a ±20% variation is considered on the values of $J$, $B$, $r$, and $G$, that is,

$$\begin{align*}
J_{\text{min}} &= 0.2305, \\
J_{\text{max}} &= 0.3457, \\
B_{\text{min}} &= 0.2205, \\
B_{\text{max}} &= 0.3307, \\
\end{align*}$$

$$\begin{align*}
\tau_{\text{min}} &= 0.0944, \\
\tau_{\text{max}} &= 0.1416, \\
G_{\text{min}} &= 18285, \\
G_{\text{max}} &= 27427. \hspace{1cm} (66)
\end{align*}$$

Given the minimum and the maximum values of $\bar{f}_{21}(x_1)$, one has the system with two local models, each one described by (8), where

$$\begin{align*}
A_{n1} &= \begin{bmatrix} 0 & 1 & 0 \\ -29.0297 & -1.0363 & 3.6156 \\ 0 & 0 & -8.8307 \end{bmatrix}, \\
A_{n2} &= \begin{bmatrix} 0 & 1 & 0 \\ -24.4981 & -1.0363 & 3.6156 \\ 0 & 0 & -8.8307 \end{bmatrix}, \\
B_{n1} &= \begin{bmatrix} 0 \\ 0 \\ 209910 \end{bmatrix}, \\
B_{n2} &= \begin{bmatrix} 0 \\ 0 \\ 209910 \end{bmatrix}, \\
C_1 &= C_2 = [1 \hspace{0.2cm} 0 \hspace{0.2cm} 0],
\end{align*}$$

and matrices $\delta A_1$ and $\delta B_1$ are described by (10), where

$$\begin{align*}
L_1 &= \begin{bmatrix} 0 & 0 & 0 \\ -0.3986 & 0.7231 & 0 \\ 0 & 0 & -1.7661 \end{bmatrix}, \\
L_2 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\
\Delta_1 &= \text{diag} [\delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}],
\end{align*}$$

$$\begin{align*}
\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \hspace{1cm} (68)
\end{align*}$$

$$\begin{align*}
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}$$
with $-1 \leq \delta_{ij} \leq 1$, for $i = 1, 2, 3, 4$, and matrices $\delta A_2$ and $\delta B_2$ are described by (10), where

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.3986 & 0.7231 & 0 & 0 \\ 0 & 0 & -1.7661 & 80734 \end{bmatrix},$$

$$\Delta_2 = \text{diag} \{ \delta_{21}, \delta_{22}, \delta_{23}, \delta_{24} \},$$

$$R_{A_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R_{B_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

with $-1 \leq \delta_{ij} \leq 1$, for $i = 1, 2, 3, 4$.

\section*{6. System Control with Norm-Bounded Uncertainties}

For the T-S fuzzy system given in (4), where each local model is described in (8), the objective is to obtain a fuzzy control law, given in (7), such that the controlled system is stable.

Theorem 1, given in [25], gives a sufficient condition for the stabilization of system (4) by the fuzzy control law (7), with the uncertain matrices described in (6).

\begin{theorem}[see [25]] Consider continuous-time T-S fuzzy system (4), with $q$ local models, where each local model is described as in (9) and (10), that is, each matrix $A_i$ and $B_i$, $i = 1, \ldots, q$, is decomposed as $A_i = A_{ii} + \delta A_i$ and $B_i = B_{ii} + \delta B_i$, respectively, with $\delta A_i = L_i \Delta_i R_{A_i}, \delta B_i = L_i \Delta_i R_{B_i}$, and $\| \Delta_i \| \leq 1$.

So, continuous-time T-S fuzzy system (4) is asymptotically stabilizable via the T-S fuzzy model-based state-feedback controller (7) if there exist a symmetric positive definite matrix, some matrices, and some scalars $\epsilon_{ij}$, for $i, j = 1, \ldots, q$ such that the following LMIs are satisfied:

\begin{equation}
\begin{bmatrix}
\Psi_{ii} & * & * \\
R_{A_i} Q - R_{B_i} M_i & - \epsilon_{ii} I & * \\
L_i^T & 0 & - \epsilon_{ii} I
\end{bmatrix} < 0, \quad (1 \leq i \leq q),
\end{equation}

\begin{equation}
\begin{bmatrix}
\Psi_{ij} & * & * & * \\
Y_{ij} & R_{A_i} Q - R_{B_i} M_j & - \epsilon_{ij} I & * \\
R_{A_i} Q - R_{B_i} M_i & 0 & - \epsilon_{ij} I & * \\
L_i^T & 0 & 0 & - \epsilon_{ij} I \\
L_j^T & 0 & 0 & - \epsilon_{ij} I
\end{bmatrix} < 0, \quad (1 \leq i \leq j \leq q),
\end{equation}

where

$$\Psi_{ii} = Q A_{ii}^T + A_{ii} Q - M_{ii}^T B_{ii}^T - B_{ii} M_{ii},$$

$$\Psi_{ij} = Q A_{ii}^T + A_{ii} Q + Q A_{jj}^T + A_{jj} Q - M_{ii}^T B_{jj}^T - B_{jj} M_{jj} - B_{ii} M_{jj} - M_{jj} B_{ii},$$

$$Q = P^{-1},$$

and $M_i = K_i P^{-1}$, where $*$ denotes the transposed elements in the symmetric positions. From the solution of the aforementioned LMIs, output feedback matrices $K_i$ are obtained from $K_i = M_i Q^{-1}$.

Multiplying (70), at left and at right, by $\text{diag}[I, I, \epsilon_{ii} I]$, the following LMI is obtained:

\begin{equation}
\begin{bmatrix}
\Psi_{ii} & * & * \\
R_{A_i} Q - R_{B_i} M_i & - \epsilon_{ii} I & * \\
\epsilon_{ii} L_i^T & 0 & - \epsilon_{ii} I
\end{bmatrix} < 0, \quad (1 \leq i \leq q),
\end{equation}

where $Q > 0$ and

$$\Psi_{ii} = Q A_{ii}^T + A_{ii} Q - M_{ii}^T B_{ii}^T - B_{ii} M_{ii},$$

Now, multiplying (71), at left and at right, by $\text{diag}[I, I, \epsilon_{ij} I, \epsilon_{ij} I]$, the following LMI is obtained:

\begin{equation}
\begin{bmatrix}
Y_{ij} & * & * & * \\
R_{A_i} Q - R_{B_i} M_j & - \epsilon_{ij} I & * & * \\
R_{A_i} Q - R_{B_i} M_i & 0 & - \epsilon_{ij} I & * \\
\epsilon_{ij} L_i^T & 0 & 0 & - \epsilon_{ij} I \\
\epsilon_{ij} L_j^T & 0 & 0 & - \epsilon_{ij} I
\end{bmatrix} < 0, \quad (1 \leq i \leq j \leq q),
\end{equation}

where $Q > 0$ and

$$Y_{ij} = Q A_{ii}^T + A_{ii} Q + Q A_{jj}^T + A_{jj} Q - M_{ii}^T B_{jj}^T$$

$$- B_{ii} M_{jj} - M_{jj} B_{ii} - B_{ii} M_{jj} - M_{jj} B_{ii},$$

So, it is observed from (74), $Y_{ij} < 0$, that T-S fuzzy system (4), with the local models described in (9) and (10) and the control law (7), is asymptotically stable if there exist matrices $Q = Q^T, M_i$ and scalars $\epsilon_{ij}$, for $i, j = 1, \ldots, q$, such that LMIs (73), (75) and $Q > 0$ hold.
For the fuzzy system of the electrostimulation for paraplegic patients, described in Sections 5 and 6, the number of rules is $q = 2$. So, to achieve the fuzzy control law (7) that stabilizes system (4), the problem consists of finding matrices $Q = Q^T, M_1$, and $M_2$ and scalars $\epsilon_{11}, \epsilon_{22}$, and $\epsilon_{12}$ such that the following LMIs hold:

\[
\begin{bmatrix}
\Psi_{11} & \ast & \ast \\
R_{A1}Q - R_{B1}M_1 & -\epsilon_{11}I & \ast \\
\epsilon_{11}L_1^T & 0 & -\epsilon_{11}I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
\Psi_{22} & \ast & \ast \\
R_{A2}Q - R_{B2}M_2 & -\epsilon_{22}I & \ast \\
\epsilon_{22}L_2^T & 0 & -\epsilon_{22}I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
Y_{12} & \ast & \ast & \ast & \ast \\
R_{A1}Q - R_{B1}M_2 & -\epsilon_{12}I & \ast & \ast & \ast \\
R_{A2}Q - R_{B2}M_1 & 0 & -\epsilon_{12}I & \ast & \ast \\
\epsilon_{12}L_1^T & 0 & 0 & -\epsilon_{12}I & \ast \\
\epsilon_{12}L_2^T & 0 & 0 & 0 & -\epsilon_{12}I
\end{bmatrix} < 0, \quad Q > 0,
\]  

(77) \hspace{1cm} (78) \hspace{1cm} (79)

where

\[
\Psi_{11} = QA_{m1}^T + A_{m1}Q - M_1^TB_{m1}^T - B_{m1}M_1,
\]

\[
\Psi_{22} = QA_{m2}^T + A_{m2}Q - M_2^TB_{m2}^T - B_{m2}M_2,
\]

\[
Y_{12} = QA_{m1}^T + A_{m1}Q + QA_{m2}^T + A_{m2}Q - M_2^TB_{m2}^T - B_{m2}M_2 - M_2^TB_{m1}^T - B_{m1}M_2.
\]

(80)

From the solution of the aforementioned LMIs, matrices $K_1$ and $K_2$ are given by

\[
K_1 = M_1Q^{-1},
\]

\[
K_2 = M_2Q^{-1}.
\]

(81)

Remark 1. For the case studied in this manuscript, with two local models and four uncertain parameters, the control design methods that consider polytopic uncertainty analysis required the solution of a set of 49 LMIs, while for the norm-bounded uncertainty analysis, only 4 LMIs are sufficient. A smaller number of LMIs requires a lower computational cost so that the LMI set is less conservative, and the solution is obtained more fastly.

6.1. Decay Rate with Norm-Bounded Uncertainties. Sometimes, only stability is not sufficient to get a suitable performance for a control system. Frequently, the transient response must also be specified. Given a linear time-invariant system, $\dot{x} = A_o x, x \in \mathbb{R}^n, A_o \in \mathbb{R}^{n \times n}$, according to [26], the decay rate is defined as the maximum value of the real constant $\gamma > 0$ such that

\[
\lim_{t \to -\infty} e^{\gamma t} \|x(t)\| = 0, \quad \forall x(0) \in \mathbb{R}^n,
\]

holds for all trajectories $x(t)$.

The condition $V(x) \leq -2\gamma V(x)$, in a system $\dot{x} = A_o x$, for all trajectories of $x(t)$, for $V(x) = x^T P x, P = P^T > 0$, is equivalent to

\[
A_o^T P + PA_o + 2\gamma P = (A_o + \gamma I)^T P + P(A_o + \gamma I) \leq 0.
\]

(82)

Then, replacing matrix $A_m$ by $A_m + \gamma I$ in (73) and (75), the LMIs that guarantee for T-S fuzzy system (4) with the local models described in (9) and (10), the fuzzy control law (7), and a decay rate greater than $\gamma$ are

\[
\begin{bmatrix}
\Psi_{ii} & \ast & \ast & \ast & \ast \\
R_{A1}Q - R_{B1}M_i & -\epsilon_{i1}I & \ast & \ast & \ast \\
R_{A2}Q - R_{B2}M_i & 0 & -\epsilon_{i1}I & \ast & \ast \\
\epsilon_{i1}L_i^T & 0 & 0 & -\epsilon_{i1}I & \ast \\
\epsilon_{i1}L_i^T & 0 & 0 & 0 & -\epsilon_{i1}I
\end{bmatrix} < 0, \quad (1 \leq i \leq q),
\]

\[
\begin{bmatrix}
Y_{ij} & \ast & \ast & \ast & \ast \\
R_{A1}Q - R_{B1}M_j & -\epsilon_{ij}I & \ast & \ast & \ast \\
R_{A2}Q - R_{B2}M_j & 0 & -\epsilon_{ij}I & \ast & \ast \\
\epsilon_{ij}L_i^T & 0 & 0 & -\epsilon_{ij}I & \ast \\
\epsilon_{ij}L_j^T & 0 & 0 & 0 & -\epsilon_{ij}I
\end{bmatrix} < 0, \quad (1 \leq i \leq j \leq q),
\]

(83) \hspace{1cm} (84) \hspace{1cm} (85)

where $Q > 0$, and

\[
\Psi_{ii} = QA_{m}^T + A_{m}Q - M_i^TB_{m}^T - B_{m}M_i + 2\gamma Q,
\]

\[
Y_{ij} = QA_{m}^T + A_{m}Q + QA_{m}^T + A_{m}Q - M_j^TB_{m}^T - B_{m}M_j - M_j^TB_{m}^T - B_{m}M_j + 4\gamma Q,
\]

\[
Q = P^{-1}, \quad M_i = K_i P^{-1}, \quad M_i, \quad M_i \text{ are obtained from } K_i = M_i Q^{-1}.
\]

For the fuzzy system of the electrostimulation for paraplegic patients, described in Sections 5 and 6, the number of rules is $q = 2$. So, to achieve a control law (7) that guarantees for system (4), where each local model is described in (9) and (10) and decay rate is greater than $\gamma$, the goal is to find matrices $Q = Q^T, M_1$, and $M_2$ and scalars $\epsilon_{11}, \epsilon_{22}$, and $\epsilon_{12}$ such that the following LMIs hold:

\[
\begin{bmatrix}
\Psi_{11} & \ast & \ast \\
R_{A1}Q - R_{B1}M_1 & -\epsilon_{11}I & \ast \\
\epsilon_{11}L_1^T & 0 & -\epsilon_{11}I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
\Psi_{22} & \ast & \ast \\
R_{A2}Q - R_{B2}M_2 & -\epsilon_{22}I & \ast \\
\epsilon_{22}L_2^T & 0 & -\epsilon_{22}I
\end{bmatrix} < 0,
\]

(86) \hspace{1cm} (87) \hspace{1cm} (88)
where

\[
\begin{align*}
\Psi_{11} &= QA_n^T + A_nQ - M_1^TB_n^T - B_nM_1 + 2yQ, \\
\Psi_{12} &= QA_n^T + A_nQ - M_1^TB_n^T - B_nM_1 + 2yQ, \\
Y_{12} &= QA_n^T + A_nQ + QA_n^T + A_nQ - M_1^TB_n^T \\
&\quad - B_nM_1 - M_1^TB_nT - B_nM_1 + 4yQ.
\end{align*}
\]

(90)

From the solution of the aforementioned LMIs, matrices \(K_1\) and \(K_2\) are given by

\[
\begin{align*}
K_1 &= M_1Q^{-1}, \\
K_2 &= M_2Q^{-1}.
\end{align*}
\]

(91)

7. System Control with Input Constraint

Consider the input control \(u(t)\) given in (4) and an initial state \(x(0)\). In various situations, it is necessary to specify a bound for the input control to avoid a too large magnitude of this input. Consider the following restriction on the input \(u(t)\) in [24]:

\[
\max\|u(t)\|_2 = \max\sqrt{u^T(t)u(t)} \leq \mu_0, \tag{92}
\]

for all \(t \geq 0\).

This condition is guaranteed by adding new LMIs in the LMI set that determines stability or decay rate. Input constraint (92) is guaranteed by the LMIs:

\[
\begin{bmatrix}
1 & x(0)^T \\
x(0) & Q \\
Q & M_i^T \\
M_i & \mu_0^2I
\end{bmatrix} \geq 0,
\]

(93)

for an initial state \(x(0)\) and \(i = 1, 2, \ldots, q\). The LMIs that guarantee the input constraint need to be added to the LMI set that determines stability or decay rate.

For the fuzzy system of the electrostimulation for paraplegic patients, described in Sections 5 and 6, the number of rules is \(q = 2\). So, the LMIs that guarantee the input constraint are

\[
\begin{bmatrix}
1 & x(0)^T \\
x(0) & Q \\
Q & M_1^T \\
M_1 & \mu_0^2I
\end{bmatrix} \geq 0,
\]

(94)

for an initial state \(x(0)\). LMIs (94)–(96) need to be added to the LMI set that determines stability or decay rate, given in (77)–(80) or (87)–(89).

For the FES, the control signal is the electric pulse width applied to the muscle that is given by

\[
P_u = P_N - \frac{M_{a0}}{G},
\]

(97)

whose value must, naturally, be positive. So, the input needs to follow the condition

\[
P_N > -\frac{M_{a0}}{G},
\]

(98)

To follow this condition, input constraint (92) is added for the input, where

\[
u(t) = P_N, \tag{99}
\]

For \(\theta_{a0} = \pi/6\), one has \(\mu_0 < 1.6638 \times 10^{-4}\). The initial state is

\[
x(0) = \begin{bmatrix} \theta_{a0} & 0 & -M_{a0} \end{bmatrix}^T = \begin{bmatrix} -\frac{\pi}{6} & 0 & -3.8028 \end{bmatrix}^T.
\]

(100)

8. Numerical Results and Simulation

After obtaining the mathematical model of FES for the knee joint of a paraplegic patient, considering norm-bounded uncertainties, as described in Sections 4 and 5, the LMIs presented in Section 6 were solved using the Control Toolbox of MATLAB, version 2012a [27].

8.1. Stability. Although the plant is already stable, the first action was to solve LMIs (77)–(80) to obtain a control law (7) that stabilizes system (4). The solution of these LMIs is

\[
\begin{bmatrix}
0.0000000033491903 & -0.000000032933051 & 0.000000000438928 \\
-0.000000032933051 & 0.00000000901932404 & -0.0000000978603726 \\
0.000000000438928 & -0.0000000978603726 & 0.003521496953692
\end{bmatrix},
\]
\[
M_1 = \begin{bmatrix}
0.000000000132633 & 0.0000000001947850 & 0.009833371724050 \\
-0.000000000065090 & -0.000000000616827 & 0.009524542815504 \\
0.00000000006293797 & 0.00000000049443200 & -0.000000000516894
\end{bmatrix},
\]
\[
M_2 = \begin{bmatrix}
-0.000000000060590 & -0.000000000616827 & 0.009524542815504 \\
0.000000000006986 & 0.0000000001080284 & 0.031657939760077 \\
0.0000000000516894 & 0.000000000234652073 & 0.011980248303538
\end{bmatrix},
\]
\[
\epsilon_{11} = 5.47606985268709 \times 10^{-7},
\]
\[
\epsilon_{22} = 5.28067137087779 \times 10^{-7},
\]
\[
\epsilon_{12} = 5.72890931362815 \times 10^{-7}.
\]

From the solution above, matrices \(K_1\) and \(K_2\) of the control law (7) are

\[
K_1 = \begin{bmatrix}
3.059460733107650 & 3.144578664119753 & 2.793257471726990 \\
2.954687616951170 & 3.042725536096676 & 2.705530990497405
\end{bmatrix},
\]
\[
K_2 = \begin{bmatrix}
2.954687616951170 & 3.042725536096676 & 2.705530990497405 \\
2.954687616951170 & 3.042725536096676 & 2.705530990497405
\end{bmatrix}.
\]

Table 2 shows the eigenvalues of \(A_{n1} - B_{n1}K_{j1}\), \(i, j = 1, 2\), with the aforementioned matrices \(K_1\) and \(K_2\), related to the nominal system, when only stability is specified. In Figure 3, the simulation result is shown for the system with the nominal values of \(B, J, \tau,\) and \(G\), from the initial state \(x(0)\) given in (100), with \(\alpha\) described in (17). The convergence of the input is very fast, reaching zero at a short time. Although some values seem to be negative, the real value is zero. A zoom on the time, from 0 to 20 s, in Figure 4, shows this fast convergence.

In Figures 5–7, the simulation result is presented for \(x_1(t), x_2(t),\) and \(x_3(t)\), respectively, considering all combinations of the minimum or maximum values of the parameters \(B, J, \tau,\) and \(G\), from the initial state given in (100), with \(\alpha\) described in (17). Each figure contains 16 curves, each one representing one combination of the extreme (minimum or maximum) values of \(B, J, \tau,\) and \(G\), as described in Table 3. The results show that the controlled system is stable for all possible values of these parameters.

8.2. Decay Rate. To obtain a faster transient response, decay rate was specified. For this purpose, LMIs (87)–(89) were solved. The solution of these LMIs is

\[
Q = \begin{bmatrix}
0.0000000006293797 & -0.0000000049443200 & -0.000000000516894 \\
-0.0000000049443200 & 0.00000000530987030 & -0.000003234652073 \\
-0.000000000516894 & -0.000003234652073 & 0.011980248303538
\end{bmatrix},
\]
\[
M_1 = \begin{bmatrix}
0.00000000071281 & 0.00000002212972 & 0.031657939760077 \\
0.00000000071281 & 0.00000000530987030 & -0.000003234652073 \\
0.00000000071281 & 0.00000000530987030 & -0.000003234652073
\end{bmatrix},
\]
\[
M_2 = \begin{bmatrix}
0.0000000006986 & 0.00000001080284 & 0.031467438088653 \\
0.0000000001080284 & 0.00000000530987030 & -0.000003234652073 \\
0.0000000001080284 & 0.00000000530987030 & -0.000003234652073
\end{bmatrix},
\]
\[
\epsilon_{11} = 1.76838667145756 \times 10^{-6},
\]
\[
\epsilon_{22} = 1.75650519970404 \times 10^{-6},
\]
\[
\epsilon_{12} = 1.8339891648706 \times 10^{-6}.
\]

Table 4 shows the eigenvalues of \(A_{n1} - B_{n1}K_{j2}\), \(i, j = 1, 2\), with the aforementioned matrices \(K_1\) and \(K_2\), related to the nominal system, when decay rate \(\gamma = 4\) is specified. In Figure 8, the simulation result is shown for the system with the nominal values of \(B, J, \tau,\) and \(G\), from the initial state given in (100), with \(\alpha\) described in (17). The convergence of the input \(u(t)\) is very fast, reaching zero at a short time. Although some values seem to be
Table 2: Eigenvalues of $A_{ni} - B_{ni}K_j$, $i, j = 1, 2$, when only stability is specified.

| Eigenvalue of $A_{n1} - B_{n1}K_1$ | Eigenvalue of $A_{n2} - B_{n2}K_2$ | Eigenvalue of $A_{n3} - B_{n3}K_2$ | Eigenvalue of $A_{n2} - B_{n2}K_1$ |
|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| $-5.863299313142837 \times 10^5$   | $-5.679155054760272 \times 10^5$   | $-5.863299313142835 \times 10^5$   | $-5.679155054760271 \times 10^5$   |
| $-2.5533482793 + j \times 1.449448541$ | $-2.5512811186 + j \times 4.6837870864$ | $-2.5533482793 + j \times 4.6838993430$ | $-2.5512811186 + j \times 5.1448427531$ |
| $-2.5533482793 - j \times 1.449448541$ | $-2.5512811186 - j \times 4.6837870864$ | $-2.5533482793 - j \times 4.6838993430$ | $-2.5512811186 - j \times 5.1448427531$ |

Figure 3: Simulation result of the nominal fuzzy system, considering only stability.

Figure 4: Time zoom on $u(t)$, for the nominal fuzzy system, considering only stability.
Figure 5: Simulation result of the uncertain fuzzy system for $x_1(t)$, considering only stability. Each curve represents one combination of the extreme (minimum or maximum) values of $B$, $J$, $r$, and $G$, as described in Table 3.

Figure 6: Continued.
Figure 6: Simulation result of the uncertain fuzzy system for $x_2(t)$, considering only stability. Each curve represents one combination of the extreme (minimum or maximum) values of $B$, $J$, $r$, and $G$, as described in Table 3.

Figure 7: Simulation result of the uncertain fuzzy system for $x_3(t)$, considering only stability. Each curve represents one combination of the extreme (minimum or maximum) values of $B$, $J$, $r$, and $G$, as described in Table 3.
Table 3: Values of the parameters $B$, $J$, $\tau$, and $G$ for each vertex.

| Vertex | $B$    | $J$    | $\tau$ | $G$    |
|--------|--------|--------|--------|--------|
| 1      | 0.2205 | 0.2305 | 0.0944 | 18285  |
| 2      | 0.2205 | 0.2305 | 0.0944 | 27427  |
| 3      | 0.2205 | 0.2305 | 0.1416 | 18285  |
| 4      | 0.2205 | 0.2305 | 0.1416 | 27427  |
| 5      | 0.2205 | 0.3457 | 0.0944 | 18285  |
| 6      | 0.2205 | 0.3457 | 0.0944 | 27427  |
| 7      | 0.2205 | 0.3457 | 0.1416 | 18285  |
| 8      | 0.2205 | 0.3457 | 0.1416 | 27427  |
| 9      | 0.3307 | 0.2305 | 0.0944 | 18285  |
| 10     | 0.3307 | 0.2305 | 0.0944 | 27427  |
| 11     | 0.3307 | 0.2305 | 0.1416 | 18285  |
| 12     | 0.3307 | 0.2305 | 0.1416 | 27427  |
| 13     | 0.3307 | 0.3457 | 0.0944 | 18285  |
| 14     | 0.3307 | 0.3457 | 0.0944 | 27427  |
| 15     | 0.3307 | 0.3457 | 0.1416 | 18285  |
| 16     | 0.3307 | 0.3457 | 0.1416 | 27427  |

Table 4: Eigenvalues of $A_{m1} - B_{m1}K_{j}$, $i, j = 1, 2$, when decay rate $\gamma = 4$ is specified.

| Eigenvalue of $A_{m1} - B_{m1}K_1$ | Eigenvalue of $A_{m2} - B_{m2}K_2$ | Eigenvalue of $A_{m1} - B_{m2}K_2$ | Eigenvalue of $A_{m2} - B_{m2}K_1$ |
|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| $-5.580376787035519 \times 10^5$ | $-5.546786114873048 \times 10^5$ | $-5.546786114873052 \times 10^5$ | $-5.580376787035501 \times 10^5$ |
| $-9.1076841424$              | $-9.0380952953$                | $-9.1077701925$                | $-9.0380260712$                |
| $-74.1012735586$             | $-74.1553615536$               | $-74.0856866552$               | $-74.1709316311$               |

Figure 8: Simulation result of the nominal fuzzy system, considering decay rate $\gamma = 4$. 

(a) $x_1(t)$
(b) $x_2(t)$
(c) $x_3(t)$
(d) $u(t)$
negative, the real value is zero. A zoom on the time, from 0 to 20s, in Figure 9, shows this fast convergence.

In Figures 10–12, the simulation result is presented for $x_1(t)$, $x_2(t)$, and $x_3(t)$, respectively, considering all combinations of the minimum or maximum values of the parameters $B, J, \tau$, and $G$, from the initial state $x(0)$ given in (100), with $\alpha$ described in (17). Each figure contains 16 curves, each one representing one combination of the extreme (minimum or maximum) values of $B, J, \tau$, and $G$, as described in Table 3. The results show that the controlled system holds the decay rate specification for all possible values of these parameters.

From the aforementioned solution, matrices $K_1$ and $K_2$ of the control law (7) are

$$K_1 = \begin{bmatrix} 0.102977078090382 & 0.428789046560496 & 0.68644946488605 \\ 0.421179157209213 & 0.683206565769427 & 0.0000000768978327 \\ -0.000000745366912 & 0.000470832034624 & 8.320872522948498 \\ -0.00000768978327 & \end{bmatrix} \times 10^4, \quad (105)$$

$$K_2 = \begin{bmatrix} 0.102515363390457 & 0.421179157209213 & 0.683206565769427 \\ 0.684676514801396 & 0.68644946488605 & 0.000000768978327 \\ 0.0000000768978327 & \end{bmatrix} \times 10^4, \quad (106)$$

Table 5 shows the eigenvalues of $A_m - B_n K_{ji}, i, j = 1, 2$, with the aforementioned matrices $K_1$ and $K_2$, related to the nominal system, when input constraint (92), with $\mu_0 = 200$, is specified. In Figure 13, the simulation result is shown for the system with the nominal values of $B, J, \tau$, and $G$, from the initial state given in (100), with $x = 0$ described in (17).

The convergence of the input $u(t)$ is very fast, reaching zero at a short time. Although some values seem to be negative, the real value is zero. A zoom on time, from 0 to 60 s, in Figure 14, shows this fast convergence.

In Figures 15–17, the simulation result is presented for $x_1(t)$, $x_2(t)$, and $x_3(t)$, respectively, considering all combinations of the minimum or maximum values of the parameters $B, J, \tau$, and $G$, from the initial state $x(0)$ given in (100), with $\alpha$ described in (17). Each figure contains 16 curves, each one representing one combination of the extreme (minimum or maximum) values $B, J, \tau$, and $G$, as described in Table 3.

When decay rate and an input constraint were specified, no solution for LMIs (87)–(89) and (94)–(96) was found.

8.3. Input Constraint. Since the pulse width (63) must be positive, input constraint (92), with $u(t) = P_N$ and $P_0 = M_m/G = 1.6638 \times 10^{-4}$, was considered in the problem. Then, LMIs (94)–(96) were added to LMIs (77)–(80).

When stability and input constraint were specified, the Matlab Control Toolbox did not find the solution for LMIs (77)–(80) and (94)–(96), with small values of $\mu_0$. Specification of a very small bound on the control input can make the problem too conservative, and it is not easy to find a solution. For $\mu_0 = 200$, the solution of these LMIs is

This fact may have occurred because the specification of decay rate and an input constraint has increased the conservativeness of the problem.

8.4. Discussion of the Results. Although the plant is already stable, the first purpose is to get a control law that guarantees the stability of the system, for any exact value of the uncertain parameters, within the considered set of values. As seen in Figure 3, for the nominal plant, and in Table 2 and Figures 5–7, considering the minimum or maximum value of each uncertain parameter, the controlled system is stable, and its state variables converge fastly to the equilibrium point. The control law converges to zero at a short time, as seen in Figures 3 and 4.

To obtain a faster transient response, decay rate was determined. As seen in Figure 8, for the nominal plant, and in Table 4 and Figures 10–12, considering the minimum or maximum value of each uncertain parameter, the convergence of the state variables is faster compared with the first
Figure 9: Time zoom on $u(t)$, for the nominal fuzzy system, considering decay rate $\gamma = 4$.

Figure 10: Simulation result of the uncertain fuzzy system for $x_1(t)$, considering decay rate $\gamma = 4$. Each curve represents one combination of the extreme (minimum or maximum) values of $B, J, \tau$, and $G$, as described in Table 3.
Figure 11: Simulation result of the uncertain fuzzy system for $x_2(t)$, considering decay rate $\gamma = 4$. Each curve represents one combination of the extreme (minimum or maximum) values of $B, J, \tau$, and $G$, as described in Table 3.

Figure 12: Continued.
Figure 12: Simulation result of the uncertain fuzzy system for $x_3(t)$, considering decay rate $\gamma = 4$. Each curve represents one combination of the extreme (minimum or maximum) values of $B, J, \tau,$ and $G$, as described in Table 3.

| Eigenvalue of $A_{m1} - B_{n1}K_1$ | Eigenvalue of $A_{m2} - B_{n2}K_2$ | Eigenvalue of $A_{m1} - B_{n1}K_2$ | Eigenvalue of $A_{m2} - B_{n2}K_1$ |
|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $-1.440973312768043 \times 10^5$ | $-1.434166564077318 \times 10^5$ | $-1.434166564077322 \times 10^5$ | $-1.440973312768037 \times 10^5$ |
| $-1.6473497848 + j5.1825399965$ | $-1.6325949036 + j5.1872191983$ | $-1.6325949036 + j5.1872191983$ | $-1.6473497851 - j4.7251599715$ |
| $-1.6473497848 - j5.1825399965$ | $-1.6325949036 - j5.1872191983$ | $-1.6325949036 - j5.1872191983$ | $-1.6473497851 - j4.7251599715$ |

Figure 13: Continued.
Figure 13: Simulation result of the nominal fuzzy system, considering input constraint $\mu_0 = 200$.

Figure 14: Time zoom on $u(t)$, for the nominal fuzzy system, considering input constraint $\mu_0 = 200$.

Figure 15: Continued.
Figure 15: Simulation result of the uncertain fuzzy system for $x_1(t)$, considering input constraint $\mu_0 = 200$. Each curve represents one combination of the extreme (minimum or maximum) values of $B, f, \tau, G$, as described in Table 3.

Figure 16: Simulation result of the uncertain fuzzy system for $x_2(t)$, considering input constraint $\mu_0 = 200$. Each curve represents one combination of the extreme (minimum or maximum) values of $B, f, \tau, G$, as described in Table 3.
Table 6: Variables used in general T-S fuzzy representation.

| Symbol     | Description                                                                 |
|------------|-----------------------------------------------------------------------------|
| $\alpha$   | $[a_1, \ldots, a_q]^T$                                                      |
| $\alpha_i(z(t))$, $i = 1, 2, \ldots, q$ | Membership function vector for the $i$-th fuzzy local model |
| $\mu_j^i(z_j(t))$, $i = 1, 2, \ldots, q$, $j = 1, 2, \ldots, p$ | Membership function of fuzzy set $\mathcal{M}_j^i$ |
| $A(a), B(a), C$ | Matrices that describe the T-S fuzzy model                               |
| $A_i, B_i, C_i$, $i = 1, 2, \ldots, q$ | Matrices that describe the $i$-th local model                              |
| $K$         | Output feedback matrix for the T-S fuzzy model                             |
| $K_j$, $i = 1, 2, \ldots, q$ | Output feedback matrix for the $i$-th local model                          |
| $\mathcal{M}_j^i$, $i = 1, 2, \ldots, q$, $j = 1, 2, \ldots, p$ | Fuzzy set $j$ of rule $i$                                               |
| $n$         | Number of state variables                                                  |

Table 6: Continued.

| Symbol     | Description                                                                 |
|------------|-----------------------------------------------------------------------------|
| $p$        | Number of premise variables                                                 |
| $q$        | Number of fuzzy rules                                                        |
| $u(t)$     | Input signal                                                                |
| $u_j^i(z(t))$, $i = 1, 2, \ldots, q$ | Membership function of the $i$-th fuzzy local model |
| $x(t)$     | State vector                                                                |
| $x(t)$     | Time derivative of the state vector                                         |
| $y(t)$     | Output signal                                                               |
| $z_j(t)$, $j = 1, \ldots, p$ | Premise variables of the T-S fuzzy model                                     |

In Tables 2 and 4, one can note that, for the second case, the closed-loop poles of the system are further from the imaginary axis than for the first case.
However, as observed in Figures 8 and 9, the control law assumes too high values, so this control signal is impracticable for the real system.

To reduce the magnitude of \( u(t) \), a constraint on the input, described in (92), was specified. However, no solution was found for small values of \( \mu_0 \). For \( \mu_0 = 200 \), the control law stabilizes the system, but the convergence of the state variables is slower than the previous cases, as seen in Figure 13, for the nominal plant, and in Table 5 and Figures 15–17, considering the minimum or maximum value of each uncertain parameter. The control law is also too high, but it converges fastly to zero, as observed in Figures 13 and 14. No solution was found when decay rate and an input constraint were specified.

### Table 7: Variables used to describe the norm-bounded uncertainties.

| \( \beta_{ij}, i = 1, \ldots, n, j = 1, \ldots, m \) | Elements of \( \delta B \) for the first local model |
| \( \beta_{ij}, i = 1, \ldots, n, j = 1, \ldots, m \) | \( \beta_{ij} \) for the second local model |
| \( \Delta \) | Diagonal matrix with the normalized uncertain variables |
| \( \Delta_1 \) | \( \Delta \) for the first local model |
| \( \Delta_2 \) | \( \Delta \) for the second local model |
| \( \varepsilon_{ik}, k = 1, \ldots, v \) | Normalized uncertain variables such that \( |\varepsilon_{ik}| \leq 1 \) |
| \( \delta_{ik}, k = 1, \ldots, v \) | \( \delta_{ik} \) for the first local model |
| \( \delta_{ik}, k = 1, \ldots, v \) | \( \delta_{ik} \) for the second local model |
| \( \delta A, \delta B \) | Matrices that describe the norm-bounded uncertainties |
| \( \delta A_1, \delta B_1 \) | \( \delta A, \delta B \) for the first local model |
| \( \delta A_2, \delta B_2 \) | \( \delta A, \delta B \) for the second local model |
| \( \phi_{ij}, i, j = 1, \ldots, n \) | \( \phi_{ij} \) for the first local model |
| \( \phi_{ij}, i, j = 1, \ldots, n \) | \( \phi_{ij} \) for the second local model |
| \( \alpha_0, B_0, C \) | Matrices that describe the nominal system |
| \( \alpha_{0,1}, \beta_{0,1}, C_1 \) | \( \alpha_0, B_0, C \) for the first local model |
| \( \alpha_{0,2}, \beta_{0,2}, C_2 \) | \( \alpha_0, B_0, C \) for the second local model |
| \( L \) | Left matrix on the decomposition of \( \Delta A \) and \( \Delta B \) |
| \( l_{ik}, i = 1, \ldots, n, k = 1, \ldots, v \) | Elements of \( L \) to be determined |
| \( l_{ik}, i = 1, \ldots, n, k = 1, \ldots, v \) | \( l_{ik} \) for the first local model |
| \( l_{ik}, i = 1, \ldots, n, k = 1, \ldots, v \) | \( l_{ik} \) for the second local model |
| \( R_A \) | Right matrix on the decomposition of \( \Delta A \) |
| \( R_{A1} \) | \( R_A \) for the first local model |
| \( R_{A2} \) | \( R_A \) for the second local model |
| \( r_{ik}, k = 1, \ldots, v, j = 1, \ldots, n \) | Elements of \( R_A \) to be determined |
| \( r_{ik}, k = 1, \ldots, v, j = 1, \ldots, n \) | \( r_{ik} \) for the first local model |
| \( r_{ik}, k = 1, \ldots, v, j = 1, \ldots, n \) | \( r_{ik} \) for the second local model |
| \( R_b \) | Right matrix on the decomposition of \( \Delta B \) |
| \( R_{b1} \) | \( R_b \) for the first local model |
| \( R_{b2} \) | \( R_b \) for the second local model |
| \( r_{ik}, k = 1, \ldots, v, j = 1, \ldots, m \) | Elements of \( R_b \) to be determined |
| \( r_{ik}, k = 1, \ldots, v, j = 1, \ldots, m \) | \( r_{ik} \) for the first local model |
| \( r_{ik}, k = 1, \ldots, v, j = 1, \ldots, m \) | \( r_{ik} \) for the second local model |

### Table 8: Knee joint parameters.

| \( \theta_i \) | \( \theta_i - \theta_{0_i} \) |
| \( \Delta \theta \) | \( \theta - \theta_0 = \theta \) |
| \( \Delta M \) | \( M_a - M_{0a} \) |
| \( \lambda \) | Coefficient of the exponential term of \( \tilde{f}_{21}(x_i) \) |
| \( \tau \) | Transfer function coefficient (time constant of the pole) |
| \( \tau_{\text{min}} \) | Minimum and maximum value of \( \tau \) |
| \( \theta \) | Knee angle between the shin and the vertical reference on the sagittal plane |
| \( \theta_{0_0} \) | Knee angle at the operation point |
| \( \theta_{0_0} = 0 \) | Null knee angular velocity at the operation point |
| \( \theta \) | Knee angular velocity |
| \( \theta \) | Resting elastic knee angle |
| \( B \) | Viscous friction coefficient |
| \( B_{\text{min}}, B_{\text{max}} \) | Minimum and maximum value of \( B \) |
| \( E \) | Coefficient of the exponential term of \( \tilde{f}_{21}(x_i) \) |
| \( \theta_{21} \) | Nonlinear term of the knee joint mathematical model |
| \( \tilde{f}_{21}(x_i) \) | Minimum and maximum value of \( \tilde{f}_{21}(x_i) \) |
| \( g \) | Gravitational acceleration |
| \( G_{\text{min}}, G_{\text{max}} \) | Minimum and maximum value of \( G \) |
| \( I \) | Inertial moment of the shank-foot complex |
| \( I_{\text{min}}, I_{\text{max}} \) | Minimum and maximum value of \( I \) |
| \( l \) | Distance between the knee and the shank-foot complex mass center |
| \( m \) | Mass of the shank-foot complex |
| \( M_s \) | Knee active torque generated by electrical stimulation |
| \( P_{\text{in}} \) | Pulse width applied on the skin of the patient |
| \( P_a \) | System input related to the electrical stimulation pulse width |

### Table 9: Variables used in the LMI control strategy.

| \( Y \) | Decay rate |
| \( \alpha_{ij}, i, j = 1, \ldots, q \) | LMI scalar variable |
| \( \mu_0 \) | Bound for input control |
| \( I \) | Identity matrix |
| \( K_i, i = 1, \ldots, q \) | Output feedback matrices |
| \( M_{\alpha}, i = 1, \ldots, q \) | LMI matrix variables |
| \( P \) | Lyapunov function candidate matrix |
| \( Q \) | LMI matrix variable |
| \( V(x) \) | Lyapunov function candidate |
| \( \dot{V}(x) \) | Time derivative of the Lyapunov function candidate |
| \( x(0) \) | Initial state vector |

### 9. Conclusion

In this manuscript, a mathematical model for the knee position control for a paraplegic patient, following the description in [1], was obtained using the parameters given in [2, 3] and considering norm-bounded uncertainties. The uncertain model is a T-S fuzzy combination of linear local models. After finding the T-S fuzzy model of the uncertain
nonlinear system, a T-S fuzzy state-feedback control has been designed for this system using LMIs. The initial objective, although the plant is already stable, is to obtain a stable system. Then, to obtain a faster transient response, decay rate was specified. Finally, an input constraint was specified to guarantee that the pulse width is always positive. The simulation results show the efficiency of the control.

Appendix

The variables used in this manuscript are described in Tables 6–9. The variables used in the T-S fuzzy representation are presented in Table 6. Table 7 lists the variables used to describe the norm-bounded uncertainties. Note that, on T-S fuzzy description, a new index can be added to these variables, corresponding to the fuzzy rule that is being used. The knee joint parameters are given in Table 8, and the variables used in the LMI control strategy are presented in Table 9.

Data Availability

The system parameter data used to support the findings of this study are published in [2, 3] and are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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