More about “short” spinning quantum strings

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Abstract

We continue investigation of the spectrum of semiclassical quantum strings in AdS\(_5\) × S\(_5\) on the examples of folded (S, J) string (with spin S in AdS\(_5\) and orbital momentum J in S\(_5\)) dual to an \text{sl}(2) sector state in gauge theory and its (J\text{'} , J) counterpart with spin J\text{'} in S\(_5\) dual to an \text{su}(2) sector state. We study the limits of small spins and large J at weak and strong coupling, pointing out that terms linear in spins provide a generalization of “protected” coefficients in the energy that are given by finite polynomials in ’t Hooft coupling \(\lambda\) (or square of string tension) for any value of \(\lambda\). We propose an expression for the coefficient of the term linear in spin J\text{'} in the (J\text{'} , J) string energy which should be the \text{su}(2) sector counterpart of the “slope function” in the \text{sl}(2) sector suggested by Basso in \texttt{arXiv:1109.3154}.

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In this paper we continue investigation of the spectrum of semiclassical quantum strings in $AdS_5 \times S^5$ in the small-spin limit ("short" strings) \cite{1,2,3,4,5,6,7}. We shall clarify the structure of the expansion of the string energy $E$ in orbital momentum $J$ in $S^5$. In particular, we shall compare the energies of folded ($S,J$) string in $AdS_3 \times S^1$ and folded ($J',J$) string in $R \times S^3$ (representing gauge-theory states in the $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$ sectors respectively). We will be interested in the small $S$ (or small $J'$) expansion of the energy at fixed $S^5$ orbital momentum $J$, i.e. in the limit $\frac{S}{J} \ll 1$ (or $\frac{S'}{J'} \ll 1$)\footnote{We shall use the following notation: the string tension is $T = \sqrt{\lambda}$, the semiclassical parameters are $S_i = \frac{S}{\sqrt{\lambda}}$, $J_i = \frac{J}{\sqrt{\lambda}}$. The spin in $S^5$ will be denoted as $J' \equiv J_1$ and the orbital momentum as $J \equiv J_2$.}. We will suggest the expression for the $J'$ "slope" function which is the $\mathfrak{su}(2)$ sector counterpart of the expression of \cite{8} in the $\mathfrak{sl}(2)$ sector.

Let us first discuss the general structure of the energy $E$ of a semiclassical string state and compare it with the corresponding expression for the gauge-theory dimension $\Delta$. For definiteness, let us consider string states with spin $S$ in $AdS_5$ and orbital momentum $J$ in
$S^5$ dual to gauge theory states from the $\mathfrak{sl}(2)$ sector represented by operators like $\text{tr}(D^S_+ \Phi J)$ (similar discussion will apply to states from the $\mathfrak{su}(2)$ sector). In perturbative planar gauge theory one first expands $\Delta = E$ in $\lambda \ll 1$ for fixed spin $S$ and $J$

$$E \equiv \Delta = S + J + \gamma(S, J, \lambda), \quad \gamma = \sum_{n=1}^{\infty} \lambda^n \gamma_n(S, J). \quad (1.1)$$

One may then further expand $\gamma_n$ in $S$ and $J$, e.g., in large $J$ for fixed $S$. The semiclassical string theory limit corresponds to first taking $\sqrt{\lambda} \gg 1$ for fixed semiclassical parameters $S = \frac{S}{\sqrt{\lambda}}, \ J = \frac{J}{\sqrt{\lambda}}$ (which means that $S$ are $J$ are assumed to be as large as $\sqrt{\lambda}$)

$$E = J + S + e(S, J, \sqrt{\lambda}), \quad e = \sum_{p=0}^{\infty} \frac{1}{(\sqrt{\lambda})^{p-1}} e_p(S, J), \quad (1.2)$$

and may then further expand $e_p$ for large or small $S, J$.

The AdS/CFT duality implies that the final expression for $E$ in (1.1) summed up in $\lambda$ and then expanded at strong coupling (i.e. in $\frac{1}{\sqrt{\lambda}} \ll 1$) should match (1.2), i.e. $\gamma(S, J, \lambda) = e(S, J, \sqrt{\lambda})$, but the two expansions are a priori very different and cannot be compared directly.

Still, as was noticed starting with \[9, 10, 11, 12, 13\], it is possible to establish a more direct relation between the perturbative gauge theory and string theory results for the few leading terms in the above expansions by considering large charge limits in which supersymmetry protection effectively comes into play.

Let us start with gauge theory and assume that $J \gg 1$ while $S$ is fixed. Ignoring wrapping corrections which should be exponentially suppressed at large $J$, the corresponding dimension should be described by the Asymptotic Bethe Ansatz (ABA) \[14\]. If one formally ignores the contribution of the dressing phase \[15, 16, 17, 18\] and starts with the original BDS Bethe Ansatz \[19\] then one finds that the $1/J$ expansion of $\gamma_n$ in (1.1) has the following structure

$$\gamma_n^{(0)} = \frac{1}{J^{2n-1}} \sum_{k=1}^{\infty} \frac{a_{nk}(S)}{J^k}, \quad a_{nk}(S) = \sum_{m=1}^{k} a_{nk;m} S^m, \quad (1.3)$$

where

$$a_{n1} = a_{n1;1} S, \quad a_{n2} = a_{n2;1} S + a_{n2;2} S^2, \quad a_{n3} = a_{n3;1} S + a_{n3;2} S^2 + a_{n3;3} S^3, \quad ... \quad (1.4)$$

This large $J$ expansion may be rewritten also as

$$\gamma_n^{(0)} = \frac{1}{J^{2n-1}} \sum_{k=0}^{\infty} \frac{a_{nk}(\frac{S}{J})}{J^k}, \quad a_{nk}(\frac{S}{J}) = \sum_{m=1}^{\infty} a_{nk;m} (\frac{S}{J})^m, \quad a_{nk;m} = a_{n,k+m;m} \quad (1.5)$$

While the functions $a_{nk}(S)$ in (1.3) are finite polynomials in $S$, the functions $a_{nk}(\frac{S}{J})$ are given by infinite series\[3\]. Eq. (1.3) implies that each $1/J^k$ term in $\gamma$ receives contributions from a

\[\text{The leading terms in this expansion of } E \text{ are consistent with the BMN scaling, i.e. depend on } J \text{ through the combination } \frac{\lambda}{\sqrt{\lambda}}.\]

\[\text{Below we shall always assume that } u = \frac{S}{J} < 1 \text{ so that the series formally converges.}\]
finite number of loop orders only, thus excluding a possibility of a non-trivial “interpolating”
functions of $\lambda$ as coefficients. In fact, the scaling (1.3) happens to be broken starting with 4
loops ($n = 4$) by the dressing phase contribution [14] that leads to additional contributions to
$\gamma_n$ with $n \geq 4$:

$$
\gamma_{1,2,3} = \gamma_{1,2,3}^{(0)}, \quad \gamma_{n \geq 4} = \gamma_{n}^{(0)} + \gamma_{n}^{(1)},
$$

(1.6)

$$
\gamma_{n}^{(1)} = \frac{1}{J^5} \sum_{k=0}^{\infty} \tilde{a}_{nk}(\frac{S}{J})^{k}, \quad \tilde{a}_{nk}(\frac{S}{J}) = \sum_{m=2}^{\infty} \tilde{a}_{nk;m}(\frac{S}{J})^{m}, \quad n \geq 4 .
$$

(1.7)

The contributions due to the presence of the phase producing the correction $\gamma_{n}^{(1)}$ appear to
start only with $S^2$ terms, i.e. they do not influence terms linear in $S$ which determine the slope
function $h_1$ in (1.25) below [8] [20] [21]. Indeed, as we shall explicitly demonstrate in Appendix
C, the first non-trivial 4-loop contribution of the phase leads to

$$
\gamma_{4}^{(1)} = \tilde{a}_{40;2} \frac{S^2}{J^7} + \ldots , \quad \tilde{a}_{40;2} = -\frac{\zeta(3)}{32\pi^2} .
$$

(1.8)

The corresponding limit on the semiclassical string theory side is $J \gg 1$ where $e_p$ in (1.2) have
the following structure

$$
e_{0,1,2,3,4} = e_{0,1,2,3,4}^{(0)}, \quad e_{p \geq 5} = e_{p}^{(0)} + e_{p}^{(1)},
$$

(1.9)

$$
e_{p}^{(0)} = \frac{1}{J^{p+1}} \sum_{q=1}^{\infty} b_{pq}(S) \frac{S^q}{J^q}, \quad b_{pq}(S) = \sum_{r=1}^{q} b_{pqr} r^r,
$$

(1.10)

where $e_{p}^{(0)}$ should have “softer” $\frac{1}{J^k}$ ($k < p + 1$) prefactor than $e_{p}^{(0)}$ and is expected to start with
$p = 5$, $e_{5}^{(0)} \sim \frac{S^2}{J^7}$. While both $e_{p}^{(0)}$ and $e_{p}^{(1)}$ receive contributions from the quantum part of the
dressing phase [16] [17] in the ABA (complementing the leading “classical” AFS part [15]),
which start with $\frac{1}{J^k}$ terms in the 1-loop $e_1$, $e_{p}^{(1)}$ would be absent if one would ignore the quantum
part of the phase in strong-coupling ABA.

For example, $e_0$ entering the classical string energy with fixed $S$ may be written as

$$
e_0 = \frac{1}{J} \left( \frac{b_{00;1} S}{J} + \frac{b_{02;1} S + b_{02;2} S^2}{J^2} + \frac{b_{03;1} S + b_{03;2} S^2 + b_{03;3} S^3}{J^3} + \ldots \right) .
$$

(1.11)

The expansion of $e_p^{(0)}$ in (1.10) may be reorganized as

$$
e_p^{(0)} = \frac{1}{J^{p+1}} \sum_{q=0}^{\infty} b_{pq}(\frac{S}{J}) \frac{S^q}{J^q}, \quad b_{pq}(\frac{S}{J}) = \sum_{r=1}^{q} b_{pqr}(\frac{S}{J})^r, \quad b_{pqr} = b_{p,q+r,r} .
$$

(1.12)

Like $a_{nk}(S)$ and $a_{nk}(\frac{S}{J})$ in (1.3), (1.5), while $b_{pq}(S)$ in (1.10) are finite polynomials, the functions
$b_{pq}(\frac{S}{J})$ are given by infinite series. Explicitly, for $e_p = e_p^{(0)}$ with $p = 0, 1, 2, 3, 4$ one has [12] [16]

$$
e_0 = \frac{1}{J} \left( b_{00} + \frac{b_{02}}{J^2} + \frac{b_{04}}{J^4} + \ldots \right) , \quad e_1 = \frac{1}{J^2} \left( b_{10} + \frac{b_{12}}{J^2} + \frac{b_{14}}{J^4} + \ldots \right) ,
$$

(1.13)

\footnote{The $\frac{1}{J^{p+1}}$ scaling of $e_p^{(0)}$ in (1.10) translates into $\frac{1}{(\sqrt{J})^{p+1}} b_{p0}(\frac{S}{J}) = \frac{1}{J^{p+1}} b_{p0}(\frac{S}{J})$ leading contribution to $e$
in (1.2).}
\[ e_2 = \frac{1}{\mathcal{J}^3} \left( b_{20} + \frac{b_{22}}{\mathcal{J}^2} + \ldots \right), \quad e_3 = \frac{1}{\mathcal{J}^4} \left( b_{30} + \frac{b_{31}}{\mathcal{J}} + \ldots \right), \quad e_4 = \frac{1}{\mathcal{J}^5} \left( b_{40} + \ldots \right) \] (1.14)

For \( e''_p \) we expect to find that
\[ e''_p = \frac{1}{\mathcal{J}^5} \tilde{b}_{p0}(S) + O \left( \frac{1}{\mathcal{J}^6} \right), \quad \tilde{b}_{p0} = \tilde{b}_{p0}/2 \frac{S^2}{\mathcal{J}^2} + O \left( \frac{S^3}{\mathcal{J}^3} \right), \quad p \geq 5. \] (1.15)

An advantage of organizing the expansion in terms of functions of the ratio
\[ u = \frac{S}{\mathcal{J}} = \frac{S}{\mathcal{J}} \] (1.16)
is that it does not explicitly depend on the coupling \( \lambda \) and thus may be compared between gauge and string theory. Due to the large underlying symmetry of the theory and the special nature of the large \( J \) limit the two expansions (1.6) and (1.9) have the same dependence of the spins and can be universally described by
\[ \gamma = e = E - S - J = \sum_{n=1}^{\infty} \frac{q_n}{\mathcal{J}^n}, \quad q_n = q_n(S, \mathcal{J}, \lambda). \] (1.17)

In the perturbative gauge theory one finds from (1.3), (1.6), (1.7)
\[ q_1 = \lambda a_{10}, \quad q_2 = \lambda a_{11}, \quad q_3 = \lambda a_{12} + \lambda^2 a_{20}, \quad q_4 = \lambda a_{13} + \lambda^2 a_{21}, \quad q_5 = \lambda a_{14} + \lambda^2 a_{22} + \lambda^3 a_{30} + \lambda^4 a_{31} + \ldots, \quad \ldots \] (1.18)
while in the perturbative string theory eqs. (1.10), (1.12), (1.15) give
\[ q_1 = \lambda b_{00}, \quad q_2 = \lambda b_{10}, \quad q_3 = \lambda^2 b_{02} + \lambda b_{20}, \quad q_4 = \lambda^2 b_{12} + \lambda b_{30}, \quad q_5 = \lambda^3 b_{03} + \lambda^2 b_{12} + \lambda^3 b_{03} + \lambda^4 b_{13} + \lambda^5 b_{20} + \lambda^6 b_{30} + \ldots, \quad \ldots \] (1.19)

Here \( b_{13}, b_{31}, \tilde{b}_{50}, \ldots \), etc., are related to the quantum phase contributions. The functions \( q_1, \ldots, q_4 \) turn out to be protected (i.e. exactly given by linear or quadratic functions of \( \lambda \) at both large and small \( \lambda \)). At the same time, \( q_5, q_6, \ldots \) are already non-trivial “interpolating” functions of \( \lambda \). For example, (1.19) and (1.21) represent weak-coupling and strong coupling expansions of the same \( q_5 \), with dots standing for further infinite number of contributions coming from the quantum dressing phase in ABA.

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5The case of large \( \mathcal{J} \) with fixed \( \frac{S}{\mathcal{J}} \) is familiar “fast string” limit. The two limits – (i) taking \( \mathcal{J} \) large for fixed \( S \) and (ii) taking \( \mathcal{J} \) large for fixed \( u = \frac{S}{\mathcal{J}} \) and then expanding in small \( u \) – lead to the same result as the dependence on \( u \) happens to be analytic.

6In particular, the dressing phase corrections are responsible for non-analytic terms with half-integer powers of \( \lambda \) and for the resolution [16, 14] of the “3-loop disagreement” [32, 33].

7This non-renormalization of \( q_1, q_2, q_3, q_4 \) should be due to the underlying supersymmetry of the large \( J \) expansion and a particular structure of the ABA [14]. Equivalently, it may be considered to be a consequence of exactness of the coefficients of the first few leading “protected” low-derivative terms in the underlying effective Landau-Lifshits type action [13, 23, 24].
The non-renormalization property of $q_1, q_2, q_3, q_4$ implies that the corresponding coefficient functions of $\frac{S}{J}$ should be the same in both the gauge-theory and the string-theory expansions, i.e.

$$a_{10} = b_{00}, \quad a_{11} = b_{10}, \quad a_{12} = b_{20}, \quad a_{20} = b_{02}, \quad a_{13} = b_{30}, \quad a_{21} = b_{12}. \quad (1.22)$$

We thus get six “non-renormalization theorems”, relating low-loop gauge theory coefficient functions to low-loop string theory ones, i.e. six infinite families of relations between coefficients in the expansions in power series in $\frac{S}{J}$. Explicitly, as follows from (1.22) and (1.15), (1.12) we find an infinite number of relations between the coefficients in (1.3) and (1.10) ($r = 1, 2, ...$)

$$a_{1q;r} = b_{q0;r}, \quad \text{i.e.} \quad a_{1,q+r;r} = b_{q;r}, \quad q = 0, 1, 2, 3,$$

$$a_{2q;r} = b_{q2;r}, \quad \text{i.e.} \quad a_{2,q+r;r} = b_{q,2+r}, \quad q = 0, 1. \quad (1.23)$$

The matching of the 1-loop gauge and tree-level string coefficient functions $a_{10} = b_{00}$ was demonstrated in [11, 12, 30]; the matching of the one-loop gauge and the one-loop string coefficients $a_{11} = b_{10}$ was seen in [25]; the matching between the 1-loop gauge and 2-loop string coefficients $a_{12} = b_{20}$ was checked (on the example of fast large-spin folded string) in [22].

The relations (1.22) should be universal, i.e. should not depend on a particular string solution (and should apply to generic states, e.g., for $\mathfrak{su}(2)$ sector states). Some of these relations will be checked below using explicit tree-level plus 1-loop string results and 1-loop and 2-loop gauge theory results for the folded string states. We may then use them to make predictions, e.g., about higher loop string coefficients from the independent knowledge of gauge-theory coefficients. For example, relations originating from $a_{12} = b_{20}, a_{13} = b_{30}$ may be used to get information about some 2-loop and 3-loop string coefficients from the knowledge of the 1-loop gauge theory coefficients.

Again, starting with $q_5$ the functions $q_n$ in (1.17) get non-trivial all-order dependence on $\lambda$ and thus their expansions at small and large $\lambda$ at fixed $u = \frac{S}{J} = \frac{S}{J}$ look different and the coefficients there cannot be matched.

The above discussion of the structure of $q_n$ in (1.17) applies for generic values of $\frac{S}{J}$, i.e. not only for $\frac{S}{J} \ll 1$ but also for $\frac{S}{J} \gg 1$, e.g., in the fast long string limit $S \gg J \gg \ln S$ [26, 27]. In this limit integer powers of $\frac{S}{J}$ in the expansion of the energy get replaced by powers of $\ln \frac{S}{J}$. For example, the analog of $a_{12} = b_{20}$ non-renormalization relation in (1.22) was checked in this limit in [28, 29]. The first unprotected function $q_5$ in (1.17) here has the structure [27]

$$q_5 = d(\lambda) \ln \frac{S}{J} + \ldots \quad d(\lambda) = \begin{cases} \lambda^3(d_0 + d_1 \lambda + \ldots), & \lambda \ll 1, \\ \lambda^3(1 + \frac{16}{3\sqrt{\lambda}} + \ldots), & \lambda \gg 1 \end{cases} \quad (1.24)$$

where $d_0 \neq 1, \ d_1 \sim \zeta(3)$.

One of our aims here is to understand the implications and possible extensions of the non-renormalization relations (1.22). A new motivation comes from the recent observation [8] of the special role of the linear in spin terms in the energy – the corresponding coefficient (“slope”

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8In particular, in (1.19), (1.21) $a_{30} \neq b_{04}$ which is an example of “3-loop disagreement”.
function) turns out not to receive contributions from the dressing phase in ABA \[20, 21\]. That means that while in general the functions \( q_5, q_6, \ldots \) in (1.17) are non-trivially renormalised, their parts linear in \( S \) are effectively protected, i.e. can be directly recovered either from the gauge theory or string theory perturbative expansions without any resummation involved. As was originally proposed in \[8\] and further discussed in \[9, 7\], we can consider the formal expansion of the string energy in small semiclassical spin parameter \( S \). Expressing \( S \) then as \( \sqrt{\lambda} \) we get a formal “small \( S \)” expansion

\[
E^2 = J^2 + h_1(J, \sqrt{\lambda}) S + h_2(J, \sqrt{\lambda}) S^2 + \ldots, \quad E = J + \frac{h_1}{2J} S + \ldots, \quad (1.25)
\]

\[
h_1 = \sqrt{\lambda} h_{1,0} + h_{1,1} + \frac{h_{1,2}}{\sqrt{\lambda}} + \ldots, \quad h_2 = h_{2,0} + \frac{h_{2,1}}{\sqrt{\lambda}} + \ldots, \quad h_{n,k} = h_{n,k}(J). \quad (1.26)
\]

Similar relation with \( h_1 = h_1(J, \lambda) \) can be found on the gauge theory side by a formal analytic continuation to the region where \( S \ll 1 \). At weak coupling, \( \lambda \ll 1 \) and for \( J \gg 1 \) we get

\[
h_1 = 2J + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{J^n}, \quad (1.27)
\]

where the functions \( c_n(\lambda) \) are finite polynomials in \( \lambda \), e.g.,

\[
c_1 = \lambda, \quad c_2 = -\lambda, \quad c_3 = \lambda - \frac{\lambda^2}{4}, \quad c_4 = -\lambda + \lambda^2, \quad c_5 = \lambda - \frac{11\lambda^2}{4} + \frac{\lambda^3}{8}, \quad (1.28)
\]

\[
c_6 = -\lambda + \frac{13\lambda^2}{2} - \lambda^3, \quad c_7 = \lambda - \frac{57\lambda^2}{4} + 5\lambda^3 - \frac{5\lambda^4}{64}, \quad c_8 = -\lambda + 30\lambda^2 - \frac{81\lambda^3}{4} + \lambda^4, \quad \ldots
\]

At strong coupling or semiclassical string theory, with \( \sqrt{\lambda} \gg 1 \) and \( J = \frac{\sqrt{\lambda}}{\lambda} \gg 1 \), we have

\[
h_1 = 2\sqrt{\lambda} J + \sum_{n=1}^{\infty} \frac{a_n(\sqrt{\lambda})}{J^n} = 2J + \sum_{n=1}^{\infty} \frac{\bar{c}_n(\lambda)}{J^n}. \quad (1.29)
\]

Due to the absence of the dressing phase contribution to \( h_1 \) \[8, 20, 21\] one may expect that \( \bar{c}_n(\lambda) \) should also given by same finite polynomials in \( \lambda \), without any resummation,

\[
c_n(\lambda) = \bar{c}_n(\lambda). \quad (1.30)
\]

This provides a non-trivial extension of the “non-renormalization” relations in (1.18), (1.20). This direct relation can indeed be proved starting from the explicit expression of the slope \( h_1 \) \[8\] valid for all \( \lambda \) and \( J \) (below \( I_k(x) \) is the modified Bessel function of the first kind)

\[
h_1(J, \lambda) = 2\sqrt{\lambda} \frac{d}{d\sqrt{\lambda}} \ln I_J(\sqrt{\lambda}) = 2J + 2\sqrt{\lambda} \frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})}. \quad (1.31)
\]

It obeys the differential equation

\[
\frac{dh_1}{d\lambda} + \frac{1}{4\lambda} h_1^2 - \frac{J^2}{\lambda} - 1 = 0. \quad (1.32)
\]
If we replace $h_1$ here by its expansion (1.27), we immediately determine the functions $c_n(\lambda)$ in (1.28). They are polynomials valid for all values of $\lambda$ since they are derived from (1.32) which is exact in $\lambda$.  

In the case of higher order functions $h_k$ with $k > 1$ in (1.25) the expansion like (1.27) will have non-trivial coefficients starting with $c_5$: they will correspond to $S^2, S^3, \ldots$ terms in $\alpha_0$ in (1.17) and thus are expected to be given by interpolating functions having different form when expanded at weak and at strong coupling.

Similar considerations apply to strings moving in $S^5$, e.g., for folded string in the $\mathfrak{su}(2)$ sector having spin $J'$ and orbital momentum $J$: the analog of (1.25) is then

$$E^2 = J^2 + \tilde{h}_1(J, \sqrt{\lambda}) J' + \tilde{h}_2(J, \sqrt{\lambda}) J'^2 + \ldots, \quad E = J + \frac{\tilde{h}_1}{2J} J' + \ldots. \quad (1.33)$$

There is a simple observation that allows one to determine the $\mathfrak{su}(2)$ sector slope $\tilde{h}_1$ in terms of the $\mathfrak{sl}(2)$ sector one $h_1$ in (1.25), (1.31). Using that the two folded string solutions are related by an analytic continuation $[30]$, it is possible to derive the following relation between the two slope functions:

$$\tilde{h}_1(J, \sqrt{\lambda}) = -h_1(J, -\sqrt{\lambda}), \quad \text{i.e.} \quad \tilde{h}_1(J, \sqrt{\lambda}) = -h_1(-J, -\sqrt{\lambda}), \quad J = \sqrt{\lambda} J'. \quad (1.34)$$

Indeed, given a classical solution for a string moving in $AdS_3 \times S^1$ with energy and spins $(\mathcal{E}, S, J)$ it can be related (by an analytic continuation converting $AdS_3$ into $S^3$) to a classical solution in $\mathbb{R} \times S^3$ with the energy and spins $(\tilde{\mathcal{E}}, J', \tilde{J})$ such that $\mathcal{E} = -J$, $\tilde{J} = -\mathcal{E}$, $\tilde{J} = J' = S$. Since this continuation involves setting the radial direction $\rho$ in $AdS_3$ equal to $i\theta$ where $\theta$ is an angle in $S^3$, the action changes sign. This sign change can be compensated by reversing the sign of the string tension $[31]$, $\sqrt{\lambda} \rightarrow -\sqrt{\lambda}$, thus ensuring that the quantum corrections to the two solutions are also in correspondence. As a result, the relations between the parameters of the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ solutions are

$$\tilde{\mathcal{E}} = -J, \quad \tilde{J} = -\mathcal{E}, \quad J' = S, \quad \sqrt{\lambda} = -\sqrt{\lambda}. \quad (1.35)$$

Expanding the two energies in respective small spins $S$ and $J'$ we then get (cf. (1.25), (1.33))

$$E^2 = J^2 + h_1(J, \sqrt{\lambda}) S + \ldots, \quad \tilde{E}^2 = \tilde{J}^2 + \tilde{h}_1(J, \sqrt{\lambda}) J' + \ldots, \quad (1.36)$$

which implies (1.34) after using (1.35). Below we shall study the consequences of (1.34) in detail, determining the exact expression for the $\mathfrak{su}(2)$ slope $\tilde{h}_1$ in terms of Bessel $K$-function and its explicit behaviour at weak and at strong coupling.

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9 From (1.32), we deduce the following recursion relation for the polynomials $c_n(\lambda)$

$$c_1 = \lambda, \quad c_n = -\lambda c'_{n-1} - \frac{1}{4} \sum_{m=1}^{n-2} c_m c_{n-m-1}.$$

10 Here we formally use the same notation for the functions of $(J, \sqrt{\lambda})$ and of $(J, \sqrt{\lambda})$.

11 A similar proposal for the $\mathfrak{su}(2)$ slope, based on $\sqrt{\lambda} \rightarrow -\sqrt{\lambda}$, $J \rightarrow -J$ in the $\mathfrak{sl}(2)$ slope was independently made in [20] by starting with the ABA at weak coupling.
The rest of this paper is organized as follows. In section 2 we shall check the “non-renormalization” relations (1.22) listing the values of $a_{nk}$ coefficients in (1.18) expanded in small spin limit and comparing them to string theory data in Appendices A and B. In section 3 we shall discuss our proposal for the slope function for the $(J', J)$ folded string state in $\mathfrak{su}(2)$ sector. Details of string-theory and gauge-theory computations are summarized in Appendices A–E.

## 2 Check of the “non-renormalization” relations

The relations (1.22) between first few leading coefficients on the string and gauge theory sides can be demonstrated explicitly in small spin (or small $u$ (1.16)) expansion for the folded string in $AdS_5$ or $S^5$. Below we list the results that follow from the perturbative data given in Appendices.

### 2.1 Folded string in $\mathfrak{sl}(2)$ sector

The gauge-theory expressions for the functions $a_{nk}$ in (1.18) entering the non-renormalization relations (1.22) for the $(S, J)$ folded string state in $\mathfrak{sl}(2)$ sector can be read from the results of Appendix C (here $u = S/J$)

\[
\begin{align*}
    a_{10} &= \frac{u}{2} - \frac{u^2}{4} + \frac{3u^3}{16} - \frac{21u^4}{128} + \frac{159u^5}{1024} + \ldots, \\
    a_{11} &= -\frac{u}{2} + \left(-\frac{1}{8} - \frac{\pi^2}{12}\right)u^2 + \left(\frac{3}{64} + \frac{\pi^2}{24} + \frac{\pi^4}{180}\right)u^3 + \left(-\frac{99}{512} + \frac{\pi^2}{384} - \frac{\pi^4}{240} - \frac{\pi^6}{1512}\right)u^4 + \ldots, \\
    a_{12} &= \frac{u}{2} + \left(-\frac{3}{16} + \frac{\pi^2}{4} - \frac{\pi^4}{90}\right)u^2 + \left(\frac{3}{16} - \frac{43\pi^2}{192} - \frac{\pi^4}{120} + \frac{11\pi^6}{3780}\right)u^3 + \ldots, \\
    a_{13} &= -\frac{u}{2} + \left(\frac{5}{32} - \frac{19\pi^2}{48} + \frac{2\pi^4}{45} - \frac{\pi^6}{315}\right)u^2 + \ldots, \\
    a_{20} &= -\frac{u}{8} - \frac{u^2}{4} + \frac{11u^3}{32} - \frac{27u^4}{64} + \frac{1041u^5}{2048} + \ldots, \\
    a_{21} &= \frac{u}{2} + \left(\frac{5}{8} + \frac{\pi^2}{24}\right)u^2 + \left(-\frac{11}{16} + \frac{\pi^2}{12}\right)u^3 + \left(\frac{561}{1024} - \frac{29\pi^2}{256} - \frac{17\pi^4}{2880} - \frac{\pi^6}{3024}\right)u^4 + \ldots.
\end{align*}
\]

These expressions can be compared with the available information about string theory coefficients $b_{nk}$ in (1.20) summarized in Appendix A. The relations (1.22) indeed hold in all cases where there is string data for comparison to be made. In addition, the above gauge-theory functions contain also terms that can be directly tested at the moment and thus provide 2- and 3-loop string predictions: such are the terms in $a_{12} = b_{20}$ and $a_{13} = b_{30}$.

In Appendix C we also compute the leading dressing phase correction to the gauge-theory anomalous dimension (1.8). The first term in (C.13) is proportional to $\frac{\lambda^4}{f^2} \left(\frac{5}{7}\right)^2$ and thus contributes to $q_5$ in (1.19). This indicates that in contrast to $q_1, q_2, q_3, q_4$ functions, starting with order $S^2$ terms (contributing to higher slope $h_2, \ldots$ functions in (1.25)) the function $q_5$ is not given by a finite polynomial as would be the case if one were to use the BDS ansatz.
2.2 Folded string in \textit{su} (2) sector

The functions $a_{nk}$ in (1.18) in the case of the $(J', J)$ folded string in \textit{su} (2) sector are computed in Appendix D (here $u = J'/J$)

\begin{align*}
a_{10} &= \frac{u}{2} - \frac{3u^2}{4} + \frac{15u^3}{16} - \frac{139u^4}{128} + \frac{1239u^5}{1024} + \ldots, \\
a_{11} &= \frac{u}{2} - \left( \frac{11}{8} + \frac{\pi^2}{12} \right) u^2 + \left( \frac{157}{64} + \frac{7\pi^2}{24} - \frac{\pi^4}{180} \right) u^3 - \left( \frac{1899}{512} + \frac{239\pi^2}{384} - \frac{17\pi^4}{720} + \frac{\pi^6}{1512} \right) u^4 + \ldots \\
a_{12} &= \frac{u}{2} + \left( - \frac{29}{16} - \frac{\pi^2}{4} + \frac{\pi^4}{90} \right) u^2 + \left( \frac{17}{4} + \frac{197\pi^2}{192} - \frac{23\pi^4}{360} + \frac{11\pi^6}{3780} \right) u^3 + \ldots, \\
a_{13} &= \frac{u}{2} + \left( - \frac{75}{32} - \frac{19\pi^2}{48} + \frac{2\pi^4}{45} - \frac{\pi^6}{315} \right) u^2 + \ldots, \\
a_{20} &= -\frac{u}{8} + \frac{u^2}{4} - \frac{9u^3}{32} + \frac{7u^4}{64} + \frac{761u^5}{2048} + \ldots, \\
a_{21} &= -\frac{u}{2} + \left( \frac{15}{8} + \frac{\pi^2}{24} \right) u^2 + \left( - \frac{67}{16} - \frac{\pi^2}{12} \right) u^3 + \left( \frac{7449}{1024} - \frac{29\pi^2}{256} + \frac{13\pi^4}{960} - \frac{\pi^6}{3024} \right) u^4 + \ldots.
\end{align*}

Again, they can be compared with the available string theory data for the $b_{nk}$ in (1.20) given in Appendix B and the relations (1.22) hold in all cases.

3 A proposal for the slope function in the \textit{su} (2) sector

Let us now study the proposal for slope function $\tilde{h}_1$ in the \textit{su} (2) sector implied by the relation (1.34) to the slope $h_1$ in the \textit{sl} (2) sector. The relation (1.34) was motivated from strong coupling so the precise definition of $\tilde{h}_1$ at weak coupling may need extra input.

Starting with the strong-coupling expansion of $h_1$ in (1.31) for fixed $J$

\begin{equation}
\begin{aligned}
h_1(J, \sqrt{\lambda}) &= 2\sqrt{J^2 + 1} \sqrt{\lambda} - \frac{1}{J^2 + 1} + \frac{4J^2 - 1}{4(J^2 + 1)^{5/2}} \frac{1}{\sqrt{\lambda}} \\
&\quad - \frac{4J^4 - 10J^2 + 1}{4(J^2 + 1)^4} \frac{1}{(\sqrt{\lambda})^2} + \frac{8J^2(8J^4 - 70J^2 + 57)}{64(J^2 + 1)^{11/2}} - \frac{25}{(\sqrt{\lambda})^3} \\
&\quad - \frac{16J^8 - 368J^6 + 924J^4 - 374J^2 + 13}{16(J^2 + 1)^7} \frac{1}{(\sqrt{\lambda})^4} + \ldots.
\end{aligned}
\end{equation}

eq. (1.34) implies that to get the corresponding expansion of $\tilde{h}_1$ one is to change $\sqrt{\lambda} \to -\sqrt{\lambda}$ and change the overall sign, i.e.

\begin{align*}
\tilde{h}_1(J, \sqrt{\lambda}) &= 2\sqrt{J^2 + 1} \sqrt{\lambda} + \frac{1}{J^2 + 1} + \frac{4J^2 - 1}{4(J^2 + 1)^{5/2}} \frac{1}{\sqrt{\lambda}} \\
&\quad + \frac{4J^4 - 10J^2 + 1}{4(J^2 + 1)^4} \frac{1}{(\sqrt{\lambda})^2} + \frac{8J^2(8J^4 - 70J^2 + 57)}{64(J^2 + 1)^{11/2}} - \frac{25}{(\sqrt{\lambda})^3} \\
&\quad - \frac{16J^8 - 368J^6 + 924J^4 - 374J^2 + 13}{16(J^2 + 1)^7} \frac{1}{(\sqrt{\lambda})^4} + \ldots.
\end{align*}
Like \( h_1 \) in (1.31), (3.1), the \( \text{su}(2) \) slope \( \tilde{h}_1(J, \lambda) \) admits also a regular expansion at large \( \sqrt{\lambda} \) and fixed \( J \), that follows also from (3.2) upon setting \( J = \frac{J}{\sqrt{\lambda}} \) and re-expanding in \( \frac{1}{\sqrt{\lambda}} \)

\[
\tilde{h}_1(J, \sqrt{\lambda}) = 2\sqrt{\lambda} + 1 - \frac{1}{\sqrt{\lambda}} \left( \frac{1}{4} - J^2 \right) + \frac{1}{(\sqrt{\lambda})^2} \left( \frac{1}{4} - J^2 \right) + \ldots ,
\]

(3.3)

which is in agreement with expectations in [7]. Since the expansion (3.2), (3.3) depends on even powers of \( J \) only, the relation (1.34) implies that it is the same as the one for the \( \text{sl}(2) \) slope \( h_1 \) in [8] but with the signs of the terms with even powers of \( \frac{1}{\sqrt{\lambda}} \) reversed.

One can compare the three-loop gauge theory data given in Appendix C for the \( \text{sl}(2) \) slope function \( h_1 \) to show that it agrees with the exact expression for the coefficients \( c_1, ..., c_8 \) given explicitly in (1.28). This is a consequence of the fact that \( h_1 \) does not receive contributions from the dressing phase. Inspecting similar three-loop data for the coefficients in the slope \( \tilde{h}_1 \) in the \( \text{su}(2) \) sector (1.33) collected in Appendix D, we find that they are in agreement with the proposed relation (1.34).

Using the explicit 3-loop gauge theory data the slopes \( h_1 \) and \( \tilde{h}_1 \) can be resummed to all orders in \( 1/J \) and take the form

\[
h_1(J, \lambda) = 2J + \frac{\lambda}{J + 1} - \frac{\lambda^2}{4(J + 1)^2(J + 2)} + \frac{\lambda^3}{8(J + 1)^3(J + 2)(J + 3)} + \cdots ,
\]

(3.4)

\[
\tilde{h}_1(J, \lambda) = 2J + \frac{\lambda}{J - 1} - \frac{\lambda^2}{4(J - 1)^2(J - 2)} + \frac{\lambda^3}{8(J - 1)^3(J - 2)(J - 3)} + \cdots .
\]

(3.5)

We observe that these two expressions are indeed related by (1.34), i.e. by \( h_1(J, \lambda) = -\tilde{h}_1(-J, \lambda) \) as functions of integer powers of \( \lambda \). The expression (3.5) is the same as found in [20].

Let us now address the question about an exact expression for \( \tilde{h}_1(J, \lambda) \) that correctly interpolates between the correct weak-coupling and strong-coupling expansion. This question turns out to be non-trivial:

(a) At strong coupling, the transformation (1.34) cannot be directly implemented by doing the replacement \( \sqrt{\lambda} \rightarrow -\sqrt{\lambda} \) in the exact expression (1.31) \(^{13}\)

(b) At weak coupling, the expansion (3.5) breaks down at some order in \( \lambda \) for any positive integer \( J \).

\(^{12}\) The expression (3.3) is of course also obtained from the exact expression in (1.31).

\(^{13}\) A simple example that explains why this is not so is the following large \( x \) expansion

\[
\frac{x^2 \pm \sqrt{x^2 + 1}}{\sqrt{x^2 + 2}} = x \pm 1 - \frac{1}{x} \pm \frac{1}{2x^2} + \frac{3}{2x^3} \pm \frac{7}{8x^4} - \frac{5}{2x^5} \pm \frac{25}{16x^6} \cdots .
\]

It shows that the transformation that changes half of the series has nothing to do with sign flip of \( x \). This is due to the branch point at infinity and to the fake odd powers of \( x \) arising from square roots.
To try to resolve these problems, we recall that the function \( Y_J(x) = I'_J(x)/I_J(x) \) (here prime is derivative over \( x = \sqrt{\lambda} \)), entering the expression (1.31) for the exact \( \mathfrak{sl}(2) \) slope \[8\] obeys the relations

\[
Y'_J = 1 + \frac{J^2}{x^2} - \frac{Y_J}{x} - Y^2_J, \quad Y_J(+\infty) = 1 .
\] (3.6)

Changing sign of \( x \) as instructed by (1.34), we are interested in the solution \( Z_J(x) \) to the following conditions

\[
-Z'_J = 1 + \frac{J^2}{x^2} + \frac{Z_J}{x} - Z^2_J, \quad Z_J(+\infty) = 1 .
\] (3.7)

Here we used that both \( h_1 \) and \( \tilde{h}_1 \),

\[
h_1 = 2\sqrt{\lambda} Y'_J , \quad \tilde{h}_1 = 2\sqrt{\lambda} Z'_J ,
\] (3.8)

should have the strong-coupling asymptotics \( 2\sqrt{\lambda} \) at fixed \( J \) (see (3.1), (3.2) where in this limit \( J = 0 \)). This implies the boundary condition \( Z_J(+\infty) = 1 . \) The unique solution of (3.7) is given in terms of the modified Bessel function of the second kind \( K_J \), i.e. \( Z_J(x) = (\ln K_J(x))' \). Thus our proposal for \( \tilde{h}_1(J, \lambda) \) is\[14\]

\[
\tilde{h}_1(J, \lambda) = -2 \sqrt{\lambda} \frac{d}{d\sqrt{\lambda}} \ln K_J(\sqrt{\lambda}) = 2J + 2\sqrt{\lambda} \frac{K_{J-1}(\sqrt{\lambda})}{K_J(\sqrt{\lambda})} .
\] (3.9)

One can check immediately that the strong coupling expansion of this function at fixed \( J \) is in agreement with (3.2), solving the above problem (a).

As an illustration, to compare the expressions for the \( \mathfrak{sl}(2) \) (1.31) and the proposed \( \mathfrak{su}(2) \) (3.9) slope functions we plotted them together for \( J = 3 \) in Figure 1.

\[14\] An equivalent form of this expression is \( h = -2 \sqrt{\lambda} \frac{d}{d\sqrt{\lambda}} \ln H^{(1)}_J(i\sqrt{\lambda}) \) where \( H^{(1)}_J \) is the first Hankel function. This follows from the relation \( K_J(x) = \frac{\pi}{2} \frac{i^{J+1}}{\sqrt{\lambda}} H^{(1)}_J(i\sqrt{\lambda}) \). We thank B. Basso for a suggestion to express \( \tilde{h}_1 \) in terms of \( K_J \). Note that \( K_J(x) = \frac{\pi}{2 \sin(\pi J)} \left[ I_{-J}(x) - I_J(x) \right] \) and that \( h_1 \) in (1.31) may be written also as \[8\] \( h_1 = -2J + 2\sqrt{\lambda} \frac{I_{J-1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} \).
Figure 1: Slope functions in \(\mathfrak{sl}(2)\) and \(\mathfrak{su}(2)\) sectors at \(J = 3\)

Here on the upper-left plot the solid line represents the \(\mathfrak{sl}(2)\) exact slope function (1.31), the dashed line is the one-loop weak coupling asymptotics \(2J + \frac{\lambda}{J+1}\) and the dot-dashed line is the four-term strong-coupling asymptotics \(2\sqrt{\lambda} - 1 - \frac{1}{\sqrt{\lambda}} (\frac{1}{4} - J^2) - \frac{1}{(\sqrt{\lambda})^2} (\frac{1}{4} - J^2)\) (cf. (3.1)). On the upper-right plot the solid line is our proposed expression for the \(\mathfrak{su}(2)\) slope function (3.9), the dashed line is the one-loop weak coupling expansion \(2J + \frac{\lambda}{J+1}\) and the dot-dashed line is the four-term strong-coupling expansion \(2\sqrt{\lambda} + 1 - \frac{1}{\sqrt{\lambda}} (\frac{1}{4} - J^2) + \frac{1}{(\sqrt{\lambda})^2} (\frac{1}{4} - J^2)\) (cf. (3.2), (3.3)). The lower plot contains both slope curves at the same time: the \(\mathfrak{sl}(2)\) one (solid lower line) and the \(\mathfrak{su}(2)\) one (dashed upper curve).

The function \(\tilde{h}_1\) defined by (3.9) is a smooth function of \(\lambda\) at either large or small \(\lambda\) for all \(J\), including integer ones (as illustrated by Figure 1 for \(J = 3\)). What happens for positive integer \(J\) is that the small \(\lambda\) expansion (3.5) becomes asymptotic rather than having finite radius of convergence (as expected for a sum of planar graphs). Indeed, expanding the function \(\tilde{h}_1\) in...
at weak coupling at fixed integer values of \( J \) to find\(^{15}\)

\[
\tilde{h}_1 = \begin{cases} 
4 + \lambda + \frac{1}{4} \lambda^2 \left( \ln \frac{\lambda}{4} + 2 \gamma_E \right) + \ldots, & J = 2 \\
6 + \frac{\lambda}{2} - \frac{\lambda^2}{16} - \frac{\lambda^3}{128} \left( 2 \ln \frac{\lambda}{4} + 4 \gamma_E - 1 \right) + \ldots, & J = 3 \\
8 + \frac{\lambda}{3} - \frac{\lambda^2}{72} + \frac{\lambda^3}{432} + \frac{\lambda^4}{20736} \left( 9 \ln \frac{\lambda}{4} + 18 \gamma_E - 8 \right) + \ldots, & J = 4 \\
10 + \frac{\lambda}{4} - \frac{\lambda^2}{192} + \frac{\lambda^3}{3072} - \frac{7\lambda^4}{147456} - \frac{\lambda^5}{2359296} \left( 16 \ln \frac{\lambda}{4} + 32 \gamma_E - 19 \right) + \ldots, & J = 5 
\end{cases}
\]  

While the regular \( \lambda^n \) terms here are in full agreement with (3.5), the appearance of non-analytic \( \lambda^J \ln \lambda \) terms is related to problem (b), i.e. a breakdown of the expansion (3.5) at positive integer \( J \). The same problem appears of course in the \( \mathfrak{sl}(2) \) slope function \( h_1 \) in (1.31), (3.4) continued to negative integer \( J \).

In general, anomalous dimensions are functions in multiparameter space of \( \lambda, J, \) spins \( S \) or \( J' \), etc., and their general behaviour is just beginning to be understood. While for generic values of the parameters one may expect that the gauge-theory dimension given by a sum of planar diagrams should have a finite radius of convergence in \( \lambda \) this expectation may break down in certain limits of the parameters (like in few known cases of IR divergences, elimination of which requires a resummation of direct perturbation theory in \( \lambda \), see below).

Indeed, one possible reason for the appearance of the above \( \ln \lambda \) terms is that the definition of the slope function in either \( \mathfrak{sl}(2) \) or \( \mathfrak{su}(2) \) sector at finite \( J \) is non-trivial in the first place, as it is based on a formal analytic continuation to small values of spin \( S \) or \( J' \) from their standard integer values. The case of \( \mathfrak{su}(2) \) sector is even more subtle since here the spin \( J' \) is bounded from above by the fixed length \( L = J' + J \) of the spin chain, implying potential problems with an analytic continuation to small \( J' \). It is possible that the continuation of the \( \mathfrak{su}(2) \) sector anomalous dimension to small \( J' \), i.e. the slope \( \tilde{h}_1 \) is defined only in the large \( J \) limit when the bound on \( J' \) becomes irrelevant (similar remark appeared in \([20]\)]\(^{16}\)). In that case (3.5), (3.9) may be viewed as a compact way of encoding the large \( J \) expansion.

It may happen though that in contrast to the \( \mathfrak{sl}(2) \) slope, the \( \mathfrak{su}(2) \) slope may actually receive wrapping contributions which also start, in general, at the \( (\lambda^L)_{J' \to 0} \sim \lambda^J \) order \([19]\). Taking them into account (by using a TBA generalization of ABA) may lead to a modification of (3.9) that will make the expansion (3.5) well-defined, i.e. cancel the \( \lambda^J \ln \lambda \) terms in (3.10). This

\(^{15}\)Here higher-order terms contain higher powers of \( \ln \lambda \), e.g., for \( J = 1 \) one gets

\[
\tilde{h}_1(1, \lambda) = 2 - \lambda \left( \ln \frac{\lambda}{4} + 2 \gamma_E \right) + \frac{1}{4} \lambda^2 \left[ \left( \ln \frac{\lambda}{4} + 2 \gamma_E \right) \left( \ln \frac{\lambda}{4} + 2 \gamma_E - 2 \right) + 2 \right] + \ldots
\]

\(^{16}\)This will be consistent with the relation (1.34) valid at strong coupling as in the semiclassical expansion with fixed \( J \) at \( \sqrt{\lambda} \ll 1 \) the orbital momentum \( J \) is always large.
may still be consistent with the strong-coupling relation (1.34) between the \( \mathfrak{sl}(2) \) and \( \mathfrak{su}(2) \) slopes as the wrapping contributions may turn out to be suppressed at strong coupling.

An alternative (and more likely) possibility already alluded to above is that the \( \lambda J \ln \lambda \) terms in (3.10) actually have a physical meaning being analogous to \( \lambda^n \ln \lambda + \ldots \) terms appearing in IR-resummed perturbation theory (see, e.g., [34, 35]). Indeed, there is a similarity between the expansion of (3.9) in (3.10) and the ladder-diagram resummed expression for the \( q-\bar{q} \) potential in [35]. A formal reason for this may be related to an analogy [36] between the expectation value of the cusp Wilson loop at small euclidean angle (\( \phi \to 0 \)) and the \( \mathfrak{sl}(2) \) slope function (1.31) at \( J = 1 \). While the \( q-\bar{q} \) potential is related to a different (\( \phi \to \pi \) or antiparallel lines) limit of the cusp Wilson loop [37, 35], the relation between the expressions in the \( \phi \to 0 \) and \( \phi \to \pi \) limits may be similar to the transformation (1.34) relating the \( \mathfrak{sl}(2) \) slope \( h_1 \) to the \( \mathfrak{su}(2) \) slope \( \tilde{h}_1 \). One may speculate that given that the cusp Wilson loop is described, for generic values of \( \lambda \) and \( \phi \), by an integrable TBA system [38, 39], it may admit an exact representation in terms of Bessel functions not only for \( \phi \to 0 \) [36, 38] but also for \( \phi \to \pi \).

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While this paper was in preparation a similar proposal for the \( \mathfrak{su}(2) \) slope was made in [20].

A String theory data for the \( \mathfrak{sl}(2) \) folded string

A.1 Classical charges and large \( J \) limit

The classical charges of the \( \mathfrak{sl}(2) \) folded string [40, 41] can be written in parametric form as the following combination of elliptic integrals depending on the cut endpoints \( a, b \) and read [30]

\[
S = \frac{1}{2\pi} \frac{ab + 1}{ab} \left[ b E \left(1 - \frac{a^2}{b^2}\right) - a K \left(1 - \frac{a^2}{b^2}\right) \right],
\]

\[
J = \frac{1}{\pi b} \sqrt{(a^2 - 1)(b^2 - 1)} K \left(1 - \frac{a^2}{b^2}\right),
\]

\[
E_0 = \frac{1}{2\pi} \frac{ab - 1}{ab} \left[ b E \left(1 - \frac{a^2}{b^2}\right) + a K \left(1 - \frac{a^2}{b^2}\right) \right],
\]

where \( E_0 = J + S + e_0(S, J) \) as in (1.2). The small \( S \) expansion of \( E_0 \) at fixed \( J \) reads

\[
E_0 = J + S + \frac{\sqrt{J^2 + 1}}{J} S - \frac{J^2 + 2}{4 J^3 (J^2 + 1)} S^2 + \frac{3 J^6 + 13 J^4 + 20 J^2 + 8}{16 J^5 (J^2 + 1)^{5/2}} S^3 + \ldots \quad (A.2)
\]

Taking the large \( J \) limit of (A.2), we obtain

\[
E_0 = J + S + \frac{S}{2 J^2} + \frac{3 S^3}{16 J^4} + \frac{21 S^4}{128 J^5} + \frac{159 S^5}{1024 J^6} + \frac{11 S^3}{32 J^6} + \ldots
\]

\[\text{We are grateful to B. Basso for this suggestion.}\]
Notice that the same expansion is obtained by considering large $J$ at fixed $S$, i.e. the limit is fully characterized by the assumption that $S/J$ is small.

We remark that (A.3) includes the contributions encoded by the functions $a_{10}(u) = b_{00}(u)$ and $a_{20}(u) = b_{02}(u)$ (see (1.5) and (1.12)). Indeed, they can be computed in string theory at classical level and, from (A.1), it is possible to derive the following elliptic parametrizations

$$a_{10}(u) = \frac{1}{2 \pi^2} \mathbb{K} \left[ (\rho^2 + 1) \mathbb{K} - 2 \mathbb{E} \right],$$  
(A.4)

$$a_{20}(u) = \frac{1}{8 \pi^4} \mathbb{K}^3 \left[ (-\rho^4 + 4 \rho^3 + 2 \rho^2 + 4 \rho - 1) \mathbb{K} - 8 \rho \mathbb{E} \right],$$  
(A.5)

where $\mathbb{E} = \mathbb{E}(1 - \rho^2)$, $\mathbb{K} = \mathbb{K}(1 - \rho^2)$, and the parameter $\rho$ is the following implicit function of the ratio $u = S/J$ 

$$u = \frac{1}{2} \left( \frac{\mathbb{E}}{\rho \mathbb{K}} - 1 \right),$$  
(A.6)

or explicitly

$$\rho = 1 - 2 \sqrt{2} \sqrt{u} + 4u - \frac{9u^{3/2}}{2 \sqrt{2}} + u^2 + \frac{25u^{5/2}}{32 \sqrt{2}} - \frac{u^3}{2} - \frac{77u^{7/2}}{256 \sqrt{2}} + \frac{11u^4}{32} + \ldots.$$  
(A.7)

Eqs. (A.4)(A.5) allow to expand $a_{10}(u)$ and $a_{20}(u)$ at any desired order with minor effort. For instance, the first terms shown in (2.1) continue as follows:

$$a_{10}(u) = \frac{u}{2} - \frac{u^2}{4} + \frac{3u^3}{16} - \frac{21u^4}{128} + \frac{159u^5}{1024} - \frac{315u^6}{2048} + \frac{321u^7}{2048} - \frac{42639u^8}{262144} + \frac{716283u^9}{4194304},$$  
(A.8)

$$a_{20}(u) = -\frac{u}{8} + \frac{u^2}{32} - \frac{11u^3}{64} - \frac{27u^4}{2048} + \frac{1041u^5}{2048} - \frac{39u^6}{64} - \frac{11937u^7}{16384} + \frac{56937u^8}{65536} + \frac{8663721u^9}{8388608},$$  
(A.9)

### A.2 One-loop quantum corrections

The one-loop energy $e_1(S,J)$ in (1.2) has been computed in the algebraic curve formalism in [3, 6]. The calculation in [6] is done at fixed $J$ and small $S$ and provides closed expressions for the coefficients $e_{1,n}(J)$ appearing in the expansion

$$e_1(S,J) = \sum_{n=1}^{\infty} e_{1,n}(J)^{S^n},$$  
(A.10)
The large $\mathcal{J}$ expansion of $e_{1,n}(\mathcal{J})$ is tricky because these coefficients are given in [6] as infinite sums. These we may choose regularize by the $\zeta$-function method as in [25, 23, 24]. This procedure is known to miss exponentially suppressed terms $\sim e^{-2\pi \mathcal{J}}$. This point has been already discussed in [44] and we shall return to this issue in Appendix E. Explicitly, we find

$$e_{1,1}(\mathcal{J}) = \frac{1}{2} \frac{1}{\mathcal{J}^3} + \frac{1}{2} \frac{1}{\mathcal{J}^5} - \frac{1}{2} \frac{1}{\mathcal{J}^7} + \frac{1}{2} \frac{1}{\mathcal{J}^9} + \ldots,$$

$$e_{1,2}(\mathcal{J}) = \frac{1}{8} - \frac{\pi^2}{12} \frac{1}{\mathcal{J}^4} + \frac{5}{24} \frac{1}{\mathcal{J}^6} + \frac{99}{64} \frac{\pi^2}{32} \frac{1}{\mathcal{J}^8} + \frac{85}{32} \frac{1}{\mathcal{J}^{10}} + \frac{5\pi^2}{192} \frac{1}{\mathcal{J}^{12}} - \frac{4025}{1024} \frac{1}{1536} + \ldots,$$

$$e_{1,3}(\mathcal{J}) = \frac{3}{64} \frac{1}{\mathcal{J}^4} + \frac{\pi^4}{180} \frac{1}{\mathcal{J}^6} + \frac{\pi^2}{12} \frac{1}{\mathcal{J}^8} + \frac{3}{16} \frac{1}{\mathcal{J}^{10}} + \frac{\pi^2}{8} \frac{1}{\mathcal{J}^{12}} + \frac{17}{8} \frac{1}{\mathcal{J}^{14}} + \frac{5\pi^2}{6} \frac{1}{\mathcal{J}^{16}} + \frac{45\pi}{64} - \frac{5\pi^2}{24} + \ldots,$$

$$e_{1,4}(\mathcal{J}) = \frac{3}{512} \frac{1}{\mathcal{J}^4} + \frac{\pi^4}{384} \frac{1}{\mathcal{J}^6} + \frac{\pi^4}{240} \frac{1}{\mathcal{J}^8} + \frac{\pi^6}{1512} \frac{1}{\mathcal{J}^{10}} + \frac{561}{1024} \frac{1}{\mathcal{J}^{12}} + \frac{29\pi^2}{256} \frac{1}{\mathcal{J}^{14}} + \frac{17\pi^4}{2880} \frac{1}{\mathcal{J}^{16}} + \frac{5\pi^4}{3024} + \ldots.$$

(A.11) 

(A.12) 

(A.13) 

(A.14) 

A.3 Large $\mathcal{J}$ expansion of Basso’s exact slope at strong coupling

The all-loop expression (1.31) of the slope $h_1(\mathcal{J}, \sqrt{\lambda})$ defined in (1.25) was proposed in [8] and later derived from the asymptotic Bethe Ansatz (ABA) in [20, 21]. Expanding $h_1(\mathcal{J}, \sqrt{\lambda})$ at large $\mathcal{J}$ we get the following ABA predictions for the $O(S)$ terms in the higher loop string energies $e_p(S, \mathcal{J})$ in (1.2)

$$e_0(S, \mathcal{J}) = S \left( \frac{1}{2} \frac{1}{\mathcal{J}^2} - \frac{1}{8} \frac{1}{\mathcal{J}^4} + \frac{1}{16} \frac{1}{\mathcal{J}^6} - \frac{5}{128} \frac{1}{\mathcal{J}^8} + \ldots \right) + O(S^2)$$

$$e_1(S, \mathcal{J}) = S \left( \frac{1}{2} \frac{1}{\mathcal{J}^3} - \frac{1}{16} \frac{1}{\mathcal{J}^5} + \frac{1}{2} \frac{1}{\mathcal{J}^7} + \frac{1}{2} \frac{1}{\mathcal{J}^9} + + \ldots \right) + O(S^2)$$

$$e_2(S, \mathcal{J}) = S \left( \frac{1}{2} \frac{1}{\mathcal{J}^4} - \frac{11}{8} \frac{1}{\mathcal{J}^6} + \frac{5}{2} \frac{1}{\mathcal{J}^8} - \frac{45}{64} \frac{1}{\mathcal{J}^{10}} + \ldots \right) + O(S^2)$$

$$e_3(S, \mathcal{J}) = S \left( \frac{1}{2} \frac{1}{\mathcal{J}^5} - \frac{13}{4} \frac{1}{\mathcal{J}^7} - \frac{81}{8} \frac{1}{\mathcal{J}^9} + \frac{23}{16} \frac{1}{\mathcal{J}^{11}} + \ldots \right) + O(S^2)$$

$$e_4(S, \mathcal{J}) = S \left( \frac{1}{2} \frac{1}{\mathcal{J}^6} - \frac{57}{8} \frac{1}{\mathcal{J}^8} + \frac{585}{16} \frac{1}{\mathcal{J}^{10}} + \ldots \right) + O(S^2),$$

Eqs. (A.15) and (A.16) of course agree with the linear part of (A.3) and with (A.11) respectively.

B String theory data for the $\mathfrak{su}(2)$ folded string

B.1 Classical charges and large $\mathcal{J}$ limit

The classical charges of the $\mathfrak{su}(2)$ folded string can be written in parametric form as the following combination of elliptic integrals depending on the complex cut endpoints $a, b$ [30]

$$\mathcal{J} = -\frac{1}{2\pi} \frac{ab}{ab} \left[ b \mathcal{E} \left(1 - \frac{a^2}{b^2}\right) - a \mathcal{K} \left(1 - \frac{a^2}{b^2}\right) \right],$$
\[
\mathcal{J}' = \frac{1}{2\pi} \frac{ab - 1}{ab} \left[ b \Re \left( 1 - \frac{a^2}{b^2} \right) + a \Re \left( 1 - \frac{a^2}{b^2} \right) \right],
\]

\[
\mathcal{E}_0 = -\frac{1}{\pi b} \sqrt{(a^2 - 1)(b^2 - 1) \Re \left( 1 - \frac{a^2}{b^2} \right)},
\]

where \( \mathcal{E}_0 = \mathcal{J} + \mathcal{J}' + \epsilon_0(\mathcal{J}', \mathcal{J}) \) as in (1.2) with the obvious replacement \( S \to \mathcal{J}' \). The small \( \mathcal{J}' \) expansion of \( \mathcal{E}_0 \) at fixed \( \mathcal{J} \) reads

\[
\mathcal{E}_0 = \mathcal{J} + \frac{\sqrt{\mathcal{J}'^2 + 1}}{\mathcal{J}} \mathcal{J}' - \frac{3 \mathcal{J}'^2 + 2}{4(\mathcal{J}'^3 + \mathcal{J}^3)} \mathcal{J}'^2 + \frac{15 \mathcal{J}'^6 + 33 \mathcal{J}'^4 + 28 \mathcal{J}'^2 + 8}{16 \mathcal{J}'^2 (\mathcal{J}' + 1)^{5/2}} \mathcal{J}'^3 + \ldots \tag{B.2}
\]

Taking the large \( \mathcal{J} \) limit of (B.2), we obtain

\[
\mathcal{E}_0 = \mathcal{J} + \mathcal{J}' + \frac{\mathcal{J}'}{2 \mathcal{J}^2} - \frac{3 \mathcal{J}'^2 - \mathcal{J}^3}{4 \mathcal{J}^4} \mathcal{J}'^2 + \frac{15 \mathcal{J}'^6 - \mathcal{J}'^7 + \mathcal{J}'^8 - 139 \mathcal{J}'^5}{128} + \frac{1239 \mathcal{J}'^5 - 32 \mathcal{J}'^6 + 9 \mathcal{J}'^7}{16} \mathcal{J}'^8 + \ldots \tag{B.3}
\]

As in the \( \mathfrak{sl}(2) \) folded string, the same expansion is obtained by expanding in large \( \mathcal{J} \) at fixed \( \mathcal{J}' \), i.e. the limit is fully characterized by the assumption that the ratio \( \mathcal{J}'/\mathcal{J} \) is small.

### B.2 One-loop quantum corrections

The one-loop energy \( \epsilon_1(\mathcal{J}', \mathcal{J}) \) has been computed in the algebraic curve approach in [7] at fixed \( \mathcal{J} \) and small \( \mathcal{J}' \). It provides closed expressions for the coefficients \( \epsilon_{1,n}(\mathcal{J}) \) appearing in the expansion

\[
\epsilon_1(\mathcal{J}', \mathcal{J}) = \sum_{n=1}^{\infty} \epsilon_{1,n}(\mathcal{J}) \mathcal{J}'^n. \tag{B.4}
\]

Doing the same calculations as for the \( \mathfrak{sl}(2) \) folded string, we find

\[
\epsilon_{1,1}(\mathcal{J}) = \frac{1}{2 \mathcal{J}^3} - \frac{1}{2 \mathcal{J}^5} + \frac{1}{2 \mathcal{J}^7} - \frac{1}{2 \mathcal{J}^9} + \frac{1}{2 \mathcal{J}^{11}} - \frac{1}{2 \mathcal{J}^{13}} + \ldots, \tag{B.5}
\]

\[
\epsilon_{1,2}(\mathcal{J}) = -\frac{11}{8} - \frac{\pi^2}{12} + \frac{15}{4} + \frac{\pi^2}{12} - \frac{183}{64} - \frac{\pi^2}{32} + \frac{65}{16} + \frac{5 \pi^2}{192} + \frac{-5565}{1024} - \frac{35 \pi^2}{384} + \ldots, \tag{B.6}
\]

\[
\epsilon_{1,3}(\mathcal{J}) = -\frac{157}{64} + \frac{5 \pi^2}{24} - \frac{\pi^4}{128} + \frac{67}{16} - \frac{\pi^2}{12} + \frac{157}{12} + \frac{\pi^2}{8} - \frac{19}{6} + \frac{\pi^2}{6} + \ldots. \tag{B.7}
\]

### C Gauge theory data for the \( \mathfrak{sl}(2) \) folded string

#### C.1 Three-loop corrections to anomalous dimension

Let us first compute the three-loop anomalous dimension of the ground state of the \( \mathfrak{sl}(2) \) spin chain, i.e. the state dual to the spinning folded string in \( AdS_3 \). The calculation is similar to
the one in Appendix B of [24], but takes into account the important technical fact that we are interested in a state with highly degenerate mode numbers. An alternative derivation could start from the results of Appendix C.1.2 of [19]. The all-order Bethe equations [45] are written in terms of the auxiliary functions

\[
x(u) = \frac{u}{2} \left( 1 + \sqrt{1 - \frac{\lambda}{4 \pi^2 u^2}} \right), \quad x^\pm(u) = x(u \pm \frac{i}{2}),
\]

where \( u \) is the rapidity of Bethe roots. Let us consider an even number \( S \) of magnons. A generic state will be specified by the \( S \) Bethe roots \( \{U_n(\lambda)\}_{n=1,\ldots,S} \) obeying the Bethe Ansatz equations

\[
J \ln \frac{x_n^+}{x_n^-} - \sum_{m\neq n}^S \ln \left( \frac{x_n^- - x_m^-}{x_n^+ - x_m^+} \right) = 2 \pi i N_n, \quad n = 1, \ldots, S, \tag{C.2}
\]

where \( x_n^\pm = x(U_n)^\pm \). Given the Bethe roots \( U_n(\lambda) \), the anomalous dimension \( \gamma(S, J, \lambda) \) in (1.1) is given by

\[
\gamma(S, J, \lambda) = \sum_{\ell=1}^\infty \lambda^\ell \gamma_\ell(S, J) = \frac{\lambda}{8 \pi^2} \sum_{n=1}^S \left( \frac{i}{x_n^+} - \frac{i}{x_n^-} \right). \tag{C.3}
\]

We will be interested in the ground state of the spin chain that is characterized by a set of Bethe roots even under \( U \to -U \)

\[
U_n = (u_1, \ldots, u_{\frac{S}{2}}, -u_1, \ldots, -u_{\frac{S}{2}}), \quad i = 1, \ldots, S. \tag{C.4}
\]

The independent variables are thus \( \{u_n\}_{n=1,\ldots,\frac{S}{2}} \). They can be found by solving (C.2) with \( n = 1, \ldots, \frac{S}{2} \) and choosing the mode numbers to be equal \( N_n = 1 \) in this range (they are \(-1\) for the remaining Bethe roots).

The large \( J \) expansion of the Bethe roots has been worked out in Appendix B.1 of [24] for the case where all \( N_n \) are distinct. In the present case, it turns out to have the form

\[
u_n(J, \lambda) = \frac{J}{2 \pi} + \frac{u_{0,n}^{(0)}}{\pi} \sqrt{J} + u_{1,n}^{(0)} + u_{2,n}^{(0)} \frac{1}{J} + u_{3,n}^{(0)} \frac{1}{J^2} + \ldots
\]

\[+ \lambda \left( \frac{u_{0,n}^{(1)}}{J} + \frac{u_{1,n}^{(1)}}{J^{3/2}} + \frac{u_{2,n}^{(1)}}{J^2} + \ldots \right) + \mathcal{O}(\lambda^2). \tag{C.5}\]

The only non-trivial problem is the determination of the constants \( u_{0,n}^{(0)} \). Indeed, all the other constants are iteratively determined by solving linear problems. Instead, the equations for \( z_n = u_{0,n}^{(0)} \) are non linear and read

\[
\sum_{m\neq n}^{\frac{S}{2}} \frac{1}{z_n - z_m} = 2 z_n, \quad n = 1, \ldots, \frac{S}{2}. \tag{C.6}
\]

\[\text{Here we are going to consider only the 3-loop corrections so the dressing phase does not contribute } \text{[14].}\]
The dressing phase \( \lambda \) starts contributing at order \( \lambda^2 \). Leading dressing phase contribution to 4-loop anomalous dimension is

\[
H_2(\sqrt{2} z_n) = 0. \tag{C.7}
\]

Working out perturbation series for various values of \( S \) we easily determine the exact \( S \)-dependence of various \( 1/J^n \) corrections to the anomalous dimension. The results for the one, two, and three-loop corrections to the anomalous dimension are

\[
\gamma_1(S, J) = \frac{S}{2J^2} - \left( \frac{S^2}{4} + \frac{S}{2} \right) \frac{1}{J^3} + \left[ - \frac{21S^4}{128} + \left( \frac{3}{64} + \frac{\pi^2}{24} + \frac{\pi^4}{180} \right) S^3 + \left( - \frac{3}{16} + \frac{\pi^2}{4} - \frac{\pi^4}{90} \right) S^2 - \frac{S}{2} \right] \frac{1}{J^4}
\]

\[
+ \left[ \frac{159S^5}{1024} + \left( - \frac{99}{512} + \frac{\pi^2}{384} - \frac{\pi^4}{240} - \frac{\pi^6}{1512} \right) S^4 + \left( \frac{3}{16} + \frac{43\pi^2}{192} - \frac{\pi^4}{120} + \frac{11\pi^6}{3780} \right) S^3
\]

\[
+ \left( \frac{5}{32} - \frac{19\pi^2}{48} + \frac{2\pi^4}{45} - \frac{\pi^6}{315} \right) S^2 + \frac{S}{2} \right] \frac{1}{J^5} + \ldots, \tag{C.8}
\]

\[
\gamma_2(S, J) = -\frac{S^2}{8J^4} + \left( \frac{3S^2}{4} + \frac{S}{2} \right) \frac{1}{J^5} + \left[ - \frac{27S^4}{64} + \left( \frac{\pi^2}{12} - \frac{11\pi^4}{16} \right) S^3 - \frac{175S^5}{16} + \frac{13S^4}{4} \right] \frac{1}{J^6}
\]

\[
+ \left[ \frac{1014S^5}{2048} + \left( \frac{561}{1024} - \frac{29\pi^2}{256} - \frac{17\pi^4}{2880} - \frac{\pi^6}{3024} \right) S^4
\]

\[
+ \left( \frac{295}{128} - \frac{131\pi^2}{384} - \frac{\pi^4}{144} + \frac{\pi^6}{7560} \right) S^3
\]

\[
+ \left( \frac{25}{16} - \frac{29\pi^2}{32} + \frac{\pi^4}{40} - \frac{\pi^6}{630} \right) S^2 - \frac{57S^6}{8} \right] \frac{1}{J^7} + \ldots, \tag{C.9}
\]

\[
\gamma_3(S, J) = \frac{S}{16J^6} + \left( \frac{3S^2}{8} - \frac{S}{2} \right) \frac{1}{J^7} + \left[ - \frac{45S^4}{128} + \left( \frac{7}{32} - \frac{\pi^2}{4} \right) S^3 + \frac{305S^5}{32} - \frac{81S^4}{8} \right] \frac{1}{J^8}
\]

\[
+ \left[ \frac{5949S^5}{8192} + \left( \frac{5219}{4096} + \frac{119\pi^2}{3072} + \frac{\pi^4}{90} + \frac{\pi^6}{12096} \right) S^4
\]

\[
+ \left( \frac{941}{512} + \frac{2059\pi^2}{1536} - \frac{37\pi^4}{2880} - \frac{11\pi^6}{30240} \right) S^3
\]

\[
+ \left( \frac{8211}{256} + \frac{165\pi^2}{128} - \frac{\pi^4}{240} + \frac{\pi^6}{2520} \right) S^2 + \frac{585S^6}{16} \right] \frac{1}{J^9} + \ldots. \tag{C.10}
\]

### C.2 Leading dressing phase contribution to 4-loop anomalous dimension

The dressing phase \( \lambda \) starts contributing at order \( \lambda^4 \). It is included as \( e^{2i\vartheta_{nm}} \) under the ln in the Bethe equations (C.2). The leading contribution to \( \vartheta_{nm} \) can be written as

\[
\vartheta_{nm} = 4 \zeta(3) \left( \frac{3}{16\pi^2} \right)^3 \left[ Q_2(u_n) Q_3(u_m) - Q_2(u_m) Q_3(u_n) \right] + O(\lambda^4) \tag{C.11}
\]
where the higher charges \( Q_r(u) \) are

\[
Q_r(u) = \frac{i}{r-1} \left[ \frac{1}{(u+i/2)^{r-1}} - \frac{1}{(u-i/2)^{r-1}} \right].
\]

(C.12)

The first few terms of the large \( J \) expansion of the 4-loop anomalous dimension \( \gamma_4(S,J) \) are found to be

\[
\gamma_4(S,J) = -\frac{\zeta(3)}{32 \pi^2 J^7} \left[ -\frac{5}{128} S + \frac{\zeta(3)}{\pi^2} \left( \frac{13}{64} S^2 + \frac{1}{128} S^3 \right) \right] \frac{1}{J^8}
\]
\[
+ \left\{ \frac{1}{2} S - \frac{7}{16} S^2 + \frac{\zeta(3)}{\pi^2} \left[ -\frac{7}{8} S^2 - \left( \frac{1}{128} - \frac{\pi^2}{32} \right) S^3 + \frac{1}{256} S^4 \right] \right\} \frac{1}{J^9} + \ldots
\]

(C.13)

Note, in particular, that the linear terms in \( S \) here match the corresponding terms in the strong-coupling expressions (A.13), (A.16) and that dressing phase contributions start with \( S^2 \) terms.

D  Gauge theory data for the \( \mathfrak{su}(2) \) folded string

The corresponding calculation in the \( \mathfrak{su}(2) \) sector is completely similar to the one in the \( \mathfrak{sl}(2) \) sector. Let \( L = J + J' \) denote the length of the \( \mathfrak{su}(2) \) spin chain. The ground state has \( J' \) magnons and its three loop anomalous dimension turns out to be\(^{19}\)

\[
\gamma_1(J', L) = \frac{J'}{2 L^2} + \left( \frac{J'^2}{4} + \frac{J'}{2} \right) \frac{1}{L^3} + \left[ \frac{3J'^3}{16} + \left( \frac{1}{8} - \frac{\pi^2}{12} \right) J'^2 + \frac{J'}{2} \right] \frac{1}{L^4}
\]
\[
+ \frac{21J'^4}{128} + \left( -\frac{3}{64} - \frac{\pi^2}{24} - \frac{\pi^4}{180} \right) J'^3 + \left( \frac{3}{16} - \frac{\pi^2}{4} + \frac{\pi^4}{90} \right) J'^2 + \frac{J'}{2} \frac{1}{L^5}
\]
\[
+ \frac{159J'^5}{1024} + \left( -\frac{99}{512} + \frac{\pi^2}{240} - \frac{\pi^4}{1512} \right) J'^4 + \left( 3 - \frac{43 \pi^2}{192} - \frac{\pi^4}{120} + \frac{11 \pi^6}{3780} \right) J'^3
\]
\[
+ \left( \frac{5}{32} - \frac{19 \pi^2}{48} + \frac{2 \pi^4}{45} - \frac{\pi^6}{315} \right) J'^2 + \frac{J'}{2} \frac{1}{L^6} + \ldots,
\]

(D.1)

\[
\gamma_2(J', L) = -\frac{J'}{8 L^4} + \left( \frac{-J'^2}{4} - \frac{J'}{2} \right) \frac{1}{L^5} + \left[ -\frac{9J'^3}{32} + \left( \frac{\pi^2}{24} - \frac{5}{8} \right) J'^2 - \frac{11J'}{8} \right] \frac{1}{L^6}
\]
\[
+ \left[ -\frac{21J'^4}{32} + \left( -\frac{\pi^2}{3} - \frac{7}{8} \right) J'^3 + \left( \frac{2 \pi^2}{3} - \frac{19}{8} \right) J'^2 - \frac{13J'}{2} \right] \frac{1}{L^7}
\]
\[
+ \left[ -\frac{807J'^5}{2048} + \left( -\frac{167}{1024} + \frac{137 \pi^2}{768} + \frac{13 \pi^4}{960} - \frac{\pi^6}{3024} \right) \right] \frac{1}{L^8}
\]
\[
+ \left( -\frac{209}{256} + \frac{349 \pi^2}{384} - \frac{11 \pi^4}{240} + \frac{11 \pi^6}{7560} \right) J'^3
\]
\[
+ \left( -\frac{31}{16} + \frac{121 \pi^2}{96} + \frac{\pi^4}{630} - \frac{\pi^6}{8} \right) J'^2 - \frac{57J'}{8} \frac{1}{L^8} + \ldots,
\]

(D.2)

\[
\gamma_3(J', L) = \frac{J'}{16 L^5} + \left( \frac{J'^2}{4} + \frac{J'}{2} \right) \frac{1}{L^6} + \left[ \frac{37J'^3}{128} + \left( \frac{93}{64} - \frac{\pi^2}{32} \right) J'^2 + \frac{5J'}{2} \right] \frac{1}{L^7}
\]

(D.3)

\(^{19}\)The notation is again that of \((1.1)\) with the obvious replacement \( S \rightarrow J' \) and with \( L \) playing here the role that \( J \) had in the \( \mathfrak{sl}(2) \) sector.
A non trivial check of these expression is the equality of the dimensions of the 2-magnon states in the \( \mathfrak{sl}(2) \) and \( \mathfrak{su}(2) \) sectors implied by superconformal invariance (see for instance \[4\])

\[
\gamma(\ell' = 2, L = 2 + J) = \gamma(S = 2, J) .
\]  

The first two terms in \( \gamma_1 \) were first computed in \[13\].

In order to compare with string theory, we are to take into account that \( L = J + J' \) and re-expand at large \( J \). The resulting expressions read \[20\]

\[
\gamma_1(J', J) = \frac{J'}{2J} + \left( \frac{J'}{4} - \frac{3J'^2}{8} \right) \frac{1}{J^3} + \left[ \frac{15J'^3}{16} + \left( - \frac{11}{8} - \frac{\pi^2}{12} \right) J'^2 + \frac{J'}{2} \right] \frac{1}{J^4}
\]

\[
+ \left[ - \frac{139J'^4}{128} + \left( \frac{157}{64} + \frac{7\pi^2}{24} - \frac{\pi^4}{180} \right) J'^3 + \left( - \frac{29}{16} - \frac{\pi^2}{4} + \frac{\pi^4}{90} \right) J'^2 + \frac{J'}{2} \right] \frac{1}{J^5}
\]

\[
+ \left[ \frac{1239J'^5}{1024} + \left( - \frac{199}{512} - \frac{23\pi^2}{384} + \frac{17\pi^4}{720} - \frac{\pi^6}{1512} \right) J'^4
\]

\[
\quad + \left( \frac{17}{4} + \frac{197\pi^2}{192} - \frac{23\pi^4}{360} + \frac{11\pi^6}{3780} \right) J'^3
\]

\[
\quad + \left( - \frac{75}{32} - \frac{19\pi^2}{48} + \frac{2\pi^4}{45} - \frac{\pi^6}{315} \right) J'^2 + \frac{J'}{2} \right] \frac{1}{J^6} + \ldots ,
\]  

\[
\gamma_2(J', J) = -\frac{J'}{8J^4} + \left( \frac{J'^2}{4} - \frac{J'}{2} \right) \frac{1}{J^5} + \left[ - \frac{9J'^3}{32} + \left( \frac{15}{8} + \frac{\pi^2}{24} \right) J'^2 - \frac{11J'}{8} \right] \frac{1}{J^6}
\]

\[
+ \left[ \frac{7J'^4}{64} + \left( - \frac{67}{16} - \frac{\pi^2}{12} \right) J'^3 + \left( \frac{113}{16} + \frac{\pi^2}{3} \right) J'^2 - \frac{13J'}{4} \right] \frac{1}{J^7}
\]

\[
+ \left[ \frac{761J'^5}{2048} + \left( \frac{7449}{1024} - \frac{29\pi^2}{256} + \frac{13\pi^4}{960} - \frac{\pi^6}{3024} \right) J'^4
\]

\[
\quad + \left( - \frac{5473}{256} - \frac{547\pi^2}{384} - \frac{11\pi^4}{240} + \frac{11\pi^6}{7560} \right) J'^3
\]

\[
\quad + \left( \frac{333}{16} + \frac{12\pi^2}{96} + \frac{4\pi^4}{40} - \frac{\pi^6}{630} \right) J'^2 + \frac{57J'^2}{8} \right] \frac{1}{J^8} + \ldots ,
\]  

\[
\gamma_3(J', J) = \frac{J'}{16J^6} + \left( \frac{J'}{2} - \frac{J'^2}{8} \right) \frac{1}{J^7} + \left[ - \frac{19J'^3}{128} + \left( - \frac{131}{64} - \frac{\pi^2}{32} \right) J'^2 + \frac{5J'}{2} \right] \frac{1}{J^8}
\]

\[
+ \left[ \frac{201J'^4}{128} + \frac{107J'^3}{32} + \left( - \frac{465}{32} - \frac{\pi^2}{2} \right) J'^2 + \frac{81J'}{8} \right] \frac{1}{J^9}
\]

\textsuperscript{20}With an abuse of the notation we do not distinguish here between \( \gamma_\ell(J', J + J') \) and \( \gamma_\ell(J', J) \).
\[ J' = 46059 J'^5 + \frac{8827}{4096} + \frac{749 \pi^2}{1024} - \frac{\pi^4}{96} + \frac{\pi^6}{12096} \]  
\[ + \left( \frac{21911}{512} + \frac{1257 \pi^2}{512} + \frac{29 \pi^4}{960} - \frac{11 \pi^6}{30240} \right) J'^3 \]  
\[ + \left( -\frac{18963}{256} - \frac{475 \pi^2}{128} - \frac{\pi^4}{240} + \frac{\pi^6}{2520} \right) J'^2 + \frac{585 J'}{16} \]  
\[ + \ldots \]  
(D.7)

### E Details of large $J$ expansion of the one-loop string correction

The coefficient $e_{1,2}(J)$ defined (A.10) is given by the exact expression [6]

\[ e_{1,2}(J) = 3 J'^4 + 11 J'^2 + 17 \frac{16 J^3}{J'^3 (J^2 + 1)^{5/2}} \sum_{n=2}^{\infty} \frac{n^2 (J^2 + 2 n^2 - 1)}{J^3 (n^2 - 1)^2 (J^2 + n^2)^{3/2}}. \]  
(E.1)

Let us discuss in detail its large $J$ expansion. The aim will be to clarify the role of $\zeta$-function regularization with respect to non-analytic and exponentially suppressed contributions. A naive expansion gives

\[ e_{1,2}(J) = \left( \frac{3}{16} - \sum_{n=2}^{\infty} \frac{n^2}{(n^2 - 1)^2} \right) \frac{1}{J'^4} + \left( \frac{7}{32} + \sum_{n=2}^{\infty} \frac{2 n^2 - n^4}{2 (n^2 - 1)^2} \right) \frac{1}{J'^6} + \ldots \]  
(E.2)

The sum in the first term is finite and gives the following coefficient of $\frac{1}{J'^4}$:

\[ \frac{3}{16} - \left( \frac{1}{16} + \frac{\pi^2}{12} \right) = \frac{1}{8} - \frac{\pi^2}{12}. \]

The sum in the second $\frac{1}{J'^6}$ term is divergent. We regularize it by the $\zeta$-function as follows

\[ \frac{7}{32} + \sum_{n=2}^{\infty} \frac{2 n^2 - n^4}{2 (n^2 - 1)^2} = \frac{7}{32} + \sum_{n=2}^{\infty} \left( \frac{1}{2 (n^2 - 1)^2} - \frac{1}{2} \right) = \]  
\[ = \frac{7}{32} + \frac{\pi^2}{24} - \frac{11}{32} - \frac{1}{2} (\zeta(0) - 1) = \frac{5}{8} + \frac{\pi^2}{24}. \]  
(E.3)

Thus the $\zeta$-function regularization provides the following result

\[ e_{1,2}(J) = \left( \frac{1}{8} - \frac{\pi^2}{12} \right) \frac{1}{J'^4} + \left( \frac{5}{8} + \frac{\pi^2}{24} \right) \frac{1}{J'^6} + \ldots, \]  
(E.4)

The details of the $\zeta$-function regularization in the general are as follows. For a rational function $R(n^2)$ we write

\[ R(n^2) = \sum_{k=0}^{p} c_k n^{2k} + \mathcal{O}(\frac{1}{n^2}). \]  
(E.5)

Then, our definition for the regularized sum is

\[ \sum_{n=2}^{\infty} R(n^2) \overset{\text{reg.}}{=} \sum_{n=2}^{\infty} \left[ R(n^2) - \sum_{k=0}^{p} c_k n^{2k} \right] + \sum_{k=0}^{p} c_k \left( \zeta(-2k) - 1 \right). \]  
(E.6)
The above procedure misses non-analytic terms with odd powers of \( \frac{1}{\mathcal{J}} \). These are due to the dressing phase in the all-order Bethe ansatz equations [14] and are not captured by the \( \zeta \)-function regularization. They affect the coefficients of \( S_n \) terms starting from \( n = 2 \).

In order to find them, at least at one-loop order, one has to compute the infinite sums in \( e_{1,n}(\mathcal{J}) \) exactly at finite \( \mathcal{J} \) and then perform the large \( \mathcal{J} \) expansion. In the specific case of \( e_{1,2}(\mathcal{J}) \), it can be written as

\[
e_{1,2}(\mathcal{J}) = e_{1,2}^{\text{anomaly}}(\mathcal{J}) + e_{1,2}^{\text{dressing}}(\mathcal{J}) + e_{1,2}^{\text{wrapping}}(\mathcal{J}),
\]

where we have used the terminology of [3, 6] and have split the correction into the so-called anomaly term, the dressing phase contribution, and the wrapping contribution. The explicit expressions for the first two can be found in Appendix A of [6]:

\[
e_{1,2}^{\text{anomaly}}(\mathcal{J}) = \frac{2\mathcal{J}^4 + 15\mathcal{J}^2 + 4}{16\mathcal{J}^3(\mathcal{J}^2 + 1)^{5/2}} - \frac{\pi^2}{12\mathcal{J}^3\sqrt{\mathcal{J}^2 + 1}},
\]

\[
e_{1,2}^{\text{dressing}}(\mathcal{J}) = \frac{(\mathcal{J}^2 + 2) \coth^{-1}(\sqrt{\mathcal{J}^2 + 1}) - \sqrt{\mathcal{J}^2 + 1}}{2\mathcal{J}^3(\mathcal{J}^2 + 1)^{3/2}}.
\]

We have computed the wrapping contribution following [44]:

\[
e_{1,2}^{\text{wrapping}}(\mathcal{J}) = \int_{\mathcal{J}}^{\infty} dt \frac{t^2}{\mathcal{J}^5(t^2 + 1)^{3/2}\sqrt{\mathcal{J}^2 - t^2}} \left[ 2\left(-\mathcal{J}^2 + (\mathcal{J}^2 + 3)t^2 + 1\right)\coth(\pi t) - 1 \right] - \pi t(t^2 + 1)(-\mathcal{J}^2 + 2t^2 + 1) \text{csch}^2(\pi t),
\]

confirming that at large \( \mathcal{J} \) it is suppressed as \( O(e^{-2\pi \mathcal{J}}) \). The expansion of \( e_{1,2}^{\text{anomaly}}(\mathcal{J}) \) is a regular power series in \( 1/\mathcal{J}^2 \)

\[
e_{1,2}^{\text{anomaly}}(\mathcal{J}) = \frac{1}{8} - \frac{\pi^2}{12} + \frac{5}{8} \frac{\pi^2}{\mathcal{J}^4} + \frac{5}{32} \frac{\pi^2}{\mathcal{J}^6} - \frac{99}{64} \frac{\pi^2}{\mathcal{J}^8} + \frac{85}{32} \frac{\pi^2}{\mathcal{J}^{10}} + \ldots,
\]

in agreement with the previous expression (A.12). At the same time, the expansion of \( e_{1,2}^{\text{dressing}}(\mathcal{J}) \) is a regular power series containing only odd powers \( 1/\mathcal{J} \)

\[
e_{1,2}^{\text{dressing}}(\mathcal{J}) = \frac{2}{3\mathcal{J}^7} - \frac{16}{15\mathcal{J}^9} + \frac{48}{35\mathcal{J}^{11}} - \frac{512}{315\mathcal{J}^{13}} + \frac{1280}{693\mathcal{J}^{15}} + \ldots.
\]

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