Robust Gain-scheduled Controller Design with a Hierarchical Structure

Yang Guo * Carsten W. Scherer **

* University of Stuttgart, Germany (e-mail: yang.guo.tu@gmail.com).
** Department of Mathematics, University of Stuttgart, Germany (e-mail: carsten.scherer@mathematik.uni-stuttgart.de)

Abstract: In the LFT framework, we propose an exact state-space solution for the problem of designing a robust gain-scheduled controller with a hierarchical structure and based on semi-definite programming. Under the hypothesis of adopting general dynamic multipliers for robustness and static D-scalings for gain-scheduling, our technique enriches the class of structured controller design problems that are tractable by convex optimization.

Keywords: Structured \( L_2 \)-gain control, robust control, gain-scheduling control, LMIs

1. INTRODUCTION

The design of hierarchically structured controllers has attracted much attention in various research areas, especially in networked control. Several approaches for the design of a structured controller have been proposed, either in the frequency domain via the Youla parameterization (see, e.g., Voulgaris (2001) and Rotkowitz and Lall (2005)) or directly in the state space (see, e.g., Schuler et al. (2011), Scherer (2013) and Rösinger and Scherer (2017)). Despite the fact that Youla techniques allow to recast the problem as an infinite-dimensional convex optimization problem, they are generally not computationally efficient and give rise to controllers of large order. Such deficiencies can be avoided by direct state space techniques. The exact convex state space solution of the optimal \( H_\infty \) problem is given in Scherer (2013) under the hypotheses that the control channel of the plant and the controller itself share the same triangular structure. Recently, Rösinger and Scherer (2017) have extend this approach to the optimal control problem of networked systems with delays.

In view of many notable results focusing on structured optimal control, it is a natural idea to also consider the synthesis of structured robust (and) gain-scheduled controllers. As for structured robust design, examples are the approaches in Adegas and Stoustrup (2011) and Menezes et al. (2016), which both rely on non-convex techniques; the former is restricted to discrete-time polytopic systems, while the latter employs non-smooth optimisation algorithms to obtain locally optimal solutions, which impedes the tuning of weighting filters as mentioned by the authors. Compared with robust synthesis, not many papers tackle the structured gain-scheduling problem for special classes of systems, such as, e.g., discrete time affine linear parameter-varying systems (Adegas et al. (2012), Emedi and Karimi (2016)) or triangular time-varying systems (Mishra et al. (2014)). To date, there are no papers addressing the problem of designing robust (and) gain-scheduled hierarchically structured controller in the \( LPV/LFT \) framework by convex optimization. This motivates the main goal of this paper, which develops a method to design a hierarchically structured robust gain-scheduled controller by solving a semi-definite program. Indeed, with Scherer (2013) and Veenman and Scherer (2012) being the starting point, we use dynamic multipliers from IQC theory (Megretski and Rantzer (1997)) for robustness and constant D-scalings for gain-scheduling (Packard (1994), Apkarian and Gahinet (1995)) to formulate the robust performance analysis and the corresponding synthesis results with linear matrix inequalities (LMIs).

The paper is organized as follows. After formally stating the structured robust gain-scheduled problem in Section 2, we introduce the linear fractional representation for the plant and the closed loop in Section 3. Then, in Section 4, we formulate the analysis results for robust stability and \( L_2 \)-gain performance and present the corresponding synthesis result in Section 5. We conclude the paper with a numerical example in Section 6.

Notations: \( L_2^* \) denotes the the space of vector valued finite energy signals on \([0, \infty)\) while \( ||d|| \) is the norm of \( d \in L_2^* \). If necessary, we use the abbreviations HeG and \((*)^*GP\) for \( G^* + G \) and \( P^*GP \) respectively. The notation \( G = \frac{AB}{C^*D} \) stands for the transfer matrix \( D + C(sI - A)^{-1}B \) and the corresponding state space realization is denoted by \( (A, B, C^*, D) \). Finally, \( C^* = i\mathbb{R} \cup \{\infty\} \).

2. PROBLEM FORMULATION

Let us consider an uncertain generalized plant

\[
\begin{pmatrix}
  e \\
  y
\end{pmatrix} = \begin{pmatrix}
P_{gg}(\delta_1, \delta_2, \Delta_1, \Delta_2) & P_{gy}(\delta_1, \delta_2, \Delta_1) \\
P_{yg}(\delta_1, \delta_2, \Delta_1) & P_{yy}(\delta_1, \delta_2)
\end{pmatrix}
\begin{pmatrix}
e \\
y
\end{pmatrix}
\]

(1)

in which \( \Delta_1, \Delta_2 \) are genuine uncertainties that do not affect the control channel \( u \rightarrow y \), while \( \delta_1, \delta_2 \) are time-varying parameters used for gain-scheduling. Several concrete control problems can be subsumed to this description.
as shown by Veenman and Scherer (2012, 2014). In this paper we deviate from existing results and assume that the block defining the control channel admits the lower block triangular structure

\[ \hat{P}_{yu}(\delta_1, \delta_2) = \begin{pmatrix} P_{11}(\delta_1, \delta_2) & 0 \\ P_{21}(\delta_1, \delta_2) & P_{22}(\delta_1, \delta_2) \end{pmatrix} \]

of dimension \( k \times m \) and partitioned according to \( k = k_1 + k_2 \) and \( m = m_1 + m_2 \). In the sequel, we systematically use \( \hat{ } \) to denote matrices with such a block triangular structure.

The signal \( d \in \mathcal{L}_2^p \) is the performance input and \( e \in \mathcal{L}_2^q \) is the performance output. Moreover, \( \Delta_1 \in \mathbb{R}^{p_1 \times q_1} \), \( \Delta_2 \in \mathbb{R}^{p_2 \times q_2} \) are time-invariant real uncertainty blocks whose values are confined to the polytopes \( \Delta_1, \Delta_2 \) that are described as the convex hulls \( \text{conv}(\Delta_1) \), \( \text{conv}(\Delta_2) \) with a finite number of generator matrices in \( \Delta_1, \Delta_2 \), respectively; we assume that both polytopes contain the zero matrix. The normalized time-varying scheduling parameters \( \delta_1, \delta_2 \) are assumed to be contained in \( \delta = [\delta : [0, \infty) \mapsto \mathbb{R}] \) \( \delta \) is piecewise continuous, \( |\delta(t)| \leq 1, \forall t \geq 0 \).

The goal is to design a gain-scheduling controller with the same block triangular structure as \( \hat{P}_{yu} \), i.e.,

\[ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \]

(3)

where \( u = \text{col}(u_1, u_2) \), \( y = \text{col}(y_1, y_2) \), which robustly stabilizes (1) and robustly achieves a guaranteed bound on the \( \mathcal{L}_2 \)-gain of the performance channel \( d \mapsto e \). Such a triangular structure is also interpreted as a hierarchical structure in Voulgaris (2001).

3. SYSTEM INTERCONNECTIONS WITH LINEAR FRACTIONAL REPRESENTATION

Let us assume that the plant (1) admits the linear fractional representation (LFR)

\[ \begin{pmatrix} z_{s1} \\ z_{s2} \end{pmatrix} = \begin{pmatrix} \hat{G}_{ss} \hat{G}_{su} \hat{G}_{1s} G_{1s} & 0 \\ \hat{G}_{2s} \hat{G}_{su} \hat{G}_{1s} G_{1s} + \hat{G}_{2u} G_{2u} & \hat{G}_{s}^{-1} \hat{G}_{su} \hat{G}_{1s} G_{1s} \end{pmatrix} \begin{pmatrix} w_{s1} \\ w_{s2} \end{pmatrix} \]

(4)

where \( u_1 = \text{col}(w_{s1}, w_{s2}) \), \( y_1 = \text{col}(z_{s1}, z_{s2}) \), \( \Delta_1 \) is the structured transfer matrix \( G \) and the scheduling block \( \Delta_2 = \text{diag}(\delta_1 I_{l_1}, \delta_2 I_{l_2}) \). Following Veenman and Scherer (2012), the sparsity pattern of \( G \) stems from the requirement that the control channel of (1) is not affected by \( \Delta_1 \) and \( \Delta_2 \); moreover, \( P_{sd} \) needs to be trilinear in some rational functions of \( (\delta_1, \delta_2) \), \( \Delta_1, \Delta_2 \) while \( P_{su} \) and \( P_{yd} \) are bilinear in rational functions of \( (\delta_1, \delta_2) \), \( \Delta_2 \), and \( (\delta_1, \omega) \), \( \Delta_1 \), respectively. If constructing the LFR of \( \hat{P}_{yu} \) by stacking those for the subblocks in (2), routine LFR manipulations allow to arrive at the above description with lower block-triangular matrices \( \hat{G}_{ss}, \hat{G}_{su}, \hat{G}_{ys} \) and \( \hat{G}_{yu} \) for \( i, j \in \{1, 2\} \). The corresponding partitions of \( \hat{G}_{ss} \) and \( \hat{G}_{ys} \) are given by \( l^1 = l_1^1 + l_2^1 \) and \( l^2 = l_1^2 + l_2^2 \), respectively.

The to-be-constructed gain-scheduled controller (3) is assumed to be a rational function of \( (\delta_1, \delta_2) \). As just argued, it can hence be described by the LFR

\[ \begin{pmatrix} y \end{pmatrix} = \begin{pmatrix} \hat{K}_{uu} & \hat{K}_{us} \\ \hat{K}_{ys} & \hat{K}_{ys} \end{pmatrix} \begin{pmatrix} u \end{pmatrix} \]

(5)

with a transfer matrix \( K \) and \( \Delta_r = \text{diag}(\delta_1 I_{l_1}, \delta_2 I_{l_2}) \), \( w_c = \text{col}(w_{c1}, w_{c2}) \) and \( z_c = \text{col}(z_{c1}, z_{c2}) \). As indicated by the hats, all subblocks of \( K \) are by themselves lower block-triangular; \( \hat{K}_{11} \) and \( \hat{K}_{22} \) carry partitions in terms of \( I_{l_1} = I_{l_1^1} + I_{l_1^2} \) and \( I_{l_2} = I_{l_2^1} + I_{l_2^2} \), respectively.

By standard computations (assuming well-posedness), the LFR of the closed loop interconnection reads as

\[ \begin{pmatrix} z \end{pmatrix} = \begin{pmatrix} P_{ss} P_{se} P_{s1} 0 \\ P_{ce} P_{c} P_{c1} 0 \end{pmatrix} \begin{pmatrix} y \end{pmatrix} \]

(6)

where the controller dependent blocks are given by

\[ K = \begin{pmatrix} G_{ss} & 0 & G_{sd} \\ G_{ce} P_{c} & G_{cc} & G_{cd} \end{pmatrix} \]

(7)

with \( G_{ss} = \text{col}(G_{ss1}, G_{ss2}) \), \( G_{sd} = \text{col}(G_{sd1}, G_{sd2}) \), \( G_{ce} = \text{col}(G_{ces1}, G_{ces2}) \), \( G_{cc} = \text{col}(G_{ces1}, G_{ces2}) \), and \( G_{cd} = \text{col}(G_{ces1}, G_{ces2}) \). In the sequel, the particular structure of (6) is of crucial importance. Since \( G_{11}, G_{22}, G_{12} \) and \( G_{1y} \) are outside of the feedback control loop, we require them to be stable as a prerequisite.

We now formulate the precise design goals. We search \( K \) in (5) which internally stabilizes \( G \) in (4) (implying internal stability of (7)) such that (6) is robustly stable and the \( \mathcal{L}_2 \)-gain of \( d \mapsto e \) is bounded by an a priori given \( \gamma > 0 \) as

\[ \int_0^\infty \left( \frac{d(t)}{e(t)} \right)^T \Pi \left( \frac{d(t)}{e(t)} \right) dt \leq 0 \ \forall d \in \mathcal{L}_2 \]

with \( \Pi : = \text{diag}(\gamma I, \frac{1}{\gamma} I) \).

4. ROBUST \( \mathcal{L}_2 \)-GAIN PERFORMANCE ANALYSIS

Our approach is based on robustness analysis with multipliers. With a fixed \( \psi \in RL_{\infty}^{(q_1 + p_1)} \) and for the uncertainties in \( \Delta_1 \), we use the set \( \Pi_1 \) of all dynamic multipliers described as \( \Pi_1 = \psi \Pi \psi^T \) such that the real symmetric \( \Pi_1 \) satisfies the frequency domain inequalities (FDIs)

\[ \psi \Pi_1 \psi^T \Pi_1 \psi \left( \frac{\Delta_1}{I} \right) > 0 \]

(8)

for all \( \Delta_1 \in \Delta_1 \). An elementary convexity argument shows that the first FDI is then valid for all \( \Delta_1 \in \Delta_1 \). With a fixed \( \psi \in RL_{\infty}^{(q_1 + p_2)} \) and for the uncertainty \( \Delta_2 \), we use the set \( \Pi_2 \) of all \( \Pi_2 = \psi \Pi_2 \psi^T \) such that
(9) hold for all $\Delta_j \in \tilde{\Delta}_j$, $j = 1, 2$. By employing the so-called partial dualization lemma from Scherer and Veenman (2012) (since $\tilde{\Pi}_1$ is non-singular and the left upper block of it is negative definite), the FDI (11) is equivalent to

\begin{equation}
(\ast)^* \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
G_{11} & 0 & G_{1d} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\end{pmatrix} < 0,
\end{equation}

which is (8) and (9) hold for all $\Delta_j \in \tilde{\Delta}_j$, $j = 1, 2$. By employing the so-called partial dualization lemma from Scherer and Veenman (2012) (since $\tilde{\Pi}_1$ is non-singular and the left upper block of it is negative definite), the FDI (11) is equivalent to

\begin{equation}
(\ast)^* \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
G_{11} & 0 & G_{1d} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\end{pmatrix} < 0,
\end{equation}

with

\begin{equation}
H_1 \\
H_2
\end{equation}

\begin{equation}
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
G_{11} & 0 & G_{1d} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\Pi_1^{I} \\
\end{pmatrix} < 0,
\end{equation}

where $\tilde{\Pi}_1 := \text{diag}(\gamma I, -\gamma I)$, $M := \text{diag}(\Pi_1, -\Pi_2)$ and $l := l + l^t$, $v := q_1 + v$. As a result, the matrices $\Pi_1$, $\Pi_2$ and $\Pi_{sc}$ which parameterize the multipliers enter the new FDIs affinely. For an unstructured controller, it has been shown in Veenman and Scherer (2014) how to overcome non-convexity (as caused by the multiplication of the controller-dependent calligraphic matrices with $\Pi_{sc}$). The purpose of this paper is to show how to proceed if the LTI part of the controller is required to be structured as in (5).

Given a controller $K$ and by KYP lemma Ranter (1996), it is well known that all the above FDIs can be converted into finite dimension tractable feasibility problems. In the sequel, we adopt the following realizations for $\Psi_1$, some subsystem of the open loop $G$ from (4), and $K$ from (5):

\begin{equation}
\Psi_1 := \begin{pmatrix}
A_0 & 0 & 0 & 0 & B_2^D & B_2^D \\
0 & 0 & 0 & 0 & 0 & 0 \\
C_v & 0 & 0 & 0 & D_v^T & D_v^T \\
\end{pmatrix} =: \begin{pmatrix}
A_0 & B_0 & C_v & D_v \\
\end{pmatrix},
\end{equation}

\begin{equation}
\begin{pmatrix}
A_H \bar{B}_{ss} & \bar{B}_{ss} & B_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} & B_{ss} & A_{ss} \\
A_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} \\
C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} \\
\end{pmatrix} = \begin{pmatrix}
A_H \bar{B}_{ss} & \bar{B}_{ss} & B_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} & B_{ss} & A_{ss} \\
A_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} & A_{ss} & B_{ss} \\
C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} & C_{ss} \\
\end{pmatrix},
\end{equation}

\begin{equation}
K := \begin{pmatrix}
A \bar{B}_1 & \bar{B}_2 & B \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_0 \bar{D}_2 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 \bar{D}_1 & D_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2 \bar{D}_2 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C F_1 & F_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\end{equation}

we can assure that $A_0 \in \mathbb{R}^{n_0 \times n_0}$ has no eigenvalues in $\mathbb{C}^0$. In addition, the matrices $A_0 \in \mathbb{R}^{n_0 \times n_0}$ and $A_H \in \mathbb{R}^{n_0 \times n_0}$ are block triangular according to $\bar{n}_1 + \bar{n}_2$ and $n_1 + n_2$, respectively. All the matrices with the superscript " conform with the partitions of their input-output channels. The reason why such realizations with particular structures in (13) exist is given by Voulgaris (2000). Hence, the realization of the inflated closed loop system $\text{col}(\hat{H}_1, \hat{H}_2)$ (resulting from (7) by extra zero rows and columns) can be extracted from those of the subsystem of $G$ and $K$ in (13). Consequently, the realization of $\text{col}(\Psi_1, \hat{H}_1, \hat{H}_2)$ can be written as

\begin{equation}
\begin{pmatrix}
A & B \\
C_0 & D_{0} \\
C_1 & D_{1} \\
C_2 & D_2 \\
C F_1 & F_2 \\
\end{pmatrix} = \begin{pmatrix}
A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\end{equation}

\begin{equation}
K \begin{pmatrix}
A \bar{B}_1 & \bar{B}_2 & B \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_0 \bar{D}_2 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 \bar{D}_1 & D_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2 \bar{D}_2 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C F_1 & F_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}^{-1} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
C & 0 & 0 & 0 & 0 & 0 \\
F_1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\end{equation}

Here $K$ stands for the realization matrix of $K$ in (13). For simplicity, let us assume $\bar{D}_{0u} = 0$, which renders the matrices of the realization of $\text{col}(\Psi_1, \hat{H}_1, \hat{H}_2)$ linearly dependent on $K$. Indeed, based on loop shifting arguments and the fact that the block triangular structure is preserved under inversion and multiplication, one can readily go back and forth between controllers with trivial and nontrivial $\bar{D}_{0u}$.
The aforementioned $\mathcal{A}$ can be alternatively expressed as diag($A_\psi, A_d$), where $A_d$ is the system matrix of the closed loop system (7). Since $A_\psi$ has no eigenvalues in $\mathbb{C}^0$, and $\mathcal{P}$ is assumed to be internally stable, the system matrix $\mathcal{A}$ has no eigenvalue in $\mathbb{C}^0$. Thus, we can apply the KYP lemma to (12) and perform a simple Schur complement and congruence transformation to equivalently reformulate it as an existence test for a symmetric certificate $X$ fulfilling

\[ (**) \left( \begin{array}{cc} M & 0 \\ 0 & \Pi_n \end{array} \right) \left( \begin{array}{c} C_\psi D_\psi 0 \\ 0 \end{array} \right) + \text{He} \left( \begin{array}{cc} XA & XB \\ C_\psi & D_\psi \end{array} \right) \left( \begin{array}{c} \frac{1}{2} Q H_{\psi 1} 0 \\ C_\psi \end{array} \right) \prec 0. \]

(15)

The left upper block of (15) reads as

\[ \text{He} \left( \begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array} \right) \left( \begin{array}{c} A_\psi 0 \\ 0 A_{ci} \end{array} \right) + (**)^T M (C_\psi 0) \prec 0. \]

Thus, stability of $A_{ci}$ is equivalent to $X_{22} \succ 0$. This can be expressed as $X + JW^{-1}J^T \succ 0$ for some $W = W^T \in \mathbb{R}^{n_0 \times n_0}$ with $J := \text{col}(I_{n_0}, 0)$. As a consequence, the certificate $X$ itself might not be invertible. However, this is crucial for synthesis in the next section. Therefore, we shift the original $X$ to the positive definite matrix $\mathcal{X} := X + JW^{-1}J^T$, which is one of our key contributions that paves the way to a structured factorization of the certificate $\mathcal{X}$. We get the following intermediate analysis result.

**Lemma 1.** The controller $K$ renders $\mathcal{P}$ internally stable and (12) valid if there exist $\mathcal{X} \succ 0$ and $W = W^T$ with

\[ \text{He} \left( \begin{array}{c} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{array} \right) \left( \begin{array}{c} A_\psi 0 \\ 0 A_{ci} \end{array} \right) + (**)^T M (C_\psi 0) \prec 0. \]

(16)

Similarly, by applying the KYP lemma to (8) and (9), we get the corresponding LMIs for $M = \text{diag}(-\Pi_1, -\Pi_2)$ which, together with Lemma 1, are the analysis conditions for characterizing robust stability and performance and form the basis for the controller synthesis in the next section.

### 5. STRUCTURED ROBUST GAIN-SCHEDULED CONTROLLER SYNTHESIS

As seen in (16), if viewing the matrices of the controller realization also as optimization variables, one loses the convexity, which is induced by terms such as, e.g., $QC_1$ and $\mathcal{X}A$. In order to render $\mathcal{X}A$ and $\mathcal{X}B$ affine in the optimization variables, a standard procedure is to resort to a suitable congruence transformation applied to (16), which is based on a factorization of $\mathcal{X}$ in the partition

\[ \mathcal{X} = \left( \begin{array}{cc} X_{11} & U_{11}^T \\ U_{31}^T & X_{31} \end{array} \right) \in \mathbb{R}^{(n_0+n_1^2+n_2^2) \times (n_0+n_1^2+n_2^2)}, \]

where $n := n_0 + n$, and where we assume w.l.o.g. that $n_1^2 = n_2^2 = n$. Furthermore, the affine dependence on the decision variables $M, W$ of the first part of (16) should be preserved after applying this congruence transformation to the first block row/column of $T_1(M, W)$. Considering these two aspects, we propose the following factorization:

\[ \mathcal{X} = \left( \begin{array}{c} Y_1 \\ V_{11} \\ V_{12} \end{array} \right) \left( \begin{array}{c} I_{n_0} \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} X_{11} \\ X_{12} \\ X_{31} \end{array} \right) \left( \begin{array}{c} I_{n_0} \\ 0 \\ U_{31} \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \\ U_{32} \end{array} \right). \]

(17)

Here $X_3$ is positive definite and unstructured, while

\[ Y_1 = \left( \begin{array}{c} I_{n_0} \\ -Z_1 \end{array} \right), \quad Y_2 = \left( \begin{array}{c} I_{n_0+\gamma_1} \\ -Z_2 \end{array} \right), \quad X_1 = \left( \begin{array}{c} X_{11} \\ 0 \\ I_n \end{array} \right), \quad X_2 = \left( \begin{array}{c} X_{31} \\ 0 \\ I_{n_2} \end{array} \right) \]

(18)

consist of the decision variables $\tilde{X}_j = \tilde{X}_j^T$ and $\tilde{Y}_j = \tilde{Y}_j^T$ and $\tilde{Z}_j$ for $j = 2, 3$. Furthermore, $V_{11}$, $U_{21}$, $V_{22}$ and $U_{32}$ have f.c.r.. The existence proof for such a factorization, the guarantee of the non-singularity of $Y$ and the symmetry of $\mathcal{X}^T Y$ follow as that in Scherer (2013). Indeed, the structured factorization for the triangular optimal control problem in Scherer (2013) has the following form

\[ \mathcal{X} = \left( \begin{array}{c} \tilde{Y}_1 \\ \tilde{V}_{11} \\ \tilde{V}_{12} \end{array} \right) \left( \begin{array}{c} I_{n_0} \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} X_{11} \\ V_{12} \end{array} \right) \left( \begin{array}{c} I_{n_0} \\ 0 \\ U_{32} \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \\ U_{32} \end{array} \right) \]

(19)

with a full symmetric matrix $\tilde{Y}_1$ and $\tilde{V}_{11}$ having f.c.r.. Actually, (17) an be obtained from (19) by a block-column Gauss elimination. The proposed factorization is also a reminiscence of the one in Rössinger and Scheler (2017), where $X_3$ is structured rather than a full matrix. By comparing the interaction between the positive definite scaling $Q$, the certificate $\mathcal{X}$ and the controller variables in (16), it is natural to cope with the terms $QC_1$ and $QD_1$ in similar vein as for the certificate $\mathcal{X}$, although $Q$ has a block diagonal structure. If there appears only one scheduling parameter $\delta_1$, then $Q$ is full and positive definite; moreover, since any congruence transformation hitting the first part of the second block row/column and the third block row/column in (16) will not ruin the affine dependence on $M, W$, unlike as is the case for $\mathcal{X}$, we do not need to modify $QC_1$ and $QD_1$ and can directly use the factorization in (19) as in Scherer (2013). Following this philosophy and since our problem involves two scheduling parameters, we factorize $Q$, analogously to (19) but in a block diagonal fashion in accordance with (10), as

\[ \left( \begin{array}{c} R_{01}^T \\ V_{11}^T \\ V_{12}^T \end{array} \right) \left( \begin{array}{c} 0 \\ I_{n_0} \\ 0 \\ I_{n_0+\gamma_1} \end{array} \right) \left( \begin{array}{c} I_{n_0} \\ 0 \\ U_{31} \end{array} \right) \left( \begin{array}{c} I_{n_0} \\ 0 \\ U_{32} \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \\ U_{32} \end{array} \right). \]

(20)
Q in combination the one in (17) for a shifted Lyapunov certificate $\mathcal{X}$ is our key enabling technical contribution.

Based on the above discussion, let us apply a congruence transformation to (16) with $T_c := \text{diag}(\sqrt[n]{r}, R, R)$ for $\tilde{R} := \text{diag}(R^T I I I)$. Then the transformed first part on the l.h.s of (16), i.e. $T_c^T T_c(W, M) T_c$, reads as

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
 1_{1\times3} & \mathcal{J}^T & 0 & 0 \\
 1_{1\times3} ⊗ A_c & \mathcal{J}^T & B_c & 0 \\
 1_{1\times3} ⊗ C_0 & D_c & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

with $\mathcal{J} := \text{col}(I_{n_x}, 0_{n_x \times n_x})$ and $1_{1\times3} := (1, 1, 1)$. Notice that all the decision variables of $T_c$ disappear in (22) due to the particular structure of $Y_1$ and $C_0$, which explicitly justifies the aforementioned choice of the factorization (17). The second part of the transformed l.h.s of (16) reads as

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

with

$$
T_c^T R = \begin{pmatrix}
 R_{11} & R_{12} & I \\
 R_{12}^T & R_{22} & I \\
 0 & 0 & 0
\end{pmatrix},
R_{11} := \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix},
S_u := \begin{pmatrix}
 0 & 0 \\
 0 & 0 \\
 0 & 0
\end{pmatrix}, u = 1, 2.
$$

Let us now set $X := (x_2, x_3)$, $Y := (y_1, y_2)$, $R := (r_1, r_2)$, $S := (s_2, s_3)$. Then the first part of (23) reads as

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

with

$$
\begin{pmatrix}
 V_{11} & V_{12} & 0 & 0 \\
 V_{21} & V_{22} & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
 G_{11} & G_{12} & 0 & 0 \\
 G_{21} & G_{22} & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

Finally, the congruence transformations to $\mathcal{X} > 0$ and $Q > 0$ with $\mathcal{Y}$ and $\mathcal{R}$, respectively, lead to

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

with

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

with

$$
\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

After laying all the necessary groundwork, we now arrive at the main synthesis result of our paper.

**Theorem 2.** There exists a structured controller $K$ in (5) such that $\Delta := \text{diag}(\delta_1, \delta_2, \delta_3)$ that ensures, for all $\Delta_1 \in \Delta_1, \Delta_2 \in \Delta_2, \delta_1, \delta_2, \delta_3 \in \delta$, the controlled system (6) is robustly stable and the $Z_\gamma$-gain of $d \rightarrow e$ is less than $\gamma$ if the LMIs (25)-(27) in the variables $W = W^T, M = \text{diag}(\Pi_1, -\Pi_2)$ constrained by $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$, $X = (x_2, x_3), Y = (y_1, y_2), R = (r_1, r_2)$ and $S = (s_2, s_3)$ structured as in (18) and (21) are feasible.

With feasible solutions, we can readily construct the controller parameters $\mathcal{K}$ iteratively, as shown in Scherer (2013). Let us emphasize that a direct application of the robust gain-scheduling framework of Veenman and Scherer (2014) would not lead to a convex solution for structured controllers; the details are left out for reasons of space.

6. NUMERICAL EXAMPLE

Let us consider a tracking problem for a system whose output is corrupted by a parameter dependent lightly damped sinusoidal disturbance as generated by an uncertain filter. After choosing some performance weighting filters, the resulting weighted generalised plant reads as

$$
\begin{pmatrix}
 \tilde{v} \\
 \tilde{u} \\
 \tilde{y}
\end{pmatrix} = \begin{pmatrix}
 -W_e & -W_e P_{\text{dia}}(\omega) & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
 \tilde{v} \\
 \tilde{u} \\
 \tilde{y}
\end{pmatrix} = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
 \tilde{v} \\
 \tilde{u} \\
 \tilde{y}
\end{pmatrix} = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix}
$$

where $y$ and $u$ are the control outputs and inputs, $\tilde{v}$ and $\tilde{u}$ are the weighted tracking error, the weighted control input and the reference, respectively. Moreover, $d$ is the input of the disturbance filter $P_{\text{dia}}$ given by

$$
P_{\text{dia}}(\omega, s) = \frac{\cos(1, 2\pi)}{(\omega + 2\pi)^2 + 1} + \frac{\sin(1\omega)}{(\omega + 2\pi)^2 + 1}
$$

and affected by the parametric uncertainty $\omega \in [-10, 2, 0.2]$. The triangular plant $P_{\text{g}}(s) = \frac{1}{s^{1+2}} + \frac{0.5}{s^{1+3}}$ is scheduled by the
time-varying parameters $\delta_1(t), \delta_2(t) \in [-0.2, 0.2]$. The goal of the design is to optimize a bound on the robust $L_2$-gain of $\text{col}(d, r) \mapsto \text{col}(\delta, \bar{u})$.

For the purpose of comparison, we have designed a robust gain-scheduled controller with a dynamic multiplier $\Pi_1$ and a nominal controller based on Scherer (2013). We then evaluated the controlled system in three scenarios, namely, (i) $\delta_1(t) = -0.2\sin(0.5t)$, $\delta_2(t) = 0.2\sin(2t)$, $\omega = 0.2$, (ii) $\delta_1(t) = -0.2\sin(0.5t)$, $\delta_2(t) = 0.2\sin(2t)$, $\omega = 0$ and (iii) $\delta_1(t) = \delta_2(t) = 0$, $\omega = 0.2$. Fig. 1 shows the tracking errors if the reference signal is $r(t) = \text{col}(0.5\sin(2t), \sin(0.9t))^T$. Fig. 2 depicts the output disturbances.

Fig. 1. Tracking errors for nominal design (black) and robust gain-scheduled design (red) in scenario (i) (thick), (ii) (thin dashed) and (iii) (thin)

Compared with the nominal design, the robust gain-scheduled controller leads to a better tracking performance for both channels in scenario (i). By comparing scenarios (ii) and (iii) with (i) in the second channel, the effects of the uncertainties on the disturbance filter and of the scheduling parameters on the plant severely deteriorate the performance with the nominal design, while the tracking behavior hardly changed with the robust controller. This demonstrates an improved robustness to the perturbations and justifies our methodology.

7. CONCLUSION

We have enriched the problem class of tractable structured controller synthesis by proposing a novel robust gain-scheduling design algorithm for hierarchically structured controllers based on semi-definite programming. The parametric uncertainties and scheduling parameters are handled by dynamic full block multipliers and static D-scalings, respectively. Key technical steps are to work with a shifted Lyapunov certificate and a structured factorization of the D-scales. The virtue of our approach over existing methods is demonstrated by a numerical example.

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