Degree-2 Abel maps for nodal curves

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Abstract

We present numerical conditions for the existence of natural degree-2 Abel maps for any given nodal curve. Scripts were written and have so far verified the validity of the conditions for numerous curves.

1 Introduction

The theory of Abel maps for curves goes back to work by Abel in the early nineteenth century. It was Riemann though, in his seminal 1857 paper [30], that introduced the maps themselves; see [23] for the history. In modern terms, given a (connected projective) curve $C$ defined over an algebraically closed field $K$, if $C$ is smooth then its degree-$d$ Abel map is a map $A_d^L : S^d(C) \to J_C$ from the $d$-th symmetric product of the curve to its Jacobian $J_C$, given by taking a sum of $d$ points of $C$, say $Q_1 + \cdots + Q_d$, to $L \otimes O_C(-Q_1 - \cdots - Q_d)$. Here, $L$ is a line bundle of degree $d$, the particular choice of which being irrelevant.

The importance of the Abel maps appears already in the work by Abel: They encode a lot of geometric information about the curve. More precisely, the fibers of $A_d^L$ are the linear equivalence classes of degree-$d$ effective divisors of $C$. Thus, all possible embeddings of $C$ in projective spaces are known once we know its Abel maps.

The 1857 paper by Riemann introduced as well the notion of “moduli,” essentially by stating, in modern terms, that the moduli space $M_g$ of genus-$g$ smooth curves has dimension $3g - 3$, if $g \geq 2$. The space itself was constructed almost a century afterwards, followed by the construction of its compactification $\overline{M}_g$, the moduli space of (Deligne–Mumford) stable curves. Those are the curves, possibly reducible, whose only singularities are ordinary nodes and whose group of automorphisms is finite. Abel maps are so naturally defined that they vary well in families of smooth curves. It is thus natural to ask what happens when smooth curves degenerate to stable ones.

Not only natural, but potentially very useful. Understanding degenerations of Abel maps is tantamount to understanding degenerations of linear series. It was through the study of these degenerations that the celebrated Brill–Noether and Gieseker–Petri Theorems were proved; see [29] and [19]. The approach to these theorems has been systematized by the theory of limit linear series on
curves of compact type, developed by Eisenbud and Harris; see [21]. Other results have been obtained through the theory; see [17].

The theory of Eisenbud and Harris points out to a partial compactification of the variety of linear series over the moduli space of curves of compact type. It is natural to ask whether one can extend this compactification over the whole $M_g$, as Eisenbud and Harris themselves asked in [18]. The supposedly intermediate step, that of constructing a compactification of the relative Picard scheme over $M_g$, has been carried out by Caporaso [3]. But the final step has proved to be very difficult.

Sketches on how to deal with limit linear series for stable curves not of compact type are sparse in the literature. Recently, Medeiros and the second author [14] were able to describe limit canonical series on curves with two components, using the theory presented in [11], the ingredients of which having already appeared in [29]. But their description is rather complicated, and relies on the strong assumption that the components intersect each other at points in general position on each component. The same assumption is present in [16], where the case of curves with more components is partially considered.

In view of the difficulties, one might ask whether we stand a better chance of compactifying the variety of linear series over $M_g$ by looking at degenerations of Abel maps. Indeed, Abel maps of all degrees have been constructed and studied for integral curves in [1]. Degree-1 Abel maps for stable curves were constructed in [6] and studied in [7]. Higher-degree Abel maps for curves of compact type appeared soon afterwards in [10].

At this point one might say that the theory of degenerations of Abel maps is on even terms with that of limit linear series. A comparison between the two theories has in fact appeared in [15], based on the more refined notion of limit linear series introduced by Osserman [26], and thus limited to two-component curves.

The present article advances the theory of degenerations of Abel maps, by presenting purely numerical conditions for the existence of degree-2 Abel maps for any given nodal curve. Whether the curve is stable or not is immaterial. The specific moduli, even the genera of its irreducible components is immaterial. Where the points of intersection of components lie on each component is immaterial, in stark contrast with [14] and [16]. In fact, for nodal curves with the same dual graph, either all of them or none of them have degree-2 Abel maps. An algorithm has been implemented by means of CoCoA scripts, available at

http://w3.impa.br/~esteves/CoCoAScripts/Abelmaps

to check the validity of these conditions for any given curve and has, so far, always returned a positive answer. Our approach is simply to extend the construction done in [9] for two-component two-node curves. Here is what we do. Assume all the singularities of $C$ are ordinary nodes. Let $C_1, \ldots, C_p$ denote the components of $C$. Instead of dealing with the curve $C$ itself, we consider a (one-parameter) smoothing $C/B$ of $C$, that is, a flat, projective map $C \to B$ to $B := \text{Spec}(K[[t]])$ whose generic fiber $X$ is smooth and whose closed fiber
is isomorphic to $C$, whence identified with $C$ by any chosen isomorphism. We assume $C$ is regular. Furthermore, instead of considering the symmetric product, we consider the ordinary Cartesian product $\hat{C}^2 := C \times B C$. The Jacobian is replaced by the so-called compactified Jacobian $\overline{\mathcal{J}}$, parameterizing torsion-free, rank-1 sheaves on the fibers of $C/B$ that are $C_1$-quasistable with respect to a fixed polarization $\underline{e} = (e_1, \ldots, e_p)$ on $C$; see Section 2.2. Its generic fiber over $B$ is isomorphic to the Jacobian of $X$. Finally, let $\mathcal{P}$ be a line bundle on $C$ of relative degree $e_1 + \cdots + e_p + 2$ over $B$.

Consider the rational map $\alpha : C^2 \dashrightarrow \overline{\mathcal{J}}$ defined over the generic point of $B$ by taking a pair of points $(Q_1, Q_2)$ on $X$ to $\mathcal{P}|_X \otimes \mathcal{O}_X(-Q_1 - Q_2)$. Using “twisters,” it is not difficult to show that $\alpha$ extends over the smooth locus of $C^2/B$, whose complement is the codimension-2 locus whose components are of the form $N \times C$ or $C \times N$, with $N$ a node of $C$; see Section 2.3. The map does not necessarily extend to the whole $C^2$; already the simple case dealt with in [9] shows this. Our goal is to produce a minimal resolution for $\alpha$.

By a minimal resolution for $\alpha$ we mean a map $\phi : \hat{C}^2 \to C^2$ which is an isomorphism away from the pairs $(R, S)$ of nodes of $C$, where either $R$ and $S$ are both reducible, or $R = S$. Furthermore, for such a pair $(R, S)$ we want that either $\phi$ be an isomorphism at it or $\phi^{-1}(R, S) \cong \mathbb{P}_K^2$, the former only if $R \neq S$ and $\alpha$ extends over $(R, S)$. We want as well that $\phi$ be symmetric, that is, that the switch involution $\iota$ of $C^2$ lift to an involution $\hat{\iota}$ of $\hat{C}^2$, as we want to take the quotient of $\hat{C}^2$ by the action of $\hat{\iota}$ to end up with a partial resolution of the relative symmetric product $S^2(C/B)$. Finally, we want a map $\alpha : \hat{C}^2 \to \overline{\mathcal{J}}$ such that $\hat{\alpha} = \alpha \hat{\iota}$ and $\hat{\alpha} = \alpha \phi$. Our Theorem 6.10 describes such a map $\phi$ when certain numerical conditions are verified on the dual graph $\Gamma_C$ of $C$.

When these conditions are verified, our $\phi$ is produced by a sequence of blowups. The first is the blowup along the diagonal $\Delta \subset C^2$, resulting in a flag scheme; see Section 2.4. Clearly, $\iota$ lifts to an involution of this blowup; the quotient by the action of the lift is the relative Hilbert scheme $\text{Hilb}^2_{C/B}$. We are thus producing a partial resolution of $\text{Hilb}^2_{C/B}$ as well. The remaining blowups are along the (strict transforms of) products $Y \times Z$ of proper subcurves of $C$, chosen according to the numerical conditions verified on $\Gamma_C$. To make sure that $\iota$ lifts to an involution on $\hat{C}^2$, if a blowup in the resolution sequence is along $Y \times Z$ for $Y \neq Z$, we require the next blowup to be along $Z \times Y$.

In Sections 3 and 4 we describe the effect of successive blowups of (modifications of) $C^2$ along (the strict transforms of) products of subcurves of $C$. Being $C$ regular, the product $C^2$ is regular away from pairs $(R, S)$ of nodes of $C$. Blowing up $C^2$ along $Y \times Z$ for proper subcurves $Y$ and $Z$ of $C$ desingularizes $C^2$ at the pairs $(R, S) \in (Y' \cap Y') \times (Z' \cap Z')$, where $Y' := \overline{C - Y}$ and $Z' := \overline{C - Z}$, by “adding” $\mathbb{P}_K^2$ over each such pair. The added $\mathbb{P}_K^2$ belongs to the strict transforms of $Y \times Z$ and $Y' \times Z'$ but not to those of $Y \times Z'$ and $Y' \times Z$. The blowups along $Y \times Z$ and $Y' \times Z'$ are the same, but different from those along $Y \times Z'$ and $Y' \times Z$.

We give a brief explanation of the nature of the numerical conditions we check on $\Gamma_C$; more details can be found in Section 6. Let $\Gamma_C^0$ be the essential
dual graph of \( C \), obtained from \( \Gamma_C \) by removing the edges with equal ends. (Thus we are ignoring the irreducible nodes of \( C \).) Let \( V \) be the set of vertices and \( E \) the set of edges of \( \Gamma_C^0 \). Then \( V = \{ C_1, \ldots, C_p \} \). We may view \( \xi \) as a map \( \xi: V \to \mathbb{Q} \) taking \( C_i \) to \( e_i \) for each \( i = 1, \ldots, p \). We may also view the multidegree of \( \mathcal{P}|_C \) as a map \( q: V \to \mathbb{Z} \) taking \( C_i \) to \( \deg(\mathcal{P}|_C) \) for each \( i = 1, \ldots, p \). Set \( v := C_1 \). We call \((\hat{\Gamma}_C^0, \xi, q, v)\) degree-2 Abel data.

A resolution of the degree-2 Abel data \((\hat{\Gamma}_C^0, \xi, q, v)\) is a map \( r: E^2 \to V^2 \) sending the diagonal of \( E^2 \) to that of \( V^2 \) and such that \( r(R, S)_1 \) is an end of \( R \) and \( r(R, S)_2 \) is an end of \( S \). We define when a resolution \( r \) is quasistable: see Subsections \( 6.4 \) and \( 6.7 \). The definition is based only on the (combinatorial) degree-2 Abel data.

The singular locus \( \Sigma \) of the Abel data is the subset of pairs \((R, S) \in E^2\) such that for any two quasistable resolutions \( r_1 \) and \( r_2 \), either \( r_1(R, S) = r_2(R, S) \) or \( r_1(R, S)_j \neq r_2(R, S)_j \) for \( j = 1, 2 \); see Definition \( 6.9 \). Notice that \( \Sigma \) contains the diagonal \( \Delta_E \) of \( E^2 \). We say that \( \Sigma \) is solvable if quasistable resolutions exist.

A blowup sequence for the Abel data is a pair \((I_1, I_2)\) of sequences \( I_1 = (I_{1,1}, \ldots, I_{1,n}) \) and \( I_2 = (I_{2,1}, \ldots, I_{2,n}) \) of equal lengths of proper nonempty subsets of \( V \). It is called symmetric if the subsets are symmetric to each other, that is, whenever \( I_{1,j} \neq I_{2,j} \) we have \((I_{1,j-1}, I_{2,j-1}) = (I_{2,j}, I_{1,j})\) or \((I_{1,j+1}, I_{2,j+1}) = (I_{2,j}, I_{1,j})\). The center \( \Xi \) of the blowup sequence is the set of pairs \((R, S) \in E^2\) such that either \( R = S \) or there is \( j \) such that one and only one end of \( R \) lies in \( I_{1,j} \) and one and only one end of \( S \) lies in \( I_{2,j} \). We call the minimum such \( j \) the order of \((R, S)\) in the blowup sequence.

We say that a blowup sequence \((I_1, I_2)\) resolves the Abel data if \( \Sigma \) is solvable, \( \Xi \supseteq \Sigma \), and for each \((R, S) \in \Sigma - \Delta_E\), any quasistable resolution \( r \) satisfies \( r(R, S) \in I_{1,j} \times (V - I_{2,j}) \) or \( r(R, S) \in (V - I_{1,j}) \times I_{2,j} \), where \( j \) is the order of \((R, S)\). We say that \((I_1, I_2)\) resolves the Abel data minimally if \( \Xi = \Sigma \).

Given a blowup sequence \((I_1, I_2)\), consider the map \( \phi: \hat{C}^2 \to C^2 \) obtained by blowing up \( C^2 \) along the diagonal and then successively along the strict transforms of \( Y_1 \times Z_{i_1}, \ldots, Y_u \times Z_{i_u} \), where \( Y_j := \cup_{i \in E_{I_{1,j}}} C_i \) and \( Z_j := \cup_{i \in E_{I_{2,j}}} C_i \) for \( j = 1, \ldots, u \). If \((I_1, I_2)\) is symmetric, so is \( \phi \). Our Theorem \( 6.10 \) says that, if \((I_1, I_2)\) resolves the Abel data minimally, then \( \phi \) is a minimal resolution for \( \alpha \).

The biggest unanswered question is whether the singular locus of a degree-2 Abel data is always solvable. Numerous examples checked by the implemented algorithm suggest that, not only is the answer positive, but also that there is a symmetric blowup sequence resolving the Abel data minimally, and thus producing a symmetric minimal resolution for \( \alpha \). Also, the third author \( [27] \) showed that, when \( \mathcal{P}|_C \) has multidegree \( (2, 0, \ldots, 0) \) and \( \xi = (0, 0, \ldots, 0) \), the singular locus is solvable. Furthermore, in this case there is a blowup sequence \((I_1, I_2)\) resolving the Abel data (possibly nonminimally), where \( I_{1,j} = I_{2,j} \) for every \( j \), and the subcurves \( Y_j := \cup_{i \in E_{I_{1,j}}} C_i \) are 2-tails or 3-tails.

A related question would be: Can we understand the minimal resolution found in a functorial way, that is, does it represent a natural functor? It might be easier to answer the first question after the second is answered.

Restricting \( \phi: \hat{C}^2 \to C^2 \) and \( \hat{\alpha} \) over the special point of \( B \) we obtain a
“resolution” \( \widehat{C}^2 \rightarrow C^2 \) and an “Abel map” \( \widehat{C}^2 \rightarrow \mathcal{J} \). The “resolution” does not depend on the smoothing \( C/B \) taken but the “Abel map” does. More precisely, the “Abel map” depends (only) on the resulting enriched structure, as defined by Mainò \[24\].

In the process leading to the proof of Theorem 6.10 we are led in Sections 3 and 4 to understand blowups of the triple product \( C^3 := C \times_B C \times_B C \). It turns out in Section 6 to be immaterial what sequence of blowups of \( C^3 \) we perform. However, the information we gather will certainly be useful for anyone trying to construct degree-3 Abel maps. We have this in mind in Section 5 as well, where we study the relationship between torsion-free, rank-1 sheaves on \( C \) and on the curve obtained from \( C \) by replacing each node by a chain of smooth rational curves of variable length (including length 0, which in fact corresponds to not replacing the node). For our immediate purposes, the length is at most 2, but anyone dealing with higher-degree Abel maps will likely profit from our study of the general case.

We worked with the compactification by quasistable sheaves produced in \[12\], which is a fine moduli space. Another compactification, that considered in \[3\] or \[31\], could have been used. Since there is a map from the former to the latter, we produce degree-2 Abel maps to the latter as well. However, by the same reason, a priori it could be easier to construct these maps directly.

In short, here is a summary of the article. In Section 2 we review the theory of compactified Jacobians, how one can use “twisters” to extend Abel maps up to codimension 2, and the construction of the flag scheme. In Section 3 we describe the blowups of \( C^2 \) and \( C^3 \) in the local analytic setting. In Section 4 we use this local analysis to describe globally blowups of \( C^2 \) and \( C^3 \). In Section 5 we study the relationship between torsion-free, rank-1 sheaves on \( C \) and on the curve obtained from \( C \) by replacing each node by a chain of smooth rational curves of variable length. In Section 6 we prove our main theorem, Theorem 6.10 already discussed above. Finally, in Section 7 we present examples of curves for which symmetric minimal resolutions for degree-2 Abel maps exist, presenting the resolution in each case.

## 2 Compactified Jacobians

### 2.1 Curves and their smoothings

A curve is a reduced, projective scheme of pure dimension 1 over an algebraically closed field \( K \). A curve may have several irreducible components, which will be simply called components. We will always assume our curves to be nodal, meaning that the singularities are nodes, that is, analytically like the origin on the union of the coordinate axes of the plane \( \mathbb{A}^2_K \). A node is called irreducible if its removal does not disconnect the curve in any Zariski neighborhood of it; otherwise, it is called reducible.

Let \( C \) be a curve defined over an algebraically closed field \( K \). Its dual graph \( \Gamma_C \) is the graph whose set of vertices \( V_C \) consists of the components of \( C \) and
whose set of edges $E_C$ consists of its singularities, the ends of an edge being the components on which the corresponding singularity lies. The essential dual graph $\Gamma_C^0$ is the subgraph of $\Gamma_C$ where the edges with equal ends, corresponding to irreducible nodes of $C$, are removed.

A subcurve of $C$ is a (nonempty, reduced) union of components of $C$. It is a curve by itself. If $Y$ is a proper subcurve of $C$, we let $Y' := \overline{C - Y}$, and call it the complementary subcurve of $Y$. We set $kY := \#(Y \cap Y')$.

A chain of rational curves is a curve whose components are smooth and rational and can be ordered, $E_1, \ldots, E_n$, in such a way that $\#E_i \cap E_{i+1} = 1$ for $i = 1, \ldots, n - 1$ and $E_i \cap E_j = \emptyset$ if $|i - j| > 1$. If $n$ is the number of components, we say that the chain has length $n$. Two chains of the same length are isomorphic. The components $E_1$ and $E_n$ are called the extreme curves of the chain. A connected subcurve of a chain is also a chain, and is called a subchain.

Let $\mathcal{N}$ be a collection of nodes of $C$, and $\eta: \mathcal{N} \to \mathbb{N}$ a function. Denote by $\tilde{C}_\mathcal{N}$ the partial normalization of $C$ at $\mathcal{N}$. For each $P \in \mathcal{N}$, let $E_P$ be a chain of rational curves of length $\eta(P)$. Let $C_\eta$ denote the curve obtained as the union of $\tilde{C}_\mathcal{N}$ and the $E_P$ for $P \in \mathcal{N}$ in the following way: Each chain $E_P$ intersects no other chain, but intersects $C_\eta$ transversally at two points, the branches over $P$ on $C_\eta$ on one hand, and nonsingular points on each of the two extreme curves of $E_P$ on the other hand. There is a natural map $\mu_\eta: C_0 \to C$ collapsing each chain $E_P$ to a point, whose restriction to $\tilde{C}_\mathcal{N}$ is the partial normalization map.

The curve $C_\eta$ and the map $\mu_\eta$ are well-defined up to $C$-isomorphism.

There are two special cases of the above construction that will be useful for us. First, if $\mathcal{N} = \{R\}$ and $\eta$ takes $R$ to 1, let $C_R := C_\eta$ and $\mu_R := \mu_\eta$. Second, if $\mathcal{N} = \mathcal{N}(C)$, where $\mathcal{N}(C)$ is the collection of reducible nodes of $C$, and $\eta: \mathcal{N}(C) \to \mathbb{N}$ is the constant function with value $m$, let $C(m) := C_\eta$ and $\mu(m) := \mu_\eta$. Set $C(0) := C$ and $\mu(0) := \text{id}_C$.

A family of (connected) curves is a proper and flat morphism $\pi: C \to B$ whose geometric fibers are connected curves. (All schemes are assumed locally Noetherian.) If $b$ is a geometric point of $B$, we put $C_b := \pi^{-1}(b)$. A smoothing of $C$ is a generically smooth family of curves $\pi: C \to B$ over an irreducible scheme $B$, together with a point $0 \in B$ whose residue field is $K$, and an isomorphism $C_0 \cong C$. We will always assume $B \cong \text{Spec}(K[[t]])$ and $C$ is regular at the reducible nodes of $C_0$. Also, the isomorphism $C_0 \cong C$ will be left implicit.

The degree, or total degree, of a coherent sheaf $\mathcal{F}$ of constant generic rank 1 on $C$ is

$$\deg(\mathcal{F}) := \chi(\mathcal{F}) - \chi(\mathcal{O}_C).$$

We will identify line bundles with invertible sheaves. If $\mathcal{F}$ is an invertible sheaf, its multidegree is the function $d_\mathcal{F}: V_C \to \mathbb{Z}$ that to each component $Y$ of $C$ associates the integer $\deg(\mathcal{F}|_Y)$. Given an ordering of the components of $C$, we may also view $d_\mathcal{F}$ as a tuple of integers.

Given a smoothing $C/B$ of $C$, a twister of $C/B$ is a special line bundle of degree 0 on $C$, of the form $\mathcal{O}_C(Z)|_C$, where $Z$ is a Cartier divisor of $C$ supported in $C$, so a formal sum of components of $C$. (Notice that each component of $C$ is a Cartier divisor of $C$ because $C$ is regular at the reducible nodes of $C$.)
A twister has degree 0 by continuity of the degree, since \( \mathcal{O}_C(Z) \) is trivial away from \( C \).

The smoothing \( C/B \) being fixed, two divisors \( Z_1 \) and \( Z_2 \) of \( C \) supported in \( C \) give isomorphic twisters, \( \mathcal{O}_C(Z_1)|_C \cong \mathcal{O}_C(Z_2)|_C \), if and only if they give twisters of the same multidegree, if and only if \( Z_1 - Z_2 \) is a multiple of \( C \); see [5], Lemma 3.4, p. 7. The multidegree of a twister \( \mathcal{O}_C(Z)|_C \) depends only on \( Z \), but the twister itself depends on the smoothing \( C/B \). As soon as the latter is fixed, we will use the notation

\[
\mathcal{O}_C(Z) := \mathcal{O}_C(Z)|_C.
\]

In her thesis [24], Mainò describes which invertible sheaves on \( C \) are twisters. A description is also given in Section 6 of [14], p. 288.

Throughout the article, fix an algebraically closed field \( K \) and a connected curve \( C \) over \( K \). Denote by \( C_1, \ldots, C_p \) its components, and fix a smoothing \( C/B \) of \( C \) with regular total space \( C \).

### 2.2 Jacobians and their compactifications

Fix an integer \( d \). Since \( C \) is a proper scheme over \( K \), by [2], Thm. 8.2.3, p. 211, there is a scheme, locally of finite type over \( K \), parameterizing degree-\( d \) line bundles on \( C \); denote it by \( J^d_C \). It decomposes as

\[
J^d_C = \coprod_{d=(d_1, \ldots, d_p) \atop d_1 + \ldots + d_p = d} J^d_C,
\]

where \( J^d_C \) is the connected component of \( J^d_C \) parameterizing line bundles \( \mathcal{L} \) such that \( \deg(\mathcal{L}|_C) = d_i \) for \( i = 1, \ldots, p \). The \( J^d_C \) are quasiprojective varieties.

The scheme \( J^d_C \) is in a natural way an open dense subscheme of \( \overline{J}^d_C \), the scheme over \( K \) parameterizing torsion-free, rank-1, simple sheaves of degree \( d \) on \( C \); see [12] for the construction of \( \overline{J}^d_C \) and its properties. (Recall that a coherent sheaf \( \mathcal{I} \) on \( C \) is torsion-free if it has no embedded components, rank-1 if it has generic rank 1 at each component of \( C \), and simple if \( \text{Hom}(\mathcal{I}, \mathcal{I}) = K \).) The scheme \( \overline{J}^d_C \) is universally closed over \( K \) but, in general, not separated and only locally of finite type. Moreover, in contrast to \( J^d_C \), the scheme \( \overline{J}^d_C \) is connected, hence not easily decomposable. Thus, to deal with a manageable piece of it, we resort to polarizations.

For our purposes, a polarization of degree \( d \) is any \( p \)-tuple of rational numbers \( \underline{e} = (e_1, \ldots, e_p) \) summing up to \( d \). Fixing a polarization \( \underline{e} \) of degree \( d \), we say that a degree-\( d \), torsion-free, rank-1 sheaf \( \mathcal{I} \) on \( C \) is semistable if for every subcurve \( Y \subseteq C \) we have

\[
\left| \deg(\mathcal{I}|_Y) - e_Y \right| \leq \frac{k_Y}{2},
\]

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where $\mathcal{I}_Y$ is the restriction of $\mathcal{I}$ to $Y$ modulo torsion, and
\[
e_Y := \sum_{C_i \subseteq Y} e_i.
\]
We say that $\mathcal{I}$ is \textit{stable} if all the inequalities in (2) are strict for proper subcurves $Y \subsetneq C$. Furthermore, we say that $\mathcal{I}$ is $C_i$-\textit{quasistable}, for a component $C_i$ of $C$, if all the inequalities in (2) hold and, moreover,
\[
\deg(\mathcal{I}_Y) > e_Y - \frac{k_Y}{2}
\]
whenever $Y$ is a proper subcurve of $C$ containing $C_i$.

The $C_i$-quasistable sheaves are simple, what can be easily proved using for instance [12], Prop. 1, p. 3049. Their importance is that they form an open subscheme $J^{d,i}_C$ of $J^d_C$ that is projective over $K$. Furthermore, there is a projective scheme $J^{d,i}_{C/B}$ over $B$ whose fiber over the special point of $B$ is (isomorphic to) $J^{d,i}_C$ and whose fiber over a geometric point $b$ of $B$ over its generic point is (isomorphic to) $J^{d,i}_C$.

Conditions (2) and (3) are purely numerical. In fact, let $\Gamma_C$ be the dual graph of $C$. A \textit{generalized multidegree} $d$ is the assignment of an integer to each vertex and each edge of $\Gamma_C$, with the condition that to an edge only 0 or 1 is assigned. The (total) \textit{degree} $d$ of $d$ is the sum of all these assigned integers. A subcurve $Y$ of $C$ corresponds to a subgraph $\Gamma_Y$ of $\Gamma_C$, whose vertices are the components of $Y$ and whose edges are the singularities of $Y$. The \textit{degree} of $d$ \textit{on} $Y$, denoted $d_Y$, is the sum of all integers assigned to the vertices and edges of $\Gamma_Y$.

Fixing a polarization $e$ of degree $d$, we say that a generalized multidegree $d$ of total degree $d$ is \textit{semistable} if for every proper subcurve $Y \subsetneq C$ we have
\[
|d_Y - e_Y| \leq \frac{k_Y}{2},
\]
and \textit{stable} if strict inequalities hold. Also, we say that $d$ is $C_i$-\textit{quasistable}, for a component $C_i$ of $C$, if $d$ is semistable and
\[
d_Y > e_Y - \frac{k_Y}{2}
\]
whenever $Y \supseteq C_i$.

To a degree-$d$ torsion-free, rank-1 sheaf $\mathcal{I}$ on $C$ there corresponds a generalized multidegree $d$ of total degree $d$ that assigns 0 to each node of $C$ around which $\mathcal{I}$ is invertible, and 1 otherwise, and assigns $\deg(\mathcal{I}_{C_i})$ to each component $C_i$. Then $d_Y = \deg(\mathcal{I}_Y)$ for each subcurve $Y$ of $C$, and thus $\mathcal{I}$ is semistable, or stable, or $C_i$-quasistable if and only if $d$ is.

\textbf{Throughout the article}, fix an integer $f$ and fix a degree-$f$ polarization $e := (e_1, \ldots, e_p)$. Set
\[
J := J^f_C \cap J^{1,1}_C, \quad J := J^1_C, \quad \overline{J} := \overline{J}^{1,1}_C.
\]
2.3 Abel maps

Let $d$ be a positive integer, and $\mathcal{P}$ a line bundle on $C$ of degree $d + f$. If $C$ is smooth, its degree-$d$ Abel map is the map $S^d(C) \to J$ induced from

$$\alpha^d: C^{\times d} \to J$$

$$(Q_1, \ldots, Q_d) \mapsto \mathcal{P} \otimes \mathcal{O}_C(-Q_1 - \cdots - Q_d),$$

where $S^d(C)$ is the quotient of $C^{\times d}$ by the group of permutations of its factors.

(Often the map found in the literature is the one obtained from $\alpha^d$ by composing with the isomorphism $J^f_C \to J^{-f}_C$ taking a line bundle to its dual. Our choice does not differ from this one in any relevant way for the purposes of this article.)

The above definition of $\alpha^d$ does not make sense if $C$ is singular. Indeed, it would be natural to think of defining the degree-$d$ Abel map of $C$ by letting

$$(Q_1, \ldots, Q_d) \mapsto \mathcal{P} \otimes I_{Q_1|C} \otimes \cdots \otimes I_{Q_d|C},$$

but two problems arise: First, the sheaf on the right might not be torsion-free; in fact, it will fail to be torsion-free if and only if two among the $Q_i$ are the same node of $C$. This is not a serious problem, as we can replace $C^{\times d}$, or rather $S^d(C)$, by the Hilbert scheme $\text{Hilb}^d_C$, parameterizing subschemes $F$ of length $d$ of $C$, and define the degree-$d$ Abel map by letting

$$F \mapsto \mathcal{P} \otimes I_{F|C}.$$

This works if $C$ is irreducible; see [1], (8.2), p. 101. But if $C$ is reducible, a second problem might nonetheless arise, namely that $\mathcal{P} \otimes I_{Q_1|C} \otimes \cdots \otimes I_{Q_d|C}$ or $\mathcal{P} \otimes I_{F|C}$ might not be $C_1$-quasistable, whence not parameterized by $\overline{J}$.

To solve the second problem, we resort to twisters arising from the smoothing $C/B$. Associate to a twister $\mathcal{O}_C(Z)$ a multidegree $\mathfrak{d}^Z_d$, that of the line bundle. Recall that the multidegree does not depend on the choice we made of the smoothing $C/B$. Also, the smoothing being fixed, there is a bijective correspondence between twisters and their multidegrees.

Now, let $\hat{C}$ denote the smooth locus of $C$. The product $\hat{C}^{\times d}$ decomposes as

$$\hat{C}^{\times d} = \coprod_{\overline{\mathfrak{l}} = (i_1, \ldots, i_d) \in \mathbb{N}^d} \hat{C}_{i_1} \times \hat{C}_{i_2} \times \cdots \times \hat{C}_{i_d},$$

where $\hat{C}_j := C_j \cap \hat{C}$ for each $j = 1, \ldots, p$. To each $d$-tuple $\overline{i}$ as above, we associate a multidegree $\mathfrak{f} := (f_1, \ldots, f_p)$ of degree $f$, by letting

$$f_j := \deg(\mathcal{P}|_{C_j}) - \# \{\ell \mid i_\ell = j\}$$

for $j = 1, \ldots, p$, and to $\mathfrak{f}$ a unique twister whose multidegree $\mathfrak{t}$ is such that $\mathfrak{f} - \mathfrak{t}$ is $C_1$-quasistable. Let $\overline{Z_\mathfrak{t}}$ be a formal sum of components of $C$ such that $\mathcal{O}_C(\overline{Z_\mathfrak{t}})$ has multidegree $\mathfrak{t}$. Then there is a natural map

$$\alpha^d_{\mathfrak{l}}: \hat{C}_{i_1} \times \cdots \times \hat{C}_{i_d} \to J$$

$$(Q_1, \ldots, Q_d) \mapsto \mathcal{P} \otimes \mathcal{O}_C(-Q_1 - \cdots - Q_d) \otimes \mathcal{O}_C(-\overline{Z_\mathfrak{t}}).$$
Putting together the $\alpha^d$, we obtain a map $\hat{\alpha}^d: \hat{\mathcal{C}}^d \to J$.

(The existence of the twister with multidegree $t$ is the numerical version of [12], Thm. 32, (4), p. 3068, applied to the restriction over the generic fiber of any extension to $C$ of any line bundle on $C$ having multidegree $f_i$. As for uniqueness, suppose there exist twisters of multidegrees $\mathfrak{L}_1$ and $\mathfrak{L}_2$ such that $g := f - \mathfrak{L}_i$ and $h := f - \mathfrak{L}_2$ are $C_1$-quasistable. By contradiction, suppose $\mathfrak{L}_1 \neq \mathfrak{L}_2$. Then $\mathfrak{L} := g - h$ is the multidegree of a nontrivial twister. Thus, as in the proof of [3], Lemma 4.1, p. 623, there exist an integer $q \geq 1$ and a decomposition $C = Y_0 \cup \cdots \cup Y_q$ in subcurves $Y_j$ containing no components in common such that $\mathfrak{L} = \sum_{j=1}^{q} \ell_j \mathfrak{L}_j$, where $\mathfrak{L}_j$ is the multidegree of $\mathcal{O}_C(Y_j)$ for $j = 1, \ldots, q$, and the $\ell_j$ are distinct positive integers. Set $Y := Y_0$. Then $by_1 \geq k_Y$ and $by_2 \leq -k_Y$. From the semistability of $\mathfrak{L}$ and $\mathfrak{L}_i$, it follows that $g_{Y'} - e_{Y'} = -\frac{k_{Y'}}{2}$ and $h_{Y'} - e_{Y'} = -\frac{k_{Y'}}{2}$. Thus, if $C_1 \subseteq Y$ (respectively, if $C_1 \not\subseteq Y$), then $\mathfrak{L}$ (respectively, $\mathfrak{L}_i$) is not $C_1$-quasistable, reaching a contradiction.)

The map $\hat{\alpha}^d$ can be extended over the generic point of $B$. To do this, first we extend the line bundle $\mathcal{P}$ to one on $C$. This is possible because $\mathcal{P}$ is associated to a Cartier divisor with support on $\hat{\mathcal{C}}$, and $B \cong \text{Spec}(K[[\ell]])$. Abusing notation, $\mathcal{P}$ will also denote the extension. For each positive integer $j$, set $C^j := \hat{\mathcal{C}} \times_B \cdots \times_B C$, the fibered product of $j$ copies of $\mathcal{C}$ over $B$. Likewise, $\hat{C}^j$ denotes the product of $j$ copies of $\hat{\mathcal{C}}$ over $B$, where $\hat{\mathcal{C}}$ denotes the smooth locus of $\mathcal{C}/B$. Denote by $\xi: \hat{C}^{d+1} \to C$ and $\rho_i: \hat{C}^{d+1} \to C^2$ the projection onto the last factor and that onto the product over $B$ of the $i$-th and last factors, for $i = 1, \ldots, d$. Let $\Delta \subset C^2$ be the diagonal subscheme, and put

$$\Delta_i := \rho_i^{-1}(\Delta)$$

for each $i = 1, \ldots, d$.

Since $\mathcal{C}$ is regular, the product $\hat{C}_{i_1} \times \cdots \times \hat{C}_{i_d} \times C_{d+1}$ is a Cartier divisor of $\hat{C}^d \times_B \mathcal{C}$ for every $i_1, \ldots, i_d+1 \in \{1, \ldots, p\}$. So, for each $i = (i_1, \ldots, i_d) \in \{1, \ldots, p\}^d$, if $Z_i = \sum_j a_j C_j$, we have a well-defined Cartier divisor:

$$\hat{C}_{i_1} \times \cdots \times \hat{C}_{i_d} \times Z_i := \sum_j a_j \hat{C}_{i_1} \times \cdots \times \hat{C}_{i_d} \times C_j.$$  

Finally, we obtain a map

$$\hat{\alpha}^d_{C/B}: \hat{\mathcal{C}}^d \to \mathcal{J}$$  

(4)

defined by the family of invertible sheaves on $\hat{\mathcal{C}}^d \times_B \mathcal{C}/\hat{\mathcal{C}}^d$ obtained as the tensor product of the “natural” sheaf

$$\left(\xi^* \mathcal{P} \otimes \mathcal{I}_{\Delta_i \cap C^{d+1}} \otimes \cdots \otimes \mathcal{I}_{\Delta_i \cap C^{d+1}}\right)_{\hat{C}^d \times_B \mathcal{C}}$$

with its “correction”:

$$\bigotimes_{i = (i_1, \ldots, i_d) \in \mathbb{N}^d, 1 \leq i_1, \ldots, i_d \leq p} \mathcal{O}_{\hat{C}^d \times_B \mathcal{C}}(-\hat{C}_{i_1} \times \cdots \times \hat{C}_{i_d} \times Z_i).$$

10
As the correction is by a Cartier divisor supported over \(0 \in B\), it follows that \(\tilde{\alpha}^d_{C/B}\) restricts to the “classical” degree- \(d\) Abel map on a smooth geometric fiber of \(C/B\).

We may ask whether \(\tilde{\alpha}^d_{C/B}\), viewed as a rational map on \(C^d\), defined in codimension 2, extends to the whole \(C^d\). Surprisingly, the answer is yes for \(d = 1\), as shown in [6] and [7], at least if \(P = \mathcal{O}_C\) or \(P = \mathcal{O}_C(P)\) for \(P \in \tilde{C}_1\).

Moreover, as shown in [6] and as we will see, a resolution is not just a matter of blowing up along all the diagonals of \(C^d\), small and large. More blowups are often necessary. We will see ahead the case \(d = 2\).

Throughout the article, fix an invertible sheaf \(P\) on \(C\) whose restriction to \(C\) has degree \(f + 2\).

### 2.4 The flag scheme

Recall that \(C^j\) is the fibered product of \(j\) copies of \(C\) over \(B\), for \(j \in \mathbb{N}\). Let \(\Delta \subset C^2\) be the diagonal subscheme, and \(\Delta_1\) and \(\Delta_2\) the two “large diagonals” of \(C^3\), inverse images of \(\Delta\) under the projections \(\rho_1,\rho_2: C^3 \to C^2\), where \(\rho_i\) is the projection onto the product over \(B\) of the \(i\)-th and last factors of \(C^3\), for \(i = 1,2\).

Our first goal is to modify the base \(C^2\) to be able to replace \(\mathcal{I}_{\Delta_1}(C) \otimes \mathcal{I}_{\Delta_2}(C)\) by a relatively torsion-free, rank-1 sheaf over the base wherever the tensor product is not so, that is, over the pairs \((R,R)\), where \(R\) is a node of \(C\). This is done by replacing \(C^2\) by the flag scheme:

\[
P_{C^2}(\mathcal{I}_{\Delta}(C^2)) := \text{Proj}_{C^2}(S(\mathcal{I}_{\Delta}(C^2))),
\]

where \(S(\mathcal{I}_{\Delta}(C^2))\) is the sheaf of symmetric algebras of \(\mathcal{I}_{\Delta}(C^2)\). In our case, another description is available.

**Proposition 2.1.** Let \(\phi: \tilde{C}^2 \to C^2\) be the blowup of \(C^2\) along \(\Delta\). Let \(p_i := p_i\phi\), where \(p_i: C^2 \to C\) is the projection onto the \(i\)-th factor for \(i = 1,2\). Let \(R \in C\). For \(i = 1,2\), let \(X_i := \rho_i^{-1}(R)\) and denote by \(\mu_i: X_i \to C\) the restriction of \(\rho_{3-i}\) to \(X_i\). Then the following statements hold:

1. \(\tilde{C}^2\) is \(C^2\)-isomorphic to \(P_{C^2}(\mathcal{I}_{\Delta}(C^2))\).

2. \(\rho_i\) is flat for \(i = 1,2\).

3. If \(R\) is not a node of \(C\) then \(\mu_i\) is an isomorphism for \(i = 1,2\).

4. If \(R\) is a node of \(C\) then \(X_i\) is \(C\)-isomorphic to \(C_R\) and \(\tilde{C}^2\) is regular along the rational component of \(X_i\) contracted by \(\mu_i\) for \(i = 1,2\).

**Proof.** As the Rees algebra of an ideal is a quotient of the symmetric algebra, \(\tilde{C}^2\) is naturally a subscheme of \(P_{C^2}(\mathcal{I}_{\Delta}(C^2))\). To show the blowup is the whole flag scheme is a local matter that need only be checked at the points where \(\Delta\) fails to be Cartier, that is, at the pairs \((R,R)\), where \(R\) is a node of \(C\). Recall that
\( \mathcal{C} \) is regular. Thus the completion of the local ring of \( \mathcal{C} \) at \( R \) is (isomorphic to) the quotient of the formal power series ring \( K[[t, x_0, x_1]] \) by the ideal \( (t - x_0 x_1) \).

That of the local ring of \( \mathcal{C}^2 \) at \( (R, R) \) is the quotient of the formal power series ring \( K[[t, x_0, x_1, y_0, y_1]] \) by the ideal \( (t - x_0 x_1, t - y_0 y_1) \), with \( \Delta \) corresponding to the ideal \( (x_0 - y_0, x_1 - y_1) \). We need to compare the symmetric and Rees algebras of this ideal. We are reduced to considering the cone \( V \) in \( \mathbb{A}^4_K \), with coordinates \( x_0, x_1, y_0, y_1 \), given by \( x_0 x_1 = y_0 y_1 \), and showing that its blowup along the subcone given by the ideal \( I := (x_0 - y_0, x_1 - y_1) \subset K[V] \) is equal to \( \mathbb{P}_V(I) \). Both are indeed equal to the subscheme \( Y \subset \mathbb{A}^4_K \times \mathbb{P}_K^1 \) given by

\[
\begin{cases}
\alpha_1 x_0 &= \alpha_0 y_1 \\
\alpha_1 y_0 &= \alpha_0 x_1
\end{cases}
\]

where \( \alpha_0, \alpha_1 \) are homogeneous coordinates of \( \mathbb{P}_K^1 \). Statement 1 is proved.

The remaining statements follow from the above description. Indeed, Statement 3 follows from the fact that \( \phi \) fails to be an isomorphism only above pairs \( (R, R) \) where \( R \) is a node of \( \mathcal{C} \). Also, to prove Statement 2 we need only observe that the map \( Y \to \mathbb{A}^2_K \) sending \( (x_0, x_1, y_0, y_1; \alpha_0 : \alpha_1) \) to \( (x_0, x_1) \) if \( i = 1 \) and \( (y_0, y_1) \) if \( i = 2 \) is flat.

(Alternatively, in more generality, flatness follows from [13], Lemma 3.4, p. 600, as the proof given there applies not only to families of integral curves but more generally to families of reduced curves with surficial singularities.)

Finally, setting \( y_0 = y_1 = 0 \) in \( Y \), we obtain the union of three lines in \( \mathbb{A}^4_K \times \mathbb{P}_K^1 \),

\[
X_0: \quad y_0 = y_1 = x_0 = \alpha_0 = 0 \\
E: \quad y_0 = y_1 = x_0 = x_1 = 0 \\
X_1: \quad y_0 = y_1 = x_1 = \alpha_1 = 0,
\]

forming a chain: \( X_0 \) and \( X_1 \) intersect \( E \) transversally at a single point, distinct for each, but do not intersect. Setting \( x_0 = x_1 = 0 \) in \( Y \), an analogous statement holds. Statement 4 follows from this description and the smoothness of \( Y \). \( \square \)

There is a natural subscheme \( F_2 \subset \mathbb{C}^2 \times_B \mathcal{C} \) which is flat of relative length 2 over \( \mathbb{C}^2 \) and contains both the inverse images of \( \Delta_1 \) and \( \Delta_2 \) under the natural map \( \mathbb{C}^2 \times_B \mathcal{C} \to \mathbb{C}^3 \). More precisely, let \( \rho_1 \) and \( \rho_2 \) be the compositions of the structure map \( \phi: \mathbb{C}^2 \to \mathbb{C}^2 \) with the projections onto the first and second factors of \( \mathbb{C}^2 \). The structure map itself can be written in different ways:

\[
\phi = (\rho_1, \rho_2) = (\rho_1 \times 1) \circ (1, \rho_2) = \sigma \circ (\rho_2 \times 1) \circ (1, \rho_1),
\]

where \( \sigma \) is the switch involution of \( \mathbb{C}^2 \). Notice that

\[
\mathbb{I}_{(\phi \times 1)^{-1}\Delta_1|\mathbb{C}^2 \times_B \mathcal{C}} = (\rho_1 \times 1)^*\mathbb{I}_\Delta \quad \text{and} \quad \mathbb{I}_{(\phi \times 1)^{-1}\Delta_2|\mathbb{C}^2 \times_B \mathcal{C}} = (\rho_2 \times 1)^*\mathbb{I}_\Delta.
\]

Pulling back the first (resp. second) sheaf above to \( \mathbb{C}^2 \) under \( (1, \rho_2) \) (resp. under \( (1, \rho_1) \)), and pushing forward under the same map, we get natural surjections:

\[
(\rho_1 \times 1)^*\mathbb{I}_\Delta \to (1, \rho_2)_*\phi^*\mathbb{I}_{\Delta|\mathbb{C}^2} \quad \text{and} \quad (\rho_2 \times 1)^*\mathbb{I}_\Delta \to (1, \rho_1)_*(\sigma \phi)^*\mathbb{I}_{\Delta|\mathbb{C}^2}.
\]
If \( \epsilon : \phi^*\mathcal{I}_{\Delta|C^2} \to \mathcal{O}_{\mathbb{C}^2}(1) \) is the tautological surjection, we obtain by composition two surjections,

\[
\begin{align*}
\mathcal{I}_{(\phi \circ \epsilon)^{-1}(\Delta\times B)} & \to (1, \rho_2)_*\mathcal{O}_{\mathbb{C}^2}(1), \\
\mathcal{I}_{(\phi \circ \epsilon)^{-1}(\Delta\times B)} & \to (1, \rho_1)_*\mathcal{O}_{\mathbb{C}^2}(1).
\end{align*}
\]

(Here we use that \( \sigma^*\mathcal{I}_{\Delta|C^2} = \mathcal{I}_{\Delta|C^2} \).) Their kernels are equal to the sheaf of ideals of \( F_2 \); see [22], Section 2, p. 109, for proof.

3 Blowups: local analysis

Recall the notation: \( C \) is a curve with components \( C_1, \ldots, C_k \), defined over \( K \), an algebraically closed field, \( \bar{C} \) is the smooth locus of \( C \), and \( \bar{C}/B \) is a smoothing of \( C \) with regular total space, the smooth locus of which is \( \bar{C} \).

Recall that \( \Delta \subset \bar{C} \) is the diagonal subscheme, and that \( \Delta_1 := \rho_1^{-1}(\Delta) \) and \( \Delta_2 := \rho_2^{-1}(\Delta) \), where \( \rho_i : \bar{C}^3 \to \bar{C}^2 \) is the projection onto the product over \( B \) of the \( i \)-th and last factors of \( \bar{C}^3 \) for \( i = 1, 2 \).

3.1 The double product

Let \( R \) and \( S \) be reducible nodes of \( C \). Let \( C_i \) and \( C_j \) be the distinct components containing \( R \), and \( C_k \) and \( C_l \) those containing \( S \). Let \( x_0 = 0 \) and \( x_1 = 0 \) be local equations for the Cartier divisors \( C_i \) and \( C_j \) of \( C \) at \( R \), respectively, and \( y_0 = 0 \) and \( y_1 = 0 \) local equations for \( C_k \) and \( C_l \) at \( S \), respectively. If \( R = S \), we assume \( i = k \) and \( j = l \), and \( x_u = y_u \) for \( u = 0, 1 \).

Recall that \( B = \text{Spec}(K[[t]]) \). We may assume that the completion of the local ring of \( \bar{C} \) at \( R \) is the quotient of \( K[[t, x_0, x_1]] \) by \( (x_0 x_1 - t) \), and that of \( \bar{C} \) at \( S \) is the quotient of \( K[[t, y_0, y_1]] \) by \( (y_0 y_1 - t) \). Thus the threefold \( \bar{C}^2 \) is described locally analytically at \( (R, S) \) as the spectrum of the quotient of \( K[[t, x_0, x_1, y_0, y_1]] \) by the ideal of the subscheme given by the equations

\[
\begin{cases}
x_0 x_1 = t, \\
y_0 y_1 = t,
\end{cases}
\]

and so it can be seen as the cone in \( \mathbb{A}^1_\mathbb{K} \), with coordinates \( x_0, x_1, y_0, y_1 \), given by \( x_0 x_1 = y_0 y_1 \).

Thus \( \bar{C}^2 \) has a quadratic isolated singularity at \( (R, S) \). This singularity can be resolved by blowing up \( \bar{C}^2 \) at \( (R, S) \), at the cost of replacing the point by a quadric surface. However, the next proposition describes resolutions of this singularity that are more efficient and, more important, better for our purposes.

If \( R = S \), the threefold \( \bar{C}^2 \) is given analytically at \( (R, S) \) in the same way as above, and the diagonal is given by the equations \( x_0 - y_0 = x_1 - y_1 = 0 \).

**Proposition 3.1.** Let \( R \) and \( S \) be reducible nodes of \( C \). Assume \( R \in C_i \cap C_j \) and \( S \in C_k \cap C_l \), for integers \( i, j, k, l \) with \( i \neq j \) and \( k \neq l \). If \( R = S \), assume \( i = k \) and \( j = l \). Let \( \phi : C^2 \to \bar{C}^2 \) denote the blowup of \( C^2 \) along \( C_i \times C_j \), or along the diagonal if \( R = S \). Put \( E := \phi^{-1}(R, S) \). Then the following statements hold:
1. $E$ is a smooth rational curve and $\overline{C^2}$ is regular in a neighborhood of $E$.

2. The strict transforms of $C_i \times C_l$ and $C_j \times C_k$ contain $E$, while those of $C_i \times C_k$ and $C_j \times C_l$ intersect $E$ transversally at a unique point, distinct for each transform.

3. If $R = S$, the strict transform of the diagonal contains $E$.

4. The composition $\overline{C^2} \to C$ of $\phi$ with the projection of $C^2$ onto any of its factors is flat.

Proof. Keep the notation prior to the statement of the proposition. Assume first that $\phi$ is the blowup along $C_i \times C_l$, whether $R = S$ or not. Here are the equations at $(R, S)$ of the products listed:

$$
\begin{align*}
C_i \times C_k: & \quad x_0 = y_0 = 0 \\
C_i \times C_l: & \quad x_0 = y_1 = 0 \\
C_j \times C_k: & \quad x_1 = y_0 = 0 \\
C_j \times C_l: & \quad x_1 = y_1 = 0
\end{align*}
$$

To prove the statements of the lemma, we may pass to the completion of the local ring of $C^2$ at $(R, S)$. So we may change $C^2$ for the cone $Z$ in $A^4_K$, with coordinates $x_0, x_1, y_0, y_1$, given by $x_0 x_1 = y_0 y_1$.

The blowup is the subscheme $Y \subset A^4_K \times P^1_K$ given by

$$
\begin{cases}
\alpha' x_0 = \alpha y_1 \\
\alpha' y_0 = \alpha x_1
\end{cases}
$$

where $\alpha, \alpha'$ are homogeneous coordinates of $P^1_K$. The inverse image $E$ of the vertex of the cone in the blowup $Y \to Z$ is given by the equations $x_0 = x_1 = y_0 = y_1 = 0$; thus $E$ is a smooth rational curve.

The strict transform of $C_i \times C_l$ is given by $x_0 = 0$ where $\alpha \neq 0$ and by $y_1 = 0$ where $\alpha' \neq 0$; the strict transform of $C_j \times C_k$ is given by $y_0 = 0$ where $\alpha \neq 0$ and by $x_1 = 0$ where $\alpha' \neq 0$. Thus both transforms contain $E$.

On the other hand, the transform of $C_i \times C_k$ is given by $\alpha = 0$, whence intersects $E$ transversally at a certain point. Analogously, the transform of $C_j \times C_l$ is given by $\alpha' = 0$, whence intersects $E$ transversally at another point.

Where $\alpha' \neq 0$, setting $u = \alpha / \alpha'$, $v = x_1$ and $w = y_1$, the blowup is given by the equations $x_0 = u w$ and $y_0 = w$. On the other hand, where $\alpha \neq 0$, setting $u = \alpha' / \alpha$, $v = y_0$ and $w = x_0$, the blowup is given by $x_1 = u v$ and $y_1 = w$. In any case, we have that the blowup is isomorphic to $A^3_K$, with coordinates $u$, $v$ and $w$, thus nonsingular. Furthermore, $\overline{C^2} \to C$ corresponds to the map $A^3_K \to A^2_K$ taking $(u, v, w)$ to $(w, v)$ or $(u, w)$ in case $\alpha' \neq 0$ and to $(w, u v)$ or $(v, u w)$ in case $\alpha \neq 0$. So, $\overline{C^2} \to C$ is flat on a neighborhood of $E$, and thus everywhere.

As for the second case, the blowup along the diagonal, the equations of the blowup are the same as above, and thus the same results hold. Also, since $E$ is given by $x_0 = x_1 = y_0 = y_1 = 0$, it follows that $E$ is contained in the strict transform of the diagonal. 

\[\square\]
The same proposition can be used, mutatis mutandis, to describe the blowup along $C_j \times C_k$ (same results) and those along $C_i \times C_k$ and $C_j \times C_l$. Also, the results of the lemma are the same if we blow up along a product of two subcurves $X \times Y$, as long as $C_j \subseteq X$ but $C_j \not\subseteq X$, and $C_l \subseteq Y$ but $C_k \not\subseteq Y$.

3.2 The triple product

Keep the notation of Subsection 3.1. Let $T$ be another reducible node of $C$, not necessarily distinct from $R$ and $S$. Let $C_m$ and $C_n$ be the distinct components of $C$ containing $T$. Let $z_0 = 0$ and $z_1 = 0$ be local equations for the Cartier divisors $C_m$ and $C_n$ of $C$ at $T$, respectively. If $T = R$ (resp. $T = S$) we set $z_u := x_u$ (resp. $z_u := y_u$) for $u = 0, 1$.

Let $\tilde{C}^2$ be the blowup of $C^2$ along $C_i \times C_l$. Let $E$ be the “exceptional” rational smooth curve on $\tilde{C}^2$ over $(R, S)$. Let $A$ (resp. $A'$) be the intersection of the strict transform of $C_i \times C_k$ (resp. $C_j \times C_l$) with $E$. Recall the proof of Proposition 3.1. The strict transform of $C_i \times C_l$ is given at $A$ by the equation $y_1 = 0$ and at $A'$ by the equation $x_0 = 0$. That of $C_j \times C_k$ is given at $A$ by $x_1 = 0$ and at $A'$ by $y_0 = 0$. Both contain $E$. The divisor given by $t = 0$ contains these strict transforms. Its residue is the strict transform of $C_i \times C_k$ at $A$, and the transform of $C_j \times C_l$ at $A'$. Thus $t = x_1 y_1 \alpha$ at $A$, with $\alpha = 0$ being an equation for $C_i \times C_k$, and $t = x_0 y_0 \alpha'$ at $A'$, with $\alpha' = 0$ being an equation for $C_j \times C_l$. Schematically,

\[
\begin{align*}
  u &= \alpha \\
  v &= x_1 \\
  w &= y_1 \\
  (\text{at } A)
\end{align*}
\quad
\begin{align*}
  u &= \alpha' \\
  v &= y_0 \\
  w &= x_0 \\
  (\text{at } A')
\end{align*}
\]

To summarize, at $A$ (resp. $A'$), we have the equation $t = uvw$, where $u = 0$ is a local equation of the transform of $C_i \times C_k$ (resp. $C_j \times C_l$), and where $v = 0$ and $w = 0$ are local equations of the transforms of $C_j \times C_k$ and $C_i \times C_l$ respectively. The functions $u, v, w$ form a regular system of parameters of $\tilde{C}^2$ at $A$ (resp. $A'$).

In case $R = S$, and $\tilde{C}^2$ is the blowup of $C^2$ along the diagonal, exactly the same summary applies. In either case, the strict transform of the diagonal, whose equations on $C^2$ are $x_0 - y_0 = x_1 - y_1 = 0$, is given by $v - w = 0$. 

Figure 1. Blowup at $(R, S)$
Thus, the following equations hold on $\mathbb{C}^2 \times_B C$ at $(A, T)$ or $(A', T)$:
\[
\begin{cases}
  uvw = t \\
  z_0z_1 = t
\end{cases}
\]

Passing to the completion of the local ring of $\mathbb{C}^2 \times_B C$ at $(A, T)$ or $(A', T)$, we may view the fourfold as the hypersurface of $\mathbb{A}^5_K$, with coordinates $u, v, w, z_0, z_1$ given by $z_0z_1 = uvw$. In particular, we see that $\mathbb{C}^2 \times_B C$ is singular at $(A, T)$ and $(A', T)$; in fact, it is singular all along $E \times \{T\}$.

In case $R = T$, the diagonal $\Delta_1$ passes through $(R, S, T)$. It is given at the point by the equations $z_0 - x_0 = z_1 - x_1 = 0$. Its inverse image in $\mathbb{C}^2 \times_B C$ is given at $(A, T)$ by $z_0 - uw = z_1 - v = 0$ and at $(A', T)$ by $z_0 - w = z_1 - uv = 0$. Likewise, if $S = T$, the diagonal $\Delta_2$ passes through $(R, S, T)$, given at the point by the equations $z_0 - y_0 = z_1 - y_1 = 0$; its inverse image in $\mathbb{C}^2 \times_B C$ is given at $(A, T)$ by $z_0 - w = z_1 - uv = 0$ and at $(A', T)$ by $z_0 - v = z_1 - uv = 0$.

To resolve the singularities of $\mathbb{C}^2 \times_B C$ along $E \times \{T\}$, we will use two blowups. The next lemma describes the effect of the two blowups in the local analytic setup of the hypersurface of $\mathbb{A}^5_K$ described above.

**Lemma 3.2.** Let $Y$ be the hypersurface of $\mathbb{A}^3_K$, with coordinates $a, b, c, z, z_1$, given by $z z_1 = abc$. Let $p: Y \rightarrow \mathbb{A}_K^3$ be the restriction of the projection
\[
(a, b, c, z, z_1) \mapsto (a, b, c).
\]

Let $X := p^{-1}(0)$. Let $X_1$ and $X_2$ be the components of $X$ defined respectively by $z = 0$ and $z_1 = 0$. Let $\psi_1: Y_1 \rightarrow Y$ be the blowup of $Y$ along the subscheme given by $z = a = 0$ and $\psi_2: Y_2 \rightarrow Y_1$ that of $Y_1$ along the strict transform of the subscheme given by $z_1 = b = 0$. Set $\rho := \psi_1 \psi_2$ and $\tilde{X} := \rho^{-1}(0)$. Then:

1. $Y_2$ is smooth, $\rho$ is flat, and $\tilde{X}$ is a nodal curve with four components, two, $\tilde{X}_1$ and $\tilde{X}_2$, isomorphic to $X_1$ and $X_2$ under $\psi_1 \psi_2$, but not intersecting each other, and (a chain of) two rational, smooth, projective components $E_1$ and $E_2$ intersecting each other at a single point, the first meeting $\tilde{X}_1$ transversally at a single point away from the second, but not meeting $\tilde{X}_2$, the second meeting $\tilde{X}_2$ transversally at a single point away from the first, but not meeting $X_1$; see Figure 2.

2. Let $\lambda: \mathbb{A}^1_K \rightarrow \mathbb{A}^3_K$ be the map sending $t$ to $(t, t, t)$, and form the Cartesian diagram:
\[
\begin{array}{ccc}
W_2 & \xrightarrow{\lambda_2} & Y_2 \\
\rho_\lambda \downarrow & & \downarrow \rho \\
\mathbb{A}^1_K & \xrightarrow{\lambda} & \mathbb{A}^3_K.
\end{array}
\]

Then $W_2$ is a smooth surface. In addition, set
\[
\tilde{X} := \lambda_2^{-1}(\tilde{X}_1), \quad X := \lambda_2^{-1}(\tilde{X}_2), \quad \tilde{E}_1 := \lambda_2^{-1}(E_1), \quad \tilde{E}_2 := \lambda_2^{-1}(E_2).
\]
Then these inverse images are prime Cartier divisors of $W_2$ summing up to $\rho_2^{-1}(0)$. Furthermore, the strict transform to $Y_2$ of the subscheme of $Y$ given by the equations heading each column of the following two tables is a Cartier divisor, and its pullback to $W_2$ is a still a Cartier divisor, a sum of $\tilde{X}, \tilde{X}_a, \tilde{E}_1$ and $\tilde{E}_2$ with the corresponding multiplicities specified in the first four entries of each column, plus, in the case of the second table only, a $\rho_3$-flat prime divisor intersecting the fiber $\rho_3^{-1}(0)$ transversally on the component specified in the fifth row:

|    | $a = 0$ | $b = 0$ | $c = 0$ | $a = 0$ | $b = 0$ | $c = 0$ |
|----|---------|---------|---------|---------|---------|---------|
| $z = 0$ | 1       | 1       | 1       | 0       | 0       | 0       |
| $\tilde{X}$ | | | | | | |
| $X_a$ | 0       | 0       | 0       | 1       | 1       | 1       |
| $\tilde{E}_1$ | 1       | 0       | 0       | 0       | 1       |
| $\tilde{E}_2$ | 1       | 0       | 1       | 0       | 0       |

|    | $z = a$ | $z = b$ | $z = c$ | $z = ab$ | $z = ac$ | $z = bc$ |
|----|---------|---------|---------|---------|---------|---------|
| $z = bc$ | 0       | 0       | 0       | 0       | 0       | 0       |
| $\tilde{X}$ | | | | | | |
| $X_a$ | 0       | 0       | 0       | 0       | 0       | 0       |
| $\tilde{E}_1$ | 0       | 1       | 1       | 0       | 0       |
| $\tilde{E}_2$ | 0       | 1       | 0       | 1       | 0       |

$E_1 \ E_1 \ E_1 \ E_2 \ E_2 \ E_2$

**Proof.** By construction of the blowup, $Y_1$ is the subscheme of $Y \times \mathbb{P}^1_K$ given in $\mathbb{A}^5_K \times \mathbb{P}^1_K$ by

\[
\begin{align*}
\gamma_1'z. &= \gamma_1 a \\
\gamma_1'bc &= \gamma_1 z.
\end{align*}
\]  

where $\gamma_1, \gamma_1'$ are homogeneous coordinates of $\mathbb{P}^1_K$. Set $E_1 = \psi_1^{-1}(0)$, thus given by the equations $a = b = c = z. = z_2 = 0$. Clearly, $E_1$ is mapped isomorphically to $\mathbb{P}^1_K$ under the second projection. Let $T_1$ and $T_1'$ be the points on $E_1$ given by $\gamma_1 = 0$ and $\gamma_1' = 0$. Note that $Y_1$ is isomorphic to $\mathbb{A}^4_K$ and $p\psi_1$ is flat where $\gamma_1 \neq 0$. On the other hand, where $\gamma_1' \neq 0$, we see that $Y_1$ is singular along the points given by $z. = z_2 = b = c = \gamma_1 = 0$. In particular, $Y_1$ is singular at $T_1$. The strict transform of $X$ is given by $a = b = c = z. = \gamma_1 = 0$ and intersects $E_1$ at $T_1$, while that of $X_a$ is given by $a = b = c = z. = \gamma_1' = 0$ and intersects $E_1$ transversally at $T_1'$.

Now, $Y_2$ is the blowup of $Y_1$ along a subscheme of codimension 1, whence $Y_2$ is isomorphic to $Y_1$ where $Y_1$ is smooth, in particular, where $\gamma_1 \neq 0$. Over $\gamma_1' \neq 0$, setting $\gamma_1' = 1$, the blowup is the subscheme of $Y_1 \times \mathbb{P}^1$ given as a subset of $\mathbb{A}^5_K \times \mathbb{A}^1_K \times \mathbb{P}^1_K$ by:

\[
\begin{align*}
z. &= \gamma_1 a \\
\gamma_2'z. &= \gamma_2 b \\
\gamma_2'c &= \gamma_2\gamma_1
\end{align*}
\]  

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where \( \gamma_2, \gamma'_2 \) are homogeneous coordinates of \( \mathbb{P}^1_K \). Notice that \( Y_2 \) is smooth and \( \rho \) is flat. Let \( E_2 := \psi_2^{-1}(T_1) \), thus given by the equations \( a = b = c = z = z_\gamma = 0 \). Clearly, \( E_2 \) is mapped isomorphically to \( \mathbb{P}^1_K \) under the third projection. Let \( T_2 \) and \( T'_2 \) be the points on \( E_2 \) given by \( \gamma_2 = 0 \) and \( \gamma'_2 = 0 \). Abusing notation, we still denote by \( E_1 \) the strict transform of \( E_1 \) to \( Y_2 \) via \( \psi_2 \): It is given where \( \gamma'_1 \neq 0 \) by the equations \( a = b = c = z = z_\gamma = \gamma'_2 = 0 \). We thus see that the “new” \( E_1 \) intersects \( E_2 \) transversally at \( T_2 \), which is the point over \( T_1 \) on the “old” \( E_1 \). The strict transform \( \tilde{X} \) of \( X \) lies where \( \gamma'_1 \neq 0 \), and is given by \( a = b = c = z = \gamma_1 = \gamma'_2 = 0 \), and thus intersects \( E_2 \) transversally at \( T'_2 \).

So, the proof of Statement 1 is complete, the curve \( \tilde{X} \) being depicted in Figure 2 below:

![Figure 2. The curve \( \tilde{X} \) (diagram)](image)

To prove Statement 2, we consider the covering of \( Y_2 \) by three open subschemes isomorphic to \( \mathbb{A}^4_K \), that isomorphic to the open subscheme of \( Y_1 \) given by \( \gamma_1 \neq 0 \), and those given by \( \gamma'_1 \gamma_2 \neq 0 \) and \( \gamma'_1 \gamma'_2 \neq 0 \). The following two tables list on each entry the equation(s) on the open subscheme of \( Y_2 \) given by the inequality heading the row of the strict transform of the subscheme of \( Y \) given by the equations heading the column:

|        | \( a = 0 \) | \( b = 0 \) | \( c = 0 \) | \( a = 0 \) | \( b = 0 \) | \( c = 0 \) |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( \gamma_1 \neq 0 \) | \( z = 0 \) | \( z = 0 \) | \( z = 0 \) | \( z = 0 \) | \( z = 0 \) | \( z = 0 \) |
| \( \gamma'_1 \gamma_2 \neq 0 \) | \( a = 0 \) | \( \gamma'_2 = 0 \) | \( c = 0 \) | \( 1 = 0 \) | \( z_\gamma = 0 \) | \( 1 = 0 \) |
| \( \gamma'_1 \gamma'_2 \neq 0 \) | \( a = 0 \) | \( 1 = 0 \) | \( \gamma_1 = 0 \) | \( 1 = 0 \) | \( b = 0 \) | \( \gamma_2 = 0 \) |

In particular, the strict transforms to \( Y_2 \) of the subschemes of \( Y \) given by the equations heading all the columns above are Cartier divisors.

Recall the equations defining \( E_1, E_2, \tilde{X} \) and \( \tilde{X}_\gamma \):

\[
\begin{align*}
E_1: \ a &= b = c = z = z_\gamma = \gamma_2 = 0 \\
E_2: \ a &= b = c = z = z_\gamma = \gamma_1 = 0 \\
\tilde{X}: \ a &= b = c = z = \gamma_1 = \gamma'_2 = 0 \\
\tilde{X}_\gamma: \ a &= b = c = z = \gamma'_1 = 0
\end{align*}
\] (8)
It must be noted that the equations for $\tilde{X}_*$ are taken in $A_K^5 \times \mathbb{P}_K^1$, whence consist of one less equation than for the other three, which are taken in $A_K^5 \times A_K^1 \times \mathbb{P}_K^1$, assuming $\gamma'_1 \neq 0$. Of these other three, $E_1$ is the only curve that has a point with $\gamma'_1 = 0$: the image of $E_1$ in $Y_1$ is given by $a = b = c = z = z_* = 0$.

Now, where $\gamma_1 \neq 0$, thus setting $\gamma_1 := 1$, viewing $Y_1$, and thus $Y_2$, as the subscheme of $A_K^6$ given by Equations (6), it follows that $W_2$ is the subscheme of $A_K^4$ given by the equations

$$\begin{cases} \gamma'_1 z = t \\ \gamma'_1 t^2 = z_.. \end{cases} \quad (9)$$

It follows that $W_2$ is smooth, isomorphic to $A_K^2$ with coordinates $\gamma'_1, z$. On the other hand, if $\gamma'_1 \neq 0$, we may view $Y_2$ as the subscheme of $A_K^5$ given by Equations (7), setting $\gamma_2 := 1$ or $\gamma'_2 := 1$. It follows that $W_2$ is the subscheme of $A_K^5$ given by

$$\begin{cases} z = \gamma_1 t \\ t = \gamma'_2 z_.. \end{cases} \quad \text{and} \quad \begin{cases} z = \gamma_2 t \\ t = \gamma'_2 \gamma_1, \end{cases} \quad (10)$$

the first set of equations where $\gamma_2 \neq 0$ and the second where $\gamma'_2 \neq 0$. So $W_2$ is smooth, isomorphic to $A_K^2$ with coordinates $\gamma'_2, z_..$ in the first case and $\gamma_1, \gamma_2$ in the second.

The following first three tables list the restrictions to $W_2$ of the equations listed in the last two tables, whereas the fourth lists on each entry the equation(s) on the open subscheme of $W_2$ given by the inequality heading the row of the intersection with $W_2$ of the subscheme of $Y_2$ heading the column:

| $a = 0$ | $b = 0$ | $c = 0$ | $a = 0$ | $b = 0$ | $c = 0$ |
|--------|--------|--------|--------|--------|--------|
| $z = 0$ | $z = 0$ | $z = 0$ | $z = 0$ | $z = 0$ | $z = 0$ |
| $\gamma_1 \neq 0$ | $\gamma'_1 = 0$ | $\gamma'_1 = 0$ | $\gamma'_1 = 0$ | $\gamma'_1 = 0$ | $\gamma'_1 = 0$ |
| $\gamma'_2 \neq 0$ | $\gamma'_2 z_.. = 0$ | $\gamma'_2 z_.. = 0$ | $\gamma'_2 z_.. = 0$ | $\gamma'_2 z_.. = 0$ | $\gamma'_2 z_.. = 0$ |
| $\gamma'_2 \neq 0$ | $\gamma_1 \gamma_2 = 0$ | $\gamma_1 \gamma_2 = 0$ | $\gamma_1 \gamma_2 = 0$ | $\gamma_1 \gamma_2 = 0$ | $\gamma_1 \gamma_2 = 0$ |

| $z = a$ | $z = b$ | $z = c$ |
|--------|--------|--------|
| $z_.. = bc$ | $z_.. = ac$ | $z_.. = ab$ |
| $\gamma_1 \neq 0$ | $\gamma'_1 = 1$ | $\gamma'_1 = 1$ |
| $\gamma'_2 \neq 0$ | $1 = (\gamma'_2)^2 z_..$ | $1 = (\gamma'_2)^2 z_..$ |
| $\gamma'_2 \neq 0$ | $\gamma_1 \gamma_2 = \gamma_1 \gamma_2$ | $\gamma_1 \gamma_2 = \gamma_1 \gamma_2$ |

| $z = ab$ | $z = ac$ | $z = bc$ |
|--------|--------|--------|
| $z_.. = c$ | $z_.. = b$ | $z_.. = a$ |
| $\gamma_1 \neq 0$ | $(\gamma'_1)^2 z_.. = 1$ | $(\gamma'_1)^2 z_.. = 1$ |
| $\gamma'_2 \neq 0$ | $\gamma'_2 = 1$ | $\gamma'_2 = 1$ |
| $\gamma'_2 \neq 0$ | $\gamma_1 \gamma_2 = \gamma_1$ | $\gamma_1 \gamma_2 = \gamma_2$ |
\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma_1 \neq 0 & 1 = 0 & \gamma'_1 = 0 & z = 0 & 1 = 0 \\
\gamma'_1 \gamma_2 \neq 0 & \gamma'_2 = 0 & 1 = 0 & 1 = 0 & z = 0 \\
\gamma_1 \gamma'_2 \neq 0 & 1 = 0 & 1 = 0 & \gamma_2 = 0 & \gamma_1 = 0 \\
\hline
\end{array}
\]

So \(\hat{X}_1, \hat{X}_2, \hat{E}_1\) and \(\hat{E}_2\) are indeed prime Cartier divisors of \(W_2\), and the intersections with \(W_2\) of the strict transforms to \(Y_2\) of the subschemes of \(Y\) given by the equations heading the columns of the above first three tables are Cartier. The two tables in Statement 2 follow from the above four tables by comparing the equations in the entries.

Observe that the strict transform to \(Y_1\) of the subscheme of \(Y\) defined by \(z_\ast = a = 0\) is a Cartier divisor, given by \(\gamma'_1 = 0\). On the other hand, the strict transform to \(Y_1\) of the subscheme defined by \(z_\ast = b = 0\) is given by \(\gamma_1 = b = 0\) and that of the subscheme defined by \(z_\ast = c = 0\) is given by \(\gamma_1 = c = 0\). Thus the blowups of \(Y_1\) along the strict transform of the subscheme given by \(z_\ast = c = 0\) and along that of the subscheme given by \(z_\ast = b = 0\) are equal. And so are the blowups along the transform of the subscheme given by \(z_\ast = b = 0\) and along \(z_\ast = c = 0\) and along that of the subscheme given by \(z_\ast = b = 0\). Thus the same proposition can be used, \textit{mutatis mutandis}, to describe all “possible” blowups.

### 4 Blowups: global analysis

Recall that \(C\) is a curve with components \(C_1, \ldots, C_p\) defined over \(K\), an algebraically closed field, \(\hat{C}\) is the smooth locus of \(C\), and \(\hat{C}/B\) is a regular smoothing of \(C\) with regular total space, the smooth locus of which is \(\hat{C}\).

Recall that \(\Delta \subset C^2\) is the diagonal subscheme, and that \(\Delta_1 := \rho_i^{-1}(\Delta)\) and \(\Delta_2 := \rho_2^{-1}(\Delta)\), where \(\rho_i:C^3 \to C^2\) is the projection onto the product over \(B\) of the \(i\)-th and last factors of \(C^3\) for \(i = 1, 2\).

#### 4.1 The double product

We will use the symbol \([X, Y]\) to describe the blowup of (a modification by blowups of) \(C^2\) along (the strict transform of) \(X \times Y\), for proper subcurves \(X, Y \subseteq C\). If \(X\) and \(Y\) are components, \(X = C_i\) and \(Y = C_l\), we shall use the shorter notation \([i, l]\) for \([C_i, C_l]\). A modification of \(C^2\) by such blowups can thus be described as a sequence

\[\Delta, [X_1, Y_1], [X_2, Y_2], \ldots, [X_u, Y_u],\]  \hspace{1cm} (11)

where \([\Delta]\) stands for blowup along the diagonal. By Proposition 2.1, the blowup along the diagonal adds a projective line on top of each pair \((R, R)\) of identical nodes, whereas by Proposition 3.1, the blowup \([X_i, Y_i]\) adds a projective line on top of each pair \((R, S) \in (X_i \cap X'_i) \times (Y_i \cap Y'_i)\), as long as a projective line has not been created before, in a preceding blowup, over the same pair. We say the projective lines created are \textit{exceptional}. On the exceptional line over each pair
(R, S) of reducible nodes there are two distinguished points, each being on the intersection of the strict transforms of three distinct products of components of C containing R and S.

By considering a sequence of blowups long and varied enough, always starting with [Δ], we obtain a scheme C^2 which is nonsingular away from the points over pairs (R, S) of distinct nodes R and S of C where either R or S is irreducible. At any rate, if this is the case, since the product X × Y of subcurves X and Y of C is Cartier away from (X ∩ X’) × (Y ∩ Y’), its strict transform to C^2 is Cartier. If this is achieved by the sequence of blowups, that is, if the strict transforms of all the products X × Y of subcurves X and Y of C are Cartier, we call C^2 a good partial desingularization of C^2.

Example 4.1. Assume C consists of four components C_1, C_2, C_3, C_4 such that

\[
\begin{align*}
C_1 \cap C_2 &= \{N_1\}, \\
C_1 \cap C_3 &= \{N_2\}, \\
C_1 \cap C_4 &= \emptyset, \\
C_2 \cap C_3 &= \{N_3, N_4, N_5\}, \\
C_2 \cap C_4 &= \{N_6\}, \\
C_3 \cap C_4 &= \{N_7, N_8\}.
\end{align*}
\]

In addition, suppose C_2, C_3, C_4 are smooth but C_1 has a node N_0. The first blowup in the sequence

\[
[\Delta, [4, 4], [2, 2], [1, 2], [2, 1], [2, 4], [4, 2], [1, 4], [4, 1]
\]
adds a \(\mathbb{P}^1_k\) over each pair \((N_i, N_j)\) for \(i = 0, \ldots, 8\), the second blowup adds a \(\mathbb{P}^1_k\) over each pair \((N_i, N_j)\) for distinct \(i, j\) such that \(6 \leq i, j \leq 8\), the third adds a \(\mathbb{P}^1_k\) over each pair \((N_i, N_j)\) for distinct \(i, j\) such that \(i, j = 1, 3, 4, 5, 6\), the fourth adds a \(\mathbb{P}^1_k\) over each pair \((N_2, N_j)\) for \(j = 1, 3, 4, 5, 6\), the fifth adds a \(\mathbb{P}^1_k\) over each pair \((N_i, N_2)\) for \(i = 1, 3, 4, 5, 6\), the sixth adds a \(\mathbb{P}^1_k\) over each pair \((N_i, N_j)\) with \(i = 1, 3, 4, 5\) and \(j = 7, 8\), the seventh adds a \(\mathbb{P}^1_k\) over each pair \((N_i, N_j)\) with \(i = 7, 8\) and \(j = 1, 3, 4, 5\), the eighth adds a \(\mathbb{P}^1_k\) over \((N_2, N_7)\) and \((N_2, N_8)\), while the last adds a \(\mathbb{P}^1_k\) over \((N_7, N_2)\) and \((N_8, N_2)\).

The above sequence desingularizes C^2 at all points not of the form \((N_0, N_i)\) or \((N_i, N_0)\) for \(i = 1, \ldots, 8\). The strict transforms of the diagonal and all the products \(C_i \times C_j\) are Cartier divisors in the resulting scheme. We have obtained a good partial desingularization of C^2. Notice that the resulting scheme would be different if we had exchanged the position of [1, 4] with that of [1, 2] in the sequence of blowups. As it is, the resulting \(\mathbb{P}^1_k\) over \((N_2, N_6)\) is contained in the strict transform of \(C_1 \times C_2\), whereas with the exchange the \(\mathbb{P}^1_k\) would intersect the transform transversally. The new resulting scheme would nonetheless still be a good partial desingularization of C^2.

Recall that \(\mathcal{N}(C)\) is the collection of reducible nodes of C.

Definition 4.2. For each \(i, k \in \{1, \ldots, p\}\), let \(\mathcal{N}_{i,k}(C)\) denote the subset of \(\mathcal{N}(C)^2\) containing pairs of nodes \((R, S)\) such that \(R \in C_i\) and \(S \in C_k\). Let
\( \phi : \tilde{C}^2 \to C^2 \) be a good partial desingularization. Denote by \( N_{i,k}(\phi) \) the subset of \( N_{i,k}(C) \) containing the pairs \((R, S)\) such that \( \phi^{-1}(R, S) \) lies on the strict transform of \( C_i \times C_k \).

**Proposition 4.3.** Let \( \phi : \tilde{C}^2 \to C^2 \) be a good partial desingularization given by

\[ [\Delta], [X_1, Y_1], [X_2, Y_2], \ldots, [X_u, Y_u]. \]

Let \( i, k \in \{1, \ldots, p\} \) and \( R, S \in C \). Then \((R, S) \in N_{i,k}(\phi) \) if and only if \( R \) and \( S \) are reducible nodes of \( C \) with \( R \in C_i \) and \( S \in C_k \) and either of the following two statements hold:

1. \( R = S \) and \( i \neq k \).

2. \( R \neq S \) and the first integer \( \nu \) such that \( R \in X_\nu \cap X'_\nu \) and \( S \in Y_\nu \cap Y'_\nu \) satisfies either \( C_i \subseteq X_\nu \) and \( C_k \subseteq Y_\nu \) or \( C_i \not\subseteq X_\nu \) and \( C_k \not\subseteq Y_\nu \).

**Proof.** Follows easily from Proposition 3.1. \( \Box \)

Recall that \( C_R \) is the curve obtained from \( C \) by replacing the node \( R \) by a smooth rational curve, and that \( C(1) \) is the one obtained by replacing each reducible node of \( C \) by a smooth rational curve; see Subsection 2.1.

**Proposition 4.4.** Let \( \phi : \tilde{C}^2 \to C^2 \) be a good partial desingularization. Let \( \rho : \tilde{C}^2 \to C \) denote its composition with the first projection \( p_1 : C^2 \to C \). Let \( R \in C \) and \( X := \rho^{-1}(R) \). Let \( \mu : X \to C \) be the restriction to \( X \) of \( \phi \) composed with the second projection \( p_2 : C^2 \to C \). Then the following statements hold:

1. \( \rho \) is flat.

2. \( \tilde{C}^2 \) is regular along each smooth rational curve of \( X \) contracted by \( \mu \).

3. If \( R \) is not a node of \( C \) then \( \mu \) is an isomorphism.

4. If \( R \) is an irreducible node of \( C \) then \( X \) is \( C \)-isomorphic to \( C_R \).

5. If \( R \) is a reducible node of \( C \) then \( X \) is \( C \)-isomorphic to \( C(1) \).

Furthermore, for each \( i, k \in \{1, \ldots, p\} \), let \( D_{i,k} \) denote the strict transform to \( \tilde{C}^2 \) of \( C_i \times C_k \). Let

\[ D = \sum_{i,k} w_{i,k} D_{i,k} \]

for given integers \( w_{i,k} \). Then, if \( R \) is a reducible node of \( X \), the restriction \( \mathcal{O}_{\tilde{C}^2}(D)|_X \) is a twister of \( X \). More specifically, for each \( i = 1, \ldots, p \), let \( \tilde{C}_i \) be the strict transform to \( X \) of \( C_i \) under \( \mu \), and for each reducible node \( S \) of \( C \), let \( ES := \mu^{-1}(S) \). Then

\[ \mathcal{O}_{\tilde{C}^2}(D)|_X \cong \mathcal{O}_X\left(\sum_{k=1}^p a_k \tilde{C}_k + \sum_{S \in N(C)} b_S ES\right), \]

where \( a_k := \sum_i w_{i,k} \), the sum over the two \( i \) such that \( R \in C_i \), and \( b_S := \sum_{i,k} w_{i,k} \), the sum over the two pairs \((i, k)\) such that \((R, S) \in N_{i,k}(\phi) \).
Proof. Since flatness is a local property, Statement 1 follows directly from Propositions 2.1 and 3.1.

Now, $C^2$ is regular except at the pairs of nodes of $C$. Thus Statement 3 holds. Furthermore, suppose $R$ is a node of $C$. Then, for each node $S \in C$, if $S \neq R$ then $\Delta$ is Cartier at $(R,S)$, and if $R$ or $S$ is irreducible then every product $Y \times Z$ of subcurves of $C$ is Cartier at $(R,S)$. So $\phi$ is an isomorphism over a neighborhood of $(R,S)$ except if $S = R$ or both $R$ and $S$ are reducible nodes of $C$.

Suppose first that $R$ is irreducible. Then $\mu$ is an isomorphism except over $R$. Now, over a neighborhood of $(R,R)$, we have that $\tilde{C}^2$ is isomorphic to $\mathbb{P}_{C^2}(I_\Delta)$. Thus $\mu^{-1}(R)$ is a smooth rational curve, $X$ is $C$-isomorphic to $C_R$ and $\tilde{C}^2$ is regular along $\mu^{-1}(R)$ by Proposition 2.1. Statement 4 and part of Statement 2 are proved.

Suppose now that $R$ is reducible. Then $\mu$ is an isomorphism except over reducible nodes of $C$. So, let $S \in \mathcal{N}(C)$. Recall the notation introduced before Proposition 3.1. We may assume without loss of generality that $(R,S) \in \mathcal{N}_{i,l}(\phi)$. So, to describe $\mu$ on a neighborhood of $E_S$, we may assume $\phi$ is simply the blowup $[i,l]$.

Then $E_S$ is a smooth rational curve and $\tilde{C}^2$ is regular along it by Proposition 3.1. The proof of Statement 2 is complete. Moreover, recall the computation done in the proof of Proposition 3.1. The fiber $X$ corresponds to the subscheme of $Y$ given by $x_0 = x_1 = 0$. Its equations in $\mathbb{A}_K^4 \times \mathbb{P}_K^1$ are thus

$$a'y_0 = 0, \quad \alpha y_1 = 0, \quad x_0 = 0, \quad x_1 = 0.$$  

This subscheme is the union of a projective line, given by $x_0 = x_1 = y_0 = y_1 = 0$, and two disjoint affine lines, given by $x_0 = x_1 = y_0 = \alpha = 0$ and $x_0 = x_1 = y_1 = \alpha' = 0$, intersecting transversally the projective line at distinct points. The first affine line corresponds to the strict transform of $C_k$ and the second to that of $C_l$. This proves Statement 5.

As for the last statements, consider the map $\lambda : B \to C$ sending the special point of $B$ to $R$ and induced by the ring homomorphism

$$\frac{K[[x_0, x_1, t]]}{(x_0x_1 - t)} \to K[[t]]$$

which sends $x_i$ to $t$ for $i = 1, 2$, and thus $t$ to $t^2$. Form the Cartesian diagram:

$$W \xrightarrow{\xi} \tilde{C}^2 \xrightarrow{\rho} C$$

Then $\rho\lambda$ is a regular smoothing of $X$.

Indeed, the special fiber of $\rho\lambda$ is isomorphic to $X$ under $\xi$, hereby identified with $X$. Let $\mathcal{C}$ denote the strict transform of $C$ under $\mu$. We need to show that $W$ is regular along $E_S \cap C$ for each $S \in \mathcal{N}(C)$. Again, recall the notation
introduced before Proposition 3.1 and that introduced in its proof. We may assume that \((R, S) \in \mathcal{N}_{i, l}(\phi)\) and that \(\phi\) is simply the blowup \([i, l]\). Thus \(\mathcal{W}\) corresponds to the base change of \(Y\) under the diagram

\[
\begin{array}{ccc}
\mathbb{A}^1_K \times \mathbb{P}^1_K & \xrightarrow{t \mapsto (t, t)} & \mathbb{A}^2_K,
\end{array}
\]

that is, to the subscheme \(\mathcal{W}\) of \(\mathbb{A}^3_K \times \mathbb{P}^1_K\), with coordinates \((t, y_0, y_1), (\alpha : \alpha')\), given by

\[
\begin{aligned}
\alpha' t &= \alpha y_1 \\
\alpha' y_0 &= \alpha t
\end{aligned}
\]

Since \(\mathcal{W}\) is smooth, so is \(\mathcal{W}\) along \(E_S\).

It follows that \(\xi^{-1}(D)\) is a Cartier divisor supported on the special fiber of \(\rho_3\), and thus \(\mathcal{O}_{\mathbb{C}^3}(D)|_X\) is a twister of \(X\).

More specifically, in order to describe \(\mathcal{O}_{\mathbb{C}^3}(D)|_X\), we need only describe the pullbacks to \(\mathcal{W}\) of \(D_{i, k}, D_{i, l}, D_{j, k}\) and \(D_{j, l}\). We need only do so in a neighborhood of \(E_S\) for each \(S \in \mathcal{N}(C)\). Thus we need only describe the pullbacks of the corresponding divisors to \(\mathcal{W}\) for \(\alpha \neq 0\) and \(\alpha' \neq 0\). If \(\alpha' \neq 0\), then \(\mathcal{W}\) corresponds to the subscheme of \(\mathbb{A}^3_K\) given by \(y_0 = \alpha t\) and \(t = \alpha y_1\), and the pullbacks to the divisors given by \(\alpha = 0, y_1 = 0, t = 0\) and \(\alpha' = 0\), respectively. On the other hand, if \(\alpha \neq 0\), then \(\mathcal{W}\) corresponds to the subscheme of \(\mathbb{A}^3_K\) given by \(y_1 = \alpha' t\) and \(t = \alpha' y_0\), and the pullbacks to the divisors given by \(\alpha = 0, t = 0, y_0 = 0\) and \(\alpha' = 0\), respectively. Thus

\[
\begin{aligned}
D_{i, k}|_{\mathcal{W}} &= \cdots + \hat{C}_k + 0E_S + 0\hat{C}_l + \cdots \\
D_{i, l}|_{\mathcal{W}} &= \cdots + 0\hat{C}_k + E_S + \hat{C}_l + \cdots \\
D_{j, k}|_{\mathcal{W}} &= \cdots + \hat{C}_k + E_S + 0\hat{C}_l + \cdots \\
D_{j, l}|_{\mathcal{W}} &= \cdots + 0\hat{C}_k + 0E_S + \hat{C}_l + \cdots
\end{aligned}
\]

Use now that \((R, S) \in \mathcal{N}_{i, l}(\phi) \cap \mathcal{N}_{j, k}(\phi)\) but \((R, S) \notin \mathcal{N}_{i, k}(\phi) \cup \mathcal{N}_{j, l}(\phi)\) to conclude.

4.2 The triple product

The blowup of (any modification of) \(\mathbb{C}^2\) along (the strict transform of) the product \(X \times Y\) of two subcurves \(X\) and \(Y\) of \(C\) is denoted by \([X, Y]\). We will also denote the blowup of (any modification of) \(\mathbb{C}^3\) along (the strict transform of) the product \(X \times Y \times Z\) of three subcurves \(X\), \(Y\) and \(Z\) of \(C\) by \([X, Y, Z]\). If \(X\), \(Y\) and \(Z\) are components, say \(X = C_i\), \(Y = C_l\) and \(Z = C_m\), we shall use the shorter notation \([i, l, m]\) for \([C_i, C_l, C_m]\). We will consider a partial desingularization of \(\mathbb{C}^3\) consisting of the base change of the sequence of blowups in \([\Pi]\), resulting in a map \(\tilde{\mathbb{C}}^2 \to \mathbb{C}^2\), followed by a sequence of blowups of \(\tilde{\mathbb{C}}^2 \times_B \mathbb{C}\).
along strict transforms of products of three subcurves of \( C \). Symbolically, the partial desingularization of \( \tilde{C}^3 \) can be described by a sequence of the form:

\[
[\Delta], [X_1, Y_1], \ldots, [X_u, Y_u], [X_{u+1}, Y_{u+1}, Z_{u+1}], \ldots, [X_{u+v}, Y_{u+v}, Z_{u+v}].
\]  

(12)

If the sequence of blowups in (12) is long and varied enough, we obtain a good partial desingularization \( \tilde{C}^2 \) of \( C^2 \), and the resulting modification \( \tilde{C}^3 \) of \( \tilde{C}^2 \times_B C \) will be such that no other blowup along the (strict transform of a) product of three subcurves of \( C \) affects it, that is, such that the strict transforms to \( \tilde{C}^3 \) of all the products \( X \times Y \times Z \) of subcurves \( X, Y \) and \( Z \) of \( C \) are Cartier. In this case, we call \( \tilde{C}^3 \) a good partial desingularization of \( \tilde{C}^2 \times_B C \).

Recall the natural subscheme \( F_2 \subset \mathbb{P}_{C^2}(\mathcal{I}_{\Delta}|C^2) \times_B C \) defined in Subsection 2.3. The map \( \phi \) factors through the structure map \( \mathbb{P}_{C^2}(\mathcal{I}_{\Delta}|C^2) \to C^2 \), whence we may consider the strict transform of \( F_2 \) to \( \tilde{C}^3 \).

**Proposition 4.5.** Let \( \phi : \tilde{C}^2 \to C^2 \) and \( \psi : \tilde{C}^3 \to \tilde{C}^2 \times_B C \) be good partial desingularizations. Let \( \rho = p_1 \circ \psi \), where \( p_1 : \tilde{C}^2 \times_B C \to C^2 \) is the projection. Then \( \rho \) is flat. In addition, let \( A \) be a closed point of \( C^2 \) and set \( (R_1, R_2) := \phi(A) \). Put \( X := \rho^{-1}(A) \). Let \( \mu : X \to C \) be the restriction to \( X \) of \( \psi \) followed by the second projection \( p_2 : \tilde{C}^2 \times_B C \to C \). Then the following statements hold:

1. If neither \( R_1 \) nor \( R_2 \) is a reducible node of \( C \), then \( \mu \) is an isomorphism.
2. If only one between \( R_1 \) and \( R_2 \) is a reducible node of \( C \), or if both are but \( A \) is not one of the two distinguished points of \( \phi^{-1}(R_1, R_2) \), then \( X \) is \( C \)-isomorphic to \( C(1) \).
3. If \( R_1 \) and \( R_2 \) are reducible nodes of \( C \) and \( A \) is one of the two distinguished points of \( \phi^{-1}(R_1, R_2) \), then \( X \) is \( C \)-isomorphic to \( C(2) \).

Finally, let \( \Delta_1, \Delta_2 \) and \( \tilde{F}_2 \) denote the strict transforms of \( \Delta_1, \Delta_2 \) and \( F_2 \) to \( \tilde{C}^3 \). Let \( T \in C \). Then the following statements hold:

4. For \( i = 1, 2 \), the strict transform \( \tilde{\Delta}_i \) is \( \rho \)-flat along \( \mu^{-1}(T) \) unless, possibly, \( R_i \) is a reducible node of \( C \) and \( T = R_i \).
5. For \( i = 1, 2 \), the strict transform \( \tilde{\Delta}_i \) is Cartier along \( \mu^{-1}(T) \) unless \( R_i \) is an irreducible node of \( C \) and \( T = R_i \).
6. \( \tilde{F}_2 \) is \( \rho \)-flat along \( \mu^{-1}(T) \) unless, possibly, \( T \) is a reducible node of \( C \) and either \( R_1 = T \) or \( R_2 = T \).

\[ \mathcal{I}_{\tilde{F}_2}|\tilde{C}^3 = \mathcal{I}_{\tilde{\Delta}_1}|\tilde{C}^3 \mathcal{I}_{\tilde{\Delta}_2}|\tilde{C}^3 = \mathcal{I}_{\tilde{\Delta}_1}|\tilde{C}^3 \otimes \mathcal{I}_{\tilde{\Delta}_2}|\tilde{C}^3 \text{ in a neighborhood of } \mu^{-1}(T), \]

unless \( T \) is an irreducible node of \( C \) and \( R_1 = R_2 = T \).

**Proof.** Let \( R_1, R_2, T \in C \) and \( A \in \phi^{-1}(R_1, R_2) \). Let \( X \) and \( \mu \) be as in the statement of the proposition. First of all, notice that Statement 7 follows from Statement 5. Indeed,

\[ \mathcal{I}_{\tilde{F}_2}|\tilde{C}^3 = \mathcal{I}_{\tilde{\Delta}_1}|\tilde{C}^3 \mathcal{I}_{\tilde{\Delta}_2}|\tilde{C}^3 = \mathcal{I}_{\tilde{\Delta}_1}|\tilde{C}^3 \otimes \mathcal{I}_{\tilde{\Delta}_2}|\tilde{C}^3. \]
in a neighborhood of $\mu^{-1}(T)$ if either $\tilde{\Delta}_1$ or $\tilde{\Delta}_2$ is Cartier along $\mu^{-1}(T)$.

To prove the remaining statements, notice first that all the products $X \times Y \times Z$ of proper subcurves $X$, $Y$ and $Z$ of $C$ are Cartier at $(R_1, R_2, T)$ if at most one among $R_1$, $R_2$ and $T$ is a reducible node of $C$. Statement 1 follows. Furthermore, the strict transform of each product $X \times Y \times Z$ to $\tilde{C}^2 \times_B C$ is the strict transform of $D \times Z$, where $D$ is the strict transform to $\tilde{C}^2$ of $X \times Y$. Since $\phi$ is good, any such $D$ is Cartier, and hence the strict transforms of all the products $X \times Y \times Z$ to $\tilde{C}^2 \times_B C$ are Cartier at $(A, T)$ if $T$ is not a reducible node of $C$. Thus $\psi$ fails to be an isomorphism over $(A, T)$ and $\rho$ fails to be flat in a neighborhood of $\mu^{-1}(T)$ only if $T$ and $R_i$ are reducible nodes of $C$ for $i = 1$ or $i = 2$.

Now, by base change, the strict transforms of $\Delta_1$, $\Delta_2$ and $F_2$ to $\tilde{C}^2 \times_B C$ are $p_1$-flat. Thus, $\tilde{\Delta}_1$ or $\tilde{\Delta}_2$ or $\tilde{F}_2$ is $p$-flat along $\mu^{-1}(T)$ unless, possibly, $T$ is a reducible node of $C$. However, $\tilde{\Delta}_i$ does not intersect $\mu^{-1}(T)$ unless $R_i = T$, for $i = 1, 2$. Statements 4 and 6 follow.

It follows as well that $\tilde{\Delta}_i$ is trivially Cartier along $\mu^{-1}(T)$ if $T \neq R_i$, for $i = 1, 2$. If $T = R_i$ and $R_i$ is a nonsingular point of $C$, then $\Delta_i$ is Cartier at $(R_1, R_2, T)$ and both $\psi$ is an isomorphism over $(A, T)$ and $\phi$ is an isomorphism over $(R_1, R_2)$. So $\tilde{\Delta}_i$ is Cartier along $\mu^{-1}(T)$ if $R_i$ is a nonsingular point of $C$.

If $T = R_i$ and $R_i$ is an irreducible node of $C$, then $\psi$ is an isomorphism over $(A, T)$. Furthermore, $\phi$ is an isomorphism over $(R_1, R_2)$ unless $R_1 = R_2$. Since $\Delta_i$ fails to be Cartier at $(R_1, R_2, T)$, it follows that $\tilde{\Delta}_i$ fails to be Cartier along $\mu^{-1}(T)$ unless $R_1 = R_2$. However, even in this case, $\tilde{\Delta}_i$ fails to be Cartier along $\mu^{-1}(T)$ as well, as a local reasoning, similar to that done before Lemma 3.2 shows.

To finish the proof of the proposition we may now assume, without loss of generality, that $R_2$ is a reducible node of $C$. We need only describe the structure of $\tilde{C}^3$ locally around $\mu^{-1}(T)$ when $T$ is a reducible node of $C$, and show that $\rho$ is flat and $\tilde{\Delta}_2$ is Cartier along $\mu^{-1}(T)$.

Assume first that $R_1$ is not a reducible node of $C$. Then $\phi$ is an isomorphism over $(R_1, R_2)$ and $\psi$ is locally around $X$ of the form $1_C \times_B \varphi: C \times_B \tilde{C}^2 \to C^3$, where $\varphi: \tilde{C}^2 \to C^2$ is a good partial desingularization. Thus $X$ is $C$-isomorphic to $C(1)$ and $\rho$ is flat along $X$ by Proposition 4.4.

If $R_2$ is not an irreducible node of $C$ either, it follows as well from the same proposition that $\tilde{C}^3$ is regular along each smooth rational curve of $X$ contracted by $\mu$. Now, $\tilde{\Delta}_2$ intersects $X$ along $\mu^{-1}(R_2)$, which is a smooth rational curve of $X$ because $R_2$ is reducible. Since $\tilde{C}^3$ is regular along $\mu^{-1}(R_2)$, it follows that $\tilde{\Delta}_2$ is Cartier along $X$.

We may now assume that $R_1$ is a reducible node of $C$. Assume as well that $T$ is a reducible node of $C$. Suppose first that $A$ is one of the distinguished points of $\phi^{-1}(R_1, R_2)$. Then $p_1$ looks locally analytically over $(A, T)$ like the map $p$ in Lemma 3.2 over the origin, and $\psi$ like $\psi_1 \psi_2$. In this case, the flatness of $\rho$ along $\mu^{-1}(T)$ and Statements 3 and 5 follow from that lemma.

Suppose now that $A$ is not any of the distinguished points of $\phi^{-1}(R_1, R_2)$. Recall the observation after the proof of Lemma 3.2. In order that $\psi$ be a good
partial desingularization, one of the blowups leading to it must be along (the
strict transform of) \( D \times Z \), where \( D \) is the strict transform of a product \( X \times Y \)
of two subcurves \( X \) and \( Y \) of \( C \) such that \( R_1 \in X \cap X' \), \( R_2 \in Y \cap Y' \) and
\( \phi^{-1}(R_1, R_2) \subseteq D \), and \( Z \) is a subcurve of \( C \) such that \( T \in Z \cap Z' \). Then \( p_1 \)
looks locally analytically over \((A,T)\) like the map \( p \) in Lemma 3.2 over a point
\((a,b,c)\) with \( b \neq 0 \), and \( \psi \) like \( \psi_1 \). In this situation, the local computations are
similar to those done in the proofs of Propositions 3.1 and 4.4. Thus we get
Statement 2, as well as the flatness of \( \rho \) at any point of \( \widehat{C}^3 \) over \((A,T)\) and the
regularity of \( \widehat{C}^3 \) along \( \mu^{-1}(T) \) for every \( T \in N(C) \). Finally, since \( \Delta_2 \) is trivial
along \( X \) away from \( \mu^{-1}(R_2) \), the regularity of \( \widehat{C}^3 \) there yields that \( \Delta_2 \) is Cartier
along all of \( X \), finishing the proof of Statement 5.

Let \( R, S \) and \( T \) be reducible nodes of \( C \). Let \( A \in \widehat{C}^2 \) be a distinguished
point of \( \phi^{-1}(R,S) \). We say that \([X,Y,Z] \) affects \((A,T)\) if

\[
R \in X \cap X', \quad S \in Y \cap Y', \quad T \in Z \cap Z',
\]

and \( A \) lies on the strict transform to \( \widehat{C}^2 \) of \( X \times Y \). Let \( C_i \) and \( C_j \) be the
components containing \( R \), let \( C_k \) and \( C_l \) be those containing \( S \), and \( C_m \) and \( C_n \)
those containing \( T \). We say that \([X,Y,Z] \) is of type \([i,k,m] \) at \((R,S,T)\) if

\[
C_i \times C_k \times C_m \subseteq X \times Y \times Z \quad \text{and} \quad C_j \times C_l \times C_n \subseteq X' \times Y' \times Z'.
\]

**Proposition 4.6.** Let \( \phi: \widehat{C}^2 \to C^2 \) and \( \psi: \widehat{C}^3 \to \widehat{C}^2 \times_B C \) be good partial desingularizations, given by the sequence of blowups

\[
[\Delta], [X_1,Y_1], \ldots, [X_u,Y_u], [X_{u+1},Y_{u+1},Z_{u+1}], \ldots, [X_{u+v},Y_{u+v},Z_{u+v}]. \tag{13}
\]

Fix reducible nodes \( R \) and \( S \) of \( C \), and a distinguished point \( A \) of \( \phi^{-1}(R,S) \).
Then there is a unique choice of integers \( i,j,k,l \in \{1, \ldots, p\} \) such that \( A \) is the
intersection of the strict transforms to \( \widehat{C}^2 \) of

\[
C_i \times C_k, \quad C_i \times C_l, \quad C_j \times C_k.
\]

Put \( X := p_1^{-1}(A) \), where \( p_1: \widehat{C}^2 \times_B C \to \widehat{C}^2 \) is the projection. For each
\( m = 1, \ldots, p \), let \( \widehat{C}_m \) denote the strict transform to \( X \) of \( C_m \) under \( p_2 \psi \), where
\( p_2: \widehat{C}^2 \times_B C \to C \) is the projection. For each reducible node \( T \) of \( C \), let \( m_T, n_T \in \{1, \ldots, p\} \) be the distinct integers such that \( T \in C_{m_T} \cap C_{n_T} \), and let \( \widehat{C}_{m_T} \) (resp.
\( E_{m_T} \)) be the rational curve in \( \psi^{-1}(A,T) \) intersecting \( \widehat{C}_{m_T} \) (resp. \( \widehat{C}_{n_T} \)). Choose
\( m_T \) such that the first blowup in \( (13) \) to affect \((A,T)\) is of type \([r_{T,1},s_{T,1},m_T]\)
for \((r_{T,1},s_{T,1}) \in \{i,j\} \times \{k,l\}\). Also, let \((r_{T,2},s_{T,2}), (r_{T,3},s_{T,3}) \in \{i,j\} \times \{k,l\}\)
distinct from each other and from \((r_{T,1},s_{T,1})\) such that

1. \( \{(r_{T,1},s_{T,1}), (r_{T,2},s_{T,2}), (r_{T,3},s_{T,3})\} = \{(i,k),(i,l),(j,k)\} \),

2. either \([r_{T,2},s_{T,2},m_T]\) or \([r_{T,3},s_{T,3},m_T]\) appears first in the sequence of
types of blowups affecting \((A,T)\).
For each \( r, s, m \in \{1, \ldots, p\} \), let \( D_{r,s,m} \) denote the strict transform to \( \widetilde{C}^3 \) of \( C_r \times C_s \times C_m \). Also, let \( \Delta_1 \) and \( \Delta_2 \) denote the strict transforms to \( \widetilde{C}^3 \) of \( \Delta_1 \) and \( \Delta_2 \). Let

\[
D := \sum_{r,s,m} w_{r,s,m} D_{r,s,m},
\]

where the \( w_{r,s,m} \) are given integers. For each \( m = 1, \ldots, p \), set

\[
w(m) := w_{i,k,m} + w_{i,l,m} + w_{j,k,m}.
\] (14)

For each reducible node \( T \) of \( C \), set

\[
w(T, m_T) := w_{rT,1,sT,1,m_T} + w_{rT,2,sT,2,n_T} + w_{rT,3,sT,3,s_T},
\]

\[
w(T, n_T) := w_{rT,1,sT,1,m_T} + w_{rT,2,sT,2,n_T} + w_{rT,3,sT,3,n_T}.
\] (15)

Finally, set

\[
(h(R,i), h(R,i)) := \begin{cases} 
(1,0) & \text{if } r_{R,1} = r_{R,2} = i \\
(1,1) & \text{if } (r_{R,1}, m_R) = (j, i) \text{ or } (r_{R,2}, n_R) = (j, i) \\
(0,0) & \text{if } (r_{R,1}, m_R) = (j, j) \text{ or } (r_{R,2}, n_R) = (j, j).
\end{cases}
\] (16)

\[
(h(S,k), h(S,l)) := \begin{cases} 
(1,0) & \text{if } s_{S,1} = s_{S,2} = k \\
(1,1) & \text{if } (s_{S,1}, m_S) = (l, k) \text{ or } (s_{S,2}, n_S) = (l, k) \\
(0,0) & \text{if } (s_{S,1}, m_S) = (l, l) \text{ or } (s_{S,2}, n_S) = (l, l).
\end{cases}
\] (17)

Let \( \lambda: B \to \widetilde{C}^2 \) be any section of \( \widetilde{C}^2/B \) sending the special point of \( B \) to \( A \) and such that the pullbacks of the strict transforms of \( C_i \times C_k, C_i \times C_1, \) and \( C_j \times C_k \) are all prime. Form the Cartesian diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{\xi} & \widetilde{C}^3 \\
\rho \downarrow & & \downarrow p \psi \\
B & \xrightarrow{\lambda} & \widetilde{C}^2.
\end{array}
\]

Then \( \rho \) is a regular smoothing of \( C \), and the pullbacks of \( \Delta_1, \Delta_2 \) and \( D \) to \( W \) under \( \xi \) are Cartier divisors. More precisely,

\[
\xi^* D = \sum_{m=1}^p w(m)\mathcal{C}_m + \sum_{T \in \mathcal{N}(C)} (w(T, m_T)E_{T,m_T} + w(T, n_T)E_{T,n_T})
\]

\[
\xi^* \Delta_1 = h(R,i)E_{R,1} + h(R,j)E_{R,j} + \Gamma
\]

\[
\xi^* \Delta_2 = h(S,k)E_{S,k} + h(S,l)E_{S,l} + \Gamma_2,
\]

where \( \Gamma \) and \( \Gamma_2 \) are relative effective Cartier divisors of \( W/B \) intersecting \( X \) transversally on \( E_{R,i} \) and \( E_{S,k} \), respectively.
Proof. Locally analytically, the map $\lambda$ is defined by the homomorphism of $K$-algebras
\[ K[[a, b, c]] \longrightarrow K[[t]], \]
sending $a$, $b$ and $c$ to $t$, where $a = 0$, $b = 0$ and $c = 0$ are local equations at $A$ of the strict transforms to $\tilde{C}^2$ of $C_i \times C_k$, $C_i \times C_l$ and $C_j \times C_k$. To show that $\rho$ is a smoothing of $C$, we need only show that $W$ is regular along $\xi^{-1} \psi^{-1}(A, T)$ for every reducible node $T$ of $X$. This follows from the regularity of $W_2$ in Lemma \[3.2\]

It follows that $\xi^* D$, $\xi^* \tilde{\Delta}_1$ and $\xi^* \tilde{\Delta}_2$ are Cartier divisors, the first supported on the special fiber of $\rho$. Also, the supports of $\xi^* \tilde{\Delta}_1$ and $\xi^* \tilde{\Delta}_2$ intersect $X$ only along $\xi^{-1} \psi^{-1}(A, R)$ and $\xi^{-1} \psi^{-1}(A, S)$, respectively. So we need only describe $\xi^* D$, $\xi^* \tilde{\Delta}_1$ and $\xi^* \tilde{\Delta}_2$ in a neighborhood of $\xi^{-1} \psi^{-1}(A, T)$ for each reducible node $T$ of $C$. Hence, we may use Lemma \[3.2\] and need only describe the restrictions to $\tilde{X}$ of the Cartier divisors on $Y_2$ corresponding to $D$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$.

So, fix a reducible node $T$. Let $z = 0$ and $z_\infty = 0$ be local equations for the Cartier divisors $C_{mT}$ and $C_{nT}$ of $C$ at $T$, respectively. We may assume $a = 0$, $b = 0$ and $c = 0$ are local equations at $A$ of the strict transforms to $\tilde{C}^2$ of $C_{rt,1} \times C_{st,1}$, $C_{rt,2} \times C_{st,2}$ and $C_{rt,3} \times C_{st,3}$ respectively.

The divisor $D$ is a sum of the $D_{r,s,m}$, and $\xi^* D_{r,s,m}$ is nonzero in a neighborhood of $\xi^{-1} \psi^{-1}(A, T)$ if and only if $m$ is either $mT$ or $nT$ and $(r, s)$ is equal to $(i, k)$, $(i, l)$ or $(j, k)$, or equivalently, to $(rt,1, st,1)$, $(rt,2, st,2)$ or $(rt,3, st,3)$. The six cases,

\[ (rt,1, st,1, mT), (rt,2, st,2, mT), (rt,3, st,3, mT), \]
\[ (rt,1, st,1, nT), (rt,2, st,2, nT), (rt,3, st,3, nT), \]
correspond exactly to the six cases in the heading of the first table of Lemma \[3.2\] in the same ordering, from which follows that the coefficients of $C_{mT}$, $C_{nT}$, $E_{T,mT}$ and $E_{T,nT}$ in the description of $\xi^* D$ are exactly those prescribed by \[14\] and \[15\].

Furthermore, as observed above, $\xi^* \tilde{\Delta}_1$ intersects $X$ only along $\xi^{-1} \psi^{-1}(A, R)$. So, assume $T = R$. The equations at $(A, T)$ of the strict transform of $\Delta_1$ to $\tilde{C}^2 \times_B C$ depend on the choice of the sequence $(rT,1, sT,1)$, $(rT,2, sT,2)$, $(rT,3, sT,3)$. There are six possible sequences:

\[ (i, k), (i, l), (j, k) \]
\[ (i, l), (i, k), (j, k) \]
\[ (i, k), (j, k), (i, l) \]
\[ (i, l), (j, k), (i, k) \]
\[ (j, k), (i, k), (i, l) \]
\[ (j, k), (i, l), (i, k) \]

There are also two cases: $mR = i$ and $nR = j$ or $mR = j$ and $nR = i$. If $mR = i$ and $nR = j$, the equations at $(A, T)$ of the strict transform of $\Delta_1$ to $\tilde{C}^2 \times_B C$
corresponding to each of the six sequences above are those on the headings of the last three columns of the second table of Lemma 3.2, each set of equations being repeated twice, more precisely,

\[
\begin{align*}
& z = ab, \quad z_\circ = c \\
& z = ab, \quad z_\circ = c \\
& z = ac, \quad z_\circ = b \\
& z = ac, \quad z_\circ = b \\
& z = bc, \quad z_\circ = a \\
& z = bc, \quad z_\circ = a.
\end{align*}
\]

On the other hand, if \(m_R = j\) and \(n_R = i\) we obtain the headings of the first three columns, just exchanging \(z\) and \(z_\circ\) in the above sequence of equations. The reasoning behind these conclusions is laid out right before the statement of Lemma 3.2. Now, it is just a matter of using the entries of the second table of Lemma 3.2 to conclude that \(\xi^* \tilde{\Delta}_1\) is of the stated form, with the coefficients of \(E_{R,i}\) and \(E_{R,j}\) as prescribed by \(10\).

The same proof works, \textit{mutatis mutandis}, to describe \(\xi^* \tilde{\Delta}_2\).

5 Admissible invertible sheaves

If \(E\) is a chain of rational curves and \(\mathcal{L}\) is an invertible sheaf on \(E\), then \(\mathcal{L}\) is determined by its restrictions to the irreducible components of \(E\), and thus by its multidegree. In particular, \(\mathcal{L} \cong \mathcal{O}_E\) if and only if \(\text{deg}(\mathcal{L}|_F) = 0\) for each component \(F \subseteq E\).

\textbf{Lemma 5.1.} Let \(E\) be a chain of rational curves of length \(n\). Let \(E_1\) and \(E_n\) denote the extreme curves. Let \(\mathcal{L}\) be an invertible sheaf on \(E\). Then the following statements hold:

1. \(\text{deg}(\mathcal{L}|_F) \geq -1\) for every subchain \(F \subseteq E\) if and only if \(\text{h}^1(E, \mathcal{L}) = 0\).
2. \(\text{deg}(\mathcal{L}|_F) \leq 1\) for every subchain \(F \subseteq E\) if and only if

\[
\text{h}^0(E, \mathcal{L}(-P - Q)) = 0
\]

for any two points \(P \in E_1\) and \(Q \in E_n\) on the nonsingular locus of \(E\).

\textit{Proof.} Let \(E_1, \ldots, E_n\) be the irreducible components of \(E\), ordered in such a way that \(#E_i \cap E_{i+1} = 1\) for \(i = 1, \ldots, n - 1\). We prove the statements by induction on \(n\). If \(n = 1\) all the statements follow from the knowledge of the cohomology of the sheaves \(\mathcal{O}_{E_i}\).

Suppose \(n > 1\). We show Statement 1. Assume that \(\text{deg}(\mathcal{L}|_F) \geq -1\) for every subchain \(F \subseteq E\). Consider the natural exact sequence

\[
0 \to \mathcal{L}|_{E_i}(-N) \to \mathcal{L} \to \mathcal{L}|_{E'_i} \to 0,
\]

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where $E'_1 := E - E_1$ and $N$ is the unique point of $E_1 \cap E'_1$. By induction, $h^1(E'_1, \mathcal{L}|_{E'_1}) = 0$. If $\deg \mathcal{L}|_{E_1} \geq 0$ then $h^1(E_1, \mathcal{L}|_{E_1}(-N)) = 0$ as well, and hence $h^1(E, \mathcal{L}) = 0$ from the long exact sequence in cohomology.

Suppose now that $\deg \mathcal{L}|_{E_1} < 0$. If $\deg \mathcal{L}|_{E_1} \geq 0$, we invert the ordering of the chain, and proceed as above. Thus we may suppose $\deg \mathcal{L}|_{E_1} < 0$ as well.

Since $\deg \mathcal{L}|_E \geq -1$, there is $i \in \{2, \ldots, n-1\}$ such that $\deg \mathcal{L}|_{E_i} \geq 1$. Let $F_1 := E_1 \cup \cdots \cup E_{i-1}$ and $F_2 := E_{i+1} \cup \cdots \cup E_n$. Consider the natural exact sequence

$$0 \to \mathcal{L}|_{E_i}(-N_1 - N_2) \to \mathcal{L} \to \mathcal{L}|_{F_1} \oplus \mathcal{L}|_{F_2} \to 0,$$

where $N_1$ and $N_2$ are the two points of intersection of $E_i$ with $E'_i := E - E_i$. By induction, $h^1(F_1, \mathcal{L}|_{F_1}) = h^1(F_2, \mathcal{L}|_{F_2}) = 0$. Also, since $\deg \mathcal{L}|_{E_i} \geq 1$, we have $h^1(E, \mathcal{L}|_{E_i}(-N_1 - N_2)) = 0$, and thus it follows from the long exact sequence in cohomology that $h^1(E, \mathcal{L}) = 0$ as well.

Assume now that $h^1(E, \mathcal{L}) = 0$. Then $h^1(F, \mathcal{L}|_F) = 0$ for every subchain $F \subseteq E$. By induction, $\deg(\mathcal{L}|_F) \geq -1$ for every proper subchain $F \subsetneq E$. Since $E$ is the union of two proper subchains, it follows that $\deg(\mathcal{L}) \geq -2$. Assume by contradiction that $\deg(\mathcal{L}) = -2$. Then $\deg(\mathcal{L}|_F) = -1$ for every proper subchain $F \subsetneq E$ containing $E_1$ or $E_n$. It follows that

$$\deg(\mathcal{L}|_{E_i}) = \begin{cases} 0 & \text{if } 1 < i < n, \\ -1 & \text{otherwise.} \end{cases}$$

But then $\mathcal{L}$ is the dualizing sheaf of $E$, and thus $h^1(E, \mathcal{L}) = 1$, reaching a contradiction. The proof of Statement 1 is complete.

Statement 2 is proved in a similar way. Alternatively, it is enough to observe that $\mathcal{O}_E(-P - Q)$ is the dualizing sheaf of $E$, and thus, by Serre Duality,

$$h^0(E, \mathcal{L}(-P - Q)) = h^1(E, \mathcal{L}^{-1}).$$

So Statement 2 follows from 1. \hfill \Box

Let $\phi: \mathcal{X} \to S$ be a family of connected curves. An $S$-flat coherent sheaf $\mathcal{I}$ on $\mathcal{X}$ is said to be a relatively torsion-free, rank-1 sheaf (of relative degree $d$) on $\mathcal{X}/S$ if the restriction of $\mathcal{I}$ to each geometric fiber of $\phi$ is torsion-free, rank-1 (of degree $d$). Let $\mathcal{E}$ be a locally free sheaf on $\mathcal{X}$ of constant rank and $\mathcal{I}$ a relatively torsion-free, rank-1 sheaf on $\mathcal{X}/S$. Let $\sigma: S \to \mathcal{X}$ be a section of $\phi$ through its smooth locus. We say that $\mathcal{I}$ is semistable (resp. stable, resp. $\sigma$-quasistable) with respect to $\mathcal{E}$ if, for every geometric point $s$ of $S$,

1. $\chi(I_s \otimes E_s) = 0$,

2. $\chi((I_s)_Y \otimes E_s|_Y) \geq 0$ for every proper subcurve $Y \subset X_s$ (resp. with equality never, resp. with equality only if $\sigma(s) \notin Y$),

where $I_s$ and $E_s$ denote the restrictions of $\mathcal{I}$ and $\mathcal{E}$ to the fiber $X_s$ of $\phi$ over $s$. Notice that it is enough to check Property 2 above for connected subcurves $Y$. 

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The above notions of stability coincide with those in Section 2.2 for an appropriate choice of component and polarization. In fact, let \( \omega \) be the dualizing sheaf of \( X_s \) and \( Z \) the component of \( X_s \) containing \( \sigma(s) \). If \( F \) is a vector bundle on \( X_s \) with \( \chi(I_s \otimes F) = 0 \), there is a a polarization \( \mathcal{L} \) for \( X_s \) satisfying

\[
e_Y = -\frac{\deg(F|_Y)}{\text{rk}(F)} + \frac{\deg(\omega|_Y)}{2}
\]

for every subcurve \( Y \) of \( X_s \). Then \( \mathcal{L} \) has degree \( \deg(I_s) \) and the sheaf \( I_s \) is semistable (resp. stable, resp. \( \sigma \)-quasistable) with respect to \( F \) if and only if \( I_s \) is semistable (resp. stable, resp. \( Z \)-quasistable) with respect to \( \mathcal{L} \). Conversely, given a polarization \( \mathcal{L} \) of \( X_s \) of degree \( \deg(I_s) \), there is a vector bundle \( F \) on \( X_s \) with \( \chi(I_s \otimes F) = 0 \) such that (18) holds for every subcurve \( Y \) of \( X_s \); see [24, Rmk. 1.16.

Let \( \psi: \mathcal{Y} \to \mathcal{X} \) be a proper morphism such that the composition \( \rho := \phi \circ \psi \) is another family of curves. We say that \( \psi \) is a semistable modification of \( \phi \) if for each geometric point \( s \) of \( S \) there are a collection of nodes \( \mathcal{N} \) of the fiber \( X_s \) of \( \phi \) over \( s \) and a map \( \eta: \mathcal{N} \to \mathbb{N} \) such that the induced map \( Y_s \to X_s \), where \( Y_s \) is the fiber of \( \rho \) over \( s \), is \( X_s \)-isomorphic to \( \mu_\eta: (X_s)_\eta \to X_s \).

Assume \( \psi \) is a semistable modification of \( \phi \). Let \( \mathcal{L} \) be an invertible sheaf on \( \mathcal{Y} \). We say that \( \mathcal{L} \) is \( \psi \)-admissible (resp. strongly \( \psi \)-admissible, resp. \( \psi \)-invertible) at a given \( s \in S \) if the restriction of \( \mathcal{L} \) to every chain of rational curves on \( \mathcal{Y} \) over a node of a geometric fiber of \( \phi \) over \( s \) has degree \(-1, 0 \) or \( 1 \) (resp. \(-1 \) or \( 0 \), resp. \( 1 \)). We say that \( \mathcal{L} \) is \( \psi \)-admissible (resp. strongly \( \psi \)-admissible, resp. \( \psi \)-invertible) if \( \mathcal{L} \) is so at every \( s \in S \). Notice that, if \( \mathcal{L} \) is strongly \( \psi \)-admissible, for every chain of rational curves on \( \mathcal{Y} \) over a node of a geometric fiber of \( \phi \), the degree of \( \mathcal{L} \) on each of the components of the chain is \( 0 \) but for at most one component where the degree is \(-1 \).

**Proposition 5.2.** Let \( \phi: \mathcal{X} \to S \) be a family of connected curves, \( \psi: \mathcal{Y} \to \mathcal{X} \) a semistable modification of \( \phi \) and \( \rho := \phi \psi \). Let \( \mathcal{L} \) be an invertible sheaf on \( \mathcal{Y} \) of relative degree \( d \) over \( S \). Then the following statements hold:

1. The points \( s \) of \( S \) at which \( \mathcal{L} \) is \( \psi \)-admissible (resp. strongly \( \psi \)-admissible, resp. \( \psi \)-invertible) form an open subset of \( S \).

2. \( \mathcal{L} \) is \( \psi \)-admissible if and only if \( \psi_* \mathcal{L} \) is a relatively torsion-free, rank-1 sheaf on \( \mathcal{X}/S \) of relative degree \( d \), whose formation commutes with base change. In this case, \( R^1 \psi_* \mathcal{L} = 0 \).

3. Let \( \sigma: S \to \mathcal{Y} \) be a section through the smooth locus of \( \rho \) such that \( \sigma(s) \) is not on any component of \( Y_s \) contracted by \( \psi_s \) for any geometric point \( s \) of \( S \). Let \( \mathcal{E} \) be a vector bundle on \( \mathcal{X} \). Then \( \mathcal{L} \) is semistable (resp. \( \sigma \)-quasistable, resp. stable) with respect to \( \psi^* \mathcal{E} \) if and only if \( \mathcal{L} \) is \( \psi \)-admissible (resp. strongly \( \psi \)-admissible, resp. \( \psi \)-invertible) and \( \psi_* \mathcal{L} \) is semistable (resp. \( (\psi \sigma) \)-quasistable, resp. stable) with respect to \( \mathcal{E} \).
Proof. All of the statements and hypotheses are local with respect to the étale topology of $S$. So we may assume $S$ is Noetherian and that there is an invertible sheaf $A$ on $\mathcal{X}$ that is relatively ample over $S$. Let $\tilde{A} := \psi^*A$.

For each geometric point $s$ of $S$, let $Y_s := \rho^{-1}(s)$ and $\psi_s := \psi|_{Y_s} : Y_s \to X_s$, with $X_s := \phi^{-1}(s)$.

We prove Statement 1 first. For each geometric point $s$ of $S$, let $E_s$ be the subcurve of $Y_s$ which is the union of all the components contracted by $\psi_s$, and let $\tilde{X}_s$ be the partial normalization of $X_s$ obtained as the union of the remaining components. Since $\psi|_{\tilde{X}_s} : \tilde{X}_s \to X_s$ is a finite map, it follows that $\tilde{A}|_{\tilde{X}_s}$ is ample, and thus $h^1(\tilde{X}_s, (\mathcal{L} \otimes \tilde{A}^{\otimes m_s})|_{\tilde{X}_s}(-\sum P_i)) = 0$ for every large enough integer $m_s$, where the sum runs over all the branch points of $\tilde{X}_s$ above $X_s$. Since $S$ is Noetherian, a large enough integer works for all $s$, that is, for every $m >> 0$,

$$h^1(\tilde{X}_s, (\mathcal{L} \otimes \tilde{A}^{\otimes m})|_{\tilde{X}_s}(-\sum P_i)) = 0 \quad \text{for every geometric point } s \text{ of } S. \quad (19)$$

So, for each large enough integer $m$ such that (19) holds, it follows from the long exact sequence in cohomology associated to the natural exact sequence

$$0 \to (\mathcal{L} \otimes \tilde{A}^{\otimes m})|_{\tilde{X}_s}(-\sum P_i) \to \mathcal{L}_s \otimes \tilde{A}_s^{\otimes m} \to (\mathcal{L} \otimes \tilde{A}^{\otimes m})|_{E_s} \to 0 \quad (20)$$

that

$$h^1(Y_s, \mathcal{L}_s \otimes \tilde{A}_s^{\otimes m}) = h^1(E_s, \mathcal{L} \otimes \tilde{A}^{\otimes m}|_{E_s}), \quad (21)$$

where $\mathcal{L}_s$ and $\tilde{A}_s$ are the restrictions of $\mathcal{L}$ and $\tilde{A}$ to $Y_s$. On the other hand, since $\tilde{A}$ is a pullback from $\mathcal{X}$, it follows that

$$h^1(E_s, \mathcal{L} \otimes \tilde{A}^{\otimes m}|_{E_s}) = \sum h^1(F, \mathcal{L}|_{F}) \quad \text{for every integer } m, \quad (22)$$

where the sum is over all the maximal chains $F$ of rational curves on $Y_s$ contracted by $\psi_s$. Putting together (21) and (22), it follows now from Lemma 5.1 that

$$h^1(Y_s, \mathcal{L}_s \otimes \tilde{A}_s^{\otimes m}) = 0 \quad (23)$$

if and only if $\deg(\mathcal{L}|_{F}) \geq -1$ for every chain $F$ of rational curves on $Y_s$ contracted by $\psi_s$. This is the case if $E_s$ is $\psi_s$-admissible.

It follows from semicontinuity of cohomology that the geometric points $s$ of $S$ such that $\mathcal{L}|_{Y_s}$ has degree at least $-1$ on every chain of rational curves of $Y_s$ contracted by $\psi_s$ form an open subset $S_1$ of $S$. Likewise, for each integer $n$, the geometric points $s$ of $S$ such that $\mathcal{L}^{\otimes n}|_{Y_s}$ has degree at least $-1$ on every chain of rational curves of $Y_s$ contracted by $\psi_s$ form an open subset $S_n$ of $S$. Then $S_1 \cap S_{-1}$ parameterizes those $s$ for which $\mathcal{L}|_{Y_s}$ is $\psi_s$-semistable, $S_1 \cap S_{-2}$ parameterizes those $s$ for which $\mathcal{L}|_{Y_s}$ is strongly $\psi_s$-semistable, and $S_2 \cap S_{-2}$ parameterizes those $s$ for which $\mathcal{L}|_{Y_s}$ is $\psi_s$-invertible.

We prove Statement 2 now. Assume for the moment that $\mathcal{L}$ is $\psi$-admissible. To show that $\psi_* \mathcal{L}$ is flat over $S$, we need only show that $\phi_*(\psi_* \mathcal{L} \otimes \tilde{A}^{\otimes m})$ is locally free for each $m >> 0$. By the projection formula, we need only show
that \( \rho_*(\mathcal{L} \otimes \hat{A}^{\otimes m}) \) is locally free for each \( m \gg 0 \). This follows from what we have already proved: For each large enough integer \( m \) such that (19) holds, also (23) holds for each geometric point \( s \) of \( S \), because \( \mathcal{L} \) is \( \psi \)-admissible.

Furthermore, taking the long exact sequence in higher direct images of \( \psi_* \) for the exact sequence (20) with \( m = 0 \), using (22) and that \( \psi|_{\tilde{X}_s} : \tilde{X}_s \to X_s \) is a finite map, it follows that \( R^1\psi_*\mathcal{L} = 0 \) for every geometric point \( s \) of \( S \). Since the fibers of \( \psi \) have at most dimension 1, the formation of \( R^1\psi_*\mathcal{L} \) commutes with base change, and thus \( R^1\psi_*\mathcal{L} = 0 \).

Another consequence of (23) holding for each geometric point \( s \) of \( S \) is that the formation of \( \rho_* (\mathcal{L} \otimes \hat{A}^{\otimes m}) \) commutes with base change for \( m \gg 0 \). We claim now that the base change map \( \lambda_T \mathcal{L} \to \psi_\phi \mathcal{L} \) is an isomorphism for each Cartesian diagram of maps

\[
\begin{array}{ccc}
\mathcal{Y}_T & \xrightarrow{\lambda_Y} & \mathcal{Y} \\
\psi_T & \downarrow & \downarrow \\
\mathcal{X}_T & \xrightarrow{\lambda_X} & \mathcal{X} \\
\phi_T & \downarrow & \downarrow \\
T & \xrightarrow{\lambda} & S.
\end{array}
\]

Indeed, since \( \mathcal{A} \) is relatively ample over \( S \), it is enough to check that the induced map

\[
\phi_{T*}(\lambda_T^* \psi_* \mathcal{L} \otimes \lambda_T^* \hat{A}^{\otimes m}) \to \phi_{T*}(\psi_{T*} \lambda_T^* \mathcal{L} \otimes \lambda_T^* \hat{A}^{\otimes m})
\]

is an isomorphism for \( m \gg 0 \). But, by the projection formula, the right-hand side is simply \( \phi_{T*} \psi_{T*} \lambda_T^* (\mathcal{L} \otimes \hat{A}^{\otimes m}) \). Also, since \( \psi_* \mathcal{L} \) is \( S \)-flat, the left-hand side is \( \lambda^* \phi_* (\psi_* (\mathcal{L} \otimes \hat{A}^{\otimes m}) \) for \( m \gg 0 \), whence equal to \( \lambda^* \phi_* (\mathcal{L} \otimes \hat{A}^{\otimes m}) \) by the projection formula. So, since the formation of \( \rho_* (\mathcal{L} \otimes \hat{A}^{\otimes m}) \) commutes with base change for \( m \gg 0 \), it follows that (24) is an isomorphism for \( m \gg 0 \), as asserted.

To prove the remainder of Statement 2 we may now assume that \( S \) is a geometric point. Let \( X := \mathcal{X} \) and \( Y := \mathcal{Y} \). We need only show that \( \psi_* \mathcal{L} \) is a torsion-free, rank-1 sheaf of degree \( d \) on \( X \) if and only if \( \mathcal{L} \) is \( \psi \)-admissible. Again, let \( E \) be the union of the components of \( Y \) contracted by \( \psi_* \), and let \( \tilde{X} \) be the union of the remaining components. Taking higher direct images under \( \psi \) in the natural exact sequences

\[
0 \to \mathcal{L}|_{\tilde{X}}(- \sum P_i) \to \mathcal{L} \to \mathcal{L}|_E \to 0,
\]
\[
0 \to \mathcal{L}|_E(- \sum P_i) \to \mathcal{L} \to \mathcal{L}|_{\tilde{X}} \to 0,
\]

where the sums run over the intersection points \( P_i \) of \( \tilde{X} \) and \( E \), and using that \( \psi|_{\tilde{X}} \) is a finite map, we get

\[
R^1\psi_* \mathcal{L} = R^1\psi_* \mathcal{L}|_E
\]
and the exact sequence
\[ 0 \to \psi_*\mathcal{L}|_E(-\sum P_i) \to \psi_*\mathcal{L} \to \psi_*\mathcal{L}|_X \to R^1\psi_*\mathcal{L}|_E(-\sum P_i) \to R^1\psi_*\mathcal{L} \to 0. \]

Since \( \psi|_X \) is also birational, \( \psi_*\mathcal{L}|_X \) is a torsion-free, rank-1 sheaf of degree \( \deg \mathcal{L}|_X + e \), where \( e \) is the number of points of \( X \) desingularized in \( X \), thus equal to the number of maximal chains of rational curves on \( Y \) contracted by \( \psi \). Since \( \psi_*\mathcal{L}|_E(-\sum P_i) \) is supported at finitely many points, it follows that \( \psi_*\mathcal{L} \) is torsion-free if and only if \( h^0(E, \mathcal{L}|_E(-\sum P_i)) = 0 \). The latter holds if and only if the degree of \( \mathcal{L} \) on each chain of rational curves in \( E \) is at most 1, by Lemma 5.1. Furthermore, if the latter holds, then \( R^1\psi_*\mathcal{L}|_E(-\sum P_i) \) has length \( e - \deg \mathcal{L}|_E \) by the Riemann–Roch Theorem. Since \( \deg \mathcal{L}|_X + \deg \mathcal{L}|_E = d \), it follows that \( \deg \psi_*\mathcal{L} = d \) if and only if \( R^1\psi_*\mathcal{L} = 0 \). By Lemma 5.1, the latter holds if and only if \( h^1(E, \mathcal{L}|_E) = 0 \), thus if and only if the degree of \( \mathcal{L} \) on each chain of rational curves in \( E \) is at least \(-1\), by Lemma 5.1. The proof of Statement 2 is complete.

We prove Statement 3 now. We may assume that \( S \) is a geometric point. Let \( X := \mathcal{X} \) and \( Y := \mathcal{Y} \). Let \( P \in Y \) be the image of the section \( \sigma \). Since \( \psi^*\mathcal{E} \) has degree 0 on every component of \( Y \) contracted by \( \psi \), and \( P \) does not lie on any of these components, it follows from the definitions that a semistable (resp. \( P \)-quasistable, resp. stable) sheaf has degree \(-1\), 0 or \( 1 \) (resp. \(-1\) or 0, resp. 0) on every chain of rational curves of \( Y \) contracted by \( \psi \).

We may now assume that \( \mathcal{L} \) is \( \psi \)-admissible. Let \( W \) be any connected subcurve of \( X \). Set \( W' := X - W \) and \( \Delta_W := W \cap W' \). Set \( \delta := \#\Delta_W \). Let \( V_1 := Y - \psi^{-1}(W') \) and \( V_2 := Y - \psi^{-1}(W) \). Let \( F_1, \ldots, F_r \) be the maximal chains of rational curves contained in \( \psi^{-1}(\Delta_W) \). Then \( 0 \leq r \leq \delta \).

Claim: \( (\psi_*\mathcal{L})_W \cong \psi_*((\mathcal{L}|_Z)_W) \) for a certain connected subcurve \( Z \subseteq Y \) such that:

1. \( V_1 \subseteq Z \subseteq \psi^{-1}(W) \).
2. For each connected subcurve \( U \subseteq Y \) such that \( V_1 \subseteq U \subseteq \psi^{-1}(W) \),
\[ \deg(\mathcal{L}|_U) \geq \deg(\mathcal{L}|_Z). \]

(Notice that Property 1 implies that \( P \in Z \) if and only if \( \psi(P) \in W \).)

Indeed, if \( W = X \), let \( Z := \psi^{-1}(W) \). Suppose \( W \neq X \). Then \( \delta > 0 \). Let \( M_1, \ldots, M_\delta \) be the points of intersection of \( V_1 \) with \( V_j := Y - V_1 \) and \( N_1, \ldots, N_\delta \) those of \( V_2 \) with \( V'_2 := Y - V_2 \).

Write \( F_i = F_{i,1} \cup \cdots \cup F_{i,e_i} \), where \( F_{i,j} \cap F_{i,j+1} \neq \emptyset \) for \( j = 1, \ldots, e_i - 1 \) and \( F_{i,1} \) intersects \( V_1 \). Up to reordering the \( M_i \) and \( N_i \), we may assume that \( F_{i,1} \) intersects \( V_1 \) at \( M_i \) and \( F_{i,e_i} \) intersects \( V_2 \) at \( N_i \) for \( i = 1, \ldots, r \). (Thus \( M_i = N_i \) for \( i = r + 1, \ldots, \delta \).) Up to reordering the \( F_i \), we may also assume that there are nonnegative integers \( u \) and \( t \) with \( u \leq t \) such that
\[
\deg(\mathcal{L}|_{F_i}) = \begin{cases} 
1 & \text{for } i = 1, \ldots, u \\
0 & \text{for } i = u + 1, \ldots, t \\
-1 & \text{for } i = t + 1, \ldots, r.
\end{cases}
\]
Up to reordering the $F_i$, we may assume there is an integer $b$ with $u \leq b \leq t$ such that, for each $i = u + 1, \ldots, t$, we have that $i > b$ if and only if $\deg(L|_{F_{i,j}}) = 0$ for every $j$ or the largest integer $j$ such that $\deg(L|_{F_{i,j}}) \neq 0$ is such that $\deg(L|_{F_{i,j}}) = -1$. Set $G_i := F_i$ for $i = b + 1, \ldots, r$. For each $i = u + 1, \ldots, b$, let $G_i := F_{i,1} \cup \cdots \cup F_{i,j-1}$, where $j$ is the largest integer such that $\deg(L|_{F_{i,j}}) = 1$, let $\hat{G}_i := F_i - G_i$ and denote by $B_i$ the point of intersection of $G_i$ and $\hat{G}_i$. (Notice that $1 < j \leq e_i$.) Let $B_i := M_i$ and $\hat{G}_i := F_i$ for $i = 1, \ldots, u$, and $B_i := N_i$ for $i = b + 1, \ldots, \delta$.

For $i = u + 1, \ldots, r$, since the degree of $L|_{\hat{G}_i}(B_i)$ on each subchain of $G_i$ is at most 1, it follows from Lemma 5.1 that

$$h^0(G_i, L|_{G_i}(-M_i)) = 0 \quad \text{for} \quad i = u + 1, \ldots, r. \quad (26)$$

Furthermore, for $i = 1, \ldots, b$, the total degree of $L|_{\hat{G}_i}$ is 1; thus, by Lemma 5.1 and the Riemann–Roch Theorem,

$$h^1(\hat{G}_i, L|_{\hat{G}_i}(-B_i - N_i)) = 0 \quad \text{for} \quad i = 1, \ldots, b. \quad (27)$$

Set

$$Z := V_1 \cup G_{u+1} \cup \cdots \cup G_r$$

and $Z' := Y - Z$. Put $\Delta_Z := Z \cap Z'$. Notice that $\Delta_Z = \{B_1, \ldots, B_\delta\}$. Also, notice that $Z$ is connected, and

$$\deg(L|_U) \geq \deg(L|_Z) = \deg(L|_{V_1}) - (b - u) - (r - t)$$

for each connected subcurve $U \subseteq Y$ such that $V_1 \subseteq U \subseteq \psi^{-1}(W)$.

We have three natural exact sequences:

$$0 \to L|_{Z'}(-\sum_{i=1}^\delta B_i) \to L \to L|_Z \to 0, \quad (28)$$

$$0 \to \bigoplus_{i=1}^b L|_{\hat{G}_i}(-B_i - N_i) \to L|_{Z'}(-\sum_{i=1}^\delta B_i) \to L|_{V_2}(-\sum_{i=1}^\delta B_i) \to 0, \quad (29)$$

$$0 \to \bigoplus_{i=u+1}^r L|_{G_i}(-M_i) \to L|_Z \to L|_{V_1} \to 0. \quad (30)$$

Since $L$ is $\psi$-admissible, so are $L|_{V_1}$ with respect to $\psi|_{V_1} : V_1 \to W$ and $L|_{V_2}$ with respect to $\psi|_{V_2} : V_2 \to W'$. Then $\psi_* L|_{V_1}$ is a torsion-free, rank-1 sheaf on $W$ and $R^1\psi_* (L|_{V_2}(-\sum B_i)) = 0$ by Statement 1.

Since $R^1\psi_* (L|_{V_2}(-\sum B_i)) = 0$, from (27) and the long exact sequence of higher direct images under $\psi$ of (29) and (28) we get that $R^1\psi_* (L|_{Z'}(-\sum B_i)) = 0$ and the natural map $\psi_* L \to \psi_* (L|_Z)$ is surjective. Also, it follows from (26) and the long exact sequence of higher direct images under $\psi$ of (30) that the natural map $\psi_* L \to \psi_* (L|_{V_1})$ is injective. Thus, since $\psi_* L|_{V_1}$ is a torsion-free, rank-1 sheaf on $W$, so is $\psi_* (L|_Z)$. And, since $\psi_* L \to \psi_* (L|_Z)$ is surjective, we get an isomorphism $(\psi_* L)_W \cong \psi_* (L|_Z)$, finishing the proof of the claim.
To prove the “only if” part of Statement 2, let $W$ be any connected subcurve of $X$. Let $Z$ be as in the claim. Since $L$ is admissible with respect to $\psi$, Statement 1 yields $R^1\psi_*L = 0$, and hence $R^1\psi_*({\mathcal L}|_Z) = 0$ from the long exact sequence of higher direct images under $\psi$ of (38). Thus, by the claim and the projection formula,

$$\chi((\psi_*{\mathcal L})_W \otimes {\mathcal E}|_W) = \chi(\psi_*({\mathcal L}|_Z) \otimes {\mathcal E}|_W) = \chi({\mathcal L}|_Z \otimes (\psi^*{\mathcal E})|_Z). \quad (31)$$

If $L$ is semistable (resp. $P$-quasistable, resp. stable) then $\chi({\mathcal L}|_Z \otimes (\psi^*{\mathcal E})|_Z) \geq 0$ (resp. with equality only if $Z = Y$ or $Z \not\ni P$, resp. with equality only if $Z = Y$). Now, if $Z = Y$ then $W = X$. Also, $P \in Z$ if and only if $\psi(P) \in W$. So (31) yields $\chi((\psi_*{\mathcal L})_W \otimes {\mathcal E}|_W) \geq 0$ (resp. with equality only if $W = X$ or $W \not\ni \psi(P)$, resp. with equality only if $W = X$).

As for the “if” part, let $U$ be a connected subcurve of $Y$. If $U$ is a union of components of $Y$ contracted by $\psi$, then $U$ is a chain of rational curves of $Y$ collapsing to a node of $X$, and hence ${\mathcal L}|_U$ has degree at least $-1$ (exactly $0$ if $L$ is $\psi$-invertible). Thus

$$\chi({\mathcal L}|_U \otimes \psi^*{\mathcal E}|_U) = \text{rk}(\mathcal E)\chi({\mathcal L}|_U) \geq 0,$$

with equality only if $L$ is not $\psi$-invertible.

Suppose now that $U$ contains a component of $Y$ not contracted by $\psi$. Then $W := \psi(U)$ is a connected subcurve of $X$. Let $\hat{U}$ be the smallest subcurve of $Y$ containing $U$ and $Y - \psi^{-1}(W')$, where $W' := X - W$. Then $\hat{U}$ is connected and contained in $\psi^{-1}(W)$. Furthermore, $\chi({\mathcal O}_U) - \chi({\mathcal O}_{\hat{U}})$ is the number of connected components of $\hat{U} - U$. Thus

$$\deg({\mathcal L}|_U) + \chi({\mathcal O}_U) \geq \deg({\mathcal L}|_{\hat{U}}) + \chi({\mathcal O}_{\hat{U}}), \quad (32)$$

with equality only if $L$ has degree $1$ on every connected component of $\hat{U} - U$. Let $Z$ be as in the claim. Notice that $\chi({\mathcal O}_{\hat{U}}) = \chi({\mathcal O}_Z)$. Since $\deg({\mathcal L}|_{\hat{U}}) \geq \deg({\mathcal L}|_Z)$ by the claim, using (31) and (32) we get

$$\chi({\mathcal L}|_U \otimes \psi^*{\mathcal E}|_U) = \text{rk}(\mathcal E)(\deg({\mathcal L}|_U) + \chi({\mathcal O}_U)) + \deg(\psi^*{\mathcal E}|_U) \geq \text{rk}(\mathcal E)(\deg({\mathcal L}|_{\hat{U}}) + \chi({\mathcal O}_{\hat{U}})) + \deg(\psi^*{\mathcal E}|_{\hat{U}}) = \text{rk}(\mathcal E)(\deg({\mathcal L}|_{\hat{U}}) + \chi({\mathcal O}_Z)) + \deg(\psi^*{\mathcal E}|_Z) \geq \text{rk}(\mathcal E)(\deg({\mathcal L}|_Z) + \chi({\mathcal O}_Z)) + \deg(\psi^*{\mathcal E}|_Z) = \chi({\mathcal L}|_Z \otimes (\psi^*{\mathcal E})|_Z) = \chi((\psi_*{\mathcal L})_W \otimes {\mathcal E}|_W).$$

Assume that $\psi_*L$ is semistable (resp. $(\psi\sigma)$-quasistable, resp. stable) with respect to $\mathcal E$. Then $\chi((\psi_*{\mathcal L})_W \otimes {\mathcal E}|_W) \geq 0$ (resp. with equality only if $W = X$ or $W \not\ni \psi(P)$, resp. with equality only if $W = X$). So $\chi({\mathcal L}|_U \otimes \psi^*{\mathcal E}|_U) \geq 0$. Suppose $\chi({\mathcal L}|_U \otimes \psi^*{\mathcal E}|_U) = 0$. Then $\chi((\psi_*{\mathcal L})_W \otimes {\mathcal E}|_W) = 0$ and equality holds in (32). If $W \not\ni \psi(P)$ then $U \not\ni P$. Suppose $W = X$. Then $\hat{U} = Y$. If $U \neq Y$ then $L$ has degree $1$ on each connected component of $Y - U$, and thus $L$ is not strongly admissible. \qed
Proposition 5.3. Let $X$ be a connected curve. Let $\psi: Y \to X$ be a semistable modification of $X$. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on $Y$ which are $\psi$-admissible. Assume that $\mathcal{M} \otimes \mathcal{L}^{-1}$ is a twister of the form

$$O_Y \left( \sum c_E E \right), \ c_E \in \mathbb{Z},$$

where the sum runs over the set of components $E$ of $Y$ contracted by $\psi$. Then $\psi_* \mathcal{L} \simeq \psi_* \mathcal{M}$.

Proof. Set $T := M \otimes L - 1$. Let $R$ be the set of smooth, rational curves contained in $Y$ and contracted by $\psi$. If $R = \emptyset$, then $T = O_Y$ and thus $L \cong M$. Suppose $R \neq \emptyset$. Let $K$ be the set of maximal chains of rational curves contained in $R$.

Claim: For every $F \in K$ and every two components $E_1, E_2 \subseteq F$ such that $E_1 \cap E_2 \neq \emptyset$, we have $|c_{E_1} - c_{E_2}| \leq 1$. In addition, if $E$ is an extreme component of $F$, then $|c_E| \leq 1$.

Indeed, let $E_1, \ldots, E_n$ be the components of $F$, ordered in such a way that $\#E_i \cap E_{i+1} = 1$ for $i = 1, \ldots, n - 1$. Since $\mathcal{L}$ and $\mathcal{M}$ are admissible, $|\deg_G T| \leq 2$ for every subchain $G$ of $F$. Set $c_{E_0} := c_{E_{n+1}} := 0$. We will reason by contradiction. Thus, up to reversing the order of the $E_i$, we may assume that $c_{E_i} - c_{E_{i+1}} \geq 2$ for some $i \in \{0, \ldots, n\}$. Then

$$c_{E_i} \leq c_{E_{i-1}} \leq \cdots \leq c_{E_1} \leq c_{E_0} = 0,$$

because, if $c_{E_j} > c_{E_{j-1}}$ for some $j \in \{1, \ldots, i\}$, then

$$\deg_{E_j \cup \cdots \cup E_i} T = c_{E_{j-1}} - c_{E_j} + c_{E_{i+1}} - c_{E_i} < -2.$$

Similarly, $c_{E_{i+1}} \geq c_{E_{i+2}} \geq \cdots \geq c_{E_n} \geq c_{E_{n+1}} = 0$. But then

$$0 \leq c_{E_{i+1}} < c_{E_i} \leq 0,$$

a contradiction that proves the claim.

Now, for each $F \in K$, let $F^\dagger$ be the (possibly empty) union of components $E \subseteq F$ such that $c_E = 0$. For each connected component $G$ of $F - F^\dagger$ and irreducible components $E_1, E_2 \subseteq G$, it follows from the claim that $c_{E_1} - c_{E_2} > 0$. Let $K^+$ (resp. $K^-$) be the collection of connected components $G$ of $F - F^\dagger$ for $F \in K$ such that $c_E > 0$ (resp. $c_E < 0$) for every irreducible component $E \subseteq G$.

Notice that, again by the claim,

$$c_E = \begin{cases} 1 & \text{if } E \text{ is an extreme component of some } G \in K_F^+ \\ -1 & \text{if } E \text{ is an extreme component of some } G \in K_F^- \end{cases} \quad (33)$$

So, being $\mathcal{L}$ and $\mathcal{M}$ admissible,

$$\deg_G \mathcal{L} = -\deg_G \mathcal{M} = \begin{cases} 1 & \text{if } G \in K^+ \\ -1 & \text{if } G \in K^- \end{cases} \quad (34)$$

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Define
\[ W^+ := Y - \cup_{G \in K^+} G, \quad W^- := Y - \cup_{G \in K^-} G, \quad W := Y - \cup_{G \in K^+ \cup K^-} G. \]

For each \( G \in K^+ \cup K^- \), let \( N_G \) and \( N'_G \) denote the points of \( G \cap Y - G \), and put
\[ D^+: = \sum_{G \in K^+} (N_G + N'_G) \quad \text{and} \quad D^- := \sum_{G \in K^-} (N_G + N'_G). \]

We may view \( D^+ \) and \( D^- \) as divisors of \( W \). Thus, by (33),
\[ M|_W \simeq L|_W (D^+ - D^-). \] (35)

Consider the natural diagram
\[
\begin{array}{cccc}
0 & \rightarrow & \bigoplus_{G \in K^+} L|_G(-N_G - N'_G) & \rightarrow & L & \rightarrow & L|_{W^+} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & L|_W(-D^-) & & L|_G & & \bigoplus_{G \in K^-} L|_G & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & & \\
\end{array}
\] (36)

where the horizontal and vertical sequences are exact. By (34) and Lemma 5.1 and using the Riemann–Roch Theorem,
\[ R^i \psi_* L|_G(-N_G - N'_G) = H^i(G, L|_G(-N_G - N'_G)) \otimes O_{\psi(G)} = 0 \]
for \( G \in K^+ \) and \( i = 0, 1 \), whereas
\[ \psi_* L|_G = H^0(G, L|_G) \otimes O_{\psi(G)} = 0 \quad \text{for} \quad G \in K^- . \]

Hence, it follows from Diagram (36), by considering the associated long exact sequences in higher direct images of \( \psi \), that
\[ \psi_* L \simeq (\psi|_W)_* L|_W(-D^-). \] (37)

Consider a second diagram, similar to Diagram (36), but with the roles of \( K^+ \) and \( K^- \), and thus of \( D^+ \) and \( D^- \), reversed, and \( M \) substituted for \( L \). As before,
\[ R^i \psi_* M|_G(-N_G - N'_G) = H^i(G, M|_G(-N_G - N'_G)) \otimes O_{\psi(G)} = 0 \]
for $G \in \mathcal{K}^-$ and $i = 0, 1$, whereas

$$
\psi_* \mathcal{M}|_G \simeq H^0(G, \mathcal{M}|_G) \otimes \mathcal{O}_{\psi(G)} = 0 \text{ for } G \in \mathcal{K}^+.
$$

Hence, taking the associated long exact sequences,

$$
\psi_* \mathcal{M} \simeq (\psi|_W)_* \mathcal{M}|_W (-D^+).
$$

Combining (35), (37) and (38), we get $\psi_* \mathcal{L} \simeq \psi_* \mathcal{M}$. □

6 The degree-2 Abel map

6.1 Corrections

Recall the notation: $C$ is a curve with irreducible components $C_1, \ldots, C_p$ defined over $K$, an algebraically closed field, $\hat{C}$ is the smooth locus of $C$, and $\mathcal{C}/B$ is a smoothing of $C$ with regular total space $\mathcal{C}$, the smooth locus of which is $\hat{C}$. Also, $\mathcal{P}$ is an invertible sheaf on $C$ and $\mathcal{N}(C)$ is the set of reducible nodes of $C$.

We would like to resolve the rational map

$$
\alpha^2_{\mathcal{C}/B} : \mathcal{C}^2 \rightarrow \mathcal{J}
$$

given by (4). We will reduce the resolution of the above map to a combinatorial question.

As seen in Subsection 2.3, for each pair of integers $(i, k)$ such that $1 \leq i, k \leq p$, there is a formal sum

$$
Z_{(i,k)} = \sum_{m=1}^{p} w_{(i,k)}(m) C_m
$$

of components of $C$ such that

$$
\mathcal{P}|_C \otimes \mathcal{O}_C(-Q_1 - Q_2) \otimes \mathcal{O}_C(-Z_{(i,k)})
$$

is $C_1$-quasistable for every $Q_1 \in \hat{C}_i$ and $Q_2 \in \hat{C}_k$, where $\hat{C}_j := C_j \cap \hat{C}$ for every $j$. The formal sum is not unique, but differs from another by a multiple of the sum of all the components of $C$. Given two components $C_m$ and $C_n$ of $C$, set

$$
\delta_{(i,k)}(m, n) := w_{(i,k)}(m) - w_{(i,k)}(n).
$$

Note that $\delta_{(i,k)}(m, n)$ remains the same if $Z_{(i,k)}$ is replaced by another formal sum having the same property. Also,

$$
\delta_{(i,k)}(m, n) = \delta_{(k,i)}(m, n) = -\delta_{(k,i)}(n, m) = -\delta_{(i,k)}(n, m).
$$

We emphasize the numerical nature of the $\delta_{(i,k)}(m, n)$ by the following Definition and Proposition:

**Definition 6.1.** Let $\Gamma$ be a connected graph without loops. Let $V$ be its set of vertices and $E$ its set of edges. For each $i \in V$, let

$$
\xi_i : V \rightarrow \mathbb{Z}
$$

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be defined by letting $c_i(j)$ be the number of edges of $\Gamma$ with ends $i$ and $j$, for $j \neq i$, and

$$c_i(i) := -\sum_{j \neq i} c_i(j).$$

Let $q: V \to \mathbb{Z}$ and $e: V \to \mathbb{Q}$ be such that

$$f := \sum_{i \in V} e(i) = -2 + \sum_{i \in V} q(i).$$

For each proper nonempty subset $I \subset V$, let $k_I$ be the number of edges of $\Gamma$ with one end inside $I$ and the other end outside. Let $v \in V$. A function $d: V \to \mathbb{Z}$ satisfying $\sum_{i \in V} d(i) = f$ is called $v$-quasistable with respect to $e$ if

$$\sum_{i \in I} (d(i) - e(i)) \geq -\frac{k_I}{2}$$

for every proper nonempty subset $I \subset V$, with equality only if $v \notin I$.

For each $u \in V$, let $\hat{\delta}_u: V \to \mathbb{Z}$ be defined by letting $\hat{\delta}_u(w) := 0$ if $w \neq u$ and $\hat{\delta}_u(u) := 1$. As pointed out in Subsection 2.3, for each pair $(i, k) \in V^2$, there exists $w_{(i, k)}: V \to \mathbb{Z}$ such that

$$q - \hat{\delta}_i - \hat{\delta}_k - \sum_{m \in V} w_{(i, k)}(m)e_m$$

is $v$-quasistable with respect to $e$. The functions $w_{(i, k)}$ are not uniquely defined, but their differences, $w_{(i, k)}(m) - w_{(i, k)}(n)$, are.

Let

$$\hat{\delta}: V^4 \to \mathbb{Z}$$

be defined by

$$\hat{\delta}(i, k, m, n) := w_{(i, k)}(m) - w_{(i, k)}(n)$$

for each $(i, k, m, n) \in V^4$. The following conditions hold:

$$\hat{\delta}(i, k, m, n) = \hat{\delta}(k, i, m, n) = -\hat{\delta}(k, i, n, m) = -\hat{\delta}(i, k, n, m)$$

for all $(i, k, m, n) \in V^4$.

The data $(\Gamma, e, q, v)$ are called degree-2 Abel data. And $\hat{\delta}$ is called their correction function.

**Proposition 6.2.** Let $\Gamma$ be the essential dual graph of $C$ and $V$ its set of vertices. Let $e: V \to \mathbb{Q}$ and $q: V \to \mathbb{Z}$ be defined by setting $e(C_i) = e_i$ and $q(C_i) := \deg(P|_{C_i})$ for each $i = 1, \ldots, p$. Set $v := C_1$. Then $(\Gamma, e, q, v)$ are degree-2 Abel data and their correction function $\hat{\delta}$ satisfies

$$\hat{\delta}(C_1, C_k, C_m, C_n) = \delta_{(i, k)}(m, n)$$

for all $i, k, m, n$ between 1 and $p$.

**Proof.** Immediate.
6.2 Admissibility

Let \( \phi : \tilde{C}^3 \to C^2 \) and \( \psi : \tilde{C}^3 \to \tilde{C}^2 \times_B C \) be good partial desingularizations. Let \( \rho : \tilde{C}^3 \to C^2 \) be the composition of \( \psi \) with the projection \( p_1 : \tilde{C}^2 \times_B C \to \tilde{C}^2 \). It follows from Proposition 4.5 that the fibers of \( \rho \) over closed points of \( \tilde{C}^2 \) are isomorphic to \( C \), to \( C(1) \) or to \( C(2) \).

For each pair \((i,k)\) such that \( 1 \leq i, k \leq p \), denote by \( Z_{(i,k)} \) the weighted sum over \( m = 1, \ldots, p \) of the strict transforms to \( \tilde{C}^3 \) of the triple products \( C_i \times C_k \times C_m \), each with weight \( w_{(i,k)}(m) \). Also, let \( \Delta_l \) be the inverse image of the diagonal \( \Delta \subseteq C^2 \) under the projection \( C^3 \to C^2 \) onto the product over \( B \) of the \( l \)-th and last factors of \( C^3 \), and \( \tilde{\Delta}_l \) be the strict transform of \( \Delta_l \) to \( \tilde{C}^3 \) for \( l = 1, 2 \). The \( Z_{(i,k)} \) are Cartier divisors of \( \tilde{C}^3 \), whereas the \( \tilde{\Delta}_l \) are so at least along the fibers of \( \rho \) isomorphic to \( C(2) \), again by Proposition 4.5.

Recall the natural subscheme \( F_2 \subset \mathbb{P} C^2(\mathcal{I}_{\Delta_l(\xi^2)}) \times_B C \) defined in Subsection 2.4. Let \( \tilde{F}_2 \) be the strict transform of \( F_2 \) to \( \tilde{C}^3 \). It follows from Proposition 4.5 that wherever \( \tilde{F}_2 \) is not \( \rho \)-flat,

\[
\mathcal{I}_{\tilde{F}_2}[3] = \mathcal{I}_{\tilde{\Delta}_1}[3] \mathcal{I}_{\tilde{\Delta}_2}[3] = \mathcal{I}_{\tilde{\Delta}_1}[3] \otimes \mathcal{I}_{\tilde{\Delta}_2}[3]
\]

and both \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_2 \) are Cartier. In any case, it follows that the sheaf of ideals \( \mathcal{I}_{\tilde{F}_2}[3] \) is a relatively torsion-free, rank-1 sheaf of relative degree \(-2\) on \( \tilde{C}^3 / \tilde{C}^2 \) agreeing with \( \mathcal{O}_{\tilde{C}^3}(\tilde{\Delta}_1 - \tilde{\Delta}_2) \) around the fibers of \( \rho \) isomorphic to \( C(2) \).

Set

\[
\mathcal{L}_\psi := \psi^* p_2^* \mathcal{P} \otimes \mathcal{I}_{\tilde{F}_2}[3] \otimes \mathcal{O}_{\tilde{C}^3}(- \sum_{i,k=1}^p Z_{(i,k)}),
\]

where \( p_2 : \tilde{C}^2 \times_B C \to C \) is the projection.

**Lemma 6.3.** Let \( \phi : C^2 \to C^2 \) and \( \psi : \tilde{C}^3 \to \tilde{C}^2 \times_B C \) be good partial desingularizations. Then \( \psi \) is a semistable modification of the projection \( p_1 : \tilde{C}^2 \times_B C \to \tilde{C}^2 \). In addition,

1. \( \mathcal{L}_\psi \) is \( \psi \)-admissible if and only if \( \mathcal{L}_\psi \) is \( \psi \)-admissible at each distinguished point of \( \phi^{-1}(R,S) \) for every \( R, S \in \mathcal{N}(C) \).

Furthermore, let \( \rho : \tilde{C}^3 \to \tilde{C}^2 \) be the composition of \( \psi \) with \( p_1 \). For each fiber \( X \) of \( \rho \) over a closed point of \( \tilde{C}^2 \), let \( \mu_X : X \to C \) be the restriction of \( \psi \) to \( X \) composed with the projection \( p_2 : \tilde{C}^2 \times_B C \to C \). If \( \mathcal{L}_\psi \) is \( \psi \)-admissible then

2. \( \mu_X^* \mathcal{L}_\psi \) defines a map to \( \mathcal{J} \) if and only if \( \mu_X^* (\mathcal{L}_\psi|_X) \) is \( C_1 \)-quasistable with respect to \( \phi \) for every fiber \( X \) over a distinguished point of \( \phi^{-1}(R,S) \) for \( R, S \in \mathcal{N}(C) \).

**Proof.** The first statement follows from Lemma 4.5, which says that the fibers of \( \rho \) over closed points of \( \tilde{C}^2 \) are isomorphic to \( C \), \( C(1) \) or \( C(2) \). There is a stratification \( \tilde{C}^2 = U_{-1} \cup U_0 \cup U_1 \cup U_2 \) by locally closed subschemes such that the fibers of \( \rho \) over \( U_i \) are isomorphic to \( C(i) \) for \( i = 0, 1, 2 \), and nonsingular
for $i = -1$. (Of course, $U_{-1}$ is the fiber of $\overline{C}^2$ over the generic point of $B$.) By Proposition 4.5, the subscheme $U_2$ is simply the collection of distinguished points on the fibers of $\phi$ over pairs of reducible nodes of $C$.

By preservation of degree, the degrees of $L_\psi|_X$ on the exceptional curves of $\mu_X : X \to C$ are the same for every fiber $X$ of $\rho$ over a point on a connected component of $U_1$. Now, every connected component of $U_1$ has a point of $U_2$ essentially torsion-free, rank-1 sheaf on $X$. Thus, again by preservation of degree, if $L_\psi|_X$ is $\mu_X$-admissible for the fibers $X$ over points on $U_2$, then so is $L_\psi|_X$ for the fibers $X$ over points on $U_1$, and thus so is $L_\psi$ with respect to $\psi$. This proves the “if” part of Statement 1. The “only if” part is trivial.

Suppose now that $L_\psi$ is $\psi$-admissible. By Proposition 5.2, $\psi_*L_\psi$ is a relatively torsion-free, rank-1 sheaf on $\overline{C}^2 \times_B C/\overline{C}^2$ of relative degree $f$, with formation commuting with base change. We prove Statement 2. Since

$$(\psi_*L_\psi)|_{\mu^{-1}_\pi(A)} \cong \mu_{\rho^{-1}(A)}^*(L_\psi|_{\rho^{-1}(A)})$$

for every $A \in \overline{C}^2$, only its “if” part is nontrivial. So assume $\mu_X^*(L_\psi|_X)$ is $C_1$-quasistable with respect to $\epsilon$ for every fiber $X$ over a point on $U_2$. Since quasistability is an open property by [12], Prop. 34, p. 3071, $\mu_X^*(L_\psi|_X)$ is $C_1$-quasistable with respect to $\epsilon$ for every fiber $X$ over a closed point of $\overline{C}^2$ on a neighborhood of $U_2$. Any such neighborhood intersects all connected components of $U_0$ and of $U_1$. But quasistability is a numerical condition, and since, by preservation of degree, the multidegree of $L_\psi|_X$ is constant as $X$ varies as a fiber on each connected component of $U_0$ or $U_1$, if $\mu_X^*(L_\psi|_X)$ is $C_1$-quasistable with respect to $\epsilon$ for some such fiber $X$, then so is $\mu_X^*(L_\psi|_X)$ for any such fiber $X$. So $\psi_*L$ defines a map to $\overline{C}^2$.

**Definition 6.4.** Let $(\Gamma, \omega, q, v)$ be degree-2 Abel data. Let $V$ be the set of vertices and $E$ the set of edges of $\Gamma$. A resolution of the degree-2 Abel data is a map

$$\mathcal{E} : E^2 \to V^2$$

that assigns to each pair of edges $(e_1, e_2)$ of $\Gamma$ a pair of vertices $(v_1, v_2)$ where $v_1$ is an end of $e_1$ and $v_2$ is an end of $e_2$, and takes the diagonal of $E^2$ to the diagonal of $V^2$.

Two resolutions $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be equivalent on $F \subseteq E^2$ if for each pair of edges $(e_1, e_2) \in F$, either $v_{1,1} = v_{2,1}$ and $v_{1,2} = v_{2,2}$ or $v_{1,1} \neq v_{2,1}$ and $v_{1,2} \neq v_{2,2}$, where $\mathcal{E}_i(e_1, e_2) = (v_{i,1}, v_{i,2})$ for $i = 1, 2$. They are simply said to be equivalent if they are equivalent on $E^2$. Given a resolution $\mathcal{E}$, an equivalent resolution $\mathcal{E}'$ is obtained by the condition that $\mathcal{E}'(e_1, e_2)_i$ and $\mathcal{E}(e_1, e_2)_i$ be the distinct vertices of $e_i$ for each $(e_1, e_2) \in E^2$ and $i = 1, 2$. We call $\mathcal{E}$ the mirror resolution of $\mathcal{E}$.
Let $\hat{\delta}$ be the correction function of $(\Gamma, q, q, v)$. Let $\Gamma$ be a resolution of the Abel data. We call $\Gamma$ admissible at $(e_1, e_2) \in E^2$ if, letting $(v_1, v_2) = \Gamma(e_1, e_2)$ and $(w_1, w_2) = \hat{\Gamma}(e_1, e_2)$, the following three conditions hold:

1. For each edge $e$ distinct from $e_1$ and $e_2$, letting $m$ and $n$ be the ends of $e$, the following inequalities hold:

$$
|\delta(v_1, v_2, m, n) - \delta(w_1, v_2, m, n)| \leq 1,
|\delta(v_1, v_2, m, n) - \delta(v_1, w_2, m, n)| \leq 1,
|\delta(w_1, v_2, m, n) - \delta(v_1, w_2, m, n)| \leq 1,
|\delta(w_1, w_2, m, n) - \delta(w_1, v_2, m, n)| \leq 1,
|\delta(w_1, w_2, m, n) - \delta(v_1, w_2, m, n)| \leq 1.
$$

2. If $e_1 \neq e_2$ then

$$
|\delta(v_1, v_2, v_1, w_1) - \delta(w_1, v_2, v_1, w_1) - 1| \leq 1,
|\delta(v_1, v_2, v_1, w_1) - \delta(v_1, w_2, v_1, w_1)| \leq 1,
|\delta(w_1, v_2, v_1, w_1) - \delta(v_1, w_2, v_1, w_1) + 1| \leq 1,
|\delta(w_1, w_2, v_1, w_1) - \delta(v_1, w_2, v_1, w_1)| \leq 1,
|\delta(w_1, w_2, v_1, w_1) - \delta(v_1, w_2, v_1, w_1) + 1| \leq 1,
|\delta(v_1, v_2, v_2, w_2) - \delta(v_1, v_2, w_2) + 1| \leq 1,
|\delta(v_1, v_2, v_2, w_2) - \delta(w_1, v_2, w_2)| \leq 1,
|\delta(w_1, v_2, v_2, w_2) - \delta(v_1, v_2, w_2) - 1| \leq 1,
|\delta(w_1, v_2, v_2, w_2) - \delta(v_1, v_2, v_2, w_2) + 1| \leq 1,
|\delta(v_1, v_2, v_2, w_2) - \delta(w_1, v_2, v_2, w_2)| \leq 1.
$$

3. If $e_1 = e_2$

$$
|\delta(v_1, v_1, v_1, w_1) - \delta(v_1, v_1, v_1, w_1) + 1| \leq 1,
|\delta(v_1, v_1, v_1, w_1) - \delta(w_1, v_1, v_1, w_1)| \leq 1,
|\delta(w_1, v_1, v_1, w_1) - \delta(v_1, v_1, v_1, w_1) - 1| \leq 1,
|\delta(w_1, v_1, v_1, w_1) - \delta(w_1, v_1, v_1, w_1)| \leq 1.
$$

(Notice that in this case $(v_1, w_1) = (v_2, w_2)$.)

We say that $\Gamma$ is admissible if $\Gamma$ is admissible at every $(e_1, e_2) \in E^2$. Notice that $\Gamma$ is admissible if and only if any other resolution equivalent to $\Gamma$ is admissible.

**Theorem 6.5.** Let $\Gamma$ be the essential dual graph of $C$. Let $V$ be its set of vertices and $E$ be its set of edges. Let $\varepsilon: V \to Q$ and $q: V \to Z$ be defined by setting $\varepsilon(C_i) := e_i$ and $q(C_i) := \deg(P|_{C_i})$ for each $i = 1, \ldots, p$. Set $v := C_1$. Let $\phi: \tilde{C}^2 \to C^2$ be a good partial desingularization. Let $\Gamma: E^2 \to V^2$ be a function “defined” by sending each pair of reducible nodes $(R, S)$ to $(C_i, C_k)$, where $(R, S) \in C_i \times C_k$ and the strict transform to $\tilde{C}^2$ of $C_i \times C_k$ does not contain $\phi^{-1}(R, S)$. Then:
1. \( \rho \) is a resolution of \((\Gamma, q, q, v)\). Any other function satisfying the same condition “defining” \( \rho \) is equivalent to it.

Let \( \psi: \tilde{\mathbb{C}}^3 \to \tilde{\mathbb{C}}^2 \times_B \mathbb{C} \) be a good partial desingularization and \( \rho: \tilde{\mathbb{C}}^3 \to \tilde{\mathbb{C}}^2 \) the composition of \( \psi \) with the first projection \( p_1: \tilde{\mathbb{C}}^2 \times_B \mathbb{C} \to \tilde{\mathbb{C}}^2 \).

2. For each \( R, S \) reducible nodes of \( C \), the sheaf \( L_\psi \) is \( \psi \)-admissible at the two distinguished points of \( \phi^{-1}(R, S) \) if and only if \( \rho \) is admissible at \((R, S)\).

3. \( L_\psi \) is \( \psi \)-admissible if and only if \( \rho \) is admissible.

Proof. Since \( \phi \) is a composition of blowups, the first along the diagonal, the strict transform to \( \tilde{\mathbb{C}}^2 \) of \( C_1 \times C_j \) does not contain \( \phi^{-1}(R, R) \) for any \( R \in \mathcal{N}(C) \). So \( \rho \) is a resolution. Furthermore, let \((R, S) \in \mathcal{N}(C)^2 \) and \( G := \phi^{-1}(R, S) \). Let \( C_1 \) and \( C_j \) be the two components containing \( R \) and \( C_k \) and \( C_l \) those containing \( S \). Then the strict transform to \( \tilde{\mathbb{C}}^2 \) of \( C_1 \times C_k \) does not contain \( G \) if and only if that of \( C_j \times C_j \) does not contain \( G \) if and only if that of \( C_j \times C_k \) contains \( G \) if and only if that of \( C_1 \times C_1 \) contains \( G \). So, any other function satisfying the same condition “defining” \( \rho \) is equivalent to it.

We prove Statement 2 now. Let \( \tilde{\rho} \) be the correction function of \((\Gamma, q, q, v)\).

Let \((R, S)\) be a pair of reducible nodes of \( C \). Let \( C_1 \) and \( C_j \) be the two components containing \( R \) and \( C_k \) and \( C_l \) those containing \( S \). Assume that \( \tilde{\rho}(R, S) = (C_1, C_k) \). Let \( A_1 \) and \( A_2 \) be the distinguished points of \( \phi^{-1}(R, S) \).

Assume that \( A_1 \) is the point of intersection of the strict transforms to \( \tilde{\mathbb{C}}^2 \) of \( C_1 \times C_k \) and \( C_j \times C_l \), whereas \( A_2 \) is the point of intersection of the strict transforms to \( \tilde{\mathbb{C}}^2 \) of \( C_j \times C_j \) and \( C_1 \times C_l \).

The components of \( C \) will be viewed as components of \( X_1 \) and \( X_2 \). As for the remaining components, let \( T \) be a reducible node of \( C \). Let \( C_m \) and \( C_n \) be the components of \( C \) containing \( T \). For each \( a = 1, 2 \), let \( E_{a,T,m} \) (resp. \( E_{a,T,n} \)) be the irreducible component of \( \psi^{-1}(A_{a,T}) \) intersecting \( C_m \) (resp. \( C_n \)). We need to understand when

\[
\begin{align*}
|\deg(L_\psi|_{E_{a,T,m}})| &\leq 1, \\
|\deg(L_\psi|_{E_{a,T,n}})| &\leq 1, \\
|\deg(L_\psi|_{E_{a,T,m}}) + \deg(L_\psi|_{E_{a,T,n}})| &\leq 1 \\
\end{align*}
\tag{40}
\]

for each \( T \in \mathcal{N}(C) \) and \( a = 1, 2 \).

Assume first that \( T \neq R \) and \( T \neq S \). Then the degree of \( L_\psi \) on \( E_{a,T,m} \) or \( E_{a,T,n} \) is the same as the degree of

\[
\mathcal{M} := \mathcal{O}_{\mathbb{C}}^3(-Z_{(i,k)} - Z_{(i,l)} - Z_{(j,k)} - Z_{(j,l)}).
\]

By Proposition 10 independently of \( \psi \), Inequalities (10) are satisfied for \( a = 1 \) if and only if

\[
\begin{align*}
|\delta_{(i,k)}(m, n) - \delta_{(j,k)}(m, n)| &\leq 1, \\
|\delta_{(i,j)}(m, n) - \delta_{(i,l)}(m, n)| &\leq 1, \\
|\delta_{(j,k)}(m, n) - \delta_{(i,l)}(m, n)| &\leq 1,
\end{align*}
\]

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and are satisfied for \( a = 2 \) if and only if
\[
|\delta_{(j,l)}(m,n) - \delta_{(j,k)}(m,n)| \leq 1, \\
|\delta_{(j,l)}(m,n) - \delta_{(i,l)}(m,n)| \leq 1, \\
|\delta_{(j,k)}(m,n) - \delta_{(i,l)}(m,n)| \leq 1.
\]

Assume now that \( T = R \) but \( T \neq S \). Use again Proposition \ref{proposition:4.6}. There are several cases to be checked but all of them yield that, independently of \( \psi \), Inequalities \eqref{eq:40} are satisfied for \( a = 1 \) if and only if
\[
|\delta_{(i,k)}(i,j) - \delta_{(j,k)}(i,j) - 1| \leq 1, \\
|\delta_{(i,k)}(i,j) - \delta_{(i,l)}(i,j)| \leq 1, \\
|\delta_{(j,k)}(i,j) - \delta_{(i,l)}(i,j) + 1| \leq 1,
\]
and are satisfied for \( a = 2 \) if and only if
\[
|\delta_{(j,l)}(i,j) - \delta_{(j,k)}(i,j)| \leq 1, \\
|\delta_{(j,l)}(i,j) - \delta_{(i,l)}(i,j) + 1| \leq 1, \\
|\delta_{(j,k)}(i,j) - \delta_{(i,l)}(i,j) + 1| \leq 1.
\]

Assume \( T = S \) but \( T \neq R \). Use Proposition \ref{proposition:4.6} Again, several cases need to be checked but all of them yield that, independently of \( \psi \), Inequalities \eqref{eq:40} are satisfied for \( a = 1 \) if and only if
\[
|\delta_{(i,k)}(k,l) - \delta_{(i,k)}(k,l) + 1| \leq 1, \\
|\delta_{(j,k)}(k,l) - \delta_{(i,l)}(k,l) - 1| \leq 1, \\
|\delta_{(i,k)}(k,l) - \delta_{(j,k)}(k,l)| \leq 1,
\]
and are satisfied for \( a = 2 \) if and only if
\[
|\delta_{(j,k)}(k,l) - \delta_{(i,l)}(k,l) - 1| \leq 1, \\
|\delta_{(j,l)}(k,l) - \delta_{(j,k)}(k,l) + 1| \leq 1, \\
|\delta_{(i,l)}(k,l) - \delta_{(j,l)}(k,l)| \leq 1.
\]

Finally, assume that \( T = S = R \). Using Proposition \ref{proposition:4.6} independently of \( \psi \), Inequalities \eqref{eq:40} are satisfied for \( a = 1 \) if and only if
\[
|\delta_{(i,j)}(i,j) - \delta_{(i,l)}(i,j) + 1| \leq 1,
\]
and are satisfied for \( a = 2 \) if and only if
\[
|\delta_{(i,j)}(i,j) - \delta_{(j,l)}(i,j) - 1| \leq 1.
\]

Comparing the above inequalities with those in Definition \ref{definition:6.4} we see that \( \mathcal{L}_\psi \) is \( \psi \)-admissible at \( A_1 \) and \( A_2 \) if and only if \( \mathcal{L}_\psi \) is admissible at \( (R, S) \).

Statement 3 follows from Statement 2 and Lemma \ref{lemma:6.3} \( \square \)

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**Proposition 6.6.** Let \( \phi : \widetilde{\mathcal{C}}^2 \rightarrow \mathcal{C}^2 \) be a good partial desingularization. Let \( \psi_1 : \mathcal{C}^3 \rightarrow \widetilde{\mathcal{C}}^2 \times_B \mathcal{C} \) and \( \psi_2 : \mathcal{C}^3 \rightarrow \mathcal{C}^2 \times_B \mathcal{C} \) be good partial desingularizations. Then, if \( \mathcal{L}_{\psi_1} \) is admissible, so is \( \mathcal{L}_{\psi_2} \). If this is the case, and \( \psi_1, \mathcal{L}_{\psi_1} \) defines a map to \( \mathcal{J} \), then so does \( \psi_2, \mathcal{L}_{\psi_2} \), and the maps are equal.

**Proof.** The first statement follows from Theorem 6.5 as \( \mathcal{L} \) depends only on \( \phi \). Suppose \( \mathcal{L}_{\psi_1} \) and thus \( \mathcal{L}_{\psi_2} \) are admissible. By Proposition 5.2 both \( \psi_1, \mathcal{L}_{\psi_1} \) and \( \psi_2, \mathcal{L}_{\psi_2} \) are relatively torsion-free, rank-1 sheaves on \( \mathcal{C}^2 \times_B \mathcal{C}^2 \) of relative degree \( f \), with formation commuting with base change.

Let \( \rho_i : \mathcal{C}^3 \rightarrow \mathcal{C}^2 \) be the composition of \( \psi_i \) with the projection \( p_i : \mathcal{C}^2 \times_B \mathcal{C} \rightarrow \mathcal{C}^2 \), for \( i = 1, 2 \). Suppose \( \psi_1, \mathcal{L}_{\psi_1} \) defines a map to \( \mathcal{J} \). For each \( (R, S) \in \mathcal{N}(C)^2 \) and each distinguished point \( A \in \phi^{-1}(R, S) \), let \( X_i(A) \) be the fiber of \( \rho_i \) over \( A \) and \( \mu_i(A) : X_i(A) \rightarrow C \) the restriction to \( X_i(A) \) of \( \psi_i \) composed with the projection \( p_i : \mathcal{C}^2 \times_B \mathcal{C} \rightarrow \mathcal{C} \), for \( i = 1, 2 \). By Lemma 6.3 the sheaf \( \psi_1, \mathcal{L}_{\psi_1} \) defines a map to \( \mathcal{J} \) if and only if \( \mu_1(A)_{\ast}(\mathcal{L}_{\psi_1}|_{X_1(A)}) \) is \( C_1 \)-quasistable with respect to \( \mathcal{L} \) for every \( A \) as above. Now, \( X_1(A) \) and \( X_2(A) \) are \( C \)-isomorphic. Identify \( X_1(A) \) with \( X_2(A) \), calling both \( X \). Then, it follows from Proposition 4.6 that \( \mathcal{L}_{\psi_1}|_{X} \) and \( \mathcal{L}_{\psi_2}|_{X} \) differ by a twister “supported” on the exceptional components of \( X \). Thus, by Proposition 5.3 we have that \( \mu_1(A)_{\ast}(\mathcal{L}_{\psi_1}|_{X}) \cong \mu_2(A)_{\ast}(\mathcal{L}_{\psi_2}|_{X}) \), and hence \( \mu_2(A)_{\ast}(\mathcal{L}_{\psi_2}|_{X_2(A)}) \) is \( C_1 \)-quasistable with respect to \( \mathcal{L} \) for each \( (R, S) \in \mathcal{N}(C)^2 \) and each distinguished point \( A \in \phi^{-1}(R, S) \). Using Lemma 6.3 again, it follows that \( \psi_2, \mathcal{L}_{\psi_2} \) defines a map to \( \mathcal{J} \). That the maps defined by \( \psi_1, \mathcal{L}_{\psi_1} \) and \( \psi_2, \mathcal{L}_{\psi_2} \) are equal is a consequence of the fact that they are equal on the fiber of \( \mathcal{C}^2 \) over the generic point of \( B \), which is dense in \( \mathcal{C}^2 \). \( \square \)

### 6.3 Quasistability

**Definition 6.7.** Let \( \Gamma \) be a connected graph without loops, with set of vertices \( V \) and set of edges \( E \). Let \( i \) be a nonnegative integer. Let \( \Gamma(i) \) be the graph obtained from \( \Gamma \) by replacing each edge with a directed graph of \( i + 1 \) edges. Then also \( \Gamma(i) \) is connected without loops. Let \( V(i) \) and \( E(i) \) denote the sets of vertices and edges of \( \Gamma(i) \), respectively. Notice that \( V \) may be viewed as a subset of \( V(i) \); call the vertices of \( V(i) \setminus V \) **exceptional**.

Recall that, for each \( v \in V(i) \),

\[
\varphi_v : V(i) \rightarrow \mathbb{Z}
\]

is defined by letting \( \varphi_v(w) \) be the number of edges of \( \Gamma(i) \) with ends \( v \) and \( w \), for \( w \neq v \), and

\[
\varphi_v(v) := - \sum_{w \neq v} \varphi_v(w).
\]

Two functions \( \varphi_1, \varphi_2 : V(i) \rightarrow \mathbb{Z} \) are said to be \( \Gamma \)-**equivalent** if

\[
\varphi_1 - \varphi_2 = \sum_{v \in V(i) \setminus V} a_v \varphi_v,
\]

where \( a_v \) is a set of integers for each \( v \in V(i) \setminus V \).
for some integers $a_e$. A function $\underline{d} : V(i) \to \mathbb{Z}$ is always $\Gamma$-equivalent to a unique function $\bar{d} : V(i) \to \mathbb{Z}$ such that, for each edge $e \in E$, if $v_1, \ldots, v_i$ are the exceptional vertices of $\Gamma(i)$ created on $e$, then $\underline{d}(v_j) = 0$ for all $j = 1, \ldots, i$ with the possible exception of a single $j$ for which $\bar{d}(v_j) = -1$. Call $\bar{d}$ the \textit{reduction} of $\underline{d}$.

Let $(\Gamma, \underline{C}, q, v)$ be degree-2 Abel data. Then there are natural degree-2 Abel data $(\Gamma(i), \underline{C'}_i, q'_i, v)$, where $\underline{C'}_i$ (resp. $q'_i$) restricts to $\underline{C}$ (resp. $q$) on the vertices of $\Gamma$ and assigns 0 to each exceptional vertex of $\Gamma(i)$.

Assume $i = 2$. Let $\bar{\Delta} : V^4 \to \mathbb{Z}$ be the correction function of $(\Gamma, \underline{C}, q, v)$. Let $\Gamma' : E^2 \to V^2$ be a resolution of the Abel data. Let $\tilde{\phi}$ denote its mirror resolution. We associate to $r$ two functions $\tilde{s}_1, \tilde{s}_2 : E^2 \times V(2) \to \mathbb{Z}$, defined on a triple $(e_1, e_2, w)$ as follows:

1. If $w \in V$, let $\tilde{s}_1(e_1, e_2, w) := \tilde{s}_2(e_1, e_2, w) := q(w)$.

2. If $w \in V(2) - V$, let $e \in E$ be the edge of $\Gamma$ on which $w$ was created. Let $m$ be the end of $e$ to which $w$ is connected by an edge of $\Gamma(2)$, and let $n$ denote the other end. Set $(v_1, v_2) := r(e_1, e_2)$ and $(w_1, w_2) := \tilde{r}(e_1, e_2)$. Then put

$$
\tilde{s}_1(e_1, e_2, w) := \tilde{\Delta}(v_1, v_2, m, n) + \tilde{\Delta}(v_1, w_2, m, n) + \tilde{\Delta}(w_1, v_2, m, n) - \epsilon_1,
$$

$$
\tilde{s}_2(e_1, e_2, w) := \tilde{\Delta}(w_1, v_2, m, n) + \tilde{\Delta}(w_1, w_2, m, n) + \tilde{\Delta}(v_1, w_2, m, n) - \epsilon_2,
$$

where

$$
\epsilon_1 := \# \{ i \in \{1, 2\} \mid v_i = m \text{ and } e_i = e \},
$$

$$
\epsilon_2 := \# \{ i \in \{1, 2\} \mid w_i = m \text{ and } e_i = e \}.
$$

We say that $r$ is \textit{quasistable at $(e_1, e_2) \in E^2$} if $r$ is admissible at $(e_1, e_2)$ and the functions $\tilde{s}_1(e_1, e_2, -)$ and $\tilde{s}_2(e_1, e_2, -)$ have $v$-quasistable reductions with respect to $\underline{C}$. And we say that $r$ is \textit{quasistable} if $r$ is quasistable at every $(e_1, e_2) \in E^2$.

\textbf{Theorem 6.8.} Let $\Gamma$ be the essential dual graph of $C$. Let $V$ be its set of vertices and $E$ its set of edges. Let $\underline{C} : V \to \mathbb{Q}$ and $q : V \to \mathbb{Z}$ be defined by setting $\underline{C}_i = e_i$ and $q(C_i) := \deg(\mathcal{P}|_{C_i})$ for every $i = 1, \ldots, p$. Set $v := C_1$.

Let $\phi : \mathbb{C} \to \mathbb{C}$ be a good partial desingularization. Let $\Gamma' : E^2 \to V^2$ be a function "defined" by sending each pair of reducible nodes $(R, S)$ to $(C_i, C_k)$, where $(R, S) \in C_i \times C_k$ and the strict transform to $\mathbb{C}^2$ of $C_i \times C_k$ does not contain $\phi^{-1}(R, S)$. Let $\psi : \mathbb{C}^3 \to \mathbb{C}^2 \times_B C$ be a good partial desingularization.

Then:

1. For each pair $(R, S)$ of reducible nodes of $C$, the resolution $r$ is quasistable at $(R, S)$ if and only if there is an open neighborhood in $\mathbb{C}^2$ of the two distinguished points of $\phi^{-1}(R, S)$ over which $\psi_* \mathcal{L}_v$ is a relatively torsion-free, rank-1 sheaf on $\mathbb{C}^2 \times_B C / \mathbb{C}^2$ with formation commuting with base change and defining a map to $\mathfrak{J}$.
2. The resolution \( \varphi \) is quasistable if and only if \( \psi_\ast \mathcal{L}_\psi \) is a relatively torsion-free, rank-1 sheaf on \( \tilde{C}^2 \times_B \mathcal{C}/\mathcal{C}^2 \) with formation commuting with base change and defining a map \( \varphi \).

**Proof.** Let \( R \) and \( S \) be reducible nodes of \( C \). Let \( A_1 \) and \( A_2 \) be the two distinguished points of \( \varphi^{-1}(R, S) \). Let \( \rho: \tilde{C}^3 \to \tilde{C}^2 \) be the composition of \( \psi \) with the projection \( p_1: \tilde{C}^2 \times_B C \to \tilde{C}^2 \). For \( a = 1, 2 \), let \( X_a := \rho^{-1}(A_a) \) and let \( \mu_a: X_a \to C \) be the restriction of \( \psi \) to \( X_a \) composed with the second projection \( p_2: \tilde{C}^2 \times_B C \to C \).

We prove the first statement. By Theorem 6.5, the resolution \( \varphi \) is admissible at \( (R, S) \) if and only if \( \mathcal{L}_\psi \) is \( \psi \)-admissible at \( A_1 \) and \( A_2 \), thus if and only if \( \mathcal{L}_\psi \) is \( \psi \)-admissible over a neighborhood \( U \) in \( \tilde{C}^2 \) of \( A_1 \) and \( A_2 \), by Proposition 5.2 so if and only if \( \psi_\ast \mathcal{L}_\psi \) is a relatively torsion-free, rank-1 sheaf on \( \tilde{C}^2 \times_B \mathcal{C}/\mathcal{C}^2 \) with formation commuting with base change over \( U \). It remains to show that \( \varphi \) is quasistable at \( (R, S) \) if and only if \( \mu_a \ast \mathcal{L}_\psi|_{X_a} \) is \( C_1 \)-quasistable with respect to \( \varphi \) for \( a = 1, 2 \).

For each \( a = 1, 2 \), the direct image \( \mu_a \ast \mathcal{L}_\psi|_{X_a} \) is \( C_1 \)-quasistable if and only if there is a twister \( T \) “supported” on the exceptional components of \( X_a \) such that \( \mathcal{L}_\psi|_{X_a} \otimes T \) is \( C_1 \)-quasistable. Indeed, as observed in Definition 6.7 there is a twister \( T \) “supported” on the exceptional components of \( X_a \) such that \( \mathcal{L}_\psi|_{X_a} \otimes T \) is strongly \( \mu_a \)-admissible. Since also \( \mathcal{L}_\psi|_{X_a} \) is \( \mu_a \)-admissible, \( \mu_a \ast (\mathcal{L}_\psi|_{X_a} \otimes T) \cong \mu_a \ast (\mathcal{L}_\psi|_{X_a}) \) by Proposition 5.2. So \( \mu_a \ast (\mathcal{L}_\psi|_{X_a}) \) is \( C_1 \)-quasistable if and only if \( \mu_a \ast (\mathcal{L}_\psi|_{X_a} \otimes T) \) is \( C_1 \)-quasistable, if and only if \( \mathcal{L}_\psi|_{X_a} \otimes T \) is \( C_1 \)-quasistable, by Proposition 5.2.

The first statement of the theorem follows now from the following claim: For each \( a = 1, 2 \), there is a twister \( T \) “supported” on the exceptional components of \( X_a \) such that \( \mathcal{L}_\psi|_{X_a} \otimes T \) is \( C_1 \)-quasistable if and only if \( \varphi_\ast (R, S, -) \) has \( C_1 \)-quasistable reduction with respect to \( \varphi \).

Let us prove the claim for \( a = 1 \) only, the other case being similar. Set \( A := A_1 \), \( X := X_1 \) and \( \mu := \mu_1 \). Let \( C_i \) and \( C_j \) be the distinct components of \( C \) containing \( R \) and \( C_k \) and \( C_l \) those containing \( S \). Assume \( \varphi(R, S) = (C_i, C_k) \). We may assume that \( A \) lies on the strict transforms of \( C_i \times C_k, C_j \times C_k \) and \( C_i \times C_l \). We will denote by \( C_m \) the strict transform under \( \mu \) of the component \( C_m \) of \( C \), for each \( m \). For each reducible node \( T \) of \( C \), let \( m_T \) and \( n_T \) be the distinct integers such that \( T \in C_{m_T} \cap C_{n_T} \) and let \( E_{T,m_T} \) and \( E_{T,n_T} \) be the exceptional components of \( X \) over \( T \), with \( E_{T,m_T} \) intersecting \( C_{m_T} \) and \( E_{T,n_T} \) intersecting \( C_{n_T} \).

As in the statement of Proposition 4.6 let \( \lambda: B \to \tilde{C}^2 \) be any section of \( \tilde{C}^2/B \) sending the special point 0 of \( B \) to \( A \) and such that the pullbacks of the strict transforms of \( C_i \times C_k, C_i \times C_l \) and \( C_j \times C_k \) are all equal to 0. Form the Cartesian diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\xi} & \tilde{C}^3 \\
\downarrow{\rho} & & \downarrow{\rho} \\
B & \xrightarrow{\lambda} & \tilde{C}^2.
\end{array}
\]
The fiber of $W/B$ over 0 is isomorphic to $X$ under $\xi$ and will also be denoted by $X$.

Recall that $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ denote the strict transforms of $\Delta_1$ and $\Delta_2$ to $\tilde{C}^3$. By Proposition 4.6, there are relative effective Cartier divisors $\Gamma_1$ and $\Gamma_2$ of $W/B$ such that $\xi^*\Delta_1 - \Gamma_1$ and $\xi^*\Delta_2 - \Gamma_2$ are effective and supported on the exceptional components of $X$. Furthermore, $\Gamma_1$ and $\Gamma_2$ intersect $X$ transversally, the first at $E_{R,1}$ and the second at $E_{S,2}$.

Let

$$Z := \sum_{i,k=1}^p Z_{i,k} = \sum_{i,k=1}^p \sum_{m=1}^p w_{i,k}(m)D_{i,k,m},$$

where $D_{i,k,m}$ is the strict transform to $\tilde{C}^3$ of $C_i \times C_k \times C_m$ for each $i, k, m$. By Proposition 4.6,

$$\xi^*Z = \sum_{m=1}^p w(m)C_m + \sum_{T \in \mathcal{N}(C)} (w(T,m)E_{T,m} + w(T,n)E_{T,n})$$

for certain integers $w(m)$ and $w(T,m)$, for $m = 1, \ldots, p$ and $T \in \mathcal{N}(C)$. More precisely,

$$w(m_T) - w(T,m_T) = \delta_2(T)$$
$$w(T,m_T) - w(T,n_T) = \delta_3(T)$$
$$w(T,n_T) - w(n_T) = \delta_1(T),$$

where $\delta_1(T), \delta_2(T)$ and $\delta_3(T)$ are

$$\delta_{(i,k)}(m_T,n_T), \quad \delta_{(j,k)}(m_T,n_T), \quad \delta_{(i,l)}(m_T,n_T)$$

in a particular order.

Set

$$N := \xi^*Z + \sum_{T \in \mathcal{N}(C)} (\delta_2(T)E_{T,m} - \delta_1(T)E_{T,n})$$

and put

$$\mathcal{L}_\psi := \xi^*\psi^*p_2^*\mathcal{P} \otimes \mathcal{O}_W(-\Gamma_1 - \Gamma_2) \otimes \mathcal{O}_W(-N).$$

Then $\xi^*\mathcal{L}_\psi$ and $\mathcal{L}_\psi'$ differ by a twister “supported” on the exceptional components of $X$. Also,

$$\deg(\mathcal{L}_\psi'|_Y) = \delta_1(R,S,Y)$$

for every component $Y$ of $X$. So, $\delta_1(R,S,-)$ has $C_1$-quasistable reduction with respect to $\varepsilon^2$ if and only if there is a twister $T$ “supported” on the exceptional components of $X$ such that $\mathcal{L}_\psi'|_X \otimes T$ is $C_1$-quasistable, thus if and only if there is a twister $T$ “supported” on the exceptional components of $X$ such that $\mathcal{L}_\psi|_X \otimes T$ is $C_1$-quasistable.

To prove the second statement of the theorem, notice that, by Lemma 6.3, the sheaf $\psi_*\mathcal{L}_\psi$ defines a map to $\mathcal{T}$ if and only if $\mu_{X*}(\mathcal{L}_\psi|_X)$ is $C_1$-quasistable with respect to $\xi$ for every fiber $X$ over a distinguished point of $\phi^{-1}(R,S)$ for $R, S \in \mathcal{N}(C)$. Then apply the first statement already proved.
three conditions are verified: blowup sequence if there is $E$ the subset of and one and only one end of $(\alpha \in \mathcal{J}^I)$ of $(\mathcal{J}^I)$ order $R, S$. It is called symmetric if the subsets are symmetric to each other, that is, whenever $I_{1,j} \neq I_{2,j}$ we have $(I_{1,j-1}, I_{2,j-1}) = (I_{2,j}, I_{1,j})$ or $(I_{1,j+1}, I_{2,j+1}) = (I_{2,j}, I_{1,j})$. We say that a pair $(R, S) \in E^2$ is affected by the blowup sequence if there is $j$ such that one and only one end of $R$ lies in $I_{1,j}$ and one and only one end of $S$ lies in $I_{2,j}$. We call the minimum such $j$ the order of $(R, S)$ in the blowup sequence. The center of the blowup sequence is the subset of $E^2$ consisting of the pairs $(R, S) \in E^2$ such that either $R = S$ or $(R, S)$ is affected.

We say that a blowup sequence $(I_1, I_2)$ resolves the Abel data if the following three conditions are verified:

1. The center of the blowup sequence contains $\Sigma$.
2. $\Sigma$ is solvable
3. For each $(R, S) \in \Sigma - \Delta_E$, any quasistable resolution $r$ satisfies $r(R, S) \in I_{1,j} \times (V - I_{2,j})$ or $r(R, S) \in (V - I_{1,j}) \times I_{2,j}$, where $j$ is the order of $(R, S)$. We say that $(I_1, I_2)$ resolves the Abel data minimally if its center is $\Sigma$.

**Theorem 6.10.** Let $\Gamma$ be the essential dual graph of $C$. Let $V$ be its set of vertices and $E$ its set of edges. Let $\mathcal{E} : V \to \mathbb{Q}$ and $q : V \to \mathbb{Z}$ be defined by setting $\mathcal{E}(C_i) := e_i$ and $q(C_i) := \deg(\mathcal{P}|_{C_i})$ for each $i = 1, \ldots, p$. Set $\nu := C_1$. Let $\Sigma$ be the singular locus of the Abel data $(\Gamma, \mathcal{E}, q, v)$. Assume that $\Sigma$ is solvable. Then the degree-2 rational map $\alpha_{\mathcal{E}/B}^2 : \mathcal{C}^2 \to \mathcal{J}$ is defined on $\mathcal{C}^2 - \Sigma$. Furthermore, let $(I_1, I_2)$ be a blowup sequence, and let $u$ denote its length. Let $\phi : \tilde{\mathcal{C}}^2 \to \mathcal{C}^2$ be the sequence of blowups

$$[\Delta], [X_1, Y_1], \ldots, [X_u, Y_u],$$

where $X_i$ is the union of those $C_j$ in $I_{1,i}$ and $Y_i$ is the union of those $C_j$ in $I_{2,i}$, for each $i = 1, \ldots, u$. Let $\tilde{\phi} : \tilde{\mathcal{C}}^2 \to \mathcal{C}^2$ and $\psi : \tilde{\mathcal{C}}^3 \to \tilde{\mathcal{C}}^2 \times_B \mathcal{C}$ be good partial desingularizations. Let $\tau : E^2 \to V^2$ be a function “defined” by sending each pair of reducible nodes $(R, S)$ to $(C_i, C_k)$, where $(R, S) \in C_i \times C_k$ and the strict transform to $\tilde{\mathcal{C}}^2$ of $C_i \times C_k$ does not contain $\tilde{\phi}^{-1}(R, S)$. Then:

1. If $(I_1, I_2)$ resolves the Abel data, then there is a map

$$\tilde{\alpha}_{\mathcal{E}/B}^2 : \tilde{\mathcal{C}}^2 \to \mathcal{J}$$
resolving $\alpha^2_{c/B}$. In addition, if $\bar{\phi} = \phi'\phi$ for $\phi' : \tilde{C}^2 \to \tilde{C}^2$, then $\iota$ is quasistable, in which case $\psi_*L_\psi$ defines a map $\bar{\alpha}^2_{c/B} : \tilde{C}^2 \to \mathcal{F}$ agreeing with $\hat{\alpha}^2_{c/B}\phi'$.

2. Conversely, if $(I_1, I_2)$ resolves the Abel data minimally and $\iota$ is quasistable, then $\bar{\phi}$ factors through $\tilde{C}^2$.

**Proof.** Assume that $\Sigma$ is solvable. Then there is a quasistable resolution $\iota'$ of the Abel data. For each $(R, S) \in E^2$, let

$$\phi^{(R,S)} : [i, k] : \mathcal{X} \to C^2$$

and $\phi^{(R,S)}_2 : [i, l] : \mathcal{Y} \to C^2$ be the two different blowups, where $\iota'(R, S) = (C_i, C_k)$ and $\iota' = (C_j, C_l)$. By Theorem 6.8, the composition $\alpha^2_{c/B} \phi^{(R,S)}_\iota$ is defined on a neighborhood of the two distinguished points of $\mathcal{Y}$ over $(R, S)$. Using preservation of the degree, as in the proof of Lemma 6.3 it follows that $\alpha^2_{c/B}$ is defined on $C^2 - E^2$.

Let now $(R, S) \in E^2 - \Sigma$. Set $\phi_i := \phi^{(R,S)}_i$ for $i = 1, 2$. By definition of $\Sigma$, we have that $\iota'$ is quasistable at $(R, S)$. Then, again by Theorem 6.8, also $\alpha^2_{c/B} \phi_i$ is defined on a neighborhood of the two distinguished points of $\mathcal{X}$ over $(R, S)$. It follows that both maps factor through a set-theoretic map $U \to \mathcal{F}$ defined on a Zariski neighborhood $U \subseteq C^2$ of $(R, S)$ and agreeing with $\alpha^2_{c/B}$ away from $(R, S)$. Since $\phi_1$ (or $\phi_2$) is proper and surjective, the map $U \to \mathcal{F}$ is continuous. To show it is a morphism of schemes, we need only show now that $\phi_1^*O_X = O_{C^2}$.

We need only show the above equality at $(R, S)$. So we may work locally analytically. In this setup, we consider the closed subscheme $Z \subset A^4_K$ given by $x_0x_1 = y_0y_1$ and the blowup $W \subset A^4_K \times P^1_K$ given by $\alpha'x_0 = \alpha y_1$ and $\alpha'y_0 = \alpha x_1$. Consider the projection $p_1 : A^4_K \times P^1_K \to A^4_K$. We need to show that $p_1^*O_W = O_Z$.

Consider the following diagram of exact sequences

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{I}_Z|_{A^4_K} & \longrightarrow & O_{A^4_K} & \longrightarrow & O_Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & p_1^*\mathcal{I}_W|_{A^4_K \times P^1_K} & \longrightarrow & p_1^*O_{A^4_K \times P^1_K} & \longrightarrow & p_1^*O_W \\
\end{array}
$$

(41)

The middle vertical map is an isomorphism. Now, $\mathcal{I}_W|_{A^4_K \times P^1_K}$ has the following presentation:

$$0 \longrightarrow O_{A^4_K \times P^1_K}(-2) \longrightarrow O_{A^4_K \times P^1_K}(-1) \oplus O_{A^4_K \times P^1_K}(-1) \longrightarrow \mathcal{I}_W|_{A^4_K \times P^1_K} \longrightarrow 0. $$

From the long exact sequence of higher direct images of $p_1$, we get

$$R^1p_1^*\mathcal{I}_W|_{A^4_K \times P^1_K} = 0 \quad \text{and} \quad p_1^*\mathcal{I}_W|_{A^4_K \times P^1_K} \cong R^1p_1^*O_{A^4_K \times P^1_K}(-2) \cong O_{A^4_K}. $$

So, the rightmost map in the bottom row of Diagram (41) is surjective. Thus, since the middle vertical map is an isomorphism, the comorphism $O_Z \to p_1^*O_W$ is surjective. On the other hand, it is clearly injective.
Assume now that \((I_1, I_2)\) resolves the Abel data. Then, for each \((R, S) \in \Sigma\), we have that \(\tilde{\tau}'(R, S) = (C_i, C_k)\), where \((R, S) \in C_i \times C_k\) and the strict transform of \(C_i \times C_k\) under \(\phi\) does not contain \(\phi^{-1}(R, S)\). To prove Statement (1), we may assume that \(\tilde{\phi}\) factors through \(\phi\). Then \(\tilde{\tau}'\) and \(\tilde{\tau}\) are equivalent on \(\Sigma\). Furthermore, by definition of \(\Sigma\), if \((R, S) \in E^2 - \Sigma\), then there is a quasistable resolution \(\tilde{\tau}''\) of the Abel data that is not equivalent to \(\tilde{\tau}'\) at \((R, S)\). Thus, either \(\tilde{\tau}\) is equivalent to \(\tilde{\tau}'\) or to \(\tilde{\tau}''\) at \((R, S)\). Since quasistability is checked on each pair of reducible nodes, it follows that \(\tilde{\tau}\) is itself quasistable.

By Theorem 6.3, since \(\tilde{\tau}\) is quasistable, \(\psi_\ast L_\psi\) defines a map \(\tilde{\alpha}_{C/B} : \tilde{C}^2 \to \tilde{\mathcal{J}}\). We need only prove that \(\tilde{\alpha}_{C/B}\) factors through \(\tilde{C}^2\). This is clearly true over the points of \(\tilde{C}^2\) where \(\phi'\) is an isomorphism, thus over \(\phi^{-1}(\Sigma)\). This is also true away from \(\Sigma\), since we have proved that \(\tilde{\alpha}_{C/B}\) is defined on \(\tilde{C}^2 - \Sigma\), and since \(\tilde{\alpha}_{C/B} \circ \phi\) agrees with \(\tilde{\alpha}_{C/B}\) over the generic point of \(B\), so wherever the former is defined.

Finally, assume that \((I_1, I_2)\) resolves the Abel data minimally and that \(\tilde{\tau}\) is quasistable. Statement (2) is local, so we need only check that \(\tilde{\phi}\) factors through \(\phi\) at pairs \((R, S)\) of distinct reducible nodes of \(C\). So, let \((R, S)\) be such a pair. Let \(C_i\) and \(C_j\) be the components of \(C\) containing \(R\) and \(C_k\) and \(C_l\) those of \(S\). In the blowup sequence defining \(\phi\), let \([W, Z]\) be the first blowup in which \(R \in W \cap W'\) and \(S \in Z \cap Z'\). Without loss of generality, we may assume that \(C_i \subseteq W\) and \(C_j \subseteq Z\). Then, in a neighborhood of \((R, S)\), the map \(\tilde{\phi}\) is equal both to the blowup \([i, l]\) and the blowup \([j, k]\). Also \(\tilde{\tau}(R, S) = (i, k)\) or \(\tilde{\tau}(R, S) = (j, l)\).

If \((R, S) \not\in \Sigma\), then \(\phi\) is an isomorphism at \((R, S)\), and thus \(\tilde{\phi}\) factors through \(\phi\) at \((R, S)\). On the other hand, if \((R, S) \in \Sigma\), then, since \(\tilde{\tau}\) is quasistable, it follows the smallest integer \(m\) such that \(R \in X_m \cap X'_m\) and \(S \in Y_m \cap Y'_m\) is such that \(\tilde{\tau}(R, S) \in I_{1,m} \times (V - I_{2,m})\) or \(\tilde{\tau}(R, S) \in (V - I_{1,m}) \times I_{2,m}\). In any case, it follows that either

\[
C_i \subseteq X_m, \quad C_j \subseteq X'_m, \quad C_l \subseteq Y_m, \quad C_k \subseteq Y'_m
\]

or

\[
C_j \subseteq X_m, \quad C_i \subseteq X'_m, \quad C_k \subseteq Y_m, \quad C_l \subseteq Y'_m.
\]

In any case, in a neighborhood of \((R, S)\), the map \(\phi\) is equal both to the blowup \([i, l]\) and the blowup \([j, k]\), and thus equal to \(\tilde{\phi}\).

### 7 Examples

There are CoCoA scripts to determine whether the singular locus of given degree-2 Abel data is solvable and, if so, whether a symmetric blowup sequence resolving minimally the Abel data exists. They are available at

[http://w3.impa.br/∼esteves/CoCoAScripts/Abelmaps](http://w3.impa.br/~esteves/CoCoAScripts/Abelmaps)

The scripts were applied to various Abel data, and the output has always been positive.
Throughout this section, \((\Gamma, q, v)\) will denote degree-2 Abel data. Denote by \(V\) the set of vertices of \(\Gamma\).

First of all, the various notions associated to degree-2 Abel data \((\Gamma, q, v)\) depend on the associated correction function \(\delta\), which in turn depends only on the difference \(q - e\). Thus we may assume that \(q\) is any fixed function \(V \rightarrow \mathbb{Z}\). We will assume that

\[
q(w) = \begin{cases} 
2 & \text{if } w = v, \\
0 & \text{if } w \neq v 
\end{cases}
\]  

(42)

In particular, \(e\) must have degree 0. A possible choice of \(e\) is the zero function. Though many other choices are possible, it follows from the definition that the correction function \(\delta\) of \((\Gamma, q, v)\) is the same as that of \((\Gamma, q, v)\) for any choice of integers \(a_1, \ldots, a_p\), where the \(c_i\) are given in Definition 6.1. We may thus assume that \(-e\) is \(v\)-quasistable with respect to any fixed polarization, say the zero polarization. In other words, we may assume that

\[
-k_I \leq \sum_{i \in I} x_i < k_I
\]

(43)

for each proper nonempty subset \(I \subset V\) containing \(v\).

Now, on \(\mathbb{R}^p\), with coordinates \(x_1, \ldots, x_p\), consider the subspace \(H_p \subset \mathbb{R}^p\) given by

\[
x_1 + \cdots + x_p = 0.
\]

The affine subspaces of \(H_p\) given by

\[
\sum_{i \in I} x_i + \frac{k_I}{2} = a_I,
\]

(44)

where \(I\) ranges through all proper subsets of \(V\) containing \(v\) and the \(a_I\) through \(\mathbb{Z}\), induce a stratification \(\Xi = \Xi(\Gamma)\) of \(H_p\) by convex strata. A stratum is a connected component of the intersection of certain subspaces of the form (44) with the complements of certain subspaces of the same form.

Let \(\Xi_0 = \Xi_0(\Gamma)\) be the collection of strata of \(\Xi\) whose points \((x_1, \ldots, x_p)\) satisfy

\[
-k_I \leq \sum_{i \in I} x_i < \frac{k_I}{2}
\]

for each proper nonempty subset \(I \subset V\) containing \(v\). Then \(\Xi_0\) is a finite collection. A polarization \(\xi\) of degree 0 satisfying (43) for each proper nonempty subset \(I \subset V\) containing \(v\) belongs to one of the strata in \(\Xi_0\). Furthermore, if \(\xi\) and \(\xi'\) are polarizations lying on the same stratum, the corresponding Abel data \((\Gamma, q, \xi, v)\) and \((\Gamma, q, \xi', v)\) have the same correction function.

In other words, for a given connected graph without loops \(\Gamma\), we need only apply our Cocoa scripts finitely many times to show that every Abel data \((\Gamma, q, v)\) supported in \(\Gamma\) has solvable singular locus and admits a symmetric blowup sequence resolving it minimally.
In the following examples, Abel data \((\Gamma, e, q, v)\) are presented in the following way: The graph \(\Gamma\) is given by its intersection matrix, \(\Xi\), which is a square matrix whose rows (and columns) are in bijective correspondence with the set of vertices of \(\Gamma\) and whose entry at position \((i, j)\) is the number of edges between the \(i\)-th and \(j\)-th vertices if \(i \neq j\), and the negative of the number of edges with end at the \(i\)-th vertex if \(i = j\). We will assume \(v\) is the first vertex and \(q\) is fixed as above. The polarization \(e\) is given as a tuple.

**Example 7.1.** Consider the graph \(\Gamma\) whose intersection matrix is

\[
\begin{bmatrix}
-2 & 1 & 1 & 0 \\
1 & -5 & 3 & 1 \\
1 & 3 & -6 & -2 \\
0 & 1 & 2 & -3 \\
\end{bmatrix}
\]

Set \(v := 1\) and \(q := (2, 0, 0, 0)\). If we set \(e := (0, 0, 0, 0)\), then the CoCaA scripts tell us that the Abel data has solvable singular locus, and the blowup sequence

\[((\{1\}, \{1\}), (\{4\}, \{4\})\]

resolves it minimally. We get the same result if we set \(e := (0, -1/2, 0, 1/2)\).

On the other hand, if we set \(e = (0, 1/2, 0, -1/2)\), the Abel data has solvable singular locus, but the blowup sequence that resolves it minimally is

\[((\{1\}, \{1\})\]

So, as expected, the resolution depends on the polarization.

**Example 7.2.** In [10], Abel maps for curves of compact type are studied. Abel maps of every degree are constructed, starting from the degree-1 Abel maps constructed in [6] and [7], using the fact that the generalized Jacobian of a curve of compact type is projective. Arguably, the simplest curves not of compact type are the “circular curves,” whose dual graph has the following intersection matrix:

\[
\begin{bmatrix}
-2 & 1 & 0 & 0 & \ldots & 0 & 1 \\
1 & -2 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & -2 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & -2 & 1 \\
1 & 0 & 0 & 0 & \ldots & 1 & -2 \\
\end{bmatrix}
\]

As before, we set \(v := 1\) and \(q := (2, 0, \ldots, 0)\). To simplify, set \(e := (0, \ldots, 0)\).

The simplest case is that of graphs with two vertices only. Then the Abel data is solvable and

\[((\{1\}, \{1\})\]

55
resolves it minimally. For three vertices, again the Abel data is solvable, and
\[\{(1), \{1\}\}, \{(2), \{2\}\}, \{(3), \{3\}\}\]
resolves it minimally. The case of four vertices is more interesting: a minimal resolution is
\[\{(1), \{1\}\}, \{(2), \{2\}\}, \{(1, 2), \{1, 2\}\}, \{(3), \{3\}\}, \{(2, 3), \{2, 3\}\}, \{(1, 3), \{1, 3\}\}\]
Finally, the case of five vertices may perhaps indicate the pattern: a minimal resolution is
\[\{(1), \{1\}\}, \{(2), \{2\}\}, \{(1, 2), \{1, 2\}\}, \{(3), \{3\}\}, \{(2, 3), \{2, 3\}\}, \{(1, 3), \{1, 3\}\}, \{(4), \{4\}\}, \{(3, 4), \{3, 4\}\}, \{(2, 4), \{2, 4\}\}, \{(1, 4), \{1, 4\}\}\]

**Example 7.3.** Assuming \(q\) is of the form \((4, 2)\), if we assume the polarization \(\mathcal{E}\) satisfies Inequalities \((43)\), heuristically, the further the inequalities are from being equalities, the smallest a minimal resolution of the Abel data is. And, indeed, if
\[2 - \frac{k_I}{2} \leq \sum_{i \in I} \mathcal{E}(i) < \frac{k_I}{2}\]
for each proper nonempty subset \(I \subset V\) containing \(v\), the empty sequence resolves the Abel data minimally. So, for instance, consider the graph \(\Gamma\) whose intersection matrix is
\[
\begin{pmatrix}
-4 & 2 & 2 & 0 \\
2 & -7 & 3 & 2 \\
2 & 3 & -7 & 2 \\
0 & 2 & 2 & -4
\end{pmatrix},
\]
a small variation of that in Example 7.1. Set \(v := 1\) and \(q := (2, 0, 0, 0)\). If we set \(\mathcal{E} := (0, 0, 0, 0)\), then the empty sequence resolves the Abel data minimally, whereas if we set \(\mathcal{E} := (-1/2, 0, 1/2, 0)\), then
\[\{(1), \{1\}\}\]
resolves it minimally, and if \(\mathcal{E} := (-1, -1, 1, 1)\), then
\[\{(1), \{1\}\}, \{(4), \{4\}\}\]
resolves it minimally.

**Remark 7.4.** The reader might have observed that in all the blowup sequences \((I_1, I_2)\) above we had \(I_{1,j} = I_{2,j}\) for every \(j\). Indeed, the \texttt{CoCoA} scripts are written in such a way that those special symmetric blowup sequences are preferred over the others.
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