DEFINABLY COMPACT GROUPS DEFINABLE IN REAL CLOSED FIELDS. II

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Abstract. We continue the analysis of definably compact groups definable in a real closed field \( \mathcal{R} \). In [3], we proved that for every definably compact definably connected semialgebraic group \( G \) over \( \mathcal{R} \) there are a connected \( R \)-algebraic group \( H \), a definable injective map \( \phi \) from a generic definable neighborhood of the identity of \( G \) into the group \( H(R) \) of \( R \)-points of \( H \) such that \( \phi \) acts as a group homomorphism inside its domain. The above result and our study of locally definable covering homomorphisms for locally definable groups combine to prove that if such group \( G \) is in addition abelian, then its o-minimal universal covering group \( \tilde{G} \) is definably isomorphic, as a locally definable group, to a connected open locally definable subgroup of the o-minimal universal covering group \( H(R)^0 \) of the group \( H(R)^0 \) for some connected \( R \)-algebraic group \( H \).

1. Introduction

This is the second paper of two papers studying definably compact groups definable in real closed fields.

This paper offers a description of the semialgebraically connected semialgebraic groups over a sufficiently saturated real closed field \( R \) through the study of their o-minimal universal covering groups (see Def. 3.3) and of their relation with the \( R \)-points of some connected \( R \)-algebraic group.

We establish a connection between the o-minimal universal covering groups of an abelian definably compact definably connected group definable in \( \mathcal{R} \) and of the semialgebraically connected component \( H(R)^0 \) of the group \( H(R) \) of \( R \)-points of some connected \( R \)-algebraic group \( H \). More precisely we show the following.

Theorem 7.2. Let \( R \) be a sufficiently saturated real closed field. Then, the o-minimal universal covering group of an abelian definably compact definably connected group definable in \( \mathcal{R} \) is an open locally definable subgroup of the o-minimal universal covering group of the semialgebraically connected component \( H(R)^0 \) of the group \( H(R) \) of \( R \)-points of some connected \( R \)-algebraic group \( H \).

To prove this result we apply Theorem 5.1 of the first paper ([3]) and some results on locally definable covering homomorphisms of locally definable groups proved in this...

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paper. Theorem 5.1 in [3] asserts that for every definably compact definably connected semialgebraic group \( G \) over \( \mathbb{R} \) there are a connected \( R \)-algebraic group \( H \) and a definable local homomorphism \( \phi \) from a generic definable neighborhood of the identity of \( G \) into the group \( H(\mathbb{R}) \) of \( \mathbb{R} \)-points of \( H \), where by a local homomorphism we mean the following.

**Definition 1.1.** Let \( G_1 \) and \( G_2 \) be two topological groups, \( X \subseteq G_1 \) a neighborhood of the identity of \( G_1 \), and \( \phi : X \to G_2 \) a map. \( \phi \) is called a local homomorphism if \( x, y, xy \in X \) implies \( \phi(xy) = \phi(x)\phi(y) \). We say that an injective map \( \phi : X \subseteq G_1 \to G_2 \) is a local homomorphism in both directions if \( \phi : X \to G_2 \) and \( \phi^{-1} : \phi(X) \to X \) are local homomorphisms.

The strategy to prove Theorem 7.2 is to use the \( R \)-algebraic group \( H \) and the local homomorphism given by [3, Theorem 5.1] to define a locally definable map \( \theta \) from some open locally definable subgroup of the o-minimal universal covering group of \( H(\mathbb{R}) \) to \( G \) such that \( \theta \) works as the o-minimal universal covering group of \( G \).

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1.1. The structure of the paper. In Section 2 we regard the locally definable spaces, \( \text{ld-spaces} \), and their ld-covering maps as are defined in [2]. The \( \bigvee \)-definable groups in \( \mathcal{M} \) are examples of such spaces. We show that any ld-covering map between ld-spaces is closed for definable subspaces (Prop. 2.12). The o-minimal universal covering homomorphism of a connected locally definable group is introduced and studied in Section 3. Sections 4 and 5 investigate the connection between abelian definably generated groups, existence of generic definable sets, convex sets, and covers of definable groups. We prove in Proposition 4.5 the existence of a convex set inside a definable generic subset of an abelian \( \bigvee \)-definable group \( U \) with \( U^{00} \). This is a crucial fact in the construction of a well defined covering map in Theorem 6.1.

In the last two sections we develop some results on local homomorphisms and their extensions to locally definable homomorphisms. Finally, in Section 7, by means of Theorems 7.1 and [3, Theorem 5.1], we prove the main result of this paper: Theorem 7.2.

**Notation.** Our notation and any undefined term that we use from model theory, topology, or algebraic geometry are generally standard. For a group \( G \) whose group operation is written multiplicatively, we use the following notation \( \prod_n X = \underbrace{X \cdot \ldots \cdot X}_{n\text{-times}} \) and \( X^n = \{ x^n : x \in X \} \) for any \( n \in \mathbb{N} \).

2. LD-SPACES AND LD-COVERING MAPS

From now until the end of this paper, unless stated otherwise, we work over a sufficiently saturated o-minimal expansion of a real closed field \( \mathbb{R} \), where by a sufficiently saturated structure we mean a \( \kappa \)-saturated structure for some sufficiently large cardinal \( \kappa \).

In [2] Baro and Otero introduced the locally definable category, which extends the locally semialgebraic one introduced by Delfs and Knebusch in [6] and is more flexible than the \( \bigvee \)-definable group category. \( \bigvee \)-definable groups are examples of locally definable spaces
and their locally definable covering homomorphisms are locally definable covering maps of locally definable spaces. Following, we will introduce some definitions of the locally definable category from [2], and then we will prove some results on locally definable covering maps that will be applied later in the study of the locally definable covering homomorphisms of locally definable groups.

2.1. Ld-spaces and ld-maps.

Definition 2.1. Let $M$ be a set. A locally definable space is a triple $(M, (M_i, \phi_i)_{i \in I})$ where

(i) $M_i \subseteq M$, $M = \bigcup_{i \in I} M_i$, and $\phi_i : M_i \to Z_i$ is a bijection between $M_i$ and a definable set $Z_i \subseteq \mathbb{R}^{d(i)}$ for every $i \in I$,

(ii) $\phi_i (M_i \cap M_j)$ is a definable relative open subset of $Z_i$ and the transition maps $\phi_{ij} = \phi_j \circ \phi_i^{-1} : \phi_i (M_i \cap M_j) \to M_i \cap M_j \to \phi_j (M_i \cap M_j)$ are definable for every $i, j \in I$.

The dimension of $M$ is $\dim (M) := \sup (\dim (Z_i) : i \in I)$. If $Z_i$ and $\phi_{ij}$ are definable over $A \subseteq R$ for all $i, j \in I$, we say that $M$ is a locally definable space over $A$.

Note that every definable space ([19, Chapter 10]) is a locally definable space with $|I| < \aleph_0$.

Every locally definable space $(M, (M_i, \phi_i)_{i \in I})$ has a unique topology on $M$ such that each $M_i$ is open and $\phi_i$ is a homeomorphism for all $i \in I$; more precisely, $\mathcal{O} \subseteq M$ is open if and only if $\phi_i (\mathcal{O} \cap M_i)$ is relatively open in $Z_i$ for every $i \in I$. Throughout this subsection any topological property of locally definable spaces refers to this topology.

Definition 2.2. Let $(M, (M_i, \phi_i)_{i \in I})$ be an ld-space.

(i) An ld-space is a Hausdorff locally definable space.

(ii) A subset $X \subseteq M$ is called a definable subspace of $M$ if there is a finite $J \subseteq I$ such that $X \subseteq \bigcup_{j \in J} M_j$ and $\phi_j (X \cap M_j)$ is definable for all $j \in J$.

(iii) A subset $Y \subseteq M$ is called an compatible subspace of $M$ if $\phi_i (Y \cap M_i)$ is definable for every $i \in I$, or equivalently, $Y \cap X$ is a definable subspace of $M$ for every definable subspace $X$ of $M$.

By Theorem 3.9 of [2], every $\forall$-definable group $\mathcal{U}$ with its $\tau$-topology (see [14, Lemma 7.5]) is an ld-space of finite dimension, and any definable subset of $\mathcal{U}$ is a definable subspace of $\mathcal{U}$.

We recall that any compatible subspace $Y$ of an ld-space $M$ inherits a natural structure of ld-space [2, Remark 2.3] given by $(Y, Y_i = Y \cap M_i, \phi_i |_{Y_i})$. And if $Y$ is a definable subspace then it inherits the structure of a definable space. Note that the only compatible subspaces of a definable space are the definable ones.

Now, we will introduce the maps between ld-spaces as in [2]. For this we note that given two ld-spaces $(M, (M_i, \phi_i)_{i \in I})$ and $(N, (N_j, \psi_j)_{j \in J})$ we can endow $M \times N$ with the structure $(M \times N, (M_i \times N_j, (\phi_i, \psi_j))_{i \in I})$ that makes it into an ld-space, and as it is defined in [19], a map $f : M \to N$ between definable spaces $M, N$ is a definable map if its graph is a definable subspace of $M \times N$. 
Definition 2.3. A map $\theta : M \to N$ between ld-spaces (locally definable spaces) $(M, (M_i, \phi_i)_{i \in I})$ and $(N, (N_j, \psi_j)_{j \in J})$ is called an ld-map (locally definable map) if $\theta(M_i)$ is a definable subspace of $N$ and $\theta|_{M_i}: M_i \to \theta(M_i)$ is definable for every $i \in I$.

2.2. Some topological notions in ld-spaces.

Definition 2.4. Let $M$ be an ld-space.

(i) $M$ is connected if $M$ has no compatible nonempty proper clopen subspace.

(ii) An ld-path in $M$ is a continuous ld-map $\alpha : [0, 1] \to M$.

(iii) $M$ is path connected if for every $x_1, x_2 \in M$ there is an ld-path $\alpha : [0, 1] \to M$ such that $\alpha(0) = x_1$ and $\alpha(1) = x_2$.

(iv) The path connected component of a point $x \in M$ is the set of all $y \in M$ such that there is an ld-path from $x$ to $y$.

By Remarks 4.1 and 4.3, and Fact 4.2 of [2], (i) an ld-space $M$ is connected if and only if $M$ is path connected if and only if every ld-map from $M$ to a discrete ld-space is constant, and (ii) every path connected component of an ld-space is a clopen compatible subspace.

Claim 2.5. Let $M = \bigcup_{i \in I} X_i$ be an ld-space such that $\{X_i : i \in I\}$ is a collection of connected compatible subspaces of $M$ and $\bigcap_{i \in I} X_i \neq \emptyset$. Then $M$ is connected.

Proof. Let $Y \subseteq M$ be a compatible nonempty clopen in $M$. Since $Y \neq \emptyset$, there is $k \in I$ such that $Y \cap X_k \neq \emptyset$. Since $X_k$ and $Y$ are compatible in $M$, so is $Y \cap X_k$, and in particular $Y \cap X_k \subseteq X_k$ is a clopen compatible set in $X_k$. By the connectedness of $X_k$, $Y \cap X_k = X_k$.

As $\bigcap_{i \in I} X_i \neq \emptyset$, $X_i \cap X_k \neq \emptyset$ for every $i \in I$, then $X_i \cap Y \neq \emptyset$, and as above we conclude that $Y \cap X_i = X_i$ for every $i \in I$. Therefore, $M = \bigcup_{i \in I} X_i = \bigcup_{i \in I} Y \cap X_i = Y$. Then $M$ has no clopen proper nonempty compatible subset.

Corollary 2.6. Let $M, N$ be two connected ld-spaces. Then the product ld-space $M \times N$ is connected.

Proof. Fix $y \in N$. For $x \in M$, let $T_x = (\{x\} \times N) \cup (M \times \{y\})$. Since $(x, y) \in (\{x\} \times N) \cap (M \times \{y\})$, Claim 2.5 implies that $T_x$ is connected. Finally, as $\bigcap_{x \in M} T_x = M \times \{y\}$, again Claim 2.5 implies that $\bigcup_{x \in M} T_x = M \times N$ is connected.

Proposition 2.7. Let $\mathcal{U}$ be a locally definable group and $X \subseteq \mathcal{U}$ a connected definable set such that the identity element $e_{\mathcal{U}} \in X$. Then the definable generated group $\langle X \rangle$ is a connected locally definable group.

Proof. By Corollary 2.6, $X \times \cdots \times X$ is a connected definable space for every $i \in \mathbb{N}$. Since the ld-map

$$p_i : \langle X \rangle \times \cdots \times \langle X \rangle \to \langle X \rangle$$

$$(x_1, \ldots, x_i) \mapsto \prod_{1 \leq j \leq i} x_j$$

is a definable map, $\langle X \rangle$ is a connected locally definable group.
is continuous (with respect to their topologies of locally definable groups) and the image of a connected ld-space by a continuous ld-map is connected, then $p_i \left( \prod_{i=1}^{n} X \right) = \prod_{i=1}^{n} X$ is connected.

Finally, as $\bigcap_{i \in \mathbb{N}} \bigcup_{\mathcal{O} \in \mathcal{L}} \bigcup_{l \in L} \mathcal{O}_l \times X^{-1} \subseteq (X \times X^{-1}) \neq \emptyset$, then $\bigcup_{i \in \mathbb{N}} \bigcup_{\mathcal{O} \in \mathcal{L}} \bigcup_{l \in L} \mathcal{O}_l \times X^{-1} = \langle X \rangle$ is connected by Claim 2.5. \(\square\)

**Definition 2.8.** Let $M$ be an ld-space and $x_0 \in M$. Let $\alpha, \gamma : [0,1] \to M$ be two ld-paths. A continuous ld-map $H(t,s) : [0,1] \times [0,1] \to M$ is a homotopy between $\alpha$ and $\gamma$ if $\alpha = H(\cdot,0)$ and $\gamma = H(\cdot,1)$. In this case, $\alpha$ and $\gamma$ are called homotopic, denoted $\alpha \sim \gamma$.

Let $\mathbb{L}(M,x_0)$ be the set of all ld-paths that start and end at the element $x_0 \in M$. Note that being homotopic $\sim$ is an equivalence relation on $\mathbb{L}(M,x_0)$. We define the o-minimal fundamental group $\pi_1(M,x_0) := \mathbb{L}(M,x_0)/\sim$. Observe that $\pi_1(M,x_0)$ is a group with the operation given by the class of the concatenation of its representatives; i.e., $[\alpha] \cdot [\gamma] = [\alpha \cdot \gamma]$. In case $M$ is a connected locally definable group, $\pi_1(M,x_0)$ is an abelian group ([7, Prop. 4.1]).

$M$ is called simply connected if $M$ is path connected and $\pi_1(M,x_0)$ is the trivial group.

### 2.3. Covering maps for ld-spaces.

The next definition of covering map for ld-spaces is taken from [2].

**Definition 2.9.** Let $(M_i, \phi_i)_{i \in I}$ and $(N_j, \psi_j)_{j \in J}$ be ld-spaces. A surjective continuous ld-map $\theta : M \to N$ is called an ld-covering map if there is a family $\{\mathcal{O}_l : l \in L\}$ of open definable subspaces of $N$ such that

1. $N = \bigcup_{l \in L} \mathcal{O}_l$,
2. the cover $\{\mathcal{O}_l \cap N_j : l \in L\}$ of every $N_j$ admits a finite subcover, and
3. for every $l \in L$ and each connected component $C$ of $\theta^{-1}(\mathcal{O}_l)$, the restriction $\theta |_C : C \to \mathcal{O}_l$ is a definable homeomorphism (so in particular both $C$ and $\theta |_C$ are definable).

We call $\{\mathcal{O}_l : l \in L\}$ a $\theta$-admissible family of definable neighborhoods.

**Remark 2.10.** Let $(M_i, \phi_i)_{i \in I}$ and $(N_j, \psi_j)_{j \in J}$ be ld-spaces, and let $\theta : M \to N$ be a surjective continuous ld-map. Then it is easy to prove that $\theta : M \to N$ is an ld-covering map if and only if there is a family $\{\mathcal{O}_l : l \in L\}$ of open definable subspaces of $N$ such that

1. $N = \bigcup_{l \in L} \mathcal{O}_l$,
2. the cover $\{\mathcal{O}_l \cap N_j : l \in L\}$ of every $N_j$ admits a finite subcover, and
3. for every $l \in L$, $\theta^{-1}(\mathcal{O}_l)$ is a disjoint union $\bigcup_{i \in L_l} \mathcal{O}'_{l,i}$ of open definable subspaces of $M$ such that for every $i \in L_l$ the restriction $\theta |_{\mathcal{O}'_{l,i}} : \mathcal{O}'_{l,i} \to \mathcal{O}_l$ is a definable homeomorphism (so in particular both $\mathcal{O}'_{l,i}$ and $\theta |_{\mathcal{O}'_{l,i}}$ are definable).

Now, we will prove that any ld-covering map between ld-spaces is closed for definable subspaces; notice that such a map is always open.
Remark 2.11. Let \(( M, ( M_i, \phi_i)_{i \in I} )\) be an ld-space and \( X \subseteq M \) a definable space. If \( y \in \text{Cl}(X) \), then there is a definable map \( g : (0, \epsilon) \to X \), for some \( \epsilon > 0 \), such that \( \lim_{t \to 0} g(t) = y \).

Proof. Since \( X \) is a definable subspace of \( M \), then there is a finite \( J \subseteq I \) such that \( X \subseteq \bigcup_{j \in J} M_j \). As \( y \in \text{Cl}(X) \), then \( y \in \text{Cl}(X \cap M_j) \) for some \( j \in J \). Also, since \( y \in M \), there is \( k \in I \) such that \( y \in M_k \). Since \( M_k \) is open in \( M \) and \( y \in \text{Cl}(X \cap M_j) \), then \( M_k \cap X \cap M_j \neq \emptyset \).

Because the definable spaces of an ld-space are closed under finite intersections, then \( \phi_k ( M_k \cap X \cap M_j ) \) is a definable set in \( Z_k \). As \( y \in \text{Cl}(M_k \cap X \cap M_j) \), then \( \phi_k(y) \in \text{Cl}(\phi_k(M_k \cap X \cap M_j)) \), then, by [16, Thm. 4.8], there is a definable map \( \gamma : (0, \epsilon) \to \phi_k(M_k \cap X \cap M_j) \) such that \( \lim_{t \to 0} \gamma(t) = \phi_k(y) \). Because \( \phi_k \) is a homeomorphism between \( M_k \) and \( Z_k \), \( g = \phi_k^{-1} \circ \gamma : (0, \epsilon) \to M_k \cap X \cap M_j \) is a definable map such that \( \lim_{t \to 0} g(t) = y \). \( \Box \)

Proposition 2.12. Let \(( M, ( M_i, \phi_i)_{i \in I} ), ( N, ( N_j, \psi_j)_{j \in J} )\) be ld-spaces, a definable subspace \( C \subseteq M \) closed in \( M \), and \( \theta : M \to N \) an ld-covering map. Then \( \theta(C) \subseteq N \) is a definable subspace closed in \( N \).

Proof. We will show that if \( y \in \text{Cl}(\theta(C)) \), then \( y \in \theta(C) \). As the image of a definable space by an ld-map is a definable space, \( \theta(C) \) is a definable space. Because \( y \in \text{Cl}(\theta(C)) \), Remark 2.11 yields the existence of a definable map \( g : (0, \epsilon) \to \theta(C) \) such that \( \lim_{t \to 0} g(t) = y \).

Now, since \( \theta : M \to N \) is an ld-covering map, there is an open definable subspace \( O \subseteq N \) such that \( y \in O \), \( \theta^{-1}(O) = \bigcup_{j \in S} O_j' \), and each \( O_j' \) is an open definable subspace in \( M \) homeomorphic to \( O \) by \( \theta \).

Since \( \lim_{t \to 0} g(t) = y \) and \( O \) is an open neighborhood of \( y \), there is \( \delta > 0 \) such that \( \delta \leq \epsilon \) and \( g((0, \delta)) \subseteq O \). Without loss of generality, we can assume that \( \text{dom} (g) = (0, \delta) \).

Let \( C' = (\theta |_C)^{-1}(g((0, \delta))) \). Since \( C' \subseteq \theta^{-1}(O) = \bigcup_{j \in S} O_j' \) and \( C' \) is definable, then saturation implies that \( C' \subseteq O_{j_i} \cup \ldots \cup O_{j_k} \) for some \( k \in \mathbb{N} \). As \( (0, \delta) = g^{-1}(\theta(C')) \), then \( (0, \delta) = g^{-1}(\theta(C')) = g^{-1}\left( \bigcup_{1 \leq i \leq k} \theta(O_{j_i} \cap C') \right) = \bigcup_{1 \leq i \leq k} g^{-1} \circ \theta \left( O_{j_i} \cap C' \right) \).

Since \( \theta |_{O_j'} : O_j' \to O \) is a definable homeomorphism for every \( j \in S \), any path \( f \) in \( O \) can be lifted through \( \theta |_{O_j'} \) to a path \( \tilde{f} \) in \( O_{j_i}' \). Therefore, in particular, for each \( i \in \{1, \ldots, k\} \), \( \tilde{g}_{j_i} = \theta |_{O_{j_i}'}^{-1} \circ g \) is a definable path in \( O_{j_i}' \cap C' \).

By \( \alpha \)-minimality, there are \( j \in \{1, \ldots, k\} \) and a positive \( \epsilon^* < \delta \) such that \( (0, \epsilon^*) \subseteq g_{j_i}^{-1}(O_{j_i}' \cap C') \). Let \( x = \theta |_{O_{j_i}'}^{-1}(y) \). Since \( g |_{(0, \epsilon^*)} (t) \to y \) as \( t \to 0 \), then the lifting \( \tilde{g}_{j_i} |_{(0, \epsilon^*)} (t) \to x \). So, \( x \in \text{Cl}(C) \). But \( C \) is closed in \( U \), so \( x \in C \); namely, \( y \in \theta(C) \). Then the image by \( \theta \) of any definable closed subspace of \( M \) is closed in \( N \). This ends the proof of Proposition 2.12. \( \Box \)
With the previous proposition we can prove the existence of a homeomorphism between simply connected definable spaces as a restriction of a given ld-covering map as we see in the following proposition.

**Proposition 2.13.** Let $M$, $N$ be ld-spaces and $\theta : M \to N$ an ld-covering map. Let $Y \subseteq N$ be a compatible subspace in $N$, then $\theta |_{\theta^{-1}(Y)}; \theta^{-1}(Y) \to Y$ is an ld-covering map of ld-spaces. If, moreover, $Y$ is definable, $n_0 \in Y$, and $m_0 \in \theta^{-1}(n_0)$, then there is a definable subspace $W \subseteq M$ open in $\theta^{-1}(Y)$ such that $m_0 \in W$ and the following hold.

(i) $\theta|_W : W \to Y$ is a definable covering map of ld-spaces.

(ii) If in addition $Y$ is simply connected, then $\theta|_{W_{m_0}} : W_{m_0} \to Y$ is a homeomorphism of definable spaces where $W_{m_0}$ is the connected component of $m_0$ in $W$.

**Proof.** Since the preimage of a compatible subspace by an ld-map is a compatible subspace, $\theta^{-1}(Y)$ is a compatible subset of $M$. As $\theta |_{\theta^{-1}(Y)}; \theta^{-1}(Y) \to Y$ is a continuous surjection, it only remains to show the existence of a $\theta |_{\theta^{-1}(Y)}$-admissible family of definable neighborhoods. Let \( \{O_{ij}\}_{i \in L} \) be a $\theta$-admissible family of definable neighborhoods such that $\theta^{-1}(O_{ij}) = \bigcup_{j \in S_i} O_{ij}$ and $O_{ij} \subseteq O_{ij}$ is ld-homeomorphic to $O_{ij}$ by $\theta$ for any $i \in L$, $j \in S_i$. Then it is easy to see that \( \{O_i \cap Y\}_{i \in L} \) is a $\theta |_{\theta^{-1}(Y)}$-admissible family of definable neighborhoods.

Now, assume that $Y$ is also a definable space. Following, we will prove (i). Let \( \{O_i \cap Y\}_{i \in L} \) be the above $\theta |_{\theta^{-1}(Y)}$-admissible family of definable neighborhoods for $\theta |_{\theta^{-1}(Y)}; \theta^{-1}(Y) \to Y$. Hence, the definability of $Y$ and the saturation of the model imply that there is $s \in \mathbb{N}$ such that $Y = \bigcup_{1 \leq i \leq s} O_i \cap Y$. For each $i \in \{1, \ldots, s\}$ fix an arbitrary finite nonempty subset $S_i \subseteq S_i$ such that if $e_{m_0} \in O_i \cap Y$, then there is $j \in S_i$ such that $e_{m_0} \in O_{ij} \cap \theta^{-1}(Y)$. Let $W = \bigcup \left\{O_{ij} \cap \theta^{-1}(Y) : i \in \{1, \ldots, s\}, j \in S_i\right\}$, which is open in $\theta^{-1}(Y)$. Then \( \{O_i \cap Y : i \in \{1, \ldots, s\}\} \) is a $\theta |_{W}$-admissible family of definable neighborhoods. So $\theta |_{W} : W \to Y$ is a definable covering map of ld-spaces.

For (ii), first we will prove that $\theta |_{W_{m_0}} : W_{m_0} \to Y$ is a definable covering map of ld-spaces, this is the next claim.

**Claim 2.14.** Let $W_{m_0}$ be the connected component of $m_0$ in $W$. Then

(i) $\theta |_{W_{m_0}} : W_{m_0} \to Y$ is surjective.

(ii) There is a $\theta |_{W_{m_0}}$-admissible family of definable neighborhoods.

Therefore, $\theta |_{W_{m_0}} : W_{m_0} \to Y$ is a definable covering map of ld-spaces.

**Proof.** (i) By Fact 4.2 of [2], $W_{m_0}$ is a clopen definable subset of $W$. By Proposition 2.12, $\theta(W_{m_0})$ is a definable space clopen in $Y$, but $Y$ is connected, so $\theta(W_{m_0}) = Y$; i.e., $\theta$ is surjective.

(ii) The same $\theta |_{W}$-admissible family of definable neighborhoods \( \{O_i \cap Y : i \in \{1, \ldots, s\}\} \) works for $\theta |_{W_{m_0}}$ because if $C$ is a connected component of $\theta |_{W}^{-1}(O_i \cap Y)$ in $W$, then $C$ is either entirely contained in $W_{m_0}$ or is disjoint from $W_{m_0}$. Therefore, $C$ is homeomorphic by $\theta |_{W_{m_0}}$ with $O_i \cap Y$.

From (i) and (ii), $\theta |_{W_{m_0}} : W_{m_0} \to Y$ is a definable covering map of ld-spaces. \(\square\)
Since $Y$ is simply connected, [10, Remark 3.8] implies that there is an ld-covering map $\beta : Y \to W_{m_0}$ such that $\text{id} = \theta |_{W_{m_0}} \circ \beta$, then $\theta |_{W_{m_0}} : W_{m_0} \to Y$ is a definable homeomorphism. □

3. THE O-MINIMAL UNIVERSAL COVERING HOMOMORPHISM OF A LOCALLY DEFINABLE GROUP

This section is devoted to introduce the notion and properties of locally definable covering homomorphism and o-minimal universal covering homomorphism.

**Definition 3.1.** Let $\mathcal{U}, \mathcal{V}$ be locally definable groups. An ld-covering map $\theta : \mathcal{U} \to \mathcal{V}$ that is also a homomorphism is called a **locally definable covering homomorphism**. As before, $\{U_i\}_{i \in I}$ is called a $\theta$-admissible family of definable neighborhoods.

Two locally definable covering homomorphisms $\theta : \mathcal{U} \to \mathcal{V}$, $\theta' : \mathcal{U}' \to \mathcal{V}$ are called **equivalent** if there are locally definable covering homomorphisms $\beta : \mathcal{U} \to \mathcal{U}'$ and $\beta' : \mathcal{U}' \to \mathcal{U}$ such that $\theta = \theta' \circ \beta$ and $\theta' = \theta \circ \beta'$, so the following diagram commutes.

\[
\begin{array}{ccc}
U & \stackrel{\beta}{\longrightarrow} & U' \\
\theta \downarrow & & \theta' \downarrow \\
\mathcal{V} & \stackrel{\beta'}{\longrightarrow} & \mathcal{V}
\end{array}
\]

In general, in our diagrams the regular arrows are maps whose existence is assumed, and the dashed arrows are maps whose existence is asserted. The inclusion map is denoted by $i$.

**Fact 3.2.** [7, Theorem 3.6] Let $\theta : \mathcal{U} \to \mathcal{V}$ be a surjective locally definable homomorphism between locally definable groups. If $\ker(\theta)$ has dimension zero, then $\theta : \mathcal{U} \to \mathcal{V}$ is a locally definable covering homomorphism.

**Definition 3.3.** Let $\mathcal{V}$ be a connected locally definable group. A locally definable covering homomorphism $\theta : \mathcal{U} \to \mathcal{V}$ with $\mathcal{U}$ connected is called an **o-minimal universal covering homomorphism** of $\mathcal{V}$ if for every locally definable covering homomorphism $\pi : Z \to \mathcal{V}$ with $Z$ connected, there exists a locally definable covering homomorphism $\beta : \mathcal{U} \to Z$ such that $\theta = \pi \circ \beta$. In this case $Z$ is called an **o-minimal universal covering group** of $\mathcal{V}$.

Note that if there are two o-minimal universal covering homomorphisms $\theta : \mathcal{U} \to \mathcal{V}$ and $\theta' : \mathcal{U}' \to \mathcal{V}$ of a connected locally definable group $\mathcal{V}$, then there exist locally definable covering homomorphisms $\beta : \mathcal{U} \to \mathcal{U}'$ and $\beta' : \mathcal{U}' \to \mathcal{U}$ such that $\theta = \theta' \circ \beta$ and $\theta' = \theta \circ \beta'$. Therefore, if $\mathcal{V}$ has an o-minimal universal covering homomorphism, then it is unique up to equivalent locally definable covering homomorphisms. Thus, we can say “the” o-minimal universal covering homomorphism of $\mathcal{V}$, and sometimes we denote the o-minimal universal covering group of $\mathcal{V}$ by $\tilde{\mathcal{V}}$.

In [9] Edmundo and Eleftheriou constructed a locally definable covering homomorphism $\theta : \mathcal{U} \to \mathcal{V}$ for a given connected locally definable group $\mathcal{V}$ that satisfies the definition of an
o-minimal universal covering homomorphism of \( V \) (Def. 3.3) (so the o-minimal universal covering homomorphism of \( V \) exists), and they showed the following.

**Fact 3.4.** [9, Thm. 3.11] For a connected locally definable group \( V \), the kernel of its o-minimal universal covering homomorphism is isomorphic, as abstract groups, to the o-minimal fundamental group \( \pi_1(V) \).

**Fact 3.5.** [10, Remark 3.8] A locally definable covering homomorphism \( \pi : U \to V \) between connected locally definable groups \( U \) and \( V \) is the o-minimal universal covering homomorphism of \( V \) if and only if \( \pi_1(U) = \{0\} \).

**Remark 3.6.** Let \( U, V \) be connected locally definable groups, and \( \theta : U \to V \) a locally definable covering homomorphism. Then

(i) \( U \) is abelian if and only if \( V \) is abelian.

(ii) Assume that \( V \) is abelian. Then \( U \) is divisible if and only if \( V \) is divisible.

*Proof.*

(i) Clearly, if \( U \) is abelian, by the surjectiveness of \( \theta \), \( V \) is abelian. Now, assume that \( V \) is abelian, and let \( \pi : \tilde{V} \to V \) be the o-minimal universal covering homomorphism of \( V \). Then there is a locally definable covering homomorphism \( \beta : \tilde{V} \to U \) such that \( \pi = \theta \circ \beta \). Since \( V \) is abelian, so is \( \tilde{V} \), then, by going through \( \beta \), \( U \) is also abelian.

(ii) It is clear that if \( U \) is divisible, by the surjectiveness of \( \theta \), \( V \) is divisible. The another implication needs the abelianness of the groups, and it is [5, Proposition 5.13].

**Fact 3.7.** [5, Proposition 5.14] The o-minimal universal covering group of a connected abelian divisible locally definable group is divisible and torsion free.

**Claim 3.8.** Let \( U \) be a connected locally definable group covering an abelian connected definable group \( G \). If \( U \) is torsion free, then \( U \) is simply connected.

*Proof.* Since \( G \) is an abelian (definably) connected definable group, then \( G \) is divisible (see, e.g., the proof of [11, Theorem 2.1]). Then \( U \) is also abelian and divisible, by Remark 3.6. So the map \( p_k : U \to U : x \mapsto x^k \) is a bijective locally definable homomorphism for any \( k \in \mathbb{N} \), so in particular \( p_k \) is a locally definable covering homomorphism. Thus, by [2, Corollary 6.12] or [7, Proposition 4.6], the induced map \( p_k : \pi_1(U) \to \pi_1(U) : [\gamma] \mapsto [p_k \circ \gamma] \) is an injective homomorphism; therefore, the \( k \)-torsion group \( U[k] \) of \( U \) satisfies that \( \{0\} = U[k] = \ker(p_{k,*}) = \pi_1(U)/p_{k,*}(\pi_1(U)) \). Then, \( \pi_1(U) = (\pi_1(U))^k \) for every \( k \in \mathbb{N} \), thus \( \pi_1(U) \) is a divisible group.

Now, let \( \theta : U \to G \) be a locally definable covering homomorphism, and \( \theta : \pi_1(U) \to \pi_1(G) \) its induced injective homomorphism, so \( \theta : (\pi_1(U)) \) is a divisible subgroup of \( \pi_1(G) \). By [11, Theorem 2.1], there is \( s \in \mathbb{N} \) such that \( \pi_1(G) \cong \mathbb{Z}^s \), then the only possible divisible subgroup of \( \pi_1(G) \) is the trivial one, so \( \pi_1(U) = \{0\} \).

From Fact 3.7 and Claim 3.8, we have that if \( G \) is a connected abelian definable group, \( G \) is torsion free if and only if \( G \) is simply connected.
Corollary 3.9. Let $U$ be a connected torsion free locally definable group, $G$ an abelian connected definable group, and $\theta : U \to G$ a locally definable covering homomorphism. Then $\theta : U \to G$ is the o-minimal universal covering homomorphism of $G$.

Proof. By [10, Remark 3.8] and Claim 3.8, $U$ is simply connected. So, by Fact 3.5, $\theta : U \to G$ is the o-minimal universal covering homomorphism of $G$. \qed

4. ABELIAN DEFINABLY GENERATED GROUPS, CONVEX SETS, AND COVERS OF DEFINABLE GROUPS

In this section we present some properties of the abelian $\vee$-definable groups in relation to their smallest type-definable subgroup of index smaller than $\kappa$, if it exists, and to some generic subsets and convex sets.

Note that if $U$ is a connected $\vee$-definable group with $U^{(0)}$, then $U$ has a definable left-generic set, thus, by Fact 2.3 in [12], $U$ is definably generated, and hence locally definable.

In the first part of this section, we point out some central facts about the existence of $U^{(0)}$ for an abelian definably generated group $U$ as well as necessary and sufficient conditions for being a cover of a definable group. The first of these facts gathers Proposition 3.5 and Theorem 3.9 of Peterzil and Eleftheriou’s work in [12].

Fact 4.1. [12] Let $U$ be a connected abelian definably generated group of dimension $d$. Then:

(i) $U$ covers a definable group if and only if the subgroup $U^{(0)}$ exists if and only if $U$ contains a definable generic set.

(ii) If $U^{(0)}$ exists, then $U^{(0)}$ is torsion free, $U$ and $U^{(0)}$ are divisible, and $U/U^{(0)}$ is a Lie group isomorphic, as a topological group, to $\mathbb{R}^k \times \mathbb{T}^r$ for some $k, r \in \mathbb{N}$ with $k + r \leq d$, where $\mathbb{T}$ is the circle group.

Definition 4.2. [5, Def. 5.3] Let $G$ be an abelian group and $X \subseteq G$.

(i) $X$ is called convex if for every $a, b \in X$ and $n, m \in \mathbb{N}$, not both null, $X$ contains every solution $x \in X$ of the equation $x^{n+m} = a^m b^n$.

(ii) The convex hull $ch(X)$ of $X$ is the set of all $x \in G$ such that $x^n = a_1 \cdots a_n$ for some $n \in \mathbb{N}$ and some $a_1, \ldots, a_n \in X$ not necessarily distinct.

(iii) A locally definable abelian group $U$ has definably bounded convex hulls if for all definable $X \subseteq U$, there is a definable $Y \subseteq U$ such that $ch(X) \subseteq Y$.

If $U$ is a divisible torsion free abelian group, then it is easy to prove that $X \subseteq U$ is convex if and only if $\prod_n X = X^n$ for every $n \in \mathbb{N}$.

Fact 4.3. [5, Theorem 5.6] Let $U$ be a connected abelian definably generated group. The following are equivalent:

(i) $U$ covers a definable group.

(ii) For every definable $X \subseteq U$, there is a definable $Y \subseteq U$ such that $\prod_n X \subseteq Y^n$ for all $n \in \mathbb{N}$.

(iii) $U$ is divisible and has definably bounded convex hulls.

The second part of this section is devoted to prove Proposition 4.5.
Claim 4.4. Let $L$ be a topological group isomorphic, as a topological group, to $\mathbb{R}^k \times \mathbb{T}^r$ for some $k, r \in \mathbb{N}$, where $\mathbb{T}$ is the circle group. Let $C \subseteq L$ be a compact neighbourhood of the identity element $e_L$ of $L$. Then there is an increasing sequence $\{n_i\}_i \subseteq \mathbb{N}$ such that $C^{n_i} \subseteq C^{n_{i+1}}$ for every $i \in \mathbb{N}$, and $L = \bigcup_{i \in \mathbb{N}} C^{n_i}$.

Proof. First, note that in $\mathbb{R}^k \times \mathbb{T}^r$ every compact neighbourhood $Y \subseteq \mathbb{R}^k \times \mathbb{T}^r$ of the identity element $e$ of $\mathbb{R}^k \times \mathbb{T}^r$, there is a neighbourhood $X \subseteq Y$ of $e$ such that $\mathbb{R}^k \times \mathbb{T}^r = \bigcup_{n \in \mathbb{N}} X^n$ and $X^n \subseteq X^{n+1}$. Therefore, as $L$ and $\mathbb{R}^k \times \mathbb{T}^r$ are isomorphic as topological groups, then there is a neighbourhood $\mathcal{O} \subseteq C$ of $e_L$ such that $L = \bigcup_{n \in \mathbb{N}} \mathcal{O}^n$, and $\mathcal{O}^n \subseteq \mathcal{O}^{n+1}$ for every $n \in \mathbb{N}$.

Let us define the sequence $\{n_i\}_i \subseteq \mathbb{N}$ inductively as follows.

Let $n_1 = 1$. Let us assume that $n_{i-1}$ is defined for $i \geq 2$. Since $C$ is compact, $C^{n_{i-1}} \cup C^i$ is compact, so $C^{n_{i-1}} \cup C^i \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{O}^n$ yields the existence of finitely many natural numbers $i_1, \ldots, i_s$ such that $C^{n_{i-1}} \cup C^i \subseteq \mathcal{O}^{i_1} \cup \ldots \cup \mathcal{O}^{i_s}$. As $\mathcal{O}^n \subseteq \mathcal{O}^{n+1}$ for every $n \in \mathbb{N}$, then $C^{n_{i-1}} \cup C^i \subseteq \mathcal{O}^{n_i}$ where $n_i = \max\{i_1, \ldots, i_s, n_{i-1}\}$.

Finally, by the definition of the $n_i$’s and $\mathcal{O} \subseteq C$, it follows directly that $C^{n_i} \subseteq C^{n_{i+1}}$ for every $i \in \mathbb{N}$, and $L = \bigcup_{i \in \mathbb{N}} C^{n_i}$.

Proposition 4.5. Let $\mathcal{U}$ be a connected abelian $\forall$-definable group such that $\mathcal{U}^{00}$ exists. Let $X \subseteq \mathcal{U}$ be a definable set such that $\mathcal{U}^{00} \subseteq X$ and $Z \subseteq \mathcal{U}$ a definable set. Then

(i) $\mathcal{U} = \bigcup_{n \in \mathbb{N}} X^n$.

(ii) There is $k \in \mathbb{N}$ such that $Z \subseteq X^k$.

(iii) There is $k \in \mathbb{N}$ such that the convex hull $\text{ch}(Z)$ of $Z$ is contained in $X^k$. If, moreover, $\mathcal{U}$ is torsion free, then $\text{ch}(Z^k) \subseteq X$.

Proof. Let $L$ denote the group $\mathcal{U}/\mathcal{U}^{00}$, let $\pi : \mathcal{U} \to L$ be the quotient homomorphism, and consider $L$ as the locally compact topological space given by the logic topology (see [14, Lemma 7.5]). By [12, Thm. 3.9], $L$ is isomorphic, as a topological group, to $\mathbb{R}^{r_1} \times \mathbb{T}^{r_2}$ for some $r_1, r_2 \in \mathbb{N}$.

As $\mathcal{U}^{00} \subseteq X$, saturation yields the existence of a definable $Y \subseteq X$ such that $\mathcal{U}^{00} \subseteq Y \subseteq Y \cdot Y \subseteq X$. Thus, by [12, Fact 2.3(2)], $Y$ generates $\mathcal{U}$. Furthermore, $\pi'(Y) = \{l \in L : \pi^{-1}(l) \subseteq Y\}$ is an open neighbourhood of the identity element $e_L$ of $L$. Therefore, $\pi(Y)$ is a compact connected neighbourhood of $e_L$ in $L$ and generates $L$.

Claim 4.4 yields the existence of an increasing sequence $\{n_i\}_i \subseteq \mathbb{N}$ such that $\pi(Y)^{n_i} \subseteq \pi(Y)^{n_{i+1}}$ for every $i \in \mathbb{N}$ and $L = \bigcup_{i \in \mathbb{N}} \pi(Y)^{n_i} = \pi\left(\bigcup_{i \in \mathbb{N}} Y^{n_i}\right)$. Hence,

$$\mathcal{U} = \bigcup_{i \in \mathbb{N}} Y^{n_i} \cdot \mathcal{U}^{00} = \bigcup_{i \in \mathbb{N}} \left(Y \cdot \mathcal{U}^{00}\right)^{n_i} \subseteq \bigcup_{i \in \mathbb{N}} X^{n_i} \subseteq \bigcup_{n \in \mathbb{N}} X^n.$$

This gives us (i).

Since $\pi(X)$ is a compact set in $L$ and $L = \bigcup_{i \in \mathbb{N}} \pi(Y)^{n_i}$, there are $k_1, \ldots, k_s \in \{n_i\}_i \subseteq \mathbb{N}$ such that $\pi(X) \subseteq \pi(Y)^{k_1} \cup \ldots \cup \pi(Y)^{k_s}$. As $\pi(Y)^{n_i} \subseteq \pi(Y)^{n_{i+1}}$, then $\pi(X) \subseteq \pi(Y)^{k^*}$ where $k^* = \max\{k_1, \ldots, k_s\}$. Therefore,

$$X \subseteq X \cdot \mathcal{U}^{00} \subseteq Y^{k^*} \cdot \mathcal{U}^{00} \subseteq (Y \cdot Y)^{k^*} \subseteq X^{k^*}.$$
By (i) and saturation, if $Z \subseteq \mathcal{U}$ is a definable set, then there are $l_1, \ldots, l_m \in \mathbb{N}$ such that $Z \subseteq X^{l_1} \cup \cdots \cup X^{l_m}$, then $Z \subseteq X^{k^*}$ where $k^* = \max \{l_1k^*, \ldots, l_mk^*\}$, which yields (ii).

Finally, let us prove (iii). By Lemma 3.7 in [12], $\mathcal{U}^{00}$ exists if and only if $\mathcal{U}$ covers a definable group, and by Theorem 5.6 in [5], if and only if $\mathcal{U}$ has definably bounded convex hulls; i.e., for every definable $Z' \subseteq \mathcal{U}$ there is a definable $W \subseteq \mathcal{U}$ containing the convex hull $\text{ch}(Z')$ of $Z'$. Then, there is a definable set $W$ such that $\text{ch}(Z) \subseteq W \subseteq \mathcal{U}$, and (ii) yields the existence of $k \in \mathbb{N}$ such that $W \subseteq X^k$, then $\text{ch}(Z) \subseteq X^k$. Since $\mathcal{U}^{00}$ exists, Proposition 3.5 in [12] implies that $\mathcal{U}$ is divisible. If in addition $\mathcal{U}$ is torsion free, then the map $x \mapsto x^k : \mathcal{U} \rightarrow \mathcal{U}$ is a group isomorphism for every $k \in \mathbb{N}$, so if $\text{ch}(Z) \subseteq X^k$, then $\text{ch}(Z^k) \subseteq X$.

\[ \square \]

5. Local homomorphisms and generic sets: some technical propositions

Below we prove some technical results that will be applied in the proofs of Theorems 6.1, and 7.1.

Proposition 5.1. Let $\mathcal{Z}$ and $\mathcal{V}$ be locally definable groups such that $\mathcal{Z}^{00}$ exists. Let $W \subseteq \mathcal{Z}$ be a definable set such that $\mathcal{Z}^{00} \subseteq W$, and $\theta : W \rightarrow \mathcal{V}$ be a definable local homomorphism. Then

(i) there is a definable symmetric set $W' \subseteq W$ such that $\mathcal{Z}^{00} \subseteq W' \subseteq \prod_i W' \subseteq W$ and $\theta(W')$ is generic in $\langle \theta(W') \rangle$.

(ii) $\theta(\mathcal{Z}^{00})$ is a type-definable subgroup of $\langle \theta(W') \rangle$ of index less than $\kappa$, and hence $\langle \theta(W') \rangle^{00} \subseteq \theta(\mathcal{Z}^{00}) \subseteq \theta(W')$.

Proof. (i) As $\mathcal{Z}^{00} \subseteq W$, saturation implies that there is a definable symmetric $W' \subseteq W$ such that $\mathcal{Z}^{00} \subseteq W' \subseteq \prod_i W' \subseteq W$. Since $W'$ is generic in $\mathcal{Z}$ and the structure is $\kappa$-saturated (with $\kappa \geq \aleph_1$), then $W'W' \subseteq \bigcup_{i<\aleph_1} w_iW'$ for some $\{w_i\}_{i<\aleph_1} \subseteq \mathcal{Z}$.

Let $I = \{i < \aleph_1 : W'W' \cap w_iW' \neq \emptyset\}$, and $i \in I$. If $xy = w_iz$ with $x, y, z \in W'$, then $w_i = xyz^{-1} \in \prod_i W'$, thus $w_iW' \subseteq \prod_i W' \subseteq W$. Therefore,

$$\theta(W'W') = \theta(W') \theta(W') \subseteq \bigcup_{i \in I} \theta(w_i) \theta(W') \subseteq \langle \theta(W') \rangle,$$

and $\theta(w_i) \in \langle \theta(W') \rangle$ for $i \in I$. Hence, $\theta(W') \theta(W')$ is covered by $\theta(W')$ by $\aleph_1$ group translates. It implies that $\theta(W')$ is a definable generic subset in $\langle \theta(W') \rangle$.

(ii) We will see that $\theta(\mathcal{Z}^{00})$ is a type-definable subgroup of $\langle \theta(W') \rangle$ of index less than $\kappa$. By saturation, $\theta(\mathcal{Z}^{00})$ is a type-definable set. Now, as $[\mathcal{Z} : \mathcal{Z}^{00}] < \kappa$, $W' \subseteq \bigcup_{j<\kappa} b_jz^{00}$ with $\{b_j\}_{j<\kappa} \subseteq \mathcal{Z}$. Let $J = \{j < \kappa : W' \cap b_jz^{00} \neq \emptyset\}$. Then, if $j \in J$ and $x = b_jz$ with $x \in W'$, $z \in \mathcal{Z}^{00}$, then $b_j = xz^{-1} \in W'\mathcal{Z}^{00} \subseteq \prod_2 W'$, so $\theta(b_j) \in \prod_2 \theta(W')$. Thus,

$$\theta(W') \subseteq \bigcup_{j \in J} \theta(b_j) \theta(\mathcal{Z}^{00})$$

and $\theta(b_j) \in \langle \theta(W') \rangle$.
In addition, by (i), \( \langle \theta (W') \rangle \subseteq \bigcup_{i<\kappa} v_i \theta (W') \) for some \( \{ v_i \}_{i<\kappa} \subseteq \langle \theta (W') \rangle \). Then,
\[
\langle \theta (W') \rangle \subseteq \bigcup_{i<\kappa} v_i \bigcup_{j \in J} \theta (b_j) \theta (Z^{00}) = \bigcup_{i<\kappa,j \in J} v_i \theta (b_j) \theta (Z^{00}) .
\]
Hence, \( \left[ \langle \theta (W') \rangle : \theta (Z^{00}) \right] < \kappa \).

Note that since \( \theta (Z^{00}) \) is a type-definable subgroup of \( \langle \theta (W') \rangle \) of index \( < \kappa \), then \( \langle \theta (W') \rangle^{00} \) exists (see [14, Prop. 7.4]), and thus \( \langle \theta (W') \rangle^{00} \subseteq \theta (Z^{00}) \subseteq \theta (W') \).

**Proposition 5.2.** Let \( \mathcal{U} , \mathcal{V} \) be locally definable groups with identities \( e_\mathcal{U} \) and \( e_\mathcal{V} \), respectively, and \( \theta : \mathcal{U} \to \mathcal{V} \) a locally definable covering homomorphism. Let \( Y \subseteq \mathcal{V} \) be a definable simply connected set, and \( Y' \subseteq Y \) a connected definable set such that \( e_\mathcal{V} \in Y' \) and \( Y' \subseteq Y \), then there is a definable set \( W' \subseteq \mathcal{U} \) such that \( e_\mathcal{U} \in W' \), \( \theta |_{W'} : W' \to Y' \) is a definable homeomorphism and a local homomorphism in both directions.

\[
\begin{array}{c}
W' \searrow \cdots \cdots \searrow i \cdots \cdots \searrow U \\
\theta |_{W'} \downarrow \quad \downarrow \theta \\
Y' \searrow i \quad Y' \subseteq Y \searrow i \quad \searrow V
\end{array}
\]

**Proof.** By Proposition 2.13, there is a definable \( W_1 \subseteq \mathcal{U} \) open in \( \theta^{-1} (Y) \) such that \( e_\mathcal{U} \in W_1 \) and \( \theta |_{W_1} : W_1^0 \to Y \) is a definable homeomorphism, where \( W_1^0 \) is the identity component of \( W_1 \). Let \( W' = \theta^{-1} |_{W_1^0} (Y') \), then \( \theta (W'W') = \theta (W') \theta (W') \subseteq Y'Y' \subseteq Y \), then \( W'W' \subseteq \theta^{-1} (Y) \subseteq W_1^0 \) ker \( \theta \).

In addition, \( W_1^0 k_1 \cap W_1^0 k_2 = \emptyset \) if \( k_1 \neq k_2 \) and \( k_1, k_2 \in \ker \theta \); otherwise, if there are \( y_1, y_2 \in W_1^0 \) such that \( y_1 k_1 = y_2 k_2 \), then \( \theta (y_1 k_1) = \theta (y_1) = \theta (y_2 k_2) = \theta (y_2) \), but \( \theta \) is injective in \( W_1^0 \), then \( y_1 = y_2 \), so \( k_1 = k_2 \), which is a contradiction since \( k_1 \neq k_2 \). Then \( \theta |_{W_2} : W_2 \to Y \) is a definable covering map of ld-spaces and \( e_\mathcal{U} \in W_2 \).

Therefore, from the connectedness of \( W'W' \) and \( W'W' \subseteq W_1^0 \) ker \( \theta \), we get \( W'W' \subseteq W_1^0 \). Thus, [3, Remark 2.12] implies that the homeomorphism \( \theta |_{W'} : W' \to Y' \) is a local homomorphism in both directions.

\[\square\]

6. EXTENSION OF A DEFINABLE LOCAL HOMOMORPHISM FROM A TORSION FREE ABELIAN LOCALLY DEFINABLE GROUP

**Theorem 6.1.** Let \( Z \) be a connected abelian torsion free locally definable group such that \( Z^{00} \) exists, and let \( \mathcal{V} \) be an abelian locally definable group. Let \( W \subseteq Z \) be a definable set such that \( Z^{00} \subseteq W \). Assume that \( \theta : W \subseteq Z \to \mathcal{V} \) is a definable local homomorphism.

Then there exists a unique locally definable homomorphism \( \overline{\theta} : Z \to \mathcal{V} \) extending \( \theta \).

If in addition \( \mathcal{V} \) is connected, \( \mathcal{V}^{00} \) exists, \( \mathcal{V}^{00} \subseteq \theta (W) \), \( \theta \) is injective and \( \theta^{-1} : \theta (W) \to W \) is a local homomorphism, then \( \overline{\theta} : Z \to \langle \theta (W) \rangle = \mathcal{V} \) is the o-minimal universal covering homomorphism of \( \mathcal{V} \).
Proof. By Proposition 5.1, there is a definable symmetric $W_1 \subseteq W$ such that $Z^{00} \subseteq W_1 \subseteq \prod_4 W_1 \subseteq W$, and $\theta(W_1)$ is generic in $\langle \theta(W_1) \rangle$. Now, note that since $Z^{00}$ exists, by [12, Proposition 3.5], $Z$ is divisible, then the map $z \mapsto z^k : Z \to Z$ is a group isomorphism. By Proposition 4.5(iii), there is $k \in \mathbb{N}$ such that the convex hull $\text{ch}(W_1^k)$ of $W_1^k$ is contained in $W_1$.

Let $y \in Z = \langle W_1 \rangle = \bigcup_{n \in \mathbb{N}} \prod_n W_1$, so there are $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in W_1$ such that $y = y_1 \cdots y_n$. Let

$$\overline{\theta}(y) = \left( \theta \left( y_1^k \right) \cdots \theta \left( y_n^k \right) \right)^k.$$

Claim 6.2. The map $\overline{\theta} : Z \to V$ defined as above satisfies the following.

(i) $\overline{\theta}$ is a well defined map.
(ii) $\overline{\theta}$ is a locally definable homomorphism.
(iii) $\overline{\theta}$ is the unique extension of $\theta : W \to V$ that is a locally definable homomorphism from $Z$ to $V$.
(iv) If, moreover, $V$ is connected, $V^{00}$ exists, $V^{00} \subseteq \theta(W)$, $\theta$ is injective and $\theta^{-1} : \theta(W) \to W$ is a local homomorphism, then $\overline{\theta}(Z) = V$ and $\overline{\theta}$ is the o-minimal universal covering homomorphism of $V$.

Proof. (i) As $\text{ch} \left( W_1^k \right) \subseteq W_1$, then for every $i, j \in \mathbb{N}$ with $j \leq i$ and $i > 0$ we have that:

$$\prod_j \left( W_1^k \right)^j \subseteq W_1.$$

And since $\theta$ is a locally homomorphism, then for every $y_1, \ldots, y_j \in W_1^k$

$$\theta \left( \frac{1}{y_1^j} \cdots \frac{1}{y_j^j} \right) = \theta \left( \frac{1}{y_1^i} \right) \cdots \theta \left( \frac{1}{y_j^i} \right).$$

(6.0.1)

Now, we will see that $\overline{\theta}$ is well defined.
Let $y \in Z = \langle W_1 \rangle = \bigcup_{n \in \mathbb{N}} \prod_n W_1$, and suppose that

$$y = y_1 \cdots y_n = x_1 \cdots x_m.$$

(6.0.2)
for some $y_1, \ldots, y_n, x_1, \ldots, x_m \in W_1$, and $m, n \in \mathbb{N}$. Additionally, assume, without loss of generality, that $m \leq n$.

\[
\bar{\theta}(y_1 \cdots y_n) = \left( \theta \left( \frac{1}{y_1^n} \right) \cdots \theta \left( \frac{1}{y_n^n} \right) \right)^k
\]
\[
= \left( \theta \left( \frac{1}{y_1^n} \right) \cdots \theta \left( \frac{1}{y_n^n} \right) \right)^k, \quad \text{by Eq. (6.0.1)}
\]
\[
= \left( \theta \left( \frac{1}{x_1^m} \cdots \frac{1}{x_n^m} \right) \right)^k, \quad \text{by Eq. (6.0.2)}
\]
\[
= \left( \theta \left( \frac{1}{x_1^m} \right) \cdots \theta \left( \frac{1}{x_n^m} \right) \right)^k, \quad \text{by Eq. (6.0.1)}
\]
\[
= \theta \left( \frac{1}{x_1} \cdots \frac{1}{x_m} \right)^k.
\]

Therefore, $\bar{\theta}$ is well defined.

(ii) Since $\bar{\theta} \mid_{\prod_n W_1}$ is a definable map for every $n \in \mathbb{N}$, then the restriction of $\bar{\theta}$ to a definable subset of $\mathcal{Z} = \langle W_1 \rangle$ is a definable map. And by definition of $\bar{\theta}$, $\bar{\theta}$ is clearly a group homomorphism.

(iii) First, we will see that $\theta \mid_{ch \left( W_1^k \right)} = \bar{\theta} \mid_{ch \left( W_1^k \right)}$.

By definition of the convex hull (Def. 4.2) and the divisibility of $\mathcal{Z}$, an element in $ch \left( W_1^k \right) \subseteq W_1$ is of the form $y_1^\frac{1}{n} \cdots y_n^\frac{1}{n}$ for some $y_1, \ldots, y_n \in W_1^\frac{1}{k}$ and $n \in \mathbb{N}$. Thus,
\[
\overline{\theta} \left( y_1^{\frac{1}{k_1}} \cdots y_n^{\frac{1}{k_n}} \right) = \left( \theta \left( y_1^{\frac{1}{k_1}} \cdots y_n^{\frac{1}{k_n}} \right) \right)^k, \text{ by Eq. (6.0.1)}
\]
\[
= \left( \theta \left( \frac{1}{k_1} \right) \right)^k \cdots \left( \theta \left( \frac{1}{k_n} \right) \right)^k
\]
\[
= \theta \left( y_1^{\frac{1}{k_1}} \right)^k \cdots \theta \left( y_n^{\frac{1}{k_n}} \right)^k, \text{ by Eq. (6.0.1)}
\]
\[
= \theta \left( y_1^{\frac{1}{k_1}} \cdots y_n^{\frac{1}{k_n}} \right), \text{ by Eq. (6.0.1)}
\]

Then \( \theta \) and \( \overline{\theta} \) agree on \( ch \left( W_1^{\frac{1}{k}} \right) \).

Now, we will verify uniqueness. Let \( \beta : Z \to V \) be a locally definable homomorphism
that is an extension of \( \theta : W \to V \), then in particular \( \beta \) and \( \overline{\theta} \) agree on \( W_1^{\frac{1}{k}} \). Let \( Z' = \{ z \in Z : \beta(z) = \overline{\theta}(z) \} \). Then \( G^{00} \subseteq W_1^{\frac{1}{k}} \subseteq Z' \) and \( Z' \) is an open locally definable subgroup of \( Z \). By [7, Lemma 2.12], \( Z' \) is a compatible subset of \( Z \). But \( Z \) is connected, then \( Z = Z' \); i.e., \( \beta = \overline{\theta} \).

Then, \( \beta = \overline{\theta} \). And hence, \( \overline{\theta} |_{W} = \theta \); i.e., \( \overline{\theta} \) is also an extension of \( \theta : W \to V \).

(iv) First, note that \( \overline{\theta}(Z) = \bigcup_{n \in \mathbb{N}} \prod_n \left( \theta \left( W_1^{\frac{1}{k}} \right) \right)^k = \left( \theta \left( W_1^{\frac{1}{k}} \right) \right)^k \). Now, by Proposition 4.5(ii), there is \( l \in \mathbb{N} \) such that \( W \subseteq W_l \), then \( \theta^{-1}(V^{00}) \subseteq W \subseteq W_l \). By the hypothesis on \( \theta^{-1} \), \( \theta^{-1}(V^{00}) \) is a type-definable subgroup of \( Z \); moreover, by [12, Proposition 3.5], \( V^{00} \) is divisible, so \( \theta^{-1}(V^{00}) \) is an abelian torsion free divisible subgroup of \( Z \). Henceforth, \( \theta^{-1}(V^{00}) \subseteq W_l \) implies that \( \theta^{-1}(V^{00}) = \theta^{-1}(V^{00})^k \subseteq W_1^{\frac{1}{k}} \), thus \( V^{00} \subseteq \left( \theta \left( W_1^{\frac{1}{k}} \right) \right)^k \), it follows that \( V = \left( \theta \left( W_1^{\frac{1}{k}} \right) \right)^k = \langle \theta(W) \rangle \). Then, \( \overline{\theta} : Z \to V \) is surjective.

On the other hand, notice that \( \dim(\ker(\overline{\theta})) = 0 \) if and only if \( \dim(\overline{\theta}(Z)) = \dim(\langle \theta(W_1) \rangle) \). Since \( W_1 \) and \( \theta(W_1) \) are generic in \( Z \) and \( \langle \theta(W_1) \rangle \), respectively, then \( \dim(\overline{\theta}(Z)) = \dim(W_1) \) and \( \dim(\langle \theta(W_1) \rangle) = \dim(\theta(W_1)) \); finally, \( \dim(\overline{\theta}(Z)) = \dim(\langle \theta(W_1) \rangle) \) follows from the injectivity of \( \theta \), so \( \overline{\theta} : Z \to V \) is a locally definable covering homomorphism by [7, Theorem 3.6].

Since \( Z \) is abelian, connected, and \( Z^{00} \) exists, then \( Z \) covers an abelian definable group ([12, Thm. 3.9]). Thus Claim 3.8 yields \( Z \) is simply connected. Therefore, by [10, Remark 3.8], \( \overline{\theta} : Z \to V \) is the \( \alpha \)-minimal universal covering homomorphism of \( V \). \( \square \)

This concludes the proof of Theorem 6.1. \( \square \)
7. Universal covers of locally homomorphic abelian locally definable groups

Theorem 7.1. Let \( U, V \) be abelian connected locally definable groups such that \( U^0 \) exists and \( U^0 \) is an intersection of \( \omega \)-many simply connected definable subsets of \( U \). Let \( X \subseteq U \) be a definable set with \( U^0 \subseteq X \), and \( \phi : X \subseteq U \rightarrow \phi(X) \subseteq V \) a definable homeomorphism and a local homomorphism. Assume that \( \pi : \tilde{V} \rightarrow V \) is the o-minimal universal covering homomorphism of \( V \).

Then,

(i) there are a connected locally definable subgroup \( W \) of \( \tilde{V} \) and a locally definable homomorphism \( \tilde{\theta} : \tilde{W} \rightarrow U \) that is the o-minimal universal covering homomorphism of \( U \), and

(ii) there is a connected symmetric definable \( X' \subseteq X \) with \( U^0 \subseteq X' \) such that \( \pi \mid_{\tilde{W}} : \tilde{W} \rightarrow \langle \phi(X') \rangle \leq V \) is the o-minimal universal covering homomorphism of \( \langle \phi(X') \rangle \).

If in addition \( X \) is simply connected, then \( W \) is a subgroup of the o-minimal universal covering group \( \langle \phi(X) \rangle \) of \( \langle \phi(X) \rangle \).

\[ \begin{array}{ccc}
\pi & \downarrow \\
\tilde{W} & \xrightarrow{i} & \tilde{V} \\
\downarrow & & \\
U & \xrightarrow{i} & X \xrightarrow{\phi} \phi(X) \xrightarrow{i} V
\end{array} \]

Proof. Since \( U^0 \) is an intersection of \( \omega \)-many simply connected definable subsets of \( U \), then \( U^0 \subseteq X \) and saturation imply that there are simply connected definable sets \( X_1 \) and \( X_2 \) such that \( U^0 \subseteq X_2 \subseteq X_2X_2 \subseteq X_1 \subseteq X \). Thus, by [3, Remark 2.12], \( \phi \mid_{X_2} \) is a local homomorphism in both directions.

Moreover, the connected definable set \( Y_2 = \phi(X_2) \) is such that \( Y_2Y_2 \subseteq Y_1 \) where \( Y_1 = \phi(X_1) \) and the identity of \( V \) belongs to \( Y_2 \). So Proposition 5.2 yields the existence of a connected definable \( \tilde{W} \subseteq \tilde{V} \) such that the identity of \( \tilde{V} \) is in \( W_2 \) and \( \pi \mid_{W_2} : W_2 \rightarrow Y_2 \) is a definable homeomorphism and a local homomorphism in both directions, hence so is \( \theta = \phi \mid_{X_2}^{-1} \circ \pi \mid_{W_2} \).

By saturation, \( U^0 \subseteq X_2 \), and Proposition 5.1, then there is a connected symmetric definable \( X' \subseteq X_2 \) such that (i) \( U^0 \subseteq X' \subseteq \prod X' \subseteq X_2 \), (ii) \( W_3 = \theta^{-1}(X') \) is generic in \( W = \langle W_3 \rangle \leq \tilde{V} \), \( W^0 \) exists, and \( W^0 \subseteq W_3 \), and (iii) \( Y_3 = \phi(X') \) is generic in \( \langle Y_3 \rangle \leq V \), \( \langle Y_3 \rangle^0 \) exists, and \( \langle Y_3 \rangle^0 \subseteq Y_3 \). Note that \( W \) is connected by Proposition 2.7.

By Theorem 6.1, there are locally definable homomorphisms \( \varphi : W \rightarrow U \) and \( \varphi : W \rightarrow \langle Y_3 \rangle \leq V \) that are the o-minimal universal covering homomorphisms of \( U \) and \( \langle Y_3 \rangle \), respectively. Moreover, \( \varphi : W \rightarrow U \) and \( \varphi : W \rightarrow \langle Y_3 \rangle \leq V \) are the unique extensions of \( \theta \mid_{W_3} : W_3 \rightarrow U \) and \( \pi \mid_{W_3} : W_3 \rightarrow Y_3 \subseteq V \), respectively, that are locally definable homomorphisms. Since \( \varphi \mid_{W_3} = \pi \mid_{W_3} \), then \( \varphi = \pi \mid_{W} \). If in addition \( X \) is simply connected, then the
o-minimal universal covering group \( \langle \phi (X) \rangle \) of \( \langle \phi (X) \rangle \) exists, and \( W \) is a closed subgroup of \( \langle \phi (X) \rangle \).

7.1. The o-minimal universal covering group of an abelian connected definably compact semialgebraic group. Applying the main results obtained so far, we present below one of the key results of this work.

**Theorem 7.2.** Let \( G \) be an abelian connected definably compact group definable in a sufficiently saturated real closed field \( R \). Then there are a connected \( R \)-algebraic group \( H \), an open connected locally definable subgroup \( W \) of the o-minimal universal covering group \( \tilde{H} (R)^0 \) of \( H(R)^0 \), and a locally definable homomorphism \( \tilde{\theta} : W \subseteq H(R)^0 \rightarrow G \) that is the o-minimal universal covering homomorphism of \( G \).

**Proof.** By [3, Theorem 5.1], there are a connected \( R \)-algebraic group \( H \) such that \( \dim (G) = \dim (H(R)) \), a definable set \( X \subseteq G \) such that \( G^{00} \subseteq X \), and a definable homeomorphism \( \phi : X \subseteq G \rightarrow \phi (X) \subseteq H(R) \) that is also a local homomorphism.

By [4, Thm. 2.2], \( G^{00} \) is an intersection of \( \omega \)-many simply connected definable subsets of \( G \). Thus, by Theorem 7.1, there are a connected locally definable subgroup \( W \leq \tilde{H} (R)^0 \) and a locally definable homomorphism \( \tilde{\theta} : W \subseteq H (R)^0 \rightarrow G \) that is the o-minimal universal covering homomorphism of \( G \), and since \( \dim (G) = \dim (W) \), \( W \) is also open in \( H (R)^0 \). □

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