Abstract. We show the optimal $C^{1,1}$ regularity of geodesics in nef and big cohomology class on Kähler manifolds away from the non-Kähler locus, assuming sufficiently regular initial data. As a special case, we prove the $C^{1,1}$ regularity of geodesics of Kähler metrics on compact Kähler varieties away from the singular locus. Our main novelty is an improved boundary estimate for the complex Monge-Ampère equation that does not require strict positivity of the reference form near the boundary. We also discuss the case of some special geodesic rays.

1. Introduction

In [27], Mabuchi introduced a Riemannian structure on the space of Kähler metrics on a compact manifold $X$ without boundary. Later, Semmes [34] and Donaldson [20] independently showed that these geodesics could be given as solutions to the Dirichlet problem for the complex Monge-Ampère operator, and since then there has been a great deal of work to establish regularity and positivity properties of such solutions – see [3, 4, 11, 12, 13, 15, 21, 31, 32, 33, 35]. In particular, the recent work of Chu-Tosatti-Weinkove [11] establishes $C^{1,1}$ regularity of solutions (based on their earlier work [10]), which is known to be optimal by examples of Lempert-Vivas [30], Darvas-Lempert [18], and Darvas [14].

An obvious follow-up question is to consider geodesics between Kähler metrics on a singular Kähler variety. Using Hironaka’s theorem [25] on resolution of singularities, one usually exchanges the singular variety with a strictly positive Kähler metric for a smooth space with a degenerate metric. One then defines a geodesic between these degenerate metrics as in Semmes/Donaldson, i.e. as a solution to a Dirichlet problem for the complex Monge-Ampère operator. It is then natural to ask about regularity for general solutions to this Dirichlet problem, not necessarily those arising as geodesics on a singular variety. In fact, we shall go a step further and investigate regularity for the Dirichlet problem when the reference form is not even semi-positive, but merely nef and big. This will not only cover the previous set up, but will also include the case of geodesics in a nef and big class. We refer the reader to [16] [21] for previous work in the semi-positive case, and [22] for the nef and big case.

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When working in such generality, there are two fundamental problems that one must overcome: the first is a lack of inherent positivity near the boundary, which prevents previously known boundary estimates [8, 23, 9] from being applied. The second is that we will need to allow our boundary data to be unbounded from below – our approach is to approximate the unbounded boundary data by smooth functions. In full generality, we will have to leave this as a technical assumption unfortunately, but we provide a construction of the approximations in the case of geodesics in a nef and big class.

Our main result follows the approach in [2, 8, 23, 6, 28] with several technical improvements – we state our main theorem now, delaying some notation and definitions until the end of the section.

**Theorem 1.1.** Suppose that \((M, \omega)\) is a compact Kähler manifold with weakly pseudoconcave boundary. Let \(\alpha\) be a smooth, real \((1, 1)\)-form on \(M\) that is \(\psi\)-big and nef, and suppose there is a function \(\varphi \in \text{PSH}(M, \alpha)\), \(\psi \leq \varphi \leq 0\), such that:

1. **(a)** There exists a sequence of smooth functions \(\varphi_\varepsilon \in \text{PSH}(\alpha + \varepsilon \omega) \cap C^\infty(M)\), decreasing to \(\varphi\), such that we have the bounds:
   \[
   |\nabla \varphi_\varepsilon|_\omega + |\nabla^2 \varphi_\varepsilon|_\omega \leq Ce^{-B_0 \psi}
   \]
   for each \(\varepsilon > 0\), with \(B_0, C\) fixed positive constants.
2. **(b)** We have the key positivity condition:
   \[
   (1.1) \quad \alpha + \varepsilon \omega + \sqrt{-1} \partial \overline{\partial} \varphi_\varepsilon \geq Ce^{B_0 \psi} \omega,
   \]
   for each \(\varepsilon > 0\), with \(B_0, c\) fixed positive constants.

Then the envelope:
\[
V := \sup \{ v \in \text{PSH}(M, \alpha) \mid v|_{\partial M} \leq \varphi|_{\partial M} \}
\]
is in \(C^{1,1}_{\text{loc}}(M \setminus \text{Sing}(\psi))\).

We then get as immediate corollaries:

**Corollary 1.2.** Given two cohomologous Kähler metrics \(\omega_1, \omega_2\) on a singular Kähler variety \(X\), the geodesic connecting them is in \(C^{1,1}_{\text{loc}}(X_{\text{reg}} \times A)\), where \(A \subset \mathbb{C}\) is an annulus and \(X_{\text{reg}}\) is the smooth part of \(X\).

Here, by a compact Kähler variety we mean a reduced, irreducible compact complex analytic space which admits a Kähler metric in the sense of Moishezon [29]. Note that we do not need \(X\) to be normal.

**Corollary 1.3.** Let \((X, \omega)\) be a smooth Kähler manifold without boundary, \([\alpha]\) a big and nef class, and \(\varphi_1, \varphi_2\) two \(\alpha\)-psh, exponentially smooth functions with the same singularity type that also satisfy the condition \((1.1)\) and \(\psi \leq \varphi_1, \varphi_2\). Then the geodesic connecting \(\varphi_1\) and \(\varphi_2\) is in \(C^{1,1}_{\text{loc}}((X \setminus E_{nK}(\alpha)) \times A)\), where again \(A \subset \mathbb{C}\) is an annulus.
Here we say that a function $f$ is more singular than $g$ if $f - C \leq g$ for some constant $C$. Similarly, we say that $f$ and $g$ has the same singularity type if $|f - g| < C$.

The most interesting case to apply Corollary 1.3 is when $\varphi_1$ and $\varphi_2$ have minimal singularities. Note however that the corollary covers more singular initial data as well, provided there is an appropriate Kähler current $\psi$ – for instance, it deals with the case when both $\varphi_1$ and $\varphi_2$ are also Kähler currents. Of course though, if $\psi$ is singular on a larger set than $E_{nK}(\alpha)$, then one gets worse estimates.

Corollary 1.2 was raised as an explicit question at the AIM workshops “The complex Monge-Ampère equation” [5, Question 7] and ”Nonlinear PDEs in real and complex geometry” [36, Question 2.8], and Corollary 1.3 confirms an expectation raised in [17, pg. 396].

Note that if $\varphi$ in Theorem 1.1 is actually smooth near the boundary, one can simply take $\varphi_\varepsilon = \max\{\varphi, v_\varepsilon - C_\varepsilon\}$ for all $\varepsilon > 0$, where the $v_\varepsilon$ come from the nef condition (see below), and the $C_\varepsilon$ are large constants such that $\varphi_\varepsilon = \varphi$ near the boundary – this is because we don’t actually need the estimates in part (a) to hold everywhere, only near the boundary. In this manner we recover, and actually improve upon, [28, Theorem 1.3].

Remark 1.4. Finally, we note that some of the estimates in this paper can be combined with the technique of [11] to improve upon the main result of [24] – the proof is straightforward but tedious, so we leave the details to the interested reader. The merit of He’s technique is that it does not require any positivity of the boundary data beyond that they be quasi-psh. However, it applies only in the setting of geodesics and the overall conclusion is weaker, establishing $C^{1,1}$ regularity in the spacial directions only, i.e. not in the annular directions.

Theorem 1.1 will be proved in Section 2. The new boundary estimate is established in Proposition 2.3 – it is an a priori bound for the tangent-normal derivatives along the Berman path [2]. We prove Corollaries 1.2 and 1.3 in Section 3 and briefly discuss the case of geodesic rays – we mainly observe that the results in [28] still apply in this generality. Finally, we include an appendix containing some estimates for the Dirichlet problem for the $\omega$-Laplacian when the boundary data is degenerating, which will be needed in the proof of Theorem 1.1.

We now set some notation and definitions, which are standard in the case of a manifold without boundary, but do not generally make sense when there is boundary – we shall adopt their use however for convenience and enhanced readability. Throughout, $(M^n, \omega)$ will be a compact Kähler manifold with non-empty boundary, of complex dimension $n$. 
First, notation – given a smooth \((1,1)\)-form \(\alpha\) on \(M\), and a function \(f : M^\circ \to \mathbb{R}\), we write:
\[
\alpha_f := \alpha + \sqrt{-1} \partial \overline{\partial} f.
\]
Abusing notation, we will also mean \(f|_{\partial M}\) to be the upper-semicontinuous extension of \(f\) to \(\partial M\) – that is, we define:
\[
f|_{\partial M}(x_0) := \limsup_{x \to x_0} f(x) \quad \forall x_0 \in \partial M.
\]
Note that if \(f\) is actually continuous up the boundary, then our definition agrees with the normal one.

We now make some definitions. First, we say that \(\varphi : M^\circ \to \mathbb{R}\) is \(\alpha\)-plurisubharmonic (or \(\alpha\)-psh) and write \(\varphi \in \text{PSH}(M, \alpha)\), if \(\varphi\) is upper-semicontinuous and satisfies:
\[
\alpha \varphi \geq 0
\]
in the sense of currents. Now, we say that a closed, real \((1,1)\)-form \(\alpha\) on \(M\) is \textbf{big} if there exists a function \(\psi\) and a constant \(\delta > 0\) such that:
\begin{enumerate}
  \item \(\psi \leq 0\).
  \item \(\psi\) is \textbf{exponentially smooth} – i.e. \(e^{C\psi} \in C^\infty(\overline{M})\) for some \(C\). Note that this forces \(\psi\) to be bounded above, but not below, even at the boundary.
  \item \(\alpha \psi \geq \delta \omega\) – i.e. \(\alpha \psi\) is a \textbf{Kähler current}.
\end{enumerate}

It is somewhat more proper to call \(\alpha \psi\)-\textbf{big}, in order to emphasize that there is no canonical \(\psi\), as there is in the boundary-less case – we will not always do so however, sometimes leaving the \(\psi\) implicit. Note also that condition (2) is weaker than assuming \(\psi\) has analytic singularities, which is usually done in the case without boundary.

Finally, we say that \(\alpha\) is \textbf{nef} if for every \(\varepsilon > 0\) there exists a bounded function \(v_\varepsilon\), smooth up the boundary, such that:
\[
\alpha + \varepsilon \omega + \sqrt{-1} \partial \overline{\partial} v_\varepsilon > 0.
\]

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2. \textbf{\(C^{1,1}\)-Estimates for Big and Nef Classes}

First, we point out that one can relax the assumptions in (1.1) slightly – throughout, we’ll work with this other function \(F\), as it makes the proof slightly easier to follow:

**Proposition 2.1.** Suppose that \(F\) is an exponentially smooth, quasi-psh function such that:
\[
(2.1) \quad \alpha + \varepsilon \omega + \sqrt{-1} \partial \overline{\partial} \varphi_\varepsilon \geq e^F \omega \quad \forall \varepsilon > 0.
\]

Then there exists an exponentially smooth, strictly \(\alpha\)-psh function \(\tilde{\psi}\) such that (1.1) holds for all \(\varepsilon > 0\) and
\[
\alpha + \sqrt{-1} \partial \overline{\partial} \tilde{\psi} \geq \frac{\delta}{2} \omega.
\]
Further, \( \tilde{\psi} \) will be singular only along \( \text{Sing}(\psi) \cup \text{Sing}(F) \).

**Proof.** Let \( \psi \) be as in the big condition. We may assume without loss of generality that \( F \leq 0 \). By assumption,
\[
\sqrt{-1} \partial \overline{\partial} F \geq -C \omega,
\]
so if we define:
\[
\tilde{\psi} := \psi + \frac{\delta}{2C} F,
\]
we have the claim, with \( c = 1 \) and \( B = 2C/\delta \). \( \square \)

Note that, if we happen to have \( \varphi_\varepsilon = \varphi \) for all \( \varepsilon > 0 \) (so that \( \varphi \) is actually smooth and \( \alpha \) is “semipositive”), then we can scale \( \omega \) such that \( \omega \geq \alpha_\varphi \). Then the function
\[
F := \log \left(\frac{(\alpha_\varphi)^n}{\omega^n}\right)
\]
is exponentially smooth and satisfies (2.1) – to see this, look at the eigenvalues \( \lambda_j \) of \( \alpha_\varphi \) in normal coordinates for \( \omega \) at a point:
\[
\lambda_j = \left( \prod_{k=1}^n \lambda_k \right) / \left( \prod_{k=1, k \neq j}^n \lambda_k \right) \geq e^F.
\]
Unfortunately, such an \( F \) will not always be quasi-psh – however, we will show in Section 3 that in the case of a geodesic between two Kähler metrics on a singular Kähler variety, we can always find an exponentially smooth \( F' \leq F \) that will be quasi-psh, and hence we will be able to apply our results in that setting.

Before moving on, observe that being exponentially smooth gives control over all derivatives of \( F \) and \( \psi \):
\[
\begin{align*}
|\nabla \psi|_g + |\nabla^2 \psi|_g &\leq Ce^{-B_0 \psi} \\
|\nabla F|_g + |\nabla^2 F|_g &\leq Ce^{-B_0 F},
\end{align*}
\]
where here \( g \) is the Riemannian metric corresponding to \( \omega \). Also, by replacing \( \varepsilon \) with \( \varepsilon/2 \) and relabeling the \( \varphi_\varepsilon \), we can improve condition (2.1) to
\[
\alpha + \varepsilon \omega + \sqrt{-1} \partial \overline{\partial} \varphi_\varepsilon \geq (e^F + \varepsilon/2) \omega,
\]
without changing the problem. Finally, we will also assume without loss of generality that:
\[
\alpha \leq \omega.
\]
Proof of Theorem 1.1. Our strategy will be the same as in [28], which is a combination of the techniques in [2] and the estimates in [12]. We begin by approximating $V$ by the following envelopes:

$$V_{\varepsilon} := \sup \{ v \in \text{PSH}(M, \alpha + \varepsilon \omega) \mid v|_{\partial M} \leq \varphi_{\varepsilon}|_{\partial M} \}.$$ 

Note that the $V_{\varepsilon}$ decrease pointwise to $V$ as $\varepsilon$ decreases to 0.

We now define the obstacle functions $h_{\varepsilon} \in C^\infty(M)$ as the solutions to the following Dirichlet problems:

$$\{ \Delta_2 h_{\varepsilon} = -n, \quad h_{\varepsilon}|_{\partial M} = \varphi_{\varepsilon}|_{\partial M} \}.$$

By Lemma A.1 and A.2, we have control over the derivatives of the $h_{\varepsilon}$:

$$|\nabla h_{\varepsilon}|_g + |\nabla^2 h_{\varepsilon}|_g + |\nabla^3 h_{\varepsilon}|_g \leq Ce^{-B_0\psi},$$

and we know that they decrease as $\varepsilon \to 0$. In particular, for all $\varepsilon \leq 1$, $h_{\varepsilon} \leq h_1 \leq C$, independent of $\varepsilon$.

Observe that for any $v \in \text{PSH}(M, \alpha + \varepsilon \omega)$ with $v|_{\partial M} \leq \varphi_{\varepsilon}|_{\partial M}$, we have:

$$v \leq h_{\varepsilon}$$

by (2.4) and the weak maximum principle for the $\omega$-Laplacian. We thus see that:

$$V_{\varepsilon} = \sup \{ v \in \text{PSH}(M, \alpha + \varepsilon \omega) \mid v \leq h_{\varepsilon} \},$$

where the inequality now holds on all of $M$.

We now approximate the $V_{\varepsilon}$ by the smooth solutions to the non-degenerate Dirichlet problem:

$$\{ (\alpha + \varepsilon \omega + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon,\beta})^n = e^\beta(u_{\varepsilon,\beta}-h_{\varepsilon}) + n \log(\varepsilon/4)\omega^n, \quad \alpha + \varepsilon \omega + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon,\beta} > 0, \quad u_{\varepsilon,\beta}|_{\partial M} = \varphi_{\varepsilon}|_{\partial M} \}.$$ 

By [2, Proposition 2.4], [28, Proposition 4.5], the $u_{\varepsilon,\beta}$ converge uniformly to $V_{\varepsilon}$ as $\beta \to \infty$ (the convergence is not uniform in $\varepsilon$, but this will not be a problem as we will send $\beta$ to infinity before sending $\varepsilon$ to zero). Note that as $u_{\varepsilon,\beta} \in \text{PSH}(M, \alpha + \varepsilon \omega)$ with $u_{\varepsilon,\beta}|_{\partial M} = \varphi_{\varepsilon}|_{\partial M}$, by (2.6), we have:

$$u_{\varepsilon,\beta} - h_{\varepsilon} \leq 0.$$

Also note that:

$$(\alpha + \varepsilon \omega + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon})^n \geq (\varepsilon/2)^n\omega^n \geq e^\beta(\varphi_{\varepsilon}-h_{\varepsilon}) + n \log(\varepsilon/4)\omega^n$$

where we used (2.3) and $\varphi_{\varepsilon} \leq h_{\varepsilon}$. This makes $\varphi_{\varepsilon}$ a subsolution of (2.7), so that $u_{\varepsilon,\beta}$ actually exists (by results of [8, 23]), and we have:

$$\varphi_{\varepsilon} \leq u_{\varepsilon,\beta}.$$

Our goal is to establish uniform $C^2$-estimates for $u_{\varepsilon,\beta}$, independent of $\varepsilon$ and $\beta$ (we drop the subscripts now for ease of notation). We have just shown the requisite $C^0$-bound:

$$\psi \leq \varphi \leq \varphi_{\varepsilon} \leq u \leq h_{\varepsilon} \leq h_1.$$
and since $\varphi_\varepsilon|_{\partial M} = u|_{\partial M} = h_\varepsilon|_{\partial M}$, it follows that the gradient is bounded on $\partial M$:

$$|\nabla u|_g \leq |\nabla \varphi_\varepsilon|_g + |\nabla h_\varepsilon|_g \leq Ce^{-B_0\psi}$$ on $\partial M$.  

We now bound the gradient on the interior by simplifying the argument in [12, Lemma 4.1, iii]):

**Lemma 2.2.** There exist uniform constants $\beta_0, B, \text{ and } C$ such that

$$|\nabla u|_g \leq Ce^{-B\psi}$$ for all $\beta \geq \beta_0$.

**Proof.** We begin by defining:

$$\tilde{\omega} := \alpha + \varepsilon \omega + \sqrt{-1} \partial \bar{\partial} u.$$

Then note that, as $u \geq \psi$ and $\psi \leq 0$, we have:

$$\tilde{u} := u - (1 + \delta)\psi \geq -\delta \psi \geq 0,$$

and:

$$\tilde{\omega} - \sqrt{-1} \partial \bar{\partial} \tilde{u} = (1 + \delta)(\alpha + \sqrt{-1} \partial \bar{\partial} \psi) - \delta \alpha + \varepsilon \omega \geq (1 + \delta)\delta \omega - \delta \alpha \geq \delta^2 \omega,$$

by (2.4). We now seek to bound the quantity:

$$Q := e^{H(\tilde{u})}|\nabla u|^2_g,$$

by a constant independent of $\varepsilon$ and $\beta$, where here $H(s)$ is defined for $s \geq 0$ as:

$$H(s) := -Bs + \frac{1}{s+1},$$

for some large constant $B$ to be determined. Let $x_0$ be a maximum point for $Q$ – it cannot be in $\text{Sing}(\psi)$, as $Q$ is zero there. If it is on the boundary of $M$, by (2.11) and $\tilde{u} \geq -\delta \psi$, we have

$$Q(x_0) \leq Ce^{-B\tilde{u}(x_0)} - B_0\psi(x_0) \leq Ce^{(B\delta - B_0)\psi(x_0)} \leq C,$$

provided we take $B > B_0/\delta$.

Thus, suppose that $x_0$ is an interior point. It suffices to prove that

$$|\nabla u|^2_g(x_0) \leq Ce^{-B\psi(x_0)},$$

for some uniform constant $C$ – by the same argument as above, this would imply $Q$ is uniformly bounded.

Next, following the estimates in [12, Lemma 4.1, iii]) (replacing $\delta$ with $\delta^2$), we see that:

$$0 \geq e^{-H} \Delta_g Q(x_0) \geq H''|\nabla u|^2_g|\nabla \tilde{u}|^2_g + \left( -\frac{\delta^2}{2} H' - C \right) |\nabla u|^2_g \text{tr}_{\tilde{g}} \omega$$

$$+ (CH' + 2\beta)|\nabla u|^2_g - 2\beta \text{Re}(\nabla h_\varepsilon, \nabla u)_g$$

$$+ 2\text{Re}(\nabla (n \log(\varepsilon/4)), \nabla u)_g + CH' e^{-B_0\psi} + CH' |\nabla \tilde{u}|^2_{\tilde{g}},$$

for some uniform constant $C > 1$, whose exact value will change from line to line (here $\tilde{g}$ is the Riemannian metric corresponding to $\tilde{\omega}$). Picking then

$$B = \max\{(2/\delta^2)(C + 1), B_0/\delta, 3\},$$
we may use the definition of $H$ to see that for $\beta \geq \beta_0 := C(B + 1) + 1$, we have:

\begin{equation}
0 \geq \frac{2|\nabla u|^2 |\nabla \bar{u}|^2}{(\bar{u} + 1)^3} + (\beta + 1)|\nabla u|^2 - 2\beta|\nabla h_\varepsilon|^2 |\nabla u|_g \\
- C(B + 1)e^{-B_0\psi} - C(B + 1)|\nabla \bar{u}|^2_g.
\end{equation}

(2.13)

We may now assume that at $x_0$ we have both:

$$|\nabla u|^2_\varepsilon \geq C(B + 1)(\bar{u} + 1)^3$$

and

$$|\nabla u|_g \geq 2|\nabla h_\varepsilon|^2.$$

If either condition fails, we obtain $|\nabla u|^2_\varepsilon(x_0) \leq C e^{-B_0\psi(x_0)}$ directly. Otherwise, we still get:

$$C(B + 1)e^{-B_0\psi} \geq |\nabla u|^2_\varepsilon(x_0),$$

from (2.13), as required. □

We now bound the Hessian near the boundary.

**Proposition 2.3.** Assume that the key condition (2.3) is satisfied for each $\varepsilon > 0$. Then there exist uniform constants $B$ and $C$ such that

$$|\nabla^2 u|_g \leq Ce^{-B(F + \psi)}$$

on $\partial M$.

**Proof.** Fix a point $p \in \partial M$, and center coordinates $(\{z_i\}_{i=1}^n, B_R)$ at $p$, where $B_R$ is a ball of radius $R$. Write $z_i = x_{2i-1} + \sqrt{-1}x_{2i}$ for $1 \leq i \leq n$. Let $r$ be a defining function for $M$ in $B_R$ (so that $\{r \leq 0\} = M \cap B_R$ and $\{r = 0\} = \partial M \cap B_R$). After making a quadratic change of coordinates, we may assume that (see [6, Remark 7.13])

$$r(z) = -x_{2n} + \Re \left( \sum_{i,j=1}^n a_{ij}z_i \overline{z_j} \right) + O(|z|^3) \text{ near } p.$$

As in [6, pg. 272], we define the tangent vector fields

$$D_\gamma = \frac{\partial}{\partial x_\gamma} - \frac{r_{x_\gamma}}{r_{x_{2n}}} \frac{\partial}{\partial x_{2n}} \text{ for } 1 \leq \gamma \leq 2n - 1$$

and the normal vector field

$$D_{2n} = -\frac{1}{r_{x_{2n}}} \frac{\partial}{\partial x_{2n}}.$$

Recall that we are writing:

$$\tilde{\omega} := \alpha + \varepsilon \omega + \sqrt{-1}\partial\overline{\partial}u$$

and

$$\tilde{\omega} = \sqrt{-1} \sum_{i,j=1}^n \bar{g}_{ij} dz_i \wedge d\overline{z}_j.$$
We also denote the inverse matrix of \((\tilde{g}^{-1})\) by \((\tilde{g}^{-1})\). Throughout, \(C\) will be a constant, independent of \(\varepsilon, \beta\), whose exact value may change from line to line.

We split the proof into three steps:

**Step 1.** The tangent-tangent derivatives.

Since \(u = \varphi_\varepsilon\) on \(\partial M\), at 0 (\(p \in \partial M\)), we have
\[
|D_\gamma D_\eta u| = |D_\gamma D_\eta \varphi_\varepsilon| \leq C e^{-C\psi} \quad \text{for } 1 \leq \gamma, \eta \leq 2n - 1,
\]
as desired.

**Step 2.** The tangent-normal derivatives.

We define
\[
U := u - h_\varepsilon
\]
and
\[
U_\gamma := D_\gamma U.
\]
Now consider the following quantity:
\[
w = \mu_1 (u - \varphi_\varepsilon) + e^{BF} \left( \mu_2 |z|^2 - e^{B\psi} |U_\gamma| - e^{B\psi} U_\gamma^2 \right),
\]
where \(1 \leq \gamma \leq 2n - 1\) and \(\mu_1, \mu_2,\) and \(B\) are large constants to be determined later. Note that the \(\gamma\) derivatives/directions are real, while the \(i\) derivatives/directions will be complex.

We claim that
\[
(2.14) \quad w \geq 0 \text{ on } B_R.
\]

Given the claim, we can control the tangent-normal derivatives as follows. Dropping the square term, we see:
\[
|U_\gamma| \leq \mu_1 e^{-B(F + \psi)} (u - \varphi_\varepsilon) + \mu_2 e^{-B\psi} |z|^2.
\]
At 0, both sides are 0, so
\[
|D_{2n} U_\gamma| \leq D_{2n} \left( \mu_1 e^{-B(F + \psi)} (u - \varphi_\varepsilon) + \mu_2 e^{-B\psi} |z|^2 \right).
\]
Combining this with \((2.2)\) and Lemma \(2.2\) we see that
\[
|D_{2n} U_\gamma| \leq C e^{-(B + B_0)(F + \psi)}.
\]
By the definition of \(U_\gamma\) and \((2.5)\), we then have
\[
|D_{2n} D_\gamma u| \leq |D_{2n} U_\gamma| + |D_{2n} D_\gamma h_\varepsilon| \leq C e^{-(B + B_0)(F + \psi)},
\]
as desired.

We now prove \((2.14)\). We first show that it holds on the boundary of \(B_R \cap M\), and then extend it to the interior by a minimum principle argument. \(\partial (B_R \cap M)\) has two components, \(\partial B_R \cap M\) and \(B_R \cap \partial M\). On the second
part, since \( u = h_\varepsilon = \varphi_\varepsilon \) on \( \partial M \), we see \( w = \mu_2 e^{BF} |z|^2 \geq 0 \). On \( \partial B_R \cap M \), recalling that \( u \geq \varphi_\varepsilon \) and taking \( B \) to be sufficiently large, we see that

\[
 w \geq e^{BF} \left( \mu_2 R^2 - e^{BF} |U_\gamma| - e^{BF} U_\gamma^2 \right) \geq e^{BF} (\mu_2 R^2 - C).
\]

Hence, after fixing \( R \) and \( \mu_2 \), we can arrange that

\[
 (2.15) \quad w \geq 0 \text{ on } \partial (B_R \cap M).
\]

Suppose then that \( x_0 \) is a minimum point of \( w \). If it is the case that either \( U_\gamma(x_0) = 0 \) or \( e^{F(x_0)} = 0 \), then clearly \( w(x_0) \geq 0 \), and hence \( w \geq 0 \) on all of \( B_R \cap M \). Thus we may assume that \( x_0 \) is an interior minimum such that \( U_\gamma(x_0) \neq 0 \) and \( e^{F(x_0)} > 0 \) – we will assume that \( U_\gamma(x_0) < 0 \) here, as the alternative is basically the same.

At \( x_0 \), we wish to compute

\[
 \Delta g w = \mu_1 \Delta g (u - \varphi_\varepsilon) + \mu_2 \Delta g (e^{BF} |z|^2)
 + \Delta g (e^{BF} + B \psi U_\gamma) - \Delta g (e^{BF} + B \psi U_\gamma^2).
\]

For the first term, we observe that:

\[
 \Delta g (u - \varphi_\varepsilon) = n - tr_\omega (\alpha + \varepsilon \omega + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon) \leq n - (e^F + \varepsilon/2)tr_\omega,
\]

by (2.13), and that

\[
 \frac{\varepsilon}{2} tr_\omega \geq n e^{-\log(\varepsilon/4)}
\]

by the arithmetic-geometric mean inequality and (2.7). As \( u - h_\varepsilon \leq 0 \), we have then that

\[
 \frac{\varepsilon}{2} tr_\omega \geq e^{-\log(\varepsilon/4)} = 2n.
\]

Thus,

\[
 (2.18) \quad \mu_1 \Delta g (u - \varphi_\varepsilon) \leq -\mu_1 (e^F + \varepsilon/4)tr_\omega.
\]

For the second term of (2.16), by (2.2) and a direct calculation, we obtain

\[
 (2.19) \quad \mu_2 \Delta g (e^{BF} |z|^2) \leq CB^2 e^{(B-B_0)F} tr_\omega.
\]

For the third term of (2.16), by (2.2), Lemma 2.2, and the Cauchy-Schwarz inequality, we have

\[
 \Delta g (e^{BF+\psi} U_\gamma)
 = e^{BF+\psi} \Delta g (U_\gamma) + U_\gamma \Delta g (e^{BF+\psi})
 + Be^{BF+\psi} \overline{g^{ij}} \left( (F + \psi)u_{\gamma j} + (F + \psi) j_{\gamma i} \right)
 \leq e^{BF+\psi} \Delta g (D_\gamma u) - e^{BF+\psi} \Delta g (D_\gamma h_\varepsilon) + CB^2 e^{(B-B_0)(F+\psi)} tr_\omega
 + Be^{BF+\psi} \overline{g^{ij}} \left( \frac{1}{2B} U_{ij} U_{\gamma j} + 2B (F + \psi) u_{\gamma j} \right)
 \leq e^{BF+\psi} \Delta g (D_\gamma u) + CB^2 e^{(B-B_0)(F+\psi)} tr_\omega + \frac{1}{2} e^{BF+\psi} \overline{g^{ij}} U_{ij} U_{\gamma j}.
\]
By the definition of $D_\gamma$, we have

$$D_\gamma = \frac{\partial}{\partial x_\gamma} + a \frac{\partial}{\partial x_{2n}}$$

where $a = -\frac{r_{x_\gamma}}{r_{x_{2n}}}$.

For the third order term $\Delta_\gamma(D_\gamma u)$, we see that (cf. [6 pg. 275])

$$\Delta_\gamma(D_\gamma u) = \text{tr}_\omega(D_\gamma \sqrt{-1} \partial \bar{\partial} u) + u_{x_{2n}} \Delta_\gamma a + 2a_{x_{2n}-1} - 2\text{tr}_\omega(da \wedge \iota_{\partial / \partial x_{2n}-1}(\alpha + \varepsilon \omega)),$$

and applying $D_\gamma$ to (2.21) gives

$$\text{tr}_\omega(D_\gamma(\alpha + \varepsilon \omega) + D_\gamma \sqrt{-1} \partial \bar{\partial} u) = \beta U_\gamma + \text{tr}_\omega(D_\gamma \omega),$$

where we are letting $D_\gamma$ act on the components $\alpha$ and $\omega$ in the (fixed) $z$-coordinates. Hence,

$$\Delta_\gamma(D_\gamma u) = \beta U_\gamma + \Gamma,$$

where $\Gamma$ denotes a term satisfying $|\Gamma| \leq Ce^{-B_0 \psi} \text{tr}_\omega \omega$. Using this, we then get the estimate:

$$\Delta_\gamma(e^{B(F+\psi)} U_\gamma) \leq \beta e^{B(F+\psi)} U_\gamma + C B^2 e^{(B-B_0)(F+\psi)} \text{tr}_\omega \omega + \frac{1}{2} e^{B(F+\psi)} g^i U_i U \gamma_i U \gamma_j.$$

For the fourth term of (2.16), using (2.20) and Lemma 2.2 at the expense of increasing $B_0$, we compute

$$-\Delta_\gamma(U_\gamma^2) = -2 U_\gamma \Delta_\gamma(U_\gamma) - 2 g^i U_i U \gamma_i U \gamma_j \leq C e^{-B_0 \psi} \text{tr}_\omega \omega - 2 \beta U_\gamma^2 - 2 g^i U_i U \gamma_i U \gamma_j,$$

so that we have

$$\Delta_\gamma(e^{B(F+\psi)} U_\gamma^2) = -U_\gamma^2 \Delta_\gamma(e^{B(F+\psi)}) - e^{B(F+\psi)} \Delta_\gamma(U_\gamma^2)$$

$$-2 B e^{B(F+\psi)} g^i \left( (F + \psi)_i U_i U \gamma_i + (F + \psi)_j U_i U \gamma_j \right) \leq C B^2 e^{(B-B_0)(F+\psi)} \text{tr}_\omega \omega - 2 \beta e^{B(F+\psi)} U_\gamma^2 - 2 e^{B(F+\psi)} g^i U_i U \gamma_i U \gamma_j$$

$$+ e^{B(F+\psi)} g^i U_i U \gamma_i U \gamma_j + 4 B^2 e^{B(F+\psi)} U_i^2 g^i (F + \psi)_j (F + \psi)_g$$

$$\leq C B^2 e^{(B-B_0)(F+\psi)} \text{tr}_\omega \omega - e^{B(F+\psi)} g^i U_i U \gamma_i U \gamma_j - 2 \beta e^{B(F+\psi)} U_\gamma^2.$$

Now, substituting (2.18), (2.19), (2.21) and (2.22) into (2.16), at $x_0$, we obtain

$$\Delta_\gamma w \leq -\mu_1 e^{(F + \varepsilon / 4) \text{tr}_\omega \omega} + C B^2 e^{(B-B_0)(F+\psi)} \text{tr}_\omega \omega$$

$$- \frac{1}{2} e^{B(F+\psi)} g^i U_i U \gamma_i U \gamma_j + \beta e^{B(F+\psi)} U_\gamma - 2 \beta e^{B(F+\psi)} U_\gamma^2.$$
Using then the fact that $U_{\gamma}(x_0) < 0$, we see that
\[
\Delta \tilde{g} w(x_0) \leq -\mu_1 e^F \text{tr} \tilde{\omega} + C B^2 e^{(B-B_0)(F+\psi)} \text{tr} \tilde{\omega}.
\]
Choosing $B, \mu_1$ sufficiently large, it then follows from the fact that $e^{F(x_0)} > 0$ that
\[
\Delta \tilde{g} w(x_0) < 0.
\]
But since $x_0$ was assumed to be an interior minimum, we must have:
\[
\Delta \tilde{g} w(x_0) \geq 0,
\]
which is a contradiction. Hence, (2.14) follows.

**Step 3.** The normal-normal derivatives.

By steps 1 and 2 we have
\[
|D_\gamma D_\eta u(p)| + |D_\gamma D_{2n} u(p)| \leq C e^{-B_0(F+\psi)} \text{ for } 1 \leq \gamma, \eta \leq 2n - 1.
\]
Thus, to bound the normal-normal derivative, it thus sufficient to bound $|u_n|$. Expanding out the determinant $\det(\tilde{g}_{ij})_{1 \leq i, j \leq n}$, we see that we already have the bound
\[
\det(\tilde{g}_{ij})_{1 \leq i, j \leq n} - \tilde{g}_{n\bar{n}} \det(\tilde{g}_{ij})_{1 \leq i, j \leq n-1} \leq C e^{-B_0(F+\psi)},
\]
Recalling (2.7) and $u - h_\varepsilon \leq 0$, it is clear that
\[
\det(\tilde{g}_{ij})_{1 \leq i, j \leq n} = e^{\beta(u-h_\varepsilon) + n \log(\varepsilon/2)} \det(g_{ij})_{1 \leq i, j \leq n} \leq C,
\]
so that (2.23) implies
\[
\tilde{g}_{n\bar{n}} \det(\tilde{g}_{ij})_{1 \leq i, j \leq n-1} \leq C e^{-B_0(F+\psi)}.
\]
Next we show that there is a uniform lower bound for $\det(\tilde{g}_{ij})_{1 \leq i, j \leq n-1}$. Note that the holomorphic tangent bundle to $\partial M$ at $p$, denoted by $T^h_{\partial M}$, is spanned by $\{ \frac{\partial}{\partial z_i} \}_{i=1}^{n-1}$. Then
\[
\tilde{\omega}|_{T^h_{\partial M}} = (\alpha + \varepsilon \omega + \sqrt{-1} \partial \bar{\partial} u)|_{T^h_{\partial M}}
\]
\[
= (\alpha + \varepsilon \omega + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon)|_{T^h_{\partial M}} + \sqrt{-1} \partial \bar{\partial} (u - \varphi_\varepsilon)|_{T^h_{\partial M}}
\]
\[
\geq e^F \omega|_{T^h_{\partial M}} + \sqrt{-1} \partial \bar{\partial} (u - \varphi_\varepsilon)|_{T^h_{\partial M}},
\]
where we used (2.3) in the last inequality. Since $u - \varphi_\varepsilon \equiv 0$ on $\partial M$, by [6, Lemma 7.3],
\[
\sqrt{-1} \partial \bar{\partial} (u - \varphi_\varepsilon)|_{T^h_{\partial M}} = (\nu \cdot (u - \varphi_\varepsilon)) L_{\partial M, \nu},
\]
where $\nu$ is an outward pointing normal vector field on $\partial M$ and $L_{\partial M, \nu}$ is the corresponding Levi-form of $\partial M$. Recalling that $\partial M$ is weakly pseudoconcave, we have $L_{\partial M, \nu} \leq 0$. Since $u - \varphi_\varepsilon \geq 0$ on $M$ and $u - \varphi_\varepsilon \equiv 0$ on $\partial M$, we have
\[
\nu \cdot (u - \varphi_\varepsilon) \leq 0,
\]
so (2.25) implies
\[ \tilde{\omega}|_{T^\perp_{\partial M}} \geq e^F \omega|_{T^\perp_{\partial M}}. \]
Taking wedges, we then get
\[ \det(\tilde{g}_{i\bar{j}})_{1 \leq i,j \leq n-1} \geq \frac{1}{C} e^{(n-1)F}. \]
Combining this with (2.24) and the definition of \( \tilde{\omega} \) we have
\[ |u_n| = |\tilde{g} - \alpha - \varepsilon g| \leq C e^{-B(F + \psi)}, \]
at \( p \), as desired. \( \square \)

We can now bound the Laplacian on the interior:

**Proposition 2.4.** Assume we are in the situation in Proposition 2.3. Then there exist uniform constants \( \beta_0, B, \) and \( C > 0 \) such that:
\[ |\Delta g u| \leq Ce^{-B\tilde{\psi}} \quad \text{for all } \beta \geq \beta_0, \]
where here \( \tilde{\psi} \) is as in Proposition 2.1.

**Proof.** We may assume that without loss of generality that \( F \leq 0 \). By the construction of \( \psi \), we have
\[ \tilde{\psi} \leq \psi \leq 0 \text{ and } \alpha + \sqrt{-1} \partial \bar{\partial} \psi \geq \frac{\delta}{2} \omega. \]
Recalling \( u - \psi \geq 0 \) and (2.4), it then follows that
\[ u - \tilde{\psi} \geq 0 \]
and
\[ \alpha + \varepsilon \omega + (1 + \delta/2) \sqrt{-1} \partial \bar{\partial} \tilde{\psi} \geq (1 + \delta/2) \frac{\delta}{2} \omega - \frac{\delta}{2} \omega \geq \frac{\delta^2}{4} \omega. \]

The trick is to again use:
\[ \tilde{u} := u - (1 + \delta/2) \tilde{\psi}. \]
By (2.26), it is clear that
\[ -\Delta g \tilde{u} \geq -n + \frac{\delta^2}{4} \text{tr} \omega. \]
Consider the following quantity:
\[ Q = \log \text{tr} \omega - B\tilde{u}, \]
where \( B \) is a constant to be determined later. We will bound \( Q \) above using the maximum principle. Let \( x_0 \) be a maximum point of \( Q \). It suffices to prove
\[ (\text{tr} \omega \tilde{\psi})(x_0) \leq Ce^{-C\tilde{\psi}(x_0)} \quad \text{for some } C, \]
as then:
\[ Q(x_0) \leq \log C - C\tilde{\psi}(x_0) - B\tilde{u}(x_0) \leq \log C + (B\delta/2 - C)\tilde{\psi}(x_0) \leq C, \]
as long as \( B \geq 2C/\delta. \)
Now, if \( x_0 \in \partial M \), then we are already done by Proposition 2.3, as:
\[
\text{tr}_{\tilde{\omega}}(\alpha + \varepsilon \omega) + \Delta_{\tilde{g}} u \leq C e^{-B(\psi + F)} \leq C e^{-C \tilde{\psi}} \quad \text{on } \partial M.
\]
Note also that \( x_0 \) cannot occur on \( \text{Sing}(\psi) \). We may then compute at \( x_0 \), using (2.27) and the estimate of [1, 37]:
\[
0 \geq \Delta_{\tilde{g}} Q(x_0) \geq \frac{1}{\text{tr}_{\tilde{\omega}}}(\text{tr}_{\tilde{\omega}}(\alpha + \varepsilon \omega) - \text{tr}_{\omega}(\tilde{\omega})) - Bn + \frac{B\delta^2}{4} \text{tr}_{\tilde{\omega}}
\]
\[
\geq \left( \frac{B\delta^2}{4} - C \right) \text{tr}_{\tilde{\omega}} - \frac{\text{tr}_{\omega}(\text{Ric}(\omega) - \beta \sqrt{-1} \partial \bar{\partial}(u - h_\varepsilon)) - Bn}{\text{tr}_{\tilde{\omega}}}
\]
\[
\geq \frac{\beta}{2} + \frac{-C \beta - C e^{-B_0 \psi}}{\text{tr}_{\tilde{\omega}}}
\]
for \( B, \beta \) sufficient large. Rearranging gives:
\[
\frac{C + C e^{-B_0 \psi(x_0)}}{\text{tr}_{\tilde{\omega}}(x_0)} \geq \frac{1}{2}.
\]
It then follows that
\[
\text{tr}_{\omega}(\tilde{\omega}(x_0)) \leq 2C(1 + e^{-B_0 \psi(x_0)}) \leq 4C e^{-B_0 \tilde{\psi}(x_0)}
\]
as required.

Thus, we conclude that:
\[
Q \leq C
\]
for a uniform \( C \). It follows that:
\[
\text{tr}_{\omega}(\tilde{\omega}) \leq C e^{-B \tilde{\psi}},
\]
and hence:
\[
(\Delta_{\tilde{g}} u) = (\text{tr}_{\omega}(\tilde{\omega}) - \text{tr}_{\omega}(\alpha + \varepsilon \omega)) \leq C e^{-B \tilde{\psi}}.
\]
\[
\square
\]

Proposition 2.5. Assume we are in the situation in Proposition 2.3. Then there exist uniform constants \( \beta_0, B, \) and \( C > 0 \) such that
\[
|\nabla^2 u|_{\tilde{g}} \leq C e^{-B \tilde{\psi}} \quad \text{for all } \beta > \beta_0,
\]
where \( \tilde{\psi} \) is as in Proposition 2.7.

Proof. We already have that the Hessian is bounded on the boundary by Proposition 2.3. We may then apply the maximum principle argument in [12] Lemma 4.3 using \( \tilde{\omega} \) instead of \( \psi \), which gives us the estimate everywhere, as desired. Note that, although it is assumed in [12] that \( \psi \) has analytic singularities, it is easy to see that the proof only needs the weaker assumption of exponential smoothness, in the specific form of (2.2).
For the reader’s convenience, we give a brief sketch here. Recalling Lemma 2.2 and Propositions 2.3 and 2.4, it is clear that

\[ (2.28) \quad \sup_M (e^{B \tilde{g} \psi} |\nabla u|^2_g + e^{B \tilde{g} \psi} |\Delta u|) + \sup_\partial M (e^{B \tilde{g} \psi} |\nabla^2 u|_g) \leq C. \]

Without loss of generality, we assume that \( \tilde{\psi} \leq -1 \). We consider the following quantity:

\[ Q = \log \lambda_1 + \rho (e^{B \tilde{g} \psi} |\nabla u|^2_g - A \tilde{u}), \]

where \( \lambda_1 \) is the largest eigenvalue of the real Hessian \( \nabla^2 u \), \( \tilde{u} \) is as in Proposition 2.4. \( A \) and \( B \) are positive constants to be determined, and the function \( \rho \) is given by

\[ \rho(s) = -\frac{1}{2} \log \left( 1 + \sup_M (e^{B \tilde{g} \psi} |\nabla u|^2_g) - s \right). \]

Let \( x_0 \) be a maximum point of \( Q \). By a similar argument to that in Proposition 2.4, it suffices to prove

\[ \lambda_1(x_0) \leq e^{-C e^{B \tilde{g} \psi}(x_0)}. \]

If \( x_0 \in \partial M \), then we are done by (2.28). Thus, we assume that \( x_0 \) is an interior point and \( Q \) is smooth at \( x_0 \) (otherwise, we just need to apply a perturbation argument as in [12]). We compute everything at \( x_0 \). Applying \( \partial_k \) to the logarithm of (2.7), we have

\[ g^{ik} \partial_k (g_{ij} \tilde{g}) = \beta (u_k - (h_\varepsilon)_k). \]

It then follows that

\[ \Delta g (e^{B \tilde{g} \psi} |\nabla u|^2_g) \]

\[ \geq e^{B \tilde{g} \psi} \frac{1}{2} \sum_k \bar{g}^{ik} (|u_{ik}|^2 + |u_{ik}|^2) - C B^2 e^{(B-C) \tilde{g}} \sum_i \bar{g}^{i} - C \bar{g}^{i} - \beta \]

\[ \geq e^{B \tilde{g} \psi} \frac{1}{2} \sum_k \bar{g}^{ik} (|u_{ik}|^2 + |u_{ik}|^2) - \sum_i \bar{g}^{i} - \beta, \]

after choosing \( B \) sufficiently large such that \( C B^2 e^{(B-C) \tilde{g}} \leq 1 \). Since \( \rho' \leq \frac{1}{2} \), we obtain

\[ \Delta g (\rho (e^{B \tilde{g} \psi} |\nabla u|^2_g)) \geq \frac{\rho' e^{B \tilde{g} \psi}}{2} \sum_k \bar{g}^{ik} (|u_{ik}|^2 + |u_{ik}|^2) \]

\[ + \rho'' \bar{g}^{ik} |\partial_i (e^{B \tilde{g} \psi} |\nabla u|^2_g)|^2 - \frac{1}{2} \sum_i \bar{g}^{i} - \beta. \]

Applying \( V_1 V_1 \) to the logarithm of (2.7) and using \( V_1 V_1 (u) = \lambda_1 \), we have

\[ g^{ik} V_1 V_1 (g_{ij} \tilde{g}) = g^{ik} g^{i\eta} \left| V_1 (g_{ij} \tilde{g}) \right|^2 + V_1 V_1 (\log \det g) + \beta V_1 V_1 (u - h_\varepsilon) \]

\[ \geq g^{ik} g^{i\eta} \left| V_1 (g_{ij} \tilde{g}) \right|^2 - C + \beta (\lambda_1 - C e^{-C \tilde{g}}), \]
where we used (2.5) and \( \tilde{\psi} \leq \psi \) in the second inequality. Without loss of generality, we assume that \( \lambda_1 \geq 4Ce^{-C\tilde{\psi}} + 4C \).

It then follows that

\[
\tilde{g}^\nu V_1 V_1(\tilde{g}^\nu) \geq \tilde{g}^\nu \frac{\partial_i (u V_1)}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}^\nu \tilde{g}^\rho |V_1(\tilde{g}^\rho)|^2}{\lambda_1^2} + \frac{\beta}{2},
\]

which implies

\[
(2.30)
\Delta \tilde{g}(log \lambda_1) \geq 2 \sum_{\alpha>1} \tilde{g}^\nu |\partial_i (u V_1)|^2 - \frac{\tilde{g}^\nu \tilde{g}^\rho |V_1(\tilde{g}^\rho)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} - \frac{\tilde{g}^\nu |\partial_i (u V_1)|^2}{\lambda_1^2} + \frac{\beta}{2}.
\]

Combining (2.27), (2.29), (2.30) and the rest of arguments of [12, Lemma 4.3], we obtain \( \lambda_1(x_0) \leq Ce^{-C_\psi(x_0)} \), as required.

\( \square \)

We may now finish as follows. By [2, Proposition 2.4], we have:

\[
u_{\varepsilon, \beta} \overset{C^0}{\rightharpoonup} V_{\varepsilon}
\]

where:

\[
V_{\varepsilon} := \sup \{ v \in \text{PSH}(X, \alpha + \varepsilon \omega) \mid v \leq h_{\varepsilon} \}.
\]

As mentioned earlier, the \( V_{\varepsilon} \) decrease pointwise to \( V \) as \( \varepsilon \) decreases to 0.

Using (2.10), Lemma 2.2 and Proposition 2.5 we establish uniform \( C^{1,1} \) estimate for \( u \) on compact subsets away from \( E_{nK}(\alpha) \), which implies \( V \in C^{1,1}_{\text{loc}}(M \setminus E_{nK}(\alpha)) \), as required.

\( \square \)

3. Geodesics between Singular Kähler Metrics

We now show that our results apply in the setting of regularity of geodesics between singular Kähler metrics.

Proof of Corollary 1.2. Let \((X_0, \omega_0)\) be a compact Kähler variety, without boundary, and:

\[
\mu : (X, \omega) \to (X_0, \omega_0)
\]

a smooth resolution of the singularities of \( X_0 \) with simple normal crossings, which exists thanks to Hironaka’s theorem [25]. Let \( \mu^{-1}(X_0, \text{Sing}) = E = \cup_k E_k \) be the exceptional divisor with smooth irreducible components \( E_k \).

Let \( \alpha_0 := \mu^* \omega_0 \geq 0 \), which will be a smooth semi-positive form. It is well-known that \( [\alpha_0] \) is a big and nef class, and that \( E_{nK}(\alpha_0) = \text{Supp}(E) \).

Consider smooth Hermitian metrics \( h_k \) on \( O(E_k) \) and defining sections \( s_k \) for each \( E_k \).

Elementary results in several complex variables will now show:

\[
(3.1) \quad \alpha_0^n \geq b \left( \prod_{k=1}^m |s_k|_{h_k}^{a_k} \right) \omega^n
\]
for fixed constants $a_k > 0$ and a $b > 0$ depending on $\omega_0$. To see this, we work locally – cover $X_0$ by open charts $U_i$ such that for each $i$ there exists an embedding:

$$\iota_i : U_i \hookrightarrow \Omega_i \subset \mathbb{C}^N,$$

with $N$ uniformly large, such that $\omega_0$ extends to a smooth Kähler form (which we will also call $\omega_0$) on the open set $\Omega_i$. Relabeling $\mu$ to be $\iota_i \circ \mu$, we have by functoriality that $\mu^* \omega_0$ is unchanged, so we may work with a holomorphic map between smooth spaces. Fix coordinates $z$ on $\mu^{-1}(\Omega_i)$ and $x$ on $\Omega_i$, and define the Jacobian of $\mu$ to be the $n \times N$ matrix:

$$\text{Jac}(\mu) := \left( \frac{\partial \mu^k}{\partial z^j} \right)_{1 \leq j \leq n, 1 \leq k \leq N},$$

where $\mu^k$ is the $k$th coordinate function of $\mu$ on $\Omega_i$.

Putting $e^j := \sqrt{-1} dz^j \wedge d\bar{z}^j$, one can then compute that:

$$\alpha_0^n(z) = \det(\text{Jac}(\mu) \cdot \omega_0(\mu(z)) \cdot \text{Jac}(\mu)^T)n!e^1 \wedge \ldots \wedge e^n$$

where we are expressing $\omega_0(x)$ as an $N \times N$ matrix in the $x$-coordinates. Letting $c > 0$ be a constant such that $\omega_0 \geq c \omega_{\text{Eucl}}$ on all charts $\Omega_i$ (which we can do, after possibly shrinking them slightly, as there are only finitely many), we then have:

$$\alpha_0^n(z) \geq c^n \det(\text{Jac}(\mu) \cdot \text{Jac}(\mu)^T)n!e^1 \wedge \ldots \wedge e^n.$$

We now use [26, Lemma, pg. 304] to see that:

$$\det \left( \text{Jac}(\mu) \cdot \text{Jac}(\mu)^T \right) = \sum \limits_{n \times n \text{ minors} \ J_k \text{ of } \text{Jac}(\mu)} | \det(J_k) |^2$$

so that:

$$\alpha_0^n \geq c^n \left( \sum \limits_{n \times n \text{ minors} \ J_k \text{ of } \text{Jac}(\mu)} | \det(J_k) |^2 \right) n!e^1 \wedge \ldots \wedge e^n.$$

Each determinant in the sum is a holomorphic function, and furthermore, we know that their common zero locus is $E_{\alpha K}(\alpha_0) = E$, as $\mu$ is a local biholomorphism if and only if $\text{Jac}(\mu)$ has full-rank, which is only true when at least one of the determinants is non-zero. Thus, by the Weierstrass preparation theorem and the fact that $E$ has simple normal crossings, we know that we can express each determinant (locally) as a product of the $s_i$ to some powers, as well as some other local holomorphic functions that do not vanish along all of $E$ – up to estimating these, the smooth Hermitian metrics, and $\omega^n$, we then see the claim (3.1).

We may now use the discussion immediately following Proposition 2.1 to see that:

$$\alpha_0 \geq c e^F \omega,$$
where:

\[ F := \log \left( b \prod_{k=1}^{m} |s_k| |a_k| |h_k| \right), \]

and \( c \) depends on an upper bound for \( \alpha_0 \) (which always exists as it is a smooth form). Up to shrinking \( b \), we may arrange that \( F \leq 0 \), and note that \( F \) has analytic singularities only along \( E_{nK}(\alpha_0) \). For a very large constant \( C \), then, we have that \( F \in \text{PSH}(X, C\omega) \), by the Poincaré-Lelong formula:

\[ \sqrt{-1}\partial \bar{\partial} F = \sum_{k=1}^{m} a_k (|E_k| - R_k) \geq -C\omega, \]

so we can apply Proposition 2.1 to get the key condition (1.1). Note that the resulting \( \tilde{\psi} \) actually has analytic singularities only along \( E_{nK}(\alpha_0) \), so our estimates will be optimal.

To now apply this to the geodesic, we will need to translate this onto the product space \( X \times A \), where \( A \) is the annulus:

\[ A := \{ \tau \in \mathbb{C} | 1 < |\tau| < e \}. \]

Let \( \pi \) be the projection onto \( X \) and \( p \) the projection onto \( A \), and define \( \alpha := \pi^* \alpha_0 \). Throughout, we will use:

\[ t := \log |\tau|. \]

Consider two Kähler metrics \( \omega_1 \) and \( \omega_2 \) on \( X_0 \) such that \( \alpha_1 := \mu^* \omega_1 \) and \( \alpha_2 := \mu^* \omega_2 \) are cohomologous to \( \alpha_0 \). Fix a Kähler form \( \omega \) on \( X \) such that \( \alpha_k \leq \omega \) for \( k = 0, 1, 2 \),

constants \( c, B > 0 \), and an exponentially smooth, strictly \( \alpha_0 \)-psh \( \tilde{\psi} \) as above such that:

\[ \alpha_k \geq ceB\tilde{\psi} \omega \text{ for } k = 1, 2. \]

There then exist two smooth functions \( \varphi_1, \varphi_2 \) such that:

\[ \alpha_k = \alpha_0 + \sqrt{-1}\partial \bar{\partial} \varphi_k, \quad k = 1, 2. \]

The geodesic between \( \alpha_1 \) and \( \alpha_2 \) is then defined to be the envelope:

\[ V := \sup \{ v \in \text{PSH}(X \times A, \alpha) | v|_{t=0} \leq \pi^* \varphi_1, \ v|_{t=1} \leq \pi^* \varphi_2 \}. \]

Fix a large constant \( C \) such that \( \varphi_2 - (C - 1) \leq \varphi_1 \leq \varphi_2 + (C - 1) \). Let \( f \) be the solution to the Dirichlet problem on \( A \):

\[ \begin{cases} \sqrt{-1}\partial \bar{\partial} f = \omega_{\text{Eucl}}, \\ f|_{\partial A} = 0. \end{cases} \]

We then define \( \varphi \) to be the following subsolution:

\[ \varphi(x, \tau) := \max \{ \pi^* \varphi_1(x) - Ct, \pi^* \varphi_2(x) - C(1 - t) \} + p^* f(\tau), \]
where \( \tilde{\max} \) is a regularized maximum function with error \( 1/2 \). Observe that, for both \( k = 1, 2 \), on \( X \times A \) we have:
\[
\alpha + \sqrt{-1} \partial \bar{\partial} (\pi^* \varphi_k + Ct + p^* f) \geq c e^{B \pi^* \tilde{\psi}} \pi^* \omega + \sqrt{-1} \partial \bar{\partial}_\tau (\pm C \log |\tau| + p^* f) \\
\geq c e^{B \pi^* \tilde{\psi}} (\pi^* \omega + p^* \omega_{\text{Eucl}}).
\]
Thus, by an elementary property of \( \tilde{\max} \) [19, Lemma 5.18], we have that:
\[
\alpha + \sqrt{-1} \partial \bar{\partial} \varphi \geq c e^{B \pi^* \tilde{\psi}} (\pi^* \omega + p^* \omega_{\text{Eucl}}).
\]
also, which is the key condition (1.1) that we require (as we just take \( \varphi_\varepsilon = \varphi \) for all \( \varepsilon > 0 \)).

**Proof of Corollary 1.3** The idea is to construct a good sequence of Kähler potentials for \( \{\alpha + \varepsilon \omega\} \), and then take regularized maximums with \( \varphi_1 \) and \( \varphi_2 \), which will preserve the estimates we need. Specifically, recall that \( (X, \omega) \) is a Kähler manifold without boundary, and \( [\alpha] \) is a nef and big class on \( X \). Our sequence of potentials will be the solutions to the following Monge-Ampère equations:
\[
(\alpha + (\varepsilon/2) \omega + \sqrt{-1} \partial \bar{\partial} v_\varepsilon)^n = e^{\beta_0 v_\varepsilon} \omega^n, \\
\alpha + (\varepsilon/2) \omega + \sqrt{-1} \partial \bar{\partial} v_\varepsilon > 0,
\]
where \( \beta_0 > 0 \) is a fixed number such that we have the estimates:
\[
|\nabla v_\varepsilon|^2_g + |\nabla^2 v_\varepsilon|_g^2 \leq Ce^{-B \psi} \text{ for all } \varepsilon > 0.
\]
We can see this by establishing \( C^0 \) bounds for the \( v_\varepsilon \) – it is immediate from the comparison principle [7, Remark 2.4] that the \( v_\varepsilon \) are decreasing as \( \varepsilon \to 0 \); if \( \varepsilon_1 > \varepsilon_2 \), then:
\[
\int_{\{v_\varepsilon_1 < v_\varepsilon_2\}} e^{\beta_0 v_\varepsilon_2} \omega^n + 2^{-n} (\varepsilon_1 - \varepsilon_2)^n \omega^n \leq \int_{\{v_\varepsilon_1 < v_\varepsilon_2\}} (\alpha + (\varepsilon_1/2) \omega + \sqrt{-1} \partial \bar{\partial} v_{\varepsilon_2})^n \\
\leq \int_{\{v_\varepsilon_1 < v_\varepsilon_2\}} (\alpha + (\varepsilon_1/2) \omega + \sqrt{-1} \partial \bar{\partial} v_{\varepsilon_1})^n = \int_{\{v_\varepsilon_1 < v_\varepsilon_2\}} e^{\beta_0 v_{\varepsilon_1}} \omega^n \leq \int_{\{v_\varepsilon_1 < v_\varepsilon_2\}} e^{\beta_0 v_{\varepsilon_2}} \omega^n,
\]
which is a contradiction unless \( \omega^n(\{v_\varepsilon_1 < v_{\varepsilon_2}\}) = 0 \), in which case \( v_{\varepsilon_2} \leq v_{\varepsilon_1} \) everywhere, by continuity. In particular, the \( v_\varepsilon \) are uniformly bounded above by \( v_1 \). Further, the same argument shows that the \( v_\varepsilon \) are bounded below by \( v_0 \) solving:
\[
\langle (\alpha + \sqrt{-1} \partial \bar{\partial} v_0)^n \rangle = e^{\beta_0 v_0} \omega^n.
\]
By [7, Theorem 6.1], \( v_0 \) has minimal singularities, so there exists a large constant \( C \) such that \( \psi - C \leq v_0 \leq v_\varepsilon \) for all \( \varepsilon > 0 \). The proofs of Lemma 2.2 and Proposition 2.3 now apply directly (One can also probably use easier proofs to obtain (3.3), but it follows immediately from what we have done already – see also [12, Section 4]).

It will be convenient to renormalize all the involved quantities to have supremum zero:
\[
\sup_X \varphi_1 = \sup_X \varphi_2 = \sup_X v_\varepsilon = 0,
\]
and assume without loss of generality that
\[ \psi \leq \varphi_1, \ 0 < \psi \leq \varphi_2. \]

This will not affect the problem in anyway, as the diligent reader may check.

We will now pull-back everything to the product space \( X \times A \) as in Corollary 1.2 – define \( A, t, \pi, p \) and \( f \) as in that proof, and again fix a constant \( C \) such that:

\[ \varphi_2 - (C - 1) \leq \varphi_1 \leq \varphi_2 + (C - 1), \]

where we used that \( \varphi_1 \) and \( \varphi_2 \) have the same singularity type. The geodesic connecting \( \varphi_1 \) and \( \varphi_2 \) is defined by

\[ V := \sup\{v \in PSH(X \times A, \pi^* \alpha) \mid v|_{\{t=0\}} \leq \pi^* \varphi_1, \ v|_{\{t=1\}} \leq \pi^* \varphi_2\}. \]

As in the proof of Corollary 1.2, we define:

\[ \varphi(x, \tau) := \max\{\pi^* \varphi_1(x) - Ct, \pi^* \varphi_2(x) - C(1 - t)\} + p^* f(\tau). \]

To apply Theorem 1.1 we define the smooth approximates

\[ \varphi_\varepsilon(x, \tau) := \max\{\pi^* \varphi_1(x) - Ct, \pi^* \varphi_2(x) - C(1 - t), \pi^* v_\varepsilon(x) - C_\varepsilon\} + p^* f(\tau) \]

where \( C_\varepsilon := -\log(\varepsilon/2) + C + 2 \). Clearly, \( \varphi_\varepsilon \) decrease pointwise to \( \varphi \) as \( \varepsilon \) decreases to 0. We claim that

\[ \pi^*(\alpha + \varepsilon \omega) + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon \geq e^{\pi^* \psi}(\pi^* \omega + p^* \omega_{\text{Eucl}}), \]

and

\[ |\nabla \varphi_\varepsilon|_g + |\nabla^2 \varphi_\varepsilon|_g \leq C^{-B\psi}. \]

Given these, Corollary 1.3 will follow from Theorem 1.1.

Let us prove (3.4) first. Let \( (x_0, \tau_0) \in X \times A \) and \( t_0 := \log|\tau_0| \). If we have

\[ \pi^* \varphi_1(x_0) - Ct_0 \leq \pi^* v_\varepsilon(x_0) - C_\varepsilon + 1, \]

then using that \( \psi \leq \varphi_1 \) and \( v_\varepsilon \leq 0 \), it follows that

\[ e^{\pi^* \psi} \leq \varepsilon/2 \]

near \( (x_0, \tau_0) \). Using (3.2), we see that

\[ \pi^*(\alpha + \varepsilon \omega) + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon \geq \frac{\varepsilon}{2} \pi^* \omega + p^* \omega_{\text{Eucl}} \geq e^{\pi^* \psi}(\pi^* \omega + p^* \omega_{\text{Eucl}}), \]

near \( (x_0, \tau_0) \), which implies (3.3) there. If

\[ \pi^* \varphi_1(x_0) - Ct_0 > \pi^* v_\varepsilon(x_0) - C_\varepsilon + 1 \]

on the other hand, then it follows from the definition of \( \max \varphi_\varepsilon \) that:

\[ \varphi_\varepsilon = \varphi \]

near \( (x_0, \tau_0) \), and so:

\[ \pi^*(\alpha + \varepsilon \omega) + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon \geq e^{\pi^* \psi}(\pi^* \omega + p^* \omega_{\text{Eucl}}), \]

by our assumptions on \( \varphi_1 \) and \( \varphi_2 \).

We now check (3.5). For ease of notation, write:

\[ b_1 := \pi^* \varphi_1 - Ct, \ b_2 := \pi^* \varphi_2 - C(1 - t), \ b_\varepsilon := v_\varepsilon - C_\varepsilon. \]
Recall then the definition of $\tilde{\max}$:

$$\tilde{\max}(a, b, c) := \int_{\mathbb{R}^3} \max\{y_1, y_2, y_3\} \theta(y_1 - a) \theta(y_2 - b) \theta(y_3 - c) \, dy_1 \, dy_2 \, dy_3,$$

where here $\theta, 0 \leq \theta \leq 1$ is a cutoff function on $\mathbb{R}$, $\theta \equiv 1$ near 0, with support in $[-1/2, 1/2]$. Thus,

$$\varphi_{\varepsilon} = \max(b_1, b_2, b_{\varepsilon}) + \pi^* f.$$

A short calculation shows that

$$|\nabla \varphi_{\varepsilon}|_g \leq C \left( \min\{|b_1|, |b_2|, |b_{\varepsilon}| \} + 1 \right) (|\nabla b_1| + |\nabla b_2| + |\nabla b_{\varepsilon}|),$$

as $b_1, b_2, b_{\varepsilon} \leq 0$. Using (3.3) and the fact that $\varphi_1$ and $\varphi_2$ are both exponentially smooth, we then see

$$|\nabla \varphi_{\varepsilon}|_g \leq C(-\pi^* \varphi_1 + C)e^{-B\pi^* \psi} \leq C(-\pi^* \psi)e^{-B\pi^* \psi} \leq Ce^{-B\pi^* \psi}.$$

A similar argument shows that $|\nabla^2 \varphi_{\varepsilon}|_g \leq Ce^{-B\pi^* \psi}$, establishing (3.5).

**Remark 3.1.** Let us briefly note that the above procedure to produce the $v_{\varepsilon}$ does not work in the case of a general $\alpha$ on a general $M$ – this is because one would need to solve a Dirichlet problem, and so would need to have knowledge/control of permissible boundary data, which is unavailable without further assumptions, e.g. as in Corollary 1.3.

We now briefly discuss the case of geodesic rays originating at singular Kähler metrics. Recall that the main result of [28] can be summarized as follows (see that paper for a more specific statement):

**Theorem 3.2.** Suppose that $M$ is a compact complex manifold with boundary. Let $\xi$ be a function with analytic singularities on $M$, such that $\xi$ is singular on a divisor $E \subset M$ with $E \cap \partial M = \emptyset$. Let $R_h$ be a smooth form cohomologous to $[E]$, and suppose that $\alpha$ is a closed, smooth, real $(1, 1)$-form on $M$ such that $\alpha - R_h$ is $\psi$-big and nef, with $\text{Sing}(\psi) \cap \partial M = \emptyset$. Finally, let $\varphi \in PSH(M, \alpha)$ be smooth near the boundary of $M$ and sufficiently regular. Then the envelope:

$$\sup\{v \in PSH(M, \alpha) \mid v|_{\partial M} \leq \varphi|_{\partial M}, \ v \leq \xi + O(1)\}$$

is in $C^{1,1}_{\text{loc}}(M \setminus \text{Sing}(\xi + \psi))$ if the boundary of $M$ is weakly pseudoconcave and $\alpha + \sqrt{-1} \partial \bar{\partial} \varphi \geq \delta \omega$ on some neighborhood of $\partial M$.

In [28], the above theorem was used to prove $C^{1,1}$ regularity of certain geodesic rays originating at a Kähler metric by taking $M = X \times \mathbb{D}$. Here, we simply remark that the results in Section 2 can be combined with the method in [28] to improve Theorem 3.2 and this can then be used to prove regularity of certain geodesic rays on a singular Kähler variety in the exact same way:

**Theorem 3.3.** Theorem 3.2 is still valid if we allow $\text{Sing}(\psi)$ to intersect $\partial M$ and we weaken the assumptions that $\varphi$ be smooth and strictly $\alpha$-psh near the boundary to just assuming the existence of a family of $\varphi_{\varepsilon}$ satisfying conditions (a) and (b) in Theorem 1.4.
Appendix A. Estimates for $\Delta \omega$

Lemma A.1. Let $\varphi_\varepsilon$ be as in Theorem 1.1 and let $h_\varepsilon$ be the solutions to:

$$\begin{cases}
\Delta^2 h_\varepsilon = -n, \\
h_\varepsilon |_{\partial M} = \varphi_\varepsilon |_{\partial M}.
\end{cases}$$

for all $\varepsilon > 0$. Then there exist positive constants $B, C$ such that:

$$|\nabla h_\varepsilon|_g + |\nabla^2 h_\varepsilon|_g \leq Ce^{-B\psi}.$$

Proof. First, without loss of generality we may scale $\psi$ such that:

$$\sup_M \psi = -1.$$

Now note that by the maximum principle, we have that the $h_\varepsilon$ are decreasing as $\varepsilon \to 0$, and that (cf. (2.10)):

(A.1) $\psi \leq \varphi_\varepsilon \leq h_\varepsilon \leq h_1$.

Let $b$ be the solution to:

$$\begin{cases}
\Delta \omega b = -1, \\
b |_{\partial M} = 0,
\end{cases}$$

and define:

$$\tilde{h}_\varepsilon := h_\varepsilon - \varphi_\varepsilon \geq 0.$$

We claim that there exists a constant $B > 0$ such that:

$$h_\varepsilon \leq \varphi_\varepsilon + e^{-B\psi} b \text{ on } M \setminus \text{Sing}(\psi).$$

Combining this with the lower bound in (A.1), it follows that $|\nabla h_\varepsilon|_g \leq Ce^{-B\psi}$ at the boundary. To see the claim, using (2.2), we compute:

$$\Delta \omega (e^{B\psi}(\tilde{h}_\varepsilon + \tilde{h}_\varepsilon^2) - b) \geq e^{B\psi}(1 + 2\tilde{h}_\varepsilon)\Delta \omega \tilde{h}_\varepsilon + 2e^{B\psi}|\nabla \tilde{h}_\varepsilon|_g^2$$

$$- BCe^{(B-C)\psi}|\nabla \tilde{h}_\varepsilon|_g - CB^2e^{(B-C)\psi}(\tilde{h}_\varepsilon + \tilde{h}_\varepsilon^2) + 1.$$

Since $\Delta_2 h_\varepsilon = -n$, it is clear that $\Delta \omega h_\varepsilon = -2n$. Combining this with (A.1) and $|\Delta \omega \varphi_\varepsilon| \leq Ce^{-C\psi}$, we see that

$$|\tilde{h}_\varepsilon| + |\Delta \omega \tilde{h}_\varepsilon| \leq Ce^{-C\psi}.$$

By the Cauchy-Schwarz inequality, we have

$$\Delta \omega (e^{B\psi}(\tilde{h}_\varepsilon + \tilde{h}_\varepsilon^2) - b) \geq 2e^{B\psi}|\nabla \tilde{h}_\varepsilon|_g^2 - e^{B\psi}(\nabla \tilde{h}_\varepsilon|_g^2 + CB^2e^{-2C\psi}) - CB^2e^{(B-C)\psi} + 1$$

$$\geq - CB^2e^{(B-C)\psi} + 1.$$

Now since $\sup_M \psi = -1$, we may choose $B$ sufficiently large such that

$$\Delta \omega (e^{B\psi}(\tilde{h}_\varepsilon + \tilde{h}_\varepsilon^2) - b) \geq - CB^2e^{(B-C)\psi} + 1 > 0.$$

It follows then from the maximum principle that

$$e^{B\psi} \tilde{h}_\varepsilon \leq b,$$
as claimed.

To bound the gradient on the interior, we consider the quantity:

\[ Q = e^{2B\psi} |\nabla h_\varepsilon|^2_g + e^{B\psi} h_\varepsilon^2, \]

where \( B \) is a constant to be determined. Suppose that \( Q \) achieves a maximum at \( x_0 \). If \( x_0 \in \partial M \), then we are already done. Otherwise, we choose holomorphic normal coordinates at \( x_0 \) for \( g \) (the Riemannian metric corresponding to \( \omega \)), so that:

\[ \sum_i (h_\varepsilon)_{kii} = \sum_i (h_\varepsilon)_{ik} = (\Delta_\omega h_\varepsilon)_k = 0. \]

This implies

\[
\Delta_\omega (e^{2B\psi} |\nabla h_\varepsilon|^2_g)
\geq 2e^{2B\psi} \text{Re} \left( \sum_{i,k} (h_\varepsilon)_{kii} (h_\varepsilon)_T \right) + e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g
\]

\[ - 2Be^{2B\psi} |\nabla \varphi_\varepsilon|_g |\nabla h_\varepsilon|_g |\nabla^2 h_\varepsilon|_g \]

\[ \geq e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g - \frac{1}{2} e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g - CB^2 e^{2B\psi} |\nabla \varphi_\varepsilon|^2_g |\nabla^2 h_\varepsilon|^2_g \]

\[ \geq \frac{1}{2} e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g - CB^2 e^{(2B-C)\psi} |\nabla^2 h_\varepsilon|^2_g. \]

By the maximum principle, at \( x_0 \) we have

\[ 0 \geq \Delta_\omega Q = \Delta_\omega (e^{2B\psi} |\nabla h_\varepsilon|^2_g) + \Delta_\omega (e^{B\psi} h_\varepsilon^2) \]

\[ \geq \frac{1}{2} e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g - CB^2 e^{(2B-C)\psi} |\nabla h_\varepsilon|^2_g + e^{B\psi} |\nabla h_\varepsilon|^2_g - CB^2 e^{(B-C)\psi}. \]

We can now choose \( B \) sufficiently large such that

\[ CB^2 e^{(2B-C)\psi} \leq \frac{1}{2} e^{B\psi}, \]

implying

\[ |\nabla h_\varepsilon|^2_g \leq Ce^{-C\psi} \text{ at } x_0. \]

Using (A.1) then shows that \( Q(x_0) \leq C \), as desired.

Finally, we bound the Hessian of the \( h_\varepsilon \). We establish the boundary estimate first. The tangent-tangent derivative estimate is obvious. The tangent-normal derivatives can be bounded in a manner analogous to the proof of Proposition 2.3 very briefly, one considers the quantity:

\[ w = b + e^{B\psi} (\mu |z|^2 - e^{B\psi} |D_\gamma h_\varepsilon| - e^{B\psi} |D_\gamma \tilde{h}_\varepsilon|^2), \]

on a ball \( B_R(p) \) of fixed radius \( R \), with \( p \in \partial M \), \( 1 \leq \gamma \leq 2n - 1 \), and \( \mu \) and \( B \) constants. Choosing \( \mu \) sufficiently large, one arranges that:

\[ w \geq 0 \text{ on } \partial (B_R \cap M), \]
and then shows that $w$ cannot have an interior minimum point. We refer the reader to the proof of Proposition 2.3 for more details. Finally, the normal-normal derivative is bounded just by the fact that $\Delta_\omega h_\varepsilon = -2n$. Thus, we have the second order estimate on the boundary:

$$|\nabla^2 h_\varepsilon|_g \leq Ce^{-C\psi} \text{ on } \partial M.$$ 

To bound the Hessian everywhere now, we consider the quantity:

$$Q = e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g + e^{B\psi} |\nabla h_\varepsilon|^2_g,$$

where $B$ is a constant to be determined. Let $x_0$ be an interior maximum point of $Q$. Choosing holomorphic normal coordinates for $g$ at $x_0$ gives

$$\sum_i (h_\varepsilon)_k \bar{\eta} = (\Delta_\omega h_\varepsilon)_k - \partial_k \partial_{\bar{k}} (g^{\bar{j}})(h_\varepsilon)_{\bar{j}} = -\partial_k \partial_{\bar{k}} (g^{\bar{j}})(h_\varepsilon)_{\bar{j}},$$

which implies $|\sum_i (h_\varepsilon)_k \bar{\eta}| \leq C|\nabla^2 h_\varepsilon|_g$. Similarly, we also have $|\sum_i (h_\varepsilon)_{k\bar{l}} \bar{\eta}| \leq C|\nabla^2 h_\varepsilon|_g$. Combining this with the Cauchy-Schwarz inequality, it follows that

$$\Delta_\omega (e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g) \geq 2e^{2B\psi} \text{Re} \left( \sum_{i,k,l} (h_\varepsilon)_k \bar{\eta} (h_\varepsilon)_{\bar{l}} + \sum_{i,k,l} (h_\varepsilon)_{k\bar{l}} \bar{\eta} (h_\varepsilon)_{\bar{l}} \right) + e^{2B\psi} |\nabla^3 h_\varepsilon|^2_g - 2Be^{2B\psi} |\nabla^2 h_\varepsilon|_g |\nabla^3 h_\varepsilon|_g \geq -Ce^{2B\psi} |\nabla^2 h_\varepsilon|^2_g - CB^2 e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g$$

$$\geq -CB^2 e^{(2B-C)\psi} |\nabla^2 h_\varepsilon|^2_g.$$ 

Using the maximum principle and (A.2), at $x_0$, we have

$$0 \geq \Delta_\omega Q \geq \Delta_\omega (e^{2B\psi} |\nabla^2 h_\varepsilon|^2_g) + \Delta_\omega (e^{B\psi} |\nabla h_\varepsilon|^2_g) \geq -CB^2 e^{(2B-C)\psi} |\nabla^2 h_\varepsilon|^2_g + \frac{1}{2} e^{B\psi} |\nabla^2 h_\varepsilon|^2_g - CB^2 e^{(2B-C)\psi} |\nabla h_\varepsilon|^2_g.$$ 

After choosing $B \geq C$ sufficiently large, we see that

$$CB^2 e^{(2B-C)\psi} \leq \frac{1}{4} e^{B\psi} \text{ and } e^{(B-C)\psi} |\nabla h_\varepsilon|^2_g \leq C.$$ 

It then follows that

$$|\nabla^2 h_\varepsilon|^2_g \leq Ce^{-C\psi} \text{ at } x_0,$$

which implies $Q(x_0) \leq C$, as desired. \qed

**Lemma A.2.** Assume we are in the situation in Lemma A.1. For each integer $k \geq 3$, there exist positive constants $B_k, C_k$ such that

$$|\nabla^k h_\varepsilon|_g \leq C_k e^{-B_k\psi}.$$
Proof. Since $\psi$ is exponentially smooth, there exists a constant $C_0$ such that $e^{C_0 \psi}$ is smooth. Using $\Delta \omega h_\varepsilon = -2n$, it is clear that
\begin{align}
\Delta \omega (e^{B C_0 \psi} h_\varepsilon) &= -2n e^{B C_0 \psi} + B (B - 1) h_\varepsilon e^{(B - 2) C_0 \psi} |\nabla (e^{C_0 \psi})|^2 \\
&+ B h_\varepsilon e^{(B - 1) C_0 \psi} \Delta \omega (e^{C_0 \psi}) + 2 B e^{(B - 1) C_0 \psi} \Re \langle \nabla (e^{C_0 \psi}), \nabla h_\varepsilon \rangle,
\end{align}
where $B$ is a constant to be determined. Using Lemma A.1 and choosing $B$ sufficiently large, we obtain
\[ \| \Delta \omega (e^{B C_0 \psi} h_\varepsilon) \|_{C^1(M)} + \| e^{B C_0 \psi} \varphi_\varepsilon \|_{C^3(\partial M)} \leq C. \]
Applying the Schauder estimate, it follows that
\[ \| e^{B C_0 \psi} h_\varepsilon \|_{C^{2,\frac{1}{2}}(M)} \leq C. \]
At the expense of increasing $B$, it follows from (A.3) that
\[ \| \Delta \omega (e^{B C_0 \psi} h_\varepsilon) \|_{C^{1,\frac{1}{2}}(M)} + \| e^{B C_0 \psi} \varphi_\varepsilon \|_{C^3(\partial M)} \leq C. \]
Using the Schauder estimate again, we obtain
\[ \| e^{B C_0 \psi} h_\varepsilon \|_{C^{3,\frac{1}{2}}(M)} \leq C. \]
Repeating the above argument, for any $k \geq 3$, there exist constants $B_k, C_k$ such that
\[ \| e^{B_k \psi} h_\varepsilon \|_{C^{k,\frac{1}{2}}(M)} \leq C_k, \]
as required. \hfill \Box

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