Littlewood-Richardson Coefficients via Yang-Baxter Equation

Oleg Gleizer
Northeastern University, Boston, MA 02115

Alexander Postnikov
University of California, Berkeley, CA 94720

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Abstract

The purpose of this paper is to present an interpretation for the decomposition of the tensor product of two or more irreducible representations of $GL(N)$ in terms of a system of quantum particles. Our approach is based on a certain scattering matrix that satisfies a Yang-Baxter type equation. The corresponding piecewise-linear transformations of parameters give a solution to the tetrahedron equation. These transformation maps are naturally related to the dual canonical bases for modules over the quantum enveloping algebra $U_q(sl_n)$. A byproduct of our construction is an explicit description for the cone of Kashiwara’s parametrizations of dual canonical bases. This solves a problem posed by Berenstein and Zelevinsky. We present a graphical interpretation of the scattering matrices in terms of web functions, which are related to honeycombs of Knutson and Tao.

1 Introduction

The aim of this paper is to further investigate the Grothendieck ring $K_N$ of polynomial representations of the general linear group $GL(N)$. Let $V_\lambda$ be the irreducible representation of $GL(N)$ with highest weight $\lambda$. The structure constants $c^\nu_{\lambda\mu}$ of the Grothendieck ring in the basis of irreducible representations are given by

$$V_\lambda \otimes V_\mu = \sum_\nu c^\nu_{\lambda\mu} V_\nu.$$ 

We also mention several alternative interpretations for the numbers $c^\nu_{\lambda\mu}$. These numbers are:

- the structure constants of the ring of symmetric polynomials in the basis of Schur polynomials;
- the coefficients of the decomposition into irreducibles of representations of symmetric groups induced from parabolic subgroups;
- the structure constants of the cohomology ring of a Grassmannian in the basis of Schubert classes.

The celebrated Littlewood-Richardson rule (see, e.g., [1]) is an explicit combinatorial description of the coefficients $c^\nu_{\lambda\mu}$. Several variations of this rule are known, including Zelevinsky pictures and Berenstein-Zelevinsky triangles [2].

*The latest version of this paper can be found on the web at http://www.math.berkeley.edu/~apost/
In this paper we present a new interpretation of the Grothendieck ring $K_N$ and the Littlewood-Richardson coefficients $c'_{\lambda\mu}$. Our construction is based on the scattering matrix $R(c)$ that acts in the tensor square of the linear space $E$ with the basis $e_0, e_1, e_2, \ldots$ by

$$R(c) : e_x \otimes e_y \rightarrow \begin{cases} e_{y+c} \otimes e_{x-c} & \text{if } c \geq x - y, \\ 0 & \text{otherwise.} \end{cases}$$

(We assume that $e_x = 0$ whenever $x < 0$.) We denote by $R_{ij}(c)$ the operator acting on $E^{\otimes m}$ as $R(c)$ on the $i$-th and $j$-th copy of $E$ and as an identity elsewhere. The tensor product of two irreducible representations $V_\lambda$ and $V_\mu$ can be written as a certain combination of the operators $R_{ij}(c)$.

Using the operators $R_{ij}(c)$ we define a new bilinear operation “∗” on the tensor algebra $T(E)$ that will correspond to the operation of tensor product of representations of $GL(N)$. It is straightforward that a Pieri type formula holds for the $*$-product of any basis element in $T(E)$ with $e_k$. The proof of the statement that tensor product $V_\lambda \otimes V_\mu$ is given by the $*$-product easily follows from this fact and the fact that “∗” is an associative operation.

The associativity of the $*$-multiplication is obtained from the following Yang-Baxter type relation for the scattering matrices. The operators $R_{12}(c_{12})$, $R_{13}(c_{13})$, and $R_{23}(c_{23})$ acting on $E^{\otimes 3}$ satisfy the relation

$$R_{23}(c_{23})R_{13}(c_{13})R_{12}(c_{12}) = R_{12}(c'_{12})R_{13}(c'_{13})R_{23}(c'_{23}),$$

where $c_{12}, c_{13}, c_{23}$ are arbitrary parameters and $c'_{12}, c'_{13}, c'_{23}$ are given by the following piecewise-linear formulas

$$c'_{12} = \min(c_{12}, c_{13} - c_{23}),$$
$$c'_{13} = c_{12} + c_{23},$$
$$c'_{23} = \max(c_{23}, c_{13} - c_{12}). \tag{1.1}$$

Surprisingly, the same piecewise-linear transformations arise in the theory of dual canonical bases for the modules over the quantum enveloping algebra $U_q(sl_n)$ (see [2, 4]). For a fixed reduced decomposition of the longest element $w_o$ in the symmetric group $S_n$, elements of the dual canonical basis (also known as the string basis) are parameterized by \( \binom{n}{2} \)-tuples of integers (strings) that belong to a certain string cone (Kashiwara’s parametrization). Two parametrizations that correspond to reduced decompositions related by a Coxeter move are obtained from each other by the formulas (1.1).

The string cone was described in [3] for a certain reduced decomposition of $w_o$. The core of our construction lies in an explicit description of the string cone for any reduced decomposition. Thus we solve a rather nontrivial problem posed in [3].

We also present a graphical (or “pseudo-physical”) interpretation of the scattering matrices and their compositions in the language of web functions and “systems of quantum particles.” Web functions are closely related to honeycombs of Knutson and Tao [3] and Berenstein-Zelevinsky triangles [3]. It is shown in [3] that integral honeycombs are in one-to-one correspondence with Berenstein-Zelevinsky patterns. We establish a simple “dual” correspondence between integral web functions and Berenstein-Zelevinsky patterns. This reveals the “hidden duality” of the Littlewood-Richardson coefficients under the conjugation of partitions.

We briefly outline the structure of the paper. In Section [3] we give some background on the representation theory of general linear groups, the Littlewood-Richardson coefficients, and the combinatorics of symmetric groups and reduced decompositions. In Section [3] we define the scattering matrices $R_{ij}(c)$ and formulate our rule for the Littlewood-Richardson coefficients. Section [3] is devoted to the Yang-Baxter type relation for the scattering matrices. In Section [3] we define and
study principal cones for any reduced decomposition of a permutation. In the case of the longest permutation these cones are exactly the string cones of parametrizations of dual canonical bases. The associativity of the $\ast$-product is deduced in Section 6. In Section 7 we define web functions and establish their relationship with the scattering matrices and Berenstein-Zelevinsky patterns.

2 Preliminaries

In this section we remind the reader the basic notions and notation related to symmetric groups and representations of general linear groups.

2.1 Representations of general linear groups

Let us recall the basics of the representation theory of the general linear group $GL(N)$.

The general linear group $GL(N)$ is the automorphism group of the $N$-dimensional complex linear space $\mathbb{C}^N$. A complex finite-dimensional linear representation $V$ of $GL(N)$ is called polynomial if the corresponding mapping $GL(N) \rightarrow \text{Aut}(V)$ is given by polynomial functions. An arbitrary holomorphic finite-dimensional representation is obtained by tensoring a polynomial representation with a determinant representation $g \mapsto \det^k(g)$ for suitable negative $k$.

An irreducible polynomial representation of $GL(N)$ is uniquely determined by its highest weight $\lambda = (\lambda_1, \ldots, \lambda_N)$, which can be any integer element of the dominant chamber given by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. We denote by $V_\lambda$ the irreducible representation with highest weight $\lambda$. Its degree is $|\lambda| = \lambda_1 + \cdots + \lambda_N$.

The collection of polynomial representations of $GL(N)$ equipped with the operations of direct sum and tensor product has the structure of an abelian category. Let $K_N = K(GL(N))$ be the Grothendieck ring of this category. Degree of representations provides a natural grading on the ring $K_N$. Slightly abusing notations, we will identify a representation with its image in the Grothendieck ring $K_N$.

The irreducible representations $V_\lambda$ form a $\mathbb{Z}$-basis of $K_N$. Our primary interest is in the structure constants of $K_N$. In other words, we would like to investigate the coefficients $c_{\lambda \mu}^\nu$ of the expansion of the tensor product of two irreducible representations into a direct sum of irreducibles:

$$V_\lambda \otimes V_\mu = \sum_{\nu} c_{\lambda \mu}^\nu V_\nu.$$

The weights $\omega_k = (1, 1, \ldots, 1, 0, \ldots, 0)$ (with $k$ ones) are called the fundamental weights. By convention $\omega_0 = (0, \ldots, 0)$. Every dominant weight $\lambda$ can be written uniquely as a sum of fundamental weights $\lambda = \omega_{x_1} + \cdots + \omega_{x_m}$, $1 \leq x_1 \leq \cdots \leq x_m \leq N$. Actually, the numbers $x_i$ are just parts of the partition $\lambda$'s conjugate to $\lambda$, i.e., $\lambda' = (x_m, x_{m-1}, \ldots, x_1)$.

The fundamental representation $V_{\omega_k}$ is the $k$-th exterior power of the tautological representation of $GL(N)$. Pieri’s formula gives an explicit rule for the tensor product of $V_{\omega_k}$ with an irreducible representation $V_\lambda$.

Proposition 2.1 (Pieri’s formula) For $\lambda = \omega_{x_1} + \cdots + \omega_{x_m}$, $1 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq N$, we have

$$V_{\omega_k} \otimes V_\lambda = \sum V_\mu,$$

where the sum is over all $\mu = \omega_{y_1} + \cdots + \omega_{y_{m+1}}$ satisfying the following interlacing conditions:

$$0 \leq y_1 \leq x_1 \leq y_2 \leq x_2 \cdots \leq y_m \leq x_m \leq y_{m+1} \leq N,$$

$$y_1 - x_1 + y_2 - x_2 + \cdots + y_m - x_m + y_{m+1} = k.$$
The Grothendieck ring $K_N$ is generated by the fundamental representations $V_{\omega_k}$. This implies the following statement that will be handy afterward.

**Lemma 2.2** Suppose that $\odot$ is a bilinear associative multiplication operation on the linear space $K_N$ such that for any fundamental weight $\omega_k$ and any dominant weight $\lambda$ the product $V_{\omega_k} \odot V_{\lambda}$ is given by Pieri’s formula (2.1) and $V_{(0,...,0)}$ is the identity element. Then $\odot$ is the usual multiplication in $K_N$ — tensor product of representations.

**Proof** — We will show that $V_{\lambda} \odot V_{\mu} = V_{\lambda} \otimes V_{\mu}$ by induction on the degree $|\lambda|$ of $V_{\lambda}$. First, $V_{(0,...,0)} \odot V_{\mu} = V_{\mu}$ by the condition of lemma. Suppose that the statement is true for any $V_{\lambda}$ with $|\lambda| < d$. For $|\lambda| = d$, we can express $V_{\lambda}$ via the generators $V_{\omega_k}$ as $\sum V_{\omega_k} \otimes W_k$, where the $W_k$ are degree $d-1$ elements of $K_N$. Then, by the inductive hypothesis,

$$V_{\lambda} \odot V_{\mu} = (\sum V_{\omega_k} \otimes W_k) \odot V_{\mu} = \sum V_{\omega_k} \odot (W_k \otimes V_{\mu}) = \sum V_{\omega_k} \otimes W_k \otimes V_{\mu} = V_{\lambda} \otimes V_{\mu}.$$ 

□

### 2.2 Symmetric group

Our constructions rely strongly on the combinatorics of reduced decompositions in the symmetric group $S_n$. This section is devoted to a brief account of this theory.

Let $s_a \in S_n$ be the adjacent transposition that interchanges $a$ and $a + 1$. Then $s_1, \ldots, s_{n-1}$ generate the symmetric group $S_n$. The generators $s_a$ satisfy the following Coxeter relations:

$$
\begin{align*}
    s_a^2 &= 1, \\
    s_as_b &= s_b s_a, & \text{for } |a - b| \geq 2, \\
    s_as_{a+1}s_a &= s_{a+1} s_a s_{a+1}.
\end{align*}
$$

(2.2)

For a permutation $w \in S_n$, an expression $w = s_{a_1} s_{a_2} \cdots s_{a_l}$ of minimal possible length $l$ is called a reduced decomposition; and $l = \ell(w)$ is the length of $w$. The corresponding sequence $a = (a_1, a_2, \ldots, a_l)$ is called a reduced word for $w$. Let $R(w)$ denote the set of all reduced words for $w$. A pair $(i, j)$, $1 \leq i < j \leq n$, is called an inversion in $w$ if $w(i) > w(j)$. By $I(w)$ we denote the set of all inversions of $w$. The number $|I(w)|$ of inversions in $w$ is equal to its length $\ell(w)$.

Let $w_o$ be the longest permutation in $S_n$ given by $w_o(i) = n + 1 - i$. Then $I(w_o)$ is the set of all pairs $1 \leq i < j \leq n$. A total ordering "<" of inversions $(i, j)$ in $I(w_o)$ is said to be a reflection ordering if for any triple $i < j < k$ we have

$$
\begin{align*}
    \text{either } (i, j) < (i, k) < (j, k) \text{ or } (j, k) < (i, k) < (i, j).
\end{align*}
$$

Also, for any $w \in S_n$, we say that a total ordering of inversions in $I(w)$ is a reflection ordering if it is a final interval of some reflection ordering of $I(w_o)$.

The set of all reflection orderings of $I(w)$ is in one-to-one correspondence with the set of reduced decompositions of $w$, cf. [3, Proposition 2.13]. Explicitly, for a reduced decomposition $w = s_{a_1} s_{a_2} \cdots s_{a_l}$, the sequence of pairs $(i_1, j_1) < \cdots < (i_t, j_t)$ such that $i_r = s_{a_1} s_{a_2} \cdots s_{a_{r-1}} (a_r)$ and $j_r = s_{a_1} s_{a_2} \cdots s_{a_{r+1}} (a_r + 1)$, $r = 1, \ldots, l$, is a reflection ordering of $I(w)$. Moreover, every reflection ordering of $I(w)$ arises in this fashion.

Graphically, we represent a reduced decomposition by its wiring diagram, which is also called a pseudo-line arrangement. For instance, the reduced decomposition $s_3 s_2 s_1 s_2$ of an element in $S_4$ is depicted by the diagram.
The nodes of this diagram correspond to the adjacent transpositions. On the other hand, each node is a crossing of $i$-th and $j$-th pseudo-lines, where $(i, j)$ forms an inversion. Reading these pairs in the wiring diagram from bottom to top gives the corresponding reflection ordering of the inversions. In the above example, the associated reflection ordering is $(1, 4) < (1, 2) < (1, 3) < (2, 3)$.

Applying the Coxeter relations to reduced decompositions results in the local transformations that are called 2-moves and 3-moves. Namely, 2-moves correspond to the second equation in (2.2) and 3-moves to the third equation in (2.2). Two reduced decompositions of the same permutation are always connected by a sequence of 2- and 3-moves. Graphically, 2- and 3-moves can be represented by the following local transformations of wiring diagrams, where $i < j < k < l$.

### 3 Scattering Matrix

Let $E$ be the linear space with a basis $e_x$, $x \in \mathbb{Z}_+$. We will always assume that $e_x = 0$ for $x < 0$.

**Definition 3.1** For $c \in \mathbb{Z}$, the scattering matrix $R(c)$ is the linear operator which acts on the space $E \otimes E$ by

$$R(c) : e_x \otimes e_y \mapsto \begin{cases} e_{y+c} \otimes e_{x-c} & \text{if } c \geq x - y, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$
The space $E$ can be viewed as the space of states of a certain quantum particle. The basis vector $e_x$ corresponds to a particle with energy level $x$. We will think of the scattering matrix $R(c)$ as the result of the interaction of two particles with energy levels $x$ and $y$. Pictorially, we can represent it by the following “Feinman diagram”

\[ e_x \xrightarrow{R(c)} e_y \]

or simply

\[ x \xrightarrow{c} y \]

Notice that the energy conservation law holds in our model, since the sum of energies of particles after the interaction $(y + c) + (x - c)$ is the same as before the interaction.

By $R_{ij}(c)$ we denote the linear endomorphism of $E^\otimes m = E \otimes \cdots \otimes E$ which acts as the transformation $R(c)$ on the $i$-th and the $j$-th copies of $E$ and as an identity operator on other copies. Let $a = (a_1, \ldots, a_l)$ be a reduced word for $w \in S_n$, which is associated with the reduced decomposition $w = s_{a_1}s_{a_2}\cdots s_{a_l}$, and let $(i_1,j_1) < \cdots < (i_l,j_l)$ be the corresponding reflection ordering of the inversion set $I(w)$. For a collection $C = (c_{ij}), (i,j) \in I(w)$, of integer parameters, we define an endomorphism $R_a(C)$ of $E^\otimes m$ as the composition of scattering matrices

\[ R_a(C) = R_{i_1j_1}(c_{i_1j_1})R_{i_2j_2}(c_{i_2j_2}) \cdots R_{i_lj_l}(c_{i_lj_l}). \]  

(3.2)

It is clear that $R_{ij}(c_{ij})$ commutes with $R_{kl}(c_{kl})$ provided that all $i,j,k,l$ are distinct. Thus the composition $R_a(C)$ stays invariant when we apply a 2-move to the reduced word $a$.

For positive integers $m$ and $n$, let $w(m,n)$ be the permutation from $S_{m+n}$ given by

\[
\begin{pmatrix}
1 & 2 & \cdots & m & m+1 & m+2 & \cdots & m+n \\
1 & 2 & \cdots & n & m+1 & m+2 & \cdots & m+n \\
& & \cdots & & & & & \\
& & & \cdots & & & & \\
& & & & \cdots & & & \\
& & & & & \cdots & & \\
& & & & & & \cdots & \\
& & & & & & & \cdots \\
& & & & & & & & \cdots \\
\end{pmatrix}.
\]

All reduced decompositions of the permutation $w(m,n)$ are related by 2-moves (cf. the diagram below). Thus the map $R_a(C)$ does not depend upon any particular choice of a reduced word $a$ for $w(m,n)$. We denote by $R(w(m,n))(C)$ this endomorphism of $E^\otimes m \otimes E^\otimes n$. It depends upon the collection of $mn$ parameters $C = (c_{ij}), 1 \leq i \leq m < j \leq m+n$.

Let $T(E)$ denote the tensor algebra of the linear space $E$. We define a new bilinear operation $M : T(E) \otimes T(E) \to T(E)$ whose restriction $M_{m,n} : E^\otimes m \otimes E^\otimes n \to E^\otimes (m+n)$ is given by

\[ M_{m,n} = \sum_C R_{w(m,n)}(C), \]  

(3.3)

where the sum is over all collections $C$ of nonnegative integer parameters $c_{ij}, 1 \leq i \leq m < j \leq m+n$, such that

\[ c_{ij} \geq c_{kl} \quad \text{whenever} \quad k \leq i < j \leq l. \]  

(3.4)

We will use the notation $A \ast B$ for $M(A,B)$, where $A,B \in T(E)$, and occasionally call this multiplication operation $\ast$-product. Although the sum in (3.3) involves infinitely many terms, only a finite number of them are nonzero in the expansion for $A \ast B$.

Let us remark that a collection $C$ of nonnegative integers that satisfy (3.4) is usually called a rectangular shaped plane partition.

The composition of scattering matrices $R_{w(m,n)}$ can be represented by the wiring diagram shown below (for $m = 4$ and $n = 3$). The summation in (3.3) is over all collections of nonnegative integer parameters $c_{ij}$ that weakly decrease downwards along the pseudo-lines of this diagram.
Theorem 3.2  The space $T(E)$ equipped with the multiplication operation $M$ is an associative ring.

Recall that $\omega_1, \ldots, \omega_N$ are the fundamental weights of $GL(N)$. By convention $\omega_0 = 0$.

Theorem 3.3  The projection $p_N : T(E) \rightarrow K_N$ defined on the basis elements by

$$p_N : e_{x_1} \otimes \cdots \otimes e_{x_m} \mapsto \begin{cases} V_\lambda, & \lambda = \omega_{x_1} + \cdots + \omega_{x_m}, \text{ provided } x_1 \leq x_2 \leq \cdots \leq x_m \leq N, \\ 0, & \text{otherwise} \end{cases}$$

is a homomorphism from the ring $(T(E), M)$ to the Grothendieck ring $K_N$ of polynomial representations of $GL(N)$. In other words, if $p_N(A) = V_\lambda$ and $p_N(B) = V_\mu$ then $p_N(A \ast B) = V_\lambda \otimes V_\mu$, the tensor product of representations.

Summarizing the above assertions and definitions, we can formulate a rule for the Littlewood-Richardson coefficients. Let us denote by $e_{x_1 \ldots x_m}$ the element $e_{x_1} \otimes \cdots \otimes e_{x_m} \in T(E)$.

Corollary 3.4  Let $\lambda = \omega_{x_1} + \cdots + \omega_{x_m}$, $\mu = \omega_{y_1} + \cdots + \omega_{y_n}$, and $\nu = \omega_{z_1} + \cdots + \omega_{z_{m+n}}$, where $x_1 \leq \cdots \leq x_m$, $y_1 \leq \cdots \leq y_n$, and $z_1 \leq \cdots \leq z_{m+n}$. The Littlewood-Richardson coefficient $c_{\lambda \mu}^{\nu}$ is equal to the number of collections $C$ of nonnegative integers $c_{ij}$, $1 \leq i \leq m$, $m+1 \leq j \leq m+n$, such that

$$c_{ij} \geq c_{kl}, \quad \text{for } i \leq k < l \leq j;$$

and

$$R_{(m,n)}(C) \cdot (e_{x_1 \ldots x_m} \otimes e_{y_1 \ldots y_n}) = e_{z_1 \ldots z_{m+n}}.$$ 

Proposition 3.5  

1. We have $e_{x_1 \ldots x_m} * e_{y_1 \ldots y_n} = 0$ unless $x_1 \leq \cdots \leq x_m$ and $y_1 \leq \cdots \leq y_n$.

2. The product $e_{x_1 \ldots x_m} * e_{y_1 \ldots y_n}$ involves only terms $e_{z_1 \ldots z_{m+n}}$ with $z_1 \leq \cdots \leq z_{m+n}$.

Proof — 1. First, we show that applying $R_{13}(c_{13})R_{23}(c_{23})$ to $e_{x_1} \otimes e_{x_2} \otimes e_{y_1}$ always results in zero provided $c_{23} \geq c_{13}$ and $x_1 > x_2$. Indeed, we have $R_{13}(c_{13})R_{23}(c_{23}) \cdot (e_{x_1} \otimes e_{x_2} \otimes e_{y_1}) = R_{13}(c_{13}) \cdot (e_{x_1} \otimes e_{y_1+c_{23}} \otimes e_{x_2-c_{23}})$ (or zero). This expression is nonzero only if $c_{13} \geq x_1 - (x_2 - c_{23})$, i.e., $c_{13} - c_{23} \geq x_1 - x_2$. Contradiction.
In general, suppose that, say, \( x_i > x_{i+1} \). The composition of operators \( R_{(m,n)}(C) \) with \( C \) satisfying \((3.4)\) involves the fragment \( R_{i+1} R_{i+1} (c_{i+1} m+1) R_{i+1} c_{i+1} m+1 \), where \( c_{i+1} m+1 \geq c_{i+1} m+1 \). By the above argument, applying these operators gives zero.

2. This statement follows by induction on \( m \) from the next Proposition \ref{prop:Pieri-type}.

Let us verify the statement of Theorem \ref{thm:Yang-Baxter type relation} for the \( \ast \)-product of \( e_x \) with an arbitrary \( e_{x_1} \cdots e_{x_m} \). This product is given by the following Pieri-type formula.

**Proposition 3.6** For \( 0 \leq x_1 \leq \cdots \leq x_m \), we have

\[
e_x \ast e_{x_1} \cdots e_{x_m} = \sum e_{y_1} \cdots e_{y_{m+1}},
\]

where the sum is over all \( y_1, \ldots, y_{m+1} \) satisfying the following interlacing conditions:

\[
0 \leq y_1 \leq x_1 \leq y_2 \leq x_2 \cdots \leq y_m \leq x_m \leq y_{m+1},
\]

\[
y_1 - x_1 + y_2 - x_2 + \cdots + y_m - x_m + y_{m+1} = x.
\]

**Proof** — By definition, \( e_x \ast e_{x_1} \cdots e_{x_m} = \sum R_{1 m+1}(c_m) R_{1 m}(c_{m-1}) \cdots R_{12}(c_1) \cdot (e_x \otimes e_{x_1} \otimes \cdots e_{x_m}) \), where the sum is over \( c_1 \geq c_2 \geq \cdots \geq c_m \geq 0 \). Each nonvanishing summand in the previous sum is equal to \( e_{x-c_1} \otimes e_{x_1+c_1-c_2} \otimes e_{x_2+c_2-c_3} \otimes \cdots e_{x_m+c_{m-1}-c_m} \otimes e_{c_m+c_1} \) provided \( c_1 \geq x - x_1 \), \( c_2 \geq (x_1 + c_1) - x_2 \), \( c_3 \geq (x_2 + c_2) - x_3 \), etc. Let us denote \( y_1 = x - c_1 \), \( y_2 = x_1 + c_1 - c_2 \), \( y_3 = x_2 + c_2 - c_3 \), \ldots, \( y_m = x_{m-1} + c_{m-1} - c_m \), \( y_{m+1} = x_m + c_m \). Then all the above inequalities are equivalent to the interlacing conditions \((3.5)\).

Due to Lemma \ref{lem:saturating} and Proposition \ref{prop:Pieri-type}, Theorem \ref{thm:Yang-Baxter type relation} and Corollary \ref{cor:Yang-Baxter type relation} would follow from Theorem \ref{thm:Yang-Baxter type relation}, which says that \( M \) is an associative operation. The proof of associativity given in Section 3 is based on a Yang-Baxter type relation for the scattering matrices \( R_{ij}(c) \) (Section 4) and on the construction of certain polyhedral cones in the space of the parameters \( c_{ij} \) (Section 5).

### 4 Yang-Baxter Equation and Tetrahedron Equation

As we mentioned before, for distinct \( i, j, k, l \), the endomorphism \( R_{ij}(c_{ij}) \) commutes with \( R_{kl}(c_{kl}) \). Thus \( R_a(C) \) does not change when we apply a 2-move to the reduced word \( a \). The relations that involve 3-moves are less trivial.

**Theorem 4.1** The operators \( R_{12}(c_{12}), R_{13}(c_{13}), \) and \( R_{23}(c_{23}) \) acting on \( E^\otimes 3 \) satisfy the relation

\[
R_{23}(c_{23}) R_{13}(c_{13}) R_{12}(c_{12}) = R_{12}(c'_{12}) R_{13}(c'_{13}) R_{23}(c'_{23}),
\]

where \( c_{12}, c_{13}, c_{23} \) are arbitrary parameters and \( c'_{12}, c'_{13}, c'_{23} \) are given by

\[
c'_{12} = \min(c_{12}, c_{13} - c_{23}),
\]

\[
c'_{13} = c_{12} + c_{23},
\]

\[
c'_{23} = \max(c_{23}, c_{13} - c_{12}).
\]

Moreover, for fixed \( c_{12}, c_{13}, c_{23} \), the collection \( c'_{12}, c'_{13}, c'_{23} \) defined by \((4.2)\) is a unique collection of parameters such that \((4.1)\) holds identically.

The following two wiring diagrams related by a 3-move illustrate the statement of the theorem.
Remark 4.2 If $c_{13} = c_{12} + c_{23}$ then $c'_{12} = c_{12}$, $c'_{13} = c_{13}$, and $c'_{23} = c_{23}$. In this case the equation (4.1) becomes the famous quantum Yang-Baxter equation with two parameters, which is well-known to the informed reader in the form $R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u)$.

Proof of Theorem 4.4 — The operator $R_{23}(c_{23})R_{13}(c_{13})R_{12}(c_{12})$ maps the basis vector $e_{x_1} \otimes e_{x_2} \otimes e_{x_3}$ either to $e_{x_1 + c_{13}} \otimes e_{x_2 + c_{12} - c_{13} + c_{23}} \otimes e_{x_1 - c_{12} - c_{23}}$ if

$$
\begin{align*}
&\begin{cases}
    c_{12} \geq x_1 - x_2, \\
    c_{13} \geq (x_2 + c_{12}) - x_3, \\
    c_{23} \geq (x_1 - c_{12}) - (x_2 + c_{12} - c_{13}),
\end{cases} \\
\text{or to zero otherwise. Likewise, the operator} \\
R_{12}(c'_{12})R_{13}(c'_{13})R_{23}(c'_{23}) \text{ maps } e_{x_1} \otimes e_{x_2} \otimes e_{x_3} \text{ either to } e_{x_1 + c'_{13}} \otimes e_{x_2 - c'_{23} + c'_{13} - c'_{12}} \otimes e_{x_1 - c_{13}} \\
\text{if}
\begin{align*}
&\begin{cases}
    c'_{23} \geq x_2 - x_3, \\
    c'_{13} \geq x_1 - (x_2 - c'_{23}), \\
    c'_{12} \geq (x_2 - c'_{23} + c'_{13}) - (x_3 - c'_{23}),
\end{cases} \\
\text{or to zero otherwise. These two operators are equal if and only if}
\end{align*}
\end{align*}
$$

or (the second identity is the difference of two others) and for any $x_1, x_2, x_3$ the condition (4.3) is equivalent to the condition (4.4). We can write these two sets of inequalities in a more compact form as follows:

$$
\begin{align*}
\begin{cases}
    \min(c_{12}, c_{23} + 2c_{12} - c_{13}) \geq x_1 - x_2, \\
    c_{13} - c_{12} \geq x_2 - x_3,
\end{cases} \text{ is equivalent to } \\
\begin{cases}
    c'_{13} - c'_{23} \geq x_1 - x_2, \\
    \min(c'_{23}, c_{12} + 2c'_{23} - c'_{13}) \geq x_2 - x_3.
\end{cases}
\end{align*}
$$

Thus

$$
\begin{align*}
\min(c_{12}, c_{23} + 2c_{12} - c_{13}) &= c'_{13} - c'_{23}, \\
c_{13} - c_{12} &= \min(c'_{23}, c_{12} + 2c'_{23} - c'_{13}).
\end{align*}
$$
These two identities together with (4.5) are equivalent to the relations (4.2).

It follows for Theorem 4.1 that, for \(i < j < k\), the operators \(R_{ij}(c_{ij})\), \(R_{ik}(c_{ik})\), and \(R_{jk}(c_{jk})\) acting on \(E^\otimes n\) satisfy the relation

\[ R_{jk}(c_{jk})R_{ik}(c_{ik})R_{ij}(c_{ij}) = R_{ij}(c'_{ij})R_{ik}(c'_{ik})R_{jk}(c'_{jk}), \]

where

\[ c'_{ij} = \min(c_{ij}, c_{ik} - c_{jk}), \]
\[ c'_{ik} = c_{ij} + c_{jk}, \]
\[ c'_{jk} = \max(c_{jk}, c_{ik} - c_{ij}). \] (4.6)

The inverse transformation \((c'_{ij}, c'_{ik}, c'_{jk}) \rightarrow (c_{ij}, c_{ik}, c_{jk})\) is given by similar formulas

\[ c_{ij} = \max(c'_{ij}, c'_{ik} - c'_{jk}), \]
\[ c_{ik} = c'_{ij} + c'_{jk}, \]
\[ c_{jk} = \min(c'_{jk}, c'_{ik} - c'_{ij}). \] (4.7)

We will denote by \(\mathbb{Z}^I(w)\) the set of all collections of integers parameters \(C = (c_{pq})\) with \((p, q) \in I(w), p < q\). For \(i < j < k\) such that \((i, j), (i, k), (j, k) \in I(w)\), we denote by \(T_{ijk}\) the local transformation of parameters

\[ T_{ijk} : \mathbb{Z}^I(w) \rightarrow \mathbb{Z}^I(w) \]
\[ T_{ijk} : (c_{pq}) \rightarrow (c'_{pq}) \]

where the \(c'_{pq}\) are given by formulas (4.6) for \(p, q \in \{i, j, k\}\) and \(c'_{pq} = c_{pq}\) otherwise.

For any two reduced words \(a, b \in R(w)\) of a permutation \(w \in S_n\), we will define a transition map \(T^b_a : \mathbb{Z}^I(w) \rightarrow \mathbb{Z}^I(w)\) as a composition of local transformation maps \(T_{ijk}\). If \(a = (\ldots, a, b, \ldots)\) and \(a' = (\ldots, b, a, \ldots), |a - b| \geq 2\), are two reduced words for \(w\) related by a 2-move, then \(T_{i}^a\) is the identity map. If \(a = (\ldots, a, a+1, a, \ldots)\) and \(a' = (\ldots, a+1, a, a+1, \ldots)\) are two reduced words related by a 3-move, then the corresponding reflection orderings of \(I(w)\) differ only in three places: \(\cdots < (j, k) < (i, k) < (i, j) < \cdots\) and \(\cdots < (i, j) < (i, k) < (j, k) < \cdots\) for certain \(i < j < k\). In this case we define \(T_{i}^a = T_{ijk}\) and \(T_{i+j+1}^{a'} = T_{ijk}^{-1}c'_{i+j+1}\). In general, we choose a chain of reduced words \(a, a^1, a^2, \ldots, a^k, b \in R(w)\) that interpolates between \(a\) and \(b\) such that any two adjacent words are related by a 2- or 3-move. Then we define \(T_{i}^a = T_{i}^{a_1} \cdots T_{i}^{a_k} T_{i}^{a_{k+1}}\).

It follows from the uniqueness part of Theorem 4.1 that the transition map \(T^b_a\) does not depend upon a choice of path of 2- and 3-moves joining the reduced words \(a\) and \(b\). Let us remark that this property amounts to verifying that the local transformation maps \(T_{ijk}\) satisfy the following tetrahedron equation.

**Theorem 4.3** (Tetrahedron equation) Let \(w_0\) be the longest element in \(S_4\). The following identity for the compositions of maps \(\mathbb{Z}^I(w_0) \rightarrow \mathbb{Z}^I(w_0)\) holds:

\[ T_{123}T_{124}T_{134}T_{234} = T_{234}T_{134}T_{124}T_{123}. \]

It is left as an exercise for the reader to verify directly that the local transformation maps \(T_{ijk}\) satisfy the tetrahedron equation.

Recall that \(R_a(C)\) is the composition of scattering matrices defined by (3.2). It is immediately clear from Theorem 4.1 that \(R_a(C) = R_{a'}(T^b_a(C))\) if \(a\) and \(a'\) are related by a 2 or 3-move. Thus, in general, we have

\[ R_a(C) = R_b(T^b_a(C)) \] (4.8)

for any two reduced words \(a\) and \(b\) for \(w\) and any collection of parameters \(C \in \mathbb{Z}^I(w)\).
5 Principal Cones

Let $a$ be a reduced word of a permutation $w \in S_n$. In this section we construct and study a certain polyhedral cone $C_a$ in the space $\mathbb{Z}^{I(w)}$. In the case when $w = w_o$ is the longest permutation in $S_n$, the cone $C_a$ is exactly the cone of Kashiwara’s parametrizations of dual canonical bases for $U_q(sl_n)$. It is the string cone in the terminology of Berenstein and Zelevinsky [2]. The explicit description of $C_a$ gives an answer to a question posed in [2].

5.1 Rigorous paths and statements of results

Let us fix a reduced word $a \in R(w)$ and an integer $0 \leq s \leq n$. We construct an oriented graph $G(a,s)$ from the wiring diagram corresponding to $a$ as follows. Denote by $v_{ij}$ the vertex of the wiring diagram which is the intersection of the $i$-th and $j$-th pseudo-lines. The vertex set of the graph $G(a,s)$ is composed of the vertices $v_{ij}$ together with $2n$ boundary vertices: $U_1, \ldots, U_n$ that mark the upper ends of pseudo-lines from left to right and $L_1, \ldots, L_n$, that mark the lower ends of pseudo-lines from left to right. Notice that $U_i$ is the upper end of the $i$-th pseudo-line. We orient downward the $s$ pseudo-lines of the wiring diagram whose lower ends are labelled $L_1, \ldots, L_s$ and we orient upward the remaining $n - s$ pseudo-lines whose lower ends are labelled $L_{s+1}, \ldots, L_n$. Two vertices are connected by an edge in the graph $G(a,s)$ if they are adjacent vertices on the same pseudo-line. Directions of edges in $G(a,s)$ agree with directions of the corresponding pseudo-lines. For example, the graph $G(121,2)$ is shown on the picture below.

![Graph G(121,2)](image)

An oriented path in the graph $G(a,s)$ is a sequence of vertices $v_0, \ldots, v_l$ connected by the oriented edges $v_0 \to v_1$, $v_1 \to v_2$, $\ldots$, $v_{l-1} \to v_l$. Notice that the graph $G(a,s)$ is acyclic, i.e., there is no closed oriented cycle in the graph. Thus there are finitely many oriented paths in $G(a,s)$. We say that an oriented path $v_0 \to v_1 \to \ldots \to v_l$ is rigorous if it satisfies the following condition: There are no three adjacent vertices $v_a \to v_{a+1} \to v_{a+2}$ in the path such that $v_a, v_{a+1}, v_{a+2}$ belong to the same $i$-th pseudo-line, $v_{a+1}$ is the intersection of the $i$-th and $j$-th pseudo-line, and either $i < j$ and both $i$-th and $j$-th pseudo-lines are oriented upwards, or $i > j$ and the $i$-th and $j$-th pseudo-lines are oriented downwards. In other words, a path is rigorous if and only if it avoids the following two fragments:

![Path fragments](image)

Here the thick lines show path fragments and the thin lines show the pseudo-lines they intersect.
For example, in the graph $G(121, 2)$ shown above all paths connecting boundary vertices are rigorous except the following two paths: $L_3 \to v_{13} \to v_{23} \to L_1$ and $U_3 \to v_{13} \to v_{23} \to L_1$.

Let $P = (v_0 \to v_1 \to \cdots \to v_l)$ be a rigorous path connecting two boundary vertices $v_0$ and $v_l$. Suppose that the edge $v_{i-1} \to v_r$ is on the $i$-th pseudo-line, for $r = 1, \ldots, l$. We denote by $c_P$ the expression

$$c_P = c_{i12} + c_{i23} + \cdots + c_{i(l-1)i},$$

where we assume that $c_{ii} = 0$ and for $i > j$ the coefficients $c_{ij}$ are given by $c_{ij} = -c_{ji}$.

**Definition 5.1** For a reduced word $a \in \mathcal{R}(w)$, we define the principal cone $C_a$ as the the polyhedral cone in the the integer lattice $\mathbb{Z}^{I(w)}$ of collections $C = (c_{ij})$ given by the inequalities $c_P \geq 0$ for all rigorous paths $P$ in the graph $G(a, s)$ from the vertex $L_{s+1}$ to $L_s$, for $1 \leq s \leq n - 1$.

**Theorem 5.2** For any two reduced words $a, b \in \mathcal{R}(w)$, the transition map $T^b_a$ bijectively maps the cone $C_a$ to the cone $C_b$.

**Example 5.3** To illustrate the definition and the theorem we describe the principal cones for two reduced decompositions of $w_0$ in $S_3$.

![Diagram](image.png)

$$C_{121} = \{ c_{12} \geq 0, \ c_{13} \geq c_{23} \geq 0 \} \quad \quad C_{212} = \{ c_{23} \geq 0, \ c_{13} \geq c_{12} \geq 0 \}$$

Indeed, for $a = 121$ we have the rigorous paths the $L_2 \to v_{23} \to L_1$, $L_3 \to v_{13} \to v_{23} \to L_2$, and $L_3 \to v_{13} \to v_{12} \to v_{23} \to L_2$. Analogously, for $a = 212$ we have the rigorous paths $L_2 \to v_{12} \to v_{23} \to v_{13} \to L_1$, $L_2 \to v_{12} \to v_{13} \to L_1$, and $L_3 \to v_{12} \to L_2$. One can easily verify that the transformation map $T_{123}$ maps the cone $C_{121}$ into the cone $C_{212}$.

In the case when $a \in \mathcal{R}(w_0)$ is a reduced word for the longest permutation in $S_n$ there are two alternative descriptions of the principal cone $C_a$.

**Theorem 5.4** For a reduced word $a \in \mathcal{R}(w_0)$, the principal cone $C_a$ is the set of all collections $C = (c_{ij}) \in \mathbb{Z}^{I(w)}$ such that for any reduced word $b \in \mathcal{R}(w_0)$ all entries $c'_{ij}$ of the collection $C' = (c'_{ij}) = T^b_a(C) \in \mathbb{Z}^{I(w)}$ are nonnegative.

For a reduced word $b \in \mathcal{R}(w_0)$, let $low(b)$ denote the pair $(i, j)$, $1 \leq i < j \leq n$, such that the lowest node of the wiring diagram of $b$ is the crossing of $i$-th and $j$-th pseudo-lines. (It is clear that $j = i + 1$.) For example, $low(121) = (2, 3)$ and $low(212) = (1, 2)$.

The principal cone can be described by a weaker set of conditions as follows.
Theorem 5.5 For a reduced word \( a \in R(w_o) \), the principal cone \( C_a \) is the set of all collections \( C = (c_{ij}) \in \mathbb{Z}^{I(w_o)} \) such that for any reduced word \( b \in R(w_o) \) the lowest entry \( c'_{\text{low}(b)} \) of \( C' = (c'_{ij}) = T_a^b(C) \) is nonnegative.

Remark 5.6 In the case of \( w_o \), we can choose either of the descriptions from Theorems 5.4 or 5.5 as the definition of the principal cone. Then Theorem 5.2 would become trivial. But this would obscure the fact that the principal cone is actually a polyhedral cone.

Before we proceed, let us consider several examples of principal cones.

Example 5.7 Let \( a_o = (1, 2, 1, 3, 2, 1, \ldots, n-1, n-2, \ldots, 1) \in R(w_o) \) be the lexicographically minimal reduced word for the longest permutation. By Definition 5.1, the principal cone \( C_{a_o} \) is given by the inequalities:

\[
c_{12} \geq 0, \quad c_{13} \geq c_{23} \geq 0, \quad c_{14} \geq c_{24} \geq c_{34} \geq 0, \quad c_{15} \geq c_{25} \geq c_{35} \geq c_{45} \geq 0, \ldots
\]

Indeed, in this case all inequalities \( c_{sp} \geq 0 \) in Definition 5.1 are of the form \( c_{si} \geq c_{s+1} \) for \( i > s + 1 \).

Berenstein and Zelevinsky [2] studied the string cone of Kashiwara’s parametrizations of dual canonical basis for \( U_q(sl_n) \). This is a cone \( \tilde{C}_a \) in the \( \binom{n}{2} \)-dimensional space of strings \( C = (c_{ij}) \) that depends upon a choice of reduced word \( a \in R(w_o) \) for the longest permutation. It follows from the definitions that if \( a \) and \( a' \) differ by a 2- or 3-move then \( \tilde{C}_{a'} = T_{a'}^a(\tilde{C}_a) \). Thus string cones \( \tilde{C}_a \) transform according to the transition maps \( T_{a'}^a \). The string cone was explicitly calculated in [2] for the lexicographically minimal reduced word \( a_o \). In this case \( \tilde{C}_{a_o} \) is given by the inequalities (5.2).

Theorem 5.2 and Example 5.7 imply the following statement.

Corollary 5.8 (String cones) For a reduced word \( a \in R(w_o) \), the principal cone \( C_a \) is exactly the string cone \( \tilde{C}_a \).

Definition 5.1 gives an explicit description of the string cone \( \tilde{C}_a \). This settles the problem of describing the string cones for any reduced word \( a \in R(w_o) \).

Example 5.9 This example is related to our construction of the \( * \)-product in Section 3. Recall that the permutation \( w(m, n) : i \mapsto i + n \pmod{m+n} \) in \( S_{m+n} \) has a unique reduced decomposition up to 2-moves. For example, for \( m = 4 \) and \( n = 3 \) we have the following wiring diagram:
By Definition 5.1, the corresponding principal cone \( C_{(m,n)} = C_a \) is given by the inequalities \(-c_{ik} + c_{ij} \geq 0\) for \( i \leq m < j < k; c_{1m+n} \geq 0; \) and \( c_{jk} - c_{ik} \geq 0\) for \( i < j \leq m < k\). These are exactly the conditions (5.4) on the parameters in the sum (3.3). Thus the \( s\)-product in \( T(E) \) can be written as the sum

\[
\sum_{C \in C_{(m,n)}} R_{(m,n)}(C) \cdot (e_{x_1 \ldots x_m} \otimes e_{y_1 \ldots y_n}).
\]

**Example 5.10** Let us also illustrate the definitions by the following example for the reduced decomposition \( s_2 s_1 s_2 s_3 s_2 s_1 \) of the longest element \( w_0 \in S_4 \).

\[
\begin{array}{cccc}
U_1 & U_2 & U_3 & U_4 \\
12 & 13 & 14 & 24 \\
23 & 24 & 23 & 34 \\
1 & 2 & 3 & 4 \\
\end{array}
\]

The rigorous paths and inequalities are:

- \( L_2 \to v_{23} \to v_{34} \to v_{24} \to L_1 \) for \( s = 1 \):
  - \( c_{34} \geq 0 \)
  - \( c_{32} + c_{24} \geq 0 \)

- \( L_3 \to v_{23} \to L_2 \) for \( s = 2 \):
  - \( c_{23} \geq 0 \)

- \( L_4 \to v_{14} \to v_{13} \to v_{12} \to v_{24} \to v_{23} \to L_3 \) for \( s = 3 \):
  - \( c_{12} \geq 0 \)
  - \( c_{13} + c_{34} + c_{24} \geq 0 \)
  - \( c_{14} + c_{43} + c_{32} \geq 0 \)

In more conventional notation the inequalities defining the cone \( C_{212321} \) can be written as

\[
C_{212321} = \begin{cases} 
  c_{12} \geq 0 \\
  c_{13} \geq c_{23} \geq 0 \\
  c_{13} + c_{34} \geq c_{24} \geq c_{23} \\
  c_{14} \geq c_{23} + c_{34} \\
  c_{34} \geq 0 
\end{cases}.
\]

To prove Theorems 5.2, 5.4, and 5.5, we need some extra notation. For two boundary vertices \( B \) and \( E \) in \( \{U_1, \ldots, U_n, L_1, \ldots, L_n\} \), let

\[
M^{a,s}_{B,E} = M^{a,s}_{B,E}(C) = \min c_P
\]

be the minimum of expressions (5.1) over all rigorous paths \( P \) in the graph \( G(a, s) \) from the vertex \( B \) to the vertex \( E \), here \( C = (c_{ij}) \). (Note that there are finitely many such paths.) If there are no rigorous paths from \( B \) to \( E \) in \( G(a,s) \) then we set \( M^{a,s}_{B,E} = +\infty \).

Using this notation, Definition 5.1 of the principal cone can be written as

\[
C_a = \left\{ C \in Z^I(w) \mid M^{a,s}_{L_i+1,L_i}(C) \geq 0, \text{ for } s = 1, \ldots, n-1 \right\}.
\]  \hspace{1cm} (5.3)

Theorem 5.2 is an immediate corollary of the following more general statement.

**Theorem 5.11** For any two reduced words \( a, b \in R(w_0) \), an integer \( 0 \leq s \leq n \), and two boundary vertices \( B \) and \( E \), we have \( M^{a,s}_{B,E}(C) = M^{b,s}_{B,E}(C') \), where \( C' = T_a^{B,E}(C) \). In other words, the expressions \( M^{a,s}_{B,E}(C) \) are invariant under the transition maps \( T_a^{B,E} \).
5.2 Proofs

Proof of Theorem 5.11 — Let us first verify the statement of the theorem for the symmetric group $S_3$. In this case we have only two reduced words $121$ and $212$ for $w_\alpha$. There are four possible cases $s = 0$, $s = 1$, $s = 2$, and $s = 3$.

Let us start with the case $s = 3$ when all pseudo-lines are oriented downwards. The graphs $G(121, 0)$ and $G(212, 0)$ are given on the picture below. We also give the transition maps $T_{121}^{212}$ and its inverse $T_{212}^{121}$ for a quick reference, cf. (4.2) and (4.7).

![Transition Maps](image)

Enumerating rigorous paths in these graphs, we obtain

$$
\begin{pmatrix}
M_{U_1, L_1}^{212, 3} & M_{U_2, L_1}^{212, 3} & M_{U_3, L_1}^{212, 3} \\
M_{U_1, L_2}^{212, 3} & M_{U_2, L_2}^{212, 3} & M_{U_3, L_2}^{212, 3} \\
M_{U_1, L_3}^{212, 3} & M_{U_2, L_3}^{212, 3} & M_{U_3, L_3}^{212, 3}
\end{pmatrix} =
\begin{pmatrix}
c_{12} + c_{23} & \min(c_{12}, c_{13} + c_{32}) & 0 \\
+\infty & c_{21} + c_{13} + c_{32} & c_{21} \\
+\infty & +\infty & c_{31}
\end{pmatrix}
$$

$$
\begin{pmatrix}
M_{U_1, L_1}^{212, 3} & M_{U_2, L_1}^{212, 3} & M_{U_3, L_1}^{212, 3} \\
M_{U_1, L_2}^{212, 3} & M_{U_2, L_2}^{212, 3} & M_{U_3, L_2}^{212, 3} \\
M_{U_1, L_3}^{212, 3} & M_{U_2, L_3}^{212, 3} & M_{U_3, L_3}^{212, 3}
\end{pmatrix} =
\begin{pmatrix}
c'_{13} & c'_{12} & 0 \\
+\infty & c'_{23} + c'_{31} + c'_{12} & \min(c'_{21}, c'_{23} + c'_{31}) \\
+\infty & +\infty & c'_{32} + c'_{21}
\end{pmatrix}.
$$

It is immediate from the formulas for the transition maps $T_{121}^{212}$ and $T_{212}^{121}$ that these two matrices are equal to each other.

In the next case ($s = 2$) the pseudo-lines with the lower ends $L_1$ and $L_2$ are oriented downward and the pseudo-line with the lower end $L_3$ is oriented upward as shown on the following picture:
In this case we have

\[
\begin{pmatrix}
M_{L_3, U_1}^{212}, M_{L_3, L_2}^{212}, M_{L_3, L_1}^{212} \\
M_{U_2, U_1}^{212}, M_{U_2, L_2}^{212}, M_{U_2, L_1}^{212} \\
M_{U_3, U_1}^{212}, M_{U_3, L_2}^{212}, M_{U_3, L_1}^{212}
\end{pmatrix} =
\begin{pmatrix}
0 & \min(c_{12}, c_{13} + c_{32}) & c_{12} + c_{23} \\
0 & c_{21} & 0 \\
0 & \min(c_{31}, c_{31} + c_{12}) & c_{31} + c_{12} + c_{23}
\end{pmatrix},
\]

\[
\begin{pmatrix}
M_{L_3, U_1}^{212}, M_{L_3, L_2}^{212}, M_{L_3, L_1}^{212} \\
M_{U_2, U_1}^{212}, M_{U_2, L_2}^{212}, M_{U_2, L_1}^{212} \\
M_{U_3, U_1}^{212}, M_{U_3, L_2}^{212}, M_{U_3, L_1}^{212}
\end{pmatrix} =
\begin{pmatrix}
0 & c'_{12} & c'_{13} \\
\min(c'_{21}, c'_{23} + c'_{31}) & 0 & \min(c'_{23}, c'_{21} + c'_{13}) \\
0 & c'_{32} + c'_{21} & c'_{32} + c_{21} + c'_{13}
\end{pmatrix}.
\]

Again, it is clear that these two matrices are equal to each other.

The cases \( s = 0 \) and \( s = 1 \) are completely symmetric to the cases \( s = 3 \) and \( s = 2 \), respectively.

We can now verify the statement of the theorem for an arbitrary \( n \). This general statement reduces to the case of \( S_3 \) (\( n = 3 \)) as follows. Clearly, it is enough to prove the statement for two reduced words \( a \) and \( a' \) related by a 3-move. The corresponding reflection orderings of inversions differ only in tree terms: \( \ldots < (j,k) < (i,k) < (i,j) < \ldots \) and \( \ldots < (i,j) < (i,k) < (j,k) < \ldots \) for some \( i < j < k \). The transition map \( T_{a}^{s} \) is the map \( T_{ijk} \) that transforms \( c_{ij}, c_{ik}, \) and \( c_{jk} \) into \( c'_{ij}, c'_{ik}, \) and \( c'_{jk} \) according formulas (4.6) and does not change other variables.

The intersection points of the pseudo-lines labelled \( i, j, \) and \( k \) form a subdiagram \( S' \) in the wiring diagram of \( a \) (respectively, a subdiagram \( S'' \) in the wiring diagram of \( a' \)) isomorphic to a wiring diagram for \( S_3 \). Let us add six auxiliary vertices \( u_1, u_2, u_3, \) and \( l_1, l_2, l_3 \) to the graph \( G(a, s) \) (respectively, in \( G(a', s) \)) that mark the upper and the lower ends of the pseudo-lines \( i, j, \) and \( k \) in this subdiagram.

If a path \( P \) in the graph \( G(a, s) \) does not pass through any of the vertices \( v_{ij}, v_{ik}, \) and \( v_{jk} \) then the expression \( (L) \) for \( c_P \) does not change under the transformation map \( T_{a}^{s} \). Otherwise, the path \( P \) arrives to the subdiagram \( S' \) via one of the six auxiliary points \( u_1, \ldots, l_3 \) and leaves the subdiagram via another of these six points.

Let us fix two vertices \( b \) and \( e \) of the six auxiliary vertices and two rigorous paths \( P_1 \) (from \( B \) to \( b \)) and \( P_2 \) (from \( e \) to \( E \)). And let \( M_{b,e,P_1,P_2}^{a,s}(C) \) (respectively, \( M_{b,e,P_1,P_2}^{a',s}(C') \)) be the minimum of the expressions \( c_P \) over rigorous paths \( P \) in \( G(a, s) \) (respectively, in \( G(a', s) \)) which are obtained by concatenation of the path \( P_1, \) a rigorous path in \( S \) (respectively, in \( S' \)) from \( b \) to \( e \), and the path \( P_2 \). Then, by our definitions,

\[
M_{b,e,P_1,P_2}^{a,s}(C) = \min_{b,e,P_1,P_2} M_{b,e,P_1,P_2}^{a,s}(C) \quad \text{and} \quad M_{b,e,P_1,P_2}^{a',s}(C') = \min_{b,e,P_1,P_2} M_{b,e,P_1,P_2}^{a',s}(C').
\]

It follows from the case of \( S_3 \) considered above that \( M_{b,e,P_1,P_2}^{a,s}(C) = M_{b,e,P_1,P_2}^{a',s}(C') \). Therefore, \( M_{b,e,P_1,P_2}^{a,s}(C) = M_{b,e,P_1,P_2}^{a',s}(C') \). This proves the theorem. \( \square \)

Proof of Theorem 5.5 — Suppose that \( low(b) = (i, j) \), the lower end of \( i \)-th pseudo-line is \( L_{s+1} \), and the lower end of \( j \)-th pseudo-line is \( L_s \). (In the case of \( w_o \), we have \( i = n - s \) and \( j = n - s + 1 \).) Then there is only one rigorous path in \( G(b, s) \) from \( L_{s+1} \) to \( L_s \), namely, \( L_{s+1} \rightarrow v_{ij} \rightarrow L_s \). In this case \( M_{L_{s+1}, L_s}^{b,s}(C') = c_{ij} \). Thus \( M_{L_{s+1}, L_s}^{a,s}(C) = M_{L_{s+1}, L_s}^{b,s}(C') = c_{ij} = c_{low(b)} \), where \( C' = T_q^b(C) \).

For any \( s = 1, \ldots, n - 1 \), there is a reduced decomposition \( b \) of the longest permutation \( w_o \) such that \( low(b) = (n - s, n - s + 1) \). Thus the inequality \( M_{L_{s+1}, L_s}^{b,s}(C) \geq 0 \) is equivalent to saying that for any reduced word \( b \in \mathcal{R}(w_o) \) such that \( low(b) = (n - s, n - s + 1) \) the lowest entry \( c_{low(b)} \) of \( C' = T_q^b(C) \) is nonnegative. The statement follows. \( \square \)
Proof of Theorem 5.4 — We deduce this theorem from Theorem 5.3. Let us fix \( a \in R(w_0) \). It is enough to show that if \( C = (c_{ij}) \) has a negative entry then there is a reduced word \( b \in R(w_0) \) such that the lowest entry \( c'_{low} = c'_{low}(b) \) of \( C' = T^b_a(C) \) is negative.

Suppose not. Let us pick a reduced word \( b \) such that \( C' = T^b_a(C) \) has a negative entry \( c'_{pq} < 0 \) located on the lowest possible level. The pair \( (p, q) \neq low(b) \) does not correspond to the lowest crossing in the wiring diagram of \( b \). Thus (possibly, after several 3-moves that don’t affect \( c'_{pq} \)) we can make a 3-move transforming three entries \( (c'_{ij}, c'_{ik}, c'_{jk}) \rightarrow (c''_{ij}, c''_{ik}, c''_{jk}) \) by the rule (4.6) such that \( (p, q) \in \{(i, j), (i, k), (j, k)\} \) but \( (p, q) \) is not the lowest pair \( (j, k) \) among these three. By our assumption, \( c''_{jk} \) is nonnegative. Then \( c''_{ij} = \min(c'_{ij}, c''_{ik} - c_{jk}) \) is negative and \( c''_{ij} \) is located on a lower level in the resulting wiring diagram than the level of \( c'_{pq} \) in \( b \). Contradiction. \( \square \)

6 Associativity

In this section we prove Theorem 3.2 which claims that the \( \ast \)-product defined by (3.3) is an associative operation.

Proof of Theorem 3.2 — We need to verify that

\[
(e_{x_1 \cdots x_m} \ast e_{y_1 \cdots y_n}) \ast e_{z_1 \cdots z_k} = e_{x_1 \cdots x_m} \ast (e_{y_1 \cdots y_n} \ast e_{z_1 \cdots z_k}), \tag{6.1}
\]

for any positive \( m, n, k \) and \( x_1 \leq \cdots \leq x_m, y_1 \leq \cdots \leq y_n, z_1 \leq \cdots \leq z_k \).

Let \( Id_k \) be the identity permutation in \( S_k \). The permutation \( w(m, n) \times Id_k \in S_{m+n} \times S_k \) is canonically embedded into \( S_{m+n+k} \). Likewise, the permutation \( Id_m \times w(n, k) \in S_m \times S_{n+k} \) is canonically embedded into \( S_{m+n+k} \). Then \( w(m+n,k) \cdot (w(m,n) \times Id_k) = w(m+n,k) \cdot (Id_m \times w(n,k)) \).

We will denote this permutation by \( w(m, n, k) \).

Remind that the permutations \( w(m+n,k) \) and \( w(m,n) \times Id_k \) have unique (up to 2-moves) reductions. Let \( a^1 \) be a reduced word for \( w(m, n, k) \) obtained by concatenation of reduced words for \( w(m+n,k) \) and \( w(m,n) \times Id_k \). Analogously, let \( a^2 \) be a reduced word for \( w(m, n, k) \) obtained by concatenation of reduced words for \( w(m+n,k) \) and \( Id_m \times w(n,k) \).

The inversion set \( I(w(m,n,k)) \) of the permutation \( w(m,n,k) \) is the union of the following three sets of pairs \( [1, m] \times [m+1, m+n], [1, m] \times [m+n+1, m+n+k], \) and \( [m+1, m+n] \times [m+n+1, m+n+k] \), where \( [a, b] = \{a, a+1, \ldots, b\} \).

By the definition of \( \ast \)-product, the left hand side of the expression (6.1) is equal to the sum \( \sum_{C \in C_a^1} R_{a^1}(C) \cdot (e_{x_1 \cdots x_m} \otimes e_{y_1 \cdots y_n} \otimes e_{z_1 \cdots z_k}) \) over all collections \( C = (c_{ij}) \in Z^I(w(m,n,k)) \) with nonnegative integer entries such that

\[
c_{ij} \geq c_{pq} \quad \text{whenever} \quad 1 \leq p \leq i \leq m, \quad m + 1 \leq j \leq q \leq m + n, \quad c_{ij} \geq c_{pq} \quad \text{whenever} \quad 1 \leq p \leq i \leq m + n, \quad m + n + 1 \leq j \leq q \leq m + n + k, \]

cf. (3.4). These are exactly the inequalities defining the principal cones \( C_a \), cf. Example 5.5. Thus the left hand side of (6.1) can be written as

\[
\sum_{C \in C_a^1} R_{a^1}(C) \cdot (e_{x_1 \cdots x_m} \otimes e_{y_1 \cdots y_n} \otimes e_{z_1 \cdots z_k}).
\]

Analogously, the right hand side of (6.1) can be written as

\[
\sum_{C \in C_a^2} R_{a^2}(C) \cdot (e_{x_1 \cdots x_m} \otimes e_{y_1 \cdots y_n} \otimes e_{z_1 \cdots z_k}).
\]

The equality of these two expressions follows from (4.8) and Theorem 5.2.

This proves Theorem 3.2 and thus completes the proof of our main statement concerning the \( \ast \)-product—Theorem 5.3. \( \square \)

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7 Web Functions, Berenstein-Zelevinsky Triangles, and Hidden Duality

In this section we give a geometric interpretation of the scattering matrix (3.1) in terms of certain web functions as well as a “physical” motivation for it. Then we establish a relationship between integral web diagrams and fillings of Berenstein-Zelevinsky triangles. We also discuss the “hidden duality” of the Littlewood-Richardson coefficients under conjugation of partitions: $c_{\lambda\mu}^\nu = c_{\lambda'\mu'}^{\nu'}$.

7.1 Web functions

It will be convenient to use the baricentric coordinates in $\mathbb{R}^2$. Namely, we represent a point in $\mathbb{R}^2$ by a triple $(\alpha, \beta, \gamma)$ such that $\alpha + \beta + \gamma = 0$. We say that a line in $\mathbb{R}^2$ is of the first (respectively, second, or third) type if its first (respectively, second, or third) baricentric coordinate is fixed. We will denote by $(a, *, *)$ the first type line given by $\{(\alpha, \beta, \gamma) | \alpha + \beta + \gamma = 0, \alpha = a\}$. Analogously, we will denote by $(*, b, *)$ and $(*, *, c)$ the lines of the second and third types given by $\{(\alpha, \beta, \gamma) | \alpha + \beta + \gamma = 0, \beta = b\}$ and $\{(\alpha, \beta, \gamma) | \alpha + \beta + \gamma = 0, \gamma = c\}$, respectively. Each of the following two pictures represents a union of three rays of first, second, and third type originating at the same point.

![Web Diagrams](image)

Notice that in both cases we have $a + b + c = 0$ and $a' + b' + c' = 0$. We say that these two types of sets are left and right forks. The central point of a fork is called its node. The node of the left (respectively, right) fork shown above is the point $(a, b, c)$ (respectively, $(a', b', c')$), in the baricentric coordinates. We say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is a fork function (left or right) if there is a fork such that $f$ is equal to 1 on three rays of the fork, to $3/2$ on its node, and 0 everywhere else.

**Definition 7.1** A web function is a function $f : \mathbb{R}^2 \to \mathbb{R}^+$ such that for every point in $\mathbb{R}^2$ there exists an open neighborhood $U$ of the point, for which the restriction $f|_U$ is either zero, or the characteristic function of a line of one of three types, or a fork function (left or right), or a finite sum of several such functions. We say that a web function is integral if all its lines are of the form $(a, *, *), (*, b, *), (*, *, c)$ with integer $a$, $b$, and $c$.

Geometrically, we represent a web function by a picture (called web diagram) composed of rays and line segments of one of three types, and left or right forks (possibly doubled, tripled, etc.). See below for examples of web diagrams.

We say that a web function is generic if it only takes values 0, 1 and $3/2$. In other words, the diagram of a generic web function is composed of noncrossing rays, line segments, and forks. An arbitrary web diagram can be obtained by degeneration of a generic web diagram, i.e., by merging several lines, line intervals, and nodes together. For example the following diagram on the left-hand side presents a generic web function. The diagram of a non-generic web function on the right-hand side is obtained by merging three nodes together. The double line shows the locus where the web function is equal to 2.
Each web diagram consists of several nodes, line intervals, semi-infinite rays, and/or infinite lines. We are only interested in web functions whose diagrams have finitely many nodes. We will refer to semi-infinite rays in a web diagram as boundary rays. It is possible that the boundary rays are doubled (as in the example above), tripled, etc. The possible directions for boundary rays are North-West and South-East (for type 1 rays), North-East and South-West (for type 2 rays), and West and East (for type 3 rays), as shown on the picture below:

Recall that we defined the scattering matrix $R(c)$ by

$$R(c) : e_x \otimes e_y \mapsto \begin{cases} e_{y+c} \otimes e_{x-c} & \text{if } c \geq x - y, \\ 0 & \text{otherwise}, \end{cases}$$

(see Definition 3.1). Here (unlike Section 3) we allow $x$, $y$, and $c$ to be any real numbers.

The first type line $(-x, *, *)$ can be thought of as the trajectory of a certain left particle of energy $x$. We will denote this particle by $l(x)$. Analogously, the second type line $(*, y, *)$ represents the trajectory of a right particle of energy $y$, denoted by $r(y)$. In both cases the trajectories go downward (from left to right for left particles and from right to left for right particles). Then the scattering matrix $R(c)$ represents an interaction of a left particle of energy $x$ with a right particle of energy $y$. The following web diagram visualizes the scattering matrix $R(c)$.

The horizontal segment in this diagram lies on the third type line $(*, *, c)$. Thus the interaction $R(c)$ happens on the level $c$. The condition $c \geq x - y$ means that the interaction happens before the trajectories of the particles $l(x)$ and $r(y)$ cross each other.
Recall that in Section \[3\] we defined the operator \(R_{(m,n)}((c_{ij}))\) as a composition of of the scattering matrices \(R_{ij}(c_{ij}), 1 \leq i \leq m, m + 1 \leq j \leq m + n\). The operator \(R_{(m,n)}((c_{ij}))\) applied to the vector \(e_{x_1} \otimes \cdots \otimes e_{x_m} \otimes e_{y_1} \otimes \cdots \otimes e_{y_n}\) and producing the vector \(e_{x_1} \otimes \cdots \otimes e_{x_{m+n}}\) can be represented by a web diagram, which is a combinations of several pieces similar to the one shown above. In our pseudophysical lexicon, this diagram represents an interaction of \(m\) left particles with \(n\) right particles. An example of such a web diagram for \(m = 4\) and \(n = 3\) is given below:

In general, such a web diagram need not be as regular as the one shown above. The edge lengths can be arbitrarily deformed.

This web diagram has the following boundary rays: North-West rays corresponding to incoming particles \(l(x_1), \ldots, l(x_m)\); North-East rays, corresponding to incoming particles \(r(y_1), \ldots, r(y_n)\); South-West rays, corresponding to outgoing particles \(r(z_1), \ldots, r(z_n)\); South-East rays, corresponding to outgoing particles \(l(z_{n+1}), \ldots, l(z_{n+m})\); and no East or West boundary rays. The \(i\)-th left particle interacts with the \(j\)-th right particle on the level \(c_{ij+m}\). In the web diagram, this interaction is represented by an interval which lies on the line \((\ast, \ast, c_{ij+m})\). Such a web diagram is integral if and only if all \(x_i, y_j, z_k\), and \(c_{ij}\) are integers.

Using the language of web diagrams, we derive the following statement from Corollary 3.4.

**Corollary 7.2** Let \(\lambda = \omega_{x_1} + \cdots + \omega_{x_m}\), \(\mu = \omega_{y_1} + \cdots + \omega_{y_n}\), and \(\nu = \omega_{z_1} + \cdots + \omega_{z_{m+n}}\) be three dominant weights in \(GL(N)\), where \(1 \leq x_1 \leq \cdots \leq x_m \leq N\), \(1 \leq y_1 \leq \cdots \leq y_n \leq N\), and \(0 \leq z_1 \leq \cdots \leq z_{m+n} \leq N\). The Littlewood-Richardson coefficient \(c_{\lambda\mu}^{\nu}\) is equal to the number of integral web diagrams which have the following fixed boundary rays:

- the North-West rays \((-x_1, \ast, \ast), \ldots, (-x_m, \ast, \ast)\);
- the North-East rays \((\ast, y_1, \ast), \ldots, (\ast, y_n, \ast)\);
- the South-West rays \((\ast, z_1, \ast), \ldots, (\ast, z_n, \ast)\);
- the South-East rays \((-z_{n+1}, \ast, \ast), \ldots, (-z_{m+n}, \ast, \ast)\);
- no East or West boundary rays.

Independently of our work a similar to a web diagram notion of a honeycomb tinkertoy recently appeared in \[4\] in relation to Klyachko’s saturation hypothesis. (The origin of the term “honeycomb” should be clear from the previous picture.) This tinkertoy is given along with a statement reminiscent of Corollary \[7.4\]. In our notation this statement can be reformulated as follows:
Theorem 7.3 Let \( \lambda = (\lambda_1, \ldots, \lambda_N) \), \( \mu = (\mu_1, \ldots, \mu_N) \), and \( \nu = (\nu_1, \ldots, \nu_N) \) be three dominant weights of \( \text{GL}(N) \). The Littlewood-Richardson coefficient \( c_{\nu|\lambda\mu} \) is equal to the number of integral web diagrams with the following fixed boundary rays:

- the North-West rays \((\lambda_1, *, *) \), \( \ldots \), \((\lambda_N, *, *) \);
- the South-West rays \((*, \mu_1, *) \), \( \ldots \), \((*, \mu_N, *) \);
- the East rays \((*, *, -\nu_1) \), \( \ldots \), \((*, *, -\nu_N) \).

In a sense, these two statements are dual to each other. The proof of Theorem 7.3 is based on a simple one-to-one correspondence (see Appendix) between integral honeycomb tinkertoys (in our notation, web diagrams satisfying the conditions of Theorem 7.3) and Berenstein-Zelevinsky patterns. This correspondence just assigns to such a web diagram the triangular array filled by lengths of edges of the diagram.

The Berenstein-Zelevinsky interpretation of the Littlewood-Richardson coefficients, among its many other virtues, makes it clear that these coefficients are symmetric with respect to the action of \( S_3 \) by permuting the three weights. Nevertheless this construction obscures the invariance of the Littlewood-Richardson coefficients under the conjugation of partitions: \( c_{\nu|\lambda\mu} = c_{\nu'|\lambda'\mu'} \). This “hidden duality” can be observed from another even simpler bijection between web diagrams and Berenstein-Zelevinsky patterns, which is “dual” to the one given in Appendix. To formulate the correspondence we have to rigorously define these patterns.

### 7.2 BZ-functions and BZ-triangles

We say that BZ-lattice \( \mathcal{L}_{\text{BZ}} \) is the set \( (\frac{1}{2} \mathbb{Z} \times \frac{1}{2} \mathbb{Z}) \setminus (\mathbb{Z} \times \mathbb{Z}) \). Using the baricentric coordinates we can describe \( \mathcal{L}_{\text{BZ}} \) as the set of points \((\alpha, \beta, \gamma), \alpha + \beta + \gamma = 0\), such that \( 2\alpha \), \( 2\beta \), and \( 2\gamma \) are integer but at least one \( \alpha \), \( \beta \), or \( \gamma \) is not integer.

Every integer point \((a, b, c), a + b + c = 0\), has six neighbours in \( \mathcal{L}_{\text{BZ}} \) that form the vertices of the following hexagon:

\[
A = (a, b - \frac{1}{2}, c + \frac{1}{2}) \quad B = (a - \frac{1}{2}, b, c + \frac{1}{2}) \\
F = (a + \frac{1}{2}, b - \frac{1}{2}, c) \quad C = (a - \frac{1}{2}, b + \frac{1}{2}, c) \\
E = (a + \frac{1}{2}, b, c - \frac{1}{2}) \quad D = (a, b + \frac{1}{2}, c - \frac{1}{2})
\]

**Definition 7.4** A function \( f : \mathcal{L}_{\text{BZ}} \to \{0, 1, 2, \ldots\} \) is called a BZ-function if for any hexagon as above it satisfies the following hexagon condition:

\[
f(A) + f(B) = f(D) + f(E), \quad f(B) + f(C) = f(E) + f(F), \quad f(C) + f(D) = f(F) + f(A).
\]

**Proposition 7.5** Integral web functions are in one-to-one correspondence with BZ-functions. This correspondence \( \kappa \) is given by restricting a web function \( f : \mathbb{R}^2 \to \mathbb{R}_+ \) to the BZ-lattice \( \mathcal{L}_{\text{BZ}} \):

\[
\kappa : \{\text{integral web functions}\} \to \{\text{BZ-functions}\} \\
\kappa : f \mapsto f|_{\mathcal{L}_{\text{BZ}}}.
\]
Proof — Restrictions of first, second, and third type lines and of left and right forks to a hexagon are the following:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

It is clear that all these five functions satisfy the hexagon condition. It is also not hard to verify that any nonnegative integer function on a hexagon that satisfies the hexagon condition is a linear combination of these five functions with nonnegative integer coefficients. Thus restrictions of integral web functions to the BZ-lattice are BZ-functions and every BZ-function can be obtained in such a way. On the other hand, an integral web function is determined by its values on \( L_{BZ} \).

Let us fix an integer \( N \geq 1 \). The BZ-triangle \( T_N \) is the triangular subset in \( L_{BZ} \) given by the inequalities \( \alpha > -N, \beta > 0, \) and \( \gamma = -\alpha - \beta > 0 \). A Berenstein-Zelevinsky pattern (BZ-pattern) of size \( N \) is the restriction of a BZ-function to the triangle \( T_N \).

For example, the values in the center of the hexagon are equal to 1 for the first three functions above and to 3/2 for the remaining two functions.

For a BZ-pattern of size \( N \), let \( a_1, \ldots, a_{2N-2} \) be the number in the lower row; \( b_1, \ldots, b_{2N-2} \) be the numbers on its left side; and \( c_1, \ldots, c_{2N-2} \) be the numbers on its right side (in all cases we count the numbers from left to right). For the triangle on the picture above, \( b_1 = a_1, b_2 = a_7, b_3 = a_10, b_4 = a_{14}, b_5 = a_{16}, b_6 = a_{18} \), and \( c_1 = a_{18}, c_2 = a_{17}, c_3 = a_{15}, c_4 = a_{13}, c_5 = a_9, c_6 = a_6 \).

Berenstein and Zelevinsky \[3\] found the following interpretation of the Littlewood-Richardson coefficients in terms of BZ-patterns.

**Theorem 7.6** \[3\] Let \( \lambda, \mu, \nu \) be three dominant weights for \( \text{GL}(N) \) such that \( |\lambda| + |\mu| = |\nu| \) and \( \lambda = l_1 \omega_1 + \cdots + l_N \omega_N, \mu = m_1 \omega_1 + \cdots + m_N \omega_N, \) and \( \nu = n_1 \omega_1 + \cdots + n_N \omega_N \). Then the Littlewood-Richardson coefficient \( c^{\nu}_{\lambda \mu} \) is equal to the number of BZ-patterns of size \( N \) with the following boundary conditions:

\[
\begin{align*}
l_1 &= b_1 + b_2, & l_2 &= b_3 + b_4, & \ldots, & l_{N-1} &= b_{2N-3} + b_{2N-2}, \\
m_1 &= c_1 + c_2, & m_2 &= c_3 + c_4, & \ldots, & m_{N-1} &= c_{2N-3} + c_{2N-2}, \\
n_1 &= a_1 + a_2, & n_2 &= a_3 + a_4, & \ldots, & n_{N-1} &= a_{2N-3} + a_{2N-2}.
\end{align*}
\]
Proposition 7.5 says that web functions are essentially BZ-patterns of infinite size. Let us fix a set of boundary rays that satisfy the conditions of Corollary 7.2. Then an integral web functions with these boundary rays is determined by its restriction to $T_N$, which is a BZ-pattern. The conditions on the rays of such web functions transform into the boundary conditions for the BZ-patterns from Theorem 7.6. Thus Corollary 7.2 is equivalent to Theorem 7.6.

Proposition 7.7 Let $\lambda$, $\mu$, and $\nu$ be three dominant weights for $GL(N)$ such that $|\lambda| + |\mu| = |\nu|$. The integral web functions that satisfy the conditions of Corollary 7.2 are in one-to-one correspondence with the BZ-patterns of size $N$ that satisfies the boundary conditions of Theorem 7.6. This correspondence $\kappa_N$ is given by restricting a web function $f : \mathbb{R}^2 \to \mathbb{R}^+$ to the BZ-triangle $T_N$:

$$\kappa_N : f \mapsto f |_{T_N}.$$ 

The following picture illustrates the statement of the proposition. It shows a web diagram and the BZ-triangle $T_{11}$. In this case the corresponding BZ-pattern has 1’s at the points that belong to the web diagram and it has 0’s everywhere else.
8 Remarks and Open Questions

There are several questions that remained outside the scope of the paper. We briefly mention them here, and they will be properly illuminated in subsequent publications.

First of all, an open problem of interest is to describe explicitly the transformation maps $T^b_a$ for any two reduced words for $w$ (see Section 4).

There is an analogy between piecewise-linear transformations $T_{ijk}$ given by (4.6) and the transformations for Lusztig’s parametrization of the canonical basis in $U^+_q(sl_n)$. Lusztig’s transformations were thoroughly investigated in [1]. The combinatorial essence of this work lies in a certain chamber ansatz. It would be interesting to find analogues of the results of [1].

In a forthcoming paper, Berenstein and Zelevinsky investigate string cones and relations between Lusztig’s and Kashiwara’s parametrizations. It would be interesting to find a relationship between our combinatorial description of the string cone in terms of rigorous paths and their construction.

Following [1], it is possible to formulate the transition maps $T_{ijk}$ and $T^b_a$ in the language of the tropical semiring—a kind of algebraic system where one is allowed to add, multiply, and divide, but not subtract. Taking the presentation of the tropical multiplication by the usual addition, tropical division by usual subtraction, and tropical addition by the operation min, we can recover piecewise-linear combinatorics. On the other hand, taking the more natural presentation of the tropical multiplication by the usual multiplication, tropical division by the usual division, and tropical addition by the usual addition, we can move to the area of rational mathematics. Hopefully, the rational expressions corresponding to the piecewise-linear transition maps $T^b_a$ can be presented by some determinant-like creatures.

Knutson and Tao [6] defined honeycombs as certain embeddings of certain graphs into $\mathbb{R}^2$. They used honeycombs in the proof of Klyachko’s saturation conjecture. Our web functions are related to honeycombs, but they are defined in a different way by means of local conditions. Sometimes this definition is more convenient. It is possible to give a proof to the saturation conjecture in terms of web functions which is simpler than Knutson and Tao’s proof.

References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky, Parametrizations of canonical bases and totally positive matrices, Adv. Math. 122 (1996), 49–149.

[2] A. Berenstein, A. Zelevinsky, String bases for quantum groups of type $A_r$, in I. M. Gelfand Seminar, Adv. Soviet Math. 16 Part 1, Amer. Math. Soc., Providence, 1993, 51–89.

[3] A. Berenstein, A. Zelevinsky, Triple multiplicities for $sl(r+1)$ and the spectrum of the exterior algebra of the adjoint representation, J. Algebraic Comb. 1 (1992), 7–22.

[4] A. Berenstein, A. Zelevinsky, Canonical bases for the quantum group of type $A_r$ and piecewise-linear combinatorics, Duke Math. J. 82 (1996), no. 3, 473–502.

[5] M. Dyer, Hecke algebras and shellings of Bruhat intervals, Comp. Math. 89 (1993), 91–115.

[6] A. Knutson, T. Tao, Apiary views of the Berenstein-Zelevinsky polytope, and Klyachko’s saturation conjecture, preprint dated July 28, 1998.

[7] I. G. Macdonald, Symmetric functions and Hall polynomials, second edition, Clarendon Press, Oxford, 1995.