The method of boundary states with perturbations as applied to the analysis of geometrically non-linear elastostatic bodies

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Abstract. This study makes a case for the application of the numeric and analytic method of boundary states with perturbations (MBSP) to analytic problems focused on the stress-strain states (SSS) of geometrically non-linear isotropic elastic bodies. The defining relations are represented through a weakly non-linear operator equation that contains (on an additive basis) a non-linear operator decomposable into a linear combination of a sequence of linear operators. A solution is proposed for a simple case of a linearly heterogeneous uniaxial loading problem for a long half-pipe with butt ends subjected to loads. Even in this load scenario, geometrical non-linearity affects the way the body alters its shape and causes buckling. This study analyzes the SSS involved, draws conclusions, and addresses application prospects.

Keywords
Method of boundary states, method of boundary states with perturbations, MBS, MBSP, geometrically nonlinear elasticity, Almansi tensor, perturbation method, Poincare method, half-pipe.

Introduction
Fundamentals of the method of boundary states (MBS). The MBS is based on the concept of medium states which in effect means any specific solution of medium-defining equations, whatever its boundary conditions [1].

In the context of a specific bounded solid, the "media state" concept gives way to that of the "internal state" ξ. A combination of all possible internal states forms the space of internal states Ξ. For linear media, an internal state becomes linear to state addition and multiplication operations. If a scalar product can be determined for two arbitrary states ξ(1), ξ(2) of the space Ξ, such space Ξ is Cartesian [2]. A closure of a space of internal states forms a complete space, and so the space of linear media states becomes Hilbertian. Consequently, the internal state can be presented as a Fourier series with the coefficients $c_k = (\xi, \xi^{(k)})_\Xi$ of an orthonormal basis set.
[2]: \( \xi^{(1)}, \xi^{(2)}...\xi^{(n)}\in \Xi, (\xi^{(i)}, \xi^{(j)}) = \delta_{ij} \):

\[
\xi = \sum_{k} c_k\xi^{(k)}. \tag{1}
\]

The indentation left on the body’s boundary by the internal state \( \xi \) is perceived as the boundary state \( \gamma \) equal to the internal one. The sum total of all boundary states forms the space of boundary states \( \Gamma \). If it is linear and allows determining a scalar product, such space is Cartesian. A closure of a space of boundary states forms a complete space, and so the space of linear media boundary states becomes Hilbertian. Consequently, the boundary state can also be presented as a Fourier series of an orthonormal basis set: \( \gamma^{(1)}, \gamma^{(2)}...\gamma^{(n)}\in \Gamma, (\gamma^{(i)}, \gamma^{(j)}) = \delta_{ij} \):

\[
\gamma = \sum_{k} c_k\gamma^{(k)}, c_k = (\gamma, \gamma^{(k)})_{\Gamma}. \tag{2}
\]

Each element \( \gamma \in \Gamma \) is matched with only one element \( \xi \in \Xi \), and the spaces of internal and boundary states are isomorphic: \( \Xi \leftrightarrow \Gamma \). If isomorphism is maintained for the results of state additions and number multiplications

\[
\xi^{(1)} + \xi^{(2)} \leftrightarrow \gamma^{(1)} + \gamma^{(2)}, \alpha \xi \leftrightarrow \alpha \gamma, \text{ } \alpha \in \mathbb{R}^1 \tag{3}
\]

and, at the same time, the scalar products of isomorphic pairs of elements equal one another,

\[
(\gamma^{(1)}, \gamma^{(2)})_{\Gamma} = (\xi^{(1)}, \xi^{(2)})_{\Xi}, \tag{4}
\]

then we are dealing with a Hilbert isomorphism. This makes it possible to reduce internal state analysis to that of the corresponding boundary state, where the basic set of the elements of the space \( \Xi \) is directly and uniquely matched by a basic set of the elements of the space \( \Gamma \). In practice, the computer-aided MBS solution of the problem is reduced to a conditioned truncated Infinite System of linear algebraic Equations (ISE) formulated based on data available on boundary conditions, due account being taken of the requirements as to the resolvability of such a boundary problem.

The theory-of-elasticity fundamentals of the MBS are set out in study [3]. The below solution uses the basis set building methods as described in [3].

The most time-consuming portion of the proposed MBS solution is the formation of an orthonormal basis set. There are several algorithms to do it, but the Cholesky decomposition has proved to be the most efficient method.

**State basis orthogonalization methods.** The classic Gram-Schmidt algorithm involves scalar products (Gram matrix \( G = [(\varphi_i, \varphi_j)] \)) computed at each orthogonalization step. Its matrix solution provides for building a Schmidt matrix \( H \) that orthogonalizes the basis \( \Phi \) based on the Gram matrix. The result is an orthonormal basis:

\[
\Psi = H\Phi, \quad H = [h_{ij}]_{n \times n}. \tag{5}
\]

The Cholesky decomposition of the positively defined Gram matrix

\[
G = X X^T, \tag{6}
\]

establishes a correlation between the Schmidt and Cholesky matrices: \( H = X^{-1} \). Yet another efficient orthogonalization method is the singular value decomposition

\[
G = U \Lambda U^T, \tag{7}
\]
where \( \Lambda = \text{diag} \{ \lambda_1, \lambda_2, ..., \lambda_n \} \), \( \lambda_j > 0 \) is a diagonal spectrum matrix built from positive eigenvalues of the Gram matrix, and \( U \) is a real unitary matrix: \( U U^T = E \). The orthogonalizing basis is not a lower-triangular Schmidt matrix, but the square matrix \( S = U \sqrt{\Lambda} \), in which spectrum eigenvalues can be ordered arbitrarily (flexibility).

A comparison study led us to conclude that the Cholesky method was the most efficient solution tactic from all viewpoints. The inversion of the matrix \( X \) does not cause any degradation in accuracy, as there is a shorter way for triangular matrices than the traditional cofactor calculations, and the determinant computation is reduced to the multiplication of diagonal elements. Although the singular value (spectral) decomposition is more time-consuming, the usage of an orthonormal basis to solve specific problems can significantly limit the dimensions of the active section of the basis set. That spills over to the dimensions for the truncation of a resolving infinite system of equations for a minor adjustment of the Bessel-type summation (the left-hand side of Bessel’s inequality), with all ensuing consequences.

Specific solution based on given volume forces. A theorem positing the existence of a basis of a separable space of polynomial volume forces is formulated and rigorously proven [4]. Namely, each polynomial \( w = x^\alpha y^\beta z^\gamma \), \( \alpha, \beta, \gamma = \{ 0, 1, 2, ... \} \) engenders the displacement vector \( u \) and a corresponding internal state \( \xi = \{ u, \hat{\varepsilon}, \hat{\sigma} \} = \begin{cases} \begin{pmatrix} w \\ \frac{w}{2} \end{pmatrix}, \begin{pmatrix} 2 \alpha x^{-1} & \beta y^{-1} & \gamma z^{-1} \\ \beta y^{-1} & 0 & 0 \\ \gamma z^{-1} & 0 & 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \lambda + 2 \mu & \alpha x^{-1} & \mu \gamma y^{-1} & \mu \gamma z^{-1} \\ \mu \beta y^{-1} & \lambda \alpha x^{-1} & 0 \\ \mu \gamma z^{-1} & 0 & \lambda \alpha x^{-1} \end{pmatrix} \begin{pmatrix} w \\ \frac{w}{2} \end{pmatrix} \end{cases} \),

which corresponds to the volume force

\[
X = -w \begin{pmatrix}
(\lambda + 2\mu)(\alpha - 1)x^{-2} + \mu(\beta - 1)y^{-2} + \mu(\gamma - 1)z^{-2} \\
(\lambda + \mu)\alpha x^{-1}y^{-1} \\
(\lambda + \mu)\alpha x^{-1}z^{-1}
\end{pmatrix}.
\]

Yet another two variations are achieved through a circular substitution of indices and a corresponding change of positions of the rows and columns. Polyvariety is obtained through the iteration of the values \( \alpha, \beta, \gamma \) within each fixed sum \( \alpha + \beta + \gamma \). Any polynomial vector is then with absolute accurately ”covered” by such a section of the dimensionality basis set as is sufficient based on the order of the components of the polynomial force. Such a linear combination of internal states isomorphic to forces provides a strict particular solution for a field corresponding to the given volume forces.

Specific algorithms are proposed for filling the countable basis of the polynomial volume forces space, which makes it possible to design an analytic solution for a problem where a solid body is subjected to a superposition of polynomial forces and the unknown is the stress-strain state [4].

The defining relations of the geometrically non-linear medium [5–7], and in particular the Cauchy formulae (8), the generalized Hooke’s law (9), and the equilibrium equations (10), are presented in tensor/index notation:

\[
2E_{ij} = \frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i} \frac{\partial U_k}{\partial X_j}, \quad (8)
\]

\[
\Sigma_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}, \quad (9)
\]

\[
\frac{\partial \Sigma_{ij}}{\partial X_j} = 0. \quad (10)
\]
Here \( U_i, E_{ij}, \Sigma_{ij} \) are the components of dimensionless displacement vectors, strain tensors, and stress tensors, respectively; \( \lambda, \mu \) are non-dimensionalized Lamé parameters, and \( \delta_{ij} \) is the Kronecker delta. A combination of tensorial and indicial notation is used, which involves the repeated index convention for summation.

A replacement of variables [8] is performed using the parameter \( 0 < \beta << 1 \):

\[
x_i = \beta X_i, E_{ij} = \beta \varepsilon_{ij}, U_i = u_i, \Sigma_{ij} = \beta \sigma_{ij},
\]

after which equations (8)–(10) rearrange to (12)–(14):

\[
2 \varepsilon_{ij} = u_{i,j} + u_{j,i} + \beta u_{k,i} u_{k,j}, \tag{12}
\]

\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2 \mu \varepsilon_{ij}, \tag{13}
\]

\[
\sigma_{ij,j} = 0. \tag{14}
\]

The presence of the parameter \( \beta \) in (12) characterizes these equations as weakly non-linear.

Using the perturbations method \([8–10]\), write out the characteristics of the state \( \xi = \{u_i, \varepsilon_{ij}, \sigma_{ij}\} \) as power series:

\[
\xi = \sum_k \beta^k \xi^{(k)}, \quad \xi^{(k)} = \{u_i^{(k)}, \varepsilon_{ij}^{(k)}, \sigma_{ij}^{(k)}\}. \tag{15}
\]

The Poincaré method was earlier used for the assessment of the state of an axially asymmetric body (semicylinder) [3].

The purpose of the study is to analyze the efficiency of the Poincaré–Lindstedt method when combined with the numeric & analytic MBS to analyze the state of a long shell-type body. In addressing the purpose, we propose a problem for a half-pipe made from a geometrically non-linear elastic material with butt ends subjected to simple uniaxial loads.

1. Weakly non-linear operator equation

Assume that the system of the body’s defining equations has weak non-linearity described through the small parameter \( \beta << 1 \), and is generally presented through the operator form

\[
L\xi + \beta K\xi = 0, \tag{16}
\]

which is linear if \( \beta = 0 \). Here \( \xi = \xi(x) \) is the set of properties of the body’s state, \( x \in \mathbb{R}^n \), \( n \) is the space’s dimensionality, \( L \) is the linear operator defining the principal part of the body equations, and \( K \) is a non-linear operator of the body’s properties. Let us apply the methods of the perturbation theory to find the state \( \xi \) presented as a series (15). Assume that “iteration expansion” can be applied to the operator \( K \)

\[
K\xi = K \left( \sum_k \beta^k \xi^{(k)} \right) = \sum_k \beta^k K^{(k)} \xi^{(k)}, \tag{17}
\]

where \( K^{(k)} \) are operators non-linear to \( \xi^{(k)} \) whose properties are fully defined by the time of step \( k \). In that case, (16) rearranges to

\[
\sum_{k=0}^{\infty} \beta^k L\xi^{(k)} + \sum_{k=1}^{\infty} \beta^k K^{*(k-1)} \xi^{(k-1)} = 0. \tag{18}
\]
A comparison of the equations where the monomials $\beta^k$ have the same powers leads to a sequence of linear operator equations

$$L\xi^{(0)} = 0,$$

$$L\xi^{(k)} + K^*_{(k-1)}\xi^{(k-1)} = 0, \quad k \in N_0 = [1, 2, ...],$$

i.e., to a homogeneous or heterogeneous linear operator equation

$$L\xi^{(k)} = f, \quad f = -K^*_{(k-1)}\xi^{(k-1)}.$$  \hspace{1cm} (19)

In equation (19) the right-hand side $f$ is the known function of the states $\xi^{(0)}, \xi^{(1)}, \xi^{(k-1)}$, that were assumed to have been built on all of the previous $k$ iteration steps.

Each step of the process is successful if both the general solution of the homogeneous equation

$$L\xi = 0$$  \hspace{1cm} (20)

and the specific solution of equation (19) are built efficiently.

2. Perturbed linear elastic medium

In linear isotropic elastostatics, problems are posed to find the stress-strain state (SSS) of a medium contained in the region $V$ with the boundary $\partial V$. The internal state $[1] \xi = \{u_{i}, \varepsilon_{ij}, \sigma_{ij}\}$ is an excessive set of information containing the components of displacement vectors, and strain and stress tensors. Contextually, the linear operator $L$ contains information on Cauchy equations (12), the generalized Hooke’s law (13), and equilibrium equations where there is no volume force at work (14).

If posed correctly, such problems are solved numerically (through finite & boundary element methods etc.). Today’s computer technology makes it possible to build numeric & analytic solutions (MBS, MBSP) [8]. The methodology for structuring full parametric solutions (FPS) is also described [4].

The general solution (20) is a linear combination of elements of a countable basis of a Hilbert space of internal states. If there is a heterogeneous component in equation

$$\sigma_{i,j,j} + X_{i} = 0,$$  \hspace{1cm} (21)

contextually define the non-linear operators of the iteration expansion in (18).
Further on, introducing the notation $s_{ij}^{(k)} = \sigma_{ij}^{(k)} - \sigma_{ij}^{*(k-1)}$, we arrive at a classic format of an isotropic elastostatics problem

$$2\epsilon_{ij}^{(k)} = u_{ij}^{(k)} + u_{ij}^{(k)}\delta_{ij} = \lambda \epsilon^{(k)}_{min} \delta_{ij} + 2\mu \epsilon_{ij}^{(k)}, \quad s_{ij}^{(k)} + \bar{X}_i^{(k)} = 0. \quad (22)$$

By solving the problem in relation to state $\{u_i^{(k)}, \varepsilon_{ij}^{(k)}, \sigma_{ij}^{(k)}\}$ we come back to the fields of these characteristics. Restoring strains and stresses at iteration $k$ in accordance with the notations introduced

$$\sigma_{ij}^{(k)} = s_{ij}^{(k)} + \sigma_{ij}^{*(k-1)},$$
$$\varepsilon_{ij}^{(k)} = \epsilon_{ij}^{(k)} + \varepsilon_{ij}^{*(k-1)} \quad (23)$$

we get the state $\varepsilon^{(k)} = \{u_i^{(k)}, \varepsilon_{ij}^{(k)}, \sigma_{ij}^{(k)}\}$. The resulting internal state of the body is the combination (17).

**Note.** The addition of volume forces to equilibrium equations does not change the solution method in any meaningful way. The force will bring some additions to $\bar{X}_i^{(0)}$ or to all $\bar{X}_i^{(k)}$.

### 3. Non-uniform loading of a cylindrical half-pipe

The below problem is posed in dimensionless form. The length scale is taken to equal the pipe’s radius $R$, and the stress scale is taken to equal the shear modulus $\mu$.

The problem studies an axial loading of a long semi-cylinder (Fig.1.a). The body’s height is much greater than its base radius ($l = 10R$). The problem’s boundary conditions are described in formula (24) (on a dimensionless basis):

$$p = \{ \{0,0,0\}, (x,y,z) \in S_1 \bigcup \bigcup S_5 \bigcup S_6, \}$$

$$S_1 = \{(x, y, z) \in R^3 | x^2 + y^2 = 0.81, x \geq 0, z \in [-5, 5]\},$$

$$S_2 = \{(x, y, z) \in R^3 | x^2 + y^2 = 1, x \geq 0, z \in [-5, 5]\},$$

$$S_{3,4} = \{(x, y, z) \in R^3 | 0.81 \leq x^2 + y^2 \leq 1, x \geq 0, z = \mp 5\},$$

$$S_5 = \{(x, y, z) \in R^3 | x = 0, y \in [-1, -0.9], z \in [-5, 5]\},$$

$$S_6 = \{(x, y, z) \in R^3 | x = 0, y \in [0.9, 1], z \in [-5, 5]\}.$$  

The unknown is the stress-strain state of the long half-pipe made from an isotropic linear-elastic material with a Poisson ratio of $\nu = 0.25$, where the strain is geometrically non-linear (1).

The SSS is built through the means of the MBSP in three iterations. Below is presented the displacement vector (rounded to a thousandth place)

$$\mathbf{u} = \begin{pmatrix}
0.05(y^2-x^2)-0.2z^2 \\
-0.1xy \\
0.4xz \\
-0.03x + 0.25y^2 + 0.21xz^2 \\
0.17x - 0.5xy \\
-0.01x - 0.20xz^2 \\
0.023x + 0.082y^2 - 0.019xz^2 \\
0.048x - 0.16xy \\
-0.11x + 0.019xz^2 \\
\end{pmatrix}^T + 0.1 \beta \begin{pmatrix}
-0.03x + 0.25y^2 + 0.21xz^2 \\
0.17x - 0.5xy \\
-0.01x - 0.20xz^2 \\
0.023x + 0.082y^2 - 0.019xz^2 \\
0.048x - 0.16xy \\
-0.11x + 0.019xz^2 \\
\end{pmatrix}^T + 0.1 \beta^2 \begin{pmatrix}
0.023x + 0.082y^2 - 0.019xz^2 \\
0.048x - 0.16xy \\
-0.11x + 0.019xz^2 \\
\end{pmatrix}^T.$$

The post-deformation shape of the half-pipe is presented on Fig.1.b Below, the displacement values $\mathbf{u}$ match those of the non-dimensionalized medium by virtue of the scaling method
chosen (11). Although the strain and stress change in scale, their distribution patterns remain unchanged. That is why there is no recalculation to real (dimensionless) values.

An efficient solution method may be the MBS applied at every iteration step of the small parameter method [3]. The MBSP algorithm is implemented in Wolfram Mathematica and so the solution is presented in a numeric & analytic form.

Table 1 shows non-zero Fourier coefficients. The saturation of the Bessel-type summation (the left-hand side of Bessel’s inequality) can indirectly signify that the solution is correct. As seen from Fig.2, 30 elements to a basis were enough at iteration 0, and its expansion through more elements had no effect on the final result.

Table 1. Non-zero Fourier coefficients.

| k   | c_k  | k   | c_k  |
|-----|------|-----|------|
| 1   | -0.218 | 10  | 0.0298 |
| 4   | -0.330 | 14  | -0.129 |
| 6   | 0.530  | 18  | 0.292  |
| 7   | 0.00855 |     |       |

Tables 2 and 3 present the distribution of displacements in axial sections.

Table 2 shows the distribution of point displacements of the body’s section $z = 0$ that provides a more detailed description of the deformation pattern. The figure’s background corresponds to a zero displacement. A comparison of a linear and a geometrically non-linear scenario shows significant influence of the non-linear addition on the body’s shape while also conveying the displacement patterns in general. White background for $u_x$ means that the half-pipe’s edge has been displaced in the direction of $x > 0$ (buckling) while the middle region is slightly crumpled, whereas regions in the vicinity of $z = 2.5$ (Table 3) and at $z = 5$ are displaced in the opposite direction (manifest buckling). A comparison of displacements $u_x$ and $u_y$ leads us to conclude

![Figure 1. Half-pipe: a) load configuration; b) post-deformation shape.](image-url)
Table 2. Displacements $u_x$, $u_y$ of the 0-th and result iterations in the section $z=0$.

| $u_x$ | $u_y$ |
|-------|-------|
| ![Linear approximation](image1.png) | ![Adjustment for geometric non-linearity](image2.png) |

that the intermediate region between zones $\varphi = 0$ (x-axis) and $\varphi = \pi/2$ (y-axis) is flattened into an oblong (not circular) shape with a large semiaxis diameter along $Oy$.

Fig. 3 presents the stresses $\sigma_{zz}$ of the result iteration in section $z = 0$.

The distribution of stress states (Fig.3) in cross section $z = 0$ ($x, y > 0$) stays virtually unchanged from iteration to iteration and corresponds to a manifest elongation of longitudinal fibres in accordance with the loading pattern.
Figure 3. Stresses $\sigma_{zz}$ of the result iteration in section $z = 0$.

Figure 4. Strains $u_x, u_y$ of the result iteration in section $y = 0$.

Table 3. Displacements $u_y$ and $u_z$ of the result iteration in section $z=2.5$ and $z=5$.

|       | $u_y$  | $u_z$  |
|-------|--------|--------|
| $z=2.5$ | ![Graph](image1) | ![Graph](image2) |
| $z=5$   | ![Graph](image3) | ![Graph](image4) |

Fig. 4 illustrates the displacements $u_x$ & $u_y$ of the result iteration in the section $y = 0$. The middle fibre displacement epures ($y = 0, x = 1, z \in [-5, 5]$) provide an accurate quantitative representation of the curve.
Conclusions
1 An analysis of the stress-strain states of a half-pipe showcased the importance of factoring in the non-linear component of the strain tensor as they lead to finite displacements of the body’s points event at relatively small values of strain components.
2 The MBSP has proven to be an efficient method for analyzing the states of geometrically non-linear elastic bodies.
3 The implementation of the MBS in conjunction with the perturbations method revealed the aspects of related algorithms and software that need further enhancement: automation of the filling of the basis of Hilbert space of internal states, automation of the design of step-by-step particular solutions of iteration problems based on fictitious polynomial forces, and automation of the graphic representation of intermediate results and final output.

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