Universality of the rainfall phenomenon

M. Ignaccolo, C. De Michele, and S. Bianco
1 University of North Texas, Center for Nonlinear Science, Department of Physics, Denton, TX, United States of America. and
2 Politecnico di Milano, Department of Hydraulic, Environmental, Roads and Surveying Engineering, Milano, Italy.

We show that the universal properties of the rainfall phenomenon are the scaling properties of the probability density function of inter-drop intervals during quiescent periods, time intervals of sparse precipitation, and the universal shape of the probability density function of drop diameters during non-quiescent periods, time intervals of active precipitation. Our results indicate that the continuous flux-like vision of rainfall based on quantities such as the rain duration, rain intensity and drought duration is ineffective in detecting the universality of the phenomenon. A comprehensive understanding of rainfall behavior must rest on the acknowledgment of its discrete drop-like nature.

I. INTRODUCTION

Rainfall is a discrete drop-like phenomenon that has often been described as a continuous flux-like phenomenon. The most common instrument used to measure the rain, the pluviometer, collects the water volume fallen through a given area per unit of time. The use of pluviometers and the importance of knowing intensity and duration of the rainfall phenomenon has led to a description based on flux-like quantities such as the rain duration, rain intensity and drought duration. Even when radar measurement are used to infer the precipitable volume of rain (or rainfall) and “rainfall” (or rain). So, are there any properties common among different rainPacific areas? Do they have the same features at large times? How is the temporal variability of rain can be described in terms of quiescent periods, periods of sparse precipitation characterized by a small drop diameter average, and non quiescent periods, periods of active precipitation characterized by a large drop diameter average diameter. Moreover, the average and variance of the sequence of drop diameters are not stationary. The probability density function of the inter drop time intervals \( \tau \) has an universal feature: a power law regime in the region \( \tau \geq 1 \text{min} \) and \( \tau \leq 1 \text{h} \). Inter drop time intervals in this range belong to quiescent periods. Finally, an universal shape for the probability density function of drop diameters during non-quiescent periods emerges upon removal of the non stationarity of the sequence of drop diameters.

II. FLUX-LIKE VIEW OF A DROP-LIKE PHENOMENON

The flux-like view of rainfall is that of a ON-OFF process. The rainfall time series is an alternating sequence of consecutive time intervals of duration \( \Delta > 0 \), the integration time of the instrument used to monitor the precipitation, with (ON) or without (OFF) detectable precipitation. The relevant quantities are the duration of ON and OFF periods (rain duration and drought duration) and the volume of rain fallen during consecutive OFF periods (rain intensity). An integration time \( \Delta > 0 \) causes all drop time intervals of duration \( \tau < \Delta \) to be lost (equivalent to be detected as a drought of null duration), and all inter drop time intervals of duration \( \tau > \Delta \) to be detected as drought of duration \( [\tau/\Delta] - 1 \) or \( [\tau/\Delta] \) (\([\cdot]\) indicates the integer part). Thus, in the limit \( \tau \gg \Delta \)

\[
P^d_{\Delta}(\tau) \propto \Delta \psi(\tau),
\]

where \( P^d_{\Delta}(\tau) \) is the distribution of drought durations and \( \psi(\tau) \) is probability density function of inter drop time intervals. Eq. 1 shows that distributions of drought durations relative to different integration times \( \Delta \) will all have the same features at large times \( \tau \gg \Delta \) as confirmed by panel (a) of Fig. 1. This property is lost for the distributions of rain durations \( P^r_{\Delta}(\tau) \), panel (b) of...
Fig. I and rain intensities \( P_d^k(v) \), panel (c) of Fig. I. As the integration time \( \Delta \) increases a larger amount of inter drop time intervals are lost and rain durations that were separated by a drought duration are now detected as a longer rain period. This effect produces larger rain durations and, together with the temporal ordering of the sequence of drop diameters, larger rain intensities.

However, some properties of the temporal variability of the rainfall phenomenon can be inferred.

### A. Quiescence

Quiescence is a way of describing the temporal variability of the rainfall phenomenon based on the relationship between inter drop time intervals and drop diameters. We divide the rainfall time series in time intervals of length \( \delta \). A couple of consecutive time intervals of detectable precipitation may be separated by \( k \geq 0 \) consecutive droughts (empty intervals) or be adjacent. In this case we say that the couple is separated by \( k=0 \) consecutive droughts. Finally, for each couple we evaluate the average number of drops \( n_{avg} \), and the average drop diameter \( d_{avg} \). Fig. 2 shows the relationship between the couple average number of drops, the couple average drop diameter and the number of consecutive droughts in between the couple for \( \delta=10s \). We see the tendency for a number of consecutive droughts \( k \geq 1 \) (interdrop time intervals \( \tau \geq 10s \)) to separate couples with a small average number of drops \( (\leq 5) \) and a small average drop diameter \( (\leq 0.6mm) \). To quantify this tendency, we introduce the concept of quiescence of order \((\delta, m, n)\). A couple of consecutive time intervals of length \( \delta \) with detectable precipitation and separated by \( k \) consecutive droughts is a quiescent couple of order \((\delta, m, n)\), if:

\[
\begin{align*}
    k & \geq m \\
    \text{or} \\
    k & < m \text{ but } n_{avg} \leq n,
\end{align*}
\]

where \( n_{avg} \) is the couple average number of drops.

### III. TEMPORAL VARIABILITY

Our data are from a Joss Wälkvoegel impact disdrometer RD-69 with a time integration \( \Delta=10s \), and 127 different diameter classes (from 0.2998 mm to 4.9952 mm). Thus, we can record neither all the inter drop time intervals less than 10s, nor the exact arrival ordering of drops.

The properties of the sequences of inter drop time intervals and drop diameters do not straightforward translate into those of the distributions of rain durations \( (P^k(\tau)) \) and intensities \( (P^k(v)) \) [e.g., 15]. Thus, these quantities are not valuable proxies to investigate the universal properties of the rainfall phenomenon.

![Fig. 1: Panel (a): Log-log plot of the probability \( P^k(\theta>\tau) \) of having a drought duration larger than \( \tau \) during the interval of time from January 24th to May 11th 2004. Different symbols indicate different values on the integration time intervals: \( \Delta=10 \text{ seconds} \) (white squares), \( \Delta=1 \text{ minute} \) (white circles), \( \Delta=10 \text{ minutes} \) (white triangles), \( \Delta=1 \text{ hour} \) (white diamonds). Panel (b): Log-log plot of the probability \( P^k(\theta>\tau) \) of having a rain duration larger than \( \tau \) during the same interval of time, and for the same the integration time intervals of panel (a). Panel (c): The probability \( P^k(V>v) \) of having a rain duration larger than \( v \) during the same interval of time, and for the same the integration time intervals of panel (a).]

![Fig. 2: The logarithm of the density \( \rho(k,n_{avg}) \) of the couples \((k,n_{avg})\) in the \( kn_{avg} \) plane (bottom x-axis and left y-axis). The logarithm of the density \( \rho(k,d_{avg}) \) of the couples \((k,d_{avg})\) in the \( kd_{avg} \) plane (top x-axis and right y-axis). The inner plots show the logarithm of the densities \( \rho(k,n_{avg}) \) and \( \rho(k,d_{avg}) \) in the region close to respective origins. Both densities are obtained using the data relative to the time interval from January 24th to May 11th 2004.]

\[
\begin{align*}
\end{align*}
\]
Quiescent (non-quiescent) periods are rainfall periods occupied by consecutive quiescent (non-quiescent) couples. During quiescent periods, large inter drop intervals \((\tau \geq m\delta)\) are preceded and followed by few \((n_{avg} \leq n)\) drops of small diameter (Fig. 2). Non-quiescent or active periods are characterized by small inter drop time intervals \((\tau < m\delta)\) separating drops with a larger range of diameters (Fig. 2). Non quiescent periods are responsible for the bulk of precipitation: for a quiescence of order \((\delta=10s, m=1, n=5)\) the precipitated volume of water belongs to non quiescent periods. Some care is necessary in choosing the duration \(\delta\) of the time intervals dividing the rainfall time series: a \(\delta\) too large \((\gtrsim 10\) minutes\) will result in a mixing quiescent with non-quiescent periods, a \(\delta\) too small \((\lesssim 1\) second\) will result in too many time integration intervals with just one drop. Two quiescence of order \((\delta, m, n)\) and \((\delta', m', n')\) are “equivalent” if \(m'=m(\delta/\delta')\) and \(n'=n(\delta/\delta')\).

### B. Drop diameter variability

[5] introduce the concept of averaged “instant” shape to characterize the variability of raindrop spectra and their departure from the exponential form observed by [11]. [14] show that the probability density function \(\eta_\theta(d)\) of drop diameters may change according to the portion (e.g.: dissipative edge, cloud base) or the type of storm (e.g.: orographic, non orographic) observed by a disdrometer. Here, we show the temporal variability of the sequence of drop diameter is characterized by a moving average and a moving variance. Fig. 4 indicates that the sequence of drop diameters does not have a constant average, as quiescent periods have a lower average diameter than non-quiescent periods. A closer examination of the sequence of drop diameters reveals that its average together with its variance are not stationary. Support for this thesis comes also from the results of [8]. They report for the autocorrelation function of the sequence of drop diameters with a slow decay (the autocorrelation function reaches zero at lag \(\approx 1250\)) followed by a long negative tail (lag \(\gtrsim 1250\)). The auto correlation function (as confirmed by simulations not reported here for brevity) of sequences of drop diameters exponentially and normally distributed with changing intensity around a moving average has the same features of that of [8].

### IV. UNIVERSALITY

In Fig. 3 we plot the probability \(P_\theta^Q(\theta>\tau)\) of having a drought duration \(\theta\) larger than \(\tau\) for several time intervals of continuous observations at Chilbolton over a period of almost 2 years. All curves show a power law regime in the region between \(\tau\gtrsim 1\) min and \(\tau\lesssim 1\) h. The extensive period of time covered by our data, together with observations in other location of Earth’s surface [7, [13], indicate that the power law regime in the region from 1 minute to 1 hour of

Fig. 3 is an universal property of the probability density function \(\psi(\tau)\) of inter drop time intervals (Eq. 1). This power law regime is a characteristic of quiescent periods: all quiescent couples (Eq. 2) of order \((\delta=10s, m=1, n=5)\) are separated by inter drop time intervals \(\tau \geq 10s\) (Fig. 2). Moreover, Fig. 3 suggests that the end of the power law regime at \(\tau \approx 1h\) signals a time scale separation between two different dynamics: the inter storm dynamics where quiescent and non-quiescent periods alternate each other, and the dynamics regulating the occurrence of different storms (meteorological dynamics). Thus, the probability density function \(\psi(\tau)\) of inter drop time intervals can be thought as the sum of three components: 1) \(\psi_{NQ}(\tau)\), the probability density function of non quiescent periods \((\tau \in [0s, \lesssim 10s])\). 2) \(\psi_\delta(\tau)\), the probability density function of quiescent periods \((\tau \gtrsim 10s)\) with a power law regime in the region between \(\tau \gtrsim 1\) min and \(\tau \lesssim 1\) h. 3) \(\psi_{QM}(\tau)\), the probability density function describing the meteorological variability \((\tau \lesssim 1\) h\) of the particular location where the measurements are done. The index \(Q\) in \(\psi_{QM}(\tau)\) indicates that all inter drop time intervals \(\tau \lesssim 1\) h belongs to quiescent periods (Eq. 2 and Fig. 2).

![Log-log plot of the probability density function](image.png)

**FIG. 3**: Log-log plot of the probability density function \(P_\theta^Q(\theta>\tau)\) of having a drought duration \(\theta\) larger than \(\tau\) as a function of \(\tau\) for different time intervals of continuous observations at Chilbolton. The period of continuous observation were: from April 1st to November 3rd 2003 (white squares), from November 5th 2003 to January 5th 2004 (black squares), from January 8th to January 20th 2004 (black circles), from January 24th to May 11th 2004 (white circles), from May 14th to July 17th 2004 (white upward triangles), from July 19 th to August 2nd 2004 (white downward triangles), from August 4th to August 19 th 2004 (white diamonds), from December 10th 2004 to February 28th 2005 (white pentagons). The curves are shifted for clarity. The ticks on the y axis indicates different decades.

In panel (a) of Fig. 4 we plot the probability density functions \(\eta_\theta(d)\) of drop diameters of quiescent periods relative to different months of observations. The observed variability is due to the non stationary character of the sequence of drop diameters (Sec. [11]). In fact, if the non stationarity is removed an universal shape

**FIG. 4**: Log-log plot of the probability density function \(\eta_\theta(d)\) of having a drought duration \(\theta\) larger than \(\tau\) for different time intervals of continuous observations at Chilbolton. The period of continuous observation were: from April 1st to November 3rd 2003 (white squares), from November 5th 2003 to January 5th 2004 (black squares), from January 8th to January 20th 2004 (black circles), from January 24th to May 11th 2004 (white circles), from May 14th to July 17th 2004 (white upward triangles), from July 19 th to August 2nd 2004 (white downward triangles), from August 4th to August 19 th 2004 (white diamonds), from December 10th 2004 to February 28th 2005 (white pentagons). The curves are shifted for clarity. The ticks on the y axis indicates different decades.
for the probability density function emerges. We consider non-overlapping time intervals of duration $T$ and remove the average in every time interval. Panel (b) of Fig. 4 shows that the probability density function $\eta_Q(d_{T,\mu})$ of the zero-average drop diameters sequence has a much smaller variability than the probability density function $\eta_Q(d)$ of the original sequence (Fig. 4 panel (a)). If together with moving average also the moving variance is eliminated (e.g. rescaling to unity the variance in each time interval), the probability density functions $\eta_Q(d_{T,\mu,\sigma})$ of the zero-average unitary-variance drop diameter sequences relative to different months “collapse” into a single curve (Fig. 4 panels (b) and (c)). The shape of this curve is not appreciably altered either by the choice of time intervals of different duration $T$ (ranging from 10 seconds to $\approx 10$ minutes) to remove the non stationarity of the sequence of drop diameters of quiescent periods, either by the use of different quiescence orders ($n$ of Eq. [2] ranging from 5 to 20) and of their “equivalence” classes ($m$ and $\delta$ of Eq. [2] changing in such a way to preserve the factors $m\delta$ and $n\delta$). The probability density function $\eta_Q(d_{T,\mu,\sigma})$ of the zero-average unitary-variance drop diameter sequence has two asymptotic exponential tails: one for the positive and one for the negative values of the rescaled zero-averaged diameters (Fig. 4 panels (b) and (c)). A least squares fit of the exponential tails produce the following values for the decay constants: $\lambda_+=2$ ($2\text{mm} \leq d_{T,\mu,\sigma} \leq 6\text{mm}$) and $\lambda_-=4.56$ ($-4\text{mm} \leq d_{T,\mu,\sigma} \leq -2\text{mm}$).

V. CONCLUSIONS

We introduce the concept of quiescence to describe the temporal variability of the rainfall phenomenon. The quiescence captures a fundamental relationship (Fig. 2) between inter drop time intervals, drop diameter and their time ordering. These properties are not detected by the flux-like quantities such as rain duration and rain intensity. Using the concept of quiescence, we identify what are the universal properties of the rainfall phenomenon. The scaling property of the probability density function of inter drop time intervals during quiescent periods (Fig. 3) and the universal shape for probability density function of drop diameters (Fig. 4). Our results suggest that the analysis of inter drop time intervals and drop diameters sequences and their properties offers a deeper insight than the analysis of the properties of flux-like quantities such as rain duration and rain intensity. A comprehensive understanding of the rainfall phenomenon must rest on its drop-like nature.

FIG. 4: Panel (a): The probability density function $\eta_Q(d)$ of drop diameters during non-quiescent periods for the months of April 2003, October 2003, March 2004 and April 2004. Panel (b): The probability density function $\eta_Q(d_{T,\mu})$ of zero-average drop diameters, and the probability density function $\eta_Q(d_{T,\mu,\sigma})$ of zero-average unitary-variance drop diameters during non-quiescent periods for the months of April 2003, October 2003, March 2004 and April 2004. Panel (c): Zoom of panel (b) for the values of $d_{T,\mu}$, $d_{T,\mu,\sigma}$ in the interval $[-2,2]$, and for values of the probability density in the interval [0.01,0.1]. The non stationarity of the sequence of drop diameters for quiescence periods was removed using time intervals of duration $T=10$.

Acknowledgments

M. I. and S. B. thankfully acknowledge Welch Foundation and ARO for financial support through Grant no. B-1577 and no. W911NF-05-1-0205, respectively. Disdrometer data have been kindly provided by British Atmospheric Data Centre Chilbolton data archive. We would like to thank Dr. P. Allegrini for his helpful comments and wish all the best to his newborn child. Many thanks to Dr. R. Vezzoli for her help and her quick messenger-course on “the psychology of the feminine gender”: sorry, we failed you. Finally, our eternal gratitude goes to Mr. F. Grosso for making us so proud and happy with his beautiful goal at 118$'$ minute of the second overtime of Italy-Germany (World Cup 2006).

[1] Atlas, D., and W. Ulbrich (1982), Assessment of the contribution of differential polarization to improved rainfall measurements, 1 – 8 pp., URSI Open symposium, Bournemouth, UK.
[2] Eagleson, P. (1970), Dynamic Hydrology, McGraw-Hill, New York.
[3] Feingold, G., and Z. Levin (1986), The lognormal distribution fit to raindrop spectra from frontal convective clouds in israel, J. Climate Appl. Meteor., 25, 1346 – 1363.
[4] Gupta, V., and E. Waymire (1990), Properties of spatial rainfall and river flow distributions, *Journal of Geophysical Research-Atmosphere*, 95(D3), 1999 – 2009.

[5] Jameson, A., and A. Kostinski (2000), Fluctuation properties of precipitation. Part VI: Observations of hyperfine clustering and drop size distribution structures in three-dimensional rain, *Journal of the Atmospheric Sciences*, 57(3), 373 – 388.

[6] Joss, J., and E. G. Gori (1978), Shapes of raindrop size distributions, *Journal of Applied Meteorology*, 17, 1054 – 1061.

[7] Lavergnat, J., and P. Golé (1998), A stochastic raindrop time distribution model, *Journal of Applied Meteorology*, 37(8), 805 – 818.

[8] Lavergnat, J., and P. Golé (2006), A stochastic model of raindrop release: Application to the simulation of point rain observations, *Journal of Hydrology*, 328(1-2), 8 – 19.

[9] Lovejoy, S., and D. Schertzer (1985), Generalised scale invariance and fractal models of rain, *Water Resources Research*, 21(8), 1233 – 1250.

[10] Lovejoy, S., and D. Schertzer (2006), Multifractals, cloud radiances and rain, *Journal of Hydrology*, 322(1-4), 59 – 88.

[11] Marshall, J. S., and W. Palmer (1948), The distribution of raindrops with size, *Journal of Meteorology*, 5, 165 – 166.

[12] Menabde, M., D. Harris, A. Seed, G. Austin, and D. Stow (1997), Multiscaling properties of rainfall and bounded random cascades, *Water Resources Research*, 33(12), 2823 – 2830.

[13] Peters, O., C. Hertlein, and K. Christensen (2001), A complexity view of rainfall, *Phys. Rev. Lett.*, 88(1), 018,701, doi:10.1103/PhysRevLett.88.018701.

[14] Pruppacher, H., and J. Klett (1997), *Microphysics of clouds and precipitation*, Atmospheric and oceanographic sciences library, vol. 18, 2nd ed., Kluwer Academic Publisher, Dordrecht.

[15] Segal, B. (1986), The influence of rain gage integration time on measured rainfall intensity distribution functions, *J. Atmos. Oceanic Technol.*, 3, 662 – 671.

[16] Smith, J. A. (1993), Marked point process models of raindrop-size distributions, *Journal of Applied Meteorology*, 32, 284 – 296.

[17] Wilson, P. S., and R. Tuomi (2005), A fundamental probability distribution for heavy rainfall, *Geophysical Research Letters*, 8, L14,812, doi:10.1029/2005GRL022464.