Two transformations of simple polytopes preserving moment-angle manifolds

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Résumé

Nous introduisons ici deux constructions assez générales sur les polytopes simples, conduisant à des polytopes ayant mêmes variétés moment-angle. Comme application, nous donnons des exemples de polytopes (dual) flag qui ne sont pas rigides.

Abstract

In this paper, we introduce two new, quite frequently occurring, constructions, yielding polytopes with diffeomorphic moment-angle manifolds. As an application, we give examples of (dual) flag polytopes that are not rigid.

Introduction

Moment-angle manifolds form a central object in toric geometry. They have been introduced by Buchstaber and Panov [B-P] to oversee the so-called quasitoric manifolds. To any simple (combinatorial) polytope is associated such a manifold that supports a toric action whose orbit space is the given polytope.

The geometry of the moment-angle manifold associated to a polytope is then completely determined by the combinatorics of the polytope. For example, we can describe the homology of a moment-angle manifold in terms of the polytope (see [Ba]) and we have more precise results in some cases ([LdM-V], [B-P], [B-M], [LdM-G], [Ir]...)

A general and very intriguing problem can (roughly) be stated as follows:

QUESTION 0.1 When do two polytopes produce ”the same” moment-angle manifolds?

Closely related to this question is the following one:

QUESTION 0.2 Given a polytope, are there other polytopes giving the same moment-angle manifold?

A polytope can be thought of as rigid if the answer to question 0.2 is negative.

Different quite natural ways of being ”the same”, more or less strong, can be considered.

Recall that any moment-angle manifold supports the natural action of a torus whose orbit space is the polytope in question.

Given two simple polytopes $P$ and $Q$, we know that $Z_P$ and $Z_Q$ are equivariantly diffeomorphic (with respect to the aforementioned actions) if and only if $P$ and $Q$ are combinatorially equivalent [B-M]. Hence, this request is too strong to involve an interesting problem.

We can weaken the request by only demanding that the moment-angle manifolds are diffeomorphic or even that their cohomology rings are isomorphic (with the natural graduation of the cohomology of a
manifold). The first request seems much stronger than the second, but the equivalence between them (and so between other notions of intermediate strength) is an open question.

We use here the notion of graded diffeomorphy. This notion is stronger than diffeomorphy, in which some important discrete invariants may vary (see [Bo1]). Nevertheless, the most widespread constructions preserving the differential structure of moment-angle manifolds also preserve the bigraduation.

In this paper, we introduce two new, quite frequently occurring, constructions, yielding graded-equivalent polytopes. The first one relies on the notion of flip of polytopes (which can be thought as ”the smallest change that a polytope can undergo”). It is called a biflip as it is a succession of two flips (not any). The second one is called ”puzzle-move” as it relies on an assembly of pieces. As an application, we give the first examples of nonrigid (dual) flag polytopes.

1 Preliminaries

Polytope  We only consider here combinatorial polytopes. We consider a simple $d$-dimensional polytope $P$ as the set of all subsets of facets of a geometric simple $d$-dimensional polytope $P_{geom}$ whose intersection are nonempty.

In this sense:
- The elements of the elements of $P$ are identified with the facets of $P_{geom}$ also called the facets of $P$. They are called the facets of $P$ and we identify a facet $F$ of $P$ with the element $\{F\}$ of $P$. We usually note $F$ the set of facets of $P$.
- The maximal elements of $P$ have $d$ elements and correspond to the subsets of facets of $P_{geom}$ containing a given vertex. They are called the vertices of $P$.
- An element of $P$ is called a face of $P$ (and corresponds to the subsets of facets of $P_{geom}$ containing a given face).

Recall just that a combinatorial polytope is completely determined by its vertices.

Moment-angle manifold

**Definition 1.1** Let $\mathcal{E}$ a finite set. We note $T^\mathcal{E}$ the real torus $S^\mathcal{E}$ where $S$ denotes the unit circle in $\mathbb{C}$, with its usual structure of topological group.

Let’s begin by recall a construction of a moment-angle manifold. We start from a simple polytope $P$, and denote $\mathcal{F}$ the set of its facets. We can consider the torus $T^\mathcal{F}$ and the product $P \times T^\mathcal{F}$, on which $T^\mathcal{F}$ acts by translation on itself.

On $P \times T^\mathcal{F}$, we identify two elements $(p, t)$ and $(p', t')$ if $p = p'$ and $p$ lies on every facet not sent on 1 by $t^{-1}t'$.

The space $Z_P$ obtained by this identification is the moment-angle manifold over $P$. The action of $T^\mathcal{E}$ descends on $Z_P$. (We can keep in mind that each facet correspond to a rotation of $Z_P$ fixing the submanifold over this facet, and that all these rotations commute).

Homology  We recall (see [Ba]) that the cohomology of $Z_P$ is endowed with a natural bigraduation, more precisely, for any subset $\mathcal{X}$ of facets of $P$, the homology of the space $\bigcup_{F \in \mathcal{X}} F$ corresponds to a part (a submodule) of the homology of $Z_P$, and we recall the following formula :

$$H_k(Z_P, \mathbb{Z}) \simeq \bigoplus_{\mathcal{X} \subset \mathcal{F}} \tilde{H}_{k-|\mathcal{X}|-1}(P_{\mathcal{X}}, \mathbb{Z})$$

We note $H^{p,q}(Z_P, \mathbb{Z})$ the subspaces of the homology of $Z_P$ on whose the bigraduation has a given value.
**Definition 1.2** A graded diffeomorphism $\phi$ between $Z_P$ and $Z_{P'}$ is a diffeomorphism between them such that, for any $p, q$, we have $\phi^*(H^{p,q}(Z_{P'}, \mathbb{Z})) = H^{p,q}(Z_P, \mathbb{Z})$.

Two polytopes $P$ and $Q$ are said Gr-equivalent if there is a graded diffeomorphism between their moment-angle manifolds.

**Flip**

**Definition 1.3** Consider a simple polytope $P$ of which $A$ is a simplicial face.

A facet $F$ of $P$ is called a bounding facet of $A$ if it meets the boundary of $A$ but not its (relative) interior.

Here, we only consider flips of polytopes at combinatorial level. Consider a polytope $P$ of dimension $d \geq 2$. Let $1 \leq p \leq d$ an integer, and $A$ a simplicial $p - 1$ face of $P$. Now consider the set of facets $\mathcal{A} = \{A_1, ..., A_{d-p+1}\}$ so that $A = \bigcap_{F \in \mathcal{A}} F$ and the set $\mathcal{B} = \{B_1, ..., B_p\}$ of bounding facets of $A$. Call now $V$ the set of vertices of the simplex on $\mathcal{A} \cup \mathcal{B}$ (i.e. the set of complements of singletons in $\mathcal{A} \cup \mathcal{B}$).

Now, consider the symmetric difference between the set of vertices of $P$ and $V$. If it is the list of vertices of a polytope $Q$, we say that $Q$ is obtained from $P$ by a flip, and setting $q = d - p + 1$, we say this flip is a $(p,q)$-flip.

If $Q$ is obtained from $P$ by a $(p,q)$-flip, then $P$ is obtained from $Q$ by a $(q,p)$-flip where the roles of $\mathcal{A}$ and $\mathcal{B}$ are inverted.

**Wedge** We give here a purely combinatorial presentation of the wedging operation. It can be also presented more geometrically, for instance in [K-W].

Let $P$ a simple $d$-polytope and $F$ a facet of $P$. We note $W_FP$, called the wedge over $P$ on $F$, the simple $d + 1$-polytope obtained by replacing $F$ by two elements $F_1$ and $F_2$, and whose vertices are:

- the sets $\{F_1; G_1; ...; G_d\}$ or $\{F_2; G_1; ...; G_d\}$ where $\{G_1; ...; G_d\}$ is a vertex of $P$
- the sets $\{F_1; F_2; G_1; ...; G_{d-1}\}$ where $\{F; G_1; ...; G_{d-1}\}$ is a vertex of $P$.

The facets $F_1$ and $F_2$ are called the main facets of the wedge.

This operation can be repeated as many times as we want over any facet, and to a map $\alpha : \mathcal{F} \rightarrow \mathbb{N}$, we note $W_\alpha P$ the polytope called multiwedge over $P$ the polytope obtained by taking $\alpha(F)$ times a wedge on $F$ for each $F$, which is well defined, i.e. does not depend on the order in which wedges are performed.

**2 Biflips**

Consider a simple polytope $P$. We know that two polytopes obtained by different flips of equal index from $P$ may not be Gr-equivalent. For instance, if we consider the pentagonal book, whose associated moment angle manifold is diffeomorphic to $\frac{3}{2} S^3 \times S^3 \# S^4 \times S^3$, an edge flip may produce either another pentagonal book or a cube, whose associated moment angle manifold is $S^3 \times S^3 \times S^3$.

There is nevertheless a simple case in which it is possible to guarantee that two flips will produce Gr-equivalent polytopes:

**Theorem 2.1** Let $P$ a simple polytope. Consider two simplicial faces $F_1$ and $F_2$ and assume that their bounding facets are the same.
Then if both faces are flippable, the two polytopes obtained after the flips are $Gr$-equivalent.

This theorem leads to the notion of biflip:

**Definition 2.1** Let $Q$ a polytope. A sequence of two flips $Q \overset{f_1}{\longrightarrow} P \overset{f_2}{\longrightarrow} R$ is called a biflip if the bounding facets of the simplicial face of $P$ appearing by $f_1$ are the same as the bounding facets of the one disappearing by $f_2$.

We immediately deduce from the theorem:

**Corollary 2.2** Two polytopes joined by a (sequence of) biflip(s) are $Gr$-equivalent.

Let's prove the theorem.

**Demonstration**

As explained in [B-M], it is possible to describe the change on the geometry of a moment angle manifold arising after a flip. This is given by an equivariant surgery. In the forecited article, it is shown that any $(1,n)$-flip on a polytope induces the same (non fully equivariant) surgery, hence the same differential structure. The proof is based on the fact that some submanifolds are isotopic inside the moment angle manifold.

In the case of more general flips of the same index, the topology of the sets that are removed and glued, as well as the glueing operation, only depend on the index. In this sense, there only remains to prove that the removed pieces are isotopic inside the moment-angle manifold $Z_P$.

If $F_1$ and $F_2$ are $(k-1)$-simplices, these removed pieces have the form $S^{2k-1} \times T^{n-d+1} \times D^{2(d-k)}$, indeed are a product neighbourhood of the submanifold $S^{2k-1} \times T^{n-d+1}$ over $F_1$ or $F_2$.

First, remark that a facet that does not bound a given simplicial face, either contains it or is disjoint from it.
Consider then the bounding facets $A$ of $F_1$ and $F_2$. Call also $B_0$ the facets containing $F_1$ and $F_2$, $B_1$ those containing $F_1$, not $F_2$, $B_2$ those containing $F_2$, not $F_1$, and $C$ those disjoint from both $F_1$ and $F_2$.

Call $k = |A|$, so $F_1$ and $F_2$ have dimension $k - 1$. Call $r = |B|$, so $|B_1| = |B_2| = d - k - r + 1$ and $|C| = n - 2d + k + r - 2$.

For a bounding facet $A$ in $A$, note $v_{A,1}$ (resp. $v_{A,2}$) the vertex of $F_1$ (resp. of $F_2$) not lying on $A$.

Consider the simplex $\Delta^{k-1}$ on $k$ vertices, each vertex $e_A$ being associated to a facet $A$ in $A$.

We construct now an explicit isotopy between the forementioned loci. We first consider the map $g$ from $\Delta^{k-1} \times [0;1]$ to $P$ that is separately affine and so that for all $A$, $g(e_A,0) = v_A$ and $g(e_A,1) = v_A'$. Notice in particular that $\Delta^{k-1} \times \{0\}$ is sent on $F_1$ and $\Delta^{k-1} \times \{1\}$ on $F_2$.

We fix a bijection $c$ between $C$ and $[1, ..., n - 2d + k + r - 2]$ as well as two bijections $b_{i}, i = 1, 2$ between respectively $B_i$ and $[1, ..., d - k - r + 1]$, then consider the map $\phi$ from the torus $T^{n-d-1}$ to $T^F$ given by $\phi(t_1, ..., t_{d+1-k}) = (x_F)_{F \in F}$ where $x_F = 1$ if $F$ is in $A$, $x_F = t_b(F)$ if $F$ is in $B$, $x_F = t_{r+b}(F)$ if $F$ is in $B_r$, $i = 1, 2$ and $x_F = t_{(d-k-r+1)+c}(F)$ if $F$ is in $C$.

Notice $\phi$ is a group morphism and an embedding. Now, we define a map $h$ from $\Delta^{k-1} \times T^{n-d+k-1}$ to $Z_P$ by considering $T^{n-d+k-1}$ as $T^A \times T^{n-d+1}$, and posing $h(x, \gamma_A, \gamma_t) = (\phi(\gamma)) \cdot g(x, t)$.

This action descends to $S^{2k-1} \times T^{n-d+1} \times [0;1]$. For $t = 0$, this amounts to the action of $T^{B_2} \times T^C$ on $S^{2k-1}$ giving the submanifold of $Z_P$ over $F_1$ and for $t = 1$, we get the action of $T^{B_1} \times T^C$ on $S^{2k-1}$ giving the submanifold of $Z_P$ over $F_2$. Thus these two submanifolds are isotopic inside $Z_P$.

Moreover, we immediately extend this isotopy to arbitrary small Collar neighbourhoods of these two submanifolds and easily extend this to a global isotopy of diffeomorphisms of $Z_P$, all being identity outside arbitrary small neighbourhoods of the image of $h$. This proves that $Z_{P_1}$ and $Z_{P_2}$ are diffeomorphic.

We now verify that our diffeomorphism preserves the bigraduation.

The homology of $Z_{P_1}$ is generated by four kinds of classes:

1. Images of classes that are represented by a cycle which is disjoint from $F'_1$.
2. The top-class.
3. Images of classes for whose any representant is homologous to the boundary of $F'_1$.
4. Images of (pure) classes whose intersection with some class of the precedent type equal 1.

This first case corresponds to the case the representing cycle is inside the locus of $Z_{P_1}$ where the diffeomorphism is identity. So the same cycle represents also the image class.

The second case is immediate, as both polytopes have the same dimension.

In the third case, The set of facets on which the representant is drawn must contain all facets bounding $F'_1$, and no bounding facet of $F_1$ otherwise it would be a boundary. Turn now in $X$ these facets into the boundary facets of $F'_2$ and the boundary of $F'_1$ into the one of $F'_2$. We preserve the degree and these classes correspond by the diffeomorphism.

In the last case, a representant of such a class must be linked with the boundary of $F'_1$ and a representant of its image must be linked with the boundary of $F'_2$. So here too the degrees must fit.

This completes the proof of the theorem. QED

**Remark 2.3** Two polytopes obtained by cutting a vertex to the same polytope $P$ are joined by a bifold as cutting a vertex produces a $(1, d)$-flip. So we recover that two such polytopes are $Gr$-equivalent.

Let’s give an application of this notion.

Recall that the list of simplicial 4-polytopes with 8 vertices appears in [G-S], where they are numbered from $P_1^8$ to $P_8^8$. We know [H] that some moment-angle manifold, viz. the one associated to the dual of the simplicial polytope $P_8^8$, is a connected sum of sphere products, with a summand being the product of
three spheres, whereas all (nontrivial) previously known examples of this kind had every summand being a product of two spheres.

**Proposition 2.4** The dual of polytope $P_{28}^8$ is not rigid, more precisely it is Gr-equivalent to the duals of $P_{27}^8$ and $P_{29}^8$ (and to no other polytope).

**Proof** Indeed, we will show that the three polytopes in question are joined by biflips.

We remark that these three polytopes are exactly the three simple polychora with four heptahedra and four hexahedra (whithout other facet).

Consider a dual cyclic polychoron with seven facets. Number its facets from 1 to 7 in the natural cyclic order. Truncate now a triangular face, say $1 \cap 2$. We could see the dual of the polychoron we get is $P_{22}^8$. The list of its vertices is the following:

$$
(1245)(1248)(1256)(1267)(1278)(1457)(1478)(1567)(2345)(2348)
$$

$$
(2356)(2367)(2378)(3456)(3467)(3478)(4567)
$$

We remark that three edges are bounded by facets 5 and 8, namely $(1245)$ (i.e. $1 \cap 2 \cap 4$), $(1478)$ and $(2345)$. We easily see that the three polytopes obtained by flipping one of these edges have only $\{1; 3\}$ and $\{6; 8\}$ as pairs of disjoint facets, so these four facets are hexahedra, whereas the other four ones are heptahedra. Moreover no two of these polychora are isomorphic. Indeed, the last one has a cubic facet (3), opposed to both others, and the second one has to facets that are pentagonal prisms (2 and 7), opposed to both others. Hence we get our three polytopes, that are, as announced, joined by biflips, hence Gr-equivalent. □

## 3 Puzzle-equivalence

We introduce here a new construction leading to Gr-equivalent polytopes. This notion that we call puzzle-equivalence is based on a generalization of the notion of blending (connected sum) of polytopes that has been introduced by Holt [Ho].

### 3.1 Wedge equivalence

**Definition 3.1** Two facets $F$ and $G$ of a polytope $P$ are called wedge-equivalent if the transposition of these two facets is an automorphism of $P$.

Clearly, wedge-equivalence is an equivalence relation, whose classes are called wedge-classes.

**Proposition 3.1** Let $F$ and $G$ two facets of a simple polytope $P$. The following assertions are equivalent:  
1) $F$ and $G$ are wedge-equivalent;  
2) any vertex of $P$ lies either on $F$ or on $G$;  
3) $F$ and $G$ belong to the same missing faces of $P$.

**Proof** Indeed, if the transposition swapping $F$ and $G$ is not an automorphism of $P$, then there are facets $H_i, 1 \leq i \leq d-1$ different from $G$ so that $F \cap H_1 \cap \ldots \cap H_{d-1}$ is a vertex and $G \cap H_1 \cap \ldots \cap H_{d-1}$ not. A minimal subset $G$ of $\{H_i\}_{1 \leq i \leq d-1}$ whose intersection with $G$ is empty yields a missing face containing $G$ and not $F$. So iii) ⇒ i).

Assume now $F$ and $G$ are wedge-equivalent. If we had a vertex neither on $F$ nor on $G$, we would have, by connexity of the graph of $P$, an edge joining such a vertex $v$ to a vertex $v'$ on $F$ or on $G$ (we can assume $F$, so $v'$ is not on $G$). The image $v''$ of $v'$ by the transposition $F \leftrightarrow G$ should also be adjacent to $v'$ and on $G$, not on $F$. Hence, $v$ and $v''$ would be two distinct vertices that are adjacent to $v'$ and not on $F$, which is impossible. So i) ⇒ ii).
Assume every vertex lies either on $F$ or on $G$, and let $\mathcal{E}$ a missing face containing $F$. Then there is an vertex on each facet of $\mathcal{E} \setminus \{F\}$. Such a vertex $v$ is on $G$ by hypothesis and let $v'$ adjacent to $v$ and not on $G$. Then $v'$ lies on $F$ and on every facet of $\mathcal{E} - \{F,G\}$, so on every facet of $\mathcal{E} - \{G\}$. As $\mathcal{E}$ is a missing face, $\mathcal{E} - \{G\}$ is not $\mathcal{E}$, so $G$ belongs to $\mathcal{E}$. Hence ii) $\Rightarrow$ iii).

This completes the proof of the proposition. □

Remark 3.2 The two main facets of a wedge are wedge-equivalent. Also if $P$ is a polytope, the facets $P \times \{0\}$ and $P \times \{1\}$ of $P \times [0;1]$ are wedge-equivalent. This in fact the only cases, and we could even consider the product with an interval as the wedge on a ghost facet.

Definition 3.2 Let $P$ be a polytope. We say that an automorphism $\phi$ of $P$ is harmless (or of wedge type) if it stabilizes wedge-classes, i.e. if, for any facet $F$ of $P$, $\phi(F)$ is wedge-equivalent to $F$.

In other words, a harmless automorphism is a composition of automorphisms that are transpositions.

In fact, harmless automorphisms of polytopes are closely related to diffeomorphisms of moment-angle manifolds:

Proposition 3.3 Let $\phi$ be an automorphism of a polytope $P$. The following assertions are equivalent:

i) The automorphism $\phi$ is harmless;

ii) the diffeomorphism $\phi$ induces on $Z_P$ is isotopic to identity;

iii) the diffeomorphism $\phi$ induces identity on the homology of $Z_P$.

Proof Let’s first prove i) $\Rightarrow$ ii). Consider a path of unitary automorphisms of $\mathbb{C}^2$ from the identity to the automorphism permutating coordinates, for instance:

$$(z, w, \theta) \rightarrow \frac{1}{2}((z + w) + e^{i\theta}(z - w), (z + w) - e^{i\theta}(z - w))$$

For $\theta = 0$, we get the identity. For $\theta = \pi$, coordinates are swapped.

This path of isometries can be restricted to any sphere centered at the origin.

Assume $\phi$ is an automorphism of a polytope $P$ which only transposes facets $F$ and $F'$. The path in $Diff(Z_P)$ preserving all coordinates but those of $F$ and $F'$ and acting on this pair of coordinates like the path thereabove clearly connects the identity to the automorphism of $Z_P$ permuting the desired coordinates. This proves that the automorphism of $Z_P$ induced by the permutation of $F$ and $F'$ is homotopic to identity.

Remark: We use here that $Z_P$ is filled by spheres on coordinates $F$ and $F'$, thanks to the model of intersection of quadrics with, in each quadric, the same coefficients for the two wedge-equivalent coordinates in question.

The implication ii) $\Rightarrow$ iii) is a basic result of homotopy theory which is valid in a much broader context [Sp].

Let’s prove iii) $\Rightarrow$ i). Assume we have an automorphism $\phi$ of a polytope $P$ whose induced diffeomorphism of $Z_P$ acts homologically trivially.

Let $\mathcal{E}$ a missing face of $P$. This missing face induces a homology class of $Z_P$. This class must be fixed by $\phi_*$, so $\phi$ must preserve $\mathcal{E}$. Then $\phi$ is harmless by proposition 3.1.

This completes the proof of the proposition. □

3.2 Puzzle-equivalence

We now introduce our manipulation on polytopes:

We consider two polytopes $P_1$ and $P_2$ embedded in a $d$-dimensional euclidean space $E$. We consider a hyperplane $H$ of $E$, whose associated half-spaces will be noted $H_+$ and $H_-$, intersecting $P_1$ and $P_2$ but
not at any vertex. For \( i = 1, 2 \), the intersection \( P_i \cap H \) is then a \( d - 1 \)-dimensional polytope \( \Delta_i \), and \( P_i \) is obtained by gluing the two parts \( P_i \cap H^+ \) and \( P_i \cap H^- \) along their common facet \( \Delta_i \).

We assume there are isomorphisms \( i_+ \) and \( i_- \) between one hand \( P_1 \cap H^+ \) and \( P_2 \cap H^+ \), on the other hand \( P_1 \cap H^- \) and \( P_2 \cap H^- \), both sending \( \Delta_1 \) on \( \Delta_2 \) (in particular \( \Delta_1 \) and \( \Delta_2 \) have to be isomorphic). We still note \( i_+ \) and \( i_- \) the restrictions of \( i_+ \) and \( i_- \) to \( \Delta_1 \), and we can consider the automorphism \( \phi = (i_-)^{-1} \circ i_+ \) of \( \Delta_1 \).

If \( \phi \) is harmless, then we say that we pass from \( P_1 \) to \( P_2 \) by a puzzle-move. In this case, we pass from \( P_2 \) to \( P_1 \) by the inverse puzzle-move.

In fact, the combinatorial data of \((P_1, \Delta^+, \phi)\) completely determines the combinatorics of \( P_2 \), where \( \Delta^+ \) is \( \Delta_1 \) with the choice of the halfspace \( H^+ \), and we sometimes define a puzzle-move with this data. In this case, noting \( P \) for \( P_1, P_2 \) is noted \( P_{\Delta^+, \phi} \).

We also can notice that when \( \phi \) is an involution, in particular a transposition, the choice of the halfspace has no importance and we can omit it.

**Definition 3.3** A puzzle-move will be called trivial or nontrivial according to whether \( P_2 \) is isomorphic to \( P_1 \) or not.

The equivalence relation between polytopes generated by puzzle-moves is called puzzle-equivalence.

Here is a picture of a puzzle-move:

![First polytope](First polytope.png)

![First facet gluing](First facet gluing.png)

![Second polytope](Second polytope.png)

![Second facet gluing](Second facet gluing.png)

**Remark 3.4** We can notice that \( P_{\Delta, \phi} \) actually depends on \( \Delta \). We can construct examples of different puzzle-moves with the same \( P \) and \( \phi \).
The puzzle-move \((P, \Delta, (AB))\) is trivial whereas \((P, \Delta', (AB))\) is not.

**Theorem 3.5** Two puzzle-equivalent polytopes are \(Gr\)-equivalent.

**Remark 3.6** In particular, if \(\Delta\) is a simplex, then any automorphism of \(\Delta\) is harmless and any connected sum can be performed, so \((P, \Delta^+, \phi)\) is always a puzzle-move. Hence we recover that two polytopes obtained by connected sums of the same polytopes \(P\) and \(Q\) on the same vertices are \(Gr\)-equivalent (let’s recall that the exact differential structure of such a moment-angle manifold is conjectured in [F-W]).

**Remark 3.7** We can notice that both polytopes on the first picture are flag polytopes. So this construction produces nonrigid flag polytopes, as announced.

Let’s now prove the theorem:

**Demonstration** Let \(P_1, P_2\) two polytopes joint by a puzzle-move.

For \(i = 1, 2\), let \(\tilde{Z}_{\Delta_i}\) the preimage of \(\Delta_i\) in \(\tilde{Z}_{P_i}\) by the natural map, \(Q_i = P_i \cap H^+, R_i = P_i \cap H^-\).

The two preimages \(\tilde{Z}_{Q_1}\) and \(\tilde{Z}_{Q_2}\) are diffeomorphic, as well as \(\tilde{Z}_{R_1}\) and \(\tilde{Z}_{R_2}\), \(\tilde{Z}_{Q_i}\) and \(\tilde{Z}_{R_i}\) having \(\tilde{Z}_{\Delta}\) as common boundary.

Now, we identify \(Z_{Q_i}\) (resp. \(Z_{R_i}\)) to a manifold \(\tilde{Z}_Q\) (resp. to a manifold \(\tilde{Z}_R\)), yielding identifications \(a_{i,+}\) (resp. \(a_{i,-}\) of \(\Delta_i\) with \(\Delta\)). With respect to the terminology thereabove, we have \(a_{i,+} = (a_{i,+})^{-1} \circ a_{i,+}\) whereas \(a_{i,-} = (a_{i,-})^{-1} \circ a_{i,-}\).

Both \(Z_{P_1}\) and \(Z_{P_2}\) are obtained by glueing \(\tilde{Z}_Q\) and \(\tilde{Z}_R\) along their common boundary \(\tilde{Z}_{\Delta}\). Indeed, \(\tilde{Z}_{\Delta}\) is given by the product of \(Z_{\Delta}\) by a real torus \(T^S\), where \(S\) corresponds to the set of facets of \(Q\) and \(R\) that are disjoint from \(\Delta\).

We can find Collar neighbourhoods \(N_Q \simeq \tilde{Z}_{\Delta} \times T^S \times [0; 1[\) in \(\tilde{Z}_Q\) and \(N_R \simeq \tilde{Z}_{\Delta} \times T^S \times [0; 1[\) in \(\tilde{Z}_R\) so that \(Z_{P_i}\) is given by identifying (glueing) their interiors in the following way:

\[
\tilde{Z}_{\Delta} \times T^S \times [0; 1[ : \tilde{Z}_{\Delta} \times T^S \times [0; 1[ \sim (\psi_i(x), \gamma, 1-t)
\]

where \(\psi_i\) is the diffeomorphism of \(Z_{\Delta}\) induced by the automorphism \(a_i = a_{i,-} \circ (a_{i,+})^{-1}\) of \(\Delta\) associated to \(P_i\). As by hypothesis \((i_-)^{-1} \circ i_+\) is harmless, the composition \(a_1 \circ (a_2)^{-1}\) is harmless too, so, by proposition 3.3, \(\psi_1 \circ (\psi_2)^{-1}\) is isotopic to identity, in other words \(\psi_1\) and \(\psi_2\) are isotopic. Hence the manifolds \(Z_{P_1}\) and \(Z_{P_2}\) are actually diffeomorphic.

Moreover, this diffeomorphism is graded.

Indeed \(\tilde{Z}_{\Delta}\) cuts \(Z_{P_i}\) into two parts. The homology of \(Z_{P_i}\) is then, by Mayer-Vietoris theory, generated by two kinds of classes:
• Classes induced by one of the two parts. In this case, this class is induced by a class that has a representant in $\partial P_1$ in one side of the hyperplane containing $\Delta$. This class is identified with a class of $P_2$ on the same side (we consider the puzzle-move fixes this side and moves the other). Obviously, these two classes have equal degrees.

• Classes (pure) yielding nontrivial classes of $\tilde{Z}_\Delta$. Such a class and its correspondant by the diffeomorphism yield "the same" class of $\tilde{Z}_\Delta$. As the degree of a representant of this class of $\tilde{Z}_\Delta$ is one less than the degree of a representant of the class itself, the two classes have equal degrees.

This proves that the diffeomorphism actually preserves bigraduation. QED

As a first application, we can again recover that two polytopes obtained by cutting a vertex to the same one are $Gr$-equivalent, as we can see on the following picture which is valid in any dimension:

![Puzzle-move diagram](image)

**Dimension 3**  We focus here on the case of puzzle-moves of 3-dimensional polytopes.

If $P$ is a 3-dimensional polytope, then its intersection with a hyperplane is a polygon. The only nontrivial harmless automorphisms of polygons are obtained for a triangle (any permutation of sides is suitable) or for a quadrilateral (each side must then be sent either on itself or on its opposite), so a nontrivial puzzle-move from $P$ must benefit from such configurations.

The case of a triangle implies that $P$ is decomposable (as a connected sum) and the permutation describes the difference between the glueings (at the same vertices).

The case of a quadrilateral is perhaps less easily described, though quite clear on a picture. Notice it requires a 4-belt on $P$, that separates it into two polytopes that are not too trivial (for example, a 4-belt surrounding just a quadrilateral facet of $P$ produces a trivial puzzle-move).

We know that polyhedra without 3- nor 4-belts (so-called Pogorelov polyhedra) are $Gr$-rigid [F-M-W], so $Gr$-equivalence between two polyhedra requires such kind of belts. Moreover, all known constructions of $Gr$-equivalent polytopes (connected sums, biflips, ...) are, in dimension 3, obtained by puzzle-moves.

Indeed, in dimension 3, a biflip of edges can be obtained by two puzzle-moves like on the following picture:
This leads to conjecture:

**Conjecture 1** For 3-dimensional polytopes, Gr-equivalence coincides with puzzle-equivalence.

### 3.3 Puzzle-rigidity

**Definition 3.4** We say that a polytope is puzzle-rigid if it is puzzle-equivalent to no other polytope.

Naturally, puzzle-rigidity is weaker than Gr-rigidity.

The following lemma may provide an easy way to guarantee puzzle-rigidity:

**Lemma 3.1** Let $P$ a polytope, $F$ and $G$ two facets of $P$. Assume that the subgraph of the 1-skeleton of $P$ induced by the vertices that are neither on $F$ nor on $G$ is connected.

Then there is no nontrivial puzzle-move induced by the transposition $(F,G)$.

If any pair of facets of $P$ satisfies the condition thereupon, the polytope $P$ will be called W-puzzle-rigid and, in particular, it is puzzle-rigid.

**Proof** Consider a puzzle-move induced by a transposition $(F,G)$. If $F$ and $G$ satisfy the condition of the lemma, then all the vertices outside $F \cup G$ lie on the same side of the separator as no vertex joining two such vertices can cross it. This side can be chosen as the right one. On the left side, every vertex lies on $F$ or $G$ and if such a vertex $v$ is not on $G$, it is adjacent to exactly one vertex $v'$ that is not on $F$. As the edge joining $v$ to $v'$ does not lie on $F$ nor $G$, it cannot cross $H$, so $v'$ must be on the left side and as it is not on $F$, it must lie on $G$.

This proves that the transposition $(F,G)$ preserves the left vertices, so it produces a trivial puzzle-move. The second part of the lemma follows immediately. □

We now use this lemma to highlight the difference between Gr-equivalence and puzzle-equivalence.

In [152], we classified Gr-rigidity among polytopes with three facets more than their dimension. Recall that generically, such a polytope is nonrigid. Hence, the following theorem shows a neat difference between Gr-rigidity and puzzle-rigidity.
Proposition 3.8 Any polytope with (at most) three facets more than its dimension is puzzle-rigid.

Proof We just have to verify $W$-puzzle-rigidity, i.e. the criterion of lemma [3.1] for such polytopes.

If $n = d + 1$, then any vertex lies on every facet but one, so in the union of any pair of facets.

If $n = d + 2$, then any vertex lies on every facet but two, so there is at most one vertex not lying on the union of a given pair of facets.

If $n = d + 3$, then any vertex of the complement of the union of two given facets lies on exactly all other facets but one, so any two such vertices are adjacent, i.e. the corresponding graph is complete, which proves the criterion. □

Puzzle-rigidity and wedges We now deal with relations between puzzle-rigidity, or $W$-puzzle-rigidity, and the wedging operation.

For the second one, the situation is particularly simple:

Proposition 3.9 Let $P$ a polytope, $F$ a facet of $P$. Then $W_F P$ is $W$-puzzle-rigid if and only if $P$ is $W$-puzzle-rigid.

Proof Assume $P$ is $W$-puzzle-rigid and $F$ is a facet of $P$. Consider now two distinct facets $\tilde{G}$ and $\tilde{H}$ of $W_F P$ corresponding to facets $G$ and $H$ of $P$. If $G = H = F$, then $\tilde{G}$ and $\tilde{H}$ are the main facets, so their union contains all vertices of $W_F P$. If $G = F, H \neq F$, then the graph on the complement of $\tilde{G} \cup \tilde{H}$ is isomorphic to the graph on the complement of $G \cup H$, which is connected by hypothesis.

If neither $G$ nor $H$ is $F$, then both intersections of the graph induced by the complement of $\tilde{G} \cup \tilde{H}$ with the main facets are connected and any vertex of any of these graphs is adjacent to one of the other graph (by swapping the main facets). Anyway, the desired graph is actually connected.

Conversely, assume $P$ is not $W$-puzzle-rigid and let $G, H$ two facets of $P$ so that the graph induced by their complement is disconnected.

Consider now facets $\tilde{G}, \tilde{H}$ over respectively $G$ and $H$. If $G$ is $F$, then the graph induced by the complement of $\tilde{G} \cup \tilde{H}$ is isomorphic to this disconnected graph. If neither $G$ nor $H$ is $F$, then, as two adjacent vertices of wedge have their projections on the basic polytope equal or adjacent, points projecting on different components of the (graph induced by the) complement of $G \cup H$ cannot be adjacent, so have to be in different components of the (graph induced by the) complement of $\tilde{G} \cup \tilde{H}$. Hence, the desired graph is disconnected. □

As we shall see, the situation is not that simple concerning puzzle-rigidity. For instance, an hexagon is puzzle-rigid, but an hexagonal book (a wedge over the hexagon) is not. We can ask:

Question Is $W$-puzzle-rigidity of a polytope equivalent to puzzle-rigidity of all multiwedges over it?

In fact, we can see that a puzzle-move on a polytope induces naturally a combinatorial operation on any (multi)wedge over it.

Definition 3.5 Let $(P, \Delta^+, \phi)$ a puzzle-move, $F$ a facet of $P$ and $H$ the space of $\Delta$. We can consider $F = F \cap H$. There is a hyperplane $H'$ of the space of $W_F P$ that separates vertices according to whether their projection on $P$ is in $H_+$ or $H_-$, and let’s make the natural choice for $H'_+$ and $H'_-$. Combinatorially, its intersection with $W_F P$ is $W_{F'} (P \cap H)$. We now consider an automorphism $\phi'$ of $W_{F'} (P \cap H)$ that fixes one of the two main facets and acts like $\phi$ on the others (throw the natural identification).

Then $\phi'$ is a harmless automorphism of $W_{F'} (P \cap H)$, and we can consider the simplicial complex on the facets of $W_F P$ whose maximal elements correspond to either the facets containing a vertex in $H'_+$ or those containing a vertex in $H'_-$ but where the ones meeting $H'$ are repaced by their image by $\phi'$ (we just mimic the combinatorial realisation of a puzzle-move).

This complex will be noted $P_{H^+, \phi'}$. 

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In many cases, we can guarantee that the passage from $P$ to $P_{H^+,\phi'}$ is a nontrivial puzzle-move, so it prevents the rigidity of the wedge, and sometimes of any multiwedge over the base polytope.

The following proposition often applies when dealing with (multi)wedges:

**Lemma 3.2** Let $P$ a polytope. Assume that there is a puzzle-move that transforms $P$ into a polytope with more wedge-equivalence classes. Then the operation induced by proposition 3.5 on a (multi)wedge over $P$ cannot be a trivial puzzle-move.

**Proof** Indeed, the number of wedge-classes of $W_\alpha P$ is equal to the one of $P$ and, calling $P_\Delta$ the polytope obtained from $P$ by the required puzzle-move, the object obtained by proposition 3.5 on $W_\alpha P$, if a polytope, contains $P_\Delta$ as face, so has at least as many wedge-classes as it, i.e. more than $W_\alpha P$. So we cannot get isomorphic polytopes. □

Realisability of moves may be more delicate, but we can settle it in two relevant cases:

**Lemma 3.3** If $F$ is disjoint from $H$ or if $\bar{F}$ is fixed by $\phi$, then the operation is a puzzle-move.

If the automorphism $\phi$ of $\Delta$ can be extended to an isometry of a neighbourhood of $\Delta$ in $P$, then the operation is a puzzle-move.

**Remark 3.10** If the original puzzle-move consists of changing the glueing of a connected sum of two polytopes (in other words if $\Delta$ is a simplex), then there is realisation of $P$ so that a neighbourhood of $\Delta$ is the product of a regular simplex and an interval. In this case the proposition thereupon applies so the operation of proposition 3.5 is actually a puzzle-move.

**Proof** In the first case, we easily notice that the produced object is simply $W_F Q$. So it is a polytope.

In the second case, the isometry of the neighbourhood of $\Delta$ in $P$ naturally extends to an isometry of a neighbourhood of $H' \cap W_F P'$ in $P'$, and the corresponding glueing also produces a convex body, i.e. a polytope. □

As an application, we have:

**Theorem 3.11** Let $n \geq 6$. Then no nontrivial multiwedge over the $n$-gon is puzzle-rigid.

**Demonstration** Indeed, if we consider a simple wedge over an hexagon, i.e. a hexagonal book, we can notice that a puzzle-move corresponding to a connected sum can turn a wedge over an hexagon into a polytope which is not a nontrivial wedge, so has one more wedge-class.

The lemmas thereupon permit to conclude. QED

Note: We can see that dual cyclic polytopes of dimension $d \geq 3$ and at least $d + 4$ facets are not puzzle-rigid, hence not rigid, neither are any (multi)wedge over such a polytope. The same question is open by replacing "dual cyclic" by "dual neighbourly".

**Puzzle-rigidity and products** The puzzle-equivalence between products of polytopes is the simplest we can expect. Precisely, we get:

**Proposition 3.12** Let $P$ and $Q$ two polytopes. Then any polytope puzzle-equivalent to $P \times Q$ is a product $P' \times Q'$ where $P'$ is puzzle-equivalent to $P$ and $Q'$ puzzle-equivalent to $Q$. 

Proof Let $P$ and $Q$ two polytopes, $F$ a facet of $P$ and $G$ a facet of $Q$. Then $F$ and $G$ satisfy the condition of lemma 3.1. Indeed, if we consider two vertices $(v_{P,1}, v_{Q,1})$ and $(v_{P,2}, v_{Q,2})$ out of $F \cup G$, it means that neither $v_{P,i}$ is on $F$ and neither $v_{Q,i}$ is on $G$. Then there is a path in $P$ from $v_{P,1}$ to $v_{P,2}$ avoiding $F$ and a path in $Q$ from $v_{Q,1}$ to $v_{Q,2}$ avoiding $G$. Using these paths, we get a path from $(v_{P,1}, v_{Q,1})$ to $(v_{P,2}, v_{Q,2})$ avoiding $F \cup G$.

So any elementary puzzle-move of $P \times Q$ permutes two facets of the same kind ($F \times Q$ or $P \times G$), we can assume to be $\tilde{F} = F \times Q$ and $\tilde{F}' = F' \times Q$. Let’s then prove that the polytope $R$ obtained after this puzzle-move has the form $\tilde{P} \times Q$.

Consider a vertex $v$ of $P$. If it lies neither on $F$ nor on $F'$, then for any $v_Q$ on $Q$, the vertex $(v, v_Q)$ is preserved by the move, so it is a vertex of $R$. Another thing to notice is this case is that all these vertices lie on the same side of $H$, otherwise a vertex joining two of them should cross $H$, which is incompatible with the existence of the puzzle-move.

If $v \in F \cap F'$, then for any $v_Q$ on $Q$, the vertex $(v, v_Q)$ is preserved by the move, whichever side of $H$ $v$ lies, so is also a vertex of $R$.

If $v$ is on $F$, not on $F'$, then there is exactly one vertex $\tilde{v}$ of $P$ that is adjacent to $v$ and not on $F$. We just distinguish two cases:

If $\tilde{v}$ is on $F'$, then for any vertex $v_Q$ of $Q$, the pair of vertices $\{(v, v_Q); (\tilde{v}, v_Q)\}$ is necessarily preserved by the puzzle-move, so both, and in particular $(v, v_Q)$ are vertices of $R$.

If now $\tilde{v}$ is not on $F'$, then, as we have mentioned, all vertices $(\tilde{v}, v_Q)$ are on the same side of $H$. As no edge joining $(v, v_Q)$ to $(v', v_Q)$ can cross $H$ as it neither lies on $F$ nor $F'$, all the vertices of the form $(v, v_Q)$ must be on the same side of $H$. So applying the puzzle-move does not break the product-by-$Q$ structure.

The same argument also being valid for vertices lying on $F' - F$, the polytope obtained after the puzzle-move is actually a product of some polytope $\tilde{P}$ by $Q$.

The end of the proof is easy, we can fix any particular vertex $v_Q$ of $Q$ and we observe that when we consider the only face $P \times \{v_Q\}$ of $P \times Q$, the global puzzle-move induces a puzzle-move from $P$ to $\tilde{P}$ induced by the transposition $(F, F')$.

The proposition is then proved. □

In particular, the product of two puzzle-rigid polytopes is also puzzle-rigid.

Notice that replacing puzzle-rigid or puzzle-equivalent by rigid or $Gr$-equivalent in the previous proposition leads to open questions.

Generalisation? Puzzle-equivalence is defined for embedded polytopes by introducing an hyperplane of the ambient space. Naturally, we can suspect that there exists and would be valuable to produce a more general, purely combinatorial version of this operation that would work on more general combinatorial objects than polytopes, and induce $Gr$-equivalence on corresponding moment-angle manifolds.

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