Integral transform methods in goodness-of-fit testing, I: the gamma distributions

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Abstract
We apply the method of Hankel transforms to develop goodness-of-fit tests for gamma distributions with given shape parameters and unknown rate parameters. We derive the limiting null distribution of the test statistic as an integrated squared Gaussian process, obtain the corresponding covariance operator and oscillation properties of its eigenfunctions, show that the eigenvalues of the operator satisfy an interlacing property, and make applications to two data sets. We prove consistency of the test, provide numerical power comparisons with alternative tests, study the test statistic under several contiguous alternatives, and obtain the asymptotic distribution of the test statistic for gamma alternatives with varying rate or shape parameters and for certain contaminated gamma models. We investigate the approximate Bahadur slope of the test statistic under local alternatives, and we establish the validity of the Wieand condition under which approaches through the approximate Bahadur and the Pitman efficiencies are in accord.

Keywords Bahadur slope · Contaminated model · Contiguous alternative · Gaussian process · Generalized Laguerre polynomial · Goodness-of-fit testing · Hankel transform · Hilbert–Schmidt operator · Lipschitz continuity · Modified Bessel function · Pitman efficiency

Mathematics Subject Classification Primary 33C10 · 62G10; Secondary 62G20 · 62H15

This paper is dedicated to Professor Norbert Henze, on the occasion of his 67th birthday.

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# Introduction

The topic of goodness-of-fit testing has been intensely studied recently. Consequently, there exists a comprehensive body of results developed, by Henze and other authors, using test statistics based on integral transforms of Fourier, Laplace, Mellin, and related types, and making astute use of related differential equations and distributional characterizations. The resulting test statistics have been shown to be superior in various ways to classical goodness-of-fit statistics, notably in comparisons of power, consistency, and in their behavior with respect to contiguous alternatives.

On reviewing the literature on goodness-of-fit tests we were motivated to develop such tests, for multivariate exponential families, based on integral transforms, and the first step in such a program is to derive such results for the classical gamma distributions. In this paper, we apply Hankel transform methods to develop goodness-of-fit tests for gamma distributions with given shape parameter $\alpha$ and unknown rate parameter. We remark that we were particularly fortuitous to have as a constant guide in our investigations the results of Baringhaus and Taherizadeh (2010) and Taherizadeh (2009) for the exponential distributions.

The gamma distributions with known shape parameters arise in queueing theory (Allen 1990), ion channel activation (Kass et al. 2014), the analysis of engineering equipment breakdowns (Czaplicki 2014; Sturgul 2015), the calculation of insurance premiums for maritime commerce (Postan and Poizner 2013), and other areas, and goodness-of-fit tests for these distributions date back to Pettitt (1978). For the case of unknown shape parameter, goodness-of-fit tests based on empirical distribution functions were provided by D’Agostino and Stephens (1986) and numerous other authors; in particular, Henze et al. (2012) developed a test based on the empirical Laplace transform and provided an extensive review of the literature.

Let $X$ be a positive random variable with probability density function (p.d.f.) $f(x)$; also, let $J_\nu$ be the Bessel function of the first kind of order $\nu$, as defined in (2.1). For $\nu \geq -1/2$, the function

$$
\mathcal{H}_{X,\nu}(t) = \Gamma(\nu + 1) \int_0^{\infty} (tx)^{-\nu/2} J_\nu(2(tx)^{1/2}) f(x) \, dx,
$$

(1.1)

$t \geq 0$, is called the Hankel transform of order $\nu$ of $X$. For $X \sim \text{Gamma}(\alpha, 1)$, a gamma distribution with shape parameter $\alpha$ and scale parameter 1, we have $\mathcal{H}_{X,\alpha-1}(t) = e^{-t/\alpha}$.

Let $X_1, \ldots, X_n$ be independent, identically distributed (i.i.d.), positive, continuous random variables with a distribution $\mathcal{P}$. We wish to test the null hypothesis, $H_0 : \mathcal{P} \in \{\text{Gamma}(\alpha, \lambda), \lambda > 0\}$ against the alternative, $H_1 : \mathcal{P} \notin \{\text{Gamma}(\alpha, \lambda), \lambda > 0\}$, where $\alpha$ is known. Since $H_0$ does not specify $\lambda$ then $X_1, \ldots, X_n$ cannot be used directly to conduct the test. Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_j$ be the sample mean and set $Y_j = X_j/\bar{X}_n$, $j = 1, \ldots, n$; under $H_0$, the distribution of $Y_1, \ldots, Y_n$ does not depend on $\lambda$, so we can base a test on them. Let $P_0$ denote the distribution function of the Gamma$(\alpha, 1)$ distribution. We define the empirical Hankel transform of order $\nu$ of $Y_1, \ldots, Y_n$ as
\[ \mathcal{H}_{n,v}(t) = \frac{\Gamma(v + 1)}{n} \sum_{j=1}^{n} (tY_j)^{-v/2} J_v(2\sqrt{tY_j}), \quad (1.2) \]

\( t \geq 0 \), and then the statistic for testing \( H_0 \) against \( H_1 \) is

\[ T^2_{n,\alpha-1} = n \int_0^{\infty} \left[ \mathcal{H}_{n,\alpha-1}(t) - e^{-t/\alpha} \right]^2 dP_0(t). \quad (1.3) \]

As the Hankel transform is one-to-one we will infer from large values of \( T^2_{n} \) that \( \mathcal{H}_{n,\alpha-1}(t) \) differs significantly from \( e^{-t/\alpha} \), hence large values of \( T^2_{n,\alpha-1} \) provide strong evidence against \( H_0 \). Therefore, we will obtain the distribution of \( T^2_{n,\alpha-1} \) and analyze its properties, e.g., consistency, behavior under contiguous alternatives, efficiency, and compare its power with alternative tests.

We now summarize our results. We give in Sect. 2 basic results on the Bessel and related special functions, and some properties and examples of Hankel transforms of some probability distributions. In Sect. 3, we state the limiting null distribution of the statistic \( T^2_n \) as an integral of the square of a centered Gaussian process \( Z \).

We present in Sect. 4 properties of \( S \), the covariance operator corresponding to \( Z \), oscillation properties of the eigenfunctions of \( S \), and interlacing properties of the eigenvalues of \( S \). In Sect. 5, we make applications to two data sets, assert the consistency of the test, and provide numerical power comparisons with the Anderson-Darling and Cramér-von Mises statistics. In Sect. 6, we consider the test statistic under various contiguous alternatives to \( H_0 \). In particular, we state the asymptotic distribution of \( T^2_n \) under gamma alternatives with varying rate or shape parameters and for a class of contaminated gamma models.

In Sect. 7, we present the Bahadur and Pitman efficiency properties of the statistic \( T^2_n \). We investigate the approximate Bahadur slope of \( T^2_n \) under certain local alternatives and establish the validity of the Wieand condition, under which the approaches through the approximate Bahadur efficiency and the Pitman efficiency are in accord.

In Sect. 8, we describe some open problems and directions for future research, while Sects. 9–11 are reserved for proofs or definitions.

### 2 Bessel functions and Hankel transforms

Throughout the paper, all needed results on the classical special functions can be found in the books by Erdélyi et al. (1953) and Olver et al. (2010), and we conform to their notation. Thus,

\[ \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx, \]

\( \text{Re}(\alpha) > 0 \), is the gamma function, and for \( \alpha \in \mathbb{C} \) and \( k \in \mathbb{N}_0 \), the set of nonnegative integers, we will make frequent use of the \textit{rising factorial}, \( (\alpha)_k = \alpha(\alpha+1) \cdots (\alpha+k-1) \).
We write $X \sim \text{Gamma}(\alpha, \lambda)$ whenever a random variable $X$ is gamma-distributed with shape parameter $\alpha > 0$, rate parameter $\lambda > 0$, and p.d.f. $f(x) = \lambda^\alpha x^{\alpha-1} e^{-\lambda x} / \Gamma(\alpha)$, $x > 0$.

For $\nu \in \mathbb{R}$, $-\nu \notin \mathbb{N}$, the **Bessel function of the first kind of order** $\nu$ is

$$J_\nu(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(v + 1 + j)} (z/2)^{2j+v}, \quad z \in \mathbb{C};$$

see Erdélyi et al. (1953, Chapter 7). In particular, the series (2.1) is continuous, converges absolutely for all $z$, and converges uniformly on compact subsets of $\mathbb{C}$.

The modified **Bessel function of the first kind of order** $\nu$ is defined for $-\nu \notin \mathbb{N}$ and $x \in \mathbb{R}$ as

$$I_\nu(x) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(v + 1 + j)} (x/2)^{2j+v},$$

Let $a, b \in \mathbb{R}$, where $-b \notin \mathbb{N}_0$. The **confluent hypergeometric function** is defined as

$$1F_1(a; b; x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j j!} x^j,$$

$x \in \mathbb{R}$. We refer to Olver et al. (2010, Chapter 13) for detailed accounts of this function. Especially, we will make repeated use of **Kummer’s formula**:

$$1F_1(a; b; x) = e^x 1F_1(b - a; b; -x).$$

Let $X$ be a positive random variable with probability density function $f(x)$ and Hankel transform $\mathcal{H}_{X,\nu}$, as defined in (1.1). Then, $\mathcal{H}_{X,\nu}$ satisfies the following properties:

**Lemma 1** For $\nu \geq -1/2$,

(i) $|\mathcal{H}_{X,\nu}(t)| \leq 1$ for all $t \geq 0$.

(ii) $\mathcal{H}_{X,\nu}(0) = 1$.

(iii) $\mathcal{H}_{X,\nu}(t)$ is a continuous function of $t$.

**Example 1** Let $X \sim \text{Gamma}(\alpha, \lambda)$, where $\alpha, \lambda > 0$. For $t \geq 0$, it follows from the definition (1.1) of the Hankel transform that

$$\mathcal{H}_{X,\nu}(t) = \frac{\Gamma(v + 1)}{\Gamma(\alpha)} \lambda^{\alpha} \int_0^{\infty} (tx)^{-\nu/2} J_\nu(2\sqrt{tx}) x^{\alpha-1} e^{-\lambda x} \, dx.$$

Writing $(tx)^{-\nu/2} J_\nu(2\sqrt{tx})$ as a power series and integrating term-by-term, we obtain

$$\mathcal{H}_{X,\nu}(t) = 1F_1(\alpha; \nu + 1; -t/\lambda).$$

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For the case in which \( \nu = \alpha - 1 \), (2.5) reduces to \( \mathcal{H}_{X,\nu}(t) = 1 F_1(\alpha; \alpha; t/\lambda) = e^{-t/\lambda}, \ t \geq 0 \). In particular, if \( \alpha = 1 \), so that \( X \) has an exponential distribution with rate parameter \( \lambda \), then \( \mathcal{H}_{X,0}(t) = e^{-t/\lambda}, \ t \geq 0 \), as shown by Baringhaus and Taherizadeh (2010, Example 2.1).

**Example 2** Let \( Z \sim \text{Gamma}(\alpha, 1) \) independently of a positive random variable \( X \). Then,

\[
\mathcal{H}_{XZ,\nu}(t) = E_X \left[ 1 F_1(\alpha; \nu + 1; -tX) \right],
\]

\( t \geq 0 \). To prove this result, we again apply (1.1), and the independence of \( X \) and \( Z \), obtaining

\[
\mathcal{H}_{XZ,\nu}(t) = E_X E_Z \left[ \Gamma(\nu + 1)(tXZ)^{-\nu/2} J_\nu(2(tXZ)^{1/2}) \right].
\]

Applying Example 1 to calculate the expectation with respect to \( Z \), we obtain

\[
\mathcal{H}_{XZ,\nu}(t) = E_X \left[ 1 F_1(\alpha; \nu + 1; -tX) \right].
\]

In particular, if \( \nu = \alpha - 1 \) then \( \mathcal{H}_{XZ,\nu}(t) = E_X \left[ e^{-tX} \right] \), the Laplace transform of \( X \), a result shown for \( \nu = 0 \) in Baringhaus and Taherizadeh (2010, Example 2.2).

The following example, which provides the Hankel transform of a function related to the gamma density, will be needed repeatedly in the sequel.

**Example 3** Suppose that \( X \sim \text{Gamma}(\alpha, 1) \). Then, for \( t \geq 0 \),

\[
E \left[ \left( tX/\alpha \right)^{1-(\alpha/2)} J_\alpha \left( 2(tX/\alpha)^{1/2} \right) \right] = \frac{1}{\Gamma(\alpha + 1)} t^{\frac{1}{2}} e^{-t/\alpha}. \tag{2.6}
\]

Here again, we write \( (tX/\alpha)^{1-(\alpha/2)} J_\alpha \left( 2(tX/\alpha)^{1/2} \right) \) as a power series in \( tX/\alpha \), integrate term-by-term, and simplify the resulting series to obtain (2.6).

The next result constitutes a characterization of the gamma distributions using Hankel transforms of arbitrary order \( \nu \), where \( \nu \geq -1/2 \). The result allows extension to the gamma case the results of Baringhaus and Taherizadeh (2013) on a supremum norm test statistic.

**Theorem 1** Let \( X \) be a positive random variable with Hankel transform \( \mathcal{H}_{X,\nu} \). If there exist \( \epsilon > 0 \) and \( \alpha > 0 \) such that \( \mathcal{H}_{X,\nu}(t) = 1 F_1(\alpha; \nu + 1; -t) \) for all \( t \in [0, \epsilon] \), then \( X \sim \text{Gamma}(\alpha, 1) \).

We refer to Hadjicosta (2019) for three proofs of this result.
3 The distribution of the test statistic

Let $X_1, \ldots, X_n$ be i.i.d., positive, continuous random variables with distribution $\mathcal{P}$. We wish to test the null hypothesis, $H_0 : \mathcal{P} \in \{\text{Gamma}(\alpha, \lambda), \lambda > 0\}$ against the alternative hypothesis, $H_1 : \mathcal{P} \notin \{\text{Gamma}(\alpha, \lambda), \lambda > 0\}$, where $\alpha$ is known. Using the empirical Hankel transform $\mathcal{H}_{n,v}$ given in (1.2), we define the test statistic

$$T^2_{n,v} = n \int_0^\infty \left[ \mathcal{H}_{n,v}(t) - 1 F_1(\alpha; v + 1; -t/\alpha) \right]^2 dP_0(t). \tag{3.1}$$

Under $H_0$, $E(X_1) = \alpha/\lambda$ and, for large $n$, $Y_j = X_j/X_n \simeq \lambda X_j/\alpha$, almost surely. By the Continuous Mapping Theorem (Billingsley 1968, p. 31), for each $t \geq 0$ and for sufficiently large $n$, the sequence of random variables $(tY_j)^{-v/2} J_v(2\sqrt{tY_j})$, $j = 1, \ldots, n$, approximates the i.i.d. sequence $(tX_j)^{-v/2} J_v(2(tX_j)^{1/2})$, $j = 1, \ldots, n$. Applying to (1.2) the Strong Law of Large Numbers we obtain, for large $n$, $\mathcal{H}_{n,v}(t) \simeq \mathcal{H}_{X_1,v}(\lambda t/\alpha)$, almost surely. By Example 1 and the Hankel Uniqueness Theorem 12, $\mathcal{H}_{X_1,v}(\lambda t/\alpha) \approx 1 F_1(\alpha; v + 1; -t/\alpha)$, $t \geq 0$, if and only if $H_0$ is valid. Therefore, large values of $T^2_{n,v}$ provide strong evidence against $H_0$.

We also remark that since the family of gamma distributions is scale-invariant then the test statistic, as a function of $X_1, \ldots, X_n$, should satisfy the same property. Since $Y_1, \ldots, Y_n$ clearly are scale-invariant in $X_1, \ldots, X_n$ then the same holds for $T^2_{n,v}$.

Henceforth, we set $v = \alpha - 1$; since $v \geq -1/2$ then $\alpha \geq 1/2$. We also denote $T^2_{n,\alpha-1}$ and $\mathcal{H}_{n,\alpha-1}$ by $T^2_n$ and $\mathcal{H}_n$, respectively. By Kummer’s formula (2.4), the statistic (3.1) reduces to (1.3).

We now evaluate the test statistic $T^2_n$ for a given random sample.

**Proposition 1** The test statistic (1.3) is a $V$-statistic of order 2. Specifically,

$$T^2_n = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n h(Y_i, Y_j)$$

where, for $x, y \geq 0$,

$$h(x, y) = \Gamma(\alpha) (xy)^{(1-\alpha)/2} \exp(-x - y) I_{\alpha-1}(2(xy)^{1/2})$$

$$- \left( \frac{\alpha}{\alpha + 1} \right)^\alpha \left[ \exp \left( - \frac{\alpha x}{\alpha + 1} \right) + \exp \left( - \frac{\alpha y}{\alpha + 1} \right) \right] + \left( \frac{\alpha}{\alpha + 2} \right)^\alpha. \tag{3.2}$$

Denote by $L^2 = L^2(P_0)$ the space of (equivalence classes of) Borel measurable functions $f : [0, \infty) \to \mathbb{C}$ that are square-integrable with respect to $P_0$, i.e. $\int_0^\infty |f(t)|^2 dP_0(t) < \infty$. The space $L^2$ is a separable Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{L^2} = \int_0^\infty f(t) \overline{g(t)} dP_0(t).$$
and the corresponding norm, $\|f\|_{L^2} = \langle f, f \rangle_{L^2}^{1/2}$, $f, g \in L^2$. Moreover, it is well-known that the normalized Laguerre polynomials $\{L_n^{(\alpha-1)} : n \in \mathbb{N}_0\}$, defined in Appendix 1, form an orthonormal basis, i.e. a complete orthonormal system, for $L^2$; see Szegö (1967, Chapter 5.7).

Define the stochastic process

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ \Gamma(\alpha)(tY_j)^{(1-\alpha)/2} J_{\alpha-1}(2\sqrt{tY_j}) - e^{-t/\alpha} \right],$$

(3.3)

$t \geq 0$. We will view $Z_n := \{Z_n(t), t \geq 0\}$ as a random element in $L^2$ since, as we will observe in Lemma 2 below, its sample paths are in $L^2$. The proof of the following result follows directly from the definition (1.3) of the statistic $T_n^2$ and the observation that $n^{1/2}[H_n(t) - e^{-t/\alpha}] = Z_n(t)$.

**Lemma 2** The test statistic (1.3) can be written as

$$T_n^2 = \int_0^\infty (Z_n(t))^2 \, dP_0(t) = \|Z_n\|_{L^2}^2.$$

In particular, $\|Z_n\|_{L^2}^2 < \infty$.

It is well-known that under $H_0$, $(Y_1, \ldots, Y_n)$ has a Dirichlet distribution which does not depend on $\lambda$. Therefore, without loss of generality, we will set $\lambda = 1$ in deriving the null distribution of $T_n^2$.

**Theorem 2** Let $X_1, X_2, \ldots$ be i.i.d. Gamma$(\alpha, 1)$ random variables, where $\alpha \geq 1/2$, and let $Z_n := \{Z_n(t), t \geq 0\}$ be the stochastic process defined in (3.3). Then there exists a centered Gaussian process $Z := \{Z(t), t \geq 0\}$, with sample paths in $L^2$ and with covariance function,

$$K(s, t) = e^{-(s+t)/\alpha} \left[ \Gamma(\alpha)(st/\alpha^2)^{(1-\alpha)/2} I_{\alpha-1}(2\sqrt{st/\alpha}) - \alpha^{-3}st - 1 \right],$$

(3.4)

$s, t \geq 0$, such that $Z_n \overset{d}{\to} Z$ in $L^2$ as $n \to \infty$. Moreover,

$$T_n^2 \overset{d}{\to} \int_0^\infty [Z(t)]^2 \, dP_0(t).$$

**Remark 1** The proof of Theorem 2 is by an approach similar to that of Baringhaus and Taherizadeh (2010) and is given in Sect. 10. As $Y_1, \ldots, Y_n$ are not independent, we cannot directly apply a Central Limit Theorem to deduce that $Z_n \to Z$. Instead, we apply a standard method of constructing auxiliary processes, $Z_{n,1}, Z_{n,2},$ and $Z_{n,3}$, and then decomposing $Z_n - Z$ into a sum of four parts, viz.,

$$Z_n - Z = (Z_n - Z_{n,1}) + (Z_{n,1} - Z_{n,2}) + (Z_{n,2} - Z_{n,3}) + (Z_{n,3} - Z).$$
Next, we show that $Z_n - Z_{n,1} - Z_{n,2} - Z_{n,3}$ each converge to 0 in probability, in $L^2$; then we apply a Central Limit Theorem to deduce that $Z_{n,3} \overset{d}{\to} Z$ in $L^2$, and so we obtain $Z_n \overset{d}{\to} Z$ in $L^2$. Finally, we apply the Continuous Mapping Theorem to conclude that $\|Z_n\|_{L^2}^2 \overset{d}{\to} \|Z\|_{L^2}^2$.

4 Eigenvalues and eigenfunctions of the covariance operator

The covariance operator $S : L^2 \to L^2$ of the random element $Z$ is defined for $s \geq 0$ and $f \in L^2$ by

$$Sf(s) = \int_0^\infty K(s, t) f(t) \, dP_0(t),$$

where $K(s, t)$ is the covariance function defined in Eq. (3.4). Let $\{\delta_k : k \geq 1\}$ be the positive eigenvalues, listed in non-increasing order, of $S$; also, let $\{\chi_{1k}^2 : k \geq 1\}$ be i.i.d. $\chi_1^2$ random variables. It follows from the Karhunen-Loève expansion of the Gaussian process $Z(t)$ that the integrated squared process, $\int_0^\infty Z^2(t) \, dP_0(t)$, has the same distribution as $\sum_{k=1}^\infty \delta_k \chi_{1k}^2$; see Le Maître and Knio (2010, Chapter 2). Therefore, under $H_0$, $T_n^2 \to \sum_{k=1}^\infty \delta_k \chi_{1k}^2$.

For $s, t \geq 0$, let

$$K_0(s, t) = e^{-(s+t)/\alpha} \Gamma(\alpha)(st/\alpha)^{(1-\alpha)/2} I_{\alpha-1}(2\sqrt{st/\alpha}),$$

the first term in the covariance function defined in Eq. (3.4); by (10.12),

$$K_0(s, t) = \int_0^\infty \Gamma(\alpha)(sx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(sx/\alpha)^{1/2}) \times \Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2}) \, dP_0(x).$$

We will find first the eigenvalues and eigenfunctions of the integral operator $S_0 : L^2 \to L^2$, defined for $s \geq 0$ and $f$ in $L^2$ by

$$S_0 f(s) = \int_0^\infty K_0(s, t) f(t) \, dP_0(t).$$

Before presenting the results on the eigenvalues and eigenfunctions of $S_0$, we state for the sake of completeness some preliminary definitions pertaining to (linear) operators on $L^2$. Note that these definitions are provided by Sunder (2015) or Young (1998).

An operator $T : L^2 \to L^2$ is called symmetric (self-adjoint) if, for all $f, g \in L^2$, $\langle Tf, g \rangle_{L^2} = \langle f, Tg \rangle_{L^2}$. A symmetric operator $T$ is called positive if $\langle Tf, f \rangle_{L^2} \geq 0$ for all $f \in L^2$. An operator $T$ is called compact if for every bounded sequence $\{f_k : k \in \mathbb{N}\}$ in $L^2$, the sequence $\{T f_k : k \in \mathbb{N}\}$ has a convergent subsequence in $L^2$. The set of eigenvalues of a compact operator is countable.
An operator $T$ is Hilbert–Schmidt if for every orthonormal basis $\{f_k : k \in \mathbb{N}\}$ in $L^2$, the series $\sum_{k=1}^{\infty} \|T f_k\|_{L^2}^2$ converges. Each Hilbert–Schmidt operator is compact (Young 1998, p. 93).

An operator $T$ is of trace class if for every orthonormal basis $\{f_k : k \in \mathbb{N}\}$ in $L^2$, the series $\sum_{k=1}^{\infty} \|T f_k\|_{L^2}$ converges. An operator $T$ is trace-class if and only if it is a product of two Hilbert–Schmidt operators (Sunder 2015, p. 74). Further, trace-class operators are Hilbert–Schmidt.

Recall that $\alpha \geq 1/2$. Throughout the remainder of the paper, we use the notation

$$\beta = \left(\frac{\alpha + 4}{\alpha}\right)^{1/2} \quad \text{and} \quad b_\alpha = \left(1 + \frac{1}{2}\alpha(1 - \beta)\right)^{1/2}. \quad (4.2)$$

We also set

$$\rho_k = \alpha \beta^{4k + 2\alpha}, \quad (4.3)$$

$k \in \mathbb{N}_0$, and for $s \geq 0$,

$$L_{k}^{(\alpha - 1)}(s) = \beta^{\alpha/2} \exp((1 - \beta)s/2)L_{k}^{(\alpha - 1)}(\beta s), \quad (4.4)$$

where $L_{k}^{(\alpha - 1)}(s)$ is the generalized Laguerre polynomial defined in (9.11).

**Theorem 3** The set $\{(\rho_k, L_{k}^{(\alpha - 1)}) : k \in \mathbb{N}_0\}$ is a complete enumeration of the eigenvalues and eigenfunctions, respectively, of $S_0$, and the eigenfunctions $\{L_{k}^{(\alpha - 1)} : k \in \mathbb{N}_0\}$ form an orthonormal basis in $L^2$. Moreover, $S_0$ is positive and of trace-class.

For the proof of this result we refer to Hadjicosta (2019) or Hadjicosta and Richards (2018).

**Theorem 4** Let $S : L^2 \to L^2$ be the covariance operator of the random element $Z$ defined as

$$Sf(s) = \int_0^\infty K(s, t) f(t) \, dP_0(t),$$

for all $s \geq 0$ and for all functions $f$ in $L^2$, where $K(s, t)$ is the covariance function defined in Eq. (3.4). Then, $S$ is positive and of trace-class.

The proof of this result is similar to the proof of Theorem 3, and the complete details are provided by Hadjicosta (2019).

Recall that a non-trivial function $f \in L^2$ is an eigenfunction of $S$ if there exists an eigenvalue $\delta \in \mathbb{C}$ such that $Sf = \delta f$. As $S$ is self-adjoint and positive, its eigenvalues are real and nonnegative. In the next result, whose proof is given in Sect. 11, we find the positive eigenvalues and corresponding eigenfunctions of $S$, and we will also show that 0 is not an eigenvalue of $S$. 

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Theorem 5 For $\delta \in \mathbb{R}$, $\delta \neq \rho_k$ for any $k \in \mathbb{N}$, define the functions
\[
A(\delta) = 1 - \beta^{\alpha} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2,
\]
\[
B(\delta) = 1 - \alpha \beta^{\alpha} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta)^2,
\]
and
\[
D(\delta) = \alpha^2 \beta^{\alpha} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta).
\]

Then the positive eigenvalues of $S$ are the positive roots of the function $G(\delta) := \alpha^3 A(\delta) B(\delta) - D^2(\delta)$. Also, the eigenfunction corresponding to an eigenvalue $\delta$ has the Fourier-Laguerre expansion
\[
\beta^{\alpha/2} \sum_{k=0}^{\infty} \frac{\rho_k}{\rho_k - \delta} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \left( c_1 + c_2 \alpha^{-1} (b_\alpha^2 - k\beta) \right) \mathcal{L}_k^{(\alpha-1)},
\]
where $c_1, c_2$ are not both equal to 0, $\alpha^3 c_1 A(\delta) = c_2 D(\delta)$, and $c_2 B(\delta) = c_1 D(\delta)$.

In the previous result, we assumed that $\delta \notin \{ \rho_k : k \in \mathbb{N}_0 \}$. As stated in the following conjecture, we believe that this assumption is valid for all $\alpha$.

Conjecture 1 For $\delta$ an eigenvalue of the operator $S$, there is no $l \in \mathbb{N}_0$ such that $\delta = \rho_l$.

Conjecture 2 There is no $l \in \mathbb{N}_0$ such that
\[
\alpha \beta^{\alpha+2} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{\rho_k^2}{\rho_k - \rho_l} (l - k)^2 = 1 + \alpha (b_\alpha^2 - l\beta)^2. \tag{4.5}
\]

We will show in Appendix 3 that Conjecture 2 implies Conjecture 1.

Remark 2 Since $b_\alpha < 1$ then $\rho_k < \rho_0$ for all $k \geq 1$. Therefore, if $l = 0$ then each term in the sum on the left-hand side of (4.5) is negative, hence the sum itself is negative. On the other hand, the right-hand side clearly is positive. Therefore, the conjecture is valid if $l = 0$.

Conjecture 1 was proved by Taherizadeh (2009) for $\alpha = 1$ and by Hadjicosta (2019) for $\alpha = 2$. In both cases, the left-hand side of (4.5) was shown to exceed the right-hand side, so we conjecture that the same holds for all $\alpha$. We have found that the method of proof for $\alpha = 1, 2$ extends to all integer $\alpha \leq 10$, however the method is inapplicable for integer $\alpha \geq 11$ or for non-integral $\alpha$. 

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A difficulty of the eigenvalues $\delta_k$ is that they have no closed form expression; hence there is no simple formula for $m$, the number of terms in the truncated series \[ \sum_{k=1}^{m} \delta_k \chi_{1k}^2 \] that should be used in practice to approximate the asymptotic distribution, \[ \sum_{k=1}^{\infty} \delta_k \chi_{1k}^2, \] of the test statistic $T_n^2$.

For $\alpha = 1$, Baringhaus and Taherizadeh (2010) calculated several $\delta_k$ numerically and found that the truncated sum $\sum_{k=1}^{10} \delta_k$ closely approximates the exact value of $Tr(S)$; hence, the distribution of the truncated sum, $\sum_{k=1}^{10} \delta_k \chi_{1k}^2$ is a good approximation to the asymptotic distribution, $\sum_{k=1}^{\infty} \delta_k \chi_{1k}^2$, of $T_n^2$. This approach is feasible since, as $S_0$ is of trace-class then by Brislawn (1991, p. 237, Corollary 3.2), $Tr(S_0)$ can be calculated by integrating the kernel $K_0$ or by evaluating the sum of all eigenvalues $\rho_k$:

\[ \int_0^\infty K_0(s, s) \, dP_0(s) = Tr(S_0) = \sum_{k=0}^{\infty} \rho_k = \alpha^2 b_\alpha^{-2} (1 - b_\alpha^{-4})^{-1}. \tag{4.6} \]

Since $S$ also is of trace-class then, using (3.4), we obtain

\[ \sum_{k=1}^{\infty} \delta_k = Tr(S) = \int_0^\infty K(s, s) \, dP_0(s) = \alpha^2 \left[ \frac{b_\alpha^2}{1 - b_\alpha^4} - \frac{1}{(\alpha + 2)^\alpha} \left( 1 + \frac{(\alpha + 1)}{(\alpha + 2)^2} \right) \right]. \tag{4.7} \]

To determine for general $\alpha$ the number of terms in the truncated series $\sum_{k=1}^{m} \delta_k \chi_{1k}^2$ that should be used in practice to approximate the asymptotic distribution of $T_n^2$, we derive bounds for the $\delta_k$ in terms of the $\rho_k$ and then obtain a general formula for $m$ as a function of $\alpha$. In this regard, we are reminded of the concept of a “scree plot” in principal component analysis; see Johnson and Wichern (1998, p. 441), so we refer to the ratio $(\sum_{k=1}^{m} \delta_k)/Tr(S)$ as the $m$th scree ratio for $T_n^2$.

Since $S$ is compact and positive then the set of all its eigenvalues is countable and contains only nonnegative values (Young 1998, p. 98, Theorem 8.12). To prove that the eigenvalues are positive and also are simple, i.e., of multiplicity 1, we will apply the theory of total positivity; see Karlin (1964). In what follows, we denote by $\det(a_{ij})$ the $r \times r$ determinant with $(i, j)$th entry $a_{ij}$.

**Proposition 2** The eigenvalues $\{\delta_k : k \geq 1\}$ of $S$ and the eigenvalues $\{\rho_k : k \geq 0\}$ of $S_0$ are positive and simple. In particular, $S$ and $S_0$ are injective. Further, the corresponding eigenfunctions $\{\phi_k : k \geq 1\}$ of $S$ satisfy the oscillation property,

\[ (-1)^{r(r-1)/2} \det(\phi_i(s_j)) \geq 0 \tag{4.8} \]

for all $r \geq 1$ and $0 \leq s_1 < \cdots < s_r < \infty$, and the same property holds for the eigenfunctions $\{\xi_k : k \geq 0\}$ of $S_0$.

We now state an interlacing property of the eigenvalues $\delta_k$ and $\rho_k$.

**Proposition 3** For $k \geq 1$, $\rho_{k-1} \geq \delta_k \geq \rho_{k+1}$. In particular, $\delta_k = O(\rho_k)$ as $k \to \infty$. 
Table 1  Values of the lower bound on $m$ for the scree ratio of $T_n^2$

| $\alpha$  | 0.5 | 0.75 | 1  | 3  | 5  | 10 | 20 | 50 |
|-----------|-----|------|----|----|----|----|----|----|
| $m$       | 15  | 12   | 10 | 6  | 4  | 3  | 2  | 1  |

Remark 3  The preceding result yields the inequalities $\rho_0 \geq \delta_1 \geq \rho_2 \geq \delta_3 \geq \cdots$ and $\rho_1 \geq \delta_2 \geq \rho_3 \geq \delta_4 \geq \cdots$. For the case in which $\alpha = 1$, we have observed from the tables of eigenvalues computed by Taherizadeh (2009, p. 28, 54) that the eigenvalues $\rho_k$ and $\delta_k$ satisfy the stronger, strict interlacing property, $\rho_k > \delta_k > \rho_{k+1}$ for all $k \geq 1$, and we therefore conjecture that the strict interlacing property holds for general $\alpha$. We have not been able to resolve this conjecture using general Hilbert space operator-theoretic methods or using specific properties of the Bessel functions, and it appears that more powerful methods are needed to resolve the problem.

There is also the issue of choosing the value of $m$ so that the $m$th scree ratio of $T_n^2$ exceeds $1 - \epsilon$, where $0 < \epsilon < 1$. Applying the interlacing inequalities for $\delta_k$, we obtain $\sum_{k=1}^{m} \delta_k \geq \sum_{k=2}^{m+1} \rho_k$. Since $Tr(S_0) > Tr(S)$, we advise that $m$ be chosen so that

$$
\sum_{k=0}^{m+1} \rho_k \geq (1 - \epsilon)Tr(S_0).
$$

This leads to a value for $m$ that is readily applicable. Substituting $\rho_k = \alpha^a b_{\alpha}^{4k+2a}$, evaluating in closed form the resulting geometric series, and substituting for $Tr(S_0)$ from (4.6), we obtain

$$
\alpha^a b_{\alpha}^{2a} \frac{1 - b_{\alpha}^{4(m+2)}}{1 - b_{\alpha}^{4}} = \sum_{k=0}^{m+1} \rho_k \geq (1 - \epsilon)Tr(S_0) = (1 - \epsilon)\alpha^a b_{\alpha}^{2a} \frac{1}{1 - b_{\alpha}^{4}}.
$$

Solving this inequality for $m$, we obtain

$$
m \geq \log \epsilon \frac{4}{4 \log b_{\alpha}} - 2.
$$

(4.9)

We illustrate this bound by calculating it for various values of $\alpha$. For $\epsilon = 10^{-10}$, which represents accuracy to ten decimal places, this results in the values displayed in Table 1.

5 Applications, consistency of the test, and numerical power calculations

The first data set (Hogg and Tanis 2009, p. 155) consists of $n = 25$ waiting times (in seconds) for a Geiger counter to observe 100 alpha-particles emitted by barium-133. As noted by Hogg and Tanis (2009, p. 464), a Kolmogorov–Smirnov test that the data
were drawn from a Gamma(α = 100, λ = 14.7) distribution failed to reject that hypothesis at the 10% level of significance.

We apply the statistic $T_n^2$ to test $H_0$, the null hypothesis that the data are drawn from a gamma distribution with $α = 100$ and unspecified $λ$. The observed value of $T_n^2$ is $6.301 \times 10^{-10}$.

We used the limiting null distribution of $T_n^2$ to estimate $T_{n;0.05}^2$. For $α = 100$, it follows from Table 1 that only one eigenvalue is needed to approximate accurately the asymptotic distribution of $T_n^2$; therefore, $T_n^2 \approx \delta_1 \chi^2_1$. By (4.7), we obtain $\delta_1 \approx T r(S) = 6.722 \times 10^{-6}$. Therefore, $T_{n;0.05}^2 \approx \delta_1 \chi^2_{1.0.05}$, where $\chi^2_{1.0.05}$ is the 95th percentile of the $\chi^2_1$ distribution, so we obtain $T_{n;0.05}^2 = 2.582 \times 10^{-5}$. As this critical value exceeds the observed value of $T_n^2$, we fail to reject the null hypothesis that the waiting times are drawn from a Gamma($α = 100, λ$) distribution.

As an alternative approach, we conducted a simulation study to approximate $T_{n;0.05}^2$, the 95th percentile of the null distribution of $T_n^2$. We generated 10,000 random samples of size $n = 25$ from the Gamma(100, 1) distribution, calculated the value of $T_n^2$ for each sample, and recorded the 95th percentile of all 10,000 simulated values of $T_n^2$. We repeated this process ten times, finally approximating $T_{n;0.05}^2$ as the 20%-trimmed mean of all 10 simulated 95th percentiles, viz., $T_{n;0.05}^2 = 2.368 \times 10^{-5}$. Since this critical value exceeds the observed value of $T_n^2$ then we fail to reject the null hypothesis at the 5% level of significance. Moreover, we derived from our simulation study an approximate P-value of 0.99 for the test.

The second data set, given by Barlow and Campo (1975), provides $n = 107$ failure times (in hours) for the right rear brakes on a sample of tractors. The data were analyzed recently by Cuparić et al. (2018), where the null hypothesis of exponentiality was rejected.

To test the hypothesis that the data are drawn from a gamma-distributed population, we assume for illustrative purposes that $α = 2.3$. This value was obtained by setting the maximum likelihood estimate of the mode of the Gamma($α, λ$) density, viz., $(α - 1)\bar{X}_n/α$, equal to a mode of the histogram, and solving the resulting equation for $α$. Then the observed value of $T_n^2$ is 0.0053.

For $α = 2.3$, it follows from (4.9) that $T_n^2 \approx \sum_{k=1}^{7} \delta_k \chi^2_{1k}$. We calculated the $δ_k$ numerically as the positive roots of the function $G(δ)$ in Theorem 5, and then we applied the results of Imhof (1961) or Kotz et al. (1967) to derive the distribution of $\sum_{k=1}^{7} \delta_k \chi^2_{1k}$ and carry out the test. A one-term approximation (Kotz et al. 1967, Eqs. (71), (79)),

$$P\left(\sum_{k=1}^{m} \delta_k \chi^2_{1k} \geq t\right) \approx P(\chi^2_{m} \geq 2t/(\delta_1 + \delta_m))$$

is well-known to be accurate for other problems (see Gupta and Richards (1983)) and leads to an explicit formula, $T_{n;0.05}^2 \approx 1/2(\delta_1 + \delta_m)\chi^2_{m;0.05}$, for an approximate critical value of $T_n^2$.

As an alternative to calculating $δ_1,\ldots, δ_M$, we can apply the interlacing inequalities in Proposition 3 to obtain a stochastic upper bound, $\sum_{k=1}^{M} \delta_k \chi^2_{1k} \leq \sum_{k=0}^{M-1} \rho_k \chi^2_{1k}$. By applying results of Kotz et al. (1967, loc. cit.) or Imhof (1961) to approximate the
critical values of this upper bound, we obtain a conservative test of $H_0$, i.e., with a level of significance at most 5%.

By performing a simulation study as for the previous data set, we obtained the approximation, $T_{n; 0.05}^2 = 0.0356$, which exceeds the observed value of the test statistic. Also, by applying the results of Imhof (1961), the limiting critical value derived from $\sum_{k=1}^{7} \delta_k X_k^{1/2}$, denoted by $T_{\infty; 0.05}^2$, equals 0.0376. Therefore, we fail to reject the null hypothesis at the 5% level of significance. Moreover, the simulation study provided an approximate P-value of 0.47.

Since it was assumed for illustrative purposes that $\alpha = 2.3$, we repeated the test for several values of $\alpha$, obtaining the results in Table 2. We note that the null hypothesis is rejected for the case in which $\alpha = 1$ where, under the null hypothesis, the data are exponentially distributed; hence, we deduce in this case the same conclusion as Cuparić et al. (2018).

With regard to the consistency of the test statistic, we now provide a result that the test is consistent against any fixed alternative distribution.

**Theorem 6** Let $X_1, X_2, \ldots$ be a sequence of positive, i.i.d., random variables with finite mean $\mu$. Let $\gamma \in (0, 1)$ denote the level of significance of the test and $c_{n, \gamma}$ be the $(1 - \gamma)$-quantile of the test statistic $T_n^2$ under $H_0$. If $X_1, X_2, \ldots$ are not Gamma-distributed then

$$\lim_{n \to \infty} P(T_n^2 > c_{n, \gamma}) = 1.$$  

With regard to a proof of this result, if we define

$$\Lambda := \int_0^\infty \left( E[\Gamma(\alpha)(tX_1/\mu)^{(1-\alpha)/2}J_{\alpha-1}(2(tX_1/\mu)^{1/2})] - e^{-t/\alpha} \right)^2 dP_0(t),$$

then the essential part of the proof is to establish that $n^{-1} T_n^2 \overset{p}{\to} \Lambda$. The extensive details required to prove this limit are provided by Hadjicosta (2019) or Hadjicosta and Richards (2018).

**Remark 4** By applying Theorem 1 of Baringhaus et al. (2017) we also find that, under fixed alternatives to the null hypothesis, $n^{1/2}(n^{-1} T_n^2 - \Lambda) \overset{d}{\to} N(0, \sigma^2)$ as $n \to \infty$, where $\sigma^2$ is a constant that is determined from the alternative distribution.

---

**Table 2** The outcome of testing the tractor brakes data with numerous values of $\alpha$

| $\alpha$ | 1.0 | 1.8 | 2.3 | 3 | 5 | 8 |
|---|---|---|---|---|---|---|
| Observed $T_n^2$ | 0.6965 | 0.0162 | 0.0053 | 0.0534 | 0.1977 | 0.3180 |
| $T_{n; 0.05}^2$ | 0.1406 | 0.0559 | 0.0356 | 0.0228 | 0.0088 | 0.0037 |
| $T_{\infty; 0.05}^2$ | 0.1420 | 0.0576 | 0.0376 | 0.0235 | 0.0091 | 0.0037 |
| $P$ value | 0.0000 | 0.3013 | 0.4702 | 0.0026 | 0.0000 | 0.0000 |
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Table 3 Critical values of $T_n^2$ for $\alpha = 2, 5, 10$

| n   | $\alpha = 2$  | $\alpha = 5$  | $\alpha = 10$ |
|-----|---------------|---------------|---------------|
|     | $1 - \gamma$ | $1 - \gamma$ | $1 - \gamma$ |
| 20  | 0.032         | 0.006         | 0.002         |
|     | 0.044         | 0.008         | 0.003         |
|     | 0.056         | 0.011         | 0.004         |
|     | 0.075         | 0.015         |               |
| 50  | 0.033         | 0.006         | 0.002         |
|     | 0.046         | 0.009         | 0.003         |
|     | 0.059         | 0.011         | 0.004         |
|     | 0.078         | 0.015         |               |
| 80  | 0.033         | 0.006         | 0.002         |
|     | 0.046         | 0.009         | 0.003         |
|     | 0.060         | 0.011         | 0.004         |
|     | 0.079         | 0.015         |               |
| 100 | 0.033         | 0.006         | 0.002         |
|     | 0.046         | 0.009         | 0.003         |
|     | 0.060         | 0.012         | 0.004         |
|     | 0.079         | 0.016         |               |
| $\infty$ | 0.048       | 0.006         | 0.002         |
|     | 0.063         | 0.009         | 0.003         |
|     | 0.080         | 0.012         | 0.004         |
|     |               |               |               |

Turning to numerical calculations of the power of the test, we provide in Table 3 simulated critical values for various $n$ and four levels of significance, denoted by $\gamma$, for $\alpha = 2, 5, 10$. The last row of Table 3 is derived using the approximate limiting null distribution $\sum_{k=1}^7 \delta_k \chi_k^2$ and the method of Imhof (1961) for calculating the distribution of such linear combinations. The eigenvalues $\{\delta_k\}$ are calculated numerically, by applying the results of Theorem 5, using the Newton–Raphson method in the software R (R Development Core Team 2007). All other entries in Table 3 are calculated as the 20%-trimmed mean of 10 simulated $(1 - \gamma)$-percentiles, each based on 10000 replications. The values in the table indicate that, as $\alpha$ increases, the critical points converge more rapidly; in particular, for $\alpha = 10$ and $n \geq 20$, the 90th and higher percentiles equal, to three decimal places, the limiting percentiles.

Next, we compare the power of the new test with the Cramér-von Mises ($C^2$) and Anderson-Darling ($A^2$) tests. We conducted a Monte Carlo study with 5000 replications at a 5% significance level for $n = 20, 50$. The critical values of $C^2$ and $A^2$ are calculated in the same way as for $T_n^2$, viz., as the 20%-trimmed mean of 10 simulated 95%-percentiles, each based on 10000 replications. In Tables 4, 5, and 6, we present for $\alpha = 2, 5, 10$ the percentage points of 5000 Monte Carlo samples found to be significant. An asterisk denotes a power of 100%, and we list in boldface the most powerful test in each case. For $\theta > 0$ and $x > 0$, the alternative distributions considered are the:

- Gamma$(\alpha)$: Gamma distribution with shape parameter $\alpha$ and rate parameter 1.
- Weibull$(\theta)$: Weibull distribution with density function $\theta x^{\theta - 1} \exp(-x^\theta)$.
- LIFR$(\theta)$: Linear Increasing Failure Rate distribution with density function $(1 + \theta x) \exp(-x - \frac{1}{2} \theta x^2)$.
- LN$(\theta)$: Lognormal distribution with density function $(\theta x)^{-1} (2\pi)^{-1/2} \exp(-\log x)^2 / 2\theta^2$.
- IG$(\theta)$: Inverse Gaussian distribution with density function $(\theta/2\pi x^3)^{1/2} \exp(-\theta(x-1)^2/2x)$.
- GO$(\theta)$: Gompertz distribution with density function $\theta e^{x+\theta} \exp(-\theta e^x)$.
- Rayleigh$(\theta)$: Rayleigh distribution with density function $(x/\theta) \exp(-x^2/2\theta)$.

These distributions were chosen from among numerous alternatives for which calculations were done by several authors, e.g., Baringhaus and Taherizadeh (2010), Henze et al. (2012), and Taherizadeh (2009).
Table 4  Percentage points of 5000 Monte Carlo samples found to be significant at 5% level of significance; $\alpha = 2$

| Distribution | $T^2_n$ | $C^2$ | $A^2$ |
|--------------|---------|-------|-------|
| $n = 20$     |         |       |       |
| Gamma(1)     | 62      | 48    | 64    |
| Gamma(1.5)   | 18      | 12    | 18    |
| Gamma(2)     | 5       | 5     | 5     |
| Gamma(2.5)   | 9       | 8     | 7     |
| Gamma(3)     | 19      | 18    | 13    |
| Gamma(3.5)   | 33      | 29    | 26    |
| Gamma(4)     | 48      | 42    | 40    |
| Weibull(1)   | 62      | 47    | 67    |
| Weibull(2)   | 27      | 24    |       |
| Weibull(2.5) | 73      | 64    |       |
| Weibull(3)   | 96      |       | 91    |
| LIFR(0.02)   | 61      | 58    | 64    |
| LIFR(0.05)   | 56      | 42    | 62    |
| LIFR(0.1)    | 53      | 36    | 58    |
| LN(0.8)      | 21      | 20    |       |
| LN(0.9)      | 39      | 37    |       |
| LN(1)        | 60      | 55    |       |
| LN(1.5)      | 98      | 97    |       |
| IG(0.5)      | 81      | 78    |       |
| IG(1)        | 32      | 30    |       |
| IG(1.5)      | 10      | 10    |       |
| IG(3)        | 31      | 29    |       |
| IG(3.5)      | 47      | 43    |       |
| IG(4)        | 61      | 57    |       |
| GO(2)        | 24      | 35    |       |
| GO(4)        | 38      | 47    |       |
| Rayleigh(1)  | 27      | 22    |       |
| $n = 50$     |         |       |       |
| Gamma(1)     | 93      | 94    |       |
| Gamma(1.5)   | 33      | 33    |       |
| Gamma(2)     | 5       | 5     |       |
| Gamma(2.5)   | 16      | 13    |       |
| Gamma(3)     | 46      | 39    |       |
| Gamma(3.5)   | 77      | 67    |       |
| Gamma(4)     | 92      | 87    |       |
| Weibull(1)   | 92      | 94    |       |
| Weibull(2)   | 69      | 61    |       |
| Weibull(2.5) | 99      | 99    |       |
In the case of the Gompertz distributions, the test based on $A^2$ is the most powerful of the three for all tabulated $n$ and $\alpha$, and the test based on $T_n^2$ is the next most powerful. For $\alpha = 2$ and $n = 20$, we see from Table 4 that the test based on $T_n^2$ is at least as powerful as the tests based on $C^2$ and $A^2$ for 17 of the 26 alternatives considered. The tables for $\alpha = 2$ and $n = 50$ also indicate that the test based on $T_n^2$ is comparable in power to the other two tests. Therefore, for small values of $\alpha$, the new test is a serious competitor to the classical tests, irrespective of the size of $n$.

For large $\alpha$ and small $n$, Tables 5 and 6 indicate that the test based on $T_n^2$ is more powerful than the tests based on $C^2$ and $A^2$ for the majority of alternatives considered here. If $n$ is large then $T_n^2$ becomes less superior to the other two tests; this is a consequence of the consistency of each test, which implies that, as $n \to \infty$, the powers of all three tests converge to 1.

### 6 Contiguous alternatives to the null hypothesis

For $n \in \mathbb{N}$, let $X_{n1}, \ldots, X_{nn}$ be a triangular array of row-wise independent random variables. As usual, let $P_0 = \text{Gamma}(\alpha, 1)$, $\alpha \geq 1/2$, and let $Q_{n1}$ be a probability measure dominated by $P_0$. We wish to test the null hypothesis

$$H_0 : \text{The marginal distribution of each } X_{ni}, i = 1, \ldots, n, \text{ is } P_0$$

against the alternative

$$H_1 : \text{The marginal distribution of each } X_{ni}, i = 1, \ldots, n, \text{ is } Q_{n1}.$$
### Table 5

Percentage points of 5000 Monte Carlo samples found to be significant at 5% level of significance; $\alpha = 5$

| Distribution  | $T^2_n$ | $C^2$ | $A^2$ |
|--------------|---------|-------|-------|
| $n = 20$     |         |       |       |
| Gamma(4)     | 15      | 9     | 13    |
| Gamma(4.5)   | 9       | 7     | 7     |
| Gamma(5)     | 5       | 5     | 5     |
| Gamma(6)     | 7       | 7     | 6     |
| Gamma(7)     | 13      | 12    | 10    |
| Gamma(8)     | 23      | 19    | 17    |
| Gamma(10)    | 47      | 37    | 35    |
| Weibull(3)   | 15      | 17    | 15    |
| Weibull(4)   | 65      | 62    | 59    |
| Weibull(5)   | 94      | 93    | 92    |
| LIFR(2)      | 94      | 72    | 93    |
| LIFR(4)      | 88      | 63    | 87    |
| LN(0.5)      | 18      | 13    | 16    |
| LN(0.7)      | 83      | 63    | 76    |
| LN(0.9)      | 98      | 92    | 97    |
| LN(1)        | 99      | 96    | 99    |
| IG(2)        | 71      | 53    | 65    |
| IG(2.5)      | 49      | 36    | 43    |
| IG(3)        | 31      | 22    | 27    |
| GO(2)        | 79      | 54    | 84    |
| GO(4)        | 54      | 35    | 67    |
| Rayleigh(1)  | 32      | 15    | 33    |
| $n = 50$     |         |       |       |
| Gamma(4)     | 23      | 13    | 21    |
| Gamma(4.5)   | 9       | 6     | 9     |
| Gamma(5)     | 5       | 5     | 5     |
| Gamma(6)     | 11      | 11    | 10    |
| Gamma(7)     | 30      | 25    | 25    |
| Gamma(8)     | 55      | 44    | 48    |
| Gamma(10)    | 91      | 79    | 82    |
| Weibull(3)   | 31      | 41    | 38    |
| Weibull(4)   | 98      | 98    | 98    |
| Weibull(5)   | *       | *     | *     |
| LIFR(2)      | *       | 98    | *     |
| LIFR(4)      | *       | 95    | *     |
| LN(0.5)      | 27      | 25    | 26    |
| LN(0.7)      | 99      | 95    | 98    |
| LN(0.9)      | *       | *     | *     |
| LN(1)        | *       | *     | *     |
We write the Radon-Nikodym derivative of $Q_{n1}$ with respect to $P_0$ in the form $dQ_{n1}/dP_0 = 1 + n^{-1/2}h_n$, and then we will need the following

**Assumption 7** We assume that:

(A1) The functions $\{h_n : n \in \mathbb{N}\}$ form a sequence of $P_0$-integrable functions converging pointwise, $P_0$-almost everywhere, to a function $h$, and

(A2) $\sup_{n \in \mathbb{N}} E_{P_0}|h_n|^4 < \infty$.

Since $\int (dQ_{n1}/dP_0) \, dP_0 = 1$ then we also have $\int h_n \, dP_0 = 0$, for all $n \in \mathbb{N}$.

Denote the indicator function of an event $A$ by $I(A)$. By applying (A2), we deduce the uniform integrability of $|h_n|^2$:

$$\lim_{k \to \infty} \sup_{n \in \mathbb{N}} |E_{P_0}|h_n|^2 I(|h_n|^2 > k) = \lim_{k \to \infty} \sup_{n \in \mathbb{N}} \int |h_n|^2 I(|h_n|^2 > k) \, dP_0$$

$$\leq \lim_{k \to \infty} k^{-1} \sup_{n \in \mathbb{N}} E_{P_0}|h_n|^4 = 0.$$ 

By Bauer (1981, p. 95, Theorem 2.11.4), the $P_0$-almost everywhere convergence of $h_n$ to $h$ implies the $P_0$-stochastic convergence of $h_n$ to $h$. Again by Bauer (1981, p. 104, Theorem 2.12.4), the uniform integrability of $h_n^2$ together with the $P_0$-stochastic convergence of $h_n$ to $h$ imply the convergence of $h_n$ in mean square, viz.

$$\lim_{n \to \infty} \int |h_n - h|^2 \, dP_0 = 0.$$ 

By the triangle and the Cauchy–Schwarz inequalities,

$$0 \leq \lim_{n \to \infty} \left| \int (h_n - h) \, dP_0 \right| \leq \lim_{n \to \infty} \int |h_n - h| \, dP_0 \leq \lim_{n \to \infty} \left( \int |h_n - h|^2 \, dP_0 \right)^{1/2} = 0,$$

therefore

$$\lim_{n \to \infty} \int h_n \, dP_0 = \int h \, dP_0 = 0.$$ 

Hadjicosta (2019) has shown that Assumptions 7 hold for several contiguous alternatives, e.g.,

Table 5 continued

| n = 50 |   |   |   |
|-------|---|---|---|
| IG(2) | 95 | 89 | 94 |
| IG(2.5) | 81 | 72 | 76 |
| IG(3) | 55 | 49 | 52 |
| GO(2) | 98 | 92 | 99 |
| GO(4) | 82 | 75 | 95 |
| Rayleigh(1) | 56 | 33 | 59 |
Table 6 Percentage points of 5000 Monte Carlo samples found to be significant at 5% level of significance; \( \alpha = 10 \)

| Distribution | \( T^2_n \) | \( C^2 \) | \( A^2 \) |
|--------------|-------------|--------|--------|
| **n = 20**   |             |        |        |
| Gamma(5)     | 68          | 37     | 58     |
| Gamma(8)     | 16          | 8      | 13     |
| Gamma(10)    | 5           | 5      | 5      |
| Gamma(12)    | 7           | 7      | 5      |
| Gamma(15)    | 19          | 14     | 12     |
| Gamma(20)    | 48          | 36     | 32     |
| Weibull(5)   | 38          | 37     | 34     |
| Weibull(6)   | 72          | 70     | 67     |
| Weibull(7)   | 91          | 90     | 90     |
| LIFR(50)     | 97          | 78     | 96     |
| LIFR(100)    | 73          | 59     | 57     |
| LN(0.2)      | 42          | 23     | 33     |
| LN(0.4)      | 97          | 86     | 95     |
| LN(0.6)      | 59          | 37     | 48     |
| IG(4)        | 39          | 24     | 32     |
| IG(5)        | 23          | 14     | 19     |
| IG(6)        | 14          | 10     | 11     |
| GO(10)       | 82          | 57     | 83     |
| GO(20)       | 59          | 42     | 67     |
| Rayleigh(1)  | 94          | 68     | 91     |
| **n = 50**   |             |        |        |
| Gamma(5)     | 96          | 72     | 89     |
| Gamma(8)     | 28          | 13     | 20     |
| Gamma(10)    | 5           | 5      | 5      |
| Gamma(12)    | 16          | 10     | 9      |
| Gamma(15)    | 52          | 31     | 34     |
| Gamma(20)    | 93          | 76     | 82     |
| Weibull(5)   | 80          | 81     | 80     |
| Weibull(6)   | 99          | 99     | 99     |
| Weibull(7)   | *           | *      | *      |
| LIFR(50)     | *           | 99     | *      |
| LIFR(100)    | *           | 99     | *      |
| LN(0.2)      | *           | 97     | 98     |
| LN(0.4)      | 73          | 50     | 61     |
| LN(0.6)      | *           | 99     | *      |
| IG(4)        | 99          | 91     | 96     |
Table 6 continued

| n = 50 |
|--------|
| IG(5)  | 90 | 72 | 82 |
| IG(6)  | 69 | 47 | 59 |
| IG(7)  | 43 | 29 | 33 |
| IG(8)  | 23 | 15 | 18 |
| GO(10) | 98 | 93 |
| GO(20) | 84 | 81 |
| Rayleigh(1) | * | 97 | * |

(1) Gamma alternatives with shape parameter $\alpha \geq 1/2$ and rate parameter $\lambda_n = 1 + n^{-1/2}$.
(2) Gamma alternatives with shape parameter $\alpha_n = \alpha + n^{-1/2}$ and rate parameter 1.
(3) Contaminated models, $Q_n = (1 - n^{-1/2})P_0 + n^{-1/2}P_1$, where the probability measure $P_1$ is dominated by $P_0$ and $\int (dP_1/dP_0)^4 \, dP_0 < \infty$.

Let $P_n = P_0 \otimes \cdots \otimes P_0$ and $Q_n = Q_{n1} \otimes \cdots \otimes Q_{n1}$, where $P_0 = \text{Gamma}(\alpha, 1)$, $\alpha \geq 1/2$, be the $n$-fold product probability measures of $P_0$ and $Q_{n1}$ respectively.

**Theorem 8** Let $X_{n1}, \ldots, X_{nn}, n \in \mathbb{N}$, be a triangular array of positive row-wise i.i.d. random variables, where $X_{nj} \equiv X_j$, $j = 1, \ldots, n$. We assume that the distribution of $X_{nj}$ is $Q_{n1}$ for every $j = 1, \ldots, n$. Further, let $Z_n := \{Z_n(t), t \geq 0\}$ be a stochastic process with

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ \Gamma(\alpha)(tX_{nj}/\bar{X}_n)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_{nj}/\bar{X}_n)^{1/2}) - e^{-t/\alpha} \right].$$

$t \geq 0$. Under Assumption 7, there exists a centered Gaussian process $Z := \{Z(t), t \geq 0\}$ with sample paths in $L^2$ and the covariance function $K(s, t)$ in (3.4), and a function

$$c(t) = \int_{0}^{\infty} \left[ \Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2}) + \alpha^{-2} t e^{-t/\alpha} x \right] h(x) \, dP_0(x),$$

$t \geq 0$, such that $Z_n \overset{d}{\to} Z + c$ in $L^2$. Moreover, as $n \to \infty$,

$$T_n^2 \overset{d}{\to} \int_{0}^{\infty} \left( Z(t) + c(t) \right)^2 \, dP_0(t).$$

A detailed proof of this theorem is provided by Hadjicosta (2019) who followed the approach of Taherizadeh (2009, pp. 79–91).

**7 The efficiency of the test**

Let $X_1, X_2, \ldots$ be i.i.d., positive random variables with a distribution $P$ that is indexed by a parameter $\theta \in \Theta := (-\eta, \eta)$ or $\Theta := [0, \eta)$, for some $\eta > 0$. We represent $H_0$
by \( \Theta_0 = \{0\} \) and \( H_1 \) by \( \Theta_1 = \Theta \setminus \{0\} \). In Sect. 3, we showed that \( T^2_n \) is scale-invariant, i.e. it does not depend on the unknown rate parameter \( \lambda \). Under the null hypothesis, we assume that \( X_1, X_2, \ldots \) are i.i.d., positive \( P_0 \)-distributed random variables; further, under the local alternative, represented by \( \theta \in \Theta_1 \), we assume that \( X_1, X_2, \ldots \) are i.i.d., positive \( P_\theta \)-distributed random variables.

The Radon-Nikodym derivative of \( P_\theta \) with respect to \( P_0 \) is \( dP_\theta / dP_0 = 1 + \theta h_\theta \). We assume that as \( \theta \to 0 \), the function \( h_\theta \) converges to some function \( h \) in \( L^2 \). Since \( \int (dP_\theta / dP_0) \ dP_0 = 1 \), we obtain \( \int_0^\infty h_\theta(x) \ dP_0(x) = 0 \), for \( \theta \in \Theta_1 \). Further, we shall assume that for \( \theta \in \Theta_1 \),

\[
\int_0^\infty x h_\theta(x) \ dP_0(x) = 0. \tag{7.1}
\]

Let \( \Theta_0 \) and \( \Theta_1 \) be null and alternative parameter spaces, respectively, and \( \{U_n : n \in \mathbb{N}\} \) be a sequence of test statistics. For \( \theta \in \Theta_0 \), \( F_n(t) = P_\theta(U_n < t), t \in \mathbb{R} \), is the null distribution of \( U_n \), and the level attained by \( U_n \) is \( L_n := 1 - F_n(U_n) \). For \( \theta \in \Theta_1 \), the exact Bahadur slope of the sequence \( \{U_n : n \in \mathbb{N}\} \) is

\[
c(\theta) = -2 \lim_{n \to \infty} n^{-1} \log L_n,
\]

whenever the limit exists (almost surely). For \( \theta \in \Theta_0 \), this limit exists with \( c(\theta) = 0 \).

For a sequence \( \{U_{1,n} : n \in \mathbb{N}\} \) of test statistics with exact Bahadur slope \( c_j(\theta) \), \( j = 1, 2 \), the exact Bahadur asymptotic relative efficiency of \( \{U_{1,n} : n \in \mathbb{N}\} \) with respect to \( \{U_{2,n} : n \in \mathbb{N}\} \) is \( e^{B_{1,2}}(\theta) = c_1(\theta)/c_2(\theta), \theta \in \Theta_1 \). If \( e^{B_{1,2}}(\theta) > 1 \), then we prefer the sequence \( \{U_{1,n} : n \in \mathbb{N}\} \).

In general, it is difficult to calculate the exact Bahadur slope (Bahadur 1971, Theorem 7.2), so we investigate the approximate Bahadur slope. We note that Bahadur (1967) showed that for \( \Theta_0 = \{\theta_0\} \), the approximate Bahadur slope is close to the exact Bahadur slope for \( \theta \) in a neighborhood of \( \theta_0 \), i.e., under local alternatives.

To obtain the approximate Bahadur slope of our test statistic \( T^2_n \) under local alternatives, we need to show that the sequence \( \{T_n : n \in \mathbb{N}\} \) is a standard sequence (Bahadur 1960, Section 2).

**Theorem 9** The sequence of test statistics \( \{T_n : n \in \mathbb{N}\} \) is a standard sequence. The approximate Bahadur slope of the test is \( c^{(a)}(\theta) := \delta_1^{-1} b^2(\theta) \), where \( \delta_1 \) is the largest eigenvalue of \( S \) and

\[
b^2(\theta) = \theta^2 \int_0^\infty \left[ \int_0^\infty \Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2})h_\theta(x) \ dP_0(x) \right]^2 \ dP_0(t). \tag{7.2}
\]

Moreover,

\[
\lim_{\theta \to 0} \frac{c^{(a)}(\theta)}{\theta^2} = \delta_1^{-1} \int_0^\infty \left[ \int_0^\infty \Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2})h(x) \ dP_0(x) \right]^2 \ dP_0(t). \tag{7.3}
\]
A complete proof of this result is given by Hadjicosta (2019) following the approach of Taherizadeh (2009, p. 98, Theorem 5.4).

If we write the squared term in (7.3) as a product of two integrals, one in \( x \) and one in \( y \), interchange the order of integration, and apply Weber’s integral (10.1), then (7.3) reduces to

\[
\lim_{\theta \to 0} \frac{c^{(a)}(\theta)}{\theta^2} = \delta_{1}^{-1} \int_{0}^{\infty} \int_{0}^{\infty} I_{a-1}(2(xy/\alpha^2)^{1/2})h(x)h(y) \, dx \, dy,
\]

and a similar result holds for (7.2). These expressions provide an alternative way to calculate the approximate Bahadur slope of the test.

We now obtain the limiting approximate Bahadur slope for several alternatives. In the following calculations, we will take \( \alpha = 2 \) as the general case can be treated similarly. Consider the contaminated models \( P_{\theta} = (1-\theta)P_{0} + \theta \cdot P_{1} \), where \( P_{1} \) is a probability measure dominated by \( P_{0} \); for \( \alpha = 1 \), these alternatives were considered earlier by Baringhaus and Taherizadeh (2010). It is straightforward to verify that assumption (7.1) is satisfied if \( \int x \, dP_{1}(x) = \int x \, dP_{0}(x) = 2 \). Also, \( h_{\theta} = (dP_{1}/dP_{0}) - 1 \). By (7.3), the limiting Bahadur slope of the sequence \( \{T_{n} : n \in \mathbb{N} \} \) is

\[
c_{T} := \lim_{\theta \to 0} \frac{c^{(a)}(\theta)}{\theta^2} = \delta_{1}^{-1} \int_{0}^{\infty} (\mathcal{H}_{P_{1}}(t/2) - e^{-t/2})^{2} \, dP_{0}(t),
\]

where \( \mathcal{H}_{P_{1}} \) denotes the Hankel transform of \( P_{1} \). Further, by applying the results of Theorem 5 for calculating the eigenvalues of \( \mathcal{S} \), we obtain \( \delta_{1}^{-1} = 83.242 \).

Consider the contaminated gamma models \( P_{\theta} = (1-\theta)P_{0} + \theta \cdot \text{Gamma}(2k, k) \), \( k \in \mathbb{N}, k \geq 2 \). By Eq. (2.5) and Kummer’s formula (2.4), we obtain

\[
\mathcal{H}_{\text{Gamma}(2k, k)}(t/2) = F_{1}(2k; 2; -t/2k) = e^{-t/2k} F_{1}(2 - 2k; 2; t/2k).
\]

In Table 7, we provide the limiting approximate Bahadur slopes for \( k = 2, 3, 4, 5, 6 \).

| \( k \) | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| \( c_{T} \) | 0.149 | 0.284 | 0.373 | 0.433 | 0.477 |

Weand (1976) showed that if two standard sequences \( \{U_{1,n} : n \in \mathbb{N} \} \) and \( \{U_{2,n} : n \in \mathbb{N} \} \) satisfy an additional criterion, now called Wieand’s condition, then the limiting approximate Bahadur efficiency is in accord with the limiting Pitman efficiency, as the level of significance decreases to 0. In the next theorem, we state that Wieand’s condition is valid for our sequence of test statistics \( \{T_{n} : n \in \mathbb{N} \} \). The proof of this...
Theorem 10 The sequence \( \{T_n : n \in \mathbb{N}\} \) satisfies Wieand’s condition: There exists a constant \( \theta^* > 0 \) such that for any \( \epsilon > 0 \) and \( \gamma \in (0, 1) \), there exists a constant \( C > 0 \) such that, for any \( \theta \in \Theta_1 \cap (-\theta^*, \theta^*) \) and \( n > C/b^2(\theta) \), \( P(|n^{-1/2}T_n - b(\theta)| < \epsilon b(\theta)) > 1 - \gamma \).

8 Concluding remarks

In constructing the test statistic \( T_n^2 \) in (1.3), we set \( \nu = \alpha - 1 \). In this case, the test statistic has a relatively simple expression as a \( V \)-statistic, so the test can be carried out in a straightforward way. The resulting test statistic also is consistent, has good power performance, and extensive results can be obtained for the eigenvalues and eigenfunctions of the corresponding covariance operator.

For general \( \nu \), the calculations in the proof of Proposition 1 can be extended. However, the final expression for the resulting \( V \)-statistic will be more complicated, for it will involve the generalized hypergeometric series. Under \( H_0 \), we will again obtain \( T_n^2 \xrightarrow{d} \sum_{k=1}^{\infty} \delta_k X_{1k}^2 \), but the eigenfunctions of the corresponding covariance operator will be more complex and may be unavailable.

We remark that there are many choices, other than \( P_0 \), for the weight measure. Our choice of \( P_0 \) is motivated by classical tests, such as the Anderson-Darling and Cramér-von Mises statistics, for which the weight measure is determined by \( H_0 \). In the gamma case, the orthogonal polynomials for \( P_0 \) are well-known; however, this may not hold for more general weight measures. We note that Henze et al. (2012) and Taherizadeh (2009, p. 65) provided results for weight measures \( \omega \) of the form \( d\omega(t) = e^{-\beta t} dP_0(t) \), where \( \beta \) is a “tuning parameter;” similar results for the testing problem considered in this paper can be obtained using our methods.

The entries in Table 2 reflect the dependency of \( T_n^2 \) on the value of \( \alpha \). This raises the problem of extending our results to the case in which \( \alpha \) is unknown. This problem appears to be formidable; if we replace each \( \alpha \) in (3.2) with a suitable estimator \( \hat{\alpha} \), then parametric bootstrap procedures can be used to estimate the resulting critical values and the power of the test. However, it may be more difficult to derive the analytical properties of the test statistic.

If \( \hat{\alpha} \) is scale-invariant, the results of Henze et al. (2012) lead us to believe that, under certain regularity conditions, the asymptotic distribution of the resulting statistic can be obtained. However, the comments of Henze et al. (2012, Remark 2.3) also apply to our problem, viz., a finite-sample implementation of the test will require knowledge of the value of \( \alpha \) which, however, is unknown. We also note that there is a substantial literature on the problem of inserting a parameter estimator into a \( V \)-statistic; cf., de Wet and Randles (1987), Leucht and Neumann (2013), and Matsui and Takemura (2008); it is an open problem to apply those approaches to our test statistic.

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Compliance with ethical standards

Conflict of interest: On behalf of both authors, the corresponding author states that there is no conflict of interest.

9 Appendix 1: Bessel functions and Hankel transforms

For the special case in which \( \nu = -\frac{1}{2} \), it follows from (2.1) that, for \( x \in \mathbb{R} \),
\[
x^{1/2} J_{-1/2}(x) = \left( \frac{2}{\pi} \right)^{1/2} \cos x,
\]
(9.1)

For \( \nu > -1/2 \), the Bessel function is also given by the Poisson integral,
\[
J_\nu(x) = \frac{(x/2)^\nu}{\pi^{1/2} \Gamma(\nu + 1/2)} \int_0^\pi \cos(x \cos \theta)(\sin \theta)^{2\nu} \, d\theta,
\]
(9.2)

\( x \in \mathbb{R} \); see Erdélyi et al. (1953, 7.12(9)), Olver et al. (2010, (10.9.4)). This result can be proved by expanding \( \cos(x \cos \theta) \) as a power series in \( x \cos(\theta) \) and integrating term-by-term.

The Bessel function \( J_\nu \) also satisfies the inequality,
\[
|J_\nu(z)| \leq \frac{1}{\Gamma(\nu + 1)} |z/2|^\nu \exp(\text{Im}(z)),
\]
(9.3)

\( \nu \geq -1/2, z \in \mathbb{C} \); see Erdélyi et al. (1953, 7.3.2(4)) or Olver et al. (2010, (10.14.4)). Henceforth, we assume that \( \nu \geq -1/2 \). For \( t, x \geq 0 \), we set \( z = 2(tx)^{1/2} \) in (9.3) to obtain
\[
\left| (tx)^{-\nu/2} J_\nu(2(tx)^{1/2}) \right| \leq \frac{1}{\Gamma(\nu + 1)}.
\]
(9.4)

Although the next two results may be known, we were unable to find them in the literature.

Lemma 3 For \( \nu \geq -1/2 \) and \( t \geq 0 \),
\[
\left| t^{-\nu} J_{\nu+1}(t) \right| \leq \frac{1}{2^\nu \pi^{1/2} \Gamma(\nu + 1/2)}.
\]
(9.5)

Proof By Olver et al. (2010, (10.6.6)),
\[
t^{-\nu} J_{\nu+1}(t) = -(t^{-\nu} J_\nu(t))',
\]
(9.6)

\( t \geq 0 \). For \( \nu > -1/2 \), it follows by differentiating the Poisson integral (9.2) that
\[
2^\nu \pi^{1/2} \Gamma(\nu + 1/2) |t^{-\nu} J_{\nu+1}(t)| = \left| \int_0^\pi \cos \theta \sin(t \cos \theta) (\sin \theta)^{2\nu} \, d\theta \right| \\
\leq \int_0^\pi |\cos \theta| |(\sin \theta)^{2\nu}| \, d\theta.
\]
By a substitution, \( s = \sin^2 \theta \), the latter integral reduces to a beta integral,

\[
\int_0^1 s^{a-1}(1 - s)^{b-1} \, ds = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)},
\]

\( a, b > 0 \). This produces (9.5).

For \( \nu = -1/2 \), it follows from (9.6) and (9.1) that

\[
J_{1/2}(t) = (2/\pi)^{1/2} \sin t; \quad (9.7)
\]

cf. Olver et al. (2010, (10.16.1)). Then, \(|t^{1/2} J_{1/2}(t)| \leq (2/\pi)^{1/2} \), as stated in (9.5). \( \Box \)

**Remark 5**

Substituting \( \nu = 0 \) in Lemma 3, we obtain

\[
|J_{0}(t)| \leq 2/\pi, \quad t \geq 0.
\]

This bound is sharper than a bound given in Olver et al. (2010, (10.14.1)), viz., \(|J_{0}(t)| \leq 2^{-1/2}, \quad t \geq 0\).

**Lemma 4**

For \( \nu \geq -1/2 \), the function \( t^{-\nu} J_{\nu+1}(t), \quad t \geq 0 \), is Lipschitz continuous, satisfying for \( u, v \in \mathbb{R} \), the inequality

\[
|u^{-\nu} J_{\nu+1}(u) - v^{-\nu} J_{\nu+1}(v)| \leq \frac{1}{2^{\nu+1} \Gamma(v + 2)} |u - v|. \quad (9.8)
\]

**Proof** For \( \nu > -1/2 \) we apply (9.6), (9.2), and the triangle inequality to obtain

\[
2^\nu \pi^{1/2} \Gamma\left(\nu + \frac{1}{2}\right)|u^{-\nu} J_{\nu+1}(u) - v^{-\nu} J_{\nu+1}(v)|
\]

\[
\leq \int_0^\pi |\sin(u \cos \theta) - \sin(v \cos \theta)| |\cos \theta| (\sin \theta)^{2\nu} \, d\theta.
\]

By a well-known trigonometric identity, and the inequality \(|\sin t| \leq |t|, \quad t \in \mathbb{R}\),

\[
|\sin(u \cos \theta) - \sin(v \cos \theta)| = 2|\sin\left(\frac{1}{2}(u - v) \cos \theta\right) \cos\left(\frac{1}{2}(u + v) \cos \theta\right)|
\]

\[
\leq |u - v| |\cos \theta| \left|\cos\left(\frac{1}{2}(u + v) \cos \theta\right)\right|
\]

\[
\leq |u - v| |\cos \theta|. \quad (9.9)
\]

Therefore,

\[
|u^{-\nu} J_{\nu+1}(u) - v^{-\nu} J_{\nu+1}(v)| \leq \frac{2}{2^\nu \pi^{1/2} \Gamma\left(\nu + \frac{1}{2}\right)} |u - v| \int_0^{\pi/2} (\cos \theta)^{2\nu} (\sin \theta)^{2\nu} \, d\theta.
\]

Substituting \( t = \sin^2 \theta \) reduces the latter integral to a beta integral, and then we obtain (9.8).

For \( \nu = -1/2 \), we apply (9.7) to obtain

\[
|u^{1/2} J_{1/2}(u) - v^{1/2} J_{1/2}(v)| = (2/\pi)^{1/2} |\sin u - \sin v| \leq (2/\pi)^{1/2} |u - v|,
\]

the latter inequality following from (9.9) with \( \theta = 0 \). Then, we obtain (9.8) for \( \nu = -1/2 \). \( \Box \)
As regards the modified Bessel function $I_\nu$, defined in (2.2), with $i = \sqrt{-1}$ we find from (2.1) that
\[ I_\nu(x) = i^{-\nu} J_\nu(ix), \]
x $\in \mathbb{R}$; hence, by (9.3),
\[ |\Gamma(v + 1)(x/2)^{-\nu} I_\nu(x)| \leq 1. \quad (9.10) \]

For $n \in \mathbb{N}_0$ and $\alpha > 0$, the (generalized) Laguerre polynomial of order $\alpha - 1$ and degree $n$ is
\[ L_n^{(\alpha-1)}(x) = \frac{(-n\alpha)_n}{n!} \sum_{k=0}^{n} \frac{(\alpha + k)(n-k)!}{(n-k)!} (-x)^k, \]
x $\in \mathbb{R}$; see Olver et al. (2010, Chapter 18) or Szegö (1967, Chapter 5). The normalized (generalized) Laguerre polynomial of order $\alpha - 1$ and degree $n$ is defined by
\[ \mathcal{L}_n^{(\alpha-1)}(x) := \left( \frac{n!}{(\alpha)_n} \right)^{1/2} L_n^{(\alpha-1)}(x), \quad (9.11) \]
x $\in \mathbb{R}$. It is well-known (see Olver et al. (2010, Chapter 18.3) or Szegö (1967, Chapter 5.1)) that the polynomials $\mathcal{L}_n^{(\alpha-1)}$ are orthonormal with respect to the Gamma($\alpha, 1$) distribution:
\[ \int_0^\infty \mathcal{L}_n^{(\alpha-1)}(x) \mathcal{L}_m^{(\alpha-1)}(x) \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} \, dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \]

**Lemma 5** For $v > 0$ and $\alpha > 0$,
\[ \int_0^\infty x^\alpha e^{-vx} \mathcal{L}_n^{(\alpha-1)}(x) \, dx = \frac{\Gamma(\alpha + n)}{n!} (v - 1)^{n-1} v^{-(\alpha+n+1)} (\alpha(v - 1) - n). \]

**Proof** Starting with the known integral (Olver et al. 2010, (18.17.34)),
\[ \int_0^\infty x^{\alpha-1} e^{-vx} \mathcal{L}_n^{(\alpha-1)}(x) \, dx = \frac{\Gamma(\alpha + n)}{n!} (v - 1)^{n} v^{-(\alpha+n)}, \]
we differentiate each side with respect to $v$ and simplify the outcome to obtain the result. \hfill \Box

**Proof of Lemma 1** (i) By (9.4) for $J_\nu(x)$, $\Gamma(v + 1)\left|J_\nu(2\sqrt{tx})\right| \leq 1$ for all $x, t > 0$. Therefore, by the triangle inequality, $|\mathcal{H}_{X,v}(t)| \leq 1$.
(ii) It follows from the series expansion (2.1) that
\[ \Gamma(v + 1)(tx)^{-v/2} J_\nu \left(2(t x)^{1/2}\right) \bigg|_{t=0} = 1, \]
for all $x$, so we obtain $\mathcal{H}_{X,v}(0) = 1$. \hfill \Box
As the function $\left(\frac{\Gamma(v+1)(2\sqrt{tx})}{\sqrt{tx}}\right)$ is a power series in $tx$, it is continuous in $t \geq 0$
for every fixed $x \geq 0$. As it is also bounded, then the conclusion follows from the Dominated Convergence Theorem.

The following Hankel transform inversion theorem is a classical result that can be obtained from many sources, e.g., Sneddon (1972, p. 309, Theorem 1).

**Theorem 11** (Hankel Inversion) Let $X$ be a positive, continuous random variable with probability density function $f(x)$ and Hankel transform $H_{X,v}$. For $x > 0$, $f(x) = \frac{1}{\Gamma(v+1)} \int_0^\infty \left(\frac{\Gamma(v+1)(2\sqrt{tx})}{\sqrt{tx}}\right) H_{X,v}(t) \, dt$.

As a consequence of the inversion formula, we obtain the uniqueness of the Hankel transform.

**Theorem 12** (Hankel Uniqueness) Let $X$ and $Y$ be positive random variables with corresponding Hankel transforms $H_{X,v}$ and $H_{Y,v}$. Then $H_{X,v} = H_{Y,v}$ if and only if $X \overset{d}{=} Y$.

The next result, on the continuity of the Hankel transform, is analogous to Theorem 2.3 of Baringhaus and Taherizadeh (2010). Therefore, we will omit the proof.

**Theorem 13** (Hankel Continuity) Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of positive random variables with corresponding Hankel transforms $\{H_n, n \in \mathbb{N}\}$. If there exists a positive random variable $X$, with Hankel transform $H$, such that $X_n \overset{d}{\to} X$, then for all $t \geq 0$,

$$\lim_{n \to \infty} H_n(t) = H(t) \quad (9.12)$$

Conversely, suppose there exists $H : [0, \infty) \to \mathbb{R}$ such that $H(0) = 1$, $H$ is continuous at 0, and (9.12) holds. Then $H$ is the Hankel transform of a positive random variable $X$, and $X_n \overset{d}{\to} X$.

10 Appendix 2: The test statistic

**Proof of Proposition 1** By squaring the integrand in (1.3), there are three terms to be calculated. First,

$$n \int_0^\infty H_n^2(t) \, dP_0(t) = \frac{1}{n} \int_0^\infty \left( \sum_{i=1}^n \Gamma(\alpha)(Y_{it})^{(1-\alpha)/2} J_{\alpha-1}(2\sqrt{tY_i}) \right)^2 \, dP_0(t)$$

$$= \frac{\Gamma(\alpha)}{n} \sum_{i=1}^n \sum_{j=1}^n (Y_{i}Y_{j})^{(1-\alpha)/2} \int_0^\infty J_{\alpha-1}(2\sqrt{tY_i})J_{\alpha-1}(2\sqrt{tY_j})e^{-t} \, dt.$$
These integrals are of the form of Weber’s exponential integral (Olver et al. 2010, (10.22.67)):

\[
\int_0^\infty \exp(-pt) J_v(2\sqrt{at}) J_v(2\sqrt{bt}) \, dt = p^{-1} \exp\left(-\frac{(a+b)}{p}\right) I_v(2\sqrt{ab}/p),
\]

valid for \(\nu > -1\) and \(a, b, p > 0\). Simplifying the resulting expressions, we obtain

\[
n \int_0^\infty H_n^2(t) \, dP_0(t) = \frac{\Gamma(\alpha)}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (Y_i Y_j)^{(1-\alpha)/2} \exp(-Y_i - Y_j) I_{\nu-1}(2(Y_i Y_j)^{1/2}).
\]

Second, by proceeding as in Example 1, it is straightforward to deduce

\[
2n \int_0^\infty H_n(t) e^{-t/\alpha} \, dP_0(t) = 2 \sum_{i=1}^{n} (1 + \alpha^{-1})^{-\alpha} e^{-\alpha Y_i/(\alpha+1)} \equiv \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\alpha}{\alpha + 1}\right) \left[ e^{-\alpha Y_i/(\alpha+1)} + e^{-\alpha Y_j/(\alpha+1)} \right].
\]

Third, we have a gamma integral:

\[
n \int_0^\infty e^{-2t/\alpha} \, dP_0(t) = n \left(\frac{\alpha}{\alpha + 2}\right)^\alpha = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\alpha}{\alpha + 2}\right)^\alpha.
\]

Collecting together all three terms, we obtain the desired result. \(\square\)

**Proof of Theorem 2** By (9.6), \((s^{1-\alpha} J_{\alpha-1}(s))' = -s^{1-\alpha} J_{\alpha}(s)\). Therefore, the Taylor expansion of order 1 of the function \(s^{1-\alpha} J_{\alpha-1}(s)\), at a point \(s_0\), is

\[
s^{1-\alpha} J_{\alpha-1}(s) = s_0^{1-\alpha} J_{\alpha-1}(s_0) + (s_0 - s)u^{1-\alpha} J_{\alpha}(u),
\]

where \(u\) lies between \(s\) and \(s_0\). Setting \(s = 2(t Y_j)^{1/2}\) and \(s_0 = 2(t X_j/\alpha)^{1/2}\), we obtain

\[
2^{1-\alpha}(t Y_j)^{(1-\alpha)/2} J_{\alpha-1}(2(t Y_j)^{1/2}) = 2^{1-\alpha}(t X_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(t X_j/\alpha)^{1/2})
\]

\[
+ 2[(t X_j/\alpha)^{1/2} - (t Y_j)^{1/2}] u^{1-\alpha} J_{\alpha}(u_j),
\]

where \(u_j\) lies between \(2(t Y_j)^{1/2}\) and \(2(t X_j/\alpha)^{1/2}\). Define

\[
W_n = \alpha^{-1/2} - \overline{X}_n^{-1/2} = \frac{\overline{X}_n - \alpha}{(\alpha \overline{X}_n)^{1/2}(\alpha^{1/2} + \overline{X}_n^{1/2})};
\]
then
\[(tX_j/\alpha)^{1/2} - (tY_j)^{1/2} = (tX_j/\alpha)^{1/2} - (tX_j/X_n)^{1/2} = W_n(tX_j)^{1/2},\]
and (10.2) reduces to
\[2^{1-\alpha}(tY_j)^{(1-\alpha)/2} J_{\alpha-1}(2(tY_j)^{1/2}) = 2^{1-\alpha}(tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_j/\alpha)^{1/2}) + 2\alpha W_n(tX_j)^{1/2}u_j^{1-\alpha} J_\alpha(u_j).\]

Multiplying both sides of (10.4) by $2^{\alpha-1}$, adding and subtracting the term
\[2(tX_j)^{1/2}W_n(tX_j/\alpha)^{(1-\alpha)/2} J_\alpha(2(tX_j/\alpha)^{1/2})\]
on the right-hand side, and then simplifying the result, we obtain

\[(tY_j)^{(1-\alpha)/2} J_{\alpha-1}(2(tY_j)^{1/2}) = (tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_j/\alpha)^{1/2}) + 2\alpha^{1/2} W_n(tX_j/\alpha)^{(1-\alpha/2)} J_\alpha(2(tX_j/\alpha)^{1/2}) + 2\alpha W_n(tX_j)^{1/2} \left( u_j^{1-\alpha} J_\alpha(u_j) - (2(tX_j/\alpha)^{1/2})^{1-\alpha} J_\alpha(2(tX_j/\alpha)^{1/2}) \right).\]

Define the processes $Z_{n,1}(t)$, $Z_{n,2}(t)$, and $Z_{n,3}(t)$, $t \geq 0$, by

\[Z_{n,1}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ \Gamma(\alpha)(tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_j/\alpha)^{1/2}) + 2\Gamma(\alpha)\alpha^{1/2} W_n(tX_j/\alpha)^{(1-\alpha)/2} J_\alpha(2(tX_j/\alpha)^{1/2}) - e^{-t/\alpha} \right],\]
\[Z_{n,2}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ \Gamma(\alpha)(tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_j/\alpha)^{1/2}) + 2\alpha^{-1/2} W_n(tX_j/\alpha)^{(1-\alpha)/2} J_\alpha(2(tX_j/\alpha)^{1/2}) - e^{-t/\alpha} \right],\]
\[Z_{n,3}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ \Gamma(\alpha)(tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_j/\alpha)^{1/2}) + \alpha^{-2}(X_j - \alpha) e^{-t/\alpha} - e^{-t/\alpha} \right].\]

We will show that
\[Z_{n,3} \overset{d}{\rightarrow} Z \text{ in } L^2,\]
\[\|Z_n - Z_{n,1}\|_{L^2} \overset{p}{\rightarrow} 0,\]
\[\|Z_{n,1} - Z_{n,2}\|_{L^2} \overset{p}{\rightarrow} 0,\]
\[\|Z_{n,2} - Z_{n,3}\|_{L^2} \overset{p}{\rightarrow} 0.\]
To establish (10.6), let
\[
Z_{n,3,j}(t) := \Gamma(\alpha) \left( t X_j / \alpha \right)^{(1-\alpha)/2} J_{\alpha-1} \left( 2 \left( t X_j / \alpha \right)^{1/2} \right) + \alpha^{-2} (X_j - \alpha) t e^{-t/\alpha} - e^{-t/\alpha},
\]
(10.10)
\[t \geq 0, \; j = 1, \ldots, n.\] Since \(X_j \sim \text{Gamma}(\alpha, 1)\) then \(E(X_j - \alpha) = 0\); also, by Example 1,
\[
E \left[ \Gamma(\alpha) (t X_j / \alpha)^{(1-\alpha)/2} J_{\alpha-1} \left( 2 \left( t X_j / \alpha \right)^{1/2} \right) \right] = e^{-t/\alpha}.
\]
Therefore \(E(Z_{n,3,j}(t)) = 0, \; t \geq 0\) and \(j = 1, \ldots, n,\) and \(Z_{n,3,1}, \ldots, Z_{n,3,n}\) clearly are i.i.d. random elements in \(L^2\). Applying the Cauchy–Schwarz inequality and (9.4), we obtain \(E(\|Z_{n,3,1}\|_L^2)^2 < \infty.\) Thus, by the Central Limit Theorem in \(L^2\) (Ledoux and Talagrand 1991, p. 281),
\[
Z_{n,3} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Z_{n,3,j} \overset{d}{\longrightarrow} Z,
\]
where \(Z := (Z(t), \; t \geq 0)\) is a centered Gaussian random element in \(L^2.\) This proves (10.6) and shows that \(Z\) has the same covariance operator as \(Z_{n,3,1}.\)

It is well-known that the covariance operator of the random element \(Z_{n,3,1}\) is uniquely determined by the covariance function of the stochastic process \(Z_{n,3,1}(t)\) (Gikhman and Skorokhod 1980, pp. 218–219). We now show that the function \(K(s, t)\) in (3.4) is the covariance function of \(Z_{n,3,1}.\) Noting that \(E[Z_{n,3,1}(t)] = 0\) for all \(t,\) we obtain
\[
K(s, t) = \text{Cov}[Z_{n,3,1}(s), Z_{n,3,1}(t)]
\]
\[= \text{Cov}[Z_{n,3,1}(s) + e^{-s/\alpha}, Z_{n,3,1}(t) + e^{-t/\alpha}]
\]
\[= E[(Z_{n,3,1}(s) + e^{-s/\alpha})(Z_{n,3,1}(t) + e^{-t/\alpha})] - e^{-(s+t)/\alpha}.
\]
By (10.10),
\[
E(Z_{n,3,1}(s) + e^{-s/\alpha})(Z_{n,3,1}(t) + e^{-t/\alpha})
\]
\[= E \left[ \Gamma(\alpha) \left( s X_1 / \alpha \right)^{(1-\alpha)/2} J_{\alpha-1} \left( 2 \left( s X_1 / \alpha \right)^{1/2} \right) + \alpha^{-2} (X_1 - \alpha) s e^{-s/\alpha} \right]
\]
\[\times \left[ \Gamma(\alpha) \left( t X_1 / \alpha \right)^{(1-\alpha)/2} J_{\alpha-1} \left( 2 \left( t X_1 / \alpha \right)^{1/2} \right) + \alpha^{-2} (X_1 - \alpha) t e^{-t/\alpha} \right],
\]
(10.11)
so the calculation of \(K(s, t)\) reduces to evaluating the four terms obtained by expanding the product on the right-hand side of (10.11).

The first term in the product in (10.11) is evaluated using Weber’s integral (10.1):
\[
E \left[ \Gamma(\alpha)^2 \left( s X_1 / \alpha \right)^{(1-\alpha)/2} \right) (t X_1 / \alpha)^{(1-\alpha)/2} \right) J_{\alpha-1} \left( 2 \left( s X_1 / \alpha \right)^{1/2} \right) J_{\alpha-1} \left( 2 \left( t X_1 / \alpha \right)^{1/2} \right)
\]
\[= \Gamma(\alpha) (st/\alpha^2)^{(1-\alpha)/2} e^{-(s+t)/\alpha} I_{\alpha-1} \left( 2 \sqrt{st} / \alpha \right).
\]
(10.12)
The second term in the product in (10.11) is a Hankel transform of the type in Example 1,

\[ E \left[ \Gamma(\alpha)(sX_1/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(sX_1/\alpha)^{1/2}) \alpha^{-2}(X_1 - \alpha)te^{-t/\alpha} \right] = -\alpha^{-3}st \exp \left( - (s + t)/\alpha \right), \]

and the third term in the product is the same as the second term but with \( s \) and \( t \) interchanged.

The fourth term in the product in (10.11) is

\[ E \left[ \alpha^{-4}(X_1 - \alpha)^2ste^{-(s+t)/\alpha} \right] = \alpha^{-4}ste^{-(s+t)/\alpha} \text{Var}(X_1) = \alpha^{-3}ste^{-(s+t)/\alpha}. \]

Combining all four terms, we obtain (3.4).

To establish (10.7), we begin by showing that

\[ (\sqrt{n}W_n)^2 = \left( \frac{\sqrt{n}(X_n - \alpha)}{(\alpha X_n)^{1/2}(\alpha^{1/2} + X_n^{1/2})} \right)^2 \xrightarrow{d} \chi_1^2/4\alpha^2, \]

where \( \chi_1^2 \) denotes a chi-square random variable with one degree of freedom. By the Central Limit Theorem, \( \sqrt{n}(X_n - \alpha) \xrightarrow{d} \mathcal{N}(0,\alpha) \), and by the Law of Large Numbers and the Continuous Mapping Theorem, \( (\alpha X_n)^{1/2}(\alpha^{1/2} + X_n^{1/2}) \xrightarrow{p} 2\alpha^{3/2} \). By Slutsky's theorem (Chow and Teicher 1988, p. 249), \( \sqrt{n}W_n \xrightarrow{d} \mathcal{N}(0, \frac{1}{4}\alpha^{-2}) \), hence \( (\sqrt{n}W_n)^2 \xrightarrow{d} \chi_1^2/4\alpha^2 \).

By the Taylor expansion in (10.5),

\[ Z_n - Z_{n,1} = \frac{\Gamma(\alpha)}{\sqrt{n}} \sum_{j=1}^{n} \left[ (tY_j)^{(1-\alpha)/2} J_{\alpha-1}(2(tY_j)^{1/2}) \right. \]

\[ - (tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_j/\alpha)^{1/2}) \]

\[ - 2\alpha^{1/2} W_n(tX_j/\alpha)^{(1-\alpha)/2} \left. J_{\alpha}(2(tX_j/\alpha)^{1/2}) \right] \]

\[ = \frac{2\alpha \Gamma(\alpha)}{n} \left( \sqrt{n}W_n \right)^2 \sum_{j=1}^{n} (tX_j)^{1/2} \left[ u_j^{1-\alpha} J_{\alpha}(u_j) \right. \]

\[ - (2(tX_j/\alpha)^{1/2})^{1-\alpha} J_{\alpha}(2(tX_j/\alpha)^{1/2}) \].

Define

\[ V_n := \frac{1}{n^2} \int_{0}^{\infty} \left[ \sum_{j=1}^{n} (tX_j)^{1/2} \left( u_j^{1-\alpha} J_{\alpha}(u_j) \right. \right. \]

\[- \left. (2(tX_j/\alpha)^{1/2})^{1-\alpha} J_{\alpha}(2(tX_j/\alpha)^{1/2}) \right] \right]^2 dP_0(t). \]
Then, \( \|Z_n - Z_{n,1}\|_{L^2}^2 = 4^\alpha [\Gamma(\alpha)]^2 (\sqrt{n} W_n)^2 V_n \). By the Cauchy–Schwarz inequality,

\[
V_n \leq \frac{1}{n} \int_0^\infty t \sum_{j=1}^n X_j \left| u_j^{1-\alpha} J_\alpha(u_j) - (2(t X_j/\alpha)^{1/2})^{1-\alpha} J_\alpha(2(t X_j/\alpha)^{1/2}) \right|^2 \, dP_0(t).
\]

Recall that \( u_j \) lies between \( 2(t Y_j)^{1/2} \) and \( 2(t X_j/\alpha)^{1/2} \), so we can write

\[
\begin{align*}
& \quad u_j = 2(1 - \theta_{n,j,t})(t X_j/\alpha)^{1/2} + 2\theta_{n,j,t}(t Y_j)^{1/2} \\
& = (2(t X_j)^{1/2} + \theta_{n,j,t}(X_n^{-1/2} - \alpha^{-1/2})),
\end{align*}
\]

where \( \theta_{n,j,t} \in [0, 1] \). By Lemma 4, the Lipschitz property of the Bessel functions,

\[
4^\alpha [\Gamma(\alpha + 1)]^2 \left| u_j^{1-\alpha} J_\alpha(u_j) - (2(t X_j/\alpha)^{1/2})^{1-\alpha} J_\alpha(2(t X_j/\alpha)^{1/2}) \right|^2 
\leq |u_j - 2(t X_j/\alpha)^{1/2}|^2 
= |2(t X_j)^{1/2} \theta_{n,j,t}(X_n^{-1/2} - \alpha^{-1/2})|^2 
\leq 4t X_j (X_n^{-1/2} - \alpha^{-1/2})^2,
\]

since \( \theta_{n,j,t} \in [0, 1] \). Therefore,

\[
V_n \leq \frac{1}{4^\alpha - 1}[\Gamma(\alpha + 1)]^2 \left( \frac{1}{n} \sum_{j=1}^n X_j^2 \right) (X_n^{-1/2} - \alpha^{-1/2})^2 \int_0^\infty t^2 \, dP_0(t).
\]

By the Law of Large Numbers, \( (X_n^{-1/2} - \alpha^{-1/2})^2 \overset{P}{\to} 0 \) and \( n^{-1} \sum_{j=1}^n X_j^2 \overset{P}{\to} E(X_1^2) = \alpha(\alpha + 1) \), so it follows that \( V_n \overset{P}{\to} 0 \). By Slutsky’s theorem, \( \|Z_n - Z_{n,1}\|_{L^2}^2 \overset{d}{=} 4^\alpha [\Gamma(\alpha)]^2 (\sqrt{n} W_n)^2 \cdot V_n \overset{d}{\to} 0 \), therefore \( \|Z_n - Z_{n,1}\|_{L^2}^2 \overset{P}{\to} 0 \), as asserted in (10.7).

To establish (10.8), define

\[
\Delta_j(t) := \Gamma(\alpha)(t X_j/\alpha)^{1-(\alpha/2)} J_\alpha(2(t X_j/\alpha)^{1/2}) - \alpha^{-1} t e^{-t/\alpha},
\]

\( t \geq 0, \ j = 1, \ldots, n \). Then it is straightforward to verify that

\[
Z_{n,1} - Z_{n,2} = \frac{2\alpha^{1/2}}{\sqrt{n}} W_n \sum_{j=1}^n \Delta_j(t)
\]

and therefore

\[
\|Z_{n,1} - Z_{n,2}\|_{L^2}^2 = (2\alpha^{1/2} W_n)^2 \int_0^\infty \left[ \frac{1}{\sqrt{n}} \sum_{j=1}^n \Delta_j(t) \right]^2 \, dP_0(t). \quad (10.13)
\]
By the Law of Large Numbers, $W_n \xrightarrow{p} 0$. Also, as shown in Example 3,

$$E\left[\Gamma(\alpha)(tX_j/\alpha)^{1-(\alpha/2)} J_\alpha(2(tX_j/\alpha)^{1/2})\right] = \alpha^{-1} t e^{-t/\alpha},$$

hence $E(\Delta_j(t)) = 0$, $t \geq 0$, $j = 1, \ldots, n$. Also, $\Delta_1(t), \ldots, \Delta_n(t)$ are i.i.d. random elements in $L^2$. We now show that $E(\|\Delta_1\|^2_{L^2}) < \infty$. We have

$$E(\|\Delta_1\|^2_{L^2}) = E \int_0^\infty \Delta_1^2(t) \, dP_0(t)$$

$$= E \int_0^\infty \left[\Gamma(\alpha)(tX_1/\alpha)^{1-(\alpha/2)} J_\alpha(2(tX_1/\alpha)^{1/2}) - \alpha^{-1} t e^{-t/\alpha}\right]^2 \, dP_0(t).$$

To show that $E(\|\Delta_1\|^2_{L^2}) < \infty$ it suffices, by the Cauchy–Schwarz inequality, to prove that

$$E \int_0^\infty \left[\Gamma(\alpha)(tX_1/\alpha)^{1-(\alpha/2)} J_\alpha(2(tX_1/\alpha)^{1/2}) \right]^2 \, dP_0(t) < \infty \quad (10.14)$$

and

$$E \int_0^\infty (\alpha^{-1} t e^{-t/\alpha})^2 \, dP_0(t) < \infty. \quad (10.15)$$

To establish (10.14), we apply the inequality (9.5) to obtain

$$|J_\alpha(2(tX_1/\alpha)^{1/2})| \leq (tX_1/\alpha)^{-(1-\alpha)/2} / \pi^{1/2} \Gamma(\alpha + \frac{1}{2}),$$

for $t \geq 0$. Therefore,

$$E \int_0^\infty \left[\Gamma(\alpha)(tX_1/\alpha)^{1-(\alpha/2)} J_\alpha(2(tX_1/\alpha)^{1/2}) \right]^2 \, dP_0(t)$$

$$\leq \left(\frac{\Gamma(\alpha)}{\pi^{1/2} \Gamma(\alpha + \frac{1}{2})}\right)^2 E(X_1/\alpha) \int_0^\infty t \, dP_0(t) < \infty.$$

As for (10.15), that expectation is a convergent gamma integral. Hence, $E(\|\Delta_1\|^2_{L^2}) < \infty$.

By the Central Limit Theorem in $L^2$, $n^{-1/2} \sum_{j=1}^n \Delta_j(t)$ converges to a centered Gaussian random element in $L^2$. Thus, by the Continuous Mapping Theorem,

$$\left\| \frac{1}{\sqrt{n}} \sum_{j=1}^n \Delta_j(t) \right\|_{L^2}^2 := \int_0^\infty \left[\frac{1}{\sqrt{n}} \sum_{j=1}^n \Delta_j(t) \right]^2 \, dP_0(t)$$

converges in distribution to a random variable which has finite variance. Since $W_n \xrightarrow{p} 0$ then by (10.13) and Slutsky’s Theorem, we obtain $\|Z_{n,1} - Z_{n,2}\|^2_{L^2} \xrightarrow{d} 0$; therefore, $\|Z_{n,1} - Z_{n,2}\|_{L^2} \xrightarrow{p} 0$. 

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To prove (10.9), we observe that
\[ Z_{n,2} - Z_{n,3} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left( 2\alpha^{-1/2} W_{n} t e^{-t/\alpha} - \alpha^{-2} (X_j - \alpha) t e^{-t/\alpha} \right) \]
\[ = t e^{-t/\alpha} \sqrt{n(X_n - \alpha)} R_n, \]
where
\[ R_n = \frac{2}{\alpha X_n^{1/2} (\alpha^{1/2} + X_n^{1/2})} - \frac{1}{\alpha^2}. \]

Therefore,
\[ \|Z_{n,2} - Z_{n,3}\|_{L^2}^2 = \left[ \sqrt{n(X_n - \alpha)} R_n \right]^2 \int_{0}^{\infty} (te^{-t/\alpha})^2 \, dP_0(t). \]

As noted earlier, \( \int_{0}^{\infty} (te^{-t/\alpha})^2 \, dP_0(t) < \infty. \) Also, by the Central Limit Theorem, \( \sqrt{n(X_n - \alpha)} \overset{d}{\to} \mathcal{N}(0, \alpha); \) and by the Law of Large Numbers, \( R_n \overset{p}{\to} 0. \) By Slutsky’s theorem, \( \left[ \sqrt{n(X_n - \alpha)} R_n \right]^2 \overset{d}{\to} 0; \) hence \( \left[ \sqrt{n(X_n - \alpha)} R_n \right]^2 \overset{p}{\to} 0, \) and therefore \( \|Z_{n,2} - Z_{n,3}\|_{L^2} \overset{p}{\to} 0. \)

Finally, by the Continuous Mapping Theorem in \( L^2, \) \( \|Z_n\|_{L^2} \overset{d}{\to} \|Z\|_{L^2}, \) i.e.
\[ T_n^2 = \int_{0}^{\infty} Z_n^2(t) \, dP_0(t) \overset{d}{\to} \int_{0}^{\infty} Z^2(t) \, dP_0(t). \]

The proof now is complete. \( \square \)

11 Appendix 3: Eigenvalues and eigenfunctions of the covariance operator

**Proof of Theorem 5** Since the set \( \{ \mathbb{I}_k^{(\alpha-1)} : k \in \mathbb{N}_0 \} \) is an orthonormal basis for \( L^2, \) the eigenfunction \( \phi \in L^2 \) corresponding to an eigenvalue \( \delta \) can be written as
\[ \phi = \sum_{k=0}^{\infty} \langle \phi, \mathbb{I}_k^{(\alpha-1)} \rangle_{L^2} \mathbb{I}_k^{(\alpha-1)}. \]

We restrict ourselves temporarily to eigenfunctions for which this series is pointwise convergent. Substituting this series into the equation \( S\phi = \delta\phi, \) we obtain
\[ \int_{0}^{\infty} K(s, t) \sum_{k=0}^{\infty} \langle \phi, \mathbb{I}_k^{(\alpha-1)} \rangle_{L^2} \mathbb{I}_k^{(\alpha-1)}(t) \, dP_0(t) = \delta \sum_{k=0}^{\infty} \langle \phi, \mathbb{I}_k^{(\alpha-1)} \rangle_{L^2} \mathbb{I}_k^{(\alpha-1)}(s). \]

\[ (11.1) \]
Substituting the covariance function $K(s, t)$ in the left-hand side of (11.1), writing $K$ in terms of $K_0$, and assuming that we can interchange the order of integration and summation, we obtain

$$\delta \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} \mathcal{L}_k^{(\alpha-1)}(s)$$

$$= \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} \int_{0}^{\infty} \left[ K_0(s, t) - e^{-(s+t)/\alpha} (\alpha^{-3} st + 1) \right] \mathcal{L}_k^{(\alpha-1)}(t) \, dP_0(t).$$

(11.2)

By Theorem 3,

$$\int_{0}^{\infty} K_0(s, t) \mathcal{L}_k^{(\alpha-1)}(t) \, dP_0(t) = \rho_k \mathcal{L}_k^{(\alpha-1)}(s).$$

On writing $\mathcal{L}_k^{(\alpha-1)}$ in terms of $L_k^{(\alpha-1)}$, the generalized Laguerre polynomial, applying the well-known formula (Olver et al. 2010, (18.17.34)) for the Laplace transform of $L_k^{(\alpha-1)}$, and making use of (4.2) and (4.3), we obtain

$$\langle e^{-t/\alpha}, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} := \int_{0}^{\infty} e^{-t/\alpha} \mathcal{L}_k^{(\alpha-1)}(t) \, dP_0(t) = \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \beta^{\alpha/2} \rho_k.$$  (11.3)

Again writing $\mathcal{L}_k^{(\alpha-1)}$ in terms of $L_k^{(\alpha-1)}$, applying Lemma 5, and (4.2) and (4.3), we obtain

$$\langle te^{-t/\alpha}, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} := \int_{0}^{\infty} te^{-t/\alpha} \mathcal{L}_k^{(\alpha-1)}(t) \, dP_0(t)$$

$$= \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \alpha^2 \beta^{\alpha/2} \rho_k (b_\alpha^2 - k\beta).$$  (11.4)

In summary, (11.2) reduces to

$$\delta \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} \mathcal{L}_k^{(\alpha-1)}(s)$$

$$= \sum_{k=0}^{\infty} \rho_k \langle \phi, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} \left[ \mathcal{L}_k^{(\alpha-1)}(s) - e^{-s/\alpha} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \beta^{\alpha/2} (\alpha^{-1} s (b_\alpha^2 - k\beta) + 1) \right].$$

(11.5)

By applying (11.3), we also obtain the Fourier-Laguerre expansion of $e^{-s/\alpha}$ with respect to the orthonormal basis $\{ \mathcal{L}_k^{(\alpha-1)} : k \in \mathbb{N}_0 \}$; indeed,
Let

\[ c_1 = \int_0^{\infty} e^{-t/\alpha} \phi(t) \, dP_0(t) = \beta^{\alpha/2} \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \rho_k, \]  

(11.6)

and

\[ c_2 = \int_0^{\infty} t e^{-t/\alpha} \phi(t) \, dP_0(t) = \alpha^2 \beta^{\alpha/2} \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \rho_k (b_\alpha^2 - k \beta). \]  

(11.7)

Combining (11.5)–(11.7), we find that (11.1) reduces to

\[ \delta \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} \mathcal{L}_k^{(\alpha^{-1})} (s) \]

\[ = \sum_{k=0}^{\infty} \rho_k \left[ \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} - \beta^{\alpha/2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \left( c_1 + c_2 \alpha^{-1} (b_\alpha^2 - \beta) \right) \right] \mathcal{L}_k^{(\alpha^{-1})} (s), \]

(11.8)

and now comparing the coefficients of \( \mathcal{L}_k^{(\alpha^{-1})} (s) \), we obtain

\[ \delta \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} = \rho_k \left[ \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} - \beta^{\alpha/2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \left( c_1 + c_2 \alpha^{-1} (b_\alpha^2 - \beta) \right) \right], \]

(11.9)

for all \( k \in \mathbb{N}_0 \). Since we have assumed that \( \delta \neq \rho_k \) for any \( k \) then we can solve this equation for \( \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} \) to obtain

\[ \langle \phi, \mathcal{L}_k^{(\alpha^{-1})} \rangle_{L^2} = \beta^{\alpha/2} \frac{\rho_k}{\rho_k - \delta} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \left( c_1 + c_2 \alpha^{-1} (b_\alpha^2 - \beta) \right). \]  

(11.10)
Substituting (11.10) into (11.6), we get
\[
c_1 = c_1 \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 + c_2 \alpha^{-1} \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta)
\]
\[
= c_1 (1 - A(\delta)) + c_2 \alpha^{-3} D(\delta);
\]
therefore,
\[
\alpha^3 c_1 A(\delta) = c_2 D(\delta). \tag{11.11}
\]

Similarly, by substituting (11.10) into (11.7), we obtain
\[
c_2 = c_1 \alpha^2 \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta) + c_2 \alpha \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta)^2
\]
\[
= c_1 D(\delta) + c_2 (1 - B(\delta));
\]
hence,
\[
c_2 B(\delta) = c_1 D(\delta). \tag{11.12}
\]

Suppose \(c_1 = c_2 = 0\); then it follows from (11.10) that \(\langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} = 0\) for all \(k\) and so \(\phi = 0\), which is a contradiction since \(\phi\) is a non-trivial eigenfunction. Hence, \(c_1\) and \(c_2\) cannot be both equal to 0. Combining (11.11) and (11.12), and using the fact that \(c_1, c_2\) are not both 0, it is straightforward to deduce that \(\alpha^3 A(\delta) B(\delta) = D^2(\delta)\). Therefore, if \(\delta\) is a positive eigenvalue of \(S\) then it is a positive root of the function \(G(\delta) = \alpha^3 A(\delta) B(\delta) - D^2(\delta)\).

Conversely, suppose that \(\delta\) is a positive root of \(G(\delta)\) with \(\delta \neq \rho_k\) for any \(k \in \mathbb{N}_0\). Define
\[
\gamma_k := \beta^{\alpha/2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \frac{\rho_k}{\rho_k - \delta} \left( c_1 + c_2 \alpha^{-1} (b_\alpha^2 - k\beta) \right), \tag{11.13}
\]
k \(\in \mathbb{N}_0\), where \(c_1\) and \(c_2\) are real constants that are not both equal to 0 and which satisfy (11.11) and (11.12). That such constants exist can be shown by following a case-by-case argument similar to Taherizadeh (2009, p. 48); for example, if \(D(\delta) \neq 0\), \(A(\delta) \neq 0\), and \(B(\delta) \neq 0\), then we can choose \(c_2\) to be any non-zero number and then set \(c_1 = c_2 B(\delta)/D(\delta)\).

Define
\[
\tilde{\phi}(s) := \sum_{k=0}^{\infty} \gamma_k \mathfrak{L}_k^{(\alpha-1)}(s), \tag{11.14}
\]
s \(\geq 0\). By applying the ratio test, we find that \(\sum_{k=0}^{\infty} \gamma_k^2 < \infty\); therefore, \(\tilde{\phi} \in L^2\).
To show also that (11.14) converges pointwise, we apply (9.11), (4.4), and a Laguerre polynomial inequality (Erdélyi et al. 1953, p. 207) to obtain
\[
|L_k(\alpha-1)(s)| = \beta^{\alpha/2} \exp((1 - \beta)s/2) \left( \frac{k!}{(\alpha)_k} \right)^{1/2} |L_k(\alpha-1)(\beta s)|
\]
\[
\leq \begin{cases} 
\frac{\beta^{\alpha/2}}{2} & 1/2 \leq \alpha < 1 \\
\frac{\beta^{\alpha/2}}{\alpha} & \alpha \geq 1
\end{cases}
\]  
(11.15)

for \( s \geq 0 \). Thus, to establish that (11.14) pointwise converges pointwise, we need to show that
\[
\sum_{k=0}^{\infty} \left( \frac{\alpha_k}{k!} \right)^{1/2} |\gamma_k| < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \left( \frac{k!}{\alpha_k} \right)^{1/2} |\gamma_k| < \infty.
\]
(11.16)

However, the convergence of each of these series follows from the ratio test.

Next, we justify the interchange of summation and integration in our calculations. By a corollary to Theorem 16.7 in Billingsley (1979, p. 224), we need to verify that
\[
\sum_{k=0}^{\infty} |\gamma_k| \int_{0}^{\infty} K(s, t) |\mathbf{I}_k^{(\alpha-1)}(t)| \, dP_0(t) < \infty.
\]
(11.17)

By (9.10) and (4.1),
\[
0 \leq K_0(s, t) \leq \exp(-(s + t)/\alpha) \exp(2\sqrt{s t}/\alpha) = \exp(-((\sqrt{s} - \sqrt{t})^2)/\alpha) \leq 1.
\]
(11.18)

By the triangle inequality and by (11.18), we have
\[
0 \leq K(s, t) \leq K_0(s, t) + (\alpha^{-3}s t + 1) \exp(-(s + t)/\alpha) \leq 2 + \alpha^{-3}s t,
\]
\( s, t \geq 0 \). Thus, to prove (11.17), we need to establish that
\[
\sum_{k=0}^{\infty} |\gamma_k| \int_{0}^{\infty} (2 + \alpha^{-3}s t) |\mathbf{I}_k^{(\alpha-1)}(t)| \, dP_0(t) < \infty.
\]

By applying the bound (11.15), we see that it suffices to prove that
\[
\sum_{k=0}^{\infty} \left( \frac{\alpha_k}{k!} \right)^{1/2} |\gamma_k| \int_{0}^{\infty} t^j \, dP_0(t) < \infty
\]
and

\[ \sum_{k=0}^{\infty} \left( \frac{k!}{(\alpha)_k} \right)^{1/2} |\gamma_k| \int_0^{\infty} t^i \, dP_0(t) < \infty, \]

\( j = 0, 1. \) As these integrals are finite, the convergence of both series follows from (11.16).

To calculate \( S\tilde{\phi}(s) \) from (11.14), we follow the same steps as before to obtain

\[ S\tilde{\phi}(s) = \int_0^{\infty} K(s, t) \sum_{k=0}^{\infty} \gamma_k \Lambda_k^{(\alpha-1)}(t) \, dP_0(t) \]

\[ = \sum_{k=0}^{\infty} \rho_k \gamma_k \Lambda_k^{(\alpha-1)}(s) - c_1 \beta^{\alpha/2} \sum_{k=0}^{\infty} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \rho_k \Lambda_k^{(\alpha-1)}(s) \]

\[ - c_2 \alpha^{-1} \beta^{\alpha/2} \sum_{k=0}^{\infty} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \rho_k (b_\alpha - k \beta) \Lambda_k^{(\alpha-1)}(s). \]

By the definition (11.13) of \( \gamma_k \), and noting that

\[ \frac{\rho_k}{\rho_k - \delta} - 1 = \frac{\delta}{\rho_k - \delta}, \]

we have

\[ S\tilde{\phi}(s) = \beta^{\alpha/2} \sum_{k=0}^{\infty} \left[ \frac{\rho_k}{\rho_k - \delta} - 1 \right] \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \rho_k (c_1 + c_2 \alpha^{-1} (b_\alpha - k \beta)) \Lambda_k^{(\alpha-1)}(s) \]

\[ = \beta^{\alpha/2} \delta \sum_{k=0}^{\infty} \frac{\rho_k}{\rho_k - \delta} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} (c_1 + c_2 \alpha^{-1} (b_\alpha - k \beta)) \Lambda_k^{(\alpha-1)}(s) \]

\[ = \delta \sum_{k=0}^{\infty} \gamma_k \Lambda_k^{(\alpha-1)}(s) = \delta \tilde{\phi}(s). \]

Therefore, \( \delta \) is an eigenvalue of \( S \) with corresponding eigenfunction \( \tilde{\phi} \).  \( \square \)

A proof that Conjecture 2 implies Conjecture 1. Suppose there exists \( l \in \mathbb{N}_0 \) such that \( \delta = \rho_l \). Substituting \( k = l \) in (11.9) and simplifying the outcome, we obtain

\[ c_1 = c_2 \alpha^{-1} (l \beta - b_\alpha^2). \]  (11.19)

Substituting \( \delta = \rho_l \) in (11.8), applying (11.19), and cancelling common terms in (11.8), we obtain

\[ \langle \phi, \Lambda_k^{(\alpha-1)} \rangle_{L^2} = c_2 \alpha^{-1} \beta^{(2+\alpha)/2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \frac{l - k}{\rho_k - \rho_l} \rho_k, \]  (11.20)

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for \( k \neq l \). Substituting this result for the inner product into (11.6), we obtain

\[
c_1 = \beta^{\alpha/2} \left[ \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \rho_k \right]^2
\]

\[
+ \langle \phi, \mathcal{L}_l^{(\alpha-1)} \rangle_{L^2} \left( \frac{(\alpha)_l}{l!} \right)^{1/2} \rho_l
\]

\[
= \beta^{\alpha/2} \left[ \sum_{k=0}^{\infty} c_2 \alpha^{-1} \beta^{(2+\alpha)/2} \frac{(\alpha)_k}{k!} \rho_k \frac{l - k}{\rho_k - \rho_l} \rho_k^2 \right]^2
\]

Similarly, substituting (11.20) into (11.7), we obtain

\[
c_2 = \alpha^2 \beta^{\alpha/2} \left[ \sum_{k=0}^{\infty} \langle \phi, \mathcal{L}_k^{(\alpha-1)} \rangle_{L^2} \left( \frac{(\alpha)_k}{k!} \right)^{1/2} \rho_k (b^2 - k \beta) \right]^2
\]

\[
+ \langle \phi, \mathcal{L}_l^{(\alpha-1)} \rangle_{L^2} \left( \frac{(\alpha)_l}{l!} \right)^{1/2} \rho_l (b^2 - l \beta)
\]

\[
= \alpha^2 \beta^{\alpha/2} \left[ c_2 \alpha^{-1} \beta^{(2+\alpha)/2} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{l - k}{\rho_k - \rho_l} \rho_k^2 (b^2 - k \beta) \right]^2
\]

On simplifying the above expressions and substituting for \( c_1 \) from (11.19), we obtain
\[
\beta^{\alpha/2} \left( \frac{(\alpha)_l}{l!} \right)^{1/2} \rho_l \langle \phi, \mathfrak{I}_l^{(\alpha-1)} \rangle_{L^2} = c_2 \left[ \alpha^{-1} (l \beta - b_\alpha^2) - \alpha^{-1} \beta^{\alpha+1} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{l - k}{\rho_k - \rho_l^2} \right], \tag{11.21}
\]
and
\[
\alpha^2 \beta^{\alpha/2} \left( \frac{(\alpha)_l}{l!} \right)^{1/2} \rho_l (b_\alpha^2 - l \beta) \langle \phi, \mathfrak{I}_l^{(\alpha-1)} \rangle_{L^2} = c_2 \left[ 1 - \alpha \beta^{\alpha+1} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{l - k}{\rho_k - \rho_l^2} (b_\alpha^2 - k \beta) \right]. \tag{11.22}
\]

Suppose that \( c_2 = 0 \) then it follows from (11.19) that \( c_1 = 0 \), which contradicts the earlier observation that \( c_1 \) and \( c_2 \) are not both zero; therefore, \( c_2 \neq 0 \). Also, by (4.2), \( b_\alpha^2 < 1 < \beta \), so \( b_\alpha^2 - k \beta \neq 0 \) for all \( k \in \mathbb{N}_0 \). Solving (11.21) and (11.22) for the inner product \( \langle \phi, \mathfrak{I}_l^{(\alpha-1)} \rangle_{L^2} \) and equating the two expressions, we obtain
\[
1 - \alpha \beta^{\alpha+1} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{l - k}{\rho_k - \rho_l^2} (b_\alpha^2 - k \beta) = \alpha (b_\alpha^2 - l \beta) \left[ (l \beta - b_\alpha^2) - \beta^{\alpha+1} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{l - k}{\rho_k - \rho_l^2} \right].
\]

Simplifying the above equation, we obtain (4.5). □

A \( C^\infty \) kernel \( K : \mathbb{R}^2 \to \mathbb{R} \) is extended totally positive (ETP) if for all \( r \geq 1 \), all \( s_1 \geq \cdots \geq s_r \), all \( t_1 \geq \cdots \geq t_r \), there holds
\[
\det \left( K(s_i, t_j) \right) / \prod_{1 \leq i < j \leq r} (s_i - s_j)(t_i - t_j) > 0, \tag{11.23}
\]
where instances of equality for the variables \( s_i \) and \( t_j \) are to be understood as limiting cases, and then L’Hospital’s rule is to be used to evaluate this ratio.

**Proof of Proposition 2** By (3.4), the kernel \( K(s, t) \) is of the form
\[
K(s, t) = e^{-(s+t)/\alpha_s^2} s^{\alpha_s^2} \sum_{k=0}^{\infty} c_k s^k t^k,
\]
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where the coefficients \( c_k \) are positive for all \( k = 0, 1, 2, \ldots \). Therefore,

\[
\det \left( K(s_i, t_j) \right) = \det \left( e^{-(s_i + t_j) / \alpha s_i^2 t_j^2} \sum_{k=0}^{\infty} c_k s_i^k t_j^k \right)
\]

\[
= \left( \prod_{i=1}^{r} e^{-(s_i + t_i) / \alpha s_i^2 t_i^2} \right) \cdot \det \left( \sum_{k=0}^{\infty} c_k s_i^k t_j^k \right).
\]

By Karlin (1964, p. 101) the series \( \sum_{k=0}^{\infty} c_k s_i^k t_j^k \) is ETP so, by (11.23), \( K(s, t) \) is ETP.

In the case of \( K_0 \), we have

\[
K_0(s, t) = e^{-(s+t)/\alpha} \sum_{k=0}^{\infty} c_k s^k t^k,
\]

where \( c_k > 0 \) for all \( k \). Then it follows by a similar argument that \( K_0(s, t) \) is ETP.

By a result of Karlin (1964), the eigenvalues of an integral operator are simple and positive if the kernel of the operator is ETP. Therefore, the eigenvalues of \( S \) and \( S_0 \) are simple and positive. In particular, 0 is not an eigenvalue of \( S \) or \( S_0 \), so both operators are injective. Also, the oscillation property (4.8) follows from Karlin (1964, Theorem 3).

Proof of Proposition 3 Define the kernels \( k_0(s, t) = -e^{-(s+t)/\alpha} \) and \( k_1(s, t) = -e^{-(s+t)/\alpha} \alpha^{-3} s t, s, t \geq 0 \). Also, define on \( L^2 \) the corresponding integral operators,

\[
\mathcal{U}_j f(s) = \int_0^\infty k_j(s, t) f(t) dP_0(t),
\]

\( j = 0, 1, s \geq 0 \). Then it follows from (3.4) that \( S = S_0 + \mathcal{U}_0 + \mathcal{U}_1 \).

Each \( \mathcal{U}_j \) clearly is self-adjoint and of rank one, i.e., the range of \( \mathcal{U}_j \) is a one-dimensional subspace of \( L^2 \). Also, \( S_0 + \mathcal{U}_0 \) is compact and self-adjoint and its kernel, \( K_0 + k_0 \), is of the form

\[
K_0(s, t) + k_0(s, t) = e^{-(s+t)/\alpha} s t \sum_{j=0}^{\infty} c_j s^j t^j,
\]

where \( c_j > 0 \) for all \( j \). Arguing as in the proof of Proposition 2, we find that the eigenvalues of \( S_0 + \mathcal{U}_0 \) are simple and positive; hence, \( S_0 + \mathcal{U}_0 \) is injective.

Let \( \omega_k, k \geq 1 \), be the eigenvalues of \( S_0 + \mathcal{U}_0 \), where \( \omega_1 > \omega_2 > \cdots \). Since \( S_0 \) is compact, self-adjoint, and injective, and since \( \mathcal{U}_0 \) is self-adjoint and of rank one then, by Hochstadt (1973) or Dancis and Davis (1987), the eigenvalues of \( S_0 \) and \( S_0 + \mathcal{U}_0 \) are interlaced: \( \rho_{k-1} \geq \omega_k \geq \rho_k \) for all \( k \geq 1 \). Also, there is exactly one eigenvalue of \( S_0 + \mathcal{U}_0 \) in one of the intervals \( [\rho_k, \rho_{k-1}], (\rho_k, \rho_{k-1}), \) or \( (\rho_k, \rho_{k-1}] \).

Since \( \mathcal{U}_1 \) is self-adjoint and of rank one then by Hochstadt’s theorem, the eigenvalues of \( S_0 + \mathcal{U}_0 \) and \( S_0 + \mathcal{U}_0 + \mathcal{U}_1 = S \) are interlaced: \( \omega_k \geq \delta_k \geq \omega_{k+1} \) for all \( k \geq 1 \). Also,
there is exactly one eigenvalue of $S$ in one of the intervals $[\omega_{k+1}, \omega_k)$, $(\omega_{k+1}, \omega_k)$, or $(\omega_k, \omega_{k+1}]$.

Combining these interlacing results, we obtain $\rho_{k-1} \geq \delta_k \geq \rho_{k+1}$, $k \geq 1$. Also, since $\rho_k = \alpha^k t^{2k+2\alpha}$ then, by the interlacing inequalities, $\delta_k = O(t^{2\alpha})$, hence $\delta_k = O(\rho_k)$.

\[\Box\]

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