Quadratic Privacy-Signaling Games, Payoff Dominant Equilibria and the Information Bottleneck Problem

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Abstract—We introduce a privacy-signaling game problem in which a transmitter with privacy concerns observes a pair of correlated random variables which are modeled as jointly Gaussian. The transmitter aims to hide one of these random variables and convey the other one whereas the objective of the receiver is to accurately estimate both of the random variables. These conflicting objectives are analyzed in a game theoretic framework where depending on the commitment conditions (of the sender), we consider Nash or Stackelberg equilibria. We show that for scalar sources a payoff dominant Nash equilibrium among all admissible policies is attained by affine policies. We prove that these Nash equilibria coincide with the Stackelberg equilibria. We also show that there always exists an informative Stackelberg equilibrium for the multidimensional parameter setting. We revisit the information bottleneck problem within our Stackelberg framework where under the information bottleneck setup the sender observes only one of the parameters. We characterize the Stackelberg equilibria under certain conditions and when these conditions are not met we establish the existence of informative equilibria.

Index Terms—Signaling games, Nash equilibrium, Stackelberg equilibrium, privacy, estimation, information bottleneck.

I. INTRODUCTION AND SYSTEM MODEL

Various applications such as social networks, networked control, smart grid and crowd sensing benefit from data collected from decision makers. In these applications, users share information with a service provider which wishes to improve the quality of service by utilizing information gathered from the users. The users as well are interested in enhanced service quality as they benefit from the service while at the same time they wish to retain a certain level of privacy. The privacy objective arises from the fact that the information they wish to convey to the service provider may be correlated with certain private information they want to protect. For instance, in smart grid applications, power usage information shared by the users with the service provider may disclose some information related to users such as their habits and behaviors [2], [3]. For that reason, privacy is a major challenge in smart grid applications and this is a current research topic in numerous studies (see [3]–[6] and references therein). The problem of preserving privacy while maintaining reasonable system performance appears in various contexts [7]–[17].

In this manuscript, we consider the following communication scenario between a sender and a receiver motivated by the aforementioned applications. There is a pair of messages at the sender and the perspective of the sender is such that one message needs to be protected and the other message needs to be conveyed. As opposed to the sender, the receiver desires to accurately estimate both messages with the aim of acquiring as much information as possible. Under this setting, we investigate the interactions between the sender and the receiver whose objectives are different from each other due to the privacy concerns of the transmitter.

Consider an information transmission scenario in which a sender encodes a pair of correlated random variables $x$ and $y$ into $z$ using an encoding function denoted by $z = \gamma^e(x, y)$ and a receiver wants to decode both of the random variables. In this communication scenario, the sender wishes to convey information contained in $y$ whereas it views $x$ as a private parameter that needs to be hidden from the receiver. Suppose that there is a channel between the sender and the receiver characterized by a conditional distribution $p(r | z)$ where $r$ denotes the observation of the receiver. The aim of the receiver is to accurately estimate both parameters given its observation $r$. Let the decoders for estimating $x$ and $y$ at the receiver be denoted by $\gamma^d_x(r)$ and $\gamma^d_y(r)$, respectively. Figure 1 illustrates the considered information transmission scenario.

We model the parameters $x$ and $y$ as jointly Gaussian distributed random variables. Let $(x, y)$ be a zero mean Gaussian distributed random vector with a positive definite covariance matrix $\Sigma \triangleq \begin{bmatrix} \sigma_x^2 & \rho \\ \rho & \sigma_y^2 \end{bmatrix}$ and a nonzero correlation, i.e., $\rho \neq 0$. It is assumed that the joint distribution of $x$ and $y$ is common knowledge, i.e., both players know $\sigma_x^2$, $\sigma_y^2$ and $\rho$. Since $x$ and $y$ are correlated, transmitting $y$ directly discloses information related to the private parameter $x$. In other words, the objectives of hiding $x$ and conveying $y$ are conflicting. These conflicting objectives at the transmitter are modeled via...
the following objective function:

\[ J^c(\gamma^e, \gamma^d_x, \gamma^d_y) = \mathbb{E}[(\gamma^d_y(r) - y)^2] - \delta \mathbb{E}[(\gamma^d_x(r) - x)^2] \]  

(1)

which is to be minimized, where \( \delta \) is a positive design parameter that determines the level of desired privacy in terms of hiding \( x \). On the other hand, the receiver aims to extract both of the parameters. Thus, the receiver wishes to minimize the following objective function:

\[ J^d(\gamma^e, \gamma^d_x, \gamma^d_y) = \mathbb{E}[(\gamma^d_x(r) - x)^2] + \mathbb{E}[(\gamma^d_y(r) - y)^2]. \]  

(2)

In [2], the mean squared errors corresponding to parameters \( x \) and \( y \) are incorporated into the objective function with equal weights since taking different weights does not alter the problem. In this work, we investigate the Nash and the Stackelberg equilibria for the described strategic information transmission scenario in which the objectives of the sender and the receiver are as defined above.

The game dynamics for the Nash equilibrium are as follows. Each player chooses its strategy simultaneously. These chosen strategies are referred to as a Nash equilibrium if no player gains by unilaterally deviating from its strategy. In other words, neither the sender nor the receiver would have any incentive to unilaterally change their strategies when they operate at a Nash equilibrium. Suppose that the set of possible strategies for the encoder is denoted by \( \Gamma^e \), i.e., \( \gamma^e \in \Gamma^e \), and those for the decoders of each parameter are denoted by \( \Gamma^d_x \) and \( \Gamma^d_y \), i.e., \( \gamma^d_x \in \Gamma^d_x \) and \( \gamma^d_y \in \Gamma^d_y \). A set of policies \( \gamma^e, \gamma^d_x, \gamma^d_y \) forms a Nash equilibrium if [18]

\[ J^c(\gamma^e, \gamma^d_x, \gamma^d_y) \leq J^c(\gamma^e, \gamma^d_{x,*}, \gamma^d_{y,*}) \]  

(3)

for all \( \gamma^e \in \Gamma^e \) and

\[ J^d(\gamma^e, \gamma^d_x, \gamma^d_y) \leq J^d(\gamma^e, \gamma^d_{x,*}, \gamma^d_{y,*}) \]  

(4)

for all \( \gamma^d_x \in \Gamma^d_x \) and \( \gamma^d_y \in \Gamma^d_y \).

On the other hand, the Stackelberg equilibrium involves a sequential game play in the sense that first the sender and then the receiver act. The sender chooses and announces its strategy and then the receiver acts upon learning the strategy of the sender. Once the sender announces its strategy, it cannot change it, i.e., the sender commits to employ this announced strategy. The receiver employs an optimal response to the announced strategy of the sender, which is known by the receiver before taking an action. A set of policies \( \gamma^e, \gamma^d_x, \gamma^d_y \) forms a Stackelberg equilibrium if [18]

\[ J^c(\gamma^e, \gamma^d_{x,*}, \gamma^d_{y,*}(\gamma^e), \gamma^d_{y,*}(\gamma^e)) \leq J^c(\gamma^e, \gamma^d_x, \gamma^d_{x,*}(\gamma^e), \gamma^d_{y,*}(\gamma^e)) \]  

(5)

for all \( \gamma^e \in \Gamma^e \), where \( \gamma^d_{x,*}(\gamma^e) \) and \( \gamma^d_{y,*}(\gamma^e) \) are such that

\[ J^d(\gamma^e, \gamma^d_x, \gamma^d_y) \leq J^d(\gamma^e, \gamma^d_x(\gamma^e), \gamma^d_y(\gamma^e)) \]  

(6)

for all \( \gamma^d_x \in \Gamma^d_x \) and \( \gamma^d_y \in \Gamma^d_y \).

A. Literature Review

For signaling games under the Nash equilibrium concept, Crawford and Sobel in their foundational paper [19] investigate a communication scenario between a better informed sender and a receiver where sender’s cost contains a bias term leading to misaligned objectives. They obtain the interesting result that under some technical conditions the sender needs to quantize the information it sends at a Nash equilibrium. To put it differently, the misalignment in the objectives results in information hiding through quantization of the sent message. In contrast to Crawford and Sobel, the Bayesian persuasion problem considers the Stackelberg equilibrium concept rather than the Nash equilibrium concept [20]. Recently, the SIT problem attracted attention also in the communication and control theory literature [21]–[29]. For instance, the work [22] considers quadratic costs with a bias term appearing in sender’s cost and investigates both scalar and multidimensional parameter settings. An interesting observation from [22] is the existence of a linear Nash equilibrium which is in contrast to the quantized nature of the equilibrium in Crawford and Sobel. In [23], the misalignment in the objectives is due to a bias term which is modeled as a random variable. The authors consider the Stackelberg equilibrium concept and focus on affine parameters. In [24], a communication scenario between prospect theoretic agents whose cost functions are distorted by subjective biases is investigated using the Stackelberg equilibrium concept.

In the literature, several studies consider the SIT problem in which the sender takes privacy of certain information into account by employing a suitable privacy measure, under either the Nash or Stackelberg criteria [30]–[33]. In these studies, a common theme is to model private and nonprivate parameters as jointly Gaussian random variables. In [30], a communication scenario between a sender and a receiver is investigated using the Stackelberg equilibrium concept in which an additional side information is assumed to be available at the receiver. The estimation errors are measured using quadratic costs and a family of Stackelberg equilibria is characterized under an a priori affine policy assumption. In contrast, here, we do not restrict the policies to be affine a priori and we consider a setting with no side information. Also, we investigate Nash equilibria as well and show that a payoff dominant equilibrium is attained by affine policies. We use this result to show that the Stackelberg equilibria are also affine even when the sender is not restricted to the affine class. The work in [31] also investigates a Stackelberg game where the utility measure for the nonprivate parameter is quadratic and the privacy measure is entropy based. Both noiseless and noisy communication scenarios are considered and (essentially) unique linear encoding and decoding policies that form a Stackelberg equilibrium are characterized. In [32], a Nash game is studied where the privacy measure is based on mutual information and the utility measure for the nonprivate parameter is quadratic. In [32], apart from the previously described Gaussian scenario, another scenario in which private and nonprivate data are treated as discrete random variables is considered.
The tradeoff between utility and privacy appears also in various other contexts \cite{8,17,34,38}. One line of related work is the differential privacy literature where the main problem of interest is to protect private information on publicly available databases \cite{36,37}. The notion of differential privacy ensures that private information provided by an individual to a database is not compromised by a third party, e.g., data analyst, who retrieves information from this database. There is a growing interest in differential privacy in many different fields including control theory \cite{9,12,34}. In this context, an interesting result from \cite{9} is the application of the Laplacian or Gaussian perturbations to guarantee differential privacy. For a comprehensive treatment of differential privacy on such problems as filtering and estimation, please see \cite{39}. Another line of work is the privacy funnel problem \cite{35,38} which have connections with the information bottleneck technique \cite{40}. In the information bottleneck technique, the aim is to compress an observed random variable while trying to preserve information related to another correlated random variable which is not observed \cite{40}. The information bottleneck problem specializing to Gaussian random variables is investigated in \cite{41} where the parameters of interest are jointly Gaussian vectors. The compression objective in the information bottleneck problem can also be viewed as a privacy objective as in our framework in the sense that the corresponding information is desired to be removed from the revealed message. In the information bottleneck problem, the costs involve mutual information and only one of the parameters is received at the sender whereas in our framework the costs include mean squared error terms and both of the parameters are observed at the sender. In order to provide an estimation theoretic perspective on the information bottleneck problem, we formulate a similar problem where we use mean squared error terms for the costs as in our original setting and we show that there are operational and consequential differences when the encoder is allowed to use both of the hidden variables. In the privacy funnel problem, it is desired to convey as much information as possible related to an observed random variable while trying to leak as low information as possible related to an unobserved private parameter. It should be noted that in the privacy funnel problem, only the nonprivate parameter is observed at the sender whereas in our framework, we assume that both the nonprivate and private parameters are observed at the sender. Another related work is \cite{17} where the tradeoff between utility and privacy is investigated through formulating constrained optimization problems that consider settings with a discrete parameter and a continuous parameter. The continuous parameter case focuses on Gaussian perturbations applied to the nonprivate parameter to protect private information.

\section*{B. Contributions of the Manuscript}

The main aim of this manuscript is to provide both Nash and Stackelberg equilibria analyses for the considered privacy-signaling game problem. In game theory, since the solution concept involves an equilibrium (Nash, Stackelberg, and refinements), one cannot talk about an optimal equilibrium in general. Nonetheless, as a main contribution of our work, we establish and compute an equilibrium, which is desirable among all, for both of the players. The main contributions of this manuscript can be summarized as follows:

- In the literature, a Nash equilibrium analysis of the privacy game problem, in which both the privacy and the utility (for the nonprivate parameter) are measured via the mean squared error cost, has not been available. In this manuscript, we consider this problem for the first time in the literature to our knowledge. More importantly, we show that the most desirable Nash equilibrium (for both players) is the affine equilibrium in Theorem 1 considering the noiseless communication setting. This payoff dominance result is obtained by first explicitly characterizing the affine Nash equilibria and then formulating an equivalent problem. In addition, we establish the uniqueness of the characterized affine Nash equilibria based on the proposed equivalent formulation.
- We show that the characterized affine Nash equilibria coincide with the Stackelberg equilibria in Theorem 2. It should be emphasized that in characterizing the Stackelberg equilibria, we do not make an affine policy restriction. In other words, if we consider the optimization problem that the encoder needs to solve while obtaining the Stackelberg equilibria, the characterized affine policies are the optimal solution among any sets of policies. In addition, our Stackelberg equilibrium analysis comprises a uniqueness result among the affine class of policies.
- We show that there always exists an informative Stackelberg equilibrium for the multidimensional parameter setting in Theorem 3.
- We provide a game theoretic perspective on the information bottleneck problem by formulating it as a Stackelberg game under the mean squared error distortion criterion for the case of multidimensional sources. We characterize the Stackelberg equilibria under certain conditions in Theorem 4. For the cases when these conditions are not met, we establish the existence of informative equilibria in Theorem 5. In the special case of scalar sources, we explicitly characterize the equilibria, which is either fully informative or noninformative depending on an explicit condition, in Corollary 1.
- We extend our results on the noiseless channel setting with scalar sources to the additive Gaussian noise channel setting which is commonly encountered in practice. Under this setting, it is shown that a payoff dominant Nash equilibrium is attained by linear policies, which are explicitly characterized, in Theorem 6. This theorem also establishes that the characterized linear Nash equilibrium is unique among the affine class. We also show that the Nash equilibria coincide with the Stackelberg equilibrium in Theorem 7. In addition, the characterized linear Stackelberg equilibrium is unique among any set of policies.
- We also establish the existence of nonlinear Nash and Stackelberg equilibria considering a discrete channel setting in which the encoding function is restricted to
take discrete values in Theorem 11 and Theorem 12 respectively.

The remainder of the manuscript is organized as follows. Section II provides Nash and Stackelberg equilibria analyses considering the noiseless channel. Section III presents our results for the multidimensional case. Section IV investigates the information bottleneck problem as an instance of our proposed framework. Section V-A and Section V-B extend the results to the Gaussian noise channel and discrete channel, respectively. Section VI provides numerical examples, and Section VII concludes the manuscript with some final remarks.

II. NOISELESS CHANNEL BETWEEN THE SENDER AND THE RECEIVER

In this section, we consider a noiseless communication scenario. The receiver has direct access to the encoded random variable $z$, i.e., $r = z$. In addition, we assume that there is a power constraint $E[z^2] \leq P$ at the transmitter where $P$, which takes a finite positive value, represents the maximum average power. Under this setting, the sender can transmit a real valued $z$ for which the average power constraint is satisfied.

A. Nash Equilibria

In the following theorem, we show that payoff dominant Nash equilibrium are affine and we explicitly characterize these equilibria. A payoff dominant Nash equilibrium is the most desirable equilibrium for both of the players (among all coding/decoding policies, including those that are non-linear) in a sense that is made explicit in the following definition.

Definition 1: A Nash equilibrium that is not Pareto dominate by any other Nash equilibrium of the game is said to be a payoff dominant Nash equilibrium.

Theorem 1:

(i) The encoding policies $\gamma^e(x,y) = C$ for all $|C| \leq \sqrt{P}$ and the decoding policy $\gamma^d_x(r) = 0$ and $\gamma^d_y(r) = 0$ form a noninformative Nash equilibrium.

(ii) There exist informative Nash equilibria in which the decoding policies, $\gamma^d_x(r) = Kr + L$ and $\gamma^d_y(r) = Mr + N$, satisfy

\[
\frac{M}{K} = \frac{\delta \sigma^2 + \sigma^2_u}{2\rho} + \frac{\sqrt{(\delta \sigma^2 + \sigma^2_u)^2 - 4\delta \rho^2}}{2\rho},
\]

with $\tilde{P}(K, L, M, N) \leq (M^2 - \delta K^2)^2P$, and $L$ and $N$ are such that either $N/L = M/K$ or $L = N = 0$ holds. Also, the corresponding encoding policy, $\gamma^e(x,y) = Ax + By + C$, is given by

\[
A = -\delta K/(M^2 - \delta K^2), \quad B = M/(M^2 - \delta K^2), \quad C = -L/K.
\]

(iii) These informative equilibria are payoff dominant Nash equilibria. In addition, they are unique among the affine class of policies.

Proof: In the rest of this subsection, we prove Theorem 11 where we focus on (i) and (ii). We first characterize affine Nash equilibria and then we prove their payoff dominance property among all encoding and decoding policies, including those that are non-linear. Towards that goal, we first need the best responses of each player specializing to affine policies.

Lemma 1: When the encoder is affine and it is expressed as $z = Ax + By + C$, the decoders are also affine and they are given by

\[
\gamma^d_x(r) = \left( \frac{A\sigma^2_X + B\rho}{A^2\sigma^2_X + B^2\sigma^2_Y + 2AB\rho} \right)(r - C),
\]

\[
\gamma^d_y(r) = \left( \frac{A\rho + B\sigma^2_Y}{A^2\sigma^2_X + B^2\sigma^2_Y + 2AB\rho} \right)(r - C).
\]

Proof: See Appendix A.

Having characterized the best response of the receiver to affine encoding policies, we next analyze the best response of the sender to affine decoding polices.

Lemma 2: Define $\tilde{P}(K, L, M, N) \triangleq \delta^2 \sigma^2_K M^2 - 2\delta \rho KM + (M^2 - \delta K^2)L^2$. When the decoders are affine in the form of $\gamma^d_x(z) = Kz + L$ and $\gamma^d_y(z) = Mz + N$, the optimal encoder is specified by

\[
\gamma^e(x,y) = \frac{M(y - N) - \delta K(x - L)}{M^2 - \delta K^2},
\]

if $\tilde{P}(K, L, M, N) \leq (M^2 - \delta K^2)^2P$ and $M^2 > \delta K^2$; and otherwise by

\[
\gamma^e(x,y) = \frac{\sqrt{\tilde{P}}(y - N) - \sqrt{\tilde{P}}K(x - L)}{\sqrt{\tilde{P}}(M, L, M, N)}.
\]

Proof: See Appendix B.

Thus far, the best responses of affine encoding and decoding policies are explored. To sum up, it is shown that the optimal decoding policies are affine when the encoder is affine and also the optimal encoding policy is affine when the decoders are affine.

Now, suppose that the system parameters and the terms that specify the decoding policies are such that the optimal encoding policy is as in (13). Namely, it is assumed that $M^2 > \delta K^2$ and $\tilde{P}(K, L, M, N) \leq P(M^2 - \delta K^2)^2$. In the following, a Nash equilibrium analysis is carried out under these two assumptions.

To begin with, by combining (11)-(13), the following fixed point equations are obtained:

\[
K = \frac{(M^2 - \delta K^2)(\rho M - \delta \sigma^2_K)}{\sigma^2_M^2 + \delta^2 \sigma_X^2 K^2 - 2\delta \rho KM},
\]

\[
L = \frac{(M^2 - \delta KL)(\rho M - \delta \sigma^2_K)}{\sigma^2_M^2 + \delta^2 \sigma_X^2 K^2 - 2\delta \rho KM},
\]

\[
M = \frac{(M^2 - \delta K^2)(\sigma^2_M M - \delta \rho K)}{\sigma^2_M M^2 + \delta^2 \sigma_X^2 K^2 - 2\delta \rho KM},
\]

\[
N = \frac{(M^2 - \delta KL)(\sigma^2_M M - \delta \rho K)}{\sigma^2_M M^2 + \delta^2 \sigma_X^2 K^2 - 2\delta \rho KM}.
\]
After some algebraic manipulations, (16) and (18) lead to the same expression \(N(\sigma_x^2 K - \rho M) = L(\delta K - \sigma_0^2 M)\). If \(L = N = 0\), this equality holds and otherwise noting that \(M/K = (\delta K - \sigma_0^2 M)/(\sigma_x^2 K - \rho M)\) from (15) and (17), we get \(\frac{M}{K} = \frac{\sigma_0^2}{\sigma_x^2}\).

Now, it remains to take (15) and (17) into account in order to characterize an equilibrium. After dividing both sides of (15) and (17) by \(K\) and defining \(q = M/K\), we obtain

\[
1 = \frac{(q^2 - \delta)(q\sigma_x^2 + \sigma_0^2)}{q^2\sigma_x^2 - 2\rho q + \sigma_0^2}\quad\text{and}\quad q = \frac{(q^2 - \delta)(q\sigma_x^2 + \sigma_0^2)}{q^2\sigma_x^2 - 2\rho q + \sigma_0^2},
\]

and these can simultaneously be solved if

\[
q^2\rho - q(\sigma_x^2 + \sigma_0^2) + \delta\rho = 0. \tag{19}
\]

Note that in order for (19) to be solvable, it is required that \((\sigma_x^2 + \sigma_0^2)^2 - 4\delta\rho^2 \geq 0\) or equivalently \((\sigma_x^2 + \sigma_0^2) - 2\sqrt{\delta}\rho \geq 0\). In the following, we will show that this condition is always satisfied. Towards that goal, define \(f(\delta) \triangleq (\sigma_x^2 + \sigma_0^2) - 2\sqrt{\delta}\rho\). Taking the derivative of \(f(\delta)\) with respect to \(\delta\) and equating it to zero, we get \(\delta = \rho^2/\sigma_x^2\). By applying the second derivative test, we can conclude that this \(\delta\) achieves the minimum of \(f(\delta)\) and the value at this minimal point is given by \(f(\delta) = \sigma_x^2 - \rho^2/\sigma_x^2\). Through the positive semi-definiteness property of covariance matrices, it follows that \(f(\delta) \geq 0\). Since any \(\delta\) satisfies \(f(\delta) \geq f(\delta)\), it is concluded that the condition for the existence of a solution to (19) is always met. Moreover, as \(\Sigma\) is assumed to be positive definite, the inequality is always strict which leads to the fact that (19) has always two roots. Therefore, the solutions of \((M/K)^2\rho - (M/K)(\sigma_x^2 + \sigma_0^2) + \delta\rho = 0\), which always exist as already shown, satisfy the fixed point equations in (15) and (17). Moreover, recall that at the beginning of this proof we assumed that \(M^2 > \delta K^2\), equivalently \(q^2 > \delta\), holds. In the following, this condition is taken into account. Note that the right hand side of (7) corresponds to one of the roots of (19).

By inserting this root into \(q^2 > \delta\), we get

\[
\frac{\delta\sigma_x^2 + \sigma_0^2}{\rho} \left( \frac{\delta\sigma_x^2 + \sigma_0^2}{\rho} + \frac{\sqrt{\delta\sigma_x^2 + \sigma_0^2}^2 - 4\delta\rho^2}{\rho} \right) > 4\delta. \tag{20}
\]

Next, after defining \(s \triangleq \frac{\delta\sigma_x^2 + \sigma_0^2}{\rho}\), (20) leads to \(s^2 - 4\delta > -s\) when \(\rho > 0\) and to \(s^2 - 4\delta > s\) when \(\rho < 0\). Since the left hand side and the right hand side of both inequalities are positive and negative, respectively, both inequalities and consequently the inequality in (20) are satisfied. Furthermore, via a similar analysis, it can be shown that the second root of (19) does not satisfy the condition of \(M^2 > \delta K^2\). Finally, the corresponding encoding policy at this equilibrium is as in (13), which leads to the expressions in (8)-(10). In fact, a similar analysis can be carried out to show that there does not exist an equilibrium when the optimal encoding policy is as in (14), which implies the uniqueness of the derived equilibria among the affine class. However, we take a more intuitive approach in the following.

Next, we establish our main results on the equilibrium behavior for the noiseless setup, in particular the property that the informative affine equilibria lead to the payoff dominant equilibrium, and such an equilibrium is in fact unique. Towards that goal, we show that in a linearly transformed orthogonal coordinate system the sender wishes to fully reveal one coordinate whereas wants to hide the other coordinate. In the following, we take the revealed message at the characterized equilibria as one coordinate and take the other coordinate as orthogonal to the revealed message. In other words, we linearly transform \((x,y)\) into \((u,v)\) such that \(u\) corresponds to the revealed parameter and \(v\) is independent of \(u\). It is noted that this transformation is invertible, i.e., one can recover \((x,y)\) from \((u,v)\). In particular, consider the following transformation from the pair \((x,y)\) to the pair \((u,v)\):

\[
u = \begin{pmatrix}
-\sigma_x^2 + \sigma_y^2 & -\sqrt{(\sigma_x^2 + \sigma_y^2)^2 - 4\delta\rho^2} \\
2\delta\rho & 2\delta\rho
\end{pmatrix}
\]

\[
u = \begin{pmatrix}
-\sigma_x^2 + \sigma_y^2 & +\sqrt{(\sigma_x^2 + \sigma_y^2)^2 - 4\delta\rho^2} \\
2\delta\rho & 2\delta\rho
\end{pmatrix}
\]

We propose an equivalent problem under this linear transformation. The encoder consists of two consecutive mappings one of which is fixed as above and the other one can arbitrarily be chosen by the sender. In other words, there is a linear mapping from \((x,y)\) to \((u,v)\) fixed as above and then an encoding function \(z = \gamma(u,v)\). At the receiver side, we also consider a similar setting. The observation at the receiver is mapped into \(\tilde{u}\) and \(\tilde{v}\) via \(\gamma^{d_u}(r)\) and \(\gamma^{d_v}(r)\), respectively, which can arbitrarily be selected by the receiver. Then, these auxiliary variables are mapped into estimates of \(x\) and \(y\) as follows:

\[
\gamma^{d_x}(r) = \frac{\gamma^{d_u}(r) + \gamma^{d_v}(r)}{2} + \frac{(\gamma^{d_u}(r) - \gamma^{d_v}(r))(\delta\sigma_x^2 + \sigma_y^2)}{2\sqrt{(\delta\sigma_x^2 + \sigma_y^2)^2 - 4\delta\rho^2}}, \tag{23}
\]

\[
\gamma^{d_y}(r) = \frac{(\gamma^{d_u}(r) - \gamma^{d_v}(r))\delta\rho}{\sqrt{(\delta\sigma_x^2 + \sigma_y^2)^2 - 4\delta\rho^2}}. \tag{24}
\]

Fig. 2 provides an illustration for the equivalent formulation. Since the proposed transformation is invertible, the problem in the transform domain is equivalent to the original problem.

Then, we express the objective functions of each player in terms of the parameters in the transformed coordinate system.
The objective function of the sender can be written as

\[ J^d(\gamma^e, \gamma^{dv}, \gamma^{dv}) = \delta \left( \frac{(\delta \sigma_X^2 + \sigma_Y^2)}{\sqrt{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}} - 1 \right) \mathbb{E}[(u - \gamma^{dv}(r))^2] \]

\[ + \frac{\delta}{2} \left( \frac{(\delta \sigma_X^2 + \sigma_Y^2)}{\sqrt{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}} - 1 \right) \mathbb{E}[(v - \gamma^{dv}(r))^2], \]

where the first coefficient is positive and the second coefficient is negative. Similarly, if we express the objective function of the receiver in terms of the parameters in the introduced coordinate system, we get

\[ J^d(u, \gamma^{dv}, \gamma^{dv}) = w_1 \mathbb{E}[(u - \gamma^{dv}(r))^2] + w_2 \mathbb{E}[(v - \gamma^{dv}(r))^2] + w_3 \mathbb{E}[(u - \gamma^{dv}(r))(v - \gamma^{dv}(r))], \]

where

\[ w_1 \triangleq \left( 1 - \frac{1}{2\sqrt{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}} \right)^2 + \frac{\delta \rho^2}{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}, \]

\[ w_2 \triangleq \left( 1 + \frac{1}{2\sqrt{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}} \right)^2 + \frac{\delta \rho^2}{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}, \]

and

\[ w_3 = -\frac{2\delta \rho^2}{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}. \]

In the proposed equivalent problem, the optimal \( \gamma^{dv}(r) \) for a given encoding policy \( \gamma^e(u, v) \) is the minimum mean squared error estimators of the corresponding parameters, as stated below.

**Proposition 1:** For a fixed encoding function \( \gamma^e(u, v) \), the optimal \( \gamma^{dv}(r) \) that minimize (26) are given by \( \mathbb{E}[u|r] \) and \( \mathbb{E}[v|r] \), respectively.

**Proof:** The result is proven using the completion of the squares technique. Suppose that \( \gamma^{dv}(r) = \mathbb{E}[u|r] + g(r) \) and \( \gamma^{dv}(r) = \mathbb{E}[v|r] + h(r) \). Inserting these expressions into the objective function of the receiver, we get

\[ J^d(g, h) = w_1 \mathbb{E}[(u - \mathbb{E}[u|r])^2] + w_2 \mathbb{E}[(v - \mathbb{E}[v|r])^2] + w_3 \mathbb{E}[(u - \mathbb{E}[u|r])(v - \mathbb{E}[v|r])] \]

\[ + \mathbb{E}[(u - \mathbb{E}[u|r])^2 + w_2 h(r)^2 + w_3 g(r)h(r)]. \]

Notice that the first three terms correspond to the objective value when the minimum mean squared error estimators are employed. Now, observe that

\[ \mathbb{E}[w_1 g(r)^2 + w_2 h(r)^2 + w_3 g(r)h(r)] = \]

\[ \mathbb{E} \left[ g(r) \left( 1 - \frac{(\delta \sigma_X^2 + \sigma_Y^2)}{2\sqrt{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}} \right) \right] \]

\[ + h(r) \left( 1 + \frac{(\delta \sigma_X^2 + \sigma_Y^2)}{2\sqrt{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2}} \right)^2 \]

\[ + \frac{\delta \rho^2}{(\delta \sigma_X^2 + \sigma_Y^2)^2 - 4\delta \rho^2} \mathbb{E}[(g(r) + h(r))^2] \geq 0. \]

It follows that \( J^d(g, h) \geq w_1 \mathbb{E}[(u - \mathbb{E}[u|r])^2] + w_2 \mathbb{E}[(v - \mathbb{E}[v|r])^2] + w_3 \mathbb{E}[(u - \mathbb{E}[u|r])(v - \mathbb{E}[v|r])] \) which shows the optimality of the minimum mean squared error estimators.

In the equivalent formulation, the objective function of the sender contains two mean squared error terms corresponding to independent random variables \( u \) and \( v \) where the sender wishes to transmit \( u \) whereas wants to protect \( v \). The receiver still employs the minimum mean squared error estimators corresponding to \( u \) and \( v \) for a given encoding policy. Next, we use the equivalent formulation to obtain the uniqueness and payoff dominance results. Towards that goal, we first present the following lemma.

**Lemma 3:** At a Nash equilibrium, the sender does not transmit any information related to the linear combination (of the private and nonprivate parameters) \( v \) specified by (27).

**Proof:** Suppose that the sender employs an encoding policy \( \gamma^e(u, v) \) which conveys information related to both parameters. In response to this encoding policy, it is optimal for the receiver to employ the minimum mean squared error estimators of each parameter, as shown in Proposition 1. Denote these estimators by \( f(r) \) and \( g(r) \) for estimating \( u \) and \( v \), respectively. Assume that the mean squared error for estimating \( u \) with the corresponding optimal estimator \( g(r) \) is less than \( \sigma^2_u \), i.e., \( \mathbb{E}[(v - g(r))^2] < \sigma^2_u \). In response to the decoding policies of \( f(r) \) and \( g(r) \), the sender can switch to the following policy to improve its objective value. Instead of sending \( z = \gamma^e(u, v) \), the sender can transmit \( z = \gamma^e(u, n) \) while keeping the encoding function \( \gamma^e(\cdot, \cdot) \) the same where \( n \) follows the same distribution as \( v \) and is independent of \( u \) and \( v \). In that case, the performance for estimating \( u \) remains the same since receiving \( \gamma^e(u, v) \) or \( \gamma^e(u, n) \) are equivalent for the decoding policy \( f(r) \). However, the performance for estimating \( v \) degrades as shown in the following:

\[ \mathbb{E}[(v - g(r))^2] = \mathbb{E}[v^2] - 2\mathbb{E}[vg(r)] + \mathbb{E}[g(r)^2] \]

\[ = \mathbb{E}[v^2] - 2\mathbb{E}[v]\mathbb{E}[g(r)] + \mathbb{E}[g(r)^2] \]

\[ = \mathbb{E}[v^2] + \mathbb{E}[g(r)^2] \geq \sigma^2_u, \]

where the second equality is due to the fact that \( v \) and \( r \) are independent in case \( \gamma^e(u, n) \) is transmitted. Since the sender wishes to hide \( v \) (see (23)), the sender gains by employing \( z = \gamma^e(u, n) \) instead of \( z = \gamma^e(u, v) \). As a result, any encoding policy which yields \( \mathbb{E}[(v - \gamma^{dv}(r))^2] < \sigma^2_u \) cannot be a Nash equilibrium in that case the sender can change its strategy to improve its objective value.

From Lemma 3, it follows that the performance of each player at a Nash equilibrium is determined by the mean squared error for estimating \( u \) and this result is valid for any coding/decoding policies including non-linear ones. Since transmitting an affine function of \( u \) yields the minimum attainable performance for both players, this is a payoff dominant Nash equilibrium, i.e., the most desirable equilibria for both players among all coding/decoding policies including non-linear ones.

Since the receiver cannot transmit information related to \( v \), the only possible affine Nash equilibria correspond to sending an affine function of \( u \) satisfying the average power constraint, which gives the equilibria in the statement of the theorem. Thus, the characterized equilibria are unique among the affine class.

**Remark 1:** Theorem 1 characterizes a family of informative affine Nash equilibria in which the transmitted information is the same for the whole family, i.e., for the encoding policy \( \gamma^e(x, y) = Ax + By + C \) the ratio \( B/A \) is always the same at the characterized equilibria, and these equilibria lead to the
same performance values. Thus, these affine equilibria may be viewed to be essentially unique.

**Remark 2:** Note that when $\delta \to 0^+$, the encoding and decoding policies satisfy $M/K \to \frac{\sigma_x^2}{\rho}$, $A \to 0$ and $B \to 1/M$ as can be deduced from (7), (8) and (9). Therefore, $\delta \to 0^+$ implies that the encoder transmits directly $y$ at a Nash equilibrium. When $\delta = 0$, the sender also transmits $y$ directly at a Nash equilibrium since it does not have any privacy concern in this case. Hence, the Nash equilibrium specified in Theorem 1 as $\delta \to 0^+$ coincides with the Nash equilibrium when $\delta = 0$.

**Remark 3:** It is seen that the ratio of $B$ and $A$ converges to $-\frac{\sigma_x^2}{\rho}$ as $\delta \to \infty$, which can be verified from (7), (8) and (9). This shows that in the high privacy regime, the encoder makes the revealed information $z$ and the private parameter $x$ essentially uncorrelated.

Theorem 1 characterizes a family of equilibria in which there is communication between the transmitter and the receiver. Hence, the considered game always admits informative affine Nash equilibria regardless of the system parameters. More importantly, these affine equilibria are the most desirable equilibria for both of the players.

### B. Convergence to Informative Affine Nash Equilibrium under Best Response Dynamics

In this subsection, we investigate the convergence behavior of fixed point iterations. In particular, we consider an initial linear encoding policy that does not yield a Nash equilibrium and then we update the policies of each player in an iterative manner by calculating the best responses. The aim is to see if an initial linear encoding strategy which does not yield a Nash equilibrium converges to an informative Nash equilibrium as a result of fixed point iterations. The following theorem provides a necessary and sufficient condition for reaching an informative Nash equilibrium. In the following, we employ the equivalent formulation used in the proof of Theorem 1 which facilitates the analysis.

**Theorem 2:** Consider an initial linear encoding policy $\tilde{\gamma}(u,v) = Au + Bv$ with $A \neq 0$ and $B \neq 0$, which does not lead to a Nash equilibrium. Suppose that best responses of the players are computed in an iterative manner starting from this initial encoding strategy. These fixed point iterations converge to an informative Nash equilibrium if and only if $\sigma_u^2 > \sigma_v^2 \delta$, where the variances $\sigma_u^2$ and $\sigma_v^2$ can be computed using the definitions of $u$ and $v$ in (21) and (22), respectively, and $\delta$ is defined as

$$\delta \triangleq \frac{(\delta \sigma_x^2 + \sigma_y^2) + \sqrt{(\delta \sigma_x^2 + \sigma_y^2)^2 - 4\delta \rho^2}}{(\delta \sigma_x^2 + \sigma_y^2) - \sqrt{(\delta \sigma_x^2 + \sigma_y^2)^2 - 4\delta \rho^2}}.$$  

**Proof:** See Appendix C.

Theorem 2 presents a necessary and sufficient condition, which depends only on the system parameters such as $\sigma_x, \sigma_y, \rho$ and $\delta$, to reach an informative Nash equilibrium as a result of fixed point iterations. This result has an important implication in terms of numerical stability. In case the encoder and decoder have slightly different initializations or priors, and the derived condition is not met, it means that it is not possible to reach an informative Nash equilibrium via updating policies in an iterative manner.

### C. Stackelberg Equilibria

The equivalent formulation employed in the proof of Theorem 1 is again useful for characterizing the Stackelberg equilibria. It is important to emphasize that in the following analysis the set of possible encoding strategies, i.e., $\Gamma_1$, is not restricted to the affine class.

**Lemma 4:** At a Stackelberg equilibrium, the sender does not transmit any information related to the linear combination (of the private and nonprivate parameters) $v$ specified by (22).

**Proof:** We will show that any encoding policy which yields $E[(v - g(r))^2] < \sigma_v^2$ cannot be a Stackelberg equilibrium via a similar analysis to that employed in Lemma 3. Towards that goal, we compare the performance of two scenarios from the perspective of the sender. Recall that in a Stackelberg equilibrium the sender chooses a policy and announces this policy to the receiver and the receiver acts with the knowledge of sender’s policy. In the first scenario, the encoder chooses a policy $\tilde{\gamma}(\cdot, \cdot)$ and the receiver employs its optimal response. Assume that $E[(v - \gamma_{dV}(r))^2] < \sigma_v^2$ with the corresponding set of policies. In the second scenario, suppose that the encoder chooses the same policy as before with the exception that the sender replaces $v$ by an independent noise following the same distribution as $v$. As the sender announces its strategy, the optimal response of the receiver for parameter $v$ becomes $\gamma_{dV}(r) = E[v|v] = E[v] = 0$ due to the independence of $v$ and $r$. Therefore, we get $E[(v - \gamma_{dV}(r))^2] = \sigma_v^2$ in this case. Notice that the mean squared error performance in estimating $u$ is the same for both scenarios. As a result, the second scenario yields better performance for the sender (see (25)). Thus, transmitting information related to $v$ is not desirable for the sender.

**Theorem 3:** The Stackelberg equilibria coincide with the informative Nash equilibria characterized in Theorem 1. In addition, this family of equilibria is unique among the affine class of policies.

**Proof:** The result follows from Lemma 4. Since the receiver cannot transmit information related to $v$, the objectives of both players reduce to the minimization of $E[(u - \gamma_{dV}(r))^2]$. The minimum can be attained by employing an affine function of $u$ at the encoder. This gives the family of equilibria characterized in Theorem 1 which is unique among the affine class of policies.

**Remark 4:** In fact, employing any invertible function $f(u)$ at the sender also yields a Stackelberg equilibrium. Since the receiver knows the commitment of the sender, it can simply employ $f^{-1}(r)$ to perfectly recover $u$.

**Remark 5:** While the Stackelberg equilibrium is always informative, i.e., the sender uses the pair of parameters $(x, y)$ in constructing its message, the Nash equilibrium can be noninformative or informative which are characterized in Theorem 1.

### III. THE MULTIDIMENSIONAL CASE

In this section, we provide a discussion on the multidimensional parameter setting. In particular, the private and nonpri-
vate parameters as well as the message to be transmitted are multidimensional. We focus on the noiseless communication setting and we impose no power constraint at the sender to simplify the problem. Similar to the scalar setting, the objective functions of the sender and receiver are defined as
\[ J^e(\gamma^e, \gamma^d, \delta^K) = -\delta E[\|x - \hat{x}\|^2] + E[\|y - \hat{y}\|^2]. \quad (31) \]
\[ J^d(\gamma^e, \gamma^d, \delta^K) = E[\|x - \hat{x}\|^2] + E[\|y - \hat{y}\|^2]. \quad (32) \]

A. Nash Equilibria

In this subsection, we investigate the multidimensional setting under the Nash equilibrium concept. In particular, we derive a pair of fixed point equations by imposing a certain constraint. We begin with characterizing the best responses.

**Lemma 5:** If the sender is affine in the form of \( z = \gamma^e(x, y) = Ax + By + C \), then the receiver is also affine and it is expressed as
\[ \gamma^d(x) = (\Sigma_{XX} A^T + \Sigma_{XY} B^T)(A\Sigma_{XX} A^T + B\Sigma_{YY} B^T)^{-1}(r - C), \quad (33) \]
\[ \gamma^d(y) = (\Sigma_{YY} B^T + \Sigma_{YX} A^T)(A\Sigma_{XX} A^T + B\Sigma_{YY} B^T)^{-1}(r - C). \quad (34) \]

**Proof:** By completing the squares technique, the optimization problem at the encoder can be expressed as
\[ \min_z \gamma^e(z, y) = \left[ \begin{pmatrix} (MT^2 M - \delta K M)z - (MT^2 y - N) \\ -\delta K^T(x - L) \end{pmatrix} \right]^T (MT^2 M - \delta K^T K)^{-1} \left[ \begin{pmatrix} (MT^2 M - \delta K M)z - (MT^2 y - N) \\ -\delta K^T(x - L) \end{pmatrix} \right]. \]
Since \((MT^2 M - \delta K^T K)\) is assumed to be positive definite, the result immediately follows.

It is noted that the best response characterization for the encoder is given under a constraint on \( M \) and \( K \). If this condition is not met, the completion of the squares technique does not work and also the problem becomes nonconvex. The condition of \((MT^2 M - \delta K^T K)\) being positive definite translates to \( M^2 - \delta K^2 > 0 \) in the scalar case, and this case yields the affine Nash equilibrium in the scalar setting (see Theorem 1).

The next step is to combine these best responses to express the fixed point equations which need to be satisfied at a Nash equilibrium. We specialize to linear policies for simplicity by setting all the elements of \( L \) and \( N \) to zero.

**Theorem 4:** Consider linear decoding policies \( \gamma^d(z) = Kz \) and \( \gamma^d(v) = Mz \). If a set of linear decoding policies with \( M^T M - \delta K^T K \) being positive definite and, \( M \) and \( K \) satisfying
\[ K = (\Sigma_{XY} M - \delta \Sigma_{XX} K)(\delta^2 K^T M + \delta^2 K^T \Sigma_{XX} K + \delta^2 K^T \Sigma_{YY} M + \delta^2 K^T \Sigma_{XY} M - \delta K^T \Sigma_{XX} K - \delta M^T \Sigma_{XX} K)^{-1}(M^T M - \delta K^T K) \quad (35) \]
\[ M = (\Sigma_{YY} M - \delta \Sigma_{XX} K)(\delta^2 K^T M + \delta^2 K^T \Sigma_{XX} K + \delta^2 K^T \Sigma_{YY} M + \delta^2 K^T \Sigma_{XY} M - \delta M^T \Sigma_{XX} K - \delta K^T \Sigma_{XY} K)^{-1}(M^T M - \delta K^T K) \quad (36) \]
exists, then these linear policies yield a Nash equilibrium assuming that the sender employs its best response.

First note that in (35) and (36), with the matrices taken to be the all zero matrices, the equations are satisfied. Thus, there always exists a Nash equilibrium, which is a babbling equilibrium. The more interesting question is whether there exist informative equilibria. That there exists an equilibrium (babbling) rules out a straightforward application of fixed point theorems, such as Brouwer’s [43], and one needs to restrict policies to a properly defined convex, compact set of policies strictly outside an open ball containing babbling policies. Having a coupled set of fixed point equations involving matrices, positive definiteness restriction on \((M^T M - \delta K^T K)\) and restricting the policies not to contain the babbling policies make it difficult the apply the standard fixed point theorems. In the most general case of arbitrary covariance matrices, these equations cannot be simplified to a single fixed point equation. In the following, we state some existence results considering special cases in terms of covariance matrices. These can be regarded as generalizations of the results for the scalar setting in Section 1 under some special circumstances.

**Theorem 5:** Consider pairs of observations \((x_1, y_1), \ldots, (x_n, y_n)\), transmitting their pairwise linear combinations \(u_{ij}, \ldots, u_{kn}\), with parameters \(x_1, y_1\), yields a payoff dominant Nash equilibrium.

(i) For independent pairs of observations \((x_1, y_1), \ldots, (x_n, y_n)\), transmitting their pairwise linear combinations \(u_{ij}, \ldots, u_{kn}\), with parameters \(u_{ij}\) as in (31) with parameters \(x_1, y_1\), yields a payoff dominant Nash equilibrium.

(ii) Consider pairs of observations \((x_1, y_1), \ldots, (x_n, y_n)\), where \(x_i = \alpha_i x_1\) and \(y_i = \alpha_i y_1\) for \(i = 2, \ldots, n\) and nonzero scalars \(\alpha_i\). Then, transmitting the linear combination \(u\) specified by (31) with parameters \(x_1, y_1\) yields a payoff dominant Nash equilibrium.

**Proof:**
(i) For independent pairs of observations \((x_1, y_1), \ldots, (x_n, y_n)\), transmitting their pairwise linear combinations \(u_{ij}, \ldots, u_{kn}\), with parameters \(u_{ij}\) as in (31) with parameters \(x_1, y_1\), yields a payoff dominant Nash equilibrium.

(ii) Consider pairs of observations \((x_1, y_1), \ldots, (x_n, y_n)\), where \(x_i = \alpha_i x_1\) and \(y_i = \alpha_i y_1\) for \(i = 2, \ldots, n\) and nonzero scalars \(\alpha_i\). Then, transmitting the linear combination \(u\) specified by (31) with parameters \(x_1, y_1\) yields a payoff dominant Nash equilibrium.
(ii) Suppose that the transformation in (21) and (22) is applied to every \((x_i, y_i)\) pair. As a result, we obtain pairs \((u_1, v_1), \ldots, (u_n, v_n)\). From Lemma 3 we know that the sender cannot transmit information related to \(v_1\) (as well as \(v_2, \ldots, v_n\) since these are simply scaled versions of \(v_1\)) at a Nash equilibrium. Then, both players want the receiver to perfectly recover \(u_1, \ldots, u_n\). If the sender transmits \(u_1\), then the receiver can perfectly recover \(u_2, \ldots, u_n\). Thus, the sender transmits \(u_1\) to obtain an informative Nash equilibrium.

Remark 6: Consider the scenario in (i) of Theorem 5. Note that it is possible to construct Nash equilibria by only sending a subset of \(u_1, \ldots, u_n\). Notice that in that case the equilibria are no longer payoff dominant equilibria.

Theorem 5 explicitly characterizes Nash equilibria considering special covariance matrix structures. Investigation of other covariance matrix structures remains a topic of future work.

It is noted that the objective functions in (31) and (32) include mean squared errors corresponding to each scalar parameter with a nonzero weighting coefficient (i.e., the coefficient is \(-\delta\) for private parameters and 1 for nonprivate parameters). A slight modification in the cost structure of the sender may lead to a babbling equilibrium under special structures of covariance matrices. In the following, we give an example involving special structures for the cost and covariance matrices where the nature of the equilibrium is an noninformative one.

Example 1: Consider \(J^c(\gamma^c, \gamma^dx, \gamma^dy) = E[(y - \hat{y})^TQ(y - \hat{y})] = \delta E[(x - \hat{x})^TP(x - \hat{x})]\), where \(Q\) is a diagonal matrix with positive elements except for the last element which is zero and \(P\) is an all zero matrix except for the last diagonal element which is positive. Suppose that the covariance matrix of \(y\) is given by \(\Sigma_Y = \text{diag}(0, \ldots, 0, a)\) for \(a > 0\). In addition, the last elements of \(x\) and \(y\) are correlated. In that case, the equilibrium is noninformative since the sender cannot transmit information related to the last element of \(y\) and the other elements of \(y\) are already deterministic. Namely, the problem reduces to a zero sum game.

B. Stackelberg Equilibria

In this subsection, our main result is to show that there always exists an informative Stackelberg equilibrium. Towards that goal, we provide a lower bound for the performance of the encoder under which the encoder conveys information related to both parameters. This implies that the Stackelberg equilibrium should be informative since the Stackelberg equilibrium cannot yield a performance which is worse than this informative lower bound. In the following, we provide an analysis building on the approach initiated in [30] where the authors consider a slightly different setting in which side information is available at the receiver.

Theorem 6: There always exists an informative Stackelberg equilibrium for the multidimensional parameter setting.

Proof: We will obtain a lower bound under which the sender conveys information to the receiver and the receiver employs its best response. Suppose that the receiver employs the following linear policies \(\gamma^d(x) = \Sigma_xz\Sigma_z^{-1}z\) and \(\gamma^d(y) = \Sigma_yz\Sigma_z^{-1}z\). Notice that these are the optimal minimum mean squared error estimators for jointly Gaussian random vectors. Thus, if the sender is linear, the optimal response of the receiver is given by these equations. Then, the mean squared errors for estimating each parameter can be written as \(E[|x - \Sigma_xz\Sigma_z^{-1}z|^2]\) and \(E[|y - \Sigma_yz\Sigma_z^{-1}z|^2]\). Thus, the aim is to minimize \(\text{Tr}(\Sigma_yy - \Sigma_yz\Sigma_z^{-1}z) + \delta\Sigma_xz\Sigma_z^{-1}z + \delta\Sigma_yz\Sigma_z^{-1}z)\) over \(x, y, z\). Since we are looking for a lower bound, let us assume that \(\Sigma_z = I\) since it is possible to scale the message without changing the objective. Namely, if \(z\) with particular values of \(x, y, z\) and \(z\) is feasible, then \(z = \Sigma_z^{-1}z\) is also feasible with \(\Sigma_z = I\), and \(z\) and \(z\) induce the same cost.

Since we are looking for a lower bound, let us assume that \(z\) is scalar. Thus, after defining \(u = [\Sigma_xz, \Sigma_yz]\) and \(Q = \Sigma_xx, \Sigma_xy, \Sigma_yy\), the optimization problem at the sender can be written as

\[
\min_u u^T\text{diag}(\delta I, -I)u, \quad \text{subject to } u^TP^{-1}u \leq 1, \tag{37}
\]

where the constraint is due to positive semi-definiteness condition expressed using Schur’s complement. If we write \(Q = P^2\) and apply a transformation of variables, we obtain an equivalent optimization problem

\[
\min_\hat{u} \hat{u}^TP\text{diag}(\delta I, -I)\hat{P}\hat{u}, \quad \text{subject to } \hat{u}^T\hat{u} \leq 1. \tag{38}
\]

Since we can show that \(P\text{diag}(\delta I, -I)P\) has at least one negative eigenvalue, the optimal solution should yield a negative objective value. For a particular \(\hat{u}\) with \(||\hat{u}|| < 1\) and \(\hat{u}^TP\text{diag}(\delta I, -I)P\hat{u} < 0\), we can apply a scaling to improve the objective as follows: If \(u = \alpha\hat{u}\) where \(\alpha = 1/||\hat{u}||\), we have that \(||u|| = 1\) and \(u^TP\text{diag}(\delta I, -I)P\hat{u} < \alpha^2 P\text{diag}(\delta I, -I)P\hat{u}\). Thus, the constraint should be satisfied with equality. Then, the problem reduces to a special instance of Courant-Fischer-Weyl minimax principle [44, p.58]. In particular, for a real and symmetric matrix \(A\), we have \(\lambda_{\min} \leq (x^TAx)/(x^Tx) \leq \lambda_{\max}\) for all nonzero \(x\) where the lower and upper bounds are attained by eigenvectors of \(A\) corresponding to the smallest and largest eigenvalues, respectively. Thus, the normalized eigenvector of \(P\text{diag}(\delta I, -I)P\) corresponding to the smallest eigenvalue solves (38). As a result, we get

\[
\begin{bmatrix}
\Sigma_xz \\
\Sigma_yz
\end{bmatrix} = P\hat{u}^* \text{ where } \hat{u}^* \text{ is the optimal solution to (38) and this can be attained via a linear encoding policy (Recall that this solution satisfies the positive semi-definiteness condition for the covariance matrix of } x, y \text{ and } z). \text{ Notice that } P\hat{u}^* \text{ cannot be identically zero since this contradicts with the optimal objective value in (38) being negative.}
\]

Hence, a set of policies in which the sender conveys information and the receiver employs its best response is found. This characterization gives a lower bound on the Stackelberg equilibrium as there is a linear policy restriction as well as a scalar message restriction. Since an informative lower bound
for the Stackelberg equilibrium is found, it follows that there always exists an informative Stackelberg equilibrium.

In addition to the existence result of Theorem 6 we can in fact explicitly characterize the equilibrium considering special covariance matrix structures. In particular, the Nash equilibrium characterized in Theorem 5 of this manuscript coincide with the Stackelberg equilibria. As in the scalar setting, these characterized equilibria do not make an a priori affine restriction at the encoder and affine equilibria arise from its optimality property.

IV. GAUSSIAN INFORMATION BOTTLENECK PROBLEM
INTERPRETED WITH THE PAPER’S FORMULATION

We now revisit the information bottleneck problem [40], which is a popular framework in the current literature, as an instance of our formulation under the Stackelberg equilibrium concept in the following sense: in contrast to privacy game setup, only the parameter $x$ is observed at the sender in the information bottleneck setup. In particular, we provide an estimation theoretic perspective on the information bottleneck problem. The information bottleneck problem considers a similar objective to that employed in this manuscript where the performance metrics involve mutual information rather than the mean squared error. In the information bottleneck problem, the aim is to compress an observed random variable while trying to preserve information related to a correlated random variable. These conflicting objectives are analyzed by formulating an optimization problem involving the mutual information between the considered random variables. More specifically, the aim is to find the optimal solution to the following optimization problem:

$$\min_{z=\gamma'(x)} I(x; z) - \beta I(y; z).$$ (39)

The information bottleneck problem considering jointly Gaussian multidimensional sources is studied in [41] where the structure of the optimal solution, which is jointly Gaussian with $x$ [45], is identified. The objective function in (39) resembles the objective function considered in this manuscript in the sense that in both problems the parameter $y$ is desired to be conveyed while information related to the parameter $x$ is desired to be removed from the displayed message.

The information bottleneck problem can in fact be viewed as a Stackelberg game between a sender and a receiver. In this game, the sender wants to compress the observed random variable and to convey the unobserved random variable. The use of mutual information as a performance metric effectively means that the receiver uses all the available information related to both of the parameters, i.e., it always employs its best response as in the Stackelberg equilibrium. Thus, the receiver is concerned with extracting information related to both of the parameters, which is also the case in our framework.

In the following, we consider a setting which is similar to the information bottleneck technique by employing mean squared error terms as our metric. As in the information bottleneck framework, the sender observes only the parameter $x$, rather than observing both parameters. Namely, the encoder has access to only partial information and is of the form $z = \gamma'(x)$. The objective functions of the sender and receiver are as defined in (31) and (32), respectively. In addition, the encoded random variable is directly observable by the receiver, i.e., $r = z$. Since the receiver is concerned with estimating both parameters, it employs the minimum mean squared error estimators of each parameter. Since the equilibrium concept is the Stackelberg equilibrium, the objective function of the sender can be written as

$$J^*(\gamma') = -\delta E[|x - E[x|z]|^2] + E[|y - E[y|z]|^2].$$ (40)

Now, observe that

$$E[|y - E[y|z]|^2] = E[|y - E[y|x]|^2] + E[|E[y|x] - E[y|z]|^2]$$

$$= E[|y - E[y|x]|^2] + E[|E[y|x] - E[y|z]|^2]$$

where the second equality follows from the fact that $y - E[y|x]$ is orthogonal to $x$ and $z$ (orthogonality with respect to $z$ is due to the fact that $y - x - z$ is a Markov chain in that order) and the third equality is due to the law of iterated expectations. By noticing that $E[y|x] = \Sigma_{XY} \Sigma_{XX}^{-1} x$, the objective function can be written as $J^*(\gamma') = \mathbb{E}[\|y - E[y|x]\|^2] - \delta E[\|x - E[x|z]\|^2] + E[(x - E[x|z])^T (\Sigma_{XX}^{-1} \Sigma_{XY} \Sigma_{YY} \Sigma_{XY}^{-1} \Sigma_{XX}^{-1}) (x - E[x|z])]$, where the first term is independent of the encoder. Thus, we obtain an optimization problem of the form

$$\min_{z=\gamma'(x)} \mathbb{E}\left[ (x - E[x|z])^T M(x - E[x|z]) \right],$$ (41)

where $M \triangleq (\Sigma_{XX}^{-1} \Sigma_{XY} \Sigma_{YY} \Sigma_{XY}^{-1} \Sigma_{XX}^{-1} - \delta I)$. Notice that $M$ is symmetric and thus it is diagonalizable with orthonormal eigenvectors. Hence, we can write $M = QAQ^T$ where $A$ is a diagonal matrix (with possibly positive and negative terms on the diagonal). If we define $\hat{x} = Q^T x$, we obtain an equivalent problem:

$$\min_{z=\gamma'(x)} \mathbb{E}\left[ \text{tr} \left( A(\hat{x} - E[\hat{x}|z])(\hat{x} - E[\hat{x}|z])^T \right) \right].$$ (42)

Under certain symmetry conditions, it is possible to solve the optimization problem in (42) and we state this result in the following theorem. Let $\lambda_i$ denote the $i$th diagonal element of $A$ and $\tilde{x}_i$ denote the $i$th element of $\hat{x}$.

Theorem 7: Depending on conditions involving $M = (\Sigma_{XX}^{-1} \Sigma_{XY} \Sigma_{YY} \Sigma_{XY}^{-1} \Sigma_{XX}^{-1} - \delta I)$, the Stackelberg equilibrium for the information bottleneck problem is characterized as follows:

(i) If the eigenvectors of $M$ and $\Sigma_X$ are completely aligned, then the sender transmits all $\tilde{x}_i$ for which $\lambda_i \geq 0$ at the Stackelberg equilibrium.

(ii) If $M$ is positive semi-definite, then there exists a fully informative Stackelberg equilibrium in which the sender transmits $x$.

(iii) If $M$ is negative definite, then the Stackelberg equilibrium is always noninformative.

(iv) Let $M$ be negative semi-definite with at least one eigenvalue equal to zero. Then, there exists an informative Stackelberg equilibrium if a component $\tilde{x}_i$ for which $\lambda_i = 0$ is uncorrelated with any component $\tilde{x}_j$ for which $\lambda_j < 0$. Otherwise, the equilibrium is always noninformative.
Proof:
(i) If we write the covariance matrix of \( \tilde{x} \), we get \( \Sigma_{X\tilde{X}} = \mathbb{E}[Q^T xx^T Q] = Q^T \Sigma_{XX} Q \). Since the eigenvectors of \( \Sigma_{X\tilde{X}} \) and \( M \) are assumed to be aligned, the covariance matrix of \( \tilde{x} \) becomes diagonal. Since the aim is to minimize \( (42) \) and each component \( \tilde{x}_i \) is independent of each other, the solution becomes transmitting all \( \tilde{x}_i \) for which \( \lambda_i \geq 0 \).
(ii) Since \( M \) is positive semi-definite, \( \Lambda \) contains nonnegative terms. As a result, the objective function in \( (42) \) is lower bounded by zero since all the coefficients are nonnegative. If the sender reveals \( x \), the receiver can perfectly recover \( \tilde{x} \) and this gives an objective value of zero. Therefore, the specified encoding policy attains the lower bound, which implies that this encoding policy yields a Stackelberg equilibrium.
(iii) In this case, \( \Lambda \) becomes a diagonal matrix with all negative elements. If the sender reveals information about any component of \( \tilde{x} \), the receiver would use this information to reduce its objective. However, as all the coefficients \( \lambda_i \) is strictly negative, revealing information about any component of \( \tilde{x} \) increases the objective in \( (42) \). Thus, transmitting information related to \( \tilde{x} \) is not desirable for the encoder, which implies that the Stackelberg equilibrium is noninformative.
(iv) First, consider the informative scenario. Under the given assumption, it is possible send \( \tilde{x}_i \) for which \( \lambda_i = 0 \) as it does not reveal any information related to any \( x_j \) for which \( \lambda_j < 0 \). Thus, revealing \( \tilde{x}_i \) for which \( \lambda_i = 0 \) does not affect the cost. As a result, we get an informative equilibrium. The noninformative scenario occurs when the receiver is able to extract information related to \( x_j \) for which \( \lambda_j < 0 \).

The only remaining case which is not covered in Theorem 7 is when \( M \) is neither positive semi-definite nor negative semi-definite. In the following, we show that in this remaining scenario, there always exists an informative equilibrium. In fact, the reduced problem in \( (42) \) can also be viewed as a multidimensional privacy-signaling game under the Stackelberg equilibrium concept where \( \tilde{x} \) is partitioned such that the indices with negative coefficient correspond to the private parameter and the indices with positive coefficient correspond to the nonprivate parameter. The following result uses this observation and Theorem 6.

Theorem 8: If \( M = (\Sigma_{X\tilde{X}}^{-1} \Sigma_{XY} \Sigma_{Y\tilde{X}} \Sigma_{X\tilde{X}}^{-1} - \delta I) \) has at least one positive and at least one negative eigenvalue, then there always exists an informative Stackelberg equilibrium.

Proof: The result is established by treating the optimization problem as a privacy game problem by appropriately partitioning \( \tilde{x} \) into private and nonprivate parameters based on the sign of diagonal terms of \( \Lambda \). The optimization problem in \( (42) \) can be written as \( \min_{z = \gamma'(x)} \sum_{i=1}^{n} \lambda_i \mathbb{E}[(\tilde{x}_i - \mathbb{E}[\tilde{x}_i|z])^2] \) where \( n \) denotes the size of the vector \( \tilde{x} \) or equivalently \( x \).

This equivalent problem can be viewed as a privacy game under the Stackelberg equilibrium concept in the following sense: We can view the components \( \tilde{x}_i \) for which \( \lambda_i \geq 0 \) as the nonprivate parameter and the components \( \tilde{x}_j \) for which \( \lambda_j < 0 \) as the private parameter. For this equivalent privacy game formulation under the Stackelberg equilibrium concept, we use the result of Theorem 6 to conclude that there always exist an informative Stackelberg equilibrium for the information bottleneck setup under the given condition in the statement of the theorem.

In the special case of scalar sources, Theorem 7 simplifies. In particular, depending on the value of \( \delta \), the equilibrium is either fully informative or noninformative and we summarize this result in the following corollary.

Corollary 1: The solution of the information bottleneck problem for scalar sources considering the Stackelberg equilibrium concept under the mean squared error distortion criterion is given by one of the following cases:

1) If \( (\rho^2/\sigma_X^4) > \delta \), then the sender completely reveals \( x \), i.e., \( z \) is an invertible function of \( x \). Thus, the equilibrium is fully informative.
2) If \( (\rho^2/\sigma_X^4) < \delta \), then the sender does not reveal information related to \( x \), i.e., the equilibrium is noninformative.
3) If \( (\rho^2/\sigma_X^4) = \delta \), then the sender either completely reveals \( x \) or does not reveal any information related to \( x \).

The following corollaries can be regarded as a generalization of the results for the scalar setting.

Corollary 2: Suppose that the multidimensional parameters are in the form \( (x_1, y_1), \ldots, (x_n, y_n) \) with \( (x_i, y_i) \) and \( (x_j, y_j) \) being independent for \( i \neq j \). Then, at the equilibrium the sender transmits random variables in \( \tilde{x} \) which correspond to positive values of \( \Lambda \).

Proof: Since \( \Sigma_{Y\tilde{X}} \) is a diagonal matrix in that case, \( M \) becomes a diagonal matrix. Then, we get the result by invoking (i) of Theorem 4.

Corollary 3: Suppose that the multidimensional parameters are in the form \( (x_1, y_1), \ldots, (\alpha_n, x_n, \alpha_n, y_n) \) for scalars \( \alpha_2, \ldots, \alpha_n \). Then, the solution becomes transmitting \( x_1 \) (or any \( x_i \)) when the privacy ratio is smaller than the threshold specified in Corollary 1 i.e., \( \delta < (\rho^2/\sigma_X^4) \). If \( \delta = (\rho^2/\sigma_X^4) \), then the fully informative and noninformative scenarios induce the same cost as in Corollary 1. When \( \delta > (\rho^2/\sigma_X^4) \), the equilibrium is noninformative.

Proof: In that case, it is easy to verify that \( M \) becomes a diagonal matrix with the same diagonal elements. Then, the result follows from Theorem 7.

Remark 7: In the information bottleneck problem, the sender uses partial information since only parameter \( x \) is available at the sender whereas in our privacy-signaling games formulation the sender has access to both of the parameters. Due to this further restriction that only partial information is available at the sender, our information bottleneck analysis provides a lower bound on the performance of our original Stackelberg game setting in which both of the parameters are observed at the sender.

Remark 8: It should be emphasized that the information bottleneck problem involving mutual information corresponds to the Stackelberg equilibrium concept since employing mutual information effectively means that the receiver uses all the available information, i.e., it employs its best response.
Consequently, it is shown that the information bottleneck problem under the mean squared error criterion can be reduced to a minimization problem involving a weighted mean squared error in terms of a transformed random vector. We characterize the Stackelberg equilibrium under certain conditions. If these conditions are not satisfied, we show that there always exists an informative Stackelberg equilibrium.

Remark 9: It is noted that while there is a complete solution for the information bottleneck problem [41], these results do not apply to the information bottleneck problem under the mean squared error cost criterion. The reason for this is as follows. In the information bottleneck problem, there is a flexibility of working in a transformed coordinate system as the mutual information is not affected by a transformation as long as it is invertible. Thus, a transformation is applied in [43] to simplify the problem. On the other hand, in our formulation, which uses mean squared error terms, if a transformation is applied on the considered random variables, then this transformation affects the costs and hence the transformation does not simplify the problem for the vector parameter case. In other words, if the covariance matrix of \( \tilde{x} \) is not diagonal in [42], it means that we cannot transform it to a random vector with a diagonal covariance matrix while keeping the matrix \( \Lambda \) diagonal.

V. Noisy Channel between the Sender and the Receiver

In this section, we extend our results considering scalar parameters to two important channel settings. (In fact, Lemma 3 and Lemma 4 apply to other channel settings.) For a given channel, the aim is to find an encoder/decoder pair that is optimal in conveying a zero mean Gaussian source over that particular channel in mean squared error sense. Once these policies are obtained, applying these policies to convey the linear combination \( u \) specified by (21) yields a Nash equilibrium as well as a Stackelberg equilibrium. The following builds on this observation.

A. Gaussian Noise Channel between the Sender and the Receiver

In this subsection, we consider the same problem as in Section II except that there is an additive Gaussian noise (e.g., measurement noise) between the transmitter and the receiver. More specifically, the sender encodes \( x \) and \( y \) into \( z \) which is subject to additive noise \( w \) independent of \( x \) and \( y \) and the receiver uses the observation \( r = z + w \) while decoding both of the random variables. The additive noise term is modeled as zero mean Gaussian with variance \( \sigma^2_w \). As before, the encoded random variable takes real values and there is an average power constraint at the sender, i.e., \( \mathbb{E}[z^2] \leq P \).

1) Nash Equilibria: Notice that the equivalent formulation employed in the proof of Theorem 1 is also applicable in the noisy communication scenario. The following theorem uses this observation to prove that a payoff dominant Nash equilibrium is attained by linear policies.

Theorem 9:

(i) The encoding policies \( \gamma^e(x, y) = C \) for all \( |C| \leq \sqrt{P} \) and the decoding policy \( \gamma^{d_x}(r) = 0 \) and \( \gamma^{d_y}(r) = 0 \) form a noninformative Nash equilibrium.

(ii) The encoding policy

\[
\gamma^e(x, y) = \alpha \left[ \frac{(\delta \sigma_x^2 + \sigma_y^2) + \sqrt{(\delta \sigma_x^2 + \sigma_y^2)^2 - 4 \delta \rho^2}}{2 \delta \rho} \right] y - x
\]

(43)

yields an informative Nash equilibrium where \( \alpha \) is a scaling coefficient that is either positive or negative such that \( \mathbb{E}[\gamma^e(x, y)^2] = P \) holds. The corresponding decoding policies at this equilibrium are given by

\[
\gamma^{d_x}(r) = \frac{A \sigma_x^2 + B \rho}{P + \sigma_u^2} r,
\]

(44)

\[
\gamma^{d_y}(r) = \frac{A \rho + B \sigma_y^2}{P + \sigma_u^2} r,
\]

(45)

where \( A \) and \( B \) denote the coefficients in front of \( x \) and \( y \), respectively, for the encoding policy in (43).

(iii) This informative equilibrium is a payoff dominant Nash equilibrium. Moreover, it is unique among the affine class of policies.

Proof: Lemma 3 implies that the sender is restricted to send \( u \). Then, it is easy to verify that sending \( u \) after scaling up to the maximum available power level yields a Nash equilibrium. The decoding policies at this equilibrium are given by the minimum mean squared error estimators corresponding to each parameter. Since the observation \( r \) is jointly Gaussian with \( x \) and \( y \), the conditional expectation formula for Gaussian distributions can be employed to obtain [44] and [45] [46, p. 155].

The proof for the payoff dominance property of the equilibrium uses the observation that the performance of both players is determined by \( \mathbb{E}[(u - \gamma^{d_y}(r))^2] \) at a Nash equilibrium. Since the source is scalar and the Gaussian noise is additive, we can employ the well-known result that the problem of transmitting a scalar Gaussian source over a scalar Gaussian channel under an average power constraint admits an optimal solution with linear encoding scaled to satisfy the power constraint with equality (see e.g. [47, p. 376]). Hence, the result immediately follows.

The proof for uniqueness is similar to the proof of Theorem 1 except that in the noisy case the encoding policy must be linear in \( u \) and all the available power needs to be used.

2) Stackelberg Equilibria:

Theorem 10: The Stackelberg equilibrium coincides with the informative Nash equilibrium characterized in Theorem 9. This equilibrium is unique among any set of policies.

Proof: Lemma 4 implies that the encoder cannot use \( v \) and it can only use \( u \) in constructing its message. As the objectives of each player then becomes the minimization of the mean squared error for estimating \( u \), the optimal strategy of the sender is to employ an encoding policy which is linear in \( u \) with average power equal to \( P \) (see the proof of Theorem 9). In addition, this encoding strategy is unique due to the fact that it is the unique solution to the problem of transmitting a
scalar Gaussian source over a scalar Gaussian channel under an average power constraint \[47\].

It is important to emphasize that the encoder is not restricted to be affine. Since the problem reduces to transmitting a scalar Gaussian source over a scalar Gaussian channel under an average power constraint, we obtain these linear policies as the optimal unique solution to this reduced problem.

### B. Discrete Noiseless Channel between the Sender and the Receiver

In this subsection, we investigate the discrete channel setting where the sender is restricted to transmit a discrete value, i.e., \(z \in \{0, \ldots, M - 1\}\) for some \(M \geq 2\). We assume that the channel is noiseless, i.e., \(r = z\).

1) Nash and Stackelberg Equilibria: While investigating the discrete channel setting, we employ the equivalent formulation which facilitates the analysis. In this equivalent formulation, after determining \(\tilde{\gamma}^c(u, v)\) for the sender and \(\gamma^d(r)\) and \(\gamma^{\nu}(r)\) for the receiver, one can completely express the encoding and decoding policies at the equilibrium using \(21-24\).

Lemma 3 and Lemma 4 imply that both players share the common objective of minimizing \(E[(u - \gamma^d (r))^2]\) under both of the equilibrium concepts. Since the sender is restricted to transmit discrete values, it is required to quantize \(u\) at the sender. Since this would correspondence to classical quantization, that an optimal quantizer exists follows from classical results in the literature, e.g., \[48\]. Namely, there exist quantization bins and reconstruction points which minimize the corresponding mean squared error. Thus, by assigning each bin to a discrete value of \(z\) and then using the corresponding optimal reconstruction points at the receiver yield a Nash equilibrium. We summarize this result in the following theorem.

**Theorem 11:** Consider the quantization of \(u\) into \(M\) bins where each bin is assigned to a discrete value of \(z\) at the encoder and the corresponding reconstruction points at the receiver such that \(E[(u - \gamma^d (r))^2]\) is minimized. This pair of encoding and decoding policies, which always exists, forms an informative Nash equilibrium. In addition, this equilibrium is a payoff dominant equilibrium.

It is worth to point out that for any number of bins less than \(M\), there exists a Nash equilibrium. In other words, even if \(z\) can take \(M\) discrete values, a quantization policy using less than \(M\) bins at the sender and the corresponding reconstruction points at the receiver is also a Nash equilibrium. In addition, the case of single bin is also a Nash equilibrium in which no information related to \(u\) is conveyed to the receiver.

It is noted that using a large number of bins yields lower \(E[(u - \gamma^d (r))^2]\). Since the mean squared error for estimating \(u\) is desired to be minimized for both players, using a large number of bins results in improved objectives for both players. This monotonicity property with respect to the number of bins implies that in the Stackelberg equilibrium there must be \(M\) bins.

**Theorem 12:** The pair of policies in Theorem 11 is also a Stackelberg equilibrium.

**Remark 10:** The performance of both players at the equilibrium in the discrete channel setting for finite \(M\) is strictly worse than the performance at the characterized family of equilibria in the noiseless setting in Section 14. This is due to the fact that the term \(E[(u - \gamma^d (r))^2]\) is strictly greater than zero in this case whereas it is zero in the noiseless case.

### VI. Numerical Examples

In this section, the proposed privacy-signaling game problem is investigated through numerical examples, where the following scenario is considered: The variances of the private and nonprivate parameters are set as \(\sigma_X^2 = \sigma_Y^2 = 1\), and the average power level constraint at the sender is given by \(P = 1\). In the following, we illustrate the performances at the characterized equilibriums. Since the informative Nash equilibriums coincide with the Stackelberg equilibriums, we do not make a distinction between them.

Fig. 3 plots the estimation errors for the private and nonprivate parameters with respect to the privacy ratio. The estimation error for the private parameter increases with the privacy ratio since the transmitter removes information related to the private parameter due to enhanced privacy concerns. This removal also distorts the information conveyed related to the nonprivate parameter and hence the corresponding estimation error also increases.
TABLE I
THE RATIO OF COEFFICIENTS AT THE ENCODER FOR THE DERIVED INFORMATIVE EQUILIBRIA WHEN $\sigma_X^2 = 1$ AND $\sigma_Y^2 = 1$.

| Scenario | $B/A$ |
|----------|--------|
| $\rho = 0.3$ and $\delta = 0.1$ | $-36.39$ |
| $\rho = 0.3$ and $\delta = 1$ | $-6.61$ |
| $\rho = 0.3$ and $\delta = 10$ | $-3.63$ |
| $\rho = 0.7$ and $\delta = 0.1$ | $-15.04$ |
| $\rho = 0.7$ and $\delta = 1$ | $-2.44$ |
| $\rho = 0.7$ and $\delta = 10$ | $-1.50$ |

Fig. 4 plots the estimation errors for the private and non-private parameters with respect to the correlation between them. In the low privacy scenario, the estimation error for $y$ does not change significantly with respect to the correlation since most of the information contained in $y$ is conveyed to the receiver regardless of the correlation. As a result, more information is leaked related to the private parameter as the correlation is increased. In contrast, in the high privacy scenario, regardless of the correlation, most of the information related to $x$ is removed from the transmitted message. Thus, the estimation error for the nonprivate parameter increases with the correlation whereas no significant changes in the estimation error for the private parameter are observed.

Finally, Table I illustrates the tradeoff between utility in terms of conveying $y$ and privacy in terms of hiding $x$ by providing the structure of the encoder at the equilibrium for various values of the privacy ratio and correlation. It can be inferred that if the privacy ratio is increased while the correlation is kept the same, the information leakage related to the private parameter reduces, as expected.

VII. CONCLUSION

A communication setting between a sender with privacy concerns and a receiver has been investigated in a game theoretic framework. The private and nonprivate parameters have been modeled as jointly Gaussian random variables. Under a noiseless communication setting, it has been shown that a payoff dominant Nash equilibria is attained by affine policies for scalar sources. It has been proven that the characterized Nash equilibria form also a Stackelberg equilibrium. These results have been further generalized to the Gaussian noisy channel setting as well as a discrete noiseless channel setting. Furthermore, it has been shown that there always exists informative Stackelberg equilibria considering the case when the parameters are multidimensional. We have also provided an estimation theoretic perspective on the information bottleneck problem under the Stackelberg equilibrium concept. We have characterized the equilibria for the multidimensional parameter setting under certain conditions and we have established the existence of informative equilibria considering the case when these conditions are not met.

APPENDIX

A. Proof of Lemma 2

Given the encoding policy, the optimization problem at the receiver decouples over $\gamma_{\theta X}(\cdot)$ and $\gamma_{\theta Y}(\cdot)$. Consequently, the optimal decoders at the receiver become minimum mean squared error estimators of each parameter which leads to the expressions in (11) and (12) [46, p. 155].

B. Proof of Lemma 2

Considering the decoders specified in the statement of the lemma, the optimization problem that the sender needs to solve becomes (see (1))

$$\min_{z=\gamma^*(x,y)} E[(Mz + N - y)^2] - \delta E[(Kz + L - x)^2].$$

(46)

Suppose first that $M^2 = \delta K^2$ holds. As a result, the optimization problem in (46) reduces to

$$\min_{z=\gamma^*(x,y)} E[z^2] \leq P \frac{(M^2 - \delta K^2)}{M^2 - \delta K^2}.$$  

(47)

Cauchy-Schwarz inequality can be employed to conclude that the optimal solution to (47) is given by the encoder expressed in (14).

Now suppose that $M^2 \neq \delta K^2$ holds. In this case, omitting the terms that do not depend on $z$, the optimization problem in (46) can be written as

$$\min_{z=\gamma^*(x,y)} E[z^2] \leq P \frac{(M^2 - \delta K^2)}{M^2 - \delta K^2}.$$  

(48)

It is seen that when $M^2 > \delta K^2$, the goal becomes minimizing the expectation term under the constraint on the average transmit power and considering this case the optimal encoding policy can be found by solving

$$\min_{z=\gamma^*(x,y)} E[z^2] - 2E \left[ \frac{E[(M(y - N) - \delta K(x - L))^2]}{M^2 - \delta K^2} \right].$$  

(49)

The minimization problem in (49) can be solved in two steps. Namely, we first solve the minimization problem for a fixed average power level $p$ and then we obtain the optimal encoding policy by searching for the optimal average power level. More explicitly, the optimization problem in (49) can equivalently be expressed as

$$\min_{p \leq P} \min_{z=\gamma^*(x,y)} E[z^2] = p - 2E \left[ \frac{E[(M(y - N) - \delta K(x - L))^2]}{M^2 - \delta K^2} \right].$$  

(50)

In order to characterize the solution to the inner minimization problem, Cauchy-Schwarz inequality can be employed. As a result, we get $z = \frac{\alpha M(y - N) - \delta K(x - L)}{M^2 - \delta K^2}$ where $\alpha$ is a nonnegative scaling term used to set the average power level to $p$. Next, based on the outer minimization problem, the optimal average power level becomes

$$\arg \min_{p \leq P} p - 2\sqrt{p} \left( \frac{E[(M(y - N) - \delta K(x - L))^2]}{M^2 - \delta K^2} \right)^{1/2}.$$  

(51)

Consequently, if the average power level of $z = \frac{M(y - N) - \delta K(x - L)}{M^2 - \delta K^2}$ is less than $P$, this becomes the optimal
encoding policy solving (49), equivalently (46), as given in (13). Otherwise, the scaling factor α needs to be such that the average power level is equal to \( P \), which gives an encoding policy as in (14).

Finally, we consider the case of \( M^2 < \delta K^2 \), which leads to a maximization instead of a minimization and consequently the optimal encoding policy can be obtained by solving
\[
\max_{z=\gamma(x,y), E[z]^2 \leq P} \mathbb{E}[z^2] - 2\mathbb{E} \left[ z \left( \frac{M(y-N) - \delta K(x-L)}{M^2 - \delta K^2} \right) \right].
\]
(52)

In this case, the solution directly follows from Cauchy-Schwarz inequality and it is given by \( z = \alpha \frac{\delta K(x-L) - M(y-N)}{M^2 - \delta K^2} \) where \( \alpha \) is a nonnegative scaling term that needs to be such that the average power constraint is satisfied with equality. Moreover, by noting that \( M^2 - \delta K^2 \) is negative in this case, the optimal encoding policy can be written as in (14).

C. Proof of Theorem 2

Since the case of \( A \neq 0 \) and \( B = 0 \) yields an informative Nash equilibrium (as long as the average power constraint is satisfied and the receiver employs its best response), it is required to show that \( A \) does not converge to zero and \( B \) converges to zero. In order to express fixed point iterations considering the equivalent formulation, Lemma 1 and Lemma 2 can be employed after making appropriate substitutions. Denote the coefficients at the encoder after the \( n \)th iteration by \( A^{(n)} \) and \( B^{(n)} \). If \( (A^{(n)})^2 \sigma_U^4 > \delta (B^{(n)})^2 \sigma_V^4 \) and the average encoding power is less than or equal to \( P \), then the fixed point iterations become
\[
A^{(n+1)} = \frac{A^{(n)} \sigma_U^4}{(A^{(n)})^2 \sigma_U^4 + (B^{(n)})^2 \sigma_V^4} \left( (A^{(n)})^2 \sigma_U^4 + (B^{(n)})^2 \sigma_V^4 \right),
\]
(53)
\[
B^{(n+1)} = -\frac{\delta B^{(n)} \sigma_V^4}{(A^{(n)})^2 \sigma_U^4 - \delta (B^{(n)})^2 \sigma_V^4} \left( (A^{(n)})^2 \sigma_U^4 - \delta (B^{(n)})^2 \sigma_V^4 \right).
\]
(54)

On the other hand, if these two conditions are not met, the fixed point iterations can be expressed as
\[
A^{(n+1)} = \frac{A^{(n)} \sigma_U^4 \sqrt{P}}{(A^{(n)})^2 \sigma_U^4 + \delta^2 (B^{(n)})^2 \sigma_V^4} \left( (A^{(n)})^2 \sigma_U^4 + \delta^2 (B^{(n)})^2 \sigma_V^4 \right)^{1/2},
\]
(55)
\[
B^{(n+1)} = -\frac{\delta B^{(n)} \sigma_V^4 \sqrt{P}}{(A^{(n)})^2 \sigma_U^4 - \delta^2 (B^{(n)})^2 \sigma_V^4} \left( (A^{(n)})^2 \sigma_U^4 - \delta^2 (B^{(n)})^2 \sigma_V^4 \right)^{1/2}.
\]
(56)

It is noted that in either case the ratio of \( B^{(n)} \) and \( A^{(n)} \) satisfies \( B^{(n+1)} / A^{(n+1)} = -B^{(n)} / A^{(n)} \sigma_V^4 / \sigma_U^4 \). It is seen that the convergence behavior depends on the value of \( \sigma_V^4 / \sigma_U^4 \). In the following, we treat each case separately.

3) \( \sigma_V^4 / \sigma_U^4 < 1 \): This is the case for which an informative Nash equilibrium is attained. In this case, \( |A^{(n)}/B^{(n)}| \) goes to infinity as \( n \to \infty \). We first consider the case when (55) and (56) are valid as \( n \to \infty \). Notice that \( |A^{(n+1)}| \) can be expressed as
\[
|A^{(n+1)}| = \frac{|A^{(n)}|}{|B^{(n)}|} \frac{\sigma_U^4 \sqrt{P}}{(A^{(n)})^2 \sigma_U^4 + \delta^2 (B^{(n)})^2 \sigma_V^4} \left( (A^{(n)})^2 \sigma_U^4 + \delta^2 (B^{(n)})^2 \sigma_V^4 \right)^{1/2}.
\]
(57)

Noting that \( |A^{(n)}/B^{(n)}| \) goes to infinity as \( n \to \infty \), it is concluded from above that \( |A^{(n)}| \to \sqrt{P} / \sigma_U \) as \( n \to \infty \). Since \( |A^{(n)}| \) goes to a finite value and \( |A^{(n)}|/B^{(n)}| \) goes to infinity, it follows that \( B^{(n)} \) goes to zero. Now, consider the case when (55) and (56) are valid as \( n \to \infty \). By manipulating (53), we get
\[
|A^{(n+1)}| = \frac{|A^{(n)}| \sigma_V^4}{(A^{(n)})^2 \sigma_U^4 + (B^{(n)})^2 \sigma_V^4} \left( (A^{(n)})^2 \sigma_U^4 + (B^{(n)})^2 \sigma_V^4 \right) \left( (A^{(n)})^2 \sigma_U^4 - \delta (B^{(n)})^2 \sigma_V^4 \right).
\]
(58)

Since \( |A^{(n)}| \) is an increasing sequence, it is guaranteed that \( A^{(n)} \) does not converge to zero. Also, due to the average power constraint at the encoder, \( A^{(n)} \) can take only a finite value. Thus, since \( |A^{(n)}/B^{(n)}| \) goes to infinity, it follows that \( B^{(n)} \) goes to zero. Consequently, an informative Nash equilibrium is reached in either case.

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