DIFFERENTIAL CALCULUS
ON THE QUANTUM HEISENBERG GROUP

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ABSTRACT. The differential calculus on the quantum Heisenberg group is constructed. The duality between quantum Heisenberg group and algebra is proved.

I. Introduction

The one dimensional deformed Heisenberg group and algebra were investigated in [1], [2]. In this paper, using Woronowicz’s theory ([3]), we construct the differential calculus on the deformed one dimensional Heisenberg group and we describe the structure of its quantum Lie algebra. Then we prove that our quantum Lie algebra is equivalent to the one dimensional deformed Heisenberg algebra.

II. The differential calculus

The quantum group $H(1)_q$ is a matrix quantum group à la Woronowicz ([4])

$T = \begin{pmatrix}
1 & \alpha & \beta \\
0 & 1 & \delta \\
0 & 0 & 1
\end{pmatrix}$

where the matrix elements $\alpha, \beta, \delta$ generate the algebra $\mathcal{A}$ and satisfy the following relations ([1])

$[\alpha, \beta] = i\lambda \alpha,$

$[\delta, \beta] = i\lambda \delta,$

$[\alpha, \delta] = 0,$

$\lambda$ being a real parameter.

* Supported by Łódź University grant No 505/445

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The coproduct, counit and antipode are given by

\[ \Delta(\alpha) = I \otimes \alpha + \alpha \otimes I, \]
\[ \Delta(\beta) = I \otimes \beta + \beta \otimes I + \alpha \otimes \delta, \]
\[ \Delta(\delta) = I \otimes \delta + \delta \otimes I, \]
\[ S(\alpha) = -\alpha, \]
\[ S(\beta) = -\beta + \alpha\delta, \]
\[ S(\delta) = -\delta, \]
\[ \varepsilon(\alpha) = \varepsilon(\beta) = \varepsilon(\delta) = 0. \]  
(3)

The main ingredient of the Woronowicz theory is the choice of a right ideal in \( \text{ker} \varepsilon \), which is invariant under the adjoint action of the group. The adjoint action is defined as follows

\[ \text{ad}(a) = \sum_k b_k \otimes S(a_k)c_k \]  
(4)

here

\[ (\Delta \otimes I) \circ \Delta(a) = \sum_k a_k \otimes b_k \otimes c_k. \]

One can prove the following

**Theorem 1.** Let \( R \subset \text{ker} \varepsilon \) be the right ideal generated by the following elements: \( \alpha^2, \delta^2, \beta\alpha, \beta\delta, \alpha\delta, \beta^2 + 2i\lambda\beta \). Then

(i) \( R \) is \( \text{ad} \)-invariant, \( \text{ad}(R) \subset R \otimes A \)

(ii) \( \text{ker} \varepsilon / R \) is spanned by the following elements: \( \alpha, \beta, \delta \).

Having established the structure of \( R \) we follow closely the Woronowicz construction. The basis of the space of the left-invariant 1-forms consists of the following elements

\[ \omega_\alpha \equiv \pi r^{-1}(I \otimes \alpha) = d\alpha, \]
\[ \omega_\beta \equiv \pi r^{-1}(I \otimes \beta) = d\beta, \]
\[ \omega_\delta \equiv \pi r^{-1}(I \otimes \delta) = d\beta - \alpha d\delta; \]  
(5)

here the mapping \( r^{-1} \) is given by

\[ r^{-1}(a \otimes b) = (a \otimes I)(S \otimes I)\Delta(b), \quad a, b \in A, \]

and the mapping \( \pi \) is given by

\[ \pi(\sum_k a_k \otimes b_k) = \sum_k a_k db_k \]

where \( \sum_k a_k \otimes b_k \in A \otimes A \) is such an element that

\[ \sum_k a_kb_k = 0. \]
The next step is to find the commutation rules between the invariant forms and generators of \( A \). The detailed calculations result in the following formulae

\[
\begin{align*}
[\alpha, \omega_\alpha] &= 0, \\
[\delta, \omega_\alpha] &= 0, \\
[\beta, \omega_\alpha] &= -i\lambda \omega_\alpha, \\
[\alpha, \omega_\delta] &= 0, \\
[\delta, \omega_\delta] &= 0, \\
[\beta, \omega_\delta] &= -i\lambda \omega_\delta, \\
[\alpha, \omega_\beta] &= 0, \\
[\delta, \omega_\beta] &= 0, \\
[\beta, \omega_\beta] &= 2i\lambda \omega_\beta.
\end{align*}
\] (6)

Then, following Woronowicz’s paper [3], we can construct the right-invariant forms

\[
\begin{align*}
\eta_\alpha &= \omega_\alpha, \\
\eta_\delta &= \omega_\delta, \\
\eta_\beta &= \omega_\beta - \omega_\alpha \delta + \omega_\delta \alpha.
\end{align*}
\] (7)

This concludes the description of the bimodule \( \Gamma \) of 1-forms on \( H(1)_q \). The external algebra can now be constructed as follows ([3]). On \( \Gamma^\otimes 2 \) we define a bimodule homomorphism \( \sigma \) such that

\[
\sigma(\omega \otimes_A \eta) = \eta \otimes_A \omega
\] (8)

for any left-invariant \( \omega \in \Gamma \) and any right-invariant \( \eta \in \Gamma \). Then by definition

\[
\Gamma^{\wedge 2} = \frac{\Gamma^\otimes 2}{\ker(I - \sigma)}.
\] (9)

Equations (7)–(9) allow us to calculate the external product of left-invariant 1-forms. The result reads

\[
\begin{align*}
\omega_\beta \wedge \omega_\alpha &= -\omega_\alpha \wedge \omega_\beta, \\
\omega_\beta \wedge \omega_\delta &= -\omega_\delta \wedge \omega_\beta, \\
\omega_\beta \wedge \omega_\beta &= 0, \\
\omega_\alpha \wedge \omega_\alpha &= 0, \\
\omega_\delta \wedge \omega_\delta &= 0, \\
\omega_\alpha \wedge \omega_\delta &= -\omega_\delta \wedge \omega_\alpha.
\end{align*}
\] (10)

To complete the external calculus, we derive the Cartan-Maurer equations

\[
\begin{align*}
d\omega_\alpha &= 0, \\
d\omega_\delta &= 0, \\
d\omega_\beta &= -\omega_\delta \wedge \omega_\beta.
\end{align*}
\] (11)
III. Quantum Lie algebra

In order to obtain the counterpart of the classical Lie algebra, we introduce the counterpart of the left-invariant vector fields. They are defined by the formula

\[ da = (\chi_\alpha \ast a)\omega_\alpha + (\chi_\beta \ast a)\omega_\beta + (\chi_\delta \ast a)\omega_\delta. \]  
(12)

In order to find the quantum Lie algebra, we apply the external derivative to both sides of (12), we use \( d^2 a = 0 \) on the left-hand side and calculate the right-hand side using (11) and again (12). Nullifying the coefficients in front of basis elements of \( \Gamma^{\wedge 2} \), we find the quantum Lie algebra

\[ [\chi_\alpha, \chi_\beta] = 0, \]
\[ [\chi_\alpha, \chi_\delta] = 0, \]
\[ [\chi_\alpha, \chi_\delta] = \chi_\beta. \]  
(13)

From the Woronowicz theory, it follows that the coproduct of the functional \( \varphi_i \) (\( \varphi_i = \chi_\alpha, \chi_\beta, \chi_\delta \)) can be written in the form

\[ \Delta \varphi_i = \sum_j \varphi_j \otimes f_{ji} + I \otimes \varphi_i \]  
(14)

where \( f_{ji} \) are the functionals entering in the commutation rules between the left-invariant forms and elements of \( \mathcal{A} \)

\[ \omega_j a = \sum_i (f_{ji} \ast a)\omega_i, \]  
(15)

Then, it follows from commutation rules (6) that the coproduct for our functionals can be written in the following form

\[ \Delta \chi_\alpha = \chi_\alpha \otimes f_\alpha + I \otimes \chi_\alpha, \]
\[ \Delta \chi_\beta = \chi_\beta \otimes f_\beta + I \otimes \chi_\beta, \]
\[ \Delta \chi_\delta = \chi_\delta \otimes f_\delta + I \otimes \chi_\delta. \]  
(16)

Using the fact that \( \Delta f_i = f_i \otimes f_i \) (\( i = \alpha, \beta, \delta \)) ([3]) and (6) and (15), we can calculate the functionals \( f_i \). After some calculations we obtain

\[ f_\alpha = (I - 2i\lambda \chi_\beta)^{\frac{1}{2}}, \]
\[ f_\beta = I - 2i\lambda \chi_\beta, \]  
\[ f_\delta = (I - 2i\lambda \chi_\beta)^{\frac{3}{2}}. \]  
(17)

Now it is easy to see that the substitution

\[ \chi_\delta = B_0, \]
\[ \chi_\beta = B_1, \]
\[ \chi_\alpha = B_2 \]  
(18)

reproduces the structure of the Hopf algebra generated by the infinitesimal generators (obtained by contraction procedure) of the quantum matrix pseudogroup \( H(1)_q \), which was described in [1]. This proves the duality between the quantum Heisenberg group and algebra.
References

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