Flavor masses and mixing in modular $A_4$ Symmetry

Mohamed Abbas,

Physics Department, Faculty of Science and Arts, Jouf University, Tabarjal, Al-Jouf, Ksa.

Physics Department, Faculty of Sciences, Ain Shams University, Abbassiyah 11566, Cairo, Egypt.

February 10, 2020

Abstract

A flavor model based on a $A_4$ modular group is proposed to account for both lepton and quark parameters (masses and mixing). We consider the inverse seesaw mechanism to produce the light neutrino masses. Both neutrino and charged lepton masses are obtained in terms of Yukawa coupling ratios and the module $\tau$ of the $A_4$ modular form. The calculated lepton and quark parameters are in good agreement with the recent data.

1 Introduction

The type-I seesaw mechanism [1] is the common scenario to explain the smallness of neutrino mass. In this mechanism, the small neutrino mass is obtained by extension of the fermion contents with three chiral supermultiplets as heavy right handed neutrinos $N^c_i$. The mass of the light neutrino states can be calculated through the relation $m_\nu = -m_D M^{-1} m_D^T$, where

*E-mail: maabbas@ju.edu.sa
$m_D$ is the Dirac mass and $M$ is the Majorana mass of right handed neutrinos $N_i$. In order to obtain the small neutrino mass of order $\mathcal{O}(10^{-2})$ eV and Dirac mass of order GeV, the mass scale of the $N^c$ will be of order $\mathcal{O}(10^{11}\mathrm{GeV})$ or so, which is far from experimental reach. In this scenario, the lepton number is violated via the large scale of the right handed neutrino masses.

On the other hand, the inverse seesaw mechanism [2, 3, 4] provides an alternative mechanism to explain tiny neutrino mass via a double suppression by the new physics scale $M_R$ via small scale $\mu_s$ through the relation $m_\nu = m_D M_R^{-1} \mu_s M_T^{-1} m_D$. In this mechanism, the singlets $S$ acquire very tiny mass $\mu_s$ which violates the lepton number. The lepton number violation (LNV) occurring by this tiny mass scale is very small compared to that in the case of type I seesaw. Thus the lepton number can be regarded as an approximate symmetry rather than an exact one. The Lepton number symmetry is enhanced when $\mu_s$ and therefore $m_\nu$ tend to zero, and lepton number violation (LNV) vanishes. [5]

Many aspects such as the differences in mixing and mass hierarchy for lepton and quark sectors force the flavor symmetry to be proposed to account for these aspects. Several models based on discrete symmetries were proposed to account for flavor aspects (see [6]). For most of these models, some additional scalars (flavons) were considered besides a lot of assumptions and extra $Z_N$ symmetries were proposed to account for experimental data.

Recently, finite modular groups $\Gamma_N$ have been proposed to interpret the flavor aspects [7, 8]. In modular groups, the coupling constants can transform non trivially and extra symmetries under modular weights are impeded into the group, so there is no need to impose other symmetries to match the data. Some of $\Gamma_N$ are isomorphic to finite permutation groups, for instance, $\Gamma_2 \cong S_3$ [9, 10, 11, 12], $\Gamma_3 \cong A_4$ [13, 14, 15, 16], $\Gamma_4 \cong S_4$ [17, 18, 19] and $\Gamma_5 \cong A_5$ [20, 21].

In this paper, we introduce a model based on modular $A_4$ symmetry to account for masses and mixing for leptons and quarks. First, we give an introduction to the modular groups and modular forms and how to use it as flavor symmetry, then we explain our $A_4$ model in the lepton sector and finally we study the quark masses and mixing.
2 Modular groups

In this section, we give a brief summary of the modular groups and modular forms. The modular group \( \bar{\Gamma} \) is defined as linear fractional transformations on the complex upper half plane \( \mathcal{H} \) and has the form \[ \gamma : z \rightarrow \gamma(\tau) = \frac{a\tau + b}{cz + d}, \] where \( a, b, c, d \in \mathbb{Z}, ad - bc = 1 \). The modular group \( \bar{\Gamma} \) is isomorphic to the projective special linear group \( PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{I, -I\} \), where \( SL(2, \mathbb{Z}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), ad - bc = 1 \} \).

The group \( \bar{\Gamma} \) is generated by two matrices \( S \) and \( T \) where their action on the complex number \( \tau \) is given by,
\[ S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1. \]

The two generators \( S, T \) can be represented by two by two matrices as
\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

Because \( I \) and \(-I\) are indistinguishable in \( PSL(2, \mathbb{Z}) \), we can say that
\[ S^2 = I, \quad (ST)^3 = I. \]

Define the infinite inhomogeneous modular groups \( \Gamma(N), N = 1, 2, 3, \ldots \) as
\[ \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{mod } N \right\}. \]
For $N = 1$, 
\[
\Gamma(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \mod 1 \right\}. \tag{7}
\]
Any integers can satisfy the conditions $a, d \equiv 1 \mod 1$ and $b, c \equiv 0 \mod 1$, so $\Gamma(1) \equiv SL(2, \mathbb{Z})$. For $N = 1, 2$, we define $\tilde{\Gamma}(N) = \Gamma(N)/\{I, -I\}$ whereas for $N > 2$, $\tilde{\Gamma}(N) = \Gamma(N)$ because $-I \notin \Gamma(N)$ for $N > 2$. It is straightforward to notice that $\tilde{\Gamma}(1) = PSL(2, \mathbb{Z}) = \tilde{\Gamma}$. The groups $\Gamma(N)$ and $\tilde{\Gamma}(N)$ are discrete but infinite, so we can construct the finite modular groups $\Gamma_N = \tilde{\Gamma}/\tilde{\Gamma}(N)$.

The modular function $f(\tau)$ of weight $2k$ is a meromorphic function of the complex variable $\tau$ satisfies 
\[
f(\gamma(\tau)) = f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} f(\tau) \quad \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(N), \tag{8}
\]
where the integer $k \geq 0$. By using Eqs. (1) and (6), it is easy to calculate
\[
\frac{d(\gamma(\tau))}{d\tau} = \frac{1}{(c\tau + d)^2}. \tag{9}
\]
From Eq. (8), one can get 
\[
f(\gamma(\tau)) = \left(\frac{d(\gamma(\tau))}{d\tau}\right)^{-k}, \quad f(\gamma(\tau))d(\gamma(\tau))^k = f(\tau)d\tau^k.
\]
From the above equation, we conclude that $f(\tau)d\tau^k$ is invariant under $\Gamma(N)$. If the modular function is holomorphic everywhere, it is called "modular form" of weight $2k$. The modular forms of level $N$ and weight $2k$ form a linear space of finite dimension. In the basis at which the transformation of a set of modular forms $f_i(\tau)$ described by a unitary representation $\rho$, one can get
\[
f_i(\gamma(\tau)) = (c\tau + d)^{2k} \rho_{ij}(\gamma) f_j(\tau), \quad \gamma \in \Gamma(N). \tag{10}
\]
Consider the superpotential $W(z, \phi)$ be written in terms of supermultiplets $\phi^I$, where $I$ refers to different sectors in the theory,
\[
W(\tau, \phi) = \sum_I \sum_n Y_{i_1 i_2 \ldots i_n}(\tau) \phi^{i_1} \phi^{i_2} \ldots \phi^{i_n}. \tag{11}
\]
The invariance of the superpotential $W(z, \phi)$ under the modular transformation requires $Y_{I_1 \ I_2 \ ... \ I_n}(z)$ to be a modular form transforming in the representation
\[
Y_{I_1 \ I_2 \ ... \ I_n}(\gamma \tau) = (cz + d)^{k \nu(n)} \rho(\gamma) Y_{I_1 \ I_2 \ ... \ I_n}(\tau).
\]  
(12)

The modular invariance forces the condition
\[
k \nu = k_{I_1} + k_{I_2} + \ldots + k_{I_n}.
\]  
(13)

2.1 Modular forms of level 3

The group $A_4$ has one triplet representation $3$ and 3 singlets $1, \ 1'$ and $1''$ and is generated by two elements $S$ and $T$ satisfying the conditions
\[
S^2 = T^3 = (ST)^3 = 1.
\]  
(14)

The modular form of level 3 has the form
\[
f_i(\gamma(\tau)) = (c\tau + d)^{2k} \rho_{ij}(\gamma) f_j(\tau), \ \gamma \in \Gamma(3).
\]  
(15)

The modular form of weight 2 and level 3 transforms as a triplet and is given by $Y^{(2)}_3 (y_1, y_2, y_3)$ where:
\[
y_1(\tau) = \frac{i}{2\pi} \left[ \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27 \eta'(3\tau)}{\eta(3\tau)} \right],
\]
\[
y_2(\tau) = -\frac{i}{\pi} \left[ \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right],
\]
\[
y_3(\tau) = -\frac{i}{\pi} \left[ \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right].
\]  
(15)

where the Dedekind eta-function $\eta(z)$ is defined as
\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}.
\]  
(16)

One can construct modular forms of higher weights using the multiplication rules of $A_4$ [8].
The lepton contents in the model are extended by adding a triplet of chiral supermultiplets $N^c$ as right handed neutrinos and three SM singlets $S_i$ to get the neutrino masses via inverse seesaw mechanism. We add a gauge singlet scalar $\chi$ transforming trivially under $A_4$ to get the masses of the singlet fermions $N^c$ and $S$. Contrary to most of the flavor symmetric models, no extra discrete symmetries are considered in our model. According to the modular invariance condition in Eq.(13), we chose the modular weights such that the following relations are satisfied:

\begin{align*}
K_L + K_{H_u} + K_E &= 2, \\
K_L + K_{H_u} + K_N &= 2, \\
2K_S + 8K_\chi &= 0, \\
K_S + K_N + K_\chi &= 0. \tag{17}
\end{align*}

If we chose $K_L = 1$, $K_{H_u} = 4$, we can get the modular weights of other fields as shown in Table (1).

The lepton modular $A_4$ invariant superpotential can be written as

\begin{equation}
\begin{aligned}
w_l &= \lambda_1 E_1^c E_1^c H_d (L \otimes Y_3^{(2)})_1 + \lambda_2 E_2^c E_2^c H_d (L \otimes Y_3^{(2)})'_1 + \lambda_3 E_3^c E_3^c H_d (L \otimes Y_3^{(2)})''_1 \\
&+ g_1 ((N^c H_u L)_{3S} Y_3^{(2)})_1 + g_2 ((N^c H_u L)_{3A} Y_3^{(2)})_1 + h (N^c \otimes S)_1 \chi \\
&+ \frac{f}{\Lambda} (S^c \otimes S)_1 \chi^8, \tag{18}
\end{aligned}
\end{equation}

where $\Lambda$ is the non-renormalizable scale and $g_1$ is the coupling constant of the term of the symmetric triplet arising from the product of the two triplets $L$ and $Y$, while $g_2$ is the coupling of the antisymmetric triplet term. Note that the chosen weights in Table (1) prevent the construction of the two terms $S^c H_u L$ and $(N^c N^c \chi)$. After spontaneous symmetry breaking, the scalar fields $H_u$, $H_d$ and $\chi$ acquire vevs namely $v_u$, $v_d$ and $v'$ respectively, where $v' \gg v_u, v_d$. We assume that $v'$ satisfies the relation $\frac{v'}{\Lambda} \sim \mathcal{O}(\lambda_c)$ where

| fields | $L$ | $E_1^c$ | $E_2^c$ | $E_3^c$ | $N^c$ | $S$ | $H_d$ | $H_u$ | $\chi$ |
|--------|-----|---------|---------|---------|-------|-----|-------|-------|-------|
| $A_4$  | 3   | 1       | 1"      | 1'      | 3     | 1   | 1     | 1     | 1     |
| $k_I$  | 1   | 1       | 1       | 1       | -3    | 4   | 0     | 4     | -1    |

Table 1: Assignment of flavors under $A_4$ and the modular weight $k_I$
\( \lambda_c = 0.225 \) is the Cabibbo angle. The mass matrices for charged leptons and neutrinos are

\[
M_e = v_d \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
y_1 & y_2 & y_3 \\
y_3 & y_1 & y_2 \\
y_2 & y_3 & y_1
\end{pmatrix},
\]

\[
\mu_s = f v' \lambda_c^7 \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

\[
M_R = h v' \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

\[
m_D = v_u \begin{pmatrix}
2g_1y_1 & (-g_1 + g_2)y_3 & (-g_1 - g_2)y_2 \\
(-g_1 - g_2)y_3 & 2g_1y_2 & (-g_1 + g_2)y_1 \\
(-g_1 + g_2)y_2 & (-g_1 - g_2)y_1 & 2g_1y_3
\end{pmatrix}.
\]

In general, the charged lepton mass matrix, \( M_e \), is not Hermitian so it can be diagonalized by two unitary matrices as,

\[
M_e^{\text{diag}} = U_e^L M_e U_e^R,
\]

where \( M_e^{\text{diag}} \) is the diagonal mass matrix of charged leptons which depends on the modulus \( \tau \) and the parameters \( \lambda_1^7 \) and \( \lambda_3^7 \) up to overall parameter. The neutrino mass matrix in the basis \( (\nu_L, N^c, S) \) is given by

\[
M_\nu = \begin{pmatrix}
0 & m_D & 0 \\
M_D^T & 0 & M_R \\
0 & M_R^T & \mu_s
\end{pmatrix}.
\]

By diagonalization of this matrix, one can get three eigenvalues, one for the light neutrino and the other two for the heavy neutrino states. The masses of the light neutrino state \( m_\nu \) can be obtained as

\[
m_\nu = m_D M_R^{-1} \mu_s M_R^{T^{-1}} m_D^T.
\]

The overall parameter \( \frac{f v'^2 g_1^2 \lambda_c^7}{h v' d} \) determines the scale of light neutrino masses and can be easily chosen to achieve the desired scale. For instance, we can set \( h \sim O(1 \text{ GeV}) \), \( f \sim O(0.01 \text{ GeV}) \), \( v' \sim O(10 \text{ TeV}) \), \( v_u \sim O(10^2 \text{ GeV}) \) and \( g_1 \sim O(0.01 \text{ GeV}) \) to get the neutrino masses of order \( O(10^{-2} \text{ eV}) \). The neutrino mass matrix \( m_\nu \) is complex and symmetric, so it is convenient to
\[
\Delta m^2_{12} \quad \Delta m^2_{23} \\
10^{-5} \text{eV}^2 \quad 10^{-4} \text{eV}^2 \\
r = \frac{\Delta m^2_{12}}{\Delta m^2_{23}} \\
\theta_{12}/^\circ \\
\theta_{23}/^\circ \\
\theta_{13}/^\circ \\
\delta_{CP}/\pi
\]

| best fit | 7.39 | 2.51 | 0.0294 | 33.82 | 49.8 | 8.6 | 1.57 |
| 3\sigma range | 6.79-8.01 | 2.41-2.61 | 0.026-0.033 | 31.61-36.27 | 40.6-52.5 | 8.27-9.03 | 1.088-2 |

Table 2: 3\sigma range for neutrino mixings and mass difference squares from \[25\] for inverted hierarchy.

diagonalize the Hermitian matrix \( M_{\nu} = m_1^\dagger m_\nu \),
\[
M_{\nu}^{\text{diag}} = U_\nu^\dagger M_{\nu} U_\nu.
\]  
(22)

The lepton mixing \( U_{PMNS} \) matrix is given by
\[
U_{PMNS} = U_R^\dagger U_\nu.
\]  
(23)

The mixing angles can be calculated from the relations
\[
\sin^2(\theta_{13}) = |(U_{PMNS})_{13}|^2, \quad \sin^2(\theta_{12}) = \frac{|(U_{PMNS})_{12}|^2}{1 - |(U_{PMNS})_{13}|^2}, \quad \sin^2(\theta_{23}) = \frac{|(U_{PMNS})_{23}|^2}{1 - |(U_{PMNS})_{13}|^2}.
\]  
(24)

The mixing angles and mass ratios are determined by the ratios \( g_2/g_1, \lambda_3^\lambda_1, \lambda_2^\lambda_1 \) and the modulus \( \tau \). The best fit values and 3\sigma ranges for the experimental results for inverted hierarchy are summarized in Table 2, in which the neutrino mass squared differences are defined as
\[
\Delta m^2_{12} = m^2_2 - m^2_1, \quad |\Delta m^2_{23}| = |m^2_3 - (m^2_2 + m^2_1)/2|.
\]

The parameters are scanned in the upper half of the complex plane by fixing \( r = \frac{\Delta m^2_{12}}{\Delta m^2_{23}} \) and the mixing angles with the 3\sigma ranges in table 2. The module \( \tau \) is scanned in the ranges \( \text{Re}[\tau] \in [-1.5, 1.5] \) and \( \text{Im}[\tau] \in [0.4, 3] \), the coupling ratio is scanned in the range \( g_2/g_1 \in [0, 2] \). We found a benchmark \( \tau = 0.507 + 0.871i, \quad g_2/g_1 = 0.1, \quad \frac{\lambda_3}{\lambda_1} = 0.0003, \quad \frac{\lambda_2}{\lambda_1} = 0.06 \), for inverted neutrino mass hierarchy, with
\[
r = 0.025, \quad \frac{m_e}{m_\tau} = 0.00029, \quad \frac{m_\mu}{m_\tau} = 0.061, \\
\theta_{12} = 33.04^\circ, \quad \theta_{23} = 44.26^\circ, \quad \theta_{13} = 19.45^\circ, \quad \delta_{CP} = 1.67\pi.
\]  
(25)
The value of $\tau = 0.507 + 0.871 i$ is close to $\tau = -\omega^2 = 0.5 + 0.866$ at which two neutrino eigenvalues are degenerate and two mixing angles are maximal. It is clear that the value of $\theta_{13}$ is so larger than its $3\sigma$ range.

We found better benchmark compatible with experimental results

$$\tau = -1.4961 + 0.525 i \quad g_2/g_1 = 0.36 \quad \frac{\lambda_1}{\lambda_3} = 41.76, \quad \frac{\lambda_2}{\lambda_3} = 588.$$  \hfill (26)

The neutrino mass spectrum is inverted hierarchical one. The mass ratio and mixing are

$$r = \frac{\Delta m^2_{12}}{|\Delta m^2_{23}|} = 0.0273, \quad \frac{m_e}{m_\tau} = 0.00029, \quad \frac{m_\mu}{m_\tau} = 0.061,$$

$$\theta_{12} = 35.6^\circ, \quad \theta_{23} = 41.6^\circ, \quad \theta_{13} = 8.45^0, \delta_{CP} = 1.9\pi.$$  \hfill (27)

which is compatible with the recent data in Table (2).

The pattern obtained at $\tau = -1.4961 + 0.525 i$ is a consequence of deviation from the value $\tau = -1.5 + .5i$ at which the neutrino mass matrix has one zero eigenvalue and two mixing angles are zero.

## 4 Quark masses

The embedding of the quark sector into a flavor model is a challenge due to the differences in the mass hierarchy and mixing for leptons and quarks. In this model, we extend the modular $A_4$ symmetry to the quark sector at the same value of the modulus $\tau = -1.4961 + 0.525 i$. All quarks transform as singlets under $A_4$. For this, we construct the modular forms of weight 4 by multiplication of two triplets of weight 2. Using $A_4$ multiplication rules of two triplets, one can get one triplet and three singlets all of weight 4 as

$$Y_3^{(4)} = \begin{pmatrix} y_1^2 - 2y_2 y_3 \\
 y_2^2 - 2y_2 y_1 \\
 y_2^2 - 2y_1 y_3 \end{pmatrix}, \quad Y_1^{(4)} = y_1^2 + 2y_2 y_3, \quad Y_2^{(4)} = y_3^2 + 2y_2 y_1, \quad Y_3^{(4)} = y_2^2 + 2y_1 y_3. \hfill (28)$$

The transformation of the above singlets are

$$Y_1^{(4)} \sim 1, \quad Y_2^{(4)} \sim 1', \quad Y_3^{(4)} \sim 1''.$$
At all values of $\tau$, the condition $Y_3^{(4)} = 0$ is satisfied. The assignments of the quark fields are shown in Table 3.

The $A_4$ invariant superpotential for down quarks can be written as

$$w_d = \frac{h_{d1}^d}{\lambda^3} d_1^c H_d Q_1 \chi^3 + \frac{h_{d2}^d}{\lambda} d_2^c H_d Q_2 \chi + \frac{h_{d3}^d}{\lambda^2} Y_2^{(4)} d_3^c H_d Q_2 \chi^2 + h_{d4}^d Y_1^{(4)} d_4^c H_d Q_3. \quad (29)$$

The chosen $A_4$ and $k$ assignments prevent the other mixing terms. Without loss of generality, we assume that $h_{d1}^d/h_{d3}^d \sim h_{d2}^d/h_{d3}^d \sim 1/2$, $h_{d3}^d/h_{d3}^d \sim 1$ and the down quark mass matrix takes the form

$$m_d = h_{d3}^d < H_d > \begin{pmatrix} \lambda^3/2 & 0 & 0 \\ 0 & \lambda/2 & 0 \\ 0 & Y_2^{(4)} \lambda^2 & Y_1^{(4)} \end{pmatrix}. \quad (30)$$

This mass matrix can be diagonalization by biunitary transformation, $V_d^L^\dagger m_d V_d^R = M_d$, where

$$V_d^L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.999 & 0.0022 \\ 0 & 0.0022 - 0.00003i & 0.999 - 0.014i \end{pmatrix},$$

$$V_d^R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.99 + 0.05i & 0.05 \\ 0 & -0.05 - 0.00017i & -0.99 - 0.0034i \end{pmatrix}, \quad (31)$$

with the corresponding eigenvalues

$$M_d = \text{diag}(\lambda^4/2, \lambda^2/2, 1) h_{d3}^d Y_1^{(4)} < H_d > . \quad (32)$$

The hierarchical spectrum of down quark masses is in good agreement with the recent data for quark masses: $^{20}$

$$m_d = 4.67^{+0.48}_{-0.17} \text{GeV}, \quad m_s = 93^{+11}_{-5} \text{GeV} \quad m_b = 4.18^{+0.03}_{-0.02} \text{GeV}.$$
For the up quarks, the invariant superpotential under modular $A_4$ can be written as

\[
W_u = h_{u1}^u Y_2(4) u_1^c H_u Q_1 \chi^4 + h_{u2}^u Y_1(4) u_1^c H_u Q_2 \chi^3 + h_{u3}^u Y_3(4) u_1^c H_u Q_3 \\
+ h_{u1}^u Y_3(4) u_2^c H_u Q_1 \chi^4 + h_{u2}^u Y_2(4) u_2^c H_u Q_2 \chi^2 + h_{u3}^u Y_3(4) u_2^c H_u Q_3 \\
+ h_{u1}^u Y_3(4) u_3^c H_u Q_1 \chi^4 + h_{u2}^u Y_2(4) u_3^c H_u Q_2 \chi + h_{u3}^u Y_3(4) u_3^c H_u Q_3 \chi^4
\]

Using the condition $Y_3(4) = 0$, and assume that the couplings $h_{ij}^u$ are of the same order, the up quark mass matrix takes the form

\[
M_u = h_{33}^u < H_u > \begin{pmatrix}
Y_2(4) \chi^4 & Y_1(4) \chi^3 & 0 \\
Y_1(4) \chi^3 & 0 & Y_2(4) \\
0 & Y_2(4) \chi^4 & Y_1(4) \chi^2
\end{pmatrix}. \quad (34)
\]

The right handed rotation of $M_u$ is found to be

\[
V_R^u = \begin{pmatrix}
-0.936 + 0.27i & -0.221 - 0.003i & 0.011 + 0.00015i \\
-0.21 + 0.06i & 0.975 + 0.017i & 0.00012 \\
-0.01 + 0.003i & -0.002 - 0.0002i & -0.9998 - 0.01i
\end{pmatrix} \quad (35)
\]

with the corresponding eigenvalues

\[
M_u^{diag} = h_{33}^u < H_u > Y_1(4) \begin{pmatrix} 0.000018 & 0.01 \end{pmatrix}, \quad (36)
\]

which are in good agreement with the up quark mass ratios $m_u/m_t = 0.000012, \ m_c/m_t = 0.008$.

The quark mixing matrix, $V_{CKM}$ takes the form

\[
|V_{CKM}| = |V_R^u V_d^R| = \begin{pmatrix}
0.975 & 0.221 & 0.0003 \\
0.22 & 0.973 & 0.051 \\
0.011 & 0.05 & 0.9986
\end{pmatrix} , \quad (37)
\]

which is close to the correct $V_{CKM}$ except the 1-3 element is one order of magnitude smaller than the correct value.
5 Conclusion

We built a model which is free from large number of flavons and or extra symmetries like $Z_N$ symmetries which were considered in many models based on the flavor symmetry. The experimental mass differences ratios and lepton mixing angles are determined in terms of the modulus $\tau$ and the coupling ratio $g_2/g_1$. The predicted parameters of mixings and mass ratios are compatible with the recent data. For the same value of $\tau = -1.4961 + 0.525 i$, we extend the modular $A_4$ symmetry to the quark sector. The calculated mass ratios are in good agreement with the experimental results. The $V_{CKM}$ matrix can be obtained up to small deviation in 1-3 element.

References

[1] P. Minkowski, “$\mu \rightarrow e\gamma$ at a Rate of One Out of $10^9$ Muon Decays?,” Phys. Lett. B 67, (1977) 421; M. Gell-Mann, P. Ramond and R. Slansky, “Complex Spinors and Unified Theories,” Conf. Proc. C 790927, (1979) 315 [arXiv:1306.4669 [hep-th]]; R. N. Mohapatra and G. Senjanovic, “Neutrino Mass and Spontaneous Parity Violation,” Phys. Rev. Lett. 44, (1980) 912; T. Yanagida, “Horizontal Symmetry and Masses of Neutrinos,” Prog. Theor. Phys. 64, (1980) 1103.

[2] D. Wyler and L. Wolfenstein, “Massless Neutrinos in Left-Right Symmetric Models,” Nucl. Phys. B 218, (1983) 205.

[3] R. N. Mohapatra and J. W. F. Valle, “Neutrino Mass and Baryon Number Nonconservation in Superstring Models,” Phys. Rev. D 34, (1986) 1642.

[4] E. Ma, “Lepton Number Nonconservation in $E(6)$ Superstring Models,” Phys. Lett. B 191, (1987) 287.

[5] G. t Hooft, Phys. Rev. Lett. 37, 8 (1976);

[6] A selective list includes: W. Grimus, A. S. Joshipura, S. Kaneko, L. Lavoura and M. Tanimoto, “Lepton mixing angle theta(13) = 0 with a horizontal symmetry D(4),” JHEP 0407,(2004) 078 [arXiv:hep-ph/0407112]; J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, “The flavor symmetry,” Prog. Theor.
Phys. 109, (2003) 795 [Erratum-ibid. 114, (2005) 287] arXiv:hep-ph/0302196; R. N. Mohapatra, M. K. Parida and G. Rajasekaran, “High scale mixing unification and large neutrino mixing angles,” Phys. Rev. D 69, (2004) 053007 arXiv:hep-ph/0301234; C. Hagedorn, M. Lindner and R. N. Mohapatra, “$S(4)$ flavor symmetry and fermion masses: Towards a grand unified theory of flavor,” JHEP 0606, (2006) 042 arXiv:hep-ph/0602244; I. de Medeiros Varzielas, S. F. King and G. G. Ross, “Neutrino tri-bi-maximal mixing from a non-Abelian discrete family symmetry,” Phys. Lett. B 648, (2007) 201 arXiv:hep-ph/0607045; E. Ma, “Near tri-bimaximal Neutrino Mixing with Delta(27) Symmetry,” Phys. Lett. B 660, (2008) 505 arXiv:0709.0507 [hep-ph]; C. Luhn, S. Nasri and P. Ramond, “Tri-Bimaximal Neutrino Mixing and the Family Symmetry $Z_7 \times Z_3$,” Phys. Lett. B 652, (2007) 27 arXiv:0706.2341 [hep-ph]; E. Ma and G. Rajasekaran, “Softly broken $A(4)$ symmetry for nearly degenerate neutrino masses,” Phys. Rev. D 64, (2001) 113012 arXiv:hep-ph/0106291; arXiv:hep-ph/0001113.

[7] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, “Finite Modular Groups and Lepton Mixing,” Nucl. Phys. B 858, (2012) 437 doi:10.1016/j.nuclphysb.2012.01.017 arXiv:1112.1340 [hep-ph].

[8] F. Feruglio, “Are neutrino masses modular forms?,” doi:10.1142/9789813238053_0012 arXiv:1706.08749 [hep-ph].

[9] T. Kobayashi, K. Tanaka and T. H. Tatsuishi, “Neutrino mixing from finite modular groups,” Phys. Rev. D 98 (2018) no.1, 016004 doi:10.1103/PhysRevD.98.016004 arXiv:1803.10391 [hep-ph].

[10] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi and H. Uchida, “Finite modular subgroups for fermion mass matrices and baryon/lepton number violation,” Phys. Lett. B 794 (2019) 114 doi:10.1016/j.physletb.2019.05.034 arXiv:1812.11072 [hep-ph].

[11] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, “Modular $S_3$ invariant flavor model in SU(5) GUT,” arXiv:1906.10341 [hep-ph].

[12] H. Okada and Y. Orikasa, “A modular $S_3$ symmetric radiative seesaw model,” arXiv:1907.04716 [hep-ph].
[13] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, “Modular $A_4$ invariance and neutrino mixing,” JHEP 1811 (2018) 196 doi:10.1007/JHEP11(2018)196 arXiv:1808.03012 [hep-ph].

[14] H. Okada and M. Tanimoto, “CP violation of quarks in $A_4$ modular invariance,” Phys. Lett. B 791 (2019) 54 doi:10.1016/j.physletb.2019.02.028 arXiv:1812.09677 [hep-ph].

[15] P. P. Novichkov, S. T. Petcov and M. Tanimoto, “Trimaximal Neutrino Mixing from Modular $A_4$ Invariance with Residual Symmetries,” Phys. Lett. B 793 (2019) 247 doi:10.1016/j.physletb.2019.04.043 arXiv:1812.11289 [hep-ph].

[16] T. Nomura and H. Okada, “A two loop induced neutrino mass model with modular $A_4$ symmetry,” arXiv:1906.03927 [hep-ph].

[17] J. T. Penedo and S. T. Petcov, “Lepton Masses and Mixing from Modular $S_4$ Symmetry,” Nucl. Phys. B 939 (2019) 292 doi:10.1016/j.nuclphysb.2018.12.016 arXiv:1806.11040 [hep-ph].

[18] P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, “Modular $S_4$ models of lepton masses and mixing,” JHEP 1904 (2019) 005 doi:10.1007/JHEP04(2019)005 arXiv:1811.04933 [hep-ph].

[19] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, “New $A_4$ lepton flavor model from $S_4$ modular symmetry,” arXiv:1907.09141 [hep-ph].

[20] P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, “Modular $A_5$ symmetry for flavour model building,” JHEP 1904 (2019) 174 doi:10.1007/JHEP04(2019)174 arXiv:1812.02158 [hep-ph].

[21] G. J. Ding, S. F. King and X. G. Liu, “Neutrino Mass and Mixing with $A_5$ Modular Symmetry,” arXiv:1903.12588 [hep-ph].

[22] J. H. Bruinier, G. V. D. Geer, G. Harder, and D. Zagier, The 1-2-3 of Modular Forms. Universitext. Springer Berlin Heidelberg, 2008.

[23] F. Diamond and J. M. Shurman, A first course in modular forms, vol. 228 of Graduate Texts in Mathematics. Springer, 2005.
[24] R. C. Gunning, *Lectures on Modular Forms*. Princeton, New Jersey USA, Princeton University Press, 1962.

[25] I. Esteban, M. C. Gonzalez-Garcia, A. Hernandez-Cabezudo, M. Maltoni and T. Schwetz, “Global analysis of three-flavour neutrino oscillations: synergies and tensions in the determination of $\theta_{23}$, $\delta_{CP}$, and the mass ordering,” JHEP 1901, (2019) 106 doi:10.1007/JHEP01(2019)106 [arXiv:1811.05487 [hep-ph]].

[26] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018) and 2019 update.