Rank three bipartite entangled states are distillable

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We prove that the bipartite entangled state of rank three is distillable. So there is no rank three bipartite bound entangled state. By using this fact, we present some families of rank four states that are distillable. We also analyze the relation between the low rank state and the Werner state.

I. INTRODUCTION

In recent years, entanglement has been regarded as a quantum resource for many novel tasks such as quantum computation, quantum cryptography, quantum teleportation and so on [1]. These quantum-information tasks cannot be carried out by classical resources and they rely on the entangled states. Although the mixed entangled states are directly used in some quantum-information tasks [2], most of them require the pure entangled states of bipartite or multipartite system to be the crucial elements. However in a lab, it turned out that the pure entangled states always become mixed by the decoherence due to the coupling with the environment. A central topic in quantum information theory is thus how to extract pure entangled states from mixed states [3].

An entangled state \( \rho \) is distillable if one can asymptotically or explicitly extract some pure entangled state from infinitely many copies of \( \rho \) by using only local operations and classical communication (LOCC). It has been proved that the entangled 2-qubit states are always distillable [4, 5, 6]. Nevertheless there exist bound entangled (BE) states which are not distillable under LOCC [7]. Concretely, a bipartite entangled state \( \rho \) in the Hilbert space \( H_A \otimes H_B \) is BE if it has positive partial transpose (PPT) with respect to system \( A \) (or \( B \)), namely \( \rho^{T_A} \) (or \( \rho^{T_B} \)) \( \geq 0 \). Such states are called PPT BE states and usually it cannot be used for quantum-information tasks under LOCC [2, 3].

A more formidable challenge is that whether a bipartite state \( \rho_{AB} \) having non-positive partial transpose (NPT) with respect to system \( A \) (or \( B \)) is always distillable. This class of states are always entangled due to the celebrated Peres-Horodecki criterion [8]. It was pointed out by [10] that any NPT state can be converted into some NPT Werner states under LOCC. Much efforts have been devoted to distilling this kind of states and there has been a common belief that NPT BE Werner states indeed exists [11, 12, 13, 14, 15, 16, 17]. In addition, it has been proved that the NPT states in \( 2 \times N \) space are distillable [18] and the rank two NPT states of bipartite systems are also distillable [19]. However, the situation becomes more complex when we distill the entangled state whose subsystems have higher dimensions or that has a higher rank.

In this paper we show that the rank three bipartite entangled states are distillable under LOCC. We give the concrete method of distilling this class of states. It helps infer the analytical calculation of distillable entanglement [20, 21]. A rank three state is entangled if and only if (iff) it is NPT, namely there is no PPT BE state of rank three [22]. So we also obtain that there are no rank-three NPT BE states and all of them can be used for quantum-information tasks. It is similar to the case of rank two states and we conclude: a rank two or three state is distillable iff it is entangled. This conclusion does not hold for the bipartite entangled states with higher ranks, e.g., there have been the rank four PPT BE states constructed by the unextendible product bases (UPB) [23].

Moreover, we will investigate the NPT states of rank four and find out some families of states that are distillable. This helps distill the NPT states which have more complex structure. In addition, we will show that locally converting the Werner state into the rank three entangled state is difficult, so our result is independent of the expectant fact that there exists NPT BE Werner state.

The rest of this paper is organized as follows. In Sec. II we prove our main result on rank three states and then we use it to distill the rank four NPT states. We also discuss the relationship between the result in this paper and the Werner state. We conclude in Sec. III.

II. DISTILLATION OF RANK THREE AND FOUR BIPARTITE STATES

Throughout this paper we will use the following notations. The rank of a bipartite state \( \rho_{AB} \) is referred to as \( r(\rho_{AB}) \), and the reduced density operator of it as \( \rho_A = Tr_B \rho_{AB}, \rho_B = Tr_A \rho_{AB} \). The range of the density operator \( \rho_{AB} \) is referred to as \( R(\rho_{AB}) \). Another useful tool is the so-called invertible local operator (ILO) (or the local filter) [24], namely the nonsingular matrix. Physically, it can be probabilistically realized through the positive operator valued measure (POVM) [1], so we can use it when distilling the NPT states.

We first consider the NPT states of rank three. Before proving our main theorem, we recall a useful lemma that was proved in [10].

Lemma 1. If \( r(\rho_{AB}) < \max[r(\rho_A), r(\rho_B)] \), then the bipartite state \( \rho_{AB} \) is distillable.

The lemma has been used to show that there is no rank two BE state [19]. It was proven by using the reduction criterion [10], i.e., a state is distillable when the reduction criterion is violated (See Eq. (6) in [13]). It follows from lemma 1 that any rank three state in \( M \times N \) space with \( \max[M, N] > 3 \) is distillable. Since an NPT state in
2 × 2 or 2 × 3 space is also distillable \[4, 5, \Box\], it suffices to consider the rank three NPT states \(\rho_{AB}\) in 3 × 3 space. Moreover, we can perform some ILO on the subsystem \(B\) such that \(\rho_B = \frac{1}{3}I\). Then only the state having the following form does not violate the reduction criterion (up to local unitary transformations)

\[
\sigma_{AB} = \frac{1}{3}|\psi_0\rangle\langle\psi_0| + \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2|, \quad \sigma_B = \frac{1}{3}I \quad (1)
\]

where the three eigenvectors satisfy \(\langle\psi_i|\psi_j\rangle = \delta_{ij}\) and

\[
|\psi_0\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle, \quad (2)
\]

\[
|\psi_1\rangle = \sum_{i,j=0}^{2} b_{ij} |ij\rangle, \quad (3)
\]

\[
|\psi_2\rangle = \sum_{i,j=0}^{2} c_{ij} |ij\rangle. \quad (4)
\]

Notice that there is always at least a Schmidt rank two state by linear combination of the eigenvectors. In addition, any spectral decomposition of the state \(\sigma_{AB}\) have the form in Eq. (1) (in which the state \(|\psi_0\rangle\) has a more general form, e.g., \(|\psi_0\rangle = \sum_{ij=0}^{2} a_{ij} |ij\rangle\)).

In what follows we will concentrate on the NPT state \(\sigma_{AB}\) in Eq. (1) because any rank three NPT state in 3 × 3 space can be locally converted into \(\sigma_{AB}\), otherwise it is distillable in terms of the reduction criterion. There is a simple situation we can treat easily as follows.

**Lemma 2.** The state \(\sigma_{AB}\) is distillable when there is a product state in its range.

**Proof.** Without loss of generality, we consider the state \(\sigma_{AB}\) with \(\theta = 0\). Then its coefficients \(b_{00}, c_{00}, i = 0, 1, 2\) equal zero because of the condition \(\sigma_B = \frac{1}{3}I\).

We project the state \(\sigma_{AB}\) by using the local projector \(I_A \otimes (|1\rangle\langle 1| + |2\rangle\langle 2|)_B\) and obtain the resulting state \(\frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2|\). It’s a rank two NPT state and hence distillable. It implies the state \(\sigma_{AB}\) is also distillable.

Lemma 2 has given a criterion that tells whether a rank three NPT state is distillable. We will generalize it to the case of rank four states later. It is also useful for the distillation of general rank three NPT state as shown below. Let us consider the state \(\sigma_{AB}\) whose range has no product state. We take the projector \(P_{AB}\) onto the 2 × 3 subspace spanned by \{|00\}, \{01\}, \{02\}, \{10\}, \{11\}, \{12\}\} and obtain the state

\[
\sigma_{AB}^1 = |\psi_0^1\rangle\langle\psi_0^1| + |\psi_1^1\rangle\langle\psi_1^1| + |\psi_2^1\rangle\langle\psi_2^1|, \quad (5)
\]

which is not normalized for convenience. The resulting states \(|\psi_1^1\rangle\) equal \(P_{AB}|\psi_1\rangle\), respectively. We will follow this notation below, e.g., \(|\psi_0^2\rangle = V_A \otimes V_B |\psi_1^2\rangle\), etc.

The state \(\sigma_{AB}^1\) is distillable if it is entangled since it is in 2 × 2 or 2 × 3 space. Let us consider the case in which \(\sigma_{AB}^1\) is separable. First, the state \(\sigma_{AB}^1\) is in 2 × 2 space iff \(b_{22} = c_{22} = 0, i = 0, 1\). In this case, the condition \(\sigma_B = \frac{1}{3}I\) leads to \(b_{22} = 0, i = 0, 1\) and \(|b_{22}|^2 + |c_{22}|^2 = 1\). When \(b_{22} = c_{22} = 0\), either the state \(|\psi_1\rangle\) or \(|\psi_2\rangle\) becomes a product state and hence \(\sigma_{AB}\) is distillable in terms of lemma 2; When \(b_{22}c_{22} \neq 0\), we can remove the coefficients \(b_{2i}, c_{2i}, i = 0, 1\) by using linear combination of the eigenvectors \(|\psi_i\rangle\), \(i = 0, 1, 2\). It is then easy to see that \(R(\sigma_{AB})\) contains a product state and thus \(\sigma_{AB}\) is distillable.

Second, we investigate the state \(\sigma_{AB}^1\) in 2 × 3 space. Notice the rank of \(\sigma_{AB}^1\) remains three, otherwise there will be a product state in \(R(\sigma_{AB})\) and it is distillable. We can always write a rank three separable state \(\rho\) in 2 × 3 space as the sum of three product states \([25, 26]\). To prove it, suppose the state has the form

\[
\rho = \sum_{i=0}^{d-1} |\phi_i\rangle\langle\phi_i|, \quad d > 3. \quad (6)
\]

Without loss of generality we choose the first three product states as a set of linearly independent vectors, so any other product state can be written as \(|\phi_j\rangle\langle\phi_j| = \sum_{i=0}^{2} k_{ij} |\phi_i\rangle\langle\phi_i|, j = 3, \ldots\). Notice the vectors \(|\phi_i\rangle, i = 0, 1, 2\), and two vectors in \(|\phi_i\rangle, i = 0, 1, 2\) are linearly independent, respectively. So the product state \(|\phi_j\rangle\langle\phi_j|, j \geq 3\) equals either one of the first three product states, or \(|\phi_j\rangle\langle\phi_j| = \sum_{i=0}^{2} k_{ij} |\phi_i\rangle\langle\phi_i|\) in which \(|\phi_0\rangle\) is proportional to \(|\phi_1\rangle\). In this case it is easy to write the state \(\rho\) as the sum of three product states.

Using the above conclusion, we can express the state \(\sigma_{AB}^1\) by means of eigenvectors \(|\psi_1\rangle = (a_{00}|0\rangle + a_{11}|1\rangle)|\phi_1\rangle + |2\rangle|\phi_2\rangle, i = 0, 1, 2\). Moreover, the vectors \(|\phi_i\rangle\)'s are linearly independent, while \(|\phi_2\rangle\)'s linearly dependent. We perform some ILO's on the state \(\sigma_{AB}\) and remove two coefficients \(a_{00}\) and \(a_{11}\). The resulting state \(\sigma_{AB}^2\) still has the form in Eq. (1), otherwise it is distillable.

For the state \(\sigma_{AB}^2\) when the condition \(a_{00}a_{11} = 0\) is satisfied, we find that \(R(\sigma_{AB}^1)\) contains a product state because of the orthogonal conditions \(|\psi_1^2\rangle\langle\psi_1^2| = \delta_{ij}\). So the state \(\sigma_{AB}\) is distillable. Let us move to investigate the state \(\sigma_{AB}^3\) satisfying the condition \(a_{00}a_{11} = 0\). By performing ILO's on \(\sigma_{AB}^2\) we greatly simplify its form such that

\[
\sigma_{AB}^3 = |\psi_0^3\rangle\langle\psi_0^3| + |\psi_1^3\rangle\langle\psi_1^3| + |\psi_2^3\rangle\langle\psi_2^3|, \quad (7)
\]

where

\[
|\psi_0^3\rangle = |00\rangle + |2\rangle|\psi\rangle, \quad (8)
\]

\[
|\psi_1^3\rangle = |11\rangle + |2\rangle|\phi\rangle, \quad (9)
\]

\[
|\psi_2^3\rangle = |10\rangle + |1\rangle|\phi\rangle + |2\rangle|\psi\rangle + |3\rangle|\phi\rangle, \quad (10)
\]

\[
|\psi\rangle = x_0|0\rangle + x_1|1\rangle + x_2|2\rangle, \quad (11)
\]

\[
|\phi\rangle = y_0|0\rangle + y_1|1\rangle + y_2|2\rangle. \quad (12)
\]

Notice the state is not normalized and the condition \(\sigma_{AB}^3 = \frac{1}{3}I\) is also not required. We project \(\sigma_{AB}^3\) by the projector \((|0\rangle + a|1\rangle + |2\rangle)\Lambda_A \otimes I_B, a \in R\) and obtain the state \(\sigma_{AB}^3\) in 2 × 3 space. It is entangled and thus distillable when its partial transpose is not positive \(\Box\). Nevertheless, there may be some cases in which the coefficients \(x_i, y_i, i = 0, 1, 2\) make that \(\sigma_{AB}^3\) is not distillable. We are going to find out such coefficients by calculating
several average values $\text{Tr}(\rho A B (T_\text{A} \rho A B^\dagger T_\text{B}^\dagger))$, where $|\omega_i\rangle = |00\rangle + b|22\rangle, |\psi_i\rangle = (1 + b)|01\rangle + b|20\rangle, |\sigma_i\rangle = |02\rangle + b|20\rangle, b \in C$. To keep the average value always positive, we find that it is necessary that $x_1 = x_2 = 0$. However, this means the state $|\psi_0\rangle$ is of product form and hence the state $\sigma_{AB}^0$ is distillable. As it can be converted into the state $\sigma_{AB}^0$ by ILOs, the latter is also distillable. Now we reach our main theorem in this paper.

**Theorem.** The rank three NPT states are distillable under LOCC. ■

So the rank three entangled states can be used for quantum-information tasks. In fact, we have proposed the method of distilling $\sigma_{AB}$ in the proof of the theorem. First, when the given state contains a product state in its range, it can be projected onto a rank two entangled state. According to the reduction criterion, we can distill it by the procedure similar to the famous BBPSSW protocol [5, 10]. It is also the method of distilling the rank three entangled states that cannot be converted into $\sigma_{AB}$. Second, when the given state $\rho$ contains no product state in $R(\rho)$, we project it by the projector $|0\rangle\langle 0| + |1\rangle\langle 1|\otimes I_B$. The resulting state is entangled and thus distillable; otherwise, we should project the initial state $\rho$ by the projector $|0\rangle\langle 0| + a|1\rangle\langle 1| + 2|2\rangle\langle 2|\otimes I_B$ after performing some ILOs on $\rho$. There will be a suitable parameter $a$ making the resulting state entangled and thus distillable.

The rank three entangled states are a quite special class of states. As there have been PPT BE states of any higher rank (e.g., rank four PPT BE states constructed by UPB [23]), we indeed have found out the lowest rank space in which there is no BE state. It also implies that when a state can be locally projected into some rank three NPT state, then it is distillable. This causes new methods of distilling quantum states having more complex structure. We will show it in terms of distilling the rank four NPT states below. One may also find other way to distill the entangled states based on the theorem. For example, the tensor product of the rank three entangled states are also entangled for certain.

On the other hand, the analytical calculation of distillable entanglement is also an important issue in quantum information theory. The problem is very difficult and there have been some optimal bounds on distillable entanglement [20, 21]. Specially, the bound is saturated if we can find a way to distill the state and get the same amount of pure entanglement as the bound. In this case we get the analytical result of distillable entanglement. As it is possible to find out whether the bound on rank three NPT state is saturated by using our method of distilling it, we indeed provide new ways to calculate the distillable entanglement.

Third, our result is also independent of the expectant fact that there exist NPT Werner states $\rho_w$. We do not know whether an NPT state $\rho$ is distillable, even it can be converted into some Werner state which is proved to be not distillable under LOCC. One can easily exemplify it by locally taking some rank three NPT state into $\rho_w$, while the latter is expected to be not distillable. Conversely, it is difficult to convert the Werner state into the state $\sigma_{AB}$, so we still do not know whether the latter is distillable. To see it, we have the Werner state in a $N \times N$ space as follows [26]

$$
\rho_w = (a + b) \sum_{i,j=0}^{N-1} |ij\rangle\langle ij| - 2b \sum_{i<j=0}^{N-1} \frac{|ij\rangle - |ji\rangle}{\sqrt{2}} \frac{|ij\rangle - |ji\rangle}{\sqrt{2}},
$$

where $a > 0, b < 0$ are two parameters satisfying $a + b \geq 0$. The most general local transformation on a quantum state $\rho$ has the form $\Lambda(\rho) = \sum_i A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger$. Because the resulting state is entangled, there must be at least one pair of Kraus operators $A_i, B_i$ that have at least rank two, respectively. In this case, the state $\Lambda(\rho)$ will have the rank not less than four when $a + b > 0$, which means a rank three NPT state cannot be output by this local channel. The only exception happens when $a + b = 0$, but it is difficult to judge whether the state $\Lambda(\rho)$ is of rank three and entangled.

Let us investigate further the problem of distilling rank four states by using the theorem in this paper. Different from the case of rank three state, it is well-known that there indeed exist PPT BE states of rank four even in the $3 \times 3$ space. It is easy to show that the NPT BE states of rank four possibly exist only in three kinds of spaces, $4 \times 4, 3 \times 4, 3 \times 3$ in terms of lemma 1. One will meet lots of difficulties when applying the technique in this paper to distill the rank four NPT state $\rho$, e.g., the resulting state from $\rho$ by projection can be $3 \times 3$ and it may be PPT BE. Besides, the Peres-Horodecki criterion is no more a sufficient condition for the separability of state in $2 \times 4$ space, etc. Nevertheless, we still can obtain some useful results on this problem when the target state has a special form.

**Lemma 3.** For a rank four NPT state in $4 \times 4$ or $3 \times 4$ space, it is distillable when there is a product state in its range.

**Proof.** By employing similar deduction for the state $\sigma_{AB}$, only the state having the following form does not violate the reduction criterion

$$
\rho_{AB} = \frac{1}{4} \sum_{i=0}^{3} |\psi_i\rangle\langle \psi_i|, \rho_B = \frac{1}{4} I, \quad \rho_A = \frac{1}{4} I,
$$

where the four eigenvectors satisfy $\langle \psi_i|\psi_j\rangle = \delta_{ij}$. Up to the local unitary transformations we have $|\psi_0\rangle = |00\rangle$. Next, we project the state $\rho_{AB}$ by the projector $I_A \otimes (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|)A$ and obtain the NPT state $\rho'_{AB} = \frac{1}{4} \sum_{i=1}^{3} |\psi_i\rangle\langle \psi_i| \otimes |2\times 3, 3\times 3, 4\times 4\rangle$. By means of the BBPSSW and Horodeckis’ protocol, our theorem and the reduction criterion, respectively, the state $\rho'_{AB}$ and hence $\rho_{AB}$ is always distillable.

So we have generalized lemma 1 to the case of rank four NPT states. Moreover, we hope that it always holds for the NPT states whose rank equal to its maximal dimension of subsystems. However, it does not hold when...
the rank of a state is larger, e.g., the PPT BE state in $3 \times 3$ space constructed in [7] contains infinitely many product states in its range, but its rank equals eight. It is also unclear that whether the rank four NPT states $\rho$ in this space are distillable. Solving this problem is more difficult since we cannot rely on the reduction criterion. However, $\rho$ is distillable when we can project it onto a rank three NPT state in terms of our theorem.

For example, the following $3 \times 3$ rank four NPT state is distillable

$$
\rho_{AB} = \lambda_0 |00\rangle\langle 00| + \lambda_1 |01\rangle\langle 01| + \lambda_2 |\psi_2\rangle\langle \psi_2| + \lambda_3 |\psi_3\rangle\langle \psi_3|,
$$

$$
|\psi_2\rangle = \sum_{i,j=0}^2 c_{ij}|ij\rangle,
$$

$$
|\psi_3\rangle = \sum_{i,j=0}^2 d_{ij}|ij\rangle, \lambda_0, \lambda_1, \lambda_2, \lambda_3 > 0.
$$

(15)

To prove it, we project the state $\rho_{AB}$ by the projector $(|1\rangle\langle 1| + |2\rangle\langle 2|)_A \otimes I_B$. When the resulting state $\rho_{AB} = \lambda_2 |\psi_2\rangle\langle \psi_2| + \lambda_3 |\psi_3\rangle\langle \psi_3|$ is entangled, it is also distillable. On the other hand when $\rho_{AB}$ is separable, we can write it as the sum of two product states since it is in a space not larger than $2 \times 3$. Besides, the rank of $\rho_{AB}$ must be two because of $r(\rho_A) = 3$. By performing some ILOs on the state $\rho_{AB}$ and linear combination of $|\psi_2\rangle$ and $|\psi_3\rangle$, we can convert them into $|\psi_2\rangle = |\phi_0\rangle + |\phi_1\rangle$ and $|\psi_3\rangle = |0\rangle\otimes |0\rangle + |2\rangle\otimes |\omega_2\rangle$, and keep the other two terms $|00\rangle$ and $|01\rangle$ unchanged.

When either of the states $|\psi_2\rangle$ and $|\psi_3\rangle$ is of product form, we easily project the state $\rho_{AB}$ onto a $2 \times 3$ subspace, the resulting state is still entangled and distillable. On the other hand when both the states $|\psi_2\rangle$ and $|\psi_3\rangle$ are entangled, we project the state $\rho_{AB}$ by the projector $[0]|a(0) + a(2)|A \otimes I_B$. The obtained state $\rho_{AB}^3$ is $2 \times 3$ and its rank is four by choosing suitable parameter $a$. This state is separable if $\hbar$ has the decomposition $\rho_{AB}^3 = |\psi\rangle_A\langle \psi| \otimes |\omega_2\rangle_B + |0\rangle_A \langle 0| \otimes \rho_B^3$ with $r(\rho_B^3) = 3$. However it is impossible, since it requires a $4 \times 4$ coefficient unitary matrix $[a_{ij}]$ in which $a_{i3} = 0, i = 1, 2, 3$, and $a_{0i}, i = 0, 1, 2$ cannot be zero simultaneously. Hence the state $\rho_{AB}^3$ is entangled and thus distillable. This also completes the proof showing that the state $\rho_{AB}$ in Eq. (15) is distillable.

As above we have given several families of states that can be distilled by means of the fact that the rank three NPT states are distillable. The main difficulty in entanglement distillation is the great amount of parameters that cannot be removed during the filtering process. For example, it is unknown that whether the rank four NPT states are distillable. All in all, more efforts are required to distill other classes of rank four NPT states.

### III. CONCLUSIONS

We have proved that the bipartite rank three NPT states and some families of rank four NPT states are distillable. So they are indeed available resource for quantum-information tasks. An open problem is that whether all rank four NPT states are distillable. Our result also gives an insight into the relationship between the low rank states and the Werner states.

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