1. Introduction

This paper recasts some of the literature which has grown out of Kim’s extension of Chabauty’s method for bounding points on curves in the language of periods. We retrieve many of the main results of the papers [24], [25], [22], [11] as a simple consequence of two constructions: the universal comparison isomorphism applied to the unipotent fundamental group, and an elementary lemma in linear algebra. We also show how the method can be made motivic, explicit, and extended to divisors.

The main idea can be summarised as follows. Suppose, for simplicity, that $X$ is a smooth affine scheme over $\mathbb{Q}$. Integration provides a natural comparison isomorphism between its algebraic de Rham and Betti cohomology

$$H^n_{dR}(X; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \sim \rightarrow H^n(X(\mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C})$$

where $\omega$ is a regular differential $n$-form on $X$ over $\mathbb{Q}$, and $\gamma$ a smooth topological chain on $X(\mathbb{C})$ of real dimension $n$. This isomorphism is transcendental, but can be made algebraic by replacing the ring $\mathbb{C}$ with a suitable ring of ‘motivic’ periods $P_m^\mathbb{C}$, where $\mathbb{C}$ is some version of a Tannakian category of motives. For its definition, and some candidates for $\mathbb{C}$, see §3.1. One deduces a universal comparison isomorphism

$$H^n_{dR}(X; \mathbb{Q}) \otimes_{\mathbb{Q}} P_m^\mathbb{C} \sim \rightarrow H^n(X(\mathbb{C}) \otimes_{\mathbb{Q}} P_m^\mathbb{C})$$

where $I_m^\gamma(\omega)$ is a ‘motivic’ version of the integral of $\omega$ along $\gamma$. The ring $P_m^\mathbb{C}$ is a $\mathbb{Q}$-algebra equipped with a period homomorphism

$$\text{per} : P_m^\mathbb{C} \rightarrow \mathbb{C}$$

satisfying $\text{per}(I_m^\gamma(\omega)) = \int_\gamma \omega$, so (1.1) can be retrieved from (1.2) by applying the map per. The point is that the isomorphism (1.2) is defined over $\mathbb{Q}$. Furthermore, standard conjectures about mixed motives suggest that the subspace of $P_m^\mathbb{C}$ generated by the motivic periods of $X$ should be tightly constrained. This implies relations between the $I_m^\gamma(\omega)$ for varying $\gamma$ and $\omega$, and can be used to infer arithmetic information about integral points on $X$ or varieties related to $X$.

Since the general theory of motives is largely conjectural, we have to make do with mixed Tate motives over number fields, for which the required bounds on motivic periods are known. This is a consequence of Borel’s deep theorems on the algebraic $K$-theory of number fields. As a consequence, we mainly consider the case when $X$ is the punctured projective line. We now state some results that can be obtained in this setting before returning to a more general situation in §3.
1. Main example: the unit equation. Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let $S$ be a finite set of rational primes. Let $X_S := X(\mathbb{Z}[S^{-1}])$ denote the set of $S$-integral points on $X$. It is the set of solutions to the unit equation

$$u + v = 1,$$

where $u, v$ are rational numbers whose numerator and denominator have prime factors contained in $S$. This set is finite, as shown by Siegel [27] and reproved by Kim [24] using the unipotent fundamental group. The problem of understanding $S$-units is related to a number of questions in diophantine geometry.

Let $\mathcal{M}_S$ denote the category of mixed Tate motives ramified only at $S$. It is a full subcategory of the Tannakian category of mixed Tate motives over $\mathbb{Q}$ [12].

The classical polylogarithms can be expressed iterated integrals on $X(\mathbb{C})$:

$$\text{Li}_n(x) = \sum_{k \geq 1} \frac{x^k}{k^n} = \int_0^x \frac{dz}{1 - z} \frac{dz}{z} \cdots \frac{dz}{z}.$$ 

For $x \in X_S$, there exists a motivic version $\text{Li}_{i_1}^{m_1}(x)$, which is an element of the ring of effective motivic periods $\mathcal{P}^m_{\mathcal{M}_S}$ of the category $\mathcal{M}_S$, satisfying $\text{per}(\text{Li}_{i_1}^{m_1}(x)) = \text{Li}_{i_1}(x)$. The ring $\mathcal{P}^m_{\mathcal{M}_S}$ is a graded $\mathbb{Q}$-vector space, finite dimensional in every weight. It follows that if there are many integral points $x_1, \ldots, x_N \in X_S$, there must exist $\mathbb{Q}$-linear relations between the $\text{Li}_{i_1}^{m_1}(x_i)$, and hence their periods $\text{Li}_{i_1}(x_i)$. This imposes a constraint on the $x_i$. One can replace the classical polylogarithms with multiple polylogarithms or indeed any iterated integrals on $X$. The entirety of this paper reduces to this simple idea. Technicalities arise only if one wants to make this constraint explicit, or if one wishes to generalise to the non mixed-Tate case [22].

1.1.1. Effectivity. The problem is that the coefficients in such a relation will depend on the points $x_i$, so this method is not effective. In order to obtain a relation which is universal for all points, one must replace $\mathcal{P}^m_{\mathcal{M}_S}$ with a graded ring $\mathcal{P}^m_{\mathcal{MT}_S}$ of unipotent de Rham periods — a key technical point being that the unipotent version of $2\pi i$ in the latter ring is trivial. This is equivalent to reducing ‘modulo $\pi$’ and increases the number of available relations. The ring $\mathcal{P}^m_{\mathcal{MT}_S}$ is simply the affine ring $\mathcal{O}(U_{\mathcal{MT}_S})$ of the unipotent radical of the Tannaka group of $\mathcal{M}_S$ with respect to the de Rham fiber functor. The Frobenius at $p$ endows it with a $p$-adic period map

$$\text{per}_p : \mathcal{P}^m_{\mathcal{MT}_S} \rightarrow \mathbb{Q}_p$$

for all $p \notin S$, and contains objects $\text{Li}_{i_1}^{m_1}(x_i)$ of degree $n$, whose periods are ‘single-valued’ $p$-adic versions of the classical polylogarithms. To gain some intuition for these objects, observe that $\text{Li}_{i_1}^1(x) = -\log^u(1-x)$, and that the unipotent logarithm satisfies the usual functional equation: for any $x \in X(\mathbb{Q})$,

$$\log^u(x) = \sum_p v_p(x) \log^u(p),$$

where the sum is over all primes $p > 0$, and $v_p$ denotes the $p$-adic valuation. The elements $\log^u(p)$ are algebraically independent over $\mathbb{Q}$ [6]. The higher unipotent polylogarithms generalize $p$-adic valuations of points $x \in X_S$.

By applying an elementary lemma in linear algebra [5,1] to the ring $\mathcal{P}^m_{\mathcal{MT}_S}$ and computing its dimensions in each degree, we deduce the following theorem.

**Theorem 1.1.** Let $s = |S|$ and let $k > 2s - 1$. There are integers $w = w(s, k)$, $N = N(s, k)$ such that, for any choice of $N$ linearly independent polynomials

$$P_1, \ldots, P_N \in \mathbb{Q}[\log^u(x), \text{Li}_{i_1}^1(x), \ldots, \text{Li}_{i_N}^1(x)]$$

we have
which are homogeneous of degree $w$, the equation

$$\det(P_i(x_j)_{1 \leq i,j \leq N}) = 0$$

is satisfied for any $N$ integral points $x_1, \ldots, x_N \in X_S$.

The integers $w, N$ are easily computable. One can also assume (§9.4) that the $P_i$ do not involve any logarithms $\log^r(x)$. The results of [24], [11] are a consequence of this theorem. For $p$ a prime not in $S$, we can take the $p$-adic period of (1.3) giving

$$\det(\text{per}_p(P_i(x_j)_{1 \leq i,j \leq N})) = 0.$$ 

It defines an alternating $p$-adic analytic function vanishing on $\wedge^N X_S$ and yields information about the disposition of integral points. For example, if $s = 1$, and $k = 2$, we can take $w = 2$, $N = 2$, set $x_1$ a tangent vector at infinity, and $x_2 = x$ to retrieve Coleman’s equation

$$\det \left( \begin{array}{cc} \text{Li}_2^p(x) & \log^p(x) \text{Li}_1^p(x) \\ 1 & 1 \end{array} \right) = \text{Li}_2^p(x) - \frac{1}{2} \log^p(x) \text{Li}_1^p(x) = 0 \quad \text{for all } x \in X_S.$$

The main result of [11] follows from an application of theorem 1.1 with $N = 3$, $w = 4$. The approach in that paper used the conjectural non-vanishing of a $p$-adic zeta value.

The theorem can be applied as follows: given $x_1, \ldots, x_{N-1}$ ‘known’ points in $X_S$, a row expansion of the determinant (1.3) gives an explicit equation

$$(1.4) \quad \sum_{i=1}^{N} p^w_i P_i(x_N) = 0$$

which can be viewed as a constraint on any further point $x_N \in X_S$. One can show, by a minimality argument, that equation (1.4) is not identically zero. Thus, one has the following dichotomy: either there are fewer than $N$ points on $X_S$, or there exists an explicit equation for the remaining points on $X_S$. In either case one reproves the fact that $X_S$ is finite [27, 24]. The method easily extends to the cases considered in [22].

Using the work of Hardy and Ramanujan on the partition function, one can estimate the value of $N$. Unfortunately, it is rather large - possibly slightly larger than the expected number of points on $X_S$. In [11] we show how to replace points $x_i \in X_S$ with divisors on $X(\mathbb{Q})$ to get around this problem.

Besser and de Jeu have shown how to compute the $p$-adic classical polylogarithms explicitly in [4], which raises the possibility of practical applications of the previous two theorems for studying integral points. If one assumes a version of Beilinson’s conjecture, the results stated above carry through to curves of higher genus. This is discussed in [11]. More precisely, for sufficiently many integral points $x_1, \ldots, x_N$ on (an integral model of) a curve $X$ of genus $\geq 2$, the determinant of $p$-adic iterated integrals along Frobenius invariant paths from $p$ to $x_i$ should always vanish.

Acknowledgements. This paper was partly written during a stay at the IHES in 2015. Many thanks to Richard Hain and Ishai Dan-Cohen for discussions on this topic. The author is partly supported by ERC grant 724638 - ‘Galois theory of periods and applications’.

2. Motivation: non-abelian and ‘motivic’ Abel-Jacobi map

We shall briefly explain, in this extended introduction, why the ideas of this paper can be thought of as a natural generalisation of the classical Abel-Jacobi map.
2.1. Abel-Jacobi. Let $X$ be a connected Riemann surface, and $p \in X$ a point. The Abel-Jacobi map relative to the point $p \in X$ is defined by

\[
\begin{align*}
X & \rightarrow \text{Hom}(H^0(X, \Omega^1_X), \mathbb{C}) / H_1(X; \mathbb{Z}) \\
x & \mapsto (\omega \mapsto \int_{\gamma_x} \omega \pmod{\Lambda})
\end{align*}
\]

where $\gamma_x$ is a continuous path from $p$ to $x$, and $\Lambda$ is the period lattice:

\[
\Lambda = \text{Im}(H_1(X; \mathbb{Z}) \rightarrow \text{Hom}(H^0(X, \Omega^1_X), \mathbb{C}))
\]

\[
[\gamma] \mapsto (\omega \mapsto \int_{\gamma} \omega)
\]

Since $H^0(X, \Omega^1_X) = F^1H^1(X; \mathbb{C})$, it is customary to write the right-hand side of (2.1) as a space of double cosets

\[
F^0 \backslash H^1(X; \mathbb{C})^\vee / H_1(X; \mathbb{Z}).
\]

The group in the middle is classical or $C^\infty$ de Rham cohomology (we shall denote algebraic de Rham cohomology by a subscript $dR$). However, the following form is better adapted to the non-abelian setting:

\[
\text{Hom}(F^1H^1(X; \mathbb{C}), \mathbb{C}) / H_1(X; \mathbb{Z})
\]

This construction can be generalised in at least three different ways: first by defining a motivic version of this construction; next by replacing cohomology with fundamental groups; and finally by changing cohomology theories (fiber functors).

2.2. ‘Motivic’ version. Now let $X$ be a smooth, geometrically connected algebraic curve over $\mathbb{Q}$ for simplicity, and suppose that $p, x \in X(\mathbb{Q})$. Define the motivic version of the line integral to be the matrix coefficient (§3.2 or [3], §2)

\[
I^m_{\gamma_x}(\omega) = [H^1(X, \{p, x\}), [\gamma_x], [\omega]]^m
\]

in the ring of periods $\mathcal{P}^m_C$ of a suitable category $C$ (for example, the category $\mathcal{H}$ of realisations of $\mathcal{P}_C$). This means that $I^m_{\gamma_x}(\omega)$ is viewed as an element of the affine ring of tensor isomorphisms from de Rham to Betti fiber functors of the category $C$. In the case $C = \mathcal{H}$, the object $H^1(X, \{p, x\})$ in $\mathcal{H}$ denotes a triple

\[
(H^1(X(\mathbb{C}), \{p, x\}), H^1_{dR}(X, \{p, x\}), \text{comp})
\]

where the first component is relative singular (Betti) cohomology, the middle component is relative algebraic de Rham cohomology (both $\mathbb{Q}$-vector spaces) and comp the canonical comparison isomorphism between their complexifications. Since its boundary is contained in $\{p, x\}$, the path $\gamma_x$ defines a relative homology class:

\[
[\gamma_x] \in H_1(X(\mathbb{C}), \{p, x\}) \cong H^1(X(\mathbb{C}), \{p, x\})^\vee.
\]

On the other hand, the holomorphic differential one-form $\omega$ restricts to zero at $x$ and $p$ and has a well-defined class in relative algebraic de Rham cohomology $H^1_{dR}(X, \{p, x\})$. One way to see this is to consider the long exact relative cohomology sequence

\[
H^0_{dR}(\{x, p\}) \rightarrow H^1_{dR}(X, \{x, p\}) \rightarrow H^1_{dR}(X) \rightarrow 0.
\]

By applying the Hodge filtration functor $F^1$, we deduce an isomorphism

\[
F^1 H^1_{dR}(X, \{x, p\}) \rightarrow F^1 H^1_{dR}(X).
\]

It is important to point out that since $F^1H^1_{dR}(X) \cong H^0(X, \Omega^1(X))$, we can indeed view the cohomology class of a holomorphic differential form, via the previous isomorphism, as a relative cohomology class in a canonical way, independently of $x$. 


Replacing the de Rham-Betti comparison isomorphism (1.1) with its universal version (1.2), we can define a ‘motivic’ version of the Abel-Jacobi map.

**Definition 2.1.** Consider the map

\[
X(Q) \to \text{Hom}(H^0(X, \Omega^1_X), P_C^m) / H_1(X; \mathbb{Z})
\]

where \(\Lambda^m\) is the lattice of motivic periods of \(X\), defined to be

\[
\Lambda^m = \text{Im}(H_1(X; \mathbb{Z}) \to \text{Hom}(H^0(X, \Omega^1_X), P_C^m)) \quad [\gamma] \mapsto (\omega \mapsto I^m_\gamma(\omega))
\]

where \(I^m_\gamma(\omega)\) is the motivic period \([H^1(X), [\gamma], [\omega]]^m\).

The classical Abel-Jacobi map (2.1) is recovered from (2.4) by composing with the period homomorphism per : \(P_C^m \to \mathbb{C}\), which induces

\[
\text{Hom}(H^0(X, \Omega^1_X), P_C^m) \to \text{Hom}(H^0(X, \Omega^1_X), \mathbb{C})
\]

Furthermore, the map (2.4) extends in the usual manner to divisors of degree zero:

\[
\text{Div}^0(X(Q)) \to \text{Hom}(H^0(X, \Omega^1_X), P_C^m) / H_1(X; \mathbb{Z})
\]

\[
\sum_i n_i(x_i - p) \mapsto (\omega \mapsto \sum_i n_iI^m_{\gamma_i}(\omega) \mod \Lambda^m)
\]

It vanishes on principle divisors if \(X\) is compact. There is an obvious extension to curves defined over a number fields \(k\), where the Betti cohomology is relative to an embedding \(k \to \mathbb{C}\). To keep the discussion simple, we have chosen to work only over \(Q\) for this introduction.

**2.3. Non-abelian version.** One can now generalize by replacing homology with a suitable notion of algebraic fundamental group. Here we shall consider the unipotent fundamental groupoid. It is a groupoid object in a suitable category \(C\) which has Betti and de Rham realisations, denoted \(\pi^B_1\) and \(\pi^dR_1\) respectively. The Betti component is simply the Mal’cev completion of the topological fundamental groupoid and is equipped with a natural map, for any points \(p, x \in X(Q)\):

\[
\pi^\text{top}_1(X, p, x) \to \pi^B_1(X, p, x)(Q),
\]

where \(\pi^\text{top}_1\) denotes the homotopy classes of paths in \(X(\mathbb{C})\). The Betti and de Rham fundamental groupoids are related by a universal comparison:

\[
\pi^B_1(X, p, x) \times P_C^m \xrightarrow{\sim} \pi^dR_1(X, p, x) \times P_C^m,
\]

which, after applying the period homomorphism, becomes an isomorphism over \(\mathbb{C}\). The literature would suggest replacing (2.2) with the higher Albanese manifolds [19]

\[
F^0 / \pi^dR_1(X, p)(\mathbb{C}) / \pi^\text{top}_1(X, p).
\]

Here \(F^0\) denotes a subgroup scheme of \(\pi^dR_1(X, p)\) defined using the Hodge filtration. There is an analogous notion for the de Rham torsor of paths \(\pi^dR_1(X, p, x)\) ([11, 10]). A problem with this construction is that although the coset spaces

\[
F^0 \setminus \pi^dR_1(X, p, x) \cong F^0 \setminus \pi^dR_1(X, p, x)
\]

are isomorphic as schemes, they are not canonically so, and the resulting constructions (including the coefficients of any Coleman function constructed in this manner) will
depend implicitly on the point \( x \). To circumvent this, we shall instead use a scheme \( \pi_1^{dR} \), also defined using the Hodge filtration, which admits a dominant morphism

\[
\pi_1^{dR}(X, p, x) \rightarrow \pi_1^{dR}(X, p, x)
\]

but is now independent of the point \( x \). It has a canonical de Rham path

\[
x 1_p \in \pi_1^{dR}(X, p, x)(k).
\]

The point is that there is now a canonical isomorphism,

\[
\mathcal{O}(\pi_1^{dR}(X, p, x)) \xrightarrow{\sim} \mathcal{O}(\pi_1^{dR}(X, p))
\]

which generalises the crucial fact \( F^1 H^{\text{dR}}_1(X, \{x, p\}) = F^1 H^{\text{dR}}_1(X) \). Furthermore, the elements of the affine ring of \( \pi_1^{dR} \) can be viewed as holomorphic iterated integrals, which again generalises the holomorphic line integrals in the classical situation (2.1).

We obtain a sequence of maps

\[
\pi_1^{\text{top}}(X, p) \rightarrow \pi_1^{B}(X, p)(\mathbb{Q}) \xrightarrow{(2.10)} \pi_1^{dR}(X, p)(\mathcal{P}_C^m)
\]

**Definition 2.2.** Out of this, we can define a map

\[
X(\mathbb{Q}) \rightarrow \text{Hom}_{\text{alg}}\left(\mathcal{O}(\pi_1^{dR}(X, p), \mathcal{P}_C^m) \right) / \pi_1^{\text{top}}(X, p)
\]

where \( \omega \) is an element of \( \mathcal{O}(\pi_1^{dR}(X, p)) \), and \( \gamma_x \) is a path from \( p \) to \( x \). The notation \( \text{Hom}_{\text{alg}} \) denotes algebra homomorphisms, and so (2.10) could also be written

\[
X(\mathbb{Q}) \rightarrow \pi_1^{dR}(X, p)(\mathcal{P}_C^m) / \pi_1^{\text{top}}(X, p).
\]

The ‘motivic period’ \( I_{\gamma_x}^m(\omega) \) is defined by the matrix coefficient

\[
I_{\gamma_x}^m(\omega) = \left[ \mathcal{O}(\pi_1(X, p, x)), \gamma_x, \omega \right]^m \in \mathcal{P}_C^m
\]

and is the motivic version of the iterated integral of \( \omega \) along the path \( \gamma_x \).

The key point is that \( \omega \in \mathcal{O}(\pi_1^{dR}(X, p)) \) is viewed as an element in the ring \( \mathcal{O}(\pi_1^{dR}(X, p, x)) \subset \mathcal{O}(\pi_1^{dR}(X, p, x)) \) via the canonical isomorphism (2.8), and illustrates why the Albanese manifolds cannot be used without introducing some hidden choices.

Applying the period homomorphism, we obtain a non-abelian Abel-Jacobi map

\[
X(\mathbb{C}) \rightarrow \pi_1^{dR}(X, p)(\mathbb{C}) / \pi_1^{\text{top}}(X, p)
\]

The iterated integral makes sense for any point \( x \in X(\mathbb{C}) \) and so this map extends to a smooth and in fact, holomorphic, map

\[
(2.10) \quad X(\mathbb{C}) \rightarrow \pi_1^{dR}(X, p)(\mathbb{C}) / \pi_1^{\text{top}}(X, p).
\]

It extends equally well to divisors of degree zero, in the obvious manner. Furthermore, there is no reason to restrict to curves: the construction works for any smooth geometrically connected algebraic variety \( X \) over a number field \( k \subset \mathbb{C} \).

**Remark 2.3.** The image of \( \pi_1^{\text{top}}(X, p) \) inside \( \pi_1^{dR}(X, p)(\mathcal{P}_C^m) \), which generalises the lattice of periods, is interesting. Already in the case when \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \), and \( p \) is a tangent vector of length 1 at the origin, the coefficients of this map are generated by the ring of motivic multiple zeta values and the motivic version of \( 2\pi i \).
A simplification of (2.9) is possible if we replace \( X \) with a simply connected open subset. In that case, the path \( \gamma_x \) is unique up to homotopy and we do not need to quotient by the non-abelian periods, giving a map \( X(\mathbb{Q}) \to \underline{x_dR}(X, p)(\mathbb{P}_m) \).

2.4. Changing cohomology theories. The classical Abel-Jacobi map is obtained from the comparison isomorphism from de Rham to Betti cohomology. The method of Chabauty and Coleman via \( p \)-adic integrals is naturally expressed using the comparison from de Rham to crystalline cohomology. More generally, given fiber functors \( \omega \) and \( \omega' \) on the category \( C \), we obtain a universal comparison, analogue of (2.7)

\[
\pi'_1(X, p, x) \times \mathcal{P}_C^m \xrightarrow{\sim} \pi'_1(X, p, x) \times \mathcal{P}_C^m',
\]

where \( \mathcal{P}_C^m \) is the ring of \((\omega, \omega')\)-periods of \( C \). We shall mainly consider the cases where \( \omega \) is the de Rham fiber functor, and \( \omega' \) is de Rham, Betti or the crystalline fiber functor. In the latter setting, the topological path \( \gamma_x \) can be replaced the unique Frobenius-invariant path \( [3] \) between the reductions of \( p \) and \( x \). For the majority of this paper, we take \((\omega, \omega')\) to be \((dR, dR)\) and focus on the mixed Tate case, for which \( \underline{x_dR} = \underline{x_dR} \). Other variants are possible: Kim’s approach uses \( \ell \)-adic and crystalline cohomology \([23]\).

3. Background on periods

3.1. \( C \)-periods. Suppose that \( C \) is a Tannakian category over \( \mathbb{Q} \) such that

1. \( C \) is equipped with a pair of fiber functors \( \omega_dR, \omega_B : C \to \text{Vec}_\mathbb{Q} \) to the category of finite dimensional vector spaces over \( \mathbb{Q} \).
2. there is an isomorphism, functorial in \( M \), and compatible with \( \otimes \)

\[
\text{comp}_{B,dR} : \omega_dR(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_B(M) \otimes \mathbb{C}.
\]

Define the ring of periods of \( C \) to be the affine ring

\[
\mathcal{P}_C^m = \mathcal{O}(\text{Isom}_C^\otimes(\omega_dR, \omega_B)).
\]

It is a \( \mathbb{Q} \)-algebra, generated by matrix coefficients \([M, \gamma, \omega]^m\), where \( \gamma \in \omega_B(M)^\vee \) and \( \omega \in \omega_dR(M) \). It is the function which to \( \phi \in \text{Isom}_C^\otimes(\omega_dR, \omega_B) \) assigns \( \gamma(\phi(\omega)) \).

The period homomorphism

\[
\text{per} : \mathcal{P}_C^m \to C
\]

is defined by evaluation on the point \( \text{comp}_{B,dR} \in \text{Isom}_C^\otimes(\omega_dR, \omega_B)(\mathbb{C}) \) defined by (2). For every object \( M \) in \( C \) there is a universal comparison map

\[
\omega_dR(M) \otimes \mathbb{Q} \mathcal{P}_C^m \xrightarrow{\sim} \omega_B(M) \otimes \mathbb{Q} \mathcal{P}_C^m
\]

\[
\omega \otimes 1 \mapsto \sum \{e_i \otimes [M, e_i^\vee, \omega]^m
\]

where \( \{e_i\} \) is a basis of \( \omega_B(M) \), and \( \{e_i^\vee\} \) the dual basis. This formula does not depend on the choice of basis. Composing the previous map with the period homomorphism, and tensoring with \( \mathbb{C} \), gives back the isomorphism (2). See \([9]\), \$2 for further details.

3.2. Examples. As a first example, let \( C \) be the category \( \mathcal{H} \) \((12), \$1.10\) of triples \((V_B, V_dR, c)\) where \( V_dR, V_B \) are finite-dimensional \( \mathbb{Q} \)-vector spaces equipped with an increasing filtration \( W \), and \( c \) is an isomorphism \( c : V_dR \otimes \mathbb{C} \cong V_B \otimes \mathbb{C} \) which respects \( W \). The vector space \( V_dR \) is furthermore equipped with a decreasing filtration \( F \), such that \( V_B \), equipped with \( cF \) and \( W \), is a graded-polarizable \( \mathbb{Q} \)-mixed Hodge structure. It can also be equipped with a functorial real Frobenius involution \( F_\infty : V_B \cong V_B \) but which will play a limited role here.
In this setting, we can define $\mathcal{P}_\mathcal{H}^{m,+}$ to be the subring of $\mathcal{P}_\mathcal{H}^m$ generated by matrix coefficients $[M, \gamma, \omega]^m$ which are effective: i.e., $W_1 M = 0$ \cite{3.2}.

For any smooth scheme $X$ and a normal crossings divisor $D \subset X$ over $\mathbb{Q}$, Deligne’s mixed Hodge theory implies that the triple

$$H^n(X, D) := (H^n(X(\mathbb{C}), D(\mathbb{C})), H^n_{dR}(X, D), \text{comp}_{B, dR})$$

is an object of $\mathcal{H}$, and depends functorially in $X$. In particular, we can define the ‘motivic’ integral to be the (effective) matrix coefficient

$$I^m_{\gamma}(\omega) = [H^n(X, D), \gamma, \omega]^m \in \mathcal{P}_\mathcal{H}^{m,+}$$

whenever $\omega \in H^n_{dR}(X, D)$ and $\gamma$ is a smooth chain on $X(\mathbb{C})$ whose boundary is contained in $D(\mathbb{C})$.

Alternatively, one can take $C$ to be a variant of Nori’s elementary category of motives. Such a formalism enables us to make sense of \cite{22} but since we presently lack control on the size of the space of motivic periods in this category, there are no unconditional applications to integral points.

Therefore, for the majority of this paper, we shall take $C$ to be a category of mixed Tate motives over a number field \cite{4.3}. In this situation, the space of motivic periods is under control, but this condition poses severe restrictions on the scheme $X$.

3.3. Variants. The category $\mathcal{H}$ can be enriched to include $\ell$-adic and crystalline components \cite{1.10}. More generally, given any two fiber functors $\omega_i$ from $C$ to the category of projective $R_i$-modules for $i = 1, 2$, we can form a ring of $\omega_1, \omega_2$ periods

$$\mathcal{P}_{C}^{\omega_1, \omega_2} = \mathcal{O}(\text{Isom}_{C}(\omega_1, \omega_2)).$$

Since $\text{Isom}_{C}(\omega_2, \omega_1)$, the scheme of tensor isomorphisms from $\omega_2$ to $\omega_1$, is a $G_C^{\omega_1} \times G_C^{\omega_2}$ bitorsor, where $G_C^{\omega_1} = \text{Aut}_{C}(\omega_1)$, it follows that $G_C^{\omega_2}$ acts on $\mathcal{P}_{C}^{\omega_1, \omega_2}$ on the left, $G_C^{\omega_1}$ acts on the right. Later in the paper, we wish to have Tannaka groups acting on the left of fundamental groups, and hence on the right of their affine rings. Since this is the opposite of the usual convention, where groups act on their representations on the left, this will have the effect of reversing ‘left’ and ‘right’ throughout this section. The ring $\mathcal{P}_{C}^{\omega_1, \omega_2}$ is generated by matrix coefficients $[M, v_1, v_2]^{\omega_1, \omega_2}$ where $v_1 \in \omega_1(M)\vee$, $v_2 \in \omega_2(M)$. Of particular relevance for the applications discussed here are the cases $(\omega_1, \omega_2) = (\text{dR}, \text{dR})$, and $(\omega_1, \omega_2) = (\text{cris}, \text{dR})$, and $(\omega_1, \omega_2) = (B, \text{dR}) =: \mathfrak{m}$.

3.4. Mixed Tate motives over number fields \cite{6.12}. Let $k$ be a number field, and $S$ a finite set of primes in $\mathcal{O}_k$. Let $MT_k$ denote the category of mixed Tate motives over $k$, and $MT_S \subset MT_k$ the full subcategory of motives unramified at $S$. It is a neutral Tannakian category over $\mathbb{Q}$ with a canonical fiber functor

$$\omega : MT_k \rightarrow \text{Vec}_\mathbb{Q}$$

which factors through the category of graded vector spaces. The de Rham fiber functor $\omega_{dR} : MT_k \rightarrow \text{Vec}_k$ is obtained from $\omega$ by tensoring with $k$: $\omega_{dR} = \omega \otimes k$. Let

$$G_S^\omega = \text{Aut}_{MT_S}^\omega(\omega)$$

be the Tannaka group relative to $\omega$. It is an affine group scheme over $\mathbb{Q}$, and if we denote by $U_S^\omega$ its pro-unipotent radical, we have a canonical splitting

$$G_S^\omega = U_S^\omega \times G_m.$$  

For every embedding $\sigma : k \hookrightarrow \mathbb{C}$ there is a Betti fiber functor $\omega_\sigma : MT_k \rightarrow \text{Vec}_\mathbb{Q}$, and hence we have a $\mathbb{Q}$-algebra of motivic periods $P_{MT_S}^{\omega_\sigma} = \mathcal{O}(\text{Isom}_{MT_S}^\omega(\omega, \omega_\sigma))$. It has a left action of $G_S^\omega$ and a period homomorphism $\text{per} : P_{MT_S}^{\omega_\sigma} \rightarrow \mathbb{C}$. It is generated by matrix coefficients $[M, \gamma, v]$ where $v \in \omega(M)$, and $\gamma \in \omega_\sigma(M)\vee$. 
3.4.1. Unipotent periods. The ring of $\omega$-periods $\mathcal{P}_S^\omega$ is defined to be the affine ring $\mathcal{O}(G_S^\omega)$, and is generated by matrix coefficients $[M, v, f]^\omega$ where $v \in \omega(M)$ and $f \in \omega(M)^\gamma$. Finally, the ring of unipotent periods is
\begin{equation}
\mathcal{P}_S^\omega = \mathcal{O}(U_S^\omega)
\end{equation}
and is generated by the restrictions of the matrix coefficients $[M, v, f]^\omega$ to $U_S^\omega$. It is equipped with the conjugation action of $G_S^\omega$, and in particular, is graded via the action of $\mathbb{G}_m$. The object $[M, v, f]^\omega$ is the function which to $u \in U_S^\omega$ assigns $f(uv) \in \mathbb{A}^1$.

3.4.2. Structure. Let $\mathcal{O}_S$ denote the ring of $S$-integers in $k$. It is known \[12 \ 26\] that
\begin{equation}
\text{Ext}^1_{\mathcal{MT}_S}(\mathbb{Q}(0), \mathbb{Q}(-n)) \cong K_{2n-1}(\mathcal{O}_S) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{equation}
and all higher Ext groups in $\mathcal{MT}_S$ vanish. Furthermore, Borel proved that
\begin{equation}
\text{dim}_\mathbb{Q} K_{2n-1}(\mathcal{O}_S) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} 
|S| & \text{if } n = 1, \\
r_2 & \text{if } n > 1 \text{ even} \\
r_1 + r_2 & \text{if } n > 1 \text{ odd}
\end{cases}
\end{equation}
where $r_1$ (resp. $r_2$) is the number of real (resp. complex) places of $k$. It follows that the graded Lie algebra of $U_S^\omega$ is free with $|S|$ generators in degree 1, $r_2$ generators in even degrees, and $r_1 + r_2$ generators in odd degrees. The affine ring of $U_S^\omega$ is the graded dual to the universal enveloping algebra of $\text{Lie}^{\omega} U_S^\omega$. This gives precise control on the dimensions of the weight-graded pieces of $\mathcal{P}_S^\omega$, and is the key input for theorem \[11\].

3.4.3. $p$-adic period. We shall assume that the category of mixed Tate motives over $\mathcal{O}_S$ is equipped with a crystalline fiber functor for all primes $p$ of $\mathcal{O}_S$, and a canonical comparison isomorphism $\text{comp}_{dR, crys} : M_{dR} \otimes k_p \xrightarrow{\sim} M_{crys}$. This was announced in \[30\]. Pulling back the Frobenius via this map defines a functorial isomorphism
\[ F_p : M_{dR} \otimes k_p \xrightarrow{\sim} M_{dR} \otimes k_p \]
and hence a canonical element $F_p \in G_S^\omega(k_p)$. It acts on the de Rham realisation of $\mathbb{Q}(-1)$ by multiplication by $p$, so its image in $G_m(k_p)$ is the element $p$. Denote its image in $G_S^\omega(k_p)$ by $p$ also. Hence one obtains an element
\[ \overline{F}_p = p^{-1} F_p \in U_S^\omega(k_p) . \]
One can also consider $F_p p^{-1} = p \overline{F}_p p^{-1} \in U_S^\omega(k_p)$. The (single-valued) $p$-adic period is the homomorphism
\[ \text{per}_p : \mathcal{P}_S^\omega \longrightarrow k_p \]
defined by evaluation at the point $\overline{F}_p$. There is a Betti analogue: $\text{per}_\sigma : \mathcal{P}_S^\omega \rightarrow k_\sigma$ where $\sigma : k \rightarrow \mathbb{R}$ is a real place of $k$. It is the ‘single-valued period’, corresponding to the isomorphisms $M_{dR} \otimes \mathbb{C} \xrightarrow{\sim} M_{B, \sigma} \otimes \mathbb{C} \rightarrow M_{B, \sigma} \otimes \mathbb{C} \xleftarrow{\sim} M_{dR} \otimes \mathbb{C}$ where the map in the middle is the real Frobenius $(-1)F_{\infty} \otimes \text{id} \sigma$, where $-1 \in G_m(\mathbb{Q})$.

4. Background on unipotent fundamental groups

Let $X$ be a smooth, geometrically connected scheme over a number field $k \subset \mathbb{C}$. 
4.0.1. **Betti fundamental groupoid.** Let $x \in X(\mathbb{C})$, and let $\pi_1^{un}(X(\mathbb{C}), y, x)$ denote the unipotent completion of the topological fundamental groupoid $\pi_1(X(\mathbb{C}), y, x)$. It is a scheme over $\mathbb{Q}$, equipped with a natural map from homotopy classes of paths from $x$ to $y$ into its set of rational points:

$$\pi_1(X(\mathbb{C}), y, x) \longrightarrow \pi_1^{un}(X(\mathbb{C}), y, x)(\mathbb{Q}) .$$

It forms a groupoid which is compatible with the previous map:

$$\pi_1^{un}(X(\mathbb{C}), z, y) \times \pi_1^{un}(X(\mathbb{C}), y, x) \longrightarrow \pi_1^{un}(X(\mathbb{C}), z, x) .$$

We shall compose paths in the functional order: $\alpha \beta$ means $\beta$ followed by $\alpha$.

The unipotent fundamental group admits the following Tannakian formulation. Consider the category $\mathcal{L}(X)$ of local systems of finite-dimensional $\mathbb{Q}$-vector spaces on $X(\mathbb{C})$, which are equipped with an exhaustive increasing filtration by local sub-systems of $\mathbb{Q}$-vector spaces, such that the associated graded pieces are trivial (constant). This is a Tannakian category, and for every point $x \in X(\mathbb{C})$ there is a fiber functor $\omega_x : \mathcal{L}(X) \to \text{Vec}_{\mathbb{Q}}$. The unipotent completions can be retrieved as follows:

$$\pi_1^{un}(X(\mathbb{C}), x) = \text{Aut}_{(\mathcal{L}(X))}^{\otimes}(\omega_x) \quad \text{and} \quad \pi_1^{un}(X(\mathbb{C}), y, x) = \text{Isom}_{(\mathcal{L}(X))}^{\otimes}(\omega_y, \omega_x) .$$

One can replace $x$ or $y$ with tangential base points, and the analogous statements remain true. The case where $X$ is a curve is treated in [13] §15.

4.0.2. **de Rham fundamental groupoid.** Let $U(X)$ be the Tannakian category of unipotent algebraic vector bundles on $X$ defined over $k$. These are equipped with a flat connection with regular singularities at infinity, and an exhaustive increasing filtration by flat $k$ sub-bundles such that the successive quotients are trivial. For any rational point $x \in X(k)$, the functor $x^*: U(X) \to U(\text{Spec } k)$ is a fiber functor $\omega_X : U(X) \to \text{Vec}_{k}$.

The fibers of the de Rham fundamental groupoid are the schemes over $k$

$$\pi_1^{dR}(X, x) = \text{Aut}_{U(X)}^{\otimes}(\omega_x) \quad \text{and} \quad \pi_1^{dR}(X, y, x) = \text{Isom}_{U(X)}^{\otimes}(\omega_y, \omega_x) .$$

They form a groupoid, and in particular a torsor

$$(4.1) \quad \pi_1^{dR}(X, y, x) \times \pi_1^{dR}(X, x) \longrightarrow \pi_1^{dR}(X, y, x)$$

As before, one can replace $x$ or $y$ with rational tangential base points.

The complexification of the affine ring of the de Rham fundamental group can be written down using the Chen’s reduced bar construction [3]. If $A(X)$ denotes the de Rham complex of smooth differential forms on $X(\mathbb{C})$, then

$$\mathcal{O}(\pi_1^{dR}(X, y, x)) \otimes_k \mathbb{C} \cong H^0(\overline{\mathcal{B}}(A(X)))$$

where $\overline{\mathcal{B}}$ is the reduced bar construction. A version of this construction is available over $k$ in the case when $X$ is the complement of a normal crossing divisor in a smooth proper variety, all defined over $k$ ([20], §14). For example, if $X$ is an affine curve, then we can choose global regular 1 forms $\omega_1, \ldots, \omega_m$ which represent the cohomology classes $H^1_{dR}(X, k)$. In this case the affine ring is isomorphic to the tensor coalgebra

$$(4.2) \quad \mathcal{O}(\pi_1^{dR}(X, y, x)) \cong T^c(\bigoplus_{i=1}^m \omega_i k)$$

equipped with the shuffle product. The groupoid structure is dual to the deconcatenation operation on tensors of one-forms.
4.0.3. Comparison. The Riemann-Hilbert correspondence restricts to an equivalence of categories \(L(X) \otimes_k \mathbb{C} \leftrightarrow U(X) \otimes_k \mathbb{C}\) and hence an isomorphism of schemes

\[
\text{comp}_{B,dR} : \pi_1^{un}(X,y,x) \times \mathbb{C} \longrightarrow \pi_1^{dR}(X,y,x) \times \mathbb{C},
\]

for all \(x,y \in X(k)\). It can be computed via the map

\[
\gamma \mapsto (\omega \mapsto \int_{\gamma} \omega)
\]

where the second map is the iterated integral of \(\omega\) along the path \(\gamma\). Chen’s \(\pi_1\)-de Rham theorem \([8]\) is equivalent to the statement that the comparison map is an isomorphism.

4.0.4. Beilinson’s construction. When \(X\) is the complement of a normal crossing divisor in a smooth projective variety, the unipotent fundamental group carries a natural mixed Hodge structure due to Morgan (via minimal models) and Hain (via the bar construction). A different approach is due to Beilinson, who proved that the unipotent fundamental groupoid is the Betti homology of a certain cosimplicial scheme constructed out of \(X,x,y\), \([29]\). See \([12]\), §3.3, for details.

In the case \(x \neq y\), consider \(X' = X \times \ldots \times X\), the product of \(n\) copies of \(X\), and let \(D_0 = \{x\} \times X^{n-1}\), \(D_i = X^{i-1} \times \Delta \times X^{n-i-1}\) for \(1 \leq i \leq n - 1\), \(D_n = X^{n-1} \times \{y\}\) where \(\Delta \subset X \times X\) is the diagonal. Then Beilinson’s theorem states that

\[
(4.3) \quad \mathcal{O}(\pi_1^{un}(X,y,x)) \cong \lim_{n \to \infty} \left( H^n(X^n(\mathbb{C}), \cup_{i=0}^n \comp D(\mathbb{C})) \right),
\]

where the morphisms between the relative cohomology groups on the right are via face maps. The multiplicative and Hopf algebroid (dual to the groupoid) structures are geometric: they arise from morphisms of algebraic varieties and hence induce morphisms of the corresponding relative cohomology groups on the right hand side (although not all these facts seem to have been proved explicitly in the literature). By replacing singular cohomology with another appropriate cohomology theory \(\omega\), one obtains the corresponding notion of fundamental group:

\[
(4.4) \quad \pi_1^X(y,x) := \text{Spec} \left( \lim_{n \to \infty} \left( H^n(X^n, \cup_{i=0}^n \comp D) \right) \right).
\]

In particular, the affine ring of the de Rham fundamental groupoid is isomorphic to its de Rham cohomology \([29]\), which is automatically endowed with a \(k\)-mixed Hodge structure. Therefore, the triple

\[
\mathcal{O}(\pi_1^{un}(X,y,x)) := (\mathcal{O}(\pi_1^B(X,y,x)), \mathcal{O}(\pi_1^{dR}(X,y,x)), \text{comp}_{B,dR})
\]

is an Ind-object of the category of realisations \(\mathcal{H}\) (in the case \(k = \mathbb{Q}\), and \(x,y \in X(\mathbb{Q})\) since, for simplicity, we only defined \(\mathcal{H}\) over \(\mathbb{Q}\)). The construction requires some modifications when \(x,y\) are tangential or coincide.

4.0.5. Torsors. We shall write \((4.4)\) as \(y \Pi_x^\omega\). Composition of paths defines morphisms

\[
(4.5) \quad \imath \Pi_x^\omega \times y \Pi_x^\omega \longrightarrow z \Pi_x^\omega
\]

with respect to which \(y \Pi_x^\omega\) is a \(z \Pi_x^\omega\)-torsor. Dually, this is given by a homomorphism

\[
(4.6) \quad \Delta : \mathcal{O}(y \Pi_x^\omega) \longrightarrow \mathcal{O}(y \Pi_x^\omega) \otimes_k \mathcal{O}(z \Pi_x^\omega)
\]

which is induced by a morphism of algebraic varieties. It has the property that \((\text{id} \otimes m)(\Delta \otimes \text{id}) : \mathcal{O}(y \Pi_x^\omega) \otimes_k \mathcal{O}(z \Pi_x^\omega) \rightarrow \mathcal{O}(y \Pi_x^\omega) \otimes_k \mathcal{O}(z \Pi_x^\omega)\) is an isomorphism, where \(m\) denotes multiplication. Furthermore, a torsor over a pro-unipotent affine group scheme
has a rational point. This follows from the fact the Galois cohomology of the additive group $H^1(\text{Gal}(k/k), \mathbb{G}_a)$ is trivial.

4.0.6. Hodge and weight filtrations. It follows from (4.4) that $O_y(y \Pi^\omega_x)$ is equipped with an increasing weight filtration $W$ which satisfies $W_{-1} = 0$ and

$$W_0 O_y(y \Pi^\omega_x) \cong k$$

when $X$ is connected. The filtration $W$ is respected by (4.6). In the de Rham realisation, $O_y(y \Pi^{dR}_x)$ is equipped, as a consequence of (4.4), with a natural decreasing Hodge filtration $F$, which satisfies $F^0 O_y(y \Pi^{dR}_x) = O_y(y \Pi^{dR}_x)$. It is respected by (4.6). Since $O_y(y \Pi^{dR}_x)$ is an ind-object in the category $\mathcal{H}$, and morphisms in the category of mixed Hodge structures are strict with respect to $W$ and $F$, we deduce that (4.6) is strict with respect to $W$, and also $F$ in the case $\omega = dR$. These filtrations are compatible with the multiplicative structure: $W_n W_m \subset W_{n+m}$, and $F^n F^m \subset F^{n+m}$.

4.0.7. Other realisations. The advantage of Beilinson’s construction is that the $\ell$-adic and crystalline realisations come for free. In fact, his construction, being geometric, defines a groupoid object in Nori’s category of mixed motives. In [11] we shall briefly mention the crystalline fundamental group [2], [28]. Under the assumptions [23] that $X$ has a good model over the ring of $S$-integers in a number field, one can define the crystalline fundamental group as a Tannaka group of the category [10] of unipotent (and hence overconvergent) isocrystals.

4.0.8. Motivic fundamental groupoid: mixed Tate case. For the applications to integral points, we require a fundamental groupoid in an actual abelian category of mixed Tate (or Artin-Tate) motives, which is highly restrictive. For example, $X$ can be a smooth unirational variety over a number field $k$. Our main example is $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathbb{Q}$. In a certain class of cases, and for $x, y \in X(k)$, Deligne and Goncharov have shown that Beilinson’s construction is the realisation of the motivic fundamental groupoid

$$\pi_1^{\text{mot}}(X, y, x)$$

which is a pro-object of $\mathcal{MT}_k$ (its affine ring is an Ind-object in $\mathcal{MT}_k$). Call $(X, x, y)$ Tate and unramified outside a finite set of primes $S$ if $\pi_1^{\text{mot}}(X, y, x)$ lies in the subcategory $\mathcal{MT}_S \subset \mathcal{MT}_k$ for some finite set of primes $S$. The set of primes $S$ can be determined from a regular model of $X$ ([12], proposition 4.17). In particular, if $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and $x, y \in X_S$ are $S$-units, then $(X, x, y)$ is unramified outside $S$.

There are at least two peculiarities in the mixed Tate case that we can exploit. The first is the existence of the canonical fiber functor, which is more precise than the de Rham fiber functor since it is over $\mathbb{Q}$. For ease of notation, we shall write

$$y \Pi^\omega_x = \omega(\pi_1^{\text{mot}}(X, y, x)) .$$

Its affine ring is defined over $\mathbb{Q}$. The other peculiarity is that $O_y(y \Pi^\omega_x)$ is graded by the weight, and hence equipped with an augmentation $\varepsilon : O_y(y \Pi^\omega_x) \to \operatorname{gr}_0^W O_y(y \Pi^\omega_x) \cong \mathbb{Q}$. Equivalently, there is a canonical $\omega$-path from $x$ to $y$:

$$y 1_x \in y \Pi^\omega_x .$$

We shall define a generalisation of the canonical de Rham path to the non-mixed Tate case in [11.5.3] using the Hodge filtration.

5. Some linear algebra

The crux of our argument is a simple observation in linear algebra. All tensors will be over a field $k$. 

5.1. **Coaction of linear maps.** Let $V, W$ be finite-dimensional $k$-vector spaces. The action of $k$-linear maps from $V$ to $W$ defines a canonical map:

$$\phi \otimes v \mapsto \phi(v) : \text{Hom}(V, W) \otimes V \longrightarrow W.$$  

Dually, this defines a coaction:

$$(5.1) \text{co} : V \longrightarrow W \otimes \text{Hom}(V, W)^\vee.$$  

Identifying $\text{Hom}(V, W) = V^\vee \otimes W$, and $\text{Hom}(V, W)^\vee = W^\vee \otimes V$, the map $\text{co}$ can equivalently be defined as the composition

$$V \longrightarrow k \otimes V \longrightarrow W \otimes W^\vee \otimes V$$

where the first map is $v \mapsto 1 \otimes v$, and the map $k \rightarrow W \otimes W^\vee$ is the map which sends 1 to the identity map $\text{id} \in W \otimes W^\vee = \text{Hom}(W^\vee, W^\vee)$. Thus, if we pick a basis $\{e_i\}$ of $W$, and let $\{e_i^\vee\}$ denote the dual basis of $W$, then

$$\text{co}(v) = \sum_i e_i \otimes [e_i^\vee, v]$$

where $[w, v]$, for $w \in W^\vee, v \in V$ is the function on $\text{Hom}(V, W)$ given by $\phi \mapsto w(\phi(v))$.

Any linear map $\phi : V \rightarrow W$ factorizes through the canonical map (5.1):

$$\phi : V \xrightarrow{\text{co}} W \otimes \text{Hom}(V, W)^\vee \xrightarrow{\text{id} \otimes \text{ev}_\phi} W,$$

where the second map $\text{ev}_\phi$ is ‘evaluation of $\phi$’. This follows from

$$(\text{id} \otimes \text{ev}_\phi) \text{co}(v) = \sum_i e_i \otimes e_i^\vee(\phi(v)) = \phi(v).$$

We shall use this construction in two different ways; firstly in the definition of the universal comparison isomorphism, and secondly by applying it to a ring of periods.

5.2. **Coaction by a scheme $A$ of linear maps.** A vector space $U$ over $k$ of finite dimension can be viewed as an affine scheme, i.e., the functor from commutative unitary $k$-algebras $R$ to sets $R \mapsto U \otimes R$ is representable. It is represented by the symmetric tensor algebra $\mathcal{O}(U) := \bigoplus_{n \geq 0} \text{Sym}^n U^\vee$, since

$$\text{Hom}_{k-\text{alg}}\left(\bigoplus_{n \geq 0} \text{Sym}^n U^\vee, R\right) = \text{Hom}(U^\vee, R) \cong U \otimes R.$$  

Furthermore, linear maps between vector spaces give rise to morphisms between the corresponding schemes. Apply this to $U = \text{Hom}(V, W)$. Let us suppose that

$$(5.2) A \subseteq \text{Hom}(V, W)$$

is a closed subscheme. There is a natural linear map $U^\vee \hookrightarrow \mathcal{O}(U) \rightarrow \mathcal{O}(A)$.

**Definition 5.1.** Denote the composition of (5.1) with the previous map by

$$(5.3) \text{co} : V \longrightarrow W \otimes \mathcal{O}(A).$$

Any element $\phi \in A(R) \subset \text{Hom}(V, W) \otimes R$ factorizes through (5.3):

$$\phi = (\text{id} \otimes \text{ev}_\phi) \text{co},$$

where $\text{ev}_\phi : \mathcal{O}(A) \rightarrow R$ is the homomorphism which represents $\phi$. From now on, we shall drop the commutative ring $R$ from the notation, and simply write $\phi \in A$, where the dependence on $R$ is understood.
5.3. **Infinite-dimensional vector spaces.** The previous construction only uses the finite-dimensionality of $W$, but extends to the case when $V$ is infinite-dimensional. Writing $V = \varinjlim V_i$ as a limit of finite-dimensional vector spaces $V_i$, we have

$$\text{Hom}(V, W) = \varinjlim \text{Hom}(V_i, W),$$

which is a projective limit of schemes and hence a scheme.

On the other hand, we can use the construction with infinite-dimensional graded vector spaces $V = \bigoplus_n V_n$ whose graded quotients $V_n$ are finite-dimensional $k$-vector spaces, if one interprets $\vee$ to be the graded dual, i.e., $V^\vee := \bigoplus_n V_n^\vee$.

5.4. **Determinant equation.** Now suppose in addition that $W$ is a commutative ring. By extending scalars from $k$ to $W$ the map $\co: W \otimes V \rightarrow W \otimes O(A)$ defines a $W$-linear map given by $w \otimes v \mapsto w\co(v)$. For any $n \geq 0$, we deduce a $W$-linear map

$$\bigwedge^n \co: W \otimes \bigwedge^n V \rightarrow W \otimes \bigwedge^n O(A).$$

Suppose that $N = \dim_k O(A) + 1$ is finite. Then $\bigwedge^N O(A)$ vanishes and so $\bigwedge^N \co$ is the zero map. Thus given $\phi_1, \ldots, \phi_N$ in $A$ the map

$$\phi_1 \wedge \ldots \wedge \phi_N: \bigwedge^N V \rightarrow W$$

vanishes since it factors through $\bigwedge^N \co$.

**Lemma 5.2.** If $N = \dim_k O(A) + 1$ is finite, then given any $\phi_1, \ldots, \phi_N \in A$,

$$\det \left( \phi_i(v_j) \right)_{1 \leq i, j \leq N} = 0$$

for all $v_1, \ldots, v_N \in V$.

**Remark 5.3.** An $\alpha \in \ker(W \otimes V \subseteq W \otimes O(A)) \subseteq W \otimes V$ defines a universal equation

$$\left( \id \otimes \phi \right)(\alpha) = 0 \quad \text{for all } \phi \in A.$$

A non-zero $\alpha$ exists if and only if $\dim_k V > \dim_k O(A)$. Note that (5.4) is intrinsic and defined over $k$, but (5.3) depends on a choice of $\alpha$, and involves coefficients in $W$.

5.5. **Filtered version.** Let $V, W$ be as in 5.2. In our applications, $V, W$ are equipped with increasing filtrations $F_n V, F_n W$ by subspaces. Let $\text{Hom}_F(V, W)$ denote the space of linear maps respecting $F$, i.e., $\phi: V \rightarrow W$ such that $\phi(F_n V) \subset F_n W$ for all $n$. We obtain an increasing filtration, also denoted by $F$, on $\text{Hom}_F(V, W)^\vee$ via

$$F_n (\text{Hom}(V, W)^\vee) := \text{Im}(\text{Hom}_F(F_n V, F_n W)^\vee \rightarrow \text{Hom}_F(V, W)^\vee)$$

where the second map is the dual of the restriction $\text{Hom}_F(V, W) \rightarrow \text{Hom}_F(F_n V, F_n W)$. Note that this is not the standard definition of a filtration on $\text{Hom}$.

The space $\text{Hom}_F(F_n V, F_n W)$ acts on every $F_n V$ on the left

$$\text{Hom}_F(F_n V, F_n W) \otimes F_n V \rightarrow F_n W$$

so dually, (5.1) restricts to a linear map

$$\co: F_n V \rightarrow F_n W \otimes F_n \text{Hom}_F(V, W)^\vee.$$

**Lemma 5.4.** The subspace $\text{Hom}_F(V, W)$ is a closed subscheme of $\text{Hom}(V, W)$. 
Proof. Since $V$ and $W$ are finite-dimensional, there exist integers $n_0, n_1$ such that $F_n = F_{n_0}$ for all $n_0 \leq n$ and $F_n = F_{n_1}$ for all $n \geq n_1$. Here, $F_i = F_j$ means that both $F_i V = F_j V$ and $F_i W = F_j W$. Consider the natural map

$$\text{Hom}(V, W) \rightarrow \bigoplus_{n_0 \leq n \leq n_1} \text{Hom}(F_n V, W/F_n W).$$

Then $\text{Hom}_F(V, W)$ is its kernel, i.e., the inverse image of the point 0. \hfill \Box

Now suppose that $A$ is a closed subscheme of $\text{Hom}_F(V, W)$. In particular, the points of $A$ respect the filtration $F$. Then we have

$$\text{co} : F_n V \rightarrow F_n W \otimes F_n \mathcal{O}(A),$$

where $F_n \mathcal{O}(A)$ is the increasing filtration on $\mathcal{O}(A)$ defined by the image of the subspace $F_n \mathcal{O}(\text{Hom}_F(V, W)) = \mathcal{O}(\text{Hom}_F(F_n V, F_n W))$ via $\mathcal{O}(\text{Hom}_F(V, W)) \rightarrow \mathcal{O}(A)$.

**Corollary 5.5.** Now suppose that $V, W$ are filtered vector spaces, possibly infinite dimensional, such that $F_n W$ is finite-dimensional for all $n$. Then $\text{Hom}_F(V, W)$ is representable, and therefore an affine scheme.

Proof. Let $V = \lim V_i$ such that $F_n V_i = V_i \cap F_n V$ is finite-dimensional for all $i$. Then $\text{Hom}_F(V, W) = \lim \text{Hom}_F(V_i, W)$ and $\text{Hom}_F(V_i, W) = \lim \text{Hom}_F(F_n V_i, F_n W)$. Therefore $\text{Hom}_F(V, W)$ is a projective limit of schemes. \hfill \Box

5.6. Graded version. Now consider the case when $V = \bigoplus V_n, W = \bigoplus W_n$ are graded by integers. Equivalently, $V, W$ admit a left action by $\mathbb{G}_m$. Define $\text{Hom}_{\mathbb{G}_m}(V, W)$ to be the subspace of $\text{Hom}(V, W)$ of linear maps which respect the gradings, and let us write $\text{Hom}_n(V, W) = \text{Hom}(V_n, W_n)$. The reader is warned that this is not the usual grading on $\text{Hom}_{\mathbb{G}_m}(V, W)$. The degree $n$ component of $\text{co}$ is the map

$$\text{co}_n : V_n \rightarrow W_n \otimes \text{Hom}_n(V, W)^\vee.$$

As in the previous paragraph, one easily checks that $\text{Hom}_{\mathbb{G}_m}(V, W)$ is a subscheme of $\text{Hom}(V, W)$. In the case when $A$ is a closed subscheme of $\text{Hom}_{\mathbb{G}_m}(V, W)$, i.e., when points of $A$ respect the gradings, then we have a coaction

$$\text{co} : V_n \rightarrow W_n \otimes \mathcal{O}(A)_n$$

where $\mathcal{O}(A)_n$ is the induced grading on $\mathcal{O}(A)$.

6. Non-abelian algebraic cocycles

We consider the space of one-cocycles of an affine group scheme with coefficients in another affine group scheme. We give necessary conditions for this to define a scheme.

6.1. Reminders on cocycles. Let $G, \Pi$ be groups such that $G$ acts on $\Pi$ on the left. This means that $g(ab) = g(a)g(b)$ for all $g \in G$ and $a, b \in \Pi$. Recall that the set of non-abelian cocycles $Z^1(G, \Pi)$ is the pointed set consisting of maps $c : G \rightarrow \Pi$ of sets satisfying the cocycle condition

$$c_{gh} = c_g \cdot c_h \quad \text{for all } g, h \in G.$$

This implies that $c_1 = 1$. The trivial cocycle is defined by $c_g = 1$ for all $g \in G$. The set of cocycles is equivalently given by

$$Z^1(G, \Pi) \cong \ker \left( \text{Hom}(G, G \times \Pi) \rightarrow \text{Hom}(G, G) \right),$$

since one verifies that given any $c : G \rightarrow \Pi$, the map $g \mapsto (g, c_g) : G \rightarrow G \times \Pi$ is a homomorphism if and only if $c$ satisfies the cocycle condition above.
Two cocycles \( c, c' \) are called equivalent if there exists \( b \in \Pi \) such that
\[
c'_g = b c_g g(b^{-1})
\]
for all \( g \in G \). The pointed set of equivalence classes is denoted by \( H^1(G, \Pi) \).

6.2. Cocycles for affine group schemes. Now let \( G, \Pi \) be two affine group schemes over a field \( k \), and suppose that \( G \) acts on \( \Pi \) on the left. There is a morphism
\[
G \times \Pi \rightarrow \Pi
\]
such that \( g(ab) = g(a)g(b) \) for all \( (R\text{-valued}) \) points \( g \in G \) and \( a, b \in \Pi \). It can equivalently be expressed by a commutative diagram, which we leave to the reader.

**Definition 6.1.** Define an algebraic cocycle over \( k \) to be a morphism of schemes
\[
c : G \rightarrow \Pi
\]
such that for all commutative unitary \( k \)-algebras \( R \), the induced map on \( R\text{-valued} \) points \( G(R) \rightarrow \Pi(R) \) is a cocycle. Denote the set of such morphisms by \( Z^1_{\text{alg}}(G, \Pi)(k) \).

Define a functor \( Z^1_{\text{alg}}(G, \Pi) \) from commutative unitary \( k \)-algebras \( k' \) to pointed sets
\[
k' \mapsto Z^1_{\text{alg}}(G \times k', \Pi \times k')(k') \quad (=: Z^1_{\text{alg}}(G, \Pi)(k')) .
\]

Modifying by a coboundary defines a natural transformation of functors
\[
\Pi \times Z^1_{\text{alg}}(G, \Pi) \xrightarrow{\sim} Z^1_{\text{alg}}(G, \Pi)
\]
\[
b, c \mapsto (g \mapsto b^{-1} c_g g(b)) .
\]

**Examples 6.2.**
1. If \( G \) is a constant group scheme of finite type, then
\[
Z^1_{\text{alg}}(G, \Pi)(k) \rightarrow Z^1(G(k), \Pi(k))
\]
is a bijection since every cocycle on points is algebraic.
2. In general, this is false. Let \( k = \mathbb{Q} \), and \( G = \Pi = \mathbb{G}_m \) with \( G \) acting trivially on \( \Pi \). Then \( Z^1(G(k), \Pi(k)) \) is the set of homomorphisms \( c : \mathbb{Q}^\times \rightarrow \mathbb{Q}^\times \). Setting \( c(-1) = 1 \), and assigning any value to \( c(p) \) for \( p \) a positive prime defines such a homomorphism. It is rarely algebraic.
3. The functor \( Z^1_{\text{alg}}(G, \Pi) \) is not necessarily representable. Let \( k = \mathbb{Q}, \Pi = \mathbb{G}_m \) and \( G = \mu_N = \text{Spec } \mathbb{Q}[x]/(x^N - 1) \), the group scheme of \( N \)-th roots of unity, with \( G \) acting trivially on \( \Pi \). Then elements of the set \( Z^1_{\text{alg}}(G, \Pi)(R) \) are given by algebra homomorphisms
\[
\phi : R[x, x^{-1}] \rightarrow R[x]/(x^N - 1) \text{ such that } \Delta \phi = (\phi \otimes \phi) \Delta ,
\]
where \( \Delta(x) = x \otimes x \). Such a homomorphism is of the form \( \phi(x) = x^n \) where \( 0 \leq n < N \). Therefore \( Z^1_{\text{alg}}(G, \Pi)(R) \) is the pointed set \( \mathbb{Z}/N\mathbb{Z} \). It is not representable since for any representable functor \( F, F(R \times S) = F(R) \times F(S) \).

6.3. Interpretations. We shall view a cocycle \( c \in Z^1_{\text{alg}}(G, \Pi)(k) \) in three ways:
1. as a morphism of schemes \( c : G \rightarrow \Pi \),
2. as a homomorphism of algebras \( c : \mathcal{O}(\Pi) \rightarrow \mathcal{O}(G) \),
3. as a point \( c \in \Pi(\mathcal{O}(G)) \),
satisfying, in each case, certain conditions which are equivalent to the cocycle relation. In the first case, this can be expressed by the commutativity of
\[
G \times G \rightarrow \Pi \times \Pi
\]
\[
\downarrow \quad \downarrow
\]
\[
G \quad \xrightarrow{c} \quad \Pi
\]
where the vertical maps are multiplication, and the map along the top is

\[ G \times G \xrightarrow{\Delta \times \text{id}} G \times G \xrightarrow{\text{id} \times c} \Pi \times G \xrightarrow{\text{id} \times \Delta} \Pi \times \Pi. \]

The conditions for (2) and (3) are left to the reader.

### 6.4. Filtrations

Let \( k, G, \Pi \) be as in \([6.2]\). Suppose that there exist increasing filtrations \( F \) on the affine rings \( \mathcal{O}(G), \mathcal{O}(\Pi) \) such that

- \( F_{-1}H = 0 \) and \( H = \bigcup_n F_nH \)
- \( \Delta F_nH \subset F_n(H \otimes \mathcal{O}(G)) \)

for \( H \in \{ \mathcal{O}(G), \mathcal{O}(\Pi) \} \). The filtration on a tensor product is defined by

\[ F_n(H \otimes H') = \sum_{a+b=n} F_aH \otimes F_bH'. \]

Therefore both the coproduct on \( \mathcal{O}(G) \) dual to the group law on \( G \), and the coaction of \( \mathcal{O}(G) \) on \( \mathcal{O}(\Pi) \) dual to the action of \( G \) on \( \Pi \), respect the filtrations \( F \).

**Definition 6.3.** Define \( Z_{\text{alg}, F}(G, \Pi)(R) \) to be the set of algebraic cocycles \( c : G \times R \to \Pi \times R \) such that \( c^* : \mathcal{O}(\Pi) \otimes R \to \mathcal{O}(G) \otimes R \) respects the filtrations \( F \).

It defines a functor from commutative unitary \( k \)-algebras to pointed sets.

**Proposition 6.4.** Let \( k \) be a field and suppose that

\[(6.3) \dim_k F_n \mathcal{O}(G) < \infty. \]

Then \( Z^1_{\text{alg}, F}(G, \Pi) \) is representable, and hence defines an affine scheme over \( k \).

**Proof.** Suppose that \( A \) is an affine group scheme, and that \( \mathcal{O}(A) \) is equipped with an increasing filtration \( F \) as above. Let \( \text{Hom}_F(G, A) \) denote the functor whose \( R \)-valued points, for \( R \) a commutative unitary \( k \)-algebra, are \( R \)-linear morphisms of Hopf algebras

\[ \phi : \mathcal{O}(A) \otimes_k R \to \mathcal{O}(G) \otimes_k R \]

such that \( \phi \) respects \( F \). Corollary \([5.10]\) implies that the functor of \( k \)-linear maps \( \text{Hom}_F(\mathcal{O}(A), \mathcal{O}(G)) \) is an affine scheme over \( k \), by \([5.3]\). The subspace

\[ \text{Hom}_F(G, A)(R) \subset \text{Hom}_F(\mathcal{O}(A), \mathcal{O}(G))(R) \]

consists of linear maps satisfying algebraic equations including:

\[ \phi(xy) = \phi(x)\phi(y) \quad \text{and} \quad \Delta(\phi(x)) = (\phi \otimes \phi)\Delta(x) \]

for all \( x, y \in \mathcal{O}(A) \). There are additionally antipode, unit and counit conditions which we omit. Since these are all algebraic and defined over \( k \), it follows that \( \text{Hom}_F(A, G) \) is a closed subscheme of \( \text{Hom}_F(\mathcal{O}(A), \mathcal{O}(G)) \).

Now apply this remark in the cases

\[ A = G \quad \text{and} \quad A = \Pi \times G \]

where the semi-direct product on the right is given by the left action \([6.2]\) of \( G \) on \( \Pi \). There is an isomorphism of algebras \( \mathcal{O}(\Pi \times G) \cong \mathcal{O}(\Pi) \otimes \mathcal{O}(G) \) which endows \( \mathcal{O}(\Pi \times G) \) with a filtration \( F \) induced from that of \( \mathcal{O}(\Pi) \) and \( \mathcal{O}(G) \). It follows that \( \text{Hom}_F(G, \Pi) \) and \( \text{Hom}_F(G, \Pi \times G) \) are both affine schemes. Since these scheme structures are natural, the morphism \( \Pi \times G \to G \) induces a morphism of affine schemes

\[ \text{Hom}_F(G, \Pi \times G) \to \text{Hom}_F(G, G) \]

and its kernel (inverse image of the identity) is also a scheme. Via \([6.1]\), the kernel can be identified, as a functor, with \( Z^1_{\text{alg}, F}(G, \Pi) \), which is therefore representable. \( \square \)
The proof shows that $Z^1_{\text{alg,}F}(G, \Pi)$ is a closed subscheme of $\text{Hom}_F(\mathcal{O}(\Pi), \mathcal{O}(G))$. By Yoneda’s lemma, the natural transformation of functors

$$Z^1_{\text{alg,}F}(G, \Pi) \times G \rightarrow \Pi$$

given on $R$-points by $(c, g) \mapsto c_g$ is a morphism of schemes. It is functorial in $G, \Pi$: given affine group schemes $G' \rightarrow G$ and $\Pi \rightarrow \Pi'$ whose affine rings are equipped with filtrations in the manner described above, the natural maps $Z^1_{\text{alg,}F}(G', \Pi) \rightarrow Z^1_{\text{alg,}F}(G, \Pi')$ and $Z^1_{\text{alg,}F}(G, \Pi) \rightarrow Z^1_{\text{alg,}F}(G', \Pi)$ are morphisms of schemes.

**Example 6.5.** Let $U$ be a pro-unipotent affine group scheme. Its affine ring $\mathcal{O}(U)$ is equipped with a canonical filtration $C_n\mathcal{O}(U)$ called the coradical filtration. It is defined by $C_{-1}\mathcal{O}(U) = 0$, $C_0\mathcal{O}(U) = k$, and for $n \geq 1$

$$x \in C_n\mathcal{O}(U)_+ \iff (\Delta')^nx = 0$$

where $\Delta'$ is the reduced coproduct on $\mathcal{O}(U)$ defined by $\Delta' = \Delta - \text{id} \otimes 1 - 1 \otimes \text{id}$, and $\mathcal{O}(U)_+$ is the kernel of the augmentation $\varepsilon : \mathcal{O}(U) \rightarrow k$. Since $U$ is connected, we have $\mathcal{O}(U) = \mathcal{O}(U)_+ \oplus k$. The space $C_i\mathcal{O}(U)_+$ is the space of primitive elements of $\mathcal{O}(U)$. Any morphism $U \rightarrow U'$ of pro-unipotent affine group schemes necessarily respects the coradical filtration since it is defined purely in terms of the coproducts.

Since the kernel of $\Delta'$ is $C_1\mathcal{O}(U)_+$, it follows from the equation

$$\Delta'C_n \subset \sum_{1 \leq i \leq n-1} C_i \otimes C_{n-i}$$

and induction on $n$ that

$$\dim C_1\mathcal{O}(U) < \infty \iff \dim C_n\mathcal{O}(U) < \infty \text{ for all } n.$$ 

Note that the space of primitive elements $C_1\mathcal{O}(U)_+$ is isomorphic to $H^1(U; k)$.

**Corollary 6.6.** Let $U, \Pi$ be pro-unipotent affine group schemes with $\dim H^1(U; k) < \infty$. Then the functor of morphisms $\text{Hom}(U, \Pi)$ is an affine scheme.

If $G$ and $\Pi$ are both pro-unipotent and $\dim H^1(G; k) < \infty$ then $Z^1_{\text{alg}}(G, \Pi)$ is an affine scheme.

6.5. **Torsors.** Now let $\Pi_x$ be a right $\Pi$-torsor with $G$-action. This means that $\Pi_x$ is an affine scheme over $k$, equipped with a morphism

$$(q, p) \mapsto q.p : \Pi_x \times \Pi \rightarrow \Pi_x$$

which defines a right action of $\Pi$ on $\Pi_x$, i.e., $q.(pq') = (qp').p'$. Furthermore, $\Pi_x, \Pi$ both admit left actions by $G$, i.e., morphisms

$$G \times \Pi_x \rightarrow \Pi_x \quad \text{and} \quad G \times \Pi \rightarrow \Pi$$

which define group actions, which are compatible with the group law on $\Pi$ and right action of $\Pi$ on $\Pi_x$. All this can be translated into commutative diagrams, which we omit. The torsor condition is the property that

$$(6.4) \quad (q, p) \mapsto (q, qp) : \Pi_x \times \Pi \rightarrow \Pi_x \times \Pi_x$$

is an isomorphism of affine schemes over $k$. Finally, one requires the existence of a point on $\Pi_x(R)$ for some $R$ faithfully flat extension of $k$.

Since $k$ is a field, let us suppose that we are given a point $1_x \in \Pi_x(k')$ for some extension $k'$ of $k$. This data defines an algebraic cocycle over $k'$ as follows. For all $g \in G(R)$, for $R$ a commutative unitary $k'$ algebra, it follows from (6.4) that there exists a unique $\alpha_g \in \Pi(R)$ such that

$$g(1_x) = 1_x \cdot \alpha_g.$$
This defines a map \( \alpha : G(R) \to \Pi(R) \). The cocycle condition follows from the uniqueness of \( \alpha \), and the equations \( gh(1_x) = 1_x \cdot \alpha_{gh} \) and
\[
g(h(1_x)) = g(1_x \cdot \alpha_h) = 1_x \cdot \alpha_g \alpha_k \, ,
\]
which imply that \( \alpha_{gh} = \alpha_g \alpha_k \). That \( \alpha \) is algebraic can be seen as follows. Denote by \( G', \Pi', \Pi'_x \) the extension of scalars of \( G, \Pi, \Pi_x \) to \( k' \). Restricting the base change of \( \alpha \) to \( \text{Spec} \ k' \times \Pi' \subset \Pi'_x \times \Pi' \) and projecting onto \( \Pi'_x \) defines an isomorphism
\[
1_x : \Pi' \to \Pi'_x \, .
\]
It is given on points by \( p \mapsto 1_x.p \). Then \( \alpha \) comes from the morphism of schemes
\[
G' \longrightarrow \Pi'_x \overset{(1_x)^{-1}}{\longrightarrow} \Pi'
\]
where the first map is given by the action on \( 1_x \), i.e., \( g \mapsto g(1_x) \). In conclusion:
\[
\alpha \in \text{Z}_{\text{alg}}^1(G, \Pi)(k') \, .
\]
This cocycle can be viewed in three different ways (6.5.3):

1. as the morphism \( \alpha : G' \to \Pi' \) as constructed above.
2. as an algebra homomorphism \( \mathcal{O}(\Pi') \to \mathcal{O}(G') \) given by
\[
\mathcal{O}(\Pi') \overset{\text{(1.5).1}}{\longrightarrow} \mathcal{O}(\Pi_x) \overset{\Delta}{\longrightarrow} \mathcal{O}(\Pi'_x) \otimes_{k'} \mathcal{O}(G') \overset{1_x \otimes \text{id}}{\longrightarrow} \mathcal{O}(G')
\]
where \( \Delta \) is the coaction dual to the action of \( G' \) on \( \Pi'_x \).
3. Setting \( R = \mathcal{O}(G') \) in (1) gives a map \( \alpha : G'(\mathcal{O}(G')) \to \Pi'(\mathcal{O}(G')) \). The image of the identity element is a point \( \alpha(\text{id}) \in \Pi'(\mathcal{O}(G')) \).

Changing the point \( 1_x \) has the effect of modifying the cocycle by a coboundary. Let \( 1'_x \in \Pi_x(k') \) be another point and \( \alpha' \) the associated cocycle. Then there exists a unique \( b \in \Pi(k') \) such that \( 1'_x = 1_x.b \), and
\[
\alpha'_g = b^{-1} \alpha_g(b) \, .
\]

6.6. **Weight filtrations.** In our examples, the affine rings \( \mathcal{O}(\Pi_x), \mathcal{O}(\Pi), \mathcal{O}(G) \) are equipped with increasing exhaustive filtrations \( W \), such that \( W_{-1} = 0 \). These filtrations are compatible in that the coactions satisfy:
\[
\Delta : W_n \mathcal{O}(\Pi) \longrightarrow W_n(\mathcal{O}(\Pi_x) \otimes \mathcal{O}(\Pi))
\]
\[
\Delta : W_n \mathcal{O}(\Pi_x) \longrightarrow W_n(\mathcal{O}(\Pi_x) \otimes \mathcal{O}(G))
\]
and similarly in the second equation with \( \Pi_x \) replaced by \( \Pi \). Furthermore, for each of these three affine rings, the filtration \( W \) is compatible with the algebra structure:
\[
W_{m.n} \subset W_{m+n} \, .
\]
Finally, the morphisms considered above are strict with respect to the filtration \( W \). Let \( 1_x \in \Pi_x(k') \). Then repeating the argument of (6.5) we find that
\[
\mathcal{O}(\Pi_x) \overset{\Delta}{\longrightarrow} \mathcal{O}(\Pi_x) \otimes \mathcal{O}(\Pi) \overset{1_x \otimes \text{id}}{\longrightarrow} k' \otimes \mathcal{O}(\Pi)
\]
preserves the filtration \( W \) and induces an isomorphism \( k' \otimes \mathcal{O}(\Pi_x) \cong k' \otimes \mathcal{O}(\Pi) \). By strictness, it defines an isomorphism for every \( n \):
\[
1_x : k' \otimes W_n \mathcal{O}(\Pi_x) \longrightarrow k' \otimes W_n \mathcal{O}(\Pi) \, .
\]
Writing \( \Pi' = \Pi \times_k k' \), and \( G' = G \times_k k' \), consider the chain of homomorphisms
\[
W_n \mathcal{O}(\Pi') \overset{1_x \otimes \text{id}}{\longrightarrow} W_n \mathcal{O}(\Pi'_x) \overset{\Delta}{\longrightarrow} W_n(\mathcal{O}(\Pi'_x) \otimes_{k'} \mathcal{O}(G)) \overset{1_x \otimes \text{id}}{\longrightarrow} W_n \mathcal{O}(G)
\]
and we deduce that the cocycle associated to the point \( 1_x \) preserves \( W \), i.e., \( \alpha \in \text{Z}_{\text{alg}}^1(G, \Pi)(k') \). The point of this construction is that if \( k \) is a field and \( W_n \mathcal{O}(G) \)
is of finite dimension for all $n$, $Z_{\text{alg}}^1 W(G, \Pi)$ will define a scheme by proposition 6.4. By replacing $G$ with the quotient through which it acts on $\Pi$, we can always assume that $G$ acts faithfully on $\Pi$. With this in mind, our criterion for representability will be satisfied in all the examples we shall need to consider.

Remark 6.7. The approach to proving representability of a functor of cocycles in [24] is different, and involves an inductive argument via long exact sequences. See also [21].

6.7. Determinant equation. Now let us suppose, as above, that $Z = Z_{\text{alg}}^1 F(G, \Pi)$ is a scheme, for some filtration $F$. Then we can apply the formalism of §5.4. Let us spell this out explicitly in this situation. We have a morphism of schemes

$$(c, g) \mapsto c \times G \to \Pi$$

whose dual is the morphism of algebras (which we again denote by $\text{co}$)

$$\text{co} : \mathcal{O}(\Pi) \to \mathcal{O}(G) \otimes_k \mathcal{O}(Z)$$

This uses, of course, the fact that $Z$ is a scheme. Given an $k'$-valued cocycle $\phi \in Z(k')$, for some commutative unitary $k$-algebra $k'$, the morphism of algebras

$$\phi : \mathcal{O}(\Pi) \to \mathcal{O}(G) \otimes_k k'$$

is obtained by composing (6.5) with $\phi : \mathcal{O}(Z) \to k'$, namely

$$\phi(v) = (\text{id} \otimes \phi) \text{co}(v).$$

Proceeding in the same manner as §5.4, (6.5) extends to an $\mathcal{O}(G)$-linear map

$$\text{co} : \mathcal{O}(G) \otimes_k \mathcal{O}(\Pi) \to \mathcal{O}(G) \otimes_k \mathcal{O}(Z)$$

and by taking exterior powers, gives an $\mathcal{O}(G)$-linear map

$$\bigwedge^N \text{co} : \mathcal{O}(G) \otimes_k \bigwedge^N \mathcal{O}(\Pi) \to \mathcal{O}(G) \otimes_k \bigwedge^N \mathcal{O}(Z).$$

In the same manner as lemma 5.2, we deduce the following proposition.

Proposition 6.8. Suppose that $N = \dim_k \mathcal{O}(Z) + 1$ is finite, and let $k'$ be a commutative unitary $k$-algebra. Given any set of $N$ cocycles

$$\phi_1, \ldots, \phi_N \in Z^1(G, \Pi)(k')$$

the following determinant equation holds:

$$\det(\phi_i(v_j))_{1 \leq i, j \leq N} = 0$$

for all $v_1, \ldots, v_N \in \mathcal{O}(\Pi)$, where $\phi_i(v_j)$ is defined via (6.6). Remark 6.9. Our scheme-theoretic formulation allows us to take $k'$ to be a ring other than $k$. It will turn out that integral points on curves will only give rise to $k$-valued cocycles. Considering a larger ring $k'$ gives considerable flexibility to the method and enables us to consider ‘virtual cocycles’ constructed out of divisors §10.

7. Main argument : mixed Tate case

Throughout we shall argue with $R$-points on affine group schemes without specifying the ring $R$. All tensor products are over $k$ unless specified otherwise. We compose paths in the functional sense, and let Tannaka groups act on fundamental groups on the left. Reversing both conventions simultaneously still leads to agreeable formulae, but changing one or other convention on its own does not.

At the risk of repeating some of the arguments in [11] we consider only the mixed Tate case in this section, since this is one of the few situations in which the results are known to hold unconditionally.
7.1. Points and cocycles. Let $\mathcal{O}_S$ be the ring of $S$-integers in a totally real number field $k$, and let $(X, x, y)$ be such that its motivic fundamental group is mixed Tate unramified over $S$. Let us fix a possibly tangential base point $0 \in X(\mathcal{O}_S)$. For any other (tangential) point $x \in X_S$, let $\pi_0^x$ denote the fundamental torsor of paths from 0 to $x$ with respect to the canonical fiber functor. Via the right-torsor structure \[ \pi_0^x \times \pi_0^x \rightarrow \pi_0^x \]
the canonical path $x_0 \in \pi_0^x(\mathbb{Q})$ defines an algebraic cocycle [6.3]
\[ c^x \in Z^1_{alg}(G^0_{\mathbb{S}}, \pi_0^x)(\mathbb{Q}) \]
Now consider any quotient $\pi_0^x \rightarrow \Pi$ in the category $\mathcal{M}\mathcal{T}(\mathcal{O}_S)$, and denote its de Rham realisation by $\Pi^\omega$. Equivalently, the affine ring $\mathcal{O}(\Pi^\omega)$ is a Hopf subalgebra of $\mathcal{O}(\pi_0^x)$ and is stable under the right $\mathcal{O}(G^{\omega}_{\mathbb{S}})$-coaction. Via the natural transformation \[ Z^1_{alg}(G^{\omega}_{\mathbb{S}}, \pi_0^x) \rightarrow Z^1_{alg}(G^{\omega}_{\mathbb{S}}, \Pi^\omega) \]
we obtain an algebraic cocycle:
\[ c^x \in Z^1_{alg}(G^0_{\mathbb{S}}, \Pi^\omega)(\mathbb{Q}) \]

7.2. Weight filtration. The fundamental groupoids $\pi_0^x, \pi_0^0, \Pi$ are equipped with weight filtrations satisfying the conditions of [6.6]. The same is true of the affine ring $\mathcal{O}(G^{\omega}_{\mathbb{S}})$, which satisfies \[ W_{-1} \mathcal{O}(G^{\omega}_{\mathbb{S}}) = 0 \text{ and } \dim W_n(\mathcal{O}(G^{\omega}_{\mathbb{S}})) < \infty \]
for all $n$. The first property was proved in [6], [3.4] (note that this is only true if the weight filtration is defined, via the Tannakian formalism, with respect to the conjugation action of $G^{\omega}_{\mathbb{S}}$ on itself) and the second follows from the fact that $S$ is finite. It follows from [6.6] that the cocycle $c^x$ satisfies \[ c^x \in Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi^\omega)(\mathbb{Q}) \]
and furthermore $Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi^\omega)$ is an affine scheme over $\mathbb{Q}$.

7.3. Weight grading. We can be more precise still using the fact that the weight filtration actually splits in the canonical realisation.

The subgroup $G_m \leq G^{\omega}_{\mathbb{S}}$ acts trivially on the path $x_0 \in [6.8]$. Since the cocycle $c^x$ is defined (on points) by $g(x_0) = x_0 \times c^x$, it follows that $c^x = 1$ for all $g \in G_m$.

Denote the space of left $G_m$-cocycles trivial on $G_m$ by \[ Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi) = \ker \left( Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi) \rightarrow Z^1_{alg, W}(G_m, \Pi) \right) \]
By proposition [6.3] this is an affine scheme over $\mathbb{Q}$, since the map on the right is a morphism of affine schemes over $\mathbb{Q}$ (as $W_n \mathcal{O}(G_m)$ is finite-dimensional).

Define $Z^1_{alg, W}(U^\omega_{\mathbb{S}}, \Pi)^{G_m}$ to be the closed subscheme of $Z^1_{alg, W}(U^\omega_{\mathbb{S}}, \Pi)$ whose points consists of cocycles satisfying the equation
\[ c_{gug^{-1}} = g(c_u) \text{ for } g \in G_m, u \in U^\omega_{\mathbb{S}}. \]
Then restriction to $U^\omega_{\mathbb{S}}$ defines an isomorphism of schemes
\[ Z^1_{alg, W}(U^\omega_{\mathbb{S}}, \Pi)^{G_m} \cong Z^1_{alg, W}(U^\omega_{\mathbb{S}}, \Pi)^{G_m}. \]
Indeed, if $c_g = 1$ for all $g \in G_m$ then the cocycle equation $c_{ph} = c_g(c_h)$ implies that (7.3) holds. Conversely, any $U^\omega_{\mathbb{S}}$-cocycle $c$ satisfying (7.3) defines an element $c' \in Z^1_{alg, W}(G^0_{\mathbb{S}}, \Pi)$ by setting $c'_{ug} = c_u$ for $u \in U^\omega_{\mathbb{S}}, g \in G_m$. 

\[
\begin{align*}
\pi_0^x & \times \pi_0^x \rightarrow \pi_0^x \\
c^x & \in Z^1_{alg}(G^0_{\mathbb{S}}, \pi_0^x)(\mathbb{Q}) \\
c^x \in Z^1_{alg}(G^0_{\mathbb{S}}, \pi_0^x)(\mathbb{Q}) \\
c^x \in Z^1_{alg}(G^0_{\mathbb{S}}, \pi_0^x)(\mathbb{Q}) \\
c^x \in Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi^\omega)(\mathbb{Q}) \\
c^x \in Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi^\omega)(\mathbb{Q}) \\
c^x \in Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi^\omega)(\mathbb{Q}) \\
c^x \in Z^1_{alg, W}(G^{\omega}_{\mathbb{S}}, \Pi^\omega)(\mathbb{Q}) \\
c_{gug^{-1}} = g(c_u) \text{ for } g \in G_m, u \in U^\omega_{\mathbb{S}}. \\
Z^1_{alg, W}(U^\omega_{\mathbb{S}}, \Pi)^{G_m} \cong Z^1_{alg, W}(U^\omega_{\mathbb{S}}, \Pi)^{G_m}. \\
\end{align*}
\]
7.4. Interpretation as a homomorphism. We can think of \( c^x \) as an algebra homomorphism \( c^x : \mathcal{O}(\Pi) \rightarrow \mathcal{O}(G^s_S) \). Its restriction to \( U^+_S \) defines a homomorphism

\[
(7.4) \quad c^x : \mathcal{O}(\Pi) \longrightarrow \mathcal{O}(U^+_S) = \mathcal{P}^u_S.
\]

Equation (7.4) is equivalent to the fact that \( c^x \) is \( G_m \)-equivariant, where \( G_m \) acts via conjugation on \( \mathcal{O}(U^+_S) \). In other words, (7.4) respects the weight-gradings on both sides. Explicitly, the canonical de Rham paths define isomorphisms:

\[
(7.5) \quad \omega_0 \Pi^c_0 \cong x \Pi^c_0 \quad \text{and} \quad \mathcal{O}(\omega_0 \Pi^c_0) \cong \mathcal{O}(x \Pi^c_0).
\]

The homomorphism (7.4) can then be defined by the composition of maps

\[
(7.6) \quad c^x : \mathcal{O}(\Pi) \subset \mathcal{O}(\omega_0 \Pi^c_0) \cong \mathcal{O}(x \Pi^c_0) \xrightarrow{\Delta} \mathcal{O}(x \Pi^c_0) \otimes \mathcal{O}(U^+_S) \xrightarrow{\otimes \text{Id}} \mathcal{O}(U^+_S),
\]

where \( \Delta \) is the coaction dual to the left action of \( U^+_S \) on \( x \Pi^c_0 \).

7.5. Interpretation via motivic iterated integrals. We can also think of \( c^x \) as a \( \mathcal{P}^u_S \)-valued point on \( \Pi \), i.e., \( c \in \Pi(\mathcal{P}^u_S) \).

Indeed, the map \( \Delta \) in the previous subsection is nothing other than the restriction to \( \mathcal{O}(x \Pi^c_0) \) of the (unipotent) universal comparison isomorphism

\[
(7.7) \quad \mathcal{O}(x \Pi^c_0) \otimes \mathcal{P}^u_S \xrightarrow{\sim} \mathcal{O}(x \Pi^c_0) \otimes \mathcal{P}^u_S.
\]

It follows that the composition

\[
\mathcal{O}(x \Pi^c_0) \xrightarrow{\Delta} \mathcal{O}(x \Pi^c_0) \otimes \mathcal{P}^u_S \xrightarrow{\otimes \text{Id}} \mathcal{P}^u_S
\]

is given for any \( w \in \mathcal{O}(x \Pi^c_0) \) as a matrix coefficient \( w \mapsto [\mathcal{O}(x \Pi^c_0), w, x 1_0]^u \).

**Definition 7.1.** For any \( w \in \mathcal{O}(\omega_0 \Pi^c_0) \) let us denote by

\[
(7.8) \quad I^u(x; w; 0) = [\mathcal{O}(\omega_0 \Pi^c_0), w, x 1_0]^u \quad \text{where} \quad w \text{ is viewed in } \mathcal{O}(\omega_0 \Pi^c_0) \text{ via the isomorphism (7.7).}
\]

It could be called a ‘unipotent de Rham motivic iterated integral’ of \( w \) from 0 to \( x \).

It follows that \( c^x \in \Pi(\mathcal{P}^u_S) \) is the homomorphism

\[
(7.9) \quad \mathcal{O}(\Pi) \longrightarrow \mathcal{P}^u_S \quad \quad w \quad \mapsto \quad I^u(x; w; 0).
\]

Viewed yet another way, \( c^x \) is the image of the point \( x 1_0 \) under the composition

\[
(7.10) \quad x \Pi^c_0(\mathcal{P}^u_S) \xrightarrow{\sim} x \Pi^c_0(\mathcal{P}^u_S) \xrightarrow{\otimes \text{Id}} \omega_0 \Pi^c_0(\mathcal{P}^u) \longrightarrow \Pi(\mathcal{P}^u_S)
\]

where the first isomorphism is the (unipotent) universal comparison isomorphism. The reason we work with unipotent instead of de Rham periods is because the unipotent version of \( 2 \pi i \) is trivial, which is not true for its de Rham version.

7.6. Determinant. Let us denote \( Z^{1}_{G_m}(G^s_S, \Pi^c) \) simply by \( Z \). It is a scheme. Its affine ring is graded by the action of \( G_m \) induced by the left action of \( G_m \) on \( Z \) given by \( h \mapsto gh : G^s_S \rightarrow G^s_S \) for \( g \in G_m \).

**Theorem 7.2.** Let \( n \geq 0 \). Let us write \( N = \dim \mathcal{O}(Z)_{2n} \) and suppose that

\[
\dim \mathfrak{g}_2^{W} \mathcal{O}(\Pi^c) > N.
\]

Let \( w_1, \ldots, w_N \) be (linearly independent) elements in \( \mathfrak{g}_2^{W} \mathcal{O}(\Pi^c) \). Then any \( N \) points \( x_1, \ldots, x_N \in X^S \) satisfy the equation

\[
(7.11) \quad \det \left( I^u(x_{ij}; w_i; 0) \right)_{1 \leq i, j \leq N} = 0.
\]
Proof. This is the weight-graded version of proposition 6.8. The map
\[ \text{co} : O(\Pi^\omega) \rightarrow \mathcal{P}_S^u \otimes O(Z) \]
where \( u \mapsto w(c_u) \in \mathcal{P}_S^u \) is viewed as a morphism from \( U_S^Z \) to \( \mathbb{A}^1 \).
Since \( O(\Pi^\omega), \mathcal{P}_S^u \) are graded by weight and have even degrees, and cocycles in \( Z \) respect the gradings,
the definition in (5.6) provides a grading \( O(Z) = \bigoplus_n O(Z)_{2n} \) on \( O(Z) \) such that
\[ \text{co} : \text{gr}_{2n}^W O(\Pi^\omega) \rightarrow \text{gr}_{2n}^W (\mathcal{P}_S^u) \otimes O(Z)_{2n} \, . \]
The grading on \( O(Z) \) is induced then, either by the action of \( \mathbb{G}_m \) on \( O(\Pi^\omega) \), or the
conjugation action of \( \mathbb{G}_m \) on \( U_S^Z \). These coincide by (7.3). The conjugation action of
\( \mathbb{G}_m \) on \( U_S^w \) is equivalent to the left action of \( \mathbb{G}_m \) on \( U_S^w \times \mathbb{G}_m = U_S^w \). Every cocycle
7.8 factorizes through this map. By the argument of proposition 6.8 we deduce that
\[ \det \left( w_i(e^{x_j}) \right)_{1 \leq i,j \leq N} = 0 \, . \]
By definition 7.1 the coefficient of \( w_i(e^{x_j}) \) is precisely \( I^u(x_j; w_i; 0) \).

7.7. Discussion.

(1) The hypothesis \( \dim \text{gr}_{2n}^W O(\Pi^\omega) > \dim O(Z)_{2n} \) of the theorem provides, via
remark 5.8 the existence of a non-explicit function
\[ \sum p_i I^u(x; w_i; 0) = 0 \]
for all \( x \in X_S \), where \( p_i \in \mathcal{P}_S^u \) and \( w_i \in O(\Pi) \) are of weight \( 2n \). Since it is
not known how to construct the elements in \( \mathcal{P}_S^u \) except in very special cases, this approach seems to be impractical. An added complication is that the
injectivity of the \( p \)-adic period homomorphism on \( \mathcal{P}_S^u \) is completely open, so it is not known that the elements \( p_i \) have non-vanishing \( p \)-adic periods. There
are multiple reasons, therefore, for preferring the explicit approach via (7.8).

(2) Equation (7.8) is a ‘motivic’ Coleman function which vanishes on \( X^N_S \). We can
retrieve a function vanishing on \( X_S \) as follows. Suppose there exist \( N - 1 \)
points \( x_1, \ldots, x_{N-1} \) on \( X_S \). Write \( x_N = x \). A row expansion of (7.8) yields
\[ \sum_{i=1}^{N-1} p_{w_i} I^u(x; w_i; 0) = 0 \, , \]
which holds for all integral points \( x \in X_S \). The unipotent period \( p_{w_i} \in \mathcal{P}_S^u \)
is the determinant of the minor of \( (I^u(x_j; w_i; 0))_{1 \leq j \leq N} \) obtained by deleting
row \( N \) and column \( i \). The equation (7.8) could be trivial if every \( p_{w_i} \) were to
vanish. But in that case, the equation \( p_{w_N} = 0 \) is equivalent to an equation
\[ \det \left( I^u(x_j; w_i; 0) \right)_{1 \leq j \leq N-1} = 0 \]
which vanishes for every set of \( N - 1 \) points \( x_1, \ldots, x_{N-1} \in X_S \). Continuing
in this way, we reach the following conclusion.

Corollary 7.3. Either \( |X_S| < N \) or every point \( x \in X_S \) satisfies an equation
of the form (7.9) where not all coefficients \( p_{w_i} \in \mathcal{P}_S^u \) are identically zero.

(3) Let \( p < \infty \) be prime not in \( S \). Denote the \( p \)-adic period of a motivic unipotent
de Rham motivic iterated integral (definition 7.1) by
\[ I^p(x; w; 0) = \text{per}_p \left( I^u(x; w; 0) \right) \, . \]
Since \( I^u(x; w; 0) \) is the unipotent de Rham period \( [O(x; \Pi^\omega_0), w, 1]_0 ]^u \), we have
\[ I^p(x; w; 0) = z_{10}(F_p w) \]
where $\overline{F}_p$ is the normalised Frobenius acting on $\mathcal{O}(\sum^{n}_{i} \omega) \otimes_{\mathbb{Q}} \mathbb{Q}_p$. It is the single-valued $p$-adic iterated integral of $w$ from 0 to $x$.

**Corollary 7.4.** Assume the above set-up, and the conditions of theorem 7.2. For any set of $N$ points $x_1, \ldots, x_N \in X_S$, we have

\begin{equation}
\det \left( I^p(x_j; w_i; 0) \right)_{1 \leq i,j \leq N} = 0.
\end{equation}

In particular, if $|X_S| \geq N$ then there exists a non-trivial $p$-adic analytic function on $X(\mathbb{Q}_p)$ which vanishes on the image of $X_S$.

**Proof.** By the same minimality argument as (2), it is enough to show that the single-valued $p$-adic iterated integral $x \mapsto I^p(x; w; 0)$ is a non-zero $p$-adic analytic function. In fact, they are linearly independent for linearly independent $w$, which follows from the differential equation ($p$-adic KZ equation) they satisfy \((\ref{LinearIndependence}), \S 3.2)\.

(4) Under the conditions of the previous corollary, the set $|X_S|$ is finite when $X$ is a curve. Even in this case, the conditions for the motivic fundamental groupoid of $X$ to be mixed Tate are highly restrictive: for example $X = \mathbb{F}^1 \setminus \Sigma$ where $\Sigma$ is a finite non-empty set of points unramified outside $S$. Note also that \((\ref{DimensionFormula})\) does not depend on $S$, but only its cardinality $|S|$.

8. Dimension Formulae

We now compute some formulae for the dimensions of the weight-graded pieces of $\mathcal{O}(\Pi)$ and $\mathcal{O}(Z)$ in theorem \((\ref{DimensionFormula}), \S 7.2) via two different methods.

**8.1. Upper bound on the space of cocycles.** The first method is via the Hopf algebra interpretation \((\ref{HopfAlg}) \S 2) of cocycles. Let $G_S^\omega, \Pi^\omega$ be as in theorem \((\ref{DimensionFormula}), \S 7.2) and let $c \in Z = Z^1_{G_S^\omega}(G_S^\omega, \Pi^\omega)$. It defines an algebra homomorphism

$$c : \mathcal{O}(\Pi) \rightarrow \mathcal{O}(U_S^\omega) = \mathcal{P}^u_S$$

which respects the weight gradings. Since $c$ is multiplicative, it is determined by its value on a choice of representatives $Q \mathcal{O}(\Pi^\omega)$ of indecomposable elements in $\mathcal{O}(\Pi^\omega)$. Let $\Delta' : \mathcal{P}^u_S \rightarrow \mathcal{P}^u_S \otimes_{\mathbb{Q}} \mathcal{P}^u_S$ denote the reduced coproduct $\Delta' = \Delta - \text{id} \otimes 1 - 1 \otimes \text{id}$, where $\Delta$ is dual to the multiplication in $U_S^\omega$. The space of primitive elements in $P^u_S$ is

$$\text{Prim} \mathcal{P}^u_S = \ker \Delta'.$$

Let $w \in \text{gr}^W_S \mathcal{O}(\Pi^\omega)$. The value of $\Delta' c(w)$ is completely determined, via the cocycle condition, by $c(w')$ for $w'$ of weight $< n$. The indeterminacy of $c(w)$ is therefore given by an element of $\text{gr}^W_S \text{Prim} \mathcal{P}^u_S$. Equivalently, for two such homomorphisms $c_1, c_2$ which agree on $W_{n-1} \mathcal{O}(\Pi^\omega)$ and satisfy $\Delta' c_1(w) = \Delta' c_2(w)$, the difference $c_1 - c_2$, restricted to $\text{gr}^W_S \mathcal{O}(\Pi^\omega)$, defines an element of

$$\text{Hom}(\text{gr}^W_S \mathcal{O}(\Pi^\omega), \text{gr}^W_S \text{Prim} \mathcal{P}^u_S)$$

where Hom denotes linear maps of vector spaces. Since the affine ring of the scheme of homomorphisms is the symmetric algebra on its dual \((\ref{SymmetricAlgebra}), \S 5.2)\ it follows that $\mathcal{O}(Z)$ has at most

$$\left( \dim \text{gr}^W_S \mathcal{O}(\Pi^\omega) \right) \times \left( \dim \text{gr}^W_S \text{Prim} \mathcal{P}^u_S \right)$$

algebra generators in degree $n$. 

8.2. Poincaré series. Since all objects are graded with weights only in even degrees, we shall hereafter divide the weights by two.

**Lemma 8.1.** Let \( \ell_m = \dim \mathcal{O}^{gr}_{2m} \mathcal{O}(\Pi^\omega) \) be the dimension of the space of algebra generators of \( \mathcal{O}(\Pi^\omega) \) in weight \( 2m \). Then

\[
(8.1) \quad \dim \mathcal{O}^{gr}_{2m} \mathcal{O}(\Pi^\omega) = \text{coeff. of } t^n \text{ in } \prod_{m \geq 1} \frac{1}{(1-t^m)^{\ell_m}}.
\]

**Proof.** The ring \( \mathcal{O}(\Pi^\omega) \) is a commutative graded Hopf algebra. By the Milnor-Moore theorem it is isomorphic to the graded polynomial algebra on its generators. \( \square \)

For all \( n \geq 1 \) we have

\[
\text{gr}^{W}_{2n} \text{Prim } \mathcal{P}^{S}_{\mathbb{P}} \cong \text{gr}^{W}_{2n} H^{1}(U^{\omega}_{S}, \mathbb{Q}) \cong \text{Ext}^{1}_{\mathcal{M}^{T}(\mathcal{O}_{S})}(\mathbb{Q}(0), \mathbb{Q}(-n)).
\]

by, for example, §6. The dimensions of the right-hand vector space are given by Borel’s theorem \( (3.4) \). Therefore \( \text{gr}^{W}_{2n} \text{Prim } \mathcal{P}^{S}_{\mathbb{P}} = |S| \). For \( m \geq 2 \), write

\[
r(m) = \text{gr}^{W}_{2m} \text{Prim } \mathcal{P}^{S}_{\mathbb{P}} = \begin{cases} r_1 + r_2 & \text{if } m \text{ odd} \\ r_2 & \text{if } m \text{ even} \end{cases},
\]

where \( r_1 \) (resp. \( r_2 \)) denotes the number of real (resp. complex) embeddings of \( k \).

8.3. By the discussion of \( (8.1) \) we deduce that \( \mathcal{O}(Z) \) has at most \( r(m) \ell_m \) algebra generators in weight \( 2m \), and therefore

\[
(8.2) \quad \dim \mathcal{O}(Z)_{2n} \leq \text{coeff. of } t^n \text{ in } \frac{1}{(1-t)^{|S|\ell_1}} \prod_{m \geq 2} \frac{1}{(1-t^m)^{r(m)\ell_m}}.
\]

It turns out that this inequality \( (8.2) \) is in fact an equality, essentially because \( U^{\omega}_{S} \) has no \( H^2 \). In any case, a sufficient condition for theorem \( (7.2) \) to apply is if, for some \( n \),

\[
\left( \text{coeff. of } t^n \text{ in } \prod_{m \geq 1} \frac{1}{(1-t^m)^{\ell_m}} \right) \geq \left( \text{coeff. of } t^n \text{ in } \frac{1}{(1-t)^{|S|\ell_1}} \prod_{m \geq 2} \frac{1}{(1-t^m)^{r(m)\ell_m}} \right)
\]

This clearly cannot happen if all \( r(m) \geq 1 \). Therefore we must assume that \( r_2 = 0 \), i.e., that \( k \) is totally real. When the previous inequality is satisfied, theorem \( (7.2) \) can be applied. The quantity \( N \) is the integer in the right-hand side of the inequality.

8.4. Nowhere have we assumed that the scheme \( Z^{1}_{\mathbb{G}_m}(G^{\omega}_{S}, \Pi^\omega) \), nor for that matter, \( \Pi^\omega \) is finite dimensional (it is not in general). The argument works without this assumption. In the finite-dimensional case, we can apply the following lemma.

**Lemma 8.2.** Consider a power series with \( M \) factors:

\[
F(t) = \prod_{i=1}^{M} \frac{1}{(1-t^{\alpha_i})} \in \mathbb{Q}[[t]]
\]

where the \( \alpha_i \) are strictly positive integers. Then the coefficients of \( t^n \) in \( F \) grow polynomially in \( n \) of degree \( M - 1 \).

**Proof.** The coefficients of \( F(t) \) are bounded above by those of

\[
\frac{1}{(1-t)^M} = \sum_{n \geq 1} \binom{n+M-1}{M-1} t^n
\]

since the geometric series expansion of \( (1-t^{\alpha})^{-1} \) has non-negative coefficients which are termwise bounded above by those of \( (1-t)^{-1} = 1 + t + t^2 + \ldots \). A lower bound for the coefficients of \( F(t) \) is likewise given by an expansion of \( (1-t^{\alpha})^{-M} \), where \( \alpha \) is
the least common multiple of $\alpha_1, \ldots, \alpha_M$. By the previous formula, the coefficient of $t^n$ in this series is $\binom{n+M-1}{M-1}$, which is again polynomial in $n$ of degree $M - 1$. □

Suppose that there exists $w > 0$ such that

$$\ell_m = 0 \quad \text{for all} \quad m > w.$$ (8.3)

This happens if we replace $\Pi$ with the quotient $\Pi/W_{-2w-1}\Pi$, whose affine ring is the Hopf subalgebra of $O(\Pi)$ generated by $W_wO(\Pi)$. In fact, (8.3) is equivalent to $W_{-2w-1}\Pi = 1$. With this assumption, the generating series (8.1) and (8.2) have finitely many factors and we can apply the lemma to deduce the following

Corollary 8.3. Suppose $k$ is totally real with $r_1$ real embeddings. A sufficient condition for some coefficient $t^n$ in (8.1) to exceed that of (8.2) is

$$\ell_1 + \ldots + \ell_w > |S|\ell_1 + r_1(\ell_3 + \ell_5 + \ldots + \ell_r)$$ where $r$ is the largest odd integer with $3 \leq r \leq w$. In this situation, (8.2) applies.

This is the theorem of [22]. The assumption (8.3) implies that $O(\Pi^w)$ and $O(Z)$ have finite transcendence degree and so $\Pi$ and $Z$ are both finite-dimensional with

$$\dim \Pi = \ell_1 + \ldots + \ell_w,$$

(8.5) $$\dim Z^1_{G_m}(G^w_S, \Pi) \leq |S|\ell_1 + r_1(\ell_3 + \ell_5 + \ldots + \ell_r).$$

The latter inequality is in fact an equality. We will show below that (8.4) is implied by the inequality $\dim \Pi \geq \dim Z^1_{G_m}(G^w_S, \Pi)$. We emphasize, however, that our approach does not require finite-dimensionality of either scheme.

8.5. Dimensions via homological algebra. We give a second approach to computing dimensions using standard methods of homological algebra. The argument is formally similar to [23].

Lemma 8.4. The natural transformation of functors

$$Z^1_{G_m}(G^w_S, \Pi) \to H^1_{G_m}(G^w_S, \Pi)$$

is bijective, where $H^1_{G_m}(G^w_S, \Pi) = \ker \left( H^1_{\text{alg}}(G^w_S, \Pi) \to H^1_{\text{alg}}(G_m, \Pi) \right)$ and $H^1_{\text{alg}}$ is the functor from commutative unitary $k$-algebras to pointed sets whose $R$-points are equivalence classes of elements in $Z^1_{G_m}(G^w_S, \Pi)$ modulo boundaries.

Proof. A class in $H^1_{G_m}(G^w_S, \Pi)(R)$ can be represented by $a : G^w_S \times R \to \Pi \times R$ which defines an algebraic cocycle. We claim that it has a unique representative whose restriction to $G_m \times R$ is trivial. For the existence, the triviality of the class of $a$ restricted to $G_m \times R$ means that there exists $c \in \Pi(R)$ such that $a_g = cg(c^{-1})$ for all $g \in G_m \times R$. By replacing $a_g$ with $g \mapsto c^{-1}a_g(g(c))$, we can assume that $a$ is trivial on $G_m \times R$. For the uniqueness, consider another representative $a'$ where $a'_g = b^{-1}a_g(b)$ for some $b \in \Pi(R)$, and such that $a'$ is also trivial on $G_m \times R$. It follows that

$$g(b) = b \quad \text{for all} \quad g \in G_m(R)$$

and hence $\log(b) \in \Lie \Pi(R)$ is $G_m(R)$-invariant. The logarithm exists since $\Pi$ is pro-unipotent. But $(\Lie \Pi(R))^{G_m(R)}$ is a quotient of $(\Lie \Pi_0(R))^{G_m(R)}$. Since $G_m(R) = R^\times$ contains $Q^\times$ and the latter acts non-trivially on all Tate objects $Q(n)_{dR}$ for all $n < 0$ it follows that $(\Lie \Pi_0(R))^{G_m(R)}$ is a quotient of $\text{gr}^W_0 \Lie \Pi_0(R) = 0$. Therefore $\log(b)$ vanishes, and hence $b$ is the unit element. □
In the proof we showed that \( \Pi^a(R) \) has trivial \( G_m(R) \)-invariants for all \( R \).

Let \( L^1 \Pi = \Pi \) and \( L^{n+1} \Pi = [\Pi, L^n \Pi] \) for \( n \geq 1 \) be the lower central series of \( \Pi \). Set \( \Pi_n = \Pi/L^{n+1} \Pi \) and consider the short exact sequence of affine group schemes

\[ 1 \longrightarrow \text{gr}^n_{LCS} \Pi \longrightarrow \Pi_n \longrightarrow \Pi_{n-1} \longrightarrow 1. \]

Taking non-abelian cohomology with respect to \( G = G_S^\omega \) or \( G_m \) leads to a sequence

\[ \Pi_n^G \longrightarrow H^1(G, \text{gr}^n_{LCS} \Pi) \longrightarrow H^1(G, \Pi_n) \longrightarrow H^1(G, \Pi_{n-1}) \longrightarrow H^2(G, \text{gr}^n_{LCS} \Pi), \]

which, on taking points, is a long exact sequence of non-abelian cohomology sets. We showed that \( \Pi_n^G \) is trivial since \( G \) contains \( G_m \) and \( \Pi^G_0 = 1 \). Since \( \text{gr}^n_{LCS} \Pi \) is abelian, and since ordinary group cohomology can be computed using cocycles,

\[ H^i(G, \text{gr}^n_{LCS} \Pi) = \text{Ext}^i_{G-\text{mod}}(\mathbb{Q}, \text{gr}^n_{LCS} \Pi). \]

The latter is trivial for \( i \geq 2 \) if \( G = G_S^\omega \) by \[8.4.2\] and vanishes for \( i \geq 1 \) if \( G = G_m \). We deduce by induction on \( n \) and the fact that \( \Pi_0 = 1 \) that \( H^1(G_m, \Pi_n) = 0 \) for all \( n \) and hence \( H^1(G_m, \Pi) \) is trivial. It follows that \( H^1_{G_m}(G_S^\omega, \Pi) = H^1(G_S^\omega, \Pi) \). By lemma \[5.4\] the former defines a representable functor and hence a scheme.

Therefore we obtain the sequence

\[ 0 \longrightarrow \text{Ext}^1_{\mathcal{MT}_S}(\mathbb{Q}, \text{gr}^n_{LCS} \Pi) \longrightarrow H^1(G_S^\omega, \Pi_n) \longrightarrow H^1(G_S^\omega, \Pi_{n-1}) \longrightarrow H^2(G, \text{gr}^n_{LCS} \Pi). \]

This means that the fibers of the morphism \( H^1(G_S^\omega, \Pi_n) \longrightarrow H^1(G_S^\omega, \Pi_{n-1}) \) of schemes are principle \( \text{Ext}^1_{\mathcal{MT}_S}(\mathbb{Q}, \text{gr}^n_{LCS} \Pi) \)-spaces. It follows from properties of the dimension of schemes that

\[ \dim H^1(G_S^\omega, \Pi_n) \leq \dim H^1(G_S^\omega, \Pi_{n-1}) + \dim \text{Ext}^1_{\mathcal{MT}_S}(\mathbb{Q}, \text{gr}^n_{LCS} \Pi). \]

This inequality does not use the surjectivity in the previous sequence, and therefore applies in the more general situation of \[111\] It is in fact an equality in the present example since the final term is trivial. Assume that \( \Pi = \Pi/W_{m+1} \) for some \( m \). In particular, the lower central series terminates. It follows by induction that

\[ \dim H^1(G_S^\omega, \Pi) \leq \sum_{n \geq 0} \dim \text{Ext}^1_{\mathcal{MT}_S}(\mathbb{Q}, \text{gr}^n_{LCS} \Pi). \]

If we write \( \text{gr}_{LCS} \Pi \cong \bigoplus_w \mathbb{Q}(-w)^{\ell_w} \) (the lower central series filtration coincides with the weight-filtration) then by lemma \[8.3\] we deduce that

\[ \dim Z^1_{G_m}(G_S^\omega, \Pi) \leq \sum_n \ell_n \dim (\text{Ext}^1_{\mathcal{MT}_S}(\mathbb{Q}(0), \mathbb{Q}(-n))). \]

which gives \[8.5\] by \[3.4\].

9. **Application: the unit equation**

Let \( k = \mathbb{Q} \), \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \), \( \mathcal{O}_S = \mathbb{Z}[\frac{1}{S}] \) where \( S \) is a finite set of rational primes, and let 0 be the tangential base-point 1 at 0. The set of integral points \( X_S \) are the solutions to the equation \( u + v = 1 \) where \( u, v \) are \( S \)-integral. We shall also consider certain tangential basepoints on \( X_S \) as solutions to this equation. For background on the motivic fundamental group, see \[13\], \[12\], \[5\].
9.1. **Unit equation.** First let \( \Pi = 0\Pi_0 \) be the entire fundamental group. Since \( k = \mathbb{Q} \), the canonical fiber functor \( \omega \) coincides with the de Rham fiber functor \( \omega_{dR} \). Then \( \mathcal{O}(\Pi^w) = T^w(\mathbb{Q}e_0 \oplus \mathbb{Q}e_1) \) is the graded vector space spanned by words in two letters \( e_0, e_1 \). It is equipped with the shuffle product \( \mathfrak{m} \) and deconcatenation coproduct whose action on generators is given by the formula

\[
\Delta^{\text{dec}}(e_{i_1} \cdots e_{i_n}) = \sum_{k=0}^{n} e_{i_1} \cdots e_{i_k} \otimes e_{i_{k+1}} \cdots e_{i_n}.
\]

The augmentation \( \varepsilon \) is projection onto the empty word. An element of \( \Pi^w(R) \) is an invertible formal power series in non-commuting variables \( R/\langle\langle e_0, e_1 \rangle\rangle \) which is group-like with respect to the completed coproduct for which \( e_0, e_1 \) are primitive.

Any point \( x \in X_S \) defines an algebraic cocycle \( c^x \in \Pi(\mathcal{P}_S^w) \). It is a homomorphism \( \mathcal{O}(\Pi^w) \to \mathcal{P}_S^w \) of graded algebras and is given by the generating series

\[
L^w(x) = \sum_w w \Pi^w_u(x)
\]

where we use the notation

\[
\Pi^w_u(x) := L^w(x; w; 0)
\]

for the unipotent iterated integral from 0 to \( x \) of the word \( w \), where \( e_0 \) stands for \( \frac{dx}{1-x} \) and \( e_1 \) for \( \frac{dx}{1-x} \). In particular, if \(-1\) is the tangent vector \(-1\) at 1, then \( L^w(-1) = \sum_w w^u(w) \) is the unipotent version of the motivic Drinfeld associator \([5], (2.10)\).

One has \( \dim \mathfrak{dr}_S^w \mathcal{O}(\Pi) = 2^w \). An explicit set of algebra generators for \( \mathcal{O}(\Pi) \) are given by the set of Lyndon words in \( e_0, e_1, \) with respect to the ordering \( e_0 < e_1 \). By Witt’s inversion formula, any set of algebra generators in weight \( w \) number

\[
\ell_w = \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 2^d
\]

where \( \mu \) is the Möbius function. In particular, \( \ell_1 = 2 \), since \( e_0 \) and \( e_1 \) are Lyndon words. Since \( r_1 = 1 \), equation \([3.4] \) is equivalent to the inequality

\[
\ell_2 + \ldots + \ell_{2k} > 2|S| - 2
\]

which is easily satisfied for all \( k \) sufficiently large, since \( \ell_w \sim 2^w \). The conditions of theorem 7.2 are satisfied for some \( \Pi/W_{-k-1}\Pi \) and it follows that \( X_S \) is finite.

9.2. **Depth one quotient.** Rather than considering all iterated integrals, it is enough only to consider the unipotent versions of the classical polylogarithms:

\[
\log^w(x) = \Pi^w_{e_0}(x) = I^w(x; e_0; 0) \\
\Pi^w_{e_1}(x) = I^w_{e_1e_0^{-1}}(x) = I^w(x; e_1e_0\ldots e_0; 0)
\]

Denote their images under \( \text{per}_p \) with a superscript \( p \) instead of \( u \). Therefore let \( \Pi \) be the length \( 2k \), depth 1 quotient of \( 0\Pi_0 \). Its affine ring \( \mathcal{O}(\Pi^w) \subset \mathcal{O}(0\Pi_0^w) \) is the Hopf subalgebra generated by words of degree at most 1 in the letter \( e_1 \) and of length \( \leq 2k \).

The latter are stable under deconcatenation. Its indecomposable elements with respect to the shuffle product are given by the \( 2k + 1 \) words

\[
\{e_0, e_1, e_1e_0, \ldots , e_1e_0^{2k-1}\}.
\]

These are Lyndon words for the ordering \( e_0 < e_1 \). Its Poincaré series is thus \([3.4] \)

\[
\sum_{n \geq 0} d_n t^n = \frac{1}{(1-t)^2(1-t^2)(1-t^3)\ldots(1-t^{2k})}
\]
where \( d_n = \dim \gr_{2n}^W \mathcal{O}(\Pi^c) \), and \( \mathcal{O}(Z) \) Poincaré series (8.2)

\[
\sum_{n \geq 0} c_n t^n = \frac{1}{(1-t)^{2|S|}(1-t^3)(1-t^5) \ldots (1-t^{2k-1})}
\]

where \( c_n = \dim \mathcal{O}(Z)_{2n} \). Therefore by applying lemma 8.2 a necessary and sufficient condition for some \( d_n \) to exceed \( c_n \) is \( 2k + 1 > 2|S| + k - 1 \), i.e.,

\[
k > 2(|S| - 1) .
\]

Therefore, for any \( |S| \), and integer \( k \) satisfying (9.2), or possibly infinite, let \( w \) be minimal such that \( d_w > c_w \). Define

\[
N(k, |S|) = c_w + 1 \quad \text{and} \quad w(k, |S|) = w .
\]

Theorem 9.2 implies the following corollary.

**Corollary 9.1.** Let \( k > 2(|S| - 1) \) as above, and write \( N = N(k, |S|) \). Then for all \( N \)-tuples \( x_1, \ldots, x_N \in X_S \), the equation

\[
det(P_j(x_i)) = 0
\]

holds, where \( P_1, \ldots, P_N \) denotes any set of distinct monomials of the form

\[
(\log^u(x))^r \prod_{i=1}^k (Li_i^u(x))^{r_i}
\]

where \( r_i \geq 0 \) are integers, and the total weight is \( w(k, |S|) \). This last condition is equivalent to \( r_0 + r_1 + 2r_2 + \ldots + kr_k = w(k, |S|) \).

Let \( p \notin S \) be a finite prime. Then in particular

\[
det(\per_p P_j(x_i)) = 0 .
\]

The corollary states that in order to detect integral points \( X_S \), one is only required to compute the \( p \)-adic polylogarithms \( \log^p(x), \Li_1^p(x), \ldots, \Li_{2k}^p(x) \). Unfortunately, it seems that taking the single-valued period at the infinite prime gives the zero equation, and hence no information about integral points. This is because the single-valued versions of \( \Li_n(x) \), for \( n \) even, vanish on \( X(\mathbb{R}) \).

### 9.3. Coproduct.

There is an explicit formula for the coaction on unipotent iterated integrals in this case which is equivalent to a formula due to Goncharov [18]. It can be deduced directly by dualising [5] an older formula of Ihara’s for the action of the absolute Galois group on the \( l \)-adic completion of the fundamental group of the projective line minus three points.

Let \( x \in X_S \), and let \( \Delta : \mathcal{O}(\Pi^0) \rightarrow \mathcal{O}(\Pi^0) \otimes \mathbb{Q} P^u_S \) denote the right coaction dual to the left action of \( U^u_S \) on \( \Pi^0 \). Let us denote by \( D = (\text{id} \otimes \pi) \Delta \), where \( \pi \) denotes the projection modulo products:

\[
\pi : P^u_S \rightarrow \frac{(P^u_S)_{>0}}{(P^u_S)_{\geq 0}} .
\]

The ‘infinitesimal’ coaction is obtained via the following formula.

**Theorem 9.2.** For any \( a_1, \ldots, a_{N+1} \in \{0, 1\} \), where \( a_{N+1} = 0 \), we have

\[
D I^u(x; a_1 \ldots a_N; a_{N+1}) = \sum_{1 \leq i < j \leq N+1} I^u(x; a_1 a_2 \ldots a_i a_j \ldots a_{N-1} a_N; a_{N+1}) \otimes \pi(I^u(a_i; a_{i+1} \ldots a_{j-1}; a_j))
\]

\[
+ \sum_{1 \leq j \leq N+1} I^u(x; a_j a_{j+1} \ldots a_{N-1} a_N; 0) \otimes \pi(I^u(x; a_1 a_2 \ldots a_{j-1}; a_j))
\]
where the right-hand term in the second (resp. third) line is suitably interpreted \( \text{[5], \S 2} \) as a unipotent period of \( \mathcal{O}(\pi_{\mathcal{O}}^{\omega}) \) (resp. \( \mathcal{O}(\pi_{\mathcal{O}}^{\omega}) \)) where \( a, b \in \{1_0, -1_1\} \).

The terms in the right-hand side of the tensor in the second line of \( \text{[5.5]} \) are unipotent versions of motivic multiple zeta values (denoted by \( \zeta^a \) in \( \text{[7]} \)).

**Corollary 9.3.** For all \( x \in X_S \),

\[
D \text{Li}_n^\omega(x) = \text{Li}_{n-1}^\omega(x) \otimes \pi(\log^a(x)) + 1 \otimes \pi(\text{Li}_{n}^\omega(x)) \,.
\]

**Proof.** Set \( a_1 = a_2 = \ldots = a_{N-1} = 0 \) and \( a_N = 1 \) in \( \text{[5.5]} \). Use the fact that \( I^\omega(a_i; a_{i+1}, \ldots a_{j-1}; a_j) \) vanishes whenever \( a_i = a_j \), or all \( a_{i+1} = \ldots = a_{j-1} = 0 \) to deduce that all terms in the second line of \( \text{[5.5]} \) vanish. By the shuffle product formula, \( I^\omega(x; 0; a_j) \) is a power of \( I^\omega(x; 0; a_j) \), so only the terms \( j = 1, 2 \) in the third line of \( \text{[5.5]} \) survive.

By exponentiating \( D - 1 \otimes \pi \), we retrieve the formula for the full coaction:

\[
(9.6) \quad \Delta \text{Li}_n^\omega(x) = 1 \otimes \text{Li}_n^\omega(x) + \sum_{i=0}^{n-1} \text{Li}_{n-i}^\omega(x) \otimes \frac{1}{i!} (\log^a(x))^i
\]

where \( \text{Li}_0^\omega(x) := 1 \). It is valid for all \( x \in X(\mathbb{Q}) \). This can also be proved by direct methods, for example using \((2.3)\) of \[6\], without passing via the previous theorem.

### 9.4. Depth one grading.

A peculiarity of the depth one quotient of \( \mathcal{O}(\pi_{\mathcal{O}}^{\omega}) \) is that the associated depth-graded is preserved by the action of \( U_{\mathcal{O}}^{\omega} \). More precisely, the coaction \( \text{[9.3]} \) restricted to the affine ring of the depth one quotient of \( \pi_{\mathcal{O}}^{\omega} \) happens to factor, by \( \text{[9.6]} \), through the deconcatenation coproduct:

\[
(9.7) \quad \Delta^{\text{dec}} e_1 e_0^{n-1} = 1 \otimes e_1 e_0^{n-1} + \sum_{i=0}^{n-1} e_1 e_0^{n-1-i} \otimes e_1^i
\]

This can also be proved directly on noting that the depth one quotient of \( \pi_{\mathcal{O}}^{\omega} \) is a \( \Pi^\omega \) torsor, where \( \Pi^\omega \) was defined in \( \text{[9.2]} \) and the fact that \( U_{\mathcal{O}}^{\omega} \) acts trivially on \( \Pi^\omega \). Since the left-hand factors are all of \( D \)-degree one (the \( D \)-degree is the degree in the letter \( e_1 \)), it follows that the depth grading is preserved by \( U_{\mathcal{O}}^{\omega} \) in this situation. A glance at \( \text{[5.3]} \) shows that this is absolutely not true in general.

Since \( e_0 \) and the \( e_1 e_0^{n-1} \) generate \( \mathcal{O}(\Pi) \) as an algebra, we can therefore grade the affine ring of the depth one quotient \( \mathcal{O}(\Pi) \) by \( D \)-degree and, by \( \text{[5.6]} \), \( \mathcal{O}(Z) \) too.

The \( (weight, D\text{-degree}) \)-bigraded version of the generating series \( \text{[8.1]} \) is

\[
(9.8) \quad \sum_{k,n \geq 0} \dim \mathcal{O}(Z)_{2n,k} s^k t^n = \frac{1}{(1-t)(1-st)(1-st^2)(1-st^3) \cdots (1-st^{2k})}.
\]

The bigraded version of \( \text{[8.2]} \) is

\[
(9.9) \quad \sum_{k,n \geq 0} \dim \mathcal{O}(Z)_{2n,k} s^k t^n = \frac{1}{(1-t)(1-st)(1-st^2)(1-st^3) \cdots (1-st^{2k-1})}
\]

where \( \mathcal{O}(Z)_{2n,k} \) is the component of \( \mathcal{O}(Z) \) of weight \( 2n \) and \( D \)-degree \( k \).

### 9.5. Examples.
9.5.1. Take $|S| = 1$, and $k = 2$. The coefficient of $st^2$ in (9.8) is 2. Its coefficient in (9.7) is 1. Let $N = 2$. The two elements $e_1 e_0$ and $e_1 w e_0 = e_0 e_1 + e_1 e_0$ are linearly independent in $\gr^W_2 \gr^D_1 \mathcal{O}(\Pi^2)$. The corresponding iterated integrals are $\Li^0_2(x)$ and $\Li^0_1(x) \log^u(x)$. Theorem (7.2) implies that for every pair of points $x_1, x_2 \in X_S$:

$$\det \begin{pmatrix} \Li^0_2(x_1) & \log^u(x_1) \Li^0_1(x_1) \\ \Li^0_2(x_2) & \log^u(x_2) \Li^0_1(x_2) \end{pmatrix} = 0 .$$

If $S = \{ p \}$, let $x_2$ be the tangential base point $-p$ at $\infty$. We have

$$\Li^0_2(x_2) = \frac{1}{n!} (\log^u(p))^n \quad \text{and} \quad \log^u(x_2) = \log^u(p) .$$

Since $\log^u(p) \neq 0$, we deduce that for every $x \in X_S$ the equation

$$\det \left( \frac{\Li^0_2(x)}{2}, \frac{1}{2} \log^u(x) \Li^0_1(x) \right) = 0$$

holds. Therefore for every $x \in X_S$, we deduce that

$$\Li^0_2(x) - \frac{1}{2} \log^u(x) \Li^0_1(x) = 0$$

which can be viewed as a ‘motivic’ equation vanishing on integral points. Taking the $p$-adic period, for some different $p \notin S$, gives the equation for all $x \in X_S$:

$$\Li^0_2(x) - \frac{1}{2} \log^u(x) \Li^0_1(x) = 0$$

This equation is due to Coleman. Note that it does not depend on $S$, so it actually proves that $| \cup_{|S| = 1} X_S |$ is finite, which is easy to verify by hand.

9.5.2. Now, with $|S| = 1$ still, set $k = 2$. The coefficient of $st^4$ in (9.8) (resp. (9.7)) is 2 (resp. 4). The three elements $e_1 e_0 e_0 e_0$ and $e_1 e_0 e_0 e_0$ and $e_0 e_1 e_0 e_1$ are linearly independent in $\gr^W_2 \gr^D_1 \mathcal{O}(\Pi^2)$. Theorem (7.2) implies that for every set of three points $x_1, x_2, x_3 \in X_S$ the following equation is satisfied:

$$\det \begin{pmatrix} \Li^0_3(x_1) & \log^u(x_1) \Li^0_1(x_1) & (\log^u(x_1))^3 \Li^0_1(x_1) \\ \Li^0_3(x_2) & \log^u(x_2) \Li^0_1(x_2) & (\log^u(x_2))^3 \Li^0_1(x_2) \\ \Li^0_3(x_3) & \log^u(x_3) \Li^0_1(x_3) & (\log^u(x_3))^3 \Li^0_1(x_3) \end{pmatrix} = 0$$

Suppose that $S = \{ 2 \}$, and let $x_3$ be the tangent vector $-2$ at infinity. Then the point $\frac{1}{2} \in X_S$ and we can set $x_2 = \frac{1}{2}$. Again by (7.2), this gives the equation

$$\det \begin{pmatrix} \Li^0_3(x_1) & \log^u(x_1) \Li^0_1(x_1) & (\log^u(x_1))^3 \Li^0_1(x_1) \\ \Li^0_3(\frac{1}{2}) & \log^u(\frac{1}{2}) \Li^0_1(\frac{1}{2}) & (\log^u(\frac{1}{2}))^3 \Li^0_1(\frac{1}{2}) \\ \frac{1}{2} & \frac{1}{2} \log^u(\frac{1}{2}) \Li^0_1(\frac{1}{2}) & \frac{1}{2} \end{pmatrix} = 0$$

using the fact that $\log^u(2)$ is non-zero. Taking the $p$-adic period (replace all superscripts $u$ with $p$), and multiplying out gives a Coleman function in weight 4 for $X_S$, which is equivalent to the main theorem of (11). The method of that paper relied on a conjectural ‘exhaustion’ property for motivic iterated integrals, and the conjectural non-vanishing of a $p$-adic zeta value. The approach taken above is unconditional.
Remark on asymptotics. Returning to the situation of §9.2, let us fix $s = |S|$, and give a crude estimate of $N = N(s, \infty)$ as defined in (9.3) by approximating the generating series (8.1) and (8.2). First of all, one knows the rough asymptotics

$$\text{coeff. of } t^n \text{ in } \prod_{k \geq 0} \frac{1}{1 - t^{2k+1}} \sim \frac{3^\frac{1}{2} e^{\frac{\pi}{\sqrt{3}} n}}{12} \frac{1}{n^\frac{3}{4}}$$

using the more precise asymptotics of the partition function due to Hardy and Ramanujan [23]. The symbol $\sim$ means that the ratio of both sides tends to 1 as $n$ goes to infinity. The asymptotics of the corresponding series multiplied by $(1 - t)^{-m}$, are obtained from the above by integrating:

$$\text{coeff. of } t^n \text{ in } \frac{1}{(1 - t)^r} \prod_{k \geq 1} \frac{1}{1 - t^{2k+1}} \sim \left(\frac{2\sqrt{3n}}{\pi}\right)^r \frac{3^\frac{1}{2} e^{\frac{\pi}{\sqrt{3}} n}}{12} \frac{1}{n^\frac{3}{4}}$$

$$\text{coeff. of } t^n \text{ in } \frac{1}{1 - t} \prod_{k \geq 1} \frac{1}{1 - t^k} \sim \frac{2^\frac{1}{2} e^{\frac{\pi}{\sqrt{6}}} n}{4\pi}$$

Set $r = 2s$. A short calculation shows that $u(s, \infty)$ is roughly of order $s^{2+2\varepsilon}$, and hence $N(s, \infty) \sim \exp(s^{1+\varepsilon})$. The condition $|X_S| < N$ in the dichotomy of corollary 7.4 thus compares unfavorably with a theorem due to Evertse [16], which states that $|X_S| < 3 \times 7^{1+2|S|}$. It is curious that the asymptotics are fairly similar. I do not know if considering the full fundamental group instead of the depth one quotient would significantly improve this estimate for $N$ or not.

10. Virtual $S$-units

Although theorem 7.2 suffices to prove finiteness results, its defect is that one requires $N-1$ points on $X_S$ in order to construct an explicit Coleman function for the remaining points on $X_S$. As shown in the previous paragraph, $N$ grows at least exponentially in $|S|$ and there may not exist the required number of integral points on $X_S$ in the first place. This would seem to make an effective application of the method impractical. In this section we propose a remedy, by introducing what could be called virtual $S$-units or virtual cocycles.

10.1. Scheme of divisors. Let $R$ be a commutative $k$-algebra. Let us denote the free $R$-module of divisors on $X_T$ with coefficients in $R$ to be

$$\text{Div}(X_T)(R) = \{ \sum_{x \in X_T} n_x x \text{ where } n_x \in R \},$$

where $T$ is a finite set of rational primes.

**Proposition 10.1.** The functor $R \mapsto \text{Div}(X_T)(R)$ is representable, and defines an affine vector space scheme of finite dimension over $k$, which we denote by $\text{Div}(X_T)$.

**Proof.** By Siegel’s theorem the set $X_T$ is finite. Therefore $\text{Div}(X_T)(R)$ is simply $\text{Hom}(X_T, k) \otimes_k R$. Since $\text{Hom}(X_T, k)$ is a finite-dimensional vector space, it defines an affine scheme as in .
Let $\Pi$ be a quotient of the fundamental groupoid $\pi_1 S$ in the category $\mathcal{MT}(\mathcal{O}_S)$. Thus $\mathcal{O}(\Pi^\omega) \subset \mathcal{O}(\pi_1 S)$ is a Hopf subalgebra stable under the action of $G^S_{\Pi}$. Choose $M \subset \mathcal{O}(\Pi^\omega)$ a graded vector space spanned by a set of algebra generators which are homogeneous in the weight. In other words, the natural map

$$M \xrightarrow{\sim} \mathcal{Q}\mathcal{O}(\Pi^\omega)$$

is an isomorphism of vector spaces, where $\mathcal{Q}$ denotes the space of indecomposables $I/I^2$ where $I \leq \mathcal{O}(\Pi^\omega)$ is the augmentation ideal. For example, if $\Pi = \pi_1 S$ is the full fundamental group, we may take $M$ to be the graded vector space spanned by the set of Lyndon words in $e_0, e_1$. The group scheme $\Pi$ can be retrieved from $M$, since $M$ determines $\mathcal{O}(\Pi^\omega)$, which in turn uniquely determines $\mathcal{O}(\Pi)$ by the Tannaka theorem.

For simplicity, write

$$(10.1) \quad H_S(M) = \text{Hom}_{k-\text{vec}}(M, \mathcal{P}^S_G)$$

for the space of $k$-linear maps from $M$ to $\mathcal{P}^S_G$ which respect the weight-gradings. We have established that $W_{2n} \mathcal{P}^S_G$ is finite-dimensional for all $n$, and therefore by corollary 5.5, $H_S(M)$ is representable and hence an affine scheme. It is in fact a projective limit of vector-space schemes and therefore has a $k$-linear structure.

By the Milnor-Moore theorem, $\mathcal{O}(\Pi^\omega)$ is a polynomial algebra on $M$, so any linear map from $M$ to an algebra extends uniquely to an algebra homomorphism on $\mathcal{O}(\Pi^\omega)$. In particular, there is a natural equivalence of functors:

$$\text{Hom}_{k-\text{alg}}(\mathcal{O}(\Pi^\omega), \mathcal{P}^S_G) \xrightarrow{\sim} H_S(M),$$

where the left-hand Hom denotes homomorphisms of algebras which respect the weight gradings and the map is restriction to $M$. It follows that given two choices $M, M'$ of generating spaces, there is a canonical isomorphism of schemes

$$H_S(M) \xrightarrow{\sim} H_S(M')$$

by extending a linear map to $\mathcal{O}(\Pi^\omega)$ by multiplicativity, and then restricting to $M'$. However, the linear structures on $H_S(M)$ and $H_S(M')$ are not equivalent. We can therefore view the choice of $M$ as endowing the space of graded algebra homomorphisms from $\mathcal{O}(\Pi^\omega)$ to $\mathcal{P}^S_G$ with a choice of $k$-linear structure.

**Definition 10.2.** The cocycle map extends by linearity to a natural $R$-linear transformation of functors from commutative rings to free $R$-modules:

$$(10.2) \quad c : \text{Div}(X_S) \rightarrow H_S(M) \quad \sum n_x x \mapsto \sum n_x c^x|_M .$$

By the Yoneda lemma, (10.2) is a morphism of schemes.

The map (10.2) is given explicitly by

$$c(\sum n_x x) = (w \mapsto \sum n_x \text{Li}^u_w(x)) \quad \text{for all } w \in M .$$

Since a cocycle is determined by its action on $M$, the space of cocycles $Z^1_{\text{gr}}(G^S_{\omega}, \Pi^\omega) \rightarrow H_S(M)$ is a closed subscheme. The point is that, although there is no linear structure on cocycles, there is a linear structure on $H_S(M)$.
Definition 10.3. Define the scheme of virtual $M$-cocycles for $X_S$ to be the fiber product
\[(10.3) \quad V^1_S(X_S, M) = \text{Div}(X_S) \times_{H_S(M)} Z^1_{\mathbb{G}_m}(G^\omega_S, \Pi^\omega) .\]

Its points consist of those divisors which satisfy the equations defining a cocycle.

Since the points of $X_S$ define divisors in $\text{Div}(X_S)(\mathbb{Q})$ and also cocycles, we have
\[X_S \subset V^1_S(X_S, M)(\mathbb{Q}) .\]
The definition is functorial with respect to $M$ in the sense that if $M \subset M'$, then the natural map $V^1_S(X_S, M') \to V^1_S(X_S, M)$ is a closed embedding.

10.2. Enlarging the set of primes $S$. Now let $T$ be a finite set of primes containing $S$. There is a natural injective map of graded Hopf algebras
\[\mathcal{P}^\omega_S \to \mathcal{P}^\omega_T\]
since $\mathcal{MT}(O_S)$ is a full subcategory of $\mathcal{MT}(O_T)$ and hence $U^\omega_T \to U^\omega_S$ is faithfully flat. In particular, we deduce a morphism of affine schemes
\[H_S(M) \to H_T(M)\]
which is a closed embedding. Via this embedding we can view $Z^1_{\mathbb{G}_m}(G^\omega_S, \Pi^\omega)$ as a closed subscheme of $H_T(M)$.

Definition 10.4. Define the scheme of $T$-virtual $M$-cocycles for $X_S$ to be
\[(10.4) \quad V^1_T(X_S, M) = \text{Div}(X_T) \times_{H_T(M)} Z^1_{\mathbb{G}_m}(G^\omega_S, \Pi^\omega) .\]

Its points are divisors which define a cocycle and are unramified at primes in $T \setminus S$.

The set of integral points $X_S$ satisfy $X_S \subset \text{Div}(X_T)(\mathbb{Q})$ and therefore
\[(10.5) \quad X_S \subset V^1_T(X_S, M)(\mathbb{Q}) .\]
This inclusion can be strict for small $M$, as the examples below will show, but we claim that for any $T$, one has $X_S = \varprojlim_M V^1_T(X_S, M)(\mathbb{Q})$.

The scheme $V^1_T(X_S, M)$ is functorial in $T$ (as well as $M$). Given $T' \supset T$ there is a natural morphism such that the following diagram commutes
\[
\begin{array}{ccc}
V^1_T(X_S, M) & \to & V^1_{T'}(X_S, M) \\
\downarrow & & \downarrow \\
Z^1_{\mathbb{G}_m}(G^\omega_S, \Pi^\omega) & = & Z^1_{\mathbb{G}_m}(G^\omega_S, \Pi^\omega) .
\end{array}
\]
The map along the top is a closed embedding.

10.3. Variants. Taking the union over all finite sets $T$ containing $S$, we can set
\[V^1_\infty(X_S, M) = \varprojlim_T V^1_T(X_S, M)\]
It is a functor from commutative $k$-algebras to sets. Elements in $V^1_\infty(X_S, M)$ are simply divisors on $X$ which give rise to cocycles for $G^\omega_S$.

Similarly, it is convenient to consider divisors with support along a given finite set $D \subset X(\mathbb{Q})$ of rational points. Then let $\text{Div}(D)$ be the scheme whose points are
\[\text{Div}(D)(R) = \{ \sum_{x \in D} n_x x : n_x \in R \} .\]
There exists $T$ such that all points in $D$ are $T$-integral, i.e. $D \subset X_T$. So we can define
\[(10.6) \quad V^1_D(X_S, M) = \text{Div}(D) \times_{H_T(M)} Z^1_{\mathbb{G}_m}(G^\omega_S, \Pi^\omega) .\]
It does not depend on the choice of $T$, and defines a closed subscheme of $V^1_T(X_S, M)$.
Its points consists of the divisors supported on $D$ which form a cocycle for $G^\omega_S$. 
10.4. Coleman functions via virtual cocycles. The point of the previous construction is that divisors provide a way to construct cocycles via the natural map
\[ V_1^1(X_S, M)(\mathbb{Q}) \rightarrow Z^1(G_S^c, \Pi^w)(\mathbb{Q}) \]
This provides the following generalisation of theorem 10.2.

**Theorem 10.5.** Let \( n \geq 0 \) and \( k \) be a finite extension of \( \mathbb{Q} \). Let \( \Pi, M \) be as above. Let us write \( N = \dim \mathcal{O}(Z)_{2n} \) and suppose that
\[ \dim \mathfrak{g}_{2n}^W \mathcal{O}(\Pi^w) > N. \]
Let \( w_1, \ldots, w_N \) be linearly independent elements in \( \mathfrak{g}_{2n}^W \mathcal{O}(\Pi^w) \). Then any \( N \) elements
\[ \xi_1, \ldots, \xi_N \in V_1^1(X_S, M)(k) \]
satisfy the following equation with coefficients in \( k \):
\[ (10.7) \quad \det (Li^n(\xi_j))_{1 \leq i, j \leq N} = 0 \]
where \( Li^n \) is extended linearly to divisors: \( Li^n(\sum n_xx) \) is defined to be \( \sum n_xLi^n(x) \).

This discussion motivates the following question:

**Question 1.** Is \( V_{1}^{1}(X_S, M) \) Zariski-dense in \( \mathbb{Z}^1_{G_m}(G_S^c, \Pi^w) \) for sufficiently large \( D \)?

10.5. Ramification. The natural map \[ c : \text{Div}(X_T) \rightarrow \text{H}_T(M) \]
Short of demanding that \( c \) define a cocycle, we can first find conditions for \( c \) to be unramified outside \( S \).

**Definition 10.6.** Define \( \text{Div}_{M,S}(X_T) \), the space of \( M \)-unramified virtual divisors, by
\[ \text{Div}_{M,S}(X_T) = \text{Div}(X_T) \times_{\text{H}_T(M)} H_S(M). \]
The space \( \text{Div}_{M,S}(X_T) \) is an affine scheme. It is a projective limit of vector space schemes. In particular, it has a \( k \)-linear structure. It is finite dimensional if \( M \) is.

There is a variant \( \text{Div}_{M,S}(D) \) in which we replace \( X_T \) with a subset \( D \subset X_T \). Its points are divisors supported on \( D \) which are \( M \)-ramified outside of \( S \).

The ramification conditions can be computed recursively. The graded Lie algebra \( \mathfrak{u}_T^w \) of \( U_S^w \) is a free Lie algebra and satisfies
\[ (\mathfrak{u}_T^w)^{ab} \cong \bigoplus_{n \geq 1} \text{Ext}^1_{\mathcal{O}_T}(\mathcal{O}(0, \mathcal{O}(n))) \otimes \mathbb{Q}(−n). \]
There are \( |T| \) generators in degree (one half of the Hodge-theoretic weight) 1 corresponding to every prime \( p \in T \), and a generator in every odd degree \( \geq 3 \). The former are dually to \( \log^w(p) \), the latter to \( \zeta^n(2n+1) \), for \( n \geq 1 \). Denote a choice of lift of these generators to \( \mathfrak{u}_T^w \) by \( \nu_p \), for \( p \in T \) and \( \sigma_{2n+1} \), for \( n \geq 1 \). They act by derivations on \( \mathcal{P}_T^w \).

In fact, the elements \( \nu_p \) are uniquely determined, but the \( \sigma_{2n+1} \) involve choices. Their action factors through the operator \( D \) given by the formula \( \mathfrak{D} \), so in particular the action of \( \nu_p \) can be written down in closed form.

**Lemma 10.7.** An element \( \xi \in \mathcal{P}_T^w \) lies in the subspace \( \mathcal{P}_S^w \) if and only if

1. \( \nu_p \xi = 0 \) for all \( p \in T \backslash S \).
2. \( \nu_p \xi \in \mathcal{P}_S^w \) for all \( p \in S \).
3. \( \sigma_{2n+1} \xi \in \mathcal{P}_S^w \) for all \( n \geq 1 \).

**Proof.** The conditions imply that the action of \( U_T^w \) on \( \xi \) factors through \( U_S^w \). \( \square \)
Since the operators $\sigma_{2n+1}$ decrease the degree by $2n+1$, and $P^n_T$ has no elements of negative degrees, there are only finitely many conditions to be satisfied in (2), and the lemma yields a finite number of conditions for $\xi$ to be unramified at primes outside $S$.

Since the ramification conditions are linear, the space $\text{Div}_{M,S}(X_T)$ is isomorphic to an affine space $A^n$ if $M$ is finite-dimensional.

**Proposition 10.8.** Let $M$ be finite-dimensional. Then the dimension of $\text{Div}_{M,S}(X_T)$ grows exponentially in $|T|$.

*Proof.* Let $m = |T\setminus S|$. The number of conditions for an element $\xi \in \text{Div}(X_T)$ to be an $M$-virtual $S$-unit is a polynomial in $m$, by lemma [10.7] since $M$ is finite. But by [15], the number of $T$ units grows faster than polynomially in $|S'|$. \(\square\)

We view the space of virtual cocycles $V^1_T(X_S, M) \subset \text{Div}_{M,S}(X_T)$ as a closed subscheme of this affine space.

**Theorem 10.9.** In the situation of theorem 10.5, there exists a finite set of primes $T$ containing $S$, a finite extension $k$ over $\mathbb{Q}$, and a motivic Coleman function such that

$$
\sum_{i=1}^m p_i \text{Li}^n_{w_i}(x) = 0 \quad \text{for all } x \in X_S,
$$

where the $p_i \in P^n_S \otimes_{\mathbb{Q}} k$ are not all zero and are given by explicit polynomials in the $\text{Li}^n_{w_i}(\xi_j)$ for some divisors $\xi_j \in \text{Div}_{X_T}(k)$ with support in $X_T$.

*Proof.* The number of equations defining $V^1_T(X_S, M)$ only depends on $M$. It follows that the dimension of $V^1_T(X_S, M)$ tends to infinity as $|T|$ tends to infinity (this lends some plausibility to question 1.) In particular, for sufficiently large $T$, we have

$$\dim V^1_T(X_S, M) > 0$$

and hence an infinite supply of virtual cocycles. Theorem 10.5 therefore overcomes the difficulties explained in [9.6] although there may not be $N$ integral points on $X_S$ to which one can apply the theorem 7.2 we can always find $N$ virtual cocycles if $T$ is sufficiently large, after passing to a finite field extension.

In particular, we can proceed as in 7.4 (2). Let us suppose, by repeatedly taking minors of the matrix (7.8) and reducing the value of the integer $N$, that every $N - 1$ minor is not identically zero on all sets of $N - 1$ elements of $V^1_M(X_S, M)(k)$. Therefore we can take $\xi_1, \ldots, \xi_{N-1} \in V^1_M(X_S, M)(k)$ and $\xi_N = x \in X_S$ an actual integral point via (10.5) to obtain a motivic Coleman function

$$
\sum p_i \text{Li}^n_{w_i}(x) = 0 \quad \text{for all } x \in X_S,
$$

where the $p_i \in P^n_S \otimes_{\mathbb{Q}} k$ are minors of the matrix (10.7). By minimality of $N$, they are not all zero. \(\square\)

**Question 2.** Is $P^n_{MT(\mathbb{Q})}$ generated by the $\text{Li}^n_w(x)$, for $x \in X(\mathbb{Q})$?

Likewise, one can ask if $P^n_{MT(\mathbb{Q})}$ is generated by the $\text{Li}^n_m(x)$?

10.6. Examples.
10.6.1. Example 1. Let $\Pi = \Pi_{0}^{ab}$. It is the weight one quotient of the motivic fundamental group. Its affine ring is the symmetric tensor algebra on $\{e_{0}, e_{1}\}$. Let $S = \{2\}$ and $S' = \{2, 3\}$. The set $X_{S'} \cap (0, 1)$ of $S'$ integral points is

$$D = \left\{ \frac{1}{1}, \frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{8}{7} \right\}.$$

The point $\left\{ \frac{1}{2} \right\}$ is a genuine integral point on $X_{S}$. Since $O(\Pi_{0}^{\omega})$ is generated by $M_{1} = \mathbb{Q}e_{0} \oplus \mathbb{Q}e_{1}$, and $\text{Li}_{u}^{u} = \log^{u}$, $\text{Li}_{n}^{u}(x) = \log^{u}(1 - x)$, the condition for a divisor $\xi$ supported on these points to be unramified is that

$$\log^{u}(\xi) \text{ and } \log^{u}(1 - \xi) \in \mathcal{P}_{S}^{u}$$

i.e., both are unramified at prime 3. Thus we are looking for linear combinations of the points in $D$ such that all $\log^{u}(3)$ terms cancel. By the functional equation of the unipotent logarithm (\cite{[6]} §5.3) we have

$$\log^{u}(\prod p_{i}^{n_{i}}) = \sum n_{i} \log^{u}(p_{i})$$

where $p_{i}$ are positive primes, and $n_{i} \in \mathbb{Z}$. We find, that in addition to $\xi_{0} = \frac{1}{2}$, the following elements are $M_{1}$-virtual $S$-units:

$$\xi_{1} = \left[ \frac{1}{7} \right] - 2\left[ \frac{1}{7} \right]$$

$$\xi_{2} = \left[ \frac{2}{7} \right] - \left[ \frac{1}{7} \right]$$

$$\xi_{3} = \left[ \frac{3}{7} \right] + \left[ \frac{1}{7} \right]$$

$$\xi_{4} = \left[ \frac{8}{7} \right] - 2\left[ \frac{1}{7} \right]$$

For example, $\text{Li}_{u}^{u}(\xi_{1}) = \log^{u}(\xi_{1}) = \log^{u}(\frac{1}{7}) - 2\log^{u}(\frac{1}{7}) = 0$ and similarly $\text{Li}_{n}^{u}(\xi_{1}) = -\log^{u}(1 - \xi_{1}) = \log^{u}(2)$, which are both unramified at the prime 3. This can also be checked using (1) and (2) of lemma 10.7. Therefore

$$\text{Div}_{M_{1}, S}(D)(\mathbb{Q}) = \mathbb{Q}\xi_{0} \oplus \mathbb{Q}\xi_{1} \oplus \mathbb{Q}\xi_{2} \oplus \mathbb{Q}\xi_{3} \oplus \mathbb{Q}\xi_{4}$$

and in this case we have $V_{D}^{1}(X_{S}, M_{1}) = \text{Div}_{M_{1}, S}(D)$ since there are no further conditions for a divisor to define a cocycle.

10.6.2. Now let $S$ and $S'$ be as above, and let $\Pi$ be the weight two quotient of $\Pi_{0}$. Its affine ring in $\omega$ is the Hopf algebra generated by $W_{2}O(\Pi_{0}^{\omega})$, and is generated by $\{e_{0}, e_{1}, e_{1}e_{0}\}$. Let $M_{2} = \mathbb{Q}e_{0} \oplus \mathbb{Q}e_{1} \oplus \mathbb{Q}e_{1}e_{0}$. Since $\text{Li}_{n}^{u}(\xi_{0}) = \text{Li}_{n}^{u}(2)$, the extra condition for a divisor $\xi$ to be $M_{2}$-unramified is that

$$\text{Li}_{n}^{u}(\xi) \in \mathcal{P}_{S}^{u} .$$

We have already computed

\begin{equation}
\text{(10.8)} \quad \Delta \text{Li}_{n}^{u}(x) = 1 \otimes \text{Li}_{n}^{u}(x) + \text{Li}_{n}^{u}(x) \otimes \log^{u}(x) + \text{Li}_{2}^{u}(x) \otimes 1 .
\end{equation}

Therefore the condition that $\text{Li}_{n}^{u}(\xi)$ be unramified at 3, where $\xi = \sum n_{x}[x]$, is the condition that the coefficients in

$$\Delta \text{Li}_{n}^{u}(\xi) = \sum n_{x} \log^{u}(1 - x) \otimes \log^{u}(x)$$

of the terms $\log^{u}(2) \otimes \log^{u}(3)$, $\log^{u}(3) \otimes \log^{u}(3)$ and $\log^{u}(3) \otimes \log^{u}(2)$ should all vanish. There are three such equations, and we find that

$$\text{Div}_{M_{2}, S}(D)(\mathbb{Q}) = \mathbb{Q}\xi_{0} \oplus \mathbb{Q}\xi_{5} ,$$

is two dimensional, where

\begin{equation}
\text{(10.9)} \quad \xi_{0} = \left[ \frac{1}{2} \right] \quad \text{and} \quad \xi_{5} = 6\left[ \frac{1}{7} \right] - 6\left[ \frac{8}{7} \right] + \left[ \frac{3}{7} \right] .
\end{equation}

We easily check that $\text{Li}_{u}^{u}(\xi_{5}) = -3\log^{u}(2)$ and $\text{Li}_{u}^{u}(\xi_{5}) = -3\log^{u}(2)$.

$$\Delta \text{Li}_{u}^{u}(\xi_{0}) = -\log^{u}(2) \otimes \log^{u}(2) \quad \text{and} \quad \Delta \text{Li}_{u}^{u}(\xi_{5}) = 0$$

The second equation implies that $\text{Li}_{2}^{u}(\xi_{5}) = 0$, and that $\text{Li}_{u}^{u}(\xi_{5}) = \alpha\zeta^{u}(2)$, for some $\alpha \in \mathbb{Q}$. By taking the period, one verifies numerically that $\alpha$ is very close to $-1$. 
10.6.3. Continuing the previous example, let $\xi$ be an $M_2$-virtual $S$-unit. Then the condition that $\xi$ define a virtual cocycle in $Z^1_{\mathbb{G}_m}(G_S^0, \Pi)$ is the equation
\[
\Delta' \text{Li}_2^n(\xi) = \text{Li}_1^n(\xi) \otimes \log^n(\xi).
\]
If we write $\xi = \sum n_i \alpha_i$, this is equivalent to the equations
\[
(10.10) \sum n_i \left( \log^n(1 - \alpha_i) \otimes \log^n(\alpha_i) \right) = \left( \sum n_i \log^n(1 - \alpha_i) \right) \otimes \left( \sum n_j \log^n(\alpha_j) \right).
\]
Let us identify $\text{Div}_{M_2,S}(D) \cong \mathbb{A}^2$ by writing $\xi = t_1 \xi_0 + t_2 \xi_5$. By earlier calculations $\Delta' \text{Li}_2^n(\xi) = -t_1 \log^n(2) \otimes \log^n(2)$, $\log^n(\xi) = (t_1 - 3t_2) \log^n(2)$ and $\text{Li}_1^n(\xi) = (t_1 - 3t_2) \log^n(2)$. Therefore the condition that $\xi$ lie in $V_D^1(X_S, M_2)$ is the equation
\[
-t_1 \log^n(2) \otimes \log^n(2) = (t_1 - 3t_2)(t_1 - 3t_2) \log^n(2) \otimes \log^n(2).
\]
Since $\log^n(2)$ is non-zero, the subscheme
\[
V_D^1(X_S, M_2) \subset \text{Div}_{M_2,S}(D)
\]
is given by the conic
\[
\frac{1}{1 - a^2} \xi_0 + \frac{a}{3(a^2 - 1)} \xi_5
\]
lie in $V_D^1(X_S, M_2)$, for any $a \neq \{0, 1\}$, where $\xi_0$ was defined in (10.9). When $a = 0$ we retrieve the genuine point $\frac{1}{2} \in X_S$, but for other values of $a$ we obtain a new cocycle.

10.6.4. Let $M_2$, $\Pi$, $D$ be as above. Let $S' = \{2, 3\}$ but this time take $S = \{3\}$. Proceeding in a similar manner, we find a vector space of two unramified divisors:
\[
\xi_6 = -6[\frac{1}{2}] + [\frac{1}{2}] - 6[\frac{3}{2}] + \frac{3}{2} \quad \text{and} \quad \xi_7 = -3[\frac{1}{2}] - 6[\frac{3}{2}] + \frac{3}{2} + [\frac{5}{2}]
\]
So $\text{Div}_{M_2,S}(D)(\mathbb{Q}) = \mathbb{Q} \xi_6 \oplus \mathbb{Q} \xi_7$. The scheme of virtual $\Pi$-cocycles is a conic in this two dimensional affine space, and contains the following rational point
\[
-\frac{6}{7} \left[ \frac{1}{2} \right] + \frac{12}{7} \left[ \frac{1}{2} \right] - \frac{9}{7} \left[ \frac{1}{2} \right] - \frac{6}{7} \left[ \frac{3}{2} \right] - \frac{24}{7} \left[ \frac{3}{2} \right] + \frac{3}{2} \left[ \frac{3}{2} \right] + [\frac{5}{2}] \in V_D^1(X_S, M_2)(\mathbb{Q}).
\]
It satisfies (10.10). As far as the dilogarithm is concerned, this divisor plays the role of a point on $X_S$, even though there are no such points since $X_S = \emptyset$.

10.7. Remarks on Zagier’s conjecture and Bloch group. Beilinson and Deligne’s interpretation of Zagier’s conjecture involves finding divisors $\xi = \sum n_x [x]$ such that
\[
\Delta' \text{Li}_n^\Pi(\xi) = 0.
\]
For $n \geq 1$, such a $\xi$ defines an element of $\text{Ext}^1_{MT(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$, and the single-valued period of the associated de Rham period is a rational multiple of $\zeta(2k+1)$ if $n = 2k+1$ is odd. The conjecture follows easily from the known theorems about mixed Tate motives over $\mathbb{Q}$ in this case. The condition (10.12) is linear, and gives an explicit description of the rational algebraic $K$-theory of $\mathbb{Q}$. This story can be extended to number fields.

Compare, on the other hand, the equations defining a virtual cocycle, which in the case of the classical polylogarithms are the non-linear equations
\[
(10.13) \Delta' \text{Li}_n^\Pi(\xi) = \sum_{i=0}^{n-1} \text{Li}_{n-i}^\Pi(\xi) \otimes \frac{1}{i!} \left( \log^n(\xi) \right)^i.
\]
Compare with (4.6). More precisely, these are the equations defining $V^1_D(X_S, M)$ inside $\text{Div}_{M,S}(D)$ where $\xi$ is any divisor supported on $D$, where $M = \mathbb{Q}e_0 \oplus \bigoplus_{0 \leq k < n} \mathbb{Q}e_1 \epsilon^k_0$. 


11. General curves

We indicate how the above arguments can be generalised to arbitrary curves over a number field $k$. The missing ingredient is the conjectural upper bound on the space of motivic periods, which is expected to follow from a version of Beilinson’s conjecture. For this reason we only provide the key steps in the argument and leave the computations of the conjectural dimensions for future applications. The main novelty in this section is the definition and use of the canonical de Rham path on the schemes $\Pi$. It is not clear how to make the argument work using the higher Albanese manifolds.

We work in a Tannakian category $\mathcal{H}$ of realisations [13], §1.10 which contains as many cohomology theories as required and sufficiently many to ensure that, at least conjecturally, the category of mixed motives over $k$ should be a full subcategory of $\mathcal{H}$. Let $X$ be a curve over $k$, and consider the situation [410.7]. In particular, Beilinson’s cosimplicial construction of the unipotent fundamental groupoid of $X$ defines a groupoid of pro-objects $\pi^1(X, y, x)$ in $\mathcal{H}$ for every $x, y \in X(k)$, whose realisations coincide with those defined in [3]. For any pair of fiber functors $\omega, \omega'$ on $\mathcal{H}$, let $\mathcal{P}^{\omega, \omega}_{\mathcal{H}}$ denote the ring of $\omega', \omega$-periods of $\mathcal{H}$, and let $\mathcal{P}^{\omega, \omega}_{X} \subset \mathcal{P}^{\omega, \omega}_{\mathcal{H}}$ denote the ring generated by the matrix coefficients of $\mathcal{O}(\pi^1(X, y, x))$ for all $x, y \in X(k)$, or possibly tangential base points over $k$. The main examples that we have in mind are when $\omega$ is the de Rham or crystalline fiber functor, and $\omega'$ is the Betti functor relative to an embedding of $k$ into $\mathbb{C}$. A version of Beilinson’s conjectures predicts some control on the dimensions of the weight-graded pieces of $\mathcal{P}^{\omega, \omega}_{X}$, which we shall not discuss here.

For $x, y$ as above, write $y \Pi^\omega_x$ for the $\omega$-realisation of $\pi^1(X, y, x)$. These form torsors:

$$y \Pi^\omega_x \times_y \Pi^\omega_y \rightarrow y \Pi^\omega_x$$

and possess a rational point, since $y \Pi^\omega_x$ is a pro-unipotent affine group scheme.

The discussion proceeds along the lines of [17]. The main difference is that we now consider two fiber functors $\omega$ and $\omega'$, which leads to a notion of bitorsors. In the mixed Tate case we used ‘canonical de Rham paths’ using the weight-grading on the de Rham or canonical fiber functor. Since these do not exist in general, they will be replaced with a new notion of canonical de Rham paths which we shall discuss at some length, and in the crystalline case by Frobenius-invariant paths, which are well-known.

11.1. Cocycles of a bi-torsor. Let $0$ be a fixed rational base point (tangential or otherwise) of $X$. Let $G^\omega = \text{Aut}_{\mathcal{H}}^\omega(\omega)$, and likewise for $\omega'$, and set

$$P = \text{Isom}_{\mathcal{H}}^\omega(\omega, \omega') = \text{Spec } \mathcal{P}^{\omega, \omega}_{\mathcal{H}}.$$ 

It is a $G^\omega \times G'^\omega$ bitorsor (since our convention is that Tannaka groups act on the left of fundamental groups, and hence on the right on their affine rings). Suppose that for every $x \in X_S$ we are given two paths

$$x c_0 \in x \Pi^\omega_0(k) \quad \text{and} \quad x \gamma_0 \in x \Pi'^\omega_0(k')$$

where $k'$ is an extension of $k$ (for example, a $p$-adic field). The scheme $P$ acts via:

$$P \times x \Pi'^\omega_0 \rightarrow x \Pi^\omega_0.$$ 

Since $x \Pi^\omega_0$ is a $x \Pi^\omega_0$-torsor we obtain a morphism of schemes over $k'$

$$a : P \times_k k' \rightarrow x \Pi^\omega_0 \times_k k'$$

given on points by the equation $\phi(x \gamma_0) = x c_0 \phi a_\phi$ for $\phi \in P$. It satisfies

$$a_{ggh} = \alpha_g \phi(a_\phi \phi(\beta_h)) \quad \text{for} \quad (g, h) \in G^\omega \times G'^\omega,$$
where
\[ \alpha \in Z^1_{\text{alg}}(G^\omega, \omega \Pi^0_0)(k) \quad \text{and} \quad \beta \in Z^1_{\text{alg}}(G^\omega, \omega \Pi^0_1)(k') \]
are the algebraic cocycles defined by \( x_0 \) and \( x_{\gamma_0} \) respectively, i.e., \( g(x_0) = x_0 \alpha_g \) and \( g(x_{\gamma_0}) = x_{\gamma_0} \beta_g \). We call the data of \( a, \alpha, \beta \) an algebraic bi-torsor cocycle.

Changing \( x_0 \) to \( x_0 b^{-1} \) and \( x_{\gamma_0} \) to \( x_{\gamma_0} c^{-1} \) modifies it by a boundary:
\[
(a_g, a_\phi, \beta_h) \mapsto (b a_\phi g(b)^{-1}, b a_\phi \phi(c)^{-1}, c \beta_h h(c)^{-1})
\]
where \((b, c) \in \omega \Pi^0_0(k) \times \omega \Pi^0_1(k')\). From now on, we shall mostly emphasise the functor \( \omega' \) and consider only the right action of \( G^\omega \) on \( P \). In other words, we shall drop \( \alpha \) from the data and consider only \((a, \beta)\).

11.2. Let \( \Pi \) be an affine group scheme in \( \mathcal{H} \). This means that its affine ring is an Ind-object of \( \mathcal{H} \). Let \( Z^1_{r, \text{alg}}(P, \Pi^\omega) \) denote the functor whose \( R \) points, where \( R \) is a commutative ring over \( k \), are pairs of morphisms of schemes
\[
a : P \times_k R \rightarrow \Pi^\omega \times_k R \quad \beta : G^\omega \times_k R \rightarrow \Pi^\omega \times_k R
\]
such that the following cocycle condition is satisfied
\[
a_{\phi h} = a_\phi \phi(\beta_h) \quad (11.6)
\]
for all \( h \in G^\omega, \phi \in P \). This implies that \( \beta \) is necessarily an algebraic \( G^\omega \)-cocycle, and therefore there is a natural transformation of functors
\[
Z^1_{r, \text{alg}}(P, \Pi^\omega) \rightarrow Z^1_{\text{alg}}(G^\omega, \Pi^\omega) \quad (11.7)
\]
Its fibers are left \( \Pi^\omega \)-torsors. To see this, let \( a, a' \in Z^1_{r, \text{alg}}(P, \Pi^\omega)(R) \) be two elements with the same image. Define \( b : P \times_k R \rightarrow \Pi^\omega \times_k R \) by \( b_\phi = a_\phi' a_\phi^{-1} \). Then by (11.6),
\[
b_{\phi h} = a_{\phi h}' a_{\phi h}^{-1} = a_{\phi h}' \phi(\beta_h) \phi(\beta_h)^{-1} a_{\phi h}^{-1} = b_\phi
\]
for all \( h \in G^\omega \times_k R \), and since \( P \) is a \( G^\omega \)-torsor, it is trivialised over some finite flat extension \( R' \) of \( R \). It follows that \( b_\phi \) is constant for all \( \phi \in P(R') \). Therefore the images of \( a, a' \in Z^1_{r, \text{alg}}(P, \Pi^\omega)(R') \) satisfy \( a' = b a \) for some \( b \in \Pi^\omega(R') \). The converse is clear. On the other hand, since \( \Pi^\omega \) is pro-unipotent, it follows that the fibers of (11.7) are either empty or trivial \( \Pi^\omega \)-torsors.

Remark 11.1. Define the space of left \( P \)-cocycles \( Z^1_{l, \text{alg}}(P, \Pi^\omega) \) to be the functor whose \( R \)-points are morphisms \( a : P \times_k R \rightarrow \Pi^\omega \times_k R \) together with a \( a \in Z^1_{\text{alg}}(G^\omega, \Pi^\omega)(R) \) such that \( a_g = a \phi a \) for all \( g \in G^\omega, \phi \in P \). There is a natural transformation
\[
Z^1_{l, \text{alg}}(P, \Pi^\omega) \rightarrow Z^1_{\text{alg}}(G^\omega, \Pi^\omega) \quad (11.8)
\]
whose fibers are right \( Z^0_{l, \text{alg}} \)-torsors, where \( Z^0_{l, \text{alg}} \) is the inverse image of the trivial cocycle. Its \( R \)-points are morphisms \( r : P \times_k R \rightarrow \Pi^\omega \times_k R \) such that \( r_{g \phi} = g(r_\phi) \) for all \( g \in G^\omega \).

The right action of \( Z^0_{l, \text{alg}}(P, \Pi^\omega) \) is given by \( a_\phi \mapsto a_{\phi r_\phi} \).

Let \( a, a' \in Z^1_{r, \text{alg}}(P, \Pi^\omega)(k') \), and suppose that \( a' = ba \) for some \( b \in \Pi^\omega(k') \) as above, it follows from (11.6) that \( a \) and \( a' \) differ by a twist by the boundary \( b \). Denote by \( a, a' \) the corresponding cocycles in \( Z^1(G^\omega, \Pi^\omega)(k') \), images of \( a, a' \) under (11.8). Then by (11.6) it follows that \( a' g = b a g b^{-1} \).

Remark 11.2. In our applications, the schemes \( P, G, \Pi \) are all equipped with increasing weight filtrations \( W \). The morphisms \( a, \alpha, \beta \) arising from paths (11.2) will respect these filtrations. From now on, our spaces of cocycles will implicitly denote those cocycles.
which respect the weight filtrations (denoting this by a subscript \( W \) would clutter the notation unnecessarily).

In this case, our sufficient condition for \( Z^1_{F,\text{alg}}(G^{\omega'}, \Pi^{x'}) \) to define a scheme, namely, the finite-dimensionality of \( W_n(\mathcal{O}(G^{\omega'})) \) for all \( n \), will also suffice to ensure that \( Z^1_{F,\text{alg}}(P, \Pi^x) \) is a scheme, since \( P \) is a \( G^{\omega'} \)-torsor and so \( W_n(\mathcal{O}(P)) \) will also be finite-dimensional. Therefore \( Z^1_{F,\text{alg}}(P, \Pi^{x}) \) will be a closed subscheme of

\[
\text{Hom}_W(\mathcal{O}(\Pi^{x}), \mathcal{O}(G^{\omega'})) \times \text{Hom}_W(\mathcal{O}(\Pi^{x}), \mathcal{O}(P)) .
\]

11.3. Now let \( \Pi \) be an affine group scheme in \( \mathcal{H} \), given by a quotient of \( o\Pi_0 \). The image of the bicocycle (11.3) corresponding to a point \( x \) and the paths \( x\gamma_0 \in o\Pi_0(k') \), \( x\gamma_0 \in o\Pi_0(k) \), define algebra homomorphisms \( a^x, \alpha^x, \beta^x \):

\[
\begin{align*}
\alpha^x : \mathcal{O}(\Pi^{x'}) & \rightarrow \mathcal{P}_X^{\omega,x} \\
\alpha^x : \mathcal{O}(\Pi^{x'}) & \rightarrow \mathcal{P}_X^{\omega,x} \otimes_k k' \\
\beta^x : \mathcal{O}(\Pi^{x'}) & \rightarrow \mathcal{P}_X^{\omega,x} \otimes_k k'
\end{align*}
\]

since \( a \) is dual to \( \mathcal{O}(\Pi^{x'}) \otimes_k k' \rightarrow \mathcal{O}(P) \otimes_k k' = \mathcal{P}_{X,G}^{\omega,x} \otimes_k k' \), and its image lands in the subring spanned by the periods of the fundamental groupoid of \( X \). The cocycle conditions above can be translated into commutative diagrams which we shall omit. The morphisms \( a, \alpha, \beta \) respect the weight filtrations.

11.4. Frobenius-invariant paths. We can reduce the size of the space of bitorsor cocycles as follows. Suppose that \( x\gamma_0 \in x\Pi_0^x(k') \) is invariant under the action of an element (‘Frobenius’) \( F \in G^{\omega'}(k') \), for some \( k \)-algebra \( k' \). In this case, the map

\[
a : \mathcal{O}(\Pi^{x'}) \rightarrow \mathcal{P}_X^{\omega,x} \otimes_k k'
\]

lands in the \( F \)-invariant subspace of \( \mathcal{P}_X^{\omega,x} \otimes_k k' \). Suppose that the trivial path \( 1 \in o\Pi_0^x(k') \) is the unique \( F \)-invariant element. Then the cocycle \( \beta \) corresponding to \( x\gamma_0 \) lies in the space \( \beta \in Z^1_{F,\text{alg}}(G^{\omega'}, \Pi^{x})(k') \) of cocycles which are trivial on \( F \in G^{\omega'}(k') \) and, by the same argument as the first paragraph of 11.3, we have

\[
Z^1_{F,\text{alg}}(G^{\omega'}, \Pi^{x}) \xrightarrow{\sim} H^1_{F,\text{alg}}(G^{\omega'}, \Pi^{x'}) .
\]

where the subscript \( F \) denotes elements which are trivial on \( F \in G^{\omega'}(k') \). This certainly applies in the case when \( \omega' \) is the crystalline fiber functor (or de Rham by transporting \( F \)), and \( x\gamma_0 \) is the unique Frobenius invariant path \( 3 \) §1.5.2.

In the case when \( \omega' \) is the Betti realisation corresponding to a real embedding of \( k \), we can choose \( x\gamma_0 \) to be invariant under complex conjugation (invariant under the real Frobenius \( F_{\infty} \)). In this situation we merely deduce that the target space of the maps \( a^x, \beta^x \) lie in the subspace of real-Frobenius invariant periods.

11.5. Canonical de Rham paths. We now explain how to choose the path \( x\gamma_0 \) in the case when \( \omega \) is the de Rham fiber functor, by exploiting the Hodge filtration.

11.5.1. Hodge filtration. The affine ring of the de Rham fundamental groupoid carries a natural Hodge filtration, denoted \( F^n \). The left coaction dual to the right-torsor structure \( x\Pi_0^x \times o\Pi_0^x \rightarrow x\Pi_0^x \) respects the Hodge filtration, i.e.,

\[
\Delta F^n \mathcal{O}(x\Pi_0^x) \subset F^n(\mathcal{O}(o\Pi_0^x) \otimes_k \mathcal{O}(x\Pi_0^x)) .
\]

Since \( F^0 \mathcal{O}(x\Pi_0^x) = \mathcal{O}(x\Pi_0^x) \) for \( \bullet \in \{ x, 0 \} \), this implies that

\[
\Delta F^n \mathcal{O}(x\Pi_0^x) \subset \sum_{p+q=n, p, q \geq 0} F^p \mathcal{O}(o\Pi_0^x) \otimes_k F^q \mathcal{O}(x\Pi_0^x) .
\]
In particular, $\Delta F^1 \subset F^0 \otimes F^1 + F^1 \otimes F^0$. We can define

$$F^{n-1} \bullet \Pi^0 = \text{Spec} \left( \mathcal{O}(\bullet \Pi^0)/F^n \right) \quad \text{for } \bullet = x, 0$$

because $F^i \cdot F^j \subset F^{i+j}$ and hence $F^n$ defines an ideal in $\mathcal{O}(\bullet \Pi^0)$. By (11.11)

$$\Delta \mathcal{O}(x \Pi^0)/F^1 \subset \mathcal{O}(0 \Pi^0)/F^1 \otimes_k \mathcal{O}(x \Pi^0)/F^1$$

and by a similar equation with $x$ replaced by 0, we deduce that $F^0_0 \Pi^0$ is a closed subgroup of $0 \Pi^0$. By contrast with [25], our $F^n$ are closed subschemes but are not subgroups for $n \geq 1$. Furthermore, the subgroup $F^0$ is not in general normal.

**Lemma 11.3.** Multiplication gives a structure of a trivial right-torsor

$$F^0_0 \Pi^0 \rightarrow F^0_0 \Pi^0_{dR} \rightarrow F^0_0 \Pi^0_{dR}.$$

In particular, there exists a rational point in $F^0_0 \Pi^0_{dR}(k)$.

**Proof.** By the torsor property (11.1), the isomorphism

$$\mathcal{O}(x \Pi^0_{dR}) \otimes_k \mathcal{O}(x \Pi^0_{dR}) \cong \mathcal{O}(0 \Pi^0_{dR}) \otimes_k \mathcal{O}(x \Pi^0_{dR})$$

induces an isomorphism $\mathcal{O} \otimes F^1 \rightarrow \mathcal{O} \otimes F^1$ and hence

$$\mathcal{O}(x \Pi^0_{dR}) \otimes_k \mathcal{O}(x \Pi^0_{dR}) \cong \mathcal{O}(0 \Pi^0_{dR}) \otimes_k \mathcal{O}(x \Pi^0_{dR}).$$

It respects the Hodge filtration on both sides. By strictness, it induces an isomorphism on the associated $\text{gr}^0_F$ of both sides of the previous equation. By (11.3) and implies that it is a torsor. To show that it is trivial, use the fact that the affine ring of $F^0_0 \Pi^0_{dR}$ is a connected filtered Hopf algebra (e.g., by the weight filtration), and so $F^0_0 \Pi^0_{dR}$ is pro-unipotent. Any non-empty torsor over a pro-unipotent affine group scheme admits a rational point. □

11.5.2. The affine scheme $\Pi$.

**Definition 11.4.** Let $x \Pi_0$ denote the largest subalgebra of $\mathcal{O}(x \Pi^0_{dR})$ such that:

(i). $W_0 \cdot x \Pi_0 \cong W_0 \mathcal{O}(x \Pi^0_{dR}) \cong k$,

(ii) $x \Pi_0$ is stable under the coaction $\Delta : x \Pi_0 \rightarrow \mathcal{O}(0 \Pi^0_{dR}) \otimes_k x \Pi_0$,

(iii) $x \Pi_0 \subset F^1 \mathcal{O}(x \Pi^0_{dR}) + W_0 \cdot x \Pi_0$.

Since $W_0 \cap F^1 \mathcal{O}(x \Pi^0_{dR}) = 0$, the sum in the right-hand side of (iii) can be replaced with a direct sum. Likewise, define $0 \Pi_0$ by relacing $x$ with 0 in the above definition.

It is not necessarily a Hopf algebra, only a left $\mathcal{O}(0 \Pi^0_{dR})$-comodule algebra.

**Definition 11.5.** For $\bullet \in \{0, x\}$ let us define an affine scheme

$$\bullet \Pi^0_{dR} = \text{Spec} \left( x \Pi_0 \right).$$

There is a natural morphism of schemes $x \Pi^0_{dR} \rightarrow \bullet \Pi^0_{dR}$ and condition (ii) implies that the following diagram commutes

$$\begin{array}{ccc}
\bullet \Pi^0_{dR} \times \Pi^0_{dR} & \rightarrow & \bullet \Pi^0_{dR} \\
\downarrow & & \downarrow \\
\bullet \Pi^0_{dR} \times \Pi^0_{dR} & \rightarrow & \bullet \Pi^0_{dR}
\end{array}$$

In other words, $\bullet \Pi^0_{dR}$ is stable under the right-action of $0 \Pi^0_{dR}$.

There is an identical construction for any affine groupoid scheme $\Pi$ in the category $\mathcal{H}$ which is a quotient of $0 \Pi_0$ (e.g., $\Pi = 0 \Pi_0/W_0$), which we again denote by $\Pi^0_{dR}$.
Examples 11.6. Suppose that $O(\bullet \Pi_0^{dR})$ is isomorphic to the tensor coalgebra on a vector space $V$ and that the coaction is given by deconcatenation of tensors, e.g. $\Pi^d$. Then $\bullet H_0$ is the subspace generated by $V \otimes \ldots \otimes V \otimes F^1 V$. It is strictly contained in $F^1 O(\bullet \Pi_0^{dR})$, which is the subspace generated by $V \otimes \ldots \otimes F^1 V \otimes V \otimes \ldots$.

The reader is warned that since the Hodge filtration is not motivic, the scheme $\Pi_0^{dR}$ does not admit an action by the de Rham Galois group $G^{dR}$. Equivalently $\bullet H_0$ is not stable under the coaction by its affine ring $O(G^{dR})$.

11.5.3. Canonical de Rham path. By definition 11.4 there is a natural map
\[ \varepsilon : xH_0 \longrightarrow xH_0/F^1 xH_0 \cong W_0 xH_0 \cong k \]
which is a homomorphism. It defines a point $\text{Spec } k \rightarrow x\Pi_0^{dR}$.

Definition 11.7. Define the canonical path $x1_0 \in x\Pi_0^{dR}(k)$ to be this point.

Consider the composition of the coaction (definition 11.4 (iii))
\[ \Delta : xH_0 \longrightarrow O(\Pi_0^{dR}) \otimes_k xH_0 \]
with id $\otimes x1_0$. It yields a homomorphism we denote by $x1_0 : xH_0 \rightarrow O(\Pi_0^{dR})$.

Lemma 11.8. It defines a canonical isomorphism of algebras
\[ x1_0 : xH_0 \sim \rightarrow 0H_0 \].

Proof. By lemma 11.3 we can choose a point $c \in F^0 x\Pi_0^{dR}(k)$. The coaction
\[ \Delta : O(\Pi_0^{dR}) \longrightarrow O(\Pi_0^{dR}) \otimes_k O(\Pi_0^{dR}) \]
satisfies $\Delta F^1 \subset F^1 \otimes F^0 + F^0 \otimes F^1$. But $c$ annihilates $F^1 O(\Pi_0^{dR})$, therefore
\[ (\text{id} \otimes c) \Delta : F^1 O(\Pi_0^{dR}) \longrightarrow F^1 O(\Pi_0^{dR}) \).

By the torsor property, and strictness, 11.13 is an isomorphism since it induces an isomorphism on the associated graded for the Hodge filtration. Furthermore, by coassociativity of $\Delta$, 11.13 is compatible with the left-coaction by $O(\Pi_0^{dR})$ (on points, 11.13 corresponds to $g \mapsto cg : O \Pi_0^{dR} \sim \rightarrow x\Pi_0^{dR}$, which respects right-multiplication by $\Pi_0^{dR}$ since $ga \mapsto cga$). The algebras $xH_0$ and $0H_0$ are characterised by definition 11.3 and are therefore mapped isomorphically onto each other by 11.13.

Remark 11.9. In the case when $O(\bullet \Pi_0^{dR})$ has Hodge numbers of type $(p, p)$ only, then definition 11.4 gives $\bullet H_0 = O(\Pi_0^{dR})$. The natural map $\bullet \Pi_0^{dR} \rightarrow \Pi_0^{dR}$ is then an isomorphism, and the path $x1_0$ is an element of $\Pi_0^{dR}(k)$. Thus $x1_0$ is the natural generalisation of the canonical de Rham path to the non mixed-Tate setting.

11.6. Comparison with Albanese manifolds. Since $F^1 O(\bullet \Pi_0^{dR})$ is the ideal of functions vanishing on $F^0 \Pi_0^{dR}$ by 11.11, we make the following definition.

Definition 11.10. For $\bullet \in \{ x, 0 \}$, define an affine scheme over $k$
\[ F^0 \Pi_0^{dR} = \text{Spec } (W_0 + F^1 O(\bullet \Pi_0^{dR})) \).

It follows from the earlier properties of the Hodge filtration that $W_0 + F^1$ is indeed an algebra. It is not the group-theoretic quotient since $F^0$ is not a normal subgroup. Its complex points are the manifolds defined by Hain 11.
The scheme $F^0 \setminus \Pi^dR_0$ is equipped with a canonical point given by the composition:
\[ \mathcal{O}(F^0 \setminus \Pi^dR_0) \to \mathcal{O}(F^0 \setminus \Pi^dR_0)/F^1 = W_0\mathcal{O}(\Pi^dR_0) \cong k \]
since $W_0 \cap F^1 = 0$. The inclusions of spaces $\cdot H_0 \subset (W_0 + F_1)\mathcal{O}(\Pi^dR_0) \subset \mathcal{O}(\Pi^dR)$, together with the previous remark, gives rise to the following commutative diagram:
\[
\begin{array}{ccc}
F^0 \setminus \Pi^dR_0 & \longrightarrow & \text{Spec } (k) \\
\downarrow & & \downarrow \\
\cdot \Pi^dR_0 & \longrightarrow & F^0 \setminus \Pi^dR_0 \\
\end{array}
\]
In particular, the image of any path in $F^0 \cdot \Pi^dR_0(k)$ is the canonical path $x_1 \in \cdot \Pi^dR_0(k)$.

**Remark 11.11.** Let $N^0$ denote the normaliser of $F^0$ in $\Pi^dR_0$. Then the quotient $N^0 \setminus \Pi^dR_0$ is the affine group scheme whose affine ring is the largest Hopf subalgebra $A \subset (W_0 + F_1)\mathcal{O}(\Pi^dR_0)$. Therefore $\Delta A \subset A \otimes_k A$, and it follows that $A \subset 0H_0$ by definition. Our space $\cdot \Pi^dR_0$ therefore satisfies
\[ F^0 \setminus \Pi^dR_0 \to 0 \cdot \Pi^dR_0 \to N^0 \setminus 0 \Pi^dR_0 \]
but in general neither morphism is an isomorphism. Indeed, in example 11.6 the algebra $A$ is the tensor algebra on $F^1V$ and consists of ‘totally holomorphic’ iterated integrals.

**Caveat 11.12.** A point $c \in F^0 \setminus \Pi^dR_0(k)$ defines, via (11.13), an isomorphism
\[ F^0 \setminus \Pi^dR_0 \cong F^0 \setminus \pi \cdot \Pi^dR_0 \]
but this isomorphism is not canonical: it depends on the choice of point $c$. On the other hand, we have shown that the schemes $\cdot \Pi^dR_0$ and $\pi \cdot \Pi^dR_0$ are canonically isomorphic.

11.7. **General situation.** Let $\omega$ be the de Rham fiber functor. Let $n$ be a positive integer. Then $W_n\mathcal{O}(\cdot \Pi^dR_0)$ is the fiber at $x \in X(k)$ of an object in the category of unipotent vector bundles on $X$. There exists an open affine subset $U \subset X$, containing 0, such that the underlying algebraic vector bundle $W_n\mathcal{O}(\Pi_0)$ (forgetting its connection) is trivial on $U$. Therefore there is a canonical isomorphism for all $x \in U(k)$:
\[ x_0 : W_n\mathcal{O}(\cdot \Pi^dR_0) \cong \Gamma(U, W_n\mathcal{O}(\Pi_0)) \cong W_n\mathcal{O}(0 \cdot \Pi^dR_0). \]

Now choose a path $x_{\gamma_0} \in x \Pi^dR_0(k')$ for $x \in U(k)$. The map $a^x$ is dual to
\[ \begin{align*}
a^x : W_n\mathcal{O}(0 \cdot \Pi^dR_0) & \to \mathcal{P}^{dR, \omega'}_{X} \otimes_k k' \\
\eta & \mapsto [\mathcal{O}(\pi_1(X, x, 0)), x_{\gamma_0}, x_{c_0^{-1}} \eta]^{dR, \omega'} \end{align*} \]
By applying the method of [2] this can be used to construct ‘motivic’ Coleman functions and hence detect points in $U(k)$ whenever the requisite inequality on dimensions is satisfied. If $\omega'$ is the Betti fiber functor and $U(\mathbb{C})$ is simply-connected, then the class of $x_{\gamma_0}$ is uniquely defined. In the case when $\omega'$ is de Rham or crystalline, we can ask that the path $x_{\gamma_0}$ be Frobenius-invariant, and again its class is canonical. In the crystalline case, the $p$-adic period is given precisely by Coleman integration by [2].

This method should in principle work well enough to bound points on local affine charts $U$ of $X$. In order to eliminate this dependence on $U$, we use the canonical de Rham paths on the schemes $\pi$ as defined above. This gives:
\[ \begin{align*}
a^x : 0H_0 & \to \mathcal{P}^{dR, \omega'}_{X} \otimes_k k' \\
\eta & \mapsto [\mathcal{O}(\pi_1(X, x, 0)), x_{\gamma_0}, x_{c_0^{-1}} \eta]^{dR, \omega'} \end{align*} \]
where $0H_0$ is canonically identified with $\cdot H_0 \subset \mathcal{O}(\cdot \Pi^dR_0)$ via lemma 11.8. The point is that one only computes iterated integrals of forms in the subspace $0H_0 \subset \mathcal{O}(\cdot \Pi^dR_0)$. 

When $\omega'$ is de Rham or crystalline, and $x\gamma_0$ is the unique Frobenius-invariant path, then the homomorphism (11.15) depends on no choices. In this case its image is contained in the subspace of Frobenius-invariant periods.

11.8. Dimensions. We now wish to apply the construction of (11.4) in the situation where $\omega$ and $\omega'$ are de Rham and crystalline fiber functors, respectively. Let $x\gamma_0 \in \pi_0^d(k)$ be the unique Frobenius-invariant path.

Let us choose a path $x_0 \in F\pi_0^d(k)$ in the following way. The action of $G^d_{\text{dR}}$ on the de Rham image of the Tate object $\mathbb{Q}(-1)$ defines a character $\chi : G^d_{\text{dR}} \to \mathbb{G}_m$. By a version of Levi's theorem, it is possible to choose a splitting of this map over $k$, which in turn induces a splitting of the weight filtration on $\omega_{dR}(M)$ for all objects $M$ in $\mathcal{H}$, in such a way that it is compatible with the Hodge filtration (in fact one can split the Hodge and weight filtrations simultaneously). Thus for every object $M$ in $\mathcal{H}$ we have fixed a functorial isomorphism $M \cong \mathcal{G}^W M$. Applying this to the Ind object $\mathcal{O}(\pi_0^d)$, we obtain a homomorphism by projecting onto weight zero:

$$x_0 : \mathcal{O}(\pi_0^d) \twoheadrightarrow \mathcal{G}^W \mathcal{O}(\pi_0^d) \twoheadrightarrow \mathcal{G}^W \mathcal{O}(\pi_0^d) \cong k$$

for any point $x$. Since this is compatible with $F$, we have $F^1 \mathcal{O}(\pi_0^d) \subset \ker x_0$ and therefore $x_0 \in F\pi_0^d(k)$. The image of $x_0$ in $\pi_0^d(k)$ is the canonical point $x_1$.

This data provides homomorphisms

$$\alpha^x : \mathcal{O}(\pi_0^d) \to \mathcal{P}^{dR}_{X^dR}$$
$$\beta^x : \mathcal{O}(\pi_0^d) \to \mathcal{P}^{dR}_{X^dR} \otimes_k k'$$

where $\beta^x$ is cocycle associated to $\pi_0^d$. Note that although $\alpha^x$ depends on our choice of weight-splitting via $x_0$, we shall only consider its restriction to the subspace $\pi_0^d$, which is canonical. In particular, the data $(\alpha^x, (\alpha^x, \beta^x))$ defines an element of

$$Z^1_{\text{alg}}(G^dR, \Pi^dR) \times Z^1_{\text{alg}}(P, \pi_0^d)$$

where $P$ is the scheme of isomorphisms from $dR$ to $\omega'$. Consider the subspace $Z$ of this product consisting of bi-cocycles which respect the weight filtration, and furthermore such that $\beta^x$ is Frobenius-invariant, and $\alpha^x$ is $\mathbb{G}_m$-invariant, where $\mathbb{G}_m$ is viewed as a subgroup of $G^dR$ via our choice of weight-splitting. The space $Z$ is a functor from commutative rings to sets, and is equipped with a natural transformation

$$Z \to Z^1_{\text{alg}}(G^dR, \Pi^dR).$$

Let us assume that $Z, Z^1_{\text{alg}}(G^dR, \Pi^dR)$ are schemes: for instance, if $W_n \mathcal{O}(\pi_0^d)$ is finite dimensional for all $n$. The fibers of (11.15) are empty or torsors over $(\pi_0^d)^{G_{\text{dR}}} \mathbb{G}_m$, by the discussion following (11.7). This is because the image of $Z$ in $Z^1_{\text{alg}}(G^dR, \Pi^dR)$ is $\mathbb{G}_m$-invariant by definition of $Z$. By the same argument as in lemma 5.4, the group $(\pi_0^d)^{G_{\text{dR}}} \mathbb{G}_m$ is trivial. Therefore we have shown that the fibers of (11.15) are empty or 0-dimensional.

Since $x\gamma_0$ is the canonical Frobenius-invariant path, we deduce by (11.9) that

$$\dim Z \leq \dim H^1_{\text{dR}}(G^dR, \pi_0^d).$$

The identical reasoning applies for any quotient $\Pi$ of $\pi_0^d$ in the category $\mathcal{H}$ (for instance, $\Pi = \pi_0^d/W_m$). Thus the methods of (5.4) can be applied whenever

$$\dim \Pi^dR > \dim H^1_{\text{dR}}(G^dR, \Pi^dR).$$
We obtain a function vanishing on integral points by applying \([\square]\) with \(V = O(\Pi^{dR})\), \(W = \mathcal{P}_{\chi,dR}\) and \(Z\) the scheme of bi-cocycles described above.

**Remark 11.13.** The condition \((11.17)\) differs from \(\dim (F^0) > \dim H^1(G^\omega, \Pi^\omega)\) stated in [25], conjecture 1.

11.9. **Case of \(\mathbb{P}^1\setminus\{0,1,\infty\}\).** The previous discussion can be applied in the case \(X = \mathbb{P}^1\setminus\{0,1,\infty\}\). We retrieve an identical version of theorem [7.2] using Coleman integration instead of the single-valued \(p\)-adic periods. Because the fundamental group is mixed Tate, we have \(F^0(\Pi^\omega) = 1\) and \(0H_0 = O(0\Pi^{dR}_0)\). Therefore \(\omega_0\) is the canonical de Rham path \(1.8\) and \(0\Pi^{dR}_0 \cong 0\Pi^{dR}_0\). Inequality \((11.17)\) therefore reduces to

\[
\dim \Pi > \dim H^1_{dR}(G^\omega_S, \Pi^\omega)\,.
\]

Under this condition we can apply the identical reasoning to \((11.15)\) to deduce the analogue of theorem [7.2] with single-valued \(p\)-adic iterated integrals replaced with Coleman integrals, since the dimension computations are identical.

11.10. If one assumes a version of Beilinson’s conjectures as mentioned above, then one checks that \((11.17)\) will be satisfied for any curve of genus \(g \geq 2\) over \(\mathbb{Q}\), for essentially the same reasons as given in [25]. Roughly, the size of the graded pieces of the unipotent fundamental group grow approximately of the order \((2g)^n\) (the difference between \(\Pi\) and \(\Pi^\omega\) is essentially negligibley by example [11.6]), and the space of motivic extensions, once the weight is sufficiently large, is expected to be a lattice in the space of extensions of mixed Hodge structures. This is easily calculated, for example, by [8] (6.4). Under this assumption, it would be interesting to carry out a precise analysis of the dimensions, as in [9] and work out some explicit versions of theorem [7.2] namely, write down a determinant of \(p\)-adic iterated integrals which vanishes conjecturally on all \(N\)-tuples of integral points, for suitable \(N\).

It would be interesting to ascertain whether one can use complex periods to detect integral points on curves of higher genus, or whether this is ruled out. For example, one might optimistically hope to take \(\omega = \omega' = dR\), but this time apply the single-valued iterated integrals as defined in [6], §4.1, §8.3. Finally, the method above can also be extended to divisors in a very similar manner to [10].

**References**

[1] A. Beilinson, P. Deligne: *Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs*, Motives (Seattle, WA, 1991), 97-121, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[2] Besser, A.: *Coleman integration using the Tannakian formalism*, Math. Ann. 322 (2002), no. 1, 19-48.

[3] Besser, A.: *Heidelberg lectures on Coleman integration*, The arithmetic of fundamental groups—PIA 2010, 3-52, Contrib. Math. Comput. Sci., 2, Springer, Heidelberg, (2012).

[4] Besser, A., de Jeu, R.: *Li\((p)\)-service? An algorithm for computing \(p\)-adic polylogarithms*, Math. Comp. 77 (2008), no. 262, 1105-1134.

[5] F. Brown: *Motivic periods and the projective line minus 3 points*, proceedings of the ICM (2014). [arXiv:1407.5165](https://arxiv.org/abs/1407.5165)

[6] F. Brown: *Notes on motivic periods*, [arXiv:1512.06410](https://arxiv.org/abs/1512.06410)

[7] F. Brown: *Decomposition of motivic multiple zeta values*, ‘Galois–Teichmuller theory and Arithmetic Geometry’, Adv. Stud. Pure Math., 63, (2012).

[8] K. T. Chen: *Iterated path integrals*, Bull. Amer. Math. Soc. 83 (1977), 891-879.

[9] R. Coleman: *Effective Chabauty*, Duke Math. J. 52 (1985), no. 3, 765-770.

[10] B. Chiarellotto, B. Le Stum: *F-isocrystals unipotents*, Compositio Math. 116 (1999), no. 1, 81-110.

[11] I. Dan-Cohen, S. Wewers: *Mixed Tate motives and the unit equation*, Int. Math. Res. Not. IMRN 2016, no. 17, 5291-5354.
[12] P. Deligne, A. B. Goncharov: *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. École Norm. Sup. 38 (2005), 1 - 56.

[13] P. Deligne: *Le groupe fondamental de la droite projective moins trois points*, Math. Sci. Res. Inst. Publ., 16, (1989) 79297.

[14] P. Deligne: *Le groupe fondamental unipotent motivique de $\mathbb{G}_m - \mu_N$, pour $N = 2, 3, 4, 6$ ou 8*, Publ. Math. Inst. Hautes Études Sci. No. 112 (2010), 101-141.

[15] P. Erdős, C. Stewart, R. Tijdeman: *Some diophantine equations with many solutions*, Comp. Math. 66 (1988), 36-56.

[16] J-H. Evertse: *On equations in $S$-units and the Thue-Mahler equation*, Invent. Math. 75 (3), 561-584, 1984.

[17] H. Furusho: *$p$-adic multiple zeta values. I. $p$-adic multiple polylogarithms and the $p$-adic KZ equation*, Invent. Math. 155 (2004), no. 2, 253-286.

[18] A. B. Goncharov: *Galois symmetries of fundamental groupoids and noncommutative geometry*, Duke Math. J. 128 (2005), 209-284.

[19] R. Hain: *Higher Albanese Manifolds*, Lecture Notes in Math., 1246, 8491 (1987).

[20] R. Hain: *Iterated integrals and algebraic cycles: examples and prospects*, Contemporary trends in algebraic geometry and algebraic topology (Tianjin, 2000), 55118, Nankai Tracts Math., 5, World Sci. Publ., (2002).

[21] R. Hain: *Remarks on Non-Abelian Cohomology of Proalgebraic Groups*, J. Algebraic Geom. 22 (2013), no. 3, 581-598.

[22] M. Hadian: *On a motivic method in diophantine geometry*, The arithmetic of fundamental groups-PIA 2010, 127-146, Contrib. Math. Comput. Sci., 2, Springer, Heidelberg, (2012).

[23] G. H. Hardy, S. Ramanujan: *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) 17 (1918), 75-115.

[24] M. Kim: *The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel*, Invent. Math. 161 (2005), no. 3, 629-656.

[25] M. Kim: *The unipotent Albanese map and Sebner varieties for curves*, Publ. Res. Inst. Math. Sci. 45 (2009), no. 1, 89-133.

[26] M. Levine: *Tate motives and the vanishing conjectures for algebraic K-theory*, Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), 167188, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluwer Acad. Publ., Dordrecht, 1993.

[27] C. Siegel: *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss. Phys.-Math. Klasse, 1929: On some applications of Diophantine approximations, 81-138, Quad./Monogr., 2, Ed. Norm., Pisa, (2014).

[28] V. Vologodsky: *Hodge structure on the fundamental group and its application to $p$-adic integration*, Mosc. Math. J. 3 (2003), no. 1, 205-247, 260.

[29] Z. Wojtkowiak: *Cosimplicial objects in algebraic geometry*, Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), 287-327, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, (1993).

[30] G. Yamashita: *Bounds for the dimensions of $p$-adic multiple $L$-value spaces*, Doc. Math. 2010, Andrei A. Suslin’s sixtieth birthday, 687-723.