THE ERROR BOUNDS AND PERTURBATION BOUNDS OF THE ABSOLUTE VALUE EQUATIONS AND SOME APPLICATIONS

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Abstract. In this paper, by introducing a class of absolute value functions, we study the error bounds and perturbation bounds of two types of absolute value equations (AVEs): \( Ax - B|x| = b \) and \( Ax - |Bx| = b \). Some useful error bounds and perturbation bounds for the above two types of absolute value equations are presented. By applying the absolute value equations, we obtain some useful error bounds and perturbation bounds for the horizontal linear complementarity problem (HLCP). Incidentally, two new error bounds for linear complementarity problem (LCP) are given, coincidentally, which are equal to the existing result. Without constraint conditions, a new perturbation bound for the LCP is given as well. Besides, without limiting the matrix type, some computable estimates for the above upper bounds are given, which are sharper than some existing results under certain conditions. Some numerical examples for the AVEs from the LCP are given to show the feasibility of the perturbation bounds.

Key words. Absolute value equations; the error bound; the perturbation bound; horizontal linear complementarity problem; linear complementarity problem

AMS subject classifications. 90C33, 65G50, 65G20

1. Introduction. Considering the following two types of the absolute value equations (AVEs)

(1.1) \[ Ax - B|x| = b \]

and

(1.2) \[ Ax - |Bx| = b, \]

where \( A, B \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \), \(| \cdot |\) denotes the componentwise absolute value of the vector. The AVEs (1.1) and (1.2), respectively, were introduced in [1] by Rohn and [2] by Wu. Clearly, when \( B = I \) in (1.1) and (1.2), where \( I \) denotes the identity matrix, the AVEs (1.1) and (1.2) reduce to the standard absolute value equations

(1.3) \[ Ax - |x| = b, \]

which was considered in [3] by Mangasarian and Meyer.

Over these years, the AVEs (1.1), (1.2) and (1.3) have excited much interest since they often occur in many significant mathematical programming problems, including linear programs, quadratic programs, bimatrix game, linear complementarity problem (LCP), see [3–6] and references therein. For instance, the AVEs (1.1) is equal to the LCP of determining a vector \( z \in \mathbb{R}^n \) such that

(1.4) \[ w = Mz + q \geq 0, \quad z \geq 0 \text{ and } z^Tw = 0 \text{ with } M \in \mathbb{R}^{n \times n} \text{ and } q \in \mathbb{R}^n. \]

By using \( z = |x| + x \) and \( w = |x| - x \) for (1.4), the AVEs (1.1) is obtained with \( A = I + M \), \( B = I - M \) and \( b = -q \).

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Although the AVEs in [3] is a NP-hard problem, so far, a large number of theoretical results, numerical methods and applications have been extensively excavated. For instance, among the theoretical results, in spite of determining the existence of a solution to the AVEs in [5,20] is NP-hard, and checking whether the AVEs has unique or multiple solutions in [6] is also NP-complete, there still exist some very important conclusions, in particular, some sufficient and necessary conditions for ensuring the existence and uniqueness of the solutions of the AVEs (1.1), (1.2) and (1.3) were established for any \( b \in \mathbb{R}^n \), see [2, 7–9].

Likewise, solving the AVEs in [5] is NP-hard as well. It may be due to the fact that the AVEs contains a non-linear and non-differential absolute value operator. Even so, some efficient numerical methods have been developed, such as the generalized Newton method [10], Newton-based matrix splitting method [11], the exact and inexact Douglas-Rachford splitting method [12], the Picard-HSS method [13], the sign accord method [14], the concave minimization method [15], the Levenberg-Marquardt method [16], and so on.

As an important application of the AVEs, for all we know, the AVEs was first viewed as a very effective tool to gain the numerical solution of the LCP in [17], called the modulus method. At present, this numerical method has achieved rapid development and its many various versions were proposed, see [18,19] and references therein. Since the modulus method has the superiorities of simple construction and quick convergence behavior, it is often regarded as a top-priority method for solving the large-scale and sparse complementarity problem (CP).

In addition to the above aspects about the AVEs, another very important problem is the sensitivity and stability analysis of the AVEs, i.e., how the solution variation is when the data is perturbed. More specifically, when \( \Delta A, \Delta B \) and \( \Delta b \) are the perturbation terms of \( A, B \) and \( b \) in (1.1) and (1.2), respectively, how do we characterize the change in the solution of the perturbed AVEs. With respect to this regard, to our knowledge, the perturbation analysis of the AVEs (1.1) and (1.2) has not been discussed. In addition, for the error bound of AVEs, under the assumption of strongly monotone property, a global projection-type error bound was provided in [20]. Obviously, this kind of conditional projection-type error bound frequently has great limitation. Therefore, based on these considerations, in this paper, we in-depth discuss the error bounds and the perturbation bounds of the AVEs. Firstly, by introducing a class of the absolute value functions, the framework of error bounds for the AVEs are presented without any constraints. Without limiting the matrix type, some computable estimates for their upper bounds are given. These bounds are sharper than the existing bounds in [20] under certain conditions. Secondly, we establish the framework of perturbation bounds for the AVEs and present some computable upper bounds. It is pointed out that when the nonlinear term \( B|x| \) in (1.1) is vanished, the presented perturbation bounds reduce to the classical perturbation bounds for the linear systems \( Ax = b \), including Theorem 1.3 in numerical linear algebra textbooks [26] and Theorem 2.1 [27]. Thirdly, as another aspect of applications, by making use of the absolute value equations, we convert the HLCP to the equal certain absolute value equations, obtain the framework of error bounds and perturbation bounds for the HLCP, and gain some computable upper bounds without limiting the matrix type. In particular, two new equal error bounds for the LCP are exploited, concomitantly, three new computable upper bounds are obtained and sharper than that in [30] for the system matrix being an \( H_+ \)-matrix under proper conditions. Further, without the conditional conditions, we display a new framework of perturbation bound of the
LCP and obtain three new computable upper bounds advantage over that in [33] for the system matrix being a symmetric positive definite matrix and an $H_+\text{-}matrix$. Fourthly, a new approach for some existing perturbation bounds in [33] for the LCP is provided as well. Of course, finally, to show the efficiency of some proposed bounds, some numerical examples for the AVEs from the LCP are investigated.

The rest of the article is organized as follows. Section 2 provides the framework of error bounds for the AVEs by introducing a class of absolute value functions. In Section 3, some perturbation bounds for the AVEs are provided. In Section 4, the frameworks of error bounds and perturbation bounds for the HLCP are presented by using the AVEs. In Section 5, some numerical examples for the AVEs from the LCP are investigated. Finally, in Section 6, we end up with this paper with some conclusions.

Finally, to end this section, we remind some notations, definitions and conclusions in [2][7][34], which will be used in the later discussion.

Let $A = (a_{ij})$ and $N = \{1, 2, \ldots, n\}$. Then we denote $|A| = (|a_{ij}|)$. $A = (a_{ij})$ is called an $M\text{-}matrix$ if $A^{-1} \geq 0$ and $a_{ij} \leq 0$ $(i \neq j)$ for $i, j \in N$; an $H\text{-}matrix$ if its comparison matrix $(A)$ (i.e., $\langle a_{ii} = |a_{ii}|, \langle a_{ij} = -|a_{ij}| i \neq j$ for $i, j \in N$) is an $M\text{-}matrix$; an $H_+\text{-}matrix$ if $A$ is an $H\text{-}matrix$ with $a_{ii} > 0$ for $i \in N$; a $P\text{-}matrix$ if all principal minors of $A$ are positive. Let $\rho(\cdot)$, $\sigma_{\min}$ and $\sigma_{\max}$ denote the spectral radius, the smallest singular value and the largest singular value of matrix, respectively. For two vectors $q, e \in \mathbb{R}^n$, by $q_+$ and $q_-$ we denote $q_+ = \max\{0, q\}$, $q_- = \max\{0, -q\}$ and $e = (1, 1, \ldots, 1)^T$. The norm $\|\cdot\|$ means $p\text{-}norm$, i.e., $\|\cdot\|_p$ with $p \geq 1$.

The AVEs (1.1) has a unique solution for any $b \in \mathbb{R}^n$ if and only if $A - BD$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $d_i \in [-1, 1]$, see Theorem 3.2 in [7]; the AVEs (1.2) has a unique solution for any $b \in \mathbb{R}^n$ if and only if $A - DB$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $d_i \in [-1, 1]$, see Theorem 3.3 in [2].

2. Error bound. In this section, without further illustration, we always assume that the matrix $A - BD$ or $A - DB$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $d_i \in [-1, 1]$ such that the AVEs (1.1) or (1.2) has the unique solution, respectively. Under this premise, we can give the framework of error bounds on the distance between the approximate solution and the exact solution of the AVEs (1.1) and (1.2).

2.1. Framework of error bounds for AVEs. In this subsection, the framework of error bounds for the AVEs is obtained. To achieve our goal, the following absolute value function is introduced, see Lemma 2.1.

**Lemma 2.1.** Let $a = (a_1, a_2, \ldots, a_n)^T$, $b = (b_1, b_2, \ldots, b_n)^T$ be any two vectors in $\mathbb{R}^n$. Then there exist $d_i \in [-1, 1]$ such that

$$(2.1) \quad |a_i| - |b_i| = d_i(a_i - b_i), i = 1, 2, \ldots, n.$$ 

**Proof.** Its proof is straightforward, which is omitted. \[\square\]

Let $r(x) = Ax - B|x| - b$. Clearly, $x^*$ is a solution of the AVEs (1.1) if and only if $r(x^*) = 0$. The function $r(x)$ is called the natural residual of the AVEs (1.1). Let $x^*$ be the unique solution of the
AVEs (1.1). Then from Lemma 2.1 we have
\[ r(x) = Ax - B|x| - b - (Ax^* - B|x^*| - b) \]
\[ = A(x - x^*) - B(|x| - |x^*|) \]
\[ = A(x - x^*) - BD(x - x^*) \]
\[ = (A - B\tilde{D})(x - x^*), \]
where \( \tilde{D} = \text{diag}(\tilde{d}_i) \) with \( \tilde{d}_i \in [-1, 1] \), which promptly results in the error bounds for the AVEs (1.1), see Theorem 2.2.

**Theorem 2.2.** Let \( x^* \) be the unique solution of AVEs (1.1). Then for any \( x \in \mathbb{R}^n \) and \( D = \text{diag}(d_i) \) with \( d_i \in [-1, 1] \),
\[
\frac{1}{\alpha} \|r(x)\| \leq \|x - x^*\| \leq \alpha \|r(x)\|,
\]
where
\[ \alpha = \max \|A - BD\| \quad \text{and} \quad \overline{\alpha} = \max \|(A - BD)^{-1}\|. \]

By using the same technique for the AVEs (1.2), we have

**Theorem 2.3.** Let \( x^* \) be the unique solution of AVEs (1.2). Then for any \( x \in \mathbb{R}^n \) and \( D = \text{diag}(d_i) \) with \( d_i \in [-1, 1] \),
\[
\frac{1}{\beta} \|r(x)\| \leq \|x - x^*\| \leq \beta \|r(x)\|,
\]
where
\[ \beta = \max \|A - DB\| \quad \text{and} \quad \overline{\beta} = \max \|(A - DB)^{-1}\|. \]

When \( B = I \) in Theorem 2.2 or Theorem 2.3, the error bounds for the AVEs (1.3) can be obtained, see Corollary 2.1.

**Corollary 2.1.** Let \( x^* \) be the unique solution of AVEs (1.3). Then for any \( x \in \mathbb{R}^n \) and \( D = \text{diag}(d_i) \) with \( d_i \in [-1, 1] \),
\[
\frac{1}{\gamma} \|r(x)\| \leq \|x - x^*\| \leq \gamma \|r(x)\|,
\]
where
\[ \gamma = \max \|A - D\| \quad \text{and} \quad \overline{\gamma} = \max \|(A - D)^{-1}\|. \]

In [20], with strongly monotone property, Chen et al. also considered the global error bound of the AVEs (1.3) and presented the following result.

**Theorem 2.4.** *(Theorem 4.1 in [20]*) Let \( x^* \) be the unique solution of AVEs (1.3). Then for any \( x \in \mathbb{R}^n \),
\[
\frac{1}{C} \|r(x)\| \leq \|x - x^*\| \leq C \|r(x)\|,
\]
where
\[ C = \|A + I\| + \|A - I\| \quad \text{and} \quad \overline{C} = \frac{\|A + I\| + \|A - I\|}{\sigma_{\min}(A)^2 - 1} \quad \text{with} \quad \sigma_{\min}(A) > 1. \]

Now, we show that the error bounds in Corollary 2.1 are sharper than that in Theorem 2.4. Here, we consider \[ \| \cdot \|_2 \] for Corollary 2.1 and Theorem 2.4.

**Theorem 2.5.** Let the assumptions of Corollary 2.1 and Theorem 2.4 be satisfied. Then
\begin{equation}
\label{eq2.6}
\|A + I\|_2 + \|A - I\|_2 \geq \max \|A - D\|_2
\end{equation}
and
\begin{equation}
\label{eq2.7}
\max \|(A - D)^{-1}\|_2 \leq \frac{\|A + I\|_2 + \|A - I\|_2}{\sigma_{\min}(A)^2 - 1}.
\end{equation}

**Proof.** Firstly, we prove (2.6). Since \[ \max \|A - D\|_2 \] is a convex polyhedron, its maximum value is obtained at the vertex of \[ D \], i.e.,
\[ \max \|A - D\|_2 = \max \{\|A + I\|_2, \|A - I\|_2\}, \]
from which we prove (2.6).

Next, we prove (2.7). From \[ \sigma_{\min}(A) > 1 \], we know \[ \|A^{-1}\|_2 < 1 \]. Further,
\[ \|A^{-1}\|_2 \|D\|_2 < 1. \]
By Banach Perturbation Lemma in [21], clearly,
\[ \|(A - D)^{-1}\|_2 \leq \frac{\|A^{-1}\|_2}{1 - \|A^{-1}\|_2 \|D\|_2} \leq \|A^{-1}\|_2. \]
By the simple computation, we have
\[ \|A^{-1}\|_2 (\sigma_{\min}(A)^2 - 1) = \|A^{-1}\|_2 \left( \frac{1}{\sigma_{\max}(A^{-1})^2} - 1 \right) \]
\[ = \|A^{-1}\|_2 \left( \frac{1}{\|A^{-1}\|_2^2} - 1 \right) \]
\[ = \frac{1 - \|A^{-1}\|_2^2}{\|A^{-1}\|_2^2}. \]
Since
\[ \|I + A^{-1}\|_2 + \|I - A^{-1}\|_2 = \|A^{-1}(I + A)\|_2 + \|A^{-1}(I - A)\|_2 \leq \|A^{-1}\|_2 (\|A + I\|_2 + \|A - I\|_2), \]
we only show that
\begin{equation}
\label{eq2.8}
1 - \|A^{-1}\|_2^2 \leq \|I + A^{-1}\|_2 + \|I - A^{-1}\|_2.
\end{equation}
In fact, the inequality (2.8) holds because
\[ 1 - \|A^{-1}\|_2^2 < 2 = \|2I\|_2 = \|I + A^{-1} + I - A^{-1}\|_2 \leq \|I + A^{-1}\|_2 + \|I - A^{-1}\|_2. \]
This proves (2.7). \( \square \)

From the proof of Theorem 2.5, we find some interested results for the matrix norm, see Proposition 2.1.

**Proposition 2.1.** The following statements hold:
(1) For $A \in \mathbb{R}^{n \times n}$ and $\alpha > 0$ in $\mathbb{R}$,

$$2\alpha - \|A\| \leq \|\alpha I + A\| + \|\alpha I - A\|.$$  

(2) Let $A \in \mathbb{R}^{n \times n}$ and $\alpha \geq 0$ in $\mathbb{R}$. Then for $\|A\| \leq \alpha$,

$$\alpha + \|A\| \leq \|\alpha I + A\| + \|\alpha I - A\|.$$  

2.2. Estimations of $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$. In the Section 2.1, we have given some error bounds for the AVEs (1.1), (1.2) and (1.3) in Theorem 2.2, Theorem 2.3, and Corollary 2.1, respectively. From the proof of Theorem 2.5, it is not difficult to find that

$$\overline{\alpha} = \overline{\beta} = \max\{|A - B|, |A + B|\} \text{ and } \overline{\gamma} = \max\{|A - I|, |A + I|\}.$$  

However, in general, it is difficult to compute quantities $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ because they contain any $D = \text{diag}(d_i)$ with $d_i \in [-1, 1]$. To overcome this disadvantage, in this subsection, we explore some computable estimations for $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$.

In the following, we focus on estimating the value of $\overline{\alpha}$. For $\overline{\beta}$, its process is completely analogous, with regard to $\overline{\gamma}$ just for their special case. To present the reasonable estimations for $\overline{\alpha}$, here we consider three aspects: (1) $\rho(|A^{-1}B|) < 1$; (2) $\sigma_{\min}(A) > \sigma_{\max}(B)$; (3) $\sigma_{\max}(A^{-1}B) < 1$. For these three cases, the AVEs (1.1) for any $b \in \mathbb{R}^n$ has a unique solution, see Theorem 2 in [23] and Theorem 2.1 in [24], Corollary 3.2 in [7]. Similarly, for $\overline{\beta}$, we display three aspects: (1) $\rho(|BA^{-1}|) < 1$; (2) $\sigma_{\min}(A) > \sigma_{\max}(B)$; (3) $\sigma_{\max}(BA^{-1}) < 1$. For these three cases, the AVEs (1.2) for any $b \in \mathbb{R}^n$ has a unique solution as well, see Corollary 3.3 in [2] and Lemma 2.2 in [25], Corollary 3.2 in [2].

2.2.1. Case I. Assume that matrices $A$ and $B$ in (1.1) satisfy

$$\rho(|A^{-1}B|) < 1.$$  

We can present a reasonable estimation for $\overline{\alpha}$, see Theorem 2.6.

**Theorem 2.6.** Let $\rho(|A^{-1}B|) < 1$ in (1.1). Then

$$\overline{\alpha} \leq \|(I - |A^{-1}B|^{-1}||A^{-1}|.$$  

**Proof.** Since $A^{-1}BD \leq |A^{-1}BD| \leq |A^{-1}B|$, by Theorem 8.1.18 of [22], we get

$$\rho(A^{-1}BD) \leq \rho(|A^{-1}BD|) \leq \rho(|A^{-1}B|) < 1.$$  

So

$$|(I - A^{-1}BD)^{-1}| = |I + (A^{-1}BD) + (A^{-1}BD)^2 + ...|$$

$$\leq |I + (A^{-1}BD) + (|A^{-1}BD|)^2 + ...|$$

$$\leq |I + (|A^{-1}B|) + (|A^{-1}B|)^2 + ...|$$

$$=(I - |A^{-1}B|)^{-1}.$$  

Combining

$$\|(A - BD)^{-1}\| = \|(I - A^{-1}BD)^{-1}A^{-1}\| \leq \|(I - A^{-1}BD)^{-1}\| \|A^{-1}\|$$
with
\[ \|(I - A^{-1}BD)^{-1}\| \leq \|\|(I - A^{-1}BD)\|^{-1}\| \leq \|(I - |A^{-1}B|)^{-1}\|, \]
the desired bound (2.9) can be gained. \[\square\]

Similar to the proof of Theorem 2.6, for $\overline{\beta}$, we have

**Theorem 2.7.** Let $\rho(|BA^{-1}|) < 1$ in (1.2). Then
\[ \overline{\beta} \leq \|A^{-1}\|\|(I - |BA^{-1}|)^{-1}\|. \]

Needless to say, for $\overline{\gamma}$, we have

**Corollary 2.2.** Let $\rho(|A^{-1}|) < 1$ in (1.3). Then
\[ \overline{\gamma} \leq \|A^{-1}\|\|(I - |A^{-1}|)^{-1}\|. \]

### 2.2.2. Case II.

If $\sigma_{\min}(A) > \sigma_{\max}(B)$ in (1.1), then for $\overline{\alpha}$ we have Theorem 2.8.

**Theorem 2.8.** Let $\sigma_{\min}(A) > \sigma_{\max}(B)$ in (1.1). Then
\[ (2.10) \quad \overline{\alpha} \leq \frac{1}{\sigma_{\min}(A) - \sigma_{\max}(B)}. \]

**Proof.** Since
\[ \sigma_{\min}(A) - \sigma_{\max}(BD) \geq \sigma_{\min}(A) - \sigma_{\max}(B)\sigma_{\max}(D) \geq \sigma_{\min}(A) - \sigma_{\max}(B) \]
and
\[ \sigma_{\min}(A - BD) \geq \sigma_{\min}(A) - \sigma_{\max}(BD), \]
we have
\[ \|(A - BD)^{-1}\|_2 = \sigma_{\max}((A - BD)^{-1}) \]
\[ = \frac{1}{\sigma_{\min}(A - BD)} \]
\[ \leq \frac{1}{\sigma_{\min}(A) - \sigma_{\max}(BD)} \]
\[ \leq \frac{1}{\sigma_{\min}(A) - \sigma_{\max}(B)}. \]

This completes the proof for Theorem 2.8. \[\square\]

For $\overline{\beta}$, we have the same as the result in Theorem 2.9.

**Theorem 2.9.** Let $\sigma_{\min}(A) > \sigma_{\max}(B)$ in (1.2). Then
\[ \overline{\beta} \leq \frac{1}{\sigma_{\min}(A) - \sigma_{\max}(B)}. \]

**Corollary 2.3.** Let $\sigma_{\min}(A) > 1$ in (1.3). Then
\[ \overline{\gamma} \leq \frac{1}{\sigma_{\min}(A) - 1}. \]
It is easy to see that the upper bound in Corollary 2.3 is still sharper than that in Theorem 2.4, i.e.,

\[
\frac{1}{\sigma_{\min}(A) - 1} \leq \frac{\|A + I\|_2 + \|A - I\|_2}{\sigma_{\min}(A)^2 - 1},
\]

which is equal to

\[
\sigma_{\min}(A) + 1 \leq \|A + I\|_2 + \|A - I\|_2.
\]

In fact, by using Proposition 2.1, we have

\[
1 + \|A^{-1}\|_2 \leq \|I + A^{-1}\|_2 + \|I - A^{-1}\|_2 \leq \|A^{-1}\|_2(\|A + I\|_2 + \|A - I\|_2).
\]

**Remark 2.10.** The conditions in Theorems 2.6 and 2.8 are not included each other, e.g., take

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.9 & -0.4 \\ 0.4 & 0.9 \end{bmatrix}.
\]

By the simple computation,

\[
\sigma_{\min}(A) = 1 > \sigma_{\max}(B) = 0.9849,
\]

but

\[
\rho(|A^{-1}B|) = 1.3 > 1.
\]

This shows that matrices A and B satisfy the condition in Theorem 2.8, do not satisfy the condition in Theorem 2.6. Now we take

\[
A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1.6 & 0 \\ 0 & 1.6 \end{bmatrix}.
\]

By the simple computation,

\[
\rho(|A^{-1}B|) = 0.8000 < 1,
\]

but

\[
\sigma_{\min}(A) = 1.5616 < \sigma_{\max}(B) = 1.6.
\]

This shows that matrices A and B satisfy the condition in Theorem 2.6, do not satisfy the condition in Theorem 2.8.

**2.2.3. Case III.** Here, we consider this case that B is nonsingular and \(\sigma_{\max}(A^{-1}B) < 1\) in (1.1). Based on this, for \(\overline{\sigma}\) we have Theorem 2.11.

**Theorem 2.11.** Let B be nonsingular and \(\sigma_{\max}(A^{-1}B) < 1\) in (1.1). Then

\[
(2.11) \quad \overline{\sigma} \leq \|A^{-1}B\|_2 \|B^{-1}\|_2.
\]

**Proof.** From Corollary 3.2 in [1], \(A + BD\) is nonsingular under the assumptions. So, we have

\[
\|(A - BD)^{-1}\|_2 = \|(B^{-1}A - D)^{-1}B^{-1}\|_2 \leq \|(B^{-1}A - D)^{-1}\|_2 \|B^{-1}\|_2.
\]
Noting that $\sigma_{\text{max}}(A^{-1}B) < 1$ is equal to $\|A^{-1}B\|_2 < 1$ and 
$\|A^{-1}BD\|_2 \leq \|A^{-1}B\|_2 \|D\|_2 < 1$.

Making use of Banach perturbation lemma in [21] leads to
$\|(B^{-1}A - D)^{-1}\|_2 \leq \frac{\|A^{-1}B\|_2^2}{1 - \|A^{-1}B\|_2 \|D\|_2} \leq \|A^{-1}B\|_2$.

Therefore, the proof of Theorem 2.11 is completed.

For $\beta$, we have

**Theorem 2.12.** Let $B$ be nonsingular and $\sigma_{\text{max}}(BA^{-1}) < 1$ in (1.2). Then
$$\beta \leq \|B^{-1}\|_2 \|BA^{-1}\|_2.$$ 

**Corollary 2.4.** Let $\sigma_{\text{max}}(A^{-1}) < 1$ in (1.3). Then
$$\gamma \leq \|A^{-1}\|_2.$$ 

**Remark 2.13.** Comparing Theorem 2.8 with Theorem 2.11, it is easy to find that
$$\sigma_{\text{max}}(A^{-1}B) \leq \sigma_{\text{max}}(A^{-1}) \sigma_{\text{max}}(B) = \frac{\sigma_{\text{max}}(B)}{\sigma_{\text{min}}(A)}.$$

Whereas, it does not show that Theorem 2.11 is weaker than Theorem 2.8 because Theorem 2.11 asks for $B$ being nonsingular. Besides, we need to point out that Theorem 2.11 sometimes performs better than Theorem 2.8, vice versa. To illustrate this, we take
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}.$$ 

Obviously, $B$ is nonsingular,
$$\sigma_{\text{max}}(A^{-1}B) = 0.5 < 1, \sigma_{\text{min}}(A) = 2 > \sigma_{\text{max}}(B) = 1.5.$$ 

This shows that the conditions in Theorem 2.8 with Theorem 2.11 are satisfied. From Theorem 2.8 and Theorem 2.11, we have
$$\|A^{-1}B\|_2 \|B^{-1}\|_2 = 0.5 < \frac{1}{\sigma_{\text{min}}(A) - \sigma_{\text{max}}(B)} = 2,$$
from which shows that the upper bound in Theorem 2.11 is sharper than that in Theorem 2.8. Now, we take
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}.$$ 

Likewise, $B$ is nonsingular,
$$\sigma_{\text{max}}(A^{-1}B) = 0.5 < 1, \sigma_{\text{min}}(A) = 2 > \sigma_{\text{max}}(B) = 1.$$ 

This implies that the conditions in Theorem 2.8 with Theorem 2.11 are satisfied as well. From Theorem 2.8 and Theorem 2.11, we have
$$\|A^{-1}B\|_2 \|B^{-1}\|_2 = \frac{5}{4} > \frac{1}{\sigma_{\text{min}}(A) - \sigma_{\text{max}}(B)} = 1,$$
which implies that the upper bound in Theorem 2.8 is sharper than that in Theorem 2.11.
3. Perturbation bound. In this section, we focus on the perturbation analysis of AVEs when $A, B$ and $b$ are perturbed. For instance, for the AVEs (1.1), when $\Delta A$, $\Delta B$ and $\Delta b$ are the perturbation terms of $A$, $B$ and $b$, respectively, how do we characterize the change in the solution of the following perturbed AVEs (3.1)

$$ (A + \Delta A)y - (B + \Delta B)|y| = b + \Delta b. $$

For the AVEs (1.1), firstly, we consider the following special case (3.2)

$$ Ay - B|y| = b + \Delta b. $$

Assume that the AVEs (3.2) has the unique solution $y^*$. Let $x^*$ be the unique solution of AVEs (1.1). Subtracting (1.1) from (3.2), we have

$$ Ax^* - B|x^*| - (Ay^* - B|y^*|) = -\Delta b, $$

which is equal to

$$ x^* - y^* = -(A - B\tilde{D})^{-1}\Delta b, $$

where $\tilde{D} = diag(\tilde{d})$ with $\tilde{d}_i \in [-1, 1]$. Making use of the norm for both sides of (3.3), noting that

$$ \|A - B\tilde{D}\|^{-1} \leq \max \|A - BD\|^{-1} $$

for any $D = diag(d)$ with $d_i \in [-1, 1]$, we have

$$ \|x^* - y^*\| \leq \max \|A - BD\|^{-1}\|\Delta b\|. $$

Moreover, from the AVEs (1.1) with $x^*$, it is easy to check that

$$ \frac{\|b\|}{\|x^*\|} \leq \|A\| + \|B\|. $$

Combining (3.5) with (3.6), we obtain

$$ \frac{\|x^* - y^*\|}{\|x^*\|} \leq \max \|A - BD\|^{-1}\|\Delta b\| \left(\frac{\|b\|}{\|x^*\|} \right) \leq \max \|A - BD\|^{-1}\|\Delta b\| \left(\frac{\|b\|}{\|x^*\|} \right) $$

from which we immediately get

**Theorem 3.1.** Let $x^*$, $y^*$ be the unique solutions of AVEs (1.1) and (3.2), respectively. Then for any $D = diag(d_i)$ with $d_i \in [-1, 1]$,

$$ \frac{\|x^* - y^*\|}{\|x^*\|} \leq \max \|A - BD\|^{-1}\|\Delta b\| \left(\frac{\|b\|}{\|x^*\|} \right) $$

Similarly, we assume that $y^*$ is the unique solution of AVEs (3.1). The following theorem is the framework of AVEs (1.1) perturbation.

**Theorem 3.2.** Let $x^*$, $y^*$ be the unique solutions of AVEs (1.1) and (3.1), respectively. Then for any $D = diag(d_i)$ with $d_i \in [-1, 1]$

$$ \frac{\|x^* - y^*\|}{\|x^*\|} \leq \max \|A - BD + (\Delta A - \Delta BD)\|^{-1}\left(\frac{\|\Delta b\|}{\|b\|} \left(\frac{\|b\|}{\|x^*\|} \right) \right). $$
\textbf{Proof.} Based on the assumptions, the AVEs (1.1) is equal to

\begin{equation}
(A + \Delta A)x^* - (B + \Delta B)|x^*| = b + \Delta Ax^* - \Delta B|x^*|.
\end{equation}

Subtracting (3.8) from (3.1) with $y^*$, we have

\begin{equation}
(A - B\hat{D} + (\Delta A - \Delta B\hat{D}))(x^* - y^*) = -\Delta b + \Delta Ax^* - \Delta B|x^*|,
\end{equation}

where $\hat{D} = \text{diag}(\hat{d}_i)$ with $\hat{d}_i \in [-1, 1]$. Using the norm for (3.9), similar to (3.8), results in

\begin{equation}
\|x^* - y^*\| \leq \max \| (A - BD + (\Delta A - \Delta BD))^{-1} \| (\|\Delta b\| + (\|\Delta A\| + \|\Delta B\|)) \|x^*\|
\end{equation}

for any $D = \text{diag}(d_i)$ with $d_i \in [-1, 1]$. By making use of (3.10) and (3.6), the desired bound (3.7) can be obtained. 

\textbf{Remark 3.3.} Here, a special case is considered, i.e., when $B = 0$ in (1.1), the AVEs (1.1) reduces to the linear systems

\[Ax = b.\]

It is not difficult to find that from Theorem 3.2 we easily obtain a classical perturbation bound for the linear systems $Ax = b$, i.e.,

\[\frac{\|x^* - y^*\|}{\|x^*\|} \leq \frac{\kappa(A)}{1 - \kappa(A)} \left( \frac{\|\Delta b\|}{\|b\|} + \frac{\|\Delta A\|}{\|A\|} \right)
\]

for $\|A^{-1}\|\|\Delta A\| < 1$, where $\kappa(A) = \|A^{-1}\||\|A\|$, see Theorem 1.3 in numerical linear algebra textbooks [20].

In addition, when $|\Delta A| \leq \epsilon|A|$, $|\Delta B| \leq \epsilon|B|$ and $|\Delta b| \leq \epsilon|b|$, we have

\textbf{Theorem 3.4.} Let $x^*$, $y^*$ be the unique solutions of AVEs (1.1) and (3.1), respectively. Assume that $|\Delta A| \leq \epsilon|A|$, $|\Delta B| \leq \epsilon|B|$ and $|\Delta b| \leq \epsilon|b|$. Then

\begin{equation}
\frac{\|x^* - y^*\|}{\|x^*\|} \leq \epsilon \max \left( \frac{(A - BD)^{-1}((|A| + |B|)|x^*|)}{1 - \epsilon \max ((A - BD)^{-1}((|A| + |B|))\|x^*\|)} \right),
\end{equation}

where $\epsilon \max \| (A - BD)^{-1}((|A| + |B|)) \| < 1$.

\textbf{Proof.} From (3.10), we have

\[x^* - y^* = (A - B\hat{D})^{-1}(-(\Delta A - \Delta B\hat{D}))(x^* - y^*) - \Delta b + \Delta Ax^* - \Delta B|x^*|,
\]

and so

\[\|x^* - y^*\| = \|(A - B\hat{D})^{-1}(-(\Delta A - \Delta B\hat{D}))(x^* - y^*) - \Delta b + \Delta Ax^* - \Delta B|x^*|\|
\]

\[\leq \|(A - B\hat{D})^{-1}((\Delta A - \Delta B\hat{D}))(x^* - y^*)\| + \|(A - B\hat{D})^{-1}((\Delta b + \Delta Ax^* + \Delta B|x^*|)\|\]

\[\leq \epsilon \|(A - B\hat{D})^{-1}((|A| + |B|)|x^* - y^*|) + \epsilon \|(A - B\hat{D})^{-1}((|A| + |B|)|x^*|)\|
\]

\[\leq \epsilon \max ((A - BD)^{-1}((|A| + |B|))|x^* - y^*| + \epsilon \max ((A - BD)^{-1}((|A| + |B|))|x^*|).
\]

Hence,

\[\|x^* - y^*\| \leq \epsilon \max ((A - BD)^{-1}((|A| + |B|)|x^* - y^*|) + \epsilon \max ((A - BD)^{-1}((|A| + |B|))|x^*|),
\]
from which we obtain the desired bound $\|x^* - y^*\|$.  

**Remark 3.5.** Similar to Remark 3.3, when $B = 0$ in (1.7), Theorem 3.4 reduces to another classical perturbation bound for the linear systems $Ax = b$, i.e.,

$$
\frac{\|x^* - y^*\|}{\|x^*\|} \leq \epsilon \frac{\|A^{-1}|A||b + |A|x^*\|}{(1 - \epsilon \|A^{-1}|A||)}||x^*||
$$

where $\epsilon \|A^{-1}|A|| < 1$, see Theorem 2.1 [27] and Theorem 1.1 in [28].

Noting that $|(I - A^{-1}BD)^{-1}| \leq (I - |A^{-1}B|)^{-1}$ for $\rho(|A^{-1}B|) < 1$ from Theorem 2.6, together with Theorem 3.4, Theorem 3.6 is obtained and its advantage successfully avoids any diagonal matrix $D = \text{diag}(d_i)$ with $d_i \in [-1, 1]$.  

**Theorem 3.6.** Let $x^*, y^*$ be the unique solutions of AVEs (1.1) and (3.1), respectively. Assume that $|\Delta A| \leq \epsilon |A|$, $|\Delta B| \leq \epsilon |B|$, $|\Delta b| \leq \epsilon |b|$ and $\rho(|A^{-1}B|) < 1$. Then

$$
(3.12) \quad \frac{\|x^* - y^*\|}{\|x^*\|} \leq \epsilon \frac{\|I - |A^{-1}B|\|^{-1}|A^{-1}|(|b + |A| |x^*|)\|}{(1 - \epsilon \|I - |A^{-1}B|\|^{-1}|A^{-1}|(|A| + |B|)||x^*||)}
$$

where $\epsilon \|I - |A^{-1}B|\|^{-1}|A^{-1}|(|A| + |B|)|| < 1$.

**Remark 3.7.** Making use of “$DB$” and/or “$D\Delta B$” instead of “$BD$” and/or “$D\Delta B$” in Theorems 3.1, 3.2 and 3.4 directly, some perturbation bounds for the AVEs (3.1) can be obtained. Of course, by using the simple modifications for Theorem 3.6, its modified version is suit for the AVEs (1.3) as well, which is omitted.  

**Remark 3.8.** It is easy to see that $B = I$ and/or $\Delta B = 0$ in Theorems 3.1, 3.2, 3.4 and 3.6, some perturbation bounds for the AVEs (1.3) can be obtained as well.  

**Remark 3.9.** By directly making use of the results in the Section 2.2, naturally, we can present some computable upper bounds for $\max \|(A - BD)^{-1}\|$ in Theorem 3.1 and $\max \|(A - BD + (\Delta A - \Delta BD))^{-1}\|$ in Theorem 3.2.

4. Application in HLCP. In this section, by making use of the AVEs (1.1), we can derive some error bounds and perturbation bounds for the HLCP. Coincidentally, two new equal forms about the error bounds for LCP are given. Without constraint conditions, a new perturbation bound for LCP is obtained, compared with Theorem 3.1 in [33].

As is known, the horizontal linear complementarity problem is to determine a pair of vectors $w, z$ in $\mathbb{R}^n$ such that

$$
(4.1) \quad Mz - Nw = q, z \geq 0, w \geq 0, z^T w = 0,
$$

where $M, N \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, respectively, are given known matrices and the vector, which is denoted by HLCP and comes from the well-known book [29] by Cottle et al. Clearly, when $N = I$, the HLCP (4.1) reduces to the well-known LCP.

4.1. Error bound for HLCP. In this subsection, we will discuss the error bound for HLCP. Our approach is to display the HLCP (4.1) in the form of absolute value equations to study the error bound for HLCP. For this goal, we take $w = \frac{|x^* - x|}{2}$ and $z = \frac{|z| + |w|}{2}$ for (1.1). Under this transformation, the HLCP (4.1) is equally converted to the following absolute value equations

$$
(4.2) \quad \frac{M + N}{2} x - \frac{N - M}{2} |x| = q.
$$

Obviously, to gain the error bound for the HLCP (4.1) is equivalent to gain the error bound for the AVEs (1.2). That is to say, we can discuss the error bound for the
HLCP (4.1) from the point of absolute value equations. This approach is completely different from the published work in [30].

To help our discussion, the following definition and lemma are required.

Definition 4.1. [31] Matrix $H = \{M, N\}$ has the column $W$-property if the determinants of all column representative matrices of $H = \{M, N\}$ are all positive or all negative.

By using Theorem 2 (b) in [31], Lemma 4.2 can be obtained, its proof is omitted.

Lemma 4.2. Matrix $H = \{M, N\}$ has the column $W$-property if and only if $M(I - T) + NT$ is nonsingular for any nonnegative diagonal matrix $T = \text{diag}(t)$ with $t \in [0, 1]^n$.

Remark 4.3. Gabriel and Moré in [37] proved that $A$ is a $P$-matrix if and only if $A + T(I-A)$ is nonsingular for any nonnegative diagonal matrix $T = \text{diag}(t)$ with $t \in [0, 1]^n$. Clearly, Lemma 4.2 is a generalization of their result.

From Theorem 2 in [31], we know that the HLCP (4.1) has the unique solution for $H$ having the column $W$-property. Without loss of generality, we always assume that the block matrix $H$ satisfies the conditions of Lemma 4.2 such that the HLCP (4.1) has the unique solution $(w^*, z^*)$. In this case, the corresponding $x^* (x^* = z^* - w^*)$ is the unique solution of the AVEs (4.2).

Let

$$r_h(x) = \frac{M + N}{2}x - \frac{N - M}{2}|x| - q.$$ 

By making use of Theorem 2.2 directly and the simple passages, we present the error bounds for the HLCP (4.1), see Theorem 4.4.

Theorem 4.4. Let $x^*$ be the unique solution of AVEs (4.2). Then for any $x \in \mathbb{R}^n$, $D = \text{diag}(d_i)$ with $d_i \in [-1, 1]$ and $\Lambda = \text{diag}(\lambda_i)$ with $\lambda_i \in [0, 1]$,

$$\frac{1}{H_l} \|r_h(x)\| \leq \|x - x^*\| \leq H_u \|r_h(x)\|,$$

where

$$H_l = \max \left\| \frac{M + N}{2} - \frac{N - M}{2}D \right\| \text{ and } H_u = \max \left\| \left( \frac{M + N}{2} - \frac{N - M}{2}D \right)^{-1} \right\|,$$

which is equal to

$$\frac{1}{\tilde{H}_l} \|r_h(x)\| \leq \|x - x^*\| \leq \tilde{H}_u \|r_h(x)\|,$$

with

$$\tilde{H}_l = \max \|N(I - \Lambda) + M\Lambda\| \text{ and } \tilde{H}_u = \max \|(N(I - \Lambda) + M\Lambda)^{-1}\|.$$

It is well known that, if $N$ is nonsingular, the HLCP (4.1) can be equivalently written as the LCP

$$w = N^{-1}Mz - N^{-1}q, z \geq 0, w \geq 0, z^Tw = 0.$$

If we apply the error bounds of Eq. (2.3) on page 516 in [30] to such LCP, then we directly obtain

$$\frac{1}{C_l} \|r(x)\| \leq \|x - x^*\| \leq C_u \|r(x)\|,$$
where

\[ C_l = \max \| I - \Lambda + \Lambda N^{-1}M \| \quad \text{and} \quad C_u = \max \| (I - \Lambda + \Lambda N^{-1}M)^{-1} \|. \]

Comparing (4.4) with (4.5), the advantages of the former are obvious. Firstly, the former does not need that \( N \) must be nonsingular matrix; secondly, when using the latter for calculating the error bounds, the inverse of matrix \( N \) has to address. Whereas, the former successfully prevent this form happening.

By making use of Theorems 2.6, 2.8 and 2.11, some computable estimates for \( H_u \) or \( \bar{H}_u \) are obtained off-hand, see Corollary 4.1.

**Corollary 4.1.** Let \( A = \frac{M+I}{2} \) and \( B = \frac{N-M}{2} \). The following statements hold:

(i) Let \( \rho([A^{-1}B]) < 1 \). Then

\[ H_u = \bar{H}_u \leq \| (I - |A^{-1}B|)^{-1} \| A^{-1} \|. \]

(ii) Let \( \sigma_{\min}(A) > \sigma_{\max}(B) \). Then

\[ H_u = \bar{H}_u \leq \frac{1}{\sigma_{\min}(A) - \sigma_{\max}(B)}. \]

(iii) Let \( \sigma_{\max}(A^{-1}B) < 1 \). Then

\[ H_u = \bar{H}_u \leq \| A^{-1}B \|_2 B^{-1} \|_2. \]

Naturally, for \( N = I \) in (4.2), combining Theorem 4.4 with Corollary 4.1, we immediately obtain the error bounds for LCP, see Corollary 4.2.

**Corollary 4.2.** Let \( x^* \) be the unique solution of AVEs (4.2) with \( N = I \). Then for any \( x \in \mathbb{R}^n \), \( D = \text{diag}(d_i) \) with \( d_i \in [-1, 1] \) and \( \Lambda = \text{diag}(\lambda_i) \) with \( \lambda_i \in [0, 1] \)

\[ \frac{1}{L_l} \| r_h(x) \| \leq \| x - x^* \| \leq L_u \| r_h(x) \|, \]

where

\[ L_l = \max \left\| \frac{M+I}{2} - \frac{I-M}{2} D \right\| \quad \text{and} \quad L_u = \max \left\| \left( \frac{M+I}{2} - \frac{I-M}{2} D \right)^{-1} \right\|, \]

which is equal to

\[ \frac{1}{L_l} \| r_h(x) \| \leq \| x - x^* \| \leq \bar{L}_u \| r_h(x) \|, \]

with

\[ \bar{L}_l = \max \| I - \Lambda + M\Lambda \| \quad \text{and} \quad \bar{L}_u = \max \| (I - \Lambda + M\Lambda)^{-1} \|. \]

Further, let \( \bar{A} = \frac{M+I}{2} \) and \( \bar{B} = \frac{L-M}{2} \). The following statements hold:

(i) Let \( \rho([\bar{A}^{-1}\bar{B}]) < 1 \). Then

\[ \bar{L}_u = L_u \leq \| (I - |\bar{A}^{-1}\bar{B}|)^{-1} \| \| \bar{A}^{-1} \|. \]

(ii) Let \( \sigma_{\min}(\bar{A}) > \sigma_{\max}(\bar{B}) \). Then

\[ \bar{L}_u = L_u \leq \frac{1}{\sigma_{\min}(\bar{A}) - \sigma_{\max}(\bar{B})}. \]
(iii) Let $\sigma_{\text{max}}(\bar{A}^{-1}\bar{B}) < 1$. Then

\begin{equation}
(4.13) \quad \bar{L}_u = L_u \leq \|\bar{A}^{-1}\bar{B}\|_2 \|\bar{B}^{-1}\|_2.
\end{equation}

Remark 4.5. Based on Theorem 2.1 in [32], the lower and upper error bounds in (4.10) are the same as that in (2.3) on page 516 in [30]. In addition, their new equal form, i.e., (4.9), is exploited as well. Moreover, from (4.9), we present some estimates of the upper bounds for three cases: (1) $\rho(|\bar{A}^{-1}\bar{B}|) < 1$; (2) $\sigma_{\text{min}}(\bar{A}) > \sigma_{\text{max}}(\bar{B})$; (3) $\sigma_{\text{max}}(\bar{A}^{-1}\bar{B}) < 1$, where $\bar{A} = \frac{M+I}{2}$ and $\bar{B} = \frac{I-M}{2}$. These estimates are not limited the special type of matrix.

Since in theory it is difficult to compare the upper bounds in Corollary 4.2 and Theorem 2.1 in [30], here, we can adopt some numerical examples to investigate the upper bounds in Corollary 4.2 and Theorem 2.1 in [30]. For the sake of simplicity, we only list a simple example, see Example 4.1.

Example 4.1 Let

\[ M = \begin{bmatrix} 1 & -0.5 \\ 0.5 & 1 \end{bmatrix}. \]

Obviously, $M$ is an $H_+$-matrix. By the simple computations, from Theorem 2.1 in [30], we get

\[ \|\langle M \rangle^{-1}\max\{D_M, I\}\|_2 = 2, \]

where $D_M$ denotes the diagonal part of $M$.

From Corollary 4.2 (i), we get

\[ \rho(|\bar{A}^{-1}\bar{B}|) = 0.2941 < 1, \quad \text{and} \quad \|\langle I - |\bar{A}^{-1}\bar{B}| \rangle^{-1}\||_2 \|\bar{A}^{-1}\||_2 = 1.3744 < 2. \]

From Corollary 4.2 (ii), we get

\[ \sigma_{\text{min}}(\bar{A}) = 1.0308 > \sigma_{\text{max}}(\bar{B}) = 0.2500, \quad \text{and} \quad \frac{1}{\sigma_{\text{min}}(\bar{A}) - \sigma_{\text{max}}(\bar{B})} = 1.2808 < 2. \]

From Corollary 4.2 (iii), we get

\[ \sigma_{\text{max}}(\bar{A}^{-1}\bar{B}) = 0.2425, \quad \text{and} \quad \|\bar{A}^{-1}\bar{B}\|_2 \|\bar{B}^{-1}\|_2 = 0.9701 < 2. \]

The above results imply that (i), (ii) and (iii) in Corollary 4.2 are sharper than that in Theorem 2.1 in [30] under proper conditions. This implies that our presented bounds, (i), (ii) and (iii) in Corollary 4.2, are meaningful in a way, and may advantage over that in Theorem 2.1 in [30] under certain conditions.

4.2. Perturbation bound for HLCP. In this subsection, we fasten on the perturbation analysis of the HLCP (4.1) when $M, N$ and $q$ are perturbed to $M + \Delta M, N + \Delta N$ and $q + \Delta q$, respectively. In this case, the perturbed HLCP is to find a pair of vectors $s, t$ in $\mathbb{R}^n$ such that

\begin{equation}
(4.14) \quad (M + \Delta M)s - (N + \Delta N)t = q + \Delta q, \quad s \geq 0, \quad t \geq 0, \quad s^T t = 0.
\end{equation}

Assume that $\{M + \Delta M, N + \Delta N\}$ satisfies the conditions of Lemma 4.2 such that the perturbed HLCP (4.14) has the unique solution.
Let 
\[ t = \frac{|y| - y}{2} \quad \text{and} \quad s = \frac{|y| + y}{2}. \]

Then the perturbed HLCP (4.14) is equally transformed into the following absolute value equations

\[ \frac{M + N + \Delta M + \Delta N}{2} y - \frac{N - M + \Delta N - \Delta M}{2} |y| = q + \Delta q. \]

By making use of Theorem 3.2 directly and the simple passages, together with Theorems 2.6, 2.8 and 2.11, and
\[ \|q\| \|x\| \leq \frac{1}{2}(\|M + N\| + \|M - N\|), \]
we present the perturbation bound for the HLCP (4.1), see Theorem 4.6.

**Theorem 4.6.** Let \( x^*, y^* \) be the unique solutions of AVEs (4.2) and (4.15), respectively. Then for any \( D = \text{diag}(d_i) \) with \( d_i \in [-1, 1] \) and \( \Lambda = \text{diag}(\lambda_i) \) with \( \lambda_i \in [0, 1] \)

\[ \frac{\|x^* - y^*\|}{\|x^*\|} \leq \pi_r \left( \frac{\|\Delta q\|}{\|q\|} \left( \frac{\|M + N\| + \|M - N\|}{2} \right) + \frac{\|\Delta M + \Delta N\| + \|\Delta M - \Delta N\|}{2} \right), \]

with

\[ \pi_r = \max \left\| \left( \frac{M + N + \Delta M + \Delta N}{2} - \frac{N - M + \Delta N - \Delta M}{2} D \right)^{-1} \right\|. \]

or,
\[ \pi_r = \max \| (N + \Delta N)(I - \Lambda) + (M + \Delta M)\Lambda \|^{-1}. \]

Moreover, let \( \hat{\Delta} = \frac{M + N + \Delta M + \Delta N}{2} \) and \( \hat{B} = \frac{N - M + \Delta N - \Delta M}{2} \). The following statements hold:

(i) Let \( \rho(|\hat{\Delta}^{-1}\hat{B}|) < 1 \). Then
\[ \pi_r \leq \| (I - |\hat{\Delta}^{-1}\hat{B}|)^{-1} \| \cdot \| \hat{\Delta}^{-1} \|. \]

(ii) Let \( \sigma_{\min}(\hat{\Delta}) > \sigma_{\max}(\hat{B}) \). Then
\[ \pi_r \leq \frac{1}{\sigma_{\min}(\hat{\Delta}) - \sigma_{\max}(\hat{B})}. \]

(iii) Let \( \sigma_{\max}(\hat{\Delta}^{-1}\hat{B}) < 1 \). Then
\[ \pi_r \leq \| \hat{\Delta}^{-1}\hat{B} \|_2 \cdot \| \hat{B}^{-1} \|_2. \]

When \( N = I \) and \( \Delta N = 0 \) in Theorem 4.6, we immediately obtain the corresponding perturbation bound for LCP, see Corollary 4.3.

**Corollary 4.3.** Let \( N = I \) and \( \Delta N = 0 \) in Theorem 4.6, and \( x^*, y^* \) be the unique solutions of AVEs (4.2) and (4.15), respectively. Then for any \( D = \text{diag}(d_i) \) with \( d_i \in [-1, 1] \) and \( \Lambda = \text{diag}(\lambda_i) \) with \( \lambda_i \in [0, 1] \)

\[ \frac{\|x^* - y^*\|}{\|x^*\|} \leq \pi_r \left( \frac{\|\Delta q\|}{\|q\|} \left( \frac{\|M + I\| + \|M - I\|}{2} \right) + 2\|\Delta M\| \right), \]
with
\[ \alpha_l = \max \left\| \left( \frac{M + I + \Delta M}{2} - \frac{I - M - \Delta M}{2} \right)^{-1} \right\|, \]
or,
\[ \alpha_l = \max \| (I - \Lambda + (M + \Delta M)\Lambda)^{-1} \| . \]

Moreover, let \( \tilde{A} = \frac{M+I+\Delta M}{2} \) and \( \tilde{B} = \frac{I-M-\Delta M}{2} \). The following statements hold:

(i) Let \( \rho(\tilde{A}^{-1}\tilde{B}) < 1 \). Then
\[ (4.19) \quad \alpha_l \leq \| (I - |\tilde{A}^{-1}\tilde{B}|)^{-1} \| \| \tilde{A}^{-1} \|. \]

(ii) Let \( \sigma_{\min}(\tilde{A}) > \sigma_{\max}(\tilde{B}) \). Then
\[ (4.20) \quad \alpha_l \leq \frac{1}{\sigma_{\min}(\tilde{A}) - \sigma_{\max}(\tilde{B})}. \]

(iii) Let \( \sigma_{\max}(\tilde{A}^{-1}\tilde{B}) < 1 \). Then
\[ (4.21) \quad \alpha_l \leq \| \tilde{A}^{-1} \tilde{B} \|_2 \| \tilde{B}^{-1} \|_2. \]

Comparing Corollary 4.3 with Theorem 3.1 in [33] (also see Theorem 4.8 in the following Section 4.3), the perturbation bound in Corollary 4.3 is more general. Not only that, our perturbation bound unlike Theorem 3.1 in [33] here does not subject to certain conditions. In a way, this new perturbation bound for the LCP perfects the work in [33]. In particular, these computable estimates of the upper bound in Corollary 4.3 are not limited the special type of matrix.

**4.3. A new approach for the existing perturbation bounds of LCP.** The approach of the above results gained in the Sections 4.1 and 4.2 is to that the HLCP is equally transformed into the corresponding absolute value equations by making use of change of variable. As is known, for the LCP, without making use of change of variable, it is also transformed into a certain absolute value equations by the fact that
\[ w = Mz + q, z \geq 0, w \geq 0, z^Tw = 0 \iff \min\{z, Mz + q\} = 0, \]
see [30, 36]. Noting that
\[ \min\{a, b\} = \frac{1}{2}(a + b - |a - b|) \text{ for } \forall a, b \in \mathbb{R}^n, \]
obviously, the LCP \( (M, q) \) is equal to the following absolute value equations
\[ (4.22) \quad \frac{1}{2}((M + I)z + q) = \frac{1}{2}((M - I)z + q). \]

In fact, by making use of the AVEs (4.22), we also obtain Theorem 2.8 and Theorem 3.1 in [33]. To achieve this goal, without loss of generality, we consider the relationship between the solution of the LCP\( (A, b) \) and the solution of LCP\( (B, p) \). Assume that \( x^* \) and \( y^* \) are the unique solutions of the LCP\( (A, b) \) and the LCP\( (B, p) \), respectively. Then
\[ (4.23) \quad \frac{1}{2}((A + I)x^* + b) = \frac{1}{2}((A - I)x^* + b). \]
and

\[(4.24)\]
\[
\frac{1}{2}((B + I)y^* + p) = \frac{1}{2}((B - I)y^* + p).
\]

Combining (4.23) with (4.24), by the simple passages, we have

\[
\frac{I + \tilde{D} + (I - \tilde{D})A}{2}(x^* - y^*) = -\frac{(I - \tilde{D})(A - B)y^* - (I - \tilde{D})(b - p)}{2},
\]

where \(\tilde{D} = \text{diag}(\tilde{d}_i)\) with \(\tilde{d}_i \in [-1, 1]\), which is equal to

\[(4.25)\]
\[
(I - \tilde{\Lambda} + \Lambda A)(x^* - y^*) = -\tilde{\Lambda}(A - B)y^* - \tilde{\Lambda}(b - p),
\]

where \(\tilde{\Lambda} = \frac{I - \tilde{D}}{2} = \text{diag}(\tilde{\lambda}_i)\) with \(\tilde{\lambda}_i \in [0, 1]\). From (4.25), we have

\[
\begin{align*}
\|x^* - y^*\| & = \|(I - \tilde{\Lambda} + \tilde{\Lambda} A)^{-1}\tilde{\Lambda}((A - B)y^* + b - p)\| \\
& \leq \|(I - \tilde{\Lambda} + \tilde{\Lambda} A)^{-1}\tilde{\Lambda}\|\|(A - B)y^* + b - p\| \\
& \leq \|(I - \tilde{\Lambda} + \tilde{\Lambda} A)^{-1}\tilde{\Lambda}\|\|(A - B\|\|y^*\| + \|b - p\|)
\end{align*}
\]

\[(4.26)\]
\[
\leq \beta(A)(\|A - B\|\|y^*\| + \|b - p\|),
\]

where \(\beta(A) = \max\{\|(I - \Lambda + \Lambda A)^{-1}\Lambda\|\text{ for any } \Lambda = \text{diag}(\lambda_i)\text{ with } \lambda_i \in [0, 1]\), or

\[(4.27)\]
\[
\|x^* - y^*\| \leq \beta(B)(\|A - B\|\|x^*\| + \|b - p\|),
\]

where \(\beta(B) = \max\{\|(I - \Lambda + \Lambda B)^{-1}\Lambda\|\text{ for any } \Lambda = \text{diag}(\lambda_i)\text{ with } \lambda_i \in [0, 1]\).

Noting that 0 is the solution of LCP\((B, p)\), repeating the above process, we obtain

\[(4.28)\]
\[
\|y^*\| \leq \beta(B)\|(-p)_+\|.
\]

Furthermore, from \(Ax^* + b \geq 0\), we obtain \((-b)_+ \leq (Ax^*)_+ \leq |Ax|\), which implies that

\[(4.29)\]
\[
\frac{1}{\|x^*\|} \leq \frac{\|A\|}{\|(-b)_+\|}.
\]

Hence, combining (4.26), (4.27), (4.28) with (4.29), we have

**Lemma 4.7.** Assume that \(x^*\) and \(y^*\) are the unique solutions of the LCP\((A, b)\) and the LCP\((B, p)\), respectively. Let \(\beta(A) = \max\{\|(I - \Lambda + \Lambda A)^{-1}\Lambda\|\}\) and \(\beta(B) = \max\{\|(I - \Lambda + \Lambda B)^{-1}\Lambda\|\}\text{ for any } \Lambda = \text{diag}(\lambda_i)\text{ with } \lambda_i \in [0, 1]\). Then

\[
\|x^* - y^*\| \leq \beta(A)(\|A - B\|\|(-p)_+\| + \|b - p\|)
\]

and

\[
\frac{\|x^* - y^*\|}{\|x^*\|} \leq \beta(B)(\|A - B\| + \|b - p\|\|A\|\|(-b)_+\|).
\]

Let

\[(4.30)\]
\[
\mathcal{M} := \{A | \beta(M)\|M - A\| \leq \eta < 1\}.
\]
Then for any $A \in \mathcal{M}$, we have
\begin{equation}
\beta(A) \leq \alpha(M) = \frac{1}{1 - \eta} \beta(M),
\end{equation}
see [33]. Together with Lemma 4.7, Theorem 4.8 can be obtained, i.e., Theorems 2.8 and 3.1 in [33].

**Theorem 4.8.** (Theorems 2.8 and 3.1 in [33]) Assume that $x^*$ and $y^*$ are the unique solutions of the LCP $(A, b)$ and the LCP $(B, p)$, respectively. Then for any $A, B \in \mathcal{M}$, where $\mathcal{M}$ is defined as in (4.30),
\[
\|x^* - y^*\| \leq \alpha(M)^2 \|A - B\| \|(-p)_+\| + \alpha(M) \|b - p\|,
\]
where $\alpha(M)$ is defined in (4.31). Further, let $A = M, B = M + \Delta M, b = q, p = q + \Delta q$ with $\|\Delta M\| \leq \epsilon \|M\|$ and $\|\Delta q\| \leq \epsilon \|(-q)_+\|, \epsilon \beta(M) \|M\| = \delta < 1$. Then
\[
\frac{\|x^* - y^*\|}{\|x^*\|} \leq \frac{2\delta}{1 - \delta}.
\]

Similarly, here we compare Corollary 4.3 with Theorem 4.8 by a simple example. For the sake of convenience, we consider two cases that $M$ is a symmetric positive definite matrix, and also is an $H_+$-matrix. In this case, it’s going to be converted to compare Corollary 3.2 (i) and (ii) in [33] with Corollary 4.3. Specifically, see Example 4.2.

**Example 4.2** Assume that $\epsilon$ is a sufficiently small positive number. Let
\[
M = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, q = \begin{bmatrix} -1 \\ \sqrt{6} \end{bmatrix}.
\]

Obviously, $M$ is a symmetric positive definite matrix, and also is an $H_+$-matrix. Taking $\Delta M = \epsilon M$ with $\|\Delta q\| \leq \epsilon \|(-q)_+\|$, it is easy to obtain that
\[
\kappa_\infty(\langle M \rangle) = \kappa_2(M) = 1,
\]
where $\kappa(\cdot)$ denotes the condition number, see Corollary 3.2 (i) and (ii) in [33]. Hence, we can obtain
\[
\frac{2\delta}{1 - \delta} = \frac{2\epsilon}{1 - \epsilon}.
\]

From Corollary 4.3 (i), we get
\[
\rho(|\tilde{A}^{-1}\tilde{B}|) = 1 + 3\epsilon \frac{5 + 3\epsilon}{5 + 3\epsilon} < 1
\]
and
\[
\|(I - |\tilde{A}^{-1}\tilde{B}|)^{-1}\|_\infty \|\tilde{A}^{-1}\|_\infty = \frac{5 + 3\epsilon}{4} \times \frac{2}{2.5 + 1.5\epsilon} = 1.
\]

Further,
\[
\frac{\|\Delta q\|_\infty}{\|q\|_\infty} \left( \frac{\|M + I\|_\infty + \|M - I\|_\infty}{2} \right) + \epsilon \|M\|_\infty
\leq \epsilon \|(-q)_+\|_\infty \left( \frac{\|M + I\|_\infty + \|M - I\|_\infty}{2} \right) + \epsilon \|M\|_\infty
= \left( \frac{1}{4\sqrt{6}} + \frac{1}{2} \right) 3\epsilon.
\]
Hence,
\[
\|(I - |\tilde{A}^{-1}\tilde{B}|)^{-1}\|_\infty \|\tilde{A}^{-1}\|_\infty \left( \frac{\epsilon \|(-q)_+\|_\infty \left( \frac{\|M + I\|_\infty + \|M - I\|_\infty}{2} \right) + \epsilon \|M\|_\infty}{\|q\|_\infty} \right)
\]
\[= (\frac{1}{4\sqrt{6}} + \frac{1}{2})3\epsilon.\]

It is easy to check that
\[
(\frac{1}{4\sqrt{6}} + \frac{1}{2})3\epsilon < \frac{2\epsilon}{1-\epsilon}.
\]

From Corollary 4.3 (ii), we get
\[
\sigma_{\min}(\tilde{A}) = \frac{2.5 + 1.5\epsilon}{2}, \sigma_{\max}(\tilde{B}) = \frac{0.5 + 1.5\epsilon}{2},
\]
from which we have
\[
\frac{1}{\sigma_{\min}(\tilde{A}) - \sigma_{\max}(\tilde{B})} = 1.
\]

Further,
\[
\frac{\|\Delta q\|_2}{\|q\|_2} \left( \frac{\|M + I\|_2 + \|M - I\|_2}{2} \right) + \epsilon \|M\|_2
\]
\[\leq \frac{\epsilon \|(-q)_+\|_2}{\|q\|_2} \left( \frac{\|M + I\|_2 + \|M - I\|_2}{2} \right) + \epsilon \|M\|_2 = 1.8\epsilon.
\]

In this case,
\[
\frac{1}{\sigma_{\min}(\tilde{A}) - \sigma_{\max}(\tilde{B})} \left( \frac{\epsilon \|(-q)_+\|_2}{\|q\|_2} \left( \frac{\|M + I\|_2 + \|M - I\|_2}{2} \right) + \epsilon \|M\|_2 \right) = 1.8\epsilon.
\]

It is easy to check that
\[
1.8\epsilon < \frac{2\epsilon}{1-\epsilon}.
\]

From Corollary 4.3 (iii), we have
\[
\sigma_{\max}(\tilde{A}^{-1}\tilde{B}) = \frac{1 + 3\epsilon}{5 + 3\epsilon} < 1 \text{ and } \sigma_{\max}(\tilde{B}^{-1}) = \frac{2}{0.5 + 1.5\epsilon}.
\]

Likewise, it is easy to check that
\[
\frac{1 + 3\epsilon}{5 + 3\epsilon} \times \frac{2}{0.5 + 1.5\epsilon} \times 1.8\epsilon < \frac{2\epsilon}{1-\epsilon}.
\]

The above numerical results show that our presented upper bounds, (i), (ii) and (iii) in Corollary 4.3, are sharper than Corollary 3.2 (i) and (ii) in [33] under certain conditions. This implies that our presented upper bounds, (i), (ii) and (iii) in Corollary 4.3, are significant in a way.
5. Numerical examples. In this section, we focus on investigating the feasibility of the relative perturbation bounds of Theorem 3.2 and Theorem 3.6 in the Section 3 by making use of some numerical examples. For the sake of convenience, we consider the AVEs (1.1)

\[ Ax - B|x| = b, \]  
which is from the LCP (1.4) by making use of change of variable \( z = |x| + x \) and \( w = |x| - x \). For the convenient comparison, here we introduce some notations, i.e.,

\[ r = \frac{\|x^* - y^*\|_2}{\|x^*\|_2}, \quad w = \frac{\|\Delta b\|_2}{\|b\|_2} (\|A\|_2 + \|B\|_2 + \|\Delta A\|_2 + \|\Delta B\|_2), \]

\[ \tau = \|(I - |A^{-1}B|)^{-1}\||A^{-1}||\Delta A\|_2 \|x^*\|_2, \quad \upsilon = \frac{1}{\sigma_{\min}(A) - \sigma_{\max}(B)} w, \quad \nu = \|A^{-1}B\|_2 \|B^{-1}\|_2 w \]

and

\[ \delta = \epsilon \frac{\|\Delta A\|_2 |\Delta A^{-1}| \|A^{-1}\| \|\Delta B\|_2 \|B^{-1}\|_2 \|x^*\|_2}{(1 - \epsilon \|\Delta A^{-1}B\|_2 \|A^{-1}B^{-1}\|_2 \|A\|_2 + \|B\|_2 \|x^*\|_2)} \]

where \( A = A + \Delta A \) and \( B = B + \Delta B \), \( x^* \) and \( y^* \) denote the solutions of the corresponding AVEs and the perturbed AVEs, respectively, which can be obtained by using the iteration method on page 364 in [3] with the initial vector being zero and the corresponding absolute error less than \( 10^{-6} \). All the computations are done in Matlab R2021b on an HP PC (Intel® Celeron® G4900, 3.10GHz, 8.00 GB of RAM).

Example 5.1 [38] Let

\[ M = \text{tridiag}(1, 4, -2) \in \mathbb{R}^{n \times n}, \quad q = -4e \in \mathbb{R}^n. \]

Obviously, \( M \) is an \( H_+ \)-matrix. This implies that the corresponding LCP has a unique solution, i.e., the equal AVEs has a unique solution.

One is interested in the perturbation error for the solution of the corresponding AVEs (1.1) caused by small changes in \( A, B \) and \( b \). The perturbation way we do it is

\[ \Delta A = \epsilon \text{tridiag}(1, 2, -1), \quad \Delta B = \epsilon \text{tridiag}(1, 1, 1), \quad \Delta b = \epsilon e. \]

| \( \epsilon \) | 0.01 | 0.015 | 0.02 | 0.025 | 0.03 |
|-----|-----|-----|-----|-----|-----|
| \( r \) | 0.0040 | 0.0061 | 0.0081 | 0.0101 | 0.0122 |
| \( \tau \) | 0.2845 | 0.4124 | 0.5321 | 0.6442 | 0.7495 |
| \( \upsilon \) | 0.0465 | 0.0692 | 0.0916 | 0.1138 | 0.1356 |
| \( \nu \) | 0.0305 | 0.0460 | 0.0617 | 0.0774 | 0.0934 |
| \( \delta \) | 0.0048 | 0.0072 | 0.0097 | 0.0121 | 0.0146 |

Table 5.1

Relative perturbation bounds of Example 5.1 with \( n = 30 \).

In Tables 5.1 and 5.2, we list the numerical results for four relative perturbation bounds with the different size and \( \epsilon \), from which we find that among four relative bounds, \( \delta < \nu, \delta < \upsilon \) and \( \delta < \tau \), i.e., \( \delta \) is closest to the true relative error. In addition, these numerical results also illustrate that the proposed bounds are very close to the
real relative value when the perturbation is very small. From Tables 5.1 and 5.2, it is easy to find that the relative perturbation bounds given by Theorem 3.2 and Theorem 3.6 are feasible and effective under some suitable conditions.

Next, we investigate another example, which is from [18].

**Example 5.2** [18] Let $M = \hat{M} + \mu I$, $q = -Mz^*$, where $\mu = 4$, $
\hat{M} = \text{blktridiag}(-I, S, -I) \in \mathbb{R}^{n \times n}$, $S = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{m \times m}$, $n = m^2$, and $z^* = (1, 2, 1, 2, \ldots, 1, 2, \ldots)^T \in \mathbb{R}^n$ is the unique solution of the corresponding LCP. Of course, the equal AVEs has a unique solution as well.

For Example 5.2, the perturbation way we do it is

$$
\Delta A = \epsilon \text{tridiag}(-1, 2, -1), \quad \Delta B = \epsilon \text{tridiag}(1, -1, 1), \quad \Delta b = \epsilon e.
$$

| $\epsilon$ | 0.01 | 0.015 | 0.02 | 0.025 | 0.03 |
|-----------|------|-------|------|-------|------|
| $r$       | 0.0028 | 0.0042 | 0.0056 | 0.0070 | 0.0084 |
| $\tau$    | 0.2571 | 0.3870 | 0.5177 | 0.6493 | 0.7817 |
| $\nu$     | 0.0422 | 0.0631 | 0.0839 | 0.1046 | 0.1252 |
| $\delta$  | 0.0268 | 0.0403 | 0.0538 | 0.0674 | 0.0810 |

Table 5.3

**Relative perturbation bounds of Example 5.2 with $n = 225$.**

| $\epsilon$ | 0.01 | 0.015 | 0.02 | 0.025 | 0.03 |
|-----------|------|-------|------|-------|------|
| $r$       | 0.0030 | 0.0045 | 0.0060 | 0.0075 | 0.0090 |
| $\tau$    | 0.2798 | 0.4212 | 0.5637 | 0.7071 | 0.8516 |
| $\nu$     | 0.0466 | 0.0697 | 0.0927 | 0.1155 | 0.1382 |
| $\delta$  | 0.0284 | 0.0426 | 0.0569 | 0.0713 | 0.0857 |

Table 5.4

**Relative perturbation bounds of Example 5.2 with $n = 400$.**

Similarly, with the different size and $\epsilon$, Tables 5.3 and 5.4 list these four relative perturbation bounds for Example 5.2. From Tables 5.3 and 5.4, we can draw the same conclusion. That is to say, among these four bounds, $\delta$ is optimal, compared with other three bounds. These numerical results in Tables 5.3 and 5.4 further illustrate that when the perturbation term is very small, the proposed bounds are very close to the real relative value. Meanwhile, this further confirms that under some suitable conditions, Theorem 3.2 and Theorem 3.6 indeed provide some valid relative perturbation bounds.
6. Conclusion. In this paper, by introducing a class of absolute value functions, the frameworks of error and perturbation bounds of two types of absolute value equations (AVEs) have been established. After then, by applying the absolute value equations, the frameworks of error and perturbation bounds for the horizontal linear complementarity problem (HLCP) are obtained. Incidentally, some equal new frameworks of error bounds for linear complementarity problem (LCP) are given and its more general perturbation bound is presented as well. Besides, without limiting the matrix type, some computable estimates for the above upper bounds are given, which are sharper than some existing results under certain conditions. In addition, some numerical examples for absolute value equations from the LCP are given to show the feasibility of the proposed perturbation bounds.

Data Availability Statement. The data that support the findings of this study are available from the corresponding author upon request.

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