On the probabilistic approach for Gaussian Berezin integrals

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Abstract
We present a novel approach to Gaussian Berezin correlation functions. A formula well known in the literature expresses these quantities in terms of submatrices of the inverse matrix appearing in the Gaussian action. By using a recently proposed method to calculate Berezin integrals as an expectation of suitable functionals of Poisson processes, we obtain an alternative formula which allows one to skip the calculation of the inverse of the matrix. This formula, previously derived using different approaches (in particular by means of the Jacobi identity for the compound matrices), has computational advantages which grow rapidly with the dimension of the Grassmann algebra and the order of correlation. By using this alternative formula, we establish a mapping between two fermionic systems, not necessarily Gaussian, with short and long range interaction, respectively.

1 Introduction

The concept and the use of anticommuting variables originates both in the contest of the functional integral approach to the quantization of fermionic systems [1, 2] and in several combinatorial problems (e.g. to represent the partition function of the planar Ising model) [3, 4]. The anticommuting character implies that these variables belong to a Grassmann algebra.

A Grassmann algebra $G_g$ of $g$ generators $\{\xi_1, \xi_2, \ldots, \xi_g\}$, is defined by the identity $1^g$ and requiring the generators obey the following anticommutation relations:

$$\{\xi_i, \xi_j\} = \xi_i\xi_j + \xi_j\xi_i = 0 \quad \forall \ i,j \in \{1, \ldots g\}. \quad (1)$$

Elements of $G_g$ of the form $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq g$ and
$k \leq g$, are called monomials. Each elements of $\mathcal{G}_g$, $F(\xi)$, can be written as a unique complex linear combination of the $2^g$ (independent) monomials:

$$F(\xi) = f_0 1^g + \sum_{k=1}^{g} \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq g} f_{i_1, i_2, \ldots, i_k} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k},$$

where $f_{i_1, i_2, \ldots, i_k} \in \mathbb{C}$. In this algebra, derivation and integration (the Berezin integral) are properly defined. The reader is referred to [1, 5, 6] for details. Let us introduce the symbols $d\xi_1, \ldots, d\xi_g$ satisfying the following formal relations

$$\{d\xi_i, d\xi_i\} = \{d\xi_i, \xi_j\} = 0. \quad (2)$$

Let us define the elementary Berezin integrals

$$\int B d\xi_i = 0, \text{ and for } j \neq i \int B \xi_j d\xi_i = \xi_j,$$

iterating this relations we arrive to

$$\int B \xi_1 \xi_2 \cdots \xi_g d\xi_1 \cdots d\xi_g = 1,$$

so, from linearity we have the following general definition of Berezin integral

$$\int B F(\xi) d\xi_1 d\xi_2 \cdots d\xi_g = f_1, \ldots, g. \quad (3)$$

The derivation is defined as

$$\frac{\delta}{\delta \xi_i} \xi_{\mu_1} \cdots \xi_{\mu_n} = \delta_{\mu_1 i} \xi_{\mu_2} \xi_{\mu_3} \cdots \xi_{\mu_n} - \delta_{\mu_2 i} \xi_{\mu_1} \xi_{\mu_3} \cdots \xi_{\mu_n} + \cdots + (-1)^{n-1} \delta_{\mu_n i} \xi_{\mu_1} \xi_{\mu_2} \cdots \xi_{\mu_{n-1}}. \quad (4)$$

Let now $\mathcal{G}_{2g}$ be a Grassmann algebra over $\mathbb{C}$, with $2g$ generators, which we label as $g$ barred and $g$ unbarred: $\xi_1, \xi_2, \ldots, \xi_g; \bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_g$. Let $Z_2^* = \{0, 1\}$ and $(Z_2^*)^g = Z_2^* \otimes Z_2^* \otimes \ldots \otimes Z_2^*$ (the direct product of $Z_2^*$ $g$-times). Let us define the normalized Gaussian Berezin integrals:

$$\int B \psi^* d\mu = \frac{\int B \exp(-\bar{\xi}_i A \xi) \xi_1^{n_1} \xi_1^{\bar{n}_1} \cdots \xi_g^{n_g} \xi_g^{\bar{n}_g} d\xi_1 d\bar{\xi}_1 \cdots d\xi_g d\bar{\xi}_g}{\int B \exp(-\xi_i A \xi) d\xi_1 d\bar{\xi}_1 \cdots d\xi_g d\bar{\xi}_g} \quad (5)$$

where $A$ is any invertible $g \times g$ matrix, $\psi = (\xi_1, \bar{\xi}_1, \ldots, \xi_g, \bar{\xi}_g)$, $\psi^* = \xi_1^{n_1} \bar{\xi}_1^{\bar{n}_1} \cdots \xi_g^{n_g} \bar{\xi}_g^{\bar{n}_g}$ with

$$s = (n_1, \bar{n}_1, \ldots, n_g, \bar{n}_g) \in (Z_2^*)^{2g} \quad (6)$$
and the definition of the Gaussian measure $d\mu$ is implicit in (5). Let $\bar{k}$ be the number of labels $i$ such that $\bar{n}_i = 1$ and $k$ the number of labels $j$ such that $n_j = 1$. Due to the parity of the Gaussian function $\exp(-\vec{\xi}, A\vec{\xi})$, it is easy to see that the integral (5) vanishes if $k \neq \bar{k}$.

When $k = \bar{k}$, the integral (5) can be evaluated by a well known formula in terms of the determinant of a submatrix of the inverse matrix $C = A^{-1}$ (see e.g. [2]):

$$\int_B \psi^* d\mu = \det C_s,$$  \hspace{1cm} (7)

where $C_s$ is the matrix obtained from $C$ erasing the rows with label $j$ if $n_j = 0$ and the columns with label $i$ if $\bar{n}_i = 0$. If $s = (0, \ldots, 0)$, we define $C_s = 1$. Equation (7) is then expressed in terms of a $k \times k$ submatrix of the inverse of $A$.

Starting from a recently developed approach to calculate Berezin integrals [7], we will derive (see Eq. (12)) a formula for (5) in terms of the determinant of a $(g - k) \times (g - k)$ sub matrix of $A$. When compared with the (7), this formula gives rise to an algebraic identity, which is nothing else than Jacobi’s theorem relating the minors of a matrix to the minors of the inverse of the same matrix, see e.g. [8, 9], see also [10]. Although this identity and the Gaussian Berezin correlation functions can be derived algebraically in a simpler manner, even not using the Jacobi’s theorem [11, 12, 13], a probabilistic derivation provides an important example of how the stochastic approach for fermions models can be applied to obtain exact results.

It is worth considering that even if this identity is well known among mathematicians, it seems that it has been rarely explored in Quantum Field Theory [11, 12, 10]. In section 5 we will explain some interesting aspects and physical applications of this identity. In particular we will find a mapping between two fermionic systems. According to this mapping the Grassmannian partition function of a short range system having $A$ as matrix for the free (Gaussian) part of the action, equals the Grassmannian partition function of a long range system having $A^{-1}$ as matrix for the free part.
In the remainder we will give the proof providing in full detail all the calculations as they may look unfamiliar to many readers.

2 Gaussian Berezin integrals

Given \( s = (n_1, \bar{n}_1, ..., n_g, \bar{n}_g) \in (\mathbb{Z}_2^*)^{2g} \), let

\[
s^* = (1 - \bar{n}_1, 1 - n_1, ..., 1 - \bar{n}_g, 1 - n_g).
\]

Probabilistically, we will show that besides

\[
\int B \exp(-\bar{\xi}, A\xi) \psi_s d\xi_1 d\bar{\xi}_1 ... d\xi_g d\bar{\xi}_g = \det C_s \det A
\]

holds also

\[
\int B \exp(-\bar{\xi}, A\xi) \psi_s d\xi_1 d\bar{\xi}_1 ... d\xi_g d\bar{\xi}_g = \det A_{s^*} (-1)^{W(s)}. \tag{10}
\]

The factor \((-1)^{W(s)}\) determines a global sign with

\[
W(s) = (n_1 + \bar{n}_1) + 2(n_2 + \bar{n}_2) + \cdots + g(n_g + \bar{n}_g). \tag{11}
\]

From the above equations we get the following expression for the normalized Gaussian Berezin integrals

\[
\int B \psi^s d\mu = \frac{\det A_{s^*}}{\det A} (-1)^{W(s)}, \tag{12}
\]

and by using Eq. (7) the Jacobi's identity follows \(^2\)

\[
\frac{\det A_{s^*}}{\det A} (-1)^{W(s)} = \det C_s. \tag{13}
\]

3 Berezin integrals as averages over Poisson processes

Let \( \mathbb{Z}_2 = \{ -1, 1 \} \) and \( \mathbb{Z}_2^g = \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes ... \otimes \mathbb{Z}_2 \) (the direct product of \( \mathbb{Z}_2 \) \( g \)-times). Let \( \epsilon \) and \( \sigma \in \mathbb{Z}_2^g \) and \( e_g = (1, ..., 1) \in \mathbb{Z}_2^g \). Let \( \{ N^\xi_{\epsilon} \}_{\epsilon \neq e_g} \) be a family of

\(^1\)According to the definition given for \( C_s \) in introduction, \( A_{s^*} \) is the \((g - k) \times (g - k)\) submatrix of \( A \) obtained from \( A \) erasing the rows with label \( i \) if \( \bar{n}_i = 1 \) and the columns with label \( j \) if \( n_j = 1 \). If \( s = (1, ..., 1) \) we define \( A_{s^*} = 1 \).

\(^2\)It is simple to observe that \((n_1 + \bar{n}_1) + 2(n_2 + \bar{n}_2) + \cdots + g(n_g + \bar{n}_g) = i_1 + j_1 + i_2 + j_2 + \cdots i_k + j_k\), where \( i_1 ... i_k \) and \( j_1 ... j_k \) are the sets of indices such that \( n_{i_\nu} = 1 \) and \( n_{j_\nu} = 1 \) respectively, for \( \nu = 1 ... k \). In this form the sign in the Jacobi's relation is normally presented in literature.
$2^g - 1$ left-continuous independent Poisson processes of pure growth with unit parameter. These Processes are then characterized by the probabilities

$$P(N_{t+\Delta t}^\varepsilon - N_t^\varepsilon = k) = \frac{(\Delta t)^k}{k!} \exp(-\Delta t)$$

(14)

Following [7], given any “action” $S(\xi) \in G_g$, we can calculate the Berezin integrals by an expectation of a suitable functionals of Poisson processes using the following formula:

$$\int B_{\xi_1^{n_1}} \cdots \xi_g^{n_g} \exp(-S(\xi))d\xi_1 \cdots d\xi_g = \Delta_g((-1)^n) \exp(|\Gamma| - s(\varepsilon_g)) \times E\left(\prod_{l=1}^g \frac{1 - \sigma_l}{2} + \frac{1 + \sigma_l}{2} \sigma_1 \cdots \sigma_{l-1}\right),$$

(15)

where:

$$\Delta_g(\sigma) = \prod_{l=1}^g \left(1 - \frac{\sigma_l}{2} + \frac{1 + \sigma_l}{2} \sigma_1 \cdots \sigma_{l-1}\right),$$

(16)

$$C_g(\varepsilon, \sigma) = \prod_{l=1}^g \left(\frac{1 + \varepsilon_l}{2} + \frac{1 - \varepsilon_l}{2} \varepsilon_1 \cdots \varepsilon_{l-1} \sigma_1 \cdots \sigma_{l-1}\right),$$

(17)

$$(-1)^n = ((-1)^{n_1}, \ldots, (-1)^{n_g}) \in \mathbb{Z}_2^{\times g},$$

(18)

$$s(\varepsilon) = \Delta(\varepsilon) \int B_{\xi_1^{\frac{1}{2}(1+\varepsilon)}} \cdots \xi_g^{\frac{1}{2}(1+\varepsilon)} \exp(-S(\xi))d\xi_1 \cdots d\xi_g,$$

(19)

Finally the stochastic integrals that appear in (15) are ordinary Stieltjes integrals:

$$\int_{[0,t]} f(s, N_s) dN_s = \sum_{s_h < t} f(s_h, N_{s_h}),$$

(22)
$s_k$ being the jump times of the process. We refer the reader to [7, 14] for proofs and details. Here we want to emphasize that: *only the trajectories such that $N^\epsilon_1 \in \{0, 1\}, \forall \epsilon \in \Gamma$ can contribute to the expectation in (15).* We will indicate such a set as the set "zero − one". This happens due to the structure of the coefficients $C_g$ present in the argument of the logarithm which occurs in Eq. (15). In fact, let us consider the expectation of the exponential occurring in (15), if we look at the definitions of the coefficients $C_g$, neglecting their moduli, we can write this expectation as a sum of terms corresponding to the following families of events with respective contributions and probabilities (the latter are obtained by integrating over the jumps times the infinitesimal probabilities (14)):

- event in which no process jumps. This event gives 1 with probability $\exp(-|\Gamma|)$;

- events in which only one process, say with a generic label $\epsilon^1$, makes a jump. Each of these events will give the value $-s(\epsilon^1) \prod_{t \in t_1^\epsilon = -1} \frac{1+(-1)^n}{2}$ with probability $\int_0^1 ds_1 \exp(-|\Gamma|) = \exp(-|\Gamma|)$;

- events in which two and only two processes with different labels, say $\epsilon^1$ and $\epsilon^2$, jump at times $s_1 < s_2$ respectively. Each of these events will give the value $(-s(\epsilon^1)) \prod_{t \in t_1^\epsilon = -1} \frac{1+(-1)^n}{2} \times (-s(\epsilon^2)) \prod_{t \in t_1^\epsilon = -1} \frac{1+(-1)^n+\frac{1}{2}(1-\epsilon)}{2}$ with probability $\int_0^1 s_1 < s_2 ds_1 ds_2 \exp(-|\Gamma|)$

- and so on . . .

- events in which one single process, say with label $\epsilon^1$, jumps two times. Each of these events gives: $-s(\epsilon^1) \prod_{t \in t_1^\epsilon = -1} \frac{1+(-1)^n}{2} \times (-s(\epsilon^1)) \prod_{t \in t_1^\epsilon = -1} \frac{1+(-1)^n+1}{2}$ with probability $\int_0^1 s_1 < s_2 ds_1 ds_2 \exp(-|\Gamma|)$

- and similarly for processes with more than two jumps.

It is then clear that the last type of contributions gives always zero. Furthermore note that events in which two or more processes jump at the same time, have probability zero, even if $C_g \neq 0.$
So, given $S(\xi)$ and then $s(\epsilon)$, one will have to determine what is the set of the effective trajectories, in general a smaller subset of the above set of trajectories zero–one. In the next section we will investigate the Gaussian case ($S$ bilinear in the $\xi$) and we will see that such a set is “very small”. The search for the effective trajectories and their phases will allow us to automatically construct the matrix $A_{s*}$ and, in the final expression, to recover its determinant up to a global sign.

4 Proof of equation (12)

The application of Eq. (15) will give us the Gaussian integral not normalized, let be $I$ (the numerator of (5)). Let us consider (15) with $g$ barred variables and $g$ unbarred variables. We will have to calculate various parts. In section 4.1 we will calculate $s(\epsilon)$ and then we will have $\Gamma$ for our bilinear case, next we will calculate partially $C_{2g}$. This result will be used in section 4.2 for calculating the effective trajectories and in section 4.3 for evaluating the phase of (15) which will provide us the final formula. We will consider the integral with respect to $d\tilde{\xi}_1...d\tilde{\xi}_g d\xi_1...d\xi_g$ (instead of $d\xi_1d\tilde{\xi}_1...d\xi_g d\tilde{\xi}_g$). Therefore, we will evaluate the following integral

$$
\tilde{I} = \int B d\tilde{\xi}_1...d\tilde{\xi}_g d\xi_1...d\xi_g \tilde{\epsilon}^{n_1}_1 \cdots \tilde{\epsilon}^{n_1}_g \xi_1^{n_1} \cdots \xi_g^{n_1} \exp(-\xi, A\xi),
$$

(23)

which is related to $I$ by a simple correcting sign, $I = \tilde{I}(-1)^Q$ which we will calculate in section 4.3. We will see that $Q$ depends on $g$ and $s$. With the chosen ordering of the generators of the Grassmann algebra, $\epsilon$ and $s$ should be changed into

$$
\tilde{\epsilon} = (\tilde{\epsilon}_1,...,\tilde{\epsilon}_g; \epsilon_1,...,\epsilon_g),
$$

(24)

$$
\tilde{s} = (\tilde{n}_1,...,\tilde{n}_g; n_1,...,n_g).
$$

(25)

However, for simplicity, we will continue to use, if not ambiguus, the initial notations $\epsilon$ and $s$ also for the new arrangement of the Grassmann generators.
At the end we will return to the true integral corresponding to the original arrangement of the Grassmann generators by using $I = \tilde{I}(-1)^Q$.

4.1 Calculation of the factors: $s(\epsilon)$, $\Delta_{2g}(\epsilon)$ and $C_{2g}$

Let us start by calculating $s(\epsilon)$ defined by Eq. (19):

$$s(\epsilon) = \Delta_{2g}(\epsilon) \int_B d\bar{\xi}_1 ... d\bar{\xi}_g d\xi_1 ... d\xi_g \xi_1^{2(1+\bar{\epsilon}_1)} ... \xi_g^{2(1+\bar{\epsilon}_g)} \xi_1^{\frac{1}{2}(1+\epsilon_1)} ... \xi_g^{\frac{1}{2}(1+\epsilon_g)} (\bar{\xi}, A\xi)$$

$$= \Delta_{2g}(\epsilon) \int B d\bar{\xi}_1 ... d\bar{\xi}_g d\xi_1 ... d\xi_g \xi_1^{2(1+\bar{\epsilon}_1)} ... \xi_g^{2(1+\bar{\epsilon}_g)} \xi_1^{\frac{1}{2}(1+\epsilon_1)} ... \xi_g^{\frac{1}{2}(1+\epsilon_g)} \sum_{ij} \xi_i A_{ij} \xi_j$$

$$= \Delta_{2g}(\epsilon) \sum_{ij} A_{ij} (-1)^{p_{ij}} \int B d\bar{\xi}_1 ... d\bar{\xi}_g d\xi_1 ... d\xi_g \xi_1^{2(1+\bar{\epsilon}_1)} ... \xi_g^{2(1+\bar{\epsilon}_g)} \xi_1^{\frac{1}{2}(1+\epsilon_1)+1} ... \xi_g^{\frac{1}{2}(1+\epsilon_g)}$$

$$\times \xi_i^{\frac{1}{2}(1+\epsilon_1)} ... \xi_j^{\frac{1}{2}(1+\epsilon_j)+1} ... \xi_g^{\frac{1}{2}(1+\epsilon_g)},$$

(26)

where $(-1)^{p_{ij}}$ accounts for the exchanges needed to bring $\bar{\xi}_i$ in front of $\xi_1^{2(1+\bar{\epsilon}_1)}$ and $\xi_j$ in front of $\xi_j^{2(1+\epsilon_j)}$. We see, that integral (26) vanishes unless $\epsilon$ has the form

$$\epsilon_{ij} = (\bar{\epsilon}_1, ...\bar{\epsilon}_g; \epsilon_1, ..., \epsilon_g), \text{ with } \bar{\epsilon}_i = \epsilon_j = -1 \text{ and } \bar{\epsilon}_l = \epsilon_l = 1 \text{ for } l \neq i,j. \quad (27)$$

Furthermore it follows from this that

$$s(e_{2g}) = 0, \text{ where } e_{2g} = (1, ..., 1; 1, ..., 1).$$

The set $\Gamma$ is then characterized. Its elements are in the set of the $2g - p\epsilon\epsilon_{ij}$ such that $A_{ij} \neq 0$. In this way we can arrange the set of Poisson processes in a $g \times g$ matrix:

$$\mathcal{N} = (N^{ij}_s), \text{ where } N^{ij}_s \equiv N^{\epsilon_{ij}}_s. \quad (28)$$

Even if not explicitly indicated, all the sums and products will have to be considered in the range such that $A_{ij} \neq 0$.

From Eq. (26) we have

$$s(\epsilon_{ij}) = \Delta_{2g}(\epsilon_{ij})(-1)^{p_{ij}} A_{ij}.$$ (29)

In appendix B it is easily proved that one has

$$s(\epsilon_{ij}) = A_{ij}.$$ (30)
As regards the calculation of $C_{2g}$, according to our choice of the ordering of the Grassmann generators, we must consider (see Eq. (21)) $\tilde{N}_s = (\tilde{N}_s; N_s)$ and $\tilde{n} = (\tilde{n}; n)$, where

$$\tilde{N}_s^i = \sum_{\varepsilon \in \Gamma} \frac{1}{2} (1 - \varepsilon^i) N_s^\varepsilon,$$  

$$N_s^i = \sum_{\varepsilon \in \Gamma} \frac{1}{2} (1 - \varepsilon^i) N_s^\varepsilon.$$  

(31)

(32)

So to calculate $C_{2g}$ one has to evaluate

$$C_{2g}(\epsilon_{ij}, -(-1)^{\tilde{N}_s + \tilde{n}}) \equiv C_{2g} \left( \epsilon_{ij}, -((-1)^{\tilde{N}_s^i + \tilde{n}_i}, ..., (-1)^{\tilde{N}_s^g + \tilde{n}_g}; (-1)^{N_s^i + n_1}, ..., (-1)^{N_s^g + n_g}) \right).$$

In appendix C it is easily proved that one has

$$C_{2g}(\epsilon_{ij}, -(-1)^{\tilde{N}_s + \tilde{n}}) = \frac{1}{2} \left( 1 + (-1)^{\tilde{N}_s^i + \tilde{n}_i} \frac{1}{2} + (-1)^{N_s^i + n_1} \frac{1}{2} \right. (-1)^{g+j+i+1} \times \exp\{i\pi[\tilde{N}_s^i + ... + \tilde{N}_s^g + N_s^1 + ... + N_s^{i-1} + \tilde{n}_i + ... + \tilde{n}_g + n_1 + ... + n_{j-1}]\},$$

where

$$\tilde{N}_s^i = \sum_{l=1}^g N_s^{il},$$

$$N_s^j = \sum_{l=1}^g N_s^{lj}.$$  

(33)

Therefore, as far as the exponential factor in (15) is concerned, a given trajectory will give the following contribution

$$\prod_{ij} \exp\left[ \int_{[0,1]} \log(-A_{ij}) dN_s^{ij} \right] \exp\left[ \int_{[0,1]} \log\left( \frac{1}{2} \left( 1 + (-1)^{\tilde{N}_s^i + \tilde{n}_i} \right) \frac{1}{2} + (-1)^{N_s^i + n_1} \right) dN_s^j \right] \times \exp\left[ \int_{[0,1]} i\pi[g + i + j + 1 + \sum_{m=i}^g (\tilde{N}_s^m + \tilde{n}_m) + \sum_{m=1}^{j-1} (N_s^m + n_m)] dN_s^{ij} \right],$$

where the range of the product is over the couples $ij$ with $A_{ij} \neq 0$.

4.2 Effective trajectories

In section 3, we have seen that the set of the effective trajectories is a subset of set zero – one (i.e. the set in which one single process can make at most one jump during the time interval $[0,1]$) and we have anticipated that in the
Gaussian case such a set is small. Let us concentrate on the second factor of the product (34). Let us define:

$$\bar{G}_i \cdot G_j = \frac{1 + (-1)^{\bar{N}_i^j + \bar{n}_i}}{2} \cdot \frac{1 + (-1)^{N_j^i + n_j}}{2}$$

and remember that the processes are left continuous. Let us note from (33) that $\bar{N}_i^j$ and $N_j^i$ are the sums of the elements of the $i$th row and the $j$th column of the matrix $\mathcal{N}$ respectively.

Let us now consider the trajectories in which in every row and in every column of $\mathcal{N}$ there is at most just one process which jumps (clearly, one process which jumps in some row, jumps in some corresponding column too). With such events, if for example $N_s^{ij}$ is a process which jumps at time $s^{ij}$, then according to Eq. (35) we have:

$$\bar{G}_i \cdot G_j|_{s=s^{ij}} = \frac{1 + (-1)^{\bar{n}_i}}{2} \cdot \frac{1 + (-1)^{n_j}}{2}.$$  (36)

Since we consider the factor corresponding to the link $(ij)$ and carry out the stochastic integral in $dN_s^{ij}$, we see that in order it does not give $\exp \log(0)$, it must be $\bar{n}_i = n_j = 0$.

Let us now consider one event in which there is at least one row in which at least two processes, $N_s^{ij}$ and $N_s^{ik}$, jump, at times $s^{ij}$ and $s^{ik}$ respectively. Let us consider the $(ij)$ factor in Eq. (34). We have to integrate in $dN_s^{ij}$, i.e. we have to evaluate the integrand at time $s^{ij}$. Let $\bar{n}_i = 1$. We must consider two possibilities:

1. if $s^{ij} < s^{ik}$ we have:
   $\bar{N}_{s^{ij}} = 0$ from which follows $\bar{G}_i = 0$

2. if $s^{ij} > s^{ik}$ we have:
   $\bar{N}_{s^{ij}} = 1$ from which follows $\bar{G}_i = 1$.

Therefore the stochastic integral can be finite in the the case 2, but as we will integrate in $dN_s^{ik}$, we will have to calculate the integrand at time $s^{ik}$ then obtaining $\bar{G}_i = 0$. Finally Let us note that the event $s^{ij} = s^{ik}$ has zero probability.

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\footnote{This implies that if $s = s^{ij}$ is the jump time of the first jump for the process $N_s^{ij}$, then $N_s^{ij} = 0$ for $s \leq s^{ij}$ and $N_s^{ij} = 1$ for $s > s^{ij}$.}
Let $\bar{n}_i = 0$. Repeating a similar argument we will arrive at the same conclusions. Of course, the same discussion can be done for the columns.

We have proved that the effective trajectories belong to the set in which in any row and in any column of the matrix $\mathbf{N}$ there is at most one jumping process. Let us call such a set:

$$C_1 = \{\text{set of trajectories in which for any row and for any column of the matrix } \mathbf{N} \text{ there is at most one jumping process}\}. \quad (37)$$

Furthermore we have proved that as $\bar{n}_i = 1$ and/or $n_j = 1$, then it must be $N_{ij}^{1} = 0$. Let us call such set:

$$C_2 = \{\text{set of the trajectories in which if } \bar{n}_i \text{ and/or } n_j = 1 \text{ then } N_{ij}^{1} = 0\}. \quad (38)$$

If now we consider (34), (regardless of the phase), imposing these restrictions by using the product of two characteristic functions, we can eliminate the stochastic integral writing (34) as:

$$\prod_{ij} (-A_{ij})^{N_{ij}^{1}} \cdot \chi(C_1) \chi(C_2) S_{ij}, \quad (39)$$

where with $S_{ij}$ we have indicated the phase (i.e. the last of the three factors appearing in (34)). Observing Eq. (15), we see that to complete the search of the effective trajectories, we must multiply the above product by the following factor

$$\bar{F} \cdot F = \prod_{l=1}^{g} \frac{1 - (-1)^{\bar{n}_l} + \bar{n}_l}{2} \frac{1 - (-1)^{N_{l}^{1} + n_l}}{2}. \quad (40)$$

This factor takes the values 0 or 1 and its presence, using the fact that in Eq. (39) the function $\chi(C_1)$ occurs, involves a third characteristic function $\chi(C_3)$ which accounts for the fact that if for some $m$, $n_m = 0$, then there must be one and only one $j$ such that $N_{1j}^{mj} = 1$. Analogously, if for some $m$, $\bar{n}_m = 0$, then there must be one and only one $k$ such that $N_{1k}^{km} = 1$. If now we use also the function $\chi(C_2)$, then for any given $s$ (6) we find that the set of the effective
trajectories $\mathcal{C}$ is given by:

$$\mathcal{C} = \{ \text{set of trajectories such that:} \}
\begin{align*}
\text{if } \bar{n}_i = 1, \text{ any process on the row } i \text{ of } \mathcal{N} \text{ does not jump;}
\text{if } n_j = 1, \text{ any process on the column } j \text{ of } \mathcal{N} \text{ does not jump;}
\text{if } \bar{n}_i = 0 \text{ and } n_j = 0, \text{ there is one and only one process jumping to 1 on the } i\text{th row and on the } j\text{th column of } \mathcal{N}\}. 
\end{align*}
$$

Clearly if $\bar{k} \neq k$ then $\mathcal{C}$ reduces to the null set and the integral (23) gives 0. From now on, we will always suppose the $\bar{k} = k$ case.

From its definition we see that $\mathcal{C}$ has $(g-k)!$ elements so that we can represent it in terms of suitable effective matrices $\mathcal{N}$ in the following way. First we must erase all the rows with label $i$ if $\bar{n}_i = 1$ and the columns with label $j$ if $n_j = 1$. Then we must consider all the ways to fill the remaining $(g-k) \times (g-k)$ sub-matrix with $(g-k)$ times “1” in an array such that two of them are never in the same row or in the same column, like for a determinant.

Let us consider as an example the case $g = 5$ and $k = 2$ with $n_1 = n_2 = n_3 = \bar{n}_1 = \bar{n}_2 = \bar{n}_3 = 0$ and $n_4 = n_5 = \bar{n}_4 = \bar{n}_5 = 1$. In this case $\mathcal{C}$ has 6 elements represented by 6 matrices $\mathcal{N}$, like e.g. (the symbol “×” means that the corresponding process never jump)

$$\mathcal{N} = \begin{pmatrix} 0 & 1 & 0 & \times & \times \\ 1 & 0 & 0 & \times & \times \\ 0 & 0 & 1 & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{pmatrix}.$$ 

Up to now we have not spoken about the event in which no process jumps (with probability $e^{-|\Gamma|}$). It is immediate to observe that such an event is effective only if $n_i = \bar{n}_i = 1$ for any $i$ from 1 to $g$ and that the integral (23) gives simply 1. So, from now on, we will ignore this trivial case.
4.3 Phases

Let us face now the problem of finding a closed expression for the $S_{ij}$ (see Eq. (39)):

$$S_{ij} = \exp \left[ \int_{[0,1]} i\pi [g + i + j + 1 + \sum_{m=j}^{g} (\bar{N}_{s}^{m} + \bar{n}_{m}) + \sum_{m=i}^{j-1} (N_{s}^{m} + n_{m})] dN_{s}^{ij} \right].$$ (42)

Then we will need to consider the product on the couples $(ij)$ for every $(ij)$ such that $N_{1}^{ij} = 1$. Let us factor the rhs of the above equation in the following form:

$$S_{ij} = \exp \left[ i\pi [g + i + j + 1] N_{1}^{ij} \right] \times \exp \left[ \int_{[0,1]} i\pi [\sum_{m=1}^{j-1} (N_{s}^{m} + n_{m}) + \sum_{m=i}^{g} (\bar{N}_{s}^{m} + \bar{n}_{m})] dN_{s}^{ij} \right].$$ (43)

Let $s_{ij}, s_{lm}, s_{ml}$ be generic jump times with their corresponding labels, then from (33) we obtain:

$$S_{ij} = \exp \left[ i\pi [g + i + j + 1] N_{1}^{ij} \right] \times (-1)^{F_{ij}}$$ (44)

where

$$F_{ij} = \left[ \sum_{m=j+1}^{g} \sum_{l=1}^{g} \theta(s_{ij} - s_{ml}) N_{1}^{ml} + \sum_{m=i}^{j-1} \sum_{l=1}^{g} \theta(s_{ij} - s_{lm}) N_{1}^{lm} \right] N_{1}^{ij} +$$

$$+ \left[ \sum_{m=i}^{g} \bar{n}_{m} + \sum_{m=1}^{j-1} n_{m} \right] N_{1}^{ij}.$$ (45)

which, taking account the definition of $C_{1}$, becomes:

$$F_{ij} = \left[ \sum_{m=i+1}^{g} \sum_{l=1}^{g} \theta(s_{ij} - s_{ml}) N_{1}^{ml} + \sum_{m=1}^{j-1} \sum_{l=1}^{g} \theta(s_{ij} - s_{lm}) N_{1}^{lm} \right] N_{1}^{ij} +$$

$$+ \left[ \sum_{m=i+1}^{g} \bar{n}_{m} + \sum_{m=1}^{j-1} n_{m} \right] N_{1}^{ij}.$$ (46)

Now in order to calculate the expectation, we must integrate over the jump times and then to sum over all the trajectories the following product:

$$\prod_{ij} (-A_{ij}) \chi(C) \exp \left[ i\pi [g + i + j + 1] N_{1}^{ij} \right] \times (-1)^{F_{ij}}.$$

(47)
The integration must be done using the probability of the Poisson processes and it is easy to see that

\[ dP = e^{-|\Gamma|} \prod_{ij:N_{ij}^1=1} ds^{ij}. \quad (48) \]

We should integrate (47) with the measure \( dP \) and then sum over all the possible events belonging to the set \( C \); we should then calculate the following integrals:

\[ \int_0^1 e^{-|\Gamma|} \prod_{ij:N_{ij}^1=1} (-1)^{H_{ij}} ds^{ij}, \quad (49) \]

where

\[ H_{ij} = [ \sum_{m=i+1}^g \sum_{l=1}^g \theta(s^{ij} - s^{ml}) N_{11}^{ml} + \sum_{m=1}^{j-1} \sum_{l=1}^g \theta(s^{ij} - s^{lm}) N_{11}^{lm}]. \]

In appendix D it is proved that

\[ H_{ij} = [ \sum_{m=i+1}^g \sum_{l=1}^g \theta_{ij}^{ml} N_{11}^{ml} + \sum_{m=1}^{j-1} \sum_{l=1}^g \theta_{ij}^{lm} N_{11}^{lm}], \quad (50) \]

where the variables with four labels \( \theta_{hk}^{ij} \) satisfy the following properties:

\[ \theta_{hk}^{ij} + \theta_{ij}^{hk} = 1 \]

\[ \theta_{hk}^{ij} + \theta_{hk}^{ij} = 0. \quad (51) \]

Now we have the following formula for the integral \( I \) (the numerator of (5))

\[ I = (-1)^Q \Delta_2 \left( (-1)^\bar{s} \right) \sum_{N_{ij}^1=0}^{1} \chi(C) \prod_{ij:N_{ij}^1=1} (-A_{ij})(-1)^{\bar{s}+1} \times (-1)^{\sum_{m=1}^{j} n_{m} + \sum_{n=1}^{j-1} n_{m} (-1)^{i+j} (-1)^{H_{ij}}}, \quad (52) \]

where \( \bar{s} = (\bar{n}, \bar{n}) \) and \( (-1)^Q \) is the sign necessary to have \( I \) from \( \bar{I} \).

Now we will face the problem to obtain a closed expression for each of the factors appearing in the above equation for \( I \).

### 4.3.1 Correcting sign \( (-1)^Q \)

To calculate \( (-1)^Q \) we must consider the exchanges needed to restore the monomial \( \tilde{\xi}_{\bar{n}_{1}} \cdots \tilde{\xi}_{\bar{n}_{g}} \tilde{\xi}_{\bar{n}_{1}} \cdots \tilde{\xi}_{\bar{n}_{g}} \) and the ordering of integrations in the original order.

The first produces a sign equals to:

\[ (-1)^{n_1(\bar{n}_1 + \bar{n}_2 + \cdots + \bar{n}_g) + n_2(\bar{n}_2 + \cdots + \bar{n}_g) + \cdots + n_g\bar{n}_g}, \]
the second:

\[ (-1)^{g(g+1)/2}. \]

Therefore

\[ (-1)^Q = (-1)^{\bar{n}_1(n_1 + \bar{n}_2 + \cdots + \bar{n}_g) + n_2(\bar{n}_2 + \cdots + \bar{n}_g) + \cdots + n_g \bar{n}_g + g(g+1)/2}. \tag{53} \]

4.3.2 \( \Delta_{2g}(-1^\bar{s}) \)

For \( \Delta_{2g}(-1^\bar{s}) \) we have:

\[
\Delta_{2g}(-1^\bar{s}) = \prod_{l=1}^{g} \left( \frac{1 + (-1)^{\bar{n}_l}}{2} + \frac{1 - (-1)^{\bar{n}_l}}{2}(-1)^{\bar{n}_1} \cdots (-1)^{\bar{n}_{l-1}} \right)
\times \prod_{l=1}^{g} \left( \frac{1 + (-1)^{n_l}}{2} + \frac{1 - (-1)^{n_l}}{2}(-1)^{n_1} \cdots (-1)^{n_{l-1}} \right)
\]

from which one easily has:

\[
\Delta_{2g}(-1^\bar{s}) = \prod_{l=1}^{g} \exp \left( (i\pi)n_l \left[ \bar{n}_1 + \cdots + \bar{n}_{l-1} + l - 1 \right] \right)
\times \prod_{l=1}^{g} \exp \left( (i\pi)n_l \left[ n_1 + \cdots + n_{l-1} + g + l - 1 \right] \right).
\]

So we have:

\[ \Delta_{2g}(-1^\bar{s}) = (-1)^R, \tag{54} \]

where

\[
R = \bar{n}_2(\bar{n}_1 + 1) + \bar{n}_3(\bar{n}_1 + \bar{n}_2 + 2) + \cdots + \bar{n}_g(\bar{n}_1 + \cdots + \bar{n}_{g-1} + g - 1)
+ n_1(k + g) + n_2(k + g + n_1 + 1) + \cdots
+ n_g(k + g + n_1 + \cdots + n_{g-1} + g - 1),
\]

i.e.:

\[
R = \bar{n}_2(\bar{n}_1 + 1) + \bar{n}_3(\bar{n}_1 + \bar{n}_2 + 2) + \cdots + \bar{n}_g(\bar{n}_1 + \cdots + \bar{n}_{g-1} + g - 1)
+ n_2(n_1 + 1) + \cdots + n_g(n_1 + \cdots + n_{g-1} + g - 1)
+ (k + g)(n_1 + \cdots + n_g).
\]
4.3.3 Constant factors

Due to the structure of the set \( \mathcal{C} \), it is easy to observe that the following relations hold:

\[
\chi(\mathcal{C}) \prod_{ij: \ n_{ij}^1 = 1} (-1)^{\sum_{m=i}^{g} \bar{n}_m + \sum_{m=1}^{j-1} n_m} = (-1)^S,
\]

with

\[
S = (1 - \bar{n}_1)(\bar{n}_2 + \cdots + \bar{n}_g) + (1 - \bar{n}_2)(\bar{n}_3 + \cdots + \bar{n}_g) + \cdots (1 - \bar{n}_{g-1})(\bar{n}_g) + (1 - \bar{n}_2)(1 - \bar{n}_3)(\bar{n}_1 + \bar{n}_2) \cdots (1 - \bar{n}_g)(\bar{n}_1 + \cdots \bar{n}_{g-1})
\]

and

\[
\chi(\mathcal{C}) \prod_{ij: \ n_{ij}^1 = 1} (-1)^{i+j} = (-1)^T,
\]

with

\[
T = \sum_i (1 - \bar{n}_i)i + \sum_j (1 - n_j)j.
\]

Bringing out of the sum the constant factor \((-1)^{g+1}(-1)\), we arrive at

\[
I = (-1)^{Q+R+S+T+g(g-k)} \sum_{\ n_{1i}^1 = 0}^{1} \cdots \sum_{\ n_{g1}^1 = 0}^{1} \chi(\mathcal{C}) \prod_{ij: \ n_{ij}^1 = 1} (A_{ij})(-1)^{H_{ij}}.(57)
\]

Now we must calculate the parity of the sum \( Q + R + S + T + g(g - k) \). We will indicate the parity of an integer \( n \) with:

\[ \mathcal{P}[n] \equiv \text{Parity}[n]. \]

Using the fact that \( \sum_i g \bar{n}_i = \sum_i g n_i = k \), it is not difficult to get the following relations:

\[
\mathcal{P}[R + S + T] = \mathcal{P}[(n_1 + \bar{n}_1) + (n_2 + \bar{n}_2)3 + \cdots + (n_g + \bar{n}_g)(2g - 1) + \bar{n}_1(\bar{n}_2 + \cdots + \bar{n}_g) + \bar{n}_2(\bar{n}_1 + \bar{n}_3 + \cdots + \bar{n}_g) + \cdots \bar{n}_g(\bar{n}_1 + \cdots + \bar{n}_{g-1}) + (\bar{n}_2 + \cdots + \bar{n}_g) + (\bar{n}_3 + \cdots + \bar{n}_g) + \cdots + (\bar{n}_g) + (n_1) + (n_1 + n_2) + \cdots + (n_1 + \cdots + n_{g-1}) + k(g - k)],
\]

16
from which one has:

$$
P[R + S + T] = P[(n_1 + \bar{n}_1) + (n_2 + \bar{n}_2)2 + \cdots + (n_g + \bar{n}_g)g + k(k - 1) + k(g - k) + k(g - 1)],
$$

so, using Eq. (53) we obtain:

$$
P[Q + R + S + T + g(g - k)] =

P[n_1(\bar{n}_1 + \bar{n}_2 + \cdots + \bar{n}_g) + n_2(\bar{n}_2 + \cdots + \bar{n}_g) + \cdots + n_g\bar{n}_g

+ \frac{g(g + 1)}{2} + (n_1 + \bar{n}_1) + (n_2 + \bar{n}_2)2 + \cdots + (n_g + \bar{n}_g)g + g(g - k)](58)
$$

In the next subsection we will extract another constant factor coming from the phases \((-1)^{H_{ij}}\) contained in the product (57).

### 4.3.4 Permutations - completion of the proof

Let us now investigate the full meaning of the \(H_{ij}\). Let be \(h = g - k\). Let \(I\) and \(J\) be the sets (of cardinality \(h\)) of the labels \(i\) and \(j\) satisfying \(\bar{n}_i = 0\) and \(n_j = 0\) respectively. We will call such sets “active”. First, let us suppose that \(I = J = \{1, \ldots, h\}\). In terms of the matrix \(A\), we are looking at the submatrix of \(A\) obtained considering the first \(h\) rows and the first \(h\) columns of \(A\).

In section 4.2 we have seen that each event of the set \(C\) is a proper realization of the matrix \(N\). Choosing the rows as reference, we can obtain the events of \(C\) considering for each of the first \(h\) row indices \(i\), a permutation \(\pi = (\pi_1, \ldots, \pi_h)\) such that \(N^{i\pi_1} = 1\). In other words, for any row \(i \in I\), \(\pi_i\) tells us which is the corresponding (unique) active column. With these definitions we can carry out the summation over the events by summing over the permutations:

$$
\sum_{N^{i\pi_1}_{1\ldots,NN^{j\omega}_{1\ldots,h}=0,1}} \chi(C)f(\cdot) = \sum_{\pi \in \Pi} f(\cdot),
$$

where \(\Pi\) is the set of permutations of \(h\) elements. Let us define also for each of the first \(h\) column indices \(j\), the corresponding permutation \(\omega = (\omega_1, \ldots, \omega_h)\) such that \(N^{j\omega_{1\ldots,h}} = 1\). If \(\pi\) and \(\omega\) are referred to the same event, it is easy to see that one has \(\omega_{\pi_j} = j\) for any \(j \in J\).
Given an effective trajectory, if $\pi$ is the corresponding permutation, one has

$$
\sum_{i,j: N_{ij}^1=1} H_{ij} = \sum_{i=1}^{h} \left[ \sum_{m=i+1}^{h} \theta_{m\pi_m}^{i\pi_i} + \sum_{m=1}^{\pi_i - 1} \theta_{\omega_{m} m}^{i\pi_i} \right].
$$

Let us now expand the r.h.s. of the above expression in the following way:

\[
\begin{align*}
\theta_{2\pi_2}^{1\pi_1} + \theta_{3\pi_3}^{1\pi_1} + \cdots + \theta_{h\pi_h}^{1\pi_1} + \left[ \theta_{\omega_1 1}^{1\pi_1} + \theta_{\omega_2 2}^{1\pi_1} + \cdots + \theta_{\omega_{(\pi_1-1)} (\pi_1-1)}^{1\pi_1} \right] \\
+ \left[ \theta_{3\pi_3}^{2\pi_2} + \cdots + \theta_{h\pi_h}^{2\pi_2} \right] + \left[ \theta_{\omega_1 1}^{2\pi_2} + \theta_{\omega_2 2}^{2\pi_2} + \cdots + \theta_{\omega_{(\pi_2-2)} (\pi_2-2)}^{2\pi_2} \right] \\
+ \cdots \\
+ \theta_{\omega_1 1}^{h\pi_h} + \theta_{\omega_2 2}^{h\pi_h} + \cdots + \theta_{\omega_{(\pi_h-1)} (\pi_h-1)}^{h\pi_h}.
\end{align*}
\]

(59)

Let us consider the first row of the above expression. If $\pi_1 > \pi_2$, then in the first row, besides $\theta_{2\pi_2}^{1\pi_1}$, $\theta_{3\pi_3}^{1\pi_1}$ occurs too and $\omega_{\pi_2} = 2$. On the other hand if $\pi_1 < \pi_2$ (the equal sign is not possible since $\pi$ is a permutation) the former pairing for $\theta_{2\pi_2}^{1\pi_1}$ is not possible in the same row, therefore the only possible pairing is with a different row where the upper labels for the $\theta$’s are all different from $1\pi_1$. So in this case the pairing will be with $\theta_{3\pi_3}^{1\pi_1}$. Then, using the relations (51) one has the following dichotomy: $\pi_1 > \pi_2 \rightarrow 1$ and $\pi_1 < \pi_2 \rightarrow 0$. Repeating the same argument for any row of the above expression one arrives at:

\[
\sum_{i,j: N_{ij}^1=1} H_{ij} = \theta(\pi_2 - \pi_1) + \theta(\pi_3 - \pi_1) + \cdots + \theta(\pi_h - \pi_1) \\
+ \theta(\pi_3 - \pi_2) + \cdots + \theta(\pi_h - \pi_2) \\
+ \cdots \\
+ \cdots \\
+ \theta(\pi_h - \pi_{h-1}).
\]

(60)

The r.h.s. of (60) is just the number of exchanges needed to bring the permutation $\pi$ to the inverted fundamental permutation i.e. to the permutation $(h, h-1, \ldots, 1)$.  

18
If now we want to express the sign of the permutation in more conventional terms, i.e. in terms of the number of exchanges needed to bring the permutation to the fundamental one \((1, 2, \ldots, h)\), it is enough to consider the relation between such number of exchanges and the former. If we define \(\bar{H}_{ij}\) in the following way:

\[
\sum_{ij: N_{i,j}^{1}} \bar{H}_{ij} = [1 - \theta(\pi_2 - \pi_1)] + [1 - \theta(\pi_3 - \pi_1)] + \cdots + [1 - \theta(\pi_h - \pi_1)]
\]

\[
+ [1 - \theta(\pi_3 - \pi_2)] + \cdots + [1 - \theta(\pi_h - \pi_2)] \ldots
\]

\[
+ \ldots
\]

\[
+ \ldots
\]

\[
+ [1 - \theta(\pi_h - \pi_{h-1})],
\]

we see that the above r.h.s. now counts the number of exchanges needed to bring the permutation to the fundamental one; on the other hand we have:

\[
\sum_{ij: N_{i,j}^{1}} H_{ij} = \sum_{ij: N_{i,j}^{1}} \bar{H}_{ij} - \frac{h(h - 1)}{2}.
\]

(61)

The term \(\frac{h(h - 1)}{2}\) is the last of the constant factors just mentioned before. It must be summed to all the others constant factors, i.e. to the r.h.s. of Eq. (58). So under our hypothesis \(I = J = \{1, \ldots, h\}\), we have the following final global phase:

\[
(-1)^{(Q + R + S + T + g(g - k) + \frac{h(h - 1)}{2})} = (-1)^{W_0}
\]

(62)

where \(W_0\) is given from

\[
W_0 = \left[ n_1(\bar{n}_1 + \bar{n}_2 + \ldots + \bar{n}_g) + n_2(\bar{n}_2 + \ldots + \bar{n}_g) + \ldots + n_g\bar{n}_g 
\right. 

\]

\[
\left. + (n_1 + \bar{n}_1) + (n_2 + \bar{n}_2)2 + \cdots + (n_g + \bar{n}_g)g + \frac{k(k + 1)}{2} \right].
\]

On the other hand if \(I = J = \{1, \ldots, h\}\) it is easy to see that \((-1)^{W_0} = 1.\)

Now we can return to the general case in which the active rows and columns are no more restricted to be the first \(h\). For this aim it is enough to consider the numerator of the (5) and to make a suitable permutation which brings
$d\xi_1 d\xi_2 \ldots d\xi_g$ in $d\xi_{p_1} d\xi_{q_1} \ldots d\xi_{p_h} d\xi_{q_h}$ with $n_{p_1} = \bar{n}_{q_1} = \ldots n_{p_h} = \bar{n}_{q_h} = 0$. So we have a again the case $\mathcal{I} = \mathcal{J} = \{1, \ldots h\}$ up to a sign related to this permutation which is $(-1)^{\sum_{v=1}^{h} p_v + q_v} = (-1)^{\sum_{v=1}^{h} i_v + j_v} = (-1)^W$ (see the footnote n.2).

Then for the Gaussian, not normalized, Berezin integral $I$ we have found:

$$I = (-1)^W \sum_{\pi \in \Pi} (-1)^\pi \prod_{i \in \mathcal{J}} A_i \pi_i,$$

where now $\pi = (\pi_1, \ldots, \pi_h)$ with $\pi_i \in \mathcal{J}$ for any $i \in \mathcal{I}$ and $(-1)^\pi$ is the sign of the permutation $\pi$, i.e. the sign determined from the parity of the number of exchanges needed to bring $\pi$ to $(j_1, \ldots, j_h)$ with $j_1 < \cdots < j_h$. So up to the global phase $(-1)^W$, $I$ is just the determinant of the $(g - k) \times (g - k)$ submatrix defined in section 2. Then we have found:

$$I = \det A_s \ast (-1)^W(s)$$

5 Algebraic, geometrical and physical aspects

Equations (12) and (13) provide interesting interpretations and applications in several contests.

5.1 Algebraic

If we choose $s = (1, 1, \ldots, 1, 1)$, Eq. (13) reduces to the trivial identity

$$\det C = \frac{1}{\det A},$$

(63)

which tells us, as it is well known, to calculate the determinant of the inverse of $A$, it is not necessary to evaluate the inverse of $A$. Eq. (13) generalizes this property to any submatrix of $C$, $C_s$, since to obtain $\det C_s$ we need just the determinants of $A$ and $A_s \ast$, with $A_s \ast$ submatrix of $A$. However it is important to note that in general $\widetilde{C}_s(-1)^W$ is not the inverse of $A(\widetilde{A}_s \ast)^{-1}$, where with $\widetilde{B}_s$ we indicate the $g \times g$ extension of $B_s$, i.e. the matrix representing the operator which is the direct sum of the operator in $k$ dimensions, represented by the matrix $B_s$, plus the identity operator in $g - k$ dimensions.
Actually we can easily prove algebraically Eq. (13) in cases in which $A_{s^*}$ is a submatrix of $A$ with contiguous labels, in the following way. Let us suppose $A_{s^*}$ is the submatrix obtained erasing the last $h$ rows and $h$ columns of $A$, where $h = g - k$. Let us decompose the matrix $A$ in four blocks: $A_{11}; A_{12}; A_{21}$ and $A_{22}$, with dimensions $k \times k; k \times h; h \times k$ and $h \times h$ respectively. Let us make the same for the matrix $C$. In particular we have $A_{22} = A_{s^*}$ and $C_{11} = C_s$. With such a decomposition it is easy to multiply any two matrices by considering the blocks like elements of matrices of dimensions $2 \times 2$. Now let us define the matrix $B = \tilde{A}_{s^*} + D$, where $D$ has $D_{22} = A_{12}$ and the others three blocks zero. Hence one has $\det B = \det \tilde{A}_{s^*} = \det A_{s^*}$. On the other hand for $E = CB$ one has $E_{11} = C_{11}; E_{12} = 0; E_{21} = C_{21}$ and $E_{22} = 1$. So, since $\det E = \det C_s$, by using

$$\frac{\det A_{s^*}}{\det A} = \det(CB),$$

we get the identity (13) (it is simple to see that in this case $W$ is even).

In the general case we will have to face the complexity of the indices and the proof of Eq. (13) can be obtained as a corollary of the Binet - Cauchy's theorem on the compound matrices [8, 9].

5.2 Geometrical

Up to a sign, the determinant of a $g \times g$ matrix represents the volume of a prism in a $g$ dimensional Euclidean space. The volume is that spanned by $g$ vectors whose $g$ components are the $g$ rows of the matrix. From Eq. (13) we have

$$\frac{|\det A|}{|\det A_{s^*}|} = \frac{1}{|\det C_s|}.$$  \hspace{1cm} (64)

Now, the l.h.s. of this equation is the ratio of two volumes: one is that of a given $g$ dimensional prism, while the other is that of a $g - k$ dimensional prism obtained by spanning the space using the $g - k$ vectors whose components are given by the $g - k$ rows of $A_{s^*}$, i.e. it is a $g - k$ dimensional section of the first prism. So this ratio is nothing else than the relative “height” with respect to this section (here with “height” we mean a generalized height as in general it
will be itself a lower dimensional volume). Hence Eq. (64) tell us that $|\det C_s|$ is the inverse of this height.

\[
H = \frac{1}{|\det C_s|}
\]

\[
S = |\det(A_s^*)|
\]

5.3 Physical
5.3.1 Gaussian averages

Equations (7), (12) and (13) show that the problem of evaluating Gaussian Berezin integrals can be addressed as follows:

- **if** $A^{-1}$ **is known**, **then** we will use Eq. (7),

- **if** $A$ **is known**, **then** we will use Eq. (12),

- **if both** $A$ **and** $A^{-1}$ **are known**, we will use Eq. (7) if $k \leq g/2$ and Eq. (12) if $k > g/2$.

The last point is obvious. Since the calculation of a determinant of a $d \times d$ matrix leads to a sum of $d!$ elements with alternate signs, the advantage in the choice of one or another representation may be factorial with respect to the dimensions as $k$ is enough different from $g/2$. The others two points, first remarked in [11], are also obvious. In fact let us suppose we are, for example, in the second case and we are interested to the not normalized Gaussian integrals (that is, we don’t need to evaluate $\det A$ for both the two representations). In order to calculate $C_s$ from $A$ we must calculate $k^2$ minors of $A$, each of them in general involving a determinant of a $(g-1) \times (g-1)$ submatrix of $A$. Therefore we should to consider $k^2[(g-1)!]$ terms in order to calculate $\det C_s$, while to calculate $\det A_s^*$ we need to consider $(g-k)!$ terms. So, in general, for any $k > 1$, knowing $A$, the use of Eq. (12) provides the following gain (calculated\}
as ratio between the number of terms present in the two representations)

\[
G\text{AIN} = k^2 (g - 1)(g - 2) \ldots (g - k + 1),
\]  

(65)

while for \( k = 1 \) the GAIN is 1.

For normalized Gaussian integrals, actually the gain can be less as the calculation of \( \det A \) could implicitly provides some of the above \( k^2 \) minors. However in general the gain will be of the same order.

5.3.2 Mapping between two fermionic systems

Besides the above practical advantage, Eq. (13) provides an interesting theoretical insight.

In terms of Gaussian Berezin averages Eq. (13) may be read as:

\[
< \psi^* > A = < \psi^* > A^{-1} \frac{(-1)^W(s)}{\det A}.
\]

(66)

\(< \cdot > A \) means a Gaussian average being \( A \) the matrix of the Gaussian action.

Let be given a fermionic system with the following action

\[
S(\xi, \bar{\xi}) = \frac{1}{2} (\bar{\xi}, A\xi) - V(\xi, \bar{\xi}),
\]

(67)

where \( V(\xi, \bar{\xi}) \) is an arbitrary element of \( G_{2g} \) which we can expand in monomials as

\[
V(\xi, \bar{\xi}) = \sum_{s \in (\mathbb{Z}_2^2)^{2g}} \psi^* V_s,
\]

(68)

with \( V_s \in \mathbb{C} \). The system is then characterized by the following Grassmannian partition function

\[
Z = \int_B \exp[-S(\xi, \bar{\xi})] d\xi_1 d\bar{\xi}_1 \ldots d\xi_g d\bar{\xi}_g.
\]

(69)

Let \( s^* = (m_1, \bar{m}_1; \ldots; m_g, \bar{m}_g) \). Let \( \hat{T} \) be a linear operator on \( G_{2g} \) whose action on monomials \( \psi^{s^*} \) is defined by

\[
\hat{T} \psi^{s^*} = \psi^{s^*} (-1)^{\sum_{i=1}^g [(m_i + \bar{m}_i) + m_i, \bar{m}_i] + g}
\]

(70)
As shown in appendix A, by using Eq. (66), it is not difficult to see that for $Z$ we have also the representation

$$Z = (\det A) \int B \exp[-S^*(\xi, \bar{\xi})] d\xi_1 d\xi_g d\bar{\xi}_1 d\bar{\xi}_g,$$

(71)

where now the action $S^*$ is given by

$$S^*(\xi, \bar{\xi}) = \frac{1}{2} (\bar{\xi}, A^{-1} \xi) - V^*(\xi, \bar{\xi}),$$

(72)

and the potential $V^*$ is related to $V$ according to the following relationship

$$V^*(\xi, \bar{\xi}) = \log \left[ T \exp[V(-\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \bar{\xi}})] \psi^e \right],$$

(73)

Let $\eta, \bar{\eta}$ be vectors of $G_2$ (Grassmann sources). If we define the generating Grassmann functional integral

$$Z(\bar{\eta}, \eta) = \int B \exp[-S(\xi, \bar{\xi}) + (\eta, \bar{\xi}) + (\xi, \bar{\eta})] d\xi_1 d\xi_g d\bar{\xi}_1 d\bar{\xi}_g,$$

(74)

we have also

$$Z(\bar{\eta}, \eta) = (\det A) \int B \exp[-S^*(\xi, \bar{\xi}; \eta, \bar{\eta})] d\xi_1 d\xi_g d\bar{\xi}_1 d\bar{\xi}_g,$$

(75)

where now $S^*$ is defined as before, but $V^*$ depends also on $\eta$ and $\bar{\eta}$ and is given by

$$V^*(\xi, \bar{\xi}; \eta, \bar{\eta}) = \log \left[ T \exp[V(-\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \bar{\xi}}) + (\eta, \frac{\partial}{\partial \xi}) + (-\frac{\partial}{\partial \bar{\eta}})] \psi^e \right].$$

(76)

Hence we now have, for example, that a fermionic system whose action is represented by a free part $A$ plus a short range interaction, as regard the functional integral, can be viewed as a fermionic system with free part $A^{-1}$ plus a “long range” interaction. Substantially $V^*$ is the complement of $V$ in $G_{2g}$. However notice that the term “long range” here has not exactly the same meaning used in physics. The above long range character is related to the fact that if $V$ connects few Grassmann variables, $V^*$ connects many.

---

5Let us observe that we adopted for Berezin integration the definition (3), which is different from $\int d\xi_1 d\xi_2 \ldots d\xi_g F(\xi) = f_{1, \ldots, k}$ as $F$, besides $\xi_1 \xi_2 \ldots \xi_g$, depends on others Grassmann variables too. In this case it is important to consider the relations (2).
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APPENDIX A

Let be

\[ S(\xi, \bar{\xi}) = \frac{1}{2}(\bar{\xi}, A\xi) - V(\xi, \bar{\xi}), \]

a generic action. Since \((\bar{\xi}, A\xi)\) is an even element of \(G_{2g}\), we have \(\exp[-S(\xi, \bar{\xi})] = \exp[-\frac{1}{2}(\bar{\xi}, A\xi)] \exp[V(\xi, \bar{\xi})]\) and we can expand the latter factor in Taylor series
with respect to \(V(\xi, \bar{\xi})\). Remembering that \([V(\xi, \bar{\xi})]^N\) is finite sum of monomials, we see that \(Z\) can be written as a finite sum of Gaussian Berezin averages
each one multiplied by a suitable coefficient given by the coefficients of the
potential (68) and by \(\det A\) (the Gaussian normalization).

Explicitly Eq. (66) reads as

\[ <\xi_1^{n_1} \bar{\xi}_1^{\bar{n}_1} \ldots \xi_g^{n_g} \bar{\xi}_g^{\bar{n}_g}> = <\xi_1^{1-\bar{n}_1} \bar{\xi}_1^{-n_1} \ldots \xi_g^{1-\bar{n}_g} \bar{\xi}_g^{-n_g}> A^{-1} \frac{(-1)^W(8)}{\det A}. \]

The r.h.s. of these relations tell us that we can calculate each Gaussian averages
considering \(A^{-1}\) instead of \(A\) as matrix associated to the Gaussian action, provide we use \(s^*\) instead of \(s\). \(\psi^s = \xi_1^{1-\bar{n}_1} \bar{\xi}_1^{-n_1} \ldots \xi_g^{1-\bar{n}_g} \bar{\xi}_g^{-n_g}\) can be obtained
from \(\psi^e = \xi_1^{\bar{n}_1} \bar{\xi}_1^{n_1} \ldots \xi_g^{\bar{n}_g} \bar{\xi}_g^{n_g}\) by multiple Grassmann derivation applied to the
highest grade monomial \(\psi^e = \xi_1 \bar{\xi}_1 \ldots \xi_g \bar{\xi}_g\):

\[ \psi^s = (\frac{\partial}{\partial \xi_1})^{\bar{n}_1} (\frac{-\partial}{\partial \xi_1})^{n_1} \ldots (\frac{\partial}{\partial \xi_g})^{\bar{n}_g} (\frac{-\partial}{\partial \xi_g})^{n_g} \psi^e, \]
or

\[ \psi^s = (-1)^{n_1 \bar{n}_1 + \ldots + n_g \bar{n}_g} (\frac{-\partial}{\partial \xi_1})^{n_1} (\frac{\partial}{\partial \xi_1})^{\bar{n}_1} \ldots (\frac{-\partial}{\partial \xi_g})^{n_g} (\frac{\partial}{\partial \xi_g})^{\bar{n}_g} \psi^e. \]
As regards $(-1)^{W(s)}$, it can be expressed in terms of $s^* = (m_1, \bar{m}_1; \ldots; m_g, \bar{m}_g)$ and one finds

$$(-1)^{W(s^*)} = (-1)^{W(s^*(s))} = (-1)^{(m_1+\bar{m}_1+2(m_2+\bar{m}_2)+\ldots+g(m_g+\bar{m}_g)}$$  \hfill (77)

Hence, we have

$$<V(\xi, \bar{\xi}) > = \sum_{s \in (\mathbb{Z}_2^*)^{2g}} <\psi^s > A \ V_s = \sum_{s \in (\mathbb{Z}_2^*)^{2g}} <\psi^s > A^{-1} (-1)^{W(s^*)} \ \frac{V_s}{\det A}$$

and by using the definition of $\hat{T}$ it is easy to see that one has

$$<V(\xi, \bar{\xi}) > = \hat{T} <V(-\frac{\partial}{\partial \bar{\xi}}, \frac{\partial}{\partial \xi})\psi^s > A^{-1} \ \frac{1}{\det A}$$ \hfill (78)

and in general

$$<V^N(\xi, \bar{\xi}) > = \hat{T} <V^N(-\frac{\partial}{\partial \bar{\xi}}, \frac{\partial}{\partial \xi})\psi^s > A^{-1} \ \frac{1}{\det A}.$$ \hfill (79)

Finally, by using Eq.s (78, 79) and the definition of $V^*$ (73) we get Eq. (71).

By using this result the expression (75) follows as corollary.

**APPENDIX B**

Concerning $(-1)^{p_{ij}}$, since it accounts of the exchanges needed to bring $\bar{\xi}_i$ in front of $\xi_1$ and $\bar{\xi}_j$ in front of $\xi_1$ and considering the structure of $\epsilon_{ij}$, we see that the first permutation involves $(g - 1) + (g - i)$ exchanges and the second one $g - j$. We have

$$(-1)^{p_{ij}} = (-1)^{g-i-j-1}.$$

Thus for any given $\epsilon \in \Gamma$

$$s(\epsilon_{ij}) = \Delta_{2g}(\epsilon_{ij})(-1)^{g-i-j-1} A_{ij}.$$

Let us now calculate $\Delta_{2g}(\epsilon_{ij})$. Due to the arrangements of the generators of the algebra, we see from (16) that we can break up its expression in one part which contains only the barred components and another which contains both:

$$\Delta_{2g}(\epsilon_{ij}) = \prod_{l=1}^{g} \left( \frac{1-\epsilon_{l}}{2} + \frac{1+\epsilon_{l}}{2} \bar{\epsilon}_{l-1} \right) \prod_{l=1}^{g} \left( \frac{1-\epsilon_{l}}{2} + \frac{1+\epsilon_{l}}{2} \bar{\epsilon}_{1} \ldots \bar{\epsilon}_{g} \epsilon_{1} \ldots \epsilon_{l-1} \right).$$
For the structure of $\epsilon_{ij}$ we obtain

$$\Delta_{2g}(\epsilon_{ij}) = (-1)^{g-i} \times (-1)^{g-1+j} = (-1)^{g-i+j-1}$$

Therefore we finally have

$$s(\epsilon_{ij}) = A_{ij}.$$  

**APPENDIX C**

Among all the $2g$ factors which are contained in $C_{2g}$, only the $i$--th and the $(g+j)$--th can be different from 1, and one has

$$C_{2g}(\epsilon_{ij}, -(1)\tilde{N}_s + \tilde{n}_s) = \frac{1 + (-1)^{\frac{N_i^i + \tilde{n}_i}{2}}}{2} \frac{1 + (-1)^{\frac{N_j^j + n_j}{2}}}{2} \times (-1)^{\frac{N_i^i + \tilde{n}_i + N_j^j + n_j + \ldots + n_1 + \ldots}{2}} \times \exp\{i\pi[\tilde{N}_s^i + \ldots + \tilde{N}_s^g + N_1^i + \ldots + N_1^j + \ldots + \tilde{n}_i + \ldots + \tilde{n}_g + n_1 + \ldots + n_{j-1}]\},$$

i.e.

$$C_{2g}(\epsilon_{ij}, -(1)\tilde{N}_s + \tilde{n}_s) = \frac{1 + (-1)^{\frac{N_i^i + \tilde{n}_i}{2}}}{2} \frac{1 + (-1)^{\frac{N_j^j + n_j}{2}}}{2} \times (-1)^{\frac{N_i^i + \tilde{n}_i + N_j^j + n_j + \ldots + n_1 + \ldots}{2}} \times \exp\{i\pi[\tilde{N}_s^i + \ldots + \tilde{N}_s^g + N_1^i + \ldots + N_1^j + \ldots + \tilde{n}_i + \ldots + \tilde{n}_g + n_1 + \ldots + n_{j-1}]\},$$

where, due the structure of $\epsilon_{ij}$ and by using Eqs. (31) and (32), one has

$$\tilde{N}_s^i = \sum_{l=1}^g N_i^il,$$

$$N_s^j = \sum_{l=1}^g N_j^lj.$$  

**APPENDIX D**

Let us suppose that $N_1^{ij} = 1$, let us consider in (49), the factor with labels $(ij)$ and let us observe the structure of $H_{ij}$. Let $N_{1hk} = 1$, we must consider four events:

1. $h < i + 1$, $k > j - 1$ from which none contribute to $H_{ij}$ from $N^{hk}$

2. $h \geq i + 1$, $k > j - 1$ from which in $H_{ij}$ appears $\theta(s^{ij} - s^{hk})$

3. $h < i + 1$, $k \leq j - 1$ from which in $H_{ij}$ appears $\theta(s^{ij} - s^{hk})$
4. \( h \geq i + 1, k \leq j - 1 \) from which in \( H_{ij} \) appears \( \theta(s^{ij} - s^{hk}) + \theta(s^{ij} - s^{hk}) \).

The case n. 1 gives \((-1)^0 = 1\). The case n. 2 involves \((-1)^\theta(s^{ij} - s^{hk})\); as one considers in (49) the factor with labels \( h, k \) one has \( i < h + 1 \) and \( j < k + 1 \) from which \( j \leq k \). But two processes cannot jump in the same row or the same column, it is effective only \( j \leq k - 1 \) and that means that in \( H_{hk} \) one has the analog of case n. 3 where \( i, j \rightarrow h, k \) and viceversa. Then one has \( \theta(s^{hk} - s^{ij}) \), which together with \( \theta(s^{ij} - s^{hk}) \) coming from the former factor, give us \((-1)^1 = -1\). The case n. 3 is exactly analogous to the case n. 2. The case n. 4 gives \((-1)^2 = 1\).

Therefore the integrals over the jump times are fictitious and it is then convenient to define the variables with four labels \( \theta_{hk}^{ij} \) satisfying the following properties:

\[
\theta_{hk}^{ij} + \theta_{ij}^{hk} = 1 \\
\theta_{hk}^{ij} + \theta_{hk}^{ij} = 0.
\]

Hence we will write:

\[
H_{ij} = \left[ \sum_{m=i+1}^{g} \sum_{l=1}^{g} \theta_{ml}^{ij} N_{1}^{ml} + \sum_{m=1}^{j-1} \sum_{l=1}^{g} \theta_{lm}^{ij} N_{1}^{lm} \right],
\]

which does not depend on the jump times.

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