COEFFICIENT QUANTIZATION IN BANACH SPACES

S. J. DILWORTH, E. ODELL, TH. SCHLUMPRECHT, AND ANDRÁS ZSÁK

Abstract. Let \((e_i)\) be a dictionary for a separable Banach space \(X\). We consider the problem of approximation by linear combinations of dictionary elements with quantized coefficients drawn usually from a ‘finite alphabet’. We investigate several approximation properties of this type and connect them to the Banach space geometry of \(X\). The existence of a total minimal system with one of these properties, namely the coefficient quantization property, is shown to be equivalent to \(X\) containing \(c_0\).

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1. Introduction

We begin with the problem which motivates this paper. Let \((X, \|\cdot\|)\) be a separable infinite-dimensional Banach space and let \((e_i)\) be a semi-normalized dictionary for \(X\) (i.e. \((e_i)\) has dense linear span in \(X\)). For a given choice of \(N \in \mathbb{N}\), consider the problem of approximating an element \(x \in X\) by an element of the ‘lattice’

\[ \mathcal{D}^N((e_i)) = \{ \sum_{i \in E} \frac{k_i}{2^N} e_i : k_i \in \mathbb{Z}, E \subset \mathbb{N} \text{ finite} \}. \]
In many situations (e.g. when \((e_i)\) is a Schauder basis for \(X\)) each coefficient \(k_i/2^N\) of an approximant from \(D^N((e_i))\) will be bounded by a constant that depends only on \((e_i)\) and \(\|x\|\). In this case the approximant will be chosen from a collection of vectors in \(D^N((e_i))\) whose coefficients are quantized by a ‘finite alphabet’.

We investigate two natural approximation properties. The first of these, which we call the Coefficient Quantization Property (abbr. CQP), is defined roughly as follows: for every prescribed tolerance there exists a quantization such that every vector \(x = \sum_{i\in E} a_i e_i\) in \(X\) that can be expressed as a finite linear combination of dictionary elements can be approximated by a quantized vector \(y = \sum_{i\in E} d_i e_i\) with the same (or possibly smaller) support \(E\). Thus, for each \(\varepsilon > 0\), there exists \(N\) such that for every \(x\) with finite support \(E\) there exists \(y \in D^N((e_i))\) supported in \(E\) such that \(\|x - y\| \leq \varepsilon\).

Precise definitions and some useful permanence properties are presented in Section 2. One of our main results (Theorem 2.4) is the perhaps surprising fact that quantization of the unit ball for some \(\varepsilon < 1\) automatically implies quantization of the whole space.

Several examples of bases with the CQP, including the Schauder system for \(C([0,1])\) and a class of bases for \(C(K)\), where \(K\) is a countable compact metric space, are discussed in Section 3. On the other hand, it is shown that the Haar basis for \(C(\Delta)\), where \(\Delta\) denotes the Cantor set, is not a CQP basis. It turns out that all of the natural examples satisfy a stronger form of the CQP which we call the Strong Coefficient Quantization Property. Roughly, this means that the quantization of each coefficient can be an arbitrary \(\delta\)-net, not necessarily a discrete subgroup of \(\mathbb{R}\).

W. T. Gowers [11] proved that every real-valued Lipschitz function on the unit sphere of \(X\) is essentially constant on the sphere of an infinite-dimensional subspace of \(X\) if (and only if by [19]) \(X\) contains an isomorph of \(c_0\). A key feature of his argument was the fact that the unit vector basis of \(c_0\) has the CQP. The main results of this paper, as summarized in the following theorem, yield an intimate connection between the CQP and containment of \(c_0\).

**Main Theorem.** Let \(X\) be a separable Banach space. Then \(X\) has a fundamental and total normalized minimal system with the CQP if and only if \(c_0\) is isomorphic to a subspace of \(X\). Moreover, if \(X\) has a basis then \(X\) has a normalized weakly null basis with the CQP if and only if \(X\) contains an isomorph of \(c_0\).

The sufficiency is proved in Section 4 (Theorem 4.1) and the necessity is proved in Section 6 (Theorem 6.1). The necessity result is stated more precisely as the following dichotomy: if \((e_i)\) is a fundamental and total minimal system with the CQP then some subsequence of \((e_i)\) is equivalent to the unit vector basis of \(c_0\) or to the summing basis of \(c_0\).

For the reader who wishes to make a beeline for the proof of the Main Theorem we suggest a shorter route through the paper. After absorbing the definitions of the CQP and SCQP in Section 2 and the NQP in Section 5, he or she should then read Section 4, Theorem 5.11 (which is very short), and Section 6.

The second natural approximation property, which we call the Net Quantization Property (abbr. NQP), is investigated in Section 5. We say that \((e_i)\) has the NQP if for every \(\varepsilon > 0\) there exists \(N\) such that \(D^N((e_i))\) is an \(\varepsilon\)-net for \(X\). We prove that the NQP is a weaker property than the CQP. In particular, while the CQP is preserved under the operation of passing to a subsequence, this is not the case for
the NQP. Indeed, we prove (Theorem 5.9) that every normalized bimonotone basic sequence may be embedded as a subsequence of a Schauder basis with the NQP. Another main result of Section 5 is related to the greedy algorithm in Banach spaces (see e.g. [2]). It is proved that the unit vector basis of $c_0$ is the only quasi-greedy NQP minimal system.

We do not know whether or not every space $X$ with an NQP basis contains $c_0$. However, we are able to prove the weaker result that if $X$ admits a minimal system with the NQP then the dual space of $X$ contains an isomorphic copy of $\ell_1$ (Theorem 5.15). In particular, $X$ is necessarily non-reflexive.

The last section contains some examples and questions of a finite-dimensional character that are related to the CQP.

Standard Banach space notation and terminology are used throughout (see e.g. [15]). For the sake of clarity, however, we recall the notation that is used most heavily. Let $(X, \|\cdot\|)$ be a real Banach space with dual space $X^*$. The unit ball of $X$ is the set $\text{Ball}(X) := \{x \in X : \|x\| \leq 1\}$. We write $Y \hookrightarrow X$ (where $(Y, \|\cdot\|)$ is another Banach space) if there exists a continuous linear isomorphism from $Y$ into $X$.

Let $(e_i)$ be a sequence in $X$. The closed linear span of $(e_i)$ is denoted $\langle (e_i) \rangle$. We say that $(e_i)$ is weakly Cauchy if the scalar sequence $(x^*(e_i))$ converges for each $x^* \in X^*$. We say that $(e_i)$ is nontrivial weakly Cauchy if $(e_i)$ is weakly Cauchy but not weakly convergent, i.e. $(e_i)$ converges weak-star to an element of $X^{**} \setminus X$. We say that a sequence $(e_i)$ of nonzero vectors is basic if there exists a positive constant $K$ such that

$$\|\sum_{i=1}^n a_i e_i\| \leq K \|\sum_{i=1}^n a_i e_i\|$$

for all scalars $(a_i)$ and all $1 \leq m \leq n \in \mathbb{N}$; the least such constant is called the basis constant. $(e_i)$ is monotone if we can take $K = 1$; $(e_i)$ is $C$-unconditional, where $C$ is a positive constant, if

$$\|\sum_{i=1}^n \varepsilon_i a_i e_i\| \leq C \|\sum_{i=1}^n a_i e_i\|$$

for all scalars $(a_i)$, all choices of signs $\varepsilon_i = \pm 1$, and all $n \geq 1$. The least such constant is called the constant of unconditionality. We say that $(e_i)$ is a (Schauder) basis for $X$ if $(e_i)$ is basic and $\langle (e_i) \rangle = X$. Two basic sequences $(e_i)$ and $(f_i)$ are said to be equivalent if the mapping $e_i \mapsto f_i$ extends to a linear isomorphism from $\langle (e_i) \rangle$ onto $\langle (f_i) \rangle$.

For $1 \leq p < \infty$, $\ell_p$ is the space of real sequences $(a_i)$ equipped with the norm $\|(a_i)\|_p = (\sum_{i=1}^\infty |a_i|^p)^{1/p}$. The space of sequences converging to zero (resp. bounded) equipped with the supremum norm $\|\cdot\|_\infty$ is denoted $c_0$ (resp. $\ell_\infty$). The linear space of eventually zero sequences is denoted $c_{00}$. For $(a_i) \in c_{00}$, the support of $x$, denoted $\text{supp} x$, is the set $\{i \in \mathbb{N} : a_i \neq 0\}$. The space of continuous functions on a compact Hausdorff space $K$ equipped with the supremum norm $\|\cdot\|_\infty$ is denoted $C(K)$. For Banach spaces $X$ and $Y$, the direct sum $X \oplus_\infty Y$ (resp. $X \oplus Y$) is equipped with the maximum norm $\|(x, y)\|_\infty = \max(\|x\|, \|y\|)$ (resp. sum norm $\|(x, y)\|_1 = \|x\| + \|y\|$). Similarly, $(\sum_{n=1}^\infty X_n)_0$ and $(\sum_{n=1}^\infty X_n)_1$ denote the $c_0$ and $\ell_1$ sums of the Banach spaces $(X_n)_{n=1}^\infty$ equipped with their usual norms.

Finally, it is worth emphasizing that we consider only real Banach spaces in this paper.
2. The Coefficient Quantization Property

Throughout, $X$ will denote a separable infinite-dimensional Banach space and $(e_i)$ will denote a semi-normalized dictionary for $X$, i.e:

(i) there exist positive constants $a$ and $b$ such that $a \leq \|e_i\| \leq b$ ($i \in \mathbb{N}$);

(ii) $(e_i)$ is a fundamental system for $X$, i.e. $[(e_i)] = X$.

We say that $(e_i)$ is a minimal system (we shall always assume that the minimal system is semi-normalized and fundamental) if there exists a biorthogonal sequence $(e_i^*)$ in $X^*$ such that $e_i^*(e_j) = \delta_{ij}$. We say that $(e_i)$ is total if $e_i^*(x) = 0$ for all $i \in \mathbb{N}$ implies that $x = 0$, and that $(e_i)$ is bounded if $\sup \|e_i^*\| = M < \infty$. Ovsepian and Pełczyński [20] showed that every separable Banach space possesses a total and bounded minimal system [20]. Pełczyński [21] proved later that one can take $M = 1 + \varepsilon$ for any $\varepsilon > 0$.

Recall that a subset $S$ of a metric space $(T, \rho)$ is a $\delta$-net for $A \subseteq T$ (and is said to be $\delta$-dense in $A$) if for every $x \in A$ there exists $y \in S$ such that $\rho(x, y) \leq \delta$. Also $S$ is said to be $\delta$-separated if the distance between distinct points of $S$ is at least $\delta$.

**Definition 2.1.** A dictionary $(e_i)$ has the $(\varepsilon, \delta)$-Coefficient Quantization Property (abbr. $(\varepsilon, \delta)$-CQP) if for every $x = \sum_{i \in E} a_i e_i \in X$ (where $E$ is a finite subset of $\mathbb{N}$) there exist $n_i \in \mathbb{Z}$ ($i \in E$) such that

$$
\|x - \sum_{i \in E} n_i \delta e_i\| \leq \varepsilon.
$$

We say that $(e_i)$ has the CQP if $(e_i)$ has the $(\varepsilon, \delta)$-CQP for some $\varepsilon > 0$ and $\delta > 0$.

**Remark 2.2.** Setting

$$
\mathcal{F}_\delta((e_i)) := \left\{\sum_{i \in E} n_i \delta e_i : E \subset \mathbb{N} \text{ finite}, n_i \in \mathbb{Z}\right\},
$$

note that [21] is equivalent to the following:

$$
\mathcal{F}_\delta((e_i)_{i \in E}) \text{ is } \varepsilon\text{-dense in } [(e_i)_{i \in E}].
$$

We begin with some elementary observations.

**Proposition 2.3.** Let $(e_i)$ be a dictionary for $X$ with the CQP and let $\varepsilon, \delta > 0$.

(a) The following are equivalent:

(i) $(e_i)$ has the $(\varepsilon, \delta)$-CQP.

(ii) $(e_i)$ has the $(\lambda \varepsilon, \lambda \delta)$-CQP for all $\lambda > 0$.

(iii) $(e_i)$ has the $(1, \delta/\varepsilon)$-CQP.

Thus, if $(e_i)$ has the CQP then there exists $c > 0$ such that $(e_i)$ has the $(\varepsilon, c \varepsilon)$-CQP for all $\varepsilon > 0$.

(b) The mapping

$$
\delta \mapsto \varepsilon(\delta) := \inf\{\varepsilon : (e_i) \text{ has the } (\varepsilon, \delta)\text{-CQP}\}
$$

is linear, i.e. $\varepsilon(\lambda \delta) = \lambda \varepsilon(\delta)$ for all $\delta > 0$ and $\lambda > 0$; moreover, if $(e_i)$ is linearly independent then $(e_i)$ has the $(\varepsilon(\delta), \delta)$-CQP.
Proof. (a) To prove the implication (i) \(\Rightarrow\) (ii), let \(\lambda > 0\) and \(x = \sum_{i \in E} a_i e_i\), where \(E\) is finite. Since \((e_i)\) has the \((\varepsilon, \delta)\)-CQP there exist \(n_i \in \mathbb{Z}\) such that \(\|x/\lambda - \sum_{i \in E} n_i \delta e_i\| \leq \varepsilon\). Hence \(\|x - \sum_{i \in E} n_i \lambda \delta e_i\| \leq \lambda \varepsilon\), which proves (ii). The proofs of the other implications are similar.

(b) The first assertion is an immediate consequence of (a), and the second is an easy compactness argument.

\[\Box\]

Now suppose that we relax Definition 2.1 by only requiring that one can approximate each element \(x\) of the unit ball of \(X\) instead of the whole space. Accordingly, for each \(\delta > 0\), we define \(\varepsilon(b)\) to be the infimum of those \(\varepsilon > 0\) such that for all finite \(E \subset \mathbb{N}\) we have that

\[F_3((e_i)_{i \in E})\text{ is } \varepsilon\text{-dense in } Ba((e_i)_{i \in E}).\]

The following theorem, which is the main result of this section, explains why the CQP has been defined in terms of quantization of the whole space instead of the unit ball.

**Theorem 2.4.** Let \((e_i)\) be a dictionary for \(X\). The following are equivalent:

(i) \((e_i)\) has the CQP;

(ii) \(\varepsilon(b)(\delta_0) < 1\) for some \(\delta_0 > 0\);

(iii) there exists \(\delta_1 > 0\) such that \(\varepsilon(\delta) = \varepsilon(b)(\delta) < \infty\) for all \(0 < \delta \leq \delta_1\).

**Proof.** The implications (i) \(\Rightarrow\) (ii) and (iii) \(\Rightarrow\) (i) are clear. To prove the nontrivial implication (ii) \(\Rightarrow\) (iii), let \(q_0 := (\varepsilon(b)(\delta_0) + 1)/2 < 1\). First we show that there exist \(0 < q_1 < 1\) and \(\delta_1 > 0\) such that for every \(0 < \delta < \delta_1\), we have \(\varepsilon(b)(\delta) < q_1\).

Indeed, choose \(n_1 \in \mathbb{N}\) and \(0 < q_1 < 1\) such that

\[n_1 + 1 \frac{n_1}{n_1} < q_1 < 1,

and set \(\delta_1 := \frac{\delta_0}{n_1}\). For \(0 < \delta \leq \delta_1\) and \(x = \sum_{i \in E} a_i e_i \in Ba(X)\), with \(E \subset \mathbb{N}\) finite, choose \(n \in \mathbb{N}\) such that \(\frac{\delta_0}{n + 1} < \delta \leq \frac{\delta_0}{n}\) (note that \(n \geq n_1\)) and choose \(k_i \in \mathbb{Z}\) \((i \in E)\) such that

\[\left\|\sum_{i \in E} a_i e_i - \frac{\delta_0}{n + 1}\right\| \leq \sum_{i \in E} k_i \delta_0 e_i < q_0.

Thus, since \(n \geq n_1\),

\[\left\|\sum_{i \in E} a_i e_i - \sum_{i \in E} k_i (n + 1) \delta e_i\right\| \leq q_0 \frac{(n + 1)\delta}{\delta_0} \leq q_0 \frac{n + 1}{n} < q_1,

which implies that \(\varepsilon(b)(\delta) < q_1\).

Suppose that \(0 < \delta, \delta \leq \delta_1\) satisfy

\[q_1 \leq \frac{\delta}{\delta_1} \leq \frac{1}{q_1}.

We claim that

\[\varepsilon(b)(\delta) \leq \frac{\varepsilon(b)(\delta)}{\delta}.

\]
Once the claim is shown, it follows, by exchanging the roles of \( \delta \) and \( \tilde{\delta} \), that we also have
\[
\frac{\varepsilon^{(b)}(\tilde{\delta})}{\delta} \leq \frac{\varepsilon^{(b)}(\delta)}{\delta},
\]
which implies local linearity and, thus, linearity of \( \varepsilon^{(b)} \) on \((0, \delta_1] \).

Let \( x = \sum_{i \in E} a_i e_i \in Ba(X) \) with \( E \) finite. There exists \( y = \sum_{i \in E} k_i \delta e_i \in \mathcal{F}_\delta((e_i)) \) such that \( \|x - y\| < q_1 \). Note that \( (\tilde{\delta}/\delta)(x - y) \in Ba(X) \) by (2.4). Hence, given \( \eta > 0 \), there exists \( z = \sum_{i \in E} m_i \delta e_i \in \mathcal{F}_\delta((e_i)) \) such that
\[
\|\tilde{\delta}/\delta(x - y) - z\| < (1 + \eta)\varepsilon^{(b)}(\tilde{\delta}),
\]
i.e.
\[
\|x - \sum_{i \in E} (k_i + m_i) \delta e_i \| < (1 + \eta)\delta \varepsilon^{(b)}(\tilde{\delta}),
\]
which yields (2.3) since \( \eta > 0 \) is arbitrary.

In order show that \( \varepsilon(\cdot) = \varepsilon^{(b)}(\cdot) \) on \((0, \delta_1] \), let \( 0 < \delta \leq \delta_1 \), let \( x = \sum_{i \in E} a_i e_i \), with \( E \subset \mathbb{N} \) finite, and let \( \eta > 0 \) be arbitrary. If \( \|x\| \geq 1 \) there exist \( k_i \in \mathbb{Z} \) (\( i \in E \)) such that
\[
\left\| \frac{x}{\|x\|} - \sum_{i \in E} k_i \frac{\delta}{\|x\|} e_i \right\| < (1 + \eta)\varepsilon^{(b)}\left(\frac{\delta}{\|x\|}\right) = (1 + \eta)\frac{\varepsilon^{(b)}(\tilde{\delta})}{\|x\|}
\]
and thus
\[
(2.4) \quad \left\| x - \sum_{i \in E} k_i \delta e_i \right\| < (1 + \eta)\varepsilon^{(b)}(\tilde{\delta}).
\]
If \( \|x\| \leq 1 \) we can of course also find \( k_i \in \mathbb{Z} \) such that (2.3) holds. Since \( \eta > 0 \) is arbitrary, it follows that \( \varepsilon(\cdot) \leq \varepsilon^{(b)}(\cdot) \) and, thus, \( \varepsilon(\cdot) = \varepsilon^{(b)}(\cdot) \) on \((0, \delta_1] \).

The following corollary is a quantitative version of the last result.

**Corollary 2.5.** Let \( 0 < \varepsilon_0 < 1 \) and \( \delta > 0 \). If \( \mathcal{F}_\delta((e_i)_{i \in E}) \) is \( \varepsilon_0 \)-dense in \( Ba([\langle e_i \rangle_{i \in E}]) \) for all finite \( E \subset \mathbb{N} \) then \( \mathcal{F}_\delta((e_i)_{i \in E}) \) is \( \varepsilon_1 \)-dense in \( [\langle e_i \rangle_{i \in E}] \) for all \( \varepsilon_1 > \left(\left\lceil \frac{\varepsilon_0}{1 - \varepsilon_0} \right\rceil + 1\right)\varepsilon_0 \).

(Here \( \lceil x \rceil \) denotes the integer part of \( x \).) In particular, if \( \varepsilon_0 < 1/2 \), then \( \mathcal{F}_\delta((e_i)_{i \in E}) \) is \( \varepsilon_1 \)-dense in \( [\langle e_i \rangle_{i \in E}] \) for all \( \varepsilon_1 > \varepsilon_0 \).

**Proof.** Using the notation of the last proof, we may take \( n_1 = \lceil \varepsilon_0/(1 - \varepsilon_0) \rceil + 1 \). The last proof yields
\[
\varepsilon(\delta/n_1) = \varepsilon^{(b)}(\delta/n_1) \leq \varepsilon^{(b)}(\delta) \leq \varepsilon_0.
\]
Thus, \( \varepsilon(\delta) \leq n_1 \varepsilon_0 \), which gives the result.

**Remark 2.6.** The assumption that \( (e_i) \) is semi-normalized is not required for the validity of Corollary 2.5. Moreover, if \( (e_i) \) is linearly independent then strict inequality in (2.4) may be replaced by non-strict inequality. Finally, the result is also valid for quasi-normed spaces.

In the finite-dimensional setting Corollary 2.5 can be formulated as a covering result of independent interest.
Theorem 2.7. Let $K \subset \mathbb{R}^n$ be a compact zero-neighborhood that is star-shaped about zero (i.e. $\lambda K \subseteq K$ for all $0 \leq \lambda \leq 1$) and let $L \subset \mathbb{R}^n$ be a lattice (i.e. a discrete subgroup of $\mathbb{R}^n$). If $K \subset L + \varepsilon_0 K$, where $0 < \varepsilon_0 < 1$, then $\mathbb{R}^n = L + \varepsilon_1 K$, where $\varepsilon_1 = (\lfloor \varepsilon_0/(1 - \varepsilon_0) \rfloor + 1)\varepsilon_0$.

Proof. The gauge functional $\|x\|_K := \min\{t > 0 : x \in tK\}$ is positively homogeneous, which is the only property of the norm that is used in the proof of Theorem 2.4. Hence, setting

$$
\varepsilon_L^{(b)}(\delta) := \min\{\varepsilon : K \subset \delta L + \varepsilon K\},
$$

the proof of Theorem 2.4 yields

$$
\varepsilon_L^{(b)}(\delta) = n_1 \delta \varepsilon_L^{(b)}(1/n_1) \leq n_1 \delta \varepsilon_0
$$

for all $0 \leq \delta \leq 1/n_1$, where $n_1 := n_1(\varepsilon_0)$ is defined as in the proof of Corollary 2.5. The proof is concluded as before. □

The examples presented in the next section all have a formally stronger version of the CQP which we now define.

Definition 2.8. Let $\varepsilon > 0$ and let $\delta > 0$.

(a) A dictionary $(e_i)$ has the $(\varepsilon, \delta)$-Strong Coefficient Quantization Property (abbr. $(\varepsilon, \delta)$-SCQP) if for every sequence $D := (D_i)$ of $\delta$-nets for $\mathbb{R}$, such that $0 \in D_i$ and for every $x = \sum_{i \in E} a_i e_i$ in $X$ (where $E$ is a finite subset of $\mathbb{N}$) there exist $d_i \in D_i$ ($i \in E$) such that

$$(2.6) \quad \|x - \sum_{i \in E} d_i e_i\| \leq \varepsilon.$$ 

(b) $(e_i)$ has the SCQP if $(e_i)$ has the $(\varepsilon, \delta)$-SCQP for some $\varepsilon > 0$ and $\delta > 0$.

Remarks 2.9.

(i) If we set

$$
\mathcal{F}_D((e_i)) := \{\sum_{i \in E} d_i e_i : E \subset \mathbb{N} \text{ finite}, d_i \in D_i\},
$$

then (2.6) is equivalent to the following:

$$
\mathcal{F}_D((e_i)) \text{ is } \varepsilon\text{-dense in } [(e_i)_{i \in E}].
$$

(ii) The obvious analogue for the SCQP of Proposition 2.3 is valid.

(iii) Note also the implication $(\varepsilon, \delta)$-SCQP $\Rightarrow (\varepsilon, 2\delta)$-CQP since $2\delta \mathbb{Z}$ is a $\delta$-net.

(iv) If $(e_i)$ has the $(\varepsilon, \delta)$-CQP, we say that $(e_i)$ is an $(\varepsilon, \delta)$-CQP dictionary, and similarly for the SCQP.

(v) To avoid repetition we shall assume henceforth that every $\delta$-net for $\mathbb{R}$ contains zero.

(vi) Unless stated otherwise all sums of the form $\sum a_i e_i$ will be assumed to be finite.

The uniformity built into the definition of the SCQP (i.e. that $\varepsilon$ depends only on $\delta$, not on the choice of $(D_i)$) is natural in view of the following uniform boundedness result.

Proposition 2.10. Let $(e_i)$ be a dictionary for $X$. The following are equivalent:

(i) $(e_i)$ has the SCQP;
(ii) For all $\delta > 0$ and for every sequence $(D_i)$ of $\delta$-nets there exists $M > 0$ such that for every $x = \sum_{i \in E} a_i e_i \in X$ (where $E$ is a finite subset of $\mathbb{N}$) there exist $d_i \in D_i$ ($i \in E$) such that
\[
\|x - \sum_{i \in E} d_i e_i\| \leq M;
\]

(2.8) \inf \{\|x - \sum_{i \in E} d_i e_i\|: d_i \in D_i\} < \delta
\]

and thus (2.8) yields $\|x - \sum_{i \in E} d_i e_i\| < \delta$.

Proof. Clearly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (i), we argue by contradiction. Suppose that (i) does not hold. Then by (ii) of Remarks 2.9, (e) fails the $(M,1)$-SCQP for all $M > 0$. First we construct by induction a sequence $(E_n)$ of finite disjoint subsets of $\mathbb{N}$, a sequence $(e_n)$ of sequences of 1-nets, and vectors $x_n = \sum_{i \in E_n} a^n_i e_i \in X$ ($n \geq 1$) such that
\[
\|x_n - \sum_{i \in E_n} d^n_i e_i\| > n \quad (n \geq 1).
\]

Suppose that $n_0 \geq 1$ and that the construction has been carried out for all $n < n_0$. Set $F := \cup_{n < n_0} E_n$. Since $(e_i)$ does not have the $(M,1)$-SCQP for $M = \text{card}(F) \max_{i \in F} \|e_i\| + n_0$ there exist a sequence $(D^{n_0}_i)$ of 1-nets, a finite set $G \subset \mathbb{N}$, and $x = \sum_{i \in G} a_i e_i \in X$ such that
\[
\|x - \sum_{i \in G} d^{n_0}_i e_i\| > \text{card}(F) \max_{i \in F} \|e_i\| + n_0.
\]

Choose $d^{n_0}_i \in D^{n_0}_i$ such that $|a_i - d^{n_0}_i| \leq 1$ for $i \in G \cap F$. Then
\[
\|\sum_{i \in G \cap F} (a_i - d^{n_0}_i) e_i\| \leq \text{card}(G \cap F) \max_{i \in G \cap F} \|e_i\|,
\]
and thus (2.7) yields
\[
\inf \{\|\sum_{i \in G \cap F} (a_i - d^{n_0}_i) e_i\|: d^{n_0}_i \in D^{n_0}_i\} > n_0.
\]

Set $E_n := G \setminus F$ and $x_n = \sum_{i \in G \setminus F} a_i e_i$ to complete the induction. Now define a sequence $(D_i)$ of 1-nets as follows:

\[
D_i = \begin{cases} 
D^n_i & \text{if there exist } n \text{ such that } i \in E_n, \\
2\mathbb{Z} & \text{otherwise.}
\end{cases}
\]

Then by (2.7) $(D_i)$ does not satisfy (iii).

\[\square\]

Our first permanence result ensures that the SCQP is preserved under linear isomorphisms.

**Proposition 2.11.** Suppose that $T : X \to Y$ is a bounded operator. Suppose also that $(e_i)$ is a dictionary for $X$ with the property that $(T(e_i))$ is a dictionary for $Y$.
(a) If $(e_i)$ is an $(\varepsilon, \delta)$-SCQP dictionary for $X$ then $(T(e_i))$ is an $(\varepsilon\|T\|, \delta)$-SCQP dictionary for $Y$.
(b) If $(e_i)$ has the SCQP then $(T(e_i))$ also has the SCQP.
Remark 2.12. The analogue of Proposition 2.11 for the CQP is also valid.

The following useful result shows that the SCQP is preserved after normalization of the dictionary.

Proposition 2.13. Suppose that $(e_i)$ has the $(\varepsilon, \delta)$-SCQP and that $a \leq \|e_i\| \leq b$. Then the normalized dictionary $(e_i/\|e_i\|)$ has the $(\varepsilon, \delta')$-SCQP for $\delta' = a\delta$.

Proof. Let $(D_i')$ be a family of $\delta'$-nets for $\mathbb{R}$. Then each $D_i = \{d_i'/\|e_i\| : d_i' \in D_i'\}$ is a $\delta$-net. Since $(e_i)$ has the $(\varepsilon, \delta)$-SCQP, it follows that for each $\sum_{i \in E} a_i |e_i/\|e_i\||$ in $X$, where $E$ is a finite subset of $\mathbb{N}$, there exist $d_i' \in D_i'$ $(i \in E)$ such that

\[ \|\sum_{i \in E} \frac{a_i}{\|e_i\|} e_i - \sum_{i \in E} \frac{d_i'}{\|e_i\|} e_i\| \leq \varepsilon. \]

We conclude this section with some open problems.

Problems 2.14. (1) For a given dictionary $(e_i)$ is the SCQP equivalent to the CQP?
(2) Does the analogue of Theorem 2.4 for the SCQP hold?
(3) Does the analogue of Proposition 2.13 for the CQP hold?

Remark 2.15. We say that a dictionary $(e_i)$ has property P if the following condition holds. There exists $\delta > 0$ such that for all $\delta$-nets $(D_i)$ and for all finite $E \subseteq \mathbb{N}$ there exist $d_i \in D_i \setminus \{0\}$ $(i \in E)$ such that $\|\sum_{i \in E} d_i e_i\| \leq 1$. To see that Property P implies the SCQP, let $(D_i)$ be a sequence of $\delta$-nets and consider $x = \sum_{i \in E} a_i e_i$. Clearly, each $D_i := \{d_i - a_i : d_i \in D_i\} \cup \{0\}$ is a $\delta$-net. Property P implies that there exist $d_i \in D_i$ $(i \in E)$ with $d_i \neq a_i$ such that $\|\sum_{i \in E} (d_i - a_i) e_i\| \leq 1$, so $(e_i)$ has the SCQP. When $(e_i)$ is linearly independent, one can also show that the converse implication holds, i.e. that the SCQP implies Property P. So for a linearly independent dictionary the first problem stated above is equivalent to the following: is the CQP equivalent to Property P?

3. Examples

3.1. The unit vector basis of $c_0$. The unit vector basis of $c_0$ has the $(\varepsilon, \varepsilon)$-SCQP. To see this, let $(D_i)$ be a sequence of $\varepsilon$-nets. Given $x = \sum_{i \in E} a_i e_i$, simply choose $d_i \in D_i$ such that $|a_i - d_i| \leq \varepsilon$. Then

\[ \|\sum_{i \in E} a_i e_i - \sum_{i \in E} d_i e_i\| = \max_{i \in E} \|a_i - d_i\| \leq \varepsilon. \]
It is instructive to note that if \((e_i)\) is a bounded minimal system then the above procedure for choosing the approximation is only effective for the unit vector basis of \(c_0\). To be precise, suppose that the \(\delta\)-nets \((D_i)\) are \(\gamma\)-separated for some \(\gamma > 0\). Consider the following algorithm: choose \(d_i\) to be the best approximation to the coefficient \(a_i\) (or the best approximation of smallest absolute value when \(a_i\) is exactly half-way between two \(d_i\) values).

**Proposition 3.1.** Let \((e_i)\) be a bounded minimal system. The following are equivalent:

1. \((e_i)\) is equivalent to the unit vector basis of \(c_0\);
2. \((e_i)\) has the SCQP and the algorithm described above implements the SCQP (when the \(\delta\)-nets are \(\gamma\)-separated);
3. \((e_i)\) has the CQP and the algorithm described above implements the CQP (for \(D_i = \mathbb{Z} \delta\)).

**Proof.** \((i) \Rightarrow (ii)\) was proved above and \((ii) \Rightarrow (iii)\) is trivial. For the proof of \((iii) \Rightarrow (i)\), suppose that the \((\varepsilon, \delta)\)-CQP for \((e_i)\) is implemented by the aforementioned algorithm, where \(0 < \varepsilon < 1\) and \(\delta > 0\). Let \(x = \sum_{i \in E} a_i e_i\) be a unit vector and suppose that \(\max |a_i| < \delta/2\). According to the algorithm, we should approximate \(x\) by taking \(d_i = 0\) for all \(i \in E\), which yields the contradiction \(1 = \|x\| \leq \varepsilon < 1\). Hence

\[
\frac{1}{M} \max |a_i| \leq \|x\| \leq \frac{2}{\delta} \max |a_i|,
\]

where \(M = \sup \|e_i^*\|\). Thus, \((i)\) holds. \(\square\)

### 3.2. The summing basis of \(c_0\).

The linear space of sequences \((a_i)\) for which \(\sum_{i=1}^{\infty} a_i\) converges is a Banach space when equipped with the following norm:

\[
\|(a_i)\|_{sb} = \sup_n \left| \sum_{i=1}^{n} a_i \right|.
\]

This space is isometrically isomorphic to the space \(c\) of convergent sequences with the supremum norm. The unit vector basis \((e_i)\) is equivalent to a conditional basis of \(c_0\) called the summing basis.

To see that \((e_i)\) has the \((\varepsilon, \varepsilon)\)-SCQP, let \((a_i) \in c_{00}\). Suppose that \((d_i)_{i=1}^{k}\) have been chosen so that \(\left| \sum_{i=1}^{j} (a_i - d_i) \right| \leq \varepsilon\) for \(1 \leq j \leq k\). Then we continue by choosing \(d_{k+1} \in D_{k+1}\) so that \(d_{k+1} = 0\) if \(a_{k+1} = 0\) and so that \(\left| \sum_{i=1}^{k+1} (a_i - d_i) \right| \leq \varepsilon\).

Let us generalize this example as follows. Let \(N \in \mathbb{N}\). For each \(1 \leq n \leq N\), let \((e_i)^{\infty}_{i=1} = \left\{a_i\right\}_{i=1}^{\infty}\) be a sequence of signs \(e_i^n = \pm 1\). Consider the following norm on \(c_{00}\):

\[
\|(a_i)\| = \max_{1 \leq n \leq N} \left\| (e_i^n a_i) \right\|_{sb}.
\]

For each \(\eta = (\eta_n)_{n=1}^{N} \in \{-1, 1\}^{N}\), let \(A_\eta = \{m \in \mathbb{N} : e_m^n = \eta_n, 1 \leq n \leq N\}\). Then \((A_\eta)\) \((\eta \in \{1, -1\}^{N})\) is a partition of \(\mathbb{N}\). Note that for \((a_i), (d_i) \in c_{00}\), the triangle inequality gives

\[
\|(a_i - d_i)\| \leq \sum_{\eta \in \{1, -1\}^{N}} \|(a_i - d_i)_{i \in A_\eta}\|_{sb}
\]

Now suppose that \((D_i)\) is a sequence of \(\varepsilon/2^{N}\)-nets for \(\mathbb{R}\). For each \(\eta \in \{1, -1\}^{N}\), choose \(d_i \in D_i\) for \(i \in A_\eta\) so that \(\|(a_i - d_i)_{i \in A_\eta}\|_{sb} \leq \varepsilon/2^{N}\). This is possible
since the summing basis has the \( \langle \varepsilon/2^N, \varepsilon/2^N \rangle \)-SCQP. It follows from (3.10) that \( \| (a_i - d_i) \| \leq \varepsilon \). Hence \( \| \cdot \| \) has the \( \langle \varepsilon, \varepsilon/2^N \rangle \)-SCQP.

3.3. The Schauder basis. Let us recall the definition of the classical Schauder basis \((f_i)_{i \geq 0}\) for \( C([0,1]) \): \( f_0(t) = 1, f_1(t) = t \), and for \( i = 2^k + l, 0 \leq l < 2^k \), \( f_i \) is the piecewise-linear function supported on \([2^{-k}, (l + 1)2^{-k}]\) satisfying \( f_i((2^{-k}) = f_i(l) = 0 \) and \( f_i((2l + 1)2^{-k-1}) = 1 \).

**Theorem 3.2.** The Schauder basis for \( C([0,1]) \) has the \( \langle \varepsilon, \varepsilon \rangle \)-SCQP for all \( \varepsilon > 0 \).

**Proof.** Let \( (D_i) \) be a sequence of \( \varepsilon \)-nets. Suppose that \( N \geq 0 \) and that \( x = \sum_{i=0}^{N} a_i f_i \). We shall prove that there exist \( d_i \in D_i \) such that

\[
\| \sum_{i=0}^{k} (a_i - d_i) f_i \|_\infty \leq \varepsilon
\]

for \( 0 \leq k \leq N \) and such that \( d_i = 0 \) if \( a_i = 0 \). Choose \( d_0 \in D_0 \) such that \( |a_0 - d_0| \leq \varepsilon \) and choose \( d_1 \in D_1 \) such that \( |a_0 + a_1 - d_0 - d_1| \leq \varepsilon \) (with \( d_i = 0 \) if \( a_i = 0 \)). This establishes (3.10) for \( k = 0 \) and \( k = 1 \). Suppose that \( 2 \leq n \leq N \) and that \( d_0, \ldots, d_{n-1} \) have been chosen so that (3.10) holds for \( 0 \leq k \leq n - 1 \). Let the support of \( f_n \) be the dyadic interval \([a,b]\) and consider

\[
g(x) = \left| \sum_{i=0}^{n} a_i f_i(x) - \sum_{i=0}^{n-1} d_i f_i(x) \right|.
\]

Then \( g \) is piecewise-linear on \([a,b]\) with nodes at \( a, b, \) and \((a + b)/2\). So \( g \) must attain its maximum at one of these three points. If the maximum occurs at either \( x = a \) or \( x = b \), then, since \( f_n(a) = f_n(b) = 0 \), it follows from the case \( k = n - 1 \) of (3.10) that

\[
\max_{x \in [a,b]} g(x) \leq \max_{x \in [0,1]} \left\| \sum_{i=0}^{n-1} (a_i - d_i) f_i \right\|_\infty \leq \varepsilon.
\]

Then, setting \( d_n = 0 \), (3.10) will be satisfied for \( k = n \). So suppose that the maximum is attained at \((a + b)/2\). Choose \( d_n \in D_n \) such that

\[
\left| \sum_{i=0}^{n-1} (a_i - d_i) f_i \frac{a + b}{2} + a_n f_n \frac{a + b}{2} - d_n \right| \leq \varepsilon.
\]

With this choice of \( d_n \), we see that (3.10) is again satisfied for \( k = n \). \( \square \)

**Remark 3.3.** Let \( K \) be an uncountable compact metric space. Then \( C(K) \) is uniformly isomorphic to \( C([0,1]) \) by Milutin’s Theorem [17]. Since the Schauder basis of \( C([0,1]) \) has the \( \langle \varepsilon, \varepsilon \rangle \)-SCQP, it follows from Propositions 2.11 and 2.13 that \( C(K) \) has a normalized \( \langle \varepsilon, c\varepsilon \rangle \)-SCQP basis for some absolute constant \( c > 0 \).

3.4. Tree spaces. By a tree we shall mean a partially ordered set \( (T, \leq) \) with the property that each node \( \alpha \in T \) has finitely many linearly ordered predecessors (with respect to \( \leq \)). We say that \( T \) is rooted if there is exactly one node without an immediate predecessor. The tree \( T_\infty \) is the rooted tree with the property that every node has countably infinitely many immediate successors. We equip \( c_{00}(T) \) with the following norm:

\[
\| x \| = \max_{\beta \in T} |S_\beta(x)|,
\]
where \( S_\beta(x) = \sum_{\alpha \leq \beta} x(\alpha) \). Let \( S(T) \) denote the completion of the normed space \((c_{00}(T), \| \cdot \|)\).

Henceforth we shall assume that \( T \) is countably infinite. Suppose that \((\alpha(i))\) is any enumeration of \( T \) which respects the ordering of \( T \), i.e. such that

\[ \alpha(i) \leq \alpha(j) \Rightarrow i \leq j. \]

Clearly, \((e_{\alpha(i)})\) is a normalized monotone basis for \( S(T) \).

**Proposition 3.4.** (a) Suppose that \( T \) is rooted. Then \( S(T) \) is isometrically isomorphic to \( C(K) \), where \( K \) is the weak-star closure of \( \{ S_\beta : \beta \in T \} \) in \( \text{Ba}(S(T)^*) \).

(b) If \( K \) is a countable compact metric space then \( C(K) \) is isometrically isomorphic to \( S(T) \) for some rooted tree \( T \).

(c) \( S(T_\infty) \) is isometrically isomorphic to \( C(\Delta) \), where \( \Delta \) denotes the Cantor set.

**Proof.** (a) It is easily seen that \( c_{00}(T) \) is a separating subalgebra of \( C(K) \). Since \( S_\alpha(e_\emptyset) = 1 \) for all \( \alpha \in T \), where \( \emptyset \) is the root node, it follows that \( \chi_K \in c_{00}(T) \), and hence by the Stone-Weierstraß theorem that \( c_{00}(T) \) is dense in \( C(K) \).

(b) It is well-known that every countable compact metric space is homeomorphic to an ordinal interval \([0, \alpha]\), for some countable ordinal \( \alpha \), with the order topology. We prove the result by transfinite induction. Suppose the result holds for \( K = [0, \beta] \) for all \( 0 \leq \beta < \alpha \). There exist \( 1 \leq n \leq \infty \) and countable ordinals \( \alpha_j < \alpha \) (\( 0 \leq j < n \)) such that \( K := [0, \alpha_i] \) is homeomorphic to the one-point compactification of the disjoint union of the ordinal intervals \( K_j := [0, \alpha_j] \) (\( 0 \leq j < n \)). By hypothesis there exist trees \( T_j \) (\( 0 \leq j < n \)) such that \( S(T_j) \) is isometrically isomorphic to \( C(K_j) \). Let \( T \) be the rooted tree which has each \( T_j \) (\( 0 \leq j < n \)) as a subtree immediately succeeding the root node. Then \( S(T) \) is easily seen to be isometrically isomorphic to \( C(K) \).

(c) In this case \( K \) is easily seen to be a perfect and totally disconnected compact metric space, and thus homeomorphic to \( \Delta \). \( \square \)

**Theorem 3.5.** \((e_\alpha)_{\alpha \in \tau} \) has the \((\varepsilon, \varepsilon) - \text{SCQP} \) in \( S(T) \) for all \( \varepsilon > 0 \).

**Proof.** Let \((\alpha(i))\) be any ordering of the basis which respects the ordering of \( T \). Let \( \varepsilon > 0 \) and let \((D_\alpha)_{\alpha \in \tau} \) be a family of \( \varepsilon \)-nets and suppose that \( \sum_{i \in K} x_{\alpha(i)} \in c_{00}(T) \). We define \( d_\alpha \in D_\alpha \) inductively. Suppose that \( n \geq 0 \) and that \( d_{\alpha(1)}, \ldots, d_{\alpha(n)} \) have been chosen such that

\[
|S_\gamma(\sum_{i=1}^{n} (x_{\alpha(i)} - d_{\alpha(i)}e_{\alpha(i)}))| \leq \varepsilon
\]

for all \( \gamma \in \tau \) (This condition is vacuous for \( n = 0 \).) If \( x_{\alpha(n+1)} = 0 \), set \( d_{\alpha(n+1)} = 0 \). Otherwise choose \( d_{\alpha(n+1)} \in D_{\alpha(n+1)} \) such that

\[
|\sum_{\beta < \alpha(n+1)} (x_\beta - d_\beta + x_{\alpha(n+1)} - d_{\alpha(n+1)})| \leq \varepsilon,
\]

noting that if \( \beta < \alpha(n+1) \) then \( \beta = \alpha(j) \) for some \( j \leq n \). Now we verify the inductive hypothesis for \( n + 1 \). If \( \gamma \geq \alpha(n+1) \) then

\[
|S_\gamma(\sum_{i=1}^{n+1} (x_{\alpha(i)} - d_{\alpha(i)}e_{\alpha(i)}))| = |\sum_{\beta \leq \alpha(n+1)} (x_\beta - d_\beta)| \leq \varepsilon.
\]
On the other hand, if \( \gamma < \alpha(n+1) \) or if \( \gamma = \alpha(n+1) \) are incomparable, then
\[
|S_{\gamma} \left( \sum_{i=1}^{n+1} (x_{\alpha(i)} - d_{\alpha(i)})e_{\alpha(i)} \right) | = |S_{\gamma} \left( \sum_{i=1}^{n} (x_{\alpha(i)} - d_{\alpha(i)})e_{\alpha(i)} \right) | \leq \varepsilon
\]
by the inductive assumption. This completes the verification of the inductive step. It follows that
\[
\| \sum_{n \in E} (x_{\alpha(n)} - d_{\alpha(n)})e_{\alpha(n)} \| \leq \varepsilon.
\]
Thus, \((e_{\alpha})_{\alpha \in \mathcal{T}}\) has the \((\varepsilon, \varepsilon)\)-SCQP. \(\square\)

**Corollary 3.6.** If \( K \) is a countable compact metric space or if \( K = \Delta \) then \( C(K) \) has a monotone basis with the \((\varepsilon, \varepsilon)\)-CQP for all \( \varepsilon > 0 \).

**Remark 3.7.** In all of the above examples the dictionary \((e_{i})\) has the neighborly CQP, i.e. for every \( x = \sum_{i \in E} a_{i}e_{i} \) with finite support, the approximation \( y = \sum_{i \in E} a_{i}\delta_{e_{i}} \) satisfies \( |a_{i} - a_{i}\delta| \leq \delta \). We do not know whether this holds in general, i.e. whether the CQP implies the neighborly CQP.

### 3.5. The Haar Basis for \( C(\Delta) \)
We have already seen that \( C(\Delta) \) has a monotone basis with the \((\varepsilon, \varepsilon)\)-CQP. Surprisingly, however, the natural basis of \( C(\Delta) \), namely the Haar basis, does not have the CQP. Let us recall the definition of the Haar basis. Let \( \Delta_{0} := \Delta \), and, for \( k \geq 0 \), let \( \Delta_{2k+1} \) and \( \Delta_{2k+2} \) be the left-hand and right-hand halves of \( \Delta_{k} \) obtained by removing the ‘middle third’ in the classical construction of the Cantor set. Then
\[
h_{i} = \begin{cases} 
    \chi_{\Delta} & \text{for } i = 0 \\
    \chi_{\Delta_{2i-1}} - \chi_{\Delta_{2i}} & \text{for } i > 0.
\end{cases}
\]
Clearly, \( (h_{i})_{i=0}^{\infty} \) is a monotone basis for \( C(\Delta) \). For \( k = 1, 2, \ldots \), we say that the \( 2^{k-1} \) Haar functions \( \{h_{i}; 2^{k-1} \leq i < 2^{k}\} \) are on the \( k \)-th level.

**Proposition 3.8.** Let \( 0 < \varepsilon < 1 \) and let \( \delta > 0 \). Then \( \mathcal{F}_{\delta}((h_{i})) \) is not an \( \varepsilon \)-net for the unit ball of \( C(\Delta) \). In particular, \((h_{i})\) does not have the CQP.

**Proof.** For \( N \in \mathbb{N} \), let \( x_{N} = (1/N) \sum_{i=1}^{2N-1} h_{i} \) and let \( y \in \mathcal{F}_{\delta}((h_{i})) \). Note that \( \|x_{N}\| = 1 \). We shall prove that \( \|x - y\| \geq 1 \) provided \( N \geq 2/\delta \). Since \((h_{i})\) is a monotone basis, we may assume that \( y \in \text{span}\{h_{i}; 0 \leq i \leq 2^{N} - 1\} \). Since \( x_{N} \) and \(-x_{N}\) have the same distribution, we may also assume that the coefficient of \( h_{0} \) in the expansion of \( y \) is \(-\alpha \), where \( \alpha \geq 0 \). Let \( k_{1} \geq 1 \) be the first level (if there are any) of the Haar system for which the leftmost Haar function has a nonzero coefficient in the expansion of \( y \). Let this Haar function be \( h_{i_{1}} \), and let \( a_{1} \) be the corresponding coefficient. Note that \( |a_{1}| \geq \delta \). By considering the left-hand and the right-hand halves of the support of \( h_{i_{1}} \), and using the monotonicity of the Haar basis, we see that
\[
\max_{t \in I_{1}}(x - y)(t) \geq \frac{k_{1} - 2}{N} + \alpha + \delta = \frac{k_{1}}{N} + \alpha + (\delta - \frac{2}{N}),
\]
where \( I_{1} \) is the (left-hand or right-hand) half of the support of \( h_{i_{1}} \), on which \( a_{1}h_{i_{1}} \) takes a negative value. Now we repeat the argument for \( I_{1} \). Suppose that the next level for which there is a nonzero coefficient in the leftmost Haar function whose support is entirely contained in \( I_{1} \) is the \( (k_{1} + k_{2}) \)-th level, where \( k_{2} \geq 1 \). Let \( h_{i_{2}} \)
denote this Haar function and let $a_2$ be the corresponding coefficient. Then, by the same reasoning as above, we get
\[
\max_{t \in I_2} (x - y)(t) > \frac{k_1 + k_2}{N} + \alpha + 2(\delta - \frac{2}{N}),
\]
where $I_2$ is the half of the support of $h_{i_2}$ on which $a_1 h_{i_2}$ takes a negative value. This process terminates after $J \geq 0$ steps at level $k_1 + \cdots + k_J$ with a set $I_J$ (half of the support of $h_{i_J}$) such that
\[
\max_{t \in I_J} (x - y)(t) > k_1 + \cdots + k_J + \alpha + J(\delta - \frac{2}{N}).
\]
Finally, let $I$ be the left-hand half of the leftmost Haar function on the $N$-th level whose support is entirely contained in $I_J$. Since the inductive process has terminated after $J$ steps, we obtain
\[
(x - y)(t) \geq 1 + \alpha + J(\delta - \frac{2}{N}) \geq 1 \quad (t \in I)
\]
provided $N \geq 2/\delta$.

**Remarks 3.9.** (i) The proof of Proposition 3.8 actually shows that if $\delta \geq 2/N$ then $F_\delta((h_i))$ is not an $\varepsilon$-net for the unit ball of $\ell_2^\infty$ for any $0 < \varepsilon < 1$.

(ii) In the terminology of Section 5 below, Proposition 3.8 shows that $(h_i)$ does not have the Net Quantization Property.

### 4. An Existence Result

**Theorem 4.1.** Suppose that $c_0 \hookrightarrow X$. Then $X$ has a bounded, total, weakly null, normalized minimal system which has the $(\varepsilon, c\varepsilon)$-SCQP for all $\varepsilon > 0$, where $c$ is an absolute constant (independent of $X$ and $\varepsilon$). Moreover, if $X$ has a basis, then $X$ has a normalized weakly null $(\varepsilon, c\varepsilon)$-SCQP basis.

First let us explain the construction that is used in the proof of Theorem 4.1. To that end, let $(e_{n,j})_{j=1}^{n+1}$ denote the unit vector basis of $\ell_2^\infty$. Define a new basis $(f_{n,j})_{j=1}^{n+1}$ as follows:
\[
f_j = e_j + \frac{e_{n+1}}{n} \quad (1 \leq j \leq n)
\]
and
\[
f_{n+1} = e_1 + e_2 + \cdots + e_n.
\]
The following lemma is easily verified.

**Lemma 4.2.** $(f_{n,j})_{j=1}^{n+1}$ is a normalized basis for $\ell_\infty^{n+1}$ with basis constant at most 3.

**Proof of Theorem 4.1.** By Sobczyk's theorem that $c_0$ is 2-complemented in any separable superspace and James's theorem that every Banach space isomorphic to $c_0$ contains an almost isometric copy of $c_0$, it follows that $X$ is uniformly isomorphic to $X \oplus_\infty c_0$. So by Proposition 2.11 and Proposition 2.13 it suffices to prove the result for $X \oplus_\infty c_0$. Let $(\phi_i)$ be a normalized total minimal system (resp. normalized basis) for $X$. For convenience, we regard $c_00$ as the space of all finitely supported sequences $(a_j^n)$ doubly indexed by $n \in \mathbb{N}$ and $1 \leq j \leq n^2 + 1$. Let $(e_j^n)$ denote the standard
basis for this realization of $c_{00}$ and order the basis elements lexicographically (i.e., $e_1^1, e_2^1, e_3^1, e_2^2, \ldots$). Define a norm $\| \cdot \|_Y$ on $c_{00}$ as follows:

$$
\| (a_j^n) \|_Y = \max \left\{ \sup_{n \geq 1} \| (a_j^n + a_{n+1}^j) \|_\infty, \left\| \sum_{n=1}^\infty \frac{1}{n^2} \left( \sum_{j=1}^{n^2} a_j^n \right) \phi_n \right\|_X \right\},
$$

and let $Y$ denote the completion of $(c_{00}, \| \cdot \|)$. It is easily seen that $Y$ is isometrically isomorphic to $X \oplus_\infty (\sum_{n=1}^\infty \ell_2^n)_{0}$, which in turn is isometrically isomorphic to $X \oplus_\infty c_0$, and that $(e_j^n)$ is a normalized bounded and total minimal system for $Y$. Moreover, for each $n \in \mathbb{N}$, $(e_j^n)_{j=1}^{n^2+1}$ is isometrically equivalent to the basis $(f_j)_{j=1}^{n^2+1}$ described above. Thus, for the case in which $(\phi_i)$ is a basis for $X$, it follows easily from Lemma 4.2 that $(e_j^n)$ is a basis for $Y$.

Let us next check that $(e_j^n)$ is weakly null. Under the isometric isomorphism of $Y$ with $X \oplus_\infty (\sum_{n=1}^\infty \ell_2^n)_{0}$, the basis vector $e_j^n$ corresponds to

$$(4.12) \begin{cases} 
(\phi_n/n^2, g_j^n), & \text{if } 1 \leq j \leq n^2 \\
(0, \sum_{i=1}^{n^2} g_j^n), & \text{if } j = n^2 + 1,
\end{cases}$$

where $(g_j^n)_{j=1}^{n^2}$ denotes the unit vector basis of $\ell_2^n$. Thus it suffices to check that the sequence defined by (4.12) is weakly null. But this is readily verified directly using the fact that $(X \oplus_\infty (\sum_{n=1}^\infty \ell_2^n)_{0})^*$ is isometrically isomorphic to $X^* \oplus_1 (\sum_{n=1}^\infty \ell_1^{2^n})_{1}$.

To see that $(e_j^n)$ has the $(\varepsilon, \delta, c\varepsilon)$-SCQP, let $\delta > 0$ and let $(D_j^n)$ be a doubly-indexed family of $\delta$-nets and let $(a_j^n) \in c_{00}$. For each $n \in \mathbb{N}$, choose $d_j^n \in D_j^n$, with $d_j^n = 0$ if $a_j^n = 0$, such that

$$(4.13) \begin{cases} 
\left| \sum_{j=1}^{k} (a_j^n - d_j^n) \right| \leq \delta & (1 \leq k \leq n^2)
\end{cases}$$

and

$$(4.14) |a_{n^2+1}^n - d_{n^2+1}^n| \leq \delta.$$ 

From (4.13) and the triangle inequality, we see that

$$(4.15) |a_j^n - d_j^n| \leq 2\delta \quad (1 \leq j \leq n^2).$$

Combining (4.13), (4.14), and (4.15), we obtain

$$
\| \sum (a_j^n - d_j^n)e_j^n \|_Y \leq \sup_n \max_{1 \leq j \leq n^2} |a_j^n - d_j^n + a_{n^2+1}^n - d_{n^2+1}^n|
\left( \frac{1}{n^2} \sum_{j=1}^{n^2} |a_j^n - d_j^n| \right) \leq 3\delta + \delta \cdot \frac{\pi^2}{6}.
$$

This shows that $(e_j^n)$ is a minimal system (resp. basis) for $Y$ with the $(\varepsilon, \delta, c\varepsilon)$-SCQP for $c = (3 + \pi^2/6)^{-1}$. \hfill \Box

Remark 4.3. The construction used in the proof of Theorem 4.1 was first used by Wojtaszczyk [24]. The dual construction was used recently in [10] to construct a quasi-greedy basis for $L_1([0,1])$. 
5. The Net Quantization Property

In this section we discuss a natural quantization property which is more general than the CQP.

**Definition 5.1.** Let $\varepsilon > 0$ and let $\delta > 0$.
(a) A dictionary $(e_i)$ has the $(\varepsilon, \delta)$-Net Quantization Property (abbr. $(\varepsilon, \delta)$-NQP) if for every $x \in X$ there exist a finite subset $E \subset \mathbb{N}$ and $n_i \in \mathbb{Z}$ ($i \in E$) such that
\[
\|x - \sum_{i \in E} n_i \delta e_i\| \leq \varepsilon.
\]
(b) $(e_i)$ has the NQP if $(e_i)$ has the $(\varepsilon, \delta)$-NQP for some $\varepsilon > 0$ and $\delta > 0$.

**Remarks 5.2.** (i) Note that (5.16) simply says that $\mathcal{F}_\delta((e_i))$ is an $\varepsilon$-net for $X$. In particular, choosing $x = \sum_{i \in F} a_i e_i$ in (5.10), it is important to emphasize that the set $E$ is not required to be contained in $F$. This suggests that the the NQP property should be weaker than the CQP property, and we prove below that this is indeed the case.

(ii) The analogue of Proposition 2.3 remains valid for the NQP.

The analogue of Theorem 2.4 for the NQP which is stated below remains valid with essentially the same proof.

**Theorem 5.3.** Let $(e_i)$ be a dictionary for $X$. The following are equivalent:
(i) $(e_i)$ has the NQP;
(ii) there exist $0 < \varepsilon < 1$ and $\delta > 0$ such that $\mathcal{F}_\delta((e_i))$ is an $\varepsilon$-net for $Ba(X)$.

**Corollary 5.4.** Let $X$ be a separable Banach space. There exists a dictionary $(e_i)$ with the NQP such that $\mathcal{F}_1((e_i))$ is $M$-dense in $X$ and $(1/M)$-separated for some $M > 0$.

**Proof.** Let $(x_n)_{n=1}^\infty$ be a semi-normalized fundamental bounded minimal system for $X$ with $\|x_n\| \leq 1/3$ for all $n$. Let $(y_n)$ be dense in the unit ball of $X$ with $y_n \in (x_i)_{i=1}^{n-1}$, and let $e_n = x_n + y_n$. Then $(e_n)$ is semi-normalized and $1/2$-dense in $Ba(X)$. So by Theorem 5.3, $\mathcal{F}_1((e_i))$ is an $M$-net for $X$ for some $M > 0$. Using the fact that $(x_i)$ is a bounded minimal system it is easily verified that $\mathcal{F}_1((e_i))$ is $(1/M)$-separated for sufficiently large $M$.

The counterpart to Corollary 2.5 takes the following form. This result seems to be of interest even when $X$ is finite-dimensional.

**Theorem 5.5.** Let $0 < \varepsilon_0 < 1$, $\delta > 0$, and let $(e_i)$ be a (not necessarily semi-normalized) fundamental system for $X$. If $\mathcal{F}_\delta((e_i))$ is $\varepsilon_0$-dense in $Ba(X)$ then $\mathcal{F}_\delta((e_i))$ is $\varepsilon_1$-dense in $X$ for all
\[
\varepsilon_1 > \left( \frac{\varepsilon_0}{1 - \varepsilon_0} \right) + 1\varepsilon_0.
\]
In particular, if $\varepsilon_0 < 1/2$, then $\mathcal{F}_\delta((e_i))$ is $\varepsilon_1$-dense in $X$ for all $\varepsilon_1 > \varepsilon_0$.

Next we introduce the analogue of the SCQP.

**Definition 5.6.** Let $\varepsilon > 0$ and let $\delta > 0$.
(a) A dictionary $(e_i)$ has the $(\varepsilon, \delta)$-Strong Net Quantization Property (abbr. $(\varepsilon, \delta)$-SNQP) if $\mathcal{F}_{\delta}(D_i)$ is an $\varepsilon$-net for $X$ for every sequence $D_i = (D_i)$ of $\delta$-nets.
(b) $(e_i)$ has the SNQP if $(e_i)$ has the $(\varepsilon, \delta)$-SNQP for some $\varepsilon > 0$ and $\delta > 0$. 

Proposition 5.7. Let \((e_i)\) be a Schauder basis for \(X\). The following are equivalent:

(i) \((e_i)\) has the SNQP;
(ii) for all \(\delta > 0\) and for every sequence \(\mathcal{D} = (D_i)\) of \(\delta\)-nets there exists \(M := M(\mathcal{D}) > 0\) such that \(\mathcal{F}_{\mathcal{D}}(e_i)\) is an \(M\)-net for \(X\);
(iii) condition (ii) for \(\delta = 1\).

Remark 5.8. The analogues of Propositions 2.4 and 2.8 remain valid for the SNQP.

Trivially, every separable Banach space has a dictionary with the \((\varepsilon, c\varepsilon)\)-SNQP for all \(0 < c < 1\). Indeed, simply take \((e_i)\) to be dense in the unit sphere of \(X\). By a more careful choice of dense set in the unit sphere of \(\ell_2\), it is not difficult to construct an NQP dictionary for \(\ell_2\) which is not a CQP dictionary. Our next result, the construction of an SNQP Schauder basis which does not have the CQP, is more involved. It is a consequence of the following general embedding theorem. (Recall that a Schauder basis is bimonotone if the basis projections \((P_n)\) satisfy \(\|P_n\| = \|I - P_n\| = 1\) for all \(n \geq 1\).)

Theorem 5.9. Let \((e_i)\) be a normalized bimonotone basis for a Banach space \(E\). Given \(\eta > 0\) there exists a Banach space \(U\) with a normalized monotone basis \((u_i)\) with the following properties:

(a) \((u_i)\) has the \((\varepsilon, \varepsilon/3)\)-SNQP;
(b) there exists a subsequence \((u_{n_i})\) of \((u_i)\) that is \((1 + \eta)\)-equivalent to \((e_i)\).

Before proceeding with the proof, let us see how it implies the existence of an SNQP basis which is not a CQP basis. The CQP is inherited by subsequences, so if we apply Theorem 5.9 to any basis \((e_i)\) which does not have the CQP (e.g. the unit vector basis of \(\ell_2\)) then the constructed basis \((u_i)\) will have the SNQP but not the CQP.

Proof of Theorem 5.9. Choose integer reciprocals \(\eta_i \downarrow 0\) such that for each \(j\) the set

\[ S_j := \{ \sum_{i=1}^{j} k_i \eta_i e_i^* : k_i \in \mathbb{Z} \} \cap Ba(E^*) \]

is \((1 - \eta)\)-norming for \(\langle e_i \rangle_{i=1}^j\). Note that if \(j \leq k\), then each element \(g\) of \(S_k\) is an “extension” of an element \(g'\) of \(S_j\) (i.e. \(g(e_i) = g'(e_i)\) for \(1 \leq i \leq j\)). Note also that \((e_i^*)_{i=1}^j \subset S_j\) since \((e_i)\) is bimonotone. We shall construct a subset \(\mathcal{G} \subset Ba(\ell_0) \cap c_{00}\) such that \(P_n(\mathcal{G}) \subset \mathcal{G}\) for all \(n \in \mathbb{N}\), where \((P_n)\) is the sequence of basis projections in \(c_{00}\). Then we define \(U\) to be the Banach space with Schauder basis \((u_i)\) whose norm is given by

\[ \| \sum a_i u_i \| = \sup_{f \in \mathcal{G}} \left| \sum f(i) a_i \right|. \]

The conditions on \(\mathcal{G}\) ensure that \((u_i)\) is a monotone basis for \(U\). The construction of \(\mathcal{G}\) and the sequence \((n_i)\) is inductive. Set \(n_1 = 1\) and

\[ \mathcal{G}_1 := \{ (k_1 \eta_1, 0, 0, \ldots) : k_1 \eta_1 e_1^* \in S_1 \}. \]
Suppose $j_0 \geq 1$ and that $n_j$ and $G_j$ have been defined for each $j \leq j_0$ such that every $f \in G_j$ is supported on $[1, n_j]$, $P_n(G_j) \subset G_j$ for all $n \in \mathbb{N}$, and $P_n(G_{j+1}) \subset G_j$, i.e., every element of $G_{j+1} \setminus G_j$ is an extension on $[n_j+1, n_{j+1}]$ of some element of $G_j$, and such that if $f \in G_j$ then there exists a $\tilde{g} := \tilde{g}(f) \in S_j$ such that $f(n_i) = \tilde{g}(e_i)$ for all $1 \leq i \leq j$ (and, conversely, for every $g \in S_j$ there exists $f \in G_j$ such that $g = \tilde{g}(f)$). We now proceed to the definition of $n_{j_0+1}$ and $G_{j_0+1}$. Let

$$T_{j_0} := \{(f, g) \in G_{j_0} \times S_{j_0+1} : g \text{ extends } \tilde{g}(f) \} \subset G_{j_0} \times S_{j_0+1}.$$ 

Let $n_{j_0+1} := n_{j_0} + \text{card } T_{j_0} + 1$ and define a bijection $(f, g) \rightarrow i((f, g))$ from $T_{j_0}$ onto $[n_{j_0} + 1, n_{j_0+1} - 1]$. For each $(f, g) \in T_{j_0}$, define $f' := f'((f, g))$ by

$$f'(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq n_{j_0} \\ 1 & \text{if } i = i((f, g)) \\ g(e_{j_0+1}) & \text{if } i = n_{j_0+1} \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$G_{j_0+1} := \{ P_n(f'((f, g))) : (f, g) \in T_{j_0}, n \leq n_{j_0+1} \}.$$ 

Finally, define $G = \cup_{j \geq 1} G_j$. Then $G$ satisfies $P_n(G) \subset G$ ($n \in \mathbb{N}$) as claimed. Thus, $(u_i)$ is a monotone basis for $U$. Moreover, since $e_j^* \in S_j$, it is easily checked that $\|u_i\| = 1$ for all $i$. Henceforth, we identify $G$ with a norming subset of $Ba(U^*)$ and use the notation $f(\sum a_i u_i) := \sum f(i) a_i$ for $f \in G$. It is clear from the construction that

$$\| \sum_{i=1}^m a_i u_i \| \leq \sup_{g \in S_m} g(\sum_{i=1}^m a_i e_i),$$

and so $(u_n)$ is $(1 + \eta)$-equivalent to $(e_i)$, which verifies (b).

Let us now turn to the verification of (a). Let $\varepsilon > 0$ and let $(D_i)$ be a sequence of $\varepsilon/3$-nets. To show that $F(u_i)$ is an $\varepsilon$-net for $U$, it suffices to show that for every $x = \sum_{i \in A} a_i u_i \in U$, where $A \subset \mathbb{N}$ is finite, there exists $y = \sum_{i \in E} d_i e_i$ ($d_i \in D_i$), where $E \subset \mathbb{N}$ is finite, such that $\|x - y\| \leq 2\varepsilon/3$ (since the collection of all such $x$ is dense in $U$). We may assume that $A \subset [1, n_j]$ for some $j$. The proof is by induction on $j$. The case $j = 1$ is clear: $n_1 = 1$, so $x = a_1 u_1$ in this case, and we simply choose $d_1 \in D_1$ with $|a_1 - d_1| \leq \varepsilon/3$, so that $\|x - d_1 u_1\| \leq \varepsilon/3$.

Suppose the inductive hypothesis holds for $j = j_0$. For the inductive step, suppose that $x = \sum_{i=1}^{n_{j_0}+1} a_i u_i$ and let $x' = \sum_{i=1}^{n_{j_0}} a_i u_i$. By the inductive hypothesis there exists $y' = \sum_{i=1}^{n_{j_0}} d_i u_i$ such that $\|x' - y'\| \leq 2\varepsilon/3$. Let $y = \sum_{i=1}^{n_{j_0}+1} d_i u_i$ be an extension of $y'$ to $[1, n_{j_0+1}]$. Then

$$|f(x) - y| = |f(x') - y'| \leq 2\varepsilon/3 \quad \text{for all } f \in G_j \text{ when } j \geq j_0.$$ 

Since $P_{n_{j_0}+1}(G_j) = G_{j_0+1}$ when $j \geq j_0 + 1$, it suffices to choose the extension $y$ such that $|f(x) - y| \leq 2\varepsilon/3$ for all $f \in G_{j_0+1} \setminus G_j$. To that end, for each $(f, g) \in T_{j_0}$, setting $i' := i((f, g))$ choose $d_{i'} \in D_{i'}$ such that

$$|f(x') - y'| + a_{i'} - d_{i'}| \leq \varepsilon/3.$$ 

This defines $d_i$ for $n_{j_0} + 1 \leq i \leq n_{j_0+1} - 1$. Finally, choose $d_{n_{j_0}+1} \in D_{n_{j_0}+1}$ such that $|a_{n_{j_0}+1} - d_{n_{j_0}+1}| \leq \varepsilon/3$. This completes the definition of $y$. Suppose that
Moreover, and formally weaker assumption that every subsequence has the NQP. When \((\text{T}_{\text{Theorem 5.10.}})\)
equiv\text{CQP.}

Let \(|\|\parallel x \parallel| \leq \varepsilon / 3 + \varepsilon / 3 = 2 \varepsilon / 3.

Moreover,

\[ |(P_n f')(x - y)| = |f(x' - y') + a_v - d_v| \leq 2 \varepsilon / 3 \quad (1 \leq n < i') \]

by inductive hypothesis since \(P_n f \in \mathcal{G}_{j_0}\) for \(1 \leq n < i'\). This completes the proof of the inductive step. \(\square\)

In some of our results in Section \[\text{[Section 4]}\] it is possible to replace the CQP by the formally weaker assumption that every subsequence has the NQP. When \((e_i)\) is a Schauder basis, however, our next result shows that this assumption is in fact equivalent to the CQP.

**Theorem 5.10.** Let \((e_i)\) be a semi-normalized basic sequence which fails the CQP. Then some subsequence fails the NQP for its closed linear span.

**Proof.** Let \(K\) be the basis constant of \((e_i)\). We may assume without loss of generality that \(||e_i|| \leq 1\) for all \(i\).

**Claim 1:** For every \(\delta > 0\) there exists \(M \subset \mathbb{N}\) such that \((e_i)_{i \in M}\) fails the \((1, \delta)\)-NQP.

**Proof of Claim 1:** Suppose not. Then there exists \(\delta > 0\) such that \((e_i)_{i \in M}\) has the \((1, \delta)\)-NQP for every \(M \subset \mathbb{N}\). Let \(x = \sum_{i \in E} a_i e_i\) and let \(n = \max E\). Since \(M := E \cup (n, \infty)\) has the \((1, \delta)\)-NQP there exists \(y \in \mathcal{F}_\delta((e_i)_{i \in M})\) such that \(||x - y|| \leq 1\). Then \(||x - P_E y|| \leq K\). Thus \((e_i)\) has the \((K, \delta)\)-CQP, which is a contradiction.

**Claim 2:** For all \(n \in \mathbb{N}\) there exist a finite set \(F_n \subset [n + 1, \infty)\) and \(x_n = \sum_{i \in F_n} a_i e_i\) such that \(||y - x_n|| > 2 K\) for all \(y \in \mathcal{F}_{1/n}((e_i)_{i \in F_n})\).

**Proof of Claim 2:** Let \(\delta_n = 1/n\). By Claim 1 there exists \(M_n \subset \mathbb{N}\) such that \((e_i)_{i \in M_n}\) fails the \((2K + 1, \delta_n)\)-NQP. So there exists \(z_n = \sum_{i \in M_n} a_i e_i\) with \(||z_n - y|| > 2K + 1\) for all \(y \in \mathcal{F}_{1/n}((e_i)_{i \in M_n})\). Let \(x_n = z_n|_{[n+1, \infty)}\). Note that every vector supported on \([1, n] \cap M_n\) (in particular, the vector \(z_n - x_n\)) can be 1-approximated by an element of \(\mathcal{F}_{1/n}((e_i)_{i \in M_n})\) simply by approximating each of the (at most \(n^2\)) nonzero coordinates to within \(\delta_n = 1/n\). Setting \(F_n := \text{supp} x_n\), it follows that \(||x_n - y|| > 2K\) for all \(y \in \mathcal{F}_{1/n}((e_i)_{i \in F_n})\). Thus, \(x_n\) and \(F_n\) verify Claim 2.

Now pass to a subsequence so that the sets \(F_{n_k}\) satisfy \(\max F_{n_k} < \min F_{n_k+1}\) for all \(k \in \mathbb{N}\). Let \(M = \bigcup_{k \geq 1} F_{n_k}\).

**Claim 3:** \((e_i)_{i \in M}\) fails the NQP.

**Proof of Claim 3:** Suppose that \((e_i)_{i \in M}\) has the \((1, \delta)\)-NQP (and hence the \((1, 1/n)\)-NQP provided \(1/n < \delta\)). Choose \(k\) with \(1/n_k < \delta\). Then there exists \(y \in \mathcal{F}_{1/n_k}((e_i)_{i \in M})\) such that \(||y - x_{n_k}|| \leq 1\). But this implies that \(||P_{F_{n_k}}(y) - x_{n_k}|| \leq 2K\), which contradicts the choice of \(x_{n_k}\) and \(F_{n_k}\). \(\square\)

We turn now to discuss the relationship between the NQP and unconditionality.
Theorem 5.11. Suppose that $X$ has a semi-normalized unconditional basis $(e_i)$ with the NQP. Then $(e_i)$ is equivalent to the unit vector basis of $c_0$.

Proof. Let $K$ be the constant of unconditionality of $(e_i)$ and choose $\varepsilon > 0$ such that $K < \frac{1 - \varepsilon}{\varepsilon}$. There exists $\delta > 0$ such that $F_\delta((e_i))$ is $\varepsilon$-dense in $X$. Suppose $x = \sum e_i^*(x) e_i \in X$ with $\|x\| = 1$ and $\|x\|_\infty := \sup |e_i^*(x)| = \alpha < \delta$. Choose $y \in F_\delta((e_i))$ with $\|x - y\| \leq \varepsilon$. Then

$$\|y\| \geq \|x\| - \|x - y\| \geq 1 - \varepsilon.$$  

Since $\sup |e_i^*(x)| \leq \alpha$ and since $y \in F_\delta((e_i))$, it follows that $y = \sum \lambda_i e_i^*(x - y) e_i$ for a multiplier sequence $(\lambda_i)$ satisfying

$$\sup |\lambda_i| \leq \frac{\delta}{\delta - \alpha}.$$  

Hence by $K$-unconditionality of $(e_i)$, we have

$$1 - \varepsilon \leq \|y\| \leq K \sup |\lambda_i| \|y - x\| \leq K \frac{\delta}{\delta - \alpha} \varepsilon.$$  

If $(e_i)$ is not equivalent to the unit vector basis of $c_0$ then $\alpha$ may be chosen to be arbitrarily small. But then (5.17) yields $K \geq \frac{1 - \varepsilon}{\varepsilon}$, which contradicts the choice of $\varepsilon$. □

Weaker notions of unconditionality (see [5]), especially that of a quasi-greedy basis, have recently attracted attention in connection with greedy algorithms for data compression. Our next goal is to show that every quasi-greedy basis with the NQP is equivalent to the unit vector basis of $c_0$. The relevant definitions are given next. For further information on the topic of greedy algorithms in Banach spaces, we refer the reader to [14, 6, 7, 5, 26].

Definition 5.12. Let $(e_i)$ be a dictionary for $X$ and let $\delta > 0$.

(a) Denote by $L((e_i), \delta)$ the least constant $L \in [1, \infty]$ with the property that whenever $\| \sum a_i e_i \| \leq 1$ and $F \subset \{i : |a_i| \geq \delta\}$ then

$$\| \sum_{i \in F} a_i e_i \| \leq L.$$  

(b) We say that $(e_i)$ is Elton-unconditional if

$L((e_i), \delta) < \infty$ for all $\delta > 0$.

(c) Denote by $K((e_i), \delta)$ the least constant $K \in [1, \infty]$ with the property that whenever $\| \sum a_i e_i \| \leq 1$ and $F = \{i : |a_i| \geq \delta\}$ then

$$\| \sum_{i \in F} a_i e_i \| \leq K.$$  

(d) We say that $(e_i)$ is quasi-greedy if

$K((e_i)) := \sup_{\delta > 0} K((e_i), \delta) < \infty$. 
Remark 5.13. Clearly, $K((e_i), \delta) \leq L((e_i), \delta))$. Note that $(e_i)$ is unconditional if and only if $\sup_{i>0} L((e_i), \delta) < \infty$. It is known that every quasi-greedy basic sequence is Elton-unconditional (in fact a semi-normalized Schauder basis $(e_i)$ is Elton-unconditional if and only if $K((e_i), \delta) < \infty$ for all $\delta > 0$) and that there exist Elton-unconditional bases which are not quasi-greedy [3].

Lemma 5.14. Let $(e_i)$ be a minimal system for $X$. Suppose that there exist $0 < \varepsilon < 1$, $\delta > 0$, and $\lambda > 0$ such that $\mathcal{F}_\delta((e_i)) \cap \lambda \text{Ba}(X)$ is an $\varepsilon$-net for $\text{Ba}(X)$ and such that $L((e_i), \delta/\lambda) < \infty$. Then $(e_i)$ is equivalent to the unit vector basis of $c_0$.

Proof. Clearly, $(e_i)$ has the NQP. So by Theorem 5.11 it suffices to show that $(e_i)$ is unconditional. Let $S := \mathcal{F}_\delta((e_i)) \cap \lambda \text{Ba}(X)$. Since $S$ is an $\varepsilon$-net for $\text{Ba}(X)$ it follows that $S$ is $(1-\varepsilon)$-norming for $X^*$, i.e.

$$\|x^*\| \leq \frac{1}{1-\varepsilon} \sup\{|x^*(x)| : x \in S\} \quad (x^* \in X^*).$$

Moreover, if $x = \sum_{i \in E} k_i \delta e_i \in S$ and $F \subseteq E$, then (since $x/\lambda \in \text{Ba}(X)$) $\|\sum_{i \in F} k_i \delta e_i\| \leq \lambda L((e_i), \delta/\lambda)$. Hence

$$\tilde{S} := \{\sum_{i \in F} k_i \delta e_i : \sum_{i \in F} k_i \delta e_i \in S, F \subseteq E\} \subseteq \lambda L((e_i), \delta/\lambda) \text{Ba}(X).$$

Now suppose that $\sum_{i \in F} a_i e_i^* \in X^*$ and that $F \subseteq E$. Then

$$\|\sum_{i \in F} a_i e_i^*\| \leq \frac{1}{1-\varepsilon} \sup\{|\sum_{i \in E} a_i e_i^*(x) : x \in S\}$$

$$\leq \frac{1}{1-\varepsilon} \sup\{|\sum_{i \in \tilde{S}} a_i e_i^*(x) : x \in \tilde{S}\}$$

$$\leq \frac{\lambda}{1-\varepsilon} L((e_i), \delta/\lambda) \|\sum_{i \in E} a_i e_i^*\|.$$

Thus, $(e_i^*)$ is $K$-unconditional for $K = \lambda L((e_i), \delta/\lambda)/(1-\varepsilon)$, and hence by duality $(e_i)$ is also $K$-unconditional.

The following substantial strengthening of Theorem 5.11 is an immediate consequence of the last result.

Theorem 5.15. Suppose that $(e_i)$ is a minimal system with the NQP. If $(e_i)$ is Elton-unconditional (in particular, if $(e_i)$ is quasi-greedy) then $(e_i)$ is equivalent to the unit vector basis of $c_0$.

The main open question of this section is the following.

Problem 5.16. Suppose that $X$ has an NQP basis. Does $c_0 \hookrightarrow X$?

In fact, we do not know whether or not $\ell_1$ provides a negative answer to Problem 5.16.

Problem 5.17. Does $\ell_1$ have an NQP basis (resp. minimal system)?

We conclude this section with some partial results concerning Problem 5.16.

Theorem 5.18. Suppose that $(e_i)$ is a bounded NQP minimal system for $X$. Then $\ell_1 \hookrightarrow X^*$. In fact, either $\ell_\infty \hookrightarrow X^*$ or a subsequence of $(e_i^*)$ is equivalent to the unit vector basis of $\ell_1$. 
Proof. Since \((e_i)\) has the NQP, there exists \(\delta > 0\) such that \(F_\delta((e_i)) \cap (3/2)Ba(X)\) is a \(1/2\)-net for \(Ba(X)\). Thus,

\[
\frac{1}{2} \|x^*\| \leq \sup \{\|x^*(x)\| : x \in F_\delta((e_i)) \cap \frac{3}{2}Ba(X)\} \leq \frac{3}{2} \|x^*\| \quad (x^* \in X^*).
\]

So \(X^* \hookrightarrow C(K)\), where \(K\) is the weak-star closure of \(F_\delta((e_i)) \cap (3/2)Ba(X)\) in \(X^{**}\). By Rosenthal’s \(\ell_1\) theorem \([22]\), \((e_i^*)\) has a subsequence equivalent to the unit vector basis of \(\ell_1\) or a weakly Cauchy subsequence \((f_i^*)\). The former obviously implies that \(\ell_1 \hookrightarrow X^*\). In the latter case, let \(g_i^* = f_{2i}^* - f_{2i-1}^*\). Then \((g_i^*)\) is weakly null in \(X^*\), and, since the range of \(f_i^*|_K \subset \mathbb{Z}\delta\), we have

\[
g_i^*(k) \neq 0 \Rightarrow |g_i^*(k)| \geq \delta \quad (i \in \mathbb{N}, k \in K).
\]

Thus, for each \(k \in K\), the sequence \((g_i^*(k))\) is eventually zero, so the series \(\sum g_i^*\) is extremely weakly unconditionally Cauchy, i.e. \(\sum_{i=1}^{\infty} |g_i^*(k)|\) converges (trivially!) for every \(k \in K\). By a theorem of Elton \([9]\) (see also \([12]\)), \(c_0 \hookrightarrow [(g_i^*)]\). But this implies that \(\ell_\infty\) is isomorphic to a subspace of \(X^*\) \([3]\), and a fortiori that \(\ell_1 \hookrightarrow X^*\). \(\square\)

**Corollary 5.19.** Suppose that \(X\) is reflexive. Then \(X\) does not contain a bounded minimal system with the NQP.

### 6. Containment of \(c_0\)

The main result of this section is the following converse to Theorem 4.1.

**Theorem 6.1.** Let \((e_i)\) be a semi-normalized basic sequence with the CQP. Then \((e_i)\) has a subsequence that is equivalent to the unit vector basis of \(c_0\) or to the summing basis of \(c_0\).

As the proof is quite long we shall break it down into several parts. We shall frequently refer to the excellent survey article \([1]\) for the proofs of certain assertions.

First we prove a result which is of independent interest.

**Theorem 6.2.** Let \((e_i)\) be a semi-normalized nontrivial weakly Cauchy basis for \(X\). Then there exists a subsequence \((e_{n_i})\) such that either

(a) \((e_{n_i})\) is equivalent to the summing basis of \(c_0\), or

(b) \((e_{n_i}^*)\) is weakly null in \([(e_{n_i})]^*\).

**Proof.** Let \(x^{**} \in X^{**} \setminus X\) be the weak-star limit of \((e_i)\). By passing to a subsequence we may assume that \((e_i)\) dominates the summing basis, i.e. \(\|\sum a_i e_i\| \geq c \|(a_i)\|_{\ell^1}\) for some \(c > 0\) \([1]\) Prop. II.1.5]. If \(x^{**} \in X^{**} \setminus D(X)\), where \(D(X)\) denotes the collection of all elements of \(X^{**} \setminus X\) whose restrictions to \(Ba(X^*)\) (equipped with the weak-star topology) are differences of semi-continuous functions (see \([1]\)), then by \([23]\) Theorem 1.8 \((e_i)\) has a strongly summing subsequence \((e_{n_i})\). In particular, \((\sum_{i=1}^{n} e_{n_i}^*)\) is a nontrivial weakly Cauchy sequence in \([(e_{n_i})]^*\) \([1]\) Lemma II.2.6], and so \((e_{n_i}^*)\) is weakly null in \([(e_{n_i})]^*\), which yields (b).

Now suppose that \(x^{**} \in D(X)\). Then there exists a sequence \((x_i)\subset X\) that is equivalent to the summing basis of \(c_0\) such that \(x_i \to x^{**}\) weak-star \([1]\) Theorem II.1.2]. Note that \((e_i-x_i)\) is weakly null. If some subsequence of \((e_i-x_i)\) is norm-null then (a) follows by a standard perturbation argument. So we may assume that \((e_i-x_i)\) is a semi-normalized weakly null sequence. By a theorem of Elton \([5]\) \([13]\), \((e_i-x_i)\) has either a subsequence equivalent to the unit vector basis of \(c_0\) or a basic subsequence whose sequence of biorthogonal functionals is weakly null (in the dual...
Suppose that \((e_{n_i} - x_{n_i})\) be the \(c_0\) subsequence. Then

\[
c\|(a_i)\|_{sB} \leq \left\| \sum a_i e_{n_i} \right\|_B \leq \| \sum a_i (e_{n_i} - x_{n_i}) \| + \| \sum a_i x_{n_i} \| \leq C_1 \sup_i |a_i| + C_2 \|(a_i)\|_{sB} \leq C_3 \|(a_i)\|_{sB},
\]

for certain constants \(C_1, C_2, C_3\). Hence \((e_{n_i})\) is equivalent to the summing basis of \(c_0\). If the second alternative holds, let \((e_{n_i} - x_{n_i})\) be a basic subsequence with weakly null biorthogonal functionals. To prove that \((e^*_{n_i})\) is weakly null in \([([e_{n_i}])^*]\), it suffices to show that \(a_i \to 0\) whenever \((a_i)\) satisfies \(\sup_m \| \sum_{i=1}^m a_i e_{n_i} \| = K < \infty\). Now

\[
\left\| \sum_{i=1}^m a_i x_{n_i} \right\| \leq C_2 \|(a_i)\|_{sB} \leq c^{-1} C_2 \left( \sum_{i=1}^m \|a_i e_{n_i}\| \right) \leq c^{-1} C_2 K,
\]

and hence by the triangle inequality

\[
\sup_m \left\| \sum_{i=1}^m a_i (e_{n_i} - x_{n_i}) \right\| \leq K + c^{-1} C_2 K.
\]

Since the sequence of biorthogonal functionals to \((e_{n_i} - x_{n_i})\) is weakly null, we deduce finally that \(a_i \to 0\).

**Proposition 6.3.** Suppose \(X\) has a minimal system \((e_i)\) with the NQP. Then no subsequence of \((e^*_i)\) is weakly null.

**Proof.** Let \(0 < \varepsilon < 1\). There exists \(\delta > 0\) such that \(\mathcal{F}_\delta((e_i)) \cap 2 \mathcal{B}a(X)\) is an \(\varepsilon\)-net for \(\mathcal{B}a(X)\). Thus,

\[
\frac{1 - \varepsilon}{2} \|x^*\| \leq \sup \{ |x^*(x)| : x \in \mathcal{F}_\delta((e_i)) \cap 2 \mathcal{B}a(X) \} \leq 2 \|x^*\| \quad (x^* \in X^*).
\]

So \(X^* \hookrightarrow C(K)\), where \(K\) is the weak-star closure of \(\mathcal{F}_\delta((e_i)) \cap 2 \mathcal{B}a(X)\) in \(X^{**}\). Suppose that \((e^*_{n_i})\) is a weakly null subsequence of \((e^*_i)\), whence \(\sup_i \|e^*_{n_i}\| = C < \infty\). Thus,

\[
\{ |e^*_{n_i}(k)| : k \in K \} \subset \{0\} \cup [\delta, 2C] \quad (i \geq 1),
\]

so \((e^*_{n_i})\) has an unconditional basic subsequence [5] Theorem 23] (see also [10] and [16]). Relabel this unconditional subsequence as \((e^*_n)\) and let \(Y := [[e^*_n]] \subset X^*\). Observe that \((e_{n_i}|_Y)\) is a semi-normalized unconditional basic sequence in \(Y^*\) whose biorthogonal sequence is \((e^*_{n_i}) \subset Y\). We claim that \((e_{n_i})\) has the NQP for its closed linear span in \(Y^*\). To prove the claim, let \(x = \sum_{i \in A} a_i e_i\), where \(A \subset \{n_i : i \geq 1\}\) is finite. Since \((e_i)\) has the NQP for \(X\) there exists \(y = \sum_{i \in B} m_i \delta e_i\) with \(|x - y| \leq \varepsilon\), where \(B \subset \mathbb{N}\) is finite and \(m_i \in \mathbb{Z}\) for each \(i\). Let \(z = \sum_{i \in B'} m_i e_i\), where \(B' = B \cap \{n_i : i \geq 1\}\). Then \(y|_Y = z|_Y\) and

\[
\|x - z\|_{Y^*} \leq \|x - y\| \leq \varepsilon,
\]

which proves the claim. Since \((e_{n_i}|_Y) \subset Y^*\) is an unconditional basic sequence with the NQP, it follows from Theorem 5.11 that \((e_{n_i}|_Y)\) is equivalent to the unit vector basis of \(c_0\). But this implies that \((e^*_{n_i})\) is equivalent to the unit vector basis of \(\ell_1\), which contradicts the assumption that \((e^*_{n_i})\) is weakly null! \(\square\)
Proposition 6.4. Suppose that \((e_i)\) is a weakly null dictionary for \(X\). If every subsequence of \((e_i)\) has the NQP for its closed linear span (in particular, if \((e_i)\) has the CQP) then \((e_i)\) has a subsequence equivalent to the unit vector basis of \(c_0\).

Proof. By the aforementioned theorem of Elton \((e_i)\) has a subsequence equivalent to the unit vector basis of \(c_0\) or a basic subsequence \((e_{n_i})\) such that \((e_{n_i}^*)\) is weakly null in \([e_{n_i}]^*\). But the latter cannot happen by Proposition 6.3. □

Proposition 6.5. Suppose that \((e_i)\) is a nontrivial weakly Cauchy dictionary for \(X\). If every subsequence of \((e_i)\) has the NQP for its closed linear span (in particular, if \((e_i)\) has the CQP) then \((e_i)\) has a subsequence equivalent to the summing basis of \(c_0\).

Proof. By Theorem 6.2 either \((e_i)\) has a subsequence equivalent to the summing basis or a basic subsequence \((e_{n_i})\) such that \((e_{n_i}^*)\) is weakly null in \([e_{n_i}]^*\). But \((e_{n_i})\) has the NQP for its closed linear span, so the latter alternative cannot happen by Proposition 6.3. □

Proof of Theorem 6.1. By Rosenthal’s \(\ell_1\) theorem [22], either \((e_i)\) has a subsequence that is equivalent to the unit vector basis of \(\ell_1\) or a weakly Cauchy basic subsequence. The first possibility cannot occur since the unit vector basis of \(\ell_1\) does not have the NQP. For the second possibility, either the subsequence is weakly null or it is nontrivial weakly Cauchy. In the former case there is a subsequence equivalent to the unit vector basis of \(c_0\) by Proposition 6.4, and in the latter there is a subsequence equivalent to the summing basis by Proposition 6.5. □

Combining Theorem 4.1 and Theorem 6.1 we obtain a new characterization of separable Banach spaces containing \(c_0\).

Theorem 6.6. Let \(X\) be a separable Banach space. The following are equivalent:

(a) \(c_0 \hookrightarrow X\);
(b) \(X\) has a weakly null bounded and total minimal system with the SCQP;
(c) \(X\) has a total minimal system \((e_i)\) with the CQP;
(d) \(X\) has a dictionary \((e_i)\) with no nonzero weak limit point such that every subsequence of \((e_i)\) has the NQP for its closed linear span.

Proof. (a) ⇒ (b) follows from Theorem 4.1, (b) ⇒ (c) is trivial, (c) ⇒ (d) follows from the fact that a total minimal system has no nonzero subsequential weak limit point. To prove (d) ⇒ (a), note that \((e_i)\) has a weakly Cauchy basic subsequence, so the result follows from Propositions 6.4 and 6.5. □

We conclude this section with some results about NQP minimal systems that are motivated by Problem 5.16 above.

Proposition 6.7. Let \((e_i)\) be a minimal system for \(X\) with the NQP. Then no subsequence of \((e_i^*)\) is nontrivial weakly Cauchy.

Proof. Suppose that \((e_i)\) has the \((\varepsilon, \delta)\)-NQP and that \((e_{n_i}^*)\) is nontrivial weakly Cauchy. After passing to a subsequence of \((e_{n_i}^*)\) we may assume that \((f_i)\) is a weakly null basis for \(Y = \{f_i\} = \{|e_{n_i}^*\}\) , where \(f_1 = e_{n_1}\) and \(f_i = e_{n_i}^* - e_{n_{i-1}}^*\) for \(i \geq 2\) [1, Prop. II.1.7]. Setting \(e = \sum e_{n_i} \mid_Y\) (the sum converging weak-star in \(Y^*\)), \(n_0 := 0\), and \(e_0 := 0\), the sequence of biorthogonal functionals \((f_i^*) \subset Y^*\) is given
by $f_i^* = e - \sum_{j=0}^{i-1} e_{n_j} | Y$. We claim that $(f_i^*)$ has the NQP for its closed linear span in $Y^*$. To check this claim, let $x = \sum_{i \in A} a_i f_i^*$, where $A \subseteq N$ is finite. Then we may rewrite the expression for $x$ in the form

$$x = b f_1^* + \sum_{i \in B} b_i e_{n_i} | Y$$

for some finite $B \subset \mathbb{N}$ and scalars $b, b_i$. Since $(e_i)$ has the $(\epsilon, \delta)$-NQP there exists $z = \sum_{i \in C} m_i \delta e_i (m_i \in \mathbb{Z})$, where $C$ is a finite subset of $\mathbb{N}$, such that $\| \sum_{i \in B} b_i e_{n_i} - z \| \leq \epsilon$. Since $e_{n_i} | Y = f_i^* - f_{i+1}^*$, it follows that

$$z | Y = \sum_{i \in C'} m'_i \delta f_i^*$$

for some finite $C' \subset \mathbb{N}$ and $m'_i \in \mathbb{Z}$. Choose $m \in \mathbb{Z}$ such that $| b - m \delta | \leq \delta$. From (6.18) and (6.19), we obtain

$$\| x - (m \delta f_1^* + \sum_{i \in C'} m'_i \delta f_i^*) \|_Y \leq |m \delta - b| \| f_1^*\| + \| \sum_{i \in B} b_i e_{n_i} - z \|_Y$$

$$\leq \delta \| f_1^* \| + \| \sum_{i \in B} b_i e_{n_i} - z \|$$

$$\leq \delta \| f_1^* \| + \epsilon,$$

which verifies the claim. Thus $(f_i^*)$ has the NQP for its closed linear span and its biorthogonal sequence $(f_i)$ is weakly null. But this contradicts Proposition 6.3. \hfill \Box

**Theorem 6.8.** Let $(e_i)$ be a seminormalized basis with the NQP. Then every subsequence of $(e_i^*)$ has a further subsequence equivalent to the unit vector basis of $\ell_1$.

**Proof.** By Proposition 6.3 no subsequence of $(e_i^*)$ is weakly null, and by Proposition 6.7 no subsequence is nontrivial weakly Cauchy. Thus, by Rosenthal’s $\ell_1$ theorem, every subsequence of $(e_i^*)$ has a further subsequence equivalent to the unit vector basis of $\ell_1$. \hfill \Box

### 7. Some Notions Related to the CQP

There seems to be very little known about the relationships between the different quantization properties introduced in the previous sections. Let us recast some of the questions we formulated in previous sections.

Throughout this section $(e_i)$ and $(f_i^*)$ is a bounded minimal system of a Banach space $X$ and we assume that $(e_i)$ (and, thus, also $(e_i^*)$) are semi-normalized.

**Question 7.1.** Let $\epsilon, \delta > 0$.

1. If $(e_i)$ satisfies the $(\epsilon, \delta)$-CQP, does it satisfy the $(\epsilon, \delta/2)$-SCQP, does it satisfy $(\epsilon, \delta)$-neighborly CQP (see Remark 3.7)?

   In the case that the answer to our aforementioned questions are negative do they at least have qualitative positive answers, i.e. does the CQP imply the SCQP, does the CQP imply the neighborly CQP?

2. In our next example we will exhibit that for some $\epsilon, \delta > 0$ the $(\epsilon, \delta)$-NQP does not imply the $(\epsilon, \delta/2)$-SNQP. But we do not know whether or not the NQP implies the SNQP.
One can reformulate these questions into finite dimensional ones. Assume that \( n \in \mathbb{N} \) and that \( K \subset \mathbb{R}^n \) is a symmetric and convex body (i.e. \( 0 \in K^\circ \)).

Let us consider the following properties \( K \) may have

\[
\begin{align*}
(\text{P1}) & \quad \bigcup_{z \in \mathbb{Z}^n} z + K = \mathbb{R}^n \\
(\text{P2}) & \quad \bigcup_{z \in \prod_{i=1}^n D_i} z + K = \mathbb{R}^n \\
& \quad \text{whenever } D_i \subset \mathbb{R}, 0 \in D_i, D_i \text{ is } \frac{1}{2}\text{-net for } i = 1, 2, \ldots, n
\end{align*}
\]

Note that \( (\text{P3}) \) means that, not only is every point of \( \mathbb{R}^n \) an element of some translate of \( K \) by some point \( p \) having integer coordinates, but that \( p \) can be chosen so that \( \max_{i=1,2,\ldots,n} |x_i - p_i| \leq 1 \).

It is easy to see that \( (e_i) \) satisfies \( (\varepsilon, \delta)-\text{CQP}, (\varepsilon, \delta/2)-\text{SCQP} \) or \( (\varepsilon, \delta)-\text{neighborly CQP}, \) if and only if for any finite \( I \subset \mathbb{N} \) the set

\[
K_I = \frac{\varepsilon}{\delta} B_X \cap [e_i : i \in I],
\]

satisfies \( (\text{P1}), (\text{P2}) \) or \( (\text{P3}) \) respectively.

If we do not assume that \( (e_i) \) is a monotone basis a similar statement for \( \text{NQP} \) and \( \text{SNQP} \) is slightly more complicated.

First if \( E = (\mathbb{R}^n, \| \cdot \|) \) is finite dimensional then the unit vector basis \( (e_i) \) has the \( (\varepsilon, \delta)-\text{NQP} \) or the \( (\varepsilon, \delta/2)-\text{SNQP} \) if and only if \( \frac{\varepsilon}{\delta} B_E \) satisfies \( (\text{P1}) \) or \( (\text{P2}) \). If for all \( n \in \mathbb{N} \) \( K_{\{1,2,\ldots,n\}} \) (defined as above) satisfies \( (\text{P1}) \) or \( (\text{P2}) \) then for any \( \eta > 0 \) \( (e_i) \) has the \( (\varepsilon, \delta - \eta)-\text{NQP} \) or the \( (\varepsilon, \delta - \eta)-\text{SNQP} \) respectively. Conversely, if \( (e_i) \) is a monotone basis which satisfies the \( (\varepsilon, \delta)-\text{NQP} \) or the \( (\varepsilon, \delta)-\text{SNQP} \), then for all \( n \in \mathbb{N} \) the set \( K_{\{1,2,\ldots,n\}} \) satisfies \( (\text{P1}) \) or \( (\text{P2}) \), respectively.

The following example shows that \( (\text{P1}) \not\Rightarrow (\text{P2}) \).

Example. In \( \mathbb{R}^2 \) let \( K \) be the convex hull of the points

\[
P_1 = \left(\frac{1}{4}, 1\right), \quad P_2 = \left(\frac{3}{4}, 1\right), \quad P_3 = \left(-\frac{1}{4}, -1\right), \quad \text{and} \quad P_4 = \left(-\frac{3}{4}, -1\right).
\]

Instead of a formal proof, we leave it to the reader to verify the following by drawing a picture:

a) \( K \) is a parallelogram which tiles \( \mathbb{R}^2 \), i.e.

\[
\bigcup_{z \in \mathbb{Z}^2} z + K = \mathbb{R}^2 \quad \text{and} \quad (z + K^\circ) \cap (z' + K^\circ) \quad \text{whenever } z \neq z' \text{ are in } \mathbb{Z}^2.
\]

b) For \( Q = \frac{3}{4}P_2 + \frac{1}{2}P_3 \) we have

\[
Q \in \left(\{0,0\} + K\right) \cap \left([1,1] + K\right).
\]

c) For small enough \( \eta > 0 \)

\[
P - (0, \eta/4) \notin \bigcup_{z \in \mathbb{Z} \times (1-\eta)\mathbb{Z}} z + K.
\]

(Thus \( K \) does not satisfy \( (\text{P2}) \).)
Question 7.2. Let $K \subset \mathbb{R}^n$ be convex and symmetric and put for $I \subset \{1, 2, \ldots, n\}$

$$K_I = \{(x_1, x_2, \ldots, x_n) \in K : x_i = 0 \text{ for } i \in \{1, 2, \ldots, n\} \setminus I\}.$$

1. If $K_I$ satisfies $(P_1)$ for all $I \subset \{1, 2, \ldots, n\}$, does it satisfy $(P_2)$ or $(P_3)$?
2. Is there at least a universal constant $c \geq 1$ so that if $K_I$ satisfies $(P_1)$ for all $I \subset \{1, 2, \ldots, n\}$, then it satisfies $(P_2)$ or $(P_3)$?
3. Is there a universal constant $c \geq 1$ so that if $K$ satisfies $(P_1)$ then it satisfies $(P_2)$?

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Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
E-mail address: dilworth@math.sc.edu

Department of Mathematics, The University of Texas, 1 University Station C1200, Austin, TX 78712, USA
E-mail address: odell@math.utexas.edu

Department of Mathematics, Texas A & M University, College Station, TX 78712, USA
E-mail address: thomas.schlumprecht@math.tamu.edu

Fitzwilliam College, Cambridge, CB3 0DG, England
E-mail address: a.zsak@dpmms.cam.ac.uk