Bias-robust Integration of Observational and Experimental Estimators

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\section*{Abstract}

We describe a simple approach for combining an unbiased and a (possibly) biased estimator, and demonstrate its robustness to bias: estimate the error and cross-correlation of each estimator, and use these to construct a weighted combination that minimizes mean-squared error (MSE). Theoretically, we demonstrate that for any amount of (unknown) bias, the MSE of the resulting estimator is bounded by a small multiple of the MSE of the unbiased estimator. In simulation, we demonstrate that when the bias is sufficiently small, this estimator still yields notable improvements in MSE, and that as the bias increases without bound, the MSE of this estimator approaches that of the unbiased estimator.

This approach applies to a range of problems in causal inference where combinations of unbiased and biased estimators arise. When small-scale experimental data is available, estimates of causal effects are unbiased under minimal assumptions, but may have high variance. Other data sources (such as observational data) may provide additional information about the causal effect, but potentially introduce biases. Estimators incorporating these data can be arbitrarily biased when the needed assumptions are violated. As a result, naive combinations of estimators can have arbitrarily poor performance.

We show how to apply the proposed approach in these settings, and benchmark its performance in simulation against recent proposals for combining observational and experimental estimators. Here, we demonstrate that this approach shows improvement over the experimental estimator for a larger range of biases than alternative approaches.

\section{Introduction}

In many causal inference problems, we are motivated to combine unbiased and potentially biased estimators of causal effects with the goal of obtaining more precise estimates. For instance, we might hope to combine a small experimental dataset with a larger observational study, or use a learned relationship (in historical data) between short-term surrogates and long-term outcomes to augment an experimental dataset where some long-term outcomes are unobserved. One straightforward strategy for combining an unbiased estimator $\hat{\theta}_u$ with a possibly biased estimator $\hat{\theta}_b$ is to use an affine combination

$$\hat{\theta}_\lambda = \lambda \hat{\theta}_b + (1 - \lambda) \hat{\theta}_u,$$

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Figure 1: The y-axis shows the relative MSE: the ratio of the mean-squared error (MSE) of $\hat{\theta}_b$ to the MSE of the unbiased estimator. The x-axis shows the bias $\mu$ of the biased estimator. In this example, the biased ($\hat{\theta}_b$) and unbiased ($\hat{\theta}_u$) estimators are independent with the same variance and sample size.

for a well-chosen $\hat{\lambda}$, designed to minimize mean-squared error. Estimators of this general form have a long history in forecasting and model averaging, dating back to Bates & Granger (1969), and variants of this strategy have been proposed in the context of estimating conditional average treatment effects using kernel regression (Cheng & Cai, 2021) and stratum-specific effects using shrinkage estimation (Rosenman et al., 2020).

However, theoretical guarantees that speak to the practical applicability of this strategy have been elusive, particularly for one-dimensional causal parameters such as the average treatment effect (ATE). In this setting, shrinkage-style results do not apply, and blanket guarantees for when the combined estimator out-performs the unbiased estimator (in terms of mean-squared error) are hard to come by.

In this work, we revisit this linear combination strategy, but adopt a different theoretical approach. A primary difficulty in understanding the behavior of the combined estimator is that it depends on the unknown bias of the biased estimator. For this reason, we focus on characterizing the MSE of the combined estimator, indexed by this bias. We give an illustrative example in Figure 1, for two estimators whose errors are Gaussian. The overall takeaway is this: If the bias is small enough, MSE can be substantially improved. On the other hand, if the bias is very large, the MSE approaches that of the unbiased estimator. The worst case scenario occurs for bias that lies in the middle range, but here the inflation in MSE is bounded.

This characterization can directly inform practice. First, in causal inference problems, we often have other tools (such as sensitivity analysis) for reasoning about the magnitude of the bias. Second, for a given problem, we have enough information to construct similar MSE curves to Figure 1 from simulation. This can be used to make a decision about whether the unbiased estimator or the combined estimator is the right choice in a given problem. Finally, we develop these results in the general case where $\hat{\theta}_u, \hat{\theta}_b$ may be correlated, considering a broader scope than prior work that focused on combinations of estimators from independent datasets.
The remainder of this work proceeds as follows: In Section 2, we give the form of the estimator that we consider, and discuss how it can be implemented in the context of biased and unbiased estimators that are regular and asymptotically linear, and give two illustrative applications. In Section 3, in addition to showing that this estimator is asymptotically consistent, we prove a finite-sample upper-bound on the MSE relative to that of the unbiased estimator. To understand when \( \hat{\theta}_\lambda \) improves upon \( \hat{\theta}_u \), we turn to simulation studies in Section 4, where we investigate the performance of the combination estimator \( \hat{\theta}_\lambda \) against a number of baseline approaches. For our approach and others, we study how parameters of the data-generating process impact the worst-case performance (for an adversarial choice of bias \( \mu \)), the best-case performance (typically achieved when \( \mu = 0 \)), and the threshold value of \( \mu \) beyond which a given estimator fails to improve upon \( \hat{\theta}_u \). We find that the estimator \( \hat{\theta}_\lambda \), which does not require hyperparameter tuning, generally strikes a more conservative trade-off, with a lower worst-case relative MSE and a larger threshold value for \( \mu \), while giving up on some possible improvement when \( \mu \) is zero. Finally, in Section 5, we seek to build intuition for whether or not the observed thresholds (analogous to the vertical line in Figure 1) are high enough to warrant practical application of the method. To do so, we construct a simulation to mimic the SPRINT Trial (SPRINT Research Group et al., 2015), alongside a much larger (but confounded) observational study. Here, we simulate a curve analogous to Figure 1, but directly parameterized by sensitivity analysis parameters, and find that the combined estimator can tolerate a moderate amount of influence of the confounder on treatment assignment.

## 2 Motivation and Setup: Biased and Unbiased Estimators

In this work we seek to estimate a causal effect \( \theta_0 \in \mathbb{R} \) with minimum mean-squared error,

\[
\text{MSE}(\hat{\theta}) := \mathbb{E}\left[(\hat{\theta} - \theta_0)^2\right],
\]

(2)

in settings where multiple candidate estimators are available. In particular, we assume the existence of two estimators, denoted \( \hat{\theta}_u, \hat{\theta}_b \). Typically, we will assume that \( \hat{\theta}_u \) is an unbiased estimator for the causal effect \( \theta_0 \), either in the asymptotic sense (i.e., it is consistent for the causal effect), or in the stronger sense that it is unbiased in finite samples. An example is an estimator derived from experimental data, whose consistency is implied by mild assumptions. On the other hand, the estimator \( \hat{\theta}_b \) is biased without strong additional assumptions, such as an estimator derived from observational data. In this work, we consider linear combinations of estimators of the form

\[
\hat{\theta}_\lambda := \lambda \hat{\theta}_b + (1 - \lambda) \hat{\theta}_u = \hat{\theta}_u + \lambda (\hat{\theta}_b - \hat{\theta}_u),
\]

(3)

for a real-valued weight \( \lambda \in \mathbb{R} \). Throughout, we assume that \( \hat{\theta}_u \xrightarrow{i.p.} \theta_0 \), where \( \xrightarrow{i.p.} \) denotes convergence in probability, and \( \hat{\theta}_b \xrightarrow{i.p.} \theta_0 + \mu \), where \( \mu \) is referred to as the bias of \( \hat{\theta}_b \). Alternatively, if we have it that \( \mathbb{E}[\hat{\theta}_u] = \theta_0, \mathbb{E}[\hat{\theta}_b] = \theta_0 + \mu \) in finite samples, then the MSE of \( \hat{\theta}_\lambda \) can be written (for a fixed value of \( \lambda \)) as

\[
\text{MSE}(\hat{\theta}_\lambda) = \lambda^2 (\mu^2 + \sigma_b^2) + (1 - \lambda)^2 \sigma_u^2 + 2\lambda (1 - \lambda) \sigma_{bu}
\]

(4)

where \( \sigma_b^2 = \text{Var}(\hat{\theta}_b), \sigma_u^2 = \text{Var}(\hat{\theta}_u), \) and \( \sigma_{bu} = \text{Cov}(\hat{\theta}_u, \hat{\theta}_b) \). From here, we can observe that the performance of this estimator depends on the trade-off between the relative variances of both estimators, as well as their
Table 1: Notation

| Notation       | Definition                        |
|----------------|----------------------------------|
| $\theta_0$     | Causal Effect                    |
| $\hat{\theta}_u, \hat{\theta}_b$ | Unbiased / Biased Estimator     |
| $\hat{\lambda}$ | Combination Estimator            |
| $\hat{\sigma}_u^2, \hat{\sigma}_b^2, \hat{\sigma}_{bu}$ | Estimators of the variance / covariance of $\hat{\theta}_u, \hat{\theta}_b$ |
| $A, X, S, Y$   | Action, Covariates, Surrogates, Outcome |
| $D \in \{O, E\}$ | Indicator of observational / experimental dataset |
| $n_{obs}, n_{exp}, n$ | Number of samples from each dataset, with $n = n_{obs} + n_{exp}$ |

The theoretically optimal weight $\lambda^*$ for minimizing the MSE is given by

$$\lambda^* = \frac{\hat{\sigma}_u^2 - \hat{\sigma}_{bu}}{\mu^2 + \hat{\sigma}_u^2 + \hat{\sigma}_b^2 - 2\hat{\sigma}_{bu}}.$$  \hspace{1cm} (5)

The optimal weight depends on several unknown quantities (including the bias) that must be estimated from data. In this work, we consider estimation of $\lambda^*$ using plug-in estimates of each quantity, using $(\hat{\theta}_b - \hat{\theta}_u)^2$ as an estimate of $\mu^2$, alongside plug-in estimates of the variance and covariance of $\hat{\theta}_u, \hat{\theta}_b$.\(^2\) We use $\hat{\lambda}$ to denote the estimator with $\hat{\lambda}$ estimated in this fashion, and refer to this as the combination estimator

$$\hat{\lambda} = \frac{\hat{\sigma}_u^2 - \hat{\sigma}_{bu}}{(\hat{\theta}_u - \hat{\theta}_b)^2 + \hat{\sigma}_u^2 + \hat{\sigma}_b^2 - 2\hat{\sigma}_{bu}}.$$  \hspace{1cm} (6)

In the remainder of this section, we illustrate the ubiquity of the motivating problem with a few selected examples. In Section 2.1 we describe an illustrative unbiased estimator, and in Sections 2.2 and 2.3 we describe two different estimators that make use of additional observational samples.

**Notation** We use $A$ to denote a binary action, $Y$ an outcome, $Y_a$ a potential outcome, and $X$ covariates. We use $V$ to denote the full set of observed variables, such that we may have $V_i = (A_i, X_i, Y_i)$. We use script characters to indicate the support of a random variable, e.g., $A = \{0, 1\}$. We use $D_{obs} = \{V_i\}_{i=1}^{n_{obs}}$ to denote an observational sample of size $n_{obs}$, and $D_{exp} = \{V_i\}_{i=1}^{n_{exp}}$ to denote an experimental sample of size $n_{exp}$, with a total sample size of $n_{obs} + n_{exp} = n$. We use the random variable $D$ to denote the population from which a sample is drawn, with $D = O$ indicating membership in an observational sample, and $D = E$ indicating membership in the experimental sample. We summarize selected notation in Table 1.

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\(^1\)We give a short proof of this claim, which is a generally known fact, in Appendix A.

\(^2\)The specifics of estimating $\hat{\sigma}_u^2, \hat{\sigma}_b^2, \hat{\sigma}_{bu}$ will vary based on the underlying estimators and application. General approaches include the non-parametric bootstrap, or using consistent estimators of the asymptotic variance / covariance (scaled by $n^{-1}$) as an approximation of the finite-sample variance / covariance. Estimators for the latter generally exist for regular and asymptotically linear (RAL) estimators.
2.1 An Unbiased Experimental Estimator

We assume throughout that there exists a consistent estimator $\hat{\theta}$ of the causal effect $\theta_0$, which is derived entirely from experimental data. For simplicity in the motivating examples that follows, we consider estimation of the causal effect in the experimental population

$$\theta_0 = \mathbb{E}[Y_1 - Y_0 \mid D = E],$$

and assume that experimental sample is randomized with a fixed (known) probability of treatment.

**Assumption 1** (Identification of ATE in the Experimental Sample). The following conditions hold

1. Consistency: $Y = Y_a$ when $A = a, D = E$
2. Ignorability: $Y_a \perp \perp A \mid D = E$
3. Positivity: $\mathbb{P}(A = a \mid D = E) > 0$ for all $a \in \{0, 1\}$.

Under Assumption 1, the causal effect can be estimated in an unbiased fashion using

$$\hat{\theta}_u = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{D_i = E\} \left( \frac{Y_i A_i}{e} - \frac{Y_i (1 - A_i)}{1 - e} \right),$$

where $e$ is the (known) probability of treatment assignment in the experimental sample, and $\hat{P}(D = E)$ is the empirical estimate $n_{\exp} / (n_{\exp} + n_{\obs})$.\footnote{Note that we write this as an average over the entire sample of size $n_{\exp} + n_{\obs}$, to be consistent with later notation, but the above is equivalent to taking the sample average of the pseudo-outcome over the experimental sample alone.}

We consider this formulation for simplicity, because it yields an experimental estimator that is not only consistent, but is unbiased in finite samples.

2.2 Combining Surrogates and Primary Outcomes in Experimental Data

There are several settings in which we can construct an alternative estimator based on observational data. We give one example here, and another in Section 2.3.

Suppose that we have additional surrogate outcomes $S$ in the experimental sample, and an observational sample $D_{\text{obs}} = \{s_i, x_i, y_i\}_{i=1}^{n_{\text{obs}}}$ containing surrogates, covariates, and outcomes, but no information on treatment. We write $S_a$ to denote the potential surrogate outcome under treatment $A = a$. In this context, we can construct an alternative estimator of the causal effect, by using a surrogate index estimator (Athey et al., 2019) that leverages the observational sample to learn the causal relationship between the short-term surrogate outcomes $S$ and the long-term outcome $Y$. In contrast to the setting of Athey et al. (2019), we assume that the outcome $Y$ is available in both experimental and observational samples, and consider using the surrogate only to obtain a higher-precision estimator. For such an approach to yield an unbiased estimator, we require a few assumptions.

**Assumption 2.** The following conditions hold, in addition to those of Assumption 1

1. Unconfounded Treatment Assignment: $A \perp \perp (Y_1, Y_0, S_1, S_0) \mid D = E$
Figure 2: (a) A causal graph consistent with Assumption 2, where the red arrow denotes an illustrative edge that is not permitted: A direct effect of the treatment $A$ on the outcome $Y$, and the dotted bi-directional arrows illustrates another potential violation of assumptions due to unmeasured confounding. (b) A causal graph consistent with Assumption 3, where the red arrows similarly indicate violations of the assumption.

2. Surrogacy: $A \perp \perp Y \mid S, X, D = E$

3. Comparability: $D \perp \perp Y \mid S, X$

4. Overlap: $P(D = E \mid S = s, X = x) > 0$ for all $s \in S, x \in X$

Under Assumption 2, which corresponds to Assumptions 1–4 of Athey et al. (2019), the following quantities are equivalent in the experimental sample

$$E[Y_a \mid D = E] = E[Y \mid A, D = E] = E[h(S, X) \mid A, D = E]$$  \hspace{1cm} (9)

where $h(S, X) := E[Y \mid S, X, D = O]$ is referred to as the surrogate index (See Theorem 1 of Athey et al., 2019). Assumption 2 is reflected in Figure 2a, which captures the fact that $Y$ is conditionally independent of both $A$ and $D$ given $S, X$, implying that there is no direct effect of treatment on the outcome $Y$.

This provides us several possible methods for estimating the causal effect using a combination of the observational and experimental sample. Here we give one simple estimator as an illustrative example

$$\hat{\theta}_b = \frac{1}{n} \sum_{i=1}^{n} \frac{1 \{D_i = E\} \hat{h}(S_i, X_i)A_i}{P(D = E)} - \frac{\hat{h}(S_i, X_i)(1 - A_i)}{1 - e}$$  \hspace{1cm} (10)

which takes the same form as $\hat{\theta}_u$, with $Y_i$ is replaced by $\hat{h}(S_i, X_i)$. Here if $\hat{h}(S_i, X_i)$ is a consistent estimator for the conditional expectation $h(S, X)$, and if Assumption 2 holds, then $\hat{\theta}_b$ is a consistent estimator of $\theta_0$. We refer to $\hat{\theta}_b$ in this section as the surrogate index estimator. Asymptotic bias in $\hat{\theta}_b$ can arise due to violations of Assumption 2, as illustrated in Figure 2a, e.g., if there is a direct effect of the treatment $A$ on the outcome $Y$, or unmeasured confounding between the surrogates and outcome.

2.3 Combining ATE estimates from Observational and Experimental Samples

Alternatively, suppose we have access to a much larger observational study (with the same treatment, outcome, and covariates), whose support covers the RCT population, and which can be used to estimate the ATE in the
RCT-population $\theta_0$ (Eq. 7). In particular, if the causal effect can be identified in the observational study (internal validity) and transported to the RCT population (external validity), then we can hope to use the observational data to construct an alternative estimator of $\theta_0$, which might be expected to have smaller variance, especially if $n_{obs}$ is substantially larger than $n_{exp}$. These assumptions are formalized in Assumption 3.

**Assumption 3** (Internal and External Validity of Observational Study). The following conditions hold, in addition to those of Assumption 1

1. Consistency: $A = a, D = O \implies Y_a = Y$
2. Unconfounded Treatment Assignment: $Y_a \perp A \mid X, D = O, \forall a \in A$
3. Positivity of Treatment Assignment: $\mathbb{P}(X = x \mid D = O) \implies \mathbb{P}(A = a \mid X = x, D = O) > 0, \forall a \in A, x \in X$
4. Unconfounded Selection: $Y_a \perp D \mid X, \forall a \in A$
5. Positivity of Selection: $\mathbb{P}(X = x) > 0 \implies \mathbb{P}(D = d \mid X = x) > 0, \forall d \in \{O, E\}, x \in X$

The particulars of Assumption 3 can be found in the literature on transportability of causal effects (see Degtiar & Rose (2021) for a recent review). Under Assumption 3, the observational data can be used to estimate the causal effect using a variety of estimators. For instance, one can estimate the causal effect as

$$\hat{\theta}_b = \frac{1}{n} \sum_{i=1}^{n} \frac{1 \{D_i = E\}}{P(D = E)} \times (\hat{g}_1(X_i) - \hat{g}_0(X_i))$$

(11)

where $\hat{g}_a(X)$ is an estimate of the conditional expectation under treatment in the observational population $g_a(X) := \mathbb{E}[Y \mid A = a, D = O, X]$ (Dahabreh et al., 2020). One can alternatively construct re-weighting estimators that use only the observational dataset, or doubly-robust estimators that combine the two (see Section 5 of Dahabreh et al. (2020) for examples). Here we note that, with the exception of re-weighting estimators that use only the observational sample, the estimators $\hat{\theta}_u$ and $\hat{\theta}_b$ are not independent, due to the shared use of experimental samples. Note that $\hat{\theta}_b$ may not be consistent for $\theta_0$ if Assumption 3 fails, e.g., due to confounding in treatment assignment or selection.

### 3 Theoretical Results

#### 3.1 Asymptotic Behavior

As a basic form of “robustness”, we may prefer that any proposed alternative estimator remains consistent for the underlying causal effect. As a starting point, we consider a general setup that enables us to state basic asymptotic results of this form.

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4For simplicity in these examples, we do not consider the causal effect in the observational population $\mathbb{E}[Y_1 - Y_0 \mid D = O]$, which requires assumptions about external validity of the RCT (e.g., that potential outcomes are conditionally independent of $D$ given $X$, that overlap holds between the studies, etc). Since we focus on the setting where one estimator is known with high confidence to be consistent or unbiased, we focus on the setting where the causal effect of interest is defined with respect to the RCT population $\mathbb{E}[Y_1 - Y_0 \mid D = E]$, requiring no additional assumptions on the RCT population beyond Assumption 1.
Theorem 1
In this setting, it is straightforward to demonstrate consistency of the estimator.

\[ \sqrt{n} \left( \frac{\hat{\theta}_u - \theta_0}{\hat{\theta}_b - (\theta_0 + \mu)} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\phi_u(Z_i)}{\phi_b(Z_i)} + o_p(1) \] (12)

where the random vector \((\phi_u(Z), \phi_b(Z))\) has zero-mean and bounded covariance, and where \(\mu\) is a finite constant. Note that \(n\) here denotes the total sample size \(n_{\text{exp}} + n_{\text{obs}}\). Similarly, we assume that \(\hat{\theta}_u, \hat{\theta}_b\) are asymptotically normal, in that

\[ \sqrt{n} \left( \frac{\hat{\theta}_u - \theta_0}{\hat{\theta}_b - (\theta_0 + \mu)} \right) \xrightarrow{d} N \left( 0, \begin{bmatrix} \text{Var}(\phi_u) & \text{Cov}(\phi_u, \phi_b) \\ \text{Cov}(\phi_u, \phi_b) & \text{Var}(\phi_b) \end{bmatrix} \right) \] (13)

This assumption requires that both \(\hat{\theta}_u\) and \(\hat{\theta}_b\) are asymptotically normal for their respective limits \(\theta_0 + \mu\) and \(\theta_0\), a condition that is broadly satisfied by commonly used estimators. This formulation also suggests a natural choice for estimating the variance of \(\hat{\theta}_u\), the variance of \(\hat{\theta}_b\), and the covariance, by first estimating

\[ \text{Var}(\phi_u) = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\phi}_u(Z_i) \right)^2 \quad \text{Var}(\phi_b) = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\phi}_b(Z_i) \right)^2 \quad \text{Cov}(\phi_u, \phi_b) = \frac{1}{n} \sum_{i=1}^{n} \hat{\phi}_u(Z_i) \hat{\phi}_b(Z_i), \]

where \(\hat{\phi}_u, \hat{\phi}_b\) are consistent plug-in estimators \(\hat{\phi}_u \xrightarrow{i.p.} \phi_u, \hat{\phi}_b \xrightarrow{i.p.} \phi_b\) for their respective influence functions. The components of \(\hat{\lambda}\) are then given by

\[ \hat{\sigma}_u^2 = n^{-1} \text{Var}(\phi_u) \quad \hat{\sigma}_b^2 = n^{-1} \text{Var}(\phi_b) \quad \hat{\sigma}_{bu} = n^{-1} \text{Cov}(\phi_u, \phi_b) \]

In this setting, it is straightforward to demonstrate consistency of the estimator.

Theorem 1 (Consistency). Let \(\hat{\theta}_b \xrightarrow{i.p.} \theta_0 + \mu, \hat{\theta}_u \xrightarrow{i.p.} \theta_0\), and let \(n\hat{\sigma}_u^2 \xrightarrow{i.p.} \nu_u, n\hat{\sigma}_b^2 \xrightarrow{i.p.} \nu_b\), and \(n\hat{\sigma}_{bu} \xrightarrow{i.p.} \nu_{bu}\) for finite constants \(\nu_u, \nu_b\), and \(\nu_{bu}\). Then, if \(\mu \neq 0\), we have it that \(\hat{\lambda} \xrightarrow{i.p.} 0\) and \(\hat{\theta}_u \xrightarrow{i.p.} \theta_0\).

All proofs can be found in Appendix A.

3.2 Finite-Sample Behavior

While consistency is a desirable quality for any estimator, the primary motivation for using the estimator \(\hat{\theta}_\lambda\) is to improve performance in finite samples. In this section, we focus on bounding the worst-case behavior of \(\hat{\theta}_\lambda\) under arbitrary values of the bias. In contrast to the asymptotic presentation in Section 3.1, we assume here that the estimator \(\hat{\theta}_u\) is unbiased.

Assumption 5 (Unbiased Experimental Estimator). The estimator \(\hat{\theta}_u\) is unbiased in finite samples, such that \(E_n[\hat{\theta}_u] = \theta_0\) for any sample size \(n\) where \(n_{\text{exp}} > 0\).

First, we make some assumptions that rule out cases where \(\hat{\lambda}\) is not well-defined, or is trivially zero.

Assumption 6 (Non-Zero Variance). The unbiased estimator has non-zero variance \(\sigma_u^2 > 0\), and the difference \(\hat{\theta}_u - \hat{\theta}_b\) has non-zero variance

\[ \text{Var}(\hat{\theta}_u - \hat{\theta}_b) = \sigma_u^2 + \sigma_b^2 - 2\sigma_{bu} > 0 \] (14)

and the estimators \(\hat{\sigma}_u^2, \hat{\sigma}_b^2, \hat{\sigma}_{bu}\) satisfy \(\hat{\sigma}_u^2 + \hat{\sigma}_b^2 - 2\hat{\sigma}_{bu} > 0\) almost surely.
Assumption 6 rules out the case where \( \hat{\theta}_b = \hat{\theta}_u + \mu \), and ensures that \( \hat{\lambda} \) is always well-defined for all \( \mu \), including \( \mu = 0 \). Our main result is Theorem 2, which bounds the relative MSE of the estimator by a constant factor that depends on the relative variance of \( \hat{\theta}_b, \hat{\theta}_u \) and the correlation between \( \hat{\theta}_b \) and \( \hat{\theta}_u \).

**Theorem 2 (Bound on MSE).** Under Assumptions 5 and 6, and where \( \sigma_b^2, \sigma_u, \sigma_{bu} \) are known, define \( c, \rho \) by \( c := \sigma_b / \sigma_u \) and \( \rho = \sigma_{bu} / \sqrt{\sigma_b^2 \sigma_u^2} \), where \( \rho = 0 \) if \( \sigma_b^2 = 0 \). Then the MSE of the estimator \( \hat{\theta}_\lambda \) is bounded by

\[
E[(\hat{\theta}_\lambda - \theta_0)^2] \leq \sigma_u^2 \left( 1 + \frac{1}{2} \frac{|1 - \rho c|}{\sqrt{1 - 2 \rho c + c^2}} \right)^2
\]  

(15)

Theorem 2, whose full proof can be found in Appendix A, simplifies presentation by assuming that the terms \( \sigma_b^2, \sigma_u^2, \sigma_{bu} \) are known. This is useful for building intuition for the behavior of the worst-case bound, in terms of the underlying variance / covariance of the estimators. For instance, if the estimators are independent (\( \rho = 0 \)), the worst-case relative MSE is a function of \( c = \sigma_b / \sigma_u \). In particular, the upper bound increases for smaller values of the variance \( \sigma_b^2 \) of the biased estimator. While Theorem 2 is useful for building intuition, it is straightforward to give a more general version that does not assume knowledge of the underlying variance/covariance, and only depends on the behavior of the estimators of these terms.

**Theorem 3 (Bound on MSE, unknown variance).** Under Assumptions 5 and 6, with estimators \( \hat{\sigma}_b^2, \hat{\sigma}_u^2, \hat{\sigma}_{bu} \) that have bounded second moments, the MSE of the estimator \( \hat{\theta}_\lambda \) is bounded by

\[
E[(\hat{\theta}_\lambda - \theta_0)^2] \leq \left( \sigma_u + \frac{1}{2} \sqrt{E[S^2]} \right)^2
\]

where \( E[S^2] = E[(\hat{\sigma}_u^2 - \sigma_{bu})^2 / (\hat{\sigma}_u^2 + \hat{\sigma}_b^2 - 2\sigma_{bu})] \).

By analyzing the MSE of \( \hat{\lambda} \) as the bias varies, we can build additional intuition for its behavior. Intuitively, for large values of the bias, \( \hat{\lambda} \) will tend to zero, and \( \hat{\theta}_\lambda \) will tend to the unbiased estimator \( \hat{\theta}_u \). We can formalize this intuition by considering the MSE of \( \hat{\lambda} \) under a sequence of biases that are increasing without limit.

**Proposition 1.** Consider a sequence of biased estimators \( \hat{\theta}_b^{(k)} \) which can be written as \( \hat{\theta}_b + \mu_k \), where \( E[\hat{\theta}_b] = \theta_0 \), where \( \text{Cov}(\hat{\theta}_b^{(k)}, \hat{\theta}_u) = \sigma_{bu} \), and \( \text{Var}(\hat{\theta}_b^{(k)}) = \sigma_b^2 \). Let \( \mu_k \rightarrow \infty \) as \( k \rightarrow \infty \). The MSE of the resulting sequence of estimators \( \hat{\theta}_\lambda^{(k)} \) converges to the MSE of the unbiased estimator \( \hat{\theta}_u \)

\[
\lim_{k \to \infty} E \left[ (\hat{\theta}_\lambda^{(k)} - \theta_0)^2 \right] = \sigma_u^2
\]  

(16)

where \( \hat{\theta}_\lambda^{(k)} = \lambda \hat{\theta}_b^{(k)} + (1 - \lambda) \hat{\theta}_u \), and where \( \lambda = (\sigma_u^2 - \sigma_{bu}) / ((\hat{\theta}_u - \hat{\theta}_b)^2 + \sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}) \).

### 4 Multivariate Gaussian Simulations

In this section, we provide empirical evidence, via simulation, that \( \hat{\lambda} \) has favorable properties relative to comparable approaches. In Section 4.1 we describe our simulation setup, in Section 4.2 we describe the baseline approaches we consider, and in Section 4.3 we provide a comparison of performance under a variety of settings.
Table 2: For each of the following parameter settings, for each value of $\mu$, the estimators are constructed from drawing $n$ samples of $(\psi_u, \psi_b)$ from a normal distribution with mean $(\theta_0, \theta_0 + \mu)$, marginal variances $\text{Var}(\psi_u)$, $\text{Var}(\psi_b)$ and covariance determined by the chosen correlation. The value of $\theta_0$ is 1 throughout.

| Parameter        | Values                                      |
|------------------|---------------------------------------------|
| $n$              | $\{500, 1000, 2000, 4000\}$                |
| $\text{Var}(\psi_u)$ | $\{1, 2, 4, 8, 16\}$                      |
| $\text{Var}(\psi_b)$ | $\{0, 1, 2, 4, 8, 16\}$                   |
| $\text{corr}(\psi_u, \psi_b)$ | $\{-0.5, -0.25, 0, 0.25, 0.5\}$   |
| $\mu$            | $[0, 1.5]$, increments of 0.002             |

4.1 Simulation Setup

We simulate $\hat{\theta}_u, \hat{\theta}_b$ via correlated sample averages: In particular, we take $\psi_u, \psi_b$ to be drawn from a multivariate normal distribution, where $\hat{\theta}_u = \frac{1}{n} \sum_i \psi_u^{(i)}$ and $\hat{\theta}_b$ are the sample averages, and where we can directly estimate quantities like the variance $\text{Var}(\psi_u)$, etc. We investigate the performance of the combination estimator for different covariance structures of $(\psi_u, \psi_b)$, as we vary the bias. In particular, we compute the squared error of $\hat{\lambda}$ and the squared error of $\hat{\theta}_u$, and for each set of simulation parameters in Table 2, we repeat this process 10000 times to estimate the mean squared error.

4.2 Baselines

Here we describe several baseline approaches, while deferring more detail to Section C in the Appendix. We note that these approaches are designed for the combination of independent estimators, typically motivated by the combination of observational and experimental estimates of treatment effects. Because we consider the setting where the estimators can be correlated, we modify these approaches accordingly.

Shrinkage Estimators Rosenman et al. (2020) proposes the use of shrinkage estimators in the setting where both estimators are independent, and the target parameter is multi-dimensional. They similarly observe that the theoretically optimal weight in this setting is given by $\lambda^* = \sigma_u^2 / (\mu^2 + \sigma_u^2 + \sigma_b^2)$, but instead of estimating each term in the denominator individually, they observe that $(\hat{\theta}_u - \hat{\theta}_b)^2$ is an unbiased estimate of the denominator.\footnote{One can make a similar observation in the case where $\hat{\theta}_u, \hat{\theta}_b$ are correlated, that $\mathbb{E}[(\hat{\theta}_b - \hat{\theta}_u)^2] = \mu^2 + \sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}$.}

Adapted to the setting of correlated estimators, their estimator takes the same form as ours, except that $\hat{\lambda}$ is estimated as

$$\hat{\lambda} = (\hat{\sigma}_u^2 - \sigma_{bu}) / (\hat{\theta}_u - \hat{\theta}_b)^2,$$

which is optionally clipped to lie in $[0, 1]$.

Hypothesis Testing Yang et al. (2020) propose a test-based procedure. In our simulation setting, this involves constructing the test statistic

$$T_n = (\hat{\theta}_u - \hat{\theta}_b)^2 / (n\hat{\sigma}),$$

where $\hat{\sigma}$ is an estimate of the standard deviation of $\hat{\theta}_u - \hat{\theta}_b$, such that $T_n$ follows a chi-square distribution under the null hypothesis that the bias is zero. For a chosen significance level, their procedure pools the data together.
if the test is not rejected, and otherwise uses only the unbiased estimator. Yang et al. (2020) propose a data-adaptive approach to choosing the significance level, by estimating the bias directly and then simulating from the asymptotic mixture distribution of their estimator under that bias to select a cutoff that yields optimal performance. We replicate this data-driven approach in our experiments, as described in Appendix C.

**Anchored Thresholding** Chen et al. (2021) attempt to estimate and correct for the bias in $\hat{\theta}_b$, estimating it via soft-thresholding as

$$
\hat{\mu} = \begin{cases} 
\text{sign}(\hat{\theta}_b - \hat{\theta}_u) \left( |\hat{\theta}_b - \hat{\theta}_u| - \lambda \sqrt{\hat{\text{Var}}(\hat{\theta}_b - \hat{\theta}_u)} \right), & \text{if } |\hat{\theta}_b - \hat{\theta}_u| \geq \lambda \cdot \sqrt{\hat{\text{Var}}(\hat{\theta}_b - \hat{\theta}_u)} \\
0, & \text{otherwise.}
\end{cases}
$$

(19)

and then combine $\hat{\theta}_u$ and $\hat{\theta}_b - \hat{\mu}$ using inverse variance-weighting. In our experiments, we take $\lambda = 0.5\sqrt{\log n}$, in keeping with their synthetic experimental setup.

### 4.3 Simulation Results

Throughout, we use the relative MSE as our measure of estimator performance, given by $\frac{\text{MSE}(\hat{\theta})}{\text{MSE}(\hat{\theta}_u)}$: An estimator $\hat{\theta}$ is said to out-perform / under-perform $\hat{\theta}_u$ when the relative MSE is less than 1 / greater than 1.

For each simulation setting, we compute the maximum value of $\mu$ (the “bias threshold”) for which the combination estimator has lower MSE than $\hat{\theta}_u$, and do similarly for each alternative estimator. For each estimator, we also compute the “worst-case relative MSE”, defined as the maximum relative MSE over all values of the bias, and the “best-case relative MSE”, defined as the minimum over all values of the bias.

#### 4.3.1 Building Intuition with a Single Example

In Figure 3a, we plot the relative MSE of each approach against the value of the bias, for the parameter setting where $\text{Var}(\psi_u) = \text{Var}(\psi_b) = 1$, $n = 1000$, $\text{Corr}(\psi_u, \psi_b) = 0$. In this setting, we can make a few observations about our proposed approach, as well as some of the baselines.\(^6\)

First, we can observe that each estimator makes a trade-off between the worst-case and best-case relative MSE. Here, our proposed estimator has among the lowest worst-case relative MSE of any approach, comparable to that of the hypothesis-testing estimator, but also achieves a lower best-case MSE. More broadly, we observe general patterns across all estimators: All of the baseline estimators have a bounded relative MSE, with the lowest relative MSE naturally occurring when $\mu = 0$.

Second, we note that for all estimators (aside from the anchored thresholding estimator of Chen et al. (2021)), the relative MSE decreases for sufficiently large values of $\mu$, approaching 1 as $\mu$ becomes large. For the hypothesis-testing estimator of Yang et al. (2020), this reflects the fact that the probability of rejecting the observational estimator approaches 1 as $\mu$ increases. For the clipped shrinkage estimator of Rosenman et al. (2020), and for our approach, this reflects the fact that $\hat{\lambda}$ approaches zero (and $\hat{\theta}_\lambda$ approaches $\hat{\theta}_u$) for sufficiently large values of $\mu$, as illustrated in Proposition 1.

\(^6\)Note that Figure 3a does not include the unclipped version of the shrinkage estimator in Rosenman et al. (2020), as this exhibits poor performance when the bias is low: We give this comparison in Figure 7 in the Appendix.
Figure 3: (a) For a fixed set of parameters from Table 2, we show \( \mu \) on the x-axis, and \( \text{MSE}(\hat{\theta})/\text{MSE}(\hat{\theta}_u) \) on the y-axis (lower is better), for each estimator described in Section 4.2. (b) For each simulation setting, we compute the maximum value of \( \mu \) (the “bias threshold”) for which the combination estimator has lower MSE than \( \hat{\theta}_u \), and do similarly for each alternative estimator. We then compute the difference of these thresholds, where a \textbf{negative} value means that our threshold is higher (better), and plot a histogram of these differences across all simulations. (c) For each simulation setting, we plot (on the y-axis) the maximum relative MSE of \( \hat{\theta} \) for each alternative approach, minus the maximum relative MSE of \( \hat{\theta}_\lambda \), and (on the x-axis) we similarly plot the difference in the minimum relative MSE, where negative values (in both cases) signify an improvement over \( \hat{\theta}_\lambda \).

Finally, the bias threshold for our approach is higher than that of the alternative estimators, occurring at around \( \mu = 0.06 \) (around 2 times \( 1/\sqrt{n} \)), while the value of \( \mu \) that attains the worst-case relative MSE tends to fall into the range 0.10 to 0.15 for all estimators except for the anchored thresholding estimator: This is equivalent to 3–5 times \( 1/\sqrt{n} \), a range where the squared bias of \( \hat{\theta}_b \) is of the same order as the variance of the unbiased estimator. Intuitively, in this regime the bias is sufficiently large that it introduces additional MSE, but small enough that it is difficult to detect.

### 4.3.2 Comparing to alternative approaches

We now report results over the entire batch of experiments, across all the parameter settings given in Table 2.

In Figure 3b, we observe that the combination estimator has the highest bias-tolerance of the estimators considered here. To do so, we compute difference of bias thresholds for each alternative estimator, where a \textbf{negative} value means that our threshold is higher, and plot a histogram of these differences across all simulations. Our threshold is the highest in every scenario, indicated by fact that all reported values are negative.

In Figure 3c, we observe that none of the alternative estimators dominates ours (achieving best and worst-case MSE that are \textbf{both} lower than ours) in any of the simulated scenarios. For each simulation setting, we plot (on the y-axis) the maximum relative MSE of \( \hat{\theta} \) for each alternative approach, minus the maximum relative MSE of \( \hat{\theta}_\lambda \), and (on the x-axis) we similarly plot the difference in the minimum relative MSE, where negative values (in both cases) signify an improvement over \( \hat{\theta}_\lambda \). However, some estimators can be observed to simply make a different
Figure 4: (a) Every simulation setting (in terms of parameters in Table 2, excluding $\mu$) corresponds to a different dot, where on the x-axis we plot the best-case relative MSE (when $\mu = 0$) and on the y-axis we plot the worst-case relative MSE (over all values of $\mu$, with other simulation parameters fixed). Figures (b-c) show variation in the minimum and maximum relative MSE of the combination estimator $\hat{\theta}_\lambda$ where $n = 1000$, as a function of (b) variance in $\psi_b$, where $\text{Var}(\psi_u) = 4, \text{corr}(\psi_b, \psi_u) = 0$, and (c) correlation between $\psi_u, \psi_b$ where $\text{Var}(\psi_u) = 4, \text{Var}(\psi_b) = 4$.

trade-off: In particular, the anchored-threshold estimator of Chen et al. (2021) always has a higher maximum relative MSE, as well as a lower minimum relative MSE. This may reflect the optimism of the approach: If the observed difference $\hat{\theta}_u - \hat{\theta}_b$ is sufficiently small, it assumes that the bias is equal to zero. The differences between the best-case MSE of $\hat{\theta}_\lambda$ and the other estimators is inconsistent, with superior best-case performance in some scenarios, and inferior best-case performance in others.

4.3.3 Understanding factors that drive performance across settings

The largest opportunities for improvement (e.g., low-variance $\hat{\theta}_b$) also have the highest worst-case error: In Figure 4a, we examine the smallest and largest values of the relative MSE of $\hat{\theta}_\lambda$ for each combination of simulation parameters, and observe that these values exhibit a nearly linear relationship: The larger the potential upside (when $\mu = 0$), the larger the potential downside (when $\mu$ is chosen adversarially).

In Figures 4b and 4c, we demonstrate that the magnitude of the smallest/largest relative MSE depends on the relative benefit of incorporating $\hat{\theta}_b$: In Figure 4b, we show that both decrease in magnitude as the variance of $\hat{\theta}_b$ increases, and in Figures 4c, we show that both decrease with increasing positive correlation of $\hat{\theta}_b$ and $\hat{\theta}_u$, and that both increase for more negative correlations, observations consistent with the worst-case bound given in Theorem 2.

The worst-case relative MSE of $\hat{\theta}_\lambda$ is empirically bounded by a small constant factor: Across all parameter settings, the largest relative MSE of $\hat{\theta}_\lambda$ is bounded, never exceeding a 27% increase in MSE over the use of $\hat{\theta}_u$ alone. Moreover, measured by relative MSE, the potential upside is also larger than the potential downside, across all parameter settings.
Bias-Variance Ratio: $\frac{\mu^2}{\text{Var}(\hat{\theta}_u - \hat{\theta}_b)}$

(a) Zero Correlation

(b) Positive Correlation

(c) Negative Correlation

Figure 5: Relative MSE as a function of the ratio $\mu^2/\text{Var}(\hat{\theta}_u - \hat{\theta}_b)$, across all sample sizes. The dashed green vertical line denotes $\mu^2 = 2\text{Var}(\hat{\theta}_u - \hat{\theta}_b)$. (a) $\text{corr}(\psi_b, \psi_u) = 0$, shown here with $\text{Var}(\psi_u) = 4$. (b) Variance fixed at $\text{Var}(\psi_u) = 16$ and correlation is fixed at 0.5. (c) Same as b, but correlation fixed at $-0.5$. We observe that when the correlation is positive, the maximum tolerable bias is lower, and higher when the correlation is negative, with the gap to predicted threshold determined by the difference in the variance of $\hat{\theta}_b, \hat{\theta}_u$.

5 Simulation based on SPRINT Trial

While the observations outlined in Section 4 build evidence for what influences the behavior of the combination estimator, we might plausibly wonder whether or not the observed thresholds (on the maximum allowable bias) are high enough to warrant practical application of the method. To provide evidence on this question, we construct a simulation where the parameters are designed to mimic the observed statistics of the SPRINT Trial (SPRINT Research Group et al., 2015).

The SPRINT Trial investigated the effectiveness of two different targets for systolic blood pressure ($<120\text{mm Hg}$, the “intensive” treatment, and $<140\text{mm Hg}$, the “standard” treatment) among non-diabetic patients with high cardiovascular risk. We take $T = 0$ to denote the standard regime, and $T = 1$ to denote the intensive regime. Several of the outcomes considered in this trial are time-to-event outcomes: The primary composite outcome is comprised of myocardial infarction, other acute coronary syndromes, stroke, heart failure, or death from cardiovascular causes. For simplicity, we consider this outcome as a binary variable. We take $Y = 1$ to
denote the presence of the primary composite outcome. Note that in this simulation, the true value of the treatment effect is \( \theta_0 = E[Y_1 - Y_0] = -0.0164 \), a decrease of 1.64% in the absolute risk of the primary outcome.

**Creating an observational dataset with realistic confounding:** To construct confounded observational data, we first define a unobserved confounder. We use the reported trial statistics to calibrate the strength of the association between this confounder and the potential outcomes. The trial reports the incidence of the primary outcome across both arms for several sub-groups (see Figure 4 of SPRINT Research Group et al. (2015)). To emulate a plausible binary confounder, we consider previous chronic kidney disease (CKD), which has the smallest p-value for an interaction effect. Taking \( U = 1 \) as presence of previous CKD, we then take the observed incidence of \( Y \) in treatment and control, across these two subpopulations (from Figure 4 of SPRINT Research Group et al. (2015)), as the values of \( E[Y_1 | U] \) and \( E[Y_0 | U] \) in our simulation. We provide additional details in Appendix C.2.

Given a pre-defined effect of \( U \) on \( Y_a \), we introduce confounding in the observational study via the following model for treatment selection

\[
P(T = 1 | U, D = O) = \logit^{-1}(\gamma(U - 1/2))
\]

where the intercept is chosen to keep the log-odds symmetric around 0 for \( U = 1, U = 0 \). For \( \gamma > 0 \), patients with a history of CKD are more likely to receive intensive management. In this setting, there are no other covariates for simplicity. This model of confounding can be viewed in the Rosenbaum sensitivity model (See 4.2 of Rosenbaum, 2010), satisfying the bound

\[
\Gamma^{-1} \leq \frac{P(T = 1 | U = 1, D = O)P(T = 0 | U = 0, D = O)}{P(T = 0 | U = 1, D = O)P(T = 1 | U = 0, D = O)} \leq \Gamma
\]

with \( \Gamma = \exp(\gamma) \).

**Simulation of estimator performance:** Based on the generative model above, we simulate \( n_{exp} = 9361 \) samples from the simulated trial (the size of the original trial), and a ten-fold larger amount from an observational study, \( n_{obs} = 100000 \). We note that the generative model is identical (e.g., the distribution of \( U \)) except for the treatment assignment mechanism, and we examine the performance of the combination estimator as we vary the confounding bias \( \gamma \). In each dataset, the estimators \( \hat{\theta}_u \), \( \hat{\theta}_b \) are given by

\[
\hat{\theta}_u = \frac{1}{n_{exp}} \sum_{i: D_i = E} Y_i \left( \frac{T_i}{\hat{e}_E} - \frac{1 - T_i}{1 - \hat{e}_E} \right)
\]

\[
\hat{\theta}_b = \frac{1}{n_{obs}} \sum_{i: D_i = O} Y_i \left( \frac{T_i}{\hat{e}_O} - \frac{1 - T_i}{1 - \hat{e}_O} \right)
\]

where \( \hat{e}_d = (\sum_i 1 \{ D_i = d \})^{-1} \sum_i T_i 1 \{ D_i = d \} \) is an empirical estimate of the treatment probability in dataset \( d \). To construct \( \hat{\lambda} \) we estimate \( \hat{\sigma}_u^2, \hat{\sigma}_b^2 \) by the variance of plug-in estimates of the corresponding influence functions, as described in Appendix C.2. These are used to construct \( \hat{\lambda} \) for each pair of observational and experimental estimators, using

\[
\hat{\lambda} = \frac{\hat{\sigma}_u^2}{(\hat{\theta}_u - \hat{\theta}_b)^2 + \hat{\sigma}_u^2 + \hat{\sigma}_b^2}
\]

For each value of \( \gamma \), we repeat this process 10000 times, where each iteration gives us one observation of the squared error for each estimator. We perform this procedure for 20 values of \( \gamma \) evenly spaced between 0 and 2. For each value of \( \gamma \), we obtain the corresponding value of \( \Gamma \) as \( \Gamma = \exp(\gamma) \).

**Results:** In Figure 6 we compare the root mean-squared error (RMSE) of \( \hat{\theta}_u \) and \( \hat{\lambda} \), for each value of \( \gamma \). In this particular scenario, we see that \( \hat{\lambda} \) improves on the performance of the unbiased estimator \( \hat{\theta}_u \) in the regime where
Figure 6: For each value of $\gamma$, and corresponding value of $\Gamma$, we show the root mean-squared error (RMSE) for each estimator, calculated over 10k simulations. 95% confidence intervals are obtained via bootstrapping.

$\gamma < 1$, and otherwise tends to perform similarly. For reference, the impact of CKD on the composite outcome in the control group, also measured on the log-odds scale, is 0.55. In Table 3, we give the RMSE of the combination estimator over different values of $\gamma$, as we additionally vary the sample size $n_{\text{obs}} \in \{10000, 20000, 50000, 100000\}$, and observe that the maximum allowable value of $\gamma$ decreases slightly as the sample size increases.

6 Related Work and Discussion

Our work fits within the broader literature of combining observational effect estimates with unbiased estimates from randomized trials, though this type of problem arises in a broader variety of settings, as demonstrated in Section 2. In Section 4, we compare to a number of related methods that were originally developed for the setting of combining observational effect estimates with those from randomized trials. We first discuss our contributions in the context of these methods, going beyond the empirical comparisons in Section 4.

Chen et al. (2021) demonstrate that their anchored-thresholding estimator achieves the performance (up to poly-log factors) of an oracle that selectively chooses an estimator based on whether or not (in our notation) the bias $\mu$ is larger or smaller than $\sigma_{u}$, and demonstrate that this performance is minimax optimal under a certain data-generating process. In contrast to their analysis, which hides constant factors, our investigation in Section 4 focuses on understanding the constant factors involved in a finite-sample setting, and understanding qualitative performance across different values of the bias. However, we do not make any claims about performance in the setting where the observational data is of a higher order than the experimental data, $n_{\text{obs}} \gg n_{\text{exp}}$, which is a main focus of their work. Yang et al. (2020) give a procedure that first tests for bias, pooling observational and randomized trial data if this test fails to reject, and otherwise using only the randomized trial estimate. They focus on the asymptotic properties of this test-based procedure, including the asymptotic regime where the bias scales as $n^{-1/2}$. Finally, Rosenman et al. (2020) use shrinkage estimators for the setting where the target
Table 3: The RMSE of the combination estimator over selected values of $\gamma$, with the full set given in Table 4. For legibility, the RMSE is multiplied by 1000, on which scale the RMSE of the unbiased estimator is 4.97. For each sample size we **bold** the largest value of which remains below the RMSE of the unbiased estimator.

| $\gamma$ | 10k  | 20k  | 50k  | 100k |
|----------|------|------|------|------|
| 0.00     | 4.24 | 3.98 | 3.72 | 3.59 |
|          |      |      |      |      |
| 0.95     | 4.74 | 4.82 | 4.92 | 4.97 |
| 1.00     | 4.78 | 4.87 | 4.99 | 5.04 |
| 1.05     | 4.82 | 4.92 | 5.05 | 5.10 |
| 1.10     | 4.85 | 4.97 | 5.10 | 5.16 |
| 1.15     | 4.88 | 5.01 | 5.15 | 5.21 |
| 1.20     | 4.91 | 5.05 | 5.20 | 5.26 |
| 1.25     | **4.94** | 5.09 | 5.24 | 5.31 |
|          |      |      |      |      |
| 2.00     | 5.22 | 5.36 | 5.45 | 5.48 |

estimand is multivariate, allowing for the use of classical results (Stein, 1981; Strawderman, 2003) which give dominance in terms of MSE across the entire vector, while our focus is on the one-dimensional setting where such results do not apply.

Yang & Ding (2020) propose a superficially similar approach to combining observational and validation data, for the setting where the latter measures all relevant confounders, while the former does not. Under the assumption that both datasets are drawn from a common distribution, they propose a variance reduction technique that makes use of both datasets: Applying the same biased estimator (using only the observational subset of covariates) to both datasets, taking the difference of these estimators (which is consistent for zero), and subtracting a rescaled version of this difference from the consistent estimator that uses only the validation data. This approach does not directly apply in the settings we consider.

Cheng & Cai (2021) adopt a similar approach as ours, taking an adaptive linear combination of observational and experimental estimators. We focus on a broader class of estimation problems that involve estimating a real-valued parameter, including the use of surrogate outcomes (as described in Section 2), while they focus on CATE estimation with kernel regression, in the context of combining experimental and trial estimators. However, while they approach a specific problem setting, their approach aligns with ours on a conceptual level, and yields a similar combination estimator in the setting of ATE estimation via combination of experimental and observational data. Their approach differs in the addition of re-scaling of the estimated bias by a factor of $n^{-\beta}$, where $\beta > 0$ is a hyperparameter. This is done to ensure that if the bias is zero, $\lambda$ will converge to the optimal inverse-variance weights.

In contrast to several relevant methods in the literature, our proposed approach has the practical advantage that it does not require the specification of hyperparameters, in contrast to the approaches of Chen et al. (2021); Yang et al. (2020); Cheng & Cai (2021). Moreover, our theoretical analysis focuses less on the asymptotic regime, and more squarely on understanding worst-case expected error in the finite sample regime, giving worst-case bounds on the relative MSE of our approach in the presence of an adversarially chosen bias. Finally, we view our
Simulation design in Section 4 as a potentially useful tool for practitioners, as they consider the use of estimators that combine experimental and observational data. In particular, they provide a heuristic method of assessing whether or not observational estimators should be combined with experimental estimators in the first place: For a given estimate of the variance / covariance of each estimator, they indicate a level of bias for which a given method will fail to out-perform the experimental estimator alone. As demonstrated in Section 5, it may be possible to translate assumptions on the size of various biases (e.g., confounding bias) into such reasoning. In practice, we rarely believe that observational estimators are completely free of bias, and this approach provides a simulation-based means of assessing whether or not plausible biases are “small enough” that making use of observational data is reasonable.

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Appendix

A Proofs

A.1 Proof of Minor Claims

In Section 2, we claimed that $\lambda^*$ takes a particular form. For completeness, we provide a proof here, but this is a known fact from the literature.

Proof. Suppose that $E[\hat{\theta}_b] = \theta_0 + \mu$, and that $E[\hat{\theta}_u] = \theta_0$. The corresponding MSE is given by

$$MSE(\lambda \hat{\theta} + (1 - \lambda)\hat{\theta}_u)$$

where we observe that $E[(\hat{\theta}_b - \theta_0)^2] = \mu^2 + Var(\hat{\theta}_b)$, and where we use the fact that

$$E[(\hat{\theta}_b - \theta_0)(\hat{\theta}_u - \theta_0)] = E[(\hat{\theta}_b - (\theta_0 + \mu))(\hat{\theta}_u - \theta_0)] + E[\mu(\hat{\theta}_u - \theta_0)]$$

$$= Cov(\hat{\theta}_b, \hat{\theta}_u)$$

Note that Equation (26) is a quadratic in $\lambda$, which is minimized by setting the derivative equal to zero

$$0 = 2\lambda(\mu^2 + Var(\hat{\theta}_b)) - 2(1 - \lambda)Var(\hat{\theta}_u) + (2 - 4\lambda)Cov(\hat{\theta}_b, \hat{\theta}_u)$$

$$= 2\lambda(\mu^2 + Var(\hat{\theta}_b)) + 2\lambda Var(\hat{\theta}_u) - 4\lambda Cov(\hat{\theta}_b, \hat{\theta}_u) - 2Var(\hat{\theta}_u) + 2Cov(\hat{\theta}_b, \hat{\theta}_u)$$

$$\implies \lambda^* = \frac{Var(\hat{\theta}_u) - Cov(\hat{\theta}_b, \hat{\theta}_u)}{\mu^2 + Var(\hat{\theta}_b) + Var(\hat{\theta}_u) - 2Cov(\hat{\theta}_b, \hat{\theta}_u)}$$

A.2 Proofs of Main Results

Theorem 1 (Consistency). Let $\hat{\theta}_b \rightarrow P \theta_0 + \mu, \hat{\theta}_u \rightarrow P \theta_0$, and let $n\sigma^2_u \rightarrow P \nu_u, n\sigma^2_b \rightarrow P \nu_b$, and $n\sigma_{bu} \rightarrow P \nu_{bu}$ for finite constants $\nu_u, \nu_b,$ and $\nu_{bu}$. Then, if $\mu \neq 0$, we have it that $\hat{\lambda} \rightarrow P 0$ and $\hat{\theta}_\lambda \rightarrow P \theta_0$.

Proof. To see that $\hat{\lambda} \rightarrow P 0$ when $\mu \neq 0$, we can observe that

$$\hat{\lambda} = \frac{n\sigma^2_u - n\sigma_{bu}}{n(\hat{\theta}_u - \hat{\theta}_b)^2 + n\sigma^2_u + n\sigma^2_b - 2n\sigma_{bu}}.$$
which converges to zero by the fact that \( \hat{\theta}_u - \hat{\theta}_b \xrightarrow{\text{i.p.}} \mu, n\hat{\sigma}_u^2 \xrightarrow{\text{i.p.}} \nu_u, n\hat{\sigma}_b^2 \xrightarrow{\text{i.p.}} \nu_u, \) and \( n\hat{\sigma}_{bu} \xrightarrow{\text{i.p.}} \nu_{bu}. \) The denominator diverges to infinity due to the extra factor of \( n \) in the denominator, i.e., \( n(\hat{\theta}_u - \hat{\theta}_b)^2 \xrightarrow{\text{i.p.}} \infty, \) and the result follows from the continuous mapping theorem. The fact that \( \hat{\lambda} \xrightarrow{\text{i.p.}} 0 \) when \( \mu \neq 0 \) is sufficient to conclude that \( \hat{\lambda} \xrightarrow{\text{i.p.}} \theta_0 \) by another application of the continuous mapping theorem to the expression \( \hat{\lambda} = \hat{\theta}_u + \hat{\lambda}(\hat{\theta}_b - \hat{\theta}_u), \) and the fact that \( \hat{\theta}_b - \hat{\theta}_u \xrightarrow{\text{i.p.}} \mu. \)

**Theorem 2 (Bound on MSE).** Under Assumptions 5 and 6, and where \( \sigma_u^2, \sigma_b^2, \sigma_{bu} \) are known, define \( c, \rho \) by \( c := \sigma_b / \sigma_u \) and \( \rho = \sigma_{bu} / \sqrt{\sigma_u^2 \sigma_b^2} \), where \( \rho = 0 \) if \( \sigma_b^2 = 0 \). Then the MSE of the estimator \( \hat{\theta}_\lambda \) is bounded by

\[
\mathbb{E}[(\hat{\theta}_\lambda - \theta_0)^2] \leq \sigma_u^2 \left( 1 + \frac{1}{2} \frac{|1 - \rho c|}{\sqrt{1 - 2\rho c + c^2}} \right)^2
\]

**Proof.** Writing \( \hat{\theta}_b(\mu), \hat{\lambda}(\mu) \) and \( \hat{\lambda}(\mu) \) to explicitly denote that they are functions of the bias \( \mu \), we have that

\[
(\hat{\theta}_\lambda(\mu) - \theta_0)^2 \leq \text{sup}_m (\hat{\theta}_\lambda(m) - \theta_0)^2,
\]

and so by the monotonicity of expectations,

\[
\mathbb{E}_P[(\hat{\theta}_\lambda(\mu) - \theta_0)^2] \leq \mathbb{E} \left[ \text{sup}_m (\hat{\theta}_\lambda(m) - \theta_0)^2 \right]
\]

\[
= \mathbb{E} \left[ \text{sup}_m (\hat{\theta}_u - \theta_0 + \hat{\lambda}(m)(\hat{\theta}_b(m) - \hat{\theta}_u))^2 \right]
\]

\[
= \mathbb{E} \left[ \text{sup}_m (\hat{\theta}_u - \theta_0)^2 + (\hat{\lambda}(m)(\hat{\theta}_b(m) - \hat{\theta}_u))^2 + 2(\hat{\theta}_u - \theta_0)(\hat{\lambda}(m)(\hat{\theta}_b(m) - \hat{\theta}_u)) \right]
\]

\[
= \mathbb{E} \left[ (\hat{\theta}_u - \theta_0)^2 \right] + \mathbb{E} \left[ \text{sup}_m (\hat{\lambda}(m)(\hat{\theta}_b(m) - \hat{\theta}_u))^2 + 2(\hat{\theta}_u - \theta_0)(\hat{\lambda}(m)(\hat{\theta}_b(m) - \hat{\theta}_u)) \right]
\]

\[
= \sigma_u^2 + \mathbb{E} \left[ \text{sup}_m (\hat{\lambda}(m)(\hat{\theta}_b(m) - \hat{\theta}_u))^2 + 2(\hat{\theta}_u - \theta_0)(\hat{\lambda}(m)(\hat{\theta}_b(m) - \hat{\theta}_u)) \right].
\]

(30)

Our strategy is then to calculate the supremum inside of the expectation, and then give the bound in terms of the remaining parameters. In what follows, we will use \( \theta_b' = \theta_b - \mu. \)

**Lemma 1.** The optimizer \( m^* \) that achieves the supremum in Equation (30) is given by

\[
m^* = \begin{cases} 
(\hat{\theta}_u - \theta_0') + \sqrt{\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}}, & \text{if } (\sigma_u^2 - \sigma_{bu})(\hat{\theta}_u - \theta_0) \geq 0 \\
(\hat{\theta}_u - \theta_0') - \sqrt{\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}}, & \text{if } (\sigma_u^2 - \sigma_{bu})(\hat{\theta}_u - \theta_0) < 0.
\end{cases}
\]

(31)

and the associated optimal value is

\[
\text{sup}_m 2 \hat{\lambda}(m + \theta_b' - \hat{\theta}_u)(\hat{\theta}_u - \theta_0) + \hat{\lambda}^2(m + \theta_b' - \hat{\theta}_u)^2 \geq \frac{\sigma_u^2 - \sigma_{bu}}{\sqrt{\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}}} \left| \frac{\hat{\theta}_u - \theta_0}{\sqrt{\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}}} \right| + \frac{\sigma_b^2 - \sigma_{bu}}{4(\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu})}
\]

(32)
We will prove Lemma 1 after we finish the main argument. Using Lemma 1, we can write Equation (30) as

\[
E\left[\left(\hat{\lambda}(\mu) - \theta_0\right)^2\right] \leq \sigma_u^2 + \frac{\left|\sigma_u^2 - \sigma_{bu}\right|}{\sqrt{\sigma_u^2 + \sigma_b^2} - 2\sigma_{bu}} \left|E\left[\hat{\theta}_u - \theta_0\right]\right| + \frac{(\sigma_u^2 - \sigma_{bu})^2}{4(\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu})}
\]

(33)

Here, we note that by application of Jensen’s inequality, \(E\left[\left|\hat{\theta}_u - \theta_0\right|\right] < \sqrt{E((\hat{\theta}_u - \theta_0)^2)}\), and because \(E[\hat{\theta}_u] = \theta_0\) by assumption, this gives us

\[
E_P\left[\left(\hat{\lambda} - \theta_0\right)^2\right] \leq \sigma_u^2 + \frac{\left|\sigma_u^2 - \sigma_{bu}\right|}{\sqrt{\sigma_u^2 + \sigma_b^2} - 2\sigma_{bu}} \sigma_u + \frac{(\sigma_u^2 - \sigma_{bu})^2}{4(\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu})}
\]

(34)

which we can recognize as \((a + b)^2\) where \(a = \sigma_u\) and \(b = \frac{1}{2} \frac{|\sigma_u^2 - \sigma_{bu}|}{\sqrt{\sigma_u^2 + \sigma_b^2} - 2\sigma_{bu}}\).

To arrive at the desired form, we define the correlation as \(\rho := \sigma_{bu}/\sqrt{\sigma_u^2\sigma_b^2}\), and define the ratio of the standard deviations as \(c = \sqrt{\sigma_b^2/\sigma_u^2}\). If \(\sigma_b^2 = 0\), we define \(\rho = 0\). Note that by Assumption 6, \(\sigma_u^2 > 0\), so \(c\) is always well-defined. This allows us to write

\[
\begin{align*}
\sigma_{bu} &= \sigma_u^2 \rho c \\
\sigma_b^2 &= \sigma_u^2 c^2
\end{align*}
\]

and observe that this allows us to rewrite Equation (34) as

\[
E_P\left[\left(\hat{\lambda} - \theta_0\right)^2\right] \leq \sigma_u^2 + \frac{\left|1 - \rho c\right| \sigma_u}{\sqrt{\sigma_u^2 (1 - 2\rho c + c^2)}} + \frac{(\sigma_u^2 (1 - \rho c))^2}{4\sigma_u^2 (1 - 2\rho c + c^2)}
\]

\[= \sigma_u^2 + \sigma_u^2 \frac{|1 - \rho c|}{\sqrt{1 - 2\rho c + c^2}} + \frac{(1 - \rho c)^2}{4(1 - 2\rho c + c^2)}
\]

\[= \sigma_u^2 \left(1 + \frac{1}{2} \frac{|1 - \rho c|}{\sqrt{1 - 2\rho c + c^2}}\right)^2
\]

(35)

which gives the desired result.

**Proof of Lemma 1:** Because the expression in Equation (30) is a differentiable function of \(\delta\), we will first enumerate all of the stationary points (where the derivative is zero), and then demonstrate that the chosen value achieves the maximum objective value over all such stationary points. We will use the simplifying expressions

\[
\Delta := m + \hat{\theta}_b' - \hat{\theta}_u
\]

\[
S_u := \sigma_u^2 - \sigma_{bu}
\]

\[
S_b := \sigma_b^2 - \sigma_{bu}
\]

\[
\hat{\lambda} = \frac{\sigma_u^2 - \sigma_{bu}}{(\hat{\theta}_b' - \hat{\theta}_u + \delta)^2 + \sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}} = \frac{S_u}{\Delta^2 + S_u + S_b}
\]

21
and we will expand the supremum in Equation (30) to write it as a function of ∆, observing that

\[
\sup_{\text{bias}} 2\lambda (m + \hat{\theta}_b - \hat{\theta}_u) (\hat{\theta}_u - \theta_0) + \lambda^2 (m + \hat{\theta}_b - \hat{\theta}_u)^2 = \sup_{\Delta} 2\lambda \Delta (\hat{\theta}_u - \theta_0) + \lambda^2 \Delta^2
\]  

\[
= \sup_{\Delta} 2 \frac{S_u \Delta (\hat{\theta}_u - \theta_0)}{\Delta^2 + S_u + S_b} + \frac{S_u^2 \Delta^2}{(\Delta^2 + S_u + S_b)^2}
\]

\[
= \sup_{\Delta} \frac{2(\Delta^2 + S_u + S_b)}{(\Delta^2 + S_u + S_b)^2} S_u \Delta (\hat{\theta}_u - \theta_0) + S_u^2 \Delta^2
\]

(36)

(37)

(38)

Before enumerating the stationary points in this expression, we demonstrate that the maximum is attained by a finite value of ∆. In particular, as \(|\Delta| \to \infty\), the entire term goes to zero, as the denominator is \(O(\Delta^4)\) while the numerator is \(O(\Delta^3)\). This justifies the use of the first-order condition to identify local maxima and minima, observing that \(\frac{d}{dm} \Delta = 1\) and \(\frac{d}{dm} S = 0\), and that taking the supremum with respect to ∆ is equivalent to doing so with respect to bias, since ∆ is simply \(m\) plus a fixed offset.

First, we compute the derivative of the numerator and denominator of the expression in Equation (38)

\[
\frac{d}{d\Delta} \left[ 2(\Delta^2 + S_u + S_b) S_u \Delta (\hat{\theta}_u - \theta_0) + S_u^2 \Delta^2 \right] = \frac{d}{d\Delta} \left[ 2(S_u \Delta^3 + S_u^2 \Delta + S_u S_b) (\hat{\theta}_u - \theta_0) + S_u^2 \Delta^2 \right]
\]

\[
= (6S_u \Delta^2 + 2S_u^2 + 2S_u S_b) (\hat{\theta}_u - \theta_0) + 2S_u^2 \Delta
\]

(39)

(40)

and then we compute the derivative of the expression in Equation (38)

\[
\frac{d}{d\Delta} \left[ \frac{2(\Delta^2 + S_u + S_b) S_u \Delta (\hat{\theta}_u - \theta_0) + S_u^2 \Delta^2}{(\Delta^2 + S_u + S_b)^2} \right] = \left( \frac{(\Delta^2 + S_u + S_b)^2}{(\Delta^2 + S_u + S_b)^4} \right) \frac{(\Delta^2 + S_u + S_b) S_u \Delta (\hat{\theta}_u - \theta_0) + S_u^2 \Delta^2}{(\Delta^2 + S_u + S_b)^4}
\]

\[
= \frac{2(\Delta^2 + S_u + S_b) S_u \Delta (\hat{\theta}_u - \theta_0) + S_u^2 \Delta^2}{(\Delta^2 + S_u + S_b)^4} \frac{4(\Delta^2 + S_u + S_b) \Delta}{(\Delta^2 + S_u + S_b)^4},
\]

(41)

(42)

and simplify this expression to find values of ∆ for which this is equal to zero. Note that by Assumption 6, \(S_u + S_b > 0\), so that the denominator term is non-zero, and we can safely remove it (and a similar term in the...
Next, we will show that for finding a global maximum, it suffices to consider Equation (48). In particular, we have

\[ 0 = (Δ^2 + S_u + S_b) \cdot [(6S_uΔ^2 + 2S_u^2 + 2S_uS_b)(\hat{θ}_u - θ_0) + 2S_u^2Δ] \]

\[-[2(Δ^2 + S_u + S_b)S_uΔ(\hat{θ}_u - θ_0) + S_u^2Δ]^2\]

\[ \iff (Δ^2 + S_u + S_b)(3S_uΔ^2 + S_u^2 + S_uS_b)(\hat{θ}_u - θ_0) + (Δ^2 + S_u + S_b)S_u^2Δ \]

\[-4(Δ^2 + S_u + S_b)S_uΔ^2(\hat{θ}_u - θ_0) - 2S_u^3Δ^3 = 0 \]

\[ \iff (Δ^2 + S_u + S_b)(S_u + S_b - Δ^2)S_u(\hat{θ}_u - θ_0) + (Δ^2 + S_u + S_b)S_u^2Δ - 2S_u^2Δ^3 = 0 \]

\[ \iff (S_u + S_b - Δ^2)S_u(\hat{θ}_u - θ_0) + S_u^2Δ = \frac{2S_u^2Δ^3}{Δ^2 + S_u + S_b} \]

\[ \iff (S_u + S_b - Δ^2)S_u(\hat{θ}_u - θ_0) = \frac{2S_u^2Δ^3 - (S_u^2Δ)(Δ^2 + S_u + S_b)}{Δ^2 + S_u + S_b} \]

\[ \iff (S_u + S_b - Δ^2)S_u(\hat{θ}_u - θ_0) = \frac{S_u^2Δ^3 - S_u^2Δ - S_u^3ΔS_u}{Δ^2 + S_u + S_b} \]

\[ \iff (S_u + S_b - Δ^2)S_u(\hat{θ}_u - θ_0) = -S_u^2Δ \frac{-Δ^2 + S_u + S_b}{Δ^2 + S_u + S_b} \]

\[ \iff S_u(\hat{θ}_u - θ_0) = \frac{-S_u^2Δ}{Δ^2 + S_u + S_b} \quad \text{(If } Δ^2 \neq S_u + S_b) \]

\[ \iff (Δ^2 + S_u + S_b)S_u(\hat{θ}_u - θ_0) + S_u^2Δ = 0 \quad \text{(47)} \]

In Equation (43) we divide by two and distribute terms, and in Equation (44) we collect terms involving \( \hat{θ}_u - θ_0 \) before simplifying further. Equation (45) reveals that \( Δ^2 = S_u + S_b \) is a stationary point, and Equation (46) implicitly defines another set of stationary points. Any stationary point is a solution to one of the following.

\[ Δ^2 = S_u + S_b \quad \text{(48)} \]

Next, we will show that for finding a global maximum, it suffices to consider Equation (48). In particular, we demonstrate that when we plug these conditions into the original expression from Equation (38),

\[ \frac{2(Δ^2 + S_u + S_b)S_uΔ(\hat{θ}_u - θ_0) + S_u^2Δ^2}{(Δ^2 + S_u + S_b)^2} = \begin{cases} \leq 0, & \text{if } Δ \text{ satisfies Eq. (47)} \\ \pm 4S_u(\sqrt{S_u + S_b}(\hat{θ}_u - θ_0) + S_u^2), & \text{if } Δ \text{ satisfies Eq. (48)} \end{cases} \]

where the stationary points satisfying Equation (48) always include a non-negative solution, while those satisfying Equation (47) are always non-positive. We prove both of these points below.

**Solutions satisfying Equation (47):** The solutions implied by Equation (47) satisfy

\[ (Δ^2 + S_u + S_b)S_u(\hat{θ}_u - θ_0) + S_u^2Δ = 0 \quad \text{(49)} \]

which includes the solution \( Δ = 0 \) when \( \hat{θ}_u = θ_0 \), in which case the value of the optimization objective is zero. When \( \hat{θ}_u \neq θ_0 \), we have it that \( Δ \neq 0 \) is no longer a solution. We can, however, use Equation (47) to observe that \((Δ^2 + S_u + S_b)S_u(\hat{θ}_u - θ_0) = -S_u^2Δ\), which implies that the value of the objective is given by

\[ \frac{2(Δ^2 + S_u + S_b)S_uΔ(\hat{θ}_u - θ_0) + S_u^2Δ^2}{(Δ^2 + S_u + S_b)^2} = \frac{2Δ(-S_u^2Δ) + S_u^2Δ^2}{(Δ^2 + S_u + S_b)^2} = \frac{-S_u^2Δ^2}{(Δ^2 + S_u + S_b)^2} \leq 0 \quad \text{(50)} \]
which is equal to zero if and only if $S_u = 0$, in which case $\hat{\lambda}$ is zero. Because this expression is non-positive, it is not a global maximum, since there exist solutions that are positive (see below).

**Solutions satisfying Equation (48):** When $\Delta^2 = S_u + S_b$, we see that Equation (38) is equal to

$$\frac{\pm 4(S_u + S_b)S_u(\sqrt{S_u + S_b})(\hat{\theta}_u - \theta_0) + S_u^2(S_u + S_b)}{4(S_u + S_b)^2} = \frac{\pm 4S_u(\sqrt{S_u + S_b})(\hat{\theta}_u - \theta_0) + S_u^2}{4(S_u + S_b)}$$

which is maximized by taking the absolute value of the first term, choosing $\Delta = \sqrt{S_u + S_b}$ if $S_u(\hat{\theta}_u - \theta_0) > 0$ and $\Delta = -\sqrt{S_u + S_b}$ if $S_u(\hat{\theta}_u - \theta_0) < 0$. If $\hat{\theta}_u = \theta_0$ the choice of sign is irrelevant, and can be chosen arbitrarily. This will always yield a non-negative solution, given by

$$\frac{4|S_u| (\sqrt{S_u + S_b}) |\hat{\theta}_u - \theta_0| + S_u^2}{4(S_u + S_b)}$$

(51)

which yields the claimed result, that the supremum is given by

$$\frac{|\sigma_u^2 - \sigma_{bu}| |\hat{\theta}_u - \theta_0|}{\sqrt{\sigma_u^2 + \sigma_b^2} - 2\sigma_{bu}} + \frac{(\sigma_u^2 - \sigma_{bu})^2}{4(\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu})}$$

(52)

This completes the proof of Lemma 1, and the proof of the main result follows.

**Theorem 3 (Bound on MSE, unknown variance).** Under Assumptions 5 and 6, with estimators $\hat{\sigma}_b^2, \hat{\sigma}_u^2, \hat{\sigma}_{bu}$ that have bounded second moments, the MSE of the estimator $\hat{\lambda}$ is bounded by

$$\mathbb{E}[(\hat{\lambda} - \lambda)^2] \leq \left(\sigma_u + \frac{1}{2}\sqrt{\mathbb{E}[S^2]}\right)^2$$

where $\mathbb{E}[S^2] = \mathbb{E}[(\hat{\sigma}_u^2 - \hat{\sigma}_{bu})^2/(\hat{\sigma}_u^2 + \hat{\sigma}_b^2 - 2\hat{\sigma}_{bu})]$.  

**Proof.** The proof starts from the same argument as Theorem 2, which yields that the MSE is bounded by

$$\sigma_u^2 + \mathbb{E} \left[ \sup_m 2\hat{\lambda}(m)(m + \hat{\sigma}_b^2 - \hat{\sigma}_u)(\hat{\theta}_u - \theta_0) + \hat{\lambda}(m)(m + \hat{\sigma}_b^2 - \hat{\sigma}_u)^2 \right].$$

(53)

where $\hat{\lambda}(m)$ now depends on the additional (random) terms $\hat{\sigma}_u^2, \hat{\sigma}_b^2, \hat{\sigma}_{bu}$. Lemma 1 can be immediately extended to this setting, replacing all references to $\sigma_u^2$ with $\hat{\sigma}_u^2$ and likewise for $\sigma_b^2, \sigma_{bu}$. This follows from the fact that Lemma 1 considers a supremum that occurs within the expectation, where $\hat{\sigma}_u^2$ and other estimated quantities can be taken as fixed values. This immediately yields the general result that

$$\mathbb{E}[(\hat{\lambda} - \lambda)^2] \leq \sigma_u^2 + \mathbb{E} \left[ \frac{|\hat{\sigma}_u^2 - \hat{\sigma}_{bu}| |\hat{\theta}_u - \theta_0|}{\sqrt{\sigma_u^2 + \hat{\sigma}_b^2 - 2\sigma_{bu}}} + \frac{(\hat{\sigma}_u^2 - \hat{\sigma}_{bu})^2}{4(\sigma_u^2 + \hat{\sigma}_b^2 - 2\sigma_{bu})} \right]$$

(54)
which can be refined by a change of notation, writing \( S := \frac{|\sigma^2 - \hat{\sigma}_{bu}|}{\sqrt{\hat{\sigma}^2_k + \sigma^2 - 2\sigma_{bu}}}, \) and observing that this yields

\[
E[(\hat{\lambda} - \theta_0)^2] \leq \sigma^2_u + E\left[S | \hat{\theta}_u - \theta_0| + \frac{S^2}{4}\right] = \sigma^2_u + E\left[S | \hat{\theta}_u - \theta_0| + \frac{1}{4}E[S^2]\right] \leq \sigma^2_u + \sqrt{E[S^2]}E\left[(\hat{\theta}_u - \theta_0)^2\right] + \frac{1}{4}E[S^2] = \sigma^2_u + \sqrt{E[S^2]}\sigma_u + \frac{1}{4}E[S^2] = \left(\sigma_u + \frac{1}{2}\sqrt{E[S^2]}\right)^2
\]

where \( E[S^2] = E[(\hat{\sigma}^2 - \hat{\sigma}_{bu})^2/(\sigma^2 + \sigma^2 - 2\sigma_{bu})]. \)

\( \square \)

**Proposition 1.** Consider a sequence of biased estimators \( \hat{\theta}_b^{(k)} \) which can be written as \( \hat{\theta}_b + \mu_k \), where \( E[\hat{\theta}_b] = \theta_0 \), where \( \text{Cov}(\hat{\theta}_b^{(k)}, \hat{\theta}_u) = \sigma_{bu} \), and \( \text{Var}(\hat{\theta}_b^{(k)}) = \sigma_b^2 \). Let \( k \to \infty \) as \( k \to \infty \). The MSE of the resulting sequence of estimators \( \hat{\theta}_\lambda^{(k)} \) converges to the MSE of the unbiased estimator \( \hat{\theta}_u \)

\[
\lim_{k \to \infty} E\left[(\hat{\theta}_\lambda^{(k)} - \theta_0)^2\right] = \sigma^2_u
\]

where \( \hat{\theta}_\lambda^{(k)} = \hat{\lambda}\hat{\theta}_b^{(k)} + (1-\hat{\lambda})\hat{\theta}_u \), and where \( \hat{\lambda} = (\sigma_u^2 - \sigma_{bu})/(\hat{\theta}_u - \hat{\theta}_b)^2 + \sigma_u^2 + \sigma_b^2 - 2\sigma_{bu} \).

**Proof.** First, we prove that the estimator \( \hat{\theta}_\lambda \) converges almost surely to the unbiased estimator \( \hat{\theta}_u \) as \( k \to \infty \)

\[
\hat{\theta}_\lambda^{(k)} \xrightarrow{a.s.} \hat{\theta}_u,
\]

which implies almost-sure convergence of the squared error

\[
\left(\hat{\theta}_\lambda^{(k)} - \theta_0\right)^2 \xrightarrow{a.s.} \left(\hat{\theta}_u - \theta_0\right)^2.
\]

Once we have established this, use the dominated convergence theorem to give the desired result.

**Almost-Sure Convergence:** Let \( \Omega \) denote the sample space, such that \( \hat{\theta}_u(\omega), \hat{\theta}_b(\omega) \) are the realized values of \( \hat{\theta}_u, \hat{\theta}_b \) for the event \( \omega \in \Omega \). For any realization of \( \hat{\theta}_u(\omega), \hat{\theta}_b(\omega) \), we will show that the estimator \( \hat{\theta}_\lambda^{(k)}(\omega) \) converges to \( \hat{\theta}_u(\omega) \) as \( k \to \infty \). In the sequel, we will drop the \( \omega \) for simplicity of presentation, and consider any realization of \( \hat{\theta}_u, \hat{\theta}_b \), and write \( \hat{\theta}_\lambda^{(k)} \) as the value \( \hat{\theta}_b + \mu_k \)

\[
\hat{\theta}_\lambda = \frac{(\hat{\theta}_b + \mu_k) \cdot (\sigma_u^2 - \sigma_{bu})}{(\hat{\theta}_u - \hat{\theta}_b - \mu_k)^2 + \sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}} + \hat{\theta}_u \cdot \frac{((\hat{\theta}_u - \hat{\theta}_b - \mu_k)^2 + \sigma_u^2 - \sigma_{bu})}{(\hat{\theta}_u - \hat{\theta}_b - \mu_k)^2 + \sigma_u^2 + \sigma_b^2 - 2\sigma_{bu}}.
\]
where we can see that as $\mu_k \to \infty$, the first term goes to zero, given $\mu_k^2$ in the denominator and $\mu_k$ in the numerator. Meanwhile, the second term converges to $\hat{\theta}_u$, as $\lambda$ converges to 0, so that $(1 - \lambda)\hat{\theta}_u$ converges to $\hat{\theta}_u$

$$\lim_{k \to \infty} \frac{(\hat{\theta}_u - \hat{\theta}_i' - \mu_k)^2 + \sigma_b^2 - \sigma_{bu}}{(\hat{\theta}_u - \hat{\theta}_i' - \mu_k)^2 + \sigma^2_u + \sigma_b^2 - 2\sigma_{bu}} = 1$$ (57)

**Dominated Convergence Theorem:** Here we apply the dominated convergence theorem, by defining a random variable $Z$ such that $(\hat{\theta}_i^{(k)} - \theta_0)^2 \leq Z$ almost surely, and where $\mathbb{E}[Z] < \infty$. First, we can observe that $(\hat{\theta}_i^{(k)} - \theta_0)^2$ is a function of $\mu_k$, and is upper bounded by the supremum over all possible values of $\mu_k \in \mathbb{R}$. In the following, we represent this by replacing $\mu_k$ with the value $\delta$.

$$(\hat{\theta}_i^{(k)} - \theta_0)^2 = (\hat{\theta}_u - \theta_0)^2 + 2\lambda(\hat{\theta}_i' + \mu_k - \hat{\theta}_u)(\hat{\theta}_u - \theta_0) + \lambda^2(\hat{\theta}_i' + \mu_k - \hat{\theta}_u)^2 \leq \sup_{\delta \in \mathbb{R}} (\hat{\theta}_u - \theta_0)^2 + 2\lambda(\hat{\theta}_i' + \delta - \hat{\theta}_u)(\hat{\theta}_u - \theta_0) + \lambda^2(\hat{\theta}_i' + \delta - \hat{\theta}_u)^2$$

The first term does not depend on $\delta$, and by Lemma 1, the remainder is maximized by taking $\delta^*$ as defined in Equation (31). This yields that

$$(\hat{\theta}_i^{(k)} - \theta_0)^2 \leq (\hat{\theta}_u - \theta_0)^2 + \frac{\sigma^2_u - \sigma_{bu}}{\sqrt{\sigma^2_u + \sigma_b^2 - 2\sigma_{bu}}} \cdot \left| \hat{\theta}_u - \theta_0 \right| + \frac{(\sigma_u^2 - \sigma_{bu})}{4(\sigma_u^2 + \sigma_b^2 - 2\sigma_{bu})}$$

the right-hand side is a random variable that does not depend on $\mu_k$, and it has a finite expectation, as shown in Theorem 2. This completes the proof.

**B Comparison to Cheng & Cai (2021)**

Cheng & Cai (2021) take a similar approach to taking an adaptive linear combination of observational and experimental estimators. We focus on a broader class of estimation problems that involve estimating a real-valued parameter, including the use of surrogate outcomes (as described in Section 2), while they focus on CATE estimation with kernel regression, in the context of combining experimental and trial estimators. However, while they approach a specific problem setting, their approach aligns with ours on a conceptual level, and is nearly equivalent in the setting of ATE estimation via combination of experimental and observational data, with the exception of an additional hyperparameter that they introduce, scaling the estimated bias by a factor of $n^{-\beta}$.

Here, we give the approach of that work, in the context of ATE estimation, combining observational and experimental data. We focus on the ATE in the trial population, which they denote as $\tau_0(\mathbf{v}) = \mathbb{E}[Y_1 - Y_0 \mid \mathbf{V} = \mathbf{v}, Z = 0]$, where $Z = 0$ denotes the trial population and $Z = 1$ denotes the observational population, and $\mathbf{V}$ denotes a set of covariates. We use the notation $\tau$ instead of $\tau(\mathbf{v})$ because there is no conditioning set for the ATE, and we will take $\tau_0$ to be the target of inference. In this case, Equation 8 of Cheng & Cai (2021) becomes

$$\hat{\tau} = \hat{\tau}^r + \eta(\hat{\tau}^o - \hat{\tau}^r)$$ (58)

and the goal is to estimate the optimal value of $\eta$ from data. Here, Cheng & Cai (2021) consider standard doubly-robust (DR) pseudo-outcomes for the treatment effect (see Equation 10 of Cheng & Cai (2021)), which
are denoted as \( \hat{\Psi}^r \) for the outcomes based on the trial data, and \( \hat{\Psi}^o \) for outcomes based on the observational data. Here we use \( \hat{\Psi} \) to denote the pseudo-outcome when we use a plug-in estimate of nuisance parameters, and \( \hat{\Psi} \) to denote the pseudo-outcome when we plug in the true values of the nuisance parameters.

Cheng & Cai (2021) consider locally constant kernel regression for estimation of CATE, which is unnecessary for ATE. As a result, under the simplifying assumption that the distribution of \( X \) is the same across the trial and observational study,\(^7\)

\[
\hat{\tau} = n^{-1} \sum_{i=1}^{n} \frac{1 \{ Z_i = 0 \}}{P(Z = 0)} \hat{\Psi}^r
\]

where \( \hat{P}(Z = 0) = n^{-1} \sum_{i=1}^{n} 1 \{ Z_i = 0 \} \) is the empirical estimate of the proportion of the total dataset in the trial, with an analogous estimator for the observational data (see Equations 11–12 of Cheng & Cai (2021)). Here, it is assumed throughout that this probability is bounded away from zero, so this work excludes the case where the number of observational samples is of a different asymptotic order than the number of trial samples.

With this in mind, both estimators can be written with an asymptotically linear representation as follows, where \( \hat{\tau} \) is used as the asymptotic limit of an estimator \( \hat{\tau} \)

\[
\sqrt{n}(\hat{\tau} - \tau^r) = n^{-1/2} \sum_{i=1}^{n} \frac{1 \{ Z_i = 0 \}}{P(Z = 0)} \left( \hat{\Psi}^r - \hat{\tau}^r \right) + o_p(1),
\]

and likewise for \( \hat{\tau}^o \), replacing the superscript \( r \) with \( o \), and \( Z = 0 \) with \( Z = 1 \). Here, the term \( \xi^r \) is the influence function. Per Lemma 5, the MSE for the target parameter \( \tau^r \) can be written as

\[
E \left[ (\hat{\tau} - \tau^r)^2 \right] = n^{-1}E \left[ (\xi^r_\tau - \eta(\xi^r_\tau - \xi^o_\tau))^2 \right] + \eta^2(\tau^o - \tau^r)^2 + o(n^{-1/2}),
\]

which suggests the following scaled empirical criterion (multiplying by \( n^2 \)), for estimating \( \eta \),

\[
\hat{Q}(\eta) = \sum_{i=1}^{n} \left( \hat{\xi}^r_\tau - \eta(\hat{\xi}^r_\tau - \hat{\xi}^o_\tau) \right)^2 + \eta^2 n^{2-\beta}(\tau^o - \tau^r)^2
\]

where \( \hat{\eta} = \arg\min_{\eta \in \mathbb{R}} \hat{Q}(\eta) \), and where the empirical influence functions are estimated via plug-in, where e.g., \( \hat{\xi}^r_\tau = \frac{1 \{ Z_i=0 \}}{\hat{P}(Z=0)} (\hat{\Psi}^r_i - \hat{\tau}^r) \). Here, the solution is given by the following

\[
\eta = \frac{E_n \left[ (\hat{\xi}^r_\tau)^2 \right] - E_n[\hat{\xi}^r_\tau \hat{\xi}^o_\tau]}{E_n[(\hat{\xi}^r_\tau - \hat{\xi}^o_\tau)^2] + n^{1-\beta}(\tau^o - \tau^r)^2}
\]

where \( E_n[\cdot] \) is the empirical average.

**Connection to our approach:** Based on the asymptotically linear form of these estimators, we can write our estimators of the variance and covariance of each estimator as

\[
\hat{\sigma}_\tau^2 := \frac{1}{n} E_n \left[ (\hat{\xi}^r_\tau)^2 \right] \quad \hat{\sigma}_o^2 := \frac{1}{n} E_n \left[ (\hat{\xi}^o_\tau)^2 \right] \quad \hat{\sigma}_{bo} := \frac{1}{n} E_n \left[ \hat{\xi}^r_\tau \hat{\xi}^o_\tau \right]
\]

\(^7\)This avoids the need for the weights \( \omega(X) \) in their equations, but this is only for the sake of notational simplicity here.
and observe that the proposed estimator of Cheng & Cai (2021), adapted to the setting of ATE estimation, would be equal to a similar affine combination as the one we present in this work, with weights

$$\eta = \frac{\hat{\sigma}_u^2 - \hat{\sigma}_{bu}}{n^{-\beta}(\hat{\tau}^r - \hat{\tau}^o)^2 + \hat{\sigma}_u^2 + \hat{\sigma}_b^2 - 2\hat{\sigma}_{bu}}$$

which differs from our choice of $\hat{\lambda}$ in the $n^{-\beta}$ term in the denominator.

C Additional Experimental Details

C.1 Baselines

Hypothesis Testing We give the methodology of Yang et al. (2020) in full generality here, before discussing how it applies to our setting. They suppose that there exists some score function $S_\psi(V)$, where $\psi$ is the parameter of interest and $V$ denotes observed data, where $\delta = 0$ corresponds to the observational data and $\delta = 1$ corresponds to the randomized data. Let there be $m$ samples in the randomized data, denoted $\mathcal{A}$, and $n$ samples in the observational data, denoted $\mathcal{B}$. Solving for $\psi$ requires solving the moment condition $E[S_\psi(V)] = 0$. The simplest example of such a score function approach is estimation of the mean of $V$, where $S_\psi(V) = V - \psi$, and solving for $\psi$ is simply given by observing that $E_n[V] = \psi$.

The core approach is to construct a statistic for testing whether or not $S_\psi(V)$ has the same average value in unbiased randomized trial data, versus in the potentially biased observational (or “real world”) data. The first step in constructing their test statistic is to estimate the parameter from the randomized data, denoting this estimate as $\hat{\psi}_{rt} = \hat{\theta}_u$, and then evaluate the score on the real-world data, giving

$$n^{-1/2} \sum_{i \in \mathcal{B}} \tilde{S}_{rw,\hat{\psi}_{rt}}(V_i) = \sqrt{n}(\hat{\theta}_u - \hat{\theta}_b)$$

(63)

which is then used to construct the test statistic (see Equation 7 of Yang et al. (2020)) as

$$T_n = \frac{(\hat{\theta}_u - \hat{\theta}_b)^2}{n\hat{\sigma}}$$

(64)

where $\hat{\sigma}$ is a consistent estimate of the asymptotic variance of $\sqrt{n}(\hat{\theta}_u - \hat{\theta}_b)$. In our setting, this asymptotic variance is given by an estimate of $\sigma_u^2 + \hat{\sigma}_b - 2\sigma_{bu}$. This test statistic converges in distribution to a chi-square random variable under the null hypothesis that no bias exists. With this in mind, their estimator can be represented as follows

$$\sum_{i \in \mathcal{A} \cup \mathcal{B}} \left\{ \delta_i \tilde{S}_{\psi}(V_i) + 1 \{T_n < c_\gamma\} (1 - \delta_i) \tilde{S}_{\psi}(V_i) \right\} = 0$$

(65)

where if $T_n \geq c_\gamma$, this reduces to using $\hat{\theta}_u$, and otherwise this reduces to pooling the data and taking a global average of $\hat{\theta}_u, \hat{\theta}_b$, weighted by sample size. In our experiments, $\hat{\theta}_u, \hat{\theta}_b$ have the same sample size, so this is just a simple average of $\hat{\theta}_u, \hat{\theta}_b$.

The asymptotic bias and MSE of this estimator (for a given threshold $c_\gamma$) depends on the underlying bias of the observational estimator. Yang et al. (2020) derive an analytical formula for these terms (see Corollary 1 of
Yang et al. (2020)), and note that $c_\gamma$ can be tuned by first estimating the bias, plugging this into these formula, and choosing a threshold that minimizes the resulting MSE. More practically, they suggest estimating the bias using a plug-in estimate, specifying a grid of values for the significance level $\gamma$, and simulating from the limiting mixture distribution to identify the significance level that minimizes the MSE.

In our experiments, we implement this data-driven selection of the hyperparameter as follows: For each setting of parameters in Table 2, we simulate performance of this approach for a grid of significance levels $\gamma \in \{0, 0.05, 0.10, \ldots, 0.95, 1\}$. For each value of the bias $\mu \in [0, 1.5]$, we record the threshold which yields minimum $\text{MSE}$. Then, we re-run the simulations, where we first estimate the bias as $\hat{\mu} = \text{sign}(\hat{\theta}_b - \hat{\theta}_u) \left( \frac{\hat{\theta}_b - \hat{\theta}_u}{\sqrt{\text{Var}(\hat{\theta}_b - \hat{\theta}_u)}} \right)$, and then look up the optimal cutoff based on our prior simulations.

**Anchored Thresholding** Given an unbiased estimate $\hat{\theta}_u$ and a biased estimate $\hat{\theta}_b$, Chen et al. (2021) always combine the estimators, but they first apply a bias correction to $\hat{\theta}_b$. In particular, they apply soft-thresholding to estimate the bias, where

$$\hat{\mu} = \begin{cases} 
\text{sign}(\hat{\theta}_b - \hat{\theta}_u) \left( \frac{\hat{\theta}_b - \hat{\theta}_u}{\sqrt{\text{Var}(\hat{\theta}_b - \hat{\theta}_u)}} \right), & \text{if } |\hat{\theta}_b - \hat{\theta}_u| \geq \lambda \cdot \sqrt{\text{Var}(\hat{\theta}_b - \hat{\theta}_u)} \\
0, & \text{otherwise.}
\end{cases} \tag{66}$$

This estimated bias is used to “correct” $\hat{\theta}_b$ by replacing it with $\hat{\theta}_b - \hat{\mu}$. At this stage, the estimators are combined on the assumption that both are unbiased, with the combination (in our setting) given by

$$\hat{w}(\hat{\theta}_b - \hat{\mu}) + (1 - \hat{w})\hat{\theta}_u \tag{67}$$

where

$$\hat{w} = \frac{\hat{\sigma}_u^2 - \hat{\sigma}_{bu}}{\hat{\sigma}_u^2 + \hat{\sigma}_b^2 - 2\hat{\sigma}_{bu}}. \tag{68}$$

In this setting $\lambda$ is a hyperparameter, which should be of asymptotic order $\lambda \approx \sqrt{\log n}$. In their experiments, they choose a constant $\lambda_1 = 0.5$ and then set $\lambda = \lambda_1 \cdot \sqrt{\log n}$, so we do the same.

### C.2 SPRINT Simulation

**Details on Generative Model**: The generative model for potential outcomes in the simulated RCT can be described as follows, consistent with data reported in SPRINT Research Group et al. (2015).

\[
\begin{align*}
P(Y_1 = 1 \mid U = 1) &= 0.081 & P(Y_1 = 1 \mid U = 0) &= 0.040 \\
P(Y_0 = 1 \mid U = 1) &= 0.096 & P(Y_0 = 1 \mid U = 0) &= 0.057
\end{align*}
\]

Here, the p-value for a heterogeneous treatment effect was not significant ($p = 0.32$), but $U = 1$ has a strong marginal association with the primary outcome. The marginal rate of $U$ in the RCT is $1330 + 1316 / 9361 \approx 28\%$, which we use as our incidence of $U$ across both the simulated RCT and simulated observational study.

\[P(U = 1) = 0.28.\]
Figure 7: An illustrative setup (same parameter settings as Figure 3a), where we have added $\hat{\theta}_S$, the shrinkage estimator of Rosenman et al. (2020) without clipping $\hat{\lambda}$ to lie in $[0, 1]$, to compare to $\hat{\theta}_{SC}$, the clipped variant presented in Figure 3a. For small values of the bias, estimating $\hat{\lambda}$ as $\hat{\sigma}_u^2/(\hat{\theta}_u - \hat{\theta}_b)^2$ can result in extremely large values of $\hat{\lambda}$, which in turn contributes to much larger values of mean-squared error.

Details on Variance Estimation: The variance of each estimator is estimated as

\[
\hat{\sigma}_u^2 = \frac{1}{n_{\text{exp}}^2} \sum_{i:D_i=E} \left( Y_i - \hat{\mu}_E(T_i) \right) \left( \frac{T_i}{\hat{e}_E} - \frac{(1 - T_i)}{1 - \hat{e}_E} \right) + (\hat{\mu}_E(1) - \hat{\mu}_E(0)) - \hat{\theta}_a \right)^2
\]

\[
\hat{\sigma}_b^2 = \frac{1}{n_{\text{obs}}^2} \sum_{i:D_i=O} \left( Y_i - \hat{\mu}_O(T_i) \right) \left( \frac{T_i}{\hat{e}_O} - \frac{(1 - T_i)}{1 - \hat{e}_O} \right) + \hat{\mu}_O(1) - \hat{\mu}_O(0) - \hat{\theta}_b \right)^2
\]

where $\hat{\mu}_d(t) = (\sum_i 1 \{D_i = d, T_i = t\})^{-1} \sum_i Y_i 1 \{D_i = d, T_i = t\}$ is the empirical mean in treatment arm $t$ in dataset $d$.

D Additional Experimental Results

In this section we present some additional empirical results. Figure 7 demonstrates that the shrinkage estimator of Rosenman et al. (2020), when applied to our one-dimensional setting, has poor performance without clipping the weights. Table 4 gives the full version of Table 3.
References

Athey, S., Chetty, R., Imbens, G. W., and Kang, H. The surrogate index: Combining Short-Term proxies to estimate Long-Term treatment effects more rapidly and precisely. *NBER Working Paper Series*, (26463), November 2019.

Bates, J. M. and Granger, C. W. J. The combination of forecasts. *The Journal of the Operational Research Society*, 20(4):451–468, December 1969.

Chen, S., Zhang, B., and Ye, T. Minimax rates and adaptivity in combining experimental and observational data. *arXiv preprint (2109.10522)*, September 2021.

Cheng, D. and Cai, T. Adaptive combination of randomized and observational data. *arXiv preprint (2111.15012)*, November 2021.

Dahabreh, I. J., Robertson, S. E., Steingrimsson, J. A., Stuart, E. A., and Hernán, M. A. Extending inferences from a randomized trial to a new target population. *Statistics in medicine*, 39(14):1999–2014, June 2020.

Degtiar, I. and Rose, S. A review of generalizability and transportability. *arXiv preprint (2102.11904)*, February 2021.

Rosenbaum, P. R. *Observational Studies*. Springer New York, December 2010.

Rosenman, E., Basse, G., Owen, A., and Baiocchi, M. Combining observational and experimental datasets using shrinkage estimators. *arXiv preprint (2002.06708)*, February 2020.

SPRINT Research Group, Wright, Jr, J. T., Williamson, J. D., Whelton, P. K., Snyder, J. K., Sink, K. M., Rocco, M. V., Reboussin, D. M., Rahman, M., Oparil, S., Lewis, C. E., Kimmel, P. L., Johnson, K. C., Goff, Jr, D. C., Fine, L. J., Cutler, J. A., Cushman, W. C., Cheung, A. K., and Ambrosius, W. T. A randomized trial of intensive versus standard Blood-Pressure control. *The New England journal of medicine*, 373(22):2103–2116, November 2015.

Stein, C. M. Estimation of the mean of a multivariate normal distribution. *The Annals of Statistics*, 9(6):1135–1151, November 1981.

Strawderman, W. E. On minimax estimation of a normal mean vector for general quadratic loss. In *Institute of Mathematical Statistics Lecture Notes - Monograph Series*, Lecture notes-monograph series, pp. 3–14. Institute of Mathematical Statistics, Beachwood, OH, 2003.

Yang, S. and Ding, P. Combining multiple observational data sources to estimate causal effects. *Journal of the American Statistical Association*, 115(531):1540–1554, 2020.

Yang, S., Zeng, D., and Wang, X. Elastic integrative analysis of randomized trial and Real-World data for treatment heterogeneity estimation. *arXiv preprint (2005.10579)*, May 2020.
Table 4: The RMSE of the combination estimator over selected values of $\gamma$. For legibility, the RMSE is multiplied by 1000, on which scale the RMSE of the unbiased estimator is 4.97. For each sample size we **bold** the largest value which remains below the RMSE of the unbiased estimator.

| $\gamma$ | 10k  | 20k  | 50k  | 100k |
|----------|------|------|------|------|
| 0.00     | 4.24 | 3.98 | 3.72 | 3.59 |
| 0.05     | 4.24 | 3.99 | 3.72 | 3.60 |
| 0.10     | 4.24 | 4.00 | 3.74 | 3.62 |
| 0.15     | 4.25 | 4.01 | 3.77 | 3.66 |
| 0.20     | 4.26 | 4.04 | 3.81 | 3.71 |
| 0.25     | 4.28 | 4.07 | 3.86 | 3.77 |
| 0.30     | 4.30 | 4.10 | 3.92 | 3.83 |
| 0.35     | 4.32 | 4.15 | 3.99 | 3.91 |
| 0.40     | 4.35 | 4.19 | 4.06 | 4.00 |
| 0.45     | 4.38 | 4.24 | 4.14 | 4.08 |
| 0.50     | 4.41 | 4.29 | 4.22 | 4.18 |
| 0.55     | 4.44 | 4.35 | 4.30 | 4.27 |
| 0.60     | 4.48 | 4.41 | 4.38 | 4.37 |
| 0.65     | 4.52 | 4.47 | 4.47 | 4.46 |
| 0.70     | 4.55 | 4.53 | 4.55 | 4.55 |
| 0.75     | 4.59 | 4.59 | 4.63 | 4.64 |
| 0.80     | 4.63 | 4.65 | 4.71 | 4.73 |
| 0.85     | 4.67 | 4.70 | 4.78 | 4.81 |
| 0.90     | 4.71 | 4.76 | 4.85 | 4.89 |
| 0.95     | 4.74 | 4.82 | **4.92** | **4.97** |
| 1.00     | 4.78 | 4.87 | 4.99 | 5.04 |
| 1.05     | 4.82 | 4.92 | 5.05 | 5.10 |
| 1.10     | 4.85 | **4.97** | 5.10 | 5.16 |
| 1.15     | 4.88 | 5.01 | 5.15 | 5.21 |
| 1.20     | 4.91 | 5.05 | 5.20 | 5.26 |
| 1.25     | **4.94** | 5.09 | 5.24 | 5.31 |
| 1.30     | 4.98 | 5.13 | 5.28 | 5.34 |
| 1.35     | 5.00 | 5.16 | 5.31 | 5.38 |
| 1.40     | 5.03 | 5.19 | 5.34 | 5.41 |
| 1.45     | 5.05 | 5.22 | 5.36 | 5.43 |
| 1.50     | 5.07 | 5.24 | 5.39 | 5.45 |
| 1.55     | 5.10 | 5.27 | 5.40 | 5.47 |
| 1.60     | 5.11 | 5.29 | 5.42 | 5.48 |
| 1.65     | 5.13 | 5.30 | 5.43 | 5.49 |
| 1.70     | 5.15 | 5.32 | 5.44 | 5.49 |
| 1.75     | 5.17 | 5.33 | 5.44 | 5.50 |
| 1.80     | 5.18 | 5.34 | 5.45 | 5.50 |
| 1.85     | 5.19 | 5.35 | 5.45 | 5.49 |
| 1.90     | 5.20 | 5.35 | 5.45 | 5.49 |
| 1.95     | 5.21 | 5.36 | 5.45 | 5.49 |
| 2.00     | 5.22 | 5.36 | 5.45 | 5.48 |