Localization via Automorphisms of the CARs.
Local gauge invariance

HENDRIK GRUNDLING  
Department of Mathematics,  
University of New South Wales,  
Sydney, NSW 2052, Australia.  
hendrik@maths.unsw.edu.au  
FAX: +61-2-93857123

KARL–HERMANN NEEB  
Fachbereich Mathematik,  
Technische Universität Darmstadt,  
Schlossgartenstrasse 7,  
D–64289 Darmstadt Germany.  
neeb@mathematik.tu-darmstadt.de

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Abstract

The classical matter fields are sections of a vector bundle $E$ with base manifold $M$, and the space $L^2(E)$ of square integrable matter fields w.r.t. a locally Lebesgue measure on $M$, has an important module action of $C^\infty_b(M)$ on it. This module action defines restriction maps and encodes the local structure of the classical fields. For the quantum context, we show that this module action defines an automorphism group on the algebra of the canonical anticommutation relations, $\text{CAR}(L^2(E))$, with which we can perform the analogous localization. That is, the net structure of the $\text{CAR}(L^2(E))$ w.r.t. appropriate subsets of $M$ can be obtained simply from the invariance algebras of appropriate subgroups. We also identify the quantum analogues of restriction maps, and as a corollary, we prove a well-known “folk theorem,” that the $\text{CAR}(L^2(E))$ contains only trivial gauge invariant observables w.r.t. a local gauge group acting on $E$.

1 Introduction

Classically, a matter field $\Psi$ on spacetime $M$ is a smooth section of an appropriate smooth vector bundle $q_E : E \to M$ with typical fiber being a finite dimensional vector space $V$ over $F \in \{\mathbb{R}, \mathbb{C}\}$. The space $\Gamma(E)$ of smooth sections of $E$ is a module for the action of pointwise multiplication with $C^\infty(M) := C^\infty(M, F)$, and this module action encodes the local structure of the fields. In particular, for an open set $U \subset M$ the submodule carried by $U$, i.e. $\Gamma_U := \{c \in \Gamma(E) \mid c(U^c) = 0\}$, is

$$\Gamma_U = \text{Span}\{f \cdot c \mid c \in \Gamma(E), f \in C^\infty(M), f(U^c) = 0\}.$$

In a quantum field theory on the other hand, one is given a $C^*$-algebra $\mathcal{F}$ (the “field algebra”) in which spacetime locality is specified by the following:

(L1) There is directed set $\Upsilon$ of open relatively compact subsets of $M$, partially ordered by set inclusion, such that $M = \bigcup\{W \mid W \in \Upsilon\}$ and $\xi(W) \in \Upsilon$ for all $W \in \Upsilon$ and $\xi \in \text{Diff } M$. Moreover each $\xi \in \text{Diff } M$ is uniquely determined by its action on $\Upsilon$. 

1
Theorem. Let \( \mathcal{A}(W_1) \subseteq \mathcal{A}(W_2) \) if \( W_1 \subseteq W_2 \).

Further relativistic structures are usually also given (cf. [Ha92, HK64]), but here we will not be concerned with those. There is usually no counterpart of the classical \( C^\infty(M) \)-module action, and given an \( A \in \mathcal{F} \), no restriction map of \( A \) to \( W \), producing an element of \( \mathcal{A}(W) \), is specified. Here we will be concerned with this issue: in particular, we will show that for the \( C^* \)-algebra of the canonical anticommutation relations, built upon the classical matter fields, that the classical module structure of the algebra \( C^\infty(M) \) (resp. \( L^\infty(M) \)) on \( L^2(E) \), defines an automorphic action \( \alpha : U(L^\infty(M,\mu)) \to \text{Aut}(L^2(E)) \), such that

- each \( \mathcal{A}(W) \) is the fixed point algebra in \( \text{CAR}(L^2(E)) \) of the automorphism group \( \alpha_{U(L^\infty(W^c)))} \), hence \( \alpha \) contains the locality information in this quantum context. In this last expression we used the natural identification of \( U(L^\infty(W^c)) \) with the unitaries in \( U(L^\infty(M)) \) which are 1 on \( W \), i.e., with \( \chi_W + U(L^\infty(W^c)) \).
- We will obtain a conditional expectation \( \nu_W : \text{CAR}(L^2(E)) \to \mathcal{A}(W) \) which is exactly the quantum restriction map of observables to \( \mathcal{A}(W) \).
- As a corollary, we are able to prove the well–known “folk theorem,” that the \( \text{CAR}(L^2(E)) \) contains only trivial gauge invariant observables w.r.t. a local gauge group acting on \( E \).

2 Commutative von Neumann Algebras and Automorphisms of the CAR

In this section, we will first prove in full generality the appropriate properties of the \( \text{CAR}(\mathcal{H}) \) which we will need, and in the subsequent sections will apply these.

2.1 Definition Let \( \mathcal{H} \) be a complex Hilbert space (not necessarily separable). A \textit{CAR-algebra} of \( \mathcal{H} \) is a \( C^* \)-algebra \( \text{CAR}(\mathcal{H}) \), together with a continuous antilinear map \( a : \mathcal{H} \to \text{CAR}(\mathcal{H}) \) whose image generates \( \text{CAR}(\mathcal{H}) \) as a \( C^* \)-algebra and which satisfies the \textit{canonical anticommutation relations}

\[
\{a(f),a(g)^*\} = \langle f,g \rangle 1 \quad \text{and} \quad \{a(f),a(g)\} = 0 \quad \text{for} \quad f,g \in \mathcal{H}
\]

where we write \( \{A,B\} := AB + BA \) for the anticommutator of two operators. This determines \( \text{CAR}(\mathcal{H}) \) up to natural isomorphism ([BR97, Thm. 5.2.5]), in particular, it is a simple \( C^* \)-algebra. In view of the naturality, there is an automorphic action \( \alpha : U(\mathcal{H}) \to \text{Aut}(\text{CAR}(\mathcal{H})) \) given by

\[
\alpha_U(a(f)) := a(Uf) \quad \text{for} \quad U \in U(\mathcal{H}), \ f \in \mathcal{H}.
\]

The pointwise continuity of the action (where \( U(\mathcal{H}) \) has the strong operator topology) is immediate from the continuity and the \( U(\mathcal{H}) \)-equivariance of the map \( a : \mathcal{H} \to \text{CAR}(\mathcal{H}) \).

The first main theorem which we want to prove in this section is the following:

\textbf{Theorem.} Let \( \mathcal{H} \) be complex Hilbert space (not necessarily separable) and let \( \mathcal{N} \subseteq B(\mathcal{H}) \) be a non-atomic commutative von Neumann algebra. Then the fixed point algebra of the action \( \alpha : U(\mathcal{N}) \to \text{Aut}(\text{CAR}(\mathcal{H})) \) is trivial, i.e. \( \text{CAR}(\mathcal{H})^{U(\mathcal{N})} = \mathbb{C}1 \).

The proof for this is long, and requires some preparatory results.
2.2 Proposition For a commutative von Neumann algebra $\mathcal{N} \subseteq B(\mathcal{H})$, the following are equivalent:

(a) $\mathcal{N}$ is non-atomic, i.e. $\mathcal{N}$ does not contain any minimal non-zero projection.

(b) There exists a weakly continuous curve $\gamma: [0,1] \to P_\mathcal{N} := \{p \in \mathcal{N} \mid p^* = p = p^2\}$ with $\gamma(0) = 1$ and $\gamma(1) = 1$, i.e. $P_\mathcal{N}$ is arcwise connected in the weak topology.

(c) For each $v \in \mathcal{H}$ and each $\varepsilon > 0$ there exists a finite set of mutually orthogonal projections $P_1, \ldots, P_r \in \mathcal{N}$ with $\sum_{j=1}^r P_j = 1$ and $\max_j \|P_j v\| < \varepsilon$.

Proof. (a) $\Rightarrow$ (b), (c): Since every non-atomic commutative von Neumann algebra $\mathcal{N}$ is an $\ell^\infty$-direct sum of algebras of the form $L^\infty(X, \mu)$, where $(X, \mu)$ is a non-atomic measure space, it suffices to prove the assertion for such an algebra $L^\infty(X, \mu)$. Now [CN01] Lemma 2.5 implies the existence of an increasing family $(X_t)_{0 \leq t \leq 1}$ of measurable subsets of $X$ with $\mu(X_t) = t$. We then put $\gamma(t) := \chi_{X_t}$ (the characteristic function). To verify that $\gamma$ is $\sigma(\mathcal{N}, \mathcal{N}_1)$-continuous, we have to show that for each $f \in L^1(X, \mu) \cong L^\infty(X, \mu)_*$, the curve $t \mapsto \langle f, \gamma_t \rangle = \int_{X_t} f \, d\mu$ is continuous, which follows from Lebesgue’s Theorem on Dominated Convergence.

To verify (c), let $v \in \mathcal{H}$ and observe that $f_v(A) := \langle Av, v \rangle$ defines an element of $\mathcal{N}$ with $\|P_v\|^2 = f_v(P)$ for each projection $P \in P_\mathcal{N}$. Since $f_v \circ \gamma: [0,1] \to \mathbb{R}$ is continuous, it is uniformly continuous, and there exists an $n \in \mathbb{N}$ with

$$\left| f_v(\gamma\left(\frac{k}{n}\right)) - f_v(\gamma\left(\frac{k-1}{n}\right)) \right| < \varepsilon \quad \text{for} \quad k = 1, \ldots, n.$$

For the projections

$$P_k := \gamma\left(\frac{k}{n}\right) - \gamma\left(\frac{k-1}{n}\right)$$

we then have

$$\|P_k v\|^2 = f_v(P_k) < \varepsilon \quad \text{and} \quad P_1 + P_2 + \cdots + P_n = 1.$$

(b) $\Rightarrow$ (a): If $\mathcal{N}$ is not non-atomic and $P$ is a minimal non-zero projection, then the curve $P\gamma(t)$ in $P_\mathcal{N}$ is weakly continuous from 0 to $P$, which contradicts the minimality of $P$.

(c) $\Rightarrow$ (a): If $\mathcal{N}$ is not non-atomic and $P$ is a minimal projection, then we pick a unit vector $v \in \text{im}(P)$. The minimality of $P$ implies that either $P_j P = 0$ or $P_j P = P$, so that either $P_j v = 0$ or $P_j v = v$. This leads to $\max_j \|P_j v\| = \|v\| = 1$ which contradicts (c).

In the case that $\mathcal{H}$ is separable, this proposition follows from the fact that non-atomic maximal abelian von Neumann algebras are isomorphic to $L^\infty([0,1])$ (cf. [KR86] Thm. 9.4.1).

2.3 Lemma Let $\mathcal{P} := \{P_1, \ldots, P_r\}$ be a set of nonzero mutually orthogonal projectors with $\sum_{j=1}^r P_j = 1$ (called a partition). Let $T := \prod_{j=1}^r e^{iR_j} \cong T^r \subseteq U(\mathcal{H})$ be the corresponding unitary group and let

$$\nu_\mathcal{P}: \text{CAR}(\mathcal{H}) \to \text{CAR}(\mathcal{H})^T, \quad \nu_\mathcal{P}(A) := \int_T \alpha t(A) \, d\mu_T(t)$$

be the fixed point projection, where $\mu_T$ is the normalized Haar measure on the torus $T$. Let

$$A(f_1, \ldots, f_n; g_1, \ldots, g_n) := a(f_1)^* \cdots a(f_n)^* a(g_1) \cdots a(g_n) \quad \text{for} \quad f_i, g_j \in \mathcal{H}$$
and

\[ \mathcal{B}_n := \{ A(f_1, \ldots, f_n; g_1, \ldots, g_n) \mid f_i, g_i \in \mathcal{H} \}, \quad \mathcal{B}_0 := \mathbb{C}1. \]

Then there exists for each \( A \in \text{span} \left( \bigcup_{n=0}^{\infty} \mathcal{B}_n \right) \subseteq \text{CAR}(\mathcal{H}) \) a constant \( C_A \), independent of \( \mathcal{P} \), such that for each \( f \in \mathcal{H} \) we have the estimate

\[ \|\nu_\mathcal{P}(A), a(f)\| \leq C_A \max_\ell \|P_\ell f\|. \]

If \( \mathcal{P} = \{1\} \), then \( T = T1 \) and \( \text{CAR}(\mathcal{H})^T = \text{GICAR}(\mathcal{H}) \) is the well-known GICAR ("gauge invariant CAR") which is the closure of span \( \left( \bigcup_{n=0}^{\infty} \mathcal{B}_n \right) \). For any other partition \( \mathcal{P} \), the algebra \( \text{GICAR}(\mathcal{H}) \) contains \( \text{CAR}(\mathcal{H})^T \) for the corresponding \( T \).

**Proof.** First we observe that for \( GICAR(\mathcal{H}) \), the element

\[ A(P_1, f_1, \ldots, P_n, f_n; P_1, g_1, \ldots, P_n, g_n) \in \text{CAR}(\mathcal{H}) \]

is an eigenvector of \( \alpha_t \), \( t \in T \), corresponding to the character

\[ t = (t_1, \ldots, t_r) \mapsto \prod_{k=1}^{n} t_{i_k} t_{j_k}^{-1}. \]

Its image under \( \nu_\mathcal{P} \) is non-zero if and only if this character is trivial, which means that

\[ |\{ik \mid i_k = \ell\}| = |\{jk \mid j_k = \ell\}| \quad \text{for each} \quad \ell \in \{1, \ldots, r\}, \]

so that there exists a permutation \( \sigma \) on \( \{1, \ldots, n\} \) with \( j_k = i_{\sigma(k)} \) for \( k = 1, \ldots, n \). We thus obtain

\[ \nu_\mathcal{P}(A(P_1, f_1, \ldots, P_n, f_n; P_1, g_1, \ldots, P_n, g_n)) = A(P_{i_1}, f_{i_1}, \ldots, P_{i_n}, f_{i_n}, P_{j_1}, g_1, \ldots, P_{j_n}, g_n) \]

when it is nonzero. Similar assertions hold for monomials in the \( A \)'s.

We now prove by induction on \( n \) that the lemma holds for \( A \in \text{span} \left( \bigcup_{j=0}^{n} \mathcal{B}_j \right) \). If \( n = 1 \), then we find for \( f_1, g_1 \in \mathcal{H} \):

\[ \nu_\mathcal{P}(A(f_1; g_1)) = \nu_\mathcal{P}\left( \sum_{i,j=1}^{r} A(P_i f_1; P_j g_1) \right) = \sum_{i=1}^{r} A(P_i f_1; P_i g_1) \]

leads to

\[ [\nu_\mathcal{P}(A(f_1; g_1)), a(f)] = \sum_{i=1}^{r} [A(P_i f_1; P_i g_1), a(f)] = -\sum_{i=1}^{r} \langle P_i f_1, f \rangle a(P_i g_1), \]

where we used the identity

\[ [a(f)^*a(g), a(h)] = a(f)^*a(g)a(h) - a(h)a(f)^*a(g) = -\{a(f)^*, a(h)\}a(g) = -\langle f, h \rangle a(g). \quad (2) \]
Thus we obtain
\[
\| \nu_p(A(f_1; g_1), a(f)) \| \leq \sum_{i=1}^{r} \| P_i f_1 \| \| P_i f \| \| P_i g_1 \| \leq (\max \| P_j f \|) \sum_{i=1}^{r} \| P_i f_1 \| \| P_i g_1 \|
\]
\[
\leq (\max \| P_j f \|) \left( \sum_{i=1}^{r} \| P_i f_1 \|^2 \right)^{1/2} \left( \sum_{k=1}^{r} \| P_k g_1 \|^2 \right)^{1/2} = (\max \| P_j f \|) \cdot \| f_1 \| \| g_1 \|,
\]
by the Cauchy–Schwarz inequality. With the choice \( C_A = \| f_1 \| \| g_1 \| \), this proves our assertion for \( A = A(f_1, g_1) \), and hence, by the triangle inequality, our assertion for \( n = 1 \) follows.

Next we assume that our assertion holds for each element of \( \text{span} \left( \bigcup_{j=0}^{k} B_j \right) \). First we observe that \( \text{Span}(B_1 \cup \cdots \cup B_{k+1}) \) is spanned by elements of the form \( A = A_k + A_{k+1} \) where
\[
A_k \in \text{Span}(B_1 \cup \cdots \cup B_k) \quad \text{and} \quad A_{k+1} = A(f_1; g_1) \cdots A(f_{k+1}; g_{k+1}).
\]
Then our induction hypothesis implies that
\[
\| [\nu_p(A), a(f)] \| \leq \| [\nu_p(A_k), a(f)] \| + \| [\nu_p(A_{k+1}), a(f)] \| \leq C_{A_k} \max \| P_\ell f \| + \| [\nu_p(A_{k+1}), a(f)] \|,
\]
so that we only need to prove our assertion for \( A_{k+1} \), i.e., we may assume that \( A = A(f_1; g_1) \cdots A(f_{k+1}; g_{k+1}) \). Then
\[
\nu_p(A) = \nu_p \left( \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_{2k+2}=1}^{r} A(P_{\ell_1} f_1; P_{\ell_{2k+2}} g_1) \cdots A(P_{\ell_{k+1}} f_{k+1}; P_{\ell_{2k+2}} g_{k+1}) \right)
\]
\[
= \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_{k+1}=1}^{r} \sum_{\sigma \in S_{k+1}} A(P_{\ell_1} f_1; P_{\ell_{\sigma(1)}} g_1) \cdots A(P_{\ell_{k+1}} f_{k+1}; P_{\ell_{\sigma(k+1)}} g_{k+1})
\]
where \( S_{k+1} \) denotes the permutation group of \( \{1, \ldots, k+1\} \). Observe that all the terms of this sum are in the image of \( \nu_p \). Now
\[
[\nu_p(A), a(f)] = \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_{k+1}=1}^{r} \sum_{\sigma \in S_{k+1}} \left\{ A(P_{\ell_1} f_1; P_{\ell_{\sigma(1)}} g_1, a(f)) A(P_{\ell_2} f_2; P_{\ell_{\sigma(2)}} g_2) \cdots A(P_{\ell_{k+1}} f_{k+1}; P_{\ell_{\sigma(k+1)}} g_{k+1}) + \right.
\]
\[
\left. \cdots + A(P_{\ell_1} f_1; P_{\ell_{\sigma(1)}} g_1) \cdots A(P_{\ell_{k+1}} f_{k+1}; P_{\ell_{\sigma(k+1)}} g_{k+1}) A(P_{\ell_1} f_1; P_{\ell_{\sigma(1)}} g_1, a(f)) \right\}
\]
\[
= \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_{k+1}=1}^{r} \sum_{\sigma \in S_{k+1}} \left\{ -(P_{\ell_1} f_1, f) a(P_{\ell_{\sigma(1)}} g_1) A(P_{\ell_2} f_2; P_{\ell_{\sigma(2)}} g_2) \cdots A(P_{\ell_{k+1}} f_{k+1}; P_{\ell_{\sigma(k+1)}} g_{k+1}) - \right.
\]
\[
\left. \cdots - A(P_{\ell_1} f_1; P_{\ell_{\sigma(1)}} g_1) \cdots A(P_{\ell_{k+1}} f_{k+1}; P_{\ell_{\sigma(k+1)}} g_{k+1}) (P_{\ell_{k+1}} f_{k+1}, f) a(P_{\ell_{\sigma(k+1)}} g_{k+1}) \right\}.
\]
We thus arrive at
\[
\left\| [\nu_{\mathcal{P}}(A), a(f)] \right\| \leq \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_{k+1}=1}^{r} \sum_{\sigma \in S_{k+1}} \left\{ \right.
\| P_{\ell_1} f_1 \| \| P_{\ell_1} f_1 \| \| P_{\sigma_1(j)} g_1 \| \| P_{\ell_2} f_2 \| \| P_{\sigma_2(j)} g_2 \| \cdots \| P_{\ell_{k+1}} f_{k+1} \| \| P_{\sigma(k+1)} g_{k+1} \| + \\
\cdots + \| P_{\ell_1} f_1 \| \| P_{\ell_1} f_1 \| \cdots \| P_{\ell_{k+1}} f_{k+1} \| \| P_{\sigma(k+1)} g_{k+1} \| \right\}
\leq \left( \max_{\ell} \| P_{\ell} f \| \right) (k+1) \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_{k+1}=1}^{r} \sum_{\sigma \in S_{k+1}} \prod_{i=1}^{k+1} \| P_{\ell_i} f_i \| \prod_{j=1}^{k+1} \| P_{\sigma(j)} g_j \| \\
= \left( \max_{\ell} \| P_{\ell} f \| \right) (k+1) \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_{k+1}=1}^{r} \sum_{\sigma \in S_{k+1}} \prod_{i=1}^{k+1} \| P_{\ell_i} f_i \| \| P_{\ell_i} g_{\sigma^{-1}(i)} \| \\
= \left( \max_{\ell} \| P_{\ell} f \| \right) (k+1) \sum_{\sigma \in S_{k+1}} k+1 \prod_{i=1}^{r} \| P_{\ell_i} f_i \| \| P_{\ell_i} g_{\sigma^{-1}(i)} \| \\
\leq \left( \max_{\ell} \| P_{\ell} f \| \right) (k+1) \prod_{i=1}^{k+1} \| f_i \| \| g_{\sigma^{-1}(i)} \| \\
= \left( \max_{\ell} \| P_{\ell} f \| \right) (k+1)(k+1) \prod_{i=1}^{k+1} \| f_i \| \| g_i \| ,
\]
where we used the Cauchy–Schwarz inequality to obtain
\[
\sum_{\ell_i=1}^{r} \| P_{\ell_i} f_i \| \| P_{\ell_i} g_{\sigma^{-1}(i)} \| \leq \left[ \sum_{i=1}^{r} \| P_{\ell_i} f_i \|^{2} \right]^{1/2} \left[ \sum_{i=1}^{r} \| P_{\ell_i} g_{\sigma^{-1}(i)} \|^{2} \right]^{1/2} = \| f_i \| \| g_{\sigma^{-1}(i)} \| .
\]
Observe that \( C_A := (k+1)(k+1)! \prod_{i=1}^{k+1} \| f_i \| \| g_i \| \) does not depend on \( \mathcal{P} \), so that this completes our induction. \( \square \)

We are now ready to prove our first main result.

2.4 Theorem Let \( \mathcal{H} \) be complex Hilbert space and let \( \mathcal{N} \subseteq B(\mathcal{H}) \) be a non-atomic commutative von Neumann algebra. Then \( \text{CAR}(\mathcal{H})^{U(\mathcal{N})} = \mathbb{C}1. \)

Proof. If an \( A \in X_0 := \text{Span}\left( \bigcup_{i=0}^{\infty} \mathcal{B}_i \right) \) is \( U(\mathcal{N}) \)-invariant, then \( \nu_{\mathcal{P}}(A) = A \) for all partitions \( \mathcal{P} \subset \mathcal{N} \). Hence the preceding lemma leads to
\[
\left\| [A, a(f)] \right\| \leq C_A \max_{\ell} \| P_{\ell} f \| \text{ for all } f \in \mathcal{H}, \ \mathcal{P} \subset \mathcal{N},
\]
where \( C_A \) does not depend on \( \mathcal{P} \) or \( f \). Since \( \mathcal{N} \) is nonatomic, Proposition 2.2 implies that \( \max_{\ell} \| P_{\ell} f \| \) can be made arbitrarily small. We conclude that \( A \) commutes with all \( a(f), f \in \mathcal{H}, \) hence with \( \text{CAR}(\mathcal{H}) \). But the center of \( \text{CAR}(\mathcal{H}) \) is \( \mathbb{C}1, \) which leads to
\[
\text{Span}\left( \bigcup_{i=0}^{\infty} \mathcal{B}_i \right) \cap \text{CAR}(\mathcal{H})^{U(\mathcal{N})} = \mathbb{C}1. 
\]
Recall that GICAR(\(\mathcal{H}\)) = CAR(\(\mathcal{H}\))^\(T_1\) is the closure of \(X_0\). Let \(A \in \text{GICAR}(\mathcal{H})_{U(\mathcal{N})} = \text{CAR}(\mathcal{H})_{U(\mathcal{N})}\), then for each \(\varepsilon > 0\) there exists an \(A_0 \in X_0\) such that \(\|A - A_0\| < \varepsilon\). For a unit vector \(f \in \mathcal{H}\) we choose the finite partition \(\mathcal{P}\) in such a way that \(\text{CAR}(A_0, \mathcal{N}) < \varepsilon\). Then we obtain

\[
\|x, a(f)\| = \|[\nu_\mathcal{P}(A), a(f)]\| = \|[\nu_\mathcal{P}(A - A_0), a(f)]\| + \|[\nu_\mathcal{P}(A_0), a(f)]\| \\
\leq 2\|f\| \|A - A_0\| + C_{A_0} \max_\ell \|P_\ell f\| \\
\leq 2\varepsilon + 3\varepsilon.
\]

Since \(\varepsilon > 0\) was arbitrary, we obtain \(A \in Z(\text{CAR}(\mathcal{H})) = C1\), and this completes the proof. \(\blacksquare\)

2.5 Remark If the von Neumann algebra \(\mathcal{N}\) is not non-atomic, then we obtain for each minimal non-zero projection \(P \in \mathcal{P}_\mathcal{N}\) a decomposition

\[
U(\mathcal{N}) \cong e^{R_\mathcal{N}} \times U((1 - P)\mathcal{N}) \cong T \times U((1 - P)\mathcal{N}),
\]

so that

\[
C1 \neq \text{GICAR}(P(\mathcal{H})) \subseteq \text{CAR}(\mathcal{H})_{U(\mathcal{N})}.
\]

Therefore the assumption of \(\mathcal{N}\) being non-atomic in the preceding theorem is necessary.

For maximal commutative subalgebras, the preceding theorem could also be obtained from the results of Wolfe in [Wo75]. However his arguments are very indirect and difficult. We think that our proof is much more transparent and direct.

2.6 Corollary If \(A \subseteq B(\mathcal{H})\) is a \(C^\ast\)-subalgebra, then \(\text{CAR}(\mathcal{H})_{U(\mathcal{A})} = \text{CAR}(\mathcal{H})_{U(\mathcal{A}''\mathcal{N})}\), and if \(\mathcal{A}'' \subseteq B(\mathcal{H})\) contains a nonatomic commutative von Neumann algebra, then \(\text{CAR}(\mathcal{H})_{U(\mathcal{A})} = C1\).

Proof. The action of \(U(\mathcal{H})\) on \(\text{CAR}(\mathcal{H})\) is continuous. Since \(U(\mathcal{A})\) is strongly dense in \(U(\mathcal{A}''\mathcal{N})\) by Kaplansky’s Density Theorem ([KR83 Cor. 5.3.7]), it follows that \(\text{CAR}(\mathcal{H})_{U(\mathcal{A})} = \text{CAR}(\mathcal{H})_{U(\mathcal{A}''\mathcal{N})}\). Let \(\mathcal{N} \subseteq \mathcal{A}''\mathcal{N}\) be a nonatomic commutative von Neumann subalgebra, then by \(U(\mathcal{N}) \subseteq U(\mathcal{A}''\mathcal{N})\) we get \(C1 = \text{CAR}(\mathcal{H})_{U(\mathcal{N})} \supseteq \text{CAR}(\mathcal{H})_{U(\mathcal{A}''\mathcal{N})} = \text{CAR}(\mathcal{H})_{U(\mathcal{A})}\). \(\blacksquare\)

2.7 Remark To prepare for our second main result, we need to recall some facts about tensor products (cf. [Ta79 Sect. IV.2, p188]). Let \(X\) and \(Y\) be Banach spaces and let \(X'\), resp., \(Y'\) be their topological duals. We have an identification of the algebraic tensor product \(X \otimes Y\) with a subspace of \(B(X', Y)\) by the linear injection \(\Phi : X \otimes Y \to B(X', Y)\) given by \(\Phi(x \otimes y)(f) := f(x)y\) for \(f \in X'\), and we have a similar identification map \(\Psi : X \otimes Y \to B(Y', X)\) by \(\Psi(x \otimes y)(f) := f(y)x\) for \(f \in Y'\). The map \(\Phi : X \otimes Y \to B(X', Y)\) is an isometry w.r.t. the minimal cross-norm \(\lambda\), hence extends as an isometry to the completion, denoted by \(X \otimes_\lambda Y\) (cf. [Ta79 Prop. IV.2.1, p189]). Explicitly \(\cdot \|\lambda\) is given by

\[
\left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\lambda := \sup \left\{ \left\| \sum_{i=1}^n f(x_i) g(y_i) \right\| : \|f\| \leq 1; \ g \in Y', \|g\| \leq 1 \right\}.
\]

It is easy to verify that \(\otimes_\lambda\) is a bifunctor on the category of Banach spaces. This implies in particular that, if \(X = X_1 \oplus X_2\) is a direct sum of two closed subspaces and \(P_j : X \to X_j\) are the corresponding projections, then we obtain a topological isomorphism

\[
X \otimes_\lambda Y \to (X_1 \otimes_\lambda Y) \oplus (X_2 \otimes_\lambda Y), \quad \Psi(x \otimes y) \mapsto (P_1 \circ \Psi(x \otimes y)) \oplus (P_2 \circ \Psi(x \otimes y))
\]
subordinated to the decomposition \( B(Y', X) \cong B(Y', X_1) \oplus B(Y', X_2) \). From that we further conclude that
\[
\Phi(X_1 \otimes \lambda Y) = \{ \phi \in \Phi(X \otimes \lambda Y) \subseteq B(X', Y) \mid X_1^\perp \subseteq \ker \phi \}
\]
\[
\Psi(X_1 \otimes \lambda Y) = \{ \psi \in \Psi(X \otimes \lambda Y) \subseteq B(Y', X) \mid \im(\psi) \subseteq X_1 \}.
\] (3)

This observation is particularly useful if \( G \subseteq GL(X) \) is a group of isometries and \( X_1 = X^G \) is the subspace of \( G \)-fixed elements. Then \( G \otimes \id_Y \) is a group of isometries of \( X \otimes \lambda Y \) for which \( \Psi((g \otimes \id_Y)(x \otimes y))(f) := f(y)g.x \) for \( f \in Y' \), i.e. \( \Psi \) is \( G \)-equivariant. In particular \( \Psi((X \otimes \lambda Y)^{G \otimes \id_Y})(f) \subseteq X^G \) for \( f \in Y' \). Thus, since \( X = X^G \oplus X_2 \), we get from (3) that
\[
(X \otimes \lambda Y)^{G \otimes \id_Y} = X^G \otimes \lambda Y.
\] (4)

Note that the minimal \( C^* \)-cross norm for nonabelian \( C^* \)-algebras does not coincide with the minimal Banach cross norm \( \lambda \) (cf. [La79] Theorem IV.4.14, p189]).

### 2.8 Proposition

Given a decomposition \( \mathcal{H} = \mathcal{K} \oplus \mathcal{L} \) of a complex Hilbert space, let \( \mathcal{U} \subset U(\mathcal{K}) \) be a group of unitaries with \( \text{CAR}(\mathcal{K})^\mathcal{U} = \mathcal{C}1 \). Then \( \text{CAR}(\mathcal{H})^\mathcal{U} = \text{CAR}(\mathcal{L}) \), considered as a subalgebra of \( \text{CAR}(\mathcal{H}) \), we identify \( \mathcal{U} \subset U(\mathcal{K}) \) with \( \mathcal{U} \odot 1 \subset U(\mathcal{H}) \).

**Proof.** The operator \( V := P_{\mathcal{K}} - P_{\mathcal{L}} \) on \( \mathcal{H} \) is unitary and the associated automorphism of \( \text{CAR}(\mathcal{H}) \) satisfies \( \alpha_V \left| \text{CAR}(\mathcal{K}) = \id = \alpha_V^\circ \right. \), and \( \alpha_V \left| \text{CAR}(\mathcal{L}) \right. \) induces the canonical grading for which the generators \( a(f) \) are odd. In particular, \( P_{\mathcal{L}}^\mathcal{U} := \frac{1}{2} \left( \id + \alpha_V \right) \) projects \( \text{CAR}(\mathcal{H}) \) onto
\[
C^* \left( \text{CAR}(\mathcal{K}) \cup \text{CAR}(\mathcal{L})_{\text{even}} \right) = \text{CAR}(\mathcal{K}) \otimes \text{CAR}(\mathcal{L})_{\text{even}}
\]
([Ta03] Exer. XIV.1.5b.p. 94)), and \( P_{\mathcal{L}}^\mathcal{U} := \frac{1}{2} \left( \id - \alpha_V \right) \) projects \( \text{CAR}(\mathcal{H}) \) onto the subspace \( \text{span}(\text{CAR}(\mathcal{K}) \cdot \text{CAR}(\mathcal{L})_{\text{odd}}) \). Since \( [\alpha_U, \alpha_V] = 0 \) for all \( U \in U(\mathcal{H}) \) preserving \( \mathcal{L} \), the relation \( A \in \text{CAR}(\mathcal{H})^{\mathcal{U}} \) is equivalent to \( P_{\mathcal{L}}^\mathcal{U} A \in \text{CAR}(\mathcal{H})^{\mathcal{U}} \supseteq P_{\mathcal{L}}^\mathcal{U} A \). Thus, for the rest of this proof, we may assume that either \( A \in P_{\mathcal{L}}^\mathcal{U} \text{CAR}(\mathcal{H}) \) or \( A \in P_{\mathcal{L}}^\mathcal{U} \text{CAR}(\mathcal{H}) \).

First, let
\[
A \in \text{CAR}(\mathcal{H})^{\mathcal{U}} \cap P_{\mathcal{L}}^\mathcal{U} \text{CAR}(\mathcal{H}) \subset \text{CAR}(\mathcal{K}) \otimes \text{CAR}(\mathcal{L})_{\text{even}} \subseteq \text{CAR}(\mathcal{K}) \otimes \lambda \text{CAR}(\mathcal{L})_{\text{even}}.
\]

Here we use the fact that the \( C^* \)-tensor product of \( C^* \)-algebras is defined by a cross norm which dominates the minimal Banach cross norm \( \lambda \), which leads to the inclusion on the right (cf. Remark 2.7). Now observe that \( \mathcal{C}1 = \text{CAR}(\mathcal{K})^{\mathcal{U}} \) is complemented, in fact a projection onto \( \mathcal{C}1 \) is given by \( P_f(A) := f(A)1 \) for any continuous functional \( f \) with \( f(1) = 1 \), in which case the complementary subspace is \( \text{Ker}(f) \). Thus (3) above implies that
\[
\left( \text{CAR}(\mathcal{K}) \otimes \lambda \text{CAR}(\mathcal{L})_{\text{even}} \right)^{\mathcal{U}} = \text{CAR}(\mathcal{K})^{\mathcal{U}} \otimes \lambda \text{CAR}(\mathcal{L})_{\text{even}} = 1 \otimes \lambda \text{CAR}(\mathcal{L})_{\text{even}},
\]
which implies that \( A \in \text{CAR}(\mathcal{L})_{\text{even}} \).

Now let \( A \in P_{\mathcal{L}}^\mathcal{U} \text{CAR}(\mathcal{H}) \) be a \( \mathcal{U} \)-invariant element. We observe that for every unit vector \( f \in \mathcal{L} \), the element \( u := a(f) + a(f)^* \) is hermitian and satisfies
\[
u^2 = a(f)a(f)^* + a(f)^*a(f) = \{a(f), a(f)^*\} = \langle f, f \rangle 1 = 1,
\]
so that it is unitary. Clearly, $u \in \text{CAR}(\mathcal{L})_{\text{odd}}$. Then $Au \in P_{\text{even}}^L \text{CAR}(\mathcal{H})$ is also $U$-invariant, hence contained in $\text{CAR}(\mathcal{L})_{\text{even}}$ by the preceding argument. This leads to $A = Auu \in \text{CAR}(\mathcal{L})_{\text{odd}}$, so that our proof is complete.

3 Automorphism Groups Encoding Locality

To build the quantum fields, we need first to add extra structure to the classical fields, which we now list. Let $M$ be a $k$-dimension $\sigma$-compact smooth manifold and let $G \subseteq U_n(\mathbb{C})$ be a closed subgroup. Further, let $q_E : E \to M$ be a complex vector bundle which is a $G$-bundle, i.e. its typical fibre is $\mathbb{C}^n$ on which $G$ acts by the defining matrices for $U_n(\mathbb{C}) \supseteq G$, and $E$ has an atlas of local trivializations for which the transition functions take their values in $G$ (the set of trivializations is called a $G$-structure). It is known that the property of $\mathbb{C}$-generated by $C$ with respect to the identical representation of $G$-bundle is equivalent to requiring $E$ to be associated to a $G$-principal bundle $q_p : P \to M$ with respect to the identical representation of $G$ on $\mathbb{C}^n$ (cf. [KM97] Cor. 37.13 and [CB94] p. 368]). This means that $E = (P \times \mathbb{C}^n)/G$, where $G$ acts on $P \times \mathbb{C}^n$ by $g.(p, v) = (pg^{-1}, g v)$ and $q_E([p, v]) := q_p(p) \in M$ where we write $[p, v]$ for the elements of $E$, i.e. the $G$-orbit of $(p, v)$ in $P \times \mathbb{C}^n$.

We obtain from the norm on $\mathbb{C}^n$ a function

$$| \cdot | : E \to [0, \infty) \quad \text{by} \quad |[p, v]| = |v| = (v \cdot v)^{1/2}.$$ 

This is well-defined because $G$ acts as unitaries on $\mathbb{C}^n$. Let $\mu$ be a locally Lebesgue measure on $M$ (this could be obtained from a nowhere vanishing smooth $k$-form on $M$ if we are concerned with smoothness) and consider the $L^2$–sections of $E$, i.e. $c : M \to E$ smooth such that $q_E \circ c = \text{id}_M$ and

$$\|c\|^2 := \int_M |c(x)|^2 d\mu(x) < \infty.$$ 

Let $L^2(E)$ denote the $L^2$–completion of the space of smooth $L^2$–sections w.r.t. this norm.

Now the $C^\infty(M)$–module action on $\Gamma(E)$ need to be restricted to the bounded smooth functions $C_b^\infty(M)$ to obtain an action on $L^2(E)$ by bounded operators. If we complete $C_b^\infty(M)$ w.r.t. the strong operator topology we get an action of $L^\infty(M, \mu)$ on $L^2(E)$. This is because the subalgebra $C_b^\infty(M)$ separates all the points of $M$ and on each point is nonzero, hence by the Stone–Weierstrass theorem it is $C^\ast$-norm dense in $C_0(M)$, and the von Neumann algebra generated by $C_0(M)$ in $B(L^2(E))$ is $L^\infty(M, \mu)$. Note that this module action of $L^\infty(M, \mu)$ on $L^2(E)$ encodes locality of the classical fields in a particularly simple way, e.g. restriction to a Borel subset $W \subset M$ with $\mu(W) \neq 0$, is just done by multiplication of the characteristic function $\chi_W \in L^\infty(M, \mu)$. We denote this submodule by $L^2(E \upharpoonright W) := \chi_W L^2(E)$. It is characterized by

$$L^2(E \upharpoonright W) = \{ c \in L^2(E) \mid f \cdot c = 0 \quad \forall f \in \chi_W, L^\infty(M, \mu) = L^\infty(W^c, \mu) \}.$$ 

Moreover, $L^\infty(M, \mu)$ is nonatomic, as $\mu$ is locally Lebesgue. Henceforth we omit $\mu$ from the notation $L^\infty(M)$. Thus, from the original $C^\ast(M)$–module action on $\Gamma(E)$ we have obtained an action of the commutative nonatomic von Neumann algebra $L^\infty(M)$ on $L^2(E)$, and it encodes locality information of the classical fields. Below we will use its unitary group to define automorphisms of the CAR–algebra of $L^2(E)$.

To quantize the matter fields, we consider the $C^\ast$-algebra $\text{CAR}(L^2(E))$, on which we have
the usual action \( \alpha : \mathcal{U}(\mathcal{H}) \to \text{Aut} \mathcal{C}(\mathcal{H}), \mathcal{H} := L^2(E) \), given by \( \alpha_q(a(f)) := a(Uf), f \in \mathcal{H}. \)

Since for classical gauge theory, the \( C^\infty(M) \)-module action on the matter fields is crucial, we will be particularly concerned with the restriction of \( \alpha \) to \( U(L^\infty_c(M)) \subset \mathcal{U}(\mathcal{H}). \)

In the quantum situation, locality is specified by the algebras

\[
\mathcal{A}(W) := C^*\{a(c) | c \in L^2(E\upharpoonright W)\} \cong \mathcal{C}(L^2(E\upharpoonright W))
\]

for any relatively compact open set \( W \subset M. \) This collection of algebras satisfies precisely the conditions (L1) and (L2) above, where we take \( \mathcal{F} = \mathcal{C}(L^2(E)) \). We now show that each \( \mathcal{A}(W) \) is in fact the fixed point algebra in \( \mathcal{C}(L^2(E)) \) of the automorphism group \( \alpha_{U(L^\infty_c(W^c))} \), hence \( \mathcal{A}(W) \) already contains the locality information in this quantum context. The natural identification of \( U(L^\infty_c(W^c)) \) with the unitaries in \( U(L^\infty_c(M)) \) which are 1 on \( W, \) i.e. with \( \chi_W \oplus U(L^\infty_c(W^c)), \) was used. In view of the preparations from the preceding sections, we can now prove our main result.

**3.1 Theorem** With respect to the action \( \alpha : U(L^\infty_c(M)) \to \text{Aut} \mathcal{C}(L^2(E)) \) from above, for each relatively compact \( W \subset M, \) the subalgebra \( \mathcal{A}(W) \) is the fixed point algebra of the subgroup \( U(L^\infty_c(W^c)) \) of \( U(L^\infty_c(M)), \) where \( W^c \) denotes the complement of \( W. \)

**Proof.** The von Neumann algebra \( L^\infty_c(W^c) \) is nonatomic since \( \mu \) is locally Lebesgue. Thus \( L^\infty_c(W^c) \) satisfies the hypotheses of Theorem 2.3. Hence the fixed point algebra of the group of automorphisms \( \alpha_{U(L^\infty_c(W^c))} \) acting on \( \mathcal{C}(L^2(E\upharpoonright W^c)) \) is just the constants. Since \( L^2(E) = L^2(E\upharpoonright W) \oplus L^2(E\upharpoonright W^c), \) it now follows from Proposition 4.1.3.4 that \( \mathcal{A}(W) \) is the fixed point algebra of \( \alpha_{U(L^\infty_c(W^c))} \subset \text{Aut} \mathcal{C}(L^2(E)) \) as claimed. \( \blacklozenge \)

### 4 Invariance Under Local Gauge Transformations

We can now use our result above to prove a well-known “folk theorem,” stating that the only invariant elements under the group of local gauge transformations of the \( \mathcal{C}(\mathcal{H}) \)-algebra are the constants. We will need the very mild condition that \( T_1 \subseteq G. \) This condition guarantees that \( \alpha_{U(C^\infty_c(M))} \) is contained in the action of the local gauge group on the \( \mathcal{C}(\mathcal{H}) \)-algebra, and we will need only enough detail of the gauge action on \( \mathcal{C}(L^2(E)) \) to verify this.

An intrinsic definition of the gauge group \( \text{Gau} \, E \) is as the group of those smooth bundle automorphisms \( \gamma \in \text{Aut}(E) \) which induce the identity on the base manifold \( M, \) i.e. \( q_E \circ \gamma = q_E, \) and which preserves the \( G \)-structure, i.e. the union of the \( G \)-structure and its composition with \( \gamma \) is a \( G \)-structure. However, the (equivalent) customary definition of \( \text{Gau} \, E \) is via the property that \( E = (P \times C^n)/G \) is an associated bundle to \( P. \) Briefly, one defines \( \text{Gau} \, E \) as those \( \gamma \in \text{Aut}(E) \) of the form \( \gamma[p, v] = [p, f(p)v] \), where \( f : P \to G \) is a smooth function satisfying \( f(p, g) = g^{-1}f(p)g \) for all \( p \in P, \) \( g \in G \) (cf. [Bl81 Thm. 3.2.2] and [Is99 Comment 4, p. 239]).

If \( f \) has values in the center of \( G, \) then it defines a function \( \tilde{f} : M \to Z(G) \) by \( f = \tilde{f} \circ q_P, \) and \( \gamma \) commutes with \( \text{Gau} \, E. \)

To obtain a unitary action on \( L^2(E) \) from the action of \( \text{Gau} \, E \) on \( E, \) observe that \( |(\gamma \cdot c)(x)| = |c(x)| \) for \( \gamma \in \text{Gau} \, E \) and \( c \in L^2(E), \) which leads us to define

\[
(V_\gamma c)(x) := (\gamma \cdot c)(x) := \gamma(c(x)).
\]

If \( f \) has values in the center of \( G, \) then \( (V_\gamma c)(x) = \tilde{f}(x)c(x). \) In particular, if \( T_1 \subseteq G \subset U(C^n), \)

10
then \( V_{\text{Gau}E} \) contains the module action of \( U(C_b^\infty(M)) \) on \( L^2(E) \). Using the action \( V : \text{Gau}E \to U(L^2(E)) \), we obtain the usual automorphic action

\[
\kappa : \text{Gau}E \to \text{Aut CAR}(L^2(E)) \quad \text{by} \quad \kappa_\gamma(a(f)) := a(V_\gamma f) \quad \text{for} \quad \gamma \in \text{Gau}E, \ f \in L^2(E).
\]

Note that \( \kappa(\text{Gau}E) \) commutes with \( \alpha(U(L^\infty(M))) \) since the multiplication with smooth functions commutes with the action \( V \) of \( \text{Gau}E \) on \( \Gamma(E) \).

**4.1 Theorem** Given the action \( \kappa : \text{Gau}E \to \text{Aut CAR}(L^2(E)) \) above, then \( \text{CAR}(L^2(E))^{\text{Gau}E} = \mathbb{C}1 \) if either:

(i) \( \mathbb{T}1 \subseteq G \), or

(ii) in the Fock representation \( \pi_F \) we have that \( \Gamma(V(\text{Gau}E))'' \supseteq \Gamma(U(L^\infty(M))) \), where \( \Gamma \) denotes the second quantization on Fock space of unitaries on \( L^2(E) \).

**Proof.** (i) If \( \mathbb{T}1 \subseteq G \), then \( \alpha_{U(C_b^\infty(M))} \subset \kappa(\text{Gau}E) \), and hence any gauge invariant element is invariant w.r.t. \( \alpha_{U(C_b^\infty(M))} \). Observe that \( C_b^\infty(M)'' = C_b(M)'' = L^\infty(M) \), hence for any selfadjoint \( H \in L^\infty(M) \), there is a net \( \{H_\nu\} \subset C_b^\infty(M) \) of selfadjoint elements such that \( H_\nu \to H \) in the strong operator topology, and with \( \|H_\nu\| \leq \|H\| \) for all \( \nu \) (cf. [KR83 Thm. 5.3.5, p. 329]).

However, any unitary \( U \in L^\infty(M) \) is of the form \( U = \exp(iH) \) for some positive \( H \in L^\infty(M) \) (cf. [KR83 Thm. 5.2.5, p. 313]). Since the map \( t \to \exp(it) \) is continuous, we have

\[
U(C_b^\infty(M)) \ni \exp(iH_\nu) \to \exp(iH) = U \in L^\infty(M)
\]

in the strong operator topology (cf. [KR83 Prop. 5.3.2, p327]). Hence by continuity of the action \( \alpha \), we have by Theorem 2.4 that

\[
\text{CAR}(L^2(E))^{\text{Gau}E} \subseteq \text{CAR}(L^2(E))^{U(C_b^\infty(M))} = \text{CAR}(L^2(E))^{U(L^\infty(M))} = \mathbb{C}1.
\]

(ii) If \( \Gamma(V(\text{Gau}E))'' \supseteq \Gamma(U(L^\infty(M))) \), then \( \Gamma(V(\text{Gau}E))' \subseteq \Gamma(U(L^\infty(M)))' \), and as

\[
\Gamma(U(L^\infty(M)))' \cap \pi_F(\text{CAR}(L^2(E))) = \pi_F(\text{CAR}(L^2(E))^{U(L^\infty(M))}) = \mathbb{C}1,
\]

it follows that

\[
\Gamma(V(\text{Gau}E))' \cap \pi_F(\text{CAR}(L^2(E))) = \pi_F(\text{CAR}(L^2(E))^{\text{Gau}E}) \subseteq \mathbb{C}1.
\]

Since \( \pi_F \) is faithful (as \( \text{CAR}(L^2(E)) \) is simple), we obtain that \( \text{CAR}(L^2(E))^{\text{Gau}E} = \mathbb{C}1 \).

In a full gauge theory, electromagnetism will be included, hence \( G = \nu(\mathbb{T} \times H) \) where \( H \) is a compact connected Lie group, and \( \nu : \mathbb{T} \times H \to U(\mathbb{C}^n) \) is a homomorphism which takes \( (T, e) \) to \( \mathbb{T}1 \) (cf. [CG07, p. 118]). Thus the assumption of the inclusion \( \mathbb{T}1 \subseteq G \) is not physically unreasonable.

Theorem 4.1 seems to be a well-known "folk theorem," and the usual strategy for finding gauge invariant elements in an appropriate representation \( \pi : \text{CAR}(L^2(E)) \to \mathcal{B}(\mathcal{H}) \) is to select them from the currents (generators of the unitary one-parameter groups implementing automorphisms of the \( \text{CAR}(L^2(E)) \)). Of course, if one enlarges \( \text{CAR}(L^2(E)) \) by additional elements which are gauge invariant, the action of \( U(L^\infty(M)) \) will not be able to select the local algebras directly.
5 Localizing Maps

There is more useful information in the automorphic action $\alpha : U(L^\infty_c(M)) \to \text{Aut} \, \text{CAR}(L^2(E))$, other than the net of local algebras. For example, we can obtain from it a set of conditional expectations which extend the restriction maps $L^2(E) \to L^2(E|S)$, $c \to c|S$ from the fields to the algebra $\text{CAR}(L^2(E))$ (for $S \subset M$ open and relatively compact). Note that the map $a(f) \to a(f|S)$ cannot extend to $\text{CAR}(L^2(E))$ as a *-homomorphism, since this will violate the canonical anticommutation relations.

Recall that a map $\nu : \text{CAR}(L^2(E)) \to \text{CAR}(L^2(E))$ is a conditional expectation if it is a positive map such that $\nu(1) = 1$ and

$$\nu(A\nu(B)) = \nu(\nu(A)B) = \nu(A)\nu(B) \quad \forall A, B \in \text{CAR}(L^2(E)).$$

A projection on $\text{CAR}(L^2(E))$ is a contractive linear map which is idempotent. In Thm. III.3.5], it is shown that all projections which preserve the identity are in fact conditional expectations. The maps $\nu_P : \mathcal{H} \to \mathcal{H}^P$ which occurred in the proof of Lemma 2.3 are typical examples.

5.1 Theorem Let $S \subset M$ be a relatively compact set. Then there is a conditional expectation $\nu_S : \text{CAR}(L^2(E)) \to \mathcal{A}(S) \subset \text{CAR}(L^2(E))$, which satisfies $\nu_S(\alpha_h(A)) = \nu_S(A)$ for all $h \in U(L^\infty_c(S^c))$, and

$$\nu_S(A(f_1, \ldots, f_n; g_1, \ldots, g_m)) = A(ps f_1, \ldots, ps f_n; ps g_1, \ldots, ps g_m) \quad \text{for} \quad f_i, g_j \in L^2(E), \quad (5)$$

where $A(f_1, \ldots, f_n; g_1, \ldots, g_m) := a^*(f_1) \cdots a^*(f_n)a(g_1)\cdots a(g_m)$ denotes a normal ordered monomial, and $ps$ is the projection of $L^2(E)$ onto the subspace $L^2(E|S)$.

Proof. We can build $\nu_S$ from the conditional expectations $\nu_P$ from Lemma [Wo75 Prop. 3.11], but it is easiest just to define it explicitly. Recall first that, given $R \in B(L^2(E))$ with $\|R\| \leq 1$, we can define a positive map $\alpha_R : \text{CAR}(L^2(E)) \to \text{CAR}(L^2(E))$, which takes each normal ordered monomial $a^*(f_1) \cdots a^*(f_n)a(g_1)\cdots a(g_m)$ to $a^*(Rf_1) \cdots a^*(Rf_n)a(Rg_1)\cdots a(Rg_m)$ and the identity to itself (cf. [HK75 Prop. 2.1]). With $R = ps$ and $\nu_S := \alpha_{ps}$, we then obtain [5]. It is uniquely determined by what it does on the monomials $A(\cdots)$ since $\text{CAR}(L^2(E))$ is topologically spanned by these. Its equivariance w.r.t. $U(L^\infty_c(S^c))$ is obvious, as well as the fact that it is idempotent, hence a conditional expectation [Ta79 Thm. III.3.5].

Another useful approach to the restriction maps is as follows. For any Borel set $S \subset M$, let $\mathcal{N}_S \subset \text{CAR}(L^2(E))$ be the closed left ideal of $\text{CAR}(L^2(E))$ generated by the set \{ $a(c) \mid c \in L^2(E|S)$ \}, and denote the generating hereditary subalgebra by $\mathcal{D}_S := \mathcal{N}_S \cap \mathcal{N}_S^\perp$. Note that $\mathcal{N}_S$ is proper since it annihilates the vacuum in the Fock representation. If $S$ is relatively compact, then $\mathcal{N}_S^\perp$ is nonzero for $M$ noncompact. Now each closed left ideal $\mathcal{J}$ of $\text{CAR}(L^2(E))$ has a unique associated open projection in the universal von Neumann algebra $P \in \text{CAR}(L^2(E))^\prime$ characterized by $\mathcal{J} = \text{CAR}(L^2(E)) \cap \text{CAR}(L^2(E))^\prime P$ (cf. [Fe89 Prop. 3.11.9, 3.11.10, Thm. 3.10.7], [AK69]), and then the hereditary $C^*$-subalgebra is

$$\mathcal{J} \cap \mathcal{J}^* = \text{CAR}(L^2(E)) \cap P \text{CAR}(L^2(E))^\prime P.$$

Hence for any state $\omega$ we have $\omega(P) = \|\omega(\mathcal{J} \cap \mathcal{J}^*)\|$.
5.2 Proposition Let $S$ be relatively compact and let $P_{S^c}$ be the open projection of $\mathcal{N}_{S^c}$, and denote its complementary closed projection by $\overline{P}_S := 1 - P_{S^c}$. Define a map
\[ \bar{\nu}_S : \text{CAR}(L^2(E)) \to C^* \left( \{ P_{S^c} \} \cup \text{CAR}(L^2(E)) \right) \]
by $\bar{\nu}_S(A) := \overline{P}_S A \overline{P}_S$. Then we have that
\[ \bar{\nu}_S(A) = \overline{P}_S \nu_S(A) \overline{P}_S \quad \text{for all} \quad A \in \text{CAR}(L^2(E)). \]
Moreover $\bar{\nu}_S$ is an isomorphism on $\mathcal{A}(S)$, i.e. $\nu_S(\text{CAR}(L^2(E))) = \mathcal{A}(S) \triangleq \overline{P}_S \mathcal{A}(S) \overline{P}_S = \bar{\nu}_S(\text{CAR}(L^2(E)))$, and furthermore
\[ \mathcal{N}_{S^c} = \{ A \in \text{CAR}(L^2(E)) \mid \nu_S(A^* A) = 0 \}. \]

Proof. Since $P_{S^c}$ acts as a right identity for the elements of $\mathcal{N}_{S^c}$, we have that $a(c)(1 - P_{S^c}) = 0$ whenever $c \in L^2(E \uparrow S^c)$. Hence, for all $c \in L^2(E)$, we have
\[ a(c)(1 - P_{S^c}) = (a(p_{S^c} + a((1 - p_S)c))(1 - P_{S^c}) = a(p_{S^c})(1 - P_{S^c}), \]
and hence
\[ (1 - P_{S^c})a(c)^* = (1 - P_{S^c})a(p_{S^c})^*. \]
Given any normal ordered monomial
\[ A(c_1, \ldots, c_n; d_m, \ldots, d_1) := a(c_1)^* \cdots a(c_n)^* a(d_m) \cdots a(d_1), \]
we can permute the $a(c_i)^*$ amongst themselves, and the $a(d_j)$ factors amongst themselves using the CAR-relationships, acquiring only ± factors in the process. Thus using such permutations to get terms adjacent to $1 - P_{S^c}$, we obtain:
\[
\bar{\nu}_S(A(c_1, \ldots, c_n; d_m, \ldots, d_1)) = (1 - P_{S^c})A(c_1, \ldots, c_n; d_m, \ldots, d_1)(1 - P_{S^c})
\]
\[ = (1 - P_{S^c})A(p_{S^c}c_1, \ldots, p_{S^c}c_n; p_{S^d}d_1, \ldots, p_{S^d}d_1)(1 - P_{S^c}) \in (1 - P_{S^c})\mathcal{A}(S)(1 - P_{S^c}). \]
Thus the positive map $\bar{\nu}_S$ will take all normal ordered polynomials in the fields to the same normal ordered polynomials of the fields restricted to $S$, conjugated by $1 - P_{S^c} = \overline{P}_S$. Thus $\bar{\nu}_S(A) = \overline{P}_S \nu_S(A) \overline{P}_S$ for all $A \in \text{CAR}(L^2(E))$. In particular, the range of $\bar{\nu}_S$ is $\overline{P}_S \mathcal{A}(S) \overline{P}_S$.

Next, we show that on $\mathcal{A}(S)$ the map $\bar{\nu}$ is in fact an isomorphism. It suffices to show that $P_{S^c}$ commutes with $\mathcal{A}(S)$, since then $\bar{\nu}_S$ is a *-homomorphism, which is an isomorphism since $\mathcal{A}(S) = \text{CAR}(L^2(E \uparrow S))$ is simple. Recall that $P_{S^c}$ is the complementary projection of the projection onto the subspace annihilated by $\{ a(c) \mid c \in L^2(E \uparrow S^c) \}$ in the universal representation space. The generating elements $\{ a(c) \mid c \in L^2(E \uparrow S) \}$ of $\mathcal{A}(S)$ and their adjoints all anticommute with $\{ a(c) \mid c \in L^2(E \uparrow S^c) \}$, hence will preserve the subspace annihilated by $\{ a(c) \mid c \in L^2(E \uparrow S^c) \}$. Thus $\{ a(c) \mid c \in L^2(E \uparrow S) \}$ (as well as $\mathcal{A}(S)$) will commute with the projection onto this subspace, hence with $P_{S^c}$. As $\bar{\nu}_S$ is thus an isomorphism on $\mathcal{A}(S)$ it follows for its range that $(1 - P_{S^c})\mathcal{A}(S)(1 - P_{S^c}) \cong \mathcal{A}(S)$.

To see that $\mathcal{N}_{S^c} = \{ A \in \text{CAR}(L^2(E)) \mid \nu_S(A^* A) = 0 \}$, note that by the previous isomorphism, $0 = \nu_S(A^* A)$ if and only if $0 = \bar{\nu}_S(A^* A) = (1 - P_{S^c})A^* A(1 - P_{S^c})$ if and only if $A = AP_{S^c}$, which characterizes $\mathcal{N}_{S^c}$. \qed
Thus we have shown that the classical module action of $C^\infty(M)$ on the matter fields defines an automorphism group on $\text{CAR}(L^2(E))$ which provides the analogous localization to what this module action does in the classical picture. We also identified the quantum analogues of restriction maps, and obtained a proof that the $\text{CAR}(L^2(E))$ has only trivial gauge invariant elements.

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