NEW LINK INVARIANTS AND YANG-BAXTER EQUATION

by

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Abstract

We have new solutions to the Yang-Baxter equation, from which we have constructed new link invariants containing more than two arbitrary parameters. This may be regarded as a generalization of the Jones’ polynomial. We have also found another simpler invariant which discriminates only the linking structure of knots with each other, but not details of individual knot.

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1. **Introduction:**

Knot or link invariants are useful to distinguish two topologically inequivalent knots and links from each other. The well-known examples are those of Conway, Jones, Kauffman, and Homfly polynomials [1]. Although these invariants can be constructed in a variety of ways, one interesting method is to begin with solutions of Yang-Baxter equations ([1] and [2]). The purpose of this note is to present some new knot and link invariants in this manner. We will show, first, in section 2, the existence of a link invariant which distinguishes only linking structure but not the individual knot configuration of each component knot contained in the link. The solution possesses as many arbitrary parameters as are desired so as to enable us in general to distinguish any two linking structures. In section 3, we will consider a more general situation to obtain a family of knot invariants containing two arbitrary integer parameters by solving the Yang-Baxter equation (hereafter referred to as YBE). The new invariants may be considered as a generalization of the one-parameter Jones’ polynomials but differs from those of Kauffman and Homfly’s.

Since we start with the YBE in our construction, we will briefly sketch the material relevant to our calculations. Let $V$ be a finite-dimensional vector space of dimension $N$, i.e.

$$N = \text{Dim } V \quad (1.1)$$

Let $e_1, e_2, \ldots, e_N$ be a basis of $V$ and consider a linear mapping $R(\theta) : V \otimes V \rightarrow V \otimes V$ by

$$R(\theta) \ e_a \otimes e_b = \sum_{c,d=1}^N R^d_{ac}(\theta)e_c \otimes e_d \quad (1.2)$$

in terms of scattering matrix elements $R^d_{ac}(\theta)$ where $\theta$ is the spectral variable which may be identified as the rapidity, if we wish. Next, set

$$V^n = V \otimes V \otimes \ldots \otimes V \quad (n-\text{times}) \quad (1.3)$$

and introduce $R_{ij}(\theta) : V^n \rightarrow V^n (i,j = 1, 2, \ldots, n, i < j)$ in the analogous fashion ([2] and [3]), which operates only in the $i$-th and $j$-th vector spaces contained in the tensor product $V^n$. Then, the $\theta$-dependent YBE is the equation

$$R_{12}(\theta)R_{13}(\theta')R_{23}(\theta'') = R_{23}(\theta'')R_{13}(\theta')R_{12}(\theta) \quad (1.4)$$
where variables $\theta$, $\theta'$, and $\theta''$ satisfy the constraint

$$\theta' = \theta + \theta''$$  \hspace{1cm} (1.5)

For our study of the knot and link invariants, the $\theta$-dependence is actually superfluous, and we need consider only $\theta$-independent YBE:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$ \hspace{1cm} (1.6)

Evidently, Eq. (1.6) may be regarded as a special case of Eq. (1.4) by setting

$$R_{ij} = R_{ij}(\theta = 0) \quad \text{or} \quad R_{ij}(\theta = \infty)$$ \hspace{1cm} (1.7)

provided that $R_{ij}(\theta)$ is not singular at $\theta = 0$ and/or $\theta = \infty$.

Let $P_{ij}$ ($i, j = 1, 2, \ldots, n, i \neq j$) : $V^n \to V^n$ be the permutation operator of the $i$-th and $j$-th vectors in $V^n$, and set

$$\sigma_j = P_{j,j+1}R_{j,j+1} \quad (j = 1, 2, \ldots, n - 1)$$ \hspace{1cm} (1.8)

Then, it is known ([2] and [3]) that the $\theta$-independent YBE (1.6) will lead to

$$\sigma_{j+1}\sigma_j\sigma_{j+1} = \sigma_j\sigma_{j+1}\sigma_j \quad (j = 1, 2, \ldots, n - 2)$$ \hspace{1cm} (1.9)

in addition to

$$\sigma_j\sigma_k = \sigma_k\sigma_j \quad \text{if} \quad |j - k| \geq 2$$ \hspace{1cm} (1.10)

Assuming hereafter that the inverse $\sigma_j^{-1}$ or equivalently $R_{ij}^{-1}$ exists, then Eqs. (1.9) and (1.10) are precisely the Artin’s relations for the braid group $B_n$ of $n$-strings, which is generated by $1, \sigma_j, \text{and} \sigma_j^{-1} (j = 1, 2, \ldots, n - 1)$.

Any link can now be constructed out of braids in view of the Alexander theorem ([1] and [4]) by identifying both ends of the strings in the braid. However, in order to construct a link invariant, we further assume the existence of the Markov trace $\phi_n(g)$ in $B_n$ for $g \in B_n$, which satisfies the Markov conditions:

(i) $\phi_n(gg') = \phi_n(g'g) \quad (g, g' \in B_n)$ \hspace{1cm} (1.11a)

(ii) $\phi_{n+1}(g\sigma_n) = \tau \phi_n(g) \quad (g \in B_n)$ \hspace{1cm} (1.11b)
(iii) \( \phi_{n+1}(g^{-1}) = \tau \phi_n(g) \quad (g \in B_n) \). \quad (1.11c)

Here \( \tau \) and \( \tau \) are some non-zero constants. The link invariant associated with the Markov trace is now given by

\[
M_n(g) = \left( \frac{1}{\tau \tau} \right)^{n-1} \left( \frac{\tau}{\tau} \right)^{\frac{1}{2} w(g)} \phi_n(g)
\]  
(1.12)

where \( w(g) \) is the exponent sum of the generators appearing in the braid \( g \) (for example, if \( g = \sigma_1^3 \sigma_2^{-1} \), then \( w(g) = 3 - 1 = 2 \)). In this paper, we identify the Markov trace to be

\[
\phi_n(g) = \text{Tr } \rho(g) \quad .
\]  
(1.13)

Here \( \rho(g) \) is the representation matrix of \( g \in B_n \) in the module \( V^n \) on which \( g \) acts. The Markov conditions are then satisfied, provided that the scattering matrix \( R_{ab}^{dc}(\equiv R_{ab}^{dc}(\theta = 0)) \) obeys

\[
\sum_{j=1}^{N} R_{aj}^{dc} = \tau \delta_a^d \quad , \quad \sum_{j=1}^{N} (R^{-1})^{jc} = \tau \delta_b^c \quad .
\]  
(1.14)

We mention also the fact that we can always set \( \tau = \tau = 1 \) for all results given in sections 2 and 3 of this note. We then have

\[
M_n(g) = \phi_n(g) = \text{Tr } \rho(g) \quad .
\]  
(1.15)

2. **Multi-parameter Solution of YBE and Link Invariant**

Let \( V \) be the \( N \)-dimensional vector space as in section 1, and consider linear mappings \( J_\mu : V \to V \) for \( \mu = 1, 2, \ldots, p \), satisfying

\[
J_\mu J_\nu = J_\nu J_\mu \quad .
\]  
(2.1)

Suppose that \( R(\theta) : V \otimes V \to V \otimes V \) is given by

\[
R(\theta)x \otimes y = \sum_{\mu,\nu=1}^p R_{\mu\nu}(\theta)J_\mu x \otimes J_\nu y
\]  
(2.2)

for some functions \( B_{\mu\nu}(\theta) \) of \( \theta \). Defining \( R_{ij}(\theta) \) similarly in \( V^n \), it is trivial to see

\[
R_{ij}(\theta)R_{kj}(\theta') = R_{ij}(\theta')R_{ij}(\theta) \quad .
\]  
(2.3)
Therefore, Eq. (2.2) furnishes not only a solution of the YBE (1.4) but also that of the classical YBE [3],

\[ [R_{12}(\theta), R_{13}(\theta')] + [R_{12}(\theta), R_{23}(\theta'')] + [R_{13}(\theta'), R_{23}(\theta'')] = 0 \quad (2.4) \]

Note that \( B_{\mu\nu}(\theta) \) for \( \mu, \nu = 1, 2, \ldots, p \) are arbitrary functions of \( \theta \).

In order to construct the link invariant, we set \( \theta = \theta' = \theta'' = 0 \) with \( R = R(\theta = 0) \) and \( B_{\mu\nu} = B_{\mu\nu}(\theta = 0) \). Expressing the operation of \( R^{-1} \) in \( V^2 \) similarly by

\[ R^{-1}x \otimes y = \sum_{\mu, \nu=1}^{p} B_{\mu\nu} J_\mu x \otimes J_\nu y \quad , \quad (2.5) \]

the Markov conditions Eqs. (1.14) are rewritten now as

\[ \sum_{\mu, \nu=1}^{p} B_{\mu\nu} J_\mu J_\nu = \tau \text{Id} \quad , \quad (2.6a) \]
\[ \sum_{\mu, \nu=1}^{p} \overline{B}_{\mu\nu} J_\mu J_\nu = \overline{\tau} \text{Id} \quad (2.6b) \]

where \( \text{Id} \) stands for the identity map in \( V \).

A simple realization satisfying all these conditions is easily found, as follows. Suppose that we have

(i) \( p = N \) \quad (2.7a)
(ii) \( J_\mu J_\nu = \delta_{\mu\nu} J_\mu \quad (\mu, \nu = 1, 2, \ldots, N) \) \quad (2.7b)
(iii) \( \sum_{\mu=1}^{N} J_\mu = \text{Id} \quad . \) \quad (2.7c)

Note that \( J_\mu \) may be identified with the projection operator of the basis vector \( e_\mu \) as

\[ J_\mu e_\nu = \delta_{\mu\nu} e_\nu \quad (2.8) \]

which we assume hereafter. Because of \( p = N \), both greek and latin indices can now take the same range of values 1, 2, \ldots, \( N \), so that we shall hereafter in this section use them interchangeably. Then, the scattering matrix can be expressed as

\[ R^{dc}_{ab}(\theta) = \delta_{b}^{d} \delta_{a}^{c} B_{ab} \quad , \quad (R^{-1})^{dc}_{ab} = \delta_{b}^{d} \delta_{a}^{c} \overline{B}_{ab} \quad (2.9) \]
with
\[ B_{ab} = (B_{ab})^{-1} \quad . \tag{2.10} \]

Assuming moreover,
\[ B_{11} = B_{22} = \ldots = B_{NN} = 1 \quad , \tag{2.11} \]

the Markov conditions Eqs. (2.6) are satisfied with
\[ \tau = \tau = 1 \quad . \tag{2.12} \]

Note that \( B_{\mu\nu} \) for \( \mu \neq \nu \) are completely arbitrary constants as long as they are non-zero.

We can now compute the Markov invariant for any link, when we note
\[ \text{Tr} \ J_\mu = 1 \quad . \tag{2.13} \]

For example, consider the link corresponding to the braid \( g = \sigma_1^2 \) with \( n = 2 \) which is depicted graphically in Fig. 1. It is easy to calculate
\[ \text{Tr} \ \sigma_1^2 = \sum_{\mu,\nu=1}^{N} B_{\mu\nu} B_{\nu\mu} \text{Tr} \ J_\mu \ (\text{Tr} \ J_\nu) = \sum_{\mu,\nu=1}^{N} B_{\mu\nu} B_{\nu\mu} \quad , \tag{2.14} \]

where we have written \( \rho(\sigma_1) = \sigma_1 \) for simplicity with the same convention hereafter.

**Figure 1.** A link corresponding to the braid \( g = \sigma_1^2 \).
In Fig. 1, we designated two independent loops contained therein as $\mu$ and $\nu$, respectively by reasoning to be explained.

We can compute other invariants in a similar fashion. However, there exists a simple graphical realization for computations of the invariant as follows: First, suppose that the link consists of $m$ interlocking loops ($m \leq n$). We name these loops as $\mu, \nu, \ldots$ etc. with directions as in Fig. 1, which can assume $N$ values $1, 2, \ldots, N$. For each intersection of the directed $\mu$-th and $\nu$-th loops, we assign a factor $B_{\mu\nu}$ or $\overline{B}_{\nu\mu} = (B_{\nu\mu})^{-1}$, depending upon whether the $\mu$-th loop at the left crosses the $\nu$-th loop at the right above or below (see Figures 2 and 3):

![Figure 2. Crossing of the string $\mu$ above $\nu$.](image)

![Figure 3. Crossing of the string $\mu$ below $\nu$.](image)

We then multiply all these factors and sum upon all loop indices $\mu, \nu, \ldots$ over the values $1, 2, \ldots, N$. Finally, we assign a factor $N$ for any unknot (i.e. an isolated simple
circle) in the link if it exists. The rule immediately gives the result of Eq. (2.14) for Fig. 1. As a more complicated example, consider the link depicted in Fig. 4 corresponding to the braid \( g = \sigma_1^2 \sigma_2^{-2} \) with \( n = 3 \) to find
\[
\text{Tr} \left( \sigma_1^2 \sigma_2^{-2} \right) = \sum_{\mu, \nu, \lambda=1}^{N} B_{\mu \nu} B_{\nu \mu} (B_{\nu \lambda})^{-1} (B_{\lambda \nu})^{-1}.
\]

(2.15)

**Figure 4.** Link corresponding to the braid \( g = \sigma_1^2 \sigma_2^{-2} \).

Although we have found the rule on the basis of the Markov trace, we can directly verify its invariances against the Reidemeister’s 3 moves ([1] and [4]). Especially, Eq. (2.11) guarantees the invariance under the 3rd Reidemeister’s move as we can observe from Fig. 5.
We remark here that Fig. 5 is the graphical realization of the Markov condition Eq. (1.14) for $\tau = \tau' = 1$. Also, we need not represent now the link in terms of the braid for the calculation, although we will do so for the sake of illustration in this note.

We note that for a pure knot, we have only a single loop, and hence that we have always the trivial result $\phi_n(g) = N$, no matter how complicated the knot is. This is because we have $B_{\mu\mu} = 1$. For example, consider a pure knot depicted in Fig. 6, corresponding to the braid $g = \sigma_3^1$ where we calculate $\text{Tr} \sigma_3^1 = \sum_{\mu=1}^{N} (B_{\mu\mu})^3 = N$.

In summary, the present invariant is useful only for determining the global interlocking nature of the link, ignoring all details of individual knot structures contained therein.

Although we have considered a particular solution given by Eqs. (2.7) and (2.8), we can proceed similarly for more general solutions of Eqs. (2.1) and (2.6). Nevertheless, the
resulting Markov invariant can tell us again only about the interlocking link structure but not on the individual knot structure. However, we will not go into the details. In this connection, it may be worthwhile to make the following comment. Suppose that again we assume \( p = N \) but
\[
J_\mu e_\nu = e_{\mu + \nu} \quad (\mu, \nu = 1, 2, \ldots, N)
\]
(2.16) instead of Eq. (2.8), where we impose the cyclic condition
\[
e_{\mu + N} = e_\mu \quad (\mu = 1, 2, \ldots, N)
\]
(2.17) for the basis vectors. We can now readily verify the validity of Eq. (2.1) with \( J_\mu J_\nu = J_{\mu + \nu} \) and \( J_N = \text{Id} \). The scattering matrix is now given by
\[
R_{ab}^{dc}(\theta) = F_{a-c}^{d-b}(\theta)
\]
(2.18) where we have set
\[
F_{a-c}^{d-b}(\theta) = B_{c-a,d-b}(\theta)
\]
(2.19) by extending the definition of \( B_{\mu\nu}(\theta) \) to satisfy \( B_{\mu\pm N,\nu}(\theta) = B_{\mu,\nu\pm N}(\theta) = B_{\mu,\nu}(\theta) \). The fact that Eq. (2.18) gives a solution of the YBE for arbitrary function \( F_\mu^{\nu}(\theta) \) \((-N < \mu, \nu < N)\) can be directly verified also from the component-wise YBE:
\[
\sum_{j,k,\ell=1}^{N} R_{a_1b_1}^{jk}(\theta) R_{k_1c_1}^{\ell a_2}(\theta') R_{j_2c_2}^{b_2}(\theta'') = \sum_{j,k,\ell=1}^{N} R_{b_1c_1}^{jk}(\theta'') R_{c_1j}^{\ell k}(\theta') R_{k_2a_2}^{b_2}(\theta)
\]
(2.20) after some calculations. We can construct Markov invariants on the basis of the solution Eq. (2.18). However, since a more general case will be discussed in the next section, we will not go into detail.

3. YBE as Triple Product and New Knot Invariants

In order to find non-trivial knot invariants, we must discover more general solutions of the YBE. For this, it is more convenient to recast the YBE as the triple product equation [5]. We will consider only the case of the \( \theta \)-independent YBE for simplicity in the following. Let \( \langle \cdot | \cdot \rangle \) be a symmetric bilinear non-degenerate form in the vector space \( V \) and set
\[
g_{jk} = g_{kj} = \langle e_j | e_k \rangle \quad , \quad (j, k = 1, 2, \ldots, N)
\]
(3.1)
We raise the indices in terms of the inverse tensor \( g^{jk} \) as
\[
e^j = \sum_{k=1}^{N} g^{jk} e_k \quad .
\] (3.2)

Any \( x \in V \) can then be expanded as
\[
x = \sum_{j=1}^{N} < x | e_j > e^j = \sum_{j=1}^{N} e_j < e^j | x > \quad .
\] (3.3)

We now introduce two triple linear products \([x, y, z]\) and \([x, y, z]^*\) in \( V \) by
\[
[e^c, e_a, e_b] = \sum_{d=1}^{N} e_d R^{dc}_{ab} \quad ,
\] (3.4a)
\[
[e^d, e_b, e_a]^* = \sum_{c=1}^{N} R^{dc}_{ab} e_c \quad (3.4b)
\]
so that we have
\[
R^{dc}_{ab} = < e^d | [e^c, e_a, e_b] > = < e^c | [e^d, e_b, e_a]^* > \quad .
\] (3.5)

Now, the \( \theta \)-independent YBE can then be shown to be rewritten as the triple-product equation
\[
\sum_{j=1}^{N} [v, [u, e_j, z], [e^j, x, y]]^* = \sum_{j=1}^{N} [u, [v, e_j, x]^*, [e^j, z, y]^*] \quad .
\] (3.6)

As we may easily see, the choice \( x = e_{a_1}, y = e_{b_1}, z = e_{c_1}, u = e_{a_2}, \) and \( v = e_{c_2} \) in Eq. (3.6) will reproduce Eq. (2.20) for \( \theta = \theta' = \theta'' = 0 \). Also Eq. (3.5) will lead to a constraint equation
\[
< u|[v, x, y] > = < v|[u, y, x]^* > \quad (3.7)
\]
in the basis-independent notation. The relationship between the triple products and the linear mapping \( R \) given in Eq. (1.2) is easily found to be
\[
R x \otimes y = \sum_{j=1}^{N} e_j \otimes [e^j, x, y] = \sum_{j=1}^{N} [e^j, y, x]^* \otimes e_j \quad .
\] (3.8)

Especially, if we define
\[
R^* = P_{12} R P_{12} \quad ,
\] (3.9)
then we obtain the symmetrical relation of

\[ R^* x \otimes y = \sum_{j=1}^{N} e_j \otimes [e^j, x, y]^* = \sum_{j=1}^{N} [e^j, y, x] \otimes e_j \quad . \] (3.10)

Note that the condition \( R^* = R \) is equivalent to have \([z, x, y]^* = [z, x, y]\).

After these preparations, we seek solutions of Eq. (3.6) with the ansatz of

\[ [x, y, z] = \sum_{\mu, \nu=1}^{p} \left\{ A_{\mu\nu} < y|J_\nu z > J_\mu x + B_{\mu\nu} < x|J_\nu y > J_\mu z \right. \]

\[ + C_{\mu\nu} < z|J_\nu x > J_\mu y \left. \} \right) \quad (3.11a) \]

\[ [x, y, z]^* = \sum_{\mu, \nu=1}^{p} \left\{ A_{\mu\nu} < y|J_\nu z > J_\mu x + B_{\nu\mu} < x|J_\nu y > J_\mu z \right. \]

\[ + C_{\nu\mu} < z, |J_\nu x > J_\mu y \left. \} \right) \quad (3.11b) \]

for some linear mapping \( J_\mu (\mu = 1, 2, \ldots, p) \) in \( V \), where \( A_{\mu\nu}, B_{\mu\nu}, \) and \( C_{\mu\nu} \) are some constants to be determined. The constraint Eq. (3.7) is satisfied by Eq. (3.11), provided that we have

\[ < x|J_\mu y > = < J_\mu x|y > \quad . \] (3.12)

The action of \( R \) in \( V \otimes V \) can be obtained from Eq. (3.8) to be

\[ R x \otimes y = \sum_{\mu, \nu=1}^{p} \left\{ A_{\mu\nu} < y|J_\nu y > \sum_{j=1}^{N} e_j \otimes J_\mu e^j \right. \]

\[ + B_{\mu\nu} J_\nu x \otimes J_\mu y + C_{\mu\nu} J_\nu y \otimes J_\mu x \left. \} \right) \quad . \] (3.13)

Comparing this with Eq. (2.2), we see that \( B_{\mu\nu} \) here stands really for \( B_{\nu\mu} \) of section 2, or equivalently we are interchanging the role of \( R \) and \( R^* \). Defining the inverse \( R^{-1} \) similarly by

\[ R^{-1} x \otimes y = \sum_{\mu, \nu=1}^{p} \left\{ A_{\mu\nu} < x|J_\nu y > \sum_{j=1}^{N} e_j \otimes J_\mu e^j \right. \]

\[ + B_{\mu\nu} J_\nu x \otimes J_\mu y + C_{\mu\nu} J_\nu y \otimes J_\mu x \left. \} \right) \quad , \] (3.14)
the Markov condition is now equivalent to

\[
\sum_{\mu, \nu=1}^{p} \{ (A_{\mu\nu} + B_{\mu\nu}) J_{\mu} J_{\nu} + (\text{Tr} J_{\nu}) C_{\mu\nu} J_{\mu} \} = \tau \text{Id} \quad (3.15a)
\]

\[
\sum_{\mu, \nu=1}^{p} \{ (\overline{A}_{\mu\nu} + \overline{B}_{\nu\mu}) J_{\mu} J_{\nu} + (\text{Tr} J_{\nu}) \overline{C}_{\nu\mu} J_{\mu} \} = \tau \text{Id} \quad (3.15b)
\]

where we have set

\[
\text{Tr} J_{\nu} = \sum_{k=1}^{N} < e^{k} | J_{\nu} e^{k} > . \quad (3.16)
\]

We have now to impose some algebraic relations among \( J_{\mu} \)'s. We will not consider, however, those given by Eqs. (2.7) and (2.8) in this note because of the following reason. Suppose that we assume the validity of Eqs. (2.7) and (2.8). Then, we can find the following solution of the YBE (1.6) or equivalently (3.6):

\[
A_{\mu\nu} = 0 \quad , \quad (3.17a)
\]

\[
C_{\mu\nu} = C \theta(\mu - \nu) = \begin{cases} C, & \text{if} \quad \mu \geq \nu \\ 0, & \text{if} \quad \mu < \nu \end{cases} \quad , \quad (3.17b)
\]

\[
B_{\mu\nu} = B \frac{g_{\mu}}{g_{\nu}} - \left\{ \frac{C}{2} + B \pm \left[ \left( \frac{C}{2} \right)^{2} + B^{2} \right]^{\frac{1}{2}} \right\} \delta_{\mu\nu} \quad (3.17c)
\]

for arbitrary constants \( B, C, \) and \( g_{\mu} (\mu = 1, 2, \ldots, N) \). Especially, if we choose \( B = 1, \ C = q - \frac{1}{q}, \) and \( g_{\mu} = 1 \) for a parameter \( q \), it will lead to the well-known solution [6] of

\[
R_{ab}^{dc} = \left( q - \frac{1}{q} \right) \delta_{a}^{d} \delta_{b}^{c} [\theta(a - b) - \delta_{ab}] + \delta_{a}^{c} \delta_{b}^{d} [q\delta_{ab} + (1 - \delta_{ab})] . \quad (3.18)
\]

However, the Markov condition Eq. (1.14) is not satisfied by this solution except for the trivial case of \( q = 1 \). In order to obtain link invariant, we must resort then to a more elaborate graphical analysis based upon the state model [6]. Unfortunately, the method does not appear to be readily extended to the more general solution Eq. (3.17).

Instead of Eqs. (2.7) and (2.8), we will assume the following relations among \( J_{\mu} \)'s:

First, we extend the range of values for Greek indices \( \mu, \nu \) etc. to all integers with periodicity conditions

\[
J_{\mu+p} = J_{\mu} \quad , \quad A_{\mu+p,\nu} = A_{\mu,\nu+p} = A_{\mu\nu} \quad (3.19)
\]
and similarly for $B_{\mu\nu}$ and $C_{\mu\nu}$. Next, we assume

(i) $J_\mu J_\nu = J_{\mu+\nu}$  \hspace{1cm} (3.20a)
(ii) $J_0 = J_p = 1d$ \hspace{1cm} (3.20b)
(iii) $\text{Tr} \ J_\mu = N \delta_{\mu,0}$ \hspace{1cm} (3.20c)

where we have set

$$\delta_{\mu\nu} = \begin{cases} 
1, & \text{if } \mu = \nu \pmod{p} \\
0, & \text{otherwise}
\end{cases} \hspace{1cm} (3.21)$$

We must then have

$$N = pm \hspace{1cm} (3.22)$$

for another positive integer $m$ by the following reason. Setting

$$P = \frac{1}{p} \sum_{\mu=0}^{p-1} J_\mu$$

it is easy to see

$$PJ_\mu J_\mu P = P \hspace{1cm} P^2 = P .$$

Especially, $\text{Tr} \ P = m$ must be a positive integer. On the other side, we calculate

$$\text{Tr} \ P = \frac{1}{p} \sum_{\mu=0}^{p-1} \text{Tr} \ J_\mu = \frac{1}{p} N$$

which leads to the validity of Eq. (3.22).

The basis vectors $e_j$ ($j = 1, 2, \ldots, N$) may be labelled now as

$$e_j = e_{\mu,A} \ (\mu = 1, 2, \ldots, p, \ A = 1, 2, \ldots, m) \hspace{1cm} (3.23)$$

with

$$< e_{\mu,A} | e_{\nu,B} > = \delta_{\mu+\nu,0} \delta_{AB} \hspace{1cm} (3.24)$$

on which $J_\mu$ acts as

$$J_\mu e_{\nu,A} = e_{\mu+\nu,A} \hspace{1cm} (3.25)$$

Note that these relations are then consistent with Eq. (3.12). Also, if $m = 1$, then Eq. (3.25) will reproduce Eq. (2.16).
Now, we insert the expressions in Eqs. (3.11) to both sides of the YBE (3.6) and use Eq. (3.3). After some calculations, we then find

\[
O = \sum_{j=1}^{N} \left\{ [v, [u, e_j, z], [e^j, x, y]]^* - [u, [v, e_j, x], [e^j, z, y]^*] \right\}
\]

\[
= \sum_{\mu, \nu, \lambda=1}^{p} \left\{ K_1 < x|J_\lambda y > < u|J_\mu z > J_\nu v - \hat{K}_1 < z|J_\lambda y > < v|J_\mu x > J_\nu u 
+ K_2 < u|J_\lambda y > < z|J_\mu x > J_\nu v - \hat{K}_2 < v|J_\mu z > J_\nu u 
+ K_3 < u|J_\lambda x > < z|J_\mu y > J_\nu v - \hat{K}_3 < v|J_\mu y > J_\nu u 
+ K_4 < u|J_\lambda u > < y|J_\mu z > J_\nu x - \hat{K}_4 < u|J_\lambda v > < y|J_\mu x > J_\nu z 
+ K_5 < v|J_\lambda z > < u|J_\mu y > J_\nu x - \hat{K}_5 < u|J_\lambda x > < y|J_\mu v > J_\nu z 
+ K_6 < v|J_\lambda y > < u|J_\mu z > J_\nu x - \hat{K}_6 < u|J_\lambda y > < v|J_\mu x > J_\nu z 
+ K_7 < v|J_\lambda u > < z|J_\mu x > J_\nu y 
+ K_8 < v|J_\lambda x > < u|J_\mu z > J_\nu y \right\} 
\]

(3.26)

Here, we have, for simplicity, suppressed the indices \(\mu, \nu\), and \(\lambda\) with

\[
K_j \equiv K_{j,\mu\nu\lambda} \quad (j = 1, 2, \ldots, 8) 
\]

We also note that the term proportional to \(< v|J_\lambda z > < u|J_\lambda x > J_\nu y\) is absent, since it will result only from \(B_{\mu\nu}\) terms in accordance with the result of section 2. The explicit values for \(K_j\) are given by

\[
K_1 = \sum_{\alpha, \beta, \gamma, \tau=1}^{p} \left\{ \delta_{\alpha+\beta+\gamma+\tau, \mu} A_{\nu\beta} A_{\gamma\lambda} A_{\alpha\tau} A_{\gamma\lambda} + B_{\alpha\tau} A_{\gamma\lambda} \right\}
\]

(3.27a)

\[
K_2 = \sum_{\alpha, \beta, \gamma, \lambda=1}^{p} \left\{ A_{\nu,\alpha-\gamma} A_{\mu-\beta} B_{\gamma-\alpha,\beta} + A_{\nu,\mu-\gamma} B_{\alpha,\lambda-\beta} C_{\gamma-\alpha,\beta} - B_{\gamma,\lambda-\beta} A_{\alpha-\gamma} C_{\nu-\alpha,\beta} \right\} 
\]

(3.27b)
\[K_3 = \sum_{\alpha, \beta, \gamma=1}^p \left\{ A_{\nu, \lambda-\gamma} A_{\alpha, \mu-\beta} C_{\gamma-\alpha, \beta} + A_{\nu, \mu-\gamma} B_{\alpha, \lambda-\beta} B_{\gamma-\alpha, \beta} - A_{\gamma-\alpha, \mu} B_{\alpha, \lambda-\beta} B_{\nu-\gamma, \beta} - A_{\gamma-\beta, \mu} A_{\alpha, \lambda-\gamma} C_{\nu-\alpha, \beta} \right\} \] (3.27c)

\[K_4 = \sum_{\alpha, \beta, \gamma=1}^p \left\{ B_{\beta, \nu-\gamma} A_{\lambda-\beta, \alpha} C_{\gamma-\mu, \alpha} - A_{\gamma-\alpha, \mu} A_{\lambda-\beta, \alpha} B_{\nu-\gamma, \beta} - A_{\gamma-\beta, \mu} B_{\lambda-\gamma, \alpha} C_{\nu-\alpha, \beta} \right\} \] (3.27d)

\[K_5 = \sum_{\alpha, \beta, \gamma=1}^p \left\{ B_{\beta, \nu-\gamma} B_{\lambda-\beta, \alpha} C_{\gamma-\mu, \alpha} - B_{\gamma-\mu, \beta} B_{\lambda-\gamma, \alpha} C_{\nu-\alpha, \beta} \right\} \] (3.27e)

\[K_6 = \sum_{\alpha, \beta, \gamma=1}^p \left\{ B_{\beta, \nu-\gamma} C_{\alpha-\beta, \mu} C_{\gamma-\lambda, \alpha} + C_{\beta, \nu-\alpha} C_{\alpha-\gamma, \mu} B_{\lambda-\beta, \gamma} - C_{\gamma, \mu-\beta} B_{\lambda-\gamma, \alpha} C_{\nu-\alpha, \beta} \right\} \] (3.27f)

\[K_7 = \sum_{\alpha, \beta, \gamma=1}^p \left\{ B_{\beta, \nu-\gamma} A_{\lambda-\beta, \alpha} B_{\gamma, \mu-\alpha} - B_{\mu-\alpha, \gamma} A_{\lambda-\beta, \alpha} B_{\nu-\gamma, \beta} \right\} \] (3.27g)

\[K_8 = \sum_{\alpha, \beta, \gamma=1}^p \left\{ B_{\beta, \nu-\gamma} C_{\alpha-\beta, \mu} B_{\gamma, \lambda-\alpha} - B_{\mu-\alpha, \gamma} C_{\lambda, \alpha-\beta} B_{\nu-\gamma, \beta} + C_{\beta, \nu-\alpha} C_{\alpha-\gamma, \mu} C_{\lambda-\beta, \gamma} - C_{\gamma, \mu-\beta} C_{\lambda, \alpha-\gamma} C_{\nu-\alpha, \beta} \right\} \] (3.27h)

Moreover \( \hat{K}_j \) \((j = 1, 2, \ldots, 6)\) are the same expression as \( K_j \) except for the interchange of \( B_{\mu\nu} \leftrightarrow B_{\nu\mu}, C_{\mu\nu} \leftrightarrow C_{\nu\mu} \).

If we are interested in the \( \theta \)-dependent YBE (1.4), then the expressions (3.26) and (3.27) are still valid, if we interpret the product ABC in that order, for example, by \( A(\theta'')B(\theta')C(\theta) \) for \( K_j \) and \( A(\theta)B(\theta')C(\theta'') \) for \( \hat{K}_j \), respectively. Actually we will have \( K_5 = \hat{K}_5 = 0 \) for the present \( \theta \)-independent case because of the following reason. We change first \( \alpha \to \nu - \alpha, \beta \to \mu - \beta, \) and \( \gamma \to \lambda - \gamma \) in the second term of \( K_5 \) and then let \( \alpha \to \gamma \to \beta \to \alpha \) to see the desired cancellation of the first term.

The YBE (3.6) is now satisfied, provided that we have

\[K_j = \hat{K}_j = 0 \quad (j = 1, 2, 3, 4, 6) \] (3.29a)

\[K_7 = K_8 = 0 \quad . \] (3.29b)
Although it is difficult to find the general solution of Eqs. (3.29), we found some special solutions which further satisfy the Markov condition (3.15) given now by

\[ \sum_{\lambda=1}^{p} (A_{\lambda, \mu} - B_{\lambda, \mu}) + N C_{\mu, 0} = \tau \delta_{\mu, 0} \]  \hspace{1cm} (3.30a)

\[ \sum_{\lambda=1}^{p} (\overline{A}_{\lambda, \mu} - \overline{B}_{\lambda, \mu}) + N \overline{C}_{0, \mu} = \tau \delta_{\mu, 0} \]  \hspace{1cm} (3.30b)

When we note

\[ \sum_{j=1}^{N} J_{\mu} e_j \otimes J_{\nu} e^j = \sum_{j=1}^{N} e_j \otimes J_{\mu+\nu} e^j , \]

then the relation \( R R^{-1} = \text{Id} \) can be expressed as

\[ N \sum_{\lambda=1}^{p} \overline{A}_{\lambda \nu} A_{\mu, -\lambda} + \sum_{\alpha, \beta=1}^{p} \{ A_{\beta \nu} [B_{\alpha, \mu-(\alpha+\beta)} + C_{\alpha, \mu-(\alpha+\beta)}] + B_{\alpha, \nu-(\alpha+\beta)} A_{\mu \beta} + C_{\alpha, \nu-(\alpha+\beta)} A_{\mu \beta} \} = 0 \]  \hspace{1cm} (3.31a)

\[ \sum_{\alpha, \beta=1}^{p} \{ B_{\mu-\alpha, \nu-\beta} B_{\alpha \beta} + \overline{C}_{\alpha, \nu-(\alpha+\beta)} A_{\mu \beta} \} = \delta_{\mu, 0} \delta_{\nu, 0} \]  \hspace{1cm} (3.31b)

\[ \sum_{\alpha, \beta=1}^{p} \{ B_{\nu-\beta, \mu-\alpha} C_{\alpha \beta} + \overline{C}_{\mu-\alpha, \nu-\beta} B_{\alpha \beta} \} = 0 \]  \hspace{1cm} (3.31c)

We seek solutions of the YBE with the ansatz of

\[ A_{\mu \nu} = \left( \delta_{\mu+\nu, 0} - \frac{1}{p} \right) A + D \]  \hspace{1cm} (3.32a)

\[ \overline{A}_{\mu \nu} = \left( \delta_{\mu+\nu, 0} - \frac{1}{p} \right) \overline{A} + \overline{D} \]  \hspace{1cm} (3.32b)

\[ C_{\mu \nu} = \left( \delta_{\mu+\nu, 0} - \frac{1}{p} \right) C + F \]  \hspace{1cm} (3.32c)

\[ \overline{C}_{\mu \nu} = \left( \delta_{\mu+\nu, 0} - \frac{1}{p} \right) \overline{C} + \overline{F} \]  \hspace{1cm} (3.32d)

for some constants \( A, \overline{A}, C, \overline{C}, D, \overline{D}, F, \) and \( \overline{F} \). Moreover, we impose the condition

\[ \sum_{\lambda=1}^{p} B_{\lambda, \mu-\lambda} = G \]  \hspace{1cm} (3.33a)

\[ \sum_{\lambda=1}^{p} \overline{B}_{\lambda, \mu-\lambda} = \overline{G} \]  \hspace{1cm} (3.33b)
as well as
\[ \sum_{\alpha,\beta=1}^p \overline{B}_{\mu-\alpha,\nu-\alpha} B_{\alpha\beta} = \delta_{\mu,0} \delta_{\nu,0} - \frac{1}{p} \delta_{\mu+\nu,0} + G\overline{G} \quad (3.33c) \]
for all \( \mu, \nu = 1, 2, \ldots, p \). A simple solution satisfying Eqs. (3.33) is for example given by
\[ B_{\mu\nu} = \frac{1}{k} \left[ \delta_{\mu,0} \delta_{\nu,0} - \frac{1}{p} \delta_{\mu+\nu,0} \right] + \frac{1}{p} G, \]
\[ \overline{B}_{\mu\nu} = \frac{1}{k} \left[ \delta_{\mu,0} \delta_{\nu,0} - \frac{1}{p} \delta_{\mu+\nu,0} \right] + \frac{1}{p} \overline{G} \]
for any constant \( k \).

We have then found the following three solutions of the YBE. First, all these solutions must satisfy the conditions:

\[ p(A^2 + C^2) + NAC = 0 \quad , \quad (3.34a) \]
\[ \tau = pA + NC = p(pD + NF + G) = -pC^2/A \quad , \quad (3.34b) \]
\[ p^2\overline{A}A = p^2\overline{C}C = \tau\overline{\tau} = 1 \quad , \quad (3.34c) \]
\[ GD = GF = 0 \quad . \quad (3.34d) \]

The rests of relations are given then by

**Solution 1**

\[ G = 0 \quad , \quad A = pD \quad , \quad C = pF \quad , \quad (3.35a) \]
\[ \overline{G} = 0 \quad , \quad \overline{A} = p\overline{D} \quad , \quad \overline{C} = p\overline{F} \quad , \quad (3.35b) \]

**Solution 2**

\[ G = -C^2/A \quad , \quad D = F = 0 \quad , \quad (3.36a) \]
\[ \overline{G} = -(\overline{C})^2/\overline{A} \quad , \quad \overline{D} = \overline{F} = 0 \quad , \quad (3.36b) \]
Solution 3

\[
G = 0 \quad , \quad D = C^4/pA^3 \quad , \quad F = C^3/pA^2 \quad ,
\]
\[
G = 0 \quad , \quad D = (C)^4/p(A)^3 \quad , \quad F = (C)^3/p(A)^2 \quad ,
\]

(3.37a) (3.37b)

The special case of \( p = 1 \) (and hence \( N = m \)) is of some interest. In that case, we have \( B_{\mu\nu} = 0 \) for solutions 1 and 3, while solution 2 will reduce to a special case of the one given in section 2. Consider the solution 1 for \( p = 1 \), where the scattering matrix is now given by

\[
R_{ab}^{dc} = A g^{dc}g_{ab} + C \delta^d_a \delta^c_b \quad ,
\]

(3.38a)

\[
(R^{-1})_{ab}^{dc} = \frac{1}{A} g^{dc}g_{ab} + \frac{1}{C} \delta^d_a \delta^c_b
\]

(3.38b)

with

\[
\frac{C}{A} + \frac{A}{C} = -N
\]

(3.38c)

If we normalize \( R_{ab}^{dc} \) by setting \( AC = 1 \), then this reduces to the solution given by Kauffman [7] who has also shown that the resulting knot invariant corresponds to the Jones’ polynomial. This fact can be seen also as follows. It is more convenient to normalize \( R_{ab}^{dc} \) now by \( \tau = \tau = 1 \) and hence \( A = -C^2 \). In that case, we find

\[
\frac{1}{C^2} R_{ab}^{dc} - C^2(R^{-1})_{ab}^{dc} = \left( \frac{1}{C} - C \right) \delta^d_a \delta^c_b
\]

(3.39)

which is the generating relation for the Jones’ polynomial.

The three solutions given by Eqs. (3.32)-(3.37) contain two arbitrary integers \( p, m \) as well as many constants in \( B_{\mu\nu} \), when we use the normalization \( \tau = \tau = 1 \). However, we will no longer have the simple relation such as Eq. (3.39) for the general case. The resulting link invariants are moreover rather complicated. For example, we calculate here \( \text{Tr} \ (\sigma_1)^{\ell} \) for any positive integer \( \ell \):

\[
\text{Tr} \ (\sigma_1)^{\ell} = (p - 1)(NA + pC)^{\ell} + (p - 1)(N^2 - p^2)p^{\ell - 2}C^\ell
\]

\[
+ p^{\ell}(ND + pF)^{\ell} + p^{2(\ell - 1)}(N^2 - p^2)F^{\ell}
\]

\[
+ \frac{1}{2} [1 - (-1)^{\ell}] Np^{\ell - 1}G^\ell + \frac{1}{2} [1 + (-1)^{\ell}] N^2 Q_{\ell} \quad , \quad (\ell \geq 1)
\]

(3.40)
while \( Q_\ell \) for \( \ell = 2s = \text{even} \geq 2 \) is given by

\[
Q_\ell = \sum_{\mu_1, \mu_2 \ldots, \mu_s = 1}^{p} \sum_{\nu_1, \ldots, \nu_s = 1}^{p} \delta_{\mu_1 + \mu_2 + \ldots + \mu_s, 0} \delta_{\nu_1 + \nu_2 + \ldots + \nu_s, 0} \times K_{\mu_1 \nu_1} K_{\mu_2 \nu_2} \ldots K_{\mu_s \nu_s}
\]

(3.41a)

\[
K_{\mu \nu} = \sum_{\alpha, \beta = 1}^{p} B_{\mu - \alpha, \nu - \beta} B_{\beta \alpha} .
\]

(3.41b)

Especially, we note that \( B_{\mu \nu} \) terms in Eq. (3.40) will not contribute for the case of \( \ell = \text{odd} \), corresponding to knots as in Fig. 6 (\( \ell = 3 \)). The present link invariants differ from these of both Kauffman and Homfly polynomials. Also its relationship to the 3-dimensional approach due to Witten [8] is not obvious.

In ending this note, we remark that we can find more solutions of YBE. One example is given by

\[
A_{\mu, \nu} = D, \quad C_{\mu, \nu} = F, \quad A_{\mu \nu} = \overline{D}, \quad C_{\mu \nu} = \overline{F}
\]

satisfying conditions

\[
p^4 \overline{D} D = p^4 \overline{F} F = \tau \tau = 1 ,
\]

\[
p(D^2 + F^2) + NF D = 0 ,
\]

\[
\tau = p(pD + NF) = -p^2 F^2 / D
\]

while \( B_{\mu \nu} \) must obey relations

\[
\sum_{\lambda = 1}^{p} B_{\lambda, \mu - \lambda} = \tau \left( \delta_{\mu, 0} - \frac{1}{p} \right)
\]

\[
\sum_{\lambda = 1}^{p} \overline{B}_{\lambda, \mu - \lambda} = \overline{\tau} \left( \delta_{\mu, 0} - \frac{1}{p} \right)
\]

\[
\sum_{\alpha, \beta = 1}^{p} \overline{B}_{\mu - \alpha, \nu - \beta} B_{\alpha \beta} = \delta_{\mu, 0} \delta_{\nu, 0} - \frac{1}{p^2} .
\]

A solution for \( B_{\mu, \nu} \) and \( \overline{B}_{\mu, \nu} \) satisfying these conditions is easily found to be

\[
B_{\mu \nu} = \tau \left( \delta_{\mu, 0} \delta_{\nu, 0} - \frac{1}{p^2} \right) ,
\]

\[
\overline{B}_{\mu \nu} = \overline{\tau} \left( \delta_{\mu, 0} \delta_{\nu, 0} - \frac{1}{p^2} \right) .
\]
There are other solutions in which we have $A_{\mu\nu} = 0$. However, the Markov conditions are satisfied only for the rather uninteresting case of $p = N$.

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