Quantum Flows as Markovian Limit of Emission, Absorption and Scattering Interactions

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Commun. Math. Phys. 254, 489-512, 2005

Abstract

We consider a Markovian approximation, of weak coupling type, to an open system perturbation involving emission, absorption and scattering by reservoir quanta. The result is the general form for a quantum stochastic flow driven by creation, annihilation and gauge processes. A weak matrix limit is established for the convergence of the interaction-picture unitary to a unitary, adapted quantum stochastic process and of the Heisenberg dynamics to the corresponding quantum stochastic flow: the convergence strategy is similar to the quantum functional central limits introduced by Accardi, Frigerio and Lu\cite{1}. The principal terms in the Dyson series expansions are identified and re-summed after the limit to obtain explicit quantum stochastic differential equations with renormalized coefficients. An extension of the Pulé inequalities\cite{2} allows uniform estimates for the Dyson series expansion for both the unitary operator and the Heisenberg evolution to be obtained.

1 Introduction

In the interaction picture, the unitary $U_t$ arising from a time-dependent perturbation $V_t$, is given by

$$U_t = \mathcal{T} \exp \left\{ -i \int_0^t ds V_s \right\}$$

(1.1)

where $\mathcal{T}$ is Dyson's time-ordering operation. A principal aim of quantum field theory is then to obtain a normal-ordered version of $U_t$. When $V_t$ involves a sum of monomials of canonical quantum fields, we may use Feynman rules to expand $U_t$: we associate a vertex to each monomial, with the number of legs corresponding with the degree; we then construct the class $\mathcal{F}$ of Feynman diagrams consisting of such vertices with certain legs contracted (internal lines).
and the remainder free (external lines); we then specify a rule for writing down an operator \( L_G(t) \) which, for each \( G \in \mathfrak{F} \), will be a normal-ordered product of the fields associated to the external lines of \( G \). We then determine a development of the form \( U_t = \sum_{G \in \mathfrak{F} } L_G(t) \). Now, if \( G \) can be decomposed as two disconnected sub-diagrams \( G_1 \) and \( G_2 \), then \( L_G = \tilde{N} L_{G_1} L_{G_2} \) where \( \tilde{N} \) is Wick’s normal-ordering operation. This leads to a second presentation of \( U_t \):

\[
U_t = \tilde{N} \exp \left\{ \sum_{G \in \mathfrak{F}} L_G(t) \right\}
\]

where \( \mathfrak{F}_C \) is the class of connected Feynman diagrams.

If, in place of quantum fields, we considered quantum white noises, then the time-ordered presentation corresponds to a Stratonovich form while the normal-ordered presentation corresponds to an Itô form. Our aim is not to justify this statement, for which there is ample support \cite{3,4,5}, but to prove an asymptotic result which, effectively, is an analogue of the Wong-Zakai theorem for classical stochastic processes. The interaction that we shall be interested in is given below as (1.8), and is quadratic in the reservoir creation/annihilation operator fields \( a_t^\pm (\lambda) \): the corresponding connected Feynman diagrams will have at most two legs and therefore will be linear chains. These describe a reservoir quanta created, subsequently multiply-scattered (i.e., at several times annihilated and immediately re-created) and finally reabsorbed: external lines may also be present.

We shall be interested, not in the S-matrix limit \( t \to \infty \), but in the more subtle van Hove \cite{6}, or weak coupling, limit where we rescale time as \( t/\lambda^2 \) with \( \lambda \) a coupling strength parameter appearing in \( V_t \) and consider the limit \( \lambda \to 0 \) with \( t \) fixed. The fields \( a_t^\pm (\lambda) \) will converge, in a sense to be spelled out below, to quantum white noises: more correctly, integrated versions of these fields converge to the fundamental quantum stochastic processes of Hudson and Parthasarathy’s theory \cite{7}. The van Hove limit turns out to have dominant contribution from Feynman diagrams where there is no overlap in the time ranges of the individual connected subgraphs: these are the so-called type I terms. All other terms (type II) are suppressed. A similar feature is observed for the limit of the dynamical flow of observables.

1.1 The Classical Wong-Zakai Theorem

Wong and Zakai \cite{8} studied Langevin type equations driven by differentiable noises \( \xi_t (\lambda) \) having correlation \( \langle \xi_t (\lambda) \xi_s (\lambda) \rangle = \frac{1}{\lambda^2} G \left( \frac{t-s}{\lambda^2} \right) \) which became delta-correlated only in the limit \( \lambda \to 0 \). They found that the limit dynamics was described by a stochastic differential equation taking the same form as the pre-limit equations in the Stratonovich calculus.

Let us specialize to the flow on a symplectic manifold generated by a random Hamiltonian

\[
\Upsilon_t^{(\lambda)} = H + \sum_\alpha F_\alpha \xi_t^\alpha (\lambda)
\]

(1.3)
where $H$ and $F_{\alpha}$ are smooth functions on phase space and $\xi^\alpha_t(\lambda)$ are differentiable stochastic processes converging to independent white noises. If $x^{(\lambda)}_t$ is the phase trajectory starting from $x_0$ then the evolution of functions is $J^{(\lambda)}_t(f) := f(x^{(\lambda)}_t)$. In the limit $\lambda \to 0$ we obtain, in accordance with the Wong-Zakai result, the Stratonovich-Fisk equation

$$dJ_t(\cdot) = J_t\{\cdot, H\} dt + \sum_{\alpha} J_t\{\cdot, F_{\alpha}\} \circ dB^\alpha_t$$

(1.4)

where $B^\alpha_t$ are independent Wiener processes and the differential is of Stratonovich type: here we may view the motion as that governed by the formal Hamiltonian $\Upsilon_t = H + \sum_{\alpha} F_{\alpha}\xi^\alpha_t$ where $\xi^\alpha_t$ are white noises. A general treatment of these problems using the van Hove limit is well-understood [9]. These are the stochastic flows that preserve the Poisson bracket structure [10]. Averaging with respect to the Wiener measure, we obtain the dynamical semigroup $E[J_t(\cdot)] \equiv \exp\{t\Upsilon(\cdot)\}$. From the Itô calculus, the generator will be the hypo-elliptic operator

$$\Upsilon(\cdot) = \sum_{\alpha} \{\{\cdot, F_{\alpha}\}, F_{\alpha}\} + \{\cdot, H\}$$

(1.5)

which is already displayed in Hörmander form.

### 1.2 Quantum Markov Approximations

It was first suggested by Spohn [11] that the weak coupling limit should be properly considered as a Markovian limit underscored by a functional central limit. The rigorous determination of irreversible semigroup evolutions has been given for specific models [12, 2]. (A detailed account of the derivation of the master equation for a class of quantum open systems is given in Davies’ book [13].) The form of the generator of quantum dynamical semigroups was deduced [14, 15] using the guiding principle that the semi-group be completely positive. Hudson and Parthasarathy [7] subsequently developed a quantum stochastic calculus giving an Itô theory of integration with respect to Bosonic Fock space processes and demonstrated how to construct dilations of the quantum dynamical semigroups mentioned above using a Fock space as auxiliary space.

The program now is to begin with a microscopic model for a system-reservoir interaction and then obtain by some Markovian limit procedure, such as the weak coupling limit, a quantum stochastic evolution. It was first noted by von Waldenfels [16] that stochastic models successfully describe the weak coupling limit regime for the Wigner-Weisskopf atom. Later, Accardi, Frigerio and Lu [1] showed how to do this for an interaction of the type $\Upsilon^{(\lambda)} = E_{10} \otimes a^+_t(\lambda) + E_{01} \otimes a^-_t(\lambda)$ where $E_{10}$ and $E_{01}$ are bounded, mutually adjoint operators on the system space $h_S$ and $a^\pm_t(\lambda)$ are creation/annihilation fields having a correlation

$$\langle a^+_t(\lambda) a^-_s(\lambda) \rangle = \frac{1}{\lambda^2} G\left(\frac{t-s}{\lambda^2}\right)$$

(1.6)
where $G(\cdot)$ is integrable. In the sense of Schwartz distributions, we have
\[
\lim_{\lambda \to 0} \langle a_t^- (\lambda) a_t^+ (\lambda) \rangle = \gamma \delta (t - s) \text{ where } \gamma = \int_{-\infty}^{+\infty} dt \ G(t) \text{ is finite. We shall also take an interest in the constants}
\]
\[
\kappa_+ := \int_0^\infty dt \ G(t) , \quad \kappa_- := \int_{-\infty}^0 dt \ G(t) \text{ and } K := \int_0^\infty dt \ |G(t)|. \quad (1.7)
\]
We shall assume that $G(-t) = G(t)^*$ so that $\kappa_\pm \equiv \frac{1}{2} \gamma \pm i \sigma$. Already in [1], several important steps were taken: to begin with, there is the anticipation of the limit algebraic structure by means of a quantum functional central limit theorem which captures the long time asymptotic behaviour; secondly, there is the identification of the principal, type I, terms in the Dyson series which survive the Markovian limit (they are the ones arising from only time-consecutive two-point contractions); finally, there is a rigorous estimate of the Dyson series expansion employing an argument due to Pulé [2].

1.3 Statement of the Problem

Our aim is to extend this result in [1] to the more general class of interactions
\[
\U^{(\lambda)}_t = E_{11} \otimes a_t^+ (\lambda) a_t^- (\lambda) + E_{10} \otimes a_t^+ (\lambda) + E_{01} \otimes a_t^- (\lambda) + E_{00} \otimes 1
\]
\[
= E_{\alpha\beta} \otimes [a_t^+ (\lambda)]^\alpha [a_t^- (\lambda)]^\beta \quad (1.8)
\]
(We introduce the summation convention that when the Greek indices $\alpha, \beta, \ldots$ are repeated then we sum each index over the values 0 and 1 - moreover we understand the index $\alpha$ in $[\cdot]^\alpha$ to represent a power.) We require only the conditions that the system operators $E_{\alpha\beta}$ are bounded with $K \| E_{11} \| < 1$, where $K$ is the constant introduced in (1.7).

The interaction includes a scattering term, $E_{11} \otimes a_t^+ (\lambda) a_t^- (\lambda)$, and a constant term. The terms involving $E_{01}$ and $E_{10}$ describe the emission and absorption of reservoir quanta and this component has been employed in models of laser interactions [17]. The constant term is of little consequence as we shall take it to commute with the free Hamiltonian. However, the scattering term is highly non-trivial: we have to contend with emission, multiple scatterings and absorption. This means that the number of terms in the Dyson series expansion of
\[
\U^{(\lambda)}_t = \mathbf{T} \exp \left\{-i \int_0^t ds \ Y^{(\lambda)}_s \right\} \quad (1.9)
\]
grows rapidly (in fact, as the Bell numbers of combinatorics [21]). However, we are able to prove a uniform estimate of the Dyson series expansion by a generalization of the Pulé inequalities, which we give in section 7. We are then able to re-sum the series to obtain an adapted, unitary process $\U_t$ of Hudson-Parthasarathy type (Theorem 8.1). The type of limit involved is of a weak character and is often referred to as convergence in matrix elements.

We show that the Heisenberg evolution $J^{(\lambda)}_t (X) = U^{(\lambda)}_t (X \otimes 1_R) U^{(\lambda)}_t$ likewise converges in weak matrix elements, for fixed bounded observables $X \in \mathcal{B}(\mathfrak{h}_S)$, to $J_t (X) = U^{(\lambda)}_t (X \otimes 1_R) U_t$ (Theorem 10.1).
We are able to obtain the quantum stochastic differential equations satisfied by \(U_t\) and by the flow \(J_t\). In particular, these equations will involve a gauge differential (due to the scattering) as well as creation, annihilation and time. In particular, we compute the Lindblad generator for the flow. We remark that interactions of the type (1.8) were considered previously in the case where the coefficients \(E_{\alpha\beta}\) were commuting operators \([18]\), and Fermionic operators \([19]\). In the former case, a strong resolvent limit was established for the common spectral resolution, while in the latter, the anti-commutation relations kill off all but type I terms.

2 Moments and Cumulants

Let \(\Gamma (\mathfrak{h})\) be the (Bose) Fock space over the one-particle Hilbert \(\mathfrak{h}\). The Fock vacuum will be denoted by \(\Phi\) and the exponential vector map by \(\varepsilon : \mathfrak{h} \mapsto \Gamma (\mathfrak{h})\). As usual \(\varepsilon (0) = \Phi\). We denote the creation fields as \(A^+ (\cdot)\), the annihilation fields as \(A^- (\cdot)\) and the differential second quantization field as \(d\Gamma (\cdot)\), as standard. The Weyl operator with test function \(f\) is \(W (f) := \exp [A^+ (f) - A^- (f)]\) and we have the Weyl map \(W (\cdot)\).

As is well-known, the fields \(Q (\cdot) = A^+ (\cdot) + A^- (\cdot)\) are Gaussian random fields when taken in the Fock vacuum state. More generally, we have \([20]\)

\[
\langle \Phi \mid \exp \left\{ it \left( d\Gamma (H) + A^+ (H f) + A^- (H f) + \langle f | H f \rangle \right) \right\} \Phi \rangle = \exp \int (e^{itx} - 1) \, d\mu_H (dx)
\]

where \(H\) is self-adjoint on \(\mathfrak{h}\) with spectral measure \(\mu_H^f\) for vector state \(f \in \mathfrak{h}\). This time, we are dealing with Poissonian fields. We remark that if \(\mu_H^f = \lambda \delta_1\), then we obtain a random variable with Poisson distribution of intensity \(\lambda > 0\):

\[
\exp \left\{ \lambda (e^{it} - 1) \right\} = \sum_n \sum_m \frac{(it)^n}{n!} S (n, m) \lambda^m.
\]

The coefficients \(S (n, m) = \frac{1}{m!} \sum_{l=1}^m (-1)^{l+m} l^n (\binom{m}{l})\) are well-known combinatorial factors: they are the Stirling number’s of the second kind \([21]\) and they count the number of ways of partitioning a set of \(n\) items into \(m\) non-empty subsets.

The expansion of Poissonian field moments in terms of cumulants, or more generally the expansion of Green’s functions in terms of their connected Green’s functions, can best be described in the language of partitions \([22]\).

A partition of the integers \(\{1, \ldots, n\}\) is a collection of non-empty, disjoint subsets (called parts) whose union is \(\{1, \ldots, n\}\). The set of all such partitions will be denoted as \(\mathfrak{P}_n\): there will be \(S (n, m)\) partitions of \(\{1, \ldots, n\}\) having exactly \(m\) parts and \(B_n = \sum_m S (n, m)\) partitions of \(\{1, \ldots, n\}\) in total. \(B_n\) are called the Bell numbers \([21]\).
Lemma (2.1) Let \( f_1, g_1, \ldots, f_n, g_n \in \mathfrak{h} \). Then

\[
\sum_{\alpha,\beta \in \{0,1\}^n} \left\langle \Phi \mid [A^+ (f_n)]^{\alpha(n)} [A^- (g_n)]^{\beta(n)} \cdots [A^+ (f_1)]^{\alpha(1)} [A^- (g_1)]^{\beta(1)} \Phi \right\rangle
\]

\[
= \sum_{\mathcal{A} \in \mathfrak{P}_n} \prod_{\{i(1),\ldots,i(k)\} \in \mathcal{A}} \left\langle g_{i(k)} | f_{i(k-1)} \right\rangle \cdots \left\langle g_{i(3)} | f_{i(2)} \right\rangle \left\langle g_{i(2)} | f_{i(1)} \right\rangle
\]

(2.1)

where we take the various sets (parts of the partition) \( \{i(1),\ldots,i(k)\} \in \mathcal{A} \) to be ordered so that \( i(1) < i(2) < \cdots < i(k) \) and if the set is a singleton it is given the factor of unity.

**Proof.** If \( \alpha(i) = 0, 1 \), then we have the absence, respectively presence, of the creator \( A^+ (f_i) \). Likewise \( \beta(i) \) gives the absence or presence of the \( i \)-th annihilator. Evidently we must have \( \alpha(n) = 0 = \beta(1) \).

Essentially we have a vacuum expectation of a product of \( n \) factors \([A^+ (f_i)]^{\alpha(i)} [A^- (g_i)]^{\beta(i)}\) and this ultimately when put to normal order will be a sum of terms each of which is a product of pair contractions \( \langle g_i | f_k \rangle \) where \( i > k \). For a given term in the sum we write \( i \sim k \) if \( \langle g_i | f_k \rangle \) appears. An equivalence relation is determined by a set of contractions as follows: we always have \( i \equiv i \) and, more generally, we have \( i \equiv k \) if there exists a sequence \( j(1), \ldots, j(r) \) such that either \( i \sim j(1) \sim j(2) \sim \cdots j(r) \sim k \) or \( k \sim j(1) \sim j(2) \sim \cdots j(r) \sim i \). A partition \( \mathcal{A} \) in \( \mathfrak{P}_n \) is then obtained by looking at the equivalence classes. (Singletons are just the unpaired labels.) The correspondence between the terms in the sum and the elements of \( \mathfrak{P}_n \) is one-to-one and the weight given to a particular partition \( \mathcal{A} \in \mathfrak{P}_n \) is just the product of \( \langle g_i | f_k \rangle \)'s given in (2.1).

There is a convenient diagrammatic way to understand the formula (2.1). We first of all associate one of four possible vertices with each component \([A^+ (f_j)]^{\alpha(j)} [A^- (g_j)]^{\beta(j)}\), \( j = 1, \ldots, n \), they are, for \( \{\alpha_j, \beta_j\} = (1,1), (1,0), (0,1) \) and \( (0,0) \) respectively.

![Scattering Emission Absorption Neutral](figure1.png)

**Figure 1**

We draw the \( n \) vertices in a line and proceed to join up the emission lines to the absorption lines (pair contractions!). A typical situation is depicted below:

![Figure 2](figure2.png)
Evidently we must again join up all creation and annihilation operators into pairs; we however get creation, multiple scattering and annihilation as the rule; otherwise we have a single neutral vertex. In the figure, we can think of a particle being created at vertex \(i(1)\) then scattered at \(i(2), i(3), i(4)\) successively before being annihilated at \(i(5)\). (This component has been highlighted using thick lines.) Now the argument: each such component corresponds to a unique part, here \(\{i(5), i(4), i(3), i(2), i(1)\}\), having two or more elements; singletons may also occur and these are just the constant term vertices. Therefore every such diagram corresponds uniquely to a partition of \(\{1, \ldots, n\}\).

We remark that (2.1) can be considered as a special case of the expansion

\[
G(x_1, \ldots, x_n) = \sum_{A \in \mathcal{P}_n} \prod_{\{i(1), \ldots, i(k)\} \in A} C(x_{i(1)}, \ldots, x_{i(k)})
\]

of an \(n\)-particle Green’s function \(G\) in terms of the connected Green’s functions \(C\).

Let us write \(\mathfrak{P}\) for the set \(\bigcup_n \mathfrak{P}_n\) of finite partitions. With each partition \(A \in \mathfrak{P}_n\) we associate a sequence of occupation numbers \(n = (n_j)_{j=1}^\infty\) where \(n_j = 0, 1, 2, \ldots\) counts the number of \(j\)-tuples making up \(A\). In general, we set

\[
E(n) := \sum_j jn_j, \quad N(n) := \sum_j n_j
\]

so that if \(A \in \mathfrak{P}_n\) leads to sequence \(n\), then \(E(n) = n\), while \(N(n)\) counts the number of parts making up the partition. We shall denote by \(\mathfrak{P}_n\) the set of all partitions having the same occupation number sequence \(n\).

Given a partition \(A \in \mathfrak{P}_n\) we use the convention \(q(j, k, r)\) to label the \(r\)-th element of the \(k\)-th \(j\)-tuple. A simple example of a partition in \(\mathfrak{P}_n\) is given by selecting in order from \(\{1, 2, \ldots, E(n)\}\) first of all \(n_1\) singletons, then \(n_2\) pairs, then \(n_3\) triples etc. The labelling for this particular partition will be denoted as \(\bar{q}(\ldots, \ldots)\) and explicitly we have

\[
\bar{q}(j, k, r) = \sum_{l<j} ln_l + (k - 1)n_j + r.
\]

**Definition (2.2):** We shall denote by \(\mathcal{S}_n^0\) the collection of Pulé permutations, that is, \(\rho \in \mathcal{S}_n, E(n) = n\), such that \(q = \rho \circ \bar{q}\) again describes a partition in \(\mathfrak{P}_n\). Specifically, \(\mathcal{S}_n^0\) consists of all the permutations \(\rho\) for which the following requirements are met:

i) the order of the individual \(j\)-tuples is preserved for each \(j\) -

\[
\rho(\bar{q}(j, k, 1)) < \rho(\bar{q}(j, k', 1)) \quad \forall j, 1 \leq k < k' \leq n_j;
\]

ii) creation always precedes annihilation in time for any contraction pair -

\[
\rho(\bar{q}(j, k, 1)) < \rho(\bar{q}(j, k, 2)) < \cdots < \rho(\bar{q}(j, k, j)) \quad \forall j, 1 \leq k \leq n_j.
\]
In these notations we may rewrite the result of the lemma (2.1) as:

Lemma (2.3) Let $f_1, g_1, \ldots, f_n, g_n \in h$. Then

$$
\sum_{\alpha, \beta \in \{0, 1\}^n} \langle \Phi \mid [A^+ (f_n)]^\alpha(n) [A^- (g_n)]^\beta(n) \cdots [A^+ (f_1)]^\alpha(1) [A^- (g_1)]^\beta(1) \Phi \rangle 
$$

$$
= \sum_{\rho \in \Theta^0_n} \sum_{j \geq 2} \prod_{k=1}^{n_j} \prod_{r=1}^{j-1} \langle g_{\rho(q(j,k,r+1))} | f_{\rho(q(j,k,r))} \rangle (2.6)
$$

To better understand this, we return to our diagram conventions. Given an arbitrary diagram, we wish to construct the Pulè permutation putting it to the basic form. For instance, we might have an initial segment of a diagram looking like the following:

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......
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Figure 3

There will exist a permutation $\sigma$ of the $n$ vertices which will reorder the vertices so that we have the singletons first, then the pair contractions, then the triples, etc., so that we obtain a picture of the following type

```
    n_3 triples    n_2 pairs    n_1 singletons
......
```

Figure 4

The permutation is again unique if we retain the induced ordering of the first emission times for each connected block.

3 A Microscopic Model

We shall consider a quantum mechanical system $S$ (state space $h_S$) coupled to a Bose quantum field reservoir $R$ over a one-particle space $h_1^R$ (state space $h_R = \Gamma (h_1^R)$). We shall take the reservoir to be in the Fock vacuum state $\Phi$. The interaction between the system and the reservoir will be given by the formal Hamiltonian

$$
H^{(\lambda)} = H_S \otimes 1_R + 1_S \otimes d\Gamma (H_1^R) + H_{\text{int}}^{(\lambda)}
$$

(3.1)

where the operators $H_S$ and $H_1^R$ are self-adjoint and bounded below on $h_S$ and $h_1^R$, respectively. The interaction is taken to be

$$
H_{\text{int}}^{(\lambda)} = E_{11} \otimes A^+ (g) A^- (g) + \lambda E_{10} \otimes A^+ (g) + \lambda E_{01} \otimes A^- (g) + \lambda^2 E_{00} \otimes 1_R
$$

(3.2)
where \( E_{\alpha\beta} \) are bounded operators on \( h \) with \( E_{11} \) and \( E_{00} \) self-adjoint and \( E_{10} = E_{01} \). The operators \( A^+ (g) \) and \( A^- (g) \) are the creation and annihilation operators with test function \( g \in h_1^R \). (The parameter \( \lambda \) is real and will later emerge as a rescaling parameter in which we hope to obtain a Markovian limit.) We shall also assume the following harmonic relations

\[
e^{i\tau H_S} E_{\alpha\beta} e^{-i\tau H_S} = e^{i\omega\tau (\beta - \alpha)} E_{\alpha\beta};
\]

\[
e^{i\tau H_R} A^\pm_R (g) e^{-i\tau H_R} = A^\pm_R (\theta \tau g).
\]

(3.3)

where \( (\theta \tau : \tau \in \mathbb{R}) \) will be the one-parameter group of unitaries on \( h_1^R \) with Stone generator \( H^1_R \).

We transfer to the interaction picture with the help of the unitary

\[
U (\tau, \lambda) = e^{i\tau (H_S \otimes 1_R + 1_S \otimes H_R)} e^{-i\tau H^{(\lambda)}}.
\]

(3.4)

In the weak coupling regime, we are interested in the behaviour at long time scales \( \tau = t/\lambda^2 \) and from our earlier specifications we see that \( U^{(\lambda)}_t = U \left( t/\lambda^2, \lambda \right) \) satisfies the interaction picture Schrödinger equation

\[
\frac{\partial}{\partial t} U^{(\lambda)}_t = -i \Upsilon_t (\lambda) U^{(\lambda)}_t
\]

(3.5)

with \( \Upsilon_t (\lambda) \) as in (1.8). Here we meet the time-dependent rescaled reservoir fields

\[
a^\pm_t (\lambda) := \frac{1}{\lambda} e^{i\omega t/\lambda^2} A^\pm (\theta \lambda \tau g).
\]

(3.7)

Specifically we have \( \gamma = \int_{-\infty}^{+\infty} d\tau \left\langle g, e^{i\tau (H^1_R - \omega)} g \right\rangle = 2\pi \left\langle g, \delta (H^1_R - \omega) g \right\rangle \) and \( \kappa_+ = \left\langle g, \frac{1}{i(H^1_R - \omega - i0^+)} g \right\rangle = \frac{1}{2} \gamma - i \text{PV} \left\langle g, \frac{1}{(H^1_R - \omega)} g \right\rangle \) where PV denotes the principle value part.

## 4 Quantum Central Limit

The limit \( \lambda \to 0 \) for the above, the two-point function becomes delta-correlated. However, it is vital to have a mathematical framework in which to interpret the limit states and observables.

For convenience we set

\[
\theta^\nu_t := \exp \left\{ i\tau (H^1_R - \omega) \right\}.
\]

(4.1)

We assume the existence of a non-zero subspace, \( \mathfrak{t} \), of \( h_1^R \) for which

\[
\int_{-\infty}^{\infty} |\left\langle f_j, \theta^\nu u f_k \right\rangle| du < \infty
\]

whenever \( f_j, f_k \in \mathfrak{t} \). (In reference [1], explicit examples of dense subspaces, \( \mathfrak{t} \), are given and correspond to “mass-shell” Hilbert spaces.) The question of
completeness can be addressed immediately: a sesquilinear form on \( \mathcal{F} \) is defined by

\[
(f_j | f_k) := \int_{-\infty}^{\infty} \langle f_j, \theta^{\omega}_u f_k \rangle \, du \equiv 2\pi \langle f_j, \delta (H_\lambda^1 - \omega) f_k \rangle \tag{4.2}
\]

and we can quotient out the null elements for this form; the completed Hilbert space will again be denoted by \( \mathcal{F} \) and \( (,\,) \) will be its inner product. The test vector \( g \) appearing in the interaction must belong to \( \mathcal{F} \) so that the constant \( \gamma \equiv (g|g) \) is finite.

Let \( W(\cdot) \) be the Weyl map from \( \mathfrak{h}_R^1 \) as before. We now fix \( f_j \in \mathcal{F} \) and \( 0 \leq S_j < T_j < \infty \) for certain indices \( j \) and introduce the rescaled operators

\[
A^\pm_\lambda (j) := \frac{1}{\lambda} \int_{S_j}^{T_j} du A^\pm_R \left( \theta^{\omega}_{u/\lambda^2} f_j \right), \quad W_\lambda(j) := W \left( \frac{1}{\lambda} \int_{S_j}^{T_j} du \theta^{\omega}_{u/\lambda^2} f_j \right). \tag{4.3}
\]

Note that, with respect to our earlier notations (3.7), if \( f_j = g \) then \( A^\pm_\lambda (j) \equiv \int_{S_j}^{T_j} du a^\pm_\lambda (\lambda) \). The following result is proved as lemma 3.2 in Accardi, Frigerio and Lu [1]. We write \( 1_{[S,T]} \) for the characteristic function of an interval \( [S,T] \).

**Lemma (4.1)** For the fields introduced in (4.3)

\[
\lim_{\lambda \to 0} \left[ A^-_\lambda (j), A^+_\lambda (k) \right] = (f_j | f_k) \langle 1_{[S_j,T_j]}, 1_{[S_k,T_k]} \rangle.
\]

The right hand side is the inner product \( \langle f_j \otimes 1_{[S_j,T_j]}, f_k \otimes 1_{[S_k,T_k]} \rangle \) on the Hilbert space \( \mathcal{F} \otimes L^2 (\mathbb{R}_+) \). This space is isomorphic in a natural way to the \( \mathcal{F} \) - valued square-integrable functions on \( \mathbb{R}_+ \) and we denote this space as \( L^2 (\mathbb{R}_+, \mathcal{F}) \).

The appropriate noise space for the limit \( \lambda \to 0 \) will in fact be the Bose Fock space \( \Gamma \left( L^2 (\mathbb{R}_+, \mathcal{F}) \right) \). Indeed, we have the following fact proved as theorem 3.4 in [1].

**Theorem (4.2)** Let \( \Psi \) be the Fock vacuum for \( \Gamma \left( L^2 (\mathbb{R}_+, \mathcal{F}) \right) \) and let \( W(\cdot) \) denote the usual Weyl mapping from \( L^2 (\mathbb{R}_+, \mathcal{F}) \) into the unitaries on \( \Gamma \left( L^2 (\mathbb{R}_+, \mathcal{F}) \right) \). Then

\[
\lim_{\lambda \to 0} \langle \Phi \rangle W_\lambda (1) \ldots W_\lambda (k) \Phi = \langle \Phi \rangle W \left( f_1 \otimes 1_{[S_1,T_1]} \right) \ldots W \left( f_k \otimes 1_{[S_k,T_k]} \right) \langle \Phi \rangle
\]

for arbitrary \( k \) and \( f_j \in \mathcal{F} \) and \( 0 \leq S_j < T_j < \infty \).

## 5 The Dyson Series Expansion of \( U_t^{(\lambda)} \)

The formal Dyson series development \( U_t^{(\lambda)} = \sum_{n=0}^{\infty} (-i)^n D_n (t, \lambda) \) involves the multiple time integrals

\[
D_n (t, \lambda) = \int_{\Delta_n (t)} ds_n \ldots ds_1 \Upsilon_n (\lambda) \ldots \Upsilon_1 (\lambda). \tag{5.1}
\]
For \( \sigma \in \mathfrak{S}_n \), we introduce the simplex

\[
\Delta^\sigma_n(t) := \left\{ (s_n, \ldots, s_1) : t > s_\sigma(n) > \cdots > s_\sigma(1) > 0 \right\}
\]

(5.2)

and \( \Delta_n(t) \) in (5.1) is the simplex corresponding to the identity permutation.

We consider matrix elements of the type

\[
\langle \phi_1 \otimes W_\lambda(1) \Phi | T_s n \ldots T_s 1 \phi_2 \otimes W_\lambda(2) \Phi \rangle
\]

with \( \phi_j \in \mathfrak{h}_S \) and \( W_\lambda(j) \) as in (4.3). Substituting for the Dyson series, we find that the \( n \)-th term can be rewritten as an expectation involving the vacuum state \( \Phi \) only:

\[
\langle \phi_1 \otimes W_\lambda(1) \Phi | T_s n \ldots T_s 1 \phi_2 \otimes W_\lambda(2) \Phi \rangle = \langle \phi_1 \otimes \Phi | T_s n \ldots T_s 1 \phi_2 \otimes \Phi \rangle \langle W_\lambda(1) \Phi | W_\lambda(2) \Phi \rangle
\]

(5.3)

where \( T_s n(\lambda) \) is obtained from \( T_s(\lambda) \) by the canonical translations

\[
a^+_t(\lambda) \rightarrow a^+_t(\lambda) + h_1(t, \lambda); \quad a^-_t(\lambda) \rightarrow a^-_t(\lambda) + h^*_2(t, \lambda)
\]

(5.4)

with

\[
h_j(t, \lambda) = \frac{1}{\lambda^2} \int_{S_j} du \left\langle \theta^\omega_{\eta/\lambda^2} f_j | \theta^\omega_{\eta/\lambda^2} g \right\rangle.
\]

(5.5)

That is,

\[
T_s(\lambda) = \tilde{E}_{\alpha\beta}(t, \lambda) \otimes \left[ a^+_t(\lambda) \right]^\alpha \left[ a^-_t(\lambda) \right]^\beta
\]

(5.6)

where

\[
\begin{align*}
\tilde{E}_{00}(t, \lambda) &= E_{00} + E_{11} h_1(t, \lambda) + E_{01} h^*_2(t, \lambda) + E_{11} h_1(t, \lambda) h^*_2(t, \lambda); \\
\tilde{E}_{10}(t, \lambda) &= E_{10} + h^*_2(t, \lambda) E_{11}; \\
\tilde{E}_{01}(t, \lambda) &= E_{01} + h_1(t, \lambda) E_{11}; \\
\tilde{E}_{11}(t, \lambda) &= E_{11}.
\end{align*}
\]

(5.7)

In this way we see that the \( n \)-th term in the Dyson series expansion of the matrix element is, up to the factor \((-i)^n \langle W_\lambda(1) \Phi | W_\lambda(2) \Phi \rangle\),

\[
\begin{align*}
\int_{\Delta_n(t)} ds_n \ldots ds_1 & \left\langle \phi_1 | \tilde{E}_{\alpha_n, \beta_n}(s_n, \lambda) \ldots \tilde{E}_{\alpha_1, \beta_1}(s_1, \lambda) \phi_2 \right\rangle \\
& \times \left\langle \Phi \left[ a^+_s(\lambda) \right]^\alpha_n \left[ a^-_s(\lambda) \right]^\beta_n \ldots \left[ a^+_1(\lambda) \right]^\alpha_1 \left[ a^-_1(\lambda) \right]^\beta_1 \Phi \right\rangle
\end{align*}
\]

(5.8)

and our summation convention is now in place. The vacuum expectation can be computed using lemmas (2.1) or (2.3). The resulting terms can be split into two types: type I will survive the \( \lambda \rightarrow 0 \) limit; type II will not. They are distinguished as follows:
**Type I:** Terms involving contractions of time consecutive annihilator-creator pairs only. (That is, under the time-ordered integral in (5.8), an annihilator \( a_{s_{j+1}} (\lambda) \) must be contracted with the creator \( a_{s_j}^+ (\lambda) \).)

**Type II:** All other cases.

The terminology used here is due to Accardi, Frigerio and Lu [1].

We again resort to a diagrammatic convention in order to describe the Dyson series expansion into sums of integrals of products of two-point functions. There is a one-to-one correspondence between the diagrams appearing in the \( n \)-th term of the Dyson series and set of partitions of the \( n \) vertices. The diagram pictured as a typical situation in that section would contribute a weight of

\[
(-1)^{17} \int_{\Delta_{17}(t)} \tilde{E}^{01} (t_{17}, \lambda) \tilde{E}^{00} (t_{16}, \lambda) \cdots \tilde{E}^{10} (t_1, \lambda) \times C_\lambda (t_{17} - t_{11}) \cdots C_\lambda (t_2 - t_1)
\]

to the series. Let us consider a typical diagram. We shall assume that within the diagram there are \( n_1 \) singleton vertices \([\cdots\cdots]\), \( n_2 \) contraction pairs \([\cdots\cdots]\), \( n_3 \) contraction triples \([\cdots\cdots]\), etc. That is the diagram has a total of \( n = \sum_j n_j \) vertices which are partitioned into \( m = \sum_j n_j \) connected subdiagrams. We see that the total number of diagrams contributing to the \( n \)-th level of the Dyson series will be given by the Bell number \( B_n \).

### 6 Principal Terms in the Dyson Series

A standard technique in perturbative quantum field theory and quantum statistical mechanics is to develop a series expansion and argue on physical grounds that certain “principal terms” will exceed the other terms in order of magnitude [23]. Often it is possible to re-sum the principal terms to obtain a useful representation of the dominant behaviour. Mathematically, the problem down to showing that the remaining terms are negligible in the limiting physical regime being considered.

Let \( n \) be a positive integer and \( m \in \{0, \ldots, n-1\} \). Let \( \{(p_j, q_j)\}_{j=1}^m \) be contractions pairs over indices \( \{1, \ldots, n\} \) such that if \( P = \{p_1, \ldots, p_m\} \) and \( Q = \{q_1, \ldots, q_m\} \) then \( P \) and \( Q \) are both non-degenerate subsets of size \( m \) and we require that \( p_j < q_j \) for each \( j \) and that \( Q \) be ordered so that \( q_1 < \ldots < q_m \). We understand that \( (p_j, q_j)_{j=1}^m \) is type I if \( q_j = p_j + 1 \) for each \( j \) and type II otherwise. The following result is an extension of lemma 4.2 in Accardi, Frigerio and Lu [1] as now \( P \cap Q \) need not be empty.

**Lemma (6.1)** Let \( (p_j, q_j)_{j=1}^m \) be a set of \( m \) pairs of contractions over indices \( \{1, \ldots, n\} \) then

\[
\left| \int_{\Delta_n(t)} ds_1 \cdots ds_n \prod_{j=1}^m \langle a_{s(p_j)} (\lambda) a_{s(q_j)}^+ (\lambda) \rangle \right| \leq \frac{\gamma^m t^{n-m}}{(n-m)!}.
\]
Moreover, as \( \lambda \to 0 \),

\[
\int_{\Delta_n(t)} ds_1 \ldots ds_n \prod_{j=1}^{m} \langle a_{s(p_j)}^- (\lambda) a_{s(q_j)}^+ (\lambda) \rangle \to \left\{ \begin{array}{ll}
\frac{a_{\infty}^- a_{\infty}^+}{(n-m)!}, & \text{type I}; \\
0, & \text{type II}.
\end{array} \right.
\]  

(6.2)

**Proof.** Let \( q = q_1 \) and set \( t(q) = [s(p) - s(q)] / \lambda^2 \) then

\[
\left| \int_{\Delta_n(t)} ds_1 \ldots ds_n \prod_{j=1}^{m} \left\langle a_{s(p_j)}^- (\lambda) a_{s(q_j)}^+ (\lambda) \right\rangle \right| = \left| \int_0^t ds (1) \ldots \int_0^{s(q-2)} ds (q-1) \int_0^{s(p)/\lambda^2} dt (q) \int_0^{s(p)-\lambda^2 t(q)} ds (q+1) \ldots \right.
\]

\[
\left. \left| \int_0^{s(n-1)} ds (n) \left\langle g, \theta_{\omega(q)}^\nu g \right\rangle \prod_{j=2}^{m} \left\langle a_{s(p_j)}^- (\lambda) a_{s(q_j)}^+ (\lambda) \right\rangle \right| \right|
\]

However, we have that \( s(p) - \lambda^2 t(p) < s(q-1) \) and so we obtain the bound

\[
\left| \int_0^t ds (1) \ldots \int_0^{s(q-2)} ds (q-1) \int_-\infty^\infty dt (q) \int_0^{s(p)-\lambda^2 t(q)} ds (q+1) \ldots \right.
\]

\[
\left. \left| \int_0^{s(n-1)} ds (n) \left\langle g, \theta_{\omega(q)}^\nu g \right\rangle \prod_{j=2}^{m} \left\langle a_{s(p_j)}^- (\lambda) a_{s(q_j)}^+ (\lambda) \right\rangle \right| \right|
\]

And so, working inductively we obtain (6.1).

Suppose now that the pairs are of **type I**, then \( p = q - 1 \) and so the lower limit of the \( t(q) \)-integral is zero. Consequently, we encounter the sequence of integrals

\[
\ldots \int_0^{s(q-2)} ds (q-1) \int_0^{s(q-1)/\lambda^2} dt (q) \int_0^{s(q-1) - \lambda^2 t(q)} ds (q+1) \ldots \left\langle g, \theta_{\omega(q)}^\nu g \right\rangle \ldots
\]

this occurs for each \( q \)-variable and so we recognize the limit as stated in (6.2) for **type I** terms.

If the pairs are of **type II**, on the other hand, then let \( j = \min \{ k : p_k < q_k - 1 \} \); setting \( q = q_k \), we encounter the sequence of integrals

\[
\ldots \int_0^{s(q-2)} ds (q-1) \int_0^{s(q-1)/\lambda^2} dt (q) \int_0^{s(q-1)} ds (q+1) \ldots \left\langle g, \theta_{\omega(q)}^\nu g \right\rangle \ldots
\]

but now, with respect to the variables \( s(1), \ldots, s(p), \ldots, s(q-1) \) we have that, since \( s(p) \neq s(q-1) \), the lower limit \([s(p) - s(q-1)] / \lambda^2 \) of the \( t(q) \)-integral is almost always negative and so, as \( t \to \left\langle g, \theta_{\omega(q)}^\nu g \right\rangle \) is continuous, we have the dominated convergence of the whole term to zero. \( \blacksquare \)
Clearly type II terms do not contribute to the $n$–th term in the series expansion in the limit. However, we must establish a uniform bound for all these terms when the sum over all terms is considered. We do this in the next section.

Before proceeding let us remark that the expression (5.8) is bounded by

$$\sum_{\Delta_n(t)} ds_n \ldots ds_1 \left\langle \Phi \left[ \left[ a_{s_n}^+ (\lambda) \right]^{\alpha_n} \left[ a_{s_n}^- (\lambda) \right]^{\beta_n} \ldots \left[ a_{s_1}^+ (\lambda) \right]^{\alpha_1} \left[ a_{s_1}^- (\lambda) \right]^{\beta_1} \right] \right\rangle$$

where

$$C_{11} = \| E_{11} \|;$$
$$C_{10} = \| E_{10} \| + \| E_{11} \| h_2;$$
$$C_{01} = \| E_{01} \| + \| E_{11} \| h_1;$$
$$C_{00} = \| E_{00} \| + \| E_{10} \| h_1 + \| E_{01} \| h_2 + \| E_{11} \| h_1 h_2$$

(6.4)

and $h_1 = \int_{-\infty}^{\infty} du \| g(\theta_2 f_1) \|, h_2 = \int_{-\infty}^{\infty} du \| g(\theta_2 f_2) \|.

Recall that we require that $KC_{11} < 1$ and that $C = \max \{ C_{11}, C_{10}, C_{01}, C_{00} \} < \infty$.

We need to do some preliminary estimation. We employ the occupation numbers introduced in section 2. The number of times that we will have $(\alpha, \beta) = (1, 1)$ in a particular term will be $\sum_{j \geq 2} (j - 2) n_j$ (that is, singletons and pairs have none, triples have one, quadruples have two, etc.) and this equals $E(n) - 2N(n) + n_1$. Therefore, we shall have

$$C_{\alpha_n \beta_n} \ldots C_{\alpha_1 \beta_1} \leq C_{1}^{E(n) - 2N(n) + n_1} C_{2}^{2N(n) - n_1}.$$  \hfill (6.5)

7 Generalized Pulé Inequalities

Putting all this together we get the bound

$$\sum_{\rho \in \Omega_n^0} \sum_{\Delta_n(t)} ds_n \ldots ds_1 \left\langle \Phi \left[ \left[ a_{s_n}^+ (\lambda) \right]^{\alpha_n} \left[ a_{s_n}^- (\lambda) \right]^{\beta_n} \ldots \left[ a_{s_1}^+ (\lambda) \right]^{\alpha_1} \left[ a_{s_1}^- (\lambda) \right]^{\beta_1} \right] \right\rangle$$

$$\leq \sum_{\rho \in \Omega_n^0} \sum_{\Delta_n(t)} ds_n \ldots ds_1 \prod_{j \geq 2} \prod_{k=1}^{n_j} G_{\lambda} \left( s_{\rho(\bar{q}(j,k,r))} - s_{\rho(\bar{q}(j,k,r+1))} \right) \prod_{j=1}^{n_j} G_{\lambda} \left( s_{\rho(\bar{q}(j,k,r))} - s_{\rho(\bar{q}(j,k,r+1))} \right)$$

(7.1)

where we use the estimate (6.5) and we obtain the sum over all relevant terms by summing over all admissible permutations of the basic $\bar{q}$ term. To estimate the simplicial integral we generalize an argument due to Pulé (lemma 3 of [2]).
Let $\tilde{\rho}$ be the induced mapping on $\mathbb{R}^n$ obtained by permuting the Cartesian coordinates according to $\rho \in \mathcal{S}_n^0$. Then the bound in (7.1) can be written as

$$E(n) = \sum_n C_{11}^{E(n) - 2N(n) + n_1} C_2^{N(n) - n_1} \sum_{n} \prod_{j \geq 2} \prod_{k=1}^{n_j} \prod_{r=1}^{j-1} G_{\lambda}(s_{\tilde{q}(j,k,r+1)} - s_{\tilde{q}(j,k,r)}) \prod_{j=2}^{n_j} \prod_{k=1}^{j-1} G_{\lambda}(s_{\tilde{q}(j,k,r+1)} - s_{\tilde{q}(j,k,r)})$$

where $R = \bigcup \{ \tilde{\rho} \Delta_n(t) : \rho \in \mathcal{S}_n^0 \}$. This is down to the fact that the image sets $\tilde{\rho} \Delta_n(t)$ will be distinct for different $\rho \in \mathcal{S}_n^0$. Now the region, $R$, of integration is a subset of $[0, t]^n$ for which the variables $s_{\tilde{q}(j,k,1)}$ are ordered primarily by the index $j$ and secondarily by the index $k$. Moreover, each of the variables

$$u_{\tilde{q}(j,k,r)} := s_{\tilde{q}(j,k,r+1)} - s_{\tilde{q}(j,k,r)}$$

are positive, $(\forall j; k = 1, \ldots, n; r = 1, \ldots, j - 1)$. (These properties of $R$ are implicit from the choice of the ordering $\tilde{q}$ and of the nature of the permutations $\rho \in \mathcal{S}_n^0$.) Consider the change of variables

$$(s_1, \ldots, s_n) \mapsto (s_{\tilde{q}(j,k,1)}; u_{\tilde{q}(j,k,r)})$$

where the ordering is first by the $j$, second by the $k$, and for the $u$’s finally by the $r = 1, \ldots, j - 1$. This defines a volume-preserving map which will take $R$ into $\Delta_{n_1} \times \Delta_{n_2} \times \cdots \times [0, \infty)^{n_2} \times [0, \infty)^{2n_3} \times \cdots$. From this we are able to find the upper estimate on (7.2) of the form

$$E(n) = \sum_n C_{11}^{E(n) - 2N(n) + n_1} C_2^{N(n) - n_1} \frac{(t \vee 1)^{n_1}}{n_1!} \left( \prod_{k=1}^{n_1} \prod_{r=1}^{j-1} G_{\lambda}(s_{\tilde{q}(j,k,r+1)} - s_{\tilde{q}(j,k,r)}) \right)$$

$$\leq \sum_n e^{AE(n) + BN(n)}$$

where $A = \ln (K C_{11})$ and $B = \ln (t \vee 1) + \ln (C^2 \vee 1) + \ln (C_{11}^{-2} \vee 1) + \ln (K^{-1} \vee 1)$. The restriction to those sequences $n$ with $E(n) = n$ can be lifted and the following estimate for the entire series obtained

$$\Omega(A, B) = e^{AE(n) + BN(n)} = \prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} e^{(kA+B)n_k} = \exp \left\{ \frac{e^{A+B}}{1 - e^A} \right\}.$$

The manipulations are familiar from, for example, the calculation of the grand canonical partition function for the free Bose gas [24]. The requirement for convergence is that $e^A < 1$, or equivalently, that $KC_{11} < 1$. 

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8 Limit Transition Amplitudes

We are now ready to re-sum the Dyson series. First of all, observe that the functions $h_j (t, \lambda)$ defined in (5.5) will have the limits

$$h_j (t) := \lim_{\lambda \to 0} h_j (t, \lambda) = 1_{[S_j, T_j]} (f_j | g).$$ (8.1)

Likewise, we obtain $\tilde{E}_{\alpha \beta} (t) = \lim_{\lambda \to 0} \tilde{E}_{\alpha \beta} (t, \lambda)$ which will be just the expressions in (5.7) with the $h_j (t, \lambda)$ replaced by their limits. Explicitly, we have

$$\tilde{E}_{11} (t) = E_{11}, \quad \tilde{E}_{01} (t) = E_{01} [h_1 (t)]^\alpha,$$
$$\tilde{E}_{10} (t) = E_{10} [h_2^* (t)]^\beta, \quad \tilde{E}_{00} (t) = [h_1 (t)]^\alpha E_{\alpha \beta} [h_2^* (t)]^\beta.$$ (8.2)

Secondly, only type I terms will survive the limit. This means that, for the $n$-th term in the Dyson series, the only sequences $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n$ appearing will be those for which $0 = \alpha_n = \beta_1$ and $\beta_l = \alpha_{l+1}$ for $l = 1, \ldots, n-1$.

Thirdly, we encounter the following limit of the two point function: $G_\lambda (t - s)$.

Let $f$ and $g$ be Schwartz functions then we will have the limit

$$\int_0^T dt_2 \int_0^{t_2} dt_1 G_\lambda (t_2 - t_1) f (t_2) g (t_1) \to \kappa_+ \int_0^T ds f (s) g (s).$$

Therefore, employing lemma (2.3), we find

$$\lim_{\lambda \to 0} \langle \phi_1 \otimes W_\lambda (1) \Phi | U (t/\lambda^2, \lambda) \phi_2 \otimes W_\lambda (2) \Phi \rangle = \langle W (f_1 \otimes 1_{[S_1, T_1]}) \Psi | W (f_2 \otimes 1_{[S_2, T_2]}) \Psi \rangle$$

$$\quad + \sum_n (-i)^n \int_{\Delta_n (t)} ds_n \ldots ds_1 \prod_{l=1}^{n-1} [\kappa_+ \vartheta_+ (s_{l+1} - s_l)]^\beta_l$$

$$\times \sum_{\beta_+ \in \{0,1\}^{n-1}} \langle \phi_1 | \tilde{E}_{0\beta_{n-1}} (s_n) \ldots \tilde{E}_{\beta_1 \beta_2} (s_2) \tilde{E}_{\beta_1 0} (s_1) \phi_2 \rangle$$ (8.3)

where we use the symbol $\vartheta_+$ for a one-sided delta function: $\int \vartheta_+ (t - s) f (s) ds = f (t^+)$.  

We now develop this series. Suppose that we have $\beta_{k+1} = 0 = \beta_k$, that is, there are no contractions to the $k$-th term, then we encounter the factor $\tilde{E}_{00} (s) = [h_1 (s)]^\alpha E_{\alpha \beta} [h_2^* (s)]^\beta$ where $s = s_k$. Otherwise, if we have contractions on the terms associated to consecutive variables $s_{k+r}, \ldots, s_{k+1}, s_k$ and we assume that $s_{k+r}$ is not paired to $s_{k+r+1}$, nor $s_k$ to $s_{k-1}$, then we encounter the factor $\tilde{E}_{01} (s_{k+r}) \tilde{E}_{11} (s_{k+r-1}) \ldots \tilde{E}_{11} (s_{k+1}) \tilde{E}_{10} (s_k)$ with the variables $s_{k+r}, \ldots, s_{k+1}, s_k$ all forced equal to a common value $s$, say. This factor will then be $[h_1 (s_k)]^\alpha E_{\alpha 1} (E_{11})^{r-2} E_{1\beta} [h_2^* (s_k)]^\beta$.

Now (8.3) involves a sum over all consecutive pairings: the corresponding partition will have all parts consisting of consecutive labels. We can list these parts in increasing order, say from 1 to $m$ if there are $m$ of them, and let $r_j$ be
the size of the $j$-th part. The number of contractions will be $\sum \beta_i$ and this will be $n - m = \sum_{j=1}^{m} (r_j - 1)$. With these observations we see that (8.3) becomes

$$
\langle W \left( f_1 \otimes 1_{[S_1,T_1]} \right) \Psi | W \left( f_2 \otimes 1_{[S_2,T_2]} \right) \Psi \rangle \sum_{n} \sum_{m} \sum_{r_1, \ldots, r_m = n} d s_m \cdot d s_1 \cdot \left( -i \sum_{j=1}^{m} r_j \Sigma_{j=1}^{m} (r_j - 1) \right) \phi_1 E_{\alpha_1, \beta_1}^{(r_1)} \cdots E_{\alpha_m, \beta_m}^{(r_m)} \phi_2
$$

where we set

$$
E_{\alpha, \beta}^{(r)} := \begin{cases} 
E_{\alpha, \beta}, & r = 1; \\
E_{\alpha_1} (E_{11})^{r-2} E_{1\beta}, & r \geq 2.
\end{cases}
$$

(8.5)

In the following, we shall encounter the coefficients

$$
L_{\alpha, \beta} := -i \sum_{r=1}^{\infty} \left( -i \kappa \right)^{r-1} E_{\alpha, \beta}^{(r)} = -i E_{\alpha, \beta} - \kappa E_{\alpha_1} \frac{1}{1 + i \kappa E_{11}} E_{1\beta}.
$$

In the following, we shall encounter the coefficients

$$
L_{\alpha, \beta} := -i \sum_{r=1}^{\infty} \left( -i \kappa \right)^{r-1} E_{\alpha, \beta}^{(r)} = -i E_{\alpha, \beta} - \kappa E_{\alpha_1} \frac{1}{1 + i \kappa E_{11}} E_{1\beta}.
$$

(8.6)

With respect to the representation $L^2(\mathbb{R}^+; \mathfrak{t}) \cong \mathfrak{t} \otimes L^2(\mathbb{R}^+)$, we introduce the four fundamental operator processes (here $\chi_{[0,t]}$ is the operator on $L^2(\mathbb{R}^+)$ corresponding to multiplication by $1_{[0,t]}$)

$$
\begin{align*}
(\text{creation}) & \quad A^{10}_t = A^+ (g \otimes 1_{[0,t]}); \\
(\text{conservation}) & \quad A^{11}_t = d \Gamma \left( |g\rangle \langle g| \otimes \chi_{[0,t]} \right); \\
(\text{annihilation}) & \quad A^{01}_t = A^- (g \otimes 1_{[0,t]}); \\
(\text{time}) & \quad A^{00}_t = t.
\end{align*}
$$

(8.8)

These are the basic quantum stochastic processes on the Hudson-Parthasarathy space $\Gamma \left( L^2(\mathbb{R}^+, \mathfrak{t}) \right)$. We note that the quantum Itô table takes the concise form

$$
d A^{a1}_t d A^{\beta}_t = \gamma d A^{a\beta}_t
$$

(8.9)

with all other pairs vanishing.

**Theorem (8.1)** Suppose the system operators $E_{\alpha, \beta}$ are bounded with $K \| E_{11} \| < 1$. Let $\phi_1, \phi_2 \in \mathfrak{h}_S$ and $f_1, f_2 \in \mathfrak{t}$. Then

$$
\lim_{\lambda \to 0} \left\langle \phi_1 \otimes W_\lambda \left( 1 \right) \Phi | U^{(\lambda)}_t \phi_2 \otimes W_\lambda \left( 2 \right) \Phi \right\rangle = \left\langle \phi_1 \otimes W \left( f_1 \otimes 1_{[S_1,T_1]} \right) \Psi | U_t \phi_2 \otimes W \left( f_2 \otimes 1_{[S_2,T_2]} \right) \Psi \right\rangle
$$

where $(U_t : t \geq 0)$ is a unitary adapted quantum stochastic process on $\mathfrak{h}_S \otimes \Gamma \left( L^2(\mathbb{R}^+, \mathfrak{t}) \right)$ satisfying the quantum stochastic differential equation

$$
d U_t = L_{\alpha, \beta} U_t \otimes d A^{a\beta}_t.
$$

(8.10)
with $U_0 = 1$ and where the coefficients are given by (8.6):

$$
\begin{align*}
L_{11} & = -iE_{11}(1 + i\kappa E_{11})^{-1}, \quad L_{10} = -i(1 + i\kappa E_{11})^{-1}E_{10} \\
L_{01} & = -iE_{01}(1 + i\kappa E_{11})^{-1}, \quad L_{00} = -iE_{00} - \kappa E_{01}(1 + i\kappa E_{11})^{-1}E_{10}.
\end{align*}
$$

Proof. The quantum stochastic differential equation (8.10) takes the form

$$
dU_t = \frac{1}{\gamma} (W - 1) U_t \otimes dA_t^{11} + LU_t \otimes dA_t^{10} - L^\dagger WU_t \otimes dA_t^{01} - \left( \frac{1}{2} \gamma L^\dagger L + iH \right) U_t \otimes dA_t^{00}
$$

where

$$
\begin{align*}
W & = \frac{1 - i\kappa - E_{11}}{1 + i\kappa + E_{11}} \text{ (unitary)} \\
L & = -i(1 + i\kappa E_{11})^{-1}E_{10} \text{ (bounded)} \\
H & = E_{00} + \text{Im} \left\{ \frac{1}{1 + i\kappa + E_{11}} E_{10} \right\} \text{ (self-adjoint).} \quad (8.11)
\end{align*}
$$

A fundamental result of quantum stochastic calculus [7] is that the process $U_t$ defined as the solution of (8.11) with initial condition $U_0 = 1$, exists and is an adapted, unitary process. With our summation convention in place, we have the chaotic expansion

$$
U_t = \sum_{m \geq 0} \int_{\Delta_m(t)} L_{\alpha(m),\beta(m)} \cdots L_{\alpha(1),\beta(1)} \otimes dA_{s(m)}^{\alpha(m)\beta(m)} \cdots dA_{s(1)}^{\alpha(1)\beta(1)} \quad (8.12)
$$

and so \( \langle \phi_1 \otimes W (f_1 \otimes 1_{[S_1,T_1]}) \Psi | U_t \phi_2 \otimes W (f_2 \otimes 1_{[S_2,T_2]}) \Psi \rangle \) can be expressed as

$$
\langle W (f_1 \otimes 1_{[S_1,T_1]}) \Psi | W (f_2 \otimes 1_{[S_2,T_2]}) \Psi \rangle \sum_{m \geq 0} \langle \phi_1 | L_{\alpha(m),\beta(m)} \cdots L_{\alpha(2),\beta(2)} L_{\alpha(1),\beta(1)} \phi_2 \rangle
$$

$$
\times \int_{\Delta_m(t)} ds_m \cdots ds_1 \left( |h_1 (s_m)^{\alpha(m)} | h_2^* (s_m)^{\beta(m)} | \cdots |h_1 (s_1)^{\alpha(1)} | h_2^* (s_1)^{\beta(1)} \right).
$$

By inspection, this evidently agrees with (8.4). \( \blacksquare \)

8.1 Re-summing the Series

Again we drop all diagrams that are type II to get the series

$$
\begin{align*}
\bigcirc & = + \big[ \ldots \big] + \big[ \big[ \ldots \big] \big] \\
& + \big[ \big[ \ldots \big] \big] + \big[ \big[ \ldots \big] \big] + \ldots
\end{align*}
$$

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We see the first appearance of scattering in the last term in the 3rd term of the series: such terms however eventually out-proliferate diagrams with no scattering. The terms have been grouped by vertex number, however, it also possible to group them by effective vertex number (equal to the number of parts, or equivalently the original simplex degree minus the number of contractions) to give

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+ 
\cdots
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\]

where now each box is the following sum over all effective one-vertex contributions:

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\cdots
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\]

which is analogous to the expression of the self-energy in quantum field theory: as a sum over irreducible terms. (As we have seen, one-vertex contributions terminate at second order when there is no scattering: as this is a form of cumulant expansion, the emission/absorption problem is Gaussian, while allowing scattering means that we must have cumulant moments to all orders!)

If the limit effective one-vertex label is \( t \) then its weight is

\[
\begin{align*}
- i \dot{E}_{00} (t) + (-i)^2 \kappa \dot{E}_{01} (t) \dot{E}_{10} (t) + (-i)^3 \kappa^2 \dot{E}_{01} (t) \dot{E}_{11} (t) \dot{E}_{10} (t) + \cdots \\
&= - i \dot{E}_{00} (t) - \kappa \dot{E}_{01} (t) \frac{1}{1 + i \kappa E_{11}} \dot{E}_{10} (t) \\
&= [h^*_1 (t)]^\alpha G_{\alpha \beta} [h_2 (t)]^\beta
\end{align*}
\]

where the geometric series can be summed since \( \| \kappa E_{11} \| < 1 \). We therefore see that

\[
\lim_{\lambda \to 0} \langle \phi_1 \otimes \varepsilon_{\lambda} (1) \Phi_R | U_t (\lambda) \phi_2 \otimes \varepsilon_{\lambda} (2) \Phi_R \rangle
\]

\[
= \left\langle \phi_1 \otimes \varepsilon (f_1 \otimes 1_{[S_1, T_1]}) \Phi [\left[ 1 + \int_0^t G_{\alpha \beta} dA_{\alpha \beta} \right] \phi_2 \otimes \varepsilon (f_2 \otimes 1_{[S_2, T_2]}) \Phi \right\rangle.
\]

The QSDE then takes the form

\[
\begin{align*}
dU_t &= G_{\alpha \beta} dA_{\alpha \beta} U_t \\
&= \frac{1}{\gamma} (W - 1) U_t \otimes dA_{11}^1 + LU_t \otimes dA_{10}^1 \\
&\quad - L^\dagger W U_t \otimes dA_{01}^0 - \left( \frac{1}{2} \gamma L^\dagger L + iH \right) U_t \otimes dA_{00}^0
\end{align*}
\]

with the coefficients \( (W, L, H) \) are as before.
9 Dynamical Evolutions

Let $X$ be a bounded operator on the system state space $\mathcal{H}_S$. We define its *Heisenberg evolute* to be

$$J_{t(\lambda)}(X) := U_{t(\lambda)}^\dagger [X \otimes 1_R] U_{t(\lambda)}.$$  \hfill (9.1)

In addition, what we term the *co-evolute* is defined to be

$$K_{t(\lambda)}(X) := U_{t(\lambda)}^\dagger [X \otimes 1_R] U_{t(\lambda)}^\dagger.$$  \hfill (9.2)

We wish to study the limits of $J_{t(\lambda)}$ and $K_{t(\lambda)}$ as quantum processes taken relative to the Fock vacuum state $\Phi \in \mathcal{H}_R$ for the Bose reservoir. To this end, we note the developments

$$K_{t(\lambda)}(X) = \sum_n (-1)^n \int_{\Delta_n(t)} ds_n \cdots ds_1 X_{\lambda_{n(\lambda)}} \cdots \lambda_{\lambda_{1(\lambda)}} (X \otimes 1_R),$$  \hfill (9.3)

$$J_{t(\lambda)}(X) = \sum_{n, \tilde{n}} (-i)^{n+\tilde{n}} \int_{\Delta_n(t)} ds_n \cdots ds_1 \int_{\Delta_{\tilde{n}}(t)} dt_{\tilde{n}} \cdots dt_1 \times Y_{s_1}^{(\lambda)} \cdots Y_{s_n}^{(\lambda)} [X \otimes 1_R] Y_{\tilde{s}_1}^{(\lambda)} \cdots Y_{\tilde{s}_{\tilde{n}}}^{(\lambda)},$$  \hfill (9.4)

where $\lambda_{\lambda_{i(\lambda)}} := \frac{1}{\Delta(t)} \lambda_{\lambda_{i(\lambda)}}$.

We note that the co-evolution has the simpler form when iterated. The evolution itself requires a separate expansion of the unitaries. (This disparity is related to the proof of unitarity for quantum stochastic processes in [7], where the isometric property requires some work while the co-isometric property is established immediately.) In fact, the same inequalities as used to establish the convergence of $U_{t(\lambda)}$ suffice for the co-evolution: in both cases we have a Picard iterated series. We remark that in [26] the co-evolution only is treated for emission/absorption interactions.

We likewise have the expansion

$$\left< \phi_1 \otimes W_{\lambda_1} (1) \Phi \mid J_{t(\lambda)}(X) \phi_2 \otimes W_{\lambda_2} (2) \Phi \right>$$

$$= \sum_{n, \tilde{n}} (-i)^{n-\tilde{n}} \int_{\Delta_n(t)} ds_n \cdots ds_1 \int_{\Delta_{\tilde{n}}(t)} dt_{\tilde{n}} \cdots dt_1 \times \left< \phi_1 \mid \tilde{E}_{\alpha_1, \beta_1} (s_1, \lambda) \cdots \tilde{E}_{\alpha_n, \beta_n} (s_n, \lambda) X \tilde{E}_{\mu_1, \nu_1} (s_{\tilde{n}}, \lambda) \cdots \tilde{E}_{\mu_{\tilde{n}}, \nu_{\tilde{n}}} (s_{\tilde{n}}, \lambda) \phi_2 \right>$$

$$\times \left< \Phi \mid [a_{s_1}^+ (\lambda)]^{\alpha_1} [a_{s_1}^- (\lambda)]^{\beta_1} \cdots [a_{s_n}^+ (\lambda)]^{\alpha_n} [a_{s_n}^- (\lambda)]^{\beta_n} [a_{s_{\tilde{n}}}^+ (\lambda)]^{\mu_1} [a_{s_{\tilde{n}}}^- (\lambda)]^{\nu_1} \Phi \right>. \hfill (9.5)$$

The vacuum average of the reservoir operators can be expressed as a sum of products of two-point functions with each summand representable as a partition of $n + \tilde{n}$ vertices. Our strategy is similar to before. We shall use diagrams
to describe the individual contributions, and attempt to obtain a uniform estimate. The Heisenberg diagrams are a more involved than last time due to the scattering, however, the general idea goes through again.

Let us consider an arbitrary Heisenberg diagram. If we considered only the \( t - t \) contractions and ignored everything else then we would have a partition of the \( n t \)–variables, let’s say with occupation numbers \( \mathbf{n} = (n_j) \). Likewise, if we looked at only the \( s - s \) contractions then we have a partition of the \( n' \) \( s \)–variables, say with occupation numbers \( \mathbf{n}' = (n'_j) \). At this stage we can then take the \( s - t \) contractions into account. The diagram below shows a quartet of \( s \) variables joined to a triple of \( t \) variables.

![Figure 5](image)

Let \( l_{jk} \) be the number of \( s - t \) contractions joining a part of \( j \) ‘s’s to a part of \( k \) ‘s: here we use an obvious abuse of terminology, as technically they are all in the same part! We also introduce the occupation numbers \( \mathbf{l} = (l_j), \mathbf{l}' = (l'_j) \) where \( l_k = \sum_j l_{jk} \) and \( l'_j = \sum_k l_{jk} \). (When no scattering was present, we only had the possibility that \( l_{11} \), previously denoted as \( l \), could be non-zero.) It is convenient to introduce the occupation numbers \( \mathbf{m} = (m_j) \) and \( \mathbf{m}' = (m'_j) \) where \( m_j = n_j - l_j \) and \( m'_j = n'_j - l'_j \). Here \( m_j \) counts the number of parts of \( t \)-variables of size \( j \) having no elements contracted with an \( s \)-variable.

The procedure adopted in the last chapter is now repeated. We consider equivalence classes of Heisenberg diagrams leading to the same set of sequences \( \mathbf{n}, \mathbf{n}', \mathbf{l}, \mathbf{l}' \), or equivalently \( \mathbf{m}, \mathbf{m}', \mathbf{l}, \mathbf{l}' \) as above. We can choose a basic Heisenberg diagram as the representative of each class, and there will be permutations \( \rho \in \mathfrak{S}_n \) and \( \rho' \in \mathfrak{S}_n' \) of the \( t \) and \( s \) variables respectively which will allow us to reorganize the basic Heisenberg diagram into any other element of the the class. (We omit the explicit choice of basic of Heisenberg diagram and leave its specification to the reader as an exercise.)

Now for each diagram in a given class there will then be chronologically ordered blocks of sizes \( m_1, m_2, \ldots, m'_1 \), \( m_2, \ldots, l_1, l_2, \ldots \) and by the type of argument encountered before we arrive at the following upper bound for the sum of absolute values for all the diagrams:

\[
\sum_{\mathbf{m}, \mathbf{m}'} C_{11}^{E(m+m'+1+l')} - 2N(m+m'+1+l') + m_1 + m'_1 + l_1 + l_2 \\
\times C^{2N(m+m'+1+l') - (m_1 + m'_1 + l_1 + l_2)} \\
\times (t \lor l)^N(m+m'+1) \\
\times (m_1!m_2! \cdots) (m'_1!m'_2! \cdots) (l_1!l_2! \cdots) \\
\times K^{E(m+m'+1+l') - N(m+m'+1+l')} \gamma^N(l). 
\]

21
Here we add sequences of occupation numbers componentwise, ie \( m + m' \) is 
\((m_j + m'_j)\), etc., and we note that \( N(l) = N'(l') \). Recalling the constants \( A \) and 
\( B \) from before, and introducing \( B' = \frac{1}{2} \ln (t \lor 1) + \ln (C^2 \lor 1) + \ln (C_{11}^{-1} \lor 1) + \ln (K^{-1} \lor 1) + \frac{1}{2} \ln (\gamma) \), we sum the series to get the upperbound

\[
\exp \left\{ \frac{e^{A+B}}{1-e^A} + \frac{e^{2A+B'}}{1-e^{2A}} \right\}
\]

which is again convergent as \( e^A < 1 \).

We now wish to determine the limit \( \lambda \to 0 \). Once again, only diagrams having time consecutive \( s-s \) and \( t-t \) contractions, as well as non-crossing \( s-t \) contractions, are going to contribute to the limit. The presence of scattering now means that we have more diagrams, however, we can reduce this using the effective vertex method and, once again we can arrive at a simple recursive formula. This time, we have

\[
\begin{align*}
& (\quad X \quad) = (\quad X \quad) \\
& \quad + (\quad X \quad) \\
& \quad + (\quad X \quad) \\
& \quad + (\quad X \quad)
\end{align*}
\]

Here we meet new effective vertices in the final diagram. On the right we have

\[
\begin{align*}
\begin{array}{c}
\quad \\
\quad \end{array}
\end{align*}
\]

which for vertex time \( t \) corresponds to the operator weight

\[
-\frac{1}{1 + \kappa \overline{E}_{11}} \frac{\kappa \overline{E}_{11}}{} (t) \frac{\kappa \overline{E}_{11}}{} (t) \frac{\kappa \overline{E}_{11}}{} (t) \frac{\kappa \overline{E}_{11}}{} (t) + \cdots
\]

While on the left we have

\[
\begin{align*}
\begin{array}{c}
\quad \\
\quad \end{array}
\end{align*}
\]

\[
\equiv G_{1\beta} [h_2 (t)]^\beta.
\]
which has the weight
\[
i\dot{E}_{01}(t) + i^2\kappa^*\dot{E}_{01}(t)\dot{E}_{11}(t) + i^3(\kappa^*)^2\dot{E}_{01}(t)\dot{E}_{11}(t) + \cdots
\]
\[
= i\dot{E}_{01}(t) \frac{1}{1 - i\kappa^*E_{11}}
\]
\[
= \left[h_1(t)^\alpha \right] G_{1\alpha}^1.
\]

The recursion relation here is
\[
\left<\phi_1 \otimes \varepsilon(1) | U_t^\dagger[X \otimes 1] U_t \phi_2 \otimes \varepsilon(2)\right> = \left<\phi_1 \otimes \varepsilon(1) | [X \otimes 1] \phi_2 \otimes \varepsilon(2)\right>
\]
\[
+ \int_{\Delta_2(t)} \left<\phi_1 \otimes \varepsilon(1) | U_{t_2}^\dagger[X \otimes 1] \left(G_{\alpha\beta} \left[h_1(t_1)^*\right]^\alpha \left[h_2(t_1)^\beta\right] dU_{t_2}\phi_2 \otimes \varepsilon(2)\right)\right>
\]
\[
+ \int_{\Delta_2(t)} \left<\phi_1 \otimes \varepsilon(1) | dU_{t_1}^\dagger \left[G_{1\alpha}^1 \left[h_1(t_1)^*\right]^\alpha \left[h_2(t_1)^\beta\right] [X \otimes 1] U_{t_2}\phi_2 \otimes \varepsilon(2)\right)\right>
\]
\[
+ \int_0^t dt_1 \left<\phi_1 \otimes \varepsilon(1) | U_{t_1}^\dagger \left[h_1(t_1)^*\right]^\alpha G_{1\alpha}^1 X G_{1\beta}^1 \left[h_2(t_1)^\beta\right] \right] U_{t_1}\phi_2 \otimes \varepsilon(2)\right>
\]
\[
= \left<\phi_1 \otimes \varepsilon(1) | \left\{1 + \int_0^t U_s^\dagger \left[(XG_{\alpha\beta} + G_{\beta\alpha}^1 X + G_{1\alpha}^1 X G_{1\beta}) \otimes dA_{s}^{\alpha\beta}\right] \otimes dA_{s}^{\alpha\beta} U_s\right\}\phi_2 \otimes \varepsilon(2)\right>,
\]
and this is the form we want!

To summarize, the pre-limit flow \( J_t^{(\lambda)} : \mathcal{B}(\mathfrak{h}_S) \rightarrow \mathcal{B}(\mathfrak{h}_S \otimes \mathfrak{h}_R) \) given by \( J_t^{(\lambda)}(X) := U_t^{\dagger(\lambda)}(X \otimes 1_R) U_t^{(\lambda)} \) converges in the sense of weak matrix elements, for fixed \( X \in \mathcal{B}(\mathfrak{h}_S) \), to the limit process \( J_t(X) = J_t(X \otimes 1) J_t \). We find that \((J_t)_{t \geq 0}\) determines a quantum stochastic flow on \( \mathfrak{h}_S \otimes \Gamma (L^2(\mathbb{R}^+, \mathfrak{f})) \) and from the quantum stochastic calculus we obtain the quantum Langevin, or stochastic Heisenberg, equation
\[
dJ_t(X) = J_t(\mathcal{L}_{\alpha\beta}(X)) \otimes dA_t^{\alpha\beta}.
\]

The super-operators \( \mathcal{L}_{\alpha\beta} \) are the required Evans-Hudson maps obtained \[27\] \( \mathcal{L}_{\alpha\beta}(X) = XG_{\alpha\beta} + G_{\beta\alpha}^1 X + G_{1\alpha}^1 X G_{1\beta} \) and these can be written in the standard form
\[
\mathcal{L}_{11}(X) = \frac{1}{\gamma} (W^\dagger XW - X);
\]
\[
\mathcal{L}_{10}(X) = W^\dagger [X, L]; \quad \mathcal{L}_{01}(X) = -[X, L^\dagger] W;
\]
\[
\mathcal{L}_{00}(X) = \frac{\gamma}{2} [L^\dagger, X] L + \frac{\gamma}{2} L^\dagger [X, L] - i[X, H].
\]

In particular, \( \mathcal{L}_{00} \) is a generator of Lindblad type \[15\]. We shall give a more detailed treatment of the convergence in the next section.
10 The Convergence of the Heisenberg Evolution

We now wish to determine the limit $\lambda \to 0$ of (9.5). We have an integration over a double simplex region and the main features emerge from examining the vacuum expectation of the product of creation and annihilation operators. Evidently, the vacuum expectation can be decomposed as a sum over products of two point functions and it is here that lemma 6.1 becomes important. What must happen for a term to survive the limit? If we have any contractions between vertices labelled by the $t$’s then the term will vanish if the times are not consecutive. The same is true for contractions between vertices labelled by the $s$’s. From our estimate in the previous section, we can ignore the terms that do not comply with this.

As a result, contractions between the $s$’s, say, will come in time-consecutive blocks: for instance, we will typically have $m$ blocks of sizes $r_1, r_2, \ldots, r_m$ (these are integers 1,2,3,..., and $\sum_{j=1}^{m} r_j = n$). With a similar situation for the $t$’s, we obtain the expansion

$$
\left\langle \phi_1 \otimes W_\lambda (1) \Phi \middle| J_t^{(\lambda)} (X) \right| \phi_2 \otimes W_\lambda (2) \Phi \right\rangle
$$

$$
= \sum_{n, \bar{n}} (-i)^{n-\bar{n}} \sum_{r=1}^{n} \sum_{t=\bar{n}} \int_{\Delta_n(t)} ds_n \cdots ds_1 \int_{\Delta_{\bar{n}}(t)} dt_{\bar{n}} \cdots dt_1
$$

$$
\times \left\langle \phi_1 \middle| \tilde{E}_{\alpha_1 \beta_1}^{(r_1)} \left( s_1^{(1)}, \ldots, s_{r_1}^{(1)} ; \lambda \right) \cdots \tilde{E}_{\alpha_m \beta_m}^{(r_m)} \left( s_1^{(m)}, \ldots, s_{r_m}^{(m)} ; \lambda \right) \right| \phi_2 \right\rangle
$$

$$
\times X \tilde{E}_{\mu_1 \nu_1}^{(l_1)} \left( t_1^{(1)}, \ldots, t_{l_1}^{(1)} ; \lambda \right) \tilde{E}_{\mu_2 \nu_2}^{(l_2)} \left( t_1^{(2)}, \ldots, t_{l_2}^{(2)} ; \lambda \right) \tilde{E}_{\mu_3 \nu_3}^{(l_3)} \left( t_1^{(3)}, \ldots, t_{l_3}^{(3)} ; \lambda \right) \tilde{E}_{\mu_4 \nu_4}^{(l_4)} \left( t_1^{(4)}, \ldots, t_{l_4}^{(4)} ; \lambda \right)
$$

$$
\times \left\langle \Phi \right| \begin{bmatrix} a_{+}^{\mu_1} (\lambda) \\ a_{-}^{\beta_1} (\lambda) \\ \vdots \\ a_{+}^{\mu_m} (\lambda) \\ a_{-}^{\beta_m} (\lambda) \end{bmatrix}^{\alpha_1} \begin{bmatrix} a_{-}^{\beta_1} (\lambda) \\ a_{+}^{\alpha_2} (\lambda) \\ \vdots \\ a_{-}^{\beta_m} (\lambda) \\ a_{+}^{\alpha_1} (\lambda) \end{bmatrix}^{\beta_1} \begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix} \Phi
$$

$$
+ \text{negligible terms} \quad (10.1)
$$

where we relabel the times as

$$
s_k^{(j)} : = s_{r_1 + \cdots + r_{j-1} + k}, \quad 1 \leq k \leq r_j;
$$

$$
t_k^{(j)} : = t_{l_1 + \cdots + l_{j-1} + k}, \quad 1 \leq k \leq l_j;
$$

and introduce the block product of system operators

$$
\tilde{E}_{\alpha \beta}^{(r_j)} \left( s_1^{(j)}, \ldots, s_{r_j}^{(j)} ; \lambda \right) := \tilde{E}_{\alpha_1 \beta_1} \left( s_1^{(1)} ; \lambda \right) \tilde{E}_{\alpha_2 \beta_2} \left( s_2^{(1)} ; \lambda \right) \cdots \tilde{E}_{\alpha_{r_j-1} \beta_{r_j-1}} \left( s_{r_j-1}^{(1)} ; \lambda \right) \tilde{E}_{\alpha_{r_j} \beta_{r_j}} \left( s_{r_j}^{(1)} ; \lambda \right).
$$
We now examine the limit of (10.1). The estimate on the series expansion of the Heisenberg evolve given in the previous section shows that we can ignore the so-called negligible terms in (10.1). The limit is rather difficult to see at this stage. However, what we can do is to recast the expression that we claim will be the limit,

\[ \langle \phi_1 \otimes W (f_1 \otimes 1_{[S, T_1]}) \Psi | J_t (X) \phi_2 \otimes W (f_2 \otimes 1_{[S, T_2]}) \Psi \rangle, \]  

with \( J_t (X) = U_t^\dagger (X \otimes 1) U_t \), in a more explicit form.

Recall the chaotic expansion of the process \( U_t \) given in (8.12), the expression (10.2) then becomes

\[ \sum_{m, \bar{m}} \int_{\Delta_m(t)} \int_{\Delta_{\bar{m}}(t)} \sum_{r_1, \ldots, r_m, t_1, \ldots, t_\bar{m}} \sum_i \langle \phi_1 | \tilde{E}_{\alpha_1 \beta_1}^{(r_1)}(s_1) \otimes \cdots \otimes \tilde{E}_{\alpha_m \beta_m}^{(r_m)}(s_m) X \tilde{E}_{\mu_1 \nu_1}^{(t_1)}(t_1) \cdots \tilde{E}_{\mu_{\bar{m}}}^{(t_{\bar{m}})}(t_{\bar{m}}) \phi_2 \rangle \]

\[ \times \langle \Psi | dA_{\alpha_1 \beta_1}^{(r_1)} \cdots dA_{\alpha_m \beta_m}^{(r_m)} \cdots dA_{\mu_1 \nu_1}^{(t_1)} \cdots dA_{\mu_{\bar{m}}}^{(t_{\bar{m}})} W (f_2 \otimes 1_{[S, T_2]}) \Psi \rangle. \]

Now the expectation between the states \( W (f_2 \otimes 1_{[S, T_2]}) \Psi \) can be converted into an expectation between the Fock vacuum state \( \Psi \) if we make the following replacements

\[ dA^{11} \rightarrow dA^{11} + h_2^2 dA^{10} + h_1 dA^{01} + h_1 h_2^2 dA^{10} \]

\[ dA^{10} \rightarrow dA^{10} + h_1 dA^{00} \]

\[ dA^{01} \rightarrow dA^{01} + h_2^2 dA^{00} \]

\[ dA^{00} \rightarrow dA^{00} \]  

(10.3)

where \( h_j (t) = 1_{[S, T_j]} (f_j | g) \) as in (8.1). This leads to the development

\[ \sum_{m, \bar{m}} \int_{\Delta_m(t)} \int_{\Delta_{\bar{m}}(t)} \sum_{r_1, \ldots, r_m, t_1, \ldots, t_{\bar{m}}} \sum_i \langle \phi_1 | \tilde{E}_{\alpha_1 \beta_1}^{(r_1)}(s_1) \otimes \cdots \otimes \tilde{E}_{\alpha_m \beta_m}^{(r_m)}(s_m) X \tilde{E}_{\mu_1 \nu_1}^{(t_1)}(t_1) \cdots \tilde{E}_{\mu_{\bar{m}}}^{(t_{\bar{m}})}(t_{\bar{m}}) \phi_2 \rangle \]

\[ \times \langle \Psi | dA_{\alpha_1 \beta_1}^{(r_1)} \cdots dA_{\alpha_m \beta_m}^{(r_m)} \cdots dA_{\mu_1 \nu_1}^{(t_1)} \cdots dA_{\mu_{\bar{m}}}^{(t_{\bar{m}})} \Psi \rangle. \]  

(10.4)

where the operators \( \tilde{E}_{\alpha \beta}^{(r)} (t) \) are given by

\[ \tilde{E}_{\alpha \beta}^{(r)} (t) = \begin{cases} \tilde{E}_{\alpha \beta} (t), & r = 1; \\ \tilde{E}_{\alpha 1} (t) \left( \tilde{E}_{11} (t) \right)^{r-2} \tilde{E}_{1 \beta} (t), & r \geq 2. \end{cases} \]

Again we note that the operators \( \tilde{E}_{\alpha \beta} (t) \) have been introduced in (8.2).

It remains to be shown that the limit of (10.1) will be (10.4). We observe that
We now require the fact that

\[
\lim_{\lambda \to 0} \left( \phi_1 \otimes W_\lambda (1) \Phi \right) J^\lambda_t (X) \phi_2 \otimes W_\lambda (2) \Phi
\]

\[
= \sum_{n, \hat{n}} (-i)^{n-\hat{n}} \sum_{m, \hat{m}} \sum_{r, \hat{r}} \sum_{t, \hat{t}} \sum_{l, \hat{l}} \sum_{s, \hat{s}} \int_{\Delta_m (t)} ds_m \cdots ds_1 \int_{\Delta_{\hat{m}} (t)} d\hat{s}_m \cdots d\hat{s}_1 \int_{[\Delta_m (t)]} \cdots \int_{[\Delta_{\hat{m}} (t)]} \cdots \int_{[\Delta_m (t)]} \cdots \int_{[\Delta_{\hat{m}} (t)]} \cdots \int_{[\Delta_m (t)]} \cdots \int_{[\Delta_{\hat{m}} (t)]}
\times \langle \phi_1 | \hat{E}^{(r_1)} (s_1) \cdots \hat{E}^{(r_m)} (s_m) X \hat{E}^{(l_{\hat{m}})} (t_{\hat{m}}) \cdots \hat{E}^{(l_1)} (t_1) \phi_2 \rangle
\times (\kappa) \sum_r \sum_{\hat{r}} \sum_t \sum_{\hat{t}} \sum_s \sum_{\hat{s}}
\times \lim_{\lambda \to 0} \langle \phi | a_{s_1}^+ (\lambda) \rangle^{\alpha_1} [a_{s_1}^- (\lambda)]^{\beta_1} \cdots [a_{s_m}^+ (\lambda)]^{\alpha_m} [a_{s_m}^- (\lambda)]^{\beta_m}
\left[ a_{t_1}^+ (\lambda) \right]^{\mu_1} [a_{t_1}^- (\lambda)]^{\nu_1} \cdots \left[ a_{t_1}^+ (\lambda) \right]^{\mu_1} [a_{t_1}^- (\lambda)]^{\nu_1} \Phi \rangle.
\]

We now require the fact that

\[
\lim_{\lambda \to 0} \int_{[\Delta_m (t)]} ds_m \cdots ds_1 dt_{\hat{m}} \cdots dt_1 \Phi \left[ a_{s_1}^+ (\lambda) \right]^{\alpha_1} [a_{s_1}^- (\lambda)]^{\beta_1} \cdots [a_{s_m}^+ (\lambda)]^{\alpha_m} [a_{s_m}^- (\lambda)]^{\beta_m}
\left[ a_{t_1}^+ (\lambda) \right]^{\mu_1} [a_{t_1}^- (\lambda)]^{\nu_1} \cdots \left[ a_{t_1}^+ (\lambda) \right]^{\mu_1} [a_{t_1}^- (\lambda)]^{\nu_1} \Phi \rangle f (s_m, \ldots, s_1, t_{\hat{m}}, \ldots, t_1)
= \int_{[\Delta_m (t)]} \Psi \left[ A_{s_m}^\alpha (\lambda) \right]^{\beta_m} \cdots \left[ A_{s_1}^\alpha (\lambda) \right]^{\beta_1} \cdots \left[ A_{t_1}^\alpha (\lambda) \right]^{\mu_1} [a_{t_1}^- (\lambda)]^{\nu_1} \Phi \rangle f (s_m, \ldots, s_1, t_{\hat{m}}, \ldots, t_1)
\]

for \( f \) continuous and \( R \) a bounded region in \( m + \hat{m} \) dimensions which is the union of simplices of the type \((5.2)\). This is readily seen, of course, by expanding the \( \Phi \)-expectation as a sum of products of two-point functions and reassembling the limit in terms of the \( \Psi \)-expectations of the processes \( A_t^{ab} \). This is evident from theorems 4.2 and 6.1 quoted earlier and from the quantum Itô calculus \([7]\).

We therefore see that the limit form as given in (10.5) agrees with the stated limit.

**Theorem (10.1)** Suppose that \( E_{\alpha \beta} \) are bounded with \( K \| E_{11} \| < 1 \), as before. Let \( \phi_1, \phi_2 \in h_S \) and \( f_1, f_2 \in t \). Then, for \( X \in B (h_S) \),

\[
\lim_{\lambda \to 0} \left( \phi_1 \otimes W_\lambda (1) \Phi \right) J^\lambda_t (X) \phi_2 \otimes W_\lambda (2) \Phi
= \left( \phi_1 \otimes W (f_1 \otimes 1_{[S, T_1]}) \Psi \right) J_t (X) \phi_2 \otimes W (f_2 \otimes 1_{[S, T_2]}) \Psi.
\]

To summarize, the pre-limit flow \( J^\lambda_t : B (h_S) \to B (h_S) \otimes h_R \) given by \( J^\lambda_t (X) := U_t^{(\lambda)} (X \otimes 1_R) U_t^{\lambda} \) converges in the sense of weak matrix elements, for fixed \( X \in B (h_S) \), to the limit process \( J_t (X) = U_t (X \otimes 1) U_t \). We find that \( (J_t)_{t \geq 0} \) determines a quantum stochastic flow on \( h_S \otimes \Gamma \left( L^2 (\mathbb{R}^+, t) \right) \) and from the quantum stochastic calculus \([7]\) we obtain the quantum Langevin, or stochastic Heisenberg, equation

\[
dJ_t (X) = J_t (\mathcal{L}_{\alpha \beta} (X)) \otimes dA_t^{\alpha \beta}
\]

where

The analogous result will hold for the co-evolution. Though, as mentioned before, there is a more immediate proof using the original estimates.
11 Conclusions

We began with a discussion of time-ordered versus normal ordered presentations of unitary operators relating to scattering dynamics. It is suggestive to write the limit unitary \( U_t \) as either

\[
U_t = \tilde{T} \exp \left\{ -i \int_0^t ds \, E_{\alpha\beta} \otimes [a_s^\dagger]^{\alpha} [a_s]^{\beta} \right\}, \tag{11.1a}
\]

or

\[
U_t = \tilde{N} \exp \left\{ \int_0^t ds \, L_{\alpha\beta} \otimes [a_s^\dagger]^{\alpha} [a_s]^{\beta} \right\}. \tag{11.1b}
\]

Here \( a_t^\# \) are just symbols and we mean nothing more than that (11.1b) is the solution of (8.10) while (11.1a) reminds us that we have the limit generated by a perturbation \( \Upsilon_t^{(\lambda)} = E_{\alpha\beta} \otimes [a_t^\dagger (\lambda)]^{\alpha} [a_t (\lambda)]^{\beta} \). (Formally, of course, we might consider \( a_t^\# \) as a limiting form of the fields \( a_t^{(\lambda)} \).)

Remarkably, these identifications (11.1a, b) can be viewed as presentations of (1.1) and (1.2) if we supplement the operators \( a_t^{\alpha} \) with the following white noise CCR

\[
[a_t, a_s^\dagger] = \kappa_+ \delta_+ (t-s) + \kappa_- \delta_- (t-s) \tag{11.2}
\]

where \( \delta_\pm \) are future/past delta functions: we would have the right hand side \( \gamma(t-s) \) if it was not for the fact that we invariably meet with simplicial integrals. The stochastic Schrödinger equation (8.10) can be written as

\[
dU_t = [a_t^\dagger]^{\alpha} L_{\alpha\beta} U_t [a_t]^{\beta} \, dt \tag{11.3}
\]

which is in normal ordered form. If we understand that \( [a_t^\dagger]^{\alpha} X_{\alpha\beta} (t) [a_t]^{\beta} \, dt \) means \( X_{\alpha\beta} (t) \otimes dA_t^{\alpha\beta} \) then we recover the Hudson-Parthasarathy calculus. The product of two quantum stochastic integrals will have to be put into normal order, using (11.2), but this will be equivalent to the usual quantum Itô rule with Itô table (8.9).

Alternatively, we could consider the equation \( dU_t = -i \Upsilon_t U_t \, dt \) with \( \Upsilon_t = E_{\alpha\beta} \otimes [a_t^\dagger]^{\alpha} [a_t]^{\beta} \): this is what is suggested by (11.1b). (The Hamiltonian \( \Upsilon_t \) plays an analogous role to the one encountered earlier for classical stochastic Hamiltonian flows leading to (1.4).) However, the expression \( \Upsilon_t U_t \) contains terms like \( a_t U_t \) which are out of normal order and so cannot be directly interpreted in the quantum Itô calculus. Nevertheless, the following purely formal manipulations can be used [5]

\[
[a_t, U_t] = [a_t, 1 - i \int_0^t \Upsilon_s U_s ds] = -i \kappa_+ \int_0^t E_{1\beta} [a_s]^{\beta} \delta_+ (t-s) U_s ds \]

leading to

\[
a_t U_t = \frac{1}{1 + i \kappa_+ E_{11}} \left\{ U_t a_t - i \kappa_+ E_{10} U_t \right\}. \tag{11.4}
\]
(Similar manipulations have been performed separately for emission-absorption and for scattering interactions in [28].)

By making the replacement (11.4), wherever it occurs, we obtain a proper normal ordered form and this turns out to be precisely (11.3). In the classical problem for the limit of the flow under the Hamiltonian (1.3), the canonical structure is never lost - though we have to look to the Stratonovich calculus to see it. We similarly have that the canonical structure is retained in the quantum problem - and we even have a formal Hamiltonian $\mathcal{Y}_t$- provided that we look at things in the appropriate way.

Acknowledgement 1 The author is greatful to Ramon van Handel for many stimulating discussions about the original paper that lead to several improvements and a revision of the Heisenberg flow convergence proof.

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