MINIMAL SURFACES IN FINITE VOLUME NONCOMPACT HYPERBOLIC 3-MANIFOLDS

PASCAL COLLIN, LAURENT HAUSWIRTH, LAURENT MAZET, AND HAROLD ROSENBERG

Abstract. We prove there exists a compact embedded minimal surface in a complete finite volume hyperbolic 3-manifold $\mathcal{N}$. We also obtain a least area, incompressible, properly embedded, finite topology, 2-sided surface. We prove a properly embedded minimal surface of bounded curvature has finite topology. This determines its asymptotic behavior. Some rigidity theorems are obtained.

1. Introduction

There has been considerable progress on the study of properly embedded minimal surfaces in Euclidean 3-space [12]. We now know all such orientable surfaces that are planar domains; they are planes, helicoids, catenoids and Riemann’s minimal surfaces. Also we understand the geometry of properly embedded periodic minimal surfaces that are finite topology in the quotient.

In hyperbolic 3-space, there is no classification of this nature. A continuous rectifiable curve in the boundary at infinity of $\mathbb{H}^3$ is the asymptotic boundary of a least area embedded simply connected surface [2].

In this paper we study the existence of periodic minimal surfaces in $\mathbb{H}^3$. More precisely, we consider surfaces in complete noncompact hyperbolic 3-manifolds $\mathcal{N}$ of finite volume. Throughout the paper, we will refer to such manifolds $\mathcal{N}$ as hyperbolic cusp manifolds. In a closed hyperbolic manifold (or any closed Riemannian 3-manifold), there is always a compact embedded minimal surface [13]. They cannot be of genus zero or one, but there are many higher genus such surfaces. The existence and deformation theory of such surfaces was initiated by K. Uhlenbeck [22].

Hyperbolic cusp manifolds play an important role in the theory of closed hyperbolic 3-manifolds. Many link complements in the unit 3-sphere have such a finite volume hyperbolic structure (see [15] and references therein). Given any $V > 0$, Jorgensen proved there are a finite number of such $\mathcal{N}$ of volume $V$. Then Thurston proved that a closed hyperbolic 3-manifold of volume less than $V$ can be obtained from this finite number of manifolds $\mathcal{N}$ given by Jorgensen by hyperbolic Dehn surgery on at least one of the cusp ends (see [3] for details).

We will prove there is a compact embedded minimal surface in any complete hyperbolic 3-manifold of finite volume. Since such a noncompact manifold $\mathcal{N}$ is “not convex at infinity”, minimization techniques do not produce such a minimal
surface. To understand this the reader can verify that on a complete hyperbolic 3-
punctured 2-sphere, there is no simple closed geodesic. In dimension 3, a min-max
technique, together with several maximum principles in the cusp ends of $\mathcal{N}$, will
produce compact embedded minimal surfaces.

We will give two existence results of embedded compact minimal surfaces.

**Theorem A.** There is a compact embedded minimal surface $\Sigma$ in $\mathcal{N}$.

**Theorem B.** Let $S$ be a closed orientable embedded surface in $\mathcal{N}$ which is not a
2-sphere or a torus. If $S$ is incompressible and nonseparating, then $S$ is isotopic to
a least area embedded minimal surface.

Concerning properly embedded noncompact minimal surfaces, there are already
existence results due to Hass, Rubinstein and Wang [9] and Ruberman [17]. Using
different arguments, we give another proof of Ruberman’s minimization result.

**Theorem.** Let $S$ be a properly embedded, noncompact, finite topology, incompress-
able, nonseparating surface in $\mathcal{N}$. Then $S$ is isotopic to a least area embedded
minimal surface.

The surfaces produced by the above theorem have bounded curvature. Actually
the techniques we develop enable us to prove:

**Theorem C.** Let $\Sigma$ be a properly embedded minimal surface in $\mathcal{N}$ of bounded
curvature. Then $\Sigma$ has finite topology.

Since stable minimal surfaces have bounded curvature we conclude:

**Corollary 1.** A properly embedded stable minimal surface in $\mathcal{N}$ has finite topology.

Finite topology is particularly interesting here due to the Finite Total Curvature
Theorem below that describes the geometry of the ends of a properly immersed
minimal surface in $\mathcal{N}$ of finite topology.

**Theorem 2** (Collin, Hauswirth, Rosenberg [5]). A properly immersed minimal
surface $\Sigma$ in $\mathcal{N}$ of finite topology has finite total curvature

$$\int_{\Sigma} K_\Sigma = 2\pi \chi(\Sigma).$$

Moreover, each end $A$ of $\Sigma$ is asymptotic to a totally geodesic 2-cusp end in an end
$C$ of $\mathcal{N}$.

We will make precise these notions.

The simplest example of a surface $\Sigma$ with finite topology appearing in the above
theorem is a 3-punctured sphere. Actually, minimal 3-punctured spheres are totally
geodesic.

**Theorem D.** A proper minimal immersion of a 3-punctured sphere in $\mathcal{N}$ is totally
geodesic.

The paper is organized as follows. In Section 2 we make some general remarks
on the geometry of cusp manifolds stating some results of Jorgensen, Thurston
and Adams. In Section 3 we consider 3-punctured spheres in hyperbolic cusp
manifolds and prove Theorem D. In Section 4 we study minimal surfaces entering
the ends of hyperbolic cusp manifolds $\mathcal{N}$. We prove two maximum principles which
govern the geometry of minimal surfaces in the ends of $\mathcal{N}$. We also establish
a transversality result which is used to study annular ends of minimal surfaces. Section 5 proves Theorems A and B, the existence of compact embedded minimal surfaces in hyperbolic cusp manifolds. Section 6 proves the minimization result in the noncompact case. Then in Section 7 we present several examples to illustrate these theorems.

2. SOME DISCUSSION OF THE MANIFOLDS $N$

In this section we recall some facts about the geometry of a noncompact hyperbolic 3-manifold $N$ of finite volume (see [3] for more details).

Such $N$ are the union of a compact connected submanifold bounded by mean concave tori with constant mean curvature one and a finite number of ends, each end isometric to a quotient of a horoball of $\mathbb{H}^3$ by a $\mathbb{Z}^2$ group of parabolic isometries leaving the horoball invariant. The horospheres in this horoball quotient to mean curvature one tori in $N$.

An end of $N$ can be parametrized by $M = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 1/2\}$ modulo a group $G = G(v_1, v_2)$, generated by two translations by linearly independent horizontal vectors $v_1, v_2 \in \mathbb{R}^2 \times \{0\}$.

The end $C = M/G$ is endowed with the quotient of the hyperbolic metric of $M$,

$$g_\mathbb{H} = \frac{1}{z^2}(dx^2 + dy^2 + dz^2) = \frac{1}{z^2}dX^2.$$  

The horospheres $\{z = c\}$ quotient to tori $T(c)$ of mean curvature one with respect to the unit normal vector $\frac{\partial}{\partial z}$. The vertical curves $\{(x, y) = \text{constant}\}$ are geodesics orthogonal to the tori $T(c)$, with arc length given by $s = \ln z$. The induced metric on $T(c)$ is flat, and lengths on $T(c)$ decrease exponentially as $s \to \infty$.

We will denote by $T(a, b)$ the subset $\{a \leq z \leq b\}$ of $C$.

The Euclidean planes $\{ax + by = c\}$ are totally geodesic surfaces in $C$. When they are properly embedded in $C$, they are the totally geodesic 2-cusp ends in $C$ that appear in the Finite Total Curvature Theorem above.

Define $\Lambda(C) = \max\{|v_1|, |v_2|\}$ with $|v|$ the Euclidean norm. We note that we have made a choice of generators $v_1, v_2$ of the group $G$, so the value $\Lambda(C)$ depends on this choice (we can minimize the value of $\Lambda$ among all choices, but it is not important in the following).

Remark 1. The above notation is well adapted to study the geometry close to $z = 1$. For $z_0$ larger than 1, let $H$ be the map $(x, y, z) \mapsto (z_0 x, z_0 y, z_0 z)$ which sends $M$ to $\mathbb{R}^2 \times [z_0/2, +\infty)$. This map gives us then a chart of $C'' = \{z \geq z_0/2\} \subset C$ parametrized by $\{z' \geq 1/2\}$ with $\Lambda(C'') = \Lambda(C)/z_0$. So, considering a part of the end that is sufficiently far away, we can always assume that $\Lambda(C)$ is small.

We mention two theorems concerning the manifolds $N$.

**Theorem 3** (Jorgensen [20]). Given $V > 0$, there exist a finite number of such manifolds $N$ whose volume is equal to $V$.

**Theorem 4** (Thurston [20]). Any compact hyperbolic 3-manifold $M^3$, $\partial M^3 = \emptyset$, with $\text{Vol}(M) < V$ is obtained from the finite number of $N$ given by Jorgensen’s theorem by hyperbolic Dehn surgery on at least one of the cusp ends.

Concerning surface theory in $N$, we mention one theorem that inspired Theorem D.
Theorem 5 (Adams [1]). Let \( \Sigma \) be a properly embedded 3-punctured sphere in \( \mathcal{N} \), \( \Sigma \) incompressible. Then \( \Sigma \) is isotopic to a totally geodesic 3-punctured sphere in \( \mathcal{N} \).

3. Minimal 3-punctured spheres are totally geodesic

In this section we prove that, under some hypotheses, a minimal surface is totally geodesic. We first have the following result.

Theorem D. A proper minimal immersion of a 3-punctured sphere in \( \mathcal{N} \) is totally geodesic. Moreover, it is \( \pi_1 \) injective.

Proof. Let \( \Sigma \subset \mathcal{N} \) be a properly immersed minimal 3-punctured sphere in \( \mathcal{N} \). Let \( x_0 \in \Sigma \) and \( \alpha, \beta, \gamma \) be three loops at \( x_0 \) that are freely homotopic to embedded loops in the different ends of \( \Sigma \). We notice that \( \alpha \cdot \beta \) is homotopic to \( \gamma \).

Let \( \pi: \mathbb{H}^3 \to \mathcal{N} \) be a universal covering map and \( \tilde{x}_0 \) be in \( \pi^{-1}(x_0) \). Let \( \tilde{\Sigma} \) be the lift of \( \Sigma \) passing through \( \tilde{x}_0 \). The choice of \( \tilde{x}_0 \) induces a monomorphism \( \varphi: \pi_1(\Sigma, x_0) \to \text{Isom}^+ (\mathbb{H}^3) \). \( \tilde{\Sigma} \) is then a proper immersion of the quotient of the universal cover of \( \Sigma \) by \( \ker \varphi \). Let \( \Gamma \) be the image of the monomorphism \( \varphi \). As a consequence \( \tilde{\Sigma} \) is properly immersed in \( \mathbb{H}^3 \). Let us denote by \( T_\alpha \), \( T_\beta \) and \( T_\gamma \) the maps in \( \Gamma \) associated by \( \varphi \) to \( [\alpha], [\beta], [\gamma] \). We notice that \( T_\alpha \circ T_\beta = T_\gamma \).

By the Finite Total Curvature Theorem (Theorem 2), we know each end is asymptotic to \( \mu \times \mathbb{R}_+ \) where \( \mu \) is a geodesic in some \( T(c) \) in a cusp end of \( \mathcal{N} \); \( \mu \times \mathbb{R}_+ \) is a totally geodesic annulus in this cusp end. The inclusion of \( T(c) \) into \( \mathcal{N} \) induces an injection of the fundamental group of \( T(c) \) into that of \( \mathcal{N} \). Hence \( \alpha, \beta \) and \( \gamma \) are sent to nonzero parabolic elements of \( \text{Isom}^+ (\mathbb{H}^3) \) by \( \varphi \).

Next we will prove the limit set of \( \tilde{\Sigma} \) is a circle \( C \) in \( \partial_{\infty} \mathbb{H}^3 \cong \mathbb{S}^2 \) (a loop in \( \partial_{\infty} \mathbb{H}^3 \) is called a circle if it is the asymptotic boundary of a totally geodesic plane). More precisely, we have the following claim, whose proof is based on Adams’ work [1].

Claim 1. There is a circle \( C \) in \( \partial_{\infty} \mathbb{H}^3 \) which is invariant by \( \Gamma \). The limit set of \( \tilde{\Sigma} \) is \( C \).

Then the maximum principle yields that \( \tilde{\Sigma} \) is the totally geodesic plane \( P \) bounded by \( C \), thus proving Theorem D. More precisely, foliate \( \mathbb{H}^3 \cup \partial_{\infty} \mathbb{H}^3 \) minus two points by totally geodesic planes and their asymptotic boundaries so that \( P \) is one leaf of the foliation. This foliation at \( \partial_{\infty} \mathbb{H}^3 \) is a foliation by circles with two “poles” \( p \) and \( q \). The circles close to \( p \) bound hyperbolic planes \( Q \) in \( \mathbb{H}^3 \) that are disjoint from \( \tilde{\Sigma} \). As the circles in the foliations of \( \partial_{\infty} \mathbb{H}^3 \) go from \( p \) to \( C \), there can be no first point of contact of the planes with \( \tilde{\Sigma} \) (\( \tilde{\Sigma} \) is proper and the limit set of \( \tilde{\Sigma} \) is \( C \)). Hence \( \tilde{\Sigma} \) is in the half space of \( \mathbb{H}^3 \setminus P \) containing \( q \). The same argument with planes coming from \( q \) to \( C \) shows that \( \overline{\tilde{\Sigma}} = P \). So \( \tilde{\Sigma} \) is simply connected, which implies \( \ker \varphi = \{1\} \) and \( \Sigma \) is \( \pi_1 \) injective.

Let us now go back to Claim [1].

Proof of Claim [1]: Using the half space model for \( \mathbb{H}^3 \) with \( \infty \) the fixed point of \( T_\alpha \) and using the \( SL_2(\mathbb{C}) \) representation of \( \text{Isom}^+ (\mathbb{H}^3) \), we can write

\[
T_\alpha = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \quad T_\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

with \( w \in \mathbb{C}^*, a, b, c, d \in \mathbb{C} \) such that \( ad - bc = 1 \) and \( a + d = 2 \). Then \( T_\gamma = T_\alpha \cdot T_\beta = \begin{pmatrix} a + cw & b + dw \\ c & d \end{pmatrix} \) is parabolic so it must satisfy \( \lambda = a + cw + d = \pm 2 \).
Since \( a + d = 2 \) we have \( c = 0 \) if \( \lambda = 2 \) or \( c = -4/w \) if \( \lambda = -2 \). If \( c = 0 \), \( T_\alpha \) and \( T_\beta \) would fix the point \( \infty \) and all elements in \( \Gamma \) would have \( \infty \) as fixed point and \( \{ \infty \} \) is the limit set of \( \Gamma \). We will rule out this possibility below.

If \( c = -4/w \), the fixed point of \( T_\beta \) is \( x_\beta = \frac{w(d-a)}{8} \) and the fixed point of \( T_\gamma \) is \( x_\gamma = \frac{w(d-a)}{8} + \frac{w}{2} \). \( T_\alpha \) leaves invariant the circle \( C = \{ \frac{w(d-a)}{8} + tw, t \in \mathbb{R} \} \cup \{ \infty \} \).

Also we have
\[
T_\beta(\infty) = -\frac{wa}{4} = \frac{w(d-a)}{8} = \frac{w}{4} \in C,
\]
\[
T_\beta(x_\gamma) = T_\alpha^{-1}(x_\gamma) = \frac{w(d-a)}{8} - \frac{w}{2} \in C.
\]

Hence \( T_\beta \) leaves \( C \) invariant. Thus \( \Gamma \) leaves \( C \) invariant. Actually, \( C \) is the limit set of \( \Gamma \).

Now let us see that the limit set of \( \tilde{\Sigma} \) is the limit set \( \partial \Gamma \) of \( \Gamma \). We split \( \Sigma \) into the union of a compact part \( K \) containing \( x_0 \) and three cusp ends \( C_i \). So \( \tilde{\Sigma} \) splits into the union of the lift \( \tilde{K} \) of \( K \) and the union of pieces contained in disjoint horoballs \( H_\alpha \) with boundary along the horospherical \( \partial H_\alpha \). Because of the asymptotic behavior of \( \Sigma \), the lift \( g_i \) of \( \partial C_i \) in \( \partial H_\alpha \) is not homeomorphic to a circle. Thus there is a nontrivial parabolic element \( \gamma \in \Gamma \) that leaves \( g_i \) and then \( H_\alpha \) invariant. If \( p \) is any point in \( \mathbb{H}^3 \), the sequence \( \gamma^n(p) \) converges to the center of \( H_\alpha \) (the asymptotic boundary of \( H_\alpha \)) and also to \( \partial \Gamma \): the center of \( H_\alpha \) belongs to \( \partial \Gamma \).

If \( \partial \Gamma \) is only one point (case \( c = 0 \)) it means that there is only one horoball \( H_\alpha \) of \( \tilde{\Sigma} \) is the limit set \( \partial \Gamma \) of \( \Gamma \). We will rule out this possibility below. Since any point in \( \tilde{K} \) is at a finite distance from \( H_\alpha \) and \( \tilde{K} \) is periodic, the \( z \) function reaches its minimum somewhere. The maximum principle then gives a contradiction. So \( c \neq 0 \) and \( \partial \Gamma = C \).

Now, let \( (p_i) \) be a proper sequence of points in \( \tilde{\Sigma} \) and assume it converges to some point \( p_\infty \in \partial_\infty \mathbb{H}^3 \); we want to prove \( p_\infty \in \partial \Gamma = C \). If all the \( p_i \) belong to \( \tilde{K} \), there is a sequence of elements \( \gamma_i \in \Gamma \) such that the distance between \( p_i \) and \( \gamma_i \cdot \tilde{x}_0 \) stays bounded. So \( (p_i) \) and \( (\gamma_i \cdot \tilde{x}_0) \) have the same limit; thus \( p_\infty \) is in the limit set of \( \Gamma \) and so in \( C \).

We can assume that \( p_i \in H_\alpha_i \) for all \( i \). If the sequence \( (\alpha_i) \) is finite, \( p_\infty \) is a center of one of the \( H_\alpha \) so it is in \( C \). If the sequence \( (\alpha_i) \) is not finite, then the distance from \( \tilde{x}_0 \) to \( H_\alpha \) goes to \( \infty \). Actually there is a decreasing sequence of neighborhoods \( N_m \) of \( C \) in \( \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3 \) such that \( N_m \) contains all horoballs centered on \( C \) whose distance to \( \tilde{x}_\infty \) is larger than \( m \) and such that \( \bigcap_{m>0} N_m = C \). Thus \( p_i \in H_\alpha_i \subset N_m \) with \( \lim m_i = +\infty \). This implies \( p_\infty \in \bigcap_{m>0} N_m = C \).}

The proof of the preceding result is based on the study of the group representation \( \varphi \). It can be controlled under some other hypotheses.

**Proposition 6.** Let \( S_0 \) and \( S_1 \) be two properly immersed minimal surfaces in \( \mathcal{N} \) such that the immersions are homotopic. If \( S_0 \) is totally geodesic, then \( S_0 = S_1 \).

**Proof.** Let \( f : S \times [0, 1] \to \mathcal{N} \) be the homotopy between the two minimal immersions. Let \( x_0 \in S \) and \( \tilde{x}_0 \) be a point in \( \pi^{-1}(f(x_0, 0)) \) where \( \pi : \mathbb{H}^3 \to \mathcal{N} \) is a covering map. Let \( g : \tilde{S} \times [0, 1] \to \mathcal{N} \) be the lift of \( f_\ast \) such that \( g_0(x_0, 0) = \tilde{x}_0 \) where \( \tilde{S} \) is the universal cover of \( S \). This defines a group representation
\[
\varphi : \pi_1(S \times [0, 1], (x_0, 0)) = \pi_1(S, x_0) \to \text{Isom}^+(\mathbb{H}^3)
\]
such that $g$ is $\varphi$-equivariant. Since $f(S, 0)$ is totally geodesic, $g(\tilde{S}, 0)$ is a totally geodesic plane $P$. Thus if $(\gamma_i)$ is a proper sequence in $\text{im} \varphi$ and $p \in P$, we have $\gamma_i(p) \in P$ and any limit point of $(\gamma_i(p))_{i \in \mathbb{N}}$ is in the asymptotic boundary of $P$, i.e. a circle $C$. So the limit set of $\text{im} \varphi$ in contained in $C$. As in the proof of Claim 1, this implies that $g_1(S)$ has $C$ as asymptotic boundary. Then, as in the proof of Theorem C, $g(\tilde{S}, 1)$ is totally geodesic and equals $g(\tilde{S}, 0) = P$. So $S_0 = f(S, 0) = f(S, 1) = S_1$.

4. Minimal surfaces in the cusp ends of $\mathcal{N}$

In this section, we will analyze the behavior of embedded minimal surfaces that enter cusp ends of $\mathcal{N}$. In dimension 2, the situation is simple. If $N^2$ is a 2-cusp (i.e. a quotient of a horodisk of $\mathbb{H}^2$ by a parabolic isometry leaving the horodisk invariant), then a geodesic that enters $N^2$ either goes straight to the cusp (i.e. it is an orthogonal trajectory of the horocycles of the cusp) or it leaves $N^2$ in a finite time. In dimension 3, for the moment we know that a properly immersed minimal annulus that enters a cusp end of $\mathcal{N}^3$ is asymptotic to a 2-cusp of the end $(\gamma \times [c, \infty)$, $\gamma$ a compact geodesic of $T(c)$), or the intersection of the minimal annulus with the end of $\mathcal{N}$ is compact.

We will establish two maximum principles in the ends of $\mathcal{N}$ which will control the geometry of embedded minimal surfaces in the ends.

Let $C = M/G(v_1, v_2)$ be an end of $\mathcal{N}$, parametrized by the quotient of $M = \{(x, y, z) \in \mathbb{R}^3 | z \geq 1/2\}$ as in Section 2 with $\Lambda(C) = \Lambda(v_1, v_2)$ the diameter of $T(1)$. Recall that we can make $\Lambda(C)$ as small as we wish by passing to a subend $C'$ of $C$ defined by $z \geq z_0$, $z_0$ large.

We modify the metric on $C$ introducing smooth functions $\Psi : [1/2, \infty) \rightarrow \mathbb{R}$ satisfying $\Psi(z) = z$ for $1/2 \leq z \leq 1$, and $\Psi$ nondecreasing. There will be other conditions on $\Psi$ as we proceed.

Let $g_\Psi = \frac{1}{\Psi^2(x)}dX^2$ be a new metric on $C$; $g_\Psi$ is the hyperbolic metric of $\mathcal{N}$ for $1/2 \leq z \leq 1$.

The mean curvature of the torus $T(z)$ in the metric $g_\Psi$ equals $\Psi'(z)$, with respect to the unit normal $\Psi(z)\partial_z$ (so points towards the cusp: perhaps it is zero). The sectional curvatures for $g_\Psi$ are

$$K_{g_\Psi} = \begin{cases} -\Psi'(z)^2 & \text{for the } (\partial_x, \partial_y) \text{ plane,} \\ \Psi(z)\Psi''(z) - \Psi'(z)^2 & \text{for the } (\partial_x, \partial_z) \text{ and } (\partial_y, \partial_z) \text{ planes.} \end{cases}$$

We will always introduce $\Psi$'s such that $|\Psi|$ and $|\Psi\Psi''|$ are bounded by some fixed constant. Hence the sectional curvatures of the new metrics will be uniformly bounded as well. Then given $\varepsilon_0 > 0$, there is a $k_0 > 0$ such that a stable minimal surface in $(C, g_\Psi)$ has curvature bounded by $k_0$ at all points at least at a distance $\varepsilon_0$ from the boundary. The bound $k_0$ depends only on the bound of the sectional curvatures and $\varepsilon_0$, not on the injectivity radius [16].

Remark 2. The pullback of the $g_\Psi$ by the map $H$ defined in Remark 1 is $H^*g_\Psi = g_{\Psi, z_0}$ where $\Psi_{z_0}(z) = \frac{1}{z_0}\Psi(z_0 z)$. This modification does not change the estimates on $\Psi'$ and $\Psi\Psi''$.

4.1. Maximum principles. In this section we prove maximum principles for a cusp end $C$ endowed with a metric $g_\Psi$. The following estimates will depend on an upper-bound on $|\Psi'|$ and $|\Psi\Psi''|$. 

We have a first result.

**Proposition 7** (Maximum principle I). Let \( k_0, \varepsilon_0 > 0 \). There is a \( \Lambda_0 = \Lambda(k_0, \varepsilon_0) \) such that if \( \Sigma \) is an embedded minimal surface in \((C, g_\Psi)\) with \( |A_\Sigma| \leq k_0 \) and \( \Lambda(C) \leq \Lambda_0 \), then if \( p \in \Sigma \) is at least an intrinsic distance \( \varepsilon_0 \) from \( \partial \Sigma \) and if \( z(q) \leq z(p) \) for all \( q \) in the intrinsic \( \varepsilon_0 \)-disk centered at \( p \), then \( \Sigma = \{ z = z(p) \} \) and \( \Psi'(z(p)) = 0 \).

**Proof.** Let \( \pi : M \to C \) be the covering projection and \( \Sigma = \pi^{-1}(\Sigma) \). We may suppose \( p = \pi(0, 0, z(p)) \).

Since the curvature of \( \Sigma \) is bounded by \( k_0 \), \( \Sigma \) is a graph of bounded geometry in a neighborhood of \( p \). Hence there exists \( \mu = \mu(k_0, \varepsilon_0) \) and a smooth function \( u : D_\mu(0, 0) \to \mathbb{R}, D_\mu(0, 0) = \{ x^2 + y^2 \leq \mu \}, u(0, 0) = z(p) \) and the graph of \( u \) in \( M \) is a subset of \( \Sigma \). \( \mu \) can be chosen such that if \( q \in \text{graph}(u) \), then \( d_\Sigma(\pi(q), p) < \varepsilon_0 \).

Hence \( z(p) \) is a maximum value of \( u \) in \( D_\mu(0, 0) \).

Now \( \Sigma \) is invariant by \( G(v_1, v_2) \), so if \( \Lambda_0 < \mu/2 \), we have \( v_1, v_2 \in D_{\mu/2}(0, 0) \). So \( D_\mu(0, 0) \cap D_\mu(v_1) = D \neq \emptyset \).

Let \( u_1 : D \to \mathbb{R} \) be \( u_1(q) = u(q - v_1) \); the graph of \( u_1 \) is contained in \( \Sigma \). Then \( u_1 v_1 = u(0) \geq u(v_1) \) since \( u \) reaches its maximum at \( O = (0, 0) \). Also \( O \in D \), \( u_1(0) = u(-v_1) \leq u(0) \). Thus the graphs of \( u \) and \( u_1 \) over \( D \) must intersect, and since \( \Sigma \) is embedded, \( u = u_1 \) on \( D \). Hence \( u_1 \) is a smooth continuation of \( u \) to \( D_\mu(0) \cup D_\mu(v_1) \). Repeating this with \( G = \mathbb{Z}v_1 + \mathbb{Z}v_2 \), we see that \( u \) extends smoothly to an entire minimal graph contained in \( \Sigma \). This graph is periodic with respect to \( G \) hence bounded below. The maximum principle at a minimum point of \( u \) implies that \( u \) is constant. Hence \( u = u(0, 0) = z(p) \) and \( T(z(p)) \) is minimal so \( \Psi'(z(p)) = 0 \). \( \square \)

Next we use the maximum principle I to prove a compact embedded minimal surface cannot go far into a cusp end, no \textit{a priori} curvature bound assumed. More precisely we have the following statement.

**Proposition 8** (Maximum principle II). Let \( 0 < t_0 < 1/2 \). There is a \( \Lambda_0 = \Lambda_0(t_0) \) such that if \( \Lambda(C) \leq \Lambda_0 \) and \( \Sigma \) is a compact embedded minimal surface in \((C, g_\Psi)\) with \( \partial \Sigma \subset T(1 - t_0) \) (\( \Sigma \) being transverse to \( T(1 - t_0) \)), then \( \Sigma \subset \{ z \leq 1 \} \).

**Proof.** First suppose \( \Sigma \) is a stable minimal surface. Then by curvature bounds for stable surfaces \([10]\), we know there is a \( k_0 \) such that \( |A_\Sigma| \leq k_0 \) on \( \Sigma \cap \{ z \geq 1 - t_0/2 \} \); \( k_0 \) depends on our assumed bounds on \( \Psi', \Psi'' \). By the maximum principle I, there is a \( \Lambda_0 \), only depending on \( t_0 \), such that if \( \Lambda(C) \leq \Lambda_0 \), then \( z \) has no maximum larger than 1. Hence \( \Sigma \subset \{ z \leq 1 \} \).

Now suppose that \( \Sigma \) is not stable. Choose \( c_0 \) and \( c \) so that \( z < c_0 < c \) on \( \Sigma \) and consider \( \Sigma \subset X = \{ 1/2 \leq z \leq c \} \). We remark that \( \Sigma \) separates \( \hat{X} \), the interior of \( X \). Indeed any loop in \( \hat{X} \) is homotopic to a loop in \( T(c_0) \) which does not intersect \( \Sigma \). So the intersection number mod 2 of a loop with \( \Sigma \) is always 0 since it is a homotopy invariant. This proves that \( \Sigma \) separates \( \hat{X} \). Then denote by \( A \) the connected component of \( X \setminus \Sigma \) which contains \( \{ z = c \} \). \textit{A priori} the boundary of \( A \) is mean convex except for \( \{ z = c \} \). But we can modify the function \( \Psi \) for \( c_0 \leq z \leq c \) such that \( \Psi'(c) = 0 \) and keeping \( \Psi \) nondecreasing and the bounds on \( \Psi' \) and \( \Psi'' \) (\( c \) should be assumed very large). Then \( T(c) \) is now minimal and \( A \) has
mean convex boundary. In $A$, there exists a least area surface $\tilde{\Sigma}$ with $\partial \tilde{\Sigma} = \partial \Sigma$. Now the maximum of the $z$ function on $\tilde{\Sigma}$ is larger than the one on $\Sigma$. Since $\tilde{\Sigma}$ is stable we already know that $\tilde{\Sigma} \subset \{ z \leq 1 \}$, so $\Sigma$ as well.

\[ \Box \]

4.2. Transversality. Now we will see that embedded minimal surfaces of bounded curvature are “strongly transversal” to $T(c)$ in $C$ endowed with the hyperbolic metric.

**Proposition 9.** Let $k_0, \varepsilon_0 > 0$ be given. There exist constants $\Lambda_0$ and $\theta_0 > 0$ such that if $\Sigma$ is an embedded minimal surface in $(C, g_Z)$, $\Lambda(C) \leq \Lambda_0$, $|A_n| \leq k_0$, with $\partial \Sigma$ at an intrinsic distance greater than $\varepsilon_0$ of the points of $\Sigma$ in $T(1)$, then the angle between $\Sigma$ and $T(1)$ is at least $\theta_0$. The constants $\Lambda_0$ and $\theta_0$ only depend on $k_0$ and $\varepsilon_0$.

**Proof.** If this proposition fails, there exist $\Sigma_n, p_n \in \Sigma_n \cap T(1)$ in a hyperbolic cusp $C_n$ satisfying the hypotheses, such that $\Lambda(C_n)$ and the angle between $\Sigma_n$ and $T(1)$ at $p_n$ go to zero. Lift $\Sigma_n$ to $M$ so that $p_n = (0, 0, 1)$. The curvature bound gives the existence of a disk $D = D_\mu(0, 0) \subset \mathbb{R}^2$ and smooth functions $u_n$ on $D$ whose graphs are contained in $\Sigma_n$ (for large $n$). These functions have bounded $C^{2, \alpha}$ norm by the curvature bound and the fact that their gradient at $(0, 0)$ converges to zero. Hence a subsequence of the $u_n$ converges to a minimal graph $u$ over $D$, and the graph of $u$ is tangent to $T(1)$ at $(0, 0, 1)$.

Let $v_1^n, v_2^n$ be the generators of the group leaving $C_n$ invariant. Let $v_0$ be in $D$. Since $\Lambda(C_n) \to 0$, there is a sequence $(a^n_1, a^n_2)_{n \in \mathbb{N}}$ in $\mathbb{Z}^2$ such that $a^n_1 v_1^n + a^n_2 v_2^n \to v_0$.

The graph of $u_n(\cdot - (a^n_1 v_1^n + a^n_2 v_2^n))$ over $D + a^n_1 v_1^n + a^n_2 v_2^n$ is also a part of a lift of $\Sigma_n$. Since $\Sigma_n$ is embedded, its lift is also embedded. So, for any $n$, we have either $u_n(\cdot) \leq u_n(\cdot - (a^n_1 v_1^n + a^n_2 v_2^n))$ or $u_n(\cdot) \geq u_n(\cdot - (a^n_1 v_1^n + a^n_2 v_2^n))$. Thus at the limit, $u(\cdot) \leq u(\cdot - v_0)$ or $u(\cdot) \geq u(\cdot - v_0)$ on $D \cap (D + v_0)$.

Let $S$ be the totally geodesic surface in $M$ tangent to $\{ z = 1 \}$ at $(0, 0, 1)$. Over $D$, $S$ can be described as the graph of a radial function $h$. We have $h(0, 0) = 1$ and there is $\alpha > 0$ such that, over $D$, $h(x, y) \leq 1 - \alpha(x^2 + y^2)$. The functions $u$ and $h$ are two solutions of the minimal surface equation with the same value and the same gradient at the origin. So the function $u - h$ looks like a harmonic polynomial of degree at least 2.

If the degree is 2, one can find $v_0 \in D \setminus \{(0, 0)\}$ such that $(u - h)(v_0) < 0$ and $(u - h)(-v_0) < 0$. Then we have

\[
\begin{align*}
u_0(0, 0) < h(v_0) < h(0, 0) = u(v_0 - v_0), \\
u_0(0, 0) - v_0 < h(-v_0) < h(0, 0) = u(0, 0).
\end{align*}
\]

This contradicts $u(\cdot) \leq u(\cdot - v_0)$ or $u(\cdot) \geq u(\cdot - v_0)$ on the whole $D \cap (D + v_0)$.

If the degree is larger than 3, the growth at the origin of $h$ implies that there is a disk $D'$ centered at the origin included in $D$ such that $u < 1$ in $D' \setminus \{(0, 0)\}$. So if $v_0 \in D' \setminus \{(0, 0)\}$ we have

\[
u_0(0, 0) = u(v_0 - v_0) \text{ and } u((0, 0) - v_0) = u(-v_0) < u(0, 0).
\]

This gives also a contradiction $u(\cdot) \leq u(\cdot - v_0)$ or $u(\cdot) \geq u(\cdot - v_0)$ on the whole $D \cap (D + v_0)$. \[ \Box \]

We notice that Remark 1 can be used to get strong transversality with $T(c)$ for $c \geq 1$.

A consequence of Proposition 9 is then the following result.
Theorem C. Let \( \Sigma \) be a properly embedded minimal surface in \( \mathcal{N} \) of bounded curvature. Then \( \Sigma \) has finite topology.

Proof. If \( k_0 \) is an upper bound of the norm of the second fundamental form of \( \Sigma \), Proposition \( \PageIndex{9} \) gives a constant \( \Lambda_0 \). Now \( \mathcal{N} \) can be decomposed as the union of a compact part \( K \) and a finite number of cusp ends \( C_i \) with \( \Lambda(C_i) \leq \Lambda_0 \). Since \( \Sigma \) is transversal to the tori \( T_i(c) \), \( \Sigma \) has the same topology as \( \Sigma \cap \overset{\circ}{K} \), so it has finite topology. \( \square \)

5. Existence of compact embedded minimal surfaces in \( \mathcal{N} \)

Producing minimal surfaces is often done by minimizing the area in a certain class of surfaces. In order to ensure the compactness of our surface in \( \mathcal{N} \), a min-max argument is more suitable in our proof of the following existence result.

Theorem A. There exists a compact embedded minimal surface in any \( \mathcal{N} \).

Proof. Let \( C_1, \ldots , C_k \) be the cusp ends of \( \mathcal{N} \). Let \( z_i \) be the \( z \)-coordinates in \( C_i \) and assume that \( \Lambda(C_i) \leq \Lambda_0 \), for \( 1 \leq i \leq k \) and \( \Lambda_0 \) the constant given by the maximum principle II. This can always be realized by Remarks \( g_0 \) and \( h_0 \).

Now we change the hyperbolic metric in each end \( C_i \) as follows. Let \( \Psi : [1/2, \infty) \to \mathbb{R} \) satisfy \( \Psi(z) = z \) for \( 1/2 \leq z \leq 1 \), \( \Psi'(z) > 0 \) and \( \lim_{z \to \infty} \Psi(z) = 3/2 \).

Let \( L \) be large (to be specified later) and modify the metric \( g_0 \) in \([L, L + 1]\) so the new metric gives a compactification of \( C_i \) by removing \( \{z_i \geq L\} \) and attaching a solid torus to \( T(L) \). The precise way to do this will be explained below. With this new metric the mean curvature of the tori \( T_i(z) \) is increasing for \( L \leq z \leq L + 1 \), going from \( \Psi'(L) \) at \( z = L \) to \( \infty \) as \( z \to L + 1 \). \( z = L + 1 \) corresponds to the core of the solid torus. We do this in each cusp and get a compact manifold \( \overset{\circ}{\mathcal{N}} \) without boundary endowed with a certain metric. We notice that the manifold does not depend on \( L \), but the metric does.

Now we can choose a Morse function \( f \) on \( \overset{\circ}{\mathcal{N}} \) such that all the tori \( T_i(z) \), \( 1/2 \leq z \leq L \), \( 1 \leq i \leq k \), are level surfaces of \( f \).

This Morse function \( f \) defines a sweep-out of the manifold \( \overset{\circ}{\mathcal{N}} \), and

\[
M_0 = \max_{t \in \mathbb{R}} H^2(f^{-1}(t))
\]

essentially does not depend on \( L \) (in fact it can decrease when \( L \) increases). Here \( H^2 \) is the 2-dimensional Hausdorff measure.

Almgren-Pitts min-max theory applies to this sweep-out and gives a compact embedded minimal surface \( \Sigma \) in \( \overset{\circ}{\mathcal{N}} \) whose area is at most \( M_0 \) (see Theorem 1.6 in \( \cite{[3]} \) ). Let us now see that \( \Sigma \) actually lies in the hyperbolic part of \( \mathcal{N} \) so in \( \mathcal{N} \).

Since \( \Psi \to 3/2 \), the metric on \( T_i(k, k + 1) \) is uniformly controlled and close to being flat. As a consequence of the monotonicity formula for minimal surfaces (see Theorem 17.6 in \( \cite{[19]} \)), if \( \Sigma \cap T_i(k + 1/2) \neq \emptyset \) (\( 1 \leq k \leq L - 1 \)), the area of \( \Sigma \cap T_i(k, k + 1) \) is at least some \( c_0 > 0 \). The constant \( c_0 \) only depends on the ambient sectional curvature bound and the \( v_j \)'s (the vectors in the end \( C_i \)).

This monotonicity formula gives at least linear growth for \( \Sigma \). More precisely, if a connected component of \( \Sigma \) intersects \( T_i(1) \) and \( T_i(L) \), it has area at least \( c_0(L - 1) \). Hence by choosing \( L \) larger than \( M_0/c_0 \) there is no component of \( \Sigma \) meeting both \( T_i(1) \) and \( T_i(L) \).
Also, no connected component lies entirely in \( \{ z_i \geq 1 \} \). Indeed, the \( z_i \) would have a minimum on the component which is impossible by the classical maximum principle and the sign of the mean curvature on \( T_i(z) \). Thus \( \Sigma \) stays out of \( \{ z_i \geq L \} \). Hence by the maximum principle II, \( \Sigma \) does not enter in any \( \{ z_i \geq 1 \} \), which completes the proof.

Let us now give the definition of the new metric on \([L, L + 1]\). The tori \( T_i(c) \) are the quotient of \( \mathbb{R}^2 \) by \( v_i^1, v_i^2 \), so they can be parametrized by \( u v_i^1 + v_i^2 \) where \((u, v) \in S^1 \times S^1 \). With this parametrization, the metric \( g_\Psi \) on \( C_i \) is then

\[
\frac{1}{4\pi^2\Psi(z_i)^2} (|v_1|^2 du^2 + 2\langle v_1, v_2 \rangle dudv + |v_2|^2 dv^2 + dz_i^2)
\]

where \( \langle v_1, v_2 \rangle \) denotes the usual scalar product in \( \mathbb{R}^2 \). Let \( \varphi \) be a smooth nonincreasing function on \([L, L + 1]\) such that \( \varphi(z) = 1 \) near \( L \) and \( \varphi(z) = ((L + 1) - z)/a \) near \( L + 1 \). We then change the metric on \( \{ L \leq z_i \leq L + 1 \} \) by

\[
1 \quad \frac{1}{\Psi^2(z_i)} (dz_i^2 + a^2\varphi(z_i)^2 du^2 + 2b\varphi(z_i) dv + c^2 dv^2).
\]

Actually, this change consists of cutting \( \{ z_i \geq L \} \) from the cusp end \( C_i \) and gluing a solid torus along \( T(L) \). To see this, let \( D \) be the unit disk with its polar coordinates \((r, \theta) \in [0, 1] \times S^1 \) and let us define the map \( h : D \times S^1 \to S^1 \times S^1 \times [L, L + 1] \) by \((r, \theta, v) \mapsto (\theta, v, L + 1 - r) \). The induced metric by \( h \) from the one in \([\Pi]\) for \( r \) near 0 is

\[
\frac{1}{\Psi^2(L + 1 - r)} (dr^2 + a^2\frac{r^2}{a^2} d\theta^2 + 2b\frac{r}{a} d\theta dv + c^2 dv^2)
\]

This is a well-defined metric on the solid torus \( D \times S^1 \).

With this new metric, the tori \( T_i(c) = \{ z_i = c \} \ (c \in [L, L + 1]) \) have constant mean curvature \( \Psi_i(c) - \frac{\varphi(c)}{2\varphi(c)} \Psi(c) > 0 \) with respect to \( \Psi(z) \partial_z \).

A minimization argument can be done under some hypotheses to produce compact minimal surfaces.

**Theorem B.** Let \( S \) be a closed orientable embedded surface in \( \mathcal{N} \) which is not a 2-sphere or a torus. If \( S \) is incompressible and nonseparating, then \( S \) is isotopic to a least area embedded minimal surface.

**Proof.** Let \( C_1, \ldots, C_k \) be the cusp ends of \( \mathcal{N} \). Let \( z_i \) be the \( z \)-coordinates in \( C_i \) such that the surface \( S \) does not enter in \( \{ z_i \geq 1 \} \). We assume that \( \lambda(C_i) \leq \Lambda_0 \), for \( 1 \leq i \leq k \) and \( \Lambda_0 \) the constant given by the maximum principle II for the function \( \Psi \) below.

Let \( \Psi : \mathbb{R}^*_+ \to \mathbb{R} \) be a smooth increasing function such that \( \Psi(z) = z \) on \((0, 1]\) and \( \Psi'(2) = 0 \).

For each \( a \geq 1 \), let \( \mathcal{N}(a) \) be \( \mathcal{N} \) with each cusp end truncated at \( z_i = a \); i.e. \( \mathcal{N}(a) = \mathcal{N} \setminus \bigcup_{1 \leq i \leq k} \{ z_i > a \} \). We remark that the \( \mathcal{N}(a) \) are all diffeomorphic to each other.
Let $n$ be an integer. In each cusp end $C_i$, we change the metric on $\mathcal{N}(2n)$ by using a function $\Psi_n : [1/2, 2n] \to \mathbb{R}; z \mapsto n\Psi(\frac{z}{n})$. So $\Psi_n(z) = z$ on $[1/2, n]$ and $\Psi_n'(2n) = 0$; the torus $T_j(2n)$ is minimal. We notice that the metric on $\mathcal{N}(n)$ is not modified.

Let us minimize the area in the isotopy class of $S$ in the manifold with minimal boundary $\mathcal{N}(2n)$. By Theorem 5.1 and remarks before Theorem 6.12 in [10], there is a least area surface $\Sigma_n$ in $\mathcal{N}(2n)$ which is isotopic to $S$. Theorem 5.1 in [10] can be applied because $\mathcal{N}(2n)$ is $P^2$-irreducible ($\mathcal{N}(2n)$ is orientable and its universal cover is diffeomorphic to $\mathbb{R}^3$). Moreover the minimization process does not produce a nonorientable surface since, in that case, $S$ would be isotopic to the boundary of the tubular neighborhood of it; hence $S$ would separate $\mathcal{N}$. Finally $\Sigma_n$ is not one connected component of $\partial \mathcal{N}(2n)$ since $S$ is not a torus.

In $\mathcal{N}(2n)$, $(\{z = c\})_{c \in [1,2n]}$ is a mean convex foliation so $\Sigma_n \cap \mathcal{N}(1) \neq \emptyset$. By the maximum principle II, it implies that $\Sigma_n \subset \mathcal{N}(1)$ so in a piece of $\mathcal{N}$ where the metric never changes. A priori, the surfaces $\Sigma_n$ could be different. But, since they all lie in $\mathcal{N}(1)$, they all appear in the minimization process in $\mathcal{N}(2)$ so they all have the same area. So $\Sigma_1$ is a least area surface in the isotopy class of $S$ in $\mathcal{N}$ with the hyperbolic metric.

Remark 3. We can notice that there is a uniform lower bound for the area of minimal surfaces in manifolds $\mathcal{N}$. The point is that the thick part of such a manifold $\mathcal{N}$ is not empty. So at each point in the thick part there is an embedded geodesic ball of radius $\varepsilon_3/2$ centered at that point where $\varepsilon_3$ is the Margulis constant of hyperbolic 3-manifolds.

Each connected component of the thin part is either a hyperbolic cusp or the tubular neighborhood of a closed geodesic. So it is foliated by mean convex surfaces, and a minimal surface $\Sigma$ cannot be included in such a component. So there is $x \in \Sigma$ in the thick part. Thus we can apply a monotonicity formula (see [19]) to conclude that the area of the part of $\Sigma$ inside the geodesic ball of radius $\varepsilon_3/2$ and center $x$ is larger than a constant $c_3 > 0$ that depends only on the geometry of the hyperbolic $\varepsilon_3/2$ ball.

When the compact minimal surface $\Sigma$ is stable, we can be more precise (see [8], where Hass attributes this estimate to Uhlenbeck). Applying the stability inequality to the constant function 1, we get that

$$\int_{\Sigma} -(\text{Ric}(N, N) + |A|^2) \geq 0.$$ 

Since $|A|^2 = -2(K_\Sigma + 1)$, the Gauss-Bonnet formula gives

$$\text{Area}(\Sigma) \geq -\frac{1}{2} \int_{\Sigma} K_\Sigma = -\frac{1}{2} \chi(\Sigma) = 2\pi(g - 1)$$

with $g$ the genus of $\Sigma$. Now the Gauss equation implies $K_\Sigma$ is less than or equal to $-1$, so the Gauss-Bonnet formula yields $g > 1$. Thus $|\Sigma|$ is greater than or equal to $2\pi$. Also, combining the Gauss equation and the Gauss-Bonnet formula yields $|\Sigma|$ less than or equal to $4\pi(g - 1)$; $\Sigma$ need not be stable.

6. Existence of noncompact embedded minimal surfaces in $\mathcal{N}$

In [9], Hass, Rubinstein and Wang construct proper minimal surfaces in manifolds $\mathcal{N}$ by a minimization argument in homotopy classes. In [17], Ruberman
constructs least area surfaces in the isotopy class. Here we make use of results in Section 4 to give a different approach on the proof of this second result.

First we remark that, in manifolds \( \mathcal{N} \), there is always a “Seifert” surface. \( \mathcal{N} \) is topologically the interior of a compact manifold \( \overline{\mathcal{N}} \) with tori boundary components and each boundary torus is incompressible. By Lemma 6.8 in [11], there is a compact embedded surface \( S \subset \overline{\mathcal{N}} \) with nonempty boundary which is incompressible and 2-sided; moreover it is nonseparating. Then \( S = S \cap \mathcal{N} \) is a properly embedded smooth surface in \( \mathcal{N} \), \( S \) incompressible, of finite topology, noncompact, nonseparating and 2-sided.

The result is the following statement.

**Theorem 10.** Let \( S \) be a properly embedded, noncompact, finite topology, incompressible, nonseparating surface in \( \mathcal{N} \). Then \( S \) is isotopic to a least area embedded minimal surface.

**Proof.** \( S \) has a finite number of annular ends \( A_1, \ldots, A_p \), each one being included in one cusp end \( C_i \) of \( \mathcal{N} \). Since \( A_j \) is incompressible in \( C_i \), we can isotope \( S \) so that each annular end \( A_j \) is totally geodesic in the end \( C_i \) it enters. We still call \( S \) this new surface and we notice that its area is finite for the hyperbolic metric.

Let \( \Psi : \mathbb{R}_{>0}^* \to \mathbb{R} \) be a smooth increasing function such that \( \Psi(z) = z \) on \((0, 1]\) and \( \Psi'(4/3) = 0 \). Let \( \Lambda_0 \) be the constant given by the maximum principle II and the transversality lemma (Propositions 8 and 9). Assume the ends of \( \mathcal{N} \) are chosen so that \( \Lambda(C_i) \leq \Lambda_0 \) for each end \( C_i \).

As in the proof of Theorem B, we denote \( \mathcal{N}(a) = \mathcal{N} \setminus \bigcup_{1 \leq i \leq k} \{ z_i > a \} \). We remark that \( \mathcal{N}(a) \) is diffeomorphic to \( \overline{\mathcal{N}} \).

Let \( n \) be a large integer. In each cusp end \( C_i \), we change the metric on \( \mathcal{N}(4n) \) by using a function \( \Psi_n : [1/2, 4n] \to \mathbb{R}; z \mapsto 3n \Psi(\frac{z}{3n}) \). So \( \Psi_n(z) = z \) on \([1/2, 3n]\) and \( \Psi'_n(4n) = 0 \); the torus \( T_j(4n) \) is minimal. We notice that the metric on \( \mathcal{N}(3n) \) is not modified.

Let \( S(4n) = S \cap \mathcal{N}(4n) \); the area of \( S(4n) \) is bounded by some constant \( A \) independent of \( n \). By Theorem 6.12 in [10], there is a least area surface \( \Sigma(4n) \) in \( \mathcal{N}(4n) \), isotopic to \( S(4n) \) and \( \partial \Sigma(4n) = \partial S(4n) \). We remark that \( \Sigma(4n) \) is stable so has bounded curvature away from its boundary (independent of \( n \)) (see [16]).

In each cusp end \( C_i \), Proposition 8 implies \( \Sigma(4n) \) is transverse to the tori \( T_j(a) \), \( 1 \leq a \leq 2n \) (see Remark 4 in order to apply Proposition 8). So each intersection \( \Sigma(4n) \cap T_j(a) \) is composed of the same number of Jordan curves for \( 1 \leq a \leq 2n \). The next claims prove that this number is equal to the number of boundary components of \( \Sigma(4n) \) on \( T_j(4n) \).

**Claim 2.** Let \( \Omega \) be a domain in \( \Sigma(4n) \) with boundary in \( T_j(a) \) \( (1 \leq a \leq n) \). Then \( \Omega \) does not enter in any \( \{ z_i \geq a \} \).

**Proof.** If \( \Sigma \) enters in one \( \{ z_i \geq a \} \), by transversality, it enters in \( \{ z_i \geq 2n \} \). So the function \( z_i \) will have a maximum larger than \( 2n \), which is impossible by Proposition 8 (see also Remark 1). \( \square \)

**Claim 3.** Let \( \gamma \) be a connected component of \( \Sigma(4n) \cap T_j(a) \) \( (1 \leq a \leq n) \). Then \( \gamma \) is not trivial in \( \pi_1(T_j(a)) \).

**Proof.** Assume that \( \gamma \) is trivial in \( \pi_1(T_j(a)) \). Since \( \Sigma(4n) \) is incompressible, \( \gamma \) bounds a disk \( \Delta \) in \( \Sigma(4n) \). By Claim 2 \( \Delta \) stays in \( \mathcal{N}(a) \) where the metric is still hyperbolic. So we can lift \( \Delta \) to a minimal disk \( \Delta' \in \mathbb{R}^2 \times \mathbb{R}_+ \) (with the
hyperbolic metric) with boundary in \( z_1 = a \) and entirely included in \( \{ z_1 \leq a \} \). This is impossible by the maximum principle since \( \{ z_1 = s \} \) has constant mean curvature one.

Claim 4. Let \( \Sigma \) be a connected component of \( \Sigma(4n) \cap \{ n \leq z_1 \leq 4n \} \). Then \( \Sigma \) is an annulus with one boundary component in \( T_1(n) \) and one in \( T_1(4n) \).

Proof. Let us first prove that the inclusion map of \( \Sigma \) in \( \{ n \leq z_1 \leq 4n \} \) is \( \pi_1 \)-injective. So let \( \gamma \) be a loop in \( \Sigma \) which bounds a disk in \( \{ n \leq z_1 \leq 4n \} \). Since \( \Sigma(4n) \) is incompressible, there is a disk \( \Delta \) in \( \Sigma(4n) \) bounded by \( \gamma \). If \( \Delta \) is in \( \Sigma \), we are done. If not, there is a subdisk \( \Delta' \) of \( \Delta \) with boundary in \( T_1(n) \), but this is impossible by Claim 3. So the inclusion map is \( \pi_1 \)-injective. We notice that \( \pi_1(\{ n \leq z_1 \leq 4n \}) \) is Abelian, so \( \pi_1(\Sigma) \) is Abelian. This implies that \( \Sigma \) is topologically a sphere, a disk, an annulus or a torus. The sphere and the torus are not possible since \( \Sigma \) has a nonempty boundary. Claim 2 implies that \( \Sigma \) must have a boundary component on \( T_1(4n) \). If the whole boundary of \( \Sigma \) is in \( T_1(4n) \), the \( z_1 \) function admits a minimum on \( \Sigma \) that is impossible by the maximum principle since the \( T_1(c), n \leq c \leq 4n \), have positive mean curvature. So \( \Sigma \) is an annulus with one boundary component in \( T_1(n) \) and one in \( T_1(4n) \).

With these claims, we have thus proved that \( \Sigma(4n) \cap N(n) \) is isotopic to \( S \cap N(n) \) (here, we allow the boundary to move). We also notice that because of the curvature estimate on \( \Sigma(4n) \) and the transversality estimate given by Proposition 9, the intersection curves \( \Sigma(4n) \cap T_1(a) \) \( (1 \leq a \leq n) \) have bounded curvature. So they have a well-controlled geometry far in the cusp. More precisely, there is \( a_0 \) such that \( \Sigma(4n) \cap \{ a_0 \leq z_1 \leq n \} \) is a graph over \( S \cap \{ a_0 \leq z_1 \leq n \} \). So the sequence \( \Sigma(4n) \cap N(n) \) is a sequence of surfaces with uniformly bounded area and curvature whose behavior in the cusps is well controlled. Thus a subsequence converges to a minimal surface \( \Sigma \). This convergence says that \( \Sigma(4n) \cap N(k) \) can be written as a graph or a double graph over \( \Sigma \cap N(k) \). In the first case, the surface \( \Sigma \) is then isotopic to \( S \). In the second case, \( \Sigma(4n) \cap N(k) \) is isotopic to the boundary of a tubular neighborhood of \( \Sigma \cap N(k) \) in \( N(k) \); this implies that \( \Sigma(4n) \cap N(k) \) is a separating surface, which is impossible by the properties of \( S \).

We notice that the area estimate given in Remark 3 is also true for noncompact minimal surfaces. Indeed, because of the asymptotic behavior of a stable minimal surface, the constant function 1 can be used as a test function even in the noncompact case.

7. Some examples

In this section, we give some “explicit” examples that illustrate the above theorems.

H. Schwarz and A. Novius [15] constructed periodic minimal surfaces in \( \mathbb{R}^3 \) by constructing minimal surfaces in a cube possessing the symmetries of the cube. These surfaces then extend to \( \mathbb{R}^3 \) by symmetry in the faces.

K. Polthier [14] constructed periodic embedded minimal surfaces in \( \mathbb{H}^3 \) in an analogous manner. Let \( P \) be a finite side polyhedron of \( \mathbb{H}^3 \) such that symmetry in the faces of \( P \) tessellate \( \mathbb{H}^3 \). If \( \Sigma_0 \) is an embedded minimal surface in \( P \), meeting the faces of \( P \) orthogonally and with the same symmetry as \( P \), then \( \Sigma \) extends to an embedded minimal surface in \( \mathbb{H}^3 \) by symmetry in the faces. Polthier makes this
work for many polyhedron $P$, *e.g.* for all the regular ideal Platonic solids whose vertices are on the spheres at infinity. Among these examples, one can obtain examples in complete hyperbolic 3-manifolds of finite volume.

We first describe how this technique yields an embedded genus 3 compact minimal surface in the figure eight knot complement $\mathcal{N}$.

Let $T$ be an ideal regular tetrahedron of $\mathbb{H}^3$; all the dihedral angles are $2\pi/3$. In the Klein model of $\mathbb{H}^3$ (the unit ball of $\mathbb{R}^3$), $T$ is a regular Euclidean tetrahedron with its four vertices on the unit sphere. Label the faces of $T$ and two vertices of $T$, as in Figure 1a. Then identify face $A$ with face $B$ by a rotation by $2\pi/3$ about $v$, and identify $D$ with $C$ by a rotation by $2\pi/3$ about $w$.

![Figure 1](attachment:figure1.png)

**Figure 1.** A minimal surface in the Gieseking manifold

The quotient of $T$ by these face matchings produces a nonorientable hyperbolic 3-manifold of finite volume. There is one vertex, and its link is a Klein bottle. This manifold $\mathcal{N}$ was discovered by Gieseking in 1912 [7].

The orientable 2-sheeted cover $\mathcal{N}'$ of the Gieseking manifold is diffeomorphic to the complement of the figure eight knot in $\mathbb{S}^3$, hence is a complete hyperbolic manifold of finite volume. In [21], Thurston explains how $\mathcal{N}'$ is homeomorphic to the complement of the figure eight knot (see also [6]).
We construct an embedded compact minimal surface in $\mathcal{N}$ that lifts to a surface of genus 3 in $\mathcal{N}'$.

The geodesics from each vertex of $T$ to its opposite face all meet at one point $p$ in $T$. Join $p$ to each edge of $T$ by the minimizing geodesic. Also join $p$ to each vertex of $T$ by a geodesic. This produces the edges of a tessellation of $T$ by 24 congruent tetrahedra.

Consider the tetrahedron $T_1$ of this tessellation as in Figure 1b. By a conjugate surface technique, Polthier proved there exists an embedded minimal disk $D_1$ in $T_1$ meeting the boundary of $T_1$ orthogonally as in Figure 1c. Symmetry by the faces of $T_1$ (and the faces of the symmetric tetrahedron of the tessellation of $T$) extend $D_1$ to an embedded minimal surface $S$ meeting each face of $T$ in one embedded Jordan curve in the interior of the face. $S$ is topologically a sphere minus 4 points.

The face identifications on $T$ send $S \cap A$ to $S \cap B$ and $S \cap D$ to $S \cap C$. Hence $S$ passes to the quotient in $\mathcal{N}$ to a compact embedded minimal surface whose topology is the connected sum of two Klein bottles. The lift of this to $\mathcal{N}'$ is a genus 3 compact embedded minimal surface.

A Seifert surface for the figure eight knot is an incompressible surface homeomorphic to a once punctured torus. Applying Theorem 10 gives a properly embedded minimal one punctured torus in the complement of the figure eight knot.

Theorem 4 of Adams [1] yields many totally geodesic properly embedded 3-punctured spheres in complete hyperbolic 3-manifolds $\mathcal{N}$ of finite volume. Suppose $\mathcal{N}$ arises as a link or knot complement that contains an embedded incompressible 3-punctured sphere (so by Adams, it is isotopic to a totally geodesic one), for example if the link or the knot contains a part as in Figure 2a such that the disk $D$ with 2 punctures is a 3-punctured incompressible sphere in $\mathcal{N}$. An example is the Whitehead link (Figure 2b).

![Figure 2](image)

**Figure 2.** Incompressible 3-punctured sphere in general position and in the complement of Whitehead link

The Borromean ring is also a hyperbolic link. Its complement contains an embedded incompressible thrice punctured sphere (Figure 3a) and an embedded once punctured torus (Figure 3b) which is isotopic to a properly embedded minimal once punctured torus by Theorem 10.

It will be interesting to estimate the areas of the minimal surfaces obtained by Theorems A, B and 10 as in Remark 3. For example, consider the figure eight knot...
Figure 3. Incompressible 3-punctured sphere and 1-punctured torus in the complement of Borromean rings and an incompressible 1-punctured torus in the figure eight knot complement $N$. We know there is a properly embedded minimal once punctured torus $\Sigma$ in $N$ by Theorem 10 (Figure 3c). The Finite Total Curvature Theorem (Theorem 2) and the Gauss equation tell us the area of $\Sigma$ is strictly less than $2\pi$ (there are no embedded totally geodesic surfaces in $N$).

What is the area of $\Sigma$? What is the properly embedded, noncompact, minimal surface of smallest area (if it exists) in $N$? And in all such manifolds $N$?

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Institut de mathématiques de Toulouse, Université Paul Sabatier, 118, route de Narbonne, F-31062 Toulouse cedex, France

E-mail address: collin@math.ups-tlse.fr

Université Paris-Est, LAMA (UMR 8050), UPEM, UPEC, CNRS, F-77454, Marne-la-Vallee, France

E-mail address: hauswirth@univ-mlv.fr

Université Paris-Est, LAMA (UMR 8050), UPEC, UPEM, CNRS, 61, avenue du Général de Gaulle, F-94010 Créteil cedex, France

E-mail address: laurent.mazet@math.cnrs.fr

Instituto Nacional de Matematica Pura e Aplicada (IMPA), Estrada Dona Castorina 110, 22460-320, Rio de Janeiro-RJ, Brazil

E-mail address: rosen@impa.br