Positive Solutions of $p$-th Yamabe Type Equations on Graphs

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Abstract

Let $G = (V, E)$ be a finite connected weighted graph, and assume $1 \leq \alpha \leq p \leq q$. In this paper, we consider the following $p$-th Yamabe type equation

$$-\Delta_p u + hu^{q-1} = \lambda f u^{\alpha-1}. $$

on $G$, where $\Delta_p$ is the $p$-th discrete graph Laplacian, $h \leq 0$ and $f > 0$ are real functions defined on all vertices of $G$. Instead of the approach in [12], we adopt a new approach, and prove that the above equation always has a positive solution $u > 0$ for some constant $\lambda \in \mathbb{R}$. In particular, when $q = p$ our result generalizes the main theorem in [12] from the case of $\alpha \geq p > 1$ to the case of $1 \leq \alpha \leq p$. It’s interesting that our new approach can also work in the case of $\alpha \geq p > 1$.

1 Introduction

As is known that, let $(M^m, g)$ be a closed Riemannian manifold of dimension $m(\geq 3)$, the Yamabe problem consists of finding metrics of constant scalar curvature in $[g] = \{fg|f : M \to \mathbb{R}_{>0}\}$, the conformal class of $g$. A metric $\tilde{g} = f^{p_m - 2}g$ conformal to $g$ has constant scalar curvature $s \in \mathbb{R}$ if and only if the positive function $f$ satisfies the Yamabe equation corresponding to $g$:

$$-a_m \Delta_g f + s_g f = sf^{p_m - 2},$$

where $\Delta_g$ denotes the Laplace-Beltrami operator corresponding to $g$, and $s_g$ denotes the scalar curvature of $g$, $a_m = \frac{4(m-1)}{m-2}$, $p_m = \frac{2m}{m-2}$ is the Sobolev critical exponent.

In the case of closed manifolds it was proved that at least one solution exists following a program introduced by H. Yamabe in [20]. Solutions of the Yamabe equations are

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critical points of the Hilbert-Einstein function \( S \) on the space of Riemannian metrics on \( M \) restricted to \([g]\), the conformal class of \( g \),

\[
S(g) = \frac{\int_M s_g d\mu_g}{Vol(M, g)^{2/\rho_m}}.
\]

This problem was also studied by Trudinger \[18\], Aubin \[1\], and completely solved by Schoen \[17\].

Recently, there are tremendous work concerning the discrete weighted Laplacians and various equations on graphs, among those we refer readers to \[2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19\]. Particularly, there have been some works on dealing with Yamabe type equations on graphs, on which please refer to \[8, 9, 12, 13\]. Grigor’yan, Lin and Yang \[9\] first studied a Yamabe type equation on a finite graph as follows

\[
-\Delta u + hu = |u|^{\alpha - 2}u, \quad \alpha > 2,
\]

where \( \Delta \) is a usual discrete graph Laplacian, and \( h \) is a positive function defined on the vertices. They show that the equation \((1.1)\) always has a positive solution. Inspired by their work, Ge \[12\] studied a Yamabe type equation on a finite graph, that is

\[
-\Delta_p u + hu^{p-1} = \lambda f u^{\alpha - 1}.
\]

We state the main result in \[12\] as follows

**Theorem 1.1.** Let \( G = (V, E) \) be a finite connected graph. Given \( h, f \in C(V) \) with \( h \leq 0, f > 0 \). Assume \( \alpha \geq p > 1 \). Then the following \( p \)-th Yamabe equation

\[
-\Delta_p u + hu^{p-1} = \lambda f u^{\alpha - 1}
\]

on \( G \) always has a positive solution \( u \) for some constant \( \lambda \in \mathbb{R} \).

From this result, one naturally wants to know:

**Problem I:** Can one solve the \( p \)-th Yamabe equation \((1.2)\) for \( 1 \leq \alpha \leq p \)?

To answer this question is our main purpose of this paper.

**Remark 1.** Ge and Jiang \[13\] also studied a following Yamabe type equation

\[
-\Delta_p u + hu^{p-1} = gu^{\alpha - 1}, \quad u > 0,
\]

on an infinite graph. The main result in \[13\] is as follows
Theorem 1.2. Consider the $p$-th Yamabe equation (1.2) on a connected, infinite and locally finite graph $G$ with $\alpha > p \geq 2$. Assume $g \geq 0$ and $g$ is bounded from above, $h$ satisfies $\inf_{x \in V} h(x) > 0$ and $\inf_{x \in V} h(x)\mu(x) > 0$. Further assume $h^{-1} \in L^\delta(V)$ for some $\delta > 0$ (or $h(x) \to \infty$ when $x \to \infty$), then (1.2) has a positive solution.

From this result one still needs to know: Can one solve this $p$-th Yamabe equation (1.2) under the assumption $2 < \alpha \leq p$?

We will solve this problem in our another paper [21].

Next note that Grigor’yan, Lin and Yang’s pioneer paper [9] also studied a similar Yamabe type equation as follows

$$-\Delta_p u + h |u|^{p-2} u = f(x, u), \quad p > 1,$$

on a finite graph under the assumption $h > 0$. They show that the equation (1.3) always has a positive solution under certain assumptions about $f(x, u)$. It is remarkable that their $\Delta_p$ considered in the equation (1.3) is different from ours when $p \neq 2$.

If we replace the $p$-th Laplacian $\Delta_p$ in equation (1.2) with $\Delta_p$ in equation (1.3), then we can ask

**Problem II:** Do positive solutions of the $p$-th Yamabe equation (1.2) still exist?

In this paper we attempt to solve both problems above on a finite graph. The answers turn out be interesting. In fact, we can prove that the equation (1.2) actually has a positive solution $u > 0$ in the case of $1 \leq \alpha \leq p$. Note that Ge’s approach in [12] can’t succeed in solving Problem I any more, so we have to adopt a new approach. Let’s outline our approach to solution of Problem I.

First, we study the following Yamabe type equation

$$-\Delta_p u + \mu u^{\alpha-1} = \lambda f u^{\alpha-1},$$

in the case of $1 \leq \alpha \leq p \leq q$.

Note that in this equation we add a new constant $\mu$ and extend exponents of the second term from $p$ to any integer $q$ satisfying $q \geq p$. In order to study the equation above, we have to make a transformation as

$$-\Delta_p u - \lambda f u^{\alpha-1} = -\mu u^{\alpha-1}.$$

Otherwise our derivation can’t go on. Then by using the constrained minimization technique, we can prove that, for any constant $\lambda \in \mathbb{R}$, the equation above actually has a positive solution $\hat{u} > 0$ at least for some constant $\mu = \mu(\lambda) \in \mathbb{R}$. 

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Then, set $\tilde{\lambda} < 0$, we can construct a positive function $u = u(\hat{u}, \tilde{\lambda})$ depending on $\hat{u}, \tilde{\lambda}$ by using the scaling technique. In fact, we can prove that this function $u = u(\hat{u}, \tilde{\lambda})$ is exactly a positive solution of the Yamabe type equation
\[-\Delta_p u + hu^{q-1} = \lambda f u^{\alpha-1},\]
for some constant $\lambda = \lambda(\hat{u}, \tilde{\lambda})$.

Note that here the constant $\mu$ added in previous equation disappear at all, which highly depends on the condition $1 \leq \alpha \leq p \leq q$.

Finally take $q = p$, we completely solve Problem I.

The key point of our approach is the assumption: $1 \leq \alpha \leq p \leq q$. In fact, our approach is still successful in the case of $1 \leq q \leq p \leq \alpha$, where taking $q = p$ particularly, we can get the main theorem in [12]. In this respect, we find that our approach is more generic.

As for Problem II, we observe that the difference between the two $p$-th Laplacian $\Delta_p$ in equation (1.2) and in equation (1.3) is not essential, therefore we can similarly prove that equation (1.2) actually has a positive solution.

We organize this paper as follows: In section 2, we introduce some notions on graphs and state our main results. In section 3, we give Sobolev imbedding. Section 4 is devoted to prove an important theorem which is key to solve both problems above. In section 5, we completely solve Problem I and Problem II.

## 2 Settings and main results

Let $G = (V, E)$ be a finite graph, where $V$ denotes the vertex set and $E$ denotes the edge set. Fix a vertex measure $\mu : V \to (0, +\infty)$ and an edge measure $\omega : E \to (0, +\infty)$ on $G$. The edge measure $\omega$ is assumed to be symmetric, that is, $\omega_{ij} = \omega_{ji}$ for each edge $i \sim j$.

Denote $C(V)$ as the set of all real functions defined on $V$, then $C(V)$ is a finite dimensional linear space with the usual function additions and scalar multiplications. For any $p > 1$, the $p$-th discrete graph Laplacian $\Delta_p : C(V) \to C(V)$ is
\[
\Delta_p f_i = \frac{1}{\mu_i} \sum_{j \sim i} \omega_{ij} |f_j - f_i|^{p-2}(f_j - f_i)
\]
for any $f \in C(V)$ and $i \in V$. $\Delta_p$ is a nonlinear operator when $p \neq 2$ (see [12] for more properties about $\Delta_p$). Now we can state an important theorem as follows

**Theorem 2.1.** Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0$, $f > 0$. Assume $1 \leq \alpha \leq p \leq q$. Then for any constant $\lambda \in \mathbb{R}$, the following $p$-th Yamabe equation
\[-\Delta_p u + hu^{q-1} = \lambda f u^{\alpha-1}\]
(2.1)
on $G$ always has a positive solution $u > 0$ for some constant $\mu = \mu(\lambda) \in \mathbb{R}$.

If we want to get rid off $\mu$, then we have

**Theorem 2.2.** Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0$, $f > 0$. Assume $1 \leq \alpha \leq p \leq q$. Then the following $p$-th Yamabe equation

$$-\Delta_p u + hu^{q-1} = \lambda f u^{\alpha-1}$$

(2.2)
on $G$ always has a positive solution $u > 0$ for some constant $\lambda \in \mathbb{R}$.

This theorem actually generalizes the Problem I and gives a positive answer. As a corollary, let’s take $q = p$, we get the following

**Corollary 2.3.** Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0$, $f > 0$. Assume $1 \leq \alpha \leq p$. Then the following $p$-th Yamabe equation

$$-\Delta_p u + hu^{p-1} = \lambda f u^{\alpha-1}$$
on $G$ always has a positive solution $u > 0$ for some constant $\lambda \in \mathbb{R}$.

This corollary completely solves Problem I. Further, combining this corollary with Theorem [1.1], we have

**Corollary 2.4.** Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0$, $f > 0$. Assume $p > 1$, $\alpha \geq 1$. Then the following $p$-th Yamabe equation

$$-\Delta_p u + hu^{p-1} = \lambda f u^{\alpha-1}$$
on $G$ always has a positive solution $u$ for some constant $\lambda \in \mathbb{R}$.

Next we consider the $p$-th Laplacian $\Delta_p$ (please refer to [9]) in equation (1.3) which is defined in distributional sense by

$$\int_V (\Delta_p u) \phi d\mu = -\int_V |\nabla u|^{p-2} \Gamma(u, \phi) d\mu, \forall \phi \in C_c(V),$$

(2.3)

where $C_c(V)$ denotes the set of all functions with compact support, obviously $C_c(V) = C(V)$ when $G$ is a finite graph. $\Gamma(u, \phi), |\nabla u|$ are defined respectively by

$$\Gamma(u, \phi)_i = \frac{1}{2\mu_i} \sum_{j \sim i} \omega_{ij}(u_j - u_i)(\phi_j - \phi_i), \forall i \in V,$$

(2.4)

and

$$|\nabla u|_i = \sqrt{\Gamma(u, u)_i}, \forall i \in V.$$
Point-wisely, $\Delta_p$ can be written as
\[
\Delta_p u_i = \frac{1}{2\mu_i} \sum_{j \sim i} (|\nabla u|_j^{p-2} + |\nabla u|_i^{p-2}) \omega_{ij} (u_j - u_i), \forall i \in V. \tag{2.6}
\]

Applying this $p$-th Laplacian $\Delta_p$ we can study the $p$-th Yamabe equation
\[-\Delta_p u + hu^{p-1} = \lambda f u^{\alpha - 1}\]
again and we obtain

**Theorem 2.5.** Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0$, $f > 0$. Assume $p > 1$, $\alpha \geq 1$. Then the following $p$-th Yamabe equation
\[-\Delta_p u + hu^{p-1} = \lambda f u^{\alpha - 1}\]
on $G$ always has a positive solution $u > 0$ for some constant $\lambda \in \mathbb{R}$.

### 3 Sobolev embedding

For any $f \in C(V)$, define the integral of $f$ over $V$ with respect to the vertex weight $\mu$ by
\[
\int_V f d\mu = \sum_{i \in V} \mu_i f_i.
\]

Set $\text{Vol}(G) = \int_V d\mu$.

For any $f \in C(V)$, define
\[
\int_V |\nabla f|^p d\mu = \sum_{i \sim j} \omega_{ij} |f_j - f_i|^p, \tag{3.1}
\]
where $|\nabla f|$ is defined as
\[
|\nabla f_i| = \left(\frac{1}{2\mu_i} \sum_{j \sim i} \omega_{ij} |f_j - f_i|^p\right)^{1/p}, \forall i \in V. \tag{3.2}
\]

Next we consider the Sobolev space $W^{1,p}(G)$ on the graph $G$. Define
\[
W^{1,p}(G) = \left\{ u \in C(V) : \int_V |\nabla u|^p d\mu + \int_V |u|^p d\mu < +\infty \right\},
\]
and
\[
\|u\|_{W^{1,p}(G)} = \left(\int_V |\nabla u|^p d\mu + \int_V |u|^p d\mu\right)^{\frac{1}{p}}.
\]

Since $G$ is a finite graph, then $W^{1,p}(G)$ is exactly $C(V)$, a finite dimensional linear space. This implies the following Sobolev embedding [12]
Lemma 3.1. (Sobolev embedding) Let $G = (V, E)$ be a finite graph. The Sobolev space $W^{1,p}(G)$ is pre-compact. Namely, if $\{\varphi_n\}$ is bounded in $W^{1,p}(G)$, then there exists some $\varphi \in W^{1,p}(G)$ such that up to a subsequence, $\varphi_n \to \varphi$ in $W^{1,p}(G)$.

Remark 2. The convergence in $W^{1,p}(G)$ is in fact pointwise convergence.

4 Proof of Theorem 2.1

In this section we focus on proving Theorem 2.1. First we consider a constrained minimization problem for the following energy functional

$$E(u) = \frac{1}{p} \int_V |\nabla u|^p d\mu - \frac{\lambda}{\alpha} \int_V fu^\alpha d\mu,$$

(4.1)
on the space $C(V)$, restricted to the subset

$$M = \{u \in C(V) : \frac{1}{q} \int_V hu^q d\mu = -1\}$$

Define

$$\beta = \inf_{u \in M} \{E(u) : u \geq 0, u \not\equiv 0\}.$$We will find a positive solution of the equation (2.1) step by step as follows.

Step 1. $E(u)$ is bounded below.

Let’s first prove the following two lemmas

Lemma 4.1. Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0$. Assume $1 \leq \alpha \leq p \leq q$, $\forall u \in M$, $u \geq 0$, then we have the following inequalities

$$\|u\|_q^q \leq \frac{q}{(-h)_m}, \|u\|_q^\alpha \leq \left[\frac{q}{(-h)_m}\right]^{\frac{\alpha}{q}}$$

(4.2)

where $(-h)_m = \min_{i \in V}(-h_i)$.

Proof. $\forall u \in M$, $u \geq 0$ and $h \leq 0$, we have

$$q = - \int_V hu^q d\mu \geq (-h)_m \int_V u^q d\mu = (-h)_m \|u\|_q^q.$$Hence

$$\|u\|_q^q \leq \frac{q}{(-h)_m}, \|u\|_q^\alpha \leq \left[\frac{q}{(-h)_m}\right]^{\frac{\alpha}{q}}.$$\qed

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Lemma 4.2. Let $G = (V, E)$ be a connected finite graph. Given $f \in C(V)$ with $f > 0$. Assume $1 \leq \alpha \leq p \leq q$, $\forall u \in C(V), u \geq 0$. Then we have the following inequality

$$0 \leq \int_V fu^\alpha d\mu \leq f_M \operatorname{Vol}(G)^{1-\frac{\alpha}{q}} \|u\|_q^\alpha$$

(4.3)

where $f_M = \max_{i \in V} (f_i)$, and $\operatorname{Vol}(G) = \int_V d\mu$.

**Proof.** $\forall u \in C(V), u \geq 0$ and $f > 0$, by Hölder inequality we have

$$\int_V fu^\alpha d\mu \leq f_M \|1 \cdot u^\alpha\|_1 \leq f_M \|\frac{q}{q-\alpha}\|u^\alpha\|_q^\alpha = f_M \operatorname{Vol}(G)^{1-\frac{\alpha}{q}} \|u\|_q^\alpha.$$

Now we can prove that the energy functional $E(u)$ is bounded below for all $u \geq 0, u \not\equiv 0$. Hence $\beta = \inf_{u \in \mathcal{M}} \{E(u) : u \geq 0, u \not\equiv 0\}$ and $\beta \in \mathbb{R}$.

**Theorem 4.3.** Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0$, $f > 0$. Assume $1 \leq \alpha < p$. Then for $\forall u \in \mathcal{M}, u \geq 0$, then the energy functional $E(u)$ is bounded below by a constant, namely we have the following inequality

$$E(u) \geq C_{\lambda, \alpha, q, h, f, G,}$$

(4.4)

where $C_{\alpha, q, h, f, G} := \frac{|\lambda|}{\alpha} f_M \left[\frac{q}{(-h)_m}\right]^{\frac{\alpha}{q}} \operatorname{Vol}(G)^{1-\frac{\alpha}{q}} \leq 0$ is a constant depending only on the information of $\lambda, \alpha, q, h, f, G$.

**Proof.** Combining the two lemmas above, we have

$$\int_V fu^\alpha d\mu \leq f_M \operatorname{Vol}(G)^{1-\frac{\alpha}{q}} \|u\|_q^\alpha \leq f_M \operatorname{Vol}(G)^{1-\frac{\alpha}{q}} \left[\frac{q}{(-h)_m}\right]^{\frac{\alpha}{q}}.$$

Hence

$$E(u) \geq -\frac{\lambda}{\alpha} \int_V fu^\alpha d\mu \geq -\frac{|\lambda|}{\alpha} f_M \left[\frac{q}{(-h)_m}\right]^{\frac{\alpha}{q}} \operatorname{Vol}(G)^{1-\frac{\alpha}{q}}.$$

**Step 2.** There exists a $\hat{u} \geq 0, \hat{u} \not\equiv 0$ such that $\beta = E(\hat{u})$.

To find such $\hat{u}$, we choose $u_n \in \mathcal{M}, u_n \geq 0$, satisfying

$$E(u_n) \to \beta$$

as $n \to \infty$ and

$$\int_V hu_n^\alpha d\mu = -q.$$

Further we can assume $E(u_n) \leq 1 + \beta$ for all $n$, then we have
Theorem 4.4. Let $G = (V, E)$ be a connected finite graph. Given $h, f \in C(V)$ with $h \leq 0, f > 0$. Assume $1 \leq \alpha \leq p \leq q$. Let $u_n$ satisfy the above conditions, then $\{u_n\}$ is bounded in $W^{1, p}(G)$. In fact, we have the following estimate
\[
\|u_n\|_{W^{1, p}(G)} \leq C_{\lambda, \alpha, \beta, p, q, h, f, G}
\]
where $C_{\lambda, \alpha, \beta, p, q, h, f, G} := p(1 + \beta) + \frac{p|\lambda|FM}{\alpha} \left[ \frac{q}{(-h)_m} \right]^\frac{\alpha}{q} \text{Vol}(G)^{1 - \frac{\alpha}{q}} + \left[ \frac{q}{(-h)_m} \right]^\frac{\beta}{q} \text{Vol}(G)^{1 - \frac{\beta}{q}}$ is a constant depending only on the information of $\lambda, \alpha, \beta, p, q, h, f, G$.

**Proof.** Observe that the Sobolev norm $\| \cdot \|_{W^{1, p}(G)}$ is related to the energy functional $E(u)$, then by (4.2) and (4.3) we get
\[
\|u_n\|_{W^{1, p}(G)} = \int_V |\nabla u_n|^p d\mu + \int_V |u_n|^p d\mu
\]
\[
= p\left( \frac{1}{p} \int_V |\nabla u|^p d\mu - \frac{\lambda}{\alpha} \int_V fu^\alpha d\mu \right) + \frac{p\lambda}{\alpha} \int_V fu^\alpha d\mu + \int_V |u_n|^p d\mu
\]
\[
\leq pE(u_n) + \frac{p\lambda}{\alpha} \int_V fu^\alpha d\mu + \|u_n\|_q^p \text{Vol}(G)^{1 - \frac{\beta}{q}}
\]
\[
\leq p(1 + \beta) + \frac{p|\lambda|FM}{\alpha} \left[ \frac{q}{(-h)_m} \right]^\frac{\alpha}{q} \text{Vol}(G)^{1 - \frac{\alpha}{q}} + \left[ \frac{q}{(-h)_m} \right]^\frac{\beta}{q} \text{Vol}(G)^{1 - \frac{\beta}{q}}
\]
\[
= C_{\lambda, \alpha, \beta, p, q, h, f, G}.
\]

Therefore $\{u_n\}$ is bounded in $W^{1, p}(G)$. Then by Lemma 3.1 there exists some $\hat{u} \in W^{1, p}(G) = C(V)$ such that up to a subsequence, $u_n \rightarrow \hat{u}$ in $W^{1, p}(G)$. We may also denote this subsequence as $u_n$. Note $u_n \geq 0$ and $\int_V hu_n^q d\mu = -q$, let $n \rightarrow +\infty$, we know $\hat{u} \geq 0$ and
\[
\int_V h\hat{u}^q d\mu = -q \neq 0,
\]
hence $\hat{u} \in M$. This also implies that $\hat{u} \neq 0$. Since $G$ is finite graph, by Remark 2 the convergence in the Sobolev space $W^{1, p}(G)$ is in fact pointwise convergence. Moreover, because $G$ is a finite graph, so the energy functional $E(u)$ is actually continuous, which yields that $E(u)$ attains its infimum exactly at the point $\hat{u} \in M$. Therefore we have
\[
\beta = E(\hat{u}).
\]

**Step 3.** $\hat{u} \geq 0, \hat{u} \neq 0$ is exactly a nontrivial solution of the Yamabe type equation (2.1)
Recall the constrained minimization problem for the following energy functional

\[ E(u) = \frac{1}{p} \int_V |\nabla u|^p d\mu - \frac{\lambda}{\alpha} \int_V fu^\alpha d\mu, \]

on the space \( C(V) \), restricted to the subset

\[ M = \{ u \in C(V) : \frac{1}{q} \int_V hu^q d\mu = -1 \}. \]

Now we derive the Euler-Lagrange equations by the Lagrange multiplier rule. First set

\[ G(u) = \frac{1}{q} \int_V hu^q d\mu + 1. \]

It is easy to see that both \( E(u) \) and \( G(u) \) are continuously Fréchet differentiable, and the Fréchet derivatives \( DE(u), DG(u) \) are respectively given by

\[
< v, DE(u) > = \int_V (|\nabla u|^{p-1} \nabla v - \lambda fu^{\alpha-1} v) d\mu \\
= \int_V (-\Delta_p u - \lambda fu^{\alpha-1}) v d\mu, \forall v \in C(V),
\]

and

\[
< v, DG(u) > = \int_V hu^{q-1} v d\mu, \forall v \in C(V).
\]

Then let

\[ \Phi(t, \mu) = E(u + tv) + \mu G(u + tv), \forall v \in C(V), \]

by direct calculation we have

\[
\frac{d\Phi(t, \mu)}{dt} \bigg|_{t=0} = < v, DE(u) + \mu DG(u) > = \int_V (-\Delta_p u - \lambda fu^{\alpha-1} + \mu hu^{q-1}) v d\mu.
\]

Therefore if \( u \) is a critical point with regard to the above constrained minimization problem, then \( u \) satisfies the following Euler-Lagrange equations

\[
\int_V (-\Delta_p u - \lambda fu^{\alpha-1} + \mu hu^{q-1}) v d\mu = 0, \forall v \in C(V).
\]

In particular, the infimum point \( \hat{u} \) satisfies the Euler-Lagrange equations

\[
\int_V (-\Delta_p \hat{u} - \lambda f\hat{u}^{\alpha-1} + \mu h\hat{u}^{q-1}) v d\mu = 0, \forall v \in C(V). \tag{4.5}
\]

This implies \( \hat{u} \geq 0 \) is a nontrivial solution of the Yamabe type equation (2.1)

\[-\Delta_p \hat{u} - \lambda f\hat{u}^{\alpha-1} + \mu h\hat{u}^{q-1} = 0.\]
Moreover, we can deduce the formula of \( \mu \). Inserting \( v = \hat{u} \) into the equality (4.5) yields that
\[
\int_V (|\nabla \hat{u}|^p - \lambda f \hat{u}^\alpha + \mu h \hat{u}^q) d\mu = \int_V (|\nabla \hat{u}|^p - \lambda f \hat{u}^\alpha) d\mu - \mu q = 0.
\]
thus \( \mu \) can be given by
\[
\mu = \mu(\lambda, \hat{u}) = \frac{1}{q} \int_V (|\nabla \hat{u}|^p - \lambda f \hat{u}^\alpha) d\mu.
\] (4.6)

**Step 4. \( \hat{u} > 0 \).**

Recall that
\[
\Phi = \Phi(t, \mu) = E(u + tv) + \mu G(u + tv),
\]
then by direct calculation, we have
\[
\frac{\partial \Phi}{\partial u_i} = \mu_i (\Delta_p u_i - \lambda f_i u_i^{\alpha-1} + \mu h u_i^{q-1}).
\] (4.7)

Note the graph \( G \) is connected, if \( \hat{u} > 0 \) is not satisfied, since \( \hat{u} \geq 0 \) and not identically zero, then there is an edge \( i \sim j \), such that \( \hat{u}_i = 0 \), but \( \hat{u}_j > 0 \). Then by the definition of the \( p \)-th Laplacian \( \Delta_p \) in section 2 we have
\[
-\Delta_p \hat{u}_i = -\frac{1}{\mu_i} \sum_{k \sim i} \omega_{ik} |\hat{u}_k - \hat{u}_i|^{p-2} (\hat{u}_k - \hat{u}_i) < 0.
\]
Therefore by (4.7), we have
\[
\left. \frac{\partial \Phi}{\partial u_i} \right|_{u=\hat{u}} = \mu_i (\Delta_p \hat{u}_i) < 0.
\]
Recall we had proved that \( \hat{u} \) is the minimum value of \( \Phi \), hence there should be
\[
\left. \frac{\partial \Phi}{\partial u_i} \right|_{u=\hat{u}} \geq 0,
\]
which is a contradiction. Hence \( \hat{u} > 0 \).

This completes the proof of Theorem 2.1.

5 Proofs of Theorem 2.2 and Theorem 2.5

5.1 Proofs of Theorem 2.2

Now we can prove Theorem 2.2 by Theorem 2.1.
First choose any negative constant $\tilde{\lambda} < 0$, and assume that $h, f \in C(V)$ with $h \leq 0$, $f > 0$, $1 \leq \alpha \leq p \leq q$, then consider the following $p$-th Yamabe equation

$$- \Delta_p u + \mu h u^{q-1} = \tilde{\lambda} f u^{\alpha-1}. \quad (5.1)$$

By Theorem 2.1, we know that this equation on $G$ always has a positive solution $\hat{u} > 0$ for some constant $\mu = \mu(\tilde{\lambda}) \in \mathbb{R}$. Equivalently, the solution $\hat{u} > 0$ satisfies

$$\int_V (-\Delta_p \hat{u} - \tilde{\lambda} f \hat{u}^{\alpha-1} + \mu h \hat{u}^{q-1}) v d\mu = 0, \forall v \in C(V). \quad (5.2)$$

Next using this positive solution $\hat{u} > 0$, we will construct a positive solution $u = u(\hat{u}, \mu) > 0$ of the $p$-th Yamabe equation

$$\int_V (-\Delta_p u - \lambda f u^{\alpha-1} + h u^{q-1}) v d\mu = 0, \forall v \in C(V). \quad (5.3)$$

Recall that $\tilde{\lambda} < 0$, $f > 0$, $\hat{u} > 0$, hence by the formula (4.6) we have

$$\mu = \frac{1}{q} \int_V (|\nabla \hat{u}|^p - \tilde{\lambda} f \hat{u}^{\alpha-1}) d\mu > 0.$$ 

then scaling with a suitable power of $\mu$, we set

$$u(s) = \mu^s \hat{u},$$

thus $\hat{u} = \mu^{-s} u(s)$, substitute it into the equation (5.2)

$$0 = \int_V (-\Delta_p \hat{u} - \tilde{\lambda} f \hat{u}^{\alpha-1} + \mu h \hat{u}^{q-1}) v d\mu$$

$$= \int_V (|\nabla \hat{u}|^{p-1} \nabla v - \tilde{\lambda} f \hat{u}^{\alpha-1} v + \mu h \hat{u}^{q-1} v) d\mu$$

$$= \int_V (\mu^{-s(p-1)} |\nabla u(s)|^{p-1} \nabla v - \mu^{-s(\alpha-1)} \tilde{\lambda} f u(s)^{\alpha-1} v + \mu^{1-s(q-1)} h u(s)^{q-1} v) d\mu$$

$$= \mu^{-s(p-1)} \int_V (-\Delta_p u(s) - \mu^{-s(\alpha-p)} \tilde{\lambda} f u(s)^{\alpha-1} + \mu^{1-s(q-p)} h u(s)^{q-1}) v d\mu.$$ 

Let $1 - s(q-p) = 0$, thus $s = \frac{1}{q-p}$. Then set

$$u = u(\hat{u}, \mu) = u(s)|_{s=\frac{1}{q-p}} = \frac{1}{\mu^{\frac{q}{q-p}}} \hat{u}, \lambda = \lambda(\tilde{\lambda}, \mu) = \mu^{-s(\alpha-p)}|_{s=\frac{1}{q-p}} \tilde{\lambda} = \mu^{\frac{p-q}{q-p}} \tilde{\lambda},$$

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By the equality above we obtain that \( \forall v \in C(V) \)
\[
\int_V (-\Delta_p u - \lambda f u^{\alpha-1} + hu^{q-1}) vd\mu \\
= \mu^{\frac{\alpha-1}{q-\alpha}} \int_V (-\Delta_p \hat{u} - \tilde{\lambda} f \hat{u}^{\alpha-1} + \mu h \hat{u}^{q-1}) vd\mu \\
= 0,
\]
which implies
\[-\Delta_p u + hu^{q-1} = \lambda f u^{\alpha-1}.
\]
Finally, by \( \hat{u} > 0, \mu > 0 \) we conclude that \( u = \mu^{\frac{1}{1-q}} \hat{u} > 0 \).

This completes the proof of Theorem 2.2.

\[\square\]

5.2 Proofs of theorem 2.5

To prove Theorem 2.5, first we observe that the unique difference between Problem I and Problem II is the definition of \( \Delta_p \). Therefore the proof in this case is totally similar. We only sketch the idea of the proof here.

First, we define the Sobolev form \( \| \cdot \|_{W^{1,p}} \) and the integration
\[
F(u) = \int_V |\nabla u|^p d\mu
\]
in energy functional by using definitions (2.4) and (2.5)
\[
|\nabla u|_i = \sqrt{\Gamma(u, u)_i}, \Gamma(u, \phi)_i = \frac{1}{2\mu_i} \sum_{j \sim i} \omega_{ij}(u_j - u_i)(\phi_j - \phi_i), \forall i \in V. \quad (5.4)
\]
Then we can use this \( |\nabla u| \) to define energy functional as follows. For \( \alpha \geq p \), we can define energy functional similarly as in [12]
\[
I(u) = \left( \int_V |\nabla u|^p d\mu - \int_V hu^p d\mu \right) \left( \int_V fu^\alpha d\mu \right)^{-\frac{p}{\alpha}}, \quad (5.5)
\]
and for \( \alpha \leq p \), our energy functional (4.1) is similarly defined by
\[
E(u) = \frac{1}{p} \int_V |\nabla u|^p d\mu - \frac{\lambda}{\alpha} \int_V fu^\alpha d\mu,
\]
on the space \( C(V) \), also restricted to the subset
\[
M = \{ u \in C(V) : \frac{1}{q} \int_V hu^q d\mu = -1 \}.
\]

Note that \( F(u) = \int_V |\nabla u|^p d\mu \) in definitions of two energy functionals above is continuously Fréchet differentiable. By the proof of Theorem 5 in [9], the Fréchet derivatives \( DF(u) \) is given by
\[
< v, DF(u) > = \int_V (-\Delta_p u)vd\mu, \forall v \in C(V), \quad (5.6)
\]
where $\Delta^p$ is exactly the $p$-th Laplacian $\Delta_p$ in (2.6).

This equality is key to guarantee that all of proofs in this paper and in [12] can be generalized in this case. This can be checked explicitly for two cases $\alpha \geq p$, $\alpha \leq p$ step by step.

\[ \square \]

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