A METHOD FOR INTEGRAL COHOMOLOGY OF POSETS

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1. Introduction and summary

Homotopy type of partially ordered sets (poset for short) play a crucial role in algebraic topology. In fact, every space is weakly equivalent to a simplicial complex which, of course, can be considered as a poset. Posets also arise in more specific contexts as homological decompositions \[10, 6, 16, 20\] and subgroups complexes associated to finite groups \[7, 26, 4\]. The easy structure of a poset has led to the development of several tools to study their homotopy type, including the remarkable Quillen’s theorems \[25\] and their equivariant versions by Thévenaz and Webb \[28\]. In spite of their apparent simplicity posets are the heart of many celebrated problems: Webb’s conjecture (proven in \[27\] and generalized in \[21\]), the unresolved Quillen’s conjecture on the $p$-subgroup complex (see \[1\]), or the fundamental Alperin’s conjecture (see \[19\]).

In this paper we propose a method to compute integral cohomology of posets. This toolbox will be applicable as soon as the poset has certain local properties. More precisely, we will require certain structure on the category under each object of the poset. By means of homological algebra of functors we prove that, in the presence of these local structures, the cohomology of the poset is that of a co-chain complex

\[
0 \to M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \ldots,
\]

where $M_n$ is free with one generator for each object of “degree $n$” of the poset. If the local structure is shared at a global level by the whole poset, further developments show that the cohomology of the poset is that of a co-chain complex

\[
0 \to B_0 \xrightarrow{\Omega_0} B_1 \xrightarrow{\Omega_1} B_2 \xrightarrow{\Omega_2} \ldots,
\]

where $B_n$ is free with one generator for each “critical” object of “degree $n$” of the poset. We also obtain the inequalities

\[(3)\quad b_n \leq \text{rk } B_n\]

and

\[(4)\quad b_n - b_{n-1} + \ldots + (-1)^n b_0 \leq \text{rk } B_n - \text{rk } B_{n-1} + \ldots + (-1)^n \text{rk } B_0.
\]

where the $b$ are the Betti numbers of the geometrical realization of the poset.

The complex \[(1)\] applies to simplex-like posets, i.e. posets such that the category under any object is isomorphic to (the inclusion poset of) a simplex. The notion of simplex-like poset is half-way between semi-simplicial complexes (as defined originally by Eilenberg and Zilber in \[11\]) and simplicial complexes in the classical sense. In
this latter case the co-chain complex (1) reduces to the usual simplicial co-chain complex for integral cohomology. Notice that all the examples in the first paragraph are simplex-like posets, but not all of them are simplicial complexes.

Equations (3) and (4) closely resemble weak and strong Morse inequalities and, in fact, whenever the poset is a simplicial complex equipped with a Morse function (see [24, 13]) our complex (2) is similar to the associated Morse complex on the critical simplices. It is interesting to point that while the Morse complex is obtained after reconstructing the poset by homotopical gluing through critical points of the Morse function, our complex here is directly achieved through homological algebra.

As first application we give an alternative proof of (the cohomological part of) Webb’s conjecture for saturated fusion systems (already proven in [21]). Further applications are related to Coxeter groups: we prove that the Coxeter complex has the cohomology of a sphere if the group is finite and that of a point if the group is infinite.

The layout of the paper is as follows: Section 2 contains preliminaries about graded posets and homological algebra on the category of functors. In Section 3 we define the local structure ("local covering family") we require on the under categories of a poset and study its properties. Further application of these features leads, through Section 4, to a sequence of functors to compute the integral cohomology of a graded poset. This culminates in the co-chain complex (1). The global structure on the poset ("global covering family") is defined and used in Section 5 to obtain co-chain complex (2) and the inequalities (3) and (4). In Section 6 we show how simplex-like posets fit in this context. As an example we give a poset model of the real projective plane. Next, Section 7 is devoted to show the interplay between Morse theory and global covering families. Finally, Webb’s conjecture is proven in Section 8 while Coxeter groups are treated in Section 9.

Notation: By the symbol $\mathcal{P}$ we denote a category which is a poset (see Section 2 below), and their objects will be denoted by $p, q, \ldots \in \text{Ob}(\mathcal{P})$. All posets will be graded, which means roughly that objects have an assigned degree (see below for precise definitions). Then $\mathcal{P}_n = \text{Ob}_n(\mathcal{P})$ denotes the set of objects of degree $n$. By $\text{Ab}^\mathcal{P}$ we denote the category of abelian groups and of functors from $\mathcal{P}$ to $\text{Ab}$ respectively. If $S$ is a set then $|S|$ denotes the number of elements in $S$. For a category $\mathcal{C}$, we denote its opposite category by $\mathcal{C}^{\text{op}}$ and its realization by $|\mathcal{C}|$ (the geometrical realization of the (simplicial set) nerve of the category $\mathcal{C}$). The functor $c_Z : \mathcal{C} \to \text{Ab}$ is the functor which sends any object to $Z$ and any morphism to the identity on $Z$.

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2. Preliminaries

In this section we introduce some homological algebra on the abelian category of functors $\text{Ab}^\mathcal{P}$ from a given category $\mathcal{P}$ to the category $\text{Ab}$ of abelian groups. We shall
work over a special kind of categories \( \mathcal{P} \) called graded partial ordered sets (defined below). As well we define the cohomology of \( \mathcal{P} \) with coefficients \( F \in \text{Ab}^\mathcal{P} \) and show a “shifting argument” to compute this cohomology. This method will be developed systematically in the next section in case \( F \) takes free abelian groups as values.

**Definition 2.1.** A poset is a category \( \mathcal{P} \) in which, given objects \( p \) and \( p' \),

- there is at most one arrow \( p \to p' \), and
- if there are arrows \( p \to p' \) and \( p' \to p \) then \( p = p' \).

Clearly a poset defined as above is exactly the same as a set endowed with a partial order. If \( \mathcal{P} \) is a poset we will write sometimes \( p \leq p' \) to denote that there is an arrow \( p \to p' \), and \( p < p' \) if this is the case and \( p \neq p' \). To define graded posets we first need to introduce preceding relations:

**Definition 2.2.** If \( \mathcal{P} \) is a poset and \( p < p' \) then \( p \) precedes \( p' \) if \( p \leq p'' \leq p' \) implies that \( p = p'' \) or \( p' = p'' \).

**Definition 2.3.** Let \( \mathcal{P} \) be a poset. \( \mathcal{P} \) is called graded if there is a function \( \text{deg} : \text{Ob}(\mathcal{P}) \to \mathbb{Z} \), called the degree function of \( \mathcal{P} \), which is order reversing and such that if \( p \) precedes \( p' \) then \( \text{deg}(p) = \text{deg}(p') + 1 \). If \( p \in \text{Ob}(\mathcal{P}) \) then \( \text{deg}(p) \) is called the degree of \( p \).

Notice that for a given poset \( \mathcal{P} \) and an object \( p_0 \in \text{Ob}(\mathcal{P}) \) the under category \( (p_0 \downarrow \mathcal{P}) \) (see [22]) is exactly the full subcategory with objects \( \{ p | p \leq p_0 \} \). We also define

**Definition 2.4.** Let \( \mathcal{P} \) be a graded poset, let \( p_0 \in \text{Ob}(\mathcal{P}) \), let \( n \in \mathbb{Z} \) and let \( S \subseteq \mathbb{Z} \). Then we define \( (p_0 \downarrow \mathcal{P})_n \), \( (p_0 \downarrow \mathcal{P})_S \) as the full subcategories of \( \mathcal{P} \) with objects \( \{ p | p \leq p_0 \} \), \( \{ p | p < p_0 \} \), \( \{ p | p \leq p, \text{deg}(p) = n \} \) and \( \{ p | p \leq p, \text{deg}(p) \in S \} \) respectively.

From the homotopy viewpoint, restricting to graded poset means no loss: any topological space is weakly homotopy equivalent to a \( CW \)-complex which, in turn, is homotopy equivalent to a simplicial complex. This last can be seen as a graded poset in which the degree function is the dimension of its simplices (more precisely and according to our definition of order reversing degree function, the opposite of the simplicial complex is the graded poset).

The category \( \text{Ab}^\mathcal{P} \), with objects the functors \( F : \mathcal{P} \to \text{Ab} \) and functors the natural transformations between them, is an abelian category in which the short exact sequences are the object-wise ones (see [23]). Because it contains enough injectives objects (see [29]) we can define the right derived functors of the inverse limit functor \( \lim : \text{Ab}^\mathcal{P} \to \text{Ab} \). For a given functor \( F \in \text{Ab}^\mathcal{P} \) we define the cohomology of \( \mathcal{P} \) with coefficients in \( F \) as

\[
H^*(\mathcal{P}; F) = \lim^* F.
\]

If \( F \) is the constant functor of value \( M \in \text{Ab} \) then \( H^*(\mathcal{P}; F) \) equals the cohomology of the topological realization \( |\mathcal{P}| \), of \( \mathcal{P} \), with trivial coefficients \( M \). A functor \( F \in \text{Ab}^\mathcal{P} \) will be called acyclic if \( H^*(\mathcal{P}; F) = 0 \) for \( * > 0 \).
Remark 2.5. In the rest of the paper we assume the following on any graded poset $\mathcal{P}$:

- the set $\{p|p_0 \leq p\}$ is finite for any $p_0 \in \text{Ob}(\mathcal{P})$, and
- the degree function $\text{deg}$ of $\mathcal{P}$ takes values $\{\ldots, 3, 2, 1, 0\}$.

The second condition above is equivalent, by definition, to the poset $\mathcal{P}$ being bounded above, as we can always consider translations of a degree function ($\text{deg}' = \text{deg} + c$). These conditions will be clearly fulfilled in the applications.

Next we introduce some acyclic objects in $\text{Ab}^\mathcal{P}$.

Definition 2.6. Let $\mathcal{P}$ be a graded poset and $F \in \text{Ab}^\mathcal{P}$. Then $F'$ is the functor defined by

$$F'(p_0) = \prod_{p \in (p_0 \uparrow \mathcal{P})} F(p)$$

on objects $p_0 \in \text{Ob}(\mathcal{P})$. For a morphism $p_1 \to p_0$ the summand $F(p)$ corresponding to $p_1 \leq p$ is mapped by the identity map to itself at the summand corresponding to $p_0 \leq p$ if $p_1 \leq p_0 \leq p$. Otherwise it is mapped to zero.

Notice that $F'$ is built in a similar way as enough injectives are shown to exist in $\text{Ab}^\mathcal{P}$ (see [8, 243ff.]). The next result summarizes some interesting properties of the functor $F'$:

Theorem 2.7. Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a graded poset. Then the following holds:

(a) for each $G \in \text{Ab}^\mathcal{P}$ there is a bijection

$$\text{Hom}_{\text{Ab}^\mathcal{P}}(G, F') \cong \prod_{p \in \text{Ob}(\mathcal{P})} \text{Hom}_{\text{Ab}}(G(p), F(p)),$$

(b) $\lim \leftarrow F' \cong \prod_{p \in \text{Ob}(\mathcal{P})} F(p)$,

(c) $F'$ is acyclic, and

(d) $F'$ is injective in $\text{Ab}^\mathcal{P}$ if and only if $F(p)$ is injective in $\text{Ab}$ for each $p \in \text{Ob}(\mathcal{P})$.

Proof. For the first part, the bijection $\varphi : \text{Hom}_{\text{Ab}^\mathcal{P}}(G, F') \to \prod_{p \in \text{Ob}(\mathcal{P})} \text{Hom}_{\text{Ab}}(G(p), F(p))$ is given by

$$\varphi(\mu)_p = \pi_p \mu_p,$$

where $\pi_p : F'(p) \to F(p)$ is the projection into the summand corresponding to $p \leq p$.

The second part is consequence of (a) and of the isomorphism of abelian groups

$$\lim \leftarrow H \cong \text{Hom}_{\text{Ab}^\mathcal{P}}(\mathbb{Z}, H)$$

for any $H \in \text{Ab}^\mathcal{P}$, where $\mathbb{Z}$ is the functor $\mathcal{P} \to \text{Ab}$ of constant value the integers. Part (c) is proven in [9]. There it is also proven that $F'$ is injective if and only if $F$ takes as values injective abelian groups. □
To finish this section we discuss shortly how to compute higher limits via a “shifting argument”. Fix $F \in \text{Ab}^\mathcal{P}$ and consider $\text{Ker}_F \in \text{Ab}^\mathcal{P}$ defined by

$$\text{Ker}_F(p_0) = \bigcap_{p \in (p_0 \downarrow \mathcal{P})_+} \text{Ker} F(p_0 \to p)$$

and such that sends non-identity morphisms to zero. By Theorem 2.7a) if we have a family of maps $\{\tau_p : F(p) \to \text{Ker}_F(p)\}_{p \in \text{Ob}(\mathcal{P})}$ then there is a natural transformation $\lambda : F \Rightarrow \text{Ker}'_F$. If $\lambda$ is object-wise injective then we obtain a short exact sequence in $\text{Ab}^\mathcal{P}$

$$0 \Rightarrow F \Rightarrow \text{Ker}'_F \Rightarrow G \Rightarrow 0,$$

where $G$ is the object-wise co-image of $\lambda$. By Theorem 2.7c) $\text{Ker}'_F$ is acyclic, and thus the long exact sequence of the derived functors $\lim^* \Rightarrow F = \lim^{i-1} G$ for $i > 1$ and $\lim^1 F = \text{Coim}\{\lim \text{Ker}'_F \to \lim G\}$. The conditions in the next definition ensure that we can build such a natural transformation $\lambda$ which is object-wise injective:

**Definition 2.8.** Let $\mathcal{P}$ be a graded poset, let $F : \mathcal{P} \to \text{Ab}$ be a functor and let $n \in \mathbb{Z}$. We say that $F$ is $n$-condensed if

(a) $F(i) = 0$ if $\text{deg}(i) < n$, and

(b) $\text{Ker}_F(i) = 0$ if $\text{deg}(i) > n$.

If the functor $F$ is $n$-condensed then we can consider the natural transformation $\lambda : F \Rightarrow \text{Ker}'_F$ given by Theorem 2.7a) for the maps $\tau_p : F(p) \to \text{Ker}_F(p)$

$$\tau_p = \begin{cases} 1_F(p) & \text{if } \text{deg}(p) = n \\ 0 & \text{otherwise.} \end{cases}$$

Notice that we have

$$\text{Ker}_F(p) = \begin{cases} F(p) & \text{if } \text{deg}(p) = n \\ 0 & \text{otherwise} \end{cases}$$

by hypotheses (a) and (b) of Definition 2.8. Moreover the functor $\text{Ker}'_F$ takes values on objects

$$\text{Ker}'_F(p_0) = \prod_{p \in (p_0 \downarrow \mathcal{P})_n} F(p). \tag{5}$$

The homomorphism $\lambda_p : F(p) \to \text{Ker}'_F(p)$ is given by

$$\lambda_i = \prod_{p \in (p_0 \downarrow \mathcal{P})_n} F(p_0 \to p) : F(p_0) \to \text{Ker}'_F(p_0) = \prod_{p \in (p_0 \downarrow \mathcal{P})_n} F(p).$$

So $\lambda_i$ is a kind of “diagonal”. An easy induction argument on $\text{deg}(p) \in \{n, n + 1, \ldots\}$ shows that $\lambda$ is a monic natural transformation and we obtain

**Lemma 2.9.** Let $F : \mathcal{P} \to \text{Ab}$ be an $n$-condensed functor. Then there is a short exact sequence

$$0 \longrightarrow F \xrightarrow{\lambda} \text{Ker}'_F \longrightarrow G \longrightarrow 0.$$
On the object $p_0$, $G$ takes the value

$$G(p_0) = \prod_{p \in (p_0 \downarrow P)_n} F(p)/\lambda_{p_0}(F(p_0)).$$

It is clear that $G$ satisfies condition (m) of Definition 2.8 for $n + 1$, but in general condition (b) does not hold for $G$ and $n + 1$. More precisely, if $\text{deg}(p_0) > n + 1$ then $\text{Ker}_G(p_0) = 0$ is equivalent to the natural map

$$F(p_0) \to \lim_{(p_0 \downarrow P)_*} F$$

being an isomorphism. This natural map is a monomorphism by condition (b) of $F$ being $n$-condensed. So, $\text{Ker}_G(p_0) = 0$ if and only if $F(p_0) \to \lim_{(p_0 \downarrow P)_*} F$ is surjective. We summarize these results in the following:

**Lemma 2.10.** Let $F : P \to \text{Ab}$ be an $n$-condensed functor. Then there is a short exact sequence

$$0 \to F \xrightarrow{\lambda} \text{Ker}'_F \to G \to 0.$$ 

Moreover, $G$ is $(n + 1)$-condensed if and only if for each object $p_0$ of degree greater than $n + 1$, we have $F(p_0) \xrightarrow{\cong} \lim_{(p_0 \downarrow P)_*} F$.

**Example 2.11.** Consider the graded poset $P$ with shape

\[
\begin{array}{c}
\cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\]

where the subindexes denote the degree of the objects. The geometrical realization $|P|$ has the homotopy type of a two dimensional sphere $S^2$. Consider now the functor $F : P \to \text{Ab}$ with values

\[
\begin{array}{ccc}
\mathbb{Z}_2 & \times & \mathbb{Z}_4 \\
\times & \mathbb{Z}_12 \\
\mathbb{Z}_2 & \times & \mathbb{Z}_12
\end{array}
\]

The functors $\text{Ker}_F$ and $\text{Ker}'_F$ have values

\[
\begin{array}{ccc}
0 & \to & 0 \\
0 & \to & 0 \\
0 & \to & 0
\end{array}
\] 

and

\[
\begin{array}{ccc}
\mathbb{Z}_{12} \oplus \mathbb{Z}_{12} & \to & \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \\
\mathbb{Z}_{12} \oplus \mathbb{Z}_{12} & \to & \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}
\end{array}
\]
respectively. As Ker\(_F\) is concentrated in degree 0 the functor \(F\) is 0-condensed. The functor \(G\) from Lemma 2.9 is given by:

\[
\begin{array}{ccc}
\mathbb{Z}_6 \oplus \mathbb{Z}_{12} & \rightarrow & 0 \\
\pi \oplus 1 & \rightarrow & 0 \\
\mathbb{Z}_3 \oplus \mathbb{Z}_{12} & \rightarrow & 0 \\
\mathbb{Z}_6 \oplus \mathbb{Z}_{12} & \rightarrow & 0 \\
\end{array}
\]

where \(\pi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_3\) is the quotient map. Notice that \(G\) is not 1-condensed as Ker\(_G\) takes the value \(\mathbb{Z}_2\) on the two objects of degree 2 of \(\mathcal{P}\). In fact, if \(\text{deg}(p_0) = 2\), then the map

\[
F(p_0) = \mathbb{Z}_2 \rightarrow \lim_{(p_0|\mathcal{P})} F = \mathbb{Z}_4
\]

clearly is not an isomorphism. A straightforward computation shows that

\[
\lim F = \mathbb{Z}_2
\]

and

\[
\lim^1 F = \text{Coim}\{\lim \text{Ker}_F \rightarrow \lim G\} = \text{Coim}\{\mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{12}\} = \mathbb{Z}_2.
\]

The next section is devoted to finding “local” conditions on the shape of the poset \(\mathcal{P}\) such that \(G\) and the subsequent functors obtained by applying Lemma 2.9 to a functor \(F\) that takes free abelian groups as values turn out to be condensed.

### 3. Local covering families

In this section we study a bit further the condition given in Lemma 2.10 when dealing with a functor \(F\) which satisfies the following condition

**Definition 3.1.** Let \(F : \mathcal{P} \rightarrow \text{Ab}\) be a functor where \(\mathcal{P}\) is a graded poset. We say that \(F\) is **free** if \(F(p)\) is a finitely generated free abelian group for each object \(p \in \text{Ob}(\mathcal{P})\) (not to be confused with a free object in the abelian category \(\text{Ab}(\mathcal{P})\)).

We also shall need the following

**Definition 3.2.** Let \(A \xrightarrow{f} B\) be a map between free abelian groups. We say that \(f\) is **pure** if Coker\((f)\) is a free abelian group (see [15]).

If \(A \cong \mathbb{Z}^n\) is a finitely generated free abelian group we call \(\text{rk}(A) \overset{\text{def}}{=} n\). The following property of pure maps is straightforward, and will be used repeatedly in what follows,

**Lemma 3.3.** Let \(A \xrightarrow{f} B\) be a map in \(\text{Ab}\) between free abelian groups of the same rank. If \(f\) is pure and injective then it is an isomorphism.

Now consider the condition in Lemma 2.10 again: fix \(p_0\) of degree greater than \(n + 1\) and consider the map given by restriction

\[
\lim_{(p_0|\mathcal{P})} F = \text{Hom}_{(p_0|\mathcal{P})} (c_\mathbb{Z}, F) \rightarrow \prod_{p \in J} F(p)
\]
over a subset $J \subseteq (p_0 \downarrow \mathcal{P})_n$. If this restriction map turns out to be injective (notice that it is injective for $J = (p_0 \downarrow \mathcal{P})_n$ because $F$ is $n$-condensed) then the composition

$$F(p_0) \to \lim_{(p_0 \downarrow \mathcal{P})^*} F \to \prod_{p \in J} F(p)$$

is also injective. If $F$ is a free functor (Definition 3.1) then both groups $F(p_0)$ and $\prod_{p \in J} F(p)$ are free abelian groups (because we are assuming Remark 2.5). If the map $F(p_0) \to \prod_{p \in J} F(p)$ is pure then, by Lemma 3.3, the condition $\text{rk} F(p_0) = \sum_{p \in J} \text{rk} F(p)$ implies that this composition is an isomorphism and so $F(p_0) \xrightarrow{\cong} \lim_{(p_0 \downarrow \mathcal{P})^*} F$. Thus we study the subsets $J \subseteq \text{Ob}(\mathcal{P})$ that make this restriction map a pure monomorphism:

**Definition 3.4.** Let $\mathcal{P}$ be a graded poset with degree function $\text{deg}$. A family of subsets $J = \{J^p_0\}_{p_0 \in \text{Ob}(\mathcal{P})}$, $0 \leq n \leq \text{deg}(p_0)$ with $J^p_0 \subseteq (p_0 \downarrow \mathcal{P})_n$ is a local covering family if

a) For each $p_0$ and $0 \leq n < \text{deg}(p_0)$ it holds that $\bigcup_{p \in J^p_0 \cup J^p_{n+1}} (p \downarrow \mathcal{P})_n = (p_0 \downarrow \mathcal{P})_n$

b) For each $p_0$, $0 \leq n < \text{deg}(p_0)$ and $p \in J^p_{n+1}$ it holds that $J^p_n \subseteq J^p_0$

Notice that the definition above does not depend on a functor defined over the category $\mathcal{P}$. Also, we have $J^p_0 = \{p_0\}$ by (3). The next definition states the relation we expect between a local covering family and an $n$-condensed free functor

**Definition 3.5.** Let $\mathcal{P}$ be a graded poset, $J$ be a local covering family and $F : \mathcal{P} \to \text{Ab}$ be an $n$-condensed free functor. We say that $F$ is $J$-determined if for any object $p_0$ of degree greater than $n + 1$ the restriction map

$$\lim_{(p_0 \downarrow \mathcal{P})^*} F \to \prod_{p \in J^p_0} F(p)$$

is a monomorphism and the map $F(p_0) \to \prod_{p \in J^p_0} F(p)$ is pure. If $\text{deg}(p_0) = n + 1$ then we require the last map above to be a pure monomorphism.

The main feature of local covering families is that they allow freeness plus $J$-determinacy to pass from $F$ to $G$. For an object $p_0$ with $\text{deg}(p_0) \geq n + 1$ notice that the map

$$F(p_0) \to \prod_{p \in J^p_0} F(p)$$

is a pure monomorphism as consequence of Definition 3.3. The condition of Definition 3.5 for $\text{deg}(p_0) = n + 1$ is added in order to obtain that $G$ is a free functor. Notice
that the following proposition restricts to functors which take free abelian groups as values.

**Proposition 3.6.** Let \( \mathcal{P} \) be a graded poset and \( \mathcal{J} \) a local covering family. Assume that \( F : \mathcal{P} \to \text{Ab} \) is \( n \)-condensed, free and \( \mathcal{J} \)-determined and consider the functor \( G \) defined by

\[
0 \longrightarrow F \xrightarrow{\lambda} \text{Ker} F' \longrightarrow G \longrightarrow 0.
\]

If for each object \( p_0 \) with \( \text{deg}(p_0) \geq n + 1 \) it holds that \( \text{rk} F(p_0) = \sum_{p \in J_{n+1}^n} \text{rk} F(p) \), then \( G \) is \( (n+1) \)-condensed, free and \( \mathcal{J} \)-determined.

**Proof.** Notice that the hypothesis implies that for any object \( p_0 \) of degree \( \text{deg}(p_0) > n + 1 \) the two maps

\[
F(p_0) \to \lim_{(p_0 \downarrow \mathcal{P})_*} F \to \prod_{p \in J_{n+1}^n} F(p)
\]

are isomorphisms. In particular, \( F(p_0) \xrightarrow{\sim} \lim_{(p_0 \downarrow \mathcal{P})_*} F \) and so \( G \) is \( (n+1) \)-condensed.

If \( \text{deg}(p_0) = n + 1 \) then the map

\[
F(p_0) \to \prod_{p \in J_{n+1}^n} F(p)
\]

is an isomorphism by hypothesis. Next we prove that \( G \) is a free functor. Consider any \( p \in \text{Ob}(\mathcal{P}) \) with \( \text{deg}(p) \geq n + 1 \) (if \( \text{deg}(p) < n + 1 \) then \( G(p) = 0 \)) and the short exact sequence of abelian groups

\[
0 \to F(p) \xrightarrow{\lambda_p} \text{Ker} F'(p) \xrightarrow{\pi_p} G(p) \to 0.
\]

Then it is straightforward that the map

\[
s_p : \text{Ker} F'(p) = \prod_{q \in (p \downarrow \mathcal{P})_n} F(q) \to \prod_{q \in J_{n+1}^n} F(q) \xrightarrow{\sim} F(p)
\]

is a section of \( \lambda_p \), i.e. \( s_p \circ \lambda_p = 1_{F(p)} \) (since the restriction map \( F(p) \to \prod_{q \in J_{n+1}^n} F(q) \) is injective). This implies that the short exact sequence above splits and so \( G(p) \) is a subgroup of the free abelian group \( \text{Ker} F'(p) \), and thus it is free as well.

Next we prove that \( G \) is \( \mathcal{J} \)-determined. Take \( p_0 \) of degree \( n = \text{deg}(p_0) \) greater than \( n + 2 \). We first check that the restriction map

\[
\lim_{(p_0 \downarrow \mathcal{P})_*} G \to \prod_{p \in J_{n+1}^n} G(p)
\]

is injective. Consider any element \( \psi \in \lim_{(p_0 \downarrow \mathcal{P})_*} G = \text{Hom}_{\text{Ab}}(\mathbb{Z}, G) \) which is in the kernel of the restriction map above. Notice that, as \( \text{deg}(p_0) > n + 2 \), we can consider the subset \( J_{n+2}^{n+1} \subseteq (p_0 \downarrow \mathcal{P})_* \). If for any \( q \in J_{n+2}^{n+1} \) it holds that \( \psi_q(1) = 0 \) then \( \psi = 0 \) because of Definition [5.4a] and because \( G \) is \( (n+1) \)-condensed.
Thus take $q \in J_{n+2}^0$. We want to see that $x \stackrel{\text{def}}{=} \psi_q(1) = 0$. Recall the short exact sequence of abelian groups

$$0 \to F(q) \xrightarrow{\lambda_{n+1}} \text{Ker}_{F}(q) \xrightarrow{\pi_{n+1}} G(q) \to 0$$

and take $y \in \text{Ker}_{F}(q)$ such that $\pi_{n+1}(y) = x$. Recall that $\text{Ker}_{F}(q) = \prod_{p \in (q \downarrow \mathcal{P})} F(p)$ and denote by $\alpha_p : q \to p$ the unique arrow from $q$ to $p$ for $p \in (q \downarrow \mathcal{P})$.

Now consider the restriction $y|_{J^q_n} \in \prod_{p \in J^q_n} F(p)$. Because $\text{deg}(q) = n + 2 > n + 1$ the map $F(q) \to \prod_{p \in \mathcal{P}} F(p)$ is an isomorphism by hypothesis. Then there exists a unique $z \in F(q)$ with $F(\alpha_p)(z) = y_p$ for each $p \in J^q_n \subseteq (q \downarrow \mathcal{P})_n$. If we prove that $F(\alpha_p)(z) = y_p$ for each $p \in (q \downarrow \mathcal{P})_n$ then $\lambda_{n+1}(z) = y$. This implies that $x = \pi_{n+1}(y) = \pi_{n+1}(\lambda_{n+1}(z)) = 0$ and completes the proof.

Thus take $p \in (q \downarrow \mathcal{P})_n$. By Definition [3.3b] there is $\beta_p : p' \to p$ with $p' \in J^q_{n+1}$. Write $\beta_p : q \to p'$ for the unique arrow from $q$ to $p'$. It holds that $\alpha_p = \beta_p \circ \beta_p'$. By Definition [3.3b] we have that $J^q_{n+1} \subseteq J^q_{n+1}$. Thus $G(\beta_p)(x) = G(\beta_p)(\psi_q(1)) = \psi_{p'}(1) = 0$ as $\psi$ is in the kernel of the restriction map. The short exact sequence

$$0 \to F(p') \xrightarrow{\lambda_{p'}} \text{Ker}_{F}(p') \xrightarrow{\pi_{p'}} G(p') \to 0$$

implies that there exists $t_{p'} \in F(p')$ such that $\lambda_{p'}(t_{p'}) = \text{Ker}_{F}(\beta_{p'})(y)$. Consider $z_{p'} = F(\beta_{p'})(z)$. We have that $z_{p'}$ and $t_{p'}$ have the same image by the restriction map

$$\lim_{p \in J^q_{n+1}} F \to \prod_{p \in J^q_{n+1}} F(p)$$

because $J^q_{n} \subseteq J^q_{n+1}$. Because $F$ is $\mathcal{J}$-determined then this restriction map is a monomorphism and so $z_{p'} = t_{p'}$. This implies that $F(\alpha_p)(z) = F(\beta_p \circ \beta_{p'})(z) = F(\beta_p)(z_{p'}) = F(\beta_p)(t_{p'}) = y_p$

and the proof of the restriction map being injective is finished.

Now we check that the map

$$\omega : G(p_0) \to \prod_{p \in J^q_{n+1}} G(p)$$

is pure. Take $z \in \prod_{p \in J^q_{n+1}} G(p)$ such that there exists $x \in G(p_0)$ with $m \cdot z = \omega(x)$ for some $m \neq 0$. We have to check that there exists $x' \in G(p_0)$ with $z = \omega(x')$, or equivalently, that $x = m \cdot x'$ for some $x' \in G(p_0)$. Recall once more the short exact sequence of abelian groups

$$0 \to F(p_0) \xrightarrow{\lambda_{p_0}} \text{Ker}_{F}(p_0) \xrightarrow{\pi_{p_0}} G(p_0) \to 0$$

and take $y \in \text{Ker}_{F}(p_0)$ with $\pi_{p_0}(y) = x$. We are going to build $h \in F(p_0)$ such that $y - \lambda_{p_0}(h) = m \cdot y'$, i.e., such that for any $p \in (p_0 \downarrow \mathcal{P})_n$ the element $(y - \lambda_{p_0}(h))_p = y_p - F(p_0 \to p)(h) \in F(p)$ is divisible by $m$. This implies that $x = m \cdot x'$ with $x' = \pi_{p_0}(y')$.

Notice that by hypothesis for each $q \in J^q_{n+1}$, $G(p_0 \to q)(x) = m \cdot z_q \in G(q)$. This implies that there exist $h_q \in F(q)$ and $y_q \in \text{Ker}_{F}(q)$ such that $\text{Ker}_{F}(p_0 \to$
q)(y) - \lambda_q(h_q) = m \cdot y_q, \text{i.e., such that for each } p \in (q \downarrow \mathcal{P})_n \subseteq (p_0 \downarrow \mathcal{P})_n \text{ we have that } y_p - F(q \rightarrow p)(h_q) = m \cdot (y_q)_p \in F(p) \text{ (it is enough to take } y_q \text{ with } \pi_q(y_q) = z_q). \]

To build \( h \) we use the map
\[
\tau : \prod_{p \in J_n^{p_0}} F(p) \xrightarrow{\cong} F(p_0)
\]
given by hypothesis, which is the inverse of the map
\[
F(p_0) \rightarrow \prod_{p \in J_n^{p_0}} F(p).
\]

For each \( p \in J_n^{p_0} \subseteq (p_0 \downarrow \mathcal{P})_n \) choose, by Definition 3.4.1, \( q(p) \in J_{n+1}^{p_0} \) such that there is an arrow \( q(p) \rightarrow p \). Then set \( \eta_p = F(q(p) \rightarrow p)(h_{q(p)}) \in F(p) \), where \( h_{q(p)} \) is built as before. Define \( h \overset{\text{def}}{=} \tau(\eta) \). By construction \( F(p_0 \rightarrow p)(h) = F(q(p) \rightarrow p)(h_{q(p)}) \) for each \( p \in J_n^{p_0} \) (but not for an arbitrary \( p \in (p_0 \downarrow \mathcal{P})_n \)).

With this definition for \( h \) we check now that \( y_p - F(p_0 \rightarrow p)(h) \) is divisible by \( m \) for each \( p \in (p_0 \downarrow \mathcal{P})_n \). This finishes the proof. Fix \( p \in (p_0 \downarrow \mathcal{P})_n \) and \( q_p \in J_{n+1}^{p_0} \) such that there is an arrow \( q_p \rightarrow p \) (we are not assuming that \( q_p = q(p) \) if \( p \in J_n^{p_0} \)). On the one hand we have by hypothesis that
\[
y_k - F(q_p \rightarrow k)(h_{q_p}) = m \cdot (y_{q_p})_k
\]
for each \( k \in \mathcal{P}_n^{q_p} \). In particular,
\[
y_k - F(q_p \rightarrow k)(h_{q_p}) = m \cdot (y_{q_p})_k
\]
for each \( k \in J_n^{q_p} \). Set \( h' \overset{\text{def}}{=} F(p_0 \rightarrow q_p)(h) \). Because \( q_p \in J_{n+1}^{p_0} \) then, by Definition 3.4.1, \( J_n^{q_p} \subseteq J_n^{p_0} \) and thus by construction for any \( k \in J_n^{q_p} \)
\[
y_k - F(p_0 \rightarrow k)(h) = y_k - F(q(k) \rightarrow k)(h_{q(k)}) = m \cdot (y_{q(k)})_k.
\]
Notice that \( F(p_0 \rightarrow k)(h) = F(q_p \rightarrow k)(F(p_0 \rightarrow q_p)(h)) = F(q_p \rightarrow k)(h') \). On the other hand, we have obtained
\[
y_k - F(q_p \rightarrow k)(h') = m \cdot (y_{q(k)})_k
\]
for each \( k \in J_n^{q_p} \).

Now write \( \eta_k = (y_{q(k)})_k - (y_{q_p})_k \) for each \( k \in J_n^{q_p} \) and write \( h'' = \tau(\eta) \in F(q_p) \) where \( \tau \) is the inverse of the map
\[
F(q_p) \rightarrow \prod_{p \in J_n^{q_p}} F(p).
\]

By Equations (7) and (8) it is straightforward that the elements \( h_{q_p} - m \cdot h'' \) and \( h' \) have the same image by this map. Then, as this map is injective by hypothesis, \( h' = h_{q_p} - m \cdot h'' \). As \( p \in J_n^{q_p} \) we have
\[
y_p - F(p_0 \rightarrow p)(h) = y_p - F(q_p \rightarrow p)(h) = y_p - F(q_p \rightarrow p)(h_{q_p} - m \cdot h''),
\]
and this equals
\[
m \cdot (y_{q_p})_p + m \cdot F(q_p \rightarrow p)(h'').
\]
Thus \( y_p - F(p_0 \rightarrow p)(h) \) is divisible by \( m \).
If $\deg(p_0) = n + 2$ we have to see that the map

$$\omega : G(p_0) \rightarrow \prod_{p \in \mathcal{P}_n^{p_0}} G(p)$$

is a pure monomorphism. To prove that $\omega$ is a monomorphism use the proof above starting where $\psi_q$ is considered for an arbitrary object $q$ of degree $n + 2$. The proof of $\omega$ being pure is exactly the same as above.

**Remark 3.7.** Notice that in the conditions of the proposition we have the following formula for the rank of the free abelian group $G(p_0)$ for $\deg(p_0) \geq n + 1$

$$\text{rk}(G(p_0)) = \sum_{p \in (p_0 \parallel \mathcal{P})_n} \text{rk } F(p) - \text{rk } F(p_0).$$

This is so because of the short exact sequence of free abelian groups

$$0 \rightarrow F(p_0) \xrightarrow{\lambda_{p_0}} \ker'_F(p_0) \xrightarrow{\pi_{p_0}} G(p_0) \rightarrow 0.$$

**Remark 3.8.** In the conditions of the proposition there are isomorphisms

$$F(p_0) \xrightarrow{\cong} \prod_{p \in \mathcal{P}_n^{p_0}} F(p)$$

for each $p_0$ with $\deg(p_0) \geq n + 1$. Moreover, we have built a map $s_{p_0} : \ker'_F(p_0) \rightarrow F(p_0)$ with $s_{p_0} \circ \lambda_{p_0} = 1_{F(p_0)}$. To the homomorphism $s_{p_0}$ corresponds the monomorphism

$$G(p_0) \xrightarrow{\delta_{p_0}} \ker'_F(p_0)$$

given by

$$\pi_{p_0}(x) \xrightarrow{\cong} x - (\lambda_{p_0} \circ s_{p_0})(x),$$

which satisfies $\pi_{p_0} \circ \delta_{p_0} = 1_{G(p_0)}$. It is straightforward that, by construction, $\text{Im } \delta_{p_0} = \prod_{p \in (p_0 \parallel \mathcal{P})_n \setminus \mathcal{J}_{n_0}^{p_0}} F(p)$, and thus

$$G(p_0) \xrightarrow{\delta_{p_0}} \prod_{p \in (p_0 \parallel \mathcal{P})_n \setminus \mathcal{J}_{n_0}^{p_0}} F(p).$$

Moreover, $x = \delta_{p_0}(y)$ is the only preimage of $y$ by $\pi_{p_0}$ which verifies $x_p = 0$ for $p \in \mathcal{J}_{n_0}^{p_0}$.

The main consequence of the previous proposition is that it reduces the problem of whether $G$ is $(n + 1)$-condensed to some integral equations. Moreover, this procedure can be applied recursively because the resulting functor $G$ turns out to be $(n + 1)$-condensed, free and $\mathcal{J}$-determined, and so the proposition applies to $G$ too. Notice again that the ranks of $G$ are given by Remark 3.7.
Example 3.9. Consider the graded poset $\mathcal{P}$ with shape

\begin{center}
\begin{array}{ccccccc}
a_2 & c_1 & f_0 & \\
& d_1 & g_0 & \\
b_2 & c_1 & h_0, \\
\end{array}
\end{center}

where the subindexes denote the degree of the objects. Notice that $\mathcal{P}$ has the homotopy type of a wedge of 4 2-dimensional spheres $\bigvee_{i=1}^4 S^2_i$. The sets

- $J'_0 = \{ f \}$, $J'_1 = \{ g \}$, $J'_h = \{ h \}$,
- $J''_i = \{ c \}$, $J''_0 = \{ f \}$, $J''_1 = \{ d \}$, $J''_d = \{ g \}$, $J''_e = \{ e \}$, $J''_c = \{ h \}$,
- $J''_a = \{ a \}$, $J''_f = \{ e \}$, $J''_b = \{ d \}$, $J''_g = \{ f \}$, $J''_h = \{ b \}$, $J''_e = \{ e \}$, $J''_h = \{ h \}$.

define a local covering family $J$ for $\mathcal{P}$. Consider now the the functor $F : \mathcal{P} \to \text{Ab}$ with values

\begin{center}
\begin{array}{ccccccc}
\Z & \Z & \Z & \\
& \Z & \Z & \\
\Z & \Z & \Z, \\
\end{array}
\end{center}

such that all the arrows arriving and departing from $c$ and $e$ are the identity and all the arrows arriving and departing from $d$ are minus the identity. Then $F$ is 0-condensed, free and $J$-determined. Moreover, equations in Proposition 3.6 hold for any object of degree greater or equal to 1. Thus we obtain a functor $G$ which is 1-condensed, free and $J$-determined. By Remark 3.7 we know the ranks of the free abelian groups that $G$ takes as values:

\begin{center}
\begin{array}{ccccccc}
\Z^2 & \Z^2 & 0 & \\
& \Z^2 & 0 & \\
\Z^2 & \Z^2 & 0, \\
\end{array}
\end{center}

Moreover, equations in Proposition 3.6 applied to $G$ hold for any object of degree greater or equal to 2, and thus we obtain a functor $H$ which is 2-condensed, free and $J$-determined:

\begin{center}
\begin{array}{ccccccc}
\Z^4 & 0 & 0 & \\
& 0 & 0 & \\
\Z^4 & 0 & 0, \\
\end{array}
\end{center}

By Remark 3.8 we can identify $G(a) \cong F(g) \oplus F(h) = \Z_g \oplus \Z_h$. Also $G(c) \cong \Z_g \oplus \Z_h$, $G(d) \cong \Z_f \oplus \Z_h$ and $G(e) \cong \Z_f \oplus \Z_g$. By the definition of $G$ as a co-image it is easy
to see that
\[
G(a \to c) : \mathbb{Z}_g \oplus \mathbb{Z}_h \to \mathbb{Z}_g \oplus \mathbb{Z}_h \quad (g, h) \mapsto (g, h).
\]
Also \(G(a \to d)(g, h) = (-g, h - g)\) and \(G(a \to c)(g, h) = (-h, g - h)\) with respect to the ordered basis mentioned above. Additional computations lead to
\[
\lim F = \mathbb{Z},
\]
\[
\lim_{\leftarrow}^1 F = \text{Coim}\{\lim \ker_{F^r} \to \lim G\} = \text{Coim}\{\mathbb{Z}^3 \to \mathbb{Z}^2\} = 0, \quad \text{and}
\]
\[
\lim_{\leftarrow}^2 F = \text{Coim}\{\lim \ker_{G^r} \to \lim H\} = \text{Coim}\{\mathbb{Z}^6 \to \mathbb{Z}^8\} = \mathbb{Z}^4.
\]

4. Integral cohomology

In this section we apply the work developed in the preceding sections to compute the cohomology with integer coefficients of the realization of a graded poset \(P\) equipped with a local covering family \(J\).

To compute the abelian group \(H^n(|P|; \mathbb{Z})\) for \(n \geq 0\) we compute the higher limit \(\lim_{\leftarrow}^n \mathbb{Z}\) where \(\mathbb{Z} : P \to \text{Ab}\) is the functor of constant value \(\mathbb{Z}\) which sends every morphism to the identity \(1_\mathbb{Z}\). We begin studying the extra conditions that the local covering family \(J\) must satisfy to apply iteratively the Proposition 3.6 beginning on \(c_Z\).

First, notice that \(c_Z\) is 0-condensed (we are assuming 2.5) and free (Definition 3.1). By Definition 3.5, \(c_Z\) is \(J\)-determined as 0-condensed functor if and only if for each \(p_0 \in \text{Ob}(P)\) with \(\deg(p_0) \geq 2\) the set \(J_{p_0}^0\) intersects each connected component of \((p_0 \downarrow P)_*\). The dimensional equation in Proposition 3.6 for \(p_0 \in \text{Ob}(P)\) with \(\deg(p_0) \geq 1\) becomes \(\text{rk}\ c_Z(p_0) = 1 = |J_{p_0}^0| = \sum_{p \in J_{p_0}^0} \text{rk}\ c_Z(p)\). Thus, \(c_Z\) is \(J\)-determined as a 0-condensed functor if and only if \((p_0 \downarrow P)_*\) is connected for \(\deg(p_0) \geq 2\) and \(|J_{p_0}^0| = 1\) for \(\deg(p_0) \geq 1\). The successive applications of Proposition 3.6 give, by the dimensional equation in the statement of the Proposition 3.6, the following:

**Definition 4.1.** Let \(P\) be a graded poset. Define, inductively on \(n\), the number \(R_{n}^{p_0}\) for each object \(p_0\) with \(\deg(p_0) \geq n\) by \(R_{0}^{p_0} = 1\) and by
\[
R_{n}^{p_0} = \sum_{p \in (p_0 \downarrow P)_{n-1}} R_{n-1}^{p} - R_{n-1}^{p_0}
\]
for \(n \geq 1\).

**Definition 4.2.** Let \(P\) be a graded poset and \(J\) be a local covering family for \(P\). We say that \(J\) is adequate if \((p_0 \downarrow P)_*\) is connected for \(\deg(p_0) \geq 2\), and if we have the equality
\[
R_{n}^{p_0} = \sum_{p \in J_{p_0}^0} R_{n}^{p}
\]
for \(n \geq 0\) and \(\deg(p_0) \geq n + 1\).
Proposition 4.3. Let \( \mathcal{P} \) be a graded poset and let \( \mathcal{J} \) be an adequate local covering family. Then there is a sequence of functors \( F_0, F_1, F_2, \ldots \) defined by \( F_0 \overset{\text{def}}{=} c_\mathbb{Z}: \mathcal{P} \rightarrow \text{Ab} \) and by the short exact sequence

\[
0 \longrightarrow F_{n-1} \xrightarrow{\lambda_{n-1}} \text{Ker}_{F_{n-1}}' \xrightarrow{\pi_n} F_n \longrightarrow 0
\]

for \( n = 1, 2, 3, \ldots \). Moreover, \( F_n \) is \( n \)-condensed, free and \( \mathcal{J} \)-determined for \( n \geq 0 \). For \( \text{deg}(p_0) \geq n \) we have \( \text{rk} F_n(p_0) = R^n_{p_0} \).

The local properties of a graded poset \( \mathcal{P} \) equipped with an adequate local covering family \( \mathcal{J} \) give rise to a sequence \( F_0 = c_\mathbb{Z}, F_1, F_2, \ldots \) of functors. Now, we study some properties of these functors which are independent of the local covering family \( \mathcal{J} \).

The first point to notice is that the sequence of functors from Proposition 4.3 does not depend on the adequate local covering family \( \mathcal{J} \). Thus, two or more adequate local covering families can be considered for the same graded poset and they still give rise to the same sequence of functors. Next we focus on the short exact sequence

\[
0 \longrightarrow F_{n-1} \xrightarrow{\lambda_{n-1}} \text{Ker}_{F_{n-1}}' \xrightarrow{\pi_n} F_n \longrightarrow 0
\]

for \( n \geq 1 \) of Proposition 4.3. The beginning of the long exact sequence of derived functors for this short exact sequence is

\[
0 \xrightarrow{\lim \text{Ker}_{F_{n-1}}'} \lim F_n \xrightarrow{\omega_n} \lim F_n \xrightarrow{H^n(\mathcal{P}; \mathbb{Z})} 0 \quad (9)
\]

by Theorem [2.7], where \( \lambda_{n-1} = \widetilde{\lambda}_{n-1} \) and \( \omega_n = \widetilde{\pi}_n \) are the induced maps. Notice that the three inverse limits appearing above are free abelian groups as the corresponding functors take free abelian groups as values. In fact, for the middle term we have the exact description

\[
\lim \text{Ker}_{F_{n-1}}' \cong \prod_{p \in \text{Ob}_{n-1}(\mathcal{P})} F_{n-1}(p) \quad (10)
\]

given by Theorem [2.7]. It turns out that there is also a simpler description for \( \lim F_n \), which can be interpreted as the analogue in the context of CW-complexes to the fact that the cohomology on degree \( n \) depends upon the \( n + 1 \) skeleton (recall that \( F_n(p) = 0 \) if \( \text{deg}(p) < n \)).

Lemma 4.4. Let \( \mathcal{P} \) be a graded poset and let \( \mathcal{J} \) be an adequate local covering family. Let \( c_\mathbb{Z}, F_1, F_2, \ldots \) be the sequence of functors given by Proposition 4.3. Then

\[
\lim F_n \cong \lim F_n|_{\mathcal{P}(n+1,n)}
\]

for each \( n \geq 0 \).

Proof. Consider the restriction map

\[
\lim F_n \rightarrow \lim F_n|_{\mathcal{P}(n+1,n)}.
\]

This map is clearly a monomorphism because \( F_n \) is an \( n \)-condensed functor. To see that it is surjective take \( \psi \in \lim F_n|_{\mathcal{P}(n+1,n)} \). We want to extend \( \psi \) to each \( p \in \text{Ob}(\mathcal{P}) \) with \( \text{deg}(p) > n + 1 \). We do it inductively on \( \text{deg}(p) \).
Notice that (see the beginning of the proof of Proposition 3.6)

\[ F_n(p) \to \lim_{(p \downarrow P)^*} F_n \]

is an isomorphism for \( \text{deg}(p) > n + 1 \). For \( \text{deg}(p) = n + 2 \) we have that \( j \in (p \downarrow P)^* \) implies that \( \text{deg}(j) \leq n + 1 \). Then there is a unique way of extending \( \psi \) to \( \psi(p) \).

Once we have extended \( \psi \) to \( P \{n+2, n+1, n\} \) we proceed with an induction argument. That the extension that we are building is actually a functor is again due to that \( F_n \) is \( n \)-condensed. \(\square\)

Also, from Equations (9) and (10), we have the following formula, analogue to that of the Euler characteristic for \( CW \)-complexes:

**Lemma 4.5.** Let \( \mathcal{P} \) be a poset for which exists an adequate local covering family. Then

\[
\sum_i (-1)^i \text{rk} H^i(\mathcal{P}; \mathbb{Z}) = \sum_i (-1)^i \sum_{p \in \text{Ob}(\mathcal{P})} R_i^p.
\]

Take up again Equations (9) and (10). We can form the sequence of abelian groups

\[
0 \to \prod_{p \in \text{Ob}(\mathcal{P})} F_0(p) \overset{d_0}{\to} \prod_{p \in \text{Ob}_1(\mathcal{P})} F_1(p) \overset{d_1}{\to} \prod_{p \in \text{Ob}_2(\mathcal{P})} F_2(p) \overset{d_2}{\to} \ldots
\]

where \( d_n = \iota_{n+1} \circ w_{n+1} \) for \( n \geq 0 \). Then it is straightforward that this sequence is a co-chain complex and its cohomology is exactly the cohomology of \( \mathcal{P} \) with integers coefficients:

**Theorem 4.6.** Let \( \mathcal{P} \) be a graded poset for which exists an adequate local covering family. Then there exists a co-chain complex

\[
0 \to \prod_{p \in \text{Ob}(\mathcal{P})} F_0(p) \overset{d_0}{\to} \prod_{p \in \text{Ob}_1(\mathcal{P})} F_1(p) \overset{d_1}{\to} \prod_{p \in \text{Ob}_2(\mathcal{P})} F_2(p) \overset{d_2}{\to} \ldots
\]

of which cohomology is \( H^*(\mathcal{P}; \mathbb{Z}) \).

5. **Global covering families**

Recall that local covering families were defined as subsets of the local categories \( (p \downarrow \mathcal{P}) \) for \( p \in \text{Ob}(\mathcal{P}) \), where \( \mathcal{P} \) is a graded poset. In this section we define global covering families by subsets of the whole category \( \mathcal{P} \), imitating some of the local features of the local covering families.

**Definition 5.1.** Let \( \mathcal{P} \) be a graded poset for which there exists an adequate local covering family, and consider the sequence of functors \( F_0 = c_{\mathbb{Z}}, F_1, F_2, \ldots \) given by Proposition 4.3. A **global covering family** is a family of subsets \( \mathcal{K} = \{K_n\}_{n \geq 0} \) with \( K_n \subseteq \text{Ob}_n(\mathcal{P}) \) such that

1. the morphism

\[
\lim_{\longrightarrow} F_n \to \prod_{p \in K_n} F_n(p)
\]

is injective for each \( n \geq 0 \), and
(2) the morphism

$$\prod_{p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(p) \to \prod_{p \in K_n} F_n(p)$$

is pure for each $n \geq 1$.

Notice that the definition does not depend on the particular adequate local covering family used to obtain the sequence of functors (as the sequence of functors does not depend on it). The map

$$\prod_{p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(p) \to \prod_{p \in K_n} F_n(p)$$

used in the definition is the composition

$$\prod_{p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(p) \hookrightarrow \prod_{p \in \text{Ob}_{n-1}(P)} F_{n-1}(p) \xrightarrow{d_{n-1}} \prod_{p \in \text{Ob}_n(P)} F_n(p) \to \prod_{p \in K_n} F_n(p).$$

We also have maps

$$\prod_{p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(p) \hookrightarrow \prod_{p \in \text{Ob}_{n-1}(P)} F_{n-1}(p) \xrightarrow{d_{n-1}} \prod_{p \in \text{Ob}_n(P)} F_n(p) \to \prod_{p \in K_n} F_n(p)$$

and

$$\prod_{p \in K_{n-1}} F_{n-1}(p) \hookrightarrow \prod_{p \in \text{Ob}_{n-1}(P)} F_{n-1}(p) \xrightarrow{d_{n-1}} \prod_{p \in \text{Ob}_n(P)} F_n(p) \to \prod_{p \in K_n} F_n(p),$$

obtained by pre and post composing the differential $d_{n-1}$ with appropriate inclusions and projections. We denote all of them by $d_{n-1}$. Also we fix the following

**Notation.** We will write elements $x \in \prod_{p \in \text{Ob}_n(P)} F_n(p)$ as $x = y \oplus z$, where $y \in \prod_{p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(p)$ and $z \in \prod_{p \in K_{n-1}} F_{n-1}(p)$. Also, $d_n(y \oplus z) = (y_1 \oplus z_1) \oplus (y_2 \oplus z_2)$ where $d_n(y \oplus 0) = y_1 \oplus y_2$ and $d_n(0 \oplus z) = z_1 \oplus z_2$.

Next we introduce the Betti numbers associated to a global covering family

**Definition 5.2.** Let $\mathcal{P}$ be a graded poset for which there exists an adequate local covering family, and let $\mathcal{K}$ be a global covering family for $\mathcal{P}$. Then we define, for $n \geq 0$, the $n$-th Betti number of the family $\mathcal{K}$ as

$$b_0^\mathcal{K} = \sum_{p \in K_0} R_0^p$$

and

$$b_n^\mathcal{K} = \sum_{p \in K_{n-1}} R_{n-1}^p - \sum_{p \in \text{Ob}_{n-1}(\mathcal{P})} R_{n-1}^p + \sum_{p \in K_n} R_n^p$$

for $n \geq 1$. 
Remark 5.3. It turns out that a local covering family gives, for each subcategory \((p_0 \downarrow P)\), a global covering family \(\mathcal{K}\) for \((p_0 \downarrow P)\) with \(b_n^\mathcal{K} = 0\) for \(n \geq 1\) and \(b_0^\mathcal{K} = 1\). More precisely, let \(P\) be a graded poset and let \(\mathcal{J}\) be an adequate local covering family for \(P\). Fix the graded poset \(P' = (p_0 \downarrow P)\) for some object \(p_0\) of \(P\). Define \(K_n = J_n \subseteq \text{Ob}_n(P')\) for \(0 \leq n \leq \text{deg}(p_0)\) and empty otherwise. Then, using that \(p_0\) is an initial object in \(P'\) and Remark 3.8, it is straightforward that \(\mathcal{K} = \{K_n\}_{n \geq 0}\) is a global covering family for \(P'\). In fact, all the the maps in Definition 5.1 become isomorphisms. The integral equalities in Definition 4.2 (adequateness) correspond exactly to the statements \(b_0^\mathcal{K} = 1\) and \(b_n^\mathcal{K} = 0\) for \(n \geq 1\).

Recall once again the exact sequence (9). From there we have that \(b_n\), the \(n\)-th Betti number of \(P\), equals

\[
b_n = \text{rk} \lim_{\leftarrow} F_{n-1} - \sum_{p \in \text{Ob}_{n-1}(P)} R^p_{n-1} + \text{rk} \lim_{\leftarrow} F_n.
\]

Because the map \(\lim_{\leftarrow} F_n \to \prod_{p \in K_n} F_n(p)\) is injective then we have

\[
\text{rk} \lim_{\leftarrow} F_n \leq \text{rk} \prod_{p \in K_n} F_n(p) = \sum_{p \in K_n} R^p_n
\]

Thus, by Definition 5.2

\[
b_n \leq b_n^\mathcal{K}
\]

for any \(n \geq 0\). An easy induction argument proves

Proposition 5.4. Let \(P\) be a poset for which exists an adequate local covering family and a global covering family \(\mathcal{K}\). Then

\[
b_n \leq b_n^\mathcal{K}
\]

and

\[
b_n - b_{n-1} + \ldots + (-1)^n b_0 \leq b_n^\mathcal{K} - b_{n-1}^\mathcal{K} + \ldots + (-1)^n b_0^\mathcal{K}
\]

for any \(n \geq 0\).

Now we focus on the main theorem of this section, which states that we can compute the cohomology of \(P\) using a cochain complex which in degree \(n\) has a free abelian group of rank \(b_n^\mathcal{K}\).

Theorem 5.5. Let \(P\) be a graded poset for which exists an adequate local covering family and a global covering family \(\mathcal{K}\). Then there exists a cochain complex

\[
0 \to B_0^\mathcal{K} \xrightarrow{\Omega_0} B_1^\mathcal{K} \xrightarrow{\Omega_1} B_2^\mathcal{K} \xrightarrow{\Omega_2} \cdots
\]

with \(B_n^\mathcal{K} \cong \mathbb{Z}^{b_n^\mathcal{K}}\) for \(n \geq 0\), of which cohomology is \(H^*(P; \mathbb{Z})\).

Before proving the proposition we prove the following definition-lemma
Lemma 5.6. Let $\mathcal{P}$ be a graded poset for which there exists an adequate local covering family and a global covering family $\mathcal{K}$. Fix $n \geq 1$, then there is a split short exact sequence

$$0 \to \prod_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus K_{n-1}} F_n(p) \to \prod_{p \in K_n} F_n(p) \to B^K_n \to 0,$$

where $B^K_n \cong \mathbb{Z} b^K_n$.

Proof. The map $\lambda : \prod_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus K_{n-1}} F_n(p) \to \prod_{p \in K_n} F_n(p)$ is pure by definition of global covering family, and thus its cokernel $B^K_n$ is free and we have a section. So, if we show that this map is a monomorphism we obtain that the sequence is exact and the appropriate rank of the cokernel by the definition of the number $b^K_n$. Take $x \in \prod_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus K_{n-1}} F_n(p)$ with $\lambda(x) = 0$. Recall the sequence

$$\lim_{\leftarrow} F_{n-1} \to \prod_{p \in \text{Ob}_{n-1}(\mathcal{P})} F_n(p) \to \lim_{\rightarrow} F_n \to \prod_{p \in \text{Ob}_n(\mathcal{P})} F_n(p),$$

which is exact at $\prod_{p \in \text{Ob}_{n-1}(\mathcal{P})} F_n(p)$. Then $\iota_n \circ \omega_n(x \oplus 0)_p = 0$ for each $p \in K_n$ and, by Definition 5.1 (1), we obtain that $\omega_n(x \oplus 0) = 0$. By exactness there exist $y \in \lim_{\rightarrow} F_n$ with $\iota_n(y) = x \oplus 0$. But, by definition, $(x \oplus 0)_p = 0$ for each $p \in K_{n-1}$. Then, by Definition 5.1 (1) again, we obtain that $y = 0$ and $x = 0$. \qed

The leftmost square in the following diagram commutes, and thus, we can find an arrow which closes the rightmost square. We denote this arrow by $\Omega_n$, and it is the differential on $B_n$ induced by the differential $d_n$:

$$0 \to \prod_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus K_{n-1}} F_n(p) \xrightarrow{d_{n-1}} \prod_{p \in K_n} F_n(p) \xrightarrow{d_n} B^K_n \to 0$$

$$0 \to \prod_{p \in \text{Ob}_n(\mathcal{P}) \setminus K_n} F_n(p) \xrightarrow{d_n} \prod_{p \in K_{n+1}} F_{n+1}(p) \xrightarrow{\Omega_n} B^K_{n+1} \to 0.$$

The maps $\Omega$ verify $\Omega_{n+1} \circ \Omega_n = 0$ for each $n \geq 0$, as can be seen by using the preceding diagram. We have then a chain complex

$$0 \to B^K_0 \xrightarrow{\Omega_0} B^K_1 \xrightarrow{\Omega_1} B^K_2 \xrightarrow{\Omega_2} \cdots,$$

and we claim that the homology of this chain complex is the same as that of the chain complex in Theorem 4.6.

Thus to prove the proposition we shall build for each $n \geq 1$ an epimorphism

$$\psi : \text{Ker} \Omega_n \to \text{Ker} \Omega_n / \text{Im} d_{n-1}$$

of which kernel is $\text{Im} \Omega_{n-1}$. We denote the class of $x \in \prod_{p \in K_n} F_n(p)$ in $B^K_n$ as $\overline{x}$, and the class of $x \in \text{Ker} d_n$ in $\text{Ker} d_n / \text{Im} d_{n-1}$ by $\overline{x}$.

Thus, take $\overline{x} \in B^K_n$ such that $\Omega_n(\overline{x}) = 0$. Then there is $y \in \prod_{p \in \text{Ob}_n(\mathcal{P}) \setminus K_n} F_n(p)$ such that $d_n(x) = d_n(y)$. If $d_n(y_1 \oplus 0) = y_1 \oplus y_2$ and $d_n(0 \oplus x) = x_1 \oplus x_2$ then $y_2 = x_2$. Define

$$\psi(\overline{x}) = -y \oplus x.$$
Then $d_n(-y \oplus x) = -(y_1 \oplus y_2) + (x_1 \oplus x_2) = x_1 - y_1 \oplus 0$. Therefore $\iota_{n+1}(\omega_{n+1}(-y \oplus x)) = x_1 - y_1 \oplus 0$ and, as the map

$$\lim \frac{\iota_{n+1}}{\iota_{n+1}} \rightarrow \bigoplus_{p \in \text{Ob}_n(\mathcal{P})} F_{n+1}(p) \rightarrow \bigoplus_{p \in \text{Ker}_{n+1}(\mathcal{P})} F_{n+1}(p)$$

is injective by definition of global covering family, we obtain that $x_1 = y_1$ and $d_n(-y \oplus x) = 0$, i.e., $-y \oplus x \in \text{Ker} d_n$. Notice that the element $y$ chosen above is unique: if $y' \in \bigoplus_{p \in \text{Ob}_n(\mathcal{P}) \setminus \text{Ker}_n} F_n(p)$ and $d_n(x) = d_n(y)$ then $d_n(y - y') = 0$ and by Lemma 5.6 $y = y'$.

Now we prove that $\psi(\bar{x})$ does not depend on the chosen representative $x$. Thus take $x' = x + z_2$, where $z \in \bigoplus_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus \text{Ker}_{n-1}} F_{n-1}(p)$ and $d_{n-1}(0 \oplus z) = z_1 \oplus z_2$. Then $0 = d_n(d_{n-1}(0 \oplus z)) = z_{1,1} + z_{1,2} \oplus z_{2,1} + z_{2,2}$. Moreover, $d_n(y - z_1 \oplus 0) = y_1 - z_{1,1} \oplus y_2 - z_{1,2}$ and thus $d_n(y - z_1) = y_2 - z_{1,2} = y_2 + z_{2,2} = x_2 + z_{2,2} = d_n(x')$. Then,

$$\psi(x') = -y + z_1 \oplus x'.$$

But notice that $d_{n-1}(z \oplus 0) = z_1 \oplus z_2 = (-y + z_1 \oplus x') - (-y \oplus x)$ and thus $-y + z_1 \oplus x' = -y \oplus x$ in $\text{Ker} d_n / \text{Im} d_{n-1}$.

It is clear that $\psi$ is a homomorphism. Next we prove that it is an epimorphism. Take $a \oplus b \in \text{Ker} d_n / \text{Im} d_{n-1}$. Then $0 = d_n(a \oplus b) = a_1 + b_1 \oplus a_2 + b_2$. Consider $b \in \bigoplus_{p \in \text{Ker}_n} F_n(p)$. Notice that $d_n(a \oplus 0) = a_1 \oplus a_2$ and $d_n(0 \oplus b) = b_1 \oplus b_2$. Hence $d_n(b) = b_2 = -a_2 = d_n(-a)$, and thus $\Omega_n(b) = 0$ and $\psi(b) = a \oplus b$.

To finish the proof of the proposition we show that $\text{Ker} \psi = \text{Im} \Omega_{n-1}$. First take $\bar{x} \in B^K_n$ with $\bar{x} = \Omega_{n-1}(\bar{y})$ and $\bar{y} \in B^K_{n-1}$. There is $z \in \bigoplus_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus \text{Ker}_{n-1}} F_{n-1}(p)$ with $d_{n-1}(z) = x - d_{n-1}(y)$. If $d_{n-1}(z \oplus 0) = z_1 \oplus z_2$ and $d_{n-1}(0 \oplus y) = y_1 \oplus y_2$ then $z_2 = x - y_2$ and $d_n(z) = x_2 = z_{2,2} + y_{2,2}$. Take $t = -z_1 - y_1 \in \bigoplus_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus \text{Ker}_{n-1}} F_{n-1}(p)$. Therefore $d_n(t, 0) = -z_{1,1} - y_{1,1} \oplus -z_{1,2} - y_{1,2} = d_n(t) = -z_{1,1} - y_{1,1} = z_{2,2} + y_{2,2} = x_2$ and $\psi(x) = -t \oplus x$ with $d_n(x \oplus y) = z_1 + y_1 \oplus z_2 + y_2 = -t \oplus x$. This shows that $\text{Im} \Omega_{n-1} \subseteq \text{Ker} \psi$.

Finally we prove that $\text{Ker} \psi \subseteq \text{Im} \Omega_{n-1}$. Take $\bar{x} \in B^K_n$ such that $\psi(\bar{x}) = 0$ in $\text{Ker} d_n / \text{Im} d_{n-1}$. This means that if $y \in \bigoplus_{p \in \text{Ob}_n(\mathcal{P}) \setminus \text{Ker}_n} F_n(p)$ is such that $d_n(x) = d_n(y)$ then there is $a \oplus b \in \bigoplus_{p \in \text{Ker}_n} F_n(p)$ with $d_{n-1}(a \oplus b) = -y \oplus x$. Hence $a \in \bigoplus_{p \in \text{Ob}_{n-1}(\mathcal{P}) \setminus \text{Ker}_{n-1}} F_{n-1}(p)$ and $b \in \bigoplus_{p \in \text{Ker}_n} F_n(p)$ are such that $d_{n-1}(b) = x - d_{n-1}(a)$ and thus $\Omega_{n-1}(\bar{b}) = \bar{x}$.

### 6. Simplex-like posets

In this section we will show that for a big family of posets, which includes simplicial complexes, there exists local covering families. We start defining posets which locally are like simplicial complexes:
Definition 6.1. The category $\triangle_n$ has as objects the faces of the standard $n$-dimensional simplex and arrows the inclusions among faces. For example $\triangle_2$ is the category:

$$
\begin{array}{c}
\vdash \\
\triangle \\
\vdash
\end{array}
$$

Definition 6.2. Let $\mathcal{P}$ be a poset. Then $\mathcal{P}$ is simplex-like if for all $p \in \text{Ob}(\mathcal{P})$ the category $(\mathcal{P} \downarrow p)$ is isomorphic to $\Delta_n$ for some $n$.

Of course simplicial complexes are simplex-like posets. In fact ([14, 3.1]), simplicial complexes are exactly the simplex-like posets such that any two elements which have a lower bound have an infimum, i.e., a greatest lower bound. Another examples of simplex-like posets are barycentric subdivisions and, in general, subdivision categories in the sense of [21]. We have:

Lemma 6.3. Let $\mathcal{P}$ be a simplex-like poset. Then $\mathcal{P}^{\text{op}}$ is a graded poset.

Proof. To see that $\mathcal{P}$ is graded recall that for any $p \in \text{Ob}(\mathcal{P})$ the subcategory $(\mathcal{P} \downarrow p)$ is isomorphic to $\Delta_n$ for some $n \geq 0$. Define $\deg(p) = n$. Then $\deg : \text{Ob}(\mathcal{P}^{\text{op}}) \to \mathbb{Z}$ is a degree function which assigns 0 to maximal elements and $\mathcal{P}$ is graded.

Next, we will define local covering families for $\Delta_n^{\text{op}}$. Due to the local shape of simplex-like posets this local covering family for $\Delta_n^{\text{op}}$ will allow us to define a local covering family for the opposite category of a simplex-like poset.

Lemma 6.4. There exists an adequate local covering family for $\Delta_n^{\text{op}}$ for each $n \geq 0$.

Proof. Fix a total order $v_0 < v_1 < \ldots < v_n$ for the vertices of $\Delta_n$. For each face $\sigma$ of $\Delta_n$ and each $0 \leq m \leq \deg(\sigma)$ we must define a subset $J_m^\sigma$ of the $m$-dimensional faces of $\sigma$. Take the greatest vertex contained in $\sigma$ and then define $J_m^\sigma$ as those $m$-dimensional faces of $\sigma$ which contain this vertex. Then it is straightforward that $J$ is an adequate local covering family of $\Delta_n^{\text{op}}$, and that

$$
R_m^\sigma = \sum_{l=0}^{m} (-1)^{(m-l)} \binom{\deg(\sigma)+1}{l}
$$

for $\sigma \in \triangle_n$ and $0 \leq m \leq \deg(\sigma)$.

Now we reach the main result of this section:

Lemma 6.5. If $\mathcal{P}$ is a simplex-like poset then there is an adequate local covering family for the graded poset $\mathcal{P}^{\text{op}}$. 
Proof. By definition there are isomorphisms of categories \((P \downarrow p)^{op} \simeq (p \downarrow P^{op}) \simeq \Delta_n^{op}\) for each \(p \in \text{Ob}(P)\), and we know that \(\Delta_n^{op}\) can be equipped with an adequate local covering family. To build an adequate local covering family

\[ \mathcal{K} = \{ K_p^i \}_{i \in \text{Ob}(P^{op}), 0 \leq p \leq \text{deg}(i) } \]

we just have to choose appropriately the isomorphisms \((P \downarrow p) \simeq \Delta_n\).

Consider the degree function \(\text{deg} : \text{Ob}(P^{op}) \to \mathbb{Z}\) defined in Lemma 6.3 and the set \(T\) of the maximal elements of \(P^{op}\), i.e., \(T = \{ p \in \text{Ob}(P^{op})| \text{deg}(p) = 0 \}\). Choose a total order \(<\) for \(T\) (suppose \(T\) is finite or use the Axiom of Choice \[\text{[13]}\]). Then, given \(p \in \text{Ob}(P^{op})\), consider the subset \((p \downarrow P^{op})_0 \subseteq T\) and the restriction \((p \downarrow P^{op})_0, <\) of the total order from \(T\). There is a unique isomorphism

\[ \varphi_p : (p \downarrow P^{op}) \simeq \Delta_{\text{deg}(p)}^{op} \]

which induces an order preserving map

\[ ((p \downarrow P^{op})_0, <) \simeq (\Delta_{\text{deg}(p)})_0 = \{ v_0, v_1, v_2, \ldots, v_{\text{deg}(p)} \}. \]

Denote by \(\mathcal{J}\) the local covering family for \(\Delta_{\text{deg}(p)}^{op}\) of Lemma 6.4 and define, for \(0 \leq m \leq \text{deg}(p)\),

\[ K^p_m = \varphi_p^{-1}(J^p_m). \]

Then \(\mathcal{K}\) fulfills condition \([3]\) of Definition 3.4 because for \(0 \leq m < \text{deg}(p)\)

\[ (p \downarrow P^{op})_m = \varphi_p^{-1}((\Delta_{\text{deg}(p)})_m) = \varphi_p^{-1} \big( \bigcup_{i \in J^p_{m+1}} (i \downarrow \Delta_{\text{deg}(p)})_m \big) = \bigcup_{\varphi_p^{-1}(i) \in \varphi_p^{-1}(K^p_{m+1})} \varphi_p^{-1}((i \downarrow \Delta_{\text{deg}(p)})_m) = \bigcup_{i \in K^p_{m+1}} (i \downarrow P^{op})_m. \]

To check condition \([3]\) of Definition 3.4 take \(q \in K^p_{m+1}\) for some \(0 \leq m < \text{deg}(p)\) and call \(\mathcal{J}'\) to the local covering family for \(\Delta_{m+1}^{op}\) of Lemma 6.4. We want to see that \(K^q_m \subseteq K^p_m\). Recalling the natural inclusion \((q \downarrow P^{op}) \subseteq (p \downarrow P^{op})\) this is equivalent to

\[ \varphi_q^{-1}(J^q_m(q)) \subseteq \varphi_p^{-1}(J^p_m(q)) \]

and to

\[ \psi(J^q_m(q)) \subseteq J^p_m(q) \]

where \(\psi = \varphi_p \circ \varphi_q^{-1}\). By construction \(\psi\) is order preserving and thus this inclusion holds. \[\square\]

Notice that comparing the general expression for \(R^p_n\) in the proof of Lemma 6.4 with the binomial expansion of \((1 - 1)^{\text{deg}(p)+1} = 0\) we obtain that \(R^p_{\text{deg}(p)} = 1\) for any object \(p\) in a simplex-like poset. The following are re-statements of results about local covering families applied to simplex-like posets:

**Lemma 6.6.** Let \(P\) be a simplex-like poset and consider the graded poset \(P^{op}\) (for which exists an adequate local covering family by Lemma 6.5). Let \(\mathcal{K}\) be a global covering family for \(P^{op}\). Then the Betti numbers of the family \(\mathcal{K}\) are given by

\[ b^\mathcal{K}_0 = |K_0|, \]
and for \( n \geq 1 \), by
\[
b^K_n = |K_{n-1}| - |\text{Ob}_{n-1}(P)| + |K_n|.
\]

Lemma 6.7. Let \( P \) a simplex-like poset. Then
\[
\sum_i (-1)^i \text{rk} H^i(P; \mathbb{Z}) = \sum_i (-1)^i |\text{Ob}_i(P)|.
\]

Again because \( R_{\text{deg}(p)}^p = 1 \), Proposition 4.3 gives \( \text{rk} F_{\text{deg}(p)}(p) = 1 \) and therefore \( F_{\text{deg}(p)}(p) = \mathbb{Z} \). We may identify the object \( p \in P_n \) with its image \( v_0 v_1 \ldots v_n \) under the isomorphism \( (P \downarrow p) \cong \Delta_n \) built in Lemma 6.6. Then the expression for the differential in Theorem 4.6 becomes the familiar expression for simplicial complexes:

Proposition 6.8. Let \( P \) be a simplex-like poset. Then there exists a co-chain complex
\[
0 \to \mathbb{Z}^{\text{Ob}_0(P)} \xrightarrow{d_0} \mathbb{Z}^{\text{Ob}_1(P)} \xrightarrow{d_1} \mathbb{Z}^{\text{Ob}_2(P)} \xrightarrow{d_2} \ldots
\]
of which cohomology is \( H^*(P; \mathbb{Z}) \). Under the identifications described above we have for \( p = v_0 v_1 \ldots v_n \in \text{Ob}_n(P), \, n \geq 1 \) and \( x \in \mathbb{Z}^{\text{Ob}_n-1(P)} \)
\[
d_{n-1}(x)_p = \sum_{j=0}^n (-1)^{n-j} x_{v_0 v_{j+1} \ldots v_n}
\]

Proof. Take \( p \) and \( x \) as in the statement. The short exact sequence considered in Remark 3.7
\[
0 \to F_{n-1}(p) \xrightarrow{\lambda_p} \text{Ker}F_{n-1}(p) \xrightarrow{\pi_p} F_n(p) \to 0
\]
takes the form
\[
0 \to F_{n-1}(v_0 v_1 \ldots v_n) \xrightarrow{\lambda_p} \prod_{j=0}^n F_{n-1}(v_0 \ldots \hat{v}_j \ldots v_n) \xrightarrow{\pi_p} F_n(v_0 v_1 \ldots v_n) \to 0.
\]
By the definition of the differential \( d_{n-1} \) we have that \( d_{n-1}(x)_p = \pi_p(y) \), where \( y = x|_{\{v_0 \ldots \hat{v}_j \ldots v_n, j = 0, \ldots, n\}} \in \prod_{j=0}^n F_{n-1}(v_0 \ldots \hat{v}_j \ldots v_n) \). As in Remark 4.8 we identify \( \pi_p(y) \) with its unique pre-image \( y' \in \prod_{j=0}^n F_{n-1}(v_0 \ldots \hat{v}_j \ldots v_n) \) such that \( y'_{v_0 \ldots \hat{v}_j \ldots v_n} = 0 \) for \( j = 0, \ldots, n-1 \). Thus, if we obtain \( z \in \prod_{j=0}^n F_{n-1}(v_0 v_1 \ldots v_n) \) such that \( \lambda_p(z)_{v_0 \ldots \hat{v}_j \ldots v_n} = y_{v_0 \ldots \hat{v}_j \ldots v_n} \) for \( j = 0, \ldots, n-1 \), we will have \( \pi_p(y) \equiv y' = y - \lambda_p(z) \).

The rest of the proof is by induction on \( n \). For \( n = 1 \) the short exact sequence is
\[
0 \to \mathbb{Z}_{v_0 v_1} \xrightarrow{\lambda_p} \mathbb{Z}_{v_0} \times \mathbb{Z}_{v_1} \xrightarrow{\pi_p} \mathbb{Z}_{v_0 v_1} \to 0,
\]
where \( \lambda_p(n) = (n, n) \). Then, if \( y = (y_{v_0}, y_{v_1}) \), we have \( z = y_{v_1} \) and
\[
y' = y - \lambda_p(z) = (y_{v_0}, y_{v_1}) - (y_{v_1}, y_{v_1}) = (y_{v_0} - y_{v_1}, 0).
\]

Now we make the inductive step for \( n \geq 2 \). Recall that we want \( z \) such that \( \lambda_p(z)_{v_0 \ldots \hat{v}_j \ldots v_n} = y_{v_0 \ldots \hat{v}_j \ldots v_n} \) for \( j = 0, \ldots, n-1 \). Again by Remark 4.8 we identify
\[
(11) \quad F_{n-1}(v_0 v_1 \ldots v_n) \cong \prod_{j=0}^{n-1} F_{n-2}(v_0 v_1 \ldots \hat{v}_j \ldots v_n).
\]
and, for \( j = 0, \ldots, n - 1, \)
\[
F_{n-1}(v_0 v_1 \ldots \hat{v}_j \ldots v_n) \cong F_{n-2}(v_0 v_1 \ldots \hat{v}_j \ldots v_{n-1}),
\]
and
\[
F_{n-1}(v_0 v_1 \ldots v_{n-1}) \cong F_{n-2}(v_0 v_1 \ldots v_{n-2}).
\]
Recall that the “diagonal” \( \lambda_p \) is given by \( \prod_{j=0}^{n} F_{n-1}(v_0 \ldots v_n \to v_0 \ldots \hat{v}_j \ldots v_n) \). For \( j_0 \in \{0, \ldots, n-1\} \), and with the identifications (11) and (12), the map \( F_{n-1}(v_0 \ldots v_n \to v_0 \ldots \hat{v}_{j_0} \ldots v_n) \) becomes exactly the projection
\[
\prod_{j=0}^{n-1} F_{n-2}(v_0 v_1 \ldots \hat{v}_j \ldots v_{n-1}) \to F_{n-2}(v_0 v_1 \ldots \hat{v}_{j_0} \ldots v_{n-1}).
\]
Thus, \( z \in F_{n-1}(v_0 v_1 \ldots v_n) \) is such that \( z_{v_0 v_1 \ldots \hat{v}_j \ldots v_{n-1}} = y_{v_0 v_1 \ldots \hat{v}_j \ldots v_{n-1}} \) (again under the identifications (11) and (12)). The only thing left to compute is \( F_{n-1}(v_0 \ldots v_n \to v_0 v_1 \ldots v_{n-1}))(z) \) under the identifications (11) and (13). Define \( q = v_0 v_1 \ldots v_{n-1} \) and consider the short exact sequence given by Remark 3.7
\[
0 \to F_{n-2}(v_0 v_1 \ldots v_{n-1}) \xrightarrow{\lambda_q} \prod_{j=0}^{n-1} F_{n-2}(v_0 v_1 \ldots \hat{v}_j \ldots v_{n-1}) \xrightarrow{\pi_q} F_{n-1}(v_0 v_1 \ldots v_{n-1}) \to 0.
\]
Now, by the induction hypothesis, \( \pi_q(z) = \sum_{j=0}^{n-1} (-1)^{n-1-j} z_{v_0 v_1 \ldots \hat{v}_j \ldots v_{n-1}} \). Thus, finally,
\[
y' = y - z = (y_{v_0 v_1 \ldots v_{n-1}} - \sum_{j=0}^{n-1} (-1)^{n-1-j} z_{v_0 v_1 \ldots \hat{v}_j \ldots v_{n-1}}, 0, \ldots, 0)
\]
which equals
\[
y' = (\sum_{j=0}^{n} (-1)^{n-j} y_{v_0 v_1 \ldots v_j \ldots v_n}, 0, \ldots, 0).
\]

Before finishing this section we show how the symmetry of the category \( \Delta_n \) is useful when checking if a given family of subsets of objects \( K = \{K_n\}_{n \geq 0} \) fulfills the properties of a global covering family. Consider a simplex like poset \( \mathcal{P} \) and call \( \mathcal{J} \) to the local covering family for \( (p \downarrow \mathcal{P}^\text{op}) \simeq \Delta_n^\text{op} \) built in Lemma 6.3. Denote \( p = v_0 v_1 \ldots v_n \) and assume \( \text{deg}(p) = n \geq 1 \), then there are isomorphisms (Remark 3.8)
\[
F_{n-1}(v_0 v_1 \ldots v_n) \cong \prod_{j=0}^{n-1} F_{n-1}(v_0 \ldots \hat{v}_j \ldots v_n)
\]
and
\[
F_n(v_0 v_1 \ldots v_n) \cong F_{n-1}(v_0 v_1 \ldots v_{n-1}).
\]
By letting the symmetric group \( \Sigma_{n+1} \) act on \( \Delta_n^\text{op} \) we map \( \mathcal{J} \) to other local covering families for \( (p \downarrow \mathcal{P}^\text{op}) \). In particular we can permute \( v_n \) with any of the objects
\{v_0, \ldots, v_{n-1}\}. Recall that the functors $F$ do not depend on the local covering family chosen. We may summarize this as

**Remark 6.9.** Let $\mathcal{P}$ be a simplex-like poset and let $p = v_0v_1 \ldots v_n$ be an object with $\text{deg}(p) = n \geq 1$. The maps

$$F_{n-1}(v_0v_1 \ldots v_n) \to \prod_{j=0, j \neq j_0}^n F_{n-1}(v_0 \ldots \hat{v}_j \ldots v_n)$$

and

$$F_n(v_0v_1 \ldots v_n) \to F_{n-1}(v_0 \ldots \hat{v}_{j_0} \ldots v_{n-1})$$

are both isomorphisms for $j_0 = 0, \ldots, n$.

**Example 6.10.** In this example we consider the real projective plane $\mathbb{R}P^2$ and a poset model $\mathcal{P}$ of it. It has four 2-cells, six edges and three vertices:

This poset is a simplex-like poset and thus there is a local covering family for its opposite category $\mathcal{P}^{\text{op}}$. Notice that it is not a simplicial complex as, for example, there are two edges ($c$ and $e$) with common vertices $v$ and $x$. Applying Proposition 6.8 we obtain that its cohomology is computed by a cochain complex:

$$0 \to \mathbb{Z}^3 \xrightarrow{d_0} \mathbb{Z}^6 \xrightarrow{d_1} \mathbb{Z}^4 \to 0.$$

Proposition 6.8 gives a description of the differentials $d_0$ and $d_1$. Now consider the following family $\mathcal{K} = \{K_0, K_1, K_2\}$ of subsets of objects of $\mathcal{P}$:

$$K_0 = \{x\},$$

$$K_1 = \{d, e, f\},$$

$$K_2 = \{A, B, C, D\}.$$

Using Remark 6.9 it is straightforward that $\mathcal{K}$ is a global covering family. Its Betti numbers are $b_0^\mathcal{K} = 1$, $b_1^\mathcal{K} = 1$ and $b_2^\mathcal{K} = 1$ and thus, by Theorem 5.5, the cohomology of $\mathbb{R}P^2$ is that of a cochain complex

$$0 \to \mathbb{Z} \xrightarrow{\Omega_0} \mathbb{Z} \xrightarrow{\Omega_1} \mathbb{Z} \to 0.$$

The induced differentials are easily computed to be $\Omega_0 \equiv 0$ and $\Omega_1 \equiv \times 2$. 
7. Morse theory

In this section we show how any Morse function on a simplicial complex gives rise to a global covering family. The setup for this section is the discrete Morse theory for $CW$-complexes that Forman introduces in [12]. We will restrict ourselves to simplicial complexes, as the same author does in the user’s guide [13].

We start introducing the concept of Morse function in this context. Suppose $P$ is a given simplicial complex (in this section all simplicial complexes are assumed to have a finite number of vertices). Also, for a simplex $p$ we write $n$ to denote that $n$ is the dimension of $p$. A function $f : P \to \mathbb{R}$ is called a Morse function if for each $p^n \in P$

$$|\{q^{n+1} > p | f(q) \leq f(p)\}| \leq 1$$

and

$$|\{q^{n-1} < p | f(p) \leq f(q)\}| \leq 1.$$  

This means roughly that $f$ increases as the dimension of the simplices increase, with at most one exception, locally, at each simplex. We reproduce here the basic result [12, Lemma 2.5]

**Lemma 7.1.** Let $P$ be a simplicial complex and $f : P \to \mathbb{R}$ a Morse function. Then for any simplex $p^n$ either

$$|\{q^{n+1} > p | f(q) \leq f(p)\}| = 0$$

or

$$|\{q^{n-1} < p | f(p) \leq f(q)\}| = 0.$$  

If both conditions in the lemma hold for a simplex then we call it critical, i.e., for $p^n$ in $P$ we say that $p$ is critical if

$$|\{q^{n+1} > p | f(q) \leq f(p)\}| = 0$$

and

$$|\{q^{n-1} < p | f(p) \leq f(q)\}| = 0.$$  

Now, we come back to the setup of posets. Because $P$ is a simplicial complex then it is a simplex-like poset, and by Lemma [6.3] there is an adequate local covering family for $P^{op}$. In the next proposition we see how any Morse function $f : P \to \mathbb{R}$ gives rise to a global covering family on $P^{op}$.

**Proposition 7.2.** Let $P$ be a simplicial complex and $f : P \to \mathbb{R}$ a Morse function. Then there is a global covering family $K$ for $P^{op}$. Moreover, for each $n \geq 0$,

$$b_n^K = |\{\text{critical simplices of dimension } n\}|.$$  

**Proof.** According to Lemma [7.1] we can divide the simplices of dimension $n \geq 0$, $P_n$, in the following disjoint sets

$$C_n = \{p^n| |\{q^{n+1} > p | f(q) \leq f(p)\}| = 0 \text{ and } |\{q^{n-1} < p | f(p) \leq f(q)\}| = 0\},$$

$$D_n = \{p^n| |\{q^{n-1} < p | f(p) \leq f(q)\}| = 1\},$$

and

$$E_n = \{p^n| |\{q^{n+1} > p | f(q) \leq f(p)\}| = 1\}.$$
The set $C_n$ consists of the $n$-dimensional critical simplices. For $n \geq 1$ there is a bijection

$$E_{n-1} \rightarrow D_n$$

which maps $p^{n-1} \in E_{n-1}$ to the unique $n$-simplex $q^n$ with $f(q) \leq f(p)$. Now define $K_n = C_n \cup D_n$ (disjoint union) for each $n \geq 0$. Notice that if $\mathcal{K} = \{K_n\}_{n \geq 0}$ were a global covering family then

$$b^K_0 = |K_0| = |C_0| + |D_0| = |C_0|$$

as $D_0 = \emptyset$ and, for $n \geq 1$,

$$b^K_n = |K_n| - |\mathcal{P}_{n-1}| + |K_n|.$$

Because of the bijection $E_{n-1} \cong D_n$ this equals

$$b^K_n = |C_{n-1}| + |D_{n-1}| - (|C_{n-1}| + |D_{n-1}| + |E_{n-1}|) + |C_n| + |D_n| = |C_n|.$$

Fix $n \geq 0$. We show that the restriction map

$$w : \lim\inf F_n \rightarrow \prod_{p \in K_n} F_n(p)$$

is a monomorphism, where $F_n : \mathcal{P}^{op} \rightarrow \text{Ab}$ are the functors obtained from Lemma 4.3 applied to $\mathcal{P}^{op}$.

Take $\psi \in \lim\inf F_n = \text{hom}_{\text{Ab}}(c_{\mathbb{Z}}, F_n)$ which goes to zero by the restriction map $w$. If we prove that $\psi(q) = 0$ for each simplex of dimension $n+1$ then $\psi(p) = 0$ for every simplex of dimension $n$. To see this consider any simplex $p^n$. If $p$ is not the face of any $(n+1)$-simplex then $p \in K_n$ and $\psi(p) = 0$ (as $\psi$ is in the kernel of $w$).

If there exists $q^{n+1}$ with $q > p$ then $\psi(q) = 0$ by hypothesis and then $\psi(p) = \psi(q \rightarrow p)(\psi(q)) = \psi(q \rightarrow p)(0) = 0$. Finally, as $F_n$ is $n$-condensed, if $\psi(p) = 0$ for each $n$-simplex $p$ then $\psi = 0$.

Now we prove that $\psi(q) = 0$ for any $(n+1)$-simplex $q$. Recall that we are assuming that the set of vertices of $\mathcal{P}$, and thus $\mathcal{P}$ itself, is finite. We consider the total ordered finite set $f(\mathcal{P}_{n+1})$ and make induction on it. The base case is a $(n+1)$-simplex $q$ such that $f(q) = \min\{f(\mathcal{P}_{n+1})\}$. There are $n+2$ $n$-dimensional faces of $q$. If at least $n+1$ of these faces are in $K_n$ then $\psi(p) = 0$ for each of these faces and, by 6.9, $\psi(q) = 0$.

Now suppose that there exists at least two $n$-dimensional faces of $q$ which are not in $K_n$. By the definition of Morse function one of the values of $f$ in these faces is strictly smaller than $f(q)$. Call this face $p$ so $f(q) > f(p)$. As $p$ belongs to $E_n$ there exists an $(n+1)$-dimensional simplex $q'$ such that $q' > p$ and $f(q') \leq f(p)$. But then we obtain $f(q) > f(p) \geq f(q')$. This contradicts the fact that $f(q)$ is minimum among $(n+1)$-simplices and thus it has to be the case that there are at least $n+1$ $n$-dimensional faces of $q$ which are in $K_n$.

Next we do the induction step: take an $(n+1)$-dimensional simplex $q$. By definition of Morse function at least $n+1$ of the $n+2$ $n$-dimensional faces $p$ of $q$ satisfy $f(q) > f(p)$. For each one of these $n+1$ faces $p$ either $p \in K_n$ and $\psi(p) = 0$ or $p \in E_n$ and there exists an $(n+1)$-dimensional simplex $q'$ with $f(q') \leq f(p)$. In this last case we obtain $f(q) > f(p) \geq f(q')$ and then by the induction hypothesis $\psi(q') = 0$ and...
\[\psi(p) = \psi(q' \to p)(\psi(q')) = 0.\] Thus, \(\psi\) is zero in at least \(n + 1\) \(n\)-dimensional faces of \(q\) and by 6.9 \(\psi(q) = 0\).

Now we prove that for \(n \geq 1\) the map

\[w : \prod_{p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(p) \to \prod_{p \in K_n} F_n(p)\]

is pure. Take \(y \in \prod_{p \in K_n} F_n(p), m \geq 1\) and \(z \in \prod_{p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(p)\) with \(m \cdot y = w(z)\). We want to prove that \(m|z_p\) for each \(p \in \text{Ob}_{n-1}(P) \setminus K_{n-1}\). The base case is a \((n-1)\)-simplex \(p\) not in \(K_{n-1}\) such that \(f(p)\) is a minimum value. As \(p\) is in \(E_{n-1}\) there is \(q^n \in D_n \subset K_n\) with \(f(q) \leq f(p)\). For any of the \(n\) faces \(p'\) of \(q\) different from \(p\) we have \(f(q) > f(p')\) and thus \(f(p) \geq f(q) > f(p')\). As \(f(p)\) is minimum then \(p' \in K_{n-1}\) and, by Remark 6.9, \(m|z_p\) as \(m|w(z)_q\). The induction step runs in a similar way to the earlier case. \(\square\)

8. Webb’s conjecture

In [7], Brown introduces the so called Brown’s complex of a finite group. Given a finite group \(G\) and a prime \(p\), its associated Brown’s complex \(S_p(G)\) is the geometrical realization of the poset of non trivial \(p\)-subgroups of \(G\). Webb conjectured that the orbit space \(S_p(G)/G\) (as topological space) is contractible. This conjecture was first proven by Symonds in [27], generalized for blocks by Barker [2, 3] and extended to arbitrary (saturated) fusion systems by Linckelmann [21].

The works of Symonds and Linckelmann prove the contractibility of the orbit space by showing that it is simply connected and acyclic, and invoking Whitehead’s Theorem. Both proofs of acyclicity work on the subposet of normal chains (introduced by Knorr and Robinson [19] for groups). Symonds uses that the subposet of normal chains is \(G\)-equivalent to Brown’s complex, as was proved by Thévenaz and Webb in [28]. Linckelmann proves on his own that, also for fusion systems, the orbit space and the orbit space on the normal chains has the same cohomology [21 Theorem 4.7]. In this chapter we shall apply the results of Section 5 to prove, in an alternative way, that the orbit space on the normal chains is acyclic.

Let \((S, F)\) be a saturated fusion system where \(S\) is a \(p\)-group [6]. Consider its subdivision category \(S(F)\) (see [21, Section 2]) and the poset \([S(F)]\). An object in \([S(F)]\) is an \(F\)-isomorphism class of chains

\([Q_0 < Q_1 < \ldots < Q_n]\)

where the \(Q_i\)’s are subgroups of \(S\). The subcategory \(([S(F)] \downarrow [Q_0 < \ldots < Q_n])\) has objects \([Q_{i_0} < \ldots < Q_{i_m}]\) with \(0 \leq m \leq n\) and \(0 \leq i_1 < i_2 < \ldots < i_m \leq n\) (see [21].
For the graded poset \([S(\mathcal{F})]\) denoted by \(S_Q\) of \(\geq n\) for \(i = 0, \ldots, n\). Also, denote by \([S_Q(\mathcal{F})]\) the subdivision category of \(S_Q(\mathcal{F})\), which is a sub-poset of \([S(\mathcal{F})]\). By \([21\text{ Theorem 4.7}], H^*([S(\mathcal{F})]; \mathbb{Z}) = H^*([S_Q(\mathcal{F})]; \mathbb{Z})\). Our goal in this section is to prove that \(H^*([S_Q(\mathcal{F})]; \mathbb{Z}) = 0\) for \(n \geq 1\) and \(H^0([S_Q(\mathcal{F})]; \mathbb{Z}) = \mathbb{Z}\). It is straightforward that \([S_Q(\mathcal{F})]\) is a simplex-like poset and thus, by Lemma \([6.5\text{ Appendix}]\) there exists an adequate covering family for the graded poset \([S_Q(\mathcal{F})]^{op}\). We shall build a global covering family \(K\) for \([S_Q(\mathcal{F})]^{op}\). This family \(K\) will verify \(b^K_n = 0\) for \(n \geq 1\) and \(b^K_0 = 1\), and so Theorem \([5.5\text{ Theorem}]\) will give the desired result.

The definition of the global covering family is as follows, and it is related with the pairing defined by Linckelmann in \([21\text{ Definition 5.5}]\). The notion of paired chains was used by Knörr and Robinson in several forms throughout \([19\text{ Definition 8.1}]\).

**Definition 8.1.** For the graded poset \([S_Q(\mathcal{F})]^{op}\) define the subsets \(K = \{K_n\}_{n \geq 0}\) by

\[
K_n = \{[Q_0 < \ldots < Q_n] \mid [Q_0 < \ldots < Q_n] = [Q'_0 < \ldots < Q'_n] \Rightarrow \cap_{i=0}^n N_S(Q_i) = Q'_n\}.
\]

The fact that \(K\) is defined through a pairing provides (see below) a bijection \(\psi : \text{Ob}_n([S_Q(\mathcal{F})]^{op}) \to K_n \to K_{n+1}\) for each \(n \geq 0\). This proves that \(b^K_n = 0\) for each \(n \geq 1\). Moreover, \(K_0 = \{[S]\}\) and thus \(b^K_0 = 1\). Next we prove that the family \(K = \{K_n\}_{n \geq 0}\) defined in \([8.1\text{ Theorem}]\) is a global covering family for \([S_Q(\mathcal{F})]^{op}\). We use terminology and results from \([6\text{ Appendix}]\).

For any chain \(Q_0 < \ldots < Q_n\) in \(S_Q(\mathcal{F})\) define the following subgroup of automorphisms of \(Q_n\)

\[
A_{Q_0 < \ldots < Q_n} = \{\alpha \in \text{Aut}(Q_n) \mid \alpha(Q_i) = Q_i, i = 0, \ldots, n\}.
\]

Then,

\[
N^A_{Q_0 < \ldots < Q_n}(Q_n) = \cap_{i=0}^n N_S(Q_i).
\]

If \([Q_0 < \ldots < Q_n] = [Q'_0 < \ldots < Q'_n]\) then there is \(\varphi \in \text{Iso}_\mathcal{F}(Q_n, Q'_n)\) with \(Q'_i = \varphi(Q_i)\) for \(i = 0, \ldots, n\) and

\[
\varphi A_{Q_0 < \ldots < Q_n} \varphi^{-1} = A_{Q'_0 < \ldots < Q'_n}.
\]

By \([6\text{ A.2(a)]}, Q_n\) is fully \(A_{Q_0 < \ldots < Q_n}\)-normalized if and only if \(|N^A_{Q_0 < \ldots < Q_n}(Q_n)|\) is maximum among \(|N^A_{Q'_0 < \ldots < Q'_n}(Q_n)|\) with \([Q'_0 < \ldots < Q'_n] = [Q_0 < \ldots < Q_n]\), i.e., if and only if \(|\cap_{i=0}^n N_S(Q_i)|\) is maximum among \(|\cap_{i=0}^n N_S(Q'_i)|\) with \([Q_0 < \ldots < Q_n] = [Q'_0 < \ldots < Q'_n]\).
\[Q_0 < \ldots < Q_n\]. Notice that in the isomorphism class of chains \([Q_0 < \ldots < Q_n]\) there is always a representative \(Q'_0 < \ldots < Q'_n\) which is fully \(A_{Q_0} \ldots Q_n\)-normalized, and that any two representatives \(Q_0 < \ldots < Q_n\) and \(Q'_0 < \ldots < Q'_n\) of \([Q_0 < \ldots < Q_n]\) which are fully \(A_{Q_0} \ldots Q_n\)-normalized and fully \(A_{Q_0} \ldots Q_n\)-normalized respectively verify
\[
| \cap_{i=0}^{n} N_S(Q'_i) | = | \cap_{i=0}^{n} N_S(Q''_i) |.
\]

Thus, Definition 8.1 is equivalent to

**Definition 8.2.** For the graded poset \([S(F)]^{op}\) define the subsets \(K_n = \{(Q'_0 < \ldots < Q'_n) \mid Q'_n \text{ fully } A_{Q'_0} \ldots Q'_n\text{-normalized and } \cap_{i=0}^{n} N_S(Q'_i) = Q'_n\}\).

**Lemma 8.3.** For any \(n \geq 0\) there is a bijection
\[\psi : \text{Ob}_n([S(F)]^{op}) \setminus K_n \rightarrow K_{n+1}.
\]

**Proof.** Take \([Q_0 < \ldots < Q_n] \in \text{Ob}_n([S(F)]^{op}) \setminus K_n\) and a representative \(Q'_0 < \ldots < Q'_n\) which is fully \(A_{Q'_0} \ldots Q'_n\)-normalized. Then \(\cap_{i=0}^{n} N_S(Q'_i) > Q'_n\). Define
\[\psi([Q_0 < \ldots < Q_n]) = [Q_0 < \ldots < Q'_n < \cap_{i=0}^{n} N_S(Q'_i)].\]

The proof is divided in four steps:

a) \(\psi\) is well defined. Take another representative \(Q''_0 < \ldots < Q''_n\) which is fully \(A_{Q''_0} \ldots Q''_n\)-normalized. Then, by [6, A.2(c)], there is a morphism
\[\varphi \in \text{Hom}_X(N_S^{A_{Q''_0} \ldots Q''_n}(Q'_n), N_S^{A_{Q''_0} \ldots Q''_n}(Q''_n))\]
with \(\varphi(Q'_i) = Q''_i\) for \(i = 0, \ldots, n\). As \(\cap_{i=0}^{n} N_S(Q'_i) = \cap_{i=0}^{n} N_S(Q''_i)\) then \(\varphi\) is an isomorphism onto \(N_S^{A_{Q''_0} \ldots Q''_n}(Q'_n)\) and thus
\[Q'_0 < \ldots < Q'_n < \cap_{i=0}^{n} N_S(Q'_i)] = [Q'_0 < \ldots < Q'_n < \cap_{i=0}^{n} N_S(Q''_i)].\]

b) \(\psi([Q_0 < \ldots < Q_n])\) belongs to \(K_{n+1}\). We have \(\psi([Q_0 < \ldots < Q_n]) = [Q_0 < \ldots < Q'_n < \cap_{i=0}^{n} N_S(Q'_i)]\) where \(Q'_0 < \ldots < Q'_n\) is fully \(A_{Q'_0} \ldots Q'_n\)-normalized. Take any representative \(Q_0 < \ldots < Q'_n < Q''_n+1\) in \([Q'_0 < \ldots < Q'_n < \cap_{i=0}^{n} N_S(Q'_i)]\). If it were the case that \(Q''_n+1 < \cap_{i=0}^{n+1} N_S(Q'_i)\) then we would have
\[\cap_{i=0}^{n} N_S(Q'_i) \cong Q''_n+1 < \cap_{i=0}^{n+1} N_S(Q'_i) = \cap_{i=0}^{n} N_S(Q''_i),\]
which is in contradiction with \(Q'_0 < \ldots < Q'_n\) being fully \(A_{Q'_0} \ldots Q'_n\)-normalized.

c) \(\psi\) is injective. Suppose we have \([Q_0 < \ldots < Q_n]\) and \([R_0 < \ldots < R_n]\) with
\[Q'_0 < \ldots < Q'_n < \cap_{i=0}^{n} N_S(Q'_i)] = [R'_0 < \ldots < R'_n < \cap_{i=0}^{n} N_S(R'_i)].\]
Then \([R_0 < \ldots < R_n] = [R'_0 < \ldots < R'_n] = [Q'_0 < \ldots < Q'_n] = [Q_0 < \ldots < Q_n].\)

d) \(\psi\) is surjective. Take \([Q_0 < \ldots < Q_n < Q_{n+1}]\) in \(K_{n+1}\). We check that
\[\psi([Q_0 < \ldots < Q_n]) = [Q_0 < \ldots < Q_n < Q_{n+1}].\]

Take a representative \(Q'_0 < \ldots < Q'_n\) in \([Q_0 < \ldots < Q_n]\) which is fully \(A_{Q'_0} \ldots Q'_n\)-normalized. Then \([Q_0 < \ldots < Q_n] \in \text{Ob}_n([S(F)]^{op}) \setminus K_n\) and \(\psi([Q_0 < \ldots < Q_n]) = [Q'_0 < \ldots < Q'_n < \cap_{i=0}^{n} N_S(Q'_i)].\)

Then, by [6, A.2(c)], there is
\[\varphi \in \text{Hom}_X(N_S^{A_{Q_0} \ldots Q_n}(Q_0), N_S^{A_{Q'_0} \ldots Q'_n}(Q'_n))\]
Lemma 8.4. The family $K = \{K_n\}_{n \geq 0}$ defined in $[S,F]$ is a global covering family for $[S_{\leq}(F)]^{op}$.

Proof. We start proving that for any $n \geq 0$ the map

$$\lim_{\longleftarrow} F_n \rightarrow \prod_{i \in K_n} F_n(i)$$

is a monomorphism. Take $\psi \in \lim_{\longleftarrow} F_n$ such that $\psi(i) = 0$ for each $i \in K_n$. If there is no object of degree greater than $n$ then $K_n = \text{Ob}_n([S_{\leq}(F)]^{op})$ and we are done. If not, we prove that $\psi(j) = 0$ for each $j$ of degree $n + 1$ by induction on $|Q_{n+1}|$. This is enough to see that $\psi$ is zero as $F_n$ is $n$-condensed. The base case is $j = [Q_0 < \ldots < Q_{n+1}]$ with $|Q_{n+1}|$ maximal. This implies that $J_j \subseteq K_n$. Then $\psi(j)$ goes to zero by the monomorphism

$$F_n(j) \rightarrow \prod_{i \in J_j} F_n(i),$$

and thus $\psi(j) = 0$. For the induction step consider $j = [Q_0 < \ldots < Q_{n+1}]$ and $j' = [Q_0 < \ldots < Q_l < \ldots < Q_{n+1}] \in J_j$ with $0 \leq l < n$. Then, either $j' \in K_n$ and $\psi(j') = 0$, or $j' \notin K_n$ and there is an arrow in $[S_{\leq}(F)]^{op}$

$$j'' = [Q_0 < \ldots < Q_l < \ldots < Q_{n+1} \cap_{i=0,i \neq l} N_S(Q_i)] \rightarrow j' = [Q_0 < \ldots < Q_l < \ldots < Q_{n+1}].$$

In the latter case $\psi(j'') = 0$ by the induction hypothesis, and thus $\psi(j') = 0$ too. As before, since the map $F_n(j) \rightarrow \prod_{i \in J_j} F_n(i)$ is a monomorphism, $\psi(j) = 0$.

Now we prove that for any $n \geq 1$ the map

$$\omega : \prod_{i \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_n(i) \rightarrow \prod_{i \in K_n} F_n(i)$$

is pure. Take $y \in \prod_{i \in K_n} F_n(i), m \geq 1$ and $x \in \prod_{i \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(i)$ with

$$m \cdot y = \omega(x).$$

We want to find $x'$ with $m \cdot x' = x$. We prove that $x_i$ is divisible by $m$ for each $i = [Q_0 < \ldots < Q_{n-1}] \in \text{Ob}_{n-1}(P) \setminus K_{n-1}$ by induction on $|Q_{n-1}|$. The base case is $i = [Q_0 < \ldots < Q_{n-1}]$ with $|Q_{n-1}|$ maximal. Consider the arrow in $[S_{\leq}(F)]^{op}$

$$j = [Q_0' < \ldots < Q_{n-1}' \cap_{i=0,i \neq l} N_S(Q_i')] \rightarrow [Q_0 < \ldots < Q_{n-1}].$$

As $Q_{n-1}$ is maximal then $J_j \subseteq K_{n-1}$. Then $m \cdot y_j = \omega(x)_j$ is the image of $(x_i, 0, \ldots, 0)$ by the map

$$\text{Ker} F_{n-1}(j) = \prod_{i \in (j \setminus S_{\leq}(F))_{n-1}} F_{n-1}(i) \xrightarrow{\pi_j} F_n(j).$$
As $F_n(j) \cong \prod_{t \in (j|S_0(\mathcal{F}))_{n-1}\setminus J_n^t} F_{n-1}(l) = F_{n-1}(i)$ by Remark 3.8 then $m$ divides $x_i$.

For the induction step consider $i = [Q_0 < \ldots < Q_{n-1}] \in \text{Ob}_{n-1}(P) \setminus K_{n-1}$ and $j = [Q'_0 < \ldots < Q'_{n-1} < \bigcap_{i=0}^{n-1} N_S(Q'_i)]$. As before, $m \cdot y_j = \omega(x)_j$ is the image of $\tilde{x} = x|\left(j \downarrow [S_0(\mathcal{F})]\right)_{n-1}$ by the map

$$\text{Ker}_{F_{n-1}}(j) = \prod_{t \in (j|S_0(\mathcal{F}))_{n-1}} F_{n-1}(i) \xrightarrow{\pi_j} F_n(j).$$

By Remark 3.8 again,

$$m \cdot y_j = \pi_j(\tilde{x} - (\lambda_j \circ s_j)(\tilde{x})) = \pi_j((x_i - (\lambda_j \circ s_j)(x|J_n^j), 0, \ldots, 0)).$$

Now, by the induction hypothesis, for each $l \in J_n^j$, either $l \in K_{n-1}$ and $x_l = 0$, either $l \notin K_{n-1}$ and thus, by the induction hypothesis, $m$ divides $x_l$. Then $m$ divides $x|J_n^j$ and so, by the equation above and the isomorphism $F_n(j) \cong F_{n-1}(i)$, $m$ divides $x_i$ too. \hfill $\square$

### 9. Coxeter groups

It is well known that the Coxeter complex associated to a finite Coxeter group $(W, S)$ is a simplicial decomposition of a sphere of dimension $|S| - 1$, and thus has the homotopy type of a sphere. In case $W$ is infinite then this complex becomes contractible. In this section we prove that the cohomology of the Coxeter complex is that of a sphere if $W$ is finite or trivial if $W$ is infinite. We do it by using the techniques of earlier sections. More precisely, we construct a global covering family $\mathcal{K}$ for the Coxeter complex with $b^K_0 = b^K_{|S|-1} = 1$ and $b^K_n = 0$, $n \neq 0$, $|S| - 1$ for $W$ finite, and with $b^K_0 = 1$ and $b^K_n = 0$, $n \neq 0$ for $W$ infinite. We will use general facts about Coxeter groups which can be found in [5], [17] or [14], as well as we will recall some basic definitions and statements.

Let $(W, S)$ be a Coxeter system where $W$ is a Coxeter group and $S = \{s_1, \ldots, s_N\}$ is a set of generators (which we always assume is finite). For any word $w \in W$ its length $l(w)$ is the minimum number of generators from $S$ that are needed to write it. If $W$ is finite there is a unique element of maximal length which we denote by $w_0$. For every subset $I \subset S$ we have the parabolic subgroup $W_I \leq W$ generated by the generators belonging to $I$. We also define the following subset

$$W^I = \{w \in W|l(ws) > l(w) \text{ for all } s \in I\}.$$  

In [17, 5.12] and [5] exercise 3, p. 37] is proven the following:

**Lemma 9.1.** Fix $I \subset S$. Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that $w = uv$. Then $l(w) = l(u) + l(v)$. Moreover, $u$ is the unique element of smallest length in the coset $wW_I$.

From this lemma it is straightforward that there is a bijection

$$W/W_I \rightarrow W^I$$

which sends the coset $wW_I$ to $W/W_I$ to the unique element $u \in W^I$ of smallest length in the coset $wW_I$. Notice that $uW_I = wW_I$ for $u$ and $w$ as in the lemma. Next we
describe the Coxeter complex associated to \((W, S)\): it is a simplicial complex which simplices are the cosets of proper parabolic subgroups with inclusion reversed:

\[ wW_I \leftarrow w'W_{I'} \iff I \subset I' \text{ and } w^{-1}w' \in W. \]

The dimension of the simplex \(wW_I\) is \(n = |S| - |I| - 1\). Thus, vertices correspond to maximal subsets \(I\) of \(S\) and facets, i.e., maximal dimensional faces, correspond to \(I = \emptyset\). Notice that it is a pure simplicial complex. By the bijection \[(14)\] we can write the coset \(wW_I\) as \(uW_I\) with \(u \in W^I\), where \(w\) and \(u\) are as in the previous lemma. We will do this in the rest of the section. For any \(u \in W\) define (cf. \[17, 1.11\])

\[ S_u = \{ s \in S | l(us) > l(u) \} \]

and let \(s_u = \max\{S_u\}\) be the maximum element in \(S_u\) with respect to the order in \(S\): \(s_1 < \ldots < s_N\). Notice that \(S_u = S\) if and only if \(u = 1\). Moreover, if \(W\) is finite then \(S_u = \emptyset\) if and only \(u = w_0\), the unique word of maximal length. It is also clear that \[(15)\]

\[ u \in W^I \iff I \subseteq S_u. \]

Before defining the global covering family notice that, by Lemma 6.5, there is an adequate local covering family for \(P^{op}\). Now, for each \(n \geq 0\) define

\[ K_n = \{ uW_I | |I| = |S| - 1 - n \text{ and } s_u \notin I \}. \]

The condition \(|I| = |S| - 1 - n\) in the definition above just states that \(uW_I\) correspond to a simplex of dimension \(n\).

**Lemma 9.2.** The family \(\mathcal{K} = \{K_n\}_{n \geq 0}\) is a global covering family for \(P^{op}\). Moreover, \(b^K_n = 1\). If \(W\) is finite then \(b^K_{|S|-1} = 1\) and \(b^K_n = 0\) for \(n \neq 0, |S| - 1\). If \(W\) is infinite then \(b^K_n = 0\) for \(n \neq 0\).

By Theorem 5.5 this lemma imply that the integral cohomology of \(P\) is that of a sphere of dimension \(|S| - 1\) if \(W\) is finite, and that of a point if \(W\) is infinite.

**Proof.** First we compute the numbers \(b^K_n\) and afterwards we will prove \(\mathcal{K}\) satisfies Definition 5.1. Fix \(u \in W\) and consider the contribution which it makes to \(P^{op}\). By Equation \((15)\) \(u \in W^I\) if and only if \(I \subseteq S_u\). Thus, denoting by \(P_u\) the sub-poset with objects \(\{uW_I, I \subseteq S_u\}\) we have that \(P\) is contained in the disjoint union \(\bigcup_{u \in W} P_u\). It is a combinatorial exercise (cf. Lemma 6.4) that there is a bijection between \(K_{n+1} \cap P_u\) and \(\{P_n \setminus K_n\} \cap P_u\) for \(n \geq 0\) if \(u \neq w_0\). This gives, in case \(W\) is infinite, \(b^K_n = 0\) for \(n \geq 1\). If \(W\) is finite then \(P_{w_0} = \{w_0W_\emptyset\}\) and \(w_0W_\emptyset\) is a facet which belongs to \(K_{|S|-1}\). Then we obtain \(b^K_{|S|-1} = 1\) and \(b^K_n = 0\) for \(|S| - 1 > n \geq 1\). Finally, it is an easy consequence of the definition that \(K_0 = \{1 \cdot W_{S \setminus \{s_n\}}\}\) and thus \(b^K_0 = 1\).

Now fix \(n \geq 0\). We show that the restriction map

\[ \lim_{\leftarrow} F_n \rightarrow \prod_{uW_I \in K_n} F_n(uW_I) \]

is a monomorphism, where \(F_n : P^{op} \rightarrow Ab\) are the functors obtained from Lemma 4.3 applied to \(P^{op}\).

Take \(\psi \in \lim_{\leftarrow} F_n = \text{hom}_{Ab}(cz, F_n)\) which is mapped to zero by the restriction map. To prove that \(\psi = 0\) it is enough to prove that \(\psi(uW_I) = 0\) for each simplex \(uW_I\) of
dimension \( n + 1 \) (as in the proof of Theorem 7.2). We do this by induction on the length \( l(u) \).

We start with \( uW_I \) with \( l(u) = 0 \), i.e., \( u = 1 \). Then \( S_u = S \). If \( s_u = s_N \notin I \), i.e., \( u \in K_{n+1} \), then there \( n+1 \) \( n \)-faces of \( u \) which are in \( K_n \). These are the cosets \( W_J \) where \( J = I \cup \{s\} \) with \( s \in S \setminus \{I \cup \{s_N\}\} \). The remaining face, i.e., \( W_J \) with \( J = I \cup \{s_N\} \), is not in \( K_n \). By Remark 6.9, \( \psi(W_J) = 0 \). Now assume that \( s_u = s_N \in I \), i.e., \( u \notin K_{n+1} \). Consider any of the \( n+2 \) \( n \)-faces \( W_J \) of \( W_I \), where \( J = I \cup \{s\} \) with \( s \in S \setminus I \). Then \( W_J \) is also a face of then \( n+1 \) simplex \( W_I\setminus\{s_N\}\cup\{s\} \). By the preceding argument for the case \( W_I\setminus\{s_N\}\cup\{s\} \in K_{n+1} \) we obtain \( \psi(W_I\setminus\{s_N\}\cup\{s\}) = 0 \), and thus \( \psi(W_J) = 0 \). Then \( \psi \) is zero in all the faces of \( W_I \). By Remark 6.9 again we obtain \( \psi(W_I) = 0 \).

Next we do the induction step: take an \( n+1 \) dimensional simplex \( uW_I \) with \( l(u) > 0 \). First assume that \( u \in K_{n+1} \), i.e., \( s_u \notin I \). The faces of \( uW_I \) are the \( n+2 \) \( n \)-simplices \( uW_J \) where \( J = I \cup \{s\} \) and \( s \in S \setminus I \). Notice that, as \( I \subseteq S_u \), \( S \setminus I = S_u \setminus I \cup S \setminus S_u \), where the union is disjoint. Take first \( s \in S \setminus S_u \). Then \( l(us) < l(u) \) and, as \( us \in uW_J \), the unique element \( u' \) of minimal length in \( uW_J \) is different from \( u \) and has smaller length. Then \( uW_J = u'W_I \) is also a face of \( uW_I \). By the induction hypothesis and because \( l(u') < l(u) \) we have \( \psi(uW_I) = 0 \) and thus \( \psi(uW_J) = 0 \). Now take \( s \in S_u \setminus I \). Then \( uW_J \in K_n \) unless \( s = s_u \). Then \( \psi \) is zero in all but one face of \( uW_I \) and, by Remark 6.9, \( \psi(uW_I) = 0 \).

Now assume that \( u \notin K_{n+1} \), i.e., \( s_u \in I \), and take a face \( uW_J \) as before. If \( J = I \cup \{s\} \) with \( s \in S \setminus S_u \) then arguing as before we obtain that \( \psi(uW_J) = 0 \). If \( J = I \cup \{s\} \) with \( s \in S_u \setminus I \) then \( uW_J \) is also a face of the \( n+1 \) simplex \( uW_I\setminus\{s_N\}\cup\{s\} \). By the preceding argument for the case \( uW_I\setminus\{s_N\}\cup\{s\} \in K_{n+1} \) we have \( \psi(uW_I\setminus\{s_N\}\cup\{s\}) = 0 \), and thus \( \psi(W_J) = 0 \). Then \( \psi \) takes the value zero in all the faces of \( uW_I \) and by Remark 6.9 \( \psi(uW_I) = 0 \).

Finally, fix \( n \geq 1 \). The proof that the map

\[
\psi : \prod_{uW_I \in \mathcal{F}_{n-1} \setminus K_{n-1}} F_{n-1}(uW_I) \to \prod_{uW_{I'} \in K_n} F_n(uW_{I'})
\]

is pure is made as in Theorem 7.2. It uses induction on the length \( l(u) \) for \( (n-1) \)-dimensional simplices \( uW_I \), and the base case is \( u = 1 \). \( \square \)

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