Control of Dams When the Input Is a Lévy Type Process

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Abstract

Zuckermann [10] considers the problem of optimal control of a finite dam using \( P_{\lambda,\tau}^M \) policies, assuming that the input process is Wiener with drift term \( \mu \geq 0 \). Lam and Lou [7] treat the case where the input is a Wiener process with a reflecting boundary at its infimum, with drift term \( \mu \geq 0 \), using the long-run average and total discounted cost criteria. Attia [3] obtains results similar to those of Lam and Lou, through simpler and more direct methods. Bae et al. [5] treat the long-run average cost case when the input process is a compound Poisson process with a negative drift. In this paper we unify and extend the results of these authors.

Keywords: \( P_{\lambda,\tau}^M \) policies; Lévy processes; exit times; Poisson processes; resolvent; total discounted and long-run-average costs.

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1. Introduction and summary

Lam and Lou [7] consider the control of a finite dam, with capacity \( V > 0 \), where the water input is a Wiener process, using \( P_{\lambda,\tau}^M \) policies. In these policies the water release rate is assumed to be zero until the water reaches level \( \lambda > 0 \). As soon as this happens the water is released at rate \( M > 0 \) until the water content reaches level \( \tau > 0 \), \( \lambda > \tau \). They use the total discounted as well as the long-run average cost criteria. Attia [3] obtains the same results of Lam and Lou, using simpler methods. Lee and Ahn [8] consider the long-run average cost case, for the \( P_{\lambda,0}^M \) policy, when the water input is a compound Poisson process. Abdel-Hameed [1] treats the case where the water input is a compound Poisson process with a positive drift. He obtains the total discounted as well as the long-run average costs. Bae et al. [4] consider the \( P_{\lambda,\tau}^M \) policy in assessing the workload of an M/G/1 queuing system. Bae et al. [5] consider the log-run average cost for \( P_{\lambda,\tau}^M \) policy in a finite dam, when the input process is a compound Poisson process, with a negative drift term. At any time, the release rate can be increased from 0 to \( M \) with a starting cost \( K_1 M \), or decreased from \( M \) to zero with a closing cost \( K_2 M \). Moreover, for each unit of output, a reward \( R \) is received. Furthermore, there is a penalty cost which is a bounded measurable function on the state space of the content process. In this paper
we treat the more general cases where the input process is assumed to be a spectrally positive \( \text{Lévy} \) or a spectrally positive \( \text{Lévy} \) process reflected at its infimum.

For any process \( Y = \{Y_t, t \geq 0\} \) with state space \( E \), any Borel set \( A \subset E \) and any functional \( f \), \( E_y(f) \) denotes the expectation of \( f \) conditional on \( Y_0 = y \), \( P_y(A) \) denotes the corresponding probability measure and \( I_A(\cdot) \) is the indicator function of the set \( A \). In the sequel we will write indifferently \( P_0 \) or \( P \) and \( E_0 \) or \( E \). Throughout, we let \( R = (-\infty, \infty), R_+ = [0, \infty), N = \{1, 2, \ldots\} \) and \( N_+ = \{0, 1, \ldots\} \). For \( x, y \in R \), we define \( x \vee y = \max \{x, y\} \) and \( x \wedge y = \min \{x, y\} \).

For every \( t \geq 0 \), we define \( Y_t = \inf_{0 \leq s \leq t} (Y_s, \wedge 0), Y_t = \sup_{0 \leq s \leq t} (Y_s, \vee 0) \).

We will use the term "increasing" to mean "non-decreasing" throughout this paper.

In Section 2, we discuss the cost functionals. In Section 3 we define the input processes and discuss their properties. In Section 4 we obtain formulas needed for computing the cost functionals using the total discounted as well as the long-run average cost cases. In section 5 we discuss the special cases where the input process is a Gaussian process, a Gaussian process reflected at its infimum and a spectrally positive \( \text{Lévy} \) process of bounded variation.

2. The cost functionals

For each \( t \in R_+ \), let \( Z_t \) be the dam content at time \( t \), \( Z = \{Z_t, t \in R_+\} \). We define the following sequence of stopping times:

\[
\begin{align*}
\hat{T}_0 &= \inf\{t \geq 0 : Z_t \geq \lambda\}, & \hat{T}_0^* &= \inf\{t \geq \hat{T}_0 : Z_t = \tau\}, \\
\hat{T}_n &= \inf\{t \geq \hat{T}_{n-1} : Z_t \geq \lambda\}, & \hat{T}_n^* &= \inf\{t \geq \hat{T}_n : Z_t = \tau\}, & n = 1, 2, \ldots \quad (2.1)
\end{align*}
\]

It follows that the process \( Z \) is a delayed regenerative process with regeneration points \( \{\hat{T}_n, n = 0, 1, \ldots\} \). The regeneration cycle is defined to be the time between successive regeneration points. During a given cycle, the release rate is either 0 or \( M \). When the release rate is zero, the process \( Z \) is either a spectrally positive \( \text{Lévy} \) process (denoted by \( I \)) or a spectrally positive \( \text{Lévy} \) process reflected at its infimum (denoted by \( Y \)), and remains so till the water reaches level \( \lambda \); from then until it reaches level \( \tau \) the content process behaves like the process \( \overset{(M)}{I} = I - M \) reflected at \( V \), we denote this process by \( \overset{*}{I} \). It follows that, for each \( t \geq 0 \),

\[
\overset{*}{I}_t = \overset{(M)}{I}_t - \sup_{0 \leq s \leq t} (\overset{(M)}{I}_t - V) \wedge 0. \quad (2.2)
\]

When the release rate is 0, the dam is maintained at a net maintenance cost rate \( g \), where \( g \) is a bounded measurable function on \( (l, \lambda) \), where \( l \) is the lower
bound of the state space of the process $I$. Furthermore, maintenance of the
dam when the release rate is $M$ is done at a cost rate $g^*$, where $g^*$ is a bounded
measurable function on $(\tau, V]$.

For $x < \lambda$ and $\alpha \in R_+$, the discounted cost during the interval $[0, T_0)$,
denoted by $C_\alpha(x, \lambda, 0)$, is given as follows

$$C_\alpha(x, \lambda, 0) = E_x \int_0^{\hat{T}_0} e^{-\alpha t} g(Z_t) dt. \quad (2.3)$$

Define, $T^*_\tau = \hat{T}_0 - \hat{T}_0$, when the release rate is $M$, starting at $x \geq \tau$,
the expected discounted cost in the interval $[0, T^*_\tau)$, denoted by $C_\alpha(x, \tau, M)$ is
given as follows

$$C_\alpha(x, \tau, M) = E_x \int_0^{T^*_\tau} e^{-\alpha t} g^*(Z_t) dt. \quad (2.4)$$

We now discuss the computations of the cost functionals using the total
discounted cost as well as the long-run average cost criteria. Let $C_\alpha(x)$ be the
expected cost during the first cycle, $[0, \hat{T}_0)$, when $Z_0 = x$. From the definition
of the $P_{\lambda, \tau}$ policy, it follows that for $\lambda < x < V$

$$C_\alpha(x) = M\{K_1 - RE_x \int_0^{T^*_\tau} e^{-\alpha t} dt\} + C_\alpha^g(\lambda, \tau), \lambda < x < V \quad (2.5)$$

and for $x \leq \lambda$

$$C_\alpha(x) = M\{K_2 + K_1 E_x[e^{-\alpha \hat{T}_0}] - \frac{R}{\alpha}\{E_x[e^{-\alpha \hat{T}_0}] - E_x[e^{-\alpha T_0}]\} + C_\alpha(x, \lambda, 0) + E_x[e^{-\alpha \hat{T}_0}C_\alpha((Z_{\lambda} \wedge \tau), \tau, M)]\}. \quad (2.6)$$

Let $C_\alpha(\lambda, \tau)$ and $C(\lambda, \tau)$ denote the total discounted cost and the long-
run average cost, respectively. By modifying the result in Abdel-Hameed [1], it
follows that

$$C_\alpha(\lambda, \tau) = C_\alpha(x) \frac{E_x[\exp(-\alpha \hat{T}_0)]C_\alpha(\tau)}{1 - E_x[\exp(-\alpha \hat{T}_0)]}, \quad (2.7)$$

and

$$C(\lambda, \tau) = \frac{M(K + RE_0[T_0^{T^*_\tau}]) + C_\alpha(x, \lambda, 0) + E_x[C_\alpha((Z_{\lambda} \wedge \tau), \tau, M)]}{E_x[\hat{T}_0]} - RM. \quad (2.8)$$
where $K = K_1 + K_2$ and $C_\alpha(\tau)$ is the total discounted cost during the interval $[0, \tilde{T}_0)$, given that $Z_0 = \tau$.

3. The input processes and their characteristics

In this paper we consider the cases where the input process is a spectrally positive Lévy process, and a spectrally positive Lévy process reflected at its infimum. In the remainder of this section we describe these processes and discuss some of their characteristics. The reader is referred to [6] for a more detailed discussion of the definitions and results mentioned in this section.

Definition. A Lévy process $L = \{L_t, t \geq 0\}$ with state space $\mathbb{R}$ is said to be spectrally positive Lévy process, it has no negative jumps. It follows that, for each $\theta \in \mathbb{R}^+$, $x \in \mathbb{R}$,

$$E[e^{-\theta L_t}] = e^{t\phi(\theta)},$$

where

$$\phi(\theta) = -a\theta + \frac{\theta^2 \sigma^2}{2} - \int_0^\infty (1 - e^{-\theta x} - \theta x I_{\{x < 1\}})\upsilon(dx).$$

The term $a \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$ are the drift and variance of the Brownian motion, respectively, and $\upsilon$ is a positive measure on $(0, \infty)$ satisfying $\int_0^\infty (x^2 \wedge 1)\upsilon(dx) < \infty$.

The function $\phi$ is known as the Lévy exponent, and it is strictly convex and tends to infinity as $\theta$ tends to infinity. For $\alpha \in \mathbb{R}^+$, we define

$$\eta(\alpha) = \sup\{\theta : \phi(\theta) = \alpha\} \quad (3.2),$$

the largest root of the equation $\phi(\theta) = \alpha$. It is seen that this equation has at most two roots, one of which is the zero root. Note that, $E(L_1) = \int_1^\infty x\upsilon(dx) + \mu$. Furthermore, $\lim_{t \to \infty} L_t = \infty$ if and only if $E(L_1) > 0$, and $\lim_{t \to \infty} L_t = -\infty$ if and only if $E(L_1) < 0$. Also, if $E(L_1) = 0$, then $\lim_{t \to \infty} L_t$ does not exist. Furthermore, $\eta(0) > 0$, if and only if $E(L_1) > 0$.

An important case is when the process $L$ is of bounded variations, i.e., $\sigma^2 = 0$ and $\int_0^\infty (x \wedge 1)\upsilon(dx) < \infty$. Let

$$\zeta = -a + \int_0^1 x\upsilon(dx).$$

In this case we can write

$$\phi(\theta) = \zeta\theta - \int_0^\infty (1 - e^{-\theta x})\upsilon(dx), \quad (3.3)$$

where necessarily $\zeta$ is strictly positive.
Definition. A Lévy process is said to be spectrally negative if it has no positive jumps.

For any spectrally positive Lévy input process $L$, we let $\hat{L} = -L$ throughout this paper. It is clear that $L$ is spectrally positive if and only if the process $\hat{L}$ is spectrally negative.

We now introduce tools, which will be central in the rest of this paper.

Definition. For any spectrally positive Lévy process with Lévy exponent $\phi$ and for $\alpha \geq 0$, the $\alpha$-scale function $W^{(\alpha)}: \mathbb{R} \to \mathbb{R}^+$, $W^{(\alpha)}(x) = 0$ for every $x < 0$, and on $[0, \infty)$ it is defined as the unique right continuous increasing function such that

$$
\int_0^{\infty} e^{-\beta x} W^{(\alpha)}(x) = \frac{1}{\phi(\beta) - \alpha}, \beta > \eta(\alpha) \quad (3.4)
$$

We will denote $W^{0}$ by $W$ throughout. For $\alpha \geq 0$, we have (see (8.24) of [6])

$$
W^{(\alpha)}(x) = \sum_{k=0}^{\infty} \alpha^k W^{*(k+1)}(x), \quad (3.5)
$$

where $W^{*(k)}$ is the kth convolution of $W$ with itself.

It follows that $W^{(\alpha)}(0+) = 0$ if and only if the process $L$ is of unbounded variation. Furthermore, $W^{\alpha}$ right and left differentiable on $(0, \infty)$. By $W^{(\alpha)}(x)$, we will denote the right derivative of $W^{(\alpha)}$ in $x$.

The adjoint $\alpha-$ scale function associated with $W^{(\alpha)}$ (denoted by $Z^{(\alpha)}$) is defined as follows:

Definition. For $\alpha \geq 0$, the adjoint $\alpha-$ scale $Z^{(\alpha)}: \mathbb{R}^+ \to [1, \infty)$ is defined as

$$
Z^{(\alpha)}(x) = 1 + \alpha \int_0^x W^{(\alpha)}(x) dx. \quad (3.6)
$$

It follows that as $x \to \infty$, for $\alpha > 0$, $W^{(\alpha)}(x) \sim \frac{e^{\eta(\alpha)x}}{\phi(\eta(\alpha))}$ and $Z^{(\alpha)}(x) \sim \frac{\alpha}{\eta(\alpha)}$.

4. Basic results

To derive $C_\alpha(x, \lambda, 0)$, $E_x[e^{-\alpha \hat{T}_0}]$, $C(\lambda, \tau)$, $E_\tau[\hat{T}_0]$ we define the input process $Z$ killed at $\hat{T}_0$, denoted by $X$ such that for every $t \geq 0$

$$
X_t = \{Z_t, t < \hat{T}_0\}. \quad (4.1)
$$

It is known that this killed process is a strong Markov process.

For the input process $X$, any Borel set $A$ contained in the state space of $X$, $t \in \mathbb{R}^+$, the probability transition function of this process is given as follows

$$
P_t(x, A) = P_x(\{Z_t \in A, t < \hat{T}_0\})
$$
and for each $\alpha \in \mathbb{R}_+$ its $\alpha$–potential is defined as follows

$$U^\alpha(x, A) = \int_0^\infty P_t(x, A)e^{-\alpha t}dt = E_x \int_0^{\hat{T}_0} e^{-\alpha t} I_{\{Z_t \in A\}}dt.$$ (4.2)

We note that for $x < \lambda$

$$C_{\alpha}(x, \lambda, 0) = U^\alpha g(x).$$ (4.3)

The following lemma will be used extensively throughout this paper.

Lemma. Let $S = \{S_t, t \geq 0\}$ be a strong Markov process. Define, $G = \{\sigma(S_u, u \leq t)\}_{t \geq 0}$, $\kappa$ to be any stopping time with respect to $G$ and $U_{\alpha,S}(x, A)$ as the $\alpha$–potential of the process $S$ killed at $\kappa$. Denote the state space of this process by $E$. Then, for $x \in E$

$$E_x[e^{-\alpha \kappa}] = 1 - \alpha U_{E}^{\alpha,S}(x).$$ (4.4)

Proof. From the definition of $U_{\alpha,S}$ and for any bounded measurable function $f$ whose domain is $E$, we have

$$U_{\alpha,S}f(x) = E_x[\int_0^{\kappa} e^{-\alpha t} f(S_t)dt] = \int_E f(y)U_{E}^{\alpha,S}(x, dy).$$

Taking $f$ to be identically equal to one, we have

$$\frac{1 - E_x[e^{-\alpha \kappa}]}{\alpha} = U_{E}^{\alpha,S}(x).$$

The required result is immediate from the last equation above. ■

First we consider the case where, during the period $[0, \hat{T}_0)$, the process $Z$ is a spectrally positive Lévy process, denoted by $I$. In this case we note that $\hat{T}_0 = \inf\{t \geq 0 : \hat{I}_t \geq \lambda\}$ and we will denote it by $T_0^\lambda$. Throughout the rest of this paper, for any $a \in \mathbb{R}$, we define $T^-_a = \inf\{t \geq 0 : \hat{I}_t \leq a\}$, $\Upsilon_\lambda^+ = \inf\{t \geq 0 : \hat{I}_t \geq \lambda\}$ and $\Upsilon^-_a = \inf\{t \geq 0 : \hat{I}_t \leq a\}$.

Proposition. For $\alpha \geq 0, a \leq \lambda$ the $\alpha$ potential $(U^\alpha)$ of the process $I$ killed at $T^\lambda_\Upsilon \wedge T^-_a$ is absolutely continuous with respect to the Lebesgue measure on $[a, \lambda]$ and a version of its density is given by

$$u^{(1)}(x, y) = W^{(\alpha)}(\lambda - x)W^{(\alpha)}(y - a) - W^{(\alpha)}(y - x), \ x, y \in [a, \lambda].$$ (4.5)

Proof. For $A \subset [a, \lambda]$
\[ U(x, A) = E_x \int_0^T e^{-\alpha t} I_{\{t_1 \in A\}} dt \]
\[ = E_x \int_{T^-}^{T^- + T^+} e^{-\alpha t} I_{\{t_1 \in (-\lambda, A)\}} dt \]
\[ = E_{\lambda-x} \int_0^{\lambda-x} e^{-\alpha t} I_{\{t_1 \in (-\lambda, A)\}} dt \]
\[ = \int_{(\lambda-A)} [W^{(\alpha)}(\lambda - x) - W^{(\alpha)}(\lambda - y)] dy, \]

where the last equation follows from Theorem 8.7 of [7], this establishes our assertion.

Corollary. For \( \alpha \geq 0 \) the \( \alpha \)-potential (\( U^{(\alpha)} \)) of the process \( X \) is absolutely continuous with respect to the Lebesgue measure on \( (-\infty, \lambda] \) and a version of its density is given by

\[ u^{(\alpha)}(x, y) = W^{(\alpha)}(\lambda - x)e^{-\eta(\alpha)(\lambda - y)} - W^{(\alpha)}(y - x), \quad x, y \in (-\infty, \lambda]. \] (4.6)

Proof: The proof follows from (4.5) by letting \( a \to -\infty \) and since, for \( \alpha \geq 0 \), \( W^{(\alpha)}(x) \sim e^{\eta(\alpha) x} \) as \( x \to \infty \). \( \blacksquare \)

With the help of the last corollary above we are now in a position to find \( E_x[e^{-\alpha T^+} \] and \( E_x[T^+]. \)

Theorem. (i) For \( \alpha > 0 \) and \( x < \lambda \) we have

\[ E_x[e^{-\alpha T^+}] = Z^{(\alpha)}(\lambda - x) - \frac{\alpha}{\eta(\alpha)} W^{(\alpha)}(\lambda - x). \] (4.7)

(ii) For \( x < \lambda \) we have

\[ E_x[T^+] = \frac{W(\lambda - x)}{\eta(0)} \quad \text{for } \eta(0) > 0, \quad \eta(0) = 0, \] (4.8)

where for every \( x \geq 0 \),

\[ \tilde{W}(x) = \int_0^x W(y) dy. \] (4.9)

Proof. We only prove (i), the proof of (ii) is easily obtained from (i) and hence is omitted. Let \( U^{(\alpha)} \) be the \( \alpha \)-potential of the process \( X \), then
\[
E_x[e^{-\alpha T_x^+}] = 1 - \alpha U_{i(-\infty,\lambda)}^\alpha(x)
\]
\[
= 1 - \alpha \int_{-\infty}^\lambda \{ W^\alpha(\lambda - y)e^{-(\lambda-y)\eta(\alpha)} - W^\alpha(y - \lambda) \} dy
\]
\[
= 1 + \alpha \int_{x}^\lambda W^\alpha(y - x) dy - \alpha W^\alpha(\lambda - x) \int_{-\infty}^\lambda e^{-(\lambda-y)\eta(\alpha)} dy
\]
\[
= Z(\alpha)(\lambda - x) - \frac{\alpha}{\eta(\alpha)} W(\alpha)(\lambda - x),
\]
where the first equation follows from (4.4), the second equation follows from (4.6), the third equation follows since \( W(\alpha)(x) = 0, x < 0 \), and the last equation follows from the definition of \( Z(\alpha) \). \[\blacksquare\]

For any Borel set \( B \subset R_+ \times R \), we let \( M(B) \) be the Poisson random measure counting the number of jumps of the process \( I \) in \( B \) with Lévy measure \( \nu \), where if \( B = [0, t] \times A, A \subset R \), then \( E[M(B)] = t\nu(A) \). We need the following to compute the last term in (2.6).

Proposition. For \( \alpha \geq 0 \) let \( u^\alpha(x, y) \) be as given in (4.5) and \( x \leq \lambda \leq z \), then

\[
E_x[e^{-\alpha T_x^+, \lambda} \in D, T_x^+ < T^-] = \int_{0}^{\lambda} \nu(dx - y)u^\alpha(x, y) dy \quad (4.10)
\]

Proof. For \( x < \lambda, \alpha \geq 0, C \subset [\lambda, \infty) \) and \( D \subset (a, \lambda) \)

\[
E_x[e^{-\alpha T_x^+, \lambda} \in C, X_{T_x^+} \in D, T_x^+ < T^-] = E_x[\int_{0, \infty} \times (0, \infty), \begin{array}{l}
e^{-\alpha t} I_{\{X_t - < \lambda, X_t - > a, X_t \in D\} \{y \in C - X_t\} M(dt, dy) \\
e^{-\alpha t} I_{\{X_t - < \lambda, X_t - > a, X_t \in D\} \{y \in C - X_t\} M(dt, dy) \}
\end{array}
\]
\[
= E_x[\int_{0, \infty} \times (0, \infty), \begin{array}{l}
e^{-\alpha t} I_{\{X_t - < \lambda, X_t - > a, X_t \in D\} \{y \in C - X_t\} M(dt, dy) \\
e^{-\alpha t} I_{\{X_t - < \lambda, X_t - > a, X_t \in D\} \{y \in C - X_t\} M(dt, dy) \}
\end{array}
\]
\[
= E_x[\int_{0, \infty} \times (0, \infty), \begin{array}{l}
e^{-\alpha t} I_{\{X_t - < \lambda, X_t - > a, X_t \in D\} \{y \in C - X_t\} M(dt, dy) \\
e^{-\alpha t} I_{\{X_t - < \lambda, X_t - > a, X_t \in D\} \{y \in C - X_t\} M(dt, dy) \}
\end{array}
\]
\[
= \int_{0}^{\lambda} \nu(dx - y)u^\alpha(x, y) dy,
\]
where the second equation follows from the compensation formula (Theorem 4.4. of [7]). Our assertion is proved by taking \( D = [a, \lambda] \). \[\blacksquare\]
The following corollary gives the formula needed to compute the last term of (2.6), when the input process is a spectrally positive Lévy process.

Corollary. Let \( u^\alpha \) be as defined in (4.6). For \( \alpha \geq 0 \) and for \( x \leq \lambda \leq z \),

\[
E_x[e^{-\alpha I^+_\lambda}, I^+_\lambda \in dz] = \int_{-\infty}^{\lambda} v(dz - y)u^\alpha(x, y)dy. \tag{4.11}
\]

Proof. The proof follows immediately from (4.6) and (4.10) by letting \( a \to -\infty \).

We now turn our attention to the case where, during the period \([0, \hat{T}_0)\), the process \( Z \) is a spectrally positive Lévy process reflected at its infimum, denoted by \( Y \). We will denote \( \hat{T}_0 \) by \( \tau_\lambda \), in this case.

The following proposition gives the \( \alpha \)-potential of the process \( X \) defined in (4.1).

Proposition. Assume that during the period \([0, \hat{T}_0)\), the process \( Z \) is a spectrally positive Lévy process reflected at its infimum. Denote the \( \alpha \)-potential of the process \( X \) by \( U^{(2)}_\alpha \). Then for any \( x, y \in [0, \lambda) \),

\[
U^{(2)}_\alpha(x, dy) = \frac{W^{(\alpha)}(\lambda - x)W^{(\alpha)}(dy)}{W^{(\alpha)'(\lambda)}} - W^{(\alpha)}(y - x)dy, \tag{4.12}
\]

where for \( x, y \in [0, \lambda) \), \( W^{(\alpha)}(dy) = W^{(\alpha)}(0)\delta_0(dy) + W^{(\alpha)'}(y)dy \), and \( \delta_0 \) is the delta measure in zero.

Proof. Note that for each \( t \geq 0 \),

\[
Y_t = I_t - I^\wedge_t \tag{4.13}
\]

\[
\hat{Y}_t = \hat{I}_t - \hat{I}^\wedge_t,
\]

where the process \( I = \{I_t, t \geq 0\} \) is a spectrally positive Lévy process. The result follows from part (ii) of Theorem 8.11 of [7], since the process \( \hat{I} \) is a spectrally negative Lévy process.

The following proposition gives \( E_x[e^{-\alpha \tau_\lambda}] \) and \( E_x[\tau_\lambda] \).

Proposition. (i) For \( \alpha \geq 0 \) and \( x < \lambda \) we have

\[
E_x[e^{-\alpha \tau_\lambda}] = Z^{(\alpha)}(\lambda - x) - W^{(\alpha)}(\lambda - x)\frac{\alpha W^{(\alpha)}(\lambda)}{W^{(\alpha)'}(\lambda)}. \tag{4.14}
\]

(ii) For \( x < \lambda \) we have

\[
E_x[\tau_\lambda] = W(\lambda - x)\frac{W(\lambda)}{W^{(\alpha)}(\lambda)} - \tilde{W}(\lambda - x). \tag{4.15}
\]
Proof. The proof of part (i) follows from (4.4) and (4.12), in a manner similar to the proof of (4.7). The proof of part (ii) follows from part (i) by direct differentiation. We omit both proofs. ■

To find a formula analogous to (4.11), for the spectrally positive Lévy process reflected at its infimum, we first need few definitions. For \( z > \lambda \) we let

\[
\begin{align*}
l_\alpha(dz) &= W^{(\alpha)}(\lambda - x) \int_0^\lambda W^{(\alpha)}(dy) v(dy - y) \quad (4.16) \\
&= -W_+^{(\alpha)}(\lambda) \int_0^\lambda dy W^{(\alpha)}(y - x) v(dy - y).
\end{align*}
\]

\[
\begin{align*}
L_\alpha(z) &= \int_{(z,\infty)} l_\alpha(du), \quad (4.17) \\
V_\alpha(\lambda) &= W_+^{(\alpha)}(\lambda)Z^{(\alpha)}(\lambda - x) - \lambda W^{(\alpha)}(\lambda - x)W^{(\alpha)}(\lambda). \quad (4.18)
\end{align*}
\]

The following proposition gives the required formula.

**Proposition.** (i) For \( \alpha \geq 0 \) and for \( z \leq \lambda < z \),

\[
E_z[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} \in dz] = \frac{l_\alpha(dz)}{W_+^{(\alpha)}(\lambda)}, \quad z > \lambda. \quad (4.19)
\]

(ii) For \( \alpha \geq 0 \)

\[
E_z[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} = \lambda] = \frac{V_\alpha(\lambda) - L_\alpha(\lambda)}{W_+^{(\alpha)}(\lambda)}. \quad (4.20)
\]

Proof. (i) From (4.13), for \( x \geq 0 \), \( Y_0 = x \) if and only if \( I_0 = x \) if and only if \( \hat{I}_0 = -x \). Furthermore, \( Y_{\tau_\lambda} = I_{T_\lambda^+} \) almost surely on \( \{T_\lambda^+ < T_0^-\} \). Therefore

\[
\begin{align*}
E_z[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} \in dz] &= E_z[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} \in dz, T_\lambda^+ < T_0^-] + E_z[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} \in dz, T_\lambda^+ > T_0^-] \\
&= E_z[e^{-\alpha T_\lambda^+}, I_{T_\lambda^+} \in dz, T_\lambda^+ < T_0^-] + E_z[e^{-\alpha T_\lambda^+}, T_\lambda^+ > T_0^-] \\
&\quad \times E_0[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} \in dz] \\
&= E_z[e^{-\alpha T_\lambda^+}, I_{T_\lambda^+} \in dz, T_\lambda^+ < T_0^-] + E_{-x}[e^{-\alpha T_0^-}, \tau^-_\lambda > T_0^+] \\
&\quad \times E_0[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} \in dz] \\
&= E_z[e^{-\alpha T_\lambda^+}, I_{T_\lambda^+} \in dz, T_\lambda^+ < T_0^-] + E_{\lambda,-x}[e^{-\alpha \tau_\lambda^-}, \tau^-_\lambda > T_0^+] \\
&\quad \times E_0[e^{-\alpha \tau_\lambda}, Y_{\tau_\lambda} \in dz], \quad (4.21)
\end{align*}
\]

where the second equation follows using the strong Markov property, the third and fourth equations follow from since \( \hat{I} = -I \) and from the definitions of \( T_\lambda^+, T_0^-, \tau^-_\lambda, \tau^+_\lambda \).
Letting \( a \to 0 \) in (4.5) and (4.10), we find that the first term in the last equation above is equal to 
\[
\lambda \int_0^\nu (dz - y) [W^{(\alpha)}(\lambda - x) \frac{W^{(\alpha)}(y)}{W^{(\alpha)}(\lambda)} - W^{(\alpha)}(y - x)] dy.
\]
The second term is equal to \( W^{(\alpha)}(\lambda - x) \) (see (8.8) of [6]) and the third term is equal to \( \frac{\theta a dz}{W^\alpha(\lambda)} \) (this follows from Theorem 4.1 of [9] by letting the \( \beta, \gamma \to 0 \) and noting that if \( \Pi(dz) \) is the Lévy measure of the process \( \hat{I} \), then (for all \( z \geq 0 \)) \( \Pi(-\infty, -z] = \nu(z, \infty) \).

Our assertion is satisfied by replacing each of the three terms in (4.21) by the corresponding value indicated in the last paragraph and after some algebraic manipulations, which we omit.

(ii) The proof is immediate from (4.14) and (4.19).

Now we turn our attention to computing \( C_\alpha(x, M, \tau), E_x[\exp(\tau - T^-)], \) and \( E_x[T^-] \). For each \( t \geq 0 \)
\[
\hat{X}_t = \{ I_t, t < T^- \}. \quad (4.22)
\]
We note that the sample paths of a spectrally positive Lévy process and a spectrally positive Lévy process reflected at its infimum behave the same way starting at any \( x > \tau \) until they reach level \( \tau \), thus \( \hat{X} \) behaves the same way in both cases. Let \( \hat{U} \) be the potential of the process \( \hat{X} \). For each \( x \in (\tau, V] \)
\[
C_\alpha(x, M, \tau) = \hat{U}^{\alpha}(x) \quad (4.23)
\]
Let \( I^{(M)} = I - M \), as defined in Section 2.1. We note that this process is a spectrally positive Lévy process with the Lévy exponent \( \phi_M(\theta) = \phi(\theta) + \theta M, \theta \geq 0 \). We denote its \( \alpha \)--scale and adjoint \( \alpha \)--scale functions by \( W^\alpha_M \) and \( Z^\alpha_M \), respectively. Note that \( W^{(\alpha)}_M \) is obtained from \( W^{(\alpha)} \), upon replacing the term \( a \) in (3.1) by \( a - M \).

Theorem. For \( \alpha > 0 \), \( \hat{U}^{\alpha} \) is absolutely continuous with respect to the Lebesgue measure on \( (\tau, V] \), and a version of its density is given by
\[
\hat{u}^{\alpha}(x, y) = \frac{Z^\alpha_M(V - x) W^\alpha_M(y - \tau)}{Z^\alpha_M(V - \tau)} - W^\alpha_M(y - x) . \quad x, y \in (\tau, V] \quad (4.24)
\]
Proof. For each \( t \geq 0 \), we define \( M_t = (M) I_t - V \). For any \( b \in R \), we define \( \gamma^+_b = \inf \{ t \geq 0 : \hat{M}_t - M_t > b \} = \inf \{ t \geq 0 : \hat{M}_t - M_t > b \} \) and
\[\gamma_b^- = \inf \{ t \geq 0 : M_t - \tilde{M}_t < b \} \]. For and Borel set \( A \subseteq (\tau, V] \) and \( x \in (\tau, V] \) we have

\[
P_x \{ \mathcal{X} t \in A \} = P_x \{ I_t \in A, t < T^- \}
\]

\[
= P_x \{ \sup_{s \leq t} (I_s - V) \vee 0) \in A, t < T^- \}
\]

\[
= P_{V-x} \{ M_t - \tilde{M}_t \in A - V, t < \tau^- V \}
\]

\[
= P_{V-x} \{ \tilde{M}_t - M_t \in V - A, t < \tau^- V \}
\]

Using Theorem 8.111 (ii) of \([6]\), the result follows. ■

The following theorem gives Laplace transform of the distribution of the stopping time \( T^- \) and \( E_x[T^-, x \in (\tau, V) \].

Theorem. (i) Let \( x \in [\tau, V) \) and \( \alpha \in R_+ \). Then we have

\[
E_x[e^{-\alpha T^-}] = \frac{Z_M^\alpha(V - x)}{Z_M^\alpha(V - \tau)}, \quad (4.25)
\]

(ii) For \( x \in [\tau, V) \)

\[
E_x[T^-] = W_M(V - \tau) - W_M(V - x), \quad (4.26)
\]

where, \( W_M(x) = \int_0^x W_M(y) dy \).

Proof. We only prove (i), the proof of (ii) follows easily from (i) and is omitted. We have

\[
E_x[e^{-\alpha T^-}] = 1 - \alpha U^\alpha I_{[\tau, V]}(x)
\]

\[
= 1 - \alpha \int_\tau^\tau \{ \frac{Z_M^\alpha(V - x)}{Z_M^\alpha(V - \tau)} - W_M^\alpha(y - x) \} dy
\]

\[
= 1 - \alpha \{ \frac{Z_M^\alpha(V - x)}{Z_M^\alpha(V - \tau)} \} \{ \frac{Z_M^\alpha(V - \tau) - 1}{\alpha} \} - \{ \frac{Z_M^\alpha(V - x) - 1}{\alpha} \}
\]

\[
= \frac{Z_M^\alpha(V - x)}{Z_M^\alpha(V - \tau)} - Z_M^\alpha(V - x) + Z_M^\alpha(V - x)
\]

\[
= \frac{Z_M^\alpha(V - x)}{Z_M^\alpha(V - \tau)}.
\]
where the third equation follows from (4.24), the fourth equation follows from the definition of the function $Z_M^{(\alpha)}$ and the fifth equation follows from obvious manipulations.

**Remark.** When $V = \infty$, for $\alpha \geq 0$ we let

$$\eta_M(\alpha) = \sup\{\theta : \phi(\theta) - \theta M = \alpha\}$$

since $Z_M^{(\alpha)}(x) = O(e^{\eta_M(\alpha)x})$ as $x \to \infty$, then we have

$$E_x[T^* - \tau] = \frac{(x - \tau)}{\eta_M(\alpha)'(0)}$$

if $M > E(I_1)$

$$= \infty$$

if $M \leq E(I_1)$.

This is consistent with the well known fact about the busy period of the M/G/1 queuing system.

To compute $E_x[\exp(-\alpha^*T_0)]$, we first observe that, for $\lambda \leq x \leq V$, $\hat{T}_0 = T^*_0$ almost everywhere. Hence $E_x[\exp(-\alpha^*T_0)]$ is given in (4.25). We now turn our attention to the case where $x < \lambda$. We first consider the case where the input process is a spectrally positive Lévy process.

**Theorem.** Assume that the input process is a spectrally positive Lévy process. For $z > \lambda$, we define

$$h_\alpha(x, dz) = \int_0^\lambda dyu^\alpha(x, y)v(dz - y),$$

$u^\alpha(x, y)$ is defined in (4.6).

Then, for $\alpha \geq 0, x < \lambda$

$$E_x[\exp(-\alpha^*T_0)] = \frac{1}{Z^{\alpha}(\lambda - \tau)} \left[ \int_\lambda^\lambda Z^{\alpha}(\lambda - z)h_\alpha(x, dz) + \int_\lambda^\infty h_\alpha(x, dz) \right] \hspace{10cm} (4.27)$$

**Proof:** We write
\[
E_x [e^{-\alpha T_0}] = E_x [e^{-\alpha T^+_0 - \alpha \tilde{T}_0 - T^+_0}]
\]
\[
= E_x [E_x [e^{-\alpha T^+_0 - \alpha \tilde{T}_0 - T^+_0} | \sigma (T^+_0, I_{T^+_0})]]
\]
\[
= E_x [e^{-\alpha T^+_0} E_{(I_{T^+_0} \wedge V)} [e^{-\alpha T^-_0}]]
\]
\[
= \frac{1}{Z^\alpha (V - \tau)} E_x [e^{-\alpha T^+_0} Z^\alpha_M (V - (I_{T^+_0} \wedge V))] \]
\[
= \frac{1}{Z^\alpha_M (V - \tau)} \int_0^V Z^\alpha_M (V - z) h_\alpha (x, dz) + \int_0^\infty h_\alpha (x, dz),
\]

where the third equation follows since, given \( T^+_0 \) and \( I_{T^+_0} \), \( \tilde{T}_0 - T^+_0 \) is equal to \( T^-_0 \) almost everywhere. The fourth equations from (4.11), the last equation follows from the fact that \( Z^\alpha (0) = 1 \), and the definition of \( h_\alpha (x, dz) \). \( \blacksquare \)

The following theorem gives a result analogous to (4.27) when the input process is a spectrally positive Lévy process reflected at its infimum.

Theorem. Assume that the input process is a spectrally positive Lévy process reflected at its infimum. For \( z \geq \lambda \), let \( I_{\alpha} (dz), L_{\alpha} (z), \) and \( V_{\alpha} (\lambda) \) be as defined in (4.16), (4.17) and (4.18), respectively. Define

\[
g_\alpha (x, dz) = \begin{cases} 
\frac{I_{\alpha} (dz)}{W^{(\alpha)}_z (\lambda)}, & z > \lambda \\
\frac{V_{\alpha} (\lambda) - L_{\alpha} (\lambda)}{W^{(\alpha)}_z (\lambda)} \delta_\lambda (dz). & 
\end{cases}
\]

Then, for \( \alpha \geq 0, x < \lambda \)

\[
E_x [e^{-\alpha \tilde{T}_0}] = \frac{1}{Z^\alpha_M (\lambda - \tau)} \left[ \int_0^V Z^\alpha_M (\lambda - z) g_\alpha (x, dz) + \int_0^\infty g_\alpha (x, dz) \right]. \tag{4.28}
\]

Proof. The proof follows in a manner similar to the proof of (4.27), using (4.19), (4.20) and (4.25). \( \blacksquare \)

5. Special Cases

In this section we consider the cases where the input process is a spectrally positive Lévy process of bounded variation, Brownian motion reflected at its infimum and Wiener process. For the first case, we extend the results of Bae et al [5] who assumed that the input process is a compound Poisson process with a negative drift. We also simplify some of their results. For the second case, we obtain results similar to those of Attia [3] and Lam and Lou [7]. In the third case we obtain results similar to those of Zuckerman [10].
negative drift, and with Lévy measure is given by $\nu$. The Lévy measure is given by $\nu$. Our $C\alpha > 0$, let $\rho = \frac{1}{\c}$. Furthermore, letting $\alpha > 0$, we get the same result given in page 523 of this reference. We note that their entities $w(x)$ and $E[L^n(\lambda, \tau)]$ given in p. 521 are nothing but our $C(x, \lambda, 0)$ and $E_x[\tau_\lambda]$, respectively. Using (4.12) and (4.15) we get the same result given in page 523 of this reference. Furthermore, their functions $E[P^\lambda(\lambda, \tau)]$ and $E[L^M(\lambda, \tau)]$ given in page 525 are our $C(x, M, \tau)$ and $E_x[T_\tau]$, respectively. Using (4.24) and (4.26) we get a simpler form for these entities. Furthermore, letting $\alpha = 0$, in (4.19) we provide a simpler form for the distribution of $L(\tau)$ (the overshoot) given on page 525 of their paper.

(c) Assume that the input process is a gamma process with a negative drift. In this case, $f(x) = \frac{G(x) - G}{m}$, where $G = 1 - G$ and $m = \int_0^\infty G(x)dx$, which is assumed to be finite. We note that their entities $w(x)$ and $E[L^n(\lambda, \tau)]$ given in p. 521 are nothing but our $C(x, \lambda, 0)$ and $E_x[\tau_\lambda]$, respectively. Using (4.12) and (4.15) we get the same result given in page 523 of this reference. Furthermore, their functions $E[P^\lambda(\lambda, \tau)]$ and $E[L^M(\lambda, \tau)]$ given in page 525 are our $C(x, M, \tau)$ and $E_x[T_\tau]$, respectively. Using (4.24) and (4.26) we get a simpler form for these entities. Furthermore, letting $\alpha = 0$, in (4.19) we provide a simpler form for the distribution of $L(\tau)$ (the overshoot) given on page 525 of their paper.

For $\alpha > 0$, $W^{(\alpha)}$ is computed using (3.5) and the above equation.
(ii) Assume that the input process is a Brownian motion with drift term 
\( \mu \in R \), variance term \( \sigma^2 \), reflected at its infimum. We will show that the results of [3] and [7] follow from our results. In this case, the Lévy measure \( \nu = 0 \), and from (3.1) we have that for \( \theta \geq 0 \), \( \varphi(\theta) = -\mu \theta + \frac{\alpha \sigma^2}{2} \). It follows that, for \( \alpha \geq 0 \), \( \eta(\alpha) = \sqrt{\frac{2 \alpha \sigma^2 + \mu^2}{\sigma^2}} \). Let \( \delta = \sqrt{\frac{2 \alpha \sigma^2 + \mu^2}{\sigma^2}} \), we have, \( W^\alpha(x) = \frac{2}{\delta} e^{\mu x/\sigma^2} \sinh(\frac{x}{\sigma^2}) \) and \( Z^\alpha(x) = e^{\mu x/\sigma^2} \left( \cosh(\frac{x}{\sigma^2}) - \frac{\delta}{\mu} \sinh(\frac{x}{\sigma^2}) \right) \). We note that \( W^\alpha(x) \) is differentiable, and \( W^\alpha(x) = \frac{\mu}{\alpha} W^\alpha(x) + \frac{\alpha}{\sigma^2} e^{\mu x/\sigma^2} \cosh(\frac{x}{\sigma^2}) \), it follows that \( \frac{W^\alpha(x)}{W^\alpha(\lambda)} = \left( \frac{\mu + \delta \coth(\frac{\lambda x}{\sigma^2})}{\mu + \delta \coth(\frac{\lambda x}{\sigma^2})} \right) \). Substituting the values of \( Z^\alpha(\lambda - x), W^\alpha(\lambda - x) \) and \( \frac{W^\alpha(x)}{W^\alpha(\lambda)} \) in (4.14), we have, for \( \alpha \geq 0 \), \( x \leq \lambda \)

\[
E_x[e^{-\alpha T}] = e^{\mu(\lambda - x)} \left[ \cosh(\frac{(\lambda - x)\delta}{\sigma^2}) - \frac{1}{\delta} \sinh(\frac{(\lambda - x)\delta}{\sigma^2}) \right] \left( \mu + \frac{2\alpha \sigma^2}{\mu + \delta \coth(\frac{\lambda x}{\sigma^2})} \right).
\]

Case 1. \( \mu \neq 0 \): It follows that, for \( x \geq 0 \), \( W(x) = \frac{2^{\alpha x/\sigma^2 - 1}}{\mu + \frac{2\alpha \sigma^2}{\mu}} \), \( W'(x) = \frac{2^{\alpha x/\sigma^2}}{\mu + \frac{2\alpha \sigma^2}{\mu}} \)
and \( \frac{\partial}{\partial y} W(y) = \frac{2^{\alpha x/\sigma^2}}{\mu} \left( e^{\mu y/\sigma^2} - 1 \right) - \frac{\mu}{\mu} \). Substituting the values of \( W(\lambda - x), W(\lambda - x), W(\lambda), W'(\lambda) \) in (4.15) we have, for \( x \leq \lambda \),

\[
E_x[\tau_\lambda] = \frac{\lambda - x}{\mu} + \frac{\sigma^2}{2\mu^2} \left[ e^{-2\mu \lambda/\sigma^2} - e^{-2\mu x/\sigma^2} \right].
\]

We note that, \( W^\alpha(0) = 0, (2) \) in (4.12) is absolutely continuous with respect to the Lebesgue measure on \([0, \lambda]\) and for \( y \in [0, \lambda) \), \( W^\alpha(dy) = W^\alpha(y)dy \). Substituting the values of \( W^\alpha(\lambda - x), W^\alpha(\lambda), W^\alpha(y - x) \), and \( W^\alpha(y) \) in (4.12) we get a version of the the density of \( U^\alpha \), denoted by \( u^\alpha \). Thus, \( C_\alpha(x, \lambda, 0) \) is computed using (4.3).

Let \( \hat{\mu} = \mu - M, \delta = \sqrt{2\sigma^2 + \mu^2} \), we note that \( W^\alpha_M(x) = \frac{2}{\delta} e^{\mu x/\sigma^2} \sinh(\frac{x}{\sigma^2}) \). We note that the input process is continuous, \( Y^\alpha_\tau = \lambda < V \), almost everywhere. Therefore, the term \( C_\alpha((Z^\alpha_{T_0} \land V), \tau, M) \) in (2.6) reduces to \( C_\alpha(\lambda, \tau, M) \) which is computed using (4.23). Furthermore,

\[
E_x[e^{-\alpha T}] = E_x[e^{-\alpha T}] E\Lambda[e^{-\alpha T^\tau}],
\]
where \( E_x[e^{-\alpha T}] \) is given above and \( E\Lambda[e^{-\alpha T^\tau}] \) is given in (4.25).

Let \( \lambda^* = V - \lambda \) and \( \tau^* = V - \tau \), then

\[
E_x[T_0] = E_x[\tau_\lambda] + E\Lambda[T^\tau],
\]
Note that
\[ E_\lambda[T^-] = \frac{\lambda^* - \tau^*}{\mu^*} + \frac{\sigma^2}{2\mu^*} \left[ e^{2\mu^* / \sigma^2} - e^{2\mu\lambda^*/\sigma^2} \right], \]

where the last equation follows from (4.26) after some tedious calculations which we omit.

Case 2. \( \mu = 0 \): In this case, \( \delta = \sqrt{2\alpha \sigma^2} \), letting \( \mu \to 0 \), in the corresponding equations in case 1 above, we have

\[ E_x[e^{-\alpha T^+}] = \cosh((\lambda - x)\delta / \sigma^2) - \sigma^2 \sinh((\lambda - x)\delta / \sigma^2) \coth(\frac{\lambda \delta}{\sigma^2}), \]

\[ E_x[T^+] = \frac{\lambda^2 - x^2}{\sigma^2}, \]

and \( E_\lambda[T^-] \) is obtained by replacing \( \mu^* \) by \( -M \) in the last equation of case 1.

(iii) Assume that the input process is a Brownian motion with drift term \( \mu > 0 \) and variance parameter \( \sigma^2 \). Substituting the values of \( W^{(\alpha)}(x) \), \( Z^{(\alpha)}(x) \), given in (ii) in (4.7) we have, for \( x \leq \lambda \), \( E_x[e^{-\alpha T^+_\lambda}] = \exp((\delta - \mu)(x - \lambda)) \).

Substituting \( \eta(0) = \frac{2\mu^*}{\sigma^2}, \frac{1}{\sigma^2}(e^{2\mu x/\sigma^2} - 1) \) and \( \frac{\sigma^2}{2\sigma^2}(e^{2\mu x/\sigma^2} - 1) - \frac{\lambda}{\sigma^2} \) for \( W(x) \) and \( \tilde{W}(x) \), respectively, in (4.8) we have, for \( x \leq \lambda \), \( E_x[T^+\lambda] = \frac{\lambda^2 - \mu x}{\sigma^2} \). These results are consistent with the results of Zuckerman [10], p.423. The computations of the other entities in the cost functionals (2.7) and (2.8) can be obtained in a manner similar to those discusses in (ii) with obvious modifications.

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