A stroll in the jungle of error bounds

Trong Phong Nguyen*

May 2, 2017

Abstract

The aim of this paper is to give a short overview on error bounds and to provide the first bricks of a unified theory. Inspired by the works of [8, 15, 13, 16, 10], we show indeed the centrality of the Lojasiewicz gradient inequality. For this, we review some necessary and sufficient conditions for global/local error bounds, both in the convex and nonconvex case. We also recall some results on quantitative error bounds which play a major role in convergence rate analysis and complexity theory of many optimization methods.

Key words: Error bounds, Kurdyka–Lojasiewicz inequality, Lojasiewicz function inequality, descent method.

1 Introduction

Let $X$ be a Banach space. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, an error bound is an inequality that bounds the distance from an arbitrary point in a test set to the level set in terms of the function values. More precisely, we shall say that $f$ has an error bound on a set $K \subset X$ if there exists an increasing function $\varphi : [0, +\infty) \to [0, +\infty)$, $\varphi(0) = 0$ such that

$$\text{dist}(x, [f \leq 0]) \leq \varphi(f(x)), \forall x \in K.$$ (1)

When $K = X$ then $f$ is said to possess global error bound; otherwise we say that $f$ has a local error bound. Error bounds have a lot of applications in many fields. They may be used to establish the rate of convergence of many optimization methods: we can think to descent methods for solving minimization problems [11, 26, 57, 55, 56, 67], to the cyclic projection algorithm [12, 18, 46], to algorithms for solving variational inequalities, see e.g., [66]. In [16, 69] the error bound theory is used to estimate the complexity of a wealth of descent methods for convex problems. Error bounds have also played a major role in the context of metric regularity [5, 37, 40] or within the field of exact penalty functions, see e.g., [21].

*TSE (Université Toulouse I Capitole), Manufacture des Tabacs, 21 allée de Brienne, 31015 Toulouse, Cedex 06, France. E-mail: trong-phong.nguyen@ut-capitole.fr. The author thanks the Air Force Office of Scientific Research, Air Force Material Command, USAF, under grant number FA9550-14-1-0056 and FA9550-14-1-0500 for their partial support.
Let us now present the two major mathematical results that are structuring the theory of error bounds and along which we will develop our own presentation. Hoffman seems to be the first to provide an error bound in the context of optimization theory. His result concerns affine function system:

**Theorem 1 (Hoffman, 1952)** \[34\] Let \( A, B \in \mathbb{R}^{m \times n} \) be some matrices and \( a, b \) are the vectors in \( \mathbb{R}^m \). Assume that
\[
S = \{ x \in \mathbb{R}^n | Ax \leq a, Bx = b \}
\]
is nonempty. Then, there exists a scalar \( c > 0 \) such that
\[
dist(x, S) \leq c (\| [Ax - a]_+ \| + \| Bx - b \|), \forall x \in \mathbb{R}^n.
\]

Around the same time, a very general and powerful result was provided in \[50\] for semi-algebraic functions. This result was developed as a positive response to a conjecture of Schwartz in the distribution theory of functions (see \[53\]). Later, in \[33\], Hironaka extended this inequality to the case of subanalytic function.

**Theorem 2 (Lojasiewicz, 1959)** \[33, 50\] Let \( \phi, \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) be two continuous subanalytic functions. If \( \phi^{-1}(0) \subset \psi^{-1}(0) \) then for each compact, subanalytic set \( K \subset \mathbb{R}^n \), there exist a constant \( c > 0 \) and a integer \( N \) such that
\[
c |\psi(x)|^N \leq |\phi(x)|, \forall x \in K.
\]

After those pioneering works, the study of error bounds has attracted numerous researches. In 1972, under a Slater’s condition and a boundedness assumption on the level sets, Robinson \[65\] extended the result of Hoffman to systems of convex differentiable inequalities. Mangasarian \[58\] established the same result for the maximum of finitely many differentiable convex functions. Later on, Auslender and Crouzeix \[5\], extended Mangasarian’s result to non-differentiable convex functions. Some other sufficient conditions were also given by Deng in \[22, 23\], by using in particular a Slater’s condition on the recession function. In \[43\], Lewis and Pang gave a characterization of Lipschitz global error bound for convex functions in terms of the directional derivatives. The work of Lewis and Pang was further generalized by Ng and Yang \[60\], by Wu and Yu in \[72, 71\], by Klatte and Li in \[38\]. In a series papers \[6, 8, 7, 9, 10\], Azé and Corvellec presented some characterizations of error bounds in terms of the strong slope in the context of metric spaces.

The first fundamental works on quantitative error bounds seem to be those of Gwozdziewicz \[30\] and Kollár \[39\] for polynomial functions. Inspired by these works many researchers have tried to provide more general types of quantitative error bounds. Li, Murdokhovich and Pham \[47\] established a local error bound for polynomial function systems in the nonconvex case. Li \[44\], and Yang \[73\] obtained some error bounds for polynomial convex functions, the work of Li was extended for piecewise convex polynomial function in \[45\], which has also improved the result of Li \[48\]. In \[61\], Ngai gave some similar results on polynomial function systems. For the quadratic function systems, Luo and Luo \[52\] seem to be the first to have studied global error bounds for such class, under the assumption of convexity. This work has been improved by Pang and Wang \[70\] and later by Luo and Sturm \[54\] who derived a global error bound for such function without assuming convexity.
The connection between error bound and Kurdyka–Lojasiewicz inequality was first settled by Bolte, Daniilidis, Ley and Mazet, in [15]. Later some of these results were improved [16, 10].

This paper is organized as follows:

In Section 3, based on the results in [8, 15, 10], we give a characterization of error bounds, specifying this result in some particular cases. We establish the connection between this result and some other previous sufficient conditions for Lipschitz error bounds.

In Section 4, we review some results on local error bounds and global error bound, respectively. We focus on the class of polynomial functions whose error bounds are of Hölder type, which play a major role in complexity theory of many optimization methods.

2 Preliminaries

Let $X$ be a Banach space, $X^*$ be topological dual space and $f : X \to \mathbb{R}$ be a lower semicontinuous function. For any $\alpha, \beta \in \mathbb{R}$, we set $[f \leq \alpha] = \{x \in X | f(x) \leq \alpha\}$, $[f = \alpha] = \{x \in X | f(x) = \alpha\}$, $[\alpha \leq f \leq \beta] = \{x \in X | \alpha \leq f(x) \leq \beta\}$, and $[\alpha]_+ = \max\{\alpha, 0\}$. For any subset $S \subset X$, denote $\text{dist}(x, S) = \inf_{u \in S} \|x - u\|$, and $\text{bd} S$, $\text{cl} S$, $\text{int} S$ respectively are the boundary, closure and interior set of $S$. For $x \in X$, $\delta > 0$, set $B_\delta(x) = \{y \in X | \text{dist}(x, y) < \delta\}$.

The Fréchet subdifferential of $f$ at $x \in \text{dom} f$, denoted $\partial^F f(x)$, is defined by

$$\partial^F f(x) = \left\{ u \in X^* | \liminf_{y \to x} \frac{f(y) - f(x) - \langle u, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

The limiting-subdifferential of $f$ at $x \in \text{dom} f$, written $\partial f(x)$ is defined as follows

$$\partial f(x) = \left\{ u \in X^* | \exists x_k \to x, f(x_k) \to f(x), u_k \in \partial^F f(x_k) \to u \right\}.$$

And

$$f'(x, d) = \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t},$$

is called the derivative of $f$ at $x$ in the direction $d \in X$.

The strong slope of $f$ at $x$ is given by

$$|\nabla f|(x) = \begin{cases} 0 & \text{if } x \text{ is a local minimum point of } f, \\ \limsup_{y \to x} \frac{f(x) - f(y)}{\|x - y\|} & \text{otherwise.} \end{cases}$$

It is easy to see that

- $\|d\| |\nabla f|(x) \geq -f'(x, d), \forall (x, d) \in X^2$.
- $|\nabla f|(x) \leq \text{dist}(0, \partial^F f(x)), \forall x \in X$.

We recall the chain rule for the strong slope.

**Lemma 3** [10] Let $-\infty < \alpha < \beta \leq +\infty$ and a function $\varphi : ]\alpha, \beta[ \to \mathbb{R}$ with $\varphi \in C^1(\alpha, \beta)$ and $\varphi'(s) > 0, \forall s \in ]\alpha, \beta[$. One has

$$|\nabla (\varphi \circ f)|(x) = \varphi'(f(x)) |\nabla f|(x), \forall x \in [\alpha < f < \beta].$$
As mentioned in the introduction, the theory of error bound can be developed, based on the theory of Lojasiewicz on the subanalytic function. Let us recall the definition of such function class.

**Definition 1**

(i) A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is real-analytic on \( S \subset \mathbb{R}^n \) if it can be represented locally on \( S \) by a convergent power series, this means that, for any \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in S \), there exists a neighborhood \( U(\bar{x}) \) such that

\[
    f(x) = \sum_{i_1, \ldots, i_n=0}^{\infty} a_{i_1, \ldots, i_n} (x_1 - \bar{x}_1)^{i_1} \cdots (x_n - \bar{x}_n)^{i_n}, \quad \forall x \in U(\bar{x}).
\]

(ii) A subset \( S \) of \( \mathbb{R}^n \) is called semianalytic if for each point \( \bar{x} \in \mathbb{R}^n \) admits a neighborhood \( U(\bar{x}) \) for which \( S \cap U(\bar{x}) \) is represented in the following form

\[
    S \cap U(\bar{x}) = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{ x \in \mathbb{R}^n | f_{ij}(x) = 0, g_{ij}(x) > 0 \},
\]

where the functions \( f_{ij}, g_{ij} : \mathbb{R}^n \rightarrow \mathbb{R} \) are real-analytic for all \( 1 \leq i \leq p, 1 \leq j \leq q \). If the graph of \( f \) is a semianalytic set in \( \mathbb{R}^{n+1} \), we say that \( f \) is a semianalytic function.

(iii) \( S \) is called subanalytic if each point \( \bar{x} \in \mathbb{R}^n \), there exist a neighborhood \( U(\bar{x}) \) and a bounded semianalytic set \( A \subset \mathbb{R}^{n+m} \), (for some \( m \in \mathbb{N}^* \)) such that \( S \cap U(\bar{x}) \) is the projection on \( \mathbb{R}^n \) of \( A \). The function \( f \) is subanalytic if its graph is a subanalytic set in \( \mathbb{R}^{n+1} \).

We give some elementary properties of subanalytic function and subanalytic set, see [13].

1. If \( S \) is subanalytic set then so are its boundary \( \text{bd} S \), its closure \( \text{cl} S \), its interior \( \text{int} S \), and its complement set.

2. The class of subanalytic sets is closed under finite union and intersection. The distance function to a subanalytic set is a subanalytic function.

3. The image of a bounded subanalytic set under a subanalytic map is subanalytic. The inverse image of a subanalytic set under a subanalytic map is a subanalytic set.

4. When \( S \) is a closed, convex subanalytic set, the Euclidean projector onto \( S \) is a subanalytic function.

In this work, we focus on the Hölder-type error bound, which is very common in practice.

**Definition 2** Let \( f \) be a function on the Banach space \( X \) and \( K \) be a subset of \( X \). We say that \( f \) admits a

1. Hölder-type error bound on \( K \) if there exists \( \tau > 0 \) and \( a, b > 0 \) such that

\[
    \text{dist}(x, [f \leq 0]) \leq \tau \left( [f(x)]^a + [f(x)]^b \right), \quad \forall x \in K. \tag{2}
\]

2. Lipschitz-type (or linear) error bound on \( K \) if the inequality (2) holds with \( a = b = 1 \), for all \( x \in K \).

When \( K \equiv X \) then \( f \) is said to have a global error bound, otherwise we say that \( f \) possesses a local error bound.
3 Characterization of error bounds

3.1 Characterizing error bounds through Kurdyka–Lojasiewicz inequality

The Lojasiewicz gradient inequality was introduced in [51]. Let $f: \mathbb{R}^n \to \mathbb{R}$ be an analytic function. For any $\bar{x} \in \text{dom } f$, there exist $\tau > 0$, $\theta \in [0, 1)$ and a neighbourhood $U(\bar{x})$ such that

$$\|\nabla f(x)\| \geq \tau |f(x) - f(\bar{x})|^{\theta}, \forall x \in U(\bar{x}).$$

In [41], Kurdyka generalized the above result to the class of $C^1$ functions whose graphs belong to an o-minimal structure (the definition of o-minimal structure can be seen in [41, Definition 1], [14, Definition 6]), this result was extended to the nonsmooth class by Bolte, Danillidis, Lewis and Shiota [14]. The corresponding generalized Lojasiewicz gradient inequality is called the Kurdyka–Lojasiewicz (KL inequality for short) inequality. In addition, the generalization for the class nonsmooth subanalytic functions has been obtained by Bolte, Danillidis and Lewis in [13]. This has opened the road to many theoretical and algorithmic developments (see [1, 2, 3, 4, 17, 16, 29]). We summarize the above extension by the following theorem.

Theorem 4 [14, 13] Let $f: \mathbb{R}^n \to \mathbb{R}$ be a definable function in an arbitrary o-minimal structure over $\mathbb{R}^n$. Then for all $\bar{x} \in \text{dom } f$, there exist $\delta > 0$ and a neighbourhood $U(\bar{x})$ of $\bar{x}$ such that

$$\varphi'(f(x) - f(\bar{x})) \text{ dist}(0, \partial f(x)) \geq 1, \forall x \in U(\bar{x}) \cap [f(\bar{x}) < f < f(\bar{x}) + \delta],$$

where $\varphi: [0, \delta] \to [0, +\infty)$ is an increasing function, which vanishes at zero and $\varphi \in C^0[0, \delta] \cap C^1(0, \delta)$. The class of such functions $\varphi$ will be denoted by $K[0, \delta]$.

The connection between error bounds and Kurdyka–Lojasiewicz inequality was established in [15] (see also [16]), this was further improved by Azé and Corvellec [10].

Azé and Corvellec have series of researches on the characterization of global error bounds for lower semicontinuous functions in terms of the strong slope, see [8, 7, 9, 10, 19]. These works are of great help for this section. Let us now give the result in [8], in which, the authors used Ekeland’s variational principle to establish the connection between the strong slope and the linear error bound.

Theorem 5 [8] Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and $-\infty < \alpha < \beta \leq \infty$. Then

$$\inf_{x \in [\alpha < f < \beta]} |\nabla f|(x) = \inf_{\alpha \leq \gamma < \beta} \left( \inf_{x \in [\gamma < f < \beta]} \frac{f(x) - \gamma}{\text{dist}(x, [f \leq \gamma])} \right).$$

We rewrite the latter theorem as a characterization of linear global error bound, which is a well known result since Ioffe’s pioneering works [36].

Theorem 6 [8] Let $\tau > 0$, the following assertions are equivalent

(i) $|\nabla f|(x) \geq \frac{1}{\tau}, \forall x \in [\alpha < f < \beta]$.

(ii) $\tau (f(x) - \gamma) \geq \text{dist}(x, [f \leq \gamma]), \forall \gamma \in [\alpha, \beta], x \in [\gamma < f < \beta]$. 5
For any \( \varphi \in K[0, \beta - \alpha] \), thanks to Lemma \(^3\), we can apply the latter result for the function \( x \mapsto \varphi(f(x) - \alpha) \), therefore we obtain a nonlinear version of Theorem \(^6\).

**Theorem 7** Assume that \( \varphi \in K[0, \beta - \alpha] \). The following statements are equivalent

(i) \( \varphi'(f(x) - \alpha)|\nabla f|(x) \geq 1, \forall x \in [\alpha < f < \beta] \).

(ii) \( \varphi(f(x) - \alpha) \geq \varphi(\gamma - \alpha) + \text{dist}(x, [f \leq \gamma]), \forall \gamma \in [\alpha, \beta], \forall x \in [\gamma < f < \beta] \).

This is content of \(^{15}\) Corollary 4, \(^{10}\) Theorem 4.2. In the latter result, if we let \( \gamma \) equal to \( \alpha \) in the assertion (ii), then we immediately obtain as a consequence, a sufficient condition for nonlinear global error bound.

**Corollary 8** We suppose that

\[
\varphi'(f(x) - \alpha)|\nabla f|(x) \geq 1, \forall x \in [\alpha < f < \beta],
\]

where \( \varphi \in K[0, \beta - \alpha] \). Then

\[
\varphi(f(x) - \alpha) \geq \text{dist}(x, [f \leq \alpha]), \forall x \in [\alpha < f < \beta].
\]

Generally, the converse of this corollary is false, as shown in \(^{42}\) Remark 3 (when \( f \) is a polynomial function) and in \(^{16}\) Theorem 28 (when \( f \) is convex). However, in some particular cases, this converse may be hold, for example:

- \( f \) is an analytic function with an isolated zero, see \(^{30}\).
- \( f \) is a convex function and an additional assumption on \( \varphi \), see \(^{16}\), (we also show this result in Theorem \(^{14}\)).

Recalling \( \|d\| |\nabla f|(x) \geq -f'(x, d), \forall (x, d) \in X^2 \), a consequence of Theorem \(^6\) is:

**Corollary 9** For any \( \tau > 0 \), suppose that for each \( x \in [\alpha < f < \beta] \), there exists a unit vector \( d_x \in X \) such that

\[
f'(x, d_x) \leq -\frac{1}{\tau}.
\]

Then

\[
\tau (f(x) - \alpha) \geq \text{dist}(x, [f(x) \leq \alpha]), \forall x \in [\alpha < f < \beta].
\]

This is the content of \(^{60}\) Theorem 2.5. A local version of Theorem \(^7\) is given as follows

**Theorem 10** \(^{10}\) Consider the following statements

(i) There exists \( \varepsilon > 0 \) such that

\[
\varphi'(f(x) - \alpha)|\nabla f|(x) \geq 1, \forall x \in B_\varepsilon(\bar{x}) \cap [\alpha < f < \beta].
\]

(ii) There exists \( \rho > 0 \) such that

\[
\varphi(f(x) - \alpha) \geq \varphi(\gamma - \alpha) + \text{dist}(x, [f(x) \leq \gamma]), \forall \gamma \in [\alpha, \beta], \forall x \in B_\rho(\bar{x}) \cap [\alpha < f < \beta].
\]
Then (i) $\Rightarrow$ (ii) with $\rho = \varepsilon/2$ and (ii) $\Rightarrow$ (i) with $\varepsilon = \rho$.

In the statement (ii), by setting $\gamma = \alpha$, we obtain a local version of Corollary \[ \ref{corollary8} \]

**Corollary 11** \[ \ref{corollary11} \] For any $\bar{x} \in [f \leq \alpha]$, suppose that there exists $\varepsilon > 0$ such that

$$
\varphi'(f(x) - \alpha)\|\nabla f\|(x) \geq 1, \forall x \in B_{2\varepsilon}(\bar{x}) \cap [\alpha < f < \beta].
$$

Then

$$
\varphi(f(x) - \alpha) \geq \text{dist}(x, [f(x) \leq \alpha]), \forall x \in B_{\varepsilon}(\bar{x}) \cap [\alpha < f < \beta].
$$

If we take $\varphi(s) = \tau s^{\theta}$, $\tau > 0$, $\theta \in [0, 1]$, then this corollary recover the result of Ngai, Thera \[ \ref{ngai-thera} \, Corollary 2 \].

As we mentioned before, the converse of the latter corollary does not always hold. The results of Corollary \[ \ref{corollary8} \] Corollary \[ \ref{corollary11} \] have been appeared in numerous works, for instance, see \[ \ref{30}, \ref{42}, \ref{62}, \ref{68}, \ref{63}, \ref{64} \]. This result gives an useful tools for establishing the quantitative error bounds, see \[ \ref{47}, \ref{46} \].

### 3.2 Equivalence in the convex case

In the sequel, we suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function. The following extra-properties are available.

- $\partial^F f(x) = \partial f(x) = \{u \in X^* | \langle u, y - x \rangle \leq f(y) - f(x), \forall y \in X\}, \forall x \in \text{dom } f$.

- $|\nabla f|(x) = \text{dist}(0, \partial f(x)), \forall x \in X$.

In the convex case, Theorem \[ \ref{theorem5} \] can be simplified by the following proposition.

**Proposition 12** \[ \ref{proposition12} \] For $-\infty < \alpha < \beta \leq +\infty$, the following assertions hold true:

(i) $|\nabla f|(x) = \sup_{f(z) \leq f(x)} \frac{f(x) - f(z)}{\text{dist}(x, z)}$, with $x$ is not a minimum point of $f$.

(ii) $\inf_{[\alpha < f \leq \beta]} |\nabla f|(x) \geq \inf_{[f = \alpha]} |\nabla f|(x)$.

(iii) $\inf_{\alpha \leq \gamma < \beta} \left( \inf_{x \in [\gamma < f \leq \beta]} \frac{f(x) - \gamma}{\text{dist}(x, [f \leq \gamma])} \right) = \inf_{x \in [\alpha < f \leq \beta]} \frac{f(x) - \alpha}{\text{dist}(x, [f \leq \alpha])}$.

Thanks to Proposition \[ \ref{proposition12} \] the convex version of Theorem \[ \ref{theorem6} \] is given as following.

**Theorem 13** Suppose $-\infty < \alpha < \beta \leq +\infty$ and $\tau > 0$. Consider the following statements

(i) $\inf_{x \in [f = \alpha]} \text{dist}(0, \partial f(x)) \geq \frac{1}{\tau}$.

(ii) $\inf_{x \in [\alpha < f < \beta]} \text{dist}(0, \partial f(x)) \geq \frac{1}{\tau}$.

(iii) $\tau (f(x) - \alpha) \geq \text{dist}(x, [f(x) \leq \alpha]), \forall x \in [\alpha < f < \beta]$.

Then (i) $\Rightarrow$ (ii) $\iff$ (iii).
We mention that the assumption (i) in the above theorem is equivalent to the condition \( 0 \notin \text{cl} (\partial f (f^{-1}(0))) \), which is called \textit{strong Slater’s condition} \cite{43,60}.

We now consider the converse of Corollary \( \S \). Assume that

\[
\varphi(f(x) - \alpha) \geq \text{dist}(x, [f \leq \alpha]), \forall x \in [\alpha < f < \beta], \varphi \in \mathcal{K}(0, \beta - \alpha),
\]

which is equivalent to

\[
\frac{\varphi(f(x) - \alpha)}{f(x) - \alpha} \frac{f(x) - \alpha}{\text{dist}(x, [f \leq \alpha])} \geq 1, \forall x \in [\alpha < f < \beta].
\]

Thanks to Proposition\( \S \) the latter inequality implies that

\[
\frac{\varphi(f(x) - \alpha)}{f(x) - \alpha} \text{dist}(0, \partial f(x)) \geq 1, \forall x \in [\alpha < f < \beta].
\]

Thus, if \( \varphi \) satisfies the condition

\[
\int_{0}^{\beta - \alpha} \frac{\varphi(s)}{s} ds < +\infty,
\]

then we get

\[
\psi'(f(x) - \alpha) \text{dist}(0, \partial f(x)) \geq 1, \forall x \in [\alpha < f < \beta],
\]

where

\[
\psi(s) = \int_{0}^{s} \frac{s}{t} dt, \forall s > 0.
\]

Therefore, when \( f \) is convex, the converse of Corollary \( \S \) is given as following.

**Theorem 14** Assume that \( \varphi(f(x) - \alpha) \geq \text{dist}(x, [f \leq \alpha]), \forall x \in [\alpha < f < \beta], \) where \( \varphi \in \mathcal{K}[0, \beta - \alpha] \) and

\[
\int_{0}^{\beta - \alpha} \frac{\varphi(s)}{s} ds < +\infty. \tag{3}
\]

Then, we get

\[
\psi'(f(x) - \alpha) \text{dist}(0, \partial f(x)) \geq 1, \forall x \in [\alpha < f < \beta].
\]

This result has been appeared in \cite[Theorem 6]{16}, \cite[Theorem 30]{15}. We remark that when \( \varphi(s) = \tau s^\theta, (\tau, \theta > 0), \) then the condition (3) holds.

We will show that the Theorem \( \S \) covers numerous results on Lipschitz global error bounds in the literature.

- In \cite{65}, Robinson proved that if \( f \) satisfies the Slater condition (there exists \( \bar{x} \) such that \( f(\bar{x}) < 0 \) and the set \( [f \leq 0] \) is bounded then \( f \) has a Lipschitz global error bound. More generally, in \cite{23}, Deng proved the following fact: If there exist \( \delta > 0, \Delta > 0 \) such that

\[
[f \leq -\delta] \neq \emptyset \quad \text{and} \quad \sup_{[f \leq 0]} \text{dist}(x, [f \leq -\delta]) \leq \Delta,
\]

then

\[
\text{dist}(x, [f \leq 0]) \leq \frac{\Delta}{\delta} [f(x)]_+, \forall x \in X.
\]
Let us show that this result is actually a consequence of Theorem 13. Indeed, take \( x \in [f = 0] \) and \( u \in \partial f(x) \). For any \( \varepsilon > 0 \), there exists \( z \in [f \leq -\delta] \) such that \( \text{dist}(x, z) \leq \Delta + \varepsilon \). Thus, we obtain
\[
\delta \leq f(x) - f(z) \leq \|u\|\|x - z\| \leq \|u\|(\Delta + \varepsilon),
\]
which implies that
\[
\inf_{[f = 0]} \text{dist}(0, \partial f(x)) \geq \frac{\delta}{\Delta}.
\]
Combining with Theorem 13, \( f \) has Lipschitz global error bound.

Note that Deng’s result [23, Theorem 1] also covers the one in [22], in which the author start form the assumption that there exist a unit vector \( u \) and a constant \( \tau > 0 \) such that
\[
f^\infty(u) = \sup_{t > 0} \frac{f(x + tu) - f(x)}{t} \leq -\frac{1}{\tau} \tag{4}
\]
to derive that \( \text{dist}(x, [f \leq 0]) \leq \tau f(x) \), \( \forall x \in X \).

**The work of Robinson was also generalized in other directions.** More precisely, instead of the boundedness assumption on the set \( [f \leq 0] \), in [58], Mangasarian used the asymptotic constraint qualification condition (this means for any sequence \( (x_k)_{k \in \mathbb{N}} \subset [f = 0] \) such that \( \lim \|x_k\| = \infty \), then the zero vector is not a limit point of any sequence \( (u_k)_{k \in \mathbb{N}}, \text{ with } u_k \in \partial f(x_k) \)) to obtain a Lipschitz global error bound for differentiable convex function. Auslender and Crouzeix in [5] extended the work of Mangasarian to the case nonsmooth convex functions. On the other hand, in [38, Theorem 2], Klatte and Li proved that, a convex function satisfies the Slater and the asymptotic qualification conditions if and only if
\[
\inf_{x \in [f = 0]} \text{dist}(0, \partial f(x)) > 0.
\]
Therefore, it is clear that the results of Mangasarian [58], Auslender and Crouzeix [5] are the consequences of Theorem 13.

**In [43], Lewis and Pang characterized Lipschitz error bounds using directional derivatives as follows.** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a lower semicontinuous and convex function. They proved that the Lipschitz global error bound holds for \( f \):
\[
\text{dist}(x, [f \leq 0]) \leq \tau f(x), \quad \forall x \in \mathbb{R}^n,
\]
if and only if
\[
f'(\bar{x}, d) \geq \tau \|d\|, \quad \forall \bar{x} \in [f = 0], \ d \in N_{[f \leq 0]}(\bar{x}),
\]
where the cone normal is defined by \( N_S(\bar{x}) = \{u \in X^* | \langle u, y - \bar{x} \rangle \leq 0, \forall y \in S \}, \forall S \subset \mathbb{R}^n \). This result has been obtained by several other researchers, we can mention here the works of Ng and Zheng [59], [60] where they characterized error bounds for lower semicontinuous functions. Consider now [60, Theorem 3.1].

Suppose that \( X \) is a reflexive Banach space. Then the following statements are equivalent
(i) \( \text{dist}(x, [f \leq 0]) \leq \tau[f(x)]_+, \) for all \( x \in X. \)

(ii) For each \( x \in [f = 0], \) we get

\[
\inf \{ f'(x, d) | d \in N_{f=0}(x), \|d\| = 1 \} \geq \frac{1}{\tau}.
\]

(iii) For each \( x \in X \setminus [f \leq 0], \) there exists \( d_x \in X, \|d_x\| = 1, \) such that

\[
f'(x, d_x) \leq -\frac{1}{\tau}.
\]

Let us prove that the (iii) above assertions are equivalent to (ii) of Theorem 13. Assume that the assumption (iii) holds, then for all \( x \in X \setminus [f \leq 0], \) we get

\[
|\nabla f|(x) \geq -f'(x, d) \geq \frac{1}{\tau}.
\]

Conversely, suppose that \(|\nabla f|(x) \geq \frac{1}{\tau}, \forall x \in X \setminus [f \leq 0]. \) Take any \( x \in X, \) we get

\[
\text{dist}(0, \partial f(x)) \geq \frac{1}{\tau}, \forall x \in X \setminus [f \leq 0],
\]

hence there exists \( d_x \in X \) such that

\[
-\tau = \inf \{ \langle x^*, d \rangle | x^* \in X^*, \|x^*\| \leq \tau \} \geq \sup \{ \langle u, d \rangle | u \in \partial f(x) \} = f'(x, d).
\]

It follows that

\[
f'(x, d_x) \leq -\frac{1}{\tau}.
\]

Similarly, by setting \( \varphi(s) = \tau s^\theta, (\tau, \theta > 0), \) we can see that the following result of Ng and Zheng [59] is also a consequence of Theorem 14:

Let \( X \) be a reflexive Banach space and \( f : X \to \mathbb{R} \cup \{+\infty\} \) a continuous function. Suppose that for each \( x \in X \setminus S, \) there exists \( d_x \in X, \|d_x\| = 1 \) and \( \tau > 0, \theta \in (0, 1) \) such that

\[
f'(x, d_x) \leq -\tau f^{1-\theta}(x).
\]

Then we get

\[
\text{dist}(x, [f \leq 0]) \leq \tau[f(x)]_+, \forall x \in X.
\]

### 3.3 Qualification conditions and error bounds

#### 3.3.1 Slater’s condition and error bounds

We recall that if there exists \( \bar{x} \) such that \( f(\bar{x}) < 0 \) then \( f \) is said to satisfy the Slater condition. This condition plays an important role for the study of error bounds. The existence of the Lipschitz global error bound usually requires the convexity and the Slater condition. We consider the following example, which shows that for a convex function without the Slater condition, the Lipschitz global error bound may fail to hold.
Example 1 \[f(x,y) = x + \sqrt{x^2 + y^2}, \ (x,y) \in \mathbb{R}^2.\]

It is easy to check that the function \( f \) is convex, nonnegative on \( \mathbb{R}^2 \) and \( [f = 0] = \{(x,0)|x \leq 0\} \) has empty interior. Take the sequence \( (z_k = (-k,1))_{k \in \mathbb{N}} \) then \( f(z_k) \) converges to 0 but \( \text{dist}(z_k,[f \leq 0]) = 1, \forall k \in \mathbb{N} \), so that there is not global error bound for \( S \).

As mentioned earlier, the Slater condition was used for the first time by Robinson \[65\].

Theorem 15 \[65\] Let \( f_1, \ldots, f_m \) be convex functions on \( \mathbb{R}^n \) and assume that there is \( \bar{x} \) such that \( f_i(\bar{x}) < 0, \ldots, f_m(\bar{x}) < 0 \). Then there exists \( \tau > 0 \) such that

\[
\text{dist}(x, [f \leq 0]) \leq \tau \|x - \bar{x}\| \sum_{i=1}^{m} [f_i(x)]_+ , \forall x \in \mathbb{R}^n,
\]

where \( f(x) = \max_{i=1, ..., m} f_i(x) \).

In additional, when \( \{x \in \mathbb{R}^n|f_i(x) \leq 0, i = 1, ..., m\} \) is bounded then there exists \( \tau > 0 \) such that

\[
\text{dist}(x, [f \leq 0]) \leq \tau \sum_{i=1}^{m} [f_i(x)]_+ , \forall x \in \mathbb{R}^n.
\]

As a consequence, we immediately deduce that the convex function systems \( f_1, \ldots, f_m \) has Lipschitz local error bound.

Luo and Luo \[52\] used the Slater condition to establish the Lipschitz global error bound for convex quadratic systems, this result has been extended by Pang and Wang \[70\]. In general, the Slater condition is not sufficient to ensure that the global error bound holds, even if \( f \) is a convex function. We consider the following example:

Example 2 \[44\] Let \( f_1, f_2: \mathbb{R}^4 \to \mathbb{R} \) be defined by \( f_1(x) = x_1 \) and

\[
f_2(x) = x_1^{16} + x_2^8 + x_3^6 + x_1^2x_2^2x_3^2 + x_1^2x_2^4 + x_1^4x_2^4 + x_1^4x_2^6 + x_1^2x_2^6 + x_1^2 + x_2^2 + x_3^2 - x_4,
\]

for all \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \). Define \( f(x) = \max\{f_1(x), f_2(x)\}, \forall x \in \mathbb{R}^4 \).

We get the following properties, (see \[44\]).

(i) \( f_1, f_2 \) are convex polynomial functions, therefore \( f \) is convex.

(ii) \( f \) satisfies the Slater condition.

(iii) For any \( \alpha, \beta \in \mathbb{R} \) with \( \alpha \leq \beta \), then \( \sup_{x \in [f \leq \beta]} \text{dist}(x, [f \leq \alpha]) = +\infty. \)

By taking \( \alpha = 0, \beta = 1 \) in the property (iii), we imply that there exists a sequence \( (x_k)_{k \in \mathbb{N}} \subset [f \leq 1] \) such that \( \text{dist}(x_k, [f \leq 0]) = +\infty \), this show that \( f \) can not possess the Hölder global error bound.

However enhancing the assumptions we can derive global error bounds from the Slater like condition. For instance, let \( f \) be a lower semicontinuous, convex function on \( \mathbb{R}^n \) which satisfies the Slater condition, then \( f \) has Lipschitz global error bound if one of the following assertions holds.

1. \( f \) can be expressed as maximum of finitely many bounded below convex polynomials function on \( \mathbb{R}^n \), i.e: \( f(x) = \max_{i=1, ..., d} f_i(x), \forall x \in \mathbb{R}^n \), where \( f_i \) is a polynomial function on \( \mathbb{R}^n \) with \( \inf f_i > -\infty \), for all \( i = 1, \ldots, d \), see \[44\] Theorem 4.1.]
2. $f$ is a separable function (in the sense that $f(x) = \sum_{i=1}^{n} f_i(x_i)$ where $x = (x_1, \ldots, x_n)$ and each $f_i$ is a lower semicontinuous function), see [44, Theorem 4.1].

3. $f$ is well-posed (for any sequence $\{x_k\}_{k \in \mathbb{N}}$ for which dist$(0, \partial f(x_k)) \to 0$ then $f(x_k) \to \inf_X f$), see [43, Corollary 1].

4. $f$ satisfies the asymptotic qualification condition, see [5].

Notice that if $f$ is a convex function, then $f$ satisfies Slater condition if and only if $0 \notin \partial f(-1(0))$. We can easily see that, if $f$ satisfies the Slater condition and the level set $[f \leq 0]$ is bounded then $f$ possesses the strong Slater condition. Furthermore, in [38], Klatte and Li proved that, for a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which satisfies the Slater condition, the following conditions are equivalent:

1. The strong Slater condition holds.
2. The asymptotic qualification condition is satisfied.
3. \( \sup_{x \in [f=0]} \inf_{y \in [f<0]} \frac{\|x-y\|}{f(y)} < +\infty. \)

### 3.3.2 Abadie qualification condition and error bounds

We begin this subsection by considering an example:

**Example 3** [49] For $(x, y) \in \mathbb{R}^2$, take $f_1(x, y) = x + y$, $f_2(x, y) = -x - y$, $f_3(x, y) = (x + y)^2$ and $f(x, y) = (f_1, f_2, f_3)(x, y)$. Then $[f \leq 0] = \{(x, -x) | x \in \mathbb{R}^n\}$ has no interior point, but we can check that dist($(x, y), [f \leq 0]) \leq 2\|f(x, y)\|_+$, for all $(x, y) \in \mathbb{R}^2$.

This means that, the global error bound may be hold without the Slater condition. In [49], Li used the Abadie qualification condition to characterize Lipschitz-type error bound for convex quadratic systems.

Recall that, the tangent cone of $S \subset \mathbb{R}^n$ is defined by

\[ T_S(x) = \{y \in \mathbb{R}^n | \langle u, y \rangle \leq 0, \forall u \in N_S(x)\}. \]

Let us now recall the definition of Abadie’s condition.

**Definition 3** [49] We say that the systems $f_1, f_2, \ldots, f_m : X \to \mathbb{R}$ satisfies the Abadie condition at $\bar{x} \in S = \{x \in X | f_i(x) \leq 0, i = 1, \ldots, m\}$ if

\[ T_S(\bar{x}) = \{u \in X | \langle f'_i(\bar{x}), u \rangle \leq 0, \forall i \in I(\bar{x})\}, \]

where $I(\bar{x}) = \{i : f_i(\bar{x}) = 0\}$.

If this property holds at every point in $S$, then we say that the systems $f_1, f_2, \ldots, f_m$ satisfies the Abadie condition on $S$.

When $X = \mathbb{R}^n$ and $f_1, \ldots, f_m$ are convex functions, we have the two following properties, see [49].
1. The system $f_1, f_2, \ldots, f_m$ satisfies the Abadie condition at $\bar{x} \in S$ if and only if

$$N_S(\bar{x}) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i f_i'(\bar{x})|\lambda_i \geq 0 \right\}.$$

2. If there exists $x \in S$ such that $f_i(x) < 0$ with $f_i$ is not affine function, for all $i = 1, \ldots, m$ then the systems $f_1, f_2, \ldots, f_m$ satisfies Abadie’s condition on $S$.

Let us now give a necessary and sufficient condition for a convex quadratic system to have a Lipschitz-type global error bound, which was established by Li [49, Theorem 4.2].

**Theorem 16** [49] Let $f_1, f_2, \ldots, f_m$ be convex quadratic functions on $\mathbb{R}^n$ such that $S = \{x \in \mathbb{R}^n|f_i(x) \leq 0, (i = 1 \ldots, m)\}$ is nonempty. The following statements are equivalent

(i) The system $(f_i)_{i=1,\ldots,m}$ satisfies the Abadie condition on $S$.

(ii) There exists $\tau > 0$ such that

$$\text{dist}(x, S) \leq \tau \sum_{i=1}^{m} [f_i(x)]_+, \forall x \in \mathbb{R}^n.$$

Later, in [62, Theorem 6], Ngai and Théra extended this result in the Banach space. In which, $f_i: X \to \mathbb{R}, i = 1, \ldots, m$ are defined by

$$f_i(x) = \frac{1}{2} \langle A_i x, x \rangle + \langle B_i, x \rangle + c_i,$$

where $A_i: X \times X \to \mathbb{R}$ be a symmetric continuous bilinear and semi-definite positive, $B_i \in X^*$ and $c_i \in \mathbb{R}$, for $i = 1, \ldots, m$. In this paper, Ngai and Théra also gave the relation between the Abadie condition and the Lipschitz local error bound for the convex function systems.

**Theorem 17** [62] Let $f_1, \ldots, f_m$, be convex continuous functions on the neighborhood of $\bar{x} \in S = \{x \in X|f_i(x) \leq 0, i = 1, \ldots, m\}$. Set $f = \max_{i=1,\ldots,m} f_i$.

(i) If there exist $\tau > 0, \varepsilon > 0$ such that

$$\text{dist}(x, S) \leq \tau \sum_{i=1}^{m} [f_i(x)]_+, \forall x \in B_\varepsilon(\bar{x}) \cap K,$$

then the Abadie condition is satisfied on $B_\delta(\bar{x}) \cap S$ for some $\delta > 0$.

(ii) If $f_i, \ldots, f_m$ are differentiable on $B_\delta(\bar{x})$, then the converse of part (i) holds.
4 Existence and quantitative results

The first result on local error bound was deduced from the result of Hörmander, in his work on the fundamental solution of partial differential equation.

**Theorem 18 (Hörmander, 1958)** \[35\] Let $f$ be a polynomial function on $\mathbb{R}^n$. With the assumption that $[f \leq 0]$ is nonempty, there exist $\tau > 0$, $a > 0$ and $b \in \mathbb{R}$ such that

$$\text{dist}(x, [f = 0]) \leq \tau (1 + \|x\|)^b |f(x)|^a, \forall x \in \mathbb{R}^n.$$  

This “error bound” has an extra factor of $(1 + \|x\|)^b$. One sees that we can remove this extra factor when restricting the error bound to a bounded region, in that case this local error bound for $f$ can be deduced from. Luo and Luo applied the above theorem to obtain the Hölder local error bound for polynomial function systems \[52, Theorem 2.2\], this result was extended for analytic systems, by Luo and Pang \[53, Theorem 2.2\]. Recently, Kurdyka and Spondziewa \[42, Corollary 10\] showed that the exponents $a, b$ in Theorem 18 can be computed explicitly:

$$b = 2, \quad a = \frac{1}{d(6d-3)^{n-1}}.$$  

**Result of Łojasiewicz**  
A very general local error bound is deduced from the result of Łojasiewicz, Theorem 2 if we take $\phi(x) = f(x)$ and $\psi(x) = \text{dist}(x, [f \leq 0])$, we get a local error bound result for subanalytic functions, also called Łojasiewicz function inequality. It also includes a special case of the polynomial equation studied by Hörmander.

**Theorem 19** \[50\] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous subanalytic function. For any compact set $K \subset \mathbb{R}^n$, there exist $\tau > 0, a > 0$ such that

$$\text{dist}(x, [f \leq 0]) \leq \tau [f(x)]^a, \forall x \in K.$$  

With a direct application of the Theorem 19 to a subanalytic system, we recover a result of Luo and Pang \[53, Theorem 2.2\], in which they obtained the similar result for an analytic system:

**Theorem 20** Let $f_1, f_2, \ldots, f_r$ and $g_1, g_2, \ldots, g_s$ be continuous subanalytic functions on $\mathbb{R}^n$, set

$$S = \{x \in \mathbb{R}^n | f_i(x) \leq 0, i = 1, \ldots, r; g_j(x) = 0, j = 1, \ldots, s\}.$$  

Then, for each compact set $K \subset \mathbb{R}^n$, there exist $\tau > 0, a > 0$ such that

$$\text{dist}(x, S) \leq \tau ([f(x)]^a + \|g(x)\|^a, \forall x \in K,$n

where $f(x) = (f_1(x), \ldots, f_r(x))$, $g(x) = (g_1(x), \ldots, g_s(x))$.

We mention that in all the above results of error bound, the Hölder exponent is not unknown, even in the result of Luo and Luo \[52\] for polynomial function systems.
4.1 Local error bound for polynomial

We are now interested in the estimation of exponents within error bounds. First, we present the result of Gwozdziewicz [30], in which a local quantitative error bound for a single real polynomial function with an isolated zero is provided.

For each \( n, d \in \mathbb{N} \), we set
\[
\kappa(n, d) = (d-1)^n + 1 \quad \text{and} \quad R(n, d) = \begin{cases} 1 & \text{if } d = 1 \\ d(3d-1)^{n-1} & \text{if } d \geq 2. \end{cases}
\]

**Theorem 21** [30] Let \( f \) be a polynomial function on \( \mathbb{R}^n \) with degree \( d \). Assume that \( x = 0 \) is an isolate zero of \( f \), this means \( f(0) = 0 \) and there is \( \delta > 0 \) with \( f(x) \neq 0 \), for all \( x \in B_\delta(0) \setminus \{0\} \). Then there exist positive constants \( \tau, \varepsilon \) such that
\[
\|x\| \leq \tau |f(x)|^{\frac{1}{\kappa(n, d)}},
\]
for all \( \|x\| \leq \varepsilon \).

A similar result for polynomial function system was given by Kollár in [39].

**Theorem 22** [39] Let \( f_1, \ldots, f_m \) be some polynomial functions on \( \mathbb{R}^n \) whose degrees do not exceed \( d \). Set \( f(x) = \max_{i=1,\ldots,m} f_i(x) \) for \( x \in \mathbb{R}^n \). Assume that there is \( \delta > 0 \) such that \( f(x) = 0 \) and \( f(x) \neq 0, \forall x \in B_\delta(0) \setminus \{0\} \). Then there exist \( \tau, \varepsilon \) such that
\[
\|x\| \leq \tau |f(x)|^{\frac{1}{n^\beta}},
\]
for all \( x \) such that \( \|x\| \leq \varepsilon \), where
\[
\beta(n-1) = \left(\frac{n-1}{\left[\frac{n-1}{2}\right]}\right).
\]

Without the assumption of isolated zero point, Kurdyka and Spodzieja [42, Corollary 4] (see also [61]) obtained an error bound for a polynomial function with the Hölder exponent \( a = R^{-1}(n, d) \).

To our knowledge, these are the first general results on error bounds with some estimations of the exponent. Some applications of these above results can be found in [18, 44, 45, 47, 61, 46].

In [47], Li, Mordukhovich and Pham, gave local error bounds for polynomial function systems in the nonconvex case, with exponents explicitly determined by the dimension of the underlying space and the degree of the involved polynomial functions. In this work, they obtained two results, one is based on Lojasiewicz gradient inequality, and the other result is proved with a technique similar to that of Theorem 20.

**Theorem 23** [47]. Let \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_s \) be real polynomial functions on \( \mathbb{R}^n \) with degree at most \( d \), and let
\[
S = \{ x \in \mathbb{R}^n | f_i(x) \leq 0, g_j(x) = 0 \}.
\]
Then for each \( \bar{x} \in S \) there exist \( \tau > 0, \varepsilon > 0 \) such that
\[
\operatorname{dist}(x, S) \leq \tau \left( \sum_{i=1}^{r} [f_i(x)]_+ \right)^{\frac{1}{R(n+r+s,d+1)}} + \sum_{j=1}^{s} |g_j(x)|, \quad \text{with } \|x - \bar{x}\| \leq \varepsilon.
\]

15
Before beginning the proof of the latter theorem, let us recall a result of D’Acunto and Kurdyka [20], which established Lojasiewicz gradient inequality for polynomial functions.

**Theorem 24** [20] Let \( f \) be a polynomial function with degree \( d \), suppose that \( f(0) = 0 \). There exists \( c > 0, \varepsilon > 0 \) such that

\[
\|\nabla f(x)\| \geq \tau|f(x)|^{1 - \frac{1}{R(n,d)}} \quad \text{with} \quad \|x\| \leq \varepsilon.
\]

Now, we apply this result to establish the Lojasiewicz gradient inequality for maximum of finitely many polynomial functions.

**Lemma 25** Let \( f(x) = \max_{i=1,...,r} f_i(x) \) where \( f_i \) are polynomial functions on \( \mathbb{R}^n \) whose degrees do not exceed \( d \), and \( \bar{x} \in \mathbb{R}^n \) with \( f(\bar{x}) = 0 \). Then, exist \( c > 0, \varepsilon > 0 \) such that

\[
\text{dist} \ (0, \partial f(x)) \geq \tau|f(x)|^{1 - \frac{1}{R(n+r-1,d+1)}} \quad \text{with} \quad \|x - \bar{x}\| \leq \varepsilon.
\]

**Proof.** Without loss of generality, suppose that \( f_i(\bar{x}) = 0 \), \( i = 1,\ldots,r \). For each subset \( I = \{i_1,\ldots,i_q\} \subset \{1,\ldots,r\} \), we define the polynomial function \( F_I: \mathbb{R}^{n+q-1} \to \mathbb{R} \) as following

\[
F_I(x,\lambda) = \begin{cases} 
\sum_{j=1}^{q-1} \lambda_j f_{i_j}(x) + \left(1 - \sum_{j=1}^{q-1} \lambda_j\right) f_{i_q}(x) & \text{if } q \geq 2 \\
 f_{i_1}(x) & \text{if } q = 1,
\end{cases}
\]

where \( \lambda = (\lambda_1,\ldots,\lambda_{q-1}) \in \mathbb{R}^{q-1} \). It is clear that \( F_I \) has degree at most \( d + 1 \) and \( F(\bar{x},\lambda) = 0, \forall \lambda \in \mathbb{R}^{q-1} \). Set

\[
P = \left\{ \lambda \in \mathbb{R}^{q-1} | \lambda_j \geq 0, \sum_{j=1}^{q-1} \lambda_j = 1 \right\}.
\]

\( P \) is a compact set. For each \( \bar{\lambda} \in P \), if \( \nabla F_I(\bar{x},\bar{\lambda}) = 0 \), then thanks to Theorem 24 there exit \( \varepsilon_I > 0, \tau_I > 0 \) such that

\[
\|\nabla F(x,\lambda)\| \geq \tau_I|F_I(x,\lambda)|^{1 - \frac{1}{R(n+q-1,d+1)}}, \quad \text{with} \quad \|\lambda - \bar{\lambda}\| \leq \varepsilon_I, \|x - \bar{x}\| \leq \varepsilon_I.
\]  \hspace{1cm} (6)

In the other case, when \( \nabla F_I(\bar{x},\bar{\lambda}) \neq 0 \) then (6) immediately holds. By the compactness of \( P \), the inequality (6) holds for all \( \lambda \in P \). Set

\[
\tau = \min \{\tau_I|I \subset \{i,\ldots,r\}, I \neq \emptyset\} > 0 \quad \text{and} \quad \varepsilon = \min \{\varepsilon_I|I \subset \{i,\ldots,r\}, I \neq \emptyset\} > 0.
\]

Take an arbitrary point \( x \in \mathbb{R}^n \) such that \( \|x - \bar{x}\| \leq \varepsilon \) and \( I(x) = \{i|f_i(x) = f(x)\} \), then there exist \( \lambda_i \geq 0, i \in I(x) \) and \( \sum_{i \in I(x)} \lambda_i = 1 \) such that

\[
\text{dist} \ (0, \partial f(x)) = \left\| \sum_{i \in I(x)} \lambda_i \nabla f_i(x) \right\|.
\]
On the other hand, for \( i \in I(x) \), we have
\[
F_{I(x)}(x, \lambda) = \sum_{i \in I(x)} \lambda_i f_i(x) = f(x)
\]
and
\[
\| \nabla F_{I(x)}(x, \lambda) \| = \| \sum_{i \in I(x)} \lambda_i \nabla f_i(x) \| = \text{dist}(0, \partial f(x)).
\]
By combining the above inequalities and (6), we have the conclusion.

\[\square\]

We now provide the proof of Theorem 23.

**Proof of Theorem 23** We consider the proof for \( \bar{x} \in \text{bd}(S) \). For any \( e = (e_i)_{i=1}^{s} \in \{-1, 1\}^s \), define the function
\[
f_e(x) = \max \{0, f_1(x), \ldots, f_r(x), e_1 g_1(x), \ldots, e_s g_s(x)\}, \quad \forall x \in \mathbb{R}^n.
\]
One can see that \( f_e \) is the maximum of \( r + s + 1 \) polynomial function with degree not exceed \( d \), and \( f_e(\bar{x}) = 0 \). Applying Lemma 25, one obtains \( \tau_e > 0 \) and \( \varepsilon_e > 0 \) such that
\[
\text{dist}(0, \partial f_e(x)) \geq \tau_e |f_e(x)|^{1 - \frac{1}{\mathcal{K}(n+r+s,d+1)}}, \quad \forall \|x - \bar{x}\| \leq \varepsilon_e.
\]
Set
\[
\tau = \min \{\tau_e |e \in \{-1, 1\}^s\} > 0, \quad \varepsilon = \{\varepsilon_e | e \in \{-1, 1\}^s\} > 0,
\]
and
\[
f(x) = \max \{0, f_1(x), \ldots, f_r(x), g_1(x), \ldots, g_s(x), -g_1(x), \ldots, -g_s(x)\}
\]
For any \( x \) with \( \|x - \bar{x}\| \leq \varepsilon \) and \( f(x) > 0 \), then we can find \( e \in \{-1, 1\}^s \) such that \( f(x) = f_e(x) \) and \( \text{dist}(0, \partial f(x)) = \text{dist}(0, \partial f_e(x)) \). Therefore,
\[
\text{dist}(0, \partial f(x)) \geq \tau |f(x)|^{1 - \frac{1}{\mathcal{K}(n+r+s,d+1)}}, \quad \forall \|x - \bar{x}\| \leq \varepsilon.
\]
By applying Corollary 11 with \( \phi(s) = s^{\frac{1}{\mathcal{K}(n+r+s,d+1)}} \), \( \forall s > 0 \), we obtain the conclusion. \[\square\]

By using the same technique as in [53, Theorem 2.1], [52, Theorem 2.2], one can obtain an error bound whose exponent is different from Theorem 23.

**Theorem 26** [47] With the assumptions of Theorem 23, we have the following local error bound.
\[
\text{dist}(x, S) \leq \tau \left( \sum_{i=1}^{r} \left| f_i(x) \right| + \sum_{j=1}^{s} \left| g_j(x) \right| \right)^{\frac{2}{\mathcal{K}(n+r+s,d+2)}}, \quad \text{with } \|x - \bar{x}\| \leq \varepsilon.
\]
4.2 Global error bounds for polynomial

4.2.1 Nonconvex case

We begin this subsection by recalling the result of Luo and Sturm [54]. The authors established the global error bound for the zero set of a quadratic function.

**Theorem 27** [54] Let \( f: \mathbb{R}^n \to \mathbb{R} \) be the quadratic function. There exists a constant \( \tau > 0 \) such that

\[
\text{dist}(x, [f = 0]) \leq \tau (|f(x)| + |f(x)|^{1/2}), \quad \forall x \in \mathbb{R}^n.
\]

This result is recovered by the works of [60, Corollary 5], [25, Corollar 2]. Remark that this theorem does not until hold for an arbitrary polynomial.

**Example 4** Let \( f(x, y) = (xy - 1)^2 + (x - 1)^2, \forall (x, y) \in \mathbb{R}^2 \).

One has \([f \leq 0] = \{(1, 1)\}\). Consider the sequence \((x_k = \frac{1}{k}, y_k = k)_{k \in \mathbb{N}}\), it is easy to check that

\[
0 < f(x_k, y_k) = \left(1 - \frac{1}{k}\right)^2 < 1, \forall k \in \mathbb{N} \text{ and } d((x_k, y_k), [f \leq 0]) \to +\infty (k \to +\infty),
\]

therefore, \( f \) does not possess Hölder global error bound.

However, when \( f \) is a polynomial convex, this result was proved by Yang [73], and we present it in Theorem 32.

Let us now present the characterization of global error bound for semi–algebraic, which is proved by Ha [31].

Suppose that \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) has a Hölder global error bound,

\[
\text{dist}(x, [f \leq 0]) \leq \tau \left([f(x)]_+^a + [f(x)]_+^b\right), \quad \forall x \in \mathbb{R}^n.
\]

(7)

We observe easily that for any sequence \((x_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n\), two following assertions hold

(i) If \( f(x_k) \to 0 \), then \( \text{dist}(x_k, [f \leq 0]) \to 0 \).

(ii) If \( \text{dist}(x_k, [f \leq 0]) \to +\infty \), then \( f(x_k) \to +\infty \).

Conversely, in [31], Ha proved that, for a polynomial function which satisfies two above conditions, then it possesses Hölder global error bound. This result was extended for the class of continuous semi-algebraic functions, see [25, Theorem 2]. The definition of the semi-algebraic function is well-known, we can see the one in [25, Definition 1].

**Theorem 28 (Characterization of global error bound for semi-algebraic)** [31, 25] Let \( f: \mathbb{R}^n \to \mathbb{R} \) be a continuous semi-algebraic function. The following statements are equivalent:

1. For any sequence \((x_k)_{k \in \mathbb{N}} \in \mathbb{R}^n \setminus [f \leq 0] \) and \( \|x_k\| \to +\infty \), we have:

   (i) If \( f(x_k) \to 0 \) then \( \text{dist}(x_k, [f \leq 0]) \to 0 \).

   (ii) If \( \text{dist}(x_k, [f \leq 0]) \to +\infty \) then \( f(x_k) \to +\infty \).

2. There exist \( \tau > 0 \) and \( a, b > 0 \) such that

\[
\text{dist}(x, [f \leq 0]) \leq \tau \left([f(x)]_+^a + [f(x)]_+^b\right), \quad \forall x \in \mathbb{R}^n.
\]
Proof. (2) ⇒ (1) is obvious, we now prove the implication (1) ⇒ (2). The proof is divided into two parts. Using (i), we shall prove that an error bound holds on the neighborhood of \([f \leq 0]\), while by using (ii) we provide a bound for large \(\text{dist}(x, [f \leq 0])\).

Assume (i) holds. Let us prove that there exist \(\tau_1 > 0, a > 0\) and \(r > 0\) such that

\[
\text{dist}(x, [f \leq 0]) \leq \tau_1 [f(x)]^a_+, \forall x \in [f \leq r].
\]

For \(t \in \mathbb{R}\), put \(\varphi(t) = \sup\{\text{dist}(x, [f \leq 0]) : f(x) = t\}\). It is a semi-algebraic function. Thanks to (i), there exists \(r > 0\) such that \(\varphi(t) < \infty\) for all \(t \in [0, r]\). We can choose \(r\) sufficiently small such that \(\varphi(t)\) is continuous and \(\varphi(t) \neq 0\) on \((0, r]\). By using Puiseux Lemma:

\[
\varphi(t) = \tau t^a + O(t^a), \ (t \to 0).
\]

From the assumption (i), it can be seen that \(\tau > 0, a > 0\). So there exist \(r > 0\) and \(\tau_1 > 0\) such that \(\varphi(t) \leq \tau_1 t^a\), for all \(t \in [f \leq r]\). It means that

\[
\text{dist}(x, [f \leq 0]) \leq \tau_1 [f(x)]^a_+, \forall x \in [f \leq r].
\]

Using (ii), let us prove that there exist \(\tau_2 > 0, b > 0\) and \(\delta > 0\) such that

\[
\text{dist}(x, [f \leq 0]) \leq \tau_2 [f(x)]^b_+, \forall x \in [\delta < f].
\]

This conclusion is clear when \(f\) is bounded from above. We assume thus that \(\sup_{\mathbb{R}^n} f = \sup_{\mathbb{R}^n} \varphi = +\infty\). It appears that \(\varphi(t) > 0\) when \(t\) is sufficiently large, so there exist \(\tau > 0\) and \(b > 0\) such that

\[
\varphi(t) = \tau t^b + O(t^b).
\]

This implies that there is \(c\tau_2 > 0, R > 0\) such that

\[
\text{dist}(x, [f \leq 0]) \leq \tau_2 [f(x)]^b_+, \forall x \in [R < f].
\]

It is easily seen that (ii) implies the existence of \(M > 0\) such that \(\text{dist}(x, S) < M\), for all \(x \in [r < f < R]\). It gives \(\text{dist}(x, [f \leq 0]) \leq \frac{M}{\tau_2} f(x)^a\). Combining with such inequality on the domain \([f \leq r]\) and \([f \geq R]\), we have the conclusion. □

The implication (2) ⇒ (1) in the latter theorem explains why do we need two exponents \([f(x)]^a_+\) and \([f(x)]^b_+\) in the global error bound \(7\). One is ensuring that the inequality \(7\) holds when \(\text{dist}(x, [f \leq 0]) \to 0\), and the other keeps such inequality holds when \(\text{dist}(x, [f \leq 0]) \to +\infty\). Generally, the exponents are different.

Example 5 [22] Let \(f(x, y) = x^2 + y^4, \forall (x, y) \in \mathbb{R}^2\).

It can be seen that \([f \leq 0] = \{(0, 0)\}\), and

\[
\text{dist}((x, y), [f \leq 0]) \leq f^\frac{1}{2}(x, y) + f^\frac{1}{4}(x, y), \forall (x, y) \in \mathbb{R}^2.
\]

On the other hands, by taking two sequences \((x_k^1 = k, y_n^1 = 0)_{k \in \mathbb{N}}\) and \((x_k^2 = 0, y_n^2 = 1/k)_{k \in \mathbb{N}}\), this follows that there does not exist \(\alpha \in \mathbb{R}\) such that

\[
\text{dist}((x, y), [f \leq 0]) \leq \tau [f(x, y)]^\alpha_+, \forall (x, y) \in \mathbb{R}^2.
\]
By using Theorem 28, Ha [31] provided a global error bound for polynomial function under a Palais–Smale condition. After that, his result was improved in [25] for continuous semi-algebraic functions.

We recall that, \( f \) is said to possess the Palais-Smale condition (PS) at \( r_0 \) if any sequence \( (x_k)_{k \in \mathbb{N}} \), for which \( f(x_k) \to r_0 \) and \( \text{dist}(0, \partial f(x_k)) \to 0 \), then \( (x_k)_{k \in \mathbb{N}} \) possesses a converging subsequence.

Theorem 29 [31, 25] Let \( f: \mathbb{R}^n \to \mathbb{R} \) be a continuous semi-algebraic function. Suppose that \( f \) satisfies the Palais-Smale condition at each \( r > 0 \), then there exist constants \( \tau > 0 \) and \( a, b > 0 \) such that
\[
\text{dist}(x, [f \leq 0]) \leq \tau \left( [f(x)]_+^a + [f(x)]_+^b \right), \forall x \in \mathbb{R}^n.
\]

Proof. It is enough to show that \( f \) satisfies the two conditions (i) and (ii) in Theorem 28. First we establish (i). By contradiction, we assume that there exists a sequence \( (x_k)_{k \in \mathbb{N}} \) and a constant \( \delta > 0 \) such that:
\[
\|x_k\| \to \infty, f(x_k) \to 0 \quad \text{and} \quad \text{dist}(x_k, [f \leq 0]) > \delta.
\]

Put \( X = \{ x | f(x) \geq 0 \} \), then \( X \) is a complete metric space. Applying Ekeland’s principle (see [27]), there is a sequence \( (y_k)_{k \in \mathbb{N}} \subset X \) such that
\[
f(y_k) \leq f(x_k) = \varepsilon_k
\]
\[
\text{dist}(x_k, y_k) \leq \sqrt{\varepsilon_k}
\]
\[
f(y_k) \leq f(x) + \sqrt{\varepsilon_k} \text{dist}(x, y_k), \forall x \in X.
\]

It is clear that \( f(y_k) \to 0 \) and \( \|y_k\| \to +\infty \). We can suppose that \( \text{dist}(y_k, [f \leq 0]) \geq \frac{\delta}{2} \), therefore \( \forall t \in (0, \frac{\delta}{2}) \) and for all \( u \in \mathbb{R}^n, \|u\| = 1 \) we obtain
\[
\frac{f(y_k + tu) - f(y_k)}{t} \geq -\sqrt{\varepsilon_k}.
\]

Thus \( |\nabla f|(y_k) \leq \sqrt{\varepsilon_k} \). On the other hands, \( ||\partial f(y_k)|| \leq |\nabla f|(y_k) \) (see [9, Remark 6.1]), therefore \( \partial f(y_k) \to 0 \), which is in contradiction with Palais-Smale’s condition.

Now, we will prove that \( f \) satisfies the condition (ii) of Theorem 28. By contradiction, suppose that there exists a sequence \( (x_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n \) such that:
\[
\|x_k\| \to \infty, \text{dist}(x_k, [f \leq 0]) \to +\infty \quad \text{and} \quad f(x_k) \to t \in \mathbb{R}.
\]

Set \( X = \{ x | f(x) \geq 0 \} \), \( X \) is a complete metric space. Applying Ekeland’s principle, there is a sequence \( (y_k)_{k \in \mathbb{N}} \subset X \) such that
\[
f(y_k) \leq f(x_k) = t_k
\]
\[
\text{dist}(x_k, y_k) \leq \frac{\text{dist}(x_k, [f \leq 0])}{2}
\]
\[
f(y_k) \leq f(x) + \frac{2f(x_k)}{\text{dist}(x_k, [f \leq 0])} \text{dist}(x, y_k), \forall x \in X
\]
Therefore, without loss of generality we can assume that the sequence $f(y_k)$ is convergent, $\|y_k\| \to \infty$ and $\text{dist}(y_k, [f \leq 0]) \to +\infty$, therefore,

$$\| \nabla f(y_k) \| \leq |\nabla f(y_k)| \leq \frac{2f(x_k)}{\text{dist}(x_k, [f \leq 0])} \to 0,$$

contradicting to Palais-Smale’s condition. \hfill \Box

### 4.2.2 Convex case

We begin this subsection by giving a result of Facchinei, Pang \cite{28}, they assert that a lower semicontinuous convex function, a Hölder-type error bound on a level set can be extended to a global error bound.

**Theorem 30** \cite{28} Let $f$ be a lower semicontinuous convex function on $\mathbb{R}^n$ with $[f \leq 0]$ nonempty. Suppose that there exist $\delta > 0$ and $\tau > 0$, $\theta > 0$ such that

$$\text{dist}(x, [f \leq 0]) \leq \tau \left( [f(x)]_+ + [f(x)]_{\theta}^\theta \right), \quad \forall x \in [f \leq \delta].$$

There exists $\tau' > 0$ such that

$$\text{dist}(x, [f \leq 0]) \leq \tau' \left( [f(x)]_+ + [f(x)]_{\theta}^\theta \right), \quad \forall x \in \mathbb{R}^n.$$

When we take $\theta = 1$, this means that for a convex function, the Lipschitz error bound on the level set can be extended to a global error bound.

**Proof.** Let $x \in \mathbb{R}^n$ such that $f(x) > \delta$ and $p = P_{[f \leq 0]} x$. It is clear that $f(p) = 0$. For any $\lambda \in (0, 1)$, we denote $x_\lambda = \lambda x + (1 - \lambda)p$. It can be seen that $p = P_{[f \leq 0]} x_\lambda$ and $\text{dist}(x_\lambda, [f \leq 0]) = \lambda \text{dist}(x, [f \leq 0])$. By convexity, we get

$$f(x_\lambda) \leq \lambda f(x) + (1 - \lambda) f(p) = \lambda f(x).$$

We deduce that

$$\text{dist}(x, [f \leq 0]) \leq \frac{\text{dist}(x_\lambda, [f \leq 0])}{f(x_\lambda)} f(x).$$

On the other hand, by choosing $\lambda = \frac{\delta}{2f(x)}$, we get

$$f(x_\lambda) \leq \lambda f(x) = \frac{\delta}{2} < \delta.$$

Therefore, thanks to the assumption on error bounds, we obtain

$$\text{dist}(x_\lambda, [f \leq 0]) \leq \tau \left( f(x_\lambda) + f^\theta(x_\lambda) \right).$$

It follows that

$$\frac{\text{dist}(x_\lambda, [f \leq 0])}{f(x_\lambda)} < \tau \left( 1 + f^\theta(x_\lambda) \right) < c \left( 1 + \left( \frac{\delta}{2} \right)^{\theta-1} \right).$$
Combining the above inequalities, we get
\[
\text{dist}(x, [f \leq 0]) \leq \tau \left( 1 + \left( \frac{\delta}{2} \right)^{\theta-1} \right) f(x).
\]
This means that
\[
\text{dist}(x, [f \leq 0]) \leq \tau \left( 1 + \left( \frac{\delta}{2} \right)^{\theta-1} \right) (f(x) + f^\theta(x)), \forall x \in \mathbb{R}^n.
\]

Combining this result with Theorem 19, we immediately obtain a result similar to [16, Theorem 3] and [24, Theorem 6].

**Theorem 31** Let \( f_i : \mathbb{R}^n \to \mathbb{R} \), \( i = 1, \ldots, m \) be continuous, convex and subanalytic functions. Assume that, the set
\[
S = \{ x \in \mathbb{R}^n | f_i(x) \leq 0, i = 1, \ldots, m \}
\]
is nonempty, compact. Then, there exist \( \tau, \theta > 0 \) such that
\[
\text{dist}(x, S) \leq \tau \left( [f(x)]_+ + [f(x)]_+^\theta \right), \forall x \in \mathbb{R}^n,
\]
where \( f(x) = \sum_{i=1}^m [f_i(x)]_+ \).

We remark that if \( f_i \) is coercive then for all \( r \in \mathbb{R} \), the set \( [f_i \leq r] \) is compact.

We recall now the definition of piecewise convex polynomial functions.

**Definition 4** [48, 45] A continuous function \( f \) on \( \mathbb{R}^n \) is called to be a piecewise convex polynomial function if there exist finitely many polyhedra \( P_1, \ldots, P_k \) with \( \bigcup_{j=1}^k P_j = \mathbb{R}^n \) such that the restriction of \( f \) on each \( P_j \), denoted by \( f_j \), is a convex polynomial function. The degree of \( f \), denoted by \( \deg(f) \), is defined as the maximum of \( \deg(f_j) \).

In [48], Li studied error bounds for a convex piecewise quadratic function. More precisely, let \( f \) be a convex piecewise quadratic function. Then, there exists \( \tau > 0 \) such that
\[
\text{dist}(x, [f \leq 0]) \leq \tau \left( [f(x)]_+ + \sqrt{[f(x)]_+} \right), \forall x \in \mathbb{R}^n.
\]
(8)

By using Theorem 22 and Theorem 30, Li [44] showed that, for a convex polynomial function \( f \) on \( \mathbb{R}^n \) with degree \( d \), there exists \( \tau > 0 \) such that
\[
\text{dist}(x, [f \leq 0]) \leq \tau \left( [f(x)]_+ + [f(x)]_{+\overline{e(n,d)}}^\frac{1}{d} \right), \forall x \in \mathbb{R}^n.
\]
(9)

This result is further improved by Yang [73].

**Theorem 32** [73] Let \( f \) be a polynomial convex with degree \( d \). There exists \( \tau > 0 \) such that
\[
\text{dist}(x, [f \leq 0]) \leq \tau \left( [f(x)]_+ + [f(x)]_{+\overline{e(n,d)}}^{\frac{1}{d}} \right), \forall x \in \mathbb{R}^n.
\]
The two above results (8), (9) have been extended by Li ([45]), for general convex piecewise polynomial function.

**Theorem 33** [45] Let $f$ be a piecewise convex polynomial function on $\mathbb{R}^n$ with degree $d$. Suppose that one of the following two conditions holds:

(i) If $\text{dist}(x, [f \leq 0]) \to +\infty$ then $f(x) \to +\infty$.

(ii) $f$ is convex.

There exists $c > 0$ such that

$$\text{dist}(x, [f \leq 0]) \leq c \left( [f(x)]_+ + \frac{1}{\kappa(n,d)} \right), \forall x \in \mathbb{R}^n.$$  

Let us now present a global error bound for convex polynomial function systems. In [52], under the Slater condition, Luo and Luo proved that a global Lipschitzian error bound holds for convex quadratic systems. After that, without the Slater condition, Pang and Wang in [70], showed that any systems of convex quadratic has a global error bound.

**Theorem 34** [70] Let $f_1, f_2, \ldots, f_m$ be convex quadratic functions. Assume that

$$S = \{ x \in \mathbb{R}^n | f_i(x) \leq 0, i = 1, \ldots, m \}$$

is not empty, then there exists a positive integer $\text{dist} \leq n + 1$ and a scalar $c > 0$ such that

$$\text{dist}(x, S) \leq c \max \left( \|f(x)_+\|, \|f(x)_+\|^{\frac{1}{2d}} \right), \forall x \in \mathbb{R}^n,$$

where $f(x) = (f_i(x))_{i=1, \ldots, m}, \forall x \in \mathbb{R}^n$.

Furthermore, if $S$ contains an interior point, then $d = 0$.

Similarly Theorem 17, the latter result is extended to the Banach space in [62], Theorem 7, with

$$f_i(x) = \frac{1}{2} \langle A_i x, x \rangle + \langle B_i, x \rangle + c_i,$$

where $A_i : X \times X \to \mathbb{R}$ is a symmetric continuous bilinear and semi-definite positive, $B_i \in X^*$ and $c_i \in \mathbb{R}$, for $i = 1, \ldots, m$.

Note that this result does not hold for a general convex polynomial function system, see Example 2. However, in some particular cases, the global error bound hold for such systems.

**Theorem 35** Let $f_1, \ldots, f_p$ be convex polynomial functions on $\mathbb{R}^n$ whose degrees are at most $d$. Let $f(x) = \max_{i=1, \ldots, m} f_i(x), \forall x \in \mathbb{R}^n$. Then, the following statements are hold

1. [44] If $f_i(x) \geq 0, \forall x \in \mathbb{R}^n, i = 1, \ldots, m$ then there exists $\tau > 0$ such that

$$\text{dist}(x, [f \leq 0]) \leq \tau \left( [f(x)]_+ + \frac{1}{\kappa(n,d)} \right), \forall x \in \mathbb{R}^n.$$
2. If \( S = \{ x \in K | f(x) \leq 0 \} \) is a nonempty compact set, where \( K \) is a convex polyhedral in \( \mathbb{R}^n \). Then, there exists \( \tau > 0 \) such that

\[
\text{dist}(x, S) \leq c \left( [f(x)]_+ + [f(x)]_{\frac{1}{\kappa(n, 2d)}} \right), \forall x \in K.
\]

3. Let \( K \) is a convex polyhedral in \( \mathbb{R}^n \) and \( S = \{ x \in K | f(x) \leq 0 \} \) is nonempty. Assume that, for each \( v \in K^\infty \): \( \max_{i=1, \ldots, p} f_i^\infty(v) = 0 \Rightarrow f_i^\infty(v) = 0 \) (see (4)). Then, there exists \( \tau > 0 \) such that

\[
\text{dist}(x, S) \leq c \left( [f(x)]_+ + [f(x)]_{\frac{1}{\kappa(n, 2d)}} \right), \forall x \in K,
\]

where \( K^\infty \) is recession cone of \( K \), defined by

\[
K^\infty = \{ v \in \mathbb{R}^n | x + tv \in K, \forall t > 0, x \in K \}.
\]

References

[1] Pierre Antoine Absil, Robert Mahony, and Benjamin Andrews. Convergence of the iterates of descent methods for analytic cost functions. *SIAM Journal on Optimization*, 16(2):531–547, 2005.

[2] Hedy Attouch and Jérôme Bolte. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features. *Mathematical Programming*, 116(1):5–16, 2009.

[3] Hedy Attouch, Jérôme Bolte, Patrick Redont, and Antoine Soubeyran. Proximal alternating minimization and projection methods for nonconvex problems: An approach based on the kurdyka–lojasiewicz inequality. *Mathematics of Operations Research*, 35(2):438–457, 2010.

[4] Hedy Attouch, Jérôme Bolte, and Benar Fux Svaiter. Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward–backward splitting, and regularized gauss–seidel methods. *Mathematical Programming*, 137(1-2):91–129, 2013.

[5] Alfred Auslender and Jean Pierre Crouzeix. Global regularity theorems. *Mathematics of Operations Research*, 13(2):243–253, 1988.

[6] Dominique Azé. A survey on error bounds for lower semicontinuous functions. In *ESAIM: Proceedings*, volume 13, pages 1–17. EDP Sciences, 2003.

[7] Dominique Azé and Jean Noël Corvellec. On the sensitivity analysis of hoffman constants for systems of linear inequalities. *SIAM Journal on Optimization*, 12(4):913–927, 2002.

[8] Dominique Azé and Jean Noël Corvellec. Characterizations of error bounds for lower semicontinuous functions on metric spaces. *ESAIM: Control, Optimisation and Calculus of Variations*, 10(3):409–425, 2004.

[9] Dominique Azé and Jean Noël Corvellec. Nonlinear local error bounds via a change of metric. *Journal of Fixed Point Theory and Applications*, 16(1-2):351–372, 2014.
[10] Dominique Azé and Jean Noël Corvellec. Nonlinear error bounds via a change of function. *Journal of Optimization Theory and Applications*, pages 1–24, 2016.

[11] Amir Beck and Shimrit Shtern. Linearly convergent away-step conditional gradient for non-strongly convex functions. *Mathematical Programming*, pages 1–27, 2015.

[12] Amir Beck and Marc Teboulle. Convergence rate analysis and error bounds for projection algorithms in convex feasibility problems. *Optimization Methods and Software*, 18(4):377–394, 2003.

[13] Jérôme Bolte, Aris Daniilidis, and Adrian Lewis. The lojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems. *SIAM Journal on Optimization*, 17(4):1205–1223, 2007.

[14] Jérôme Bolte, Aris Daniilidis, Adrian Lewis, and Masahiro Shiota. Clarke subgradients of stratifiable functions. *SIAM Journal on Optimization*, 18(2):556–572, 2007.

[15] Jérôme Bolte, Aris Daniilidis, Olivier Ley, and Laurent Mazet. Characterizations of lojasiewicz inequalities: subgradient flows, talweg, convexity. *Transactions of the American Mathematical Society*, 362(6):3319–3363, 2010.

[16] Jérôme Bolte, Trong Phong Nguyen, Juan Peypouquet, and Bruce Suter. From error bounds to the complexity of first-order descent methods for convex functions. *Mathematical Programming*, pages 1–37, 2015.

[17] Jérôme Bolte, Shoham Sabach, and Marc Teboulle. Proximal alternating linearized minimization for nonconvex and nonsmooth problems. *Mathematical Programming*, 146(1-2):459–494, 2014.

[18] Jonathan Borwein, Guoyin Li, and Liangjin Yao. Analysis of the convergence rate for the cyclic projection algorithm applied to basic semialgebraic convex sets. *SIAM Journal on Optimization*, 24(1):498–527, 2014.

[19] Jean Noël Corvellec and Viorica Motreanu. Nonlinear error bounds for lower semicontinuous functions on metric spaces. *Mathematical Programming*, 114(2):291, 2008.

[20] Didier D’Acunto and Krzysztof Kurdyka. Explicit bounds for the lojasiewicz exponent in the gradient inequality for polynomials. *Annales Polonici Mathematici*, 87(1):51–61, 2005.

[21] Jean Pierre Dedieu. Approximate solutions of analytic inequality systems. *SIAM Journal on Optimization*, 11(2):411–425, 2000.

[22] Sien Deng. Computable error bounds for convex inequality systems in reflexive banach spaces. *SIAM Journal on Optimization*, 7(1):274–279, 1997.

[23] Sien Deng. Global error bounds for convex inequality systems in banach spaces. *SIAM journal on control and optimization*, 36(4):1240–1249, 1998.
[24] Sien Deng. Perturbation analysis of a condition number for convex inequality systems and global error bounds for analytic systems. *Math. Program.*, 83:263–276, 1998.

[25] Si Tiep Dinh, Huy Vui Ha, and Tien Son Pham. Hölder-type global error bounds for non-degenerate polynomial systems. *arXiv preprint arXiv:1411.0859*, 2014.

[26] Dmitriy Drusvyatskiy and Adrian Lewis. Error bounds, quadratic growth, and linear convergence of proximal methods. *arXiv preprint arXiv:1602.06661*, 2016.

[27] Ivar Ekeland et al. Nonconvex minimization problems. *Bull. AMS*, 1(3), 1979.

[28] Francisco Facchinei and Jong Shi Pang. *Finite-dimensional variational inequalities and complementarity problems*. Springer Science & Business Media, 2007.

[29] Pierre Frankel, Guillaume Garrigos, and Juan Peypouquet. Splitting methods with variable metric for kl functions. *arXiv preprint arXiv:1405.1357*, 2014.

[30] Janusz Gwoździewicz. The lojasiewicz exponent of an analytic function at an isolated zero. *Commentarii Mathematici Helvetici*, 74(3):364–375, 1999.

[31] Huy Vui Ha. Global holderian error bound for nondegenerate polynomials. *SIAM Journal on Optimization*, 23(2):917–933, 2013.

[32] Huy Vui Ha and Tien Son Pham. *Genericity in Polynomial Optimization*, volume 3. World Scientific, 2016.

[33] Heisuke Hironaka. *Introduction to real-analytic sets and real-analytic maps*. Istituto matematico” L. Tonelli” dell Universita di Pisa, 1973.

[34] Alan Hoffman. On approximate solutions of systems of linear inequalities. *Selected Papers Of Alan J Hoffman: With Commentary*, pages 174–176, 2003.

[35] Lars Hörmander. On the division of distributions by polynomials. *Arkiv för matematik*, 3(6):555–568, 1958.

[36] Aleksandr Davidovich Ioffe. Metric regularity and subdifferential calculus. *Russian Mathematical Surveys*, 55(3):501–558, 2000.

[37] Alexander Davidovich Ioffe. Regular points of lipschitz functions. *Transactions of the American Mathematical Society*, 251:61–69, 1979.

[38] Diethard Klatte and Wu Li. Asymptotic constraint qualifications and global error bounds for convex inequalities. *Mathematical programming*, 84(1):137–160, 1999.

[39] János Kollár. An effective lojasiewicz inequality for real polynomials. *Periodica Mathematica Hungarica*, 38(3):213–221, 1999.

[40] Alexander Kruger. Error bounds and metric subregularity. *Optimization*, 64(1):49–79, 2015.
[41] Krzysztof Kurdyka. On gradients of functions definable in o-minimal structures. *Annales de l’institut Fourier*, 48(3):769–783, 1998.

[42] Krzysztof Kurdyka and Stanisław Spodzieja. Separation of real algebraic sets and the Łojasiewicz exponent. *Proceedings of the American Mathematical Society*, 142(9):3089–3102, 2014.

[43] Adrian Lewis and Jong Shi Pang. Error bounds for convex inequality systems. In *Generalized convexity, generalized monotonicity: recent results*, pages 75–110. Springer, 1998.

[44] Guoyin Li. On the asymptotically well behaved functions and global error bound for convex polynomials. *SIAM Journal on Optimization*, 20(4):1923–1943, 2010.

[45] Guoyin Li. Global error bounds for piecewise convex polynomials. *Mathematical Programming*, pages 1–28, 2013.

[46] Guoyin Li, Boris Mordukhovich, TTA Nghia, and Tien Son Pham. Error bounds for parametric polynomial systems with applications to higher-order stability analysis and convergence rates. *Mathematical Programming*, pages 1–34, 2015.

[47] Guoyin Li, Boris Mordukhovich, and Tien Son Pham. New fractional error bounds for polynomial systems with applications to Hölderian stability in optimization and spectral theory of tensors. *Mathematical Programming*, 153(2):333–362, 2015.

[48] Wu Li. Error bounds for piecewise convex quadratic programs and applications. *SIAM Journal on Control and Optimization*, 33(5):1510–1529, 1995.

[49] Wu Li. Abadie’s constraint qualification, metric regularity, and error bounds for differentiable convex inequalities. *SIAM Journal on Optimization*, 7(4):966–978, 1997.

[50] Stanisław Łojasiewicz. Sur le probleme de la division. *Studia Mathematica*, 18, 1959.

[51] Stanisław Łojasiewicz. Une propriété topologique des sous-ensembles analytiques réels. *Les équations aux dérivées partielles*, 117:87–89, 1963.

[52] Xiao Dong Luo and Zhi Quan Luo. Extension of hoffmans error bound to polynomial systems. *SIAM Journal on Optimization*, 4(2):383–392, 1994.

[53] Zhi Quan Luo and Jong Shi Pang. Error bounds for analytic systems and their applications. *Mathematical Programming*, 67(1-3):1–28, 1994.

[54] Zhi Quan Luo and Jos Sturm. Error bounds for quadratic systems. In *High performance optimization*, pages 383–404. Springer, 2000.

[55] Zhi Quan Luo and Paul Tseng. Error bound and convergence analysis of matrix splitting algorithms for the affine variational inequality problem. *SIAM Journal on Optimization*, 2(1):43–54, 1992.
[56] Zhi Quan Luo and Paul Tseng. On the linear convergence of descent methods for convex essentially smooth minimization. *SIAM Journal on Control and Optimization*, 30(2):408–425, 1992.

[57] Zhi Quan Luo and Paul Tseng. Error bounds and convergence analysis of feasible descent methods: a general approach. *Annals of Operations Research*, 46(1):157–178, 1993.

[58] Olvi Mangasarian. A condition number for differentiable convex inequalities. *Mathematics of Operations Research*, 10(2):175–179, 1985.

[59] Kung Fu Ng and Xi Yin Zheng. Global error bounds with fractional exponents. *Mathematical Programming*, 88(2):357–370, 2000.

[60] Kung Fu Ng and Xi Yin Zheng. Error bounds for lower semicontinuous functions in normed spaces. *SIAM Journal on Optimization*, 12(1):1–17, 2001.

[61] Huynh Van Ngai. Global error bounds for systems of convex polynomials over polyhedral constraints. *SIAM Journal on Optimization*, 25(1):521–539, 2015.

[62] Huynh van Ngai and Michel Théra. Error bounds for convex differentiable inequality systems in Banach spaces. *Mathematical Programming*, 104(2):465–482, 2005.

[63] Tien Son Pham. The Łojasiewicz exponent of a continuous subanalytic function at an isolated zero. *Proceedings of the American Mathematical Society*, 139(1):1–9, 2011.

[64] Tien Son Pham. An explicit bound for the Łojasiewicz exponent of real polynomials. *Kodai Mathematical Journal*, 35(2):311–319, 2012.

[65] Stephen Robinson. An application of error bounds for convex programming in a linear space. *SIAM Journal on Control*, 13(2):271–273, 1975.

[66] Paul Tseng. On linear convergence of iterative methods for the variational inequality problem. *Journal of Computational and Applied Mathematics*, 60(1-2):237–252, 1995.

[67] Paul Tseng and Sangwoon Yun. A coordinate gradient descent method for nonsmooth separable minimization. *Mathematical Programming*, 117(1):387–423, 2009.

[68] Huynh Van Ngai and Michel Théra. Error bounds for systems of lower semicontinuous functions in Asplund spaces. *Mathematical Programming*, 116(1):397–427, 2009.

[69] Po Wei Wang and Chih Jen Lin. Iteration complexity of feasible descent methods for convex optimization. *Journal of Machine Learning Research*, 15(1):1523–1548, 2014.

[70] Tao Wang and Jong Shi Pang. Global error bounds for convex quadratic inequality systems. *Optimization*, 31(1):1–12, 1994.

[71] Zili Wu and Jane Ye. On error bounds for lower semicontinuous functions. *Mathematical Programming*, 92(2):301–314, 2002.
[72] Zili Wu and Jane Ye. Sufficient conditions for error bounds. *SIAM Journal on Optimization*, 12(2):421–435, 2002.

[73] Wein Hong Yang. Error bounds for convex polynomials. *SIAM Journal on Optimization*, 19(4):1633–1647, 2009.