ITÔ’S FORMULA FOR THE $L_p$-NORM OF STOCHASTIC $W_p^1$-VALUED PROCESSES

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Abstract. We prove Itô’s formula for the $L_p$-norm of a stochastic $W_p^1$-valued processes appearing in the theory of SPDEs in divergence form.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete with respect to $(\mathcal{F}, P)$ $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$. Denote by $P$ the predictable $\sigma$-field in $\Omega \times (0, \infty)$ associated with $\{\mathcal{F}_t\}$. Let $w^k_t$, $k = 1, 2, \ldots$, be independent one-dimensional Wiener processes with respect to $\{\mathcal{F}_t\}$. Let $\mathcal{D}$ be the space of generalized functions on the Euclidean $d$-dimensional space $\mathbb{R}^d$ of points $x = (x^1, \ldots, x^d)$. We consider processes with values in $\mathcal{D}$ whose stochastic differential is given by

$$du_t = (D_if^i_t + f^0_t) \, dt + g^k_t \, dw^k_t,$$

where $f^i_t$, $g^k_t$ are $L_p$-valued processes, $u_t$ is a $W_p^1$-valued process, and the summation convention over repeated indices is enforced. Our main goals are to give conditions on $u$, $f^i$, and $g^k$, which are sufficient to assert that $u_t$ is a continuous $L_p$-valued process, and to derive Itô’s formula for $||u_t||_{L_p}^p$.

This was never done before, no matter how strange it may look. The hardest step is showing that $u_t$ is continuous as an $L_p$-valued function. More or less standard fact is that under natural conditions one can estimate

$$E \sup_t ||u_t||_{L_p}^p$$

and from here and equation (1.1), implying that $(u_t, \varphi)$ is continuous in $t$ for any test function $\varphi \in C_0^\infty$, one used to derive that $u_t$ is only a weakly continuous $L_p$-valued process. Even though the above mentioned Itô’s formula was not proved, the fact that, actually, $u_t$ is indeed continuous as an $L_p$-valued process was known and proved by different methods for $p = 2$, on the basis of abstract results for SPDEs in Hilbert spaces, and for $p > 2$, on the basis of embedding theorems for stochastic Banach spaces. In this way of arguing proving the continuity of $||u_t||_{L_p}^p$ required a full blown theory of

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SPDEs with constant coefficients (cf. [4] and [2]). We present a “direct” and self-contained proof of the formula and the continuity.

Finally, we mention that there are many situations in which Itô’s formula is known for Banach space valued processes. See, for instance, [1] and the references therein. These formulas could be more general in some respects but they do not cover our situation and are closer to our Lemma 5.1 where the term $D_i f^i$ is not present in (1.1).

2. MAIN RESULT

We take a stopping time $\tau$ and fix a number $p \geq 2$.

Denote $L_p = L_p(\mathbb{R}^d)$. We use the same notation $L_p$ for vector- and matrix-valued or else $\ell_2$-valued functions such as $g_t = (g^k_t)$ in (1.1). For instance, if $u(x) = (u^1(x), u^2(x), \ldots)$ is an $\ell_2$-valued measurable function on $\mathbb{R}^d$, then

$$
\|u\|_{L_p}^p = \int_{\mathbb{R}^d} |u(x)|^p dx = \int_{\mathbb{R}^d} \left( \sum_{k=1}^{\infty} |u^k(x)|^2 \right)^{p/2} dx.
$$

Introduce

$$
D_i = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, d.
$$

By $Du$ we mean the gradient with respect to $x$ of a function $u$ on $\mathbb{R}^d$.

As usual,

$$
W^1_p = \{ u \in L_p : Du \in L_p \}, \quad \|u\|_{W^1_p} = \|u\|_{L_p} + \|Du\|_{L_p}.
$$

If $\tau$ is a stopping time, then

$$
\mathbb{L}_p(\tau) := L_p((0, \tau], \mathcal{P}, L_p), \quad \mathbb{W}^1_p(\tau) := L_p((0, \tau], \mathcal{P}, W^1_p).
$$

We also need the space $W^1_p(\tau)$, which is the space of functions $u_t = u_t(\omega, \cdot)$ on $\{(\omega, t) : 0 \leq t \leq \tau, t < \infty\}$ with values in the space of generalized functions on $\mathbb{R}^d$ and having the following properties:

(i) We have $u_0 \in L_p(\Omega, \mathcal{F}_0, L_p)$;
(ii) We have $u \in \mathbb{W}^1_p(\tau)$;
(iii) There exist $f^i \in \mathbb{L}_p(\tau), i = 0, \ldots, d$, and $g = (g^1, g^2, \ldots) \in \mathbb{L}_p(\tau)$ such that for any $\varphi \in C^\infty_0$ with probability 1 for all $t \in [0, \infty)$ we have

$$
(u_{t \wedge \tau}, \varphi) = (u_0, \varphi) + \sum_{k=1}^{\infty} \int_0^t I_{s \leq \tau}(g^k_s, \varphi) \, dw^k_s
$$

$$
+ \int_0^t I_{s \leq \tau}(f^0_s, \varphi) - (f^i_s, D_i \varphi) \, ds. \quad (2.1)
$$

In particular, for any $\phi \in C^\infty_0$, the process $(u_{t \wedge \tau}, \phi)$ is $\mathcal{F}_t$-adapted and continuous. In case that property (iii) holds, we say that (1.1) holds for $t \leq \tau$. 
The reader can find in [2] a discussion of (ii) and (iii), in particular, the fact that the series in (2.1) converges uniformly in probability on every finite subinterval of \([0, \tau]\). This will also be seen from the proof of Lemma 4.2.

Here is our main result.

**Theorem 2.1.** Let \(u \in \mathcal{W}_p^1(\tau), f^j \in L_p(\tau), g = (g^k) \in L_p(\tau)\) and assume that (1.1) holds for \(t \leq \tau\) in the sense of generalized functions. Then there is a set \(\Omega' \subset \Omega\) of full probability such that

1. \(u_{t \wedge \tau} I_{\Omega'}\) is a continuous \(L_p\)-valued \(\mathcal{F}_t\)-adapted function on \([0, \infty)\);
2. for all \(t \in [0, \infty)\) and \(\omega \in \Omega'\) itô’s formula holds:

\[
\int_{\mathbb{R}^d} |u_{t \wedge \tau}|^p dx = \int_{\mathbb{R}^d} |u_0|^p dx + p \int_0^{t \wedge \tau} \int_{\mathbb{R}^d} |u_s|^{p-2} u_s g^k_s dx dw^k_s
\]

\[
+ \int_0^{t \wedge \tau} \left( \int_{\mathbb{R}^d} \left[ |u_t|^p u_t f_0 - p(p-1)|u_t|^{p-2} f_t D_i u_t + (1/2)p(p-1)|u_t|^{p-2} g^2_t \right] dx \right) dt. \tag{2.2}
\]

Furthermore, for any \(T \in [0, \infty)\) and

\[
E \sup_{t \leq \tau \wedge T} \|u_t\|_{L_p}^p \leq 2E\|u_0\|_{L_p}^p + NT^{p-1} \|f_0\|_{L_p(\tau)}^p
\]

\[
+ NT^{(p-2)/2} \left( \sum_{i=1}^d \|f^i\|_{L_p(\tau)}^p + \|g\|_{L_p(\tau)}^p + \|Du\|_{L_p(\tau)}^p \right), \tag{2.3}
\]

where \(N = N(d, p)\).

We prove Theorem 2.1 in Section 6 after we prepare the necessary tools in Sections 3-5.

Here is an “energy” estimate.

**Corollary 2.2.** Under the conditions of Theorem 2.1

\[
E \int_{\mathbb{R}^d} |u_0|^p dx + E \int_0^\tau \left( \int_{\mathbb{R}^d} \left[ |u_t|^p u_t f_0 - p(p-1)|u_t|^{p-2} f_t D_i u_t + (1/2)p(p-1)|u_t|^{p-2} g^2_t \right] dx \right) dt \geq EI_{T < \infty} \int_{\mathbb{R}^d} |u_{\tau}|^p dx. \tag{2.4}
\]

Furthermore, if \(\tau\) is bounded then there is an equality instead of inequality in (2.4).

The proof of the corollary is given in Section 6.

3. Auxiliary results

We need two well-known results (see, for instance, Lemma 6.1 and Corollary 6.2 in [3]), which we prove for completeness of presentation.
Lemma 3.1. Let \((E, \Sigma, \mu)\) be a measure space, \(r \in [1, \infty)\), \(u_n, u \in L_r(\mu)\), and \(u_n \to u\) in measure. Finally, let
\[
\|u_n\|_{L_r(\mu)} \to \|u\|_{L_r(\mu)}.
\]
Then
\[
\|u_n - u\|_{L_r(\mu)} \to 0.
\]

Proof. We have
\[
|u|^r - |u_n|^r = (|u|^r - |u_n|^r)_+ - (|u|^r - |u_n|^r).
\]
Upon integrating through this equation and observing that \((|u|^r - |u_n|^r)_+ \leq |u|^r\) we conclude by the dominated convergence theorem that
\[
\int_E |u|^r - |u_n|^r \mu(dx) \to 0. \quad (3.2)
\]
Next, if \(|u_n - u| \geq 3|u|\), then \(|u_n| + |u| \geq 3|u|\), \(|u| \leq (1/2)|u_n|\), \(|u|^r \leq (1/2)|u_n|^r\),
\[
|u_n|^r - |u|^r \geq (1/2)|u_n|^r, \quad |u_n - u| \leq |u_n| + |u| \leq 2|u_n|,
\]
which along with (3.2) imply that
\[
\int_E |u_n - u|^r I_{|u_n - u| \geq 3|u|} \mu(dx) \to 0.
\]
Furthermore,
\[
\int_E |u_n - u|^r I_{|u_n - u| < 3|u|} \mu(dx) \to 0
\]
by the dominated convergence theorem. By combining the last two relations we come to (3.1). The lemma is proved.

Corollary 3.2. Let \((E, \Sigma, \mu)\) be a measure space, \(r, s \in (1, \infty)\), \(r^{-1} + s^{-1} = 1\), \(u_n, u \in L_r(\mu)\), \(v_n, v \in L_s(\mu)\), \(u_n \to u\) and \(v_n \to v\) in measure. Finally, let
\[
\|u_n\|_{L_r(\mu)} \to \|u\|_{L_r(\mu)}, \quad \|v_n\|_{L_s(\mu)} \to \|v\|_{L_s(\mu)}.
\]
Then
\[
\int_E |u_n v_n - uv| \mu(dx) \to 0, \quad \int_E u_n v_n \mu(dx) \to \int_E uv \mu(dx).
\]
Indeed, it suffices to use Hölder’s inequality and the formula
\[
u_n v_n - uv = (u_n - u)v + (v_n - v)u + (u_n - u)(v_n - v).
\]
4. Integrating $L^p$ Functions

Most likely a big part of what follows in this section can be obtained from some abstract constructions in [1]. However, it does not look easy to obtain estimate (4.2). In any case, it is worth giving all rather simple arguments for completeness. Set

$$L^p = L^p(\mathbb{R}^d)$$

and for Borel subsets $\Gamma$ of a Euclidean space denote by $\mathcal{B}(\Gamma)$ the $\sigma$-field of Borel subsets of $\Gamma$.

**Definition 4.1.** By $U^p$ we denote the set of functions $u = u_t(x) = u_t(\omega, x)$ on $\Omega \times [0, \infty) \times \mathbb{R}^d$ such that

1. $u$ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{R}^d)$;
2. for each $x$, the function $u_t(x)$ is $\mathcal{F}_t$-adapted;
3. $u_t(x)$ is continuous in $t \in [0, \infty)$ for each $(\omega, x)$;
4. the function $u_t(\omega, \cdot)$ as a function of $(\omega, t)$ is $L^p$-valued, $\mathcal{F}_t$-adapted, and continuous in $t$ for any $\omega$.

**Lemma 4.2.** Let $g = (g^k) \in L^p$. Then there exists a function $u \in U^p$ such that for any $\phi \in C_0^\infty$ the equation

$$\langle u_t, \phi \rangle = \sum_{k=1}^{\infty} \int_0^t g_k^s \phi \, dw_s$$

holds for all $t \in [0, \infty)$ with probability one. Furthermore, for any $T \in [0, \infty)$ we have

$$E \int_{\mathbb{R}^d} \sup_{t \leq T} |u_t(x)|^p \, dx \leq NT^{(p-2)/2} E \int_0^T \|g_s\|^p_{L^p} \, ds,$$

where $N = N(p)$.

Proof. First assume that there is an integer $j \geq 1$, (nonrandom) functions $g^k \in C_0^\infty$, and bounded stopping times $\tau_0 \leq \tau_1 \leq \ldots \leq \tau_j$ such that $g^k \equiv 0$ for $k > j$ and

$$g^k_t(x) = \sum_{i=1}^j g^{ik}(x)I_{(\tau_{i-1}, \tau_i]}(t)$$

for $k \leq j$.

Then define

$$u_t(x) = \sum_{i,k=1}^j g^{ik}(x)(w^k_{t \wedge \tau_i} - w^k_{t \wedge \tau_{i-1}}).$$

Obviously, $u \in U^p$. Furthermore, (4.1) holds for any $\phi \in C_0^\infty$ for all $t$ with probability one since its right-hand side equals

$$\sum_{i,k=1}^j \langle g^{ik}, \phi \rangle (w^k_{t \wedge \tau_i} - w^k_{t \wedge \tau_{i-1}})$$
for all \( t \) with probability one. Next, by the Burkholder-Davis-Gundy inequalities for each \( x \)

\[
E \sup_{t \leq T} |u_t(x)|^p = E \sup_{t \leq T} \left| \sum_k \int_0^t g^k_s(x) \, dw^k_s \right|^p \leq NE \left( \int_0^T |g_s(x)|_{L^2} \, ds \right)^{p/2},
\]

which after applying Hölder’s inequality \((p \geq 2)\) yields

\[
E \sup_{t \leq T} |u_t(x)|^p \leq NT^{(p-2)/2} E \int_0^T |g_s(x)|_{L^p} \, ds.
\]

We integrate this inequality over \( \mathbb{R}^d \) and use the fact that the measurability properties of \( g, u \) and the continuity of \( u_t \) in \( t \) allow us to use Fubini’s theorem. Then we come to (4.2).

By Theorem 3.10 of [2] the set of \( g \)’s like the one above is dense in \( L^p \).

Therefore, to prove the lemma it suffices to show that the set of \( g \)’s for which the statements of the lemma are true is closed in \( L^p \).

Take a sequence \( g^n = (g^{nk}) \in \mathbb{L}_p \), \( n = 1, 2, \ldots \), such that for each \( n \) there is a function \( u^n \) corresponding to \( g^n \) and possessing the asserted properties. Assume that for a \( g \in \mathbb{L}_p \) we have \( g^n \to g \) in \( \mathbb{L}_p \) as \( n \to \infty \). Using a subsequence of \( g^n \) we may assume that for any \( T \in [0, \infty) \)

\[
E \int_{\mathbb{R}^d} \sup_{t \leq T} \left| u^{n+1}_t(x) - u^n_t(x) \right|^p \, dx \leq T^{(p-2)/2} 2^{-n}.
\]  

(4.3)

Introduce

\[
A_n = \{ (\omega, x) : \sup_{t \leq n} |u^{n+1}_t(x) - u^n_t(x)| \geq n^{-2} \}.
\]

Then

\[
\sum_{n=1}^{\infty} E \int_{\mathbb{R}^d} \sup_{t \leq n} \left| u^{n+1}_t(x) - u^n_t(x) \right| I_{A_n}(x) \, dx
\]

\[
\leq \sum_{n=1}^{\infty} n^{2(p-1)} E \int_{\mathbb{R}^d} \sup_{t \leq n} \left| u^{n+1}_t(x) - u^n_t(x) \right|^p \, dx
\]

\[
\leq \sum_{n=1}^{\infty} n^{2(p-1)} n^{(p-2)/2} 2^{-n} < \infty,
\]

implying that

\[
\sum_{n=1}^{\infty} \sup_{t \leq n} \left| u^{n+1}_t(x) - u^n_t(x) \right| I_{A_n}(x) < \infty
\]

for almost all \((\omega, x)\). The series with the complements of \( A_n \) in place of \( A_n \) obviously converges everywhere. We conclude that the \( \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable set

\[
G = \{ (\omega, x) : \sum_{n=1}^{\infty} \sup_{t \leq n} \left| u^{n+1}_t(x) - u^n_t(x) \right| < \infty \}.
\]
has full measure. By Fubini’s theorem the function \( P((\omega, x) \in G) \) is a Borel function of \( x \) equal to 1 for almost all \( x \). Accordingly we introduce a Borel set of full measure
\[
\Gamma = \{ x : P((\omega, x) \in G) = 1 \}
\]
and the \( \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable set \( G' \) of full measure by
\[
G' = \{ (\omega, x) : x \in \Gamma, \sum_{n=1}^{\infty} \sup_{t \leq n} |u_{t}^{n+1}(x) - u_{t}^{n}(x)| < \infty \}.
\]

Now define
\[
u_{t}^{'}(x) = \lim_{n \to \infty} u_{t}^{n}(x) \quad (4.4)
\]
for \((\omega, x) \in G'\), \( t \geq 0 \) and set \( u_{t}^{'}(x) \equiv 0 \) for \((\omega, x) \not\in G'\). Also set
\[
v_{t}(x) = \begin{cases} 
\lim_{n \to \infty} u_{t}^{n}(x) & \text{if the limit exists}, \\
0 & \text{otherwise}.
\end{cases}
\]

Obviously,
\[
u_{t}^{'}(\omega, x) = v_{t}(\omega, x)I_{G'}(\omega, x). \quad (4.5)
\]
Furthermore, \( v \) is known to be \( \mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable since \( u^{n} \) possess this property. It follows that \( u^{'} \) is \( \mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable. For each \( x \), the functions \( u_{t}^{n}(x) \) are \( \mathcal{F}_{t} \)-adapted and so is \( v_{t}(x) \). Also \( I_{G'}(\omega, x) \) is \( \mathcal{F}_{0} \)-measurable (and hence \( \mathcal{F}_{t} \)-adapted) for each \( x \) since the \( \mathcal{F}_{t} \) are complete and
\[
P(I_{G'}(\omega, x) = 1) = P((\omega, x) \in G, x \in \Gamma)
\]
equals zero if \( x \not\in \Gamma \) and one if \( x \in \Gamma \) by the choice of \( \Gamma \). Now equation (4.5) allows us to conclude that \( u_{t}^{'}(x) \) is \( \mathcal{F}_{t} \)-adapted for each \( x \).

Since the limit in (4.4) is uniform in \( t \) on any finite interval, we see that \( u_{t}^{'} \) is continuous in \( t \) for any \((\omega, x)\). In particular,
\[
\sup_{t \leq T} |u_{t}^{'}(x)|
\]
is \( \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable, estimate (1.2) with \( u^{'} \) in place of \( u \) makes sense and holds owing to Fatou’s lemma and the assumption on \( u^{n} \).

Estimate (1.2) shows that there is a set \( \Omega' \in \mathcal{F}_{0} \) of full probability such that \( u_{t}^{'}(\omega, \cdot) \in L_{p} \) for all \( t \) if \( \omega \in \Omega' \) and moreover
\[
\int_{\mathbb{R}^d} \sup_{t \leq T} |u_{t}^{'}(\omega, x)|^{p} \, dx < \infty
\]
for any \( T \in [0, \infty) \) if \( \omega \in \Omega' \). This fact, the continuity of \( u_{t}^{'} \) in \( t \), and the dominated convergence theorem imply that \( u_{t}^{'} \) is continuous as an \( L_{p} \)-valued function of \( t \) for any \( \omega \in \Omega' \). We now set
\[
u_{t}(\omega, x) = u_{t}^{'}(\omega, x)I_{\Omega'}(\omega).
\]
Then we see that to show that \( u \in U_{p} \) it suffices to prove that \( u_{t}(\omega, \cdot) \) is \( \mathcal{F}_{t} \)-adapted as an \( L_{p} \)-valued function.
Obviously, to do this step it suffices to prove the assertion of the lemma related to (4.1). By the Burkholder-Davis-Gundy inequalities for any \( T \in [0, \infty) \)

\[
E \sup_{t \leq T} \left| \sum_k \int_0^t (g_s^k - g_s^{nk}, \phi) \, dw_s \right|^p \leq NE \left( \int_0^T \sum_{k=1}^\infty (g_s^k - g_s^{nk}, \phi)^2 \, ds \right)^{p/2}
\]

\[
\leq NE \left( \int_0^T \sum_{k=1}^\infty (|g_s^k - g_s^{nk}|^2, |\phi|)^2 \| \phi \|^2_{L_2} \, ds \right)^{p/2}
\]

\[
\leq NE \int_0^T \int_{\mathbb{R}^d} |g_s - g_s^n|_{L_2}^p \, dx \, ds \leq N \| g - g^n \|_{L_p},
\]

(4.6)

where \( N \) is independent of \( n \). In addition, estimate (4.3) easily imply that

\[
E \sup_{t \leq T} \left| (u_t - u^n_t, \phi) \right| \to 0
\]

as \( n \to \infty \) for any \( T \in [0, \infty) \). By combining these fact and passing to the limit in (4.1) with \( u^n \) in place of \( u \) we get the desired result and the lemma is proved.

Remark 4.3. It is tempting to assert that \( u_t(x) \) is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable since it is \( \mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable and, for each \( x \), it is predictable. However, we do not know if this assertion is true.

In a similar way, without using the Burkholder-Davis-Gundy inequalities, the following result is established.

**Lemma 4.4.** Let \( f \in L_p(\infty) \). Then there exists a function \( u \in \mathcal{U}_p \) such that for any \( \phi \in C_0^\infty \) the equation

\[
(u_t, \phi) = \int_0^t (f_s, \phi) \, ds
\]

(4.7)

holds for all \( t \in [0, \infty) \) with probability one. Furthermore, for any \( T \in [0, \infty) \) we have

\[
E \int_{\mathbb{R}^d} \sup_{t \leq T} |u_t(x)|^p \, dx \leq NT^{p-1}E \int_0^T \| f_s \|^p_{L_p} \, ds,
\]

(4.8)

where \( N = N(p) \).

Remark 4.5. Observe that the integral on the right in (4.7) need not exist for each \( \omega \) since \((f_t, \phi)\) is generally only measurable with respect to the completion of \( \mathcal{P} \) and the function \((f_t, \phi)\) need not be Lebesgue measurable in \( t \) for each \( \omega \). In case of Lemma 4.2 this moment does not arise because of freedom in defining the stochastic integrals.
Lemma 5.1. Let \( f \in L_p(\tau), \ g = (g^k) \in L_p(\tau) \) and assume that we are given a function \( u_t \) on \( \Omega \times [0, \infty) \) with values in the space of distributions on \( \mathbb{R}^d \) such that \( u_0 \in L_p(\Omega, F_0, L_p) \) and for any \( \phi \in C_0^\infty \) with probability one for all \( t \in [0, \infty) \) we have

\[
(u_t \wedge \tau, \phi) = (u_0, \phi) + \int_0^t I_{s \leq \tau} (f_s, \phi) \, ds + \sum_{k=1}^\infty \int_0^t (g^k_s, \phi) I_{s \leq \tau} \, dw^k_s. \tag{5.1}
\]

Then, there is a set \( \Omega' \in F_0 \) of full probability such that

(i) \( u_{t \wedge \tau} I_{\Omega'} \) is an \( L_p \)-valued \( F_1 \)-adapted continuous process on \([0, \infty)\),

(ii) for all \( t \in [0, \infty) \) and \( \omega \in \Omega' \)

\[
\|u_t \wedge \tau\|_{L_p} = \|u_0\|_{L_p} + \int_0^{t \wedge \tau} \left[ p \int_{\mathbb{R}^d} |u_s|^{p-2} u_s f_s \, dx \, ds \right. \\
+ (1/2) p(p-1) \int_{\mathbb{R}^d} |u_s|^{p-2} |g_s|^2 \, dx \, ds \right] \\
+ p \int_0^{t \wedge \tau} \int_{\mathbb{R}^d} |u_s|^{p-2} u_s g^k_s \, dx \, dw^k_s. \tag{5.2}
\]

Proof. First observe that the right-hand sides of (5.1) and (5.2) will be affected only on the set of probability zero independent of \( t \) if we replace \( f \) and \( g \) with \( L_p \)-valued predictable functions \( \hat{f} \) and \( \hat{g} \) such that

\[
\|f_t - \hat{f}_t\|_{L_p} + \|g_t - \hat{g}_t\|_{L_p} = 0
\]

for almost all \((\omega, t)\). It follows that without losing generality we may assume that \( f \) and \( g \) are predictable as \( L_p \)-valued functions.

Lemmas 4.2 and 4.4 allow us to find a \( v \in U_p \) such that for any \( \phi \in C_0^\infty \) equation (5.1) with \( v_t \) in place of \( u_{t \wedge \tau} \) holds for all \( t \) with probability one. It follows that for any countable set \( A \subset C_0^\infty \) there exists a set \( \Omega' \) of full probability such that for any \( \omega \in \Omega' \), \( \phi \in A \), and \( t \geq 0 \) we have \( (u_{t \wedge \tau}, \phi) = (v_t, \phi) \). If the set \( A \) is chosen appropriately, then we conclude that \( u_{t \wedge \tau} = v_t \) in the sense the distributions, whenever \( \omega \in \Omega' \) and \( t \geq 0 \). In particular, assertion (i) holds with this \( \Omega' \).

This argument allows us to assume that \( \tau = \infty \) and \( u \in U_p \) and concentrate on proving (5.2). This argument also shows that for any \( T \in [0, \infty) \)

\[
E \int_{\mathbb{R}^d} \sup_{t \leq T} |u_t(x)|^p \, dx < \infty
\]

implying that there exists a set \( \Omega'' \) of full probability such that for any \( \omega \in \Omega'' \) we have

\[
\int_{\mathbb{R}^d} \sup_{t \leq T} |u_t(x)|^p \, dx < \infty, \quad \forall T \in [0, \infty). \tag{5.3}
\]
Now, take a nonnegative function $\zeta \in C^\infty_0(\mathbb{R}^d)$ with unit integral, for $\varepsilon > 0$, define $\zeta_\varepsilon = \varepsilon^{-d}\zeta(x/\varepsilon)$, and for any locally summable $h$ given on $\mathbb{R}^d$ introduce the notation $h^{(\varepsilon)} = h \ast \zeta_\varepsilon$. Then (5.1) implies that for each $x$

almost surely for all $t \in [0, \infty)$

$$u_t^{(\varepsilon)}(x) = u_0^{(\varepsilon)}(x) + \int_0^t f_s^{(\varepsilon)}(x) \, ds + \int_0^t g_s^{k(\varepsilon)}(x) \, dw_s^k.$$  \hspace{1cm} (5.4)

By Itô’s formula, for each $x$

$$|u_t^{(\varepsilon)}|^p = |u_0^{(\varepsilon)}|^p + \int_0^t p|u_s^{(\varepsilon)}|^{p-2} u_s^{(\varepsilon)} g_s^{k(\varepsilon)} \, dw_s^k$$

$$+ \int_0^t \left[p|u_s^{(\varepsilon)}|^{p-2} u_s^{(\varepsilon)} f_s^{(\varepsilon)} \right] \, ds + \left(1/2\right) p(p-1)|u_s^{(\varepsilon)}|^{p-2}|g_s^{(\varepsilon)}|^2 \, ds$$

(a.s.), where we dropped the argument $x$ for simplicity. We want to integrate this equality over $\mathbb{R}^d$ and use the stochastic and deterministic Fubini’s theorems. We will see that there is no difficulties with the integral with respect to $ds$. However, in order to be able to apply the stochastic version of Fubini’s theorem we need at least that the resulting stochastic integral make sense, that is we need at least the inequality

$$\int_0^t \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^d} |u_s^{(\varepsilon)}|^{p-1} |g_s^{k(\varepsilon)}| \, dx \right)^2 \, ds < \infty$$

to hold (a.s.). The computations below show that, actually, for a sequence of stopping times $\tau_n \uparrow \infty$,

$$E \int_0^{t \wedge \tau_n} \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^d} |u_s^{(\varepsilon)}|^{p-1} |g_s^{k(\varepsilon)}| \, dx \right)^2 \, ds < \infty$$

(5.6)

and this is known to be sufficient to apply the stochastic version of Fubini’s theorem. By the way, notice that $u_t^{(\varepsilon)}(x)$ is continuous (infinitely differentiable) in $x$ for any $(\omega,t)$. Therefore, it is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Since it is also continuous in $t$ for each $(\omega,x)$, the function $u_t^{(\varepsilon)}(x)$ if $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and there is no measurability obstructions in applying Fubini’s theorems.

To deal with the integral with respect to $s$ observe that by Young’s inequality for any $t \in [0, \infty)$

$$\int_0^t |u_s^{(\varepsilon)}|^{p-1} f_s^{(\varepsilon)} \, ds \leq \frac{\gamma^{p/(p-1)}}{t(t-1)} \int_0^t |u_s^{(\varepsilon)}|^p \, ds + \frac{\gamma^{p-1}}{t} \int_0^t |f_s^{(\varepsilon)}|^p \, ds,$$

$$\int_0^t |u_s^{(\varepsilon)}|^{p-2} g_s^{(\varepsilon)} g_s^{(\varepsilon)} \, ds \leq \frac{\gamma^{p/(p-2)}}{t(2t-1)} \int_0^t |u_s^{(\varepsilon)}|^p \, ds + \frac{\gamma^{(p-2)/2}}{t} \int_0^t |g_s^{(\varepsilon)}|^p \, ds,$$

(5.7)

where $\gamma > 0$ is any number (however, if $p = 2$ we set $\gamma = 1$ in the second inequality). Actually, below in this proof we only need (5.7) with $\gamma = 1$. More general $\gamma$’s will appear in the proof of Theorem 2.1.
By Minkowski’s inequality
\[
\sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^d} |u^{(\epsilon)}_k|^p |g^{(\epsilon)}_k| \, dx \right)^2 \leq \left( \int_{\mathbb{R}^d} |u^{(\epsilon)}_k|^p \, dx \right)^2 \left( \int_{\mathbb{R}^d} |g^{(\epsilon)}_k| \, dx \right)^2.
\]  
(5.8)

By Hölder’s inequality the right-hand side of (5.8) is less than
\[
\left( \int_{\mathbb{R}^d} |g^{(\epsilon)}_k| \, dx \right)^{2(p-1)/p} \left( \int_{\mathbb{R}^d} |u^{(\epsilon)}_k|^p \, dx \right)^{2/p} \leq \left( \int_{\mathbb{R}^d} |g^{(\epsilon)}_k| \, dx \right)^{2(p-1)/p} \left( \int_{\mathbb{R}^d} \sup_{s \leq t} |u^{(\epsilon)}_k|^p \, dx \right)^{2/p}.
\]

Here
\[
|u^{(\epsilon)}_k|^p \leq (|u^{(\epsilon)}|^p), \quad \sup_{s \leq t} |u^{(\epsilon)}_k|^p \leq (\sup_{s \leq t} |u^{(\epsilon)}|^p),
\]

\[
\int_{\mathbb{R}^d} (\sup_{s \leq t} |u^{(\epsilon)}|^p) \, dx = \int_{\mathbb{R}^d} \sup_{s \leq t} |u^{(\epsilon)}|^p \, dx.
\]

It follows from here and (5.3) that the process
\[
\xi^{(\epsilon)}_t := \int_0^t \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^d} |u^{(\epsilon)}_k|^p |g^{(\epsilon)}_k| \, dx \right)^2 \, ds
\]
is well defined, \(\mathcal{F}_t\) adapted, and is continuous in \(t\) (a.s.). Hence, the stopping times
\[
\tau_n = \tau \wedge \inf\{t \geq 0 : \xi^{(\epsilon)}_t \geq n\},
\]
\(n = 1, 2, \ldots\), are well defined, \(\tau_n \uparrow \infty\), and, obviously, (5.6) holds.

Estimates (5.7) show that there is no trouble in applying the deterministic Fubini’s theorem to the integrals with respect to \(ds\) in (5.5). Estimate (5.6) implies that for each fixed \(t\) we can apply the stochastic Fubini’s theorem to the stochastic term in (5.5) with \(t \wedge \tau_n\) in place of \(t\). Hence we obtain that with probability one
\[
\int_{\mathbb{R}^d} |u^{(\epsilon)}|_t^p \, dx = \int_{\mathbb{R}^d} |u^{(\epsilon)}_0|^p \, dx + \int_0^{t \wedge \tau_n} \left[ p \int_{\mathbb{R}^d} |u^{(\epsilon)}_s|^p - 2 u^{(\epsilon)}_s f^{(\epsilon)}_s \, dx \, ds \right.

\quad \left. + (1/2)p(p - 1) \int_{\mathbb{R}^d} |u^{(\epsilon)}_s|^p - 2 |g^{(\epsilon)}_s|^2 \, dx \right] \, ds

\quad \left. + p \int_0^{t \wedge \tau_n} \int_{\mathbb{R}^d} |u^{(\epsilon)}_s|^p - 2 u^{(\epsilon)}_s g^{k(\epsilon)}_s \, dx \, dw^k_s \right.
\]

Since \(\tau_n \uparrow \infty\), we have also
\[
\int_{\mathbb{R}^d} |u^{(\epsilon)}|_t^p \, dx = \int_{\mathbb{R}^d} |u^{(\epsilon)}_0|^p \, dx + \int_0^t \left[ p \int_{\mathbb{R}^d} |u^{(\epsilon)}_s|^p - 2 u^{(\epsilon)}_s f^{(\epsilon)}_s \, dx \, ds \right.

\quad \left. + (1/2)p(p - 1) \int_{\mathbb{R}^d} |u^{(\epsilon)}_s|^p - 2 |g^{(\epsilon)}_s|^2 \, dx \right] \, ds

\quad \left. + p \int_0^t \int_{\mathbb{R}^d} |u^{(\epsilon)}_s|^p - 2 u^{(\epsilon)}_s g^{k(\epsilon)}_s \, dx \, dw^k_s \right)
\]

(a.s.) for each \(t\).
We now pass to the limit as $\varepsilon \to 0$ in (5.9). Observe that for any $h \in L_p$ we have $\|h^{(\varepsilon)}\|_{L_p} \leq \|h\|_{L_p}$ and $h^{(\varepsilon)} \to h$ in $L_p$ as $\varepsilon \to 0$. Therefore, the left-hand side of (5.9) tends to the left-hand side of (5.2) with $(\tau = \infty)$ for all $(\omega,t)$. The same is true (a.s.) for the first term on the right in (5.9).

To prove the convergence in probability of stochastic integrals it suffices to prove that

$$
\int_0^t \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^d} \left| u_s^{(\varepsilon)} \right|^{p-2} u_s^{(\varepsilon)} g_s^{(\varepsilon)} - \left| u_s \right|^{p-2} u_s g_s \right) dx \right)^2 ds \to 0 \quad (5.10)
$$

as $\varepsilon \to 0$ (a.s.). Notice that for $s \leq t$ by Minkowski’s and Hölder’s inequalities

$$
\sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^d} \left| u_s^{(\varepsilon)} \right|^{p-2} u_s (g_s^{(\varepsilon)} - g_s) dx \right)^2 ds \leq \left( \int_{\mathbb{R}^d} \left| u_s \right|^{p-1} |g_s^{(\varepsilon)} - g_s|\ell_2 dx \right)^2 \leq \sup_{r \leq t} \|u_r\|_{L_p} 2^{(p-1)} \|g_s^{(\varepsilon)} - g_s\|_{L_p}.
$$

The integral of the last term in $s$ tends to zero as $\varepsilon \to 0$ (a.s.) owing to the above mentioned properties of mollifiers and the fact that $\sup_{r \leq t} \|u_r\|_{L_p} < \infty$ (for all $\omega$).

Hence to prove (5.10) it suffices to show that

$$
J^\varepsilon := \int_0^t \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^d} \left| u_s^{(\varepsilon)} \right|^{p-2} u_s^{(\varepsilon)} g_s^{(\varepsilon)} - \left| u_s \right|^{p-2} u_s g_s \right) dx \right)^2 ds
$$

tends to zero (a.s.). By Minkowski’s inequality

$$
J^\varepsilon \leq \int_0^t (I_s^\varepsilon)^2 ds, \quad (5.11)
$$

where

$$
I_s^\varepsilon := \int_{\mathbb{R}^d} \left| u_s^{(\varepsilon)} \right|^{p-2} u_s^{(\varepsilon)} - \left| u_s \right|^{p-2} u_s \right| g_s^{(\varepsilon)} dx.
$$

Observe that on a set of full probability for almost any $s$ we have

$$
|g_s^{(\varepsilon)} - g_s|\ell_2 \to 0, \quad |g_s^{(\varepsilon)}|\ell_2 \to |g_s|\ell_2, \quad u_s^{(\varepsilon)} \to u_s
$$

in $L_p$. In particular,

$$
\| u_s^{(\varepsilon)} \|_{L_p/(p-1)} = \| u_s^{(\varepsilon)} \|_{L_p} \to \| u_s \|_{L_p/(p-1)}.
$$

By Lemma 3.1

$$
|u_s^{(\varepsilon)}|^{p-2} u_s^{(\varepsilon)} - |u_s|^{p-2} u_s \to 0
$$

in $L_p/(p-1)$. It follows that $I_s^\varepsilon \to 0$ as $\varepsilon \to 0$ for almost all $s$ if $\omega \in \Omega'$ with $P(\Omega') = 1$. Furthermore,

$$
I_s^\varepsilon \leq \sup_{r \leq t} \|u_r^{(\varepsilon)}\|_{L_p} \|g_s^{(\varepsilon)}\|_{L_p} + \sup_{r \leq t} \|u_r\|_{L_p} \|g_s^{(\varepsilon)}\|_{L_p}
$$

$$
\leq 2 \sup_{r \leq t} \|u_r\|_{L_p} \|g_s\|_{L_p},
$$
which is square integrable over \([0, t]\) (a.s.). By the dominated convergence theorem and (5.11) we have \(I^\varepsilon \to 0\) (a.s.) yielding the desired convergence in probability of the stochastic integrals in (5.9) as \(\varepsilon \to 0\).

The integral with respect to \(ds\) on the right in (5.9) presents no difficulty owing to Corollary 3.2 and is treated similarly to what is done above after (5.11).

Thus, for each \(t\) equation (5.2) holds with probability one. Since both parts are continuous in \(t\), it also holds for all \(t\) at once on the set of full probability and this finally brings the proof of the lemma to an end.

6. PROOF OF THEOREM 2.1 AND COROLLARY 2.2

First we prove the theorem. We use the notation \(h^{(\varepsilon)}\) as in the proof of Lemma 5.1 taking there a nonnegative \(\xi \in C_0^\infty\) with unit integral. By substituting \(\phi \ast \zeta\), where \(\zeta(x) = \zeta(-x)\), in place of \(\phi\) in (2.1) we see that \(u^{(\varepsilon)}_t\) satisfies

\[
(u^{(\varepsilon)}_{t \wedge \tau}, \varphi) = (u_0^{(\varepsilon)}, \varphi) + \sum_{k=1}^\infty \int_0^t I_{s \leq \tau}(g^k_s, \varphi) \, dw^k_s
\]

\[
+ \int_0^t I_{s \leq \tau}(- (f^i_s, D_i \varphi) + (f^0_s, \varphi)) \, ds,
\]

which is (5.1) with \(u_0^{(\varepsilon)}\) and \(g^k_t\) in place of \(u_0\) and \(g^k_t\), respectively, and with

\[
D_i (f^i)^{(\varepsilon)} + (f^0)^{(\varepsilon)}
\]

in place of \(f_t\).

From Lemma 5.1 we obtain that, for an \(\Omega^{\varepsilon}\) with \(P(\Omega^{\varepsilon}) = 1\), \(u^{(\varepsilon)}_{t \wedge \tau}\) \(I_{\Omega^{\varepsilon}}\) is a continuous \(L_p\)-valued \(F_t\)-adapted process on \([0, \infty)\) and the corresponding counterpart of (5.2) holds, integrating by parts in which leads to the fact that with probability one for all \(t \geq 0\)

\[
\int_{\mathbb{R}^d} |u^{(\varepsilon)}_{t \wedge \tau}|^p \, dx = \int_{\mathbb{R}^d} |u_0^{(\varepsilon)}|^p \, dx + p \int_0^{t \wedge \tau} \int_{\mathbb{R}^d} |u_s^{(\varepsilon)}| |u_s^{(\varepsilon)} - 2u_s^{(\varepsilon)} f^0_s - p(p-1)|u_s^{(\varepsilon)}| - 2f^i_s D_i u_s^{(\varepsilon)}
\]

\[
+ (1/2)p(p-1)|u_s^{(\varepsilon)}|^{p-2} |g^k_s|_{L_{p/2}}^2 \, dx \, ds.
\]

We take the supremums with respect to \(t\) of both parts and repeat a standard argument which was introduced by E. Pardoux. We will be using (5.7) and the fact that, by the inequality \(ap^{p-2}bc \leq a^p + b^p + c^p\), \(a, b, c \geq 0\), we have

\[
\int_{\mathbb{R}^d} |u_s^{(\varepsilon)}| |u_s^{(\varepsilon)} - 2f^i_s D_i u_s^{(\varepsilon)} \, dx \leq \frac{\gamma^{p/(p-2)}}{T} \int_{\mathbb{R}^d} |u_s^{(\varepsilon)}|^p \, dx
\]

\[
+ \frac{T^{(p-2)/2}}{\gamma^{p/2}} \int_{\mathbb{R}^d} [f^i_s |p + |D u_s^{(\varepsilon)}| p] \, dx,
\]

(6.3)
where \( f^{(e)} = (f_1^{(e)}, \ldots, f_d^{(e)}) \) and \( \gamma > 0 \) is any number. We also use (5.8) and the Burkholder-Davis-Gundy inequalities. Then for an appropriate choice of the parameter \( \gamma \) we find from (6.2) that

\[
E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} \leq E \sup_{t \leq \tau \wedge T} \| u_0^{(e)} \|^p_{L^p} + (1/4) E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} + \frac{N T^{(p-2)/2}}{p} \int_0^{T \wedge \tau} \left( \| g_s^{(e)} \|^p_{L^p} + \| f_s^{(e)} \|^p_{L^p} + \| Du_s^{(e)} \|^p_{L^p} \right) \, ds
\]

\[
+ N T^{p-1} \int_0^{T \wedge \tau} \| f_0^{(e)} \|^p_{L^p} \, ds + N E \left( \int_0^{T \wedge \tau} \left( \int_{\mathbb{R}^d} |u_s^{(e)}|^p \left| g_s^{(e)} \right| \varepsilon \, dx \right)^2 \, ds \right)^{1/2},
\]

where the last expectation is estimated by

\[
E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} \left( \int_0^{T \wedge \tau} \| g_s^{(e)} \|^2_{L^p} \, ds \right)^{1/2}
\]

\[
\leq T^{(p-2)/(2p)} E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} \left( \int_0^{T \wedge \tau} \| g_s^{(e)} \|^p_{L^p} \, ds \right)^{1/p}
\]

\[
\leq (1/4) E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} + N T^{(p-2)/2} \int_0^{T \wedge \tau} \| g_s^{(e)} \|^p_{L^p} \, ds.
\]

Hence,

\[
E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} \leq E \sup_{t \leq \tau \wedge T} \| u_0^{(e)} \|^p_{L^p} + (1/4) E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} + \frac{N T^{(p-2)/2}}{p} \int_0^{T \wedge \tau} \left( \| g_s^{(e)} \|^p_{L^p} + \| f_s^{(e)} \|^p_{L^p} + \| Du_s^{(e)} \|^p_{L^p} \right) \, ds
\]

\[
+ N T^{p-1} \int_0^{T \wedge \tau} \| f_0^{(e)} \|^p_{L^p} \, ds.
\]

Upon collecting like terms we come to

\[
E \sup_{t \leq \tau \wedge T} \| u_t^{(e)} \|^p_{L^p} \leq 2 E \| u_0^{(e)} \|^p_{L^p} + N T^{p-1} \int_0^{T \wedge \tau} \| f_0^{(e)} \|^p_{L^p} \, ds
\]

\[
+ N T^{(p-2)/2} \int_0^{T \wedge \tau} \left( \| g_s^{(e)} \|^p_{L^p} + \| f_s^{(e)} \|^p_{L^p} + \| Du_s^{(e)} \|^p_{L^p} \right) \, ds. \quad (6.4)
\]

One can lawfully object that the last step leads to estimate (6.4) only if its left-hand side is finite. However, by Lemma 5.1 the process \( \| u_t^{(e)} \|^p_{L^p} \) is a continuous \( \mathcal{F}_t \)-adapted process which starts at \( \| u_0^{(e)} \|^p_{L^p} \) and we can stop it at time \( \tau_n \) when it first reaches the level \( \| u_0^{(e)} \|^p_{L^p} + n \) with \( n > 0 \) or at time \( \tau \) whichever comes first. Then

\[
E \sup_{t \leq \tau \wedge \tau_n} \| u_t^{(e)} \|^p_{L^p} \leq E \| u_0^{(e)} \|^p_{L^p} + n < \infty.
\]

Hence, the left-hand side of (6.4) will be finite if we replace there \( \tau \) with \( \tau_n \). Therefore, thus modified (6.4) holds and sending \( n \) to infinity yields (6.4) as is.
By applying this result to \( u_t^{(\varepsilon_1)} - u_t^{(\varepsilon_2)} \) we conclude that
\[
E \sup_{t \leq \tau \wedge T} \|u_t^{(\varepsilon_1)} - u_t^{(\varepsilon_2)}\|_{L^p} \to 0
\]
as \( \varepsilon_1, \varepsilon_2 \to 0 \). It follows that there exists a function \( v_t = v_t(\omega, x), 0 \leq t \leq \tau(\omega), t < \infty, x \in \mathbb{R}^d \), such that \( v_{t \wedge \tau} \) is continuous in \( t \) and \( \mathcal{F}_t \)-adapted as an \( L_p \)-valued function and
\[
E \sup_{t \leq \tau \wedge T} \|u_t^{(\varepsilon)} - v_t\|_{L^p} \to 0 \tag{6.5}
\]
as \( \varepsilon \to 0 \). In particular, in probability, for any \( \phi \in C_0^\infty \), we have
\[
(u_{t \wedge \tau}^{(\varepsilon)}, \phi) \to (v_{t \wedge \tau}, \phi) \tag{6.6}
\]
uniformly on \([0, T]\) for any \( T \in [0, \infty) \). Also in probability,
\[
|\int_0^T \left( |(f_s^{0(\varepsilon)} - f_s^0, D_i \phi)| + |(f_s^{0(\varepsilon)} - f_s^0, \phi)| \right) ds|^p \leq N \int_0^T \sum_{j=0}^d \|f_s^{j(\varepsilon)} - f_s^j\|_{L^p}^p ds \to 0,
\]
and (cf. \( \text{(4.6)} \))
\[
|\int_0^T \sum_{k=1}^\infty (g_{s_k}^{k(\varepsilon)} - g_{s_k}^k, \phi)^2 ds|^{p/2} \leq N \int_0^T \|g_{s}^{(\varepsilon)} - g_s\|_{L^p}^p ds \to 0.
\]
Therefore, we can pass to the limit in \((6.1)\) and conclude that \((1.1)\) holds with \( v \) in place of \( u \). The same argument as in the proof of Lemma \((5.1)\) now shows that with probability one the generalized functions \( v_{t \wedge \tau} \) and \( u_{t \wedge \tau} \) coincide for all \( t \in [0, \infty) \). This proves the assertion (i) of the theorem. After that \((2.3)\) is obtained by sending \( \varepsilon \to 0 \) in \((6.4)\). Finally, the argument in the proof of Lemma \((5.1)\) can also be repeated almost literally to obtain formula \((2.2)\) from \((6.2)\). The theorem is proved.

**Proof of Corollary 2.2** Denote by \( J(\tau) \) the left-hand side of \((2.4)\). Estimates similar to \((5.7)\) and \((6.3)\) show that \( J(\tau) \to \infty \) and if we have a sequence of stopping times \( \tau_n \uparrow \tau \), then \( J(\tau_n) \to J(\tau) \) as \( n \to \infty \). By taking a sequence which localizes the stochastic integral in \((2.2)\), then taking expectations of both sides of \((2.2)\), and, finally, using Fatou’s lemma, we obtain the first assertion of the corollary. If \( \tau \) is bounded, then the described procedure will yield the second assertion as well, due to \((2.3)\) and the dominated convergence theorem.

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