RIESZ MEANS ON SYMMETRIC SPACES

A. FOTIADIS AND E. PAPAGEORGIOU

To the memory of Professor Michel Marias.

Abstract. Let $X$ be a non-compact symmetric space of dimension $n$. We prove that if $f \in L^p(X)$, $1 \leq p \leq 2$, then the Riesz means $S^z_R(f)$ converge to $f$ almost everywhere as $R \to \infty$, whenever $\Re z > (n - \frac{1}{2}) \left( \frac{2}{p} - 1 \right)$.

1. Introduction and statement of the results

In this article we study the almost everywhere convergence of the Riesz means on a noncompact symmetric space of arbitrary rank. To state our results, we need to introduce some notation.

Let $G$ be a semi-simple, noncompact, connected Lie group with finite center and let $K$ be a maximal compact subgroup of $G$. We consider the $n$-dimensional symmetric space of noncompact type $X = G/K$, and let $\dim X = n$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ its dual. If $\dim \mathfrak{a} = l$, we say that $X$ has rank $l$.

The Killing form on $\mathfrak{g}$ restricts to a positive definite form on $\mathfrak{a}$, which in turn induces a positive inner product and hence a norm $\| \cdot \|$ on $\mathfrak{a}^*$. Denote by $\rho$ the half sum of positive roots, counted with their multiplicities. Fix $R \geq \|\rho\|^2$ and $z \in \mathbb{C}$ with $\Re z \geq 0$, and consider the bounded function

$$S^z_R(\lambda) = \left( 1 - \frac{\|\rho\|^2 + \|\lambda\|^2}{R} \right)^z, \quad \lambda \in \mathfrak{a}^*.$$  

Denote by $\kappa^z_R$ its inverse spherical Fourier transform in the sense of distributions and consider the so-called Riesz means operator $S^z_R$:

$$S^z_R(f)(x) = \int_G f(y)\kappa^z_R(y^{-1}x)dy = (f * \kappa^z_R)(x), \quad f \in C_0(X).$$
For every pair \( p, q \) such that \( 1 \leq p, q \leq \infty \), denote by \((L^p + L^q)(X)\) the Banach space of all functions \( f \) on \( X \) which admit a decomposition \( f = g + h \) with \( g \in L^p \) and \( h \in L^q \). The norm of \( f \in (L^p + L^q)(X) \) is given by

\[
\|f\|_{(p,q)} = \inf \{ \|f\|_p + \|g\|_q : \text{for all decompositions } f = g + h \}.
\]

For \( q \geq 1 \), denote by \( q' \) its conjugate. In the present work we prove the following results.

**Theorem 1.** Let \( z \in \mathbb{C} \) with \( \text{Re} z \geq n - \frac{1}{2} \) and consider \( q > 2 \). Then, for every \( p \) such that \( 1 \leq p \leq q' \), and for every \( r \in [q'/(p' - q), \infty] \), \( S^*_R \) is uniformly bounded from \( L^p(X) \) to \((L^p + L^r)(X)\).

Next we deal with the maximal operator \( S^*_s \) associated with Riesz means:

\[
S^*_s(f)(x) = \sup_{R > \|\rho\|^2} |S^*_R(f)(x)|, \quad f \in L^p(X), \quad 1 \leq p \leq 2.
\]

Set

\[
Z_0(n, p) = \left( n - \frac{1}{2} \right) \left( \frac{2}{p} - 1 \right).
\]

We have the following result.

**Theorem 2.** Let \( 1 \leq p \leq 2 \) and consider \( q > 2 \). If \( \text{Re} z > Z_0(n, p) \), then for every \( s \geq pq/(2 - p + pq - q) \), there is a constant \( c(z) > 0 \), such that for every \( f \in L^p(X) \),

\[
\|S^*_s f\|_{(p,s)} \leq c(z)\|f\|_p.
\]

Note that the \((p, s)\) norm is defined in (3). As a corollary of the Theorem 2 we obtain the almost everywhere convergence of Riesz means.

**Theorem 3.** Let \( 1 \leq p \leq 2 \). If \( \text{Re} z > Z_0(n, p) \), then for \( f \in L^p(X) \),

\[
\lim_{R \to +\infty} S^*_R f(x) = f(x), \quad \text{a.e.}.
\]

Our result treats the general case of noncompact symmetric spaces of all ranks. It is interesting that the index \( Z_0(n, p) \) only depends on the Euclidean dimension of \( X \) and not on the rank of \( X \). The only known results studying the Riesz means on noncompact symmetric spaces are \([18, 37]\), where the authors treat the case of rank one noncompact symmetric spaces, as well as the case of arbitrary rank when \( G \) is complex, and the case of \( SL(3, \mathbb{H})/Sp(3) \) respectively.

Here we treat the general case of noncompact symmetric spaces of all ranks, by using the inverse Abel transform. This way we can study the general case of a noncompact symmetric space, an area that remained inactive since the seminal work \([18]\) in 1991. The price we pay is that
our result is valid for \( \operatorname{Re} z \) larger than \( Z_0(n, p) = (n - \frac{1}{2}) \left( \frac{2}{p} - 1 \right) \).

Note that in the setting of \( \mathbb{R}^n \), [31], as well as in case of the rank one symmetric spaces, [18], (4) is valid for \( \operatorname{Re} z \) larger than the critical index \( z_0(n, p) = \left( \frac{n-1}{2} \right) \left( \frac{2}{p} - 1 \right) \). Thus, we can treat the arbitrary rank case but our result is not optimal, as a consequence of the lack of an explicit formula for the inverse Abel transform in the general case of a symmetric space.

Many authors have investigated the almost everywhere convergence of Riesz means. They have already been extensively studied in the case of \( \mathbb{R}^n \) ([8, 9, 31, 32] as well as in the book [14]). In the case of elliptic differential operators on compact manifolds they are treated in ([6] [10] [19] [24] [30] [34]). The case of Lie groups of polynomial volume growth and of Riemannian manifolds of nonnegative curvature is studied in [1] [27] and the case of compact semisimple Lie groups in [11].

To prove Theorem 1, we split the Riesz means operator in the sum of two convolution operators: \( S^z_R = S^z_{R,0} + S^z_{R,\infty} \). The local part \( S^z_{R,0} \) has a compactly supported kernel around the origin, while the kernel of the part at infinity \( S^z_{R,\infty} \) is supported away from the origin. To treat the local part, we follow the approach of [1] [29]. More precisely, we express the kernel of \( S^z_{R,0} \) via the heat kernel \( p_t \) of \( X \), and we make use of its estimates. Let \(-\Delta \) be the Laplace-Beltrami operator on \( X \). Then, combining the with the fact that the wave operator \( \cos(t \sqrt{-\Delta - \|\rho\|^2}) \) of \( X \) propagates with finite speed, allows us to prove that \( S^z_{R,0} \) is continuous on \( L^p(X) \) for all \( p \geq 1 \). To treat the part at infinity of the operator, we proceed as in [25], and obtain estimates of its kernel by using the support preserving property of the Abel transform.

This paper is organized as follows. In Section 2 we present the necessary ingredients for our proofs. In Section 3 we deal with the local part and the part at infinity, of the Riesz mean operator and we prove Theorem 1. In Section 4 we prove Theorem 2 and we deduce Theorem 3.

2. Preliminaries

In this section we recall some basic facts about symmetric spaces. For details see for example [2] [17] [22] [26].

2.1. Symmetric spaces. Let \( G \) be a semisimple Lie group, connected, noncompact, with finite center and let \( K \) be a maximal compact subgroup of \( G \). We denote by \( X \) the noncompact symmetric space \( G/K \). In the sequel we assume that \( \dim X = n \). Denote by \( \mathfrak{g} \) and \( \mathfrak{k} \) the Lie
algebras of $G$ and $K$. Let also $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ which is orthogonal to $\mathfrak{k}$ with respect to the Killing form. The Killing form induces a $K$-invariant scalar product on $\mathfrak{p}$ and hence a $G$-invariant metric on $X$. Denote by $\Delta$ the Laplace-Beltrami operator on $X$, by $d(.,.)$ the Riemannian distance and by $dx$ the associated Riemannian measure on $X$. Denote by $\Delta$ the Laplace-Beltrami operator on $X$, by $d(.,.)$ the Riemannian distance and by $dx$ the associated Riemannian measure on $X$. Denote by $\overline{B}(x,r)$ the volume of the ball $B(x,r)$, $x \in X$, $r > 0$, and recall that there is a $c > 0$, such that

$$|B(x,r)| \leq cr^n \text{ for all } r \leq 1,$$

[35, p.117].

Fix a maximal abelian subspace of $\mathfrak{p}$ and denote by $\mathfrak{a}^*$ the real dual of $\mathfrak{a}$. If $\dim \mathfrak{a} = l$, we say that $X$ has rank $l$. We also say that $\alpha \in \mathfrak{a}^*$ is a root vector, if the space

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X, \text{ for all } H \in \mathfrak{a}\} \neq \{0\}.$$

Let $A$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{a}$. Let $\mathfrak{a}_+ \subset \mathfrak{a}$ be a positive Weyl chamber and let $\overline{\mathfrak{a}_+}$ be its closure. Set $A^+ = \exp \mathfrak{a}_+$. Its closure in $G$ is $A_+ = \exp \overline{\mathfrak{a}_+}$. We have the Cartan decomposition

$$G = K\overline{A_+}K = K\exp \overline{\mathfrak{a}_+}K.$$

Then, each element $x \in G$ is written uniquely as $x = k_1(\exp H)k_2$. We set

$$|x| = |H|, \quad H \in \overline{\mathfrak{a}_+},$$

the norm on $G$ [34, p.2]. Denote by $x_0 = eK$ a base point of $X$. If $x, y \in X$, there are isometries $g, h \in G$ such that $x = gx_0$ and $y = hx_0$. Because of the Cartan decomposition (6), there are $k, k' \in K$ and a unique $H \in \overline{\mathfrak{a}_+}$ with $g^{-1}h = k \exp Hk'$. It follows that

$$d(x,y) = |H|,$$

where $d(x,y)$ is the geodesic distance on $X$ [36].

Normalize the Haar measure $dk$ of $K$ such that $\int_K dk = 1$. Then, from the Cartan decomposition, it follows that

$$\int_G f(g)dg = \int_K dk_1 \int_{\overline{\mathfrak{a}_+}} \delta(H)dH \int_K f(k_1 \exp(H)k_2)dk_2,$$

where the modular function $\delta(H)$ satisfies the estimate

$$\delta(H) \leq ce^{2\rho(H)}.$$

We identify functions on $X = G/K$ with functions on $G$ which are $K$-invariant on the right, and hence bi-$K$-invariant functions on $G$ are
identified with functions on $X$, which are $K$-invariant on the left. Note that if $f$ is $K$-bi-invariant, then by (8),

$$\int_G f(g) \, dg = \int_X f(x) \, dx = \int_{a_+} f(\exp H) \delta(H) \, dH. \tag{10}$$

2.2. The spherical Fourier transform. Denote by $S(K\backslash G/K)$ the Schwartz space of $K$-bi-invariant functions on $G$. For $f \in S(K\backslash G/K)$, the spherical Fourier transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(\lambda) = \int_G f(x) \varphi_{-\lambda}(x) \, dx, \quad \lambda \in a^*,$$

where $\varphi_{\lambda}$ is the elementary spherical function of index $\lambda$ on $G$. Note that from [22] we have the following estimate

$$\varphi_0(\exp H) \leq c(1 + |H|)^d e^{-\rho(H)}, \tag{11}$$

where $d$ is the cardinality of the set of positive indivisible roots.

Let $S(a^*)$ be the usual Schwartz space on $a^*$. Denote by $W$ the Weyl group associated to the root system of $(g, a)$ and denote by $S(a^*)^W$ the subspace of $W$-invariant functions in $S(a^*)$. Then, by a celebrated theorem of Harish-Chandra, $\mathcal{H}$ is an isomorphism between $S(K\backslash G/K)$ and $S(a^*)^W$ and its inverse is given by

$$(\mathcal{H}^{-1} f)(x) = c \int_{a^*} f(\lambda) \varphi_{-\lambda}(x) \frac{d\lambda}{|c(\lambda)|^2}, \quad x \in G, \quad f \in S(a^*)^W,$$

where $c(\lambda)$ is the Harish-Chandra function and $c$ is a normalizing constant independent of $f$, [22, Theorem 7.5].

2.3. The heat kernel on $X$. Set

$$w_t(\lambda) = e^{-t(\|\lambda\|^2 + \|\rho\|^2)}, \quad t > 0, \quad \lambda \in a^*,$$

Then the heat kernel $p_t(x)$ of $X$ is given by $(\mathcal{H}^{-1} w_t)(x)$ [4].

The heat kernel $p_t$ on symmetric spaces has been extensively studied, see for example [4, 5]. Sharp estimates of the heat kernel have been obtained by Davies and Mandouvalos in [15] for the case of real hyperbolic space, while Anker and Ji [4] and later Anker and Ostellari [5], generalized the results of [15] to all symmetric spaces of noncompact type.

Denote by $\Sigma^+_0$ the set of positive indivisible roots $\alpha$ of $(g, a)$ and by $m_\alpha$ the dimension of the root space $g^\alpha$. In [5, Main Theorem] it is
proved the following sharp estimate:

\[
p_t(\exp H) \leq ct^{-n/2} \left( \prod_{\alpha \in \Sigma_0^+} (1 + t + \langle \alpha, H \rangle)^{m_{\alpha} + m_{2\alpha} - 1} \right) \times \times e^{-\|\rho\|^2t - \langle \rho, H \rangle - |H|^2/4t}, \quad t > 0, \ H \in \overline{a_+},
\]

where \( n = \dim X \).

From (12), we deduce the following crude estimate

\[
p_t(\exp H) \leq ct^{-n/2} e^{-\|H\|^2/4t}, \quad t > 0, \ H \in \overline{a_+},
\]

which is sufficient for our purposes.

Note also that (13) yields the on-diagonal upper bound

\[
p_t(e) \leq ct^{-n/2}.
\]

As it is shown in [21, Lemma 3.1], estimate (14) implies that there is an absolute constant \( D > 0 \), sufficiently large, such that for every \( a > 0 \), there holds

\[
\int_{d(x,x_0) > a} p_t^2(x) dx \leq ct^{-n/2} e^{-a^2/Dt}.
\]

3. Proof of Theorem 1

Let \( \kappa_R^z \) be the kernel of the Riesz means operator. We start with a decomposition of \( \kappa_R^z \):

\[
\kappa_R^z = \zeta \kappa_R^z + (1 - \zeta) \kappa_R^z := \kappa_R^{z,0} + \kappa_R^{z,\infty},
\]

where \( \zeta \in C^\infty(K \setminus G/K) \) is a cut-off function such that

\[
\zeta(x) = \begin{cases} 
1, & \text{if } |x| \leq 1/2, \\
0, & \text{if } |x| \geq 1.
\end{cases}
\]

Denote by \( S_R^{z,0} \) (resp. \( S_R^{z,\infty} \)) the convolution operator on \( X \) with kernel \( \kappa_R^{z,0} \) (resp. \( \kappa_R^{z,\infty} \)).

3.1. The local part. We shall prove the following proposition.

**Proposition 4.** Assume that \( \Re z > n/2 \). Then the operator \( S_R^{z,0} \) is bounded on \( L^p(X) \), \( 1 \leq p \leq \infty \), and \( \|S_R^{z,0}\|_{p \to p} \leq c(z) \), for some constant \( c(z) > 0 \).

The proof is lengthy and it will be given in several steps. First, we shall express the kernel \( \kappa_R^z \) in terms of the heat kernel \( p_t \) of \( X \). Then, we shall use the heat kernel estimates (13) to prove that \( \kappa_R^z \) is integrable in the unit ball \( B(0,1) \) of \( X \). This implies that \( S_R^{z,0} \) is
bounded on $L^\infty(X)$. We then prove that $S^z_R$ is bounded on $L^2(X)$, and an interpolation argument between $L^\infty(X)$ and $L^2(X)$ allows us to conclude.

To express the kernel $\kappa^z_R$ in terms of $p_t$, we follow [1] and we write

\begin{equation}
S^z_R(\lambda) = \left(1 - \frac{||\lambda||^2 + ||\rho||^2}{R}\right)^z.
\end{equation}

Set $r = \sqrt{R}, \xi = ||\lambda||$ and consider the function

\begin{equation}
h^z_R(\lambda) = h^z_R(\xi) := \left(1 - \frac{\xi^2 + ||\rho||^2}{r}\right)^z e^{\frac{\xi^2 + ||\rho||^2}{r^2}}.
\end{equation}

Then, from (18) and (19) we have

\begin{equation}
s^z_R(\lambda) = h^z_R(\lambda) e^{-\frac{||\lambda||^2 + ||\rho||^2}{r^2}},
\end{equation}

and thus

\begin{equation}
s^z_R(\sqrt{-\Delta - ||\rho||^2}) = h^z_R(\sqrt{-\Delta - ||\rho||^2}) e^{-1/r^2(-\Delta)}.
\end{equation}

Next, we recall the construction of the partition of unity of [1, p.213] we shall use for the splitting of the operator $s^z_R(-\Delta)$. For that we set $\psi(\xi) = e^{-\xi^2/2}, \xi \geq 0$, and $\psi_1(\xi) = \psi(\xi)\psi(1 - \xi)$. Then $\psi_1 \in C^\infty(\mathbb{R})$ and supp $\psi_1 = [0, 1]$. Set also $\phi(\xi) = \psi_1(\xi + \frac{5}{4})$, and

$\phi_j(\xi) = \phi(2^j(\xi - 1)), \quad j \in \mathbb{N}.$

Then $\phi_j(\xi)$ is a $C^\infty$ function with support in $I_j = [1 - 5/2^{j+2}, 1 - 1/2^{j+2}]$. The functions

$\chi_j(\xi) = \frac{\phi_j(\xi)}{\sum_{i \geq 0} \phi_i(\xi)},$

form the required partition of unity.

Set

$\chi_{j,r}(\xi) = \chi_j((\xi/r)^2),$

and

$h^z_{j,r}(\xi) := h^z_R(\xi)\chi_{j,r}(\xi).$

Consider the operator

\begin{equation}
T^z_{j,r} := s^z_{j,r}(\sqrt{-\Delta - ||\rho||^2}) = h^z_{j,r}(\sqrt{-\Delta - ||\rho||^2}) e^{-1/r^2(-\Delta)}.
\end{equation}

Note that by (22) and (21),

\begin{equation}
\sum_{j \in \mathbb{N}} T^z_{j,r} = \sum_{j \in \mathbb{N}} h^z_{j,r}(\sqrt{-\Delta - ||\rho||^2}) e^{-1/r^2(-\Delta)} = h^z_R(\sqrt{-\Delta - ||\rho||^2}) e^{-1/r^2(-\Delta)} = s^z_R(\sqrt{-\Delta - ||\rho||^2}).
\end{equation}
Denote by \( \kappa_{j,r}^z \) the kernel of the operator \( T_{j,r}^z \). Then, (22) implies that
\[
\kappa_{j,r}^z(x) = T_{j,r}^z \delta_{x_0}(x) = h_{j,r}^z(\sqrt{-\Delta} - \|\rho\|^2)e^{-1/r^2(-\Delta)}\delta_{x_0}(x)
\]
(24)
where \( x_0 \) is the basepoint on \( X \). Consequently, (23) and (24) imply that
\[
(25) \quad \kappa_R^z = \sum_{j\in\mathbb{N}} \kappa_{j,r}^z.
\]
So, to estimate the kernel \( \kappa_R^z \), it suffices to estimate the kernels \( \kappa_{j,r}^z \), which by (24) are expressed in terms of the heat kernel \( p_t \) of \( X \) and the functions \( h_{j,r}^z \).

There is a \( c > 0 \) such that
\[
(26) \quad |\text{supp} \ h_{j,r}^z| \leq cr^{2-j},
\]
[1, p.214]. Note that the functions \( \chi_j \), as well as \( h_{j,r}^z \), are radial and thus invariant by the Weyl group [2, p.612].

Note also that for every \( k \in \mathbb{N} \), there is a \( c_k > 0 \), such that for every \( r > 0 \), it holds
\[
(27) \quad \|\chi_{j,r}^{(k)}\|_\infty \leq c_k r^{-k} 2^{kj}, \quad \|h_{j,r}^z(k)\|_\infty \leq c_k r^{-k} 2^{-(\text{Re} \ z-k)j}.
\]
As it is mentioned in [1, p.214], the estimates (26) and (27) imply that for every \( k \in \mathbb{N} \), there is a \( c_k > 0 \) such that
\[
(28) \quad \int_{|t| \geq s} |\hat{h}_{j,r}^z(t)| dt \leq c_k s^{-k} r^{-k} 2^{(k-\text{Re} \ z)j}, \quad s > 0,
\]
where \( \hat{h}_{j,r}^z \) is the euclidean Fourier transform of \( h_{j,r}^z \).

**Lemma 5.** Let \( \kappa_R^z \) be the kernel of the Riesz mean operator \( S_R^z \). Then, there is \( c > 0 \), independent of \( R \), such that for \( \text{Re} \ z > n/2 \),
\[
\|\kappa_R^z\|_{L^1(B(0,1))} \leq c.
\]

**Proof.** For the proof we shall consider different cases. Recall that \( R \geq \|\rho\|^2 \).

**Case 1:** \( \|\rho\|^2 \leq R \leq \|\rho\|^2 + 1 \).

Combining (13) and the heat semigroup property, we get that
\[
(29) \quad \|p_t\|_{L^2(X)} = \left( \int_X p_t(x,y)p_t(y,x)dy \right)^{1/2} \leq p_{2t}(x,x)^{1/2} \leq ct^{-n/4}.
\]
Thus, using (27), (24), (29) and (5) we have
\[
\|\kappa_{j,r}^z\|_{L^1(B(0,1))} \leq |B(0,1)|^{1/2}\|\kappa_{j,r}^z\|_{L^2(X)} \\
\leq c\|h_{j,r}^z(\sqrt{-\Delta} - \|\rho\|^2)\|_{L^2 \to L^2} \|p_{1/r^2}\|_{L^2(X)} \\
\leq c\|h_{j,r}^z\|_{\infty}(1/r^2)^{-n/4} \\
\leq c(n, \|\rho\|)2^{-j \text{Re} z}.
\]
(30)

So,
\[
\|\kappa_{j,r}^z\|_{L^1(B(0,1))} \leq \sum_{j \in \mathbb{N}} \|\kappa_{j,r}^z\|_{L^1(B(0,1))} \leq c \sum_{j \in \mathbb{N}} 2^{-j \text{Re} z} \leq c,
\]
\[
\text{since Re } z > 0.
\]

Case 2: \(R \geq \|\rho\|^2 + 1\).

Recall that \(r = \sqrt{R}\). So, the ball \(B(0,1/r)\) is contained in the unit ball. Next, let \(i \geq 0\) be such that \(2^i - 1 < r \leq 2^i\) and consider the annulus \(A_p = \{x \in X : 2^p \leq |x| \leq 2^{p+1}\}\), with \(p \geq -i\). We write
\[
B(0,1) \subseteq \bigcup_{p=-i}^{0} A_p.
\]

Applying (27), (24), (29) and (5) and proceeding as in Case 1, we have
\[
\|\kappa_{j,r}^z\|_{L^1(B(0,1/r))} \leq |B(0,1/r)|^{1/2}\|\kappa_{j,r}^z\|_{L^2(X)} \\
\leq c_n r^{-n/2}\|h_{j,r}^z(\sqrt{-\Delta} - \|\rho\|^2)\|_{L^2 \to L^2} \|p_{1/r^2}\|_{L^2(X)} \\
\leq c_n r^{-n/2}\|h_{j,r}^z\|_{\infty}(1/r^2)^{-n/4} \\
= c_n \|h_{j,r}^z\|_{\infty} \\
\leq c_n 2^{-j \text{Re} z},
\]
that is
\[
\|\kappa_{j,r}^z\|_{L^1(B(0,1/r))} \leq c 2^{-j \text{Re} z}.
\]
(31)

So, to finish the proof of the lemma it remains to prove estimates of the kernels \(\kappa_{j,r}^z\) on the annulus \(A_p\). For that, we shall use the fact that the kernel \(G_t(x,y), x,y \in X\), of the wave operator \(\cos(t \sqrt{-\Delta} - \|\rho\|^2)\), propagates with finite speed \([7, \text{p.19}]\), that is
\[
\text{supp}(G_t) \subset \{(x,y) : d(x,y) \leq |t|\}.
\]
(32)

As observed by the authors, \([7, \text{pp.39-40}]\), we may use the following formula for even functions \(f(\lambda)\):
\[
f(\sqrt{-\Delta} - \|\rho\|^2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t) \cos(t \sqrt{-\Delta} - \|\rho\|^2) dt.
\]
(33)
Since \( h_{j,r}^z \) is even, by (33) we have

\[
\kappa_{j,r}^z(x) = [h_{j,r}^z(\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)](x)
\]

\[
= (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{h}_{j,r}^z(t)[\cos(t\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)](x)dt.
\]

So, if \( x \in A_p \), then

\[
\kappa_{j,r}^z(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{h}_{j,r}^z(t)[\cos(t\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)1_{\{|y|\leq 2^{p-1}\}}](x)dt \\
+ (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{h}_{j,r}^z(t)[\cos(t\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)1_{\{|y|> 2^{p-1}\}}](x)dt.
\]

(35)

\[
= (2\pi)^{-1/2} \int_{|t|\geq 2^{p-1}} \hat{h}_{j,r}^z(t)[\cos(t\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)1_{\{|y|\leq 2^{p-1}\}}](x)dt \\
+ (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{h}_{j,r}^z(t)[\cos(t\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)1_{\{|y|> 2^{p-1}\}}](x)dt,
\]

where in the last equality we have used the finite propagation speed of the wave operator: if \(|y| \leq 2^{p-1}\) and \(|x| \geq 2^p\), then (32) implies that \(|t| \geq 2^{p-1}\).

So, using (34), equality (35) rewrites

\[
\kappa_{j,r}^z(x) = (2\pi)^{-1/2} \int_{|t|\geq 2^{p-1}} \hat{h}_{j,r}^z(t)[\cos(t\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)1_{\{|y|\leq 2^{p-1}\}}](x)dt \\
+ \hat{h}_{j,r}^z(\sqrt{-\Delta - \|p\|^2})p_{r-2}(\cdot)1_{\{|y|> 2^{p-1}\}}(x).
\]

(36)

Applying Cauchy-Schwarz to (36) and using the fact that \( \|\cos t\sqrt{-\Delta}\|_{2\to 2} \leq 1 \), as well as the spectral theorem, we obtain

\[
\|\kappa_{j,r}^z\|_{L^1(A_p)} \leq c|A_p|^{1/2} \int_{|t|\geq 2^{p-1}} |\hat{h}_{j,r}^z(t)| \|p_{r-2}\|_{2} dt \\
+ c|A_p|^{1/2}\|\hat{h}_{j,r}^z\|_{\infty} \|p_{r-2}1_{\{|y|> 2^{p-1}\}}\|_2 := I_1 + I_2.
\]

(37)

From (27), (15) and the fact that \( 2^{i-1} < r \leq 2^i \), it follows that

\[
I_2 \leq c2^{p/2}2^{-j}\Re z(r^{-1/2})^{-n/4}e^{-2^{p-1}/2Dr^{-2}} \\
\leq c2^{-j}\Re z2^{p/2}r^{-n/2}e^{-2^{p-2}/4D} \\
\leq c2^{-j}\Re z2^{(p+1)n/2}e^{-Dr2^{p-1}}.
\]

Using the elementary estimate

\[
e^{-Dr2^{p-1}x/n/2} \leq c_k x^{-k}, \text{ for all } x > 1, k \in \mathbb{N},
\]
we obtain
\begin{equation}
I_2 \leq 2^{-j \Re z} 2^{-k(p+i)}.
\end{equation}

Also, from (29) we have that
\begin{equation}
I_1 \leq c 2^{p/2} (r^{-2})^{-n/4} \int_{|t| \geq 2^{p-1}} |\hat{h}_{j,r}^z(t)| dt.
\end{equation}

Then, applying (28) for $k > n/2$, we obtain
\begin{equation}
I_1 \leq c n 2^{(p+i)n/2} 2^{-pk} 2^{(k-\Re z)j} \leq c 2^{-(p+i)(k-n/2) - j(\Re z - n/2)}.
\end{equation}

Finally, using (38) and (39), (37) implies that
\begin{equation}
\|\kappa_{j,r}^z\|_{L^1(A_p)} \leq c 2^{-(p+i)(k-n/2) - j(\Re z - n/2)}.
\end{equation}

End of proof of Lemma 5.

It follows from (31) and (40) that
\begin{equation}
\|\kappa_{j,r}^z\|_{L^1(B(0,1))} \leq c 2^{-j \Re z + c \sum_{p=0}^0 2^{-(p+i)(k-n/2) - j(\Re z - n/2)}} \leq c 2^{-j(\Re z - n/2)}.
\end{equation}

So, for $\Re z > n/2$,
\begin{equation}
\|\kappa_{j,r}^z\|_{L^1(B(0,1))} \leq \sum_{j \geq 0} \|\kappa_{j,r}^z\|_{L^1(B(0,1))} \leq c \sum_{j \geq 0} 2^{-j(\Re z - n/2)} \leq c.
\end{equation}

\[ \square \]

Lemma 6. $S^z_R$ is bounded on $L^2(X)$.

Proof. Set
\begin{equation}
\kappa_{j,r}^{z,0} = \zeta_{j,r}^z, \ T_{j,r}^{z,0} = *\kappa_{j,r}^{z,0} \text{ and } s_{j,r}^{z,0} = \mathcal{H}(\kappa_{j,r}^{z,0}),
\end{equation}

where $\zeta$ is the cut-off function given in (17).

By Plancherel theorem and using (42), we get that
\begin{equation}
\|T_{j,r}^{z,0}\|_{L^2 \to L^2} \leq \|s_{j,r}^{z,0}\|_{L^\infty(a^*)} = \|\mathcal{H}(\kappa_{j,r}^{z,0})\|_{L^\infty(a^*)} = \|\mathcal{H}(\zeta)\|_{L^\infty(a^*)} \leq c(\zeta) < \infty.
\end{equation}

But $\zeta \in S(K\backslash G/K)$. Therefore, its spherical Fourier transform $\mathcal{H}(\zeta)$, belongs in $S(a^*)^W \subset L^1(a^*)$, (see Section 2). So,
\begin{equation}
\|\mathcal{H}(\zeta)\|_{L^1(a^*)} \leq c(\zeta) < \infty.
\end{equation}
From (43), (22) and (27) it follows that
\[ \| T_{j,r}z,0 \|_{L^2 \to L^2} \leq c(\zeta) \| h_{j,r}^z(\sqrt{\cdot}) e^{-1/j^2(\cdot)} \|_{L^\infty(a^*)} \]
(44)
Further, by (44) and the fact that \( S_{z,0}^R = \sum_{j \geq 0} T_{j,r}^z \) it follows that
\[ \| S_{R}^z,0 \|_{L^2 \to L^2} \leq \sum_{j \geq 0} \| T_{j,r}^z \|_{L^2 \to L^2} \leq c \sum_{j \geq 0} 2^{-j \Re z} \leq c < \infty. \]
(45)

**End of the proof of Proposition 4** Since \( \kappa_{z,0}^r = \sum_{j \geq 0} \kappa_{j,r}^z \), by Lemma 5, we have
\[ \| \kappa_{R}^z,0 \|_{L^1(X)} = \| \zeta \kappa_{R}^z \|_{L^1(X)} \leq c \| \kappa_{R}^z \|_{L^1(B(0,1))} < c. \]
This implies that
\[ \| S_{R}^z,0 \|_{L^\infty \to L^\infty} \leq c(z). \]
(46)

By interpolation and duality, it follows from (46) and (45), that for all \( p \in [1, \infty], \| S_{R}^z,0 \|_{p \to p} \leq c(z), \) with \( \Re z > n/2. \)

### 3.2. The part at infinity.
For the part at infinity \( S_{R}^z,\infty \) of the operator, we proceed as in [25] to obtain estimates of its kernel \( \kappa_{R}^z,\infty \). Let \( l = \text{rank}(X). \)
To begin with, recall that \( \kappa_{R}^z = \mathcal{H}^{-1}s_{R}^z. \) Recall also the following result from [25, p.650], based on the Abel transform conservation property.

**Lemma 7.** For \( x = k_{1}(\exp H)k_{2} \in G, \) with \( |x| > 1 \) and \( k \in \mathbb{N} \) with \( k > \frac{n}{2} - \frac{l}{4}, \) we have that
\[ \| \kappa_{R}^z,0 \|_{L^1(X)} = \| \zeta \kappa_{R}^z \|_{L^1(X)} \leq c \| \kappa_{R}^z \|_{L^1(B(0,1))} < c. \]
(46)

Thus, to estimate the kernel for \( |x| > 1, \) it suffices to obtain estimates for the derivatives of the euclidean inverse Fourier transform of \( s_{R}^z(\lambda). \)
Denote by \( J_{\nu}(t) = r^{-\nu} J_{\nu}(t), \) \( t > 0, \) where \( J_{\nu} \) is the Bessel function of order \( \nu. \) Then, it holds
\[ (\mathcal{F}^{-1}s_{R}^z)(\exp H) = c(n, z) R^{-z} (R - \| \rho \|^2)^{z+l/2} J_{z+l/2} \left( \sqrt{R - \| \rho \|^2} |H| \right), \]
[14, 18], and we shall need the following auxiliary lemma.
Lemma 8. For every multi-index \( \alpha \), it holds that
\[
|\partial_H^\alpha \mathcal{J}_{z+t/2}(\sqrt{R-\|\rho\|^2}|H|)| \leq c(R-\|\rho\|^2)^{|\alpha|/2} \cdot (\frac{\text{Re}z + l+1}{2}) |H|^{-(\text{Re}z+l+1/2)}.
\]

Proof. Using the identity \( \mathcal{J}'_\nu(t) = -t\mathcal{J}_{\nu+1}(t) \), it is straightforward to get that
\[
\mathcal{J}'_\nu(t) = (-1)^{\alpha} t^\alpha \mathcal{J}_{\nu+a}(t) + \sum_{j=1}^{[\alpha/2]} c_j^\alpha t^{a-2j} \mathcal{J}_{\nu+a-j}(t), \quad a \in \mathbb{N},
\]
for some constants \( c_j^\alpha \), where \([a]\) denotes the integer part of \( a \). Applying the inequality
\[
|\mathcal{J}_\nu(t)| \leq c\mu t^{-(\text{Re} \mu + 1/2)}, \quad \text{for all } t > 0,
\]

\[18\], it follows that
\[
|\partial_H^\alpha \mathcal{J}(\sqrt{R-\|\rho\|^2}|H|)| \leq c(R-\|\rho\|^2)^{|\alpha|/2} \cdot (\frac{\text{Re}z + l+1}{2}) |H|^{-(\text{Re}z+l+1/2)}
\]
and (49) follows by taking \( \nu = z + l/2 \). \( \square \)

Lemma 9. If \( R \geq \|\rho\|^2 + 1 \), then
\[
|\kappa^z_R(x)| \leq c\varphi_0(x) R^{-\frac{1}{2}(\text{Re}z-n+\frac{1}{2})} |x|^{-\text{Re}z-\frac{1}{2}}, \quad |x| > 1.
\]

Proof. From (49), we get that
\[
I^2 := \int_{|H|>|x|-\frac{1}{2}} \left( \sum_{|\alpha|\leq 2k} |\partial_H^\alpha \mathcal{J}_{z+t/2}(\sqrt{R-\|\rho\|^2}|H|)| \right)^2 dH
\]
\[
\leq c \left( \sum_{|\alpha|\leq 2k} (R-\|\rho\|^2)^{a/2} \right)^2 \times
\]
\[
\int_{|H|>|x|-\frac{1}{2}} \left( (R-\|\rho\|^2)^{-\frac{\text{Re}z + l+1}{4}} |H|^{-(\text{Re}z+l+1/2)} \right)^2 dH
\]
\[
\leq c(R-\|\rho\|^2)^{-2(\text{Re}z + l+1)+2k} \int_{u>|x|-\frac{1}{2}} u^{-(l+1)-2\text{Re}z} u^{l-1} du
\]
\[
(52) \quad I \leq c(R-\|\rho\|^2)^{-2(\text{Re}z + l+1)+2k} \left( |x| - \frac{1}{2} \right)^{-2\text{Re}z-\frac{1}{2}}.
\]

For \( R \geq \|\rho\|^2 + 1 \), since \( k > \frac{n}{2} - \frac{l}{4} \), we have that
\[
(53) \quad I \leq c(R-\|\rho\|^2)^{-\frac{(\text{Re}z-1/2)+1}{2}} \left( |x| - \frac{1}{2} \right)^{-\text{Re}z-\frac{1}{2}}.
\]
Using (53) and (48), from (47) we obtain that
\[ |\kappa_{z,R}(x)| \leq c\varphi_0(x)R^{-\Re z}(R - \|\rho\|^2)^{\Re z + \frac{1}{2}} \times \]
\[ \times (R - \|\rho\|^2)^{-\left(\frac{\Re z + \frac{1}{2}}{2} + \frac{1}{4}\right)} \left(\frac{|x|}{2} - \frac{1}{2}\right)^{-\Re z - \frac{1}{2}} \]
\[ \leq c\varphi_0(x)R^{-\frac{1}{2}(\Re z - \frac{1}{2})}|x|^{-\Re z - \frac{1}{2}}, \quad |x| > 1. \]

Using the estimate (53) and proceeding as above, one can prove the following result.

**Lemma 10.** If \( \|\rho\|^2 \leq R \leq \|\rho\|^2 + 1 \), then
\[ |\kappa_{z,R}(x)| \leq c\varphi_0(x)|x|^{-\Re z - \frac{1}{2}}, \quad |x| > 1. \]

Finally, we shall prove the following result, which, combined with Proposition 4, finishes the proof of Theorem 1.

**Proposition 11.** Let \( \Re z \geq n - \frac{1}{2} \) and consider \( q > 2 \). Then for every \( p \) such that \( 1 \leq p \leq q' \), \( S_{z,\infty}^{\ast} \) is continuous from \( L^p(X) \) to \( L^r(X) \) for every \( r \in [qp'/2 - q, \infty) \), and \( \|S_{z,\infty}^{\ast}\|_{p \to r} \leq c(z) \) for all \( R \geq \|\rho\|^2 \).

**Proof.** Recall that \( \kappa_{z,\infty}^{\ast}(x) = \kappa_{z,R}(x) \) for every \( |x| > 1 \). Using the estimates of \( \kappa_{z,R} \) from Lemmata 9 and 10, as well as the estimate (11), it follows that \( \kappa_{z,\infty}^{\ast} \) is in \( L^q(X) \) for every \( q > 2 \). Thus, by Young’s inequality, the operator \( f \to |f| * \kappa_{z,\infty}^{\ast} \) maps \( L^p(X) \), \( p \in [1, q'] \), continuously into \( L^r(X) \), for every \( r \in [qp'/2 - q, \infty) \).

Further, for \( z \geq n - \frac{1}{2} \), in Lemmata 9 and 10 the estimates of the kernel \( \kappa_{z,\infty}^{\ast} \) are uniform with respect to \( R \). This implies that the norm \( \|S_{z,\infty}^{\ast}\|_{p \to r} \) is bounded by a constant, uniform with respect to \( R \). \( \square \)

### 4. Proof of Theorem 2 and Theorem 3

In this section we give the proof of Theorem 2 which deals with the \( L^p \)-continuity of the maximal operator \( S_{z}^{\ast} \) associated with the Riesz means. This allows us to deduce the almost everywhere convergence of Riesz means \( S_{z}^{\ast}(f) \) to \( f \), as \( R \to +\infty \).

Recall first that
\[ S_{z}^{\ast}(f) = \sup_{R > \|\rho\|^2} |S_{z,R}^{\ast}(f)|, \quad f \in L^p(X). \]

The following proposition holds true, [18, Lemma 4.1].

**Proposition 12.** Let \( \Re z > 0 \). Then, \( S_{z}^{\ast} \) is continuous on \( L^2(X) \).
Recall the following decomposition of the kernel $\kappa^z_R$ of the operator $S^z_R$:

\begin{equation}
\kappa^z_R = \zeta \kappa^z_R + (1 - \zeta) \kappa^z := \kappa^z,0 + \kappa^z,\infty,
\end{equation}

where $\zeta \in C^\infty(K \setminus G/K)$ is a cut-off function such that

\begin{equation}
\zeta(x) = \begin{cases} 1, & \text{if } |x| \leq 1/2, \\ 0, & \text{if } |x| \geq 1. \end{cases}
\end{equation}

Denote by $S^z,0_R$ (resp. $S^z,\infty_R$) the convolution operators on $X$ with kernel $\kappa^z,0_R$ (resp. $\kappa^z,\infty_R$). Then,

$$S^z_* f \leq \sup_{R \geq \|\rho\|^2} |S^z,0_R f| + \sup_{R \geq \|\rho\|^2} |S^z,\infty_R f|.$$ 

The following holds true for the part at infinity $S^z,\infty_R$ of the operator $S^z_*$.

**Proposition 13.** Let $\Re z \geq n - \frac{1}{2}$. Then, for every $q > 2$ and $p \in [1, q']$, $S^z,\infty_R$ is continuous from $L^p(X)$ to $L^r(X)$ for every $r \in [qp'/(p' - q), \infty]$.

The proof relies on the uniform kernel estimates for $\kappa^z,\infty_R$ implied by Lemmata 9 and 10. It is similar to the proof of Proposition 11, thus omitted.

We shall now prove the following result concerning the local part $S^z,0_R$ of the Riesz means maximal operator.

**Proposition 14.** Let $\Re z \geq n - \frac{1}{2}$. Then, $S^z,0_R$ is continuous on $L^p(X)$, for every $p \in (1, \infty)$, and it maps $L^1(X)$ continuously into $L^{1,w}(X)$.

Denote by $e^{t\Delta}$, $t > 0$, the heat operator on $X$. Then, $e^{t\Delta} = *p_t$, where $p_t$ is the heat kernel on $X$. Recall that $p_t$ is given as the inverse spherical Fourier transform of $w_t(\lambda) = e^{-t(\|\lambda\|^2 + \|\rho\|^2)}$, $\lambda \in a^*$. Consider the radial multiplier

\begin{equation}
M(R^{-1}\lambda) := s^z_R(\lambda) - w_{R^{-1}}(\lambda), \ R \geq \|\rho\|^2.
\end{equation}

Denote by $K_R(x)$ the kernel of the operator $M(-R^{-1}\Delta)$ and set $K^0_R(x) := \zeta(x)K_R(x)$. Similarly, set $s^z_R := \mathcal{H}(\zeta \kappa^z_R) = \mathcal{H}(\kappa^z,0_R)$ and $w^0_{R^{-1}} = \mathcal{H}(\zeta p_{R^{-1}}) = \mathcal{H}(p^0_{R^{-1}})$. Then, using (57), we have that

\begin{equation}
\mathcal{H}(\kappa^z_R) := M^0(-R^{-1}\cdot) = s^z_R - w^0_{R^{-1}},
\end{equation}
From (58) we have that
\begin{equation}
S_{*}^{z,0}f = \sup_{R \geq \|\rho\|^2} |s_{R}^{z,0}(-\Delta)f| \leq \sup_{R \geq \|\rho\|^2} |M^{0}(-R^{-1}\Delta)f| + \sup_{R \geq \|\rho\|^2} |f * p_{R^{-1}}^{0}|.
\end{equation}

Consider the operator \((-\Delta)^{i\gamma}, \gamma \in \mathbb{R}\), which in the spherical Fourier transform variables is given by
\begin{equation*}
\mathcal{H}((-\Delta)^{i\gamma}f) = (\|\lambda\|^2 + \|\rho\|^2)^{i\gamma} \mathcal{H}(f), \lambda \in \mathfrak{a}^{*}.
\end{equation*}
Denote by \(\kappa^{\gamma} = \mathcal{H}^{-1}(\|\lambda\|^2 + \|\rho\|^2)^{i\gamma}\) the kernel of \((-\Delta)^{i\gamma}\). As in [1, 18], using the Mellin transform \(M(\gamma)\) of the radial function \(M(\lambda)\), one can express the operator \(M(-R^{-1}\Delta)\) as follows:
\begin{equation}
M(-R^{-1}\Delta) = \int_{-\infty}^{+\infty} \mathcal{M}(\gamma) R^{-i\gamma} (-\Delta)^{i\gamma} d\gamma,
\end{equation}
where
\begin{equation}
|\mathcal{M}(\gamma)| \leq c(1 + |\gamma|)^{-(\Re z+1)},
\end{equation}
[18]. Using (60), the kernel \(K_{R}\) of \(M(-R^{-1}\Delta)\) is given by
\begin{equation*}
K_{R} = \int_{-\infty}^{+\infty} \mathcal{M}(\gamma) R^{-i\gamma} \kappa^{\gamma} d\gamma,
\end{equation*}
and thus
\begin{equation*}
K_{R}^{0}(x) = \zeta(x)K_{R}(x) = \int_{-\infty}^{+\infty} \mathcal{M}(\gamma) R^{-i\gamma} \zeta(x) \kappa^{\gamma}(x) d\gamma = \int_{-\infty}^{+\infty} \mathcal{M}(\gamma) R^{-i\gamma} \kappa^{0}(x) d\gamma.
\end{equation*}
It follows that
\begin{equation*}
M^{0}(-R^{-1}\Delta) = \int_{-\infty}^{+\infty} \mathcal{M}(\gamma) R^{-i\gamma} (-\Delta)^{i\gamma,0} d\gamma.
\end{equation*}
Hence,
\begin{equation}
\sup_{R \geq \|\rho\|^2} |M^{0}(-R^{-1}\Delta)f| \leq \int_{-\infty}^{+\infty} |\mathcal{M}(\gamma)||(-\Delta)^{i\gamma,0}f| d\gamma
\end{equation}

**Lemma 15.** The operator \((-\Delta)^{i\gamma,0}\) is bounded on \(L^{p}\), \(p \in (1, \infty)\), with
\begin{equation}
\|(-\Delta)^{i\gamma,0}\|_{L^{p}\to L^{p}} \leq c_{p}(1 + |\gamma|)^{\left\lfloor n/2 \right\rfloor + 1}.
\end{equation}
Moreover, the operator \((-\Delta)^{i\gamma,0}\) is also \(L^{1} \to L^{1,w}\) bounded, with
\begin{equation}
\|(-\Delta)^{i\gamma,0}\|_{L^{1}\to L^{1,w}} \leq c(1 + |\gamma|)^{\left\lfloor n/2 \right\rfloor + 1}.
\end{equation}
Proof. To prove the lemma, we shall proceed as in [2]. More precisely, by using a smooth, radial partition of unity (and thus invariant by the Weyl group), we decompose the multiplier $m^\gamma(\lambda) = (\|\lambda\|^2 + \|\rho\|^2)^i\gamma$ as follows

$$m^\gamma(\lambda) = \sum_{k=0}^{+\infty} m_k^\gamma(2^{-k}\lambda),$$

where $\text{supp } m_0^\gamma \subset \{\|\lambda\| \leq 2\}$ and $\text{supp } m_k^\gamma \subset \{1/2 \leq \|\lambda\| \leq 2\}$ for $k \geq 1$. Then, for every $p \in (1, +\infty)$, we have

$$\|(\Delta)^{i\gamma,0}\|_{p\rightarrow p} \leq c_p \sup_{k\geq 0} \|m_k^\gamma\|_{H^{\sigma/2}},$$

with $\sigma > n$ and $H_2^{\sigma/2}$ the usual Sobolev space, [2, Corollary 17, ii]. Note that the same upper bound also holds for the $L^1 \rightarrow L^{1,w}$ norm of $\|(\Delta)^{i\gamma,0}\|$, [2]. A straightforward computation yields

$$\|m_k^\gamma\|_{H^{\sigma/2}} \leq c(1 + |\gamma|)^{\sigma/2},$$

for $\sigma/2$ an integer, and Lemma 15 follows from (65). □

End of the proof of Proposition 14. We shall complete the proof for the $L^p$ boundedness of $S_0^*, p \in (1, \infty)$; the $L^1 \rightarrow L^{1,w}$ result is similar, thus omitted. Recall that (59) states that

$$S_0^* f \leq \sup_{R \geq \|\rho\|^2} |M^0(-R^{-1}\Delta)f| + \sup_{R \geq \|\rho\|^2} |f \ast p_{R-1}|.$$

Note that since $p_t(x) \geq 0$, for every $x \in X$, we have $p_0^t(x) \leq p_t(x)$. Thus,

$$|(f \ast p_0^t)(x)| \leq (|f| \ast p_t)(x).$$

Also, it is known (see for example [3, Corollary 3.2]) that the heat maximal operator $\sup_{t>0} |e^{t\Delta}f|$ is $L^p$-bounded and also $L^1 \rightarrow L^{1,w}$ bounded. This implies that the operator $\sup_{R \geq \|\rho\|^2} |f \ast p_{R-1}|$ is also $L^p$-bounded and $L^1 \rightarrow L^{1,w}$ bounded. Thus, from (59), it follows that to prove the $L^p$-boundedness of the operator $S_0^*$, it suffices to prove the $L^p$-boundedness of the operator $\sup_{R \geq \|\rho\|^2} |M^0(-R^{-1}\Delta)|$, and similarly for the $L^1 \rightarrow L^{1,w}$ boundedness.
From (62) and (66), we have that
\[
\| \sup_{R \geq \|\rho\|^2} |M^0(-R^{-1}\Delta)| \|_p \leq \int_{-\infty}^{+\infty} |\mathcal{M}(\gamma)||(-\Delta)^{i\gamma,0}|_{p\rightarrow p} \|f\|_p d\gamma
\]
\[
\leq c\|f\|_p \int_{-\infty}^{+\infty} (1 + |\gamma|)^{-(\text{Re}z+1)}(1 + |\gamma|)^{\lfloor n/2 \rfloor + 1} d\gamma
\]
\[
\leq c\|f\|_p \int_{-\infty}^{+\infty} (1 + |\gamma|)^{-(\text{Re}z-\lfloor n/2 \rfloor)} d\gamma \leq c\|f\|_p,
\]
whenever \( \text{Re}z \geq n - \frac{1}{2} \). This completes the proof of Proposition 14. □

Proof of Theorem 2. The proof of Theorem 2 follows from Stein’s complex interpolation, between the \( L^p \) result for \( p \) close to 1 and the \( L^2 \) result (Propositions 12, 13 and 14).

Proof of Theorem 3. As it is already mentioned in the Introduction, from Theorem 2 and Propositions 13 and 14, and well-known measure theoretic arguments (see for example [20, Theorem 2.1.14]), we deduce the almost everywhere convergence of Riesz means: if \( 1 \leq p \leq 2 \) and \( \text{Re}z > (n - \frac{1}{2}) \left( \frac{2}{p} - 1 \right) \), then
\[
\lim_{R \to +\infty} S^z_R(f)(x) = f(x), \text{ a.e., for } f \in L^p(X).
\]

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*Current address*: Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54.124, Greece