EQUIVARIANT COMPACTIFICATIONS OF VECTOR GROUPS WITH HIGH INDEX

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Abstract. In this note, we classify smooth equivariant compactifications of $G^n_a$ which are Fano manifolds with index $\geq n - 2$.

1. Introduction

Let $G^n_a$ be the complex vector group of dimension $n$, i.e., $\mathbb{C}^n$ equipped with the additive group structure. A smooth equivariant compactification of $G^n_a$ (SEC in abbreviation) is a $G^n_a$-action $A : G^n_a \times X \to X$ on a projective manifold $X$ of dimension $n$ with an open orbit $O \subset X$. Since $G^n_a$ contains no nontrivial finite subgroup, the open orbit $O$ is isomorphic to $\mathbb{C}^n$. When our interest is on the underlying manifold $X$, we say that $X$ is a SEC.

The study of SEC is started from [HT], where a classification in dimension 3 and of Picard number one is obtained. Recently a full classification of Fano 3-folds which are SEC is obtained in [HM], while it seems difficult to pursue further in higher dimension. In [FH3], the first author and J.-M. Hwang introduced the notion of Euler-symmetric projective varieties, namely nondegenerate projective varieties admitting many $\mathbb{C}^*$-actions of Euler type. It is shown in loc. cit. that they are equivariant compactifications of vector groups and they are classified by certain algebraic data (called symbol systems), while it remains the problem to translate the smoothness in terms of these algebraic data.

Recall that for a Fano manifold $X$ of dimension $n$, its index $i_X$ is the greatest integer such that $-K_X = i_X H$ for some divisor $H$ on $X$. It is well-known that $i_X \leq n + 1$. By a series of works of Fujita ([F1], [F2], [F3]), Mella ([Me]), Mukai ([M]) and Wiśniewski ([W1]), the classification of Fano $n$-folds with index $i_X \geq n - 2$ is known. Based on this, we will give a classification of SEC $n$-folds with index $\geq n - 2$. In the case of Picard number one, our result reads:

Theorem 1.1. Let $X$ be an $n$-dimensional SEC with Picard number one. Assume that $i_X \geq n - 2$, then $X$ is isomorphic to one of the following:

1. 6 homogeneous varieties: $\mathbb{P}^n, \mathbb{Q}^n, \text{Gr}(2, 5), \text{Gr}(2, 6), S_5, \text{Lag}(6)$

2. 5 non-homogeneous ones:
   - (2-a) Smooth linear sections of $\text{Gr}(2, 5)$ of codimension 1 or 2.
   - (2-b) $\mathbb{P}^4$-general linear sections of $S_5$ of codimension 1, 2 or 3.

\footnote{See [FH2](Definition 2.5).}
Here we use the notation of the homogeneous varieties in p.466 of [FH1]. In the last section, we will give the classification of SEC $n$-folds with index $\geq n - 2$ and higher Picard number (cf. Proposition 3.4). As a corollary of our result, we obtain the following

**Corollary 1.2.** Let $X$ be a SEC $n$-fold of Picard number one. Assume that the VMRT at a general point of $X$ is smooth (e.g. $X$ is covered by lines) and $n \leq 5$. Then $X$ is isomorphic to one of the following:

$\mathbb{P}^n, \mathbb{Q}^n$, smooth linear sections of codimension 1 or 2 of $\text{Gr}(2, 5)$.

It is expected that for a SEC of Picard number one, its VMRT at a general point is always smooth ([FH3]). By our results in the present paper, to classify Fano 4-folds which are SEC, the only remaining case is Fano 4-folds of index 1 and with Picard number at least 2.

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2. Picard number one case

Let $X$ be a uniruled projective manifold. An irreducible component $K$ of the space of rational curves on $X$ is called a minimal rational component if the subvariety $K_x$ of $K$ parameterizing curves passing through a general point $x \in X$ is non-empty and proper. Curves parameterized by $K$ will be called minimal rational curves. Let $\rho : U \rightarrow K$ be the universal family and $\mu : U \rightarrow X$ the evaluation map. The tangent map $\tau : U \rightarrow \mathbb{P}T(X)$ is defined by $\tau(u) = [T_{\mu(u)}(\mu(\rho^{-1}(\mu(u))))] \in \mathbb{P}T_{\mu(u)}(X)$. The closure $C \subset \mathbb{P}T(X)$ of its image is the VMRT structure on $X$. The natural projection $C \rightarrow X$ is a proper surjective morphism and a general fiber $C_x \subset \mathbb{P}T_x(X)$ is called the variety of minimal rational tangents (VMRT for short) at the point $x \in X$. It is well-known that $\dim C_x = -K_X \cdot \ell - 2$, where $\ell \in K$ is a general minimal rational curve through $x \in X$.

**Examples 2.1.** An irreducible Hermitian symmetric space of compact type is a homogeneous space $M = G/P$ with a simple Lie group $G$ and a maximal parabolic subgroup $P$ such that the isotropy representation of $P$ on $T_x(M)$ at a base point $x \in M$ is irreducible. The highest weight orbit of the isotropy action on $\mathbb{P}T_x(M)$ is exactly the VMRT at $x$. The following table (e.g. Section 3.1 [FH1]) collects basic information on these varieties. By [A], these are all SEC among rational homogeneous manifolds $G/P$ of Picard number one.
Examples 2.2. Let $\Sigma$ be an $n$-dimensional vector space endowed with a skew-symmetric 2-form $\omega$ of maximal rank. The symplectic Grassmannian $M = \text{Gr}_\omega(k, \Sigma)$ is the variety of all $k$-dimensional isotropic subspaces of $\Sigma$, which is not homogeneous if $n$ is odd. Let $W$ and $Q$ be vector spaces of dimensions $k \geq 2$ and $m$ respectively. Let $t$ be the tautological line bundle over $\mathbb{P}W$. The VMRT $C_x \subset \mathbb{P}^x(M)$ of $\text{Gr}_\omega(k, \Sigma)$ at a general point is isomorphic to the projective bundle $\mathbb{P}((Q \otimes t) \oplus t^{\otimes 2})$ over $\mathbb{P}W$ with the projective embedding given by the complete linear system

$$H^0(\mathbb{P}W, (Q \otimes t^*) \oplus (t^*)^{\otimes 2}) = (W \otimes Q)^* \oplus \text{Sym}^2W^*.$$ 

Alternatively, $C_x$ is isomorphic to the blowup of $\mathbb{P}^{m+k-1}$ along some linear space, hence it is a SEC.

Recall that a subvariety $X \subset \mathbb{P}V$ is called conic-connected if through two general points there passes an irreducible conic.

**Proposition 2.3.** Let $X \subset \mathbb{P}V$ be a conic-connected smooth subvariety. Then $X$ is a SEC if and only if $X \subset \mathbb{P}V$ is isomorphic to one of the following or their biregular projections:

1. The VMRT of an irreducible Hermitian symmetric space of compact type.
2. The VMRT of a symplectic Grassmannian $\text{Gr}_\omega(k, k+n+1)$ for $2 \leq k \leq n$.
3. A nonsingular linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ of codimension $\leq 2$.
4. A $\mathbb{P}^4$-general linear section of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ of codimension $\leq 3$.

**Proof.** Assume first that $X$ is not prime Fano, then by Theorem 2.2 [IR], $X \subset \mathbb{P}V$ is projectively equivalent to one of the following or their biregular projections:

1. The second Veronese embedding of $\mathbb{P}^n$.
2. The Segre embedding of $\mathbb{P}^a \times \mathbb{P}^n-a$ for $1 \leq a \leq n-1$.
3. The VMRT of the symplectic Grassmannian $Gr_\omega(k, k+n+1)$ for $2 \leq k \leq n$.
4. A hyperplane section of the Segre embedding $\mathbb{P}^a \times \mathbb{P}^{n+1-a}$ with $2 \leq a, n+1-a$.

The case (a4) is not a SEC by the proof of Proposition 6.3 [FH1].

Now assume that $X$ is a prime Fano manifold, then it is an Euler symmetric variety by Corollary 5.6 [FH3]. Let $r$ be the rank of $X$. By Theorem 3.7 [FH3], the $r$-th fundamental form at a general point $x \in X$ is non-zero. This implies that there exists a hyperplane $H$ such that $\text{mult}_x(H) = r$. Hence for any curve $C \not\subset H$ lying on $X$ through $x$, we have $H \cdot C \geq r$. By our assumption, $X$ is conic-connected, hence the conics through $x$ cover $X$. Let $C$ be a general such conic, then we get $2 = H \cdot C \geq r$. This implies that $r = 2$, hence $X \subset \mathbb{P}V$ is quadratically symmetric. Our claim follows then from Proposition 7.11 and Theorem 7.13 of [FH2].

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| Type | I.H.S.S. $M$ | VMRT $S$ | $S \subset \mathbb{P}T_x(M)$ |
|------|-------------|----------|-------------------------------|
| I    | $\text{Gr}(a, a+b)$ | $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$ | Segre |
| II   | $\mathbb{S}_n$ | $\text{Gr}(2, n)$ | Plücker |
| III  | $\text{Lag}(2n)$ | $\mathbb{P}^{n-1}$ | Veronese |
| IV   | $\mathbb{Q}^n$ | $\mathbb{Q}^{n-2}$ | Hyperquadric |
| V    | $\mathbb{Q}\mathbb{P}^2$ | $\mathbb{S}_5$ | Spinor |
| VI   | $\mathbb{E}_7/(\mathbb{E}_6 \times U(1))$ | $\mathbb{O}\mathbb{P}^2$ | Severi |
Recall that for a Fano manifold $X$ of dimension $n$, its index $i_X$ is the greatest integer such that $-K_X = i_X H$ for some divisor $H$ on $X$. By Kobayashi-Ochiai’s theorem, we have $i_X \geq n$ if and only if $X$ is either $\mathbb{P}^n$ or $\mathbb{Q}^n$. A Fano manifold $X$ is called del Pezzo (resp. Mukai) if $i_X = n - 1$ (resp. $i_X = n - 2$).

**Proposition 2.4.** Let $X$ be a SEC of $\rho_X = 1$. Assume that the VMRT at a general point of $X$ is smooth, then $i_X \geq 3$.

**Proof.** Let $D \subset X$ be the boundary divisor, which is irreducible since $\rho_X = 1$. By Theorem 2.5 [HT], we have Pic($X$) = $\mathbb{Z} D$, hence $-K_X = i_X D$. By Theorem 2.7 [HT], we have $i_X \geq 2$. Let $\ell$ be a minimal rational curve through a general point, then we have $\ell \cdot D = 1$ by [FH3] (Proposition 5.4 (v)). This implies that $i_X = -K_X \cdot \ell$. Assume that $i_X = 2$, then this implies that $-K_X \cdot \ell = 2$, hence there exists only finitely many minimal rational curves passing through a general point, namely the VMRT $C_x$ consists of points for $x \in X$ general. By Proposition 5.4 (ii) [FH3], $C_x$ is irreducible and non-degenerate, which is not possible. This gives that $i_X \geq 3$. □

**Remark 2.5.** It is expected that the assumption on the smoothness of the VMRT at a general point of $X$ in Corollary 1.2 and Proposition 2.4 is always fulfilled. See [FH3] (Conjecture 5.7).

**Proposition 2.6.** Let $X$ be an $n$-dimensional del Pezzo manifold of Picard number one. Then $X$ is a SEC if and only if $X$ is a smooth linear section of $\text{Gr}(2,5)$ of codimension $\leq 2$.

**Proof.** As $\rho_X = 1$, $X$ is isomorphic to one of the following by Fujita’s classification ([F1], [F2], [F3]),

1. cubic hypersurface in $\mathbb{P}^{n+1}$;
2. complete intersection of 2 quadrics in $\mathbb{P}^{n+2}$;
3. smooth linear sections of $\text{Gr}(2,5)$;
4. a hypersurface of degree 4 in $\mathbb{P}(2,1,\cdots,1)$;
5. a hypersurface of degree 6 in $\mathbb{P}(3,2,1,\cdots,1)$;

It’s well-known that for $X$ in (1) and (2), we have Aut$(X) = 0$, hence $X$ is not a SEC. For case (4), it is a double cover of $\mathbb{P}^n$ ramified along a smooth hypersurface of degree 4, hence its automorphism group is finite by [LP] (Theorem 4.5). For case (5), the blowup of $X$ along some point is an elliptic fibration over $\mathbb{P}^{n-1}$ by [F3], hence it is not rational. Consider case (3), then its smooth linear sections of codimension $\leq 2$ are SEC by Proposition 2.11 [FH2] since they are Euler symmetric. By Proposition 2.4 we have $i_X = n - 1 \geq 3$, hence $n \geq 4$, which shows that codimension 3 or 4 smooth linear sections of $\text{Gr}(2,5)$ are not SEC. □

**Lemma 2.7** ([F], Satz 8.11). Let $X$ be a smooth variety of dimension $n$, which is a complete intersection in a weighted projective space. Then $H^p(X, \Omega^q_X(t)) = 0$ if $p + q < n$ and $t < q - p$.

**Corollary 2.8.** Let $X$ be a Mukai manifold of $\rho_X = 1$. Assume that $X$ is a complete intersection in a weighted projective space, then Lie(Aut($X$)) = aut($X$) = 0.
Proof. By assumption, we have $-K_X = O_X(n-2)$. As $T_X \simeq \Omega_X^{n-1} \otimes K_X = \Omega_X^{n-1}(n-2)$, we have $H^0(X, T_X) = 0$ by Lemma 2.7.

The following lemma is well-known for general sections, but we need it for any (smooth) section.

**Lemma 2.9.** Any linear section of codimension $\leq 5$ (resp. $\leq 3$) of $S_5 \subset \mathbb{P}^{15}$ (resp. $\text{Gr}(2,6) \subset \mathbb{P}^{14}$) is conic-connected.

**Proof.** By Proposition 2.19 (Chap. III, [Z]), any two points of $S_5 \subset \mathbb{P}^{15}$ can be joined by a smooth quadric of dimension 6 on it, which implies the claim. For $X := \text{Gr}(2,6) \subset \mathbb{P}^{14}$, it is a Severi variety. By Theorem 2.4 b) (Chap. IV [Z]), for any two points $x, y \in X$ such that the line $\overline{xy}$ is not on $X$, then $x, y$ are joined by a smooth quadric of dimension 4. Take any linear section $X'$ of $X$ of codimension $\leq 3$. For two general points $x', y' \in X'$, if the line $\overline{x'y'}$ is on $X$, then it is also contained in $X'$ as $X'$ is a linear section of $X$. This implies that $X'$ is a projective space of dimension at least 5, which is not possible since $X$ does not contain any $\mathbb{P}^5$. Hence $x', y'$ are joined by a $\mathbb{Q}^4$ on $X$, hence they are joined by a conic on $X'$.

**Proposition 2.10.** Let $X$ be an $n$-dimensional Mukai variety with $\rho_X = 1$. Then $X$ is a SEC if and only if $X$ is one of the following

(i) a $\mathbb{P}^4$-general linear section of the 10-dimensional spinor variety $S_5$ of codimension $\leq 3$.

(ii) the 8-dimensional Grassmanian $\text{Gr}(2,6)$.

(iii) the 6-dimensional Lagrangian Grassmanian $\text{Lag}(6)$.

**Proof.** When $n = 3$, then $X$ is isomorphic to $\mathbb{P}^3$ or $\mathbb{Q}^3$ by [HT]. Now assume $n \geq 4$. By Mukai’s classification [M], $X$ is either a complete intersection in a weighted projective space or a smooth linear section of one of the following varieties:

(a) a quadric section of the cone over $\text{Gr}(2,5) \subset \mathbb{P}^9$;

(b) the 10-dimensional spinor variety $S_5 \subset \mathbb{P}^{15}$;

(c) the 8-dimensional Grassmanian $\text{Gr}(2,6) \subset \mathbb{P}^{14}$;

(d) the 6-dimensional Lagrangian Grassmanian $\text{Lag}(6) \subset \mathbb{P}^{13}$;

(e) the 5-dimensional Fano contact manifold $G_2/\mathbb{P}_2 \subset \mathbb{P}^{13}$.

By Corollary 2.8, we only need to consider cases (a)-(e). For case (a), their smooth linear sections are called Gushel-Mukai varieties. By Proposition 3.19 (c) [DK], they have finite automorphism groups, hence they are not SEC.

For the remaining cases, $X$ is covered by lines, so it has smooth VMRT at general points. By Proposition 2.4, we have $i_X = n - 2 \geq 3$, hence $n \geq 5$. For cases (b) and (c), they are conic-connected by Lemma 2.9 hence by Proposition 2.3 $X$ is as in (i) and (ii).

For case (d), a smooth hyperplane section $X$ of $\text{Lag}(6)$ is a compactification of a symmetric variety with $\text{Aut}^0 = \text{SL}_3$ by [R] (Theorem 3). As $\text{SL}_3$ does not contain any subgroup isomorphic to $\mathbb{G}_a^5$, the variety $X$ is not a SEC. Hence only $\text{Lag}(6)$ itself is a SEC.

For case (e), its VMRT at a general point is degenerate, hence it cannot be a SEC by [FH3] (Proposition 5.4 (ii)).
This concludes the proof of Theorem 1.1. This implies Corollary 1.2 by Proposition 2.4.

3. Higher Picard number case

**Proposition 3.1.** Let $X$ be a Fano manifold of dimension $n$ with index $i_X \geq (n+1)/2$. If $\rho_X \geq 2$, then $X$ is a SEC if and only if it is one of the following:

$$\mathbb{P}^2 \times \mathbb{P}^2, \ \mathbb{P}^{\frac{n-1}{2}} \times \mathbb{Q}^{\frac{n+1}{2}}, \ \mathbb{P}_{\mathbb{F}_{n+1}}(\mathcal{O}(1) \oplus \mathcal{O}^{\frac{n+1}{2}}).$$

**Proof.** By [W2], a Fano manifold with $i_X \geq (n+1)/2$ and $\rho_X \geq 2$ is one of the varieties in the list or the homogeneous variety $\mathbb{P}^T_{\mathbb{P}^{n+1}}$, while the latter is not a SEC by [A]. The projective bundle $\mathbb{P}_{\mathbb{F}_{n+1}}(\mathcal{O}(1) \oplus \mathcal{O}^{\frac{n+1}{2}})$ is isomorphic to the blowup of $\mathbb{P}^n$ along a linear $\mathbb{P}^{\frac{n-3}{2}}$, which is a SEC. □

As an immediate corollary, we have

**Corollary 3.2.** Let $X$ be a del Pezzo manifold with $\rho_X \geq 2$, then $X$ is a SEC if and only if $X$ is one of the following: (a) Blowup of $\mathbb{P}^2$ at 1 or 2 points, (b) $\mathbb{P}^2 \times \mathbb{P}^2$, (c) blowup of $\mathbb{P}^3$ at 1 point, (d) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

**Corollary 3.3.** Let $X$ be a Mukai manifold with $\rho_X \geq 2$. Assume that $\dim X \geq 5$, then $X$ is a SEC if and only if $X$ is one of the following: (a) $\mathbb{P}^3 \times \mathbb{P}^3$; (b) $\mathbb{P}^2 \times \mathbb{Q}^3$, (c) blowup of $\mathbb{P}^4$ along 1 point.

Note that Fano SEC in dimension 3 are fully classified in [HM], while Mukai manifolds of dimension 4 are classified by Wiśniewski [W1]. To complete the picture, it remains to determine which are SEC in the list of [W1].

**Proposition 3.4.** Let $X$ be a 4-dimensional Mukai manifold with $\rho_X \geq 2$. Then $X$ is a SEC if and only if $X$ is one of the following: (a) $\mathbb{P}^1 \times \mathbb{P}^3$; (b) $\mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O}(1) \oplus \mathcal{O})$; (c) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; (d) the blowup of $\mathbb{Q}^4$ along a line; (e) $\mathbb{P}_{\mathbb{Q}^2}(\mathcal{O}(-1) \oplus \mathcal{O})$; (f) $\mathbb{P}_{\mathbb{Q}^3}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$.

**Proof.** By Wiśniewski’s classification [W1] (see [IP, Table 12.7]), $X$ is isomorphic to one of the following varieties,

1. $\mathbb{P}^1 \times V$, where $V \cong \mathbb{P}T_{\mathbb{P}^3}$ or $V \cong V_d$ is a del Pezzo threefold of degree $1 \leq d \leq 5$ and $\rho_{V_d} = 1$,
2. $\mathbb{P}^1 \times V$, where $V$ is isomorphic to $\mathbb{P}^3$, $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O})$ (the blowup of $\mathbb{P}^3$ along 1 point), or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,
3. a Verra fourfold, that is, a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2,2)$,
4. a divisor on $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(1,2)$,
5. an intersection of two divisors of bidegree $(1,1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$,
6. a divisor on $\mathbb{P}^2 \times \mathbb{Q}^3$ of bidegree $(1,1)$,
7. the blowup of $\mathbb{Q}^4$ along a conic which is not contained in a plane lying on $\mathbb{Q}^4$,
8. $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N})$, where $\mathcal{N}$ is the null-correlation bundle on $\mathbb{P}^3$,
9. the blowup of $\mathbb{Q}^4$ along a line,
In case (1), it follows from Blanchard’s lemma \([B]\) (Theorem 7.2.1) that an effective action of \(G^4_a\) on \(X \cong \mathbb{P}^1 \times V\) descends in a unique way to an action of \(G^4_a\) on \(V\) making the second projection an equivariant morphism. The coimage of the latter action is isomorphic to \(G^3_a\) making \(V\) a SEC, which contradicts Corollary \([L2]\). The variety \(X\) is not a SEC. In case (2), the listed varieties are clearly SEC.

In case (3), an effective action of \(G^4_a\) on \(X\) induces an inclusion \(G^4 \subseteq \text{Aut}_L(B) \subseteq \text{Aut}(\mathbb{P}^2 \times \mathbb{P}^2)\), where \(\text{Aut}_L(B)\) stands for the group of automorphisms of the branch locus \(B\) induced by automorphisms of \(\mathbb{P}^2 \times \mathbb{P}^2\). Indeed, this follows verbatim from the proof of \([LP]\) (Lemma 4.1, Lemma 4.2, Proposition 4.3) replacing \(\mathbb{P}^n\) by \(\mathbb{P}^n \times \mathbb{P}^n\). In particular, we obtain in this way an effective action of \(G^4_a\) on \(\mathbb{P}^2 \times \mathbb{P}^2\) that fixes the branch locus \(B\). We know on the other hand, after Hassett and Tschinkel \([HT]\) (Proposition 3.2), that the boundary divisors for the possible actions of \(G^4_a\) on \(\mathbb{P}^2 \times \mathbb{P}^2\) are of bidegree \((1, 0)\) and \((0, 1)\), hence \(X\) cannot be a SEC.

In cases (4), (5) and (6), the variety \(X\) is isomorphic to the projectivization \(\mathbb{P}_Y(B)\) of a reflexive non-locally free sheaf \(B\) on a smooth variety \(Y\). Such sheaves are called Bânică sheaves in \([BW]\)(Section 2), where it is shown that the canonical map \(X \to Y\) is a Mori contraction with connected but not equidimensional fibers. The general strategy to prove that none of these cases give rise to a SEC will be to analyze the points where the dimension of the fibers of \(X \to Y\) jumps.

It follows from \([BW]\)(Theorem 6.8) that for \(X\) as in case (4) we have that \(Y \cong \mathbb{P}^3\) and the canonical fibration \(X \to \mathbb{P}^3\) has 8 fibers isomorphic to \(\mathbb{P}^2\). By Blanchard’s lemma, an effective action of \(G^4_a\) on \(X\) induces a unique action on \(\mathbb{P}^3\) for which \(X \to \mathbb{P}^3\) is equivariant. As before, there is an induced effective action of \(G^3_a\) making \(\mathbb{P}^3\) a SEC. On one hand, we notice that the 8 points \(p_1, \ldots, p_8 \in \mathbb{P}^3\) having 2-dimensional fibers are invariant and hence contained in the boundary hyperplane divisor \(H \cong \mathbb{P}^2 \subset \mathbb{P}^3\). On the other hand, if we write

\[
X = \left\{ \sum_{i=0}^{2} x_iq_i(y_0, y_1, y_2, y_3) = 0 \right\} \subset \mathbb{P}^2_x \times \mathbb{P}^3_y,
\]

where the \(q_i\) are quadratic forms, we have that \(X \to \mathbb{P}^3\) is induced by the second projection and hence it has 2-dimensional fibers over the set \(S = \{ q_0(y) = q_1(y) = q_2(y) = 0 \} = \{ p_1, \ldots, p_8 \}\). We claim that \(S\) is not contained in a hyperplane \(H\) and hence \(X\) is not a SEC. Indeed, if we assume that \(S \subset H\) and we denote by \(Q_i\) the hyperquadric \(\{ q_i(y) = 0 \}\), then \(L_i = Q_i \cap H\) is a (possibly reducible) curve of degree 2 in \(H\) and \(L_{ij} = Q_i \cap Q_j\) is a curve of degree 4 in \(\mathbb{P}^3\) for \(i \neq j\). Since \(S \subset L_{ij} \cap H\), and \(S\) is a 0-dimensional scheme of length 8, it follows that \(L_{ij}\) has a common component, say \(N_{ij}\), contained in \(H\) and thus \(L_i\) and \(L_j\) have a common component for \(i \neq j\). Since each of the \(L_i\) is a curve of degree 2, we have that if \(L_0 \cap L_1 \cap L_2\) is 0-dimensional then each \(L_i\) is reducible and given by the union of two lines in \(H\), from which we can easily verify that \(L_0 \cap L_1 \cap L_2\) is of length 3 and contains \(S\), which is absurd. We conclude therefore that \(Q_0 \cap Q_1 \cap Q_3\) have a common component, but this is not possible as \(X\) is irreducible. The cases (5) and (6) are similar but easier: in the former case we

\[(10) \mathbb{P}^3(\mathcal{O}(-1) \oplus \mathcal{O});\]

\[(11) \mathbb{P}^3(\mathcal{O}(-1) \oplus \mathcal{O}(1)).\]
have that $Y \cong \mathbb{P}^3$ and the canonical fibration $X \to \mathbb{P}^3$ has 4 fibers isomorphic to $\mathbb{P}^2$ by [BW] (Theorem 6.8). More precisely, for each $p_1, \ldots, p_4 \in \mathbb{P}^3$ having 2-dimensional fiber, the fiber is given by the dual hyperplane $H_{p_i}$ in $(\mathbb{P}^3)^\vee$ determined by $p_i$. Since each of the points $p_i$ is invariant we deduce that each $H_{p_i}$ is invariant by the induced action of $G_a^3$ in $(\mathbb{P}^3)^\vee$, from which we would get four different invariant divisors, a contradiction. In the latter case we have that $Y \cong \mathbb{Q}^3$ and the canonical fibration $X \to \mathbb{Q}^3$ has 2 fibers isomorphic to $\mathbb{P}^2$, and hence the result follows from the fact [HT] (Theorem 6.1) that there is a unique effective action of $G_a^3$ making $\mathbb{Q}^3$ a SEC, for which there is only one invariant point.

In cases (7) and (9) the variety $X$ is isomorphic to the blowup of $\mathbb{Q}^4$ along a smooth curve $C \subset \mathbb{Q}^4$. By Blanchard’s lemma, $X$ is a SEC if and only if $C$ is invariant by the unique effective action of $G_a^4$ on $\mathbb{Q}^4$. A simple computation in coordinates shows that the only invariant smooth curves on $\mathbb{Q}^4$ are lines. This shows that in case (7) $X$ is not a SEC, while $X$ as in case (9) is a SEC.

In case (8), it follows from [CP] (Theorem 3.1) that $X \cong \mathbb{P}_3(\mathcal{N})$ is isomorphic an homogeneous space $G/P$ and hence it follows from [A] that $X$ is not a SEC.

In cases (10) and (11) there is a blowdown $X \to Z$ sending the divisor corresponding to a section of the $\mathbb{P}^1$-bundle structure of $X$ to a point $z \in Z$, where $Z \cong \mathbb{Q}_a^3$ is the cone over $\mathbb{Q}_a^3$ in $\mathbb{P}^5$ and $Z \cong \mathbb{P}(1,1,1,1,2)$, respectively, and $z \in Z$ is the only singular point of each of these varieties. In both cases $Z$ is a SEC (cf. [AP] (Section 6), [AR] (Proposition 2)) and $z \in Z$ is an invariant point for the respective actions since is the only singular point. We conclude therefore that in both cases (10) and (11) $X$ is a SEC.

\[\square\]

References

[A] Arzhantsev, Ivan V.: Flag varieties as equivariant compactifications of $G_a^n$. Proc. Amer. Math. Soc. 139 (2011), no. 3, 783-786

[AP] Arzhantsev, Ivan; Popovskiy, Andrey: Additive actions on projective hypersurfaces. Automorphisms in birational and affine geometry, Springer Proc. Math. Stat. 79 (2014), 17–33

[AR] Arzhantsev, Ivan; Romaskevich, Elena: Additive actions on toric varieties. Proc. Amer. Math. Soc. 145 (2017), no. 5, 1865–1879

[B] Brion, M.: Some structure theorems for algebraic groups. Proc. Sympos. Pure Math. 94 (2017), 53–126

[BW] Ballico, Edoardo; Wiśniewski, Jarosław A.: On Bănică sheaves and Fano manifolds. Compositio Math. 102 (1996), no. 3, 313–335

[CP] Campana, Frédéric; Peternell, Thomas: 4-folds with numerically effective tangent bundles and second Betti numbers greater than one. Manuscripta Math. 79 (1993), no. 3-4, 225–238

[DK] Debarre, Olivier; Kuznetsov, Alexander: Gushel-Mukai varieties: classification and b irrationalities. Algebr. Geom. 5 (2018), no. 1, 15-76

[F] Flenner, Hubert: Divisorenklassengruppen quasihomogener Singularitäten. J. Reine Angew. Math. 328 (1981), 128160.

[FH1] Fu, B. and Hwang, J.-M.: Classification of non-degenerate projective varieties with non-zero prolongation and application to target rigidity, Invent. math. 189 (2012) 457-513

[FH2] Fu, B. and Hwang, J.-M.: Special birational transformations of type $(2,1)$. J. Algebraic Geom. 27 (2018), no. 1, 55–89

[FH3] Fu, B. and Hwang, J.-M.: Euler-symmetric projective varieties, arXiv:1707.06764
[F1] Fujita, Takao: On the structure of polarized manifolds with total deficiency one. I. J. Math. Soc. Japan 32 (1980), no. 4, 709725.

[F2] Fujita, Takao: On the structure of polarized manifolds with total deficiency one. II. J. Math. Soc. Japan 33 (1981), no. 3, 415-434

[F3] Fujita, Takao: On the structure of polarized manifolds with total deficiency one. III. J. Math. Soc. Japan 36 (1984), no. 1, 75-89

[HM98] Hwang, J.-M. and Mok, N.: Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kahler deformation. Invent. Math. 131 (1998) 393–418

[HM] Huang, Z. and Montero, P.: Fano threefolds as equivariant compactifications of the vector group, arXiv:1802.08090

[HT] Hassett, B. and Tschinkel, Y.: Geometry of equivariant compactifications of $G^a_n$. Internat. Math. Res. Notices 22(1999), 1211–1230

[IR] Ionescu, P. and Russo, F.: Conic-connected manifolds, J. reine angew. Math. 644 (2010), 145–157

[IP] Iskovskikh, V. A.; Prokhorov, Yu. G.: Fano varieties. Algebraic geometry, V, 1247, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999

[LP] Lyu R., Pan, X. : Remarks on Automorphism and Cohomology of Cyclic Coverings, arXiv:1705.06618

[Me] Mella, Massimiliano: Existence of good divisors on Mukai varieties. J. Algebraic Geom. 8 (1999), no. 2, 197–206

[M] Mukai, Shigeru: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. Proc. Nat. Acad. Sci. U.S.A. 86 (1989), no. 9, 3000-3002

[R] Ruzzi, Alessandro: Geometrical description of smooth projective symmetric varieties with Picard number one. Transform. Groups 15 (2010), no. 1, 201-226

[W1] Wiśniewski, Jarosław A.: Fano 4-folds of index 2 with $b_2 \geq 2$. A contribution to Mukai classification. Bull. Polish Acad. Sci. Math. 38 (1990), no. 1-2, 173184

[W2] Wiśniewski, Jarosław A.: On Fano manifolds of large index. Manuscripta Math. 70 (1991), no. 2, 145-152

[Z] Zak, F.L.: *Tangents and secants of algebraic varieties*. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993

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