Bifurcation analysis in SIR epidemic model with treatment

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Abstract. We investigated the bifurcation analysis of nonlinear system of SIR epidemic model with treatment. It is accepted that the treatment is corresponding to the quantity of infective which is below the limit and steady when the quantity of infective achieves the limit. We analyze about the Transcritical bifurcation which occurs at the disease free equilibrium point and Hopf bifurcation which occurs at endemic equilibrium point. Using MATLAB we show the picture of bifurcation at the disease free equilibrium point.

1. Introduction.
Mathematical models in modern times play an answer role between policy making, together with health-economic aspects, toughness assessment, control-program evaluation, and optimizing a range of detection. A normally epidemic model is susceptible, infectious and recovered model (SIR). [1, 2] Treatment plays an necessary position in controlling or decreasing the extent of ailments certain as much measles, tuberculosis. In this paper, the treatment dosage is insincere in imitation of keep proportional in imitation of the wide variety regarding infective when the capability on cure is no longer reached and otherwise, takes the maximal ability. Consider that treatment rate of a disease. $T(I) = \begin{cases} I & \text{if } 0 \leq I \leq I_0 \\ K & \text{if } I > I_0 \end{cases}$ where $K$ is a positive constant and $I$ is the number of infected individuals.[5,7] Parameters concerning the regulation construct the biological concerns for a given model. If one of the Eigen value has zero real-part is called the bifurcation point. Eigen values that decide stability of an equilibrium point [9] are functions of these parameters. As opposed to utilizing eigenvalues, Liu created a criterion to discovering basic Hopf bifurcation In light of just those Hurwitz determinants computed from that polynomial. Furthermore, Yu (2005) enlarged the utilization of Hurwitz determinants to Figure k-Hopf bifurcations, the crossing for different pairs of eigenvalues at the same period. Those advantage to these two systems is that one doesn't
necessity should understand those nonlinear characteristic equation with find the bifurcations. [3, 4, 6, 8] These techniques best describe discovering bifurcating results. We requirement on apply more propelled techniques, for example, extension and the center manifold theorem so as should know those stability of these bifurcation results. Bifurcations need aid as a rule dictated Eventually by the Eigen values of the Jacobian matrix, yet all the they are often challenging to figure out high dimensional systems. Though the researchers studied the asymptotic behavior of epidemic models, in recent papers bifurcation behavior has been seen more.

2. Existence of equilibrium point.
Consider an epidemic population with a limited reserve for treatment. The population is divided into classes of susceptible \( S(t) \), infected \( I(t) \) and the recovered \( R(t) \). The following system of nonlinear ordinary equations represented the dynamics of SIR epidemic model with a limited reserve for treatment

\[
\begin{align*}
\dot{S} &= \pi - \beta SI - \mu S \\
\dot{I} &= \beta SI - (\mu + \gamma + \mu_t + r)I \\
\dot{R} &= (\gamma + r)I - \mu R \\
\end{align*}
\]

(1)

However susceptible population recruitment rate is \( \pi, \beta > 0 \) as an effective infection rate, the removal rate \( \gamma \) is the number of infectious individuals transfer to recovered section (iv) \( \mu \) is taken as natural death rate and \( \mu_t \) as disease related death rate (v) \( r \) is consider as treatment rate.

Now, all the solutions of above system are uniformly bounded. The system has two possible equilibrium points, disease free equilibrium point \( E_1 = (S_1, 0, 0) \) and \( E_2 = (S_2, I_2, R_2) \) called endemic equilibrium point and denoted by where

\[
\begin{align*}
S_1 &= \frac{\pi}{\mu} \\
S_2 &= \frac{\pi}{\beta I_2 + \mu} \\
I_2 &= \frac{\pi}{(\mu + \mu_t + \gamma + r)} \frac{\mu}{\beta} \\
R_2 &= \frac{(\gamma + r)\pi}{(\mu + \mu_t + \gamma + r)} \frac{(\gamma + r)}{\beta} \\
\end{align*}
\]

(2)

3. Transcritical Bifurcation Analysis.
We analyze the dynamical behavior of the above system by varying the parameter around each equilibrium point. Applications of the Sotomayor’s theorem for local bifurcation are modified. According to this theorem, the Jacobian matrix of the above system at disease free equilibrium point \( E_1 \) has transcritical bifurcation.

It is easily verify that the Jacobian matrix of system at \( (E_1, \beta^*) \) can be calculated as \( J = Df(E_1, \beta^*) \)
\[ J = \begin{pmatrix} -\mu & -\beta^* S_1 & 0 \\ \beta I & 0 & 0 \\ 0 & \gamma + r & -\mu \end{pmatrix} \]  

(3)

where \( \beta^* = \frac{(\mu + \gamma + r + \mu_s)}{S_i} \)

From the above Jacobian matrix (3), the second \( \lambda_s \) Eigen value \( \lambda_i \) in the direction of \( I \) is zero while \( \lambda_s \) and \( \lambda_r \) are negative.

Further the Eigen vector \( Z = (z_1, z_2, z_3)^T \) corresponding to \( \lambda_i \) satisfy the condition \( JZ = \lambda_i Z \) then \( JZ = 0 \)

\[ \begin{pmatrix} -\mu & -\beta^* S_1 & 0 \\ \beta I & 0 & 0 \\ 0 & \gamma + r & -\mu \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]  

(4)

From which we get

\[-\mu z_1 - \beta^* S_1 z_2 = 0 \]

\[(\gamma + r) z_2 - \mu z_3 = 0 \]  

(5)

After solving the above system (5) of equations we get

\[ z_1 = q z_2, \quad z_3 = w z_2 \]

Where

\[ q = \frac{-\beta^* S_1}{\mu}, \quad w = \frac{(\gamma + r)}{\mu} \]

Thus \[ Z = \begin{pmatrix} q z_2 \\ z_2 \\ w z_2 \end{pmatrix} \]. Similarly the Eigen vector \( w = (w_1, w_2, w_3)^T \) related to \( \lambda_i \) of \( J^T \) can be written as

\[ \begin{pmatrix} -\mu & 0 & 0 \\ -\beta^* S_1 & \gamma + r & 0 \\ 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0. \]

From this we have

\[-\mu w_1 = 0 \]

\[-\beta^* S_1 w_1 + (\gamma + r) w_3 = 0 \]

\[-\mu w_3 = 0 \]
Then by solving the above equations we get the solution 
\[ w = \begin{pmatrix} 0 \\ w_2 \\ 0 \end{pmatrix} \]

Now the system (1) can be written as in a vector form
\[ \frac{dx}{dt} = f(x) \]

Here \( X = (S, I, R)^T \) and \( f \left( f_1, f_2, f_3 \right)^T \) with \( f_i = i = 1, 2, 3 \) then determine \( \frac{df}{d\beta} = f_\beta \)

We get that
\[ f_\beta = \begin{pmatrix} -SI \\ SI \\ 0 \end{pmatrix} \]

Then
\[ f_\beta(E_1, \beta^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ w^T f_\beta(E_1, \beta^*) = 0 \]

\[ Df_\beta(E_1, \beta^*) = \begin{pmatrix} 0 & -S_i & 0 \\ 0 & S_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ w^T \left[ Df_\beta(E_1, \beta^*) . Z \right] = w_2 z_2 S_i \neq 0 \]

According to Sotomayor theorem when the parameter \( \beta \) passing through the bifurcation value \( \beta^* \) then the transcritical bifurcation occurs at disease free equilibrium.

### 4. Hopf Bifurcation Analysis.

Hopf –bifurcation around the endemic equilibrium point \( E_2 \) is satisfied by the system at \( \beta = \frac{3\mu + \alpha + \gamma + r}{S_2 + I_2} \)

The necessary and sufficient condition of the system has a Hopf bifurcation at \( E_2 \) given by
\[ T(\beta) = 0 \]
\[ \frac{dT}{d\beta} \neq 0 \text{ at } \beta = \bar{\beta} \]

Now the jacobian matrix of the system at endemic equilibrium point can be calculated as
\[ J(E_2) = \begin{pmatrix} -(\beta I_2 + \mu) & -\beta S_2 & 0 \\ \beta I_2 & \beta S_2 - (\mu + \gamma + r) & 0 \\ 0 & \gamma + r & -\mu \end{pmatrix} \]

Then the characteristic equation can be written as 
\[ \lambda^3 + S_1 \lambda^2 + S_2 \lambda + S_3 = 0 \]
By solving the above equation the system has one negative eigen value and the remaining two eigen values are complex conjugate. Clearly, the condition for the Hopf-bifurcation is satisfied at $\beta = \beta$. Now we check the second condition in the following

$$\frac{dT}{d\beta} = S_2 \neq 0$$

Hence, the system at the parameter $\beta = \beta$ has a Hopf-bifurcation around the endemic equilibrium point $E_2$.

5. Numerical Simulation

In this numerical simulation, we confirm our analytical results and show the effects bifurcation value on the dynamic of SIR model. We solved numerically the above system by assuming the initial condition as:

$S = 10, I = 4, R = 4, \pi = 15, \beta = 0.001, \gamma = 0.1, \mu = 1, r = 1, \mu = 0.5$

Clearly, Figure (1) explain near the disease free equilibrium point $E$, the transcritical bifurcation of the system (1) occurs. When the infection rate increases the system (1) becomes unstable at the disease free equilibrium point and the trajectory of system (1) approaches asymptotically to the endemic equilibrium point.

![Figure 1](image)

**Figure 1** Transcritical bifurcation occurs at the disease free equilibrium point at $\beta = 0.001$
$S = 10, I = 4, R = 4, \pi = 15, \beta = 0.1, \gamma = 0.1, \mu_i = 1, r = 1, \mu = 0.5$

Figure 2: Transcritical bifurcation occurs at the disease free equilibrium point with $\beta = 0.1$

Figure 1 and 2 shows that transcritical bifurcation occurs at the disease free equilibrium point by varying the infection rate. By varying the infection rate increases from $\beta = 0.001$ to $\beta = 0.1$ the trajectory of the system approaches asymptotically to the endemic equilibrium point.

6. Conclusions and Discussion
When increases in the infection rate the asymptotic behavior of the system near the disease free equilibrium point approaching to the endemic equilibrium point and the system has a transcritical bifurcation. The system loses its stability at the time of the treatment rate increases and the system has Hopf bifurcation occurs at the endemic equilibrium point. Using the different types of Bifurcation at the equilibrium point we analyze the dynamical behavior of the system.

References
[1]. Fayeldi, Trija, Agus Suryanto, and Agus Widodo 2013 Dynamical behaviors of a discrete SIR epidemic model with nonmonotone incidence rate Int. Journal App.Math.Stat 47 pp 416- 423
[2]. Carlo B, Massimilano F, Luca G 2013 Hopf Bifurcation in a Delayed-Energy-Based Model of Capital Accumulation Journal of Applied Mathematics Inf. Sci. Lett. 7 pp 1-5
[3]. Wang.W.D and Ruan.S.G 2004 Bifurcation in an epidemic model with constant removal rate of the infectives, J. Math. Anal.Appl. 291 pp 775-793
[4]. Zhang.T.L, Liu.J.L and Teng.Z.D 2010 Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure Nonlinear Anal. 11 pp 293-306
[5]. Al-Sheikh and Sarah A 2012 Modeling and analysis of an SEIR epidemic model with a limited resource for treatment Global Journal of Science Frontier Research Mathematics and Decision Sciences 12 pp 56-66
[6]. Feng and Liping 2012 Hopf bifurcation analysis of a delayed viral infection model in computer networks Mathematical and Computer Modelling 56 pp 167-179
[7]. Zaman, Gul, Yong Han Kang and Il Hyo Jung 2009 Optimal treatment of an SIR epidemic model with time delay BioSystems 98 pp 43-50
[8]. Wang, Wendi, and Shigui Ruan 2004 Bifurcations in an epidemic model with constant removal rate of the infectives Journal of Mathematical Analysis and Applications 291 pp 775-793.
[9]. Balamuralitharan S and Radha M, 2017 Stability analysis of cholera-carrier dependent infectious Disease International journal of pure and applied mathematics 113 pp 234-242