Three remarks on a question of Aczél

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Abstract. At ISFE 2009 J. Aczel asked for the monotonic solutions of a certain functional equation. We show that when a parameter appearing in the equation takes a certain value there is a unique monotonic solution, and when it takes other values there are infinitely many monotonic solutions. Further, we comment on continuous and continuously differentiable solutions.

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1. Introduction

Consider the functional equation

\[ f(x^2 R) = \frac{k}{2xR} f(x), \quad x > \frac{1}{R}, \]  

where \( R, k \) are positive constants. In a personal communication, we learned from Ali Abbas that he gives this equation the interpretation of describing the probability density function for power consumption in a wire.

At the 47th International Symposium on Functional Equations 2009, Janos Aczél [1, p. 195] presented the general solution of this equation as

\[ f(x) = \frac{(\ln(xR))^c}{xR} p(\log_2(\ln(xR))), \]

where \( c = \log_2(k/2) \) and \( p \) is an arbitrary periodic function of period 1 on \( \mathbb{R} \) (for proof define \( p(s) = (\frac{k}{2})^s \exp(2^s f(\frac{\exp(s^2)}{R})) \)), and then asked for the monotonic solutions.

A special solution for the functional equation (1) is

\[ \varphi_c(x) = \frac{(\ln(xR))^c}{xR}, \]

obtained by taking \( p \equiv 1 \) in (2). This function is monotonic precisely when \( c \leq 0 \), that is, when \( k \leq 2 \). Are all monotonic solutions of (1) scalar multiples
of $\varphi_c$? We show that when $k = 2$ the answer is yes, and that it is no when $k < 2$. We then consider the effects of imposing continuity and differentiability conditions at $1/R$. Here, too, the results depend on the value of $k$.

2. Monotonicity

First, assume that $k = 2$.

**Theorem 1.** Let $p$ be periodic of period 1. The function

$$f(x) = \frac{1}{xR} \cdot p(\log_2(\ln(xR)))$$

is monotonic on $(1/R, \infty)$ if, and only if, $p$ is constant.

**Proof.** If $p$ is not constant, then it takes two distinct values $M > m$. For any $x_1, x_2$ sufficiently close to $1/R$,

$$M \frac{1}{x_1 R} > m \frac{1}{x_2 R}.$$  

Let $x_1$ be sufficiently close to $1/R$ which satisfies $p(\log_2(\ln(x_1 R))) = M$, and let $x_2 > 1/R$ be smaller than $x_1$ such that $p(\log_2(\ln(x_2 R))) = m$. Then $x_2 < x_1$, but $f(x_1) > f(x_2)$, so $f$ is not monotonic decreasing. In other words, if $f$ is monotonic decreasing, then $p$ is constant.

If $f$ is an increasing function then $-f$ is a decreasing function of the same form, so $p$ must be constant.  

**Corollary 2.** When $k = 2$, the only monotonic solutions to (1) are $f(x) = \frac{\lambda}{xR} = \lambda \varphi_0(x)$, $\lambda \in \mathbb{R}$.

Now we consider the case $k < 2$.

**Theorem 3.** There are infinitely many differentiable functions $p$ such that (2) is a monotonic solution of (1) with $k < 2$.

**Proof.** Indeed, differentiating (2) we find

$$f'(x) = \frac{\ln(xR)^{c-1}}{x^2 R} \left( (c - \ln(xR)) \cdot p(\log_2(\ln(xR))) + \log_2 e \cdot p'(\log_2(\ln(xR))) \right).$$  

(4)

Keeping in mind that $c < 0$ in this case, it is easy to see that, so long as $p$ is bounded away from 0 and $|p'|$ is bounded from above by a small enough number, $f'$ has a constant sign in $(1/R, \infty)$.  

Thus, there are many monotonic — even differentiable — solutions other than $\varphi_c$. 