THE PRINCIPLE DESCRIBING POSSIBLE COMBINATIONS OF SINGULARITIES IN DEFORMATIONS OF A FIXED SINGULARITY

TOHSUKE URABE

Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-03 Japan
(E-mail: urabe@math.metro-u.ac.jp)

This is a review of my recent work.

What kinds of combinations of singularities can appear in small deformation fibers of a fixed singularity? We consider this problem for hypersurface singularities on complex analytic spaces of dimension 2.

Recall first that a connected Dynkin graph of type $A$, $D$ or $E$ corresponds to a surface singularity called a rational double point. (Durfee [3].)

When the fixed singularity is a rational double point corresponding to a Dynkin graph $\Gamma_0$, the answer to our problem is well-known.

1. Any small deformation fiber has only rational double points as singularities.
2. A combination of rational double points corresponding to a Dynkin graph $\Gamma$ appears on a small deformation fiber if and only if $\Gamma$ is a subgraph of $\Gamma_0$.

(The type of each component of $\Gamma$ corresponds to the type of the singularity on a small deformation fiber $Y$ and the number of components of each type corresponds to the number of singularities of each type on $Y$.)

This follows from the description of the semi-universal deformation family of a rational double point due to Grothendieck and Brieskorn. In this case the deformation family can be identified with a subspace of the corresponding simple Lie algebra, and the monodromy covering of the parameter space of the family can be identified with the Cartan subalgebra of the corresponding type. (Slodowy [6].) Thus the above fact follows.

We would like to consider more complicated singularity in the higher hierarchies in Arnold’s classification list. (Arnold [1]. There we can find the defining polynomials of singularities considered below.)

Let $\Xi$ be a class of surface singularities. By $PC(\Xi)$ we denote the set of Dynkin graphs $\Gamma$ with several components such that there exists a small deformation fiber $Y$ of a singularity belonging to $\Xi$ satisfying the following conditions:

1. $Y$ has only rational double points as singularities.
2. The combination of rational double points on $Y$ corresponds exactly to $\Gamma$.

When $\Xi$ is one of three kinds $P_8$, $X_9$, $J_{10}$ of simple elliptic singularities, the answer to the above problem is the following: If a small deformation fiber has a singularity which is not a rational double point, then the singularity is unique and it is a simple elliptic singularity of the same type $\Xi$. Besides, $\Gamma$ belongs to $PC(\Xi)$.
if and only if \( \Gamma \) can be made by elementary transformations repeated twice from the corresponding basic graph \( \Gamma_0 \). The basic graph \( \Gamma_0 \) is the Dynkin graph of type \( E_6 \) if \( \Xi = F_8 \), of type \( E_7 \) if \( \Xi = X_9 \), or of type \( E_8 \) if \( \Xi = J_{10} \). (Looijenga [5], Urabe [7].)

An elementary transformation is an operation by which we can make a new Dynkin graph from a given Dynkin graph. We give the definition below.

**Definition.** (An elementary transformation) The following procedure is called an elementary transformation of a Dynkin graph.

1. Replace each connected component by the corresponding extended Dynkin graph.
2. Choose in arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with edges issuing from them.

We can find the definition of the extended Dynkin graph in any book on Lie algebras. (Bourbaki [2].) They can be made by adding one vertex and one or two edges to each connected component of the Dynkin graph. The position of the added vertex and edges depends on the type of the component.

**Example.** We start from \( E_7 \). Removing a vertex of the extended Dynkin graph of type \( E_7 \) as in the following figure, we can make the graph \( D_6 + A_1 \):

![Diagram](image)

Applying another elementary transformation to \( D_6 + A_1 \), we can make the graph \( D_4 + 3A_1 \).

Since \( E_7 \) is the basic graph for \( \Xi = X_9 \), we have \( D_4 + 3A_1 \in PC(X_9) \)

In the case where \( \Xi \) is the class of cusp singularities \( x^p + y^q + z^r + \lambda xyz = 0 \) of type \( (p, q, r) \), (The indices \( p, q \) and \( r \) are positive integers with \( 1/p + 1/q + 1/r < 1 \). We may assume \( p \leq q \leq r \). The parameter \( \lambda \) is a non-zero constant.) Looijenga gave the answer. We call the graph below the Gabriéllov graph of type \( (p, q, r) \). The numbers of vertices in the three arms are \( p, q \) and \( r \) respectively including the common central one. A subgraph is called a Dynkin subgraph, if all of its components are Dynkin graphs of type \( A, D \) or \( E \).
According to Loijenga, $\Gamma \in PC(\Xi)$ if and only if $\Gamma$ can be made by one elementary transformation from a Dynkin subgraph of the Gabr"{e}lov graph of type $(p, q, r)$. (Loojenga [5].)

Recently we have succeeded in extending the similar descriptions to fourteen triangle singularities and six quadrilateral singularities.

The following fourteen singularities are called triangle singularities or exceptional singularities. (Arnold [1].)

$$\begin{align*}
E_{12}, & \quad Z_{11}, \quad Q_{10}, & & W_{12}, \quad S_{11}, \quad U_{12} \\
E_{13}, & \quad Z_{12}, \quad Q_{11}, & & W_{13}, \quad S_{12}, \quad U_{13} \\
E_{14}, & \quad Z_{13}, \quad Q_{12}, & & W_{14}, \quad S_{13}, \quad U_{14}
\end{align*}$$

For each $\Xi$ of the above fourteen classes the corresponding Gabr"{e}lov graph is defined. The type $(p, q, r)$ of the corresponding Gabr"{e}lov graph is as in the following list (Gabr"{e}lov [4]):

$$\begin{align*}
(2, 3, 7), & \quad (2, 4, 5), \quad (3, 3, 4), & & (2, 5, 5), \quad (3, 4, 4), & & (4, 4, 4) \\
(2, 3, 8), & \quad (2, 4, 6), \quad (3, 3, 5), & & (2, 5, 6), \quad (3, 4, 5) \\
(2, 3, 9), & \quad (2, 4, 7), \quad (3, 3, 6),
\end{align*}$$

**Theorem 1.** (Urabe [10].) Let $\Xi$ be one of the above fourteen classes of triangle singularities. The following two conditions are equivalent.

(A) $\Gamma \in PC(\Xi)$.

(B) Either (B-1) or (B-2) holds.

(B-1) $\Gamma$ can be made by an elementary transformation or a tie transformation from a Dynkin subgraph of the corresponding Gabr"{e}lov graph to $\Xi$. 


(B-2) Γ is one of the following exceptions:

For Ξ = Z_{13}, A_7 + A_4.
For Ξ = S_{11}, 2A_4 + A_1.
For Ξ = U_{12}, 2D_4 + A_2 + A_6 + A_4 + A_5 + A_1 + 2A_4 + A_1.
For the other eleven classes, no exceptions.

A tie transformation in (B-1) is another operation by which we can make a new Dynkin graph from a given Dynkin graph.

**Definition.** (A tie transformation) Assume that applying the following procedure to a Dynkin graph Γ, we have obtained the Dynkin graph \( \Gamma' \). Then we call the following procedure a tie transformation of a Dynkin graph.

1. Add one vertex and a few edges to each component of Γ and make it into the extended Dynkin graph of the corresponding type. Moreover attach the corresponding coefficient of the maximal root to each vertex.
2. Choose in an arbitrary manner subsets \( A, B \) of the set of the vertices of the extended graph \( \tilde{\Gamma} \) satisfying the following conditions:
   - \( \langle a \rangle \ A \cap B = \emptyset. \)
   - \( \langle b \rangle \ Let V be the set of vertices of an arbitrarily chosen component \( \tilde{\Gamma}' \) of Γ. Let \( \ell \) be the number of elements in \( V \cap A \) and \( n_1, n_2, ..., n_\ell \) be the numbers attached to \( V \cap A \). Furthermore let \( N \) be the sum of numbers attached to \( V \cap B \). (If \( V \cap B = \emptyset \), then \( N = 0 \).) Then the greatest common divisor of the \( \ell + 1 \) numbers \( N, n_1, n_2, ..., n_\ell \) is 1.
3. Erase all attached integers and remove vertices belonging to \( A \) together with edges issuing from them.
4. Draw another new vertex \( \circ \) corresponding to a root \( \alpha \) with \( \alpha^2 = 2 \). Connect this new vertex \( \circ \) and each vertex in \( B \) by an edge.

**Remark.** Often the resulting graph \( \Gamma \) after the above procedure (1) – (4) is not a Dynkin graph. We consider only the cases where the resulting graph \( \Gamma \) is a Dynkin graph and then we call the above procedure a tie transformation. Under this restriction the number \#(B) of elements in the set \( B \) satisfies \( 0 \leq \#(B) \leq 3. \) \( \ell = \#(V \cap A) \geq 1. \)

Any book on Lie algebras contains the definition of integers in (1) called the coefficients of the maximal root. (Bourbaki [2].)

**Example.** We consider the case \( \Xi = W_{13} \). The Gabrielov graph in this case is the following and it has a Dynkin subgraph of type \( E_8 + A_2 \):

First we apply a tie transformation to \( E_8 + A_2 \). In the second step of the transformation we can choose subsets \( A \) and \( B \) as follows:
For the component of type $E_8$, $\ell = 1$, $n_1 = 4$, $N = 1$, and thus $G.C.D. (n_1, N) = 1$. For the component $A_2$, $\ell = 1$, $n_1 = 1$, $N = 1$, and thus $G.C.D. (n_1, N) = 1$. One sees that the condition $(b)$ is satisfied. Adding a new vertex $\theta$ in the fourth step, one gets a graph of type $A_6 + D_5$ as the result of the transformation. By Theorem 1 one can conclude $A_6 + D_5 \in PC(W_{13})$.

Second we apply an elementary transformation to $E_8 + A_2$.

As in the above figure we can make $E_6 + 2A_2$. Thus $E_6 + 2A_2 \in PC(W_{13})$.

Now, it is very strange that Theorem 1 has a few exceptions in a few cases. Perhaps this is because our theory has a missing part.

**Problem.** Find the missing part of our theory and give a simple characterization of the set $PC(\Xi)$ without exceptions.

This problem may be very difficult, but I believe that there exists a solution.

Now, for the following nine singularities of the above fourteen ones we can state another theorem:

\begin{align*}
E_{12}, & \quad Z_{11}, & \quad Q_{10}, \\
E_{13}, & \quad Z_{12}, & \quad Q_{11}, \\
E_{14}, & \quad Z_{13}, & \quad Q_{12}.
\end{align*}
Theorem 2. (Urabe [9].) Let $\Xi$ be one of the above nine classes of singularities. The following two conditions are equivalent.

(A) $\Gamma \in PC(\Xi)$.

(B) The Dynkin graph $\Gamma$ has only components of type $A$, $D$ or $E$, and can be made from the essential basic graph depending on $\Xi$ by a combination of two of elementary transformations and tie transformations.

The respective essential basic graph corresponding to the above nine singularities

\[ E_8, \quad E_7, \quad E_6 \]

\[ E_8 + BC_1, \quad E_7 + BC_1, \quad E_6 + BC_1 \]

\[ E_8 + G_2, \quad E_7 + G_2, \quad E_6 + G_2 \]

In the condition (B) four kinds of combinations – “elementary” twice, “tie” twice, “elementary” after “tie”, and “tie” after “elementary” – are all permitted. Even when $\Xi = Z_{13}$ no exception appears in Theorem 2. Note that Dynkin graphs of type $BC_1$ or $G_2$ appear and the number of repetitions of transformations is not one but two.

Here we give some explanation on Dynkin graphs and root systems of type $BC$. A root system $R$ is a finite subset of a Euclidean space satisfying axioms on symmetry. Usually we assume moreover the following axiom (*) of the reduced condition:

\[ \text{If } \alpha \in R, \text{ then } 2\alpha \notin R \]

Under these axioms we obtain irreducible root systems of type $A$, $B$, $C$, $D$, $E$, $F$ and $G$ as in any book on Lie algebras. However, under the absence of the axiom (*) we have further a series of irreducible root systems, which are called of type $BC_k$ ($k = 1, 2, 3, \ldots$). (Bourbaki [2].) It is easy to generalize the concept of Dynkin graphs to root systems of type $BC$. (Urabe [8].) The Dynkin graph of type $BC_1$ is the following: $\otimes$

We explain the meaning of this $BC_1$ graph. Recall first the meaning of Dynkin graphs. Let $R$ be an irreducible root system and $\Delta \subset R$ be the root basis. We can assume that the longest root $\alpha \in R$ satisfies $\alpha^2 = 2$ after normalizing the inner product of the ambient Euclidean space. The Dynkin graph $\Gamma$ of $R$ is the graph drawn by the following rules: (1) The vertices of $\Gamma$ have one-to-one correspondence with the set $\Delta$ (the root basis). (2) Two vertices in $\Gamma$ corresponding to two elements $\alpha, \beta \in \Delta$ are connected by an edge in $\Gamma$ if and only if the inner product $(\alpha, \beta) \neq 0$.

If $R$ is of type $A$, $D$ or $E$, then $R$ consists of only roots $\alpha$ with $\alpha^2 = 2$, and every $\alpha \in \Delta$ satisfies $\alpha^2 = 2$. Therefore in these cases every vertex in the Dynkin graph can be denoted by a small white circle $\bigcirc$.

If $R$ is of type $BC_1$, then $\Delta$ consists of a unique root $\delta$ with $\delta^2 = 1/2$ and $R = \{-2\delta, -\delta, \delta, 2\delta\}$. The vertex in the Dynkin graph corresponding to a root $\delta$ with $\delta^2 = 1/2$ is denoted by $\otimes$. The $BC_1$ graph is the graph consisting of a unique vertex of this kind. In this case the maximal root $\eta$ is equal to $2\delta$, and thus the extended Dynkin graph, i.e., the graph corresponding to $\Delta^+ = \Delta \cup \{-\eta\}$ is the following: $\otimes$ (The edge is bold. The numbers are the coefficients of the maximal root.)

If $R$ is of type $G_2$, then $\Delta$ consists of two elements $\alpha$ with $\alpha^2 = 2$ and $\gamma$ with $\gamma^2 = 2/3$. We denote the vertex corresponding to a root $\gamma$ with $\gamma^2 = 2/3$ by $\otimes$. 
Our Dynkin graph of type $G_2$ is the following; and our extended Dynkin graph of type $G_2$ is the following (The numbers are the coefficients of the maximal root):

Note that as a result of an elementary or a tie transformation, a graph consisting of a unique vertex corresponding to $\gamma$ with $\gamma^2 = 2/3$ can appear. We call the graph the Dynkin graph of type $G_1$. This corresponds to the root system $R = \{-\gamma, \gamma\}$ with $\gamma^2 = 2/3$. The extended Dynkin graph of type $G_1$ is the following:

We can explain why we do not use the standard expression of the $G_2$ graph. (Bourbaki [2].) If we use the standard expression, we cannot define the concept of the $G_1$ graph.

Note that since we have assumed that the Dynkin graph $\Gamma$ in Theorem 2 has only components of type $A$, $D$ or $E$, any Dynkin graph with a component of type $G_2$, $G_1$ or $BC_1$ made by two transformations has no meaning, and is to be thrown out.

In the next hierarchy of Arnold’s list six kinds of quadrilateral singularities $J_{3,0}$, $Z_{1,0}$, $Q_{2,0}$, $W_{1,0}$, $S_{1,0}$, $U_{1,0}$ appear. Also for them we can show similar theorems to Theorem 2. This is the theme of Urabe [8]. Though the theorems for them are also quite simple, we have to introduce several additional concepts such as Dynkin graphs of type $B$ or $F$, obstruction components and so forth. Since they have been treated in Urabe [8], we omit the further explanation for them here.

References

1. Arnold, Vladimir Igorevic, Local normal forms of functions, Invent. Math. 35 (1976), 87–109.
2. Bourbaki, Nicolas, Groupes et algèbre de Lie. Chaps. 4-6, Hermann, Paris, 1968.
3. Durfee, Alan H., Fifteen characterization of rational double points and simple critical points, Enseign. Math. II 25 (1979), 131–163.
4. Gabriélov, A. M., Dynkin diagrams for unimodular singularities, Funkt. Anal. i Ego Prilozh. 8:3 (1974), 1–6.
5. Looijenga, Eduard, Rational surfaces with an anti-canonical cycle, Ann. of Math. 114 (1981), 267–322.
6. Slodowy, Peter, Simple singularities and simple algebraic groups, Lecture Notes in Mathematics, vol. 815, Springer, 1980.
7. Urabe, Tohsuke, On singularities on degenerate Del Pezzo surfaces of degree 1, 2, Proc. Symp. Pure Math. 40 part 2 (1983), 587-591.
8. Urabe, Tohsuke, Dynkin graphs and quadrilateral singularities, Lecture Notes in Mathematics, vol. 1548, Springer, 1993.
9. Urabe, Tohsuke, Dynkin graphs and triangle singularities, preprint (1993), to appear in the Proceedings of Workshop on Topology and Geometry (Hanoi, March, 1993).
10. Urabe, Tohsuke, Dynkin graphs, Gabriélov graphs and triangle singularities, preprint (1993), submitted to the Proceedings of International Geometrical Colloquium (Moscow, May, 1993).