Temperature Power Law of Equilibrium Heavy Particle Density

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ABSTRACT

A standard calculation of the energy density of heavy stable particles that may pair-annihilate into light particles making up thermal medium is performed to second order of coupling, using the technique of thermal field theory. At very low temperatures a power law of temperature is derived for the energy density of the heavy particle. This is in sharp contrast to the exponentially suppressed contribution estimated from the ideal gas distribution function. The result supports a previous dynamical calculation based on the Hartree approximation, and implies that the relic abundance of dark matter particles is enhanced compared to that based on the Boltzmann equation.
Estimate of the relic number density of dark matter particles such as LSP (Lightest Supersymmetric Particle) is conventionally made using a thermally averaged Boltzmann equation [1]. When the freeze-out temperature $T_f$ is relatively high, for instance $T_f > M/5$ with $M$ the mass of the annihilating heavy particle, this procedure is justified, but only after a careful examination (as we shall do in this paper). On the other hand, if the temperature is low, e.g., for $T_f < M/30$ (a case typical for the cold dark matter), the use of the Boltzmann equation is dubious, and a more general quantum kinetic equation becomes necessary [2]. A basic reason for this is that a finite time behavior of quantum system should be analyzed beyond the Boltzmann equation which uses the on-shell S-matrix element, hence is not fundamental at the full quantum mechanical level. As is well known, quantum mechanics at finite time cannot be described in terms of the S-matrix alone.

Some model calculation using a kinetic equation based on the Hartree approximation was performed in [2], assuming slow variation of the occupation number. In cosmology the temperature variation is given by the adiabatic law, $\dot{T}/T = -\frac{1}{3}$ the Hubble rate, and it was found that the heavy particle number density follows the equilibrium value until a freeze-out temperature below which the annihilation is frozen. This picture of the sudden freeze-out [1] is valid both for the Boltzmann equation and for our new kinetic equation. Thus, estimate of the freeze-out temperature using the equilibrium abundance is crucial even when one includes off-shell effects. It has been shown [3], [4] that for the equilibrium occupation number higher order term in coupling dominates at low temperatures over that given by the ideal gas distribution function ($1/(e^{E/T} - 1)$ for bosons). A temperature power dependence was thus derived for the equilibrium heavy particle abundance. A larger relic abundance than previously thought of emerged, hence a more restrictive region for the model parameter space is anticipated.

In the present work we employ a more familiar technique of the thermal field theory (imaginary-time formalism). Although there is no real time in the thermal field theory that governs the out-of-equilibrium dynamics, the thermal average one computes here deals with manifestly off-shell quantities. Hence the present method departs from the S-matrix approach of the thermally averaged Boltzmann equation. (In fairness we should point out that our kinetic approach [2] justifies the thermally averaged Boltzmann equation at high temperatures, but not at low temperatures.) We shall derive for the observable energy density a basically similar, but numerically
different equilibrium result, from the previous result \[2\]; the temperature power is different. Since definition of the occupation number is somewhat ambiguous in interacting field theory, a direct computation of the energy density is desirable, which we do in the present work. In a companion paper \[4\] we derive a new form of the kinetic equation, the one much simpler in form than that of ref.\[2\]. The idea there is the Hartree approximation using the influence functional method, and we shall clarify how the detailed balance equation there agrees with the result given in the present work.

For simplicity, we take throughout this paper a relativistic boson model, assuming the annihilation interaction
\[ \frac{\lambda}{4} \varphi^2 \chi^2, \] (1)
with \( \varphi \) being the heavy particle field and \( \chi \) being the light field taken as making up a part of the thermal bath. The thermal average of any operator \( O \) is done by using the Gibbs weight \( e^{-\beta H} \) with \( \beta = 1/T \) the inverse temperature,
\[ \langle O \rangle \equiv \frac{\text{tr} O e^{-\beta H}}{\text{tr} e^{-\beta H}} - \text{(T = 0 contribution)} . \] (2)
The total Hamiltonian \( H \) contains both contribution from the thermal bath and interaction. Both for consistent renormalization and for thermalization of light \( \chi \) particles, we need self-interaction of the form,
\[ \frac{\lambda_\varphi}{4!} \varphi^4, \quad \frac{\lambda_\chi}{4!} \chi^4. \] (3)
We numerically assume \( |\lambda_\varphi| \) to be much less than \( \lambda^2 \), but \( \lambda_\chi \) of arbitrary order provided \( |\lambda| \ll |\lambda_\chi| < 1 \). This hierarchy of coupling constants is assumed keeping in mind that the annihilation interaction is weak, while lighter particles can be kept in thermal equilibrium due to a stronger self-interaction given by \( \lambda_\chi \). The mass hierarchy \( m_\chi \ll m_\varphi \) also helps in favor of this picture.

We compute the \( \varphi \) energy density \( \rho_\varphi \) including renormalization counter terms;
\[ \rho_\varphi = \langle H_\varphi + \text{(counter terms)} \rangle, \] (4)
\[ H_\varphi = -\frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} M^2 \varphi^2, \] (5)
where \( \dot{\varphi} \) is the Euclidean time derivative. Note that by our definition of the thermal trace the zero-temperature value is automatically subtracted in this formula.
To leading order $O[\lambda^0]$ the $\varphi$ energy density given by this definition contains the "free field" result:

$$
\rho^{(0)}_\varphi = \int \frac{d^3 p}{(2\pi)^3} \frac{\sqrt{p^2 + M^2}}{e^{\sqrt{p^2 + M^2}/T} - 1}.
$$

(6)

The quantity $\rho_\varphi$ contains product of field operators at the same spacetime point and require renormalization at the composite operator level, separately from the Lagrangian counter terms. Complication due to the operator mixing also occurs. For instance, the operator mixing of composite operators occurs among nine second rank tensor operators of mass dimension up to 4,

$$
g_{\mu\nu} \left( \varphi^2, \chi^2, \varphi^2 \chi^2, \varphi^4, \chi^4, (\partial \varphi)^2, (\partial \chi)^2 \right), \partial_\mu \varphi \partial_\nu \varphi, \partial_\mu \chi \partial_\nu \chi.
$$

(7)

The renormalized Hamiltonian $H_\varphi$ is then of the form,

$$
H_\varphi = H_\varphi^{\text{bare}} + \sum_i^2 Z_i^{(2)} O_{2,i} + \sum_i^7 Z_i^{(4)} O_{4,i}
$$

(8)

to $O[\lambda^2]$. There are two operators $O_{2,i}$ of mass dimension 2, while there are seven operators $O_{4,i}$ of mass dimension 4. There is no $Z^{(1)}$ factor of order $\lambda$ due to the assumed interaction hierarchy.

Renormalization constants $Z_i$ are derived by assigning a canonical value to the zero-temperature Fourier transformed correlation function at one external momentum point. (We use the on-shell point for renormalization.) For example, for $\partial_\mu \varphi \partial_\nu \varphi$ the correlators with field operators up to mass dimension 4

$$
\langle 0 | \partial_\mu \varphi \partial_\nu \varphi(x) \psi(x_1) \psi(x_2) | 0 \rangle,
$$

(9)

$$
\langle 0 | \partial_\mu \varphi \partial_\nu \varphi(x) \psi(x_1) \psi(x_2) \psi(x_3) \psi(x_4) | 0 \rangle,
$$

(10)

with $\psi$ either $\varphi$ or $\chi$, should be taken into account. (Higher dimensional correlators are all finite and irrelevant to renormalization.) The free field limit values are used to define $Z_i$.

To order $O[\lambda^2]$ there are three types of diagrams (two topologically distinct) contributing to the energy density $\langle H_\varphi \rangle$. See Fig.1 and Fig.2. The temperature power dependence arises from the amputated sunset diagram (Fig.1), which gives in the configuration space

$$
\lambda^2 \int_0^\beta d^4 x d^4 y \ \Delta_\varphi(x) \Delta_\varphi(-y) \Delta_\varphi(y - x) \Delta_\chi(x - y) \Delta_\chi(y - x),
$$

(11)
prior to renormalization. The propagator in thermal medium $\Delta_{\phi, \chi}(y)$ is periodic in the Euclidean time $y_0$ with a period $\beta = 1/T$, hence giving the range of integration $0 \sim \beta$ as explicitly indicated. The Fourier transformed propagator has the well known form, $\Delta(\omega_n, \vec{p}) \sim 1/( -\omega_n^2 + \vec{p}^2 + M^2)$ with discrete $\omega_n = 2\pi n/\beta$ ($n = 0, \pm 1, \pm 2, \cdots$). The other contributions from Fig.2 are exponentially suppressed by $e^{-M/T}$ with $M$ the heavy $\phi$ particle mass, the factor familiar in the conventional approach.

As usual, one rewrites eq.(11) using the Fourier transform. The resulting discrete energy sum over $\omega_n$ can be converted to a contour integral of this variable $z = \omega_n$, using the function $1/(e^{\beta z} - 1)$. After some algebraic manipulation, one finds that to $O[\lambda^2]$

$$\rho^{(2)}_\phi \sim - \lambda^2 \int \frac{dk \, dp \, dp' (2\pi)^3 \delta(\vec{p} + \vec{p}' + \vec{k} + \vec{k}')}{2\omega_p} 2\omega_p$$

$$+ \frac{f_p f_{p'} (1 + f_k)(1 + f_{k'}) - (1 + f_p)(1 + f_{p'})(1 + f_k)}{(\omega_p + \omega_{p'} - \omega_k - \omega_{k'})^2}$$

$$+ \frac{2 f_p f_{p'} (1 + f_k)f_{k'} - (1 + f_p)(1 + f_{p'})(1 + f_{k})}{(\omega_p + \omega_{p'} - \omega_k + \omega_{k'})^2}$$

$$+ \frac{f_p f_{p'} f_k f_{k'} - (1 + f_{p'})(1 + f_{k'})(1 + f_k)(1 + f_{k})}{(\omega_p + \omega_{p'} + \omega_k + \omega_{k'})^2}$$

$$+ \frac{2 f_p f_{p'} f_{p'} f_k f_{k'} - (1 + f_{p'})(1 + f_{k'})(1 + f_k)(1 + f_{k})}{(\omega_p - \omega_{p'} - \omega_k - \omega_{k'})^2}$$

$$+ \frac{2 f_p f_{p'} f_k (1 + f_{k'}) - (1 + f_p)(1 + f_{p'})(1 + f_k)f_{k'}}{(\omega_p - \omega_{p'} + \omega_k - \omega_{k'})^2}$$

$$+ \frac{2 f_p f_{p'} f_k (1 + f_{k'}) - (1 + f_p)(1 + f_{p'})(1 + f_k)f_{k'}}{(\omega_p - \omega_{p'} + \omega_k - \omega_{k'})^2}.$$  \hfill (12)

We dropped minor Boltzmann suppressed terms to obtain this result. A shorthand notation for the phase space integral $dk = d^3k/(2\pi)^32\omega_k$ was used here, and $f_{p, p'}$ are the occupation number for the heavy $\phi$ particle, while $f_{k, k'}$ are that for the light $\chi$ particle;

$$f_p = \frac{1}{e^{\sqrt{p^2 + M^2}/T} - 1}, \quad f_k = \frac{1}{e^{k/T} - 1}. \hfill (13)$$

A similar form to eq.(12) was derived for the proper self-energy in ref.\cite{5}. For simplicity we assume the $\chi$ mass $m_\chi \ll T$, and indeed take $m_\chi = 0$ here. Terms containing $f_p$ or $f_{p'}$ in eq.(12) are Boltzmann suppressed by $e^{-M/T}$. Drop-
ping all these Boltzmann suppressed terms, one obtains after removal of the infinity

\[
\frac{\lambda^2}{16\pi^2} \int dk \, dk' \, f_k \, f_{k'} \, (\omega_k^2 + \omega_{k'}^2) \int_{4M^2}^{\infty} ds \, \frac{1}{s^2} \sqrt{1 - \frac{4M^2}{s}}.
\]

This gives to leading order of \(T/M\)

\[
\rho^{(2)}_\varphi = c \frac{\lambda^2 \, T^6}{M^2}, \quad c = \frac{1}{69120} \sim 1.4 \times 10^{-5}.
\]

Terms of higher temperature-power are subleading, and neglected here.

There is a simple reason how the temperature dependence \(\propto T^6\) arises. Since \(\mathcal{H}_\varphi\) has both dimension 2 and 4 operators, terms of order \(T^2\) and \(T^4\) are divergent and they are cancelled by counter terms \(Z_i O_i\). The reason why one does not have divergent \(O[\lambda^2 T^6/M^2]\) terms is that \(Z_i\) is already of order \(\lambda^2\) and one only needs \(\langle O_i \rangle\) to \(O[\lambda^0]\) which is either Boltzmann-suppressed or has no \(T^6 M^{-2}\) term. The remaining finite term is then of order \(T^6\) unless some special cancellation mechanism works. In our companion paper \[4\] we give a separate computation of the equilibrium energy density from the kinetic approach, which exactly agrees with the present result.

We separately computed the interaction energy density given by Fig.3 to get

\[
\rho_{\text{int}} = \langle \frac{\lambda}{4} \varphi^2 \chi^2 + \text{(counter terms)} \rangle \sim -\frac{\pi^2}{64800} \lambda^2 \frac{T^8}{M^4},
\]

dropping \(O[\lambda]\) Boltzmann suppressed terms. Thus, the interaction makes a minor contribution in the pair-annihilation model.

How about terms of order \(\lambda \chi^2\)? All these turn out to give Boltzmann suppressed contributions to \(\rho_\varphi\) and \(\rho_{\text{int}}\). This makes our result insensitive to the self-interaction among light \(\chi\) particles.

A possibility that the interaction makes at low temperatures a comparable contribution to the heavy particle energy was noted by Singh and Srednicki \[6\] who explicitly calculated the interaction energy in the simple solvable model of \[3\]. However their suggestion that quantum kinetic approach should be discarded and we should go back to the Boltzmann equation is not valid for a number of reasons. First of all, there is no solid justification for the Boltzmann equation at low temperatures. Next, the amount of interaction energy is model dependent; indeed our annihilation model gives a negligible contribution of interaction. We are neither confident of their claim that the solvable gaussian model gives a comparable contribution of interaction, because renormalization is not taken into account in their computation.
Moreover, as we shall show below, one can make in the annihilation model a clear
distinction between various forms of energy by taking into account the cosmological
expansion. This argument makes clear which part should be regarded as the dark
matter energy density.

We would however like to point out some peculiarity; cancellation of order $\lambda^2 T^6$
terms for the total energy, $\rho_{\text{tot}} = \rho_\varphi + \rho_\chi + \rho_{\text{int}}$. Namely, a term in $\rho_\chi$ of order $\lambda^2 T^6$
exactly cancels the same order term in $\rho_\varphi$ to give a $\lambda^2 T^8$, and no $\lambda^2 T^6$, term to $\rho_{\text{tot}}$. This
however does not mean that both, $\rho_\chi$ and $\rho_{\text{tot}}$, are numerically of order $\lambda^2 T^8$. There are more important,
larger terms of order, for instance, $\lambda \lambda^n \chi$ for these. With a larger $\lambda_\chi$ coupling, one has a consistent picture for thermalization of $\chi$ particles.

Although it is technically difficult to compute the momentum distribution of relic
particles, it is relatively easy to compute the pressure of heavy particles in thermal
equilibrium. It turns out that the pressure $p$ is one fifth of the energy, $p = \frac{1}{5} \rho$. It is neither of completely non-relativistic (in which case $p = 0$) nor of relativistic
($p = \frac{1}{3} \rho$) form. Thus, right after the freeze-out the equation of state implies that the
energy density follows $\rho \propto a^{-18/5}$. But the well known redshift effect makes the high
momentum component energetically subdominant after rapid cosmological expansion.
It is thus reasonable to suppose that at later epochs the heavy particle is essentially
non-relativistic, behaving like $\rho_\varphi \propto a^{-3}$. On the other hand, the interaction energy
density $\approx \lambda \varphi^2 \chi^2$ decreases much more rapidly like $a^{-5}$ in thermal equilibrium. This
decrease is faster than the high momentum part of $\rho_\varphi$. We thus find that the correct
dark matter density decreasing with the volume factor should be identified as $\rho_\varphi$.

Since our formula for the energy density is valid only at low temperatures, $T \ll M$, meaning that the individual particle energy $E \sim M$, one can also deduce for the
heavy particle number density at $T \ll M$

$$n_\varphi = \frac{\rho_\varphi(0) + \rho_\varphi(2)}{M} \sim \frac{\rho_\varphi(0)}{M} + c \lambda^2 \frac{T^6}{M^3}. \quad (17)$$

There is a critical temperature $T_{\text{cr}}$ below which the temperature power term
dominate the usual Boltzmann term. This may easily be estimated by equating the two formulas;

$$\left(\frac{MT_{\text{cr}}}{2\pi}\right)^{3/2} e^{-M/T_{\text{cr}}} = c \lambda^2 \frac{T_{\text{cr}}^6}{M^3}. \quad (18)$$

Numerically, the value of $T_{\text{cr}}/M$ ranges from 1/28 to 1/33 for $\lambda = 0.1 - 0.01$. Thus,
in this $\lambda$ range $T_{\text{cr}} \approx M/30$, very crudely. A useful empirical formula of the critical
temperature in the coupling range of $\lambda = 0.1 - 10^{-4}$ is

$$\frac{M}{T_{cr}} = 23 - 2.3 \ln \lambda. \quad (19)$$

Is our result reliable at the zero temperature limit, $T \to 0$? We argue that this is not so from the following reason. Our method of using the equilibrium value for the freeze-out abundance is based on available sufficient time for relaxation towards equilibrium. In cosmology expansion makes this time limited to the Hubble time. The physical time scale towards equilibrium is the inverse of the pair-creation rate for $\chi \chi \rightarrow \varphi \varphi$,

$$\langle \sigma v \rangle n_\varphi \approx O[10^{-7}] \lambda^4 \left( \frac{T}{M} \right)^5 T. \quad (20)$$

This should be compared to the Hubble rate, $O[1] N^{1/2} T^2 / m_{pl}$ where $N$ is the relativistic degrees of freedom contributing to the energy density. The condition for relaxation in cosmology is then

$$\frac{T}{M} \gg O[10^{-3}] \frac{N^{1/8}}{\lambda} \left( \frac{M}{100 \text{GeV}} \right)^{5/4}. \quad (21)$$

For a very small $\lambda$ this condition may violate $T < M$, in which case the use of the ordinary Boltzmann equation is justified. We thus need a relatively large coupling and/or a relatively small mass $M$ for our new result to be dominant over the Boltzmann suppressed number density. In our published calculation of \cite{2} there is a technical mistake in computation of the $\varphi$ number density so that the correct condition for the off-shell effect becomes more stringent. The numerical result there should thus be corrected, and the parameter region for the new effect is much more reduced, as seen more fully in our subsequent analysis.

We now estimate the freeze-out temperature $T_f$. The simplest way is to equate the Hubble rate given as a function of the temperature $T$ to the annihilation rate $\langle \sigma v \rangle n_\varphi$, where $\sigma$ is the annihilation cross section of order $\lambda^2 / M^2$. Using the number density, eq.(17) and equating to the Hubble rate

$$H = 1.66 \times \sqrt{N} \frac{T^2}{m_{pl}}, \quad (22)$$

we find for $T_f < T_{cr}$

$$T_f \approx 0.3 \text{GeV} \frac{N^{1/8}}{\lambda} \left( \frac{M}{100 \text{GeV}} \right)^{5/4}. \quad (23)$$
This holds only when the temperature power term dominates over the exponential $e^{-M/T}$ term for $n_\varphi$. The relic mass density is then given by

\[
\left( \frac{n_\varphi}{T^3} \right)_f \approx 4 \times 10^{-13} \frac{N^{3/8}}{\lambda} \left( \frac{M}{100 \text{GeV}} \right)^{3/4}.
\]

(24)

Demanding that this is smaller than the present critical density gives a constraint on the model parameter, the coupling $\lambda$ and the heavy mass $M$. The $\varphi$ mass density relative to the critical density is at present

\[
\frac{\rho_\varphi^{(2)}}{\rho_c} \approx 10^{-4} \frac{N^{3/8}}{\lambda} \left( \frac{M}{100 \text{GeV}} \right)^{7/4}.
\]

(25)

Numerical estimate of the freeze-out temperature including the Boltzmann suppressed region is given in Fig.4. The relativistic degree of freedom $N$ is taken 10.75 throughout this paper. At a given $\lambda$ this freeze-out temperature substantially differs from the naive estimate obtained by using the Boltzmann factor, when the heavy particle mass is small.

A more precise, yet approximate estimate is possible by using time evolution equation in the expanding universe. From the equation for the number density

\[
\frac{dn_\varphi}{dt} + 3 H n_\varphi = - \langle \sigma v \rangle (n_\varphi^2 - n_{\varphi}^{eq}) ,
\]

(26)

one has for the yield $Y = n_\varphi / T^3$

\[
\frac{dY}{dT} = d \langle \sigma v \rangle m_{\text{pl}} (Y^2 - Y_{eq}^2) ,
\]

(27)

\[
d = 1.66 \sqrt{N} , \quad \langle \sigma v \rangle \approx \frac{\lambda^2}{32\pi M^2} ,
\]

(28)

where the temperature-time relation $t \propto T^{-2}$ was used. The equilibrium value is approximately a sum of two terms valid at high (but $T/M \ll 1/\sqrt{c\lambda} \approx 3 \times 10^2/\lambda$) and low temperatures,

\[
Y_{eq} = \frac{1}{T^3} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{\sqrt{p^2 + M^2}/T} - 1} + c \lambda^2 \left( \frac{T}{M} \right)^{3/2}.
\]

(29)

In Fig.5 we plot an example of numerical computation of the time evolution equation (27). Among two examples of coupling $\lambda = 0.05, 0.3$ both with $M = 100 \text{GeV}$, the smaller coupling case gives indistinguishable new effect and the ordinary Boltzmann approach is numerically correct. On the other hand, the larger coupling case gives a substantially different relic abundance from the conventional result.
Time evolution obtained from eq. (26) supports the picture of sudden freeze-out as in the Lee-Weinberg analysis [1]. The major difference here is the new equilibrium abundance $n_{eq}$.

The final relic abundance of dark matter particles is shown as a contour plot in the parameter $(M, \lambda)$ plane. In Fig. 6 we show the present $\phi$ mass density relative to the critical density, for computation both with and without our new effect. Our new effect tends to show up for a larger coupling and a smaller mass.

Since our new effect gives an additional positive contribution to an energy integral, the relic density is always enhanced from the conventional result without our effect. Thus the allowed parameter region in the model parameter space gets always smaller by our result. The real question is then how much of the previously known region is excluded by this new effect. How our result of the relic density affects the presumably best motivated case of the SUSY dark matter remains to be studied numerically. Especially in the $Z$ and the Higgs resonance region the effective coupling is large [1], and there the off-shell effect may be very large. This should be checked in more realistic calculation beyond our boson model.

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Figure caption

Fig.1
Amputated sunset diagram for the composite operator. The solid line and the broken line are the heavy $\varphi$ particle and the light $\chi$ particle propagators in thermal equilibrium. The crossed circle represents the insertion of the composite operator.

Fig.2
The rest of diagrams contributing to the composite operator as in Fig.1 of $O[\lambda^2]$.

Fig.3
Diagrams contributing to the interaction energy.

Fig.4
The inverse freeze-out temperature $T_f^{-1}$ vs the mass $M$ of the heavy particle, computed for two cases of coupling, $\lambda = 0.1, 0.05$. Dotted curves are result using the Boltzmann factor.

Fig.5
Time evolution of the relative energy density of heavy particles, $M n_\varphi/n_\gamma$, computed for two cases of coupling, $\lambda = 0.3, 0.05$. For the larger coupling ($\lambda = 0.3$) our new result and old result based on the thermally averaged Boltzmann equation do substantially differ, while the smaller coupling case gives indistinguishable result.

Fig.6
Contour plot of the present mass density of relic particles. The lines shown correspond to $1, 10^{-1}, 10^{-2} \times$ the critical mass density $\rho_c^0$ (taken here $1.05 \times 10^{-5} h_0^2$ GeV cm$^{-3}$ with $h_0 = 0.7$), along with results obtained using the Boltzmann suppression factor.
Fig. 4

\[ \frac{M}{T_f} \] vs. \( M \, (\text{GeV}) \)

\( (\text{FREEZE-OUT TEMPERATURE})^{-1} \)

- 0.1
- 0.05

\( M \, (\text{GeV}) \) from 1 to 10^4
Fig. 5

TIME EVOLUTION

M = 100 GeV
Fig. 6

OLD

$\rho_c^0$ 

$\times 0.01$ 

$\times 0.1$ 

EXCLUDED

$\lambda$

$M$ (GeV)