On the Weight Enumerator and the Maximum Likelihood Performance of Linear Product Codes

Mostafa El-Khamy and Roberto Garello

Abstract

Product codes are widely used in data-storage, optical and wireless applications. Their analytical performance evaluation usually relies on the truncated union bound, which provides a low error rate approximation based on the minimum distance term only. In fact, the complete weight enumerator of most product codes remains unknown. In this paper, concatenated representations are introduced and applied to compute the complete average enumerators of arbitrary product codes over a field $F_q$. The split weight enumerators of some important constituent codes (Hamming, Reed-Solomon) are studied and used in the analysis. The average binary weight enumerators of Reed Solomon product codes are also derived. Numerical results showing the enumerator behavior are presented. By using the complete enumerators, Poltyrev bounds on the maximum likelihood performance, holding at both high and low error rates, are finally shown and compared against truncated union bounds and simulation results.

I. Introduction

Product codes were introduced by Elias [1] in 1954, who also proposed to decode them by iteratively (hard) decoding the component codes. With the invention of turbo codes [2], soft iterative decoding techniques received wide attention [3]: low complexity algorithms for turbo decoding of product codes were first introduced by Pyndiah in [4]. Other efficient algorithms were recently proposed in [5] and in [6].

For product codes, an interesting issue for both theory and applications regards the analytical estimation of their maximum likelihood performance. Among other, this analytical approach allows to: (i) forecast the code performance without resorting to simulation; (ii) provide a benchmark for testing sub-optimal iterative decoding algorithms; (iii) establish the goodness of the code, determined by the distance from theoretical limits.

The analytical performance evaluation of a maximum-likelihood decoder requires the knowledge of the code weight enumerator. Unfortunately, the complete enumerator is unknown for most families of product codes. In these years, some progress has been made in determining the first terms of product code weight enumerators. The multiplicity of low weight codewords for an arbitrary linear product code were computed by Tolhuizen [7]. (In our paper, these results will be extended to find the exact input output weight enumerators of low weight codewords.)

Even if the first terms can be individuated, the exact determination of the complete weight enumerator is very hard for arbitrary product codes [7, 8]. By approximating the number of the remaining codewords by that of a normalized random code, upper bounds on the performance of binary product codes using the ubiquitous union bound were shown in [9]. However, this approximation is not valid for all product codes.

In this paper, we will consider the representation of a product codes as a concatenated scheme with interleaver, and we will derive the average input-output weight enumerator for linear product codes over a generic field $F_q$. When combined with the extended Tolhuizen’s result, this will provide a complete approximated enumerator for the product code. We will show how it closely approximates the exact weight enumerator.

This work was presented in part at the WirelessCom Symposium on Information Theory, Maui, Hawaii, USA in June 2005. Mostafa El-Khamy is with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125, USA, email: mostafa@systems.caltech.edu. Roberto Garello is with the Dipartimento di Elettronica, Politecnico di Torino, Italy, email: garello@polito.it. This research was supported by NSF grant no. CCF-0514881 and grants from Sony, Qualcomm, and the Lee Center for Advanced Networking.
Previous work in the literature (see for example [10], and reference therein) focused on estimating the product code performance at low error rates via the truncated union bound, using the enumerator low-weight terms only. By using the complete approximate enumerator, it is possible to compute the Poltyrev bound [11], which establish tight bounds on the maximum likelihood performance at both high and low error rates.

The outline of the paper is as follows. In section II, we introduce the basic notation and definitions. In section III, we extend Tollhuizen results and derive the exact input-output weight enumerator for product code low-weight codewords. Product code representation as serial and parallel concatenated codes with interleavers are introduced in section IV-A. Uniform interleavers on finite fields with arbitrary size are discussed in section IV-B. The average weight enumerators of product codes are then computed in sections IV-C. The merge with exact low-weight terms, and the discussion of the combined enumerator properties are performed in section V.

The computation of product code average enumerators relies on the knowledge of the input-redundancy weight enumerators of the component codes. For this reason, we derive in section VI closed form formulas for the enumerator functions of some linear codes commonly used in the construction of product codes: Hamming, extended Hamming, and Reed Solomon codes. We proceed in section VII to derive the average binary weight enumerators of Reed Solomon product codes defined on finite fields of characteristic two.

To support our theory, we present some numerical results. Complete average enumerators are depicted and discussed in section VIII-A. Analytical bounds on the maximum likelihood performance are shown at both high and low error rates, and compared against simulation results in section VIII-B. Finally, we conclude the paper in section IX.

II. Preliminaries

Let \( F_q \) be a finite field of \( q \) elements, and \( C \) a \((n_c, k_c, d_c)\) linear code over \( F_q \) with codeword length \( n_c \), information vector length \( k_c \) and minimum Hamming distance \( d_c \).

The weight enumerator (WE) of \( C \), \( E_C(h) \), (also called multiplicity) is the number of codewords with Hamming weight \( h \):

\[
E_C(h) = |\{ c \in C : w(c) = h \}|
\]

where \( w(\cdot) \) denotes the symbol Hamming weight.

For a systematic code \( C \), the input-redundancy weight enumerator (IRWE), \( R_C(w, p) \), is the number of codewords with information vector weight \( w \), whose redundancy has weight \( p \):

\[
R_C(w, p) = |\{ c = (i|p) \in C : w(i) = w \ w(p) = p \}|
\]

If \( T = (n_1, n_2) \) is a partition of the \( n \) coordinates of the code into two sets of size \( n_1 \) and \( n_2 \), the split weight enumerator \( A^T(w_1, w_2) \) is number of codewords with Hamming weights \( w_1 \) and \( w_2 \) in the first and second partition, respectively. If \( T \) is an \((k, n-k)\) partition such the first set of cardinality \( k \) constitutes of the information symbol coordinates, \( R(w_1, w_2) = A^T(w_1, w_2) \).

The input-output weight enumerator (IOWE), \( O_C(w, h) \), is the number of codewords whose Hamming weight is \( h \), while their information vector has Hamming weight \( w \):

\[
O_C(w, h) = |\{ c \in C : w(i) = w \ w(c) = h \}|
\]

For a systematic code,

\[
O_C(w, h) = R_C(w, h - w) .
\]

(1)

It is also straight forward that

\[
E_C(h) = \sum_{w=0}^{k_c} O_C(w, h) .
\]

(2)
The $WE$ function of $C$ is defined by this polynomial in invariant $Y$:

$$E_C(Y) = \sum_{h=0}^{n_c} E_C(h)Y^h$$

while the $IRWE$ function and the $IOWE$ function of $C$ are defined by these bivariate polynomials in invariants $X$ and $Y$:

$$R_C(X, Y) = \sum_{w=0}^{k_c} \sum_{p=0}^{n_c-k_c} R_C(w, p)X^wY^p,$$

$$O_C(X, Y) = \sum_{w=0}^{k_c} \sum_{h=0}^{n_c} O_C(w, h)X^wY^h.$$  \hspace{1cm} (3)

$$O_C(X, Y) = R_C(1, Y).$$  \hspace{1cm} (4)

These functions are related by

$$O_C(X, Y) = R_C(X, Y) \times R_C(Y, Y).$$  \hspace{1cm} (5)

and

$$E_C(Y) = R_C(Y, Y) = O_C(1, Y).$$  \hspace{1cm} (6)

In the following, we will denote the coefficient of $X^wY^h$ in a bivariate polynomial $f(X, Y)$ by the coefficient function $\Lambda(f(X, Y), X^wY^h)$. For example, $O_C(w, h) = \Lambda(O_C(X, Y), X^wY^h)$. Similarly, $\Lambda(O(X, Y), Y^w)$ is the coefficient of $Y^w$ in the bivariate polynomial $O(X, Y)$ and is a univariate polynomial in $X$.

Let the code $C$ be transmitted by a 2-PSK constellation over an Additive White Gaussian Noise (AWGN) channel with a signal-to-noise ratio (SNR) $\gamma$. The symbols $\Phi_c(\gamma)$ and $\Phi_b(\gamma)$ will denote the corresponding codeword error probability (CEP) and bit error probability (BEP) of a maximum likelihood (ML) decoder, respectively. These ML performance can be estimated by computing analytical bounds based on the code enumerators.

The truncated union bound, taking into account the minimum distance term only, provides a heuristic approximation commonly used at high SNR/low CEP:

$$\Phi_c(\gamma) \simeq \frac{1}{2} E_C(d_c) \text{erfc} \sqrt{\frac{k_c}{n_c} d_c \gamma}. $$  \hspace{1cm} (7)

This formula provides a simple way for predicting the code performance at very high SNR, where maximum likelihood error events are mostly due to received noisy vectors lying in the decoding regions of codewords nearest to the transmitted one. Anyway, it is not useful in predicting the performance at low SNR.

Tight bounds on the maximum likelihood codeword error probability of binary linear codes for AWGN and binary symmetric channel (BSC), holding at both low and high SNR, were derived by Poltyrev in [11]. Other bounds such as the Divsalar simple bound and the variations on the Gallager bounds are also tight for AWGN and fading channels [12, 13]. These bounds usually require knowledge of the complete weight enumerator $E_C(h)$. In this paper, we will apply the Poltyrev bound by using a complete approximate weight enumerator of the considered product codes.

Given the codeword error probability, the computation of the bit error probability may pose a number of technical problems. Let $\Phi_c(E_C(h), \gamma)$ denote the CEP over a channel with an SNR $\gamma$ computed by using the weight enumerator $E_C(h)$. The bit error probability $\Phi_b(\gamma)$ is derived from the CEP by computing $\Phi_b(\gamma) = \Phi_c(I_C(h), \gamma)$, where $I_C(h) = \sum_{w=1}^{k_c} \frac{w}{k_c} O(w, h)$.  


A common approximation in the literature is \( I_C(h) \approx \frac{h}{nc} E_C(h) \). This approximation is useful if the IOWE \( O(\cdot, \cdot) \) is not known but the weight enumerator WE \( E(\cdot) \) is. Some codes satisfy this approximation with equality: they are said to possess the \textit{multiplicity property}. This is the case, for example, of all codes with transitive automorphism groups (including Hamming and extended Hamming codes) \([10]\) or all maximum distance separable codes (including Reed-Solomon codes) \([14]\).

Let \( R \) and \( C \) be \((n_r, k_r, d_r)\) and \((n_c, k_c, d_c)\) linear codes over \( F_q \), respectively. The product code whose component codes are \( R \) and \( C \), \( P = R \times C \), consists of all matrices such that each row is a codeword in \( R \) and each column is a codeword in \( C \). \( P \) is an \((n_p, k_p, d_p)\) linear code, with parameters

\[
n_p = n_r n_c, \quad k_p = k_r k_c, \quad d_p = d_r d_c
\]

### III. Exact IOWE of Product Codes for Low Weight Codewords

In \([7]\), Tolhuizen showed that in a linear product code \( P = R \times C \) the number of codewords with symbol Hamming weight \( 1 \leq w < h_o \) is:

\[
E_P(h) = \frac{1}{q-1} \sum_{i|h} E_C(i) E_R(h/i),
\]

where, given

\[
w(d_r, d_c) = d_r d_c + \max(d_r \left\lfloor \frac{d_c}{q} \right\rfloor, d_c \left\lfloor \frac{d_r}{q} \right\rfloor),
\]

the weight \( h_o \) is

\[
h_o = \begin{cases} 
  w(d_r, d_c) + 1, & \text{if } q = 2 \text{ and both } d_r \text{ and } d_c \text{ are odd} \\
  w(d_r, d_c), & \text{otherwise}
\end{cases}
\]

In particular, the minimum distance multiplicity of a product code is given by

\[
E_P(d_p) = \frac{E_R(d_p) E_C(d_c)}{q-1}.
\]

These results are based on the properties of \textit{obvious} (or \textit{rank-one}) codewords of \( P \), i.e., direct product of a row and a column codeword \([7]\). Let \( r \in R \) and \( c \in C \), then an obvious codeword, \( p \in P \), is defined as

\[
p_{i,j} = r_i c_j,
\]

where \( r_i \) is the symbol in the \( i \)-th coordinate of \( r \) and \( c_j \) is the symbol in the \( j \)-th coordinate of \( c \). It follows that the rank of the \( n_c \times n_r \) matrix defined by \( p \) is one and the Hamming weight of \( p \) is clearly the product of the Hamming weights of the component codewords, i.e.,

\[
 w(p) = w(r) w(c).
\]

Tolhuizen showed that any codeword with weight smaller than \( w(d_r, d_c) \) is obvious (Theorem 1, \([7]\)) (smaller or equal if \( q = 2 \) and both \( d_r \) and \( d_c \) are odd (Theorem 2, \([7]\))). The term \( \frac{1}{q-1} \) in \((8)\) and \((10)\) is due to the fact \( (\lambda r_i)(c_j/\lambda) \) are equal for all nonzero \( \lambda \in F_q \).

A generalization of Tolhuizen’s result to input output weight enumerators is given in the following theorem.

**Theorem 1**

Let \( R \) and \( C \) be \((n_r, k_r, d_r)\) and \((n_c, k_c, d_c)\) linear codes over \( F_q \), respectively. Given the product code \( P = R \times C \), the exact IOWE for codewords with output Hamming weight \( 1 < h < h_o \) is given by

\[
O_P(w, h) = \frac{1}{q-1} \sum_{i|h} \sum_{j|h} O_R(i,j) O_C(w/i, h/j),
\]
where the sum extends over all factors \( i \) and \( j \) of \( w \) and \( h \) respectively, and \( h_o \) is given by \( \Box \).

\[ \Box \]

Proof: Let \( p \in P \) be a rank-one codeword, then there exists a codeword \( r \in R \) and a codeword \( c \in C \) such that \( p_{i,j} = r_i c_j \). The \( k_r k_c \) submatrix of information symbols in \( p \) could be constructed from the information symbols in \( c \) and \( r \) by \( \Box \) for \( 1 \leq i \leq k_r \) and \( 1 \leq j \leq k_c \). It thus follows that the input weight of \( p \) is the product of the input weights of \( c \) and \( r \) while its output (total) weight is given by \( \Box \). Since all codewords with weights \( h < h_o \) are rank-one codewords, the theorem follows.

These results show that both the weight enumerators and the input-output weight enumerators of product code low-weight codewords are determined by the constituent code low-weight enumerators. This is not the case for larger weights, where the enumerators of \( P \) are not completely determined by the enumerators of \( R \) and \( C \) \( \mathbb{[7]} \).

It is important to note the number of rank-one low-weight codewords is very small, as shown by the following corollary regarding Reed Solomon (RS) product codes.

**Corollary 1**

Let \( C \) be an \((n, k, d)\) Reed Solomon code over \( F_q \). The weight enumerator of the product code \( P = C \times C \) has the following properties,

\[
E_P(h) = \begin{cases} 
1, & h = 0; \\
(q - 1) \binom{n}{d^2}, & h = d^2; \\
0, & d^2 < h < d(d + 1).
\end{cases}
\]

\[ (14) \]

**Proof:** Let us apply \( \Box \). From the maximum distance separable (MDS) property of RS codes, \( d = n - k + 1 \) and \( n < q \). It follows that \( w(d, d) = d(d + 1) \). Also \( E_C(d) = (q - 1) \binom{n}{d} \). The first obvious codeword of nonzero weight has weight \( d^2 \). The next possible nonzero obvious weight is \( d(d + 1) \) which is \( w(d, d) \).

**Example 1**

Let us consider the \((7, 5, 3)\) RS code. The number of codewords of minimum weight is \( E_C(d) = 245 \). The complete IOWE function of \( C \) is equal to (this will be discussed in more detail in section \( \mathbb{[7]} \):

\[
\mathbb{O}_C(X, Y) = 1 + 35XY^3 + 140X^2Y^3 + 70X^3Y^3 + 350X^2Y^4 + 700X^3Y^4 + 175X^4Y^4 + 2660X^3Y^5 + 2660X^4Y^5 + 266X^5Y^5 + 9170X^4Y^6 + 3668X^5Y^6 + 12873X^5Y^7.
\]

Let \( P \) be the square product code \( P = C \times C \). The minimum distance of \( P \) is \( d_p = 9 \). By \( \Box \), its multiplicity is \( E_P(d_p) = 8575 \). By applying Theorem \( \Box \) the input-output weight enumerator for codewords in \( P \) with output weight \( d_p = 9 \) is given by

\[
\Lambda(\mathbb{O}_P(X, Y), Y^9) = 175X + 1400X^2 + 700X^3 + 2800X^4 + 2800X^5 + 700X^9.
\]

\[ (15) \]

By Corollary \( \Box \) there are no codewords in \( P \) with either weight 10 or 11. No information is available for larger codeword weights \( 12 \leq w \leq 49 \).

The following theorem shows that rank-one codewords of a product code maintain the multiplicity property.

**Theorem 2**

If the codes \( C \) and \( R \) have the multiplicity property and \( P = R \times C \) is their product code, then the subcode constituting of the rank-one codewords in \( P \) has this property.

\[ \Box \]
Proof: It follows from Th. 1 that, for $h \leq h_o$

\[ I_p(h) = \frac{1}{q-1} \sum_{w=1}^{k_c k_r} w \sum_{i|w} \sum_{j|h} O_R(i,j) O_C(w/i, h/j) \]

\[ = \frac{1}{q-1} \sum_{j|h} i \sum_{i=1}^{k_r} O_R(i,j) \sum_{t=1}^{k_c} t O_C(t, h/j) \]

\[ = \frac{1}{q-1} \sum_{j|h} E_R(j) E_C(h/j) \]

\[ = \frac{h}{n_p} E_p(h), \]

which proves the assertion.

IV. Average IOWE of Product Codes

In the previous section, we have shown how to exactly compute the product code IOWE, for low weight codewords. For higher codeword weights, it is very hard to find the exact enumerators for an arbitrary product code over $F_q$.

In this section, we will relax the problem of finding the exact enumerators, and we will focus on the computation of average weight enumerators over an ensemble of proper concatenated schemes. To do this:

1. We will represent a product code as a concatenated scheme with a row-by-column interleaver. Two representations will be introduced. The first one is the typical serial interpretation of a product code, while the second one is a less usual parallel construction.

2. We will replace the row-by-column interleavers of the schemes by uniform interleavers [15], acting as the average of all possible interleavers. To do this, we will introduce and discuss uniform interleavers for codes over $F_q$.

3. We will compute the average enumerator for these concatenated schemes, which coincide with the scheme enumerators if random interleavers were used instead of row-by-column ones.

A code constructed using a random interleaver is no longer a rectangular product code. However, as we shall see, the average weight enumerator gives a very good approximation of the exact weight enumerator of the product code. This will confirm the experimental results by Hagenauer et al. that the error performance of linear product codes did not differ much if the row-column interleaver is replaced with a random interleaver [3, Sec. IV B]. We also confirm that numerically in section VIII.

A. Representing a Product Code as a Concatenated Code

Let us first study the representation of a product code as a concatenated scheme with a row-by-column interleaver.

Construction 1

Given the code $R(n_c, k_c, d_c)$, the augmented code $R^{k_c}$ is obtained by independently appending $k_c$ codewords of $R$. The code $R^{k_c}$ has codeword length $k_c n_r$ and dimension $k_c k_r$. Moreover, its IOWE function is given by

\[ O_c^k(X, Y) \triangleq O_c(X, Y) = (O_c(X, Y))^k. \]  

(16)

See Fig. [1]. The encoding process may be viewed as if we are first generating a codeword of $R^{k_c}$, with length $k_c n_r$ symbols. The symbols of this codeword are read into an $k_c \times n_r$ matrix by rows and read out column by column. In other words, the symbols of the augmented codeword are interleaved
by a row-by-column interleaver. Each column is then encoded into a codeword in $C$. The augmented columns form a codeword in $P$ of length $n_r n_c$.

**Observation 1:** An $(n_r n_c, k_r n_c, d_r, d_c)$ product code $P = R \times C$ is the serial concatenation of an $(k_c n_r, k_c, r_c)$ outer code $R^{k_c}$ with an $(n_c n_r, k_c n_r)$ inner code $C^{n_r}$ through a row-by-column interleaver $\pi$ with length $N = k_c n_r$. (Equivalently $P = R \times C$ is the serial concatenation of an $(k_r n_c, k_r k_c)$ outer code $C^{k_r}$ with an $(n_c n_r, k_r n_c)$ inner code $R^{n_c}$ through a row-by-column interleaver with length $k_r n_c$ respectively.)

**Construction 2**

As an alternative, let the coordinates of a systematic product code be partitioned into four sets as shown in Fig. 2. We can introduce the following parallel representation.

**Observation 2:** See Figure 3. An $(n_r n_c, k_r n_c, d_r, d_c)$ product code can be constructed as follows:
1. Parallel concatenate the $(n_r k_c, k_c k_r)$ code $R^{k_c}$, with the $(n_c k_r, k_c k_r)$ code $C^{k_r}$ through a row-by-column interleaver $\pi_1$ of length $N_1 = k_c n_r$.
2. Interleave the parity symbols generated by $R^{k_c}$ with a row-by-column interleaver $\pi_2$ of length $N_2 = k_c (n_r - k_r)$.
3. Serially concatenate these interleaved parity symbols with the $(n_c (n_r - k_r), k_c (n_r - k_r))$ code $C^{n_r - k_r}$.

**B. Uniform Interleavers over $F_q$**

Given the two product code representations just introduced, we would like to substitute the row-by-column interleavers with uniform interleavers. In this section, we then investigate the uniform interleaver properties, when the interleaver is a symbol based interleaver and the symbols are in $F_q$. The concept of uniform interleaver was introduced in [15] and [16] for binary vectors in order to study turbo codes: it is a probabilistic object acting as the average of all possible interleavers of the given length. In the binary case, the number of possible permutations of a vector of length $L$ and Hamming weight $w$ is $\binom{L}{w}$. Let us denote by $V(L, w)$ the probability that a specific vector is output by the interleaver when a vector of length $L$ and input $w$ is randomly interleaved. In this binary case we have

$$V(L, w) = \frac{1}{\binom{L}{w}}.$$  \hspace{1cm} (17)

If $v$ is a vector on $F_q$ of length $L$ and the frequency of occurrence of the $q$ symbols is given by $l_0, l_1, \ldots, l_{q-1}$ respectively, then the number of possible permutations is given by the multinomial coefficient \[17\]

$$\frac{L!}{l_0! l_1! \ldots l_{q-1}!}.$$  \hspace{1cm} (18)

However, this requires the knowledge of the occurrence multiplicity of each of the $q$ symbols in the permuted vector.

We then introduce here the notion of uniform codeword selector (UCS). Let us suppose a specific vector of symbol weight $w$ and length $L$ is output from the interleaver corresponding to a certain interleaver input with the same weight. This vector is encoded by an $(N, L)$ code $C$ following the interleaver.

We assume that all the codewords of $C$ with input weight $w$ have equal probability of being chosen at the encoder’s output. The UCS picks one of these codewords (with input weight $w$) at random. Thus the probability that a specific codeword is chosen by the UCS is

$$V(L, w) = \frac{1}{\sum_h O_C(w, h)} = \frac{1}{(q - 1)^w \binom{L}{w}}.$$  \hspace{1cm} (18)

where $\sum_h O_C(w, h)$ is the total number of codewords with input weight $w$. This is equivalent to a uniform interleaver over $F_q$ which identifies codewords by their Hamming weights. It is noticed that
for the binary case, the uniform interleaver (17) is equivalent to the UCS (18). The UCS has the property of preserving the cardinality of the resulting concatenated code.

C. Computing the Average Enumerators

Construction 1

Given the Construction 1 of Obs. 1 and Fig. 1, let us replace the row-by-column interleaver \( \pi \) of length \( N = k_c n_r \) with a uniform interleaver over \( F_q \) of the same length. It is easy to show that the average IOWE function of the product code \( \mathcal{P} \) is given by

\[
\bar{O}_\mathcal{P}(X, Y) = \sum_{w=0}^{k_c n_r} V(k_c n_r, w) \Lambda \left( \bar{O}_\mathcal{R}(X, Y), Y^w \right) \Lambda \left( \bar{O}_\mathcal{C}^{n_r}(X, Y), X^w \right).
\]  

(19)

The average weight enumerator function \( \bar{E}_\mathcal{P}(Y) \) can be computed from \( \bar{O}_\mathcal{P}(X, Y) \) by applying (17).

Construction 2

Given the Construction 2 of Obs. 2 and Fig. 3, let us replace the two row-by-column interleavers \( \pi_1 \) of length \( N_1 = k_r k_c \) and \( \pi_2 \) of length \( N_2 = k_c (n_r - k_r) \), with two uniform interleaver overs \( F_q \) of length \( N_1 \) and \( N_2 \), respectively.

We begin by finding the partition weight enumerator (PWE) of the code \( \mathcal{D} \) resulting from the parallel concatenation of \( \mathcal{R}^{k_c} \) with \( \mathcal{C}^{k_r} \). We have:

\[
\bar{P}_{\mathcal{D}}(W, X, Y) = \sum_{w=0}^{k_c k_r} \sum_{x=0}^{k_c (n_r - k_r)} \sum_{y=0}^{k_c (n_r - k_r)} \bar{P}_{\mathcal{D}}(w, x, y) W^w X^x Y^y.
\]  

(21)

(Note that \( \bar{R}_{\mathcal{D}}(W, X) = \bar{P}_{\mathcal{D}}(W, X, X) \) gives the average IRWE function of a punctured product code with the checks on checks deleted.)

The partition weight enumerator function of the product code \( \mathcal{P} \) is then given by

\[
\bar{P}_{\mathcal{P}}(W, X, Y, Z) = \sum_{x=0}^{k_c (n_r - k_r)} V(k_c (n_r - k_r), x) \Lambda \left( \bar{R}_\mathcal{C}^{n_r-k_r}(X, Z), X^x \right) \Lambda \left( \bar{P}_{\mathcal{D}}(W, X, Y), X^x \right) Z^x.
\]  

(22)

The PWE, \( \bar{P}_{\mathcal{P}}(w, x, y, z) \), enumerates the codewords with a weight profile shown in Fig. 2 and is given by expanding the PWE function \( \bar{P}_{\mathcal{P}}(W, X, Y, Z) \) as follows,

\[
\bar{P}_{\mathcal{P}}(W, X, Y, Z) = \sum_{w=0}^{k_c k_r} \sum_{x=0}^{k_c (n_r - k_r)} \sum_{y=0}^{k_c (n_r - k_r)} \sum_{z=0}^{k_c (n_c - k_c)} \bar{P}_{\mathcal{P}}(w, x, y, z) W^w X^x Y^y Z^z.
\]  

(23)

It follows that the average IRWE function of \( \mathcal{P} \) is \( \bar{R}_{\mathcal{P}}(X, Y) = \bar{P}_{\mathcal{P}}(X, Y, Y, Y) \). Consequently, the IOWE function \( \bar{O}_{\mathcal{P}}(X, Y) \) can be obtained via (17) and the weight enumerator function \( \bar{E}_{\mathcal{P}}(Y) \) via (19). By using (19), the cardinality of the code given by \( \bar{E}_{\mathcal{P}}(Y) \) is preserved to be \( q^{k_c k_r} \).
V. Merging Exact and Average Enumerators into Combined Enumerators

The results in the previous section are now combined with those of section IV reflecting our knowledge of the exact IOWE of product codes for low weights. Let \( h_o \) be defined as in (8). We introduce a complete IOWE which is equal to:

- the exact IOWE for \( h < h_o \);
- the average IOWE for \( h \geq h_o \):

\[
\tilde{O}_P(X,Y) = \sum_{w=0}^{k_e k_r n_r n_c} \sum_{h=0}^{k_e k_r n_r n_c} \tilde{O}_P(w,h) X^w Y^h, \quad (24)
\]

such that

\[
\tilde{O}_P(w,h) = \begin{cases} O_P(w,h), & h < h_o; \\ O_P(w,h), & h \geq h_o, \end{cases}
\]

(25)

where \( O_P(w,h) \) is given by Th. 11 while \( \tilde{O}_P(w,h) = \Lambda(\tilde{O}_P(X,Y), X^w Y^h) \) is derived as in section IV-C. We will call \( \tilde{O}_P(X,Y) \) the combined input output weight enumerator (CIOWE) of \( P \). The corresponding combined weight enumerator function \( \tilde{E}_P(Y) \) can be computed by (6).

Let us now discuss some properties of the CIOWE. Let \( W(C) \triangleq \{ h : E_C(h) \neq 0 \} \) be the set of weights \( h \), such that there exists at least one codeword \( c \in C \) with weight \( h \). Observe that the weight of a product codeword \( p \in P \) is simultaneously equal to the sum of the row weights and to the sum of the column weights. We define an integer \( h \) a plausible weight of \( p \in P \), if \( h \) could be simultaneously partitioned into \( n_c \) integers restricted to \( W(R) \) and into \( n_r \) integers restricted to \( W(C) \).

Note however, that not all plausible weights are necessarily in \( W(P) \).

**Theorem 3**

Suppose \( P = R \times R \), (the row code \( R \) is the same as the column code \( C \)), then the set of weights with a non-zero coefficient in the average weight enumerator of \( P \) derived by either (10) or (22) are plausible weights for the product code.

**Proof:** The set of plausible weights of a product code is the set of weights \( h \) such the coefficients of \( Y^h \) in both \( (E_C(Y))^{n_r} \) and \( (E_R(Y))^{n_c} \) is non-zero. When \( R = C \), it suffices to show that for any weight \( h \) if the coefficient of \( Y^h \) in \( E_P(Y) \) is non-zero, then it is also non-zero in \( (E_C(Y))^{n_r} \).

For Construction 1, let \( \tilde{E}_P(Y) \) be the average weight enumerator derived from (10) by \( \tilde{E}_P(Y) = \tilde{O}_P(1,Y) \). Since all output weights that appear in \( \tilde{O}_P(1,Y) \) are obtained from \( \Lambda(\tilde{O}_C^{n_r}(X,Y), X^w) \) then, by (16), they have nonzero coefficients in \( (E_C(Y))^{n_r} \) and we are done.

For Construction 2, let \( \tilde{E}_P(Y) \) be the average weight enumerator derived from (22) by \( \tilde{E}_P(Y) = \tilde{P}_P(Y,Y,Y,Y) \). Let \( \Upsilon(W,Y) = \Lambda(\tilde{P}_D(W,X,Y), X^z) \). From (20), it follows that any exponent with a nonzero coefficient in \( \Upsilon(Y,Y) \) also has a non-zero coefficient in \( E_C^{k_r}(Y,Y) \) or equivalently \( E_C^{k_r}(Y) \). Similarly if \( \Upsilon'(X,Z) = \Lambda(\tilde{E}_C^{n_r-k_r}(X,Z), X^z) \) \( X^z \), then any exponent with a non-zero coefficient in \( \Upsilon'(Y,Y) \) also has a non-zero exponent in \( E_C^{n_r-k_r} \). It follows from (22) that any exponent with a non-zero coefficient in \( \tilde{E}_P(Y) \) also has a non-zero coefficient in \( E_C^{n_r-k_r} \) and we are done.

In [9], the authors approximated the weight enumerator of the product code by a binomial distribution for all weights greater than \( h_o \). Our approach has the advantage that only plausible weights appear in the combined enumerators of the product code.

VI. Split weight enumerators

As seen in the previous section, deriving the CIOWE of the product code requires the knowledge of the IRWE of the component codes. In this section we discuss the weight enumerators of some codes which are typically used for product code schemes. In particular, we show closed form formulas for the IRWE of Hamming, extended Hamming, Reed Solomon codes. To do this, it is sometimes easier.
to work with the split weight enumerator (SWE, see definition in section 13) of the dual code. The connection between the IRWE of a code and its dual was established in [13]. The following theorem gives a simplified McWilliams identity relating the SWE of a linear code with that of its dual code in terms of Krawtchouk polynomials.

**Theorem 4**

Let \( C \) be an \((n, k)\) linear code over \( F_q \) and \( C^\perp \) be its dual code. Let \( A(\alpha, \beta) \) and \( A^\perp(\alpha, \beta) \) be the SWEs of \( C \) and \( C^\perp \) respectively for an \((n_1, n_2)\) partition of their coordinates, then

\[
A^\perp(\alpha, \beta) = \frac{1}{|C|} \sum_{w=0}^{n_1} \sum_{v=0}^{n_2} A(w, v)K_\alpha(w, n_1)K_\beta(v, n_2),
\]

such that for \( \beta = 0, 1, \ldots, n_1, K_\beta(v, \gamma) = \sum_{j=0}^{\beta} \binom{\gamma - v}{\beta - j}(-1)^j(q - 1)^{\beta - j} \) is the Krawtchouk polynomial.

**Proof:** By a straightforward manipulation of the MacWilliams identity for the split weight enumerator [19] Ch. 5, Eq. 52 [20], it follows that for linear codes and \( r = q - 1 \),

\[
A^\perp(X, Y) = \frac{1}{|C|} (1 + rX)^{n_1}(1 + rY)^{n_2}A(X, Y) \left( \frac{1 - X}{1 + rX}, \frac{1 - Y}{1 + rY} \right) \]

\[
= \frac{1}{|C|} \sum_{w=0}^{n_1} \sum_{v=0}^{n_2} A(w, v)(1 - rX)^{w_1 - w}(1 - X)^w(1 - rY)^{w_2 - v}(1 - Y)^v,
\]

(26)

where \( A(X, Y) \) and \( A^\perp(X, Y) \) are the SWE functions of \( C \) and \( C^\perp \) respectively. Observing that for a positive integer \( \gamma \) and \( 0 \leq \beta \leq \gamma \), \( (1 - rY)^{\gamma - v}(1 - Y)^v = \sum_{\beta=0}^{\gamma} K_\beta(v, \gamma)Y^\beta \) is the generating function for the Krawtchouk polynomial [19] Ch. 5, Eq. 53 and that \( A^\perp(\alpha, \beta) \) is the coefficient of \( X^\alpha Y^\beta \) in the right-hand side of (26), the result follows.

By observing that the roles of the input and the redundancy are interchanged in the code and its dual, we have:

**Corollary 2**

The IRWEs of \( C \) and \( C^\perp \) are related by

\[
R^\perp(\alpha, \beta) = \frac{1}{|C|} \sum_{v=0}^{n_1} \sum_{w=0}^{n_2} R(w, v)K_\beta(w, n_1)K_\alpha(v, n - k).
\]

\(\nabla\)

A. Hamming Codes

The IRWE function of systematic Hamming codes could be derived by observing that they are the dual code of simplex codes [19], [21]. A recursive equation for evaluating the IRWE of Hamming codes was given in [22]. The following theorem gives a closed form formula for the IRWE function of Hamming codes in terms of Krawtchouk polynomials.

**Theorem 5**

The IRWE of \((2^m - 1, 2^m - m - 1, 3)\) (systematic) Hamming codes is

\[
R_H(\alpha, \beta) = \frac{1}{2^m} \left( \sum_{w=1}^{m} \binom{m}{w} K_\beta(w, m)K_\alpha(2^m - 1 - w, 2^m - m - 1) + \binom{m}{\beta} \left( \frac{2^m - m - 1}{\alpha} \right) \right).
\]

**Proof:** By observing that the IRWEF of the \((2^m - 1, m, 2^m - 1)\) simplex code is \( R_s(X, Y) = 1 + \sum_{w=1}^{m} \binom{m}{w} X^w Y^{2^m - 1 - w} \). Using Cor. 2 and observing that \( K_\beta(0, m) = \binom{m}{\beta} \), we obtain the result. \(\nabla\)
B. Extended Hamming Codes

Extended Hamming codes were studied in [10], where it was shown they possess the multiplicity property, and closed-form formulas for their input-output multiplicity were provided. The following theorem shows a closed expression for their IR WE function in terms of Krawtchouk polynomials.

**Theorem 6**
A closed form formula for the IR WE of the \((2^m, 2^m - 2^{-m} - 1, 4)\) Extended Hamming codes is

\[
R_{EH}(\alpha, \beta) = \frac{1}{2^{m+1}} \left( \sum_{w=1}^{m} \binom{m+1}{w} K_\beta(w, m+1) K_\alpha(2^{m-1} - w, 2^m - m - 1) \right. \\
\left. + \binom{m+1}{\beta} \binom{2^m - m - 1}{\alpha} (1 + (-1)^{\alpha+\beta}) \right).
\]

**Proof:** By observing that the extended Hamming codes are the duals of the \((2^m, m+1, 2^m-1)\) first order Reed Muller (RM) codes whose IRWE function could be shown to be \(R(X, Y) = 1 + X^{m+1}Y^{2^m-m-1} + \sum_{\alpha=1}^{m} \binom{m+1}{\alpha} X^\alpha Y^{2^{m-1}-\alpha}\). By Cor. 2 and observing that \(K_\beta(\gamma, \gamma) = \binom{\gamma}{\beta}(-1)^{\beta}\) the result follows. \(\Box\)

Note that the WE of extended Hamming (EH) codes could also be derived from that of Hamming (H) codes by using the well known relation [19], \(E_{EH}(h) = E_H(h) + E_H(h-1)\) if \(h\) is even and is zero otherwise. It follows that

\[
R_{EH}(\alpha, \beta) = \begin{cases} 
R_H(\alpha, \beta) + R_H(\alpha, \beta - 1), & \alpha + \beta \text{ is even;} \\
0, & \text{otherwise}
\end{cases}
\]  
(27)

C. Reed Solomon Codes

Reed Solomon codes are maximum distance separable (MDS) codes [19]. The SWE of MDS codes was recently studied by El-Khamy and McEliece in [14], where this theorem was proved:

**Theorem 7**
[14] The SWE of MDS codes is given by

\[
A^T(w_1, w_2) = E(w_1 + w_2) \binom{n_1}{w_1} \binom{n_2}{w_2} \binom{n}{w_1 + w_2}.
\]

It follows that the IRWE of an \((n, k)\) systematic RS code is given by:

\[
R_{RS}(\alpha, \beta) = E(\alpha + \beta) \binom{k}{\alpha} \binom{n-k}{\beta} \binom{n}{\alpha+\beta}.
\]

VII. Average Binary IRWE of product Reed Solomon Codes

Recently, new techniques for decoding Reed Solomon codes beyond half the minimum distance were derived in [23], and algebraic soft decision algorithms were proposed (see [24] and references therein). In this section we derive a number of results on RS product codes and their binary image.

Let \(d(X) = \sum_{i=0}^{k-1} d_i X^i\) be a data polynomial over \(F_q\). Then an \((n, k)\) Reed Solomon code is generated by evaluating the data polynomial \(d(X)\) at \(n\) distinct elements of the field forming a set called the support set of the code \(S = \{\alpha_0, \alpha_1, ..., \alpha_{n-1}\}\). The generated codeword is \(c = \{d(\alpha_0), d(\alpha_1), ..., d(\alpha_{n-1})\}\). The resulting code is not systematic. In the following theorem we show how a product of two RS codes can be generated by polynomial evaluation of a bivariate polynomial.
Theorem 8
Let the $k_r k_c$ data symbols $d_{i,j}$ be given by the bivariate polynomial $D(X, Y) = \sum_{i=0}^{k_r-1} \sum_{j=0}^{k_c-1} d_{i,j} X^i Y^j$.
Let the support set of the row and column codes, $\mathcal{R}$ and $\mathcal{C}$ respectively, be given by $S_r = \{\alpha_0, \alpha_1, ..., \alpha_n\}$ and $S_c = \{\beta_0, \beta_1, ..., \beta_n\}$ respectively. Then the product code $\mathcal{P} = \mathcal{R} \times \mathcal{C}$ is generated by $\mathcal{P}_{i,j} = D(\alpha_i, \beta_j)$ for $i = 0, ..., n_r - 1$ and $j = 0, ..., n_c - 1$.

\[
\text{Proof: } \quad \text{To prove that } \mathcal{P} \text{ is really the product of } \mathcal{R} \text{ and } \mathcal{C}, \text{ we prove that each row, } r, \mathcal{P}_{r,*} = \{P_{i,j} : i = \{0, ..., n_r - 1\} & j = r\} \text{ is a codeword in } \mathcal{R} \text{ and each column } c, \mathcal{P}_{*,c} = \{P_{i,j} : j = \{0, ..., n_c - 1\} & i = c\} \text{ is a codeword in } \mathcal{C}. \text{ The } r\text{-th row is given by } \mathcal{P}_{r,*} = \{D(\alpha_0, \beta_r), D(\alpha_1, \beta_r), ..., D(\alpha_n, \beta_r)\}. \text{ Observe that } D(\alpha_i, \beta_r) = \sum_{i=0}^{k_r-1} \sum_{w=0}^{k_c-1} d_{v,w}(\alpha_i) d_{v,w}(\beta_r) w = \sum_{w=0}^{k_c-1} \left( \sum_{v=0}^{k_r-1} d_{v,w}(\beta_r) w \right) (\alpha_i)^v. \text{ Let } \gamma_r^v = \sum_{w=0}^{k_c-1} d_{v,w}(\beta_r) w \text{ and } \gamma_r^v = \{\gamma_r^v : v = 0, ..., k_r - 1\}, \text{ then } \gamma_r^v \text{ forms the information vector which is encoded into the } r\text{-th row } \mathcal{P}_{r,*} = \{D'(\alpha_0), D'(\alpha_1), ..., D'(\alpha_n)\} \text{ by } D'(X) = \sum_{i=0}^{k_r-1} \gamma_i^r X^i. \text{ This proves that } \mathcal{P}_{r,*} \text{ is a codeword in } \mathcal{R}.

Similarly, any column $c$ could be expressed as $\mathcal{P}_{c,*} = \{D''(\beta_0), D''(\beta_1), ..., D''(\beta_n)\}$, where $D''(X) = \sum_{j=0}^{k_c-1} \delta_j^c X^j$ and $\delta_j^c = \sum_{i=0}^{k_r-1} d_{i,j}(\alpha_c)^i$ is the $j$-th element in the information vector for column $c$. Thus each column is a codeword in $\mathcal{C}$.

Since the cardinality of this code is $q^{k_r k_c}$ we are done.

(\text{This proof also gives insight how algebraic soft decision decoding can be used for iteratively decoding the component RS codes of the product code.})

Given the product of Reed Solomon codes defined over a field of characteristic two, it is often the case that the binary image of the code is transmitted over a binary-input channel. The performance would thus depend on the binary weight enumerator of the component RS codes, which in turn depends on the basis used to represent the $2^m$-ary symbols as bits. Furthermore, it is very hard to find the exact binary weight enumerator of a RS code for a specific basis representation \cite{25}. The average binary image of a class of generalized RS codes has been studied in \cite{26}. The average binary image for codes, defined over finite fields of characteristic two, was derived by assuming a binomial distribution of the bits in the non-zero symbols in \cite{27}. Let $\mathcal{C}_b$ denote the binary image of an $(n, k)$ code $\mathcal{C}$ which is defined over the finite field $F_{2^m}$. Let $E_C(Y)$ be the weight enumerator function of $\mathcal{C}$. Then the average weight enumerator of the $(nm, km)$ code $\mathcal{C}_b$ is given by \cite{27}

$$E_{C_b}(Y) = E_C(E(Y)), \quad (28)$$

where $E(Y) = \frac{1}{2^{m-1}}((1 + Y)^m - 1)$ is the generating function of the bit distribution in a non-zero symbol. We assume that the distribution of the non-zero bits in a non-zero symbol follows a binomial distribution and that the non-zero symbols are independent. If the coordinates of the code $\mathcal{C}$ are split into $p$ partitions, then there is a corresponding $p$-partition of the coordinates of $\mathcal{C}_b$, where each partition in $\mathcal{C}_b$ is the binary image of a partition in $\mathcal{C}$. By independently finding the binary image of each partition, the average partition weight enumerator of $\mathcal{C}_b$ could be derived as in the following lemma.

Lemma 1
\cite{14}Let $\mathcal{P}(W, X, Y, Z)$ be the PWE function of a code $\mathcal{P}$ defined over $F_{2^m}$. The average PWE of the binary image $\mathcal{P}_b$ is $\mathcal{P}_b(W, X, Y, Z) = \mathcal{P}(\Psi(W), \Psi(X), \Psi(Y), \Psi(Z))$.

\[
\n
\text{Corollary 3}
\n\text{If } \mathcal{P}_b(X, Y) \text{ is the combined IRWE of the } (n_p, k_p) \text{ product code $\mathcal{P}$ defined over $F_{2^m}$, then the combined IRWE of its binary image is}

$$\tilde{\mathcal{P}}_b(X, Y) = \tilde{\mathcal{P}}_b(\Psi(X), \Psi(Y)),$$

where

$$\Psi(X) = \frac{1}{2^{m-1}}((1 + X)^m - 1)$$
and
\[
\tilde{\mathbb{E}}_{P_b}(X, Y) = \sum_{x=0}^{k_p m} \sum_{y=0}^{n_p m-k_p m} R_{P_b}(x, y) X^x Y^y.
\]

Note this same formula does not hold in the case of the IOWE. However, the binary IOWE could be derived from the binary IRWE by using \((\text{5})\).

VIII. Numerical Results

In this section we show some numerical results supporting our theory. The combined input output enumerators of some product codes are investigated in section \(\text{VIII-A}\). Analytical bounds to ML performance are computed and discussed in section \(\text{VIII-B}\). Hamming codes, extended Hamming codes and Reed Solomon codes are considered as constituent codes.

A. Combined Input Output Weight Enumerators

Example 2
Let us consider the extended Hamming code \((8,4)\). From Th. \(\text{6}\) its IOWE function is
\[
\mathbb{O}_{EH}(X, Y) = 1 + 4XY^4 + 6X^2Y^4 + 4X^3Y^4 + X^4Y^8.
\]

Let us now study the square product code \((8,4)^2\). By applying \((\text{19})\) we can derive the average weight enumerator function obtained with the serial concatenated representation. By rounding to the nearest integer, we obtain:
\[
\mathbb{E}_{P}(Y) = 1 + 3Y^8 + 27Y^{12} + 107Y^{16} + 604Y^{20} + 3153Y^{24} + 13653Y^{28} + 30442Y^{32} + 13653Y^{36} + 3153Y^{40} + 604Y^{44} + 107Y^{48} + 27Y^{52} + 3Y^{56} + Y^{64}.
\]

By \((\text{22})\), we can derive the average weight enumerator function obtained with the parallel concatenated representation. We obtain:
\[
\mathbb{E}_{P}(Y) = 1 + 2Y^8 + 26Y^{12} + 98Y^{16} + 568Y^{20} + 3116Y^{24} + 13780Y^{28} + 30353Y^{32} + 13780Y^{36} + 3116Y^{40} + 568Y^{44} + 98Y^{48} + 26Y^{52} + 2Y^{56} + Y^{64}.
\]

(For space limitations we do not show the IOWE functions.) Note that all codewords are of plausible weights as expected from Th. \(\text{3}\). It could be checked that in both cases, the cardinality of the code (without rounding) is preserved to be \(2^{16}\). In general, the parallel representation gives more accurate results than the serial one, and will be used for the remaining results in this paper.

For low-weight codewords, we can compute the exact IOWE. By Th.\(\text{11}\) the exact IOWE of the product code for weights less than \(h_o = 24\) is equal to
\[
\mathbb{O}_{P}(X, Y) = 1 + 16XY^{16} + 48X^2Y^{16} + 32X^3Y^{16} + 36X^4Y^{16} + 48X^6Y^{16} + 16X^9Y^{16}.
\]

It follows that the combined weight enumerator function for this product code is
\[
\tilde{\mathbb{E}}_{P}(Y) = 1 + 196Y^{16} + 3116Y^{24} + 13781Y^{28} + 30353Y^{32} + 13781Y^{36} + 3116Y^{40} + 568Y^{44} + 98Y^{48} + 26Y^{52} + 2Y^{56} + Y^{64}.
\]
A symmetric weight enumerator of the component codes implies a symmetric one for the product code. Thus, by the knowledge of the exact coefficients of exponents less than 24, $\tilde{E}_p(Y)$ could be improved by setting the coefficients of $Y^{54}, Y^{52}$ and $Y^{56}$ to be zero and adjusting the coefficients of the middle exponents such that the cardinality of the code is preserved. We obtain:

$$\tilde{E}_p'(Y) = 1 + 196Y^{16} + 3164Y^{24} + 13995Y^{28} + 30824Y^{32} + 13995Y^{36} + 3164Y^{40} + 196Y^{48} + Y^{64}, \quad (30)$$

In this case, the exact weight enumerator can be found by exhaustively generating the 65536 codewords of the product code, and it is equal to:

$$E_p(Y) = 1 + 196Y^{16} + 4704Y^{24} + 10752Y^{28} + 34230Y^{32} + 10752Y^{36} + 4704Y^{40} + 196Y^{48} + Y^{64}.$$ 

It can be verified that the combined weight enumerator (30) gives a very good approximation of this exact weight enumerator.

Example 3
The combined weight enumerator of the extended Hamming product code $(16,11)^2$, computed by applying (20) and (22), is depicted in Fig. 4. It is observed that for medium weights, the distribution is close to that of random codes, which is given by

$$E(w) = q^{-(n_p-k_p)} \left( \begin{array}{c} n_p \\ w \end{array} \right) (q-1)^w,$$

except that only plausible weights exist.

Example 4
The combined symbol weight enumerator of the $(7,5)^2$ product RS codes over $F_8$, computed by applying (20) and (22), is shown in Fig. 5. It can be observed that the weight enumerator approaches that of a random code over $F_8$ for large weights. The average binary weight enumerator of the $(147,75)$ binary image, obtained by applying Corollary 3, is shown in Fig. 6. It is superior to a random code at low weights and then, as expected, approaches that of a binary random code.

B. Maximum Likelihood Performance

In this section, we investigate product code performance. The combined weight enumerators are used to compute the Poltyrev bound [11], which gives tight analytical bounds to maximum likelihood performance at both high and low error rates. For proper comparison, truncated union bound approximation and simulation results are also considered.

Example 5
The codeword error rate (CER) and the bit error rate (BER) performance of two Hamming product codes ($(7,4)^2$ and $(31,26)^2$) are shown in Fig. 7. We have depicted:

- The Poltyrev bounds on ML performance (P on the plots), obtained by using the combined weight enumerator computed via (22).
- The truncated union bound (L on the plots), approximating the ML performance at low error rates, and computed from the minimum distance term via (7).
• The simulated performance of iterative decoding (S on the plots), corresponding to 15 iterations of
the BCJR algorithm on the constituent codes trellises ([8], [10]).
By looking at the results, we can observe that:
• The combined weight enumerators derived in this paper, in conjunction with the Poltyrev bound,
provide very tight analytical bounds on the performance of maximum likelihood decoding also at low
SNRs (where the truncated union bound does not provide useful information).
• For the (7, 4)^2 code the exact enumerator can be exhaustively computed, and the exact Poltyrev
bound is shown in the figure. It is essentially identical to the bound computed with the combined
weight enumerator.
• The ML analytical bounds provide very useful information also for iterative decoding performance.
In fact, the penalty paid by iterative decoding with respect to ideal ML decoding is very limited, as
shown in the figure (feedback coefficients for weighting the extrinsic information and improve iterative
decoding has been employed, as explained in [10]).

Example 6
The performance of the extended Hamming product code (32, 26)^2 is investigated in Fig. 8. Also in
this case, the tightness of the bounds is demonstrated, for both the CER and the BER. With the aid
of the Poltyrev bound for the BSC channel, hard ML bounds have also been plotted. It is shown that
soft ML decoding on the AWGN channel offers more than 2 dB coding gain over hard ML decoding.

Example 7
In Fig. 9, the performance of soft and hard ML decoding of various Hamming and extended Hamming
codes are studied and compared. As expected, the EH product codes show better performance than
Hamming product codes of the same length due to their larger minimum distance and lower rate. (For
the (7, 4)^2 Hamming product code and the (8, 4)^2 extended Hamming product code, it is observed
that the bounds using our combined weight enumerator overlapped with ones using the exact weight
enumerators, which can be calculated exhaustively in these cases.)

Example 8
The performance of the binary image of some Reed Solomon product codes, for both soft and hard
decoding, are investigated in Fig. 10 where the Poltyrev bound has been plotted. As expected, soft
decoding has about 2 dB of gain over hard decoding. It can be observed that these product codes have
good performance at very low error rates (BER lower than 10^-9), where no error floor appears.

It is well known that the sphere packing bound provides a lower bound to the performance achiev-
able by a code with given code-rate and codeword length [28]. The discrete-input further limitation
occurring when using a given PSK modulation format was addressed in [29]. The distance of the code
performance from this theoretical limit can be used an indicator of the code goodness.

Let us consider, for example, the (15, 11)^2 RS product code, corresponding to a (900, 484) binary
code. By looking at the Poltyrev bound plotted in Fig. 10 this code achieves a BER=10^-10 for a
signal-to-noise ratio γ ≃ 2.2 dB. By computing the PSK sphere packing bound for this binary code,
we obtain a value of about 1.9 dB for BER=10^-10. This means that this RS product code is within
0.3 dB from the theoretical limit, which is a very good result at these low error rates.

IX. Conclusions
The average weight enumerators of product codes were studied in this paper. The problem was
relaxed by considering proper concatenated representations, and assuming random interleavers over
F_q instead of row-by-column interleavers. The exact IOWE for low-weight codewords were also derived
by extending Tolhuizen results. By combining exact values and average values, a complete combined
weight enumerator was computed. This enables us to study the ML performance of product codes at both low and high SNRs by applying the Poltyrev bound. The computation of average enumerators requires knowledge of the constituent code enumerators. Closed form formulas for the input redundancy enumerators of some popular codes were shown. The binary weight enumerator of ensemble of product RS codes was also derived.

The combined weight enumerators of Hamming and Reed Solomon product codes were numerically computed and discussed. Using the combined enumerators, tight bounds on the ML performance of product codes over AWGN channels were derived by using the Poltyrev bounds. The tightness of the bounds were demonstrated by comparing them to both truncated union bound approximations and simulation results.

In particular, Reed Solomon product codes show excellent performance. Reed Solomon codes are widely used in wireless, data storage, and optical systems due to their burst-error correction capabilities. The presented techniques allow to analytically estimated Reed Solomon product codes performance, and show they are very promising as Shannon-approaching solutions down to very low error rates without error floors. This suggests the search for low-complexity soft decoding algorithms for Reed Solomon codes as a very important research area in the near future.

ACKNOWLEDGMENT

The authors are grateful to Robert J. McEliece for very useful discussions.

REFERENCES

[1] P. Elias, “Error-free coding,” IRE Trans. Inform. Theory, vol. IT-4, pp. 29–37, Sept 1954.
[2] C. Berrou and A. Glavieux, “Near-optimum correcting code and decoding: Turbo codes,” IEEE Trans. Commun., vol. 44, pp. 1261–1271, Oct. 1996.
[3] J. Hagenauer, E. Offer, and L. Papke, “Iterative decoding of binary block and convolutional codes,” IEEE Trans. Inform. Theory, vol. 42, pp. 129–149, Mar 1996.
[4] R. Pyndiah, “Near optimum decoding of product codes: block turbo codes,” IEEE Trans. Commun., vol. 46, no. 8, pp. 1003–1010, August 1998.
[5] S.A. Hirst, B. Honary, and G. Markarian, “Fast Chase Algorithm with an Application in Turbo Decoding”, IEEE Trans. Communications, pp. 1693–1699, Oct. 2001.
[6] C. Argon and S. W. McLaughlin, “An efficient chase decoder for turbo product codes,” IEEE Trans. Commun., vol. 52, no. 6, pp. 896–898, June 2004.
[7] L. Tolhuizen, “More results on the weight enumerator of product codes,” IEEE Trans. Inform. Theory, vol. 48, no. 9, pp. 2573–2577, Sep. 2002.
[8] M. El-Khamy, “The average weight enumerator and the maximum likelihood performance of product codes,” in International Conference on Wireless Networks, Communications and Mobile Computing, WirelessCom Information Theory Symposium, June 2005.
[9] L. Tolhuizen, S. Baggen, and E. Hekstra-Nowacka, “Union bounds on the performance of product codes,” in Proc. of ISIT 1998, Cambridge, MA, USA, 1998.
[10] F. Chiaraluce and R. Garello, “Extended hamming product codes analytical performance evaluation for low error rate applications,” IEEE Trans. on Wireless Commun., vol. 3, pp. 2353–2361, Nov. 2004.
[11] G. Poltyrev, “Bounds on the decoding error probability of binary linear codes via their spectra,” IEEE Trans. Inform. Theory, vol. 40, no. 4, pp. 1284–1292, July 1994.
[12] D. Divsalar, “A simple tight bound on error probability of block codes with application to turbo codes,” TMO Progress Report, NASA/JPL, Tech. Rep. 42–139, 1999.
[13] I. Sason, S. Shamai, and D. Divsalar, “Tight exponential upper bounds on the ML decoding error probability of block codes over fully interleaved fading channels,” IEEE Trans. Commun., vol. 51, no. 8, pp. 1296–1305, Aug. 2003.
[14] M. El-Khamy and R. J. McEliece, “The partition weight enumerator of MDS codes and its applications,” in 2005 IEEE International Symposium on Information Theory, Adelaide, Australia, Sept 2005.
[15] S. Benedetto and G. Montorsi, “Unveiling turbo codes: Some results on parallel concatenated coding schemes,” IEEE Trans. Inform. Theory, vol. 42, no. 3, pp. 409–428, Mar. 1996.
[16] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, “Serial concatenation of interleaved codes: Design and iterative decoding,” IEEE Trans. Inform. Theory, vol. 44, no. 3, pp. 909–926, May 1998.
[17] J. H. van Lint and R. M. Wilson, A Course in Combinatorics, 2nd ed. Cambridge: Cambridge U. Press, 2001.
[18] C. Weiβ, C. Bettstetter, and S. Riedel, “Code Construction and Decoding of Parallel Concatenated Tail-Biting Codes”, IEEE Trans. Inform. Theory, pp. 366–386, Jan. 2001.
[19] F. J. MacWilliams and N. J. Sloane, The Theory of Error Correcting Codes. Amsterdam: North Holland, 1977.
[20] T.-Y. Hwang, “A relation between the row weight and column weight distributions of a matrix,” IEEE Trans. Inform. Theory, vol. 27, no. 2, pp. 256 – 257, Mar. 1981.
[21] H. feng Lu, P. V. Kumar, and E. hui Yang, “On the input-output weight enumerators of product accumulate codes,” IEEE Commun. Lett., vol. 8, no. 8, Aug 2004.
[22] I. Sason and S. Shamai, “Bounds on the error probability for block and turbo-block codes,” *Annals of Telecommunications*, vol. 54, no. 3-4, pp. 183-200, March-April 1999.

[23] V. Guruswami and M. Sudan, “Improved decoding of Reed-Solomon codes and algebraic geometry codes,” *IEEE Trans. Inform. Theory*, vol. 45, no. 6, pp. 1757-1767, Sept. 1999.

[24] M. El-Khamy and R. J. McEliece, “Interpolation multiplicity assignment algorithms for algebraic soft-decision decoding of Reed-Solomon codes,” *AMS-DIMACS volume on Algebraic Coding Theory and Information Theory*, vol. 68, 2005.

[25] T. Kasami and S. Lin, “The binary weight distribution of the extended \((2^m, 2^m - 4)\) code of the Reed Solomon code over \(\text{GF}(2^m)\) with generator polynomial \((x - \alpha)(x - \alpha^2)(x - \alpha^3)\).” *Linear Algebra Appl.*, pp. 291-307, 1988.

[26] C. Retter, “The average binary weight enumerator for a class of generalized Reed-Solomon codes,” *IEEE Trans. Inform. Theory*, vol. 37, no. 2, pp. 346-349, March 1991.

[27] M. El-Khamy and R. J. McEliece, “Bounds on the average binary minimum distance and the maximum likelihood performance of Reed Solomon codes,” in *42nd Allerton Conf. on Communication, Control and Computing*, 2004.

[28] A. Valembois and M. Fossorier, “Sphere-Packing Bounds Revisited for Moderate Block Lengths,” *IEEE Trans. Inform. Theory*, vol. 50, no. 12, pp. 2998-3014, Dec. 2004.

[29] G. Beyer, K. Engdahl, and KSh. Zigangirov, “Asymptotic analysis and comparison of two coded modulation schemes using PSK signaling - Part I,” *IEEE Trans. Inform. Theory*, vol. 47, no. 7, pp. 2782-2792, Nov. 2001.
Fig. 1
Construction 1: serial concatenation.
The four set partition of the coordinates of a product codeword used in Construction 2.

|          | $k_r$                             | $n_r-k_r$                           |
|----------|-----------------------------------|-------------------------------------|
| $k_c$    | Information, Weight=$w$          | Checks on Rows, Weight=$x$          |
| $n_c-k_c$| Check on Columns, Weight=$y$     | Checks on Checks, Weight=$z$        |

A product codeword of weight $h = w + x + y + z$
Fig. 3

Construction 2: parallel concatenation.
The combined weight enumerator of the $(16, 11)^2$ extended Hamming product code is compared with that of a random binary code of the same dimension.
The combined symbol weight enumerator of the \((7,5)^2\) Reed Solomon product code is compared with that of a random code over \(F_8\) with the same dimension.
The combined binary weight enumerator of the binary image of the \((7,5)^2\) Reed Solomon product codes is compared with that of a random binary code with the same dimension.
Performance of Hamming product codes

CER and BER performance of some Hamming product codes for soft decoding over AWGN channel. The Poltyrev bound $P$, and the truncated union bound approximation $L$, are compared to simulated performance of iterative decoding $S$. For the $(7,4)^2$ code, the Poltyrev bound computed with the exact weight enumerator is also reported.
Fig. 8

CER and BER performance of the $(32, 26)^2$ extended Hamming product code for Soft and Hard decoding over AWGN channel. The Poltyrev bound $P$ and the truncated union bound approximation $L$ are compared to simulated performance of iterative decoding $S$. 
BER performance of Hamming product codes (upper) and extended Hamming product codes (lower) over AWGN channel for both soft decision (SD) and (HD) hard decision. The Poltyrev upper bound (UB) and the truncated union bound approximation (LB) are used for SD, while the Poltyrev bound for the BSC is used for HD.
BER performance of some Reed Solomon product codes over the AWGN channel for both soft decision (SD) and hard decision (HD) decoding, obtained by plotting the Poltyrev bound computed via the combined weight enumerators.