As known, many key features of continuous-time chaotic dynamics can be treated in terms of two-dimensional or even one-dimensional discrete-time maps (see e.g. [1]). Among them, most simple and perfect ones are the famous Bernoulli map, $X_{n+1} = 2X_n \mod 1$, and the tent map, $X_{n+1} = 1 - |1 - 2X_n|$, both defined at the interval $0 \leq X \leq 1$. Almost any trajectory of both maps is chaotic, that is dense in all this interval.

In computer, however, any trajectory finishes at zero after $\leq D$ steps only, where $D$ is number of bits under use. This contraction of the discretized phase space, from total $N = 2^D$ points to single point $X = 0$, happens because the maps are non-invertible, i.e. relation between $X_n$ and $X_{n+1}$ is not an one-to-one correspondence. But a suitable slight distortion can make a discretized map invertible. Evidently, any such map is nothing but a permutation of invertible. Evidently, any such map is nothing but a permutation of invertible. Evidently, any such map is nothing but a permutation of invertible. Evidently, any such map is nothing but a permutation of invertible. Evidently, any such map is nothing but a permutation of invertible.

In particular, the invertible discrete versions of the Bernoulli map describe the Mersenne digital auto-generators of pseudo-random 0-1-sequences (see e.g. [2]). For example, the Mersenne map in Fig.1 represents transitions between 16 states of a particular “toy” 4-bit Mersenne generator ($D = 4$). The general formula of such maps is

$$
\begin{align*}
Y &= 2X + 1 - \sigma(X) , \quad X < N/2 ; \\
Y &= 2X - N + \sigma(2X - N) , \quad X \geq N/2 ,
\end{align*}
$$

(1)

where $Y \equiv NX_{n+1}$, $X \equiv NX_n$, and any of $\sigma(k)$, $k = 0, 1, ..., N/2 - 1$, equals to either zero or unit, $\sigma(k) = 0, 1$ (hence this is class of $2^{N/2}$ different maps). To see corresponding unitary evolution matrix, it is necessary to turn the map picture headfirst and insert zeros and ones in place of the empty cells and crosses, respectively.

The map in Fig.1 has two immovable points ($X = 0$ and $X = 15$) and two cycles (periodic orbits) with equal lengths $L = 7$. In applications of much greater Mersenne maps (with $D \sim 70$ [2]), the main issue is construction of those having most long cycles, with $L \sim N$.

![FIG. 1: Example of the discrete Bernoulli map (the Mersenne map, see body text).](image)

Here, we will be interested in modest problem about statistics of cycles of discrete maps, to be concrete, definite class of discrete tent maps.

II. CYCLES OF CONTINUOUS TENT MAPS

First consider the variety of generally asymmetric continuous tent maps (CTM) described by

$$
\begin{align*}
X_{n+1} &= X_n/a , \quad X \leq a ; \\
X_{n+1} &= (1 - X_n)/(1 - a) , \quad X > a ,
\end{align*}
$$

(2)

where $0 < a < 1$. The inversion of (2) reads as

$$
X_{n-1} = S_n + (a - S_n)X_n , \quad S_n = 0 \text{ or } 1 ,
$$

(3)

with $S_n = 0$ and $S_n = 1$ representing two variants of the inversion. Let us write, with the help of (3), $X_0$ as a function of $X_L$ and then equate $X_0$ to $X_L$ at various possible values $S_1, ..., S_L = 0, 1$. Thus we obtain $2^L$ linear equations which determine $2^L$ points belonging to all possible periodic orbits (cycles) of length $L$. That are either irreducible (indivisible) orbits or, if $L$ has a...
Two examples of discrete tent maps at $D = 4$ and $a = 1/2$ (left) and $a = 1/3$ (right).

In particular, if $L$ is a prime number then (4) yields

$$N_l = \frac{2^L - 2}{L} \left( L - \left\lfloor \frac{L}{2} \right\rfloor \right),$$

(5)

where $N_l$ is number of different irreducible periodic orbits with length $l$, and the sum is taken over all the dividers of $L$.

In general, it is not hard to derive from (4) that

$$2^L - 2 \left( 2^{\left\lfloor L/2 \right\rfloor} - \left\lfloor L/2 \right\rfloor \right) \leq LN_l \leq 2^L - 2$$

(6)

Here $\left\lfloor L/2 \right\rfloor = L/2$ if $L$ is even, and $\left\lfloor L/2 \right\rfloor = (L - 1)/2$ if $L$ is odd. At any $L \gg 1$, according to (5),

$$N_L \approx \frac{2^L}{L}$$

(7)

At relatively small $L$, from (4) one finds: $N_2 = 1$, $N_3 = 2$, $N_4 = 3$, $N_5 = 6$, $N_6 = 9$, $N_7 = 18$, $N_8 = 30$, $N_9 = 56$, $N_{10} = 99$, $N_{11} = 186$, ...

III. DISCRETE TENT MAPS

Next, consider invertible discrete tent maps (DTM) representing discrete analogues of the continuous tent maps (2). To be concrete, let us introduce the class of discrete maps defined by

$$Y = \left\lfloor \frac{AX}{N} \right\rfloor, \quad X < A;$$

$$Y = \left\lfloor \frac{N(N - X)}{N - A} \right\rfloor - 1, \quad X \geq A$$

(8)

Here, like in (2), $X = NX_n$ and $Y = NX_{n+1}$ are integers, $0 \leq X, Y \leq N - 1$; $A$ is analogue of $a$ in (2), $0 \leq A \leq N - 1$; $\lfloor x \rfloor$ is an integer closest to $x$ from below (with $\lfloor x \rfloor = x$ if $x$ is integer), and $\{x\}$ means closest integer greater than (or equal to) $x$ (that is $\{x\} = x$ if $x$ is integer). Two examples of maps defined by formula (5), at $D = 4$, are shown in Fig. 2.

Let us prove that (5) prescribes invertible maps, i.e. establishes an one-to-one correspondence between $X$ and $Y$. It is sufficient to prove that the upper r.h.s. and lower r.h.s. in (5) can not produce one and the same number. Indeed, in such a case we would have

$$N X' / A = Y = \{N(N - X'' - (N - A))\} - 1,$$

with some $X' < A$ and $X'' \geq A$. This would mean that

$$N X' = A Y + u, \quad 0 \leq u < A,$$

$$N(N - X'') = (N - A) Y + v, \quad 0 < v \leq N - A,$$

($u$ and $v$ are some integers), therefore,

$$N + X' - X'' = Y + (u + v)/N$$

$$0 < u + v < N$$

But, obviously, the latter inequality contradicts the preceding equality. The proof is finished.

It is easy to see also, that absolute value of a deviation of any DTM (5) from its CTM prototype (2), with $a = A/N$, (as well as deviation of (4) from Bernoulli map) does not exceed 1 lowest bit only ($\pm 1/N$, in terms of $X_n$). In this sense, any DTM tends to corresponding CTM, when $N \to \infty$. However, the inverted map remains strongly discontinuous, which is a payment for its unambiguity.

Hence, in essence, the limit of DTM at $N \to \infty$ does not coincide with corresponding CTM.
IV. CYCLES OF DISCRETE TENT MAPS

Naturally, there is no simple rule for the cycles (periodic orbits) of the DTMs. A number of various irreducible cycles and lengths of these cycles are extremely irregular functions of $A$ and $N$. In contrary to (4), now

$$\sum_{1 \leq L \leq N} L N_L = N = 2^D ,$$

where $N_L$ is a number of different irreducible periodic orbits with length $L$ (possibly, $N_L = 0$). For example, at $D = 12$ and some three next values of $A$ this expansion looks as

$$1 + 4095 = N ,$$
$$1 * 2 + 2 * 2047 = N ,$$
$$1 + 13 * 315 = N ,$$

where the second multiplier (if any) in each term is number of different cycles of a length represented by the first multiplier. In other example, for $D = 15$,

$$1 * 2 + 2 * 4 * 3 + 8 * 30 + 16 * 2032 = N \ (A = N/2) ,$$
$$1 + 32767 = N \ (A = N/2 - 1) ,$$
$$1 * 2 + 2 + 4 * 8 * 5 + 16 * 65 + 24 * 2 + 32 + 48 * 9 +$$
$$+ 60 * 2 + 64 + 72 + 103 + 112 * 20 + 120 + 128 * 10 +$$
$$+ 144 + 176 + 192 * 6 + 224 * 4 + 240 + 1200 +$$
$$+ 1570 + 8792 + 12999 = N \ (A = N/2 + 1) .$$

Typically, a DTM have a long cycle whose length is comparable with the total number of points, $N$, i.e. maximal possible length. This is illustrated by Fig.3 relating to $D = 11$. Quite similar pictures take place also at greater $D$. We see that practically any DTM has a cycle with length $L \gtrsim N/4 \div N/3$. Such long cycles, whose length $L$ is comparable with $N$, can be treated as discrete analogues of the chaotic trajectories of CTM.

The left plot in Fig.3 demonstrates highly irregular dependence of the maximal cycle length, $L_{\text{max}}$, on the asymmetry parameter $A$. At the same time, the right-hand plot there shows that even slight smoothing of the data produces rather regular results. Therefore, it is reasonable to describe the cycles of DTMs in statistical language, considering some specific subclasses of DTMs instead of individual maps.

V. STATISTICS OF CYCLES OF SYMMETRIC DISCRETE TENT MAPS

Let us consider the family of nearly (asymptotically) symmetric DTMs determined by the conditions

$$\frac{N - M}{2} \leq A < \frac{N + M}{2} ; \ N, M \to \infty ; \ \frac{M}{N} \to 0$$

Since $M \to \infty$, we obtain the growing statistical ensemble, which is all the more representative one because all the maps defined by (10) do tend to the same limit $A/N \to a = 1/2$.

It would be interesting to investigate such the family of asymptotically symmetric discrete tent maps (ASDTM), in comparison with the usual symmetric CTM. Here, simplest statistical characteristics of cycles (periodic orbits) of the ASDTM will be under our attention.

Let $W(L)$ designates a density of probability distribution of the cycle lengths in this family of maps. In other words, practically,

$$W(L) = N_L / \sum_{l=1}^{N} N_l ,$$

FIG. 3: (On the left) The length, $L_{\text{max}}$, of most long periodic orbit of the discrete tent map, as a function of the asymmetry parameter of the map, $A$, at $N = 2048 \ (D = 11)$. (On the right) The corresponding histogram of $L_{\text{max}}/N$. 
FIG. 4: (Left plot) The probability, $P(L)$, that an arbitrarily chosen point belongs to some periodic orbit (of the nearly symmetric discrete tent maps (10)) with lengths less than or equal to $L$, as a function of $L/N$, at $D = 14$ and $M \sim 200$. (Right plot) The binary logarithm of the probability, $W(L)$, that arbitrarily chosen periodic orbit (of a similar maps family) has the length $L$, via binary logarithm of $L$, at $D = 15$ and $M \sim 500$, in comparison with the analytical estimate of the $W(L)$ dependence (the straight line). The inset demonstrates the same comparison for small lengths.

Where $N_L$ is now summary number of periodic orbits of length $L$ in all the maps of the ASDTM family. It is useful to introduce also the quantity

$$P(L) = \frac{1}{MN} \sum_{l=1}^{L} lN_l,$$  \hspace{1cm} (12)

that is relative (probability) measure of points which belong to all periodic orbits with lengths $\leq L$ at all the maps.

In reality, with the help of an ordinal PC only, it would take rather long time to obtain all the periodic orbits if $D \geq 16 \div 17$. But our computations, performed at $D = 12 \div 16$, showed that already $D = 13 \div 14$ are satisfactory values, because next $D$’s increases do not change the picture qualitatively (although, of course, providing better numeric accuracy).

Therefore, it is reasonable to prefer calculations at not high $D$ ($D = 14 \div 15$) but apply slight smoothing over $L$’s values. Concretely, the plot $W(L)$ in Fig.3 represents the result of averaging of the exact $W(L)$ (defined by (11)) over the intervals $[L, L + \min(\delta L, 1)]$, with $\delta \sim 0.01$.

Such smoothed probability density, $W(L)$, is represented by the curved line at right-hand side of Fig.3. It is easy to find that the best fitting for it is nothing but the inverse proportional law:

$$W(L) \approx \frac{1}{L \ln N},$$  \hspace{1cm} (13)

where factor $(\ln N)^{-1}$ ensures the normalization,

$$\sum_{L} W(L) \approx \int_{1}^{N} W(L)dL = 1$$

The dependence (13) is shown by the straight line, and by smooth curve in the inset which demonstrates good quality of this fitting for short cycles too.

The curved line at left side of Fig.3 shows an example of the probability (12). It is not smoothed, therefore, formed by many steps with very different heights (like the famous devil’s staircase). Its closeness to the thinner straight line, which corresponds to $P = L/N$, says that the total $MN$ points are distributed approximately equally between cycles with different lengths. This is just about what the approximation (13) says.

If combining (11), (12), (13) and the approximation $P = L/N$, we obtain the estimate of mean number of cycles per one map of the family:

$$\frac{1}{M} \sum_{L=1}^{N} N_L \approx \ln N,$$  \hspace{1cm} (14)

(let us recollect that $N_L$ is summary value for all the ASDTM). At the same time, as the above examples of map expansions into cycles do show, fluctuations in number of cycles from one particular map to another are significantly greater than the mean value (14). According to these examples, as well to the $W(L)$ plots, especial contribution to the fluctuations comes from cycles whose lengths are powers of two.
Nevertheless, when rising \( D \) from 12 to 16 a definite decrease of the fluctuations was noticed. This observation pushes us to the hypothesis that formula (13) represents a true asymptotics of the cycles length distribution in the limit (10).

VI. DISCUSSION AND RESUME

There is a simple naive explanation of the hypothetical asymptotics (13). In above derivation of the estimate (7) for the cycles numbers of continuous tent maps (CTM), the factor \( 2^L \) (with \( L \) being cycle lengths) arises from the two-valued property of their inverse maps (i.e. due to their irreversibility). Since the discrete tent maps (DTM) under consideration have univalent inversions, in their case this factor must disappear. Thus one deduces the inverse proportional dependence of number of the cycles (periodic orbits) on their lengths, i.e. comes to (13) (in other words, cycles of different lengths involve approximately equal amounts of points of the discrete phase space).

In view of such reasonings, the inverse proportional law can be expected in case of any sufficiently reach family of unitary DTM, not only the ASDTM family defined by (10) (moreover, in unitary discrete analogues of multimodal piecewise linear CTM, not unimodular tent maps only).

In factual finite-precision computer simulations of more or less general CTM, distinguished from the “pathological” map \( X_{n+1} = 1 - |1 - 2X_n| \) (about it see Introduction), e.g. asymmetrical CTM (2) or the map \( X_{n+1} = 1 - c|1 - 2X_n| \) with e.g. \( c = 0.9999999999\ldots \), it can be expected that the law (13) must manifest itself sooner than (7). Indeed, if the latter rule was true, then the very short orbits with lengths \( L \leq D \) only (recollect that \( D \) is number of bits under operations) would take all \( N = 2^D \) points of the discrete phase space. In reality, with no doubts, almost any choice of an initial point results in a long orbit (“chaotic trajectory”), whose length \( L \) is comparable with \( N \), while it is hard to hit casually into a short orbit.

To resume, we performed computer statistical analysis of periodic orbits of unitary (reversible) discrete tent maps, and found that probability distribution of the orbits lengths well obeys the inverse proportional law (13).

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[1] A. J. Lichtenberg and M. A. Lieberman. Regular and stochastic motion. Springer-Verlag, N.-Y., 1983.

[2] R. C. Dixon. Spread spectrum systems. Wiley-Interscience Publ.