Research Article
Multivalued Fixed Point Results for Two Families of Mappings in Modular-Like Metric Spaces with Applications

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1. Introduction and Preliminaries

If the image of a point $x$ under two mappings is $x$ itself, then $x$ is called a common fixed point of those mappings. Theory of fixed point has a basic role in analysis (see [1–39]). Chistyakov [7] established the concept of modular metric spaces and showed briefly about modular convergence, convex modular, equivalent metrics, abstract convex cone, and metric semigroup. Padcharoen et al. [16] introduced the concept of $\alpha$-type $F$-contractions in modular metric spaces and discussed some related results. Further results on these spaces in different directions can be seen in [6, 13, 14].

In this paper, we establish some common fixed point theorems for two families of set-valued mappings satisfying a generalized contraction on a sequence only in a more generalized setting of modular-like metric spaces. New results can be established in dislocated metric spaces, ordered spaces, partial metric spaces, and metric spaces as a consequence of our findings. To support our results, some applications and examples are discussed. We give the following preliminary concepts which will be used in our results.

Definition 1. (see [16]). Let $A \neq \emptyset$. A function $w: (0, \infty) \times A \times A \rightarrow [0, \infty)$ is called a modular-like metric on $A$ if for all $a, b, c \in A$, $l > 0$, and $w_l(a, b) = w(l, a, b)$, it satisfies

(i) $w_l(0, 0) = w_l(0, 0)$ for all $l > 0$
(ii) $w_l(a, b) = 0$ for all $l > 0$ and then $a = b$
(iii) $w_{l+u}(a, b) \leq w_l(a, c) + w_u(c, b)$ for all $l, u > 0$

If we replace (ii) by $w_l(a, b) = 0$ for all $l > 0$ if and only if $a = b$, then $w$ becomes a modular metric on $A$. If we replace (ii) by $w_l(a, b) = 0$ for some $l > 0$ and then $a = b$, then $w$ becomes a regular modular metric on $A$. For $g \in A$ and $\varepsilon > 0$, $B_{w_l}(g, \varepsilon) = \{p \in A: w_l(g, p) \leq \varepsilon\}$ is a closed ball in $(A, w)$.
We will use \(m.l.m.\) space instead of modular-like metric space.

**Definition 2.** (see [16]). Let \((A, w)\) be an \(m.l.m.\) space.

(i) \(E \subseteq A\) is known as \(w\)-complete if for any sequence \((a_n)_{n \in \mathbb{N}}\) in \(E\) and for some \(l > 0\), \(w_l(a_n, a_m) \to 0\) as \(m,n \to \infty\) implies \(w_l(a_n, a) \to 0\) as \(n \to \infty\) for some \(a \in E\)

(ii) The sequence \((a_n)_{n \in \mathbb{N}}\) in \(A\) is known as \(w\)-Cauchy for some \(l > 0\) if \(w_l(a_n, a_m) \to 0\) as \(m,n \to \infty\)

(iii) The sequence \((a_n)_{n \in \mathbb{N}}\) in \(A\) is known as \(w\)-convergent to \(a \in A\) for some \(l > 0\) if and only if \(w_l(a_n, a) \to 0\) as \(n \to \infty\)

**Definition 3.** Let \((A, w)\) be an \(m.l.m.\) space and \(E \subseteq A\). A member \(p_0\) which belongs to \(E\) is said to be a best approximation in \(E\) for \(g \in A\) if

\[
    w_l(g, E) = \inf_{p \in E} w_l(g, p) = w_l(g, p_0).
\]

If each \(g \in A\) has a best approximation in \(E\), then \(E\) is known as a proximinal set. \(P(A)\) is equal to the family of proximinal sets in \(A\). Let \(A = \mathbb{R}^* \cup \{0\}\) and \(w_l(g, p) = 1/(g + p)\) for all \(l > 0\). Define a set \(E = [4, 6]\); then, for each \(y \in A\),

\[
    w_l(y, E) = \inf_{a \in \{4, 6\}} w_l(y, a) = w_l(y, 4).
\]

Hence, 4 is the best approximation in \(E\) for each \(y \in A\). Also, \([4, 6]\) is a proximinal set.

**Definition 4.** The set-valued mapping \(H_w: P(A) \times P(A) \to [0, \infty)\), defined by

\[
    H_w(N, M) = \max \left\{ \sup_{n \in N} w_l(n, M), \sup_{m \in M} w_l(N, m) \right\},
\]

is known as \(w\)-Hausdorff metric. The pair \((P(A), H_w)\) is called the \(w\)-Hausdorff metric space. Let \(A = \mathbb{R}^* \cup \{0\}\) and \(w_l(g, p) = (1/l)(g + p)\) for all \(l > 0\). If \(P = [5, 6]\) and \(O = [9, 10]\), then \(H_w(P, O) = (13/l)\).

**Definition 5.** Let \((X, w)\) be a modular-like metric space. Then, we will say that \(w\) satisfies the \(\Delta_M\)-condition if it is the case that \(\lim_{m \to +\infty} w_p(x_m, x_m) = 0\), for \(p = m + n\) implies \(\lim_{m \to +\infty} w_l(x_m, x_m) = 0(m, n \in \mathbb{N}, m > n)\) for some \(l > 0\).

**Definition 6.** (see [33]). Let \(A \neq \emptyset, \xi: A \to P(A)\) be a set-valued mapping, \(B \subseteq A, \) and \(a: A \times A \to [0, +\infty)\). Then, \(\xi\) is called \(\alpha_*\)-admissible on \(K\) if \(\alpha_*\xi(a, \xi(c)) = \inf\{a(u, v): u \in \xi(a), v \in \xi(c)\} \geq 1\) whenever \(a(u, v) \geq 1\), for all \(a, c \in B\).

**Definition 7.** Let \(A \neq \emptyset, \xi: A \to P(A)\) be a set-valued mapping, \(M \subseteq A, \) and \(a: A \times A \to [0, +\infty)\). Then, \(\xi\) is called \(\alpha_*\)-dominated on \(M\) if for all \(a \in M, \alpha_*\xi(a, b) = \inf\{a(l, b): l \in \xi(a)\} \geq 1\).

**Definition 8.** (see [39]). Consider a metric space \((W, d)\). A function \(H: W \to W\) is said \(A\)-contraction if for all \(c, k \in W\), there exists \(\tau > 0\) such that \(d(H(a, Ha), H(c, kc)) \leq \tau\) for all \(c, k \in W\).

\[
    \tau + A(d(Ha, Hc)) \leq A(d(a, c)),
\]

where \(A: \mathbb{R}^+ \to \mathbb{R}\) is a mapping that satisfies the following:

- (F1) There exists \(k \in (0, 1)\) such that \(\lim_{m \to +\infty} A^k(a, c) = 0\)
- (F2) For all \(a, c \in \mathbb{R}\) such that \(a < c, A(a) < A(c)\), that is, \(A\) is strictly increasing
- (F3) \(\lim_{m \to +\infty} A^m(a, c) = -\infty\), for each sequence \(\{\alpha_n\}_{n=1}^\infty\) of positive numbers

The family of all mappings satisfying conditions (F1) to (F3) is denoted by \(F\).

**Lemma 1.** Let \((A, w)\) be an \(m.l.m.\) space. Let \((P(A), H_w)\) be a Hausdorff \(w\)-metric-like space. Then, for each \(a \in K\) and for all \(K, M \in P(A)\), there exists \(b_* \in M\) such that \(H_w(K, M) \geq w_l(a, b_*).

Proof: If \(H_w(K, M) = \sup_{a \in K} w_l(a, M)\), then \(H_w(K, M) \geq w_l(a, M)\) for each \(a \in K\). As \(M\) is the proximinal set, for any \(a \in A\), there exists at least one best approximation \(b_* \in M\) which satisfies \(w_l(a, M) = w_l(a, b_*)\). Now, we have \(H_w(K, M) \geq w_l(a, b_*)\). Now,

\[
    H_w(K, M) = \sup_{a \in K} w_l(a, M) \geq \sup_{a \in K} w_l(a, b_*)
\]

for some \(b_* \in M\).

Hence proved.

**Example 1.** (see [21]). Let \(A = \mathbb{R}\). Define \(B: A \times A \to [0, +\infty)\) by

\[
    B(b, t) = \begin{cases} 
        1 & \text{if } b > t, \\
        4 & \text{if } b \leq t.
    \end{cases}
\]

Define \(G, M: A \to P(A)\) by

\[
    Gb = [-4 + b, -3 + b],
\]

\[
    Mt = [-2 + t, -1 + t].
\]

Then, \(G\) and \(M\) are not \(\alpha_*\)-admissible, but they are \(\alpha_*\)-dominated.

**2. Main Results**

Let \((\mathfrak{S}, w)\) be an \(m.l.m.\) space and \(c_0 \in \mathfrak{S}\); let \([S_\beta: \beta \in \Omega]\) and \([T_\phi: \phi \in \Phi]\) be two families of multifunctions from \(\mathfrak{S}\) to \(P(\mathfrak{S})\). Let \(c_1 \in S_\xi c_0\) be an element such that \(w(c_0, S_\xi c_0) = w(c_0, c_1)\). Let \(c_2 \in T_\phi c_1\) be such that \(w(c_1, T_\phi c_1) = w(c_1, c_2)\). Let \(c_3 \in S_\xi c_2\) such that...
\(w(c_2, S(c_2)) = w(c_2, c_3)\). In this way, we get a sequence \(\{T^\Phi S^\Phi(c_n)\}\) in \(\mathcal{S}\), where \(c_{2m+1} \in S(c_{2m}), c_{2m+2} \in T^\Phi(c_{2m+1}), n \in \mathbb{N}, i \in \Omega, \) and \(j \in \Phi\). Also, \(w(c_{2m}, S(c_{2m})) = w(c_{2m}, c_{2m+1})\) and \(w(c_{2m+1}, T(c_{2m+1}) = w(c_{2m+1}, c_{2m+2})\). \(\{T^\Phi S^\Phi(c_n)\}\) is said to be a sequence in \(\mathcal{S}\) generated by \(c_0\). If \(\{S_\sigma: \sigma \in \Omega\} = \{T^\Phi: \beta \in \Phi\}\), then we denote \(\{S_\sigma(c_n)\}\) instead of \(\{T^\Phi S^\Phi(c_n)\}\).

\[
\tau + U(H_{w_1}(S_\sigma t, T^\Phi g)) \leq U\left(\max\left\{w_1(t, g), w_1(t, S_\sigma t) - \frac{w_2(t, T^\Phi g)}{2}, \frac{w_2(t, T^\Phi g)}{1 + w_1(t, g)}\right\}\right),
\]

whenever \(t, g \in \mathcal{S} \land \{T^\Phi S^\Phi(c_n)\}, \alpha(t, g) \geq 1, \sigma \in \Omega, \beta \in \Phi, \) and \(H_{w_1}(S_\sigma t, T^\Phi g) > 0\).

Then, the sequence \(\{T^\Phi S^\Phi(c_n)\}\) generated by \(c_0\) converges to \(c \in \mathcal{S}\), and for each \(n \in \mathbb{N}\), a \((c_n, c_{n+1}) \geq 1\). Also, if \(c\) satisfies (8) and either \(\alpha(c_n, c) \geq 1\) or \(\alpha(c_{n+1}, c) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\), then \(S_\sigma\) and \(T^\Phi\) have a common fixed point \(c \in \mathcal{S}\) for all \(\sigma \in \Omega\) and \(\beta \in \Phi\).

\[
\tau + U(w_1(c_{2i+1}, c_{2i+2})) \leq \tau + U(H_{w_1}(S^\Phi c_2, T^\Phi c_{2i+1}))
\]
\[
\leq U\left(\max\left\{w_1(c_2, c_{2i+1}), w_1(c_2, S^\Phi c_2), \frac{w_2(c_2, T^\Phi c_{2i+1})}{2}, \frac{w_2(c_2, S^\Phi c_{2i+1})}{1 + w_1(c_2, c_{2i+1})}\right\}\right)
\]
\[
\leq U\left(\max\left\{w_1(c_2, c_{2i+1}), w_1(c_2, c_{2i+2}), \frac{w_1(c_{2i+1}, c_{2i+2}) + w_1(c_{2i+1}, c_{2i+2})}{2}, \frac{w_1(c_{2i+1}, c_{2i+2})}{1 + w_1(c_{2i+1}, c_{2i+2})}\right\}\right)
\]

This implies
\[
\tau + U(w_1(c_{2i+1}, c_{2i+2})) \leq U(\max\{w_1(c_2, c_{2i+1}), w_1(c_{2i+1}, c_{2i+2})\}).
\]

Now, if
\[
\max\{w_1(c_2, c_{2i+1}), w_1(c_{2i+1}, c_{2i+2})\} = w_1(c_{2i+1}, c_{2i+2}),
\]
then from (10), we have
\[
U(w_1(c_{2i+1}, c_{2i+2})) \leq U(w_1(c_{2i+1}, c_{2i+2})) - \tau,
\]
a contradiction. Therefore,
\[
\max\{w_1(c_2, c_{2i+1}), w_1(c_{2i+1}, c_{2i+2})\} = w_1(c_2, c_{2i+1}),
\]
for all \(i \in \mathbb{N} \cup \{0\}\). Hence, from (10), we have
\[
U(w_1(c_{2i+1}, c_{2i+2})) \leq U(w_1(c_{2i+1}, c_{2i+2})) - \tau.
\]

Similarly, we have
\[
U(w_1(c_{2i+1}, c_{2i+1})) \leq U(w_1(c_{2i+1}, c_{2i+2})) - \tau.
\]

**Theorem 1.** Let \((\mathcal{S}, w)\) be a complete m.l.m. space. Assume that \(w\) is regular and satisfies the \(\Delta^+\)-condition. Let \(c_0 \in \mathcal{S}\), \(\alpha: \mathcal{S} \times \mathcal{S} \rightarrow (0, \infty), \{T^\Phi: \beta \in \Phi\}\), and \(\{S_\sigma: \sigma \in \Omega\}\) be the families of \(\alpha\)-dominated set-valued functions on \(w\). Suppose there exist \(\tau > 0\) and \(U \in \mathcal{F}\) such that
\[
\tau + U(H_{w_1}(S_\sigma t, T^\Phi g)) \leq U\left(\max\left\{w_1(t, g), w_1(t, S_\sigma t) - \frac{w_2(t, T^\Phi g)}{2}, \frac{w_2(t, T^\Phi g)}{1 + w_1(t, g)}\right\}\right),
\]

Proof. Consider a sequence \(\{T^\Phi S^\Phi(c_n)\}\). Obviously, \(c_n \in \mathcal{S}\) for each \(n \in \mathbb{N}\). If \(j = 2i + 1\) for some \(i \in \mathbb{N}\). By the definition of \(\alpha\)-dominated mappings, we have \(\alpha(c_2, S(c_2)) \geq 1\) and \(\alpha(c_{2i+1}, T(c_{2i+1})) \geq 1\) for all \(\sigma \in \Omega\) and \(\beta \in \Phi\). As \(\alpha(c_2, S(c_2)) \geq 1\), this implies
\[
\liminf_{(a(c_n, b): b \in S(c_2)) \geq 1. \text{ Also, } c_{2i+1} \in S(c_2) \text{ for some } f \in \mathcal{S}, \text{ so } \alpha(c_2, c_{2i+1}) \geq 1. \text{ Also, } c_{2i+2} \in T(c_{2i+1}) \text{ for some } g \in \mathcal{F}. \text{ Now, by using Lemma 1, we have }
\]
\[
U(w_1(c_{2i+1}, c_{2i+2})) \leq U(w_1(c_{2i+1}, c_{2i+2})) - 2\tau.
\]

Repeating these steps, we get
\[
U(w_1(c_{2i+1}, c_{2i+2})) \leq U(w_1(c_0, c_1)) - (2i + 1)\tau.
\]

Similarly, we have
\[
U(w_1(c_2, c_{2i+1})) \leq U(w_1(c_0, c_1)) - 2\tau.
\]

Inequalities (17) and (18) can jointly be written as
\[
U(w_1(c_n, c_{n+1})) \leq U(w_1(c_0, c_1)) - nr.
\]

Taking the limit as \(n \rightarrow \infty\) in (19), we have
\[
\lim_{n \rightarrow \infty} U(w_1(c_n, c_{n+1})) = -\infty.
\]

Since \(U \in \mathcal{F}\),
\[
\lim_{n \rightarrow \infty} w_1(c_n, c_{n+1}) = 0.
\]

Applying the property (F1) of \(\mathcal{F}\), there exists \(k \in (0, 1)\) such that
we have
\[ (w_1(c_n, c_{n+1}))^k (U(w_1(c_n, c_{n+1}))) = 0. \] (22)

Considering (21) and (22) and letting \( n \to \infty \) in (23), we have
\[ \lim_{n \to \infty} n(w_1(c_n, c_{n+1}))^k = 0. \] (24)

Since (24) holds, there exists \( n_1 \in \mathbb{N} \) such that \( n(w_1(c_n, c_{n+1}))^k \leq 1 \) for all \( n \geq n_1 \) or
\[ w_p(c_n, c_m) \leq w_1(c_n, c_{n+1}) + w_1(c_{n+1}, c_{n+2}) + \cdots + w_1(c_m, c_{m+1}) \leq \frac{1}{n} \left( \frac{1}{k} + \frac{1}{(n+1)^{1/k}} + \cdots + \frac{1}{m^{1/k}} \right). \] (26)

By (19), for all \( n \in \mathbb{N} \), we obtain
\[ w_1(c_n, c_{n+1}) \leq \frac{1}{n^{1/k}} \quad \text{for all } n \geq n_1. \] (25)

Take \( p > 0 \) and \( m = n + p > n > n_1 \); then,
\[ \lim_{n \to \infty} w_1(c_n, c_{n+p}) = 0. \] (31)

As \( k \in (0, 1) \), then \( (1/k) > 1 \), and so,
\[ \lim_{n \to \infty} \left( \frac{1}{n^{1/k}} \right) = \lim_{n \to \infty} \frac{1}{(n+1)^{1/k}} = \lim_{n \to \infty} \frac{1}{m^{1/k}} = 0. \] (28)

Now, by Lemma 1, we have
\[ \tau + U \left( w_1(c_{2n+1}, T_{\beta}c) \right) \leq \tau + U \left( H_{w_1}(S_{2n}, T_{\beta}c) \right). \] (32)

Hence, \( \{T_{\beta}S_\nu(c_n)\} \) is a Cauchy sequence in \( w \). Since \( (\mathfrak{F}, w) \) is a regular complete modular-like metric space,
\[ \lim_{n \to \infty} w_1(c_n, c_{n+p}) = 0. \] (33)

Letting \( n \to \infty \) and using (31), we get
\[ \tau + U \left( w_1(c_{2n+1}, T_{\beta}c) \right) \leq \frac{1}{U} \left( \left( w_1(c_{2n+1}, T_{\beta}c) \right) + \frac{w_1(c_{2n+1}, T_{\beta}c)}{U} \right) \] (34)

Since \( U \) is strictly increasing, (32) implies
\[ \tau + U \left( w_1(c_{2n+1}, T_{\beta}c) \right) \leq U \left( w_1(c_{2n+1}, T_{\beta}c) \right). \] (35)

This is not true. So, our assumption is wrong. Hence, \( w_1(c, T_{\beta}c) = 0 \) or \( c \in T_{\beta}c \) for each \( \beta \in \Phi \). Similarly, by proceeding with Lemma 1 and inequality (8), we can prove that \( w_1(c, S_{\sigma}c) = 0 \) or \( c \in S_{\sigma}c \) for all \( \sigma \in \Omega \). Hence, \( c \) is a
common fixed point of both the mappings $S_\sigma$ and $T_\beta$ in $\mathfrak{F}$ for all $\sigma \in \Omega$ and $\beta \in \Phi$.

Example 2. Let $\mathfrak{F} = \mathbb{R}_+ \cup \{0\}$. Take $w_2(q, g) = (q + g)$ and $w_1(q, g) = 1/2(q + g)$ for all $q, g \in \mathfrak{F}$. Suppose that $S_\sigma, T_\beta: \mathfrak{F} \times \mathfrak{F} \rightarrow P(\mathfrak{F})$ are two families of multivalued mappings defined by

$$S_mv = \left[ \frac{v}{3m} \frac{2v}{3m} \right] \text{ if } v \in \mathfrak{F},$$

where $m = 1, 2, 3, \ldots$, and

$$T_nv = \left[ \frac{v}{4n} \frac{3v}{4n} \right] \text{ if } v \in \mathfrak{F},$$

where $n = 1, 2, 3, \ldots$. Suppose that $v_0 = 1$ and $w_1(v_0, S_1v_0) = w_1(1, S_1) = w_1(1, (1/3))$. So, $v_1 = 1/3$. Now, $w_1(v_1, T_1v_1) = w_1((1/3), T_1(1/3)) = w_1((1/3), (1/12))$. So, $v_2 = 1/(112)$. Now, $w_1(v_2, S_2v_2) = w_1((1/12), S_2(1/12)) = w_1((1/112), (1/72))$. So, $v_3 = 1/72$. Continuing in this way, we have \{ $T_nS_m(v_n) \}$ = \{1, 1/3, 1/12, 1/72, \ldots\}. Consider the mapping $\alpha: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ defined by

$$\alpha(r, t) = \begin{cases} 1, & \text{if } r > t, \\ 1/2, & \text{otherwise}. \end{cases}$$

Now, if $v, y \in \mathfrak{F}$, then

$$H_{w_1}(S_mv, T_ny) = \max \left\{ \sup_{a \in S_m} w_1(a, T_ny), \sup_{b \in T_ny} w_1(S_mv, b) \right\} = \begin{cases} \max \left\{ w_1 \left( \frac{2v}{3m} \left[ \frac{v}{4n} \frac{3v}{4n} \right] \right), \\ w_1 \left( \frac{v}{3m} \frac{3v}{3m} \frac{3v}{4n} \right) \right\} \right\} = \frac{1}{2} \max \left\{ \frac{2v}{3m} \frac{3v}{4n}, \frac{2y}{3m} \frac{3y}{4n} \right\} = \frac{1}{2} \max \left\{ \frac{2v}{3m} + \frac{2y}{4n}, \frac{3v}{3m} + \frac{3y}{4n} \right\}, \right.$$
\[ \tau + U(H_w(S_{\alpha}, T_{\beta}g)) \leq U \left( \max \left\{ w_1(t, g), w_1(t, S_{\alpha}t), \frac{w_1(t, S_{\alpha}t)}{1 + w_1(t, g)}, \frac{w_1(t, T_{\beta}y) \cdot w_1(t, S_{\alpha}t) \cdot w_1(y, T_{\beta}y)}{2} \right\} \right). \]  

whenever \( t, g \in \mathfrak{F} \cap \{ S_{\alpha}(c_n) \}, \ a(t, g) \geq 1, \ \alpha, \beta \in \Omega, \ and \ H_w(S_{\alpha}, T_{\beta}g) > 0 \), then the sequence \( \{ \mathfrak{F}_{\alpha}(c_n) \} \) generated by \( c_0 \) converges to \( c \in \mathfrak{F} \), and for each \( n \in \mathbb{N} \), \( a(c_n, c_{n+1}) \geq 1 \). Also, if \( c \) satisfies (44) and either \( a(c_n, c) \geq 1 \) or \( a(c, c_n) \geq 1 \) for every \( n \in \mathbb{N} \cup [0] \), then \( \{ S_{\alpha}, \sigma \in \Omega \} \) has a common fixed point \( c \) in \( \mathfrak{F} \).

3. Applications in Graph Theory

Jachymski [11] developed a relation between fixed point theory and graph theory by the induction of graphic contractions. Hussain et al. [9] established some results for the new type of contractions endowed with a graph and also showed an application. Further useful results on the graph can be seen in [34, 35, 40].

\[ \tau + U(H_w(S_{\alpha}, T_{\beta}y)) \leq U \left( \max \left\{ w_1(t, y), w_1(t, S_{\alpha}t), \frac{w_2(t, T_{\beta}y) \cdot w_1(t, S_{\alpha}t) \cdot w_1(y, T_{\beta}y)}{2} \right\} \right), \]  

whenever \( t, y \in \mathfrak{F} \cap \{ T_{\beta}S_{\alpha}(c_n) \}, \ (t, y) \in L(Y), \ \sigma \in \Omega, \ \beta \in \Phi, \) and \( H_w(S_{\alpha}, T_{\beta}y) > 0 \).

Assume that \( \mathfrak{F} \) is regular and satisfies the \( \Delta_M \)-condition. Then, \( \{ T_{\beta}S_{\alpha}(c_n) \} \) is a sequence in \( \mathfrak{F} \), \( (c_n, c_{n+1}) \in L(Y) \), and \( \{ T_{\beta}S_{\alpha}(c_n) \} \rightarrow g^* \). Also, if \( g^* \) satisfies (45) and \( (c_n, g^*) \in L(Y) \) or \( (g^*, c_n) \in L(Y) \), for each \( n \in \mathbb{N} \cup [0] \), then \( S_{\alpha} \) and \( T_{\beta} \) have common fixed point \( g^* \) in \( \mathfrak{F} \) for all \( \sigma \in \Omega \) and \( \beta \in \Phi \).

\[ \tau + U(H_w(S_{\alpha}, T_{\beta}y)) \leq U \left( \max \left\{ w_1(t, y), w_1(t, S_{\alpha}t), \frac{w_2(t, T_{\beta}y) \cdot w_1(t, S_{\alpha}t) \cdot w_1(y, T_{\beta}y)}{2} \right\} \right), \]  

whenever \( t, y \in \mathfrak{F} \cap \{ T_{\beta}S_{\alpha}(c_n) \}, \ a(t, y) \geq 1, \) and \( H_w(S_{\alpha}, T_{\beta}y) > 0 \). Also, (ii) holds. Then, by Theorem 1, we have \( \{ T_{\beta}S_{\alpha}(c_n) \} \) is a sequence in \( \mathfrak{F} \), and \( \{ T_{\beta}S_{\alpha}(c_n) \} \rightarrow g^* \) in \( \mathfrak{F} \). Now, \( c_n, g^* \in \mathfrak{F} \), and either \( (c_n, g^*) \in L(Y) \) or \( (g^*, c_n) \in L(Y) \) implies that either \( a(c_n, g^*) \geq 1 \) or \( a(g^*, c_n) \geq 1 \). So, all the requirements of Theorem 1 are satisfied. Hence, from Theorem 1, \( S_{\alpha} \) and \( T_{\beta} \)

have a common fixed point \( g^* \) in \( \mathfrak{F} \), and \( w_1(g^*, g^*) = 0 \). \( \Box \)

4. Results on Single-Valued Mappings

In this section, some consequences of our results related to single-valued mappings in \( m.l.m. \) spaces have been discussed. Let \( (\mathfrak{F}, w) \) be an \( m.l.m. \) space, \( c_0 \in \mathfrak{F} \), and \( S_{\alpha}, T_{\beta}, \mathfrak{F} \rightarrow \mathfrak{F} \) be two families of mappings. Let \( c_1 = S_{\alpha}c_0, \)
c_2 = T^0c_1$ and $c_3 = S_0c_2$. Adopting this way, we make a sequence $c_n$ in $\mathcal{X}$ so that $c_{2n+1} = S_n c_{2n}$ and $c_{2n+2} = T^0 c_{2n+1}$, where $n = 0, 1, 2, \ldots$. We represent this kind of iterative sequence by $\{T^0S_n(c_0]\}$. We say that $\{T^0S_n(c_0]\}$ is a sequence generated by $c_0$. If $\{S_\sigma; \sigma \in \Omega\} = \{T^0; \beta \in \Phi\}$, then we denote $\{\mathcal{X}S_\sigma(c_0]\}$ instead of $\{T^0S_n(c_0]\}$.

\[
\tau + U(H_{w_1}(S_{\sigma}t, T^0g)) \leq U \left( \max \left\{ w_1(t, g), w_1(t, S_{\sigma}t), \frac{w_1(t, T^0g) \cdot w_1(t, S_{\sigma}t) \cdot w_1(g, T^0g)}{1 + w_1(t, g)} \right\} \right),
\]

(47)

whenever $t, g \in \mathcal{X} \cap \{T^0S_n(c_0]\}$, $\alpha(t, g) \geq 1$, $\sigma \in \Omega$, $\beta \in \Phi$, and $w_1(S_{\sigma}t, T^0g) > 0$. Then, $\{\mathcal{X}S_\sigma(c_0]\}$ is a sequence in $\mathcal{X}$, $\alpha(c_n, c_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, and $\{T^0S_n(c_0]\} \rightarrow h \in \mathcal{X}$. Also, if $u$ satisfies (47) and either $\alpha(c_n, h) \geq 1$ or $\alpha(h, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $S_\sigma$ and $T^0$ have a common fixed point $h$ in $\mathcal{X}$ for every $\sigma \in \Omega$ and $\beta \in \Phi$.

**Proof.** The proof is similar to the proof of Theorem 1.

\[
\tau + U(H_{w_1}(S_{\sigma}t, S_\beta g)) \leq U \left( \max \left\{ w_1(t, g), w_1(t, S_{\sigma}t), \frac{w_1(t, S_\beta g) \cdot w_1(t, S_{\sigma}t) \cdot w_1(g, S_\beta g)}{1 + w_1(t, g)} \right\} \right),
\]

(48)

whenever $t, g \in \mathcal{X} \cap \{\mathcal{X}S_\sigma(c_0]\}$, $\alpha(t, g) \geq 1$, $\sigma \in \Omega$, $\beta \in \Phi$, and $w_1(S_{\sigma}t, S_\beta g) > 0$. Then, $\{\mathcal{X}S_\sigma(c_0]\}$ is a sequence in $\mathcal{X}$, $\alpha(c_n, c_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, and $\{\mathcal{X}S_\sigma(c_0]\} \rightarrow h \in \mathcal{X}$. Also, if $h$ satisfies (48) and either $\alpha(c_n, h) \geq 1$ or $\alpha(h, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $h$ is the fixed point of $S_\sigma$ in $\mathcal{X}$ for every $\sigma \in \Omega$.

### 5. Application in Integral Equations

In this section, we discuss the application of our work in integral equations. First of all, we present our main result

\[
\tau + U(H_{w_1}(S_{\sigma}t, T^0g)) \leq U \left( \max \left\{ w_1(t, g), w_1(t, S_{\sigma}t), \frac{w_1(t, T^0g) \cdot w_1(t, S_{\sigma}t) \cdot w_1(g, T^0g)}{1 + w_1(t, g)} \right\} \right),
\]

(49)

whenever $t, g \in \{T^0S_\sigma(c_n]\}$, $\sigma \in \Omega$, $\beta \in \Phi$, and $w_1(S_{\sigma}t, T^0g) > 0$. Then, $\{T^0S_\sigma(c_n]\} \rightarrow h \in \mathcal{X}$. Also, if inequality (49) holds for $t, g \in [h]$, then $S_\sigma$ and $T^0$ have a unique common fixed point $h$ in $\mathcal{X}$ for all $\sigma \in \Omega$ and $\beta \in \Phi$.

**Proof.** The proof is similar to the proof of Theorem 1. We prove only uniqueness. Let $S_\sigma$ and $T^0$ have another common fixed point $v$. Suppose $w_1(S_{\sigma}u, T^0v) > 0$. Then,

\[
\tau + U(w_1(S_{\sigma}u, T^0v)) \leq U \left( \max \left\{ w_1(u, v), w_1(u, S_{\sigma}u), \frac{w_1(u, T^0v) \cdot w_1(u, S_{\sigma}u) \cdot w_1(v, T^0v)}{1 + w_1(u, v)} \right\} \right).
\]

(50)
This implies that
\[ w_1(u, v) < \mu w_1(u, v) < w_1(u, v), \] (51)
which is not true. So, \( w_1(S_u, T_{\beta}v) = 0 \). Hence, \( u = v \).

Let \( W = C([0, 1], \mathbb{R}_+) \) be the set of all continuous functions on \([0, 1] \). Consider the families of integral equations
\[ u(k) = \int_{0}^{1} H_\sigma(k, h, u(h))dh + \epsilon, \] (52)
\[ c(k) = \int_{0}^{1} G_\beta(k, h, c(h))dh + \epsilon, \] (53)
for all \( k \in [0, 1], \sigma \in \Omega, \beta \in \Phi, \) and \( H_\sigma, G_\beta \) are the functions from \([0, 1] \times [0, 1] \times W \) to \( \mathbb{R} \). For \( \sigma \in C([0, 1], \mathbb{R}_+) \), define the supremum norm as \( \|c\|_\sigma = \sup_{k \in [0, 1]}\|c(k)e^{-\eta\tau}\| \), where \( \eta > 0 \) is arbitrarily taken. Define
\[ w_1(c, p) = \frac{1}{2} \sup_{k \in [0, 1]}\|c(k) + p(k)e^{-\eta\tau}\| = \frac{1}{2}\|c + p\|_\tau, \] (54)
for all \( c, p \in C([0, 1], \mathbb{R}_+) \); with these settings, \((C([0, 1], \mathbb{R}_+), d_{\sigma})\) becomes a complete \( m.l.m. \) space.

Now, we prove the following theorem to ensure the uniqueness and existence of a solution of families of non-linear integral equations (52) and (53).

**Theorem 5.** Assume the following constraints are satisfied:
1. \( \{H_\sigma, \sigma \in \Omega\} \) and \( \{G_\beta, \beta \in \Phi\} \) are two families of mappings from \([0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+) \) to \( \mathbb{R} \).
2. \( \|H_\sigma\|_{\sigma} \leq 1 \), \( \|G_\beta\|_{\beta} \leq 1 \), \( \sup_{k \in [0, 1]}\|c(k)e^{-\eta\tau}\| \leq \frac{1}{2}\|c\|_\tau \) for all \( \sigma \in \Omega, \beta \in \Phi, c(k) \in C([0, 1], \mathbb{R}_+) \).

Then, integral equations (52) and (53) have a unique solution.

**Proof.** By assumption (ii),
\[ (S_uu(k) + T_{\beta}c(k)) \leq \frac{\|u + S_uu\|_\tau}{\|T_{\beta}c\|_\tau} + \frac{\|c\|_\tau}{\|T_{\beta}c\|_\tau} \] (55)
Suppose there exists \( \tau > 0 \) such that
\[ \|H_\sigma(k, h, u) + G_\beta(k, h, c)\| \leq \frac{\tau E_{(\sigma, \beta)}(u, c)}{\|T_{\beta}c\|_\tau}, \] (56)
for all \( k, h \in [0, 1] \) and \( u, c \in C([0, 1], \mathbb{R}_+) \), where
\[ E_{(\sigma, \beta)}(u, c) = \max \left\{ \frac{1}{2}, \frac{\|u + S_uu\|_\tau}{\|u + S_uu\|_\tau} + \frac{\|c + T_{\beta}c\|_\tau}{\|c + T_{\beta}c\|_\tau} \right\} \] (57)

Then, integral equations (52) and (53) have a unique solution.

**6. Conclusion**
In this article, we have achieved some new results for set-valued mappings belonging to two families which satisfy
generalized rational-type Wardowski’s contraction. Dominated mappings are applied to find out the fixed point results. Applications in the subject of integral equations and graph theory are presented. Moreover, we investigate our results in new generalized modular-like metric spaces. Many consequences of our results in dislocated metric spaces, metric spaces, and partial metric spaces (even with a partial order) can be established easily.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Each author contributed equally to this paper, read, and approved the final manuscript.

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