A numerical method for solving stochastic differential equations with noisy memory

Kristina R. Dahl *

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Abstract

Stochastic differential equations with noisy memory are often impossible to solve analytically. Therefore, we derive a numerical Euler-Maruyama scheme for such equations and prove that the mean-square error of this scheme is of order $\sqrt{\Delta t}$. This is, perhaps somewhat surprisingly, the same order as the Euler-Maruyama scheme for regular SDEs, despite the added complexity from the noisy memory. To illustrate this numerical method, we apply it to a noisy memory SDE which can be solved analytically.

Key words: Euler method · Stochastic differential equation · Noisy memory · Mean-square convergence

MSC classification codes: 65C30 · 60H35 · 60H10 · 65C20

1 Introduction

In this paper, we study how noisy memory stochastic differential equations (SDEs), introduced in Dahl et al. [6], are connected to Volterra equations. We also discuss existence and uniqueness of solutions to noisy memory SDEs. Since such equations usually cannot be solved analytically, we derive an Euler-Maruyama scheme for a numerical approximation of the solution. We prove that this scheme has mean square order of convergence $\sqrt{\Delta t}$.

One should note the following unique features of the analysis:

- The stochastic differential equation (SDE) is driven by generalized noisy memory. The evolution of the state $X$ at any time $t$ is dependent on its past history $\int_{t-\delta}^{t} \phi(t, s)X(s) dB(s)$ where $\delta$ is the memory span and $dB$ is white noise.

- Noisy memory SDEs where the memory does not include a time-dependent function can be rewritten as two dimensional SDEs with delay (see Dahl et al. [6]). Hence, one may solve such equations using numerical methods for delay SDEs, see e.g., Buckwar [3], Carletti [5], Mao and Sabanis [14] and Milstein and Tretyakov [15]. However, scaling the memory by a time-dependent function implies that generalized noisy memory SDEs cannot be

*Department of Mathematics, University of Oslo. P.O box 1053, Blindern, 0316 OSLO, Norway. Email: kristrd@math.uio.no. Phone: +47-22854183
rephrased as SDEs with delay. To the best of our knowledge, no current numerical methods work for approximating the solutions of generalized noisy memory SDEs. However, our numerical scheme works for all noisy memory SDEs, including the generalized ones.

- We prove that the Euler-Maruyama scheme has mean square order of convergence $\sqrt{\Delta t}$. This is the same as the Euler-Maruyama method for classical SDEs. Hence, the added complexity from the noisy memory in the SDE does not reduce the order of convergence of the Euler-Maruyama scheme.

Noisy memory SDEs can be applied to model animal populations where the population growth depends in some stochastic way on the previous population states, as well as the current number of animals. This kind of memory effect can be useful in the modeling of species where there is a natural delay in the population growth caused by e.g., hatching of eggs for fish, or larva becoming butterflies. This delay may depend on time, such as seasonal weather effects. This motivates generalized noisy memory SDEs. For applications of stochastic delay equations connected to population dynamics, see [11]. Other applications of stochastic delay equations include spread of infectious diseases, see Beretta et al. [2], applications in physics and engineering, see Kolmanovskii and Myshkis [10] and financial applications, see Øksendal and Sulem [20].

Stochastic systems with memory, and the related field of stochastic systems with delay, has been an important field of research over the last years, see for example Mohammed and Zhang [18], Mohammed [17] and Øksendal, Sulem and Zhang [21]. Introducing a noisy memory $Z(t)$, as opposed to a perfect memory, is a natural generalization of this research. Verriest and Florchinger [24], Verriest [25] and Verriest and Michiels [26] all consider stochastic delay differential equations and derive corresponding stability results on the solution. Li and Cao [12] derive a two-step Euler-Maruyama method for a nonlinear neural stochastic delay differential equation.

The structure of the paper is as follows: In Section 2, we introduce the noisy memory SDE, show a connection between noisy memory SDEs and stochastic Volterra equations and give the Euler scheme to approximate the solution of the noisy SDE. Then, in Section 3 we state our main result on the order of convergence of the Euler method and prove several lemmas needed to prove this. In Section 4 we complete the proof of the main theorem. Finally, in Section 5 we derive an analytical solution to a noisy SDE, and give a numerical example illustrating the convergence of the Euler method.

## 2 The noisy memory SDE and the Euler scheme

In this section, we introduce a stochastic differential equation with noisy memory, and derive the corresponding Euler scheme.

Let $B_t(\omega) = B(t, \omega); (t, \omega) \in [-\delta, \infty) \times \Omega$ be a Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. We assume that $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ is the filtration generated by $\{B_t\}_{t \geq 0}$ (augmented with the $P$-null sets).

Let $\delta > 0$ be a time-delay determining the length of the memory-interval. Also, consider the functions $b : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \sigma : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$
ξ : [0, T] → R and φ : [0, T] × [0, T] → R. We will develop a numerical method for the following stochastic differential equation with noisy memory:

\[
\begin{align*}
    dX(t) &= b(t, X(t), Z(t))dt + σ(t, X(t), Z(t))dB(t), \quad t ∈ (0, T] \\
    X(t) &= \xi(t), \quad t ∈ [-δ, 0],
\end{align*}
\]

where the stochastic process

\[
    Z(t) := \int_{t-δ}^{t} φ(t, s)X(s)dB(s)
\]

(2)

is the (generalized) noisy memory of \(X(t)\), see also Dahl et al. [6]. If \(φ(t, s) = 1\) for all \(t, s ∈ [0, T]\), equation (1) is a non-generalized (or regular) noisy memory SDE. For more on stochastic differential equations in general, see for instance Øksendal [19].

The parameter \(δ\) is the memory parameter which gives the length of the memory-interval. Note that the memory is noisy due to the Itô integral in the definition. Intuitively, this means that the system does not have a perfect memory, but a slightly distorted one. The (deterministic) function \(φ\) inside the noisy memory Itô integral allows this noisy memory to vary with time, both the time of the memory, but also the current time.

Remark 2.1 Buckwar [4] considers the same type of memory as in (2), but with a deterministic Lebesgue integral instead of a stochastic Itô integral. That is, they consider a deterministic distributed memory instead of a stochastic distributed memory like us.

Note that the Brownian motion is defined for negative times is \([-δ, 0]\). For a detailed presentation of how this is done, see Holden et al. [8] (Section 2.1.1).

2.1 A connection between noisy memory SDEs and stochastic Volterra equations

Consider a very simple SDE with noisy memory,

\[
    dX(t) = Z(t)dt = \left(\int_{t-δ}^{t} X(u)dB(u)\right) dt
\]

(3)

where \(Z\) is given as in equation (2) with \(φ(t, s) = 1\) for all \(t, s ∈ [0, T]\). By changing the order of integration in (3), we see that

\[
    X(t) = X(0) + \int_{0}^{t} \int_{u}^{\min\{t, u+δ\}} X(u)dB(u)ds
    = X(0) + \int_{0}^{t} \int_{u}^{\min\{t-u, δ\}} X(u)dB(u)
\]

(4)

This is a linear stochastic Volterra equation, see Øksendal and Zhang [22]. Such equations do not have an simple analytical solution. However, they can be solved using an iterative method, see Øksendal and Zhang [22] and [23].

Now, consider equation (1). If we assume that \(b\) may be decomposed as \(b(t, X(t), Z(t)) = b(t, X(t)) + aZ(t)\), where \(a ∈ R\) and \(σ(t, X(t), Z(t)) = σ(t, X(t))\), then we can rewrite equation (1) as a stochastic Volterra equation.
In the (non-generalized) case where Theorem 2.2 equation (1).

In this section, we prove some results on the existence of a unique solution to 2.2 Existence of solution least as difficult to solve as stochastic Volterra equations. Conditions for the existence of a unique solution to such equations can be found in e.g., Wang [27].

Together with the Lipschitz and linear growth conditions, this ensures that there exists a unique càdlàg adapted solution for the existence of a unique solution to equation (1) have predictable versions, whenever $X$ is càdlàg and adapted. Together with the Lipschitz and linear growth conditions, this ensures that there exists a unique càdlàg adapted solution $X$ to the equation (1), satisfying

$$
E[ \sup_{t \in [-\delta, T]} |X(t)|^2 ] < \infty.
$$

\[ \begin{align*}
X(t) & = X(0) + \int_0^t b(s, X(s), Z(s))ds + \int_0^t \sigma(s, X(s))dB(s) \\
& = X(0) + \int_0^t \tilde{b}(s, X(s))ds + a \int_0^t \int_{s-\delta}^t \phi(s, u)X(u)dB(u)ds \\
& \quad + \int_0^t \sigma(s, X(s))dB(s) \\
& = X(0) + \int_0^t \tilde{b}(s, X(s))ds + a \int_0^t \int_{u}^{\min\{u+\delta,t\}} \phi(s, u)dsX(u)dB(u) \\
& \quad + \int_0^t \sigma(s, X(s))dB(s) \\
& = X(0) + \int_0^t \tilde{b}(s, X(s))ds + \int_0^t [\tilde{\phi}(t, s)X(s) + \sigma(s, X(s))]dB(s)
\end{align*} \]

where the third equality follows from the same kind of calculations as in [4] and $\tilde{\phi}(t, s) := a \int_0^{\min\{s+\delta,t\}} \phi(u, s)du$. This is a stochastic Volterra equation.

The previous argument shows that non-trivial noisy memory SDEs are at least as difficult to solve as stochastic Volterra equations.

2.2 Existence of solution

In this section, we prove some results on the existence of a unique solution to equation (1).

**Theorem 2.2** In the (non-generalized) case where $\phi(t, s) = 1$ for all $t, s \in [0, T]$, the following assumptions on the coefficient functions $b$ and $\sigma$ are sufficient for the existence of a unique solution to equation (1):

(i) The functions $b(\omega, t, \cdot)$ and $\sigma(\omega, t, \cdot)$ are assumed to be $C^1$ for each fixed $\omega \in \Omega$, $t \in [0, T]$.

(ii) The functions $b(\cdot, x, z)$ and $\sigma(\cdot, x, z)$ are predictable for each $x, z$.

(iii) Lipschitz condition: The functions $b$ and $\sigma$ are Lipschitz continuous in the variables $x$ and $z$ with a Lipschitz constant $D$ which is independent of the variables $t, \omega$, i.e.:

$$
|b(t, x_1, z_1) - b(t, x_2, z_2)| \leq D(|x_1 - x_2| + |z_1 - z_2|)
$$

$$
|\sigma(t, x_1, z_1) - \sigma(t, x_2, z_2)| \leq D(|x_1 - x_2| + |z_1 - z_2|).
$$

(iv) Linear growth condition: The functions $b$ and $\sigma$ satisfy the linear growth condition in the variables $x$ and $z$ with the linear growth constant $C$ independent of the variables $t, \omega$, i.e.:

$$
|b(t, x, z)| + |\sigma(t, x, z)| \leq C(1 + |x| + |z|).
$$

**Proof.** Assumptions (i) and (ii) are sufficient to ensure that the integrands in equation (1) have predictable versions, whenever $X$ is càdlàg and adapted. Together with the Lipschitz and linear growth conditions, this ensures that there exists a unique càdlàg adapted solution $X$ to the equation (1), satisfying

$$
E[ \sup_{t \in [-\delta, T]} |X(t)|^2 ] < \infty.
$$

4
This can be seen by regarding equation [1] as a stochastic functional differential equation in the sense of Mohammed [16]. □

However, we are also interested in having conditions for there to exist a unique solution to equation (1) for some general function \( \phi(t, s) \):

**Theorem 2.3** Consider the generalized case, where \( \phi(t, s) \) is some arbitrary function. Assume that

\[ b(t, X(t), Z(t)) = \tilde{b}(t, X(t)) + a Z(t), \]

where \( a \in \mathbb{R} \) and

\[ \sigma(t, X(t), Z(t)) = \sigma(t, X(t)) \]

Then, under some fairly weak additional regularity conditions (see Wang [27]), there exists a unique solution to the noisy memory SDE (1).

**Proof.** In this setting, the derivation of Section 2.1 combined with the conditions in Wang [27] guarantees existence of a unique solution. □

Note that it may be possible to prove the existence and uniqueness of a solution to equation (1) in general (without assumptions on the functions \( b \) and \( \sigma \)). However, this is beyond the scope of this paper. The Euler method presented here holds for all SDEs of the form (1).

### 2.3 The Euler scheme

Let \( \omega \in \Omega \) be a scenario. Let \( N > 0 \) be a (large) natural number and let the time step in the approximation, \( \Delta t := T/N \). Then, \( \Pi_{\text{pos}} := \{n\Delta t\}_{n=0,1,...,N} \) is a partition of the time interval \( [0, T] \). Similarly, one can partition the interval \( [-\delta, 0] \) as \( \Pi_{\text{neg}} := \{-\delta, -\delta + \Delta t, ..., -\delta + k\Delta t\} \), where \( k \) is the largest integer such that \(-\delta + k\Delta t \leq 0\). For \( i = 1, \ldots, N \), let \( \Pi_i \) denote the partition of the interval \([t_i - \delta, t_i] \) given by

\[ \Pi_i := (\Pi_{\text{pos}} \cup \Pi_{\text{neg}}) \cap [t_i - \delta, t_i], \]

i.e., the partition of \([t_i - \delta, t_i] \) coming from the partition of the whole time interval.

A natural generalization of the Euler scheme for standard SDEs (see for instance Iacus [9]) to the noisy memory SDE case is the following:

\[ X_{i+1}(\omega) = X_i(\omega) + b(t_i, X_i(\omega), Z_i(\omega))\Delta t + \sigma(t_i, X_i(\omega), Z_i(\omega))\Delta B_i(\omega) \]  

where \( \Delta B_i(\omega) := B(t_{i+1}, \omega) - B(t_i, \omega) \) with distribution \( N(0, \sqrt{\Delta t}) \) are increments of the Brownian motion and \( Z_i(\omega) := \sum_{j \in \Pi} \phi(t_i, t_j) X_j(\omega) \Delta B_j(\omega) \) approximates the noisy memory process. Note also that this is a pathwise (i.e., \( \omega \)-wise) approximation. However, in the next section, we will study the mean square error of the approximation in order to determine the convergence properties of this approximation to the exact solution.

Throughout the paper, we will assume that \( \delta > \Delta t \). This assumption is valid, as we are interested in what happens for small time steps.
3 The main result

It turns out that the noisy memory Euler scheme has mean-square order of convergence $\sqrt{\Delta t}$, which is the same as for ordinary SDEs (see Allen [1] and Mao [13], Theorem 7.3). We summarize this in the following main result:

**Theorem 3.1** The Euler approximation scheme for the solution of the stochastic noisy memory SDE (1) with constant time steps $\Delta t = \frac{T}{N}$ has mean-square order of convergence $\sqrt{\Delta t}$. That is, there exists a constant $\tilde{C}(T)$ such that if $X$ is the exact solution of the noisy memory SDE and $X_i$ is the approximated solution (at the same point), then

$$E[(X(t_i) - X_i)^2] \leq \tilde{C}(T)\Delta t$$

in all the approximation points $t_i$, $i = 1, \ldots, N$.

The rest of this section is devoted to some lemmas which are needed to prove this theorem. The final proof of Theorem 3.1 will be given in Section 4.

3.1 Some lemmas

In this section we prove some lemmas concerning the solution of the noisy memory SDE, which will be used later on in order to compute the order of convergence for the Euler approximation scheme.

We need some Lipschitz-type conditions on the given functions. Assume that there exists constants $K_1, K_2 > 0$ (independent of $\omega \in \Omega$) such that

$$|b(t, x_1, z_1) - b(s, x_2, z_2)|^2 \leq K_1(|t - s| + |x_1 - x_2|^2 + |z_1 - z_2|^2)$$

and

$$|\sigma(t, x_1, z_1) - \sigma(s, x_2, z_2)|^2 \leq K_2(|t - s| + |x_1 - x_2|^2 + |z_1 - z_2|^2).$$

(7)

We also assume that there exists constants $K_3, K_4 > 0$ such that

$$b(t, x, z)^2 \leq K_3(1 + x^2 + z^2)$$

and

$$\sigma(t, x, z)^2 \leq K_4(1 + x^2 + z^2).$$

(8)

For notational simplicity, we let $k = \max\{K_1, K_2, K_3, K_4\}$. In addition, we assume that the (real valued, deterministic) function $\phi$ is square integrable, so there exists a constant $\tilde{K}$ such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t, s)^2 dtds \leq \tilde{K}.$$  

(9)

In the following, let $X$ be the solution of the noisy memory SDE (1) and let $Z$ be the corresponding noisy memory process. In the following proofs, we will often use the inequality

$$2|ab| \leq a^2 + b^2.$$  

(10)

Note that inequality (10) implies that $(a + b)^2 \leq 2a^2 + 2b^2$.

**Lemma 3.2** There exists a constant $M > 0$ such that $E[X(t)^2] \leq M$ for all $t \in [0, T]$.
Lemma 3.3 There exists a constant $c > 0$ such that for all $t, s \in [0, T]$,

$$E[(X(t) - X(s))^2] \leq c|t - s|.$$
Lemma 3.4 follows from the Itô isometry and the fourth inequality follows from Lemma 3.2. The third inequality follows from assumption (8), the second equality uses the Itô isometry and the Cauchy-Schwartz inequality, the second inequality uses the Itô isometry (see e.g., Øksendal [19]), and the first inequality uses some algebra and inequality [10].

Proof. Assume that \( t > s \). If not, change the roles of \( t \) and \( s \).

We consider two cases:

(i) \( s \notin [t - \delta, t] \), i.e., \( s < t - \delta \): Then,

\[
E[|Z(t) - Z(s)|^2] = E[|\int_t^s \phi(t,u)X(u)dB(u) - \int_s^{t - \delta} \phi(s,u)X(u)dB(u)|^2]
\]

\[
\leq 2E[|\int_t^s \phi(t,u)X(u)dB(u)|^2] + 2E[|\int_s^{t - \delta} \phi(s,u)X(u)dB(u)|^2]
\]

\[
= 2E[\int_t^s \phi(t,u)^2X(u)^2du] + 2E[\int_s^{t - \delta} \phi(s,u)^2X(u)^2du]
\]

\[
\leq 2\int_t^{t - \delta} K M du + 2\int_s^{t - \delta} K M du \leq 4M K (t - s)
\]

where the first inequality uses inequality [10], the second equality uses the Itô isometry, the second inequality uses Lemma 3.2 and the final inequality follows from \( s < t - \delta \), i.e., \( \delta < t - s \).

Lemma 3.5

It holds that \( E[Z(t)^2] \leq \tilde{K} M \delta \) for all \( t \in [0, T] \).

Proof.

\[
E[Z(t)^2] = E[\int_{t - \delta}^t \phi(t,s)X(s)dB(s)^2]
\]

\[
= E[\int_{t - \delta}^t \phi(t,s)^2X(s)^2ds]
\]

\[
\leq \int_{t - \delta}^t \tilde{K} M ds = \tilde{K} M \delta,
\]

where the second equality uses the Itô isometry (see e.g., Øksendal [19]) and the inequality follows from Lemma 3.2.

Lemma 3.5

There exists a constant \( \tilde{N} > 0 \) such that

\[
E[|Z(t) - Z(s)|^2] \leq \tilde{N}|t - s|.
\]

Proof. Assume that \( t > s \). If not, change the roles of \( t \) and \( s \).

We consider two cases:

(i) \( s \notin [t - \delta, t] \), i.e., \( s < t - \delta \): Then,

\[
E[|Z(t) - Z(s)|^2] = E[|\int_t^s \phi(t,u)X(u)dB(u) - \int_s^{t - \delta} \phi(s,u)X(u)dB(u)|^2]
\]

\[
\leq 2E[|\int_t^s \phi(t,u)X(u)dB(u)|^2] + 2E[|\int_s^{t - \delta} \phi(s,u)X(u)dB(u)|^2]
\]

\[
= 2E[\int_t^s \phi(t,u)^2X(u)^2du] + 2E[\int_s^{t - \delta} \phi(s,u)^2X(u)^2du]
\]

\[
\leq 2\int_t^{t - \delta} \tilde{K} M du + 2\int_s^{t - \delta} \tilde{K} M du \leq 4M \tilde{K} (t - s)
\]

where the first inequality uses inequality [10], the second equality uses the Itô isometry, the second inequality uses Lemma 3.2 and the final inequality follows from \( s < t - \delta \), i.e., \( \delta < t - s \).
(ii) $s \in [t-\delta, t]:$ In this case,

\[
E[|Z(t) - Z(s)|^2] = E[\int_{t-\delta}^{t} \phi(t, u)X(u)dB(u) - \int_{s-\delta}^{s} \phi(s, u)X(u)dB(u)]^2
\]

\[
= E[\int_{t-\delta}^{t} \phi(t, u)X(u)dB(u) + \int_{s-\delta}^{s} \phi(s, u)X(u)dB(u)
+ \int_{t-\delta}^{t} \phi(t, u)X(u)dB(u)]^2
\]

\[
\leq 2E[\int_{t-\delta}^{t} \phi(t, u)X(u)dB(u) - \phi(s, u)X(u)dB(u)]^2
\]

\[
\leq 4E[(\int_{t-\delta}^{t} \phi(t, u)X(u)dB(u))^2] + 4E[(\int_{s-\delta}^{s} \phi(s, u)X(u)dB(u))^2]
\]

\[
+ 2\int_{t-\delta}^{t} \phi(t, u)X(u)dB(u) - \phi(s, u)X(u)dB(u)
\]

\[
\leq 4\int_{s}^{t} KE[X(u)^2]du + 4\int_{t-\delta}^{s} \phi(s, u)X(u)dB(u) - \phi(s, u)X(u)dB(u)
\]

\[
\leq 2(t-s)KM(4+\delta)
\]

where the second equality follows from $s \in [t-\delta, t]$, the first and second inequality follows from inequality \[10\], the third inequality follows from the Itô isometry and assumptions \{??\}. The final inequality follows from Lemma \[3\].

Hence, by combining the two items above, we see that

\[
E[|Z(t) - Z(s)|^2] \leq \max\{2M\tilde{K}(4+\delta), 4M\tilde{K}\}(t-s).
\]

The lemma follows by defining $\tilde{N}$ to be this maximum. \hfill \square

4 Error analysis and proof of the main theorem

In this section, we derive an error bound for the Euler approximation method for SDEs with generalized noisy memory. We shall see that the approximation converges to the solution of the noisy memory SDE and find the order of convergence, and thereby prove our main result, Theorem \[3.3\].

Similarly to Allen \[1\], for $t \in [t_i, t_{i+1})$, $i = 1, \ldots, N$, define

\[
\hat{X}(t) := X_i + \int_{t_i}^{t} b(t_i, X_i, Z_i)ds + \int_{t_i}^{t} \sigma(t_i, X_i(\omega), Z_i(\omega))dB(s).
\]

Note that $\hat{X}(t_i) = X_i$ for $i = 1, 2, \ldots, N$, i.e., in the time nodes, the process $\hat{X}$ equals the approximation to the solution of the noisy memory process.

We study the error

\[
\epsilon(t) = X(t) - \hat{X}(t)
\]

where $X$ is the exact solution to the noisy memory SDE \[1\]. The goal of this section is to prove that there exists a constant $C$ such that $E[|\epsilon(t_i)|^2] \leq \tilde{C}\Delta t$ for $t_i \in \Pi_{\text{pos}}$.

From the definitions,

\[
de(t) = (b(t, X(t), Z(t)) - b(t_i, X_i, Z_i))dt + \sigma(t, X(t), Z(t)) - \sigma(t_i, X_i, Z_i))dB(t)
\]
and \( \epsilon(t_i) = X(t_i) - X_i \) for \( i = 1, 2, \ldots, N \).

From Itô’s formula applied to the function \( g(t, \epsilon) = \epsilon^2 \), we see that

\[
d[\epsilon(t_i)^2] = 2(X(t) - \tilde{X}(t))(b(t, X(t), Z(t)) - b(t_i, X_i, Z_i))dt + 2(X(t) - \tilde{X}(t))(\sigma(t, X(t), Z(t)) - \sigma(t_i, X_i, Z_i))dB(t) + (\sigma(t, X(t), Z(t)) - \sigma(t_i, X_i, Z_i))^2 dt.
\]

Hence,

\[
E[\epsilon(t_{i+1})^2] = E[\epsilon(t_i)^2] + 2E[\int_{t_i}^{t_{i+1}} (X(s) - \tilde{X}(s))(b(s, X(s), Z(s)) - b(t_i, X_i, Z_i))ds] + E[\int_{t_i}^{t_{i+1}} (\sigma(t, X(t), Z(t)) - \sigma(t_i, X_i, Z_i))^2 dt]
\]

\[
\leq E[\epsilon(t_i)^2] + \int_{t_i}^{t_{i+1}} E[(X(s) - \tilde{X}(s))^2]ds + \int_{t_i}^{t_{i+1}} E[(\sigma(t, X(t), Z(t)) - \sigma(t_i, X_i, Z_i))^2]dt
\]

(13)

where the inequality follows from inequality (10).

Note that

\[
E[(b(t, X(t), Z(t)) - b(t_i, X_i, Z_i))^2]
\]

\[
= E[(b(t, X(t), Z(t)) - b(t_i, X_i, Z_i) + b(t_i, X_i, Z_i) - b(t_i, X_i, Z_i))^2]
\]

\[
\leq 2E[(b(t, X(t), Z(t)) - b(t_i, X_i, Z_i))^2] + 2E[(b(t_i, X_i, Z_i) - b(t_i, X_i, Z_i))^2]
\]

\[
\leq 2kE[|t - t_i| + |X(t) - X(t_i)|^2 + |Z(t) - Z(t_i)|^2 + |X(t_i) - X_i|^2 + |Z(t_i) - Z_i|^2]
\]

where the first inequality follows from the triangle inequality and inequality (10). The final inequality follows from the assumption (7). Similarly, one can prove that

\[
E[(\sigma(t, X(t), Z(t)) - \sigma(t_i, X_i, Z_i))^2] \leq 2kE[|t - t_i| + |X(t) - X(t_i)|^2 + |Z(t) - Z(t_i)|^2 + |X(t_i) - X_i|^2 + |Z(t_i) - Z_i|^2].
\]

Therefore, combining this with inequality (13) and using the definition of the error \( \epsilon(t) \),

\[
E[\epsilon(t_{i+1})^2] \leq E[\epsilon(t_i)^2] + \int_{t_i}^{t_{i+1}} \epsilon(t) dt + 4k \int_{t_i}^{t_{i+1}} E[|t - t_i| + |X(t) - X(t_i)|^2 + |Z(t) - Z(t_i)|^2 + |X(t_i) - X_i|^2 + |Z(t_i) - Z_i|^2]dt.
\]

(14)

Due to the noisy memory process, there is an additional source of error, compared to approximation of regular SDEs. In the following, let \( X(t) \) be an exact solution of the noisy memory SDE (11), and let \( X_j, t_j \in [0, T] \) be its approximation from the Euler method (6). For \( i = 1, \ldots, N \), define \( Z^B := \sum_{j \in \Pi} X(t_j) \Delta B_j \), i.e., the approximated noisy memory process involving the exact solution \( X \).

Lemma 4.1 For a time \( t_i \) in the partition of the time interval and \( Z^B_i = \sum_{j \in \Pi} \phi(t_i, t_j) X(t_j) \Delta B_j \), we have

\[
E[|Z(t_i) - Z^B_i|^2] \leq \Delta t \delta \bar{K}(M + c).
\]
Lemma 4.1. where the first inequality uses inequality (10) and the second inequality uses the discrete Itô isometry. Therefore,

where the final equality uses the discrete Itô isometry. Therefore,

Lemma 4.2

Proof. From the definitions,

\[
E[|Z(t_i) - Z(t_i)|^2] = E[\int_{t_i}^{t_{i+1}} \Phi(t_i, s) X(s) dB(s) - \sum_{j \in \Pi} \phi(t_i, t_j) X(t_j) \Delta B_j]^2]
\]

\[
= E[\sum_{j \in \Pi} \int_{t_i}^{t_{i+1}} (\phi(t_i, s) X(s) - \phi(t_i, t_j) X(t_j)) dB(s)]^2]
\]

\[
= E[\sum_{j \in \Pi} \int_{t_i}^{t_{i+1}} (\phi(t_i, s) X(s) - \phi(t_i, t_j) X(t_j)) dB(s)]^2
\]

\[
+ 2E[\sum_{j \in \Pi} \int_{t_i}^{t_{i+1}} (X(s) - X(t_j))^2 \phi(t_i, t_j)^2 ds]
\]

\[
\leq 2 \sum_{j \in \Pi} \int_{t_i}^{t_{i+1}} M \tilde{K}(s - t_j) ds + 2 \sum_{j \in \Pi} \int_{t_i}^{t_{i+1}} c(s - t_j) \tilde{K} ds
\]

\[
= \delta \Delta t \tilde{K}(M + c)
\]

where the fourth equality follows from the Itô isometry (see e.g. Øksendal [19]) and the inequality from Lemma 3.2, Lemma 3.3 and assumptions (?)-(??).

We can now prove the following lemma which relates the error in the noisy memory process \( Z \) to the error in the solution process \( X \).

**Lemma 4.2** For \( i = 1, \ldots, N \), let \( Z(t_i) \) be the noisy memory process, and \( Z_i \) the approximated noisy memory process, then

\[
E[|Z(t_i) - Z_i|^2] \leq 2\Delta t \delta \tilde{K}(M + c) + \tilde{K} \Delta t \sum_{j \in \Pi} E[\epsilon(t_j)^2].
\]

**Proof.** First, note that

\[
E[|Z_i - Z_i|^2] = E[|\sum_{j \in \Pi} X(t_j) \phi(t_i, t_j) \Delta B_j - \sum_{j \in \Pi} X_j \phi(t_i, t_j) \Delta B_j|^2]
\]

\[
= \sum_{j \in \Pi} \phi(t_i, t_j)^2 E[(X(t_j) - X_j)^2] \Delta t
\]

where the final equality uses the discrete Itô isometry. Therefore,

\[
E[|Z(t_i) - Z_i|^2] = E[|Z(t_i) - Z_i + Z_i - Z_i|^2]
\]

\[
\leq 2E[|Z(t_i) - Z_i|^2] + 2E[|Z_i - Z_i|^2]
\]

\[
\leq 2\Delta t \delta \tilde{K}(M + c) + \tilde{K} \Delta t \sum_{j \in \Pi} E[\epsilon(t_j)^2]
\]

where the first inequality uses inequality [10] and the second inequality uses Lemma 4.1.

\[ \square \]

We are nearly ready to prove our main result, Theorem 3.4. However, we need one more lemma:

**Lemma 4.3** Let \( x > 0 \) and \( n \in \mathbb{N} \). Then,

\[
(1 + \frac{x}{n})^n \leq e^x.
\]

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Proof. The exponential function is convex, and therefore it dominates its first order Taylor approximation at 0, so \( e^y \geq 1 + y \) for all \( y \). By inserting \( y = x/n \) and taking the \( n \)’th power, the desired inequality follows.

Finally, using all of these lemmas, we are ready to prove our main result.

Proof of Theorem 2.3

Recall from Theorem 3.1 that we would like to prove that the expected squared error of the numerical scheme is bounded by some constant (depending on the terminal time) times the time step. That is, we want to prove that \( \mathbb{E}[e(t_i)^2] \leq \hat{C}(T)\Delta t \). By combining inequality (14) with Lemma 3.3, Lemma 3.5 and Lemma 1.2, we see that

\[
\begin{align*}
\mathbb{E}[e(t_{i+1})^2] & \leq \mathbb{E}[e(t_i)^2] + \int_{t_i}^{t_{i+1}} \mathbb{E}[e(t)^2] dt + 4k \mathbb{E}[e(t_i)^2] \Delta t + 2k(\Delta t)^2 \\
& \quad + 4kc \int_{t_i}^{t_{i+1}} (t-t_i) dt + 4k \int_{t_i}^{t_{i+1}} \tilde{N}(t-t_i) dt \\
& \quad + 4k \int_{t_i}^{t_{i+1}} (2\Delta t \delta(M \tilde{K} + \tilde{K}c) + \tilde{K} \Delta t \sum_{j \in \Pi_i} \mathbb{E}[e(t_j)^2]) dt \\
& = \mathbb{E}[e(t_i)^2](1 + 4k\Delta t) + 2k(\Delta t)^2(1 + c + \tilde{N} + 2(2\delta(M \tilde{K} + \tilde{K}c)) \\
& \quad + \tilde{K} \sum_{j \in \Pi_i} \mathbb{E}[e(t_j)^2]) + \int_{t_i}^{t_{i+1}} \mathbb{E}[e(t)^2] dt.
\end{align*}
\]

Hence, by using the Bellman-Gronwall inequality, we see that

\[
\begin{align*}
\mathbb{E}[e(t_{i+1})^2] & \leq \mathbb{E}[e(t_i)^2](1 + 4k\Delta t) + 2k(\Delta t)^2(1 + c + \tilde{N} + 2(2\delta(M \tilde{K} + \tilde{K}c)) \\
& \quad + \tilde{K} \sum_{j \in \Pi_i} \mathbb{E}[e(t_j)^2]) + \int_{t_i}^{t_{i+1}} \mathbb{E}[e(t)^2] dt \\
& = e^{\Delta t} \{ \mathbb{E}[e(t_i)^2](1 + 4k\Delta t) + 2k(\Delta t)^2(1 + c + \tilde{N} + 4\delta(M \tilde{K} + \tilde{K}c)) \\
& \quad + 2\tilde{K} \sum_{j \in \Pi_i} \mathbb{E}[e(t_j)^2]) \}
\end{align*}
\]

For \( i = 1, \ldots, N \), define \( a_i := \mathbb{E}[e(t_i)^2] \). From the previous computations, we know that

\[
a_{i+1} \leq Ra_i + S \sum_{j \in \Pi_i} a_j + A
\]

where

\[
R := e^{\Delta t}(1 + 4k\Delta t) > 0, \quad S := 4\tilde{K}k(\Delta t)^2e^{\Delta t} > 0
\]

and

\[
A := 2k(\Delta t)^2e^{\Delta t}(1 + c + \tilde{N} + 4\delta(M \tilde{K} + \tilde{K}c)) > 0.
\]

Note that,

\[
a_{i+1} \leq Ra_i + A + S \sum_{j \in \Pi_i} a_j \\
\leq R \max_{j \in \Pi_i} a_j + S \sum_{j \in \Pi_i} \max_{j \in \Pi_i} a_j + A \\
= (R + S \frac{\delta}{M}) \max_{j \in \Pi_i} a_j + A \\
=: \tilde{R} \max_{j \in \Pi_i} a_j + A
\]

where \( \tilde{R} := R + S \frac{\delta}{M} = e^{\Delta t}(1 + 4k\Delta t(1 + \delta\tilde{K})) \).
By induction, and the fact that the initial approximation error is 0, inequality \([16]\) implies that \(a_n \leq \bar{A}^{\frac{n}{R-1}}\) for \(n = 1, \ldots, N\), i.e.,

\[
E[e(t_n)^2] \leq 2e^{\Delta t(k(\Delta t)^2(1 + c + N + 4\delta(M\bar{K} + \bar{K}c)))} e^{Kc(\Delta t)^{n-1}} = \Delta t e^{\frac{c^2}{2(1+\delta K)}} (1 + k(1 + \delta\bar{K})) \Delta t^n (1 + c + k + 4\delta(M\bar{K} + \bar{K}c)).
\]

(17)

Now, note that \(e^{4k(1+\delta\bar{K})T} \geq (1 + \frac{4k(1+\delta\bar{K})T}{n})^n\) because of Lemma \([4.3]\). By combining Lemma \([4.3]\) with the inequality \((17)\) and recalling that \(\Delta t = \frac{T}{n}\), we reach our goal

\[
E[e(t_n)^2] \leq \Delta t e^{\frac{c^2}{2(1+\delta K)}(1 + c + N + 4\delta(M\bar{K} + \bar{K}c))}
\]

where \(\bar{C}(T) := e^{\frac{c^2}{2(1+\delta K)}(1 + c + N + 4\delta(M\bar{K} + \bar{K}c))}.\) This completes the proof of Theorem \([5.1]\). \(\square\)

5 A noisy memory SDE with an analytical solution and a numerical example

In this section, we will compare the exact solution of a (very simple) SDE with noisy memory to the approximation given by the Euler method. We consider the following SDE with noisy memory:

\[
\begin{align*}
\frac{dX(t)}{dt} &= Z(t)dB(t) \text{ for } t \in [0, T] \\
X(t) &= 1, \ t \in [-\delta, 0].
\end{align*}
\]

(18)

where \(Z(t) := \int_{t-\delta}^{t} X(s)dB(s)\), so \(\phi(t,s) = 1\) for all \(t,s \in [0, T]\). We can solve \([18]\) analytically by using a technique from Dahl et al. \([6]\), based on rewriting the noisy SDE \([18]\) as a two-dimensional SDE with delay. This kind of delay equation can be solved iteratively for each \(\delta\)-interval.

First, we rewrite the noisy SDE \([18]\) by defining \(X_1(t) := X(t)\) and \(X_2(t) := \int_{t-\delta}^{t} X_1(s)dB(s), \ t \in [-\delta, T]\). Note that from these definitions, \(Z(t) = X_2(t) - X_2(t-\delta), \ t \in [-\delta, T]\). Then, the noisy SDE \([18]\) can be rewritten as a two-dimensional SDE with delay:

\[
\begin{align*}
\frac{dX_1(t)}{dt} &= (X_2(t) - X_2(t-\delta))dB(t), t \in (0, T], \\
\frac{dX_2(t)}{dt} &= X_1(t)dB(t), t \in [0, T], \\
X_1(t) &= 1, t \in [-\delta, 0], \\
X_2(-\delta) &= 0.
\end{align*}
\]

(19)

Note that \(X_1(t)\) and \(X_2(t)\) are known from the initial conditions for \(t \in [-\delta, 0]\). We write \([19]\) in matrix form. Define \(Y(t) := (X_1(t), X_2(t))^{\top}\) (where \((\cdot)^\top\) denotes the transpose), \(t \in [-\delta, T]\). Then, from \([19]\)

\[
\begin{align*}
\frac{dY(t)}{dt} &= aY(t)dB(t) + bY(t-\delta)dB(t), t \in [0, T] \\
Y(t) &= (1, B(t) - B(-\delta))^{\top}, t \in [-\delta, 0],
\end{align*}
\]

(20)
where $a$ and $b$ are in $\mathbb{R}^{2 \times 2}$ and defined by

$$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

For $t \in [0, \delta]$, we may rewrite (20) as

$$dY(t) = aY(t)dB(t) + \begin{bmatrix} B(\delta) - B(t) - \delta \\ 0 \end{bmatrix} dB(t),$$

which is a regular SDE without delay. For notational simplicity, define $K(t - \delta) := B(t - \delta) - B(\delta)$. Note that for $t \in [0, \delta]$, $K(t - \delta)$ is a known process which is independent of everything after time 0.

To solve this equation, define

$$F(t) := \exp(-aB(t) + \frac{1}{2}a^2t)$$

where we (in general) define the matrix exponential for a matrix $A \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$, as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

By this definition, we find (by analyzing the infinite sum) that

$$F(t) = e^{\frac{1}{2}t} \begin{bmatrix} \cosh B(t) & -\sinh B(t) \\ -\sinh B(t) & \cosh B(t) \end{bmatrix}.$$

Note also that

$$a^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which clearly commutes with the matrices $F(t)$ (for all $t$) and $a$. Also, the matrices $a$ and $F(t)$ commute (for all $t$). This justifies the following calculations:

By the two-dimensional Itô formula,

$$dF(t) = F(t)(a^2 dt - adB(t)).$$

Hence, by the Itô product rule,

$$d(F(t)Y(t)) = F(t) \begin{bmatrix} -K(t - \delta) \\ 0 \end{bmatrix} dB(t) - aF(t) \begin{bmatrix} -K(t - \delta) \\ 0 \end{bmatrix} dt.$$ 

By integrating between times 0 and $t$,

$$Y(t) = e^{aB(t) - \frac{1}{2}a^2 t} \left( Y(0) + \int_0^t F(s) \begin{bmatrix} -K(s - \delta) \\ 0 \end{bmatrix} dB(s) - \int_0^t aF(s) \begin{bmatrix} -K(s - \delta) \\ 0 \end{bmatrix} ds \right),$$

where $Y(0) = (1, -B(-\delta))^T$.

The first component of this solution $Y(t)$ is the exact solution $X(t)$ of the noisy memory SDE (18) for times $t \in [0, \delta]$. Furthermore, one can continue and iteratively solve (18) for the interval $[\delta, 2\delta]$ using the solution based on (21) as
an initial condition. By continuing like this, one can solve the equation on the entire interval \([0, T]\). We will not calculate more solutions, as the one calculated above is sufficient for our goal of illustrating the Euler method.

We now compare the exact solution just derived to the numerical approximation based on the Euler method. Let \(\delta = 1\) and \(T = 1\). It would perhaps be more realistic to choose \(T\) larger than \(\delta\) (i.e., the time span of interest is greater than the time of memory). However, as the previous exact solution gets very complicated for \(\delta < T\), we restrict ourselves to the case \(\delta = T\). Figure 1 shows 1000 different simulations of the paths of the exact solution (21) have been plotted against the corresponding paths of the approximated Euler solution using time steps of size \(\Delta t = 1/100\). In addition, the corresponding mean square error has been computed by Monte Carlo simulation (with these 1000 simulations), and this error has also been plotted. As seen by the dashed line (representing the mean square error of the Euler approximation method) in Figure 1, the Euler method approximates the exact solution well in a mean square sense.

![Plot of 1000 paths of the exact solution of noisy SDE and the corresponding Euler approximation paths as well as the mean square error computed by Monte Carlo simulation.](image)

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