Distribution of Schmidt-like eigenvalues for Gaussian ensembles of the random matrix theory

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Received 3 November 2012, in final form 15 January 2013
Published 25 February 2013
Online at stacks.iop.org/JPhysA/46/115002

Abstract
We study the probability distribution function $P_n^{(β)}(w)$ of the Schmidt-like random variable $w = x_j^2 / \left( \sum_{j=1}^{n} x_j^2 / n \right)$, where $x_j$, ($j = 1, 2, \ldots, n$), are unordered eigenvalues of a given $n \times n$ β-Gaussian random matrix, $β$ being the Dyson symmetry index. This variable, by definition, can be considered as a measure of how any individual (randomly chosen) eigenvalue deviates from the arithmetic mean value of all eigenvalues of a given random matrix, and its distribution is calculated with respect to the ensemble of such β-Gaussian random matrices. We show that in the asymptotic limit $n \to \infty$ and for arbitrary $β$ the distribution $P_n^{(β)}(w)$ converges to the Marčenko–Pastur form, i.e. is defined as $P_n^{(β)}(w) \sim \sqrt{(4 - w)/w}$ for $w \in [0, 4]$ and equals zero outside of the support, despite the fact that formally $w$ is defined on the interval $[0, n]$. Furthermore, for Gaussian unitary ensembles ($β = 2$) we present exact explicit expressions for $P_n^{(β=2)}(w)$ which are valid for arbitrary $n$ and analyse their behaviour.

PACS numbers: 02.50.−r, 02.10.Yn

(Some figures may appear in colour only in the online journal)

1. Introduction
Random covariance matrices were introduced by Wishart in his studies of multivariate populations [1]. In the physical literature, statistical properties of the eigenvalues of random matrices have attracted a great deal of attention since the seminal works of Wigner [2], Dyson and Mehta [3, 4] and Mehta [5]. Various random variables associated with the eigenvalues of random matrices have been analysed, such as, e.g., gaps in the eigenvalue spectra, number of eigenvalues in a given interval, largest or smallest eigenvalues, etc, with a special emphasis put on their typical or atypical behaviour. A variety of results and their relevance to physical systems have been recently discussed in [6, 7].
One of such variables is the so-called Schmidt eigenvalue, used to characterize, e.g., the degree of entanglement of random pure states in bipartite quantum systems. It can be defined as one of the eigenvalues of a given random matrix divided by the trace, i.e. the sum of all eigenvalues. On physical grounds, this variable can be therefore considered as a measure of heterogeneity of the eigenvalues and shows how any individual (e.g., a randomly chosen) eigenvalue deviates from the arithmetic mean of all eigenvalues of a given random matrix. A number of significant results on the distributions of such eigenvalues and their extreme values for $\beta$-Laguerre–Wishart matrices have been obtained (see, e.g., [8–12] and references therein), as well as in [13] for Hermitian matrices. Such random variables have also been considered recently within a different context as probes of an effective broadness of the first-passage time distributions in bounded domains [14, 15].

In this paper, we evaluate the probability distribution function $P_n^{(\beta)}(w)$ of a Schmidt-like random variable of the form

$$w = \frac{x_k^2}{\sum_{j=1}^{n} x_j^2/n},$$

where $x_j, (j = 1, 2, \ldots, n)$, are unordered eigenvalues of a given $n \times n \beta$-Gaussian random matrix, $\beta$ being the Dyson symmetry index. Since $x_j$ are not ordered, without any lack of generality we set $k = 1$ in what follows. Note that we use a term ‘Schmidt-like random variable’ since here we define $w$ as the ratio of a squared eigenvalue over the sum of all squared eigenvalues. Within such a definition $w$ is always positive definite and has a support on $[0, n]$.

The probability distribution function $P_n^{(\beta)}(w)$ is given by

$$P_n^{(\beta)}(w) = \left\langle \delta \left( w - \frac{nx_1^2}{\sum x_j^2} \right) \right\rangle,$$

where the angle brackets denote averaging with respect to the weight:

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{\mathcal{K}_n} \exp \left( -\beta \frac{1}{2} \sum_{k=1}^{n} x_k^2 \right) \prod_{\langle i, j \rangle} |x_j - x_i|^\beta,$$

$\mathcal{K}_n$ being a known normalization constant [5] and the symbol $\langle i, j \rangle$ signifies that the product is taken over all possible distinct pairs $(i, j)$, $i, j = 1, 2, \ldots, n$. Note that the individual eigenvalue density $\rho_n^{(\beta)}(x_1)$ is defined, in the standard way, as

$$\rho_n^{(\beta)}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{l=2}^{n} dx_l P(x_1, x_2, \ldots, x_n)$$

$$= \frac{\exp \left( -\beta x_1^2/2 \right)}{\mathcal{K}_n} \prod_{l=2}^{n} dx_l \exp \left( -\beta \frac{1}{2} x_l^2 \right) \prod_{\langle i, j \rangle} |x_j - x_i|^\beta.$$ (4)

Recall that $\rho_n^{(\beta)}(x_1)$ is known explicitly for arbitrary $\beta$ [5].

Our aim is to determine an asymptotic behaviour of $P_n^{(\beta)}(w)$ for arbitrary $\beta$ and $n \to \infty$. Apart of this, we will present an exact, explicit results for Gaussian unitary ($\beta = 2$) ensembles (GUE) valid for arbitrary $n$.

The paper is outlined as follows. In section 2, we provide some general results for $P_n^{(\beta)}(w)$ and analyse its asymptotic forms when $n \to \infty$. In section 3, we present explicit results for the distribution function of the Schmidt-like eigenvalues for GUE. Finally, in section 4 we conclude with a brief recapitulation of our results.
2. Asymptotic behaviour for arbitrary \( \beta \)

Taking advantage of the Fourier cosine representation of the delta-function, we can conveniently rewrite equation (2) as

\[
P^{(\beta)}_n (w) = \frac{2}{\pi^2} \int_0^\infty dy \cos(wy) \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{i=1}^n dx_i \exp \left( -\frac{\beta}{2} x_i^2 \right)
\]

\[
= \frac{2}{\pi^2 K_n} \int_0^\infty dy \cos(wy) \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{i=1}^n dx_i \exp \left( -\frac{\beta}{2} x_i^2 \right)
\]

\[
\times \prod_{(i,j)} |x_j - x_i|^\beta \cos \left( \frac{nyx_i^2}{\sum_{j=1}^n x_j^2} \right).
\]

(5)

Expanding next the cosine into the Taylor series

\[
\cos \left( \frac{nyx_i^2}{\sum_{j=1}^n x_j^2} \right) = \sum_{k=0}^\infty \frac{(-1)^k (nyx_i^2)^{2k}}{(2k)! \left( \sum_{j=1}^n x_j^2 \right)^{2k}}.
\]

(6)

we rewrite equation (5) as

\[
P^{(\beta)}_n (w) = \frac{2}{\pi^2 K_n} \int_0^\infty dy \cos(wy) \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{i=1}^n dx_i \exp \left( -\frac{\beta}{2} x_i^2 \right)
\]

\[
\times \prod_{(i,j)} |x_j - x_i|^\beta \sum_{k=0}^\infty \frac{(-1)^k (nyx_i^2)^{2k}}{(2k)! \left( \sum_{j=1}^n x_j^2 \right)^{2k}}.
\]

(7)

At the next step, it is convenient to single out the integration over \( dx_1 \) and to rewrite the latter equation formally as

\[
P^{(\beta)}_n (w) = \frac{2}{\pi^2} \int_0^\infty dy \cos(wy) \int_{-\infty}^\infty dx_1 \mathcal{Q}(x_1, w),
\]

(8)

where the integration kernel \( \mathcal{Q}(x_1, w) \) is defined as

\[
\mathcal{Q}(x_1, w) = \exp \left( -\frac{\beta x_1^2}{2} \right) \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{i=2}^n dx_i \exp \left( -\frac{\beta}{2} x_i^2 \right)
\]

\[
\times \prod_{(i,j)} |x_j - x_i|^\beta \sum_{k=0}^\infty \frac{(-1)^k (nyx_1^2)^{2k}}{(2k)! \left( \sum_{j=1}^n x_j^2 \right)^{2k}}.
\]

(9)

Furthermore, using the integral identity

\[
\left( \sum_{j=1}^n x_j^2 \right)^{-2k} = \frac{1}{(2k-1)!} \int_0^\infty d\xi \exp \left( -\xi \sum_{j=1}^n x_j^2 \right) \xi^{2k-1}.
\]

(10)

we can cast equation (9) into the following form:

\[
\mathcal{Q}(x_1, w) = \sum_{k=1}^\infty \frac{(-1)^k (nyx_1^2)^{2k}}{(2k)! \left( 2k-1 \right)!} \int_0^\infty d\xi \xi^{2k-1}
\]

\[
\times \left[ \exp \left( -\frac{\beta x_1^2}{2} \right) \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{i=2}^n dx_i \exp \left( -\left( \frac{\beta + \xi}{2} \right) x_i^2 \right) \prod_{(i,j)} |x_j - x_i|^\beta \right].
\]

(11)
One immediately notes that upon a simple change of the integration variables \( x_l \rightarrow z_l / \sqrt{1 + 2k / B} \), \( l = 2, 3, \ldots, n \), the expression in the square brackets becomes equal to, up to a numerical factor, the eigenvalue density defined by equation (4). This means that the multiple integrals over \( dx_l \) (with \( l = 2, 3, \ldots, n \)) can be straightforwardly performed, as well as the integration over \( d\xi \), to give

\[
P_n^{(\beta)}(w) = \frac{2}{\pi} \int_0^\infty dy \cos(\omega y) \int_{-\infty}^{\infty} dx \rho_n^{(\beta)}(x) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(f_\beta/2)}{(2k)! \Gamma(2k + f_\beta/2)} \left( \frac{\beta ny^2}{2} \right)^{2k},
\]

(12)

where \( \rho_n^{(\beta)}(x) \) is the eigenvalue density\(^3\) in equation (4) and \( f_\beta = n + \beta n(n - 1)/2 \). Next, performing in equation (12) the summation over \( k \), we obtain

\[
P_n^{(\beta)}(w) = \frac{\Gamma(f_\beta/2)}{\pi} \int_0^\infty dy \cos(\omega y) \int_{-\infty}^{\infty} dx \rho_n^{(\beta)}(x) (I_{f_\beta/2-1}(x\sqrt{2\beta ny}) + \text{c.c.}),
\]

(13)

where \( I_\nu(\nu z) = (\sqrt{\nu z}/2)^{-\nu} I_\nu(\sqrt{\nu z}) \), \( \phi = x\sqrt{2\beta ny} \), \( \nu = f_\beta/2 - 1 \), \( I_\nu(\sqrt{\nu z}) \) is the modified Bessel function and ‘c.c.’ stands for the complex conjugate of \( I_\nu(\sqrt{\nu z}) \). Equation (13) constitutes our main general result valid for arbitrary \( \beta \).

The result in equation (13) allows us to establish, for arbitrary \( \beta \), the limiting asymptotic behaviour of the distribution \( P_n^{(\beta)}(w) \) when \( n \rightarrow \infty \). To do this, recall that the modified Bessel function \( I_\nu(\nu z) \) has a uniform asymptotic expansion at large values of the order \( \nu \) (see, e.g., [16, chapter 9.77, p 378]). Its leading, in the limit \( \nu \rightarrow \infty \), term is given by

\[
I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi \nu} \ (1 + z^2)^{1/4}} (1 + O(1/\nu)),
\]

(14)

where

\[
\eta = \sqrt{1 + z^2} + \ln \left( \frac{z}{1 + \sqrt{1 + z^2}} \right).
\]

(15)

Consequently, for \( \nu \rightarrow \infty \) we have

\[
I_\nu(\sqrt{\nu} \phi) \sim \frac{1}{\sqrt{2\pi \nu}} \left( \frac{2}{\sqrt{\nu}} \right)^\nu \exp(\nu \eta) \left( 1 + O(1/\nu) \right),
\]

(16)

where \( \eta \) can be written explicitly as

\[
\eta = \sqrt{1 + \nu \phi^2 / \nu^2} - \ln(\nu) + \ln(\sqrt{\nu} / 2) - \ln \left( \frac{1 + \sqrt{1 + \nu \phi^2 / \nu^2}}{2} \right).
\]

(17)

Next, note that since \( \nu \sim n^2 \), the ratio \( \sqrt{\nu} / \nu \sim 1/n^{3/2} \) as \( n \rightarrow \infty \), and hence, the \( \sqrt{\nu} / \nu \) vanishes in this limit for any fixed \( \phi \). In consequence, \( \nu \eta \) in the exponential in equation (16) exhibits the following large-\( n \) behaviour:

\[
\nu \eta \sim -v \ln(\nu) + v + \ln(\sqrt{\nu} / 2)^v + \frac{\nu \phi^2}{4v} + O \left( \frac{n^2}{\nu^3} \right),
\]

(18)

so that

\[
I_\nu(\sqrt{\nu} \phi) \sim \frac{1}{\sqrt{2\pi v}} \exp \left( -v \ln(\nu) + v + i \frac{\beta ny^2}{2v} \right).
\]

(19)

Performing a similar analysis of the asymptotic large-\( n \) behaviour of the complex conjugate of \( I_\nu(\sqrt{\nu} \phi) \) and taking into account that

\[
\ln \Gamma(f_\beta/2) \sim \exp \left( v \ln(\nu) - v + \frac{1}{2} \ln(2\pi v) + O(1/\nu) \right),
\]

(20)

\(^3\) Note that we drop the subscript ‘1’ in the variable \( x_1 \).
we arrive eventually at the following large-\(n\) representation of the probability distribution

\[
P^{(\beta)}_{n \to \infty}(w) \sim \frac{2}{\pi} \int_0^\infty dy \cos(\omega y) \int_{-\infty}^\infty dx \, \rho^{(\beta)}_{n \to \infty}(x) \cos \left(\frac{2yx^2}{n}\right).
\]

This latter equation yields, after performing the integrations, the following asymptotic form:

\[
P^{(\beta)}_{n \to \infty}(w) \sim \sqrt{\frac{n}{2\pi w}} \rho^{(\beta)}_{n \to \infty}\left(\sqrt{\frac{nw}{2}}\right),
\]

which expresses the asymptotic behaviour of the desired probability distribution of the Schmidt-like random variable \(w\) through the asymptotic behaviour of the eigenvalue density, equation (4), with an appropriately rescaled variable. The asymptotic behaviour of the latter is well known and is defined by the Wigner semi-circle distribution [2, 5], so that we find straightforwardly the following asymptotic form for the normalized probability distribution function:

\[
P^{(\beta)}_{\infty}(w) = \frac{1}{2\pi} \begin{cases} 
\sqrt{\frac{4-w}{w}}, & \text{for } 0 < w < 4, \\
0, & \text{for } w > 4.
\end{cases}
\]

Equation (23) holds for any value of the Dyson symmetry index \(\beta\). It might seem surprising at the first glance that the limiting distribution in equation (23) has the form of the Marčenko–Pastur law [17]. On the other hand, recall that as \(n \to \infty\), the eigenvalues tend to be equidistantly spaced so that the sum \(\sum_j x_j^2/n\) tends to a constant. Then, it becomes clear why the distribution \(P^{(\beta)}_{n \to \infty}(w)\) converges to an appropriately normalized single eigenvalue density, defined by the semi-circle distribution [2, 5], so that its squared value is distributed according to the Marčenko–Pastur law. Note that the limiting distribution in equation (23) has been also found in [13] for Schmidt eigenvalues of Hermitian random matrices.

3. Gaussian unitary ensemble

We turn now to the GUE case (\(\beta = 2\) and hence, \(f_2 = n^2\)), aiming to evaluate an explicit expression for the probability distribution \(P^{(2)}_{n}(w)\), valid for an arbitrary value of \(n\). In this case, the eigenvalue density in equation (4) is explicitly given by

\[
\rho^{(2)}_n(x) = \frac{\exp(-x^2)}{2^{n/2} \sqrt{\pi} n!} \left[H_n^2(x) - H_{n+1}(x)H_{n-1}(x)\right],
\]

where \(H_n(x)\) denotes the Hermite polynomial [16]. Furthermore, we find that

\[
\int_{-\infty}^\infty dx \, \rho^{(2)}_n(x) I_{f_2/2-1}(2x\sqrt{\pi}y) = \frac{1}{2\sqrt{\pi}} \int_{-1}^1 dr \frac{1-r^2}{2^{n/2} \Gamma(n/2 - 1/2)} f(t\sqrt{\pi}y),
\]

where

\[
f(t\sqrt{\pi}y) = \int_{-\infty}^\infty dx \, e^{-(x-i\sqrt{\pi}y)^2} \left[H_n^2(x) - H_{n+1}(x)H_{n-1}(x)\right].
\]

This function can be expressed in terms of the associated Laguerre polynomials \(L_m^{(\alpha)}(p)\) since for \(n \geq m\) [16]

\[
\int_{-\infty}^\infty dx \, e^{-(x-i\sqrt{\pi}y)^2} H_n(x)H_m(x) = 2^n \sqrt{\pi} m! \pi^{n-m} L_m^{(n-m)}(-2\sqrt{\pi}y),
\]

which leads to

\[
f(t\sqrt{\pi}y) = 2^n \sqrt{\pi} L_{n-1}^{(1)}(-2t^2\pi y),
\]
where we made use of the following recurrence relation between the associated Laguerre polynomials: \( nL_n(p) + pL_n^{(2)}(p) = L_{n-1}^{(1)}(p) \). Then, the probability distribution function \( P_n^{(2)}(w) \) becomes

\[
P_n^{(2)}(w) = C \int_0^\infty dy \cos(wh) \sqrt{v} \int_0^1 \frac{dv}{\sqrt{v}} (1 - v)^{(n - 3)/2} e^{-i\pi y} L_{n-1}^{(1)}(-2i\pi v) + \text{c.c.},
\]

where the normalization constant \( C \) is given explicitly by

\[
C = \frac{\pi^{3/2} \Gamma(n^2/2)}{n\Gamma((n^2 - 1)/2)},
\]

and ‘c.c.’ here stands for the complex conjugate of \( e^{-i\pi y}L_{n-1}^{(1)}(-2i\pi v) \). Using next the series representation of the Laguerre polynomials, the integration over \( dv \) reduces to the calculation of the following integrals:

\[
\sum_{k=0}^{n-1} a_k \int_0^\infty dy \cos(wh)[e^{i\pi y}(-2i\pi v)^k + e^{-i\pi y}(2i\pi v)^k],
\]

or

\[
2 \sum_{k=0}^{n-1} a_k \int_0^\infty dy \cos(wh) \cos\left(\pi y + k\frac{\pi}{2}\right)(-2i\pi v)^k,
\]

which, using the fact that \( \cos(x + k\pi/2) = \frac{d}{dx} \cos x \), can be put into the following form:

\[
\pi L_{n-1}^{(1)}(-2i\pi v) \delta(w - v/n).
\]

Next, the integration over \( dv \) can be performed by parts, taking into account that \( v^{k-1/2}(1 - v)^{(n - 3)/2} \) with \( k > 0 \) vanishes at the integration limits. Then, the operator expression

\[
P_n^{(2)}(w) = \frac{\Gamma(n^2/2)}{\sqrt{\pi} n^2 \Gamma((n^2 - 1)/2)} L_{n-1}^{(1)} \left( 2 \frac{d}{dv} v \right) \left( 1 - v \right)^{(n - 3)/2} \]

is obtained where powers of the polynomial operator are understood as \( 2^{1/2} \frac{d}{dv} v^j \). Explicitly, the action of the Laguerre polynomial operator is defined as

\[
L_{n-1}^{(1)} \left( 2 \frac{d}{dv} v \right) \left( 1 - v \right)^{(n - 3)/2} = \sum_{k=0}^{n-1} \frac{(-1)^k n!}{(n - 1 - k)! (k + 1)! k!} \frac{d^k}{dv^k} (v^{k-1/2}(1 - v)^{(n - 3)/2}),
\]

where the derivatives can be identified with the Rodrigues formula for the Jacobi polynomials \( P_n^{(a,b)}(p) \) as

\[
\frac{d^k}{dv^k} (v^{k-1/2}(1 - v)^{(n - 3)/2}) = \frac{(-1)^k (1 - v)^{(n - 3)/2 - k}}{\sqrt{v}} P_k^{(n-3)/2-k,-1/2}(2v - 1).
\]

Consequently, recalling that \( v = w/n \), we find the following explicit result for the distribution \( P_n^{(2)}(w) \):

\[
P_n^{(2)}(w) = \frac{\Gamma(n^2/2)}{\sqrt{\pi} n^2 \Gamma((n^2 - 1)/2) n^{3/2}} \left( 1 - w/n \right)^{(n - 2n - 1)/2} \sqrt{w} \]

\[
\times \sum_{k=0}^{n-1} \binom{n}{k+1} 2^k \left( 1 - w/n \right)^{n-k-1} P_k^{(n-3)/2-k-1/2}(2w/n - 1).
\]
Furthermore, the sum on the right-hand side of the latter equation can also be represented, after some straightforward calculations, as a polynomial of \( w \), which yields

\[
P^{(2)}_n(w) = \frac{2\Gamma(n^2/2)}{\pi^{3/2} \Gamma((n^2 - 1)/2)n^{1/2}} \frac{(1 - w/n)(n^2 - 2n - 1)/2}{\sqrt{w}} \sum_{m=0}^{n-1} \frac{(-1)^m}{m^m} \binom{n - 1}{m} \alpha_m w^m, \tag{38}
\]

where the coefficients \( \alpha_m \) are defined by

\[
\alpha_m = \int_0^1 \frac{\sqrt{1-t}}{\sqrt{t}} (1 - 2t)^{n-m-1} \mathbf{2F1} \left( -m, \frac{n^2}{2} - 1, \frac{1}{2}, 2t \right), \tag{39}
\]

\( \mathbf{2F1} (\cdot) \) being the Gauss hypergeometric function. Equations (37) and (38) constitute our principal results for the case of the GUE.

Before we turn to the analysis of the asymptotic behaviour of the distribution function, it might be expedient to present several first \( P^{(2)}_n(w) \) explicitly. Below we display \( P^{(2)}_n(w) \) for \( n = 2, 3, 4, 5 \) and 6:

\[
P^{(2)}_2(w) = \frac{1}{\pi} \frac{1}{\sqrt{(2 - w)w}}, \tag{40}
\]

\[
P^{(2)}_3(w) = \frac{35(3 - w)}{576\sqrt{3w}} (3 - 2w + 3w^2), \tag{41}
\]

\[
P^{(2)}_4(w) = \frac{(4 - w)^{7/2}}{1716\pi \sqrt{w}} (12 + 30w - 53w^2 + 38w^3), \tag{42}
\]

\[
P^{(2)}_5(w) = \frac{2028117(5 - w)^7}{8192000000000\sqrt{5w}} (375 - 300w + 4490w^2 - 5996w^3 + 2711w^4), \tag{43}
\]

and

\[
P^{(2)}_6(w) = \frac{32768(6 - w)^{23/2}}{25113523969051155\pi \sqrt{w}} (810 + 3780w - 18090w^2 + 52878w^3 - 49567w^4 + 16144w^5). \tag{44}
\]

One notes that the expressions in equations (40)–(44) all diverge as \( 1/\sqrt{w} \) when \( w \to 0 \). Next, all these expressions vanish (for \( n > 2 \)) as a power-law at the other edge of the support \([0, n]\), with an exponent dependent on \( n \). The case \( n = 2 \) is special: \( P^{(2)}_{n=2}(w) \) diverges at both edges and has a minimum at \( w = 1 \), which signifies that in \( 2 \times 2 \) Gaussian random matrices the two eigenvalues are most probably very different from each other. Furthermore, in figure 1 we plot these explicit forms together with more lengthy expressions for \( n = 7 \) and \( n = 12 \). One observes that for \( w < 1 \) the distributions of arbitrary order are very close to the asymptotic result in equation (23). The distributions are multimodal indicating a set of probable and improbable values of \( w \), which mirrors certain structuring of the eigenvalues. As \( n \) gets progressively larger, the distributions become closer to the asymptotic result, equation (23), for any \( w \in [0, 4] \). Curiously enough, despite a rather complicated form of the polynomials in the second line of equations (37) and (38), they all show an appreciable variation with \( w \) only for \( w < 4 \) and are indistinguishable from zero for larger values of \( w \), despite the fact that formally their support extends to larger than 4 values of \( w \).

For arbitrary \( n \), an asymptotic behaviour of \( P^{(2)}_n(w) \) for \( w \ll 1 \) and \( w \) close to \( n \) can be readily deduced from equation (37). As we have already remarked, one finds that for \( w \to 0 \), the distribution shows a generic singular behaviour of the form

\[
P^{(2)}_n(w) \sim \frac{\Gamma(n^2/2)\mathbf{2F1} \left( -n + 1, \frac{1}{2}, \frac{1}{2}, 2 \right)}{\sqrt{\pi n} \Gamma((n^2 - 1)/2)} \frac{1}{\sqrt{w}}, \tag{45}
\]
where the amplitude
\[
\frac{\Gamma(n^2/2) \text{F}_1 \left( \left(-n + 1, \frac{1}{2}, 2, 2\right) \right)}{\sqrt{\pi n} \Gamma((n^2 - 1)/2)} \to \frac{1}{\pi}
\]  
when \( n \to \infty \), in agreement with the general result in equation (23). This implies, in turn, that for a given random matrix a randomly chosen eigenvalue will most probably be much less than the arithmetic mean of all eigenvalues. Furthermore, on the opposite extremity of the support, when \( w \) is close to \( n \), we have from equation (37) that
\[
P_n^{(2)}(w) \sim \frac{2^{n-1} \Gamma(n^2/2)}{\sqrt{\pi (n-1)!} n^{(n^2-2n+2)/2} \Gamma((n-1)^2/2)} (n - w)^{(n^2-2n+1)/2},
\]  
i.e. \( P_n^{(2)}(w) \) attains a zero value as a power-law when \( w \to n \) with an exponent which grows in proportion to \( n^2 \) when \( n \to \infty \). This implies, in turn, that for \( w \) sufficiently close to \( n \) the value of \( P_n^{(2)}(w) \) decays faster than exponentially with \( n \).

Finally, we address the question how the Marčenko–Pastur law in equation (23) can be derived from our equation (37). Clearly, this is a rather non-trivial question in view of a complicated form of this result. Below we briefly outline the steps involved in such a derivation.

Note first that for \( w < n \), one has
\[
\left( 1 - \frac{w}{n} \right)^{(n^2-2n-1)/2} \to \exp \left( -\frac{nw}{2} \right),
\]  
as \( n \to \infty \), so that
\[
\frac{\Gamma(n^2/2)}{\sqrt{\pi} \Gamma(n^2 - 1)/2)^{n^2/2}} \left( 1 - \frac{w}{n} \right)^{(n^2-2n+1)/2} \sqrt{\frac{1}{nw}} \to \exp(-nw/2) \frac{\Gamma((n^2 - 1)/2)n^{n/2}}{\sqrt{2\pi n w}}.
\]  
Furthermore, one has that, as \( n \to \infty \), [16]
\[
P_k^{(n^2-3)/(2-k-1/2)} \left( 2 \frac{w}{n} - 1 \right) \to \frac{1}{4^k} H_{2k} \left( \sqrt{\frac{nw}{2}} \right).
\]
Using next the integral representation of the Hermite polynomials

\[ H_{2k}\left(\frac{n w}{2}\right) = \exp\left(\frac{nw}{2}\right) \sqrt{\frac{2}{\pi}} \left(\frac{2}{nw}\right)^{k+1/2} \int_{0}^{\infty} dy y^{2k} \exp\left(-\frac{y^{2}}{2nw}\right) \cos(y - \pi k), \]

we can resummate the series in equation (37) to find that, as \( n \to \infty \), the sum in the latter equation converges to

\[ \sum_{k=0}^{n-1} \binom{n}{k+1} 2^{k} \left(1 - \frac{w}{n}\right)^{n-k-1} p_{k}^{(w^{2}-3)/2-k-1/2} \left(2 \frac{w}{n} - 1\right) \to \sqrt{\frac{2}{\pi nw}} \exp\left(\frac{nw^{2}}{2}\right) \int_{0}^{\infty} dy \exp\left(-\frac{y^{2}}{2nw}\right) \cos(y) L_{n-1}^{(1)}\left(\frac{y^{2}}{nw}\right). \]

Note next that as \( n \to \infty \) [16]

\[ L_{n-1}^{(1)}\left(\frac{y^{2}}{nw}\right) \to n \sqrt{w} J_{1}(2y/\sqrt{w}), \]

so that the integral on the right-hand side of equation (52) converges to

\[ \sqrt{\frac{2n}{\pi}} \exp\left(\frac{nw^{2}}{2}\right) \int_{0}^{\infty} dy \exp\left(-\frac{y^{2}}{2nw}\right) \cos(y) J_{1}\left(\frac{2y}{\sqrt{w}}\right), \]

where \( J_{1}(x) \) is the Bessel function. Noting finally that

\[ \exp\left(-\frac{y^{2}}{2nw}\right) \to 1 \]

as \( n \to \infty \), we can perform the integral in equation (54) in this limit, to obtain

\[ \sqrt{\frac{2n}{\pi}} \exp\left(\frac{nw^{2}}{2}\right) \left\{ \begin{array}{ll} \sqrt{1-w/4}, & \text{for } 0 < w < 4, \\ 0, & \text{for } w > 4. \end{array} \right. \]

On combining the latter equation with equation (49), we arrive at the Marˇcenko–Pastur law in equation (23).

4. Conclusions

To recap, we analysed the probability distribution function \( P_{n}^{(\beta)}(w) \) of the Schmidt-like random variable \( w = x_{j}^{2}/\left(\sum_{j=1}^{n} x_{j}^{2}/n\right) \), equation (1), where \( x_{j} \) are the eigenvalues of a given \( n \times n \) \( \beta \)-Gaussian random matrix. This variable, by definition, can be considered as a measure of how any individual eigenvalue deviates from the arithmetic mean value of all eigenvalues of a given random matrix, and its distribution is calculated with respect to the ensemble of such \( \beta \)-Gaussian random matrices. We showed that for arbitrary Dyson symmetry index \( \beta \) in the asymptotic limit \( n \to \infty \) the distribution \( P_{n}^{(\beta)}(w) \) converges to the Marˇcenko–Pastur form, i.e. is defined as \( P_{n}^{(\beta)}(w) \sim \sqrt{(4-w)/w} \) for \( w \in [0,4] \) and equals zero outside of the support. For Gaussian unitary (\( \beta = 2 \)) ensembles, we presented exact explicit expressions for \( P_{n}^{(\beta=2)}(w) \) valid for arbitrary \( n \). We realized that, in general, \( P_{n}^{(\beta=2)}(w) \) has a multimodal form indicating probable and unprobable values of \( w \), which mirrors certain structuring of the eigenvalues. We realized that the convergence to the Marˇcenko–Pastur form is rather fast, so that already for \( n = 12 \) the exact result appears to be quite close to the asymptotic form.

Acknowledgments

The authors acknowledge helpful discussions with Oriol Bohigas and Satya N Majumdar. This work is supported by the Brazilian agencies CNPq and FAPESP. GO is partially supported by the ESF Research Network ‘Exploring the Physics of Small Devices’.
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