ERGODIC BANACH SPACES

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Abstract

We show that any Banach space contains a continuum of non isomorphic subspaces or a minimal subspace. We define an ergodic Banach space $X$ as a space such that $E_0$ Borel reduces to isomorphism on the set of subspaces of $X$, and show that every Banach space is either ergodic or contains a subspace with an unconditional basis which is complementably universal for the family of its block-subspaces. We also use our methods to get uniformity results; for example, in combination with a result of B. Maurey, V. Milman and N. Tomczak-Jaegermann, we show that an unconditional basis of a Banach space, of which every block-subspace is complemented, must be asymptotically $c_0$ or $l_p$.

The following question was asked the authors by G. Godefroy: how many non isomorphic subspaces must a given Banach space contain? By the results of W.T. Gowers, W.T. Gowers - B. Maurey and R. Komorowski - N. Tomczak solving the homogeneous space problem, if $X$ is not isomorphic to $l_2$ then it must contain at least two non-isomorphic subspaces. Except $l_2$, no examples of spaces with only finitely, or even countably many, isomorphism classes of subspaces are known, so we may ask what the possible number of non-isomorphic subspaces of a given Banach space is, supposing it being non isomorphic to $l_2$. This question may also be asked in the setting of the classification of analytic equivalence relations up to Borel reducibility. If $X$ is not isomorphic to $l_2$, when can we classify the relation of isomorphism on subspaces of $X$?

A topological space $X$ is said to be Polish if it is separable and its topology can be generated by a complete metric. Its Borel subsets are those belonging to the smallest $\sigma$-algebra containing the open sets. A subset is analytic if it is the continuous direct image of a Polish space or equivalently of a Borel
set in a Polish space. All uncountable Polish spaces turn out to be Borel isomorphic, i.e., isomorphic by a function that is Borel bimeasurable. A $C$-measurable set is one belonging to the smallest $\sigma$-algebra containing the open sets and closed under the Souslin operation, in particular all analytic sets are $C$-measurable.

All $C$-measurable sets are universally measurable, i.e., measurable with respect to any $\sigma$-finite Borel measure on the space. Furthermore, they have the Baire property, i.e., can be written on the form $A = U \triangle M$, where $U$ is open and $M$ is meagre and are completely Ramsey. In fact they satisfy almost any regularity property satisfied by Borel sets.

Most results contained in this article are centered around the notion of Borel reducibility. This notion turns out to be extremely useful as a mean of measuring complexity in analysis. It also gives another refined view of cardinality, in that it provides us with a notion of the number of classes of an equivalence relation before everything gets muddled up by the wellorderings provided by the axiom of choice.

**Definition 1** Suppose that $E$ and $F$ are analytic equivalence relations on Polish spaces $X$ and $Y$ respectively, then we write $E \leq_B F$ iff there is a Borel function $f : X \rightarrow Y$, such that $xEy \iff f(x)Ff(y)$. Moreover, we denote by $E \sim_B F$ the fact that the relations are Borel bireducible, i.e., $E \leq_B F$ and $F \leq_B E$.

Then $E \leq_B F$ means that there is an injection from $X/E$ into $Y/F$ admitting a Borel lifting. Intuitively, this says that the objects in $X$ are simpler to classify with respect to $E$ than the objects in $Y$ with respect to $F$. Or again that $Y$ objects modulo $F$ provide complete invariants for $X$ objects with respect to $E$-equivalence, and furthermore, these invariants can be calculated in a Borel manner from the initial objects.

We call an equivalence relation $E$ on a Polish space $X$ **smooth** if it Borel reduces to the identity relation on $\mathbb{R}$, or in fact to the identity relation on any uncountable Polish space. This is easily seen to be equivalent to admitting a countable separating family $(A_n)$ of Borel sets, i.e., such that for any $x, y \in X$ we have $xEy \iff \forall n \ (x \in A_n \iff y \in A_n)$.

A Borel probability measure $\mu$ on $X$ is called $E$-**ergodic** if for any $\mu$-measurable $A \subset X$ that is $E$-invariant, i.e., $x \in A \land xEy \rightarrow y \in A$, either $\mu(A) = 0$ or $\mu(A) = 1$. We call $\mu$ $E$-**non atomic** if every equivalence class has measure 0.
Suppose $\mu$ was $E$-ergodic and $(A_n)$ a separating family for $E$. Then by ergodicity and the fact that the $A_n$ are invariant either $A_n$ or $A_n^c$ has measure 1, so $\cap \{A_n \mid \mu(A_n) = 1\} \cap \cap \{A_n^c \mid \mu(A_n) = 0\}$ is an $E$ class of full measure and $\mu$ is atomic. So a smooth equivalence relation cannot carry an ergodic, non atomic probability measure.

The minimal non smooth Borel equivalence relation is the relation of eventual agreement of infinite binary sequences, $E_0$. This is defined on $2^{\omega} = \{0, 1\}^\mathbb{N}$ by

$$xE_0y \iff \exists n \forall m \geq n x_m = y_m$$

To see that $E_0$ is non smooth just notice that the usual coinflipping measure on $2^\omega$ is $E_0$ atomic and ergodic by the zero-one law. Furthermore, any perfect set of almost disjoint infinite subsets of $\mathbb{N}$ shows that $E_0$ has a perfect set of classes.

If $E$ is an equivalence relation on a set $X$ and $A \subset X$ then we call $A$ a transversal for $E$ on $X$ if it intersects every $E$-equivalence class in exactly one point. We notice that if $E$ is an equivalence relation and $A$ a transversal for $E$, both of them analytic, then $E$ is smooth.

Our general reference for descriptive set theory and Ramsey theory is [13], we adopt his notation wholesale.

It is natural to try to distinguish some class of Banach spaces by a condition on the number of non isomorphic subspaces. A step up from homogeneity would be when the subspaces would at least admit some classification in terms of real numbers, i.e., something resembling type or entropy. This would say that in some sense the space could not be too wild and one would expect such a space to have more regularity properties than those of a more generic space, in particular than those of a hereditarily indecomposable space.

With the discoveries in Banach space theory in the seventies and nineties the hope for a general classification of the subspaces of a Banach space faded and even the hope of finding some nice class of subspaces contained in every Banach space also turned out to be doomed to failure, by the examples of Tsirelson and Gowers-Maurey. The strongest positive result was Gowers’ dichotomy saying that either a Banach space contains a hereditarily indecomposable subspace or subspace with an unconditional basis, that is either a very rigid space (with few isomorphisms and projections) or a somewhat nice space (with many isomorphisms and projections).
We isolate another class of separable Banach spaces, namely those on which the isomorphism relation between subspaces does not reduce $E_0$, the non ergodic ones, in particular this class includes those admitting classification by real numbers, and show that if a space belongs to this class, then it must satisfy some useful regularity properties. Let $\mathcal{B}_X$ be the space of closed linear subspaces of a Banach space $X$, equipped with its Effros-Borel structure (see [13] or [5]). We note that isomorphism is analytic on $\mathcal{B}_X^2$. Let us define a Banach space $X$ to be \textit{ergodic} if the relation $E_0$ Borel reduces to isomorphism on subspaces of $X$. In [5], [21], the authors studied spaces generated by subsequences of a space $X$ with a basis: for $X$ a Banach space with an unconditional basis, either $X$ is ergodic or $X$ is isomorphic to its hyperplanes, to its square, and more generally to any direct sum $X \oplus Y$ where $Y$ is generated by a subsequence of the basis, and satisfies other regularity properties.

Note that it is easily checked that Gowers’ construction of a space with a basis, such that no disjointly supported subspaces are isomorphic ([6],[10]), provides an example of a space for which the complexity of isomorphism on subspaces generated by subsequences is exactly $E_0$.

In the main part of this article, we shall consider a Banach space with a basis, and restrict our attention to subspaces generated by block-bases. As long as we consider only block-subspaces, there are more examples of spaces with low complexity, for example $l_p$, $1 \leq p$ or $c_0$ has only one class of isomorphism for block-subspaces. After noting a few facts about the number of non-isomorphic subspaces of a Banach space, that come as consequences of Gowers’ dichotomy theorem (Lemma 2 to Theorem 4), we prove that block-subspaces in a non-ergodic Banach space satisfy regularity properties (Theorem 9, Corollary 10, Theorem 11). We then show how our methods yield uniformity results and in particular find new characterizations of the classical spaces $c_0$ and $l_p$ (Corollary 18). Finally we show how to generalize our results to subspaces with a finite dimensional decomposition on the basis (Theorem 21, Proposition 22).

Let us recall a definition from H. Rosenthal: we say that a space $X$ is \textit{minimal} if $X$ embeds in any of its subspaces. Minimality is hereditary. In the context of block-subspaces, there are two natural definitions: we define a space $X$ with a basis to be \textit{block-minimal} if every block-subspace of $X$ has
a further block-subspace isomorphic to \( X \); it is \textit{equivalence block-minimal} if every block-subspace of \( X \) has a further block-subspace equivalent to \( X \).

The second property is hereditary, but the first one is not, so we also define a \textit{hereditarily block-minimal} space as a space \( X \) with a basis such that any of its block-subspaces is block-minimal.

Let \( X \) be a Banach space with a basis \( \{e_i\} \). If \( y = (y_n)_{n \in \mathbb{N}} \) is a block-sequence of \( X \), we denote by \( Y = [y_n]_{n \in \mathbb{N}} = [y] \) the closed linear span of \( y \).

For two finite or infinite block-bases \( z \) and \( y \) of \( \{e_i\} \), write \( z \leq y \) if \( z \) is a blocking of \( y \) (and write \( Z \leq Y \) for the corresponding subspaces). If \( y = (y_i)_{i \in \mathbb{N}} \), \( t = (t_1, \ldots, t_k) \) are two finite block-bases, i.e., \( \text{supp}(s_i) < \text{supp}(s_{i+1}) \) and \( \text{supp}(t_i) < \text{supp}(t_{i+1}) \), then we write \( s \preceq t \) iff \( s \) is an initial segment of \( t \), i.e., \( n \leq k \) and \( s_i = t_i \) for \( i \leq n \). In that case we write \( t \setminus s \) for \( (t_{n+1}, \ldots, t_k) \). If \( s \) is a finite block-basis and \( y \) is a finite or infinite block-basis supported after \( s \), denote by \( s \rhd y \) the concatenation of \( s \) and \( y \).

We denote by \( bb(X) \) the set of normalized block-bases on \( X \). This set can be equipped with the product topology of the norm topology on \( X \), in which case it becomes a Polish space that we denote by \( bb_N(X) \).

Sometimes we want to work with blocks with rational coordinates, though we no longer can demand these to be normalized (by rational, we shall always mean an element of \( \mathbb{Q} + i\mathbb{Q} \) in the case of a complex Banach space). We identify the set of such blocks with the set \( \mathbb{Q}_+^\omega \) of finite, not identically zero, sequences of rationals. We shall denote by \( (\mathbb{Q}_+^\omega)'^\omega \) the set of (not necessarily successive) infinite sequences of rational blocks. Again when needed we will give \( \mathbb{Q}_+^\omega \) the discrete topology and \( (\mathbb{Q}_+^\omega)'^\omega \) the product topology. The set of rational block-bases may be seen as a subset of \( (\mathbb{Q}_+^\omega)'^\omega \) and is denoted by \( bb_{\mathbb{Q}} \). The set of finite rational block-bases is then denoted by \( fb_{bb_{\mathbb{Q}}} \).

Finally for the topology that interests us the most: let \( \mathbb{Q} \) be the set of normalized blocks of the basis that are a multiple of some block with rational coordinates; we denote by \( bb_{\mathbb{Q}}(X) \) the set of block-bases of vectors in \( \mathbb{Q} \), equipped with the product topology of the discrete topology on \( \mathbb{Q} \). As \( \mathbb{Q} \) is countable, this topology is Polish and epsilon matters may be forgotten until the applications; when we deal with isomorphism classes, they are not relevant since a small enough perturbation preserves the class. Note also that the canonical embedding of \( bb_{\mathbb{Q}}(X) \) into \( \mathcal{B}_X \) is Borel, and this allows us to forget about the Effros-Borel structure when checking ergodicity. Unless
specified otherwise, from now on we work with this topology.

We first prove a Lemma about uniformity for these properties. For \( C \geq 1 \), we say that a space \( X \) with a basis is \( C \) block-minimal (resp. \( C \) equivalence block-minimal) if any block-subspace of \( X \) has a further block-subspace which is \( C \)-isomorphic (resp. \( C \)-equivalent) to \( X \).

**Lemma 2** (i) Let \( X \) be a Banach space with a basis and assume \( X \) is (equivalence) block-minimal. Then there exists \( C \geq 1 \) such that \( X \) is \( C \) (equivalence) block-minimal.

(ii) Suppose \( \{e_n\} \) is a basis in a Banach space, such that any subsequence of \( \{e_n\} \) has a block-sequence equivalent to \( \{e_n\} \). Then there is a subsequence \( \{f_n\} \) of \( \{e_n\} \) and a constant \( C \geq 1 \), such that any subsequence of \( \{f_n\} \) has a block-sequence \( C \)-equivalent to \( \{e_n\} \).

**Proof:** We will only prove (i) as the proof of (ii) is similar. Let for \( n \in \mathbb{N} \) \( c(n) \) denote a constant such that any \( n \)-codimensional subspaces of any Banach space are \( c(n) \)-isomorphic (Lemma 3 in [5]). Let \( X \) be block-minimal. We want to construct by induction a decreasing sequence of block-subspaces \( X_n, n \geq 1 \) and successive block-vectors \( x_n \) such that the first vectors of \( X_n \) are \( x_1, \ldots, x_{n-1} \) and such that no block-subspace of \( X_n \) is \( n \)-isomorphic to \( X \). Assume we may carry out the induction: then for all \( n \in \mathbb{N} \), no block-subspace of \( [x_n]_{n \in \mathbb{N}} \) is \( n \)-isomorphic to \( X \), and this contradicts the block-minimality of \( X \). So the induction must stop at some \( n \), meaning that every block-subspace of \( X_n \) whose first vectors are \( x_1, \ldots, x_n \) has a further block-subspace \( n \) isomorphic to \( X \). Then by definition of \( c(n) \), every block-subspace of \( X_n \) has a further block-subspace \( nc(n) \) isomorphic to \( X \). By block-minimality we may assume that \( X_n \) is \( K \)-isomorphic to \( X \) for some \( K \). Take now any block-subspace \( Y \) of \( X \), it is \( K \)-isomorphic to a subspace of \( X_n \); by standard perturbation arguments, we may find a block-subspace of \( Y \) which is \( 2K \)-equivalent to a block-subspace of \( X_n \), and by the above, an even further block-subspace \( 2Kc(n) \) equivalent to a \( nc(n) \)-isomorphic copy of \( X \); so finally \( Y \) has a block-subspace \( 2Knc(n) \) isomorphic to \( X \) and so, \( X \) is \( 2Knc(n) \) block minimal.

We may use the same proof for equivalence block-minimality, using instead of \( c(n) \) a constant \( d(n) = (1 + (n+1)c)^2 \), such that any two normalized block-sequences differing by only the \( n \) first vectors are \( d(n) \)-equivalent (\( c \) stands for the constant of the basis). \( \square \)
Let us recall a version of the Gowers' game \( G_{A,Y} \) shown to be equivalent to Gowers' original game by Bagaria and Lopez-Abad: Player I plays in the \( k \)'th move a normalized block vector \( y_k \) of \( Y \) such that \( y_{k-1} < y_k \) and Player II responds by either doing nothing or playing a normalized block vector \( x \in [y_{i+1}, \ldots, y_k] \) if \( i \) was the last move where she played a vector. Player II wins the game if in the end she has produced an infinite sequence \( (x_k)_{k \in \mathbb{N}} \) which is a block sequence in \( A \). If Player II has a winning strategy for \( G_{A,Y} \) we say that she has a winning strategy for Gowers' game in \( Y \) for producing block-sequences in \( A \). Gowers proved that if \( A \) is analytic in \( bb_N(X) \), such that any normalized block sequence contains a further normalized block sequence in \( A \), then II has a winning strategy in some \( Y \) to produce a block-sequence arbitrarily close to a block-sequence in \( A \).

As an application of Gowers' theorem one can mention that if \( X \) is (resp. equivalence) block-minimal, then there is a constant \( C \), such that for every block subspace \( Y \leq X \), Player II has a winning strategy for Gowers' game in some \( Z \leq Y \) for producing block-sequences spanning a space \( C \)-isomorphic to \( X \) (resp. \( C \)-equivalent to the basis of \( X \)).

We recall that a space with a basis is said to be quasi-minimal if any two block-subspaces have further isomorphic block-subspaces. On the contrary, two spaces are said to be totally incomparable if no subspace of the first one is isomorphic to a subspace of the second. Using his dichotomy theorem, Gowers proved the following result about Banach spaces.

**Theorem 3 (Gowers’ ”trichotomy”)** Let \( X \) be a Banach space. Then \( X \) either contains
- a hereditarily indecomposable subspace,
- a subspace with an unconditional basis such that no disjointly supported block-subspaces are isomorphic,
- a subspace with an unconditional basis which is quasi-minimal.

Using his game we prove:

**Theorem 4** Let \( X \) be a separable Banach space. Then

(i) \( X \) is ergodic or contains a quasi-minimal subspace with an unconditional basis.
(ii) $X$ contains a perfect set of mutually totally incomparable subspaces or a quasi-minimal subspace.

(iii) $X$ contains a perfect set of non isomorphic subspaces or a block minimal subspace with an unconditional basis.

Proof: First notice that because of the hereditary nature of the properties, each of the subspaces above may be chosen to be spanned by block-bases of a given basis. Rosendal proved that any hereditarily indecomposable Banach space $X$ is ergodic, and this can be proved using subspaces generated by subsequences of a basic sequence in $X$ [21]. Following Bossard (who studied the particular case of a space defined by Gowers, [1]), we may prove that a space $X$ such that no disjointly supported block-subspaces are isomorphic is ergodic (map $\alpha \in 2^{\omega}$ to $[e_{2n + \alpha(n)}]_{n \in \mathbb{N}}$, where $(e_n)$ is the unconditional basis of $X$). This takes care of (i).

A space such that no disjointly supported block-subspaces are isomorphic contains $2^{\omega}$ totally incomparable block-subspaces (take subspaces generated by subsequences of the basis corresponding to a perfect set of almost disjoint infinite subsets of $\omega$). Also any hereditarily indecomposable space is quasi-minimal so (ii) follows.

Finally for the proof of (iii) we will first show that the statement we want to prove is $\Sigma^1_2$, which will be done by showing that given a block-minimal space $X$, there is a further block subspace $Y$ such that for $Z \leq Y$ we can find continuously in $Z$ an $X' \leq Z$ and an isomorphism of $X'$ with $X$. The proof uses some ideas of Lopez-Abad [17] of coding with asymptotic sets.

By standard metamathematical facts it will then be sufficient to show the statement under Martin’s axiom and the negation of the continuum hypothesis. This was almost done by Bagaria and Lopez-Abad who showed it to be consistent relative to the existence of a weakly compact cardinal, see [2], but we will see that it can be done in a simple manner directly from $\text{MA} + \neg \text{CH}$.

Note first that having a perfect set of non isomorphic subspaces or containing a copy of $c_0$ are both $\Sigma^1_2$, and that if $X$ contains a copy of $c_0$ then it has a block sequence equivalent to the unit vector basis of $c_0$, in which case the theorem holds. If on the contrary it does not contain $c_0$ then by the solution to the distortion problem by Odell and Schlumprecht it must contain two closed, positively separated, asymptotic subsets of the unit sphere $A_0$ and $A_1$, [19]. Suppose that $Y = [y] \leq X$ is block minimal. Fix a bijection $\pi$ between $\omega$ and $\mathbb{Q}^{<\omega}$, the set of finite sequences of rationals not identically
zero, then for any \( \alpha = 0^{n_0}10^{n_1}10^{n_2}1 \ldots \) in \( 2^\omega \) there is associated a unique sequence \( (\pi(n_0), \pi(n_1), \pi(n_2), \ldots) \) in \( (\mathbb{Q}_*^<\omega)^\omega \). Furthermore any element of \((\mathbb{Q}_*^<\omega)^\omega\) gives a unique sequence of block-vectors of \( Y \) simply by taking the corresponding finite linear combinations.

Let \( D := \{(z_n) \in bb_N(X) \mid (z_n) \leq y \land \forall n \ z_n \in A_0 \cup A_1 \land \exists n \ z_n \in A_1\} \) which is Borel in \( bb_N(X) \). Then if \( (z_n) \in D \) it codes a unique infinite sequence of block vectors (not necessarily consecutive) of \( Y \), by first letting \( (z_n) \mapsto \alpha \in \{0,1\}^\omega \) where \( \alpha(n) = 1 \iff z_n \in A_1 \) and then composing with the other coding. Notice that this coding is continuous from \( D \) to \((\mathbb{Q}_*^<\omega)^\omega\), when \( \mathbb{Q}_*^<\omega \) is taken discrete.

Let \( E \) be the set of \( (z_n) \leq y \) such that \( (z_{2n+1}) \in D \) and the function sending \( (z_{2n}) \) to the sequence of block-vectors of \( Y \) coded by \( (z_{2n+1}) \) is an isomorphism of \([z_{2n}]_{n \in \mathbb{N}}\) with \( Y \).

\( E \) is clearly Borel in \( bb_N(X) \) and we claim that any block sequence contains a further block sequence in \( E \). For suppose that \( z \leq y \) is given, then we first construct a further block sequence \( (z_n) \) such that \( z_{3n+1} \in A_0 \) and \( z_{3n+2} \in A_1 \). By block minimality of \( Y \) there are a block sequence \( (x_n) \) of \( (z_n) \) isomorphic to \( Y \) and a sequence \( \alpha \in \{0,1\}^\omega \) coding a sequence of block vectors \( (y_n) \) of \( Y \) such that \( x_n \mapsto y_n \) is an isomorphism of \([x_n]\) with \( Y \) (a standard perturbation argument shows that we can always take our \( y_n \) to be a finite rational combination on \( Y \)).

Now in between \( x_n \) and \( x_{n+1} \) there are \( z_{3m+1} \) and \( z_{3m+2} \), so we can code \( \alpha \) by a corresponding subsequence \( (z'_n) \) of these such that \( x_n < z'_n < x_{n+1} \). The combined sequence is then in \( E \). So by Gowers' theorem there is for any \( \Delta > 0 \) a winning strategy \( \tau \) for \( II \) for producing blocks in \( E_\Delta \) in some \( Y' \leq Y \).

By choosing \( \Delta \) small enough and modifying \( \tau \) a bit we can suppose that the vectors of odd index played by \( II \) are in \( A_0 \cup A_1 \). So if \( \Delta \) is chosen small enough, a perturbation argument shows that \( \tau \) is in fact a strategy for playing blocks in \( E \). By changing the strategy again we can suppose that \( II \) responds to block-bases in \( bb_\xi(Y') \) by block-bases in \( bb_\xi(Y') \). So finally we see that \( X \) has a block-minimal subspace iff there are \( Y' = [y'] \) and \( Y = [y] \) with \( Y' \leq Y \leq X \) and a continuous function \( (f_1, f_2) = f : bb_\xi(Y') \to bb_\xi(Y') \times (\mathbb{Q}_*^<\omega)^\omega \) such that for all \( z \leq y' \), \( f_2(z) \) codes a sequence \( (w_n) \) of blocks of \( Y \) such that \([w_n]_{n \in \mathbb{N}} = Y \), and \( f_1(z) = (v_n) \leq z \) with \( v_n \mapsto w_n \) being an isomorphism between \([v_n]_{n \in \mathbb{N}} \) and \( Y \).

The statement is therefore \( \Sigma^1_2 \), and to finish the proof we now need the following lemma:
Lemma 5 (MA$_{\sigma}$-centered) Let $A \subset bb_\mathbb{Q}$ be linearly ordered under $\leq^*$ of cardinality strictly less than the continuum, then there is an $y_\infty \in bb_\mathbb{Q}$ such that $x_0 \leq^* y$ for all $y \in A$.

Proof: For $s \in fbb_\mathbb{Q}$ and $y \in bb_\mathbb{Q}$, denote by $(s, y)$ the set of block-bases in $bb_\mathbb{Q}$ of the form $s \sim z$ for $z \leq y$. Let $\mathcal{P} = \{(s, y) \mid s \in fbb_\mathbb{Q} \land y \in A\}$, ordered by the inclusion. As a preliminary remark, note that if $(t, z) \subset (s, y)$, then $s \leq t$, $t \setminus s \leq y$, and $z \leq^* y$; conversely, if $s \in fbb_\mathbb{Q}$ and $z \leq^* y$, then extensions $t$ of $s$, with $t \setminus s \leq z$ far enough, are such that $(t, z) \subset (s, y)$.

Put $D_n = \{(s, y) \in \mathcal{P} \mid |s| \geq n\}$ and $D_y = \{(t, z) \in \mathcal{P} \mid z \leq^* y\}$. Then $D_n$ and $D_y$, for $y \in A$, are dense in $\mathcal{P}$, i.e., any element in $\mathcal{P}$ has a minorant in $D_n$ (resp. $D_y$). To see that $D_n$ is dense, just take for any given $(s, y) \in \mathcal{P}$ some extension $s'$ of $s$ such that $s' \setminus s \leq y$ and $|s'| \geq n$, then $(s', y) \in D_n$ and $(s', y) \subset (s, y)$. On the other hand to see that $D_y$ is dense for $y \in A$ suppose $(s, z) \in \mathcal{P}$ is given, then as $A$ is linearly ordered by $\leq^*$, let $w$ be the minimum of $z$ and $y$. By the preliminary remark, $(s', w) \subset (s, z)$ for a long enough extension $s'$ of $s$ such that $s' \setminus s \leq w$, and as $w \leq^* y$, $(s', w)$ is in $D_y$.

Let $\mathcal{P}_s = \{(s, y) \mid y \in A\}$, which is centered in $\mathcal{P}$, i.e., every finite subset of $\mathcal{P}_s$ has a common minorant in $\mathcal{P}$. This follows from the same argument as above, using the preliminary remark. So since $s$ is supposed to be rational, we see that $\mathcal{P}$ is $\sigma$-centered, i.e., a countable union of centered subsets. Notice that as $|A| < 2^\omega$, there are less than continuum many dense sets $D_n$ and $D_y$. So by MA$_{\sigma}$-centered there is a filter $G$ on $\mathcal{P}$ intersecting each of these sets.

Suppose that $(s, y)$ and $(t, z) \in G$ then as $G$ is a filter, they have a common minorant $(v, w) \in G$, but then $s \leq v$ and $t \leq v$, so either $s \leq t$ or $t \leq s$. Therefore $y_\infty := \bigcup\{s \in fbb_\mathbb{Q} \mid \exists y (s, y) \in G\}$ is a block-basis. Furthermore as $G$ intersects all of $D_n$ for $n < \omega$ we see that $y_\infty$ is an infinite block-basis.

We now prove that $y_\infty \leq^* y$ for all $y \in A$. Since $G$ intersects $D_y$, without loss of generality we may assume that $(s, y) \in G$ for some $s$. Then $y_\infty \setminus s \leq y$. For if $t \leq y_\infty$ and $(t, z) \in G$, take $(v, w) \in G$ such that $(v, w) \subset (t, z)$ and $(v, z) \subset (s, y)$, then $s, t \leq v \leq y_\infty$ and $t \setminus s \leq v \setminus s \leq y$, and therefore as $t$ was arbitrary $y_\infty \setminus s \leq y$. □

Suppose now that $X$ does not have a perfect set of non isomorphic subspaces, then by Burgess' theorem (see [13] (35.21)), it has at most $\omega_1$ isomorphism classes of subspaces, and in particular as we are supposing the continuum hypothesis not to hold, less than continuum many. Let $(X_\xi)_{\xi < \omega_1}$.
be an enumeration of an element from each class. Then if none of these are minimal, we can construct inductively a $\le^*$ decreasing sequence $(Y_\xi)_{\xi<\omega_1}$ of rational block-subspace such that $X_\xi$ does not embed into $Y_\xi$ and using the above lemma find some $Y_{\omega_1}$ diagonalizing the whole sequence. By taking, e.g., the subsequence consisting of every second term of the basis of $Y_{\omega_1}$ one can suppose that $Y_{\omega_1}$ embeds into every term of the sequence $(Y_\xi)_{\xi<\omega_1}$ and that therefore in particular $Y_{\omega_1}$ is isomorphic to no $X_\xi$, $\xi < \omega_1$, which is impossible. This finishes the proof of the theorem.

We remark that in the case of $X$ not containing a minimal subspace, then there is in fact a perfect set of subspaces such each two of them do not both embed into each other, which is slightly stronger than saying that they are not isomorphic.

We recall our result from [5],[21] in a slightly modified form.

**Theorem 6** Let $X$ be a Banach space with a basis $\{e_i\}$. Then $E_0$ Borel reduces to isomorphism on subspaces spanned by subsequences of the basis, or there exists a sequence $(F_n)_{n \geq 1}$ of successive finite subsets of $\mathbb{N}$ such that for any infinite subset $N$ of $\mathbb{N}$, if $N \cap [\min(F_n), \max(F_n)] = F_n$ for infinitely many $n$’s, then the space $[e_i]_{i \in N}$ is isomorphic to $X$. It follows that if $X$ is non ergodic with an unconditional basis, then it is isomorphic to its hyperplanes, to its square, and more generally to $X \oplus Y$ for any subspace $Y$ spanned by a subsequence of the basis.

Indeed, by [5], improved in [21], either $E_0$ Borel reduces to isomorphism on subspaces spanned by subsequences of the basis, or the set of infinite subsets of $\mathbb{N}$ spanning a space isomorphic to $X$ is residual in $2^\omega$; the characterization in terms of finite subsets of $\mathbb{N}$ is then a classical characterization of residual subsets of $2^\omega$ (see [13], or the remark at the end of Lemma 7 in [5]). Both proofs are similar to (and simpler than) the following proofs of Proposition 7 and Proposition 8 for block-subspaces. The last part of the theorem is specific to the case of subspaces spanned by subsequences and is also proved in [5].

We now wish to extend this result to the set of block-bases, for which it is useful to use the Polish space $bb_d(X)$. Unless stated otherwise this is the topology refered to.
As before, the notation $x = (x_n)_{n \in \mathbb{N}}$ will be used to denote an infinite block-sequence; $\tilde{x}$ will denote a finite block-sequence, and $|\tilde{x}|$ its length as a sequence, $\text{supp}(\tilde{x})$ the union of the supports of the terms of $\tilde{x}$. For two finite block-sequences $\tilde{x}$ and $\tilde{y}$, write $\tilde{x} < \tilde{y}$ to mean that they are successive. For a sequence of successive finite block-sequences $(\tilde{x}_i)_{i \in I}$, we denote the concatenation of the block-sequences by $\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \ldots$ if the sequence is finite or $\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \ldots$ if it is infinite, and we denote by $\text{supp}(\tilde{x}_i, i \in I)$ the support of the concatenation, by $[\tilde{x}_i]_{i \in I}$ the closed linear span of the concatenation. For a finite block sequence $\tilde{x} = (x_1, \ldots, x_n)$, we denote by $N(\tilde{x})$ the set of elements of $bb_d(X)$ whose first $n$ vectors are $(x_1, \ldots, x_n)$.

**Proposition 7** Let $X$ be a Banach space with a Schauder basis. Then either $X$ is ergodic, or there exists $K \geq 1$ such that a residual set of block-sequences in $bb_d(X)$ span spaces mutually $K$-isomorphic.

**Proof**: The relation of isomorphism is either meagre or non-meagre in $bb_d(X)^2$. First assume that it is meagre. Let $(U_n)_{n \in \mathbb{N}}$ be a decreasing sequence of dense open subsets of $bb_d(X)^2$ so that $\cap_{n \in \mathbb{N}}U_n$ does not intersect $\simeq$. We build by induction successive finite blocks $\{\tilde{a}_n^0, n \in \mathbb{N}\}$ and $\{\tilde{a}_n^1, n \in \mathbb{N}\}$ such that for all $n$, $|\tilde{a}_n^0| = |\tilde{a}_n^1|$, and $\text{supp}(\tilde{a}_n^0) < \text{supp}(\tilde{a}_n^1)$ for all $(i, j) \in 2^2$. For $\alpha \in 2^n$, we let $x(\alpha)$ be the concatenated infinite block-sequence $\tilde{a}_n^0 \tilde{a}_n^1 \tilde{a}_n^2 \tilde{a}_n^3 \ldots$; for $n \in \mathbb{N}$ and $\beta \in 2^n$, we let $\tilde{x}(\beta)$ be the concatenated finite block-sequence $\tilde{a}_0^1 \tilde{a}_1^1 \tilde{a}_2^1 \tilde{a}_3^1 \ldots \tilde{a}_{n-1}^1$. We require furthermore from the sequences $\{\tilde{a}_n^0\}$ and $\{\tilde{a}_n^1\}$ that for each $n \in \mathbb{N}$, each $\beta$ and $\beta'$ in $2^n$,

$$N(\tilde{x}(\beta^0)) \times N(\tilde{x}(\beta'^1)) \subset U_n.$$

Before explaining the construction, let us check that with these conditions, the map $\alpha \rightarrow x(\alpha)$ realizes a Borel reducing of $E_0$ into $(bb_d(X), \simeq)$. Indeed, when $\alpha E_0 \alpha'$, the corresponding sequences differ by at most finitely many vectors, and since we took care that $|\tilde{a}_n^0| = |\tilde{a}_n^1|$ for all $n$, $x(\alpha)$ and $x(\alpha')$ span isomorphic subspaces. On the other hand, when $\alpha$ and $\alpha'$ are not $E_0$-related, without loss of generality there is an infinite set $I$ such that for all $i \in I$, $\alpha(i) = 0$ and $\alpha'(i) = 1$; it follows that for all $i \in I$, $(x(\alpha), x(\alpha'))$ belongs to $U_i$, and so by choice of the $U_n's$, $(x(\alpha), x(\alpha'))$ does not belong to $\simeq$.

Now let us see at step $n$ how to construct the sequences: given a pair $\beta_0, \beta'_0$ in $(2^n)^2$, using the fact that $U_n$ is dense and open, the pair $\tilde{x}(\beta_0), \tilde{x}(\beta'_0)$
may be extended to a pair of finite successive block-sequences of the form $(\tilde{x}(\beta_0) - \tilde{z}_0, \tilde{x}(\beta'_0) - \tilde{z}'_0)$ such that $N(\tilde{x}(\beta_0) - \tilde{z}_0) \times N(\tilde{x}(\beta'_0) - \tilde{z}'_0) \subset U_n$, and we may require that $\text{supp}(\tilde{x}(\beta_0)) \cup \text{supp}(\tilde{x}(\beta'_0)) \not< \text{supp}(\tilde{z}_0) \cup \text{supp}(\tilde{z}'_0)$. Given another pair $\beta_1, \beta'_1$ in $(2^n)^2$, the pair $(\tilde{x}(\beta_1) - \tilde{z}_1, \tilde{x}(\beta'_1) - \tilde{z}'_1)$ may be extended to a pair of finite successive block-sequences $(\tilde{x}(\beta_1) - \tilde{z}_1, \tilde{x}(\beta'_1) - \tilde{z}'_1)$ such that $N(\tilde{x}(\beta_1) - \tilde{z}_1) \times N(\tilde{x}(\beta'_1) - \tilde{z}'_1) \subset U_n$. Here with our notation $\tilde{z}$ extends $\tilde{z}_0$ and $\tilde{z}'$ extends $\tilde{z}'_0$. Repeat this $(2^n)^2$ times to get $\tilde{z}_{4^n-1}, \tilde{z}'_{4^n-1}$ such that for all $\beta$ and $\beta'$ in $2^n$,

$N(\tilde{x}(\beta) - \tilde{z}_{4^n-1}) \times N(\tilde{x}(\beta') - \tilde{z}'_{4^n-1}) \subset U_n$.

Finally extend $(\tilde{z}_{4^n-1}, \tilde{z}'_{4^n-1})$ to $(\tilde{a}^0_n, \tilde{a}^1_n)$ such that $|\tilde{a}^0_n| = |\tilde{a}^1_n|$; we still have for all $\beta$ and $\beta'$ in $2^n$, $N(\tilde{x}(\beta) - \tilde{a}^0_n) \times N(\tilde{x}(\beta') - \tilde{a}^1_n) \subset U_n$, i.e. with our notation,

$N(\tilde{x}(\beta - 0)) \times N(\tilde{x}(\beta' - 1)) \subset U_n$.

Now assume the relation of isomorphism is non-meagre in $bb_d(X)^2$. As the relation is analytic it has the Baire property and $bb_d(X)$ is Polish, so by Kuratowski-Ulam (Theorem 8.41 in [13]), there must be some non-meagre section, that is, some isomorphism class $A$ is non-meagre. Fix a block-sequence $x$ in this class, then clearly, for some constant $C$, the set $A_C$ of blocks-sequences spanning a space $C$-isomorphic to $[x]$ is non-meagre. Now being analytic, this set has the Baire property, so is residual in some basic open set $U$, of the form $N(\tilde{x})$, and we may assume $n \geq 1$.

We now prove that $A_k$ is residual in $bb_d(X)$ for $k = Cc(2\max(\supp(\tilde{x})))$; the conclusion of the proposition then holds for $K = k^2$. Recall that for $n \in \mathbb{N}$, $c(n)$ denotes a constant such for any Banach space $X$, any $n$-codimensional subspaces of $X$ are $c(n)$-isomorphic (Lemma 3 in [5]). So let us assume $V = N(\tilde{y})$ is some basic open set in $bb_d(X)$ such that $A_k$ is meagre in $V$. We may assume that $|\tilde{y}| > |\tilde{x}|$ and write $\tilde{y} = \tilde{x}' - \tilde{z}$ with $\tilde{x} < \tilde{z}$ and $|\tilde{x}'| \leq \max(\supp(\tilde{x}))$. Choose $\tilde{u}$ and $\tilde{v}$ to be finite sequences of blocks such that $\tilde{u}, \tilde{v} > \tilde{z}$, $|\tilde{u}| = |\tilde{x}'|$ and $|\tilde{v}| = |\tilde{x}|$, and such that $\max(\supp(\tilde{u})) = \max(\supp(\tilde{v}))$. Let $U'$ be the basic open set $N(\tilde{x}' - \tilde{z} - \tilde{u})$ and let $V'$ be the basic open set $N(\tilde{x}' - \tilde{z} - \tilde{v})$. Again $A_C$ is residual in $U'$ while $A_k$ is meagre in $V'$.

Now let $T$ be the canonical map from $U'$ to $V'$. For all $u$ in $U'$, $T(u)$ differs from at most $|\tilde{x}| + \max(\supp(\tilde{x})) \leq 2\max(\supp(\tilde{x}))$ vectors from $u$, so $|T(u)|$ is $c(2\max(\supp(\tilde{x})))$ isomorphic to $[u]$. Since $k = Cc(2\max(\supp(\tilde{x})))$

13
it follows that $A_k$ is residual in $V' \subset V$. The contradiction follows by choice of $V$. \hfill \Box

By analogy with the definition of atomic measures, we may see Proposition 7 as stating that a non-ergodic Banach spaces with a basis must be "atomic" for its block-subspaces.

We now want to give a characterization of residual subsets of $bd(X)$. If $A$ is a subset of $bd(X)$ and $\Delta = (\delta_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers, we denote by $A_\Delta$ the usual $\Delta$-expansion of $A$ in $bd(X)$, that is $x = (x_n) \in A_\Delta$ iff there exists $y = (y_n) \in A$ such that $\|y_n - x_n\| \leq \delta_n, \forall n \in \mathbb{N}$. Given a finite block-sequence $\tilde{x} = (x_1, \ldots, x_n)$, we say that a (finite or infinite) block-sequence $(y_i)$ passes through $\tilde{x}$ if there exists some integer $m$ such that $\forall 1 \leq i \leq n, y_{m+i} = x_i$.

**Proposition 8** Let $A$ be residual in $bd(X)$. Then for all $\Delta > 0$, there exist successive finite block-sequences $(\tilde{x}_n), n \in \mathbb{N}$ such that any element of $bd(X)$ passing trough infinitely many of the $\tilde{x}_n$’s is in $A_\Delta$.

**Proof:** Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of dense open sets, which we may assume to be decreasing, such that $\cap_{n \in \mathbb{N}} U_n \subset A$. Without loss of generality we may also assume $\Delta$ to be decreasing. In the following, block-vectors are always taken in $\mathbb{Q}$, in the intention of building elements of $bd(X)$.

First $U_0$ is open so there exists $\tilde{x}_0$ a finite block sequence such that $N(\tilde{x}_0) \subset U_0$. Now let us choose some $n_1 > \max(supp(\tilde{x}_0))$ and let us take an arbitrary block-vector $z_1$, and let $N_1 = \min(supp(z_1))$. Let $F_{<1}$ be a finite set of finite block sequences forming an $\delta_{N_1}$- net for all finite block sequences supported before $N_1$ and let $F_{01}$ be a finite set of finite block sequences forming an $\delta_{N_1}$- net for all finite block sequences supported after $\tilde{x}_0$ and before $N_1$. Let $G_1 = \{ \tilde{x}_0 \tilde{y}, \tilde{y} \in F_{01} \}$ and let $F_1 = F_{<1} \cup G_1$. Using the fact that $U_1$ is dense open, we may construct successively a finite block sequence $\tilde{x}_1$ which prolonges $z_1$, so that $\min(supp(\tilde{x}_1)) = N_1 > \max(supp(\tilde{x}_0))$, and such that for any $\tilde{f}_1 \in F_1$, $N(\tilde{f}_1^{-1} \tilde{x}_1)$ is a subset of $U_1$.

Let us now write what happens at the $k$-th step. We choose some $N_k > \max(supp(\tilde{x}_{k-1}))$ and an arbitrary block $z_k$ whose support starts at $N_k$. We let $F_{<k}$ be a finite set of finite block sequences forming an $\delta_{N_k}$- net for all finite block sequences supported before $n_k$ and for all $i < k$, we let $F_{ik}$ be a finite set of finite block sequences forming an $\delta_{N_k}$- net for all finite block sequences
supported after $\tilde{x}_i$ and before $N_k$. For any $I = \{i_1 < i_2 < \cdots < i_m = k\}$, we let $G_I$ be the set of finite block sequences $\tilde{z}$ passing through every $i$ in $I$, such that the finite sequence of blocks of $\tilde{z}$ supported before $\tilde{x}_{i_0}$ is in $F_{<i_0}$ and such that for all $j < m$, the finite sequence of blocks of $\tilde{z}$ supported between $\tilde{x}_{i_j}$ and $\tilde{x}_{i_{j+1}}$ is in $F_{i_j,i_{j+1}}$. And we let $F_k$ be the union of all $G_I$ over all possible subsets of $\{1, 2, \ldots, k\}$ containing $k$. Using the fact that $U_k$ is dense open, we may construct successively a finite block sequence $\tilde{x}_k$ which prolonges $z_k$, so that $\min(supp(\tilde{x}_k)) = N_k > \max(supp(\tilde{x}_{k-1}))$, and such that for any $\tilde{f}_k \in F_k$, $N(\tilde{f}_k \tilde{x}_k)$ is a subset of $U_k$.

Repeat this construction by induction, and now let $z$ be a block-sequence passing through $\tilde{x}_n$ for $n$ in an infinite set $\{n_k, k \in \mathbb{N}\}$. We may write $z = \tilde{y}_0 \tilde{x}_{n_0} \tilde{y}_1 \tilde{x}_{n_1} \ldots$, where $\tilde{y}_0$ is supported before $\tilde{x}_{n_0}$ (we may assume that $n_0 > 0$) and for $k > 0$, $\tilde{y}_k$ is supported between $\tilde{x}_{n_{k-1}}$ and $\tilde{x}_{n_k}$.

Let $\tilde{f}_0 \in F_{<n_0}$ be $\delta_{N_{n_0}}$ distant from $\tilde{y}_0$; and for any $k > 0$, let $\tilde{f}_k \in F_{n_{k-1}n_k}$ be $\delta_{N_{n_k}}$ distant from $\tilde{y}_k$. Then it is clear that $z$ is $\Delta$ distant from $f = \tilde{f}_0 \tilde{x}_{n_0} \tilde{f}_1 \tilde{x}_{n_1} \ldots$. Indeed, consider a term $z_n$ of the block-sequence $z$: if it appears as a term of some finite sequence $\tilde{x}_{n_k}$ then its distance to the corresponding block $f_n$ of $f$ is 0; if it appears as a term of some $\tilde{y}_k$ then it is less than $\delta_{N_k}$-distant from the block $f_n$, and $N_k > \max(supp(z_n)) \geq n$, so it is less than $\delta_n$-distant from $f_n$.

It remains to check that $f$ is in $\mathcal{A}$. But for all $K$, the finite sequence $\tilde{g}_K = \tilde{f}_0 \tilde{x}_{n_0} \ldots \tilde{f}_k \tilde{x}_{n_k}$ is an element of $G_{\{n_1, \ldots, n_K\}}$ so is an element of $F_K$; it follows that $N(\tilde{g}_K)$ is a subset of $U_{n_K}$ and so that $f$ is in $U_{n_K}$. Finally $f$ is in $\bigcap_{k \in \mathbb{N}} U_{n_k}$ so is in $\mathcal{A}$. \hfill $\square$

Conversely, given successive blocks $\tilde{x}_n$, the set of block-sequences passing through infinitely many of the $\tilde{x}_n$'s is residual: for a given $\tilde{x}_n$, "$(y_k)_{k \in \mathbb{N}}$ passes through $\tilde{x}_n$" is open and "$(y_k)_{k \in \mathbb{N}}$ passes through infinitely many of the $\tilde{x}_n$'s" is equivalent to "\(\forall m \in \mathbb{N}, \exists n > m \in \mathbb{N}: (y_k) \text{ passes through } \tilde{x}_n\)”, so is $G_{\delta}$, and clearly dense. If the set $\mathcal{A}$ considered is an isomorphism class, then it is invariant under small enough $\Delta$-perturbations, and so we get an equivalence: $\mathcal{A}$ is residual iff there exist successive finite block-sequences $(\tilde{x}_n), n \in \mathbb{N}$ such that any element of $bb_d(X)$ passing through infinitely many of the $\tilde{x}_n$'s is in $\mathcal{A}$.

Finally, as any element of $bb(X)$ is arbitrarily close to an element of $bb_d(X)$, the following theorem holds:
Theorem 9 Let $X$ be a Banach space with a basis. Then either $X$ is ergodic or there exists $K \geq 1$, and a sequence of successive finite block-sequences $\{\tilde{x}_n\}$ such that all block-sequences passing through infinitely many of the $\{\tilde{x}_n\}$’s span mutually $K$-isomorphic subspaces.

If in addition the basis is unconditional, then we may use the projections to get further properties of the residual class.

Corollary 10 Let $X$ be a non ergodic Banach space with an unconditional basis. Denote by $A$ an element of the residual class of isomorphism in $\mathbb{b}_d(X)$. Then for any block-subspace $Y$ of $X$, $A \simeq A \oplus Y$. If $X$ is hereditarily block-minimal, then all residual classes in $\mathbb{b}_d(Y)$, for block-subspaces $Y$ of $X$, are isomorphic.

Proof : Let $\{e_i\}$ be the unconditional basis of $X$ and let $\{\tilde{x}_n\}$ be given by Theorem 9. Consider an arbitrary block-subspace $Y$ of $X$. Its natural basis is unconditional and $Y = [y_i]_{i \in \mathbb{N}}$ is not ergodic as well. Let, by Theorem 6, $(F_n)_{n \geq 1}$ be successive finite subsets of $\mathbb{N}$ such that for any infinite subset $N$ of $\mathbb{N}$, if $N \cap [\min(F_n), \max(F_n)] = F_n$ for infinitely many $n$’s, then the space $[y_i]_{i \in N}$ is isomorphic to $Y$. Passing to subsequences we may assume that for all $n \in \mathbb{N}$, $\tilde{x}_n < \cup_{i \in F_n} supp(y_i) < \tilde{x}_{n+1}$. Then

$$A \simeq [\tilde{x}_n]_{n \in \mathbb{N}} \oplus [y_i]_{i \in \cup_{n \in N} F_n} \simeq A \oplus Y.$$ If now $X$ is hereditarily block-minimal, and $B$ belongs to the residual class in $\mathbb{b}_d(Y)$, for $Y \leq X$, then by the above $A \simeq A \oplus B$; but also $A$ is block-minimal so some copy of $A$ embeds as a block-subspace of $Y$, so $B \simeq B \oplus A$. \qed

Consider the property that every block-subspace $Y$ satisfies $A \simeq A \oplus Y$. We may think of this property as an algebraic property characterizing large subspaces in the sense that a large subspace should intuitively ”contain” other subspaces, and more importantly, a space should have at most one large subspace (here if $A$ and $A’$ satisfy the property, $A \simeq A \oplus A’ \simeq A’$). Notice that as $X$ is not ergodic, all block-subspaces are isomorphic to their squares by [5] and so the property above is equivalent to saying that every
block of $X$ embeds complementably in $A$ (i.e. $A$ is complementably universal for $bb(X)$). Generally, a space $A$ is said to be complementably universal for a class $C$ of Banach spaces if every element of $C$ is isomorphic to a complemented subspace of $X$. It is known that no separable Banach space is complementably universal for the class of all separable Banach spaces ([16], Th. 2.d.9), but there exists a Banach space $X_U$ with an unconditional basis which is complementably universal for the class of all Banach spaces with an unconditional basis ([20]), and so for $bb(X_U)$ in particular.

Combining Theorem 4 and Corollary 10, we get

**Theorem 11** Any Banach space is ergodic or contains a subspace with an unconditional basis which is complementably universal for the family of its block-subspaces.

We now study this property in more details. We also see how Theorem 9 may be used to obtain uniformity results.

**Definition 12** Let $X$ and $Y$ be Banach spaces such that $X$ has a Schauder basis. $Y$ is said to be complementably universal for $bb(X)$ if every block subspace of $X$ embeds complementably in $Y$.

**Lemma 13** Let $X$ be a Banach space. Any Banach space complementably universal for $bb(X)$ is decomposable.

**Proof** : Let $A$ be complementably universal for $bb(X)$ and indecomposable. First note that $X$ embeds complementably in $A$, so must be isomorphic to a finite-codimensional subspace of $A$. As well, any block subspace of $X$ is isomorphic to a finite-codimensional subspace of $A$ and so none of them is decomposable either. It follows easily that no subspace of $X$ is decomposable. In other words, $X$ is hereditarily indecomposable. It follows also that $X$ is isomorphic to a proper (infinite-dimensional) subspace, and this is a contradiction with properties of hereditarily indecomposable spaces. $\square$

To quantify the property of complementable universality, let us define $dec_X(Y) = \inf KK'$, where the infimum runs over all couples $(K, K')$ such that $Y$ is $K$-isomorphic to a $K'$-complemented subspace of $X$. Of course, $dec_X(Y) = +\infty$ iff $Y$ does not embed complementably in $X$. We shall say
that a space $A$ is $C$-complementably universal for $bb(X)$ if every block-subspace of $X$ is $K$-isomorphic to some $K'$-complemented subspace of $A$, for some $K$ and $K'$ such that $KK' \leq C$, that is, if $\sup_{Y \preceq X} dec_A(Y) \leq C$.

**Lemma 14** Assume $A$, $B$, $C$ are Banach spaces with bases. Then

$$dec_A(C) \leq dec_A(B)^2 dec_B(C).$$

**Proof :** Let $\epsilon$ be positive. Let $P_B$ be a projection defined on $B$ and $\alpha_{BC}$ be an isomorphism from $P_B(B)$ onto $C$ such that $\|P_B\| \cdot \|\alpha_{BC}\| \cdot \|\alpha_{BC}^{-1}\| \leq dec_B(C) + \epsilon$. Let $P_A$ be a projection defined on $A$ and $\alpha_{AB}$ be an isomorphism from $P_A(A)$ onto $B$ such that $\|P_A\| \cdot \|\alpha_{AC}\| \cdot \|\alpha_{AC}^{-1}\| \leq dec_A(B) + \epsilon$.

We let $P = \alpha_{AB} P_B \alpha_{AB} P_A$, defined on $A$; it is easily checked that $P$ is a projection. We let $\alpha = \alpha_{BC} \alpha_{AB}$: it is an isomorphism from $P(A)$ onto $C$. Then

$$dec_A(C) \leq \|P\| \cdot \|\alpha\| \cdot \|\alpha^{-1}\| \leq (dec_A(B) + \epsilon)^2 (dec_B(C) + \epsilon).$$

\[\square\]

**Proposition 15** Let $X$ be a Banach space with a Schauder basis and let $A$ be complementably universal for $bb(X)$. Then there exists $C \geq 1$ such that every finite dimensional block-subspace of $X$ $C$-embeds complementably in $A$.

**Proof :** First it is clear that it is enough to restrict ourselves to elements of $bb_d(X)$ with the previously defined topology. We let for $k \in \mathbb{N}$, $\mathcal{A}_k$ denote the set of block-subspaces of $X$ which are $k$-isomorphic to some $k$-complemented subspace of $A$. Now it is clear that one of the $\mathcal{A}_k$ must be non-meagre. This set is analytic, so has the Baire property, so is residual in some basic open set $U$, of the form $N(\tilde{u})$. We now show that $\mathcal{A}_K$ is residual for $K = kc(2 \max(\text{supp}(\tilde{u})))$. Otherwise, as in Proposition 7, we may assume $\mathcal{A}_K$ is meagre in $V = N(\tilde{y})$, and $\mathcal{A}_k$ is residual in $U' = N(\tilde{x})$ where $\tilde{x}$ extends $\tilde{u}$, $|\tilde{x}| \leq 2 \max(\text{supp}(\tilde{u}))$ and $\max(\text{supp}(\tilde{x})) = \max(\text{supp}(\tilde{y}))$.

Now let $T$ be the canonical map from $U'$ to $V$. For all $u$ in $U'$, $T(u)$ differs from at most $q \leq 2 \max(\text{supp}(\tilde{u}))$ vectors from $u$, so the space $[T(u)]$ is $c(2 \max(\text{supp}(\tilde{u}))$ isomorphic to $[u]$. So $T(u)$ is in $\mathcal{A}_K$ whenever $u$ is in $\mathcal{A}_k$. It follows that $\mathcal{A}_K$ is residual in $V$, a contradiction.
Now consider any finite dimensional space $F$ generated by a finite block-sequence of $A$, $F = [x_1, \ldots, x_p]$: it may be extended to a block-sequence $x = (x_i)_{i \in \mathbb{N}}$ in $A_K$, that is $\text{dec}_A([x]) \leq K^2$. But also $\text{dec}_{[x]}(F) \leq c$, where $c$ is the constant of the basis, so by Lemma 14, $\text{dec}_A(F) \leq cK^4$. □

**Proposition 16** Let $X$ be a space with an unconditional basis. If $A$ is complementably universal for $bb(X)$ and isomorphic to its square, then $A$ is $C$-complementably universal for $bb(X)$ for some $C \geq 1$.

**Proof:** The first part of the proof is as above to get $K \in \mathbb{N}$ such that $A_K$, the set of block-subspaces of $X$ which are $K$-isomorphic to some $K$-complemented subspace of $A$, is residual. So by Proposition 8, there exists a sequence $\tilde{x}_n$ of successive finite blocks such that any block passing through infinitely many of the $\tilde{x}_n$'s is in $A_{2K}$.

Let now $Y = [y_n]_{n \in \mathbb{N}} = [y]$ be an arbitrary block-subspace of $X$. We may define a sequence $(\tilde{y}_i)$ of finite block-sequences with $\tilde{y}_1 \tilde{y}_2 \ldots = y$ and a subsequence of $\{\tilde{x}_n\}$, denoted $\{\tilde{x}_i\}$, such that for all $i$,

$$\text{supp}(\tilde{y}_{i-1}) < \text{supp}(\tilde{x}_i) < \text{supp}(\tilde{y}_{i+1}).$$

We let $w = \tilde{y}_1 \tilde{x}_2 \tilde{y}_3 \ldots$ and $w' = \tilde{x}_1 \tilde{y}_2 \tilde{x}_3 \ldots$. It is clear that $Y_1 = [\tilde{y}_{2i-1}]_{i \in \mathbb{N}}$ is $c$-complemented in $[w]$, where $c$ is the constant of unconditionality of the basis; so $\text{dec}_{[w]}(Y_1) \leq c$. But we know that $\text{dec}_A([w]) \leq 4K^2$, so by Lemma 14, $\text{dec}_A(Y_1) \leq 16K^4c$. Likewise if we denote $[\tilde{y}_{2i}]_{i \in \mathbb{N}}$ by $Y_2$ and use $[w']$, we prove that $\text{dec}_A(Y_2) \leq 16K^4c$. It follows that $Y_1 \oplus Y_2$ satisfies $\text{dec}_{A \oplus 1A}(Y_1 \oplus Y_2) \leq (16K^4c)^2$ (here $\oplus_1$ denotes the $l_1$-sum), and, if $D$ is such that $A$ is $D$ isomorphic to its square $A \oplus_1 A$, that $\text{dec}_A(Y) \leq 2^9D^2K^8c^3$. □

In view of the fact that by Theorem 6, any unconditional basic sequence in a non-ergodic Banach space spans a space isomorphic to its square, the previous proposition may be applied to Theorem 11: a non-ergodic Banach space contains a subspace $X$ with an unconditional basis which is *uniformly* complementably universal for $bb(X)$.

Consider now spaces with a basis with the stronger property that every block-subspace is complemented. It is well-known that every block-subspace of $l_p$ or $c_0$ is complemented, and the same is true for Tsirelson’s spaces $T_{(p)}$ (see [3]) or for spaces $(\sum_{n=1}^{+\infty} \oplus l_n^s)_p$, the relevant case being $s \neq p$ (see [16]).
All these examples are asymptotically \( l_p \) or \( c_0 \), and using results of Maurey, Milman and Tomczak-Jaegermann ([18]), we shall prove that this is not by chance.

We recall the definition of an asymptotically \( l_p \) space with a basis. Consider the so-called \emph{asymptotic game} in \( X \), where Player I plays integers \((n_k)\) and Player II plays successive unit vectors \((x_k)\) in \( X \) such that \( \text{supp}(x_k) > n_k \) for all \( k \). Then \( X \) is asymptotically \( l_p \) if there exists a constant \( C \) such that for any \( n \in \mathbb{N} \), Player I has a winning strategy in the asymptotic game of length \( n \) for producing a sequence \( C \)-equivalent to the unit basis of \( l_p^n \).

**Proposition 17** Let \( X \) be a space with an unconditional basis such that every block-subspace of \( X \) is complemented. Then there exists \( C > 0 \) such that every block-subspace of \( X \) is \( C \)-complemented.

**Proof:** Once again, we may and do restrict ourselves to block-bases in \( bb_d(X) \) with our usual topology. Let \( c \) denote the unconditional constant of the basis of \( X \). We note a fact: if an infinite block-sequence \( x = (x_n)_{n \in \mathbb{N}} \) (resp. \( x' = (x'_n)_{n \in \mathbb{N}} \) ) spans a \( C \)-complemented (resp. \( C' \)-complemented) subspace of \( X \), and if \( x \) and \( x' \) are disjointly supported, then \([x] \oplus [x']\) is \( c^2(C + C') \)-complemented in \( X \). Indeed, if \( q \) (resp. \( q' \)) is the canonical projection onto \( \bigcup_n \text{supp}(x_n) \) (resp. \( \bigcup_n \text{supp}(x'_n) \)), and if \( p \) (resp. \( p' \)) is a projection onto \( [x] \) of norm less than \( C \) (resp. onto \( [x'] \) of norm less than \( C' \)), then \( qp + q'p'q' \) is a projection onto \( [x] \oplus [x'] \) of norm less than \( c^2(C + C') \).

It follows that if \([x]\) is \( C \)-complemented in \( X \), then for any finite block-sequence \( y = (y_1, \ldots, y_k) \) with \( \max(\text{supp}(y_k)) < \min(\text{supp}(x_{k+1})) \), the space \([y\prec(x_n)_{n \geq k+1}]\) is \((k + cC)c^2\)-complemented in \( X \): we just apply the previous fact noting that \([x_{n \geq k+1}]\) is \( k\)-complemented in \( X \) and that the \( k\)-dimensional space \([y]\) is \( k\)-complemented in \( X \).

Now as before we find \( K \) such that the set of \( K \)-complemented block-subspaces is non-meagre, so residual in some open set \( N(\tilde{u}) \). By the above, we have a uniform control on the constant when we modify a given number of vectors of the subspace, so as in Proposition 7, we find \( K' \) such that the set of block-sequences spanning a \( K' \)-complemented subspace of \( X \) is residual in \( bb_d(X) \). Let \( \{\tilde{x}_n\} \) be a sequence given by Proposition 8; any block-sequence passing through infinitely many \( \tilde{x}_n \)'s spans a \( 2K' \)-complemented block-subspace.
Let now \( Y = \{y\} \) be an arbitrary block-subspace of \( X \). We easily find successive intervals of integers \( \{E_i, i \in \mathbb{N}\} \), a subsequence \( (\tilde{x}_i) \) of \( (\tilde{x}_n) \), and a sequence \( (\tilde{y}_i) \) of finite block-sequences with \( \tilde{y}_1 \tilde{y}_2 \ldots = y \), such that for all \( i \), \( \text{supp}(\tilde{x}_i) \cup \text{supp}(\tilde{y}_i) \subset E_i \). By construction, \( [\tilde{y}_1 \tilde{x}_2 \tilde{y}_3 \ldots] \) is \( 2K' \)-complemented in \( X \), so \( Y_1 = [\tilde{y}_1 \tilde{y}_3 \ldots] \) is \( 2cK' \)-complemented in \( X \). Likewise, \( Y_2 = [\tilde{y}_2 \tilde{y}_4 \ldots] \) is \( 2cK' \)-complemented in \( X \). Applying the fact stated at the beginning of the proof, we deduce that the space \( Y = Y_1 \oplus Y_2 \) is \( 4c^3K' \)-complemented in \( X \). 

Note that we needed to assume unconditionality for our result. If we don’t, we at least get unconditionality in some block-subspace. Indeed, if every block-subspace is complemented, then every block-subspace \( Y = \{y_n\}_{n \in \mathbb{N}} \) has a non-trivial complemented subspace, \( \{y_{2n}\}_{n \in \mathbb{N}} \) for example. So \( X \) contains no hereditarily decomposable subspace, and by Gowers’ dichotomy theorem, it must contain an unconditional block-sequence.

It follows from Proposition 17 and from the results of Maurey, Milman and Tomczak-Jaegermann in [18] that \( c_0 \) and \( l_p \) are the only subsymmetric unconditional examples.

**Corollary 18** Let \( X \) be a space with an unconditional basis \( \{e_i\} \) such that every block-subspace of \( X \) is complemented. Then \( X \) is asymptotically \( c_0 \) or \( l_p \).

**Proof:** Again \( c \) denotes the unconditional constant of \( \{e_i\} \). Let \( K \) be given by Proposition 17. It follows that all finite block-sequences span \( cK \)-complemented subspaces of \( X \) (this could also have been proved directly in the spirit of [4]).

With Maurey, Milman, Tomczak-Jaegermann’s terminology, in particular all permissible subspaces of \( X \) far enough are \( cK \)-complemented. We may then apply their Theorem 5.3 in the unconditional case (Remark 5.4.1) to deduce that \( X \) is asymptotically \( c_0 \) or \( l_p \). 

**Proposition 19** Let \( \{e_i\} \) be an unconditional basic sequence asymptotically \( c_0 \) or \( l_p \) such that every subsequence of \( \{e_i\} \) has a block-sequence equivalent to \( \{e_i\} \) (in particular, if \( \{e_i\} \) is subsymmetric or equivalence-block minimal), then \( \{e_i\} \) is equivalent to the unit basis of \( c_0 \) or \( l_p \).
Proof : Let $p$ be such that $X = [e_i]_{i \in \mathbb{N}}$ is asymptotically $l_p$ (the case of $c_0$ is similar). Assume every subsequence of $\{e_i\}$ has a block-sequence equivalent to $\{e_i\}$. Then as shown we may (by passing to a subsequence) assume that for some $C \geq 1$, every subsequence of $\{e_i\}$ has a block-sequence $C$-equivalent to $\{e_i\}$.

We fix $n \in \mathbb{N}$ and build a winning strategy for Player II in the asymptotic game of infinite length for producing a block-sequence $2C$-equivalent to $(e_i)$. This strategy may then be opposed to a winning strategy for Player I for producing length $n$ block-sequences $C'$-equivalent to the unit basis of $l_p^n$. We get that $e_1, \ldots, e_n$ is $2CC'$-equivalent to $l_p^n$ for all $n$ which will conclude the proof.

Let $A = \{(n_k) \in [\omega]^{\omega} : \exists (x_k) \in bb_N(X) \forall k \ n_{2k} < x_k < n_{2k+1} \land (x_k) \sim^C (e_k)\}$. We claim that any sequence $(m_k) \in [\omega]^{\omega}$ contains a further subsequence in $A$, for we can suppose that $I_k = \{m_{2k}, m_{2k+1} \neq \emptyset\}$ for all $k$ and therefore take $(y_i) \sim^C (e_i)$ with $supp(y_i) \subset \bigcup I_n$. Then in between $y_i$ and $y_{i+1}$ there are $n_{2i+1} := m_{2k+1}$ and $n_{2i+2} := m_{2k+2}$, whereby $(n_k) \in A$. So by the infinite Ramsey theorem there is some infinite $A \subset \omega$ such that $[A]^{\omega} \subset A$ and there is a $C$-measurable $f : [A]^{\omega} \rightarrow bb_N(X)$ choosing witnesses $(x_k)$ for being in $A$.

Let $\Delta = (\delta_n)$, $\delta_n > 0$ be such that if two normalized block sequences are less than $\Delta$ apart, then they are $2$-equivalent. We choose inductively $n_i < m_i < n_{i+1}$ and sets $B_i \subset \omega$ such that $n_i, m_i \in B_i \subset B_{i-1}/m_{i-1}$ and such that for all $C, D \in \{\{n_{j_1}, m_{j_1}, \ldots, n_{j_k}, m_{j_k}\}, B_{j_k+1}\} (j_1 < \ldots < j_k)$ we have $\|f(C)(k) - f(D)(k)\| < \delta_k$. This can be done as the unit sphere of $[e_{n_{j_k+1}}, \ldots, e_{m_{j_k-1}}]$ is compact.

Now in the asymptotic game of infinite length, we can demand that I plays numbers from the sequence $(n_i)$ and then II replies to $n_{j_1}, \ldots, n_{j_k}$ played by I with some $n_{j_k} < x < m_{j_k}$ such that for all $C \in \{\{n_{j_1}, m_{j_1}, \ldots, n_{j_k}, m_{j_k}\}, B_{j_k+1}\}$ we have $\|f(C)(k) - x\| < \delta_k$.

Then in the end of the infinite game, supposing that I has played $(n_{j_k})$ and II has followed the above strategy responding by $(x_k)$, we have $n_{j_k} < x_k < m_{j_k}$. Let $(y_k) := f(\{n_{j_k}, m_{j_k}\}^{\omega})$, then $\|x_k - y_k\| < \delta_k$ and $(x_k) \sim^2 (y_k)$, so II wins. 

A space is said to be countably homogeneous if it has at most countably
many non-isomorphic subspaces. By Corollary 4, a countably homogeneous
space has a block-minimal subspace with an unconditional basis, and one
easily diagonalizes to get a hereditarily block-minimal subspace.

**Proposition 20** Let $X$ be a countably homogeneous, hereditarily block-mini-
mal Banach space with an unconditional basis. Then elements in the residual
class of isomorphism for $bbd(X)$ are isomorphic to a (possibly infinite) direct
sum of an element of each class.

**Proof**: We write the proof in the denumerable case. We partition $X$ in
a direct sum of subspaces $X_n, n \in \mathbb{N}$ by partitioning the basis. So each
$X_n$ embeds into $X$. For each $n$, choose a representative $E_n$ of the $n$-th
isomorphism class which is a block of $X_n$ (it is possible because $X_n$ is block-
minimal as well). By applications of Gowers’ Theorem in each $X_n$, we may
pick each vector forming the basis of each $E_n$ far enough, to ensure that
$E = \sum_{n \in \mathbb{N}} E_n$ is a block-subspace of $X$. We show that $E$ is in the residual
class $A$. Indeed, if $m$ is such that $E_m \in A$, then $E \simeq E_m \oplus \sum_{n \neq m} E_n \simeq E_m$
by Corollary 10.

It follows from the proof above that for any two block-subspaces $A$ and
$B$ of $X$, $A \oplus B$ may be embedded as a block-subspace of $X$; i.e., under the
assumptions of Proposition 20, isomorphism classes of block-subspaces of $X$
form a countable (commutative) semi-group.

We now want to generalize the previous results, considering more general
types of subspaces. So we fix $X$ to be a Banach space with a basis $\{e_n\}$ and
we denote by $c$ the constant of the basis.

We first notice that we could consider subspaces generated by disjointly
supported (but not necessarily successive) vectors, and get similar results:
we find a residual class characterized by a ”passing through” property. It
follows that in a non ergodic Banach space with an unconditional basis,
subspaces generated by disjointly supported vectors embed complementably
in a given element of the residual class. We recall that if a Banach space
has an unconditional basis such that any subspace generated by disjointly
supported vectors is complemented, then the space is $c_0$ or $l_p$ ([16]); however
there does exist a space $X_U$ cited above from [20], not isomorphic to $c_0$ or
$l_p$, and such that every subspace generated by disjointly supported vectors
embeds complementably in $X_U$. 

23
Then we want to represent any subspace of $X$, possibly up to small perturbations, on the basis $\{e_n\}$, and get similar results as for the case of block subspaces. We shall call triangular sequences of blocks the normalized sequences of (possibly infinitely supported) vectors in the product $X^\omega$’s, satisfying for all $k$, $\min(\text{supp}(x_k)) < \min(\text{supp}(x_{k+1}))$ equipped with the product of the norm topology on $X$. The set of triangular sequences of blocks will be denoted by $tt$. By Gaussian elimination method, it is clear that any subspace of $X$ may be seen as the closed linear space generated by some sequence of $tt$. Once again it is possible to discretize the problem by considering the set $tt_d$ of sequences of vectors in $Q$ such that for all $k$, $\min(\text{supp}(x_k)) < \min(\text{supp}(x_{k+1}))$, and by showing that for any $x \in tt$ and any $\epsilon > 0$, there exists $x_d$ in $tt_d$ such that $[x_d]$ is $1 + \epsilon$-isomorphic to $[x]$.

Our usual method does generalize to this setting. However the characterization of a residual set turns out to be only expressed in terms of particular subspaces of $tt$, namely those with a finite dimensional decomposition on the basis (or F.D.D.). So it gives more information, and it is actually easier, to work directly with spaces with a finite dimensional decomposition on the basis.

Note that a space with a F.D.D. on the basis must have the bounded approximation property. So our methods only allow us to study subspaces with that property. In fact, it is easy to check that the set of sequences in $tt_d$ spanning a space with an F.D.D. on the basis is residual in $tt_d$, and so, the set of spaces without the (bounded) approximation property is meagre in our topology, which explains why with our methods, we do not seem to be able to "see" spaces without the approximation property.

We say that two finite-dimensional spaces $F$ and $G$ are successive, and write $F < G$, if for any $x \in F$, $y \in G$, $x$ and $y$ are successive. A space with a finite dimensional decomposition on the basis is a space of the form $\bigoplus_{k \in \mathbb{N}} E_k$, with successive, finite-dimensional spaces $E_k$; such a space passes through $E$ if $E_k = E$ for some $k$. We let $fdd$ be the set of infinite sequences of successive finite-dimensional subspaces, and $fdd_d$ be the set of infinite sequences of successive finite-dimensional subspaces which are spanned by a collection of vectors with rational coordinates - equipped with the product of the discrete topology on the set of finite collections of rational vectors.

**Theorem 21** Let $X$ be a Banach space with a basis. Then either $X$ is
ergodic, or there exists $K \geq 1$, and a sequence of successive finite dimensional spaces $F_n$ such all spaces with finite dimensional decomposition on the basis passing through infinitely many $F_n$‘s are mutually $K$-isomorphic.

**Proof :** Most of the previous proof may be taken word by word; instead of working with block-subspaces, i.e. subspaces with 1-dimensional decomposition on the basis, we work with subspaces with arbitrary F.D.D. on the basis; we just have to define the $\Delta$-expansion of a subspace with a F.D.D. on the basis using an appropriate distance between finite-dimensional subspaces. We then end up with characterization in terms of ”passing through” some finite sequences of finite-dimensional spaces $\{\tilde{E}_i, i \in \mathbb{N}\}$, and this may be simplified to get Theorem 21, noting that we may choose these sequences to be of length 1 (replace each $\tilde{E}_i = (E_1^i, \ldots, E_n^i)$ by $E_1^i \oplus \ldots \oplus E_n^i$).

If the basis is unconditional then we can use the many projections to get additional results concerning the residual class.

**Proposition 22** Let $X$ have an unconditional basis and be non-ergodic. Let $A$ belong to the residual class in $fdd_d$. Then every subspace with a finite dimensional decomposition on the basis embeds complementably in $A$. Either $A$ fails to have l.u.s.t. or $X$ is $C$-$l_2$ saturated.

**Proof :** Let $(F_n)$ be given by Theorem 21. let $Y = \sum \oplus B_n$ be an arbitrary space with a finite dimensional decomposition on the basis. Passing to a subsequence of $(F_n)$, we may assume that there is a partition of $\mathbb{N}$ in successive intervals $J_i, i \in \mathbb{N}$, such that for all $i$,

$$F_{i-1} < B_i' = \oplus_{n \in J_i} B_n < F_{i+1}.$$  

We have that

$$A \simeq (\sum \oplus F_i)_{i \in \mathbb{N}} \simeq (\sum \oplus F_{2k-1})_{k \in \mathbb{N}} \oplus (\sum \oplus F_{2k})_{k \in \mathbb{N}},$$

and so it follows that

$$A \oplus Y \simeq (\sum \oplus F_{2k-1})_{k \in \mathbb{N}} \oplus (\sum \oplus B_{2k}')_{k \in \mathbb{N}} \oplus (\sum \oplus F_{2k})_{k \in \mathbb{N}} \oplus (\sum \oplus B_{2k-1}')_{k \in \mathbb{N}}$$

$$\simeq A \oplus A \simeq A.$$
For the last part of the Proposition, if $X$ is $C$-$l_2$ saturated for no $C$, then it follows from the Theorem of Komorowski and Tomczak-Jaegermann ([14],[15], and [23] for a survey and an improved result) that $X$ has a finitely supported subspace $L_n$ without $n$-l.u.s.t. for each $n$: indeed Komorowski-Tomczak-Jaegermann’s result takes care of the finite cotype case, and if $X$ doesn’t have finite cotype then it contains $l^n_{\infty}$’s uniformly; so it contains $1 + \epsilon$-isomorphic, finitely supported if you wish, copies of any finite dimensional space, for example without $n$-l.u.s.t.. We may by Theorem 21 extend $L_n$ to a space with a finite dimensional decomposition on the basis which is $K$ isomorphic to $A$. Then if $c$ is the unconditional constant of the basis, $A$ must fail $n/cK$-l.u.s.t. and as $n$ was arbitrary, $A$ fails l.u.s.t.. ⊡

Note that it is a consequence of the solution of Gowers and Komorowski-Tomczak to the Homogeneous Banach Space Problem, that the following strengthening holds: a Banach space isomorphic to all its subspaces with the Bounded Approximation Property must be isomorphic to $l_2$. This can also be seen as a consequence of the previous Proposition. Note also that by results of Kadec and Pelczynski (Theorem 2.d.8. in [16]), there exists a Banach space which is complementably universal for the set of Banach spaces with the Bounded Approximation Property; but as mentioned before, there is no complementably universal space for the class of separable Banach spaces. The Bounded Approximation Property seems to draw the fine line for positive results with our methods.

We conjecture that $l_2$ is the only non-ergodic Banach space. However, we are not even able to prove that $c_0$ and $l_p$, $p \neq 2$ are ergodic - although it is known that those spaces have at least $\omega_1$ non-isomorphic subspaces ([16]). To answer this question, evidently one would have to consider other types of subspaces than those generated by successive blocks, or disjointly supported blocks; one could consider spaces of the form $(\sum_{n=1}^{+\infty} \oplus B_n)_p$, with carefully chosen finite-dimensional $B_n$ so that this direct sum is isomorphic to a subspace of $l_p$, and play with the possible choices for $(B_n)$ (see [16], Proposition 2.d.7.).

On the other hand, it is more relevant to restrict the question of ergodicity to block-subspaces, if one is looking for a significant dichotomy between ”regular” and ”wild” spaces with a basis: in this setting, $c_0$ and $l_p$ are, as they should be, on the regular (i.e. non-ergodic) side of the dichotomy.
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