Research Article
A New Fractional Gradient Representation of Birkhoff Systems

Peng Wang

School of Civil Engineering and Architecture, University of Jinan, Jinan, Shandong 250022, China

Correspondence should be addressed to Peng Wang; cea_wangp@ujn.edu.cn

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A fractional generalization of the gradient system to the Birkhoff mechanics, that is, a new fractional gradient representation of the Birkhoff system is investigated, in this paper. The definitions of the fractional gradient system are generalized to Birkhoff mechanics. Based on the definition, a general condition that a Birkhoff system can be a fractional gradient system is derived, the former studies for fractional gradient representation of the Birkhoff system are special cases of this paper. As applications of the results, the Birkhoff equations and fractional gradient expression of several classical nonlinear models are derived, such as the Hénon–Heiles equation, the Duffing oscillator model, and the Hojman–Urrutia equations. The results indicate, different from the former studies, that only gave the second-order gradient (integer order) expression for the Birkhoff system, an arbitrary fractional order gradient representation, exists for the Birkhoff system. The fractional potential function obtained from the general condition can determine the stationary states of these models in Birkhoffian expression.

1. Introduction

Fractional integrals and derivatives are increasingly important in the modeling of science and engineering problems because they are more suitable for describing complex phenomena in science and engineering. The discussion of noninteger order derivatives dates back to Leibniz and L'Hopital in 1695, but until 1974 the first monography [1] written by Oldham and Spanier about fractional integrals and derivatives was published. Since then, fractional calculus has been applied in many fields such as physics, mechanics, etc. [2–15].

A gradient system is an important dynamical system. The characteristics of a gradient system are suitable, especially to determine the stability of a system by using the Lyapunov function [16–20]. If a mechanical system can be transformed into a gradient system, the stability of the mechanical system can be studied through the characteristics of the gradient system. Many researchers have performed the gradient representations of constrained mechanical systems such as the Lagrange system, the Hamilton system, and the Birkhoff system and studied the stability of their solutions [21–27]. Tarasov [28–30] generalized the Hamilton system and the Lagrange system into fractional gradient expressions. Mei et al. [31, 32] studied the fractional gradient representation of the Birkhoff system, Chen et al. [33] gave a fractional gradient representation of Poincaré equations. However, their studies only provide a special condition with second order (\(\alpha = 2\)) and only can provide a second order gradient system, which limits its application in constrained mechanical systems. In this paper, our goal is to generalize the fractional gradient representation of the Birkhoff system to any order.

The Birkhoff system is an important dynamical system; it is an extension of the Hamilton system. Birkhoff mechanics has important applications in hadronic physics, statistic mechanics, space mechanics, biophysics, and engineering [34]. Birkhoff’s mechanical system is a more general, constrained mechanical system [35–37]. Conservative and nonconservative systems and holonomic and nonholonomic systems can all be expressed by Birkhoffian formulations, which have a generalized symplectic structure [38]. So, generalizing the fractional gradient system into the Birkhoff system has general significance. Since former studies about the fractional gradient representation of the Birkhoff system only gave its second-order expression. In this paper, we will show the order of the fractional gradient Birkhoff system is not only with \(\alpha = 2\) but also can be other orders (integer or
noninteger order), and give a general condition for the Birkhoff system transforming into a fractional gradient Birkhoff system.

The structure of this paper is as follows: In Section 2, we will fix the notations of fractional derivative and fractional exterior derivative and give a brief review of the definition of the fractional gradient system and its formulation. In Section 3, we will discuss under what conditions a Birkhoff system can transform into a fractional gradient system. In Section 4, we will apply the results that we got in Section 3 to several examples, such as the Hénon–Heiles equation, the Duffing oscillator model, and the Hojman–Urrutia equations. Section 5 is the conclusion.

2. Review of the Fractional Gradient System

Fractional derivatives have different definitions. In this paper, we will adopt the Caputo and Riemann–Liouville fractional derivative definitions. Based on the definition of the Riemann–Liouville fractional derivative, we have the definition of $D_x^a$ in the Riemann–Liouville fractional derivative sense.

$$
D_x^a f(x) = \frac{1}{\Gamma (m - \alpha)} \left( \frac{d^m}{dx^m} \right) \int_0^x \frac{f(y)dy}{(x - y)^{\alpha - m + 1}} \quad (m - 1 \leq \alpha < m, x > 0),
$$

where $m$ is the first whole number greater than $a > 0$. The initial point of a fractional derivative is set to be zero. The Riemann–Liouville fractional derivative has some disadvantages in physical applications, such as the hyper-singular improper integral, where the order of singularity is higher than the dimension, and the nonzero of the fractional derivative of constants, which would make dissipation not vanish for a system in equilibrium. The Caputo fractional derivatives can choose the same initial value as the integer derivatives, which are more suitable for applying to physical problems. The Caputo fractional derivative has the following definition:

$$
D_x^a f(x) = \frac{1}{\Gamma (m - \alpha)} \int_a^x \frac{f^{(m)}(y)dy}{(x - y)^{\alpha - m + 1}} \quad (m - 1 < \alpha \leq m, x > a),
$$

where

$$
f^{(m)}(y) = \frac{d^m}{dy^m} (f(y)) (m = [\alpha] + 1).
$$

We can introduce fractional exterior derivatives as

$$
d^a = (dx_i)^a (R^a D_x^a) or d^a = (dx_i)^a (R^a D_x^a).
$$

For the convenience of expression, we use $D_x^a$ to denote a order fractional derivative, and only point out whether it is the Riemann–Liouville fractional derivative or the Caputo fractional derivative as calculation.

The differential equations of a gradient system have the form as

$$
\dot{x}_i = -\frac{\partial V(x)}{\partial x_i} (i = 1, 2, \ldots, n),
$$

where $x = (x_1, x_2, \ldots, x_n)$, and $V$ is a potential function.

Definition 1. For a dynamical system,

$$
\frac{dx_i}{dt} = F_i(x) (i = 1, 2, \ldots, n)
$$

If a fractional differential 1-form,

$$\omega_a = F_i(x) (dx_i)^a.
$$

This is an exact form, that is, $\omega_a = -d^a V$, where $V = V(x)$ is a continuous differential function, then system (7) is a fractional gradient system. Thus, we have $F_i(x) = -D_x^a V(x)$.

It is known that to be exact, it is a sufficient condition to be closed; however, the converse statement is not necessarily true. From the Poincaré theorem that is verified to be true for fractional exterior derivatives, it can be deducted that any smooth 1-form (8) that is closed is also exact on a contractible open subset $W$ of $R_n$ [28]. We have the following proposition.

Proposition 1. If smooth functions $F_i = F_i(x)$ on a contractible open subset $W$ of $R^n$ satisfy the relations

$$D_{x_i}^a F_i - D_{x_j}^a F_j = 0, (i, j = 1, 2, \ldots, n),
$$

then dynamical system (7) is a fractional gradient system with form

$$\dot{x}_i = -D_x^a V(x) (i = 1, 2, \ldots, n),
$$

where $V = V(x)$ is a continuous differential function and $D_x^a V(x) = -F_i(x)$. Because the Riemann–Liouville fractional derivative of a constant is not necessary to be zero, so, $V(x) = C$ cannot define the stationary state of a fractional system (10), where $C$ is constant. The stationary state of the fractional gradient system with the Riemann–Liouville fractional derivative can be defined by

$$V(x) = \sum_{i=0}^{n} x_i \sum_{k_i=0}^{a-m-1} \sum_{k_{i-1}=0}^{m-1} C_{k_{i-1}} \cdot \sum_{k_0=0}^{n} \left( \prod_{l=1}^{n} \left( x_l \right)^{k_l} \right) = 0,
$$

where $C_{k_{i-1}}$ are constants and $m$ is the first whole number greater than or equal to $a$.

If the fractional exact 1-form equals to zero that is $d^a V = 0$ in the Caputo derivative, we can get $V(x) = C = 0$, which defines the stationary state of the fractional gradient system (10), where $C$ is constant.
3. Fractional Gradient Representation of the Birkhoff System

An automatic Birkhoff system has form
\[
\Omega_{\mu \nu} (a) \dot{a}^\nu - \frac{\partial B(a)}{\partial a^\mu} = 0, \quad (\mu, \nu = 1, 2, 3, \ldots, 2n),
\] (12)
where \( B = B(a) \) is a Birkhoffian, and
\[
\Omega_{\mu \nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}
\] (13)
is called the Birkhoffian tensor and \( R_\mu(a) \) are Birkhoff’s functions. If the system is nonsingularity, that is \( \det(\Omega_{\mu \nu}) \neq 0 \), the automatic Birkhoff system (12) can be expressed as
\[
\dot{a}^\nu = \Omega_{\mu \nu} \frac{\partial B}{\partial a^\mu} (\mu, \nu = 1, 2, \ldots, 2n),
\] (14)
where
\[
\Omega^{\nu \mu} \Omega_{\mu \nu} = \delta^\nu_\nu.
\] (15)

**Definition 2.** For a Birkhoff system (12) or (14), if the fractional differential 1-form
\[
\omega_\alpha = \Omega^{\nu \mu} \frac{\partial B}{\partial a^\mu} (da^\mu)^\alpha
\] (16)
is an exact form, that is \( \omega_\alpha = -d^\alpha V \), then it is a fractional gradient expression of the Birkhoff system.

Make exterior derivative to fractional differential 1-form (16), we have
\[
\mathcal{D}^\alpha (\omega_\alpha) = \frac{1}{2} \mathcal{D}^\alpha_a (\Omega^{\nu \mu} \frac{\partial B}{\partial a^\mu} - \Omega^{\mu \nu} \frac{\partial B}{\partial a^\mu}) (da^\mu)^\alpha (da^\nu)^\alpha.
\] (17)
If \( \mathcal{D}^\alpha (\omega_\alpha) = 0 \) is closed, we have
\[
\mathcal{D}^\alpha_a (\Omega^{\nu \mu} \frac{\partial B}{\partial a^\mu} - \Omega^{\mu \nu} \frac{\partial B}{\partial a^\mu}) = 0.
\] (18)
So, we have the following proposition.

**Proposition 2.** If equation (14) on a contractible open subset \( W \) of \( R^n \) satisfies condition (18), it is an \( \alpha \) order fractional gradient system.

**Remark 1.** When \( \alpha = 1 \), condition (18) degenerates to an integer derivative case, which gives the condition for a Birkhoff system transforming into a classical gradient expression [21].

When \( \alpha > 1 \) (including integer and noninteger numbers), we call it an \( \alpha \) order fractional gradient expression of the Birkhoff system.

**Remark 2.** When \( \alpha = 2 \), condition (18) gives the condition for a Birkhoff system transforming into a second-order fractional gradient system [31–33].

We point out that \( 0 < \alpha < 1 \) cannot be a fractional gradient system for the Birkhoff system, it may be a fractional gradient system for the fractional Birkhoff system, which question we will discuss in future works. From (10), we have
\[
\Omega^{\nu \mu} \frac{\partial B}{\partial a^\mu} = -d^\alpha_a V(a), \quad (\mu, \nu = 1, 2, \ldots, 2n).
\] (19)

According to Birkhoff 1-form (19), we can get the expression of the potential function \( V(a) \).

\[
V(a) = I^\alpha_0 \Omega^{\nu \mu} \frac{\partial B}{\partial a^\mu},
\] (20)
where \( I^\alpha_0 \) is the fractional Riemann–Liouville integral symbol with definition
\[
I^\alpha_0 f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(y) y^{\alpha-1} dy (x > 0, \alpha > 0).
\] (21)

**Remark 3.** Here \( \alpha > 1 \) can be any order, but we should point out that only (20) makes the Birkhoff 1-form is exact, and (19) is a fractional order gradient expression of the Birkhoff system.

4. Results

**4.1. Application A: Hénon–Heiles Equation as a Birkhoff System and Fractional Gradient System.** In this section, we will give the Birkhoff expression of the Hénon–Heiles equation and prove it can be a fractional gradient system with the Riemann–Liouville or the Caputo fractional derivatives, respectively, in the Birkhoff mechanics frame.

The Hénon–Heiles equation is a sort of nonlinear, nonintegrable Hamilton system, which has broad applications in physics and applied mathematics [10, 40, 41]. The Hénon–Heiles equation’s form is
\[
\dot{x} = -x - 2xy,
\]
\[
\dot{y} = -y + x^2 - y^2.
\] (22)

Let Birkhoff variables be \( a^1 = x, a^2 = y, a^3 = \dot{x}, a^4 = \dot{y} \); a Birkhoff expression of H e non-Heiles problem is
\[
B = \frac{1}{2} (a^1)^2 + (a^2)^2 + (a^3)^2 + (a^4)^2 + 2a^2 (a^3)^2 + 2 \left[ (a^4)^3 \right].
\]
\[
R_1 = R_2 = 0 R_3 = -a^1 R_4 = a^2.
\] (23)

Therefore, we can get
when 0 ≤ α ≤ 1, (29) becomes
\[
\delta^\alpha \omega_a = \frac{1}{2 \Gamma (2 - \alpha)} \left[ (-\alpha)_{2 - \alpha} - (1 + 2 \alpha^2)(a^1)^{2 - \alpha} \right] (da^1)^\alpha \wedge (da^3)^\alpha
\]
\[
+ \frac{1}{2 \Gamma (2 - \alpha)} \left[ 2(a^1)^{2 - \alpha} \right] (da^3)^\alpha \wedge (da^4)^\alpha
\]
\[
+ \frac{1}{2 \Gamma (2 - \alpha)} \left[ 2a^1(a^2)^{1 - \alpha} (da^3)^\alpha \wedge (da^4)^\alpha \right]
\]
\[
\left[ (a^3)^{1 - \alpha} - 2(a^4)^{1 - \alpha} - (a^4)^{1 - \alpha} \right] (da^3)^\alpha \wedge (da^4)^\alpha.
\]
(29)

So \( \delta^\alpha \omega_a \neq 0 \), then (26) cannot be expressed as a fractional gradient system.

When \( 1 < \alpha \leq 2 \), we can get
\[
\delta^\alpha \omega_a = \frac{2}{\Gamma (2 - \alpha)} \left[ -(a^1)^{2 - \alpha} (da^1)^\alpha \wedge (da^4)^\alpha \right]
\]
\[
+ \left[ (a^2)^{2 - \alpha} (da^3)^\alpha \wedge (da^4)^\alpha \right],
\]
(30)

So \( \delta^\alpha \omega_a \neq 0 \), then (26) cannot be expressed as a fractional gradient system.

When \( m < \alpha \leq m + 1 \) with \( m \geq 2 \), \( \delta^\alpha \omega_a = 0 \), so (26) can be expressed as an \( \alpha \) order Caputo type fractional gradient system, where \( (m < \alpha \leq m + 1, m \geq 2) \). From (19), we can get
\[
a^3 = C D^\alpha_{\alpha} V (a),
\]
\[
a^4 = C D^\alpha_{\alpha} V (a),
\]
\[
-a^1 - 2a^2 a^3 = C D^\alpha_{\alpha} V (a),
\]
\[
-a^2 - (a^4)^2 = C D^\alpha_{\alpha} V (a).
\]
(31)

According to (20), we can get the potential function as
\[
V (a) = \frac{\Gamma (\beta + 1)}{\Gamma (\beta + 1 - \beta + 1 - \beta) m + \beta} \left[ -a^3 (a^1)^{\beta - \alpha} - a^4 (a^2)^{\beta - \alpha} + (a^1 + 2a^2 a^3)^{\beta - \alpha} \right] (a^1)^{\beta - \alpha}
\]
\[
+ \left[ a^2 + (a^1)^{\beta - \alpha} - (a^4)^{\beta - \alpha} \right] (a^2)^{\beta - \alpha}
\]
(32)

where \( \beta > 0 \). We can directly calculate that \( D^\alpha_{\alpha} V (a) \) is not the Birkhoff system (27). In order to make the potential function \( V (33) \) is the potential function of the Birkhoff system (27), we have to modify (33) as
\[
V (a) = \frac{\Gamma (\beta + 1 - m - \alpha)}{\Gamma (1 - m + \beta)} \left[ -a^3 (a^1)^{\beta - \alpha} - a^4 (a^2)^{\beta - \alpha} + (a^1 + 2a^2 a^3)^{\beta - \alpha} \right] (a^1)^{\beta - \alpha}
\]
\[
+ \left[ a^2 + (a^1)^{\beta - \alpha} - (a^4)^{\beta - \alpha} \right] (a^2)^{\beta - \alpha}
\]
(33)

Next, we will discuss its fractional gradient expression by the Caputo and Riemann–Liouville fractional derivatives, respectively.

4.2. Fractional Gradient Representation of the Birkhoff Equation (27) with the Caputo Fractional Derivative.

Firstly, we study its fractional gradient system using the Caputo fractional derivative. As we take the Caputo fractional derivative, that is, \( D^\alpha_{\alpha} = C D^\alpha_{\alpha}, \rho = 1, 2, 3, 4 \) in (28), when \( 0 < \alpha \leq 1 \), (29) becomes
\[
\delta^\alpha \omega_a = \frac{1}{2 \Gamma (2 - \alpha)} \left[ (-\alpha)_{2 - \alpha} - (1 + 2 \alpha^2)(a^1)^{2 - \alpha} \right] (da^1)^\alpha \wedge (da^3)^\alpha
\]
\[
+ \frac{1}{2 \Gamma (2 - \alpha)} \left[ 2(a^1)^{2 - \alpha} \right] (da^3)^\alpha \wedge (da^4)^\alpha
\]
\[
+ \frac{1}{2 \Gamma (2 - \alpha)} \left[ 2a^1(a^2)^{1 - \alpha} (da^3)^\alpha \wedge (da^4)^\alpha \right]
\]
\[
\left[ (a^3)^{1 - \alpha} - 2(a^4)^{1 - \alpha} - (a^4)^{1 - \alpha} \right] (da^3)^\alpha \wedge (da^4)^\alpha.
\]
(29)
\[ V(a) - C = 0, \]  

(34)

where \( C \) is constant.

4.3. Fractional Gradient Representation of the Birkhoff Equation (27) with the Riemann–Liouville Fractional Derivative. Secondly, we study the fractional gradient representation of the Birkhoff system (27) using the Riemann–Liouville fractional derivative. When we take the Riemann–Liouville fractional derivative that is \( D_{\alpha}^\rho = (R^L D_{\alpha}^\rho)^p; \rho = 1, 2, 3, 4 \), (29) becomes

\[
d^\alpha \omega = \frac{1}{2(1 - \alpha)} \left[ (-a_1)(\alpha_2 - a_1^3) + a_1(\alpha_1 - a_1^3) \right] (d\alpha) \wedge (d\beta) + 
- \frac{1}{2(2 - \alpha)} \left[ (\alpha_1 - a_1^3) + \left(1 + 2a_1^3\right)(\alpha_1^4 - a_1^3) \right] (d\alpha) \wedge (d\beta) + 
\frac{1}{2} \frac{1}{\Gamma(2 - \alpha)} \left( \alpha_1^{(\alpha_1 - a_1^3)} + (a_1^2 + (a_1^3 - a_1^3)^3)(a_1^3 - a_1^3)^3 \right) - 
\frac{2}{\Gamma(3 - a_1)^{\alpha_1 - a_1^3}} \left( \alpha_1^{(\alpha_1 - a_1^3)} + (a_1^2 + (a_1^3 - a_1^3)^3)(a_1^3 - a_1^3)^3 \right) - 
\frac{1}{2(1 - \alpha)} \left( (a_1^2 + (a_1^3 - a_1^3)^3)(a_1^3 - a_1^3)^3 \right) - 
\frac{1}{2} \frac{1}{\Gamma(2 - \alpha)} \left( (\alpha_1^{(\alpha_1 - a_1^3)} + (a_1^2 + (a_1^3 - a_1^3)^3)(a_1^3 - a_1^3)^3) \right) - 
\frac{1}{2(3 - \alpha)} \left( \alpha_1^{(\alpha_1 - a_1^3)} \right) \left( d\alpha \right) \wedge (d\beta). \]

(35)

When \( \alpha = 1 \), we have

\[
d^1 \omega_1 = -\left( 1 + a_1^2 \right) da_1 \wedge da_3 - a_1^4 da_1 \wedge da_4 - a_1^2 da_2 \wedge da_3 + \left( a_1^2 - 1 \right) da_2 \wedge da_4 \neq 0. \]

(36)

So, Birkhoff system (27) is not a gradient system. When \( \alpha = 2 \), we have

\[
d^2 \omega_2 = (da_1^3) \wedge (da_4^3) - (da_1^4) \wedge (da_2^4) \neq 0 \]

(37)

So (27) is not an \( \alpha = 2 \) order fractional gradient system. When \( \alpha = 3 \), \( d^3 \omega_3 = 0 \), so (27) is an \( \alpha = 3 \) order fractional gradient system.

From (19), we can get

\[
a_1^3 = R^L D_\alpha^3 V(a),
\]

\[
a_1^4 = R^L D_\alpha^4 V(a),
\]

\[
a_1^2 - 2a_1^4 a_2^2 = R^L D_\alpha^3 V(a),
\]

\[
a_1^2 - (a_1^3) + (a_1^4) = R^L D_\alpha^3 V(a).
\]

(38)

The potential function can be written as

\[
V(a) = \frac{\Gamma(\beta + 1 - \alpha)}{\Gamma(1 + \beta)} \left[ -a_1^\beta(a_1^3)^\beta - a_1^\beta(a_1^3)^\beta + (a_1^3 + 2a_1^4 a_2)(a_1^3)^\beta \right] - 
\frac{1}{2} \left( a_1^2 + (a_1^3)^2 \right)^2 \left( a_1^3 \right)^\beta.
\]

(39)

with \( \beta = \alpha \).

4.4. Conclusion. In this section, we give the Birkhoff expression of the Hénon-Heiles equation and give its fractional gradient representation using the Caputo fractional derivative and the Riemann–Liouville fractional derivative, respectively. We can conclude that from this example, the Hénon-Heiles equation in the Birkhoff mechanics frame can be expressed as an \( \alpha \) (noninteger) order fractional gradient system using the Caputo fractional derivative, or as a third-order fractional gradient system using the Riemann–Liouville fractional derivative. However, it cannot be a second-order gradient system, so our results are more general than those in [31–33]. We also conclude that a fractional gradient system is not a classical gradient system in general. From the calculus, we find that using the Riemann–Liouville fractional derivative can only give an integer order gradient system, but using the Caputo fractional derivative can give an integer or noninteger order gradient system, so the Caputo fractional derivative is more suitable to study fractional gradient systems than the Riemann–Liouville fractional derivative.

5. Application B: The Birkhoff Equation of Duffing Oscillator Models and Its Fractional Gradient System

In this section, we will give the Birkhoff equation expression of the Duffing oscillator models and prove it can be a fractional gradient system with the Riemann–Liouville or the Caputo fractional derivatives, respectively, in the Birkhoff mechanics frame.

Duffing oscillator models and their equation are widely discussed in nonlinear science:

\[
\ddot{y} + y + \epsilon y^3 = 0,
\]

(40)

where \( \epsilon \) is the nonlinear stiffness coefficient of the Duffing oscillator. Take Birkhoff’s variables as

\[
a_1 = y, \quad \dot{a}_1 = \dot{y}.
\]

(41)

We can get Birkhoff expression of the Duffing oscillator equation

\[
R_1 = -a_2, 
R_2 = 0,
\]

(42)

\[
B = \frac{1}{2} (a_1^2) + \frac{1}{2} (a_1^3)^2 + \frac{1}{4} \epsilon (a_1^4),
\]
with Birkhoff equations
\[
\begin{align*}
\dot{a}_1 &= a_2, \\
\dot{a}_2 &= -a_1 - \epsilon (a_1)^3.
\end{align*}
\tag{43}
\]

From (16), we can get a fractional differential 1-form of Birkhoff equation (43).

\[
\omega_a = a_2 (da_1)^a + \left(-a_1 - \epsilon (a_1)^3 (da_2)^a\right).
\tag{44}
\]

Its fractional exterior derivative is
\[
\frac{1}{2} \left[D_{a_1}^\rho \left(-a_2^\rho + D_{a_2}^\rho \left(-a_1^\rho - \epsilon (a_1^\rho)\right)\right)\right] (da_1)^a \wedge (da_2)^a.
\tag{45}
\]

Next, we will discuss its fractional gradient expression using the Caputo and the Riemann–Liouville fractional derivatives, respectively.

5.1. Fractional Gradient Representation of Birkhoff System (43) with the Caputo Fractional Derivative. Firstly, we study its fractional gradient system using the Caputo fractional derivative. As we take the Caputo fractional derivative, that is, the fractional gradient system using the Caputo fractional derivative, respectively. We conclude that, from this example, the Duffing oscillator model in the Birkhoff mechanics frame can be expressed as a noninteger order fractional gradient system.

\[
V(a) = \frac{1}{\Gamma(1 + m + \alpha)} \left[-\rho a_1^\alpha + a_1^\alpha (a_2^\alpha) + \epsilon (a_1^\alpha) (a_2^\alpha)\right].
\tag{50}
\]

The stationary state of system 43 in the Caputo derivative is defined by equation (35).

5.2. Fractional Gradient Representation of Birkhoff System (43) with the Riemann–Liouville Fractional Derivative. Secondly, we study the fractional gradient representation of the Birkhoff system (43) using the Riemann–Liouville fractional derivative. When we take the Riemann–Liouville fractional derivative, that is, \(D_{a_1} = (RLD)^\rho, \rho = 1, 2\), (45) becomes
\[
d^\alpha \omega_a = \frac{1}{2} - \frac{1}{\Gamma(2 - \alpha)} \left((a_1^\alpha) - a_1^\alpha\right) - \frac{6\epsilon}{\Gamma(4 - \alpha)} (a_1^\alpha) \wedge (da_2)^a.
\tag{51}
\]

When \(\alpha = 1\), we have
\[
d^1 \omega_1 = \frac{1}{2} \left(-2 - 3\epsilon (a_1^\alpha) (da_1)^{1-\alpha} \wedge (da_2)^a\right).
\tag{52}
\]

Then \(d^1 \omega_a \neq 0\), so (43) is not a fractional gradient system. When \(\alpha = 2\), we have
\[
d^1 \omega_1 = -3a_1^\alpha (da_1)^{1-\alpha} \wedge (da_2)^a.
\tag{53}
\]

Then \(d^1 \omega_a \neq 0\), so (43) is not a fractional gradient system. When \(\alpha = 3\), we have
\[
d^1 \omega_1 = -3a_1^\alpha (da_1)^{1-\alpha} \wedge (da_2)^a.
\tag{54}
\]

Then \(d^2 \omega_a \neq 0\), so (43) is not a fractional gradient system. When \(\alpha = 4\), \(d^2 \omega_a = 0\), so (43) is an order fractional gradient system. From equation (19), we can get
\[
-a_1^\alpha - a_1^\alpha = -a_1^\alpha = RL D_{a_1}^\rho V(a),
\tag{55}
\]

The potential function is
\[
V(a) = \frac{1}{\Gamma(1 + m + \alpha)} \left[-\rho a_1^\alpha + a_1^\alpha (a_2^\alpha) + \epsilon (a_1^\alpha) (a_2^\alpha)\right].
\tag{56}
\]

5.3. Conclusion. In this section, we give the Birkhoff equation expression of the Duffing oscillator problem and give its fractional gradient representation using the Caputo fractional derivative and the Riemann–Liouville fractional derivative, respectively. We conclude that, from this example, the Duffing oscillator model in the Birkhoff mechanics frame can be expressed as a noninteger order fractional gradient system using the Caputo fractional derivative or a fourth order fractional gradient system using the Riemann–Liouville fractional derivative. However, it cannot be a second-order gradient system, so our results are
more general than those in [31–33]. We also conclude that a fractional gradient system is not a classical gradient system in general. From the calculus, we find that using the Riemann–Liouville fractional derivative can only give an integer order gradient system, but using the Caputo fractional derivative can give an integer or noninteger order gradient system, so the Caputo fractional derivative is more suitable to study fractional gradient systems than the Riemann–Liouville fractional derivative.

### 6. Application C: Hojman-Urrutia Equations as a Birkhoff System and Fractional Gradient System

In this section, we will give the Birkhoff equation expression of the Hojman–Urrutia equations, and prove it can be a fractional gradient system with the Riemann–Liouville or Caputo fractional derivatives, respectively, in the Birkhoff mechanics frame.

The Hojman–Urrutia equations have an important status in the development of inverse problems in Lagrangian mechanics and Birkhoffian mechanics [36]. Their equations are written in the following form:

\[
\begin{align*}
\ddot{x} + y &= 0, \\
\ddot{y} + y &= 0. 
\end{align*}
\] (57)

Let \(a^1 = x, a^2 = y, a^3 = a^1, a^4 = a^2\); we can get one type of its Birkhoff expression:

\[
\begin{align*}
R_1 &= a^3 + a^3, \\
R_2 &= 0, \\
R_3 &= a^4, \\
R_4 &= 0, \\
B &= \frac{1}{2} \left[ (a^3)^2 + 2a^2a^3 - (a^4)^2 \right].
\end{align*}
\]

So we can get

\[
(\Omega_{\mu\nu}) =
\begin{pmatrix}
0 & -1 & -1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\] (59)

and

\[
(\Omega^{\mu\nu}) =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0
\end{pmatrix},
\] (60)

with Birkhoff equations

\[
\begin{align*}
a^1 &= a^3, \\
a^2 &= a^4, \\
a^3 &= -a^4, \\
a^4 &= -a^2,
\end{align*}
\] (61)

From (16), we can get a fractional differential 1-form of Birkhoff (57).

\[
\omega_a = \alpha^3(\alpha^1)^a + \alpha^4(\alpha^1)^a - \alpha^4(\alpha^4)^a - \alpha^2(\alpha^4)^a.
\] (62)

Its fractional exterior derivative is

\[
\begin{align*}
d^a\omega_a &= \frac{1}{2} \left( D^a_\alpha a^1 - D^a_\alpha a^3 \right)(\alpha^1)^a \wedge (\alpha^3)^a - \frac{1}{2} \left( D^a_\alpha a^4 + D^a_\alpha a^3 \right)(\alpha^4)^a \\
&\quad - \frac{1}{2} \left( D^a_\alpha a^4 + D^a_\alpha a^1 \right)(\alpha^4)^a \wedge (\alpha^1)^a \\
&\quad - \frac{1}{2} \left( D^a_\alpha a^2 + D^a_\alpha a^4 \right)(\alpha^2)^a \wedge (\alpha^4)^a.
\end{align*}
\] (63)

### 6.1. Fractional Gradient Representation of Birkhoff System (60) with the Caputo Fractional Derivative

Next, we will discuss its fractional gradient expression by the Caputo and the Riemann–Liouville fractional derivative, respectively.

Firstly, we study its fractional gradient system using the Caputo fractional derivative. As we take the Caputo fractional derivative, that is, \(D^\alpha = (CD^\alpha)^p, p = 1, 2, 3, 4\) in (59), when \(0 < \alpha < 1\), (62) becomes

\[
\begin{align*}
d^a\omega_a &= \frac{1}{2} \left( (2 - \alpha)^{1-a} \alpha^3(\alpha^1)^a \wedge (\alpha^3)^a - \frac{1}{2(2 - \alpha)}(\alpha^1)^a \right) \\
&\quad - \frac{1}{2} \left( (2 - \alpha)^{1-a} \alpha^4(\alpha^4)^a \wedge (\alpha^4)^a + \frac{1}{2(2 - \alpha)}(\alpha^4)^a \right) \wedge (\alpha^3)^a.
\end{align*}
\] (64)

Then, \(d^a\omega_a \neq 0\), so the Hojman–Urrutia equation expressed by Birkhoff (57) is not a fractional gradient system.

When \(m - 1 < \alpha \leq m\) with \(m \geq 2\), we can get \(d^a\omega_a = 0\), so (57) can be expressed as an \(\alpha\) order Caputo type fractional gradient system, where \((m - 1 < \alpha \leq m, m \geq 2)\). From (19), we can get

\[
\begin{align*}
a^3 &= C D^\alpha a V(a), \\
a^4 &= C D^\alpha a V(a), \\
-a^4 &= C D^\alpha a V(a), \\
-a^2 &= C D^\alpha a V(a).
\end{align*}
\] (65)

According to (20) and condition (18), we can get the potential function as

\[
V(a) = \frac{1}{\Gamma(1 - n + \alpha)} \left[ (-a^3)(a^1)^a - (a^4)(a^3)^a + (a^4)(a^3)^n + (a^2)(a^4)^n \right].
\] (66)
6.2. Fractional Gradient Representation of Birkhoff System (60) with the Riemann–Liouville Fractional Derivative.

Secondly, we study fractional gradient representation of the Birkhoff system (60) using the Riemann–Liouville fractional derivative. When we take the Riemann–Liouville fractional derivative that is $D_{\alpha}^{\mu}p(a,t) = \left(\frac{d}{dt}\right)^{\alpha}p(a,t)$ with the Riemann–Liouville fractional derivative.

6.2. Fractional Gradient Representation of Birkhoff System (60) with the Riemann–Liouville Fractional Derivative. Secondly, we study fractional gradient representation of the Birkhoff system (60) using the Riemann–Liouville fractional derivative. When we take the Riemann–Liouville fractional derivative that is $D_{\alpha}^{\mu}p(a,t) = \left(\frac{d}{dt}\right)^{\alpha}p(a,t)$ with the Riemann–Liouville fractional derivative.

When $\alpha = 1$, we get
\[
d^1\omega_1 = \frac{1}{2} \left( (a^i)^{\alpha} \right) \wedge (da^i)^{\alpha} - (da^i)^{\alpha} \wedge (da^i)^{\alpha}.
\]

Then $d^1\omega_1 \neq 0$, so (60) is not a fractional gradient system. When $\alpha = 2$, $d^2\omega_2 = 0$, so (60) is an $\alpha$ order fractional gradient system with potential function with $\alpha = 2$. From equation (19), we can get
\[
a^3 = C D_{a^3} V(a),
a^4 = C D_{a^4} V(a),
-a^3 = C D_{a^3} V(a),
a^2 = C D_{a^2} V(a).
\]

The potential function is
\[
V(a) = \frac{\Gamma(1)}{\Gamma(1 + \alpha)} \left[ (-a^3)(a^3)^{\alpha} - (a^4)(a^3)^{\alpha} + (a^4)(a^3)^{\alpha} + (a^2)(a^4)^{\alpha} \right].
\]

6.3. Conclusion. In this section, we give the Birkhoff expression of the Hojman–Urrutia equation and give its fractional gradient representation using the Caputo fractional derivative and the Riemann–Liouville fractional derivative, respectively. We can conclude that from this example, the Hojman–Urrutia equation in the Birkhoff mechanics frame can be expressed as an $\alpha$ (noninteger) order fractional gradient system using the Caputo fractional derivative or a second-order fractional gradient system using the Riemann–Liouville fractional derivative, so our results are more general than those in [31–33]. We also conclude that a fractional gradient system is not a classical gradient system in general. From the calculus, we find that using the Riemann–Liouville fractional derivative can only give an integer order gradient system, but using the Caputo fractional derivative can give an integer or noninteger order gradient system, so the Caputo fractional derivative is more suitable to study fractional gradient systems than the Riemann–Liouville fractional derivative.

7. Conclusion

In this paper, we give a definition and a general condition of a Birkhoff system to be a fractional gradient system with the Caputo fractional derivative and the Riemann–Liouville fractional derivative, respectively. When $\alpha = 1$, the definition and condition degenerate into those of the classical gradient system, and when $\alpha = 2$, our results goes back to those of the second order gradient representation [31–33], so the classical gradient system and a second-order gradient system can be considered as special cases of the fractional gradient representation of the Birkhoff system obtained in this paper.

As applications, we apply the results we get in this paper to the Henon–Heiles equation, the Duffing oscillator model, and the Hojman–Urrutia equations. The Birkhoff equation expression and fractional gradient expression of these classical equations are given. It shows that the orders of the fractional gradient Birkhoff system are not only with $\alpha = 2$ but can also be other integers or noninteger orders in this paper, so our results are more general.

The limitation of this paper is that we only discuss the fractional gradient representation of an autonomous Birkhoff system. The more general cases of the fractional gradient representation of nonautonomous Birkhoff systems and the fractional gradient representation of the fractional Birkhoff system [42] will be considered in future work.

Abbreviations

- $R\mathcal{L} D_{a^\mu}$: Symbol of Riemann–Liouville fractional derivative
- $C D_{a^\mu}$: Symbol of Caputo fractional derivative
- $R\mathcal{G}$: Riemann–Liouville fractional integral symbol
- $\Gamma(\alpha)$: Symbol of Gamma function
- $d^\alpha$: Symbol of fractional exterior derivatives
- $B(a)$: Symbol of Birkhoffian
- $\Omega_{\alpha}$: Symbol of Birkhoffian tensor
- $R_{\alpha} / R_{\alpha}$: Symbols of Birkhoff’s functions.

Data Availability

All data generated or analyzed during this study are included in this article.
Conflicts of Interest
The author declares that there are no conflicts of interest.

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References
[1] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, NY, UK, 1974.
[2] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon & Breach, Pennsylvania, USA, 1993.
[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Application of Fractional Differential Equations, Elsevier, Amsterdam, The Netherlands, 2006.
[4] F. Riewe, "Mechanics with fractional derivatives," Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics, vol. 55, no. 3, pp. 3581–3592, 1997.
[5] O. P. Agrawal, "Formulation of Euler-Lagrange equations for fractional variational problems," Journal of Mathematical Analysis and Applications, vol. 272, no. 1, pp. 368–379, 2002.
[6] D. Baleanu, S. I. Muslih, and E. M. Rabei, "On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative," Nonlinear Dynamics, vol. 53, no. 1-2, pp. 67–74, 2008.
[7] A.-R. Ei-Nabulsi, "A fractional approach to nonconservative Lagrangian dynamical systems," Fizika A, vol. 14, no. 4, pp. 289–298, 2005.
[8] M. Naber, "Time fractional Schrödinger equation," Journal of Mathematical Physics, vol. 45, no. 8, pp. 3339–3352, 2004.
[9] N. Laskin, "Fractional schrödinger equation," Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics, vol. 66, no. 5, Article ID 056108, 2002.
[10] S. K. Luo and Y. L. Xu, "Fractional birkhoffian mechanics," Acta Mechanica, vol. 226, no. 3, pp. 829–844, 2015.
[11] L. L. Wang and J. L. Fu, "Non-Noether symmetries of Hamiltonian systems with conformable fractional derivatives," Chinese Physics B, vol. 25, no. 1, Article ID 014501, 2016.
[12] H. B. Zhang and H. B. Chen, "Generalized variational problems and Birkhoff equations," Nonlinear Dynamics, vol. 83, no. 1-2, pp. 347–354, 2016.
[13] Y. D. Jia and Y. Zhang, "Fractional birkhoffian mechanics based on quasi-fractional dynamics models and its noether symmetry," Mathematical Problems in Engineering, vol. 2021, pp. 1–17, Article ID 6694709, 2021.
[14] C. J. Song and O. P. Agrawal, "Hamiltonian formulation of systems described using fractional singular Lagrangian," Acta Applicandae Mathematica, vol. 172, no. 1, 9 pages, 2021.
[15] D. Baleanu, S. Sadat Sajjadi, A. Jajarmi, and J. H. Asad, "New features of the fractional Euler-Lagrange equations for a physical system within non-singular derivative operator," European Physical Journal A: Hadrons and Nuclei, vol. 134, no. 4, p. 181, 2019.
[16] M. W. Hirsch, S. Smale, and R. L. Devaney, Differential Equations, Dynamical Systems, and an Introduction to Chaos, Elsevier, Singapore, 2008.
[17] R. I. McLachlan, G. R. W. Quispel, and N. Robidoux, "Geometric integration using discrete gradients," Philosophical Transactions of the Royal Society of London, Series A: Mathematical, Physical and Engineering Sciences, vol. 357, no. 1754, pp. 1021–1045, Article ID 1021C1045, 1999.
[18] B. de la Calle Ysern, "Asymptotically stable equilibria of gradient systems," The American Mathematical Monthly, vol. 126, no. 10, pp. 936–939, 2019.
[19] H. Yamashita, K. Aihara, and H. Suzuki, "Accelerating numerical simulation of continuous-time Boolean satisfiability solver using discrete gradient," Communications in Nonlinear Science and Numerical Simulation, vol. 102, Article ID 105908, 2021.
[20] S. Eidnes, "Order theory for discrete gradient methods," BIT Numerical Mathematics, 2022.
[21] F. X. Mei and H. B. Wu, Gradient Representations of Constrained Mechanical systems(LII), Science Press, Beijing, China, 2015.
[22] F. X. Mei and H. B. Wu, "The gradient system and generalized Hamilton system," Science China Physics, Mechanics & Astronomy, vol. 43, no. 4, pp. 538–540, 2013.
[23] F. X. Mei and H. B. Wu, "The generalized Birkhoff system and a type of combined gradient system," Acta Physica Sinica, vol. 64, no. 18, 5 pages, Article ID 184501, 2015.
[24] F. X. Mei and H. B. Wu, "Skew-gradient representation of generalized Birkhoffian system," Chinese Physics B, vol. 24, no. 10, 3 pages, Article ID 104502, 2015.
[25] F. X. Mei and H. B. Wu, "Gradient systems and mechanical systems," Acta Mechanica Sinica, vol. 32, no. 5, pp. 935–940, Article ID 935C940, 2016.
[26] X. W. Chen, Y. Zhang, and F. X. Mei, "An application of a combined gradient system to stabilize a mechanical system," Chinese Physics B, vol. 25, no. 10, Article ID 100201, 2016.
[27] C. Liu, S. X. Liu, and F. X. Mei, "Stability analysis of a simple rheonomic nonholonomic constrained system," Chinese Physics B, vol. 25, no. 12, Article ID 124501, 2016.
[28] V. E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, fields and media, Higher Education Press, Beijing, China, 2010.
[29] V. E. Tarasov, "Fractional generalization of gradient systems," Letters in Mathematical Physics, vol. 73, no. 1, pp. 49–58, 2005.
[30] V. E. Tarasov, "Fractional generalization of gradient and Hamiltonian systems," Journal of Physics A: Mathematical and General, vol. 38, no. 26, pp. 5929–5943, 2005.
[31] F. X. Mei, J. C. Cui, and H. B. Wu, "A gradient representation and a fractional gradient representation of Birkhoff system," Transactions of Beijing Institute of Technology, vol. 32, no. 12, pp. 1290–1300, 2012.
[32] J. Chen, Y. X. Guo, and F. X. Mei, "New methods to find solutions and analyze stability of equilibrium of nonholonomic mechanical systems," Acta Mechanica Sinica, vol. 34, no. 6, pp. 1136–1144, 2018.
[33] X. W. Chen, G. L. Zhao, and F. X. Mei, "A fractional gradient representation of the Poincaré equations," Nonlinear Dynamics, vol. 73, no. 1-2, pp. 579–582, 2013.
[34] R. M. Santilli, Foundations of Theoretical Mechanics II, Springer-Verlag, NY, USA, 1983.
[35] A. S. Galiullin, G. G. Gafarov, R. P. Malaishka, and A. M. Khwan, Analytical Dynamics of Helmholtz, Birkhoff and Nambu Systems, UFN, Moscow, 1997.
[36] F. X. Mei, R. C. Shi, Y. F. Zhang, and H. B. Wu, Dynamics of Birkhoffian Systems BIT Press, Beijing, China, 1996.
[37] P. Wang, J. H. Fang, and X. M. Wang, "A generalized Mei conserved quantities and Mei symmetry for Birkhoff systems," Chinese Physics B, vol. 18, no. 4, pp. 1312–1315, 2009.
[38] Y. X. Guo, S. K. Luo, M. Shang, and F. X. Mei, “Birkhoffian formulations of nonholonomic constrained systems,” *Reports on Mathematical Physics*, vol. 47, no. 3, pp. 313–322, 2001.

[39] K. Cottrill-Shepherd and M. Naber, “Fractional differential forms,” *Journal of Mathematical Physics*, vol. 42, no. 5, pp. 2203–2212, 2001.

[40] S. X. Liu, C. Liu, and Y. X. Guo, “Discrete variational calculation of Hénon-Heiles equation in the Birkhoff sense,” *Acta Physica Sinica*, vol. 60, no. 6, Article ID 064501, 2011.

[41] M. Hénon and C. Heiles, “The applicability of the third integral of motion: some numerical experiments,” *The Astronomical Journal*, vol. 69, no. 1, pp. 73–79, 1964.

[42] S. K. Luo, J. M. He, and Y. L. Xu, “Fractional Birkhoffian method for equilibrium stability of dynamical systems,” *International Journal of Non-linear Mechanics*, vol. 78, pp. 105–111, 2016.