The Quantum Emergence of Chaos

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The dynamical status of isolated quantum systems, partly due to the linearity of the Schrödinger equation is unclear: Conventional measures fail to detect chaos in such systems. However, when quantum systems are subjected to observation – as all experimental systems must be – their dynamics is no longer linear and, in the appropriate limit(s), the evolution of expectation values, conditioned on the observations, closely approaches the behavior of classical trajectories. Here we show, by analyzing a specific example, that microscopic continuously observed quantum systems, even far from any classical limit, can have a positive Lyapunov exponent, and thus be truly chaotic.

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There can be no chaos in the dynamics of isolated or closed quantum systems, a result which follows primarily from the linearity of the Schrödinger equation and the linear Hilbert space structure of the theory which, by virtue of the uncertainty principle, prevents the formation of fine-scale structure in phase space precluding chaos in the sense of classical trajectories. This leads to a widely recognized difficulty, as classical mechanics, which manifestly exhibits chaos, must emerge from quantum mechanics in an appropriate macroscopic limit.

The key to the resolution of this apparent paradox lies in the fact that all experimentally accessible situations necessarily involve measured, open systems: the central importance of such situations in the context of chaos was first emphasized by Chirikov. In a closely connected development, continuous quantum measurement theory has led to the successful understanding of the emergence of classical dynamics from the underlying quantum physics, and inequalities have been derived that encapsulate the regime under which classical motion, and thus classical chaos, exists.

The transition to classical mechanics results from the localization of the quantum density matrix due to the information continuously provided by the measurement (itself mediated by an environmental interaction), and the balancing of this against the unavoidable noise from the quantum backaction of the measurement. For a macroscopic system, the Ehrenfest theorem holds as a result of localization and, simultaneously, the backaction noise is negligible, resulting in a smooth classical trajectory.

While it has been established that observed quantum systems can be chaotic when they are macroscopic enough that classical dynamics has emerged, can they be chaotic outside this limit? This is the question we address here. By defining and computing the Lyapunov exponent for an observed quantum system deep in the quantum regime, we are able show that the system dynamics is chaotic. Further, the Lyapunov exponent is not the same as that of the classical dynamics that emerges in the classical limit. Since the quantum system in the absence of measurement is not chaotic, this chaos must emerge as the strength of the measurement is increased, and we examine the nature of this emergence.

The rigorous quantifier of chaos in a dynamical system is the maximal Lyapunov exponent. The exponent yields the (asymptotic) rate of exponential divergence of two trajectories which start from neighboring points in phase space, in the limit in which they evolve to infinity, and the neighboring points are infinitesimally close. The maximal Lyapunov exponent characterizes the sensitivity of the system evolution to changes in the initial condition: if the exponent is positive, then the system is exponentially sensitive to initial conditions, and is said to be chaotic. We apply this notion below to the observation-conditioned evolution of quantum expectation values.

The evolution of a simple single-particle quantum system under an ideal continuous position measurement is given by the nonlinear stochastic master equation (SME) for the system density matrix:

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho]dt - k[x, [x, \rho]]dt + 4k(x \rho + \rho x - 2\langle x \rangle)(dy - \langle x \rangle dt), \]

where the first term on the right hand side is due to unitary evolution, \( H \) being the Hamiltonian, and the second term represents diffusion from “quantum noise” due to the unavoidable quantum backaction of the measurement. The position operator is \( x \), and the parameter \( k \) characterizes the rate at which the measurement extracts information about the observable, and which we will refer to as the strength of the measurement. The final term represents the change in the density matrix as a result of the information gained from the measurement. Here, \( dy \) is the infinitesimal change in the continuous output of the measuring device in the time \( dt \). The continuous output of the measuring device, \( y(t) \), referred
to usually as the measurement record, is determined by $dy = \langle x \rangle dt + dW/\sqrt{8\overline{E}}$ where $dW$ is the Wiener increment, describing driving by Gaussian white noise $dW$. The noise $dW$ is due to the fact that the results of the measurement are necessarily random. (Note that the backaction and $dW$ are uncorrelated with each other.) Thus on a given experimental run, the system will be driven by a given realization of the noise process $dW$. We will label the possible noise realizations by $s$.

A single quantum mechanical particle is in principle an infinite dimensional system. However, for the purpose of defining an observationally relevant Lyapunov exponent, it is sufficient to use a single projected data stream: Here we choose the expectation value of the position, $\langle x(t) \rangle$. The important quantity is thus the divergence, $\Delta(t) = |\langle x(t) \rangle - \langle x_{\text{fid}}(t) \rangle|$, between a fiducial trajectory and a second trajectory infinitesimally close to it. It is important to keep in mind that the system is driven by noise. Since we wish to examine the sensitivity of the system to changes in the initial conditions, and not to changes in the noise, we must hold the noise realization fixed when calculating the divergence. The Lyapunov exponent is thus

$$\lambda \equiv \lim_{t \to \infty} \lim_{\Delta_s(0) \to 0} \frac{\ln \Delta_s(t)}{t} \equiv \lim_{t \to \infty} \lambda_s(t)$$

where the subscript $s$ denotes the noise realization. This definition is the obvious generalization of the conventional ODE definition to dynamical averages, where the noise is treated as a drive on the system. Indeed, under the conditions when (noisy) classical motion emerges, and thus when localization holds (Fig. 1), it reduces to the conventional definition, and yields the correct classical Lyapunov exponent. To combat slow convergence, we measure the Lyapunov exponent by averaging over an ensemble of finite-time exponents $\lambda_s(t)$ instead of taking the asymptotic long-time limit for a single trajectory.

A key result now follows: In unobserved, i.e., isolated quantum dynamical systems, it is possible to prove, by employing unitarity and the Schwarz inequality, that $\lambda$ vanishes; the finite-time exponent, $\lambda(t)$, decays away as $1/t$ [14]. This theorem codifies the expectation that, since the evolution is linear, any measure of chaos applied to it should yield a null result. As we have emphasized earlier, however, once measurement is included the evolution becomes nonlinear and the Lyapunov exponent need not vanish. We now address this crucial question for a specific example.

The system we consider is the quantum Duffing oscillator [10], which is a single particle in a double-well potential, with sinusoidal driving. The Hamiltonian for the Duffing oscillator is

$$H = \frac{p^2}{2m} + Bx^4 - Ax^2 + \Lambda x \cos(\omega t)$$

where $p$ is the momentum operator, $m$ the particle mass, and $A$, $B$ and $\Lambda$ determine the potential and the strength of the driving force. We fix the values of the parameters to be $m = 1$, $B = 0.5$, $A = 10$, $\Lambda = 10$ and $\omega = 6.07$. The action of a system relative to $\hbar$ can be varied either by changing parameters in the Hamiltonian, or by introducing scaled variables so that the Hamiltonian remains fixed, but the effective value of $\hbar$ becomes a tunable parameter. Here we employ the latter choice as it captures the notion of system size with a single number; the smaller $\hbar$ the larger the system size, and vice versa.

To examine the emergence of chaos we will first choose $\hbar = 10^{-2}$, which is small enough so that the system makes a transition to classical dynamics when the measurement is sufficiently strong. In this way, as we increase the measurement strength, we can examine the transformation from essentially isolated quantum evolution all the way to the (known) chaos of the classical Duffing oscillator. To examine the emergence of chaos, we simulate the evolution of the system for $k = 5 \times 10^{-4}, 10^{-3}, 0.01, 0.1, 1, 10$. When $k \leq 0.01$, the distribution is spread over the entire accessible region, and Ehrenfest’s theorem is not satisfied. Conversely, for $k = 10$, the distribution is well-localized (Fig. 1), and Ehrenfest’s theorem holds throughout the evolution. Since the backaction noise, characterized by the momentum diffusion coefficient, $D_p = \hbar^2 k$, remains small, at this value of $k$ the motion is that of the classical system, to a very good approximation.

Stroboscopic maps help reveal the global structural transformation in phase space in going from quantum to classical dynamics (Fig. 2). The maps consist of points through which the system passes at time intervals separated by the period of the driving force. For very small $k$, $(x)$ and $(p)$ are largely confined to a region in the center of phase space. Somewhat remarkably, at $k = 0.01$, although the system is largely delocalized, as shown in Fig. 1 nontrivial structure appears, with considerable time being spent in certain outer regions. By $k = 1$ the

FIG. 1: Position distribution for the Duffing oscillator with measurement strengths $k = 0.01$ (red) and $k = 10$ (green), demonstrating measurement-induced localization ($k = 10$). The momentum distribution behaves similarly.
localized regions have formed into narrower and sharper swirling coherent structures. At $k = 10$ the swirls disappear, and we retrieve the uniform chaotic sea of the classical map (the small ‘holes’ are periodic islands). The swirls in fact correspond to the unstable manifolds of the classical motion. Classically, these manifolds are only visible at short times, as continual and repeated folding eventually washes out any structure in the midst of a uniform tangle. In the quantum regime, however, the weakness of the measurement, with its inability to crystallize the fine structure, has allowed them to survive: we emphasize that the maps result from long-time integration, and are therefore essentially time-invariant.

To calculate the Lyapunov exponent we implement a numerical version of the classical linearization technique \cite{15}, suitably generalized to quantum trajectories. The method was tested on a classical noisy system with small measurement strengths, $k = 5 \times 10^{-4}$, 0.01 and 1, 10 (bottom). Contour lines are superimposed to provide a measure of local point density at relative density levels of 0.05, 0.15, 0.25, 0.35, 0.45, and 0.55.

The Lyapunov exponent increases over two orders of magnitude in an approximately power-law fashion as $k$ is varied from $5 \times 10^{-4}$ to 10, before settling to the classical value, $\lambda_{Cl} = 0.57$. The results in Figs. 3 and 4 show clearly that chaos emerges in the observed quantum dynamics well before the limit of classical motion is obtained.

We now compute the Lyapunov exponent for the quantum system when its action is sufficiently small that smooth classical dynamics cannot emerge, even for strong measurement. Taking a value of $\hbar = 16$, we find that for $k = 5 \times 10^{-3}$, $\lambda = 0.029 \pm 0.008$, for $k = 0.01$, $\lambda = 0.046 \pm 0.01$ and for $k = 0.02$, $\lambda = 0.077 \pm 0.01$. Thus the system is once again chaotic, and becomes more strongly chaotic the more strongly it is observed. From these results, it is clear that there exists a purely quantum regime in which an observed system, while behaving in a fashion quite distinct from its classical limit, nevertheless evolves chaotically with a finite Lyapunov exponent, also distinct from the classical value.

It is worth pointing out that an analogous analysis can also be carried out for a continuously observed classical system. First one notes that an unobserved probabilistic classical system also has provably zero Lyapunov exponent: the average of $x$ for an ensemble of classical particles does not exhibit chaos, due to the linearity of the Liouville equation \cite{14}. If we consider a noiseless observed chaotic classical system – possible since classical measurements are by definition passive (no backaction

FIG. 2: Phase space stroboscopic maps shown for 4 different measurement strengths, $k = 5 \times 10^{-4}$, 0.01 (top), and 1, 10 (bottom). Contour lines are superimposed to provide a measure of local point density at relative density levels of 0.05, 0.15, 0.25, 0.35, 0.45, and 0.55.

FIG. 3: Finite-time Lyapunov exponents $\lambda(t)$ for measurement strengths $k = 5 \times 10^{-4}$, 0.01, 10, averaged over 32 trajectories for each value of $k$ (linear scale in time, top, and logarithmic scale, bottom; bands indicate the standard deviation over the 32 trajectories). The (analytic) $1/t$ fall-off at small $k$ values, prior to the asymptotic regime, is evident in the bottom panel. The unit of time is the driving period.
different Lyapunov exponents, especially when different from will also not be localized, and may well have an exponent temperature, like a quantum system outside the classical regime, however, the noise is not weak, an observed classical system the external noise is not connected to the dynamics driven by weak noise – must tend to – which, to a very good approximation, is just classical. If, however, the noise is not weak, an observed classical system, like a quantum system outside the classical regime, will also not be localized, and may well have an exponent different from $\lambda_{Cl}$. In addition, one may expect the non-localized quantum and classical evolutions to have quite different Lyapunov exponents, especially when $h$ is large on the scale of the phase space, as quantum and classical evolutions generated by a given nonlinear Hamiltonian are essentially different [17]. The nature of the Lyapunov exponent for non-localized classical systems, and its relationship to the exponent for quantum systems is a very interesting open question.

Finally, we emphasize that the chaos identified here is not merely a formal result - even deep in the quantum regime, the Lyapunov exponent can be obtained from measurements on a real system as in next-generation cav-

ity QED and nanomechanics experiments [18]. Experimentally, one would use the known measurement record to integrate the SME [1]: this provides the time evolution of the mean value of the position. From this fiducial trajectory, given the knowledge of the system Hamiltonian, the Lyapunov exponent can be obtained by following the procedure described here.

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