CURVATURE ESTIMATES FOR GRAPHS IN WARPED PRODUCT SPACES

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Abstract. We prove local and global upper estimates for the infimum of the mean curvature, the scalar curvature and the norm of the shape operator of graphs in a warped product space. Using these estimates, we obtain some results on pseudo-hyperbolic spaces and space forms.

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1. Introduction

The study of submanifolds in a Riemannian manifold is a central subject on Differential Geometry. In many results, global properties of a submanifold are drawn from hypothesis on its topology and curvature. Just to exemplify, Aleksandrov’s theorem [Ale62] states that the round spheres are the only compact embedded hypersurfaces with constant mean curvature in Euclidean space. For a sample of results of the same nature see for example [KO67, NS69, Ros87, Ros88, Che02, MT12, Nn17, FNn18, BFH18] and references therein.

In recent years, many results of the nature described in the previous paragraph have been obtained in the case of hypersurfaces of a warped product space

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For example, Montiel [Mon99] obtained conditions for a compact hypersurface with constant mean curvature in such a space to be a slice (see [AD07a] for generalizations of this result).

An important class of hypersurfaces in warped product spaces is that of the entire graphs. A breakthrough result concerning this class is the Bernstein Theorem, which states that the planes are the only minimal entire graphs in the Euclidean 3-space. Many Bernstein-type theorems in warped product spaces have appeared in the late years (e.g. [AD06, AD07a, ADRo7, CdLo9, CCdLi1, AIR13]). Using the Alexandrov’s reflection method, Frensel [Fre83] proved that the only graphs with constant mean curvature in the half-plane model of the hyperbolic space are the horospheres (see also [dCL83, Theorem A]). Aquino and de Lima [AdLi11] obtained a Bernstein-type theorem on a particular warped product space with additional assumptions on the second fundamental form of the graph.

Unfortunately, determining the curvature of hypersurfaces in arbitrary spaces is a task that in general is either difficult or requires elaborated calculations. For this reason, obtaining curvature estimates of hypersurfaces is all that can be done in many situations. In 1955, Heinz [Hei55] obtained estimates for the mean curvature $H$ and Gaussian curvature $K$ of a surface in $\mathbb{R}^3$ which is the graph of a smooth function defined on a open disc of radius $r$ in the plane. He showed that

$$\inf |H| \leq \frac{1}{r} \quad \text{and} \quad \inf |K| \leq \frac{3e^2}{r^2}.$$  

Later on, Chern [Che65] and Flanders [Fla66], independently, extended the above inequality of the mean curvature to higher dimensions. After that, it was generalized by Finn [Fin65] to a broader class of domains of the plane and by Salavessa [Sal87, Sal89] for graphs over Riemannian manifolds. Inspired by these works, Fontenele and the second author [Fon10, CF20] established estimates of curvature for graphs in the Riemannian product $M \times \mathbb{R}$, from which they deduced sharp estimates for the infimum of the mean curvature for graphs over a complete Riemannian manifold with Ricci curvature bounded below.

In this work we establish several results of the same nature for graphs in a warped product space contained or not in a slab. For example, Corollary 3.5 provides a version of [AD07a, Theorem 2.9] for graphs in the warped product $M_\psi \times \mathbb{R}$, in which instead of making assumption on the Ricci curvature of the graph we make assumption on the sectional curvature of the base $M$. Theorem 4.5 provides an estimate for the mean curvature of a graph in a warped product space, similar in spirit to [AD07a, Proposition 2.10]. In that estimate we do not assume that the graph is contained in a slab, which allows us to apply it for any graph in the space $M_{\cosh t} \times \mathbb{R}$, providing that the sectional curvature of $M$ is bounded from below.

Our approach is based in a known relation between the principal curvatures of a graph in a warped product space $M_\psi \times I$ and the principal curvatures of a related graph in the Riemannian product $M \times J$. It is worth to point out that the curvature of the slices have strong influence in the estimates presented here.
This paper is organized as follows. In Section 2, we fix the notation and present basic results that will be used in the entire work. In Section 3, we present curvature estimates for graphs contained in a slab, and in Section 4 for any graph. In Section 5, we use the estimates of the previous sections to obtain results on pseudo-hyperbolic spaces and space forms. For example, we show that the only entire graphs with constant mean curvature contained in a slab of \( \mathbb{H}^{m+1} \equiv \mathbb{H}^m \times \mathbb{R} \) or \((\mathbb{S}^m_\cosh t) \times (0, \infty) \equiv \mathbb{H}^{m+1} \) are the slices (see Corollaries 5.2 and 5.7).

2. Preliminary and auxiliary concepts

In this section we will fix the notation and present general results about warped product spaces and graphs. Our results can be applied for \( C^2 \)-functions, however, for simplicity, we assume that all manifolds, functions, etc., are smooth.

Let \((M^m, \langle \cdot, \cdot \rangle_M)\) and \((N^n, \langle \cdot, \cdot \rangle_N)\) be Riemannian manifolds and let \( \pi_M : M \times N \to M \), \( \pi_N : M \times N \to N \) be the projections maps over \( M \) and \( N \), respectively. Let \( \psi : N \to (0, \infty) \) be a positive function. The product \( M \times N \) equipped with the Riemannian metric
\[
\langle \cdot, \cdot \rangle := (\psi \circ \pi_N)^2 \pi_M^* \langle \cdot, \cdot \rangle_M + \pi_N^* \langle \cdot, \cdot \rangle_N,
\]
is called warped product space and denoted by \( M_\psi \times N \). The relationship between the curvature tensors \( \mathcal{R}_\psi, \mathcal{R}_N \) and \( \mathcal{R}_M \), of \( M_\psi \times N \), \( N \) and the fiber \( M \), respectively, are given by following proposition (see [O’N83, p. 210]).

**Proposition 2.1.** If \( U, V, W \in \mathcal{X}(M) \) and \( X, Y, Z \in \mathcal{X}(N) \) are vector fields, then
\[
\begin{align*}
(1) \quad & \mathcal{R}_\psi(\tilde{X}, \tilde{Y})\tilde{Z} = \mathcal{R}_N(X, Y) Z, \\
(2) \quad & \mathcal{R}_\psi(\tilde{V}, \tilde{X})\tilde{Y} = - \frac{\text{Hess } \psi(X,Y)}{\psi} \tilde{V}, \\
(3) \quad & \mathcal{R}_\psi(\tilde{X}, \tilde{Y})\tilde{V} = \mathcal{R}(\tilde{V}, \tilde{W})\tilde{X} = 0, \\
(4) \quad & \mathcal{R}_\psi(\tilde{X}, \tilde{Y})\tilde{W} = - \frac{(\tilde{V}, \tilde{W})}{\psi^2} \tilde{X} \nabla \psi, \\
(5) \quad & \mathcal{R}_\psi(\tilde{V}, \tilde{W})\tilde{U} = \mathcal{R}(\tilde{V}, \tilde{W})\tilde{U} - \frac{(\nabla \psi, \nabla \psi)_N}{\psi^2} \left\{ \langle \tilde{W}, \tilde{U} \rangle_\psi \tilde{V} - \langle \tilde{V}, \tilde{U} \rangle_\psi \tilde{W} \right\},
\end{align*}
\]
where \( \tilde{\cdot} \) represents the lift of the field to \( M_\psi \times N \).

**Remark 2.2.** Differently from [O’N83], we are using the following definition for the curvature tensor of a Riemannian manifold
\[
\mathcal{R}(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\langle X, Y \rangle} Z.
\]
This choice justify the sign difference in some of the previous relations.

In the special case where \( N \) is an open interval \( I \subset \mathbb{R} \), the slices \( M \times \{ t \} \) are totally umbilical hypersurfaces with constant mean curvature (see [O’N83, p.206])
\[
\mathcal{H}(t) := (\ln \psi)'(t) = \frac{\psi'}{\psi}(t),
\]
where \( \psi \) is the conformal factor.
with respect to the vector field $-\partial_t$, where $\partial_t$ is the canonical unit normal vector field. By Proposition 2.1 item (5), we obtain that
\[
K_{M \times I}(p, t)(\bar{u}, \bar{v}) = \frac{K_M(p)(u, v)}{\psi^2(t)} - \mathcal{H}^2(t),
\]  
(2.3)
where $u, v \in T_pM$ and $K_M$ is the sectional curvature of $M$. By Eq. (2.3) and the Gauss equation, the sectional curvature of a slice at a point $(p, t)$, in the planed spanned by $\bar{u}, \bar{v}$, is given by
\[
\mathcal{K}(p, t)(\bar{u}, \bar{v}) := \frac{K_M(p)(u, v)}{\psi^2(t)}.
\]
Again by Proposition 2.1, we have that the sectional curvature $K_{M \times I}(\bar{u}, \partial_t)$ does not depend on $u \in T_pM$ and $p \in M$. Therefore we define the normal sectional curvature of $M \times \{t\}$ by
\[
\mathcal{K}^\perp(t) := K_{M \times I}(\bar{u}, \partial_t) = -\frac{\psi''}{\psi}(t).
\]
(2.4)
It follows from Proposition 2.1, Eq. (2.3) and (2.4) that, $M_{\phi} \times I$ has constant sectional curvature $\kappa$ if, and only if, $M$ has constant sectional curvature $K_M$, the normal sectional curvature is constant equal to $\kappa$ and $\mathcal{H}^2 + \kappa = \mathcal{K}$.

The main object studied here is the graph of a function $f : M \rightarrow I \subset \mathbb{R}$. The graph of $f$ is defined by
\[
\Gamma_f := \{(x, f(x)) : x \in M\}.
\]
The canonical parametrization of the graph of $f$ will be denoted by $\phi : M \rightarrow \Gamma_f \subset M_{\psi} \times I$, where $\phi(x) := (x, f(x))$. It is easy to see that
\[
d\phi_x(v) = \tilde{v} + \langle \nabla f(x), v \rangle_M \cdot \partial_t, \quad \forall x \in M, v \in T_xM.
\]  
(2.5)

We choose the following unit normal vector field to the graph of $f$ in $M_{\psi} \times I$
\[
\eta(\phi(x)) = \frac{\psi(f(x))}{W(x)} \left( \frac{\nabla f(x)}{\psi^2(f(x))} - \partial_t (\phi(x)) \right),
\]  
(2.6)
where $W(x) := \sqrt{||\nabla f(x)||^2 + (\psi \circ f)^2(x)}$ and $\nabla f$ is the gradient vector field in $M$, cf. [ACdL15, Eq.(3.3)].

The shape operator of a hypersurface in $M_{\psi} \times I$ was described in [ACdL15, Eq.(2.14)]. For the convenience of the reader, we state below this formula according our notations. For every tangent vector $v, w \in T_xM$ and $x \in M$, the shape operator of the graph of $f$ in $M_{\psi} \times I$, with respect to $\eta$, satisfies
\[
\langle A(d\phi_x(v)), d\phi_x(w) \rangle_{\psi} = -\frac{\psi(f(x))}{W(x)} \text{Hess}_x(v, w) + \frac{(\psi^2 \cdot \mathcal{H})(f(x))}{W(x)} (v, w)_M + 2 \frac{(\psi \cdot \mathcal{H})(f(x))}{W(x)}(\nabla f(x), v)_M \cdot (\nabla f(x), w)_M.
\]  
(2.7)

The following proposition is consequence from the last equation.
**Proposition 2.3.** The mean curvature of \( \Gamma_f \) in \( M_\psi \times I \) with respect to the unit normal vector field \( \eta \) is given by

\[
mH(\phi(x)) = -\text{Div}_\mathcal{M} \left( \frac{\nabla f}{\psi(f)} \right)(x) + \frac{(\psi \cdot \mathcal{H})(f(x))}{\psi(x) W(x)} \left( m - \frac{|\nabla f(x)|^2}{\psi^2(f(x))} \right),
\]

where \( \text{Div}_\mathcal{M} \) is the divergence taken with respect to the metric on \( \mathcal{M} \).

### 3. Curvature estimates for graphs contained in a slab

In this section, we establish curvature estimates for graphs contained in a slab of \( M_\psi \times I \). We say that a graph \( \Gamma_f \) of a function \( f : M \to I \) is contained in a slab, if, and only if, the function \( f \) is bounded on \( I \), i.e., there exist \( a, b \in I \) such that \( f(x) \in [a, b] \) for all \( x \in M \).

Our main ingredient to obtain these estimates is the following technical proposition.

**Proposition 3.1.** Let \( M^m \) be a complete Riemannian manifold with sectional curvature bounded below and \( \Gamma_f \subset M_\psi \times I \) a graph.

(i) If \( f \) is bounded from below, then there is a sequence \( \{x_n\} \subset M \) such that the principal curvatures \( \lambda_i \) of the graph of \( f \) satisfies

\[
\lambda_i(\phi(x_n)) < \frac{1}{n\psi(f(x_n))} + |\mathcal{H}(f(x_n))|, \quad \forall i = 1, \ldots, m,
\]

for \( n \) sufficiently large. Moreover, \( f(x_n) \to \inf f \).

(ii) If \( f \) is bounded from above, then there is a sequence \( \{y_n\} \subset M \) such that the principal curvatures \( \lambda_i \) of the graph of \( f \) satisfies

\[
-\lambda_i(\phi(y_n)) < \frac{1}{n\psi(f(y_n))} + |\mathcal{H}(f(y_n))|, \quad \forall i = 1, \ldots, m,
\]

for \( n \) sufficiently large. Moreover, \( f(y_n) \to \sup f \).

In order to establish the proposition above we need to introduce some notation. Given \( t_0 \in I \) and \( \sigma_0 \in \mathbb{R} \), consider the function \( \sigma : I \to J := \sigma(I) \subset \mathbb{R} \) defined by

\[
\sigma(t) := \sigma_0 - \int_{t_0}^t \frac{1}{\psi(\tau)} \, d\tau.
\]

The map \( \tau : M_\psi \times I \to M \times J \subset M \times \mathbb{R} \), defined by \( (x, t) := (x, \sigma(t)) \), is a conformal reversing orientation diffeomorphism. More precisely,

\[
\langle v, w \rangle_\psi = \psi^2(t)\langle d\tau(x,t)(v), d\tau(x,t)(w) \rangle_{M \times \mathbb{R}},
\]

for all \( v, w \in T_{(x,t)}(M \times I) \) (cf. [AD07b, Section 2.3]), where \( \langle \cdot, \cdot \rangle_{M \times \mathbb{R}} \) is the usual product metric. Define the function \( \hat{f} := \sigma \circ f \) and let \( \hat{\phi} \) be the canonical parametrization of \( \Gamma_f \), the graph of \( \hat{f} \), in \( M \times \mathbb{R} \). Since

\[
\hat{\phi}(x) = (x, (\sigma \circ f)(x)) = \tau(x, f(x)) = (\sigma \circ\phi)(x),
\]

it follows from Eq. (3.4) that

\[
\langle d\hat{\phi}(\cdot), d\hat{\phi}(\cdot) \rangle_{M \times \mathbb{R}} = \frac{1}{(\psi \circ f)^2} \langle d\phi(\cdot), d\phi(\cdot) \rangle_{\psi}.
\]
We choose the following unit normal vector to the $\Gamma_{\hat{f}}$
\[ \hat{n}(\hat{\phi}(x)) := \tau_{\epsilon} ((\psi \circ f)(x) \cdot \eta(\phi(x))) = \frac{1}{\hat{W}(x)} \left( -\nabla_{\hat{f}}(x) + \partial_{t}(\hat{\phi}(x)) \right), \] (3.6)
where
\[ \hat{W}(x) := \sqrt{1 + |\nabla_{\hat{f}}(x)|^2} = \frac{W(x)}{\psi(f(x))}. \] (3.7)

Let $\Theta : \Gamma_{\hat{f}} \to [-1, 0]$ be the angle function $\Theta := (\eta, \partial_{t})$. Using equations (2.7), (3.4) and (3.5), the relation between $\Lambda$ and $\hat{\Lambda}$, the second fundamental forms of $\Gamma_{\hat{f}}$ in $M_{\psi} \times I$ and $\Gamma_{\hat{f}}$ in $M \times \mathbb{R}$ with respect to the $n$ and $\hat{n}$, is given by
\[ \hat{\Lambda}(df_{\phi}(v)) = (\psi \circ f)(x) \cdot d\tau_{\phi}[x] \left( (\Lambda + ((\mathcal{H} \circ f) \cdot (\Theta \circ \phi))(x) \cdot \text{Id}) (df_{\phi}(v)) \right), \] (3.8)
for all $x \in M$ and $v \in T_{x}M$, where $\text{Id}$ is the identity operator of $T_{\phi(\tau)} \Gamma_{\hat{f}}$. From this equation we easily deduce that the principal curvatures $\lambda_{i}$ of $\Gamma_{\hat{f}}$ in $M_{\psi} \times I$ and $\hat{\lambda}_{i}$ of $\Gamma_{\hat{f}}$ in $M \times \mathbb{R}$ are related by (see Guan and Spruck [GS00, Section 2] for similar formulas)
\[ \hat{\lambda}_{i} \circ \hat{\phi} = (\psi \circ f) \left( \lambda_{i} \circ \phi + (\mathcal{H} \circ f) \cdot (\Theta \circ \phi) \right). \] (3.9)
As a consequence, the mean curvatures $\mathcal{H}$ of $\Gamma_{\hat{f}}$ and $\hat{\mathcal{H}}$ of $\Gamma_{\hat{f}}$ satisfy
\[ \hat{\mathcal{H}} \circ \hat{\phi} = (\psi \circ f) \left( \mathcal{H} \circ \phi + (\mathcal{H} \circ f) \cdot (\Theta \circ \phi) \right). \] (3.10)

Now we are in condition to present the proof of Proposition 3.1.

**Proof of Proposition 3.1.** Consider the function $\hat{f} = \sigma \circ f$ as defined before.

(i) First we assume that $f$ is bounded from below, so $\hat{f}$ is bounded from above. Using the Omori-Yau maximum principle [Omo67, Yau75] there is a sequence $(x_{n}) \subset M$ such that $\hat{f}(x_{n}) \to \sup \hat{f}, \|\nabla \hat{f}(x_{n})\| \to 0$ and
\[ \text{Hess} \hat{f}_{x_{n}}(v, v) < \frac{||v||^2}{n}, \quad \forall v \in T_{x_{n}}M, \quad v \neq 0, \]
for $n$ sufficiently large. Let $\hat{\lambda}_{i}$ be the principal curvatures of $\Gamma_{\hat{f}}$ at $\hat{\phi}(x_{n})$ for $i = 1, \ldots, m$. Consider $[\hat{e}_{i}]_{i=1}^{m}$ be an orthonormal base of eigenvectors of $\hat{\Lambda}$ associated to the principal curvatures. For each $i$ denote by $v_{i} \in T_{x_{n}}M$ the unique vector such that $df_{\phi}(x_{n})(v_{i}) = \hat{e}_{i}$. Then,
\[ \hat{\lambda}_{i}(\hat{\phi}(x_{n})) = (\hat{\Lambda})_{i} \cdot \hat{\phi}(x_{n}) = \frac{\text{Hess} \hat{f}_{x_{n}}(v_{i}, v_{i})}{\hat{W}(x_{n})} \leq \frac{1}{n}, \quad \forall i = 1, \ldots, m. \]
Then from Eq. (3.9) we obtain Eq. (3.1). We observe that, since $\sigma$ is a decreasing function $\sup f = \sigma(\inf f)$, thus $f(x_{n}) \to \inf f$.

(ii) To prove item (ii), let $\mathcal{M} : I \to [-1]$ be the reverse orientation diffeomorphism $\mathcal{M}(t) = -t$, where $-1 = \{ t \in \mathbb{R} \mid -t \in I \}$, and define $\hat{\Psi} = \psi \circ \mathcal{M}^{-1}$. Consider the reverse orientation isometry (see [Miro9, pag. 123]),
\[ \varphi : M_{\psi} \times I \to M_{\bar{\psi}} \times (-I), \]
given by \( \varphi(x,t) = (x, M(t)) \). Now define \( \overline{T} := M \circ f \) and \( \overline{\varphi} := \varphi \circ \phi \). As \( \varphi \) is a reversing orientation isometry, the principal curvature of \( \Gamma_T \) and \( \Gamma_f \) with respect to unit normal vector field \( \phi \) (resp. non-positive) and \( \inf H \) below and \( \sup H \) is not true when \( \overline{\varphi} \) is given by \( \ln \psi(t) \)'s, see Eq. (2.2).

**Theorem 3.2.** Let \( M \) be a complete Riemannian manifold with sectional curvature bounded below. If \( \Gamma_f \subset M_{\psi} \times \mathbb{R} \) is contained in a slab, then its mean curvature \( H \) satisfies

\[
|H| \leq \sup_{f(M)} |H|. \tag{3.11}
\]

In particular,

\[
\inf_{f(M)} |H| \leq \sup_{f(M)} |H|. \tag{3.12}
\]

**Proof.** By Proposition 3.1, there is a sequence \( (x_n) \subset M \) such that \( \widehat{f}(x_n) \to \sup \widehat{f} \), and \( |\widehat{H}(\widehat{f}(x_n))| \to \inf |H| = 0 \). Since \( \|\nabla f(x_n)\| \to 0 \), we have that \( \|\nabla f(x_n)\| \to 0 \). Using Eq. (3.10) and taking the limit for \( n \to \infty \), we obtain that

\[
\lim_{n \to \infty} H(\phi(x_n)) = \lim_{n \to \infty} \left( \frac{\widehat{H}(\phi(x_n))}{\psi(f(x_n))} + \frac{\psi(f(x_n))}{\psi(f(x_n))} H(f(x_n)) \right) = H(\inf f). \tag{3.13}
\]

Consequently,

\[
\inf_{f(M)} |H| \leq \lim_{n \to \infty} |H(\phi(x_n))| = |H(\inf f)| \leq \sup_{f(M)} |H|. \]

As in the proof of Proposition 3.1 item (ii), consider the function \( \overline{T} := M \circ f \) in the place of \( f \). Note that, if \( (y_n) \) is a sequence such that \( \overline{T}(y_n) \to \inf \overline{T} \) then \( f(y_n) \to \sup f \). Thus, using \( \overline{T} \) instead of \( f \) we obtain the other claim of the statement.

**Remark 3.3.** When \( f \) is only bounded from below (resp. above), it is possible to obtain the estimate (3.12), supposing that the mean curvature of \( \Gamma_f \) is non-negative (resp. non-positive) and \( \inf \psi \circ f > 0 \). In this case, we only have

\[
\inf_{f(M)} |H| \leq H(\inf f) \quad \text{(resp. } \inf_{f(M)} |H| \leq |H(\sup f)|). \tag{3.14}
\]

**Remark 3.4.** In [Sal89, Theorem 2], it is constructed a bounded from below function, \( f : \mathbb{H}^2 \to \mathbb{R} \), whose the graph \( \Gamma_f \subset \mathbb{H}^2 \times \mathbb{R} \) has constant mean curvature \( H = -1/2 \), with respect to the unit normal vector field \( \eta \). This shows that the Remark 3.3 is not true when \( f \) is bounded above and \( H \geq 0 \) (resp. \( f \) bounded below and \( H \leq 0 \)).

**Corollary 3.5.** Let \( M \) be a complete Riemannian manifold with sectional curvature bounded below. Suppose that no two slices have the same mean curvature. Then the unique graphs with constant mean curvature contained in a slab are the slices.
Proof. Let $\Gamma_f$ be a graph contained in a slab with constant mean curvature. By the last theorem $\mathcal{H}(\inf f) = \mathcal{H}(\sup f)$. Therefore, $f$ is constant.

In a similar way that the mean curvature of the slices has strong influence on the mean curvature of graphs, we will see that the scalar curvature and the norm of the shape operator of the latter is also greatly influenced by the sectional curvature and the normal sectional curvature of the former.

The following lemma shows that, under appropriate conditions on the sectional curvature of the fiber, it is possible to relate directly the scalar curvature of a graph and the normal sectional curvature of the fiber, we will see that the scalar curvature and the norm shape operator of the later is also greatly influenced by the sectional curvature and the normal sectional curvature of the former.

Lemma 3.6. Suppose there exist $\alpha \leq 0$ and $\beta \geq 0$ such that
\[
\alpha \leq \mathcal{K} \circ \phi + \mathcal{H}' \circ f \leq \beta,
\]
for all $p \in M$. Then
\[
\alpha \leq R \circ \phi - H_2 \circ \phi - \mathcal{K}^{-1} \circ f \leq \beta,
\]
where $H_2$ is 2-th mean curvature and $R$ is the scalar curvature of $\Gamma_f$ in $M_\phi \times I$.

Proof. Fix $p \in M$ and consider an orthonormal basis $\{e_1, ..., e_m\}$ of $T_{\phi(p)}\Gamma_f$ such that $A(e_i) = \lambda_i e_i$ for all $i = 1, \ldots, m$, and let $v_i = (d\phi_p)^{-1}(e_i)$. Denote by $\Re_\phi$, $\Re^M$ and $\Re_\phi$ the curvature tensors of $\Gamma_f$, $M$ and $M_\phi \times I$, respectively. It follows from the Gauss equation that
\[
(m - 1) \text{Ric}_{\phi(p)} (e_i) = \sum_{i=1, i \neq j}^m \langle \Re_\phi (e_i, e_j) e_j, e_i \rangle_{\phi} \]
\[
= \sum_{i=1, i \neq j}^m \langle \Re_\phi (e_i, e_j) e_j, e_i \rangle_{\phi} + \lambda_j(\phi(p)) \langle mH - \lambda_j(\phi(p)) \rangle,
\]
where $\text{Ric}_{\phi(p)}$ is the Ricci curvature of $\Gamma_f$ at $\phi(p)$. For all $i \neq j$ we have from Eq. (2.5) and Proposition 2.1 that
\[
\langle \Re_\phi (e_i, e_j) e_j, e_i \rangle_{\phi} = \left( \frac{K_M(v_i, v_j)}{\psi^2(f(p))} + \mathcal{H}'(f(p)) - \left( \frac{\psi''}{\psi}(f(p)) \right) \right) \|v_i \wedge v_j\|^2_{\phi} \]
\[- \left( \frac{\psi''}{\psi}(f(p)) \right) \left( \langle v_i \wedge v_j, \langle \nabla f(p), v_j \rangle M \right) + \|v_i \wedge v_j\|^2_{\phi} \langle \nabla f(p), v_j \rangle M \}
\]
\[+ 2 \left( \frac{\psi''}{\psi}(f(p)) \right) \langle v_i, v_j \rangle_{\phi} \langle \nabla f(p), v_i \rangle_M \langle \nabla f(p), v_j \rangle_M, \]
where $\|v_i \wedge v_j\|^2_{\phi} = \|v_i \wedge v_j\|^2_{\phi} - \langle v_i, v_j \rangle_{\phi}^2$. By direct computation we have
\[1 = \langle e_i \wedge e_j \rangle_{\phi} = \|v_i \wedge v_j\|^2_{\phi} + \|v_i \wedge v_j, \langle \nabla f(p), v_j \rangle M \} + \|v_i \wedge v_j\|^2_{\phi} \langle \nabla f(p), v_j \rangle M \}
\]
\[- 2 \langle v_i, v_j \rangle_{\phi} \langle \nabla f(p), v_i \rangle_M \langle \nabla f(p), v_j \rangle_M, \]
which implies that
\[
\langle \Re_\phi (e_i, e_j) e_j, e_i \rangle_{\phi} = (K(\phi(p))\langle v_i, v_j \rangle + \mathcal{H}'(f(p))) \|v_i \wedge v_j\|^2_{\phi} - \left( \frac{\psi''}{\psi}(f(p)) \right). \]
From the hypothesis and Eq. (2.4) we obtain that
\[ \alpha | v_i \wedge v_j |^2 \leq \langle R_{\phi} (e_i, e_j), e_i, e_j \rangle_{\phi} - \mathcal{K}^\perp (f(p)) \leq \beta | v_i \wedge v_j |^2. \]

It follows from the relation \( 1 = | e_i |^2 = \psi^2(f(p))| v_i |^2_M + \langle \nabla f(p), v_i \rangle_M^2 \) that
\[ | v_i \wedge v_j |^2 = \psi^4(p) \left| (v_i |^2_M \cdot | v_j |^2_M - \langle v_i, v_j \rangle_M^2) \right| \leq \psi^4(p) \cdot | v_i |^2_M \cdot | v_j |^2_M \leq 1. \]

Therefore,
\[ \alpha \leq \langle R_{\phi} (e_i, e_j), e_i, e_j \rangle_{\phi} - \mathcal{K}^\perp (f(p)) \leq \beta. \]  
\[ (3.20) \]

By Equations (3.17) and (3.20),
\[ \alpha \leq \text{Ric}_{\phi (p)} (e_j) - \frac{1}{(m-1)} \lambda_i (\phi (p)) (mH - \lambda_j (\phi (p))) - \mathcal{K}^\perp (f(p)) \leq \beta. \]  
\[ (3.21) \]

Now remembering that
\[ H_2 (\phi (p)) := \frac{1}{m(m-1)} \sum_{j \neq i} \lambda_i (\phi (p)) \lambda_j (\phi (p)), \]
summing over \( j \) from 1 to \( m \) on inequality (3.21) and dividing by \( m \), we prove the result.

We point out that the hypothesis on previous lemma holds for graphs in the hyperbolic space models \( \mathbb{R}^n_x \times \mathbb{R} \) and \( \mathbb{H}^n_{\cosh t} \times \mathbb{R} \) for any \( \alpha \leq 0, \beta \geq 0 \). We will return to this discussion in Section 5.

The hypothesis in Lemma 3.6 is similar to the \textit{convergence condition}, that was used by many authors [Mon99, ACdl15, dLdl12, Mn16, AdL11, A1R13] in formulating criteria for a graph, or more generally a hypersurface, in a warped product space to be a slice.

The next theorem present estimates for the scalar curvature \( R \) of a graph contained in a slab.

**Theorem 3.7.** Let \( M \) be a complete Riemannian manifold with sectional curvature bounded below and \( \Gamma_t \subset M_{\phi} \times 1 \) a graph contained in a slab. If there are \( \alpha \leq 0 \) and \( \beta \geq 0 \) satisfying inequality (3.15) over \( M \), then the scalar curvature \( R \) of \( \Gamma_t \) satisfies
\[ \inf_{\Gamma_t} | R | \leq \beta - \alpha + \min \left\{ 4H^2 (\inf f) + | \mathcal{K}^\perp (\inf f)|, 4H^2 (\sup f) + | \mathcal{K}^\perp (\sup f)| \right\}. \]  
\[ (3.22) \]

**Proof.** We can assume that the scalar curvature of the graph \( \Gamma_t \) of \( f \) does never vanish, otherwise there is nothing to prove. Then, by continuity of \( R \) and connectedness of \( \Gamma_t \), \( R \) is always positive or always negative. Moreover, from Lemma 3.6,
\[ R \circ \phi \leq \beta + H_2 \circ \phi + \mathcal{K}^\perp \circ f \quad \text{over} \quad M. \]

We have three possibilities:

i) \( R > 0 \) in \( \Gamma_t \).

ii) \( R < 0 \) and there exists \( p_0 \in M \) such that
\[ \beta + H_2 (\phi (p_0)) + \mathcal{K}^\perp (f(p_0)) \geq 0. \]

iii) \( R < 0 \) and \( \beta + H_2 (\phi (x)) + \mathcal{K}^\perp (f(x)) < 0 \) for all \( x \in M \).
First consider the sequence \((x_n)\) from Proposition 3.1. In this case \(f(x_n) \to \inf f\). Assuming \(i\), applying the inequality (3.16) at a point \(x_n\), we consider the limit \(n \to \infty\) and obtain

\[
|R| \leq \beta + H^2(\inf f) + \mathcal{K}^\perp(\inf f).
\]

Assuming \(ii\) we obtain, by Lemma 3.6,

\[
\alpha - \beta \leq \alpha + \mathcal{K}^\perp(f(p_0)) + H_2(\phi(p_0)) \leq R(\phi(p_0)) < 0,
\]

which implies

\[
\inf_{P} |R| \leq |R(\phi(p_0))| = -R(\phi(p_0)) \leq \beta - \alpha.
\]

Assuming \(iii\). The proof of this case follows along the lines of the proof of [Fon10, Theorem 1.2] (see [CF20, Theorem 1.4]), we present only the main differences.

If there is a subsequence \((x_{n_k})\) of \((x_n)\) such that \(H_2(\phi(x_{n_k})) \geq 0\) for all \(x_{n_k}\), as \(R < 0\) and \(\alpha \leq 0\) one has

\[
0 < -R(\phi(x_{n_k})) \leq -\alpha - \mathcal{K}^\perp(f(x_{n_k})) - H_2(\phi(x_{n_k})) \leq -\alpha - \mathcal{K}^\perp(f(x_{n_k})),
\]

and so, \(\inf_{P} |R| \leq -\alpha + |\mathcal{K}^\perp(\inf f)|\).

On the other hand, if \(H_2(\phi(x_n)) < 0\) for all \(n\), we have principal curvatures of both signs at every point of the sequence \((x_n) \subset M\). Moreover, by our assumption \(H_2(\phi(x_n)) < -\mathcal{K}^\perp(f(x_n)) - \beta\). Denoting by \(l\) the number of negative principal curvatures at \(\phi(x_n), n\) fixed, we have

\[
\lambda_1(\phi(x_n)) \leq \ldots \leq \lambda_l(\phi(x_n)) < 0 \leq \lambda_{l+1}(\phi(x_n)) \leq \ldots \leq \lambda_m(\phi(x_n)),
\]

where \(1 \leq l \leq m - 1\). Then,

\[
\frac{m(m-1)}{2} H_2(\phi(x_n)) \geq \sum_{i=1}^{l+1} \lambda_i(\phi(x_n)) \lambda_{j}(\phi(x_n)),
\]

and

\[
0 > \frac{m(m-1)}{2} R(\phi(x_n)) \geq \left( mH(\phi(x_n)) - \sum_{i=1}^{m} \lambda_i(\phi(x_n)) \right) \sum_{i=1}^{m} \lambda_i(\phi(x_n))
\]

\[
+ \frac{m(m-1)}{2} \left( \alpha + \mathcal{K}^\perp(f(x_n)) \right).
\]

Hence,

\[
\frac{m(m-1)}{2} \inf_{P} |R| \leq \left( m|H(\phi(x_n))| + \sum_{i=1}^{m} \lambda_i(\phi(x_n)) \right) \sum_{i=1}^{m} \lambda_i(\phi(x_n)) - \frac{m(m-1)}{2} \left( \alpha + \mathcal{K}^\perp(f(x_n)) \right).
\]

Then, we apply Proposition 3.1, take the limit \(n \to \infty\) and use Eq. (3.13) to obtain that

\[
\inf_{P} |R| \leq 4H^2(\inf f) - \alpha + |\mathcal{K}^\perp(\inf f)|.
\]
Comparing the estimates from the three cases we conclude that
\[ \inf_{\Gamma_f} |R| \leq \beta - \alpha + 4H^2(\inf f) + |K^\perp(\inf f)|. \]

Analogously, using the sequence \((y_n)\) of Proposition 3.1 instead the sequence \((x_n)\) and following the same steps as before, we obtain
\[ \inf_{\Gamma_f} |R| \leq 4H^2(\sup f) + \beta - \alpha + |K^\perp(\sup f)|. \]
Then, the result follows. \(\square\)

**Remark 3.8.** If we add the hypothesis that \(H\) does not change sign we can replace the boundedness hypothesis on \(f\) by boundedness above or below. More precisely, if \(f\) is only bounded below and the mean curvature \(H\) of the graph does not change the sign we are able to prove that
\[ \inf_{\Gamma_f} |R| \leq 2H^2(\inf f) + \beta - \alpha + |K^\perp(\inf f)|. \]

Now we pass to estimate the norm of the shape of \(\Gamma_f\).

**Theorem 3.9.** Let \(M^m\) be a complete Riemannian manifold with sectional curvature bounded from below and \(\Gamma_f \subset M_\psi \times I\) a graph contained in a slab. Denote by \(A\) the shape operator of \(\Gamma_f\).

(i) If \(\dim M = m \geq 3\) and the inequality
\[ \text{Ric}_{\Gamma_f} - \inf_{f(M)} K^\perp < \inf_{M}(K \circ \phi + H' \circ f) \leq 0, \] \(3.24\)
holds on \(M\), then
\[ \inf_{\Gamma_f} |A| \leq 3(m - 2) \cdot \min\{|H(\inf f)|, |H(\sup f)|\}. \]

(ii) If the mean curvature of the graph \(H\) does not change sign, then
\[ \inf_{\Gamma_f} |A| \leq m \cdot \min\{|H(\inf f)|, |H(\sup f)|\}. \]

**Proof.** (i) Let us denote by \(\lambda_1, \ldots, \lambda_m\) the principal curvatures of the graph \(\Gamma_f\), labelled by the condition \(\lambda_1 \leq \ldots \leq \lambda_m\). From equations (3.17), (3.19) and inequality (3.24) we obtain
\[ \lambda_i(mH - \lambda_i) < 0, \quad i = 1, \ldots, m. \] \(3.25\)
The above inequality implies that \(\lambda_j \neq 0, \forall j\), and that there exist principal curvatures of both signs at every point of \(\Gamma_f\). Denoting by \(l\) the number of negative principal curvatures, we have at any point of \(\Gamma_f\),
\[ \lambda_1 \leq \ldots \leq \lambda_l < 0 < \lambda_{l+1} \leq \ldots \leq \lambda_m. \] \(3.26\)
Changing \(f\) in \(M_\psi \times I\) to \(\overline{f} = M \circ f\) in \(M_{\psi} \times (-1)\), if necessary, we can assume that \(2 \leq l \leq m - 1\). For each \(i = 1, \ldots, l\), one has by inequality (3.25) that
\[ \lambda_1 + \ldots + \lambda_l + \ldots + \lambda_1 + \lambda_{l+1} + \ldots + \lambda_m > 0, \]
where \(\hat{\varphi}\) means that we omit this principal curvature in the sum. Thus,

\[
1 (\lambda_{l+1} + \ldots + \lambda_m) > (l - 1) \sum_{i=1}^{l} |\lambda_i|.
\] (3.27)

By Proposition 3.1, there exists a sequence \((x_n) \subset M\) satisfying the inequality (3.1). Using this information on inequality (3.27), we obtain

\[
\sum_{i=1}^{l} |\lambda_i| < \frac{l(m - 1)}{l - 1} \left( \frac{1}{n\psi(f(x_n))} + |\mathcal{H}(f(x_n))| \right),
\] (3.28)

and so

\[
\sum_{i=1}^{m} |\lambda_i| < \frac{(m - 1)(2l - 1)}{l - 1} \left( \frac{1}{n\psi(f(x_n))} + |\mathcal{H}(f(x_n))| \right).
\] (3.29)

Thus one has the following estimate for the square of the norm of second fundamental form at \(\phi(x_n)\):

\[
|\mathcal{A}|^2(\phi(x_n)) \leq \left( \sum_{i=1}^{m} |\lambda_i| \right)^2 < \left[ \frac{(m - 1)(2l - 1)}{l - 1} \left( \frac{1}{n\psi(f(x_n))} + |\mathcal{H}(f(x_n))| \right) \right]^2.
\] (3.30)

Using that \(\frac{(m - 1)(2l - 1)}{l - 1}\) is a decreasing function on \(l\) one obtain

\[
\inf_{\mathcal{M}} |\mathcal{A}| < 3(m - 2) \left( \frac{1}{nc} + |\mathcal{H}(f(x_n))| \right),
\]

where \(c = \inf_{\mathcal{M}} (\psi \circ f)\). By taking \(n \to +\infty\), we conclude that

\[
\inf_{\mathcal{M}} |\mathcal{A}| \leq 3(m - 2) |\mathcal{H}(\inf f)|.
\] (3.31)

If we use the sequence \((y_n)\) of Proposition 3.1 instead of the sequence \((x_n)\) and following the steps above we reach at

\[
\inf_{\mathcal{M}} |\mathcal{A}| \leq 3(m - 2) |\mathcal{H}(\sup f)|.
\] (3.32)

With the equations (3.31) and (3.32) we prove the item (i).

(ii) We use similar ideas as in the proof of [Fon10, Theorem 1.7]. Without loss of generality we can assume that \(\mathcal{H} \geq 0\). We have two possibilities, or all principal curvatures are non-negative or there are principal curvatures with negative values. First consider the sequence \((x_n)\) of Proposition 3.1.

If all principal curvatures are non-negative then

\[
|\mathcal{A}|^2(\phi(x_n)) = \sum_{i=1}^{m} \lambda_i^2(\phi(x_n)) \leq m \left( \frac{1}{nc} + |\mathcal{H}(f(x_n))| \right)^2,
\]

where \(c = \inf_{\mathcal{M}} (\psi \circ f)\). Which implies that

\[
\inf_{\mathcal{M}} |\mathcal{A}| \leq \sqrt{m} \cdot |\mathcal{H}(\inf f)| < m \cdot |\mathcal{H}(\inf f)|.
\]
Now suppose that there are principal curvatures with negative values. Let \( l \) be the number of negative principal curvatures such that
\[
\lambda_1(\phi(x_n)) \leq \ldots \leq \lambda_l(\phi(x_n)) < 0 \leq \lambda_{l+1}(\phi(x_n)) \leq \ldots \leq \lambda_m(\phi(x_n)).
\]
Since \( H \geq 0 \) we have
\[
\sum_{i=1}^{l} |\lambda_i(\phi(x_n))| = -\sum_{i=1}^{l} \lambda_i(\phi(x_n)) \leq \sum_{i=l+1}^{m} \lambda_i(\phi(x_n)) \leq (m-1) \left( \frac{1}{nc} + |\mathcal{H}(f(x_n))| \right),
\]
where \( c = \inf_M \psi \circ f \). Consequently,
\[
|A(\phi(x_n))|^2 = \sum_{i=1}^{l} \lambda_i(\phi(x_n))^2 + \sum_{i=l+1}^{m} \lambda_i(\phi(x_n))^2 \\
\leq \left( \sum_{i=1}^{l} |\lambda_i(\phi(x_n))|^2 \right) + \sum_{i=l+1}^{m} \lambda_i(\phi(x_n))^2 \\
\leq \left( (m-1)^2 + (m-1) \left( \frac{1}{nc} + |\mathcal{H}(f(x_n))| \right) \right)^2 \\
\leq m(m-1) \left( \frac{1}{nc} + |\mathcal{H}(f(x_n))| \right)^2 \leq m^2 \left( \frac{1}{nc} + |\mathcal{H}(f(x_n))| \right)^2.
\]
From this inequality we obtain
\[
\inf_{\Gamma_f} |A| \leq m \cdot |\mathcal{H}(\inf f)|.
\]
If we use the sequence \((y_n)\) of Proposition 3.1 instead the sequence \((x_n)\), we obtain the following inequality
\[
\inf_{\Gamma_f} |A| \leq m \cdot |\mathcal{H}(\sup f)|,
\]
and the item (ii) follows.

4. Curvature estimates for graphs not necessarily contained in a slab

In this section, we will make curvature estimates for graphs \( \Gamma_f \subset M_\psi \times I \) in the case where \( f \) is unbounded. We will divide the discussion in two cases, depending on whether the warped function \( \psi \) has reciprocal with finite area or not.

4.1. Warped functions whose reciprocal has finite area. The bound hypothesis on \( f \) is not always achieve in a general situation. In this section we remove this condition. Our methods presented in the last section are based in the Omori-Yau maximum principle that is used in the Riemannian product after we apply the function \( \sigma \) (see Eq. (3.3)) on \( f \). Therefore it is natural to ask about the boundedness of \( \sigma \) instead \( f \). By the definition of \( \sigma \), it is clear his relation with the warped function \( \psi \). However, it is not easy to find suitable conditions on \( \psi \) to obtain boundedness on \( \sigma \). There are classical examples that satisfy the boundedness on \( \sigma \) as the Hyperbolic spaces (see Section 5). Let us present some examples about the boundedness of the function \( \sigma \).
Example 4.1. Consider $\psi(t) = \cosh t$, $\psi : \mathbb{R} \to \mathbb{R}$. Taking the constant $\sigma_0 = \pi/2$ one has
\[
\sigma(t) = \pi/2 - \int_0^t \frac{1}{\cosh u} \, du = \pi - 2 \arctan(e^t),
\]
which is bounded on $\mathbb{R}$.

Example 4.2. Consider $\psi(t) = e^t$, $\psi : \mathbb{R} \to \mathbb{R}$. Taking $\sigma_0 = 1$ we obtain the function $\sigma(t) = e^{-t}$ that is only bounded below.

Example 4.3. Let $\psi : (0, \infty) \to \mathbb{R}$ be the identity function $\psi(t) = t$. Then, taking $\sigma_0 = 0$, we obtain that $\sigma(t) = \ln t$, thus $\sigma$ is an unbounded function.

Motivated by these examples, we define.

Definition 4.4. We say that a function $\rho : I \subset \mathbb{R} \to (0, \infty)$ has reciprocal with finite area when $1/\rho \in L^1(I)$, i.e.,
\[
\int_I \frac{1}{\rho(x)} \, dx < \infty.
\]

If $\psi$ has reciprocal with finite area, then the function $\hat{\sigma} = \sigma \circ f$ is a bounded function for any function $f$ on $M$. This simple observation allows us obtain curvature estimates for any graph $\Gamma_f \subset M \psi \times I$, similar to the estimates in the previous section. Below we will only state the results, once the proofs are similar.

The Proposition 3.1 has an analogue when $\psi$ has reciprocal with finite area. More precisely, if $\psi$ has reciprocal with finite area, then there are sequences $(x_n), (y_n) \subset M$ such that the principal curvatures $\lambda_i$ of the graph of $f$ satisfies
\[
\lambda_i(\phi(x_n)) < \frac{1}{n \psi(f(x_n))} + |\mathcal{H}(f(x_n))|, \quad \forall i = 1, \ldots, m,
\]
and
\[
-\lambda_i(\phi(y_n)) < \frac{1}{n \psi(f(y_n))} + |\mathcal{H}(f(y_n))|, \quad \forall i = 1, \ldots, m,
\]
for $n$ sufficiently large. As consequence, we obtain the following estimate for the mean curvature $H$ of the graph $\Gamma_f$.

Theorem 4.5. Let $M$ be a complete Riemannian manifold with sectional curvature bounded from below. Assume that $\inf(\psi \circ f) > 0$. If $\psi$ has reciprocal with finite area, then
\[
\inf_{\Gamma_f} |H| \leq \sup_{f(M)} |H|.
\]

Using Lemma 3.6 we can estimate the scalar curvature $R$ of a graph $\Gamma_f$ non-necessarily contained in a slab.

Theorem 4.6. Let $M$ be a complete Riemannian manifold with sectional curvature bounded below and assume there are $\alpha \leq 0$ and $\beta \geq 0$ satisfying inequality (3.15) over $M$. If $\inf(\psi \circ f) > 0$ and $\psi$ has reciprocal with finite area, then
\[
\inf_{\Gamma_f} |R| \leq \beta - \alpha + 4 \sup_{f(M)} |\mathcal{H}|^2 + \sup_{f(M)} |\mathcal{K}^\perp|.
\]
Remark 4.7. Even in the case that $\psi$ does not have reciprocal with finite area, we can obtain estimates for the mean curvature $H$ and the scalar curvature $R$ of a graph. If the function $\sigma$ (see (3.3)) is only bounded from above (resp. below) we can obtain the same estimate of (4.3) under the assumption that the mean curvature of $\Gamma_f$ is non-negative (resp. non-positive).

Similarly, if $\psi$ does not have reciprocal with finite area, but the function $\sigma$ is bounded from above or bounded from below, it is possible to prove the following estimate
\[
\inf_{\Gamma_f} |R| \leq \beta - \alpha + 2 \sup_{f(M)} |H|^2 + \sup_{f(M)} |K^\perp|,
\]
under the assumption that the mean curvature of $\Gamma_f$ is non-negative (resp. non-positive).

We now present estimates for the norm of the shape operator of a graph of an unbounded function. In Theorem 4.8 below we relax the condition that $\psi$ has reciprocal with finite area and assume only partial boundedness on $\sigma$ (see (3.3)).

Theorem 4.8. Let $M$ be a complete Riemannian manifold with sectional curvature bounded below and $\Gamma_f \subset M_\psi \times I$ a graph. Assume that $\inf(\psi \circ f) > 0$ and the function $\sigma$ is bounded from below or above.

(i) If $\dim M = m \geq 3$ and the inequality Eq. (3.24) holds on $M$, then the norm of the shape operator $A$ of $\Gamma_f$ satisfies
\[
\inf_{\Gamma_f} |A| \leq 3(m - 2) \cdot \sup_{f(M)} |H|.
\]

(ii) If the mean curvature $H$ of the graph does not change sign, then the norm of the shape operator $A$ of $\Gamma_f$ satisfies
\[
\inf_{\Gamma_f} |A| \leq m \cdot \sup_{f(M)} |H|.
\]

The following example shows that the assumption in the last theorem that the function $\sigma$ is bounded from below or above can not be dropped.

Example 4.9. Fix $a \in \mathbb{R}$ and consider the hyperbolic model space given by the upper half-space $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric $(1/y^2)(dx^2 + dy^2)$. Define the function $f : \mathbb{H}^2 \to \mathbb{R}$ by $f(x, y) = a \ln y$, with $a > 0$. The function $\sigma(t) = t$ is not bounded and $H(t) = 0$ for all $t$. The graph of $f$ has principal curvatures $\lambda_1 = \frac{-a}{\sqrt{1 + a^2}}$, $\lambda_2 = 0$ calculated with respect to $\eta$. Moreover, $|A| = \frac{|a|}{\sqrt{1 + a^2}}$ (see [AdLLJ14, Example 1] and [dLLJL14, Example 10]). This shows that the mean curvature of the graph $H$ is constant and negative, the mean curvature of the slices are trivial and $|A| > 0$.

4.2. Warped functions whose reciprocal has not finite area. In Remarks 3.3, 3.8 and 4.7 we observed that we can obtain curvature estimates with partial boundedness hypotheses. If we are in the position that $f$ is unbounded, not even partially bounded, and $\psi$ has not reciprocal with finite area, $\sigma$ is not even partially
bounded, we still can obtain estimates for the curvature elements using local estimates that we describe below. The idea behind the proofs of the next results is to use the equations \((3.9), (3.10)\) to compare the curvature of a graph in a warped product space with the curvature of a graph in a Riemannian product and then to apply the [CF20, Proposition 3.3]. We will omit the proofs here.

For every closed metric ball with radius \(r\), \(\overline{B}_r \subset M\), there are \(p, q \in \overline{B}_r\) such that

\[-\lambda_i(\phi(q)), \lambda_i(\phi(p)) \leq \frac{\nu_c(r)}{\inf_{f(B_r)} (\psi \circ f)} + \sup_{f(M)} |H|,\]

for all \(i = 1 \ldots, m\), where \(c = \inf_{\overline{B}_r} K_M\), \(K_M\) denotes the sectional curvature of \(M\) and

\[\nu_i(t) := \begin{cases} \frac{t^{-1}}{\sqrt{-s \coth (t \sqrt{-s})}}, & s \geq 0, \ t > 0, \\ \sqrt{-s}, & s < 0, \ t > 0. \end{cases}\]

With these estimates and [CF20, Theorem 1.1] we obtain

\[\inf_{\phi(B_r)} |H| \leq \frac{1}{m \inf_{\mathfrak{g}} (\psi \circ f)} \left( (m - 1) \nu_d(r) + \frac{1}{r}\right) + \sup_{f(M)} |H|,\]

where \(d = \inf_{\overline{B}_r} \text{Ric}_M\) and \(\text{Ric}_M\) is the Ricci curvature of \(M\).

When the Ricci curvature of \(M\) is bounded from below we can take the limit for \(r \to \infty\) in the last estimate to obtain the following result.

**Theorem 4.10.** Let \(M^m\) be a complete Riemannian manifold with Ricci curvature bounded from below by a constant \(d\). If \(\inf (\psi \circ f) > 0\), then the mean curvature \(H\) of the graph \(\Gamma_f \subset M \times \mathbb{R}\) satisfies

\[\inf_{\Gamma_f} |H| \leq \frac{1}{\inf_{\mathfrak{g}} (\psi \circ f)} \frac{(m - 1) \sqrt{-d}}{m} + \sup_{f(M)} |H|.\]

This theorem is an extension of [CF20, Corollary 1.2] to entire graphs in warped product spaces.

We conclude the section with comments about local estimates for the scalar curvature and the norm of the shape operator of a graph (compare with [CF20, Theorems 1.4, 1.6, 1.8]). We use local estimate for the principal curvatures given by \((4.5)\) and applying the same proof’s strategies that we used in Section 3, we obtain the following local estimates.

\[\inf_{\phi(\overline{B}_r)} |R| < 2 \left( \sup_{\phi(\overline{B}_r)} |H| + \frac{\nu_d(r)}{\inf_{\mathfrak{g}} (\psi \circ f)} + \sup_{f(M)} |H| \right) \left( \frac{\nu_d(r)}{\inf_{\mathfrak{g}} (\psi \circ f)} + \sup_{\phi(B_r)} |H| \right) + \beta - \alpha + \sup_{f(\overline{B}_r)} |K|,\]

where \(d = \inf_{\overline{B}_r} \text{Ric}_M\), \(\alpha \leq 0\) and \(\beta \geq 0\) are constants such that Eq. \((3.15)\) holds over \(\overline{B}_r\).
For the norm of the shape operator we have two estimates. First, if \( m \geq 3 \) and Eq. (3.24) holds over \( \overline{B}_r \), then

\[
\inf_{\phi(\overline{B}_r)} |\lambda| \leq 3(m - 2) \left( \frac{\nu_e(r)}{\inf f(\overline{B}_r)} \right) + \sup f(\overline{B}_r)|H|,
\]

where \( K_M \geq c \) on \( \overline{B}_r \). For the second estimate, assume that \( H \) does not change sign thus the norm of second fundamental form of \( \Gamma_f \) satisfies

\[
\inf_{\phi(\overline{B}_r)} |\lambda| \leq m \left( \frac{\nu_e(r)}{\inf f(\overline{B}_r)} \right) + \sup f(\overline{B}_r)|H|.
\]

We observe that these estimates generalize Theorems 1.2, 1.6 and 1.7 of [Fon10], respectively. If we have global hypothesis, then we can take the limit \( r \to \infty \) on the inequalities above to obtain another global estimates for the scalar curvature and the norm of the shape operator.

5. Pseudo-Hyperbolic Spaces and Space Forms

In this section we collect some consequences of the previous results.

5.1. Pseudo-Hyperbolic spaces. When \( M^m \) is a complete Riemannian manifold and the warped function \( \psi \) is either the exponential or the hyperbolic cosine, following the terminology introduced by Tashiro [Tas65, Section 3], we call the corresponding warped product space a pseudo-hyperbolic space. The geometric quantities of these spaces are presented in the table below.

| Space           | \( \mathcal{H}(t) \) | \( \mathcal{K}(p, t) \) | \( \mathcal{K}^+(p, t) \) | \( \sigma(t) \) |
|-----------------|------------------------|---------------------------|-----------------------------|-----------------|
| \( M^m_{\cosh t} \times \mathbb{R} \) | \( \tanh(t) \) | \( \frac{K_M}{\cosh^2(t)} \) | \( -1 \) | \( \pi - 2 \arctan(e^t) \) |
| \( M^m_{e^t} \times \mathbb{R} \) | \( 1 \) | \( \frac{K_M}{e^{2t}} \) | \( -1 \) | \( e^{-t} \) |

**Table 5.1.** Curvature elements of the slices and the function \( \sigma \) on the pseudo-hyperbolic spaces.

An important problem in warped product spaces is to classify the graphs with constant mean curvature. A natural question is whether the slices are the only graphs with this property. The Corollaries 5.1 and 5.2, that follow from Table 5.1 and Theorems 3.2 and 4.5, are related to this question on the particular case of pseudo-hyperbolic spaces (see [CdLo9, Theorem 5.2]).

**Corollary 5.1.** Let \( M \) be a complete Riemannian manifold with sectional curvature bounded from below and \( \Gamma_f \subset M^m_{e^t} \times \mathbb{R} \) a graph.

(i) If \( \Gamma_f \) has constant mean curvature \( H \) and is contained in a slab, then \( H = 1 \).
(ii) If $\Gamma_f$ is bounded from below by a slice, then
$$\inf_{\Gamma_f} |H| \leq 1.$$ 

(iii) If $\Gamma_f$ is contained in a slab, then its mean curvature $H$ is positive at some point (with orientation given by Eq. (2.6)).

The item (iii) above is a consequence of Eq. (3.13).

**Corollary 5.2.** Let $M^m$ be a complete Riemannian manifold with sectional curvature bounded below and $\Gamma_f \subset M^m_{\cosh_t} \times \mathbb{R}$ a graph.

(i) The mean curvature of $\Gamma_f$ satisfies
$$\inf_{\Gamma_f} |H| \leq 1.$$  

In particular, if $\Gamma_f$ has constant mean curvature, then $|H| \leq 1$.

(ii) If $\Gamma_f$ is contained in a slab, then
$$\inf_{\Gamma_f} |H| \leq |\tanh(\inf f)| < 1.$$ 

Moreover, if $\Gamma_f$ has constant mean curvature, then $\Gamma_f$ is a slice.

We now focus our discussion in the special warped product spaces $\mathbb{H}^m_{\cosh_t} \times \mathbb{R}$ and $\mathbb{R}^m_{\cosh_t} \times \mathbb{R}$. It is well known that these spaces are isometric to the hyperbolic space $\mathbb{H}^{m+1}$ (see [Mon99, Example 4.3, p. 725]). The slices in $\mathbb{R}^m_{\cosh_t} \times \mathbb{R}$ are horospheres and then its mean curvature is equal to 1 (see [AD07a, p. 512]). In the case of $\mathbb{H}^m_{\cosh_t} \times \mathbb{R}$, the slices are equidistant hypersurfaces (hyperspheres) and so its mean curvature is $\tanh t$.

The Corollary 5.2 applied to the model $\mathbb{H}^m_{\cosh_t} \times \mathbb{R}$ shows that the slices are the only graphs with constant mean curvature contained in a slab. From [AD07b, Section 2.3], [Fre83, Theorem 3.4] and [dCL83, Theorem A], we know that the only graphs in $\mathbb{R}^m_{\cosh_t} \times \mathbb{R} \equiv \mathbb{H}^{m+1}$ with constant mean curvature are the slices. In particular, there are no entire minimal graphs in such model.

As consequence of Theorem 3.7 one has the following result concerning the scalar curvature of graphs in the model $\mathbb{H}^m_{\cosh_t} \times \mathbb{R}$. We observe that in this model Eq. (3.15) holds with $\alpha = \beta = 0$.

**Corollary 5.3.** The scalar curvature $R$ of $\Gamma_f \subset \mathbb{H}^m_{\cosh_t} \times \mathbb{R} \equiv \mathbb{H}^{m+1}$ satisfies
$$\inf_{\Gamma_f} |R| \leq 5.$$  

Furthermore, when $H$ does not change sign, we obtain
$$\inf_{\Gamma_f} |R| \leq 3.$$ 

From Theorem 4.8 we obtain the following estimates for the norm of the shape operator of a graph.

**Corollary 5.4.** Suppose that $m \geq 3$. If $\text{Ric}_{\Gamma_f} < -1$, then the norm of the shape operator of $\Gamma_f \subset \mathbb{H}^m_{\cosh_t} \times \mathbb{R} \equiv \mathbb{H}^{m+1}$ satisfies
$$\inf_{\Gamma_f} |A| \leq 3(m - 2).$$
Corollary 5.5. If the mean curvature $H$ of a graph $\Gamma_f \subset \mathbb{H}^m_{\cosh t} \times \mathbb{R} \equiv \mathbb{H}^{m+1}$ does not change sign, then the norm of its shape operator satisfies
\[ \inf_{\Gamma_f} |\mathcal{A}| \leq m. \] (5.5)

Remark 5.6. With the same hypothesis, the estimates (5.3), (5.4), (5.5) still hold for entire graphs in $\mathbb{R}^m \times \mathbb{R} \equiv \mathbb{H}^{m+1}$, under the additional assumption that the graph is bounded from below by a slice.

Another warped product model for the Hyperbolic space is construct in the following way. Let $S^m$ be the $m$-dimensional sphere with radius one, i.e., $S^m := \{ x \in \mathbb{R}^{m+1} | |x| = 1 \}$. The hyperbolic space $\mathbb{H}^{m+1}$ can also be seen as the model $(S^m)_{\sinh t} \times (0, \infty)$ with metric
\[ \langle \cdot, \cdot \rangle_{\mathbb{H}^{m+1}} = \sinh^2 t \langle \cdot, \cdot \rangle_{S^m} + dt^2. \]
The slices have mean curvature and sectional curvature given by $\mathcal{H}(t) = \coth t$ and $\mathcal{K}(p, t) = \csc h^2(t)$, respectively. The normal sectional curvature of the slice is constant, $\mathcal{K}^\perp(p, t) = -1$ and $\sigma(t) = \log \coth \frac{t}{2}$. Let $f : S^m \to (0, \infty)$ be a smooth function. By Eq. (3.13) the supremum of the mean curvature of $\Gamma_f$ is always greater than 1. Moreover, Corollary 3.5 implies the following result.

Corollary 5.7. If $\Gamma_f \subset (S^m)_{\sinh t} \times (0, \infty) \equiv \mathbb{H}^{m+1}$ has constant mean curvature, then $\Gamma_f$ is a slice. In particular, there are no minimal graph in the model $(S^m)_{\sinh t} \times (0, \infty) \equiv \mathbb{H}^{m+1}$.

We observe that in model $\mathbb{H}^{m+1}$ described above, Eq. (3.15) holds with $\alpha = \beta = 0$. Then by Theorem 3.7, one has the following estimate for the scalar curvature of the graph of a function $f : S^m \to (0, \infty)$
\[ \inf_{\Gamma_f} |\mathcal{R}| \leq 4 \coth^2(\sup \, f) + 1. \]
When $m \geq 3$, Theorem 3.9 implies
\[ \inf_{\Gamma_f} |\mathcal{A}| \leq 3(m - 2) \coth(\sup \, f). \]

5.2. Spheres. Consider the $(m + 1)$-dimensional sphere $S^{m+1} \subset \mathbb{R}^{m+2}$ with the induced metric
\[ \langle \cdot, \cdot \rangle_{S^{m+1}} = \sin^2 \theta_{m+1} \langle \cdot, \cdot \rangle_{S^m} + d\theta^2_{m+1}, \]
where $\theta_{m+1} \in (0, \pi)$. Therefore, we can view $S^{m+1} - \{ p_N, p_S \}$ as the warped product $(S^m)_{\sin \theta_{m+1}} \times (0, \pi)$, where $p_N = (0, \ldots, 1)$ and $p_S = (0, \ldots, -1)$. The curvatures of the slices that we need to apply our results are given by
\[ \mathcal{H}(\theta_{m+1}) = \cot \theta_{m+1}, \quad \mathcal{K}(p, \theta_{m+1}) = \csc^2(\theta_{m+1}), \quad \mathcal{K}^\perp(p, \theta_{m+1}) = 1. \]
Note that $\sigma(\theta_{m+1}) = \log \cot \left( \frac{\theta_{m+1}}{2} \right)$ is unbounded.

Using the Corollary 3.5 we obtain.

Corollary 5.8. Any graph $\Gamma_f \subset (S^m)_{\sin \theta_{m+1} \times (0, \pi)}$ with constant mean curvature is a slice.
It is not difficult to see that the numbers $\alpha$ and $\beta$ of Eq. (3.15) are null, then applying Theorem 3.7 and Theorem 3.9 we obtain

**Corollary 5.9.** The scalar curvature of a graph $\Gamma_t \subset (S^m)_{\sin \theta_{m+1}} \times (0, \pi)$ satisfies

$$\inf_{\Gamma_t} |R| \leq 4 \min \left\{ \cot^2 (\inf f), \cot^2 (\sup f) \right\} + 1.$$  

Moreover, if $m \geq 3$ the norm of the shape operator of $\Gamma_t$ satisfies the following estimative

$$\inf_{\Gamma_t} |A| \leq 3(m - 2) \min \left\{ |\cot (\inf f)|, |\cot (\sup f)| \right\}.$$  

5.3. **Euclidean Spaces.** The expression of the Euclidean metric in spherical coordinates

$$\langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}} = t^2 \langle \cdot, \cdot \rangle_{S^m} + dt^2.$$  

show that $\mathbb{R}^{m+1} - \{0\}$ can be identified with the warped product $(S^m)_t \times (0, \infty)$.

The mean curvature, the sectional curvature and the normal sectional curvature at $(p, t) \in S^m \times \{t\}$ of the slices are given by, respectively,

$$H(t) = \frac{1}{t}, \quad K(p, t) = \frac{1}{t^2}, \quad K^\perp(p, t) = 0.$$  

We observe that $\sigma(t) = -\log t$ is unbounded in this model.

It is clear from Corollary 3.5 the following result.

**Corollary 5.10.** The only graphs with constant mean curvature in $(S^m)_t \times (0, \infty)$ are the slices. Moreover, there are no minimal graphs in this model.

Note that the mean curvature of a graph in $(S^m)_t \times (0, \infty)$ cannot be negative since we always have a sequence in $S^m$ along which the mean curvature of the graph converges to a positive number (see Eq. (3.13)). Since in this model Eq. (3.15) holds with $\alpha = \beta = 0$ we can apply Theorem 3.7 and item ii) from Theorem 3.9 to obtain the next corollary.

**Corollary 5.11.** The scalar curvature of $\Gamma_t \subset (S^m)_t \times (0, \infty)$ satisfies

$$\inf_{\Gamma_t} |R| \leq (\frac{2}{\sup f})^2.$$  

Moreover, if the mean curvature of the graph of $f$ does not change sign, then

$$H \geq 0 \quad \text{and} \quad \inf_{\Gamma_t} |A| \leq \frac{m}{\sup f}.$$  

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