THE AMBIGUOUS CLASS NUMBER FORMULA REVISITED

FRANZ LEMMERMEYER

Abstract. We will give a simple proof of the ambiguous class number formula.

INTRODUCTION

Let $K$ be a number field, $L/K$ a cyclic extension of prime degree $\ell$, and $\sigma$ a generator of its Galois group $G = \text{Gal}(L/K)$. We say that an ideal class $[a] \in \text{Cl}(L)$ is

- ambiguous if $[a] = [a]$, that is, if there exists an element $\alpha \in L^\times$ such that $a\sigma^{-1} = (\alpha)$, or more briefly, if $a\sigma^{-1} \in H_L$.
- strongly ambiguous if $a\sigma^{-1} = (1)$.

The group $\text{Am}(L/K)$ of ambiguous ideal classes is the subgroup of the class group $\text{Cl}(L)$ consisting of ambiguous ideal classes. Its subgroup of strongly ambiguous ideal classes is denoted by $\text{Am}^{st}(L/K)$.

The following “ambiguous class number formula” is well known: it gives the number of (strongly) ambiguous ideal classes in terms of the class number $h(K)$ of the base field, the number $t$ of ramified primes (including those at infinity), and the index of the units that are norms (of integers and units, respectively) inside the unit group $E_K$ of $K$:

**Theorem 1.** The number of ambiguous ideal classes is given by

$$\# \text{Am}(L/K) = h(K) \cdot \frac{\ell^{t-1}}{(E_K : E_K \cap NL^\times)};$$

$$\# \text{Am}^{st}(L/K) = h(K) \cdot \frac{\ell^{t-1}}{(E_K : NE_L)},$$

where $E_K$ is the unit group of $K$, and $E_K \cap NL^\times$ the subgroup of units that are norms of elements of $L$.

The units in $E_K \cap NL^\times$, i.e., units in $K$ that are norms of elements in $L$, coincide with the units that are local norms everywhere (i.e., norms in every completion of $L/K$). This means that the index $(E_K : E_K \cap NL^\times)$ can be computed in a purely local way.

Since the classical approach to class field theory has gone out of fashion, there are almost no modern books on class field theory in which the ambiguous class number formula is proved; exceptions are Lang [6] and Gras [4]. For this reason we would like to present a modern and yet very elementary way of deriving the ambiguous class number formula from a result that can be found in any introduction to class field theory: the calculation of the Herbrand index of the unit group.

Our approach is close to that used in Lang [6], but does not use cohomology of groups. Readers familiar with cohomology will easily recognize some of the exact
sequences below as pieces of the long exact cohomology sequence. For example, \(4\) is the beginning of the cohomology sequence attached to

\[ 1 \longrightarrow H_L \longrightarrow D_L \longrightarrow \text{Cl}(L) \longrightarrow 1, \]

which is essentially the definition of the class group. In fact, taking fixed modules gives

\[ 1 \longrightarrow H^G_L \longrightarrow D^G_L \longrightarrow \text{Am}(L/K), \]

and we obtain \(4\) below by replacing \(\text{Am}(L/K)\) with the image of \(D^G_L\), which is \(\text{Am}_{st}(L/K)\). We do not gain any additional insight, however, by interpreting the almost obvious exact sequence \(4\), which essentially is the definition of the strongly ambiguous ideal class group, as a part of the long cohomology sequence.

We take this opportunity to invite the readers to look at a few articles that provide an excellent background to our computations, or show how to do everything in cohomological terms. In particular we recommend Martinet [7] concerning discriminant bounds and the cohomological approach to investigations of class groups given by Cornell & Rosen [3].

### 1. Ambiguous and Strongly Ambiguous Ideals Classes

Let \(K\) be a number field, and let \(L/K\) be a cyclic extension of prime degree \(\ell\). We will use the following notation:

- \(D_K, D_L\) are the groups of nonzero fractional ideals in \(K\) and \(L\);
- \(H_K, H_L\) are the groups of nonzero principal fractional ideals in \(K\) and \(L\);
- \(\tilde{D}_K\) (and \(\tilde{H}_K\)) are the groups of all (resp. of all principal) ideals in \(L\) generated by ideals (principal ideals) in \(K\);
- \(E_K, E_L\) are the unit groups in \(K\) and \(L\);
- \(\text{Cl}(K), \text{Cl}(L)\) denote the ideal class groups in \(K\) and \(L\);
- \(\sigma\) is a generator of the Galois group \(G = \text{Gal}(L/K)\);
- \(e(p)\) is the ramification index of the prime \(p\) in \(L/K\);
- \(N\) denotes the relative norm for \(L/K\);
- \(A[N] = \{a \in A : Na = 1\}\) is the subgroup of \(A\) killed by the norm;
- \(A^G = \{a \in G : a^\sigma = a\}\) is the fix module of the \(G\)-module \(A\);

The following result shows that \(1\) and \(2\) are equivalent:

**Proposition 1.** The difference between ambiguous and strongly ambiguous ideal classes is measured by the exact sequence

\[ 1 \longrightarrow \text{Am}_{st}(L/K) \longrightarrow \text{Am}(L/K) \longrightarrow (E_K \cap NL^\times)/NE_L \longrightarrow 1. \]

**Proof.** We will first construct the map

\[ (3) \quad \nu : \text{Am}(L/K) \longrightarrow (E_K \cap NL^\times)/NE_L \]

and then prove exactness by computing the image and the kernel of \(\nu\).

An ideal class \([a] \in \text{Cl}(L)\) is ambiguous if and only if \(a^\sigma = (\alpha)a\) for some \(\alpha \in L^\times\). Taking norms and canceling \(Na\) yields \((N\alpha) = (1)\), which shows that \(N\alpha\) is a unit and thus belongs to \(E_K \cap NL^\times\).

The map sending \([a] \in \text{Am}(L/K)\) to \(N\alpha \in E_K \cap NL^\times\) is not well-defined since we are allowed to change \(\alpha\) by a unit. Thus we obtain a well-defined homomorphism \(\nu\) as in \(3\).

For showing that \(\nu\) is surjective assume \(\beta \in E_K \cap NL^\times\) and write \(\beta = N\alpha\). Then \((N\alpha) = (1)\), so by Hilbert’s Theorem 90 for ideals (see Artin’s Göttingen lectures in
we can actually imitate the proof of Hilbert 90 for elements by setting $a + b = \gcd(a, b)$ we have $(\alpha) = a^{\sigma - 1}$ for some ideal $a$, i.e., $a^\sigma = (\alpha)a$, and this implies $[a] \in \text{Am}_{\sigma}(L/K)$ and $\nu([a]) = \beta$.

The kernel of $\nu$ consists of all ideal classes $[a] \in \text{Am}(L/K)$ for which $a^\sigma - 1 = (\alpha)$ and $N\alpha = N\eta$ for some unit $\eta \in E_L$. Thus we can write $a^\sigma = (\alpha/\eta)a$ in the form $a^\sigma = \gamma^{1-\sigma}a$, or $(a\gamma)^{\sigma} = a\gamma$. The last equation tells us that $[a] = [a\gamma]$ is strongly ambiguous. Conversely, every strongly ambiguous class is easily seen to be in the kernel of $\nu$, and we have shown that $\ker \nu = \text{Am}_{\sigma}(L/K)$. This completes the proof. □

2. A Group-Theoretical Lemma

The following simple but useful lemma will be our main new tool in our version of the proof of Thm. 1:

**Lemma 1.** Assume that the diagram

\[
1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1 \\
\alpha \downarrow \quad \beta \downarrow \quad \gamma \downarrow \\
1 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 1
\]

is commutative with exact rows, and that the vertical maps are injective. If two out of the indices $(A' : A)$, $(B' : B)$ and $(C' : C)$ are finite, then all three are, and we have

\[
(B' : B) = (A' : A)(C' : C).
\]

We can drop the assumption that the vertical maps be injective if we assume that the vertical maps have finite kernel and cokernel, and define the index $(A' : A)$ (formally) by $(A' : A) = \frac{\# \text{coker } \alpha}{\# \text{ker } \alpha}$. In our applications, the vertical maps are always injective. Our proof below is for the general case:

**Proof of Lemma 1.** Since the kernel and the cokernel of $\alpha$ are finite, the index $(A' : A) = \frac{\# \text{coker } \alpha}{\# \text{ker } \alpha}$ is a rational number. Since the alternating product of the orders of groups in an exact sequence of finite abelian groups equals 1, the Snake Lemma gives

\[
\# \text{ker } \alpha \cdot \# \text{ker } \gamma \cdot \# \text{coker } \beta = \# \text{ker } \beta \cdot \# \text{coker } \alpha \cdot \# \text{coker } \gamma,
\]

or

\[
\frac{\# \text{coker } \alpha}{\# \text{ker } \alpha} \cdot \frac{\# \text{coker } \gamma}{\# \text{ker } \gamma} = \frac{\# \text{coker } \beta}{\# \text{ker } \beta},
\]

which immediately implies our claim.

The fact that all indices are finite when two of them are is easily seen to hold by following the proof above. □

3. Proof of Theorem 1

The basic idea for proving the ambiguous class number formula is the same as in the classical index calculations of class field theory: we transform indices of groups involving ideals into indices involving principal ideals, and then compute these from indices of unit groups and field elements.
The exact sequence
\[(4) \quad 1 \longrightarrow H^G_L \longrightarrow D^G_L \longrightarrow \text{Am}_{\text{st}}(L/K) \longrightarrow 1\]
tells us that
\[
\# \text{Am}_{\text{st}}(L/K) = \left( D^G_L : H^G_L \right) = \left( \frac{D^G_L : \widetilde{H}_K}{H^G_L : \widetilde{H}_K} \right),
\]
where \(\widetilde{H}_K\) denotes the group of principal ideals in \(L\) generated by elements of \(K^\times\).

Now
\[
\left( D^G_L : \widetilde{H}_K \right) = \left( D^G_L : \widetilde{D}_K \right) \left( \widetilde{D}_K : \widetilde{H}_K \right) = h_K \cdot \left( D^G_L : \widetilde{D}_K \right)
\]
since \(\widetilde{D}_K/\widetilde{H}_K \cong D_K/H_K = \text{Cl}(K)\). The index on the right is easily computed:

**Lemma 2.** Let \(L/K\) be a cyclic extension. Then
\[
\left( D^G_L : \widetilde{D}_K \right) = \prod_{p \text{ fin}} e(p),
\]
where the product of over all finite primes \(p\), and where \(e(p)\) is the ramification index of \(p\) in \(L/K\).

Since modern introductions to class field theory usually use the idel theoretic language, some readers may not be familiar with this formulation of Lemma 2. Here we would like to show that the formula can be proved quite easily. References are Artin’s lectures on class field theory in the appendix of Cohn’s book [2, p. 286] and Lang [6, p. 308].

We begin our proof with the observation that the group \(D^G_L/D_K\) is the fix module of \(D_L/D_K\); the last group is the direct sum (as a \(G\)-module) of the groups
\[I_p = \langle \mathfrak{P}_1, \ldots, \mathfrak{P}_g \rangle / \langle p \rangle,\]
where \(p D_L = (\mathfrak{P}_1 \cdots \mathfrak{P}_g)^{e(p)}\). Since taking fix modules commutes with direct sums, we find
\[D^G_L/D_K = \bigoplus_p I^G_p.\]

Assume now that
\[\mathfrak{P}_1^{a_1} \cdots \mathfrak{P}_g^{a_g} \in I^G_p.\]
Since \(\sigma\) acts transitively on the \(\mathfrak{P}_j\), invariance implies \(a_1 = \cdots = a_g =: a\). Thus every ideal in \(I^G_p\) has the form \((\mathfrak{P}_1 \cdots \mathfrak{P}_g)^a\), and each such ideal is \(G\)-invariant. The map \(I_p \longrightarrow \mathbb{Z}/e(p)\mathbb{Z}\) defined by sending \((\mathfrak{P}_1 \cdots \mathfrak{P}_g)^a\) to the residue class \(a + e(p)\mathbb{Z}\) is easily seen to be an isomorphism, and now our claim follows:
\[D^G_L/D_K \simeq \bigoplus_p I^G_p \simeq \bigoplus_p \mathbb{Z}/e(p)\mathbb{Z}.\]
This completes the proof of Lemma 2.

Collecting everything so far we have
\[(5) \quad \# \text{Am}_{\text{st}}(L/K) = h_K \cdot \prod_{p \text{ fin}} e(p) \frac{H^G_L}{H^G_L : \widetilde{H}_K},\]
and so it remains to compute the index in the denominator. We will do this by reducing it to an index involving numbers instead of ideals. To this end we introduce the group
\[\Delta = \{ \alpha \in L^\times : \alpha^{1-\sigma} \in E_L \}.\]
Applying Lemma 1 to the diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & E_L & \longrightarrow & K^\times E_L & \longrightarrow & \tilde{H}_K & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & E_L & \longrightarrow & \Delta & \longrightarrow & H^G_L & \longrightarrow & 1 \\
\end{array}
\]

where the maps onto the groups of principal ideals is the one sending elements to the ideals they generate, shows that

\[(H^G_L : H_K) = (\Delta : K^\times E_L).\]

Applying Lemma 1 to the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & K^\times & \longrightarrow & K^\times E_L & \longrightarrow & E_L^{1-\sigma} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K^\times & \longrightarrow & \Delta & \longrightarrow & E_L[N] & \longrightarrow & 1 \\
\end{array}
\]

shows that

\[(\Delta : K^\times E_L) = (E_L[N] : E_L^{1-\sigma}).\]

Observe that \(E_L \cap (L^\times)^{1-\sigma} = E_L[N]\): a unit killed by the norm is an element of \((L^\times)^{1-\sigma}\) by Hilbert’s Theorem 90, and the converse is trivial.

Combining (6) and (7) shows that

\[(H^G_L : H_K) = (E_L[N] : E_L^{1-\sigma}),\]

and plugging this into (5) we obtain

\[
\# \text{Am}_{st}(L/K) = h_K \cdot \frac{\prod_{p \mid \infty} e(p)}{(E_L[N] : E_L^{1-\sigma})}.
\]

Determining the index in the denominator is a nontrivial task, and the result, in the classical literature, is called the

**Theorem 2** (Unit Principal Genus Theorem). For cyclic extensions \(L/K\) we have

\[
\frac{(E_K : NE_L)}{(E_L[N] : E_L^{1-\sigma})} = \frac{1}{(L : K)} \prod_{p \nmid \infty} e(p).
\]

Theorem 2 is a well known result from class field theory (it follows painlessly from the unit theorem of Herbrand and Artin using the Herbrand index machinery; see e.g. Lang [5, p. 192, Cor. 2] or Childress [1, Prop. 5.10].

Plugging this into our expression for \(\text{Am}_{st}(L/K)\) we get the desired formula, and our proof is complete.

**Acknowledgements**

I thank the referee for the careful reading of the manuscript and for many helpful suggestions.
References

[1] N. Childress, Class field theory, Springer-Verlag 2009
[2] H. Cohn, A classical invitation to algebraic numbers and class fields, Springer-Verlag 1978
[3] G. Cornell, M. Rosen, Cohomological analysis of the class group extensions problem, Proc. Queen’s N. Th. Conf. 1979, 287–308, Kingston 1980
[4] G. Gras, Class field theory. From theory to practice, Springer-Verlag 2003
[5] S. Lang, Algebraic Number Theory, Springer-Verlag 1986
[6] S. Lang, Cyclotomic Fields I, II, Springer-Verlag 1990
[7] J. Martinet, Tours de corps de classes et estimations de discriminants, Séminaire de Théorie des Nombres Bordeaux 1974, Astérisque 24/25 57–67; Invent. Math. 44 (1978), 65–73