Exceptional solutions in two-mode quantum Rabi models

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Abstract

We study two models describing the interaction of a two-level system with two quantum fields in a parallel or orthogonal setup. We show that both models present a partial two-mode SU(2) symmetry and that they can be solved in the exceptional case of fields with the same frequency. We study their ground state configurations; that is, we find the quantum precursors of the corresponding semi-classical phase transitions, as well as their whole spectra to infer their integrability. We show that the first model is isomorphic with the quantum Rabi model and shows the standard crossover from a vacuum to a non-vacuum ground state configuration. The second model shows a crossover involving four ground state configurations: one vacuum, two non-vacuum single modes and one non-vacuum dual mode. We give analytic and numerical pointers that may suggest its integrability in certain regimes. We also show that, in the single excitation subspace, an excited two-level system may entangle two initial vacuum fields even in the ultra-strong coupling regime.

Keywords: quantum optics, cavity QED, circuit QED

1. Introduction

The so-called quantum Rabi Hamiltonian [1, 2],

$$\hat{H}_R = \frac{\omega_0}{2} \hat{\sigma}_z + \omega \hat{a}^\dagger \hat{a} + g (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_z,$$

(1)

modeling the interaction of a two-level system, described by the transition frequency \( \omega_0 \) and the Pauli operators \( \hat{\sigma}_j \) with \( j = x, y, z \), with a boson field, described by the field frequency \( \omega \) and the annihilation (creation) operators \( \hat{a} \) (\( \hat{a}^\dagger \)), can be seen as a single-field version of the dissipative two-state system [3] in the absence of bias, \( \Delta = 0 \),

$$\hat{H}_L = \frac{\omega_0}{2} \hat{\sigma}_z - \frac{\Delta}{2} \hat{\sigma}_x + \sum_j \omega_j \hat{a}_j^\dagger \hat{a}_j + \sum_j g_j (\hat{a}_j^\dagger + \hat{a}_j) \hat{\sigma}_z.$$

(2)

The dissipative two-state model is characterized by a spectral function, \( J(\omega) = \pi \sum_j \delta(\omega_j - \omega) \) and is solvable, for example, for sub-Ohmic, Ohmic and super-Ohmic spectral functions, \( J(\omega) \propto \omega^s \) with \( s < 1 \), \( s = 1 \) and \( s > 1 \), in that order [3]. On the other hand, the solvability and integrability of the quantum Rabi model has been recently discussed for any given parameter set [2, 4–8]. An equivalent approach has been used to explore the integrability and exceptional solutions of the two-qubit quantum Rabi model [9–13].

Here, we are interested in exploring specific sets of parameters where two models describing a single qubit coupled to just two boson fields can be solved analytically. Our motivation is twofold. First, we want to explore the well-known approaches to the single- [2, 4–8] or two-qubit [9–13] quantum Rabi models and see if they help in rendering these two-mode models integrable for some exceptional parameter sets, with the hope of gaining insight that may help in the search for general solutions to these models and setups including more fields. Second, circuit quantum electrodynamics (circuit QED) may provide a direct testing ground for one model from weak to ultra-strong couplings [14, 15].

$$\hat{H}_I = \frac{\omega_0}{2} \hat{\sigma}_z + \sum_{j=1}^2 \omega_j \hat{a}_j^\dagger \hat{a}_j + \sum_{j=1}^2 g_j (\hat{a}_j^\dagger + \hat{a}_j) \hat{\sigma}_z$$

(3)
and cavity QED may provide an experimental realization for the other by Raman adiabatic driving of a four-level atom coupled to two cavity electromagnetic field modes [16],
\[
\hat{H}_2 = \frac{\omega_0}{2} \hat{\sigma}_z + \sum_{j=1}^{2} \omega_j \hat{a}_j^{\dagger} \hat{a}_j + g_1 (\hat{a}_1^{\dagger} + \hat{a}_1) \hat{\sigma}_z \\
+ ig_2 (\hat{a}_1^{\dagger} - \hat{a}_2) \hat{\sigma}_y.
\]
(4)

In the following, we will refer to Hamiltonians \( \hat{H}_1 \) and \( \hat{H}_2 \) as parallel and orthogonal two-mode quantum Rabi models, in that order, because they can be seen as an ideal two-level system interacting with two standing field modes in a collinear or cross arrangement, respectively. Note that both models conserve parity, \( \hat{\Pi} = e^{-i\hat{N}} \) defined in terms of the total number of excitations, \( \hat{N} = \hat{\sigma}_z/2 + \hat{a}_1^{\dagger} \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2 + 1/2. \)

This manuscript is structured as follows. First, we will study possible candidates for symmetries of these models and regimes where they are equivalent to well-known models. Here, we will introduce the exceptional case of two fields with the same frequency, where Hamiltonian \( \hat{H}_1 \) reduces to the standard quantum Rabi model and Hamiltonian \( \hat{H}_2 \) is invariant to a SU(2) ⊗ SU(2) transformation for identical couplings. Then, we will focus on this exceptional case of two fields with the same frequency to study the ground state structure of both models. While Hamiltonian \( \hat{H}_1 \) shows a simple ground state configuration that includes just a vacuum and a non-vacuum ground state, Hamiltonian \( \hat{H}_2 \) shows a more interesting ground state configuration landscape with four possible configurations: vacuum, two non-vacuum single modes, and one non-vacuum dual mode. Next, we will discuss the integrability of Hamiltonian \( \hat{H}_1 \) due to the isomorphism with the quantum Rabi model and give analytic and numeric arguments that point in the same direction for Hamiltonian \( \hat{H}_2 \). Finally, we will demonstrate that the partial SU(2) symmetry, shown by both models in the exceptional case of two fields with the same frequency, allows us to construct closed form evolution operators in the weak coupling regime. We will use these evolution operators to show that these models may be used to entangle two boson fields, for single-photon dynamics, even in the ultra-strong coupling regime for short evolution times.

2. Symmetries and equivalence with other models

First, we want to state that both Hamiltonians \( \hat{H}_1 \) and \( \hat{H}_2 \) are invariant to full rotations, \( \theta = 2n\pi \) with \( n = 0, 1, 2, \ldots \), under the unitary transformation,
\[
\hat{U}(\theta) = e^{i\theta(\hat{a}_1^{\dagger} \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2 + h/2)},
\]
in other words,
\[
\hat{U}(2n\pi) \hat{H}_i \hat{U}^\dagger(2n\pi) = \hat{H}_i.
\]
(5)

The field part of this transformation is related to the Schwinger two-mode representation of SU(2) [17], \( \hat{J}_+ = \hat{a}_1^{\dagger} \hat{a}_2, \hat{J}_- = \hat{a}_1 \hat{a}_2^{\dagger}, \hat{J}_0 = (\hat{a}_1^{\dagger} \hat{a}_1 - \hat{a}_2^{\dagger} \hat{a}_2)/2. \) This does not provide us with any information but note that in the case of identical qubit-field couplings, \( g_1 = g_2, \) Hamiltonian \( \hat{H}_2 \) is invariant to any given rotation parameter [16],
\[
\hat{U}(\theta) \hat{H}_2 \hat{U}^\dagger(\theta) = \hat{H}_2, \quad g_1 = g_2.
\]
(7)

On the other hand, it is well known that there exists an exact unitary transformation that maps the dissipative two-level model, \( \hat{H}_1, \) into an infinitely long linear nearest-neighbor chain of coupled bosonic modes where just the first one of them is coupled to the qubit [18]. It is not surprising that such transformation in the finite case is related to the SU(2) unitary displacement operator,
\[
\hat{D}(\xi) = e^{i(\hat{a}_1^{\dagger} \hat{a}_1 - \hat{a}_1 \hat{a}_1^{\dagger})}, \quad \tan \xi = \frac{g_2}{g_1}.
\]
(8)

This two-mode displacement yields an effective model where the qubit couples to only the first boson field in the usual quantum Rabi model form, and the first and second boson field couple between them with a beam splitter form,
\[
\hat{H}_{1D} = \hat{D}(\xi) \hat{H}_1 \hat{D}^\dagger(\xi),
\]
(9)
\[
= \frac{\omega_0}{2} \hat{\sigma}_z + \sum_{j=1}^{2} \Omega_j \hat{a}_j^{\dagger} \hat{a}_j + \lambda (\hat{a}_1^{\dagger} \hat{a}_2 + \hat{a}_1 \hat{a}_2^{\dagger}) \\
+ g (\hat{a}_1^{\dagger} + \hat{a}_1) \hat{\sigma}_z.
\]
(10)

Here, we have defined effective field frequencies \( \Omega_1 = (\omega_1 g_1^2 + \omega_2 g_2^2)/g^2 \) and \( \Omega_2 = (\omega_1 g_2^2 + \omega_2 g_1^2)/g^2, \) an effective field coupling constant \( \lambda = (\omega_2 - \omega_1) g_1 g_2/g^2, \) and effective qubit-field coupling \( g = \sqrt{g_1^2 + g_2^2}. \) Note that the choice of \( \tan \xi = -g_1/g_2 \) as a two-mode displacement parameter just interchanges the boson field modes.

A set of unitary transformations cannot bring Hamiltonian \( \hat{H}_2 \) into an expression similar to \( \hat{H}_{1D} \), thus we are reduced to exploring regimes where they may be equivalent. We can start by using the same two-mode displacement on \( \hat{H}_2 \) and find
\[
\hat{H}_{2D} = \hat{D}(\xi) \hat{H}_2 \hat{D}^\dagger(\xi),
\]
(11)
\[
= \frac{\omega_0}{2} \hat{\sigma}_z + \sum_{j=1}^{2} \Omega_j \hat{a}_j^{\dagger} \hat{a}_j + \lambda (\hat{a}_1^{\dagger} \hat{a}_2 + \hat{a}_1 \hat{a}_2^{\dagger}) \\
+ \left[ g \hat{a}_1^{\dagger} + \frac{g_2^2 - g_1^2}{g} \hat{a}_1 - \frac{2g_1 g_2}{g} \hat{a}_2 \right] \hat{\sigma}_z \\
+ \left[ g_1^2 - g_2^2 \right] \frac{g_2}{g} \hat{a}_1^{\dagger} + g_1 \hat{a}_1 + \frac{2g_1 g_2}{g} \hat{a}_2^{\dagger} \hat{\sigma}_z,
\]
(12)

where the effective frequencies and couplings are the same as in the previous case. At most, we may obtain a similar form to the effective Hamiltonian \( \hat{H}_{1D} \) if we require that \( g_2^2 + g_1^2 \sim g_1^2 - g_2^2 \) and \( g_2^2 + g_1^2 \gg 2g_1 g_2. \) These restrictions yield the somewhat obvious regime \( g_1 \gg g_2 \) where we
can approximate

\[ \hat{H}_{2D} \approx \frac{\omega_0}{2} \hat{\sigma}_z + \sum_{j=1}^{2} \Omega_j \hat{a}_j^\dagger \hat{a}_j + \lambda \left( \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger \right) + g_1 \left( \hat{a}_1^\dagger + \hat{a}_1 \right) \hat{\sigma}_z. \]  

(13)

Again, it is possible to interchange the fields via the transformation parameter \( \xi \). Thus, Hamiltonian \( \hat{H}_2 \) will share the properties shown by \( \hat{H}_1 \) in the particular regions \( g_1 \gg g_2 \) and \( g_2 \gg g_1 \). Furthermore, in the case of identical couplings, where \( \hat{H}_2 \) is invariant to rotations \( \hat{U}(\theta) \),

\[ \hat{H}_{2D}|_{\theta=\pi} = \frac{\omega_0}{2} \hat{\sigma}_z + \frac{1}{2} (\omega_1 + \omega_2) \sum_{j=1}^{2} \hat{a}_j^\dagger \hat{a}_j + (\omega_2 - \omega_1) \left( \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger \right) + \sqrt{2} g_1 \left[ \left( \hat{a}_1^\dagger - \hat{a}_2 \right) \hat{\sigma}_z + \left( \hat{a}_1 - \hat{a}_1^\dagger \right) \hat{\sigma}_z \right]. \]  

(14)

This Hamiltonian is equivalent to two-field with the same frequency, one of them interacting under the Jaynes–Cummings dynamics with the qubit and the other under anti-Jaynes–Cummings dynamics [19]. For field modes with the same frequency, \( \omega_1 = \omega_2 \), it conserves the following quantities defined in the displaced frame, \( \hat{N} = -\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{\sigma}_z / 2 + 1/2 \) and the parity \( \hat{\Pi} = e^{-i \hat{N}} \) with \( \hat{N} = \hat{\sigma}_z / 2 + \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1/2 \). Thus, we can label all eigenstates of this Hamiltonian by their mean value of operators \( \hat{N} \) and \( \hat{\Pi} \), as well as their position, related to the ordering by their energy, within these subspaces.

### 3. Ground state configuration

It has been reported that the spin-\( N/2 \) version of Hamiltonian \( \hat{H}_2 \) in the semi-classical limit, \( N \gg 1 \), allows four types of ground state phases. Fan et al characterize these phases by four order parameters: the qubit energy difference, \( \langle \hat{\sigma}_z \rangle \), the two mean photon numbers, \( \langle \hat{a}_j^\dagger \hat{a}_j \rangle \), and the mean two-mode photon number, \( \langle \chi \rangle \) with \( \chi = (\hat{a}_1^\dagger + \hat{a}_2^\dagger)(\hat{a}_1 + \hat{a}_2) \) [16]. These semi-classical ground state phases as defined in [16] are: (i) a normal phase, where all order parameters are zero, \( \langle \hat{\sigma}_z \rangle = \langle \hat{a}_j^\dagger \hat{a}_j \rangle = \langle \chi \rangle = 0 \), and corresponds to a separable ground state with zero excitation, (ii) two single-mode superradiant phases, where the qubit energy difference and one of the mean photon numbers are different from zero, \( \langle \hat{\sigma}_z \rangle = 0 \), \( \langle \hat{a}_1^\dagger \hat{a}_1 \rangle = 0 \) and \( \langle \hat{a}_2^\dagger \hat{a}_2 \rangle = 0 \) or \( \langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0 \) and \( \langle \hat{a}_2^\dagger \hat{a}_1 \rangle = 0 \), while, supposedly, the mean two-photon number is zero, \( \langle \chi \rangle = 0 \), (iii) a two-mode superradiant phase, where all the order parameters are nonzero, \( \langle \hat{\sigma}_z \rangle \neq 0 \), \( \langle \hat{a}_1^\dagger \hat{a}_j \rangle \neq 0 \), and \( \langle \chi \rangle \neq 0 \). For example, in [16], an exceptional solution is found under fields with the same frequency, \( \omega_1 = \omega_2 = \omega \), the two-mode superradiant phase appears just for equal couplings above a critical coupling, \( g_1 = g_2 = g_c = \sqrt{\omega_0 \omega} / 2 \). It has recently been shown that quantum precursors, that is, the smooth transitions between ground state configurations in finite size systems, of semiclassical phase transitions can provide tools for quantum state engineering [20]. Thus, we are going to study the ground state configuration for the single qubit models in order to find the quantum precursors of the semi-classical phase transitions. In the following, we are going to discard the two-mode photon number operator as the order parameter and use the more adequate two-mode \( SU(2) \) operator \( \hat{J}_z = (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) / 2 \) that describes hopping between field modes, as we already know that the systems show such a partial symmetry.

In order to find the ground state configuration for the single-qubit models presented here, we study the exceptional case of fields with the same frequency, \( \omega_1 = \omega_2 = \omega \), which simplifies the problem by eliminating the field–field coupling in the effective models,

\[ \hat{H}_{1E} = \hat{H}_{1D} |_{\omega_1=\omega_2}, \]

\[ = \frac{\omega_0}{2} \hat{\sigma}_z + \omega \sum_{j=1}^{2} \hat{a}_j^\dagger \hat{a}_j + g (\hat{a}_1^\dagger + \hat{a}_1) \hat{\sigma}_z \]

\[ = \omega \hat{a}_2^\dagger \hat{a}_2 + \hat{H}_R. \]  

(16)

(17)

At this point, it is straightforward to recover the effective coupling transition for the quantum Rabi model [21, 22],

\[ g_{1E} = \frac{1}{2} \sqrt{\omega_0 \omega}, \]  

(18)

and conclude that the model presents a smooth transition from a vacuum ground state configuration, \( |g_0 \rangle |0 \rangle_1 |0 \rangle_2 \), at \( g_1 = g_2 = 0 \), through a near-vacuum ground state configuration; that is, a state with a total number of excitations close to zero, for coupling parameters in the range \( 0 < g_{1E} < g_{2E} \leq g_{1c} \), to a non-vacuum ground state configuration where the total number of excitations in the system is large.
compared to the near-vacuum configuration. This behavior is a quantum, finite size precursor of the so-called quantum phase transition in the semi-classical limit, for an infinitely large ensemble of qubits, and can be seen in figure 1(a), where the mean qubit energy level difference is shown for $\hat{H}_{LE}$ calculated with standard numerical methods [20, 23]. Figures 1(b) and (c) show the mean photon number in the first and second field, in that order, and figure 1(d) shows the mean of the two-mode $SU(2)$ hopping operator. It is also straightforward to borrow the deep-strong coupling, $g \gg \omega$, result from the literature [2, 7] and realize that the ground state will be twofold degenerate with ground state energy proportional to $-g^2/\omega$, corresponding to the two parity separable ground states, $|\beta\rangle = \frac{1}{\sqrt{2}} \left( |e\rangle_q + |g\rangle_q \right) \pm [\pm \beta_1]_1 \pm [\pm \beta_2]_2$, with the fields in coherent states, $|\beta\rangle = \sum_n (\beta^n/\sqrt{n!}) |n\rangle$, with parameters $\beta = g/\sqrt{2}$. This ground state configuration leads to mean values of $\langle \hat{\sigma}_z \rangle = 0$, $\langle \hat{\sigma}_1 \rangle = |\beta_1|^2$, $\langle \hat{\sigma}_2 \rangle = |\beta_2|^2$ and $\langle J \rangle = 2 \Re (\beta_1^* \beta_2)$. In order to find the critical coupling for the second model, for fields with the same frequency,

$$\hat{H}_{LE} = \frac{\omega_0}{2} \hat{b}^\dagger \hat{b} + \omega \left( \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \right)$$

$$+ \left[ \frac{g^2 - g^2}{g} \hat{a}_1^\dagger \hat{a}_2^\dagger - \frac{2g g}{g} \hat{a}_2 \hat{a}_2^\dagger \right] \hat{\sigma}_x,$$

$$+ \left[ \frac{g_1^2 - g_2^2}{g} \hat{a}_1^\dagger \hat{a}_1 + \frac{2g g}{g} \hat{a}_1 \hat{a}_1^\dagger \right] \hat{\sigma}_z.$$  \hspace{1cm} (19)

it is simpler to work with $\hat{H}_C$ in the zero and single-excitation subspaces, than just half the critical coupling found [21, 22], to obtain

$$g_{2c} = g_{1c}.$$  \hspace{1cm} (20)

Thus, the ground state configuration will also show a crossover from the vacuum state, $|g\rangle_q |0\rangle_1 |0\rangle_2$ at $g_1 = g_2 = 0$, through a near-vacuum ground configuration in the parameter range $0 < \sqrt{g_1^2 + g_2^2} \ll g_{2c}$, to three different types of excited ground state configurations related to the result in [16] for resonant fields and qubits, $\omega_j = \omega$ with $j = 0, 1, 2$. Again, this is simpler to see in the mean qubit population inversion in figure 2(a). Figure 2 shows the four order parameters defined above for Hamiltonian $\hat{H}_C$. Note that we recover the four ground state configurations described in [16] if we just exchange their two-mode photon number, $\chi$, for the two-mode $SU(2)$ hopping operator, $\chi$. Thus, we are still able to see a change in the ground state configuration for just the single qubit in the case of identical couplings, $g_1 = g_2 > g_{2c}$, as expected from the semi-classical model analysis [16]. We can write some of the different ground states for Hamiltonian $\hat{H}_C$. In the cases $g_2 \ll g_1$ and $g_1 \gg g_2$, or $g_2 \ll g_1$ and $g_2 > g_1$, the ground states will be given by

$$\frac{1}{\sqrt{2}} \left( |e\rangle_q + |g\rangle_q \right) |0\rangle_1 |0\rangle_2 \text{ or } \frac{1}{\sqrt{2}} \left( |e\rangle_q + |g\rangle_q \right) |0\rangle_1 |\pm g_2 \rangle_2$$

where the field states $|\pm g_2 \rangle$ are coherent states. In the regions $g_1 \gg g_2$ and $g_2 \gg g_1$ the ground state configuration will be given by that of Hamiltonian $\hat{H}_C$ in the deep strong coupling regime, $\frac{1}{\sqrt{2}} \left( |e\rangle_q + |g\rangle_q \right) \pm [\pm \beta_1]_1 \pm [\pm \beta_2]_2$ with the coherent parameters as defined beforehand.

Note that, as expected, in these single-qubit models the transition between ground state configurations is smooth and can be understood as a quantum precursor of the phase transitions observed in the classical limit with an infinitely large ensemble of qubits.

4. Spectra

Here, we will continue our analysis of the exceptional case of fields with identical frequencies, $\omega_1 = \omega_2 = \omega$, to find a solution for the spectra of the models. In the case of Hamiltonian $\hat{H}_C$, it is straightforward to see that the model is tractable in this exceptional case as it can be written as $\hat{H}_{LE}$ in the displaced frame. The proper basis for this Hamiltonian in the displaced frame is given by $\{ |n_2|_2 \pm |j_{2k}\rangle \};$ thus, each and every eigenstate can be labeled by the displaced mean photon number of the second field, $n_\beta = 0, 1, 2, \ldots$, parity, $\pm$, and position, $j = 0, 1, 2, \ldots$ in the parity eigenbases of the Rabi model [2, 5–7, 24, 25], $\hat{H}_C |\pm, j_{2k}\rangle_1 = E_{2k+1}^{(j)} |\pm, j_{2k}\rangle_1$. Figure 3 shows the first twelve proper values of Hamiltonian $\hat{H}_C$ on resonance, $\omega_k = \omega = \omega_0 + k = 0, 1, 2$. Note that each and every one of them can be labeled at any given effective coupling strength and the crossings in the spectra are always between different subspaces. The eigenvalues shown correspond to the following states, $\{ |0\rangle_2 |+ \pm j_{2k}\rangle_1 \}$ with $j = 0, 1, 2$ in the solid red lines, $\{ |0\rangle_2 | - j_{2k}\rangle_1 \}$ with $j = 0, 1, 2$ in the solid blue lines, $\{ |1\rangle_2 |+ \pm j_{2k}\rangle_1 \}$ with $j = 0, 1$ in the dashed red lines, $\{ |1\rangle_2 | - j_{2k}\rangle_1 \}$ with $j = 0, 1$ in the dashed blue lines, $\{ |2\rangle_2 |+ , 0\rangle_1 \}$ in the dot-dashed red lines, and $\{ |2\rangle_2 | - , 0\rangle_1 \}$ in the dot-dashed blue lines. As expected for large values of the effective coupling constant, $g \gg \omega$, the ground state will be twofold degenerate and the
degeneracy of the rest will increase in twofold steps; e.g., in this regime the twofold degenerate ground state corresponds to \( \{ |0\rangle_2|\pm, 0\rangle_{k,1}\} \), the fourfold first excited state, \( \{ |0\rangle_2|\pm, 1\rangle_{k,1}, |1\rangle_2|\pm, 0\rangle_{k,1}\} \), the sixfold second excited state, \( \{ |0\rangle_2|\pm, 2\rangle_{k,1}, |1\rangle_2|\pm, 1\rangle_{k,1}, |2\rangle_2|\pm, 0\rangle_{k,1}\} \), and so on.

As shown previously, Hamiltonian \( \hat{H}_1 \) has two regimes, \( \{ g_1 \gg g_2, g_2 \gg g_1 \} \), where it can be approximated by the collinear two-mode quantum Rabi Hamiltonian, \( \hat{H}_1 \). The spectra in these regimes can be labeled by the displaced photon number of one of the fields, and the Rabi basis for the other field and the qubit, as we have just shown above. There exists a third regime, the case of equal couplings, \( g_1 = g_2 \), where the spectra can be constructed. As mentioned in section 2, the Hamiltonian \( \hat{H}_2 \) for the exceptional case of two fields with the same frequency and equal couplings conserves the excitation operator \( \hat{N} = -\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{\sigma}_+ / 2 + 1 \). This partitions the whole Hilbert space in subspaces of infinite dimension with the same mean value \( \langle \hat{N} \rangle = 0, \pm 1, \pm 2, \ldots \). Note that we have the three labels we need to uniquely identify each eigenstate, \( \{ \pm, n_d, j \} \), the total parity of the state in the displaced basis, \( \langle \hat{1}\rangle = \pm, \) the displaced excitation operator defining the subspace, \( n_d = \langle \hat{N} \rangle \), and the ordering in the \( n_d \) subspace, \( j \). Figure 4 shows six members of the spectra on resonance, \( \omega_l = \omega \). Following this convention the ground state will always be \( |+, 0, 0\rangle \) and will be completely degenerate at large effective coupling values \( g \gg \omega \). In figure 3 we also show the eigenvalues corresponding to \( |+, 0, j\rangle \) with \( j = 0, 1, 2 \) in solid and dashed red lines, respectively, as well as those related to \( |-, 1, j\rangle \) with \( j = 0, 1 \) and \( |-, -1, 0\rangle \) in solid and dashed blue lines, in that order.

5. Single excitation dynamics

Dynamics of the quantum Rabi model are well studied in the most relevant regimes; the weak [26, 27], \( g \ll \omega \), ultra-strong [28], \( g \gg 0.1 \omega \), and deep-strong coupling [29], \( g \ll \omega \), regimes. Furthermore, it has been shown in the circuit QED laboratory that it is possible to realize a system equivalent to a single qubit coupled to multiple field modes under quantum Rabi model dynamics [14, 15]. Such experimental setups can be modified to realize the parallel two-mode Rabi model, \( \hat{H}_1 \), and it may even be possible to realize the orthogonal two-mode Rabi model, \( \hat{H}_2 \), with circuit QED.

Entanglement is a valuable asset for quantum information processing. Here, as a practical example, we show that it is possible to entangle the first and second field modes with a single excitation near the deep-strong coupling regime. In the weak coupling regime, \( g \ll \omega \), the time evolution operator for the resonant quantum Rabi model is well known. The dynamics of the parallel [30–32] and orthogonal [33, 34] two-mode Jaynes–Cummings models, that is, under the rotating wave approximation in the weak coupling regime, have also been reported in the past. In the single-excitation subspace, \( \{ |\psi_{0}\rangle |0\rangle_1 |0\rangle_2, |\psi_{e}\rangle |0\rangle_1 |0\rangle_2, |\psi_{g}\rangle |0\rangle_1 |1\rangle_2 \} \) on resonance, \( \omega_j = \omega \) with \( j = 0, 1, 2 \), and weak coupling regime, \( g \ll \omega \), the time evolution for an initial state consisting of an excited atom and vacuum fields, \( |\psi(0)\rangle = |\psi_{e}\rangle |0\rangle_1 |0\rangle_2 \)

\[
\begin{align*}
\left| \tilde{\psi}_1(t) \right| & \approx \cos gt \left| \psi_{e}\right| |0\rangle_1 |0\rangle_2 - i \sin gt \left| \psi_{g}\right| \left[ \cos \xi |1\rangle_1 |0\rangle_2 \\
& \quad + \sin \xi |0\rangle_1 |1\rangle_2 \right].
\end{align*}
\]

\[
\left| \tilde{\psi}_2(t) \right| \approx \cos gt \left| \psi_{e}\right| |0\rangle_1 |0\rangle_2 - i \sin gt \left| \psi_{g}\right| \left[ \cos \xi |1\rangle_1 |0\rangle_2 \\
& \quad - \sin \xi |0\rangle_1 |1\rangle_2 \right].
\]

oscillates coherently between the original state and an entangled state of the two fields. Thus, an initially excited qubit can entangle the two initial vacuum fields in both
configurations at times \( t = n\pi/(2g) \). At these times, in the weak coupling regime, the qubit will be at the ground state and the cavities will be sharing a photon in a ratio \( \cos \xi^2/\sin \xi^2 = g_2^2/g_1^2 \) given by the ratio between the two qubit–field couplings, \( \tan \xi = g_2/g_1 \).

In the weak coupling regime, this process is slow and we need to go for stronger couplings in order to obtain a fast single-photon entangler. Sadly, it is not possible to provide exact dynamics outside the weak coupling regime. In the following, we numerically study the dynamics in the ultra-strong coupling regime and realize that it is still possible to entangle to initial vacuum fields with an initially excited two-level system at times close to \( t = n\pi/(2g) \). Figure 5 shows the time evolution of the mean qubit population inversion and mean photon numbers of the fields in the ultra-strong coupling regime, \( g_1 = g_2 = 0.15\omega \), that gives \( g = 0.212\omega \). The numerical evolution is compared to the result obtained in the weak coupling regime and it is possible to see that they are in close agreement during the first oscillation. If we stay with this effective coupling regime, \( g = 0.212\omega \), we can keep this level of agreement for different coupling parameter ratios; figure 6 presents the case \( g_1^2 = 3g_2^2 \). In both cases, well into the ultra-strong coupling regime, we can see that the qubit does not reach complete transfer to the ground state; however, a conditional measurement of the qubit in the ground state at times \( t = n\pi/(2g) \) delivers a state close enough to the ideal entangled single-photon state.

Figures 5(d) and 6(d) show the fidelity between the weak coupling evolution and the exact numerical ultra-strong coupling evolution, \( F = |\langle \tilde{\psi}(t) | e^{-iHt} | \psi(0) \rangle |^2 \). Figures 7 and 8 show the equivalent results, under Hamiltonian \( \hat{H}_2 \), to those shown in figures 5 and 6, in that order. Note that the fidelity in these cases is better than those under Hamiltonian \( \hat{H}_1 \) dynamics for short evolution times but it seems to degrade faster for longer evolution times.

6. Conclusions

We have studied exceptional solutions for two models describing a single two-level system coupled to two boson field modes in either a parallel or orthogonal configuration. The technology to implement the first of these configurations
with circuit QED has already been demonstrated [14, 15]. It may be possible that modifications to the current setups can bring the latter model to the circuit QED laboratory. Furthermore, both models may be feasible for experimental realization through Raman adiabatic driving in cavity QED [16, 35].

The models conserve parity and show a partial SU(2) symmetry involving the two boson modes; thus, we explored regimes where they may be related to well-known models with a similar structure, like the quantum Rabi model. We focused on the exceptional case of two fields with the same frequency where the models are analytically tractable. Although only one of the models can be transformed to a form including the quantum Rabi model, we found that the ground state configurations of both models present the same critical coupling as the quantum Rabi model. Around this critical coupling, the ground state goes from the so-called critical coupling as the quantum Rabi model. Around this ground state configuration including the quantum Rabi model, we found that the frequency where the models are analytically tractable.

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