SECOND ORDER CASIMIRS FOR THE AFFINE KRICHEVER–NOVIKOV ALGEBRAS $\hat{\mathfrak{gl}}_{g,2}$ AND $\hat{\mathfrak{sl}}_{g,2}$

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Abstract. The second order casimirs for the affine Krichever–Novikov algebras $\hat{\mathfrak{gl}}_{g,2}$ and $\hat{\mathfrak{sl}}_{g,2}$ are described. More general operators which we call semi-casimirs are introduced. It is proven that the semi-casimirs induce well-defined operators on conformal blocks and, for a certain moduli space of Riemann surfaces with two marked points and fixed jets of local coordinates, there is a natural projection of its tangent space onto the space of these operators. It is (non-formally) explained how semi-casimirs appear in course of operator quantization of the second order Hitchin integrals.

1. Introduction

The description of Casimir operators (casimirs, laplacians) is one of the central questions of the representation theory of any Lie algebra. It is difficult to list all the applications of casimirs. The theory of special functions as the eigenfunctions of casimirs, the construction of Hamiltonians possessing symmetries and integrable systems, the investigation of spectral properties of systems with symmetries are some of these applications. The second order casimirs are of special interest in all these questions. In what follows “casimir” always means “second order casimir”.

The casimirs for a Lie algebra $\mathfrak{g}$ can be characterized as the operators which

1. commute with all the operators $\rho(g)$ for any (perhaps, from a given class) representation $\rho$ of $\mathfrak{g}$ and $g \in \mathfrak{g}$,
2. can be expressed via $\rho(g)$ in a certain way.

The casimirs belong to the wider class of the intertwining operators, which can be defined by omitting the second requirement (i.e., by imposing only the requirement of commutativity). For finite-dimensional semi-simple Lie algebras, the investigation of casimirs is mainly based

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on I. M. Gelfand’s theorem about the center of the universal enveloping algebra. The development of the theory of Kac–Moody algebras suggested a new approach to constructing casimirs [6]. This idea is closely related to the following fundamental phenomenon. With each (so-called admissible) representation of an affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) a certain representation of the Virasoro algebra \( \text{Vir} \) is canonically associated. The latter is called the Sugawara representation. It acts in the same linear space as the initial representation of \( \hat{\mathfrak{g}} \). Denote by \( \mathcal{D}^1 \) the sum of \( \text{Vir} \) and \( \hat{\mathfrak{g}} \) with their centers identified. In an admissible representation of \( \mathcal{D}^1 \), each element \( e \) of \( \text{Vir} \) acts first in the usual way and second via its Sugawara operator. For a semi-simple \( \mathfrak{g} \) and its loop algebra \( \hat{\mathfrak{g}} \), a certain linear combination \( \Delta_e \) of these two actions always produces an intertwining operator. One of them is called the \emph{casimir}, namely the one that corresponds to the vector field of zero degree (here we do not consider the degenerate case of so-called critical level, in which there is an infinite number of casimirs of this form).

Krichever and Novikov introduced and investigated [10, 9, 12] a certain type of tensors on Riemann surfaces, namely the meromorphic tensors that have poles only at two fixed points \( P_\pm \) of the Riemann surface. We call such tensors Krichever–Novikov tensors. For example, we speak of Krichever–Novikov functions (which form the associative algebra \( \mathcal{A} \)), Krichever–Novikov vector-fields, \( \lambda \)-forms, etc. Two new classes of Lie algebras were introduced in [10, 9, 12]. These are the centrally extended Lie algebras of Krichever–Novikov vector-fields and \( \mathfrak{g} \)-valued Krichever–Novikov functions, where \( \mathfrak{g} \) is a finite-dimensional complex semisimple or reductive Lie algebra. Let \( \hat{\mathcal{L}} \) denote the first of them (it is called the \emph{Virasoro-type} Krichever–Novikov algebra) and \( \hat{\mathfrak{g}} \) denote the second one (it is called the \emph{affine type} Krichever–Novikov algebra). For zero genus, they coincide with the usual Virasoro algebra and affine Kac–Moody algebra, respectively. For affine Krichever–Novikov algebras, the above-mentioned idea of constructing casimirs as the \( \Delta_e \)-operators was first realized in [17].

This paper is devoted to the description of the second order casimirs (and a certain generalization) for affine Krichever–Novikov algebras. We only consider Krichever–Novikov algebras which correspond to \( \mathfrak{g} = \mathfrak{sl}(l) \) (the semi-simple case: properties of simplicity are of no importance here, see Lemma 4.1 below) and \( \mathfrak{g} = \mathfrak{gl}(l) \) (the reductive case). We use the more detailed notation for \( \hat{\mathfrak{g}} \) in these cases: \( \hat{\mathfrak{sl}}_{g,2} \) for the first case and \( \hat{\mathfrak{gl}}_{g,2} \) for the second one. Here \( g \) stands for the genus of the Riemann surface in question and the index 2 corresponds to the number of marked points.
Our description of casimirs is based on the above construction of the $\Delta_e$'s where $Vir$ and the affine Kac–Moody algebra are replaced by their Krichever–Novikov counterparts $\hat{L}$ and $\hat{g}$. The questions that immediately arise are as follows: How many independent (second order) casimirs exist? Why only one of the intertwining operators described above is considered as a casimir for the Kac–Moody algebra? The usual argumentation is that one of the elements of $Vir$ is distinguished because it defines a grading of the affine algebra. This doesn’t work for Krichever–Novikov case, because there is no reason to distinguish any element of $L$ here (in particular, Krichever–Novikov algebras are not graded). In what follows we relate the question about the number of independent casimirs with a certain cocycle $\gamma$ on $D^1$. It turns out that only the elements $e \in L$ such that $\gamma(e,A) = 0$ for any $A \in A$ produce casimirs. Using this, we prove that the affine Krichever–Novikov algebras $\hat{sl}_{g,2}$ and $\hat{gl}_{g,2}$ possess only one second order casimir. In particular, this holds for Kac–Moody algebras ($g = 0$).

Further, we also consider weaker conditions than those which define casimirs. Let $\mathcal{A}_\pm \subset A$ be the subalgebra consisting of elements of positive order at the point $P_\pm$. For a certain subspace $A_0 \subset A$ (see the definition below) the following decomposition holds [10]: $A = A_- \oplus A_0 \oplus A_+$. Consider vector fields $e$ such that $\gamma(e,A) = 0$ for any $A \in A_- \oplus A_0$. We call the operators $\Delta_e$ for such vector fields semi-casimirs.

It turns out to be that there is an interesting geometrical connection between semi-casimirs and certain moduli spaces of Riemann surfaces. Let $\mathcal{M}_{g,2}^{(p)}$ be the moduli space of Riemann surfaces of genus $g$ with 2 marked points $P_\pm$ and fixed jets of local coordinates (of order 1 for $P_+$ and of order $p$ for $P_-$). For a point $\Sigma \in \mathcal{M}_{g,2}^{(p)}$, let $T_\Sigma \mathcal{M}_{g,2}^{(p)}$ denote the tangent space to this moduli space at $\Sigma$. Consider also the space of co-invariants (or conformal blocks) of the regular subalgebra $\mathfrak{g}_r$ (see Section 4(c) for the definition) in some representation of $\hat{g}$ for $g = gl(l)$. It turns out to be that semi-casimirs are well defined on conformal blocks and (by Theorem 4.2 below) for a proper $p$ there is a natural projection of $T_\Sigma \mathcal{M}_{g,2}^{(p)}$ onto the space of operators induced by semi-casimirs on conformal blocks over $\Sigma$.

The paper is organized as follows. In Section 2 we systematically introduce affine Krichever–Novikov algebras and their representations. For the proofs we refer to [21]. In Section 3 we introduce the Krichever–Novikov algebras of vector fields. We also define the Sugawara representation and consider its commutation relations with the representation of the affine algebra. These two sections can be regarded as an introduction to Krichever–Novikov algebras and their representations.
In Section 4 we introduce casimirs and semi-casimirs and formulate the results mentioned above (Theorems 4.1 and 4.2). In Section 5 we restate some of the results of Section 4 from another point of view. Note that in Section 4 we do not use the classification theorem for 2-cocycles on $D^1$. Such a theorem is proved in [1] for genus zero. As for positive genus, we don’t know any published proof of the theorem. Nevertheless the common opinion is that the result is true\footnote{In a private communication, B. Feigin mentioned that he has a proof of the result.}. To demonstrate that both approaches give the same results, in Section 5 we prove some results of Section 4 assuming the classification of 2-cocycles on $D^1$ to be known.

The setting of the problems and some approaches in this paper were significantly stimulated by my joint work with M. Schlichenmaier. In particular for a commutative $\mathfrak{g}$ he realized that the cocycles on $D^1$ are actually obstructions for the $\Delta_e$’s to be casimirs (see Lemma 4.2). I am thankful to M. Schlichenmaier for stimulating discussions and for the hospitality at the University of Mannheim. I am also thankful to B. Feigin and S. Loktev for discussions about cocycles on $D^1$ and co-invariants.
2. Affine Krichever–Novikov algebras and their representations

(a). Affine Krichever–Novikov algebras. Let \( \Sigma \) be a compact algebraic curve over \( \mathbb{C} \) of genus \( g \) with two marked points \( P_{\pm} \), \( A(\Sigma, P_{\pm}) \) be the algebra of meromorphic functions on \( \Sigma \) which are regular outside the points \( P_{\pm} \), \( \mathfrak{g} \) be a complex semi-simple or reductive Lie algebra. Then
\[
\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} A(\Sigma, P_{\pm}) \oplus \mathbb{C} c
\]
is called the affine Krichever–Novikov algebra [10, 20]. The bracket on \( \hat{\mathfrak{g}} \) is given by the relations
\[
[x \otimes A, y \otimes B] = [x, y] \otimes AB + \gamma(x \otimes A, y \otimes B)c, \quad [x \otimes A, c] = 0,
\]
where \( \gamma \) is the cocycle defined by the formula
\[
\gamma(x \otimes A, y \otimes B) = (x, y) \text{ res}_{P_{\pm}}(AdB);
\]
here \( (\cdot, \cdot) \) is a nondegenerate invariant bilinear form on \( \mathfrak{g} \) (e.g. for \( \mathfrak{g} = \mathfrak{gl}(l) \) we take \( (x, y) = \text{tr}(xy) \)). We will usually suppress the symbol \( \otimes \) in our notation. We also often write \( \mathcal{A} \) instead of \( A(\Sigma, P_{\pm}) \).

In [21] the fermion (or wedge) representations of affine Krichever–Novikov algebras were introduced. They form a rather representative class. Here we use them as a basic example and a model for our constructions. In the remainder of this section, we systematically describe these representations but refer to [21] for proofs.

(b). Holomorphic bundles. Tjurin parameters. Each fermion representation is related to a holomorphic vector bundle on the Riemann surface \( \Sigma \). For this reason consider a holomorphic bundle \( F \) of rank \( r \) and degree \( gr \) on \( \Sigma \). By the Riemann–Roch theorem, the bundle \( F \) possesses \( r \) holomorphic sections \( \Psi_1, \ldots, \Psi_r \) which form a basis over each point except for \( gr \) points. For a generic situation (which is only considered here), one can choose these points to be mutually different. We call these points the degeneration points and denote them \( \gamma_1, \ldots, \gamma_{gr} \). Following the terminology of [7, 8] we call the set of sections introduced a framing and we call a bundle with a given framing a framed bundle.

In a local trivialization of \( F \) one can consider the sections \( \Psi_1, \ldots, \Psi_r \) as vector-valued functions (say columns). These functions can be arranged into a matrix \( \Psi \). This matrix is invertible everywhere except for the points \( \gamma_i, \: i = 1, \ldots, gr \), in which \( \det \Psi \) has simple zeroes: \( \det \Psi(\gamma_i) = 0 \), \( (\det \Psi)'(\gamma_i) \neq 0 \). We call \( D = \gamma_1 + \cdots + \gamma_{gr} \) the Tjurin divisor of the bundle \( F \). Impose one more condition of genericity: \( \text{rank} \Psi(\gamma_i) = r-1, \: i = 1, \ldots, gr \). Then for each \( i = 1, \ldots, gr \)
a non-trivial solution to the system of linear equations $\Psi(\gamma_i)\alpha_i = 0$ exists and is unique up to a scalar factor. Let us introduce notation $\alpha_i = (\alpha_{ij})_{j=1}^{r}$, $i = 1, \ldots, gr$. The Tjurin divisor of the bundle and the numbers $(\alpha_{ij})_{j=1}^{r}$ are called the Tjurin parameters of the bundle $F$ [7, 8]. The framing is defined uniquely up to action of the group $GL(r)$ on $\Psi$ by right multiplication in contrary with gluing functions which act on the left. We see that this $GL(r)$-action commutes with the action of gluing functions; hence it sends sections into sections. This is why the Tjurin parameters are determined uniquely up to a scalar factor and left $GL(r)$-action. Let us emphasize that equivalent framed bundles have the same set $\gamma_1, \ldots, \gamma_{gr}$ while for non-framed bundles only the class of the divisor is invariant. According to [22] the Tjurin parameters determine the bundle uniquely up to equivalence.

In [7, 8] the following description of the space of meromorphic sections of the bundle $F$ in terms of its Tjurin parameters is proposed (only those meromorphic sections are considered which are holomorphic outside the points $P_{\pm}$). In the fibre over an arbitrary point $P$ outside the support of the divisor $D$ the elements $\Psi_j(P)$ ($j = 1, \ldots, r$) form a base. Hence for each meromorphic section $S$ its value $S(P)$ can be expanded in terms of this basis. Thus, to each section $S$ one can assign a vector-valued function $f = (f_1, \ldots, f_r)^T$ on the Riemann surface $\Sigma$ so that outside $D$ one has

$$S(P) = \sum_{j=1}^{r} \Psi_j(P)f_j(P).$$

By Cramer’s rule $f_j = \det(\Psi_1, \ldots, \Psi_{j-1}, S, \Psi_{j+1}, \ldots, \Psi_r) (\det \Psi)^{-1}$. This shows that the functions $f_j$ can be continued to the points of the divisor $D$. They will have there at most simple poles because $\Psi_1, \ldots, \Psi_r$ are holomorphic at the points of the divisor $D$ and $\det \Psi$ has simple zeroes there. By (2.3) in local coordinates in a neighborhood of a point $\gamma_i$ one has $S(z) = \Psi(\gamma_i)(\text{res}_{\gamma_i} f)z^{-1} + O(1)$. The left-hand side of the latter relation is holomorphic. Hence the residues of the functions $f_j$, $j = 1, \ldots, r$, at the point $\gamma_i$ satisfy the system of linear equations $\Psi(\gamma_i)(\text{res}_{\gamma_i} f) = 0$. This is just the system which determines the Tjurin parameters at the point $\gamma_i$. By assumption the rank of the matrix $\Psi(\gamma_i)$ is equal to $r - 1$. Hence the vectors $\text{res}_{\gamma_i} f$ and $\alpha_i$ are proportional.

**Proposition 2.1** ([7, 8]). Consider the space of meromorphic sections of the bundle $F$ which are holomorphic outside the marked points $P_{\pm}$. This space is isomorphic to the space of meromorphic vector-valued functions $f = (f_1, \ldots, f_r)^T$ (on the same Riemann surface) which are
holomorphic outside the points $P_\pm$ and the divisor $D$, have at most simple poles at the points of the $D$ and satisfy the relations

$$(\text{res}_P f_j)_{\alpha_{ik}} = (\text{res}_P f_k)_{\alpha_{ij}}, \quad i = 1, \ldots, gr, \quad j = 1, \ldots, r.$$  

Let us denote the space of vector-valued functions just defined by $F_{KN}$.

(c). The Krichever–Novikov bases. Let us introduce a basis in $F_{KN}$ having in mind constructing semi-infinite monomials on this space. For each pair of integers $n, j$ ($0 \leq j < r$) let us construct a vector-valued function $\psi_{n,j} \in F_{KN}$. The integer $n$ is called the degree of $\psi_{n,j}$. The function $\psi_{n,j}$ is specified by its asymptotic behavior at the points $P_\pm$.

Consider $\psi_{n,j}$ as a column and arrange the matrix $\Psi_n$ of these columns. We require

$$\Psi_n(z_+) = z^n_+ \sum_{s=0}^{\infty} \xi_{n,s}^+ z_+^s, \quad \xi_{n,0}^+ = \begin{pmatrix} 1 & * & \ldots & * \\ 0 & 1 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

and

$$\Psi_n(z_-) = z^{-n}_- \sum_{s=0}^{\infty} \xi_{n,s}^- z_-^s, \quad \xi_{n,0}^- = \begin{pmatrix} * & 0 & \ldots & 0 \\ * & * & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & * \end{pmatrix},$$

where $z_\pm$ is a local coordinate at the point $P_\pm$ and *’s denote arbitrary complex numbers.

Thus the matrix $\Psi_n$ has a zero of order $n$ at one of the points $P_\pm$ and the pole of order $n$ at the other one. Its determinant has $gr$ simple poles at the points of the divisor $D$ and an additional apriory not fixed divisor of zeroes outside the points $P_\pm$. We call $\{\psi_{n,j}\}$ the Krichever–Novikov basis in $F_{KN}$.

The case $r = 1$ is exceptional and needs special prescriptions for the Krichever–Novikov bases.

**Example 2.1.** The prescription for the Krichever–Novikov basis in the algebra $A(\Sigma, P_\pm)$ (that is for a rank 1 trivial bundle $F$) as introduced in [10] is as follows:

$$A_m = \alpha_m^+ z_+^{m+\varepsilon} (1 + O(z_\pm)), \quad \alpha_m^+ \in \mathbb{C}, \quad \alpha_m^+ = 1,$$

where $\varepsilon_+ = 0$ for any $m \in \mathbb{Z}$, $\varepsilon_- = -g$ for $m > 0$ or $m < -g$, and $\varepsilon_- = -g - 1$ for $-g \leq m \leq 0$. For $m > 0$ or $m < -g$ the sum of orders at the marked points is equal to $(-g)$ (notice that $r = 1$ in this case). Therefore there are exactly $g$ zeroes (and no poles) outside $P_\pm$. 
Denote by $\mathcal{A}_+$ (respectively, $\mathcal{A}_-$, $\mathcal{A}_0$) the linear space spanned by $A_m$'s, $m > 0$ (respectively, $m < -g$, $-g \leq m \leq 0$). The following decomposition holds [10]:

\begin{equation}
\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+.
\end{equation}

Another example is given in Section 3(a).

**Proposition 2.2** ([21]). 1°. There exists a unique matrix-valued function $\Psi_n$ which satisfies conditions (2.4), (2.5).

2°. The dimension of the space generated by the vector-valued functions $\psi_{n,j}$ ($n$ being fixed) is equal to $r$.

The algebra $\mathcal{A}(\Sigma, P_\pm)$ naturally acts in the space $F_{KN}$ multiplying its elements by functions. We introduce the structure constants of that action by means of the following relation [21, Proposition 2.3]:

\begin{equation}
A_m \psi_{n,j} = \sum_{k=m+n}^{m+n+\bar{g}} \sum_{j'=0}^{r-1} C_{k,n,j}^{j,j'} \psi_{k,j'},
\end{equation}

where $\bar{g} = g + 1$ if $-g \leq m \leq 0$ and $\bar{g} = g$ otherwise. This relation expresses the fact that the $\mathcal{A}(\Sigma, P_\pm)$-module $F_{KN}$ is almost graded, i.e., $k$ in (2.8) satisfies $|m+n-k| < \text{const}$ and the constant does not depend on $m$, $n$.

Let $\tau$ be the representation of the algebra $\mathfrak{g}$ in the linear space $\mathbb{C}^l$. We write $\tau(x) = (x^i_{\nu})$, where $x \in \mathfrak{g}$, $(x^i_{\nu}) \in \mathfrak{gl}(l)$ is the matrix which represents the element $x$, and the indices $i$ and $\nu$ run over $\{1,\ldots,l\}$.

To each basis element $\psi_{n,j}$ ($n \in \mathbb{Z}$, $j = 0,\ldots,r-1$) let us assign a set of basis elements $\{\psi^i_{n,j}: i = 1,\ldots,l\}$ of $F_{KN}^\tau = F_{KN} \otimes \mathbb{C}^l$. Define an action of the Lie algebra $\mathfrak{g} \otimes \mathcal{A}(\Sigma, P_\pm)$ on $F_{KN}^\tau$:

\begin{equation}
(x A_m) \psi^i_{n,j} = \sum_{k=m+n}^{m+n+\bar{g}} \sum_{j'=0}^{r-1} \sum_{\nu'} C_{k,n,j}^{j,j'} x^i_{\nu} \psi^\nu_{k,j'}.
\end{equation}

The action of $\mathfrak{g}$ does not change the indices $n$, $j$; and, for given $n$ and $j$, it is $\psi^i_{n,j}$ that plays the role of the highest weight vector. It follows from the definitions that this action is almost graded.

**(d). Fermion representations with highest weight.** Let us enumerate the symbols $\psi^i_{n,j}$ in the lexicographic order of the triples $(n, j, i)$. Let $N = N(n, j, i)$ be the number of a triple $(n, j, i)$. We normalize this numbering by the condition $N(-1, 0, l) = 0$. Introduce the following notation: $\psi^i_N = \psi^i_{n,j}$.

Introduce the fermion representation which corresponds to $F$, $\tau$, as follows. The space $V_F$ of the representation is generated over $\mathbb{C}$ by the formal expressions (semi-infinite monomials) of the form $\psi_{N_0} \wedge \psi_{N_1} \wedge
\[\ldots, \text{where } N_0 < N_1 < \ldots \] and the sign of the monomial changes under the transposition of \(\psi_N, \psi_{N'}\). We also require that \(N_k = k + m\) for \(k\) sufficiently large \((k \gg 1)\). Following [6] we call \(m\) the \textit{charge} of the monomial.

For a monomial \(\psi\) of charge \(m\) the degree of \(\psi\) is defined as follows:

\[
\deg \psi = \sum_{k=0}^{\infty} (N_k - k - m).
\]

Observe that there is an arbitrariness in the numbering of the \(\psi_{n,j}^i\)'s for \(n\) fixed; the degree of a monomial just defined does not depend on this arbitrariness.

The representation \(\pi_{F,\tau}\) in the linear space \(V_F\) is defined as follows. By (2.9) each element of \(\hat{g}\) acts on the symbols \(\psi_N\) by linear substitutions in an almost graded way. Moreover, the number of the symbols \(\psi_N\) of a fixed degree does not depend on this degree. This exactly means that if the symbols \(\psi_N\) are considered as a formal basis of an infinite-dimensional linear space then the action of an element of the algebra \(\hat{g}\) can be given by an infinite matrix with only a finite number of nonzero diagonals in this basis. Following [5], we denote the algebra of such matrices by \(a_\infty\).

\textbf{Remark 2.1.} In other terms, \(a_\infty\) is the algebra of difference operators on a 1-dimensional lattice. This remark yields a relation to the results of [14].

Thus, fixing the basis \(\{\psi_N\}\) provides us with an imbedding of \(\mathfrak{g} \otimes \mathcal{A}\) into \(a_\infty\). Since \(a_\infty\) has a standard action in \(V_F\), we obtain a representation of \(\mathfrak{g} \otimes \mathcal{A}\) in \(V_F\). Recall [6, 21] that the action of the basis element \(E_{IJ} \in a_\infty\) on a semi-infinite monomial \(\psi = \psi_{I_0} \wedge \psi_{I_1} \wedge \ldots\) is defined by the Leibniz rule:

\[
(2.11) \quad r(E_{IJ})\psi = (E_{IJ}\psi_{I_0}) \wedge \psi_{I_1} \wedge \cdots + \psi_{I_0} \wedge (E_{IJ}\psi_{I_1}) \wedge \cdots + \cdots.
\]

Due to the condition \(I_k = k + m\) \((k \gg 1)\) the action (2.11) is well defined for \(I \neq J\). For \(I = J\), the right-hand side of (2.11) contains infinitely many terms. In this case the standard regularization is used [6, 21]; it results in a projective representation \(\pi_{F,\tau}\) of \(\mathfrak{g} \otimes \mathcal{A}\). This is an almost graded representation, as it follows from almost-gradedness of the action (2.9) in the space \(F_{K\mathcal{N}}\). Another standard procedure enables us to transform \(\gamma\) to any cohomological cocycle. This can be achieved by adding certain scalars to the operators \(\pi_{F,\tau}(x_\alpha A_m)\) where \(x_\alpha\) are the generators of \(\mathfrak{g}\). Such modification of the representation results in adding a coboundary to \(\gamma\). Hence, by [16] \(\gamma\) can be written as \((2\pi i)^{-1}(x, y) \oint A dB\) where the notation is the same as in (2.2). A
2-cocycle on \( g \otimes A \) is called \textit{local} if there exist \( L \in \mathbb{Z}_+ \) such that 
\[ \gamma(xA_i, yA_j) = 0 \]
for any \( i, j \in \mathbb{Z} \) such that \( |i - j| > L \) and any \( x, y \in g \).
Since \( \gamma \) is obviously local, by [11, 13] it is cohomological to the multiple of the cocycle (2.2). Therefore we can consider \( \pi_{F, \tau} \) as a representation of \( \hat{g} \) without loss of generality.

\textbf{Conjecture 2.1.} The equivalence classes of fermion representations of \( \hat{g} \) are in one-to-one correspondence with the following pairs: an equivalence class of a holomorphic bundle of rank \( r \) and degree \( gr \) on \( \Sigma \) and an equivalence class of an \( l \)-dimensional representation of the algebra \( g \).

For \( l = r \) this conjecture is proven in [21] (for the definition of equivalence of fermion representations see also [21]).

Obviously, the action of \( a_\infty \) respects the charge of a monomial because the infinite “tails” of the monomials both on the right-hand and on the left-hand sides of (2.11) are the same. Hence the fermion space can be decomposed into a direct sum of \( \hat{g} \)-invariant subspaces of certain charges. For \( g = gl(l) \), the space of charge \( m \) is generated by the monomial \( |0\rangle = \psi_m \wedge \psi_{m+1} \wedge \cdots \) under the action of the universal enveloping algebra \( U(\hat{g}) \). This monomial is called a \textit{vacuum (monomial)} of charge \( m \). Each vacuum monomial possesses the following property:

\[ \pi_{F, \tau}(xA)|0\rangle = 0 \quad \text{for} \quad A \in A_+, \quad \text{and also for} \quad A = 1 \quad \text{and any strictly upper-triangular matrix} \quad x \in g. \]

3. \textsc{Krichever–Novikov algebras of vector fields and their representations}

\textit{(a). Algebras of vector fields and their central extensions.} Let \( \mathcal{L} \) be the Lie algebra of meromorphic vector fields on \( \Sigma \) which are allowed to have poles only at the points \( P_\pm [10, 9, 12] \). As it was first noticed in [10], for each integer \( s \geq 0 \) there is a pair of subalgebras \( \mathcal{L}^{(s)}_\pm \) in \( \mathcal{L} \) which consist of vector fields of order not less than \( s \) at the points \( P_\pm \) respectively. The following decomposition into a direct sum of subspaces holds [10]: 
\[ \mathcal{L} = \mathcal{L}^{(s)}_+ \oplus \mathcal{L}^{(s)}_0 \oplus \mathcal{L}^{(s)}_- \]
where \( \mathcal{L}^{(s)}_0 \) is a complementary space. In what follows we are mainly interested in the case \( s = 2 \) in connection with deformations of moduli of Riemann surfaces.

Considered as a linear space, \( \mathcal{L} \) has the Krichever–Novikov basis \( \{e_m: m \in \mathbb{Z}\} \). In the case \( g > 1 \) the basis vector field \( e_m \) is given by its asymptotic behavior in the neighborhoods of the points \( P_\pm [10] \):

\[ e_m(z_\pm) = e_m^\pm z_\pm^{m+\epsilon_\pm} (1 + O(z_\pm)) \frac{\partial}{\partial z_\pm}, \quad e_m^\pm \in \mathbb{C}, \quad \epsilon_m^+ = 1, \]
where $\varepsilon_+ = 1$, $\varepsilon_- = 1 - 3g$, $z_\pm$ is a local coordinate at $P_\pm$. Thus
the subalgebra $L^{(s)}_+$ is generated by the elements $e_m$, $m \geq s - 1$, the
subalgebra $L^{(s)}_-$ by the elements $e_m$, $m \leq -s - 3g + 1$ and the subspace
$L^{(s)}_0$ can be chosen to be generated by the elements $e_m$, $-s - 3g + 1 < m < s - 1$.

We will consider central extensions of the algebra $\mathcal{L}$. Each central
extension is given by a 2-cocycle of the form

$$\chi(e,f) = \text{res}_{P_\pm} \left( \frac{1}{2} (e'' f - ef'') - R(e' f - ef') \right)$$

(cf. Lemma 5.1) where $R$ is a projective connection, i.e., such a function
of a point and a local coordinate that transforms under a change of
coordinates as follows:

$$R(u)u_z^2 = R(z) + \frac{u_{zzz}}{u_z} - \frac{3}{2} \left( \frac{u_{zz}}{u_z} \right)^2.$$  

This ensures that the right hand side of (3.2) is indeed the residue
of a well-defined 1-form. Let $\mathcal{L}$ denote a central extension of $\mathcal{L}$ given
by (3.2).

(b). The action of vector fields in the space $V_F$. Consider the
action of the algebra $\mathcal{L}$ in the space $F_{KN}$. Observe that we cannot apply
naively some vector field $e \in \mathcal{L}$ considered as a first order differential
operator to a vector-valued function $\psi \in F_{KN}$. The reason is that a
generic vector field $e \in \mathcal{L}$ is of order zero at the points $\gamma_1, \ldots, \gamma_{gl}$ of
the Tjurin divisor of the bundle. Hence, $e\psi$ generically has poles of
order two at these points. Thus, $e\psi \notin F_{KN}$. Nevertheless an $\mathcal{L}$-action
on $F_{KN}$ can be defined.

According to [2] each holomorphic bundle, in particular $F$, can be
endowed with a meromorphic (therefore flat) connection $\nabla$ which has
logarithmic singularities at the points $P_\pm$. Since $\nabla$ is flat, $[\nabla_e, \nabla_f] - \nabla_{[e,f]} = 0$ for any $e, f \in \mathcal{L}$. Hence $\nabla_{[e,f]} = [\nabla_e, \nabla_f]$, and $\nabla$ defines a
representation of $\mathcal{L}$ in the space of holomorphic sections of $F$. Being
conjugated by the matrix $\Psi$ (see Section 2(b)), this representation can
be transferred to the space $F_{KN}$ of Krichever–Novikov vector-valued
functions. In what follows we fix an arbitrary connection $\nabla$ which is
logarithmic at $P_\pm$ and regular elsewhere. Applying the procedure
described above (Section 2) of lifting the representation from the space
$F_{KN}$ to the space $V_F$ we obtain the representation of the central extension
of $\mathcal{L}$.

In local coordinates, let $e = E(z)\partial$ where $\partial = \frac{\partial}{\partial z}$ and $\nabla_e = E(z)(\partial + \omega)$. Then the action of the vector field $e$ on $\psi$ can be written in the
following two equivalent forms:

\begin{align}
  e\psi &= \Psi^{-1}\nabla_{\epsilon}\Psi\psi, \\
  e\psi &= E(\partial + \omega_{\Psi})\psi,
\end{align}

(3.3)

where \( \omega_{\Psi} = \Psi^{-1}\omega\Psi + \Psi^{-1}\Psi' \) (\( \Psi \) being introduced in Section 2(b)). Due to (3.3), \( e \mapsto e\psi \) is a well-defined first order differential operator in Krichever–Novikov vector-valued functions (indeed, \((\partial + \omega_{\Psi})\psi \) is a vector-valued 1-form; multiplication by \( E \) returns it into the space of vector-valued functions).

**Lemma 3.1.** 1°. The space \( F_{KN} \) is invariant under the action (3.3).

2°. The action (3.3) is almost graded.

*Proof.* The 1° follows from the definition of the action (3.3).

To prove 2°, observe that \( \Psi \) is regular and nondegenerate at the points \( P_{\pm} \). Hence \( \omega_{\Psi} \) also has simple poles at \( P_{\pm} \). Hence \( \partial \) and \( \omega_{\Psi} \) equally decrease the order of \( \psi \) while acting in local coordinates in neighborhoods of \( P_{\pm} \).

Let us make use of (3.1). For \( n \neq 0 \), the order of \( e_{m}\psi_{n,j} \) is equal to \( n + m \) at \( P_{+} \) and \( -n - m - 3g \) at \( P_{-} \); for \( n = 0 \), these orders are equal to \( n + m \) and \( -n - m - 3g \) respectively. Hence the following relations hold for some constants \( D_{m,n,j}^{k,j'} \):

\begin{equation}
  e_{m}\psi_{n,j}^{i} = \sum_{k=m+n+\varepsilon}^{m+n+3g+\varepsilon} \sum_{j'=0}^{l-1} D_{m,n,j}^{k,j'}\psi_{k,j'}^{i},
\end{equation}

(3.4)

where \( \varepsilon = 0 \) for \( n \neq 0 \) and \( \varepsilon = 1 \) for \( n = 0 \).

Since the \( \mathcal{L} \)-action on \( F_{KN} \) is almost graded, it can be continued to the space of the fermion representation as representation of \( \hat{\mathcal{L}} \) (by the scheme described in detail in Section 2).

**Lemma 3.2.** The elements of the subalgebra \( \mathcal{L}_{+}^{(2)} \) annihilate the vacuum vectors of the fermion representations (of highest weight).

*Proof.* By definition (see Subsection (a)) \( e_{m} \in \mathcal{L}_{+}^{(2)} \) if and only if \( m \geq 1 \). In this case, by (3.4) the action of the vector field \( e \) increases the value of the index \( n \): \( k > n \) for all \( \psi_{k,j'}^{i} \) which occur in (3.4) on the right-hand side. But since the vacuum monomial contains \( \psi_{n,j}^{i} \), it also contains all the \( \psi_{k,j'}^{i} \), \( k > n \) on the right of it. This is why \( e_{m} \) annihilates the vacuum. \( \square \)
(c). **Sugawara representation.** The following is one of the most fundamental facts of representation theory of affine algebras. Each admissible representation of an affine algebra canonically defines a representation of the (corresponding) Virasoro type algebra in the same space. The latter is called the Sugawara representation. For Krichever–Novikov algebras the Sugawara representation is considered in [3, 9, 17]. Here we recall the basic facts about the Sugawara representation.

Let us introduce the following notation. For \( x \otimes A \in \hat{\mathfrak{g}} \), we denote the corresponding operator of representation by \( x(A) \). For a basis element \( A_n \in A \), we denote \( x(A_n) \) by \( x(n) \).

A module (representation) \( V \) over the Lie algebra \( \hat{\mathfrak{g}} \) is said to be admissible if \( x(n)v = 0 \), for each \( v \in V \), \( x \in \mathfrak{g} \) and \( n \) sufficiently large.

Let \( V \) be an admissible module. It is assumed that the central element of \( \hat{\mathfrak{g}} \) acts by multiplication by a scalar \( c \in \mathbb{C} \). In this paper \( V \) can be thought of as one of fermion representations which were introduced in Section 2.

From now on, till the end of this section, \( \mathfrak{g} \) is either simple or abelian Lie algebra. The reason is that the existence of a nondegenerate invariant bilinear form on \( \mathfrak{g} \) is crucial. Any simple \( \mathfrak{g} \) has such a form; if \( \mathfrak{g} \) is abelian, we require it to be equipped with such a form.

Take a basis \( u_i, i = 1, \ldots, \dim \mathfrak{g} \), of \( \mathfrak{g} \) and the corresponding dual basis \( u^i, i = 1, \ldots, \dim \mathfrak{g} \), with respect to the invariant nondegenerate symmetric bilinear form \( (\cdot, \cdot) \). The Casimir element \( \Omega^0 = \sum_{i=1}^{\dim \mathfrak{g}} u_i u^i \) of the universal enveloping algebra \( U(\mathfrak{g}) \) is independent of the choice of the basis. In what follows the summation over \( i \) is assumed in all cases when \( i \) occurs in both super and subscripts.

Let \( 2k \) be the eigenvalue of \( \Omega^0 \) in the adjoint representation. For a simple \( \mathfrak{g} \), \( k \) is the dual Coxeter number. In abelian case, \( k = 0 \).

Introduce bases in the spaces of Krichever–Novikov 1-forms and quadratic differentials. Denote their elements by \( \{ \omega_m | m \in \mathbb{Z} \} \) and \( \{ \Omega^k | k \in \mathbb{Z} \} \) respectively. We define the basis elements by the following duality relations:

\[
\text{res}_{P_\pm} A_m \omega^m = \delta^m_n, \quad \text{res}_{P_\pm} e_m \Omega^k = \delta^k_{m,n},
\]

where \( \delta^m_n \) is the Kronecker symbol. The \( \omega^m \)'s have the following asymptotic behavior [9]:

\[
\omega^m(z_{\pm}) = \mu^\pm_n z_{\pm}^{n+\varepsilon_{\pm}} (1 + O(z_{\pm})) dz_{\pm}, \quad \mu^\pm_n \in \mathbb{C}, \quad \mu^\pm_n = 1,
\]

where \( \varepsilon_{\pm} = -1, \varepsilon_- = 0 \), \( z_{\pm} \) is a local coordinate at \( P_\pm \).

For \( Q \in \Sigma \), define the formal sum (“generating function”)

\[
\hat{x}(Q) = \sum_n x(n) \cdot \omega^n(Q), \quad Q \in \Sigma.
\]
From now on we always assume that while considering series of 1-forms or 2-forms the index of summation runs over \( \mathbb{Z} \) unless otherwise stated.

Introduce the operator-valued quadratic differential \( T(Q) \) (the energy-momentum tensor) as follows:

\[
T(Q) := \frac{1}{2} \sum_{i} :\hat{u}_{i}(Q)\hat{u}^{i}(Q): = \frac{1}{2} \sum_{n,m} \sum_{i} :u_{i}(n)u^{i}(m): \omega^{n}(Q)\omega^{m}(Q).
\]  

(3.7)

Here \( :\ldots:\) denotes the normal ordering. Expand the quadratic differential \( T(Q) \) over the basis quadratic differentials:

\[
T(Q) = \sum_{k} L_{k} \cdot \Omega^{k}(Q),
\]

(3.8)

where \( L_{k} \) are operator-valued coefficients. Using duality relations we obtain

\[
L_{k} = \oint_{c_{0}} T(Q)e_{k}(Q) = \frac{1}{2} \sum_{n,m} \sum_{i} :u_{i}(n)u^{i}(m): l_{k}^{nm},
\]

(3.9)

where

\[
l_{k}^{nm} = \oint_{c_{0}} \omega^{n}(Q)\omega^{m}(Q)e_{k}(Q).
\]

Notice that for a fixed \( k \) the pairs \((n,m)\) such that \( l_{k}^{nm} \neq 0 \) satisfy the inequality of the form \( C_{2} \leq m + n \leq C_{1} \) where \( C_{1}, C_{2} \) are constants which depend only on \( k \) and \( g \). Thanks to this property and to the normal ordering, the \( L_{k} \)'s are well defined on admissible representations. For instance, if \( m, n > 0 \) or \( m, n < -g \) then \( k \leq m + n \leq k + g \). For \( g = 0 \) this implies \( l_{k}^{nm} = e_{k}^{m+n} \) which leads to the usual definition of the Sugawara operator for \( g = 0 \) (see [6] and references therein).

We will consider the class of normal orderings which satisfy the following requirement:

\[
:x(n)y(m): = x(n)y(m), \quad \text{if} \quad n \leq 0 < m \quad \text{or} \quad n < -g \leq m.
\]

(3.10)

The normal ordering determines the cohomology class of the cocycle in Theorem 3.1 below. In all other respects the following doesn’t depend on the choice of the normal ordering.

**Theorem 3.1 ([17]).** Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra, either abelian or simple. Let \( 2k \) be the eigenvalue of its casimir in the adjoint representation and \( \hat{\mathfrak{g}} \) the corresponding affine Krichever–Novikov algebra. Let \( V \) be an admissible representation of \( \hat{\mathfrak{g}} \) in which the central element acts as \( c \cdot \text{id} \). If \( c + k \neq 0 \), then the normalized Sugawara
The second order Casimirs $L_k^* = -(c + k)^{-1} L_k$ define a representation of the Lie algebra $\hat{\mathcal{L}}$ which has a "geometrical" cocycle

$$\chi(e, f) = \frac{c \cdot \dim g}{(c + k)} \oint c_0 \left( \frac{1}{2} (e'' f - e f'') - R \cdot (e' f - e f') \right) dz,$$

where $e, f \in \mathcal{L}$, $R$ is a projective connection which has poles only at the points $P_\pm$.

For each $e = \sum \lambda_k e_k \in \mathcal{L}$ (a finite sum) introduce $T(e) = \sum \lambda_k L_k^*$. By Theorem 3.1 $e \mapsto T(e)$ is a representation of $\hat{\mathcal{L}}$. We call it the Sugawara representation.

In Section 4 below we will need the following statement:

**Lemma 3.3** \((17)\).

1. $[L_k, x(r)] = -(c + k) x(e_k A_r)$.
2. $[L_k, \dot{x}(Q)] = (c + k) e_k : \dot{x}(Q) :$

where $e \cdot \dot{x}(Q) := \sum_n x(n)(e \omega^n)(Q)$, $e A$ is the derivative of a function $A$, and $e \omega$ is the Lie derivative of a 1-form $\omega$ in the direction of a vector field $e$.

**Lemma 3.4.** The Sugawara operators of the elements of the subalgebra $\mathcal{L}_+^{(2)}$ annihilate the vacuum vectors of highest weight representations of $\hat{\mathfrak{g}}$.

**Proof.** By Subsection (a), $e_k \in \mathcal{L}_+^{(2)}$ if and only if $k \geq 1$. Let us consider the Sugawara operators $L_k$ for $k \geq 1$. We want to show that if $l_{km}^{mn} \neq 0$ then either $m \geq 1$ or $n \geq 1$. Provided it is true and taking into account the normal ordering, one finds an operator of representation of the subalgebra $\hat{\mathfrak{g}}_+$ in the second position of the term $u(m)u(n)$. Hence this term annihilates the vacuum.

Consider the relation (3.9) for $l_{km}^{mn}$. By (3.5) and (3.1) $\text{ord}_{p^+} \omega = -m - 1$, $\text{ord}_{p^+} \omega^n = -n - 1$, $\text{ord}_{p^+} e_k = k + 1$. Thus, $\text{ord}_{p^+} \omega^n \omega^m e_k = -m - n + k - 1$. To have a nonzero residue at the point $P_+$ the 1-form $\omega^m \omega^n e_k$ must have a pole there, at least. Therefore $-m - n + k - 1 \leq -1$. This implies $m + n \geq k \geq 1$ and hence either $m > 0$ or $n > 0$. \qed

### 4. Casimirs, semi-casimirs, and moduli spaces

In this section we give description of the second order casimirs for the Lie algebra $\hat{\mathfrak{g}}$. Let $C_2$ denote the space of these casimirs. We also introduce the semi-casimirs and establish their connection with the moduli space $\mathcal{M}_{g,2}^{(p)}$ and coinvariants.
(a). **The second order casimirs.** For any affine Kac–Moody algebra there is only one second order casimir — the sum of certain element of the Virasoro algebra and its Sugawara operator \([6]\). The main property of this operator is as follows: it commutes with all the operators of the representation of the affine algebra under consideration.

Based on this idea we will construct the second order casimirs for \(\hat{g}\) as operators of the form \(\Delta_e := \hat{e} - T(e)\), where \(e \in \mathcal{L}\), \(\hat{e}\) is the operator of a representation of the vector field \(e\), and \(T(e)\) is its Sugawara operator. The words “the second order” mean that the operators under consideration depend quadratically on the operators of representation of the Lie algebra \(\hat{g}\). Observe that even if we deal with the representation of \(\hat{gl}_{g,2}\), the Sugawara representation \(T\) corresponds to the restriction of the latter onto the subalgebra \(\hat{sl}_{g,2}\).

On the Riemann surface \(\Sigma\), consider the Lie algebra \(D^1_{\mathfrak{g}}\) of differential operators of the form \(e + xA, e \in \mathcal{L}, x \in \mathfrak{g}, A \in \mathcal{A}(\Sigma, P_\pm)\) (i.e., the algebra of Krichever–Novikov differential operators of order less than or equal to 1). As a linear space \(D^1_{\mathfrak{g}} = \mathcal{L} \oplus \mathfrak{g} \otimes \mathcal{A}(\Sigma, P_\pm)\). In particular, for \(\mathfrak{g} = \mathfrak{gl}(1)\) one has \(D^1_{\mathfrak{g}} = D^1\). The commutation relations between vector fields and currents in \(D^1_{\mathfrak{g}}\) are well known:

\[
[e, x \otimes A] = x \otimes (eA).
\]

Below, we will consider projective representations of \(D^1_{\mathfrak{g}}\) (projective \(D^1_{\mathfrak{g}}\)-modules). Such a representation is defined as a representation of some central extension \(\widehat{D}^1_{\mathfrak{g}}\) of \(D^1_{\mathfrak{g}}\). Assuming the action of the central element to be the identity operator, we call the cocycle of this central extension the *cocycle of the projective \(D^1_{\mathfrak{g}}\)-module (representation)*. As it is stated in Section 2(d), the cocycle of the fermion representation, while being restricted onto \(\mathfrak{g} \otimes \mathcal{A}(\Sigma, P_\pm)\), gives the multiple of the cocycle \((2.2)\). We call a projective \(D^1_{\mathfrak{g}}\)-module possessing this property *normalized*, and its cocycle as well. We call a projective representation of \(D^1_{\mathfrak{g}}\) *admissible* if its restrictions to \(\mathfrak{g} \otimes \mathcal{A}(\Sigma, P_\pm)\) and \(\mathcal{L}\) are admissible.

**Lemma 4.1.** For a semisimple \(\mathfrak{g}\) and an admissible normalized projective \(\widehat{D}^1_{\mathfrak{g}}\)-module \(V\) we have

1°. \([\hat{e}, x(A)] = x(eA)\) for any \(A \in \mathcal{A}, e \in \mathcal{L}\).

2°. \([\Delta_e, x(A)] = 0\) for any \(A \in \mathcal{A}\).

**Proof.** 1°. From (4.1) and the definition of \(\widehat{D}^1_{\mathfrak{g}}\)-module, it follows that

\[
[\hat{e}, x(A)] = x(eA) + \gamma(e, xA) \circ \text{id},
\]

where \(\gamma\) is a cocycle on \(D^1_{\mathfrak{g}}\).
For a semisimple $\mathfrak{g}$ each cocycle on $D^1_\mathfrak{g}$ satisfies
\begin{equation}
\gamma(e, xA) = 0 \quad \text{for any } e \in \mathcal{L}, A \in \mathcal{A}, x \in \mathfrak{g}.
\end{equation}
Indeed, by definition of a cocycle
\begin{equation}
\gamma(e, [xA, yB]) + \gamma(yB, [e, xA]) + \gamma(xA, [yB, e]) = 0
\end{equation}
for any $e \in \mathcal{L}, A, B \in \mathcal{A}, x, y \in \mathfrak{g}$. The latter is equivalent to
\begin{equation}
\gamma(e, [x, y]AB) + \gamma(yB, x(eA)) + \gamma(xA, -y(eB)) = 0.
\end{equation}
Take $B \equiv 1$ in the latter equality. Then $\gamma(yB, x(eA)) = \gamma(y, x(eA)) = 0$, since by the assumption of the Lemma $\gamma$ is a multiple of (2.2) and the latter vanishes if one of the arguments is a constant. Further on, $eB \equiv 0$, therefore $\gamma(xA, -y(eB)) = 0$. The first summand in (4.4) is equal to $\gamma(e, [x, y]A)$. Hence $\gamma(e, [x, y]A) = 0$ for all $e \in \mathcal{L}, A \in \mathcal{A}, x, y \in \mathfrak{g}$. If $\mathfrak{g}$ is semisimple it coincides with its derived algebra. Hence (4.3) follows from the latter equality.

2° follows immediately from Lemma 3.3(1) and from 1°. ☐

Lemma 4.1.2° was proved in [17] where our part 1° above appeared as a hypothesis.

**Lemma 4.2 (M. Schlichenmaier²).** For a commutative $\mathfrak{g}$ and an arbitrary admissible representation $V$ of the corresponding (Heisenberg type) affine algebra, we have
\begin{equation}
[\Delta_e, x(A)] = \gamma(e, A) \cdot id, \quad \text{for any } e \in \mathcal{L}, A \in \mathcal{A},
\end{equation}
where $\gamma$ is a cocycle on $D^1_\mathfrak{g}$.

*Proof.* The relation (4.2) is always true (but this time $\gamma$ can be a nontrivial cocycle). Comparing (4.2) with Lemma 3.3(1) proves the claim. ☐

Now, let us consider the typical case of a reductive algebra.

**Lemma 4.3.** For $\mathfrak{g} = \mathfrak{gl}(l)$ and $V$ and $e$ from Lemma 4.1, the following commutation relation holds:
\begin{equation}
[\hat{e}, x(A)] = x(eA) + \lambda(x)\gamma(e, A) \circ id,
\end{equation}
where $\gamma$ is a cocycle on $D^1_\mathfrak{g}$ and $\lambda(x) = l^{-1}\text{tr} x$.

*Proof.* Again the relation (4.2) is true. An arbitrary $x \in \mathfrak{g}$ can be represented as follows: $x = x_0 + \lambda(x)1_l$ where $x_0 \in \mathfrak{sl}(l)$, $\lambda \in \mathfrak{g}^*$, $1_l$ stays for the identity matrix of rank $l$. By Lemma 4.2, $\gamma(e, x_0A) = 0$. Hence $\gamma(e, xA) = \lambda(x)\gamma(e, 1_lA)$. Obviously the correspondence

²Private communication.
e + A \mapsto e + 1_A A$ is a Lie algebra homomorphism $D^1 \rightarrow D^1_\mathfrak{g}$. Hence $\gamma(e, 1_A A)$ defines a cocycle on $D^1$.

**Lemma 4.4.** Let $\mathfrak{g} = \mathfrak{gl}(l)$ and let a representation of the corresponding algebra $\hat{\mathfrak{g}}$ be admissible. Then for arbitrary $e \in \mathcal{L}$, $x \in \mathfrak{g}$, $A \in \mathcal{A}$ we have

$$[\Delta_e, xA] = \lambda(x) \gamma(e, A) \circ id,$$

where $\gamma$ is a cocycle on $D^1$ and $\lambda(x) = l^{-1} \text{tr} x$.

**Proof.** By Lemma 3.3 one has $[T(e), x(A)] = x(e A)$. Now the lemma follows from comparison of the latter relation with Lemma 4.3. \qed

**Definition 4.1.** The operators $\Delta_e$ which commute with all the operators of representations of $\hat{\mathfrak{g}}$ and $A$ in $V$ are called (second order) **casimirs** of the Lie algebra $\hat{\mathfrak{g}}$ in the representation $V$.

**Remark 4.1.** The requirement of commutativity of $\Delta_e$'s with $A$ has a different meaning for $\hat{\mathfrak{sl}}_{g,2}$ and for $\hat{\mathfrak{gl}}_{g,2}$. Consider fermion representations as a typical example. The space of monomials of a given charge is generically irreducible with respect to $\hat{\mathfrak{gl}}_{g,2}$ (see [6] for the genus 0 case) but reducible with respect to $\hat{\mathfrak{sl}}_{g,2}$. Indeed, the elements of the form $d(\lambda) A \in \hat{\mathfrak{g}}$ (here $d(\lambda) = \text{diag}(\lambda, \ldots, \lambda)$, where $\lambda \in \mathbb{C}$) belong to the center of $\hat{\mathfrak{g}}$. They commute with all the operators of $\hat{\mathfrak{sl}}_{g,2}$ and hence make reducible its fermion representation. In this case the requirement of commutativity of $\Delta_e$'s with $A$ means that $\Delta_e$'s are well defined on the $\hat{\mathfrak{sl}}_{g,2}$-subrepresentations of the fermion representation.

The following lemma is an easy corollary of Definition 4.1 and Lemma 4.4.

**Lemma 4.5.** $\Delta_e$ is a casimir for $\hat{\mathfrak{g}}$ (in a certain representation) if and only if $\gamma(e, A) = 0$ for all $A \in \mathcal{A}$, where $\gamma$ is the cocycle on $D^1$ which corresponds to the representation in question.

**Remark 4.2.** The following statement shows that being a casimir is a very restrictive condition: **given a cocycle $\gamma$ on $D^1$ the vector fields which satisfy the condition of Lemma 4.5 form a Lie subalgebra in $\mathcal{L}$.** To prove this suppose $e_1, e_2 \in \mathcal{L}$ to be such vector fields that $\gamma(e_1, A) = \gamma(e_2, A) = 0$ for all $A \in \mathcal{A}$. By definition of a cocycle, $\gamma([e_1, e_2], A) + \gamma(e_2, [e_1, A]) - \gamma(e_1, [e_2, A]) = 0$ for any two such vector fields and arbitrary $A$. Two terms in the latter relation vanish because $[e_1, A], [e_2, A]$ are functions, hence $\gamma([e_1, e_2], A) = 0$.

Denote the cocycle of the representation $V$ by $\gamma_V$. According to the proof of Lemma 4.2, $\gamma_V$ defines the cocycle $\gamma_V(e, 1_A A)$ on $D^1$. We will keep the notation $\gamma_V$ for this cocycle as well.
(b). **Casimirs of fermion representations.** Let us show that the fermion representations satisfy all the requirements of the last subsection. The main point is that each fermion representation is an admissible projective $\mathcal{D}^1$-module.

It was shown in Sections 2, 3 that a fermion representation is a projective module over $\mathcal{A}$, $\mathfrak{g} \otimes \mathcal{A}$ and $\mathcal{L}$. The similar arguments show that it is also a projective module over both $\mathcal{D}^1$ and $\mathcal{D}_\mathfrak{g}^1$. Again let the fermion representation be given by a holomorphic bundle $F$ with a meromorphic (therefore flat) connection $\nabla$ which has logarithmic singularities at the points $P_{\pm}$ (see Section 3(b)), and an irreducible representation $\tau$ of $\mathfrak{g}$. By flatness, $[\nabla_e, \nabla_f] - \nabla_{[e,f]} = 0$ for any $e, f \in \mathcal{L}$. Hence, $\nabla_{[e,f]} = [\nabla_e, \nabla_f]$ and $\nabla$ defines a representation of $\mathcal{L}$ in the space of holomorphic sections of $F$. By definition of a connection, for each holomorphic section $s$ of the bundle, each $e \in \mathcal{L}$ and each $A \in \mathcal{A}$ we have $\nabla_e(As) = (eA)s + A\nabla_e s$ where $eA$ is defined by the relation (4.1). In other words, $[\nabla_e, A] = eA$, i.e., the mapping $e + A \rightarrow \nabla_e + A$ gives rise to a representation of $\mathcal{D}^1$. Being conjugated by the matrix $\Psi$ (see Sections 2(b), 3(b)), this representation can be transferred to the space $F_{KN}$ of Krichever–Novikov vector-valued functions. We will write down the action of the element $e \in \mathcal{L}$ on $\psi \in F_{KN}$ as $e\psi$. The space $V_F$, as introduced in Sections 2(c), 2(d), is actually spanned by semi-infinite monomials over $F^{\tau}_{KN} \otimes V_{\tau}$ with the following $\mathcal{D}^1_{\mathfrak{g}}$-action:

$$ (xA)(\psi \otimes v) = A\psi \otimes xv \quad \text{and} \quad (e(\psi \otimes v) = e\psi \otimes v, $$

where $V_\tau$ is the space of the representation $\tau$, $x \in \mathfrak{g}$, $A \in \mathcal{A}$, $\psi \in F_{KN}$ and $v \in V_\tau$. Applying the procedure described above (Sections 2, 3) of lifting representations from the space $F_{KN}$ to the space $V_F$, we obtain projective representations of the algebras $\mathcal{D}^1$, $\mathcal{D}_\mathfrak{g}^1$. It is easy to show that these projective representations are admissible. So in the case of fermion representations we are in the set-up of the previous subsection. In particular, we can study cocycles on $\mathcal{D}^1$ and $\mathcal{D}_\mathfrak{g}^1$ arising from the fermion representations.

Let us consider the cocycles on $\mathcal{D}^1$ in fermion representations in more detail. Take $|0\rangle = \psi_M \wedge \psi_{M+1} \wedge \ldots$ for a vacuum. Let

$$ (4.6) \quad \nabla_{e_{n}} = z^{k} \left( \partial + \omega_{1} \frac{dz}{z} + O(1) dz \right) $$

be the local behavior of $\nabla$ at $P_{+}$. For any $N \in \mathbb{Z}$, let $n(N)$ and $j(N)$ denote the first two components of the triple $(n, j, i)$ such that $N = N(n, j, i)$ (Section 2(b)). Let $\omega_j$ be the $j$-th diagonal element of $\omega_{-1}$. Our main observation is the following
Lemma 4.6. For a fermion representation of \(D^1_\mathfrak{g}\), assume that \(M < 0\), \(\nabla\) satisfies (4.6) and \(\sum_{N=M}^{-1} \omega_j(N) \notin \mathbb{Z}\). The cocycle \(\gamma\) of such a representation possesses the following properties:

1°. \(\gamma(A_{-k}, e_k) \neq 0\), for any \(k \in \mathbb{Z}\), \(k \neq 0\);
2°. \(\gamma(A_{-k}, e_m) = 0\), for any \(m > k\);
3°. \(\gamma(A_0, e_m) = 0\), for any \(m \in \mathbb{Z}\).

Proof. \(\gamma(A_{-m}, e_m) = A_{-m} \circ \nabla_{e_m} - \nabla_{e_m} \circ A_{-m} - e_m A_{-m}\). This expression can be evaluated on the vacuum vector of the representation.

1°. For \(m = k\) one of the summands \(A_{-m} \circ \nabla_{e_k}\) or \(\nabla_{e_k} \circ A_{-m}\) annihilates the vacuum. For example, for \(k > 0\) the first summand does. Under assumptions of the lemma the \(-\nabla_{e_k} \circ A_{-m} - e_k A_{-m}\) doesn’t annihilate the vacuum vector, as follows from direct calculations with power series, for example, at \(P_+\).

Locally, we have \(A_{-m} = z^{-k}(1 + O(z))\), \(e_k = z^{k+1}(1 + O(z)) \frac{\partial}{\partial z}\). Let us act by these objects on a Krichever–Novikov vector-valued function \(\psi_N\) of degree \(n\). While considering the orders at the points \(P_{\pm}\) one can forget for simplicity about the conjugation of the vector field by the matrix \(\Psi\).

By (4.6) we have

\[
\nabla_{e_k} \circ A_{-k} \psi_N = \left(z^{k+1} \frac{\partial}{\partial z} + z^k \omega_{-1}\right) \left(z^{-k}z^n(1 + O(z)) = (n - k + \omega_{-1})z^n(1 + O(z))\right).
\]

For the other term we have \(e_k A_{-m} = (-k)(1 + O(z))\). Hence,

\[
(-\nabla_{e_k} \circ A_{-m} - e_k A_{-m})\psi_N = (2k - n + \omega_j)\psi_N + \cdots,
\]

where dots denote the terms of higher degree, \(n = n(N), j = j(N)\).

By regularization

\[
(4.7) \quad (-\nabla_{e_k} \circ A_{-m} - e_k A_{-m}) |0\rangle = \left(\sum_{N=M}^{-1} (2k - n(N)) + \omega_j(N)\right) |0\rangle.
\]

For \(k < 0\) the term \(A_{-m} \circ \nabla_{e_k}\) enters the game instead of \(\nabla_{e_k} \circ A_{-m}\). This gives rise to

\[
(4.8) \quad (A_{-m} \circ \nabla_{e_k} - e_k A_{-m}) |0\rangle = \left(\sum_{N=M}^{-1} (k + n(N)) + \omega_j(N)\right) |0\rangle.
\]

If \(\sum_{N=M}^{-1} \omega_j(N) \notin \mathbb{Z}\) then both right hand sides of (4.7), (4.8) do not vanish for any \(k \in \mathbb{Z}\). Thus 1° is proven.

The proof of 2°, 3° is similar. \(\square\)

By Definition 4.1 and Lemma 4.4, \(\Delta(e)\) is a casimir if and only if

\[
(4.9) \quad \gamma(A_k, e) = 0, \quad \text{for any} \quad k \in \mathbb{Z}.
\]
We will seek for the solutions to (4.9) in the form
\[ e = \sum_{m \geq m_0} a_m e_m, \] (4.10)
where \( m_0 \in \mathbb{Z} \).

**Lemma 4.7.** For each fermion representation such that its cocycle \( \gamma \) satisfies conditions of Lemma 4.6, equation (4.9) has one-dimensional space of solutions of the form (4.10). For the generator of this space one has \( m_0 = 0 \).

**Proof.** For the vector fields of the form (4.10) the relations (4.9) read as
\[ \sum_{m \geq m_0} a_m \gamma(A_k, e_m) = 0, \quad \text{for all } k \in \mathbb{Z}, k \neq 0. \] (4.11)
This is an infinite system of linear equations for \( a_m \)'s. By Lemma 4.6 it has a triangular matrix, hence, for each \( k \neq 0 \), \( a_k \) can be expressed via the \( a_m \)'s with \( m < k \) from the \( k \)-th equation. On the contrary, for \( k = 0 \) we have \( \gamma(A_0, e_m) = 0 \) for any \( m \in \mathbb{Z} \), because \( A_0 = \text{const} \) (Lemma 4.6). Thus \( a_0 \) is an independent constant. For \( m < 0 \) one has \( a_m = 0 \) because the latter is true for sufficiently large (negative) values of \( m \). For \( m > 0 \) \( a_m \)'s can be expressed via \( a_0 \). This proves the claim. \( \square \)

Bringing together Lemmas 4.5, 4.7 we obtain the following theorem.

**Theorem 4.1.** For each fermion representation such that its cocycle \( \gamma \) satisfies conditions of Lemma 4.6 there exists exactly one (up to a scalar factor) casimir. The corresponding vector field has a simple zero at \( P_\pm \).

Let us normalize a casimir by the condition \( a_0 = 1 \). Then the eigenvalue of this casimir equals to the one of \( \Delta(e_0) \), because by Lemmas 3.2, 3.4 \( \Delta(L_e^{(2)}) \) annihilates the vacuum.

For an arbitrary vector field of the form (4.10) we can claim that both the action of the vector field and its Sugawara operator are well defined. To see the first, let us recall that for each \( v \in V \) and \( n \) sufficiently large, \( e_n v = 0 \). For Sugawara operators one has \( L_e = \sum_{m,n} l_e^{m,n} :u_m u_n: \), where \( l_e^{m,n} = \text{res}_{P_\pm}(\omega^m \omega^n e) \). But for given \( m,n \) \( \text{res}_{P_\pm}(\omega^m \omega^n e) \neq 0 \) only for a finite number of \( k \)'s. Hence the coefficients \( l_e^{m,n} \) are also well defined.

(c). **Semi-casimirs and moduli spaces \( \mathcal{M}^{(p)}_{g,2} \).** From Lemma 4.5 and the previous subsection one can conclude that the condition for \( \Delta_e \) to commute with all the elements of the algebra \( \mathcal{A} \) imposes very strong restrictions on \( e \). Here we consider weaker conditions.
Definition 4.2. We call an operator of the form $\Delta_e$ a semi-casimir if $[\Delta_e, A_k] = 0$ for each $k < 0$.

Let $\tilde{\mathcal{A}}_- \subset \mathcal{A}$ be the subspace spanned by all $A_k$, $k < 0$. Observe that, as a Lie algebra, $\tilde{\mathcal{A}}_-$ is a subalgebra of $\mathcal{A}$ (while, as an associative algebra, it is not). For $g = \mathfrak{gl}(l)$, $\tilde{\mathcal{A}}_-$ can be also considered as a Lie subalgebra of $g \otimes \mathcal{A}$ (but not of $\hat{g}$) by interpreting elements of $\mathcal{A}$ as scalar matrices. As a subalgebra of $g \otimes \mathcal{A}$, $\tilde{\mathcal{A}}_-$ commutes with $\mathfrak{sl}(l) \otimes \mathcal{A}_-$. Hence $g_r = \mathfrak{sl}(l) \otimes \mathcal{A}_- \oplus \tilde{\mathcal{A}}_-$ is a Lie subalgebra of $g \otimes \mathcal{A}$. We call $g_r$ a regular subalgebra (note that a different choice of $g_r$ is possible, see [19]).

Define coinvariants of the regular subalgebra in a $\hat{g}$-module $V$ as the quotient space $V/U(g_r)V$, where $U(g_r)$ denotes the subalgebra of the universal enveloping algebra of $\hat{g}$ corresponding to the subspace $g_r$ (in particular, it does not contain the unit).

By Definition 4.2, $\Delta_e$ is well defined on the space of coinvariants of $g_r$. Thus, this kind of condition is important for conformal field theory. On the other hand, Remark 4.2 is not applicable to semi-casimirs, because the corresponding vector fields do not form a subalgebra but only a linear subspace.

For a vector field $e$ defining a semi-casimir one has

\begin{equation}
\gamma(A_{-k}, e) = 0, \quad \text{for any } k \in \mathbb{Z}, k > 0,
\end{equation}

instead of (4.9). For semi-casimirs one has system of equations similar to (4.11) but only for $k > 0$. Therefore, the coefficients $a_m$ with $m \leq 0$ appear to be independent and all the others can be expressed via them.

Let $\tilde{\mathcal{L}}_- \subset \mathcal{L}$ be the subspace spanned by $\{e_k: k \leq 0\}$. Introduce the map $\Gamma: \tilde{\mathcal{L}}_- \mapsto \mathcal{L}$ as follows: take $e \in \tilde{\mathcal{L}}_-$ and represent it in the form (4.10); then substitute the corresponding $a_m$ ($m \leq 0$) into (4.11) and calculate $a_m$, $m > 0$. Denote by $\Gamma(e)$ the vector field which corresponds to the full set of $a_m$’s. We claim the following.

Lemma 4.8. The space of semi-casimirs coincides with $\Delta(\Gamma(\tilde{\mathcal{L}}_-))$. It is spanned by the elements $\Delta(\Gamma(e_k))$, where $k \leq 0$.

As it was mentioned above, semi-casimirs are well defined on the space of coinvariants of the subalgebra $g_r$. For $e \in \tilde{\mathcal{L}}_-$, let $\overline{\Delta}(e)$ be the operator induced by $\Delta(\Gamma(e))$ on coinvariants. The map $\overline{\Delta}$ is defined on $\tilde{\mathcal{L}}_-$ and by Lemma 4.8 its image is the space $C_2^s$ of semi-casimirs considered as operators on the space of coinvariants.

Our next step is to show that only a finite number of basis semi-casimirs are nonzero on coinvariants and, for a proper natural $p$, to
establish the correspondence between the tangent space to $\mathcal{M}_{g,2}^{(p)}$ and the space of semi-casimirs (considered on coinvariants).

**Lemma 4.9.** For a fermion representation $V$ there exists such $p \in \mathbb{Z}_+$ that $\mathcal{L}_{g_1}^{(p)} \subseteq \ker \Delta$.

**Proof.** For a fermion representation of certain charge, the space of coinvariants of $\mathfrak{g}_1$ is finite-dimensional. It is a known fact for affine Kac–Moody algebras. It is easy to reduce the statement in the almost graded case in question to the known case just by considering the associated graded objects.

Consider the decomposition

$$V = \bigoplus_{n \leq 0} V_n$$

defining the structure of an almost graded $\mathcal{D}_{g_1}^1$-module on $V$. By finiteness of coinvariants there exists such $s$ that

$$\bigoplus_{n \leq s} V_n \subseteq \mathcal{U}(\mathfrak{g}_r)V.$$

Since $V$ is an almost graded $\mathcal{L}$-module there exists such $\nu \in \mathbb{Z}_+$ that $\hat{e}_k V_n \subseteq \bigoplus_{m \leq k+n+\nu} V_m$. Obviously, $k + n + \nu < s$ for any $n \leq 0$ if $k < s - \nu$. Hence, $\hat{e}_k V \subseteq \mathcal{U}(\mathfrak{g}_r)V$ for any $k < s - \nu$.

Let us find such $k' \in \mathbb{Z}$ that $T(e_k)V \subseteq \mathcal{U}(\mathfrak{g}_r)V$ for any $k < k'$. Since $V$ is an almost graded $(\mathfrak{g} \otimes \mathcal{A})$-module there exists such $\nu' \in \mathbb{Z}_+$ that $u(i)V_n \subseteq \bigoplus_{m \leq i+n+\nu'} V_m$ for any $u \in \mathfrak{g}, i \in \mathbb{Z}$. Hence, $u^\mu(i)u_\mu(j)V_n \subseteq \bigoplus_{m \leq i+j+n+2\nu'} V_m$. The term $u^\mu(i)u_\mu(j)$: occurs in the series (3.9) for $T(e_k)$ if $l^{\nu \mu}_{ij} \neq 0$. The latter holds for

$$i + j \leq k + g.$$  (4.13)

To obtain this observe that, assuming $g > 1$, for the basis elements $A_m \in \mathcal{A}, \omega_m$ (the 1-form), $e_m \in \mathcal{L}$, one has the following behaviour at $P_-$: $A_m = O(z^{-m-g}), \omega_m = O(z^{m+g-1})dz, e_m = O(z^{m-5g+1})\partial/\partial z$. These relations also hold for $g = 1$ if one sets $\omega_m = A_{1-m}dz, e_m = A_{m+1}\partial/\partial z$, and also for $g = 0$. By (3.9) $l^{\nu \mu}_{ij} = -\text{res}_{P_-} e_k \omega^i \omega^j$. At the point $P_-$ one has $e_k \omega^i \omega^j = O(z^{-k+i+j-g-1})$. If $l^{\nu \mu}_{ij} \neq 0$ then $-k + i + j - g - 1 \leq -1$, which implies (4.13). Therefore,

$$T(e_k)V_n \subseteq \bigoplus_{m \leq n+k+g+2\nu'} V_m.$$

Hence, we can take $k' = s - g - 2\nu'$.

Obviously, for $p = \max(|\nu - s, |k'|)$ and $e \in \mathcal{L}_{g_1}^{(p)}$ we have $\Delta(e)V \subseteq \mathcal{U}(\mathfrak{g}_r)V$, thus $e \in \ker \Delta$. □
Let $\mathcal{M}_{g,2}^{(p)}$ be the moduli space of curves of genus $g$ with two marked points $P_\pm$, fixed 1-jet of local coordinate at $P_+$ and fixed $p$-jet of local coordinate at $P_-$. There is a canonical mapping $\theta: \mathcal{L} \mapsto T_{\Sigma} \mathcal{M}_{g,2}^{(p)}$. This mapping goes back to [15]. The cohomological and geometrical versions of this mapping are given in [18] and [4], respectively. Let $\tilde{\theta}$ denote the restriction of $\theta$ onto the subspace $\tilde{\mathcal{L}}$. Let $V$ be a fermion representation of $D_{1g}$ and $\gamma_V$ be the cocycle of this representation. Let also $C_{2}^{s} = C_{2}^{s}(V)$ denote the second order semi-casimirs of $\hat{g}$ in the representation $V$. We assume semi-casimirs to be restricted onto coinvariants.

**Theorem 4.2.** Take $p$ as in Lemma 4.9.

1°. The mapping $\tilde{\theta}: \tilde{\mathcal{L}} \mapsto T_{\Sigma} \mathcal{M}_{g,2}^{(p-1)}$ is surjective and $\ker \tilde{\theta} = \mathcal{L}_{-}^{(p)}$.

2°. For such $V$ that $\gamma_V$ satisfies the conditions of Lemma 4.6, the mapping $\Delta: \tilde{\mathcal{L}} \mapsto C_{2}^{s}(V)$ is surjective and $\mathcal{L}_{-}^{(p)} \subseteq \ker \Delta$.

3°. The mapping $\Delta \circ \tilde{\theta}^{-1}: T_{\Sigma} \mathcal{M}_{g,2}^{(p-1)} \mapsto C_{2}^{s}(V)$ is surjective.

**Proof.** 1° is an easy corollary of the known results just mentioned. By Proposition 4.4 of [18], the mapping $\theta: \mathcal{L} \mapsto T_{\Sigma} \mathcal{M}_{g,2}^{(p-1)}$ is surjective and $\ker \theta = \mathcal{L}_{+}^{(2)} \oplus \mathcal{L}_{-}^{(p)}$. The claim follows due to the decomposition $\mathcal{L} = \mathcal{L}_{+}^{(2)} \oplus \tilde{\mathcal{L}}$.

2° follows from Lemmas 4.8, 4.9. Part 3° is an immediate consequence of 1° and 2°. \(\square\)

5. Casimirs in terms of “geometrical” cocycles

The goal of this section is to repeat some of the results of the previous section in another set-up. A 2-cocycle on $\mathcal{D}_{1}$ is called local if there exists $L \in \mathbb{Z}$ such that $\gamma(A_i, A_j) = \gamma(e_i, e_j) = \gamma(A_i, e_j) = 0$ for any $i, j \in \mathbb{Z}$ such that $|i - j| > L$. The local cocycles on $\mathcal{D}_{1}$ have the following description.

**Lemma 5.1.** Any local cocycle on $\mathcal{D}_{1}$ in local coordinates is of the form

$$\gamma(f_1 \partial + g_1, f_2 \partial + g_2) = \text{res}_{P_+} \left[ a_1 (f_1 f''_2 - f''_1 f_2) + R(f_1 f'_2 - f'_1 f_2) + a_2 (f_1 g''_2 - f''_1 g_2) + T(f_1 g'_2 - f'_1 g_2) + a_3 g_1 dg_2 \right],$$

where $a_1, a_2, a_3 \in \mathbb{C}$.

For $g = 0$ this statement and its proof are contained in the proof of the Proposition 2.1 form [1]. For $g > 1$ the author doesn’t know any published proof of Lemma 5.1 (see Introduction). Nevertheless the statement is wellknown; it gives rise to another approach to (semi-)casimirs. Here, we consider this approach.
Lemma 5.2. For any local cocycle $\gamma$ on $\mathcal{D}^1$, the following relation in local coordinates holds:

$$\gamma(e, A) = \text{res}_{P_\gamma}(afA'' + TfA'),$$

where $e = f\partial$, $a \in \mathbb{C}$, the behavior of $T$ under an arbitrary change of local coordinates can be described by any of the following three equivalent relations:

1. $a^{-1}T(u) = a^{-1}T(z)u^{-1}_z + u_z^2u_z^{-2}$;
2. $a^{-1}T(u)du = a^{-1}T(z)dz + d\ln u_z$;
3. there exists a $v$ such that $T(z) = a\frac{\partial}{\partial z}\ln v(z)$,

where $u, z$ are local parameters, $u_z = u'(z)$, $v\partial \in \mathcal{L}$ is the local representation of a vector field.

Proof. The expression for $\gamma$ can be obtained immediately by taking $f_1 = f$, $g_1 = 0$, $f_2 = 0$, $g_2 = A$ in Lemma 5.1.

Now, let us find the transformation law that a pair $a,T$ must satisfy in order to make the expression $\gamma_{a,T}(e, A) = \text{res}_{P_\gamma}(afA'' + TfA')$ independent of the choice of local coordinates. This independence is obviously equivalent to the requirement that $\Omega_{a,T}(A) := aA'' + TA'$ be a quadratic differential for each $A \in \mathcal{A}$. It is easy to show that the desired transformation law can be written in the three equivalent forms above. For example let us briefly outline the implication $2^o \Rightarrow 3^o$. By integration of both parts of the equality $2^o$ from any fixed point to the current point $P$ one obtains $a^{-1}\int^P T(u)du = a^{-1}\int^P T(z)dz + \ln u_z$. By exponentiating, the latter equality implies $\exp(a^{-1}\int^P T(u)du) = \exp(a^{-1}\int^P T(z)dz)u_z$. Now it is easy to notice that $v(z) = \exp(a^{-1}\int^P T(z)dz)$ transforms like a vector field. \qed

Remark 5.1. Part $3^o$ of Lemma 5.2 makes sure that the quantities $T$ with the required transformation law do exist. One can take an arbitrary vector field from $\mathcal{L}$ and construct $T$ as prescribed by the relation in part $3^o$.

Using Lemma 5.2, Lemma 4.4 can be made more precise as follows.

Lemma 5.3. Let $\mathfrak{g} = \mathfrak{gl}(l)$ and suppose the representation of the corresponding algebra $\hat{\mathfrak{g}}$ is admissible. Then for arbitrary $e \in \mathcal{L}$, $x \in \mathfrak{g}$, $A \in \mathcal{A}$ one has

$$[\Delta_e, xA] = \lambda(x)\gamma(e, A) \circ \text{id},$$

where in local coordinates for $e = f\partial$ the following relation holds:

$$\gamma(e, A) = \text{res}_{P_\gamma}(afA'' + TfA') (a \in \mathbb{C})$$

and $T$ satisfies the conditions of Lemma 5.2.
From Lemmas 5.6, 5.3 one can see that only the cocycles of the form
\[ \gamma_a, T(f_1\partial + g_1, f_2\partial + g_2) = \text{res}_{P_+}[a(f_1g_2'' - f_2g_1'') + T(f_1g_2' - f_2g_1')], \quad a \in \mathbb{C}^x, \]
are responsible for commutativity of the operators \( \Delta_e \) and \( x(A) \).

By Lemma 5.2 there exist \( T_V \) and \( a \in \mathbb{C} \) such that \( \gamma(e, A) = \text{res}_{P_+}[aA'' + T_V A'] \) and \( aA'' + T_V A' \) is a quadratic differential for each \( A \in A \). Let \( \Omega_V \) denote the linear space of all quadratic differentials of this form. Introduce also the notation \( \Omega^{(2)} \) for the space of all meromorphic quadratic differentials and \( \langle \cdot, \cdot \rangle \) for the natural pairing between vector fields and quadratic differentials:
\[ \langle e, \Omega \rangle := \text{res}_{P_+}[e \Omega] \quad (e \in L, \Omega \in \Omega^{(2)}). \]

**Theorem 5.1.** \( \Delta_e \) is a casimir of \( \hat{\mathfrak{g}} \) in the representation \( V \) if and only if \( \langle e, \Omega \rangle = 0 \) for each \( \Omega \in \Omega_V \).

**Proof.** By definition of \( \langle \cdot, \cdot \rangle \), Lemma 5.3 reads as
\[ [\Delta_e, xA] = \lambda(x) \langle e, \Omega \rangle \circ \text{id}, \]
where \( \Omega = aA'' + T_V A' \). It can be easily extracted from the proof of Lemma 5.3 that \( \lambda(1) = 1 \), hence,
\[ [\Delta_e, A] = \langle e, \Omega \rangle \circ \text{id}. \]

These two relations mean that \( \Delta_e \) commute with all the elements of the form \( A \) and \( xA \) \( (x \in \mathfrak{g}, A \in A) \) iff \( \langle e, \Omega \rangle = 0 \) for each \( \Omega = aA'' + T_V A' \). If \( A \) runs over \( A \) then \( \Omega \) runs over \( \Omega_V \). This proves the theorem. \( \square \)

Let \( \Omega^+_V := \{ e \in L : \langle e, \Omega \rangle = 0, \forall \Omega \in \Omega_V \} \).

**Corollary 5.1.** \( C_2 \cong \Omega^+_V / (\Omega^+_V \cap \mathcal{L}^{(2)}) \).

This follows from Theorem 5.1 and Lemmas 3.2, 3.3.

(a). **Description of casimirs.** In a local coordinate \( z \) on \( \Sigma \) a vector field \( e \) can be represented in the form \( e = E\partial \) where \( E = E(z) \) is a local function, \( \partial = \frac{\partial}{\partial z} \). In a generic situation \( T \) can be chosen to have simple poles at the points \( P_\pm \) (Lemma 5.2.3). Example 5.1 below shows that this is the case also for those \( aT'' \)’s which come from fermion representations. Assume that a representation \( V \) is fixed and \( T = T_V \).

By Theorem 5.1 the condition for \( \Delta_e \) to be a casimir reads
\[ \oint_{c_0} E(aA'' + TA')dz = 0, \quad \text{for any } A \in A. \]

Integrating “by parts” and taking into account the arbitrariness of \( A \) one obtains the differential equation
\[ aE'' - (ET)' = 0. \]
We will seek the solutions to (5.2) of the form
\[ E = \sum_{n \geq N} a_n E_n, \]  
where \( e_n = E_n \frac{\partial}{\partial z} \) near the point \( P_+ \).

**Lemma 5.4.** For a generic \( T \) such that \( T(z) = O(z^{-1}) \) at the point \( P_+ \) the equation (5.2) has the one-dimensional space of solutions of the form (5.3).

**Proof.** The proof is nothing but expanding equation (5.2) in power series in the neighbourhood of the point \( P_+ \). Suppose for simplicity that \( \text{ord}_{P_+} e = -1 \). By assumptions we have \( E = \epsilon_{-1} z^{-1} + \epsilon_0 + \epsilon_1 z + \cdots \), \( T = \tau_{-1} z^{-1} + \tau_0 + \tau_1 z + \cdots \). Hence, for the power series the relation (5.2) reads as
\[
-2\epsilon_{-1}(1 + \tau_{-1}) = 0, \\
\epsilon_0 \tau_{-1} + \epsilon_{-1} \tau_0 = 0, \\
\epsilon_1 \tau_0 - (2 - \tau_{-1}) \epsilon_2 = 0, \\
\ldots . . . . . . . . \]
For a generic \( T \) one has \( 1 + \tau_{-1} \neq 0 \), hence the first relation implies \( \epsilon_{-1} = 0 \). Similarly \( \epsilon_0 = 0 \). From the third relation we obtain \( \epsilon_2 = \frac{\tau_0}{2 - \tau_{-1}} \epsilon_1 \). So we have exactly one independent constant \( \epsilon_1 \). All the remaining relations express \( \epsilon_k \)'s \( (k > 1) \) via \( \epsilon \)'s with smaller numbers. This remains true if \( \text{ord}_{P_+} e < -1 \). \( \square \)

The following theorem is an immediate consequence of Lemma 5.4.

**Theorem 5.2.** For a generic \( T \) such that \( T(z) = O(z^{-1}) \) at the point \( P_+ \), there is only one (up to a scalar factor) second order casimir. This casimir corresponds to the vector field which behaves at the point \( P_+ \) as follows: \( e(z) = z(1 + O(z)) \frac{\partial}{\partial z} \).

(b). **Description of semi-casimirs.** By Theorem 5.1 and in analogy with (4.12), for a vector field which defines a semi-casimir one has
\[ \text{res}_{P_+}(aE'' - (ET)')A dz = 0, \quad \text{for any } A \in \tilde{A}_-. \]
Let us suppose \( a = 1 \) for simplicity and take \( E'' - (ET)' = \sum a_i z^i \) at the point \( P_+ \) (the sum is finite from the left). We deal only with local expansions of the objects at the point \( P_+ \) here. For the element \( A_{-1} = \alpha_{-1} z^{-1} + \alpha_0 + \cdots \) of the highest degree in \( \tilde{A}_- \), the relation (4.12) gives \( \beta_{-1} a_0 + \beta_0 a_{-1} \) which enables one to express \( a_0 \) via \( a_{-1} \). Similarly for the next basis element \( A_{-2} \in \tilde{A}_- \), (4.12) expresses \( a_1 \) via \( a \)'s with
the smaller numbers. Thus the coefficients \(a_i, i < 0\) are independent and all the others can be expressed via them.

We have the following relations for the semi-casimirs instead (5.4):

\[
-2\epsilon_{-1}(1 + \tau_{-1}) = a_{-3}, \\
\epsilon_0\tau_{-1} + \epsilon_{-1}\tau_0 = a_{-2}, \\
\epsilon_{-1}\tau_2 + \epsilon_0\tau_1 + \epsilon_1\tau_0 + \epsilon_2\tau_{-1} - 2\epsilon_2 = a_0,
\]

The first two relations express \(\epsilon_{-1}, \epsilon_0\) via independent constants \(a_{-3}, a_{-2}\). The third relation expresses \(\epsilon_2\) via \(\epsilon_1\). The other relations express \(\epsilon_i\)'s via the \(\epsilon\)'s with smaller numbers \((i \geq 2)\). So the \(\epsilon_i\)'s are independent if and only if \(i \leq 1\). Thus we have obtained Lemma 4.8 again.

The following example shows that highest weight representations which satisfy the conditions of Theorem 5.3 actually exist.

**Example 5.1.** Consider the fermion representation which corresponds to a generic 2-dimensional bundle on an elliptic curve. We are going to show that the cocycle of this representation corresponds to the affine connection \(T\) which has the pole of order 1 at the point \(P_+\).

Let \(\gamma\) stand for this cocycle. First of all let us show that if \(\text{ord}_{P_+} T < -1\) then there exists \(m \geq 0\) such that \(\gamma(A_1, e_m) \neq 0\). Suppose \(\text{ord}_{P_+} T = -m - 2, m \geq 0\). Since \(\text{ord}_{P_+} e_m = m + 1\) and both \(A''_1\) and \(A'_1\) are holomorphic \((A'_1(P_+) \neq 0\) for a generic situation), one has \((A''_1 + TA'_1)e_m = O(z^{-1})dz\). Hence, for a generic situation \(\text{res}(A''_1 + TA'_1)e_m \neq 0\).

But it is easy to show for a highest weight representation that \(\gamma(A_1, e_m) = 0\) for each \(m \geq 0\). One has \([A_1, e_m] = -e_mA_1 + \gamma(A_1, e_m)\). For an elliptic curve one can take \(e_m = A_{m+1}\frac{\partial}{\partial z}\). Hence, \(\gamma(A_1, e_m) = -[A_1, e_m] - e_mA_1 = e_m\circ A_1 - A_1\circ e_m - A_{m+1}\frac{\partial A_1}{\partial z}\). Apply both parts of the latter equality to the vacuum. Then one has \(A_1|0\rangle = 0, e_m|0\rangle = \lambda|0\rangle\) where \(\lambda \in \mathbb{C}\). Evidently \(A_{m+1}\frac{\partial A_1}{\partial z}\) is an element of the subalgebra \(A_+\). Hence, \(\gamma(A_1, e_m) = 0\). What we have obtained is

\[
\text{ord}_{P_+} T \geq -1.
\]

Therefore we have to decide which of the two possibilities takes place: \(T\) is regular at the point \(P_+\) or \(T\) has a pole of order 1. It is easy to check that in the first case \(\text{res}_{P_+}(A''_1 + TA'_1)e_{-1} = 0\) while in the second case this residue generically doesn’t vanish. Hence, looking at the value of \(\gamma(A_1, e_{-1})\) we can understand which one of the two above possibilities takes place. By Lemma 4.6 generically \(\gamma(A_1, e_{-1}) \neq 0\) hence, \(\text{ord}_{P_+} T = -1\).
6. Semi-casimirs and quantization of the second order Hitchin integrals

In this section we non-formally show how the semi-casimirs appear in course of operator quantization of the second order Hitchin integrals.

The second order Hitchin integrals $\chi_i$ are defined by the expansion $\text{tr} \phi^2 = \sum \chi_i \Omega^i$ where $\phi$ is the Higgs field and $\{\Omega^i\}$ is a base of the cotangent space to $\mathcal{M}_{g,2}^{(g-1)}$ realized as a certain space of quadratic differentials.

In course of operator quantization the Higgs field $\phi$ should be replaced by the current $I$. $I$ is an arbitrary Krichever-Novikov 1-form (on the Riemann surface) which has values in the representation operators of $\hat{g}$. Thus $I = \sum u_k \omega^k$ where $\omega^k$ are the basis Krichever-Novikov 1-forms, $u_k$ are the operator-valued coefficients ($k \in \mathbb{Z}$). The $\phi^2$ should be replaced by :$I^2:$. The trace $\text{tr} :I^2:$ (where $\text{tr}$ is the ”finite-dimensional” trace which means that it is linear over the function algebra $\mathcal{A}$) is just the energy-momentum tensor $T$ which was introduced in Section 3. The expansion $T = \sum L_i \Omega^i$ (cf. (3.8)) is the quantum analog of the above expansion $\text{tr} \phi^2 = \sum \chi_i \Omega^i$. We will consider the normalized form $-T(e_i)$ of an operator $L_i$ (see Section 3). What is usually being done for compensating a normal ordering is adding certain cartanian elements to the normal ordered quantity. For example, for Kac-Moody algebras the vector field $z \frac{\partial}{\partial z}$ is being added, i.e. the casimir equals to $z \frac{\partial}{\partial z} - T(z \frac{\partial}{\partial z})$. Observe that making use of this idea exactly results in semi-casimirs $\Delta_i = e_i - T(e_i)$. Since there is only finite number of Hitchin integrals and the infinite set of semi-casimirs $\Delta_i$ we must formulate some selection rule for the latters. We propose to consider only those $\Delta_i$ which induce nontrivial operators on conformal blocks. Then by Theorem 4.2 we obtain the natural mapping of the $\chi_i$’s to the $\Delta_i$’s.

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