THE INTEGRAL COHOMOLOGY RINGS OF PETERSON VARIETIES IN TYPE A

HIRAKU ABE AND HAOZHI ZENG

Abstract. In this paper, we study the ring structure of the integral cohomology of the Peterson variety of type $A_{n-1}$. We give two kinds of descriptions: (1) we show that it is isomorphic to the $\mathfrak{S}_n$-invariant subring of the integral cohomology ring of the permutohedral variety, (2) we determine the ring structure in terms of ring generators and their relations.

1. Introduction

Let $n(\geq 2)$ be a positive integer and $Fl_n = Fl(C^n)$ the flag variety of $C^n$ which is the collection of nested sequence of linear subspaces of $C^n$:

$$Fl_n = \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = C^n) \mid \dim_{C} V_i = i \ (1 \leq i \leq n)\}.$$ 

Let $N$ be an $n \times n$ regular nilpotent matrix viewed as a linear map $N: C^n \to C^n$. The Peterson variety (of type $A_{n-1}$) is a subvariety of $Fl_n$ defined by

$$\text{Pet}_{n} := \{V_\bullet \in Fl_n \mid NV_j \subseteq V_{j+1} \text{ for all } 1 \leq j < n\},$$

where $NV_j$ denotes the image of $V_j$ under the map $N: C^n \to C^n$. It was introduced by Dale Peterson to study the quantum cohomology ring of $Fl_n$, and it has appeared in several contexts (e.g., [3, 7, 17, 22, 25]).

The cohomology ring $H^*(\text{Pet}_n; C)$ has been studied in Harada-Tymoczko ([18]), Fukukawa-Harada-Masuda ([11]), and Harada-Horiguchi-Masuda ([17]). Moreover, a natural basis of $H^*(\text{Pet}_n; C)$ which has certain positivity and integrality was discovered ([5, 14, 15, 18]), and it is now actively studied in connection with mixed Eulerian numbers in combinatorics ([5, 14, 15, 16, 19]). In this paper, we study the ring structure of the cohomology of $\text{Pet}_n$ with $\mathbb{Z}$ coefficients.

To state the first theorem of this paper, we need to introduce a toric variety which is called the permutohedral variety. Let $S$ be an $n \times n$ regular semisimple matrix viewed as a linear map $S: C^n \to C^n$ as above. The permutohedral variety is defined by

$$\text{Perm}_n := \{V_\bullet \in Fl_n \mid SV_j \subseteq V_{j+1} \text{ for all } 1 \leq j < n\}.$$ 

It is known that $\text{Perm}_n$ is the non-singular projective toric variety associated with the fan consisting of the set of Weyl chambers of type $A_{n-1}$ ([3, Theorems 6 and 11]). The symmetric group $\mathfrak{S}_n$ of $n$-letters permutes the set of Weyl chambers, and hence there is a natural $\mathfrak{S}_n$-action on the cohomology ring $H^*(\text{Perm}_n; \mathbb{Z})$ which preserves the grading and the cup product. This implies that the invariant subgroup $H^*(\text{Perm}_n; \mathbb{Z})^{\mathfrak{S}_n}$ is in fact a graded ring with respect to the cup product.

The first theorem of this paper is the following.

Theorem 1.1. As graded rings, we have $H^*(\text{Pet}_n; \mathbb{Z}) \cong H^*(\text{Perm}_n; \mathbb{Z})^{\mathfrak{S}_n}$. 

We note that the corresponding claim for cohomology rings with $\mathbb{C}$ coefficients is known as mentioned in [4, Sect. 1] based on the explicit presentations for the rings.
As the second theorem, we give an explicit presentation of the ring structure of $H^*(\text{Pet}_n; \mathbb{Z})$ in terms of ring generators and their relations. For simplicity, we assume that the regular nilpotent matrix $N$ appearing in the definition of $\text{Pet}_n$ is in Jordan canonical form. Let $L_i$ be the $i$-th tautological line bundle over $Fl_n$ ($1 \leq i \leq n$). By abusing notation, we denote the restriction of $L_i$ over $\text{Pet}_n$ by the same symbol. Let $\mathbb{Z}[y_1, y_2, \ldots, y_n]$ be the polynomial ring over $\mathbb{Z}$ with indeterminates $y_1, y_2, \ldots, y_n$. We regard this polynomial ring as a graded ring with $\deg y_i = 2$ for $1 \leq i \leq n$. Let $\phi: \mathbb{Z}[y_1, y_2, \ldots, y_n] \to H^*(\text{Pet}_n; \mathbb{Z})$ be the ring homomorphism which sends $y_i$ to the first Chern class $c_1(L^*_i)$, where $L^*_i$ is the dual line bundle of $L_i$ ($1 \leq i \leq n$). We introduce the following homogeneous ideals of $\mathbb{Z}[y_1, y_2, \ldots, y_n]$:

\[
I := (e_k(y_1, y_2, \ldots, y_n) \mid 1 \leq k \leq n),
\]

\[
I' := ((y_i - y_{i+1})e_k(y_1, \ldots, y_i) \mid 1 \leq i \leq n-1, 1 \leq k \leq \min\{i, n-i\}),
\]

where $e_k$ denotes the $k$-th elementary symmetric polynomial. We now state the second theorem of this paper.

**Theorem 1.2.** The map $\phi$ induces an isomorphism

$H^*(\text{Pet}_n; \mathbb{Z}) \cong \mathbb{Z}[y_1, y_2, \ldots, y_n]/(I + I')$

as graded rings.

Explicit presentations of the cohomology ring $H^*(\text{Pet}_n; \mathbb{C})$ were given in [11, 17, 18] as mentioned above, and the definition of the ideal $I'$ is motivated algebraically by [11, 17] and geometrically by [5]. See Section 4 for details.

**Acknowledgments.** We are grateful to Mikiya Masuda, Hideya Kuwata, and Tatsuya Horiguchi for valuable discussions. This research is supported in part by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics): Topology and combinatorics of Hessenberg varieties. The first author is supported in part by JSPS Grant-in-Aid for Early-Career Scientists: 18K13413. The second author is supported in part by NSFC: 11901218.
Example 2.1. If \( n = 5 \) and \( h: [5] \to [5] \) is given by
\[
(h(1), h(2), h(3), h(4), h(5)) = (3, 3, 4, 5, 5),
\]
then \( h \) is a Hessenberg function corresponding to the configuration of the shaded boxes drawn in Figure 1.

![Figure 1. The configuration of shaded boxes for Example 2.1](image)

Let \( Fl_n = Fl(C^n) \) be the flag variety of \( C^n \). For an \( n \times n \) matrix \( X \) (viewed as a linear map \( X: C^n \to C^n \)) and a Hessenberg function \( h: [n] \to [n] \), the Hessenberg variety associated with \( X \) and \( h \) is defined by
\[
Hess(X, h) := \{V \in Fl_n \mid XV_j \subseteq V_{h(j)} \text{ for all } 1 \leq j \leq n\}.
\]
The Peterson variety and the permutohedral variety are both special cases of Hessenberg varieties as we will see in the next subsection.

We denote by \( GL_n(C) \) the complex general linear group of degree \( n \). There is a natural action of \( GL_n(C) \) on \( Fl_n \), and we have \( Hess(gXg^{-1}, h) = g \cdot Hess(X, h) \) in \( Fl_n \). This implies that
\[
Hess(gXg^{-1}, h) \cong Hess(X, h)
\]
for \( g \in GL_n(C) \) so that taking conjugation of the matrix \( X \) does not change the isomorphism class of \( Hess(X, h) \).

2.2. Peterson varieties. Let \( N \) be an \( n \times n \) regular nilpotent matrix (i.e., a nilpotent matrix consisting of a single Jordan block), and let \( h_2: [n] \to [n] \) be the Hessenberg function given by
\[
h_2(j) = j + 1 \quad \text{for } 1 \leq j < n.
\]
The Peterson variety \( Pet_n \) is defined as a special case of Hessenberg varieties:
\[
(2.3) \quad Pet_n := Hess(N, h_2) = \{V \in Fl_n \mid NV_j \subseteq V_{j+1} \text{ for all } 1 \leq j < n\}.
\]
For simplicity, we assume that \( N \) is in Jordan canonical form in the rest of this paper. It is well-known (cf. \cite{24} or \cite{6} Lemma 7.1) that
\[
(2.4) \quad \dim_C Pet_n = n - 1.
\]
For a topological space \( X \), we denote by \( H_s(X; Z) \) and \( H^s(X; Z) \) the singular homology group of \( X \) and the singular cohomology ring of \( X \), respectively. We set \( H_{\text{odd}}(X; Z) := \oplus_{k \geq 0} H_{2k+1}(X; Z) \).

Proposition 2.2. (\cite{24} and \cite{29} Theorem 7.1)

\begin{enumerate}
  \item \( H_s(Pet_n; Z) \) is a torsion free \( Z \)-module of rank \( 2^{n-1} \).
  \item \( H_{\text{odd}}(Pet_n; Z) = 0 \).
\end{enumerate}
2.3. Permutahedral varieties. Let $S$ be an $n \times n$ regular semisimple matrix (i.e., an $n \times n$ matrix with $n$ distinct eigenvalues), and let $h_2 : [n] \to [n]$ be the Hessenberg function defined in (2.2). The permutahedral variety $\Perm_n$ is also a special case of Hessenberg varieties:

\begin{equation}
\Perm_n := \Hess(S, h_2) = \{ V_j \in \Flag_n \mid SV_j \subseteq V_{j+1} \text{ for all } 1 \leq j < n \}.
\end{equation}

It is known that $\Perm_n$ is the non-singular projective toric variety associated with the fan consisting of the set of Weyl chambers of type $A_n$ (\cite[Theorems 6 and 11]{9}). This implies that the isomorphism class of $\Hess(S, h_2)$ does not depend on a choice of a regular semisimple matrix $S$. It also follows from \cite[Theorem 11]{3} that

$$\dim \mathbb{C} \Perm_n = n - 1.$$  

**Proposition 2.3.** (\cite[Section III]{9})

(i) $H_\ast(\Perm_n; \mathbb{Z})$ is a torsion free $\mathbb{Z}$-module of rank $n!$.

(ii) $H_{\text{odd}}(\Perm_n; \mathbb{Z}) = 0$.

The Weyl group $\mathfrak{S}_n$ permutes the set of Weyl chambers of type $A_{n-1}$, and hence it induces an $\mathfrak{S}_n$-action on the cohomology ring $H^\ast(\Perm_n; \mathbb{Z})$ of the toric variety $\Perm_n$. It is known (e.g., \cite[Sect. 1]{5}) that this $\mathfrak{S}_n$-module can also be constructed as a special case of the dot action due to Tymoczko \cite{30} which we briefly review in what follows.

Recalling that we have \cite{22}, we may assume that the matrix $S$ in the diagonal form. Let $h : [n] \to [n]$ be an arbitrary Hessenberg function. We denote by $T$ the maximal torus of $\text{GL}_n(\mathbb{C})$ consisting of diagonal matrices. There is a natural action of $\text{GL}_n(\mathbb{C})$ on $\Flag_n$, and hence $T$ acts on $\Flag_n$ through the action of $\text{GL}_n(\mathbb{C})$. This $T$-action preserves $\text{Hess}(S, h) \subseteq \Flag_n$ since the matrix $S$ and elements of $T$ commute. In this way, we obtain a $T$-action on $\text{Hess}(S, h)$. In \cite{30}, Tymoczko constructed a representation of $\mathfrak{S}_n$ on the $T$-equivariant cohomology $H^\ast_T(\text{Hess}(S, h); \mathbb{C})$ by using its GKM presentation, and she showed that it induces a representation of $\mathfrak{S}_n$ on the ordinary cohomology ring $H^\ast(\text{Hess}(S, h); \mathbb{C})$ which preserves the degree and the cup product. As mentioned in \cite[Remark 2.4]{2}, the same construction works for the integral cohomology ring $H^\ast(\text{Hess}(S, h); \mathbb{Z})$ as well. Since we have $\Perm_n = \text{Hess}(S, h_2)$ by definition, we regard $H^\ast(\Perm_n; \mathbb{Z})$ as an $\mathfrak{S}_n$-module by this way throughout the paper. For this $\mathfrak{S}_n$-module, the following claim is deduced from \cite{4, 21, 26}.

**Proposition 2.4.**

(i) The image of the restriction map $H^\ast(\Flag_n; \mathbb{Z}) \to H^\ast(\Perm_n; \mathbb{Z})$ lies in the invariant submodule $H^\ast(\Perm_n; \mathbb{Z})^{\mathfrak{S}_n}$.

(ii) $\text{rank } H^\ast(\Perm_n; \mathbb{Z})^{\mathfrak{S}_n} \leq 2^{n-1}$.

**Proof.** The claim (i) for $\mathbb{Q}$ coefficients follows from \cite{21} or \cite[Sect. 8]{4}. Since the argument of \cite[Sect. 8]{4} works verbatim for $\mathbb{Z}$ coefficients as well, we explain only the outline of the proof. Let $h_n : [n] \to [n]$ be the Hessenberg function given by $h_n(j) = n$ for $1 \leq j \leq n$. Since $\Flag_n = \text{Hess}(S, h_n)$, the cohomology ring $H^\ast(\Flag_n; \mathbb{Z})$ also admits Tymoczko’s $\mathfrak{S}_n$-action. By the construction of this $\mathfrak{S}_n$-action, it follows that the restriction map $H^\ast(\Flag_n; \mathbb{Z}) \to H^\ast(\Perm_n; \mathbb{Z})$ is a homomorphism of $\mathfrak{S}_n$-modules (\cite[Lemma 8.1]{4}). Also, it is known from \cite[Proposition 4.4]{30} that the $\mathfrak{S}_n$-action on $H^\ast(\Flag_n; \mathbb{Z})$ is trivial. This proves the claim (i).
For the claim (ii), it is clear that we have the inclusion map
\[ H^\ast(\text{Perm}_n; \mathbb{Z})^S_n \hookrightarrow H^\ast(\text{Perm}_n; \mathbb{Z}). \]
Since both of these are free \( \mathbb{Z} \)-modules by Proposition 2.3, this map induces an injective linear map over \( \mathbb{C} \):
\[ H^\ast(\text{Perm}_n; \mathbb{Z})^S_n \otimes_{\mathbb{Z}} \mathbb{C} \hookrightarrow H^\ast(\text{Perm}_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}. \]
Here, the target vector space \( H^\ast(\text{Perm}_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \) has a natural structure of an \( S_n \)-representation induced by that of \( H^\ast(\text{Perm}_n; \mathbb{Z}) \), and it is isomorphic to \( H^\ast(\text{Perm}_n; \mathbb{C}) \) as \( S_n \)-representations by construction (cf. [2, Remark 2.4]). The image of the map (2.6) lies on the \( S_n \)-invariant subspace of \( H^\ast(\text{Perm}_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \sim H^\ast(\text{Perm}_n; \mathbb{C}) \) so that we obtain an injective linear map
\[ H^\ast(\text{Perm}_n; \mathbb{Z})^S_n \otimes_{\mathbb{Z}} \mathbb{C} \hookrightarrow H^\ast(\text{Perm}_n; \mathbb{C})^S_n. \]
Thus, it follows that
\[ \text{rank} H^\ast(\text{Perm}_n; \mathbb{Z})^S_n \leq \dim_{\mathbb{C}} H^\ast(\text{Perm}_n; \mathbb{C})^S_n = 2^{n-1}, \]
where the last equality follows from [26, Theorem 3.1].

**Remark 2.5.** For the second claim of Proposition 2.4, we show that the equality \( \text{rank} H^\ast(\text{Perm}_n; \mathbb{Z})^S_n = 2^{n-1} \) holds in the next section. See Remark 3.10 for details.

2.4. A connection between Peterson varieties and permutohedral varieties.

Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C} \) be distinct complex numbers. For \( t \in \mathbb{C} \), we consider an \( n \times n \) matrix \( S_t \) given by
\[
S_t = \begin{pmatrix}
t\lambda_1 & 1 \\
t\lambda_2 & 1 \\
& \ddots \\
t\lambda_{n-1} & 1 \\
t\lambda_n & 1
\end{pmatrix} \quad (t \in \mathbb{C}).
\]
For a (fixed) Hessenberg function \( h: [n] \to [n] \), this leads us to consider a family of Hessenberg varieties over the 1-dimensional base space \( \mathbb{C} \) such that the fiber over \( t \in \mathbb{C} \) is \( \text{Hess}(S_t, h) \) (see [1, Section 4] for details). For our purpose, we take \( h = h_2 \), where \( h_2 \) is the Hessenberg function given in (2.2). When \( t \neq 0 \), the matrix \( S_t \) is a regular semisimple matrix, and hence we have \( \text{Hess}(S_t, h_2) \cong \text{Perm}_n \) by (2.1). When \( t = 0 \), it is clear that \( \text{Hess}(S_0, h_2) = \text{Pet}_n \). Thus, we obtain a degeneration from \( \text{Perm}_n \) to \( \text{Pet}_n \). This family was studied in [1] to prove the following.

**Proposition 2.6.** ([1, Corollary 4.3]) We have
\[ [\text{Pet}_n] = [\text{Perm}_n] \text{ in } H_\ast(\text{Fl}_n; \mathbb{Z}), \]
where \([\text{Pet}_n]\) and \([\text{Perm}_n]\) are the cycles representing the subvarieties \( \text{Pet}_n \) and \( \text{Perm}_n \) in \( \text{Fl}_n \), respectively.
2.5. Combinatorics on Dynkin diagrams of type A. Recall from our notation that \([n - 1] = \{1, 2, \ldots, n - 1\}\). We regard it as the set of vertices of the Dynkin diagram of type \(A_{n-1}\). Namely, two vertices \(i, j \in [n - 1]\) are connected by an edge if and only if \(|i - j| = 1\). See Figure 2.

\[
\begin{array}{cccccc}
\circ & - & - & \cdots & - & \circ \\
1 & 2 & 3 & \cdots & n - 1
\end{array}
\]

**Figure 2.** The Dynkin diagram of type \(A_{n-1}\).

We also regard each subset \(J \subseteq [n - 1]\) as a full-subgraph of the Dynkin diagram. We may decompose it into the connected components:

\[
J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m,
\]

where \(J_k (1 \leq k \leq m)\) is the set of vertices of a maximal connected subgraph of \(J\). To determine each \(J_k\) uniquely, we require that elements of \(J_k\) are less than elements of \(J_{k'}\) when \(k < k'\).

**Example 2.7.** Let \(n = 10\) and \(J = \{1, 2, 4, 5, 6, 9\}\). Then we have

\[
J_1 = \{1, 2\}, \quad J_2 = \{4, 5, 6\}, \quad J_3 = \{9\}
\]

so that \(J = J_1 \sqcup J_2 \sqcup J_3 = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\}\).

For \(J \subseteq [n - 1]\), let us consider the associated Young subgroup

\[
\mathfrak{S}_J := \mathfrak{S}_{J_1} \times \mathfrak{S}_{J_2} \times \cdots \times \mathfrak{S}_{J_m} \subseteq \mathfrak{S}_n,
\]

where \(\mathfrak{S}_{J_k} (1 \leq k \leq m)\) is the subgroup of \(\mathfrak{S}_n\) generated by the simple reflections \(s_i\) for all \(i \in J_k\). Let \(w_J\) be the longest element of \(\mathfrak{S}_J\), i.e.,

\[
w_J := w_0^{(J_1)} w_0^{(J_2)} \cdots w_0^{(J_m)} \in \mathfrak{S}_J,
\]

where \(w_0^{(J_k)}\) is the longest element of \(\mathfrak{S}_{J_k} (1 \leq k \leq m)\).

**Example 2.8.** If \(n = 10\) and \(J = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\} = J_1 \sqcup J_2 \sqcup J_3\) as above, then the permutation \(w_J\) in the form of its permutation matrix is given by

\[
w_J = w_0^{(J_1)} w_0^{(J_2)} w_0^{(J_3)} = \begin{pmatrix}
1 & & & & & & & & & \\
& 1 & & & & & & & & \\
& & 1 & & & & & & & \\
& & & 1 & & & & & & \\
& & & & 1 & & & & & \\
& & & & & 1 & & & & \\
& & & & & & & 1 & & \\
& & & & & & & & & 1
\end{pmatrix}.
\]
For \( J \subseteq [n-1] \), there is a natural Hessenberg function which is determined by \( J \) as follows. Let \( h_J: [n] \to [n] \) be a function given by

\[
h_J(j) = \begin{cases} 
  j + 1 & \text{if } j \in J, \\
  j & \text{if } j \notin J.
\end{cases}
\]  

We note that

\[
h_J(j) \leq h_2(j) \quad \text{for } 1 \leq j \leq n,
\]

where \( h_2 \) is the Hessenberg function defined in (2.2).

**Example 2.9.** If \( n = 10 \) and \( J = \{1, 2, 4, 5, 6, 9\} \) as above, then the configuration of boxes of \( h_J \) is given in Figure 3 (cf. Example 2.8).

![Figure 3. The Hessenberg function \( h_J \)](image)

### 3. The relation between \( H^*(Pet_n; \mathbb{Z}) \) and \( H^*(Perm_n; \mathbb{Z}) \)

The aim of this section is to prove that there is an isomorphism

\[ H^*(Pet_n; \mathbb{Z}) \cong H^*(Perm_n; \mathbb{Z})^{S_n} \]

as graded rings.

**3.1. The Schubert varieties \( X_{w_J} \) associated with \( w_J \).** Let \( J \subseteq [n-1] \). Recall that we have the decomposition \( J = J_1 \sqcup \cdots \sqcup J_m \) into the connected components. For \( 1 \leq k \leq m \), we define \( \overline{J_k} \subseteq [n] \) by

\[ \overline{J_k} := J_k \cup \{\text{max } J_k + 1\}. \]

We also set

\[ n_k := |\overline{J_k}| = |J_k| + 1 \quad \text{for } 1 \leq k \leq m. \]

The permutation \( w_J \in S_n \) defined in (2.7) determines the corresponding Schubert variety \( X_{w_J} \subseteq Fl_n \). Since \( w_J \) is a product of longest permutations of smaller ranks (see also Example 2.8), it follows that the associated Schubert variety \( X_{w_J} \) is isomorphic to a product of flag varieties of smaller ranks:

\[
X_{w_J} \cong \prod_{k=1}^{m} Fl_{n_k}.
\]  

(3.1)
Although this is well-known, let us construct an explicit isomorphism (3.1) to use it in the next subsection. We begin with the map

\[(3.2) \quad \prod_{k=1}^{m} \text{GL}_{n_k}(\mathbb{C}) \to \text{GL}_{n}(\mathbb{C}) ; (g_1, \ldots, g_m) \mapsto g_J,\]

where \(g_J\) is an \(n \times n\) block-diagonal matrix defined as follows. For \(1 \leq k \leq m\), the \(J_k \times J_k(\subseteq [n] \times [n])\) diagonal block of \(g_J\) is \(g_k\), and the remaining diagonal blocks are matrices of size 1 having 1 as their entries.

**Example 3.1.** Let \(n = 10\) and \(J = \{1, 2, 4, 5, 6, 9\} = \{1, 2\} \cup \{4, 5, 6\} \cup \{9\}\) as above. Then we have \(J_1 = \{1, 2, 3\}, J_2 = \{4, 5, 6, 7\},\) and \(J_3 = \{9, 10\}\) so that \(n_1 = 3, n_2 = 4,\) and \(n_3 = 2\). The map (3.2) sends an element \((g_1, g_2, g_3) \in \text{GL}_{3}(\mathbb{C}) \times \text{GL}_{4}(\mathbb{C}) \times \text{GL}_{2}(\mathbb{C})\) to the block-diagonal matrix

\[
g_J = \begin{pmatrix}
g_1 & & 1 \\
& g_2 & \\
& & g_3
\end{pmatrix} \in \text{GL}_{10}(\mathbb{C})
\]

(cf. Example 2.8).

Let \(B_n \subseteq \text{GL}_n(\mathbb{C})\) be the Borel subgroup consisting of upper triangular matrices. We then have the standard identification \(F_{l_n} = \text{GL}_n(\mathbb{C})/B_n\) as is well-known. For \(g \in \text{GL}_n(\mathbb{C})\), we write \([g] = gB_n \in \text{GL}_n(\mathbb{C})/B_n\) for simplicity. It is clear that the map (3.2) induces an embedding

\[(3.3) \quad \phi_J : \prod_{k=1}^{m} F_{l_{n_k}} \to F_{l_n} ; ([g_1], \ldots, [g_m]) \mapsto [g_J].\]

We now show that the image of \(\phi_J\) coincides with the Schubert variety \(X_{w_J}\). For simplicity, we identify the permutation \(w_J\) and the element of \(F_{l_n}\) represented by its permutation matrix (see Example 2.8). Under this identification, it is straightforward to see that the embedding \(\phi_J\) sends \((w_0^{(J_1)}, w_0^{(J_2)}, \ldots, w_0^{(J_m)}) \in \prod_{k=1}^{m} F_{l_{n_k}}\) to \(w_J \in F_{l_n}\). This means that the image of \(\phi_J\) contains \(w_J\). It also follows from the definition that the image of \(\phi_J\) is stable under the action of \(B_n(\subseteq \text{GL}_n(\mathbb{C}))\), where \(B_n\) acts on \(F_{l_n} = \text{GL}_n(\mathbb{C})/B\) by restricting the left multiplication of \(\text{GL}_n(\mathbb{C})\) on \(\text{GL}_n(\mathbb{C})/B\). Therefore, the image of \(\phi_J\) is a \(B\)-stable (Zariski-)closed subset of \(F_{l_n}\) containing \(w_J\). This means that \(X_{w_J} \subseteq \text{Im} \phi_J\). Since the product \(\prod_{k=1}^{m} F_{l_{n_k}}\) is irreducible, so is the
We also know that the dimensions of $X_{w_{J}}$ and $\text{Im} \phi_{J}$ coincide since
\[
\dim_{\mathbb{C}} X_{w_{J}} = \ell(w_{J}) = \sum_{k=1}^{m} \ell(w_{0}^{(J_{k})}) = \dim_{\mathbb{C}} \left( \prod_{k=1}^{m} \text{Fl}_{n_{k}} \right) = \dim_{\mathbb{C}} \text{Im} \phi_{J},
\]
where $w_{0}^{(J_{k})}$ is the permutation appeared in (2.7). Hence, we conclude that
\[X_{w_{J}} = \text{Im} \phi_{J}.\]

Therefore, we verified that the map (3.3) is an embedding onto the Schubert variety $X_{w_{J}}$. This gives us the isomorphism in (3.1).

### 3.2. Varieties associated with $J$

For each $J \subseteq [n-1]$, we introduce varieties $\text{Fl}_{J}$, $\text{Pet}_{J}$, $\text{Perm}_{J}$ associated with $J$ in what follows. First, we set
\[(3.4)\]
\[\text{Fl}_{J} := X_{w_{J}} \cong \prod_{k=1}^{m} \text{Fl}_{n_{k}},\]
where the last isomorphism is given by (3.1).

**Example 3.2.** Let $n = 10$ and $J = \{1, 2, 4, 5, 6, 9\} = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\}$ as above. Then we have
\[\text{Fl}_{J} \cong \text{Fl}_{3} \times \text{Fl}_{4} \times \text{Fl}_{2}\]
(cf. Example 3.1).

Recall that $h_{J} : [n] \to [n]$ is the Hessenberg function defined in (2.8). Associated with $h_{J}$, we consider two varieties $\text{Hess}(N, h_{J})$ and $\text{Hess}(S, h_{J})$, where we note that $\text{Hess}(S, h_{J})$ is not connected when $J \neq [n-1]$ ([11 Corollary 9] or [28 Lemma 3.12]). It is clear that the identity flag
\[\langle e_{1} \rangle \subset \langle e_{1}, e_{2} \rangle \subset \cdots \subset \langle e_{1}, e_{2}, \ldots, e_{n} \rangle = \mathbb{C}^{n}\]
belongs to $\text{Hess}(S, h_{J})$ by definition. We denote by $\text{Hess}^{*}(S, h_{J})$ the connected component of $\text{Hess}(S, h_{J})$ containing the identity flag. We set
\[\text{Pet}_{J} := \text{Hess}(N, h_{J}) \subseteq \text{Fl}_{n},\]
\[\text{Perm}_{J} := \text{Hess}^{*}(S, h_{J}) \subseteq \text{Fl}_{n}.\]

Recalling that $\text{Pet}_{n} = \text{Hess}(N, h_{2})$ and $\text{Perm}_{n} = \text{Hess}(S, h_{2})$ from (2.3) and (2.5), it follows that
\[\text{Pet}_{J} \subseteq \text{Pet}_{n} \quad \text{and} \quad \text{Perm}_{J} \subseteq \text{Perm}_{n}\]
by (2.9).

**Lemma 3.3.** For $J \subseteq [n-1]$, the following hold.

(i) $\text{Pet}_{J}$ and $\text{Perm}_{J}$ are irreducible.

(ii) $\dim_{\mathbb{C}} \text{Pet}_{J} = \dim_{\mathbb{C}} \text{Perm}_{J} = |J|$.

**Proof.** The irreducibility of $\text{Pet}_{J} (= \text{Hess}(N, h_{J}))$ follows from [11 Sect. 7]. For $\text{Perm}_{J} (= \text{Hess}^{*}(S, h_{J}))$, it is non-singular (see section 2.3) and connected so that it is irreducible. This proves the claim (i).
For the claim (ii), we have
\[
\dim \mathbb{C} \text{Pet}_J = \sum_{j=1}^{n} (h_J(j) - j) = \dim \mathbb{C} \text{Perm}_J
\]
by [27, Theorem 10.2] and [9, Theorem 8] (see also [6, Sect. 7]). It is clear that this value is equal to \(|J|\) by the definition of \(h_J\).

We now use the embedding \(\phi_J \colon \prod_{k=1}^{m} \text{Fl}_{n_k} \to \text{Fl}_n\) given in (3.3) to study the structure of \(\text{Pet}_J\) and \(\text{Perm}_J\) for \(J \subseteq [n-1]\). We begin with considering the image of \(\prod_{k=1}^{m} \text{Pet}_{n_k}\) under \(\phi_J\). It follows from the construction of \(\phi_J\) that an arbitrary element \(V \in \phi_J(\prod_{k=1}^{m} \text{Pet}_{n_k})\) satisfies
\[
NV_j \subseteq V_{h_J(j)} \quad (1 \leq j \leq n)
\]
(see also Example 3.1). Namely, we have
\[
\phi_J \left( \prod_{k=1}^{m} \text{Pet}_{n_k} \right) \subseteq \text{Hess}(N, h_J) = \text{Pet}_J.
\]
Here, we know that \(\dim \mathbb{C} \phi_J(\prod_{k=1}^{m} \text{Pet}_{n_k})\) is equal to \(\dim \mathbb{C} \text{Pet}_J\) since
\[
\dim \mathbb{C} \phi_J \left( \prod_{k=1}^{m} \text{Pet}_{n_k} \right) = \sum_{k=1}^{m} (n_k - 1) = \sum_{i=1}^{n} (h_J(i) - i) = \dim \mathbb{C} \text{Pet}_J
\]
by (2.4) and (2.8). Since \(\text{Pet}_J\) is irreducible, we obtain
\[
(3.5) \quad \phi_J \left( \prod_{k=1}^{m} \text{Pet}_{n_k} \right) = \text{Pet}_J.
\]
To obtain a similar result for \(\text{Perm}_J\), recall that \(\text{Perm}_J = \text{Hess}^*(S, h_J)\) is the connected component of \(\text{Hess}(S, h_J)\) containing the identity flag. We also recall that \(\text{Perm}_J\) is irreducible from Lemma 3.3. Also, it is clear that the image \(\phi_J(\prod_{k=1}^{m} \text{Perm}_{n_k})\) contains the identity flag in \(\text{Fl}_n\). Thus, by an argument similar to that above, we obtain
\[
(3.6) \quad \phi_J \left( \prod_{k=1}^{m} \text{Perm}_{n_k} \right) = \text{Perm}_J.
\]
Since the image of \(\phi_J\) is \(\text{Fl}_J(= X_{w_J})\), the equalities (3.5) and (3.6) imply the following claim.

**Lemma 3.4.** For \(J \subseteq [n-1]\), both of \(\text{Pet}_J\) and \(\text{Perm}_J\) are contained in \(\text{Fl}_J\).

Since \(\phi_J\) is an embedding, the equalities (3.5) and (3.6) also imply the following decompositions into products (cf. [10, Theorem 4.5] and [28, Proposition 3.13]):
\[
(3.7) \quad \text{Pet}_J \cong \prod_{k=1}^{m} \text{Pet}_{n_k} \quad \text{and} \quad \text{Perm}_J \cong \prod_{k=1}^{m} \text{Perm}_{n_k}.
\]
It is clear from the construction that these decompositions are compatible with the one in (3.4).
Example 3.5. If \( n = 10 \) and \( J = \{1, 2, 4, 5, 6, 9\} = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\} \) as above, then we have

\[
\text{Pet}_J \cong \text{Pet}_3 \times \text{Pet}_4 \times \text{Pet}_2,
\]
\[
\text{Perm}_J \cong \text{Perm}_3 \times \text{Perm}_4 \times \text{Perm}_2
\]

which are compatible with the decomposition of \( \text{Fl}_J \) given in Example 3.2.

The following is a direct implication of Proposition 2.6.

Lemma 3.6. For \( J \subseteq [n-1] \), we have

\[
[\text{Pet}_J] = [\text{Perm}_J] \quad \text{in } H_*(\text{Fl}_n; \mathbb{Z}),
\]

where \([\text{Pet}_J]\) and \([\text{Perm}_J]\) are the cycles representing the subvarieties \( \text{Pet}_J \) and \( \text{Perm}_J \) in \( \text{Fl}_n \), respectively.

Proof. Since \( \text{Pet}_J \) and \( \text{Perm}_J \) are both subvariety of \( \text{Fl}_J(\subseteq \text{Fl}_n) \) by Lemma 3.4, it suffices to prove the equality in \( H^*(\text{Fl}_J; \mathbb{Z}) \). The decomposition \( \text{Fl}_J \cong \prod_{k=1}^{m} \text{Fl}_{n_k} \) given in (3.4) induces an isomorphism

\[
\xi: H_*(\prod_{k=1}^{m} \text{Fl}_{n_k}; \mathbb{Z}) \cong H_*(\text{Fl}_J; \mathbb{Z}).
\]

By [13] Example 1.10.2], we also have an isomorphism

\[
\zeta: \bigotimes_{k=1}^{m} H_*(\text{Fl}_{n_k}; \mathbb{Z}) \cong H_*(\prod_{k=1}^{m} \text{Fl}_{n_k}; \mathbb{Z})
\]

such that \( \zeta(\bigotimes_{k=1}^{m}[V_k]) = \prod_{k=1}^{m} V_k \) for irreducible subvarieties \( V_k \subseteq \text{Fl}_{n_k} \). By composing these two isomorphisms, we have

\[
\begin{align*}
\xi \circ \zeta(\bigotimes_{k=1}^{m}[\text{Pet}_{n_k}]) &= \xi(\prod_{k=1}^{m} \text{Pet}_{n_k}) = [\text{Pet}_J], \\
\xi \circ \zeta(\bigotimes_{k=1}^{m}[\text{Perm}_{n_k}]) &= \xi(\prod_{k=1}^{m} \text{Perm}_{n_k}) = [\text{Perm}_J]
\end{align*}
\]

since the isomorphisms in (3.7) are compatible with the isomorphism \( \text{Fl}_J \cong \prod_{k=1}^{m} \text{Fl}_{n_k} \).

By Proposition 2.6, we have the following equalities:

\[
[\text{Pet}_{n_k}] = [\text{Perm}_{n_k}] \quad \text{in } H_*(\text{Fl}_{n_k}; \mathbb{Z}) \quad (1 \leq k \leq m).
\]

Therefore, (3.8) implies that

\[
[\text{Pet}_J] = [\text{Perm}_J] \quad \text{in } H_*(\text{Fl}_J; \mathbb{Z}).
\]

\[\square\]

3.3. A proof of Theorem 1.1. Let

\[
i: \text{Pet}_n \hookrightarrow \text{Fl}_n, \quad \text{and } j: \text{Perm}_n \hookrightarrow \text{Fl}_n
\]

be the inclusion maps. We recall the following claim from [20].

Proposition 3.7. ([20 Theorem 17]) The induced map \( i_*: H_*(\text{Pet}_n; \mathbb{Z}) \to H_*(\text{Fl}_n; \mathbb{Z}) \) is an injective map whose image is a direct summand of \( H_*(\text{Fl}_n; \mathbb{Z}) \).

Recall from Proposition 2.2 that \( H_*(\text{Pet}_n; \mathbb{Z}) \) and \( H^*(\text{Fl}_n; \mathbb{Z}) \) are torsion free. Thus, the restriction map \( i^*: H^*(\text{Fl}_n; \mathbb{Z}) \to H^*(\text{Pet}_n; \mathbb{Z}) \) on the cohomology groups is the dual map of \( i_* \) in Proposition 3.7.
Corollary 3.8. ([20]) The restriction map \( i^*: H^*(Fl_n; \mathbb{Z}) \to H^*(\text{Pet}_n; \mathbb{Z}) \) is surjective.

We now prove the following which gives us Theorem 1.1 in Section 1.

Theorem 3.9. There exists a unique isomorphism \( \varphi: H^*(\text{Pet}_n; \mathbb{Z}) \to H^*(\text{Perm}_n; \mathbb{Z})^S_n \) as graded rings such that the following diagram commutes.

\[
\begin{array}{ccc}
H^*(Fl_n; \mathbb{Z}) & \xrightarrow{i^*} & H^*(\text{Pet}_n; \mathbb{Z}) \\
\downarrow & & \downarrow \varphi \\
H^*(\text{Perm}_n; \mathbb{Z}) & \xrightarrow{j^*} & H^*(\text{Perm}_n; \mathbb{Z})^S_n
\end{array}
\]

Proof. The uniqueness of \( \varphi \) follows from the commutativity of the diagram and the surjectivity of \( i^* \) (Corollary 3.8). We construct such an isomorphism \( \varphi \).

Let us begin with studying the induced maps on the homology groups:

\[ i_*: H_*(\text{Pet}_n; \mathbb{Z}) \to H_*(Fl_n; \mathbb{Z}), \]
\[ j_*: H_*(\text{Perm}_n; \mathbb{Z}) \to H_*(Fl_n; \mathbb{Z}). \]

We first claim that

\[ \text{Im } i_* = \text{Im } j_* \text{ in } H_*(Fl_n; \mathbb{Z}). \] (3.9)

Let us prove this in the following. For this purpose, we take a particular basis of \( H_*(\text{Pet}_n; \mathbb{Z}) \) as follows. For each \( J \subseteq [n-1] \), we have a cycle \([\text{Pet}_J]\) in \( H_*(\text{Pet}_n; \mathbb{Z}) \), and it is shown in [5, Proposition 3.4 and Proposition 4.1] that these cycles form a \( \mathbb{Z} \)-basis of \( H_*(\text{Pet}_n; \mathbb{Z}) \):

\[ H_*(\text{Pet}_n; \mathbb{Z}) = \bigoplus_{J \subseteq [n-1]} \mathbb{Z}[\text{Pet}_J]. \]

By Lemma 3.6, we have \( i_*[\text{Pet}_J] = j_*[\text{Perm}_J] \in \text{Im } j_* \text{ for } J \subseteq [n-1] \). This implies that

\[ \text{Im } i_* \subseteq \text{Im } j_. \] (3.10)

Let us prove that \( \text{Im } i_* = \text{Im } j_* \). Since the map \( i_* \) is injective by Proposition 3.7, it follows from (3.10) and Proposition 2.2 that

\[ 2^{n-1} = \text{rank}(\text{Im } i_*) \leq \text{rank}(\text{Im } j_*). \] (3.11)

As for \( j_* \), the dual map of \( j_* \) is precisely the restriction map

\[ j^*: H^*(Fl_n; \mathbb{Z}) \to H^*(\text{Perm}_n; \mathbb{Z}) \]

on the cohomology groups. For this map, we know from Proposition 2.4 that

\[ \text{rank}(\text{Im } j^*) \leq \text{rank } H^*(\text{Perm}_n; \mathbb{Z})^S_n \leq 2^{n-1}. \] (3.12)

Since \( j^* \) is the dual map of \( j_* \), we have

\[ \text{rank}(\text{Im } j_*) = \dim_{\mathbb{Q}}(\text{Im } j^Q) = \dim_{\mathbb{Q}}(\text{Im } j^Q_*) = \text{rank}(\text{Im } j^*), \]

where the maps \( j^Q_* \) and \( j^Q_* \) are the homomorphisms \( H_*(\text{Perm}_n; \mathbb{Q}) \to H_*(Fl_n; \mathbb{Q}) \) and \( H^*(Fl_n; \mathbb{Q}) \to H^*(\text{Perm}_n; \mathbb{Q}) \) induced by the inclusion map \( j: \text{Perm}_n \to Fl_n \), respectively. Thus, the inequalities in (3.11) and (3.12) must be equalities, and we
obtain \( \text{rank}(\text{Im } i_*) = \text{rank}(\text{Im } j_*) \). Hence, by (3.10) and Proposition 3.7, it follows that \( \text{Im } i_* = \text{Im } j_* \) as we claimed in (3.9).

Since \( i_* : H_*(\text{Pet}_n; \mathbb{Z}) \to H_*(\text{Fl}_n; \mathbb{Z}) \) is an isomorphism onto its image, (3.9) means that there exists a surjective group homomorphism

\[
\psi : H_*(\text{Perm}_n; \mathbb{Z}) \to H_*(\text{Pet}_n; \mathbb{Z})
\]

which satisfies the following commutative diagram.

\[
\begin{array}{ccc}
H_*(\text{Fl}_n; \mathbb{Z}) & \xrightarrow{i_*} & H_*(\text{Pet}_n; \mathbb{Z}) \\
\downarrow{j_*} & & \downarrow{\psi} \\
H_*(\text{Perm}_n; \mathbb{Z}) & \xleftarrow{\psi} & H_*(\text{Pet}_n; \mathbb{Z})
\end{array}
\]

Now, we consider the following commutative diagram on the cohomology groups, where we denote by \( \psi^* \) the dual map of \( \psi \).

\[
\begin{array}{ccc}
H^*(\text{Fl}_n; \mathbb{Z}) & \xrightarrow{i^*} & H^*(\text{Pet}_n; \mathbb{Z}) \\
\downarrow{j^*} & & \downarrow{\psi^*} \\
H^*(\text{Perm}_n; \mathbb{Z}) & \xleftarrow{\psi^*} & H^*(\text{Perm}_n; \mathbb{Z})
\end{array}
\]

Since \( i^* \) is surjective by Corollary 3.8, we have \( \text{Im } \psi^* = \text{Im } j^* \) by the commutativity of this diagram. Also, we know from Proposition 2.4 (i) that \( \text{Im } j^* \subseteq H^*(\text{Perm}_n; \mathbb{Z})^{S_n} \). Thus, we obtain the following commutative diagram.

\[
\begin{array}{ccc}
H^*(\text{Fl}_n; \mathbb{Z}) & \xrightarrow{i^*} & H^*(\text{Pet}_n; \mathbb{Z}) \\
\downarrow{j^*} & & \downarrow{\psi^*} \\
H^*(\text{Perm}_n; \mathbb{Z}) & \xleftarrow{\psi^*} & H^*(\text{Perm}_n; \mathbb{Z})^{S_n}
\end{array}
\]

Since the map \( i^* \) is a surjective ring homomorphism, it is the quotient map by an ideal of \( H^*(\text{Fl}_n; \mathbb{Z}) \). Hence, it follows that the map \( \psi^* \) must be a ring homomorphism by the commutativity of this diagram. Namely, \( \psi^* \) is the (graded) ring homomorphism induced by \( j^* \). Since \( \psi \) is surjective, the dual map \( \psi^* \) is an injective map whose image is a direct summand of \( H^*(\text{Perm}_n; \mathbb{Z})^{S_n} \). Thus, the quotient \( H^*(\text{Perm}_n; \mathbb{Z})^{S_n}/\text{Im } \psi^* \) is a free \( \mathbb{Z} \)-module, and its rank is less than or equal to 0 by Proposition 2.2 (i) and Proposition 2.4 (ii). Therefore, it follows that \( \text{Im } \psi^* = H^*(\text{Perm}_n; \mathbb{Z})^{S_n} \) so that \( \psi^* \) is surjective. Letting \( \varphi := \psi^* \), we complete the proof.

\[\square\]

**Remark 3.10.** Since we proved that \( \psi^* \) is an isomorphism, it follows that the equality

\[
\text{rank } H_*(\text{Perm}_n; \mathbb{Z})^{S_n} = 2^{n-1}
\]

holds for the inequality of Proposition 2.4 (ii).
Remark 3.11. The invariant subring $H^*(_{\text{Perm}_n};\mathbb{Q})^{S_n}$ with $\mathbb{Q}$ coefficients was studied by Klyachko ([22]), and he gave an explicit presentation of $H^*(_{\text{Perm}_n};\mathbb{Q})^{S_n}$ (for arbitrary Lie types). See Nadeau-Tewari ([23 Sect. 8]) for an exposition of Klyachko’s results. One can verify that it coincides with the presentation of $H^*(_{\text{Pet}_n};\mathbb{Q})$ given by Fukukawa-Harada-Masuda ([11]). See [3, 17] for a generalization to arbitrary Lie types.

4. AN EXPLICIT PRESENTATION OF THE RING $H^*(_{\text{Pet}_n};\mathbb{Z})$

The aim of this section is to give an explicit presentation of the ring $H^*(_{\text{Pet}_n};\mathbb{Z})$ in terms of ring generators and their relations.

4.1. A ring presentation of $H^*(_{Fl_n};\mathbb{Z})$. We review the following well-known presentation of the cohomology ring of the flag variety $Fl_n$. Our main reference is [12 Sect. 10.2]. For $1 \leq i \leq n$, let $E_i$ be the tautological vector bundle over $Fl_n$ whose fiber over a point $V \in Fl_n$ is $V_i$. As a convention, let $E_0$ be the sub-bundle of $E_1$ of rank 0. Set

$$\tau_i := c_1((E_i/E_{i-1})^*) \in H^2(Fl_n;\mathbb{Z}) \quad (1 \leq i \leq n),$$

where $c_1((E_i/E_{i-1})^*)$ is the first Chern class of the dual line bundle of the tautological line bundle $E_i/E_{i-1}$. By definition, we have short exact sequences

$$0 \to (E_i/E_{i-1})^* \to E_i^* \to E_{i-1}^* \to 0 \quad (1 \leq i \leq n).$$

From these sequences, it follows that

$$e_k(E_n^*) = e_k(\tau_1, \tau_2, \ldots, \tau_n) \quad (1 \leq k \leq n),$$

where $e_k(\tau_1, \tau_2, \ldots, \tau_n)$ is the $k$-th elementary symmetric polynomial in $\tau_1, \tau_2, \ldots, \tau_n$. Since $E_n^*$ is a trivial bundle of rank $n$, this implies that we have

$$(4.1) \quad e_k(\tau_1, \tau_2, \ldots, \tau_n) = 0 \quad \text{in } H^*(Fl_n;\mathbb{Z}) \quad (1 \leq k \leq n).$$

Let $\mathbb{Z}[y_1, y_2, \ldots, y_n]$ be the polynomial ring over $\mathbb{Z}$ with indeterminates $y_1, y_2, \ldots, y_n$. The ring $H^*(Fl_n;\mathbb{Z})$ is generated by $\tau_1, \tau_2, \ldots, \tau_n$, and hence we have a surjective ring homomorphism

$$\mathbb{Z}[y_1, y_2, \ldots, y_n] \to H^*(Fl_n;\mathbb{Z})$$

which sends $y_i$ to $\tau_i$ $(1 \leq i \leq n)$. By (4.1), this induces a surjective ring homomorphism

$$(4.2) \quad \mathbb{Z}[y_1, y_2, \ldots, y_n]/(e_1(y), e_2(y), \ldots, e_n(y)) \to H^*(Fl_n;\mathbb{Z}),$$

where $(e_1(y), e_2(y), \ldots, e_n(y))$ is the ideal of $\mathbb{Z}[y_1, y_2, \ldots, y_n]$ generated by $e_k(y) = e_k(y_1, y_2, \ldots, y_n)$ for $1 \leq k \leq n$. It is well-known that (4.2) is an isomorphism.

---

The line bundle $L_i$ appeared in Section 4 is $E_i/E_{i-1}$ for $1 \leq i \leq n$. 

---

\[ The line bundle \text{ } L_i \text{ } \text{appeared in Section 4 is } E_i/E_{i-1} \text{ for } 1 \leq i \leq n. \]
4.2. A module basis of $H^*(Pet_n; \mathbb{Z})$. We set
\begin{equation}
    x_i := c_1((E_i/E_{i-1})^*|_{Pet_n}) \in H^2(Pet_n; \mathbb{Z}) \quad (1 \leq i \leq n).
\end{equation}
Namely, $x_i$ is the image of $\tau_i$ under the restriction map $i^*: H^*(Fl_n; \mathbb{Z}) \to H^*(Pet_n; \mathbb{Z})$ for $1 \leq i \leq n$. We also set
\begin{equation}
    \omega_i := x_1 + x_2 + \cdots + x_i \in H^2(Pet_n; \mathbb{Z}) \quad (1 \leq i \leq n-1).
\end{equation}
For $J \subseteq [n-1]$, we have the decomposition $J = J_1 \sqcup \cdots \sqcup J_m$ into the connected components (see section 2.5), and we set
\begin{equation}
    \omega_J := \frac{1}{m_J} \prod_{i \in J} \omega_i,
\end{equation}
where $m_J$ is the positive integer defined by $m_J := |J_1||J_2|\cdots|J_m|$. This $\omega_J$ is defined in $H^{2,|J|}(Pet_n; \mathbb{Q})$, but we have the following theorem.

**Theorem 4.1.** ([5, Theorem 4.13]) For each $J \subseteq [n-1]$, the cohomology class $\omega_J$ is an element of the integral cohomology group $H^{2,|J|}(Pet_n; \mathbb{Z})$, and the set
\[ \{ \omega_J \in H^{2,|J|}(Pet_n; \mathbb{Z}) \mid J \subseteq [n-1] \} \]
is a $\mathbb{Z}$-basis of $H^*(Pet_n; \mathbb{Z})$.

In what follows, we give an integral expression for $\omega_J$ in terms of $x_1, x_2, \ldots, x_n$ (see Corollary 4.7 below). For this purpose, we prepare two technical lemmas.

**Lemma 4.2.** For $1 \leq i \leq n-1$, we have
\[ x_i^{d+1} + x_{i+1}^{d+1} + \cdots + x_n^{d+1} = (x_i^d + x_{i+1}^d + \cdots + x_n^d)x_{i+1} \quad (d = 1, 2, \ldots). \]

**Proof.** We first prove the claim for the case $d = 1$ with $1 \leq i \leq n-1$. Recall from [5, Lemma 4.1] (cf. [11] and [17]) that we have
\begin{equation}
    \alpha_j \omega_j = 0 \quad (1 \leq j \leq n-1),
\end{equation}
where $\alpha_j := x_j - x_{j+1}$. Since we have $\omega_j = \omega_{j-1} + x_j$ by definition (with the convention $\omega_0 = 0$), we have from (4.6) that
\[ (x_j - x_{j+1})(\omega_{j-1} + x_j) = 0 \quad (1 \leq j \leq n-1). \]
From this, we obtain
\begin{equation}
    x_j^2 = (\omega_{j-1} + x_j)x_{j+1} - \omega_{j-1}x_j \quad \text{(by $\omega_j = \omega_{j-1} + x_j$ again)}
\end{equation}
for $1 \leq j \leq n-1$. Thus, we obtain
\[ x_1^2 + x_2^2 + \cdots + x_i^2 = (\omega_1 x_2 - \omega_0 x_1) + (\omega_2 x_3 - \omega_1 x_2) + \cdots + (\omega_i x_{i+1} - \omega_{i-1} x_i) \]
\[ = -\omega_0 x_1 + \omega_1 x_{i+1} \]
\[ = (x_1 + x_2 + \cdots + x_i)x_{i+1} \]
which gives the claim for $d = 1$.

We assume that $d \geq 2$ in what follows, and we prove the claim of this lemma by induction on $d$. Assume by induction that
\begin{equation}
    x_i^{\ell+1} + x_{i+1}^{\ell+1} + \cdots + x_n^{\ell+1} = (x_1^\ell + x_2^\ell + \cdots + x_i^\ell)x_{i+1} \quad (1 \leq \ell \leq d-1),
\end{equation}

and we prove the claim for the case \( \ell = d \). To begin with, notice that
\[
x_j^{d+1} = x_j^2 \cdot x_j^{d-1} = \varpi_j x_j^{d-1} x_{j+1} - \varpi_{j-1} x_j^d \quad (1 \leq j \leq n - 1)
\]
by (4.7). Thus, we have
\[
(4.9) \quad x_1^{d+1} + x_2^{d+1} + \cdots + x_i^{d+1} = \sum_{j=1}^{i} \varpi_j x_j^{d-1} x_{j+1} - \sum_{j=1}^{i} \varpi_{j-1} x_j^d.
\]
The second summand in the right hand side can be written as
\[
- \sum_{j=1}^{i} \varpi_{j-1} x_j^d = - \sum_{j=0}^{i-1} \varpi_j x_j^{d+1} = - \sum_{j=1}^{i} \varpi_j x_{j+1}^d + \varpi_i x_{i+1}^d
\]
since \( \varpi_0 = 0 \) by convention. Applying this to (4.9), we obtain
\[
(4.10) \quad x_1^{d+1} + x_2^{d+1} + \cdots + x_i^{d+1} = \sum_{j=1}^{i} \varpi_j (x_j^{d-1} - x_{j+1}^{d-1}) x_{j+1} + \varpi_i x_{i+1}^d.
\]
In this equality, the first summand in the right hand side vanishes since it can be computed as
\[
\sum_{j=1}^{i} \varpi_j (x_j^{d-1} - x_{j+1}^{d-1}) x_{j+1} = \sum_{j=1}^{i} \varpi_j (x_j - x_{j+1})(x_j^{d-2} + x_j^{d-3} x_{j+1} + \cdots + x_{j+1}^{d-2}) x_{j+1}
\]
\[
= 0 \quad \text{(by (4.6))},
\]
where we take the convention \( x_j^{d-2} + x_j^{d-3} x_{j+1} + \cdots + x_{j+1}^{d-2} = 1 \) when \( d = 2 \). Therefore, (4.10) and \( \varpi_i = x_1 + x_2 + \cdots + x_i \) imply that
\[
x_1^{d+1} + x_2^{d+1} + \cdots + x_i^{d+1} = (x_1 + x_2 + \cdots + x_i) x_{i+1}^d.
\]
Applying the inductive hypothesis (4.8) to the right hand side repeatedly, we obtain
\[
x_1^{d+1} + x_2^{d+1} + \cdots + x_i^{d+1} = (x_1 + x_2 + \cdots + x_i) x_{i+1}^d
\]
\[
= (x_1^2 + x_2^2 + \cdots + x_i^2) x_{i+1}^{d-1}
\]
\[
= \cdots
\]
\[
= (x_1^d + x_2^d + \cdots + x_i^d) x_{i+1}^1
\]
which gives (4.8) for the case \( \ell = d \), as desired. \( \square \)

For \( 1 \leq i \leq n \) and a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) consisting of a weakly decreasing sequence of non-negative integers, let \( m_\lambda(x_1, x_2, \ldots, x_i) \) be the monomial symmetric polynomial in \( x_1, x_2, \ldots, x_i \in H^2(Pet_n; \mathbb{Z}) \). That is, \( m_\lambda(x_1, x_2, \ldots, x_i) \) is the sum of all distinct monomials obtained from \( x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_i^{\lambda_i} \) by permuting the indices (e.g., [12 Sect. 6.1]). For a positive integer \( d \) and a non-negative integer \( k \), we denote by \( (d, 1^k) \) the partition given by
\[
(d, 1^k) = (d, 1, 1, \ldots, 1, 0, 0, \ldots).
\]
For $1 \leq k < i \leq n$, we have the following identity:

\begin{equation}
(4.11)
(x^1 + \cdots + x^d)e_k(x_1, x_2, \ldots, x_i)
= m_{d+1,k-1}(x_1, x_2, \ldots, x_i) + m_{d,1}(x_1, x_2, \ldots, x_i) \quad (d = 2, 3, \ldots).
\end{equation}

One may obtain this identity by expanding the product in the left hand side and rearranging terms, without using any non-trivial algebraic relations for $x_1, x_2, \ldots, x_n$ in $H^*(\text{Pet}_n; \mathbb{Z})$. The following example illustrates the idea of the proof.

**Example 4.3.** Let $d = 5$, $i = 4$, and $k = 3$. Then we have

\[
\begin{align*}
(x_1^5 + x_2^5 + x_3^5 + x_4^5)e_3(x_1, x_2, x_3, x_4) &= (x_1^5 + x_2^5 + x_3^5 + x_4^5) \cdot (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\
&= m_{6,1,1}(x_1, x_2, x_3, x_4) + m_{5,1,1,1}(x_1, x_2, x_3, x_4),
\end{align*}
\]

where we have

\[
m_{6,1,1}(x_1, x_2, x_3, x_4) = x_1^5x_2x_3 + x_1^5x_2x_4 + x_1^5x_3x_4 + x_1^5x_1x_3 + x_1^5x_1x_4 + x_1^5x_3x_4 + x_3^5x_1x_2 + x_3^5x_1x_4 + x_3^5x_2x_4 + x_3^5x_1x_3 + x_3^5x_1x_4 + x_3^5x_1x_3 + x_3^5x_2x_3
\]

and

\[
m_{5,1,1,1}(x_1, x_2, x_3, x_4) = x_1^5x_2x_3x_4 + x_2^5x_1x_3x_4 + x_3^5x_1x_2x_4 + x_3^5x_1x_2x_4 + x_3^5x_1x_2x_4.
\]

We note that we have a slightly different identity when $d = 1$. Namely, we have

\begin{equation}
(4.12)
(x_1 + \cdots + x_i)e_k(x_1, x_2, \ldots, x_i)
= m_{2,1,k-1}(x_1, x_2, \ldots, x_i) + (k + 1)m_{1,k}(x_1, x_2, \ldots, x_i)
\end{equation}

for $1 \leq k < i \leq n$. Similarly to (4.11), this identity can be obtained by expanding the product in the left hand side as illustrated in the following example.

**Example 4.4.** Let $i = 4$ and $k = 2$. Then we have

\[
(x_1 + x_2 + x_3 + x_4)e_2(x_1, x_2, x_3, x_4)
= (x_1 + x_2 + x_3 + x_4)(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)
\]

which is equal to the sum of

\[
m_{2,1}(x_1, x_2, x_3, x_4) = x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_1 + x_2^2x_3 + x_2^2x_4 + x_3^2x_1 + x_3^2x_2 + x_3^2x_4 + x_4^2x_1 + x_4^2x_2 + x_4^2x_3
\]

and

\[
x_1(x_2x_3 + x_3x_4 + x_2x_4) + x_2(x_1x_3 + x_1x_4 + x_3x_4) + x_3(x_1x_2 + x_1x_4 + x_2x_4) + x_4(x_1x_2 + x_1x_3 + x_2x_3) = 3m_{1,1,1}(x_1, x_2, x_3, x_4).
\]

The next claim generalizes the previous lemma.

**Lemma 4.5.** For $0 \leq k < i \leq n - 1$, we have

\[
m_{d+1,k}(x_1, x_2, \ldots, x_i) = \begin{cases} 
m_{d,1,k}(x_1, x_2, \ldots, x_i)x_{i+1} & (d \geq 2), \\
(k + 1)m_{1,k+1}(x_1, x_2, \ldots, x_i)x_{i+1} & (d = 1).
\end{cases}
\]
Proof. When $k = 0$, the claim is

$$m_{d+1,1^0}(x_1, x_2, \ldots, x_i) = \begin{cases} 
  m_{d,1^0}(x_1, x_2, \ldots, x_i)x_{i+1} & (d \geq 2), \\
  m_{1^1}(x_1, x_2, \ldots, x_i)x_{i+1} & (d = 1)
\end{cases}$$

which is equivalent to

$$x_1^{d+1} + x_2^{d+1} + \cdots + x_i^{d+1} = (x_1^d + x_2^d + \cdots + x_i^d)x_{i+1} \quad (d \geq 1).$$

Thus, the claim follows by the previous lemma when $k = 0$.

We assume that $1 \leq k < i$ in what follows, and we prove the claim of this lemma by induction on $k$. Assume by induction that

$$m_{d+1,1^k-1}(x_1, x_2, \ldots, x_i) = \begin{cases} 
  m_{d,1^k-1}(x_1, x_2, \ldots, x_i) x_{i+1} & (d \geq 2), \\
  km_{1^1}(x_1, x_2, \ldots, x_i) x_{i+1} & (d = 1)
\end{cases}$$

Multiplying $e_k(x_1, x_2, \ldots, x_i)$ to the both sides of (4.13), we have

$$\left(x_1^{d+1} + x_2^{d+1} + \cdots + x_i^{d+1}\right)e_k(x_1, x_2, \ldots, x_i) = \left(x_1^{d} + x_2^{d} + \cdots + x_i^{d}\right)e_k(x_1, x_2, \ldots, x_i) x_{i+1}.$$

By (4.11), the left hand side of (4.15) is equal to

$$m_{d+2,1^{k-1}}(x_1, x_2, \ldots, x_i) + m_{d+1,1^k}(x_1, x_2, \ldots, x_i)$$

since $d + 1 \geq 2$. By (4.11) and (4.12), the right hand side of (4.15) is equal to

$$\begin{cases} 
  m_{d+1,1^{k-1}}(x_1, x_2, \ldots, x_i) x_{i+1} + m_{d,1^k}(x_1, x_2, \ldots, x_i) x_{i+1} & (d \geq 2), \\
  m_{2,1^{k-1}}(x_1, x_2, \ldots, x_i) x_{i+1} + (k + 1)m_{1^{k+1}}(x_1, x_2, \ldots, x_i) x_{i+1} & (d = 1)
\end{cases}$$

In both cases of $d \geq 2$ and $d = 1$, the first summands in (4.16) and (4.17) coincide because of the first case of the inductive hypothesis (4.14). Thus, the equality (4.15) implies that

$$m_{d+1,1^k}(x_1, x_2, \ldots, x_i) = \begin{cases} 
  m_{d,1^k}(x_1, x_2, \ldots, x_i)x_{i+1} & (d \geq 2), \\
  (k + 1)m_{1^{k+1}}(x_1, x_2, \ldots, x_i)x_{i+1} & (d = 1)
\end{cases}$$

as desired. \qed

For the next proposition, we recall that $\omega_i = x_1 + x_2 + \cdots + x_i$ for $1 \leq i \leq n - 1$ from (4.4).

**Proposition 4.6.** For $1 \leq a \leq b \leq n - 1$, we have

$$\underbrace{\omega_a \omega_{a+1} \cdots \omega_k}_{k} = k! e_k(x_1, x_2, \ldots, x_k),$$

where $k = b - a + 1$. 

Proof. For the case \( k = 1 \) (i.e., \( b = a \)), the claim is obvious by (4.4). We assume that \( k \geq 2 \), and we prove the claim by induction on \( k \). By the inductive hypothesis, the left hand side can be computed as
\[
(k - 1)!e_{k-1}(x_1, x_2, \ldots, x_{b-1}) \cdot \varpi_b
\]
where \( J \) and \( m_{1k}(x_1, x_2, \ldots, x_{b-1}) = \varpi_k(x_1, x_2, \ldots, x_{b-1}) \) \( e_{k-1}(x_1, x_2, \ldots, x_{b-1}) \) and \( 0 = \varpi_{b-1}(x_1, x_2, \ldots, x_{b-1}) \) again for the last equality. In the last expression, the sum of the first and the third summands is equal to 0 by the case \( d = 1 \) of Lemma 4.5 since \( 0 \leq k - 2 < b - 1 \). Thus, the last expression is equal to \( k!e_{k}(x_1, x_2, \ldots, x_b) \).

We now obtain the following formula which expresses \( \varpi_J \) in (4.5) as an integer coefficient polynomial in \( x_1, x_2, \ldots, x_n \).

Corollary 4.7. For \( J \subseteq [n - 1] \), we have
\[
\varpi_J = \prod_{k=1}^{m} e_{|J_k|}(x_1, x_2, \ldots, x_{\max J_k}) \quad \text{in } H^{|J|}(Pet_n; \mathbb{Z}),
\]
where \( J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m \) is the decomposition into the connected components.

Proof. The previous proposition implies that
\[
\prod_{i \in J} \varpi_i = \prod_{k=1}^{m} \left( \prod_{i \in J_k} \varpi_i \right) = \prod_{k=1}^{m} \left( |J_k| e_{|J_k|}(x_1, x_2, \ldots, x_{\max J_k}) \right).
\]
because each component \( J_k \) is of the form \( J_k = \{a, a + 1, \ldots, b\} \) for some \( a, b \in [n - 1] \). Since \( H^*(\text{Pet}_n; \mathbb{Z}) \) is torsion free by Proposition 2.2 we obtain the claim by dividing both sides of this equality by \( m_J = |J_1||J_2| \cdots |J_m| \).

**Example 4.8.** Let \( n = 7 \). Then we have

\[
\begin{align*}
\varpi_{(3,4,5)} &= \frac{1}{6} \varpi_3 \varpi_4 \varpi_5 = e_3(x_1, x_2, x_3, x_4, x_5), \\
\varpi_{(2,4,5)} &= \frac{1}{2} \varpi_2 \varpi_4 \varpi_5 = e_1(x_1, x_2)e_2(x_1, x_2, x_3, x_4, x_5).
\end{align*}
\]

4.3. A ring presentation of \( H^*(\text{Pet}_n; \mathbb{Z}) \). Recall from (4.3) that we have

\[ x_i = c_1((E_i/E_{i-1})^*|_{\text{Pet}_n}) \in H^2(\text{Pet}_n; \mathbb{Z}) \]

and that it is the image of \( \tau_i = c_1((E_{i}/E_{i-1})^*) \in H^2(\text{Fl}_n; \mathbb{Z}) \) under the restriction map

\[ \iota^*: H^*(\text{Fl}_n; \mathbb{Z}) \to H^*(\text{Pet}_n; \mathbb{Z}). \]

The ring \( H^*(\text{Fl}_n; \mathbb{Z}) \) is generated by \( \tau_1, \tau_2, \ldots, \tau_n \) as we saw in section 4.1.1 and the map \( \iota^* \) is surjective by [20] (see Corollary 3.8). Thus, we have a surjective ring homomorphism

\[ \phi: \mathbb{Z}[y_1, y_2, \ldots, y_n] \to H^*(\text{Pet}_n; \mathbb{Z}) \]

which sends \( y_i \) to \( x_i \) (\( 1 \leq i \leq n \)), where \( \mathbb{Z}[y_1, y_2, \ldots, y_n] \) is the polynomial ring over \( \mathbb{Z} \) with indeterminates \( y_1, y_2, \ldots, y_n \). We regard this polynomial ring as a graded ring with \( \deg y_i = 2 \) for \( 1 \leq i \leq n \). By construction, this map factors \( H^*(\text{Fl}_n; \mathbb{Z}) \), and hence it maps \( e_k(y_1, y_2, \ldots, y_n) \) to 0 in \( H^*(\text{Pet}_n; \mathbb{Z}) \) for \( 1 \leq k \leq n \) (see section 4.1.1).

To give a ring presentation of \( H^*(\text{Pet}_n; \mathbb{Z}) \), we introduce the following homogeneous ideals of \( \mathbb{Z}[y_1, y_2, \ldots, y_n] \):

\[
\begin{align*}
I &= \langle e_k(y_1, y_2, \ldots, y_n) \mid 1 \leq k \leq n \rangle, \\
I' &= \langle (y_i - y_{i+1})e_k(y_1, \ldots, y_i) \mid 1 \leq i \leq n - 1, \ 1 \leq k \leq \min\{i, n - i\} \rangle.
\end{align*}
\]

**Example 4.9.** Let \( n = 4 \). The ideal \( I \) of \( \mathbb{Z}[y_1, y_2, y_3, y_4] \) is generated by

\[ e_1(y_1, y_2, y_3, y_4), \ e_2(y_1, y_2, y_3, y_4), \ e_3(y_1, y_2, y_3, y_4), \ e_4(y_1, y_2, y_3, y_4), \]

and the ideal \( I' \) is generated by

\[ (y_1 - y_2)y_1, \ (y_2 - y_3)(y_1 + y_2), \ (y_2 - y_3)y_1y_2, \ (y_3 - y_4)(y_1 + y_2 + y_3). \]

From what we saw above, it is clear that the map \( \phi \) sends the ideal \( I \) to 0 in \( H^*(\text{Pet}_n; \mathbb{Z}) \). It follows that \( \phi \) also sends \( I' \) to 0. To see that, it suffices to show that

\[ (x_i - x_{i+1})e_k(x_1, \ldots, x_i) = 0 \quad (1 \leq k \leq i \leq n - 1) \]

in \( H^*(\text{Pet}_n; \mathbb{Z}) \), where we note that the range of \( k \) is larger than that of in (4.19). For this purpose, let \( 1 \leq k \leq i \leq n - 1 \). By Proposition 4.6 we have

\[ k!e_k(x_1, \ldots, x_i) = \varpi_{i-k+1}\varpi_{i-k+2} \cdots \varpi_i. \]
Recalling that $x_i - x_{i+1} = \alpha_i$, we obtain
\[ k!(x_i - x_{i+1})e_k(x_1, \ldots, x_i) = \alpha_i \varpi_{i-k+1} \varpi_{i-k+2} \cdots \varpi_i. \]

The right hand side of this equality is 0 by (4.14) so that we obtain
\[ k!(x_i - x_{i+1})e_k(x_1, \ldots, x_i) = 0. \]

Since $H^*(\text{Pet}_n; \mathbb{Z})$ is torsion free by Proposition 2.2, this implies the equality (4.22). Hence, $\phi$ sends $I'$ to 0, as we claimed above.

Remark 4.10. Geometric meaning of the relation (4.22) can be explained as follows. By Corollary 4.7, it can be expressed as
\[ \alpha_i \cdot \varpi_J = 0, \]
where we set $J = \{i - k + 1, i - k + 2, \ldots, i\}$. In [5], two kinds of closed subsets $X_J(= \text{Pet}_J)$ and $\Omega_J$ in $\text{Pet}_n$ are introduced, and this equality can be explained from the corresponding geometric equality
\[ X_{\{i\}} \cap \Omega_J = \emptyset \]
by an argument similar to that in the proof of [5, Lemma 4.7]. See [5, Sect. 3 and 4] for details.

Since the map $\phi$ sends both of $I$ and $I'$ to 0, it induces a surjective ring homomorphism
\[ \overline{\phi}: \mathbb{Z}[y_1, y_2, \ldots, y_n]/(I + I') \to H^*(\text{Pet}_n; \mathbb{Z}) \]
which sends $y_i$ to $x_i$ ($1 \leq i \leq n$). Here, we use the same symbol $y_i$ for its image in the quotient $\mathbb{Z}[y_1, y_2, \ldots, y_n]/(I + I')$ by abusing notation. We adopt this notation in the rest of this paper.

The next claim gives Theorem 1.2 in Section 1 which describes the ring structure of $H^*(\text{Pet}_n; \mathbb{Z})$.

Theorem 4.11. The induced homomorphism
\[ \overline{\phi}: \mathbb{Z}[y_1, y_2, \ldots, y_n]/(I + I') \to H^*(\text{Pet}_n; \mathbb{Z}) \]
sending $y_i$ to $x_i = c_1((E_i/E_{i-1})^*|_{\text{Pet}_n})$ ($1 \leq i \leq n$) is an isomorphism as graded rings, where $I$ and $I'$ are the ideals of $\mathbb{Z}[y_1, y_2, \ldots, y_n]$ defined in (1.19).

The rest of this paper is devoted for the proof of Theorem 4.11.

Remark 4.12. The relation (4.22) for $k = 1$ takes the form
\[ \alpha_1 \cdot \varpi_i = (x_i - x_{i+1})(x_1 + \cdots + x_i) = 0 \text{ in } H^*(\text{Pet}_n; \mathbb{Z}) \]
which appears as the fundamental relations of the presentation of the cohomology ring $H^*(\text{Pet}_n; \mathbb{C})$ in [11, Corollary 3.4] and [17, Theorem 4.1] (cf. [5, Remark 4.8]).
Remark 4.13. Let \( n = 4 \). If we remove \((y_2 - y_3)y_1y_2\) from the list (4.21) of the generators of \( I' \), then the quotient ring \( \mathbb{Z}[y_1, y_2, y_3, y_4]/(I + I') \) is not isomorphic to \( H^*(\text{Pet}_4; \mathbb{Z}) \). This can be verified by a computer assisted calculation. Similarly, if we remove \( e_k(y_1, y_2, y_3, y_4) \) for some \( 1 \leq k \leq 4 \) from the list (4.20) of the generators of \( I \), then the quotient ring \( \mathbb{Z}[y_1, y_2, y_3, y_4]/(I + I') \) is not isomorphic to \( H^*(\text{Pet}_4; \mathbb{Z}) \).

Set

\[ M := \mathbb{Z}[y_1, y_2, \ldots, y_n]/(I + I'). \]

If we can construct a subset \( \{\pi_J \mid J \subseteq [n - 1]\} \) of \( M \) which generates \( M \) as a \( \mathbb{Z} \)-module and satisfies \( \bar{\phi}(\pi_J) = \varpi_J \) for \( J \subseteq [n - 1] \), then it follows that the map \( \bar{\phi} \) is an isomorphism since \( \varpi_J \) for \( J \subseteq [n - 1] \) form a \( \mathbb{Z} \)-basis of \( H^*(\text{Pet}_n; \mathbb{Z}) \) by Theorem 4.11.

Motivated by Corollary 4.17, we define \( \pi_J \in M \) for each \( J \subseteq [n - 1] \) as follows. For a subset \( J \subseteq [n - 1] \) having the decomposition \( J = J_1 \cup \cdots \cup J_m \) into the connected components (see section 2.3), we set

\[ \pi_J := \prod_{k=1}^m e_{|J_k|}(y_1, y_2, \ldots, y_{\max J_k}) \in M, \]

where we take the convention

\[ \pi_{\emptyset} = 1 \in M. \]

We also define \( \pi_J \) for all \( J \subseteq \mathbb{Z} \) by taking the convention

\[ \pi_J = 0 \quad \text{unless} \quad J \subseteq [n - 1]. \]

Example 4.14. Let \( n = 7 \). Then we have

\[ \pi_{(3, 4, 5)} = e_3(y_1, y_2, y_3, y_4, y_5), \]

\[ \pi_{(2, 4, 5)} = e_1(y_1, y_2)e_2(y_4, y_2, y_3, y_4, y_5) \]

in \( M \) (cf. Example 4.18). We also have \( \pi_{(0, 2)} = 0 \) and \( \pi_{(4, 5, 7)} = 0 \) by (4.25).

Recall that we have \( \overline{\phi}(y_i) = x_i \) for \( 1 \leq i \leq n \) by definition. Hence, it is clear that we have \( \bar{\phi}(\pi_J) = \varpi_J \) for \( J \subseteq [n - 1] \) by Corollary 4.17. Thus, to prove Theorem 4.11 it is enough to show the following claim as we discussed above.

Proposition 4.15. The \( \mathbb{Z} \)-module \( M = \mathbb{Z}[y_1, y_2, \ldots, y_n]/(I + I') \) is generated by the subset \( \{\pi_J \mid J \subseteq [n - 1]\} \).

We prove this in the next subsection.

4.4. A Proof of Proposition 4.15. Before giving a proof of Proposition 4.15 we first establish some basic properties of \( \pi_J \) for \( J \subseteq [n - 1] \). We begin with the following identity in \( M \):

\[ e_k(y_1, y_2, \ldots, y_n) = \sum_{0 \leq p \leq i, \ 0 \leq q \leq n-i} e_p(y_1, y_2, \ldots, y_i)e_q(y_{i+1}, y_{i+2}, \ldots, y_n) \]
for \(1 \leq k \leq i \leq n\), where we take the convention \(e_0 = 1\). One may obtain this identity by decomposing the index set \([n]\) of monomials in the left hand side into two parts: \([n] = \{1, 2, \ldots, i\} \sqcup \{i + 1, i + 2, \ldots, n\}\).

For \(1 \leq i \leq n - 1\), recall from the definition of \(M\) that we have
\[
(y_i - y_{i+1}) e_k(y_1, \ldots, y_i) = 0 \quad (1 \leq k \leq \min\{i, n - i\}).
\]
The following claim means that the same equalities hold for a wider range of \(k\).

**Lemma 4.16.** For \(1 \leq i \leq n - 1\), we have
\[
(y_i - y_{i+1}) e_k(y_1, \ldots, y_i) = 0 \quad (1 \leq k \leq i)
\]
in \(M\).

**Proof.** If \(1 \leq k \leq n - i\), then we have \(k \leq \min\{i, n - i\}\) (since \(k \leq i\)), and the claim holds by the definition of \(M\) as we saw above.

If \(n - i < k \leq i\), then we prove the claim by induction on \(k\), where we note that we already verified the claim for \(1 \leq k \leq n - i\). Hence, we assume by induction that the claim holds when \(k \leq \ell\) for some positive integer \(\ell\) satisfying \(n - i \leq \ell < i\), and we prove the claim when \(k = \ell + 1\). Since we have \(e_k(y_1, \ldots, y_n) = 0\) by the definition \(M\), we know that
\[
(y_i - y_{i+1}) e_k(y_1, \ldots, y_n) = 0.
\]
By (4.26), this equality can be written as
\[
\sum_{0 \leq p \leq i, 0 \leq q \leq n - i} (y_i - y_{i+1}) e_p(y_1, \ldots, y_i) e_q(y_{i+1}, \ldots, y_n) = 0.
\]
Because of the condition \(1 \leq k \leq i\), it follows that the summand for \(q = 0\) (i.e., \(p = k\)) appears in the left hand side of this equality. Thus, we can separate it to obtain
\[
(y_i - y_{i+1}) e_k(y_1, \ldots, y_i) + \sum_{0 \leq p \leq i, 1 \leq q \leq n - i} (y_i - y_{i+1}) e_p(y_1, \ldots, y_i) e_q(y_{i+1}, \ldots, y_n) = 0.
\]
In the second summand, we have \(p = k - q > (n - i) - q \geq 0\) since we are considering the case \(n - i < k \leq i\). This implies that \(p \geq 1\) in the second summand. Noticing that \(1 \leq p < k\), the inductive hypothesis implies that the second summand vanishes. Thus, we obtain
\[
(y_i - y_{i+1}) e_k(y_1, \ldots, y_i) = 0.
\]

\(\Box\)

For non-negative integers \(a\) and \(b\), we use the notation
\[
[a, b] := \{ c \in \mathbb{Z} \mid a \leq c \leq b \}
\]
in what follows. For example, we have \([2, 5] = \{2, 3, 4, 5\}, [3, 2] = \emptyset,\) and \([0, 1] = \{0, 1\}\) so that
\[
\pi_{[2, 5]} = e_4(y_1, y_2, \ldots, y_5), \quad \pi_{[3, 2]} = 1, \quad \text{and} \quad \pi_{[0, 1]} = 0
\]
by (4.23), (4.24), and (4.25), respectively. Noticing that we have
\[
e_k(y_1, \ldots, y_b) = e_k(y_1, \ldots, y_b-1) + e_{k-1}(y_1, \ldots, y_{b-1}) y_b \quad (1 \leq k \leq b \leq n - 1),
\]
it follows that
\[(4.27)\quad \pi_{[a,b]} = \pi_{[a-1,b-1]} + \pi_{[a,b-1]}y_b \quad (1 \leq a \leq b \leq n - 1)\]
since we have \(\pi_{[a,b]} = e_{b-a+1}(y_1, y_2, \ldots, y_b)\). When \(a = b\), we obtain
\[
\pi_{[a,a]} = \pi_{[a-1,a-1]} + y_a \quad (1 \leq a \leq n - 1)
\]
since \(\pi_{[a,a-1]} = \pi_0 = 1\). The next claim generalizes Lemma 4.16.

**Lemma 4.17.** For \(1 \leq a \leq b \leq n - 1\), we have
\[(4.28)\quad (y_i - y_{i+1}) \cdot \pi_{[a,b]} = 0 \quad (a \leq i \leq b)\]
in \(M\).

**Proof.** When \(i = b\), the claim follows by Lemma 4.16 since we have
\[
(y_i - y_{i+1})\pi_{[a,i]} = (y_i - y_{i+1})e_{i-a+1}(y_1, \ldots, y_i) = 0
\]
in this case. We prove the claim by induction on \(b - i \geq 0\). Let \(\ell(<n - 1)\) be a non-negative integer, and assume by induction that \((4.28)\) holds for \(b - i = \ell\). We prove that \((4.28)\) holds for \(b - i = \ell + 1(\geq 1)\). By \((4.27)\), we have
\[
(4.29)\quad (y_i - y_{i+1}) \cdot \pi_{[a,b]} = (y_i - y_{i+1})\pi_{[a-1,b-1]} + (y_i - y_{i+1})\pi_{[a,b-1]}y_b.
\]
We compute the right hand side by taking cases. If \(a = 1\), then the first summand is equal to zero by \((1.25)\), and the second summand is also equal to zero by the inductive hypothesis since \(b - 1 \geq i\). If \(a > 1\), both summands are equal to zero by the inductive hypothesis. Thus, in either case, the right hand side of \((4.29)\) is equal to zero so that we obtain \((y_i - y_{i+1}) \cdot \pi_{[a,b]} = 0\).

In particular, for \(1 \leq a \leq b \leq n - 1\), we obtain
\[(4.30)\quad y_i \pi_{[a,b]} = y_{b+1} \pi_{[a,b]} \quad (a \leq i \leq b)\]
in \(M\) by applying \((4.28)\) repeatedly.

To state the next claim, let us recall a basic property of \(\varpi_j\) in the cohomology \(H^*(\text{Pet}_n; \mathbb{Z})\): it is clear from the definition \((1.1)\) that, for \(1 \leq a \leq i < b \leq n - 1\), we have
\[
\varpi_{[a,i]} \cdot \varpi_{[i+1,b]} = \frac{1}{(i - a + 1)!} \cdot \frac{1}{(b - i)!} \cdot \varpi_a \varpi_{a+1} \cdots \varpi_b = \binom{b-a+1}{i-a+1} \varpi_{[a,b]}
\]
in \(H^*(\text{Pet}_n; \mathbb{Z})\), where \(\binom{b-a+1}{i-a+1}\) is a binomial coefficient. As the following claim shows, the analogous equalities hold in \(M\) as well.

**Proposition 4.18.** For \(1 \leq a \leq i < b \leq n - 1\), we have
\[(4.31)\quad \pi_{[a,i]} \cdot \pi_{[i+1,b]} = \binom{b-a+1}{i-a+1} \pi_{[a,b]}\]
in \(M\).
Proof. We prove the claim by induction on \(b \geq 2\). When \(b = 2\), we have \(a = i = 1\) so that the claim is
\[
\pi[1,1] \cdot \pi[2,2] = 2\pi[1,2].
\]
The left hand side can be computed as
\[
\pi[1,1] \cdot \pi[2,2] = \pi[1,1] \cdot \pi[2] = \pi[1,1] \cdot (y_1 + y_2) = \pi[1,1] \cdot 2y_2 = 2\pi[1,2],
\]
where the third equality follows from (4.30), and the fourth equality follows from (4.27) with \(\pi[0,1] = 0\). Thus, we obtain the claim for \(b = 2\).

Let \(2 \leq \ell < n - 1\) be a positive integer, and assume by induction that the claim (4.31) holds for \(b = \ell\). We prove the claim (4.31) for \(b = \ell + 1\). The left hand side of (4.31) can be written as
\[
\pi[a,i] \cdot \pi[i+1,b] = \pi[a,i] \pi[i,b-1] + \pi[a,i] \pi[i+1,b-1] y_b
\]
by (4.27). Applying (4.27) again to \(\pi[a,i]\) in the first summand of the right hand side, we obtain
\[
(4.32) \quad \pi[a,i] \cdot \pi[i+1,b] = \pi[a-1,i-1] \pi[i,b-1] + \pi[a,i-1] \pi[i,b-1] y_i + \pi[a,i] \pi[i+1,b-1] y_b.
\]
We now compute each summands of the right hand side separately. For the first summand, we have
\[
\pi[a-1,i-1] \pi[i,b-1] = \left(\frac{b - a + 1}{i - a + 1}\right) \pi[a-1,b-1]
\]
which we prove by taking cases as follows. If \(a = 1\), the claim is obvious since both sides are equal to 0 by (4.25). If \(a > 1\), then the claim follows by the inductive hypothesis. For the second summand of the right hand side of (4.32), we have
\[
\pi[a,i-1] \pi[i,b-1] y_i = \left(\frac{b - a}{i - a}\right) \pi[a,b-1] y_i
\]
which we prove by taking cases as follows. If \(a = i\), the claim is obvious since both sides are equal to \(\pi[i,b-1] y_i\) by (4.24). If \(a < i\), then the claim follows by the inductive hypothesis. For the third summand of the right hand side of (4.32), we have
\[
\pi[a,i] \pi[i+1,b-1] y_b = \left(\frac{b - a}{i - a + 1}\right) \pi[a,b-1] y_b
\]
which we prove by taking cases as follows. If \(b = i + 1\), the claim is obvious since both sides are equal to \(\pi[a,i] y_b\) by (4.24). If \(b > i + 1\), then the claim follows by the inductive hypothesis.

Combining the above computations of the summands of the right hand side of (4.32), we obtain
\[
\pi[a,i] \cdot \pi[i+1,b] = \left(\frac{b - a + 1}{i - a + 1}\right) \pi[a-1,b-1] + \left(\frac{b - a}{i - a}\right) \pi[a,b-1] y_i + \left(\frac{b - a}{i - a + 1}\right) \pi[a,b-1] y_b.
\]
By applying (4.30) to the second summand of the right hand side, it follows that
\[
\pi_{[a,i]} \cdot \pi_{[i+1,b]} = \left( \frac{b-a+1}{i-a+1} \right) \pi_{[a-1,b-1]} + \left( \frac{b-a}{i-a+1} \right) \pi_{[a,b-1]} y_b + \left( \frac{b-a}{i-a+1} \right) \pi_{[a,b-1]} y_b
\]
\[
= \left( \frac{b-a+1}{i-a+1} \right) \pi_{[a-1,b-1]} + \left( \frac{b-a+1}{i-a+1} \right) \pi_{[a,b-1]} y_b
\]
where we used (4.27) for the last equality. This completes the proof. \(\square\)

For simplicity, we write
\[
\pi_i := \pi_{\{i\}} = y_1 + y_2 + \cdots + y_i \quad (1 \leq i \leq n-1)
\]
(cf. (4.4)). As for the previous proposition, the following is an analogue of [5, Lemma 5.1] which is a claim for \(\varpi_J\) in the cohomology \(H^*(Pet_n; \mathbb{Z})\).

**Proposition 4.19.** For \(1 \leq a \leq i \leq b \leq n-1\), we have
\[
\pi_i \cdot \pi_{[a,b]} = (b-i+1)\pi_{[a-1,b]} + (i-a+1)\pi_{[a,b+1]}
\]
in \(M\) with the convention \(\pi_{[0,b]} = \pi_{[a,n]} = 0\) (See (4.25)).

Before giving a proof, we recall that the identity (4.12) was obtained simply by rearranging terms, without using any non-trivial relations between the cohomology classes \(x_1, x_2, \ldots, x_n\). Thus, it follows that the same identity holds for \(y_1, y_2, \ldots, y_n\) in \(M\) as well:
\[
(y_1 + y_2 + \cdots + y_b) \cdot e_k(y_1, y_2, \ldots, y_b)
\]
\[
= m_{2,1}^{k-1}(y_1, y_2, \ldots, y_b) + (k+1)m_{1,2}^{k+1}(y_1, y_2, \ldots, y_b)
\]
\[
= m_{2,1}^{k-1}(y_1, y_2, \ldots, y_b) + (k+1)e_{k+1}(y_1, y_2, \ldots, y_b)
\]
for \(1 \leq k \leq b \leq n\). We also recall that the claim of Lemma 4.5 was derived only by the relations \(\alpha_i \varpi_i = 0\) \((1 \leq i \leq n-1)^2\). By the definitions of \(M\) and \(I'\), we have the corresponding relations \((y_i - y_{i+1})\pi_i = 0\) \((1 \leq i \leq n-1)\) in \(M\). Thus, the claim of Lemma 4.5 holds for \(y_1, \ldots, y_n\) as well. Its claim for \(d = 1\) gives us that
\[
m_{2,1}^{k-1}(y_1, \ldots, y_b) = km_{1}^{k}(y_1, \ldots, y_b)y_{b+1} = ke_k(y_1, \ldots, y_b)y_{b+1}.
\]

Applying this to the last expression in (4.35), we obtain that
\[
(y_1 + y_2 + \cdots + y_b) \cdot e_k(y_1, y_2, \ldots, y_b)
\]
\[
= ke_k(y_1, \ldots, y_b)y_{b+1} + (k+1)e_{k+1}(y_1, y_2, \ldots, y_b)
\]
for \(1 \leq k \leq b \leq n\). This can be expressed as
\[
\pi_b \cdot \pi_{[a,b]} = (b-a+1)\pi_{[a,b]}y_{b+1} + (b-a+2)\pi_{[a-1,b]} \quad (1 \leq a \leq b \leq n).
\]

\(^2\)In section 4.2, the condition that \(H^*(Pet_n; \mathbb{Z})\) is torsion free was used only in the proof of Corollary 4.1.
Proof of Proposition 4.19. We first prove the equality for the case $i = b$. In this case, the claim is

$$\pi_b \cdot \pi_{[a,b]} = \pi_{[a-1,b]} + (b - a + 1) \pi_{[a,b+1]}.$$  

The left hand side is equal to

$$\pi_b \cdot \pi_{[a,b]} = (b - a + 1) \pi_{[a,b]} y_{b+1} + (b - a + 2) \pi_{[a-1,b]}$$

by (4.36). Thus, we obtain

$$\pi_b \cdot \pi_{[a,b]} = (b - a + 1) \pi_{[a,b]} y_{b+1} + (b - a + 2) \pi_{[a-1,b]}$$

(4.38)

$$= \pi_{[a-1,b]} + (b - a + 1) \left( \pi_{[a-1,b]} + \pi_{[a,b]} y_{b+1} \right).$$

In the parenthesis of the last expression, we have

$$\pi_{[a-1,b]} + \pi_{[a,b]} y_{b+1} = \pi_{[a,b+1]}$$

(4.39)

which we prove by taking cases. If $b < n - 1$, it is obvious by (4.27). If $b = n - 1$, it can be shown as

$$\pi_{[a-1,b]} + \pi_{[a,b]} y_{b+1} = \pi_{[a-1,n-1]} + \pi_{[a,n-1]} y_n$$

$$= e_{n-a+1}(y_1, y_2, \ldots, y_{n-1}) + e_{n-a}(y_1, y_2, \ldots, y_{n-1}) y_n$$

$$= e_{n-a+1}(y_1, y_2, \ldots, y_n)$$

$$= 0 \quad \text{(by the definitions of } M \text{ and } I)$$

$$= \pi_{[a,b+1]} \quad \text{(by } b + 1 = n \text{ and } (4.25))$$

in this case as well. Thus, we obtain from (4.38) and (4.39) that

$$\pi_b \cdot \pi_{[a,b]} = \pi_{[a-1,b]} + (b - a + 1) \pi_{[a,b+1]}.$$  

This verifies (4.37) which is the desired claim (4.34) for the case $i = b$.

Now we prove the claim (4.34) by induction on $b - i \geq 0$. Let $\ell(< n - 1)$ be a non-negative integer, and assume by induction that (4.34) holds when $b - i = \ell$. We prove (4.34) for the case $b - i = \ell + 1(\geq 1)$. In this case, the left hand side of (4.34) can be computed as

$$\pi_i \cdot \pi_{[a,b]} = \pi_i \pi_{[a-1,b-1]} + \pi_i \pi_{[a,b-1]} y_b$$

by (4.27). We compute each summands of the right hand side separately. For the first summand, we have

$$\pi_i \pi_{[a-1,b-1]} = (b - i) \pi_{[a-2,b-1]} + (i - a + 2) \pi_{[a-1,b]}$$

which we prove by taking cases as follows. If $a = 1$, then both sides are equal to 0 by (4.25). If $a > 1$, then the claim follows by the inductive hypothesis since $b - 1 \geq i$. For the second summand, we have

$$\pi_i \pi_{[a,b-1]} y_b = \left( (b - i) \pi_{[a-1,b-1]} + (i - a + 1) \pi_{[a,b]} \right) y_b$$
by the inductive hypothesis since \( b - 1 \geq i \). Combining the computations of the summands of the right hand side of (4.40), we obtain
\[
\pi_i \cdot \pi_{[a,b]} = \left( (b - i) \pi_{[a-2,b-1]} + (i - a + 2) \pi_{[a-1,b]} \right) + (b - i) \pi_{[a-1,b-1]} y_b
\]

\[
= (b - i) \left( \pi_{[a-2,b-1]} + \pi_{[a-1,b-1]} y_b \right) + \pi_{[a-1,b]} + (i - a + 1) \left( \pi_{[a-1,b]} + \pi_{[a,b]} y_b \right)
\]

\[
= (b - i) \left( \pi_{[a-2,b-1]} + \pi_{[a-1,b-1]} y_b \right) + \pi_{[a-1,b]} + (i - a + 1) \left( \pi_{[a-1,b]} + \pi_{[a,b]} y_{b+1} \right),
\]

(4.41)

where we used (4.39) to \( \pi_{[a,b]} y_b \) for the last equality. We now compute the first and the third summands of the last expression separately. For the first summand, we have
\[
(b - i) \left( \pi_{[a-2,b-1]} + \pi_{[a-1,b-1]} y_b \right) = (b - i) \pi_{[a-1,b]}
\]

which we prove by taking cases as follows. If \( a > 1 \), then the claim is obvious by (4.27). If \( a = 1 \), the claim follows since both sides are equal to 0 by (4.25). For the third summand of the last expression of (4.41), we have
\[
(i - a + 1) \left( \pi_{[a-1,b]} + \pi_{[a,b]} y_{b+1} \right) = (i - a + 1) \pi_{[a,b+1]}
\]

by (4.39). Applying (4.42) and (4.43) to the last expression of (4.41), we obtain
\[
\pi_i \cdot \pi_{[a,b]} = (b - i + 1) \pi_{[a-1,b]} + (i - a + 1) \pi_{[a,b+1]}
\]

This verifies (4.34) for the case \( b - i = \ell + 1 \), as desired. \( \square \)

We now prove Proposition 4.15 which completes the proof of Theorem 4.11 as we discussed in section 4.3.

**Proof of Proposition 4.15** Let \( f \in M \) be a polynomial in \( y_1, y_2, \ldots, y_n \). We prove that \( f \) can be written as a linear combination of \( \pi_J \) for \( J \subseteq [n - 1] \). Recalling that \( \pi_i = y_1 + y_2 + \cdots + y_i \), we have
\[
y_i = \pi_i - \pi_{i-1} \quad (1 \leq i \leq n - 1),
\]

\[
y_n = -\pi_{n-1}
\]

with the convention \( \pi_0 = 0 \), where the second equality follows since we have an equality \( y_1 + y_2 + \cdots + y_n = \epsilon_1(y_1, y_2, \ldots, y_n) = 0 \) in \( M \). Hence, we can express \( f \) as a polynomial in
\[
\pi_1, \pi_2, \ldots, \pi_{n-1}.
\]

For our purpose, we may assume that \( f \) is a monomial in these variables with coefficient 1 without loss of generality. Namely, we have
\[
f = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_d}
\]

for some \( d \geq 1 \) and \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d < n \). We prove that this monomial can be expressed as a linear combination of \( \pi_J \) for \( J \subseteq [n - 1] \).
First suppose that \( d \leq n - 1 \). We prove the claim by induction on \( d \geq 1 \). When \( d = 1 \), the claim is obvious since \( \pi_{i_1} = \pi_J \) with \( J = \{i_1\} \) by (3.33). Assume by induction that

\[
\pi_{i_2} \cdots \pi_{i_d} = \sum_{J \subseteq [n-1]} c_J \pi_J
\]

for some integers \( c_J \) \((J \subseteq [n-1])\). Then we have

\[
(4.45) \quad f = \pi_{i_1} \sum_{J \subseteq [n-1]} c_J \pi_J = \sum_{J \subseteq [n-1]} c_J \pi_{i_1} \pi_J.
\]

It suffices to show that each product \( \pi_{i_1} \pi_J \) is expanded as a linear combination of \( \pi_L \) for \( L \subseteq [n-1] \). For this, we take the decomposition \( J = J_1 \sqcup \cdots \sqcup J_m \) into the connected components. By (4.23), we have

\[
\pi_{i_1} \pi_J = \pi_{i_1} \cdot \pi_{J_1} \cdots \pi_{J_m}
\]

since each \( J_k \) consists of a single connected component \((1 \leq k \leq m)\). Set \( J' := J \cup \{i_1\} \).

We consider the case for \( J' \supseteq J \) and the case for \( J' = J \) separately.

If \( J' \supseteq J \), then we have \( i_1 \notin J \). We denote by \( J'(i_1) \subseteq J' \) the connected component of \( J' \) containing \( i_1 \). We have the following three cases:

1. \( J'(i_1) = \{i_1\} \),
2. \( J'(i_1) = J_k \cup \{i_1\} \) for some \( 1 \leq k \leq m \),
3. \( J'(i_1) = J_k \cup \{i_1\} \cup J_{k+1} \) for some \( 1 \leq k < m \).

In the case (1), we have

\[
\pi_{i_1} \pi_J = \pi_{i_1} \cdot \pi_{J_1} \cdots \pi_{J_m} = \pi_J'
\]

by the definition of \( \pi_{J'} \) (see (4.23)). In the case (2), we have \( i_1 = \min J_k - 1 \) or \( i_1 = \max J_k + 1 \). In either case, we have

\[
\pi_{i_1} \pi_J = \pi_{J_1} \cdots \pi_{J_{k-1}} (\pi_{i_1} \pi_{J_k}) \pi_{J_{k+1}} \cdots \pi_{J_m}
\]

\[
= \pi_{J_1} \cdots \pi_{J_{k-1}} \left( \binom{|J_k| + 1}{1} \pi_{J'(i_1)} \right) \pi_{J_{k+1}} \cdots \pi_{J_m}
\]

by Proposition 4.18 and \((|J_k|+1) = (|J_k|+1)\). Hence, we obtain

\[
\pi_{i_1} \pi_J = \left( \binom{|J_k| + 1}{1} \right) \pi_{J'}
\]

in this case. In the case (3), we have

\[
\pi_{i_1} \pi_J = \pi_{J_1} \cdots \pi_{J_{k-1}} (\pi_{J_k} \cdot \pi_{i_1} \cdot \pi_{J_{k+1}}) \pi_{J_{k+2}} \cdots \pi_{J_m}
\]

\[
= \pi_{J_1} \cdots \pi_{J_{k-1}} \left( \binom{|J_k| + 1}{|J_k|} \pi_{J_k \cup \{i_1\}} \cdot \pi_{J_{k+1}} \right) \pi_{J_{k+2}} \cdots \pi_{J_m}
\]

\[
= \pi_{J_1} \cdots \pi_{J_{k-1}} \left( \binom{|J_k| + 1}{|J_k|} \left( \binom{|J_k| + 1 + |J_{k+1}|}{|J_k| + 1} \pi_{J_k \cup \{i_1\} \cup J_{k+1}} \right) \pi_{J_{k+2}} \cdots \pi_{J_m}
\]

\[
= \left( \binom{|J_k| + 1}{|J_k|} \right) \left( \binom{|J_k| + 1 + |J_{k+1}|}{|J_k| + 1} \right) \pi_{J'}
\]

by Proposition 4.18 again. Thus, in either case of (1)-(3), we see that the product \( \pi_{i_1} \pi_J \) is an integer multiple of \( \pi_{J'} \).
If \( J' = J \), then we have \( i_1 \in J_k \) for some \( 1 \leq k \leq m \). In this case, let us write \( J_k = [a, b] \) for some \( 1 \leq a \leq b \leq n - 1 \). Then, Proposition 4.19 implies that

\[
\pi_{i_1} \pi_J = \pi_{J_1} \cdots \pi_{J_{k-1}} (\pi_{i_1} \pi_{J_k}) \pi_{J_{k+1}} \cdots \pi_{J_m}
\]

(4.46)

\[
= \pi_{J_1} \cdots \pi_{J_{k-1}} ((b - i_1 + 1) \pi_{[a-1, b]} + (i_1 - a + 1) \pi_{[a, b+1]}) \pi_{J_{k+1}} \cdots \pi_{J_m}
\]

(4.46)

\[
= (b - i_1 + 1) \pi_{J_1} \cdots \pi_{J_{k-1}} \pi_{[a-1, b]} \pi_{J_{k+1}} \cdots \pi_{J_m} + (i_1 - a + 1) \pi_{J_1} \cdots \pi_{J_{k-1}} \pi_{[a, b+1]} \pi_{J_{k+1}} \cdots \pi_{J_m}.
\]

For the first summand in the last expression, we have

\[
\pi_{J_1} \cdots \pi_{J_{k-1}} \pi_{[a-1, b]} \pi_{J_{k+1}} \cdots \pi_{J_m}
\]

(4.47)

\[
= \begin{cases} 
\pi_{J_{\cup\{a-1\}}} & \text{(if } \max J_{k-1} < a - 2), \\
\left( |J_{k-1}| + b - a + 2 \right) \pi_{J_{\cup\{a-1\}}} & \text{(if } \max J_{k-1} = a - 2),
\end{cases}
\]

(4.47)

where the first case follows from the definition of \( \pi_{J_{\cup\{a-1\}}} \), and the second case follows by applying Proposition 4.18 to \( \pi_{J_{k-1}} \pi_{[a-1, b]} \) in the left hand side. Here, we take the convention \( J_0 = \{-\infty\} \) since only the first case of (4.47) holds when \( k = 1 \). Similarly, for the second summand in the last expression of (4.46), we have

\[
\pi_{J_1} \cdots \pi_{J_{k-1}} \pi_{[a, b+1]} \pi_{J_{k+1}} \cdots \pi_{J_m}
\]

(4.48)

\[
= \begin{cases} 
\pi_{J_{\cup\{b+1\}}} & \text{(if } \min J_{k+1} > b + 2), \\
\left( |J_{k+1}| + b - a + 2 \right) \pi_{J_{\cup\{b+1\}}} & \text{(if } \min J_{k+1} = b + 2).
\end{cases}
\]

(4.48)

Again, we take the convention \( J_{m+1} = \{+\infty\} \) since only the first case of (4.48) holds when \( k = m \). Thus, it follows from (4.46) that \( \pi_{i_1} \pi_J \) is a linear combination of \( \pi_{J_{\cup\{a-1\}}} \) and \( \pi_{J_{\cup\{b+1\}}} \). Applying this result to the right hand side of (4.45), we see that \( f \) is a linear combination of \( \pi_J \) for \( J \subseteq [n - 1] \).

Finally, we consider the case \( d > n - 1 \). In this case, the number of \( \pi_{i_d} \) appearing in (4.44) is greater than \( n - 1 \). Hence, we can write

\[
f = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_d} = (\pi_{i_1} \pi_{i_2} \cdots \pi_{i_{n-1}}) \cdot (\pi_{i_n} \pi_{i_{n+1}} \cdots \pi_{i_d}).
\]

Applying the argument used in the previous case to the product \( \pi_{i_1} \pi_{i_2} \cdots \pi_{i_{n-1}} \), we see that it is an integer multiple of \( \pi_{[n-1]} \). Since we have

\[
\pi_{[n-1]} \cdot \pi_{i_n} = 0
\]

by Proposition 4.19 we see that \( f = 0 \) in this case. This completes the proof. \qed

References

[1] H. Abe, L. DeDieu, F. Galetto, and M. Harada, Geometry of Hessenberg varieties with applications to Newton-Okounkov bodies, Selecta Math. (N.S.) 24 (2018), no. 3, 2129–2163.
[2] H. Abe, T. Horiguchi, and M. Masuda, The cohomology rings of regular semisimple Hessenberg varieties for \( h = (h(1), n, \ldots, n) \), J. Comb. 10 (2019), no. 1, 27–59.
[3] T. Abe, T. Horiguchi, M. Masuda, S. Murai, and T. Sato, Hessenberg varieties and hyperplane arrangements, J. Reine Angew. Math. 764 (2020), 241–286.
[4] H. Abe, M. Harada, T. Horiguchi, and M. Masuda, The cohomology rings of regular nilpotent Hessenberg varieties in Lie type $A$, Int. Math. Res. Not., (2019), no.17, 5316–5388.
[5] H. Abe, T. Horiguchi, H. Kuwata, H. Zeng, Geometry of Peterson Schubert calculus in type $A$ and left-right diagrams, arXiv:2104.02914.
[6] D. Anderson and J. Tymoczko, Schubert polynomials and classes of Hessenberg varieties, J. Algebra 323 (2010), no. 10, 2605–2623.
[7] A. Bălibanu, The Peterson Variety and the Wonderful Compactification, Represent. Theory 21 (2017), 132–150.
[8] S. Cho, J. Hong, and E. Lee, Bases of the equivariant cohomologies of regular semisimple Hessenberg varieties, arXiv:2104.14083v2.
[9] F. De Mari, C. Procesi and M. A. Shayman, Hessenberg varieties, Trans. Amer. Math. Soc. 332 (1992), no. 2, 529–534.
[10] E. Drellich, Combinatorics of equivariant cohomology: Flags and regular nilpotent Hessenberg varieties, PhD thesis, University of Massachusetts, 2015.
[11] Y. Fukukawa, M. Harada, and M. Masuda, The equivariant cohomology rings of Peterson varieties, J. Math. Soc. Japan 67 (2015), no. 3, 1147–1159.
[12] W. Fulton, Young Tableaux, London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge.
[13] W. Fulton, Intersection theory, Second edition. Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin, 1998.
[14] R. Goldin and B. Gorbutt, A positive formula for type $A$ Peterson Schubert calculus, arXiv:2008.12500v2.
[15] R. Goldin, L. Mihalcea, R. Singh, Positivity of Peterson Schubert Calculus, arXiv:2106.10372.
[16] R. Goldin and R. Singh, Equivariant Chevalley, Giambelli, and Monk Formulae for the Peterson Variety, arXiv:2111.15663.
[17] M. Harada, T. Horiguchi, and M. Masuda, The equivariant cohomology rings of Peterson varieties in all Lie types, Canad. Math. Bull. 58 (2015), no. 1, 80–90.
[18] M. Harada and J. Tymoczko, A positive Monk formula in the $S^1$-equivariant cohomology of type $A$ Peterson varieties, Proc. Lond. Math. Soc. (3) 103 (2011), no. 1, 40–72.
[19] T. Horiguchi, Mixed Eulerian numbers and Peterson Schubert calculus, arXiv:2104.14083v2.
[20] E. Insko, Schubert calculus and the homology of the Peterson variety, Electron. J. Combin. 22 (2015), no.2, Paper 2.26, 12 pp.
[21] A. Klyachko, Orbits of a maximal torus on a flag space, Functional Anal. Appl. 19 (1985), no. 2, 65–66.
[22] B. Kostant, Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight $\rho$, Selecta Math. (N.S.) 2 (1996), no. 1, 43–91.
[23] P. Nadeau and V. Tewari, The permutohedral variety, mixed Eulerian numbers, and principal specializations of Schubert polynomials, Int. Math. Res. Not. (2023), no.5, 3615–3670.
[24] D. Peterson, Quantum cohomology of $G/P$, Lecture course, M.I.T., Spring term, 1997.
[25] K. Rietsch, A mirror construction for the totally nonnegative part of the Peterson variety, Nagoya Math. J. 183 (2006), 105–142.
[26] J. Stembridge, Some permutation representations of Weyl groups associated with the cohomology of toric varieties, Adv. Math. 106 (1994), no. 2, 244–301.
[27] E. Sommers and J. Tymoczko, Exponents for B-stable ideals, Trans. Amer. Math. Soc. 358 (2006), no. 8, 3493–3509.
[28] N. Teff, Representations on Hessenberg varieties and Young’s rule, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), 903–914, Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011.
[29] J. Tymoczko, Linear conditions imposed on flag varieties, Amer. J. Math. 128 (2006), no. 6, 1587–1604.
[30] J. Tymoczko, Permutation Actions on Equivariant Cohomology of Flag Varieties, Toric Topology, 365–84, Contemp. Math., 460, Amer. Math. Soc., Providence, RI, 2008.
