TOROIDAL AND PROJECTIVE COMMUTING AND NON-COMMUTING GRAPHS

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Abstract. In this paper, all finite groups whose commuting (non-commuting) graphs can be embed on the plane, torus or projective plane are classified.

1. Introduction

Let $G$ be a non-abelian group. The commuting graph associated to $G$ is an undirected graph with vertex set $G \setminus Z(G)$ such that two distinct vertices $x$ and $y$ are adjacent if $xy = yx$. We denote this graph by $\Gamma_G$. Also, the non-commuting graph of $G$, which is denoted by $\Gamma'_G$, is an undirected graph with vertex set $G \setminus Z(G)$ such that two distinct vertices $x$ and $y$ are adjacent if $xy \neq yx$. Indeed, $\Gamma'_G$ is the complement of $\Gamma_G$. Commuting graphs as well as non-commuting graphs have many interesting properties, for instance it is known that (non-)commuting graphs characterize non-abelian finite simple groups among all finite groups (see [12]).

Recall that a graph is planar if it can be drawn in the plane such that its edges intersect only at their end points. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski’s Theorem [9] states that a graph is planar if and only if it contains no subdivisions of $K_5$ and $K_{3,3}$, where $K_n$ is the complete graph with $n$ vertices and $K_{m,n}$ is the complete bipartite graph with parts of sizes $m$ and $n$.

It is well-known that a compact surface is homeomorphic to a sphere, a connected sum of $g$ tori, or a connected sum of $k$ projective planes (see [10, Theorem 5.1]). We denote $S_g$ for the sphere and $S_g$ ($g \geq 1$) for the surface formed by a connected sum of $g$ tori, and $N_k$ for the one formed by a connected sum of $k$ projective planes. The number $g$ is called the genus of the surface $S_g$ and $k$ is called the crosscap of $N_k$. When considering the orientability, the surfaces $S_g$ and sphere are among the orientable class of surfaces and the surfaces $N_k$ are among the non-orientable one.

A simple graph which can be embedded in $S_g$ but not in $S_{g-1}$ is called a graph of genus $g$. Similarly, if a simple graph can be embedded in $N_k$ but not in $N_{k-1}$, then we call it a graph of crosscap $k$. The notations $\gamma(\Gamma)$ and $\gamma(\Gamma)$ stand for the genus and crosscap of a graph $\Gamma$, respectively. It is easy to see that $\gamma(\Gamma_0) \leq \gamma(\Gamma)$ and $\gamma(\Gamma_0) \leq \gamma(\Gamma)$, for all subgraphs $\Gamma_0$ of $\Gamma$. Clearly, a graph $\Gamma$ is planar if $\gamma(\Gamma) = 0$. A graph $\Gamma$ such that $\gamma(\Gamma) = 1$ is called a toroidal graph. Also, a graph $\Gamma$ such that $\gamma(\Gamma) = 1$ is called a projective graph.

2000 Mathematics Subject Classification. Primary 05C25; Secondary 05C10.

Key words and phrases. Genus, crosscap, commuting graph, non-commuting graph.

This research was in part supported by grants from IPM (No. 91050011) and (No. 900130063).
The aim of this paper is to determine finite non-abelian groups such that their commuting (or non-commuting) graphs are planar, toroidal or projective.

In this paper, \( G \) is a finite non-abelian group. In the following, we remind some useful theorems that will be used frequently in our proofs. We note that \( \lceil x \rceil \) denotes the smallest integer greater than or equal to the given real number \( x \).

**Theorem 1.1** ([11]). For positive integers \( m \) and \( n \), we have

1. \( \gamma(K_n) = \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil \) if \( n \geq 3 \),
2. \( \gamma(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil \) if \( m, n \geq 2 \).

**Theorem 1.2** ([3]). Let \( \Gamma \) be a simple graph with \( v \) vertices \((v \geq 4)\) and \( e \) edges. Then \( \gamma(\Gamma) \geq \left\lceil \frac{1}{6}(e-3v) + 1 \right\rceil \).

**Theorem 1.3** ([5][11]). For positive integers \( m \) and \( n \), we have

1. \( \tau(K_n) = \begin{cases} \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil, & n \geq 3 \text{ and } n \neq 7, \\ 3, & n = 7, \end{cases} \)
2. \( \tau(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil \) if \( m, n \geq 2 \).

A block in a graph is a maximal subgraph with no cut point. The following theorem gives a formula for computing the genus of a graph using its blocks genus.

**Theorem 1.4** ([2]). If \( \Gamma \) is a graph with blocks \( B_1, \ldots, B_n \), then

\[
\gamma(\Gamma) = \gamma(B_1) + \cdots + \gamma(B_n).
\]

Although there is no similar formula for crosscap number of a graph in terms of its blocks crosscap numbers, it is shown in [7] that \( 2K_5 \) is not projective, the fact that will be used in our proofs.

All over this paper, \( \overline{\ast} : G \to G/Z(G) \) denotes the natural homomorphism for a given group \( G \), hence \( \overline{G} = G/Z(G) \) will denote the image group. Also, \( \omega(G) = \{ |x| : x \in G \} \), \( \exp(G) = \text{lcm}(\omega(G)) \), \( Z(G) \) and \( S_p(G) \) (\( p \) prime) denote the spectrum of \( G \), the exponent of \( G \), the center of \( G \) and a Sylow \( p \)-subgroup of \( G \), respectively. In what follows, \( S_n, A_n, D_{2n} \) and \( Q_8 \) stand for the symmetric group of degree \( n \), alternating group of degree \( n \), dihedral group of order \( 2n \) and the quaternion group of order 8. Moreover, the union of \( n \) disjoint copies of a graph \( \Gamma \) will be denoted by \( n\Gamma \).

## 2. Commuting graphs

In this section, we will classify all finite non-abelian groups whose commuting graphs can be embedded in the plane, torus or projective plane. We begin with a simple lemma.

**Lemma 2.1.** Let \( G \) be a \( p \)-group of order \( p^n \), where \( n > 1 \). Then

1. If \( p > 2 \), then \( G \setminus \{1\} \) has a commuting subset with \( p^2 - 1 \geq 8 \) elements.
2. If \( p = 2 \), \( n \geq 5 \) and \( G \) is non-abelian, then \( G \setminus Z(G) \) has two disjoint commuting subsets with 6 elements

**Proof.**

1. Let \( x \) be a central element of \( G \) of order \( p \) and consider the subgroup generated by \( \{x, y\} \) for any \( y \in G \setminus \{x\} \).

2. If \( |Z(G)| \geq 8 \), then consider two distinct cosets of \( Z(G) \). Assume \( |Z(G)| \leq 4 \). Let \( H \) be a subgroup of \( G \) of order 32 containing \( Z(G) \). If \( H \) contains an abelian subgroup \( K \) of order 16, then \( K \setminus Z(G) \) contains two disjoint commuting subsets
with 6 elements. Hence, we may assume that \( H \) does not have abelian subgroups of order 16. Using the following codes in GAP [6], one can easily see that \(|Z(H)| = 2\) and consequently \(|Z(G)| = 2\).

```
for i in [1..NrSmallGroups(32)] do
    H:=SmallGroup(32,i);
    if Maximum(List(Filtered(AllSubgroups(H),IsAbelian),Order))<16 then
        Print(Order(Center(H)),"n");
    fi;
od;
```

Now, by using following codes, it follows that \( H \setminus Z(G) \) contains two disjoint commuting subsets with 6 elements.

```
for i in [1..NrSmallGroups(32)] do
    H:=SmallGroup(32,i);
    L:=Filtered(AllSubgroups(H),IsAbelian);
    counterexample:=true;
    if Maximum(List(L,Order))<16 then
        for A in L do
            for B in L do
                if Order(A)=8 and Order(B)=8 and Order(Intersection(A,B))=2 then
                    counterexample:=false;
                fi;
            od;
        od;
    fi;
    if counterexample=true then
        Print(i,"n");
    fi;
od;
```

The proof is complete. \(\square\)

**Theorem 2.2.** Let \( G \) be a finite non-abelian group. Then \( \Gamma_G \) is planar if and only if \( G \) is isomorphic to one of the following groups:

1. \( S_3, D_8, Q_8, A_4, D_{10}, D_{12}, D_8 \times Z_2, D_8 \times Z_2, S_4, SL(2,3), A_5 \),
2. \( \langle a, b : a^3 = b^4 = 1, a^b = a^{-1} \rangle \cong Z_3 \times Z_4 \),
3. \( \langle a, b : a^4 = b^4 = 1, a^b = a^{-1} \rangle \cong Z_4 \times Z_4 \),
4. \( \langle a, b : a^8 = b^2 = 1, a^b = a^{-3} \rangle \cong Z_8 \times Z_2 \),
5. \( \langle a, b : a^4 = b^2 = (ab)^4 = [a^2, b] = 1 \rangle \cong (Z_4 \times Z_2) \rtimes Z_2 \),
6. \( \langle a, b, c : a^2 = b^2 = c^2 = [a, c] = [b, c] = 1, [a, b] = c^2 \rangle \cong (Z_4 \times Z_2) \rtimes Z_2 \),
7. \( \langle a, b : a^5 = b^4 = 1, a^b = a^3 \rangle \cong Z_5 \times Z_4 \).

**Theorem 2.3.** Let \( G \) be a finite non-abelian group. Then \( \Gamma_G \) is toroidal if and only if \( G \) is isomorphic to one of the following groups:

1. \( D_{11} \),
2. \( D_{16} \),
3. \( Q_{16} \),
4. \( QD_{16} \),
5. \( A_4 \times Z_2 \),
Proof of Theorems 2.2 and 2.3. We will show that there are only finitely many groups whose commuting graph have no subgraphs isomorphic to $K_8$ or $2K_5$ and among them, we will cross out those whose commuting graph is not planar, toroidal or projective. We proceed in some steps.

(1) $|Z(G)| \geq 8$. Then $xz(G)$ induces a complete subgraph for each $x \in G \setminus Z(G)$, which is a contradiction. So, we have $|Z(G)| \leq 7$.

(2) $|Z(G)| \geq 4$. If $\varpi \in \overline{G}$ such that $|\varpi| > 2$, then $xz(G) \cup x^{-1}Z(G)$ induces a complete subgraph with at least 8 elements, which is a contradiction. Thus $\overline{G}$ is an elementary abelian 2-group and hence $G$ is nilpotent. Clearly, $|Z(G)| \neq 5, 7$. If $|Z(G)| = 6$, then $G \cong Z_3 \times H$, where $H$ is an extra special 2-group. Let $\langle x \rangle$ be the Sylow 3-subgroup of $G$. If $A \subseteq H \setminus Z(H)$ is a commuting set, then $\langle x \rangle \times A$ is a commuting set in $G \setminus Z(G)$. Thus $\Gamma_G$ has a subgraph isomorphic to $K_{3,4}$. Hence $|A| \leq 2$ and this is possible only if $H \cong D_8$ or $Q_8$. Therefore $G \cong Z_3 \times D_9$ or $Z_3 \times Q_8$, which is impossible for $\Gamma_G \cong 3K_6$. If $|Z(G)| = 4$, then $G$ is a 2-group and, by Lemma 2.1 it follows that $|G| = 16$.

(3) $|Z(G)| = 3$. If $\varpi \in \overline{G}$ is an element of order $> 3$, then $xz(G) \cup x^2Z(G) \cup x^3Z(G)$ induces a complete subgraph isomorphic to $K_9$, which is impossible. Thus $\omega(\overline{G}) \subseteq \{1, 2, 3\}$. With a same argument one can show that $C_G(x) = \langle Z(G), x \rangle$ for all $x \in G \setminus Z(G)$. Now, we have three cases. If $\overline{G}$ is a 2-group, then $G$ is abelian, which is a contradiction. Also, if $\overline{G}$ is a 3-group and $x, y \in G$ are such that $xy \neq yx$, then $xZ(G) \cup x^{-1}Z(G) \cup yZ(G) \cup y^{-1}Z(G)$ induces a subgraph isomorphic to $2K_6$, which is a contradiction. Therefore, $\overline{G}$ is neither a 2-group nor a 3-group. Then, by [4], either $\overline{G} \cong (Z_2 \times Z_2)^m \times Z_3$ or $\overline{G} \cong Z_3^m \times Z_2$. If $\overline{G} \cong (Z_2 \times Z_2)^m \times Z_3$, then $Z(G)S_2(G) \setminus Z(G)$ induces a complete subgraph with at least 9 elements, which is a contradiction. Thus $G \cong Z_3^m \times Z_2$. By previous arguments, $S_3(G)$ must be abelian, which implies that $|S_3(G) \setminus Z(G)| \leq 7$. Hence, $|S_3(G)| = 9$ and so $|G| = 18$.

(4) $|Z(G)| = 2$. If there is an element $\varpi \in \overline{G}$ with $|\varpi| \geq 5$, then $xz(G) \cup x^2Z(G) \cup x^3Z(G) \cup x^4Z(G)$ induces a subgraph isomorphic to $K_8$, which is impossible. Therefore, $\omega(\overline{G}) \subseteq \{1, 2, 3, 4\}$. Since $Z(G) \subseteq S_2(G)$, by Lemma 2.1 $|G| \leq 2^4 \cdot 3$.

(5) $|Z(G)| = 1$. Clearly, $\omega(\overline{G}) \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$. By Lemma 2.1 $|G| \leq 2^4 \cdot 3 \cdot 5 \cdot 7$. Also, if $7 \in \omega(\overline{G})$, then $S_7(G) \subseteq G$, which implies that $|G| \geq 48$.

Now, the result follows by a simple computation with GAP [6]. The converse is straightforward. 

\section{3. Non-commuting graphs}

In this section, we shall determine all finite non-abelian groups whose non-commuting graphs can be embedded in the plane, torus or projective plane. The following theorem of Abdollahi, Akbari and Maimani gives all planar non-commuting graphs.

**Theorem 3.1** ([1]). Let $G$ be a finite non-abelian group. Then $\Gamma_G$ is planar if and only if $G$ is isomorphic to one of the groups $S_3$, $D_8$ or $Q_8$.

**Theorem 3.2.** There is no toroidal non-commuting graph.

**Proof.** Assume on a contrary that $G$ is a finite group with toroidal non-commuting graph. Let $k(G)$ be the number of conjugacy classes of $G$. Since $|V(\Gamma_G)| = |G| -
$|Z(G)|$ and 

$$2|E(\Gamma'_G)| = |G|^2 - |\{(x, y) \in G \times G : xy = yx\}|$$

$$= |G|^2 - |G||k(G)|,$$

by Theorem 1.2, it follows that $|G|(|G| - k(G) - 6) + 6|Z(G)| \leq 0$. Hence $k(G) \geq |G| - 5$. On the other hand, $k(G)/|G| \leq 5/8$ (see [5]), from which it follows that $|G| \leq 13$. A simple verification shows that $S_3$, $D_8$ and $Q_8$ are the only groups with these properties each of which has a planar non-commuting graph, a contradiction.

□

Theorem 3.3. There is no projective non-commuting graph.

Proof. Suppose on the contrary that $G$ is a finite group with projective non-commuting graph. If $x, y \in G$ are such that $xy \neq yx$, then the subgraph induced by $xZ(G) \cup yZ(G)$ is isomorphic to $K_{|Z(G)|, |Z(G)|}$, which implies that $|Z(G)| \leq 3$.

On the other hand, if $x \in G \setminus Z(G)$, $y \in G \setminus C_G(x)$ and $X$ is the set of all generators of $\langle x \rangle$, then the subgraph induced by $X \cup \langle x \rangle y$ is isomorphic to $K_{|\langle x \rangle|, |\langle x \rangle|}$, where $\varphi$ is the Euler totient function, from which it follows that $|x| \leq 4$ or $|x| = 6$. If $|x| = 6$ then there exists a suitable power $x^i$ of $x$ such that $x^i \in G \setminus C_G(y)$ and the subgraph induced by $\{x, x^{-1}, x^i\} \cup \langle x \rangle y$ is isomorphic to $K_{3, 6}$, which is a contradiction. Therefore, $\omega(G) \subseteq \{1, 2, 3, 4\}$. On the other hand, if $x \in G$ such that $|\langle x \rangle| = 4$, then the subgraph induced by $\{x, x^{-1}, x^2\} \cup (G \setminus C_G(x^2))$ has a graph isomorphic to $K_{3, |G \setminus C_G(x)|}$, which implies that $|G \setminus C_G(x^2)| \leq 4$. Hence, $|G| = 8$ and consequently $\exp(G) = 2$, which is a contradiction. Therefore, $\omega(G) \subseteq \{1, 2, 3\}$. Since $G$ has no elements of order 6, it follows that $G$ is a 3-group, $G$ is a 2-group or $Z(G) = 1$. Thus, we have the following cases:

Case 1. $G$ is a 3-group. If $x \in G \setminus Z(G)$ and $y \in G \setminus C_G(x)$, then the subgraph induced by $xZ(G) \cup y^{-1}Z(G) \cup yZ(G)$ is isomorphic to $K_{3, 6}$, which is a contradiction.

Case 2. $G$ is a 2-group. Then $|Z(G)| = \exp(G) = 2$, which implies that $G$ is an extra special 2-group. So, $G = G_1 \circ \cdots \circ G_n$ is the central product of $G_1, \ldots, G_n$, where $G_i \cong D_8$ or $Q_8$, for $i = 1, \ldots, n$. Let $x, y \in G_1$ with $xy \neq yx$. If $n > 1$, then the subgraph induced by $xG_2 \cup yG_2$ is isomorphic to $K_{8, 8}$, which is impossible. Thus $n = 1$ and subsequently $G \cong D_8$ or $Q_8$, a contradiction.

Case 3. $|Z(G)| = 1$. Let $P = S_2(G)$, $Q = S_3(G)$ and $x, y \in G$ be elements of orders 2 and 3, respectively. By Case 2, either $P$ is abelian, or $P \cong D_8$ or $Q_8$. If $P$ is abelian, then the subgraph induced by $(P \setminus \{1\}) \cup Py$ is isomorphic to $K_{|P| - 1, |P|}$, which implies that $|P| \leq 4$. Hence, $|P| \geq 8$ in all cases. On the other hand, $Q$ is abelian, which implies that the subgraph induced by $(Q \setminus \{1\}) \cup Qx$ is isomorphic to $K_{|Q| - 1, |Q|}$. So, we have $|Q| = 3$. Therefore $|G| \geq 24$. The only groups with these properties are $S_3$, $A_4$ and $S_4$ each of which has a non-projective non-commuting graph. The proof is complete.

Acknowledgments. The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

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