Fluctuations of random matrix products and 1D Dirac equation with random mass

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(Dated: February 27, 2014)

We study the fluctuations of certain random matrix products $\Pi_N = M_N \cdots M_2 M_1$ of $\text{SL}(2, \mathbb{R})$, describing localisation properties of the one-dimensional Dirac equation with random mass. We obtain convenient integral representations for the variance $\Gamma_2 = \lim_{N \to \infty} \text{Var}(\ln |\Pi_N|)/N$. The case studied exhibits a saturation of the variance at low energy $\varepsilon$ along with a vanishing Lyapunov exponent $\Gamma_1 = \lim_{N \to \infty} \ln |\Pi_N|/N$, leading to the behaviour $\Gamma_2/\Gamma_1 \sim \ln(1/|\varepsilon|) \to \infty$ as $\varepsilon \to 0$.

Our continuum description sheds new light on the Kappus-Wegner (band center) anomaly.

PACS numbers: 72.15.Rn ; 02.50.-r

Transfer matrices lead to a convenient formulation of many statistical physics problems and have been extensively used since their introduction in the context of the Ising model [1]. In the presence of randomness, most of the physics is captured by the Lyapunov exponent $\Gamma_1$ which quantifies the growth rate of the matrix elements of a random matrix product (RMP) $\Pi_N = M_N \cdots M_2 M_1$. Given the measure characterizing the independent and identically distributed random matrices $M_n$’s, the Furstenberg formula allows one to obtain, at least in principle, the Lyapunov exponent $\Gamma_1 = \lim_{N \to \infty} \ln |\Pi_N|/N$, where $| \cdot |$ is a norm, in terms of the solution of the Furstenberg’s integral equation [2]. Besides the Lyapunov exponent, which describes the mean free energy of the random Ising model [3, 4], the fluctuations of RMP (Fig. 1) might also play an important role and are the main subject of this letter. They have been considered in the mathematical literature (see chapter V of Ref. [2] for a review on a generalized central limit theorem). The analysis of fluctuations is related to the generalised Lyapunov exponent analysis [3, 4] and the multifractal formalism introduced by Paladin and Vulpiani [5, 6]. Fluctuations are of particular importance in the context of quantum localisation, where they dominate several physical quantities, like the local density of states [7] or the Wigner time delay [8]. Their precise characterisation is an important issue at the heart of the scaling approach used in the justification of the single parameter scaling (SPS) hypothesis (cf. [9, 10] and references therein). In the last decade, this question has been re-examined more precisely for a lattice model [11, 13]. Recently Lyapunov exponents have been analytically obtained for general RMPs of $\text{SL}(2, \mathbb{R})$ in the continuum limit [14]. This has significantly improved our understanding of RMPs and of one-dimensional (1D) quantum localisation models due to their close connection [15]. The present work is a first attempt to generalise this approach for the fluctuations. We will consider matrices belonging to two particular subgroups of $\text{SL}(2, \mathbb{R})$:

$$M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} e^{\eta_n} & 0 \\ 0 & e^{-\eta_n} \end{pmatrix}$$

(1)
or

$$M_n = \begin{pmatrix} \cosh \tilde{\theta}_n & \sinh \tilde{\theta}_n \\ \sinh \tilde{\theta}_n & \cosh \tilde{\theta}_n \end{pmatrix} \begin{pmatrix} e^{\eta_n} & 0 \\ 0 & e^{-\eta_n} \end{pmatrix} ,$$

(2)

which are shown below to be transfer matrices for the 1D Dirac equation

$$[\sigma_2 i \partial_x + \sigma_1 m(x)] \Psi(x) = \varepsilon \Psi(x) ,$$

(3)

where $\Psi = (\psi, \chi)$ is a bi-spinor and $\sigma_i$ is a Pauli matrix. The Dirac equation (3) with random mass is a relevant model in several contexts of condensed matter, e.g. random spin chains or organic conductors (see references in [16, 17]). It can also be exactly mapped onto supersymmetric quantum mechanics [18] and the Sinai problem of 1D classical diffusion in a random force field [16, 19]. Many properties of the model can be obtained exactly when the mass is chosen to be a Gaussian white noise : $\langle m(x) \rangle = \mu g$ and $\langle m(x)m(x') \rangle = g \delta(x-x')$, where $\langle XY \rangle = \langle X \rangle \langle Y \rangle$ for random variables $X$ and $Y$. For example the Lyapunov exponent, defined in the localisation problem as $\gamma_1 = \lim_{x \to \infty} \ln |\psi(x)/x|$, is known [19] $\gamma_1 = -\mu g + \varepsilon H^{(1)}_\mu (\varepsilon/g)/H^{(1)}_\mu (\varepsilon/g)$, where $H^{(1)}_\mu (z)$ is a Hankel function.
\( \varepsilon \to 0 \), indicating a \textit{delocalisation} point in the spectrum. In this unusual case, the characteristic of \( \gamma_2 = \lim_{x \to \infty} \text{Var}[\ln|\psi(x)|]/x \) (i.e. \( \lim_{N \to \infty} \text{Var}[\ln|\Pi_N|]/N \)) is thus crucial. We will show that the fluctuations saturate as \( \varepsilon \to 0 \), and thus dominate localisation properties.

\textit{Mapping.}— The mapping between RMP and 1D localisation models like the random Kronig-Penney model \([3]\), was recently extended to general RMPs of \( \text{SL}(2, \mathbb{R}) \) \([14]\). For the case of interest here, the mapping works as follows: consider a random mass given as a superposition of delta-functions \( m(x) = \sum_n \eta_n \delta(x - x_n) \). Matching conditions across each impurity read \( \psi(x_n^+) = \psi(x_n^-) e^{\eta_n} \) and \( \chi(x_n^+) = \chi(x_n^-) e^{-\eta_n} \), hence the diagonal matrix in \([1]\), while the rotation of angle \( \theta_n = \varepsilon (x_{n+1} - x_n) \) stands for the free evolution between two impurities. If we consider the Dirac equation with a purely imaginary energy \( \varepsilon \in i \mathbb{R} \), the matrix \([2]\) with \( \theta_n = -i \theta_n \in \mathbb{R} \), relates \((\psi, \chi) = (\psi, -i \chi)\) at \( x_n \) and \( x_{n+1} \) \([20]\). The product \( \Pi_N = M_N \cdots M_2 M_1 \) thus controls the value of the spinor, \( \Psi(x_{N+1}) = \Pi_N \Psi(x_1) \), and the study of the growth of the RMP characterizes the localisation properties of the wave function. It is convenient to introduce the Riccati variable \( z(x) = -\varepsilon \chi(x)/\psi(x) \); from Eq. \([5]\), we find

\[
\frac{d}{dx} z(x) = -\varepsilon^2 - z(x)^2 - 2z(x) m(x). \tag{4}
\]

If the lengths \( \ell_n = x_{n+1} - x_n > 0 \) are either equal (lattice) or distributed with an exponential law \( P(\ell) = \rho e^{-\rho \ell} \), the stochastic differential equation (SDE) defines a Markov process. Then the Lyapunov exponent \( \gamma_1 \) of the localisation problem and \( \Gamma_1 \) of the RMP are simply related by \( \gamma_1 = \rho \Gamma_1 \) \([15]\). In the following we will consider the continuum limit of the RMP problem when the random parameters are small \( \theta_n = \varepsilon \ell_n \to 0 \) and \( \eta_n \to 0 \), i.e. the matrices \( M_n \) are close to the identity matrix, in such a way that \( \langle \eta_n \rangle \to 0 \) and \( g = \langle \eta_n^2 \rangle / \langle \ell_n \rangle \) is fixed; this limit corresponds to the case where \( m(x) \) is a Gaussian white noise with zero mean \([14]\) \([22]\). The SDE \([4]\) must be interpreted in the Stratonovich convention as is usual in physical problems \([23]\). The study of the fluctuations of the RMP can be performed by introducing the generalised Lyapunov exponent \( \gamma_2 \) \([4]\) \([5]\)

\[
\Lambda(q) = \lim_{x \to \infty} \frac{\ln(|\psi(x)|^q)}{x} = \sum_{n=1}^{\infty} \frac{q^n}{n!} \gamma_n, \tag{5}
\]

which is the generating function for the cumulants of \( \ln|\psi(x)| \). In the following discussion we focus on \( \gamma_2 \). From the definition of the Riccati variable, we may write \( \ln|\psi(x)| = \int_0^x dt \ [z(t) + m(t)] \), hence

\[
\gamma_2 = g + 2 \lim_{x \to \infty} \int_0^x dt \ [z(x) [z(t) + m(t)]] \tag{6}
\]

It is convenient to use the relation

\[
\int_{x_0}^{x} dt \ [z(t) + m(t)] = -\ln \frac{z(x)}{z(x_0)} + \int_{x_0}^{x} dt \left( z(t) - \frac{\varepsilon^2}{z(t)} \right), \tag{7}
\]

obtained by integration of \([4]\). Finally we get

\[
\gamma_2 = g - (z \ln |z/\varepsilon|) + \int dx dz' z G(z'|z) \left( z' - \frac{\varepsilon^2}{z'} \right) f(z'). \tag{8}
\]

The propagator is defined as \( G(z'|z) = \int_0^\infty dx [P_z(z'|z) - f(z)] \) where \( P_z(z'|z) \) is the conditional probability solution of the Fokker-Planck equation \( \partial_z \Pi_x(z'z) = \mathcal{G}^T \Pi_x(z'z) \) and \( f(z) = \lim_{x \to \infty} P_x(z'z) \) is the stationary distribution of the Riccati variable, with \( \mathcal{G}^T = 2g \partial_z z \partial_z z + \partial_z (z^2 + \varepsilon^2) \) being the forward generator of the diffusion (adjoint of the generator).

Eq. \([8]\) is one of our main results: \( f \) can be explicitly obtained as the normalisable solution of \( \mathcal{G}^T f = 0 \) and \( G \) solves \( \mathcal{G}^T G(z'|z) = f(z) - \delta(z - z') \). Note that, in the derivation of the second term of \([8]\), we have used the underlying supersymmetry of the Dirac equation \([15]\) \([22]\) \( f(z) = f(-\varepsilon^2/z) |\varepsilon|/z^2 \), leading to \( \langle \ln|z| \rangle = |\varepsilon| \). Solving the equation for \( G \), we can obtain an explicit representation for \( \gamma_2 \) in terms of multiple integrals. We prefer to proceed in a different manner in order to derive limiting values for \( \gamma_2 \).

\textit{Large real energy} \( \varepsilon \gg g \).— The large energy limit is the \textit{universal} regime where SPS holds \([10]\) : a unique scale controls the average and the fluctuations \( \gamma_2 \approx \gamma_1 \). The variance was explicitly calculated for this model in \([24]\) and coincides with the known value for the Lyapunov exponent \([19]\), recalled above, that saturates at high energy: \( \gamma_2 \approx g/2 \) (Fig. \(3\)).

\textit{Small real energy} \( \varepsilon \ll g \).— The process \( z(x) \) flows through the full interval \( \mathbb{R} \) and it is convenient to consider the variable \( \zeta = \mp \ln(|z|)/|\varepsilon|/2 \) for \( z \in \mathbb{R}_\pm \). When \( z(x) \) goes from \(+\infty\) to \(-\infty\) the process \( \zeta(x) \) crosses \( \mathbb{R} \) once, and a second time when \( z(x) \) goes from \( 0 \) to \(-\infty\). The new process obeys the SDE

\[
\frac{d}{dx} \zeta(x) = -\mathcal{U}(\zeta(x)) + m(x) \tag{9}
\]

for the unbounded potential \( \mathcal{U}(\zeta) = -(|\varepsilon|/2) \sinh 2\zeta \). Rewriting \([8]\) in terms of the new variable, we get \([25]\)

\[
\gamma_2 = g - 2(\mathcal{U}(\zeta) + 8) \int d\zeta d\zeta' \mathcal{G}(\zeta|\zeta') \mathcal{G}(\zeta'|\zeta') \mathcal{P}(\zeta') \tag{10}
\]

where \( \mathcal{G}^T P = 0 \) and \( \mathcal{G}^T \mathcal{G}(\zeta'|\zeta) = \mathcal{P}(\zeta) - \delta(\zeta - \zeta') \) for the forward generator \( \mathcal{G}^T = (g/2) \partial_\zeta^2 + \partial_\zeta \mathcal{U}(\zeta) \). The variable \( \zeta \) is appropriate for the low energy analysis: the exponential dependence of the potential clearly illustrates the decoupling between the “deterministic force” \( \mathcal{U}(\zeta) \) and Langevin “force” \( m(x) \). We can map the problem onto an effective free diffusion problem in the interval
of the Lyapunov exponent coinciding with the asymptotic of the exact result [19] and where the constant term in (13,15) is sensitive to the pre-

position at one boundary, \( P(\zeta^+) = 0 \), with reinjection of the current at the other boundary, \( P'(\zeta^-) = P'(\zeta^+) \). The stationary distribution takes the approximate form

\[
P(\zeta) \simeq 2(\zeta^+ - \zeta)(\zeta^+ - \zeta^-)^2
\]

and suggest a logarithmic behaviour, although it is difficult to precisely fit this logarithmic correction (Fig. 3).

Large complex energy \(-i \varepsilon \gg g\).—In this case, we have found it convenient to develop a perturbative approach based on the SDE [4]. We perform the rescaling \( x = u/|\varepsilon| \) and \( z(x) = |\varepsilon| + \sqrt{g|\varepsilon|} y(u) \), leading to

\[
\frac{dy(u)}{du} = -2y(u) - 2\eta(u) - \alpha \left[ y(u)^2 + 2y(u)\eta(u) \right],
\]

where \( \eta(u) \) is a normalised Gaussian white noise, \( \langle \eta(u)\eta(u') \rangle = \delta(u - u') \). The perturbative parameter is \( \alpha = \sqrt{g/|\varepsilon|} \). Expansion of the process in powers of \( \alpha \), as \( y = y_0 + y_1 + y_2 + \cdots \), leads to

\[
y_0(u) = -2\int_0^u dt e^{-2(u-t)} \eta(t), \tag{18}
\]

\[
y_1(u) = -\alpha \int_0^u dt e^{-2(u-t)} \left[ y_0(t)^2 + 2y_0(t)\eta(t) \right], \tag{19}
\]

\[
y_2(u) = -2\alpha \int_0^u dt e^{-2(u-t)} \left[ y_0(t) + \eta(t) \right] y_1(t), \tag{20}
\]

where transient terms have been neglected. Order zero is the Ornstein-Uhlenbeck process. These expressions are suitable to compute \( \gamma_1 = \langle z \rangle \simeq g/2 \) and the correlator \( \langle \eta(u) \rangle \). Using the stochastic calculus functionalities of Mathematica 9.0 we obtain \( \gamma_2 \simeq g^2/(4|\varepsilon|) \).

Monte Carlo simulations confirm this power law decay (Fig. 2), however we observe a saturation for \( |\varepsilon| \gg \rho \) that we understand as a deviation from the continuum limit (the simulations are performed with a non-Gaussian white noise). We have found that the saturation value is \( \gamma_2 \simeq \rho \langle \eta^4 \rangle /4 = 3g^2/(2\rho) \) [25], in agreement with the numerics (dotted lines in Fig. 2).

Low energy localisation.—The saturation of the fluctuations concomitant with the vanishing of the Lyapunov exponent has important consequences for the localisation. While the Lyapunov exponent is usually introduced as a
After a rotation $\Psi = \exp(-i\varphi)$, the expansion gives $[3,31]$ the expansion for the tight-binding (Anderson) model $[17]$ (Thouless criterion). This is a new indication that the disorder cannot be treated perturbatively for a given small energy $|\varepsilon| \ll g$, the fluctuations dominate $\sqrt{\gamma_2} \gg \gamma_1 x$ for $x \lesssim \xi_x = (1/g) \ln^2(g/|\varepsilon|)$, i.e., on a scale $\xi_x$ much larger than the inverse Lyapunov exponent $1/\gamma_1 \sim (1/g) \ln(g/|\varepsilon|)$. The scale $\xi_x$ has appeared in other studies: in the average Green’s function $[19]$ (see discussion and references in $[27]$), in the distribution of the distances between consecutive nodes of the wave function $[17]$, or in the boundary sensitive average local $[28]$ and global density of states $[17]$ (Thouless criterion). This is a new indication that the Lyapunov exponent cannot be interpreted as the inverse localisation length in this case $[17,22]$.

**Band center anomaly.** — The standard weak disorder expansion for the tight-binding (Anderson) model $-t\varphi_{n+1} + V_n \varphi_n - t\varphi_{n-1} = \varepsilon \varphi_n$ is known to break down in the band center ($\varepsilon = 0$) $[20,30]$. Whereas the standard expansion gives $[3,31]$ $\gamma_1 \simeq a \langle V_n^2 \rangle / (8t^2 \sin^2 k)$ at $\varepsilon = -2t \cos k$ for uncorrelated potentials $V_n$ (a is the lattice spacing), the correct behaviour in the band center is $\gamma_1 \simeq a \Gamma(3/4)/\Gamma(1/4)^2 \langle V_n^2 \rangle / t^2$ $[31]$. This small difference, 0.125 vs 0.114…, and those of other physical quantities, have been referred to as band center “anomalies” $[32]$. This phenomenon may be easily analysed within our continuum description: the continuum limit of the Anderson model near the band center is the random Dirac equation $[-i\sigma_3 \partial_x + V_0(x) + \sigma_1 V_n(x)] \Psi = \varepsilon \Psi$ for $2at = 1$, where $V_0(x)$ and $V_n(x)$ describe forward and backward (unklapp) scattering, respectively. This makes it clear that the disorder cannot be treated perturbatively for $\varepsilon = 0$. After a rotation $\Psi = (1/\sqrt{2})(1 - i\sigma_1) \Psi$ and choosing $V_0(x) = \sum_n v_n \delta(x - x_n)$ and $V_n(x) = \sum_n \eta_n \delta(x - x_n)$, this disordered model can be described by transfer matrices $[1]$, by setting the angles $\theta_n = \varepsilon (x_{n+1} - x_n) - v_n$. For $\varepsilon = 0$, the Lyapunov exponent in the continuum limit is expressed in terms of elliptic integrals $[14]$:

\[
\gamma_1 = g \left[ \frac{1}{k^2} \left( \frac{\mathbf{E}(k)}{\mathbf{K}(k)} - 1 \right) + 1 \right] \quad \text{with} \quad k = \frac{1}{\sqrt{1 + g_0/g}}
\]

where $g = \rho \langle \eta_n^2 \rangle$ and $g_0 = \rho \langle v_n^2 \rangle$. For uncorrelated site potentials, $\langle V_n V_m \rangle \propto \delta_{n,m}$, we have $g_0 = g$; Eq. (21) leads to $\gamma_1 = g \left[ 2\Gamma(3/4)/\Gamma(1/4)^2 \right]$, in perfect correspondence with Ref. $[31]$, and $\gamma_2/\gamma_1 \simeq 1.047$ $[12]$, i.e., one finds small anomalies. On the other hand, the suppression of forward scattering ($g_0 \ll g$) leads to the model studied in the present letter with a strong anomaly $\gamma_2/\gamma_1 \sim \ln(g/g_0)$ at $\varepsilon = 0$ and $\gamma_2/\gamma_1 \sim \ln(g/|\varepsilon|)$ for $g_0 \ll |\varepsilon| \ll g$.

**Conclusion.** — In this letter we have characterized the statistical properties of random matrix products $\Pi_N = M_N \cdots M_2 M_1$ for two subgroups of $\text{SL}(2,\mathbb{R})$, by making use of the fact that, for a certain choice of the distribution of the angles in $[1]$ and $[2]$, the transfer matrix interpretation leads to a Markov process. We have deduced the variance explicitly: the integral representations Eqs. (8,10) were demonstrated to be convenient for extracting limiting behaviours. Following Ref. $[6]$ and making use of $[7]$, the generalised Lyapunov exponent $\Lambda(q)$ may be obtained as the largest eigenvalue of the operator $\hat{\gamma}^q + q (z - z^2/2)/2$. The cumulants could be obtained by using the perturbative method used in $[12,33]$, however, apart from $\gamma_1$, this leads to integral representations that seem less convenient to handle. We have also obtained an integral representation similar to $[8]$ in the case of two other particular subgroups of $\text{SL}(2,\mathbb{R})$ $[25]$, corresponding to the model studied in $[33]$, however we have not been able to extend our method to the whole group. It remains a challenging issue to obtain a resolution of this problem, in the spirit of the general classification of Lyapunov exponents provided recently in $[14]$.

**Acknowledgements.** — We acknowledge stimulating discussions with Alain Comtet, Bernard Derrida, Thierry Jolicoeur and Satya Majumdar, and a helpful suggestion of Anupam Kundu.

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Fluctuations of random matrix products and 1D Dirac equation with random mass
Supplementary material

I. Details of the derivation of Eq. (10)

A. $\varepsilon^2 > 0$

For real energy, we see from the SDE (4) that the process $z(x)$ flows towards $\mathbb{R}$. We have introduced the change of variable $\zeta = \mp \ln(z/|z|)/2$ for $z \in \mathbb{R}_\pm$, implying that the new process $\zeta(x)$ crosses $\mathbb{R}$ twice when $z(x)$ does once. Hence the change of variable maps the SDE (8) onto the couple of SDEs

$$\frac{d\zeta}{dx} = |z| \cosh 2\zeta \pm m(x) = -U_\pm'(\zeta) + \sqrt{g} \eta(x). \quad (22)$$

In the main text we used $\langle m(x) \rangle = 0$ and the local nature of the mass correlation to disregard the sign. Here we consider for the moment the general case $\langle m(x) \rangle = \mu g$ and introduce a couple of potentials $U_\pm(\zeta) = -(|z|/2) \sinh 2\zeta \mp \mu \zeta$ related to the cases $z(x) > 0$ and $z(x) < 0$, respectively. $\eta(x)$ is a normalised Gaussian white noise with zero mean. The process is characterised by two stationary distributions $P(\pm) (\zeta)$, each normalized, related to $f(z)$ for $z \in \mathbb{R}_\pm$. For example, the

Lyapunov exponent is given by

$$\gamma_1 = \langle z + m(x) \rangle \quad (23)$$

$$= \mu g + |z| \left[ \int d\zeta P(\zeta) e^{-2\zeta} - \int d\zeta P^-(\zeta) e^{+2\zeta} \right].$$

Now considering the case $\mu = 0$ for which $P^+ = P^-$, leads to $\gamma_1 = -|z| \langle \sinh 2\zeta \rangle$, i.e.

$$\gamma_1 = 2 \langle U(\zeta) \rangle. \quad (24)$$

Fluctuations may be discussed in a similar way. A crucial observation is that, in the original SDE (4), the diffusion effectively vanishes at $z = 0$, implying the absence of correlations between different coordinates $x$ and $x'$ associated to $z(x) > 0$ and $z(x') < 0$. It follows that the contributions of the fluctuations related to the two intervals $z \in \mathbb{R}_+$ and $z \in \mathbb{R}_-$ simply add. The second term of (8) takes the form

$$\int \frac{1}{2} \left( \int d\zeta d\zeta' |z| e^{-2\zeta} G^+(\zeta|\zeta') (-2|z|) \sinh 2\zeta' P^+(\zeta') \right. \quad (26)$$

$$\left. + \int d\zeta d\zeta' (-|z|) e^{+2\zeta} G^-(\zeta|\zeta') (2|-z|) \sinh 2\zeta' P^-(\zeta') \right).$$

For $\mu = 0$ we have $G^+ = G^-$ leading to Eq. (10).

B. $\varepsilon^2 < 0$

For imaginary energy the analysis is slightly different: the process $z(x)$ is trapped on $\mathbb{R}_+$ and $\zeta(x)$ does not flow across $\mathbb{R}$. The change of variable is simply $\zeta = -(1/2) \ln(z/|z|)$. The new process is trapped by the potential well $U(\zeta) = (|z|/2) \cosh 2\zeta - \mu \zeta$. The equilibrium distribution is $\mathcal{P}(\zeta) \propto \exp [- (2/g) U(\zeta)]$. When $\mu = 0$ the potential is symmetric. We can symmetrize the expression $\gamma_1 = |z| \langle e^{-2\zeta} \rangle$, leading to $\gamma_1 = |z| \langle \cosh 2\zeta \rangle$, i.e. again to (24). Eq. (9) leads to

$$\gamma_2 = g + 2|z| \langle \zeta e^{-2\zeta} \rangle \quad (25)$$

$$+ 2|z|^2 \int d\zeta d\zeta' e^{-2\zeta} G(\zeta|\zeta') \cosh 2\zeta' \mathcal{P}(\zeta').$$

The second term can be obviously symmetrized, which gives the second term of (10). Symmetrization of the third integral term works as follows: the propagator may be decomposed over the left/right eigenvectors of the forward generator $\mathcal{G}$ as

$$\mathcal{G}(\zeta|\zeta') = \sum_{n>0} \frac{\Phi_n R(\zeta) \Phi_n^L(\zeta')}{\mathcal{E}_n} \quad (26)$$

where $\mathcal{G} \Phi_n R(\zeta) = -\mathcal{E}_n \Phi_n R(\zeta)$ and $\mathcal{G} \Phi_n L(\zeta) = -\mathcal{E}_n \Phi_n L(\zeta)$. Because the potential is symmetric, the eigenvectors have a symmetry property $\Phi_n^L(-\zeta) = -1^n \Phi_n^R(\zeta)$. Integration over $\zeta'$ in (25) selects only the contributions of even eigenvectors which allows one to symmetrize the integrand with respect to $\zeta \rightarrow -\zeta$, leading to Eq. (10).

It is remarkable that despite the dynamics of the process $\zeta(x)$ being quite different for real and imaginary $\varepsilon$, we have found a unique representation for both $\gamma_1$, Eq. (12), and $\gamma_2$, Eq. (10), expressed in terms of the potential $U(\zeta)$.

II. Direct calculation of $\gamma_2$ for $\varepsilon = 0$

The study of the case $\varepsilon = 0$ shows some subtlety related to the choice of the norm of the matrix. In the usual case, the statistical properties of the RMP are independent of the precise definition of the norm [2, 3]. Bougerol and other authors propose

$$||M|| = \text{Sup} \{ |M| : x \in \mathbb{R}^2 ; |x| = 1 \} \quad (27)$$

where $|x|$ is the norm on the vector space.
In the numerical calculation (next section), we parametrize the spinor as \( \Psi = e^\xi (\sin \Theta, -\cos \Theta) \), in the spirit of the phase formalism [3], and study the statistical properties of \( \xi(x) = (1/2) \ln |\Psi(x)\rangle \langle \Psi(x)|, \) usually setting \( \Theta(0) = 0 \). Let us discuss the general case where \( \Theta_0 \) may differ from 0. Since \( \Psi(x_{N+1}) = \Pi_N \Psi(x_1) \), the numerical procedure corresponds to considering the norm
\[
||\Pi_N||_{\Psi_0} = ||\Pi_N \Psi_0|| \quad \text{with} \quad \Psi_0 = \begin{pmatrix} \sin \Theta_0 \\ -\cos \Theta_0 \end{pmatrix},
\]
i.e. \( \xi(x_{N+1}) = \ln ||\Pi_N||_{\Psi_0} \). We also introduce another possible definition of the norm
\[
|||\Pi_N||| = \int_{|\Psi_0|=1} d\Psi_0 ||\Pi_N||_{\Psi_0},
\]
closer to the spirit of (27).

For \( \varepsilon = 0 \), the matrix product \( \Pi_N \) can be studied rather directly : the angles vanish \( \Theta_n = 0 \) and the matrices \( M_n \) commute. Hence we can write
\[
\Pi_N = \begin{pmatrix} e^\Lambda & 0 \\ 0 & e^{-\Lambda} \end{pmatrix} \quad \text{with} \quad \Lambda = \sum_{n=1}^N \eta_n.
\]
The distribution of the random variable \( \Lambda \) is given by the central limit theorem : \( \langle \Lambda \rangle = \rho x \langle \eta_n \rangle = 0 \) and \( \text{Var}(\Lambda) = \rho x \langle \eta_n^2 \rangle = gx \) (we consider that \( x \) is fixed and \( N \) fluctuates with \( \langle N \rangle = \rho x \)). We have
\[
\ln ||\Pi_N||_{\Psi_0} = \frac{1}{2} \ln \left[ \cosh 2\Lambda - \cos 2\Theta_0 \sinh 2\Lambda \right].
\]

We examine first the particular case \( \Theta_0 = 0 \), leading to \( \ln ||\Pi_N||_{\Psi_0} = -\Lambda \). We immediately deduce that \( \langle \ln ||\Pi_N||_{\Psi_0} \rangle = 0 \) and \( \text{Var} \langle \ln ||\Pi_N||_{\Psi_0} \rangle = gx \), which would lead to \( \gamma_1 = 0 \) and, incorrectly, to \( \gamma_2 = g \). The choice \( \Theta_0 = \pi/2 \) leads to a similar conclusion. This reflects the statistical properties of the two particular zero energy solutions
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{f x'} dx' m(x') \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-f x'} dx' m(x')
\]
selected by the choices \( \Theta_0 = \pi/2 \) and \( \Theta_0 = 0 \), respectively.

We now consider the case of an arbitrary initial vector, with \( \Theta_0 \notin \{0, \pi/2\} \). In the \( N \to \infty \) limit, the large \( \Lambda \) behaviour of the norm is selected : \( \ln ||\Pi_N||_{\Psi_0} \simeq |\Lambda| + \theta_1(\Lambda) \ln |\sin \Theta_0| + \theta_2(-\Lambda) \ln |\cos \Theta_0| \). Some algebra gives, for \( gx \gg 1 \),
\[
\langle \ln ||\Pi_N||_{\Psi_0} \rangle \simeq \frac{2gx}{\pi} + \frac{1}{\ln 2} \sin 2\Theta_0
\]
and
\[
\text{Var} \langle \ln ||\Pi_N||_{\Psi_0} \rangle \simeq gx \left( 1 - \frac{2}{\pi} \right) + \frac{1}{4} \ln^2 |\tan \Theta_0|.
\]
Note that the average value is reminiscent of the average of the logarithm of the transmission probability [4] (this calculation was first performed in Ref. [6] in another context). This leads to
\[
\gamma_1 = 0 \quad \text{(35)}
\]
\[
\gamma_2 = g \left( 1 - \frac{2}{\pi} \right) = g \times 0.363380... \quad \text{(36)}
\]

We can easily repeat this calculation with the second norm. Averaging of (31) gives
\[
|||\Pi_N||| = \frac{2|\Lambda|}{\pi} E \left( 1 - e^{-4|\Lambda|} \right),
\]
where \( E(k) \) is the elliptic integral [7]. We deduce the asymptotic behaviours \( \ln |||\Pi_N||| \simeq (3/4)\Lambda^2 \) for \( |\Lambda| \ll 1 \) and \( \ln |||\Pi_N||| \simeq |\Lambda| - \ln(\pi/2) \) for \( |\Lambda| \gg 1 \), leading again to (35-36).

In conclusion : for \( \varepsilon \neq 0 \), the calculation of the cumulants \( \gamma_n \) is insensitive to the precise definition of the norm, i.e. to the precise choice of the initial spinor. In the Monte Carlo simulation, we have chosen \( \Theta_0 = 0 \) in order to set a Dirichlet boundary condition for the first component of the spinor. On the other hand, setting \( \varepsilon = 0 \), the behaviour of \( \gamma_2 \) as a function of \( \Theta_0 \) presents two discontinuities precisely at 0 and \( \pi/2 \). We understand these singular values as resulting from a lack of ergodicity in the matrix space when considering the Abelian subgroup describing the case \( \varepsilon = 0 \). Hence, the value \( g \) found for \( \Theta_0 = 0 \) or \( \pi/2 \) should not be taken as the correct result.

### III. Numerical calculation

#### A. Method

We have performed a Monte Carlo simulation of the matrix problem, i.e. of the Dirac equation. We consider
\[
m(x) = \sum_n \eta_n \delta(x - x_n)
\]
where the impurities are independently and uniformly dropped on the line with a mean density \( \rho \). This corresponds to an exponential distribution \( P(\ell) = \rho e^{-\rho \ell} \) for the distance \( \ell_n = x_{n+1} - x_n > 0 \) between consecutive impurities. The mass is uncorrelated in space, i.e. is a non-Gaussian white noise. The limit of the Gaussian white noise corresponds to \( \rho \to \infty \) and \( \eta_n \to 0 \) with \( \langle \eta_n \rangle = 0 \) and \( g = \rho \langle \eta_n^2 \rangle \) fixed. This is a continuum model that is easy to implement numerically.

**Real energy.**– We parametrize the spinor as \( \Psi = e^\xi (\sin \Theta', \cos \Theta') \) and study the evolution of the two variables. Between impurities \( n \) and \( n+1 \) we have obviously \( \Theta_{n+1} - \Theta_n = \varepsilon \ell_n \) and \( \xi_{n+1} - \xi_n = 0 \), where we have introduced the notation \( \Theta_n = \Theta(x_n) \) and \( \xi_n = \xi(x_n) \).
Across the impurity \[ \tan (\Theta^+_n) = \tan (\Theta^-_n) e^{2\eta_n^2} \] and \[ \xi^+_n - \xi^-_n = (1/2) \ln (e^{2\eta_n^2} \sin^2 \Theta^+_n + e^{-2\eta_n^2} \cos^2 \Theta^+_n) \]. The norm of the RMP is identified with the norm of the spinor

\[
\ln ||\Pi_N||_{\Psi_0} = \xi(x_{N+1}) = \frac{1}{2} \ln [\Psi(x_{N+1})] \Psi(x_{N+1})^\dagger \psi(x_{N+1}) . \tag{39}
\]

**Complex energy.** For \( \varepsilon \in \mathbb{i} \mathbb{R} \), we write the spinor as \( \psi = e^{\xi}(\sin \Theta, -i \cos \Theta) \). Evolution of the two variables due to the rotation of complex angle is \[ \tan (\Theta^+_n + \pi/4) = \tan (\Theta^-_n + \pi/4) e^{2i\varepsilon n} \] and \( \xi^+_n - \xi^-_n = (1/2) \ln (\cos 2\Theta^+_n/\cos 2\Theta^-_n) \). Evolution across an impurity are similar to the case of real energy.

**B. The saturation of \( \gamma_2 \) for \( \varepsilon \to i\infty \)**

For large but finite density \( \rho \) we show that the fluctuations saturate for large complex energy \( |\varepsilon| \gg \rho \).

Expansion of the previous equations in the limit \( |\varepsilon| \ell_n \to \infty \) gives \( \xi^+_n - \xi^-_n = |\varepsilon| \ell_n + \ln \cosh \eta_n + O(e^{-2|\varepsilon|\ell_n}) + O(e^{-2|\varepsilon|\ell_n}) \). We deduce the following representation for the process \( \xi(x) \)

\[
\xi(x) = |\varepsilon|x + \sum_{n=1}^{N(x)} \ln \cosh \eta_n + O\left(\frac{\rho}{|\varepsilon|} \rho x\right) \tag{40}
\]

where \( N(x) \) is the number of impurities on the interval \([0, x] \). \( N(x) \) is a Poisson process and \( \xi(x) \) a compound Poisson process (see for instance the introduction of [11] and references therein). Using standard properties of compound Poisson processes, we obtain

\[
\gamma_1 = \lim_{x \to \infty} \frac{\xi(x)}{x} \approx |\varepsilon| + \rho \langle \ln \cosh \eta_n \rangle \tag{41}
\]

\[
\gamma_2 = \lim_{x \to \infty} \frac{\text{Var}(\xi(x))}{x} \approx \rho \langle \ln^2 \cosh \eta_n \rangle \tag{42}
\]

(note that the *cumulants* of \( \xi(x) \) involves the *moments* of the jump amplitudes). Hence for \( |\varepsilon| \gg \rho \) with \( \eta_n \ll 1 \) we obtain \( \gamma_1 \approx |\varepsilon| + \rho \langle \eta_n^2 \rangle / 2 = |\varepsilon| + g/2 \) and \( \gamma_2 \approx \rho \langle \eta_n^4 \rangle / 4 = 3g^2/(2\rho) \) (corresponding to weights distributed according to a symmetric exponential law like in [10]). For \( g \ll |\varepsilon| \ll \rho \) we recover the decay \( \gamma_2 \approx g^2/(4|\varepsilon|) \).

**IV. ANOTHER SUBGROUP OF RANDOM MATRICES OF SL(2, \mathbb{R})**

It is well-known that the random Kronig-Penney model \( H = -\frac{d^2}{dx^2} + \sum_n v_n \delta(x-x_n) \) is related to transfer matrices of the form [3]

\[
M_n = \begin{pmatrix} 1 & v_n/k \\ \sin k \ell_n & 1 \end{pmatrix} \tag{43}
\]

where \( E = k^2 \) and \( \ell_n = x_{n+1} - x_n > 0 \). The negative energy case involves rotations by complex angles \( \ell_n \).

The study of the continuum limit, \( \ell_n \to 0 \) and \( v_n \to 0 \) with \( \langle v_n \rangle = 0 \) and \( \sigma = \langle v_n^2 \rangle / (\ell_n) \) can be done along the same lines as in the letter. In this more simple case, the Riccati variable \( z(x) = \psi'(x)/\psi(x) \) obeys the SDE \( z'(x) = -E - z(x)^2 + V(x) \). In the continuum limit \( V(x) \) is a Gaussian white noise of variance \( \sigma \) and the process is characterized by the (backward) generator \( \mathcal{G} = (\sigma/2) \partial^2_x - (E + z^2) \partial_z \). We arrive at

\[
\gamma_2 = 2 \int dzdz' \cdot G(z|z') \cdot z' \cdot f(z') \tag{44}
\]

where

\[
f(z) = \frac{2N}{\sigma} f_0(z) \int_{-\infty}^{z} \frac{dt}{f_0(t)} \quad \text{with} \quad f_0(z) = e^{-\mathcal{G}t(z)} \tag{45}
\]

is the stationary distribution, involving the potential \( U(z) = Ez + (1/3)z^3 \) and the integrated density of states \( N(E) \), given in [12] for instance (also recalled in [11]). The equation

\[
\mathcal{G}^\dagger G(z|z') = f(z) - \delta(z-z') \tag{46}
\]

for the propagator can be solved:

\[
G(z|z') = \frac{1}{N(E)} \left\{ f(z) \left[ c(z') + \int_{-\infty}^{z} dt f(t) \right] - f_0(z) \int_{-\infty}^{z} dt \frac{f(t)^2}{f_0(t)} + \frac{f_0(z) f(z)}{f_0(z')} \right\} \tag{47}
\]

where

\[
c(z') + \frac{1}{2} = \frac{\sigma}{2N(E)} \left[ \int_{-\infty}^{+\infty} dz'' f(z'')^2 f(-z'') - f(-z') f(z') \right] \tag{48}
\]

We can analyse the limiting behaviours of the variance. In the high energy regime, \( k = \sqrt{E} \gg \sigma^{1/3} \) we obtain that

\[
f(z) = \frac{k/\pi}{z^2 + k^2} + \frac{\sigma k}{\pi} \frac{z}{(z^2 + k^2)^{3/2}} + O(\sigma^2) \tag{49}
\]
Recalling that \( N(E) = k/\pi + \mathcal{O}(\sigma^2) \) and
\[
G(z|z') = \left[ \frac{1}{z^2 + k^2} + \sigma \frac{z}{(z^2 + k^2)^{3/2}} \right] \Omega(z, z') + \frac{3\sigma}{16\pi k^3} \frac{1}{(z^2 + k^2)^{3/2}} + \frac{4k^4}{(z^2 + k^2)^3} + \frac{\theta_H(z - z')}{z^2 + k^2} + \mathcal{O}(\sigma^2)
\]
where \( \Omega(z, z') = \frac{1}{2} \text{sign}(z' - z) + \frac{1}{\pi} \left[ \arctan(z/k) - \arctan(z'/k) \right] \). When introducing these expressions in \([44]\), the term \( O(\sigma^0) \) seems at first sight logarithmically divergent but is eliminated by symmetry (i.e., integrals must be understood as principal parts). We get
\[
\gamma_2 = \frac{k\sigma}{\pi} \int dz \frac{z^2}{[\mathcal{U}'(z')]^3} + \mathcal{O}(\sigma^2) = \frac{\sigma}{8E} + \mathcal{O}(\sigma^2)
\]
i.e. we have recovered the asymptotic relation \( \gamma_2 \simeq \gamma_1 \) for \( E \to \infty \) (SPS).

The limit \( k = \sqrt{-E} \gg \sigma^{1/3} \) is more easy to handle. In this case the potential \( \mathcal{U}(z) \) develops a deep well at \( z = k \), where the process is most of the “time” trapped. This dominates the fluctuations that are found to be
\[
\gamma_2 \simeq \frac{\sigma}{4E^{1/2}} \quad (52)
\]
Recalling that \( \gamma_1 \simeq \sqrt{-E} \) in this case shows that \( \gamma_2 \ll \gamma_1 \).

Monte Carlo simulations are in perfect agreement with these behaviours.

\[
\varphi_1(z) = N \left( \frac{2}{\sigma} \int_{-\infty}^{z} f_0(z') \frac{dz'}{\sigma} \right) \int_{-\infty}^{z} \frac{dz''}{f_0(z'')} \cdot (54)
\]

Although it is not straightforward to prove the equivalence between \([44]\) and \([53]\), they seem to give similar results (see Fig. 1 of \([33]\)).

This problem was studied earlier in \([13]\) in another context and with a slightly different method : the generalised Lyapunov exponent \( \Lambda(q) \) is obtained as the largest eigenvalue of the operator \( \mathcal{Q}^q + q^2 \). \([13]\). The perturbative treatment \([13]\) gives a different integral representation
\[
\gamma_2 = 2 \int dz (z - \gamma_1) \varphi_1(z) \quad (53)
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