A REMARK ON WEIGHTED BERGMAN KERNELS ON ORBIFOLDS

XIANZHE DAI, KEFENG LIU, AND XIAONAN MA

Abstract. In this note, we explain that Ross-Thomas’ result [4, Theorem 1.7] on the weighted Bergman kernels on orbifolds can be directly deduced from our previous result [1]. This result plays an important role in the companion paper [5] to prove an orbifold version of Donaldson Theorem.

In two very interesting papers [1, 5], Ross-Thomas describe a notion of ampleness for line bundles on Kähler orbifolds with cyclic quotient singularities which is related to embeddings in weighted projective spaces. They then apply the results in [4] to prove an orbifold version of Donaldson Theorem [5]. Namely, the existence of an orbifold Kähler metric with constant scalar curvature implies certain stability condition for the orbifold. In these papers, the result [4, Theorem 1.7] on the asymptotic expansion of Bergman kernels plays a crucial role.

In this note, we explain how to directly derive Ross-Thomas’ result [4, Theorem 1.7] from Dai-Liu-Ma [1, (5.25)], provided Ross-Thomas condition [4, (1.8)] on $c_i$ holds. Since in [1, §5], we state our results for general symplectic orbifolds, in what follows, we will just use the version from [2, Theorem 5.4.11], where Ma-Marinescu wrote them in detail for Kähler orbifolds. We will use freely the notation in [2, §5.4]. We assume also the auxiliary vector bundle $E$ therein is $\mathbb{C}$.

Let $(X, J, \omega)$ be a compact $n$-dimensional Kähler orbifold with complex structure $J$, and with singular set $X_{\text{sing}}$. Let $(L, h^L)$ be a holomorphic Hermitian proper orbifold line bundle on $X$. Let $\nabla^L$ be the holomorphic Hermitian connections on $(L, h^L)$ with curvature $R^L = (\nabla^L)^2$.

We assume that $(L, h^L, \nabla^L)$ is a prequantum line bundle, i.e.,

\begin{equation}
R^L = -2\pi\sqrt{-1}\omega.
\end{equation}

Let $g^{TX} = \omega(\cdot, J\cdot)$ be the Riemannian metric on $X$ induced by $\omega$. Let $\nabla^{TX}$ be the Levi-Civita connection on $(X, g^{TX})$. We denote by $R^{TX} = (\nabla^{TX})^2$ the curvature, by $r^X$ the scalar curvature of $\nabla^{TX}$. For $x \in X$, set $d(x, X_{\text{sing}}) := \inf_{y \in X_{\text{sing}}} d(x, y)$ the distance from $x$ to $X_{\text{sing}}$. 


For $p \in \mathbb{N}$, the Bergman kernel $P_p(x, x')$ $(x, x' \in X)$ is the smooth kernel of the orthogonal projection from $C^\infty(X, L^p)$ onto $H^0(X, L^p)$, with respect to the Riemannian volume form $dv_X(x')$.

**Theorem 0.1** ([1, Theorem 1.4], [2, Theorem 5.4.10]). There exist smooth coefficients $b_r(x) \in C^\infty(X)$ which are polynomials in $R^T X$, and its derivatives with order $\leq 2r - 2$ at $x$ and $C_0 > 0$ such that for any $k, l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $M \in \mathbb{N}$ with

\[
(0.2) \quad \left| \frac{1}{p^k} P_p(x, x) - \sum_{r=0}^{k} b_r(x) p^{-r} \right|_{C^k} \leq C_{k,l} \left( p^{-k-1} + p^{l/2} (1 + \sqrt{pd(x, X_{\text{sing}})})^M e^{-\sqrt{C_0}d(x, X_{\text{sing}})} \right),
\]

for any $x \in X$, $p \in \mathbb{N}^*$. Moreover

\[
(0.3) \quad b_0 = 1, \quad b_1 = \frac{1}{8\pi} r^X.
\]

In local coordinates, there is a more precise form ([1, (5.25)]), see also [2, Theorem 5.4.11]. Let $\{x_i\}_{i=1}^l \subset X_{\text{sing}}$. For each point $x_i$ we consider corresponding local charts $(G_{x_i}, \tilde{U}_{x_i}) \to U_{x_i}$ with $\tilde{U}_{x_i} \subset \mathbb{C}^n$, such that $0 \in \tilde{U}_{x_i}$ is the inverse image of $x_i \in U_{x_i}$, and $0$ is a fixed point of the finite stabilizer group $G_{x_i}$ at $x_i$, which acts $\mathbb{C}$-linearly and effectively on $\mathbb{C}^n$ (cf. [2, Lemma 5.4.3]). We assume moreover that

\[
B^{\tilde{U}_{x_i}}(0, 2\varepsilon) \subset \tilde{U}_{x_i}, \quad \text{and } X_{\text{sing}} \subset W := \bigcup_{i=1}^l B^{\tilde{U}_{x_i}}(0, 1/4 \varepsilon)/G_{x_i}.
\]

Let $\tilde{U}_{x_i}^g$ be the fixed point set of $g \in G_{x_i}$ in $\tilde{U}_{x_i}$, and let $\tilde{N}_{x_i,g}$ be the normal bundle of $\tilde{U}_{x_i}^g$ in $\tilde{U}_{x_i}$. For each $g \in G_{x_i}$, the exponential map $\tilde{N}_{x_i,g,\tilde{x}} \ni Y \to \exp_{x_i}^\tilde{g} \tilde{U}_{x_i}(Y)$ identifies a neighborhood of $\tilde{U}_{x_i}^g$ with $\tilde{W}_{x_i,g} = \{ Y \in \tilde{N}_{x_i,g}, |Y| \leq \varepsilon \}$. We identify $L|_{\tilde{W}_{x_i,g}}$ with $L|_{\tilde{U}_{x_i}^g}$ by using the parallel transport along the above exponential map. Then the $g$-action on $L|_{\tilde{W}_{x_i,g}}$ is the multiplication by $e^{i\theta_g}$, and $\theta_g$ is locally constant on $\tilde{U}_{x_i}^g$.

Let $\nabla^{\tilde{N}_{x_i,g}}$ be the connection on $\tilde{N}_{x_i,g}$ induced by the Levi-Civita connection via projection. We trivialize $\tilde{N}_{x_i,g} \simeq \tilde{U}_{x_i}^g \times \mathbb{C}^{n_g}$ by the parallel transport along the curve $[0, 1] \ni t \to t \tilde{Z}_{1,g}$ for $\tilde{Z}_{1,g} \in \tilde{U}_{x_i}^g$, which identifies also the metric on $\tilde{N}_{x_i,g}$ with the canonical metric on $\mathbb{C}^{n_g}$. If $\tilde{Z} \in \tilde{W}_{x_i,g}$, we will write $\tilde{Z} = (\tilde{Z}_{1,g}, \tilde{Z}_{2,g})$ with $\tilde{Z}_{1,g} \in \tilde{U}_{x_i}^g$, $\tilde{Z}_{2,g} \in \mathbb{C}^{n_g}$. We will denote by $Z$ the corresponding point on the orbifold.
Theorem 0.2 ([1, (5.25)], [2, Theorem 5.4.11]). On $\tilde{U}_x$, as above, there exist polynomials $\mathcal{K}_{r,\tilde{Z}_1,g}(\tilde{Z}_2,g)$ in $\tilde{Z}_2,g$ of degree $\leq 3r$, of the same parity as $r$, whose coefficients are polynomials in $R^{TX}$ and its derivatives of order $\leq r - 2$, and a constant $C_0 > 0$ such that for any $k,l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $N \in \mathbb{N}$ such that

\[
\left| \frac{1}{p^n} P_p(\tilde{Z}, \tilde{Z}) - \sum_{r=0}^{k} b_r(\tilde{Z}) p^{-r} \right| \\
- \sum_{r=0}^{2k} p^{-\frac{r}{2}} \sum_{1 \neq g \in G_x} e^{i\theta p} \mathcal{K}_{r,\tilde{Z}_1,g}(\sqrt{p}\tilde{Z}_2,g) e^{-2\pi p(1-g^{-1})\tilde{z}_2,g} \\
\leq C_{k,l} \left( p^{-k-1} + p^{-k+l+1} (1 + \sqrt{p}d(Z, X_{\text{sing}}))^N e^{-\sqrt{C_0 p d(Z, X_{\text{sing}})}} \right),
\]

for any $|\tilde{Z}| \leq \varepsilon/2$, $p \in \mathbb{N}$, with $b_r(\tilde{Z})$ as in Theorem 0.1 and $\mathcal{K}_{0,\tilde{Z}_1,g} = 1$.

Given a function $f(p, x)$ in $p \in \mathbb{N}$ and $x \in X$, we write $f = O_{C^j}(p^l)$ if the $C^j$-norm of $f$ is uniformly bounded by $C p^l$.

Theorem 0.3. Let $(X, \omega)$ be a compact $n$-dimensional Kähler orbifold with cyclic quotient singularities (i.e., the stabilizer group $G_x$ is a cyclic group for any $x \in X$), and $L$ be a proper orbifold line bundle on $X$ equipped with a Hermitian metric $h_L$ whose curvature form is $-2\pi \sqrt{-1} \omega$, such that for any $x \in X$, the stabilizer group $G_x$ acts on $L_x$ as $\mathbb{Z}_{|G_x|}$-order cyclic group. Fix $N \geq 0$, and $r \geq 0$ and suppose $c_i$ are a finite number of positive constants chosen so that if $X$ has an orbifold point of order $m$ then

\[
\frac{1}{m} \sum_{i \equiv u \mod{m}} i^k c_i = \sum_{i \equiv u \mod{m}} i^k c_i \quad \text{for all } u \text{ and all } k = 0, \cdots, N + r.
\]

Then the function

\[
B_{p}^{\text{orb}}(x) := \sum_{i} c_i P_{p+i}(x, x).
\]

admits a global $C^{2r}$-expansion of order $N$. That is, there exist smooth functions $b_0, \cdots, b_N$ on $X$ such that

\[
B_{p}^{\text{orb}} = \sum_{j=0}^{N} b_j p^{n-j} + O_{C^{2r}}(p^{n-N-1}).
\]
Furthermore, $b_j$ are universal polynomials in the constants $c_i$ and the derivatives of $\omega$; in particular

$$b_0 = \sum_i c_i, \quad b_1 = \sum_i c_i \left( n_i + \frac{1}{8\pi r^X} \right).$$

(0.8)

**Remark 0.4.** Theorem 0.3 recovers [4, Theorem 1.7] of Ross-Thomas, where the remainder estimate is $O_{\omega^r}(p^{n-N-1})$.

We improve here their remainder estimate to $O_{\omega^r}(p^{n-N-1})$ and we get Theorem 0.3 directly from Theorems 0.1, 0.2.

**Remark 0.5.** By Ma-Marinescu [3, (3.30), Remark 3.10], [2, Theorem 4.1.3, Remark 5.4.13], Theorem 0.3 generalizes to any $J$-invariant metric $g_{TX}$ on $TX$. Set $\Theta := g_{TX}(J^*, \cdot)$. The only change is that the coefficients in the expansion become

$$b_0 = \frac{\omega^n}{\Theta^n} \sum_i c_i, \quad b_1 = \frac{\omega^n}{\Theta^n} \sum_i c_i \left[ n_i + \frac{r^X}{8\pi} - \frac{1}{4\pi} \Delta_\omega \log \left( \frac{\omega^n}{\Theta^n} \right) \right],$$

(0.9)

where $r^X$, $\Delta_\omega$ are the scalar curvature and the Bochner Laplacian associated to $g_{TX} = \omega(\cdot, J^*)$. Moreover, (0.7) can be taken to be uniform as $(h^L, g_{TX})$ runs over a compact set.

**Proof of Theorem 0.3.** Recall that now $G$ is a cyclic group of order $m$. Let $\zeta$ be a generator of $G$. From the local condition for orbi-ample line bundles, $\zeta$ acts on $L_{x_i}$ as a primitive $m$-th root of unity $\lambda$. Thus in (0.4), $e^{i\theta_i u} = \lambda^u$. For $u \in \{1, \cdots, m-1\}$, set

$$\eta_u = e^{-2\pi i (1-\zeta^{-u})\bar{z}_2, \zeta^u \bar{z}_2, \zeta^u},$$

$$S_u(\bar{Z}) = \sum_i c_i \sum_{j=0}^{2N+2r+1} (p+i)^{n-j} \mathcal{K}_{J^*, \bar{Z}^\zeta u} \left( \sqrt{p+i} \bar{Z}^\zeta u \right) \lambda^{u(p+i)} \eta^{p+i},$$

$$S_2 = \sum_{u=1}^{m-1} S_u, \quad S_1 = \sum_i c_i \sum_{j=0}^{N+r} b_j(\bar{Z})(p+i)^{n-j}.$$

Here $Z = z + \bar{z}$, and $z = \sum_i z_i \frac{\partial}{\partial z_i}$, $\bar{z} = \sum_i \bar{z}_i \frac{\partial}{\partial \bar{z}_i}$ when we consider them as vector fields, and $\left| \frac{\partial}{\partial z_i} \right|^2 = \left| \frac{\partial}{\partial \bar{z}_i} \right|^2 = \frac{1}{2}$. Similarly for $\bar{Z}$ (and those with subscripts).
Applying (0.4) for \( k = N + r + 1 \) we obtain for \( |Z| \leq \varepsilon / 2 \),

\[
(0.11) \quad \left| B_p^{\alpha \beta}(Z) - S_1 - S_2 \right|_{\psi} \leq C_1 p^{n-N-r-2} \left( 1 + p^{l+1} \right) \left( 1 + \sqrt{p d(Z, X_{\text{sing}})} \right)^M e^{-\sqrt{\lambda_0 p d(Z, X_{\text{sing}})}} \\
+ \sum_i c_i (p+i)^{n-N-r-1} \left( \sum_{u=1}^{n-1} \left| i \mathcal{K}_{2N+2r+2, Z_1, u} \left( \sqrt{p+i} Z_2, u \right) \eta_u^{p+i} \right|_{\psi} + b_{N+r+1}(Z) \right). 
\]

In what follows, we write for simplicity \( \tilde{Z}_{1,u} \) as \( Z_{1,u} \) and \( \tilde{Z}_{2,u} \) as \( Z_{2,u} \). For a function \( f(p, Z) \) with \( p \in \mathbb{N} \) and \( |Z| \leq \varepsilon / 2 \) we write \( f = O_{\psi}(g(p, Z)) \) if the \( \psi \)-norm of \( f \) in \( Z \) can be uniformly controlled by \( C \| g(p, Z) \| \).

Note that \( \mathcal{K}_{j, Z_{1,u}}(Z_{2,u}) \) is a polynomial in \( Z_{2,u} \) with the same parity as \( j \) and \( \deg \mathcal{K}_{j, Z_{1,u}} \leq 3j \). Denote by \( \mathcal{K}_{j, Z_{1,u}} \) the \( l \)-homogeneous part of \( \mathcal{K}_{j, Z_{1,u}} \). Then \( \mathcal{K}_{j, Z_{1,u}} = 0 \) if \( l \) and \( j \) are not in the same parity or \( l > 3j \). By (0.10),

\[
(0.12) \quad S_u(Z) = \sum_i c_i \sum_{j=0}^{2N+2r+1} \sum_{l=0}^{N+r} (p+i)^{n-l} \mathcal{K}_{j, Z_{1,u}}(Z_{2,u}) \lambda^{u(p+i)} \eta_u^{p+i} \\
= \lambda^{u} \sum_{j=0}^{2N+2r+1} \left\{ \left( \sum_{l \geq j-2n} + \sum_{l < j-2n} \right) \mathcal{K}_{j, Z_{1,u}} \left( \sqrt{p} Z_{2,u} \right) \right\} \times \left( \sum_{q=0}^{N+r} \frac{q}{q} \right) \sum_i c_i^{\lambda^u} \eta_u^{p+i} + O_{\psi^2}(p^{n-N-1}).
\]

Here we used \( (p+i)^{\gamma} = \sum_{q=0}^{N+r} p^{\gamma-q} \left( \frac{\gamma}{q} \right) i^{q} + O(p^{-N-r-1}) \) for \( \gamma < 0 \) and the following relations for \( r', r'' \in \mathbb{N} \), \( r'' \leq l \),

\[
(0.13) \quad \mathcal{K}_{j, Z_{1,u}} \left( \sqrt{p} Z_{2,u} \right) \eta_u^{p} = O_{\psi^{r'}}(p^{r'} \eta_u^{p/2}), \\
\mathcal{K}_{j, Z_{1,u}} \left( \sqrt{p} Z_{2,u} \right) = O_{\psi^{r''}}(p^{r''} |Z_{2,u}|^{l-r'}). 
\]

In order to prove (0.7) it is sufficient to show that for \( 0 \leq l \leq N + r \), \( r' \leq 2r \),

\[
(0.14) \quad w_{l,p} := \sum_i c_i^{l} \lambda^{u} \eta_u^{p+i} = O_{\psi^{r'}}(p^{l-N-r-1+\frac{r'}{2}} \eta_u^{p/2}).
\]

In fact, we will prove that \( w_{l,p} = O_{\psi^{r'}}(p^{l-N-r-1+\frac{r'}{2}} \eta_u^{p/2}) \) for \( r' \leq 2r \).
Since $dw_{l,p} = \frac{dw}{\eta_u}(pw_{l,p} + w_{l+1,p})$, and $\frac{dw}{\eta_u}$ has a term $z_{2,u}$ or $\bar{z}_{2,u}$ which can be absorbed by $\eta_u^{\frac{2r}{p}}$ to get a factor $p^{-1/2}$, we see by induction that it is sufficient to prove $w_{l,p} = O_{\bar{q}^0}(p^{l-N-r-1}\eta_u^{\frac{2r}{p}})$. To this end, write

$$w_{l,p} = \left[ \sum_i c_i i^l \lambda^u \eta^i \right] (\eta_u - 1)^{N+r-l+1} \eta_u^{\frac{2r}{p}}. \tag{0.15}$$

Since $\lambda$ is a primitive $m$-th root of unity, $\lambda^u \neq 1$ if $u \in \{1, \cdots, m-1\}$. From [4, Lemma 3.5], under the condition (0.5), the function $\eta \to \sum_i c_i i^l \lambda^u \eta^i$ has a root of order $N + r - l + 1$ at $\eta = 1$ and so the term in square brackets is bounded. For $|z_{2,u}| \leq \varepsilon$, we have by (0.10),

$$|\eta_u - 1| \leq C|z_{2,u}|^2. \tag{0.16}$$

By using (0.10) and (0.16) and the fact that $[0, \infty) \ni x \mapsto x^s e^{-x}$ is bounded for any $s \geq 0$, we get

$$\eta_u^{p/4} = O(p^{-s}) \quad \text{for } s \geq 1. \tag{0.17}$$

Thus, $w_{l,p} = O_{\bar{q}^0}(p^{l-N-r-1}\eta_u^{\frac{2r}{p}})$ and (0.14) follows.

Back in (0.12), for $q > N + r$, the corresponding contribution is certainly $p^{N+\frac{2}{p}-q} \cdot O_{\bar{q}^{2r}}(p^r \eta_u^{p/2}) = O_{\bar{q}^{2r}}(p^{n-N-1}\eta_u^{p/2})$, by (0.13). On the other hand, if $0 \leq q \leq N + r$, then, by (0.13) and (0.14), the corresponding contribution is $p^{N+\frac{2}{p}-q} \cdot O_{\bar{q}^{2r}}(p^r \eta_u^{p/2}) = O_{\bar{q}^{2r}}(p^{n-N-1}\eta_u^{p/4})$ again. Thus $\mathcal{S}_u = O_{\bar{q}^{2r}}(p^{n-N-1})$.

From (0.10) and the above argument, $\mathcal{S}_2 = O_{\bar{q}^{2r}}(p^{n-N-1})$. Combining with (0.10), (0.11) and (0.13), we get (0.7) and (0.8). \hfill \square

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