MODIFIED QUASILINEAR EQUATIONS WITH STRONGLY SINGULAR AND CRITICAL EXPONENTIAL NONLINEARITY

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Abstract

In this paper, we study global multiplicity result for a class of modified quasilinear singular equations involving the critical exponential growth:

\[
\begin{aligned}
-\Delta u - \Delta (u^2) u &= \lambda (\alpha(x) u^{-q} + f(x, u)) \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \), \( 0 < q < 3 \) and \( \alpha : \Omega \to (0, +\infty) \) such that \( \alpha \in L^\infty(\Omega) \). The function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is continuous and enjoys critical exponential growth of the Trudinger–Moser type. Using a sub-super solution method, we show that there exists some \( \Lambda^* > 0 \) such that for all \( \lambda \in (0, \Lambda^*) \) the problem has at least two positive solutions, for \( \lambda = \Lambda^* \), the problem achieves at least one positive solution and for \( \lambda > \Lambda^* \), the problem has no solutions.

Key words: Modified quasilinear operator, singular nonlinearity, sub-super solution, critical exponential nonlinearity, Trudinger-Moser inequality.

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1 Introduction and statement of main results

In this article, we study the existence, nonexistence and multiplicity of the positive solutions for the following modified quasilinear equation:

\[
\begin{aligned}
-\Delta u - \Delta (u^2) u &= \lambda (\alpha(x) u^{-q} + f(x, u)) \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \), \( 0 < q < 3 \) and the function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is defined as \( f(x, s) = g(x, s) \exp(|s|^4) \), where \( g \in C(\bar{\Omega} \times \mathbb{R}) \) satisfies some appropriate assumptions described later. We also have the following assumption on the function \( \alpha : \Omega \to \mathbb{R} \):

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(\(\alpha\)) \(\alpha \in L^\infty(\Omega)\) and \(\alpha_0 := \inf_{x \in \Omega} \alpha(x) > 0\).

The study on the equations driven by the modified quasilinear operator \(-\Delta u - \Delta(u^2)u\) is quite popular for long because of their wide range of applications in the modeling of the physical phenomenon such as in plasma physics and fluid mechanics [6], in dissipative quantum mechanics [18], etc. Solutions of such equations (called soliton solutions) are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

\[
iu_t = -\Delta u + V(x)u - h_1(|u|^2)u - C\Delta h_2(|u|^2)h_2'(|u|^2)u, \quad x \in \mathbb{R}^N,
\]

where \(V\) is a potential function, \(C\) is a real constant, \(h_1\) and \(h_2\) are real valued functions. Equations of the form (1.1) appear in the study of mathematical physics. Each different type of the function \(h_2\) represents different physical phenomenon. For example, if \(h_2(s) = s\), then (1.1) is used in the modeling of the super fluid film equation in plasma physics (see [23]). When \(h_2(s) = \sqrt{1 + s^2}\), (1.1) attributes to the study of self-channeling of a high-power ultra short laser in matter (see [32]). Because of the quasilinear term \(\Delta(u^2)u\), present in the problems of type \((P_s)\), the natural energy functional associated to such problems is not well defined. Hence, we have a restriction in applying variational method directly for studying such problems. To overcome this shortcoming, researchers developed several methods and arguments, such as the perturbation method (see for e.g., [25, 28]) a constrained minimization technique (see for e.g., [26, 27, 31, 33]), and a change of variables (see for e.g., [8, 11, 13, 12, 21, 22]).

Without the quasilinear term \(\Delta(u^2)u\), the problem \((P_s)\) goes back to the original semilinear equation

\[-\Delta u = \lambda \alpha(x)u^{-q} + f(x, u) \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]

(1.2)

Such kind of equations have significant applications in the physical modelling related to the boundary layer phenomenon for viscous fluids, non-Newtonian fluids, etc. If \(f \equiv 0\), (1.2) becomes purely singular problem, for which existence, uniqueness, non-existence and regularity results are extensively studied in [9, 20] with suitable \(\alpha(x)\) and for different ranges of \(q\).

One of the main features of problem \((P_s)\) is that the nonlinear term \(f(x, s)\) enjoys the critical exponential growth with respect to the following Trudinger–Moser inequality (see [29]):

**Theorem 1.1.** (Trudinger–Moser inequality) Let \(\Omega\) be a smooth bounded domain in \(\mathbb{R}^2\). Then for \(u \in H^1_0(\Omega)\) and \(p > 0\) we have

\[
\exp(p|u|^2) \in L^1(\Omega).
\]

Moreover,

\[
\sup_{\|u\| \leq 1} \int_{\Omega} \exp(p|u|^2) \, dx < +\infty
\]

if and only if \(p \leq 4\pi\).

Here the Sobolev space \(H^1_0(\Omega)\) and the corresponding norm \(\| \cdot \|\) are defined in Section 2. The study on the critical growth exponential problems associated to this inequality were initiated with the work of Adimurthi [1] and de Figueiredo et al. [10]. Furthermore, the problem of type \((P_s)\) without the singular term, that is the equation

\[-\Delta u - \Delta(u^2)u + V(x)u = f(x, u) \text{ in } \mathbb{R}^2,
\]

(1.3)
where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential and $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a continuous function with some suitable assumption and is having critical exponential growth (exp($ps^4$)), was studied by do Ó et al. in [11] for the first time. Note that, unlike as in the case of semilinear critical exponential problem involving Laplacian, where the critical exponential growth is given as exp($|s|^2$), in problem $(P_*)$ and in (1.3), the form of critical exponential nonlinearity $f$ is considered as exp($|s|^4$) because of the quasilinear term $\Delta(u^2)u$.

Motivated by the seminal work of Ambrosetti et al. [3], many researchers studied the global multiplicity results for singular-convex problems. In [19], the author proved the global multiplicity result for (1.2), considering critical polynomial growth as $f(x, s) = |s|^{q+2}/(q+2)$, and $0 < q < 1$ in a general smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N > 2$. Then, in the critical dimension $N = 2$, Adimurthi and Giacomoni [2] discussed the global multiplicity of $H^1_0$ solutions for (1.2) by taking the optimal range of $q$ as $1 < q < 3$ and considered $f$ to be having critical exponential growth as $f(x, s) = g(x, s)\exp(4\pi s^2)$, where $g$ is some appropriate continuous function. Later, in [17], the authors showed the similar results as in [19] for (1.2) while considering $f(x, s) = |s|^{q+2}/(q+2)$, $0 < q < 3$ and adding a smooth perturbation with sub-critical asymptotic behavior at $+\infty$. In [14, 15], Dhanya et al. studied the global multiplicity result for the problem of type (1.2) with critical growth nonlinearities combined with a discontinuous function multiplied with the singular term $u^{-q}$, $0 < q < 3$.

On the other hand, for modified quasilinear Schrödinger equations involving singular nonlinearity, there are a few results in the existing literature. Authors in [5, 30, 34] discussed existence of multiple solutions of such equations with singular nonlinearity $u^{-q}$, $0 < q < 1$ in combination with some polynomial type perturbation. Very recently, in [35], the authors studied the global multiplicity results for the equations of type $(P_*)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N > 2$, for the first time, by assuming critical polynomial growth $f(x, s) = b(x)|s|^{\frac{2N+2}{N-2}}$, where $b$ is a sign changing continuous function.

Inspired by all these aforementioned works, in this article, we investigate the existence, multiplicity and non-existence results (that is, the global multiplicity result) for problem $(P_*)$ driven by modified quasilinear operator involving singular and critical exponential nonlinearity, which was an open question. The main mathematical difficulties we face in studying problem $(P_*)$ occur in three folds as following:

1. The modified quasilinear term $\Delta(u^2)u$ prohibits the natural energy functional corresponding to problem $(P_*)$ to be well defined for all $u \in H^1_0(\Omega)$ (defined in Section 2).

2. The critical exponential nature of $f$ induces non-compactness of the Palais-Smale sequences.

3. The growth of the singular nonlinearity falls into the range $0 < q < 3$, which again prevents the associated energy functional to problem $(P_*)$ to be well defined for all $u \in H^1_0(\Omega)$.

First, we transform the problem $(P_*)$ by using a change of variable as in [8, 35]. Then, applying variational and sub-super solution methods on the transformed problem, we show that there exists some $\Lambda^* > 0$ such that for the range of the parameter $0 < \lambda < \Lambda^*$, the transformed problem has at least two positive solutions, for $\lambda = \Lambda^*$ it achieves at least one positive solution and for $\lambda > \Lambda^*$, it has no solutions. In Section
3, we prove the existence of a non trivial weak solution for the range \(0 < \lambda \leq \Lambda^*\) and non existence of solutions for \(\lambda > \Lambda^*\) by constructing a suitable sub-super solution argument. We also prove the asymptotic boundary behavior of such solutions. Then in Section 4, we investigate the existence of a second solution in a cone around the first solution for \(0 < \lambda < \Lambda^*\) by using the Ekeland variational principle and a mountain pass argument with a min-max level. There we find the first critical level, below which we prove the compactness condition of the Palais-Smale sequences. This gives rise to the existence of a second positive solution. In this whole process, we prove many technically involved delicate estimates due to the several complexities present in the problem \((P_\ast)\). We would like to mention that to the best of our knowledge, there is no work in the literature addressing the question of global multiplicity of positive solutions involving modified quasilinear operator and singular and exponential nonlinearity. In this article, we study the global multiplicity results for the problem \((P_\ast)\) up to the optimal range for the singularity \((0 < q < 3)\).

We now state all the hypotheses imposed on the continuous function \(f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\), given by \(f(x, s) = g(x, s) \exp(\left|s \right|^4)\):

\[(f1)\] \(g \in C^1(\overline{\Omega} \times \mathbb{R})\) such that for each \(x \in \overline{\Omega}\), \(g(x, s) = 0\) for all \(s \leq 0\); \(g(x, s) > 0\) for all \(s > 0\) and \(\frac{g(x, s)}{s^3}\) is non decreasing in \(s > 0\), for all \(x \in \overline{\Omega}\).

\[(f2)\] Critical growth assumption: For any \(\epsilon > 0\),

\[
\lim_{s \to +\infty} \sup_{x \in \Omega} g(x, s) \exp(-\epsilon \left|s \right|^4) = 0 \quad \text{and} \quad \lim_{s \to +\infty} \inf_{x \in \Omega} g(x, s) \exp(\epsilon \left|s \right|^4) = +\infty.
\]

\[(f3)\] There exists a constant \(\tau > 4\) such that \(0 < \tau F(x, s) \leq f(x, s)\), for all \((x, s) \in \Omega \times (0, +\infty)\).

\[(f4)\] There exists a constant \(M_1 > 0\) such that \(F(x, s) \leq M_1(1 + f(x, s))\) for all \(s > 0\).

**Example 1.2.** Consider \(f(x, s) = g(x, s)e^s\), where \(g(x, s) = \begin{cases} t^{a_0 + 1}\exp(d_0 s^r), & \text{if } s > 0 \\ 0, & \text{if } s \leq 0 \end{cases}\) for some \(a_0 > 0, 0 < d_0 \leq 4\pi\) and \(1 \leq r < 4\). Then \(f\) satisfies all the conditions from \((f1)-(f4)\).

**Remark 1.3.** From the condition \((f1)\), we deduce

\[
0 \leq \lim_{s \to 0^+} \frac{f(x, s)}{s} \leq \lim_{s \to 0^+} \frac{f(x, s)}{s^3} s^2 \leq \lim_{s \to 0^+} f(x, 1)s^2 = 0 \quad \text{uniformly in } x \in \Omega,
\]

since \(g\) is continuous, which implies that

\[
\lim_{s \to 0} \frac{f(x, s)}{s} = 0 \quad \text{uniformly in } x \in \Omega. \tag{1.4}
\]

Moreover, the condition \((f1)\) yields that \(g(x, s)\) is non decreasing in \(s\) and hence, \(g'(x, s) := g_s(x, s) \geq 0\) so that

\[
f'(x, s) := f_s(x, s) = (g'(x, s) + 4s^3g(x, s)) \exp(s^4) \geq 4s^3g(x, s) \exp(s^4) = 4s^3f(x, s).
\]

Thus, for any \(M_0 > \sqrt{2}\), there exists \(L > 0\) such that

\[
f'(x, s) \geq M_0 f(x, s) - L \quad \text{for all } s > 0. \tag{1.5}
\]
Now for any \( \phi \in C(\overline{\Omega}) \) with \( \phi > 0 \) in \( \Omega \), we define the set
\[
C_\phi := \{ u \in C_0(\overline{\Omega}) \mid \text{there exists } c \geq 0 \text{ such that } |u(x)| \leq c\phi \text{ for all } x \in \Omega \}
\]
equipped with the norm \( \|u\|_{C_\phi(\overline{\Omega})} := \|u/\phi\|_\infty \). Next, we define the following open convex set of \( C_\phi(\overline{\Omega}) \) as
\[
C_\phi^+(\Omega) = \{ u \in C_\phi(\Omega) \mid \inf_{x \in \Omega} \frac{u(x)}{\phi(x)} > 0 \}.
\]
That is, the set \( C_\phi^+(\Omega) \) is consisting of all those functions \( u \in C(\Omega) \) such that \( C_1\phi \leq u \leq C_2\phi \) in \( \Omega \) for some \( C_1, C_2 > 0 \). Let us also define the distance function
\[
\delta(x) := \text{dist}(x, \partial \Omega) = \inf_{y \in \partial \Omega} |x - y|, \text{ for any } x \in \overline{\Omega}.
\]
We consider the following eigenvalue problem:
\[
-\Delta u = \lambda u \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega.
\] (1.6)
Let \( \varphi_{1,\Omega} \in H^1_0(\Omega) \) be a positive solution (first eigenfunction) of the above equation corresponding to the first eigenvalue \( \tilde{\lambda}_{1,\Omega} \) with \( \|\varphi_{1,\Omega}\|_\infty < 1 \). We recall that \( \varphi_{1,\Omega} \in C^{1,\theta}(\overline{\Omega}) \) for some \( \theta \in (0,1) \) and \( \varphi_{1,\Omega} \in C^+_\delta(\Omega) \). For more properties of \( \varphi_{1,\Omega} \), one can refer to [16]. Then, we define the function \( \varphi_q \) as follows:
\[
\varphi_q = \begin{cases} 
\varphi_{1,\Omega} & \text{if } 0 < q < 1, \\
\varphi_{1,\Omega} \left( \log \left( \frac{2}{\varphi_{1,\Omega}} \right) \right)^{\frac{1}{q+1}} & \text{if } q = 1, \\
\varphi_{1,\Omega} & \text{if } q > 1.
\end{cases}
\]

Now we state the main results of this article:

**Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain. Suppose that the hypotheses (f1) – (f4) and (a1) hold. Then there exists \( \Lambda^* > 0 \) such that for every \( \lambda \in (0, \Lambda^*) \), problem (P_\ast) has at least one solution in \( H^1_0(\Omega) \cap C^+_\varphi(\Omega) \) and for \( \lambda > \Lambda^* \), problem (P_\ast) has no solutions.

**Theorem 1.5.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain. Assume that the hypotheses (f1) – (f4) and (a1) hold. Then for every \( \lambda \in (0, \Lambda^*) \), problem (P_\ast) has at least two solutions in \( H^1_0(\Omega) \cap C^+_\varphi(\Omega) \).

## 2 Variational framework

For \( u : \Omega \to \mathbb{R} \), measurable function, and for \( 1 \leq p \leq +\infty \), we define the Lebesgue space \( L^p(\Omega) \) as
\[
L^p(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable} \mid \int_\Omega |u|^p \, dx < +\infty \}
\]
equipped with the usual norm denoted by \( \|u\|_p \). Now the Sobolev space \( H^1_0(\Omega) \) is defined as
\[
H^1_0(\Omega) = \{ u \in L^2(\Omega) \mid \int_\Omega |\nabla u|^2 \, dx < +\infty \}
\]
endowed with the norm
\[ \|u\| := \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}. \]
Since \( \Omega \subset \mathbb{R}^2 \) is a smooth bounded domain, we have the continuous embedding \( H^1_0(\Omega) \hookrightarrow L^p(\Omega) \) for \( p \in [1, +\infty) \). Moreover, the embedding \( H^1_0(\Omega) \ni u \mapsto \exp(|u|^\beta) \in L^1(\Omega) \) is compact for all \( \beta \in [1, 2) \) and is continuous for \( \beta = 2 \). Consequently, the map \( \mathcal{M} : H^1_0(\Omega) \rightarrow L^p(\Omega) \) for \( p \in [1, +\infty) \), defined by \( \mathcal{M}(u) := \exp \left( |u|^\beta \right) \) is continuous with respect to the \( H^1_0 \)-norm topology.

The natural energy functional associated to problem \((P_s)\) is the following:
\[
I_\lambda(u) = \begin{cases} 
\frac{1}{2} \int_{\Omega} (1 + 2|u|^2)|\nabla u|^2 \, dx - \lambda \int_{\Omega} \frac{1}{1-q} \int_{\Omega} \alpha(x) u^{1-q} \, dx - \lambda \int_{\Omega} F(x, u(x)) \, dx & \text{if } q \neq 1; \\
\frac{1}{2} \int_{\Omega} (1 + 2|u|^2)|\nabla u|^2 \, dx - \lambda \int_{\Omega} \alpha(x) \log |u| \, dx - \lambda \int_{\Omega} F(x, u(x)) \, dx & \text{if } q = 1.
\end{cases}
\]
(2.1)

Observe that, the functional \( I_\lambda \) is not well defined in \( H^1_0(\Omega) \) because of the nature of the singularity \((0 < q < 3)\) as well as, due to the fact that \( \int_{\Omega} u^2|\nabla u|^2 \, dx \) is not finite for all \( u \in H^1_0(\Omega) \). So, it is difficult to apply variational methods directly in our problem \((P_s)\). In order to get rid of this inconvenience, first we apply the following change of variables introduced in \([8]\), namely, \( w := h^{-1}(u) \), where \( h \) is defined by
\[
\begin{align*}
&\quad h'(s) = \frac{1}{(1 + 2|h(s)|^2)^{1/2}} \text{ in } [0, +\infty), \\
&h(s) = -h(-s) \text{ in } (-\infty, 0].
\end{align*}
\]
(2.2)

Now we gather some properties of \( h \), which we follow throughout in this article. For the detailed proofs of such results, one can see \([8, 11]\).

**Lemma 2.1.** The function \( h \) satisfies the following properties:

1. \( h \) is uniquely defined, \( C^\infty \) and invertible;
2. \( h(0) = 0; \)
3. \( 0 < h'(s) \leq 1 \) for all \( s \in \mathbb{R}; \)
4. \( \frac{1}{2} h(s) \leq sh'(s) \leq h(s) \) for all \( s > 0; \)
5. \( |h(s)| \leq |s| \) for all \( s \in \mathbb{R}; \)
6. \( |h(s)| \leq 2^{1/4}|s|^{1/2} \) for all \( s \in \mathbb{R}; \)
7. \( \lim_{s \to +\infty} h(s)/s^{1/2} = 2^{1/2}; \)
8. \( |h(s)| \geq h(1)|s| \) for \( |s| \leq 1 \) and \( |h(s)| \geq h(1)|s|^{1/2} \) for \( |s| \geq 1; \)
9. \( h''(s) = -2h(s)(h'(s))^4, \) for all \( s \geq 0 \) and \( h''(s) < 0 \) when \( s > 0, h''(s) > 0 \) when \( s < 0. \)
10. the function \( \frac{(h(s))^\gamma h'(s)}{s} \) is strictly increasing for \( \gamma \geq 3; \)
11. \( \lim_{s \to 0} h(s)/s = 1; \)
(h_{12}) |h(s)h'(s)| < 1/\sqrt{2} for all s ∈ \mathbb{R};

(h_{13}) the function h(s)^{-\gamma}h'(s) is decreasing for all s > 0, where γ > 0.

After employing the change of variable \( w = h^{-1}(u) \) in (2.1), we define the new functional \( J_\lambda : H^1_0(\Omega) \to \mathbb{R} \) as

\[
J_\lambda(w) = \begin{cases} 
\frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \lambda \left( \frac{1}{2-q} - 1 \right) \int_{\Omega} \alpha(x)|h(w)|^{1-q} \, dx - \lambda \int_{\Omega} F(x, h(w)) \, dx & \text{if } q \neq 1; \\
\frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \lambda \int_{\Omega} \alpha(x) \log |h(w)| \, dx - \lambda \int_{\Omega} F(x, h(w)) \, dx & \text{if } q = 1.
\end{cases}
\]

(2.3)

Remark 2.2. The functional \( J_\lambda(w) \) is well defined for \( w \in H^1_0(\Omega) \cap C^+(\varphi_\Omega) \). Indeed, let us define the set \( \mathcal{D}(J_\lambda) := \{ w \in H^1_0(\Omega) : J_\lambda(w) < +\infty \} \). Now we intend to show that this set is non empty. Let \( 1 < q < 3 \).

Then using the fact that \( h(s) \) is increasing in \( s > 0, \varphi_1, \varphi_\Omega \in C^+_0(\Omega) \) and Lemma 2.1-(h8), we have

\[
h(w) > h(C_1\varphi_1, \varphi_{\Omega}) > h(C_2\varphi_{\Omega}) > \begin{cases} 
C_2h(1)\frac{\delta^{2(\frac{q}{q+1})+1}}{\delta^{\frac{2}{q+1}}} & \text{if } C_2\delta^{\frac{2}{q+1}} \leq 1, \\
h(1) & \text{if } C_2\delta^{\frac{2}{q+1}} > 1,
\end{cases}
\]

where \( C_2 \) is a positive constant. Thus, for any \( \psi \in H^1_0(\Omega) \) and \( w \in C^+_{\varphi_\Omega}(\Omega) \), we deduce the following by applying Lemma 2.1-(h3), Hölder’s inequality, Hardy’s inequality, and the Sobolev embedding:

\[
\int_{\Omega} h(w)^{1-q}h'(w)\psi \, dx = \int_{\Omega \cap \{x : C_2\delta(x)^{\frac{2}{q+1}} < 1\}} h(w)^{1-q}h'(w)\psi \, dx + \int_{\Omega \cap \{x : C_2\delta(x)^{\frac{2}{q+1}} \geq 1\}} h(w)^{1-q}h'(w)\psi \, dx
\]

\[
\leq C_3 \left( \int_{\Omega} \frac{dx}{2^{(q-1)}} \right)^{\frac{1}{2}} \left( \int_{\Omega} \psi^2 \, dx \right)^{\frac{1}{2}} + C_4h(1)^{-q} \int_{\Omega} \psi \, dx
\]

\[
< C_5\|\psi\| < +\infty,
\]

(2.4)

since \( \frac{2(q-1)}{q+1} < 1 \) for \( 1 < q < 3 \), where \( C_3, C_4, C_5 \) are positive constants. For the case \( 0 < q \leq 1 \), arguing is a similar manner as above and following [19, 35], we get (2.4). In view of this, we obtain that the set \( \mathcal{D}(J_\lambda) \neq \emptyset \) for \( 0 < q < 3 \).

Next, we check for the Gateaux differentiability of the functional \( J_\lambda \). For \( 0 < q < 1 \) and \( w \in H^1_0(\Omega) \) with \( w \geq c_0 \), using the idea of [19] combining with the properties of \( h \) described in Lemma 2.1, we can show that the functional \( J_\lambda \) is Gateaux differentiable at \( w \). For the range \( 1 \leq q < 3 \), we have the following lemma regarding the similar property of \( J_\lambda \).

Lemma 2.3. Let \( \mathcal{S} := \{ w \in H^1_0(\Omega) : w_1 \leq w \leq w_2 \} \), where \( w_1 \in C^+_{\varphi_\Omega}(\Omega) \) and \( w_2 \in H^1_0(\Omega) \). Then \( J_\lambda \) is Gateaux differentiable at \( w \) in the direction \( v - w \) for \( v, w \in \mathcal{S} \).

Proof. We have to prove that

\[
\lim_{t \to 0} \frac{J_\lambda(w + tv - w) - J_\lambda(w)}{t} = \int_{\Omega} \nabla w \nabla (v - w) \, dx - \lambda \int_{\Omega} \alpha(x)h(w)^{-q}h'(w)(v - w) \, dx
\]

\[
- \lambda \int_{\Omega} f(x, h(w))h'(w)(v - w) \, dx.
\]
For the first term and third term in the right hand side of the above expression, the proof follows in a standard way. So, we are left to show for the second term, that is, for the singular term. Since \( S \) is a convex set, for any \( t \in (0, 1) \), \( w + t(v - w) \in S \). Let us define the function \( H : H_0^1(\Omega) \to \mathbb{R} \) as

\[
H(w) = \begin{cases} 
\frac{1}{1-q} \int_{\Omega} h(w)^{1-q} \, dx & \text{if } q \neq 1 \\
\int_{\Omega} \log(h(w)) \, dx & \text{if } q = 1.
\end{cases}
\]

Now using mean value theorem, for any \( 0 < q < 3 \), it follows that

\[
\frac{H(w + t(v - w)) - H(w)}{t} = \int_{\Omega} h(w + t\theta(v - w))^{-q}h'(w + t\theta(v - w))(v - w) \, dx
\]

for some \( \theta \in (0, 1) \). Since \( w + t\theta(v - w) \in S \), using Lemma 2.1-(h)\(_{13}\), it follows that

\[
h(w + t\theta(v - w))^{-q}h'(w + t\theta(v - w))(v - w) \, dx \leq h(w_1)^{-q}h'(w_1)(v - w) \, dx.
\]

Now recalling (2.4), we get

\[
\int_{\Omega} h(w_1)^{-q}h'(w_1)(v - w) \, dx < +\infty.
\]

Thus, applying the Lebesgue dominated convergence theorem and passing to the limit \( t \to 0^+ \) in (2.5), we have

\[
\lim_{t \to 0} \frac{H(w + t(v - w)) - H(w)}{t} = \int_{\Omega} h(w)^{-q}h'(w)(v - w) \, dx.
\]

This completes the proof. \( \square \)

Thus, using the properties of the functions \( f, h \) and the above results, it can be derived that (2.3) is the associated energy functional to the following problem:

\[
\begin{cases} 
-\Delta w = \lambda \left( \alpha(x)h(w)^{-q}h'(w) + f(x, h(w))h'(w) \right) & \text{in } \Omega, \\
w > 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial\Omega.
\end{cases}
\]

(2.7)

Moreover, applying Lemma 2.1 and following the idea as in [36, Proposition 2.2], one can show that if \( w \) is a solution to (2.7), then \( u = h(w) \) is a solution to the problem (\( P_\ast \)). Thus, our main objective is now reduced to proving the existence of solutions to the new transformed equation (2.7).

**Definition 2.4.** A function \( w \in H_0^1(\Omega) \) is said to be a weak solution to (2.7) if for every compact set \( K \subset \Omega \), there exists a constant \( m_K > 0 \) such that \( w > m_K \) holds in \( K \) and for every \( \phi \in H_0^1(\Omega) \), we have

\[
\int_{\Omega} \nabla w \nabla \phi \, dx - \lambda \int_{\Omega} \alpha(x)h(w)^{-q}h'(w)\phi \, dx - \lambda \int_{\Omega} f(x, h(w))h'(w)\phi \, dx = 0.
\]

(2.8)

In the next lemma, we discuss some comparison type result related to our problem (2.7).

**Lemma 2.5.** Let \( w_1, w_2 \in H_0^1(\Omega) \cap C_0^1(\Omega) \) satisfy

\[
-\Delta w_1 \leq \gamma(x)h(w_1)^{-q}h'(w_1), \quad x \in \Omega;
-\Delta w_2 \geq \gamma(x)h(w_2)^{-q}h'(w_2), \quad x \in \Omega,
\]

where \( \gamma \in L^\infty(\Omega) \) with \( \gamma > 0 \). Then \( w_1 \leq w_2 \) a.e. in \( \Omega \).
Proof. The proof of this lemma follows in a similar fashion as in [35, Lemma 2.2].

Notations. In the next subsequent sections, we make use of the following notations:

- If $u$ is a measurable function, we denote the positive and negative parts by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, respectively.
- For any function $f$, supp $f = \{x : f(x) \neq 0\}$.
- If $A$ is a measurable set in $\mathbb{R}^2$, we denote the Lebesgue measure of $A$ by $|A|$.
- The arrows $\rightarrow$, $\rightarrow$ denote weak convergence, strong convergence, respectively.
- The arrow $\hookrightarrow$ denotes continuous embedding.
- $B_r$ denotes the ball of radius $r > 0$ centered at $0 \in H^1_0(\Omega)$.
- $\overline{B}_r$ denotes the closure of the ball $B_r$ with respect to $H^1_0(\Omega)$-norm topology.
- $\partial B_r$ denotes the boundary of the ball $B_r$.
- $B_r(x)$ denotes the ball of radius $r > 0$ centered at $x \in H^1_0(\Omega)$.
- For any $p > 1$, $p'^{-1} := \frac{p}{p-1}$ denotes the conjugate of $p$.
- $c, C_0, C_1, C_2, \ldots, \tilde{C}_1, \tilde{C}_2, \ldots, C$ and $\tilde{C}$ denote positive constants which may vary from line to line.

3 Proof of Theorem 1.4 : Existence and non-existence results

In this section, to prove the existence and non-existence results for the problem (2.7), we first need to study the existence and regularity result for the following purely singular problem:

\[
\begin{aligned}
-\Delta w & = \lambda \alpha(x)h(w)^{-q}h'(w) \text{ in } \Omega, \\
w & > 0 \text{ in } \Omega, \\
w & = 0 \text{ on } \partial\Omega.
\end{aligned}
\]

Now, we have the following result for the problem (3.1).

Theorem 3.1. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Assume that $\lambda > 0$, $q > 0$, $\alpha$ satisfies $(\alpha_1)$ and the function $h$ is defined in (2.2). Then,

(i) the problem (3.1) has a unique solution for each $\lambda > 0$, say $w_\lambda$, in $H^1_0(\Omega) \cap C^1_{\alpha q}(\Omega)$, for $q < 3$;
(ii) the solution $w_\lambda \in C^1(\Omega)$ if $q < 1$, $w_\lambda \in C^{1-\epsilon}(\Omega)$ for any small $\epsilon > 0$ if $q = 1$ and $w_\lambda \in C(\Omega)$ if $q < 3$;
(iii) the map $\lambda \rightarrow w_\lambda$ is non-decreasing and continuous from $\mathbb{R}^+$ to $C(\Omega)$. 

A global multiplicity result for singular and critical elliptic equation

Proof. Let us set \( \rho(s) := \lambda \alpha(x) h(s)^{-q} h'(s) \). Then \( \rho(w) \) verifies the hypotheses of Proposition 4.1 in [2]. Hence, using [2, Proposition 4.1], we can infer that there exists a unique solution to (3.1), say \( \overline{w} \), such that \( \overline{w} \in H^1_0(\Omega) \) for \( 1 < q < 3 \). Next, following the proof of [9, Theorem 2.2], there exist two positive constants \( c_1 := c_1(\lambda, q) << 1 \), and \( c_2 := c_2(\lambda, q) \) such that

\[
\begin{align*}
&c_1(\lambda, q) \delta(x) \leq \overline{w} \leq c_2(\lambda, q) \delta(x) \quad \text{if} \quad 0 < q < 1, \\
&c_1(\lambda, q) \delta(x)^{\frac{2}{q-1}} \leq \overline{w} \leq c_2(\lambda, q) \delta(x)^{\frac{2}{q-1}} \quad \text{if} \quad 1 < q < 3.
\end{align*}
\]

Furthermore, for the case \( q = 1 \), again recalling [9, Theorem 2.2], we can find that there exists a constant \( c(\lambda) > 0 \) and for any \( \epsilon > 0 \) small enough, there exists a constant \( c_\epsilon(\lambda) > 0 \) such that

\[
c(\lambda) \delta(x) \leq \overline{w} \leq c_\epsilon(\lambda) \delta(x)^{1-\epsilon}.
\]

This, in combination with standard elliptic regularity theory, implies that \( \overline{w} \in C^{\phi_\epsilon}(\Omega) \). Thus, (i) follows. Again, using [2, Proposition 4.1] (also see [9, Theorem 2.2]), we get (ii).

Finally, using (i) – (ii) and the maximum principle, we obtain (iii). This completes the proof.

\[\blacksquare\]

Remark 3.2. One can check that (3.1) is the transformed form (with the transformation \( u = h(w) \)) of the following purely singular problem with the modified quasilinear operator corresponding to the problem (P) :

\[
\begin{cases}
-\Delta u - \Delta(u)^2 u = \lambda \alpha(x) u^{-q} \text{ in } \Omega, \\
\quad u > 0 \text{ in } \Omega, \\
\quad u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where \( \lambda, \alpha, \Omega \) are as in Theorem 3.1. So, by the properties of \( h \) and following the idea of the proof of the Proposition 2.2 in [36], we can deduce that (3.5) has a solution \( h(\overline{w}) \) for every \( \lambda > 0 \), which satisfies all the properties in Theorem 3.1.

From the assumption (f2) and (1.4), we obtain that for any \( \epsilon > 0 \), \( r \geq 1 \), there exist \( \tilde{C}(\epsilon) \) and \( C(\epsilon) > 0 \) such that

\[
\begin{align*}
|f(x, s)| &\leq \epsilon |s| + \tilde{C}(\epsilon) |s|^{r-1} \exp \left( (1 + \epsilon) |s|^4 \right) \quad \text{for all } (x, s) \in \Omega \times \mathbb{R}, \\
|F(x, s)| &\leq \epsilon |s|^2 + C(\epsilon) |s|^r \exp \left( (1 + \epsilon) |s|^4 \right) \quad \text{for all } (x, s) \in \Omega \times \mathbb{R}.
\end{align*}
\]

For any \( w \in H^1_0(\Omega) \), in light of the Sobolev embedding, we have \( w \in L^q(\Omega) \) for all \( q \in [1, +\infty) \).

Let us define the set \( \mathcal{Q} \) as

\[
\mathcal{Q} = \{ \lambda > 0 : \text{ the problem (2.7) has a weak solution in } H^1_0(\Omega) \}
\]

and let \( \Lambda^* := \sup \mathcal{Q} \). Then we have the following result:

Lemma 3.3. Assume that the conditions in Theorem 1.4 hold and let \( h \) be defined as in (2.2). Then the set \( \mathcal{Q} \) is non empty.
Proof. First, we consider the case $0 < q < 1$. Using (3.7), Lemma 2.1-$(h_5), (h_6)$ with the Sobolev embedding and Hölder’s inequality, from (2.3), we get

$$J_\lambda(w) \geq \frac{1}{2} \|w\|^2 - \frac{\lambda}{q-1} \int_\Omega \alpha(x) h(w)^{1-q} \, dx - \lambda \epsilon \int_\Omega h(w)^2 \, dx - \lambda C(\epsilon) \int_\Omega |w|^r \exp((1 + \epsilon) h(w)^4) \, dx$$

$$\geq \frac{1}{2} \|w\|^2 - \frac{\lambda \|\alpha\|_\infty}{q-1} \int_\Omega |w|^{1-q} \, dx - \lambda \epsilon \int_\Omega w^2 \, dx - \lambda C(\epsilon) \int_\Omega |w|^r \exp(2(1 + \epsilon) w^2) \, dx$$

$$\geq \frac{1}{2} \|w\|^2 - \frac{\lambda \|\alpha\|_\infty}{q-1} C_1 \|w\|^{1-q} - \lambda \epsilon C_2 \|w\|^2 - \lambda C_3(\epsilon) \|w\|^r \left( \int_\Omega \exp(2p(1 + \epsilon) w^2) \, dx \right)^{1/p}$$

$$\geq \left( \frac{1}{2} - \lambda \epsilon C_2 \right) \|w\|^2 - \frac{\lambda \|\alpha\|_\infty}{q-1} C_1 \|w\|^{1-q} - \lambda C_4(\epsilon) \|w\|^r \left( \int_\Omega \exp \left( 2p(1 + \epsilon) \|w\|^2, \frac{w^2}{\|w\|^2} \right) \, dx \right)^{1/p},$$

for any $\epsilon > 0$, $p > 1$ and $r > 2$. Choose $\|w\| = r_0$ with $0 < r_0 < 1$ sufficiently small, $0 < \epsilon < \frac{1}{2c_2}$ sufficiently small and $p > 1$ very near to 1 such that $2(1 + \epsilon)p r_0 < 4\pi$. Then by using Theorem 1.1, from (3.7) (3.9), we obtain

$$\frac{1}{2} \|w\|^2 - \lambda \int_\Omega F(x, h(w)) \, dx \geq 2\delta_0 \quad \text{for all} \ w \in \partial B_{r_0};$$

$$\frac{1}{2} \|w\|^2 - \lambda \int_\Omega F(x, h(w)) \, dx \geq 0 \quad \text{for all} \ w \in B_{r_0}.$$ 

Now we can choose $\lambda = \lambda_0 > 0$ sufficiently small so that the last two relations yield that

$$J_{\lambda_0}|_{\partial B_{r_0}} \geq \delta_0 > 0.$$ 

Set $m_0 := \inf_{w \in B_{r_0}} J_{\lambda_0}(w)$. Since for $t > 0$ very small and $w \neq 0$, from (2.3), we have

$$J_{\lambda_0}(tw) \leq \frac{1}{2} \|tw\|^2 - C\lambda_0 \|tw\|^{1-q},$$

which implies that $m_0 < 0$. Let $\{w_k\} \subset B_{r_0}$ be a minimizing sequence such that as $k \to +\infty$,

$$J_{\lambda_0}(w_k) \to m_0;$$

$w_k \rightharpoonup w_0$ weakly in $H^1_0(\Omega);$  

$w_k \to w_0$ strongly in $L^p(\Omega), \ p \geq 1$ and  

$w_k(x) \to w_0(x)$ point-wise a.e. in $\Omega.$

Now, without loss of generality, let us assume that $w_k \geq 0$ due to the fact that $J_\lambda(w) = J_\lambda(|w|)$. Then, using the Sobolev and the Hölder’s inequality, one can easily deduce that for $0 < q < 1$,

$$\int_\Omega w_k^{1-q} \, dx \to \int_\Omega w_0^{1-q} \, dx \quad \text{and} \quad \int_\Omega |w_k - w_0|^{1-q} \, dx \to 0 \quad \text{as} \ k \to +\infty. \quad (3.10)$$

Next, by the mean value theorem, there exists $\tilde{w}_k$ in between $w_0$ and $w_k$ such that using $|h'(\tilde{w}_k)| \leq 1$ and
Now using Lemma 3.1, we deduce
\[
\int_{\Omega} \alpha(x)h(w_k)^{1-q} \, dx \leq \int_{\Omega} \alpha(x)h(w_0)^{1-q} \, dx + \int_{\Omega} \alpha(x)|h(w_k) - h(w_0)|^{1-q} \, dx
\]
\[
\leq \int_{\Omega} \alpha(x)h(w_0)^{1-q} \, dx + \int_{\Omega} \alpha(x)|w_k - w_0|^{1-q}|h'(\bar{w}_k)|^{1-q} \, dx
\]
\[
\leq \int_{\Omega} \alpha(x)h(w_0)^{1-q} \, dx + \|\alpha\|_{\infty} \int_{\Omega} |w_k - w_0|^{1-q} \, dx
\]
\[
\leq \int_{\Omega} \alpha(x)h(w_0)^{1-q} \, dx + o(1),
\]
as \(k \to +\infty\). Similarly, we get \(\int_{\Omega} \alpha(x)h(w_0)^{1-q} \, dx \leq \int_{\Omega} \alpha(x)h(w_k)^{1-q} \, dx + o(1)\) as \(k \to +\infty\). Therefore, from the last two relations, we obtain
\[
\int_{\Omega} \alpha(x)h(w_k)^{1-q} \, dx \to \int_{\Omega} \alpha(x)h(w_0)^{1-q} \, dx \text{ as } k \to +\infty.
\] 
(3.11)

Next, using Lemma 2.1-(h4), (3.6) and then borrowing the similar argument as in the third and fourth terms in (3.8), for sufficiently small \(\epsilon > 0\) and for any \(p > 1\), we deduce
\[
\int_{\Omega} f(x, h(w_k))h'(w_k)w_k \, dx \leq \int_{\Omega} f(x, h(w_k))h(w_k) \, dx
\]
\[
\leq \int_{\Omega} \left( \epsilon|h(w_k)|^2 + \tilde{C}(\epsilon)|h(w_k)|^p \exp \left( (1 + \epsilon)|h(w_k)|^4 \right) \right) \, dx
\]
\[
\leq \epsilon \tilde{C}_2 \|w_k\|^2 + \tilde{C}_4 \|w_k\|^p \left( \int_{\Omega} \exp \left( 2p(1 + \epsilon)\|w_k\|^2 \frac{w_k^2}{\|w_k\|^2} \right) \, dx \right)^{1/p}.
\]

In the last relation, using Theorem 1.1, with sufficiently small \(\|w_k\| < r_0 \ll 1\) and \(p > 1\) very close to 1 so that \(2(1 + \epsilon)p r_0 < 4\pi\), we get
\[
\limsup_{k \to +\infty} \int_{\Omega} f(x, h(w_k))h'(w_k)w_k \, dx < +\infty.
\] 
(3.12)

Now using Lemma 2.1-(h8) and (3.12), for some large \(N(>1) \in \mathbb{N}\), we deduce
\[
\int_{\Omega \cap \{x : h(w_k)(x) > N\}} f(x, h(w_k)) \, dx \leq \frac{1}{N} \int_{\Omega \cap \{x : h(w_k)(x) > N\}} f(x, h(w_k))h(w_k) \, dx
\]
\[
\leq \frac{2}{N} \int_{\Omega \cap \{x : h(w_k)(x) > N\}} f(x, h(w_k))h'(w_k)w_k \, dx
\]
\[
= O \left( \frac{1}{N} \right).
\]

The last relation, together with the Lebesgue dominated convergence theorem, implies that
\[
\int_{\Omega} f(x, h(w_k)) \, dx = \int_{\Omega \cap \{x : h(w_k)(x) \leq N\}} f(x, h(w_k)) \, dx + \int_{\Omega \cap \{x : h(w_k)(x) > N\}} f(x, h(w_k)) \, dx
\]
\[
= \int_{\Omega \cap \{x : h(w_k)(x) \leq N\}} f(x, h(w_k)) \, dx + O \left( \frac{1}{N} \right)
\]
\[
\to \int_{\Omega} f(x, h(w_0)) \, dx, \quad \text{as } k \to +\infty \text{ and } N \to +\infty.
\] 
(3.13)
Since by \( (f4) \) and \((3.13)\), \( F(x, h(w_k)) \leq M_1(1 + f(x, h(w_k))) \in L^1(\Omega) \), for all \( k \in \mathbb{N} \), using the Lebesgue dominated convergence theorem, we obtain
\[
\int_{\Omega} F(x, h(w_k)) \, dx \to \int_{\Omega} F(x, h(w_0)) \, dx \quad \text{as} \quad k \to +\infty. \tag{3.14}
\]

Now from the weak lower semi-continuity of the norm, we have
\[
r_0 \geq \liminf_{k \to +\infty} \|w_k\| \geq \|w_0\|, \tag{3.15}
\]
which yields that \( w_0 \in B_{r_0} \). Therefore,
\[
J_{\lambda_0}(w_0) \geq m_0.
\]
Furthermore, recalling \((3.11)\) and \((3.14)\), we get
\[
m_0 = \lim_{k \to +\infty} J_{\lambda_0}(w_k) \geq J_{\lambda_0}(w_0) \geq m_0.
\]
Thus,
\[
J_{\lambda_0}(w_0) = m_0 < 0.
\]
This yields that \( w_0(\neq 0) \) is a local minimizer of \( J_{\lambda_0} \) in \( H^1_0(\Omega) \).

Next, we claim that \( w_0 \) is a weak solution to the problem \((2.7)\). Note that, for any \( \phi \geq 0, \phi \in H^1_0(\Omega), \)
\[
\liminf_{t \to 0^+} \frac{J_{\lambda_0}(w_0 + t\phi) - J_{\lambda_0}(w_0)}{t} \geq 0. \tag{3.16}
\]
It can be derived from the last expression that \( -\Delta w_0 \geq 0 \) in \( \Omega \) in the weak sense and hence, by the strong maximum principle, \( w_0 > 0 \) in \( \Omega \). Furthermore, employing Fatou’s lemma in \((3.15)\), we infer that
\[
\int_{\Omega} \nabla w_0 \nabla \psi \, dx \geq \lambda \int_{\Omega} \alpha(x) h(w_0)^{-q} h'(w_0) \psi \, dx + \lambda \int_{\Omega} f(x, h(w_0)) h'(w_0) \psi \, dx \quad \text{for all} \quad \psi \in H^1_0(\Omega), \ \psi \geq 0. \tag{3.17}
\]
Now for any \( \phi \in H^1_0(\Omega) \) and \( \epsilon > 0 \), taking \( \psi = (w_0 + \epsilon \phi)^+ \) as a test function in \((3.15)\) and dividing it by \( \epsilon > 0 \), we obtain
\[
\int_{\Omega} \nabla w_0 \nabla \phi \, dx + \epsilon \int_{\Omega} |\nabla \phi|^2 \, dx \geq \frac{\lambda}{\epsilon} \int_{\Omega} \alpha(x) (h(w_0 + \epsilon \phi)^{1-q} - h(w_0)^{1-q}) \, dx
\]
\[
+ \frac{\lambda}{\epsilon} \int_{\Omega} (F(x, h(w_0 + \epsilon \phi)) - F(x, h(w_0))) \, dx.
\]
Letting the limit \( \epsilon \to 0^+ \) in the last expression, we deduce
\[
\int_{\Omega} \nabla w_0 \nabla \phi \, dx \geq \lambda \int_{\Omega} \alpha(x) h(w_0)^{-q} h'(w_0) \phi \, dx + \lambda \int_{\Omega} f(x, h(w_0)) h'(w_0) \phi \, dx.
\]
Again, by taking \( -\phi \) in place of \( \phi \), we get the reverse inequality in the last relation. Therefore, \( w_0 \) is a weak solution to the problem \((2.7)\).

Next, we discuss the case \( 1 \leq q < 3 \). For that, we consider the following problem:
\[
\begin{cases}
-\Delta w &= \lambda \alpha(x) (h(w) + \epsilon)^{-q} h'(w) + \lambda f(x, h(w)) h'(w) \quad \text{in} \ \Omega, \\
w &> 0 \quad \text{in} \ \Omega, \\
w &= 0 \quad \text{on} \ \partial \Omega,
\end{cases} \tag{3.17}
\]
A global multiplicity result for singular and critical elliptic equation

where $0 < \epsilon < 1$ is sufficiently small. We show the existence of solution to (3.17) by constructing a sub-solution and a super-solution to (3.17). Let $W$ be a solution to

$$
-\Delta W = 1 \text{ in } \Omega; \quad W = 0 \text{ on } \partial \Omega.
$$

Then by the maximum principle, $W > 0$ in $\Omega$ and by the standard elliptic regularity theory, $W \in C^1(\overline{\Omega})$ and hence, $W$ is bounded on $\overline{\Omega}$. Set

$$
\overline{w} := w_{\lambda} + MW,
$$

where $M > 0$ is a sufficiently large real constant and $w_{\lambda}$ is the solution to (3.1). Therefore,

$$
-\Delta \overline{w} = \lambda \alpha(x) h(w_{\lambda})^{-q} h'(w_{\lambda}) + M. \quad (3.18)
$$

So, using Lemma 2.1-(h3), (h6) combining with (f2) and the continuity of $f$, for any $\epsilon > 0$ there exists some constant $c(\epsilon, \Omega)$, such that

$$
\lambda f \left( x, h \left( w_{\lambda} + MW \right) \right) h' \left( w_{\lambda} + MW \right)
< \lambda c \exp \left( (1 + \epsilon) h \left( w_{\lambda} + MW \right)^4 \right)
< \lambda c \exp \left( 2(1 + \epsilon) (w_{\lambda} + MW)^2 \right)
< \lambda c \exp \left( 2(1 + \epsilon) (\|w_{\lambda}\|_{\infty} + M\|W\|_{\infty})^2 \right)
= \lambda C(M)
< M \quad \text{for sufficiently small } \lambda > 0. \quad (3.19)
$$

Plugging (3.19) in (3.18), we infer that $\overline{w}$ is a super-solution to (3.17). Now let us set

$$
\overline{w} := MW
$$

for some sufficiently small constant $m = m(\lambda) > 0$ such that $\overline{w} < 1$. Therefore,

$$
-\Delta \overline{w} = m.
$$

We claim that $\overline{w}$ is a sub-solution to (3.17). To prove the claim, it is enough to show

$$
m < \frac{\lambda \alpha_0 h'(mW)}{(h(mW) + \epsilon)^q}. \quad (3.20)
$$

In virtue of Lemma 2.1-(h4), (h5), (h8), we deduce

$$
\frac{\lambda \alpha_0}{m(h(mW) + \epsilon)^q} h'(mW) \geq \frac{\lambda \alpha_0}{m(h(mW) + \epsilon)^q} \frac{h(mW)}{2mW}
\geq \frac{\lambda \alpha_0}{m(mW + 1)^q} \frac{h(1)}{2}
\geq \frac{\lambda \alpha_0}{m^q(\|W\|_{\infty} + 1/m)^q} \frac{h(1)}{2} > 1,
$$

for sufficiently small $0 < m < 1$. This establishes the claim.

Now for $\lambda > 0$ sufficiently small, choosing $M$ sufficiently large and $m$ sufficiently small so that $M >> m$,
from (3.19) and (3.20), we obtain $0 < \underline{w} \leq w \leq \overline{w}$. Hence, there is a solution $w_\varepsilon \leq \overline{w} \leq \overline{w}$ to (3.17) in $H^1_0(\Omega)$. Since $w, \overline{w}$ do not depend on $\varepsilon$,

$$w_\varepsilon(x) \to w_\lambda(x) \quad \text{point-wise in } \Omega \text{ as } \varepsilon \to 0^+.$$  

Next, we will show that $w_\lambda$ is a weak solution to (2.7). From Lemma 2.1-(h4), (h5), (h8), we deduce

$$\alpha(x)h(w_\varepsilon) + \varepsilon^{-q}h'(w_\varepsilon)w_\varepsilon \leq \alpha(x)h(w_\varepsilon) + \varepsilon^{-q}h(w_\varepsilon) \leq \alpha(x)h(\overline{w})^{-q}$$

$$\leq \left\{ \begin{array}{ll}
\|\alpha\|_{\infty}w^{1-q}, & \text{if } 0 \leq q \leq 1, \\
\|\alpha\|_{\infty}h(1)^{1-q}\overline{w}^{1-q}, & \text{if } 1 \leq q \leq 3.
\end{array} \right. \tag{3.21}$$

On the other hand, by (f1) and Lemma 2.1-(h10), it follows that $f(x, h(s))h'(s) = \frac{f(x, h(s))}{h(s)}h(s)\overline{h}(s)$.s is increasing in $s > 0$. Using this together with Lemma 2.1-(h4), we find

$$f(x, h(w_\varepsilon))h'(w_\varepsilon)w_\varepsilon \leq f(x, h(\overline{w}))h(\overline{w}). \tag{3.22}$$

Testing (3.17) against the test function $w_\varepsilon$ and then using (3.24), (3.25), we get

$$\int_{\Omega} |\nabla w_\varepsilon|^2 \, dx = \lambda \int_{\Omega} \alpha(x)h(w_\varepsilon) + \varepsilon^{-q}h'(w_\varepsilon)w_\varepsilon \, dx + \lambda \int_{\Omega} f(x, h(w_\varepsilon))h'(w_\varepsilon)w_\varepsilon \, dx$$

$$\leq \lambda C(\alpha) \int_{\Omega} w^{1-q} \, dx + \lambda \int_{\Omega} f(x, h(\overline{w}))h(\overline{w}) \, dx < +\infty.$$

Thus, $\{w_\varepsilon\}$ is bounded in $H^1_0(\Omega)$. So, up to some sub-sequence, $w_\varepsilon \to w_\lambda$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$. Again, testing (3.17) against any $\phi \in H^1_0(\Omega)$, we deduce

$$\int_{\Omega} \nabla w_\varepsilon \nabla \phi \, dx = \lambda \int_{\Omega} \alpha(x)h(w_\varepsilon) + \varepsilon^{-q}h'(w_\varepsilon)\phi \, dx + \lambda \int_{\Omega} f(x, h(w_\varepsilon))h'(w_\varepsilon)\phi \, dx. \tag{3.23}$$

For $\phi \geq 0$, by Lemma 2.1-(h13), (h5), (h8),

$$\alpha(x)h(w_\varepsilon) + \varepsilon^{-q}h'(w_\varepsilon)\phi \leq \alpha(x)h(\overline{w})^{-q}h'(\overline{w})\phi \leq \|\alpha\|_{\infty}\|\phi\|_{\infty}h(\overline{w})^{-q} \leq Ch(1)^{-q}\overline{w}^{-q}. \tag{3.24}$$

Again for $\phi \geq 0$, using the fact that $f(x, h(s))h'(s)$ is increasing in $s > 0$, we have

$$f(x, h(w_\varepsilon))h'(w_\varepsilon)\phi \leq \|\phi\|_{\infty}f(x, h(\overline{w}))h'(\overline{w}). \tag{3.25}$$

Then, (3.23), (3.24) and (3.25) combining with the Lebesgue dominated convergence theorem yield that

$$\int_{\Omega} \nabla w_\lambda \nabla \phi \, dx = \lambda \int_{\Omega} \alpha(x)h(w_\lambda)\phi \, dx + \lambda \int_{\Omega} f(x, h(w_\lambda))h'(w_\lambda)\phi \, dx \quad \text{as } \varepsilon \to 0. \tag{3.26}$$

Since for any $\phi \in H^1_0(\Omega)$, $\phi = \phi^+ - \phi^-$, the relation in (3.26) holds for all $\phi \in H^1_0(\Omega)$. Thus $w_\lambda$ is a solution to (2.7). This completes the proof of the lemma.

In the next lemma, we aim to discuss the regularity result for the solutions of (2.7).

**Lemma 3.4.** Assume that (f1) – (f4) and (a1) hold. Let $h$ be defined as in (2.2). If $w \in H^1_0(\Omega)$ is any weak solution to (2.7) for $\lambda \in (0, \Lambda^*)$, then $w \in L^\infty(\Omega) \cap C^+_{\phi_q}(\Omega)$. 

Proof. Let \( w \in H^1_0(\Omega) \) be a weak solution to (2.7). First, in spirit of [17, Lemma A.4], we show that \( w \) is in \( L^\infty(\Omega) \). For that, let us define a \( C^1 \) cut-off function \( \psi : \mathbb{R} \to [0, 1] \) as

\[
\psi(s) = \begin{cases} 
0 & \text{if } s \leq 0, \\
1 & \text{if } s \geq 1,
\end{cases}
\]

with \( \psi'(s) \geq 0 \). Now for any \( \epsilon > 0 \), define

\[
\psi_\epsilon(s) = \psi \left( \frac{s - 1}{\epsilon} \right) \text{ for } s \in \mathbb{R}.
\]

Note that \( \nabla(\psi_\epsilon \circ w) = (\psi_\epsilon' \circ w) \nabla w \). Hence, \( \psi_\epsilon \circ w \in H^1_0(\Omega) \). Let \( v \in C^\infty_c(\Omega) \) with \( v \geq 0 \). Now using \( \phi := (\psi_\epsilon \circ w)v \) as a test function in (2.8), we obtain

\[
\int_\Omega \nabla w \nabla(\psi_\epsilon \circ w)v \, dx - \lambda \int_\Omega \alpha(x)h(w)^{-q}h'(w)(\psi_\epsilon \circ w)v \, dx \\
- \lambda \int_\Omega f(x, h(w)(x))h'(w)(\psi_\epsilon \circ w)v \, dx = 0. \tag{3.27}
\]

Since

\[
\nabla w \nabla(\psi_\epsilon \circ w)v = |\nabla w|^2(\psi_\epsilon' \circ w)v + (\nabla w \nabla)(\psi_\epsilon \circ w),
\]

from (3.27), we deduce

\[
\int_\Omega (\nabla w \nabla)(\psi_\epsilon \circ w) \, dx \leq \lambda \int_\Omega \alpha(x)h(w)^{-q}h'(w)(\psi_\epsilon \circ w)v \, dx + \lambda \int_\Omega f(x, h(w)(x))h'(w)(\psi_\epsilon \circ w)v \, dx.
\]

In the last relation, letting \( \epsilon \to 0^+ \) and using Lemma 2.1-(h\(_1\)), (h\(_3\)), we get

\[
\int_\Omega \nabla(w - 1)^+ \nabla v \, dx \leq \lambda \int_{\Omega \cap \{|x : w(x) > 1\}} \alpha(x)h(1)^{-q}h'(1)v \, dx + \lambda \int_{\Omega \cap \{|x : w(x) > 1\}} f(x, h(w))h'(w)v \, dx \\
\leq C + \lambda \int_\Omega f(x, h(w))v \, dx.
\]

Now following the arguments as in [20, Lemma 10, Theorem C] combining with Theorem 1.1, from the last relation, we infer that \( (w - 1)^+ \in L^\infty(\Omega) \). Hence \( w \in L^\infty(\Omega) \).

Now we claim that \( w_\lambda \leq w \) a.e. in \( \Omega \). Suppose the claim is not true. Then using \( (w_\lambda - w)^+ \) as the test function in \( -\Delta(w_\lambda - w) \leq \lambda \alpha(x)(h(w_\lambda)^{-q}h'(w_\lambda) - h(w)^{-q}h'(w)) \) in \( \Omega \) and recalling Lemma 2.1-(h\(_{1\lambda}\)), we deduce

\[
0 \leq \int_\Omega |\nabla(w_\lambda - w)^+|^2 \, dx \leq \lambda \int_\Omega \alpha(x)(h(w_\lambda)^{-q}h'(w_\lambda) - h(w)^{-q}h'(w))(w_\lambda - w)^+ \, dx \leq 0.
\]

Hence, the claim holds. Next, let \( z_\lambda \) be a solution to the problem,

\[
\begin{cases}
-\Delta z_\lambda = \lambda(\alpha(x)h(z_\lambda)^{-q}h'(z_\lambda)) + \|g(w)\|_\infty \exp(2\|w\|_\infty^2) \text{ in } \Omega, \\
z_\lambda > 0 \text{ in } \Omega, \\
z_\lambda = 0 \text{ on } \partial \Omega.
\end{cases} \tag{3.28}
\]
Then, arguing similarly as above, we get \( w \leq z_\lambda \). Now, it can be checked that the results in Theorem 3.1 hold for the problem (3.28). Therefore, \( z_\lambda \) is unique and \( w_\lambda \leq w \leq z_\lambda \). So, in the light of Theorem 3.1-(i), there exist two positive constants \( C_1(\lambda, q) < 1 \), and \( C_2(\lambda, q) \) such that

\[
C_1(q, \lambda) \delta(x) \leq w \leq C_2(q, \lambda) \delta(x) \quad \text{if } 0 < q < 1, \tag{3.29}
\]

\[
C_1(q, \lambda) \delta(x)^{\frac{2}{q+2}} \leq w \leq C_2(q, \lambda) \delta(x)^{\frac{2}{q+2}} \quad \text{if } 1 < q < 3. \tag{3.30}
\]

Moreover, if \( q = 1 \), there exists a constant \( C(\lambda) > 0 \) and for any \( \epsilon > 0 \) small enough, there exists a constant \( C_\epsilon(\lambda) > 0 \) such that

\[
C(\lambda) \delta(x) \leq w \leq C_\epsilon(\lambda) \delta(x)^{1-\epsilon}. \tag{3.31}
\]

Finally, combining (3.29), (3.30), (3.31) and recalling the standard elliptic regularity theory, it follows that \( w \in C^+_{\varphi q}(\Omega) \).

**Remark 3.5.** Using Lemma 3.4, and adapting the proof of Corollary 1.1 in [2] (also see [17]), one can show that if \( w \in H^1_0(\Omega) \) is any weak solution to (2.7), then \( w \in C(\Omega) \). Moreover, when \( 0 < q < 1 \), \( w \in C^1(\Omega) \).

The following lemma basically ensures the non-existence of solution to (2.7) for \( \lambda > \Lambda^* \).

**Lemma 3.6.** Let the conditions in Theorem 1.4 hold and let \( h \) be defined as in (2.2). Then, \( 0 < \Lambda^* < +\infty \).

**Proof.** From Lemma 3.3, we can infer that \( \Lambda^* > 0 \). Thus, we are left to show that \( \Lambda^* < +\infty \). Suppose this is not true. Then, there exists a sequence \( \{\lambda_k\} \subset Q \) such that \( \lambda_k \to +\infty \) as \( k \to +\infty \). For \( s > 0 \), let us define the function

\[
N_{\lambda}(s) := \lambda[h(s)^{-q} + f(x, h(s))] \frac{h'(s)}{s},
\]

We claim that there exist \( k_0 \in \mathbb{N} \) sufficiently large and \( \beta = \beta(\lambda_{k_0}) > 0 \) such that for all \( s > 0 \),

\[
N_{\lambda_{k_0}}(s) = \lambda_{k_0} [h(s)^{-q} + f(x, h(s))] \frac{h'(s)}{s} \geq \beta > \bar{\lambda}_1(\Omega), \tag{3.32}
\]

where \( \bar{\lambda}_1(\Omega) \) is the first eigenvalue of the problem (1.6) with \( \varrho(x) = \min\{1, \alpha(x)\} \). Indeed, for any arbitrary \( k \in \mathbb{N} \) and for \( s \in [1/n, n] \), \( n \in \mathbb{N} \), let us consider \( N_{\lambda_k}(s) \). Since \( N_{\lambda_k} \) is a continuous function, there exists \( s_n := s_{n,k} \in [1/n, n] \) such that

\[
N_{\lambda_k}(s_n) \leq \lambda_{\lambda_k}(s) \text{ for all } s \in [1/n, n].
\]

Now we show that, up to some sub-sequence, \( s_n \to s_{0,k} \in (0, +\infty) \) as \( n \to +\infty \). If not, we have either \( s_n \to 0 \) or \( s_n \to +\infty \) as \( n \to +\infty \). For both the cases, applying Lemma 2.1-(h6), (h8), we obtain

\[
\lim_{n \to +\infty} N_{\lambda_k}(s) \geq \lim_{n \to +\infty} N_{\lambda_k}(s_n) = +\infty.
\]

That is, \( N_{\lambda_k} \geq +\infty \) for all \( s \in (0, +\infty) \) and for all \( k \in \mathbb{N} \), which is absurd. Hence, \( s_n \to s_{0,k} \in (0, +\infty) \) as \( n \to +\infty \) and

\[
N_{\lambda_k}(s) \geq \lambda_k [h(s_{0,k})^{-q} + f(x, h(s_{0,k}))] \frac{h'(s_{0,k})}{s_{0,k}} \text{ for all } s > 0. \tag{3.33}
\]
Arguing in a similar manner as in (3.33), it can be deduced that $s_k := s_{0,k} \to s_0 \in (0, +\infty)$, up to some sub-sequence, as $k \to +\infty$. Using this fact, from (3.33), we get (3.32). Thus, the claim follows.

Since $\lambda_{k_0} \in \mathcal{Q}$, for $\lambda = \lambda_{k_0}$, let $w_{\lambda_{k_0}}$ be a solution to (2.7). So, it follows that

$$-\Delta w_{\lambda_{k_0}} - \beta g(x) w_{\lambda_{k_0}} \geq -\Delta w_{\lambda_{k_0}} - N_{\lambda_{k_0}} q(x) w_{\lambda_{k_0}}$$

$$= -\Delta w_{\lambda_{k_0}} - q(x) w_{\lambda_{k_0}} \lambda_{k_0} [h(w_{\lambda_{k_0}})^{-q} + f(x, h(w_{\lambda_{k_0}}))] \frac{h'(w_{\lambda_{k_0}})}{w_{\lambda_{k_0}}}$$

$$\geq -\Delta w_{\lambda_{k_0}} - \lambda_{k_0} [\alpha(x) h(w_{\lambda_{k_0}})^{-q} - f(x, h(w_{\lambda_{k_0}}))] h'(w_{\lambda_{k_0}}) = 0.$$ 

This implies that $-\Delta w_{\lambda_{k_0}} \geq \beta g(x) w_{\lambda_{k_0}} > 0$ in $\Omega$, which in view of strong maximum principle yields that $w_{\lambda_{k_0}} > 0$ in $\Omega$. Now by applying Picone’s identity for $\varphi_1, \Omega$ and $w_{\lambda_{k_0}}$, we derive

$$0 \leq \int_{\Omega} |\nabla \varphi_{1,\Omega}|^2 dx - \int_{\Omega} \nabla \left( \frac{\varphi_{2,\Omega}^2}{w_{\lambda_{k_0}}} \right) \nabla w_{\lambda_{k_0}} \, dx$$

$$\leq \int_{\Omega} |\nabla \varphi_{1,\Omega}|^2 dx - \int_{\Omega} \beta g(x) \varphi_{1,\Omega}^2 \, dx$$

$$= (\tilde{\lambda}_{1,\Omega}(g) - \beta) \int_{\Omega} q(x) \varphi_{1,\Omega}^2 \, dx.$$ 

Therefore, $\tilde{\lambda}_{1,\Omega}(g) \geq \beta$, which contradicts (3.32). Thus, the proof of the lemma follows.

In the next result, using a sub-super solution technique, we show the existence of at least one solution to (2.7).

**Proposition 3.7.** Let the conditions in Theorem 1.4 be satisfied and let $h$ be defined as in (2.2). Then for each $\lambda \in (0, \Lambda^*)$, (2.7) admits a nontrivial solution in $H^{1}_0(\Omega) \cap C^+_0\varphi_1(\Omega)$.

**Proof.** Let $\lambda \in (0, \Lambda^*)$ and $\lambda' \in (\lambda, \Lambda^*)$. Then from the definition of $\Lambda^*$ and Lemma 3.3, one can see that $W_{\lambda'} \in H^{1}_0(\Omega)$ forms a weak solution to (2.7) for $\lambda = \lambda'$. Let $\underline{w}_\lambda$ be as in Theorem 3.1. Then

$$-\Delta w_{\lambda} = \lambda \alpha(x) h(w_{\lambda})^{-q} h'(w_{\lambda}) \leq \lambda \alpha(x) h(w_{\lambda})^{-q} h'(w_{\lambda}) + \lambda f(x, h(w_{\lambda})) h'(w_{\lambda}), \quad x \in \Omega. \quad (3.34)$$

Thus, $\underline{w}_\lambda$ is a weak sub-solution to (2.7). Therefore, $W_{\lambda'}$ and $\underline{w}_\lambda$ satisfy the following:

$$\begin{cases} 
-\Delta w_{\lambda'} \geq \lambda \alpha(x) h(w_{\lambda'})^{-q} h'(w_{\lambda'}) \text{ in } \Omega, \\
-\Delta w_{\lambda} \leq \lambda \alpha(x) h(w_{\lambda})^{-q} h'(w_{\lambda}) \text{ in } \Omega.
\end{cases}$$ 

Hence, Lemma 2.5 yields that $\underline{w}_\lambda \leq w_{\lambda'}$. Now we consider the closed convex subset $Y_\lambda$ of $H^{1}_0(\Omega)$ as

$$Y_\lambda := \{ w \in H^{1}_0(\Omega) : \underline{w}_\lambda \leq w \leq w_{\lambda'} \}. \quad (3.35)$$

Let $\{ w_k \} \subset Y_\lambda$ be such that $w_k \to w_0$ in $H^{1}_0(\Omega)$ as $k \to +\infty$. Then, up to a sub-sequence, $w_k(x) \to w_0(x)$ point-wise a.e. in $\Omega$. Since $w_{\lambda'}$ is a solution of (2.7), by Lemma 3.4, $w_{\lambda'} \in C^+_0\varphi_1(\Omega)$. Now for $1 < q < 3$, using (3.29) and Lemma 2.1-(h3), we get

$$\alpha(x) h(w_k)^{1-q} \leq \alpha(x) w_k^{1-q} \leq \alpha(x) w_{\lambda'}^{1-q} \leq \| \alpha \|_\infty (C_2(\lambda', q))^{1-q} \| \delta^{1-q} \in L^1(\Omega). \quad (3.36)$$
Next, for \( q = 1 \), by Lemma 2.1-(h\(_8\)) and (3.31), for any sufficiently small \( \varepsilon > 0 \)
\[
\alpha(x) \log(h(w_k)) \leq \alpha(x) \log(w_k) \leq \alpha(x) \log(w_\lambda) \leq \alpha(x) w_\lambda \leq \| \alpha \|_{\infty}C\(\lambda\)'\(\delta^{-1-\varepsilon}\) \in L^1(\Omega). \tag{3.37}
\]
For \( 1 < q < 3 \), using \( h \) is increasing, (3.3) with \( c_1(q, \lambda) > 0 \) small enough such that \( c_1\delta^{\frac{q}{1+q}} < 1 \) and Lemma 2.1-(h\(_8\)), we obtain
\[
\alpha(x)h(w_k)^{-q} \leq \alpha(x)h(w_\lambda)^{-q} \leq \alpha(x)h(c_1\delta^{\frac{q}{1+q}})^{-q} \leq \| \alpha \|_{\infty}c_1^{-q}h(1)^{-q}\delta^{\frac{2(1-q)}{1+q}} \in L^1(\Omega), \tag{3.38}
\]

since \( \frac{2(1-q)}{1+q} > -1 \). Furthermore, using (f2) in combination with Lemma 2.1-(h\(_8\)) and Theorem 1.1, we deduce
\[
F(x, h(w_k)) < C \exp((1 + \varepsilon)h(w_k)^4) \leq C \exp(2(1 + \varepsilon)w_\lambda^2) \in L^1(\Omega). \tag{3.39}
\]

Therefore, by the Lebesgue dominated convergence theorem,
\[
\int_{\Omega} \alpha(x)h(w_k)^{-q} \, dx \to \int_{\Omega} \alpha(x)h(w_0)^{-q} \, dx, \quad \text{if } q \neq 1;
\]
\[
\int_{\Omega} \alpha(x) \log(h(w_k)) \, dx \to \int_{\Omega} \alpha(x) \log(h(w_0)) \, dx, \quad \text{if } q = 1;
\]
\[
\int_{\Omega} F(x, h(w_k)) \, dx \to \int_{\Omega} F(x, h(w_0)) \, dx.
\]

Using the last three limits and the weak lower semicontinuity property of the norm, it follows that \( J_\lambda \) is weakly lower semicontinuous on \( Y_\lambda \). Since \( Y_\lambda \) is weakly sequentially closed subset of \( H^1_0(\Omega) \), there exists \( w_\lambda \in Y_\lambda \) such that
\[
\inf_{w \in Y_\lambda} J_\lambda(w) = J_\lambda(w_\lambda). \tag{3.40}
\]

Now we show that \( w_\lambda \) is a weak solution to (2.7).
For \( \varphi \in H^1_0(\Omega) \) and \( \varepsilon > 0 \) small enough, we define
\[
v_\varepsilon := \min\{w_\lambda, \max\{w_\lambda, w_\lambda + \varepsilon \varphi\}\} = w_\lambda + \varepsilon \varphi - \varphi^\varepsilon + \varphi_\varepsilon \in Y_\lambda,
\]
where \( \varphi^\varepsilon := \max\{0, w_\lambda + \varepsilon \varphi - w_\lambda\} \) and \( \varphi_\varepsilon := \max\{0, w_\lambda - w_\lambda - \varepsilon \varphi\} \). By construction, \( v_\varepsilon \in Y_\lambda \) and \( \varphi^\varepsilon, \varphi_\varepsilon \in H^1_0(\Omega) \). Since \( w_\lambda + t(v_\varepsilon - w) \in Y_\lambda \), for each \( 0 < t < 1 \), using (3.40), Lemma 2.3 and mean value theorem, we obtain
\[
0 \leq \lim_{t \to 0^+} \frac{J_\lambda(w_\lambda + t(v_\varepsilon - w_\lambda)) - J_\lambda(w_\lambda)}{t}
\]
\[
= \int_{\Omega} \nabla w_\lambda \nabla (v_\varepsilon - w_\lambda) \, dx - \lambda \lim_{t \to 0^+} \int_{\Omega} \alpha(x)h(w_\lambda + \theta t(v_\varepsilon - w_\lambda))^{-q}h'(w_\lambda + \theta t(v_\varepsilon - w_\lambda))(v_\varepsilon - w_\lambda) \, dx
\]
\[
- \lambda \int_{\Omega} f(x, h(w_\lambda))h'(w_\lambda)(v_\varepsilon - w_\lambda) \, dx
\]
for some \( 0 < \theta < 1 \). From the definition of \( \varphi^\varepsilon, \varphi_\varepsilon \), we get that \( |v_\varepsilon - w_\lambda| \in H^1_0(\Omega) \), which yields that
\[
|(h(w_\lambda)^{-q}h'(w_\lambda))(v_\varepsilon - w_\lambda)| \in L^1(\Omega).
\]
Moreover, using Lemma 2.1-(h13), we obtain
\[
|h(w_\lambda + \theta t(v_\epsilon - w_\lambda))^{-q}h'(w_\lambda + \theta t(v_\epsilon - w_\lambda))(v_\epsilon - w_\lambda)| \leq |(h(w_\lambda)^{-q}h'(w_\lambda))(v_\epsilon - w_\lambda)|
\]
for all \( t \in (0, 1) \). From the last relation, it follows that
\[
\int_\Omega \nabla w_\lambda \nabla \varphi \, dx - \lambda \int_\Omega \alpha(x) h(w_\lambda)^{-q} h'(w_\lambda) \varphi \, dx - \lambda \int_\Omega f(x, h(w_\lambda)) h'(w_\lambda) \varphi \, dx \geq \frac{1}{\epsilon} (E^\epsilon - E_\epsilon), \quad (3.41)
\]
where
\[
E^\epsilon := \int_\Omega \nabla w_\lambda \nabla \varphi^\epsilon \, dx - \lambda \int_\Omega \alpha(x) h(w_\lambda)^{-q} h'(w_\lambda) \varphi^\epsilon \, dx - \lambda \int_\Omega f(x, h(w_\lambda)) h'(w_\lambda) \varphi^\epsilon \, dx;
\]
\[
E_\epsilon := \int_\Omega \nabla w_\lambda \nabla \varphi_\epsilon \, dx - \lambda \int_\Omega \alpha(x) h(w_\lambda)^{-q} h'(w_\lambda) \varphi_\epsilon \, dx - \lambda \int_\Omega f(x, h(w_\lambda)) h'(w_\lambda) \varphi_\epsilon \, dx.
\]

We define the set \( \Omega^\epsilon := \{ x \in \Omega : (w_\lambda + \epsilon \varphi)(x) \leq w_\lambda(x) \} \) so that \( |\Omega^\epsilon| \rightarrow 0 \) as \( \epsilon \rightarrow 0^+ \). Next, using the fact that \( w_\lambda \) is a super-solution to (2.7) together with Lemma 2.1-(h13), we estimate the following:
\[
\frac{1}{\epsilon} E^\epsilon = \frac{1}{\epsilon} \left[ \int_\Omega \nabla (w_\lambda - w_\lambda^\epsilon) \nabla \varphi \, dx + \int_\Omega \nabla w_\lambda \nabla \varphi^\epsilon \, dx - \lambda \int_\Omega (\alpha(x) h(w_\lambda)^{-q} + f(x, h(w_\lambda))) h'(w_\lambda) \varphi^\epsilon \, dx \right]
\]
\[
\geq \frac{1}{\epsilon} \int_{\Omega^\epsilon} |\nabla (w_\lambda - w_\lambda^\epsilon)|^2 \, dx + \int_{\Omega^\epsilon} \nabla (w_\lambda - w_\lambda^\epsilon) \nabla \varphi \, dx + \frac{\lambda}{\epsilon} \int_{\Omega^\epsilon} \alpha(x) h(w_\lambda)^{-q} h'(w_\lambda) \varphi^\epsilon \, dx
\]
\[
- h(w_\lambda)^{-q} h'(w_\lambda) \varphi^\epsilon \, dx + \frac{\lambda}{\epsilon} \int_{\Omega^\epsilon} (f(x, h(w_\lambda^\epsilon)) h'(w_\lambda^\epsilon) - f(x, h(w_\lambda)) h'(w_\lambda)) \varphi^\epsilon \, dx
\]
\[
+ \frac{1}{\epsilon} \int_{\Omega^\epsilon} \nabla (w_\lambda - w_\lambda^\epsilon) \nabla \varphi \, dx - \lambda \int_{\Omega^\epsilon} \alpha(x) h(w_\lambda)^{-q} h'(w_\lambda^\epsilon) - h(w_\lambda)^{-q} h'(w_\lambda) |\varphi| \, dx
\]
\[
- \lambda \int_{\Omega^\epsilon} |f(x, h(w_\lambda^\epsilon)) h'(w_\lambda^\epsilon) - f(x, h(w_\lambda)) h'(w_\lambda)| |\varphi| \, dx
\]
\[
= o(1) \text{ as } \epsilon \rightarrow 0^+.
\]

Arguing similarly, we have
\[
\frac{1}{\epsilon} E_\epsilon \leq o(1) \text{ as } \epsilon \rightarrow 0^+.
\]

Thus, from (3.41), we get
\[
\int_\Omega \nabla w_\lambda \nabla \varphi \, dx - \lambda \int_\Omega \alpha(x) h(w_\lambda)^{-q} h'(w_\lambda) \varphi \, dx - \lambda \int_\Omega f(x, h(w_\lambda)) h'(w_\lambda) \varphi \, dx \geq o(1) \text{ as } \epsilon \rightarrow 0^+
\]
for all \( \varphi \in H_0^1(\Omega) \). Considering \( -\varphi \) in place of \( \varphi \) and following the similar arguments as above, we infer that \( w_\lambda \) is a weak solution to (2.7). Moreover, from the construction of \( w_\lambda \) and Lemma 3.4, it follows that \( w_\lambda \in C_{\varphi_0}^+ \). This concludes the proof of the proposition. \( \square \)

**Lemma 3.8.** Assume that the conditions in Theorem 1.4 hold and let \( h \) be defined as in (2.2). Let \( \lambda \in (0, \Lambda^*) \). Then any weak solution to (2.7) obtained in Proposition 3.7 is a local minimizer for the functional \( J_\lambda \).
Proof. We prove this lemma for the case $q \neq 1$. For $q = 1$, the proof follows in a similar fashion. Now suppose the statement of the lemma does not hold. So, let us assume that $w_{\lambda}$ is not a local minimum of $J_{\lambda}$, where $w_{\lambda}$ is a solution to (2.7) obtained in Lemma 3.7. Then there exists a sequence $\{w_k\} \subset H^1_0(\Omega)$ such that

$$
\|w_k - w_{\lambda}\| \to 0 \text{ as } k \to +\infty \text{ and } J_{\lambda}(w_k) < J_{\lambda}(w_{\lambda}).
$$

(3.42)

Next, we define $w := w_{\lambda}$ and $\overline{w} := w_{\lambda}'$ as a sub-solution and a super-solution to (2.7), respectively, as defined in the proof of Proposition 3.7. Furthermore, we define

$$
v_k := \max\{w, \min\{w_k, \overline{w}\}\} = \begin{cases} w, & \text{if } w_k < w, \\ w_k, & \text{if } w \leq w_k \leq \overline{w}, \\ \overline{w}, & \text{if } w_k > \overline{w}, \end{cases}
$$

$$
u_k := (w_k - w)^-, \quad \overline{u}_k := (w_k - \overline{w})^+,
$$

$$
S_k := \text{supp}(u_k), \quad \overline{S}_k := \text{supp}(\overline{u}_k).
$$

Then, $w_k = v_k - u_k + \overline{u}_k$ and $v_k \in Y_{\lambda}$, where the set $Y_{\lambda}$ is defined in (3.35). Then we can express $J_{\lambda}(w_k)$ as

$$
J_{\lambda}(w_k) = J_{\lambda}(v_k) + A_k + B_k,
$$

(3.43)

where

$$
A_k := \frac{1}{2} \int_{S_k} (|\nabla w_k|^2 - |\nabla \overline{w}|^2) \, dx - \frac{\lambda}{1-q} \int_{S_k} \alpha(x) (h(w_k)^{1-q} - h(\overline{w})^{1-q}) \, dx 
$$

$$
- \lambda \int_{S_k} (F(x, h(w_k)) - F(x, h(\overline{w}))) \, dx,
$$

(3.44)

$$
B_k := \frac{1}{2} \int_{\overline{S}_k} (|\nabla w_k|^2 - |\nabla \overline{w}|^2) \, dx - \frac{\lambda}{1-q} \int_{\overline{S}_k} \alpha(x) (h(w_k)^{1-q} - h(\overline{w})^{1-q}) \, dx 
$$

$$
- \lambda \int_{\overline{S}_k} (F(x, h(w_k)) - F(x, h(\overline{w}))) \, dx.
$$

(3.45)

From Proposition 3.7, we get $J_{\lambda}(w_k) \geq J_{\lambda}(w_{\lambda}) + A_k + B_k$. We intend to show that $A_k, B_k \geq 0$ for large $k$, as we will see later

$$
\lim_{k \to +\infty} |\overline{S}_k| = 0 \quad \text{and} \quad \lim_{k \to +\infty} |S_k| = 0.
$$

(3.46)

Let us prove (3.46). For any $b > 0$, let us define the set

$$
\Omega_b := \{x \in: \delta(x) > b\}.
$$

Now recalling the proof of Proposition 3.7, we have

$$
\overline{w} \geq w_{\lambda} > \overline{w} := w_{\lambda} > 0.
$$
On the other hand, by (f1) and Lemma 2.1, we have \( f(x, h(s))h'(s) \) is increasing in \( s > 0 \). Therefore, combining the above facts together with Lemma 2.1-(h3), (h12) and using the mean value theorem and (3.2), (3.3), for \( x \in \Omega_{\frac{3}{2}} \), we get

\[
-\Delta(\overline{w} - w_\lambda) \geq \lambda \alpha(x)[h(\overline{w})^{-q}h'(\overline{w}) - h(w_\lambda)^{-q}h'(w_\lambda)] \\
= \lambda \alpha(x)[-q(h'(w_\lambda))^2h(w_\lambda)^{-q-1} + h(w_\lambda)^{-q}h''(w_\lambda)](\overline{w} - w_\lambda), \quad \text{for some } w_* \in (w_\lambda, \overline{w}) \\
\geq \lambda \alpha(x)[-qh(w_\lambda)^{-q-1} - \sqrt{2}h(w_\lambda)^{-q}](\overline{w} - w_\lambda) \\
\geq -\lambda \alpha(x) \left[ qh \left( c_1(q, \lambda) \left( \delta(x) \right)^a \right)^{-q-1} + \sqrt{2}h \left( c_1(q, \lambda) \left( \frac{b}{2} \right)^a \right)^{-q} \right] (\overline{w} - w_\lambda),
\]

where \( a = 1, \) if \( 0 < q < 1 \) and \( a = \frac{2}{q-1}, \) if \( 1 < q < 3. \) Applying [7, Theorem 3], from the last relation, we infer that for any \( b > 0, \) there exists a constant \( C_* > 0 \) such that

\[ \overline{w} - w_\lambda \geq C_* \frac{b}{2} > 0 \quad \text{in } \Omega_b. \]

Given \( \epsilon > 0, \) choose \( b > 0 \) such that \( |\Omega \setminus \Omega_b| < \frac{\epsilon}{2}. \) Since we assumed that \( w_k \to w_\lambda \) in \( H^1_0(\Omega), \) for sufficiently large \( k \in \mathbb{N}, \) we obtain

\[
|\mathcal{S}_k| \leq |\Omega \setminus \Omega_b| + |\mathcal{S}_k \cap \Omega_b| \\
\leq \frac{\epsilon}{2} + \int_{\mathcal{S}_k \cap \Omega_b} \frac{w_k - w_\lambda}{\overline{w} - w_\lambda} \, dx \\
\leq \frac{\epsilon}{2} + \frac{4}{C_*^2 b^2} \int_{\overline{\mathcal{S}_k} \setminus \Omega_b} (w_k - w_\lambda)^2 \, dx \\
< \epsilon + C||w_k - w_\lambda||^2.
\]

This yields that \( |\mathcal{S}_k| \to 0 \) as \( k \to +\infty. \) In a similar fashion as above, considering \( \overline{w} - w_\lambda, \) we get \( |\mathcal{S}_k| \to 0 \) as \( k \to +\infty. \) Therefore, as \( k \to +\infty \)

\[
||\overline{w}_k||^2 = \int_{\mathcal{S}_k} |\nabla(w_k - \overline{w})|^2 \, dx \\
\leq 2 \left( ||w_k - w_\lambda||^2 + \int_{\mathcal{S}_k} |\nabla(w_\lambda - \overline{w})|^2 \, dx \right) \to 0. \quad (3.47)
\]

Similarly, \( ||v_k||^2 \to 0 \) as \( k \to +\infty. \) Since \( \overline{w} \) is a super-solution to (2.7), using Lemma 2.1-(h9) and mean
value theorem, from (3.44), we obtain
\begin{align*}
A_k &= \frac{1}{2} \int_{S_k} (|\nabla(\overline{w} + \overline{v}_k)|^2 - |\nabla \overline{w}|^2) \, dx - \frac{\lambda}{1 - q} \int_{S_k} \alpha(x)(h(\overline{w} + \overline{v}_k))^{1-q} - h(\overline{w})^{1-q}) \, dx \\
&\quad - \lambda \int_{S_k} (F(x, h(\overline{w} + \overline{v}_k)) - F(x, h(\overline{w}))) \, dx, \\
&= \frac{1}{2} \|\overline{v}_k\|^2 + \int_{S_k} \nabla \overline{w} \nabla \overline{v}_k \, dx - \frac{\lambda}{1 - q} \int_{S_k} \alpha(x)h(\overline{w} + \theta \overline{v}_k)^{-q}h'(\overline{w} + \theta \overline{v}_k) \overline{v}_k \, dx \\
&\quad - \lambda \int_{S_k} f(x, h(\overline{w} + \theta \overline{v}_k))h'(\overline{w} + \theta \overline{v}_k) \overline{v}_k \, dx, \quad \theta \in (0, 1) \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 + \lambda \int_{S_k} \alpha(x)h(\overline{w})^{-q}h'(\overline{w}) \overline{v}_k \, dx + \lambda \int_{S_k} f(x, h(\overline{w}))h'(\overline{w}) \overline{v}_k \, dx \\
&\quad - \frac{\lambda}{1 - q} \int_{S_k} \alpha(x)h(\overline{w} + \theta \lambda \overline{v}_k)^{-q}h'(\overline{w} + \theta \overline{v}_k) \overline{v}_k \, dx - \lambda \int_{S_k} f(x, h(\overline{w} + \theta \overline{v}_k))h'(\overline{w} + \theta \overline{v}_k) \overline{v}_k \, dx \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 + \lambda \theta \int_{S_k} \left( f'(x, h(\overline{w} + \theta \overline{v}_k))(h(\overline{w} + \theta \overline{v}_k))^2 + f(x, h(\overline{w} + \theta \overline{v}_k))h''(\overline{w} + \theta \overline{v}_k) \right) \overline{v}_k^2 \, dx, \quad \theta \in (0, 1) \\
&= \frac{1}{2} \|\overline{v}_k\|^2 + \lambda \theta \int_{S_k} \left( f'(x, h(\overline{w} + \theta \overline{v}_k))(h(\overline{w} + \theta \overline{v}_k))^2 + f(x, h(\overline{w} + \theta \overline{v}_k))h''(\overline{w} + \theta \overline{v}_k) \right) \overline{v}_k^2 \, dx - 2f(x, h(\overline{w} + \theta \overline{v}_k))h(\overline{w} + \theta \overline{v}_k)h'(\overline{w} + \theta \overline{v}_k) \overline{v}_k^2 \, dx. \quad (3.48)
\end{align*}

Now by the definition of the function $f$, we have
\[ f'(x, h(s)) = (g'(x, h(s)) + 4h(s)^3 g(x, h(s))) \exp(h(s)^4) \geq 4h(s)^3 f(x, h(s)). \]

Using this combining with Lemma 2.1-(h13), (h12), (h3), (h6) and Hölder’s inequality, from (3.48), we deduce
\begin{align*}
A_k &\geq \frac{1}{2} \|\overline{v}_k\|^2 + \lambda \theta \int_{S_k} \left( 4f(x, h(\overline{w} + \theta \overline{v}_k))h(\overline{w} + \theta \overline{v}_k)^3(h(\overline{w} + \theta \overline{v}_k))^2 \right) \overline{v}_k^2 \, dx \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 - \lambda \theta \int_{S_k} \left( 2f(x, h(\overline{w} + \theta \overline{v}_k))h(\overline{w} + \theta \overline{v}_k)^3(h(\overline{w} + \theta \overline{v}_k))^3 \right) \overline{v}_k^2 \, dx \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 - \lambda \theta \frac{1}{\sqrt{2}} \int_{S_k} f(x, h(\overline{w} + \theta \overline{v}_k)) \overline{v}_k \, dx \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 - \lambda \theta \int_{S_k} \exp((1 + \epsilon)h(\overline{w} + \theta \overline{v}_k)^4) \overline{v}_k^2 \, dx, \quad \text{by (f2) for } \epsilon > 0 \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 - \lambda \theta \int_{S_k} \exp(2(1 + \epsilon)(\overline{w} + \theta \overline{v}_k)^2) \overline{v}_k^2 \, dx \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 - \lambda \theta \int_{S_k} \exp(4(1 + \epsilon)(\overline{w} + \theta \overline{v}_k)^2) \, dx \frac{1}{2} \left( \int_{S_k} \overline{v}_k^4 \, dx \right)^{\frac{1}{2}} \\
&\geq \frac{1}{2} \|\overline{v}_k\|^2 - \lambda \theta \int_{S_k} \exp(\overline{v}_k^4) \, dx \geq 0, \quad \text{for large } k \in \mathbb{N},
\end{align*}
where in the last line, we used Theorem 1.1, (3.47) and (3.46). Similarly, from (3.45), we can show that \( B_k \geq 0 \) for sufficiently large \( k \in \mathbb{N} \). Thus from (3.43), for large \( k \in \mathbb{N} \), it yields that

\[ J_\lambda(w_k) \geq J_\lambda(w_\lambda), \]

which is a contradiction to (3.42). Hence, \( w_\lambda \) is a local minimum of \( J_\lambda \) over \( H^1_0(\Omega) \).

The next result is a consequence of Lemma 3.8, which yields that at the threshold level of \( \lambda \), that is for \( \lambda = \Lambda^* \), we have a weak solution to (2.7).

**Proposition 3.9.** Let the conditions in Theorem 1.4 be satisfied and let \( h \) be defined as in (2.2). Then for \( \lambda = \Lambda^* \), (2.7) admits a weak solution in \( H^1_0(\Omega) \cap C^+_{\varphi q}(\Omega) \).

**Proof.** From the definition of \( \Lambda^* \), there exists an increasing sequence \( \{\lambda_k\} \in \mathbb{Q} \) such that \( \lambda_k \uparrow \Lambda^* \) as \( k \to +\infty \). Hence, by Proposition 3.7, \( \{w_{\lambda_k}\} \in H^1_0(\Omega) \cap C^+_{\varphi q}(\Omega) \cap Y_{\lambda_k} \) is a sequence of positive weak solutions to (2.7) with \( \lambda = \lambda_k \), where \( Y_{\lambda_k} \) is defined as in (3.35). Therefore,

\[
\int_\Omega |\nabla w_{\lambda_k}|^2 \, dx - \lambda_k \int_\Omega \alpha(x) h(w_{\lambda_k})^{-q} h'(w_{\lambda_k}) w_{\lambda_k} \, dx - \lambda_k \int_\Omega f(x, h(w_{\lambda_k})) h'(w_{\lambda_k}) w_{\lambda_k} \, dx = 0. \tag{3.49}
\]

For \( \lambda = \lambda_k \), let \( w_{\lambda_k} \) denote the unique solution to (3.1), which is a sub-solution to (2.7) as in (3.34). So, using Lemma 2.1-(h4), we obtain

\[
\int_\Omega |\nabla w_{\lambda_k}|^2 \, dx = \lambda_k \int_\Omega \alpha(x) h(w_{\lambda_k})^{-q} h'(w_{\lambda_k}) w_{\lambda_k} \, dx \leq \lambda_k \int_\Omega \alpha(x) h(w_{\lambda_k})^{1-q} \, dx. \tag{3.50}
\]

Now Lemma 3.8 yields that \( w_{\lambda_k} \) is a local minimizer of \( J_{\lambda_k} \) for each \( k \in \mathbb{N} \). So,

\[
J_{\lambda_k}(w_{\lambda_k}) = \min_{w \in Y_{\lambda_k}} J_{\lambda_k}(w) \leq J_{\lambda_k}(w_{\lambda_k}) \leq \left\{ \begin{array}{ll}
\frac{1}{2} \int_\Omega |\nabla w_{\lambda_k}|^2 \, dx - \frac{\lambda_k}{1-q} \int_\Omega \alpha(x) h(w_{\lambda_k})^{1-q} \, dx & \text{if } q \neq 1; \\
\frac{1}{2} \int_\Omega |\nabla w_{\lambda_k}|^2 \, dx - \lambda_k \int_\Omega \alpha(x) \log h(w_{\lambda_k}) \, dx & \text{if } q = 1.
\end{array} \right. \tag{3.51}
\]

Plugging (3.50) in (3.51), we get

\[
J_{\lambda_k}(w_{\lambda_k}) \leq \beta_k := \left\{ \begin{array}{ll}
\lambda_k \left( \frac{1}{2} - \frac{1}{1-q} \right) \int_\Omega \alpha(x) h(w_{\lambda_k})^{1-q} \, dx & \text{if } q \neq 1; \\
\frac{\lambda_k}{2} \int_\Omega \alpha(x) \, dx - \lambda_k \int_\Omega \alpha(x) \log h(w_{\lambda_k}) \, dx & \text{if } q = 1.
\end{array} \right. \tag{3.52}
\]

Thus, for all \( 0 < q < 3 \), since \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \leq \Lambda^* \), from (3.2) and (3.3) and Theorem 3.1-(iv), we infer that

\[
\sup_k \beta_k < +\infty. \tag{3.53}
\]

**Case-I:** \( 0 < q < 1 \).

Now (3.49) and (3.52) combining with Lemma 2.1-(h4) and (f3) imply that

\[
\beta_k + \lambda_k \left( \frac{1}{1-q} - \frac{1}{2} \right) \int_\Omega \alpha(x) h(w_{\lambda_k})^{1-q} \, dx \geq \frac{\lambda_k}{2} \left( \int_\Omega f(x, h(w_{\lambda_k})) h'(w_{\lambda_k}) w_{\lambda_k} \, dx - \int_\Omega F(x, h(w_{\lambda_k})) \, dx \right) \\
\geq \lambda_k \left( \frac{1}{4} - \frac{1}{\tau} \right) \int_\Omega f(x, h(w_{\lambda_k})) h(w_{\lambda_k}) \, dx. \tag{3.54}
\]
Plugging (3.54) and (3.53) in (3.49) and using Lemma 2.1-(h4), (h5) together with the fact that \( \tau > 4 \), the Sobolev inequality and the Hölder inequality, we obtain

\[
\|w_{\lambda_k}\|^2 \leq C_1 + C_2 \Lambda^* \int_\Omega \alpha(x)h(w_{\lambda_k})^{1-q} \, dx \leq C_1 + C_2 \Lambda^* \|\alpha\|_\infty \int_\Omega w_{\lambda_k}^{1-q} \, dx \leq C_1 + C_3 \|w_{\lambda_k}\|^{1-q}. \tag{3.55}
\]

**Case-II:** \( q = 1 \).

Again using (3.49), (3.52), Lemma 2.1-(h4) and (f3) combining with the inequality \( \log h(s) \leq \log s < s \), for \( s > 0 \), we get

\[
\beta_k + \lambda_k \int_\Omega \alpha(x) \, dx \\
\geq \frac{\lambda_k}{2} \int_\Omega \alpha(x) h(w_{\lambda_k})^{-1} h'(w_{\lambda_k}) w_{\lambda_k} \, dx + \frac{\lambda_k}{2} \left( \int_\Omega f(x, h(w_{\lambda_k})) h'(w_{\lambda_k}) w_{\lambda_k} \, dx - \int_\Omega F(x, h(w_{\lambda_k})) \, dx \right) \\
\geq \frac{\lambda_k}{4} \int_\Omega \alpha(x) \, dx + \lambda_k \left( \frac{1}{4} - \frac{1}{\tau} \right) \int_\Omega f(x, h(w_{\lambda_k})) h(w_{\lambda_k}) \, dx. \tag{3.56}
\]

This gives that

\[
\beta_k + \frac{3}{4} \lambda_k \int_\Omega \alpha(x) \geq \lambda_k \left( \frac{1}{4} - \frac{1}{\tau} \right) \int_\Omega f(x, h(w_{\lambda_k})) h(w_{\lambda_k}) \, dx. \tag{3.57}
\]

Using (3.57) and (3.53), from (3.49), we deduce

\[
\|w_{\lambda_k}\|^2 \leq C_4 (\beta_k + \Lambda^* \|\alpha\|_\infty |\Omega|) < +\infty. \tag{3.58}
\]

**Case-III:** \( 1 < q < 3 \).

From (3.54), it follows that

\[
\lambda_k \left( \frac{1}{4} - \frac{1}{\tau} \right) \int_\Omega f(x, h(w_{\lambda_k})) h(w_{\lambda_k}) \, dx \leq \beta_k + \lambda_k \left( \frac{1}{q - 1} + \frac{1}{2} \right) \int_\Omega \alpha(x) h(w_{\lambda_k})^{1-q} \, dx.
\]

Employing the last relation in (3.49) and using Theorem 3.1-(iv), Lemma 2.1-(h8) and (3.3) for \( \lambda_2 \) with \( c_1(q, \lambda_2) > 0 \) small enough such that \( c_1(q, \lambda_2) \delta^{\frac{2}{1+q}} < 1 \), we get

\[
\|w_{\lambda_k}\|^2 \leq C_5 + C_6 \lambda_k \int_\Omega h(w_{\lambda_k})^{1-q} \, dx \\
\leq C_5 + C_6 \lambda_k \int_\Omega h(w_{\lambda_k})^{1-q} \, dx \\
\leq C_5 + C_6 \Lambda^* \int_\Omega h(w_{\lambda_k})^{1-q} \, dx \\
\leq C_5 + C_6 \Lambda^* \int_\Omega h \left( c_1(q, \lambda_2) \delta^{\frac{2}{1+q}} \right)^{1-q} \, dx \\
\leq C_5 + C_6 \Lambda^* h(1)^{1-q} \left( c_1(q, \lambda_2) \right)^{1-q} \int_\Omega \delta^{\frac{2(1-q)}{1+q}} \, dx < +\infty, \tag{3.59}
\]

since \( \frac{2(1-q)}{1+q} > -1 \).

Thus, each of the expressions in (3.55), (3.59), (3.58) from Case-I, II, III, respectively, yields that

\[
\limsup_{k \to +\infty} \|w_{\lambda_k}\| < +\infty. \tag{3.60}
\]
Therefore, up to a sub-sequence, there exists $w_{\lambda^*} \in H^1_0(\Omega)$ such that $w_{\lambda_k} \rightharpoonup w_{\lambda^*}$ weakly in $H^1_0(\Omega)$ and $w_{\lambda_k}(x) \rightharpoonup w_{\lambda^*}(x)$ a.e. in $\Omega$ as $k \to +\infty$. Also, by the construction, it follows that $w_{\lambda_k} \geq w_{\lambda_k} \geq w_{\lambda^*}$. So,

$$w_{\lambda^*}(x) = \lim_{k \to +\infty} w_{\lambda_k}(x) > w_{\lambda_1}(x) > 0 \text{ a.e. in } \Omega.$$

Thus, by the Lebesgue dominated convergence theorem, for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \alpha(x)h(w_{\lambda_k})^{-q}h'(w_{\lambda_k})\phi \, dx \to \int_\Omega \alpha(x)h(w_{\lambda^*})^{-q}h'(w_{\lambda^*})\phi \, dx \quad \text{as } k \to +\infty. \quad (3.61)$$

Next, we show that for any $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega f(x,h(w_{\lambda_k}))h'(w_{\lambda_k})\phi \, dx \to \int_\Omega f(x,h(w_{\lambda^*}))h'(w_{\lambda^*})\phi \, dx \quad \text{as } k \to +\infty. \quad (3.62)$$

To prove (3.62), first observe that for $1 < q < 3$, using Lemma 2.1-(h4), and arguing similarly as in (3.38), $\alpha(x)h(w_{\lambda_k})^{-q}h'(w_{\lambda_k})w_{\lambda_k} \leq \|\alpha\|_\infty h(w_{\lambda_k})^{1-q} \leq \|\alpha\|_\infty h(w_{\lambda_1})^{1-q} \in L^1(\Omega)$. Thus, by the Lebesgue dominated convergence theorem, we have

$$\int_\Omega \alpha(x)h(w_{\lambda_k})^{-q}h'(w_{\lambda_k})w_{\lambda_k} \, dx \to \int_\Omega \alpha(x)h(w_{\lambda^*})^{-q}h'(w_{\lambda^*})w_{\lambda^*} \, dx \quad \text{as } k \to +\infty. \quad (3.63)$$

Furthermore, for $0 < q \leq 1$, (3.63) follows similarly as in (3.11). Hence, from (3.49), (3.60) and (3.63), we obtain

$$\limsup_{k \to +\infty} \int_\Omega f(x,h(w_{\lambda_k}))h'(w_{\lambda_k})w_{\lambda_k} \, dx < +\infty.$$  

Then repeating a similar argument as in (3.13), we get (3.62). Gathering (3.62) and (3.63), we finally infer that $w_{\lambda^*} \in H^1_0(\Omega)$ is a positive weak solution to (2.7). Moreover, in light of Lemma 3.4, we have $w_{\lambda^*} \in H^1_0(\Omega) \cap C^+_\varphi(\Omega)$. Hence, the proof of the proposition is complete. 

**Proof of Theorem 1.4** : Combining Lemma 3.3, 3.4, 3.6, 3.8 along with Proposition 3.7, 3.9, we infer that $w_\lambda \in H^1_0(\Omega) \cap C^+_\varphi(\Omega)$ is a weak solution to (2.7). Now by Lemma 2.1-(h1), we have $h$ is a $C^\infty$ function and Lemma 2.1-(h8), (h11) ensure that $h(s)$ behaves like $s$ when $s$ is close to 0. Therefore, we can conclude that $h(w_\lambda) \in H^1_0(\Omega) \cap C^+_\varphi(\Omega)$ forms a weak solution to the problem $\left( P_\star \right)$.

### 4 Proof of Theorem 1.5 : Multiplicity result

This section is dedicated toward establishing the existence of second solution of (2.7) using the mountain pass lemma in combination with Ekeland variational principle. Let us define the set

$$T := \{ w \in H^1_0(\Omega) : w \geq w_\lambda \text{ a.e. in } \Omega \}.$$  

Since by Lemma 3.8, $w_\lambda$ is a local minimizer for $J_\lambda$, it follows that $J_\lambda(w) \geq J_\lambda(w_\lambda)$ whenever $\|w_\lambda - w\| \leq \sigma_0$, for some small constant $\sigma_0 > 0$. Then, one of the following cases holds:

**ZA** (Zero Altitude): $\inf \{ J_\lambda(w) \mid w \in T, \|w - w_\lambda\| = \sigma \} = J_\lambda(w_\lambda)$ for all $\sigma \in (0, \sigma_0)$.

**MP** (Mountain Pass): There exists a $\sigma_1 \in (0, \sigma_0)$ such that $\inf \{ J_\lambda(w) \mid w \in T, \|w - w_\lambda\| = \sigma_1 \} > J_\lambda(w_\lambda)$. 

Now for the case \((ZA)\), inspired by [19] and [2], we prove the existence of second weak solution to \((2.7)\) in the following result.

**Proposition 4.1.** Let the conditions in Theorem 1.5 hold and let \(\lambda \in (0, \Lambda^*)\). Suppose that \((ZA)\) holds. Then for all \(\sigma \in (0, \sigma_0)\), \((2.7)\) admits a second solution \(v_\lambda \in H^1_0(\Omega) \cap C^4_{\psi_\lambda}(\Omega)\) such that \(v_\lambda \geq w_\lambda\) and \(\|v_\lambda - w_\lambda\| = \sigma\).

**Proof.** Let us fix \(\sigma \in (0, \sigma_0)\) and \(r > 0\) such that \(\sigma - r > 0\) and \(\sigma + r < \sigma_0\). Define the set
\[
A := \{w \in T : 0 < \sigma - r \leq \|w - w_\lambda\| \leq \sigma + r\}.
\]
Clearly \(A\) is closed in \(H^1_0(\Omega)\) and by \((ZA)\), \(\inf_{w \in A} J_\lambda(w) = J_\lambda(w_\lambda)\). So, for any minimizing sequence \(\{w_k\} \subset A\) satisfying \(\|w_k - w_\lambda\| = \sigma\), by Ekeland variational principle, we get another sequence \(\{v_k\} \subset A\) such that
\[
\begin{align*}
J_\lambda(v_k) & \leq J_\lambda(w_k) \leq J_\lambda(w_\lambda) + \frac{1}{k} \\
\|w_k - v_k\| & \leq \frac{1}{k} \\
J_\lambda(v_k) & \leq J_\lambda(v) + \frac{1}{k}\|v - v_k\| \quad \text{for all } v \in A.
\end{align*}
\]
(4.1)

For \(z \in T\), we can choose \(\epsilon > 0\) small enough so that \(v_k + \epsilon(z - v_k) \in A\). So, from (4.1), we obtain
\[
\frac{J_\lambda(v_k + \epsilon(z - v_k)) - J_\lambda(v_k)}{\epsilon} \geq -\frac{1}{k}\|z - v_k\|.
\]
Letting \(\epsilon \to 0^+\), using the fact that \(v_k \geq w_\lambda\) for each \(k \in \mathbb{N}\) and following the similar arguments as in (2.5) and (2.6) for the singular term in the last relation, we obtain
\[
-\frac{1}{k}\|z - v_k\| \leq \int_\Omega \nabla v_k \nabla (z - v_k) \, dx - \lambda \int_\Omega \alpha(x) h(v_k)^{-q} h'(v_k) (z - v_k) \, dx - \lambda \int_\Omega f(x, h(v_k)) h'(v_k) (z - v_k) \, dx
\]
(4.2)
for all \(z \in T\). Since \(\{v_k\}\) is a bounded sequence in \(H^1_0(\Omega)\), there exists \(v_\lambda \in H^1_0(\Omega)\) such that, up to a sub-sequence, \(v_k \rightharpoonup v_\lambda\) weakly in \(H^1_0(\Omega)\) and point-wise a.e. in \(\Omega\) as \(k \to +\infty\). Since \(v_k \geq w_\lambda\) for each \(k\), \(v_\lambda \geq w_\lambda\) a.e. in \(\Omega\).

**Claim:** \(v_\lambda\) is a weak solution to \((2.7)\).

For \(\phi \in H^1_0(\Omega)\) and \(\epsilon > 0\), we set
\[
\psi_{k,\epsilon} := (v_k + \epsilon \phi - w_\lambda)^-, \quad \psi_\epsilon := (v_\lambda + \epsilon \phi - w_\lambda)^-.
\]
Clearly, \(\psi_{k,\epsilon} \in H^1_0(\Omega)\). This gives that \((v_k + \epsilon \phi + \psi_{k,\epsilon}) \in T\). Taking \(z = v_k + \epsilon \phi + \psi_{k,\epsilon}\) in (4.2), we deduce
\[
\begin{align*}
-\frac{1}{k}\|(\epsilon \phi + \psi_{k,\epsilon})\| & \leq \int_\Omega \nabla v_k \nabla (\epsilon \phi + \psi_{k,\epsilon}) - \lambda \int_\Omega \alpha(x) h(v_k)^{-q} h'(v_k) (\epsilon \phi + \psi_{k,\epsilon}) \, dx \\
& \quad - \lambda \int_\Omega f(x, h(v_k)) h'(v_k) (\epsilon \phi + \psi_{k,\epsilon}) \, dx.
\end{align*}
\]
(4.3)
Note that \(|\psi_{k,\epsilon}| \leq w_\lambda + \epsilon|\phi|\). Hence, using the Sobolev embedding and the Lebesgue dominated convergence theorem, as \(k \to +\infty\), \(\psi_{k,\epsilon} \rightharpoonup \psi_\epsilon\) in \(L^p(\Omega)\), \(p \in [1, +\infty)\); \(\psi_{k,\epsilon} \rightharpoonup \psi_\epsilon\) weakly in \(H^1_0(\Omega)\) and \(\psi_{k,\epsilon}(x) \to \psi_\epsilon(x)\) point-wise a.e. in \(\Omega\). Now using Lemma 2.1-(h13), we obtain
\[
\alpha(x) h(v_k)^{-q} h'(v_k) (\epsilon \phi + \psi_{k,\epsilon}) \leq ||\alpha||_{\infty} h(v_\lambda)^{-q} h'(v_\lambda) (v_\lambda + 2\epsilon|\phi|).
\]
Furthermore, using the fact that both $h$ and $f(x, \cdot)$ are non-decreasing functions, we get
\[
f(x, h(v_k))h'(v_k)(\epsilon \phi + \psi_{k, \epsilon}) \leq f(x, h(w_\lambda + \epsilon |\phi|))(w_\lambda + \epsilon |\phi|).
\]
Hence, employing the Lebesgue dominated convergence theorem, as $k \to +\infty$, we infer
\[
\int \alpha(x)h(v_k)^{-q}h'(v_k)(\epsilon \phi + \psi_{k, \epsilon}) \, dx \to \int \alpha(x)h(\lambda)^{-q}h'(\lambda)(\epsilon \phi + \psi_e) \, dx, \\
\int f(x, h(v_k))h'(v_k)(\epsilon \phi + \psi_{k, \epsilon}) \, dx \to \int f(x, h(\lambda))h'(\lambda)(\epsilon \phi + \psi_e) \, dx.
\]
We define the sets
\[
\Omega_{k, \epsilon} := \text{supp } \psi_{k, \epsilon}, \quad \Omega_{\epsilon} := \text{supp } \psi_e \quad \text{and} \quad \Omega_0 := \{x \in \Omega : v_\lambda(x) := w_\lambda(x)\}.
\]
Then,
\[
|\Omega_{\epsilon} \setminus \Omega_0| \to 0 \quad \text{as } \epsilon \to 0; \quad (4.6) \\
|\Omega_{k, \epsilon} \setminus \Omega_{\epsilon}| + |\Omega_{\epsilon} \setminus \Omega_{k, \epsilon}| \to 0 \quad \text{as } k \to +\infty. \\
(4.7)
\]
Therefore, as $k \to +\infty$,
\[
\int \nabla v_k \nabla \psi_{k, \epsilon} \, dx = \int \nabla v_k \nabla \psi_e \, dx - \int |\nabla (v_k - v_\lambda)|^2 \, dx + \int \nabla v_\lambda \nabla (v_\lambda - v_k) \, dx + o(1) \\
\leq \int \nabla v_k \nabla \psi_e \, dx + \int \nabla v_\lambda \nabla (v_\lambda - v_k) \, dx + o(1) = \int \nabla v_k \nabla \psi_e \, dx + o(1). \\
(4.8)
\]
Combining (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), letting $k \to +\infty$ and using Lemma 2.1-(h13), we obtain
\[
\int \nabla v_\lambda \nabla \phi \, dx - \lambda \int \alpha(x)h(\lambda)^{-q}h'(\lambda)\phi \, dx - \lambda \int f(x, h(\lambda))h'(\lambda)\phi \, dx \\
\geq -\frac{1}{\epsilon} \left[ \int \nabla v_k \nabla \psi_e \, dx - \lambda \int \alpha(x)h(v_k)^{-q}h'(v_k)\psi_e \, dx - \lambda \int f(x, h(v_k))h'(v_k)\psi_e \, dx \right] \\
= \frac{1}{\epsilon} \left[ \int -\nabla (v_\lambda - w_\lambda) \nabla \psi_e \, dx + \lambda \int \alpha(x) \left[ h(v_\lambda)^{-q}h'(v_\lambda) - h(w_\lambda)^{-q}h'(w_\lambda) \right] \psi_e \, dx \right. \\
+ \lambda \int \left[ f(x, h(v_\lambda))h'(v_\lambda) - f(x, h(w_\lambda))h'(w_\lambda) \right] \psi_e \, dx \right] \\
= \frac{1}{\epsilon} \left[ \int -\nabla (v_\lambda - w_\lambda) \nabla (w_\lambda - v_\lambda - \epsilon \phi) \, dx \\
+ \lambda \int \alpha(x) \left[ h(v_\lambda)^{-q}h'(v_\lambda) - h(w_\lambda)^{-q}h'(w_\lambda) \right] (w_\lambda - v_\lambda - \epsilon \phi) \, dx \right. \\
+ \lambda \int \left[ f(x, h(v_\lambda))h'(v_\lambda) - f(x, h(w_\lambda))h'(w_\lambda) \right] (w_\lambda - v_\lambda - \epsilon \phi) \, dx \right] \\
\geq \int \nabla (v_\lambda - w_\lambda) \nabla \phi \, dx - \lambda \int \alpha(x) \left[ h(v_\lambda)^{-q}h'(v_\lambda) - h(w_\lambda)^{-q}h'(w_\lambda) \right] \phi \, dx \\
- \lambda \int \left[ f(x, h(v_\lambda))h'(v_\lambda) - f(x, h(w_\lambda))h'(w_\lambda) \right] \phi \, dx \\
+ \frac{\lambda}{\epsilon} \int \left[ f(x, h(v_\lambda))h'(v_\lambda) - f(x, h(w_\lambda))h'(w_\lambda) \right] (w_\lambda - v_\lambda) \, dx. \\
(4.9)
\]
Since \( v_\lambda \geq w_\lambda \), using Lemma 2.1-(\( h_9 \)) and the mean value theorem, if \( x \in \Omega_\varepsilon \),

\[
\left[ f(x, h(v_\lambda)) h'(v_\lambda) - f(x, h(w_\lambda)) h'(w_\lambda) \right] (w_\lambda - v_\lambda) \, dx \\
\geq -(v_\lambda - w_\lambda)^2 \left[ f'(x, h(\xi_\lambda))(h'(\xi_\lambda))^2 + f(x, h(\xi_\lambda)) h''(\xi_\lambda) \right], \quad \xi_\lambda \in (w_\lambda, v_\lambda) \\
\geq -\epsilon^2 f'(x, h(\xi_\lambda)) \phi^2.
\] (4.10)

Plugging (4.10) into (4.9), letting \( \epsilon \to 0^+ \) and using (4.6), we get

\[
\int_\Omega \nabla v_\lambda \nabla \phi \, dx - \lambda \int_\Omega \alpha(x) h(v_\lambda)^{-q} h'(v_\lambda) \phi \, dx - \lambda \int_\Omega f(x, h(v_\lambda)) h'(v_\lambda) \phi \, dx \\
\geq o(1) - \lambda \epsilon \int_\Omega f'(x, h(\xi_\lambda)) \phi^2 \, dx = o(1).
\]

Considering \(-\phi\) in place of \(\phi\) and arguing similarly as above, we get the reverse inequality in the last relation. Therefore,

\[
\int_\Omega \nabla v_\lambda \nabla \phi \, dx - \lambda \int_\Omega \alpha(x) h(v_\lambda)^{-q} h'(v_\lambda) \phi \, dx - \lambda \int_\Omega f(x, h(v_\lambda)) h'(v_\lambda) \phi \, dx = 0
\]
for all \( \phi \in H_0^1(\Omega) \). So, \( v_\lambda \in H_0^1(\Omega) \) is a weak solution to (2.7) and thus, the claim is proved. Moreover, by Lemma 3.4, \( v_\lambda \in C_{\phi q}^+ (\Omega) \).

Now we show that \( v_\lambda \neq w_\lambda \). For that, it is enough to prove that

\[
v_k \rightharpoonup v_\lambda \quad \text{in} \quad H_0^1(\Omega) \quad \text{as} \quad k \to +\infty.
\] (4.11)

Applying the Brézis-Lieb lemma, we get

\[
\|v_k\|^2 - \|v_k - v_\lambda\|^2 = \|v_\lambda\|^2 + o(1).
\]

Putting \( z = v_\lambda \) in (4.2) and using the fact that \( v_k \rightharpoonup v_\lambda \) in \( T \) as \( k \to +\infty \), we obtain that

\[
\int_\Omega |\nabla (v_k - v_\lambda)|^2 \, dx \leq o(1) - \lambda \int_\Omega \alpha(x) h(v_k)^{-q} h'(v_k)(v_\lambda - v_k) \, dx - \lambda \int_\Omega f(x, h(v_k)) h'(v_k)(v_\lambda - v_k) \, dx. \quad (4.12)
\]

Let us denote \( u_k := \frac{v_k - w_\lambda}{\|v_k - w_\lambda\|} \). Now recalling (f2) and (3.6) for any \( \epsilon > 0 \), Lemma 2.1-(\( h_3 \), \( h_4 \), \( h_5 \)) and
using Hölder’s inequality, for some large $N >> 1$, we deduce
\[
\int_{\Omega \cap \{x : v_k(x) > N\}} f(x, h(v_k)) h'(v_k) v_k \, dx \\
\leq C \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left((1 + \epsilon) h(v_k)^4\right) h'(v_k) v_k \, dx \\
\leq C \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left(2(1 + \epsilon) v_k^2\right) v_k \, dx \\
\leq C \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left(3(1 + \epsilon) v_k^2\right) \, dx \\
= C \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left(- (1 + \epsilon) v_k^2\right) \exp \left(4(1 + \epsilon)(v_k - w_\lambda + w_\lambda)^2\right) \, dx \\
\leq C \exp \left(- (1 + \epsilon) N^2\right) \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left(8(1 + \epsilon)(v_k - w_\lambda)^2 + w_\lambda^2\right) \, dx \\
\leq C \exp \left(- (1 + \epsilon) N^2\right) \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left(8(1 + \epsilon) u_k^2\|v_k - w_\lambda\|^2\right) \exp \left(8(1 + \epsilon) w_\lambda^2\right) \, dx \\
\leq C \exp \left(- (1 + \epsilon) N^2\right) \left(\int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left(8p(1 + \epsilon) u_k^2\|v_k - w_\lambda\|^2\right) \, dx\right)^{1/p} \\
\times \left(\int_{\Omega} \exp \left(8p'(1 + \epsilon) w_\lambda^2\right) \, dx\right)^{1/p'} \\
\leq C \exp \left(- (1 + \epsilon) N^2\right) \left(\int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left(8p(1 + \epsilon) u_k^2(\sigma + r)^2\right) \, dx\right)^{1/p} \\
\times \left(\int_{\Omega} \exp \left(8p'(1 + \epsilon) w_\lambda^2\right) \, dx\right)^{1/p'} . \tag{4.13}
\]

Now choosing $p > 1$ and $\sigma_0 > 0$ appropriately small so that $8(1 + \epsilon)p(\sigma + r) < 8(1 + \epsilon)p\sigma_0 < 4\pi$ and then applying Theorem 1.1, from (4.13), we derive
\[
\int_{\Omega \cap \{x : v_k(x) > N\}} f(x, h(v_k)) h'(v_k) v_k \, dx = O \left(\exp \left(- (1 + \epsilon) N^2\right)\right) . \tag{4.14}
\]

Since $f(x, h(v_k)) h'(v_k) v_k \to f(x, h(v_\lambda)) h'(v_\lambda) v_\lambda$ point-wise a.e. in $\Omega$ as $k \to +\infty$, using (4.14) and applying the Lebesgue dominated convergence theorem, it follows that
\[
\int_{\Omega} f(x, h(v_k)) h'(v_k) v_k \, dx = \int_{\Omega \cap \{x : v_k \leq N\}} f(x, h(v_k)) h'(v_k) v_k \, dx + \int_{\Omega \cap \{x : v_k > N\}} f(x, h(v_k)) h'(v_k) v_k \, dx \\
= \int_{\Omega \cap \{x : v_k \leq N\}} f(x, h(v_k)) h'(v_k) v_k \, dx + O \left(\exp \left(- (1 + \epsilon) N^2\right)\right) \\
\to \int_{\Omega} f(x, h(v_\lambda)) h'(v_\lambda) v_\lambda \, dx \quad \text{as } k \to +\infty \quad \text{and } N \to +\infty . \tag{4.15}
\]

In a similar way, we also have
\[
\int_{\Omega} f(x, h(v_k)) h'(v_k) v_\lambda \, dx \to \int_{\Omega} f(x, h(v_\lambda)) h'(v_\lambda) v_\lambda \, dx \quad \text{as } k \to +\infty .
\]

This, together with (4.15), yields that
\[
\int_{\Omega} f(x, h(v_k)) h'(v_k)(v_k - v_\lambda) \, dx \to 0 \quad \text{as } k \to +\infty . \tag{4.16}
\]
Next, we show that

$$\int_{\Omega} \alpha(x)h(v_k)^{-q}h'(v_k)(v_k - v_\lambda) \, dx \to 0 \quad \text{as} \quad k \to +\infty. \quad (4.17)$$

By the construction, we have $v_k, v_\lambda \geq w_\lambda \geq w$. Next, we consider the three cases separately as following:

**Case I:** $0 < q < 1$. Using Lemma 2.1-(h3), (h8), (3.2) with $c_1(q, \lambda) > 0$ small enough such that $c_1(q, \lambda)\delta < 1$ and (3.29) for $v_\lambda \in L^q(\Omega)$, we get

$$\alpha(x)h(v_k)^{-q}h'(v_k)w \leq \alpha(x)h(w_\lambda)^{-q}w \leq \alpha(x)h(c_1(q, \lambda)\delta)^{-q}C_2(q, \lambda)\delta \leq \|\alpha\|_\infty C(q, \lambda)h(1)^{-q}\delta^{1-q} \in L^1(\Omega). \quad (4.18)$$

Moreover, by Lemma 2.1-(h4), (h5), it follows that

$$\alpha(x)h(v_k)^{-q}h'(v_k)v_k \leq \alpha(x)h(v_k)^{1-q} \leq \|\alpha\|_\infty v_k^{1-q} \in L^1(\Omega) \quad (4.19)$$

due to the Sobolev inequality and the Hölder inequality. Now using (3.10), $\int_{\Omega} v_k^{-q}dx \to \int_{\Omega} v_\lambda^{-q}dx$ as $k \to +\infty$. So, by the Lebesgue dominated convergence theorem, as $k \to +\infty$, combining (4.18) and (4.19), we get (4.17).

**Case II:** $q = 1$. Again using Lemma 2.1-(h13), (h3), (h8), (3.4) with $c(\lambda) > 0$ small enough such that $c(\lambda)\delta < 1$ and (3.31) for $v_\lambda$ with any small $0 < \epsilon < 1$,

$$h(v_k)^{-1}h'(v_k)v_\lambda \leq \alpha(x)h(w_\lambda)^{-1}h'(w_\lambda)v_\lambda \leq h(c(\lambda)\delta)C_2(\lambda)\delta^{1-\epsilon} \leq c(\lambda)C_2(\lambda)h(1)^{-1}\delta^{-1-\epsilon} = C(\lambda, \epsilon)\delta^{-\epsilon} \in L^1(\Omega). \quad (4.20)$$

From this and the hypothesis $(\alpha 1)$, (4.17) follows, thanks to the Lebesgue dominated convergence theorem.

**Case III:** $1 < q < 3$. In view of (3.3) with $c_1(q, \lambda) > 0$ small enough such that $c_1(q, \lambda)\delta^{\frac{2}{1+q}} < 1$, recalling (3.30) for $v_\lambda$ and using Lemma 2.1-(h13), (h3), (h8), it yields that

$$h(v_k)^{-q}h'(v_k)v_\lambda \leq h(w_\lambda)^{-q}h'(w_\lambda)v_\lambda \leq h\left(c_1\delta^{\frac{2}{1+q}}\right)^{-q}v_\lambda \leq (c_1h(1))^{-q}\delta^{-2q}C_2(q, \lambda)\delta^{\frac{2}{1+q}} \leq C(\lambda, \delta)\delta^{\frac{2(1-q)}{1+q}} \in L^1(\Omega) \quad (4.21)$$

since $\frac{2(1-q)}{1+q} > -1$. Again, repeating a similar argument as in (3.38), it follows that

$$h(v_k)^{-q}h'(v_k)v_k \leq h(w_\lambda)^{1-q} \in L^1(\Omega). \quad (4.22)$$

Thus, (4.21) and (4.22), combining with the hypothesis $(\alpha 1)$ and the Lebesgue dominated convergence theorem, yield (4.17).

Therefore, taking into account (4.16) and (4.17), from (4.12), we finally get (4.11). This completes the proof of the proposition.

Now we prove the existence of second solution in the case where $(MP)$ occurs.

Now for $k \in \mathbb{N}$, we define the Moser function $\mathcal{M}_k : \Omega \to \mathbb{R}$ as

$$\mathcal{M}_k(x) = \begin{cases} 
\frac{1}{\sqrt{2\pi}} & \text{if } 0 \leq |x| \leq \frac{1}{k}, \\
\log \left(\frac{1}{|x|}\right) & \text{if } \frac{1}{k} \leq |x| \leq 1, \\
0 & \text{if } |x| \geq 1.
\end{cases}$$
Then, $\mathcal{M}_k \in H_0^1(\Omega)$ and supp $\mathcal{M}_k \subseteq B_1$. Moreover, we define the function

$$\mathcal{M}_k^\ell(x) := \mathcal{M}_k \left( \frac{x - x_0}{\ell} \right).$$

We choose $x_0$ and $\ell$ is such a way so that supp $\mathcal{M}_k^\ell \subseteq \Omega$. Note that $\| \mathcal{M}_k^\ell \| = 1$.

**Lemma 4.2.** Let the conditions in Theorem 1.5 and (MP) hold. Then

(i) $J_\lambda(w_\lambda + t\mathcal{M}_k^\ell) \to -\infty$ as $t \to +\infty$ uniformly for $k$ large.

(ii) $\sup_{t \geq 0} J_\lambda(w_\lambda + t\mathcal{M}_k^\ell) < J_\lambda(w_\lambda) + \pi$ for large $k$.

**Proof.** We prove this lemma for the case $q \neq 1$. When $q = 1$, the proof follows similarly.

(i). By (f2), there exist two positive constants $C_1, C_2$ such that

$$F(x, s) \geq C_1 \exp(h(s)^4) - C_2$$

for all $x \in \Omega, \ s \geq 0$. Using this combining with Lemma 2.1-(h8) for large $t >> 1$ and the Hölder's inequality, we get

$$J_\lambda(w_\lambda + t\mathcal{M}_k^\ell)$$

$$= \frac{1}{2} \int_{\Omega} |\nabla (w_\lambda + t\mathcal{M}_k^\ell)|^2 \, dx - \frac{\lambda}{1 - q} \int_{\Omega} \alpha(x) h(w_\lambda + t\mathcal{M}_k^\ell)^{1-q} \, dx - \lambda \int_{\Omega} F(x, h(w_\lambda + t\mathcal{M}_k^\ell)) \, dx$$

$$\leq \frac{1}{2} ||w_\lambda||^2 + t ||w_\lambda|| \| \mathcal{M}_k^\ell \| + \frac{1}{2} t^2 \| \mathcal{M}_k^\ell \|^2 + C_2 \lambda |\Omega| - C_1 \lambda \int_{\Omega} \exp \left( h(w_\lambda + t\mathcal{M}_k^\ell)^4 \right)$$

$$\leq t^2 + C_3(\lambda) t - C_4(\lambda) \int_{\Omega} \exp \left( \frac{t^2 h(1)^4}{2\pi} \log k \right) \, dx$$

$$= t^2 + C_3(\lambda) t - C_4(\lambda) \, \lambda^{\frac{2}{t^2-2}} \to -\infty \quad \text{uniformly in} \ k \ \text{as} \ t \to +\infty.$$  

(ii). Suppose (ii) does not hold. So, there exists a sub-sequence of $\{\mathcal{M}_k^\ell\}$, still denoted by $\{\mathcal{M}_k^\ell\}$, such that

$$\max_{t \geq 0} J_\lambda(w_\lambda + t\mathcal{M}_k^\ell) \geq J_\lambda(w_\lambda) + \pi. \quad (4.23)$$

Then,

$$J_\lambda(w_\lambda + t\mathcal{M}_k^\ell) = \frac{1}{2} ||w_\lambda||^2 + t \int_{\Omega} \nabla w_\lambda \nabla \mathcal{M}_k^\ell \, dx + \frac{t^2}{2} - \frac{\lambda}{1 - q} \int_{\Omega} \alpha(x) h(w_\lambda + t\mathcal{M}_k^\ell)^{1-q} \, dx$$

$$- \lambda \int_{\Omega} F(x, h(w_\lambda + t\mathcal{M}_k^\ell)) \, dx$$

$$= J_\lambda(w_\lambda) + \frac{\lambda}{1 - q} \int_{\Omega} \alpha(x) h(w_\lambda)^{1-q} \, dx + \lambda \int_{\Omega} F(x, h(w_\lambda)) \, dx$$

$$+ t\lambda \int_{\Omega} \alpha(x) h(w_\lambda)^{-q} h'(w_\lambda) \mathcal{M}_k^\ell \, dx + t\lambda \int_{\Omega} f(x, h(w_\lambda)) h'(w_\lambda) \mathcal{M}_k^\ell \, dx$$

$$+ \frac{t^2}{2} - \frac{\lambda}{1 - q} \int_{\Omega} \alpha(x) h(w_\lambda + t\mathcal{M}_k^\ell)^{1-q} \, dx - \lambda \int_{\Omega} F(x, h(w_\lambda + t\mathcal{M}_k^\ell)) \, dx. \quad (4.24)$$
From (i) and (4.23), it follows that there exists $t_k \in (0, +\infty)$ and $L_0 > 0$ satisfying $t_k \leq L_0$, such that
\[
\max_{t \geq 0} J_\lambda(w_\lambda + tM_k^\ell) = J_\lambda(w_\lambda + t_kM_k^\ell) \geq J_\lambda(w_\lambda) + \pi. \tag{4.25}
\]

Now (4.24) and (4.25) imply that
\[
\frac{\lambda}{1 - q} \int_\Omega \alpha(x)h(w_\lambda)^{1-q} \, dx + \lambda \int_\Omega F(x, h(w_\lambda)) \, dx \\
+ t_k \lambda \int_\Omega \alpha(x)h(w_\lambda)^{-q}h'(w_\lambda)M_k^\ell \, dx + t_k \lambda \int_\Omega f(x, h(w_\lambda))h'(w_\lambda)M_k^\ell \, dx \\
+ \frac{t_k^2}{2} - \frac{\lambda}{1 - q} \int_\Omega \alpha(x)h(w_\lambda + t_kM_k^\ell)^{1-q} \, dx - \lambda \int_\Omega F(x, h(w_\lambda + t_kM_k^\ell)) \, dx \geq \pi.
\]

Using mean value theorem and Lemma 2.1-(h9), we deduce
\[
\frac{\lambda}{1 - q} \left[ \int_\Omega \alpha(x)h(w_\lambda)^{1-q} \, dx \right. \\
- \left. \int_\Omega \alpha(x)h(w_\lambda + t_kM_k^\ell)^{1-q} \, dx \right] + t_k \lambda \int_\Omega \alpha(x)h(w_\lambda)^{-q}h'(w_\lambda)M_k^\ell \, dx \\
= -\lambda t_k \left[ \int_\Omega \alpha(x)h(w_\lambda + t_k\theta_kM_k^\ell)^{-q}h'(w_\lambda + t_k\theta_kM_k^\ell)M_k^\ell \, dx - \int_\Omega \alpha(x)h(w_\lambda)^{-q}h'(w_\lambda)M_k^\ell \, dx \right] \\
= -\lambda t_k^2 \theta_k \int_\Omega \alpha(x)|M_k^\ell|^2 \left[ -qh(w_\lambda + t_k\xi_kM_k^\ell)^{-q-1}h'(w_\lambda + t_k\xi_kM_k^\ell) \\
+ h(w_\lambda + t_k\xi_kM_k^\ell)^{-q}h''(w_\lambda + t_k\xi_kM_k^\ell) \right] \\
+ 2h(w_\lambda + t_k\xi_kM_k^\ell)^{-1-q}(h'(w_\lambda + t_k\xi_kM_k^\ell))^4, \tag{4.26}
\]
where $\theta_k, \xi_k \in (0, 1)$. Now by Lemma 2.1-(h3), (h12),
\[
2h(s)^{-1-q}(h'(s))^4 = 2h(s)^{-1-q}(h(s)h'(s))^2(h'(s))^2 \leq h(s)^{-1-q}h'(s).
\]

Plugging the last relation in (4.26) and using Lemma 2.1-(h13), (h3), (h8) together with the fact that $w_\lambda + t_k\xi_kM_k^\ell \geq w_\lambda > 0$ in $B_{\frac{1}{k}}(x_0)$, we obtain
\[
\frac{\lambda}{1 - q} \left[ \int_\Omega \alpha(x)h(w_\lambda)^{1-q} \, dx \right. \\
- \left. \int_\Omega \alpha(x)h(w_\lambda + t_kM_k^\ell)^{1-q} \, dx \right] + t_k \lambda \int_\Omega \alpha(x)h(w_\lambda)^{-q}h'(w_\lambda)M_k^\ell \, dx \\
\leq \lambda t_k^2 \theta_k(q + 1) \int_\Omega \alpha(x)|M_k^\ell|^2h(w_\lambda + t_k\xi_kM_k^\ell)^{-q-1}h'(w_\lambda + t_k\xi_kM_k^\ell) \, dx \\
\leq \lambda t_k^2 \theta_k(q + 1)||\alpha||_{\infty} \int_\Omega |M_k^\ell|^2h(w_\lambda)^{-q-1}h'(w_\lambda) \, dx \\
\leq \lambda t_k^2 \theta_k(q + 1)||\alpha||_{\infty}(h(1))^{-1-q} \int_\Omega |M_k^\ell|^2\frac{w_\lambda}{h(w_\lambda)}^{-q-1} \, dx = t_k^2O \left( \frac{1}{\log k} \right). \tag{4.27}
\]
On the other hand, again using the mean value theorem and Lemma 2.1-(h₂), (h₁₂), (h₃), we infer
\[
t_k \int_{\Omega} f(x, h(w_\lambda))h'(w_\lambda)M_k^\ell \, dx + \int_{\Omega} F(x, h(w_\lambda)) \, dx - \int_{\Omega} F(x, h(w_\lambda + t_kM_k^\ell)) \, dx
\]
\[
= t_k \int_{\Omega} f(x, h(w_\lambda))h'(w_\lambda)M_k^\ell \, dx - \int_{\Omega} f(x, h(w_\lambda + t_k\tilde{\theta}_kM_k^\ell))h'(w_\lambda + t_k\tilde{\theta}_kM_k^\ell) \, dx
\]
\[
= t_k^2 \int_{\Omega} -\tilde{\theta}_k|M_k^\ell|^2 \left[ f'(x, h(w_\lambda + t_k\tilde{\xi}_kM_k^\ell))(h'(w_\lambda + t_k\tilde{\xi}_kM_k^\ell))^2 \right. \\
\left. + f(x, h(w_\lambda + t_k\tilde{\xi}_kM_k^\ell))h''(w_\lambda + t_k\tilde{\xi}_kM_k^\ell) \right] \, dx
\]
\[
= t_k^2\tilde{\theta}_k \int_{\Omega} |M_k^\ell|^2 \left[ -f'(x, h(w_\lambda + t_k\tilde{\xi}_kM_k^\ell))(h'(w_\lambda + t_k\tilde{\xi}_kM_k^\ell))^2 \right. \\
\left. + 2f(x, h(w_\lambda + 2t_k\tilde{\xi}_kM_k^\ell))h(w_\lambda + t_k\tilde{\xi}_kM_k^\ell)(h'(w_\lambda + t_k\tilde{\xi}_kM_k^\ell))^2 \right] \, dx
\]
\[
\leq t_k^2\tilde{\theta}_k \int_{\Omega} |M_k^\ell|^2 \left[ (\sqrt{2} - M_0)f'(x, h(w_\lambda + t_k\tilde{\xi}_kM_k^\ell)) + L \right] h'(w_\lambda + t_k\tilde{\xi}_kM_k^\ell) \, dx \quad \text{[by (1.5)]}
\]
\[
\leq Lt_k^2\tilde{\theta}_k \int_{\Omega} |M_k^\ell|^2 \, dx = t_k^2O \left( \frac{1}{\log k} \right).
\]
where \(\tilde{\theta}_k, \tilde{\xi}_k \in (0, 1)\). Now gathering (4.23), (4.27) and (4.28), it follows that \(\frac{t_k^2}{2} + O \left( \frac{1}{\log k} \right) t_k^2 \geq \pi\). That is,
\[
t_k^2 \geq 2\pi - O \left( \frac{1}{\log k} \right).
\]
(4.29)

In virtue of the relation \(\frac{d}{dt}J_\lambda(w_\lambda + tM_k^\ell)|_{t=t_k} = 0\), we have
\[
t_k^2 + t_k \int_{\Omega} \nabla w_\lambda \nabla M_k^\ell \, dx - \lambda \int_{\Omega} \alpha(x)h(w_\lambda + t_kM_k^\ell)^{-q}h'(w_\lambda + t_kM_k^\ell)t_kM_k^\ell \, dx
\]
\[
= \lambda \int_{\Omega} f(x, h(w_\lambda + t_kM_k^\ell))h'(w_\lambda + t_kM_k^\ell)t_kM_k^\ell \, dx.
\]
(4.30)

Now we estimate the right hand side in (4.30).
Using the fact that \(w_\lambda \geq C > 0\) on \(B_{\frac{r}{2}}(x_0)\) combining with (f1) and Lemma 2.1-(h₄), we obtain
\[
\int_{\Omega} f(x, h(w_\lambda + t_kM_k^\ell))h'(w_\lambda + t_kM_k^\ell)t_kM_k^\ell \, dx
\]
\[
\geq \int_{B_{\frac{r}{2}}(x_0)} f(x, h(C + t_kM_k^\ell))h'(C + t_kM_k^\ell)t_kM_k^\ell \, dx
\]
\[
\geq \frac{1}{2} \int_{B_{\frac{r}{2}}(x_0)} f(x, h(C + t_kM_k^\ell))h(C + t_kM_k^\ell) \frac{t_kM_k^\ell}{C + t_kM_k^\ell} \, dx.
\]
(4.31)
By \((f1)\), since \(g(x, s)\) is non-decreasing in \(s\), we get
\[
\lim_{s \to +\infty} \frac{sf(x, s)}{\exp(s^4)} = \lim_{s \to +\infty} sg(x, s) = +\infty. \quad(4.32)
\]

Therefore, for any \(b > 0\) there exists some constant \(M_b > 0\) such that
\[
f(x, h(C + t_kM_k^t))h(C + t_kM_k^t) > b \exp \left( h(C + t_kM_k^t)^4 \right) \quad \text{for all } t_k > M_b. \quad(4.33)
\]

On the other hand, by Lemma 2.1-(h_7), for any \(\epsilon > 0\) there exists \(m_\epsilon > 0\) such that
\[
h(C + t_kM_k^t)^4 > (C + t_kM_k^t)^2(2 - \epsilon) \quad \text{for all } t_k \geq m_\epsilon. \quad(4.34)
\]

Plugging (4.33) and (4.34) in (4.31) and using (4.29), for \(t_k > \max\{M_b, m_\epsilon\}\), we get
\[
\int_{B_{\frac{t_k}{C\pi\log k}}(x_0)} f(x, h(w_\lambda + t_kM_k^t))h'(w_\lambda + t_kM_k^t)t_kM_k^t \, dx
\geq \frac{b}{2} \int_{B_{\frac{t_k}{C\pi\log k}}(x_0)} \exp(h(C + t_kM_k^t)^4) \frac{t_kM_k^t}{C + t_kM_k^t} \, dx
\geq \frac{b}{2} \int_{B_{\frac{t_k}{C\pi\log k}}(x_0)} \exp((C + t_kM_k^t)^2(2 - \epsilon)) \frac{t_kM_k^t}{C + t_kM_k^t} \, dx
\geq \frac{b}{2} \frac{t_kM_k^t(x_0)}{C + t_kM_k^t(x_0)} \int_{B_{\frac{t_k}{C\pi\log k}}(x_0)} \exp((C + t_kM_k^t)^2(2 - \epsilon)) \, dx
= \frac{b}{2} \frac{t_kM_k^t(x_0)}{C + t_kM_k^t(x_0)} \exp \left( \left[ C^2 + 2Ct_kM_k^t(x_0) + t_k^2|M_k^t(x_0)|^2 \right](2 - \epsilon) \right) |B_{\frac{t_k}{C\pi\log k}}(x_0)|
= \frac{b}{2} \frac{t_kM_k^t(x_0)}{C + t_kM_k^t(x_0)} C_0 \exp \left( 2Ct_kM_k^t(x_0) + t_k^2|M_k^t(x_0)|^2 \right) |B_{\frac{t_k}{C\pi\log k}}(x_0)| \quad \text{as } \epsilon \to 0^+
= \frac{b}{2} \frac{1}{1 + \frac{C}{C\pi\log k}} \frac{C_1}{C_0} \pi \ell^2 \exp \left( 2Ct_kM_k^t(x_0) + 2 \log k \left( \frac{t_k^2}{2\pi} - 1 \right) \right)
\geq \frac{b}{2} \frac{1}{1 + \frac{C\sqrt{2\pi}}{C_0\pi\log k}} \frac{C_1}{C_0} \pi \ell^2 \exp \left( 2Ct_kM_k^t(x_0) \right). \quad(4.35)
\]

Now incorporating (4.35) and (4.31) in (4.30) and using Hölder’s inequality, we get
\[
+\infty > C_2 := L_0^2 + \|w_\lambda\|L_0 \geq t_k^2 + \|w_\lambda\|t_k \geq \frac{b}{2} \left( \frac{\lambda}{C\sqrt{2\pi}} + \frac{1}{C\pi\log k} \right) \frac{C_1}{C_0} \pi \ell^2 \exp \left( 2Ct_kM_k^t(x_0) \right).
\]

Since \(\{t_k\}\) is bounded away from 0, for some \(C_3 > 0\), we have \(t_kM_k^t(x_0) > C_3(\log k)^{1/2} \to +\infty\) as \(k \to +\infty\). Hence, letting \(k \to +\infty\) in the last relation, it yields that
\[
C_2 \geq \frac{b}{2} \lambda C_1 \pi \ell^2 \exp(2C_3 \sqrt{\log k}) \to +\infty.
\]

This is absurd since \(C_2 < +\infty\). Hence, (ii) follows. This completes the proof of the lemma.

Next, we recall the following result due to P. L. Lions ([24]).
**Theorem 4.3.** Let \( \{u_k : \|u_k\| = 1\} \) be a sequence in \( H_0^1(\Omega) \) converging weakly in \( H_0^1(\Omega) \) to a non zero function \( u \). Then, for every \( 0 < p < (1 - \|u\|)^{-1} \)

\[
\sup_k \int_\Omega \exp(4\pi pu_k^2) \, dx < +\infty.
\]

**Proposition 4.4.** Let the conditions in Theorem 1.5 hold and let \( \lambda \in (0, \Lambda^*) \). Suppose that \( (MP) \) holds. Then (2.7) admits a second solution \( v_\lambda \in H_0^1(\Omega) \cap C_0^+(\Omega) \) satisfying \( v_\lambda \geq w_\lambda \) and \( \|v_\lambda - w_\lambda\| = \sigma_1 \).

**Proof.** First, we define the complete metric space

\( \Gamma := \{\eta \in C([0,1], T) : \eta(0) = w_\lambda, \|\eta(1) - w_\lambda\| > \sigma_1, J_\lambda(\eta(1)) < J_\lambda(w_\lambda)\} \)

with the metric

\[
d(\eta, \tilde{\eta}) = \max_{t \in [0,1]} \{\|\eta(t) - \tilde{\eta}(t)\|\} \quad \text{for all } \eta, \tilde{\eta} \in \Gamma.
\]

Using Lemma 4.2-(i), we have \( \Gamma \neq \emptyset \). Let us set

\[
\gamma_0 = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} J_\lambda(\eta(t)).
\]

Then, in light of Lemma 4.2-(ii) and the condition \( (MP) \), we obtain

\[
J_\lambda(w_\lambda) < \gamma_0 \leq J_\lambda(w_\lambda) + \pi.
\]

Set \( \Phi(\eta) := \max_{t \in [0,1]} J_\lambda(\eta(t)) \) for \( \eta \in \Gamma \). Then employing Ekeland’s variational principle to the functional \( \Phi \) on \( \Gamma \), we get a sequence \( \{\eta_k\} \subset \Gamma \) such that

\[
\Phi(\eta_k) < \gamma_0 + \frac{1}{k} \quad \text{and} \quad \Phi(\eta_k) < \Phi(\eta) + \frac{1}{k} \|\Phi(\eta) - \Phi(\eta_k)\| \quad \text{for all } \eta \in \Gamma.
\]

Again, applying Ekeland’s variational principle to \( J_\lambda \) on \( \Gamma \) and arguing exactly similar to [4, Lemma 3.5], one can show that there exists a sequence \( \{v_k\} \subset \Gamma \) such that

\[
\left\{
\begin{array}{l}
J_\lambda(v_k) \to \gamma_0 \quad \text{as} \quad k \to +\infty \\
\int_\Omega \nabla v_k \nabla (w - v_k) \, dx - \lambda \int_\Omega \alpha(x)h(v_k) - qh'(v_k)(w - v_k) \, dx - \lambda \int_\Omega f(x, h(v_k))h'(v_k)(w - v_k) \, dx \\
\quad \quad \geq - \frac{C}{k}(1 + \|w\|) \quad \text{for all } w \in \Gamma.
\end{array}
\right.
\]

Now choosing \( w = 2v_k \) in (4.36), we get

\[
\int_\Omega |\nabla v_k|^2 \, dx - \lambda \int_\Omega \alpha(x)h(v_k) - qh'(v_k)v_k \, dx - \lambda \int_\Omega f(x, h(v_k))h'(v_k)v_k \, dx \geq - \frac{C}{k}(1 + \|2v_k\|).
\]

In virtue of Lemma 2.1-(h4), from (4.37), we obtain

\[
\int_\Omega |\nabla v_k|^2 \, dx - \frac{\lambda}{2} \int_\Omega \alpha(x)h(v_k)^{1-q} \, dx - \frac{\lambda}{2} \int_\Omega f(x, h(v_k))h(v_k) \, dx \geq - \frac{C}{k}(1 + \|2v_k\|).
\]

We claim that

\[
\limsup_{k \to +\infty} \|v_k\| < +\infty.
\]
For $0 < q < 1$, using (4.36), (4.38), (f3), Lemma 2.1-(h5) and the Sobolev embedding, as $k \to +\infty$, it follows that

$$\gamma_0 + \frac{4C}{\tau} \|v_k\| + o(1) \geq \left(\frac{1}{2} - \frac{2}{\tau}\right) \|v_k\|^2 - \lambda \left(\frac{1}{1-q} - \frac{1}{\tau}\right) \int_{\Omega} \alpha(x) h(v_k)^{1-q} \, dx$$

$$- \frac{\lambda}{\tau} \int_{\Omega} \left[\tau F(x, h(v_k)) - f(x, h(v_k)) h(v_k)\right] \, dx$$

$$\geq \left(\frac{1}{2} - \frac{2}{\tau}\right) \|v_k\|^2 - \lambda C(q, \Omega) \left(\frac{1}{1-q} - \frac{1}{\tau}\right) \|\alpha\|_{\infty} \|v_k\|^{1-q} \, dx.$$

(4.40)

Since $\tau > 4$ and $2 > 1 - q > 0$, the last relation yields that $\{v_k\}$ is bounded in $H_0^1(\Omega)$ and hence, (4.39) holds.

Next, for $q = 1$, using (4.40) and following the arguments in similar way as in (3.56) and (3.57), we obtain (4.39).

Lastly, for the case $1 < q < 3$, again using (4.40) and the exact arguments in (3.59) used for estimating the singular term, as $k \to +\infty$, we get

$$\left(\frac{1}{2} - \frac{2}{\tau}\right) \|v_k\|^2 \leq (1 + C) \|v_k\| + \gamma_0 + o(1),$$

which gives (4.39). Therefore, there exists a $v_\lambda \in H_0^1(\Omega)$ such that $v_k \rightharpoonup v_\lambda$ weakly in $H_0^1(\Omega)$ and pointwise a.e. in $\Omega$ as $k \to +\infty$. Arguing similarly as in the proof of Proposition 4.1, we infer that $v_\lambda$ is a weak solution to (2.7) and $v_\lambda \in H_0^1(\Omega) \cap C_0^+ (\Omega)$.

Claim: $v_\lambda \neq w_\lambda$.

In order to establish this claim, it is sufficient to prove that $v_k \rightharpoonup v_\lambda$ in $H_0^1(\Omega)$ as $k \to +\infty$. For that, we need to establish

$$\int_{\Omega} f(x, h(v_k)) h'(v_k) v_k \, dx \to \int_{\Omega} f(x, h(v_\lambda)) h'(v_\lambda) v_\lambda \, dx \quad \text{as } k \to +\infty. \quad (4.41)$$

Now borrowing the similar arguments as in (4.17), we have

$$\int_{\Omega} \alpha(x) h(v_k)^{1-q} \, dx \to \int_{\Omega} \alpha(x) h(v_\lambda)^{1-q} \, dx \quad \text{as } k \to +\infty. \quad (4.42)$$

Next, we show that

$$\int_{\Omega} F(x, h(v_k)) \, dx \to \int_{\Omega} F(x, h(v_\lambda)) \, dx \quad \text{as } k \to +\infty. \quad (4.43)$$

Since $\{v_k\}$ is bounded in $H_0^1(\Omega)$, from (3.38), it follows that

$$\limsup_{k \to +\infty} \int_{\Omega} f(x, h(v_k)) h'(v_k) v_k \, dx < +\infty. \quad (4.44)$$

Now using (4.44) and Lemma 2.1-(h8), for some large $N >> 1$, we get

$$\int_{\Omega \cap \{x: h(v_k) > N\}} f(x, h(v_k)) \, dx \leq \frac{1}{N} \int_{\Omega \cap \{x: h(v_k) > N\}} f(x, h(v_k)) h(v_k) \, dx$$

$$\leq \frac{2}{N} \int_{\Omega \cap \{x: h(v_k) > N\}} 2f(x, h(v_k)) h'(v_k) v_k$$

$$= O \left(\frac{1}{N}\right).$$
The last relation, combining with the Lebesgue dominated convergence theorem, yields

\[
\int_{\Omega} f(x, h(v_k)) \, dx = \int_{\Omega \cap \{ x : h(v_k)(x) \leq N \}} f(x, h(v_k)) \, dx + \int_{\Omega \cap \{ x : h(v_k)(x) > N \}} f(x, h(v_k)) \, dx
\]

\[
= \int_{\Omega \cap \{ x : h(v_k)(x) \leq N \}} f(x, h(v_k)) \, dx + O\left( \frac{1}{N} \right)
\]

\[
\rightarrow \int_{\Omega} f(x, h(v_k)) \, dx,
\]

(4.45)
as \( k \rightarrow +\infty \) and \( N \rightarrow +\infty \). Since by (f4), \( F(x, h(v_k)) \leq M_1(1 + f(x, h(v_k))) \) for all \( k \in \mathbb{N} \), using (4.45) and the Lebesgue dominated convergence theorem, (4.43) follows. Therefore, using (4.42) and (4.43) with the weak lower semicontinuity property of the norm, we derive

\[
J_\lambda(v_k) \leq \liminf_{k \rightarrow +\infty} J_\lambda(v_k).
\]

(4.46)

Supposing the contrary, let us assume that \( \{v_k\} \) does not converge in \( H_0^1(\Omega) \) strongly. Then (4.42), (4.43) and (4.46) imply that \( J_\lambda(v_k) < \gamma_0 \). By the hypothesis, \( J_\lambda(w_\lambda) \leq J_\lambda(v_k) \). So, for sufficiently small \( \epsilon > 0 \), by Lemma 4.2, we have

\[
(\gamma_0 - J_\lambda(v_k))(1 + \epsilon) \leq \left( \max_{t \in [0,1]} J(w_\lambda + tM_k) - J_\lambda(w_\lambda) \right)(1 + \epsilon) < \pi.
\]

(4.47)

Set

\[
\zeta_0 := \lim_{k \rightarrow \infty} \left( \int_{\Omega} \alpha(x)h(v_k)^{1-q} \, dx + \lambda \int_{\Omega} F(x, h(v_k)) \, dx \right).
\]

Then,

\[
\lim_{k \rightarrow +\infty} \|v_k\|^2 = \lim_{k \rightarrow +\infty} \left[ J_\lambda(v_k) + \lambda \int_{\Omega} \alpha(x)h(v_k)^{1-q} \, dx + \lambda \int_{\Omega} F(x, h(v_k)) \, dx \right]
\]

\[
= 2(\gamma_0 + \zeta_0).
\]

(4.48)

Taking into account (4.47) and (4.48), we deduce

\[
(1 + \epsilon)\|v_k\|^2 < \frac{2\pi(\gamma_0 + \zeta_0)}{\gamma_0 - J_\lambda(v_k)} = \frac{2\pi(\gamma_0 + \zeta_0)}{\gamma_0 + \zeta_0 - \frac{\pi}{2} \|v_\lambda\|^2} = 2\pi \left( 1 - \frac{1}{2} \left( \frac{\|v_\lambda\|^2}{\gamma_0 + \zeta_0} \right) \right)^{-1}.
\]

Hence, we choose \( r > 0 \) such that \( \frac{(1+\epsilon)}{2\pi} \|v_k\|^2 = \frac{r}{2\pi} < (1 - \|u_\lambda\|^2)^{-1} \), where \( u_\lambda := \frac{v_k}{\|v_k\|} \). Also, note that \( u_k := \frac{v_k}{\|v_k\|} \rightarrow \frac{v_\lambda}{(2(\gamma_0 + \zeta_0))^{1/2}} = u_\lambda \) weakly in \( H_0^1(\Omega) \) as \( k \rightarrow +\infty \). Therefore, using Theorem 4.3, for any \( \epsilon > 0 \), we obtain

\[
\sup_k \int_{\Omega} \exp \left( 4\pi pu_k^2 \right) \, dx < +\infty,
\]

which together with the fact that \( p > \frac{(1+\epsilon)}{2\pi} \|v_k\|^2 \) yields

\[
\sup_k \int_{\Omega} \exp \left( 2(1 + \epsilon)u_k^2 \right) \, dx < +\infty.
\]

(4.49)
Now using Lemma 2.1-(h₆), (f2) and (4.49), we get
\[
\begin{align*}
\int_{\Omega} f(x, h(v_k)) h'(v_k)v_k \, dx \\
= \int_{\Omega \cap \{x : v_k(x) > N > 1\}} f(x, h(v_k)) h'(v_k)v_k \, dx + \int_{\Omega \cap \{x : v_k(x) \leq N\}} f(x, h(v_k)) h'(v_k)v_k \, dx \\
= O \left( \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left( \left( 1 + \frac{\epsilon}{2} \right) h(v_k)^4 \right) \, dx \right) + \int_{\Omega \cap \{x : v_k(x) \leq N\}} f(x, h(v_k)) h'(v_k)v_k \, dx \\
= O \left( \exp \left( -\epsilon N^2 \right) \int_{\Omega \cap \{x : v_k(x) > N\}} \exp \left( 2(1 + \epsilon)v_k^2 \right) \, dx \right) + \int_{\Omega \cap \{x : v_k(x) \leq N\}} f(x, h(v_k)) h'(v_k)v_k \, dx \\
= O \left( \exp \left( -\epsilon N^2 \right) \right) + \int_{\Omega \cap \{x : v_k(x) \leq N\}} f(x, h(v_k)) h'(v_k)v_k \, dx \\
\rightarrow \int_{\Omega} f(x, h(v_\lambda)) h'(v_\lambda)v_\lambda \, dx, \quad \text{as } k \rightarrow +\infty \text{ and } N \rightarrow +\infty,
\end{align*}
\]

thanks to the Lebesgue dominated convergence theorem. Thus, we obtain (4.41). Hence, taking into account (4.42) and (4.41) and arguing similarly as in the proof of Proposition 4.1, we can infer that \(v_k \rightarrow v_\lambda\) strongly in \(H_0^1(\Omega)\) as \(k \rightarrow +\infty\) and \(v_\lambda \neq w_\lambda\). Hence, the claim is verified. Thus, the proof of the proposition follows.

**Proof of Theorem 1.5:** From Proposition 4.4, it follows that \(v_\lambda \in H_0^1(\Omega) \cap C_{\varphi_q}^+(\Omega)\) is a weak solution to (2.7). Moreover, \(v_\lambda \neq w_\lambda\), where \(w_\lambda\) is another weak solution to (2.7). Now by Lemma 2.1-(h₁), we have \(h\) is a \(C^\infty\) function and Lemma 2.1-(h₆), (h₁₁) ensure that \(h(s)\) behaves like \(s\) when \(s\) is close to 0. Therefore, \(h(v_\lambda) \in H^1_0(\Omega) \cap C_{\varphi_q}^+(\Omega)\) forms a weak solution to the problem \((P_\lambda)\) and \(h(v_\lambda) \neq h(w_\lambda)\), where \(h(w_\lambda)\) is another solution to the problem \((P_\lambda)\) obtained in Theorem 1.4. This concludes the proof of Theorem 1.5.

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A global multiplicity result for singular and critical elliptic equation

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