AVERAGES OF LONG DIRICHLET POLYNOMIALS

BRIAN CONREY AND JON KEATING

1. Introduction and statement of results

It has been conjectured by Keating and Snaith that

\[ \frac{1}{T} \int_{0}^{T} |\zeta(1/2 + it)|^{2k} \, dt \sim \frac{g_{k}a_{k}(\log T)^{k}}{k!} \]

where for positive integer \( k \),

\[ a_{k} = \prod_{p} \left( 1 - \frac{1}{p} \right)^{(k-1)^2 \sum_{j=0}^{k-1} \left( \frac{k-1}{j} \right)^2 \frac{1}{p^{j}}} \]

and

\[ g_{k} = \frac{k!}{1^{1} \cdot 2^{2} \cdots k^{k} \cdot (k+1)^{k-1} \cdots (2k-1)^{1}}. \]

This has been proven for \( k = 1 \) and \( k = 2 \). The method of proof involves approximating \( \zeta(s) \) or \( \zeta(s)^2 \) by appropriate Dirichlet polynomials and analyzing the mean-square of such. In the pursuit of proving the above conjecture for values of \( k \) larger than 2, it may be of some interest to consider in general the mean square of Dirichlet polynomials with coefficients \( d_{k}(n) \) where

\[ \zeta(s)^{k} = \sum_{n=1}^{\infty} \frac{d_{k}(n)}{n^{s}}. \]

Thus, we consider

\[ I_{k}(T, N) = \int_{0}^{T} \sum_{n=1}^{N} \left| \frac{d_{k}(n)}{n^{1/2+it}} \right|^{2} \, dt \]

for various values of \( k \) and \( N \).

Here we present a method which will lead to conjectural values for

\[ M_{k}(\alpha) = \lim_{T \to \infty} \frac{(k!)^2}{\alpha_{k}T^{(\log T)k^{2}}} I_{k}(T, N) \]

for integer values of \( k \) and \( N = T^{\alpha} \) with \( \alpha > 0 \). In particular we are interested in unit intervals of \( \alpha \) between 0 and \( k \). For example it can be shown that

\[ \frac{1}{T} \int_{0}^{T} \left| \sum_{n \leq N} \frac{1}{n^{1/2+it}} \right|^{2} \, dt \sim \begin{cases} \log N & \text{if } N \leq T \\ \log T & \text{if } N > T \end{cases} \]
This translates to

\[ M_1(\alpha) = \begin{cases} 
\alpha & \text{if } 0 \leq \alpha \leq 1 \\
1 & \text{if } 1 < \alpha 
\end{cases} \]

Also, it can likely be proven that

\[ M_2(\alpha) = \begin{cases} 
\alpha^4 & \text{if } 0 \leq \alpha \leq 1 \\
-\alpha^4 + 8\alpha^3 - 24\alpha^2 + 32\alpha - 14 & \text{if } 1 < \alpha \leq 2 \\
2 & \text{if } 2 < \alpha 
\end{cases} \]

Next, we conjecture that

\[ M_3(\alpha) = \begin{cases} 
\alpha^9 & \text{if } 0 \leq \alpha \leq 1 \\
-2\alpha^9 + 27\alpha^8 - 324\alpha^7 + 2268\alpha^6 - 8694\alpha^5+19278\alpha^4 - 25452\alpha^3 + 19764\alpha^2 - 8343\alpha + 1479 & \text{if } 1 < \alpha \leq 2 \\
\alpha^9 - 27\alpha^8 + 324\alpha^7 - 2268\alpha^6 + 10206\alpha^5 -30618\alpha^4 + 61236\alpha^3 - 78732\alpha^2 + 59049\alpha - 19641 & \text{if } 2 \leq \alpha \leq 3 \\
42 & \text{if } 3 < \alpha 
\end{cases} \]

This is a consequence of the conjecture of [CFKRS] known as “the recipe.” We will sketch its derivation later.

The polynomials here are interesting because of their smoothness properties. The graphs of \( M_2(\alpha) \) and \( M_3(\alpha) \) are included. Notice that they are very smooth, monotonic, and are symmetric.
In fact, $M_3(\alpha)$ is 9-times continuously differentiable at $\alpha = 0$ and $\alpha = 3$ and is 5-times differentiable at $\alpha = 1$ and $\alpha = 2$. It can be proven that the only piecewise polynomial $f(\alpha)$ (with pieces of degree at most 9) which is 0 for $\alpha < 0$ is 42 for $\alpha \geq 3$, is monotonic, and satisfies $f(3 - \alpha) = 42 - f(\alpha)$ and has the same smoothness properties as $M_3(\alpha)$ is $f(\alpha) = M_3(\alpha)$. Note that the symmetry together with $M_3(\alpha) = \alpha^9$ for $0 < \alpha < 1$ implies that for $2 < \alpha < 3$ we have

$$M_3(\alpha) = (\alpha - 3)^9 + 42,$$

which only leaves the range $1 < \alpha < 2$ in question. Let $P(\alpha)$ be the polynomial that agrees with $M_3(\alpha)$ in the range $1 < \alpha < 2$. Then it satisfies $P(\alpha) + P(3 - \alpha) = 42$; this determines half of its 10 coefficients. Then the 5 times smoothness at $\alpha = 1$ determine the other 5.

2. A PROOF OF THE $k = 2$ CASE

We sketch a possible proof of the $k = 2$ case. First of all, with $s = 1/2 + it$ and $\alpha, \beta, \gamma, \delta \ll (\log T)^{-1}$ it is a theorem (but whose proof is not written down in full details anywhere) that

$$(1) \int_0^T \zeta(s + \alpha)\zeta(s + \beta)\zeta(1 - s + \gamma)\zeta(1 - s + \delta) \, dt = \int_0^T Z_t(\alpha, \beta, \gamma, \delta) \, dt + O(T^{2/3+\epsilon}),$$
where

\[ Z_t(\alpha, \beta, \gamma, \delta) = Z(\alpha, \beta, \gamma, \delta) + \left( \frac{t}{2\pi} \right)^{-\alpha-\gamma} Z(-\gamma, \beta, -\alpha, \delta) + \left( \frac{t}{2\pi} \right)^{-\alpha-\delta} Z(-\delta, \beta, \gamma, -\alpha) \]

\[ + \left( \frac{t}{2\pi} \right)^{-\beta-\gamma} Z(\alpha, -\gamma, -\beta, \delta) + \left( \frac{t}{2\pi} \right)^{-\beta-\delta} Z(\alpha, -\delta, \gamma, -\beta) \]

\[ + \left( \frac{t}{2\pi} \right)^{-\alpha-\beta-\gamma-\delta} Z(-\gamma, -\delta, -\alpha, -\beta) \]

where

\[ Z(\alpha, \beta, \gamma, \delta) = \frac{\zeta(1 + \alpha + \gamma)\zeta(1 + \alpha + \delta)\zeta(1 + \beta + \gamma)\zeta(1 + \delta)}{\zeta(2 + \alpha + \beta + \gamma + \delta)} \]

This theorem, possibly with a weaker error term, could be proven in the case that real parts of the \( \alpha, \beta, \gamma, \delta \) are small but that the imaginary parts can be as large as \( T \); (Sandro Bettin did this for the mean square case, see [B]). We will assume this uniform version of the fourth moment. By Perron’s formula we have

\[ I_2(T, N) = \int_0^T \frac{1}{(2\pi i)^2} \int_{z,w} \zeta(s + w)^2 \zeta(1 - s + z)^2 \frac{N^w N^z}{w z} \, dw \, dz \, dt \]

\[ = \frac{1}{(2\pi i)^2} \int_{z,w} \frac{N^{w+z}}{w z} \int_0^T \zeta(s + w)^2 \zeta(1 - s + z)^2 \, dt \, dw \, dz \]

We evaluate the inner integral over \( t \) using a limiting case of (1) with \( \alpha = \beta = w \) and \( \gamma = \delta = z \). Also, we are only interested in the leading order term, so, for example, the denominator in the recipe formula above just becomes \( \zeta(2) = \pi^2/6 \) and we replace \( \zeta(1 + x) \) by \( 1/x \), \( (t/2\pi)^{-\alpha} \) by \( T^{-\alpha} \), etc. In this context then, we have

\[ \frac{\zeta(2)^{-1}}{T} \int_0^T \zeta(s + w)^2 \zeta(1 - s + z)^2 \, dt \sim \frac{1 - (2 + (w + z)^2)T^{-w-z} \log^2 T - 2T^{-2w-2z}}{(w + z)^4} \]

Inserting this above we find that

\[ I_2(T, N) \sim \frac{\zeta(2)^{-1}T}{(2\pi i)^2} \int_{z,w} \frac{N^{w+z}}{w z} \frac{(1 - (2 + (w + z)^2)T^{-w-z} \log^2 T - 2T^{-2w-2z})}{(w + z)^4} \, dw \, dz. \]

The integrals over \( z \) and \( w \) are for the real parts of \( z \) and \( w \) being small but positive. We can see from this formula that we will get different answers when \( N < T, T < N < T^2, \) and \( T^2 < N \). For example, if \( T < N < T^2 \) we will move the paths of integration to the right (and so get 0) for the terms which involve \( T^{-2w-2z} \). If \( N < T \) then we do likewise for the terms which involve \( T^{-z-w} \) or \( T^{-2w-2z} \). For the rest of the terms we move the paths to the
left and collect the residues at \( w = 0 \) and \( z = 0 \). In this way we find that \( I_2(T, N) \sim \frac{T}{4\zeta(2)} \times \)

\[
\times \left\{ \begin{array}{ll}
\log^4 N & \text{if } N < T \\
8\log^3 N \log T + 32\log N \log^3 T - 24\log^2 N \log^2 T - \log^4 N - 14\log^4 T & \text{if } T < N < T^2 \\
2\log^4 T & \text{if } T^2 < N
\end{array} \right.
\]

The result about \( M_2(\alpha) \) follows.

3. Derivation of the case \( k = 3 \)

We use the conjecture of [CFKRS]. Let

\[
Z_\zeta(A; B) = \prod_{\alpha \in A, \beta \in B} \zeta(1 + \alpha + \beta)
\]

and

\[
A(A; B) = \prod_p \prod_{\alpha \in A, \beta \in B} \left( 1 - \frac{1}{p^{1+\alpha+\beta}} \right)
\times \int_0^1 \prod_{\alpha \in A} z_{p,\theta}(1/2 + \alpha) \prod_{\beta \in B} z_{p,-\theta}(1/2 + \beta) \, d\theta
\]

where \( z_{p,\theta}(x) = 1/(1 - e(\theta)/p^x) \). Then

\[
\int_0^T \prod_{\alpha \in A} \zeta(1/2 + i\tau + \alpha) \prod_{\beta \in B} \zeta(1/2 - i\tau + \beta) \, d\tau
= \int_0^T \sum_{\substack{S \subseteq A \cup B \mid |S| = |T|}} e^{-\ell(S+S+\ell)} AZ_\zeta(S \cup (-T); T \cup (-S)) \, d\tau
+ O(T^{1/2+\epsilon}).
\]

where \( \ell = \log \frac{T}{2\pi} \).

We use the above with \( A \) and \( B \) being sets of cardinality 3. A limiting argument that allows for \( A \) and \( B \) to be multisets \( A = \{ w, w, w \} \) and \( B = \{ z, z, z \} \) implies that

\[
\frac{1}{T} \int_0^T \zeta(s+w)^3\zeta(1-s+z)^3 \, dt \sim F_3(w, z)
\]
where
\[
F_3(w, z) = \frac{1}{4}(w + z)^{-9}(4 + T^{-w-z}(-9w^4 \log^4(T) - 4w^3z \log^4(T) + 4w^3 \log^3(T)) - 6w^2z^2 \log^4(T) + 12w^2z \log^3(T) - 12w^2 \log^2(T) - 4w^2 \log^3(T) + 12wz^2 \log^3(T) - 24wz \log^2(T) - z^4 \log^4(T) + 4z^3 \log^3(T) - 12z^2 \log^2(T) - 12 + T^{-2w-2z}(w^4 \log^4(T) + 4w^3z \log^4(T) + 4w^3 \log^3(T) + 6w^2z^2 \log^4(T) + 12w^2z \log^3(T) + 12w^2 \log^2(T) + 4wz^2 \log^4(T) + 12wz^2 \log^3(T) + 24wz \log^2(T) + z^4 \log^4(T) + 4z^3 \log^3(T) + 12z^2 \log^2(T) + 12) - 4T^{-3w-3z})
\]

We compute
\[
I_3(T) = \frac{1}{(2\pi i)^2} \int_{w+z} \frac{N_{w+z}}{wz} F_3(w, z) \, dw \, dz
\]
for various ranges of \(N\). If \(N < T\) only the first term matters; if \(T < N < T^2\) then the terms with \(T^{-w-z}\) also contribute; if \(T^2 < N < T^3\) then we must also include the terms with \(T^{-2w-2z}\); if \(N > T^3\) then we include all of the terms. Computing residues at \(w = 0\) and \(z = 0\) leads to the above result for \(M_3\).

4. \(k = 4\)

We know that \(M_4(\alpha) = \alpha^{16}\) for \(0 < \alpha < 1\). We know also that \(M_4(\alpha) = 24024\) for \(\alpha \geq 4\) and that \(M_4(\alpha) = 24024 - M_4(4 - \alpha)\) for all \(\alpha\), so that determines \(M_4(\alpha)\) for \(3 < \alpha < 4\). One might guess that it will be 9 times differentiable at \(\alpha = 1\) and \(\alpha = 3\). And 7 times differentiable at \(\alpha = 2\).

We can use a result in [CG] to conjecturally determine \(M_4(\alpha)\) (and indeed any \(M_k(\alpha)\)) for \(1 < \alpha < 2\). From that paper, which is based on the predicted behavior of divisor correlations

\[
\sum_{n \leq x} d_k(n) d_k(n + h)
\]

we have

**Conjecture 1.** For any positive integer \(k\), we conjecture that \(M_k(\alpha)\) exists and that

\[
M_k(\alpha) = \alpha^{k^2} \left( 1 - \sum_{n=0}^{k^2-1} (-1)^n (1 - \alpha^{-n-1}) \binom{k^2}{n+1} \gamma_k(n) \right)
\]

for \(1 < \alpha < 2\) where

\[
\gamma_k(n) = \sum_{1 \leq i, j \leq k} \binom{k}{i} \binom{k}{j} \binom{n-1}{i-1, j-1, n-i-j+1};
\]

also

\[
\gamma_k(0) = k.
\]
In [CK1-3] we confirm that the correlation method of [CG] agrees with the prediction of the recipe in [CFKRS]. These independent lines of reasoning thus both give

\[
M_4(\alpha) = -3\alpha^{16} + 64\alpha^{15} - 1920\alpha^{14} + 35840\alpha^{13} - 393120\alpha^{12} + 2725632\alpha^{11} - 12684672\alpha^{10} \\
+ 41367040\alpha^9 - 97348680\alpha^8 + 168351040\alpha^7 - 215767552\alpha^6 + 204701952\alpha^5 \\
- 141989120\alpha^4 + 70035840\alpha^3 - 23281920\alpha^2 + 4679424\alpha - 429844
\]

for \(1 < \alpha < 2\), which does satisfy the aforementioned smoothness conditions. With this information we can construct all of \(M_4(\alpha)\).

5. Remarks

With a lot more work we could find an explicit formula for \(M_k(\alpha)\) for \(2 < \alpha < 3\), or indeed for any initial interval, using the recipe method. Also, we have a preliminary version of a new method - the convolution coefficient correlation method - which would give an independent avenue into determining the \(M_k(\alpha)\). However, we suspect that there are simple smoothness conditions which would completely characterize \(M_k(\alpha)\). We are not sure exactly what these are. However, the following may be a start.

**Conjecture 2.** For any positive integer \(k\) the function \(M_k(\alpha)\) is \((k - 1)^2\) times continuously differentiable at \(\alpha = 1\).

We have checked this conjecture for \(k \leq 7\) using the proposed formulas above for \(M_k(\alpha)\) for \(0 < \alpha < 2\).
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