The quasiparticle Landau Fermi-liquid and periodic orbit theories are presented for the semiclassical description of collective excitations in nuclei, which are close to one of the main topics of the fruitful activity of S. T. Belyaev. Density-density response functions are studied at low temperatures within the temperature-dependent collisional Fermi-liquid theory in the relaxation time approximation. The isothermal, isolated (static) and adiabatic susceptibilities for nuclear matter show the ergodicity property. Temperature corrections to the response function, viscosity and thermal conductivity coefficients have been derived, also in the long wave-length (hydrodynamic) limit. The relaxation and correlation functions are obtained through the fluctuation-dissipation theorem and their properties are discussed in connection to the static susceptibilities and ergodicity of the Fermi systems. Transport coefficients, such as nuclear friction and inertia as functions of the temperature for the hydrodynamic (heat-pole and first sound) and Fermi-surface-distortion zero sound modes are derived within the Fermi-liquid droplet model. They are shown to be in agreement with the semi-microscopical calculations based on the nuclear shell model (SM) for large temperatures. This kinetic approach is extended to the study of the neutron-proton correlations in asymmetric neutron-rich nuclei. The surface symmetry binding-energy constants are presented as functions of the Skyrme force parameters in the approximation of a sharp edged proton-neutron asymmetric nucleus and applied to calculations of the isovector giant dipole resonance. The energies, sum rules and transition densities of these resonances obtained by using analytical expression for these surface constants in terms of the Skyrme force parameters are in fairly good agreement with the experimental data. An analysis of the experimental data, in particular the specific structure of these resonances in terms of a main, and some satellite peaks, in comparison with our analytical approach and other theoretical semi-microscopical models, might turn out to be of capital importance for a better understanding of the values of the fundamental surface symmetry-energy constant. The semiclassical collective moment of inertia is derived analytically beyond the quantum perturbation approximation of the cranking model for any potential well as a mean field. It is shown that this moment of inertia can be approximated by its rigid-body value for the rotation with a given frequency within the ETF and more general periodic orbit theories in the nearly local long-length approximation. Its semiclassical shell-structure components are derived in terms of the periodic-orbit free-energy shell corrections. An enhancement of the moment of inertia near the symmetry-breaking bifurcation deformations was found. We obtained good agreement between the semiclassical and quantum shell-structure components of the moment of inertia for several critical bifurcation deformations for the completely analytically solved example of the harmonic oscillator mean field.
pole deformations. We can then study the response to an external field of the dynamical quantities describing the nuclear collective motion in terms of these variables. Thus, we get important information on the transport properties of nuclei. For such a theoretical description of the collective motion it is very important to take into account the temperature dependence of the dissipative nuclear characteristics as the friction coefficient, as shown in [24, 28, 31]. The friction depends strongly on the temperature and its temperature dependence can therefore not be ignored in the description of the collective excitations in nuclei. Concerning the temperature dependence of the nuclear friction, one of the most important problems is related to the properties of the static susceptibilities and ergodicity of the Fermi systems like nuclei.

However, the quantum description of dissipative phenomena in nuclei is rather complicated because we have to take into account the residual interactions beyond the mean-field approximation. Therefore, more simple models [26, 31, 52] accounting for some macroscopic properties of the many-body Fermi-system are helpful to understand the global average properties of the collective motion. Such a model is based on the Landau Fermi-liquid theory [34, 35], applied for the nuclear interior and simple macroscopic boundary conditions on the nuclear surface [26, 27, 30, 37] (see also macroscopic approaches with different boundary conditions [41–45]). In [32], the response-function theory can be applied to describe collective nuclear excitations as the isoscalar quadrupole mode. The transport coefficients, such as friction and inertia, are simply calculated within the macroscopic Fermi-liquid droplet model (FLDM) [31, 33] and their temperature dependence can be clearly discussed (see also earlier works [27, 28, 40, 41]). The asymmetry of heavy nuclei near their stability line and the structure of the isovector dipole resonances are studied in [32, 57, 58] (see also [53, 54]). In this way, the giant multipole resonances were described, and, with increasing temperature [31, 32], a transition from zero sound modes to the hydrodynamic first sound. The friction in [31, 32] is due to the collisions of particles, which were taken into account in the relaxation-time approximation [35, 36, 55, 58] with a temperature and frequency dependence (retardation effects) [31, 34].

The most important results obtained in [32, 53] are related to the overdamped surface excitation mode for the low energy region and its dissipative characteristics as friction. For the low excitation energy region these investigations can be completed by the additional sources of the friction related to a more precise description of the heated Fermi liquids presented in [57, 58] for the infinite matter. Following [57], we should take into account the thermodynamic relations along with the dynamical Landau–Vlasov equation and introduce the local equilibrium distribution instead of the one of global statics, used earlier in [32, 53] for the linearization procedure of this equation. These new developments of the Landau theory are especially important for the further investigations of the temperature dependence of the friction. For the first step we have to work out in more details the theory of the heated Fermi liquids for nuclear matter to apply then it for the dynamical description of the collective motion in the interior of nuclei in the macroscopic FLDM [31, 32]. Our purpose is also to find the relations to some general points of the response function theory and clarify them taking the example of the analytically solved model based on the non-trivial temperature-dependent Fermi-liquid theory. One of the most important questions which would be better to clarify is the above mentioned ergodicity property, temperature dependence of the friction and coupling constant.

Another important extension of this macroscopic theory is to study the structure of the isovector giant dipole resonance (IVGDR) as a splitting phenomenon due to the nuclear symmetry interaction between neutrons and protons near the stability line [33, 40, 53, 54]. The neutron skin of exotic nuclei with a large excess of neutrons is also still one of the exciting subjects of nuclear physics and nuclear astrophysics [2, 60, 61]. Simple and accurate solutions for the isovector particle density distributions were obtained within the nuclear effective surface (ES) approximation [25, 26, 39, 41]. It exploits the saturation of nuclear matter and a narrow diffuse-edge region in finite heavy nuclei. The ES is defined as the location of points of the maximum density gradient. The coordinate system, connected locally with the ES, is specified by the distance from the given point to the surface and by tangent coordinates at the ES. The variational condition for the nuclear energy with some additional fixed integrals of motion in the local energy-density theory [70, 71] is significantly simplified in these coordinates. In particular, in the extended Thomas–Fermi (ETF) approach [72, 73] (with Skyrme forces [74–79]) this can be done for any deformations by using an expansion in a small lepton-demic parameter. The latter is of the order of the diffuse edge thickness of heavy enough nucleus over its mean curvature radius, or the number of nucleons in power one third under the distortion constraint in the case of deformed nuclei. The accuracy of the ES approximation in the ETF approach without spin-orbit (SO) and asymmetry terms was checked [25, 27] by comparing results of Hartree–Fock (HF) [53, 54] and ETF calculations [72, 73] for some Skyrme forces. The ES approach (ESA) [25, 27] was then extended by taking SO and asymmetry effects into account [59, 60]. Solutions for the isoscalar and isovector particle densities and energies at the quasi-equilibrium in the ESA of the ETF approach were applied to analytical calculations of the neutron skin and isovector stiffness coefficients in the leading order of the lepton-demic parameter and to the derivations of the macroscopic boundary conditions [40]. Our results are compared with the fundamental researches [2, 60, 62] in the liquid droplet model (LDM). These analytical expressions for the energy surface constants can be used for IVGDR calculations within the FLDM [33, 49, 52].
A further interesting application of the semiclassical response theory would consist in the study of the properties of collective rotation bands in heavy deformed nuclei. One may consider nuclear collective rotations within the cranking model as a response to the Coriolis external-field perturbation. The moment of inertia (MI) can be calculated as a susceptibility with respect to this external field. The rotation frequency of the rotating Fermi system in the cranking model is determined for a given nuclear angular momentum through a constraint, as for any other integral of motion, as in particular the particle number conservation [51]. In order to simplify such a rather complicated problem, the Strutinsky shell correction method (SCM) [8] was adjusted to the collective nuclear rotations in [8]. The collective MI is expressed as function of the particle number and temperature in terms of a smooth part and an oscillating shell correction. The smooth component can be described by a suitable macroscopic model, like the dynamical ETF approach [72, 73, 83–88] similar to the FLDM, which has proven to be both simple and precise. For the definition of the MI shell correction, one can apply the Strutinsky averaging procedure to the single-particle (s.p.) MI, in the same way as for the well-known free-energy shell correction.

For a deeper understanding of the quantum results and the correspondence between classical and quantum physics of the MI shell components, it is worth to analyze these shell components in terms of periodic orbits (POs), what is now well established as the semiclassical periodic-orbit theory (POT) [73, 89–94] (see also its extension to a given angular momentum projection along with the energy of the particle [95] and to the particle densities [96, 97] and pairing correlations [97]). Gutzwiller was the first who developed the POT for completely chaotic Hamiltonians with only one integral of motion (the particle energy) [89]. The Gutzwiller approach of the POT extended to potentials with continuous symmetries for the description of the nuclear shell structure can be found in [73, 91, 93, 98]. The semiclassical shell-structure corrections to the level density and energy have been tested for a large number of s.p. Hamiltonians in two and three dimensions (see, for instance, [72, 99]). For the Fermi gas the entropy shell corrections of the POT are a sum of periodic orbits were derived in [91], and with its help, simple analytical expressions for the shell-structure energies in cold nuclei were obtained there following a general semiclassical theory [73]. These energy shell corrections are in good agreement with the quantum SCM results, for instance for elliptic and spheroidal cavities, including the superdeformed bifurcation region [100, 102]. In particular in three dimensions, the superdeformed bifurcation nanostructure leads as function of deformation to the double-humped shell-structure energy with the first and second potential wells in heavy enough nuclei [73, 94, 98, 102, 104], which is well known as the double-humped fission barriers in the region of actinide nuclei. At large deformations the second well can be understood semiclassically, for spheroidal type shapes, through the bifurcation of equatorial orbits into equatorial and the shortest 3-dimensional periodic orbits, because of the enhancement of the POT amplitudes of the shell correction to the level density near the Fermi surface at these bifurcation deformations.

For finite heated fermionic systems, it was also shown within the POT that the shell-structure of the entropy, the thermodynamical (grand-canonical) potential and the free-energy shell corrections can be obtained by multiplying the terms of the POT expansion by a temperature-dependent factor, which is exponentially decreasing with temperature. For the case of the so called “classical rotations” around the symmetry z axis of the nucleus, the MI shell correction is obtained, for any rotational frequency and at finite temperature, within the extended Gutzwiller POT through the averaging of the individual angular momenta aligned along this symmetry axis [95, 100, 107]. A similar POT problem, dealing with the magnetic susceptibility of fermionic systems like metallic clusters and quantum dots, was worked out in [109, 107].

It was suggested in [110] to use the spheroidal cavity and the classical perturbation approach to the POT by Creagh [73, 111] to describe the collective rotation of deformed nuclei around an axis (x axis) perpendicular to the symmetry z axis. The small parameter of the POT perturbation approximation turns out to be proportional to the rotational frequency, but also to the classical action (in units of \( \hbar \)), which causes an additional restriction to Fermi systems (or particle numbers) of small enough size, in contrast to the usual semiclassical POT approach.

In [112, 113], the nonperturbative extended Gutzwiller POT was used for the calculation of the MI shell corrections within the mean-field cranking model for both the collective and the alignment rotations. In these works, for the statistical equilibrium nuclear rotations, the semiclassical MI shell corrections were obtained in good agreement with the quantum results in the case of the harmonic-oscillator potential. We extend this approach for collective rotations perpendicular to symmetry axis to the analytical calculations of the MI shell corrections for the case of different mean fields, in particular with spheroidal shapes and sharp edges. The main purpose is to study semiclassically the enhancement effects in the MI within the improved stationary phase method (improved SPM or shortly ISP M) [94, 103, 102, 103, 103], due to the bifurcations of the periodic orbits in the superdeformed region.

In the present review in Section II A we present some basic formulas of the temperature-dependent Fermi-liquid theory [57]. We consider in Sec. II B the particle number and momentum conservation equations and derive from them the energy conservation and general transport equations, in particular, the expressions for the viscosity, shear modulus and thermal conductivity coefficients. In Sec. II C we determine the density-density and density-temperature response functions with the low
temperature corrections. Section II.D shows the long wave-length (LWL, or hydrodynamic) limit for the response functions, and the specific expressions for the transport coefficients. In Sec. II.E, one obtains the static isolated, isothermal, and adiabatic susceptibilities to clarify some important points of the general response function theory, mainly, the ergodicity property of the Fermi systems [24,114]. We study the relaxation and correlation functions on the basis of the fluctuation-dissipation theorem and establish their relations to the ergodicity of the Fermi-liquid system in section II.F General aspects of the response function theory for the collective motion in nuclei are presented in Sec. III A in line with [24,29]. Section III.B shows the basic ingredients and the collective response function of the nuclear FLDM. Section III.C is devoted to the derivation of the temperature dependence of the transport coefficients, such as friction, inertia, and stiffness for the density modes for slow collective motion. The numerical illustrations are given in Sec. III.D. In Sec. IV, the semiclassical theory is extended to neutron-proton asymmetric nuclei and applied for the calculations of IVGDRs. In Sec. V, the smooth ETF and fluctuating shell-structure components of the moments of inertia are derived for collective rotations of heavy nuclei. The MI shell component is analytically presented in terms of the periodic orbits and their bifurcations within the POT. This component is compared with the quantum results for the simplest case of the deformed harmonic oscillator Hamiltonian. Comments and conclusions are finally given in Sec. VI. Some details of the thermodynamical, FLDM (in the LWL limit) and POT calculations, such as the analytical derivations of the in-compressibility, viscosity, thermoconductivity, coupling, and surface symmetry-energy constants, as well as the semiclassical MI are presented in Appendices A-E.

II. THE QUASIPARTICLE KINETIC THEORY

A. Equations of motion for the heated Fermi liquid

In the semiclassical approximation the dynamics of a Fermi liquid may be described by the distribution function \( f(r, p, t) \) in the one body phase-space. Restricting to small deviations of particle density \( \rho(r, t) \) and temperature \( T \), from their values in a thermodynamic equilibrium one may apply the linearized Landau–Vlasov equation [35,57]:

\[
\frac{\partial}{\partial t} \delta f(r, p, t) + \frac{\partial \varepsilon_{p,e}}{\partial p} \nabla_p \delta f(r, p, t) - \nabla_r \left[ \delta \varepsilon(r, p, t) \right] + V_{ext} \nabla_p f_{g.e.}(\varepsilon_{p,e}) = \delta S t. \tag{2.1}
\]

The right hand side (r.h.s.) represents the dynamic component of the integral collision term \( \delta S t \), and \( V_{ext} \) stands for an external field. We introduce here the Fermi distribution

\[
f_{g.e.}(\varepsilon_{p,e}) = \left[ 1 + \exp \left( \frac{\varepsilon_{p,e} - \mu}{T} \right) \right]^{-1} \tag{2.2}
\]

of the global equilibrium (g.e.), with \( \mu \) being the chemical potential, the temperature \( T \) is given, as usually in nuclear physics, in the energy (MeV) units (without Boltzmann’s constant), and \( \delta f(r, p, t) \) measures the deviation

\[
\delta f(r, p, t) = f(r, p, t) - f_{g.e.}(\varepsilon_{p,e}). \tag{2.3}
\]

For the sake of simplicity, the s.p. energy \( \varepsilon_{p,e} \) will be assumed to be of the form \( \varepsilon_{p,e} = p^2/2m^\text{*} \) with \( m^\text{*} \) being the effective nucleonic mass. In [24], \( \delta \varepsilon(r, p, t) \) stands for the variation of the quasiparticle energy \( \varepsilon(r, p, t) \),

\[
\delta \varepsilon(r, p, t) = \varepsilon(r, p, t) - \varepsilon_{p,e} = \frac{1}{N(T)} \int \frac{2dp'}{(2\pi\hbar)^3} F(p, p') \delta f(r, p', t). \tag{2.4}
\]

The quasiparticles’ density of states \( N(T) \) at the chemical potential \( \mu \) is given by

\[
N(T) = \int \frac{2dp'}{(2\pi\hbar)^3} \left( -\frac{\partial f_{p,e}}{\partial \varepsilon_{p,e}} \right)_{g.e}. \tag{2.5}
\]

Evidently, because of our linearization the density \( N(T) \) here is the one of equilibrium. In the sequel such a convention will inherently be applied to any coefficient of quantities of order \( \delta f \). The factor 2 accounts for the spin degeneracy. The amplitude of the quasiparticle interaction, \( F(p, p') \), commonly is written in terms of the Landau parameters \( F_0 \) and \( F_1 \), according to

\[
F(p, p') = F_0 + F_1 \hat{p} \cdot \hat{p}' , \quad \hat{p} = p/p. \tag{2.6}
\]

These two constants may be related to the two properties of nuclear matter, namely the isothermal incompressibility \( K^T \) (see Appendix A.1),

\[
K^T = 9 \rho g_0 / N(T), \tag{2.7}
\]

and the effective mass \( m^* \),

\[
m^* = G_1 m, \quad G_n = \left( 1 + \frac{F_n}{2n + 1} \right) \tag{2.8}
\]

\( (n = 0, 1) \). The equation for the effective mass \( m^* \) is known [32,57] to be valid for systems obeying Galilean invariance, which shall be assumed here.

In principle, the Landau parameters \( F_0 \) and \( F_1 \) might vary with the momenta \( p \) and \( p' \). Such a dependence will be neglected henceforth. This approximation appears to be reasonable as we are going to stick to small excitations near the Fermi surface and to temperatures \( T \), which are small as compared to the chemical potential \( \mu \). Likewise, we shall discard any temperature dependence of the effective mass. Notice that in addition to the ratio \( (T/\mu)^2 \), this dependence would be governed by the additional factor \( |m^*/m - 1| \) which is small for nuclear matter. These
assumptions will allow us to simplify further the theory \[57\] and to get more explicit results by making use of the temperature expansion for the response functions in the small parameter \(T/\mu\), as well as of the standard perturbation approach to eigenvalue problems needed later for the hydrodynamic (long-wave length) limit. We will follow \[53\] in neglecting higher order terms in the expansion \[2.28\] in Legendre polynomials.

Later, we want to study motion of the system which can be classified as an excitation on top of the local equilibrium. Following \[53, 57\], the collision term \(\delta St\) can be considered in the relaxation time approximation,

\[\delta St = -\frac{\delta f_{l.e.}(p, t)}{\tau}, \quad f_{l.e.}(\varepsilon^l, e) = \left[1 + \exp\left(\frac{\varepsilon^l - \mu(p, t) - pu(r, t)}{T(r, t)}\right)\right]^{-1}.\]  

(2.9)

Here, \(f_{l.e.}(\varepsilon^l, e)\) is the distribution function of a local equilibrium (l.e.), and \(\varepsilon^l, e\) is the associated quasiparticle energy. \(\mu(p, t)\) represents the chemical potential, \(u(r, t)\) the mean velocity field, and \(T(r, t)\) the temperature, all defined in the local sense. Like in \[32\], the relaxation time \(\tau\) is assumed to be independent of the quasiparticle momentum \(p\). However, it will be allowed to depend \(\tau\) on \(T\) as well as on the frequency of the motion (thus, accounting for retardation effects in collision processes). In \[2.30\], \(\delta f_{l.e.}(p, t)\) is defined as

\[\delta f_{l.e.}(p, t) = f(p, t) - f_{l.e.}(\varepsilon^l, e).\]  

(2.10)

It differs from \(\delta f(p, t)\) of \[2.28\] by the variations of local quantities. For the latter, we may write

\[\delta f(p, t) = \delta f_{l.e.}(p, t) + \delta f_{g.e.}(\varepsilon^g, e)\]  

(2.11)

with

\[\delta f_{l.e.}(\varepsilon^l, e) = f_{l.e.}(\varepsilon^l, e) - f_{g.e.}(\varepsilon^g, e)\]

\[= \left(\frac{\partial f_{p}}{\partial \varepsilon^l}\right)_{g.e.} \left(\frac{\partial \varepsilon^l}{\partial \mu} - \frac{\partial u}{\partial p} + \frac{\varepsilon^l - \mu}{T} \delta T\right).\]  

(2.12)

For the l.e. quasiparticle energy \(\varepsilon^l, e\), one has

\[\varepsilon^l, e = \varepsilon^g, e + \delta \varepsilon^l, e,\]  

(2.13)

where \(\delta \varepsilon^l, e\) is defined like in \[2.1\] with only \(\delta f(p, t)\) replaced by \(\delta f_{l.e.}(p, t)\). According to \[2.11\] and \[2.12\], for the simplified interaction \[2.10\], one gets

\[\delta \varepsilon_{p}^l = \delta \varepsilon(r, p, t) = \frac{F_{0}}{N(T)}e^{l} + \frac{F_{mp}}{N(T)\mu_e}p_{u},\]  

(2.14)

where \(\delta \rho\) is the dynamical component of the particle density

\[\rho(r, t) = \int \frac{2d\mathbf{p}}{(2\pi\hbar)^3} f(r, p, t) = \rho_{\infty} + \delta \rho(r, t)\]  

(2.15)

with \(\rho_{\infty}\) being its g.e. value associated to \(f_{g.e.}(\varepsilon^g, e)\) for the infinite Fermi liquid. The vector of the mean velocity \(u\) can be expressed in terms of the first moment of the distribution function (current density) and the particle density \[2.14\],

\[u(r, t) = \frac{1}{\rho} \int \frac{2d\mathbf{p}}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} \delta f(r, p, t).\]  

(2.16)

The definition of the collision term in the form \[2.10\] is incomplete without posing conditions for the conservation of the particle number, momentum, and energy (for simplicity of notations, we shall omit index \(\infty\) in the static nuclear-matter density component \(\rho\) at second order terms in the energy density variations). Notice that to the order considered, in the equation for energy conservation, \(\varepsilon\) may be replaced by \(\varepsilon^g, e\) (see also \[58\]). Incidentally, for the quasiparticle interaction \[2.6\], this substitution even becomes exact, as the dynamical part \(\delta \varepsilon\) would drop out of the last integral (as follows from \[2.11\], \[2.13\]).

They allow one to find unique solutions as functionals of the external field \(V_{ext}(t)\). Below we shall solve these equations in terms of response functions. It may be noted that, due to the conditions \[2.17\], the first variation of the distribution function \(\delta f(r, p, t)\) \[2.11\] disappears from the dynamical component \(\delta \rho(r, t)\) of the density \(\rho(r, t)\) and of the velocity field \(u(r, t)\). As one knows (see, e.g., \[35, 57, 58\]), the equation for the velocity field reduces to an identity if one takes into account the definition of the effective mass \(m^*\) given by \[2.8\].

**B. The conserving equations**

In this section, we like to deduce conserving equations for the particle number, momentum, and energy, which later on will turn out helpful to find appropriate solutions of the Landau–Vlasov equation \[2.31\]. The procedure, which relies on a moment expansion, is well known from textbooks \[55, 58, 113\]. We will follow more closely the version of \[31, 32\] (see also \[48\]).
1. THE MOMENT EXPANSION

Whereas particle number conservation implies to have
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (2.18)
\]
the momentum conservation is reflected by the following set of equations
\[
m \frac{\partial u_\alpha}{\partial t} + \sum_\beta \frac{\partial \Pi_{\alpha\beta}}{\partial r_\beta} = - \frac{\partial V_{\text{ext}}}{\partial r_\alpha}. \quad (2.19)
\]

Besides quantities introduced before, they involve
\[
\Pi_{\alpha\beta} = \int \frac{2dp}{(2\pi\hbar)^3} \frac{p_\alpha p_\beta}{m^*} \delta f(r, p, t) + \frac{F_0}{N(T)} \delta \rho(r, t) \delta_{\alpha\beta}. \quad (2.20)
\]
Substituting for \(\delta f(r, p, t)\) \((2.11)\) into \((2.20)\), one gets
\[
\Pi_{\alpha\beta} = -\sigma_{\alpha\beta} + \delta P \delta_{\alpha\beta}. \quad (2.21)
\]
The first component \(\sigma_{\alpha\beta}\), which results from the first term \(\delta f_{\text{i.e.}}(r, p, t)\) on the right of \((2.11)\), determines the dynamic shear stress tensor,
\[
\sigma_{\alpha\beta}(r, t) = - \int \frac{2dp}{(2\pi\hbar)^3} \frac{p_\alpha p_\beta}{m^*} \delta f_{\text{i.e.}}(r, p, t), \quad (2.22)
\]
whose trace vanishes. For a linearized dynamics, the non-diagonal components of the momentum flux tensor \(\Pi_{\alpha\beta}\) equal the corresponding stress tensor (but with the opposite sign), with correction terms being proportional to \(u_{\alpha u\beta}\) in \(\delta f\), and thus, of higher order, see \((2.16)\) for \(u_\alpha\).

The second component of the momentum flux tensor of \((2.21)\) can be derived from the variation \(\delta f_{\text{i.e.}}(\varepsilon_{\text{p}}^{\text{L}})\) as given by \((2.12)\). It represents the compressional part of the momentum flux tensor,
\[
\int \frac{2dp}{(2\pi\hbar)^3} \frac{p_\alpha p_\beta}{m^*} \delta f_{\text{i.e.}}(\varepsilon_{\text{p}}^{\text{L}}) + \frac{F_0}{N(T)} \delta \rho \delta_{\alpha\beta} = \delta P \delta_{\alpha\beta}. \quad (2.23)
\]
with \(\delta P \equiv \delta \rho(r, t) \) \(\int \frac{2dp}{(2\pi\hbar)^3} \delta f_{\text{i.e.}}(\varepsilon_{\text{p}}^{\text{L}})\) (2.23) [mind \((2.11)\) and \((2.17)\)]. Notice, that here only the diagonal parts survive. The only non-diagonal ones could come from the terms in \((2.12)\) involving \(\mathbf{p}\); but they vanish when integrating over angles in momentum space. Traditionally, \(\delta P\) in \((2.23)\) is referred to as the scalar pressure, see \([116]\). Using \((2.12)\) for the distribution \(\delta f_{\text{i.e.}}(\varepsilon_{\text{p}}^{\text{L}})\) and its properties mentioned above, after some simple algebraic transformations, one gets
\[
\delta P = \frac{2}{3} \int \frac{2dp}{(2\pi\hbar)^3} \frac{p^2}{2m^*} \delta f_{\text{i.e.}}(\varepsilon_{\text{p}}^{\text{L}}) + \frac{F_0}{N(T)} \delta \rho = K^T \delta \rho + \rho \left( \zeta - \frac{M}{N} \right) \delta T, \quad (2.24)
\]
with \(K^T\) being the isothermal in-compressibility \((2.7)\). For the derivation of the second equation in \((2.24)\), one can use (i) the transformation of \(\delta \mu\) to the variations of \(\delta \rho\) and \(\delta T\) [see \((2.49)\)], and (ii) the relations \((A.13)\), \((A.14)\), and \((A.10)\) for the entropy per particle \(\zeta\), the particle density \(\rho\) as well as for the quantity \(M\) \((A.10)\), respectively. Inspecting \((A.13)\) and \((A.8)\), it becomes apparent that the expression on the very right of \((2.24)\) may indeed be interpreted as an expansion of the static pressure to the first order in \(\delta \rho\) and \(\delta T\). It is thus seen that the truly non-equilibrium component \(\delta f_{\text{i.e.}}(r, p, t)\) only appears in the shear stress tensor \(\sigma_{\alpha\beta}\) given in \((2.22)\).

Note, here and below within Sec. II B we omit immaterial constants related to the global equilibrium (static) components of the moments to simplify the notations and adopt them to the ones of the standard textbooks when it will not lead to misunderstanding. We should emphasize that the Landau quasiparticle theory which is a basis of our derivations is working in a self-consistency way with small deviations from (small excitations near) the Fermi surface which are denoted by symbol "," and takes above mentioned static components as those of the external phenomenological (experimental) data. Therefore, all relations discussed below in this section should be understood as the ones between such close-to-Fermi-surface quantities within our linearized Landau–Vlasov phase space dynamics after exclusion of all above mentioned immaterial constants. Nevertheless, we keep the symbol \(\delta\) with the scalar pressure \(\delta P\) to avoid possible misunderstanding related to the linearization procedure, see more comments below after \((2.38)\).

2. THE STRESS TENSOR

It may be worthwhile to relate the stress tensor \(\sigma_{\alpha\beta}\) given in \((2.22)\) to the standard form in terms of the coefficients of the shear modulus \(\lambda\) and the viscosity \(\nu\),
\[
\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{(\lambda)} + \sigma_{\alpha\beta}^{(\nu)}. \quad (2.25)
\]
Here, the first term \(\sigma_{\alpha\beta}^{(\lambda)}\) is the conservative part of the stress tensor \(\sigma_{\alpha\beta}\),
\[
\sigma_{\alpha\beta}^{(\lambda)} = \lambda \left( \frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha} - \frac{2}{3} \nabla \mathbf{w} \delta_{\alpha\beta} \right). \quad (2.26)
\]
with \(\mathbf{u} = \partial \mathbf{w}/\partial t\) and \(\mathbf{w}\) being the displacement field. The second term in \((2.25)\) can be written as
\[
\sigma_{\alpha\beta}^{(\nu)} = \nu \left( \frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha} - \frac{2}{3} \nabla \mathbf{u} \delta_{\alpha\beta} \right), \quad (2.27)
\]
where \(\nu\) is the coefficient of the shear viscosity (or the first viscosity). For more details see Appendix A.2, in particular for expressions of the coefficients \(\lambda\) \((B.12)\) and \(\nu\) \((B.13)\) in terms of Fermi liquid interaction parameters.

To obtain microscopic expressions for the shear modulus \(\lambda\) and the viscosity \(\nu\), one needs to exploit the solution \(\delta f_{\text{i.e.}}(r, p, t)\) of the Landau–Vlasov equation \((2.1)\)
for the stress tensor $\sigma_{\alpha\beta}(r, t)$ \ref{2.22}, reducing the latter to the form \ref{2.25}. Such a calculation of $\lambda$ and $\mu$ in terms of the Landau Fermi-liquid parameters is discussed in Appendix A.2, in which Fourier transforms are exploited \cite{31}. Equivalently, one may express functions of space and time by plane waves, which for the distribution function reads \cite{31, 46}.

\[
\delta f(q, p, t) = \delta \tilde{f}(q, p, \omega) \exp[i(qr - \omega t)] \tag{2.28}\]

with $q$ being the wave vector and $\omega$ the frequency of the vibrational modes of nuclear matter. Such a plane-wave representation is to be applied to both sides of \ref{2.22} and \ref{2.25}. The amplitudes for the velocity $u$ and the displacement $w$ field then satisfy $\tilde{w} = \tilde{\bar{u}}/(-i\omega)$.

Using \ref{2.22}, \ref{2.23} and \ref{2.24} for the stress tensor $\sigma_{\alpha\beta}$ and \ref{2.21} for the scalar pressure $\delta P$,

\[
\delta P = \frac{K_{\text{tot}}}{9} \delta \rho, \tag{2.29}\]

one finally may write down a general expression for the momentum flux tensor $\Pi_{\alpha\beta}(r, t)$ \ref{2.21},

\[
\Pi_{\alpha\beta} = -\lambda \left( \frac{\partial u_{\alpha}}{\partial r_{\beta}} + \frac{\partial u_{\beta}}{\partial r_{\alpha}} - \frac{2}{3} \nabla w \delta_{\alpha\beta} \right) - \nu \left( \frac{\partial u_{\alpha}}{\partial r_{\beta}} + \frac{\partial u_{\beta}}{\partial r_{\alpha}} - \frac{2}{3} \nabla u \delta_{\alpha\beta} \right) + \frac{K_{\text{tot}}}{9} \delta \rho \delta_{\alpha\beta}. \tag{2.30}\]

A total effective in-compressibility $K_{\text{tot}}$ includes the change of the pressure due to variations of the temperature with density. With the help of \ref{A.18}, the incompressibility $K_{\text{tot}}$ can be expressed through the specific heat per particle $C_V$ \ref{A.9},

\[
K_{\text{tot}} = K^T + 6C_V \rho \frac{\delta T}{\delta \rho} \tag{2.31}\]

Again, $\delta T$ and $\delta \rho$ are the Fourier components of $\delta T(r, t)$ and $\delta \rho(r, t)$. Like all other kinetic coefficients, such as $\lambda$ and $\nu$ given in \ref{A.12} and \ref{A.13}, respectively, this effective, total in-compressibility modulus $K_{\text{tot}}$, too, depends on $\omega$ and $q$. Later on we shall discuss in more detail these quantities in the LWL limit. In this limit, the total in-compressibility $K_{\text{tot}}$ will be seen to become identical to the adiabatic one $K^T$ given in \ref{A.20}.

3. ENERGY CONSERVATION AND THE GENERAL TRANSPORT EQUATION

So far we have not looked at the energy conservation. For this purpose, one needs to consider thermal aspects as they appear in equations for the change of entropy and temperature. To do this we will follow standard procedures. We first built the scalar product of the mean velocity $u$ with the vector equation, whose component $\alpha$ is given by \ref{2.19}. Making use of the continuity equation \ref{2.18}, after some manipulations, one gets

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} m \rho u^2 + \rho E \right) = \rho \left[ \frac{1}{2} \frac{\partial}{\partial r_{\beta}} \delta_{\alpha\beta} + \rho W_{\alpha\beta} - \sigma_{\alpha\beta} - \kappa \frac{\partial T}{\partial r_{\alpha}} \delta_{\alpha\beta} \right] \tag{2.36}\]

\[
+ \rho T \left[ \frac{\partial \xi}{\partial t} + u \nabla \xi \right] - \nabla \left( \kappa \nabla T \right) - \frac{\nu}{2} \sum_{\alpha\beta} \left( \frac{\partial u_{\alpha}}{\partial r_{\beta}} + \frac{\partial u_{\beta}}{\partial r_{\alpha}} - \frac{2}{3} \nabla u \delta_{\alpha\beta} \right)^2 - \rho u \nabla \nu_{\text{ext}}. \tag{2.32}\]

On the left hand side, there appears the mean kinetic energy density and the internal energy density $\rho E$ per unit volume (defined again up to an immaterial constant). The density $E$ itself may be split in three different components,

\[
E = E^{(\lambda)} + E^{(K)} + T \delta \xi. \tag{2.33}\]

The first one,

\[
E^{(\lambda)} = \frac{\lambda}{4 \rho} \sum_{\alpha\beta} \left( \frac{\partial w_{\alpha}}{\partial r_{\beta}} + \frac{\partial w_{\beta}}{\partial r_{\alpha}} - \frac{2}{3} \nabla w \delta_{\alpha\beta} \right)^2, \tag{2.34}\]

is related to shear deformations, which is known from the solid state physics and for Fermi liquids as coming from distortions of the Fermi surface \cite{34}. The second one may be written as

\[
E^{(K)} = \frac{K_{\text{tot}}}{18} \left( \delta \rho \right)^2 ; \tag{2.35}\]

it represents the compressional component, associated to the effective total in-compressibility $K_{\text{tot}}$, which is in line of the known thermodynamic relations. Equation \ref{2.33} resembles the expression found in \ref{26, 27}, except for a generalization of the physical meaning of the incompressibility modulus $K_{\text{tot}}$ as function of $\omega$ and $q$ given in \ref{A.20}, as compared to the quasistatic adiabatic case. The third one in \ref{2.33} represents the change of heat part resulting from a change of entropy. We keep here the dynamical variation symbol $\delta$ for the entropy $\xi$ (also for the pressure $P$ here and below) to remember that all quantities of the Landau Fermi-liquid theory are presented for small dynamical deviations near the Fermi surface in the linear (or quadratic after multiplying \ref{2.19} by $u$) form in $\delta f$. We avoid here a misunderstanding with following transformations of the energy $E$ \ref{2.19}, say Legendre ones, to the differential form in line of a general comment at the beginning of this section. On the r.h.s. of \ref{2.32} the enthalpy $W_{\alpha\beta}$ per particle has been introduced,

\[
W_{\alpha\beta} = E \delta_{\alpha\beta} + \frac{1}{3} \left( \sigma_{\alpha\beta}^{(\xi)} + \delta \rho \delta_{\alpha\beta} \right), \tag{2.36}\]

(see the comment above concerning $\delta P$). Furthermore, the thermodynamic relation for the dynamical variations
of the internal energy $\mathcal{E}$ in terms of those $\zeta$ for the entropy per particle $\zeta$, the density $\rho$ and the displacement tensor $w_{\alpha\beta}$, is given by

$$d\mathcal{E} = Td\zeta + \frac{\delta P}{\rho^2} d\rho + \frac{1}{\rho} \sum_{\alpha\beta} \sigma_{\alpha\beta} dw_{\alpha\beta}. \quad (2.37)$$

The displacement tensor $w_{\alpha\beta}$ is defined as

$$w_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial w_{\alpha}}{\partial \beta} + \frac{\partial w_{\beta}}{\partial \alpha} \right). \quad (2.38)$$

Note that equation (2.39) for the pressure $\delta P$ is important to get (2.33) by integration of (2.37). According to (2.36), we get the standard relation of linearized thermodynamics of liquids, for instance, between the enthalpy and entropy $\zeta$ up to the second order term in $\delta P$.

In (2.32), we also added and subtracted the term $\nabla j_T$ containing the heat current,

$$j_T = -\kappa \nabla T, \quad (2.39)$$

with the coefficient $\kappa$ for the thermal conductivity. We may now write the equation for energy conservation as

$$\frac{\partial}{\partial t} \left( \frac{1}{2} m \rho u^2 + \rho \mathcal{E} \right) = - \sum_{\alpha\beta} \frac{\partial}{\partial r_{\beta}} \left[ u_{\alpha} \left( \frac{1}{2} m \rho u^2 \delta_{\alpha\beta} + \rho W_{\alpha\beta} - \sigma_{\alpha\beta}^{(\nu)} - \kappa \frac{\partial T}{\partial r_{\alpha}} \delta_{\alpha\beta} \right) \right] - \rho \mathbf{u} \nabla \mathbf{V}_{\text{ext}}. \quad (2.40)$$

In this way, it is seen that from (2.32) and (2.40), together with the continuity equation (2.18) and the definition of the heat current $j_T$ (2.39), one gets for the change of entropy:

$$\frac{\partial (\rho \delta \zeta)}{\partial t} = - \nabla \left( \rho \delta \zeta \mathbf{u} + \frac{1}{T} j_T \right) + \frac{\kappa}{T^2} (\nabla T)^2$$

$$+ \frac{\nu}{2T} \sum_{\alpha\beta} \left( \frac{\partial u_{\alpha}}{\partial r_{\beta}} + \frac{\partial u_{\beta}}{\partial r_{\alpha}} - \frac{2}{3} \nabla \mathbf{u} \delta_{\alpha\beta} \right)^2. \quad (2.41)$$

[again the variation $\delta$ in $\delta \zeta$ is not omitted because of the following derivations of the Fourier equation (B.15) and thermal conductivity (B.20) in Appendix A.2]. These two equations have a very clear physical meaning for normal liquids and amorphous solids (a very viscose liquids with some shear modulus $\lambda$ in our notations, i.e., solids without any crystal structure). The first equation (2.40) claims that the change of the collective and internal energy, concentrated in unit volume per unit of time and presented as the sum of the collective kinetic and internal parts, equals the corresponding energy flux through its surface and work of the external field. The second equation (2.41) is usually called as a general heat transport equation. This equation states that the change of entropy in the unit volume per unit of time equals the entropy flux through its surface (heat energy flux). Two other terms show the entropy increase related to the gradient of the temperature and dissipation due to the shear ($\nu$) viscosity. Note that there is no explicit dependence on the external field in (2.41). This dependence is manifested only through the solutions of the dynamical equations in terms of the moments. For zero external field (closed system) the entropy is increasing because of the basic thermodynamic law. Therefore, according to (2.41), the shear viscosity $\nu$ (B.13) and the thermal conductivity $\kappa$ should be positive. The energy conservation equation for the Fermi liquid (2.40) differs from the one for classical hydrodynamics by the same Fermi-surface distortions related to the shear modulus $\lambda$ (B.12) as discussed above. That is similar to the amorphous solids (in the above mentioned sense of very viscose liquids). However, in contrast to the latter, one obtains the energy conservation condition (2.40) for the dynamical variations of the Fermi-liquid collective and internal energy with the specific constants $\lambda$ (B.12), $\nu$ (B.13) and $K_{\text{tot}}$ (2.31) found from the relation to the Landau–Vlasov equation (2.1). Our way of the derivation of the energy conservation equation (2.40) for the Fermi liquids within the linearized Landau–Vlasov dynamics (2.1), as for normal liquids and solids, leads to a more explicit form of the energy conservation equation than that suggested in (2.11). In this way, we get rather simple expressions for the collective and internal components of the energy output the hydrodynamical limit.

4. POTENTIAL FLOW: FERMI LIQUID VERSUS HYDRODYNAMICS

Below, we shall be interested in the case of a viscous potential flow, for which one has

$$\mathbf{u} = \nabla \varphi, \quad \mathbf{w} = \nabla \varphi_{\text{w}} \quad \text{with} \quad \varphi = \varphi_{\text{w}} \quad (2.42)$$

[cf. the second equation with (2.26)]. With the help of these definitions, the momentum equation (2.19) and the flux tensor (2.30) can be brought to the following forms:

$$m \rho \frac{\partial \varphi}{\partial t} - 4 \frac{\lambda}{3} \Delta \varphi - 4 \frac{\lambda}{3} \Delta \varphi_{\text{w}} + K_{\text{tot}} \frac{\delta \rho}{9} = -V_{\text{ext}} \quad (2.43)$$

and

$$\Pi_{\alpha\beta} = -2 \left( \frac{\lambda}{-i\omega} + \nu \right) \left( \frac{\partial^2 \varphi}{\partial r_{\alpha} \partial r_{\beta}} - \Delta \varphi \delta_{\alpha\beta} \right)$$

$$- \left( m \rho \frac{\partial \varphi}{\partial t} + V_{\text{ext}} \right) \delta_{\alpha\beta}. \quad (2.44)$$

The diagonal term given on the very right of (2.44) had been used to remove $\delta \rho$ which still appears in (2.39). With the continuity equation (2.18) for the plane wave solutions (2.28), one has from (2.43) the equation for the velocity potential $\varphi$ (B.32).

$$m \rho \frac{\partial^2 \varphi}{\partial t^2} - \frac{\rho}{9} (K_{\text{tot}} + 12 \lambda / \rho) \Delta \varphi - 4 \frac{\nu}{3} \frac{\partial \varphi}{\partial t} = -\frac{V_{\text{ext}}}{\partial \varphi}. \quad (2.45)$$

The structure of (2.45) for the potential flow is similar to that of the Navier-Stokes equation for the velocity potential $\varphi$. The difference to the case of the common classical
liquid, is seen in the terms proportional to $\lambda$, viz in the presence of the anisotropy term (2.20), which actually represents a reversible motion. Such a term is known from the dynamics of amorphous solids. We emphasize that for Fermi liquids, this term arises only in the presence of the Fermi surface distortions, which survive even in the non-viscous limit; they will turn out important for our applications below. The shear modulus $\lambda$ may be interpreted as a measure of those distortions which are related to a reversible anisotropy of the momentum flux tensor. They disappear in the hydrodynamic limit, and so does $\lambda$, in which case all formulas of this section turn into those for normal liquids; for more details see section IV.D.

At this place an important remark is in order. It should be noted that in contrast to classical hydrodynamics our system of equations for the moments is not closed to the first few ones, namely particle density $\delta \rho$ and velocity field $u$. This is true in particular for (2.43) for the potential flow $\varphi$. Indeed, the coefficients $\lambda, \nu$ and $K_{tot}$ depend on the variable $\omega/q$ which yet is unknown. The latter is determined from a dispersion relation, which in turn has to be derived from the Landau–Vlasov equation (2.1). Such a procedure goes back to [54] where the dispersion relation was exploited for the collisionless case at $T = 0$. A collision term in the relaxation approximation has been taken into account in [55]. The extension to heated Fermi liquids and low excitations, in the way, which we are going to use later on, has been developed in [57]. It may be noted that this version of the dispersion relation, which we are aiming at, differs essentially from the one obtained in the “truncated” (scaling model) versions of the Fermi-liquid theory of [113, 114] where the momentum flux tensor is not influenced by higher moments of the distribution function. We take into account all other multipolarities (larger the quadrupole one) of the Fermi-surface distortions when there is no convergence in multipolarity expansion of the distribution function for finite and large $\omega t\tau$ or for finite $K_{tot}$, for instance for nuclear matter with small $F_0$, in contrast to the Fermi liquid $^3\text{He}$.

C. Response functions

1. DYNAMIC RESPONSE

As mentioned earlier, we want to solve the linearized equations of motion in terms of response functions. We concentrate on two quantities, namely particle density $\rho(r,t)$ and temperature $T(r,t)$ and examine how they react to the external field $V_{ext}(r,t)$ introduced earlier. This may be quantified by the following two response functions: The density-density response $\chi^{coll}_{DD}(q,\omega)$ and the temperature-density response $\chi^{coll}_{TD}(q,\omega)$ defined as

$$\chi^{coll}_{DD}(q,\omega) = -\frac{\delta \rho(q,\omega)}{V_{ext}(q,\omega)},$$

(2.46)

and

$$\chi^{coll}_{TD}(q,\omega) = -\frac{\delta T(q,\omega)}{V_{ext}(q,\omega)},$$

(2.47)

respectively. To keep the notation simple, we will omit the tilde characterizing the Fourier transform of the distribution function (2.28), (it should suffice to only show the arguments $q, \omega$). The definition of the response functions is identical to the one of (2.4), except that we have introduced the suffix “coll”. This was done adopting a notation used in the literature of nuclear physics when the dynamics of a finite nucleus is expressed in terms of shape variables, to which we will come below. Notice, however, that $V_{ext}(q,\omega)$ is only proportional to the density, $V_{ext}(q,\omega) = q_{ext}(\omega)\rho(q,\omega)$, with $q_{ext}(\omega)$ being some externally determined function. Often, one therefore defines response functions in a slightly modified way, in that the functional derivatives are performed with respect to $q_{ext}(\omega)$ instead of $V_{ext}(q,\omega)$ (see, e.g., [30]).

As will be seen below, these functions only depend on the wave number $q$ but not on the angles of the wave vector $q$. For this reason, it is convenient to introduce the dimensionless quantities $s$ and $\tau$ (with $v_p = p_e/m^*$)

$$s = \frac{\omega}{v_p q}, \quad \tau = \frac{\tau v_p q}{s}, \quad \text{implying} \quad \omega = s \tau q,$$

(2.48)

instead of the frequency $\omega$ and the wave number $q$.

To calculate the response functions (2.46) and (2.47), we follow the procedure of (2.44) and (2.45). As any further details may be found there, it may suffice to outline briefly the main features. In short, the strategy is as follows. Firstly, one rewrites the Landau–Vlasov equation (2.1) in terms of the Fourier coefficients introduced in (2.24). Evidently, in the spirit of the separation specified in (2.11), we need to evaluate explicitly only the first component $\delta f_{i,c}(r,p,t)$ which enters the conditions (2.17).

By a straightforward calculation, one may then express $\delta f_{i,c}(r,p,t)$ in terms of the unknown quantities $\delta \rho, \delta \mu$ and $\delta T$ for any given external field $V_{ext}$. The form is given in (13.4). The continuity equation (2.13) in the Fourier representation through (2.28), $q u = \omega \delta \rho / \rho$, may be used to eliminate the velocity field $u$. Furthermore, the thermodynamic relation [see (A17, A18, and 2.31)]

$$\delta \mu = \left(\frac{\partial \mu}{\partial \rho}\right)_T \delta \rho + \left(\frac{\partial \mu}{\partial T}\right)_\rho \delta T = \frac{K_T}{9 \rho} \delta \rho - \frac{M(T)}{N(T)} \delta T,$$

(2.49)

allows one to express the chemical potential $\delta \mu$ in terms of the two unknown variables $\delta \rho$ and $\delta T$. Next, one may exploit the conditions (2.17). As the second (set of) equation(s) is just an identity, provided one uses the appropriate definition of the effective mass (2.28), it is only the first and the third equation which matter. They may determine the remaining two variables $\delta \rho$ and $\delta T$ in terms of the external field,

$$\left(\frac{i s \tau_q}{is \tau_q - 1 - \nu(s) \chi_0}\right) \delta \rho + \frac{1}{1 - is \tau_q} \chi_1 \delta T = -\chi_0 \delta V_{eff}$$

(2.50)
\[-\phi(s)\chi_1 \delta \rho + \frac{1}{1 - is\tau_q} \left( \chi_2 - is\tau_q \frac{C_v}{T} \right) \delta T = -\chi_1 \delta V_{\text{eff}}. \]  

Here, the quantity

\[\phi(s) = \frac{1}{N(T)} \frac{1}{i\tau_q - 1} - \frac{3is}{\tau_q N(0)} \]  

has been introduced with \(N(0)\) being the level density of the quasiparticles at \(T = 0\),

\[N(0) = \frac{\rho_0 m^*}{\pi^* \hbar^3} = \frac{3 \rho_0}{2 \varepsilon_p}, \tag{2.53}\]

and \(\varepsilon_p = p^2 f^*/2m^*\). The functions \(\chi_n\) are given by

\[\chi_n = -N(T) \left\langle \frac{qV_p}{p} \right\rangle \frac{\varepsilon_p - \mu}{T - \mathcal{M}(T)} \right\rangle^n \tag{2.54}\]

with \(n = 0, 1, 2, \ldots\),

\[D_p = \omega - qV_p + i/\tau, \quad \nu_p = p/m^*. \tag{2.55}\]

Furthermore, in (2.50) and (2.51) a short hand notation \(\delta V_{\text{eff}}\) has been used for the sum of two terms, namely

\[\delta V_{\text{eff}} = V_{\text{ext}} + k(\omega, T) \delta \rho \tag{2.56}\]

with

\[k(\omega, T) = \frac{1}{N(T)} \left\{ \mathcal{F}_0 + \mathcal{F}_1 \left( \frac{\omega}{\nu_q} \right)^2 \right\}. \tag{2.57}\]

In (2.50), \(\delta V_{\text{eff}}\) may be considered as an effective field which includes the true external field \(V_{\text{ext}}\) and the screened field \(k(\omega, T) \delta \rho\). Our notation follows the one often used for finite nuclei: The second term in (2.57) plays the role of the collective variable and \(k\) represents the coupling constant (see, e.g., [1, 24, 29]).

The response function \(\chi_{DD}^{\text{coll}}\) of (2.40) can be now obtained from (2.50) and (2.51),

\[\chi_{DD}^{\text{coll}}(\tau_q, s) = \frac{\delta\mathcal{F}(\tau_q, s)}{D(\tau_q, s)}, \tag{2.58}\]

where

\[D(\tau_q, s) = D_0(\tau_q, s) + k(\tau_q, s) N(\tau_q, s) \tag{2.59}\]

with

\[D_0(\tau_q, s) = \left( \frac{i\tau_q}{is\tau_q - 1} - \phi(s) \chi_0 \right) \left( \chi_2 - i\tau_q \frac{C_v}{T} \right) + \phi \chi_2. \tag{2.60}\]

In (2.58), \(\delta\mathcal{F}(\tau_q, s)\) finally is given by

\[\delta\mathcal{F}(\tau_q, s) = \chi_0 \left( \chi_2 - i\tau_q \frac{C_v}{T} \right) - \chi_1^2. \tag{2.61}\]

It is worth noticing that the collective response function for the density-density mode, as given by (2.40) or (2.58), can be expressed as

\[\chi_{\text{coll}}^{\text{DD}}(q, \omega) = \frac{\chi(q, \omega)}{1 + k(\omega, T) \chi(q, \omega)}. \tag{2.62}\]

This form is analogous to the form used to describe the dynamics of shape variables [4, 24, 29]. We omit here the suffix \(DD\) because the \(TD\) response function takes on a similar form (with some modification of the numerator). It is here where the coupling constant \(k\) appears, as defined in (2.55), together with the intrinsic (or unscreened) (see [30, 57]) response function \(\chi\),

\[\chi(\tau_q, s) = -\frac{\delta \rho}{\delta V_{\text{eff}}} = \frac{\delta\mathcal{F}(\tau_q, s)}{D_0(\tau_q, s)}. \tag{2.63}\]

Both expressions can be found already in [57]. However, later we will find the form (2.58) more convenient for our applications, in particular for the case of the low frequency limit \(\omega \tau \ll 1\). When we shall expand first \(\chi\) in (2.62) in small \(\omega \tau\) near the poles of \(\chi_{\text{coll}}^{\text{DD}}\) (see next section), one should assume that the singularities of \(\chi\) related to zeros of \(D_0\) in (2.63) are far away from zeros of \(D\) in (2.58), i.e., a smoothness of \(\chi\) as function of \(\omega \tau\) near these poles. After the cancellation of a possible singularity source \(D_0\) in (2.58) we are free from such an assumption.

Finally, let us turn to the temperature-density response function \(\chi_{TD}^{\text{coll}}\). It is determined by the same system of equations (2.50) and (2.51) and can be written in the form (2.62) but with another intrinsic response function \(\chi_{TD}\) appearing in the numerator,

\[\chi_{TD}(\tau_q, s) = -\frac{\delta \mathcal{T}}{\delta V_{\text{eff}}} \tag{2.64}\]

From (2.50), (2.51) and (2.64), one obtains

\[\chi_{TD}(\tau_q, s) = \frac{i\tau_q \chi_1}{D_0(\tau_q, s)}, \tag{2.65}\]

where \(D_0(\tau_q, s)\) is given by (2.60). As compared to the one printed in [57], this expression contains an additional factor \(i\tau_q/\chi_1\), which later on will turn out important, for instance, when calculating susceptibilities and the incompressibility \(K_{\text{tot}}\) (2.31). (We are grateful to H. Heiselberg for confirming this misprint.) Substituting (2.65) into the numerator of (2.62) instead of \(\chi\), one gets the temperature-density response function (2.47) in the form similar to (2.58),

\[\chi_{TD}^{\text{coll}}(\tau_q, s) = \frac{i\tau_q \chi_1}{D(\tau_q, s)}. \tag{2.66}\]

Notice that according to (2.65) and (2.66), both response functions (2.40) and (2.47) have the same set of poles, which lie at the roots of the equation

\[D(\tau_q, s) = 0. \tag{2.67}\]

This is identical to the condition of zero determinant for the system of the linear equations (2.50) and (2.51).
2. LOW TEMPERATURE LIMIT

The expressions for the collective response functions become much simpler at low temperatures $T \ll \mu$. In this case, one may calculate $\chi_n$ of [2.54] by expanding in powers of $T/\mu$. For those applications to nuclear physics we have in mind the temperature is sufficiently small such that it suffices to mainly stick to order two. Fourth order terms shall be shown only when necessary.

A basic element for the quantities which we need to evaluate is the derivative $\frac{\partial f_p}{\partial \varepsilon_p}$ taken at global equilibrium:

$$\frac{\partial f_p}{\partial \varepsilon_p}_{\text{g.e.}} = - \left[ 4T \cosh^2 \left( \frac{\varepsilon_p - \mu}{2T} \right) \right]^{-1}_{\text{g.e.}}, \quad (2.68)$$

It appears in $\mathcal{N}(T)$ of [2.5] [see also [A.15]], which in turn is needed for $\chi_n$ of [2.54]. For small $T$, this derivative is a sharp bell-shaped function of $\varepsilon_p$, such that one may evaluate the averaging integrals [2.54] and [A.15] by expanding the smooth functions in terms of $\varepsilon_p$ near $\varepsilon_p = \mu_{\text{g.e.}}$. In this way, the Fourier-Bernoulli integrals over the dimensionless variable $[(\varepsilon_p - \mu)/T]_{\text{g.e.}}$ appear (see, e.g., [57]) which lead to

$$\chi_0 = \left[ -Q_1(\zeta) + \frac{\pi^2T^2}{12} (Q_1(\zeta) - \zeta Q'_1(\zeta)) - \frac{1}{2} \zeta^2 Q'_1(\zeta) + \mathcal{O}(T^4) \right] \mathcal{N}(0), \quad (2.69)$$

$$\chi_1 = \left[ \frac{\pi^2T}{6} \zeta Q'_1(\zeta) + \mathcal{O}(T^3) \right] \mathcal{N}(0), \quad (2.70)$$

and

$$\chi_2 = \left\{ \frac{-\pi^2}{3} \left[ Q_1(\zeta) + \frac{\pi^2T^2}{120} (36Q_1(\zeta) - 46\zeta Q'_1(\zeta)) - 21\zeta^2 Q'_1(\zeta) + \mathcal{O}(T^4) \right] \right\} \mathcal{N}(0). \quad (2.71)$$

Here $Q_1(\zeta)$ is the Legendre function of second kind with $\zeta = s + i/\tau_q$ and $T = T/\varepsilon_p$ is used also in Appendix A.3. These quantities may now be used to calculate the response functions [2.46] and [2.47], or more specifically [2.58] and [2.60]. For zero temperature, one gets the standard solutions [35, 57]. So far no assumption has been made concerning the parameter $\omega \tau$ which specifies the importance of collision in various regimes of the collective modes [35]. In particular, the formulas obtained in this section are valid both for the regimes of zero sound ($\omega \tau \gg 1$) and hydrodynamics ($\omega \tau \ll 1$). For $\omega \tau > 1$ our solutions agree with those of [35, 57]. However, below we shall be interested mainly in collective excitations of low frequencies. The notion "low frequencies" is meant to indicate that the corresponding excitation energies are smaller than those of the giant resonances. Next we will turn to the hydrodynamic regime where $\omega \tau \to 0$. As we shall see, at low temperatures our solutions approach the ones of normal classical liquids, in agreement with [113, 116].

D. Hydrodynamic regime

1. DISPERSION RELATION

The response functions can be simplified significantly in the long-wave length limit. Using $\tau_q$ introduced in [2.48], this (LWL) limit may be defined as $\tau_q \ll 1$. It can be reached in two ways, namely for small wave numbers $q$ and finite collision time $\tau$ or for small $\tau$ but finite $q$. Both cases imply that the dimensionless parameter $\omega \tau = \sigma q$, which determines the collision rate in comparison to the frequency of the modes, becomes small for any finite value $s$ of [2.48] ($|s| \lesssim 1$). As will be shown below for nuclear matter at low temperatures, this quantity is not enough large, in distinction to the case of liquid $^3$He. Therefore, a small $\tau_q$ implies hydrodynamic behavior, in contrast to the zero sound regime; where $\tau_q \gg 1$, or $\omega \tau \gg 1$.

The Landau–Vlasov equation (2.44) is an integral equation. Its solution may be sought for in terms of an eigenvalue problem with the distribution function $\delta f$ being the eigenfunctions and the sound velocity $s$ being the eigenvalues, see also [55, 56]. This eigenvalue problem may be solved perturbatively with $\tau_q$ being the smallness parameter [53, 56]. It may be noted in passing that this method may be applied to some extent as well to the eigenvalue problem of the Schrödinger equation. We shall use it to get the hydrodynamic sound velocities from the kinetic equation, see [55, 56]. To this end, we expand the solutions for $s$ and $\delta f$ into power series with respect to $\tau_q$, but restricted to linear order. Thus, we may write

$$s = s_0 + is_1\tau_q, \quad (2.72)$$

where $s_0$ and $s_1$ are independent of the expansion parameter $\tau_q$. In Appendix A.2, it is shown how the density-density response function may be calculated in the LWL limit. There two non-linear equations for the coefficients $s_0$ and $s_1$ are obtained from the dispersion relation (2.67), namely (13.20) and (13.27). The first equation [see (13.20)] has one obvious solution $s_0 = s_0^{(0)} = 0$ and two others $s_0 = \pm s_0^{(1)}$ with the same modulus,

$$s_0^{(1)} = \sqrt{\frac{g_0g_1\mathcal{N}(0)}{3N(T)}} \left( 1 + \frac{\pi^2T^2}{3g_0} \right),$$

Substituting $s_0^{(0)}$ and $s_0^{(1)}$ into the second equation (13.27), one finds the two solutions for $s_1$, $s_1^{(0)}$ and $s_1^{(1)}$, respectively. These solutions for $s$ (2.72) can be written in terms of the dimensional frequency $\omega$ by means of
and corresponds to the overdamped excitations of the first root \( \omega \approx \frac{1}{2} \), \( \Gamma \approx s_1 q \), \( \Gamma(0) = s_1 q \), \( \omega_0 = s_1 q \), \( \omega_\pm = \pm \omega_0 - i \frac{\Gamma(1)}{2} \) with \( \omega_0 = s_1 q \), \( \omega_\pm = \pm \omega_0 - i \frac{\Gamma(1)}{2} \), \( \omega_\pm = \pm \omega_0 - i \frac{\Gamma(1)}{2} \)

where \( \Gamma(0) = s_1 q \), \( \Gamma(0) = s_1 q \), \( \Gamma(0) = s_1 q \), \( \Gamma(0) = s_1 q \)

The first root \( \omega(0) \) given in (2.74) is purely imaginary and corresponds to the overdamped excitations of the hydrodynamic Raleigh mode [114, 115]. The second and third ones \( \omega_\pm \) correspond to the usual first sound mode, expressed in terms of the (macroscopic) parameters of viscosity and thermal conductivity of normal liquids [115, 116]. In (2.73) and (2.76), small corrections of the order of the product of the two small quantities \( T^2 \) and \( \gamma_0 \) have been neglected, along with \( T^2 |(m* - m)/m| = T^2 \gamma F_1 \). This procedure should be valid for nuclear matter; where the relevant parameters are small, both \( |F_0| \) and \( |(m* - m)/m| \) being of order \( \approx 0.2 \). Discarding such small corrections, our results for the sound frequencies \( \omega_\pm \) in (2.75) are in agreement with [57]. In particular, up to these small corrections, the volume (or second) viscosity disappears, as it is the case in [57]. In the expressions (2.74) to (2.76) more explicit temperature corrections are given for \( \omega(0) \) and \( \omega_\pm \) than those discussed in [57]. This will turn out important for the thermal conductivity \( \kappa \), which we shall address in Sec. 11.3 [see (2.93)]. The “widths” \( \Gamma(0) \) and \( \Gamma(1) \) are proportional to \( \tau_\kappa \), and thus, to the relaxation time \( \tau \) which represents the effects of two-body collisions. For nuclear matter, the Landau parameters \( F_0 \) and \( F_1 \) are small \( [\gamma_0 \text{ and } \gamma_1 \text{ are close to unity, see (2.8)] \} \). This reason, according to the last equation in (2.78), the sound velocities cannot be large [see the approximation (2.79)]. So, the LWL limit (\( \tau_\kappa \approx 1 \)) may be identified with the hydrodynamic collision regime \( \omega \tau = s_1 \tau_\kappa \approx 1 \). Note that for the Fermi liquid \( ^3 He \), for instance, the parameters \( \gamma_0 \) and \( \gamma_1 \) are large and second order equation of (2.79) can not be applied. Moreover, according to the first line in (2.73), the sound velocity is large. Therefore, in this case a smallness \( \tau_\kappa \) does not mean yet that \( \omega \tau \) is also small, i.e., the LWL condition is not enough for the hydrodynamical collision regime.

2. RESPONSE FOR INDIVIDUAL MODES

In the following, we are going to examine the collective response function \( \chi_{DD}^{\text{coll}} \) (2.65), in particular its behavior in the neighborhood of the individual modes given by (2.74) and (2.75). To simplify the notation, we shall at times omit the lower index “DD” and move down the upper index “coll”. Near any of the sound poles \( \omega_0 \), given in (2.75), the collective response function \( \chi_{DD}^{\text{coll}} \) may be written as

\[
\chi_{\text{coll}}^{(1)}(q, \omega) = \frac{1}{2} \left[ \frac{\Gamma(1)}{(\omega - \omega_0)^2 + (\Gamma(1))^2/4} - \frac{\Gamma(1)}{(\omega + \omega_0)^2 + (\Gamma(1))^2/4} \right]
\]

(2.78)

and

\[
\chi_{\text{coll}}^{(1)}(q, \omega) = \frac{1}{2} \left[ \frac{\omega + \omega_0}{(\omega + \omega_0)^2 + (\Gamma(1))^2/4} - \frac{\omega - \omega_0}{(\omega - \omega_0)^2 + (\Gamma(1))^2/4} \right].
\]

(2.79)

Notice that for \( \tau_\kappa = +0 \) the Lorentzians in (2.78) turn into \( \delta \)-functions.

The relaxation time \( \tau_\kappa \), which determines the dimensionless quantity \( \tau_\kappa = \tau_\kappa q \), might depend on temperature and frequency. A useful form is found in

\[
\tau = \frac{\tau_\kappa c_0 (\hbar \omega)^2}{T^2 c_0 (\hbar \omega)^2} \approx \frac{\tau_\kappa}{T^2} \text{ for } c_0 (\hbar \omega)^2 \ll T^2,
\]

(2.80)
\[\Gamma^{(1)} = \frac{4\tau_a v_s^2 q^2}{15T^2} \left[ 1 + \frac{5\pi^2 T^2}{12} \left( 1 + \frac{1}{9q_0 q^2} \right) \right]. \quad (2.81)\]

To leading order, this gives the expected dependence on temperature commonly associated to hydrodynamics, namely \(\Gamma^{(1)} \propto 1/T^2\).

Finally, we may note that in the long-wave limit the effective, total in-compressibility \(K_{\text{tot}}^{(2.81)}\) becomes identical to the adiabatic in-compressibility \(K^c\) (A.7) specified in Appendix A,

\[K^c = K^T \left( 1 + \frac{4TC_V}{K^T} \right), \quad (2.82)\]

see (2.7) for the isothermal in-compressibility \(K^T\) and (A.29) at small temperatures. For the derivation of this identity, it is more easy to consider variations \(\delta T\) and \(\delta \rho\) as caused formally by some external field \(V_{\text{ext}}\). Then, one can represent \(\delta T/\delta \rho\) in (2.81) in terms of the ratio of the temperature-density \(\chi_{GD}(\tau_q, s)\) (2.63) to density-density \(\chi_{DD}(\tau_q, s)\) (2.63),

\[\frac{\delta T}{\delta \rho} = \frac{\delta T/V_{\text{ext}}}{\delta \rho/V_{\text{ext}}} = \frac{\chi_{GD}(\tau_q, s)}{\chi_{DD}(\tau_q, s)}. \quad (2.83)\]

Using then the LWL expansions (B.22) and (B.23) up to the third order terms in \(\tau_q\), from (2.83) we get

\[\frac{\delta T}{\delta \rho} \approx \frac{T}{N(0)} \left( 1 - \frac{i\tau_q}{3s_0^{(1)}} \right) \quad \text{for} \quad \tau_q \to 0, \quad (2.84)\]

where \(s_0^{(1)}\) was defined in (2.73). As the specific heat \(C_V\) (A.27) is proportional to \(T\), we need in (2.84) only linear terms to get the temperature correction of the second order in \(T\) in the total in-compressibility \(K_{\text{tot}}\) (2.81). Substituting (2.81), (A.27) and (A.28) into (2.81) for the total in-compressibility \(K_{\text{tot}}\), one obtains identically the same as in (A.29) for the adiabatic in-compressibility \(K^c\). The same result (2.81) in the LWL limit can be obtained also from (B.15).

Let us address now the pole at \(\omega = 0\) [see (2.73)]. Near the latter, the collective response function \(\chi_{\text{coll}}^{(0)}\) begins [as may be checked with the help of (B.24), (B.25), (2.57) and (2.72)]

\[\chi_{\text{coll}}^{(0)} = \frac{ia^{(0)}}{\omega - \omega(0)} = \frac{ia^{(0)}}{\omega + \frac{\omega^2}{2}} \quad \text{with} \quad a^{(0)} = \frac{\pi^4 T^2 \tau_q (8 - 3G_0)}{108G_0^2} v_s q N(0), \quad (2.85)\]

and \(\Gamma^{(0)}\) being defined in (2.74). It may be rewritten in a more traditional form, see [114], [115] and [116]. Introducing the "diffusion coefficient"

\[D_T = \frac{\kappa}{C_P \rho} = \frac{\tau_T^2}{3} \left[ 1 - \frac{\pi^2 T^2 (80 - 29G_0)}{120G_0} \right] \quad (2.86)\]

\[\left[ D_T = \Gamma^{(0)}/(2q^2) \right], \quad \text{one gets} \quad \chi_{\text{coll}}^{(0)} = \frac{iD_T q^2}{\omega + iD_T q^2} \chi_{\text{coll}}^{(0)}(q, \omega = 0). \quad (2.87)\]

Note that according to (2.71) and (2.80), the temperature dependence of \(\Gamma^{(0)}\) becomes similar to the one found in (2.81),

\[\Gamma^{(0)} = \frac{2\tau_c v_s^2 q^2}{3T^2} \left[ 1 - \frac{\pi^2 T^2 (80 - 29G_0)}{120G_0} \right]. \quad (2.88)\]

For the dissipative and reactive parts of the response function \(\chi_{\text{coll}}^{(0)}\), from (2.87) one gets

\[\chi_{\text{coll}}^{(0)}'(q, \omega) = a^{(0)}(\omega^2 + (\Gamma^{(0)})^2/4), \quad \chi_{\text{coll}}^{(0)}''(q, \omega) = a^{(0)}(\omega^2 + (\Gamma^{(0)})^2/2). \quad (2.89)\]

The strength distribution \(\chi_{\text{coll}}^{(0)}''\) has a maximum at \(\omega = \Gamma^{(0)}/2\) and a width \(\Gamma^{(0)}/2 \propto \tau_q\). In the LWL limit \(\tau_q \ll 1\) this distribution becomes quite sharp with the maximum lying close to \(\omega = 0\). As may be inferred with the help of (2.85) and (2.77), the maximal value does not depend on \(\tau_q\) and is proportional to \(T^2\). It will be demonstrated shortly that the pole at \(\omega = 0\) (2.74) is related to the heat conduction, for which reason it sometimes is called "heat pole". Notice that the reactive response function \(\chi_{\text{coll}}^{(0)}'\) is finite at \(\omega = 0\), with a value independent of \(\tau_q\).

In the hydrodynamic regime with \(\tau_q \ll 1\), the response function \(\chi_{\text{coll}}^{(0)}\) found for the Fermi liquid becomes identical to the one for normal liquids [115, 116]. This can be made more apparent after introducing the dimensional sound velocity \(c\), a width parameter \(\Gamma\), determined as

\[c = v_s s_0^{(1)}(1), \quad \Gamma = \Gamma^{(1)}/q^2, \quad (2.90)\]

as well as the diffusion coefficient \(D (2.86)\) and the specific heats. The sum of the two contributions discussed above may then be written as

\[\chi_{\text{coll}}'' = \rho \left( \frac{\partial \rho}{\partial \rho} \right)_T \left[ \frac{(C_V/C_P)\epsilon^2 q^4 \Gamma \omega}{(\omega^2 - c^2 q^4)^2 + (\omega q^2 \Gamma)^2} + \frac{(1 - C_V/C_P) q^2 |D_T | \omega}{\omega^2 + (q^2 D_T)^2} \right]. \quad (2.91)\]

Traditionally, the peaks related to the first and second terms are called Brillouin and Rayleigh (or Landau–Placzek) peak, respectively. The ratio of the specific heats \(C_P\) and \(C_V\) per particle is discussed in Appendix A.1, see (A.22) and (A.32). Note that the sound speed \(s_0^{(1)}\), see (2.73), is identical to the adiabatic sound velocity found in Appendix A.1, see (A.31) (c in dimensional units for normal liquids), as it should be for normal liquids [115, 116]. The structure of (2.71) is identical to that discussed in the literature (see, e.g., (4.44a) of [115], if...
one only expresses the quantities appearing here in terms of viscosity and thermal conductivity. As a matter of fact, the alert reader might expect a third term (as in (4.44a) of [113]), but this one is of the order of $\tau_q^2$ and thus is neglected here. The specific temperature dependence of these parameters (in the LWL limit) will be discussed in the next subsection, with respect to the specific heats, see also Appendix A.1.

Note that in the derivation of the both amplitudes $a^{(0)}$ (2.85) and $a^{(1)}$ (2.77) we took $D(s)$ (2.69) at low temperatures using (2.69) to (2.71); and then, expand first it near the poles (2.74) and (2.73), respectively; and second, in small $\tau_q$ of the LWL limit. This way of the calculation is much more simpler because the two last operations can be exchanged only when we shall take into account next order terms in $\tau_q$, that takes much hard work. If we exchange the last two operations, expanding first in $\tau_q$ in the linear LWL approximation (2.72), and then, doing expansion near the poles, some important terms will be lost.

3. SHEAR MODULUS, VISCOSITY AND THERMAL CONDUCTIVITY

As explained in Appendix A.2, these coefficients may be obtained by applying expansions to $\chi_n$ within the perturbation theory mentioned above, for low temperatures (with $T \ll 1$); see in particular [121], [122] and [123]. They specify the stress tensor $\sigma_{\alpha\beta}$ (2.25)-(2.27) and the heat current $j_r$ (2.39).

The shear modulus $\lambda$ (3.12) in the time-reversible part $\sigma^{(x)}_{\alpha\beta}$ (2.29) of the stress tensor $\sigma_{\alpha\beta}$ (2.25) turns into zero in the long wave-length approximation linear in $\tau_q$ as in (57) up to immaterial corrections of the order of $T^4$. By another words, in this case, $\lambda$ is a small quantity of the order of $\tau_q^2$ because such corrections were neglected everywhere. It means a disappearance of the Fermi-surface distortions in our linear approach (2.72) which are the main peculiarity of Fermi liquids compared to the normal ones.

For the shear viscosity $\nu$ (3.13) taken at the first sound frequency $\omega = \omega_0^{(1)}$ (2.76), one obtains

$$\nu = \nu^{(1)} + \nu^{(2)},$$

where

$$\nu^{(1)} = \frac{2}{5} \rho \varepsilon \tau \left(1 + \frac{5\pi^2 T^2}{12}\right)$$

(2.93)

and

$$\nu^{(2)} = \frac{13\pi^4 \rho \varepsilon T^4}{720 \varepsilon q^2 \tau}.$$

(2.94)

The first term $\nu^{(1)}$ (2.93) in (2.92) is proportional to the relaxation time $\tau$ and coincides mainly with that obtained earlier for mono-atomic gases and for a Fermi liquid by using another method [57], except for the specific explicit dependence on temperature presented here. The temperature dependence of the shear viscosity $\nu^{(1)}$ (2.93) is mainly the same as for the rate of the sound damping $\Gamma^{(1)}$ (2.81), $\nu^{(1)} \propto 1/T^2$, with the temperature dependence of the relaxation time $\tau$ (2.88). Although the viscosity component $\nu^{(2)}$, too, is related to the first sound solution $\omega_0^{(1)}$, it is proportional to $1/\tau$, similar to the viscosity of zero sound but in contrast to the standard first sound viscosity $\nu^{(1)}$. The $\nu^{(2)}$ component (2.94) of the viscosity (2.92) increases with temperature as $T^3$, thus it may be considered small under usual conditions, it may become important for small wave numbers $q$ (or frequencies $\omega$) [for more details, see the discussion to come below in Sec. III C 2]. This component of the viscosity was not discussed in [57].

Let us finally turn to the thermal conductivity $\kappa$ which shows up in the equation for variations of temperature $T(r, t)$ with $r$ and $t$ [see Appendix A.2]. The form (3.20) for the heat mode $\omega = \omega_0^{(1)}$ of (2.74) may be rewritten as

$$\kappa = \frac{C_P \Gamma^{(0)}}{2q^2} \approx \frac{1}{3} \rho C_P T^2 \left[1 - \frac{\pi^2 T^2}{120 G_0} \right]$$

(2.95)

We present here also explicitly the temperature corrections up to the terms of the order of $T^2$. Our expression for the thermal conductivity $\kappa$ (2.95) differs from the one found in [115] and [57] by small $T^2$ corrections. However, they are not important in the calculations of the damping coefficient $\Gamma^{(1)}$ for the first sound mode defined in [115], and [116], see also the comment before (2.91),

$$\Gamma^{(1)} = \frac{q^2}{m \rho} \left[4 \frac{\nu^{(1)}}{3} + \frac{m \kappa}{C_P \left(C_v - 1\right)} \right]$$

(2.96)

Here, $\nu^{(1)}$ is the part of the shear viscosity coefficient related to the first sound mode, see (2.93): $C_P/C_v$ is the adiabatic ratio of the specific heats, see (3.3) and (A.10). We omitted here corrections related to the second viscosity in line of the second approximation in (2.70). In (2.96), $\kappa$ is multiplied by a small quantity of the order of the $T^2$ as follows from (A.32) and the temperature corrections to $\kappa$ written explicitly in (2.93) can be neglected in (2.96). The expression for the damping coefficient $\Gamma^{(1)}$ (2.96) with the viscosity coefficient $\nu^{(1)}$ (2.93), thermal conductivity $\kappa$ (2.95) and specific heats from (A.32) and (A.33) for viscose normal liquids is in agreement with our result for $\Gamma^{(1)}$ (2.74) including the temperature corrections.

Thus, up to the temperature corrections discussed above, we have agreement with the results of [57] for the dispersion equation, viscosity and thermal conductivity coefficients in the hydrodynamic limit. Our derivations are more strict and direct within the perturbation theory for the eigenvalue problem. We have the transition to the
hydrodynamics of normal liquids discussed in [114] and [116] in terms of the macroscopic parameters mentioned above.

E. Susceptibilities

In this section, we want to address the calculation of the static susceptibilities, for which one distinguishes isolated, isothermal and adiabatic ones [114] [116]. Their comparison is relevant for ergodicity properties, see [114] [116]. Here we will concentrate on the density mode of nuclear matter considered as an infinite Fermi-liquid system.

1. ADIABATIC AND ISOThERMAL SUSCEPTIBILITIES

The isolated susceptibility $\chi_{\text{DD}}(0)$ is defined as the static limit of the response function $\chi_{\text{DD}}(q,\omega)$ (or $\chi_{\text{DD}}(\tau_q,s)$ of (2.63) in dimensionless variables), for which one first has to take the limit $q \to 0$ (or $\tau_q \to 0$), and then, $\omega \to 0$ (or $s \to 0$) (see, e.g., [113]).

$$\chi_{\text{DD}}(0) = \lim_{\omega \to 0} \left[ \lim_{q \to 0} \chi_{\text{DD}}(q,\omega) \right] = \lim_{s \to 0} \left[ \lim_{\tau_q \to 0} \chi_{\text{DD}}(\tau_q,s) \right].$$  (2.97)

Apparently, $\chi_{\text{DD}}(0)$ satisfies the relation

$$\delta \rho = -\chi_{\text{DD}}(0)\delta V_{\text{eff}},$$  (2.98)

where $\delta V_{\text{eff}}$ and $\delta \rho$ are quasistatic variations. They can be considered as independent of time, in contrast to the ones discussed in Sec. IV C 1 see [256].

The isothermal susceptibility $\chi_{\text{DD}}^T$ is defined as the density-density response at constant temperature $T$, and the adiabatic one, $\chi_{\text{DD}}^\omega$, as that at constant entropy (per particle $\zeta$). Suitable variables for studying the variations of the density $\rho$ are therefore pressure $P$ and temperature $T$ in the first case, and pressure $P$ and entropy per particle $\zeta$ in the second one. These two representations of $\delta \rho$ can be written as

$$\delta \rho = \left( \frac{\partial \rho}{\partial P} \right)_T \delta P + \left( \frac{\partial \rho}{\partial T} \right)_P \delta T = \left( \frac{\partial \rho}{\partial \mu} \right)_\zeta \delta \mu + \left( \frac{\partial \rho}{\partial \zeta} \right)_P \delta \zeta.$$  (2.99)

For the isothermal and adiabatic susceptibilities $\chi_{\text{DD}}^T$ and $\chi_{\text{DD}}^\omega$, one thus gets the following two relations:

$$\frac{\chi_{\text{DD}}^T}{\chi_{\text{DD}}^\omega} = \frac{K^\zeta}{C_T} = \frac{\chi_{\text{DD}}^T(0)}{\chi_{\text{DD}}^\omega(0)}.$$  (2.100)

and

$$\frac{\chi_{\text{DD}}^T}{\chi_{\text{DD}}^\omega} = \frac{K^\zeta}{C_T} = \frac{\chi_{\text{DD}}^T(0)}{\chi_{\text{DD}}^\omega(0)}.$$  (2.101)

This is a general relation from thermodynamics where we have not only replaced the system’s total entropy [110] by the entropy per particle $\zeta$ applied for the intensive systems as normal and Fermi liquids.

We are interested more in the calculation of the differences between the isothermal susceptibility $\chi_{\text{DD}}^T$ defined by the relations in (2.100) (or adiabatic one $\chi_{\text{DD}}^\omega$, see (2.101)) and isolated (static) susceptibility $\chi_{\text{DD}}(0)$ presented by (2.98) [29]. For this purpose, we find first the ratio of the isothermal-to-isolated susceptibilities $\chi_{\text{DD}}^T/\chi_{\text{DD}}(0)$ in terms of the ratio of the static “intrinsic” temperature-density response function to the isolated one $\chi_{\text{DD}}(0)$ (2.63). The static temperature-density susceptibility $\chi_{\text{DD}}^T$ is defined in the same way (2.97) as the static limit of the “intrinsic” temperature-density response function $\chi_{\text{DD}}^T(\tau_q,s)$ given by (2.64). Note that the limits $\omega \to 0$ (or $s \to 0$) and $q \to 0$ (or $\tau_q \to 0$) which we consider to get the static response functions are not commutative [113]. Taking the second equations in (2.100) and (2.98) for the intensive systems as liquids, one gets

$$\frac{\chi_{\text{DD}}^T}{\chi_{\text{DD}}(0)} = 1 + \left( \frac{\partial \rho}{\partial \mu} \right)_T \left( \frac{\partial \rho}{\partial \zeta} \right)_P \chi_{\text{DD}}^T(0).$$  (2.103)

We used here the definitions (2.63) and (2.64) for the density-density and temperature-density response functions and (2.97) for their static limits $\chi_{\text{DD}}(0)$ and $\chi_{\text{DD}}^T(0)$. We then applied the thermodynamic relations of Appendix A.1 for the transformations of the derivative $\left( \frac{\partial \rho}{\partial P} \right)_T$. This derivative appears from the definition of the isothermal susceptibility $\chi_{\text{DD}}^T$ in (2.100) to another simpler thermodynamic derivatives for the application to Fermi liquids, see below. For this aim, we transform the variables $(T,P)$ to the new ones $(T,\mu)$. The derivatives of pressure $P$ over these two new variables can be then reduced to the ones of the density $\rho$ shown in the r.h.s. of (2.100) with the help of (A.18).

So, the calculations of the susceptibilities are resulted in the derivation of the static limits defined by (2.97) for the temperature-density $\chi_{\text{DD}}^T(\tau_q,s)$ and density-density $\chi_{\text{DD}}(\tau_q,s)$ response functions, see (2.63) and (2.64), and their ratio $\chi_{\text{DD}}^T(0)/\chi_{\text{DD}}(0)$ for the case of a heated Fermi liquid. We can then calculate the two ratios of the susceptibilities (2.103) and (2.102), which both determine separately each considered susceptibilities.

2. FERMI-LIQUID SUSCEPTIBILITIES

The expression for the ratio of the isothermal-to-static susceptibilities (2.103) can be simplified my making use
of the specific properties of Fermi liquids given by (A.17) and second equation in (A.18),
\[
\frac{\chi^{(c)}_{DD}(0)}{\chi_{DD}(0)} = 1 + \left( \frac{C_F}{C_V} - 1 \right) \frac{\chi_{DD}(0)}{T \chi_{DD}(0)} N(0).
\] (2.104)

According to the definition (2.97) of the static response functions applied to the ones in the ratio $\chi_{DD}(0)/\chi_{DD}(0)$ of (2.104), we shall use (2.63) and (2.65) for the corresponding intrinsic susceptibilities ($F_0 = F_1 = 0$ there). The static limit (2.97) of the response functions $\chi_{DD}(\tau_q, s)$ and $\chi_{DD}(\tau_q, s)$ in (2.104) can be found by using the LWL expansions over a small parameter $\tau_q \ll 1$ at low temperatures, see Sec. [114] and Appendix A.2 for the first limit ($\tau_q \to 0$) in (2.97). We substitute now the perturbation theory expansions for small $\tau_q$ for the quantities $\chi_{DD}(\tau_q, s)$ and $\chi_{DD}(\tau_q, s)$ in (2.104). We get this limit as functions of $s_0$ and $s_1$, and then, we shall take the second limit of $s_0 \to 0$ and $s_1 \to 0$ (\(s \to 0\) in (2.97)). Finally, we arrive then at the very simple result
\[
\frac{\chi_{DD}(0)}{\chi_{DD}(0)} = \frac{T}{N(0)}
\] (2.105)

neglecting small cubic terms in $T$, which correspond to $T^3$ corrections in susceptibilities and do not matter in this section. Note that the sequence of the limit transitions defined in (2.97) and recommended in [114] is important for the calculation of this ratio: We get zero for this ratio if we take first $s \to 0$, and then, $\tau_q \to 0$.

Substituting now the ratio (2.105) of the susceptibility into (2.104), one obtains
\[
\frac{\chi^{T}_{DD}(0)}{\chi_{DD}(0)} = \frac{C_F}{C_V} = 1 + \frac{\pi^2 T^2}{3 G_0},
\] (2.106)

see also (A.52) for the second equation. We compare now this result with (2.102) and get that our Fermi-liquid system satisfies the ergodicity property:
\[
\chi^{(c)}_{DD} = \chi_{DD}(0).
\] (2.107)

This ergodicity property was proved at low temperatures, for which the Landau Fermi-liquid theory can be applied. It is related to the adiabaticity of the velocity of the sound mode $s_0^{(c)}$, see (2.73) and discussion after (2.51). Moreover, we got the normal liquid (hydrodynamic) limit from the Fermi-liquid dynamics, and therefore, the ergodicity property is general for heated Fermi liquids and normal (classical) ones.

Another aspect of the discussed ergodicity property might be the relation to the non-degeneracy of the excitation spectrum in the infinite Fermi liquids beside of the spin degeneracy. We have only the two-fold degenerate quasiparticle states, due to the spin degeneracy. However, it does not influence on our results concerning the ergodicity relations because we consider the density-density excitations, which do not disturb the spin degree of freedom. We have only the multiplication factor two in all susceptibilities, due to the spin degeneracy, that does not change the ratios of the susceptibilities which are only important for the ergodicity discussed here.

Our susceptibilities obtained above satisfy the Kubo relations, see (4.2.32) of [114]:
\[
\chi^T \geq \chi^c \geq \chi(0)
\] (2.108)

with the equal sign for the second relation because of the ergodicity property. To realize this, we should take into account that $C_F > C_V$ (or $\zeta^S > \zeta^T$), according to (A.22), because all quantities on the r.h.s. of this equation are positive for the stable modes $G_0 = 1 + F_0 > 0$. The equal sign for the first relation in (2.108) becomes true in the two limit cases: For the temperature $T$ going to 0 or for the incompressible matter when the interaction constant $F_0$ tends to $\infty$. In both limit cases we made obvious equality $C_F = C_V$ and all susceptibilities are identical [equal signs in both relations of (2.108)].

Note that now that namely the specific Fermi-liquid expression of the static susceptibility $\chi_{DD}(0)$, see (2.104) with $F_0 = F_1 = 0$ for the case of the intrinsic response functions, depends on the sequence of the limit transitions discussed near (2.97), (2.103), (2.105) above and in [115]. For the definition (2.97) of [115], one gets
\[
\chi_{DD}(0) = \left( 1 - \frac{5\pi^2 T^2}{12} \right) N(0), \quad \chi_{DD}^T = N(T).
\] (2.109)

In the last equation, we used also (2.106). Taking the opposite sequence of the limit transitions, first $s \to 0$, one has the result $N(T)$ (2.97) for the isolated susceptibility $\chi_{DD}(0)$ like for the isothermal one $\chi_{DD}(0)$. The difference is in $T^2$ corrections. Ignoring them, the both versions of the limit transitions coincide, and we come to the result independent on temperature discussed in [36]. The ergodicity property (2.107), Kubo’s relations (2.108) and relation of the isothermal susceptibility to adiabatic one (2.102) do not depend on the specific peculiarities of the static limit of the response function discussed here in connection to Fermi liquids.

F. Relaxation and correlation functions

1. Relaxation function

Coming back to the dynamical problem, we note that the dissipative part of the response function $\chi''(\omega)$ is related to the relaxation function $\Phi''(\omega)$ [114] by
\[
\chi'' = \omega \Phi''(\omega).
\] (2.110)

We follow the notations of [24, 29] and omit the index “coll” in this section. For the comparison with the microscopic results of [29] we need really the relaxation and correlation functions related to the intrinsic response
functions. According to (2.72) and (2.77), all these intrinsic functions can be formally obtained from the collective ones at the zero Landau constants \( F_0 \) and \( F_1 \). Taking into account also (2.73) and (2.89), one has

\[
\Phi''(\omega) = \frac{a^{(1)}_{(0)}}{2\omega^{(0)}_0} \left[ \frac{\Gamma^{(1)}}{(\omega - \omega^{(1)}_0)^2 + (\Gamma^{(1)})^2/4} \right]
+ \frac{\Gamma^{(1)}}{(\omega + \omega^{(1)}_0)^2 + (\Gamma^{(1)})^2/4} + \frac{a^{(0)}}{\Gamma^{(0)}_0} \frac{\Gamma^{(0)}}{\omega^2 + (\Gamma^{(0)})^2/4}.
\]  

(2.111)

This equation can be re-written in the same way like to (2.391) in terms of the parameters \( c, \Gamma \) and \( D_T \), see (2.391) and (2.390),

\[
\Phi''(\omega) = \chi' \left[ \frac{(C_V/C_P)C^2 q^4 T}{(\omega^2 - C^2 q^2)^2 + (\omega q^2 T)^2} \right]
+ \frac{1 - C_V/C_P}{\omega^2 + (q^2 D_T)^2}.
\]  

(2.112)

We used here the Jacobian relations and (A.3), (2.7) for the transformation of the coefficient in front of the square brackets in (2.391) to the one, the intrinsic isothermal susceptibility \( \chi' \) (2.109) \((F_0 = 0)\). We also neglected terms of the order of \( \tau_q \) as in the derivation of (2.391). Equation (2.112) for the relaxation function \( \Phi''(\omega) \) is identical to the imaginary part of the r.h.s. of (28.29) in [117] with the transparent physical meaning as (2.411). The first term in the square brackets of (2.111) and (2.112) is the first sound Brillouin component with the poles (2.75) associated to the finite frequencies \( \pm \omega^{(1)}_0 \) of the time-dependent relaxation-function oscillations and their damping rate \( 1/\Gamma^{(1)} \) \((\pm \omega_q \) and \( 1/\gamma_q \) in the notation of [110], respectively, see more complete discussion of properties of the time-dependent relaxation function as a Fourier transform of the relaxation function \( \Phi(\omega) \) in [110]). The second term in (2.111) and (2.112) describes the pure damped Raleigh mode corresponding to the overdamped pole \( \omega^{(0)}_0 \) (2.47) defined by the diffuseness coefficient \( D_T \propto \Gamma^{(0)}_0 \) \((\propto \gamma_q \) in the notation of [110]). As noted in [110], the strength of this peak is a factor \( 1 - C_V/C_P \) smaller than for the two sound peaks. According to (A.32), in the zero temperature limit \( T \to 0 \), the Raleigh peak disappears but the Brillouin ones become dominating because of \( \Gamma \propto \Gamma^{(1)} \propto 1/T^2 \); see the second equation of (2.50) for the relation of \( \Gamma \) to \( \Gamma^{(1)} \) and (2.51). Note also that the coefficient in front of the square brackets in (2.111) is finite in the limit \( T \to 0 \).

2. CORRELATION FUNCTION

We like to present also the correlation function, partly for the sake of completeness and partly to allow for comparisons with calculations of the function in the nuclear SM approach of [28, 30], see also [24, 29], to the collective motion of finite nuclei. Let us use now the fluctuation-dissipation theorem [114] to get the correlation function \( \psi''(\omega) \),

\[
\psi''(\omega) \to \hbar \omega \coth \left( \frac{\hbar \omega}{2T} \right) \Phi''(\omega) = \hbar \coth \left( \frac{\hbar \omega}{2T} \right) \chi''(\omega).
\]

(2.113)

In the semiclassical limit \( \hbar \to 0 \) considered here, one has

\[
\psi''(\omega) = \frac{2T}{\omega} \chi''(\omega) = 2T \Phi''(\omega)\,.
\]

(2.114)

According to (2.91), (2.112), this correlation function can be split into the two components as in (2.91)

\[
\psi''(\omega) = \psi''_R(\omega) + \psi''_H(\omega)\,.
\]

(2.115)

Here, \( \psi''_R(\omega) \) is the heat pole part,

\[
\psi''_H(\omega) = \frac{2T}{\omega} \chi'(0)''(\omega) = 2T \chi' \left( \frac{(C_V/C_P)q^2T}{\omega^2 - q^2T^2} + \frac{T^2}{T^2} \right).
\]

(2.116)

\( \chi'(0)''(\omega) \) is given by the first equation in (2.80) and is related to the second heat pole terms in the square brackets of (2.91) and (2.112) \([\text{through (2.110)}]\). This part is singular at the zero frequency point \( \omega = 0 \) for \( \tau_q \to 0 \), see (2.86) and (2.47). The other term \( \psi''_H(\omega) \) is associated with the first sound component in the square brackets of (2.91) if (2.112),

\[
\psi''_R(\omega) = \frac{2T}{\omega} \chi'(0)''(\omega) = 2T \chi' \left( \frac{(C_V/C_P)q^2T}{\omega^2 - q^2T^2} \right).
\]

(2.116)

This component has no such singularity at \( \omega = 0 \) for \( \tau_q \to 0 \), as seen from (2.91), (2.75) and (2.81) \([\text{see (2.75) for } \chi'(0)''(\omega) \text{ in the middle of (2.411)}]\). According to the second equation in (2.113), the heat pole part \( \psi''_H(\omega) \) of (2.115) for the intrinsic correlation function can be written as in (2.91)

\[
\psi''_H(\omega) = \psi''_H(0) = \frac{\hbar \Gamma_T}{(\hbar \omega)^2 + \Gamma_T^2/4}.
\]

(2.118)

where

\[
\Gamma_T = 2\hbar q^2 D_T = \hbar \Gamma^{(0)}_0,
\]

(2.119)

and

\[
(1/T)\psi''_H(0) = \chi'' - \chi' = \chi''' - \chi(0).
\]

(2.120)
We applied here (2.102) in the first equation of (2.120) and ergodicity condition (2.107) for the second one. The specific expressions for the quantities Γ(0), χ′ and χ(0) in the last two equations (2.119) and (2.120) can be found in (2.73), (2.88) and (2.109). Note that the correlation function (2.118), corresponding to the heat pole, has the Lorentzian multiplier. This multiplier approaches the δ(ω) function in the hydrodynamic limit τq → 0 because of Γ → 0, according to (2.119) and (2.73) (Γ(0) → 0), i.e.,

\[ \psi(0)(\omega) = 2\pi\psi(0)(\omega) \delta(\omega) \quad \text{for} \quad \tau_q \rightarrow 0. \]  (2.121)

The relations (2.118), (2.120) and (2.121) confirm the smaller than the Fermi energy of the collective excitations at finite temperatures much better than the Fermi-system approximation for external field \( V_{\text{ext}} \)

The total Hamiltonian \( \hat{H}_{\text{tot}} \) is given by

\[ \hat{H}_{\text{tot}} = \hat{H} + q_{\text{ext}}(t) \hat{F}. \]  (3.8)

As shown in [24, 29], a conservation of the nuclear energy \( \langle \hat{H} \rangle \) for the Hamiltonian \( \hat{H} \) (3.6) leads to the equation of motion which is the secular equation in the Fourier representation,

\[ k^{-1} + \chi(\omega) = 0. \]  (3.9)

The coupling constant \( k \) is given by

\[ -k^{-1} = C(0) + \chi_{FF}(0), \]  (3.10)

where \( C(0) = \langle \partial^2 E(Q,S)/\partial Q^2 \rangle_{Q=0} \) is the stiffness coefficient of the internal energy \( E(Q,S) \) for the constant nuclear entropy, \( S_0 \), and \( \chi_{FF}(0) \) is the static (isolated) susceptibility. \( \langle \hat{F} \rangle_{\omega} \) and \( Q_\omega \) are related then each other by the self-consistency condition

\[ k\langle \hat{F} \rangle_{\omega} = Q_\omega. \]  (3.11)

with \( Q_\omega \) being the Fourier component of the collective variable \( Q(t) \). The ergodicity condition,

\[ \chi_{FF}(0) = \chi_{FF}^{\text{ad}}, \]  (3.12)
Thus, the intrinsic response function $\chi_{\text{response}}(\omega)$ is related to the collective response function $\chi_{\text{coll}}^{(\text{FLDM})}(\omega)$ through the relation (2.62) [4, 29]. Within the FLDM formulated below, it is simpler to derive first the collective response function $\chi_{\text{coll}}^{(\text{FLDM})}(\omega)$ by making directly use of the definition (3.3). For comparison with the microscopic quantum theory [29] and for study of the susceptibilities and of the ergodicity property, it is helpful to present the intrinsic response function $\chi(\omega)$ in terms of the collective response function $\chi_{\text{coll}}^{(\text{FLDM})}(\omega)$ found from (2.62) as

$$\chi_{\text{coll}}^{(\text{FLDM})}(\omega) = \frac{\chi_{\text{coll}}^{(\text{FLDM})}(\omega)}{1 - k\chi_{\text{coll}}^{(\text{FLDM})}(\omega)}.$$  

### B. Fermi-Liquid Droplet Model

In this section, we follow [31, 32] for the basic grounds of the FLDM [31, 32] for heavy nucleus taking into account the quasiparticle Landau–Vlasov theory for the collective dynamics of the heated Fermi liquids described in [31] and developed in the previous sections in more details for nuclear matter. The main idea is to apply this semiclassical theory for the distribution function inside the nucleus with the macroscopic boundary conditions [26, 37] like for normal liquids at its moving surface. These boundary conditions are used for the solutions of the dynamical collisional Landau–Vlasov equation (2.1), coupled with the thermodynamic relations for motion in the Fermi-liquid drop interior. Our derivations are based on the conception of the linearized dynamics near the local equilibrium instead of the global one considered earlier in [31, 32]. This is important for a low frequency region of the nuclear excitations which we are interested in this review.

We shall consider below small isoscalar vibrations of the nuclear surface near a spherical shape, which are induced by the external field $V_{\text{ext}}(t)$. To this end, we define a collective variable $Q(t)$ in the usual way:

$$R = R_0 [1 + Q(t) Y_{L0}(\hat{r})],$$  

where $R_0$ is the equilibrium radius of nucleus, and $Y_{L0}(\hat{r})$ is the spherical harmonics which represent the axially symmetric shapes as functions of the radius vector angles $\hat{r}$. For $Q(t)$ we expect the form

$$Q(t) = Q_\omega \exp(-i\omega t)$$  

with the same frequency $\omega$ as for the external field [31].

### 1. EQUATIONS OF MOTION INSIDE THE NUCLEUS

Quasiparticle conceptions of the Landau Fermi-liquid theory can be justified in the nuclear volume, where variations of the density $\rho(r, t)$ [21, 13] are small. Therefore, in the interior of sufficiently heavy nuclei, one may describe the semiclassical phase-space dynamics in terms of the distribution function $\delta f(r, p, t)$ (2.11) which satisfies the collisional Landau–Vlasov equation (2.1). We recall now the equations of Sec. II which present the collective dynamics linearized with respect to the local equilibrium (2.1). Our interior nuclear collective dynamics is then described by 6 equations, see (2.11) and (2.12), for the 6 local quantities $\delta \rho(r, t)$, $\delta \mu(r, t)$, $\mathbf{u}(r, t)$ and $\delta T(r, t)$ defined inside of the nucleus as for the nuclear matter. The conserving equations (2.18), (2.19) or (2.24) for a potential flow, (2.40) and (2.41) are helpful to find them in the semiclassical approximation.

For the isoscalar multipole vibrations of the Fermi-liquid drop surface [3.3], we shall look for the solutions of (2.1), (2.17) in terms of a superposition of the plane sound waves (2.28) over all angles $\hat{q}$ of the unit wave vector $\hat{q}$ with the amplitude $A_L(\hat{q})$.

$$\delta f(r, p, t) = \int d\Omega_\hat{q} A_L(\hat{q}) \delta f(q, p, \omega) \exp[i(qr - \omega t)]$$  

with $A_L(\hat{q}) = Y_{L0}(\hat{q})$. (3.17)

Here $L$ is the multipolarity of collective vibrations, $\hat{q}$ is the projection of the unit vector $\hat{q} = q/|q|$ on the symmetry $z$-axis. The Fourier amplitudes $\delta f(q, p, \omega)$ are presented as a spherical harmonic expansion in momentum space,

$$\delta \tilde{f}(q, p, \omega) = \left( \frac{\partial f_{\text{e.e.}}(p)}{\partial p} \right)_{\text{g.e.}} \sum_{l, \mu} A_l(\omega, q) Y_{l\mu}(\hat{p} \cdot \hat{q}),$$  

where $A_l$ are small vibration amplitudes. For such solutions, the velocity field $\mathbf{u}$ corresponds to the potential flow (2.12).

The relaxation time $\tau$ in (2.9) is assumed to be frequency and temperature dependent as in (2.80). Following (20, 31, 32), we take the form:

$$\tau(\omega, T) = \frac{\hbar}{\Gamma(\omega, T)},$$  

where

$$\Gamma(\omega, T) = \frac{\pi^2}{T_0^2} \left[ T^2 + c_0(\hbar \omega)^2 \right].$$  

For $c_0$ one has several values. For instance, $c_0 = 1/4\pi^2$, according to [34, 117], $c_0 = 1/\pi^2$ follows from [24, 36], $3/4\pi^2$ from [31] and several numbers near these constants were suggested in [55, 56]. Formula (3.20) with the $c_0 = 1/\pi^2$ and finite cut-off constant $c$ which weakens the dependence on both frequency $\omega$ and temperature $T$ at large values of these quantities may in some sense
be compared with the expressions suggested in [29] for the imaginary part of the self-energy to be used in microscopic computations [24, 29] [c in (5.20)] should not be confused with the sound velocity c used for the description of normal liquids [see, e.g., (2.90) and (2.91)]. In line of these computations, we shall use \( \Gamma_\alpha = 33.3 \) MeV and \( c = 20 \) MeV in our FLDM calculations. The value of the parameter \( c_0 = 3/4\pi^2 \) is taken as in [31, 32].

The specific value of this parameter is not important for the following derivations and results in this section because we shall apply the temperature-dependent Fermi-liquid theory for low frequencies and large temperatures. Note that for \( c \to \infty \) the expression (3.20) was derived in [31, 34, 36, 55, 56].

2. BOUNDARY CONDITIONS AND COUPLING CONSTANT

The dynamics in the surface layer of nucleus can be described by means of the macroscopic boundary conditions as in [37] by using the effective surface approximation [20, 27, 35]. For small vibration amplitudes, they read:

\[
\begin{align*}
    u_r \bigg|_{r=R_0} &= \dot{R}(t) \equiv R_0 \dot{Q}(t)Y_{L0}(\hat{r}), \\
    \Pi_{rr} \bigg|_{r=R_0} &= P_S + P_{ext},
\end{align*}
\]

where \( u_r \) and \( \Pi_{rr} \) are the radial components of the velocity field \( \mathbf{u} \) [24, 26] and the momentum flux tensor \( \Pi_{\alpha\beta} \) [22, 29], which are determined in the nuclear volume, see [38, 41, 43] for other (mirror and diffused) boundary conditions used directly for the distribution function as a solution of the Landau–Vlasov equation. In the case of the potential flow [24, 29], we shall use the specific expression for the momentum flux tensor \( \Pi_{\alpha\beta} \) [24, 26] with the shear modulus \( (\lambda) \) and viscosity \( (\nu) \) coefficients given by [B.12] and [B.13], respectively. The surface pressure \( P_S \), which is due to the tension forces for the isoscalar motion in symmetric nuclei, is given by

\[
P_S = \frac{\alpha}{R_0} (L - 1)(L + 2) Q(t)Y_{L0}(\hat{r}),
\]

where \( \alpha \) is the surface tension coefficient, see Sec. IV and Appendix D for the isovector asymmetric modes. For the tension coefficient \( \alpha \), we used an expression found in [27] within the ESA. This approximation is based on expansion of the nuclear characteristics, such as the particle density and the total energy in small parameter \( \alpha/R_0 \sim A^{-1/3} \), where \( \alpha \) is the diffuseness parameter and \( R_0 \) is the mean curvature radius of the nuclear surface [26, 27], see also [33, 40] and Appendix D. In this way, one derives the nuclear energy expansion [Wiezsäcker formula [D.8], (D.9)], \( E = E_V + E_S + ... \), with the volume part of the energy \( E_V \) proportional to the particle number \( A \), and the surface energy \( E_S = b_2 A^{2/3} \) (\( b_2 = 4\pi\gamma_0^2\alpha \) corresponds to the surface tension constant \( \alpha \), \( b_2 \approx 20 \) MeV, \( r_0 = R_0/A^{1/3} \approx 1.1-1.2 \) fm) and so on, see \[26, 27, 33, 40\] and Appendices A.A (symmetrical nuclei) and D (asymmetrical ones) for more details (the suffix “+” is omitted here). According to [17.7] of [21, 40],

\[
\alpha \approx 2C \int_0^\infty dr \left( \frac{\partial\rho_{qs}}{\partial r} \right)^2. \tag{3.24}
\]

Here and below we neglect the relatively small corrections of the order of \( A^{-1/3} \) of the ESA, which are in particular related to the semiclassical \( \hbar \) corrections and external field. The coefficient \( C \) appears earlier in front of the term which is proportional to \( (\nabla\rho_{qs}(r))^2 \) in the nuclear energy-density formula [see (D.1)], \( C = 40-60 \) MeV \( \cdot \) fm\(^5 \) [40].

An external pressure \( P_{ext} \) appears in (3.22), where we make connection to the external potential \( V_{ext} \) [31, 32, 44],

\[
P_{ext} = -\int_0^\infty dr \rho_{qs}(r) \frac{\partial V_{ext}}{\partial r}. \tag{3.25}
\]

For the density in equilibrium, one has

\[
\rho_{qs}(r) = \rho_0 w(\xi), \quad \xi = \frac{r - R}{a}, \quad a = \sqrt{\frac{C_+ \rho_\infty K}{30 b_2^2}}, \tag{3.26}
\]

This density is expressed in terms of the profile function \( w(\xi) \) with a sharp decrease from one to zero in the narrow region of the order of the diffuseness parameter \( a \) near \( \xi = 0 \) as in a step function \( (w(\xi) \to 0(R - r) \text{ for } a \to 0) \), \( b_V \approx 16 \) MeV is the separation energy per nucleon [26, 27, 37, 40]. The value of equilibrium density inside the nucleus \( \rho_0 \) [26] is given by

\[
\rho_0 = \rho_\infty \left( 1 + \frac{6b_2 r_0}{K R_0} \right), \tag{3.27}
\]

where \( \rho_\infty \) is the particle density of the infinite nuclear matter, \( \rho_\infty = 3/(4\pi r_0^3) \). The surface energy constant, \( b_2 \), and in-compressibility modulus, \( K \), in (3.27) depend on the condition of the constant temperature, entropy and of the static limit, as shown in Appendix C. In (3.27) and below, we omit the index \( X \) of these quantities which specifies one of these conditions, see Appendix C. For instance, the in-compressibility in (3.27) is denoted simply as \( K = K_{tot}(\omega = 0) = K^+ \), as shown above through [231], [281] and [D.29]. The surface energy constant \( b_2 \) in (3.27) is also identical to the adiabatic one as the in-compressibility (see Appendix C). The second term in the circle brackets of (3.27) is a small correction proportional to \( A^{-1/3} \), due to the surface tension. Boundary conditions [3.21] and [3.22] were re-derived here from [2.18] and [2.19] where all quantities are now extended to the surface region with a sharp coordinate
dependence of the particle density as in the approach \textsuperscript{37}. However, we used the specific properties of the heated Fermi-liquid drop following the same ESA \textsuperscript{26, 27, 37}. For the derivation of \textsuperscript{32}, e.g., the key equation (A.38) for the Gibbs thermodynamic potential per particle \(g\), which satisfies the thermodynamic relations (A.33), was applied instead of the entropy per particle \(s\) of \textsuperscript{37}. The result \textsuperscript{32} has the same form as in \textsuperscript{32, 37} because in its derivation we have simultaneously to use (A.37) of the temperature-dependent Fermi-liquid theory (with the entropy term \(Td_s\), in contrast to the adiabatic equation (17) of \textsuperscript{37}, see Appendix A.4 for details.

The external field \(V_{\text{ext}}\) in \textsuperscript{32} is determined by the operator \(\hat{F}(r)\) \textsuperscript{34}, and hence, \(V_{\text{ext}}\) is concentrated in the surface region of the nucleus. Indeed, for the operator \(\hat{F}(r)\) \textsuperscript{32} in the FLDM, one gets the form

$$\hat{F}(r) = \left( \frac{\partial V}{\partial \rho} \right)_{Q=0} = -R_0 \left( \frac{\partial V}{\partial r} \right)_{R=R_0} Y_{L0}(\hat{r}),$$

(3.28)

see \textsuperscript{34}. After substitution of (3.28) into (3.25) we have

$$P_{\text{ext}} = -\frac{1}{kR_0^3} q_{\text{ext}}(t) Y_{L0}(\hat{r}),$$

(3.29)

where

$$k^{-1} = \frac{K\alpha R_0^3}{18C\rho_\infty} \left[ 1 + \mathcal{O} \left( A^{-1/3} \right) \right] \approx \frac{K b_0 r_0^3}{54C} A^{4/3}. \quad (3.30)$$

The integration by parts in \textsuperscript{32} for the quasi-static coupling constant \(k^{-1}\) were used in the derivation of \textsuperscript{32, 33}, see the second equation of \textsuperscript{34}, and also applications to calculations of the collective vibration modes in \textsuperscript{142, 155}.

3. COLLECTIVE RESPONSE FUNCTION

As shown in \textsuperscript{26, 27, 14}, the linearized dynamic part of the nucleonic density \(\delta \rho(r, t)\) for the isoscalar modes can be represented as a sum of the "volume" and the "surface" term

$$\delta \rho(r, t) = \delta \rho^{\text{vol}}(r, t) w(\xi) - \frac{\partial w}{\partial r} \rho_0 \delta R,$$

(3.31)

where \(\delta R\) is the variation of nuclear radius \textsuperscript{43, 14}, \(\delta R = R_0 Q(t) Y_{L0}(\hat{r})\), \(w\) is defined around (3.26) and in Appendix D. For isovector vibration modes of the odd multipolarity (dipole), one has to account for the mass center conservation \textsuperscript{33, 22} [see \textsuperscript{43, 14}]. The upper index "vol" in \(\delta \rho^{\text{vol}}(r, t)\) of (3.31) denotes that the dynamical particle-density variation is determined by the equations of motion in the nuclear volume and is given in terms of the local part \(\delta f_{1,0}(t, r, \omega)\) \textsuperscript{22, 14} of the distribution function \(\delta f(r, p, t)\) \textsuperscript{22, 14} through (2.15).

Solving \textsuperscript{24, 45} with the first boundary condition \textsuperscript{32}, one gets the potential \(\varphi\) in the form

$$\varphi(r, t) = \frac{1}{q^2} \frac{q R_0}{j'_{L}(q R_0)} Q(t) j_L(q r) Y_{L0}(\hat{r}),$$

(3.32)

where \(j_L(x)\) is the spherical Bessel function and \(j'_L(x) = d j_L(x)/dx\). From the continuity equation (2.18) with \textsuperscript{33, 32}, one has

$$\delta \rho^{\text{vol}}(r, t) = \rho_0 \frac{q R_0}{j'_{L}(q R_0)} Q(t) j_L(q r) Y_{L0}(\hat{r}).$$

(3.33)

Therefore, according to (3.31) and (3.32), one finds

$$\delta \rho(r, t) = \rho_0 Q(t) Y_{L0}(\hat{r}) \left[ \frac{q R_0}{j'_{L}(q R_0)} j_L(q r) w(\xi) - \frac{\partial w}{\partial r} R_0 \right].$$

(3.34)

With this solution, we may now proceed to calculate the response function \(\chi_{FF}^{\text{coll}}(\omega)\) by expressing the integral over the coordinates \(r\) for the average \(\langle F^*_r \rangle\) \textsuperscript{32} in the numerator of (3.33) in terms of our collective variable \(Q_\omega\) given by (3.10). Indeed, substituting the Fourier transform of (3.34) together with \(\hat{F}\) from (3.28) into (3.4), we obtain

$$\chi_{FF}^{\text{coll}}(\omega) = -\frac{Q_\omega}{k_{\text{ext}}^2}.$$  

(3.35)

Using \textsuperscript{24, 44, 25} \textsuperscript{33} \textsuperscript{24}, (3.29) and (3.32), one may write the second boundary condition \textsuperscript{32} in terms of the collective variable \(Q(t)\) and periodic time dependence of the external field \(V_{\text{ext}}\) in the form of the equation of motion

$$B_L(x) \dot{Q} + C_L(x) Q + Z_L(x) \dot{Q} = -q_{\text{ext}}.$$  

(3.36)

We have introduced various new quantities,

$$x = \frac{\omega}{\Omega} = \frac{\omega R_0}{\nu_s}, \quad \Omega = \frac{\nu_s}{R_0} \sim \frac{\varepsilon_p}{A^{1/3} k},$$

(3.37)

which is a complex function of \(\omega\) by means of \textsuperscript{24, 44} \textsuperscript{25} for the sound velocity \(s\) with \textsuperscript{25, 59} \textsuperscript{24, 61, 59, 74} \textsuperscript{24}. In \textsuperscript{24, 47}, \(\Omega\) is the characteristic frequency of the classical particle rotation in a mean potential well of the radius \(R_0\) with the energy near \(\varepsilon_p\), as a convenient frequency unit. Other quantities are defined as

$$B_L(x) = m \rho_0 R_0^3 \frac{d j_{1,0}(x)}{dx} j'_{L}(x),$$

(3.38)

$$C_L(x) = C_L^{(S)} + C_L^{(A)}(x),$$

(3.39)

$$C_L^{(S)} = \alpha R_0^2 (L-1)(L+2) = \frac{b_0}{4\pi} A^{2/3}(L-1)(L+2),$$

(3.40)

$$C_L^{(A)}(x) = 2\lambda R_0^3 \frac{x}{j'_{L}(x)} (j'_{L}(x) + j_L(x)).$$

(3.41)
and

\[ Z_L(x) = 2\nu R^a_0 \frac{x}{j_L(x)} (j_L''(x) + j_L(x)). \] (3.42)

From (3.36), one has

\[ \frac{Q_\omega}{q_{\text{ext}}^2} = -\frac{1}{kD_L(\omega)} \] (3.43)

with

\[ D_L(\omega) \equiv -[B_L(x)\omega^2 - C_L(x) + i\omega Z_L(x)] \]

\[ = \frac{C_L^{(S)}}{j_L'(x)} \left\{ j_L''(x) - \frac{6A^{1/3} \lambda}{b_1(L - 1)(L + 2)\rho_s \varepsilon} [(i\nu \omega - \lambda) \right. \]

\[ \times \left. j_L'(x) + \left( \frac{s^2 \rho_s \varepsilon}{\bar{T}} - \lambda + i\nu \omega \right) j_L(x) \right\}. \] (3.44)

In (3.43), \( k \) is the coupling constant (3.30), see (3.38)-(3.32). The kinetic coefficients \( \lambda \) and \( \nu \) are the shear modulus \( \lambda \) and viscosity \( \nu \) given by (3.12) and (3.13), respectively. The two latter quantities enter (3.44) in the following combination

\[ \lambda - i\nu \omega = s\chi_x \rho_s \varepsilon \] (3.45)

through a function \( \chi_x \) defined by (3.16). Finally, the response function \( \chi_{\text{coll}}(\omega) \) (3.35) with (3.43) writes

\[ \chi_{\text{coll}}(\omega) = \frac{1}{k^2 D_L(\omega)}. \] (3.46)

The poles of this collective response function are determined by the following equation, see (3.44) for \( D_L(\omega) \),

\[ -B_L(x)\omega^2 + C_L(x) = i\omega Z_L(x) = 0, \] (3.47)

with \( x \) defined by (3.37). The complex solution of the dispersion equation (3.47) for \( s \) has two branches of the solutions. They are related asymptotically to the Landau–Placzek heat \( s^{(0)} \) and the sound \( s^{(1)} \) solutions considered all in the hydrodynamic limit for the infinite nuclear matter in Sec. II D see (2.74)-(2.76) for the corresponding frequencies \( \omega^{(0)} \) and \( \omega^{(1)} \). For each branch denoted below by the same upper index \( n = 0, 1 \) as well in Sec. II D we have the roots of the secular equation (3.47) written as

\[ \omega^{(n)} = \omega^{(n)}_i - i\tau^{(n)}_i / 2, \quad i = 0, 1, ..., \] (3.48)

where \( i \) numbers \( \omega_i^{(n)} \) in order of their increasing magnitude. We shall consider these roots with \( \omega_i^{(n)} \) in the frequency region of about \( \hbar \omega \lesssim 2\Omega \) which overlaps the low frequency energy region discussed below. We shall consider enough large temperatures \( T > \sqrt{\epsilon_c \hbar \Omega} \gtrsim \sqrt{\epsilon_c \hbar \omega} \) but smaller than the Fermi energy. This approximately means \( 2MeV \lesssim T \lesssim 10MeV \) for \( \epsilon_c = 3/4\pi^2 \) of (31) \((A \sim 200)\). (Low temperature limit is about 1 MeV for \( \epsilon_c = 1/4\pi^2 \) of (34)). For above mentioned frequencies \( \omega \) and temperatures \( T \), for which the quasiparticle and local-equilibrium conceptions of the theory for the heated Fermi liquids can be applied, the only lowest solutions have been found in the infinite sequence (3.45). They are associated with \( i = 0, 1 \) and 2 for the "first sound" branch \( n = 1 \), and that with \( i = 0 \) for the "Landau–Placzek" branch \( n = 0 \). (Quote marks show that the corresponding names are realized in fact only asymptotically in the hydrodynamical limit.) The total response function is the sum of the two branches mentioned above. The response function (3.46) contains all important information concerning the excitation modes of the Fermi-liquid drop. One of the ways of the receipt of this information is to analyze the response function poles (3.48) and their residua. However, this way is often not so convenient and too complicate in the case when a few poles are close to each other or they belong (or are close) to the imaginary axis of the complex plane \( \omega \). More transparent way which is free from such disadvantages is to describe the response function in terms of the transport coefficients (29).

C. Transport properties for a slow collective motion

The macroscopic response of nucleus to an external field is a good tool for calculations of the transport coefficients. To achieve this goal we follow the lines of (24) (29). For instance, in cranking model type approximations, one assumes the collective motion to be sufficiently slow such that the transport coefficients can be evaluated simply in the "zero-frequency" limit. For a such slow collective motion we shall study here the transport coefficients within the FLDM having a look at excitation energies smaller than the distance between gross shells (91),

\[ \hbar \Omega = \frac{\hbar \nu}{R} \approx \frac{\varepsilon_c}{A^{1/3}} = 7 - 10 \text{ MeV} \] (3.49)

in heavy nuclei \( \Omega \) is the particle rotation frequency (337) \( \hbar \omega \lesssim \hbar \Omega \), i.e., less than or of the order of the giant multipole resonance energies. Within the low collective motion (\( \omega \lesssim \Omega \)), we shall deal with first more simple case of the hydrodynamic approximation which can be applied for frequencies much smaller than the characteristic "collisional frequency" \( 1/\tau \) related to the relaxation time (3.19), \( \tau \ll 1 \). Using this hydrodynamic expansion of the macroscopic response function (3.46) in small parameter \( \omega \tau \), we shall look for in Sec. III C 1 the relation to the "zero frequency limit" discussed in (24). Another problem of our interest in this section is related to the correlation functions, "heat pole friction" and ergodicity property, see (24) (29). We shall consider in the next Sec. III C 2 a smaller frequency region where the nuclear heat pole like the Landau–Placzek mode for the infinite matter appears within the hydrodynamic approximation. This subsection will be ended by a more general treatment of the transport coefficients in terms of the parameters of the oscillator response function. The method of (24) (29) can be applied for the low frequency excitations, \( \omega \lesssim \Omega \),
but also in the case when the hydrodynamic approach fails, i.e. for \( \omega \tau \gtrsim 1 \).

Following [24, 29], we shall study the ”intrinsic” response function \( \chi\text{FF}(\omega) \) related to the collective one \( \chi_{\text{FF}}^{\text{coll}}(\omega) \) by the relation (3.31). The collective response function \( \chi_{\text{FF}}^{\text{coll}}(\omega) \) in the FLDM was derived straightly from (3.34) in terms of the solution for the transition density \( \delta \rho \). The ”intrinsic” response function can be then got with help of (3.41). This way is more convenient in the FLDM with respect to the opposite one used usually in the microscopic quantum calculations based on the shell model [29].

By making use of expansion of the denominator \( D_L(\omega) \) of the response function \( \chi_{\text{FF}}^{\text{coll}}(\omega) \) up to fourth order terms in small parameter \( \omega \tau \) in the low frequency region \( \omega/\Omega \ll 1 \), and then, of (3.31), one gets the response function in the \( F \) mode in the form:

\[
\chi(\omega) = k^{-2} \left( -M \omega^2 - i \gamma \omega + C_{\text{in}} - i \frac{TC}{2\omega} \right)^{-1},
\]

\[ C_{\text{in}} = C - k^{-1}, \quad (3.50) \]

\( M, C \) and \( \gamma \) can be defined as the \( Q \)-mode mass, the stiffness and the friction coefficients which are the values of \( B_L(x) \) for \( x = 0 \) (\( \omega = 0 \)). Here and below we omit the index ”FF” in the ”FF”-response functions everywhere when it will not lead to misunderstanding. Note that the formulas which we derive here and below for the ”intrinsic” response function \( \chi(\omega) \) can be applied also to the collective response function \( \chi_{\text{coll}}(\omega) \) if we only omit the index ”in” in \( C_{\text{in}} \) and in functions of \( C_{\text{in}} \) denoted by the same index (except for some approximations based on the specific properties of \( C_{\text{in}} \) compared to \( C \) and noted below if necessary). Another argument of the presentation of our results in terms of the ”intrinsic” response functions is to compare them more strictly with the discussed ones in [29] in connection to correlation functions and ergodicity. For the inertia \( M \) and stiffness \( C \), we obtain the parameters of the classic hydrodynamic model, namely:

\[ M = B_L(0) = \frac{1}{L} m_{b_0} R_0^2 \equiv M_{LD}, \quad (3.51) \]

the inertia of irrotational flow, and

\[ C = C_L(0) = C_L^{(S)} \equiv C_{LD} \quad (3.52) \]

with \( C_L^{(S)} \) being the stiffness coefficient of the surface energy (3.20). (We introduced here more traditional notations labeled by index ”LD” which means the relation to the usual liquid-drop model of irrotational flow). For friction \( \gamma \), we arrive at the temperature dependence typical for hydrodynamics,

\[
\gamma = \frac{3\Lambda(L-1)\nu_{LD}(T)}{5\pi} \frac{\nu_{LD}(T)}{\nu_{LD}(0)} \tau \equiv \gamma_{LD}, \quad (3.53)
\]

Here, \( \nu_{LD} \) is the classical hydrodynamic limit \( \nu^{(1)} \), for the viscosity coefficient, \( \tau \) the relaxation time \( \Gamma(0) \), for \( \omega = 0 \),

\[ \tau \equiv \Gamma(0, T) = \frac{\hbar}{\Gamma(0, T)}, \quad \Gamma(0, T) = \frac{\pi^2 T^2}{\Gamma_0 (1 + \pi^2 T^2/c^2)}. \quad (3.54) \]

However, our result (3.54) for the classical liquid-drop model of irrotational flow, if only extended to include the two-body viscosity, differs from the one found in [118] by an additional factor of \( (2L+1)/L \). We neglected the fourth order terms in \( \Gamma \) (Sec. III C 2) in (3.50), (3.32) and (3.53) because of the presence of more important lower order terms there. For the coefficient \( \Upsilon \) in the term proportional to \( 1/\omega \) in (3.50), one obtains

\[ \Upsilon = \frac{13A^{1/3}\pi^4T^4s_0^2}{60b_0(L-1)(L+2)\tau} = \frac{24\nu_0s_0^2A_{1/3}^{1/3}}{b_0(L-1)(L+2)} \nu^{(2)}. \quad (3.55) \]

The expression in the middle of these equations turns into zero for the Landau–Placzek kind \( (n = 0) \) of the solutions (2.74) of dispersion equation (2.47) for the velocity \( s_0 \). It is, however, finite for the first sound mode \( n = 1 \) presented in (2.75) for \( s_0^1 \). The second equation being true only for the first sound mode was obtained by making use of (2.78) for the first sound velocity \( s_0^1 \) and (2.48) for the viscosity component \( \nu^{(2)} \) up to small temperature corrections of the next order. The both equations (3.55) show the main term in the temperature expansion of the coefficient \( \Upsilon \) in front of \( 1/\omega \) in (3.50). Note that it appears in the order \( T^4 \) and can not be neglected for enough small frequencies \( \omega \). As seen from (3.50) considered for the case of the collective response, i.e., with omitted index ”in” in \( C_{\text{in}} \) of (3.50), for enough small frequencies \( \omega \), there is the pole which equals approximately \( i\sqrt{2}\Gamma \). Therefore, the physical meaning of the parameter \( \Upsilon \) (3.55) is a ”width” of the overdamped pole in the asymptotic collective response function \( \chi_{\text{coll}}(\omega) \) for enough low frequencies. As shown below, this pole and corresponding pole of the intrinsic response function (3.50) is overdamped. It is similar to the Landau–Placzek pole in the infinite nuclear matter and to the nuclear heat pole found in [29], see more detailed discussion in Sec. III C 2. The ”width” \( \Upsilon \) (3.55) of such ”heat pole” is inversely proportional to the relaxation time \( \tau \) and increases with temperature and particle number. Note also that this ”width” is proportional to the component \( \nu^{(2)} \) of the viscosity discussed in Sec. III D 3. It is somewhat similar to the viscose part of the standard expression for the first sound ”width” \( \Gamma \) in terms of the first component \( \nu^{(1)} \) of the viscosity coefficient (2.92), \( \Gamma \propto \nu^{(1)} \). However, there is in (3.55) the surface energy constant \( b_0 \) and particle number factor \( A_{1/3} \) which are both the specific parameters of a finite Fermi-liquid drop.

Thus, the denominator of the hydrodynamical response function (3.50) contains the two friction terms. One of them is proportional to the friction coefficient.
γ, γ ∝ ν(1), and another one which is proportional to Υ (Υ ∝ ν(2)). We shall consider in the next two Secs. 3.21 and 3.22 the two limit cases neglecting first the heat pole Υ term for enough large frequencies ω within the hydrodynamic approximation ωτ ≪ 1, and then, the γ friction one for smaller frequencies with the dominating heat pole, respectively.

1. HYDRODYNAMIC SOUND RESPONSE

For enough large frequencies ω within the frequent collisional (hydrodynamic) regime,

$$\omega_{\text{crit}} \tau ≪ \pi^2 \sqrt{3\hbar_0 m_0 \gamma \nu_{\text{ld}}(0)} / (L-1) \nu_{\text{ld}}(T) T^2,$$

(3.56)

one finds the first sound (i = 1) solution [27]. In this case, one can neglect the last term proportional to 1/ω compared to the friction term in the denominator of the asymptotic expression (3.50). The critical frequency ω_{\text{crit}} is defined in the second equation of (3.56) as a frequency for which these two compared terms coincide, ω_{\text{crit}} = \sqrt{C/T} \gamma = \sqrt{M \gamma / 2\gamma} (ω_{\text{ld}} = \sqrt{C/M} is the frequency of the surface liquid-drop vibrations). The critical value ω_{\text{crit}}τ increases with increasing temperature as T^2 and does not depend on particle number for the first sound mode n = 1. It equals zero for the Landau–Placzek branch n = 0, according to (3.55) for Υ. For the n = 1 mode, ω_{\text{crit}}τ is small for all temperatures T ≪ 10 MeV, ω_{\text{crit}}τ ≈ 0.6 T^2 ≪ 1 at typical values of the parameters, ε_p = 40 MeV and r_0 = 1.2 fm, and for a value C of the Skyrme forces considered in [27], C = 80 MeV-fm^5, which is somewhat larger than those of 10 MeV-fm^5 of Section IV and Appendix D) in the ESA, where A^{-1/3} is assumed to be small. We took here and below L = 2 for the quadrupole vibrations, F_0 = -0.2, F_1 = -0.6 for the Landau constants which are close to the values common used for the calculations of the nuclear giant multipole resonances [124, 125], a little more “realistic” than in 31 32. For frequencies ω within the condition (3.55), we arrive at the oscillator-like response function,

$$\chi(\omega) = k^2 \chi_{\text{osc}}(\omega) = k^2 \left( -M \omega^2 - i\gamma \omega + C_{\text{in}} \right)^{-1},$$

(3.57)

with all hydrodynamic transport coefficients presented in (3.51), (3.52) and (3.53). In the middle of (3.50), χ_{\text{osc}}(ω) is the "intrinsic" oscillator response function which describes the dynamics in terms of the Q(t) variable for the collective harmonic oscillator potential. As seen now, the constants M, C and γ were naturally called above as the transport coefficients: The collective response function χ_{\text{coll}}(ω) within the approximation (3.50) is the same (3.57) but with omitted index "in" in the stiffness coefficient, as noted above. This remark is related also to the oscillator QQ- response function χ_{\text{osc}}(ω) useful for the following analysis of the response functions in terms of the transport coefficients,

$$\chi_{\text{osc}}(\omega) = \left( -M \omega^2 - i\gamma \omega + C \right)^{-1}.$$  

(3.58)

We obtain the QQ- response functions from the FF-ones, for instance, from χ(ω) (3.57), simply multiplying by the constant k^2 because of the self-consistency condition (3.11). Note also that the condition (3.50) for the Landau–Placzek branch of the solutions for the sound velocity s [see (2.74)], is always fulfilled for ωτ ≪ 1.

In order to compare our results with those of previous calculations [29], we introduce the dimensionless quantity

$$\eta = \gamma / \left( 2 \sqrt{M |C|} \right) = 2 \varepsilon_p / \left[ 5 \partial \nu_{\text{ld}}(T) \right] \sqrt{6(L-1)\xi G_1 \nu_{\text{ld}}(0)^2 / (L+2)b_s},$$

(3.59)

see (3.58), (3.51) and (3.52) for γ, M and C, respectively. The quantity η in (3.59) characterizes the effective damping rate of the collective motion. Neglecting small temperature corrections of the viscosity coefficient ν_{ld} = ν(1) in (3.59), see [29], and substituting (3.13), (5.24) for the relaxation time τ at ω = 0, one writes

$$\eta \approx 2 \varepsilon_p \gamma_0 / \left[ 5 \pi^2 \partial \nu_{\text{ld}}(T) \right] \sqrt{\left[ 6(L-1)\xi G_1 \nu_{\text{ld}}(0)^2 / (L+2)b_s \right] + \pi^2 T^2 \nu_{\text{ld}}(0)^2 / T^2}.$$  

(3.60)

This hydrodynamic effective friction η mainly decreases with temperature as 1/T^2. For large temperatures and finite cut-off parameter c, the dimensionless friction parameter η (3.60) approaches the constant.

We have the two kind of poles of the response function (3.57) as roots of the quadratic polynomial in the denominator, the overdamped poles, see [29],

$$\omega_{\text{over}} = i\gamma_{\text{in}} / 2,$$

$$\Gamma_{\text{in}} = 2 \varepsilon_{\text{in}} \left( \eta_{\text{in}} + \sqrt{\eta_{\text{in}}^2 - 1} \right), \quad \eta_{\text{in}} > 1,$$

(3.61)

and the underdamped ones,

$$\omega_{\text{under}} = \varepsilon_{\text{in}} \left( \pm \sqrt{1 - \eta_{\text{in}}^2 - i\eta_{\text{in}}} \right), \quad \eta_{\text{in}} < 1.$$

(3.62)

These solutions depend on the two parameters,

$$\varepsilon_{\text{in}} = \sqrt{|C_{\text{in}}| / M}, \quad \eta_{\text{in}} = \gamma / \left( 2 \sqrt{M |C_{\text{in}}|} \right).$$

(3.63)

Note also that the two hydrodynamic poles in (3.57) coincide approximately for the both branches n = 0 and 1 of solutions to the dispersion equation (2.74) for the velocity s. The difference between these two modes is related only to the last term proportional to Υ in the brackets of r.h.s. of (3.57), and it was neglected under the condition (3.55).
For the real and imaginary parts of the response function $\chi(\omega)$ (3.57), with the help of (3.63) for the overdamped case (3.61), for instance, one gets for completeness [see (3.61)–(3.63)]

$$
\chi(\omega) = \frac{1}{4MK^2\omega in\sqrt{\eta in} + 1} \left( \frac{\Gamma in}{\omega^2 + (\Gamma in)^2/4} \right)
$$

(3.64)

$$
\chi''(\omega) = \frac{\omega}{4MK^2\omega in\sqrt{\eta in} + 1} \left( \frac{1}{\omega^2 + (\Gamma in)^2/4} \right)
$$

(3.65)

For a more simple case of the collective response in the FLDM, we omit index "in" in formulas of this section [see the comment after (3.50)]. From (3.50) for $\eta$ with the parameters used above for the estimate of $\omega_{\text{crit}}\tau$ of (3.56), and the "standard" $\Gamma_0 = 33.3$ MeV [29], one has an overdamped motion, $\eta > 1$, for all temperatures $T \lesssim 10$ MeV and particle numbers $A \lesssim 230$, as seen from Fig. 1. Moreover, for such temperatures and particle numbers, one can expand the "width" $\Gamma_\pm$ in small parameter $MC/\gamma^2 = (4\eta)^{-1}$, see (3.51) omitting index "in". From (3.51) (without index "in") one gets approximately

$$
\Gamma_\pm = 4\omega\eta \left\{ 1 - \frac{(4\eta)^{-1}}{(4\eta)^{-1}} \right\} = 2 \left\{ \frac{\gamma/M}{C/\gamma} \right\}, \quad \eta^2 \gg 1.
$$

(3.66)

Fig. 1 shows that the above mentioned parameter $1/(4\eta^2)$ for the expansion in (3.66) is really small for all considered temperatures. Using (3.51), (3.52) and (3.53) for the transport coefficients and the definition of $\tau$ (3.54), as in the derivation of (3.59), (3.60), one obtains from (3.66)

$$
\Gamma_+ = \frac{16G_1L(L-1)\varepsilon F^2}{5(p_0\tau_0)^2 A^{2/3}} \frac{\nu_{LD}(T)}{A^{2/3}} \frac{\tau}{\nu_{LD}(0)} \approx \frac{16\rho_0^2G_1L(L-1)\varepsilon F^2}{5\pi^2(p_0\tau_0)^2 A^{2/3}} \frac{1 + \pi^2 T^2/c^2}{T^2},
$$

(3.67)

$$
\Gamma_- = \frac{5b_0(L + 2)}{6r A^{2/3}} \frac{\nu_{LD}(0)}{A^{1/3}} \frac{\tau}{\nu_{LD}(T)} \approx \frac{5\rho_0^2b_0(L + 2)}{6\rho_0^2G_0\tau_0 A^{1/3}} \frac{T^2}{1 + \pi^2 T^2/c^2}.
$$

(3.68)

One of the "widths" specified by $\Gamma_+$ (3.67) is mainly the decreasing function of temperature, $\Gamma_+ \propto \tau \propto 1/T^2$ at low temperatures. It is typical for the hydrodynamic modes as the first sound vibrations in normal liquids; in contrast to another "width" $\Gamma_-$ (3.68), $\Gamma_- \propto 1/\tau \propto T^2$, similar to the zero sound damping in relation to the $\tau$-dependence. They both become about a constant for high temperatures, due to the cut-off factor $c$.

Note, $\Gamma_+ (3.67)$ decreases with particle number as $A^{-2/3}$ while $\Gamma_- \propto A^{-1/3}$, see (3.68). The different $A$-dependence of the "widths" $\Gamma_-$ (3.68) and $\Gamma_+ (3.67)$ can not be nevertheless referred even formally to the so-called "one- and two-body dissipation", respectively. (Collisions with potential walls without the integral collision term in the Landau–Vlasov equation but with the mirror or diffused boundary conditions might lead to the "widths" proportional to $\Omega$ in (3.67), $\Omega \propto A^{-1/3}$, as in equation (49) of [44] or through the wall formula [129–128].) They both depend on the collisional relaxation time $\tau$ and correspond to the "two-body" dissipation. The latter means here the collisional damping of the viscose Fermi liquid as in [31, 32]. The physical source of the damping in the both cases is the same collisions of particles in the nuclear volume, due to the integral collision term (2.9) with the relaxation time $\tau$. We would like to emphasize, however, that the collisional $\Gamma_-$ (3.68) depends on the surface energy constant $b_0$ and disappears proportionally to $A^{-1/3}$ with increasing particle number $A$ like $\Omega$ of (3.67), because we took into account a finite size of the system through the boundary conditions (3.21), (3.22). An additional overdamped pole with the "width" $\Gamma_-$ (3.68) appears because of the finiteness of the system and collisions inside the nucleus. This looks rather in contrast to the wall friction [127, 128] coming from the collisions with the only walls of the potential well.

We shall come back now to the intrinsic response function $\chi(\omega)$ (3.57). For the "intrinsic stiffness" $C_{\text{in}}$, one has

$$
C_{\text{in}} = -(1 - kC)/k = -1/k.
$$

(3.69)

In the last equation, we neglected a small parameter $kC$,

$$
kC = \frac{54(L-1)(L+2)C}{4\pi Kr_0 A^{2/3}} \approx \frac{9(L-1)(L+2)C}{4\pi G_0\varepsilon F^2 A^{2/3}} \approx \frac{3}{A^{2/3}},
$$

(3.70)

for the typical values of the parameters mentioned above before (3.57). We neglected also small temperature corrections of (3.29) for the in-compressibility modulus $K$, $K = K^*$, in the second equation of (3.70).

Using a smallness of the parameter $kC$ (3.70), we shall get now the relation of the coupling constant $k^{-1}$ with the isolated susceptibility $\chi(0)$ and stiffness $C$ as in equation (3.1.26) of [29]. For this purpose, we take the limit $\omega \to 0$ in (3.57) for the "intrinsic" response function $\chi(\omega)$ and expand then the obtained expression for $\chi(0)$, $\chi(0) = k^{-2}C_{\text{in}}^{-1} = -k^{-2}(1 - kC)^{-1}$, in powers of the small parameter $kC$ (3.70) up to second order terms. As result, we arrive at the relation

$$
-k^{-1} = \chi(0) + C.
$$

(3.71)

The liquid-drop transport coefficients $M$ (3.51), $C$ (3.52) and $\gamma$ (3.53) can be now compared with the ones
in the "zero frequency limit" $M(0)$, $C(0)$ and $\gamma(0)$, respectively, defined by equations (3.1.84)-(3.1.86) in [29]:

$$C(0) = - (1/k + \chi(0)) = C,$$  

$$\gamma(0) = -i (\partial \chi(\omega)/\partial \omega)_{\omega=0} = \gamma,$$  

$$M(0) = \left(\frac{1}{2} \partial^2 \chi(\omega)/\partial \omega^2\right)_{\omega=0} = M \left(1 + \gamma^2 k/M \right) = M \left(1 + 4\eta^2_{in}\right). \quad \text{(3.74)}$$

Expanding $\chi(\omega)$ near the zero frequency $\omega = 0$ in the secular equation (3.53), see [29], we assumed here and will show below that the "intrinsic" response function $\chi(\omega)$ is a smooth function of $\omega$ for small frequencies $\omega$ within the hydrodynamic condition (3.56). The second and third equations in (3.61), (3.62) and (3.64) were got approximately in the ESA from (3.57) up to small corrections in the parameter $kC$ with help of (3.71), second equation in (3.63), (3.69). As the liquid drop stiffness $C$ equals approximately the stiffness in the "zero frequency limit" $C(0)$, according to (3.72), the equation (3.71) is identical to the relation (3.10) of the general response-function theory [29] within the same ESA. As seen from (3.72)-(3.74), the stiffness $C(0)$ and friction $\gamma(0)$ equal to the liquid drop parameters, but the inertia $M(0)$ differs from the liquid-drop mass value $M$ by a positive correction.

For the definition of transport coefficients in "the zero frequency limit" (3.62), we needed to know also the properties of the "intrinsic" response function in the secular equation (3.53), concerning its pole structure. For the "intrinsic" case the quantity $\eta_{in}$, see (3.63), plays a role similar to the effective damping $\eta$ for the collective motion. Moreover, $\eta_{in}$ determines the correction to the liquid drop mass parameter $M$ in (3.74) for the inertia $M(0)$ in "the zero frequency limit". Due to a smallness of the parameter $kC$ (3.70), $\eta_{in}$ is much smaller than $\eta$ (3.63) for large particle numbers $A \approx 200 - 230$, as seen from Fig. 1.

$$\eta_{in} = \frac{\gamma}{2 \sqrt{M |C_{in}|}} \approx \gamma^2 kC$$  

$$\approx \frac{3(L - 1) h \Gamma_{0} \psi}{\pi \gamma_{0} r_{0}} \sqrt{\frac{6 \lambda_{G_{1}}}{\pi \gamma_{0} r_{0}} A} \frac{1 + \pi^2 T^2/c^2}{T^2}, \quad \text{(3.75)}$$

see (3.53), (3.54), (3.52), (3.69), and (3.70).

For such heavy nuclei ($A \approx 200 - 230$) and enough large temperatures, $T \gtrsim 5$ MeV, one has formally the "underdamped" pole structure (3.62), $\eta_{in} < 1$ for the parameters selected above. Using the expansion of the poles $\omega_{in}^\pm$ of the intrinsic response function in powers of small $\eta_{in}^2$ up to terms of the order of $\eta_{in}^4$, one writes

$$\omega_{in}^\pm = \omega_{in} \left[ \pm \left(1 - \frac{1}{2} \eta_{in}^2\right) - i\eta_{in}\right]$$  

$$\approx \pm \omega_{LD}/\sqrt{kC} - i\Gamma_{+}/4 \quad \text{for} \quad \eta_{in}^2 \ll 1; \quad \text{(3.76)}$$

see (3.63), (3.69) (for $\Gamma_{+}$ on the very r.h.s.), and (3.69) (for $kC$ there) in the derivation of the second equation. The "underdamped" poles $\omega_{in}^\pm$ approach the real axis on a large distance from the imaginary one as compared to the liquid drop frequency $\omega_{LD} = \sqrt{C/M}$, $|\omega_{in}^\pm| \gg \omega_{LD}$.

They have a small "width" $2\omega_{in} \eta_{in} = \gamma/M \times 1/T^2$ for our choice of large temperatures ($T \gtrsim 5$ MeV); see (3.63), (3.69), (3.61), and (3.54). By this reason, for the "underdamped" case of small $\eta_{in}^2$ and low frequencies $\omega \lesssim \omega_{LD}$, the intrinsic response function $\chi(\omega)$ is a smooth function of $\omega$.

For smaller temperatures $T \lesssim 4$ MeV and for our parameters used in (3.75), one has the "overdamped" poles (3.61) of the intrinsic response function $\chi(\omega)$, $\eta_{in} > 1$. For such temperatures, $\eta_{in}$ (3.70) is enough large. We can use therefore the expansion of the "widths" $\Gamma_{+}$ of (3.61) in a small parameter $(M \omega_{in}/\gamma)^2 = (4\eta_{in}^2)^{-1}$ (see Fig. 1).

$$\Gamma_{+} = 4 \omega_{in} \eta_{in} \left\{ 1 - \frac{(4\eta_{in}^2)^{-1}}{1/(k\gamma)} \right\}$$  

$$\approx 2 \left\{ \frac{\gamma/M}{1/(k\gamma)} \right\} \quad \text{for} \quad (4\eta_{in}^2)^{-1} \ll 1, \quad \text{(3.77)}$$

see (3.63), (3.69) and (3.70). The "intrinsic width" $\Gamma_{+}$ in the upper row of (3.77) and the "collective width" $\Gamma_{-}$ (3.64) [see (3.66)] are the same. $\Gamma_{+}$ in the low row has the temperature dependence as for $\Gamma_{-}$ in (3.64) but a different $A$-dependence, $\Gamma_{+} \propto A^{1/3}$ [see (3.50) and (3.63)]. Moreover, $\Gamma_{+} \gg \Gamma_{-}$ because of smallness of the parameter $kC$ (3.70). It becomes clear after dividing and multiplying the last expression for the $\Gamma_{+}$ in (3.77) by the factor $C$ and using (3.92) and (3.53).

The "intrinsic width" $\Gamma_{+}$, see (3.77), is mainly larger than $\Gamma_{+}$. They become comparable when increasing temperature, i.e., $\Gamma_{+} \gtrsim \Gamma_{+}$. As $\Gamma_{+}$ from (3.77),

$$\Gamma_{+} = \frac{10 \pi G_{b} r_{0}^{3} A^{1/3}}{27(L - 1) \sqrt{C}}, \quad \text{(3.78)}$$

is large compared to the characteristic collisional frequency $1/\tau$ (for the same choice of the parameters) the both poles are far away from the zero, see more discussions of the "intrinsic widths" below in connection with the heat pole in the next section (3.11). Therefore, the intrinsic response function $\chi(\omega)$ (3.57) is a smooth function of $\omega$ for the "overdamped" case of large $\eta_{in}$ used in the derivations of (3.77) as for the "underdamped" one discussed above. Thus, we expect that the "zero frequency limit" based on the expansion of the intrinsic response function $\chi(\omega)$ is a good approximation for low frequencies larger the critical value $\omega_{cr} \approx \gamma/\omega_{cr}$ within the hydrodynamic condition (3.56) for all considered temperatures. It means that the definition of the transport coefficients in this limit (3.72), (3.74) is justified within the hydrodynamic approximation (3.56).

The correction to the liquid drop mass parameter in the inertia $M(0)$ (3.74) is always positive. This correction
is the decreasing function of the temperature and particle number which can be presented approximately as

\[
(M(0) - M)/M = k\gamma^2/M = 4\eta_{in}^2
\]

\[
\propto (1 + \pi^2 T^2/c^2)^2 / (A T^4), \tag{3.79}
\]

see (3.75). For smaller temperatures when the expansion in (3.77) is justified, this correction is equal approximately to the ratio of the "intrinsic widths" \(\Gamma_{in}/\Gamma_{in}^\prime\) from (3.77). The relative mass correction \(\eta^2\) and the "intrinsic width" ratio \(\Gamma_{in}^\prime/\Gamma_{in}^\prime\), see (3.77), decreases with temperature \(T\) mainly as \(1/T^4\) if \(T\) is not too big, as shown in Fig. 1. The dimensionless inertia correction (3.79) is proportional approximately to \(1/T\). The zero frequency mass \(M(0)\) exceeds much the liquid drop inertia and turns asymptotically to the latter for high temperatures, see Fig. 1 Note, for enough large temperatures \(T \gtrsim 5\text{MeV}\) and particle numbers \(A \sim 200\) when \(\eta_{in}^2\) terms can be neglected in accordance with (3.75), all zero frequency transport coefficients \(C(0), \gamma(0)\) and \(M(0)\) [see (3.72), (3.73) and (3.74)] approach the corresponding liquid drop parameters.

It would be interesting now to get the "overdamped" correlation function \(\psi''(\omega)\) determined by the imaginary part of the corresponding response function \(\chi''(\omega)\) \(\approx \omega^2 / (\omega^2 + \Gamma_{in}^2)\) through the fluctuation-dissipation theorem, see (2.113) and (2.114). In the semiclassical approximation \(3.65\) for the first sound mode, one writes

\[
\frac{1}{T} \psi''(\omega) = \frac{2}{\omega} \chi''(\omega) \approx \frac{2}{4Mk^2\omega_{in}\sqrt{\eta_{in}^2 + 1}}\times \left(\frac{\omega^2 + (\Gamma_{in})^2/4}{\omega^2 + \Gamma_{in}^2/4}\right). \tag{3.80}
\]

Using the approximations as in (3.77) and (3.69), one gets from (3.80)

\[
\frac{1}{T} \psi''(\omega) = \frac{1}{k} \left(\frac{\Gamma_{in}}{\omega^2 + (\Gamma_{in})^2/4}\right) - \frac{1}{4\eta_{in}^2} \frac{\Gamma_{in}^2}{\omega^2 + (\Gamma_{in})^2/4} \approx \frac{1}{k} \left(\frac{\Gamma_{in}^2}{\omega^2 + (\Gamma_{in})^2/4}\right). \tag{3.81}
\]

The second Lorentzian in the middle is negligibly small compared to the first one because

\[
\Gamma_{in}^\prime \gtrsim \Gamma_{in} \gg 1/\tau \gg \omega, \tag{3.82}
\]

and \(4\eta_{in}^2\) is large in these derivations, see the discussions in between (3.77) and (3.78). It seems that we are left with the Lorentzian term of this correlation function on very right of (3.81) which looks as the Landau–Placzek heat-pole correlation function (2.118) and equation (4.3.30) of [29] with obvious constants \(\psi(0)\) and \(\Gamma_T\). However, we can not refer the found correlation function \(\psi''(\omega)\) of (3.81) to the heat pole one. The "width" \(\Gamma_{in}^\prime\) of (3.77) in (3.81) is finite and large compared to the characteristic collision frequency \(1/\tau\) which, in turn, is much larger considered frequencies \(\omega\), as shown above, see (3.82). The limit \(\Gamma_{in}^\prime \to 0\) for a fixed finite \(\omega\) and the corresponding \(\delta(\omega)-\)function which would show the relation to the heat pole correlation function do not make sense within the approximation (3.80) used in (3.81). In particular, the response \(\psi''(\omega)\) and the correlation \(3.81\) functions were derived for enough large frequencies \(\omega \gg \omega_{crit}\) due to the condition (3.58). Note also that the inertia parameter \(M(3.74)\) is not zero, as it should be for the heat pole.

2. HYDRODYNAMIC CORRELATIONS AND HEAT POLE

For lower frequencies \(\omega\), which are smaller the critical value \(\omega_{crit}\), we should take into account the last additional term in the denominator of (3.50) for the response function. For such small frequencies, this friction term being proportional to \(\Upsilon\) (3.59) becomes dominating as compared to the liquid-drop one \(\gamma = \gamma_{LD}\). Within this approximation, we shall derive the heat-pole response and correlation functions, and relate \(\Upsilon\) (3.59) of (3.50) with the corresponding heat pole friction. This subsection will be ended by discussions of the nuclear ergodicity.

For smaller frequencies,

\[
\omega\tau \ll \omega_{crit} \ll 1, \tag{3.83}
\]

(see the second equation in (3.56) for the critical frequency \(\omega_{crit}\), one can neglect the friction \(i\gamma \omega\) term in the denominator of the asymptotic response function (3.50) as compared to the last one, \(\gamma \omega \ll TC/2\omega\). The mass term there is even smaller than the friction one for frequencies \(\omega \ll \omega_{crit}\) for the considered parameters and will be neglected too, \(M\omega^2 \ll \gamma\omega\). In this approximation, from (3.50) one obtains the heat pole response function \(\chi(\omega) \approx \chi_{hp}(\omega)\), which is similar to (3.83), (3.84) for the infinite nuclear matter,

\[
\chi_{hp}(\omega) = \frac{\omega}{k^2C_{in}\left[\omega + i\Gamma_{hp}/2\right]} \approx -\frac{\omega}{k(\omega + i\Gamma_{hp}/2)}, \tag{3.84}
\]

where

\[
\Gamma_{hp} = -C\Upsilon/C_{in} \approx kC\Upsilon. \tag{3.85}
\]

In these derivations, we used the specific properties of the intrinsic response functions which we now are interested in for analysis of the correlation functions and ergodicity conditions [29]. In (3.83) and in all approximate equations below in this subsection, we applied also the expansion in small parameter \(kC\) (3.70) as in (3.69).

The real and imaginary parts of the response function \(\chi_{hp}(\omega)\) (3.84) are, respectively,

\[
\chi_{hp}(\omega) = \frac{\omega^2}{k^2C_{in}\left[\omega^2 + (\Gamma_{hp})^2/4\right]} \approx -\frac{\omega^2}{k\left[\omega^2 + (\Gamma_{hp})^2/4\right]}, \tag{3.86}
\]
\[ \chi_{hp}''(\omega) = -\frac{\omega \Gamma_{hp}}{2k^2 C_{in} [\omega^2 + (\Gamma_{hp})^2/4]} \]
\[ \approx \frac{\omega \Gamma_{hp}}{2k [\omega^2 + (\Gamma_{hp})^2/4]} \quad (3.87) \]

up to small \( kC \) corrections, see (3.70).

We shall derive now the correlation function \( \psi_{hp}''(\omega) \) applying the fluctuation-dissipation theorem (2.111) to the "intrinsic" response function \( \chi_{hp}''(\omega) \) (3.57) obtained in the asymptotic limit (3.83). From (2.117) and (3.87) one gets
\[ \frac{1}{T} \psi_{hp}''(\omega) = \frac{1}{2\omega} \chi_{hp}''(\omega) = -\frac{\Gamma_{hp}}{k^2 C_{in} [\omega^2 + (\Gamma_{hp})^2/4]} \approx \frac{\Gamma_{hp}}{k [\omega^2 + (\Gamma_{hp})^2/4]} \quad (3.88) \]

This correlation function looks as the Landau–Placzek peak for the infinite Fermi liquid, see (2.118).

\[ \psi_{hp}''(\omega) = \psi_{hp}^{(0)} \frac{\hbar \Gamma_{hp}}{\hbar \omega^2 + (\Gamma_{hp})^2/4} \quad (3.89) \]

It is identical to the r.h.s. of equation (4.3.30) in (29), but with the specific parameters \( \psi^{(0)} = \psi_{hp}^{(0)} \) and \( \Gamma_T = \Gamma_{hp} \),
\[ \frac{1}{T} \psi_{hp}^{(0)} = -\frac{1}{k C_{in}} \approx \frac{1}{k \Gamma_{hp}} \Gamma_{hp} = h \Gamma_{hp} \approx \hbar k C \gamma. \quad (3.90) \]

The "width" \( \Gamma_{hp} \) in (3.90) is much smaller than the characteristic collision frequency \( \gamma \),
\[ \Gamma_{hp} = \frac{13\pi^4 G_1 C T^4}{20\pi^4 b_5 r_5^2 \Gamma_0 A^{1/3}} \ll \gamma, \quad (3.91) \]
see (3.55) and (3.70). The relationship (3.91) for \( \Gamma_{hp} \) is in contrast to the one (3.82) for the "intrinsic overdamped widths" \( \Gamma_{in} \) (3.77) which are much larger than the collision frequency \( 1/\gamma \) for the same selected parameters at all temperatures \( T \lesssim 10 \text{ MeV} \) and particle numbers \( A = 200 - 230 \).

For the following discussion of the friction coefficients, we compare now the "width" \( \Gamma_{hp} \) (3.83), (3.91) with \( \Gamma_{in} \) in (3.77) [see (3.55), (3.52), (3.55), (2.75), (3.69) and (3.30)],
\[ \frac{\Gamma_{hp}}{\Gamma_{in}} = \frac{\gamma C \gamma}{2k C_{in}^2} \approx \frac{1}{2} \gamma C k^2 \gamma. \quad (3.92) \]

For all temperatures and particle numbers which we discuss here, this ratio is small,
\[ \frac{\Gamma_{hp}}{\Gamma_{in}} = \frac{351 \pi^2 (L - 1) C^2}{800 \pi^4 b_5^{10} A^{2/3}} T^4 \approx 15 T^4 A^{2/3} \quad (3.93) \]

In the second equation of (3.93), we used the same values of the parameters as in (3.71). Note that the "width" of the Landau–Placzek peak \( \Gamma^{(0)} \), \( \Gamma^{(0)} \approx \gamma^2 \ll 1/\gamma \) for \( \gamma \approx 1 \), is similar to \( \Gamma_{hp} \) and is unlike \( \Gamma_{hp} \) (3.77) in (3.81) for the hydrodynamical sound correlation function. In contrast to the hydrodynamical sound case, see (3.81), we can consider (3.88) for the correlation function approximation in the zero width limit \( \Gamma_{hp} \to 0 \) (or in the zero temperature limit \( T \to 0 \)) taking any small but finite frequency \( \omega \) under the condition (3.83). Therefore, for such frequencies \( \omega \), the correlation function (3.83) can be approximated by \( \delta(\omega) \)-like function as in (2.111) for the correlation function of the infinite Fermi liquid (2.113).

Because of a very close analogy of equation for the correlation function \( \psi_{hp}''(\omega) \) (3.89) to the Landau–Placzek peak for the infinite Fermi-liquids in the hydrodynamic limit, see (2.118), and to equation (4.3.30) of (29), we associate the pole (8.35) and corresponding asymptotics of the response (3.84) and correlation (3.88) functions with the "heat pole". As in the case of the infinite nuclear matter, this pole for the finite Fermi-liquid drop is situated at zero frequency \( \omega = 0 \). Moreover, they are both called as the "heat pole" because they disappear in the zero temperature limit \( T \to 0 \) in line of the discussions near equation (4.3.30) of (29) and after. In the case of the infinite matter, we can see this property from (2.110) because \( C_V / C_T \to 1 \) (or due to (2.120) for \( \psi^{(0)} \) in (2.118)). For the finite Fermi-liquid drop, the reason is that \( T \to 0 \) in the zero temperature limit \( T \to 0 \), see (3.55), and the only hydrodynamical sound condition (3.56) is then satisfied with the response function (3.57) and correlation function (3.84) where the heat pole is absent, see the discussion after (3.84).

To get more explicit expressions for \( \psi^{(0)} \) and \( \Gamma_T \) of (3.90), we use now (3.30), (3.55), (3.53) and (2.72) for the coupling constant \( k^{-1} \), parameter \( \gamma \), friction \( \gamma \) and stiffness \( C(0) \), respectively. With these expressions, one obtains approximately from (3.50)
\[ \frac{1}{T} \psi_{hp}^{(0)} = \frac{G_0 \delta b_5 r_5^2 A^{1/3}}{9 C} \approx 2 A^{4/3}, \quad (3.94) \]
\[ \Gamma_{hp} = \hbar \Gamma_{hp} = \frac{13\pi^4 G_1 C T^4}{20\pi^4 b_5 r_5^2 \Gamma_0 A^{1/3}} \frac{T^6}{1 + \pi^2 T^2 / c^2}. \quad (3.95) \]

In the derivation of (3.93), we used \( \Gamma_{hp} \) for the first sound solution \( s_0 = s_0^{(1)} \) \( (\nu = 1) \) in (3.55) for \( \gamma \) and (3.54) for the relaxation time \( \tau \). For simplicity, we neglected small temperature corrections in the viscosity coefficient \( \nu^{(1)} \) (3.93) and in the first sound velocity \( s_0^{(1)} \) (2.75). Other approximations are the same as well in the derivation of (3.70) for \( kC \) used in (3.95) through (3.85). The temperature dependences of the "intrinsic overdamped width" \( \Gamma_{in} \) (3.77) and "heat pole one" \( \Gamma_{hp} \) (3.95), (3.91) are different, namely \( \Gamma_{hp} \propto T^4 / \tau(0, T) \) and \( \Gamma_{in} \propto 1 / \tau(0, T) \) where the temperature dependence of the relaxation time \( \tau(0, T) \) can be found in (3.54). The both "widths" are the growing function of temperature as in (29) but with a different power. The dependence on particle number \( A \) completely differs for these compared poles being the growing function of \( A \) for the "width"
$\Gamma_{\text{in}}, \Gamma_{\text{in}} \propto A^{1/3}$, and decreasing function of $A$ for the $\Gamma_{\text{bp}}, \Gamma_{\text{bp}} \propto A^{-1/3}$. As noted above, like for the Landau–Placzek peak [see \ref{2.118}, \ref{2.119} and \ref{2.209}], the heat pole with the "width" $\Gamma_{\text{bp}}$ \ref{3.92} exists only in heated systems with a temperature $T \neq 0$. However, in contrast to the result \ref{2.119} \ref{2.209}, the $\Gamma_{T}$ of the heat pole in the infinite Fermi liquid, the heat pole "width" $\Gamma_{T}$ \ref{3.95} disappears with increasing particle number $A$, i.e., $\Gamma_{T} \rightarrow 0$ for $A \rightarrow \infty$. It allows us to emphasize also that this kind of the heat pole appears only in a finite Fermi system.

The correlation function $\psi_{\text{hp}}''(\omega)$ \ref{3.89} was obtained approximately near the pole $-i\Gamma_{\text{bp}}/2$, see \ref{3.85}. The corresponding $QQ$ correlation function $\psi_{QQ}''(\omega) = k^2 \omega \psi_{\text{hp}}''(\omega)$ is identical to the oscillator correlation function $\psi_{\text{osc}}''(\omega)$ defined through the imaginary part of $\chi_{\text{osc}}(\omega)$ from the second equation of \ref{3.57} at the zero mass parameter $M, M = 0$, see \ref{29},

$$\frac{1}{T} \psi_{\text{osc}}''(\omega) = \frac{2}{\omega} \psi_{\text{hp}}''(\omega) = \frac{2}{C_{\text{in}}} \frac{|C_{\text{in}}/\gamma_{\text{hp}}|}{\omega^2 + (C_{\text{in}}/\gamma_{\text{hp}})^2} \approx 2k \frac{1}{\omega^2 + 1/(k\gamma)^2}. \tag{3.96}$$

see again \ref{3.69} for the last approximation. The response \ref{3.83} and correlation \ref{3.88} functions are identical to the corresponding oscillator ones \ref{3.90} with a friction coefficient $\gamma_{\text{hp}}$,

$$\gamma_{\text{hp}} = 2|C_{\text{in}}|/\Gamma_{\text{hp}} \approx 2k^{-1} (\Gamma_{\text{bp}})^{-1} \approx 2/(k^2 C \gamma). \tag{3.97}$$

Here, the same equation \ref{3.69} was used, $\Gamma_{\text{hp}}$ is given by \ref{3.85}, \ref{3.95}, $k^{-1}$ is the coupling constant \ref{3.30}. [For $C$ and $\gamma$ in \ref{3.97}, one has \ref{3.82} and \ref{3.95}, respectively.] According to \ref{3.81} and \ref{3.88}, for the correlation functions $\psi''(\omega)$ and $\psi_{\text{hp}}''(\omega)$, with the help of \ref{3.55}, \ref{3.71} and \ref{3.73}, one gets

$$\frac{1}{2T} \psi''(0) = \gamma = \gamma(0), \tag{3.98}$$

$$\frac{1}{2T} \psi_{\text{hp}}''(\omega) = 2/(k^2 C \gamma) = \gamma_{\text{hp}}. \tag{3.99}$$

in line of the last right equation in \ref{3.1.85} of \ref{29}.

For the friction $\gamma_{\text{hp}}$ \ref{3.97} related to the "heat pole width" $\Gamma_{\text{hp}}$ \ref{3.85}, one approximately writes

$$\gamma_{\text{hp}} = \frac{40 G_{0} \Lambda_{\text{G1}}^{5} \Lambda_{\text{G2}}^{10} A}{117 \pi^{6} G_{1}^{2} T^{6}} \left(1 + \frac{\pi^{2} T^{2}}{c^{2}}\right), \tag{3.99}$$

see \ref{3.95} for $\Gamma_{\text{hp}}$ and \ref{3.30} for $k^{-1}$ in \ref{3.97}. We neglected here small temperature corrections in the adiabatic in-compressibility modulus $K = K_{s}$ \ref{3.29}. The heat pole friction $\gamma_{\text{hp}}$ \ref{3.99} is proportional to $1/T^6$ for smaller temperatures and $1/T^4$ for larger ones (due to the cut-off parameter $c$). This decreasing temperature dependence is much more sharp compared to the liquid drop one $\gamma$ \ref{3.53}, \ref{3.54}: $\gamma \propto 1/T^2$ for smaller temperatures, and $\gamma$ is a constant for large ones. Notice, according to \ref{3.97}, the "width" ratio $\Gamma_{\text{hp}}/\Gamma_{\text{in}}$ \ref{3.92} \ref{3.93} has a clear physical meaning as the ratio of the hydrodynamic friction coefficient $\gamma$ \ref{3.53} to the heat pole one $\gamma_{\text{hp}}$ \ref{3.99}.

$$\Gamma_{\text{hp}}/\Gamma_{\text{in}} \approx \gamma/\gamma_{\text{hp}} \approx \gamma(0)/\gamma_{\text{hp}}. \tag{3.100}$$

A smallness of this ratio shown above claims that the heat pole friction $\gamma_{\text{hp}}$ is much larger than the typical hydrodynamic one $\gamma$, see more discussions concerning this comparison of different friction coefficients below.

As seen from the inequalities \ref{3.83} with the definition of $\omega_{\text{crit}}$ from \ref{3.56}, the heat pole appears only in the "sound" branch $n = 1$ and does not exist for the Landau–Placzek branch of the solutions of \ref{2.209} for $s$. We realize it immediately noting that the width parameter $\gamma$ \ref{3.95} is proportional to $s_0$ which is finite for $n = 1$ and zero for $n = 0$ case, see \ref{2.74} and \ref{2.75}, respectively.

As shown in \ref{29}, for enough small $\Gamma_{T}$, the coefficient $\psi_{l}(0)$ in front of the Lorentzian-like correlation function, see \ref{3.118} and \ref{3.89}, is related to the difference of susceptibilities,

$$1/T \psi_{l}(0) = \chi_{T} - \chi_{0} = \chi_{T} - \chi_{\text{ad}} + \chi_{\text{ad}} - \chi(0). \tag{3.101}$$

Neglecting a small difference $\chi_{T} - \chi_{\text{ad}}$ according to \ref{3.19}, \ref{3.61}, see Appendix C and \ref{29} for details, one notes that the ergodicity condition \ref{3.12} means smallness of the $(1/T)\psi_{l}(0)$ compared to the stiffness $C$.

However, from \ref{3.90}, \ref{3.91} one gets a large quantity $(1/T)\psi_{l}(0)/C \approx 1/(kC)$ \ref{3.100}. Note, in the derivations of \ref{3.88}, \ref{3.90}, we took first $\omega \rightarrow 0$ (small $\omega T$) for the finite $\Gamma_{T}$; see also \ref{3.88} for $\gamma$ in the second equation of \ref{3.90}, and then, considered $\Gamma_{T} \rightarrow 0$ (small temperature limit $T \rightarrow 0$). We emphasize that the limits $\omega \rightarrow 0$ and $\Gamma_{T} \rightarrow 0$ are not commutative, i.e., the result of the correlation function calculations depends on the order of executing of these two operations like for the infinite Fermi-liquid matter \ref{115}. This is obvious if we take into account that the "heat pole" last term in the denominator of \ref{3.50} appears in the next $(T^4)$ order in $T$ and is proportional to $1/(\omega T)$ in contrast to the other classical (sound) hydrodynamic terms, i.e., this $\gamma$-term turns into zero for $\Gamma_{T} \rightarrow 0$ ($T \rightarrow 0$).

The relation \ref{3.100} was derived in \ref{29} using the opposite sequence of the above mentioned limits, namely, first $\Gamma_{T} \rightarrow 0$ and then $\omega \rightarrow 0$ in line of the recommendations of Forster \ref{115} [first $\Gamma_{T} \propto q^2 \rightarrow 0$ (or $\gamma \rightarrow 0$, see \ref{2.119}, \ref{2.74}], and then, $\omega \rightarrow 0$ ($s \rightarrow 0$) for the infinite Fermi-liquid]. In this case there is no contradiction with the ergodicity for the finite Fermi-liquid drop. In the limit $\Gamma_{T} \rightarrow 0$ ($T \rightarrow 0$) for a finite value of $\omega$, the condition \ref{3.50} is fulfilled instead of \ref{3.83}, and the "heat pole" term proportional to $1/\omega$ in the denominator of the response function \ref{3.50} disappears within the ESA used in the FLDM, as noted above. It means formally that one can neglect $\psi_{l}(0)$ in \ref{3.89}, and we have small quantities on the both sides of \ref{3.101} taking into account the ergodicity condition \ref{3.12} derived in Appendix C. It is not obvious that the relation \ref{3.101} can
be also derived for the opposite consequence of the above mentioned limit transitions unlike the Forster recommendations, i.e., taking first limit $\omega \to 0$ for a finite $\Gamma_T$, and then, considering the limit $\Gamma_T \to 0$. In particular, \((3.81)\) for the overdamped correlation function was obtained for the last choice of the limit sequences. Equation \((3.81)\) does have also the Lorentzian-like shape but it is not related to the "heat pole" because the coefficient in front of the Lorentzian is not equal to $\chi^T - \chi(0)$. This equation was derived only for large $\Gamma_T$ compared to the $1/\tau$, see \((3.82)\), and is true only under these conditions and within inequalities \((3.50)\). There is no a $\delta(\omega)$ function-like peak in \((3.81)\) for all possible variations of the parameters for which this equation was derived. The overdamped shape of the correlation function like \((3.81)\) does not mean yet that this function is the "heat pole" one though the opposite statement is true. We point out again that \((1/T)\psi^{(0)}(0)\) is really large compared to the stiffness $C$, \((1/T)\psi^{(0)} = 1/k$, and the ergodicity condition \((3.12)\) is fulfilled rather than the relation \((3.101)\) between \((1/T)\psi^{(0)}(0)\) and $\chi^T - \chi(0)$ within the hydrodynamic conditions \((3.50)\).

Following the Forster's recommendations \([115]\), i.e., take first the limit of small $\Gamma_T$ ($\Gamma_T \to 0$) or small temperature ($T \to 0$), one gets the typical hydrodynamic response function \((3.57)\) without "heat pole" terms. The next limit $\omega \to 0$ ($\omega \tau \to 0$) in \((3.57)\) leads to the finite value,

$$\chi(0) = \frac{1}{k^2C_{in}} \approx -\frac{1}{k} - C,$$

(3.102)

up to the relatively small corrections of higher order in parameter $kC$ \((3.70)\). This is in line of Appendix C, and the ergodicity condition \((3.12)\) is fulfilled for the finite Fermi-liquid drop within the ESA. Note that we accounted above for the $kC$ correction at the second order in \((3.102)\). In this way, we got the relation \((3.71)\) between the coupling constant $k^{-1}$, isolated susceptibility $\chi(0)$ and stiffness $C$ provided that the condition \((3.50)\) is true, see also \((3.10)\) with the stiffness $C(0) = C$ of the "zero frequency limit". Note also that the "heat pole" response function $\chi_{hp}(\omega)$ \((3.83)\) has a sharp peak near the zero frequency, and hence, is not smooth, i.e., "the zero frequency limit" for the transport coefficients can not be applied in the case \((3.83)\).

Thus, all properties of the finite Fermi liquids within the ESA concerning the ergodicity relation \((3.12)\), as applied to \((3.101)\), are quite similar to the ones for the infinite nuclear matter [besides the expressions \((3.08)\), $\Gamma_\perp \propto b/y A^{1/3}$, and \((3.91)\), $\Gamma_{hp} \propto 1/(b_y A^{1/3})$, themselves depending on $b_y$]. Our study of these properties is helpful for understanding the microscopic shell-model approach \([24, 29, 50]\). We point out that the strength function corresponding to the asymptotics \((3.54)\) is the curve with the two maxima which are related to the "heat pole" and standard (sound) hydrodynamic modes. However, for intermediate frequencies $\omega$ of the order of $\omega_{crit}$ in the low frequency region, see \((3.50)\) and \((3.53)\), the asymptotic response function \((3.50)\) can not be presented exactly in terms of a sum of the two oscillator response functions like \((3.58)\). For instance, in this case we have the transition from the "heat pole" mode to the sound hydrodynamic peak, and the response function \((3.50)\) is more complex. We have a similar problem when the hydrodynamic condition $\omega \tau \ll 1$ becomes not valid. However, as shown in the next subsection, such problems can be overcome approximately using an alternative definition for the transport coefficients suggested in \([28]\).

For larger frequencies, i.e., for $\omega \tau$ larger or of the order of 1, but within the low frequencies $\omega$ smaller than $\Omega$, see \((3.37)\), the equation for the collective motion becomes more complicate. It is not reduced generally speaking to the second order differential equation with the constant coefficients as in the zero frequency limit of the hydrodynamic approach \((3.50)\). As shown and applied in \([28, 29]\) (see also \([32]\) in connection to the FLDM), the problem of the definition of transport coefficients can be nevertheless overcome by defining them through a procedure of fitting an oscillator response function \((3.58)\) to selected peaks of the collective response function $\chi_{QQ}(\omega)$ of \((3.46)\) with respect to the parameters $M$, $C$ and $\gamma$. Here such a fitting procedure would also be adequate for temperatures mentioned above, especially because our response function \((3.40)\) has several poles \((3.48)\), for instance, with $i = 0, 1, 2; n = 1$ and $i = 0; n = 0$. Some of them are the overdamped poles close to the imaginary axis in the $\omega$-complex plane. This procedure can be done analytically in the zero frequency limit provided that the response function \((3.40)\) can be approximated by the oscillator response functions as in \((3.58)\) or by $\chi_{osc}^{hp}(\omega)$ in \((3.40)\). In this case, we have analytical fitting of the collective response function \((3.40)\) by these oscillator response functions and get the expressions for the transport coefficients \((3.72) - (3.74)\) in the zero frequency limit \((3.50)\) or \((3.91)\) for the heat pole friction in a smaller frequency region \((3.83)\). For larger frequencies, we need to carry out the fitting procedure numerically.

We should also comment a little more the definition of the transport coefficients in the zero frequency limit in connection to the one through the fitting procedure to avoid some possible misunderstanding. The transport coefficients in the zero frequency limit can be related to the "intrinsic" response function and its derivatives taken at $\omega \to 0$ \([24, 29]\); see \((3.72), (3.73), \text{and} (3.74)\). For application of this method of the transport coefficient calculations, we should be careful in the case when we have several peaks in the strength function but we need to get the transport coefficient, for instance, for the second or more high peaks. In these cases the zero frequency limit might be applied also, but we have first to remove all lower peaks in the collective response function and take then the corresponding "intrinsic" response function and its derivatives without these lower peaks. In practical applications, this limit for the transport coefficients obtained in a such way is close to the same limit for the oscillator response function which fits the selected peak.
The latter could be also the second or more high one. We shall consider now the hydrodynamical approximation $\omega \tau \ll 1$ for the response function, see (3.50), for the two cases: The sound response function (3.57) for the sound condition (3.50) and the heat-pole response function (3.84) for the heat pole condition (3.85). The corresponding correlation functions are the sound correlation function (3.51) and the heat-pole correlation one (3.85). These two different approximations are realized for different consequence of the limit transitions, i.e., the approximate result depends on the consequence of their applying. The heat pole case (3.50) is realized when we take first the limit $\omega \rightarrow 0$ for a finite width $\Gamma_T$, and then, $\Gamma_T \rightarrow 0$ (or zero temperature limit $T \rightarrow 0$). This leads approximately to the $\delta(\omega)$-like function for the correlation function. In contrast to this, the sound pole case (3.85) is realized when we take first $\Gamma_T \rightarrow 0$ (or $T \rightarrow 0$) to remove the last heat pole term proportional to $\Upsilon$ in the hydrodynamical response (3.50), and then, $\omega \rightarrow 0$. We like to follow this last consequence of the limit transition in line of the Forster recommendations [115] when we have the response (3.57) and correlation (3.51) functions without heat pole. In this case the transport coefficients for $\omega \tau \ll 1$ are the standard hydrodynamical ones (3.72), (3.73) and (3.74) related to the parameters of the standard hydrodynamical model (3.52), (3.53) and (3.55), respectively. Exception should be done for the modified mass parameter in (3.74) which turns into the irrotational flow inertia (3.51) for high temperatures.

D. Discussion of the results

In this subsection, we discuss the results of the FLDM calculations for the collective response function and transport coefficients. We shall explain now in more details the application of the general fitting procedure for the definition of the transport coefficients. We discuss also the stiffness and inertia parameters found within the FLDM. This subsection will be ended by the discussion of the friction versus temperature. One of the important points of this discussion is the ”heat pole” friction and comparison with the quantum shell-model calculations [24, 28, 29].

We show first the imaginary part of the response function $\chi_{QQ}^{\text{coll}}(\omega)$ (3.49) (its strength) for different temperatures in Fig. 4. The total collective response function $\chi_{QQ}$ is presented in Fig. 2 as a sum of the two branches $n = 0$ and 1 of eigen-frequencies $\omega^{(n)}$, see (3.48), in the imaginary part (strength) of the response function (3.49). They are related to the two different solutions of the dispersion equation (2.57) for the sound velocity $s^{(n)}$. These solutions are similar to the Landau–Placzek (Raleigh) and the sound (Brillouin) ones in normal liquids. The latter are approached exactly by $s^{(0)}$ and $s^{(1)}$ solutions for sound velocity $s$ in the hydrodynamic limit $\omega \tau \rightarrow 0$, which are related to the eigen-frequencies of the infinite-matter vibrations $\omega^{(0)}$ (2.74) and $\omega^{(1)}$ (2.75), respectively. The integral collision term is parametrized in terms of the relaxation time $\tau(\omega, T)$ [3.19], [3.20] with $c = 20$ MeV. We took the nucleus Pu-230 with particle numbers $A = 230$ as an example of enough heavy nucleus.

For the intermediate temperatures $4$ MeV $\lesssim T \lesssim 6$ MeV we have the three peak structure. More detailed plots for smaller frequencies are shown in Fig. 3 for the temperature $T = 6$ MeV for which the first two peaks ("heat pole" and usual hydrodynamic ones) are seen better in a normal scale. In Fig. 3 we show also the separate contributions of the two branches $n = 0$ (dotted line) and $n = 1$ (dashed one) for the eigen-frequencies $\omega^{(n)}$ (3.48) calculated from the secular equation (3.47) at each $s^{(n)} (n = 0, 1)$ as in Fig. 2. We present also the imaginary part of the asymptotic response function (3.50) obtained analytically above in the hydrodynamic frequent-collision limit. As seen from Fig. 4 we found from (3.48) the $n = 1$ mode with the two ($i = 0, 1$) peaks and the $n = 0$ mode with one peak ($i = 0$) for small frequencies $\omega$ and small parameter $\omega \tau$ in agreement with asymptotics (3.50). The heat pole contribution is shown separately by the dotted curve. Note that the two curves for $i = 0$ and 1 at $n = 1$ in Fig. 3 coincide because they both were calculated without the last $\Upsilon$ term in (3.50). For the dotted curve, one has $\Upsilon \propto s_0^{(0)} = 0$, and for the dashed one, the last $\Upsilon$ term in (3.50) is omitted under the asymptotical sound condition (3.50). Therefore, the upper asymptotical data (thin solid) marked also by the condition (3.56) are in factor about two larger than the dotted, or dashed, or asymptotical (3.50) ones.

The third peak in Fig. 2 appears for intermediate temperatures and larger frequencies. This peak is coming from the third pole $i = 2$ which belongs to the branch $n = 1$ in (3.48). This is the essentially Fermi-liquid underdamped mode due to the Fermi-surface distortions related to the shear modulus $\lambda$ given by (3.12). Such a peak is moving from a large zero-sound-frequency region of the giant resonances to smaller frequencies with increasing temperature. The second ($i = 1$) peak in the $n = 1$ branch and first ($i = 0$) peak in the $n = 0$ one in the low frequency region ($\omega \tau \ll 1$) are related to the overdamped motion described approximately by the overdamped oscillator response function like (3.58) for the same cut-off parameter $c = 20$ MeV. For $c = \infty$ the overdamped motion turns into the underdamped one for large temperatures $T \gtrsim 7$ MeV. The next (third) peak in a more high frequency region ($\omega \tau \gtrsim 1$) corresponds to the underdamped mode for the both c values. The first lowest peak in Figs. 2 and 3 which is not seen in Fig. 2 being too close to the ordinate axis and studied separately in Fig. 3 is due to the overdamped "heat pole" $i\Upsilon/2$ in the collective response function, see (3.55) for $\Upsilon$. The most remarkable property of this "heat pole" peak for smaller temperatures is that it has mainly a very narrow width (3.55) which increases with the temperature as $T^6$, see the comments concerning the heat pole "width"
after \(\frac{32}{33}\) and \(\frac{32}{33}\). This is in contrast to the temperature behavior of the width \(\Gamma_{-}\) like \(T^2\) for the hydrodynamic sound peak at large temperatures. Fig. 2 shows the three peaks only for the intermediate temperatures \(4 \lesssim T \lesssim 6\) MeV because for smaller temperatures the third peak moves to the high frequency region larger \(\Omega\) corresponding to the giant resonances and first peak is very close to the ordinate axis.

The transport coefficients for such two- or three resonance structure were calculated by a fitting procedure of the oscillator response functions to the selected peaks. We subtract first the "heat pole" peak known analytically, see (3.84), from the total response function (3.46). We are left then with the two-humped curve and fit then it by the sum of the two oscillator response functions as (3.85). One of them which fits the first (hydrodynamic) peak in the curve with the remaining two maxima is the overdamped oscillator response function (\(\eta > 1\)) and other one (more high in the low energy region) corresponds to the underdamped motion (\(\eta < 1\)). In this way, we get the two consequences of the transport coefficients presented in Figs. 4, 7. In these figures, the heavy squares are related to the second, hydrodynamic-sound peak of Figs. 2, 6 for the mostly overdamped modes with the effective friction \(\eta > 1\). The open squares show the third Fermi-liquid peak (see Fig. 2) related to the underdamped motion (\(\eta < 1\)) and Fermi-surface distortions, very specific for the Fermi liquids, in contrast to the normal liquids.

For the temperatures smaller about 6 MeV the second peak \(i = 1\) in the total response function is overdamped and is coming from the two poles (\(i = 1, n = 1\)) and (\(i = 0, n = 0\)) which are close to the standard hydrodynamic approach. The third peak, due to the Fermi-surface distortions as noted above, can not be found in principle in the hydrodynamic limit. The main difference between the second and third peaks can be found in the comparison of the stiffness coefficient \(C\) with the liquid-drop value \(C_{LD}\) obtained both from the fitting procedure mentioned above. For the third ("Fermi liquid" in sense of the relation to the Fermi surface distortions specific for the Fermi liquids, in contrast to normal ones) peak the stiffness \(C\) is much high than the liquid drop value \(C_{LD}\) in contrast to the second (typical hydrodynamical) one for which the stiffness \(C\) is very close to \(C_{LD}\) almost for all temperatures, see Fig. 4. It means that the third peak is essentially of different nature than the second one because exists only due to the Fermi-surface distortions. A measure of these distortions is the anisotropy (or shear modulus) coefficient \(\lambda\), see (15.12), which disappears in the hydrodynamic limit.

For enough large temperature (larger than or of the order of 7 MeV) all three peaks are not distinguished in Fig. 2. For such large temperatures the fitting procedure is a little modified to select these three peaks which are close to each other. For the finite \(c = 20\) MeV and all large temperatures presented in Fig. 2 nearly \(7 - 10\) MeV, we have one wide peak which can be analyzed as the superposition of the three peaks, namely the "heat-pole", usual overdamped hydrodynamic and underdamped "Fermi-liquid" ones. Subtracting the first "heat pole" peak [see (3.84)] as for lower temperatures, we fit then the remaining curve by the only one overdamped oscillator function like (3.55) for \(\eta > 1\). We subtract then again this overdamped fitted oscillator function from the response function (3.46) without the heat pole one (3.84) and fit the rest by the single underdamped oscillator. The found parameters of the two last oscillator response functions are used as initial values for the iteration fitting procedure of the sum of the two oscillator response functions of the same types to the response function (3.46) (without the heat pole). The found transport coefficients are presented in Figs. 4, 7. For enough large temperature nearly 10 MeV in the case \(c = \infty\) the only one underdamped oscillator can be used for fitting procedure of one peak [after an exclusion of the heat pole from (3.46)].

We show also the mass parameters found from the above described fitting procedure for several selected peaks in Fig. 5. For the third "Fermi-liquid" peaks the mass parameter \(M\) is close to the liquid drop values \(M_{LD}\) related to the irrotational flow. The mass parameter of the second "hydrodynamic" peak, due to the mixture of the identical (\(i = 1, n = 1\)) and (\(i = 0, n = 0\)) poles, is significantly smaller than the liquid drop value \(M_{LD}\) but finite. For the first "heat pole" (\(i = 0, n = 0\)) peak the mass parameter can be approximated only by zero. As noted above, the stiffness parameter for the third peak is much larger than the one for other (hydrodynamic) poles which is mainly close to the liquid drop value (see Fig. 4). As shown in Figures 4 and 5 for enough large temperatures the temperature dependences of the stiffness (\(\Gamma\)) and mass (\(M\)) parameters are close to their zero frequency limit, see (3.72) for \(\Gamma(0)\) and (3.74) for \(M(0)\). For smaller temperatures, the inertia \(M(0)\) (Fig. 5) becomes essentially larger than that found from the response function (3.46). It is in contrast to the stiffness \(\Gamma(0)\) which is identical to the liquid-drop quantity in the semiclassical limit \(h \to 0\) when \(\Gamma(0)\) does not contain quantum shell corrections.

Figs. 6 and 7 show the results for the friction coefficient \(\gamma / h\) versus the temperature for the collective response function \(\chi_{QQ}(\omega) = \kappa^2(T)\chi_{FF}(\omega)\) related to the \(\chi_{FF}(\omega)\) (3.40). We used here the same parameters as well in Figs. 2 and 3 for the response function. The solid line for the friction \(\gamma\) (3.58) corresponds to the response function (3.59) in the hydrodynamic limit (3.65), the same as for the zero frequency approach (3.72). The heavy squares show the result of the fit of (3.40) to the oscillator response function (3.58). We presented also the "heat-pole" contribution to the friction obtained from the fitting procedure by one "heat pole" (overdamped) oscillator response function (3.84), see circles in Fig. 7. We might compare the results of this fit to the friction analytically found in terms of the heat pole asymptotics (3.97) valid for smaller temperatures and shown by solid thin
lines in Figs. 6 and 7. They are in a good agreement for smaller temperatures where the overdamped "heat pole" with the "width" $\Gamma\propto T^2$ is more important. This "heat pole" friction is too big as compared to other friction components related to the hydrodynamical-sound (full squares) and "Fermi-liquid" poles in the usual scale of Fig. 6. Therefore, we use the logarithmic scale in Fig. 7.

Our FLDM friction, except for the "heat pole" one, is similar to the corresponding result of SM calculations 23, 24, see Fig. 8. A large SM friction coming from the diagonal matrix elements in Fig. 8 and standard hydrodynamic friction 3.53, as well as heavy squares shown in Figs. 6 and 7, are obviously similar. All these curves for temperatures $T \gtrsim 2$ MeV show the mainly diminishing friction, $\gamma \propto T^{-1}$ roughly like in hydrodynamics, see 3.53. Some deflection of the friction temperature dependence in Fig. 6 for large temperatures $T$ from usual hydrodynamic one $1/T^2$, i.e., a constant asymptotics is related to a different temperature behavior of the $\Gamma(0, T)$ dependence for a finite and infinite cut-off parameter $c$. This $\Gamma(0, T)$ goes to a constant for large temperatures if $c$ is finite and to zero for $c = \infty$, see the solid and dashed lines in Fig. 6.

It is noted also a similarity concerning the third ("Fermi-liquid") peak presented by the lower open squares with mainly increasing friction in Figs. 6, 7 and by joint full squares in Fig. 8. For $c = 20$ MeV and temperatures smaller about 10 MeV the friction of this mode increases, see Figs. 6, 7 in contrast to the standard hydrodynamic behavior (for $c = \infty$ this friction increases first up to about 60–7 MeV, and then, decreases at larger temperatures). In Fig. 8 the lower curve with growing dependence on the temperature for $c = 20$ MeV was obtained by excluding the contribution of the diagonal terms in the response function within the quantum approach based on the SM, see 24, 29 for the detailed explanations. Within the conceptions of the FLDM and classical hydrodynamics of the normal liquid drops the first "heat pole" friction obtained for small frequencies 3.53, within the hydrodynamic collision regime $\omega \tau \ll 1$ at finite temperature is the physical mode which can be excited when this regime might be realized like the Landau–Placzek pole for normal liquids. However, the hydrodynamic collision regime being still within a low frequency region (enough small collision frequency $1/\tau$) is expected to be not achieved in fission experiments. Therefore, the friction is related mainly to another Fermi-liquid mode corresponding to the only third peak owing to the Fermi-surface distortions. The friction of this mode is much smaller than the hydrodynamic one for small temperatures, and they become comparable for high ones. The Fermi-surface distortion friction can be characterized by completely other, mainly growing temperature behaviour, see the lower curve marked by open squares in Figs. 6 and 7. Concerning the SM calculations, it seems that we should omit the diagonal matrix elements, see 24, because of similar arguments: The hydrodynamic collision regime seems to be not realized for nuclear fission processes. (These diagonal matrix elements might correspond to the physical hydrodynamic mode if it is excited, say in another systems like a normal liquid drop). The quantum shell-model friction without contributions of diagonal matrix elements is related probably to another non-hydrodynamic mode, such as the third peak for a Fermi-liquid drop, and this might be the physical reason for an exclusion of these matrix elements.

Note that in the SM response-function derivations the diagonal matrix elements mentioned above do not contribute in the Forster’s sequence of the limit transitions discussed at the end of the previous section, first $\Gamma_T \rightarrow 0$, for exclusion of the diagonal matrix elements at finite $\omega$, and then, $\omega \rightarrow 0$ limit. In this case, we have not contribution of the diagonal matrix elements in the friction, and we are left with the low friction curves with increasing temperature dependence shown in Figs. 6, 8. For the opposite limit sequence if we consider first the small frequency limit $\omega \rightarrow 0$ for the finite (large) $\Gamma_T$ we have the contribution of diagonal matrix elements to the friction shown by the curves decreasing with temperature which correspond to the hydrodynamic limit here. As noted above, the exclusion of diagonal matrix elements for this last case could be justified because the physical condition of the hydrodynamic limit $\omega \tau \ll 1$ is not probably realized in fission processes. In that case, we expect the increasing friction; which has essentially other, non-hydrodynamic nature. We might interpret it within the FLDM as related to the third peak, due to the Fermisurface distortions.

IV. NEUTRON-PROTON CORRELATIONS AND IVGDR

A. Extensions to the asymmetric nuclei

The FLDM was successfully applied for studying the global properties of the isoscalar multipole giant resonances having nice agreement of their basic characteristics, such as the energies and sum rules, with experimental data for collective excitations of heavy nuclei 31, 46. For the collective excitation modes in asymmetric neutron-proton nuclei, the FLDM was straightforwardly extended in particular for calculations of the IVGDR structure 33, 49, 51, 52. In this case, one has the two coupled (isoscalar and isovector) Landau–Vlasov equations for the dynamical variations of distribution functions, $\delta f_{\pm}(r, p, t)$, in the nuclear phase-space volume 33:

$$\frac{\partial}{\partial t} \delta f_{\pm}(r, p, t) + \frac{p}{m_{\pm}} \nabla_r [\delta f_{\pm}(r, p, t)] + \delta(\varepsilon - \varepsilon_f) \delta \varepsilon_{\pm} + V_{\text{ext}}^{\pm} = \delta S t_{\pm}, \quad (4.1)$$

Here $m_{\pm}^*$ are the isoscalar $(\pm)$ and isovector $(-)$ effective masses, $\varepsilon = p^2/(2m_{\pm}^*)$, $\varepsilon_f = (p F^2)/(2m_{\pm}^*)$ is the Fermi energy. The splitting between the Fermi momenta $p F^\pm$ is
originated by the difference of the neutron and proton potential well depths, due to the Coulomb interaction
\[ p_F^p = p_k (1 + \Delta), \quad \Delta = 2(1 + F'_0)T/3, \] (4.2)
where \( F'_0 = 3J/\varepsilon_p - 1 \) is the isotropic isovector Landau constant of the quasiparticle interaction [4.0], \( J \) is the volume symmetry energy constant [2]. The asymmetry parameter \( T = (N - Z)/A \) is assumed to be small near the nuclear stability line, \( N \) and \( Z \) are the neutron and proton numbers in the nucleus \((A = N + Z)\). In [4.1], for the dynamical variations of the self-consistent quasiparticle (mean-field) interaction \( \delta \varepsilon_\pm (r, p, t) \), one has
\[ \delta \varepsilon_\sigma = \pi^2 \hbar^3 \sum \sigma' \left[ F_{0, \sigma\sigma'} p_{\sigma'} m_{\sigma'}^* \delta \rho_{\sigma'} + \frac{m F_{1, \sigma\sigma'}}{m_{\sigma}^* p_F^p (p_F^p)} \mathbf{p} \cdot \mathbf{j}_{\sigma'} \right]. \] (4.3)
The sum is taken over the sign index \( \sigma = \pm \). The dynamical variations of the quasiparticle interaction \( \delta \varepsilon_\pm \) at the first order with respect to the equilibrium energy \( p^2/(2m_{\pm}^*) \) is defined through those of the particle density,
\[ \delta \rho_\pm (r, t) = \int \frac{2dp}{(2\pi \hbar)^3} \delta f_\pm (r, p, t) \] (4.4)
[zero \( p \)-moments of the dynamical distribution functions \( \delta f_\sigma (r, p, t) \) [2.28], and the current density,
\[ \mathbf{j}_\pm (r, t) = \int \frac{2dp}{(2\pi \hbar)^3} \frac{p}{m} \delta f_\pm (r, p, t) \] (4.5)
(their first \( p \)-moments). The Landau interaction constants \( F_{l, \sigma\sigma'} \) in [133] are defined by expansion of the scattering quasiparticle's interaction amplitude \( F_{\sigma\sigma'} (\mathbf{p}, \mathbf{p}') \) in the Legendre polynomial series,
\[ F_{\sigma\sigma'} (\mathbf{p}, \mathbf{p}') = F_{0, \sigma\sigma'} + F_{l, \sigma\sigma'} \mathbf{p} \cdot \mathbf{p}' + \ldots, \quad \mathbf{p}' = \mathbf{p}/p. \] (4.6)
For the sake of simplicity, we assume that \( F_{l, \sigma\sigma'} \) is a symmetrical matrix \((l \leq 1)\) and \( F_{l, pp} - F_{l, nn} \) is of the second order in parameter \( \Delta \) [see below (4.1)], and can be neglected in the linear approximation with respect to \( \Delta \),
\[ F_{l, pp} = F_{l, nn}, \quad F_{l, pn} = F_{l, np}. \] (4.7)
Thus, we arrive at usual simple definitions for the isoscalar \( F_0 \) and \( F_1 \) and isovector \( F'_0 \) and \( F'_1 \) Landau interaction constants [133],
\[ F_1 = (F_{l, pp} + F_{l, pn})/2, \]
\[ F'_1 = (F_{l, pp} - F_{l, pn})/2, \quad l = 0, 1. \] (4.8)
These constants are related to the Skyrme interaction constants in the usual way [130]. The isoscalar \( (F_0) \) and isovector \( (F'_0) \) isotropic interaction constants are associated with the volume incompressibility modulus \( K \) and symmetry energy constant \( J \), respectively. The anisotropic interaction constants \( F_1 \) and \( F'_1 \) correspond to the effective masses by equations \( m_\pm^* = m(1 + F'_1/3) \) and \( m_\pm^* = m(1 + F_1/3) \). The periodic time-dependent external field in (4.1) is given by \( V_{ext} \propto \exp(-i\omega t) \) as in [33]. The collision term \( \delta S_{\pm} \) is taken in the simplest \( \tau_\pm \)-relaxation time approximation [2.9].

Solutions of these equations (4.1) associated with the dynamic multipole particle-density variations, \( \delta \rho_\pm (r, t) \propto Y_{l0}(\hat{r}) \) in the spherical coordinates \( r, \theta, \varphi \), can be found in terms of a superposition of the plane waves [2.28] over angles of the wave vector \( \mathbf{q} \) as
\[ \delta f_\pm = \delta \left( \epsilon - (p_{\pm}^2)/2m_{\pm}^* \right) \]
\[ \times \int d\Omega \mathbf{q} A_\pm Y_{l0}(\hat{q}) \exp \left[ i(\mathbf{q} \cdot \mathbf{r} - \omega t) \right], \quad \hat{q} = \mathbf{q}/q. \] (4.9)
\[ \omega = p_{\pm}^s q \sqrt{NZ/A^2}/m_{\pm}^* = |q|. \] The factor \( \sqrt{NZ/A^2} \) ensures the conservation of the center-of-mass position for the odd vibration multipolarities \( L \) [129], in particular, for the dipole modes \((L = 1)\). The amplitudes of the Fermi surface distortions \( A_\pm \) are determined by (4.1). For the simplest case of the zero anisotropic interaction \((F_1 = F'_1 = 0)\) in the collisionless limit \( \omega_\tau \to \infty \), the dispersion equation for the sound velocity \( s \) takes the form:
\[ 4F_0 F'_0 (F_0 Q_1(s) - 1) - \frac{1}{4} \Delta^2 F_0^2 F'_0 \left( \frac{s^2}{s^2 - 1} + Q_1(s) \right)^2 = 0, \] (4.10)
(We accounted for a small \( \Delta \) and large \( \omega_\tau \) at the zero temperature.) This equation has the two solutions \( s = s_n \) related to the main peak \( n = 1 \) and 2 for its satellite, see (26) of [33] for the finite \( \omega_\tau \) and nonzero \( F_1 \) and \( F'_1 \). In the limit \( \Delta \to 0 \), the dispersion equations given by (25) of [33] with our definitions for \( s_1 \) and \( s_2 \) modes \( n = 1 \) and 2 are resulted in the two (isovector and isoscalar) equations for the equations for the separated zero sounds, respectively,
\[ Q_1(s) = 1/F'_0, \quad \text{and} \quad Q_1(s) = 1/F_0. \] (4.11)

For the finite Fermi-liquid drop with a sharp ES [27, 37, 38], the macroscopic boundary conditions for the pressures and those for the velocities were derived in [33, 39, 40]. For small isovector vibrations near spherical shape, the radial mean-velocity \( u_r \) and momentum-flux-tensor \( \Pi_{rr} \) components, defined through the moments of the distribution function \( \delta f_\sigma \) as solutions of the kinetic equation (4.1) [see (2.16) and (2.20)] are given by (3.21) and (3.22) with \( u_r = u_0^+ - u_0^- \) and \( \Pi_{rr} = \Pi_{rr}^+ - \Pi_{rr}^- \). The r.h.s of these boundary conditions are the isovector ES velocity \( u_0 = R \dot{Q}_S Y_{L0}(\hat{r}) \) and capillary pressure exceed
\[ \delta P_S = 2Q_S b_S^0 p_0 A^{1/3} Y_{10}(\hat{r})/3, \] (4.12)
given through the isovector surface energy constant \( b_{\perp}^{\perp} \sim a_{\perp} \) [see (D.7) and (D.10)], where \( Q_{S} \) is the dynamical isovector-dipole \( (L = 1) \) amplitude of the motion of the neutron drop ES against the proton one \([3.15]\), keeping also the volume and the position of the center of mass conserved). Note that another interpretation of the surface symmetry-energy constant \( b_{\perp}^{\perp} \) in \([4.12]\) is considered in \([10, 53]\). This constant essentially differs from the isovector stiffness introduced in \([2]\) for the description of the neutron skin as a collective variable, see more detailed discussions in \([40, 51]\).

The energy constant, \( D = \hbar \omega A^{1/3} \), and energy weighted sum rules (EWSR),

\[
S_{1} = \frac{\hbar^{2}}{\pi} \int d\omega \omega \text{Im} \chi^{\text{coll}}(\omega), \quad (4.13)
\]

for the IVGDR can be found from the collective response function \( \chi^{\text{coll}}(\omega) \). The response function \([4.4]\) is determined by the transition density \([3.31]\) generalized to the dynamic isoscalar and isovector components \([52]\):

\[
\delta \rho_{\pm}(r, t) = \delta \rho_{\pm}^{v}(r, t) w_{\pm}(\xi) - \frac{1}{d\xi} \frac{dw_{\pm}(\xi)}{d\xi} [\delta R_{\pm} - \delta N_{L}^{\pm} Y_{\text{LO}(\hat{r})}] \quad (4.14)
\]

where \( \delta N_{L}^{\pm} \) is defined by the mass center conservation \((\int dr \, \delta \rho_{\pm} = 0)\), \( w_{\pm}(\xi) \) is given by \([D.2]\) and \([D.4]\). In Fig. \([9]\) a strong SO dependence of the isovector density \( w_{+}(\xi) \) is compared with that of the isoscalar one \( w_{-}(\xi) \) (low index “−” is omitted here and below) for the SLy7 force as a typical example \([40, 40]\). As shown in \([40]\), the isoscalar \( w(\xi) \), and therefore, the isovector \( w_{-}(\xi) \) densities depend rather strongly on the most of the Skyrme forces \([74, 75]\) near the ES. In Fig. \([10]\) (in logarithmic scale), one observes notable differences in the isovector densities \( w_{\pm}(\xi) \) derived from different Skyrme forces within the edge diffuseness. In particular, this is important for the calculations of the neutron skins of nuclei \([40]\). We emphasize that the dimensionless densities, \( w(x) \) \([D.2]\) and \( w_{-}(x) \) \([D.4]\), shown in Figs. \([9]\) and \([10]\) were obtained in the leading ES approximation \((a/R \ll 1)\) as functions of the specific combinations of the Skyrme force parameters, such as \( \beta \) and \( c_{\text{sym}} \) of \([D.3]\). Therefore, they are the universal distributions independent of the specific properties of the nucleus such as the neutron and proton numbers, and the deformation and curvature of the nuclear ES; see also \([25, 27, 35]\). These distributions yield approximately the spatial coordinate dependence of local densities in the normal-to-ES direction \( \xi \). With the correct asymptotical behavior outside of the ES layer for any ES deformation, they satisfy the leptodynamic condition \( a/R \ll 1 \), in particular, for the semi-infinite nuclear matter.

The universal functions \( w_{\pm}(\xi) \) \([D.2]\) and \( w_{-}(x) \) \([D.4]\) of the leading order in the ESA can be used [explicitly analytically in the quadratic approximation for \( \epsilon(w) \)] for the calculations of the surface energy coefficients \( b_{\perp}^{\perp} \) \([D.7]\), the neutron skin and isovector stiffness \([40]\). As shown in Appendices B and C of \([40]\), only these particle-density distributions \( w_{\pm}(\xi) \) within the surface layer are needed through their derivatives \([\text{the lower limit of the integration over } \xi \text{ in } (D.7) \text{ can be approximately extended to } -\infty \text{ because of no contributions from the internal volume region in the evaluation of the main surface terms of the pressure and energy}]\). Therefore, the surface symmetry-energy coefficient \( b_{\perp}^{\perp} \) \([D.10]\) and \([D.12]\) (also the neutron skin and the isovector stiffness \([40]\)) can be approximated analytically in terms of the functions of the definite critical combinations of the Skyrme parameters such as \( \beta, c_{\text{sym}}, a \) [see \([D.5]\)], and the parameters of the infinite nuclear matter \((b_{\nu}, \rho_{\infty}, K)\). Thus, they are independent of the specific properties of the nucleus (for instance, the neutron and proton numbers), and the curvature and deformation of the nuclear surface in the considered ESA.

Solving the Landau–Vlasov equations \([4.1]\) in terms of the zero sound plane waves \([1.9]\) with using the dispersion equations \([26]\) in \([33]\) for the sound velocities \( s_{n} \) and macroscopic boundary conditions \([8.21]\) and \([8.22]\) with \([4.12]\) on the nuclear ES, from \([D.3]\) and \([D.14]\) one obtains

\[
\chi^{\text{coll}}(\omega) = \sum_{n} \frac{A_{n}(q)}{D_{n}(\omega - i\Gamma/2)} = \sum_{n} \frac{A_{n}(q)}{D_{n}(\omega - i\Gamma/2)} = \frac{3e_{s}qR}{2b_{\perp}^{\perp}A^{1/3}} \text{Im} \chi_{L}^{\perp}(\omega), \quad (4.15)
\]

Here, \( c_{1} \approx 1 - s_{n}^{2} + F_{0}^{'}, d_{1} \approx 1 - s_{n}^{2} + F_{0}^{'}, F_{0} \) for the main \((n = 1)\) IVGDR peak. Small anisotropic \( F_{1} \) and \( F_{1}^{'}, c_{1} \) corrections and more bulky expressions for \( s_{0} \) of the satellite \((n = 2)\) peak of a smaller \((\propto I)\) strength were omitted (see \([D.11]\) in \([33]\) for more precise expressions). We present here also the simplest expressions for the amplitudes, \( A_{1}(q) \approx -\rho_{\infty}R^{3} j_{1}(qR)/(m\omega^{2}) \) and \( A_{1}(q) \approx \Delta \propto I \) for the \( n = 1 \) and 2 modes [see a more complete equation \([60]\) in \([32]\)]. The Bessel functions \( j_{1}(z) \) and its derivative \( j_{1}'(z) \) were defined after \([4.32]\) \((L = 1)\). The poles of the response function \( \chi^{\text{coll}}(\omega) \) \([4.15]\) (roots \( \omega_{n} \) of the equation \( D(\omega - i\Gamma/2) = 0 \) or \( q_{n} \)) determine the IVGDR energies \( \hbar \omega \) as their real part (the IVGDR width \( \Gamma \) is determined by their imaginary part). The residue \( A_{n} \) is important for the calculations of the EWSR \([4.13]\) at a small width of the IVGDR \( \Gamma \). Note that the expression like \([4.15]\) for the only one main peak (without the IVGDR structure) in symmetrical nuclei \((N = Z)\) with using the phenomenological boundary conditions which have the same form as \([8.21]\) and \([8.22]\), however where the isovector neutron-skin stiffness was applied instead of the surface symmetry-energy constant \( b_{\perp}^{\perp} \) in the capillary pressure exceed \([4.12]\) was obtained earlier in \([40]\).
B. Discussions of the asymmetry effects

The isovector surface energy constants \( k_{\beta} \) in the ESA using the simplest quadratic approximation of the energy density \( D \) are shown in Table 1 for several critical Skyrme forces \( 74, 75 \). These constants are rather sensitive to the choice of the Skyrme forces. The modulus of \( k_{\beta} \) for the Lyon Skyrme forces SLY4-7 74 is significantly larger than for other forces, all of them much smaller than those related to \( 2, 60, 62 \). For T6 74, one has \( C_\beta = 0 \), and therefore, \( k_{S} = 0 \), in contrast to all of other forces shown in Table 1. Notice that the isovector gradient terms which are important for the consistent derivations within the ESA \( 40 \) are not also included \( (C_\beta = 0) \) into the energy density in \( 60, 62, 67, 72 \). For RATP 74, the isovector stiffness \( (\propto -1/k_\beta) \), corresponding inversed \( k_\beta \) with the opposite sign \( 40 \), is even negative as \( C_\beta > 0 \) \( (k_\beta > 0) \). The reason of significant differences in these values might be related to those of the critical isovector Skyrme parameter \( C_\beta \) in the gradient terms of the energy density \( D \). Different experiments used for fitting this parameter were found to be almost insensitive in determining uniquely its value, and hence, \( k_{S} \) [or \( b_{S}^{(-)} \)], see \( 10, 12 \), as compared to the well-known isoscalar surface-energy constant \( b_0^{(+)} \). The isovector surface-energy constant \( k_{\beta} \) and the corresponding stiffness depend much on the SO parameter through the constant \( J_\beta \) in \( 12 \).

The IVGDR energy constants \( D = \hbar \omega (-)^{A^{1/3}} \) of the hydrodynamic model (HDM) are roughly in good agreement with the well-known experimental value \( D_{\exp} \approx 80 \) MeV for heavy nuclei within a precision better or of the order of 10\%, as shown in \( 10, 51 \) (see also \( 33, 49, 131 \)). More precise \( A^{-1/3} \) dependence of \( D \) seems to be beyond the accuracy of these HDM calculations. This takes place even accounting more consistently for the ES motion because of several other reasons (the macroscopic Fermi-surface distortions \( 49 \), also including structure of the IVGDR \( 33, 50, 52, 54, 131 \), curvature, Coulomb, quantum-shell, and pairing \( 0 \) effects towards the realistic self-consistent calculations based on the Skyrmie HF approach \( 132, 136 \). Larger values 30-80 MeV of the isovector stiffness \( 2 \) [smaller \( k_{\beta} \)] were found in \( 60, 62, 67, 72 \). With smaller \( |k_{\beta}| \) (see Table 1, or larger the isovector stiffness) the fundamental parameter of the LDM expansion in \( 2, 60, 62 \) is really small for \( A \approx 40 \), and therefore, the results obtained by using this expansion are justified \( 40 \).

Table 1 shows also the mean IVGDR energies \( D \) obtained \( 40, 51 \) within a more precised FLDM \( 33 \). The IVGDRs even for the spherical nuclei have a double-resonance structure, the main peak \( n = 1 \) which exhausts mainly the EWSR for almost all Skyrme forces and the satellite one \( n = 2 \) with the significantly smaller EWSR contributions proportional to the asymmetry parameter \( I \), typical for heavy nuclei. The last row shows the average \( D(A) \) weighted by their EWSR distribution in rather good agreement with the experimental data within the same accuracy about 10\%, and in agreement with the results of different other macroscopic IVGDR models \( 49, 53, 54, 131 \). Exclusion can be done (see Table 1) for the Skyrme forces SIII \( 74 \) and SkL3 \( 72 \), where we obtained a little larger IVGDR energies. Note that the main characteristics of the IVGDR described by mean \( D \) are almost insensitive to the isovector surface-energy constant \( k_{\beta} \) \( 40, 51 \). Therefore, we suggested \( 10, 52 \) to study the IVGDR two-peak (main and satellite) structure in order to fix the ESA value of \( k_{\beta} \) from comparison with the experimental data \( 157, 139 \) and theoretical results \( 132, 136, 140 \).

V. NUCLEAR COLLECTIVE ROTATIONS

A. General ingredients of the cranking model

Within the cranking model, the nuclear collective rotation of the Fermi independent-particle system associated with a many-body Hamiltonian, \( H = \omega = H + H_{\text{CF}} \), can be described, to a good approximation \( 129 \), in the restricted subspace of Slater determinants, by the eigenvalue problem for a s.p. Hamiltonian, usually called the Routhian. For this Routhian, in the body-fixed rotating frame \( 4, 15 \), one has

\[
h_{\omega} = h + h_{\omega}^{\text{CF}}, \quad h_{\omega}^{\text{CF}} = -\omega \cdot (\ell + s),
\]

where \( h_{\omega}^{\text{CF}} \) is the s.p. cranking field which is approximately equal to the Coriolis interaction (neglecting a smaller centrifugal term, \( \propto \omega^2 \)). The Lagrange multiplier \( \omega \) (rotation frequency of the body-fixed coordinate system) is defined through the constraint on the nuclear angular momentum \( I \), evaluated through the quantum average \( \langle \ell + s \rangle \omega = I \), of the total s.p. operator, \( \ell + s \), where \( \ell \) is the orbital angular momentum and \( s \) is the spin of the quasiparticle, thus defining a function \( \omega = \omega(I) \). The quantum average of the total s.p. operator \( \ell + s \) is obtained by evaluating expectation values of the many-body Routhian \( H_{\text{CF}} \) in the subspace of Slater determinants. For the specific case of a rotation around the \( x \) axis \( (\omega = \omega_x) \) which is perpendicular to the symmetry \( z \) axis of the axially-symmetric mean field \( V \), one has (dismissing for simplicity spin (spin-isospin) variables),

\[
\langle \ell_x \rangle \omega = d_s \sum_i n_i^{+} \int dr \, \psi_i^{\omega}(r) \, \ell_x \tilde{\psi}_i^{\omega}(r) = I_x,
\]

where \( d_s \) as the spin (spin-isospin) degeneracy in the case of the corresponding symmetry of the mean potential \( V \). The occupation numbers \( n_i^{\omega} \) for the Fermi system of independent nucleons are given by

\[
n_i^{\omega} = n(\varepsilon_i^{\omega}) = \{1 + \exp\left[(\varepsilon_i^{\omega} - \mu^{\omega})/T\right]\}^{-1}.
\]

In \( 52 \), \( \psi_i^{\omega}(r) \) are the eigenfunctions and \( \tilde{\psi}_i^{\omega}(r) \) their complex conjugate, \( \varepsilon_i^{\omega} \) the eigenvalues of the Routhian.
\( h\omega \) [5.1]. \( \mu^w \) is the chemical potential. For relatively small frequencies \( \omega \) and temperatures \( T \), \( \mu^w \) is to a good approximation equal to the Fermi energy, \( \mu^w \approx \varepsilon_F = h^2 k_F^2 / 2m^* \), where \( k_F \) is the Fermi momentum in units of \( h \). From (5.2), the rotation frequency \( \omega \) can be specifically expressed in terms of a given angular momentum of \( \approx \hbar / I_x \). Within the same approach, one approximately has for the particle number

\[
A = d_s \sum_i n_i \int d\varphi \psi_i^*(\varphi) \psi_i(\varphi) \approx d_s \int d\varphi n(\varphi), \tag{5.4}
\]

which determines the chemical potential \( \mu^w \) for a given number of nucleons \( A \). As we introduce the continuous parameter \( \omega \) and ignore the uncertainty relation between the angular momentum and angles of the body-fixed coordinate system, the cranking model is semiclassical in nature [81]. Thus, we may consider the collective MI \( \Theta_x \) (for a rotation around the \( x \) axis, and omitting, to simplify the notation, spin and isospin variables) as a response of the quantum average \( \delta(\ell_x)^\omega \) [52], to the external cranking field \( h\omega \) in (5.1). Similarly to the magnetic or isolated susceptibilities [108, 109, 111, 112], one can write

\[
\delta(\ell_x)^\omega = \Theta_x(\omega)\delta\omega, \tag{5.5}
\]

where

\[
\Theta_x(\omega) = \partial(\ell_x)^\omega / \partial \omega = \partial^2 E(\omega) / \partial \omega^2, \quad E(\omega) = (h) = d_s \sum_i n_i \int d\varphi \psi_i^*(\varphi) h \psi_i(\varphi). \tag{5.6}
\]

Traditionally [5, 110, 113], another parallel (alignment) rotation with respect to the symmetry \( z \) axis can be also considered as presented in Appendix A of [113].

As was shown in [4, 11, 12], one can treat the term \(-\omega \cdot \ell = -\omega \ell_x \) as a perturbation for a nuclear rotation around the \( x \) axis. With the constraint (5.2) and the MI [5, 6] treated in second order perturbation theory, one obtains the well-known Inglis cranking formula. Instead of carrying out the rather involved calculations presented above, one could, to obtain the yrast line energies \( E(I_x) \) for small enough temperatures \( T \) and frequencies \( \omega \), approximate the angular frequency by \( \omega = I_x / \Theta_x \) and write the energy in the form

\[
E(I_x) = E(0) + \frac{I_x^2}{2\Theta_x}. \tag{5.7}
\]

As usually done, the rotation term above needs to be quantized through \( I_x^2 \rightarrow I_x(I_x + 1) \) in order to study the rotation bands.

**B. Self-consistent ETF description of nuclear rotations**

Following reference [87], a microscopic description of rotating nuclei was obtained in the Skyrme Hartree–Fock formalism, within the Extended Thomas–Fermi density-functional theory up to order \( \hbar^2 \). Within a variational space restricted to Slater determinant, the minimization of the expectation value of the nuclear Hamiltonian lead to the s.p. Routhian \( h^2 \) [5.1] that is determined by a one-body potential \( V_q(r) \), a spin-orbit field \( W_q(r) \) and an effective mass form factor \( f_q^{\text{eff}}(r) = m / m_q^* \) (see also [72]). In addition, in the case when the time reversal symmetry is broken, a cranking field form factor \( \alpha_q(r) \) and a spin field form factor \( S_q(r) \) also appear. In this subsection the (roman) subscript \( q \) refers to the nucleon isospin \( (q = \{n,p\}) \) and should not be confused with the wave number \( q \) in other sections. All these fields can be written as functions of local densities and their derivatives, like the neutron-proton particle densities \( \rho_q(r) \), the kinetic energy densities \( \tau_q(r) \), the spin densities (also referred to as spin-orbit densities) \( \mathbf{J}_q(r) \), the current densities \( \mathbf{j}_q(r) \), and the spin-vector densities \( \rho_V(r) \). Note that in the present subsection, \( \tau_q(r) \) stands for the kinetic energy density which should not be confused with the relaxation time in previous sections (here, however, with a different subscript \( q \) as compared to \( q \) in sections [6, 111, 114] and Appendices A,B). In principle, two additional densities appear, a spin-vector kinetic energy density \( \tau_V(r) \) and a tensor coupling \( J_{q2}(r) \) between spin and gradient vectors, which have, however, been neglected since their contribution should be small, as suggested by [143].

The cranking-field form factor \( \alpha_q(r) \) contains two contributions. One of them is coming from the orbital part of the constraint, \(-\omega \cdot \ell \), which has been shown in [144] to correspond to the Inglis cranking formula [7]. The other, a Thouless–Valatin self-consistency contribution [145] has its origin in the self-consistent response of the mean field to the time-odd part of the density matrix generated by the cranking term of the Hamiltonian. The aim is now to find functional relations for the local densities \( \tau_q(r) \), \( \mathbf{J}_q(r) \), \( \mathbf{j}_q(r) \) and \( \rho_V(r) \) in terms of the particle densities \( \rho_q(r) \), in contrast to those given by Grammaticos and Voros [144] in terms of the form factors \( V_q \), \( J_q^{\text{eff}} \), \( W_q \), \( \alpha_q \) and \( S_q \). Taking advantage of the fact that, at the leading Thomas–Fermi order, the cranking field form factor is given by [87]

\[
\alpha_q^{\text{TF}} = f_q^{\text{eff}}(r \times \omega), \tag{5.8}
\]

one simply obtains the rigid-body value for the Thomas–Fermi current density

\[
\mathbf{j}_q^{\text{TF}} = \frac{m}{\hbar} (\omega \times \mathbf{r}) \rho_q. \tag{5.9}
\]

This result is not that trivial, since it is only through the effect of the Thouless–Valatin self-consistency terms that such a simple result is obtained. Notice also that (5.9) corresponds to a generalization of the case \( f_q^{\text{eff}} \neq 1 \) of a result already found by Bloch [147].
average of the particle velocity, \( p_{\text{rot}} / m = \omega \times r \), rotating with the frequency \( \omega \). In particular, the re-normalization of the cranking field form factor \( \alpha(q)^{\text{TF}} = f_q^{\text{eff}} \alpha_q \) with \( \alpha_q = (r \times \omega) \), by (4.8) can be also explained as related to the effective mass corrections, \( f_q^{\text{eff}} \neq 1 \), obtained by Landau [34] with using both the Galileo principle and the Thouless–Valatin self-consistency corrections to a particle mass \( m \) due to the quasi-particles’ (self-consistent) interaction through a mean field. They lead in (5.7) to the self-consistent TF angular momentum of the quasi-particle \( \ell_q = f_q^{\text{eff}} \ell_o \) with the classical angular momentum \( \ell_o = r \times p \) of the particle, so that \( -\omega \cdot \ell_q = \alpha_q \cdot p \). This effect is similar to that for the kinetic energies of the quasi-particles, \( \varepsilon_q = p^2 / (2m_\alpha^*) = f_q^{\text{eff}} \varepsilon_o \) where \( \varepsilon_o = p^2 / (2m) \), see after (4.1). With this transparent connection to the Landau quasi-particle theory, it is clear that there is no contradictions with the TF limit of the current densities (4.5), \( \hbar \to 0 \), accounting for the particle densities (4.4), as well as with the definitions in subsections [\( \nabla \Lambda \)] and [\( \nabla \mathbf{C} \)] because \( \hbar \) in (5.9) appears formally due to a traditional use of the dimensionless units for the angular momenta in the quantum-mechanical picture to compare with experimental nuclear data. Another reason is related to a consistent treatment of the essentially quantum spin degrees of freedom, beyond the Landau quasi-particle approach to the description of Fermi liquids, which have no straight classical limit, in contrast to the orbital angular momentum \( \ell \). The convergence in the TF limit \( \hbar \to 0 \) can be realized for smooth already quantities after the statistical (macroscopic) averaging over many s.p. (more generally speaking, many-body) quantum states to remove the fluctuating (shell) effects which appear in the denominators of the exponents within the POT (see Sec. [\( \nabla \mathbf{C} \)] for more detailed discussions). Finally, the spin paramagnetic effect can be considered as a macroscopic one in the MI like the orbital diamagnetic contribution. For instance, the spin-vector density does not have a straight classical analogue, such as the orbital angular momentum, and is considered as the object of leading order \( \hbar \).

Starting from these results and taking advantage of the fact that in the functional ETF expressions up to the order \( \hbar^2 \), it is sufficient to replace quantities, such as the cranking field form factor \( \alpha_q \), by their Thomas–Fermi expressions (after the statistical averaging mentioned above). In order to obtain a semiclassical expression, that is correct to that order in \( \hbar \), one obtains for the spin-vector densities \( \rho_n \) and \( \rho_p \), which are of order \( \hbar \) in the considered ETF expansion, a system of linear equations. They can be easily resolved (5.7). One also notices from this system of equations that the spin-vector densities are proportional to the angular velocity \( \omega \). Exploiting the well known analogy of the microscopic Routhian problem with electromagnetism, one may then define spin susceptibilities \( \chi_q \):

\[
\rho_q = \hbar \chi_q \omega .
\]

(5.10)

The key question now is to assess the sign of these susceptibilities and to decide whether or not the corresponding alignment is of a “Pauli paramagnetic” character. The study of (5.7) shows that this is the case, i.e., that the spin polarization is, indeed, of paramagnetic character, thus confirming the conclusions of the work performed by Dabrowski [148] in a simple model of non-interacting nucleons.

Since the cranking field factor \( \alpha_q \) is, apart from that of the constraining field \( \alpha_\perp \) determined only by the current densities \( j_\perp \) and the spin-vector densities \( \rho_q \), one can then write down (5.7) the contributions to the current densities \( j_\perp \) going beyond the Thomas–Fermi approach. The semiclassical corrections of order \( \hbar^2 \) can be split into contributions \( (\delta j_\perp)_q \) and \( (\delta \rho_q)_q \) coming respectively from the orbital motion and the spin degree of freedom. It is found (5.7) that the orbital correction \( (\delta j_\perp)_q \) corresponds to a surface-peaked counter-rotation with respect to the rigid-body current proportional to \( (\omega \times r) \), thus recovering the Landau diamagnetism characteristic of a finite Fermi gas. With the expressions of the current densities \( j_\perp \) and the spin-vector densities \( \rho_q \) up to order \( \hbar^2 \), one can write down the corresponding ETF expressions for the kinetic energy density \( \tau_\perp(r) \) and spin-orbit density \( J_\perp(r) \).

Having now at hand the ETF functional expressions up to order \( \hbar^2 \) of all the densities entering our problem, one is able to write down the energy of the nucleus in the laboratory frame as a functional of these local densities,

\[
E = \int dr \rho \mathcal{E} \left[ \rho_q, \tau_\perp, J_\perp, j_\perp, \rho_\parallel \right],
\]

(5.11)

where \( \rho = \rho_n + \rho_p \) as in Appendix D, \( \rho \approx \rho_\infty w_+ \). Upon some integration by parts, one finds that \( \mathcal{E} \) can be written as a sum of the energy density per particle of the non-rotating system \( \mathcal{E}(0) \) and its rotational part, in line of (5.7).

Within the ETF approach, one has from (5.11)

\[
E_{\text{ETF}} = \int d\mathbf{r} \rho(0) \mathcal{E}(0) + \frac{1}{2} \Theta_{\text{ETF}}^{(\text{dyn})} \omega^2,
\]

(5.12)

where \( \Theta_{\text{ETF}}^{(\text{dyn})} \) is the ETF dynamical moment of inertia for the nuclear rotation with the frequency \( \omega \). This MI is given in the form:

\[
\Theta_{\text{ETF}}^{(\text{dyn})} = m \sum_q \int dr \left\{ r^2 \rho_q - (3\pi^2)^{-2/3} f_q^{\text{eff}} \rho_q^{1/3} \right. \left. + \left[ \frac{\hbar^2}{2m} + W_0 (\rho + \rho_q) \right] \chi_q \right\},
\]

(5.13)

where \( r_\perp \) is the distance of a given point to the rotation axis and \( W_0 \) is the Skyrme-force strength parameter of the spin-orbit interaction (72).

One notices that the Thomas–Fermi term which comes from the orbital motion turns out to be the rigid-body moment of inertia. Semiclassical corrections of order \( \hbar^2 \) come from both the orbital motion \( (\Theta_{\text{orb.}}^{(\text{spin})}) \) and from the spin degrees of freedom \( (\Theta_{\text{spin}}^{(\text{dyn})}) \). The contribution \( \Theta_{\text{orb.}}^{(\text{spin})} \) is negative corresponding to a surface-peaked
counter rotation in the rotating frame. Such a behavior is to be expected for a N-particle system bound by attractive short-range forces (see [149]). The spin contribution $\Theta_{\text{spin}}^{(\text{dyn})}$ turns out to be of the paramagnetic type, thus leading to a positive contribution which corresponds to an alignment of the nuclear spins along the rotation axis. It can also be shown (see [149]) that the ETF kinematic moment of inertia,

$$\Theta_{\text{ETF}}^{(\text{kin})} = \frac{(\ell + s)^2}{\omega},$$

(5.14)

is identical to the ETF dynamical moment of inertia presented above.

It is now interesting to study the importance of the Thouless–Valatin self-consistency terms. This has accomplished by calculating the moment of inertia in the Thomas–Fermi approximation but omitting, this time, the Thouless–Valatin terms. One then finds [87] the following expressions for the dynamical moment of inertia, in what is simply the Inglis cranking (IC) limit

$$\Theta_{\text{IC}}^{(\text{dyn})} = m \sum_q \int dr \left[ \frac{\rho_q}{f_q^{\text{eff}}} \right]^2 + \frac{mB_3}{\hbar^2} \rho_q \rho_{\bar{q}} \left( \frac{1}{f_{\bar{q}}^{\text{eff}}} - \frac{1}{f_q^{\text{eff}}} \right)^2 r_{\perp}^2,$$

(5.15)

where $\bar{q}$ is the other charge state ($\bar{q}=p$ when $q=n$ and vice-versa) and $B_3$ is defined through the Skyrme force parameters $t_1, t_2, x_1$ and $x_2$ (see [57]). Apart from the corrective term in $\rho_q \rho_{\bar{q}}$, one notices that the first term in the expression above, which is the leading term, yields, at least for a standard HF-Skyrme force where $f_{\text{eff}}^{\text{q}} \geq 1$, to a smaller moment of inertia than the corresponding term in [149] containing the Thouless–Valatin corrections. It is also worth noting that in this approximate case, the kinematic moment of inertia is given by

$$\Theta_{\text{IC}}^{(\text{kin})} = m \sum_q \int dr \frac{\rho_q}{f_q^{\text{eff}}} r_{\perp}^2,$$

(5.16)

which turns out to be quite different from the above given dynamical moment of inertia, [57], obtained in the same limit (Thomas–Fermi limit, omitting the Thouless–Valatin self-consistency terms).

To investigate the importance of the different contributions to the total moment of inertia, we have performed self-consistent ETF calculations up to order $\hbar^4$ for 31 non-rotating nuclei, imposing a spherical symmetry, and using the SkM* Skyrme effective nucleon-nucleon interaction [151]. Such calculations yield variational semiclassical density profiles for neutrons and protons [72] which are then used to calculate the above given moments of inertia. The nuclei included in our calculations are $^{16\text{O}}, ^{56\text{Ni}}, ^{90\text{Zr}}, ^{140\text{Ce}}, ^{240\text{Pu}}$ and three isotopic chains for Ca ($A=36−50$), Sn ($A=100−132$) and Pb ($A=186−216$). The results of these calculations are displayed in figure 11 taken from [87].

One immediately notices the absence of any significant isovector dependence. The good reproduction of the total ETF moment of inertia obtained by the Thomas–Fermi (rigid-body) value is also quite striking. One finds that the orbital and spin semiclassical corrections are not small individually but cancel each other to a large extent. To illustrate this fact the ETF moments obtained by omitting only the spin contribution are also shown on the figure. One thus obtains a reduction of the Thomas–Fermi result that is about 6% in $^{240}\text{Pu}$ but as large as 43% in $^{16\text{O}}$.

The Inglis cranking approach performed at the Thomas–Fermi level underestimates the kinematic moment of inertia by as much as 25% and the dynamical moment of inertia by about 50% in heavy nuclei, demonstrating in this way the importance of the Thouless–Valatin self-consistency terms.

In [57], a crude estimate of the semiclassical corrections due to orbital and spin degrees of freedom has been made by considering the nucleus as a piece of symmetric nuclear matter (no isovector dependence as already indicated by the self-consistent results shown in figure 11 above). It turns out that these semiclassical corrections have an identical $A$ dependence ($A^{-2/3}$ relative to the leading order Thomas–Fermi, i.e. rigid-body, term)

$$\Theta_{\text{ETF}} = \Theta_{\text{RB}}^{(\text{spin})} \left[ 1 + (\eta_\ell + \eta_\gamma) A^{-2/3} \right].$$

(5.17)

A fit of the parameters $\eta_\ell$ and $\eta_\gamma$ to the numerical results displayed in Fig. 11 yields $\eta_\ell = -1.94$ and $\eta_\gamma = 2.63$ giving a total (orbital + spin) corrective term of $0.69A^{-2/3}$. For a typical rare-earth nucleus ($A=170$) all this would correspond to a total corrective term equal to 2.2% of the rigid-body value, resulting from a -6.3% correction for the orbital motion and a 8.5% correction for the spin degree of freedom.

Whereas in the calculations that lead to figure 11 above, spherical symmetry was imposed, fully variational calculations have been performed in [88], imposing however the nuclear shapes to be of spheroidal form. In this way, the nuclear rotation clearly impacts on the specific form of the matter densities relative to the $\beta$ and $\gamma$ sections, by the standard quadrupole parameters $\eta_\ell$ and $\gamma$, which, in turn, in the framework of the ETF approach determine all the other local densities, as explained above.

Trying to keep contact with usual shape parametrizations, by the standard quadrupole parameters $\beta$ and $\gamma$ equating the semi-axis lengths of the spheroids with the lengths of a standard quadrupole drop.

As a result, figure 12 shows the evolution of the equilibrium solutions (the ones that minimize the energy for given angular momentum $I$) as a function of $I$. One clearly observes that at low values of the angular momentum (in the range between 0 and 50 $\hbar$) the nuclear drop takes on an oblate shape, corresponding to increasing values of the quadrupole parameter $\beta$ with increasing $I$ values, but keeping the non-axiality parameter fixed at $\gamma = 60^\circ$. For larger values of the total angular momentum (I beyond 55 $\hbar$), one observes a transition into triaxial shapes,
where the nucleus evolves rapidly to more and more elongated shapes. For even higher values of \( I \) (I beyond 70 \( \hbar \)) the nucleus approaches the fission instability. These results are in excellent qualitative agreement with those obtained by Cohen, Plasil and Swiatecki in a rotating LDM.

It is amusing to observe here a backbending phenomena at the semiclassical level when one is plotting, as usual, the moment of inertia \( \Theta_{\text{ETF}} \) vs the rotational angular momentum, see Fig. 13. One should, however, insist on the fact that this backbending has strictly nothing to do with the breaking of a Cooper pair. The rapid increase of the moment of inertia at about \( I = 60\hbar \) with a practically constant (or even slightly decreasing) rotational frequency \( \omega \) comes simply from the fact that at such a value of \( I \) (between \( I \approx 60 \) and \( I \approx 70 \)) the nucleus elongates substantially increasing in this way its deformation and at the same time its moment of inertia.

It is therefore interesting to notice that the semiclassical ETF approach leads to a moment of inertia that is very well approximated by its Thomas–Fermi, i.e. rigid-body value. Thouless–Valatin terms which arise from the self-consistent response of the mean field to the time-odd part of the density matrix generated by the cranking piece of the Hamiltonian are naturally taken care of in this approach. Semiclassical corrections of order \( \hbar^2 \) coming from the orbital motion and the spin degree of freedom are not small individually, but compensate each other to a large extent. One has, however, to keep in mind that the shell and pairing effects, that go beyond the ETF approach, are not included in this description. These effects are not only both present, but influence each other to a large extent, especially for collective high-spin rotations of strongly deformed nuclei, as shown in 10, 22, 153.

**C. MI shell structure and periodic orbits**

We shall outlook first the basic points of the POT for the semiclassical level-density and free-energy shell corrections 3, 52, 94. We apply then the POT for the derivation of the MI through the rigid-body MI (with the shell corrections, see Appendix E) in the NLLLA related to the equilibrium collective rotation with a given frequency \( \omega \). For simplicity, we shall discard the spin and isospin degrees of freedom, in particular, the spin-orbit and asymmetry interaction.

Notice also that from the results presented in Figs. 11 and 13 (with the help of Fig. 12), one may conclude that the main contribution to the moment of inertia of the strongly deformed heavy nuclei can be found within the ETF approach to the rotational problems as a smooth rigid body MI.

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**1. GREEN’S FUNCTION TRAJECTORY EXPANSION**

For the derivations of shell effects 52 within the POT, it turns out to be helpful to use the coordinate representation of the MI through the Green’s functions \( G(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) \) 112, 113, 141, 142, 154.

\[
\Theta_x = \frac{2d_x}{\pi} \int_0^\infty d\varepsilon \, n(\varepsilon) \int d\mathbf{r}_1 \int d\mathbf{r}_2 \, \ell_x(\mathbf{r}_1) \, \ell_x(\mathbf{r}_2) \times \text{Re}[G(\mathbf{r}_1, \mathbf{r}_2; \varepsilon)] \text{Im}[G(\mathbf{r}_1, \mathbf{r}_2; \varepsilon)].
\]

(5.18)

The Fermi occupation numbers \( n(\varepsilon) \) are approximately considered at \( \omega = 0 \) (\( \varepsilon = \varepsilon_0 \)). In (5.18), \( \ell_x(\mathbf{r}_1) \) and \( \ell_x(\mathbf{r}_2) \) are the s.p. angular-momentum projections onto the perpendicular rotation \( x \) axis at the spatial points \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), respectively. With the usual energy-spectral representation for the one-body Green’s function \( G \) in the mean-field approximation, one finds the standard cranking model expression, which however includes the diagonal matrix elements of the operator \( \ell_x \). In this sense, equation (5.18) looks more general beyond the standard perturbation approximation, see 113. Moreover, the quantum criterion of the application of this standard cranking model approximation, which is a smallness of the cranking field perturbation \( \hbar \kappa_{\varepsilon}^2 \) in (5.1) as compared to the distance between the neighboring states of the non-perturbative spectrum, becomes weaker in the semiclassical approach, see more comments below in relation to 10, 21.

For the MI calculations by (5.18), through the Green’s function \( G \), one may use the semiclassical Gutzwiller trajectory expansion 83 extended to continuous symmetry 73, 91, 93, 95, 98, 99 and symmetry breaking 73, 94, 102, 103 problems,

\[
G(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) = \sum_{\text{CT}} G_{\text{CT}}(\mathbf{r}_1, \mathbf{r}_2; \varepsilon),
\]

(5.19)

where

\[
G_{\text{CT}}(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) = A_{\text{CT}}(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) \times \exp \left[ \frac{i}{\hbar} S_{\text{CT}}(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) - \frac{i\pi}{2} \sigma_{\text{CT}} - i\phi_\alpha \right].
\]

(5.20)

The sum runs over all isolated classical trajectories (CTs) or their families inside the potential well \( V(r) \) which, for a given energy \( \varepsilon \), connect the two spatial points \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). Here \( S_{\text{CT}} \) is the classical action along such a CT, and \( \sigma_{\text{CT}} \) denotes the phase associated with the Maslov index through the number of caustic and turning points along the path CT, \( \phi_\alpha \) is the constant phase depending on the dimension of the problem 73, 91, 94, 102. The amplitudes \( A_{\text{CT}} \) of the Green’s function depend on the classical stability factors and trajectory degeneracy, due to the symmetries of that potential 73, 91, 93, 95, 98, 99, 102, 103.

For the case of the isolated CTs 73, 91, one has the explicit semiclassical expression for the amplitudes through the stability characteristics of classical dynamics,

\[
A_{\text{CT}}(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) = -\frac{1}{2\pi\hbar^2} \sqrt{|J_{\text{CT}}(\mathbf{p}_1, t_{\text{CT}}; \mathbf{r}_2, \varepsilon)|}.
\]

(5.21)
Here, $J_{CT}(p_1, t_{CT}; r_2, \varepsilon)$ is the Jacobian for the transformation between the two sets of variables $p_1, t_{CT}$ and $r_2, \varepsilon$: $p_1$ and $t_{CT}$ are the initial momentum and time of motion of the particle along a CT, $t_{CT} = \partial S_{CT}/\partial \varepsilon$, $r_2$ and $\varepsilon$ are its final coordinate and energy. In more general case, if the mean field Hamiltonian $\hat{h}$ obeys a higher symmetry like that of spherical or harmonic-oscillator potentials with rational ratios of frequencies, one has to use other expressions for the amplitude $A_{CT}(r_1, r_2; \varepsilon)$ for close trajectories of a finite action (with reflection from the potential boundary), taking into account such symmetries. They account for an enhancement in the ISPM [102, 103], especially for superdeformed shapes of the potential. Some examples of the specific amplitudes for the degenerate families of closed POs in the harmonic oscillator (HO) potential are given in Appendix E of [113]. Note that (5.21) can be applied for any potential wells for the contributions of closed and non-closed trajectories which can be considered as isolated (no PO families) ones for the given end points $r_1$ and $r_2$.

Among all of CTs in (5.19), we may single out CT that is given by

$$G = G_{CT_0} + G_1 \approx G_0 + G_1.$$  \hspace{1cm} (5.22)

In the NLLLA [113, 154],

$$s_2 \ll \hbar/p,$$  \hspace{1cm} (5.23)

the first term $G_{CT_0}$ of the splitting in the middle of (5.22) is given by

$$G_{CT_0} \approx G_0(s_2, p) = \frac{m}{2\pi\hbar^2 s_2} \exp \left[ \frac{i}{\hbar} s_2 p(r) \right],$$  \hspace{1cm} (5.24)

where $p(r) = \sqrt{2m[\varepsilon - V(r)]}$, $V(r)$ is a mean nuclear potential,

$$s_2 = |r_2 - r_1|, \quad r = (r_1 + r_2)/2,$$  \hspace{1cm} (5.25)

$p = |p|$, $p = (p_1 + p_2)/2$. The second term $G_1$ in (5.22) is the fluctuating part of the Green’s function determined by all other trajectories CT that are not in the sum (5.19) with reflection points at the potential surface (see one of such trajectories CT in Fig. 14).

$$G_1(r_1, r_2; \varepsilon) = \sum_{CT_1} G_{CT_1}(r_1, r_2; \varepsilon),$$  \hspace{1cm} (5.26)

where $G_{CT_1}$ is the Green’s function component taken at the CT $\neq CT_0$, i.e., CT_1.

### 2. Level-Density and Energy Shell Corrections

The level density, $g(\varepsilon) = \sum_i \delta(\varepsilon - \varepsilon_i)$, where $\varepsilon_i$ is the quantum spectrum, is identically expressed in terms of the Green’s function $G$ as

$$g(\varepsilon) = -\frac{1}{\pi} \text{Im} \int dr \langle G(r_1, r_2; \varepsilon) \rangle_{r_1 \to r_2 \to r}.$$  \hspace{1cm} (5.27)

According to (5.22), this level density can be presented semiclasically as a sum of the smooth and oscillating components [73, 89, 91, 94],

$$g_{\text{sc}}(\varepsilon) = g_{\text{TF}}(\varepsilon) + g_{\text{osc}}(\varepsilon),$$  \hspace{1cm} (5.28)

where $g_{\text{TF}}(\varepsilon)$ is given by the ETF approach related to the component $G_0$ in (5.22) in the NLLLA [113, 154]. The local part of $g_{\text{TF}}(\varepsilon)$ is the main simplest Thomas–Fermi (TF) level density [73]. The second oscillating term $g_{\text{osc}}(\varepsilon)$ of the level density (5.28) corresponds to the fluctuating $G_1$ in the sum (5.22) for the Green’s function $G$ near the Fermi surface. The stationary phase conditions for the (standard or improved) SPM evaluation of the integral taken from $G_1$ over the spatial coordinates $r$ are the PO equations. As a result, one arrives at the sum over PO sum for this oscillating level density [73, 89, 91, 94].

$$\delta g_{\text{sc}}(\varepsilon) = \text{Re} \sum_{\text{PO}} \delta g_{\text{PO}}(\varepsilon)$$

with

$$\delta g_{\text{PO}}(\varepsilon) = B_{\text{PO}} \exp \left[ \frac{i}{\hbar} S_{\text{PO}}(\varepsilon) - \frac{i\pi}{2} \sigma_{\text{PO}} - i\phi_d \right],$$  \hspace{1cm} (5.29)

where $B_{\text{PO}}$ is an amplitude of the oscillating PO terms, see [73, 89, 91, 94, 102]. The above sum runs over the isolated POs and, in the case of degeneracies owing to the symmetries of a potential well, over all families of POs. $B_{\text{PO}}$ is the oscillation amplitude depending on the stability factors, $S_{\text{PO}}(\varepsilon)$ the action integral along a given PO, and $\sigma_{\text{PO}}$ is the Maslov phase associated with the turning and caustic points along the PO, see [73, 91, 103] for the detailed explanations.

The semiclassical free-energy shell corrections, $\delta F_{\text{sc}}$ at finite temperature $(T \ll \hbar \Omega \ll \varepsilon_0)$, can be expressed through the PO components of the energy shell corrections $\delta U_{\text{sc}}$ [73, 91, 103] (see Appendix E.1),

$$\delta U_{\text{sc}} = \text{Re} \sum_{\text{PO}} \delta U_{\text{PO}},$$

$$\delta U_{\text{PO}} = d_s \frac{\hbar^2}{2} \delta g_{\text{PO}}(\mu),$$  \hspace{1cm} (5.30)

with the exponentially decreasing temperature-dependent factor [73, 91, 103, 104, 113],

$$\delta F_{\text{sc}} = \text{Re} \sum_{\text{PO}} \frac{\pi \tau_{\text{PO}} T/\hbar}{\sinh (\pi \tau_{\text{PO}} T/\hbar)} \delta U_{\text{PO}}.$$  \hspace{1cm} (5.31)
Finally through (5.30), the shell corrections $\delta F_{scl}$ and $\delta U_{scl}$ are determined by the PO level-density shell-correction components $\delta G_C(z)$ of (5.29) at the chemical potential, $\varepsilon = \mu \approx \varepsilon_g$. In (5.30), one has the additional factor, $\propto 1/t_{PO}^2$, which yields the convergence of the PO sum (without averaging of $\delta g(z)$ over the s.p. spectrum), $t_{PO}$ is the time of motion along the PO (PO period). Another (exponential) convergence of $\delta F_{scl}$ (5.31) with increasing the period $t_{PO}$ and temperature $T$ is given by the temperature-dependent factor in front of $\delta U_{scl}$.

3. FROM CRANKING MODEL TO THE RIGID BODY ROTATION

Substituting (5.22) into (5.18), one has a sum of several terms,

$$\Theta_{x\,scl} \approx \Theta_{x\,scl}^{00} + \Theta_{x\,scl}^{01} + \Theta_{x\,scl}^{10} + \Theta_{x\,scl}^{11},$$  \hspace{1cm} (5.32)

where

$$\Theta_{x\,scl}^{n\prime n} = \frac{2d_x}{\pi} \int \left. \frac{d \varepsilon}{n(\varepsilon)} \right| \frac{d \varepsilon}{\pi} \int d \varepsilon \left. \frac{G_{n,1}(r_{12}, r_{21})}{G_{n,n}(r_{12}, r_{21})} \right| \Re \left[G_{n,1}(r_{12}, r_{21}; \varepsilon) \right] \Im \left[G_{n,n}(r_{12}, r_{21}; \varepsilon) \right].$$  \hspace{1cm} (5.33)

Indexes $n$ and $n'$ run independently the two integers 0 and 1. As shown in Appendix E.2a, the main smooth part of the semiclassical MI $\Theta_{x\,scl}$ (5.32) is associated with the TF (ETF) rigid-body component through the first term $\Theta_{x\,scl}^{00}$ averaged over the phase-space variables; see section V B also 88, 113, 154, and previous publications 83–86. The statistical averaging over phase space coordinates removes the non-local long-length correlations. The $\hbar$ corrections of the smooth ETF approach to the TF approximation were obtained in [86–88], see Sect. V B for the review of these works.

Using the transformation of the coordinates $r_1$ and $r_2$ to the center-of-mass and relative ones $r$ and $s_{x\,z}$,

$$r = \frac{r_1 + r_2}{2} \quad \text{and} \quad s_{x\,z} = r_2 - r_1, \hspace{1cm} (5.34)$$

in (5.32), respectively, one simplifies much the calculations of the oscillating terms, $\Theta_{x\,scl}^{01} + \Theta_{x\,scl}^{10} + \Theta_{x\,scl}^{11}$. In this way, one finds that the shell component $\delta G_C^{01}$ of $\Theta_{x\,scl}^{01}$ [see (5.33) at $n = 0$ and $n' = 1$] is dominating in the MI shell correction $\delta \Theta_{x\,scl}$ within the NLLLA (5.22), see Appendix E.2b. Indeed, in this approximation, substituting the components, $G_{00}$ and $G_{11}$, of the Green’s function (5.22) for $G_{00}$ into (5.33) for $\Theta_{x\,scl}^{01}$, and using the averaging over the phase-space variables in the fluctuating (shell) part $\delta \Theta_{x\,scl}$, one results in the relationship for the corresponding shell corrections (see Appendix E.2b):

$$\delta \Theta_{x\,scl} \approx \delta \Theta_{x\,scl}^{01} \approx \delta \Theta_{x\,scl}^{01}(RB).$$  \hspace{1cm} (5.35)

Here, $\delta \Theta_{x\,scl}^{01}(RB)$ is the shell correction to the rigid-body MI $\Theta_{x\,scl}^{01}(RB)$, which is related to the semiclassical particle-density $\rho(r)$ through

$$\Theta_{x\,scl}^{01}(RB) = m \int dr \, r_{1x}^2 \rho(r), \hspace{1cm} (5.36)$$

with

$$r_{1x}^2 = y^2 + z^2.$$

The particle density $\rho(r)$, and therefore, the MI (5.36), can be expressed in terms of the Green’s function $G$,

$$\rho(r) = -\frac{d_s}{\pi} \left[ \int \frac{d \varepsilon}{n(\varepsilon)} \right] [G(r_1, r_2; \varepsilon)]_{r_1 \rightarrow r_2 \rightarrow r}. \hspace{1cm} (5.38)$$

With the splitting of the Green’s function (5.22), one obtains the semiclassical sum of the smooth and oscillating (shell) components 96, 97,

$$\rho(r) \approx \rho_{\text{scl}}(r) = \rho_{\text{ETF}}(r) + \delta \rho_{\text{scl}}(r). \hspace{1cm} (5.39)$$

The integration over $\varepsilon$ in (5.38) is performed over the whole s.p. energy spectrum. For the Green’s function $G$, we applied the semiclassical expansion 5.19 in terms of the sum (5.22) of CTs in the last equation for the semiclassical particle density $\rho_{\text{scl}}(r)$. The first term in (5.39) is the (extended) Thomas–Fermi component (see Appendix E.2a). Substituting the particle density splitting (5.39) into (5.36), one has the corresponding semiclassical expression of the rigid-body MI,

$$\Theta_{x\,scl}^{01}(RB) \approx \Theta_{x\,scl}^{01} = \Theta_{x\,scl}^{01}(RB) + \delta \Theta_{x\,scl}^{01}(RB). \hspace{1cm} (5.40)$$

We introduced the shell corrections $\delta \rho$ (see [97]) to the particle density $\rho$ and $\delta \Theta_{x\,scl}^{01}(RB)$ to the rigid-body MI $\Theta_{x\,scl}^{01}(RB)$, and their semiclassical counterparts,

$$\delta \Theta_{x\,scl}^{01}(RB) \approx \delta \Theta_{x\,scl}^{01} = m \int \frac{dr}{r_{1x}^2} \delta \rho_{\text{scl}}(r), \hspace{1cm} (5.41)$$

where

$$\delta \rho_{\text{scl}}(r) = -\frac{d_s}{\pi} \left[ \int \frac{d \varepsilon}{n(\varepsilon)} \right] G_{CCT_1}(r_1, r_2; \varepsilon),$$

where $G_{CCT_1}$ is given by (5.20) with CT being the closed CT, i.e., $G_{CCT_1}(r_1 \rightarrow r_2 \rightarrow r)$. With the smooth (extended) TF MI component 113, see also the section V B, the equation (5.39) yields semiclassically

$$\Theta_{x\,scl} \approx \Theta_{x\,scl}^{01}(RB),$$

that is in agreement with the adiabatic picture of the statistically equilibrium rotation 113. Note that the non-adiabatic MI at arbitrary rotation frequencies for the HO mean field by Zelevinsky 113 was extended to the finite temperatures in 113.

We emphasize that due to an averaging over the phase space variables, one survives with the NLLLA. Note also that the classical angular-momentum projection 116 in the rotating body-fixed coordinate system is caused by the global rotation with a given frequency $\omega$ rather than by the motion of particles along the trajectories inside the nucleus with respect to this system, considered usually in the cranking model. According to the
time-reversible symmetry of the Routhian, the particles are, indeed, moving in the non-rotating coordinate system along these trajectories in both opposite directions. Their contributions to the total angular momentum of the nucleus turns out to be zero. Performing then the integration over \( s \) in (5.18) in the spherical coordinate system, one obtains the rigid-body shell correction \( \delta \Theta_x^{(RB)} \) in the NLLLA as explained in Appendix E.2. Note that the cranking model for the nuclear rotation implies that the \( \delta \) function \( \Theta^0 \) and \( \Theta^1 \), as referred to the fluctuation (non-local) correction to the rigid body MI found semiclassically to be negligibly small in the NLLLA due to the averaging over phase-space variables, see Appendix E.2b. In particular, for the HO Hamiltonian, it was shown that there is almost no contribution of the \( \delta \Theta_x^{11} \) at leading order in \( h \) in (113). Thus, with the semiclassical precision, from the adiabatic cranking model expression (5.18) we come to the MI of the statistically equilibrium rotation (5.43), which must be the rigid-body MI, according to the general theorem of the statistical physics. This is in agreement with the ETF approach of section V C.

Our semiclassical derivations, valid for the rotation frequencies \( h \omega \ll h \Omega \), are beyond the quantum criterion of the application of the standard 2nd order perturbation approach within the cranking model where \( h \omega \) is small as compared to the distance between the neighboring levels of quantum spectra. We point out that this weakness of the perturbation theory criterion is similar to that with the statistical averaging in the heated Fermi systems and with accounting for the pairing correlations (19, 21), where the role of the distance between the quantum neighboring energy levels plays the temperature and the pairing gap, as distance between gross shells \( \hbar \Omega \) (3.49) in the POT (91), respectively.

4. SHELL CORRECTIONS TO THE RIGID-BODY MI

Using (5.42) for calculations of the MI rigid-body shell correction \( \delta \Theta_{x \text{scl}}^{(RB)} \) (5.41), one may exchange the order of integrations over the coordinate \( r \) and energy \( \varepsilon \). By making use also of the semiclassical trajectory expansion (5.19) for the oscillating Green’s function component \( G_1 (r_1, r_2; \varepsilon) \) of the sum (5.22), one finds

\[
\delta \Theta_x^{(RB)} = -\frac{m}{\pi} \text{Im} \sum_{\text{CCT}_1} \int d \varepsilon \, n(\varepsilon) \times \int dr \left\{ r_{1,x}^2 A(r, r; \varepsilon) \right. \\
	imes \exp \left[ \frac{i}{\hbar} S(r_1, r_2; \varepsilon) - \frac{i}{2} \sigma - i \phi_d \right] \left. \right\}_{\text{CCT}_1}. \tag{5.44}
\]

As usually, with the semiclassical precision, we evaluate the spatial integral by the SPM extended to continuous symmetries (73, 91, 94) and the bifurcation phenomena (ISPM) (94, 100, 102, 103, 105). The SPM (ISPM) condition writes

\[
\left[ \frac{\partial S(r_1, r_2; \varepsilon)}{\partial r_1} + \frac{\partial S(r_1, r_2; \varepsilon)}{\partial r_2} \right]_{\text{CCT}_1}^* \equiv (-p_1 + p_2)_{\text{CCT}_1}^*, \tag{5.45}
\]

where the asterisk means the SPM value of the spatial coordinates and momenta, \( r_j = r_j^* \) and \( p_j = p_j^* \), \( j = 1, 2 \) at the closed CTs in the phase space, \( r_j^* = r_2^* \) and \( p_j^* = p_2^* \). Thus, with the standard relations for the canonical variables by using the action as a generating function, one arrives at the PO condition on right of (5.45). Within the simplest ISPM (94, 100, 102, 103, 105), the other smooth factors \( r_{1,x}^2 \) and \( A_{\text{CCT}_1}(r, r, \varepsilon) \) of the integrand in (5.44) can be taken off the integral over \( r \) at these stationary points. Assuming that the quantum averages \( \langle (y^2 + z^2)^2/\varepsilon \rangle \) are smooth enough functions of \( \varepsilon \) as compared to other factors, for instance, \( \delta n \), one may take them approximately also off the integral over \( \varepsilon \) at the chemical potential, \( \varepsilon = \mu \). For example, for the HO potential (see [113]), they are simply exact.

Their contributions to the total angular momentum of the internal motion of particles. Other contributions, except for a smooth rigid-body part coming from \( \Theta^0 \), like \( \Theta^{10} \) and \( \Theta^{11} \), as referred to the fluctuation (non-local) correction to the rigid body MI are found semiclassically to be negligibly small in the NLLLA due to the averaging over phase-space variables, see Appendix E.2b. In particular, for the HO Hamiltonian, it was shown that there is almost no contribution of the \( \delta \Theta_x^{11} \) at leading order in \( h \) in (113). Thus, with the semiclassical precision, from the adiabatic cranking model expression (5.18) we come to the MI of the statistically equilibrium rotation (5.43), which must be the rigid-body MI, according to the general theorem of the statistical physics. This is in agreement with the ETF approach of section V C.

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4. SHELL CORRECTIONS TO THE RIGID-BODY MI

Using (5.42) for calculations of the MI rigid-body shell correction \( \delta \Theta_x^{(RB)} \) (5.41), one may exchange the order of integrations over the coordinate \( r \) and energy \( \varepsilon \). By making use also of the semiclassical trajectory expansion (5.19) for the oscillating Green’s function component
where \( \langle r_{\perp x}^2 \rangle_{PO, \mu} \) is the average given by

\[
\langle r_{\perp x}^2 \rangle_{PO, \mu} = \frac{\int dr \mathcal{A}_{PO}(r, \mathbf{r}; \varepsilon) r_{\perp x}^2}{\int dr \mathcal{A}_{PO}(r, \mathbf{r}; \varepsilon)}
\]

(5.47)

at \( \varepsilon = \mu \), \( \mathcal{A}_{PO}(r, \mathbf{r}; \varepsilon) \) are the Green’s function amplitudes for a closed CT \( T \) in the phase space, i.e., PO. Integration over \( \mathbf{r} \) is performed over the classically accessible region of the spatial coordinates. Semiclassical expression (5.46) is general for any potential well. Shorter POs are dominating in the PO sum (5.46) \( \omega_1, \omega_2, 93, 91, 106, 113 \), see (5.31), (5.30). Therefore, according to (5.31) for \( \delta F_{scl} \), we obtain approximately the relation

\[
\delta \Theta_{x,scl}^{(RB)} \approx \frac{m}{\mu} \langle r_{\perp x}^2 \rangle_{\mu} \delta F_{scl},
\]

(5.48)

where \( \langle r_{\perp x}^2 \rangle_{\mu} \) is an average value of the quantity (5.47), independent of the specific PO, at \( \varepsilon = \mu \) over short dominating POs.

For the axially symmetric HO potential well with the commensurable frequencies \( \omega_1 \) and \( \omega_2 \), as the simplest example, the integration in (5.47) over \( \mathbf{r} \) for the 3D contribution means over the 3D volume occupied by the 3D families of orbits. For the EQ component the integral is taken over the 2D spatial region filled by the EQ families in the equatorial \( (z = 0) \) plane \( 113 \). In the incommensurable-frequency case (irrational \( \omega_1/\omega_2 \)), one has only EQ-orbit contributions. The average (5.47) can be easily calculated by using the Green’s function amplitudes \( \mathcal{A}_{PO} \) for 3D and for EQ orbits, which are given in [54, 113]. Finally, for the considered HO potential, one may arrive at

\[
\delta \Theta_{x,scl} \approx \delta \Theta_{x,scl}^{(RB)} = \frac{1 + \eta_{HO}^2}{3\omega_1^2} \delta F_{scl},
\]

(5.49)

where \( \delta F_{scl} \) is the semiclassical PO sum (5.31), 5.30 for the semiclassical free-energy shell-corrections, \( \eta_{HO} = \omega_1/\omega_2 \) is the deformation parameter. For the parallel (alignment) rotations around the symmetry axis, one finds similar relations of the MI through the rigid-body MI to the free-energy shell corrections. Moreover, one has such relations for the smooth TF parts, in particular for the HO case, see Appendices E.2.1 here and D1 in [113]. Thus, for the total moment \( \Theta_x \) [see (5.32)], one may prove semiclassically within the POT, up to the same \( \hbar \) corrections in a smooth TF part, that the shell MI and free-energy shell corrections are approximately proportional, in particular exactly for that HO Hamiltonian [113]:

\[
\Theta_{scl} = \frac{1 + \eta_{HO}^2}{3\omega_1^2} F_{scl}, \quad F_{scl} = F_{ETF} + \delta F_{scl}.
\]

(5.50)

We emphasize that the POT expressions (5.49) for \( \delta \Theta_{x,scl} \) and (5.50) of \( \Theta_{x,scl} \) were derived without a direct use of the statistically equilibrium rotation condition \( \Theta \) [113].

Substituting the semiclassical PO expansion (5.30) for the free-energy shell correction \( \delta F_{scl} \) (5.31) (after [102]) for 3D orbit families and for EQ POs into (5.49), one arrives finally at the explicit POT expressions for the MI shell corrections \( \delta \Theta_x \) in terms of the characteristics of the classical POs. For the mean field with the spheroidal shapes and sharp edges (spheroid cavity), these derivations can be performed similarly as for the HO Hamiltonian in [113] but with accounting for the specific PO degeneracies. Note that the parallel, \( \delta \Theta_z \), and perpendicular, \( \delta \Theta_x \), MI shell components are expressed through the 3D and EQ POs through the free-energy shell correction which contains generally speaking both them for the deformations larger the bifurcation ones. The dominating contributions of one of these families or coexistence of both together depend on the surface deformation parameter (semi-axis ratio of spheroid). For the critical deformations and on right of them, one observes the significant enhancement of the MI shell corrections through the PO level-density amplitudes \( \delta F_{scl} \) [see (5.29)] of the free-energy shell corrections (5.31), (5.30).

5. COMPARISON OF SHELL STRUCTURE CORRECTIONS WITH QUANTUM RESULTS

Fig. 15 shows the semiclassical free-energy shell correction \( \delta F_{scl} \) [5.31], 5.30, see also [94, 113] vs the particle-number variable, \( A^{1/3} \), at a small temperature of \( T = 0.1 \hbar \omega_1 \) for different critical symmetry-breaking and bifurcation deformations \( \eta_{HO} = 1, 6/5, 2 \) of the HO potential \( 73, 113 \) with the corresponding quantum SCM results for the same deformations. This comparison also shows practically a perfect agreement between the semiclassical, 5.31 and 5.30, and quantum results. For the spherical case \( \eta_{HO} = 5 \), one has only contributions of the families of 3D orbits with the highest degeneracy \( K = 4 \). At the bifurcation points \( \eta_{HO} = 6/5 \) and 2 of the relatively simple families of these 3D POs appear along with EQ orbits of smaller degeneracy. For \( \eta_{HO} = 6/5 \), one mainly has the contributions from EQ POs because the 3D orbits are generally too long in this case. For the bifurcation point \( \eta_{HO} = 2 \), one finds an interference of the two comparably large contributions of EQ and 3D orbits with essentially the different time periods \( t_{EQ} \) and \( t_{3D} \), respectively.

The quantum (QM) and semiclassical (SCL) shell corrections to the MI \( \delta \Theta_x \) of (5.49) are compared in Fig. 16. An excellent agreement is observed between the semiclassical and quantum results as for the free-energy shell corrections \( \delta F \). It is not really astonishing because of the proportionality of the \( \Theta_{x,scl} \) to \( \delta F \) [see (5.49)]. One finds in particular the same clear interference of contributions of 3D and EQ POs in the shell corrections to the MI at \( \eta_{HO} = 2 \). The exponential decrease of shell oscillations with increasing temperature, due to the temperature factor in front of the PO energy-shell correction
components $\delta E_{PO}$ in $[55,31]$ is clearly seen in Fig. 10.

As the MI and free-energy shell corrections are basically proportional [see $[55,40]$] for any mean potential well, we may emphasize the amplitude enhancement of the MI near the bifurcation deformations due to that for the energy-shell corrections found in $[94, 100, 102, 103, 105]$. The critical temperature for a disappearance of shell effects in the MI is found for prolate deformations ($\eta > 1$) and particle numbers $A \sim 100 \sim 200$, approximately at $T_c = \hbar \delta E_{PO} / \pi \sim \hbar \omega_0 / \pi \approx 2 \sim 3$ MeV just as for $\delta F$, see $[74, 97, 113]$. This effect is also general for any potentials.

The particle-number dependence of the shell corrections $\delta \Theta_x$ to the total MI $\Theta_x$ (alignment) is not shown because it is similar to that of $\delta \Theta_x$ through their approximate relations, $\delta \Theta_x \propto \delta \Theta_x \propto \delta F$.

VI. CONCLUSIONS

We derived the dynamical equations of motion, such as the conservation of the particle number, momentum and energy as well as the general transport equation for the entropy for low frequency excitations in nuclear matter within the Landau quasiparticle theory of heated Fermi-liquids. Our approach is based essentially on the Landau–Vlasov equation for the distribution function, and it includes all its moments in phase space, in contrast to several truncated versions of fluid dynamics similar to the hydrodynamic description in terms of a few first moments. From the dynamics of the Landau–Vlasov equation for the distribution function, linearized near the local equilibrium, we obtained the momentum flux tensor and heat current in terms of the shear modulus, viscosity, in-compressibility and thermal conductivity coefficients as for very viscose liquids called sometimes amorphous solids. We obtain the dependence of these coefficients on the temperature, the frequency and the Landau interaction parameters. We derived the temperature expansions of the density-density and temperature-density response functions for nuclear matter and got their specific expressions for small temperatures as compared to the chemical potential. The hydrodynamic limit of normal liquids for these response functions within the perturbation theory was obtained from the Landau–Vlasov equation for both distribution function and sound velocity, as for an eigenvalue problem. In this way we found the Landau–Placzek and first sound peaks in the corresponding strength functions as the hydrodynamic limit of the Fermi-liquid theory for heated Fermi-systems. The former (heat pole) peak was obtained only because of the use of the local equilibrium in the Landau–Vlasov linearized dynamics instead of the global static Fermi-distribution of the giant multipole-resonance physics. This is very important for the dispersion equation and its wave velocity solutions.

We got the isolated, isothermal and adiabatic susceptibilities for the Fermi-liquids and showed that they satisfy the ergodicity condition of equivalence of the isolated and adiabatic susceptibilities as well as the general Kubo inequality relations. We found the correlation function using the fluctuation-dissipation theorem and discussed its relation to the susceptibilities and Landau–Placzek "heat pole" in the hydrodynamic limit.

We applied the theory of heated Fermi-liquids to the Fermi-liquid drop model of finite nuclei within the Landau–Vlasov dynamics in the nuclear interior and macroscopical boundary conditions in the effective sharp surface approximation. Solutions of this problem in terms of the response functions and transport coefficients were obtained. We considered the hydrodynamic limit of these solutions and found the "heat pole" correlation function for frequencies smaller than some critical frequency. The latter was realized only because of using the local equilibrium for the distribution function. The isolated, isothermal and adiabatic susceptibilities for finite nuclei within the FLDM in the ESA were derived. We showed that the ergodicity condition is satisfied also for finite Fermi-systems as for infinite nuclear matter in the same ESA.

We found a three-peak structure of the collective strength function: the "heat", standard hydrodynamic and essentially Fermi-liquid peaks. The conditions for the existence of such modes were analyzed and the temperature dependence of their transport coefficients such as friction, stiffness and inertia were obtained in particular, in the hydrodynamic limit. We arrived at the increasing temperature dependence of the friction coefficient for the specific Fermi-liquid mode which exist due to the Fermi-surface distortions. At enough large temperatures, we showed a nice agreement with the results for the friction which were obtained earlier within the microscopic shell-model approach of $[24]$. The correlation functions found in the FLDM and quantum shell models were discussed in relation to the susceptibilities and ergodicity properties of finite nuclei.

The expression for the surface symmetry-energy constant $k_4$ was derived from simple isovector solutions of the particle density and energies in the leading ES approximation. We used them for the calculations of the energies, sum rules of the IVGDR strength and the transition densities within the HDM and FLDM $[32]$ for several Skyrme-force parameters. The surface symmetry-energy constant depends much on the fundamental well-known parameters of the Skyrme forces, mainly through the coefficient in the density gradient terms of the isovector part of the energy density. The value of this isovector constant is rather sensitive also on the SO interaction. The IVGDR strength is split into the two main and satellite peaks. The mean energies and EWSRs within both HDM and FLDM are in fairly good agreement with the experimental data.

Semiclassical functional expressions were derived in the framework of the Extended Thomas–Fermi approach. We used these analytical expressions to obtain a self-consistent description of rotating nuclei where the rotation velocity impacts on the structure of the nucleus. It
has been shown that such a treatment leads, indeed, to the Jacobi phase transition to triaxial shapes as already predicted in \cite{152} within the rotating LDM. We emphasize that the rigid-body moment of inertia gives a quite accurate approximation for the full ETF value. Being aware of the mutual influence between rotation and pairing correlations \cite{11,24,153}, it would be especially interesting to work on an approach that is able to determine the nuclear structure depending on its angular velocity, as we have done here in the ETF approach, but taking pairing correlations and their rotational quenching into account.

We derived also the shell corrections of the MI in terms of free-energy shell corrections within the nonperturbative extended POT through those of the rigid-body MI of the equilibrium rotations, which is exact for the HO potential. For the HO, we extended to the finite temperature case the Zelevinsky derivation of the non-adiabatic MI at any rotation frequency. For the deformed HO potential, one finds a perfect agreement between the semiclassical POT and quantum results for the free-energy and the MI shell corrections at several critical deformations and temperatures. For larger temperatures, we show that the short EQ orbits are mostly dominant. For small temperatures, one observes a remarkable interference of the short 3D and EQ orbits in the superdeformed region. An exponential decrease of all shell corrections with increasing temperature is observed, as expected. We point out also the amplitude enhancement of the MI shell corrections due to the bifurcation catastrophe phenomenon.

As further perspectives, it would be worth to apply our results to calculations of the IVGDR structure within the Fermi-liquid droplet model to determine the value of the fundamental surface symmetry-energy constant from comparison with experimental data for the pygmy resonance \cite{153,156} and theoretical calculations \cite{52,132–136}. For further extensions to the description of the isovector low-lying collective states, one has first to use the POT for including semiclassically the shell effects \cite{73,91,155,157}. It would be also worth to apply this semiclassical theory to the shell corrections of the MI for the spheroid cavity and for the inertia parameter of the low-lying collective excitations in nuclear dynamics involving magic nuclei \cite{110,141,142,155}. One of the most attractive subject of the semiclassical periodic orbit theory, in line of the main works of S.T. Belyaev \cite{11,10,12,13}, is its extension to the pairing correlations \cite{14,97}, and their influence on the collective vibrational and rotational excitations in heavy deformed neutron-rich nuclei \cite{12,22,153} (see also \cite{158} for the semiclassical phase-space dynamical approach to the Hartree–Fock–Bogoliubov theory).

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**Appendix A: Elements of Landau theory for equilibrized systems**

**A.1. Thermodynamic relations**

Let us begin recalling the fundamental equations \( TdS = dE - \mu dN + P dV \) and \(-SdT = dF - \mu dN + P dV\), which are related to each other by the Legendre transformation \( F = E - TS \). They imply the following relations for the chemical potential \( \mu \) and pressure \( P \):

\[
\mu = -T \left( \frac{\partial S}{\partial N} \right)_{E,V} = \left( \frac{\partial E}{\partial N} \right)_{S,V} = \left( \frac{\partial F}{\partial N} \right)_{T,V}, \tag{A.1}
\]

\[
P = -\left( \frac{\partial E}{\partial V} \right)_{S,N} = -\left( \frac{\partial F}{\partial V} \right)_{T,N}. \tag{A.2}
\]

For homogeneous systems the *intensive* quantities depend only on two independent variables. For instance, the entropy per particle \( S/N = \zeta (E/N, V/N) \) only depends on the energy and volume per particle, \( E/N \) and \( V/N \) respectively. For such systems, the adiabaticity condition may simply be expressed as \( \zeta = \text{const} \). Commonly in nuclear physics, one uses the particle density \( \rho = N/V \), in which case the chemical potential can be expressed as

\[
\mu = \left( \frac{\partial \phi}{\partial \rho} \right)_T, \tag{A.3}
\]

with \( \phi = F/V \) being the free internal energy per unit volume.

For differential quantities there exist various variants of the Gibbs-Duheim relation

\[
d\phi = -\zeta \rho dT + \mu d\rho \quad \text{or} \quad \tag{A.4}
\]
\[ dP = \zeta \rho dT + \rho d\mu \quad \text{implying} \quad \left( \frac{\partial \zeta}{\partial \mu} \right)_T \]

\[ = \frac{1}{\rho} \left[ \left( \frac{\partial \rho}{\partial T} \right)_\mu - \zeta \left( \frac{\partial \rho}{\partial \mu} \right)_T \right], \quad (A.5) \]

as follows from Legendre transformations. Thus for the derivatives of the pressure \( P \), considered as functions of \( T \) and \( \rho \), one gets from \( (A.5) \):

\[ \frac{\partial P}{\partial \rho} = \rho \left( \frac{\partial \mu}{\partial \rho} \right)_T, \quad \frac{\partial P}{\partial T} = \rho \left( \frac{\partial \mu}{\partial T} \right)_\rho \]

\[ = \rho \left( \zeta - \left( \frac{\partial \rho}{\partial T} \right)_\mu \left( \frac{\partial \mu}{\partial \rho} \right)_T \right), \quad (A.6) \]

In deriving such relations, it is useful to employ special properties of the Jacobian, which allows one to perform transformations between different variables (see e.g., [116]). These relations will be used below to get the specific heats as well as the isothermal and adiabatic incompressibilities, together with the corresponding susceptibilities. At first, we shall look at incompressibilities defined by the derivative of the pressure over the particle density (multiplied by a factor of 9). At constant entropy per particle \( \zeta \), the adiabatic in-compressibility \( K^c \) writes

\[ K^c \equiv 9 \left( \frac{\partial P}{\partial \rho} \right)_\zeta = 9 \left( \frac{\partial P}{\partial \mu} \right)_\zeta \left( \frac{\partial \mu}{\partial \rho} \right) \zeta. \quad (A.7) \]

To get the corresponding quantity at constant temperature \( K^T \), one only needs to replace \( \zeta \) by \( T \). According to \( (A.6) \) and \( (A.3) \), one obtains

\[ K^T \equiv 9 \left( \frac{\partial P}{\partial \rho} \right)_T = 9 \rho \left( \frac{\partial \mu}{\partial \rho} \right)_T = 9 \rho \left( \frac{\partial^2 \phi}{\partial \rho^2} \right)_T. \quad (A.8) \]

Next we turn to the specific heats at constant volume and constant pressure. If measured per particle, they can be defined in terms of the entropy per particle \( \zeta \) as

\[ c_V = T \left( \frac{\partial \zeta}{\partial T} \right)_V = T \left( \frac{\partial \zeta}{\partial T} \right)_\rho, \quad c_P = T \left( \frac{\partial \zeta}{\partial T} \right)_P. \quad (A.9) \]

They obey the following, well known relation to the incompressibilities \[115, 116]\]

\[ \left( \frac{c_P}{c_V} \right) = \left( \frac{\partial P/\partial \rho}{\partial P/\partial T} \right)_T = \frac{K^c}{K^T}. \quad (A.10) \]

For the variation of the entropy \( \zeta \) per particle, one finds

\[ d\zeta = -\frac{1}{\rho} \left[ \zeta + \left( \frac{\partial \mu}{\partial T} \right)_\rho \right] d\rho + \frac{c_V}{T} dT; \quad (A.11) \]

after using \( (A.4) \) and the specific heat \( c_V \) of \( (A.9) \). To get the first term we applied

\[ - \left( \frac{\partial (\rho \phi)}{\partial \rho} \right)_T = \left( \frac{\partial \mu}{\partial T} \right)_\rho = \frac{\partial^2 \phi}{\partial \rho \partial T}. \quad (A.12) \]

which is a consequence of \( (A.4) \).

### A.2. Landau theory proper

In the following, we will repeat some important relations discussed in [57] without arguing much about their proofs. These relations will be needed to derive some specific thermodynamic properties for quantities, as the entropy or the specific heats. A basic element in Landau theory is the microscopic expression for the entropy per particle,

\[ \zeta = -\frac{1}{\rho} \int \frac{2dp}{(2\pi\hbar)^3} \left[ f_p \ln f_p + (1 - f_p) \ln (1 - f_p) \right] \]

\[ = \left< \frac{\rho^2}{3m^*} \left( \frac{\varepsilon_p - \mu}{T} \right) \right> / \left< \frac{\rho^2}{3m^*} \right>, \quad (A.13) \]

in terms of the Fermi distribution \( f_p \) [c.f. (2.2)]. The (static) quasiparticle density \( \rho \) in \( (A.13) \) may be expressed as

\[ \rho = \frac{N}{V} = \frac{N^p}{3\pi^2 \hbar^3} = \int \frac{2dp}{(2\pi\hbar)^3} f_p = \mathcal{N}(T) \left< \frac{\rho^2}{3m^*} \right>, \quad (A.14) \]

with the density of states \( \mathcal{N}(T) \) [24]. The additional factor 2 in the integration measure accounts for the spin degeneracy. The expressions on the right in both \( (A.13) \) and \( (A.14) \) are obtained after integrating by parts. The brackets \( < \cdots > \) denote some kind of average, which if written for any quantity \( A(r, p, t) \) is defined as

\[ \left< A(r, p, t) \right> = \frac{1}{\mathcal{N}(T)} \int \frac{2dp}{(2\pi\hbar)^3} \left( \frac{\partial f_p}{\partial \varepsilon_p} \right)_t A(r, p', t). \quad (A.15) \]

In addition to the \( \mathcal{N}(T) \), one needs

\[ \mathcal{M}(T) = \mathcal{N}(T) \left< \frac{\varepsilon_p - \mu}{T} \right> \quad (A.16) \]

From \( (A.14) \) one may derive (see (2.9) and (2.11) of [57])

\[ d\rho = \frac{N(T)}{\mathcal{G}_0} \, d\mu + \frac{\mathcal{M}(T)}{\mathcal{G}_0} \, dT, \quad (A.17) \]

[see also (2.5) for \( \mathcal{G}_0 \)] which allows one to express the isothermal in-compressibility \( K^T \) \( (A.8) \) by \( (A.14) \). For the variation of the pressure with temperature, one gets from \( (A.6) \) and \( (A.17) \)

\[ \left( \frac{\partial P}{\partial T} \right)_\rho = \left( \zeta - \frac{\mathcal{M}(T)}{\mathcal{N}(T)} \right) \rho = \frac{2}{3} \rho c_V. \quad (A.18) \]

For the proof of the second equation, we refer to (3.35) of [57] (mind however a difference in the notations for the specific heat: Our \( \rho c_V \) is identical to the \( c_V \) of [57]). For our \( c_V \), one may derive the formula (see (3.34) of [57])

\[ c_V = \frac{T \mathcal{N}(T)}{\rho} \left< \left[ \frac{\varepsilon_p - \mu}{T} - \frac{\mathcal{M}(T)}{\mathcal{N}(T)} \right]^2 \right>. \quad (A.19) \]
Collecting Eqs. (A.8), (A.17) and (A.18), one can write the variation of the pressure as
\[
d\bar{P} = \rho \left( \frac{G_0}{N(T)} \right) d\rho + \frac{2}{3} C_V dT. \tag{A.20}
\]

Thermodynamic quantities such as in-compressibilities and susceptibilities are calculated under different conditions as fixed temperature or entropy. As one knows (see, e.g., [115]), these (in)-compressibilities may be associated to different sound velocities. To make use of the adiabaticity condition mentioned earlier, we need the derivatives of the entropy per particle \(s(\rho, T)\). The ones arising in (A.11) can be simplified by exploiting the specific Fermi-liquid expressions given in (A.17) and the second relation of (A.18) between the entropy per particle \(s\) and the specific heat \(C_V\),
\[
ds = -\frac{2}{3\rho} C_V d\rho + \frac{C_V}{T} dT. \tag{A.21}
\]

Next we turn to the adiabatic in-compressibility \(K^s\) (A.17). It may be expressed by the isothermal one \(K^T\) given in (2.7), see (2.82). To derive this relation, the Jacobian transformation from \((\rho, s)\) to \((\rho, T)\) for the derivatives of the pressure in (A.7) has been applied [mind also (A.18), (A.8) and (A.21)]. Finally, for the ratio of the specific heats, we find from (A.10), (2.7) and (A.21)
\[
\left( \frac{C_p}{C_V} \right) = 1 + \frac{4 T C_V N(T)}{9 \rho G_0}. \tag{A.22}
\]

### A.3. Low temperature expansion

In this subsection, we address the temperature dependence of the quantities introduced above. It may be derived as discussed in [57] and conveniently expressed by expansions in terms of \(\bar{T} = T/\bar{\varepsilon}_F\) with \(\bar{\varepsilon}_F\) being the Fermi energy at zero temperature, \(\bar{\varepsilon}_F = \sqrt{2m^*}/(2m^*) = (3\pi^2 h^3 \rho)^{2/3}/(2m^*)\). For some of the quantities discussed below we shall include terms of third order in \(\bar{T} = T/\bar{\varepsilon}_F\), which are not considered in [57].

From (A.14) one gets for the particle density \(\rho(\mu, T)\)
\[
\rho = \left( \frac{2m^* \mu}{3\pi^2 h^3} \right)^{3/2} \left( 1 + \frac{\pi^2 \bar{T}^2}{8} \right) \tag{A.23}
\]
as function of the chemical potential \(\mu\) and the temperature \(T\). For the chemical potential \(\mu\), one obtains
\[
\mu = \bar{\varepsilon}_F \left( 1 - \frac{\pi^2 \bar{T}^2}{12} \right), \tag{A.24}
\]
which is typical for a system of independent fermions. At this stage it may be worth while to mention that the formulas presented here remain largely unchanged in case of the presence of a density dependent potential \(V(\rho)\). As long as such a potential does not depend on the momentum, we may just change our s.p. energy \(\varepsilon^s_p\) to \(p^2/(2m^*) + V(\rho)\) and the chemical potential \(\mu\) to the \(\mu' = \mu - V(\rho)\) of [57].

For the density of states \(N(T)\) of the quasiparticles, one finds from (2.5)
\[
N(T) = N(0) \left( 1 - \frac{\pi^2 \bar{T}^2}{12} \right), \tag{A.25}
\]
where \(N(0)\) is given by (2.82). Similarly, for \(M(T)\) defined in (A.10), one gets
\[
M(T) = \frac{\pi^2}{6} N(0) \bar{T} \left( 1 + \frac{13\pi^2 \bar{T}^2}{60} \right). \tag{A.26}
\]

As different to [57], we include a temperature correction here, which is of interest for some of the quantities described in the text. The specific heat \(C_V\) (A.19) per particle for the constant volume becomes
\[
C_V = \frac{\pi^2 T}{2} \left( 1 - \frac{3\pi^2 \bar{T}^2}{10} \right). \tag{A.27}
\]

For the isothermal in-compressibility \(K^T\), one gets from (2.7) and (A.25)
\[
K^T = 6 \bar{\varepsilon}_F G_0 \left( 1 + \frac{\pi^2 \bar{T}^2}{12} \right). \tag{A.28}
\]

Likewise, for the in-compressibility modulus \(K^s\) (2.82) at constant entropy \(s\) per particle, one obtains
\[
K^s = 6 \bar{\varepsilon}_F G_0 \left[ 1 + \frac{\pi^2 \bar{T}^2}{12} \left( 1 + \frac{4}{G_0} \right) \right]. \tag{A.29}
\]

Using (A.29), the adiabatic sound velocity \(v^{(s)}\) (cf. [115, 116]) can be expressed as
\[
v^{(s)} = \sqrt{\frac{K^s}{9m}} = v_F s^s, \tag{A.30}
\]
where
\[
s^s = \sqrt{\frac{G_0 G_1}{3} \left[ 1 + \frac{\pi^2 \bar{T}^2}{12} \left( 1 + \frac{4}{G_0} \right) \right]}. \tag{A.31}
\]

The ratio of the specific heats (A.10) may be calculated using either the expansions of the in-compressibilities (A.29) and (A.28) or (A.22) together with (A.25) and (A.27). Finally, one gets
\[
\left( \frac{C_P}{C_V} \right) = 1 + \frac{\pi^2 \bar{T}^2}{3 G_0}. \tag{A.32}
\]
Thus, from (A.27) and (A.32),
\[
C_P = \frac{\pi^2 T}{2} \left[ 1 - \frac{3\pi^2 \bar{T}^2}{10} \left( 1 - \frac{10}{9 G_0} \right) \right] \tag{A.33}
\]
is the specific heat at the fixed pressure.
A.4. Thermodynamic relations for a finite Fermi-liquid drop

In this subsection, we apply the formulas derived above to extend the derivations of the boundary conditions in [26, 37, 38] to the case of equilibrium at a finite $T$. Like in these papers, the finite Fermi-liquid drop is treated in the effective sharp surface approximation, see subsection III B 2 and Appendix D. Applying to the standard thermodynamic relations $dE = TdS - PdV - \Pi_dQ$ and $dG = -SdT + VdP - \Pi_dQ$, we include the change of the collective variable $Q$ (see, e.g., [24, 24]). $G$ is the Gibbs free energy $G = F + PV = E + TS + PV$, defined similarly to the free energy $F$ with only the volume $V$ replaced by the pressure $P$. For the FLDM it is more convenient to use $G$ rather than $F$, simply because in general volume may not be conserved but the pressure has to be fixed by the boundary condition (3.22). The Gibbs free energy is used for deriving these boundary conditions as well as for the calculations of the coupling constants and susceptibilities associated to the operator $F(r)$ (3.28).

For the following derivations, we need the relations for the thermodynamical potentials per particle. The Gibbs free energy per particle $G/N$ which is identical to the chemical potential $\mu$ is related to the corresponding free energy $F/N$ by the relation $G/N \equiv \mu = F/N + P/\rho$. For a finite Fermi-liquid drop where the particle density $\rho$ is function of the coordinates (smooth inside and sharp decreasing in the surface region) they are written as in [26, 37, 38] through the variational derivatives $\delta g/\delta \rho$ and $\delta \phi/\delta \rho$ with the thermodynamical potentials densities $g$ and $\phi$ per unit of volume, respectively, and this relation reads now

$$\frac{\delta g}{\delta \rho} \equiv \mu = \frac{\delta \phi}{\delta \rho} + \frac{P}{\rho}. \tag{A.34}$$

These densities depend on the coordinates through $\rho$ and its gradients. Their calculation is carried out from the variations of the corresponding total integral quantities $G$ and $F$ with the following integration by parts, see [26, 37, 38] for details. Taking into account also that the particles in the Fermi-liquid drop move in a mean field $V$ with the coordinate dependence similar to the density $\rho$, one gets from (A.34)

$$\frac{d\delta g}{d\rho} \equiv d\mu = -\varsigma dT + \frac{1}{\rho} dP + dV \quad \text{with} \quad dV = -(PQ/N) dQ. \tag{A.35}$$

From (A.35) one has

$$\nabla \mu = -\varsigma \nabla T + \frac{1}{\rho} \nabla P + \nabla V. \tag{A.36}$$

For the derivation of the boundary condition (3.22), we used (A.36) for the transformations of (2.19) instead of (17) of [37]. The one-to-one correspondence of this derivation with that explained in [26, 37, 38] becomes obvious if we note that equation (17) was found from

$$\nabla \varepsilon = T \nabla \varsigma + \frac{1}{\rho} \nabla P + \nabla V, \tag{A.37}$$

for the adiabatic condition of a constant entropy per particle ($\varepsilon$ here is the same as $\delta \varepsilon/\delta \rho$ in the notation of [37]). The variational derivative $\delta g/\delta \rho$ (A.34) (or the chemical potential $\mu$) appears now in the following key equation for the derivation of the surface condition (3.22):

$$\rho \sim \left( \frac{\delta g}{\delta \rho} \right)_{\text{vol}}^S = -b_V \rho \varepsilon_s + 2a H, \tag{A.38}$$

where $b_V$ is the nucleon binding energy in the infinite nuclear matter, $H$ the mean curvature of the nuclear surface, $\varepsilon_s = 1/R_0$ for the spherical shape at equilibrium. Index ”vol” means that the Gibbs free energy per particle is considered as that found in the nuclear interior. Hence, is a smooth quantity taken at the nuclear surface as the quantities in the l.h.s. of the boundary conditions (3.21) and (3.22) within the precision of the ESA.

The temperature $T$ and chemical potential $\mu$ in (A.35) and (A.38) are constants as function of the coordinates $r$ within our Fermi-liquid-drop interior at equilibrium. With these properties, one gets

$$\nabla V = -\frac{1}{\rho} \nabla P = \frac{K}{9r_b} \nabla \rho. \tag{A.39}$$

In the second equation, we applied (A.20), which shows that the expression in the middle of (A.39) is proportional to the gradient of the particle density with some smooth coefficient related to the incompressibility $K$. The relation (A.39) will be used in the Appendix C for the calculation of several coupling constants and susceptibilities for the constant temperature and entropy, as well as for the static limit $\omega \to 0$, with the corresponding incompressibility modulus and particle density in the last equation (A.39).

For the derivations of the susceptibilities in Appendix C and ratio of the surface energy constants ($C.21$), we need here also the following thermodynamic relation:

$$\left( \frac{\partial^2 G}{\partial Q^2} \right)_T \left( \frac{\partial^2 E}{\partial Q^2} \right)_S = \left[ \left( \frac{\partial^2 G}{\partial T^2} \right)_Q \left( \frac{\partial^2 G}{\partial Q^2} \right)_Q \right]^{-1} \left( \frac{\partial^2 G}{\partial Q^2} \right)_Q \left( \frac{\partial^2 E}{\partial T^2} \right)_Q. \tag{A.40}$$

We obtained this relation as explained in Appendix A1 in [29] with the only one change of the free energy $F$ to the Gibbs free energy $G$. The derivatives in these equations should be considered for the constant pressure instead of the volume of the Fermi-liquid drop.
Appendix B: Stress tensor and heat current

We shall derive the specific expressions for the shear modulus and viscosity in the stress tensor $\sigma_{\alpha\beta}$ \((2.22)\) representing it in the form \((2.25)\) with \((2.26)\) and \((2.27)\) in subsection B.2. We are going also to obtain the expression for the thermal conductivity in the heat current in subsection B.1. The next subsection B.3 is devoted to the long wave approximation for the above mentioned coefficients. In the latter subsection, we derive some basic formulas for this approximation which are used for the response function in whole section II.D beside the above mentioned coefficients, in particular equations for poles of the response function.

B.1. Stress tensor, shear modulus and viscosity

For the calculation of the stress tensor $\sigma_{\alpha\beta}$ \((2.22)\), we shall show first that it really has the form given in \((2.25)\), \((2.26)\), \((2.27)\) with some coefficients $\lambda$ and $\nu$ in Appendix B.1a, and then, find their specific expressions in B.1b.

a. STRESS TENSOR FOR FERMI LIQUIDS

First, after a short calculation of the r.h.s. of \((2.26)\) and \((2.27)\), one simply gets

$$\sigma_{\alpha\beta} = -\left(\frac{1}{\omega} - i\nu\right) \left( q_\beta \dot{u}_\alpha + q_\alpha \dot{u}_\beta - \frac{2}{3} q \delta_{\alpha\beta} \right).$$  \(\text{(B.1)}\)

To simplify more these expressions we note now, that the Fourier components $\tilde{\sigma}_{\alpha\beta}$ \((B.1)\) of the stress tensor $\sigma_{\alpha\beta}$ \((2.22)\) is a symmetric tensor with the two independent components $\tilde{\sigma}_{zz}$ and $\tilde{\sigma}_{xx}$ in the Cartesian coordinate system $(x, y, z)$ with the axis $z$ directed to the wave vector $q$ because of axial symmetry. The tensor \((B.1)\) has also zero trace. Hence, from the set of equations \((B.1)\) only two independent ones survive, namely,

$$\tilde{\sigma}_{zz} = -\frac{4}{3} \left(\frac{1}{\omega} - i\nu\right) q u_z, \quad \tilde{\sigma}_{xx} = -\left(\frac{1}{\omega} - i\nu\right) q u_x,$$

$$\text{with}$$

$$\tilde{\sigma}_{xx} = \tilde{\sigma}_{yy} = -\frac{1}{2} \tilde{\sigma}_{zz}, \quad \tilde{\sigma}_{yz} = \tilde{\sigma}_{xz} \quad \text{and} \quad \tilde{\sigma}_{xy} = 0.$$  \(\text{(B.2)}\)

On the other hand, the stress tensor $\sigma_{\alpha\beta}$ \((2.22)\) in the l.h.s. of \((B.2)\) is determined by the distribution function $\delta f_{1,e}(q, p, \omega)$ in the plane-wave representation, see \((3.10)\) from $\text{[57]}$,

$$\delta f_{1,e}(q, p, \omega) = \left( \frac{\partial f_{1,p}}{\partial \varepsilon_p} \right)_{g.e.} \left( \frac{\omega}{D_p} \delta \mu + \frac{m}{m^*} \tilde{p} \right) + \left( \frac{\varepsilon_p - \mu}{T} \right)_{g.e.} \delta \tilde{T} - \frac{F_0}{N(T)} \delta \tilde{\rho}$$

$$- \left. \frac{q v_p}{D_p} \left[ \delta \mu + \mathbf{p} \tilde{\mu} + \left( \frac{\varepsilon_p - \mu}{T} \right)_{g.e.} \delta \tilde{T} \right] \right).$$  \(\text{(B.4)}\)

This expression can be easily derived from \((2.1)\) after not too lengthy and simple transformations \([24]\) besides of the adaptation to our notations. We substitute then the distribution function $\delta f_{1,e}(q, p, \omega)$ given by \((B.4)\) to the l.h.s. of \((B.2)\) through \((B.22)\) in the considered representation. In this way, we easy realize that the stress tensor \((2.22)\) has the above mentioned symmetry properties, and its components $\tilde{\sigma}_{zz}$ and $\tilde{\sigma}_{xx}$ of l.h.s. of \((B.2)\) with some shear modulus $\lambda$ and viscosity $\nu$ are indeed proportional to $q u_z$ and $q u_x$, respectively. As result, these stress tensor components can be represented for convenience in terms of the two dimensionless quantity $\chi_{zz}$ and $\chi_{xz}$ independent of the mean velocity $\mathbf{u}$,

$$\tilde{\sigma}_{zz} = -\frac{\rho_0 \varepsilon_p}{\nu_p} \chi_{zz} \tilde{u}_z, \quad \tilde{\sigma}_{xx} = -\frac{\rho_0 \varepsilon_p}{\nu_p} \chi_{xz} \tilde{u}_x,$$  \(\text{(B.5)}\)

where

$$\chi_{zz} = J_1 \frac{\rho_0}{\mu} \tilde{T} \mathbf{\hat{\sigma}} + J_2,$$  \(\text{(B.6)}\)

$$J_1 = \frac{2i N(T)}{s \tau \varepsilon_p^2 N(0)} \left( p_z \frac{\varepsilon_p}{D_p} \left( \frac{\varepsilon_p - \mu}{T} - \frac{M(T)}{N(T)} \right) \right) g.e.,$$  \(\text{(B.7)}\)

$$J_2 = -\frac{4 N^2(T)}{3 s \tau \varepsilon_p^2 N(0)} \left( \varepsilon_p P_2(\tilde{p}_z) \left[ \frac{\omega}{D_p} \left( \frac{N(0)}{N(T)} + \frac{3 \rho_0}{2 N} \tilde{p}_z \right) \right) - \frac{q v_p}{D_p} \left( \frac{g_0 N(0)}{N(T)} + \frac{3 \rho_0}{2 N} \tilde{p}_z \right) \right) g.e.,$$  \(\text{(B.8)}\)

$P_2(x)$ is the Legendre polynomial, $\tilde{p}_\alpha$ is the $\alpha$ component of the unit vector $\tilde{p}$ defined in \((2.40)\). Other quantities were defined in Secs. II.A, II.C and Appendix D, see also \((2.3)\), \((2.5)\), \((2.16)\), \((2.53)\), \((2.55)\) and \((2.48)\). $\chi_{zz}$ in \((B.5)\) is given by

$$\chi_{zz} = \frac{3 \omega N(T)}{2 s p_0^2 N(0)} \left( \frac{p_\alpha p_\beta p_\gamma p_\delta}{D_p} \left( \frac{s}{g_1} - \frac{p_\alpha}{p_\beta} \frac{p_\gamma}{p_\delta} \right) \right)$$

$$= -\frac{3 \omega N(T)}{2 s p_0^2 N(0)} \left( \frac{p_\alpha (1 - p_\beta^2) p_\gamma p_\delta}{D_p} \left( \frac{s}{g_1} - \frac{p_\gamma}{p_\delta} \frac{p_\alpha}{p_\beta} \right) \right).$$  \(\text{(B.9)}\)

In the second equation, we used the invariance of the average in the first equation with respect to the replace $\tilde{p}_z \rightarrow \tilde{p}_y$, due to the axial symmetry and the equation $\sum_\alpha \tilde{p}_\alpha^2 = 1$ for the unit vector $\tilde{p}$. We applied also the thermodynamic relation \((2.10)\) for $\delta \mu$ in the distribution function \((B.4)\) in these derivations.
b. THE SHEAR MODULUS AND VISCOSITY

The shear modulus $\lambda$ and viscosity $\nu$ can now be found from the comparison of (B.2) for continuous matter and explicit expressions (B.5) obtained above from the Fermi-liquid distribution function $\delta f_{k,e}(q, p, \omega)$ (B.3) for the same stress tensor components $\sigma_{zz}$ and $\sigma_{xz}$. Indeed, substituting (B.5) to the l.h.s. of (B.2) and canceling the velocity field components from their both sides, one finds

$$J_1 \frac{\rho}{\mu_{e,e}} \frac{\delta T}{\delta \rho} + J_2 = \frac{4}{3} \chi_{xz}, \quad \lambda - i \nu \partial_s \nu = \rho_0 \delta \nu_s \chi_{xz}. \quad \text{(B.10)}$$

From the first equation one has the ratio

$$\delta \frac{\hat{T}}{\hat{\rho}} = \frac{\mu_{e,e}}{\rho_0 J_1} \left( \frac{4}{3} \chi_{xz} - J_2 \right). \quad \text{(B.11)}$$

Separating real and imaginary parts in the second equation, one obtains the shear modulus $\lambda$ and viscosity $\nu$:

$$\lambda = \left| \frac{s^2}{s^t} \chi_{xz} \right| \rho_0 \delta \nu_t q \quad \text{(B.12)}$$

and

$$\nu = -\left( \frac{s}{s^t} \chi_{xz}^t + \frac{s^t}{s} \chi_{xz} \right) \frac{\rho_0 \delta \nu_t}{v_0}. \quad \text{(B.13)}$$

With these constants $\lambda$ and $\nu$, the equations (2.25), (2.20), (2.27) and (2.22) are identities.

The aim of the following derivations of the shear modulus and the viscosity is to simplify $J_1$, $J_2$ and $\chi_{xz}$ (B.9). For this aim, we make use of transformations of the averages like $\langle \rho^k p^l e^{\nu \int (qv)^m} T_{p, e} \rangle_{q, e}$ with some integer numbers $0 \leq k \leq 4$, $0 \leq l \leq 4$, $m = 0, 1$ and $n = 0, 1$ in terms of more simpler functions $\chi_{n}$ ($n = 0, 1, 2$) introduced in (2.54) for the response functions, see (2.54). For these functions, one has simple temperature and hydrodynamic expansions presented below at the end subsection of this Appendix B. Using such enough lengthy and simple algebraic derivations, one finally gets

$$J_1 = \frac{1}{s(1 - is \tau_q)} \left[ \left( 1 + \frac{2}{3} \chi_{xz} \right) \rho_0 \delta \nu_0 \tau_q \chi_{xz} \right] \frac{\chi_1}{\chi_0} \quad \text{(B.14)}$$

$$J_2 = \frac{2i \chi_0 \chi_{xz} T M(T)}{3 \tau_0 \chi_{xz}} \left[ 3 s \left( 1 - is \tau_q \right)^2 \right]$$

$$+ \frac{3i \tau_q}{\chi_0} \left( 1 - is \tau_q \right) \left( s^2 - \frac{\rho_0 \chi_{xz} N(T)}{3 \tau_0} \right) + \frac{s \chi_{xz} N(T)}{\tau_0} \quad \text{(B.15)}$$

$$\times \left[ \left( 3 \frac{N_0}{N(T)} \frac{\chi_1}{\chi_0} \right) \left( 1 - is \tau_q \right)^2 + 1 + \frac{2 M(T)}{N(T)} \right]$$

$$- 1 \frac{M(T)}{T} \tau_0 \frac{\chi_1}{\chi_0} \quad \text{(B.16)}$$

Note that the shear modulus $\lambda$ (B.12) and viscosity $\nu$ (B.13) depend on the sound velocity $s$, and hence, on the solution of the Landau-Vlasov equation (2.11) for $s$ (B.14), (B.15) and (B.16).

B.2. Heat current

For the following derivations of the thermal conductivity $\kappa$ in Fermi liquids, we need to derive the equation for the temperature $T$ from the general transport equation (2.11). The latter equation (2.11) in the linear approximation with respect to the dynamical variations $\delta f$ in terms of the moments, such as the velocity field $u_{\sigma}$, particle density $\delta \rho$, entropy density per particle $\delta S$ and so on, writes

$$\rho T \frac{\partial \kappa}{\partial t} + \nabla \cdot j_T = 0, \quad \text{(B.17)}$$

where $j_T$ is the heat current given in terms of the thermal conductivity $\kappa$ and temperature gradient by (2.39). By making use of the thermodynamic relation for the entropy $\varsigma$ per particle,

$$d\kappa = \left( \frac{\partial \varsigma}{\partial P} \right)_T dP + \left( \frac{\partial \varsigma}{\partial T} \right)_P dT, \quad \text{(B.18)}$$

and the well known arguments to get the thermal conductivity equation, we consider the process with the constant pressure rather than the constant of particle density. (We again omitted the symbol variation $\delta$ as in Sec. (1.11). With the help of (B.18), one then results in the Fourier thermal conductivity equation

$$\rho C_p \frac{\partial T}{\partial t} - \kappa \Delta T = 0, \quad \text{(B.19)}$$

where $C_P$ is the specific heat for the constant pressure per particle, see (A.33). (Equation (2.39) was also used in (B.19) for the heat current $j_T$). Solving equation (B.19) for the temperature $T(r, t) = T_{e,e} + s \Delta T$ in terms of the plane waves for the dynamical part of the temperature $\delta T(r, t)$ as in (2.28) and using the relations (2.18), one gets

$$\kappa = i \rho C_P v_0 s / q. \quad \text{(B.20)}$$

Notice, the thermal conductivity $\kappa$ (B.20) depends on the sound velocity $s$ as the shear modulus $\lambda$ (B.12) and viscosity $\nu$ (B.13), and therefore, on the solution of the Landau-Vlasov equation (2.11) for $s$. 
B.3. Long wave-length limit

As shown in section II.C and subsections B.1a and B.1b, many physical quantities, such as the response functions, see (2.53), the shear modulus (B.12) and viscosity (B.13) can be expanded in terms of the same helpful functions \( \chi_n \) (2.54). By this reason, it is easy to get their LWL limit by expanding the only \( \chi_n \) in small parameter \( \tau_q \).

For small \( \tau_q \), one can use asymptotic expansions for the Legendre function of second kind \( Q_1(\zeta) \) and its derivatives which enter \( \chi_n \) with its derivatives, according to (2.53), (2.54) and (2.55). This approximation is valid for large arguments \( \zeta \). Substituting these expansions into the functions \( \chi_n \) (2.54) and \( \varphi \) (2.55), one gets to fourth order in \( \tau_q \):

\[
\chi_0 = \frac{\tau_q^2}{3} \left[ 1 + 2i s_0 \tau_q - \left( 2s_1 + 3s_0^2 + \frac{3}{5} \right) \tau_q^2 \right] N(0),
\]

\[
\chi_1 = \frac{\tau_q^2}{9} \left[ 1 + 2i s_0 \tau_q - \left( 2s_1 + 3s_0^2 + \frac{6}{5} \right) \tau_q^2 \right] N(0),
\]

\[
\chi_2 = \frac{\tau_q^2}{9} \left[ 1 + 2i s_0 \tau_q - \left( 2s_1 + 3s_0^2 + \frac{3}{5} \right) \tau_q^2 - \frac{\pi^2 T^2}{20} \right] \left[ (1 + 2i s_0 \tau_q - (2s_1 + 3s_0^2 - 60) \tau_q^2) \right] N(0).
\]

With these expressions, one obtains the collective response function \( \chi_{DLB} \) of (2.58), through

\[
\mathcal{N}(s) \equiv \frac{\pi^2 T^2}{27} \left\{ -3i s_0 + (1 + 6s_0^2 + 3s_1) \tau_q \right\} \left[ 93i - (2 + 186s_0^2 + 93s_1) \tau_q \right] N^2(0),
\]

\[
D_0(s) \equiv D_0(\tau_q, s_0, s_1) = \frac{\tau_q^2}{9} \left\{ -i s_0 \left( 1 - 3s_0^2 \right) \right\} + \frac{1}{15} \left[ 5 + 3s_0^2 \left( 1 - 30s_0^2 \right) + 15s_1 \left( 1 - 9s_0^2 \right) \right] \tau_q
\]

\[
+ \frac{\pi^2 T^2}{120} \left\{ -i s_0 \left( 19 + 93s_0^2 \right) + \frac{1}{15} \left( -160 + 18s_0^2 \right) \right\} \left[ (54 + 155s_0^2 + 5s_1 \left( 57 + 837s_0^2 \right)) \tau_q \right\} N^2(0),
\]

[also for the temperature-density response function (2.60). These two quantities determine the expansion of the function \( D(s) \equiv D(\tau_q, s_0, s_1) \) in powers of \( \tau_q \), and then approximately, the excitation modes given by the dispersion relation (2.54). Indeed, equalizing zero the coefficients which appear in front of the each power of \( \tau_q \) in this expansion of \( D(\tau_q, s_0, s_1) \), we get equations for the unknown quantities \( s_0 \) and \( s_1 \) of (2.72),

\[
\frac{i s_0}{G_1} \left[ G_0 G_1 \frac{G_1}{3} + \pi^2 T^2 \left( -40 G_1 + 21 G_0 G_1 \right) \right] - 63s_0^3 - 30 G_1 G_0 s_0^2) \right\} = 0.
\]

Solving these equations, one obtains the position of the poles as given in (2.73) and (2.74).

The shear modulus (\( \lambda \)) and viscosity (\( \nu \)) coefficients enter the response function \( \chi_{DLB} \) (3.34) and (3.34) in terms of the sum \( \lambda = 4 \nu \omega / \bar{\rho} \bar{\varepsilon}_\tau \). The LWL expansion of this sum can be obtained with help of (B.21), (B.22) and (B.23) and expansions of all static quantities in \( T \) there, see Appendix A, but with taking into account fourth order terms,

\[
\frac{(\lambda - i \nu \omega) / \bar{\rho} \bar{\varepsilon}_\tau}{s \chi_{zz}} = - \frac{2}{5} \left( 1 + \frac{5 \pi^2 T^2}{12} \right) \omega \tau + \frac{\pi^4 T^4}{2160} \left[ 13 \left( G_0 - \frac{4}{5} G_1 - G_0 G_1 \right) - \frac{13i \eta G_1 \eta G_1}{\omega \tau} + \frac{i \omega \tau}{25 G_0} (780 + 2815 G_1 + 52 G_1 + 260 G_0 G_1) \right].
\]

Separating the real and imaginary parts in these equations, one gets the LWL approximation of the both real coefficients \( \lambda \) and \( \nu \). The terms linear in \( \omega \tau \) determine the hydrodynamic viscosity \( \nu^{(1)} \) (2.93), and the terms proportional to \( 1 / \omega \tau \) are related to \( \nu^{(2)} \) (2.94), see discussions of the “heat pole” for the FLDM transport coefficients in Sec. II.C.

The LWL approximation for the thermal conductivity \( \kappa \) is determined by equation (2.20) and solutions (2.74) for the heat pole and (2.73) for the sound velocity \( s \) of the dispersion equations (2.20) and (2.23).

The explicit final expressions for the viscosity \( \nu \) and the thermal conductivity \( \kappa \) are presented and discussed in subsection II.D in the LWL limit in connection with the first sound and overdamped (heat pole) modes, see (2.72) and (2.73). As seen immediately from (2.28), the linear terms in \( \tau_q \) for the shear modulus \( \lambda \) appear as high temperature corrections proportional to \( T^4 \). They are regular in \( \omega \tau \), and therefore, are totally immaterial, see more discussions in the subsection mentioned above. In the linear approximation in \( \tau_q \) of the LWL limit, it is easy to check that the derivative \( \delta T / \delta \rho \) (2.11) is the same as obtained in terms of the response functions in (2.84), and therefore, the in-compressibility \( K_{\text{tot}} \) (2.31) turns into the adiabatic one.

Appendix C: Coupling constants and susceptibilities

Let us consider the change of average \( \langle \hat{F}(r) \rangle \) of the operator \( \hat{F}(r) \) (3.37) due to a quasistatic variation of the particle density \( \rho_{qs}(r, Q, T) \),

\[
\delta \langle \hat{F}(r) \rangle_{qs}^X = \int dr \hat{F}^X(r) \delta Q_{qs} = -X_{EF} \delta Q,
\]

(C.1)
The index "X" shows one of the conditions of the constant temperature (X = T), entropy (X is "ad"), or static limit \( \omega \to 0 \) (X is "\( \omega = 0 \)").

We shall follow the notations of [24, 29] but omitting the index \( k_{\omega=0} \equiv k \), surface energy constant \( b_{\omega=0} \equiv b \), and in-compressibility \( K_{\omega=0} \equiv K = K_{\text{tot}} \) for \( \omega = 0 \), see (2.31).

We write it as the zero argument for the isolated susceptibility, see (2.31)]]. We write it as the zero argument for the isolated susceptibility, \( \chi_{\omega=0} = \chi(0) \), and stiffness coefficient, \( C_{\omega=0} = C(0) \). \( f^X \) and \( \delta f^X \) denote the quantity \( f \) and its variation provided that the index \( X \) condition is carried out. The index "qs" stands for the quasistatic quantities as in [29] and will be omitted within this Appendix. Note that the operator \( \hat{F}(r) \) (3.28) depends in the FLDM on \( X \) through the derivatives of the particle density, and by this reason, the upper index \( X \) appears in \( \hat{F}(r) \) of (3.11). From the first of (3.1) with (3.28) and (3.29), one gets the self-consistency condition (3.11),

\[
\delta \left\langle \hat{F}(r) \right\rangle^X = k_{X}^{-1} \delta Q, \quad (C.3)
\]

with the following expression for the coupling constant,

\[
k_{X}^{-1} = -R_0 \int \frac{dV}{dr} Y_{L,0}(\bar{r}) \left[ \left( \frac{\partial \rho}{\partial \bar{Q}} \right)_T \right. \left. + \left( \frac{\partial \rho}{\partial \bar{T}} \right)_Q \frac{\delta T}{\delta \bar{Q}} \right]^{X} O_1, \quad (C.4)
\]

within the ESA parameter of smallness \( a/R \sim A^{-1/3} \ll 1 \), \( O_1 = 1 + O \left( a/R \right) = 1 + O \left( A^{-1/3} \right) \). For the susceptibilities \( \chi_{F,F} \) defined by (C.1) up to small corrections of the order \( A^{-1/3} \) in the same approximation, from (3.1) for \( k_{X}^{-1} \) one gets

\[
\chi^X = -k_{X}^{-1} O_1. \quad (C.5)
\]

We omit also the low indices "FF" for the susceptibilities. Note that we have not identities of \(-k_{X}^{-1} \) to \( \chi^X \) because we neglected earlier high order \( A^{-1/3} \) corrections in the derivation of the operator \( \hat{F} \) (3.28), in particular, in the FLDM approximation (3.29) for the quasistatic particle density \( \rho_{qs} \). The equation (C.5) is in agreement with [31, 10] (identical to equation (3.1.26) of [29]), see also (3.27), (3.102), for the specific relation between the coupling constant \( k_{X}^{-1} \) and isolated susceptibility \( \chi(0) \) with presence of the stiffness term \( C(0) \) in "the zero frequency limit" within the FLDM. As shown in Sec. III C 1 through (3.40) by using the expansion in small parameter \( kC \) (3.40) up to the second order terms in \( kC \), the isolated susceptibility \( \chi(0) \), see (4.11) at \( \omega = 0 \), is related to the coupling constant \( k_{X}^{-1} \) by (3.10) with the stiffness term \( C(0) \). The correction related to the stiffness \( C(0) \) appears in (C.5) in a higher order than \( A^{-1/3} \) because it is of the order of the small parameter \( kC \sim A^{-2/3} \), see (3.70) and discussion near this equation. The zero frequency stiffness \( C(0) \) is equal approximately to the liquid drop one \( C(0) \) in the FLDM for the considered enough large temperatures for which the quantum shell effects can be neglected.

The derivatives of the quasistatic particle density in (C.2) and (C.3) and (C.4) can be found from (3.29),

\[
\begin{align*}
\left( \frac{\partial \rho}{\partial \bar{Q}} \right)_T^X &= - \left( \frac{\partial \rho}{\partial \bar{T}} \right)^X R_0 Y_{L,0}(\bar{r}), \\
\left( \frac{\partial \rho}{\partial \bar{T}} \right)_Q^X &= \frac{1}{\rho_0^0} \frac{\partial^3 R_0}{\partial \bar{T}^2} \left( \frac{\partial \rho}{\partial \bar{T}} \right)^X, \\
\end{align*} \quad (C.6)
\]

for \( Q = 0 \) with

\[
\rho_0^X = \rho_\infty \left( 1 + \frac{6b R_0^3}{K^2 R_0^3} \right), \quad (C.7)
\]

as in (3.27). We emphasize that the surface energy constant \( b_\infty \) (or the surface tension coefficient \( \alpha_\infty \)) depends also on the type of the process specified by index \( X \) as the in-compressibility \( K^X \) because of the \( X \)-dependence of the particle density derivative in the integrand of (3.70) for the tension coefficient. The total quasistatic energy is the sum of the volume and surface parts determined by the in-compressibility \( K^X \) and surface \( b_\infty \) constants, respectively. The in-compressibility modulus \( K^X \) (responsible for the change of the volume energy) is given by \( A^{X} \) for \( X = T \), and \( A^{ad} \) for \( X \) is "ad", see also (2.7), (2.82) or (2.28), (2.29) of their more specific expressions for nuclear matter. The in-compressibility \( K \) equals the adiabatic one \( K^a \) as shown through (2.81) and (2.29), \( K = K_{\text{tot}}(\omega = 0) = K^a \). In the derivations of (3.6), we took into account that \( \rho_0^X \) (C.7) does not depend on \( Q \), and the density \( \rho_\infty \) (or \( R_0 \)) is assumed to be approximately independent of index "X" in (C.6).

Substituting (C.6) into (C.1) for the coupling constant \( k_{X}^{-1} \), one writes

\[
k_{X}^{-1} = R_0^2 \int \frac{dV}{dr} \frac{\partial \rho}{\partial \bar{r}} Y_{L,0}(\bar{r}) \\
\times \left[ Y_{L,0}(\bar{r}) - \frac{1}{3\rho_\infty} \frac{\delta T}{\delta \bar{Q}} \right]^{X} O_1. \quad (C.8)
\]

The first term proportional to the density \( \partial \rho/\partial \bar{T} \) leads to small \( A^{-1/3} \) corrections to the coupling constant \( k_{X}^{-1} \) (C.8) with respect to the second component depending on the coordinate derivative \( \partial \rho/\partial \bar{T} \). However, all terms including these corrections related to the variation of the temperature \( \delta T \) in (C.4) (or (C.8)) can be neglected as compared to the first term in the square brackets. (It comes from the variation of the collective variable \( \delta Q \) up to the same relatively small corrections of the order of \( A^{-1/3} \).) Indeed, for the isothermal case "X = T" one has it exactly by its definition. For other "X" the quantity \( \delta T/\delta Q \) in (C.3) and (C.8) with the density
can be transformed within the ES precision
\[
\left(\frac{\delta T}{\delta Q}\right)^\chi = \left(\frac{\delta T}{\delta \rho} \frac{\partial \rho}{\partial Q}\right)^\chi = \left(\frac{\partial T}{\partial r}\right)^\chi R_0 Y_0(\hat{r}). \tag{C.9}
\]

For instance, for the constant entropy (adiabatic) condition \(S = \int \rho_x = S(\rho, T) = \text{const.} \), with the quasistatic particle density \(\rho \) \((3.26)\), the derivative \(\delta T/\delta Q\) can be calculated through a variation of this density \(\rho\) as shown in the middle of (C.9). In the quasistatic limit \(\omega \rightarrow 0\) all quantities of the equilibrium state can be considered also as a functional of the only density \(\rho\) \((3.26)\) in the ESA and one has again (C.9). We have used already this property in the derivation of the operator \(\hat{F}(r)\) for transformations of the derivatives of a mean field \(V\) in \((3.28)\). As noted and used for the derivations in Appendix A1, the temperature \(T(r)\) is approximately independent of the spatial coordinates \(r\) at equilibrium. Therefore, according to (C.9), the second terms in (C.3) and (C.8), which appear due to the temperature variation \(\delta T\), turn into zero with the FLDM precision.

After the simple integration over the angles \(\hat{r}\) in (C.8) for the coupling constant \(k^{-1}\), one then arrives at
\[
k^{-1}_T = R_0^3 \int_0^\infty dr \left[\frac{\delta V}{\delta \rho} \left(\frac{\partial \rho}{\partial r}\right)^2\right]_{Q=0}^\chi O_1. \tag{C.10}
\]

According to (3.26), the integrands in (C.10) contains the sharp bell function \(\partial \rho/\partial r\) of \(r\). Therefore, the integrals converges there to a small spatial region near the effective nuclear surface defined as the positions of maxima of this quantity as \(r^2\). Thus, the integrals up to small corrections of the order of \(\lambda^{-1/3}\) within the same ESA. This is like for the derivations of the boundary conditions \((3.21), (3.22)\), see [26, 27, 37] and of \((3.28)\) for the operator \(\hat{F}(r)\). In this way, we get the expansion of the coupling constant \(k^{-1}\) \((C.9)\), in powers of the \(\lambda^{-1/3}\) with the leading term shown in the second equation, there, \((C.10)\).

For the following derivations, we specify now the quasistatic derivative \((\delta V/\delta \rho)^\chi\) at \(Q = 0\) taken it from (A.36).
\[
\nabla V^\chi = \frac{K^\chi}{9 \rho_0} \nabla \rho^\chi, \tag{C.11}
\]

where index \(X\) in \(\nabla f^X\) means the gradient of the quantity \(f\) taken for the condition marked by \(X\) as in the variation \(\delta f^X\). The proportionality of the gradients in (C.11) shows the self-consistency within the ESA precision, see [A.37] for more general relations of the self-consistency in the FLDM.

Using (A.39) and (3.24) in (C.10) for the coupling constants and (C.8) for the susceptibilities, one obtains the identical results for these quantities with small corrections of the order of \(A^{-1/3}\),
\[
\chi^\chi = -k^{-1}_T O_1 = \frac{K^b b_0^b R_0^3}{12 \pi^2 \rho_0^b \sigma c^c} C O_1. \tag{C.12}
\]

We shall show now from (C.12) that the adiabatic susceptibility \(\chi^a\) and coupling constant \(k^{-1}_{ad}\) are equal to the isolated \((\chi(0))\) and quasistatic \((k^{-1})\) ones, respectively, up to small \(\lambda^{-1/3}\) corrections within the ESA.

As noted above, for the adiabatic \((K)\) and quasistatic \((K)\) in-compressibility modula, we got \(K^a = K\), see after (2.81). The surface energy constant \(\beta_s\) equals the adiabatic one \(\beta_s^a\), according to (A.38). Indeed, the volume energy per particle is also approximately the same for these cases, \(b_0^a = \beta_s\), because of its relation \(\beta_s \approx K/18\) to the in-compressibility modulus \(b_0^a = K/18\) within the ESA \((27)\) and equivalence of the corresponding incompressibility modula. The functional derivative in (A.39) is the quasichemical potential \(\mu\) which does not depend obviously on the type of the process \(X\). From (A.39), one gets now \(\alpha^ad = \alpha\) for the surface tension coefficient or \(\beta_s = \beta_s^ad\) for the surface energy constant. Namely, this quantity should be identified with the experimental value \(\beta_s = 17 - 19\text{ MeV}\) in the FLDM computations. Thus, from (C.12) one obtains the ergodicity condition (3.12) for the susceptibilities within the ESA precision,
\[
\chi(0) = \chi^ad = \frac{K^b b_0^b R_0^3}{54 \pi^2} A^{4/3}, \tag{C.13}
\]

up to small \(\lambda^{-1/3}\) corrections. As seen from (C.12), one gets also \(k^{-1} = k^{-1}_{ad}\) for the coupling constants within the same approximation, see (3.30) for \(k^{-1}\). The index "ad" for the coupling constant will be omitted below in line of 29.

We are interested also in the discussion of the difference between the susceptibilities \(\chi^T\) and \(\chi^ad\). From (C.4), one has
\[
\chi^T - \chi^ad = (k_T^{-1} - k^{-1}) O_1. \tag{C.14}
\]

It is useful to re-derive this relation by applying Appendix A1 of [29] for the specific Fermi-liquid drop thermodynamics, see (A.40). As noted in Appendix A.4, it is more convenient to use the Gibbs free energy \(G\) instead of the free energy \(F\). As the derivations of the \(\chi^T - \chi^ad\) in Appendix A1 of [29] do not contain any change in the volume and pressure variables, we can use all formulas in A1 of [29] here with the replace of the free energy \(F\) by the Gibbs free energy \(G\), in particular (A.40). There is a specific property of the FLDM with respect to the microscopic shell model with the residue interaction of [29]. The second derivatives of the Hamiltonian \((\partial^2 H/\partial Q^2)_{Q=0}\) in equations (A.1.6) and (A.1.7) of [29] depend in the FLDM on the type of the process \(X\), isothermal and adiabatic one, respectively. The first derivative of the Hamiltonian \(\partial H/\partial Q\) in equations (A.1.2) and
(A.1.3), see [29], is proportional to the derivatives of the density \( \frac{\partial \rho}{\partial Q} \) like in the susceptibilities (A.1.8),

\[
\left( \frac{\partial H}{\partial Q} \right)_x = \left( \frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial Q} \right)_x = - \left( \frac{\partial V}{\partial r} \right)_x R_0 Y_{\alpha 0}(\hat{r}).
\]

We used also this self-consistent dependence of mean potential \( V \) through the particle density \( \rho \) in the derivations of the operator \( \hat{F}(r) \) in the FLDM: The derivatives of \( V \) are proportional to the ones of the density \( \rho \) [see (C.11)], which both depend on \( x \), i.e., whether we consider the latter for the fixed temperature or entropy. Therefore, (A.1.6) and (A.1.7) with the definitions (A.1.8) of [29] as applied to the FLDM, see (A.10), should be a little modified to

\[
\left( \frac{\partial^2 G}{\partial Q^2} \right)_T = \left( \frac{\partial^2 H}{\partial Q^2} \right)_{Q=0}^T - \chi_T,
\]

\[
\left( \frac{\partial^2 E}{\partial Q^2} \right)_{S} = \left( \frac{\partial^2 \hat{H}}{\partial Q^2} \right)_{Q=0}^{\text{ad}} - \chi_{\text{ad}}.
\]

The derivatives of the thermodynamic potential \( G \) are considered for the constant pressure instead of the constant volume as used for the free energy case. Similar calculations of the average value of the second derivative of the Hamiltonian \( \langle \frac{\partial^2 \hat{H}}{\partial Q^2} \rangle \) as for the coupling constants lead to

\[
\langle \frac{\partial^2 \hat{H}}{\partial Q^2} \rangle_{Q=0}^x = \int dr \left( \frac{\partial^2 V}{\partial Q^2} \right)_{Q=0}^x \rho
\]

\[
= -R_0^4 \int_0^\infty dr \left( \frac{\partial V}{\partial r} \frac{\partial \rho}{\partial r} \right)_{Q=0}^x O_1.
\]

We integrated in the second equation of (C.17) by parts. Taking then (3.26) for the quasistatic density \( \rho_{qs} \), one gets

\[
\langle \frac{\partial^2 \hat{H}}{\partial Q^2} \rangle_{Q=0}^{qs} = -k^{-1} \chi_{\text{ad}}.
\]

with the coupling constant \( k \) (3.30). Using the same transformations of the thermodynamic derivatives as in Appendix A1 of [29], see (A.10), from (C.18) and (C.19) up to relatively small \( A^{-1/3} \) corrections of the ESA, one gets

\[
\chi_T - \chi_{\text{ad}} = - \left[ \left( \frac{\partial^2 G}{\partial T^2} \right)_Q \right]^{-1} \left( \frac{\partial^2 G}{\partial T \partial Q} \right)_Q \bigg|_{Q=0}
\]

\[
+ k_T^{-1} - k^{-1}.
\]

Applying then the relation of the Gibbs free energy \( G = A \mu \) to the chemical potential \( \mu \), we note that there is the factor \( A^{-1} \) which suppress much the contribution of the first term compared to second one, \( k^{-1} - k^{-1} \), see (C.12),

\[
k_T^{-1} - k^{-1} = \frac{5 \pi^4}{4} A^{1/3} \frac{K b_s}{56 c} \left( 1 - \frac{K T}{K b_s} \right) \chi_{\text{ad}}.
\]

Moreover, the terms in the square brackets of (C.19) are zero because the second derivative \( \left( \frac{\partial \mu}{\partial Q} \right)_T \) is zero within the precision of the FLDM. To show this, let us take the equation as for the temperature (C.2) with the only replace of the temperature \( T \) by the chemical potential \( \mu \). The above mentioned statement becomes now obvious because the chemical potential \( \mu \) is a constant as function of the spatial coordinates at the equilibrium as the temperature \( T \) independently on the type of the process \( x \) within the FLDM. As the result, we obtain the same relation (C.14) with the difference of the coupling constants shown in (C.20).

We can evaluate the ratio of the surface energy coefficients \( b_T^S/b_s \) of (C.12) using in (C.14) the fundamental relation (A.102) for the ratio of the susceptibilities \( \chi_T/\chi_{\text{ad}} \) in terms of the in-compressibility modula \( K/K^T \)

\[
\frac{b_T^S}{b_s} = \frac{K}{K^T} \left( 2 - \frac{K}{K^T} \right) \approx 1 + \frac{2 \pi^2 T^2}{3G_0}.
\]

In the last equation, we used the temperature expansions for the in-compressibilities \( K (A.29) \) and \( K^T (A.28) \). Thus, the surface energy constant \( b_T^S \) for the constant temperature is larger than adiabatic (or quasistatic) \( b_s \) and their difference is small as \( T^2 \).

**Appendix D: Symmetry-energy density functional and boundary conditions**

The nuclear energy, \( E = \int \! dr \, \rho_+ \, E(\rho_+,\rho_-) \), in the local density approach [72, 74, 75] can be calculated through the energy density \( E(\rho_+\rho_-) \) per particle,

\[
E(\rho_+\rho_-) = -b_0 + JT^2 + \frac{K}{18} \epsilon_{I}^{\text{ad}}(\rho_+) - J T^2 \epsilon_-(\rho_+ \rho_-) + \left( \frac{C_+}{\rho_+} + D_+ \right) (\nabla \rho_+)^2 + \left( \frac{C_-}{\rho_-} + D_- \right) (\nabla \rho_-)^2.
\]

Here, \( \rho_{\pm} = \rho_{n} \pm \rho_{p} \) are the isoscalar, \( \rho_{\pm} \), and isovector, \( \rho_{-} \), particle densities, \( I = (N - Z)/A \) is the asymmetry parameter, \( N \) and \( Z \) are the neutron and proton numbers in the nucleus, \( A = N + Z \). The particle separation energy \( b_0 \approx 16 \text{ MeV} \) and the symmetry energy constant of nuclear matter \( J \approx 30 \text{ MeV} \) are introduced also in (D.1). The in-compressibility modulus of the symmetric nuclear matter \( K \approx 220 - 260 \text{ MeV} \) is shown in Table I of (40, 72, 73). Equation (D.1) can be applied approximately to the realistic Skyrme forces [71, 72], in particular by neglecting small semiclassical \( h \) corrections and Coulomb terms [29, 27, 39, 72]. \( C_\pm \) and \( D_\pm \) are constants defined by the basic Skyrme force parameters. The isoscalar surface energy-density part, independent explicitly of the density gradient terms, is determined by the dimensionless function \( \epsilon_{+}(\rho_+) \) satisfying the standard saturation conditions [27, 39, 40]. For the derivation of the explicitly analytical results, we use the quadratic approximation \( \epsilon_{+}(\rho_+) = (1 - \rho_+ / \rho_{\infty})^2 = (1 - w_+)^2 \), where \( \rho_{\infty} \approx \)
0.16 fm$^{-3}$ is the density of infinite nuclear matter [see around (3.20)]. The isovector component can be simply evaluated as $\epsilon_\pm = 1 - \rho_{\Sigma}^2/(\rho_{\Sigma} + \rho_0)^2 = 1 - w_\pm^2/w_\Sigma^2$. In both these energies $\epsilon_\pm$, $w_\pm = \rho_\Sigma/(\rho_\Sigma + \rho_0)$ are the dimensionless density parameters, $I_\pm = 1$ and $T_\pm = T$. The isoscalar SO gradient terms in (D.1) are defined with a constant: $D_\pm = -9mW_0^2/16\hbar^2$, where $W_0 \approx 100 - 130$ MeV fm$^5$ and $D_\pm$ is relatively small [22, 74, 75].

From the condition of the minimum energy $E$ under the constraints of the fixed particle number $A = \int d\mathbf{r} \rho_\Sigma(r)$ and neutron excess $N - Z = \int d\mathbf{r} \rho_{\Sigma}(\mathbf{r})$, one arrives at the Lagrange equations for $\rho_\pm$ with the corresponding multipliers being the isoscalar and isovector chemical potentials with the surface corrections at the first order, $\Lambda_\pm \propto I_\pm \alpha/R \sim A^{-1/3}$ [20, 24, 33, 40].

The isoscalar and isovector particle densities $w_\pm$ can be derived from (D.1) first at the leading approximation in a small parameter $a/R$. For the isoscalar particle density $w_\pm = w_\pm(\xi)$ [\xi is the distance of the given point $r$ from the ES in units of the diffuseness parameter $a$] in the local ES coordinates, see Appendix B of [40], $\xi = (r-R)/a$, one finds (Appendix B of [40] and [22, 74]),

$$\xi = -\int_{w_\pm}^w dy \sqrt{1 + \beta y} y e(y),$$  \hspace{1cm} (D.2)

below the turning point $\xi(w = 0)$ and $w = 0$ for $\xi \geq \xi(w = 0)$, $\beta = D_+\rho_\infty/C_+$ is the dimensionless SO parameter (for simplicity of the notations, we omit the low index “+” in $w_\pm$). For $w_\pm = w(\xi = 0)$, one has the boundary condition, $d^2 \rho(\xi)/d\xi^2 = 2$ at the ES ($\xi = 0$):

$$\epsilon(w_\pm) + w_\pm(1 + \beta w_\pm) [d\epsilon(w_\pm)/dw] = 0.$$  \hspace{1cm} (D.3)

(see Appendix B of [40] where the specific solutions for $\xi(w)$ in the quadratic approximation for $\epsilon_\pm(w)$ in terms of elementary functions were derived). For $\beta = 0$ (i.e. without SO terms), it simplifies to the solution $w(\xi) = \tanh[(\xi - \xi_0)/2]$ for $\xi \leq \xi_0 = 2 \arctanh(1/\sqrt{3})$ and zero for $\xi$ outside the nucleus ($\xi > \xi_0$). For the same leading term of the isovector density, $w_\pm(\xi)$, one approximately (for large enough constants $c_{\text{sym}}$ of all desired Skyrme forces [22, 74]) finds (Appendix A of [40])

$$w_\pm = w \left(1 - \frac{\tilde{w}^2[w(1 + \tilde{c}w)]^2}{2(1 + \beta)}\right).$$  \hspace{1cm} (D.4)

where

$$\tilde{w} = 1 - w, \quad c_{\text{sym}} = a \sqrt{\frac{J}{\rho_\infty |C_-^\pm|}}, \quad \tilde{c} = \beta c_{\text{sym}}/2 - 1/(1 + \beta),$$  \hspace{1cm} (D.5)

and $a \approx 0.5 - 0.6$ fm is the diffuseness parameter [see 32, 20].

Simple expressions for the constants $b_\pm$ (D.10) can be easily derived in terms of the elementary functions in the quadratic approximation to $\epsilon_\pm(w)$, given explicitly in Appendix A [40]. Note that in these derivations we neglected curvature terms and being of the same order shell corrections. The isovector energy terms were obtained within the ES approximation with high accuracy up to the product of two small quantities, $T^2$ and $(a/R)^2$.

Within the improved ES approximation accounting also for next order corrections in a small parameter $a/R$, we derived the macroscopic boundary conditions (Appendix B of [40])

$$\delta P_{\pm} \bigg|_{\text{ES}} = (\rho_\infty I_\pm \Lambda_\pm)_{\text{ES}} = \delta P_{\pm}^\pm,$$

where $\delta P_{\pm}^\pm = 2\alpha_\pm \delta H$ (D.6) are the isovector and isoscalar surface-tension (capillary) pressures, $\delta H \approx -\delta R_0^{\pm}/R_0^{\pm}$ are small variations of mean ES curvatures $H_0$ [32, 33], $\delta R_0^\pm$ are radius variations [32, 33], and $\alpha_\pm$ are the tension coefficients, respectively,

$$\alpha_\pm = b_\pm^\pm/4\pi r_0^2, \quad b_\pm^\pm = \frac{8\pi}{a}(\rho_0 I_\pm)^2 C_\pm^\pm \times \int_0^\infty d\xi \left(1 + \frac{D_\pm + \rho_0}{C_-^\pm} w_\pm^\pm \right) \left(Dw_{\pm}/D\xi\right)^2. \hspace{1cm} (D.7)$$

The conditions (D.6) ensure the equilibrium through the equivalence of the volume and surface pressure (isoscalar or isovector) variations, see detailed derivations in Appendix B of [40]. As shown in Sec. III [26, 24, 33], the pressures $\delta P_{\pm}$ can be obtained through moments of dynamical variations of the corresponding phase-space distribution functions $\delta f_\pm(r, p, t)$ [22, 74] in the nuclear volume.

For the nuclear energy $E$ in this improved ESA (Appendix C of [40]), one obtains

$$E \approx -b_\nu A + J(N - Z)^2/A + E_{\text{S}}^{(+)} + E_{\text{S}}^{(-)}, \hspace{1cm} (D.8)$$

with the following isoscalar (+) and isovector (-) surface energy components,

$$E_{\text{S}}^{(+)} = \alpha_\pm S = b_{\text{S}}^\pm S/(4\pi r_0^2), \hspace{1cm} (D.9)$$

and the ES area $S$. These energies are determined by the isoscalar and isovector surface energy constants $b_{\text{S}}^\pm$ through the solutions for $w_\pm(\xi)$ taken at the leading order in $a/R$.

For the energy surface coefficients $b_{\text{S}}^\pm$ (D.7) with $D_\pm \approx 0$ in the quadratic approximation $\epsilon_\pm(w) = (1 - w)^2$, we finally arrived at the following explicit analytical expressions in terms of the Skyrme force parameters (Appendix C of [40])

$$b_{\text{S}}^{(+)} = \frac{6C_\rho \rho_\infty J_+}{r_0 a}, \quad b_{\text{S}}^{(-)} = k_s T, \quad k_s = 6\rho_\infty C_- J_-/(r_0 a),$$  \hspace{1cm} (D.10)

where

$$J_+ = \int_0^1 dw \sqrt{w(1 + \beta w)} (1 - w) = \frac{1}{24} (-\beta)^{-5/2} \left[J_+^{(1)} \sqrt{-\beta(1 + \beta)} + J_+^{(2)} \arcsin \sqrt{-\beta}\right], \hspace{1cm} (D.11)$$

$$J_+^{(1)} = 3 + 4\beta(1 + \beta), \quad J_+^{(2)} = -3 - 6\beta, \hspace{1cm} (D.11)$$
and
\[ J_- = -\frac{1}{1 + \beta} \int_0^1 dw \sqrt{w(1 + \beta w)(1 - w)(1 + \tilde{c} w)^2} \]
\[ = \frac{c^2}{1920(1 + \beta)(-\beta)^{3/2}} \left[ J_-^{(1)}(c_{sym}/\tilde{c}) \sqrt{-\beta(1 + \beta)} + J_-^{(2)}(c_{sym}/\tilde{c}) \arcsin(\sqrt{-\beta}) \right], \]
\[ J_-^{(1)}(z) = 105 - 4\beta \{95 + 75z + \beta [119 + 10z(19 + 6z) + 8z^2 (5z(3 + 2z) - 6)] \}, \]
\[ J_-^{(2)}(z) = 15 \{7 + 2\beta [5(3 + 2z) + 8\beta (1 + z)]\}, \quad (D.12) \]

see also \[D.3\] for \( \tilde{c}, c_{sym} \) and \( \tilde{w} \). For the limit \( \beta \to 0 \) from \[D.11\] and \[D.12\], one has \( J_- \to 4/15 \). In the limit \( C_- \to 0 \), one obtains \( k_S \to 0 \).

Appendix E: POT calculations of the MI

E.1. Energy shell corrections

The energy shell corrections \( \delta E \) can be expressed approximately through the oscillating level density component \( \delta g_\epsilon(\epsilon) \), averaged locally by using the convolution (folding) integral with a small averaging parameter \( \Gamma \) of the Gaussian weight function \[3,82\]. As shown in \[3\], neglecting small corrections of the order of the squares of the Fermi energy shell fluctuations \( (\delta\epsilon_p)^2 \) at \( \Gamma \ll \hbar\Omega \approx \epsilon_p/A^{1/3} \) (see \[6.39\], also \[91\]), one has
\[ \delta E = \int d\epsilon n(\epsilon) (\epsilon - \epsilon_p) \delta g_\epsilon(\epsilon), \]
with \( N = \int d\epsilon n(\epsilon), \quad n(\epsilon) = \theta(\epsilon_p - \epsilon). \) (E.1)
Substituting
\[ \delta g_\epsilon(\epsilon) = \text{Re} \sum_{PO} \delta g_{\epsilon,PO} \exp \left[ -\frac{t_{\epsilon,PO} \Gamma}{\hbar} \right] \] (E.2)
with \[5.29\] for \( \delta g_{\epsilon,PO} \) into \[5.1\], one can expand a smooth action in exponent at the linear order,
\[ S_{\text{PO}}(\epsilon) \approx S_{\text{PO}}(\epsilon_p) + t_{\epsilon,0} (\epsilon - \epsilon_p), \quad (E.3) \]
and pre-exponent amplitude at zero order over \( \epsilon \) near the Fermi energy \( \epsilon_p \) \( |t_{\epsilon,0}| = \partial S_{\text{PO}}(\epsilon_p)/\partial \epsilon \) (see a similar derivation of the averaged density \( \delta g_\epsilon(\epsilon) \) in \[73,91,94\]). These expansions are valid for a small enough width \( \Gamma \) mentioned above to get a sharpened bell-shaped Gaussian averaging function near \( \epsilon_p \). Calculating then simple Gaussian integrals over the energy \( \epsilon \) by integration by parts, one arrives at \[6.39\]. In these derivations at the leading order in expansion in \( (\delta\epsilon_p)^2 \), we accounted for the zero value originated by the lower limit \( \epsilon = 0 \) in \[E.1\] by using that \( t_{\epsilon,0}(\epsilon) \) is relatively large at small but finite \( \Gamma \). Thus, one stays with the only contribution (independent of \( \Gamma \)) at the upper limit \( \epsilon = \epsilon_p \), in line of the basic concepts that the energy shell correction is determined by the quantum s.p. states near the Fermi surface \[3,82\]. Similarly, the same result can be obtained by using the Lorentzian weight function [the summand in \(15.2\) is proportional to \(\exp(-t_{\epsilon,0} \Gamma/\hbar)\) in the Lorentzian case, instead of the Gaussian exponent]. In this case, the local convolution averaging of the oscillation level density component with the Lorentzian width parameter \( \Gamma \) is resulted in a formal shift of the energy \( \epsilon \to \epsilon + \Gamma t \) (\( \Gamma \ll \hbar\Omega \)). Thus, the straightforward calculations by the residue method also gives \[5.30\].

E.2. Derivation of the rigid-body MI

\[ a. \text{TF COMPONENT} \]

We substitute approximately the Green’s function \( \langle G_0 \rangle_{\Gamma_p} \), locally averaged over the momentum \( p \) by using the Gaussian weight function with a finite small width \( \Gamma_p \), into \( \Theta^{(0)} \) [see \(6.39\) at \( \nu = \nu' = 0 \)] instead of \( G_0 \),
\[ \langle G_0 \rangle_{\Gamma_p} = \frac{1}{\Gamma_p \sqrt{\pi}} \int_{-\infty}^{\infty} dp' G_0(s,p') \exp \left[ -\left( \frac{p' - p}{\Gamma_p} \right)^2 \right] \]
\[ \approx G_0(s,p) \exp \left[ \frac{s^2 \Gamma_p^2}{4 \hbar^2} \right], \quad (E.4) \]
as in \[91\] for the level density. Transforming then the integration variables \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) to the canonical average \( \mathbf{r} \) and difference \( s = s_{21} \) ones \[6.34\], for the corresponding locally averaged MI component \( \langle \Theta^{(0)}_{\Gamma_p} \rangle_{\Gamma_p} \), one approximately gets
\[ \langle \Theta^{(0)}_{\Gamma_p} \rangle_{\Gamma_p} \approx d_s m^2 \int d\epsilon n(\epsilon) \int dr \int \frac{ds}{s^2} \ell_x (r + \frac{s}{2}) \]
\[ \times \ell_x (r - \frac{s}{2}) \sin \left( \frac{2sp}{\hbar} \right) \exp \left[ -\frac{s^2 \Gamma_p^2}{2 \hbar^2} \right]. \] (E.5)
For simplicity, we omit here and below the index in \( s_{21} \) within this Appendix E.2 because it will not interfere with different notations. As shown below (at the end of this Appendix E.2a), the final result for \( \langle \Theta^{(0)}_{\Gamma_p} \rangle_{\Gamma_p} \) does not depend approximately on \( \Gamma_p \), that looks as a plateau of the SCM (without correction polynomials). Within the NLLLA \[5.29\], used already in \[5.24\] \[113,154\] after the averaging over the phase-space variables, the main contribution is given by small distance \( s_{21} \) with respect to the wave length \( \hbar/p_k \) of the particle near the Fermi surface. In this approximation at the leading zero order, due to the exponential cut-off factor decreasing with \( s \) and \( \Gamma_p \), one may expand smooth classical quantities in \( sp/\hbar \) in the argument of exponent and pre-exponent amplitude factors in \( E.5 \) at the leading order, in particular, applying
\[ \ell_x (r + s/2) \ell_x (r - s/2) \approx \ell_x^2 (r) \approx \ell^2 t_{\perp x}. \] (E.6)
In \( [5.5] \), we integrate over \( s \) in the spherical coordinates, 
\[ ds = s^2 ds \sin\theta d\theta d\varphi, \]
with the polar axis \( z_p \) directed along \( \mathbf{p}(\mathbf{r}) \) (see Fig. 12). Then, for the CT_0 momentum \( \mathbf{p}(\mathbf{r}) \), i.e., along the \( s_p \), one takes into account that the integrand and limits of the integration over angles \( \theta, \varphi \), are constants independent of other variables. Therefore, this integration over all angles gives simply \( 4\pi \), and we arrive at
\[
\langle \Theta^0_x \rangle_p \approx \frac{d_s m^2}{2\pi^2 \hbar} \int d r^2_{L} \int_0^{r_p} d \varepsilon \left[ I_{00}(s_{\max}, \varepsilon, \Gamma_p) - I_{00}(0, \varepsilon, \Gamma_p) \right].
\]
Here, we exchanged the order of integrations over \( \varepsilon \) and \( \mathbf{r} \). The remaining indefinite integral \( I_{00}(s, \varepsilon, \Gamma_p) \) over \( s \) as function of \( s, \varepsilon \) and \( \Gamma_p \) can be approximately (within the NLLLA) taken analytically,
\[
I_{00}(s, \varepsilon, \Gamma) = \int ds \sin \left( \frac{2sp}{\hbar} \right) \exp \left[ \frac{s^2 \Gamma^2}{2\hbar^2} \right] = \sqrt{\frac{\pi}{2}} \frac{i \hbar}{2} \exp \left[ -2 \left( \frac{p}{\Gamma} \right)^2 \right] \left[ \text{erf} \left( \frac{i \sqrt{2} \pi \Gamma}{\hbar} \right) + \text{erf} \left( \frac{i \sqrt{2} \pi \Gamma}{\hbar} \right) \right],
\]
where \( \text{erf}(z) \) is the standard error function, \( \text{erf}(z) = (2/\sqrt{\pi}) \int_0^z dt \exp(-t^2) \). This integral, taken at the upper limit \( s = s_{\max} \), is rather a complicated function of \( \mathbf{r} \), especially near the ES of the potential well. However, the Gaussian factor in the integrand with any small but a finite Gaussian parameter \( \Gamma_p \),
\[
h/R \ll \Gamma_p \ll p_v,
\]
removes the oscillating contribution arising from the upper limit \( s_{\max} \) (\( R \) is the mean nuclear radius). The reason is due to the exponential asymptotics at a large argument \( s \), as such
\[
\exp \left[ \frac{\Gamma^2 s^2}{2p^2} \right], \text{ or } \exp \left[ \frac{2p^2}{s^2} \right], \text{ at } p \sim p_v,
\]
or even strongly as the product of these exponents. Then, according to another asymptotics for small \( s \to 0 \),
\[
I_{00}(s, \varepsilon, \Gamma) = -\frac{\hbar}{2p} + i \sqrt{2\pi} \frac{\hbar}{\Gamma} \exp \left( -\frac{p^2}{2\Gamma^2} \right) + \frac{p^2 s^2}{\hbar} \left[ 1 + \mathcal{O} \left( \left( \frac{p^2 s^2}{\hbar} \right)^2 \right) \right],
\]
we are left with the only constant contribution from the lower limit \( s = 0 \), independent of \( s \) and of the Gaussian averaging parameter \( \Gamma_p \), satisfying the conditions \( \text{E.4} \),
\[
I_{00}(s, \varepsilon, \Gamma) \approx -\hbar/(2p).
\]

Finally, from \( \text{E.7} \) and \( \text{E.12} \) one obtains
\[
\langle \Theta^0_x \rangle_p = \frac{d_s m^2}{2\pi^2 \hbar^3} \left[ \int d \varepsilon n(\varepsilon) \int d r^2_{L} p(r) \right] = d_s m \int d r^2_{L} \rho_{TF}(r) = \Theta^{(\text{RB})}_{x, \Gamma_p}.
\]

We used also the expression for the TF particle density through \( G_0 \) \( [5.24] \),
\[
\rho_{TF}(r) = -\frac{1}{\pi} \int_0^{r_p} d \varepsilon \frac{\Gamma G_0(\varepsilon)}{p_r^2 + \Gamma^2} = \rho(\varepsilon) \left[ \frac{\Gamma G_0(\varepsilon)}{p_r^2 + \Gamma^2} \right].
\]

Similarly, using the Lorentzian weight function for the averaging in \( \Gamma_p \) instead of the Gaussian one,
\[
\langle G_0 \rangle_p = \frac{1}{\pi} \int_{-\infty}^{\infty} dp_r \frac{\Gamma G_0(p_r)}{p_r^2 + \Gamma^2} = G_0(\varepsilon) \left[ \frac{\Gamma G_0(\varepsilon)}{p_r^2 + \Gamma^2} \right].
\]

one obtains the same result \( \text{E.13} \) independently of the choice of the averaging function \( \Gamma = \Gamma_p \) in this Appendix E2a). In these derivations, we used the residue technics for the analytical evaluations of the integrals, that means formally the replace of such a local averaging by the shift of the momentum, \( p \to p + i \Gamma_p \) [see \( \text{E.23} \)] at \( \nu = \nu' = 0 \) and \( \text{E.10} \),
\[
\langle \Theta^0_x \rangle_p \approx \frac{d_s m^2}{2\pi^2 \hbar^3} \left[ \int d \varepsilon n(\varepsilon) \int d r^2_{L} p(r) \right] \exp \left( -\frac{p^2}{\Gamma^2} \right) \left[ \frac{\Gamma G_0(\varepsilon)}{p_r^2 + \Gamma^2} \right].
\]

Again, according to \( \text{E.16} \), the second strongly oscillating term of the integrand coming from the upper limit \( s = s_{\max} \) in the last line can be neglected as exponentially small, instead of the Gaussian behavior above. Then, we are left with the main first TF term [see \( \text{E.13} \)] independent of \( \Gamma_p \), as in the case of the Gaussian averaging.

b. MI SHELL CORRECTIONS

To average the oscillating component \( \delta \Theta^{01} \), of the sum \( \text{E.32} \) (see \( \text{E.33} \) at \( n = 0 \) and \( n' = 1 \)) over the phase space variables, one may use the Green’s function \( \langle G_0 \rangle_p \) \( \text{E.4} \), locally averaged with a Gaussian weight instead of \( G_0 \), and similarly, instead of \( G_1 \) \( [9] \),
\[
\langle G_1 \rangle_p = \frac{1}{\Gamma \sqrt{\pi}} \int d \varepsilon' G_1(r_1, r_2, \varepsilon') \exp \left[ -\frac{(\varepsilon' - \varepsilon)^2}{\Gamma^2} \right] = \sum_{CT_1} A_{CT_1} \exp \left[ \frac{i \sqrt{2} \pi \sigma_{CT_1} - i \phi_d - \frac{\Gamma^2 r^2}{2 \hbar^2} \right].
\]
Transforming also the integration variables $r_1$ and $r_2$ to the canonical ones \( \frac{\hbar}{4\pi^2} \) at zero temperature, one finds

$$
\langle \delta \Theta_{\delta}^{(1)} \rangle = -\frac{d_x m}{\pi^2 \hbar^2} \int_{0}^{\infty} \frac{d\varepsilon}{\varepsilon} \int_{0}^{\varepsilon} \frac{d\xi}{\xi} \int_{0}^{\infty} \frac{ds}{\hbar^2} \ell_x \left( r - \frac{s}{2} \right)
$$

$$
\times \ell_x \left( r + \frac{s}{2} \right) \cos \left( \frac{s}{\hbar} \rho(r) \right) \exp \left[ -\frac{s^2 \Gamma_p^2}{4\hbar^2} \right]
$$

$$
\times A_{CT_1}(r - \frac{s}{2}, r + \frac{s}{2}, \varepsilon) \exp \left[ \frac{i}{\hbar} S_{CT_1}(r - \frac{s}{2}, r + \frac{s}{2}, \varepsilon) \right]
$$

$$
- \frac{i}{2} \delta_S - \frac{i}{2} \delta_p - \frac{t_{CT_1}^2 \Gamma_p^2}{4\hbar^2} \right].
$$

(E.18)

We shall put $\Gamma$ and $\Gamma_p$ to be zero in the final expressions for this average \( \langle \delta \Theta_{\delta}^{(1)} \rangle \), as far as $\pi$ is much smaller than the distance between gross shells $\hbar \Omega$ (E.39) and $\Gamma_p$ satisfies inequalities (E.24). Expanding then the action phase of the second exponent and its pre-exponent factors in small $\hbar \rho / \hbar$ up to the first nonzero terms (i.e., up to the first and zeroth order ones, respectively), due to the first sharp-peaked exponential Gaussian factor in the second line of (E.18), one applies (E.6) and

$$
A_{CT_1}(r - \frac{s}{2}, r + \frac{s}{2}, \varepsilon) \approx A_{CT_1}(r, r; \varepsilon),
$$

$$
S_{CT_1}(r - \frac{s}{2}, r + \frac{s}{2}, \varepsilon) \approx S_{CT_1}(r, r; \varepsilon) + ps.
$$

(E.19)

With these expansions in (E.18), for the integration over $ds = s^2 ds \frac{dx}{d\rho(r)}$ in (E.18), we use the same spherical coordinate system \((s, \theta_s, \varphi_s)\) with the polar axis $z_s$ directed again along the momentum vector \( p(r) = (p_1 + p_2)/2 \), $x_s = \cos \theta_s$ (Fig. 4). The integral over the azimuthal angle $\varphi_s$ gives simply $2\pi$ due to the azimuthal symmetry. The integration over $x_s$ may be considered as from -1 to 1 within the NLLLA (E.23) (neglecting thus the dependence of limits for the integration over angles $\theta_s$ on $s_{max}$ and $r$), one approximately finds from (E.18)

$$
\langle \delta \Theta_{\delta}^{(1)} \rangle \approx -\frac{d_x m}{\pi^2 \hbar^2} \int_{0}^{\infty} \frac{d\varepsilon}{\varepsilon} \int_{0}^{\varepsilon} \frac{d\xi}{\xi} A_{CT_1} \times \exp \left[ \frac{i}{\hbar} S_{CT_1} - \frac{i}{2} \delta_S - \frac{i}{2} \delta_p - \frac{t_{CT_1}^2 \Gamma_p^2}{4\hbar^2} \right] \right].
$$

(E.20)

where

$$
I_{01} = \int_{0}^{s_{max}} \frac{ds}{s} \int_{-1}^{1} \frac{ds \cos \theta_s}{d\rho(r)/d\theta_s} \exp \left[ -s^2 \Gamma_p^2 / 4\hbar^2 \right]
$$

$$
\times \exp \left[ \frac{i}{\hbar} \rho \frac{x_s}{d\rho(r)/d\theta_s} \right] \exp \left[ -s^2 \Gamma_p^2 / 4\hbar^2 \right]
$$

$$
\approx \left[ I_{00} \left( s_{max}, \varepsilon, \frac{\Gamma_p}{\sqrt{2}} \right) - \frac{1}{\hbar^2} \right]
$$

$$
\times \exp \left[ -s^2 \Gamma_p^2 / 4\hbar^2 \right] \approx \hbar \frac{2p}{2\hbar} \exp \left[ -s^2 \Gamma_p^2 / 4\hbar^2 \right].
$$

(E.21)

The sum runs all of $CT_1$'s (closed $CT_1$'s). Taking then the integral over the angle variable $x_s$ in the NLLLA (E.23), one then integrate over the modulus $s$ within integration limits from 0 to $s_{max}$. Note that with the approximation $t_{CT_1} \approx t_{CT_1}^C$, due to $\Gamma \ll \hbar \Omega$ (but significantly larger than a distance between neighboring energy levels), this integral is reduced to $s = 0$ and $s_{max}$ boundaries of $I_{00}(s, \varepsilon, \Gamma_p/\sqrt{2})$ (E.8), see the third line in (E.21). Calculating approximately the integral over $s$ as in the subsection E.2a of this Appendix, and using the same asymptotics (E.10) at large upper integration limit $s = s_{max}$ and (E.11) at small lower one $s = 0$, one obtains the nonzero contribution only from the lower integration limit $s = 0$ as in the previous subsection E.2a. Other contributions of the upper limit $s_{max}$ can be neglected because the integral over $s$ contains rapidly oscillating functions, and after a local averaging in the phase space variables (even with a small but finite Gaussian averaging parameter), they exponentially disappear under the condition (E.23) for $\Gamma_p$ as in the calculations of the Thomas-Fermi MI component (Appendix E.2a). Finally, by making use of (E.21) in (E.20), one obtains

$$
\langle \delta \Theta_{\delta}^{(1)} \rangle \approx -\frac{d_x m}{\pi^2 \hbar^2} \int_{0}^{\infty} \frac{d\varepsilon}{\varepsilon} \int_{0}^{\varepsilon} \frac{d\xi}{\xi} A_{CT_1} \times \exp \left[ \frac{i}{\hbar} S_{CT_1} - \frac{i}{2} \delta_S - \frac{i}{2} \delta_p - \frac{t_{CT_1}^2 \Gamma_p^2}{4\hbar^2} \right] \left[ 1 + \exp \left( -\frac{\Gamma_p S_{max}}{\hbar} \right) \right] \int_{0}^{\infty} \frac{d\varepsilon}{\varepsilon} A_{CT_1} \times \exp \left[ \frac{i}{\hbar} S_{CT_1} - \frac{i}{2} \delta_S - \frac{i}{2} \delta_p - \frac{t_{CT_1}^2 \Gamma_p^2}{4\hbar^2} \right] \right].
$$

(E.22)

In these derivations, we used (5.27) for the perpendicular coordinate $r_{i\perp}$, and (5.44) for the oscillating shell component $\delta \Theta_{x_{scl}}^{(1)}$ of the semiclassical MI (5.40). This component is related to the oscillating shell part $\delta \rho_{osc}(r)$ [see (5.42) and (5.40) with a closed $CT_1$] in the semiclassical particle density (6.33). Like in the previous subsection of Appendix E.2, we obtain the same result (E.22) by using the Lorentzian weight function for the local average ($\varepsilon \to \varepsilon + i\Gamma$). Indeed, using its definition (E.15) for both Green function components $G_0$ and $G_1$, and performing the same integrations in the NLLLA (E.23), one gets

$$
\langle \delta \Theta_{\delta}^{(1)} \rangle \approx -\frac{d_x m}{\pi^2 \hbar^2} \int_{0}^{\infty} \frac{d\varepsilon}{\varepsilon} \int_{0}^{\varepsilon} \frac{d\xi}{\xi} A_{CT_1} \times \exp \left[ \frac{i}{\hbar} S_{CT_1} - \frac{i}{2} \delta_S - \frac{i}{2} \delta_p - \frac{t_{CT_1}^2 \Gamma_p^2}{4\hbar^2} \right] \right].
$$

(E.23)

As transparently seen from this explicit expression, one has exponential disappearance of the oscillating contributions on the upper integration limit $s_{max}$ under the conditions (E.9) for $\Gamma_p$, see the second term in figure brackets of the last two lines of (E.23). Therefore, the first constant term in these brackets (coming from the lower integration limit $s = 0$) yields immediately the finite $\Gamma \to 0$ rigid-body limit (E.22) for $\langle \delta \Theta_{\delta}^{(1)} \rangle$.\]
Using analogous analytical calculations of the other terms $\langle \delta\Theta_{10} \rangle$ and $\langle \delta\Theta_{11} \rangle$ [see (5.33)], one finds the essentially different integrals over $s$, such as

$$I_{10} = \int_0^{s_{\text{max}}} ds \sin^2(ps/h) \exp \left[ -\frac{s^2 \Gamma_p^2}{4\hbar^2} - \frac{t_{\text{CCT}}^2 \Gamma_2^2}{4\hbar^2} \right]$$

$$\approx \exp \left[ -\frac{t_{\text{CCT}}^2 \Gamma_2^2}{4\hbar^2} \right] \int_0^{s_{\text{max}}} ds \left[ 1 - \cos \left( \frac{2ps}{h} \right) \right]$$

$$\times \exp \left[ -\frac{s^2 \Gamma_p^2}{4\hbar^2} \right],$$

and

$$I_{11} = \int_0^{s_{\text{max}}} ds \sin(2ps/h) \exp \left[ -\frac{s^2 \Gamma_p^2}{4\hbar^2} - \frac{t_{\text{CCT}}^2 \Gamma_2^2}{4\hbar^2} \right]$$

$$\approx \exp \left[ -\frac{t_{\text{CCT}}^2 \Gamma_2^2}{4\hbar^2} \right] \int_0^{s_{\text{max}}} ds \sin \left( \frac{2ps}{h} \right)$$

$$\times \exp \left[ -\frac{s^2 \Gamma_p^2}{4\hbar^2} \right],$$

respectively. Integrating analytically in (E.24) and (E.25), one can see that any contributions coming from the upper limit $s_{\text{max}}$ exponentially disappear as shown in (E.10) (with the formal replace $\Gamma_p$ by $\Gamma_p/\sqrt{2}$) as above. However, in contrast to the calculations of $\langle \Theta_{10} \rangle$ and $\langle \Theta_{11} \rangle$, the contributions from the lower integration limit $s = 0$ turn into zero too, according to the asymptotics at the 4th order in distance $s$ in units of the wave-length $\hbar/p$:

$$I_{10} = \frac{2p^2 s^3}{3\hbar^2} \exp \left[ -\frac{t_{\text{CCT}}^2 \Gamma_2^2}{4\hbar^2} \right] \left[ 1 + O \left( \frac{ps}{\hbar} \right) \right],$$

$$I_{11} = \frac{2p s^3}{3\hbar} \exp \left[ -\frac{t_{\text{CCT}}^2 \Gamma_2^2}{4\hbar^2} \right] \left[ 1 + O \left( \frac{ps}{\hbar} \right) \right].$$

Therefore, in addition to (E.22), independently of the weight function for averaging, the two components associated with integrals (E.24) and (E.25) do not contribute at both integration limits within the NLLLA, as explained above. Thus, for all nonzero terms of the oscillating part of the MI, $\langle \delta\Theta_{10} \rangle$, one finally approximately arrives at the same rigid-body MI shell component in the NLLLA (E.23). This result does not depend on the choice of the weight (Gaussian and Lorentzian) functions for the local averaging over the phase space.

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**CAPTIONS TO FIGURES AND TABLE:**

**Fig. 1.** Inertia correction $(M(0) - M_{LD})/M_{LD}$ are shown by solid; dashed line shows the effective damping parameter $\eta^2$, see (3.59). $\varepsilon_0 = 40$ MeV, $r_1 = 1.2$ fm, $F_1 = -0.6$, $\tau_0 = 2.2 \times 10^{-21}$ Mev$^2$ - sec as in [32]; $b_s = 17$ MeV, $L = 2$, $I_0 = 53.3$ MeV, $c = 20$ MeV.

**Fig. 2.** Response function $\Im^{\text{coll}}_{AQQ}$ [the sum of the response-function $(s^{(n)})$ branches over $n$ $(n = 0, 1)$ in (3.40) without $k^2$ factor] versus frequency $\omega$ in units $\Omega = \varepsilon_0/R$ for different temperatures $T$ shown by numbers; $\tau_0 = -0.2$; other parameters are the same as in Fig. 1.

**Fig. 3.** Response function for the hydrodynamic collision regime and heat pole of Fig. 1, for small frequencies and temperature $T = 6$ MeV. Numbers $n = 0$ and/or 1 show the sum of the response-function $(s^{(n)})$ branches or one of them (the latter curves coincide).

**Fig. 4.** Stiffness coefficient $C$ in units of the LDM value $C_{LD}$ versus temperature $T$; full squares are obtained by fitting for the second (hydrodynamic-sound) response function peak in (3.40) as explained in sections III C and III D; joined open squares are the same for the third (Fermi-surface-distortion) peak; open circles are the heat pole stiffness; thick solid shows $C(0) = C_{LD}$; parameters are the same as in Figs. 1 and 2.

**Fig. 5.** Mass parameter $M$ in units of the LDM value $M_{LD}$ versus temperature; thin and thick solids show $M(0)/M_{LD}$ and unit LDM value; other notations and parameters are the same as in Fig. 1.

**Fig. 6.** Temperature dependence of the friction $\gamma$ in $h$ units; thick solid is the LDM value $\gamma(0) = \gamma_{LD}$; dashed line is $\gamma(0)$ for $c = \infty$; other notations and parameters are the same as in Fig. 1.

**Fig. 7.** Friction $\gamma$ (logarithmic scale, $h$ units) as function of temperature; short dashed is asymptotical heat pole friction $\gamma_{hp}/h$; other notations and parameters are the same as in Figs. 4 and 6.

**Fig. 8.** Contribution of the “heat pole” to friction for the non-ergodic system: for the fully drawn curve $\Gamma_T$ is evaluated for $c = 20$ MeV and for dashed curve $1/c = 0$ ; as reference values, the result of the wall formula (line with stars) and the contribution from the non-diagonal matrix elements (line with squares) are shown (after [28, 29]).

**Fig. 9.** Isovector $w_-$ particle density versus $\xi$ with and without $\beta = 0$ SO terms for the Skyrme force SLy7 as a typical example; the isoscalar $w$ [see (1.2) at $\varepsilon = (1 - w)^2$] is also shown by solid lines (after [40]).

**Fig. 10.** Isovector particle densities $w_-(\xi)$ as functions of $\xi$ within the quadratic approximation to $\varepsilon(w)$ for several critical Skyrme forces [74, 75] in the logarithmic scale (after [40]).

**Fig. 11.** Semiclassical moments of inertia $\Theta_x$ (divided
by $\hbar^2$ and expressed in MeV$^{-1}$) as functions of the mass number $A$. Extended Thomas–Fermi results correspond to black full dots whereas those obtained upon neglecting the spin degrees of freedom are represented by crosses. Thomas–Fermi MI are plotted as open circles. Plus signs refer, finally, to the Inglis cranking dynamical MI (after [87]).

**Fig. 12.** Equilibrium deformations of $^{90}$Zr in the $(\beta, \gamma)$ plane for different angular momentum values $I$ (after [88]).

**Fig. 13.** Variational ETF moment of inertia $\Theta_{ETF}$ (in $\hbar^2$ MeV$^{-1}$) of $^{90}$Zr as function of the rotational energy $\hbar \omega$ (in MeV) (after [88]).

**Fig. 14.** Trajectories connecting points $r_1$ and $r_2$ without (CT$_0$; solid line) and with (CT$_1$; dashed line) reflections; the initial $p_1^{(0)}$ and $p_1^{(1)}$, and the final $p_2^{(0)}$ and $p_2^{(1)}$ momenta of a particle at these points; $s_{12} = r_2 - r_1$; polar axises $z$ and $z_s$ and the corresponding angles $\theta_1$ and $\theta_2$ are shown, respectively.

**Fig. 15.** Shell-structure free energy $\delta F$ (in HO units $\hbar \omega_0$) as function of the particle number variable $A^{1/3}$ for the critical deformations $\eta = 1, 1.2$ and $2$ at a temperature of $T = 0.1h \omega_0$; The SCM smoothing parameters are $\gamma/h \omega_0 = 1.5 - 2.5$, and $M = 4 - 8$. Thin dots show the contribution of the 3D orbits, and the thin dashed curves the EQ orbit contributions (after [113]).

**Fig. 16.** Moment of inertia shell correction $\delta \Theta_x$ (in the same units as in Fig. 15) for the perpendicular rotation as function of the particle number variable, $A^{1/3}$, temperatures $T = 0.1$ and $0.2h \omega_0$. The thin dotted line shows the contribution of 3D orbits, the thin dashed line the contribution of EQ orbits for a temperature $T = 0.1h \omega_0$, and broad dashed line the one of EQ orbits for $T = 0.2h \omega_0$ (after [113]).

**TABLE 1.** Isovector energy $k_S$ and coefficients $C_{-}$ for several Skyrme forces [74, 75]; $D(A)$ is the mean IVGDR energy constants for particle numbers $A = 50 - 200$ within the FLDM and last within the hydrodynamic (Steinwendel-Jensen) model; experimental data is about 80 MeV (after [40]).
Quadrupole inertia, $\eta^2$

$A=230, c=20 MeV$

$\frac{M(0)-M}{M}$

$\omega/\Omega$

Quadrupole response

$A=230, c=20 MeV$

$T=10 MeV$

$\text{Im}\chi_Q^{\text{pol}}(\omega)$

FIG. 1.

FIG. 2.
TABLE I.

|   | SkM* | SkM | SIII | SGII | RATP | SkP | T6 | Ski3 | SLy5 | SLy7 |
|---|------|-----|------|------|------|-----|----|------|------|------|
| C | -1.79 | -4.69 | -5.59 | -0.94 | 13.9 | -20.2 | 0  | 12.6 | -22.8 | -13.4 |
| kS| -0.77 | -1.90 | -0.52 | -0.21 | 1.42 | -1.93 | 0  | 4.88 | -6.96 | -6.32 |
| DH | 73-82 | 71-76 | 79-104 | 74-77 | 87  | 70-69 | 86-88 | 105-100 | 76-84 | 77-89 |
| DE | 85-86 | 85-86 | 82  | 82  | 90-89 | 87  | 88  | 101-106 | 79-83 | 81-84 |

FIG. 3.

FIG. 4.
FIG. 5. Quadrupole inertia
\[ A=230, \quad c=20 \text{ MeV} \]

FIG. 6. Quadrupole friction
\[ A=230, \quad c=20 \text{ MeV} \]

FIG. 7. Quadrupole friction
\[ A=230, \quad c=20 \text{ MeV} \]
FIG. 8.

FIG. 9.

FIG. 10.
FIG. 11.

FIG. 12.

FIG. 13.
FIG. 14.

FIG. 15.
FIG. 16.