Gauge symmetries for a coupled Korteweg-de Vries system

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Abstract. We introduce a gauge invariant Hamiltonian which generalizes one of the Hamiltonian structures of a parametric KdV system, which includes the complex KdV system. The associated canonical field equations have an infinite sequence of conserved quantities which are also gauge invariant. The gauge symmetry relates a wide class of integrable systems to the complex KdV system.

1. Introduction

In this work we introduce a gauge invariant Hamiltonian \( H \) which generalizes one of the Hamiltonian structures of a parametric KdV system. The canonical field equations \( \frac{dF}{dt} = \{F,H\} \) associated to \( H \) define an integrable system, in the sense that it has an infinite sequence of conserved quantities, which reduces to the original parametric KdV system in a particular gauge. The interest of having a gauge invariant formulation arises because it allows to relate a wide class of integrable systems, obtained by different gauge fixing procedures, to the KdV system. Moreover it is well known that all physical theories describing the fundamental forces in nature are formulated in terms of gauge invariant actions. In particular, several of the properties of those theories are obtained by considering the theories in different gauges. We expect also that the other Hamiltonian structure of the KdV system can also be formulated in a gauge invariant way. If so, the Virasoro structure of the KdV system would be also compatible with the gauge symmetry we are proposing, as it occurs in String theory and Super Yang Mills theory.

2. A parametric coupled KdV system

The Korteweg-de Vries integrable system has a wide range of applications in low energy as well as in high energy physics. The Poisson structure arising from the Hamiltonian formulation via a Miura transformation coincides with the Virasoro algebra with a central extension which is directly related to the conformal symmetry in String Theory. Many integrable extensions of the KdV equation have been considered in the literature. The most straightforward extension arises by considering the KdV equation with the field taking values on the complex algebra. Further integrable extensions arises by considering the field with values in a general Cayley-Dickson algebra [1]. Although the complex extension seems a trivial one, it has remarkable properties which are not present in the KdV equation with a real field. See [2, 3, 4] for applications to a two-layer liquid model. In [5] it was shown to have solutions developing singularities on a
finite time. Also, solitonic solutions have been reported in [6, 7, 8]. The existence of an infinite sequence of Hamiltonians with a pencil of Poisson structures has been discussed in [9].

In the present work we consider a parametric coupled KdV system [2] which contains for a particular value of the parameter, $\lambda = -1$, the complex valued KdV equation. It is given by

$$u_t + uu_x + u_{xxx} + \lambda vv_x = 0$$  \hspace{1cm} (1)$$

$$v_t + u_x v + v_{xxx} + \lambda v_x u = 0$$  \hspace{1cm} (2)$$

In what follows we will assume that the fields $u$ and $v$ belong to the Schwartz space defined by

$$w \in C^\infty(\mathbb{R})/ \lim_{x \to \pm \infty} x^p \frac{\partial^q}{\partial x^q} w = 0; p,q \geq 0.$$  \hspace{1cm} (3)$$

We will consider prepotential fields $w, y$ defined as

$$u(x,t) = w_x(x,t)$$

$$v(x,t) = y_x(x,t).$$  \hspace{1cm} (4)$$

where the subindex $x$ denotes derivative with respect to the spatial coordinate $x$. We also use a subindex $t$ or a dot to denote derivative with respect to the time coordinate $t$.

3. The gauge invariant Hamiltonian

The parametric KdV system described previously can be obtained from the following Hamiltonian

$$H_1 = \int_{-\infty}^{+\infty} dx \mathcal{H}_1,$$  \hspace{1cm} (5)$$

$$\mathcal{H}_1 = \frac{1}{6} w_x^3 - \frac{1}{2} w_{xx}^2 + \frac{\lambda}{2} w_x y_x^2 - \frac{\lambda}{2} y_{xx},$$  \hspace{1cm} (6)$$

subject to the following two second class constraints [9]

$$\phi_1 \equiv p + \frac{1}{2} w_x = 0,$$  \hspace{1cm} (7)$$

$$\phi_2 \equiv q + \frac{\lambda}{2} y_x = 0.$$  \hspace{1cm} (8)$$

The Poisson structure on the complete phase space is the canonical one given by

$$\{u(x), p(\hat{x})\}_{PB} = \delta(x - \hat{x})$$

$$\{v(x), q(\hat{x})\}_{PB} = \delta(x - \hat{x}),$$  \hspace{1cm} (9)$$

with all other brackets between these variables being zero.

In order to present the Poisson structure on the constrained submanifold, we introduced in [9] the Dirac brackets associated to the second class constraints. The resulting Dirac brackets among the original independent fields are

$$\{u(x), u(\hat{x})\}_{DB} = -\partial_x \delta(x - \hat{x}),$$

$$\{v(x), v(\hat{x})\}_{DB} = -\frac{\lambda}{2} \partial_x \delta(x - \hat{x}),$$  \hspace{1cm} (10)$$

where we have assumed that $\lambda \neq 0$.

The canonical equations for the fields $u(x,t)$ and $v(x,t)$ are

$$u_t = \{u, H_1\}_{DB} = -uu_x - u_{xxx} - \lambda v v_x$$

$$v_t = \{v, H_1\}_{DB} = -u_x v - v_x u - v_{xxx}.$$  \hspace{1cm} (11)$$
which of course are the same as the original equations describing the parametric system.

We may now extend the Hamiltonian and Lagrangian of the above formulation to a more
general theory, which becomes invariant under a gauge transformation on the canonical fields.
The new Hamiltonian is restricted only by a first class constraint, which is the generator of the
gauge symmetry. On a particular gauge fixing the new field equations reduce to the equations
of the original parametric system.

The interesting point is that we may consider different gauge fixing procedures and all of
them are integrable systems, in the sense that the field equations have an infinite sequence of
conserved quantities.

To obtain the gauge invariant Hamiltonian we first notice that
\[ \{ \phi_1(x), \phi_1(\hat{x}) \} = \partial_x \delta \]
\[ \{ \phi_1(x), \phi_2(\hat{x}) \} = 0 \]
\[ \{ \phi_2(x), \phi_2(\hat{x}) \} = \lambda \partial_x \delta, \]  
(12)
hence we can combine \( \phi_1 \) and \( \phi_2 \) in a way that the combination commutes with itself,
\[ \{ \phi(x) + k\phi(x), \phi_1(\hat{x}) + k\phi_2(\hat{x}) \} = 0. \]  
(13)

This is so for any real parameter satisfying
\[ k^2 = -\frac{1}{\lambda}, \lambda \neq 0. \]  
(14)
\( \phi_1(x,t) + k\phi_2(x,t) \) is thus a candidate to be a first class constraint. The other combination
\( \phi_1(x,t) - k\phi_2(x,t) \) does not commute with \( \phi_1(x,t) + k\phi_2(x,t) \) and is a candidate to fix the gauge
of the gauge symmetry generated by \( \phi_1(x,t) + k\phi_2(x,t) \).

In order to have a first class constraint \( \phi_1(x,t) + k\phi_2(x,t) \) must commute, in a weak sense,
with a new Hamiltonian. The new Hamiltonian can be constructed by adding to the original
\( \mathcal{H}_1 \) a finite number of powers of \( \phi_1(x,t) - k\phi_2(x,t) \) with suitable coefficients, which may be
functions of the fields and its derivatives. The formulation is in terms of the prepotentials \( w \)
and \( y \). Although, the Hamiltonian can be expressed in terms of \( u \) and \( v \) and the canonical
field equations for \( u \) and \( v \) may be obtained in terms of the Poisson brackets without reference
to \( w \) and \( y \), the quantum formulation of the system on sure has to be formulated in terms of
\( w, y \) and not in terms of \( u, v \). There are several examples of field theories where analogous
formulations available, and in all case the quantization has to be performed in terms of the
original prepotentials.

The Hamiltonian we introduce is given by
\[ H = \int_{-\infty}^{+\infty} dx \mathcal{H}, \]  
(15)
\[ \mathcal{H} = \frac{1}{6} w_x^3 - \frac{1}{2} w_x^2 + \frac{\lambda}{2} w_x y_x^2 - \frac{\lambda}{2} y_x^2 + f(\phi_1 - k\phi_2) + g(\phi_1 - k\phi_2)^2 + \frac{1}{24}(\phi_1 - k\phi_2)^3, \]  
(16)
\[ f = -\frac{1}{2} \left( \frac{1}{2} w_x^2 + w_{xxx} + \frac{\lambda}{2} y_x^2 + k\lambda w_x y_x + k\lambda y_{xxx} \right), \]  
(17)
\[ g = \frac{1}{4} k\lambda y_x, \]  
(18)
where \( \phi_1 \) and \( \phi_2 \) are defined as previously and \( k \) is a real parameter satisfying \( k^2 = -\frac{1}{\lambda} \), we
assume \( \lambda \neq 0. \)
The infinitesimal gauge transformation for a general field $F$ is defined as

$$ F \rightarrow F + \delta F $$

$$ \delta F = \left\{ F, \int_{-\infty}^{+\infty} dx \, \eta(x,t)(\phi_1 + k\phi_2) \right\}_{PB} $$

where $\eta(x,t)$ is the infinitesimal gauge parameter.

It turns out that

$$ \delta H = 0 $$

which means that the Hamiltonian given by (16) is invariant under the gauge symmetry. Besides, we obtain

$$ \delta w = \eta(x,t) $$

$$ \delta y = k\eta(x,t) $$

$$ \delta p = \frac{1}{2} \partial_x \eta(x,t) $$

$$ \delta q = \frac{1}{2} k\lambda \partial_x \eta(x,t). $$

It turns out that $u - \phi_1$ and $v - k\phi_1$ are gauge invariant fields and all conserved quantities can be formulated in terms of them. The first few conserved quantities are given by

$$ h_1 = \int_{-\infty}^{+\infty} dx \, (u - \phi_1), \quad h_2 = \int_{-\infty}^{+\infty} dx \, (v - k\phi_1), $$

$$ h_3 = \int_{-\infty}^{+\infty} dx \, [(u - \phi_1)^2 + \lambda(v - k\phi_1)^2], $$

$$ h_4 = \int_{-\infty}^{+\infty} dx \, (u - \phi_1)(v - k\phi_1). $$

The general proof of the existence of infinite conserved quantities will be presented elsewhere. All of them commute with the Hamiltonian $H$, hence

$$ [h_i, h_j]_{PB} = 0. $$

Notice that, since $h_i, i = 1, \ldots$, are gauge invariant, they commute with the first class constraint (in the sense of the Poisson bracket), hence in the above equation one can use $H$ or $H + \int_{-\infty}^{+\infty} dx \, \mu(x,t)(\phi_1 + k\phi_2)$, with $\mu(x,t)$ being an independent Lagrange multiplier.

Conclusions

We started from one of the Hamiltonian structures of a parametric KdV system, which includes the complex KdV system, and extended it to a gauge invariant Hamiltonian formulation, which has an infinite sequence of gauge invariant conserved quantities. In a particular gauge fixing they reduce to the already known conserved quantities of the original parametric KdV system. The gauge invariant Hamiltonian is explicitly expressed in terms of a canonical pair of conjugate fields, which are prepotentials of the fields defining the original system. The generator of the gauge symmetry is explicitly obtained. In particular it maps a wide class of integrable systems to the complex KdV system.

The construction we present cannot be reduced to the real KdV equation, however we expect it can be extended to the class of Cayley-Dickson integrable systems beyond the complex algebra. The extension of the second Hamiltonian structure to a gauge invariant formulation is also being under research.

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