CONSTRUCTION OF ALMOST DISJUNCT MATRICES FOR GROUP TESTING

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ABSTRACT. In a group testing scheme, a set of tests is designed to identify a small number \( t \) of defective items among a large set (of size \( N \)) of items. In the non-adaptive scenario the set of tests has to be designed in one-shot. In this setting, designing a testing scheme is equivalent to the construction of a disjunct matrix, an \( M \times N \) matrix where the union of supports of any \( t \) columns does not contain the support of any other column. In principle, one wants to have such a matrix with minimum possible number \( M \) of rows (tests). One of the main ways of constructing disjunct matrices relies on constant weight error-correcting codes and their minimum distance. In this paper, we consider a relaxed definition of a disjunct matrix known as almost disjunct matrix. This concept is also studied under the name of weakly separated design in the literature. The relaxed definition allows one to come up with group testing schemes where a close-to-one fraction of all possible sets of defective items are identifiable. Our main contribution is twofold. First, we go beyond the minimum distance analysis and connect the average distance of a constant weight code to the parameters of an almost disjunct matrix constructed from it. Our second contribution is to explicitly construct almost disjunct matrices based on our average distance analysis, that have much smaller number of rows than any previous explicit construction of disjunct matrices. The parameters of our construction can be varied to cover a large range of relations for \( t \) and \( N \). As an example of parameters, consider any absolute constant \( \epsilon > 0 \) and \( t \) proportional to \( N^\delta, \delta > 0 \). With our method it is possible to explicitly construct a group testing scheme that identifies \( (1 - \epsilon) \) proportion of all possible defective sets of size \( t \) using only \( O\left( t^{3/2} \sqrt{\log(N/\epsilon)} \right) \) tests. On the other hand, to form an explicit non-adaptive group testing scheme that works for all possible defective sets of size \( t \), one requires \( O(t^2 \log N) \) tests.

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Combinatorial group testing is an old and well-studied problem. In the most general form it is assumed that there is a set of $N$ elements among which at most $t$ are defective, i.e., special. This set of defective items is called the defective set or configuration. To find the defective set, one might test all the elements individually for defects, requiring $N$ tests. Intuitively, that would be a waste of resource if $t \ll N$. On the other hand, to identify the defective configuration it is required to ask at least $\log \sum_{i=0}^{t} \binom{N}{i} \approx t \log \frac{N}{t}$ yes-no questions. The main objective is to identify the defective configuration with a number of tests that is as close to this minimum as possible.

In the group testing problem, a group of elements are tested together and if this particular group contains any defective element the test result is positive. Based on the test results of this kind one identifies (with an efficient algorithm) the defective set with minimum possible number of tests. The schemes (grouping of elements) can be adaptive, where the design of one test may depend on the results of preceding tests. For a comprehensive survey of adaptive group testing schemes we refer the reader to [9].

In this paper we are interested in non-adaptive group testing schemes: here all the tests are designed together. If the number of designed tests is $M$, then a non-adaptive group testing scheme is equivalent to the design of a so-called binary test matrix of size $M \times N$ where the $(i, j)$th entry is 1 if the $i$th test includes the $j$th element; it is 0 otherwise. As the test results, we see the Boolean OR of the columns corresponding to the defective entries.

Extensive research has been performed to find out the minimum number of required tests $M$ in terms of the number of elements $N$ and the maximum number of defective elements $t$. The best known lower bound says that it is necessary to have $M = O(t^2 \log N)$ tests [10, 12]. The existence of non-adaptive group testing schemes with $M = O(t^2 \log N)$ is also known for quite some time [9, 17].

Evidently, there is a gap by the factor of $O(\log t)$ in these upper and lower bounds. It is generally believed that it is hard to close the gap. On the other hand, for the adaptive setting, schemes have been constructed with as small as $O(t \log n)$ tests, optimal up to a constant factor [9, 15].

A construction of group testing schemes from error-correcting code matrices and using code concatenation appeared in the seminal paper by Kautz and Singleton [19]. Code concatenation is a way to construct binary codes from codes over a larger alphabet [22]. In [19], the authors concatenate a $q$-ary ($q > 2$) Reed-Solomon code with a unit weight code to use the resulting codewords as the columns of the testing matrix. Recently in [28], an explicit construction of a scheme with $M = O(t^2 \log N)$ tests is provided. The construction of [28] is based on the idea of [19]: instead of the Reed-Solomon code, they take a low-rate code that achieves the Gilbert-Varshamov bound of coding theory [22, 29]. Papers, such as [11, 33], also consider construction of non-adaptive group testing schemes.

In this paper we explicitly construct a non-adaptive scheme that requires a number of test proportional to $t^{3/2}$. However, we needed to relax the requirement of identifications of defective elements in a way that makes it amenable for our analysis. This relaxed requirement schemes were considered under the name of weakly separated designs in [23] and [34]. Our definition of this relaxation appeared previously in the paper [21]. We (and [21, 23, 34]) aim for a scheme that successfully identifies a large fraction of all possible defective configurations. Non-adaptive group testing has found applications in multiple different areas, such as, multi-user communication [3, 32], DNA screening [26], pattern finding [20] etc. It can be
observed that in many of these applications it would have been still useful to have a scheme that identifies almost all different defective configurations if not all possible defective configurations. It is known (see, [34]) that with this relaxation it might be possible to reduce the number of tests to be proportional to $t \log N$. However this result is not constructive. The above relaxation and weakly separated designs form a parallel of similar works in compressive sensing (see, [5,24]) where recovery of almost all sparse signals from a generic random model is considered. In the literature, other relaxed versions of the group testing problem have been studied as well. For example, in [14] it is assumed that recovering a large fraction of defective elements is sufficient. There is also effort to form an information-theoretic model for the group testing problem where test results can be noisy [2]. In other versions of the group testing problem, a test may carry more than one bit of information [4,16], or the test results are threshold-based (see [6] and references therein). Algorithmic aspects of the recovery schemes have been studied in several papers. For example, papers [18] and [27] provide very efficient recovery algorithms for non-adaptive group testing.

1.1. Results. The constructions of [19,28] and many others are based on so-called constant weight error-correcting codes, a set of binary vectors of same Hamming weight (number of ones). The group-testing recovery property relies on the pairwise minimum distance between the vectors of the code [19]. In this work, we go beyond this minimum distance analysis and relate the group-testing parameters to the average distance of the constant weight code. This allows us to connect weakly separated designs to error-correcting codes in a general way. Previously the connection between distances of the code and weakly separated designs was only known for the very specific family of maximum distance separable codes [21], where much more information than the average distance is evident.

Based on the newfound connection, we construct an explicit (constructible deterministically in polynomial time) scheme of non-adaptive group testing that can identify all except an $\epsilon > 0$ fraction of all defective sets of size at most $t$. To be specific, we show that it is possible to explicitly construct a group testing scheme that identifies $(1-\epsilon)$ proportion of all possible defective sets of size $t$ using only $8e^{-t^3/2} \log N \frac{\log(2(N-t))}{(t^2-\log \log(2N/\epsilon))}$ tests for any $\epsilon > 2(N-t)e^{-t}$. It can be seen that, with the relaxation in requirement, the number of tests is brought down to be proportional to $t^{3/2}$ from $t^2$. This allows us to operate with a number of tests that was previously not possible in explicit constructions of non-adaptive group testing. For a large range of values of $t$, namely $t$ being proportional to any positive power of $N$, i.e., $t \sim N^\delta$, and constant $\epsilon$ our scheme has number of tests only about $8e^{-t^{3/2} \sqrt{\log(2N/\epsilon)}}$. Our construction technique is same as the scheme of [19,28], however with a finer analysis relying on the distance properties of a linear code we are able to achieve more.

In Section 2 we provide the necessary definitions and state one of the main results: we state the connection between the parameters of a weakly separated design and the average distance of a constant weight code. In Section 3 we discuss our construction scheme. The proofs of our claims can be found in Sections 4 and 4.

2. DISJUNCT MATRICES

2.1. Lower bounds. It is easy to see that, if an $M \times N$ binary matrix gives a non-adaptive group testing scheme that identify up to $t$ defective elements, then, $\sum_{i=0}^{t} \binom{N}{i} \leq 2^M$. This
means that for any group testing scheme,

\[(1) \quad M \geq \log \sum_{i=0}^{t} \binom{N}{i} \geq t \log \frac{N}{t}.
\]

Consider the case when one is interested in a scheme that identifies all possible except an \(\epsilon\) fraction of the different defective sets. Then it is required that,

\[(2) \quad M \geq \log \left( (1 - \epsilon) \binom{N}{t} \right) \geq t \log \frac{N}{t} + \log(1 - \epsilon).
\]

Although (1) is proven to be a loose bound, it is shown in [23, 34] that (2) is tight.

2.2. Disjunct matrices. The \textit{support} of a vector \(x\) is the set of coordinates where the vector has nonzero entries. It is denoted by \(\text{supp}(x)\). We use the usual set terminology, where a set \(A\) contains \(B\) if \(B \subseteq A\).

\textbf{Definition 1.} An \(M \times N\) binary matrix \(A\) is called \(t\)-disjunct if the support of any column is not contained in the union of the supports of any other \(t\) columns.

It is not very difficult to see that a \(t\)-disjunct matrix gives a group testing scheme that identifies any defective set up to size \(t\). On the other hand any group testing scheme that identifies any defective set up to size \(t\) must be a \((t - 1)\)-disjunct matrix [9]. To a great advantage, disjunct matrices allow for a simple identification algorithm that runs in time \(O(Nt)\). Below we define \textit{relaxed} disjunct matrices. This definition appeared very closely in [23, 34] and independently exactly in [21].

\textbf{Definition 2.} For any \(\epsilon > 0\), an \(M \times N\) matrix \(A\) is called type-1 \((t, \epsilon)\)-disjunct if the set of \(t\)-tuple of columns (of size \(\binom{N}{t}\)) has a subset \(B\) of size at least \((1 - \epsilon)\binom{N}{t}\) with the following property: for all \(J \in B\), \(\cup_{\kappa \in J} \text{supp}(\kappa)\) does not contain support of any column \(v \notin J\).

In other words, the union of supports of a randomly and uniformly chosen set of \(t\) columns from a type-1 \((t, \epsilon)\)-disjunct matrix does not contain the support of any other column with probability at least \(1 - \epsilon\). It is easy to see the following fact.

\textbf{Proposition 1.} A type-1 \((t, \epsilon)\)-disjunct matrix gives a group testing scheme that can identify all but at most a fraction \(\epsilon > 0\) of all possible defective configurations of size at most \(t\).

The definition of disjunct matrix can be restated as follows: a matrix is \(t\)-disjunct if any \(t + 1\) columns indexed by \(i_1, \ldots, i_{t+1}\) of the matrix form a sub matrix which must have a row that has exactly one 1 in the \(i_j\)th position and zeros in the other positions, for \(j = 1, \ldots, t + 1\). Recall that, a \textit{permutation matrix} is a square binary \(\{0, 1\}\)-matrix with exactly one 1 in each row and each column. Hence, for a \(t\)-disjunct matrix, any \(t + 1\) columns form a sub-matrix that must contain \(t + 1\) rows such that a \((t + 1) \times (t + 1)\) permutation matrix is formed of these rows and columns. A statistical relaxation of the above definition gives the following.

\textbf{Definition 3.} For any \(\epsilon > 0\), an \(M \times N\) matrix \(A\) is called type-2 \((t, \epsilon)\)-disjunct if the set of \((t + 1)\)-tuples of columns (of size \(\binom{N}{t+1}\)) has a subset \(B\) of size at least \((1 - \epsilon)\binom{N}{t+1}\) with the following property: the \(M \times (t + 1)\) matrix formed by any element \(J \in B\) must contain \(t + 1\) rows that form a \((t + 1) \times (t + 1)\) permutation matrix.
In other words, with probability at least $1 - \epsilon$, any randomly and uniformly chosen $t + 1$ columns from a type-2 $(t, \epsilon)$-disjunct matrix form a sub-matrix that must have $t + 1$ rows such that a $(t + 1) \times (t + 1)$ permutation matrix can be formed. It is clear that for $\epsilon = 0$, the type-1 and type-2 $(t, \epsilon)$-disjunct matrices are same (i.e., $t$-disjunct). In the rest of the paper, we concentrate on the design of an $M \times N$ matrix $A$ that is type-2 $(t, \epsilon)$-disjunct. Our technique can be easily extended to the construction of type-1 disjunct matrices.

2.3. Constant weight codes and disjunct matrices. A binary $(M, N, d)$ code $C$ is a set of size $N$ consisting of $\{0, 1\}$-vectors of length $M$. Here $d$ is the largest integer such that any two vectors (codewords) of $C$ are at least Hamming distance $d$ apart. $d$ is called the minimum distance (or distance) of $C$. If all the codewords of $C$ have Hamming weight $w$, then it is called a constant weight code. In that case we write $C$ is an $(M, N, d, w)$ constant weight binary code.

Constant weight codes can give constructions of group testing schemes. One just arranges the codewords as the columns of the test matrix. Kautz and Singleton proved the following in [19].

**Proposition 2.** An $(M, N, d, w)$ constant weight binary code provides a $t$-disjunct matrix where, $t = \left\lfloor \frac{w - 1}{w - d/2} \right\rfloor$.

**Proof.** The intersection of supports of any two columns has size at most $w - d/2$. Hence if $w > t(w - d/2)$, support of any column will not be contained in the union of supports of any $t$ other columns. \hfill \Box

2.4. $(t, \epsilon)$-disjunct matrices from constant weight codes. We extend Prop. 2 to have one of our main theorems. However, to do that we need to define the average distance $D$ of a code $C$:

$$D(C) = \frac{1}{|C| - 1} \min_{x \in C} \sum_{y \in C \setminus \{x\}} d_H(x, y).$$

Here $d_H(x, y)$ denotes the Hamming distance between $x$ and $y$.

**Theorem 3.** Suppose, we have a constant weight binary code $C$ of size $N$, minimum distance $d$ and average distance $D$ such that every codeword has length $M$ and weight $w$. The test matrix obtained from the code is type-2 $(t, \epsilon)$-disjunct for the largest $t$ such that,

$$\alpha \sqrt{t \ln \frac{2(t + 1)}{\epsilon}} \leq \frac{w - 1 - t(w - D/2)}{w - d/2}$$

holds. Here $\alpha$ is any absolute constant greater than or equal to $\sqrt{2}(1 + t/(N - 1))$.

The proof of this theorem is deferred until after the following remarks.

**Remark:** By a simple change in the proof of the Theorem 3, it is possible to see that the test matrix is type-1 $(t, \epsilon)$-disjunct if,

$$\alpha \sqrt{t \ln \frac{2(N - t)}{\epsilon}} \leq \frac{w - 1 - t(w - D/2)}{w - d/2},$$

for an absolute constant $\alpha$.

One can compare the results of Prop. 2 and Theorem 3 to see the improvement achieved as we relax the definition of disjunct matrices. This will lead to the final improvement on the parameters of Porat-Rothschild construction [28], as we will see in Section 4.
3. Proof of Theorem

This section is dedicated to the proof of Theorem. Suppose, we have a constant weight binary code $C$ of size $N$ and minimum distance $d$ such that every codeword has length $M$ and weight $w$. Let the average distance of the code be $D$. Note that this code is fixed: we will prove a property of this code by probabilistic method.

Let us now choose $(t + 1)$ codewords randomly and uniformly from all possible $\binom{N}{t+1}$ choices. Let the randomly chosen codewords are $\{c_1, c_2, \ldots, c_{t+1}\}$. In what follows, we adapt the proof of Prop. 2 in a probabilistic setting.

Define the random variables for $i = 1, \ldots, t + 1$, $Z^i = \sum_{j=1}^{t+1} \left( w - \frac{d_H(c_i, c_j)}{2} \right)$. Clearly, $Z^i$ is the maximum possible size of the portion of the support of $c_i$ that is common to at least one of $c_j, j = 1, \ldots, t + 1, j \neq i$. Note that the size of support of $c_i$ is $w$. Hence, as we have seen in the proof of Prop. 2 if $Z^i$ is less than $w$ for all $i = 1, \ldots, t + 1$, then the $M \times (t + 1)$ matrix formed by the $t + 1$ codewords must contain $t + 1$ rows such that a $(t + 1) \times (t + 1)$ permutation matrix can be formed. Therefore, we aim to find the probability $\Pr(Z^i \leq t + 1) \leq (t + 1) \Pr(Z^1 \geq w)$ and show it to be bounded above by $w$ under the condition of the theorem.

As the variable $Z^i$'s are identically distributed, we see that,

$$\Pr(\exists i \in \{1, \ldots, t + 1\} : Z^i \geq w) \leq (t + 1) \Pr(Z^1 \geq w).$$

In the following, we will find an upper bound on $\Pr(Z^1 \geq w)$.

Define,

$$Z_i = \mathbb{E} \left( \sum_{j=2}^{t+1} \left( w - \frac{d_H(c_i, c_j)}{2} \right) \mid d_H(c_1, c_k), k = 2, 3, \ldots, i \right).$$

Clearly, $Z_1 = \mathbb{E} \left( \sum_{j=2}^{t+1} \left( w - \frac{d_H(c_1, c_j)}{2} \right) \right)$, and $Z_{t+1} = \sum_{j=2}^{t+1} \left( w - \frac{d_H(c_1, c_j)}{2} \right) = Z^1$. Now,

$$Z_1 = \mathbb{E} \left( \sum_{j=2}^{t+1} \left( w - \frac{d_H(c_1, c_j)}{2} \right) \right) = tw - \frac{1}{2} \mathbb{E} \sum_{j=2}^{t+1} d_H(c_1, c_j),$$

where the expectation is over the randomly and uniformly chosen $(t + 1)$ codewords from all possible $\binom{N}{t+1}$ choices. Note,

$$\mathbb{E} \sum_{j=2}^{t+1} d_H(c_1, c_j) = \sum_{i_1 < i_2 < \ldots < i_{t+1}} \frac{1}{(t+1)!} \sum_{j=2}^{t+1} d_H(c_{i_1}, c_{i_j})$$

$$= \frac{1}{(t + 1)! \binom{N}{t+1}} \sum_{1 \leq i \neq m \leq t+1} \sum_{j=2}^{t+1} d_H(c_{i_1}, c_{i_j}) = \frac{1}{N(N - 1)} \sum_{j=2}^{t+1} \sum_{i_1 = 1}^{N} \sum_{i_j \neq i_1} d_H(c_{i_1}, c_{i_j})$$

$$= \sum_{j=2}^{t+1} \mathbb{E}d_H(c_{i_1}, c_{i_j}) \leq tD,$$

where the expectation on the last but one line is over a uniformly chosen pair of distinct random codewords of $C$. Hence,

$$Z_1 \leq tw - D/2.$$
We start with the lemma below.

**Lemma 4.** The sequence of random variables $Z_i, i = 1, \ldots, t+1,$ forms a martingale.

The statement is true by construction. For completeness we present a proof that is deferred to Appendix A. Once we have proved that the sequence is a martingale, we show that it is a bounded-difference martingale.

**Lemma 5.** For any $i = 2, \ldots, t+1,$

$$|Z_i - Z_{i-1}| \leq (w - d/2) \left(1 + \frac{t - i + 1}{N - t}\right).$$

The proof is deferred to Appendix B.

Now using Azuma’s inequality for martingale with bounded difference [25], we have,

$$\Pr(|Z_{t+1} - Z_1| > \nu) \leq 2 \exp\left(-\frac{\nu^2}{2(w - d/2)^2 \sum_{i=2}^{t+1} c_i^2}\right),$$

where, $c_i = 1 + \frac{t - i + 1}{N - i}$. This implies,

$$\Pr(|Z_{t+1} - Z_1| > \nu + t(w - D/2)) \leq 2 \exp\left(-\frac{\nu^2}{2(w - d/2)^2 \sum_{i=2}^{t+1} c_i^2}\right).$$

Setting, $\nu = w - 1 - t(w - D/2)$, we have,

$$\Pr\left(Z^1 > w - 1\right) \leq 2 \exp\left(-\frac{(w - 1 - t(w - D/2))^2}{2(w - d/2)^2 \sum_{i=2}^{t+1} c_i^2}\right).$$

Now,

$$\sum_{i=2}^{t+1} c_i^2 \leq t \left(1 + \frac{t - 1}{N - 2}\right)^2.$$

Hence,

$$\Pr(\exists i \in \{1, \ldots, t+1\} : Z^i \geq w) \leq 2(t + 1) \exp\left(-\frac{(w - 1 - t(w - D/2))^2}{2t(w - d/2)^2 \left(1 + \frac{t - 1}{N - 2}\right)^2}\right) < \epsilon,$$

when,

$$d/2 \geq w - \frac{w - 1 - t(w - D/2)}{\alpha \sqrt{t \ln \frac{2(t+1)}{\epsilon}}} ,$$

and $\alpha$ is a constant greater than $\sqrt{2}\left(1 + \frac{t - 1}{N - 2}\right)$.

4. **Construction**

As we have seen in Section 2, constant weight codes can be used to produce disjunct matrices. Kautz and Singleton [19] gives a construction of constant weight codes that results in good disjunct matrices. In their construction, they start with a Reed-Solomon (RS) code, a $q$-ary error-correcting code of length $q - 1$. For a detailed discussion of RS codes we refer the reader to the standard textbooks of coding theory [22, 29]. Next they replace the $q$-ary symbols in the codewords by unit weight binary vectors of length $q$. The mapping from $q$-ary symbols to length-$q$ unit weight binary vectors is bijective: i.e., it is $0 \rightarrow 100 \ldots 0; 1 \rightarrow 010 \ldots 0; \ldots; q - 1 \rightarrow 0 \ldots 01$. We refer to this mapping as $\phi$. As a result, one obtains a set
of binary vectors of length $q(q-1)$ and constant weight $q$. The size of the resulting binary code is same as the size of the RS code, and the distance of the binary code is twice that of the distance of the RS code.

4.1. Consequence of Theorem 3 in Kautz-Singleton construction. For a $q$-ary RS code of size $N$ and length $q-1$, the minimum distance is $q-1-\log_q N + 1 = q - \log_q N$. Hence, the Kautz-Singleton construction is a constant-weight code with length $M = q(q-1)$, weight $w = q-1$, size $N$ and distance $2(q - \log_q N)$. Therefore, from Prop. 2 we have a $t$-disjunct matrix with,

$$t = \frac{q - 1}{q - 1 - q + \log_q N} = \frac{q - 2}{\log_q N - 1} \approx \frac{q \log q}{\log N} \approx \frac{\sqrt{M} \log M}{2 \log N}.$$ 

On the other hand, note that, the average distance of the RS code is $\frac{N}{N-1}(q-1)(1-1/q)$. Hence the average distance of the resulting constant weight code from Kautz-Singleton construction will be

$$D = \frac{2N(q-1)^2}{q(N-1)}.$$ 

Now, substituting these values in Theorem 3 we have a type-1 $(t, \epsilon)$ disjunct matrix, where,

$$\alpha \sqrt{t \ln \frac{2(N-t)}{\epsilon}} \leq (q-t) \log q \frac{\log q}{\log N} \approx \frac{(\sqrt{M} - t) \log M}{2 \log N}.$$ 

Suppose $t \leq \sqrt{M}/2$. Then,

$$M(\ln M)^2 \geq 4\alpha^2 t(\ln N)^2 \ln \frac{2(N-t)}{\epsilon}.$$ 

This basically restricts $t$ to be about $O(\sqrt{M})$. Hence, Theorem 3 does not obtain any meaningful improvement from the Kautz-Singleton construction except in special cases.

There are two places where the Kautz-Singleton construction can be improved: 1) instead of Reed-Solomon code one can use any other $q$-ary code of different length, and 2) instead of the mapping $\phi$ any binary constant weight code of size $q$ might have been used. For a general discussion we refer the reader to [9, §7.4]. In the recent work [28], the mapping $\phi$ is kept the same, while the RS code has been changed to a $q$-ary code that achieve the Gilbert-Varshamov bound [22, 29].

In our construction of disjunct matrices we follow the footsteps of [19,28]. However, we exploit some property of the resulting scheme (namely, the average distance) and do a finer analysis that was absent from the previous works such as [28].

4.2. $q$-ary code construction. We choose $q$ to be a power of a prime number and write $q = \beta t$, for some constant $\beta > 2$. The value of $\beta$ will be chosen later. Next, we construct a linear $q$-ary code of size $N$, length $M_q$ and minimum distance $d_q$ that achieves the Gilbert-Varshamov bound [22,29], i.e.,

$$\frac{\log_q N}{M_q} \geq 1 - h_q \left( \frac{d_q}{M_q} \right) - o(1),$$

where $h_q$ is the $q$-ary entropy function defined by,

$$h_q(x) = x \log_q \frac{q-1}{x} + (1-x) \log_q \frac{1}{1-x}.$$
Porat and Rothschild [28] show that it is possible to construct in time $O(M_q N)$ a $q$-ary code that achieves the Gilbert-Varshamov (GV) bound. To have such construction, they exploit the following well-known fact: a $q$-ary linear code with random generator matrix achieves the GV bound with high probability [29]. To have an explicit construction of such codes, a derandomization method known as the method of conditional expectation [1] is used. In this method, the entries of the generator matrix of the code are chosen one-by-one so that the minimum distance of the resulting code does not go below the value prescribed by (3). For a detail description of the procedure, see [28].

With the above construction with proper parameters we can have a disjunct matrix with the following property.

**Theorem 6.** Suppose $\epsilon > 2(t + 1)e^{-at}$ for some constant $a > 1$. It is possible to explicitly construct a type-2 $(t, \epsilon)$-disjunct matrix of size $M \times N$ where

$$M = O\left(t^{3/2} \ln N \sqrt{\frac{2(2+t)}{\epsilon}} \frac{\ln t}{t - \ln 2(t+1) \ln(4a)} \right).$$

To prove this theorem we need the following identity implicit in [28]. We present the proof here for completeness.

**Lemma 7.** For any $q > s$,

$$1 - h_q(1 - 1/s) = \frac{1}{s \ln q} \left( \ln \frac{q}{s} + \frac{s}{q} - 1 \right) - o\left(\frac{1}{s \ln q}\right).$$

The proof of this is deferred to Appendix C. Now we are ready to prove Theorem 6.

**Proof of Theorem 6.** We follow the Kautz-Singleton code construction. We take a linear $q$-ary code $C'$ of length $M_q \triangleq M_q^q$, size $N$ and minimum distance $d_q \triangleq d_q^2$. Each $q$-ary symbol in the codewords is then replaced with a binary indicator vector of length $q$ (i.e., the binary vector whose all entries are zero but one entry, which is 1) according to the map $\phi$. As a result we have a binary code $C$ of length $M$ and size $N$. The minimum distance of the code is $d$ and the codewords are of constant weight $w = M_q = M_q q$. The average distance of this code is twice the average distance of the $q$-ary code. As $C'$ is linear (assuming it has no all-zero coordinate), it has average distance equal to

$$\frac{1}{N - 1} \sum_{j=1}^{M_q} j A_j = \frac{N}{N - 1} \sum_{j=0}^{M_q} j \left( \frac{M_q}{j} \right) (1 - 1/q)^j (1/q)^{M_q - j} = \frac{N}{N - 1} M_q (1 - 1/q),$$

where $A_j$ is the number of codewords of weight $j$ in $C'$. Here we use the fact that the average of the distance between any two randomly chosen codewords of a nontrivial linear code is equal to that of a binomial random variable [22]. Hence the constant weight code $C$ has average distance

$$D = \frac{2N}{N - 1} M_q (1 - 1/q).$$

The resulting matrix will be $(t, \epsilon)$-disjunct if the condition of Theorem 3 is satisfied, i.e.,

$$d_q \geq M_q - \frac{M_q - 1 - t(M_q - \frac{N}{N-1} M_q (1 - 1/q))}{\alpha \sqrt{t \ln \frac{2(2+t)}{\epsilon}}} = M_q - \frac{M_q - 1 - \frac{tM_q}{N-1} (N/q - 1)}{\alpha \sqrt{t \ln \frac{2(2+t)}{\epsilon}}}$$
or if, \( d_q \geq M_q - M_q - \frac{1}{\alpha \ln \frac{2(t+1)}{\epsilon}} \).

To construct a desired \( q \)-ary code, we use the ideas of [28] where the explicitly constructed codes meet the Gilbert-Varshamov bound. It is possible to construct in time polynomial in \( N, M_q \), a \( q \)-ary code of length \( M_q \), size \( N \) and distance \( d_q \) when

\[
\frac{\log_q N}{M_q} \leq 1 - h_q \left( \frac{d_q}{M_q} \right).
\]

Therefore, explicit polynomial time construction of a type-2 \((t, \epsilon)\)-disjunct matrix will be possible whenever,

\[
\frac{\log_q N}{M_q} \leq 1 - h_q \left( 1 - \frac{1 - 1/M_q - \frac{\epsilon}{\alpha \sqrt{t \ln \frac{2(t+1)}{\epsilon}}}}{1} \right).
\]

(4)

Let us now use the fact that we have taken \( q = \beta t \) to be a prime power for some constant \( \beta \). Let us chose \( \beta > 2e^\alpha \sqrt{a} + 1 \).

Hence,

\[
\frac{1 - 1/M_q - \frac{\epsilon}{\alpha \sqrt{t \ln \frac{2(t+1)}{\epsilon}}}}{1} = \frac{1}{\gamma \sqrt{t \ln \frac{2(t+1)}{\epsilon}}} = \frac{1}{\gamma \sqrt{t \ln \frac{2(t+1)}{\epsilon}}} = \frac{1}{s} \text{ (say)},
\]

for an absolute constant \( \gamma \approx \frac{\alpha \beta}{\beta - 1} \). At this point, we see,

\[
\frac{q}{s} = \frac{\beta t}{\gamma \sqrt{t \ln \frac{2(t+1)}{\epsilon}}} > \frac{\beta}{\gamma \sqrt{a}} > 2e,
\]

from the condition on \( \epsilon \) and the values of \( \beta, \gamma \). Now, using Lemma 7 the right hand side of Eqn. (4) equals to

\[
\frac{1}{s \ln q} \left( \ln \frac{q}{s} + s - 1 \right) - o \left( \frac{1}{s \ln q} \right) \geq \frac{\ln \frac{q}{s} - 1 - o(1)}{s \ln q}.
\]

Then explicit polynomial time construction of a type-2 \((t, \epsilon)\)-disjunct matrix will be possible whenever,

\[
\frac{\log_q N}{M_q} \leq \frac{\ln \frac{\beta t}{\gamma \sqrt{t \ln \frac{2(t+1)}{\epsilon}}}}{\gamma \sqrt{t \ln \frac{2(t+1)}{\epsilon}}} - 1 - o(1)
\]

or,

\[
M = qM_q \geq \beta t \ln N \left( \frac{\ln \frac{\beta t}{\gamma \sqrt{t \ln \frac{2(t+1)}{\epsilon}}} \ln(\beta t)}{\gamma \sqrt{t \ln \frac{2(t+1)}{\epsilon}} \ln(\beta t)} - 1 - o(1) \right)
\]

\[
= \beta \gamma t^{3/2} \ln N \frac{\sqrt{\ln \frac{2(t+1)}{\epsilon}}}{\frac{1}{2}(\ln t - \ln \ln \frac{2(t+1)}{\epsilon}) + \ln \frac{\beta}{\gamma} - 1 - o(1)}.
\]
The condition on $\epsilon$ and the value chosen for $\beta$ ensure that the denominator is strictly positive. Hence it suffices to have,

$$M \geq 2\beta \gamma^{3/2} \ln N \sqrt{\frac{2(t+1)}{\epsilon}} \ln t - \ln \ln \frac{2(t+1)}{\epsilon} + \ln(4a) - o(1).$$

\[\Box\]

Note that the implicit constant in Theorem 6 is proportional to $\sqrt{a}$. We have not particularly tried to optimize the constant. However even then the value of the constant is about $8e\sqrt{a}$.

**Remark:** As in the case of Theorem 3 with a simple change in the proof, it is easy to see that one can construct a test matrix that is type-1 $(t, \epsilon)$-disjunct if,

$$M = O\left(t^{3/2} \ln N \sqrt{\frac{2(N-t)}{\epsilon}} \ln t - \ln \ln \frac{2(N-t)}{\epsilon} + \ln(4a)\right),$$

for any $\epsilon > 2(N-t)e^{-at}$, and a constant $a$.

It is clear from Prop. 1 that a type-1 $(t, \epsilon)$ disjunct matrix is equivalent to a group testing scheme. Hence, as a consequence of Theorem 6 (specifically, the remark above), we will be able to construct a testing scheme with

$$O\left(t^{3/2} \log N \sqrt{\frac{\log 2(N-t)}{\epsilon}} \log t - \log \log \frac{2(N-t)}{\epsilon}\right)$$

tests. Whenever the defect-model is such that all the possible defective sets of size $t$ are equally likely and there are no more than $t$ defective elements, the above group testing scheme will be successful with probability at least $1 - \epsilon$.

Note that, if $t$ is proportional to any positive power of $N$, then $\log N$ and $\log t$ are of same order. Hence it will be possible to have the above testing scheme with $O(t^{3/2} \sqrt{\log(N/\epsilon)})$ tests, for any $\epsilon > 2(N-t)e^{-t}$.

5. CONCLUSION

In this work we show that it is possible to construct non-adaptive group testing schemes with small number of tests that identify a uniformly chosen random defective configuration with high probability. To construct a $t$-disjunct matrix one starts with the simple relation between the minimum distance $d$ of a constant weight code and $t$. This is an example of a scenario where a pairwise property (i.e., distance) of the elements of a set is translated into a property of $t$-tuples.

Our method of analysis provides a general way to prove that a property holds for almost all $t$-tuples of elements from a set based on the mean pairwise statistics of the set. Our method will be useful in many areas of applied combinatorics, such as digital fingerprinting or design of key-distribution schemes, where such a translation is evident. For example, with our method new results can be obtained for the cases of cover-free codes [13,19,31], traceability and frameproof codes [7,30]. This is the subject of our ongoing work.
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APPENDIX A. PROOF OF LEMMA 4

We have created a sequence here that is a martingale by construction. This is a standard method due to Doob [8, 25]. Let,

\[ w - \frac{d_H(c_1, c_j)}{2} = Y_j. \]

Consider the \( \sigma \)-algebras \( \mathcal{F}_k \), \( k = 0, \ldots, t + 1 \), where \( \mathcal{F}_0 = \{ \emptyset, [N] \} \) and \( \mathcal{F}_k \) is generated by the partition of the set of \( \binom{N}{t+1} \) possible choices for \( (t+1) \)-sets into \( \binom{N}{k} \) subsets with the fixed value of the first \( k \) indices, \( 1 \leq k \leq t + 1 \). The sequence of increasingly refined partitions \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{t+1} \) forms a filtration such that \( Z_k \) is measurable with respect to \( \mathcal{F}_k \) (is constant on the atoms of the partition).

We have,

\[
Z_i = \mathbb{E}\left( \sum_{j=2}^{t+1} Y_j \mid Y_2, \ldots, Y_i \right) \\
= \sum_{j=2}^{i} Y_j + \mathbb{E}\left( \sum_{j=i+1}^{t+1} Y_j \mid Y_2, \ldots, Y_i \right) \\
= Z_{i-1} + Y_i + \mathbb{E}\left( \sum_{j=i+1}^{t+1} Y_j \mid Y_2, \ldots, Y_i \right) - \mathbb{E}\left( \sum_{j=i}^{t+1} Y_j \mid Y_2, \ldots, Y_{i-1} \right).
\]

We then have,

\[
\mathbb{E}\left( Z_i \mid Z_1, \ldots, Z_{i-1} \right) = Z_{i-1} + \mathbb{E}\left( Y_i \mid Z_1, \ldots, Z_{i-1} \right) \\
+ \mathbb{E}\left( \mathbb{E}\left( \sum_{j=i+1}^{t+1} Y_j \mid Y_2, \ldots, Y_i \right) \mid Z_1, \ldots, Z_{i-1} \right) \\
- \mathbb{E}\left( \mathbb{E}\left( \sum_{j=i}^{t+1} Y_j \mid Y_2, \ldots, Y_{i-1} \right) \mid Z_1, \ldots, Z_{i-1} \right) \\
= Z_{i-1} + \mathbb{E}\left( Y_i \mid Z_1, \ldots, Z_{i-1} \right)
\]
Proof.

We have,

\[ + \mathbb{E} \left( \sum_{j=i+1}^{t+1} Y_j \mid Z_1, \ldots, Z_{i-1} \right) - \mathbb{E} \left( \sum_{j=i}^{t+1} Y_j \mid Z_1, \ldots, Z_{i-1} \right) = Z_{i-1}. \]

**APPENDIX B. PROOF OF LEMMA 5**

Let us again assume that,

\[ w - \frac{d_H(c_1, c_i)}{2} = Y_j. \]

We have,

\[
\begin{align*}
|Z_i - Z_{i-1}| &= \left| \mathbb{E} \left( \sum_{j=2}^{t+1} Y_j \mid Y_2, \ldots, Y_i \right) - \mathbb{E} \left( \sum_{j=2}^{t+1} Y_j \mid Y_2, \ldots, Y_{i-1} \right) \right| \\
&\leq \max_{0 \leq a, b, \leq w-d/2} \left| \mathbb{E} \left( \sum_{j=2}^{t+1} Y_j \mid Y_2, \ldots, Y_i = a \right) - \mathbb{E} \left( \sum_{j=2}^{t+1} Y_j \mid Y_2, \ldots, Y_{i-1}, Y_i = b \right) \right| \\
&= \max_{0 \leq a, b, \leq w-d/2} \left| \sum_{j=1}^{t+1} \left( \mathbb{E} \left( Y_j \mid Y_2, \ldots, Y_i = a \right) - \mathbb{E} \left( Y_j \mid Y_2, \ldots, Y_{i-1}, Y_i = b \right) \right) \right| \\
&= \max_{0 \leq a, b, \leq w-d/2} \left| a - b + \sum_{j=i+1}^{t+1} \left( \mathbb{E} \left( Y_j \mid Y_2, \ldots, Y_i = a \right) - \mathbb{E} \left( Y_j \mid Y_2, \ldots, Y_i = b \right) \right) \right| \\
&\leq \max_{0 \leq a, b, \leq w-d/2} \left| w - d/2 + \sum_{j=i+1}^{t+1} \left[ \mathbb{E} \left( w - \frac{d_H(c_1, c_j)}{2} \mid d_H(c_1, c_2), \ldots, d_H(c_1, c_i) = 2(w - a) \right) \\
- \mathbb{E} \left( w - \frac{d_H(c_1, c_j)}{2} \mid d_H(c_1, c_2), \ldots, d_H(c_1, c_i) = 2(w - b) \right) \right] \right| \\
&\leq \left| w - d/2 + \sum_{j=i+1}^{t+1} \frac{(w - d/2)}{N - 1 - (i - 1)} \right| \\
&= (w - d/2) \left( 1 + \frac{t - i + 1}{N - i} \right) \\
&= (w - d/2)c_i,
\end{align*}
\]

where \( c_i = 1 + \frac{t - i + 1}{N - i} \).

**APPENDIX C. PROOF OF LEMMA 7**

Proof. The proof is straight-forward and uses the following approximation:

\[ \ln x - \ln(x - 1) = \frac{1}{x} + o \left( \frac{1}{x} \right). \]
We have,

\[
1 - h_q(1 - 1/s) = \frac{1}{\ln q} \left( 1 - \frac{1}{s} \right) \left( (\ln q - \ln(q - 1)) - (\ln s - \ln(s - 1)) \right) \\
+ \frac{1}{s \ln q} \ln \frac{q}{s} \\
= \frac{1}{\ln q} \left( \frac{1}{q} - \frac{1}{q s} - \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s} \ln \frac{q}{s} \right) - o\left( \frac{1}{\ln q} \left( \frac{1}{s} - \frac{1}{q} \right) \right) \\
= \frac{1}{s \ln q} \left( \frac{s}{q} - 1 + \ln \frac{q}{s} \right) - o\left( \frac{1}{s \ln q} \right).
\]
