Uniformity-independent minimum degree conditions for perfect matchings in hypergraphs

Asaf Ferber *Vishesh Jain†

Abstract

In this note, we prove that there exists a universal constant $c = \frac{43}{50}$ such that for every $k \in \mathbb{N}$ and every $d < k/2$, every $k$-uniform hypergraph on $n$ vertices and with minimum $d$-degree at least $(c + o_n(1)) \left( \frac{n-2}{k} \right)$ contains a perfect matching. This is the first such bound which is independent of $k$, and therefore, improves all previously known bounds when $k$ is large. Our approach is based on combining the seminal work of Alon et al. with known bounds on a conjectured probabilistic inequality due to Feige.

1 Introduction

In this note, we observe, following [1], the equivalence (Theorem 1.13) between the fractional, asymptotic version of the Erdős matching conjecture in extremal hypergraph theory, and a probabilistic inequality that controls deviations from expectation of sums of non-negative i.i.d. random variables under only a first moment constraint. As a consequence, we are able to use existing work around this probabilistic inequality to significantly improve on the bounds of the extremal hypergraph theoretic problem (Theorem 1.5) for certain choices of parameters, and to use existing work on the extremal hypergraph theoretic problem in other parameter ranges to strengthen the probabilistic inequality (Corollary 1.14). We will discuss these problems in more detail below.

1.1 Perfect matchings in hypergraphs

For $k \in \mathbb{N}$ with $k \geq 2$, a $k$-uniform hypergraph (or $k$-graph for short) $H$ is a pair $H = (V, E)$, where $V := V(H)$ is a finite set of vertices, and $E := E(H) \subseteq \binom{V(H)}{k}$ is a family of $k$-element subsets of $V$, referred to as the edges of $H$. Given a $k$-graph $H$ and a subset $S \subseteq V$ with $0 \leq |S| \leq k-1$, we let $\deg_H(S)$ be the number of edges of $H$ containing $S$. That is,

$$\deg_H(S) := |\{e \in E(H) \mid S \subseteq e\}|.$$ 

The minimum $d$-degree of $H$, $\delta_d(H)$, is defined as

$$\delta_d(H) := \min \left\{ \deg_H(S) \mid S \in \binom{V(H)}{d} \right\}.$$ 

A collection of vertex disjoint edges of $H$ is called a matching, and the number of edges in a matching is called the size of the matching. We say that a matching $M \subseteq E(H)$ is a perfect matching if $|M| = |V|/k$ (in particular, this requires $|V|$ to be divisible by $k$).

For integers $n, k, d, s$ satisfying $0 \leq d \leq k-1$ and $0 \leq s \leq n/k$, let $m_d^s(k, n)$ be the smallest integer $m$ such that every $n$-vertex $k$-graph $H$ with $\delta_d(H) \geq m$ has a matching of size $s$. Of particular interest is the case $s = n/k$ (that is, a minimum $d$-degree condition to enforce a perfect matching); for convenience, we will denote $m_d^n(k, n)$ by $m_d(k, n)$. The problem of determining $m_d(k, n)$ is a central theme in extremal graph theory and has attracted a lot of attention in the last few decades (see, e.g., the surveys [12, 16] and the references therein). The case $m_1(2, n)$ goes back to a classical work of Dirac [2] from the 1950s, where he proved that every graph on $n$ vertices with minimum degree at least $n/2$ contains a Hamiltonian cycle (that

*Massachusetts Institute of Technology. Department of Mathematics. Email: ferber@mit.edu. Research is partially supported by an NSF grant 6935855.

†Massachusetts Institute of Technology. Department of Mathematics. Email: visheshj@mit.edu
is, a cycle of length \( n \), and therefore, by keeping every other edge (assuming that \( n \) is even), also contains a perfect matching. Ever since, problems of this type (that is, minimum degree conditions which guarantee existence of a perfect matching/Hamiltonian cycle in hypergraphs) are referred to as Dirac-type problems for hypergraphs. One specific conjecture that has attracted a considerable amount of attention is the following:

**Conjecture 1.1.** For all integers \( 1 \leq d \leq k - 1 \) and \( n \) which is divisible by \( k \),

\[
m_{d}(k, n) = \left( \max \left\{ \frac{1}{2} n - 1 - (1 - \frac{1}{k})^{k-d} \right\} + o_{n}(1) \right) \left( \frac{n-d}{k-d} \right),
\]

where \( o_{n}(1) \) stands for some function that tends to 0 as \( n \) tends to infinity.

**Remark 1.2.** Simple explicit constructions (see, e.g., [1]) show that the right hand side is a lower bound on \( m_{d}(k, n) \), so the content of the conjecture is that it is also an upper bound.

It is readily checked that for \( d \geq k/2 \), the maximum of the two terms appearing in Conjecture 1.1 is \( 1/2 \). In this case, Conjecture 1.1 is even known to hold in a stronger exact form due to Treglown and Zhao [15] (generalizing and improving on previous work of Rödl, Ruckiński, and Szemerédi [13] and Pikhurko [11]). On the other hand, for \( d < k/2 \), Conjecture 1.1 has been verified for only a few cases (see the discussion in [7]) and, for example, even the case \( d = 1 \) and \( k = 6 \) is open. For \( 1 \leq d < k/2 \), the best known general upper-bounds on \( m_{d}(k, n) \) are the following (improving on earlier results by Hán, Person, and Schacht [6] and Markström and Ruckiński [10]):

**Theorem 1.3** (Kühn, Osthus, and Townsend [9]). For integers \( k \geq 3 \), \( 1 \leq d \leq k/2 \) and \( n \in \mathbb{N} \),

\[
m_{d}(k, n) \leq \left( \frac{k - d}{k} - \frac{k - d - 1}{k^{k-d}} + o_{n}(1) \right) \left( \frac{n-d}{k-d} \right).
\]

**Theorem 1.4** (Han, [7]). For integers \( k \geq 3 \), \( 1 \leq d \leq k/2 \) and \( n \in \mathbb{N} \),

\[
m_{d}(k, n) \leq \max \left\{ (1 + o_{n}(1)) \left( \frac{n-d}{k-d} \right), \delta(n, k, d) + 1 \right\},
\]

where \( \delta(n, k, d) = (1/2 + o_{n}(1)) \left( \frac{n-d}{k-d} \right) \) is some explicitly described function, and

\[
g(k, d) := 1 - \left( 1 - \frac{(k-d)(k-2d-1)}{(k-1)^{2}} \right) \left( \frac{1}{k} \right)^{k-d} + \Omega(n^{d}).
\]

For a detailed comparison of these bounds, we refer the reader to [7]. Here, we emphasize that if we think of \( k \) as large and \( d = o(k) \), then both the above bounds only show that

\[
m_{d}(k, n) \leq (1 - o_{k}(1)) \left( \frac{n-d}{k-d} \right),
\]

whereas Conjecture 1.1 predicts that

\[
m_{d}(k, n) \leq \left( 1 - e^{-1} + o_{k}(1) \right) \left( \frac{n-d}{k-d} \right).
\]

In this note, we take a first step towards Conjecture 1.1 by showing that the above statement holds with a non-zero absolute constant (unfortunately, smaller than \( e^{-1} \)), thereby substantially improving the above bounds in the particularly interesting case when \( d \) is small compared to \( k \).

**Theorem 1.5.** For integers \( k \geq 3 \), \( 1 \leq d \leq k/2 \) and \( n \) which is divisible by \( k \),

\[
m_{d}(k, n) \leq \left( 1 - \frac{7}{50} + o_{n}(1) \right) \left( \frac{n-d}{k-d} \right).
\]

Even though our proof of this theorem does not involve any significant new ideas, we believe that the statement itself is important enough. For the remainder of this note, we define

\[
m_{d}(k) := \limsup_{n \in \mathbb{N}, n \to \infty} \frac{m_{d}(k, n)}{\left( \frac{n-d}{k-d} \right)}.
\]
1.2 Fractional perfect matchings in hypergraphs

A fractional matching in a $k$-graph $H = (V, E)$ is a function $w : E \rightarrow [0, 1]$ such that for every $v \in V$, $\sum_{e \ni v} w(e) \leq 1$ (observe that if $w : E \rightarrow \{0, 1\}$, then the same condition gives a matching). The size of a fractional matching is defined to be $\sum_{e \in E} w(e)$. We say that $w$ is a perfect fractional matching if its size is $|V|/k$ (or equivalently, if $\sum_{e \ni v} w(e) = 1$ for all $v \in V$). Note that as every matching is also a fractional matching (but not vice versa), it follows that the size of the largest fractional matching in $H$ is at least as large as the size of the largest matching in $H$.

Analogously to the (integer) matching case, for an integer $0 \leq d \leq k - 1$ and a real number $0 \leq s \leq n/k$, we let $f_d^s(k, n)$ denote the smallest integer $m$ such that every $n$-vertex $k$-graph $H$ with $\delta_d(H) \geq m$ has a fractional matching of size $s$. As before, we will denote $f_d^{n/k}(k, n)$ simply by $f_d(k, n)$. In their seminal work, Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov [1] showed that in order to prove Conjecture 1.1, it suffices to prove the corresponding variant for fractional perfect matchings. More precisely, setting

$$ f_d(k) := \limsup_{n \to \infty} f_d(k, n)/\binom{n-d}{k-d}, $$

they proved the following:

**Theorem 1.6 (Theorem 1.1 in [1]).** Fix integers $k, d$ (with $k \geq 3$ and $1 \leq d < k/2$) and a real $\alpha > 0$. Then, there exists $n_0 := n_0(k, d, \alpha) \in \mathbb{N}$ such that for all $n \geq n_0$, all $k$-graphs $H$ on $n$ vertices with $\delta_d(H) \geq (f_d(k) + \alpha)\binom{n-d}{k-d}$ contain a perfect matching.

In particular, it follows that for the above range of parameters

$$ m_d(k) = f_d(k) + o(1). \quad (2) $$

Hence, it suffices to study the following (presumably easier) analogous conjecture for fractional perfect matchings.

**Conjecture 1.7.** For integers $k \geq 3$ and $1 \leq d \leq k - 1$,

$$ f_d(k) = 1 - \left(1 - \frac{1}{k}\right)^{k-d}. $$

Once again, the lower bound on $f_d(k)$ follows by a simple, explicit construction. As a means to prove the above conjecture, the authors in [1] provided a further reduction to the ‘$d = 0$’ case.

**Proposition 1.8 (Proposition 1.1 in [1]).** For integers $k \geq 3$, $1 \leq d \leq k - 1$, and $n \geq k$,

$$ f_d(k, n) \leq f_0^{n/k}(k-d, n-d). $$

In particular, this shows that it suffices to prove the following, which is a fractional version of the classical matching conjecture of Erdős [3] (see [1] for a more general statement and further discussion).

**Conjecture 1.9.** For integers $\ell, d \geq 1$ and $s := (m + d)/(\ell + d)$,

$$ \limsup_{m \to \infty} \frac{f_0^s(\ell, m)}{\binom{m}{\ell}} \leq 1 - \left(1 - \frac{1}{\ell + d}\right)^{\ell}. $$

Henceforth, we will denote the quantity appearing on the left hand side of the above inequality by $f_d^{\ell}(\ell)$; this should not be confused with $f_d(\ell)$, which was defined earlier. With this notation, Proposition 1.8 shows that for any $k \geq 3$ and $1 \leq d \leq k - 1$,

$$ f_d(k) \leq f_d^{d}(k-d). \quad (3) $$
1.3 Small deviation inequalities for sums of independent random variables

Using an elegant LP duality argument (which we will essentially replicate in the next section), the authors in [1] showed that Conjecture 1.9 follows from a conjectured probabilistic inequality due to Samuels [14]. Since Samuels proved his conjecture in some special cases, a corresponding resolution of Conjecture 1.9 in a few cases was obtained in [1]. Here, based on the approach from [1], we will show that Conjecture 1.9 is, in fact, equivalent to special cases of a conjectured probabilistic inequality (of a similar nature to Samuels’ conjecture) due to Feige [4]. Given this, our main result will follow by using known bounds on Feige’s conjecture, as we now discuss.

In [4], Feige asked the following question:

**Question 1.10.** Suppose that $X_1, \ldots, X_n$ are independent, non-negative random variables, each of which has mean 1. How large can $\Pr[X_1 + \cdots + X_n \geq n + 1]$ be?

Note that Markov’s inequality gives the upper bound $n/(n + 1) = 1 - 1/(n + 1)$ which is essentially useless. Indeed, Feige conjectured that a much better bound should hold.

**Conjecture 1.11** (Feige, [4]). Let $X_1, \ldots, X_n$ be independent, non-negative random variables, each of which has mean 1. Then, for all $d \geq 1$,

$$\Pr[X_1 + \cdots + X_n \geq n + d] \leq 1 - \left(\frac{1}{n + d}\right)^n .$$

Observe that the conjectured bound is attained when $X_1, \ldots, X_n$ are i.i.d. random variables which take on the value $n + d$ with probability $1/(n + d)$, and the value 0 otherwise. In the same paper [4], Feige also proved that Conjecture 1.11 holds with $1 - (1 - 1/(n + d))^n$ replaced by $12/13$, and this was later improved to $7/8$ by He, Zhang, and Zhang [8]. To the best of our knowledge, the current best bound in this direction is 43/50 due to Garnett [5]. We remark that [4, 8] appeared much before [1].

For finding perfect matchings in hypergraphs, the only case in Feige’s inequality which is of interest to us is when $X_1, \ldots, X_n$ are i.i.d. random variables. Accordingly, for $\ell \in \mathbb{N}$ and $d > 0$, we define the following quantity:

$$\Theta^d(\ell) := \sup \Pr[X_1 + \cdots + X_\ell \geq \ell + d],$$

where the supremum is taken over all collections of non-negative i.i.d. random variables $X_1, \ldots, X_\ell$ with mean 1. In particular, it follows from Garnett’s bound that for all $\ell \in \mathbb{N}$ and $d \geq 1$,

$$\Theta^d(\ell) \leq \frac{43}{50} .$$ (4)

**Remark 1.12.** For later use, we note that $\lim \inf_{\ell \downarrow 0} \Theta^{d - \epsilon}(\ell) = \Theta^d(\ell)$. Indeed, the direction $\geq$ is immediate, whereas for the direction $\leq$, we use the following observation: for any $\delta > 0$, and any non-negative mean 1 random variable $X$, let $Y_\delta$ denote the random variable which takes on the value 0 with probability $\delta$, and is distributed as $(1 - \delta)^{-1} \cdot X$ otherwise. Then, $Y_\delta$ is a non-negative mean 1 random variable, and letting $Y_1, \ldots, Y_\ell$ (resp. $X_1, \ldots, X_\ell$) denote i.i.d. copies of $Y_\delta$ (resp. $X$),

$$\Pr[Y_1 + \cdots + Y_\ell \geq \ell + d] \geq (1 - \delta)^\ell \Pr[X_1 + \cdots + X_\ell \geq (\ell + d)(1 - \delta)] \geq (1 - \delta)^\ell \lim \inf_{\epsilon \downarrow 0} \Theta^{d - \epsilon}(\ell),$$

so that the desired inequality follows by taking $\delta \downarrow 0$.

In the next section, we will prove the following.

**Theorem 1.13.** For any integers $\ell, d \geq 1$ we have

$$f^d(\ell) = \Theta^d(\ell).$$

It is clear that this proposition implies Theorem 1.5. Indeed, for integers $k \geq 3$ and $1 \leq d \leq k/2$, we have

$$m_d(k) = f_d(k) = f^d(k - d) = \Theta^d(k - d) \leq \frac{43}{50} ,$$
where the first equality is Eq. (2), the second inequality is Eq. (3), the third equality is Theorem 1.13, and the last inequality is Eq. (4).

On the other hand, we can use Theorem 1.13 along with (the proof of) Theorem 1.4 to obtain improved bounds for the i.i.d. version of Feige’s inequality for sufficiently large deviations. It is shown in [7] (see the last computation in the proof of Theorem 1.5 there) that for integers \(n, k\) with \(n > 3k \geq 6\),

\[
f_0^{n/k}(k-d, n-d) \leq (g(k, d) + o_n(1)) \left( \frac{n-d}{k-d} \right)
\]

(where \(g(k, d)\) is the function defined in Eq. (1)), from which it immediately follows that

\[
f^d(\ell) \leq g(\ell + d, d).
\]

Hence, we obtain:

**Corollary 1.14.** Let \(X_1, \ldots, X_\ell\) be i.i.d. non-negative random variables each of which has mean 1. Then, for all integers \(d \geq 1\),

\[
Pr[X_1 + \cdots + X_\ell \geq \ell + d] \leq g(\ell + d, d).
\]

Note that this bound does indeed improve on Markov’s inequality/Feige’s inequality/Hoeffding’s inequality for \(d\) which is sufficiently large compared to \(\ell\). For instance, one may directly check (as is done in Corollary 1.4 in [7]) that for \(0.73\ell \leq d < \ell\), \(g(\ell + d, d) < 1/2\), whereas the above mentioned inequalities give a bound which is worse than 1/2.

### 2 Proof of Theorem 1.13

We begin by showing that \(f^d(\ell) \leq \Theta^d(\ell)\). The proof is exactly the same as the proof of Theorem 2.1 in [1], with the only change being replacing the application of Samuels’ conjecture in the last step with Feige’s conjecture. Before proceeding with the details we need to introduce some notation. For an \(\ell\)-graph \(H\), we will denote the size of the largest fractional matching by \(\nu^\ast(H)\). Then, by LP-duality, we have

\[
\nu^\ast(H) = \tau^\ast(H),
\]

where \(\tau^\ast(H)\) denotes the size of the smallest fractional vertex cover in \(H\). Recall that a fractional vertex cover of \(H = (V, E)\) is a function \(t : V \to [0, 1]\) such that for every \(e \in E\), we have \(\sum_{v \in e} t(v) \geq 1\). The weight of a fractional vertex cover is defined to be \(\sum_{v \in V} t(v)\).

Let \(H\) be an \(\ell\)-graph on a vertex set \(V\) of size \(m\), and suppose that \(\nu^\ast(H) = xm\), where

\[
0 < x \leq \frac{1 + \frac{d}{\ell + d}}{\ell + d}.
\]

Then, since \(\tau^\ast(H) = xm\) there exists a weight function \(t : V \to [0, 1]\) such that \(\sum_{v \in V} t(v) \leq xm\) and for every edge \(e \in E(H)\) we have \(\sum_{v \in e} t(v) \geq 1\). Let \(v_1, \ldots, v_\ell\) be a sequence of vertices of \(H\), chosen independently and uniformly from \(V\). For each \(i \in [\ell]\), we define a random variable \(X_i \equiv t(v_i)\). Hence, \(X_1, \ldots, X_\ell\) are i.i.d. random variables with mean

\[
\mu := \frac{1}{m} \sum_{v \in V} t(v) = x.
\]

We will now use bounds on the deviation of the sum \(X_1 + \cdots + X_\ell\) to bound the number of edges in \(H\).

Since \(\sum_{e \in E} t(v) \geq 1\) for all \(e \in E\), the number \(N\) of \(\ell\)-element subsets of \(V\) with \(\sum_{v \in S} t(v) \geq 1\) is an upper bound on the number of edges of \(H\). Let \(N_1\) denote the number of all \(\ell\)-element sequences of vertices of \(V\) whose sum of weights is at least 1, and let \(N_2\) denote the number of all \(\ell\)-element sequences of distinct vertices of \(V\) whose sum of weights is at least 1. Note that \(N_1 - N_2 \leq \binom{\ell}{2} m^{\ell-1} = O_{\ell}(m^{\ell-1})\) (since \(N_1 - N_2\) is at most the number of \(\ell\)-element sequences in which at least one vertex appears twice) and \(N_2 = \ell!N_1\). Therefore,

\[
Pr \left[ \sum_{i=1}^\ell t(v_i) \geq 1 \right] = \frac{N_1}{m^\ell} = \frac{N_2 + O_{\ell}(m^{\ell-1})}{\binom{m}{\ell} \ell!} = \left(1 + o_m(1)\right) \frac{N_2}{\binom{m}{\ell}}.
\]
Moreover, we also have
\[
\Pr \left[ \sum_{i=1}^{\ell} t(v_i) \geq 1 \right] = \Pr \left[ \sum_{i=1}^{\ell} X_i \geq 1 \right] \\
= \Pr \left[ \sum_{i=1}^{\ell} x^{-1}X_i \geq x^{-1} \right] \\
\leq \Pr \left[ \sum_{i=1}^{\ell} x^{-1}X_i \geq \ell + d - o_m(1) \right] \\
\leq \Theta^{d-o_m(1)}(\ell).
\]

The desired conclusion now follows from Remark 1.12 by taking the liminf of the right hand side as \( m \to \infty \).

We will now prove the reverse inequality \( \Theta^d(\ell) \leq f^d(\ell) \), which follows by ‘reversing’ the above proof. Let \( X_1, \ldots, X_\ell \) be non-negative i.i.d. random variables with mean 1. For the purpose of bounding \( \Theta^d(\ell) \) from above, it clearly suffices to assume that each \( X_i \) is supported in \([0, \ell + d]\). By a standard approximation argument, we may further restrict our attention only to those distributions which are supported on finitely many points, and which take on each value with probability equal to some rational number. Hence, \( X_1, \ldots, X_\ell \sim X \), where \( X \) is supported on \( x_1, \ldots, x_\ell \in [0, \ell + d] \), and \( \Pr[X = x_i] = b_i/m' \), for some \( m', b_i \in \mathbb{N} \) with \( \sum_{i=1}^{\ell} b_i = m' \).

We now construct an \( \ell \)-graph \( H = (V, E) \) as follows. Let \( V = [m] \), where \( m = rm' \) for \( r \in \mathbb{N} \). Let \( t: V \to [0, 1] \) be defined by sending the first \( rb_1 \) elements of \([m]\) to \( x_1/(\ell + d) \), the next \( rb_2 \) elements of \([m]\) to \( x_2/(\ell + d) \) and so on, and define
\[
E := \left\{ S \in \binom{\ell}{\ell} : \sum_{v \in S} t(v) \geq 1 \right\}.
\]

Then, by construction, \( t \) is a fractional vertex cover of \( H \), so that
\[
\nu^*(H) = \tau^*(H) \leq \sum_{v \in V} t(v) = \frac{E[X]}{\ell + d} \cdot m = \frac{1}{\ell + d} m.
\]

As before, let \( v_1, \ldots, v_\ell \) be a sequence of vertices of \( H \), chosen independently and uniformly from \( V \), and for each \( i \in [\ell] \), let \( X_i := t(v_i) \). Then, \( X_1, \ldots, X_\ell \) are i.i.d. random variables, and observe from the definition of \( t \) that \( (\ell + d) \cdot X_i \sim X \). Moreover, by the same argument as above, it follows that
\[
\Pr \left[ \sum_{i=1}^{\ell} (\ell + d) \cdot X_i \geq \ell + d \right] = \Pr \left[ \sum_{i=1}^{\ell} t(v_i) \geq 1 \right] \\
= (1 + o_m(1)) \frac{|E|}{\binom{m}{\ell}} \\
\leq (1 + o_m(1)) \frac{f_0^{(m+d)/(\ell+d)}(\ell, m)}{\binom{m}{\ell}}.
\]

Finally, the desired conclusion follows by noting that the liminf of the right hand side (as \( r \to \infty \)) is at most \( f^d(\ell) \).

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