THE GRAHAM CONJECTURE IMPLIES THE ERDŐS-TURÁN CONJECTURE

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Abstract. Erdős and Turán once conjectured that any set \( A \subset \mathbb{N} \) with \( \sum_{a \in A} 1/a = \infty \) should contain infinitely many progressions of arbitrary length \( k \geq 3 \). For the two-dimensional case Graham conjectured that if \( B \subset \mathbb{N} \times \mathbb{N} \) satisfies
\[
\sum_{(x,y) \in B} \frac{1}{x^2 + y^2} = \infty,
\]
then for any \( s \geq 2 \), \( B \) contains an \( s \times s \) axes-parallel grid. In this paper it is shown that if the Graham conjecture is true for some \( s \geq 2 \), then the Erdős-Turán conjecture is true for \( k = 2s - 1 \).

1. Introduction

One famous conjecture of Erdős and Turán [2] asserts that any set \( A \subset \mathbb{N} \) with \( \sum_{a \in A} 1/a = \infty \) should contain infinitely many progressions of arbitrary length \( k \geq 3 \). There are two important progresses towards this direction due to Szemerédi [7] and Green and Tao [5] respectively, which assert that if \( A \) has positive upper density or \( A \) is the set of the prime numbers, then \( A \) contains infinitely many progressions of arbitrary length.

If one considers the similar question in the two-dimensional plane, Graham [4] conjectured that if \( B \subset \mathbb{N} \times \mathbb{N} \) satisfies
\[
\sum_{(x,y) \in B} \frac{1}{x^2 + y^2} = \infty,
\]
then \( B \) contains the four vertices of an axes-parallel square. More generally, for any \( s \geq 2 \) it should be true that \( B \) contains an \( s \times s \) axes-parallel grid. Furstenberg and Katznelson [3] proved the two-dimensional Szemerédi theorem, that is, any set \( B \subset \mathbb{N} \times \mathbb{N} \) with positive upper density contains an \( s \times s \) axes-parallel grid. In another words, such a set \( B \) contains any finite pattern.

The purpose of this paper is to show that if the Graham conjecture is true, then the Erdős-Turán conjecture is also true.

2. The Graham conjecture implies the Erdős-Turán conjecture

Suppose that the Erdős-Turán conjecture is false for \( k = 3 \). Then there exists a set
\[
A = \{a_1 < a_2 < a_3 < \cdots \} \subset \mathbb{N}
\]
with \( \sum_{n \in \mathbb{N}} 1/a_n = \infty \) such that \( A \) contains no arithmetic progression of length 3. Define a set \( B \subset \mathbb{N} \times \mathbb{N} \) by

\[
B = \left\{ (a_n + m, m) : n \in \mathbb{N}, m \in \mathbb{N} \right\}.
\]

Then

\[
\sum_{(x,y) \in B} \frac{1}{x^2 + y^2} = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \frac{1}{(a_n + m)^2 + m^2}
\geq \sum_{n \in \mathbb{N}} \sum_{m=1}^{a_n} \frac{1}{(a_n + m)^2 + m^2}
\geq \sum_{n \in \mathbb{N}} \frac{a_n}{(a_n + a_n)^2 + a_n^2}
= \sum_{n \in \mathbb{N}} \frac{1}{5a_n}
= \infty.
\]

In the sequel we indicate that \( B \) contains no square and argue it by contradiction. This would mean that the Graham conjecture is false for \( s = 2 \). Suppose that for some \( n, m, l \in \mathbb{N} \), \( B \) contains a square of the following form:

\[
(a_n + m, m + l), \quad (a_n + m + l, m + l),
\]
\[
(a_n + m, m), \quad (a_n + m + l, m).
\]

It follows easily from the construction of \( B \) that \( a_n - l, a_n, a_n + l \in A \), which yields a contradiction since \( A \) contains no arithmetic progression of length 3 according to the initial assumption.

Similarly, if the Graham conjecture is true for some \( s \geq 2 \), then the Erdős-Turán conjecture is true for \( k = 2s - 1 \). The interested reader can easily provide a proof.

3. Concluding Remarks

Let \( r(k, N) \) be the maximal cardinality of a subset \( A \) of \( \{1, 2, \ldots, N\} \) which is free of \( k \)-term arithmetic progressions. Behrend [1] and Rankin [6] had shown that

\[
r(k, N) \geq N \cdot \exp(-c(\log N)^{1/(k-1)}).
\]

Similarly, let \( \bar{r}(s, N) \) be the maximal cardinality of a subset \( B \) of \( \{1, 2, \ldots, N\}^2 \) which is free of \( s \times s \) axes-parallel grids. For any set \( A \subset \{1, 2, \ldots, N\} \), define

\[
\Theta(A) = \{(a + m, m) : a \in A, m = 1, 2, \ldots, N\} \subset \{1, 2, \ldots, 2N\}^2.
\]

Following the discussion in Section 2, one can easily deduce that if \( A \) is free of \( 2s-1 \) term of arithmetic progression, then \( \Theta(A) \) is free of \( s \times s \) axes-parallel grid. Hence

\[
\bar{r}(s, 2N) \geq r(2s - 1, N) \cdot N
\geq N^2 \exp(-c(\log N)^{1/(2s-2)}).
\]

We end this paper with a question. Does the Erdős-Turán conjecture imply the Graham conjecture?
References

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