Algebro-geometric solutions for the two-component Hunter-Saxton hierarchy

Yu Hou, Engui Fan

To cite this article: Yu Hou, Engui Fan (2014) Algebro-geometric solutions for the two-component Hunter-Saxton hierarchy, Journal of Nonlinear Mathematical Physics 21:4, 473–508, DOI: https://doi.org/10.1080/14029251.2014.975520

To link to this article: https://doi.org/10.1080/14029251.2014.975520

Published online: 04 January 2021
Algebro-geometric solutions for the two-component Hunter-Saxton hierarchy

Yu Hou
School of Mathematical Sciences, Institute of Mathematics and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, P.R. China

Engui Fan ∗
School of Mathematical Sciences, Institute of Mathematics and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, P.R. China

Received 5 April 2014
Accepted 29 May 2014

This paper is dedicated to provide theta function representations of algebro-geometric solutions and related crucial quantities for the two-component Hunter-Saxton (HS2) hierarchy through an algebro-geometric initial value problem. Our main tools include the polynomial recursive formalism, the hyperelliptic curve with finite number of genus, the Baker-Akhiezer functions, the meromorphic function, the Dubrovin-type equations for auxiliary divisors, and the associated trace formulas. With the help of these tools, the explicit representations of the algebro-geometric solutions are obtained for the entire HS2 hierarchy.

Keywords: Hyperelliptic curve; Baker-Akhiezer function; Algebro-geometric solutions; Trace formulas.

2000 Mathematics Subject Classification: 35Q53, 58F07, 53C35

1. Introduction

In this paper, we consider the following two-component Hunter-Saxton (HS2) system

\[
\begin{align*}
    m_t + 2u_t m + um_x + \sigma \rho \rho_x &= 0, \\
    \rho_t + (u \rho)_x &= 0,
\end{align*}
\]

(1.1)

where \( m = -u_{xx}, \) \( \sigma = \pm 1. \) The system (1.1) was recently introduced by Constantin and Ivanov in [9]. The variable \( u(x,t) \) describes the horizontal velocity of the fluid and the variable \( \rho(x,t) \) is in connection with the horizontal deviation of the surface from equilibrium, all measured in dimensionless units [9]. The HS2 system (1.1) is the short-wave (or high-frequency) limit of the two-component Camassa-Holm shallow water system [9, 17, 39, 43]. This system, reading as (1.1) with \( m \) replaced by \( (1 - \partial_{xx}^2)u \), has recently been the object of intensive study (see [8, 9, 25, 26, 45]). The HS2 system (1.1) is integrable, since it has a Lax pair [9] and a bi-Hamiltonian structure [9, 39]. It is also a particular case of the Gurevich-Zybin system describing the dynamics in a model of non-dissipative dark matter [24, 40, 44]. The mathematical properties of the system (1.1) have been studied further in a series of works (see [9, 23, 24, 36, 42–44]).

∗Corresponding author: faneg@fudan.edu.cn

Co-published by Atlantis Press and Taylor & Francis
Copyright: the authors
473
For $\rho = 0$, the system (1.1) becomes the Hunter-Saxton (HS) equation [30], which models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field. Here, $u(x,t)$ describes the director field of a nematic liquid crystal, $x$ is a space variable in a reference frame moving with the linearized wave velocity, and $t$ is a slow time variable. The field of unit vectors $(\cos u(x,t), \sin u(x,t))$ describes the orientation of the molecules [23, 30, 44].

The HS equation also arises in a different physical context as the high-frequency limit [13, 31] of the Camassa-Holm (CH) equation—a model equation for shallow water waves [7, 11, 32] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [10] with a bi-Hamiltonian structure [18], which is completely integrable [12]. The HS equation is also a completely integrable system with a bi-Hamiltonian structure, and hence, it possesses a Lax pair, an infinite family of commuting Hamiltonian flows, as well as an associated sequence of conservation laws [5, 30, 31, 39, 41]. It also describes the geodesic flow on the homogeneous space related to the Virasoro group [33–35]. Recently, algebro-geometric solutions for the HS hierarchy were investigated in [28]. Moreover, peakon solutions, global weak solutions and the Cauchy problem of the system (1.1) were discussed in [9, 23, 36, 44]. However, within the knowledge of the authors, the algebro-geometric solutions of the entire HS hierarchy are not studied yet.

This paper concerns algebro-geometric quasi-periodic solutions of the whole HS hierarchy associated with (1.1). The algebro-geometric solution, as an important feature of integrable system, is a kind of explicit solution closely related to the inverse spectral theory [1, 6, 14–16]. In a degenerated case of the algebro-geometric solution, the multi-soliton solution and periodic solution in elliptic function type may be obtained [6, 37, 38]. A systematic approach, proposed by Gesztesy and Holden to construct algebro-geometric solutions for integrable equations, has been extended to the whole (1+1) dimensional integrable hierarchy, such as the AKNS hierarchy, the CH hierarchy, etc. [19–22]. Recently, we investigated algebro-geometric solutions for the modified CH hierarchy and the Degasperis-Procesi hierarchy [27, 29].

The outline of the present paper is as follows.

In section 2, based on the polynomial recursion formalism, we derive the HS hierarchy associated with the $2 \times 2$ spectral problem. A hyperelliptic curve $\mathcal{K}_n$ of arithmetic genus $n$ is introduced with the help of the characteristic polynomial of Lax matrix $V_n$ for the stationary HS hierarchy.

In Section 3, we decompose the stationary HS equations into a system of Dubrovin-type equations. Moreover, we obtain the stationary trace formulas for the HS hierarchy.

In Section 4, we present the first set of our results, the explicit theta function representations of the potentials $u, \rho$ for the entire stationary HS hierarchy. Furthermore, we study the initial value problem on an algebro-geometric curve for the stationary HS hierarchy.

In Sections 5 and 6, we extend the analyses of Sections 3 and 4, respectively, to the time-dependent case. Each equation in the HS hierarchy is permitted to evolve in terms of an independent time parameter $t_r$. As initial data, we use a stationary solution of the $n$th equation and then construct a time-dependent solution of the $r$th equation of the HS hierarchy. The Baker-Akhiezer function, the analogs of the Dubrovin-type equations, the trace formulas, and the theta function representations in Section 4 are all extended to the time-dependent case.

Finally, we remark that although our focus in this paper is on the case $\sigma = 1$ in (1.1), all of the arguments presented here can be adapted to study the corresponding case $\sigma = -1$. 
2. The HS2 hierarchy

In this section, we provide the construction of HS2 hierarchy and derive the corresponding sequence of zero-curvature pairs using a polynomial recursion formalism. Moreover, we introduce the underlying hyperelliptic curve in connection with the stationary HS2 hierarchy.

Throughout this section, we make the following hypothesis.

Hypothesis 2.1. In the stationary case, we assume that

\[ u, \rho \in C^\infty(\mathbb{R}), \partial_x^k u, \partial_x^k \rho \in L^\infty(\mathbb{R}), \ k \in \mathbb{N}_0. \]  

(2.1)

In the time-dependent case, we suppose

\[ u(\cdot, t), \rho(\cdot, t) \in C^\infty(\mathbb{R}), \partial_x^k u(\cdot, t), \partial_x^k \rho(\cdot, t) \in L^\infty(\mathbb{R}), \ k \in \mathbb{N}_0, \ t \in \mathbb{R}, \]

\[ u(x, \cdot), u_{xx}(x, \cdot), \rho(x, \cdot), \rho_x(x, \cdot) \in C^1(\mathbb{R}), \ x \in \mathbb{R}. \]  

(2.2)

We first introduce the basic polynomial recursion formalism. Define \( \{f_l\}_{l \in \mathbb{N}_0}, \{g_l\}_{l \in \mathbb{N}_0}, \{h_l\}_{l \in \mathbb{N}_0} \) recursively by

\[
\begin{align*}
    f_0 &= \frac{1}{2}, \\
    f_l &= -2G(2\rho^2 f_{l-2, x} + 2uu_{x} f_{l-1, x} + 2\rho \rho_x f_{l-1, x} + uu_{xxx} f_{l-1, x} - \rho^2 f_{l, x} - u_{xx} f_{l-1, x}), \quad l \in \mathbb{N}, \\
    g_l &= \frac{1}{2} f_{l+1, x}, \quad l \in \mathbb{N}_0, \\
    h_l &= -g_{l+1, x} - \rho^2 f_l - u_{xx} f_{l+1}, \quad l \in \mathbb{N}_0,
\end{align*}
\]

(2.3)

where \( G \) is given by

\[
G : C^\infty_0(\mathbb{R}) \to C^\infty_0(\mathbb{R}),
\]

\[
(Gv)(x) = \int_{-\infty}^{x} \int_{-\infty}^{x} v(y) \, dy \, dx_1, \quad x \in \mathbb{R}, \ v \in C^\infty_0(\mathbb{R}).
\]

(2.4)

One observes that \( G \) satisfies

\[
\frac{d^2}{dx^2} \circ G = v.
\]

(2.5)

Explicitly, one computes

\[
\begin{align*}
    f_0 &= \frac{1}{2}, \\
    f_1 &= -u + c_1, \\
    f_2 &= G(u_x^2 - \rho^2 + 2uu_{xx}) + 2u(-u) + c_2, \\
    g_0 &= -\frac{1}{2} u_x, \\
    g_1 &= \frac{1}{2} G(4uu_{xx} + 2uu_{xxx} - 2\rho \rho_x) + 2c_1(-u_x), \\
    h_0 &= -\frac{1}{2} f_{2, xx} - \rho^2 f_0 - u_{xx} f_1, \ etc.,
\end{align*}
\]

(2.6)

where \( \{c_l\}_{l \in \mathbb{N}} \subset \mathbb{C} \) are integration constants.
Next, it is convenient to introduce the corresponding homogeneous coefficients $\hat{f}_l, \hat{g}_l,$ and $\hat{h}_l,$ defined by the vanishing of the integration constants $c_k, k = 1, \ldots, l,$

$$\begin{align*}
\hat{f}_0 &= f_0 = \frac{1}{2}, & \hat{f}_l &= f_l|_{c_k=0}, k=1,\ldots,l, & l \in \mathbb{N}, \\
\hat{g}_0 &= g_0 = -\frac{1}{2}u_x, & \hat{g}_l &= g_l|_{c_k=0}, k=1,\ldots,l, & l \in \mathbb{N}, \\
\hat{h}_0 &= h_0, & \hat{h}_l &= h_l|_{c_k=0}, k=1,\ldots,l, & l \in \mathbb{N}.
\end{align*}$$

(2.7)

Hence,

$$\begin{align*}
f_l &= \sum_{k=0}^{l} 2c_{l-k}\hat{f}_k, \quad g_l &= \sum_{k=0}^{l} 2c_{l-k}\hat{g}_k, \quad h_l &= \sum_{k=0}^{l} 2c_{l-k}\hat{h}_k, \quad l \in \mathbb{N}_0,
\end{align*}$$

(2.8)

defining

$$c_0 = \frac{1}{2}.$$  

(2.9)

Now, given Hypothesis 2.1, one introduces the following $2 \times 2$ matrix $U$ by

$$\psi_x = U(z,x)\psi = \begin{pmatrix} 0 & 1 \\ -z^{-2}\rho^2 - z^{-1}u_{xx} & 0 \end{pmatrix} \psi,$$

(2.10)

and for each $n \in \mathbb{N}_0$, the following $2 \times 2$ matrix $V_n$ by

$$\psi_{x^n} = V_n(z)\psi,$$

(2.11)

with

$$V_n(z) = \begin{pmatrix} -G_n(z) & F_n(z) \\ z^{-2}H_n(z) & G_n(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N}_0,$$

(2.12)

assuming $F_n, G_n,$ and $H_n$ to be polynomials with respect to $z$ and $C^\infty$ in $x$. The compatibility condition of linear system (2.10) and (2.11) yields the stationary zero-curvature equation

$$-V_{n,x} + [U, V_n] = 0,$$

(2.13)

which is equivalent to

$$\begin{align*}
F_{n,x} &= 2G_n, \\
H_{n,x} &= 2(\rho^2 + zu_{xx})G_n, \\
z^2G_{n,x} &= -H_n - (\rho^2 + zu_{xx})F_n.
\end{align*}$$

(2.14, 2.15, 2.16)

$^a$F_n, G_n, H_n are polynomials of degree $n+1, n, n+1$, respectively.
For simplicity, we denote it by $a$ which indicates

$$\frac{d}{dx} \det(V_n(z,x)) = -\frac{1}{z} \frac{d}{dx} \left( z^2 G_n(z,x)^2 + F_n(z,x) H_n(z,x) \right) = 0, \quad (2.17)$$

and hence

$$z^2 G_n(z,x)^2 + F_n(z,x) H_n(z,x) = R_{2n+2}(z), \quad (2.18)$$

where the polynomial $R_{2n+2}$ of degree $2n+2$ is $x$-independent. In another way, one can write $R_{2n+2}$ as

$$R_{2n+2}(z) = \left( \frac{1}{4} u_x^2 + \frac{1}{2} h_0 \right) \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0, \ldots, 2n+1} \in \mathbb{C}. \quad (2.19)$$

Here, we emphasize that the coefficient $\left( \frac{1}{4} u_x^2 + \frac{1}{2} h_0 \right)$ is a constant. In fact, equation (2.17) is equivalent to

$$2z^2 G_n G_{n,x} + F_n H_{n,x} + H_n F_{n,x} = 0. \quad (2.20)$$

Then comparing the coefficient of powers $z^{2n+2}$ yields

$$2g_0 g_{0,x} + f_0 h_{0,x} + h_0 f_{0,x} = 0, \quad (2.21)$$

which indicates

$$\frac{1}{2} u_x u_{xx} + \frac{1}{2} h_{0,x} = 0. \quad (2.22)$$

Therefore,

$$\frac{1}{4} u_x^2 + \frac{1}{2} h_0 = \partial_{-1} \left( \frac{1}{2} u_x u_{xx} + \frac{1}{2} h_{0,x} \right) = \text{constant}. \quad (2.23)$$

For simplicity, we denote it by $a^2$, $a \in \mathbb{C}$. Then, $R_{2n+2}(z)$ can be rewritten as

$$R_{2n+2}(z) = a^2 \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0, \ldots, 2n+1} \in \mathbb{C}. \quad (2.24)$$

Next, we compute the characteristic polynomial det$(yI - zV_n)$ of Lax matrix $zV_n$,

$$\det(yI - zV_n) = y^2 - z^2 G_n(z)^2 - F_n(z) H_n(z)$$

$$= y^2 - R_{2n+2}(z) = 0, \quad (2.25)$$

and then introduce the (possibly singular) hyperelliptic curve $\mathcal{C}_n$ of arithmetic genus $n$ defined by

$$\mathcal{C}_n: \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0. \quad (2.26)$$

In the following, we will occasionally impose further constraints on the zeros $E_m$ of $R_{2n+2}$ introduced in (2.24) and assume that

$$E_m \in \mathbb{C}, \quad E_m \neq E_{m'}, \quad \forall m \neq m', \quad m, m' = 0, \ldots, 2n + 1. \quad (2.27)$$

The stationary zero-curvature equation (2.13) implies polynomial recursion relations (2.3).

Introducing the following polynomials $F_n(z), G_n(z), H_n(z)$ with respect to the spectral parameter
null
Algebro-geometric solutions for the HS2 hierarchy

\[ \hat{H}_0(z) = -\rho^2 \hat{f}_n - u_{xx} \hat{f}_{n+1} + \sum_{k=0}^{n-1} \hat{h}_k z^{-k}, \]

\[ \hat{H}_{n+1}(z) = -\rho^2 \hat{f}_{n+1} + z \hat{H}_n(z). \]

In accordance with our notation introduced in (2.7) and (2.35)-(2.39), the corresponding homogeneous stationary HS2 equations are then defined by

\[ s\text{-HS2}_n(u, \rho) = s\text{-HS2}_n(u, \rho)|_{c_l=0, l=1,\ldots,n} = 0, \quad n \in \mathbb{N}_0. \]  

At the end of this section, we turn to the time-dependent HS2 hierarchy. In this case, \( u, \rho \) are considered as functions of both space and time. We introduce a deformation parameter \( t_n \in \mathbb{R} \) in \( u \) and \( \rho \), replacing \( u(x), \rho(x) \) by \( u(x, t_n), \rho(x, t_n) \), for each equation in the hierarchy. In addition, the definitions (2.10), (2.12), and (2.28)-(2.30) of \( U, V_n \) and \( F_n, G_n, \) and \( H_n \), respectively, still apply. The corresponding zero-curvature equation reads

\[ U_n - V_{n,x} + [U, V_n] = 0, \quad n \in \mathbb{N}_0, \]

which results in the following set of equations

\[ F_{n,x} = 2G_n, \quad n \in \mathbb{N}_0, \]

\[ z^2 G_{n,x} = -H_n - (\rho^2 + zu_{xx}) F_n, \]

\[ -2\rho \rho_n - zu_{xxt} - H_{n,x} + 2(\rho^2 + zu_{xx}) G_n = 0. \]

For fixed \( n \in \mathbb{N}_0 \), inserting the polynomial expressions for \( F_n, G_n, \) and \( H_n \) into (2.42)-(2.44), respectively, first yields recursion relations (2.3) for \( f_l |_{l=0,\ldots,n+1}, g_l |_{l=0,\ldots,n}, h_l |_{l=0,\ldots,n-1} \) and

\[ h_n = -\rho^2 f_n - u_{xx} f_{n+1}, \quad h_{n+1} = -\rho^2 f_{n+1}, \]

Moreover, using (2.44), one finds

\[ -u_{xxt} - h_{n,x} + \rho^2 f_n + u_{xx} f_{n+1,x} = 0, \quad n \in \mathbb{N}_0, \]

\[ 2\rho \rho_n + h_{n+1,x} - \rho^2 f_{n+1,x} = 0, \quad n \in \mathbb{N}_0. \]

Hence, using (2.45) and (2.46) permits one to write the time-dependent HS2 hierarchy as

\[ \text{HS2}_n(u, \rho) = \left( -u_{xxt} + 2u_{xx} f_{n+1,x} + u_{xxx} f_{n+1} + 2\rho \rho_n f_n + 2\rho^2 f_{n,x} \right) = 0, \quad n \in \mathbb{N}_0. \]

For convenience, we record the first equation in this hierarchy explicitly,

\[ \text{HS2}_0(u, \rho) = \left( -u_{xxt} - 2u_{xx} u_{xx} - uu_{xxx} + \rho \rho_n + c_1 u_{xxx} \right) = 0. \]

The first equation \( \text{HS2}_0(u, \rho) = 0 \) (with \( c_1 = 0 \)) in the hierarchy represents the HS2 system as discussed in section 1. Similarly, one can introduce the corresponding homogeneous HS2 hierarchy by

\[ \text{HS2}_n(u, \rho) = \text{HS2}_n(u, \rho)|_{c_l=0, l=1,\ldots,n} = 0, \quad n \in \mathbb{N}_0. \]

In fact, since the Lenard recursion formalism is almost universally adopted in the contemporary literature, we thought it might be worthwhile to use the Gesztesy and Holden’s method, the polynomial recursion formalism, to construct the HS2 hierarchy.
3. The stationary HS2 formalism

This section is devoted to a detailed study of the stationary HS2 hierarchy. We first define a fundamental meromorphic function $\phi(P,x)$ on the hyperelliptic curve $\mathcal{K}_n$, using the polynomial recursion formalism described in section 2, and then study the properties of the Baker-Akhiezer function $\psi(P,x,x_0)$, Dubrovin-type equations, and trace formulas.

For major parts of this section, we assume (2.1), (2.3), (2.6), (2.10)-(2.16), (2.26)-(2.30), and (2.33), keeping $n \in \mathbb{N}_0$ fixed.

Recall the hyperelliptic curve $\mathcal{K}_n$

$$\mathcal{K}_n: \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0,$$

$$R_{2n+2}(z) = a^2 \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0}^{2n+1} \subset \mathbb{C},$$

(3.1)

which is compactified by joining two points at infinity $P_{\infty_+}$, with $P_{\infty_+} \neq P_{\infty_-}$. But for notational simplicity, the compactification is also denoted by $\mathcal{K}_n$. Hence, $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of arithmetic genus $n$. Points $P$ on $\mathcal{K}_n \backslash \{P_{\infty_\pm}\}$ are denoted by $P = (z,y(P))$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z,y) = 0$.

The complex structure on $\mathcal{K}_n$ is defined in the usual way by introducing local coordinates

$$\zeta_{Q_0}: P \rightarrow (z - z_0)$$

near points $Q_0 = (z_0,y(Q_0)) \in \mathcal{K}_n$, which are neither branch nor singular points of $\mathcal{K}_n$; near the branch and singular points $Q_1 = (z_1,y(Q_1)) \in \mathcal{K}_n$, the local coordinates are

$$\zeta_{Q_1}: P \rightarrow (z - z_1)^{1/2};$$

near the points $P_{\infty_\pm} \in \mathcal{K}_n$, the local coordinates are

$$\zeta_{P_{\infty}_\pm}: P \rightarrow z^{-1}.$$

The holomorphic map $\ast$, changing sheets, is defined by

$$\ast: \left\{ \begin{array}{ll}
\mathcal{K}_n \rightarrow \mathcal{K}_n, \\
P = (z,y_j(z)) \rightarrow P^\ast = (z,y_{j+1(\text{mod} 2)}(z)), & j = 0, 1,
\end{array} \right.$$

$$P^{\ast\ast} := (P^\ast)^\ast, \quad \text{etc.}, \quad (3.2)$$

where $y_j(z), j = 0, 1$ denote the two branches of $y(P)$ satisfying $\mathcal{F}_n(z,y) = 0$, namely,

$$\quad (y - y_0(z))(y - y_1(z)) = y^2 - R_{2n+2}(z) = 0. \quad (3.3)$$

Taking into account (3.3), one easily finds

$$y_0 + y_1 = 0,$$

$$y_0 y_1 = -R_{2n+2}(z), \quad (3.4)$$

$$y_0^2 + y_1^2 = 2R_{2n+2}(z).$$

Co-published by Atlantis Press and Taylor & Francis

Copyright: the authors
Moreover, positive divisors on $\mathcal{K}_n$ of degree $n$ are denoted by

$$\mathcal{D}_{P_1,\ldots,P_n} : \mathcal{K}_n \to \mathbb{N}_0,$$

$$P \mapsto \mathcal{D}_{P_1,\ldots,P_n} = \begin{cases} k & \text{if } P \text{ occurs } k \text{ times in } \{P_1,\ldots,P_n\}, \\ 0 & \text{if } P \notin \{P_1,\ldots,P_n\}. \end{cases} \quad (3.5)$$

Next, we define the stationary Baker-Akhiezer function $\psi(P,x_0)$ on $\mathcal{K}_n \setminus \{P_{\omega_+},P_{\omega_-},P_0 = (0,ipf_{n+1})\}$ by

$$\psi_1(P,x_0) = \begin{pmatrix} \psi_1(P,x_0) \\ \psi_2(P,x_0) \end{pmatrix},$$

$$\psi_2(P,x_0) = U(u(x),\rho(x),z(P))\psi(P,x_0),$$

$$zV_n(u(x),\rho(x),z(P))\psi(P,x_0) = y(P)\psi(P,x_0),$$

$$\psi_1(P,x_0,x_0) = 1;$$

$$P = (z,y) \in \mathcal{K}_n \setminus \{P_{\omega_+},P_{\omega_-},P_0 = (0,ipf_{n+1})\}, (x_0) \in \mathbb{R}^2.$$ (3.6)

Closely related to $\psi(P,x_0)$ is the following meromorphic function $\phi(P,x)$ on $\mathcal{K}_n$ defined by

$$\phi(P,x) = z^{-1} \frac{\psi_1(P,x_0)}{\psi_1(P,x_0)} = P_1 \in \mathcal{K}_n, x \in \mathbb{R}$$ (3.7)

such that

$$\psi_1(P,x_0) = \exp \left( z^{-1} \int_{x_0}^x \phi(P,x') \, dx' \right), \quad P \in \mathcal{K}_n \setminus \{P_{\omega_+},P_{\omega_-},P_0\}. \quad (3.8)$$

Then, based on (3.6) and (3.7), a direct calculation shows that

$$\phi(P,x) = \frac{y + zG_n(z,x)}{F_n(z,x)} = \frac{H_n(z,x)}{y - zG_n(z,x)},$$

and

$$\psi_2(P,x_0,x_0) = \psi_1(P,x_0)\phi(P,x)/z. \quad (3.9)$$

In the following, the roots of polynomials $F_n$ and $H_n$ will play a special role, and hence, we introduce on $\mathbb{C} \times \mathbb{R}$

$$F_n(z,x) = \frac{1}{2} \prod_{j=0}^n (z - \mu_j(x)), \quad H_n(z,x) = h_0 \prod_{l=0}^n (z - \nu_l(x)). \quad (3.10)$$

Moreover, we introduce

$$\hat{\mu}_j(x) = (\mu_j(x),-\mu_j(x)G_n(\mu_j(x),x)) \in \mathcal{K}_n, \quad j = 0,\ldots,n, \quad x \in \mathbb{R}, \quad (3.11)$$

and

$$\hat{\nu}_l(x) = (\nu_l(x),\nu_l(x)G_n(\nu_l(x),x)) \in \mathcal{K}_n, \quad l = 0,\ldots,n, \quad x \in \mathbb{R}.$$

(3.12)

(3.13)
Yu Hou and Engui Fan

Due to assumption (2.1), \( u \) and \( \rho \) are smooth and bounded, and hence, \( F_n(z,x) \) and \( H_n(z,x) \) share the same property. Thus, one concludes

\[
\mu_j, \nu_l \in C(\mathbb{R}), \quad j, l = 0, \ldots, n,
\]

(3.14)

taking multiplicities (and appropriate reordering) of the zeros of \( F_n \) and \( H_n \) into account. From (3.9), the divisor \((\phi(P,x))\) of \( \phi(P,x) \) is given by

\[
(\phi(P,x)) = \mathcal{D}_{\mu_0(x)}(P) - \mathcal{D}_{\nu_0(x)}(P).
\]

(3.15)

Here, we abbreviated

\[
\hat{\mu} = \{\mu_1, \ldots, \mu_n\}, \quad \hat{\nu} = \{\nu_1, \ldots, \nu_n\} \in \text{Sym}^n(\mathcal{H}_n).
\]

(3.16)

Further properties of \( \phi(P,x) \) are summarized as follows.

**Lemma 3.1.** Suppose (2.1), assume the \( n \)th stationary HS2 equation (2.33) holds, and let \( P = (z,y) \in \mathcal{H}_n \setminus \{P_{m_1}, P_{m_2}, \ldots, P_0\} \), \((x,x_0) \in \mathbb{R}^2\). Then \( \phi \) satisfies the Riccati-type equation

\[
\phi_x(P) + z^{-1} \phi(P)^2 = -z^{-1} \rho^2 - u_{\text{ex}},
\]

(3.17)

as well as

\[
\phi(P) \phi(P^*) = -\frac{H_n(z)}{F_n(z)},
\]

(3.18)

\[
\phi(P) + \phi(P^*) = \frac{2zG_n(z)}{F_n(z)},
\]

(3.19)

\[
\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}.
\]

(3.20)

**Proof.** Equation (3.17) follows using the definition (3.9) of \( \phi \) as well as relations (2.14)-(2.16). Relations (3.18)-(3.20) are clear from (2.18), (3.4), and (3.9). □

The properties of \( \psi(P,x,x_0) \) are summarized in the following lemma.

**Lemma 3.2.** Suppose (2.1), assume the \( n \)th stationary HS2 equation (2.33) holds, and let \( P = (z,y) \in \mathcal{H}_n \setminus \{P_{m_1}, P_{m_2}, \ldots, P_0\} \), \((x,x_0) \in \mathbb{R}^2\). Then \( \psi_1(P,x,x_0), \psi_2(P,x,x_0) \) satisfy

\[
\psi_1(P,x,x_0) = \left( \frac{F_n(z,x)}{F_n(z,x_0)} \right)^{1/2} \exp \left( \frac{y}{z} \int_{x_0}^{x} \frac{F_n(z,x')}{F_n(z,x_0)}^{-1} dx' \right),
\]

(3.21)

\[
\psi_1(P,x,x_0) \psi_1(P^*,x,x_0) = \frac{F_n(z,x)}{F_n(z,x_0)},
\]

(3.22)

\[
\psi_2(P,x,x_0) \psi_2(P^*,x,x_0) = -\frac{H_n(z,x)}{z^2F_n(z,x_0)},
\]

(3.23)

\[
\psi_1(P,x,x_0) \psi_2(P^*,x,x_0) + \psi_1(P^*,x,x_0) \psi_2(P,x,x_0) = \frac{2G_n(z,x)}{F_n(z,x_0)},
\]

(3.24)

\[
\psi_1(P,x,x_0) \psi_2(P^*,x,x_0) - \psi_1(P^*,x,x_0) \psi_2(P,x,x_0) = \frac{-2y}{z^2F_n(z,x_0)}.
\]

(3.25)

**Proof.** Equation (3.21) is a consequence of (2.14), (3.8), and (3.9). Equation (3.22) is clear from (3.21) and (3.23) is a consequence of (3.10), (3.18), and (3.22). Equation (3.24) follows using (3.10), (3.19), and (3.22). Finally, (3.25) follows from (3.10), (3.20), and (3.22). □
In Lemma 3.2, we denote by
\[ \psi_l(P) = \psi_{1,+}, \quad \psi_l(P^*) = \psi_{1,-}, \quad \psi_2(P) = \psi_{2,+}, \quad \psi_2(P^*) = \psi_{2,-}, \]
and then (3.22)-(3.25) imply
\[ (\psi_{1,+} \psi_{2,-} - \psi_{1,-} \psi_{2,+})^2 = (\psi_{1,+} \psi_{1,-} + \psi_{2,-} \psi_{2,+})^2 - 4 \psi_{1,+} \psi_{2,-} \psi_{1,-} \psi_{2,+}, \] (3.26)
which is equivalent to the basic identity (2.18), \( \rho^2 G_n^2 + F_n H_n = R_{2n+2} \). This fact reveals the relations between our approach and the algebro-geometric solutions of the HS2 hierarchy.

**Remark 3.3.** The Baker-Akhiezer function \( \psi \) of the stationary HS2 hierarchy is formally analogous to that defined in the context of KdV or AKNS hierarchies. However, its actual properties in a neighborhood of its essential singularity will feature characteristic differences to standard Baker-Akhiezer functions (cf. Remark 4.2).

Next, we derive Dubrovin-type equations, that is, first-order coupled systems of differential equations that govern the dynamics of \( \mu_j(x) \) and \( \nu_l(x) \) with respect to variations of \( x \).

**Lemma 3.4.** Assume (2.1) and the \( n \)th stationary HS2 equation (2.33) holds subject to the constraint (2.27).

(i) Suppose that the zeros \( \{ \mu_j(x) \}_{j=0,\ldots,n} \) of \( F_n(z,x) \) remain distinct for \( x \in \Omega_\mu \), where \( \Omega_\mu \subseteq \mathbb{R} \) is an open interval, then \( \{ \mu_j(x) \}_{j=0,\ldots,n} \) satisfy the system of differential equations,
\[ \mu_{j,x} = 4 \frac{\gamma(\hat{\mu}_j)}{\mu_j} \prod_{k=0}^{n} (\mu_j(x) - \mu_k(x))^{-1}, \quad j = 0, \ldots, n, \] (3.27)
with initial conditions
\[ \{ \hat{\mu}_j(x_0) \}_{j=0,\ldots,n} \in \mathcal{H}_n, \] (3.28)
for some fixed \( x_0 \in \Omega_\mu \). The initial value problem (3.27), (3.28) has a unique solution satisfying
\[ \hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{H}_n), \quad j = 0, \ldots, n. \] (3.29)

(ii) Suppose that the zeros \( \{ \nu_l(x) \}_{l=0,\ldots,n} \) of \( H_n(z,x) \) remain distinct for \( x \in \Omega_\nu \), where \( \Omega_\nu \subseteq \mathbb{R} \) is an open interval, then \( \{ \nu_l(x) \}_{l=0,\ldots,n} \) satisfy the system of differential equations,
\[ \nu_{l,x} = -2 \frac{(\rho^2 + u_\nu \nu_l)}{\nu_l} \gamma(\hat{\nu}_l) \prod_{k=0}^{n} (\nu_l(x) - \nu_k(x))^{-1}, \quad l = 0, \ldots, n, \] (3.30)
with initial conditions
\[ \{ \hat{\nu}_l(x_0) \}_{l=0,\ldots,n} \in \mathcal{H}_n, \] (3.31)
for some fixed \( x_0 \in \Omega_\nu \). The initial value problem (3.30), (3.31) has a unique solution satisfying
\[ \hat{\nu}_l \in C^\infty(\Omega_\nu, \mathcal{H}_n), \quad l = 0, \ldots, n. \] (3.32)
Proof. It suffices to prove (3.27) and (3.29) since the proof of (3.30) and (3.32) follow in an identical manner. Differentiating (3.11) with respect to \( x \) then yields

\[
F_{n,x}(\mu_j) = -\frac{1}{2} \mu_j, x \prod_{k=0, k \neq j}^{n} (\mu_j(x) - \mu_k(x)).
\] (3.33)

On the other hand, taking into account equation (2.14), one finds

\[
F_{n,x}(\mu_j) = 2G_n(\mu_j) = 2y(\hat{\mu}_j) - \mu_j.
\] (3.34)

Then combining equation (3.33) with (3.34) leads to (3.27). The proof of smoothness assertion (3.29) is analogous to the KdV case in [21].

Next, we turn to the trace formulas of the HS2 invariants, that is, expressions of \( f_l \) and \( h_l \) in terms of symmetric functions of the zeros \( \mu_j \) and \( \nu_l \) of \( F_n \) and \( H_n \), respectively. For simplicity, we just record the simplest case.

Lemma 3.5. Suppose (2.1), assume the nth stationary HS2 equation (2.33) holds, and let \( x \in \mathbb{R} \). Then

\[
u_j(x) = \frac{1}{2} \sum_{j=0}^{n} \mu_j(x) - \frac{1}{2} \sum_{m=0}^{2n+1} E_m.
\] (3.35)

Proof. Equation (3.35) follows by considering the coefficient of \( z^n \) in \( F_n \) in (2.28) and (3.11), which yields

\[-u + c_1 = -\frac{1}{2} \sum_{j=0}^{n} \mu_j.
\] (3.36)

The constant \( c_1 \) can be determined by a long straightforward calculation considering the coefficient of \( z^{2n+1} \) in (2.18), which results in

\[c_1 = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m.
\] (3.37)

4. Stationary algebro-geometric solutions of HS2 hierarchy

In this section, we obtain explicit Riemann theta function representations for the meromorphic function \( \phi \), and especially, for the solutions \( u, \rho \) of the stationary HS2 hierarchy.

We begin with the asymptotic properties of \( \phi \) and \( \psi_j, j = 1, 2 \).

Lemma 4.1. Suppose (2.1), assume the nth stationary HS2 equation (2.33) holds, and let \( P = (\zeta, y) \in \mathbb{R}^2 \setminus \{P_m, P_{-m}, P_0\}, (x, x_0) \in \mathbb{R}^2 \). Then

\[
\phi(P) = -u(x) + O(\zeta), \quad P \to P_m, \quad \zeta = z^{-1},
\] (4.1)

\[
\phi(P) = \frac{iu(x) + iu_{xx}(x) - \rho(x)}{2\rho(x)} \zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z,
\] (4.2)
and

\[
\psi_1(P, x, x_0) = \exp\left((u(x_0) - u(x))\zeta + O(\zeta^2)\right), \quad P \to P_{m_+}, \quad \zeta = z^{-1},
\]

\[
\psi_2(P, x, x_0) = O(\zeta) \exp\left((u(x_0) - u(x))\zeta + O(\zeta^2)\right), \quad P \to P_{m_+}, \quad \zeta = z^{-1},
\]

\[
\psi_1(P, x, x_0) = \exp\left(\frac{i}{\zeta} \int_{x_0}^{x} dx' \varphi(x') + O(1)\right), \quad P \to P_0, \quad \zeta = z,
\]

\[
\psi_2(P, x, x_0) = O(\zeta^{-1}) \exp\left(\frac{i}{\zeta} \int_{x_0}^{x} dx' \varphi(x') + O(1)\right), \quad P \to P_0, \quad \zeta = z.
\]

**Proof.** The existence of the asymptotic expansions of \( \phi \) in terms of the appropriate local coordinates \( \zeta = z^{-1} \) near \( P_{m_+} \) and \( \zeta = z \) near \( P_0 \) is clear from its explicit expression in (3.9). Next, we compute the coefficients of these expansions utilizing the Riccati-type equation (3.17). Indeed, inserting the ansatz

\[
\phi = \phi_0 + \phi_1 z^{-1} + O(z^{-2})
\]

into (3.17) and comparing the same powers of \( z \) then yields (4.1). Similarly, inserting the ansatz

\[
\phi = \phi_0 + \phi_1 z + O(z^2)
\]

into (3.17) and comparing the same powers of \( z \) then yields (4.2). Finally, expansions (4.3)-(4.6) follow from (3.8), (3.10), (4.1), and (4.2). \( \square \)

**Remark 4.2.** We note the unusual fact that \( P_0 \), as opposed to \( P_{m_+} \), is the essential singularity of \( \psi_j, j = 1, 2 \). In addition, one easily finds the leading-order exponential term in \( \psi_j, j = 1, 2 \), near \( P_0 \) is \( x \)-dependent, which makes matters worse. This is in sharp contrast to standard Baker-Akhiezer functions that typically feature a linear behavior with respect to \( x \) in connection with their essential singularities of the type \( \exp(c(x - x_0)\zeta^{-1}) \) near \( \zeta = 0 \).

Next, we introduce the holomorphic differentials \( \eta_l(P) \) on \( \mathcal{X}_n \)

\[
\eta_l(P) = \frac{a_l z_l^{-1}}{y(P)} dz_l, \quad l = 1, \ldots, n,
\]

and choose a homology basis \( \{a_j, b_j\}_{j=1}^n \) on \( \mathcal{X}_n \) in such a way that the intersection matrix of the cycles satisfies

\[
a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \ldots, n.
\]

Associated with \( \mathcal{X}_n \), one introduces an invertible matrix \( E \in GL(n, \mathbb{C}) \)

\[
E = (E_{j,k})_{n \times n}, \quad E_{j,k} = \int_{a_k} \eta_j,
\]

\[
\zeta(k) = (c_1(k), \ldots, c_n(k)), \quad c_j(k) = (E^{-1})_{j,k},
\]

and the normalized holomorphic differentials

\[
\omega_j = \sum_{l=1}^n c_j(l) \eta_l, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \tau_{j,k}, \quad j, k = 1, \ldots, n.
\]

Apparently, the matrix \( \tau \) is symmetric and has a positive-definite imaginary part.
Yu Hou and Engui Fan

We choose a fixed base point $Q_0 \in \mathcal{X}_n \setminus \{\hat{\mu}_0(x), \hat{v}_0(x)\}$. The Abel maps $\mathcal{A}_{Q_0}(\cdot)$ and $\mathcal{G}_{Q_0}(\cdot)$ are defined by

\[ \mathcal{A}_{Q_0} : \mathcal{X}_n \to J(\mathcal{X}_n) = \mathbb{C}^n / L_n, \]

\[ P \mapsto \mathcal{A}_{Q_0}(P) = (A_{Q_0,1}(P), \ldots, A_{Q_0,n}(P)) = \left( \int_{Q_0}^{P} \omega_1, \ldots, \int_{Q_0}^{P} \omega_n \right) \quad (\text{mod } L_n) \tag{4.12} \]

and

\[ \mathcal{G}_{Q_0} : \text{Div}(\mathcal{X}_n) \to J(\mathcal{X}_n), \]

\[ \mathcal{D} \mapsto \mathcal{G}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{X}_n} \mathcal{D}(P) \mathcal{A}_{Q_0}(P), \tag{4.13} \]

where $L_n = \{ z \in \mathbb{C}^n \mid z = N + \tau M, \, N, \, M \in \mathbb{Z}^n \}$.

Let $\Phi^{(j)}_n(\hat{\mu}), \Psi_{n+1}(\hat{\mu})$ be the symmetric functions of $\{\mu_j\}_{j=0, \ldots, n}$,

\[ \Phi^{(j)}_n(\hat{\mu}) = (-1)^n \prod_{p=0}^{n} \mu_p, \quad \Psi_{n+1}(\hat{\mu}) = (-1)^{n+1} \prod_{p=0}^{n} \mu_p. \tag{4.14} \]

Here, $\hat{\mu}(x) = (\mu_0(x), \mu_1(x), \ldots, \mu_n(x)) = \mu_0(x)\mu(x)$.

The following result shows the nonlinearity of the Abel map with respect to the variable $x$, which indicates a characteristic difference between the HS2 hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

**Theorem 4.3.** Assume (2.27) and suppose that $\{\hat{\mu}_j(x)\}_{j=0, \ldots, n}$ satisfies the stationary Dubrovin equations (3.27) on an open interval $\Omega_{\mu} \subseteq \mathbb{R}$ such that $\mu_j(x), j = 0, \ldots, n$, remain distinct and nonzero for $x \in \Omega_{\mu}$. Introducing the associated divisor $\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}$, one computes

\[ \partial_x \mathcal{A}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}) = -\frac{4a}{\Psi_{n+1}(\hat{\mu}(x))} c(1), \quad x \in \Omega_{\mu}. \tag{4.15} \]

In particular, the Abel map does not linearize the divisor $\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}$ on $\Omega_{\mu}$.

**Proof.** Let $x \in \Omega_{\mu}$. Then, using

\[ \frac{1}{\mu_j} = \frac{\prod_{p=0}^{n} \mu_p}{\prod_{p=0}^{n} \mu_p} = -\frac{\Phi^{(j)}_n(\hat{\mu})}{\Psi_{n+1}(\hat{\mu})}, \quad j = 0, \ldots, n, \tag{4.16} \]

one obtains

\[ \partial_x \mathcal{A}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}) = \partial_x \left( \sum_{j=0}^{n} \int_{Q_0}^{\hat{\mu}_j} \omega \right) = \sum_{j=0}^{n} \mu_j \sum_{k=1}^{n} c(k) \frac{a \mu_{j-1}^{k-1}}{\gamma(\hat{\mu}_j)} \]

\[ = \sum_{j=0}^{n} \sum_{k=1}^{n} 4a \frac{\mu_{j-1}^{k-1}}{\mu_j} \frac{1}{\prod_{l \neq j}^{n} (\mu_j - \mu_l)} c(k) \]

\[ = -\frac{4a}{\Psi_{n+1}(\hat{\mu})} \sum_{j=0}^{n} \sum_{k=1}^{n} c(k) \frac{\mu_{j-1}^{k-1}}{\prod_{l \neq j}^{n} (\mu_j - \mu_l)} \Phi^{(j)}_n(\hat{\mu}) \]
Algebro-geometric solutions for the HS2 hierarchy

\[
= - \frac{4a}{\Psi_{n+1}(\tilde{\mu})} \sum_{j=0}^{n} \sum_{k=1}^{n} c(k)(U_{n+1}(\tilde{\mu}))_{k,j}^{-1}(U_{n+1}(\tilde{\mu}))_{j,1}
\]

\[
= - \frac{4a}{\Psi_{n+1}(\tilde{\mu})} \sum_{k=1}^{n} c(k)\delta_{k,1}
\]

\[
= - \frac{4a}{\Psi_{n+1}(\tilde{\mu})} c(1), \quad (4.17)
\]

where we used the notation \(\omega = (\omega_1, \ldots, \omega_n)\), and the relations (cf.(E.13), (E.14) [21]),

\[
U_{n+1}(\tilde{\mu}) = \left( \frac{\mu_j^{k-1}}{\prod_{j=0}^{n} (\mu_j - \mu_l)} \right)^n, \quad U_{n+1}(\tilde{\mu})^{-1} = \left( \Phi_n^{(j)}(\tilde{\mu}) \right)^n_{j=0}. \quad (4.18)
\]

The analogous results hold for the corresponding divisor \(D_{\phi_0(x)}\) associated with \(\phi(P,x)\).

Next, we introduce

\[
\hat{B}_{Q_0} : \mathscr{K} \setminus \{ P_\infty, P_- \} \to \mathbb{C}^n,
\]

\[
P \mapsto \hat{B}_{Q_0}(P) = (\hat{B}_{Q_0,1}, \ldots, \hat{B}_{Q_0,n})
\]

\[
= \begin{cases} 
\int_{Q_0}^{P} \omega^{(3)}_{p_-, p_-}, & n = 1, \\
\int_{Q_0}^{P} \eta_2, \ldots, \int_{Q_0}^{P} \eta_n, \int_{Q_0}^{P} \omega^{(3)}_{p_-, p_-}, & n \geq 2,
\end{cases}
\]

\[
(4.19)
\]

where \(\omega^{(3)}_{p_-, p_-} = a \, dz/dy(P)\) (cf.(F.53) [21]) and

\[
\hat{B}_{Q_0} : \text{Sym}^n(\mathscr{K} \setminus \{ P_\infty, P_- \}) \to \mathbb{C}^n,
\]

\[
\mathcal{Q} \mapsto \hat{B}_{Q_0}(\mathcal{Q}) = \sum_{j=1}^{n} B_{Q_0}(Q_j), \quad \mathcal{Q} = \{ Q_1, \ldots, Q_n \} \in \text{Sym}^n(\mathscr{K} \setminus \{ P_\infty, P_- \}),
\]

choosing identical paths of integration from \(Q_0\) to \(P\) in all integrals in (4.19) and (4.20). Then, one obtains the following result.

**Corollary 4.4.** Assume (2.27) and suppose that \(\{ \tilde{\mu}_j(x) \}_{j=0,\ldots,n}\) satisfies the stationary Dubrovin equations (3.27) on an open interval \(\Omega_\mu \subseteq \mathbb{R}\) such that \(\mu_j(x), j = 0, \ldots, n,\) remain distinct and nonzero for \(x \in \Omega_\mu\). Then one computes

\[
\partial_{x} \int_{Q_0}^{\tilde{\mu}_j(x)} \eta_1 = - \frac{4a}{\Psi_{n+1}(\tilde{\mu}(x))}, \quad x \in \Omega_\mu, \quad (4.21)
\]

\[
\partial_{x} \hat{B}(\mathcal{Q}_{\tilde{\mu}}(x)) = \begin{cases} 
4a, & n = 1, \\
4a (0, \ldots, 0, 1), & n \geq 2,
\end{cases} \quad x \in \Omega_\mu. \quad (4.22)
\]

**Proof.** Equation (4.21) is a special case of (4.15). Equation (4.22) follows from (4.17). 

The fact that the Abel map does not provide the proper change of variables to linearize the divisor \(\mathcal{Q}_{\tilde{\mu}_0(x)}\) in the HS2 context is in sharp contrast to standard integrable soliton equations.
such as the KdV and AKNS hierarchies. However, the change of variables
\[ x \mapsto \xi = \int^x dx' \left( \frac{4a}{\Psi_{n+1}(\hat{\mu}(x'))} \right) \] (4.23)
linearizes the Abel map \( A_{\hat{\theta}_0}(\hat{\varphi}_{\hat{\mu}_0}(\xi)) \), \( \hat{\mu}_j(\xi) = \mu_j(x), j = 0, \ldots, n \). The intricate relation between the variable \( x \) and \( \xi \) is detailed in (4.34).

Next, given the Riemann surface \( \mathcal{X}_n \) and the homology basis \( \{ a_j, b_j \}_{j=1,\ldots,n} \), one introduces the Riemann theta function by
\[ \theta(z) = \sum_{\nu \in \mathbb{Z}^n} \exp \left( 2\pi i (\nu, \zeta) + \pi i (\nu, \tau) \right), \quad \zeta \in \mathbb{C}^n, \]
where \( (A, B) = \sum_{j=1}^n \bar{A}_j B_j \) denotes the scalar product in \( \mathbb{C}^n \).

Let
\[ \omega^{(3)}_{\hat{\theta}_0(x), \hat{\vartheta}_0(x)}(P) = \left( \frac{y - \mu_0 G_n(x)}{z - \mu_0} - \frac{y + \nu_0 G_n(x)}{z - \nu_0} \right) \frac{dz}{2y} + \frac{\lambda_n}{y} \prod_{j=1}^{n-1} (z - \lambda_j) dz \] (4.24)
be the normalized differential of the third kind holomorphic on \( \mathcal{X}_n \setminus \{ \hat{\theta}_0(x), \hat{\vartheta}_0(x) \} \) with simple poles at \( \hat{\theta}_0(x) \) and \( \hat{\vartheta}_0(x) \) and residues 1 and \( -1 \), respectively,
\[ \omega^{(3)}_{\hat{\theta}_0(x), \hat{\vartheta}_0(x)}(P) = \binom{\xi^{-1} + O(1)}{\xi^{-1} + O(1)} d\xi, \quad \text{as } P \to \hat{\theta}_0(x), \] (4.25)
\[ \omega^{(3)}_{\hat{\theta}_0(x), \hat{\vartheta}_0(x)}(P) = \binom{-\xi^{-1} + O(1)}{-\xi^{-1} + O(1)} d\xi, \quad \text{as } P \to \hat{\vartheta}_0(x), \] (4.26)
where \( \xi \) in (4.25) and (4.26) denotes the corresponding local coordinate near \( \hat{\theta}_0(x) \) and \( \hat{\vartheta}_0(x) \) (cf. Appendix C [21]). The constants \( \{ \lambda_j \}_{j=1,\ldots,n} \) in (4.24) are determined by the normalization condition
\[ \int_{\hat{\theta}_0}^{\hat{\vartheta}_0} \omega^{(3)}_{\hat{\theta}_0(x), \hat{\vartheta}_0(x)}(P) = 0, \quad k = 1, \ldots, n. \]
Then
\[ \int_{\hat{\theta}_0}^{\hat{\vartheta}_0} \omega^{(3)}_{\hat{\theta}_0(x), \hat{\vartheta}_0(x)}(P) = \ln \xi + e_0 + O(\xi), \quad \text{as } P \to \hat{\theta}_0(x), \] (4.27)
\[ \int_{\hat{\theta}_0}^{\hat{\vartheta}_0} \omega^{(3)}_{\hat{\theta}_0(x), \hat{\vartheta}_0(x)}(P) = -\ln \xi + d_0 + O(\xi), \quad \text{as } P \to \hat{\vartheta}_0(x), \] (4.28)
for some constants \( e_0, d_0 \in \mathbb{C} \). We also record
\[ A_{\hat{\theta}_0}(P) - A_{\hat{\theta}_0}(P_{\pm}) = \pm U \xi + O(\xi^2), \quad \text{as } P \to P_{\pm}, \quad U = \zeta(n). \] (4.29)
In the following, it will be convenient to introduce the abbreviations
\[ \zeta(P, Q) = \Xi_{\hat{\theta}_0} - A_{\hat{\theta}_0}(P) + a_{\hat{\theta}_0}(P, Q), \]
P \in \mathcal{X}_n, \ Q = (Q_1, \ldots, Q_n) \in \text{Sym}^n(\mathcal{X}_n), \] (4.30)
where \( \Xi_{\hat{\theta}_0} \) is the vector of Riemann constants (cf. (A.45) [21]). It turns out that \( \zeta(\cdot, Q) \) is independent of the choice of base point \( \hat{\theta}_0 \) (cf. (A.52), (A.53) [21]).
Algebro-geometric solutions for the HS2 hierarchy

Based on above preparations, we will give explicit representations for the meromorphic function $\phi$ and the stationary HS2 solutions $u, \rho$ in terms of the Riemann theta function associated with $\mathcal{X}_n$.

**Theorem 4.5.** Suppose (2.1), and assume the $n$th stationary HS2 equation (2.33) holds on $\Omega$ subject to the constraint (2.27). Moreover, let $P = (z, y) \in \mathcal{X}_n \setminus \{P_0\}$ and $x \in \Omega$, where $\Omega \subseteq \mathbb{R}$ is an open interval. In addition, suppose that $\mathcal{D}_\Omega(x)$, or equivalently, $\mathcal{D}_\Omega(x)$ is nonspecial for $x \in \Omega$. Then, $\phi, u, \rho$ admit the following representations

$$\phi(P, x) = ip(x) \frac{\theta(z(P, \check{v}(x))) \theta(z(P, \hat{\mu}(x)))}{\theta(z(P_0, \check{v}(x))) \theta(z(P_0, \hat{\mu}(x)))} \exp \left( e_0 - \int_{\Omega_0}^p \omega^{(3)}_{\mu_0(x), \nu_0(x)} \right), \quad (4.31)$$

$$u(x) = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=1}^n \lambda_j - \frac{1}{2} \sum_{j=1}^n U_j \partial_\theta \ln \left( \frac{\theta(z(P_{m+1}, \hat{\mu}(x)) + \omega)}{\theta(z(P_{m+1}, \hat{\mu}(x)) + \omega)} \right) \bigg|_{\omega=0}, \quad (4.32)$$

$$\rho(x) = i u_n(x) \frac{\theta(z(P, \check{v}(x))) \theta(z(P_{m+1}, \hat{\mu}(x)))}{\theta(z(P_{m+1}, \check{v}(x))) \theta(z(P_{m+1}, \hat{\mu}(x)))}. \quad (4.33)$$

Moreover, let $\hat{\Omega} \subseteq \Omega$ be such that $\mu_j, j = 0, \ldots, n$, are nonvanishing on $\hat{\Omega}$. Then, the constraint

$$4a(x - x_0) = -4a \int_{\Omega_0}^x \frac{dx'}{\prod_{k=0}^{n} \mu_k(x')} \sum_{j=1}^n \left( \int_{\alpha_j}^{\hat{\Omega}_j} \omega^{(3)}_{\mu_{m+1}, \mu_{m+1}} \right) c_j(1)$$

$$+ \ln \left( \frac{\theta(z(P_{m+1}, \check{v}(x_0))) \theta(z(P_{m+1}, \check{v}(x)))}{\theta(z(P_{m+1}, \check{v}(x_0))) \theta(z(P_{m+1}, \check{v}(x)))} \right), \quad x, x_0 \in \hat{\Omega}. \quad (4.34)$$

holds, with

$$\hat{\zeta}_\Omega(x) \mathcal{D}_{\mu_0(x) \check{v}(x)} = \hat{\zeta}_\Omega(x) \mathcal{D}_{\mu_0(x) \check{v}(x)} - 4a \int_{\Omega_0}^x \frac{dx'}{\Psi_{m+1}(\mu(x'))} \zeta(1)$$

$$= \hat{\zeta}_\Omega(x) \mathcal{D}_{\mu_0(x) \check{v}(x)} - \zeta(1)(x - x_0), \quad x \in \hat{\Omega}. \quad (4.35)$$

**Proof.** First, we temporarily assume that

$$\mu_j(x) \neq \mu_{j'}(x), \quad v_k(x) \neq v_{k'}(x) \quad \text{for } j \neq j', k \neq k' \text{ and } x \in \hat{\Omega}, \quad (4.36)$$

for appropriate $\hat{\Omega} \subseteq \Omega$. Since by (3.15), $\hat{\mathcal{D}}_{\check{v}} \sim \mathcal{D}_{\hat{\mu}},$ and $(\mu_0)^* \notin \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}$ by hypothesis, one can use Theorem A.31 [21] to conclude that $\mathcal{D}_{\check{v}} \in \tilde{\text{Sym}}^n(\mathcal{X}_n)$ is nonspecial. This argument is of course symmetric with respect to $\hat{\mu}$ and $\check{v}$. Thus, $\mathcal{D}_{\check{v}}$ is nonspecial if and only if $\mathcal{D}_{\check{v}}$ is.

Next, we derive the representations of $\phi, u, \rho$ in terms of the Riemann theta function. A special case of Riemann’s vanishing theorem (cf. Theorem A.26 [21]) yields

$$\theta(z_{\Omega_0} - A_{\Omega_0}(P) + a_{\Omega_0} \mathcal{D}_{\check{v}}) = 0 \quad \text{if and only if } P \in \{Q_1, \ldots, Q_n\}. \quad (4.37)$$

Therefore, the divisor (3.15) shows that $\phi(P, x)$ has expression of the type

$$C(x) \frac{\theta(z_{\Omega_0} - A_{\Omega_0}(P) + a_{\Omega_0} \mathcal{D}_{\check{v}(x)})}{\theta(z_{\Omega_0} - A_{\Omega_0}(P) + a_{\Omega_0} \mathcal{D}_{\check{v}(x)})} \exp \left( e_0 - \int_{\Omega_0}^p \omega^{(3)}_{\mu_0(x), \nu_0(x)} \right), \quad (4.38)$$

where $C(x)$ is independent of $P \in \mathcal{X}_n$. Then taking into account the asymptotic expansion of $\phi(P, x)$ near $P_0$ in (4.2), we obtain (4.31). The representation (4.32) for $u$ on $\hat{\Omega}$ follows from trace formula (3.35) and the expression (F.59) [21] for $\sum_{j=0}^n \mu_j$. The representation (4.33) for $\rho$ on $\hat{\Omega}$ is clear from
Yu Hou and Engui Fan

(4.1) and (4.31). By continuity, (4.31), (4.32), and (4.33) extend from \(\tilde{\Omega}\) to \(\Omega\). Assuming \(\mu_j \neq 0\), \(j = 0, \ldots, n\), the constraint (4.34) follows by combining (4.21), (4.22), and (F.58) [21]. Equation (4.35) is clear from (4.15). Finally, (4.34) and (4.35) extend to \(\tilde{\Omega}\) by continuity. \(\square\)

**Remark 4.6.** We note that the stationary HS2 solutions \(u, \rho\) in (4.32) and (4.33) are meromorphic quasi-periodic functions with respect to the new variable \(\tilde{x}\) in (4.23). In addition, the Abel map in (4.35) linearizes the divisor \(\mathcal{D}_{\tilde{\mu}(x)}\) with respect to \(\tilde{x}\) under the constraint (4.34).

**Remark 4.7.** Since by (3.15) \(\mathcal{D}_{\tilde{\mu}(x)}\) and \(\mathcal{D}_{\hat{\mu}(x)}\) are linearly equivalent, that is

\[
A_{\bar{Q}_0}(\hat{\mu}_0(x)) + \mathcal{A}_{\bar{Q}_0}(\mathcal{D}_{\hat{\mu}(x)}) = A_{\bar{Q}_0}(\tilde{\nu}_0(x)) + \mathcal{A}_{\bar{Q}_0}(\mathcal{D}_{\tilde{\nu}(x)}),
\]

Then one infers

\[
\mathcal{A}_{\bar{Q}_0}(\mathcal{D}_{\tilde{\nu}(x)}) = \Delta + \mathcal{A}_{\bar{Q}_0}(\mathcal{D}_{\hat{\mu}(x)}), \quad \Delta = A_{\bar{Q}_0}(\hat{\mu}_0(x)).
\]

Hence, one can eliminate \(\mathcal{D}_{\tilde{\nu}(x)}\) in (4.31), in terms of \(\mathcal{D}_{\hat{\mu}(x)}\) using

\[
\hat{z}(P, \hat{\nu}) = \hat{z}(P, \hat{\mu}) + \Delta, \quad P \in \mathcal{X}_n.
\]

**Remark 4.8.** We emphasized in Remark 4.2 that \(\psi\) in (3.8) and (3.10) differs from standard Baker-Akhiezer functions. Hence, one can not expect the usual theta function representation of \(\psi_j, j = 1, 2\), in terms of ratios of theta functions times an exponential term containing a meromorphic differential with a pole at the essential singularity of \(\psi_j\) multiplied by \((x - x_0)\). However, using (E.3) and (F.59) [21], one computes

\[
F_n(z) = \frac{1}{2} z^{n+1} + \frac{1}{2} \sum_{k=0}^{n} \Psi_{n+1-k}(\hat{\mu}(x)) z^k
\]

\[
= \frac{1}{2} z^{n+1} + \frac{1}{2} \sum_{k=1}^{n} \left( \Psi_{n+1-k}(\hat{\lambda}) - \sum_{j=1}^{n} c_j(k) \partial_{\omega_j} \ln \left( \frac{\theta(z(\bar{P}_{n-j} \hat{\mu}(x)) + \omega)}{\theta(z(\bar{P}_{n-j} \hat{\mu}(x)) + \omega)} \right) \bigg|_{\omega=0} \right) z^k
\]

\[
= \frac{1}{2} \prod_{j=0}^{n} (z - \lambda_j) - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} c_j(k) \partial_{\omega_j} \ln \left( \frac{\theta(z(\bar{P}_{n-j} \hat{\mu}(x)) + \omega)}{\theta(z(\bar{P}_{n-j} \hat{\mu}(x)) + \omega)} \right) \bigg|_{\omega=0} z^k,
\]

and hence obtains the theta function representation of \(\psi_1\) upon inserting (4.42) into (3.21). Then, the corresponding theta function representation of \(\psi_2\) follows by (3.10) and (4.31).

At the end of this section, we turn to the initial value problem for the stationary HS2 hierarchy. We will show that the solvability of the Dubrovin equations (3.27) on \(\Omega_\mu \subseteq \mathbb{R}\) in fact implies the stationary HS2 equation (2.33) on \(\Omega_\mu\).

**Theorem 4.9.** Fix \(n \in \mathbb{N}_0\), assume (2.27), and suppose that \(\{\hat{\mu}_j\}_{j=0,\ldots,n}\) satisfies the stationary Dubrovin equations (3.27) on \(\Omega_\mu\) such that \(\mu_j, j = 0, \ldots, n\), remain distinct and nonzero on \(\Omega_\mu\),
where $\Omega_{\mu} \subseteq \mathbb{R}$ is an open interval. Then, $u, \rho \in C^\infty(\Omega_{\mu})$, defined by

$$u = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=0}^{n} \mu_j,$$

and

$$\rho^2 = u_t^2 + 2uu_{tx} - \frac{d^2}{dx^2} \left( \Psi_2(\mu) - u \sum_{m=0}^{2n+1} E_m \right),$$

satisfy the $n$th stationary HS2 equation (2.33), that is,

$$\text{s-HS}_2(u, \rho) = 0 \text{ on } \Omega_{\mu}. \quad (4.45)$$

**Proof.** Given the solutions $\hat{\mu}_j = (\mu_j, y(\mu_j)) \in C^\infty(\Omega_{\mu}, \mathbb{R}^n), j = 0, \ldots, n$ of (3.27), we introduce

$$F_n(z) = \frac{1}{2} \prod_{j=0}^{n} (z - \mu_j),$$

$$G_n(z) = \frac{1}{2} F_{n,x}(z), \quad (4.46)$$

on $\mathbb{C} \times \Omega_{\mu}$. Taking into account (4.46), the Dubrovin equations (3.27) imply

$$y(\hat{\mu}_j) = \frac{1}{4} \mu_j \mu_{j,k} \prod_{k=0}^{n} (\mu_j - \mu_k) = -\frac{1}{2} \mu_j F_{n,x}(\mu_j) = -\mu_j G_n(\mu_j). \quad (4.47)$$

Hence

$$R_{2n+2}(\mu_j)^2 - \mu_j^2 G_n(\mu_j)^2 = y(\hat{\mu}_j)^2 - \mu_j^2 G_n(\mu_j)^2 = 0, \quad j = 0, \ldots, n. \quad (4.48)$$

Next, we define a polynomial $H_n$ on $\mathbb{C} \times \Omega_{\mu}$ such that

$$R_{2n+2}(z) - z^2 G_n(z)^2 = F_n(z) H_n(z). \quad (4.49)$$

Such a polynomial $H_n$ exists since the left-hand side of (4.49) vanishes at $z = \mu_j, j = 0, \ldots, n$, by (4.48). To determine the degree of $H_n$, using (4.46), one computes

$$R_{2n+2}(z) - z^2 G_n(z)^2 \left|_{z \to 0} \right. = \frac{1}{2} h_0 z^{2n+2} + O(z^{2n+1}). \quad (4.50)$$

Then combining (4.46), (4.49), and (4.50), one infers that $H_n$ has degree $n + 1$ with respect to $z$. Hence, we may write

$$H_n(z) = h_0 \prod_{i=0}^{n} (z - v_i), \quad \text{on } \mathbb{C} \times \Omega_{\mu}. \quad (4.51)$$

Next, one defines the polynomial $P_n$ by

$$P_n(z) = H_n(z) + (\rho^2 + u_{tx}z) F_n(z) + z^2 G_{n,x}(z). \quad (4.52)$$

Using (4.46) and (4.51), one infers that indeed $P_n$ has degree at most $n$. Differentiating (4.49) with respect to $x$ yields

$$2z^2 G_n(z) G_{n,x}(z) + F_n(z) H_n(z) + F_n(z) H_{n,x}(z) = 0. \quad (4.53)$$
Then multiplying (4.52) by $G_n$ and replacing the term $G_nG_{n,x}$ with (4.53) leads to
\[
G_n(z)P_n(z) = F_n(z)((\rho^2 + u_{xx}z)G_n(z) - \frac{1}{2}H_{n,x}(z)) + (G_n(z) - \frac{1}{2}F_{n,x}(z))H_n(z),
\]
and hence
\[
G_n(\mu_j)P_n(\mu_j) = 0, \quad j = 0, \ldots, n
\]
on $\Omega_{\mu}$. Restricting $x \in \Omega_{\mu}$ temporarily to $x \in \tilde{\Omega}_{\mu}$, where
\[
\tilde{\Omega}_{\mu} = \{x \in \Omega_{\mu} \mid G(\mu_j(x), x) = -\frac{y(\mu_j(x))}{\mu_j(x)} \neq 0, \quad j = 0, \ldots, n\}
\]
\[
= \{x \in \Omega_{\mu} \mid \mu_j(x) \notin \{E_m\}_{m=0, \ldots, 2n+1}, \quad j = 0, \ldots, n\},
\]
one infers that
\[
P_n(\mu_j(x), x) = 0, \quad j = 0, \ldots, n, \quad x \in \tilde{\Omega}_{\mu}.
\]
Since $P_n(z)$ has degree at most $n$, (4.57) implies
\[
P_n = 0 \quad \text{on } \mathbb{C} \times \tilde{\Omega}_{\mu},
\]
and hence (2.16) holds, that is,
\[
z^2G_{n,x}(z) = -H_n(z) - (\rho^2 + u_{xx}z)F_n(z)
\]
on $\mathbb{C} \times \tilde{\Omega}_{\mu}$. Inserting (4.59) and (4.46) into (4.53) yields
\[
F_n(z)(H_{n,x}(z) - 2(\rho^2 + u_{xx}z)G_n(z)) = 0,
\]
amely
\[
H_{n,x}(z) = 2(\rho^2 + u_{xx}z)G_n(z)
\]
on $\mathbb{C} \times \tilde{\Omega}_{\mu}$. Thus, we obtain the fundamental equations (2.14)-(2.16) and (2.18) on $\mathbb{C} \times \tilde{\Omega}_{\mu}$. In order to extend these results to $\Omega_{\mu}$, we next investigate the case where $\mu_j$ hits a branch point $(E_{m_0}, 0)$. Hence, we suppose
\[
\mu_{j_1}(x) \to E_{m_0} \quad \text{as } x \to x_0 \in \Omega_{\mu},
\]
for some $j_1 \in \{0, \ldots, n\}$, $m_0 \in \{0, \ldots, 2n + 1\}$. Introducing
\[
\zeta_{j_1}(x) = \sigma(\mu_{j_1}(x) - E_{m_0})^{1/2}, \quad \sigma = \pm 1,
\]
\[
\mu_{j_1}(x) = E_{m_0} + \zeta_{j_1}(x)^2,
\]
for some $x$ in an open interval centered near $x_0$, the Dubrovin equation (3.27) for $\mu_{j_1}$ becomes
\[
\zeta_{j_1,x}(x) = \frac{c(\sigma) 2a}{E_{m_0}^{2n+1}} \left( \prod_{m=0 \atop m \neq m_0}^{2n+1} (E_{m_0} - E_m) \right)^{1/2}
\]
\[
\times \prod_{k=0 \atop k \neq j_1}^{n} (E_{m_0} - \mu_k(x))^{-1}(1 + O(\zeta_{j_1}(x)^2))
\]
\[
(4.64)
\]
Co-published by Atlantis Press and Taylor & Francis
Copyright: the authors
for some $|c(\sigma)| = 1$. Hence, (4.58)-(4.61) extend to $\Omega_\mu$ by continuity. We have now established relations (2.14)-(2.16) on $\mathbb{C} \times \Omega_\mu$, and one can proceed as in Section 2 to obtain (4.45). □

**Remark 4.10.** Although we formulated Theorem 4.9 in terms of $\{\mu_j\}_{j=0,\ldots,n}$ only, the analogous result (and strategy of proof) obviously works in terms of $\{v_j\}_{j=0,\ldots,n}$.

**Remark 4.11.** A closer look at Theorem 4.9 reveals that $u, \rho$ are uniquely determined in an open neighborhood $\Omega$ of $x_0$ by $\mathcal{H}_n$ and the initial condition $(\hat{\mu}_0(x_0), \hat{\mu}_1(x_0), \ldots, \hat{\mu}_n(x_0))$, or equivalently, by the auxiliary divisor $\mathcal{D}_{\hat{\mu}_n(x_0)} \hat{\mu}(x_0)$ at $x = x_0$. Conversely, given $\mathcal{H}_n$ and $u, \rho$ in an open neighborhood $\Omega$ of $x_0$, one can construct the corresponding polynomial $F_n(z, x)$, $G_n(z, x)$ and $H_n(z, x)$ for $x \in \Omega$, and then recover the auxiliary divisor $\mathcal{D}_{\hat{\mu}_n(x)} \hat{\mu}(x)$ for $x \in \Omega$ from the zeros of $F_n(z, x)$ and from (3.12). In this sense, once the curve $\mathcal{H}_n$ is fixed, elements of the isospectral class of the HS2 potentials $u, \rho$ can be characterized by nonspecial auxiliary divisors $\mathcal{D}_{\hat{\mu}_n(x)} \hat{\mu}(x)$.

5. The time-dependent HS2 formalism

In this section, we extend the algebro-geometric analysis of Section 3 to the time-dependent HS2 hierarchy.

Throughout this section, we assume (2.2) holds.

The time-dependent algebro-geometric initial value problem of the HS2 hierarchy is to solve the time-dependent $r$th HS2 flow with a stationary solution of the $n$th equation as initial data in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, based on the solution $u^{(0)}, \rho^{(0)}$ of the $n$th stationary HS2 equation $\text{s-HS}_2^n(u^{(0)}, \rho^{(0)}) = 0$ associated with $\mathcal{H}_n$ and a set of integration constants $\{c_l\}_{l=1,\ldots,n} \subset \mathbb{C}$, we want to construct a solution $u, \rho$ of the $r$th HS2 flow $\text{s-HS}_2^r(u, \rho) = 0$ such that $u(t_0, r) = u^{(0)}$, $\rho(t_0, r) = \rho^{(0)}$ for some $t_0, r \in \mathbb{R}$, $r \in \mathbb{N}_0$.

To emphasize that the integration constants in the definitions of the stationary and the time-dependent HS2 equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence, we employ the notation $\tilde{V}_r, \tilde{F}_r, \tilde{G}_r, \tilde{H}_r, \tilde{f}_r, \tilde{g}_r, \tilde{h}_r, \tilde{c}_r$ in order to distinguish them from $V_n, F_n, G_n, H_n, f_1, g_1, h_1, c_1$ in the following. In addition, we mark the individual $r$th HS2 flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution $u, \rho$ of the time-dependent algebro-geometric initial value problem

\begin{align*}
\text{s-HS}_2^r(u^{(0)}, \rho^{(0)}) &= \left( \begin{array}{c}
\tilde{u}_{xx} + 2 \tilde{u}_{xx} \tilde{f}_{r+1, x} + \tilde{u}_{xxx} \tilde{f}_{r+1} + 2 \rho \tilde{f}_x \tilde{f}_{r} + 2 \rho^2 \tilde{f}_{r,x} \\
2 \rho \tilde{f}_x - 2 \rho \tilde{f}_x \tilde{f}_{r+1} - 2 \rho^2 \tilde{f}_{r+1,x}
\end{array} \right) = 0, \\
(u, \rho)|_{t_r = t_0} &= (u^{(0)}, \rho^{(0)}),
\end{align*}

\begin{align*}
\text{s-HS}_2^n(u^{(0)}, \rho^{(0)}) &= \left( \begin{array}{c}
2 \tilde{u}_{xx} f_{n+1, x} + \tilde{u}_{xxx} f_{n+1} + 2 \rho \tilde{f}_x f_n + 2 \rho^2 f_{n,x} \\
-2 \rho \tilde{f}_x f_{n+1} - 2 \rho^2 f_{n+1,x}
\end{array} \right) = 0,
\end{align*}

for some $t_0, r \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where $u = u(x, t_r)$, $\rho = \rho(x, t_r)$ satisfy (2.2), and the curve $\mathcal{H}_n$ is associated with the initial data $(u^{(0)}, \rho^{(0)})$ in (5.2). Noticing that the HS2 flows are isospectral, we further assume that (5.2) holds not only for $t_r = t_0, r$, but also for all $t_r \in \mathbb{R}$. Hence, we start with the zero-curvature equations

\begin{align*}
U_{t_r} - \tilde{V}_{r,x} + [U, \tilde{V}_r] &= 0, \\
- V_{n,x} + [U, V_n] &= 0,
\end{align*}
Yu Hou and Engui Fan

where

\[
U(z) = \begin{pmatrix} 0 & 1 \\ -z^{-2} \rho^2 - z^{-1} u_{xx} & 0 \end{pmatrix},
\]

\[
V_n(z) = \begin{pmatrix} -G_n(z) & F_n(z) \\ z^{-2} H_n(z) & G_n(z) \end{pmatrix},
\]

\[
\tilde{V}_r(z) = \begin{pmatrix} -\tilde{G}_r(z) & \tilde{F}_r(z) \\ z^{-2} \tilde{H}_r(z) & G_r(z) \end{pmatrix},
\]

and

\[
F_n(z) = \sum_{l=0}^{n+1} f_l z^{n+1-l} = f_0 \prod_{j=0}^{n}(z - \mu_j),
\]

(5.6)

\[
G_n(z) = \sum_{l=0}^{n+1} g_l z^{n-l},
\]

(5.7)

\[
H_n(z) = \sum_{l=0}^{n+1} h_l z^{n+1-l} = h_0 \prod_{l=0}^{n}(z - v_l),
\]

(5.8)

\[
\tilde{F}_r(z) = \sum_{s=0}^{r+1} \tilde{f}_s z^{r+1-s},
\]

(5.9)

\[
\tilde{G}_r(z) = \sum_{s=0}^{r} \tilde{g}_s z^{-s},
\]

(5.10)

\[
\tilde{H}_r(z) = \sum_{s=0}^{r+1} \tilde{h}_s z^{r+1-s},
\]

(5.11)

for fixed \( n, r \in \mathbb{N}_0 \). Here, \( \{f_l\}_{l=0}^{n+1}, \{g_l\}_{l=0}^{n}, \{h_l\}_{l=0}^{n+1}, \{\tilde{f}_s\}_{s=0}^{r+1}, \{\tilde{g}_s\}_{s=0}^{r}, \) and \( \{\tilde{h}_s\}_{s=0}^{r+1} \) are defined as in (2.3), with \( u(x), \rho(x) \) replaced by \( u(x,t_r), \rho(x,t_r) \) etc., and with appropriate integration constants. Explicitly, (5.3) and (5.4) are equivalent to

\[
-2 \rho u_{xx} - zu_{xxt} - \tilde{H}_{r,x} + 2(\rho^2 + zu_{xx})\tilde{G}_r = 0,
\]

(5.12)

\[
\tilde{F}_{r,x} = 2\tilde{G}_r,
\]

(5.13)

\[
z^2 \tilde{G}_{r,xx} = -\tilde{H}_r - (\rho^2 + zu_{xx})\tilde{F}_r
\]

(5.14)

and

\[
F_{n,x} = 2G_n,
\]

(5.15)

\[
H_{n,x} = 2(\rho^2 + zu_{xx})G_n,
\]

(5.16)

\[
z^2 G_{n,xx} = -H_n - (\rho^2 + zu_{xx})F_n.
\]

(5.17)

From (5.15)-(5.17), one finds

\[
\frac{d}{dx} \det(V_n(z)) = -\frac{1}{z^2} \frac{d}{dx} \left( z^2 G_n(z)^2 + F_n(z)H_n(z) \right) = 0,
\]

(5.18)

and meanwhile (see Lemma 5.2)

\[
\frac{d}{dt_r} \det(V_n(z)) = -\frac{1}{z^2} \frac{d}{dt_r} \left( z^2 G_n(z)^2 + F_n(z)H_n(z) \right) = 0.
\]

(5.19)
Hence, \( z^2 G_n(z)^2 + F_n(z)H_n(z) \) is independent of variables both \( x \) and \( t_r \), which implies the fundamental identity (2.18) holds,

\[
z^2 G_n(z)^2 + F_n(z)H_n(z) = R_{2n+2}(z),
\]

and the hyperelliptic curve \( \mathcal{K}_n \) is still given by (2.26).

Next, we define the time-dependent Baker-Akhiezer function \( \psi(P,x,x_0,t_0,r_0) \) on \( \mathcal{K}_n \setminus \{P_{\infty}, P_0\} \) by

\[
\psi(P,x,x_0,t_0,r_0) = \begin{pmatrix} \psi_1(P,x,x_0,t_0,r_0) \\ \psi_2(P,x,x_0,t_0,r_0) \end{pmatrix},
\]

\[
\psi_1(P,x,x_0,t_0,r_0) = U(u(x,t_0), \rho(x,t_0), z(P)) \psi(P,x,x_0,t_0,r_0),
\]

\[
\psi_2(P,x,x_0,t_0,r_0) = \tilde{V}_r(u(x,t_0), \rho(x,t_0), z(P)) \psi(P,x,x_0,t_0,r_0),
\]

\[
zV_0(u(x,t_0), \rho(x,t_0), z(P)) \psi(P,x,x_0,t_0,r_0) = y(P) \psi(P,x,x_0,t_0,r_0),
\]

\[
\psi_1(P,x,x_0,t_0,r_0) = 1;
\]

\[
P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty}, P_0\}, \ (x,t_r) \in \mathbb{R}^2.
\]

Closely related to \( \psi(P,x,x_0,t_0,r_0) \) is the following meromorphic function \( \phi(P,x,t_r) \) on \( \mathcal{K}_n \) defined by

\[
\phi(P,x,t_r) = z \frac{\psi_1(P,x,x_0,t_0,r_0)}{\psi_1(P,x,x_0,t_0,r_0)}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty}, P_0\}, \ (x,t_r) \in \mathbb{R}^2
\]

\[
\psi_1(P,x,x_0,t_0,r_0) = \exp \left( \int_{t_0}^{t_r} ds(z^{-1} \tilde{F}_r(z,x_0,s) \phi(P,x_0,s)
\]

\[
- \tilde{G}_r(z,x_0,s)) + z^{-1} \int_{x_0}^{t} dx' \phi(P,x',t_r) \right),
\]

\[
P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty}, P_0\}.
\]

Then, using (5.21) and (5.22), one infers that

\[
\phi(P,x,t_r) = \frac{\psi_1(P,x,x_0,t_0,r_0)}{\psi_1(P,x,x_0,t_0,r_0)} \phi(P,x,t_r)/z.
\]

In analogy to (3.12) and (3.13), we introduce

\[
\hat{\mu}_j(x,t_r) = (\mu_j(x,t_r), -\mu_j(x,t_r)G_n(\mu_j(x,t_r), x,t_r)) \in \mathcal{K}_n,
\]

\[
j = 0, \ldots, n, \ (x,t_r) \in \mathbb{R}^2,
\]

\[
\hat{\nu}_l(x,t_r) = (\nu_l(x,t_r), \nu_l(x,t_r)G_n(\nu_l(x,t_r), x,t_r)) \in \mathcal{K}_n,
\]

\[
l = 0, \ldots, n, \ (x,t_r) \in \mathbb{R}^2.
\]
The regularity properties of \( F_n, H_n, \mu_j, \) and \( v_t \) are analogous to those in Section 3 due to assumptions (2.2). Similar to (3.15), the divisor \( (\phi(P,x,t_r)) \) of \( \phi(P,x,t_r) \) reads

\[
(\phi(P,x,t_r)) = \mathcal{D}_{\hat{\nu}_0(x,t_r)}\mathcal{D}_{\hat{\mu}_0(x,t_r)}(P) - \mathcal{D}_{\hat{\mu}_0(x,t_r)}\mathcal{D}_{\hat{\nu}_0(x,t_r)}(P)
\]  

(5.28)

with

\[
\hat{\mu} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}, \quad \hat{\nu} = \{\hat{\nu}_1, \ldots, \hat{\nu}_n\} \in \text{Sym}^n(\mathcal{X}_{n}).
\]  

(5.29)

The properties of \( \phi(P,x,t_r) \) are summarized as follows.

**Lemma 5.1.** Assume (2.2) and suppose that (5.3), (5.4) hold. Moreover, let \( P = (z,y) \in \mathcal{X}_{n} \backslash \{P_{0, \ldots, P_0}\} \) and \( (x,t_r) \in \mathbb{R}^2 \). Then \( \phi \) satisfies

\[
\phi_{x}(P) + z^{-1}\phi(P)^2 = -z^{-1}\rho^{2} - u_{xx},
\]  

(5.30)

\[
\phi_{y}(P) = (-z\tilde{G}_{r}(z) + \tilde{F}_{r}(z)\phi(P))_{x},
\]  

(5.31)

\[
\phi_{x}(P) = -z^{-1}\tilde{F}_{r}(z) + 2\tilde{G}_{r}(z)\phi(P) - z^{-1}\tilde{F}_{r}(z)\phi(P)^2,
\]  

(5.32)

\[
\phi(P)\phi(P^*) = -\frac{H_{n}(z)}{F_{n}(z)},
\]  

(5.33)

\[
\phi(P) + \phi(P^*) = 2\frac{zG_{n}(z)}{F_{n}(z)},
\]  

(5.34)

\[
\phi(P) - \phi(P^*) = \frac{2y}{F_{n}(z)}.
\]  

(5.35)

**Proof.** Equations (5.30) and (5.33)-(5.35) can be proved as in Lemma 3.1. Using (5.21) and (5.22), one infers that

\[
\phi_{x} = z(\ln \psi_{1})_{x} + z(\ln \psi_{1})_{x,x} = z\left(\frac{\psi_{1,x}}{\psi_{1}}\right)_{x} = z\left(-\tilde{G}_{r}\psi_{1} + \tilde{F}_{r}\psi_{2}\right)_{x} = (-z\tilde{G}_{r} + \tilde{F}_{r}\phi)_{x}.
\]  

(5.36)

Insertion of (5.14) into (5.36) then yields (5.31). To prove (5.32), one observes that

\[
\phi_{x} = z\left(\frac{\psi_{2}}{\psi_{1}}\right)_{x} = z\left(\frac{\psi_{2}}{\psi_{1}} - \frac{\psi_{2} \psi_{1,x}}{\psi_{1}^2} - z^{-1}\phi - \tilde{G}_{r}\psi_{1} + \tilde{F}_{r}\psi_{2}\right)
\]  

\[
= -z^{-2}\tilde{H}_{r}\psi_{1} + \tilde{G}_{r}\psi_{2} - z^{-1}\phi - \tilde{G}_{r}\psi_{1} + \tilde{F}_{r}\psi_{2} = -z^{-1}\tilde{H}_{r} + 2\tilde{G}_{r}\phi - z^{-1}\tilde{F}_{r}\phi^{2},
\]  

(5.37)

which leads to (5.32). Alternatively, one can also insert (5.12)-(5.14) into (5.31) to obtain (5.32). \( \square \)

Next, we determine the time evolution of \( F_n, G_n, \) and \( H_n, \) using relations (5.12)-(5.14) and (5.15)-(5.17).

**Lemma 5.2.** Assume (2.2) and suppose that (5.3), (5.4) hold. Then

\[
F_{n,t_r} = 2(G_{n}\tilde{F}_{r} - \tilde{G}_{r}F_{n}),(5.38)
\]
\[ (\phi(P) - \phi(P^*))_{t_r} = -2yF_{n,t_r}F_n^{-2}. \]  

Proof. Differentiating (5.35) with respect to \( t_r \) naturally yields 
\[ (\phi(P) - \phi(P^*))_{t_r} = 2F_n(z, x, t_r) + 2z^{-1}\tilde{F}_n(z, x, t_r) = 2z(z^2 G^2 + F_n H_n) = R_{2n+2}(z), \]  

using (5.32), (5.33), and (5.34). Equation (5.39) is clear from (5.44) and (5.45). Then, (5.40) follows 
from combining (5.42) and (5.43) then proves (5.38). Similarly, differentiating (5.34) with respect to \( t_r \), one finds 
\[ (\phi(P) + \phi(P^*))_{t_r} = 2z(G_{n,t_r}F_n - G_n F_{n,t_r})F_n^{-2}. \]  

Meanwhile, the left-hand side of (5.44) also equals 
\[ (\phi(P)_{t_r} + \phi(P^*)_{t_r}) = 2\tilde{G}_n(z, x, t_r) + 2z^{-1}\tilde{F}_n(z, x, t_r) = 2z(z^2 G^2 + F_n H_n), \]  

using (5.32), (5.33), and (5.34). Equation (5.39) is clear from (5.44) and (5.45). Then, (5.40) follows 
by differentiating (2.18), that is, \( z^2 G^2 + F_n H_n = R_{2n+2}(z) \), with respect to \( t_r \), and using (5.38) and (5.39). Finally, a direct calculation shows (5.41) holds.

Basic properties of \( \psi(P, x, t_r, t_0, r) \) are summarized as follows.

Lemma 5.3. Assume (2.2) and suppose that (5.3), (5.4) hold. Moreover, let \( P = (z, y) \in \mathcal{H}_n \setminus \{P_n, P_0\} \) and \((x, x_0, t_r, t_0, r) \in \mathbb{R}^4\). Then, the Baker-Akhiezer function \( \psi \) satisfies 
\[ \psi_1(P, x, x_0, t_r, t_0, r) = \left( \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_0, r)} \right)^{1/2} \exp \left( \frac{y}{z} \int_{t_0}^{t_r} ds \tilde{F}_n(z, x, t_r) F_n(z, x, s)^{-1} \right) + \frac{y}{z} \int_{t_0}^{t_r} ds \tilde{F}_n(z, x, t_r) F_n(z, x, s)^{-1} \),  

\[ \psi_2(P, x, x_0, t_r, t_0, r) \psi_1(P^*, x, x_0, t_r, t_0, r) = \psi_1(P^*, x, x_0, t_r, t_0, r) \psi_2(P, x, x_0, t_r, t_0, r) = \psi_1(P, x, x_0, t_r, t_0, r) \psi_2(P^*, x, x_0, t_r, t_0, r) = \frac{2 G_n(z, x, t_r)}{F_n(z, x_0, t_0, r)} \],  

\[ \psi_1(P, x, x_0, t_r, t_0, r) \psi_2(P^*, x, x_0, t_r, t_0, r) - \psi_1(P^*, x, x_0, t_r, t_0, r) \psi_2(P, x, x_0, t_r, t_0, r) = -\frac{2y}{z F_n(z, x_0, t_0, r)} \].
Lemma 5.4. Assume the remaining statements are direct consequences of (5.25), (5.33)-(5.35), and (5.47). Using the similar procedure in Lemma 3.2. Then combining (5.53) and (5.54) readily leads to (5.46).

The integrand in the above integral equals

\[
\exp \left( \int_{h_0}^{t} ds \left( z^{-1} \tilde{F}_r(z,x_0,s) \phi(P,x_0,s) - \tilde{G}_r(z,x_0,s) \right) \right). \tag{5.51}
\]

Using (5.24) and (5.38), hence, (5.51) can be expressed as

\[
\left( \frac{F_n(z,x_0,\tau)}{F_n(z,x_0,t_0,\tau)} \right)^{1/2} \exp \left( \frac{y}{z} \int_{h_0}^{t} ds \tilde{F}_r(z,x_0,s) F_n(z,x_0,s)^{-1} \right). \tag{5.53}
\]

On the other hand, the part of space variable in (5.23) can be written as

\[
\left( \frac{F_n(z,x,\tau)}{F_n(z,x,t_0,\tau)} \right)^{1/2} \exp \left( \frac{y}{z} \int_{x_0}^{x} dx' F_n(z,x',\tau)^{-1} \right), \tag{5.54}
\]

using the similar procedure in Lemma 3.2. Then combining (5.53) and (5.54) readily leads to (5.46). Evaluating (5.46) at the points \( P \) and \( P^* \) and multiplying the resulting expressions yields (5.47). The remaining consequences are direct consequences of (5.25), (5.33)-(5.35), and (5.47).

In analogy to Lemma 3.4, the dynamics of the zeros \( \{\mu_j(x,\tau)\}_{j=0,\ldots,n} \) and \( \{v_l(x,\tau)\}_{l=0,\ldots,n} \) of \( F_n(z,x,\tau) \) and \( H_n(z,x,\tau) \) with respect to \( x \) and \( \tau \) are described in terms of the following Dubrovin-type equations.

Lemma 5.4. Assume (2.2) and suppose that (5.3), (5.4) hold subject to the constraint (2.27).

(i) Suppose that the zeros \( \{\mu_j(x,\tau)\}_{j=0,\ldots,n} \) of \( F_n(z,x,\tau) \) remain distinct for \((x,\tau) \in \Omega_{\mu}, \) where \( \Omega_{\mu} \subseteq \mathbb{R}^2 \) is open and connected, then \( \{\mu_j(x,\tau)\}_{j=0,\ldots,n} \) satisfy the system of differential equations,

\[
\mu_{j,x} = 4 \left( \frac{\hat{\mu}_j}{\mu_j} \right) \prod_{k=0 \atop k \neq j}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 0,\ldots,n, \tag{5.55}
\]

\[
\mu_{j,\tau} = 4 \frac{\tilde{F}_r^{(\mu_j)}(\hat{\mu}_j)}{\mu_j} \prod_{k=0 \atop k \neq j}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 0,\ldots,n. \tag{5.56}
\]
with initial conditions

\[ \{ \hat{\mu}_j(x_0, t_0, r) \}_{j=0, \ldots, n} \in \mathcal{X}_n, \quad (5.57) \]

for some fixed \((x_0, t_0, r) \in \Omega_\mu\). The initial value problem \((5.56), (5.57)\) has a unique solution satisfying

\[ \hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{X}_n), \quad j = 0, \ldots, n. \quad (5.58) \]

(ii) Suppose that the zeros \(\{ v_l(x, t_r) \}_{l=0, \ldots, n}\) of \(H_\mu(z, x, t_r)\) remain distinct for \((x, t_r) \in \Omega_v\), where \(\Omega_v \subseteq \mathbb{R}^2\) is open and connected, then \(\{ v_l(x, t_r) \}_{l=0, \ldots, n}\) satisfy the system of differential equations,

\[ v_{l,x} = -2 \left( \frac{\rho^2 + u_{\mu} v_l}{h_0 v_l} \right) y(\hat{v}_l) \prod_{k=0, k \neq l}^n (v_l - v_k)^{-1}, \quad l = 0, \ldots, n, \quad (5.59) \]

\[ v_{l,r} = 2 \frac{\hat{H}_r(\hat{v}_l)}{h_0 v_l} \prod_{k=0, k \neq l}^n (v_l - v_k)^{-1}, \quad l = 0, \ldots, n, \quad (5.60) \]

with initial conditions

\[ \{ \hat{v}_l(x_0, t_0, r) \}_{l=0, \ldots, n} \in \mathcal{X}_n, \quad (5.61) \]

for some fixed \((x_0, t_0, r) \in \Omega_v\). The initial value problem \((5.60), (5.61)\) has a unique solution satisfying

\[ \hat{v}_l \in C^\infty(\Omega_v, \mathcal{X}_n), \quad l = 0, \ldots, n. \quad (5.62) \]

**Proof.** It suffices to prove \((5.56)\) since the argument for \((5.60)\) is analogous and that for \((5.55)\) and \((5.59)\) has been given in the proof of Lemma 3.4. Differentiating \((5.6)\) with respect to \(t_r\) yields

\[ F_{n,t_r}(\mu_j) = -\frac{1}{2} \hat{\mu}_{j,t_r} \prod_{k=0, k \neq j}^n (\mu_j - \mu_k). \quad (5.63) \]

On the other hand, inserting \(z = \mu_j\) into \((5.38)\) and using \((5.26)\), one finds

\[ F_{n,t_r}(\mu_j) = 2G_n(\mu_j) \hat{F}_r(\mu_j) = 2 \frac{y(\hat{\mu}_j)}{-\mu_j} \hat{F}_r(\mu_j). \quad (5.64) \]

Combining \((5.63)\) and \((5.64)\) then yields \((5.56)\). The rest is analogous to the proof of Lemma 3.4. \(\square\)

Since the stationary trace formulas for HS2 invariants in terms of symmetric functions of \(\mu_j\) in Lemma 3.5 extend line by line to the corresponding time-dependent setting, we next record the \(t_r\)-dependent trace formulas without proof. For simplicity, we confine ourselves to the simplest one only.

**Lemma 5.5.** Assume \((2.2)\), suppose that \((5.3), (5.4)\) hold, and let \((x, t_r) \in \mathbb{R}^2\). Then,

\[ \mu(x, t_r) = \frac{1}{2} \sum_{j=0}^n \mu_j(x, t_r) - \frac{1}{2} \sum_{m=0}^{2n+1} E_m. \quad (5.65) \]
6. Time-dependent algebro-geometric solutions of HS2 hierarchy

In our final section, we extend the results of section 4 from the stationary HS2 hierarchy, to the time-dependent case. We obtain Riemann theta function representations for the meromorphic function ϕ, and especially, for the algebro-geometric solutions u, ρ of the whole HS2 hierarchy.

We first record the asymptotic properties of ϕ in the time-dependent case.

Lemma 6.1. Assume (2.2) and suppose that (5.3), (5.4) hold. Moreover, let P = (z, y) ∈ ℜn \ \{P_m, P_0\}, (x, t_r) ∈ ℝ2. Then,

$$\phi(P) = -u_s(x, t_r) + O(\zeta), \quad P \to P_m, \quad \zeta = z^{-1}, \quad (6.1)$$

$$\phi(P) = iρ(x, t_r) + \frac{iu_s(x, t_r) - ρ_s(x, t_r)}{2ρ(x, t_r)} \zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z. \quad (6.2)$$

Since the proof of Lemma 6.1 is identical to the corresponding stationary results in Lemma 4.1, we omit the corresponding details.

Next, we investigate the properties of the Abel map. To do this, let $\bar{μ} = (μ_0, \ldots, μ_n) ∈ ℂ^{n+1},$ we define the following symmetric functions by

$$Ψ_{k+1}(\bar{μ}) = (-1)^{k+1} \sum_{l ∈ Σ_{k+1}} μ_1 \cdots μ_{k+1}, \quad k = 0, \ldots, n, \quad (6.3)$$

where $Σ_{k+1} = \{l = (l_1, \ldots, l_{k+1}) ∈ ℤ_{0}^{k+1} \mid l_1 < \cdots < l_{k+1} ≤ n\};$

$$Φ_{k+1}^{(j)}(\bar{μ}) = (-1)^{k+1} \sum_{l ∈ Σ_{k+1}^{(j)}} μ_1 \cdots μ_{k+1}, \quad k = 0, \ldots, n-1, \quad (6.4)$$

where $Σ_{k+1}^{(j)} = \{l = (l_1, \ldots, l_{k+1}) ∈ Σ_{k+1} \mid l_m ≠ j\}, \quad j = 0, \ldots, n.$ For the properties of $Ψ_{k+1}(\bar{μ})$ and $Φ_{k+1}^{(j)}(\bar{μ}),$ we refer to Appendix E [21].

Introducing

$$d_{r+1, k}(E) = \sum_{s=0}^{r+1-k} \tilde{c}_{r+1-k-s}E^s, \quad k = 0, \ldots, r + 1 \land n + 1, \quad (6.5)$$

for a given set of constants $\{\tilde{c}_t\}_{t=1}^{r+1} \subset ℂ,$ the corresponding homogeneous and nonhomogeneous quantities $\tilde{F}_r(μ_1)$ and $\tilde{F}_s(μ_1)$ in the HS2 case are then given by $^b$

$$\tilde{F}_r(μ_1) = \sum_{s=(r-n)\land 0}^{r+1} \tilde{c}_s(E)Φ_{r+1-s}^{(j)}(\bar{μ}), \quad (6.6)$$

$$\tilde{F}_r(μ_1) = \sum_{s=0}^{r+1} \tilde{c}_{r+1-s} \tilde{F}_s(μ_1) = \sum_{k=0}^{(r+1) \land (n+1)} d_{r+1,k}(E)Φ_k^{(j)}(\bar{μ}), \quad r ∈ ℤ_0, \quad \tilde{c}_0 = 1,$$

using (D.59) and (D.60) [21]. Here, $\tilde{c}_s(E), \ s ∈ ℤ_0,$ is defined by (D.2) [21].

$^b m \land n = \min\{m, n\}, \ m \lor n = \max\{m, n\}$
We now state the analog of Theorem 4.3, which indicates marked differences between the HS2 hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

**Theorem 6.2.** Assume (2.27) and suppose that \( \{ \mu_j \}_{j=0, \ldots, n} \) satisfies the Dubrovin equations (5.55), (5.56) on an open set \( \Omega_\mu \subseteq \mathbb{R}^2 \) such that \( \mu_j, \ j = 0, \ldots, n \), remain distinct and nonzero on \( \Omega_\mu \) and that \( \hat{F}_r(\mu_j) \neq 0 \) on \( \Omega_\mu, \ j = 0, \ldots, n \). Introducing the associated divisor \( \mathcal{D}_{\mu_j(x,t)} \), one computes,

\[
\partial_t a_{Q_0}(\mathcal{D}_{\mu_j(x,t)}\mathcal{Q}(x,t)) = -\frac{4a}{\Psi_{n+1}(\mu(x,t))} \zeta(1), \quad (x,t) \in \Omega_\mu, \tag{6.7}
\]

\[
\partial_t a_{Q_0}(\mathcal{D}_{\mu_j(x,t)}\mathcal{Q}(x,t)) = -\frac{4a}{\Psi_{n+1}(\mu(x,t))} \left( \sum_{k=0}^{(r+1)(n+1)} \hat{d}_{r+1,k}(E)\Psi_{k}(\mu(x,t)) \right) \zeta(1) + 4a \left( \sum_{\ell=0}^{n+1} \hat{d}_{r+1,n+2-\ell}(E) \zeta(\ell) \right), \quad (x,t) \in \Omega_\mu. \tag{6.8}
\]

In particular, the Abel map does not linearize the divisor \( \mathcal{D}_{\mu_j(x,t)}\mathcal{Q}(x,t) \) on \( \Omega_\mu \).

**Proof.** Let \( (x,t) \in \Omega_\mu \). It suffices to prove (6.8), since (6.7) is proved as in the stationary context of Theorem 4.3. We first recall a fundamental identity (E.10) [21], that is,

\[
\Phi_{k+1}^{(j)}(\mu) = \mu_j \Phi_k^{(j)}(\mu) + \Psi_{k+1}(\mu), \quad k = 0, \ldots, n, \ j = 0, \ldots, n. \tag{6.9}
\]

Then, applying (4.16), (6.6), and (6.9), one finds

\[
\tilde{F}_r(\mu_j) = \mu_j^{-1} \sum_{m=0}^{(r+1)(n+1)} \hat{d}_{r+1,m}(E)\Phi_m^{(j)}(\mu) \tag{6.10}
\]

\[
= \mu_j^{-1} \sum_{m=0}^{(r+1)(n+1)} \hat{d}_{r+1,m}(E) \left( \mu_j \Phi_m^{(j)}(\mu) + \Psi_m(\mu) \right) = \sum_{m=1}^{(r+1)(n+1)} \hat{d}_{r+1,m}(E)\Phi_m^{(j)}(\mu) - \sum_{m=0}^{(r+1)(n+1)} \hat{d}_{r+1,m}(E)\Psi_m(\mu) \frac{\Phi_m^{(j)}(\mu)}{\Psi_{n+1}(\mu)}. \]

Hence, using (5.56), (6.10), (E.4), (E.13), and (E.14) [21], one infers that

\[
\partial_t \left( \sum_{j=0}^{n} \int_{Q_0}^{\mu_j} \omega \right) = \sum_{j=0}^{n} \mu_j \partial_t \sum_{k=1}^{n} c(k) \frac{\alpha \mu_j^{k-1}}{y(\mu_j)}
\]

\[
= 4a \sum_{j=0}^{n} \sum_{k=1}^{n} c(k) \prod_{l=0}^{j-1} (\mu_j - \mu_l) \hat{F}_r(\mu_j) \mu_j^{k-1} \prod_{l=0}^{j} (\mu_j - \mu_l)
\]

\[
= 4a \sum_{j=0}^{n} \sum_{k=1}^{n} c(k) \frac{\mu_j^{k-1}}{\prod_{l=0}^{j-1} (\mu_j - \mu_l)} \times \left( - \sum_{m=0}^{(r+1)(n+1)} \hat{d}_{r+1,m}(E)\Psi_{n+1}(\mu) \frac{\Phi_m^{(j)}(\mu)}{\Psi_{n+1}(\mu)} + \sum_{m=1}^{(r+1)(n+1)} \hat{d}_{r+1,m}(E)\Phi_m^{(j)}(\mu) \right)
\]

\[
= -4a \sum_{m=0}^{(r+1)(n+1)} \hat{d}_{r+1,m}(E)\frac{\Psi_m(\mu)}{\Psi_{n+1}(\mu)} \sum_{k=1}^{n} c(k) (U_{n+1}(\mu))_{k,j} (U_{n+1}(\mu))^{-1}_{j,1}.
\]
The analogous results hold for the corresponding divisor $\mathcal{D}_{\Omega}(x,t_r)$ associated with $\phi(P,x,t_r)$.

Next, recalling the definition of $\hat{B}_{Q_0}$ and $\hat{\tilde{B}}_{Q_0}$ in (4.19) and (4.20), one obtains the following result.

**Corollary 6.3.** Assume (2.27) and suppose that $\{\hat{\mu}_j\}_{j=0,\ldots,n}$ satisfies the Dubrovin equations (5.55), (5.56) on an open set $\Omega_\mu \subseteq \mathbb{R}^2$ such that $\mu_j$, $j = 0,\ldots,n$, remain distinct and nonzero on $\Omega_\mu$ and that $\hat{F}_r(\mu_j) \neq 0$ on $\Omega_\mu$, $j = 0,\ldots,n$. Then, one computes

\[
\partial_s \sum_{j=0}^n \hat{B}_{\mu_j}(x,t_r) \Omega_\mu = -\frac{4a}{\Omega_\mu} \sum_{k=0}^{n+1} \delta_{r+1,k}(E) \sum_{k=0}^{n+1} \hat{d}_{r+1,k}(E) \sum_{m=0}^{r+1} \hat{d}_{r+1,n+1}(E) \delta_{n+1,n+1} \partial_s \hat{\tilde{B}}_{Q_0}(x,t_r) \Omega_\mu, \quad (x,t_r) \in \Omega_\mu.
\]

The fact that the Abel map does not effect a linearization of the divisor $\mathcal{D}_{\Omega}(x,t_r)$ in the time-dependent HS2 context, which is well known and discussed (using different approaches) by Constantin and McKeaen [12], Alber, Camassa, Fedorov, Holm, and Marsden [2], Alber and Fedorov [3,4]. The change of variables

\[
x \mapsto \hat{x} = \int x^\prime \left( \frac{4a}{\Omega_\mu} \right) \Omega_\mu(x^\prime,t_r)
\]

and

\[
t_r \mapsto \hat{t}_r = \int t_r \left( \frac{4a}{\Omega_\mu} \right) \sum_{k=0}^{n+1} \hat{d}_{r+1,k}(E) \delta_{n+1,n+1} \partial_s \hat{\tilde{B}}_{Q_0}(x,t_r) \Omega_\mu.
\]
linearizes the Abel map $\Delta_{Q_0}^{\bar{\rho}_0(\bar{x},t_0)}(\bar{\mu}(x,t_0))$, $\tilde{\mu}_j(x,t_0) = \mu_j(x,t_0)$, $j = 0, \ldots, n$. The intricate relation between the variables $(x,t_0)$ and $(\tilde{x},t_0)$ is detailed in (6.21). Our approach follows a route similar to Gesztesy and Holden’s treatment of the CH hierarchy [21].

Next, we shall provide the explicit representations of $\phi$, $u$, $r$ in terms of the Riemann theta function associated with $\mathcal{X}$, assuming the affine part of $\mathcal{H}$ to be nonsingular. Recalling (4.24)-(4.30), the analog of Theorem 4.5 in the stationary case then reads as follows.

**Theorem 6.4.** Assume (2.2) and suppose that (5.3), (5.4) hold on $\Omega$ subject to the constraint (2.27). In addition, let $P = (\zeta,y) \in \mathcal{X}$ \{0\} and $(x,t_0), (x_0,t_0,r) \in \Omega$, where $\Omega \subseteq \mathbb{R}^2$ is open and connected. Moreover, suppose that $\mathcal{D}_{\mathcal{H}(x,t_0)}$, or equivalently, $\mathcal{D}_{\mathcal{H}(x,t_0)}$ is nonspecial for $(x,t_0) \in \Omega$.

Then, $\phi$, $u$, and $r$ admit the representations

\[
\phi(P,x,t_0) = \mathcal{I}(x,t_0) \cdot \mathcal{I}(P_0,\tilde{\mu}(x,t_0)) \mathcal{I}(P_0,\tilde{\mu}(x,t_0)) \exp \left( \frac{1}{2} \sum_{j=1}^{n} \frac{1}{2} \sum_{j=1}^{n} U_j \partial_{\omega_j} \ln \left( \frac{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega}{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega} \right) \bigg|_{\omega = 0} \right),
\]

\[
u(x,t_0) = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \sum_{j=1}^{n} \sum_{j=1}^{n} U_j \partial_{\omega_j} \ln \left( \frac{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega}{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega} \right) \bigg|_{\omega = 0},
\]

\[
\rho(x,t_0) = i u(x,t_0) \cdot \frac{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega}{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega} \bigg|_{\omega = 0}.
\]

Moreover, let $\tilde{\Omega} \subseteq \Omega$ be such that $\mu_j$, $j = 0, \ldots, n$, are nonvanishing on $\tilde{\Omega}$. Then, the constraint

\[
4a(x-x_0) + 4a(t_r-t_0) \sum_{j=1}^{n} \left( \int_{a_j}^{\partial_{P_{x_0}}(x,t_0)} \frac{dx'}{\psi_{n+1}(\tilde{\mu}(x,t_0))} \right) c_j(1)
\]

\[
\times \left( -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \sum_{j=1}^{n} \sum_{j=1}^{n} U_j \partial_{\omega_j} \ln \left( \frac{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega}{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega} \right) \bigg|_{\omega = 0} \right) = 0
\]

\[
+ \sum_{r=0}^{n} \left( \int_{a_j}^{\partial_{P_{x_0}}(x,t_0)} \frac{dx'}{\psi_{n+1}(\tilde{\mu}(x,t_0))} \right) c_j(1)
\]

\[
\times \left( -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \sum_{j=1}^{n} \sum_{j=1}^{n} U_j \partial_{\omega_j} \ln \left( \frac{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega}{\theta(\tilde{\omega}(P_{x_0},\tilde{\mu}(x,t_0))) + \omega} \right) \bigg|_{\omega = 0} \right) = 0
\]

\[
\bigg( (x,t_0),(x_0,t_0,r) \bigg) \in \tilde{\Omega}
\]

holds, with

\[
\Delta_{Q_0}^{\bar{\rho}_0(\bar{x},t_0)}(\bar{\mu}(x,t_0)) = \Delta_{Q_0}^{\bar{\rho}_0(\bar{x},t_0)}(\bar{\mu}(x,t_0)) - 4a \left( \int_{x_0}^{x} \frac{dx'}{\psi_{n+1}(\tilde{\mu}(x',t_0))} \right) c(1)
\]

\[
= \Delta_{Q_0}^{\bar{\rho}_0(\bar{x},t_0)}(\bar{\mu}(x,t_0)) - 4a \left( \sum_{k=0}^{r+1} \left( \int_{a_j}^{\partial_{P_{x_0}}(x,t_0)} \frac{dx'}{\psi_{n+1}(\tilde{\mu}(x,t_0))} \right) c_j(1) \right)
\]

\[
+ 4a(t_r-t_0) \left( \sum_{r=0}^{n} \left( \int_{a_j}^{\partial_{P_{x_0}}(x,t_0)} \frac{dx'}{\psi_{n+1}(\tilde{\mu}(x,t_0))} \right) c_j(1) \right),
\]
We first assume that $\mu_j$, $j = 0, \ldots, n$, are distinct and nonvanishing on $\tilde{\Omega}$ and $\tilde{F}_r(\mu_j) \neq 0$ on $\tilde{\Omega}$, $j = 0, \ldots, n$, where $\tilde{\Omega} \subseteq \Omega$. Then, the representation (6.18) for $\phi$ on $\tilde{\Omega}$ follows by combining (5.28), (6.1), (6.2), and Theorem A.26 [21]. The representation (6.19) for $u$ on $\tilde{\Omega}$ follows from the trace formulas (5.65) and (F.59) [21]. The representation (6.20) for $\rho$ on $\tilde{\Omega}$ is clear from (6.18) and (6.1). In fact, since the proofs of (6.18), (6.19), and (6.20) are identical to the corresponding stationary results in Theorem 4.5, which can be extended line by line to the time-dependent setting, here we omit the corresponding details. By continuity, (6.18), (6.19), and (6.20) extend from $\tilde{\Omega}$ to $\Omega$. The constraint (6.21) then holds on $\tilde{\Omega}$ by combining (6.12)-(6.15) and (F.58) [21]. Equations (6.22) and (6.23) are clear from (6.7) and (6.8). Again by continuity, (6.21)-(6.23) extend from $\tilde{\Omega}$ to $\Omega$. \hfill \Box

Remark 6.5. One observes that (6.22) and (6.23) are equivalent to

\begin{align*}
\hat{\alpha}_{Q_0} (\mathcal{D}_{\mu_0(x,t)} \hat{u}(x,t)) = & \hat{\alpha}_{Q_0} (\mathcal{D}_{\mu_0(x_0,t)} \hat{u}(x_0,t)) - \zeta(1) (\hat{x}_0 - \hat{x}) \\
= & \hat{\alpha}_{Q_0} (\mathcal{D}_{\mu_0(x_0,t)} \hat{u}(x_0,t)) - \zeta(1) (\hat{t}_0 - \hat{t}),
\end{align*}

(6.24)

and

\begin{align*}
\hat{\alpha}_{Q_0} (\mathcal{D}_{\mu_0(x_0,t)} \hat{u}(x_0,t)) = & \hat{\alpha}_{Q_0} (\mathcal{D}_{\mu_0(x_0,t)} \hat{u}(x_0,t)) - \zeta(1) (\hat{t}_0 - \hat{t})_0, \\
= & \hat{\alpha}_{Q_0} (\mathcal{D}_{\mu_0(x_0,t)} \hat{u}(x_0,t)) - \zeta(1) (\hat{t}_0 - \hat{t}_0),
\end{align*}

(6.25)

under the change of variables $x \mapsto \hat{x}$ and $t_r \mapsto \hat{t}_r$ in (6.16) and (6.17). Hence, the Abel map linearizes the divisor $\mathcal{D}_{\mu_0(x,t)} \hat{u}(x,t)$ on $\Omega$ with respect to $\hat{x}, \hat{t}$.

Remark 6.6. Remark 4.7 applies in the present time-dependent context. Moreover, to obtain the theta function representation of $\psi_j$, $j = 1, 2$, one can write $\tilde{F}$ in terms of $\Psi_k(\hat{\mu})$ and use (5.46), in analogy to the stationary case discussed in Remark 4.8. Here we omit further details.

At the end of this section, we turn to the time-dependent algebro-geometric initial value problem of HS2 hierarchy. We will show that the solvability of the Dubrovin equations (5.55) and (5.56) on $\Omega_\mu \subseteq \mathbb{R}^2$ in fact implies equations (5.3) and (5.4) on $\Omega_\mu$.

Theorem 6.7. Fix $n \in \mathbb{N}$, assume (2.27), and suppose that $\{\mu_j\}_{j=0}^{n}$ satisfies the Dubrovin equations (5.55), (5.56) on an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$, with $\tilde{F}_r(\mu_j)$ in (5.56) expressed in terms of $\mu_k$, $k = 0, \ldots, n$, by (6.6). Moreover, assume that $\mu_j$, $j = 0, \ldots, n$, remain distinct and nonzero on $\Omega_\mu$. Then, $u, \rho \in C^\infty(\Omega_\mu)$, defined by

\begin{align*}
u = & -\frac{1}{2} \sum_{m=0}^{n+1} E_m + \frac{1}{2} \sum_{j=0}^{n} \mu_j, \\
(6.26)
\end{align*}

and

\begin{align*}
\rho^2 = & u_x^2 + 2uu_{xx} - \frac{d^2}{dx^2} \left( \Psi_2(\mu) - u \sum_{m=0}^{n+1} E_m \right), \\
(6.27)
\end{align*}

satisfy the rth HS2 equation (5.1), that is,

\begin{align*}
\text{HS2}_r(u, \rho) = & 0 \quad \text{on } \Omega_\mu, \\
(6.28)
\end{align*}

with initial values satisfying the nth stationary HS2 equation (5.2).
Algebro-geometric solutions for the HS2 hierarchy

Proof. Given the solutions $\tilde{\mu}_j = (\mu_j, y(\tilde{\mu}_j)) \in C^\infty(\Omega_{\mu}, \mathcal{X}_{\mu})$, $j = 0, \ldots, n$ of (5.55) and (5.56), we define polynomials $F_n$, $G_n$, and $H_n$ on $\Omega_{\mu}$ as in the stationary case (cf. Theorem 4.9) with properties

\begin{align}
F_n(z) &= \frac{1}{2} \prod_{j=0}^{n} (z - \mu_j), \quad (6.29) \\
G_n(z) &= \frac{1}{2} F_{n,x}(z), \quad (6.30) \\
z^2 G_{n,x}(z) &= -H_n(z) - (\rho^2 + u_{xx}z)F_n(z), \quad (6.31) \\
H_n(z) &= 2(\rho^2 + u_{xx}z)G_n(z), \quad (6.32) \\
R_{2n+2}(z) &= z^2 G_n(z) + F_n(z)H_n(z), \quad (6.33)
\end{align}

treating $t_r$ as a parameter. Define the polynomials $\tilde{G}_r$ and $\tilde{H}_r$ by

\begin{align}
\tilde{G}_r(z) &= \frac{1}{2} \tilde{F}_{r,x}(z) \quad \text{on } \mathbb{C} \times \Omega_{\mu}, \quad (6.34) \\
\tilde{H}_r(z) &= -z^2 \tilde{G}_{r,x}(z) - (\rho^2 + u_{xx}z)\tilde{F}_r(z) \quad \text{on } \mathbb{C} \times \Omega_{\mu}, \quad (6.35)
\end{align}

respectively. Next, we claim that

\begin{equation}
F_{n,t_r}(z) = 2(G_n(z)\tilde{F}_r(z) - F_n(z)\tilde{G}_r(z)) \quad \text{on } \mathbb{C} \times \Omega_{\mu}. \quad (6.36)
\end{equation}

To prove (6.36), one computes from (5.55) and (5.56) that

\begin{align}
F_{n,x}(z) &= -F_n(z) \sum_{j=0}^{n} \mu_{j,x}(z - \mu_j)^{-1}, \quad (6.37) \\
F_{n,t_r}(z) &= -F_n(z) \sum_{j=0}^{n} \tilde{F}_r(\mu_j) \mu_{j,x}(z - \mu_j)^{-1}. \quad (6.38)
\end{align}

Using (6.30) and (6.34), one concludes that (6.36) is equivalent to

\begin{equation}
\tilde{F}_{r,x}(z) = \sum_{j=0}^{n} (\tilde{F}_r(\mu_j) - \tilde{F}_r(z)) \mu_{j,x}(z - \mu_j)^{-1}. \quad (6.39)
\end{equation}

Equation (6.39) is proved in Lemma F.9 [21]. This in turn proves (6.36).

Next, differentiating (6.30) with respect to $t_r$ yields

\begin{equation}
F_{n,x,t_r} = 2G_{n,t_r}. \quad (6.40)
\end{equation}

On the other hand, taking the derivative of (6.36) with respect to $x$, and using (6.30), (6.31), (6.34), one obtains

\begin{align}
F_{n,t_r,x} &= -2z^{-2} H_n \tilde{F}_r - 2z^{-2} \rho^2 + u_{xx}z^{-1} \tilde{F}_n \tilde{F}_r + 2G_n \tilde{F}_{r,x} \\
&\quad - 2G_{r,x}F_n - 4G_tG_n. \quad (6.41)
\end{align}
Combining (6.30), (6.34), (6.40), and (6.41), one concludes
\[ z^2 G_{n,t}(z) = \tilde{H}_t(z)F_n(z) - H_n(z)\tilde{F}_t(z) \quad \text{on } \mathbb{C} \times \Omega_\mu. \]  
(6.42)

Next, differentiating (6.33) with respect to \( t_r \), and using expressions (6.36) and (6.42) for \( F_{n,t_r} \) and \( G_{n,t_r} \), respectively, one obtains
\[ H_{n,t_r}(z) = 2(H_n(z)\tilde{G}_r(z) - G_n(z)\tilde{H}_r(z)) \quad \text{on } \mathbb{C} \times \Omega_\mu. \]  
(6.43)

Finally, taking the derivative of (6.42) with respect to \( x \), and using expressions (6.30), (6.32), and (6.34) for \( F_{n,x} \), \( H_{n,x} \), and \( \tilde{F}_{r,x} \), respectively, one infers that
\[ z^2 G_{n,t,x} = F_n\tilde{H}_{t,x} + 2G_n\tilde{H}_r - 2(\rho^2 + u_{xt}z)G_n\tilde{F}_r - 2H_n\tilde{G}_r. \]  
(6.44)

On the other hand, differentiating (6.31) with respect to \( t_r \), using (6.36) and (6.43) for \( F_{n,t_r} \) and \( H_{n,t_r} \), respectively, leads to
\[ z^2 G_{n,x,t} = 2G_n\tilde{H}_r - 2H_n\tilde{G}_r - 2(\rho \rho_n + u_{xt}z)F_n - 2(\rho^2 + u_{xx}z)(G_n\tilde{F}_r - \tilde{G}_r F_n). \]  
(6.45)

Combining (6.44) and (6.45) then yields
\[ -2\rho \rho_n - u_{xt}z - \tilde{H}_{r,x} + 2(\rho^2 + u_{xx}z)\tilde{G}_r = 0. \]  
(6.46)

Thus, we have proved (5.12)-(5.17) and (5.38)-(5.40) on \( \mathbb{C} \times \Omega_\mu \) and hence conclude that (6.28) holds on \( \mathbb{C} \times \Omega_\mu \).

**Remark 6.8.** Again we formulated Theorem 6.7 in terms of \( \{\mu_j\}_{j=0,...,n} \) only. Obviously, the analogous result (and strategy proof) works in terms of \( \{v_j\}_{j=0,...,n} \).

The analog of Remark 4.11 directly extends to the current time-dependent setting.

**Acknowledgments**

The authors would like to express their sincerest thanks to referee and editor for constructive suggestions and helpful comments. The work described in this paper was supported by grants from the National Science Foundation of China (Project No. 11271079; 11075055) and Doctoral Programs Foundation of the Ministry of Education of China.

**References**

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, The inverse scattering transform–Fourier analysis for nonlinear problems, *Stud. Appl. Math.* 53 (1974) 249–315.

[2] M. S. Alber, R. Camassa, Y. N. Fedorov, D. D. Holm, and J. E. Marsden, The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDEs of shallow water and Dym type, *Commun. Math. Phys.* 221 (2001) 197–227.

[3] M. S. Alber, and Y. N. Fedorov, Wave solutions of evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians, *J. Phys. A.* 33 (2000) 8409–8425.

[4] M. S. Alber, and Y. N. Fedorov, Algebraic geometrical solutions for certain evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians, *Inverse Problems.* 17 (2001) 1017–1042.

[5] R. Beals, D. Sattinger, and J. Szmigielski, Inverse scattering solutions of the Hunter-Saxton equations, *Appl. Anal.* 78 (2001) 255–269.
Algebro-geometric solutions for the HS2 hierarchy

[6] E. D. Belokolos, A. I. Bobenko, V. Z. Enol’skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations.*, (Springer, Berlin, 1994).

[7] R. Camassa, and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993) 1661–1664.

[8] M. Chen, S. Q. Liu, and Y. Zhang, A two-component generalization of the Camassa-Holm equation and its solutions, *Lett. Math. Phys.* **75** (2006) 1–15.

[9] A. Constantin, and R. I. Ivanov, On an integrable two-component Camassa-Holm shallow water system, *Phys. Lett. A* **372** (2008) 7129–7132.

[10] A. Constantin and B. Kolev, On the geometric approach to the motion of inertial mechanical systems, *J. Phys. A* **35** (2002) R51–R79.

[11] A. Constantin, and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Ration. Mech. Anal.* **192** (2009) 165–186.

[12] A. Constantin, and H. P. McKean, A shallow water equation on the circle, *Commun. Pure Appl. Math.* **52** (1999) 949–982.

[13] H. H. Dai, and M. Pavlov, Transformations for the Camassa-Holm equation, its high-frequency limit and the Sinh-Gordon equation, *J. Phys. Soc. Japan*, **67** (1998) 3655–3657.

[14] B. A. Dubrovin, Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties, *Funct. Anal. Appl.* **11** (1977) 265–277.

[15] B. A. Dubrovin, Theta functions and nonlinear equations, *Russian Math. Surv.* **36** (1981) 11–92.

[16] B. A. Dubrovin, Matrix finite-zone operators, *Rev. Sci. Technol.* **23** (1983) 20–50.

[17] J. Escher, O. Lechtenfeld and Z. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation, *Discrete Contin. Dyn. Syst.* **19** (2007) 493–513.

[18] A. S. Fokas, and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Phys. D* **4** (1981) 47–66.

[19] F. Gesztesy, and H. Holden, Algebro-geometric solutions of the Camassa-Holm hierarchy, *Rev. Mat. Iberoam.* **19** (2003) 73–142.

[20] F. Gesztesy, and H. Holden, Real-valued algebro-geometric solutions of the Camassa-Holm hierarchy, *Philos. Trans. R. Soc. Lond. Ser. A.* **366** (2008) 1025–1054.

[21] F. Gesztesy, and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions. vol. I: (1+1)-Dimensional Continuous Models.* (Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2003).

[22] F. Gesztesy, and R. Ratneseelan, An alternative approach to algebro-geometric solutions of the AKNS hierarchy, *Rev. Math. Phys.* **10** (1998) 345–391.

[23] C. Guan and Z. Yin, Global weak solutions and smooth solutions for a two-component Hunter-Saxton system, *J. Math. Phys.* **52** (2011) 103707.

[24] C. Guan and Z. Yin, Global weak solutions for a periodic two-component Hunter-Saxton system, *Q. Appl. math.* **62** (2012) 285–297.

[25] G. L. Gui, and Y. Liu, On the global existence and wave-breaking criteria for the two-component Camassa-Holm system, *J. Funct. Anal.* **258** (2010) 4251–4278.

[26] D. D. Holm, and L. O’Náraigh and C. Tronci, Singular solutions of a modified two-component Camassa-Holm equation, preprint, (2012) arXiv: 1205.6060.

[27] Y. Hou, E. G. Fan, and Z. J. Qiao, The algebro-geometric solutions for the modified Camassa-Holm hierarchy, (2012) arXiv: 1205.6060.

[28] Y. Hou, E. G. Fan, and P. Zhao, Algebro-geometric solutions for the Hunter-Saxton hierarchy, *Z. Angew. Math. Phys.* (2013) Springer Basel DOI 10.1007/s00033-013-0339-8.

[29] Y. Hou, P. Zhao, E. G. Fan, and Z. J. Qiao, Algebro-geometric solutions for Degasperis-Procesi hierarchy, *SIAM J. Math. Anal.* **45** (2013) 1216–1266.

[30] J. K. Hunter, and R. Saxton, Dynamics of director fields, *SIAM J. Appl. Math.* **51** (1991) 1498–1521.

[31] J. K. Hunter, and Y. X. Zheng, On a completely integrable nonlinear hyperbolic variational equation, *Phys. D* **79** (1994) 361–386.

[32] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **457** (2002) 63–82.
Yu Hou and Engui Fan

[33] B. Khesin, and G. Misiołek, Euler equations on homogeneous spaces and Virasoro orbits, Adv. Math. 176 (2003) 116–144.

[34] J. Lenells, Weak geodesic flow and global solutions of the Hunter-Saxton equation, Discret. Contin. Dyn. Syst. 18 (2007) 643–656.

[35] J. Lenells, The Hunter-Saxton equation describes the geodesic flow on a sphere, J. Geom. Phys. 57 (2007) 2049–2064.

[36] J. Liu and Z. Yin, Blow-up phenomena and global existence for a periodic Hunter-Saxton system, Preprint, (2011) arXiv:1012.5448v3.

[37] V. B. Matveev, and M. I. Yavor, Solutions presque périodiques et à N-solitons de l’équation hydrodynamique non linéaire de Kaup. Ann. Inst. H. Poincaré Sect. A 31 (1979) 25–41.

[38] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, Theory of Solitons, the Inverse Scattering Methods. (Consultants Bureau, New York, 1984).

[39] P. Olver, and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, Phys. Rev. E 53 (1996) 1900–1906.

[40] M. V. Pavlov, The Gurevich-Zybin system, J. Phys. A 38 (2005) 3823–3840.

[41] E. G. Reyes, The soliton content of the Camassa-Holm and Hunter-Saxton equations. In: Nikitin, A.G., Boyko, V.M., Popovych, R.O. (eds.), Proceedings of the Fourth International Conference on Symmetry in Nonlinear Mathematical Physics, Proceedings of the Institute of Mathematics of the NAS of Ukraine, vol. 43, pp. 201–208. Kyiv (2002)

[42] J. F. Song, and C. Z. Qu, Geometric integrability of two-component Camassa-Holm and Hunter-Saxton systems, Commun. Theor. Phys. 55 (2011) 955–959.

[43] H. Wu, and M. Wunsch, Global existence for the generalized two-component Hunter-Saxton system, J. Math. Fluid Mech. 14 (2012) 455–469.

[44] M. Wunsch, On the Hunter-Saxton system, Discret Contin. Dyn. Syst. Ser. B 12 (2009) 647–656.

[45] P. Zhang, and Y. Liu, Stability of solitary waves and wave-breaking phenomena for the two-component Camassa-Holm system, Int. Math. Res. Not. 2010 (2010) 1981–2021.