A STUDY OF COUSIN COMPLEXES THROUGH THE DUALIZING COMPLEXES

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Abstract. For the Cousin complex of certain modules, we investigate finiteness of cohomology modules, local duality property and injectivity of its terms. The existence of canonical modules of Noetherian non-local rings and the Cousin complexes of them with respect to the height filtration are discussed.

Introduction

This Paper is the continuation of [DT1] and [DT2]. We have seen in [DT2] that if $M$ is a finitely generated module over a local ring $A$ which possesses the fundamental dualizing complex $I^\bullet$, then, under certain conditions on $M$, $\text{Hom}_A(M, I^\bullet)$ represents the Cousin complex of the module $H^{\text{dim}A - \text{dim}A(M)}(\text{Hom}_A(M, I^\bullet))$, the $(\text{dim}A - \text{dim}A(M))$-th cohomology module of the complex $\text{Hom}_A(M, I^\bullet)$, with respect to an appropriate filtration of $\text{Spec}(A)$; and that we can reconstruct the Cousin complex of the module $M$ by means of the fundamental dualizing complex (see the proof of [DT2, Lemma 3.1]). In section 2, we pursue our expectation that the Cousin complexes of such modules will inherit some properties of the dualizing complex of the ring itself. We will show that if $(A, \mathfrak{m})$ is a Noetherian local ring (not necessarily possessing a dualizing complex) such that all of its formal fibres are Cohen-Macaulay rings, $M$ is a finitely generated $A$-module which satisfies the condition $(S_2)$ of Serre and $\text{Min}_A(\widehat{M}) = \text{Ass}_A(\widehat{M})$, then all the cohomology modules of $C_A(M)$, the Cousin complex of $M$ with respect to the $M$-height filtration, are finitely generated $A$-modules (a result proved also, under different hypotheses, by T. Kawasaki in [K, Theorem 1.1]), and also they satisfy a local duality property which is analogous to that of the Grothendieck local duality. Here, $\widehat{M}$ denotes the completion of $M$ with respect to the $\mathfrak{m}$-adic topology. We present, in section 3, a number of applications.

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which come out of these results and those of [DT1] and [DT2].

In the remainder of the paper we study the Cousin complex of certain modules over Noetherian non-local ring $R$. In section 4 we recall the notion of canonical modules for such a ring $R$ and prove the existence of them when $R$ possesses a dualizing complex and satisfies $(S_2)$. As a result we present a partial generalization of [BH, Proposition 3.3.18]. In section 5, we generalize [DT1, Corollary 3.4] for non-local case and give a characterization for the Cousin complex of a canonical module w.r.t. the height filtration to be a dualizing complex. Finally, we give an explicit description for all indecomposables injective modules which improves [DT1, Corollary 3.3].

1. Preliminaries

Throughout $A$ is a Noetherian local ring of dimension $d$ with the maximal ideal $m$, and $M$ is a finitely generated $A$–module of dimension $s$. A finitely generated $A$-module $K_M$ (if it exists) is called the canonical module of $M$ if $K_M \otimes_A \tilde{A} \cong \text{Hom}_A(H^s_{m}(M), E(A/m))$, where $H^s_{m}(M)$ is the $s$–th local cohomology module of $M$ w.r.t. $m$, and $E(A/p)$ is the injective envelope of the $A$–module $A/p$ with $p \in \text{Spec}(A)$. The canonical module of $M$ (if exists) is unique up to isomorphism (see [HK, Lemma 5.8]).

1.1. Some remarks. If $A$ possesses a dualizing complex, then it possesses the fundamental dualizing complex

$$I^\bullet : 0 \longrightarrow I^0 \overset{\delta^0}{\longrightarrow} I^1 \overset{\delta^1}{\longrightarrow} \cdots \overset{\delta^{d-1}}{\longrightarrow} I^d \longrightarrow 0,$$

which we call “ the dualizing complex ” (see [H]), with the following properties:

(i) for each $i \geq 0$, $H^i(I^\bullet)$, the $i$-th cohomology module of $I^\bullet$, is finitely generated.

(ii) $I^i = \bigoplus_{p \in \text{Spec}(A), \dim(A/p) = d-i} E(A/p)$, $i = 0, 1, \ldots, d$.

If $A$ possesses the dualizing complex $I^\bullet$, then the module $K_M := H^{d-s}(\text{Hom}_A(M, I^\bullet))$ is the canonical module of $M$. If $K_M$ is the canonical module of $M$, it is easy to see that $(\overline{K_M}) \cong K_M$ is the canonical module of $\tilde{M}$, as $\tilde{A}$-module. For the module $M$, we set $\text{Min}_A(M)$ to denote the set of all minimal elements of $\text{Supp}_A(M)$, and

$$\text{Assh}_A(M) = \{p \in \text{Supp}_A(M) : \dim_A(A/p) = \dim_A(M)\}.$$ 

Also $M$ is said to satisfy $(S_n)$ if $\text{depth}_{A_p}(M_p) \geq \min\{n, \text{ht}_M(p)\}$ for all $p \in \text{Supp}_A(M)$. A filtration of $\text{Spec}(A)$ is a descending sequence $F = (F_i)_{i \geq 0}$ of subsets of $\text{Spec}(A)$, so that,

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq \cdots,$$
with the property that, for each \( i \geq 0 \), each member of \( \partial F_i = F_i - F_{i+1} \) is a minimal member of \( F_i \), with respect to inclusion. We say that \( \mathcal{F} \) admits \( M \) if \( \text{Supp}_A(M) \subseteq F_0 \). Suppose that \( \mathcal{F} \) is a filtration of \( \text{Spec}(A) \) that admits \( M \). The Cousin complex \( C(\mathcal{F}, M) \) for \( M \) with respect to \( \mathcal{F} \) has the form

\[
0 \to M^{d-2} \to M^{d-1} \to M^0 \to M^1 \to \cdots \to M^{n-1} \to M^n \to \cdots
\]

with \( M^n = \bigoplus_{p \in \partial F_n} (\text{Coker} d^{n-2})_p \) for all \( n \geq 0 \), and with differentiation \( d^n \), as recalled in [T].

Set \( H_M = (H_i)_{i \geq 0} \) to be the \( M \)-height filtration of \( \text{Spec}(A) \), i.e. \( H_i = \{ p \in \text{Supp}_A(M) : \text{ht}_A(p) \geq i \} \). We denote the Cousin complex of \( M \) with respect to \( H_M \) by \( C_A(M) \).

Set \( \tilde{A} = A/0 : A \). Then \( M \) has a natural structure as \( \tilde{A} \)-module. It is straightforward to see that each term of the complex \( C_A(M) \) has a natural \( \tilde{A} \)-module structure and each differentiation of \( C_A(M) \) is an \( \tilde{A} \)-homomorphism. Moreover, it is straightforward to see that:

1.2. **Lemma.** If \( M \) is a finitely generated \( A \)-module and \( \tilde{A} := A/0 : A \), then there exists an isomorphism of complexes \( C_A(M) \cong C_A(M) \).

The following lemma will be used later.

1.3. **Lemma.** [P, Theorem 3.5] Suppose that all formal fibres of \( A \) are Cohen-Macaulay. If \( M \) is a finitely generated \( A \)-module, then there is a morphism of complexes \( u^* : C_A(M) \otimes_A \tilde{A} \to C_{\tilde{A}}(\tilde{M}) \) which is a monomorphism. Moreover the quotient complex \( Q^* \), in the exact sequence

\[
0 \to C_A(M) \otimes_A \tilde{A} \xrightarrow{u^*} C_{\tilde{A}}(\tilde{M}) \to Q^* \to 0,
\]

is an exact complex, so that, for each \( i \geq 0 \), there exists an \( \tilde{A} \)-isomorphism \( H^i(C_A(M)) \otimes_A \tilde{A} \cong H^i(C_{\tilde{A}}(\tilde{M})) \).

1.4. **Convention.** For a complex \( C^* : 0 \to C^{-1} \xrightarrow{\theta^{-1}} C^0 \xrightarrow{\theta^0} C^1 \xrightarrow{\theta^1} \cdots \) of \( A \)-module and \( A \)-homomorphisms, we denote \( C' : 0 \to C^0 \xrightarrow{\theta^0} C^1 \xrightarrow{\theta^1} \cdots \) and \( (C')^* : 0 \to H^0(C') \to C^0 \to C^1 \to \cdots \).

2. **Some properties of Cousin complexes**

In this section we establish some properties of certain complexes by means of dualizing complexes. First we show that these Cousin complexes have finitely generated cohomologies.
2.1. Theorem. Let \( A \) be a ring with Cohen-Macaulay formal fibres. Assume that \( M \) satisfies \((S_2)\) and \( \text{Min}_A(\widehat{M}) = \text{Assh}_A(\widehat{M}) \). Then \( C_A(M)' \) has finitely generated cohomology modules.

Proof. Since \( M \) satisfies \((S_2)\), the Cousin complex \( C_A(M) \) is exact at \( M \) and \( M^0 \) (see [SSc, Example 4.4]). Thus \( H^0(C_A(M)) = M \). So it is enough to prove that \( H^i(C_A(M))' \) is finitely generated for all \( i > 0 \). Note that, for \( i > 0 \), we have \( H^i(C_A(M))' = H^i(C_A(M)) \). By 1.3, we have \( H^i(C_A(M)) \otimes_A \widehat{A} \cong H^i(C_A(\widehat{M})) \). Therefore \( C_A(\widehat{M}) \) is also exact at \( \widehat{M} \) and \( (\widehat{M})^0 \); so that \( \widehat{M} \) satisfies \((S_2)\) as \( \widehat{A} \)-module. Since \( \text{Min}_A(\widehat{M}) = \text{Assh}_A(\widehat{M}) \), by [DT2, Theorem 3.2], all cohomology modules \( H^i(C_A(\widehat{M})) \) are finitely generated \( \widehat{A} \)-module. Now, by [M, Exercise 7.3], the claim follows. □

2.2. Corollary. Assume that the ring \( A \) satisfies \((S_2)\) and all formal fibres of \( A \) are Cohen-Macaulay. Then \( C_A(A) \), the Cousin complex of \( A \), has finitely generated cohomology modules.

Proof. By [M, Theorem 23.9], \( \widehat{A} \) satisfies \((S_2)\) and thus \( \text{Min}(\widehat{A}) = \text{Assh}(\widehat{A}) \) (see[DT1, Remark 1.3]). □

For a ring \( A \) and a property \( P \), the \( P \) locus of \( A \) is defined to be the set \( P(A) = \{ p \in \text{Spec}(A) : P \text{ holds for } A_p \} \). We show that the \((S_n)\) locus of any \((S_2)\) local ring with Cohen-Macaulay formal fibres is an open subset of \( \text{Spec}(A) \) for all \( n \geq 2 \).

2.3. Corollary. If \( A \) satisfies \((S_2)\) and all formal fibres of \( A \rightarrow A \) are Cohen-Macaulay, then for each \( n \geq 0 \), \( S_n(A) \) is an open subset of \( \text{Spec}(A) \), in the Zariski topology. In particular, \( CM(A) \) is an open subset of \( \text{Spec}(A) \).

Proof. It follows that \( \widehat{A} \) is \((S_2)\). We assume that \( n \geq 3 \). Set \( U_i = \text{Spec}(A - \text{Supp}_A(H^i(C_A(A)))) \), \( 1 \leq i \leq n - 2 \). Each \( U_i \) is an open subset of \( \text{Spec}A \), because \( \text{Supp}_A(H^i(C_A(A))) = V(0 :_A H^i(C_A(A))) \) by 2.2. Set \( W = \cap_{i=1}^{n-2} U_i \). We show that \( S_n(A) = W \). Let \( p \in S_n(A) \); so that \( A_p \) is \((S_n)\). Thus, by [SSc, Example 4.4], \( H^n(C_A(A_p)) = 0 \) for \( 1 \leq i \leq n - 2 \). Therefore, by [S1, Theorem 3.5], we have that \( p \in U_n \) for all \( i, 1 \leq i \leq n - 2 \); that is \( p \in W \). In a similar way, we have \( W \subseteq S_n(A). □ \)

Next, we state a local duality property for the Cousin complexes of certain modules.

2.4. Theorem. (Local duality for certain Cousin complexes). Assume that all formal fibres of \( A \) are Cohen-Macaulay, \( M \) satisfies \((S_2)\), and that \( \text{Min}_A(\widehat{M}) = \).
Assh \( \hat{A}(\hat{M}) \). Then, for each \( i \geq 0 \), \( D_{\hat{A}}H^i(C_{\hat{A}}(M)') \cong H_{\hat{m}}^{s-i}(K_{\hat{M}}) \), where \( D_\hat{A} := \text{Hom}_\hat{A}(-, E(\hat{A}/\hat{m})) \). Moreover, if \( M \) admits a canonical module, then the completion signs on the right hand side of the above isomorphism can be removed.

**Proof.** Set \( \tilde{A} = A/0 :_A M \) and \( \hat{\tilde{A}} = \hat{A}/0 :_{\hat{A}} \hat{M} \). It is straightforward to see that \( \tilde{A} \) and \( \hat{\tilde{A}} \) are isomorphic rings. Let \( J^\bullet \) be the dualizing complex for \( \tilde{A} \) and assume that \( I^\bullet = \text{Hom}_{\hat{\tilde{A}}}((\hat{\tilde{A}}, J^\bullet) \) such that \( I^0 = \text{Hom}_{\hat{\tilde{A}}}((\hat{\tilde{A}}, J^{d-s}) \). Hence \( I^\bullet \) is the dualizing complex for \( \hat{\tilde{A}} \). As seen in the proof of 2.1, \( \hat{M} \) satisfies \( (S_2) \) as \( \hat{A} \)-module. It is easy to see that \( \hat{M} \) also satisfies \( (S_2) \) as \( \hat{A} \)-module. Since \( \dim_{\hat{\tilde{A}}}((\hat{\tilde{A}}, \hat{M}) = \dim(\hat{\tilde{A}}) \), we have, by the proof of [DT2, Lemma 3.1], the isomorphism of complexes

\[
C_{\hat{\tilde{A}}}((\hat{\tilde{A}}, \hat{M})') \cong \text{Hom}_{\hat{\tilde{A}}}(K_{\hat{\tilde{M}}}, I^\bullet).
\]

Therefore, by 2.1 and [B-ZS, Corollary 2.5], we have

(1) \[ D_{\hat{\tilde{A}}}H^i(C_{\hat{\tilde{A}}}((\hat{\tilde{A}}, \hat{M})')) \cong H_{\hat{\tilde{m}}}^{s-i}(K_{\hat{\tilde{M}}}) \]

for all \( i \geq 0 \).

On the other hand each formal fibre of \( \tilde{A} \) is also a formal fibre of \( A \) and \( \hat{\tilde{A}} \cong \hat{\tilde{A}} \). Hence, from 1.3, we have

(2) \[ H^i(C_{\hat{\tilde{A}}}((\hat{\tilde{A}}, \hat{M})')) \cong H^i(C_{\hat{\tilde{A}}}((\hat{\tilde{A}}, \hat{M}))) \otimes_{\hat{\tilde{A}}} \hat{\tilde{A}}, \]

for all \( i > 0 \). From (1) and (2), we obtain

(3) \[ \text{Hom}_{\hat{\tilde{A}}}(H^i(C_{\hat{\tilde{A}}}((\hat{\tilde{A}}, \hat{M}))) \otimes_{\hat{\tilde{A}}} \hat{\tilde{A}}, \hat{\tilde{E}}(\hat{\tilde{A}}/\hat{\tilde{m}})) \cong H_{\hat{\tilde{m}}}^{s-i}(K_{\hat{\tilde{M}}}). \]

The left hand side of (3) is isomorphic to

\[ \text{Hom}_{\hat{\tilde{A}}}(H^i(C_{\hat{\tilde{A}}}(M)'), \hat{\tilde{E}}(\hat{\tilde{A}}/\hat{\tilde{m}})) \]

which, in turn, is isomorphic to \( \text{Hom}_{\hat{\tilde{A}}}(H^i(C_{\hat{\tilde{A}}}(M)'), \hat{\tilde{E}}(\hat{\tilde{A}}/\hat{\tilde{m}})) \). Thus, we have from (3), the isomorphism

(4) \[ \text{Hom}_{\hat{\tilde{A}}}(H^i(C_{\hat{\tilde{A}}}(M)'), \hat{\tilde{E}}(\hat{\tilde{A}}/\hat{\tilde{m}})) \cong H_{\hat{\tilde{m}}}^{s-i}(K_{\hat{\tilde{M}}}). \]

Assume that \( N \) is an \( A, \hat{\tilde{A}} \)-bimodule such that, for \( a \in A \) and \( x \in N \), \( ax = \hat{a}x \), where \( - : A \longrightarrow \hat{\tilde{A}} \) is the natural map. Then we have

\[
\text{Hom}_{\hat{\tilde{A}}}(N, \hat{\tilde{E}}(\hat{\tilde{A}}/\hat{\tilde{m}})) \cong \text{Hom}_{\hat{\tilde{A}}}(N, \text{Hom}_{\hat{\tilde{A}}}(\hat{\tilde{A}}, E(\hat{A}/\hat{m})))
\cong \text{Hom}_{\hat{\tilde{A}}}(N \otimes_{\hat{\tilde{A}}} \hat{\tilde{A}}, E(\hat{A}/\hat{m}))
\cong \text{Hom}_{\hat{\tilde{A}}}(N, E(\hat{A}/\hat{m})).
\]
By Independence Theorem for the local cohomologies, we have $H^s_{\hat{m}}(K_{\hat{M}}) \cong H^{s-i}_{\hat{m}}(K_{\hat{M}})$. Put all these together, we obtain, from (4) and 1.2, that
\[
\text{Hom}_A(H^i(C_A(M)'), E(A/\hat{m})) \cong H^s_{\hat{m}}(K_{\hat{M}}), \quad i = 0, 1, \ldots,
\]
as $A$ and $\hat{A}$–modules.

If $M$ admits a canonical module $K_M$, we then have $(K_{\hat{M}}) \cong K_{\hat{M}}$, and by the Artinianness of $H^s_{\hat{m}}(K_{\hat{M}})$, we get the final claim. □

3. Applications

First we show that over a local ring with Cohen–Macaulay formal fibres, certain $f$–modules are also generalized Cohen–Macaulay modules. Recall that $M$ is called generalized Cohen–Macaulay (abbr. g.CM) if there exists $r \geq 1$ such that, for each system of parameters $x_1, \ldots, x_s$ for $M$ and for all $i = 1, \ldots, s$,
\[
m^r[((x_1, \ldots, x_{i-1})M : x_i)/(x_1, \ldots, x_{i-1})M] = 0.
\]
Note that, by [ScTC, (3.2) and (3.3)], $M$ is a g.CM module if and only if $H^i_{m}(M)$ is of finite length for all $i = 0, 1, \ldots, s-1$.

An $A$–module $M$ is called an $f$–module if for each system of parameters $x_1, \ldots, x_s$ for $M$ 
\[
\text{Supp}_A[((x_1, \ldots, x_{i-1})M : x_i)/(x_1, \ldots, x_{i-1})M] \subseteq \{m\}
\]
for all $i = 1, \ldots, s$. It is clear that if $M$ is g.CM module then it is an $f$–module.

3.1. Theorem. (Compare [ScTC, (3.8)]). Assume that all formal fibres of $A$ are Cohen–Macaulay. Let $M$ be an $A$–module such that $\text{Min}_A(\hat{M}) = \text{Assh}_A(\hat{M})$. If $M$ is an $f$–module with depth$_A(M) \geq 2$, then $M$ is a g.CM module.

Proof. By a straightforward argument and using the equivalent definition of $f$–module [T, Lemma 1.2 (ii)], it can be shown that $M$ is $(S_2)$ and that $\text{Min}_A(M) = \text{Assh}_A(M)$.

Now, by [T, Lemma 1.2 (iv)], the $M$–height filtration of Spec$(A)$ is the same as the $M$–dimension filtration $\mathcal{D}$ of Spec$(A)$, where $\mathcal{D} = (D_i)_{i \geq 0}, D_i = \{p \in \text{Supp}_A(M) : \text{dim}(A/p) \leq s - i\}$. Thus, by [DT1, Lemma 3.1], there exists an isomorphism
\[
C_A(M) = C(\mathcal{D}, M) \cong C(\mathcal{U}, M)
\]
(over $\text{Id}_M$), where $C(\mathcal{U}, M)$ is the complex of modules of generalized fractions on $M$ with respect to the chain of triangular subsets $\mathcal{U} = (U_i)_{i \geq 1}$ on $A$, defined by
\[
U_i = \{(x_1, \cdots, x_i) \in A^i : \text{there exists } j \text{ with } 0 \leq j \leq i \text{ such that } x_1, \cdots, x_j \text{ is an s.s.o.p. for } M \text{ and } x_{j+1} = \cdots = x_i = 1\}
\]
(See [DT1] for details). By [SZ, Corollary 2.3 and Theorem 2.4],
\[ H^{i-1}(C_A(M)) \cong H^i_m(M), i = 1, \ldots, s - 1. \]

Therefore, by Theorem 2.1, \( H^i_m(M) \) is of finite length for all \( i = 0, 1, \ldots, s - 1 \). □

3.2. **Corollary.** Assume that all formal fibres of \( A \) are Cohen-Macaulay. If \( A \) is an \( f \)-ring with \( \text{depth}(A) \geq 2 \), then \( A \) is a \( g.CM \) ring.

**Proof.** As we have seen in the proof of 3.1, \( A \) is \( (S_2) \). By [M, Theorem 23.9], \( \hat{A} \) satisfies \( (S_2) \). Thus \( \text{Min}(\hat{A}) = \text{Assh}(\hat{A}) \) (see [AG, Lemma 1.1]). Now the result follows from Theorem 3.1. □

Our next application studies the injectivity of the terms of the Cousin complex \( C_A(M) \).

In [S2], a finitely generated \( A \)-module \( M \) is defined to be a Gorenstein \( A \)-module whenever its Cousin complex provides a minimal injective resolution. It is also proved that if \( A \) admits a canonical module \( \Omega \), then any Gorenstein \( A \)-module is isomorphic to the direct sum of a finite number of copies of \( \Omega \) [S3, Theorem 2.1].

It is known that if \( A \) does not have a canonical module and has a Gorenstein module, then it has a unique indecomposable Gorenstein module \( G \) and every Gorenstein \( A \)-module is isomorphic to a direct sum of a finite number of copies \( G \) (see [FFGR and S2]). Here we extend this result and show that for any finitely generated module \( M \), over a complete \( (S_2) \) local ring \( A \) which satisfies \( (S_2) \), if \( 0 :_A M = 0 \) and \( C_A(M)' \) is an injective complex, then \( M \) is isomorphic to a direct sum of copies of a uniquely determined indecomposable one.

3.3. **Theorem.** Let \( A \) satisfy \( (S_2) \) and suppose that it possesses a dualizing complex. Assume that \( M \) satisfies \( (S_2) \) and \( 0 :_A M = 0 \). The following statements are equivalent:

(i) \( C_A(M)' \) is an injective complex;

(ii) \( M \) is isomorphic to a direct sum of a finite number of copies of the canonical module \( K \) of the ring \( A \).

**Proof.** \( (i) \Rightarrow (ii) \). We do not need \( A \) to satisfy \( (S_2) \) in this part. The proof is a straightforward adaptation of the argument in [S3, Theorem 2.1(v)]. Let \( K \) denote the canonical module of \( A \).

Let

\[ I^\bullet : 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^d \rightarrow 0 \]
be the dualizing complex for \( A \) so that \( K = H^0(I^\bullet) \). By the proof of [DT2, Lemma 3.1], \( C_A(M) \cong \text{Hom}_A(K_M, I^\bullet)^* \), where \( K_M = \text{Hom}_A(M, K) \). Hence all cohomology modules of \( C_A(M) \) are finitely generated (see [S4, Lemma 3.4(ii)]). By [S5, Theorem], \( \text{Hom}_A(K_M, I^d) \cong H^d_m(M) \). As \( H^d_m(M) \) is an Artinian injective \( A \)-module, we may write \( H^d_m(M) \cong \oplus_{i=1}^n E(A/m) \), say. Using the Matlis functor \( \text{Hom}_A(\cdot, E(A/m)) \) and that \( I^d = E(A/m) \), we obtain \( K_M \otimes_A \hat{A} \cong (\oplus_{i=1}^n A) \otimes_A \hat{A} \). This implies, by [HK, Lemma 5.8], that \( K_M \cong \oplus_{i=1}^n A \). Hence we have \( H^d_m(K_M) \cong \oplus_{i=1}^n H^d_m(A) \).

On the other hand, by Grothendieck’s Local Duality Theorem [B-ZS, Corollary 2.5] and the fact that \( M \) satisfies \( (S_2) \) so \( C_A(M) \) is exact at point \(-1, 0 \) (see [SSc, Example 4.4]), we obtain

\[
H^d_m(K_M) \cong \text{Hom}_A(H^0(C_A(M)'), E(A/m)) \cong \text{Hom}_A(M, E(A/m)).
\]

By applying the Matlis functor again, we get \( M \otimes_A \hat{A} \cong \text{Hom}_A(H^d_m(K_M), E(A/m)) \cong \text{Hom}_A(\oplus_{i=1}^n H^d_m(A), E(A/m)) \cong (\oplus_{i=1}^n K) \otimes_A \hat{A} \). Now, by [HK, Lemma 5.8], \( M \cong \oplus_{i=1}^n K \).

(i) \ ;-) (ii). We have \( \text{Supp}_A(M) = \text{Supp}_A(K) = \text{Spec}(A) \) (see [A, (1.8)] and [AG, Lemma 1.1]). It is routine to check that \( C_A(M) \cong \oplus_{i=1}^n C_A(K) \). As \( \text{Min}A = \text{Assh}A \) and the dimension filtration and the height filtration of \( \text{Spec}(A) \) are the same (see [A, (1.9)]), the claim follows by [DT1, Corollary 3.4]. \( \square \)

3.4. Corollary. Assume that \( \hat{A} \) satisfies \( (S_2) \). Then the following statements are equivalent:

(i) \( C_\hat{A}(\hat{A})' \) is an injective complex of \( \hat{A} \)-modules;

(ii) \( A \) is the canonical module of \( A \).

Moreover, if \( A \) satisfies one of the above equivalent conditions, then \( A \) is Gorenstein if and only if \( \hat{A} \) satisfies \( (S_n) \) for some \( n \geq (1/2)\text{dim}A + 1 \).

Proof. (i) \ ;-) (ii). Set \( \Omega \) for the canonical module of \( \hat{A} \). By 3.3, \( \hat{A} \cong \Omega^n \) for some \( n \). Thus \( H^d_m(\hat{A}) \cong \oplus_{i=1}^n H^d_m(\Omega) \cong \oplus_{i=1}^n E(\hat{A}/\hat{m}) \) and, by applying \( \text{Hom}_\hat{A}(\cdot, E(\hat{A}/\hat{m})) \), we get \( \Omega \cong \hat{A}^n \). Thus \( \hat{A}^{n^2} = \hat{A} \), which implies \( n = 1 \) and so \( A \) is the canonical module of \( A \).

(ii) \ ;-) (i). As \( \hat{A} \) is the canonical module of \( \hat{A} \) and \( \hat{A} \) satisfies \( (S_2) \), \( C_\hat{A}(\hat{A})' \) is the dualizing complex of \( \hat{A} \) [DT1, Corollary 3.4].

For the last part, we may assume that \( A \) is complete. By [SSc, Example 4.4], \( C_A(A) \) is exact at points \(-1, 0, 1, \ldots, n - 2 \), from which it follows, by Theorem 2.4, that \( H^d_m(A) = 0 \) for \( 0 < i \leq n - 2 \). On the other hand, as \( A \) satisfies \( (S_n) \), \( H^d_m(A) = 0 \) for all \( i < \min\{d, n\} \). As \( \text{dim} A - (n - 2) \leq n \), it follows that \( H^i_m(A) = 0 \)
for all \( i < d \), which imply the exactness of \( C_A(A) \). The other side is trivial. □

4. Canonical modules of non–local rings

Recall that, for a Noetherian (not necessarily local) ring \( R \), the canonical module of \( R \) (if it exists) is a finite \( R \)– module \( K \) such that \( K_m \), the localization of \( K \) at any maximal ideal \( m \) of \( R \), is the canonical module of \( R_m \). In order to generalize our results to the non–local case one might ask whether a canonical module exists even when \( R \) possesses a dualizing complex. We will show that, if \( R \) satisfies \((S_2)\) and all formal fibres of \( R_m \), for any maximal ideal \( m \) of \( R \), are Cohen-Macaulay, then existence of a canonical module for \( R \) is equivalent to the statement that \( R \) possesses a dualizing complex.

Throughout, \( R \) is a Noetherian ring of finite dimension which is not necessarily local.

Assume that \( R \) possesses a dualizing complex \( I^{\bullet} \) and \( t(p; I^{\bullet}), p \in \text{Spec} R \), denotes the unique integer \( i \) for which \( p \) occurs in \( I^i \) (see [H, page 23]).

4.1. Proposition. Assume that \( R \) satisfies \((S_2)\) and that it possesses a dualizing complex \( I^{\bullet} \). If \( p, q \in \text{Min}(R) \) such that \( p \subseteq r \) and \( q \subseteq r \) for some \( r \in \text{Spec}(R) \), then \( t(p; I^{\bullet}) = t(q; I^{\bullet}) \).

Proof. We may assume that \( R \) is a local ring and that its maximal ideal is \( r \). As \( R \) satisfies \((S_2)\) and possesses a dualizing complex, then \( \text{Min}(R) = \text{Assh}(R) \) [A; 1.1]. Therefore \( t(p; I^{\bullet}) = t(q; I^{\bullet}) \). □

4.2. Notation. Assume that \( R \) satisfies \((S_2)\) and that

\[
I^{\bullet} : 0 \rightarrow I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{i-1}} I^I \rightarrow 0
\]

is a dualizing complex for \( R \). It follows that \( \text{Ass}_R(I^0) \subseteq \text{Min} R \). Assume that \( \text{Ass}_R(I^0) \neq \text{Min} R \). Let \( r \) be the greatest integer such that \( X := \text{Min}(R) \cap \text{Ass}_R(I^r) \neq \emptyset \). Set, for each \( i \geq 0 \),

\[
X_i = \{ p \in \text{Ass}_R(I^i) : p \text{ contains some element of } X \};
X'_i = \text{Ass}_R(I^i) \setminus X_i;
I_i^1 = \bigoplus_{p \in X_i} E(A/p), I_i^2 = \bigoplus_{p \in X'_i} E(A/p),
\]

so that we have \( I^i = I^i_1 \oplus I^i_2 \).
4.3. Proposition. With the notations as in 4.2,

\[ \text{Hom}_R(I_1^i, I_2^{-1}) = 0 = \text{Hom}_R(I_2^i, I_1^i). \]

Proof. If \( \text{Hom}_R(I_1^i, I_2^{-1}) \neq 0 \), then \( \text{Hom}_R(I_1^i, E(R/p)) \neq 0 \) for some \( p \in X_i^i \). Assume that \( f : I_1^i \rightarrow E(R/p) \) is an \( R \)-homomorphism and that \( x \in I_1^i \). Let \( x = x_1 + \cdots + x_s \), where \( x_j \in E(R/p_j), 1 \leq j \leq s \). By definition of \( X_i^i \), we have \( p_1 \cap \cdots \cap p_s \not\subseteq p \). Take \( t \in p_1 \cap \cdots \cap p_s \setminus p \). Hence \( t^m x = 0 \) for some positive integer \( m \). On the other hand the map \( E(R/p) \xrightarrow{t^m} E(R/p) \) is an isomorphism. Thus \( t^m f(x) = f(t^m x) = 0 \) implies that \( f(x) = 0 \).

To show that \( \text{Hom}_R(I_2^i, I_1^{-1}) = 0 \), we may assume, on the contrary, that \( \text{Hom}_R(I_2^i, E(R/p)) \neq 0 \) for some \( p \in X_i^i \). So we may assume that \( p' \subseteq p \), for some \( p' \in X_i^i \). By localizing at \( p \), we get \( \text{Min}(R_p) = \text{Assh}(R_p) \), because \( R_p \) satisfies \( (S_2) \) and \( R_p \) possesses a dualizing complex. As \( p'R_p \) contains a minimal element \( q \in X. \) This contradicts with the definition of \( X_i^i \). \( \Box \)

4.4. Theorem. Assume that \( R \) satisfies \( (S_2) \) and that it possesses a dualizing complex. Then \( R \) possesses a dualizing complex

\[ J^\bullet : 0 \rightarrow J^0 \rightarrow J^1 \rightarrow \cdots \rightarrow J^d \rightarrow 0, d = \dim R, \]

such that \( \text{Ass}_R(J^0) = \text{Min}(R) \). In particular \( R \) admits a canonical module.

Proof. The proof is influenced by [H, Lemma 3.1]. Suppose that

\[ I^\bullet : 0 \rightarrow I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{d-1}} I^d \rightarrow 0 \]

is the dualizing complex for \( R \). Assume further that \( \text{Ass}(I^0) \neq \text{Min}(R) \) and that \( r \) is the greatest integer with \( X := \text{Min}(R) \cap \text{Ass}_R(I^r) \neq 0 \). We set \( X_i^i, I_1^i, I_2^i \) as in 4.2. Note that \( I_1^i = 0 \) and \( I_2^i = I_i \) for \( 0 \leq i < r \) (see [S6, Lemma 3.3]).

We construct a dualizing complex

\[ J^\bullet : 0 \rightarrow J^0 \xrightarrow{\eta^0} J^1 \xrightarrow{\eta^1} \cdots \]

as follows. Set \( J^i = I_1^{i+r} \oplus I_2^i \) for all \( i \geq 0 \), and define \( \eta^i : J^i \rightarrow J^{i+1} \)

by \( \eta^i(x + y) = \delta_1^{i+r}(x) + \delta_2^i(y) \) for \( x \in I_1^{i+r}, y \in I_2^i \), where \( \delta_1^i := \delta_j^i |_{I_1^i} \) and \( \delta_2^j := \delta_j^i |_{I_2^i}, j \geq 0 \). It follows from Proposition 4.3 that \( J^\bullet \) is a complex. To show that \( H^i(J^\bullet) \) is a finitely generated \( R \)-module for all \( i \geq 0 \), we note that, by a straightforward argument, there are two natural isomorphisms

\[ H^i(I^\bullet) \cong (\text{Ker}\delta_1^i/\text{Im}\delta_1^{i-1}) \oplus (\text{Ker}\delta_2^i/\text{Im}\delta_2^{i-1}), \]

\[ H^i(J^\bullet) \cong (\text{Ker}\delta_1^{i+r}/\text{Im}\delta_1^{i+r-1}) \oplus (\text{Ker}\delta_2^i/\text{Im}\delta_2^i), i \geq 0. \]
Therefore $J^\bullet$ is a dualizing complex for $A$. Now we have $J^0 = I^r_1 \oplus I^0$ and thus $\text{Ass}_R(I^0) \subsetneq \text{Ass}_R(J^0)$. So after a finite number of steps we are finished.

Finally, let $J^\bullet$ be a dualizing complex with $\text{Min}(R) = \text{Ass}_R(J^0)$. For each $m \in \text{Max}(R)$, the complex

$$0 \longrightarrow (J^0)_m \longrightarrow (J^1)_m \longrightarrow \cdots \longrightarrow (J^{t(m;J^\bullet)})_m \longrightarrow 0$$

is the dualizing complex for $R_m$, so that, by Grothendieck’s Local Duality Theorem [B-ZS, Corollary 2.5], $(H^0(J^\bullet))_m$ is the canonical module of $R_m$. Thus $H^0(J^\bullet)$ is a canonical module of $R$. □

As an application, we can give a partial generalization of [BH, Proposition 3.3.18].

4.5. Theorem. Assume that $R$ satisfies $(S_2)$ and possesses a dualizing complex

$$I^\bullet : 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^d \longrightarrow 0,$$

with $d = \dim R$, $I^i = \bigoplus_{ht_p = i} E(R/p)$, $i = 0, 1, \cdots$ and that $K_R$ is a canonical module of $R$.

(a) The following conditions are equivalent:

(i) $K_R$ has a rank;

(ii) $\text{rank} K_R = 1$;

(iii) $R$ is generically Gorenstein (that is $R_p$ is a Gorenstein ring for all minimal prime ideals $p$ of $R$).

(b) If $K_R$ satisfies $(S_3)$ and the equivalent conditions of (a) hold, then $K_R$ can be identified with an ideal of height 1 or equals $R$. In the first case $R/K_R$, is an $(S_2)$ ring with the canonical module $R/K_R$, the ring itself.

Proof. The proof is parallel to that of [BH, Proposition 3.3.18] and we present it for the convenience of the reader.

(a). (i)⇒(ii)⇒(iii). Set $Q$ to be the ring of total fractions of $R$ and let $K_R \otimes_R Q$ be a free $Q$–module of rank $r$, say. Let $p \in \text{Min}(R)$. As $\text{Min}(R) = \text{Ass}(R) = \text{Ass}_R(K_R)$ (see [DT1, 1.3]), we know that $K_{R_p} \cong (K_R)_p$ and, by [BH, Proposition 1.4.3], the $R_p$–module $(K_R)_p$ is free of rank $r$. As the dualizing complex of $R_p$ is $(I^\bullet)_p : 0 \longrightarrow I^0_p \longrightarrow 0$, we get $(R_p)^r \cong (K_R)_p \cong E(R/p)$ from which it follows that $r = 1$, and thus $R_p$ is Gorenstein.

(iii)⇒(i). Note that $\text{Min}(R) = \text{Ass}(R)$, and thus [BH, Proposition 1.4.3] implies that $K_R$ has rank 1.

As $\text{Ass}R = \text{Ass}(K_R), K$ is torsion free. Thus [BH, 1.4.18] implies that $K_R$ is isomorphic to a sub–module of a free $R$–module of rank 1, and it may be identified with an ideal of $R$ which we again denote by $K_R$. 
If \( \dim R = 0 \), we get \( K_R \cong R \), so we may assume \( \dim R > 0 \), and also \( K_R \) is a proper ideal of \( R \). By [BH, Proposition 1.4.3], \( K_R \) has a free sub–module \( a \), which is also an ideal of \( R \) of rank 1. Assuming \( a = xR \) with \( x \) is a base for \( a \), \( x \) is \( R \)–regular and \( K_R \)–regular. Let \( p \) be a prime ideal containing \( K_R \). Applying the functor \( \text{Hom}_{R_p}(\cdot, (I^*)_p) \) on the exact sequence \( 0 \rightarrow K_R R_p \rightarrow R_p \rightarrow R_p/K_R R_p \rightarrow 0 \), we get the exact sequence

\[
0 \rightarrow H^0(\text{Hom}_{R_p}(R_p/K_R R_p, (I^*)_p)) \rightarrow H^0((I^*)_p) \rightarrow H^0(\text{Hom}_{R_p}(K_R R_p, (I^*)_p)) \rightarrow H^1(\text{Hom}_{R_p}(R_p/K_R R_p, (I^*)_p)) \rightarrow H^1((I^*)_p) \rightarrow \cdots.
\]

Note that \( H^1(I^*)_p = 0 \) as \( K_R \) satisfies \((S_3)\) (see [DT2, Proposition 2.5]).

On the other hand, we have \( H^0(\text{Hom}_{R_p}(R_p/K_R R_p, (I^*)_p)) \cong \mathbb{Z} :_{K_R R_p} K_R R_p \leq 0 :_{K_R R_p} K_R R_p = 0 \) because \( K_R R_p \) is the canonical module of \( R_p \) and \( R_p \) satisfies \((S_2)\). Therefore we get the exact sequence

\[
o \rightarrow K_R R_p \rightarrow R_p \rightarrow H^1(\text{Hom}_{R_p}(R_p/K_R R_p, (I^*)_p)) \rightarrow 0
\]

which implies that \( H^1(\text{Hom}_{R_p}(R_p/K_R R_p, (I^*)_p)) \cong (R/K_R)_p \). It follows that \( \dim(R_p/K_R R_p) = \text{ht} p - 1 \). Using the Grothendieck local duality shows that \( R_p/K_R R_p \) is the canonical module of \( R_p/K_R R_p \). As \( K_R R_p \) contains an \( R_p \)–regular element, we have \( \text{ht}_{R_p}(K_R R_p) \geq 1 \). Since \( \dim(R_p/K_R R_p) = \text{ht} p - 1 \), we get \( \text{ht}(K_R) = 1 \).

For the final part, we may assume that \( \dim R > 3 \). As \( R \) is \((S_2)\) and \( K_R \) is \((S_3)\), from the exact sequence \( H^i_{pR_p}(R_p) \rightarrow H^i_{pR_p}(R_p/K_R R_p) \rightarrow H^{i+1}_{pR_p}(K_R R_p) \), we get \( H^i_{pR_p}(R_p/K_R R_p) = 0 \) for \( i = 0, 1 \). This shows that \( R_p/K_R R_p \) satisfies \((S_2)\). □

We can also generalize [DT1, Corollary 3.4].

4.6. Theorem. Assume that \( R \) satisfies \((S_2)\), and that \( \dim R < \infty \). The following statements are equivalent.

(i) \( R \) possesses a dualizing complex;

(ii) \( R \) admits a canonical module \( K \), and \( C(\mathcal{H}, K)' \), the induced complex of the Cousin complex of \( K \) with respect to the height filtration \( \mathcal{H} = (H_i)_{i \geq 0} \) with \( H_i = \{ p \in \text{Spec}(R) : \text{ht}(p) \geq i \} \), is a dualizing complex for \( R \);

(iii) \( R \) admits a canonical module \( K \) and \( H^1(C(\mathcal{H}, K)') \) is finitely a generated \( R \)–module for all \( i \geq 1 \).

Proof. (i) ⇒ (ii). By 4.4, there exists a dualizing complex

\[
I^\bullet : 0 \rightarrow I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots
\]
for $R$ such that $\text{Ass}_R(I^0) = \text{Min}(R)$. Set $K = \text{Ker}\delta^0$. As seen in 4.4, $K$ is a canonical module of $R$ and $\text{Ass}_R(K) = \text{Min}(R)$. Now, by [DT1, Theorem 2.4(iv)], $C(\mathcal{H}, K)'$ is a dualizing complex for $R$.

(ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(i). For each $m \in \text{Max}(R)$, we have, by [S1, Theorem 3.5], $C(\mathcal{H}, K)_m \cong C(\mathcal{H}_m, K_m)$, where $\mathcal{H}_m$ is the height filtration of $R_m$. Therefore, by [DT1, Corollary 3.4], $C(\mathcal{H}, K)'_m$ is a dualizing complex for $R_m$. Since $R_p$ satisfies $(S_2)$ for all $p \in \text{Spec}(R)$, by the same argument as in the proof of [DT1, Corollary 3.4], each term of $C(\mathcal{H}, K)'$ is an injective module. Thus, by [S4, Theorem 4.2], $C(\mathcal{H}, K)'$ is a dualizing complex for $R$. □

5. Indecomposable injective modules structure

In this section, by using of a particular dualizing complex for an $(S_2)$ Noetherian ring $R$ of finite dimension, we give an explicit description for the structure of all indecomposable injective modules. In [DT1, Corollary 3.3], it is shown that for each $p \in \text{Spec}(R)$, there exists a finitely generated $R$–module $T$, depending on $p$, such that $E(R/p)$ is a module of generalized fractions of $T$. Here we will show that $T$ can be replaced by a canonical module of $R$ and that it does not depend on $p$.

Our approach involves the concept of a chain of triangular subsets on $R$ explained in [O, page 420]. Such a chain $\mathcal{U} = (U_i)_{i \geq 1}$ determines a complex $C(\mathcal{U}, M)$ of modules of generalized fractions on an $R$–module $M$, that is

$$C(\mathcal{U}, M) : 0 \longrightarrow M \overset{e^0}{\longrightarrow} U_1^{-1}M \overset{e^1}{\longrightarrow} \cdot \cdot \cdot \overset{e^{i-1}}{\longrightarrow} U_i^{-1}M \overset{e^i}{\longrightarrow} U_{i+1}^{-1}M \overset{e^{i+1}}{\longrightarrow} \cdot \cdot \cdot$$

in which $e^0(m) = \frac{m}{(1)}$ for all $m \in M$ and $e^i(\frac{m}{(u_1, \cdots, u_i)}) = \frac{m}{(u_1, \cdots, u_i, 1)}$ for all $i \geq 1$, $m \in M$, and $(u_1, \cdots, u_i) \in U_i$. Note that in the complex $C(\mathcal{U}, M)$, $U_{i+1}^{-1}M$ is regarded as the $i$–th term, so that $H^i(C(\mathcal{U}, M)) = \text{Ker}e^{i+1}/\text{Im}e^i$, $i \geq 0$, and $H^{-1}(C(\mathcal{U}, M)) = \text{Ker}^0$.

Assume that $R$ satisfies $(S_2)$ and possesses a dualizing complex, so that $R$ possesses a dualizing complex

$$I^\bullet : 0 \longrightarrow I^0 \overset{\delta^0}{\longrightarrow} I^1 \overset{\delta^1}{\longrightarrow} \cdot \cdot \cdot \overset{\delta^{d-1}}{\longrightarrow} I^d \longrightarrow 0, d = \text{dim}R,$$

such that $\text{Ass}_R(I^0) = \text{Min}(R)$. Set $K = \text{Ker}\delta^0$, and consider the induced extended complex

$$I^* : 0 \longrightarrow K \hookrightarrow I^0 \overset{\delta^0}{\longrightarrow} I^1 \overset{\delta^1}{\longrightarrow} \cdot \cdot \cdot \overset{\delta^d}{\longrightarrow} I^d \longrightarrow 0.$$

For each $p \in \text{Ass}_R(I^0)$, the complex $0 \longrightarrow (I^0)_p \longrightarrow 0$ is the dualizing complex for $R_p$, so that $K_p \cong E(R/p)$. Hence $\text{Ass}_R(K) = \text{Min}(R)$. Thus, by [DT1, Proposition
there is a unique isomorphism of complexes (over \(Id_K\)) from \(I^*\) to \(C(\mathcal{V}, K)\), the complex of modules of generalized fractions on \(K\) with respect to the chain of triangular subsets \(\mathcal{V} = (V_i)_{i \geq 1}\) on \(R\), defined by

\[
V_i = \{(v_1, \ldots, v_i) \in R^i : \text{ht}_R((v_1, \ldots, v_j)) \geq j \text{ for all } j \text{ with } 1 \leq j \leq i\}.
\]

Now, we restate [DT1, Corollary 3.3] in a more appropriate form.

5.1. **Corollary.** Assume that \(R\) satisfies \((S_2)\) and that it possesses a dualizing complex, so that \(R\) admits a canonical module \(K\), say. Then, for each \(p \in \text{Spec}(R)\),

\[
E(R/p) \cong (V_{htp} \times (R \setminus p))^{htp-1}K
\]

where \(V_r\) is the triangular subset of \(R^r\) defined in the paragraph just before the corollary.

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