Charged lepton mixing and oscillations from neutrino mixing in the early Universe.

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Charged lepton mixing as a consequence of neutrino mixing is studied for two generations $e,\mu$ in the temperature regime $m_\nu \ll T \ll M_W$ in the early Universe. We state the general criteria for charged lepton mixing, critically reexamine aspects of neutrino equilibration and provide arguments to suggest that neutrinos may equilibrate as mass eigenstates in the temperature regime prior to flavor equalization. We assume this to be the case, and that neutrino mass eigenstates are in equilibrium with different chemical potentials. Charged lepton self-energies are obtained to leading order in the electromagnetic and weak interactions. The upper bounds on the neutrino asymmetry parameters from CMB and BBN without oscillations, combined with the fit to the solar and KamLAND data for the neutrino mixing angle, suggest that for the two generation case there is resonant charged lepton mixing in the temperature range $T \sim 5\text{GeV}$. In this range the charged lepton oscillation frequency is of the same order as the electromagnetic damping rate.

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Neutrinos play a fundamental role in cosmology and astrophysics $^1$, and there is now indisputable experimental confirmation that neutrinos are massive and that different flavors of neutrinos mix and oscillate $^2,1$. Neutrino oscillations in extreme conditions of temperature and density are an important aspect of Big Bang Nucleosynthesis (BBN), in the generation of the lepton asymmetry in the early Universe$^1,2,3,4,5,6,7,36$, and in the physics of core collapse supernovae$^2,11$. An important aspect of neutrino oscillations is lepton number violation, leading to the suggestion that leptogenesis can be a main ingredient in an explanation of the cosmological baryon asymmetry$^12$. Early studies of neutrino propagation in hot and dense media focused on the neutrino dispersion relations and damping rates in the temperature regime relevant for stellar evolution or big bang nucleosynthesis$^1,3,13$. This work has been extended to include leptons, neutrinos and nucleons in the medium$^14$. Matter effects of neutrino oscillations in the early universe were investigated in $15,16$, and more recently a field theoretical description of mixing and oscillations in real time has been provided in ref.$17$. While there is a large body of work on the study of neutrino mixing in hot and dense environments, much less attention has been given to the possibility of mixing and oscillation of charged leptons. Charged lepton number non-conserving processes, such as $\mu \rightarrow e\gamma;\mu \rightarrow 3e$ mediated by massive mixed neutrinos have been studied in the vacuum in refs.$18,19,20$. For Dirac neutrinos the transition probabilities for these processes are suppressed by a factor $m_\nu^2/\sqrt{2}M_W$ $18,19,20$. The WMAP$21$ bound on the neutrino masses $m_\alpha < 1\text{eV}$ yields typical branching ratios for these processes $B \lesssim 10^{-4}$ making them all but experimentally unobservable. In this article we explore the possibility of charged lepton mixing in the early Universe at high temperature and density. In section II we discuss the general arguments for charged lepton mixing as a result of neutrino mixing and establish the necessary conditions for this mixing to be substantial. We suggest that large neutrino chemical potentials may lead to substantial charged lepton mixing.

Without oscillations BBN and CMB provide a stringent constraint on the neutrino chemical potentials$22,23$, $\xi_\alpha$, with $-0.01 \leq \xi_\alpha \leq 0.22, |\xi_{\mu,\tau}| \leq 2.6$. Detailed studies$24,25,26,27$ show that oscillations and self-synchronization lead to flavor equilibration before BBN, beginning at a temperature $T \sim 30\text{MeV}$ $25$ with complete flavor equilibration among the chemical potentials at $T \sim 2\text{MeV}$ $21,27$. Thus prior to flavor equalization for $T > 30\text{MeV}$ there could be large neutrino asymmetries consistent with the BBN and CMB bounds in the absence of oscillations. We study whether this possibility could lead to charged lepton mixing focusing on two flavors of Dirac neutrinos corresponding to $e,\mu$ and for simplicity in the temperature regime where both are ultrarelativistic, with $m_\mu \ll T \ll M_W$. In section III we discuss general arguments within the realm of reliability of perturbation theory suggesting that the equilibrium state is described by a density matrix nearly diagonal in the mass basis. In this section we also discuss caveats and subtleties in the kinetic approach to neutrino equilibration in the literature and argue that results on the equilibrium state are in agreement with an interpretation of an equilibrium density matrix diagonal in the mass basis. Our main and only assumption is that for $T > 30\text{MeV}$ neutrinos are in equilibrium and the density matrix is nearly diagonal in the mass basis, with distribution functions of mass eigenstates that feature different and large chemical
potentials. While this is not the *only*, it is *one* possible scenario for substantial charged lepton mixing that can be explored systematically. In section III we explore charged lepton mixing in lowest order in perturbation theory as a consequence of large asymmetries in the equilibrium distribution functions of mass eigenstates. In this section we also critically discuss possible caveats and suggest a program to include non-perturbative corrections in a systematic expansion. In section IV we summarize the main aspects and results of the article.

I. CHARGED LEPTON MIXING: THE GENERAL ARGUMENT

Charged lepton mixing is a consequence of neutrino mixing in the charged current contribution to the charged lepton self energy. This can be seen as follows: consider the one-loop self energy for the charged leptons. The off-diagonal self-energy $\Sigma_{e\mu}$ is depicted in fig. 11 for the case of electron-muon mixing. The internal line in fig. 11 (a) is a neutrino propagator off-diagonal in the flavor basis, which is non-vanishing if neutrinos mix. In Fermi’s effective field theory obtained by integrating out the vector bosons the effective interaction that gives rise to charged lepton mixing is

$$H_{\text{eff}} = \frac{2G_F}{\sqrt{2}} \left[ \bar{e}_L \gamma_\mu \nu_{e L} \right] \left[ \bar{\nu}_{\mu L} \gamma_\mu \nu_{\mu L} \right].$$

(1)

where the brackets stand for average in the density matrix of the system. Eqn. 2 gives the charged-lepton mixing part of the self energy as

$$\Sigma_{e\mu} = \frac{2G_F}{\sqrt{2}} \gamma_\mu < \nu_{e L} \bar{\nu}_{\mu L} > \gamma_\mu \nu_{\mu L}.$$

(3)

The Fermi effective field theory contribution to the self-energy is depicted in fig. 11 (b).

The focus of this article is to study two aspects that emerge from this observation:

- **Mixing:** The propagating modes in the medium are determined by the poles of the full propagator with a self-energy that includes radiative corrections in the medium. The full self-energy for the charged leptons is a $2 \times 2$ matrix (in the simple case of two flavors), and eqn. 3 yields the off-diagonal matrix element in the flavor basis. This is precisely the main study in this article: we obtain the charged lepton propagator including radiative corrections in the medium up to one loop in the electromagnetic and weak interactions. Neutrino mixing leads to off diagonal components of the propagator in the charged lepton flavor basis. We find the dispersion relation of the true propagating modes in the medium by diagonalization of the *full* propagator including one loop radiative corrections. The true propagating modes in the medium are *admixtures* of electron and muon states: this is
precisely what we identify as mixing. The results given by equations [23] state quite generally that electron and muon states are mixed whenever the neutrino propagator is off-diagonal in the flavor basis. We highlight that this is precisely the condition for flavor neutrino oscillations since the propagator yields the transition amplitude from an initial to a final state. Therefore we state quite generally that provided flavor neutrinos oscillate, namely if the neutrino propagator is off diagonal in the flavor basis, charged leptons associated with these flavor neutrinos will mix. The true propagating modes of charged leptons are linear superpositions of the charged leptons associated with the flavor neutrinos. We emphasize these statements because even though they are a straightforward consequence of flavor neutrino mixing, this precise point, and its consequences, have not been previously addressed in the literature.

**Oscillations:** Consider the decays $W \to \nu_e e$ or neutron beta decay $n \to p e \bar{\nu}$ in the medium. The electron produced in the medium at the decay vertex propagates as a linear combination of the true propagating modes in the medium, each with a different dispersion relation. Upon time evolution this linear superposition will have non-vanishing overlap with a muon state yielding a typical oscillation pattern. We study this oscillation between the electron and muon charged lepton by considering the evolution of an electron wave-packet produced locally at the decay vertex. These oscillations are akin to the typical oscillation between flavor neutrino states and are a consequence of the one loop radiative correction depicted in fig.1 with neutrinos in the medium. The transition probability from an initial electron to a muon packet oscillates in time. While in the case of almost degenerate neutrinos oscillations are associated with macroscopic quantum coherence because the oscillation lengths are macroscopically large, this is not a necessary condition for oscillations, which occur whenever the initial state is a linear superposition of the propagating modes. The example of neutron beta decay gives a precise meaning to the statement of charged lepton mixing: in the decay of the neutron the charged lepton that is produced is identified with the electron. This is the initial state, which in a medium will propagate as a linear combination of the propagating modes with an oscillatory probability of finding a muon. These oscillations are fundamentally different from the space-time oscillations possibly associated with quantum entanglement and discussed in references [25].

Eqn. [43] generally states that there is charged lepton mixing when the density matrix is off diagonal in the flavor basis. This is equivalent to the statement of neutrino mixing. A simple example of a density matrix off diagonal in the flavor basis is $\hat{\rho} = |0_m \rangle \langle 0_m| \neq |0_m \rangle \langle 0_m|$ with $|0_m \rangle >$ being the vacuum state in absence of weak interactions but with a neutrino Hamiltonian with an off diagonal mass matrix in the flavor basis. This is the interaction picture vacuum of the standard model augmented by a neutrino mass matrix with flavor mixing. In the two flavor case with

$$\nu_e(\vec{x}, t) = \cos \theta \nu_1(\vec{x}, t) + \sin \theta \nu_2(\vec{x}, t) \, , \, \nu_\mu(\vec{x}, t) = \cos \theta \nu_2(\vec{x}, t) - \sin \theta \nu_1(\vec{x}, t) \tag{4}$$

with $\nu_{1,2}(\vec{x}, t)$ the fields associated with the mass eigenstates,

$$\langle 0_m | \nu_e(\vec{x}, t) \bar{\nu}_\mu(\vec{x}', t') | 0_m \rangle = \cos \theta \sin \theta \langle 0_m | \nu_2(\vec{x}, t) \bar{\nu}_2(\vec{x}', t') | 0_m \rangle - \langle 0_m | \nu_1(\vec{x}, t) \bar{\nu}_1(\vec{x}', t') | 0_m \rangle . \tag{5}$$

If the propagators for the mass eigenstates only differ in the masses, this difference leads to a very small self-energy. In a medium the flavor off diagonal expectation value [33] could be enhanced by temperature and or density. Therefore the general criterion for substantial charged lepton mixing in a medium hinges on just one aspect: a large off diagonal matrix element $\langle \nu_e L \bar{\nu}_\mu L \rangle$. One possible case for which this condition is fulfilled is if the density matrix is nearly diagonal in the mass basis with large chemical potentials for the different mass eigenstates.

We emphasize that this is only one condition for substantial charged lepton mixing and by no means unique, the analysis above shows that the most general condition is simply that $\langle \nu_e L \bar{\nu}_\mu L \rangle$ be large.

In the general case a full solution of a kinetic equation should yield the value of $\langle \nu_e L \bar{\nu}_\mu L \rangle$. If an equilibrium state of mixed neutrinos is described by a density matrix nearly diagonal in the mass basis with distribution functions for the different mass eigenstates with large and different chemical potentials, then simple expressions for the equilibrium propagators allow an assessment of the charged lepton mixing self energy. Can this be the case?

**II. ON NEUTRINO EQUILIBRATION**

**A. Equilibration in the mass basis**

A system is in equilibrium if

$$[\hat{\rho}, H] = 0 , \tag{6}$$
where $\hat{\rho}$ is the density matrix of the system and $H$ the total Hamiltonian $H = H_0 + H_{\text{int}}$ with $H_0$ the Hamiltonian in the absence of weak interactions but with a mass matrix and $H_{\text{int}} = H_{\text{NC}} + H_{\text{CC}}$. In the absence of weak interactions, an equilibrium density matrix $\hat{\rho}_0$ commutes with $H_0$, therefore it is diagonal in the mass basis. The equilibrium density matrix cannot have off-diagonal matrix elements in the mass basis because these oscillate in time. Of course without interactions the system will not reach an equilibrium state, however, as is the usual assumption in statistical mechanics, provided the interactions are sufficiently weak but lead to an equilibrium state, an almost free gas of particles in equilibrium is a suitable description, and the canonical density matrix in such case is of the form $\hat{\rho}_0 = e^{-H_0/T}$ (the grand canonical could also include a chemical potential for conserved quantities). Examples of this are abundant, a ubiquitous one is the cosmic microwave background radiation: the Planck distribution function describes free photons in equilibrium, although photons reach equilibrium by undergoing collisions with charged particles in a plasma with cross sections much larger than those of neutrinos.

Consider how the density matrix is modified from the “free field” form by “switching on” the weak interactions in perturbation theory. A perturbative expansion in the interaction picture of $H_0$ begins by writing the interaction vertices in terms of neutrino fields in the mass basis. Neglecting sterile neutrinos, neutral current interaction vertices are diagonal and only the charge current interactions induce off-diagonal correlations in the mass basis. Let us write the full density matrix as $\hat{\rho} = \hat{\rho}_0 + \delta\hat{\rho}$ where $\delta\hat{\rho}$ has a perturbative expansion in the weak coupling. The equilibrium condition leads to the following identity

$$[\delta\hat{\rho}, H_0] = - [\hat{\rho}_0, H_{\text{int}}] - [\delta\hat{\rho}, H_{\text{int}}] \tag{7}$$

Taking matrix elements in the mass eigenstates of $H_0$ the solution of eqn. (7) for the matrix elements of $\delta\hat{\rho}$ in the mass basis can be found in a perturbative expansion. The matrix elements of $\delta\hat{\rho}$ may feature non-vanishing off diagonal correlations which are perturbatively small. Namely, the equilibrium density matrix is nearly diagonal in the mass basis.

Expanding the field operators associated with the mass eigenstates in terms of Fock creation and annihilation operators of mass eigenstates, the spatial Fourier transform of the field operators is given by

$$\nu_i(\vec{k}, 0) = \sum_{\lambda} a_i(\vec{k}, \lambda) U_i(\vec{k}, \lambda) + b_i^\dagger(-\vec{k}, \lambda) V_i(-\vec{k}, \lambda) ; \quad i = 1, 2$$

where the spinors $U, V$ are orthonormalized positive and negative energy solutions solutions of the Dirac equation with mass $m_i$. If the density matrix is diagonal in the mass basis, then $\langle a_i^\dagger(\vec{k}) a_j(\vec{k}) \rangle \propto \delta_{ij}$ and the distribution functions for the mass eigenstates are $\langle a_i^\dagger(\vec{k}) a_j(\vec{k}) \rangle ; (b_i^\dagger(\vec{k}) b_i(\vec{k}))$ for neutrinos and antineutrinos of mass $i$ respectively. Switching on the neutral current interaction which is diagonal in the mass basis (provided there are no sterile neutrinos) will lead to the equilibration of neutrinos and the equilibrium distribution functions will be the usual Fermi-Dirac with a possible chemical potential. The charged current interactions yield vertices that are off-diagonal in the mass basis and induce cross correlations of the form $\langle a_i^\dagger a_j \rangle$ with $i \neq j$. In free field theory this equal time correlation function, if non-vanishing, oscillates with a time dependence $e^{i(\omega_1 - \omega_2)t}$, however, in equilibrium there cannot be a time dependence of these off diagonal correlations as the following argument shows

$$\langle a_i^\dagger(t) a_j(t) \rangle = \text{Tr} \left( \hat{\rho} e^{iHt} a_i^\dagger(0) a_j(0) e^{-iHt} \right) = \text{Tr} \left( \hat{\rho} a_i^\dagger(0) a_j(0) \right)$$

where we used eqn. (8). Either the charged current interactions that generate these off-diagonal correlations exactly cancel the free field time dependence for all values of momentum $k$ or, more likely, they lead to the decay of these off diagonal correlations to asymptotically perturbatively small expectation values as expected from the general arguments following eqn. (7).

This observation leads to the conclusion that if the perturbative expansion is reliable, the weak interactions lead to an equilibrium state described by a density matrix which is nearly diagonal in the mass basis but for possible perturbatively small off-diagonal elements. In perturbation theory the equilibrium distribution functions are diagonal in the mass basis and may feature a chemical potential for each mass eigenstate. Weak interaction vertices involve the flavor fields, but these are linear combinations of the fields that create and annihilate mass eigenstates, the true in-out states. Consider a far off-equilibrium initial state with a population of vector bosons (or neutrinos) and no neutrinos, the decay of the vector bosons (or neutrinos) results in the creation of a linear superposition of mass eigenstates, which propagate independently after production. Collisional processes via the weak interaction lead to the decoherence of the mass eigenstates and ultimately to a state of equilibrium in which equal time expectation values
in the density matrix cannot depend on time. Neutral and charged current interactions yield different relaxational dynamics: in the mass basis the neutral current interaction is diagonal and relaxation processes via neutral currents lead to equilibration in the mass basis. Charged currents feature both diagonal and off-diagonal contributions in the mass basis, the diagonal ones yield relaxation dynamics similar to the neutral current interaction. The off diagonal contributions induce correlations between different mass eigenstates, but also lead to the relaxation of these off diagonal correlations. These two types of processes leading to relaxation dynamics for diagonal and off-diagonal correlations in the mass basis are akin to the different processes that lead to the relaxation times $T_1$ (diagonal) and $T_2$ (transverse) in spin systems in nuclear magnetic resonance. These concepts are manifest in Stodolsky’s effective Bloch equation description of neutrino oscillations in a medium with a damping coefficient in the “transverse” direction, whose inverse is the equivalent of the $T_2$ relaxation time in spin systems.

By the above arguments this asymptotic equilibrium density matrix must be nearly diagonal in the mass basis at least within the realm of reliability of perturbation theory. Equilibrium correlation functions of operators at different times must be functions of the time difference. Of particular relevance to the discussion below is the flavor off diagonal $\rho_{\nu\nu'}(\vec{k},t) = \rho_{\nu\nu'}(\vec{k},t')$ where $\nu,\nu'\in\nu,\bar{\nu}$.

Writing the full density matrix as $\hat{\rho} = |0><0|$ where $|0>$ is the exact ground state of $H$. This state can be constructed systematically in perturbation theory from the ground state $|0_m>$ of the Hamiltonian $H_0$ in absence of weak interactions, namely the interaction picture ground state in the basis of mass eigenstates,

$$|0> = |0_m> + \sum_{n} |n_m> \frac{<n_m|H_{int}|0_m>}{-E_m} + \cdots. \quad (12)$$

where $|n_m>$ are Fock eigenstates of $H_0$ ("mass eigenstates") with energy $E_n$. Writing the full density matrix as $\hat{\rho} = |0_m><0_m| + \delta\hat{\rho}$ one can find $\delta\hat{\rho}$ systematically in perturbation theory. The off-diagonal flavor propagator

$$S_{\nu\nu'}(\vec{k},t-t') = <0|\nu_{\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k},t')|0> = <0|\nu_{\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k},t')|0_m> + O(g) + \cdots$$

$$= \cos\theta \sin\theta \left[ <0_m|\nu_{\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k},t')|0_m> - <0_m|\nu_{\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k},t')|0_m> \right] + O(g) + \cdots. \quad (13)$$

This propagator is the lowest order intermediate state in the process $W e \rightarrow W \mu$, and is also the internal fermion line, along with W-vector boson exchange in the off-diagonal self-energy contribution for charged leptons, see fig. 14. This simple example also leads to conclude that if there is an equilibrium state for which the equal time “distribution function” $\rho_{\nu\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k}) < 0$ then the unequal time correlation function

$$S_{\nu\nu'}(\vec{k},t-t') = <\nu_{\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k},t')> = \cos\theta \sin\theta \left[ <\nu_{\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k},t')> - <\nu_{\nu'}(\vec{k},t)\rho_{\nu\nu'}(\vec{k},t')> \right] \neq 0 \quad (14)$$

As it will become clear below, this is the correlation function that describes the charged lepton mixing. These arguments rely on the validity of the perturbative expansion and require revision in the case where perturbation theory in the mass basis must be reassessed. In section III.C we discuss this possibility and propose a method to re-arrange the perturbative expansion.

### B. On the kinetic approach

Early kinetic approaches to the dynamics of oscillating neutrinos in thermal environments were proposed by Dolgov, Stodolsky and Manohar. Dolgov introduced density matrices in flavor space, whereas Manohar and Stodolsky used a single particle density matrix of flavor states leading to Bloch-like equations and a similar description was studied in ref. 18. Stodolsky argued that decoherence between flavor states emerged from a “transverse” relaxation akin to the relaxation time $T_2$ in nuclear magnetic resonance. A Boltzmann equation for mixing and decoherence was established by Raffelt, Sigl and Stodolsky in terms of a “matrix of densities” in the
nonrelativistic domain instead of the density matrix. In this approach the field operators for different flavors were truncated to only the annihilation operators and obtained a Boltzmann equation in a perturbative expansion. A fully relativistic treatment was presented in ref. 33 introducing “matrices of densities” defined by the expectation value of bilinears of creation and annihilation operators of flavor states. A quantum kinetic description of oscillating neutrinos was presented in ref. 33 with an approach similar to those of refs. 34, 35, 36, 36, 37 in terms of single particle flavor states. However, while there is a formal unitary transformation that relates the flavor and mass Fock operators to flavor states, the notion of flavor Fock states, either in terms of single particle flavor neutrino states or by expanding field operators in terms of creation and annihilation Fock operators for flavor states. However, flavor states and the precise definition of Fock operators associated with these states are very subtle and controversial 34, 35, 36, 37, 38, 39, 40, 41.

Precisely, what are flavor states? While in the literature there is no precise definition of a “flavor state”, a proper definition of such state should begin by expanding the flavor field operator in terms of Fock creation-annihilation operators of flavor states. In such an expansion the spatial Fourier transform of the flavor neutrino field operator, for example the electron neutrino at \( t = 0 \), is given by

\[
\nu_e(\vec{k}, 0) = \sum_s \alpha_e(\vec{k}, s) U_e(\vec{k}, s) + \beta_e(\vec{k}, s) V_e(-\vec{k}, s)
\]

where the spinors \( U_e \) and \( V_e \) are orthonormalized positive and negative frequency solutions of a Dirac operator with some mass and define a basis. A Fock state of an electron neutrino can be defined by

\[
|\nu_e(\vec{k}, s)\rangle = \alpha_e^+ (\vec{k}, s) |0_m\rangle
\]

where \( |0_m\rangle \) is the vacuum of the non-interacting theory, namely the vacuum of mass eigenstates. However, the expansion requires a definite basis corresponding to a definite choice of the Dirac spinors \( U, V \), which can be chosen to be solutions of a Dirac operator for any arbitrary mass. Each possible choice of mass gives a different definition of “particle”. One possible choice is zero mass 34, another choice would be the diagonal elements of the mass matrix in the flavor basis or masses \( m_1 \) to the electron neutrino and \( m_2 \) to the muon neutrino 35, 36. Any of these choices is just as good and physically motivated but obviously arbitrary. The creation and annihilation operators are extracted by projection 42, for example \( \alpha_e^1 (\vec{k}, s) = \nu_e^1 (\vec{k}, 0) U_e(\vec{k}, s) \). Writing the electron neutrino field operator as a linear combination of the field operators that create and annihilate mass eigenstates one finds

\[
\alpha_e^1(\vec{k}, s) = \cos \theta \left[ \sum_\lambda a_e^1(\vec{k}, \lambda) U_e^1(\vec{k}, \lambda) U_e(\vec{k}, s) + b_1(-\vec{k}, \lambda) V_e^1(-\vec{k}, \lambda) U_e(\vec{k}, s) \right] + \\
\sin \theta \left[ \sum_\lambda a_e^2(\vec{k}, \lambda) U_e^2(\vec{k}, \lambda) U_e(\vec{k}, s) + b_2(-\vec{k}, \lambda) V_e^2(-\vec{k}, \lambda) U_e(\vec{k}, s) \right]
\]

The transformation between the set of “flavor” operators and those that create and annihilate mass eigenstates is unitary and the scalar products of the spinors yield generalized Bogoliubov coefficients. It is clear that there is no single choice of spinor \( U_e \) that will make

\[
U_e^1(\vec{k}, \lambda) U_e(\vec{k}, s) = 1; \ U_e^2(\vec{k}, \lambda) U_e(\vec{k}, s) = 1 \\
V_e^1(-\vec{k}, \lambda) U_e(\vec{k}, s) = 0; \ V_e^2(-\vec{k}, \lambda) U_e(\vec{k}, s) = 0
\]

A surprising result of the above identification is that the annihilation operator \( \alpha_e(\vec{k}, s) \) creates a linear combination of antineutrino mass eigenstates out of the vacuum \( |0_m\rangle \) 37, 38, 39, 40, 41, 42.

This observation indicates that any choice of the solutions for the spinors \( U_e, V_e \) to define a flavor Fock creation operator leads to \( |\nu_e\rangle \neq \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle \). A possible definition of flavor Fock states would be to define the flavor vacuum \( |0_f\rangle \) as the state annihilated by the flavor annihilation operators (defined for example for zero mass states) and to construct a Fock Hilbert space out of this vacuum by successive application of flavor Fock creation operators. However, while there is a formal unitary transformation that relates the flavor and mass Fock operators via the Bogoliubov coefficients, such transformation is not unitarily implementable in the infinite dimensional Hilbert space 37, in particular < 0_f |0_m > = 0. Finally one can simply define flavor states as

\[
|\nu_e\rangle \equiv \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle
\]

However, because of the ambiguities with the definition of flavor Fock creation-annihilation operators discussed above these single particle states are indirectly related to the flavor fields \( \nu_{\alpha , \mu}(\vec{x}, t) \) that enter in the standard model Lagrangian. Furthermore, this single particle definition does not yield any information on a Fock representation of many
particle flavor states. A quantum statistical description of a neutrino gas is intrinsically a many body description, the total wave function of an n-fermion system must be completely antisymmetric under pairwise exchange. The second quantized Fock representation allows a systematic treatment of the many particle aspects, in particular in quantum statistical mechanics a distribution function is an expectation value of Fock number operator in the density matrix. Therefore simply defining flavor states as in eqn. (19) does not yield a complete information on the many particle nature of a neutrino gas. A systematic study of the many particle aspect of the dynamical evolution of a dense gas of flavor neutrinos with a physically motivated definition of flavor states even in the non-interacting theory was presented in ref.[12] wherein subtle but important effects associated with the non-trivial Bogoliubov coefficients in the dynamics were studied. Another subtlety emerges when a chemical potential is assigned to “flavor states”, a chemical potential is a thermodynamic variable conjugate to a conserved particle number. Even in the free theory, in absence of weak interactions, flavor number is not conserved if neutrinos mix and as a result a chemical potential for flavor neutrinos is not a well defined quantity even for free mixed neutrinos. Dynamical aspects associated with this issue were also studied in ref.[12].

A counter argument to this critique would hinge on the fact that neutrino masses are small on the relevant energy scales and mass differences are even smaller, therefore one can approximately take all of the Dirac spinors to be practically massless. Of course if all spinors \( \mathcal{U}, \mathcal{V} \) for mass and flavor eigenstates are taken to be massless the overlaps \( U \mathcal{U}^\dagger = 1; U \mathcal{V}^\dagger = 0 \) and eqn. (19) becomes an identity. This point will become relevant below.

While this approximation may be justified, it glosses over the main conceptual aspects and avoids the fundamental question of what precisely is a distribution function of flavor states. Such function includes information over all scales, it yields the average occupation for all values of the momenta, not just the high energy limit.

The main point of this discussion is that there are subtleties and caveats in the kinetic description based on “flavor states” or flavor matrix of densities, which involve Fock operators for flavor states. While these subtleties and caveats may not invalidate the broad aspects of the kinetic results, they cloud the interpretation of the equilibrium state of neutrinos.

To highlight this point consider an equilibrium situation in which \( n_{e\mu}(\vec{k}) \equiv \langle \nu_1^\dagger(\vec{k})\nu_{\mu}(\vec{k}) \rangle = 0 \), for a density matrix diagonal in the mass basis this means

\[
\cos \theta \sin \theta \left[ \langle \nu_1^\dagger(\vec{k})\nu_1(\vec{k}) \rangle - \langle \nu_2^\dagger(\vec{k})\nu_2(\vec{k}) \rangle \right] = 0 \Rightarrow \langle \nu_1^\dagger(\vec{k})\nu_1(\vec{k}) \rangle = \langle \nu_2^\dagger(\vec{k})\nu_2(\vec{k}) \rangle = 0
\]

which in turn leads to the result

\[
n_{ee} = \langle \nu_e^\dagger(\vec{k})\nu_e(\vec{k}) \rangle = \langle \nu_\mu^\dagger(\vec{k})\nu_\mu(\vec{k}) \rangle = n_{\mu\mu}
\]

These conditions of “flavor equalization” are the same as those obtained in ref.[15] for the equilibrium solution of the kinetic equations, although in that reference one active and one sterile neutrino were studied. The condition (20) is consistent with identical chemical potentials for the mass eigenstates in the limit \( m_1 = m_2 = 0 \). As discussed above, it is precisely taking \( m_1 = m_2 = 0 \) that yields the correspondence between the definition of the flavor states (19) and the relation between the flavor and mass eigenstates fields when all spinors are taken to be massless. This is also the approximation used in ref.[34] where flavor fields are expanded in the basis of massless spinors. These are precisely the approximations invoked in the kinetic approach and correspond to neglecting the neutrino masses. Restoring neutrino masses the off diagonal correlation in the mass basis would be \( n_{e\mu}(\vec{k}) \propto (m_1^2 - m_2^2)/k^2 \). Thus an interpretation of the kinetic results is that the equilibrium state is described by a density matrix diagonal in the mass basis with equal chemical potential for the mass eigenstates with an off diagonal correlation \( n_{e\mu} \propto (m_1^2 - m_2^2)/k^2 \) which is neglected in the kinetic approach.

Thus the equilibrium solution of the kinetic equation in ref.[17] (see eqn. (12) in this reference) can be interpreted as a confirmation of the statement of equilibration in the mass basis when the neutrino masses are neglected, although in the “flavor” formulation of the kinetic equations this information is not readily available.

The discussion in the previous section based on general aspects of the full density matrix and a systematic perturbative expansion avoids the caveats associated with the intrinsic ambiguities in the definition of flavor states and suggests that equilibration leads to a density matrix nearly diagonal in the mass basis.

Quantum Zeno effect: References[31, 12, 44] discuss the fascinating phenomenon of the Quantum Zeno effect or “Turing’s paradox”[30]. In the case of neutrino mixing, this effect arises when the scattering rate is larger than the oscillation rate. Since neutrinos are produced in weak interaction vertices as “flavor eigenstates” when rapid collisions via the weak interactions which are diagonal in the flavor basis prevent oscillations, the states are effectively “frozen” in the flavor basis[30]. This situation may be expected at high temperature. An order of magnitude estimate reveals that such a possibility is not available in the case under consideration, with a large difference in the neutrino
asymmetries. The argument is the following: the oscillation frequency is given by
\[ \Omega \sim \frac{\delta m^2}{k} \left[ \left( \cos 2\theta - \frac{V(k)}{\delta m^2} \right)^2 + \left( \sin 2\theta \right)^2 \right]^{1/2}, \tag{22} \]
where the matter potential
\[ V(k) \approx k G_F T^3 L, \tag{23} \]
and \( L \) is the neutrino asymmetry difference between the two generations of neutrinos. Our study relies on the possibility of large asymmetries, namely \( L \sim 1 \). The decay rates are of the order
\[ \Gamma \approx G_F^2 T^5. \tag{24} \]

The “quantum zeno effect” would operate provided \( \Gamma \gg \Omega \). Even if \( V(k) \gg \delta m^2 \) which can occur at high temperatures, the oscillation frequency
\[ \Omega \sim G_F T^3 L \tag{25} \]
and \( \frac{\Gamma}{\Omega} \sim G_F T^2/L \ll 1 \) under the assumptions invoked in this article, namely: i) \( L \sim 1 \), ii) perturbation theory is valid in Fermi’s effective field theory. A reversal of this bound would entail a breakdown of the perturbative expansion, and of the Fermi effective field theory. In perturbation theory the bound is even stronger if \( \delta m^2 \gg V(k) \). Thus we conclude that, under the conditions studied in this article, the quantum Zeno effect is not effective in “freezing” the states in the flavor basis and that oscillations and relaxation may indeed result in a density matrix which is off diagonal in the flavor basis.

C. Main assumptions

After the above discussion on the general aspects of equilibration and the kinetic approach, we are in position to clearly state our main assumptions. These are the following:

- i): for \( T >> 1 \) MeV the electromagnetic and weak interaction rates ensure that leptons are in equilibrium in the early Universe, namely their distribution functions are time independent.

- ii): the results from the kinetic approach in refs.\[25\] indicate that for \( T >> 30 \) MeV neutrino oscillations are suppressed and flavor equilibration via oscillations is not operational. Therefore for \( T >> 30 \) MeV neutrinos are in equilibrium, and there could be large asymmetries in the neutrino sector consistent with the BBN and CMB bounds in the absence of oscillations.

- iii): the arguments presented above lead us to assume that the equilibrium state of the neutrino gas is described by a density matrix which is nearly diagonal in the mass basis, and allow the distribution functions of mass eigenstates to feature different chemical potentials. As per the discussion above, this is consistent with the interpretation of the equilibrium state after equalization of the chemical potentials discussed after eqn. (20).

Equation (14) entails that the equilibrium off-diagonal flavor propagator does not vanish. The numerical study in ref.\[25\] shows that flavor equilibration with \( |\xi_\nu| \lesssim 0.07 \) is established but for \( T \sim 2 \) MeV, well below the scale of interest in our study, clearly leaving open the possibility, which we assume here, of large asymmetries in the neutrino sector at a temperature much higher than that of flavor equalization. In summary: the combination of the general arguments suggesting that the equilibrium density matrix is nearly diagonal in the mass basis, at least within the framework of perturbation theory, along with the results of the kinetic approach lead us to assume that at high temperature \( T \gtrsim 30 \) MeV neutrinos are in equilibrium, the density matrix is nearly diagonal in the mass basis and there could be large asymmetries for the different mass states consistent with the bounds in absence of oscillations. Equilibration of mass eigenstates implies that the neutrino propagators in the mass basis only depend on the time difference (translational invariance in time) and are determined by the equilibrium distribution functions, the off-diagonal flavor propagator is given by eqn. (14).

The dynamics that leads to equilibration and the mechanism by which substantial chemical potentials emerge is of course very important and require a much deeper and detailed investigation as well as an understanding of initial conditions for lepton asymmetries. A consistent study should address the subtleties and caveats associated with the
the leading electromagnetic contribution to the charged lepton self-energy $\Sigma^{em}_{\alpha\beta}(\omega, k)$ for temperatures much larger than the lepton masses is dominated by one-photon exchange in the hard-thermal loop approximation\,[15, 16]. As discussed below, the temperature region of interest for substantial charged-lepton mixing is $T \sim \text{Gev}$, therefore we will neglect corrections of order $M^2/L^2; M^2/L^2 \ll 10^{-2}$ (which already multiply one power of $\alpha$) to leading order.

Quark-lepton chemical equilibrium may lead to charged lepton chemical potentials as large as those for neutrinos, therefore we allow for arbitrary charged lepton chemical potentials with the possibility that $\mu_e, \mu_\mu$ the effective Dirac equation in the medium for the space-time Fourier transforms of these fields is the following\,[17]

$$\left[\left(\gamma^0 \omega - \gamma^i \cdot \vec{k}\right) \delta_{\alpha\beta} - M_{\alpha\beta} + \Sigma^{em}_{\alpha\beta}(\omega, k) + \Sigma^{NC}_{\alpha\beta}(\omega, k) + \Sigma^{NC}_{\alpha\beta}(\omega, k) L + \Sigma^{NC}_{\alpha\beta}(\omega, k) L\right] \psi_{\beta}(\omega, k) = 0,$$

[26]

where $M = \text{diag}(M_e, M_\mu)$ is the charged lepton mass matrix and $L = (1 - \gamma^5)/2$.

### A. Electromagnetic Self-Energy

The leading electromagnetic contribution to the charged lepton self-energy $\Sigma^{em}_{\alpha\beta}(\omega, k)$ for temperatures much larger than the lepton masses is dominated by one-photon exchange in the hard-thermal loop approximation\,[15, 16]. As discussed below, the temperature region of interest for substantial charged-lepton mixing is $T \sim \text{Gev}$, therefore we will neglect corrections of order $M^2/L^2; M^2/L^2 \ll 10^{-2}$ (which already multiply one power of $\alpha$) to leading order.

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[26]
A collisional contribution to the charged lepton damping rate $\Gamma$ is of order $\alpha^2 T$ and will be neglected to leading order in $\alpha$.

### B. Charged and neutral currents self-energy

As depicted in fig. 1 to lowest order in perturbation theory in the interaction picture of $H_0$, there is a flavor off-diagonal contribution to the charged lepton self energy, $\Sigma_{e,\mu}$ given by a W-boson exchange and an internal off-diagonal fermion propagator $\langle \bar{\nu}_e(\vec{x},t)\nu_\mu(\vec{x}',t') \rangle$. In the mass basis this propagator is given by eqn. 14 and the general form of the corresponding self energy contribution is depicted in fig. 2 where the internal fermion line corresponds to a mass eigenstate neutrino $\nu_a$ in equilibrium with chemical potential $\mu_a = T \xi_a$.

We focus on the temperature regime $T \ll M_W$ and obtain the charged current contribution to the self-energy up to leading order in a local expansion in the frequency and momentum of the external leptons neglecting terms proportional to $m_a/M_W \lesssim 10^{-17}$. The general expressions for charged and neutral current self-energies are given in references [13, 17] where we refer the reader for more details.

Denoting by $\Sigma_{CC}^a(\omega, k)$ the one-loop charged current self-energy with internal neutrino line corresponding to a mass eigenstate $\nu_a$ displayed in fig. 2, the matrix $\Sigma_{a,\beta}^{CC}(\omega, k)$ in eqn. (26) has the following entries

$$
\begin{align*}
\Sigma_{e,e}^{CC} &= C^2 \Sigma_1 + S^2 \Sigma_2 \\
\Sigma_{\mu,\mu}^{CC} &= S^2 \Sigma_1 + C^2 \Sigma_2 \\
\Sigma_{e,\mu}^{CC} &= \Sigma_{\mu,e}^{CC} = -CS(\Sigma_1 - \Sigma_2)
\end{align*}
$$

(32)

consistently with a perturbative calculation in the mass basis to lowest order in the weak interactions $C = \cos \theta$; $S = \sin \theta$ and $\theta$ is the neutrino vacuum mixing angle. A fit to the solar and KamLAND data yields $\tan^2 \theta \approx 0.40$.

The off diagonal element $\Sigma_{e,\mu}^{CC}$ in eqn. 62 is responsible for charged lepton mixing. The flavor off-diagonal propagator in this self-energy is precisely given by eqn. 14. The form of this off-diagonal self-energy makes clear that there is mixing provided $\theta \neq 0, \pi/2$ and $\Sigma_1 - \Sigma_2 \neq 0$. The calculation of the self-energy $\Sigma_{e,\mu}^{CC}$ in the vacuum is standard: it is performed in the interaction picture of the true basis of in-out states, these are mass eigenstates. In this case the difference of the self-energies is determined solely by the neutrino mass difference, therefore in the vacuum $\Sigma_{e,\mu}^{CC} \propto G_F \Delta m^2 \sim 10^{-27}$ and the mixing between charged leptons is negligible.

The main point of our study is that in the medium in equilibrium with large neutrino asymmetries for the mass eigenstates, charged lepton mixing may be substantial. The propagating modes in the medium are determined by the poles of the exact propagator. An off diagonal self-energy $\Sigma_{e,\mu}$ entails that the charged lepton propagating modes are admixtures of electron and muon degrees of freedom. We now study this possibility in detail when the temperature is much larger than the lepton masses. We focus on this case for simplicity in order to extract the main features of the phenomenon and to highlight the main steps in the calculation.

We are only interested in the real part of the self-energies since at this order the imaginary part vanishes on the charged lepton mass shells. From the results obtained in 14, for any loop with a lepton with mass $m$ in the high temperature limit $T \gg m$ we obtain

$$
Re \Sigma(\omega, k) = \gamma^0 \sigma^0(\omega, k) - \gamma^1 \cdot \hat{k} \sigma^1(\omega, k).
$$

(33)
For $M_{W,Z} \gg T, \omega, k$ we find

$$\sigma^0(\omega,k) = -\frac{3}{\sqrt{2}} \frac{g G_F n_\gamma}{L} + \frac{7 \pi^2}{15 \sqrt{2}} \frac{g G_F \omega T^4}{M_B^2} I,$$

$$\sigma^1(\omega,k) = -\frac{7 \pi^2}{45 \sqrt{2}} \frac{g G_F k T^4}{M_B^2} I,$$

where $g$ is the appropriate factor for charged or neutral currents, $L$ is the asymmetry for the corresponding lepton and $M_B = M_{W,Z}$ for charged or neutral currents, $n_\gamma = 2 \zeta(3) T^3/\pi^2$, and

$$I = \frac{120}{7 \pi^4} \int_0^\infty \frac{x^3}{e^x - 1} dx.$$

The neutrino asymmetries are given by $L_\alpha = (\pi^2/12 \zeta(3))(\xi_\alpha + \xi_\alpha^3/\pi^2)$. In the absence of oscillations the combined analysis from CMB and BBN yield an upper bound on the asymmetry parameters $|\xi_e| \lesssim 0.1, |\xi_\mu| \sim 1$ for flavor neutrinos which we assume to imply a similar bound on the asymmetries for the mass eigenstates, $\xi_\alpha$. The validity of this assumption in free field theory is confirmed by the analysis in ref. [42]. From (32) and the above results the following general form for the charged and neutral current self-energies is obtained

$$\text{Re} \Sigma(\omega,k) = 3 \frac{G_F n_\gamma}{2 \sqrt{2}} [\gamma^0 A(\omega) - \gamma \cdot \vec{k} B(k)],$$

where $A(\omega,k)$ and $B(\omega,k)$ are $2 \times 2$ matrices in the charged lepton flavor basis given by

$$A(\omega) = \begin{pmatrix} A_{ee}(\omega) & A_{e\mu}(\omega) \\ A_{e\mu}(\omega) & A_{\mu\mu}(\omega) \end{pmatrix}, \quad B(k) = \begin{pmatrix} B_{ee}(k) & B_{e\mu}(k) \\ B_{e\mu}(k) & B_{\mu\mu}(k) \end{pmatrix},$$

where the matrix elements are given by

$$A_{ee}(\omega) = \left[ L_+ + \cos 2\theta L_- - \frac{7 \pi^4}{90 \zeta(3)} \frac{\omega T}{M_W^2} (I_+ + \cos 2\theta I_-) \right]$$

$$A_{\mu\mu}(\omega) = \left[ L_+ - \cos 2\theta L_- - \frac{7 \pi^4}{90 \zeta(3)} \frac{\omega T}{M_W^2} (I_+ - \cos 2\theta I_-) \right]$$

$$A_{e\mu}(\omega) = \sin 2\theta \left[ L_- - \frac{7 \pi^4}{90 \zeta(3)} \frac{\omega T}{M_W^2} I_- \right]$$

and

$$B_{ee}(k) = -\frac{7 \pi^4}{270 \zeta(3)} \frac{k T}{M_W^2} (I_+ + \cos 2\theta I_-)$$

$$B_{e\mu}(k) = -\frac{7 \pi^4}{270 \zeta(3)} \frac{k T}{M_W^2} (I_+ - \cos 2\theta I_-)$$

$$B_{\mu\mu}(k) = \frac{7 \pi^4}{270 \zeta(3)} \frac{k T}{M_W^2} \sin 2\theta I_-,$$

where we have introduced $L_\pm = L_1 \pm L_2, I_\pm = I_1 \pm I_2$. The neutral current contributions are flavor diagonal and
given by

\[
A_{ee}^{NC}(\omega) = -(1 - 4 \sin^2 \theta_w) \left[ \mathcal{L}_e - \frac{7 \pi^4}{90 \zeta(3)} \frac{\omega T}{M_W^2} \cos^2 \theta_w I_e \right] - \frac{4}{3} \sum_f g_f^e L_f \tag{45}
\]

\[
A_{\mu\mu}^{NC}(\omega) = -(1 - 4 \sin^2 \theta_w) \left[ \mathcal{L}_\mu - \frac{7 \pi^4}{90 \zeta(3)} \frac{\omega T}{M_W^2} \cos^2 \theta_w I_\mu \right] - \frac{4}{3} \sum_f g_f^\mu L_f \tag{46}
\]

\[
\tilde{A}_{ee}^{NC}(\omega) = -4 \sin^4 \theta_w \left[ \mathcal{L}_e - \frac{7 \pi^4}{90 \zeta(3)} \frac{\omega T}{M_W^2} \cos^2 \theta_w I_e \right] + \frac{8}{3} \sin^2 \theta_w \sum_f g_f^e L_f \tag{47}
\]

\[
\tilde{A}_{\mu\mu}^{NC}(\omega) = -4 \sin^4 \theta_w \left[ \mathcal{L}_\mu - \frac{7 \pi^4}{90 \zeta(3)} \frac{\omega T}{M_W^2} \cos^2 \theta_w I_\mu \right] + \frac{8}{3} \sin^2 \theta_w \sum_f g_f^\mu L_f \tag{48}
\]

and

\[
B_{ee}^{NC}(k) = -\frac{7 \pi^4}{270 \zeta(3)} \frac{k T}{M_W^2} (1 - 4 \sin^2 \theta_w) \cos^2 \theta_w I_e \tag{49}
\]

\[
B_{\mu\mu}^{CC}(k) = -\frac{7 \pi^4}{270 \zeta(3)} \frac{k T}{M_W^2} (1 - 4 \sin^2 \theta_w) \cos^2 \theta_w I_\mu \tag{50}
\]

\[
\tilde{B}_{ee}^{NC}(k) = \frac{4 \sin^4 \theta_w B_{ee}^{NC}(k)}{(1 - 4 \sin^2 \theta_w)} \text{; } \tilde{B}_{\mu\mu}^{NC}(k) = \frac{4 \sin^4 \theta_w B_{\mu\mu}^{NC}(k)}{(1 - 4 \sin^2 \theta_w)} \tag{51}
\]

where \( g_f^e, L_f \) are the vector coupling and asymmetry of fermion species \( f \), \( \theta_w \) is the Weinberg angle and \( \mathcal{L}_{e,\mu} \) are the charged lepton asymmetries. The non-vanishing off-diagonal matrix elements \( A_{eC}^{CC}, B_{CC}^{CC} \) lead to charged lepton mixing and oscillations. It is convenient to combine the charged leptons into a doublet of Dirac fields

\[
\psi(\omega, k) = \begin{pmatrix} \psi_e(\omega, k) \\ \psi_\mu(\omega, k) \end{pmatrix} . \tag{52}
\]

In the chiral representation the left and right handed components of the Dirac doublet are written as linear combinations of Weyl spinors \( v^{(h)} \) eigenstates of the helicity operator \( \vec{\sigma} \cdot \vec{k} \) with eigenvalues \( h = \pm 1 \), as follows\[17\]

\[
\psi_L = \sum_{h=\pm 1} \begin{pmatrix} v^{(h)} \\ 0 \end{pmatrix} ; \psi_R = \sum_{h=\pm 1} \begin{pmatrix} 0 \\ v^{(h)} \end{pmatrix} \tag{53}
\]

The left handed doublet

\[
\varphi^{(h)}(\omega, k) = \begin{pmatrix} l_1^{(h)}(\omega, k) \\ l_2^{(h)}(\omega, k) \end{pmatrix} , \tag{54}
\]

obey the following effective Dirac equation in the medium to leading order in \( \alpha, G_F \)[17]

\[
\left\{ \left[ (\omega + i\Gamma)^2 - k^2 \right] \mathbb{1} + \frac{3 G_F n_e}{2\sqrt{2}} \left( 2\omega \tilde{A}^{NC} - 2k \tilde{B}^{NC} + (\omega - h k)(\tilde{A} + h \tilde{B}) \right) - \tilde{M}^2 \right\} \varphi^{(h)}(\omega, k) = 0 , \tag{55}
\]

where \( \tilde{M}^2 = \text{diag}(M_0^2 + 2m_0^2, M_\mu^2 + 2m_\mu^2) \) where \( m_0, m_\mu \) are given by eqn. \[28\] and to avoid cluttering of notation \( \tilde{A}, \tilde{B} \) are the sums of the charged and neutral current contributions. To leading order the right handed doublet is determined by the relation \[17\]

\[
\xi^{(h)}(\omega, k) = -\mathbb{M} \frac{\omega + h k}{\omega^2 - k^2} \varphi^{(h)}(\omega, k) . \tag{56}
\]

The propagating modes in the medium are found by diagonalization of the above Dirac equation. Let us introduce a doublet of collective modes in the medium

\[
\chi^{(h)}(\omega, k) = \begin{pmatrix} l_1^{(h)}(\omega, k) \\ l_2^{(h)}(\omega, k) \end{pmatrix} , \tag{57}
\]
related to the flavor doublet $\varphi^{(h)}(\omega, k)$ by a unitary transformation $U^{(h)}_m$ with

$$U^{(h)}_m = \begin{pmatrix} \cos \theta^{(h)}_m & \sin \theta^{(h)}_m \\ -\sin \theta^{(h)}_m & \cos \theta^{(h)}_m \end{pmatrix},$$

and a similar transformation for the right handed doublet $\xi^{(h)}(\omega, k)$, where the mixing angle $\theta^{(h)}_m$ depend on $h, k$ and $\omega$. The eigenvalue equation in diagonal form is given by

$$\left\{ (\omega + i \Gamma)^2 - k^2 + \frac{1}{2} S_h(\omega, k) - \frac{1}{2} (M^2_e + M^2_\mu + 2 m^2_e + 2 m^2_\mu) + \frac{1}{2} \Omega_h(\omega, k) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \chi^{(h)}(\omega, k) = 0,$$

where $S_h(\omega, k), \Delta_h(\omega, k)$ and $\Omega_h(\omega, k)$ are respectively given by

$$S_h(\omega, k) = \frac{3 G_F n_\gamma}{2 \sqrt{2}} \left\{ (\omega - h k) [A_{\mu\mu}(\omega) + A_{ee}(\omega) + h(B_{ee}(k) + B_{\mu\mu}(k))] + 2 \omega (\tilde{A}^{NC}_{\mu\mu} + \tilde{A}^{NC}_{ee}) - 2 k (\tilde{B}^{NC}_{\mu\mu} + \tilde{B}^{NC}_{ee}) \right\},$$

$$\Delta_h(\omega, k) = \frac{3 G_F n_\gamma}{2 \sqrt{2}} \left\{ (\omega - h k) [A_{\mu\mu}(\omega) - A_{ee}(\omega) + h(B_{ee}(k) - B_{\mu\mu}(\omega))] + 2 \omega (\tilde{A}^{NC}_{\mu\mu} - \tilde{A}^{NC}_{ee}) - 2 k (\tilde{B}^{NC}_{\mu\mu} - \tilde{B}^{NC}_{ee}) \right\},$$

$$\Omega_h(\omega, k) = \left( \left[ (\delta \tilde{M}^2 - \Delta_h(\omega, k))^2 + [2(\omega - h k)] (A_{ee} + h B_{ee}) \right] \right)^{\frac{1}{2}},$$

where $\delta \tilde{M}^2 = M^2_\mu - M^2_e + 2 m^2_\mu - 2 m^2_e$. The mixing angle in the medium is determined by the relations

$$\sin 2\theta^{(h)}_m = -\frac{2(\omega - h k) (A_{ee} + h B_{ee})}{\Omega_h(\omega, k)}; \quad \cos 2\theta^{(h)}_m = \frac{\delta \tilde{M}^2 - \Delta_h(\omega, k)}{\Omega_h(\omega, k)},$$

where $\omega$ must be replaced by the solution of the eigenvalue equation (60) for each collective mode. A remarkable convergence of scales emerges for $T \sim 5$ GeV: if the neutrino asymmetry $|L_{-}| \sim 1$ then for nearly thermalized relativistic charged leptons with $\omega - h k$ with $k \sim T$, all of the terms in the expression for $\Omega_h(\omega, k)$ are of the same order, namely, $|\Delta_h(k, k)| \sim |TA_{ee}| \sim \delta \tilde{M}^2$. For relativistic leptons $|\omega|$ can be replaced by $k$ in the arguments of the functions $A, B$ to leading order in $\alpha, G_F$.

For $M_\mu \ll \omega, k, T \ll M_W$ we find that the leading contribution to $\Delta_h$ is given by

$$\Delta_h(\omega, k) \simeq 1.2 \times 10^{-5} \left( \frac{T}{\text{GeV}} \right)^4 \left[ \left( \frac{\omega - h k}{2T} \right) L_{-} \cos 2\theta - 0.29 \left( \frac{6.3 \omega - h k}{7.3T} \right) L_{\mu} - L_{ee} \right] (\text{GeV}^2),$$

where we have used the value $\sin^2 \theta_w = 0.23$. A resonance in the mixing angle occurs for $\Delta_h = \delta \tilde{M}^2$. The typical momentum of a lepton in the plasma is $k \sim T$, therefore in the temperature regime $T \ll M_W$ wherein our calculation is reliable, a resonance is available $\omega_{e,\mu}(k) \sim -h k \sim -h T$ if the neutrino asymmetry is close to the upper bound. Taking the values for $|\xi_a|$ inferred from the upper bounds from combined CMB and BBN data in absence of oscillations and the fit $\tan^2 \theta \sim 0.40$ from the combined solar and KamLAND data suggest the upper bound $|L_{-} \cos 2\theta| \sim 1$. With the asymmetry parameters for the charged leptons $|\xi_f|$ smaller than or of the same order of $|\xi_a|$, resonant mixing may occur in the temperature range $T \sim 5$ GeV. Even when the asymmetries from charged leptons do not allow for a resonance or at lower temperature, it is clear that at high temperature and for large neutrino asymmetries such
that $L \sim 1$ there is a large mixing angle because of the convergence of scales. Hence at high temperature and large differences in the chemical potential for mass eigenstates, the propagating charged lepton collective excitations in the medium will be large admixtures of $e;\mu$ states. Consider a slightly off-equilibrium disturbance in the medium corresponding to an initial state describing an inhomogeneous wave packet of electrons. The real time evolution of this state in the medium has to be studied as an initial value problem. Following the real time analysis presented in ref. [17], we find that if an initial state describes a wave-packet of left handed electrons of helicity $h$, with amplitude $l^h_e(0;k); k \gg M_\mu$ but no muons, the persistence and transition probabilities are given by

$$P_{e\rightarrow e}(t;k) = |l^h_e(0;k)|^2 e^{-2t} \left[ 1 - \sin^2(2\theta_m(k)) \sin^2 \left( \frac{\Omega(k)}{4k} t \right) \right]$$

(66)

$$P_{e\rightarrow \mu}(t;k) = |l^h_e(0;k)|^2 e^{-2t} \sin^2(2\theta_m(k)) \sin^2 \left( \frac{\Omega(k)}{4k} t \right) ; \Omega(k) = \Omega_{\pm}(k,k).$$

(67)

The exponential prefactor reveals the equilibration of the charged lepton distribution with the equilibration rate $2\Gamma_{\pm}$ [10, 12]. It is also remarkable that $\Gamma \sim \Omega(k)/k$ in the temperature and energy regime of relevance for the resonance $k \sim T \sim 5$ GeV. Therefore we conclude that during the equilibration time scale of charged leptons, there is a substantial transition probability. Collisional contributions are of order $\alpha^2 T$ or $G_5^2 T^5$ for electromagnetic or weak interaction processes leading to collisional relaxation time scales far larger than the oscillation scale for $T \sim 5$ GeV.

In the radiation dominated phase for $M_W \gg T$ as discussed here, we find that the ratio of the oscillation to the expansion time scale $\mathcal{H}_{\text{osc}} \lesssim 10^{-16} (T/\text{GeV})^3 < 1$, namely oscillation of mixed charged leptons occur on time scales much shorter than the expansion scale and the cosmological expansion can be considered adiabatic.

### C. Remarks: beyond perturbation theory

We have focused on the high temperature limit to provide a detailed calculation within a simpler scenario, to extract the main aspects of the phenomenon and to highlight the main steps of the calculation. However, it is clear that a similar calculation can be performed at much lower temperature and the point of principle is still valid under the assumption of an equilibrium density matrix diagonal in the mass basis: there could be substantial charged lepton mixing if there are large chemical potential differences between the distribution functions of mass eigenstates. As per the discussion above, this is not the only scenario that yields substantial charged lepton mixing, the general condition is that the off diagonal self energy $\Sigma_{\mu L}$ in eqn. [8] be non-zero (and large). The off diagonal expectation value $\langle \nu_{\mu L} \gamma \nu_{\mu L} \rangle$ must in general be found from the equilibrium solution of a kinetic equation, but with a consistent treatment that avoids the caveats and subtleties discussed in section [14, 15].

The arguments in favor of an equilibrium density matrix diagonal (or nearly so) in the mass basis, and the specific calculation described above relied on a perturbative expansion in the interaction picture of the unperturbed Hamiltonian $H_0$ which includes the neutrino mass matrix. There are possible caveats in the validity of perturbation theory, particularly in the case where medium effects lead to large corrections to the single particle states. For example, a large “index of refraction” arising from forward scattering with particles in the medium may lead to non-perturbative changes in the properties of the single particle basis. The lowest order contribution to the self-energy from forward scattering has been obtained in ref. [12], these are in general dependent on the energy of the neutrinos. Including these corrections in a perturbative approach entails summing the geometric Dyson series for the one-particle irreducible self energy in the neutrino propagators. This case is akin to the generation of a thermal mass from forward scattering in a scalar $\phi^4$ field theory at finite temperature [16], when this thermal mass is larger than the zero temperature mass there is a large modification in the propagating single particle modes in the medium. In the scalar field theory case a self-consistent re-arrangement of the perturbative expansion consists in adding the thermal mass to the unperturbed Hamiltonian and at the same time a counterterm in the interaction part. The free single particle propagators now include the thermal mass term, and in order to avoid double counting, the counterterm in the interaction Hamiltonian cancels the contributions that yield the thermal mass corrections systematically order by order in the perturbative expansion. We propose a similar strategy to include the medium modifications to the propagating single particle modes in the medium. In references [13, 17] it is found that the forward scattering contributions to the effective Hamiltonian in the medium are of the form

$$\delta H = \gamma^0 A(k) - \gamma \cdot \vec{k} B(k)$$

(68)

with $A(k); B(k)$ momentum dependent matrices in flavor space, their explicit expressions are given in refs. [13, 17]. A
re-arrangement of the perturbative expansion results by writing

\[ H = \tilde{H}_0 + \tilde{H}_{int} \tag{69} \]

where \( \tilde{H}_0 = H_0 + \delta H \) and \( \tilde{H}_{int} = H_{int} + H_c \) where the “counterterm” Hamiltonian \( H_c = -\delta H \) systematically cancels the forward scattering corrections to the self-energies consistently in the perturbative expansion. The new “free” Hamiltonian \( \tilde{H}_0 \) includes self-consistently the modifications to the propagating single particle states from the in-medium index of refraction. The field operators are now written in the basis of the solutions of the Dirac equation from the new Hamiltonian and finally the interaction is written in terms of these fields. Thus the perturbative expansion is re-organized in terms of the single particle propagating modes in the medium. The main complication in this program is that the mixing angles in the medium which determine the single particle propagating modes, are energy dependent, this introduces a non-locality in the interaction vertices which now become dependent on the energy and momentum flowing into the vertex.

If there is substantial charged lepton mixing, such phenomenon in turn affects the index of refraction for neutrinos in the medium and possibly the equilibration dynamics of neutrinos. A self-consistent treatment of the charged lepton mixing and neutrino mixing and relaxation would be required to understand the dynamical aspects of neutrino and charge lepton equilibration. This task is beyond the goals and focus of this article.

IV. CONCLUSIONS AND DISCUSSIONS

In this article we focused on studying the possibility of charged lepton mixing as a consequence of neutrino mixing at high temperature and density in the early Universe. There are three main points in this article:

• (I) We establish that a general criterion for charged lepton mixing as a consequence of neutrino mixing is that there must be off diagonal correlation of flavor fields in the density matrix. We identified one possible case in which there could be charged lepton mixing: that the equilibrium density matrix be nearly diagonal in the mass basis. This is the case in the vacuum, but in this case the smallness of the neutrino masses entails that charged lepton mixing is negligible. We argued that this effect can be enhanced in a medium if the density matrix is nearly diagonal in the mass basis with large and different chemical potential for mass eigenstates. While this is not the only case in which charged lepton mixing can occur, it is one in which we can provide a definite calculation to assess charged lepton mixing.

• (II) We have given general arguments to suggest that within the realm of validity of perturbation theory, the equilibrium density matrix must be nearly diagonal in the mass basis. We have critically re-examined the kinetic approach to neutrino mixing and relaxation in a medium at high temperature and density and highlighted several caveats and subtleties with flavor states, and or Fock operators associated with these states that cloud the interpretation of the density matrix. We argue that an equilibrium solution of the kinetic equations describing “flavor equalization” can be interpreted as a confirmation that the density matrix is nearly diagonal in the mass basis. This interpretation leads us to the main and only assumption, namely that before “flavor equalization” for \( T \gtrsim 30 \text{ MeV} \) neutrinos are in equilibrium, the density matrix is nearly diagonal in the mass basis but with distribution functions for mass eigenstates with large and different chemical potentials in agreement with the bounds from BBN and CMB in absence of oscillations.

• (III) Under this assumption and the validity of perturbation theory we have provided a definite calculation of charged lepton mixing. While the general criterion for charged lepton mixing does not imply that this is the only case in which charged leptons mix, it is a scenario that allows a definite calculation to assess the phenomenon in a quantitative manner.

In conclusion, under the assumption that the mass eigenstates of mixed neutrinos in the early Universe are in thermal equilibrium with different chemical potentials for \( T >> 30 \text{ MeV} \), before oscillations establish the equalization of flavor asymmetries neutrino mixing leads to charged leptons mixing with large mixing angles in the plasma. We explored this possibility by obtaining the leading order contributions to the charged lepton self energies in the high temperature limit. If the upper bounds on the neutrino asymmetry parameters from BBN and CMB without oscillations is assumed along with the fit for the vacuum mixing angle for two generations from the KamLAND data, we find that charged leptons mix resonantly in the temperature range \( T \sim 5 \text{ GeV} \) in the early Universe. The electromagnetic damping rate is of the same order as the oscillation frequency in the energy and temperature regime relevant for the resonance suggesting a substantial transition probability during equilibration. The cosmological expansion scale is much larger than the time scale of charged lepton oscillations. Although we assumed the validity
of perturbation theory we recognized possible caveats in the high temperature limit arising from potentially large corrections to the single-particle propagating modes from the in-medium index of refraction. We proposed a re-organization of the perturbative expansion that includes the correct single-particle propagators self-consistently. We have focused on the high temperature limit as a simpler scenario to assess charged lepton mixing, however, the calculation can be performed at lower temperatures with the corresponding technical complications associated with the lepton masses. While at much lower temperatures there is no resonant mixing of charged leptons, the results of the calculation establish a point of principle, namely that for large chemical potential differences in the distribution function of mass eigenstates of neutrinos, the charged lepton propagating modes in the medium will be admixtures of the electron and muon degrees of freedom with non-vanishing mixing angle.

We believe that the phenomenon of charged lepton mixing in a medium warrants a deeper and thorough investigation. Our study also raises relevant questions on the kinetic approach: a consistent description of the kinetics of neutrino oscillation and relaxation avoiding the caveats associated with flavor states and or Fock operators associated with these states. Furthermore, substantial charged lepton mixing also suggests that a dynamical description should include self-consistently both neutrino and charged lepton mixing in a full non-equilibrium treatment. These aspects as well as possible consequences of charged lepton mixing for leptogenesis will be explored elsewhere.

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