EXPECTED EXPONENTIAL UTILITY MAXIMIZATION OF INSURERS WITH A GENERAL DIFFUSION FACTOR MODEL: THE COMPLETE MARKET CASE

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ABSTRACT. In this paper, we consider the problem of optimal investment by an insurer. The insurer invests in a market consisting of a bank account and m risky assets. The mean returns and volatilities of the risky assets depend nonlinearly on economic factors that are formulated as the solutions of general stochastic differential equations. The wealth of the insurer is described by a Cramér–Lundberg process, and the insurer preferences are exponential. Adapting a dynamic programming approach, we derive Hamilton–Jacobi–Bellman (HJB) equation. And, we prove the unique solvability of HJB equation. In addition, the optimal strategy is also obtained using the coupled forward and backward stochastic differential equations (FBSDEs). Finally, proving the verification theorem, we construct the optimal strategy.

1. INTRODUCTION

Recently, optimization problems of insurance companies have been studied by many works. Most of these works were solved by analyzing Hamilton–Jacobi–Bellman (HJB) equations using the dynamic programming approach. We shall introduce these studies below.

Problems of minimizing ruin probabilities has been studied by [1, 5, 12, 25, 26, 44, 45, 46, 51, 52, 59]. [1, 5, 12, 25, 26] treated optimal investment problems, and [46, 51, 59] studied optimal reinsurance problems. And, [44, 45, 52] considered optimal investment and reinsurance problems.

[7, 18, 23, 61, 64, 65] studied optimal investment problems for maximizing expected exponential utilities. [18, 61, 65] employed Black-Scholes models. And, [7, 8, 22] adopted stochastic factor models that can be expected to compensate for the disadvantages of Black-Scholes model.

Mean-variance insurer’s optimal investment-reinsurance problems have been studied by [37, 38, 39, 56, 68]. [21] studied optimal investment-reinsurance problem for maximizing a power utility criterion. [4, 30, 40, 41, 57, 64, 62, 66, 67] treated optimal investment-reinsurance problems for maximizing exponential utility criterions.

Moreover, [11] studied an optimal investment and risk control problem.

On the other hand, [63, 69, 70] use the martingale method based on equivalent martingale measures and martingale representation theorems. Wang et al. [63]
treated an expected exponential utility maximization with Black–Scholes model. And, Zhou [70] considered a counterpart of [63] with the risky asset described by a Lévy process. Further, Zhou-Cadenillas [69] studied an optimal investment and risk control problem.

In addition, the stochastic maximum principle and forward backward stochastic differential equations (FBSDEs) are the main tools to obtain the solutions, see [50, 60]. Hu et al. [28] studied the utility optimization problem in incomplete market through the FBSDE approach. Cheridito-Hu [13] analyzed the optimal consumption and investment problem with general constraints in complete market. Horst et al. [27] considered the utility optimization problem with liability. The portfolio optimization under nonlinear utility is discussed in [24]. Sekine [53] discussed exponential hedging by applying BSDEs approach. Shen-Zeng [56] solved mean-variance optimal investment-reinsurance problem by adapting BSDEs approach.

The existence and uniqueness of BSDEs with the Lipschitz generators and the squared integrable terminal condition are proved in [54]. However, based on the feature of the Merton type problems, the corresponding FBSDEs with quadratic growth are obtained. The existence of quadratic backward stochastic differential equations (QBSDEs) with the bounded terminal condition is discussed in [31, 58]. In particular, Bahlali et al [2] and Briand and Hu [9, 10] analyzed QBSDE with unbounded terminal values and $L^2$-terminal data respectively. Barrieu and El Karoui [3] studied the unbounded quadratic BSDEs. Imkeller and Reis [31] discussed the path regularity for BSDEs with truncated quadratic growth and Frei et al. [20], Cheridito and Nam [15], Hu and Tang [29], and Jammeshan et al. [32] considered the multi-dimensional backward stochastic differential equations. The results of one dimensional superquadratic BSDEs are shown in [14, 17].

Peng and Wu [55] proposed the existence and uniqueness of fully coupled FBSDEs based on the monotonicity conditions. Delarue [16] discussed the existence and uniqueness of FBSDEs in a non-degenerate case based on the connection with the quasi-linear parabolic system of PDEs. The well posedness of FBSDEs where the coefficients are uniformly Lipschitz are analyzed in [47] using the decoupling random field. The solvability of coupled FBSDEs with quadratic growth is studied in [42, 43] through the Bounded Mean Oscillation (BMO)-martingale technique conditional on the small time duration. Kupper et al. [36] analyzed the local and global existence and uniqueness results for multidimensional coupled FBSDEs with superquadratic growth using the Malliavin calculus and PDE technique. Note that in the previous discussion, the coupled FBSDEs do not have the jumps. The uniqueness of the coupled FBSDEs with jumps is difficult to be verified using the PDE approach due to the challenge given by the regularity for partial integro-differential equations (PIDE). In addition, in order to show the existence and uniqueness of FBSDEs using the BMO method, the jump terms is needed to be bounded See [48]. This condition is not suitable for the proposed problem since the claims for the insurer are unbounded. In this paper, we study the particular coupled FBSDEs with jumps motivated by the optimization portfolio problem for the insurer. The uniqueness of the corresponding FBSDEs with jumps can be obtained using the Jensen’s inequality and martingale technique.

In particular, we state Badaoui-Fernández [7], Badaoui et al. [8], Fernández et al. [18], Hata-Yasuda [23], Wang [61], Yang-Zhang [65], Zhou [70]. These treated the
problems of optimal investment by insurance companies when the utility function is of exponential type.

These problems can be often solved by using dynamic programming approach. 
Browne [12], Fernández et al. [18], Wang [61], and Yang-Zhang [65] considered Black-Scholes models. In [12] the risk process follows a Brownian motion with drift. In [18] the classical Cramér–Lundberg model was adopted as the risk process. In [61] the claim process was a pure jump process. In [65] the risk process was a compound Poisson process perturbed by a standard Brownian motion.

Badaoui-Fernández [7] and Badaoui et al. [8] considered stochastic volatility models as a counterpart of [18]. Note that in [7], the correlation between the risky asset and the factor process is zero. [8] allowed that the risky assets and factor processes are correlated. Hata-Yasuda [23] treated a linear Gaussian stochastic asset and the factor process is zero. [8] allowed that the risky assets and factor processes are correlated. Hata-Yasuda [23] treated a linear Gaussian stochastic asset and the factor process is zero. [8] allowed that the risky assets and factor processes are correlated.

Our objective is to extend previous work [23] to a more general stochastic factor model that the riskless interest rate is not constant. Indeed, we consider a market defined by

\[ S^0_t = S^0_0 r(Y_t) dt, \quad S^0_0 = s_0, \]

\[ dS^i_t = S^i_t \left\{ \mu^i(Y_t) dt + \sum_{k=1}^{m} \sigma^i_{pk}(Y_t) dW^k_t \right\}, \quad S^i_0 = s^i_0, \]

\[ dY_t = g(Y_t) dt + \sigma_f(Y_t) dW_t, \quad Y_0 = y \in \mathbb{R}^n, \]

where \((W_t)_{t \geq 0}\) is an \(m\) dimensional standard Brownian motion process defined on an underlying probability space \((\Omega, \mathcal{F}, P)\). And, \(\sigma_p\) and \(\sigma_f\) are \(m \times m\), \(n \times m\) matrix-valued functions respectively and \(\mu\) and \(g\) are \(\mathbb{R}^m\)-valued, \(\mathbb{R}^n\)-valued functions respectively.

Consider an insurer who invests at time \(t\) the amount \(\pi^i_t\) of wealth \(X^i_t\) in the \(i\)th risky asset with price \(S^i_t\), \(i = 1, \ldots, m\). With \(\pi_t = (\pi^1_t, \ldots, \pi^m_t)^*\) chosen, the amount of his wealth invested in the bank account is

\[ X^\pi_t - \pi^* t 1 = X^\pi_t - \sum_{i=1}^{m} \pi^i_t. \]

Here \(1 = (1, \ldots, 1)^*\). Then, the insurer’s wealth has the dynamics:

\[ X^\pi_t = x + ct - J_t + \int_0^t \left\{ \sum_{i=1}^{m} \pi^i_s \frac{dS^i_s}{S^i_s} + (X^\pi_s - \pi^* s 1) \frac{dS^0_s}{S^0_s} \right\} ds \]

\[ = x + \int_0^t \left\{ c + \pi^* s (\mu(Y_s) - r(Y_s) 1) + r(Y_s) X^\pi_s \right\} ds + \int_0^t \pi^* s \sigma_p(Y_s) dW_s - J_t, \]

where \(x\) is the initial surplus and \(c > 0\) is the premium rate. And, the process \(J_t\) is defined by

\[ J_t := \sum_{i=1}^{m} Z_i, \]
where \((p_t)_{t \geq 0}\) is a Poisson process with a constant intensity \(\lambda > 0\), and \((Z_i)_{i \geq 1}\), the claim sizes, is a sequence of independent non-negative random variables with identical distribution \(\nu\). Moreover, we assume that
\[
\int_{z > 0} z \nu(dz) < \infty.
\]
This gives the moment condition of claim size which will be cited later in this paper. A stronger moment condition will be stated later (see (A6) in Theorem 2.1). We also assume that \((W_t)_{t \geq 0}, (p_t)_{t \geq 0}\) and \((Z_i)_{i \geq 1}\) are mutually independent. Moreover, for each \(t > 0\) the filtration \((\mathcal{F}_t)_{t \geq 0}\) is defined by
\[
\mathcal{F}_t := \sigma\{W_s, p_s, Z_j 1_{j \leq p_s}; s \leq t, j \geq 1\}.
\]
Then, we write Poisson random measure of \(J\) on \([0, \infty) \times [0, \infty)\) as
\[
N([0, t] \times U) := \sum_{0 \leq s \leq t} 1_U(\Delta J_s) = \sum_{i=1}^{p_t} 1_U(Z_i)
\]
for a Borel set \(U \subset [0, \infty)\), where \(\Delta J_s := J_s - J_{s-}\). We have
\[
J_t = \int_0^t \int_{z > 0} z N(ds, dz).
\]
Then, we also define the compensated Poisson random measure
\[
\tilde{N}(dt, dz) := N(dt, dz) - \lambda \nu(dz)dt.
\]
For simplicity, we always assume \(r, \mu, g, \sigma_p\) and \(\sigma_f\) are sufficiently smooth. We also assume the following conditions:
(A1) \(r, \mu, g, \sigma_p\) and \(\sigma_f\) are globally Lipschitz smooth such that their first order and second order derivatives are of linear growth.
(A2) \(\sigma_p(x)\) is invertible.
(A3) There are constants \(\mu_1, \mu_2 > 0\) such that for \(x, \eta \in \mathbb{R}^n, \xi \in \mathbb{R}^m\),
\[
\mu_1 |\xi|^2 \leq \xi^* \sigma_p(x) \sigma_p(x)^* \xi \leq \mu_2 |\xi|^2,
\]
\[
\mu_1 |\eta|^2 \leq \eta^* \sigma_f(x) \sigma_f(x)^* \eta \leq \mu_2 |\eta|^2.
\]
(A4) \(r\) satisfies
\[
0 \leq r(y) \leq \bar{r},
\]
where \(\bar{r}\) is a positive constant.

Remark 1.1. The smooth of the coefficients will be needed to show the solution of HJB equation is smooth. In particular, in several places we will use Theorem 10, Section 9, Chapter 2 in [35]. See the proof of Theorem 2.2, where we also need the growth condition in (A1).

In this paper, define an expected utility of the terminal wealth : for a given constant \(T > 0\),
\[
J(T, x, y; \pi) := E[U(X_T^\pi)].
\]
Here \(U : \mathbb{R} \to \mathbb{R}\) is an exponential type utility function, i.e.
(A5) \(U(x) := -e^{-\alpha x}, \alpha > 0\).

Using dynamic programming approach and FBSDE approach, we consider the following problem:
\[ \sup_{\pi \in \mathcal{A}_T} J(T, x, y; \pi). \]

Here \( \mathcal{A}_T \subset L^2(T) \) is the set of admissible strategies, where \( L^2(T) \) is defined by
\[ L^2(T) := \{ (\pi_s)_{s \in [0, T]}; \pi_s \text{ is an } \mathbb{R}^m \text{-valued } \mathcal{F}_s \text{-progressively measurable } \}
\text{stochastic process such that } \int_0^T |\pi_s|^2 ds < \infty, P - a.s. \}.

The precise definition will be given later in this paper.

In Section 2, applying dynamic programming approach, we consider the problem \((P)\). For that, we derive the HJB equation. In Subsection 2.1 the property of the complete market will help us to get the solution of the HJB equation. In Subsection 2.2 we prove the verification theorem using the HJB equation and its solution. In Section 3, using FBSDE approach, we consider the problem \((P)\). First, using the Pontryagin maximum principle, we derive the EBSDEs. In Subsection 3.1 we obtain the solution of FBSDEs using the property of the complete market. One of our contributions is to show the uniqueness of the solution. Our method is analytical and technical, but it is an unusual method not seen elsewhere. In Subsection 3.2 using FBSDE and their solutions, we prove the verification. Our method is analytical and a good use of the nature of our setting. This is one of our contributions. Note that the set of admissible strategy in FBSDE approach is different from that in the dynamic programming approach. This will come from the difference between each other’s approaches. Finally, we check the identity for solutions, optimal strategies and optimal values from dynamic programming approach and FBSDE approach.

2. Dynamic programming approach

In this section, we derive Hamilton-Jacobi-Bellman (HJB) equation for \((P)\). We prepare the dynamic version:
\[ (P') \quad V(t, x, y) := \sup_{\pi \in \mathcal{A}_{t,T}} E \left[ U(X_{t,T}^\pi) | \mathcal{F}_t \right], \]

where \( \mathcal{A}_{t,T} \) is the restriction of the space \( \mathcal{A}_T \) on the interval \([t, T]\) and \( X_{t,T}^\pi := X_{T}/X_{t}^\pi \).

Following a standard argument ([19], Chapter IV), we can formally derive the HJB equation for \((P')\) with dynamics \([1.1]\) and \([1.2]\). The HJB equation is given by:
\[ \sup_{\pi \in \mathbb{R}^m} \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \pi^* \sigma_p(y) \sigma_p(y)^* \pi V_{xx} + \frac{1}{2} \text{tr}(\sigma_f(y)\sigma_f(y)^* V_{yy}) \right. \]
\[ \left. + \pi^* \sigma_p(y) \sigma_f(y)^* V_{xy} + \{ c + \pi^* (\mu(y) - r1) + r(y)x \} V_x + g(y)^* V_y \right] = 0, \]
\[ V(T, x, y) = U(x). \]

Here \( V_x, V_y, V_{xx}, V_{yy}, V_{xy} \) are the first order and second partial derivatives of \( V \) with respect to \( x, y \). If we assume
\[ (2.2) \quad V(t, x, y) := -e^{-v(t,x,y)}, \]
then \( v \) solves the following partial differential equation:

\[
\begin{align*}
(2.3) \quad & \sup_{\pi \in \mathbb{R}^m} \mathcal{L}^\pi v(t, x, y) = 0, \quad v(T, x, y) = \alpha x, \\
(2.4) \quad & \text{where } \mathcal{L}^\pi v(t, x, y) \text{ is defined by} \\
& \mathcal{L}^\pi v(t, x, y) := \frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}((\sigma_f(y)^*)v_{yy}) - \frac{1}{2} v_y^* \sigma_f(y)^* v_y + g(y)^* v_y \\
& + \{c + r(y)x\}v_x + \frac{v_{xx} - (v_x)^2}{2} \pi^* \sigma_p(y)^* \pi + \pi^* \{ (\mu(y) - r(y))1 \} v_x \\
& + \sigma_p(y)^* (v_{xy} - v_x v_y) \} - \lambda \int_{z > 0} \left( e^{-v(t,x-z,y) + v(t,x,y) - 1} \right) \nu(dz),
\end{align*}
\]

If \( v_{xx} - (v_x)^2 < 0 \) holds, we see that the maximizer is

\[
\hat{\pi}(t, x, y) := - (\sigma_p(y)^*)^{-1} \frac{\{\theta(y) - \sigma_f(y)^* v_y\} v_x + \sigma_f(y)^* v_{xy}}{v_{xx} - (v_x)^2}
\]

Set

\[
(2.5) \quad \theta(y) := \sigma_p(y)^{-1} (\mu(y) - r(y)1).
\]

Then, we rewrite (2.3) as

\[
\begin{align*}
(2.6) \quad & \frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}((\sigma_f(y)^*)v_{yy}) - \frac{1}{2} v_y^* \sigma_f(y)^* v_y + g(y)^* v_y \\
& + \{c + r(y)x\}v_x - \frac{1}{2} \frac{1}{2} \frac{v_{xx} - (v_x)^2}{2} \left( (\theta(y) - \sigma_f(y)^* v_y) v_x + \sigma_f(y)^* v_{xy} \right)^2 \\
& - \lambda \int_{z > 0} \left( e^{-v(t,x,z,y) + v(t,x,y) - 1} \right) \nu(dz) = 0, \\
& v(T, x, y) = \alpha x.
\end{align*}
\]

**Remark 2.1.** If the riskless interest rate is constant as in [7, 23], we use suitable translations. Then, we rewrite HJB equations as parabolic partial integro-differential equations which do not depend on \( x \). We can solve the equation. However, if the riskless interest rate is not constant, this approach does not work to obtain a solution of HJB equation. In this paper, since we treat the complete market model, we are able to obtain the solution by a different approach. See Section 2.1. However, the new approach will fail if the market is not complete. It is still a challenge to solve the HJB equation.

2.1. The solution of HJB equation. By direct calculations, we have the following.

**Theorem 2.1.** Assume (A1) \( \sim (A5) \). Assume also

\[
(2.6) \quad \int_{z > 0} e^{ae^{-t+z}} \nu(dz) < \infty.
\]

Then,

\[
(2.7) \quad \bar{v}(t, x, y) := a(t, y)x + b(t, y)
\]
is a solution of (2.6). Here, \(a(t,y)\) and \(b(t,y)\) solve

\[
\begin{align*}
\frac{\partial a}{\partial t} + \frac{1}{2} \text{tr}(\sigma_f(y)\sigma_f(y)^* D^2 a) + \{g(y) - \sigma_f(y)\theta(y)\}^* Da \\
- \frac{1}{a}(Da)^* \sigma_f(y)\sigma_f(y)^* Da + r(y)a = 0,
\end{align*}
\]

(2.8)

\[a(T, y) = \alpha,\]

and

\[
\begin{align*}
\frac{\partial b}{\partial t} + \frac{1}{2} \text{tr}(\sigma_f(y)\sigma_f(y)^* D^2 b) + \{g(y) - \sigma_f(y)\theta(y)\}^* Db \\
- \sigma_f(y)\sigma_f(y)^* \frac{Da}{a} \right)^* Db + \frac{1}{2} \left| \theta(y) + \sigma_f(y)^* \frac{Da}{a} \right|^2 \\
+ ca - \lambda \int_{z > 0} \left( e^{\theta(y)z} - 1 \right) \nu(dz) = 0,
\end{align*}
\]

(2.9)

\[b(T, y) = 0.\]

Here and in the rest, \(Df = (f_{y_1}, f_{y_2}, \cdots, f_{y_n})\) is the gradient of \(f(y)\).

**Theorem 2.2.** Assume \((A1) \sim (A6)\). Then, we have the following.

1. (2.8) has a solution :

\[
a(t, y) = \alpha \hat{E}_{t,y} \left[ e^{-\int_t^T r(\hat{Y}_s)ds} \right]^{-1},
\]

(2.10)

where \(\hat{E}[:]\) denotes the expectation with respect to the probability measure \(\hat{P}\) defined by

\[
\frac{d\hat{P}}{dP}|_{\mathcal{F}_T} = \hat{E}_T,
\]

(2.11)

where

\[
\hat{E}_t := \exp \left( -\int_0^t \theta(\hat{Y}_s)^* dW_s - \frac{1}{2} \int_0^t |\theta(\hat{Y}_s)|^2 ds \right).
\]

(2.12)

And, \(\hat{Y}_s\) is the solution of

\[
d\hat{Y}_s = \left\{ g(\hat{Y}_s) - \sigma_f(\hat{Y}_s)\theta(\hat{Y}_s) \right\} ds + \sigma_f(\hat{Y}_s)d\hat{W}_s, \quad s \geq t,
\]

(2.13)

\[\hat{Y}_t = y,\]

where \(\hat{W}_s\) is a Brownian motion under \(\hat{P}\) :

\[
\hat{W}_s = W_s + \int_t^s \theta(Y_u)du.
\]

2. (2.9) has a solution :

\[
b(t, y) = \hat{E}_{t,y} \left[ \int_t^T \left\{ \frac{1}{2} \left| \theta(\hat{Y}_s) + \sigma_f(\hat{Y}_s)^* Da(s, \hat{Y}_s)a(s, \hat{Y}_s)^{-1} \right|^2 \\
+ ca(s, \hat{Y}_s) - \lambda \int_{z > 0} \left( e^{\theta(s, \hat{Y}_s)z} - 1 \right) \nu(dz) \right\} ds \right],
\]

(2.14)
where \( \tilde{E}[\cdot] \) denotes the expectation with respect to the probability measure \( \tilde{P} \) defined by
\[
\frac{d\tilde{P}}{dP} \bigg|_{\mathcal{F}_T} = \tilde{E}_T,
\]
where
\[
\tilde{E}_t := \exp \left( -\int_0^t \left\{ \theta(Y_s) + \sigma_f(Y_s)^* Da(s, \bar{Y}_s)a(s, \bar{Y}_s)^{-1} \right\}^* dW_t - \frac{1}{2} \int_0^t \left| \theta(Y_s) + \sigma_f(Y_s)^* Da(s, \bar{Y}_s)a(s, \bar{Y}_s)^{-1} \right|^2 ds \right).
\]
And, \( \bar{Y}_s \) is the solution of
\[
d\bar{Y}_s = \{ g(\bar{Y}_s) - \sigma_f(\bar{Y}_s)\theta(\bar{Y}_s) - \sigma_f(\bar{Y}_s)\sigma_f(\bar{Y}_s)^* Da(s, \bar{Y}_s)a(s, \bar{Y}_s)^{-1} \} ds + \sigma_f(\bar{Y}_s)dW_s, \quad s \geq t, \quad \bar{Y}_t = y,
\]
where \( \bar{W}_t \) is a Brownian motion under \( \tilde{P} : \)
\[
\bar{W}_t = W_t + \int_0^t \left\{ \theta(\bar{Y}_s) + \sigma_f(\bar{Y}_s)^* Da(s, \bar{Y}_s)a(s, \bar{Y}_s)^{-1} \right\} ds.
\]
Before showing the proof of this theorem, we prepare the following lemma.

**Lemma 2.1.** (Lemma 4.1.1 of [3]) Use [1.1]. For a given continuous function \( H(t, y) \) satisfying \( |H(t, y)| \leq C(1 + |y|) \) we define
\[
\rho_t := \exp \left( -\int_0^t H(s, \bar{Y}_s)^* dW_s - \frac{1}{2} \int_0^t |H(s, \bar{Y}_s)|^2 ds \right).
\]
Then, we have
\[
E[\rho_t] = 1 \quad \text{for } t \in [0, T].
\]

**Proof of Theorem 2.2.** Setting
\[
\hat{a}(t, y) = a(t, y)^{-1},
\]
then \( \hat{a} \) solves
\[
\frac{\partial \hat{a}}{\partial t} + \frac{1}{2} \text{tr} \left( \sigma_f(y)\sigma_f(y)^* D^2 \hat{a} \right) + \{ g(y) - \sigma_f(y)\theta(y) \}^* D\hat{a} - r(y)\hat{a} = 0,
\]
\[
\hat{a}(T, y) = \alpha^{-1}.
\]
From Lemma 2.1, we see that \( \tilde{P} \) is a probability measure. Hence, using Theorem 10, Section 9, Chapter 2 in [35], we have
\[
\hat{a}(t, y) = \alpha^{-1} \tilde{E}_{t,y} \left[ e^{-\int_t^T r(\bar{Y}_s)ds} \right].
\]
Therefore, (2.10) is obtained.

On the other hand, from (A4) we have
\[
0 < a(t, y) \leq \alpha e^{\rho T}.
\]
And, from Lemma 2.2 below, there is \( C > 0 \) such that
\[
|Da(t, y)|a(t, y)^{-1} \leq C(1 + |y|).
\]
Hence, using Lemma 2.1 again, we see that \( \tilde{P} \) is well-defined. And, we can check the solvability of (2.17). Using Theorem 10, Section 9, Chapter 2 in [35] again, we obtain (2.13).
Lemma 2.2. Assume (A1) \sim (A5). Then, there are \( C_1 > 0 \) and \( C_2 > 0 \) such that (2.20) is satisfied.

Proof. Setting \( \varphi(t, y) := \log a(t, y) \), we have

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} + \frac{1}{2} \tr(\sigma_f(y)\sigma_f(y)^*D^2\varphi) - \frac{1}{2}(D\varphi)^*\sigma_f(y)\sigma_f(y)^*D\varphi \\
+ (g(y) - \sigma_f(y)\theta(y))^*D\varphi + r(y) = 0,
\end{align*}
\]

(2.21)

\( \varphi(T, y) = \log \alpha \).

In the rest of the proof, we may use the result in Krylov [35] (Theorem 10, Section 9 in Chapter 2). Noting that \( \frac{\partial \varphi}{\partial t} < 0 \) holds, we see that \( \frac{\partial \varphi}{\partial y} < 0 \) holds. Using \( \frac{\partial \varphi}{\partial y} < 0 \) and following the arguments of Lemma 3.5 of [22] or Theorem 2.1 of [49], we see that

\[
|D\varphi(t, y)|^2 - \frac{\partial \varphi}{\partial t}(t, y) \leq C' \left\{ |D(\sigma_f\sigma_f^*)|^2_{2r} + |\sigma_f\sigma_f^*|^2_{2r} \\
+ |g - \sigma_f\theta|^2_{2r} + |D(g - \sigma_f\theta)|_{2r} + |r|_{2r} + |Dr|_{2r} + 1 \right\}, \quad y \in B_r, t \in [0, T],
\]

where \( C' \) is a positive constant independent of \( r \) and \( T \), \( \kappa \) is sufficiently large constant, and \( \| \cdot \|_{2r} := \| | \cdot \|_{L^\infty(B_{2r})} \). Hence, there is \( C > 0 \) such that \( |D\varphi(t, y)| \leq C(1 + |y|) \) on \( [0, T] \times \mathbb{R}^n \).

\[ \square \]

2.2. Verification theorem. In this section, we show the verification theorem for (P). Define the set of admissible strategies defined by

\[ A_T := \{ (\pi_t)_{t \in [0, T]} \in \mathcal{L}_{2, T}; E[\mathcal{E}_T(\pi)] = 1 \}, \]

where \( \mathcal{E}_t(\pi) \) is defined by

\[ \mathcal{E}_t(\pi) := \exp \left( -\int_0^t h(s, X^\pi_s, Y_s, \pi_s)^*dW_s - \frac{1}{2} \int_0^t |h(s, X^\pi_s, Y_s, \pi_s)|^2ds \right) \]

\[ \cdot \exp \left( \int_0^t \int_{s \geq 0} a(s, Y_s)zN(ds, dz) + \lambda \int_0^t \int_{s \geq 0} (1 - e^{a(s, Y_s)}z) \nu(dz)ds \right). \]

Here, \( h(t, x, y, \pi) \) is defined by

\[ h(t, x, y, \pi) := \sigma_f(y)^*\{ Da(t, y)x + Db(t, y)\} + \sigma_p(y)^*\pi a(t, y). \]

Then, we have the following.

Theorem 2.3. Assume (A1) \sim (A6). Assume also

\[ \int_{s > 0} e^{\alpha s^2 r} z \nu(dz) < \infty. \]

Define

\[ \tilde{\pi}(t, x, y) := a(t, y)^{-1}(\sigma_p(y)^*)^{-1} \left[ \sigma_f(y)^*Da(t, y)(-x + a(t, y)^{-1}) \right] \\
+ \theta(y) - \sigma_f(y)^*Db(t, y) \].

Then, the strategy \( \tilde{\pi} \in A_T \) defined by

\[ \tilde{\pi}_t := \tilde{\pi}(t, X^\tilde{\pi}_t, Y_t) \]

is optimal for (P). Moreover, we have

\[ V(0, x, y) = \tilde{V}(0, x, y), \]

where

\[ V(t, x, y) := \tilde{V}(t, x, y) \].
where $\hat{V}(t, x, y)$ is defined by

$$
\hat{V}(t, x, y) := -e^{a(t,y)x-b(t,y)}.
$$

(2.26)

Here, $a(t,y)$ and $b(t,y)$ are defined by (2.10) and (2.11) respectively.

**Proof.** We write $\hat{v}(t, x, y) = a(t,y)x + b(t,y)$. For any $\pi \in \mathcal{A}_T$, using (1.1) and (1.2), we have

$$
d\hat{v}(t, X_t^\pi, Y_t) = \left[ L^\pi \hat{v}(t, X_t^\pi, Y_t)dt + h(t, X_t^\pi, Y_t, \pi_t)^*dW_t - a(t, Y_t) \int_{z>0} zN(dt, dz) \right],
$$

where $L^\pi \hat{v}(t, x, y)$ is defined by

$$
L^\pi \hat{v}(t, x, y) = \frac{\partial \hat{v}}{\partial t} + \frac{1}{2} \text{tr}(\sigma_f(y)\sigma_f(y)^*v_{yy}) + g(y)^*v_y + \{c + (\mu(y) - r(y))\pi_t\}v_x
$$

$$
+ \frac{1}{2} \pi \sigma_p(y)\sigma_p(y)^*\pi v_{xx} + \pi \sigma_p(y)\sigma_f(y)^*v_{xy}.
$$

This can be rewritten as

$$
d\{ -\hat{v}(t, X_t^\pi, Y_t) \} = -L^\pi \hat{v}(t, X_t^\pi, Y_t)dt + d\log \mathcal{E}_T(\pi).
$$

where $L^\pi \hat{v}(t, x, y)$ is given in (2.4). Hence, using the definition of $\mathcal{A}_T$ and the fact that $L^\pi \hat{v}(t, x, y) \leq 0$ holds for $(t, x, y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$, we have

$$
J(T, x, y; \pi) = \hat{V}(0, x, y)E \left[ e^{-\int_0^T L^\pi \hat{v}(t, X_t^\pi, Y_t)dt} \mathcal{E}_T(\pi) \right]
$$

$$
\leq \hat{V}(0, x, y)E[\mathcal{E}_T(\pi)]
$$

$$
= \hat{V}(0, x, y).
$$

(2.27)

Take $\pi = \hat{\pi}$. Then, in Appendix A we observe that

$$
E[\mathcal{E}_T(\hat{\pi})] = 1.
$$

(2.28)

Namely, $\hat{\pi} \in \mathcal{A}_T$. Note that $L^\pi \hat{v}(t, x, y) = 0$ holds for $(t, x, y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$. Then, we have

$$
J(T, x, y; \hat{\pi}) = \hat{V}(0, x, y)E \left[ e^{-\int_0^T L^\pi \hat{v}(t, X_t^\pi, Y_t)dt} \mathcal{E}_T(\hat{\pi}) \right]
$$

$$
= \hat{V}(0, x, y)E[\mathcal{E}_T(\hat{\pi})]
$$

$$
= \hat{V}(0, x, y).
$$

(2.29)

$$\square$$

### 3. FBSDE Approach

In this section, we study the optimal strategy using the coupled FBSDEs based on the wealth process written as

$$
X_t^\pi = x + \int_0^t \{ r(Y_s)X_s^\pi + \pi_s^*(\mu(Y_s) - r(Y_s)1) + c \} ds
$$

$$
+ \int_0^t \pi_s^*\sigma(Y_s)dW_s - \int_0^t \int_{z>0} zN(dz, ds),
$$

(3.1)

where the factor $Y$ is given by (1.1). Recall our problem, namely

$$
\underset{\pi \in \mathcal{A}_T}{\text{sup}} J(T, x, y; \pi),
$$

(\text{P})
where \( J \) is given by (14). Here \( \tilde{A}_T(\subset L_{2,T}) \) is the set of admissible strategies described later.

We further assume \( r, \mu, b, \sigma_p \), and \( \sigma_f \) satisfying the conditions (A1) \( \sim \) (A6). The related optimization portfolio problems using the FBSDE approach are studied in \( [27, 28] \).

Following the ideas of \( [27] \), given \( \tilde{x} = U(x) = e^{-\alpha x} \), applying Itô formula to \( \tilde{X}_t^\pi = U(X_t^\pi) \), and using

\[
U^{-1}(\tilde{x}) = -\frac{1}{\alpha} \log(-\tilde{x}), \quad U(U^{-1}(\tilde{x})) = \tilde{x},
\]

\[
\partial_x U((U^{-1}(\tilde{x})) = -\alpha \tilde{x}, \quad \partial_{xx} U((U^{-1}(\tilde{x})) = \alpha^2 \tilde{x}.
\]

and

\[
d\tilde{X}_t^\pi = \partial_x U(U^{-1}(\tilde{X}_t^\pi)) \left( r(Y_t)U^{-1}(\tilde{X}_t^\pi) + \pi_t^* (\mu(Y_t) - r(Y_t)1) + c \right) dt
\]

\[
+ \frac{1}{2} \partial_{xx} U(U^{-1}(\tilde{X}_t^\pi)) \pi_t^* \sigma_p(Y_t) \sigma_p^*(Y_t) \pi_t dt
\]

\[
+ \partial_x U(U^{-1}(\tilde{X}_t^\pi)) \pi_t^* \sigma_p(Y_t) dW_t + \tilde{X}_t^\pi - \int_{z>0} (e^{\alpha z} - 1) N(dt, dz)
\]

(3.2)

\[
= -\alpha \tilde{X}_t^\pi \left( -\frac{r(Y_t)}{\alpha} \log(-\tilde{X}_t^\pi) + \pi_t^* (\mu(Y_t) - r(Y_t)1) + c \right) dt
\]

\[
+ \frac{1}{2} \alpha^2 \tilde{X}_t^\pi \pi_t^* \sigma_p(Y_t) \sigma_p^*(Y_t) \pi_t dt + \tilde{X}_t^\pi - \int_{z>0} (e^{\alpha z} - 1) \lambda \nu(dz) dt
\]

\[
- \alpha \tilde{X}_t^\pi \pi_t^* \sigma_p(Y_t) dW_t + \tilde{X}_t^\pi - \int_{z>0} (e^{\alpha z} - 1) \tilde{N}(dt, dz), \quad \tilde{X}_0 = U(x).
\]

Here, \( \tilde{N}(dt, dz) = N(dt, dz) - \lambda \nu(dz) dt \). The Pontryagin maximum principle leads to the corresponding Hamiltonian written as

\[
H(t, \pi, \tilde{x}, y, p, q^0, q) := \left( -\frac{r}{\alpha} \log(-\tilde{x}) + \pi^* (\mu - r1) + c \right)
\]

\[
+ \frac{1}{2} \alpha^2 \tilde{x} \pi^* \sigma_p \sigma_p^* \pi + \tilde{x} \int_{z>0} (e^{\alpha z} - 1) \lambda \nu(dz) \right) p - \tilde{x} (\alpha \pi^* \sigma_p q)
\]

\[
+ \tilde{x} \int_{z>0} (e^{\alpha z} - 1) q^0(z) \lambda \nu(dz),
\]

where we omit the dependence of \( r, \mu, \sigma_p, \sigma_f \) on \( y \) in this display. The first order condition

\[
\frac{\partial H}{\partial \pi}(\tilde{\pi}) = 0
\]

gives

\[
\tilde{\pi} = \frac{1}{\alpha} (\sigma_p \sigma_p^*)^{-1} \sigma_p \left( \theta + \frac{q}{p} \right).
\]

Denote

\[
\tilde{\pi}_t = \frac{1}{\alpha} (\sigma_p(Y_t) \sigma_p(Y_t)^*)^{-1} \sigma_p(Y_t) \left( \theta(Y_t) + \frac{q_t}{p_t} \right),
\]

by the abuse of the notations. In the following, we also use the notations \( \tilde{X}_t = \tilde{X}_t^\pi \) and \( X_t = X_t^\pi \). Here we do not assume that \( \sigma_p \) is invertible at this moment. If \( \sigma_p \)
is invertible, then the market is complete. We will have
\begin{equation}
\hat{\pi} = \frac{1}{\alpha} (\sigma_{Y_t}^2)^{-1} \left( \theta + \frac{q_t}{p_t} \right),
\end{equation}
and
\begin{equation}
\hat{\pi}_t = \frac{1}{\alpha} (\sigma_p(Y_t)^2)^{-1} \left( \theta(Y_t) + \frac{q_t}{p_t} \right),
\end{equation}
Then (3.2) becomes
\begin{equation}
d\tilde{X}_t = D_p H(\hat{\pi}_t) dt + D_q H(\hat{\pi}_t) dW_t + \int_{z>0} D_{\phi} H(\hat{\pi}; z) \tilde{N}(dt, dz),
\end{equation}
where
\begin{align*}
D_p H(\hat{\pi}_t), & \quad D_q H(\hat{\pi}_t), \quad D_{\phi} H(\hat{\pi}; z)
\end{align*}
are the short version of
\begin{align*}
D_p H(\hat{\pi}_t, \tilde{X}_t, p_t, q_t, q_0^0(z)), \quad D_q H(\hat{\pi}_t, \tilde{X}_t, p_t, q_t, q_0^0(z)), \quad D_{\phi} H(\hat{\pi}_t, \tilde{X}_t, p_t, q_t, q_0^0(z)).
\end{align*}
There is a dependence of $H$ on $Y_t$. In addition, $p_t$ satisfies the dual equation given by
\begin{equation}
dp_t = -D_{\phi} H(\hat{\pi}_t) dt + q_t^* dW_t + \int_{z>0} q_0^0(z) \tilde{N}(dt, dz), \quad p_T = 1,
\end{equation}
with $D_{\phi} H(\hat{\pi}_t)$ again the short version of
\begin{align*}
D_{\phi} H(\hat{\pi}_t, \tilde{X}_t, p_t, q_t, q_0^0(z)).
\end{align*}
Observe that (3.6) is a forward equation which poses the initial condition, (3.7) is a backward equation which poses the terminal condition, and (3.4) and (3.7) together is called a pair of FBSDEs (forward-backward stochastic differential equations.) A solution is given by $\tilde{X}_t, p_t, q_t$, and $q_0^0(z)$. The processes $\tilde{X}_t, p_t$, and $q_t$ are progressively measurable with respect to $\{\mathcal{F}_t\}$ in the usual sense and $q_0^0(z)$ is progressively measurable in the sense that the restriction of the function on $[0, s] \times [0, \infty) \times \Omega$ is measurable with respect to the product $\sigma$-field given by $\mathcal{B}_s \otimes \mathcal{B}(\{0, \infty\}) \otimes \mathcal{F}_s$.

When we plug (3.4) in the equations (3.6) and (3.7), the FBSDEs is the pair of equations with complicated nonlinearity. Indeed, we obtain the corresponding FBSDEs written as
\begin{equation}
d\tilde{X}_t^* = -\tilde{X}_t^* \left\{ ac - r(Y_t) \log(-\tilde{X}_t) + \frac{1}{2} \theta(Y_t)^* \sigma_p(Y_t)^* (\sigma_p(Y_t) \sigma_p(Y_t))^{-1} \sigma_p(Y_t) \theta(Y_t) \\
- \frac{1}{2} q_t^* \sigma_p(Y_t)^* (\sigma_p(Y_t) \sigma_p(Y_t))^{-1} \sigma_p(Y_t) q_t \right\} dt + \tilde{X}_t^* \int_{z>0} (e^{az} - 1) N(dt, dz) \\
- \tilde{X}_t^* \left( \theta(Y_t) + \frac{q_t}{p_t} \right)^* \sigma_p(Y_t)^* (\sigma_p(Y_t) \sigma_p(Y_t))^{-1} \sigma_p(Y_t) dW_t,
\end{equation}
\begin{align*}
dp_t = \left\{ ac - r(Y_t) (\log(-\tilde{X}_t) + 1) \right\} - \int_{z>0} (e^{az} - 1) \lambda(dz) \\
+ \frac{1}{2} \left( \theta(Y_t) + \frac{q_t}{p_t} \right)^* \sigma_p(Y_t)^* (\sigma_p(Y_t) \sigma_p(Y_t))^{-1} \sigma_p(Y_t) \left( \theta(Y_t) + \frac{q_t}{p_t} \right) \right\} p_t dt \\
- \int_{z>0} (e^{az} - 1) q_t^0(z) \lambda(dz) dt + q_t^* dW_t + \int_{z>0} q_t^0(z) \tilde{N}(dt, dz),
\end{align*}
with \( \tilde{X}_0 = U(x) \) and \( p_T = 1 \). Choosing \( \tilde{q}^0 = \frac{x}{p} \) and \( \tilde{q} = \frac{x}{p} \) and using \( x = -\frac{1}{\alpha} \log(-\tilde{x}) \) and \( \tilde{p} = \log p \), we have

\[
(3.10) \quad dX_t^\pi = \left\{ r(Y_t)X_t^\pi + c + \frac{1}{\alpha} \left( \theta(Y_t) + \tilde{q}_t \right)^* \sigma_p(Y_t)(\sigma_p(Y_t)^*)^{-1} \sigma_p(Y_t)\theta(Y_t) \right\} dt \\
+ \frac{1}{\alpha} \left( \theta(Y_t) + \tilde{q}_t \right)^* \sigma_p(Y_t)(\sigma_p(Y_t)^*)^{-1} \sigma_p(Y_t)\left( \theta(Y_t) + \tilde{q}_t \right) - \frac{1}{2} \left| \tilde{q}_t \right|^2 \right\} dt \\
- \lambda \int_{z>0} \left\{ e^{\alpha z} \left( \tilde{q}^0(z) + 1 \right) \nu(dz) dt + \tilde{q}^0_t dW_t + \int_{z>0} \log(1 + \tilde{q}_t^0(z)) N(dt, dz), \right.
\]

with the initial condition \( X_0 = x \) and the terminal condition \( \tilde{p}_T = 0 \).

In the case of complete markets, that is, the number of risky assets and the number of Brownian motions are the same and \( \sigma_p(y) \) is invertible for any \( y \). Then

\[
\sigma_p(y)^*(\sigma_p(y)\sigma_p(y)^*)^{-1}\sigma_p(y) = I.
\]

The equations can be simplified as follows.

\[
(3.12) \quad dX_t^\pi = \left\{ r(Y_t)X_t^\pi + c + \frac{1}{\alpha} \left( \theta(Y_t)^2 + \theta(Y_t)^* \tilde{q}_t \right) \right\} dt \\
+ \frac{1}{\alpha} \left( \theta(Y_t) + \tilde{q}_t \right)^* \sigma_p(Y_t)\left( \theta(Y_t) + \tilde{q}_t \right) - \frac{1}{2} \left| \tilde{q}_t \right|^2 \right\} dt \\
- \lambda \int_{z>0} \left\{ e^{\alpha z} \left( \tilde{q}^0(z) + 1 \right) \nu(dz) dt + \tilde{q}^0_t dW_t + \int_{z>0} h_t(z) N(dt, dz), \right.
\]

with the initial condition \( X_0 = x \) and \( \tilde{p}_T = 0 \). Here, \( h_t(z) := \log(1 + \tilde{q}_t^0(z)) \).

3.1. The solution of FBSDE. In this subsection, we shall obtain the solution of FBSDEs \((3.12)-(3.13)\). Now, we define the following spaces:

\[
\mathcal{S}^p := \{(X_s)_{s\in[0,T]}: X_s \text{ is a } \mathbb{R}\text{-valued } \mathcal{F}_s\text{-progressively measurable stochastic process such that } E\left[ \sup_{t\in[0,T]} |X_s|^2 \right] < \infty \},
\]

\[
\mathbb{L}^p := \{(X_s)_{s\in[0,T]}: X_s \text{ is a } \mathbb{R}^m\text{-valued } \mathcal{F}_s\text{-progressively measurable stochastic process such that } E\left[ \int_0^T |X_s|^p ds \right] < \infty \},
\]

\[
\mathbb{L}^p_N := \{(X_s)_{s\in[0,T]}: X_s: [0, T] \times \mathbb{R}_+ \to \mathbb{R} \text{ is a } \mathcal{F}_s\text{-progressively measurable stochastic process such that } E\left[ \int_0^T \int_{z>0} |X_s|^p \nu(dz) ds \right] < \infty \}.
\]
Then, we have the following lemma for the solution of the coupled FBSDEs.

**Lemma 3.1.** Define \((X_t^\pm, \tilde{p}_t, \tilde{q}_t, h_t(z))\) as

\[
(3.14) \quad X_t^\pm := \begin{cases} 
1 - \frac{1}{\alpha - \eta(t, Y_t)} [(\alpha - \eta(0,y))x_0 + \phi(t, Y_t) - \phi(0,y)] \\
+ \frac{1}{\alpha - \eta(t, Y_t)} \left[ \int_0^t r(Y_s)ds + \int_0^t \theta(Y_s^*)dW_s + \int_0^t \frac{1}{2} |\theta(Y_s)|^2 ds \\
- \int_0^t \int_{z>0} \{\alpha - \eta(s,Y_s)\} zN(ds,dz) \\
+ \lambda \int_0^t \int_{z>0} \left(e^{(\alpha-\eta(s,Y_s)z}-1\right) \nu(dz)ds \right],
\end{cases}
\]

\[
(3.15) \quad \tilde{p}_t := \eta(t, Y_t)X_t^\pm + \phi(t, Y_t),
\]

\[
(3.16) \quad h_t(z) := -\eta(t, Y_t)z,
\]

\[
(3.17) \quad \tilde{q}_t := \frac{\alpha}{\alpha - \eta(t, Y_t)} \left(X_t^\pm \sigma_f(Y_t)^*D\eta(t, Y_t) + \sigma_f(Y_t)^*D\phi(t, Y_t) + \frac{1}{\alpha - \eta(t, Y_t)\theta(Y_t)} \right),
\]

where \(\eta(t, y)\) and \(\phi(t, y)\) must satisfy

\[
(3.18) \quad \frac{\partial \eta}{\partial t} + \frac{1}{2} \text{tr}(\sigma_f(y)\sigma_f(y)^*D^2 \eta) + \{g(y) - \sigma_f(y)\theta(y)\}^*D\eta
- \frac{1}{\eta - \alpha}(D\eta)^*\sigma_f(y)^*D\eta - r(y)(\alpha - \eta) = 0,
\]

\[
\eta(T, y) = 0,
\]

and

\[
(3.19) \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} \text{tr}(\sigma_f(y)\sigma_f(y)^*D^2 \phi) + \{g(y) - \sigma_f(y)\theta(y)\}
+ \frac{1}{\alpha - \eta} \sigma_f(y)^*D\eta + \frac{1}{\alpha - \eta}(D\eta)^*\sigma_f(y)\theta(y)
- \frac{1}{2} |\theta(y)|^2 + r(y) - c(\alpha - \eta) + \int_{z>0} \left(e^{(\alpha-\eta)z}-1\right) \nu(dz) = 0,
\]

\[
\phi(T, y) = 0.
\]

Then, FBSDEs (3.12), (3.13) has a solution \((X_t^\pm, \tilde{p}_t, \tilde{q}_t, h_t(z))\) \(\in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_N\) satisfying there is \(C_1 > 0\) such that

\[
(3.20) \quad h_t(z) \leq C_1 z.
\]

**Proof.** The proof is given in Appendix B. 

In Lemma 3.1 we have

\[
(3.21) \quad p_t = \exp(\eta(t, Y_t)X_t + \phi(t, Y_t)).
\]

We can see \(\alpha - \eta(t, y)\) and \(a(t, y)\) have the same equation (see (2.8)). Moreover, from the equation of \(\varphi(t, Y) = \log a(t, y)\) given in (2.21), we can see \(\varphi(t, y) - \phi(t, y)\) and \(b(t, y)\) have the same equation (see (2.9)), but with different terminal conditions, given by

\[
\varphi(T, y) - \phi(T, y) = \log \alpha, \ b(T, y) = 0.
\]
We can conclude
\[ \eta(t, y) = \alpha - a(t, y), \quad \phi(t, y) = \log a(t, y) - \log \alpha - b(t, y). \]

We have the following result.

**Theorem 3.1.** Assume \((A1) \sim (A6)\). Then, \((3.18)\) has a solution:
\begin{equation}
(3.22) \quad \eta(t, y) = \alpha - a(t, y),
\end{equation}
where \(a(t, y)\) is given by \((2.10)\). And, \((3.19)\) has a solution \(\phi(t, y) = \log a(t, y) - \log \alpha - b(t, y)\), which has the expression:
\begin{equation}
(3.23) \quad \phi(t, y) = \hat{E}_{t,y} \left[ \int_t^T \left\{ -a(s,Y_s)\eta_s^{-1}(Da(s,Y_s))\sigma_f(Y_s)\theta(Y_s) - ca(s,Y_s) \right. \\
+ r(Y_s) - \frac{1}{2}\|\theta(Y_s)\|^2 + \int_{z>0} \left( e^{a(s,Y_s)z} - 1 \right) \lambda\nu(dz) \bigg\} ds \right],
\end{equation}
where \(Y_s\) is given in \((2.17)\).

**Proof.** Letting \(\hat{\eta} = \frac{1}{\alpha - \eta}\), we obtain \(\hat{\eta}\) satisfying
\begin{equation}
(3.24) \quad \frac{\partial \hat{\eta}}{\partial t}(t, y) + (g(y)^* - \theta(y)\sigma_f(y)) D\hat{\eta}(t, y) + \frac{1}{2} \text{tr}(\sigma_f(y)\sigma_f(y)^* D^2\hat{\eta}(t, y)) - r(y)\hat{\eta}(t, y) = 0,
\end{equation}
with the terminal condition \(\hat{\eta}(T, y) = \frac{1}{\alpha}\). Then, \((3.24)\) accords with \((2.18)\). Hence, we have
\begin{equation}
(3.25) \quad \hat{\eta}(t, y) = \alpha^{-1} \hat{E}_{t,y} \left[ e^{-\int_t^T r(\hat{Y}_s) ds} \right],
\end{equation}
where \(\hat{Y}_s\) is given by \((2.13)\).

Consider \(\xi(t, y) := \log(\alpha - \eta(t, y))\). Then, observing \(\frac{\partial(\alpha - \eta)}{\partial \eta} \leq 0\) and \(\frac{\partial^2}{\partial \eta^2} \leq 0\) and following the arguments of Lemma 2.2 for the PDE of \(\xi\) we see that
\[ \frac{|D\eta(t, y)|}{\alpha - \eta(t, y)} = |D\xi(t, y)| \leq C(1 + |y|). \]

Hence, using the conditions \((A1) \sim (A6)\) and Theorem 10, Section 9, Chapter 2 in [35], we obtain \((3.26)\). \(\square\)

**Theorem 3.2.** Assume \((A1) \sim (A6)\). Then, FBSDEs \((3.12) - (3.13)\) has a unique solution of \((X^\hat{\xi}, \hat{p}_t, \hat{q}_t, h_t(z)) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{L}_\lambda^2\) satisfying \((3.20)\).

The proof is given in Appendix C.

### 3.2. Verification theorem.

In this subsection, we show the verification theorem. Define the probability measure \(\hat{P}^N\) as
\begin{equation}
(3.26) \quad \frac{d\hat{P}^N}{dP} \bigg|_{\mathcal{F}_T} = \hat{E}_T^N,
\end{equation}
where \(\hat{E}_t^N\) is defined by
\begin{equation}
(3.27) \quad \hat{E}_t^N := \hat{E}_t \cdot \exp \left( \int_0^t \int_{z>0} \left\{ \alpha - \eta(s, Y_s) \right\} z \cdot N(ds, dz) \right.
\end{equation}
\[ + \lambda \int_0^t \int_{z>0} \left( 1 - e^{(\alpha - \eta(s, Y_s))z} \right) \nu(dz) ds \bigg), \]
Then, from (3.5) we have
\[(3.35)\]
where
\[(3.28)\]
and by (3.13) and (3.16),
\[(3.34)\]
We have
\[(3.33)\]
\[(3.32)\]
Define the set of admissible strategies defined by
\[(3.30)\]
\[\hat{\pi} \in \hat{A}_T \text{ defined by} \]
\[(3.31)\]
Then the strategy \(\hat{\pi}_t \in \hat{A}_T\) defined by
\[(3.32)\]
is optimal for \((\hat{P})\). Moreover, we have
\[(3.33)\]
Proof. To prove (3.33), we consider \(\hat{X}_t^\pi p_t\). We use the relation,
\[\hat{X}_t^\pi p_t = -\exp(\hat{p}_t - \alpha X_t^\pi).\]
We have
\[(3.34)\]
and by (3.13) and (3.16),
\[d\hat{p}_t = \left\{\alpha c + \alpha r(Y_t)X_t^\pi - r(Y_t) + \frac{1}{2}[\theta(Y_t)]^2 + \theta(Y_t)^* \hat{q}_t - \lambda \int_{z>0} \left(e^{(\alpha - \eta(t,Y_t))z} - 1\right) \nu(dz)\right\} dt + \hat{q}_t^* dW_t - \eta(t,Y_t) \int_{z>0} z N(dt,dz).\]
Then
\[(3.35)\]
From (3.35) we have
\[\hat{q}_t = -\theta(Y_t) + \alpha \sigma_p(Y_t)^* \hat{\pi}_t.\]
From this and \((3.35)\), we have
\[
\begin{align*}
\theta(Y_t)^* \hat{q}_t - \alpha(\mu(Y_t) - r(Y_t)^*) \pi_t \\
= \theta(Y_t)^* (\hat{q}_t - \alpha \sigma_p(Y_t)^* \pi_t) \\
= \theta(Y_t)^* (-\theta(Y_t) + \alpha \sigma_p(Y_t)^* \hat{\pi}_t - \alpha \sigma_p(Y_t)^* \pi_t) \\
= - \left\{ \theta(Y_t)^2 + \alpha \theta(Y_t)^* \sigma_p(Y_t)^* (\hat{\pi}_t - \pi_t) \right\}.
\end{align*}
\]
(3.36)

By \((3.35), (3.36)\), we have
\[
d(\hat{p}_t - \alpha X_t^\pi) = \left\{ \alpha r(Y_t)(X_t^\pi - X_t^\pi) - r(Y_t) - \frac{1}{2} \theta(Y_t)^2 \right\} dt
\]
\[
+ \alpha \theta(Y_t)^* \sigma_p(Y_t)^* (\hat{\pi}_t - \pi_t) \right\} dt + (-\theta(Y_t) + \alpha \sigma_p(Y_t)^* (\hat{\pi}_t - \pi_t)) \, dW_t
\]
\[
- \lambda \int_{z > 0} (e^{(\alpha - \eta(t, Y_t)) z} - 1) \nu(dz) \, dt + \int_{z > 0} (\alpha - \eta(t, Y_t)) z N(dt, dz).
\]
(3.37)

Using \((3.34)\), we have
\[
d(X_t^\pi - X_t^\pi)
\]
\[
= (r(Y_t)(X_t^\pi - X_t^\pi) + \theta(Y_t)^* \sigma_p(Y_t)(\hat{\pi}_t - \pi_t)) \, dt + (\hat{\pi}_t - \pi_t)^* \sigma_p(Y_t) \, dW_t
\]
\[
=r(Y_t)(X_t^\pi - X_t^\pi) \, dt + (\hat{\pi}_t - \pi_t)^* \sigma_p(Y_t) \, d\hat{W}_t^N,
\]
where \(\hat{W}_t^N\) is given by \((3.29)\). From \((3.29)\) and \((3.34)\), we have
\[
\hat{p}_t - \alpha X_t^\pi = \hat{p}_0 - \alpha x + \alpha (X_t^\pi - X_t^\pi) - \int_0^T r(Y_t) \, dt - \int_0^T \theta(Y_t) \, dW_t
\]
\[
- \frac{1}{2} \int_0^T |\theta(Y_t)|^2 \, dt - \lambda \int_0^T \int_{z > 0} (e^{(\alpha - \eta(t, Y_t)) z} - 1) \nu(dz) \, dt
\]
\[
+ \int_0^T \int_{z > 0} (\alpha - \eta(t, Y_t)) z N(dt, dz)
\]
\[
= \hat{p}_0 - \alpha x + \alpha (X_t^\pi - X_t^\pi) - \int_0^T r(Y_t) \, ds + \log \hat{E}^N_t,
\]
where \(\hat{E}^N_t\) is given in \((3.27)\). Therefore,
\[
\hat{X}_t^\pi p_t = U(x)p_0 e^{\alpha (X_t^\pi - X_t^\pi)} - f_0^t r(Y_t) \, dt \cdot \hat{E}^N_t.
\]
Then
\[
E[\hat{X}_t^\pi p_t] = U(x) p_0 E \left[ e^{\alpha (X_t^\pi - X_t^\pi)} - f_0^T r(Y_t) \, dt \cdot \hat{E}^N_t \right] = U(x) p_0 \hat{E}^N \left( e^{\alpha (X_t^\pi - X_t^\pi)} - f_0^T r(Y_t) \, dt \right),
\]
(3.39)
where \(\hat{E}^N[\cdot]\) is the expectation with respect to \(\hat{P}^N\). From \((3.38)\), we have, under \(\hat{P}^N\)
\[
e^{- f_0^T r(Y_t) \, dt} (X_t^\pi - X_t^\pi) = \int_0^T e^{- f_s^t r(Y_s) \, ds} (\hat{\pi}_t - \pi_t)^* \sigma_p(Y_t) \, d\hat{W}_t^N.
\]
Using this and Jensen's inequality, we have
\[
\frac{1}{E^N} \left[ e^{-\int_0^T r(Y_t)dt} \right] \geq \exp \left( \frac{1}{E^N} \left[ e^{-\int_0^T r(Y_t)dt} \right] \right) = \exp \left( \frac{1}{E^N} \left[ e^{-\int_0^T r(Y_t)dt} \right] \right) = 1,
\]
if \( \hat{\pi} \in \tilde{A}_T \), see (3.30).

Hence, using (3.30), we have
\[
E[\tilde{X}_T ] = U(x) p_0 E^N \left[ e^{\alpha(X_T^\pi - X^\pi_T)} - \int_0^T r(Y_t)dt \right] 
\leq U(x) p_0 E^N \left[ e^{-\int_0^T r(Y_t)dt} \right] = E[\tilde{X}_T ].
\]
Therefore, we see that \( \hat{\pi} \) is optimal if \( \hat{\pi} \in \tilde{A}_T \) holds. The fact that \( \hat{\pi} \in \tilde{A}_T \) holds is proved in Appendix \[D\].

We calculate \( E[\tilde{X}_T ] \). Since we also observe
\[
\text{"} \hat{Y}_t \text{"} \text{ given in (2.13) under } \tilde{P}^N = \text{"} Y_t \text{"} \text{ given in (3.28) under } \tilde{P}^N,\]
we see that from (3.22)
\[
E^N \left[ e^{-\int_0^T r(Y_t)dt} \right] = \tilde{E} \left[ e^{-\int_0^T r(\hat{Y}_t)dt} \right] = \frac{\alpha}{\alpha - \eta(0, y)}.
\]
Therefore, recalling \( \tilde{X}_0^\pi = -e^{-\alpha x} \) and \( p_0 = e^{\psi(0, y)X^\pi_0 + \phi(0, y)} \), we have
\[
E \left[ \tilde{X}_T \right] = E \left[ e^{-\{a - \alpha(x + \phi(0, y))\} + \phi(0, y)} \tilde{E} \left[ e^{-\int_0^T r(Y_t)dt} \right] \right] 
= E \left[ e^{-\{a - \alpha(x + \phi(0, y))\} + \phi(0, y)} \right] \frac{\alpha}{\alpha - \eta(0, y)}.
\]

Corollary 3.1. By comparing (2.8) (2.9) and (3.18) (3.19), we verify the identity for solutions from the PDE approach and the FBSDE approach based on the relation
\[
\eta(t, y) = a - a(t, y), \\
\phi(t, y) = \log \frac{a(t, y)}{\alpha} - b(t, y),
\]
such that the optimal strategy (3.31) and the optimal value (3.33) through the FBSDE approach are identical to the optimal strategy
\[
\hat{\pi}(t, X_t^\pi, Y_t) = (a(t, Y_t)^{-1}(\sigma_p(Y_t)^*)^{-1}) \cdot \{ \theta(Y_t) - \alpha a(t, Y_t) \}
\]
\[
\cdot \{ \theta(Y_t) - \alpha a(t, Y_t) \} \cdot \{ \theta(Y_t) - \alpha a(t, Y_t) \},
\]
\[
\cdot \{ \theta(Y_t) - \alpha a(t, Y_t) \} \cdot \{ \theta(Y_t) - \alpha a(t, Y_t) \},
\]
and the optimal value is given
\[ V(0, x) = -e^{-a(0, y)x - b(0, y)}, \]
using the HJB approach.

**Appendix A. The proof of (2.28).**

Recall that
\begin{equation}
(A.1) \quad h(s, X_s^\pi, Y_s, \tilde{\pi}_s) = \sigma f(Y_s)^* Da(s, Y_s)a(s, Y_s)^{-1} + \theta(Y_s).
\end{equation}

We consider
\[ \int_0^t \int_{s \geq 0} a(s, Y_s)zN(ds, dz) = \sum_{i=1}^{p_t} a(T_i, Y_{T_i})Z_i, \]
where \( Z_1, Z_2, \ldots \) are iid with distribution \( \nu(\cdot) \) and
\[ T_i = S_1 + S_2 + \cdots + S_i, \]
\( S_1, S_2, \ldots \) are iid having exponential distribution with parameter \( \lambda \). Then, we have
\[ E \left[ \exp \left( \sum_{i=1}^{N} a(T_i, Y_{T_i})Z_i \right) \right] = \sum_{N=0}^{\infty} E \left[ \exp \left( a(T_i, Y_{T_i})Z_i \right) ; p_t = N \right]. \]

The expectation of
\[ \exp \left( \sum_{i=1}^{N} a(T_i, Y_{T_i})Z_i \right) \right) 1_{p_t = N} \]
with respect to \( S_1, S_2, \ldots \) and \( Z_1, Z_2, \ldots \) while keeping \( W_t, Y_t \) fixed is given by
\[ \int_0^t \lambda e^{-\lambda t_1} \int_{z_1 > 0} e^{\alpha(t_1, Y_{t_1}) z_1} \nu(dz_1) dt_1 \int_t^t \lambda e^{-\lambda(t_2 - t_1)} \int_{z_2 > 0} e^{\alpha(t_2, Y_{t_2}) z_2} \nu(dz_2) dt_2 \]
\[ \cdots \int_t^{t_{N-1}} \lambda e^{-\lambda(t_{N-1} - t_{N-2})} e^{-\lambda(t - t_{N-1})} \int_{z_N > 0} e^{\alpha(t_{N}, Y_{t_{N}}) z_N} \nu(dz_N) dt_N = \lambda^N e^{-\lambda t} \int_0^t \int_{z_1 > 0} e^{\alpha(t_1, Y_{t_1}) z_1} \nu(dz_1) dt_1 \int_t^t \int_{z_2 > 0} e^{\alpha(t_2, Y_{t_2}) z_2} \nu(dz_2) dt_2 \]
\[ \cdots \int_t^{t_{N-1}} \int_{z_N > 0} e^{\alpha(t_{N}, Y_{t_{N}}) z_N} \nu(dz_N) dt_N = \frac{1}{N!} e^{-\lambda t} \left\{ \lambda t \int_0^t ds \int_{z > 0} \exp(a(s, Y_s)z) \nu(dz) \right\}^N. \]

Therefore, the expectation of
\[ \exp \left( \int_0^t \int_{s \geq 0} a(s, Y_s)zN(ds, dz) \right) \]
with respect to \( T_1, T_2, \cdots \) and \( Z_1, Z_2, \cdots \) keeping \( W_s, Y_s, 0 \leq s \leq t \) fixed is given by
\[ e^{-\lambda t} \exp \left( \lambda \int_0^t \int_{z > 0} \exp(a(s, Y_s)z) \nu(dz) ds \right). \]

Using the expression of \( \mathcal{E}_t(\tilde{\pi}) \) and the fact that \( h(s, X_s^\pi, Y_s, \tilde{\pi}_s) = \tilde{h}(s, Y_s) \) is a function of \( (s, Y_s) \), we only need to work on the proof of
\begin{equation}
(A.2) \quad E[\mathcal{E}_T^\pi] = 1,
\end{equation}
where $\mathcal{E}_t^c$ is defined by

$$\mathcal{E}_t^c := \exp \left( -\int_0^t \tilde{h}(s,Y_s)dW_s - \frac{1}{2} \int_0^t \tilde{h}(s,Y_s)^2 ds \right).$$

Then, we observe that $|\tilde{h}(s,y)| \leq C(1+|y|)$. Using Lemma 2.1, we have (4.2).

**Remark A.1.** The above argument may not work for general $\pi$, since $h(s,X_s^\pi,Y_s,\pi_s)$ depends on $N(dt,dz)$ in general. In the argument, we can not fix $\pi_s$ and take expectation with respect to $N(dt,dz)$ in the very beginning. However, if $h(s,X_s^\pi,Y_s,\pi_s)$ depends only on $Y_s$, then we can fix $W$ in the calculation and take expectation on $N(\cdot,\cdot)$ first as in Appendix A.

For general $\pi$, we use Itô’s formula to derive

$$d\mathcal{E}_t(\pi) = \mathcal{E}_{t-}(\pi) \left\{ -h(t,X^\pi_t,Y_t,\pi_t)dW_t + \int_{z>0} \left( e^{\alpha(t,Y_t)z} - 1 \right) \tilde{N}(dt,dz) \right\}.$$  

Hence, it is a positive local martingale, and hence is a supermartingale and may not be a martingale unless additional conditional on $\pi$ is imposed.

**Appendix B. The Proof of Theorem 3.1**

Applying Itô formula to (3.15) implies

(B.1)

$$d\tilde{p}_t = \left\{ X^\pi_t \left( \frac{\partial \eta}{\partial t}(t,Y_t) + \frac{1}{2} \text{tr}(\sigma_f(Y_t)\sigma_f(Y_t)^* D^2 \eta(t,Y_t)) + g(Y_t)^* D \eta(t,Y_t) 
+ r(Y_t)\eta(t,Y_t) + \frac{\partial \phi}{\partial t}(t,Y_t) + \frac{1}{2} \text{tr}(\sigma_f(Y_t)\sigma_f(Y_t)^* D^2 \phi(t,Y_t)) + g(Y_t)^* D \phi(t,Y_t) 
+ c \eta(t,Y_t) \right) + \frac{1}{\alpha} \left( \eta(t,Y_t) \theta(Y_t) + \sigma_f(Y_t)^* D \eta(t,Y_t) \right) \right) dt 
+ \left\{ X^\pi_t \left( D \eta(t,Y_t) \right)^* \sigma_f(Y_t) + (D \phi(t,Y_t))^* \sigma_f(Y_t) + \eta(t,Y_t) \frac{1}{\alpha} \left( \theta(Y_t) + \tilde{q}_t \right) \right\} dW_t 
- \int_{z>0} \eta(t,Y_t) z \tilde{N}(dt,dz).$$

Identifying the diffusion terms in (3.13) and (B.1), we obtain (3.16) and (3.17). Inserting (3.16) and (3.17) into (3.13) and (B.1) leads to

(B.2)

$$d\tilde{p}_t = \left\{ X^\pi_t \left( \alpha \sigma_f(Y_t) + \frac{\alpha}{\alpha - \eta(t,Y_t)} \theta(Y_t)^* \sigma_f(Y_t)^* D \eta(t,Y_t) \right) + \alpha c - r(Y_t) 
+ \frac{\alpha}{\alpha - \eta(t,Y_t)} \theta(Y_t)^* \left( \sigma_f(Y_t)^* D \phi(t,Y_t) + \frac{1}{\alpha} \eta(t,Y_t) \theta(Y_t) \right) \right) + \frac{1}{2} \theta(Y_t)^2 
- \lambda \int_{z>0} \left( e^{(\alpha - \eta(t,Y_t))z} - 1 \right) \nu(dz) \right) dt + \tilde{q}_t^2 dW_t + \int_{z>0} h_t(z) \tilde{N}(dt,dz),$$
and

\begin{equation}
(\text{B.3})
\begin{align*}
d\tilde{p}_t &= \left[ X^\pi_t \left\{ \frac{\partial \eta}{\partial t}(t, Y_t) + \frac{1}{2} \text{tr}(\sigma_f(Y_t)\sigma_f(Y_t)^*D^2\eta(t, Y_t)) + g(Y_t)^*D\eta(t, Y_t) \right. \\
&\quad + \frac{1}{2} \text{tr}(\sigma_f(Y_t)\sigma_f(Y_t)^*D^2\phi(t, Y_t)) + g(Y_t)^*D\phi(t, Y_t) + c\eta(t, Y_t) \\
&\quad + \frac{1}{2} \text{tr}(\sigma_f(Y_t)\sigma_f(Y_t)^*D^2\phi(t, Y_t)) + g(Y_t)^*D\phi(t, Y_t) + c\eta(t, Y_t) \\
&\quad + \frac{\eta(t, Y_t)}{\alpha - \eta(t, Y_t)} \left( \theta(Y_t) + \frac{D\eta(t, Y_t)}{\eta(t, Y_t)} \right)^* \left( \theta(Y_t) + \frac{D\eta(t, Y_t)}{\eta(t, Y_t)} \right) \right] dt \\
&\quad + h_t^* dW_t + \int_{z>0} h_t(z) N(dt, dz).
\end{align*}
\end{equation}

By comparing the coefficients of the first order and the zero order terms of \(X^\pi_t\) in (B.2) and (B.3), we see that \(\eta(t, Y_t)\) and \(\phi(t, Y_t)\) must satisfy (3.18) and (3.19).

Finally, we solve the solution of the optimal trajectory. The ansatz for \(p_t\) and \(\tilde{X}^\pi_t = -e^{-\alpha X^\pi_t}\) imply

\begin{equation}
(\text{B.4})
\log(-\tilde{X}^\pi_t p_t) = -(\alpha - \eta(t, Y_t))X_t^\pi + \phi(t, Y_t).
\end{equation}

By using (3.12) and (3.13), we have

\begin{equation}
(\text{B.5})
\begin{align*}
\log(-\tilde{X}^\pi_t p_t) &= \log(-\tilde{X}^\pi_0 p_0) - \int_0^t r(Y_s)ds - \int_0^t \theta(Y_s)^*dW_s \\
&\quad - \int_0^t \frac{1}{2} |\theta(Y_s)|^2 ds - \int_0^t \int_{z>0} \{\alpha - \eta(s-, Y_{s-})\} N(ds, dz) \\
&\quad - \lambda \int_0^t \int_{z>0} \left( e^{(\alpha-\eta(s-, Y_{s-}))z} - 1 \right) \nu(dz)ds.
\end{align*}
\end{equation}

Identifying (B.4) and (B.5) gives the optimal trajectory written as (3.14).

**APPENDIX C. THE PROOF OF THEOREM 3.2**

We prove the uniqueness of the solution of FBSDEs (3.12)-(3.13). Using (3.12) and (3.13), we have

\begin{align*}
d(\tilde{p}_t - \alpha X^\pi_t) &= - \left\{ r(Y_t) + \frac{1}{2} |\theta(Y_t)|^2 + \lambda \int_{z>0} \{e^{\alpha z + h_t(z)} - 1\} \nu(dz) \right\} dt \\
&\quad - \theta(Y_t)^*dW_t + \int_{z>0} \{h_t(z) + \alpha z\} N(dt, dz).
\end{align*}

Therefore, we have

\begin{equation}
U(X^\pi_T) = p_T U(X^\pi_T) = -e^{\tilde{p}_T - \alpha X^\pi_T} = -e^{\tilde{p}_0 - \alpha X^\pi_0} e^{-\int_0^T r(Y_s)dt} \eta^h_T,
\end{equation}

where \(\eta^h_T\) is defined by

\begin{equation}
\begin{align*}
\eta^h_t := & e^{-\int_0^t \theta(Y_s)^*dW_s - \frac{1}{2} \int_0^t |\theta(Y_s)|^2 ds} \\
& \cdot e^{\int_0^t \int_{z>0} (h_t(z) + \alpha z) N(ds, dz) - \lambda \int_0^t \int_{z>0} (e^{h_t(z)} + \alpha z) \nu(dz)ds}.
\end{align*}
\end{equation}
Here, using (3.20) and the arguments of proving (2.28), we see that $E \left[ \eta^h_T \right] = 1$. Hence, we can define the probability measure $P^h$ by

$$\left. \frac{dP^h}{dP} \right|_{F_T} = \eta^h_T,$$

Then, we have

$$E \left[ U(X_T^\xi) \right] = e^{\bar{\varphi}_0} U(x) E^h \left[ e^{-\int_0^T r(Y_t) dt} \right],$$

where $E^h[\cdot]$ is the expectation of the probability measure $P^h$.

Prepare another solution $(X_t, \tilde{p}_t, \tilde{q}_t, h_t(z)) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2$ of FBSDEs (3.12) and (3.13):

(C.2) \(dX_t' = \left\{ r(Y_t)X_t' + \frac{1}{\alpha} \left( \left( \theta(Y_t) \right)^2 + \theta(Y_t) \right) \right\} dt + \frac{1}{\alpha} (\theta(Y_t) + \tilde{q}_t)^* dW_t - \int_{z>0} zN(dt, dz),\)

(C.3) \(d\tilde{p}_t = \left\{ \alpha c + r(Y_t)(\alpha X_t^\xi - 1) + \frac{1}{2} \left( \theta(Y_t) \right)^2 + \theta(Y_t) \right\} dt + \alpha d(X_t^\xi - X_t') - r(Y_t) dt + d\log \eta^h_t + \alpha d(X_t^\xi - X_t').\)

Further, we have

$$\alpha d(X_t^\xi - X_t') = \alpha r(Y_t)(X_t^\xi - X_t') dt + \tilde{q}_t^* dW_t^N,$$

where $W_t^N$ is given by (3.29) and is a Brownian motion under $P^h$ given here. Hence, we have

(C.4) \(e^{-\int_0^T r(Y_t) dt} \alpha (X_t^\xi - X_T^\xi) = \int_0^T e^{-\int_0^s r(Y_t) dt} (\tilde{q}_t^* - \tilde{q}_t')^* dW_t^N.\)

Using this and Jensen's inequality, we have

$$\frac{1}{E^h \left[ e^{-\int_0^T r(Y_t) dt} \right]} E^h \left[ e^{-\int_0^T r(Y_t) dt} e^{\alpha (X_T^\xi - X_T')} \right] \geq \exp \left( \frac{1}{E^h \left[ e^{-\int_0^T r(Y_t) dt} \right]} E^h \left[ \int_0^T \alpha (X_t^\xi - X_t')^* dW_t^N \right] \right),$$

(C.6) \(= \exp \left( \frac{1}{E^h \left[ e^{-\int_0^T r(Y_t) dt} \right]} E^h \left[ e^{-\int_0^T r(Y_t) dt} \alpha (X_t^\xi - X_t')^* dW_t^N \right] \right) = 1.\)

On the other hand, by (C.4), we have

$$U(X_T^\xi) = -e^{\bar{\varphi}_0 - \alpha c} \int_0^T r(Y_t) dt \eta^h_T e^{\alpha (X_T^\xi - X_T')} e^{-\int_0^T r(Y_t) dt} \eta^h_T.$$
and
\[ E[U(X'_T)] = e^{\tilde{p}_0} U(x) E^h \left[ e^{- \int_0^T r(Y_t) dt} e^{\alpha (X^x - X'_t)} \right]. \]
Therefore, we have
\[ E[U(X'_T)] \leq E[U(X^\#_T)]. \]
Here, we use (C.2) and (C.6). Furthermore, we apply the above arguments to
\[ d(\tilde{p}'_t - \alpha X^\#_t), \]
and
\[ d(\tilde{p}'_t - \alpha X^\#_t) = d(\tilde{p}'_t - \alpha X'_t) + \alpha d(X'_t - X^\#_t). \]
Hence, we have
\[ E[U(X'_T)] \geq E[U(X^\#_T)]. \]
As a result, we have
\[ E[U(X^\#_T)] = E[U(X'_T)]. \]
This is equivalent to the relation
\[ \frac{1}{E^h \left[ e^{- \int_0^T r(Y_t) dt} \right]} E^h \left[ e^{- \int_0^T r(Y_t) dt} \alpha (X^\#_T - X'_T) \right] \]
\[ = \exp \left( \frac{1}{E^h \left[ e^{- \int_0^T r(Y_t) dt} \right]} E^h \left[ e^{- \int_0^T r(Y_t) dt} \alpha (X^\#_T - X'_T) \right] \right) \]
\[ = 1. \]
Hence, \( X^\#_T - X'_T = 0, \text{ a.s.} \). Using (C.5), we see that \( \tilde{q}_t - \tilde{q}'_t = 0, \text{ a.s.} \). And, we have
\[ 0 = E^h \left[ e^{- \int_0^T r(Y_t) dt} (X^\#_T - X'_T) | \mathcal{F}_t \right] = e^{- \int_0^t r(Y_s) ds} (X^\#_t - X'_t), \]
which leads to
\[ X^\#_T - X'_T = 0, \text{ a.s.} \]
Here we use the property that the martingale property from (C.5). Now, we compare (3.13) with (C.3). Recall
\[ e^{\tilde{p}_t - \alpha X^x} + \int_0^t r(Y_s) ds = e^{\tilde{p}_0 - \alpha x} \eta^h_t, \]
\[ e^{\tilde{p}'_t - \alpha X'_t} + \int_0^t r(Y_s) ds = e^{\tilde{p}_0 - \alpha x} \eta'^h_t, \]
where \( \eta'^h_t \) is given in (C.11) where \( h \) is replaced by \( h' \). Recalling that \( X^\#_T = X'_T \) and \( \tilde{p}_T = \tilde{p}'_T \), we see that \( \eta^h_T = \eta'^h_T \). Since \( \eta^h_t \) and \( \eta'^h_t \) are martingale, we have
\[ \eta^h_t = E[\eta^h_T | \mathcal{F}_t] = E[\eta'^h_T | \mathcal{F}_t] = \eta'^h_t. \]
From (C.7) we have
\[ \tilde{p}_t = \tilde{p}'_t, \quad 0 \leq t \leq T. \]
Moreover, we have
\[
\eta_T^h = 1 - \int_0^T \eta_t^h \theta(Y_t)^* dW_t + \int_0^T \eta_t^h \int_{z>0} \{ h_t^-(z) + \alpha z \} \tilde{N}(dt, dz),
\]
\[
\eta_T^{\hat{h}} = 1 - \int_0^T \eta_t^{\hat{h}} \theta(Y_t)^* dW_t + \int_0^T \eta_t^{\hat{h}} \int_{z>0} \{ h_t^{\hat{-}}(z) + \alpha z \} \tilde{N}(dt, dz).
\]

Using (C.8), we have
\[
\int_0^T \eta_t^h \int_{z>0} \{ h_t^-(z) - h_t^{\hat{-}}(z) \} \tilde{N}(dt, dz) = 0,
\]
which leads to \( h_t(z) - h_t^{\hat{}}(z) = 0, \ a.s. \ dt \times \nu(dz). \)

**Appendix D. The Proof of Theorem 3.3**

We verify \( \hat{\pi}_t \in \tilde{A}_T \). Under \( \hat{\tilde{P}}^N, \tilde{N}^N(t, E), E \in \mathcal{B}((0, \infty)) \) defined by
\[
\tilde{N}^N(t, E) := \tilde{N}(t, E) - \lambda \int_0^t \int_E \left( 1 - e^{(\alpha - \eta(s,y))z} \right) \nu(ds, dz)ds
\]
is a \( \hat{\tilde{P}}^N \)-martingale. Under \( \hat{\tilde{P}}^N \) we recall that
\[
dX_t = r(Y_t)X_t^\pi dt + c dt + \hat{\pi}_t^* \sigma_p(Y_t) d\hat{W}_t^N - \int_{z>0} z \tilde{N}^N(dt, dz)
\]
\[
- \lambda \int_{z>0} z \left( 2 - e^{(\alpha - \eta(t,y))z} \right) \nu(dt)dz.
\]
And we get
\[
X_T^\pi = e^{\int_0^T r(Y_t)dt} x + \int_0^T e^{\int_t^T r(Y_s)ds} c dt + \int_0^T e^{\int_t^T r(Y_s)ds} \hat{\pi}_t^* \sigma_p(Y_t) d\tilde{W}_t^N
\]
\[
- \int_0^T e^{\int_t^T r(Y_s)ds} \int_{z>0} z \tilde{N}^N(dt, dz) - \lambda \int_0^T e^{\int_t^T r(Y_s)ds} \int_{z>0} z \left( 2 - e^{(\alpha - \eta(t,y))z} \right) \nu(dt)dz.
\]
Hence, we have
\[
\hat{E}^N \left[ \int_0^T \hat{\pi}_t^* \sigma_p(Y_t) \sigma_p(Y_t)^* \hat{\pi}_t dt \right] \leq K \left[ x^2 + 1 + \hat{E}^N[(X_T^\pi)^2] + \int_0^T \int_{z>0} z^2 \nu(dz, dt) \right].
\]
From (D.1) we recall, under \( \hat{\tilde{P}}^N \)
\[
X_t = X_t^\pi = \frac{1}{\alpha - \eta(t, Y_t)} \left( (\alpha - \eta(0,0))x + \phi(t, Y_t) - \phi(0, y) \right)
\]
\[
+ \frac{1}{\alpha - \eta(t, Y_t)} \left( \int_0^t r(Y_s)ds + \int_0^t \theta(Y_s)^* d\tilde{W}_s^N - \int_0^t \frac{1}{2} |\theta(Y_s)|^2 ds \right)
\]
\[
- \int_0^t \int_{z>0} \{ \alpha - \eta(s, Y_s) \} z \tilde{N}^N(dt, dz)
\]
\[
- \int_0^t \int_{z>0} \{ \alpha - \eta(s, Y_s) \} \left( 2 - e^{(\alpha - \eta(s,y))z} \right) \nu(dz)ds
\]
\[
+ \lambda \int_0^t \int_{z>0} \left( e^{(\alpha - \eta(s,y))z} - 1 \right) \nu(dz)ds.
\]
From (3.23) we have $|\phi(t, y)| \leq K(1 + |y|^2)$. Then, we can see that $\hat{E}^N[(X_T^2) < \infty]$. Therefore, we have

$$\hat{E}^N \left[ \int_0^T \pi_t^* \sigma_p(Y_t) \sigma_p(Y_t^*) \pi_t dt \right] < \infty,$$

which leads to $\pi_t \in \tilde{A}_T$.

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