The corona algebra of stabilized Jiang-Su algebra

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Abstract

Let \( Z \) be the Jiang-Su algebra and \( K \) the \( C^* \)-algebra of compact operators on an infinite dimensional separable Hilbert space. We prove that the corona algebra \( M(Z \otimes K)/Z \otimes K \) has real rank zero. We actually prove a more general result.

1 Introduction

The Jiang-Su algebra \( Z \) is a projectionless unital simple separable infinite dimensional amenable \( C^* \)-algebra with the same \( K \)-theory as that of the complex field \( \mathbb{C} \) (see [16]). It had been an interesting problem to find projectionless unital infinite dimensional simple \( C^* \)-algebras ([17], [2], [3], [8]). However, the Jiang-Su algebra is the nicest that one can get. In fact \( Z \) is an inductive limit of sub-homogeneous \( C^* \)-algebras. It is one of many products of the decade of 1990’s in the Elliott program, the program of classification of amenable \( C^* \)-algebras. However, more recently, \( Z \) plays a much important role in the study of \( C^* \)-algebras. It becomes increasingly important to fully understand the structure of the Jiang-Su algebra.

In many ways, \( Z \) behaves like \( \mathbb{C} \), as its \( K \)-theory suggests. One of indirect, but important features of unital simple \( C^* \)-algebra \( \mathbb{C} \) is the simplicity (as well as complexity) of the Calkin algebra \( M(\mathbb{C} \otimes K)/\mathbb{C} \otimes K \). Let \( A \) be a non-unital but \( \sigma \)-unital \( C^* \)-algebra. The quotient \( M(A)/A \) is called the corona algebra of \( A \). The Calkin algebra is the corona algebra of \( \mathbb{C} \otimes K \), the stabilization of \( \mathbb{C} \). One of the important consequences of the simplicity of the Calkin algebra is that it has real rank zero. As we know, many important results in operator algebras (as well as operator theory) are related to the Calkin algebra, such as Fredholm index theory, the BDF-theory, \( C^* \)-algebra extension theory and the \( KK \)-theory. The fact that the Calkin algebra has real rank zero plays the crucial role in all these even though the term of real rank zero were invented much late. In that point of view, the corona algebra \( \mathbb{C} \otimes K \), inevitably, is also important, given the central role \( Z \) is playing in the study of classification of amenable \( C^* \)-algebras. However, by now, we know that the corona algebra of \( \mathbb{C} \otimes K \) is not simple. In fact it has a unique proper closed ideal. Nevertheless, we would like to know whether the corona algebra of \( \mathbb{C} \otimes K \) has some other similar structure to that of \( \mathbb{C} \otimes K \). The problem whether the corona algebra \( M(\mathbb{C} \otimes K)/\mathbb{C} \otimes K \) has the real rank zero has been around for many years and were specifically raised at the American Institute of Mathematics in 2009, as well as other places ([19]). Moreover, there is a long history of similar questions (e.g., ([6], [52], [15], [22], [24], [23], [36], [50], [51], [39]). The purpose of this paper is to establish that the corona algebra of \( \mathbb{C} \otimes K \) does have real rank zero, despite the fact it is not simple and the fact that \( Z \) has no proper projection.

One of the earliest applications of \( K \)-theory in operator algebras is the proof of the following theorem: an extension of AF-algebras is again an AF-algebra ([5] and [12]). This extends to a more general form: Let \( 0 \to B \to E \to A \to 0 \) be a short exact sequence of \( C^* \)-algebras, where the ideal and quotient have real rank zero. Then \( E \) has real rank zero if and only if \( K_1(B) \) is trivial (see [6]; also see [52]). Denote by \( Q(Z) \) the corona algebra of \( \mathbb{C} \otimes K \). Let \( J \) be the closed ideal of \( M(\mathbb{C} \otimes K) \) such that \( \pi(J) \) is the unique proper nontrivial ideal of the corona algebra, where \( \pi : M(\mathbb{C} \otimes K) \to Q(Z) \) is the quotient map. We have that \( Q(Z)/\pi(J) \) is purely infinite.
and simple (this is well-known; an explicit reference is [19]; it also follows immediately from [27] Theorem 3.5; see also [47] Theorem 2.2 and its proof). By a result of S. Zhang ([48]), it has real rank zero. It is also known that \( \pi(J) \) is also purely infinite and simple (this is also well-known; an explicit reference is [19]; it also follows immediately from the definition of \( J \) in [21] 2.2, Remark 2.9 and Lemma 2.1; see also [47] Theorem 2.2 and its proof). Therefore, from the above mentioned result, if \( K_1(J) \) is trivial, then \( Q(Z) \) has real rank zero. Much of the work of this study is to show just that.

The general strategy to show that \( K_1(J) \) is trivial is taken from an original idea of Elliott ([11]). However, unlike the case that Elliott considered, \( A \otimes \mathcal{K} \) (in particular, in the case that \( A = Z \)) lacks sufficiently many projections. We need to use positive elements to start. The method to adapt Cuntz relation to compare positive elements in the multiplier algebra of a non-unital simple \( C^\ast \)-algebra was initially used in [21] (and [25]). When \( A \) has real rank zero, one has the uniform bound for the length of path of unitaries in \( A \) which connects to the identity. This fact plays important role in many of earlier study of multiplier algebras. In general, however, for a unital simple \( C^\ast \)-algebra \( A \), the exponential length could be infinite. In other words, there will not be any control of the length. The idea to controlling the exponential length of unitaries in the multiplier algebras via the closure of commutator subgroup instead of controlling the exponential length in \( U_0(A) \) directly is new. We need results of exponential length of unitaries in the unitization of a non-unital simple \( C^\ast \)-algebra. The method to actually controlling the exponential length of unitaries in the closure of commutators for unital simple \( C^\ast \)-algebras was taken from [33] which is based on the results in the connection to the Elliott program of classification of amenable simple \( C^\ast \)-algebras.

The next section serves mainly for the notations and terminologies which will be used in the proof. Section 3 contributes to the understanding of exponential length of unitaries in the closure of commutator subgroup of non-unital simple \( C^\ast \)-algebras. The computation of the exponential length plays critical role in dealing with unitary group of some ideals in the multiplied algebras and corona algebras of some simple \( C^\ast \)-algebras, in particular, those of \( Z \otimes \mathcal{K} \). Section 4 contains a number of technical lemmas that use the Cuntz relation to produce and deal with projections in the multiplier algebra of certain non-unital simple \( C^\ast \)-algebras. The main technical lemma is [1,13] which allows us to connect a unitary to one which is nontrivial only in a small corner. Section 5 contains the main result. We show that the corona algebra of \( A \otimes \mathcal{K} \) has real rank zero for a class of unital simple \( C^\ast \)-algebras \( A \) which includes the projectionless simple \( C^\ast \)-algebra \( Z \). We expect a number of direct applications of the main result that \( M(Z \otimes \mathcal{K})/Z \otimes \mathcal{K} \) has real rank zero. However, these would be the subject of future projects.

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2 Notations

**Definition 2.1.** For any \( C^\ast \)-algebra \( C \), denote by \( T(C) \) the tracial state space of \( C \). Let \( \tau \in T(C) \) and let \( n \geq 1 \) be an integer. We will also use \( \tau \) for the trace \( \tau \otimes Tr \), where \( Tr \) is the standard (non-normalized) trace on \( M_n \).

**Definition 2.2.** Let \( A \) be a unital \( C^\ast \)-algebra. Denote by \( U(A) \) the unitary group of \( A \), by \( U_0(A) \) the path connected component of \( U(A) \) containing the identity. Denote by \( CU(A) \) the closure of the commutator group of \( U_0(A) \).
Definition 2.3. Let $A$ be a unital $C^*$-algebra with $T(A) \neq \emptyset$. Let $u \in U_0(A)$. Suppose that 
\[ \{u(t) : t \in [0, 1]\} \] 
is a continuous path of unitaries which is also piece-wisely smooth such that 
\[ u(0) = u \text{ and } u(1) = 1. \] 
Define de la Harpe-Skandalis determinant as follows:
\[
\begin{align*}
\text{Det}(u) := \text{Det}(u(t)) := \int_{[0,1]} \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt \quad \text{for all } \tau \in T(A).
\end{align*}
\]

Note that, if $u_1(t)$ is another continuous path which is piece-wisely smooth with $u_1(0) = u$ and 
\[ u_1(1) = 1, \] 
than $\text{Det}((u(t)) - \text{Det}(u_1(t)) \in \rho_A(K_0(A))$. Suppose that $u, v \in U(A)$ and 
\[ uv^* \in U_0(A). \] 
Let $\{w(t) : t \in [0, 1]\} \subset U(A)$ be a piece-wisely smooth and continuous path such that 
\[ w(0) = u \text{ and } w(1) = v. \] 
Define
\[
R_{u,v}(\tau) = \text{Det}(w(t))(\tau) = \int_{[0,1]} \tau\left(\frac{dw(t)}{dt} w(t)^*\right) dt \quad \text{for all } \tau \in T(A).
\]

Note that $R_{u,v}$ is well-defined (independent of the choices of the path) up to elements in 
$\rho_A(K_0(A))$.

Definition 2.4. Let $p$ be a supernatural number. Denote by $M_p$ the UHF-algebra associated 
with $p$. Denote by $Q_p$ the $K_0(M_p)$ which is identified with the subgroup of $Q$. Denote by $Q$ the 
UHF-algebra with $K_0(Q) = Q$ and $[1_Q] = 1$.

Definition 2.5. Let $A$ be a unital simple $C^*$-algebra. We write $TR(A) = 0$ if the tracial rank of $A$ is zero (see [28]).

Denote by $A_0$ the class of unital simple $C^*$-algebras such that $TR(A \otimes U) = 0$ for some 
infinite dimensional UHF-algebra $U$. It follows from [37] that if $A \in A_0$ then $TR(A \otimes U) = 0$ for all 
infinite dimensional simple AF-algebras $U$.

Denote by $Z$ the Jiang-Su algebra of projectionless simple ASH-algebra with $K_0(Z) = Z$ 
and $K_1(Z) = 0$. Note that $Z$ has a unique tracial state and $Z \in A_0$.

A unital separable $C^*$-algebra $A$ is said to be $Z$-stable if $A \otimes Z \cong A$.

Let $p$ and $q$ be two relative prime supernatural numbers. Define
\[
Z_{p,q} = \{ f \in C([0,1], M_{pq}) : f(0) \in A \otimes M_p \text{ and } f(1) \in A \otimes M_q \}.
\]

By Theorem 3.4 of [44], there is a trace-collapsing unital $*$-embedding $\varphi : Z_{p,q} \rightarrow Z_{p,q}$. Moreover, $Z$ is the stationary $C^*$-inductive limit $Z = \lim(Z_{p,q}, \varphi)$ (each building block is isomorphic to $Z_{p,q}$ and each connecting map is the same as $\varphi$).

2.6. Let $A$ be a unital $C^*$-algebra and say that $\tau$ is a tracial state of $A$. Let $a \in M_n(A)_+$. Then 
$d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$.

For a $C^*$-algebra $A$ and $b, c \in B_+, b \preceq c$ if $b$ is Cuntz subequivalent to $c$, i.e., there exists a 
sequence $\{x_n\}$ in $A$ such that $x_n c x_n^* \rightarrow b$.

2.7. Let $C$ be a non-unital but $\sigma$-unital nonelementary simple $C^*$-algebra. In [21], it is proven that 
there exists a unique smallest ideal $J$ of $M(C)$ which properly contains $C$. $J$ has the form 
$J = \mathcal{J}_0$ where
\[
J_0 = \{ x \in M(C) : \forall a \in C_+ - \{0\}, \exists n_0 \geq (e_m - e_n)x^* x(e_m - e_n) \leq a, \forall m > n \geq n_0 \}
\]

In the above, $\{e_n\}$ is an approximate identity for $C$. $J$ is independent of the choice of the 
approximate identity $\{e_n\}$. (See [21] 2.2, Lemma 2.1 and Remark 2.9. See also [25].)

If, in addition, $C$ has the form $C = A \otimes K$ where $A$ is unital $C^*$-algebra with unique tracial 
state $\tau$ and if also $C$ has strict comparison of positive elements, then $J$ (as defined above) is the
unique ideal of $M(C)$ which sits properly between $C$ and $M(C)$. Moreover, in this case, $J$ can be characterized by

$$J = \{ x \in M(C) : \tau(x^*x) < \infty \}. $$

(See [43]; see also [11], [20], [13], and [49].)

**Definition 2.8.** Recall also, in the above, $C$ is said to have continuous scale if for all $x \in C_+ - \{0\}$, there exists $n_0$ such that for all $m > n \geq n_0$, $(e_m - e_n) \leq x$. (21 Definition 2.5.) (Note that this implies that $M(C) = J_0 = J$.) If $C$ has continuous scale, for any strict positive element $a \in C$, $d_\tau(a)$ is a continuous function on $T(C)$.

Let $A$ be a unital simple $C^*$-algebra and let $C \subset A \otimes K$ be a non-unital hereditary $C^*$-subalgebra. Suppose that $C$ has continuous scale. Suppose also that $T(A) \neq \emptyset$. Then

$$f(t) = \sup \{ t(a) : a \in C \text{ and } 0 \leq a \leq 1 \}$$

(for $t \in T(A)$) is a continuous affine function in $\text{Aff}(T(A))$. Since $A$ is simple,

$$\inf \{ f(t) : t \in T(A) \} > 0.$$  

For each $t \in T(A)$, define

$$\tau(c) = t(c)/f(t) \text{ for all } c \in C.$$  

Then $\tau$ is a normalized trace on $C$. Note that every $\tau \in T(C)$ has this form. This also implies that $T(C)$ is compact.

Let $C$ be a non-unital and $\sigma$-unital nonelementary simple $C^*$-algebra. Let $J \subseteq M(C)$ be the ideal as defined in [27] and let $a \in J_+ \setminus \{0\}$. Then $C_1 = aCa$ has continuous scale.

### 3 Exponential length

The following is known.

**Lemma 3.1.** (cf. 10.8 of [31]) Let $\epsilon > 0$ and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing map. There exists $\eta > 0$, $\delta > 0$ and a finite subset $\mathcal{G} \subset C(\mathbb{T})_{s.a.}$ satisfying the following: Suppose that $A$ is a unital separable simple $C^*$-algebra with $TR(A) = 0$, and suppose that $u, v \in U(A)$ are two unitaries such that $sp(u) = sp(v) = \mathbb{T}$,

$$\mu_{\tau_{\varphi}}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \quad (e.3.2)$$

and for all arcs $I_a$ with length at least $a \geq \eta$, where $\varphi : C(\mathbb{T}) \to A$ is defined by $\varphi(f) = f(u)$ for all $f \in C(\mathbb{T})$ and

$$|\tau(g(u)) - \tau(g(v))| < \delta \text{ for all } g \in \mathcal{G}, \text{ for all } \tau \in T(A),$$

$$[u] = [v] \text{ in } K_1(A),$$

Then there exists a unitary $w \in U(A)$ such that

$$\|w^*uw - v\| < \epsilon. \quad (e.3.4)$$

The following is a non-unital version of the above.

**Lemma 3.2.** Let $\epsilon > 0$ and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing map. There exists $\eta > 0$, $\delta > 0$ and a finite subset $\mathcal{G} \subset C(\mathbb{T})_{s.a.}$ satisfying the following: Suppose that $A$ is a unital separable simple $C^*$-algebra with $TR(A) = 0$, $C$ is a non-unital hereditary $C^*$-subalgebra of
$A \otimes K$ with continuous scale, $B = \tilde{C}$ and suppose that $u, v \in U(B)$ are two unitaries such that $sp(u) = sp(v) = T$,

$$\mu_{\tau}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(C) \tag{e3.5}$$

and for all arcs $I_a$ with length at least $a \geq \eta$, where $\varphi : C(T) \to B$ is defined by $\varphi(f) = f(u)$ for all $f \in C(T)$ and

$$|\tau(g(u)) - \tau(g(v))| < \delta \text{ for all } g \in G \text{ and } \tau \in T(C), \tag{e3.6}$$

$$|u| = |v| \text{ in } K_1(B) \text{ and } \pi(u) = \pi(v), \tag{e3.7}$$

where $\pi : B \to B/C = C$ is the quotient map. Then there exists a unitary $w \in U(B)$ such that

$$\|w^*uw - v\| < \epsilon. \tag{e3.8}$$

Proof. Without loss of generality, we may assume that $\pi(u) = \pi(v) = 1$. Let $\epsilon > 0$ and let $\{e_n\}$ be an approximate identity for $C$ consisting of projections. Let $\Delta_1(a) = (1/3)\Delta(3a/4)$ for $a \in (0, 1)$ and $\Delta_2 = \Delta_1/2$.

Let $\delta > 0$ and $\eta_1 > 0$ (in place of $\eta$) and finite subset $G \subset C(T)$ be required by Lemma 3.1 for $\epsilon/2$ and $\Delta_2$. Put $\eta = \eta_1/4$. Suppose that $u$ and $v$ are two unitaries in $B$ satisfy the assumption for the above $\delta$ and $\eta$.

Let $\epsilon/4 > \epsilon_0 > 0$. Since $C$ has continuous scale, there exists $N \geq 1$ and unitaries $u_1, v_1 \in e_N Ae_N = e_N Ce_N$ such that

$$\|u - ((1 - e_N) + u_1)\| < \epsilon_0 \text{ and } \|v - ((1 - e_N) + v_1)\| < \epsilon_0,$$

and

$$\tau(1 - e_N) < \min\{\delta/16, \Delta(\eta)/16\} \tag{e3.9}$$

for all $\tau \in T(C)$.

By choosing sufficiently small $\epsilon_0$, we may assume that

$$|\tau(g(u_1)) - \tau(g(v_1))| < \delta \text{ for all } \tau \in T(e_N Ae_N) \tag{e3.10}$$

and for all $g \in G$. Moreover (by Lemma 3.4 of [32]),

$$\mu_{\tau}(I_a) \geq \Delta_1(a) \text{ for all } \tau \in T(C) \tag{e3.11}$$

and for all arcs $I_a$ with length $a \geq \eta$, where $\psi(f) = f((1 - e_N) + u_1)$ for all $f \in C(T)$. It follows from (e3.9) and (e3.11) that

$$\mu_{\tau}(I_a) \geq \Delta_2(a) \text{ for all } \tau \in T(e_N Ae_N)$$

and for all $a \geq \eta$, where $\psi_1(f) = f(u_1)$ for all $f \in C(T)$. It follows from 3.1 that there exists a unitary $w_1 \in e_N Ae_N$ such that

$$\|u_1 - w_1^*v_1w_1\| < \epsilon/2.$$

Let $w = (1 - e_N) + w_1$. Then

$$\|u - w^*uw\| < \epsilon.$$
**Proposition 3.3.** Let $A$ be a unital simple $C^*$-algebra, $C \subset A \otimes K$ be a non-unital hereditary $C^*$-subalgebra with continuous scale and let $B = \hat{C}$. Let $u \in U(B)$ with $\text{sp}(u) = \mathbb{T}$. Then there exists a nondecreasing function $\Delta : (0, 1) \to (0, 1)$ such that

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(C)$$

and for all arcs $I_a$ with length $a \in (0, 1)$, where $\varphi : C(\mathbb{T}) \to B$ defined by $\varphi(f) = f(u)$ for all $f \in C(\mathbb{T})$. Moreover, one may also require that $\lim_{a \to 0^+} \Delta(a) = 0$.

**Proof.** Let $\pi : B \to B/C = \mathbb{C}$ be the quotient map. Let $\lambda = \pi(u)$. So $\lambda \in \mathbb{T}$. Fix $a \in (0, 1)$. Consider finitely many open arcs $I_{a,1}, I_{a,2}, \ldots, I_{a,m} \in C(\mathbb{T})$ with radius $a/2$ such that $\lambda \in I_{a,1}$ and every arcs $I_a$ with length at least $a$ contains one of $I_{a,j}$ and $\bigcup_{j=1}^{m} I_{a,j} = \mathbb{T}$. Choose a non-zero positive function $f_{a,j} \in C(\mathbb{T})$ such that the support of $f_{a,j}$ contained in $I_{a,j}$ and $1 \geq f_{a,j}(t) > 0$ for all $t \in I_{a,j}$, $j = 1, 2, 3, \ldots, m$. Since $f_{a,j} \in C$ for $j = 2, 3, \ldots, m$, and $T(C)$ is compact (since $C$ has continuous scale),

$$d_{a,j} = \inf\{\tau(f_{a,j}) : \tau \in T(C)\} > 0.$$

For $f_{a,1}$, it dominates a non-zero positive element in $C$. It follows that

$$d_{a,1} = \inf\{\tau(f_{a,1}) : \tau \in T(C)\} > 0.$$

Define $D(a) = \min\{a, d_{a,j} : j = 1, 2, \ldots, m\}$. Then define

$$\Delta(a) = \sup\{D(\eta) : 0 < \eta \leq a\}.$$

Note that, for each $a \in (0, 1)$, $I_a \supset I_{b,j}$ for some $j$ and any $b \leq a$. Therefore

$$\mu_{\tau \circ \varphi}(I_a) \geq \mu_{\tau \circ \varphi}(I_{b,j})$$

for all $\tau \in T(C)$. It follows that

$$\mu_{\tau \circ \varphi}(I_a) \geq D(b) \text{ for all } \tau \in T(C) \text{ and for all } a \geq b.$$

Consequently

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(C).$$

\[ \Box \]

**Lemma 3.4.** Let $A$ be a unital simple separable $C^*$-algebra with $\text{TR}(A) = 0$, $C \subset A \otimes K$ be a non-unital hereditary $C^*$-subalgebra with continuous scale and let $B = \hat{C}$. Let $u \in U_0(B)$ with $\text{sp}(u) = \mathbb{T}$ and $\pi(u) = 1$, where $\pi : B \to \mathbb{C}$ is the quotient map. Then, for any $\epsilon > 0$, there exists a selfadjoint element $h \in C$ with $\text{sp}(h) = [-2\pi, 2\pi]$ such that

$$\|u - \exp(ih)\| < \epsilon \text{ and } \tau(h) = 0 \text{ for all } \tau \in T(C).$$

Moreover, we may assume that

$$\sup_{\tau \in T(C)} \lim_{n \to \infty} \tau(|h|^{1/n}) < 1.$$ 

**Proof.** First we note that the assumption that $\text{sp}(u) = \mathbb{T}$ implies that $A$ is infinite dimensional. Let $\epsilon > 0$. Let $\varphi : C(\mathbb{T}) \to B$ be the homomorphism defined by $\varphi(f) = f(u)$ for all $f \in C(\mathbb{T})$. It follows from [3.3] that there exists a non-decreasing function $\Delta : (0, 1) \to (0, 1)$ such that

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(C)$$

and for all arcs $I_a$ with length $a \in (0, 1),$ where $\varphi : C(\mathbb{T}) \to B$ defined by $\varphi(f) = f(u)$ for all $f \in C(\mathbb{T})$. Moreover, one may also require that $\lim_{a \to 0^+} \Delta(a) = 0$.
and for all arcs $I_a$ with length $a \in (0, 1)$. Let $\eta > 0$, $\delta > 0$ and $G \subset C(\mathbb{T})_{s.a.}$ a finite subset be required by \[3.2\] for $\epsilon/4$ and $\Delta$. To simplify the notation, without loss of generality, we may assume that $\|g\| \leq 1$ for all $g \in G$.

Let $\epsilon/4 > \epsilon_0 > 0$. Let $\{e_n\}$ be an approximate identity for $C$ consisting of projections. We choose an integer $N \geq 1$ and unitary $u_1 \in e_NC_{e_N}$ such that $u_1 \in U_0(e_NC_{e_N})$ and

$$\|u - ((1 - e_N) + u_1)\| < \epsilon_0/2 \quad \text{and} \quad \tau(1 - e_N) < \min\{\epsilon_0, \Delta(\eta/2)/4\}$$

for all $\tau \in T(C)$. Since $e_NC_{e_N}$ has real rank zero and infinite dimension, $u_1 \in CU(e_NC_{e_N})$. It follows from Theorem 4.5 of \[33\] that there exists a selfadjoint element $b_1 \in e_NC_{e_N}$ with $\|b_1\| \leq 2\pi$ such that

$$\|u_1 - e_N \exp(ib_1)\| < \epsilon_0/2 \quad \text{and} \quad \tau(b_1) = 0 \quad \text{for all} \quad \tau \in T(e_NC_{e_N}).$$

(This can be derived directly from the fact that $C$ had real rank zero). We assume that $e_{N+1} - e_N \neq 0$. Let $q_1, q_2 \in (e_{N+1} - e_N)C(e_{N+1} - e_N)$ be mutually orthogonal and mutually equivalent projections such that

$$|\tau(q_2)| = |\tau(q_1)| < \min\{\epsilon_0/32\pi, \Delta_1(\eta/2)/32\pi\} \quad \text{for all} \quad \tau \in T(C).$$

We choose $\epsilon_0$ sufficiently small such that $\epsilon_0 < \delta/16\pi$ and

$$|\tau(g(u)) - \tau(g(u'))| < \delta/2 \quad \text{for all} \quad g \in G$$

where $u' = (1 - e_N) + u_1$. Let $b_2 \in q_1Cq_1$ with $\text{sp}(b_2) = [-2\pi, 2\pi]$. Let $z \in U(B)$ such that $z^*q_1z = q_2$. Let $b_3 = -z^*b_2z$. Note that $\tau(b_2 + b_3) = 0$ for all $\tau \in T(C)$. Define $u_2 = (1 - e_{N+1}) + (e_{N+1} - e_N - q_1 - q_2) + (q_1 + q_2) \exp(i(b_2 + b_3)) + u_1$. By \[3.18\], \[3.17\] and the fact that $\|g\| \leq 1$ for all $g \in G$, we estimate that

$$|\tau(g(u)) - \tau(g(u_2))| < \delta \quad \text{for all} \quad g \in G.$$

It follows from \[3.2\] that there exists a unitary $w \in B$ such that

$$\|u - w^*w_2w\| < \epsilon/2.$$

Now let $h = w^*(b_1 + b_2 + b_3)w$. Then $\text{sp}(h) = [-2\pi, 2\pi]$,

$$\|u - \exp(ih)\| < \epsilon \quad \text{and} \quad \tau(h) = 0 \quad \text{for all} \quad \tau \in T(C).$$

Moreover \[3.14\] also holds since $h \in (e_N + q_1 + q_2)C(e_N + q_1 + q_2)$. \[3.21\]

**Corollary 3.5.** Let $A$ be a unital simple infinite dimensional separable $C^\ast$-algebra of tracial rank zero and let $C \subset A \otimes K$ be a non-unital hereditary $C^\ast$-subalgebra with continuous scale. Suppose that $u \in U(\hat{C})$ with $\text{sp}(u) = \mathbb{T}$ and suppose that $\{e_n\} \subset C$ is an approximate identity consisting of projections. Then, for any $\epsilon > 0$ and any $\sigma > 0$, there exists $k \geq 1$ and a unitary $w \in e_kC_{e_k}$ with $\text{sp}(w) = \mathbb{T}$ such that

$$\|u - (1 - e_k + w)\| < \epsilon \quad \text{and} \quad \tau(1 - e_k) < \sigma \quad \text{for all} \quad \tau \in T(C).$$

**Proof.** In the above proof, let

$$w = (e_{N+1} - e_N - q_1 - q_2) + (q_1 + q_2) \exp(i(b_2 + b_3)) + u_1$$

Then, since $\text{sp}(b_2) = [-2\pi, 2\pi]$, $\text{sp}(w) = \mathbb{T}$. Let $k = N + 1$. The corollary then follows. \[3.21\]
The following is a non-unital version of Cor. 3.9 of [28].

**Lemma 3.6.** Let $\epsilon > 0$. There exists $\delta > 0$ satisfying the following: For any unital separable simple $C^*$-algebra $A$ with real rank zero and stable rank one, any hereditary $C^*$-subalgebra $C \subset A \otimes \mathcal{K}$ with continuous scale, and any unitary $u \in \tilde{C}$ with $\text{sp}(u) = T$, if $v \in \tilde{C}$ is another unitary with $[v] = 0$ in $K_1(C)$ such that

$$
\|[u, v]\| < \delta \text{ and } \text{bott}_1(u, v) = 0, \quad (e3.22)
$$

there exists a continuous path of unitaries $\{V(t) : t \in [0, 1]\}$ in $\tilde{C}$ with $V(0) = v$ and $V(1) = 1$ such that

$$
\|[u, V(t)]\| < \epsilon \text{ for all } t \in [0, 1] \text{ and } \text{length}(V(t)) \leq \pi + \epsilon. \quad (e3.23)
$$

Moreover, if $\pi(v) = 1$, one can require that $\pi(V(t)) = 1$ for all $t \in [0, 1]$.

**Proof.** Without loss of generality, we may assume that $\pi(u) = 1$. Put $\tilde{C} = B$. Let $1/2 > \epsilon > 0$. From Corollary 3.9 of [28], one has the following statement: There exists $\delta > 0$ and $\sigma > 0$ satisfying the following: For any unital separable simple $C^*$-algebra $A_0$ with real rank zero and stable rank one, and unitary $u' \in A_0$ with $\text{sp}(u')$ being $\sigma$-dense in $T$ and any unitary $v' \in A_0$ with $[v'] = 0$ in $K_1(A_0)$ such that $\|[u, v']\| < \delta$ and $\text{bott}_1(u', v') = 0$, there exists a continuous path of unitaries $\{v'(t) : t \in [0, 1]\} \subset A_0$ such that $v'(0) = v'$, $v'(1) = 1$ and $\|[u', v'(t)]\| < \epsilon/4 \text{ for all } t \in [0, 1]$ and $\text{length}(\{v'(t) : t \in [0, 1]\}) \leq \pi + \epsilon/2$. Choose such $\delta$ and $\sigma$.

Choose $\delta_1 > 0$ satisfying the following: if $u'$ and $v'$ are two unitaries with $\text{sp}(u') = T$, then $\text{sp}(v')$ is $\sigma/2$-dense in $T$, provided that $\|u' - v'\| < \delta_1$.

Let $\{e_n\}$ be an approximate identity for $C$ consisting of projections. Choose $\theta > 0$ satisfying the following: for any unitaries $w_1, w_2, w_3$, $\text{bott}_1(w_1, w_3)$ and $\text{bott}_1(w_2, w_3)$ are well defined and

$$
\text{bott}_1(w_1, w_3) = \text{bott}_1(w_2, w_3)
$$

provided that $\|[w_j, w_3]\| < \theta$ and $\|w_1 - w_2\| < \theta$.

Choose $\epsilon_1 = \min\{\epsilon/4, \delta/4, \delta_1/2, \theta\}$. Choose an integer $N \geq 1$ and unitaries $u_1, v_1 \in e_N A e_N = e_N C e_N$ such that

$$
\|u - ((1 - e_N) + u_1)\| < \epsilon_1, \quad \|v - (\lambda(1 - e_N) + v_1)\| < \epsilon_1 \text{ and } \|[u_1, v_1]\| < \epsilon_1 \quad (e3.24)
$$

where $\lambda = \pi(v)$. Moreover, by the choice of $\theta$,

$$
\text{bott}_1(u, v) = \text{bott}_1(u_1', v_1'), \quad (e3.25)
$$

where

$$
u_1' = (1 - e_N) + u_1 \text{ and } v_1' = \lambda(1 - e_N) + v_1.
$$

Note that

$$
\text{bott}_1(u_1', v_1') = \text{bott}_1(u_1, v_1). \quad (e3.26)
$$

It follows from (e3.25), and (e3.26)

$$
\text{bott}_1(u_1, v_1) = 0. \quad (e3.27)
$$

By the choice of $\delta_1$, we conclude that $\text{sp}(u_1')$ is $\sigma/2$-dense in $T$. Thus $\text{sp}(u_1)$ is $\sigma$-dense in $T$. By applying the statement at the beginning of this proof (from Cor.3.9 of [28]), we obtain a continuous path of unitaries $\{w(t) : t \in [1/2, 1]\} \subset e_N A e_N$ such that

$$
w(1/2) = v_1, \quad w(1) = e_N \text{ and } \|[u_1, w(t)]\| < \epsilon/4 \text{ for all } t \in [1/2, 1]. \quad (e3.28)
$$
Moreover,
\[
\text{length}\{w(t) : t \in [1/2, 1]\} \leq \pi + \epsilon/2.
\] (e 3.29)

By (e 3.24), there exists \(h \in B_{s.a.}\) such that
\[
\|h\| < 2 \arcsin(\epsilon/4) \quad \text{and} \quad v \exp(ih) = v'.
\] (e 3.30)

Define \(w_0(t) = v \exp(i2th)\) for \(t \in [0, 1/2]\). Then
\[
w_0(0) = v, \quad w_0(1/2) = v' \quad \text{and} \quad \text{length}\{w_0(t) : t \in [0, 1/2]\} < \epsilon/2.
\]

Now define \(V(t) = w_0(t)\) if \(t \in [0, 1/2]\) and \(V(t) = \lambda(1 - eN) + w(t)\) for \(t \in [1/2, 1]\). Then \(V(t)\) is continuous and
\[
\text{length}\{V(t)\} \leq \pi + \epsilon/2 + \epsilon/2 = \pi + \epsilon.
\]

One also verifies that
\[
\|[u, V(t)]\| < \epsilon \quad \text{for all} \quad t \in [0, 1].
\] (e 3.31)

For the very last part of the lemma, assume that \(\pi(v) = 1\). Then \(\lambda = 1\). By (e 3.30), \(\pi(h) = 0\). It follows that \(\pi(w_0(t)) = 1\) for all \(t \in [0, 1/2]\). One then checks that
\[
\pi(V(t)) = 1 \quad \text{for all} \quad t \in [0, 1].
\]

\[\square\]

**Lemma 3.7.** Let \(C\) be a non-unital hereditary \(C^*\)-subalgebra of a separable \(C^*\)-algebra with stable rank one, and let \(B = \tilde{C}\). Suppose that \(u\) and \(v\) are two unitaries in \(B\) such that \(uv^* \in U_0(B)\) and \(\pi(u) = \pi(v)\), where \(\pi : B \to B/C = \mathbb{C}\) be the quotient map. Then, we can always assume that
\[
R_{u,v}(t_0) = 0
\]
where \(t_0\) is the tracial state of \(B\) such that \((t_0)|_C = 0\).

**Proof.** We may write
\[
uv^* = \prod_{j=1}^{k} \exp(\pi a_j)
\]
where \(a_j \in B_{s.a.}\). Since \(\pi(uv^*) = 1\),
\[
\sum_{j=1}^{k} \pi(a_j) = 2m
\]
for some integer \(m\). Define \(h_1 = a_1 - 2m, h_j = a_j, j = 2, 3, \ldots, k\). Define
\[
U(t) = \prod_{j=1}^{k} \exp(\pi i(1 - t)h_j)v \quad t \in [0, 1].
\]

Then \(U(t)\) is a continuous piecewise smooth path with \(U(0) = u\) and \(U(1) = v\). Moreover,
\[
\sum_{j=1}^{k} \pi(h_j) = 0.
\]

It follows that
\[
\frac{1}{2\pi i} \int_{0}^{1} t_0(\frac{dU(t)}{dt}U(t)^*)dt = 0.
\]
\[\square\]
The following is a non-unital version for a special case of Lemma 3.5 of \[33\].

**Lemma 3.8.** Let $\epsilon > 0$ and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing function. There is $\delta > 0$, $\eta > 0$, $\sigma > 0$ and there is a finite subset $\mathcal{G} \subset C(T)_{s.a.}$ satisfying the following: For any unital separable simple C*-algebra $A$ with $TR(A) = 0$, any non-unital hereditary C*-subalgebra $C \subset A \otimes K$ with continuous scale, any pair of unitaries $u,v \in C$ such that $sp(u) = T$ and $[u] = [v]$ in $K_1(C)$, $\pi(u) = \pi(v)$,

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(C)$$

for all intervals $I_a$ with length at least $\eta$, where $\varphi : C(T) \to \hat{C}$ is the homomorphism defined by $\varphi(f) = f(u)$ for all $f \in C(T)$,

$$|\tau(g(u)) - \tau(g(v))| < \delta \text{ for all } \tau \in T(C)$$

(e.3.32)

and for all $g \in \mathcal{G}$, for any $a \in Aff(T(C))$ with $a - R_{u,v}|_{T(C)} \in \rho_C(K_0(C))$ and $\|a\| < \sigma$ and $y \in K_1(C)$, there is a unitary $w \in C$ such that

$$[w] = y, \|u - w^*vw\| < \epsilon \text{ and}$$

$$\frac{1}{2\pi i} \tau(\log(u^*w^*vw)) = a(\tau) \text{ for all } \tau \in T(C).$$

(e.3.33)

(e.3.34)

**Proof.** Note that, since $sp(u) = T$, $A$ has infinite dimension. Since $TR(A) = 0$, $CU(A) = U_0(A)$ and $\rho_A(K_0(A)) = Aff(T(A))$. We will use these facts in the following proof. Without loss of generality, we may assume that $\pi(u) = \pi(v) = 1$. By [3.7] we may assume that

$$R_{u,v}(t_0) = 0,$$

(e.3.35)

where $t_0$ is the tracial state of $B$ that vanishes on $C$.

Let $\epsilon > 0$ and $\Delta$ be given. Choose $\epsilon/2 > \theta > 0$ such that, $\log(u_1), \log(u_2)$ and $\log(u_1u_2)$ are well defined and

$$\tau(\log(u_1u_2)) = \tau(\log(u_1)) + \tau(\log(u_2))$$

(e.3.36)

for every tracial state $\tau$ and for any unitaries $u_1, u_2$ such that

$$\|u_j - 1\| < \theta, \; j = 1,2.$$

We may choose even smaller $\theta$ such that

$$bott_1(u_1, v_1) = bott_1(u_2, v_1)$$

(e.3.37)

provided that $\|u_1, v_1\| < \theta$ and $\|u_1 - u_2\| < \theta$. Let $\delta' > 0$ (in place of $\delta$) be required by Lemma 3.1 of \[33\] for $\theta/2$ (in place of $\epsilon$). Put $\sigma = \delta'/2$. Let $1/2 > \delta > 0$ and $\eta$ be required by (3.2) for $\min\{\sigma, \theta/2, 1\}$ (in place of $\epsilon$) and $\Delta$. Suppose $u$ and $v$ satisfy the assumption for the above $\delta$, $\eta$ and $\sigma$. Then, by (3.2) there exists a unitary $z \in U(C)$ such that

$$\|u - z^*vz\| < \min\{\sigma, \theta/2, 1\}.$$ 

(e.3.38)

Let $b = \frac{1}{2\pi i} \log(u^*z^*vz)$. Then $\|b\| < \min\{\theta, \sigma, 1\}$. By Lemma 3.2 of [33], $\hat{b} - R_{u,v} \in \rho_C(K_0(C))$. Note that, by (e.3.35), $\|\log(u^*z^*vz)\| < \pi$. Since $\pi(u) = \pi(v) = 1$, $\pi(u^*z^*vz) = 1$. It follows that $\pi(\log(u^*z^*vz)) = 0$. In particular, $t_0(b) = 0$. 

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Let $a \in \text{Aff}(T(C))$ be such that $\|a\| < \sigma$ and $a - R_{a,v} \in \rho_C(K_0(C))$ as given by the lemma. It follows that $a - \hat{b} \in \rho_C(K_0(C))$. Moreover, $\|a - \hat{b}\| < 2\sigma < \delta'$. Let $p,q \in M_k(C)$ be projections such that

$$\tau(p-q) = a(\tau) - \tau(b) \quad \text{for all } \tau \in T(C).$$

Let $\{e_n\}$ be an approximate identity for $C$ consisting of projections. Choose an integer $N \geq 1$ and unitaries $u_1, v_1 \in e_N Ae_N = e_N Ce_N$ such that

$$\|u - ((1 - e_N) + u_1)\| < \min \{\delta/4, \theta/2\}, \quad \|v - ((1 - e_N) + v_1)\| < \min \{\delta/4, \theta/2\}$$

and $\tau(1 - e_N) < \eta$ for all $\tau \in T(C)$.

By \ref{3.5} we may also assume that $\text{sp}(u_1) = T$. By \ref{3.39} and \ref{3.41},

$$\|\tau(p-q)\| < \sigma \quad \text{for all } \tau \in T(e_N Ae_N).$$

Note that $K_1(\tilde{C}) = K_1(C) = K_1(e_N Ae_N)$. It follows from Lemma 3.1 of \cite{33} that there exists a unitary $z_1 \in e_N Ae_N$ such that

$$[z_1] = y - [z], \quad \|[u_1, z_1]\| < \theta/2 \quad \text{and} \quad \text{bott}_1(u_1, z_1)(\tau) = \tau(p-q)$$

for all $\tau \in T(e_N Ae_N)$. Put $z_2 = (1 - e_N) + z_1$, $w = zz_2$ and $u'_1 = (1 - e_N) + u_1$. It follows that

$$[w] = y \quad \text{and} \quad \|u - w^*vw\| < \theta < \epsilon$$

$$\text{bott}_1(u'_1, z_2)(\tau) = a(\tau) - \tau(b) \quad \text{for all } \tau \in T(C).$$

Put $u_0 = u((1 - e_N) + u'_1)$ and $v_0 = v(\tilde{\lambda} (1 - e_N) + v'_1)$. So

$$\|u_0 - 1\| < \min \{\delta/4, \theta/2\} \quad \text{and} \quad \|v_0 - 1\| < \min \{\delta/4, \theta/2\}.$$  

By the choice of $\theta$, we have

$$\text{bott}_1(u, z_2) = \text{bott}_1(u'_1, z_2).$$

We compute that (using \ref{3.36}, \ref{3.37}, \ref{3.47} and the Exel formula (see Lemma 3.5 of \cite{30}))

$$\frac{1}{2\pi i} \tau(\log(u^*w^*vw)) = \frac{1}{2\pi i} \tau(\log(u^*z_2^*z^*vz_2)) \quad \text{(e.3.48)}$$

$$= \frac{1}{2\pi i} \tau(\log(u^*z_3^*(uu^*)z^*vz_2)) \quad \text{(e.3.49)}$$

$$= \frac{1}{2\pi i} \tau(\log((zz_2 u^*z_2^*u)(u^*z^*vz))) \quad \text{(e.3.50)}$$

$$= \frac{1}{2\pi i} (\tau(\log(z_2 u^*z_2^*u)) + \tau(\log(u^*z^*vz))) \quad \text{(e.3.51)}$$

$$= \frac{1}{2\pi i} (\tau(\log(u^*z_2^*u)) + \tau(b)) \quad \text{(e.3.52)}$$

$$= \text{bott}_1(u, z_2)(\tau) + \tau(b) \quad \text{(e.3.53)}$$

$$= a(\tau) - \tau(b) + \tau(b) = a(\tau) \quad \text{for all } \tau \in T(C). \quad \text{(e.3.54)}$$

The above proof also contains the following
Lemma 3.9. (see 3.1 of [33]) Let $\epsilon > 0$. There exists $\delta > 0$ satisfying the following: Suppose that $A$ is a unital separable simple $C^*$-algebra with $TR(A) = 0$, $C \subset A \otimes K$ is a non-unital hereditary $C^*$-subalgebra with continuous scale and suppose that $u \in U(\tilde{C})$ with $sp(u) = T$. Then, for any $x \in K_0(C)$ with $p_C(x) < \delta$ and any $y \in K_1(C)$, there exists a unitary $v \in A$ such that

$$[v] = y, \quad [u, v] < \epsilon \quad \text{and} \quad \text{bott}_1(u, v) = x.$$  

(3.55)

Lemma 3.10. Let $A$ be a unital separable simple $C^*$-algebra which is $\mathbb{Z}$-stable and let $C \subset A \otimes K$ be a hereditary $C^*$-subalgebra with continuous scale. Then $C \otimes \mathbb{Z} \cong C$.

Proof. Let $C = a(A \otimes K)a$ for some positive element $a \in A \otimes K$. By the assumption, we may assume that

$$a \lesssim E_N,$$

where $E_N = \text{id}_{M_N(A)}$. Since $A$ is $\mathbb{Z}$-stable, $A$ has stable rank one. We may assume that $a \in M_N(A)$. So $C \otimes \mathbb{Z}$ is a hereditary $C^*$-subalgebra of $M_N(A) \otimes \mathbb{Z} \cong M_N(A)$. Let $b(A \otimes M_N)b = C \otimes \mathbb{Z}$, where $b = a \otimes 1_{\mathbb{Z}}$. Then

$$[a] = [b]$$

in the Cuntz semigroup. Therefore $C \otimes \mathbb{Z} \cong C$.

Lemma 3.11. Let $\epsilon > 0$ and let $\Delta : (0, 1) \rightarrow (0, 1)$ be a non-decreasing map. There exists $\eta > 0$, $\delta > 0$ and a finite subset $\mathcal{G} \in C(\mathbb{T})_{s.a.}$ satisfying the following: Suppose that $A$ is a $\mathbb{Z}$-stable unital separable simple $C^*$-algebra such that $TR(A \otimes Q) = 0$, $C$ is a non-unital hereditary $C^*$-subalgebra of $A \otimes K$ with continuous scale, $B = \tilde{C}$, and suppose that $u, v \in U(B)$ are two unitaries such that $sp(u) = T$,

$$\mu_{\tau \otimes \varphi}(I_a) \geq \Delta(a) \quad \text{for all} \quad \tau \in T(C)$$

and for all arcs $I_a$ with length at least $a \geq \eta$, where $\varphi : C(\mathbb{T}) \rightarrow B$ is defined by $\varphi(f) = f(u)$ for all $f \in C(\mathbb{T})$ and

$$|\tau(g(u)) - \tau(g(v))| < \delta \quad \text{for all} \quad g \in \mathcal{G}, \quad \text{for all} \quad \tau \in T(C),$$

$$[u] = [v] \quad \text{in} \quad K_1(C), \quad uv^* \in CU(B) \quad \text{and} \quad \pi(u) = \pi(v),$$

where $\pi : B \rightarrow B/C = \mathbb{C}$ is the quotient map. Then there exists a unitary $w \in U(\tilde{B} \otimes \mathbb{Z})$ such that

$$\|w^*(u \otimes 1)w - (v \otimes 1)\| < \epsilon.$$  

(3.58)

Proof. Without loss of generality, we may assume that $\pi(u) = \pi(v) = 1$. We first note, by [34], that $TR(A \otimes M_\tau) = 0$ for any supernatural number $\tau$. Let $\varphi : C(\mathbb{T}) \rightarrow B$ be the monomorphism defined by $\varphi(f) = f(u)$.

It follows from 3.3 that there is a non-decreasing function $\Delta : (0, 1) \rightarrow (0, 1)$ such that

$$\mu_{\tau \otimes \varphi}(O_a) \geq \Delta(a) \quad \text{for all} \quad \tau \in T(C)$$

(3.59)

for all open balls $O_a$ of $\mathbb{T}$ with radius $a \in (0, 1)$.

Let $\epsilon > 0$. As in 3.10, we may assume that $C \subset M_n(A)$ for integer $n \geq 1$.

Let $p$ and $q$ be a pair of relatively prime supernatural numbers of infinite type with $Q_p + Q_q = \mathbb{Q}$. Denote by $M_p$ and $M_q$ the UHF-algebras associated to $p$ and $q$ respectively. Let $\iota_\tau : C \rightarrow C \otimes M_\tau$ be the embedding defined by $\iota_\tau(c) = c \otimes 1$ for all $a \in C$, where $\tau$ is a supernatural number.
Note $\text{Tr}(M_n(A) \otimes M_t) = 0$. Define $u_t = \iota_t(u)$ and $v_t = \iota_t(v)$. Denote $B_t = C \otimes M_t$, $\tau = p \cdot q$. Denote by $\varphi_t : C(\mathbb{T}) \to B_t$ the homomorphisms defined by $\varphi_t(f) = f(u_t)$ for all $f \in C(\mathbb{T})$, $\tau = p \cdot q$.

Let $\delta_1 > 0$ (in place of $\delta$) be required by [3.6] for $\epsilon/6$. Without loss of generality, we may assume that $\delta_1 < \epsilon/12$ and is small enough such that $\text{bott}_1(u_1, z_j)$ and $\text{bott}_1(u_1, w_j)$ are well defined and

$$\text{bott}_1(u_1, w_j) = \text{bott}_1(u_1, z_1) + \cdots + \text{bott}_1(u_1, z_j)$$

(e 3.60)

if $u_1$ is a unitary and $z_j$ is any unitaries with $||u_1, u_j|| < \delta_1$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, 3, 4$. Let $\delta_2 > 0$ (in place $\delta$) be require by [3.9] for $\delta_1/8$ (in place of $\epsilon$).

Furthermore, one may assume that $\delta_2$ is sufficiently small such that for any unitaries $z_1, z_2$ in a $C^*$-algebra with tracial states, $\tau((1/2\pi i) \log(z_1 z_2^*))$ is well defined and

$$\tau((1/2\pi i) \log(z_1 z_2^*)) = \tau((1/2\pi i) \log(z_1 z_3^*)) + \tau((1/2\pi i) \log(z_2 z_4^*))$$

for any tracial state $\tau$, whenever $||z_1 - z_3|| < \delta_2$ and $||z_2 - z_4|| < \delta_2$. We may further assume that $\delta_2 < \min\{\delta_1, \epsilon/6, 1\}$.

Let $\delta > 0$, $\eta > 0$ and $\delta_3 > 0$ (in place of $\sigma$) required by [3.8] for $\delta_2$ (in place of $\epsilon$).

Now assume that $u$ and $v$ are two unitaries which satisfy the assumption of the lemma with above $\delta$ and $\eta$.

Since $uv^* \in CU(B)$, $R_{u,v} \in \rho_B(K_0(B))$. Since $\pi(u) = \pi(v) = 1$, we may assume that $R_{u,v}(t_0) = 0$ (see [3.7]). It follows that there is $a \in \text{Aff}(T(C))$ with $||a|| < \delta_3/2$ such that $a - R_{u,v}|T(C) \in \rho_C(K_0(C))$. Then the image of $a_p - R_{u,v}|T(C)$ is in $\rho_{C \otimes M_p}(K_0(C \otimes M_p))$, where $a_p$ is the image of $a$ under the map induced by $\gamma_p$. The same holds for $q$. Note that

$$\mu_{T \circ \varphi_t}(I_a) \geq \Delta(a) \quad \text{for all } \tau \in T(C)$$

(e 3.61)

and for all $a > 0$ (and certainly holds for all $a \geq \eta$). By Lemma [3.8] there exist unitaries $z_p \in B_p$ and $z_q \in B_q$ such that

$$[z_1] = 0 \quad \text{in } K_1(B_t), \quad \tau = p \cdot q, \quad ||u_p - z_p^* v_p z_p|| < \delta_2 \quad \text{and} \quad ||q - z_q^* v_q z_q|| < \delta_2.$$

Moreover,

$$\tau((1/2\pi i) \log(u_p^* z_p^* v_p z_p)) = a_p(\tau) \quad \text{for all } \tau \in T(C_p) \quad \text{and}$$

(e 3.62)

$$\tau((1/2\pi i) \log(u_q^* z_q^* v_q z_q)) = a_q(\tau) \quad \text{for all } \tau \in T(C_q).$$

(e 3.63)

We then identify $u_p, u_q$ with $u \otimes 1$ and $z_p$ and $z_q$ with the elements in the unitization of $C \otimes M_p \otimes M_q$ which is also identified with the unitization of $C \otimes Q$. In the following computation, we also identify $T(C)$ with $T(C_p)$, $T(C_q)$, and $T(C_p)$, or $T(C_q)$ with $T(C \otimes Q)$ by identifying $\tau$ with $\tau \otimes t$, where $t$ is the unique tracial state on $M_p$, or $M_q$, or $Q$. In particular,

$$a_p(\tau \otimes t) = \tau(a) \quad \text{for all } \tau \in T(C)$$

(e 3.64)

$$a_q(\tau \otimes t) = \tau(a) \quad \text{for all } \tau \in T(C)$$

(e 3.65)
We compute that by the Exel formula (see 3.5 of [30]),

\[
(\tau \otimes t)(\text{bott}_1(u \otimes 1, z^*_p z_q)) = (\tau \otimes t)(\frac{1}{2\pi i} \log(z^*_p z_q(u \otimes 1)z^*_q z_p(u \otimes 1))) \quad (e.3.66)
\]

\[
= (\tau \otimes t)(\frac{1}{2\pi i} \log(z_q(u^* \otimes 1)z^*_q z_p(u \otimes 1)z^*_p) \quad (e.3.67)
\]

\[
= (\tau \otimes t)(\frac{1}{2\pi i} \log(z_q(u^* \otimes 1)z^*_q (v \otimes 1)) + (\tau \otimes t)(\frac{1}{2\pi i} \log((v^* \otimes 1)z_p(u \otimes 1)z^*_p) \quad (e.3.68)
\]

\[
= (\tau \otimes t)(\frac{1}{2\pi i} \log(u^*_q z^*_q v_q z_q) + (\tau \otimes t)(\frac{1}{2\pi i} \log(u^*_p z^*_p u_p z^*_p))) \quad (e.3.69)
\]

\[
= (\tau(a) - \tau(a) = 0 \quad (e.3.70)
\]

for all \( \tau \in T(C) \). It follows that

\[
\tau(\text{bott}_1(u \otimes 1, z^*_p z_q)) = 0. \quad (e.3.71)
\]

for all \( \tau \in T(C \otimes Q) \).

Since the UHF-algebra \( D := Q, M_p \) or \( M_q \) have unique trace, the map \( \rho_C \otimes \text{id}_{K_0(D)} \) is the same as the map \( \rho_{C \otimes D} \) if \( K_0(C \otimes D) \) is identified as \( K_0(C) \otimes K_0(D) \) respectively.

Let \( y = \text{bott}_1(u \otimes 1, z^*_p z_q) \in \ker \rho_{C \otimes Q} \).

Since \( Q, Q_p \) and \( Q_q \) are flat \( Z \) modules, as in the proof of 5.3 of [35],

\[
\ker \rho_{C \otimes Q} = \ker \rho_C \otimes Q \quad (e.3.72)
\]

\[
\ker \rho_{C \otimes M_t} = \ker \rho_C \otimes Q_{\tau}, \quad \tau = p \text{ and } \tau = q. \quad (e.3.73)
\]

It follows that there are \( x_1, x_2, ..., x_l \in \rho_C(K_0(C)) \) and \( r_1, r_2, ..., r_l \in Q \) such that

\[
y = \sum_{j=1}^l x_j \otimes r_j. \quad (e.3.74)
\]

Since \( Q = Q_p + Q_q \), one has \( r_{j,p} \in Q_p \) and \( r_{j,q} \in Q_q \) such that \( r_j = r_{j,p} - r_{j,q} \). So

\[
y = \sum_{j=1}^l x_j \otimes r_{j,p} - \sum_{j=1}^l x_j \otimes r_{j,q}. \quad (e.3.75)
\]

Put \( y_p = \sum_{j=1}^l x_j \otimes r_{j,p} \in \ker \rho_{C \otimes M_p} \) and \( y_q = \sum_{j=1}^l x_j \otimes r_{j,q} \in \ker \rho_{C \otimes M_q} \).

It follows from [3.5] that there are unitaries \( u_p \in B_p \) and \( u_q \in B_q \) such that

\[
[w_t] = 0 \text{ in } K_1(B_t), \quad \tau = p, q, \quad ||u_p, w_p|| < \delta_1/8, \quad ||u_q, w_q|| < \delta_1/8 \text{ and } \quad (e.3.76)
\]

\[
\text{bott}_1(u_p, w_p) = y_p \text{ and } \text{bott}_1(u_q, w_q) = y_q. \quad (e.3.77)
\]

Put \( W_p = z^*_p u_p \in C \otimes M_p \) and \( W_q = z^*_q u_q \in C \otimes M_q \). Then

\[
||u_p - W_p^* v_p W_p|| < \delta_2 + \delta_1/8 < \epsilon/6 \text{ and } ||u_q - W_q^* v_q W_q|| < \delta_2 + \delta_1/8 < \epsilon/6. \quad (e.3.78)
\]

Suppose that \( \pi(W_p) = \lambda_p \) and \( \pi(W_q) = \lambda_q \). Replacing \( W_p \) by \( \tilde{\lambda}_p W_p \) and \( W_q \) by \( \tilde{\lambda}_q W_q \), we may assume that \( \pi(W_p) = \pi(W_q) = 1. \)
Note, again, that \( u_t = u \otimes 1 \) and \( v_t = v \otimes 1 \), \( t = p, q \). With identification of \( W_t, w_t, z_t \) with unitaries in the unitization \( C \otimes Q \), we also have

\[
||[u \otimes 1, W^*_p W_q]|| < \delta_1/4 \quad \text{and} \quad (e 3.79)
\]

\[
bott_1(u \otimes 1, W^*_p W_q) = bott_1(u \otimes 1, w^*_p z_p^* w_q) = bott_1(u \otimes 1, w^*_p) + bott_1(u \otimes 1, z_p^*) + bott_1(u \otimes 1, w_q) (e 3.81) = -y_p + (y_p - y_q) + y_q = 0. \quad (e 3.82)
\]

Let \( Z_0 = W^*_p W_q \). Then \( Z_0 \in U_0(C \otimes Q) \) since \( C \otimes Q \) has stable rank one. Note \( \pi(Z_0) = \pi(W^*_p) \pi(W_q) = 1 \). Then it follows from the choice of \( \delta_1 \), \( (e 3.61) \), \( (e 3.82) \) and \( 3.6 \) that there is a continuous path of unitaries \( \{ Z(t) : t \in [0, 1] \} \subset C \otimes Q \) such that \( Z(0) = Z_0 \) and \( Z(1) = 1 \) and

\[
||[u \otimes 1, Z(t)]|| < \epsilon/6 \quad \text{for all} \quad t \in [0, 1]. \quad (e 3.83)
\]

Moreover, by \( 3.6 \) we may assume that

\[
\pi(Z(t)) = 1 \quad \text{for all} \quad t \in [0, 1]. \quad (e 3.84)
\]

Define \( U(t) = W_q Z(t) \). Then

\[
U(0) = W_p, \quad U(1) = W_q \quad \text{and} \quad \pi(U(t)) = 1 \quad \text{for all} \quad t \in [0, 1]. \quad (e 3.85)
\]

So, in particular, \( U(0) \in B_p \) and \( U(1) \in B_q \). Therefore, by \( (e 3.85) \), \( U \in C \otimes \mathcal{Z}_{p,q} \subset C \otimes \mathcal{Z} \) is a unitary and, by \( (e 3.78) \) and \( (e 3.83) \),

\[
\|u \otimes 1 - U^* (v \otimes 1) U\| < \epsilon/3. \quad (e 3.86)
\]

The following is a non-unital version of Theorem 4.6 of \([33]\).

**Theorem 3.12.** Let \( A \) be a unital separable simple \( \mathcal{Z} \)-stable \( C^* \)-algebra in \( A_0 \). Let \( C \) be a non-unital hereditary \( C^* \)-subalgebra of \( A \otimes K \) with continuous scale and let \( B = \bar{C} \). Suppose that \( u \in CU(B) \). Then, for any \( \epsilon > 0 \), there exists a self-adjoint element \( h \in B \) with \( \|h\| \leq 1 \) such that

\[
\|u - \exp(i 2 \pi h)\| < \epsilon. \quad (e 3.87)
\]

**Proof.** Note that the assumption that \( u \in CU(B) \) implies that \( \pi(u) = 1 \), where \( \pi : B \to \mathbb{C} \) is the quotient map. Since \( C \) has continuous scale, without loss of generality, we may assume that \( C \subset M_k(A) \) for some \( k \geq 1 \). To simplify notation, we may further assume, without loss of generality, that \( C \subset A \). We assume that \( \text{sp}(u) = \mathbb{T} \), otherwise \( u = \exp(ig(u)) \) for some (real-valued) continuous branch of logarithm with \( \|g(u)\| \leq 2\pi \). Let \( \epsilon > 0 \). Let \( \varphi : C(\mathbb{T}) \to A \) be defined by \( \varphi(f) = f(u) \). It is a unital monomorphism. It follows from \( 3.3 \) that there is a non-decreasing function \( \Delta_1 : (0, 1) \to (0, 1) \) such that

\[
\mu_{\tau \circ \varphi}(I_a) \geq \Delta_1(a) \quad \text{for all} \quad \tau \in T(C) \quad (e 3.88)
\]

for all arcs \( I_a \) of \( \mathbb{T} \) with length \( a \in (0, 1) \). Define \( \Delta(a) = (1/3) \Delta_1(3a/4) \) for all \( a \in (0, 1) \).

Choose \( \delta_1 > 0 \) satisfying the following: If \( h_1, h_2 \) are two selfadjoint elements in any unital \( C^* \)-algebra with \( \|h_j\| \leq 3\pi, j = 1, 2 \), such that

\[
\|h_1 - h_2\| < \delta_1,
\]
Then
\[ \| \exp(ih_1) - \exp(ih_2) \| < \epsilon/4. \]

We may assume that \( \delta_1 < \epsilon/4 \). Note, by [34], for any supernatural number \( p \) of infinite type, \( TR(A \otimes M_p) = 0 \). Let \( p \) and \( q \) be two relatively prime supernatural numbers of infinite type. Consider \( u \otimes 1 \). Denote by \( u_p \) for \( u \otimes 1 \) in \( A \otimes M_p \). For any \( \delta_1/2 > \epsilon_0 > 0 \), by [3.4] there is a selfadjoint element \( h_p \in C\otimes M_p \) with \( sp(h_p) = [-2\pi, 2\pi] \) such that
\[ \| u_p - \exp(ih_p) \| < \epsilon_0 \text{ and } \tau(h_p) = 0 \text{ for all } \tau \in T(C \otimes M_p). \] (e 3.89)

Moreover, for some \( 1 > r > 0 \),
\[ \lim_{n \to \infty} \tau(|h_p|^{1/n}) < 1 - r \text{ for all } \tau \in T(C). \] (e 3.90)

Let \( \Gamma : C([-2\pi, 2\pi]) \to \text{Aff}(T(C)) \) be defined by
\[ \Gamma(f)(\tau) = (\tau \otimes t)(f(h_p)) \text{ for all } f \in C([-2\pi, 2\pi]) \text{ s.a.} \] (e 3.91)
and for all \( \tau \in T(C) \), where \( t \) is the unique tracial state on \( M_p \).

Let (note we now assume that \( C \subset A \))
\[ d(\tau) = \sup\{ \tau(c) : 0 \leq c \leq 1 \text{ and } c \in C \} \text{ for all } \tau \in T(A). \] (e 3.92)

Note that \( \inf_{\tau \in T(A)}(d(\tau)) > 0 \) since \( A \) is simple. Since \( C \) has continuous trace, \( d \in \text{Aff}(T(A)) \). Define \( \Gamma_1 : C([-2\pi, 2\pi]) \to \text{Aff}(T(A)) \) by
\[ \Gamma_1(f)(\tau) = d(\tau)\Gamma(f)(\tau/d(\tau)) \text{ for all } \tau \in T(A). \] (e 3.93)

Let \( \eta > 0, \delta > 0 \) and let \( \mathcal{G} \) be a finite subset as required by [3.11] for \( \delta_1/4 \) (in place of \( \epsilon \)) and \( \Delta \). Note \( \Delta(a) = (1/3)\Delta_1(3a/4) \) for all \( a \in (0,1) \). Choose \( \epsilon_0 \) sufficiently small, so the following holds (see Lemma 3.4 of [30]): For any unitary \( v \in C \otimes M_p \), if \( \| u_p - v \| < \epsilon_0 \), then
\[ \mu_{\tau \psi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(C) \] (e 3.94)
and for all arcs \( I_a \) with length \( a \geq \eta \), where \( \psi : C(T) \to A \otimes M_p \) is the homomorphism defined by \( \psi(g) = g(v) \) for all \( g \in C(T) \), and
\[ |\tau(g(u_p)) - \tau(g(v))| < \delta \text{ for all } \tau \in T(C) \] (e 3.95)
and for all \( g \in \mathcal{G} \). Note each \( \tau \in C \otimes M_p \) may be written as \( s \otimes t \), where \( s \in T(C) \) is any tracial state and \( t \in T(M_p) \) is the unique tracial state. It follows from 3.8 of [33] that there exists a selfadjoint element \( h_0 \in A \) with \( sp(h) = [-2\pi, 2\pi] \) such that
\[ \tau(f(h_0)) = \Gamma_1(f)(\tau) = (\frac{\tau \otimes t}{d(\tau)})f(h_p)d(\tau) \text{ for all } f \in C([-2\pi, 2\pi]) \] (e 3.96)
and for all \( \tau \in T(A) \). It follows that
\[ d_r(|h_0|) < d(\tau)(1 - r). \]
for all \( \tau \in T(A) \) (see [e.3.90]). Let \( \{ e_n \} \) be an approximate identity for \( C \) (\( e_n \) may not be projections). Since \( C \) has continuous scale, we may assume
\[ d_r(|h_0|) < \tau(e_n) \text{ for all } \tau \in T(A) \]

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and for some \( n \geq 1 \). Note that \( 0 \leq e_n \leq 1 \). So, in particular,
\[
d_r(|h_0|) < d_r(e_n) \quad \text{for all} \quad \tau \in T(C).
\]

The strict comparison implies that we may assume that \( h_0 \in e_n C e_n \). Note that
\[
\tau(h_0) = 0 \quad \text{for all} \quad \tau \in T(C). \tag{3.97}
\]

Define \( v_1 = \exp(ih_0) \in A \) and denote by \( \psi_1 : C(T) \to A \) the homomorphism given by \( \psi_1(f) = f(v_1) \) for all \( f \in C(T) \). Note that, by \( \mu_{C} \),
\[
\tau(g(v_1)) = (\tau \otimes t)(g(\exp(ih_0))) \quad \text{for all} \quad \tau \in T(C) \tag{3.98}
\]
and for all \( f \in C(T) \). By the choice of \( \epsilon_0 \), as in \( \mu \) and \( \mu_0 \),
\[
\mu_{\tau \circ \psi_1}(I_a) = \mu_{(\tau \otimes t) \circ \psi_1}(I_a) \geq \Delta(a) \quad \text{for all} \quad \tau \in T(C) \tag{3.99}
\]
and for all arcs \( I_a \) with length \( a \geq \eta \). Moreover,
\[
|\tau(g(u)) - \tau(g(v_1))| = |(\tau \otimes t)(g(u)) - (\tau \otimes t)(\exp(ih_0)))| < \delta \tag{3.100}
\]
for all \( \tau \in T(A) \) and for all \( g \in G \). Note since \( \tau(h_0) = 0 \) for all \( \tau \in T(C), \ v_1 \in CU(C) \). It follows from \( \delta \) that there exists a unitary \( W \in C \otimes \mathcal{Z} \) such that
\[
\|(u \otimes 1) - W^*(v_1 \otimes 1)W\| < \delta/4. \tag{3.101}
\]

Denote by \( \iota : A \to A \otimes \mathcal{Z} \) defined by \( \iota(a) = a \otimes 1 \) and define \( j : A \otimes \mathcal{Z} \to A \) such that \( j \circ \iota \) is approximately inner. Since \( A \) is separable, there is \( e \in C_+ \) with \( 0 \leq e \leq 1 \) which is strictly positive in \( C \). Let \( f_c(t) = 1 \) if \( t \in [c,1] \), \( f_c(t) = 0 \) if \( t \in [0,c/2] \) and linear in \( [c/2,c] \). So \( \{f_{1/n}(e)\} \) and \( \{f_{1/2}(e) \otimes 1\} \) form approximate identities for \( C \) and for \( C \otimes \mathcal{Z} \), respectively. Choose \( 1 > c > 0 \) such that
\[
\|f_c(e) \otimes 1)W^*(v_1 \otimes 1)W - W^*(v_1 \otimes 1)W\| < \delta_1/16\pi \quad \text{and} \quad \|W^*(v_1 \otimes 1)Wf_c(e) \otimes 1) - W^*(v_1 \otimes 1)W\| < \delta_1/16\pi \tag{3.102, 3.103}
\]

Let \( z \in U(A) \) such that
\[
\|z^*j \circ \iota(a)z - a\| < \delta_1/16\pi \tag{3.104}
\]
for \( a \in \{u,v_1,h,f_c(e),e,f_{c/2}(e)\} \). Then,
\[
\|u - z^*j(W^*(v_1 \otimes 1)W)z\| < \delta_1/16\pi. \tag{3.105}
\]

Moreover,
\[
\|f_{c/2}(e)z^*j(f_c(e) \otimes 1)z - z^*j(f_c(e) \otimes 1)z\| < \delta_1/8\pi \tag{3.106}
\]

Put \( H = z^*j(W^*)j(h_0 \otimes 1)j(W)z \), \( H_1 = z^*j(f_c(e) \otimes 1)j(W^*)j(h_0 \otimes 1)j(W)j(f_c(e) \otimes 1)z \) and \( h = f_{c/2}(e)z^*j(W^*)j(h_0 \otimes 1)j(W)z f_{c/2}(e) \). Then \( \|H\| \leq 2\pi, \|H_1\| \leq 2\pi \) and \( \|h\| \leq 2\pi \). We note that \( z^*j(W^*)j(h_0 \otimes 1)j(W)z \in A \) and \( C \) is hereditary, \( h \in C \) (recall that we assume that \( C \subset A \) at the beginning of this proof). Since \( v_1 = \exp(ih_0) \), by \( \mu \),
\[
\|u - \exp(iH)\| < \delta_1/16\pi. \tag{3.107}
\]
It follows from (e 3.102) and (e 3.103) that
\[ \| H - H_1 \| < 4\delta_1/16 = \delta_1/4 \]  
(e 3.108)
and by (e 3.106),
\[ \| H_1 - h \| < 4\delta_1/8 = \delta_1/2. \]  
(e 3.109)
Therefore
\[ \| H - h \| < \delta_1. \]  
(e 3.110)
By the choice of \( \delta_1 \), we have
\[ \| \exp(iH) - \exp(ih) \| < \epsilon/4. \]  
(e 3.111)
It follows from (e 3.107) and (e 3.111) that
\[ \| u - \exp(ih) \| < \epsilon. \]  
(e 3.112)
The point is that now \( h \in C \).

4 Unitaries in a small corner

In what follows, \( A \) will be a simple unital separable stably finite C*-algebra such that all quasitraces extend to traces and \( A \otimes K \) has strict comparison of positive elements. \( J \) will be the unique smallest ideal in \( M(A \otimes K) \) which properly contains \( A \otimes K \). (See 2.7)

Lemma 4.1. Let \( a, b \in J_+ \setminus (A \otimes K) \) such that \( \| b \| \leq 1 \) and \( b \) induces a continuous function on \( T(A) \). Let
\[ \inf \{ \tau(b) - d_+(a) : \tau \in T(A) \} > 0. \]
Then \( a \preceq b \).

Proof. We may assume that \( \| b \| = \| b + A \otimes K \| = \| a \| = \| a + A \otimes K \| = 1. \)

Let \( \epsilon > 0 \) be given, and let \( \{ e_k \}_{k=1}^\infty \) be an approximate identity for \( A \otimes K \) consisting of an increasing sequence of projections. We may assume that \( \epsilon < 1. \) Let \( \delta > 0 \) be such that
\[ 100\delta = \inf \{ \tau(b) - d_+((a - \epsilon/10)_+) : \tau \in T(A) \} > 0. \]
for all \( \tau \in T(A) \). Note that since \( b \) induces a continuous function on \( T(A) \), we must have that for every \( c \in (A \otimes K)_+ \setminus \{0\} \), there exists \( N' \geq 1 \) such that for all \( k \geq l \geq N' \),
\[ (e_k - e_l)b(e_k - e_l) \preceq c. \]  
(e 4.113)
Note also that by the definition of \( J \) (see 2.7), for all \( c \in (A \otimes K)_+ \setminus \{0\} \), there exists \( N'' \geq 1 \) such that for all \( k \geq l \geq N'' \),
\[ (e_k - e_l)(a - \epsilon/10)_+(e_k - e_l) \preceq c. \]  
(e 4.114)
By (e 4.113), let \( N \geq 1 \) be such that for all \( k \geq l \geq N \) and for all \( \tau \in T(A) \),
\[ d_+((e_k - e_l)b(e_k - e_l)) < \delta/100. \]
It follows that
\[ \tau(e_{N+2}b) > d_{\tau}((a - \epsilon/10)_+ + \delta/100 \text{ for all } \tau \in T(A). \]

By choosing \( \delta_1 > 0 \) small enough, we may assume that
\[ \tau((b^{1/2}e_{N+2}b^{1/2} - \delta_1)_+) > d_{\tau}((a - \epsilon/10)_+ + \delta/1000 \text{ for all } \tau \in T(A). \] (e.4.115)

Let \( h : [0, \infty) \to [0, 1] \) be the unique continuous function such that
\[
h(t) = df \begin{cases} 1 & t \in [9/10, \infty) \\ 0 & t \in [0, 8/10] \\ \text{linear on } [8/10, 9/10]. \end{cases}
\]

We now construct a subsequence \( \{k_j\}_{j=0}^\infty \) of the positive integers and a sequence \( \{p_n\}_{n=0}^\infty \) of pairwise orthogonal projections in \( A \otimes K \) which have the following properties:

1. \( N + 2 \leq k_j < k_j + 10 < k_{j+1} \) for all \( j \).
2. \( h(b^{1/2}(e_{k_{2n}} - e_{k_{2n-1}})b^{1/2}) > 0 \) for all \( n \geq 1 \).
3. \( p_m \perp p_n \) for all \( m \neq n \),
4. for each \( n \), there exists \( L' \) such that \( \sum_{k=1}^n p_n \leq e_{L'} \),
5. \( \|p_0b^{1/2}e_{N+2}b^{1/2}p_0 - b^{1/2}e_{N+2}b^{1/2}\| < \min\{\epsilon/1000, \delta_1 \epsilon/1000 \} \) and \( \tau((p_0b^{1/2}e_{N+2}b^{1/2}p_0 - \delta_1)_+) > d_{\tau}((a - \epsilon/10)_+ + \delta/10000 \text{ for all } \tau \in T(A), \)
6. \( \|p_n(b^{1/2}(e_{k_{2n}} - e_{k_{2n-1}})b^{1/2})p_n - b^{1/2}(e_{k_{2n}} - e_{k_{2n-1}})b^{1/2}\| < \min\{\epsilon/1000^{n+10}, \delta_1 \epsilon/1000^{n+10} \} \) for all \( n \geq 1 \),
7. \( h(p_n(b^{1/2}(e_{k_{2n}} - e_{k_{2n-1}})b^{1/2})p_n) > 0 \) for all \( n \geq 1 \).

The construction is by induction on \( n \) (and \( j = 2n - 1, 2n \) for \( n \geq 0 \); we don’t define \( k_{-1} \)).

**Basis step** \( n = 0 \).

We take
\[ k_0 = df N + 2. \]

By (e.4.115), find an integer \( L \geq N + 10 \) such that
\[
\|e_{L}^{1/2}e_{N+2}^{1/2}e_{L} - b^{1/2}e_{N+2}^{1/2}\| < \min\{\epsilon/1000, \delta_1 \epsilon/1000 \} \text{ and } \tau((e_{L}^{1/2}e_{N+2}^{1/2}e_{L} - \delta_1)_+) > d_{\tau}((a - \epsilon/10)_+ + \delta/10000 \text{ for all } \tau \in T(A), \] (e.4.116)

Take
\[ p_0 = df e_{L}. \]

We have also shown (5) above holds.

**Induction step.** Suppose that for \( n \geq 0 \), and for all \( l \leq n, k_{2l-1}, k_{2l} \) and \( p_l \) have been constructed. We now construct \( k_{2n+1}, k_{2n+2} \) and \( p_{n+1} \) which satisfy the condition (1)-(7) above.

By the previous steps in the induction, there exists an integer \( L' \geq 1 \) such that \( p_l \leq e_{L'} \) for all \( l \leq n \). Increasing \( L' \) if necessary, we may assume that \( L' \geq k_{2n} \).

Now choose an integer \( k_{2n+1} > k_{2n} + 10 \) such that
\[
\|e_{L'}^{1/2}b^{1/2}(1 - e_{k_{2n+1}})\| < (1/10) \min\{\epsilon/1000^{n+11}, \delta_1 \epsilon/1000^{n+11} \}. \]
Since \( \|b + A \otimes K\| > 99/100\),
\[
    h(b^{1/2}(1 - e_{k_{2n+1}}))b^{1/2} > 0 \quad \text{and} \quad (e.4.117)
\]
\[
    h((1 - e_{L'+10})b^{1/2}(1 - e_{k_{2n+1}}))b^{1/2}(1 - e_{L'+10}) > 0. \quad (e.4.118)
\]
Hence, choose \( k_{2n+2} > k_{2n+1} + 10 \) such that
\[
    h(b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}}))b^{1/2} > 0 \quad (e.4.119)
\]
(so (2) above holds for \( n + 1 \)) and
\[
    h((1 - e_{L'+10})b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}}))b^{1/2}(1 - e_{L'+10}) > 0.
\]
Note that
\[
    \|e_{L'+10}b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}})\| < (1/10) \min\{\epsilon/1000^{n+1}, \delta_1 \epsilon/1000^{n+1}\}. \quad (e.4.120)
\]
(Here we also use that \( \|b\| \leq 1 \).)
Hence, choose \( L'' > L' + 20 \) so that
\[
    h((e_{L''} - e_{L'+10})b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}}))b^{1/2}(e_{L''} - e_{L'+10}) > 0 \quad (e.4.121)
\]
\[
    \|e_{L''} - e_{L'+10}b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}}))b^{1/2}(e_{L''} - e_{L'+10}) - b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}}))b^{1/2} < \min\{\epsilon/1000^{n+1}, \delta_1 \epsilon/1000^{n+1}\}. \quad (e.4.122)
\]
It is clear that (1) follows.
Let
\[
    p_{n+1} = d_i e_{L''} - e_{L'+10}.
\]
Then (3) and (4) follow (for \( n + 1 \)). Moreover, (6) follows from (\( e.4.122 \)) and (7) follows from (\( e.4.120 \)).
This completes the inductive construction.
Note also, by (4) above \( \sum_{n=1}^{\infty} p_n \) converges strictly in \( M(A \otimes K) \).
We now construct two subsequences \( \{m_n\}_{n=0}^{\infty} \) and \( \{L_n\}_{n=0}^{\infty} \) of the positive integers and a sequence \( \{q_n\}_{n=0}^{\infty} \) of projections in \( A \otimes K \) so that the following hold:
(i) \( q_n \perp q_m \) whenever \( |n - m| \geq 2 \).
(ii) \( q_n \leq e_{L_n} - e_{L_{n-2}} \) (We define \( L_{-2} = L_{-1} = e_{L_{-2}} = e_{L_{-1}} = 0 \).)
(iii) For all \( n \geq 1 \),
\[
    \|e_{L_{n-1}}(a - e)^{1/2}(1 - e_{m_n})\| < (1/10)(\epsilon/10^{n+1}).
\]
(iv) \( \sum_n q_n \) converges strictly.
(v) \( q_n(a - \epsilon/10)^{1/2}(e_{m_n} - e_{m_{n-1}})(a - \epsilon/10)^{1/2} q_n \approx \epsilon/1000^{n+10} (a - \epsilon/10)^{1/2}(e_{m_n} - e_{m_{n-1}})(a - \epsilon/10)^{1/2} \)
for all \( n \geq 1 \).
(vi) \( q_0(a - \epsilon/10)^{1/2} e_{m_0}(a - \epsilon/10)^{1/2} q_0 \approx \epsilon/1000^{10} (a - \epsilon/10)^{1/2} e_{m_0}(a - \epsilon/10)^{1/2} \)
(vii) For all \( n \geq 1 \),
\[
    (a - \epsilon/10)^{1/2}(e_{m_n} - e_{m_{n-1}})(a - \epsilon/10)^{1/2} \leq h(p_n b^{1/2}(e_{k_{2n}} - e_{k_{2n-1}}))b^{1/2} \]p_n)
The construction is by induction on \( n \).
\textbf{Basis step} $n = 0$. 

By (e 4.114), choose $m_0 \geq 1$ so that for all $k > l \geq m_0$,

\[(a - \epsilon/10)_+^{1/2}(e_k - e_l)(a - \epsilon/10)_+^{1/2} \leq h(p_1b^{1/2}(e_{k_2} - e_{k_1})b^{1/2}p_1).
\]

Choose an integer $L_0 \geq 1$ so that

\[e_{L_0}(a - \epsilon/10)_+^{1/2} e_{m_0}(a - \epsilon/10)_+^{1/2} e_{L_0} \approx \epsilon/100^{n_0} (a - \epsilon/10)_+^{1/2} e_{m_0}(a - \epsilon/10)_+^{1/2}.
\]

Let $q_0 = df e_{L_0}$. 

So (vi) holds. 

\textit{Induction step.} Suppose that $m_l$ and $q_l$ have been constructed for all $l \leq n$. We now construct $m_{n+1}$ and $q_{n+1}$.

By the induction hypothesis,

\[
\begin{align*}
\sum_{j=0}^{n-1} q_j &\leq e_{L_{n-1}}, \quad \sum_{j=0}^{n} q_j \leq e_{L_n} \quad \text{and} \quad ||e_{L_{n-1}}(a - \epsilon/10)_+^{1/2}(1 - e_{m_n})|| < (1/10)(\epsilon/100^{n+11}). 
\end{align*}
\]

(e 4.123)

Choose $m_{n+1} > m_n$ so that the following hold: For all $k > l \geq m_{n+1}$,

\[(a - \epsilon/10)_+^{1/2}(e_k - e_l)(a - \epsilon/10)_+^{1/2} \leq h(p_{n+1}b^{1/2}(e_{k_{2n+4}} - e_{k_{2n+3}})b^{1/2}p_{n+1}).
\]

(Here, we are using (e 4.114).)

\[
\begin{align*}
||e_{L_n}(a - \epsilon/10)_+^{1/2}(1 - e_{m_{n+1}})|| < (1/10)(\epsilon/100^{n+12}).
\end{align*}
\]

Thus (iii) holds. By the induction hypothesis, we have that

\[(a - \epsilon/10)_+^{1/2}(e_{m_{n+1}} - e_{m_n})(a - \epsilon/10)_+^{1/2} \leq h(p_{n+1}b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}})b^{1/2}p_{n+1}).
\]

Also, since

\[
\begin{align*}
||e_{L_{n-1}}(a - \epsilon/10)_+^{1/2}(1 - e_{m_n})|| < (1/10)(\epsilon/100^{n+11}),
\end{align*}
\]

\[
\begin{align*}
||e_{L_{n-1}}(a - \epsilon/10)_+^{1/2}(e_{m_{n+1}} - e_{m_n})|| < (1/10)(\epsilon/100^{n+11}).
\end{align*}
\]

(Here we use that $||a|| \leq 1$.) Thus (vi) holds.

Hence, we can find $L_{n+1} > L_n$ and a projection $q_{n+1} \in (e_{L_{n+1}} - e_{L_{n-1}})(A \otimes K)(e_{L_{n+1}} - e_{L_{n-1}})$ such that

\[
\begin{align*}
||q_{n+1}(a - \epsilon/10)_+^{1/2}(e_{m_{n+1}} - e_{m_n})(a - \epsilon/10)_+^{1/2} q_{n+1} - (a - \epsilon/10)_+^{1/2}(e_{m_{n+1}} - e_{m_n})(a - \epsilon/10)_+^{1/2}|| < \epsilon/100^{n+11}.
\end{align*}
\]

Thus (v) holds. We also have (ii) holds.

This completes the inductive construction. Note that (iv) follows from (ii).

Now, by (vii) and the Cuntz relation, for all $n \geq 1$, there exists $x_n \in q_n(A \otimes K)p_n$ such that

\[
\begin{align*}
||x_n|| \leq 10/8 \quad \text{and} \quad q_n(a - \epsilon/10)_+^{1/2}(e_{m_n} - e_{m_{n-1}})(a - \epsilon/10)_+^{1/2} q_n \approx_{\epsilon/100} x_n(p_{n}b^{1/2}(e_{k_{2n+2}} - e_{k_{2n+1}})b^{1/2}p_{n})x_n^*.
\end{align*}
\]

(e 4.125)

(In the above, the norm estimate $||x_n|| \leq 10/8$ comes from the fact that we are using the function $h$ in (vii), and from the continuous functional calculus.)
Since \( \|b\| \leq 1 \), by (5) above and the assumption that \( A \) has strict comparison for positive element, we have
\[
(a - \epsilon/10)_+ \preceq (p_0 b^{1/2} e_{N+2} b^{1/2} p_0 - \delta_1)_+.
\] (e 4.126)

It follows from (e 4.126) and (vi) there exists \( x_0 \in q_0(A \otimes K)e_{N+2} \) such that
\[
\|x_0\| \leq 100/\delta_1 \quad \text{and} \quad q_0(a - \epsilon/10)^{1/2} e_{m_0} (a - \epsilon/10)^{1/2} q_0 \approx_{\epsilon/100} x_0 (p_0 b^{1/2} e_{N+2} b^{1/2} p_0) x_0^*.
\] (e 4.127)

Let \( x \in M(A \otimes K) \) be given by
\[
x = df \sum_{n=1}^{\infty} x_n.
\]

Note, by (3) and (i), \( \sum_{k=0}^{\infty} x_{2k} \) and \( \sum_{k=0}^{\infty} x_{2k+1} \) converges in the strict topology. It follows from (5),(6) and (e 4.127) that
\[
x b^{1/2} (e_{N+2} + \sum_{n=1}^{\infty} (e_{k_{2n}} - e_{k_{2n-1}})) b^{1/2} x^* \approx_{\epsilon/100} x (p_0 b^{1/2} e_{N+2} b^{1/2} p_0 + \sum_{n=1}^{\infty} p_n b^{1/2} (e_{k_{2n}} - e_{k_{2n-1}})) b^{1/2} p_n x^*
\]
\[
= x_0 p_0 b^{1/2} e_{N+2} b^{1/2} p_0 x^* + \sum_{n=1}^{\infty} x_n p_n b^{1/2} (e_{k_{2n}} - e_{k_{2n-1}}) b^{1/2} p_n x_n^*
\]
\[
\approx_{\epsilon/100} (a - \epsilon/10)^{1/2} (e_{m_0} + \sum_{n=1}^{\infty} (e_{m_n} - e_{m_{n-1}})) (a - \epsilon/10)^{1/2}
\]
\[
= (a - \epsilon/10)_+ \approx_{\epsilon/10} a
\]

Hence,
\[
x b^{1/2} (e_{N+2} + \sum_{n=1}^{\infty} (e_{k_{2n}} - e_{k_{2n-1}})) b^{1/2} x^* \approx_{\epsilon} a.
\]

Since \( b \geq b^{1/2} (e_{N+2} + \sum_{n=1}^{\infty} (e_{k_{2n}} - e_{k_{2n-1}})) b^{1/2} \), there exists \( y \in M(A \otimes K) \) such that
\[
y b y^* \approx_{\epsilon} a.
\]

Since \( \epsilon > 0 \) was arbitrary,
\[
a \preceq b.
\]

\[ \square \]

**Proposition 4.2.** Suppose, in addition, that \( A \) has stable rank one.

Suppose that \( p, q \in M(A \otimes K) \backslash A \otimes K \) are two projections such that \( \tau(p) = \tau(q) < \infty \) for all \( \tau \in T(A) \). Then there exists a partial isometry \( v \in M(A \otimes K) \) such that
\[
v^* p v = q.
\]

**Proof.** Put \( C = A \otimes K \). Let \( a \in p C p \) and \( b \in q C q \) be two strictly positive elements. Then neither are projections. By the assumption,
\[
d_\tau(a) = d_\tau(b) \quad \text{for all} \quad \tau \in T(A).
\]

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Since $C$ has strict comparison for positive elements and $A$ has stable rank one, one obtains a partial isometry $v \in C^{**}$ such that $xv \in C$ for all $x \in \overline{Ca}$ and $vy \in C$ for all $y \in \overline{C}$. Moreover,

$$v^*\overline{aC}v = \overline{bCb}.\$$

(See, for example, [29].) For any $c \in C$, since $q \in M(C)$, $qc \in C$. It follows that $qc \in \overline{bC}$.

Similarly, $cv = cpv \in C$, Therefore $v \in M(C)$. We verify that $v^*pv = q$.

Before proceeding with the next result, we both recall and introduce some notation, and also recall some preliminaries, which will be specific to the next result.

Let $K$ be the compact operators and let $\{e_{j,k}\}_{1 \leq j,k < \infty}$ be a system of matrix units for $K$. Recall that by identifying $A$ with $A \otimes e_{1,1}$, for every $\tau \in T(A)$, there is a natural extension of $\tau$ to a trace (which we also denote by “$\tau$”) on $(A \otimes K)_+$ and hence $M(A \otimes K)_+$. Similarly, by identifying $A$ with $A \otimes e_{1,1} \otimes e_{1,1}$, there is a natural extension of $\tau$ to a trace (which we also denote by “$\tau$”) on $(A \otimes K \otimes K)_+$ and hence $M(A \otimes K \otimes K)_+$.

Let $C =_{df} A \otimes K$. Note that for all $c \in M(C)_+$, for all $\tau \in T(A)$ and for all $j \geq 1$,

$$\tau(c \otimes e_{j,j}) = \tau(c).$$

(e 4.128)

Next, let $\psi : K \to K \otimes K$ be a *-isomorphism. This induces a *-isomorphism

$$\Psi =_{df} id_A \otimes \psi : C \to C \otimes K.$$

$\Psi$ induces a *-isomorphism (which we also denote by “$\Psi$’’)

$$\Psi : M(C) \to M(C \otimes K).$$

Moreover, for all $\tau \in T(A)$, we have that

$$\tau \circ \Psi = \tau.$$ (e 4.129)

Next, we introduce some notation concerning Hilbert modules. (For more information (and notation) concerning Hilbert modules, we refer the reader to [18], [45] Chapter 14 and [4] Chapter 13.) For a Hilbert $C$-module $E$, for all $x, y \in E$, let $\theta_{x,y} \in K(E)$ be the element that is given by $\theta_{x,y} =_{df} x < y, >$; i.e., $\theta_{x,y}(z) =_{df} x < y, z >$ for all $z \in E$. Recall that $K(E)$ is the closed linear span of all the $\theta_{x,y}$ for $x, y \in E$.

Recall (see, for example, [18] Lemma 4 and Theorem 1, or [45] 15.2.11 and 15.2.12) that we have a *-isomorphism

$$\Phi : K(H_C) \to C \otimes K$$

given by

$$\Phi : \eta \eta, \xi \mapsto \sum_{1 \leq j,k < \infty} (b_j c^*_k) \otimes e_{j,k}$$

where $\eta, \xi \in H_C$ have the form $\eta = (b_1, b_2, b_3, ...)$ and $\xi = (c_1, c_2, c_3, ...)$, where $b_j, c_k \in C$, $\sum_{j=1}^{\infty} b_j^* b_j$ and $\sum_{k=1}^{\infty} c_k^* c_j$ converges in norm. Moreover, $\Phi$ induces a *-isomorphism (which we also denote by “$\Phi$’’)

$$\Phi : B(H_C) \to M(C \otimes K).$$

The above *-isomorphism is a special case of Kasparov’s Theorem ([18] Theorem 1).
Proposition 4.3. Suppose, in addition, that $A$ has the property that, for every bounded strictly positive affine lower semicontinuous function $f : T(A) \to (0, \infty)$, there exists a non-zero $a \in (A \otimes K)_+$ which is not Cuntz equivalent to a projection such that $d_\tau(a) = f(\tau)$ for all $\tau \in T(A)$.

Then for every bounded, strictly positive, affine, lower semicontinuous function $f : T(A) \to (0, \infty)$, there exists a projection $p \in M(A \otimes K) \setminus (A \otimes K)$ such that

\[ \tau(p) = f(\tau) \text{ for all } \tau \in T(A). \]

Proof. We will use the notation introduced before this proposition. Let $a \in C_+ ((= (A \otimes K)_+)$ be such that $a$ is not Cuntz equivalent to a projection and $d_\tau(a) = f(\tau)$ for all $\tau \in T(A)$. By the Kasparov Absorption Theorem ([18] Theorem 2), let $q \in B(H_C) \setminus K(H_C)$ be a projection such that $qH_C \cong aC.$

Hence, let

\[ U : aC \to qH_C \]  

be an (unitary) isomorphism of Hilbert $C$-modules. Let $\eta \in qH_C \subseteq H_C$ be a vector such that $\eta = U(a)$. Hence,

\[ \langle \eta, \eta \rangle = \langle U(a), U(a) \rangle = \langle a, a \rangle = a^2 \text{ and } qH_C = \eta_C. \]

Since $\eta \in qH_C \subseteq H_C$, $\eta$ has the form

\[ \eta = (a_1, a_2, a_3, ....) \]

where $a_j \in C$ for all $j \geq 1$. Note that by (e 4.132),

\[ \sum_{j=1}^{\infty} a_j^* a_j = \langle \eta, \eta \rangle = a^2. \]

Note that

\[ K(qH_C) = qK(H_C)q \]

and under the isomorphism $\Phi$ (see the notation introduced before this proposition),

\[ \Phi(K(qH_C)) = \Phi(q)(C \otimes K)\Phi(q). \]

Also, $K(qH_C)$ is the closed linear span of elements of the form $\theta_{\eta c, \eta d}$ for $c, d \in C$. Moreover, the image, under $\Phi$, of such an element is:

\[ \Phi(\theta_{\eta c, \eta d}) = \sum_{1 \leq j, k < \infty} (a_j c d^* a_k^*) \otimes e_{j,k} \quad (c, d \in C). \]

Hence,

\[ \Phi(K(qH_C)) = b(C \otimes K) b^* \]

where $b \in C \otimes K$ is the element that is given by

\[ b = df \sum_{j=1}^{\infty} a_j \otimes e_{j,1}. \]
Hence, for all $\tau \in T(A)$,

$$\tau(\Phi(q)) = \lim_{n \to \infty} \tau((bb^*)^{1/n}) = d_\tau(bb^*).$$  \hfill (e 4.134)

But by the definition of $b$ and by (e 4.133),

$$b^*b = \sum_{j=1}^{\infty} a_j^*a_j \otimes e_{1,1} = a^2 \otimes e_{1,1}.$$  \hfill (e 4.135)

Therefore, by (e 4.130), (e 4.134) and (e 4.128), for all $\tau \in T(A)$,

$$f(\tau) = d_\tau(a) = d_\tau(a \otimes e_{1,1}) = \tau(\Phi(q)).$$  \hfill (e 4.135)

If we let $p = d_f \Psi^{-1} \circ \Phi(q)$, then, by (e 4.129), we are done.

**Remark 4.4.** We note that by [7] Theorem 5.5, if $A$ is a unital simple exact finite and $\mathcal{Z}$-stable C*-algebra then $A$ satisfies the hypotheses of Proposition 4.3.

We can generalize Proposition 4.3 to the case where the lower semicontinuous function is unbounded or takes the value $\infty$.

**Corollary 4.5.** Suppose, in addition, that $A$ has the property that, for every bounded strictly positive affine lower semicontinuous function $f : T(A) \to (0, \infty]$, there exists a nonzero $a \in (A \otimes \mathcal{K})_+$ which is not Cuntz equivalent to a projection such that $d_\tau(a) = f(\tau)$ for all $\tau \in T(A)$.

(E.g., $A$ can be unital simple exact finite and $\mathcal{Z}$-stable.)

Then for every strictly positive, affine, lower semicontinuous function $f : T(A) \to (0, \infty]$, there exists a projection $p \in M(A \otimes \mathcal{K}) - (A \otimes \mathcal{K})$ such that

$$\tau(p) = f(\tau) \quad \text{for all} \quad \tau \in T(A).$$

**Proof.** Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $M(A \otimes \mathcal{K})$ such that $q_n \sim 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} q_n = 1$, where the sum converges in the strict topology on $M(A \otimes \mathcal{K})$.

Next, $f$ is the pointwise limit of a strictly increasing sequence of strictly positive affine continuous functions on $T(A)$ (see [1] Lemma 5.3; see also [14] Theorem 11.12 or [10] (Edwards’ Separation Theorem)). In other words, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of strictly positive affine continuous functions on $T(A)$ with $f_n < f_{n+1}$ for all $n$ such that $f_n \to f$ pointwise (where $\infty$ is allowed to be the limit at a point).

For all $n \geq 1$, let $g_n$ be the strictly positive affine continuous function on $T(A)$ that is given by $g_n = d_f f_n - f_{n-1}$ (where $f_0 = d_f 0$). For each $n \geq 1$, apply Proposition 4.3 to get a projection $p_n \in q_n M(A \otimes \mathcal{K}) q_n$ such that

$$\tau(p_n) = g_n(\tau) \quad \text{for all} \quad \tau \in T(A).$$

Let $p = d_f \sum_{n=1}^{\infty} p_n$. Then the sum converges strictly and $p$ is a projection in $M(A \otimes \mathcal{K}) - (A \otimes \mathcal{K})$ such that

$$\tau(p) = f(\tau) \quad \text{for all} \quad \tau \in T(A).$$

\hfill \Box

The next lemma is standard (e.g., see Lemma 2.1 in [11]).
Lemma 4.6. For every $\epsilon > 0$, there exists $\delta > 0$ such that the following holds: If $C$ is a unital C*-algebra and $p,q \in C$ projections such that

$$pq \approx_{\delta} p$$

then there exists a unitary $u \in C$ such that

$$upu^* \leq q \text{ and } \|u - 1\| < \epsilon.$$  

The next lemma is an observation of Rordam ([42] section 4).

Lemma 4.7. Let $C$ be a unital C*-algebra and let $x \in C$ be a nilpotent element (i.e., $x^2 = 0$).

Then

$$x \in \text{GL}(C)$$

where the closure is in the norm topology.

NOTE: For the rest of this section, unless otherwise stated, $A$ is a unital separable simple $\mathbb{Z}$-stable C*-algebra with unique tracial state $\tau$ which is the only normalized quasitrace and $J$ is the unique proper C*-ideal of $M(A \otimes \mathcal{K})$ which properly contains $A \otimes \mathcal{K}$ (see [2.8]).

Lemma 4.8. Let $p \in J \setminus A \otimes \mathcal{K}$ be a projection, and suppose that $a \in pJp$ is a positive element such that

$$2d_{\tau}(a) < \tau(p)$$

and there exists positive $c \in pJp \setminus A \otimes \mathcal{K}$ such that $c \perp a$.

Then for every $\epsilon > 0$, there exists a projection $q \in pJp \setminus p(A \otimes \mathcal{K})p$ such that

$$d_{\tau}(a) < \tau(q) < d_{\tau}(a) + \epsilon \text{ and } qa \approx_{\epsilon} a.$$  

Proof. For simplicity, let us assume that $\|a\| \leq 1$ and $\epsilon < 1$.

Let $h : [0,1] \rightarrow [0,1]$ be the unique continuous function such that

$$h(t) = \begin{cases} 
1 & t \in [\epsilon/10000,1] \\
0 & t \in [0,\epsilon/100000] \\
\text{linear on } [\epsilon/100000,\epsilon/10000] 
\end{cases}$$

Hence, $p - h(a) \in p(A \otimes \mathcal{K})p$ is a positive element such that

$$\tau(p - h(a)) = \tau(p) - \tau(h(a)) > \tau(p) - d_{\tau}(a) > d_{\tau}(a).$$

Also, since $c \in \text{Her}(p - h(a))$, $p - h(a) \notin A \otimes \mathcal{K}$.

Hence, by Lemma [4.4] and Proposition [4.3], let $r \in pJp - p(A \otimes \mathcal{K})p$ be a projection such that

$$d_{\tau}(a) < \tau(r) < d_{\tau}(a) + \epsilon \text{ and } r \in \text{Her}(p - h(a)).$$

Hence,

$$r \perp (a - \epsilon/1000)_+. $$

By Lemma [4.1] again, let $x \in pJp$ be an element such that

$$x^*x = (a - \epsilon/50)_+ \text{ and } xx^* \leq r.$$
Let \( x = v|x| \) be the polar decomposition of \( x \) (in \( A^{**} \)). Then
\[
(a - \epsilon/50)_+ = |x|^2 \quad \text{and} \quad v|x|^{2}v^* \leq r.
\]
Note that since \((a - \epsilon/50)_+ \perp r, x^2 = 0. Hence, by Lemma 4.7
\[
x \in GL(pJp).
\]
Hence, by Theorem 5 in [40], let \( u \in pJp \) be a unitary such that
\[
u(a - 3\epsilon/100)_+u^* = v(a - 3\epsilon/100)_+v^* \leq r.
\]
Hence,
\[
(a - 3\epsilon/100)_+ \leq u^*ru.
\]
In particular,
\[
u^*rua \simeq \epsilon a.
\]
Taking \( q = df u^*ru \), we are done. \( \Box \)

**Lemma 4.9.** Let \( p \in J \setminus (A \otimes K) \) be a projection and let \( u \in pJp \) be a unitary.
Suppose that \( p_1 \in pJp \setminus p(A \otimes K)p \) is a projection such that
\[
100\tau(p_1) < \tau(p).
\]
Then for every \( \epsilon > 0 \) and for every projection \( q \in J \setminus A \otimes K \) with \( q \perp p \), there exist projections
\[
p_1 = e_1 < e_2 < e_3 < p + q
\]
where the inequalities are strict, and there exists a unitary \( w \in (p + q)J(p + q) \) such that
\[
\|w - (u + q)\| < \epsilon \quad \text{and} \quad we_1w^* \leq e_2 \leq we_3w^* < p + q.
\]
Moreover,
\[
e_2 \preceq p + q - e_2.
\]

**Proof.** By the virtue of 4.3 replacing \( q \) by a subprojection if necessary, we may assume that
\[
\tau(q) < (\tau(p) - \tau(p_1))/10^{10000}.
\]
By Lemma 4.1 Proposition 4.3 and Proposition 4.2 we can decompose \( q \) into a direct sum of non-zero projections
\[
q = q' + q'' + q'''
\]
such that \( q', q'', q''' \in J \setminus (A \otimes K) \).

Let \( \delta_5 > 0 \) (in place of \( \delta \)) associated with \( \epsilon/100 \) (in place of \( \epsilon \)) be given by 4.6. Let \( \delta_{j-1} > 0 \) (in place of \( \delta \)) associated with \( \delta_j/100 \) (in place of \( \epsilon \)) given by 4.6 \( j = 5, 4, 3, 2, 1 \).
Take \( e_1 = df p_1 \).
By hypothesis, \[ \tau(e_1) < \tau(p)/100. \]

Hence, \[ e_1 + ue_1 u^* \in pJp \] and \[ d_\tau(e_1 + ue_1 u^*) < \tau(p)/50. \]

It follows from Lemma 4.8 that there exists a projection \( q_1 \in (p + q')J(p + q') \) such that
\[
\begin{align*}
d_\tau(e_1 + ue_1 u^*) &< \tau(q_1) < d_\tau(e_1 + ue_1 u^*) + \min\{\delta_1, \tau(p)\}/10^{100}, \\
q_1 e_1 &\approx_{\delta_1} e_1 \quad \text{and} \quad q_1 ue_1 u^* \approx_{\delta_1} ue_1 u^*.
\end{align*}
\]

By Lemma 4.6 and the definition of \( \delta_1 \), we can find a projection \( e_2 \in (p + q')J(p + q') \) such that
\[ \|e_2 - q_1\| < \delta_2/50 \]
(and hence \( e_2 \sim q_1 \)) and
\[ e_1 \leq e_2. \]

As a consequence, (since \( \delta_1 < \delta_2/100 \)),
\[ e_2(ue_1 u^*) \approx_{\delta_2} ue_1 u^*. \]

It follows from Lemma 4.6 and the definition of \( \delta_2 \) that there is a unitary \( w_1 \in (p+q')J(p+q') \) such that
\[ \|w_1 - (u + q')\| < \delta_3/50 \]
and
\[ w_1 e_1 w_1^* \leq e_2. \]

Since
\[
d_\tau(e_2 + w_1^* e_2 w_1) \leq 2(\tau(p)/50 + \min\{\delta_1, \tau(p)\}/10^{100}) \leq \tau(p)/25 + 2\tau(p)/10^{100},
\]
and by Lemma 4.8 let \( q_2 \in (p + q' + q'')J(p + q' + q'') \) be a projection such that
\[
\begin{align*}
d_\tau(e_2 + w_1^* e_2 w_1) &< \tau(q_2) < d_\tau(e_2 + w_1^* e_2 w_1) + \min\{\delta_2, \tau(p)\}/10^{100}, \\
q_2 e_2 &\approx_{\delta_2/100} e_2
\end{align*}
\]
and
\[ q_2 w_1^* e_2 w_1 \approx_{\delta_2/100} w_1^* e_2 w_1. \]

By Lemma 4.6 and the definition of \( \delta_2 \), there exists a projection \( e_3 \in (p+q'+q'')J(p+q'+q'') \) such that
\[ \|e_3 - q_2\| < \delta_3/50 \]
(and hence \( e_3 \sim q_2 \), and
\[ e_2 \leq e_3. \]

Hence,
\[ e_3 (w_1^* e_2 w_1) \approx_{\delta_3} w_1^* e_2 w_1. \]

Hence,
\[ (w_1 + q'')e_3 (w_1 + q'')^* e_2 \approx_{\delta_3} e_2. \]

Note that since
\[ e_1 \leq e_2 \leq e_3, \]

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\[(w_1 + q'')e_1(w_1 + q'')^* \leq (w_1 + q'')e_3(w_1 + q'')^* \]

Also, by our construction of \(e_2\),

\[(w_1 + q'')e_1(w_1 + q'')^* \leq e_2\]

Hence,

\[(w_1 + q'')(e_3 - e_1)(w_1 + q'')^*(e_2 - (w_1 + q'')e_1(w_1 + q'')^*) \approx_{\delta_3} (e_2 - (w_1 + q'')e_1(w_1 + q'')^*)\]

Hence, by Lemma 4.6 and by the definition of \(\delta_3\), there exists a unitary \(w_2' \in (p + q' + q'' - w_1e_1w_1^*)J(p + q' + q'' - w_1e_1w_1^*)\) with

\[
\|w_2' - (p + q' + q'' - w_1e_1w_1^*)\| < \delta_4/100
\]

and

\[e_2 - w_1e_1w_1^* \leq w_2'(w_1 + q'')(e_3 - e_1)(w_1 + q'')^*w_2'^*\]

Hence, if we take

\[w = _{df} (w_2' + w_1e_1w_1^*)(w_1 + q'') + q''\]

then \(w \in (p + q)J(p + q)\) is a unitary,

\[
\|w - (u + q)\| < \epsilon, \quad p_1 = e_1 < e_2 < e_3 < p, \\
\text{and } we_1w^* < e_2 < we_3w^* < p \tag{e.4.138}
\]

as required. Finally, since \(q_1 \sim e_2\), by (e.4.137) and by applying Lemma 4.1 we also have

\[e_2 \leq p + q - e_2.\]

\[\square\]

**Lemma 4.10.** Let \(p \in J \setminus (A \otimes K)\) be a projection and \(u \in pJp\) be a unitary. Let \(1/2^{10} > r > 0\). Then for every projection \(q \in J \setminus A \otimes K\) with \(q \perp p\), there is a (norm-) continuous path of unitaries \(\{u(t) : t \in [0, 1]\}\) in \(U((p + q)J(p + q))\) such that

\[u(0) = u + q, \quad u(1) = ((p + q - f_2) + u_1)((p + q - f_1) + v_1)\]

where \(v_1 \in U(f_1Jf_1), \quad u_1 \in U(f_2Jf_2), \quad \tau(p + q - f_j) > r\tau(p)\) for and \(f_j \in (p + q)J(p + q) \setminus A \otimes K\) is a projection, \(j = 1, 2\).

**Proof.** By Proposition 4.3 and Lemma 4.1 there is a projection \(p_1 \in pJp \setminus p(A \otimes K)p\) such that \(\tau(p_1) > r\tau(p)\) and \(100\tau(p_1) < \tau(p)\) (equivalently, \(\beta < \tau(p_1) < \tau(p)/100\) for some \(\beta < \tau(p)/2^{10}\)).

Since \(q \in J \setminus (A \otimes K)\) is a projection, we can decompose \(q\) into orthogonal projections

\[q = q' \oplus q''\]

where \(q', q'' \in J \setminus (A \otimes K)\).

It follows from Lemma 4.9 that there exists a unitary \(w_0 \in (p + q')J(p + q')\) with

\[
\|w_0 - (u + q')\| < \epsilon/10000
\]

(which implies that \(w_0\) is path-connected to \(u + q'\) in \((p + q')J(p + q')\)) and there are projections \(e_1, e_2, e_3 \in (p + q')J(p + q') \setminus (A \otimes K)\) such that

\[p_1 = e_1 < e_2 < e_3 < p + q' < p + q\]
(where the inequalities are strict), and

\[ w_0 e_1 w_0^* < e_2 < w_0 e_3 w_0^* < p + q' < p + q. \]

Moreover,

\[ e_2 \preceq p + q - e_2. \]  \hspace{1cm} (e 4.139)

It follows that

\[ e_1 \preceq p + q - e_1. \]  \hspace{1cm} (e 4.140)

Define \( w = w_0 + q'' \). Hence, we have that

\[ \| w - (u + q) \| < \epsilon/10000, \]  \hspace{1cm} (e 4.141)

\[ p_1 = e_1 < e_2 < e_3 < e_4 =_{df} p + q, \]  \hspace{1cm} (e 4.142)

\[ w e_1 w^* < e_2 < w e_3 w^* < p + q, \]  \hspace{1cm} (e 4.143)

and there exists a positive \( c \in (p + q) J(p + q) - (A \otimes K) \) such that

\[ c \perp e_3 + w e_3 w^*. \]

From the above and by Proposition 4.2

\[ e_1 \sim w e_1 w^*, \quad e_2 - e_1 \sim e_2 - w e_1 w^*, \]

\[ e_3 - e_2 \sim w e_3 w^* - e_2 \quad \text{and} \quad p + q - e_3 \sim p + q - w e_3 w^*. \]

Recall that \( e_4 =_{df} p + q \). Put \( e_0 =_{df} 0 \). Hence, for \( j = 1, 2 \), there is a unitary \( w_{2j} \in (e_2_j - e_2_{j-2}) J(e_2_j - e_2_{j-2}) \) such that

\[ w_{2j}(e_2_j - e_2_{j-2}) w_{2j}^* = w e_{2j-1} w^* - e_{2j-2}. \]

Define

\[ z =_{df} w_2 \oplus w_4 \quad \text{and} \quad y =_{df} z^* w, \]

two unitaries in \((p + q) J(p + q)\). Then \( w = z y \). Clearly, by definition,

\[ (e_{2n} - e_{2j-2}) z = w_{2j} = z(e_{2j} - e_{2j-2}) \quad \text{and} \]

\[ \tau(e_2 - e_0) = \tau(e_2) \geq \tau(e_1) > r \tau(p). \]

Also, since

\[ z e_1 z^* = w e_1 w^*, \]

\[ e_1 = z^* w e_1 w^* z. \]

In other words,

\[ e_1 = y e_1 y^* \quad \text{or} \quad e_1 y = y e_1. \]

Hence,

\[ y = y_1 \oplus y_3 \]

where \( y_1 \in e_1 J e_1 \) is a unitary and \( y_3 \in (p + q - e_1) J(p + q - e_1) \) is a unitary. Moreover,

\[ \tau(e_2) > \tau(e_1) = \tau(p_1) > r \tau(p). \]  \hspace{1cm} (e 4.144)

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By (e.4.139) and (e.4.140), there are partial isometries \(Z_1, Z_2 \in (p + q)J(p + q)\) such that

\[
Z_i^* Z_i = e_i, \quad Z_i Z_i^* = q_i e_i' \leq p + q - e_i, \quad i = 1, 2.
\]

Define a unitary \(U_i \in (p + q)J(p + q)\) by

\[
U_i = Z_i + Z_i^* + (p + q - e_i'), \quad i = 1, 2.
\]

We may write

\[
w = zy = (w_2 \oplus U_2 w_2^* U_2 w_2^* w_4) (y_1 \oplus U_1 y_1^* U_1 y_1 U_1^* y_2).
\]

(e.4.145)

Put \(f_i = p + q - e_i, \quad i = 1, 2\), so \(f_i \in (p + q)J(p + q) \setminus A \otimes K\) are projections.

\[
u_1 = U_2 w_2 U_2^* w_4 \in (p + q - f_2)J(p + q - f_2) \text{ and } v_1 = U_1 y_1 U_1^* y_2 \in (p + q - f_1)J(p + q - f_1).
\]

Then, by (e.4.145) and (e.4.141), \(u + p\) is connected to

\[(e_2 + u_1)(e_1 + v_1)\]

via a norm continuous path in \(U(p + q)J(p + q)\). Moreover, we compute that, by (e.4.144),

\[
\tau(p + q - f_i) = \tau(e_i) > r \tau(p), \quad i = 1, 2.
\]

Lemma 4.11. Let \(p \in J \setminus (A \otimes K)\) be a projection and \(u \in pJp\) be a unitary. Let \(1 > r > 0\). Then for every projection \(q \in J \setminus A \otimes K\) with \(q \perp p\), there is an integer \(m \geq 1\) and a (norm-) continuous path of unitaries \(\{u(t) : t \in [0, 1]\}\) in \(U((p + q)J(p + q))\) such that

\[
u(0) = u + q, \quad u(1) = ((p + q - f_1) + u_1) ((p + q - f_2) + u_2) \ldots ((p + q - f_m) + u_m)
\]

where \(u_j \in U(f_j)Jf_j\), \(\tau(p + q - f_j) > r \tau(p)\), and \(f_j(p + q)J(p + q) \setminus A \otimes K\) is a projection, \(j = 1, 2, \ldots, m\).

Proof. There is an integer \(n \geq 1\) such that \(1 - (1 - s)^n > r\), where \(s = (3/4)^2.10\). By Lemma 4.11 Proposition 4.2 and Proposition 4.3 there are mutually orthogonal projections \(q_l \in (1 - p)J(1 - p) \setminus A \otimes K, \quad l = 1, 2, \ldots, n\). Put \(q = \sum_{l=1}^n q_l\).

It suffices to prove the following statement: there are projections

\[
f_1, f_2, \ldots, f_k \in (p + \sum_{j=1}^k q_j)J(p + \sum_{j=1}^k q_j) \setminus A \otimes K, \quad (e.4.146)
\]

unitaries \(u_i \in U(f_j)Jf_j\) such that \(\tau(f_j) < \tau(\sum_{j=1}^k q_j) + (1 - s)^k \tau(p)\),

(e.4.147)

\(j = 1, 2, \ldots, k\), and \(u\) is connected to \(\prod_{j=1}^k ((p + q - f_j) + u_j)\) by a norm continuous path of unitaries in \((p + q)J(p + q)\) for all \(1 \leq k \leq n\).

We prove this by induction on \(k\).

The case \(k = 1\) follows immediately from Lemma 4.11 with \(q_1\) in place of \(q\) and \(s\) in place of \(r\). Suppose that we have proven the statement for \(k\). We now prove the statement for \(k + 1\).

Now for \(1 \leq j \leq k\), apply Lemma 4.11 to \(u_j\) (in place of \(u\)), \(f_j\) (in place of \(p\)), \(s\) (in place of \(r\)) and \(q_{k+1}\) (in place of \(q\)) to get projections \(f_{j,l} \in (f_j + q_{k+1})J(f_j + q_{k+1}) \setminus A \otimes K\) (\(l = 1, 2\) and
unitaries $u_{j,l} \in f_{j,l} J f_{j,l}$ such that $u_j$ is connected to \((f_{j,1} + u_{j,1})(f_{j,2} + q_{k+1}) + u_{j,2}\) by a norm continuous path of unitaries in \((f_j + q_{k+1}) J (f_j + q_{k+1})\) and

\[
\tau(f_j + q_{k+1}) - f_{j,l} > s \tau(f_j).
\]

It follows that

\[
\tau(f_{j,l}) < \tau(f_j) + \tau(q_{k+1}) - s \tau(f_j)
\]

\[
= \tau(q_{k+1}) + (1 - s) \tau(f_j)
\]

\[
< \sum_{j=1}^{k+1} \tau(q_j) + (1 - s)^{k+1} \tau(p).
\]

The statement above holds for $k + 1$ by renaming $f_{j,l}'s$ and $u_{j,l}'s$ which completes the induction. \qed

**Lemma 4.12.** Let $p, q, r, s \in J \ A \otimes K$ be projections such that

\[
q, r, s \leq p, q, r \perp s \text{ and } \tau(q) \leq \tau(s) \leq \tau(r).
\]

Suppose that $u \in qJq$ is a unitary. Then there is a continuous path of unitaries \(\{u(t) : t \in [0, 1]\} \subset pJp\) such that

\[
\begin{align*}
u(0) &= u + (p - q) \text{ and } u(1) = v + (p - r),
\end{align*}
\]

where where $v \in rJr$ is a unitary.

**Proof.** Since $s \perp q$ and $q \leq s$, we may view $u + (p - q)$ as $u + (p - q) = \text{diag}(u, s, p - q - s) = \text{diag}(u, u^* u, p - q - s)$ (with coordinates in $(q, s, p - q - s)$) which is norm path connected to $\text{diag}(q, u, p - q - s)$, which has the form $w + (p - s)$ for some unitary $w \in sJs$.

But since $s \perp r$ and $s \leq r$, $w + (p - s)$ can be viewed as $w + (p - s) = \text{diag}(w, r, p - s - r) = \text{diag}(w, w^* w, p - s - r)$ (with coordinates in $(s, r, p - s - r)$) which is norm path connected to $\text{diag}(s, w, p - s - r)$, which has the form $v + (p - r)$ for some unitary $v \in rJr$. \qed

**Lemma 4.13.** Let $p \in J \ A \otimes K$ be a projection and let $q \in J \ A \otimes K$ be another projection with $p \perp q$ and $q \preceq p$.

Let $u \in pJp$ be a unitary. Then there exists a continuous path of unitaries \(\{u(t) : t \in [0, 1]\} \subset U((p + q) J (p + q))\) such that

\[
\begin{align*}
u(0) &= u + q \text{ and } u(1) = p + v,
\end{align*}
\]

where $v \in U(qJq)$.

**Proof.** The result of the lemma follows immediately from Lemma 4.11 and Lemma 4.12.

By Proposition 4.13, Proposition 4.2 and Lemma 4.14, we can decompose $q$ into pairwise orthogonal projections

\[
q = q' \oplus q'' \oplus q'''
\]

where $q', q'', q''' \in J \ (A \otimes K)$ and $\tau(q') < \tau(q'') < \tau(q''')$, which implies that $q' \preceq q'' \preceq q'''$.

By applying Lemma 4.11 and by passing to a unitary which connects with $u + q'$ by a continuous path of unitaries in $U((p + q) J (p + q))$, without loss of generality, we may write

\[
u + q' = \prod_{j=1}^{m} ((p + q' - f_{j}) + u_{j})
\]

(e 4.153)
such that $u_i \in U(f_j J f_j)$, where $f_j \leq p + q'$ and $f_j \leq q''$, $j = 1, 2, \ldots, m$. Note that

$$f_j, q'' \leq p + q, \ f_j \perp q'', \ f_j \perp q''' \perp q'' \ 	ext{and} \ f_j \leq q'' \leq q'', \ j = 1, 2, \ldots, m.$$ 

Thus, by Lemma 4.12, for each $j$, there is a continuous path of unitaries $\{w_j(t) : t \in [0, 1] \} \subset (p + q)J(p + q)$ such that

$$w_j(0) = (p + q' + q'' - f_j) + u_j \ 	ext{and} \ w_j(1) = v_j + (p + q' + q''),$$

where $v_j \in U(q'' J q'''$, $j = 1, 2, \ldots, m$. Define

$$u(t) = \prod_{j=1}^{m} w_j(t) \ 	ext{for all} \ t \in [0, 1].$$

Then $\{u(t) : t \in [0, 1] \}$ is a continuous path of unitaries in $(p + q)J(p + q)$. Moreover,

$$u(0) = u + q \ 	ext{and} \ u(1) = v + p,$$

where $v = \prod_{j=1}^{m} (v_j + q' + q'') \in U(qJq)$. \hfill \qed

5 The main results

**Lemma 5.1.** Let $A$ be a unital $C^*$-algebra and $B = A \otimes K$. Then $M(B)$ contains a sequence of mutually orthogonal projections $\{p_n\}$ each of which is equivalent to the identity $1_{M(B)}$, $p_n \notin B$, $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} p_n$ converges strictly to $1_{M(B)}$.

**Proof.** This is known. One can have a partition of $\mathbb{N}$ into a sequence $\{I_n\}$ of mutually disjoint infinite subsets such that $1 \in I_n$ and $I_n \cap \{1, 2, \ldots, n - 1\} = \emptyset$. Let $\{e_{ij}\}$ be a system of matrix unit for $K$. We identify $e_{ii}$ with $1_A \otimes e_{ii}$. Define

$$p_n = \sum_{j \in I_n} e_{jj}.$$ 

The convergence is in the strict topology. Let $e_n = \sum_{i=1}^{n} e_{ii}$. Then $\{e_n\}$ forms an approximate identity for $B$. Note that, by the construction,

$$p_m e_n = 0 \ 	ext{for all} \ m > n.$$ 

It follows that $\sum_{k=1}^{n} p_k$ converges strictly to $1_{M(B)}$. \hfill \qed

The following holds in much general situation.

**Lemma 5.2.** Let $A$ be a unital separable simple $\mathcal{Z}$-stable $C^*$-algebra with unique tracial state $\tau$, and let $B = A \otimes K$. Let $J$ be the smallest ideal of $M(B)$ which properly contains $B$.

Then, for any sequence of positive numbers $\{\alpha_n\}$ with $\alpha_n \geq \alpha_{n+1}$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, there exists a sequence of mutually orthogonal projections $\{q_n\} \in M(B)$ such that $q_{n+1} \leq q_n$, $\tau(q_n) = \alpha_n$, $n = 1, 2, \ldots$, such that $\sum_{n=1}^{\infty} q_n$ converges in the strict topology and $p = \sum_{n=1}^{\infty} q_n \in J$.

**Proof.** Let $\{p_n\}$ be as in 5.1.

By Proposition 4.3 and since $p_n \sim 1_{M(B)}$, let $q_n \in J$ be such that $q_n \leq p_n$ and $\tau(q_n) = \alpha_n$ for all $n \geq 1$. Then $\sum_n q_n$ converges strictly to an element of $J$. \hfill \qed
Corollary 5.3. Let $A$ be a unital separable simple $\mathcal{Z}$-stable $C^*$-algebra with unique tracial state $\tau$, let $B = A \otimes \mathcal{K}$, let $J$ be the smallest ideal of $M(B)$ which properly contains $B$, and let $p \in M(B)$ be a projection such that $\tau(p) = \infty$. Then, for any sequence of positive numbers $\{\alpha_n\}$ with $\alpha_n \geq \alpha_{n+1}$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, there exists a sequence of mutually orthogonal projections $\{q_n\} \subseteq M(B)$ such that $q_{n+1} \subseteq q_n$, $\tau(q_n) = \alpha_n$, $n = 1, 2, \ldots$, such that $\sum_{n=1}^{\infty} q_n$ converges in the strict topology to a projection $q \subseteq J$ such that $q \leq p$.

Lemma 5.4. Let $B$ be a non-unital and $\sigma$-unital simple $C^*$-algebra, $J \subseteq M(B)$ be an ideal containing $B$ such that $J/B$ is purely infinite and simple and let $p \in J$ be a non-zero projection. Suppose that $u \in pM(B)p$ is a unitary such that $|u| = 0$ in $K_1(pM(B)p)$. Then, for any $\epsilon > 0$, there are two selfadjoint elements $H_1, H_2, \subseteq pM(B)p$ and a unitary $w \subseteq p + pBp$ such that $\|H_1\| \leq \pi$, $\|H_2\| < \epsilon$ and

$$u = w \exp(iH_1) \exp(iH_2).$$

(e 5.155)

In particular, $w^*u \subseteq U_0(pM(B)p)$.

Proof. Note that $pM(B)p/pBp$ is purely infinite and simple. Let $\pi : pM(B)p \rightarrow pM(B)p/pBp$ be the quotient map. Suppose that $u$ is as in the lemma. Then $\pi(u) \subseteq U_0(pJp/pBp)$. Note that $pJp/pBp$ is a unital purely infinite simple $C^*$-algebra. It follows from a result of Chris Phillips ([11]) that, for any $\epsilon > 0$, there are selfadjoint elements $h_1, h_2 \subseteq pM(B)p/pBp$ such that

$$\pi(u) = \exp(ih_1) \exp(ih_2) \text{ and } \|h_1\| \leq \pi \text{ and } \|h_2\| < \epsilon.$$  

(e 5.156)

There are selfadjoint elements $H_1, H_2 \subseteq pM(B)p$ such that $\|H_1\| \leq \pi$, $\|H_2\| \leq \epsilon$ and $\pi(H_j) = h_j$, $j = 1, 2$. Let $u_j = \exp(iH_j)$, $j = 1, 2$. Then

$$w = uu_2^*u_1^* \subseteq p + pBp.$$  

(e 5.157)

It follows that $u = w \exp(iH_1) \exp(H_2)$ and $w^*u \subseteq U_0(pM(B)p)$.

Lemma 5.5. Let $A \subseteq A_0$ be a unital separable simple $\mathcal{Z}$-stable $C^*$-algebra with unique tracial state $\tau$, let $B = A \otimes \mathcal{K}$ and let $J \subseteq M(B)$ be an ideal containing $B$ such that $J/B$ is purely infinite and simple. Suppose that $p \subseteq J$ and $u \subseteq pM(B)p \setminus B$ be a unitary. Then, for any projection $q \subseteq J \setminus B$ and $pq = 0 = qp$, any $\epsilon > 0$, there is a unitary $v \subseteq qM(B)q$ and selfadjoint elements $h_1, h_2 \subseteq (p + q)M(B)(p + q)$ and $h_3 \subseteq \mathbb{C}(p + q) + (p + q)B(p + q)$ such that

$$\|h_1\| < \epsilon/2, \|h_2\| \leq \pi, \|h_3\| \leq 2\pi,$$

$$(u + v) - (p + q)\exp(ih_1)\exp(ih_2)\exp(ih_3) < \epsilon \text{ and}$$

$$(u + v) - (p + q)\exp(ih_1)\exp(ih_2)\exp(ih_3) \subseteq (p + q)B(p + q).$$

(e 5.158)

Proof. There is a projection $q_1 \subseteq J \setminus B$ such that $q_1 \leq q$ and there is a unitary $z \subseteq (p + q)M(B)(p + q)$ such that

$$z^*q_1z \subseteq p.$$  

It follows from Lemma 4.13 that there is a unitary $v_1 \subseteq q_1Jq_1$ such that

$$u + v_1 + (q - q_1) \subseteq U_0((p + q)J(p + q)).$$

(e 5.161)

Put $P = p + q$. It follows from 5.4 that there are selfadjoint elements $h_1, h_2 \subseteq PJP$ with $\|h_1\| < \epsilon/2$ and $\|h_2\| \leq \pi$ such that

$$u + v_1 + (q - q_1) = \exp(ih_1)\exp(ih_2)w,$$
where \( w \in P + PB_P \). Since \( B \) is simple and has stable rank one, there exists \( w_1 \in q + qBq \) such that \( [w_1] = -[w] \). Thus \( w(w_1 + p) \in U_0(P + PB_P) \). Let

\[ w(w_1 + p) = \prod_{k=1}^{m} \exp(ia_k), \]

where \( a_k \in (\mathbb{C}P + PB_P)_{s.a.} \). Let \( \pi(a_k) = \alpha_k, k = 1, 2, \ldots, m \). Since \( w(w_1 + p) \in P + PB_P \), \( \sum_{k=1}^{m} \alpha_k = 2N\pi \) for some integer \( N \). By replacing \( a_k \) by \( a_k - \alpha_k P \), we may assume that \( \pi(a_k) = 0, k = 1, 2, \ldots, m \). In other words, \( a_k \in PB_P, k = 1, 2, \ldots, m \). Choose \( b \in (qBq)_{s.a.} \) such that

\[ \tau(b) = \sum_{k=1}^{m} \tau(a_k). \]  

(e 5.162)

Set

\[ v_2 = q \exp(-ib). \]

(e 5.163)

Now let \( v = (v_1 + (q - q_1))w_1v_2 \) Then

\[ u + v = \exp(ih_1) \exp(ih_2)w(w_1 + p)(v_2 + p). \]

Note that

\[ w(w_1 + p)(v_2 + p) = P \prod_{k=1}^{m} \exp(ia_k) \exp(-ib). \]

But

\[ \tau(-b) + \sum_{k=1}^{m} \tau(a_k) = 0. \]

It follows that

\[ w(w_1 + p)(v_2 + p) \in CU(\mathbb{C}P + PB_P). \]

By 3.12 there is \( h_3 \in PB_P_{s.a.} \) with \( \|h_3\| \leq 2\pi \) such that

\[ \|w(w_1 + p)(v_2 + p) - P \exp(ih_3)\| < \epsilon. \]

It follows that

\[ \|(u + v) - P \exp(ih_1) \exp(ih_2) \exp(ih_3)\| < \epsilon. \]

\[ \Box \]

**Lemma 5.6.** Let \( A \) be a unital separable simple \( \mathcal{Z} \)-stable \( C^* \)-algebra with unique tracial state \( \tau \) and let \( J \) be the smallest ideal in \( M(A \otimes K) \) that properly contains \( A \otimes K \).

Then \( J \) has an approximate identity consisting of projections.

**Proof.** This follows immediately from Proposition 4.3 and Lemma 4.1. \( \Box \)

**Theorem 5.7.** Let \( A \in A_0 \) be a unital separable simple \( \mathcal{Z} \)-stable \( C^* \)-algebra with unique tracial state \( \tau \) and let \( J \) be the smallest ideal in \( M(A \otimes K) \) that properly contains \( A \otimes K \).

Then \( K_1(J) = 0 \). In fact, for any \( u \in J \),

\[ \text{cel}(u) \leq 7\pi. \]
Moreover, by (e 5.171), (e 5.172) and (e 5.173), one computes that

\[ U = (1 - P) + u_0, \]

where \( u_0 \in PJ \) is a unitary.

By reconsidering an approximate identity of \( J \) consisting of projections, one finds a non-zero projection \( e_0 \) in \((1 - P)J(1 - P)\). Then, by applying [5,2] one obtains a sequence of mutually orthogonal nonzero projections \( \{q_n\} \) such that \( \sum_{n=1}^{\infty} q_n \) converges in the strict topology and \( \sum_{n=1}^{\infty} q_n \in (1 - P)J(1 - P) \). By [5,3] putting \( P = p_0 \), there are a sequence of unitaries, \( v_n \in q_nJq_n \) and sequences of selfadjoint elements \( \{h_{2n-1}^n\} \subset (q_{2n-1} + q_{2n})J(q_{2n-1} + q_{2n}) \) and \( \{h_{2n}^n\} \subset (q_{2n} + q_{2n+1})J(q_{2n} + q_{2n+1}) \) such that

\[ \|h_1^{(n)}\| \leq \pi, \quad \|h_2^{(n)}\| \leq 2\pi, \quad (e \ 5.164) \]
\[ \|(v_{2n-1} + v_{2n}) - \exp(ih_1^{(n)})\| < \epsilon/2^n \quad \text{and} \]
\[ \|(v_{2n} + v_{2n+1}) - \exp(ih_2^{(n)})\| < \epsilon/2^n, \quad (e \ 5.166) \]

\( n = 0, 1, 2, \ldots \). Since \( \sum_{n=1}^{\infty} q_n \) converges in the strict topology and by (e 5.164), so do \( \sum_{n=0}^{\infty} v_n \), \( \sum_{n=1}^{\infty} v_n \), \( \sum_{n=1}^{\infty} h_1^{(n)} \), \( \sum_{n=0}^{\infty} h_1^{(n)} \), \( \sum_{n=0}^{\infty} h_2^{(n)} \) and \( \sum_{n=1}^{\infty} h_2^{(n)} \). Define

\[ U_0 = \sum_{n=0}^{\infty} v_n, \quad U_1 = \sum_{n=1}^{\infty} v_n \quad \text{(e 5.167)} \]
\[ H_{0,1} = \sum_{n=0}^{\infty} h_1^{(n)}, \quad H_{0,2} = \sum_{n=0}^{\infty} h_2^{(n)} \quad \text{(e 5.168)} \]
\[ H_{1,1} = \sum_{n=1}^{\infty} h_1^{(n)} \quad \text{and} \quad H_{1,2} = \sum_{n=1}^{\infty} h_2^{(n)} \quad \text{(e 5.169)} \]

Then

\[ U_0 = u_0 + U_1, \quad (e \ 5.170) \]
\[ \|U_1 - \exp(iH_{1,1})\| < \epsilon \quad (e \ 5.171) \]
\[ \|U_0 - \exp(iH_{0,1})\| < \epsilon \quad (e \ 5.172) \]
\[ \|H_{0,1}\| \leq \pi \quad \text{and} \quad \|H_{0,2}\| \leq 2\pi. \quad (e \ 5.173) \]

It follows from (e 5.171) that \( [U_1] = 0 \) in \( K_1(\tilde{J}) \). Then, by (e 5.172) that \( [u_0] = 0 \) in \( K_1(\tilde{J}) \). It follows that \( [u] = 0 \) in \( K_1(\tilde{J}) \). Since \( u \) is arbitrarily chosen, we conclude that

\[ K_1(\tilde{J}) = 0. \]

Moreover, by (e 5.171), (e 5.172) and (e 5.173), one computes that

\[ \text{cel}(u) \leq 3\pi + 3\pi = 6\pi. \]

So in general,

\[ \text{cel}(u) \leq 7\pi. \]

\[ \square \]
**Theorem 5.8.** Let $A \in A_0$ be a unital separable simple $\mathcal{Z}$-stable C*-algebra with unique tracial state. Then $M(A \otimes K)/(A \otimes K)$ has real rank zero.

**Proof.** Let $B =_{df} A \otimes K$, and let $\pi : M(B) \to M(B)/B$ be the natural quotient map. Let $J$ be the smallest ideal in $M(B)$ which properly contains $B$. Consider the six-term exact sequence:

\[
\begin{array}{c}
K_0(J) \to K_0(M(B)) \to K_0(M(B)/J) \\
\uparrow \\
K_1(M(B)/J) \leftarrow K_1(M(B)) \leftarrow K_1(J)
\end{array}
\]

From Theorem 5.7, $K_1(J) = 0$. The above six-term exact sequence implies that the map $K_0(M(B)) \to K_0(M(B)/J)$ is surjective. But it is also known that $K_0(M(B)) = 0$ (Proposition 12.2.1; see [38]). Hence, $K_0(M(B)/J) = 0$. Since $M(B)/J \cong \pi(M(B))/\pi(J)$, $K_0(\pi(M(B))/\pi(J)) = 0$. Therefore the map $K_0(\pi(M(B))) \to K_0(\pi(M(B))/\pi(J))$ is surjective.

Since both $\pi(J)$ and $\pi(M(B))/\pi(J)$ are simple purely infinite (this is well-known; an explicit reference can be found in [19]; it also follows immediately from [27] Theorem 3.5 and the definition of $J$ in [21] 2.2, Remark 2.9 and Lemma 2.1; see also [47] Theorem 2.2 and its proof), both $\pi(J)$ and $\pi(M(B))/\pi(J)$ have real rank zero. It follows from [6] Theorem 3.14 and Proposition 3.15 that $\pi(M(B))$ has real rank zero.

**Theorem 5.9.** $M(\mathcal{Z} \otimes K)/(\mathcal{Z} \otimes K)$ has real rank zero.

**Remark 5.10.** We note that Theorem 5.8 includes many other C*-algebras (see [34])—including crossed products coming from uniquely ergodic minimal homeomorphisms on a compact metric space with finite topological dimension (e.g., see [40], [8]).

Finally, we end the paper with some K-theory computations that follow immediately from our work.

**Corollary 5.11.** Let $\pi : M(\mathcal{Z} \otimes K) \to M(\mathcal{Z} \otimes K)/(\mathcal{Z} \otimes K) = Q(\mathcal{Z})$ be the natural quotient map. Let $J$ be the unique smallest ideal of $M(\mathcal{Z} \otimes K)$ which properly contains $\mathcal{Z} \otimes K$.

Then we have the following: $K_0(J) = \mathbb{R}$, $K_1(J) = 0$, $K_0(\pi(J)) = \mathbb{R}/\mathbb{Z} \cong T$, $K_1(\pi(J)) = 0$, $K_0(Q(\mathcal{Z})/\pi(J)) = 0$ and $K_1(Q(\mathcal{Z})/\pi(J)) = \mathbb{R}$.

**Proof.** Let $B =_{df} \mathcal{Z} \otimes K$.

That $K_0(J) = \mathbb{R}$ follows immediately from Proposition 1.2 and Proposition 1.3.

That $K_1(J) = 0$ follows immediately from Theorem 5.7.

That $K_0(Q(\mathcal{Z})/\pi(J)) = 0$ was computed in the proof of Theorem 5.8.

Hence, the six-term exact sequence (e 5.174) from Theorem 5.8 thus becomes the following six-term exact sequence:

\[
\begin{array}{c}
\mathbb{R} \to 0 \to 0 \\
\uparrow \\
K_1(M(B)/J) \leftarrow 0 \leftarrow 0
\end{array}
\]

Hence, $K_1(Q(\mathcal{Z})/\pi(J)) = K_1(M(B)/J) = \mathbb{R}$.

Next, the exact sequence

\[
0 \to B \to J \to J/B \to 0
\]

results in the six-term exact sequence.
\[ K_0(B) \rightarrow K_0(J) \rightarrow K_0(J/B) \]
\[ K_1(J/B) \leftarrow K_1(J) \leftarrow K_1(B) \]

which, by the above results, becomes

\[ \mathbb{Z} \rightarrow \mathbb{R} \rightarrow K_0(J/B) \]
\[ K_1(J/B) \leftarrow 0 \leftarrow 0 \]

Since the map \( \mathbb{Z} \rightarrow \mathbb{R} \) is the natural inclusion, \( K_1(\pi(J)) = 0 \) and \( K_0(\pi(J)) = \mathbb{R}/\mathbb{Z} \cong \mathbb{T} \). □

References

[1] R. Antoine, J. Bosa and F. Perera, Completi on of monoids with applications to the Cuntz semigroup, Internat. J. Math., 22 (2011), no. 6, 837–861. A copy is available at http://arxiv.org/pdf/1003.2874.

[2] B. E. Blackadar, A simple \( C^* \)-algebra with no nontrivial projections. Proc. Amer. Math. Soc. 78 (1980), no. 4, 504–508.

[3] B. E. Blackadar, A simple unital projectionless \( C^* \)-algebra. J. Operator Theory, 5 (1981), no. 1, 63–71.

[4] B. E. Blackadar, K-Theory for operator algebras. Second edition. Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.

[5] L. G. Brown, Extensions of \( AF \)-algebras, Operator Algebras and Applications, (Kingston, Ont. 1980) R. V. Kadison (ed.), Proc. Symp. Pure Math., 38 (1982) 175–176.

[6] L. G. Brown and G. K. Pedersen, \( C^* \)-algebras of real rank zero, Journal of Functional Analysis, 99 (1991), 131–149.

[7] N. P. Brown, F. Perera and A. S. Toms, The Cuntz semigroup, the Elliott conjecture, and dimension functions on \( C^* \)-algebras, J. Reine Angew. Math. 621 (2008), 191–211. A copy is available at http://arxiv.org/pdf/0609182v2.

[8] A. Connes, An analogue of the Thom isomorphism for crossed products of a \( C^* \)-algebra by an action of \( \mathbb{R} \). Adv. in Math. 39 (1981), no. 1, 31–55.

[9] K. T. Coward, G. A. Elliott and C. Ivanescu, The Cuntz semigroup as an invariant for \( C^* \)-algebras, J. Reine Angew. Math. 623 (2008), 161–193. A copy is available at http://arxiv.org/pdf/0705.0341.

[10] D. A. Edwards, Séparation des fonctions réelles définies sur un simplexe de Choquet, C. R. Acad. Sci. Paris, 261 (1965), 2798-2800.

[11] G. A. Elliott, Derivations of matroid \( C^* \)-algebras, II, Annals of Mathematics, Second Series 100 (1974), no. 2, 407–422.

[12] G. A. Elliott, Automorphisms determined by multipliers on ideals of a \( C^* \)-algebra, J. Functional Analysis, 23 (1976) no. 1, 1–10.
[13] G. A. Elliott, The ideal structure of the multiplier algebra of an AF-algebra, C. R. Math. Rep. Acad. Sci. Canada, 9 (1987), 225–230.

[14] K. Goodearl, Partially ordered abelian groups with interpolation, Math. Surveys and Monographs, 20, Amer. Math. Soc., Providence, 1986.

[15] N. Higson and M. Rordam, The Weyl–von Neumann Theorem for multipliers, Canad. J. Math., 43 (1991) no. 2, 322–330.

[16] X. Jiang and H. Su, On a simple unital projectionless C*-algebra, Amer. J. Math., 121 (1999), 359–413.

[17] I. Kaplansky, Functional analysis, some aspects of analysis and probability, Wiley, New York, 1958, pp. 1–34.

[18] G. G. Kasparov, Hilbert C*-modules: Theorems of Stinespring and Voiculescu, J. Operator Theory, 4 (1980), 133–150.

[19] D. Kucerovsky, P. W. Ng and F. Perera, Purely infinite corona algebras of simple C*-algebras, Math. Ann. 346 (2010), no. 1, 23–40.

[20] H. Lin, Ideals of multiplier algebras of simple AF C*-algebras, Proc. Amer. Math. Soc., 104 (1988), 239–244.

[21] H. Lin, Simple C*-algebras with continuous scales and simple corona algebras, Proc. Amer. Math. Soc., 112 (1991), no. 3, 871–880.

[22] H. Lin, Generalized Weyl–von Neumann Theorems, Internat. J. Math. 2 (1991) no. 6, 725–739.

[23] H. Lin, Exponential rank of C*-algebras with real rank zero and the Brown–Pedersen Conjectures, Journal of Functional Analysis, 114 (1993), 1–11.

[24] H. Lin, Generalized Weyl–von Neumann Theorems. II. Math. Scand., 77 (1995) no. 1, 129–147.

[25] H. Lin, Simple corona C*-algebras, Proc. Amer. Math. Soc. 132 (2004), 3215–3224.

[26] H. Lin, The tracial topological rank of C*-algebras, Proc. London Math. Soc., 83 (2001), 199–234.

[27] H. Lin, Full extensions and approximate unitary equivalence, Pacific J. Math., 229 (2007) no. 2, 389–428.

[28] H. Lin, Approximate homotopy of homomorphisms from C(X) into a simple C*-algebra, Mem. Amer. Math. Soc., 205 (2010), no. 963, vi+131 pp. ISBN: 978-0-8218-5194-4.

[29] H. Lin, Cuntz semigroups of C*-algebras of stable rank one and projective Hilbert modules, preprint. A copy is available at http://arxiv.org/pdf/1001.4558.

[30] H. Lin, Asymptotic unitary equivalence and classification of simple amenable C*-algebras, Invent. Math. 183 (2011), 385–450.

[31] H. Lin, Approximate unitary equivalence in simple C*-algebras of tracial rank one, Trans. Amer. Math. Soc. 364 (2012), no. 4, 2021–2086.
[32] H. Lin, *Homomorphisms from AH-algebras*, preprint, arXiv:1102.4631.

[33] H. Lin, *Exponential rank and exponential length for Z-stable simple C*-algebras*, preprint. A copy is available at http://arxiv.org/pdf/1301.0356

[34] H. Lin and Z. Niu, *The range of a class of classifiable separable simple amenable C*-algebras*, J. Funct. Anal., 260, (2011), 1–29.

[35] H. Lin and Z. Niu, *Homomorphisms into a simple Z-stable C*-algebras*, preprint (arXiv:1003.1760).

[36] H. Lin and H. Osaka, *Real rank of multiplier algebras of C*-algebras of real rank zero. Operator algebras and their applications (Waterloo, ON, 1994/1995)*, 235–241. Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997.

[37] H. Lin and W. Sun, *Tensor products of classifiable C*-algebras*, preprint (arXiv:1203.3737).

[38] J. A. Mingo, *K-theory and multipliers of stable C*-algebras*, Transactions of the American Mathematical Society, 299 (1987), no. 1, 397–411.

[39] G. J. Murphy, *Diagonality in C*-algebras*, Math. Z., 199 (1988) no. 2, 279–284.

[40] G. K. Pedersen, *Unitary extensions and polar decompositions in a C*-algebra*, J. Operator Theory 17 (1987), 357–364.

[41] N.C. Phillips, *Approximation by unitaries with finite spectrum in purely infinite C*-algebras*, J. Funct. Anal. 120 (1994), 98106.

[42] M. Rordam, *On the structure of simple C*-algebras tensored with a UHF-algebra*, Journal of Functional Analysis, 100 (1991), 1–17.

[43] M. Rordam, *Ideals in the multiplier algebra of a stable C*-algebra*, J. Operator Theory, 25 (1991), 283–298.

[44] M. Rordam, *The Jiang–Su algebra revisited*, J. Reine Angew. Math., 642 (2010), 129–144.

[45] N. E. Wegge-Olsen, *K-theory and C*-algebras*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[46] W. Winter, *Decomposition rank and Z-stability*. Invent. Math. 179 (2010), no. 2, 229–301.

[47] S. Zhang, *On the structure of projections and ideals of corona algebras*, Canadian J. Math. 41 (1989), 721–742.

[48] S. Zhang, *A property of purely infinite simple C*-algebras*. Proc. Amer. Math. Soc. 109 (1990), no. 3, 717–720.

[49] S. Zhang, *A Riesz decomposition property and ideal structure of multiplier algebras*, J. Operator Theory, 24 (1990), 209–226.

[50] S. Zhang, *K1-groups, quasidiagonality, and interpolation by multiplier projections*, Trans. Amer. Math. Soc. 325 (1991) no. 2, 792–818.

[51] S. Zhang, *C*-algebras with real rank zero and their corona and multiplier algebras. IV*. Internat. J. Math. 3 (1992) no. 2, 309–330.

[52] S. Zhang, *Certain C*-algebras with real rank zero and their corona and multiplier algebras. I*. Pacific J. Math. 155 (1992), no. 1, 169–197.