ON LINEAR CHAOS IN FUNCTION SPACES

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Abstract. We show that, in $L^p(0, \infty)$ ($1 \leq p < \infty$), bounded weighted translations as well as their unbounded counterparts are chaotic linear operators. We also extend the unbounded case to $C_0(0, \infty)$ and describe the spectra of the weighted translations provided the underlying spaces are complex.

1. Introduction
Extending the classical Rolewicz's result [1] and the results of [2] for the sequence spaces $l_p$ ($1 \leq p < \infty$), we show that, in the space $L^p(0, \infty)$ ($1 \leq p < \infty$), the bounded weighted left translations
\[(T_{w,a}x)(t) = wx(t + a) \quad (|w| > 1, \ a > 0)\]
as well as their unbounded counterparts
\[(T_{w,a}x)(t) = w^t x(t + a)(w > 1, \ a > 0)\]
are chaotic linear operators (the latter forecasted in [2, Remark 3.1]).

The chaoticy of the bounded weighted left translations in $C_0(0, \infty)$ established in [3], we stretch the unbounded case from the sequence space $c_0$ [2] to the space $C_0[0, \infty)$ of real- or complex-valued functions continuous on $[0, \infty)$ and vanishing at infinity, which is Banach relative to the norm
\[C_0[0, \infty) \ni x \mapsto \|x\|_{\infty} := \sup_{t \geq 0} |x(t)|\]
(also forecasted in [2, Remark 3.1]) and describe the spectra of the weighted translations provided the underlying spaces are complex.

2. Preliminaries

2.1. Hypercyclicity and Chaoticity.
For a (bounded or unbounded) linear operator $T$ in a (real or complex) Banach space $X$, a nonzero vector
\[x \in C^\infty(T) := \bigcap_{n=0}^{\infty} D(T^n)\]
($D(\cdot)$ is the domain of an operator, $T^0 := I$, $I$ is the identity operator on $X$) is called hypercyclic if its orbit under $T$
\[\text{orb}(x,T) := \{T^n x\}_{n \in \mathbb{Z}_+}\]

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(\mathbb{Z}_+ := \{0, 1, 2, \ldots\}) is the set of nonnegative integers) is dense in \(X\).

Linear operators possessing hypercyclic vectors are said to be hypercyclic.

If there exist an \(N \in \mathbb{N} \) (\(\mathbb{N} := \{1, 2, \ldots\}\) is the set of natural numbers) and a vector
\[ x \in D(T^N) \quad \text{with} \quad T^N x = x, \]
such a vector is called a periodic point for the operator \(T\) of period \(N\). If \(x \neq 0\), we say that \(N\) is a period for \(T\).

Hypercyclic linear operators with a dense in \(X\) set \(\text{Per}(A)\) of periodic points are said to be chaotic.

See [4–6].

**Remarks 2.1.**

- In the definition of hypercyclicity, the underlying space is necessarily infinite-dimensional and separable (see, e.g., [9]).

- For a hypercyclic linear operator \(T\), the set \(HC(T)\) of all its hypercyclic vectors is necessarily dense in \(X\), and hence, the more so, is the subspace \(C^\infty(T) \supset HC(T)\).

- Observe that
\[ \text{Per}(A) = \bigcup_{N=1}^{\infty} \text{Per}_N(A), \]
where
\[ \text{Per}_N(A) = \ker(A^N - I), \quad N \in \mathbb{N} \]
is the subspace of \(N\)-periodic points of \(A\).

Prior to [6, 7], the notions of linear hypercyclicity and chaoticity had been studied exclusively for continuous linear operators on Fréchet spaces, in particular for bounded linear operators on Banach spaces (for a comprehensive survey, see [8, 9]).

In [1], S. Rolewicz provides the first example of hypercyclic bounded linear operators on Banach spaces (see also [9]), which on the (real or complex) sequence space \(l_p\) (\(1 \leq p < \infty\)) of \(p\)-summable sequences or \(c_0\) of vanishing sequences, the latter equipped with the supremum norm
\[ c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \|x\|_\infty := \sup_{k \in \mathbb{N}} |x_k|, \]
are the weighted backward shifts
\[ T_w(x_k)_{k \in \mathbb{N}} := w(x_{k+1})_{k \in \mathbb{N}} \]
with \(w \in \mathbb{F} \) (\(\mathbb{F} := \mathbb{R}\) or \(\mathbb{F} := \mathbb{C}\)) such that \(|w| > 1\). Furthermore, Rolewicz’s shifts are established to be chaotic [5].

In [2] (see also [10]), it is shown that the weighted backward shifts
\[ T_w x := (w^k x_{k+1})_{k \in \mathbb{N}} \]
with \( w \in F \) such that \( |w| > 1 \) and maximal domain in the (real or complex) sequence spaces \( l_p \ (1 \leq p < \infty) \) and \( c_0 \) are chaotic unbounded linear operators and, provided the underlying space is complex, each \( \lambda \in \mathbb{C} \) is a simple eigenvalue for \( T_w \).

When establishing hypercyclicity, we obviate explicit construction of hypercyclic vectors by applying the subsequent version of the classical Birkhoff Transitivity Theorem \([9, \text{Theorem 1.16}]\) or the following Sufficient Condition for Hypercyclicity \([6, \text{Theorem 2.1}]\), which is an extension of Kitai’s criterion \([11,12]\).

**Theorem 2.1** (Birkhoff Transitivity Theorem).
A bounded linear operator \( T \) on a (real or complex) infinite-dimensional separable Banach space \( X \) is hypercyclic iff it is topologically transitive, i.e., for any nonempty open subsets \( U \) and \( V \) of \( X \), there exists an \( n \in \mathbb{Z}_+ \) such that
\[
T^n(U) \cap V \neq \emptyset.
\]
Cf. \([9, \text{Theorem 2.19}]\).

**Theorem 2.2** (Sufficient Condition for Hypercyclicity).
Let \( X \) be a (real or complex) infinite-dimensional separable Banach space and \( T \) be a densely defined linear operator in \( X \) such that each power \( T^n \ (n \in \mathbb{N}) \) is a closed operator. If there exists a set
\[
Y \subseteq C^\infty(T) := \bigcap_{n=1}^{\infty} D(T^n)
\]
dense in \( X \) and a mapping \( S : Y \to Y \) such that
\begin{enumerate}
\item \( \forall x \in Y : TSx = x \) and
\item \( \forall x \in Y : T^n x, S^n x \to 0, \ n \to \infty, \)
\end{enumerate}
then the operator \( T \) is hypercyclic.

### 2.2. Resolvent Set and Spectrum.
For a linear operator \( T \) in a complex Banach space \( X \), the set
\[
\rho(A) := \{ \lambda \in \mathbb{C} \mid \exists (T - \lambda I)^{-1} \in L(X) \}
\]
(\( L(X) \) is the space of bounded linear operators on \( X \)) and its complement \( \sigma(T) := \mathbb{C} \setminus \rho(T) \) are called the resolvent set and the spectrum of \( T \), respectively.

The spectrum \( \sigma(T) \) of a closed linear operator \( T \) in a complex Banach space \( X \) is the union of the following pairwise disjoint sets:
\[
\sigma_p(T) := \{ \lambda \in \mathbb{C} \mid T - \lambda I \text{ is not injective, i.e., } \lambda \text{ is an eigenvalue of } T \},
\]
\[
\sigma_c(T) := \left\{ \lambda \in \mathbb{C} \mid T - \lambda I \text{ is injective, not surjective, and } R(T - \lambda I) = X \right\},
\]
\[
\sigma_r(T) := \left\{ \lambda \in \mathbb{C} \mid T - \lambda I \text{ is injective and } R(T - \lambda I) \neq X \right\}
\]
(\( R(\cdot) \) is the range of an operator, and \( \overline{\sigma} \) is the closure of a set), called the point, continuous and residual spectrum of \( T \), respectively (see, e.g., \([13,14]\)).
3. BOUNDED WEIGHTED TRANSLATIONS ON $L_p(0, \infty)$

**Theorem 3.1** (Bounded Weighted Translations on $L_p(0, \infty)$).

On the (real or complex) space $L_p(0, \infty)$ ($1 \leq p < \infty$), the weighted left translation

$$(T_{w,a}x)(t) := wx(t + a), \quad x \in L_p(0, \infty), \quad t \geq 0,$$

with $w \in \mathbb{F}$ such that $|w| > 1$ and $a > 0$ is a chaotic bounded linear operator.

Furthermore, provided the underlying space is complex,

$$(3.1) \quad \sigma(T_{w,a}) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \}$$

with

$$(3.2) \quad \sigma_p(T_{w,a}) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \} \quad \text{and} \quad \sigma_c(T_{w,a}) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \}.$$

**Proof.** Let $1 \leq p < \infty$, $w \in \mathbb{F}$ such that $|w| > 1$, and $a > 0$ be arbitrary and, for the simplicity of notation, let $T := T_{w,a}$.

The linearity of $T$ is obvious. Its boundedness immediately follows from the fact that

$$T = wB,$$

where

$$(Bx)(t) := x(t + a), \quad x \in L_p(0, \infty), \quad t \geq 0,$$

is a left translation operator with $\|B\| = 1$, and hence,

$$(3.3) \quad \|T\| = |w| \|B\| = |w|$$

(here and wherever appropriate, $\|\cdot\|$ also stands for the operator norm).

Suppose that

$$U, V \subseteq L_p(0, \infty)$$

are arbitrary nonempty open sets.

By the denseness in $L_p(0, \infty)$ of the equivalence classes represented by $p$-integrable on $(0, \infty)$ eventually zero functions (see, e.g., [15]), there exist equivalence classes

$$x \in U \quad \text{and} \quad y \in V$$

represented by such functions $x(\cdot)$ and $y(\cdot)$, respectively. Since the representative functions are eventually zero,

$$\exists N \in \mathbb{N} \forall t > Na : x(t) = 0 \quad \text{and} \quad y(t) = 0.$$

For an arbitrary $n \geq N$, the $p$-integrable on $(0, \infty)$ eventually zero function

$$z_n(t) := \begin{cases} x(t), & t \in [0, Na), \\ w^{-n}y(t - an), & t \in [na, Na + na), \\ 0, & \text{otherwise}, \end{cases}$$

represents an equivalence class $z_n \in L_p(0, \infty)$.

Observe that, for all $n \geq N$,

$$(T^n z_n)(t) = y(t), \quad t \geq 0,$$
and

\[ \|z_n - x\|_p = |w|^{-n}\|y\|_p \to 0, \ n \to \infty \]

Hence, for all sufficiently large \( n \in \mathbb{N} \),

\[ z_n \in U \quad \text{and} \quad T^n z_n = y \in V \]

(see Figure 1).

\[ \text{Figure 1.} \]

By the Birkhoff Transitivity Theorem (Theorem 2.1), we infer that the operator \( T \) is hypercyclic.

To prove that \( T \) has a dense set of periodic points, let us first show that each \( N \in \mathbb{N} \) is a period for \( T \).

For an arbitrary \( N \in \mathbb{N} \), let

\[ x \in \ker T^N \setminus \{0\} , \]

where

\[ \ker T^N = \{ f \in L_p(0, \infty) \mid f(t) = 0, \ t > Na \} . \]

Then the \( p \)-integrable on \((0, \infty)\) function

\[ x_N(t) := w^{-kN}x(t - kNa), \ t \in D_k := [kNa, (k + 1)Na), k \in \mathbb{Z}_+, \]

represents an \( N \)-periodic point \( x_N \) of \( T \).

Indeed, in view of \(|w| > 1\),

\[ \int_0^\infty |x_N(t)|^p \, dt = \sum_{k=0}^\infty \int_{D_k} |w^{-kN}x(t - kNa)|^p \, dt = \sum_{k=0}^\infty (|w|^{-pN})^k \int_0^{Na} |x(t)|^p \, dt \]

\[ = \sum_{k=0}^\infty (|w|^{-pN})^k \|x\|_p^p = \frac{1}{1 - |w|^{-pN}} \|x\|_p^p < \infty, \]

and hence, \( x_N \in L_p(0, \infty) \).

Further, since

\[ (T^N x_N)(t) = w^N x_N(t + Na) = w^N w^{-kN} x(t + Na - kNa) \]

\[ = w^{-k(k-1)N} y(t - (k-1)Na), \ t \in D_{k-1}, k \in \mathbb{N}, \]
we infer that 

\[ T^N x_N = x_N. \]

Suppose that \( x \in L_p(0, \infty) \) is an arbitrary equivalence class represented by a \( p \)-integrable on \((0, \infty)\) eventually zero function \( x(\cdot) \). Then 

\[ \exists M \in \mathbb{N} : x(t) = 0, \ t > Ma. \]

Let \( x_N \) be the periodic point of the operator \( T \) of an arbitrary period \( N \geq M \) defined based on \( x \) by (3.4). Then 

\[ \| x_N - x \|_p^p = \sum_{k=0}^{\infty} \int_{D_k} |x_N(t) - x(t)|^p \, dt = \sum_{k=1}^{\infty} \int_{D_k} \left| w^{-kN} x(t - kNa) \right|^p \, dt \]

\[ = \sum_{k=1}^{\infty} \left( |w|^{-pN} \right)^k \int_0^{Na} |x(t)|^p \, dt = \sum_{k=1}^{\infty} \left( |w|^{-pN} \right)^k \| x \|_p^p \]

\[ = \frac{|w|^{-pN}}{1 - |w|^{-pN}} \| x \|_p^p \to 0, \ N \to \infty. \]

By the denseness in \( L_p(0, \infty) \) \((1 \leq p < \infty)\) of the subspace 

\[ Y := \bigcup_{n=1}^{\infty} \ker T^n, \]

where 

\[ \ker T^n = \{ f \in L_p(0, \infty) \mid f(t) = 0, \ t > na \}, \ n \in \mathbb{N}, \]

of the equivalence classes represented by \( p \)-integrable on \((0, \infty)\) eventually zero functions, we infer that the set \( \text{Per}(T) \) of periodic points of \( T \) is dense in \( L_p(0, \infty) \) as well, and hence, the operator \( T \) is chaotic.

Now, assuming that the space \( L_p(0, \infty) \) is complex, let us prove (3.1) and (3.2).

In view of (3.3), by Gelfand’s Spectral Radius Theorem [14], 

\[ \sigma(T) \subseteq \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \}. \]

For an arbitrary \( \lambda \in \mathbb{C} \) with \( |\lambda| < |w| \), let 

\[ x \in \ker T \setminus \{ 0 \} \subseteq Y \setminus \{ 0 \}, \]

where 

\[ \ker T = \{ f \in L_p(0, \infty) \mid f(t) = 0, \ t > a \} \]

(see (3.6)).

Then the \( p \)-integrable on \((0, \infty)\) function 

\[ x_\lambda(t) := \left( \frac{\lambda}{w} \right)^k x(t - ka), \ t \in [ka, (k + 1)a), k \in \mathbb{Z}_+, \ (0^0 := 1) \]

is an eigenvector for \( T \) associated with \( \lambda \).

Indeed, in view of \(|\lambda| < |w|\), 

\[ 0 < \| x_\lambda \|_p^p = \int_0^{\infty} |x_\lambda(t)|^p \, dt = \sum_{k=0}^{\infty} \int_{ka}^{(k+1)a} \left( \left( \frac{\lambda}{w} \right)^k x(t - ka) \right)^p \, dt \]
\[ \begin{align*}
\sum_{k=0}^{\infty} \left( \frac{\lambda}{w} \right)^p & \left( \int_{ka}^{(k+1)a} |x(t-ka)|^p dt \right) = \sum_{k=0}^{\infty} \left( \frac{\lambda}{w} \right)^p \int_0^a |x(t)|^p dt \\
= \sum_{k=0}^{\infty} \left( \frac{\lambda}{w} \right)^p k |x|^p_p < \infty,
\end{align*} \]

and hence, \( x_{\lambda} \in L_p(0, \infty) \setminus \{0\} \).

Further,

\[ (Tx_{\lambda})(t) = wx_{\lambda}(t + a) = w \left( \frac{\lambda}{w} \right)^k x(t + a - ka) \]

\[ = \lambda \left( \frac{\lambda}{w} \right)^{k-1} x(t - (k-1)a), \ t \in [(k-1)a, ka), k \in \mathbb{N}, \]

which implies that (3.9)

\[ Tx_{\lambda} = \lambda x_{\lambda}, \]

and hence, \( \lambda \in \sigma_p(T) \).

Conversely, let \( \lambda \in \sigma_p(T) \) be an arbitrary eigenvalue for \( T \) with an associated eigenvector \( x_{\lambda} \in L_p(0, \infty) \setminus \{0\} \). Then, for

\[ x_k(t) := x_{\lambda}(t), \ t \in [ka, (k+1)a), \ k \in \mathbb{N}, \]

by (3.9), we have:

\[ \lambda x_{k-1}(t) = wx_k(t + a), \ t \in [ka, (k+1)a) \pmod{\lambda_1}, \ k \in \mathbb{N}, \]

\((\lambda_1 \text{ is the Lebesgue measure on } \mathbb{R})\).

Whence,

\[ x_k(t) = \left( \frac{\lambda}{w} \right)^k x_{\lambda}(t - ka), \ t \in [ka, (k+1)a) \pmod{\lambda_1}, \ k \in \mathbb{N}, \]

which, in view of \( x_{\lambda} \neq 0 \), implies that

\[ 0 < \int_0^a |x_{\lambda}(t)|^p dt \leq \int_0^{\infty} |x_{\lambda}(t)|^p dt = \|x_{\lambda}\|_p^p < \infty \]

and

\[ \infty > \|x_{\lambda}\|_p^p = \int_0^\infty |x_{\lambda}(t)|^p dt = \sum_{k=0}^{\infty} \int_{ka}^{(k+1)a} \left( \frac{\lambda}{w} \right)^k x_{\lambda}(t - ka) |x_{\lambda}(t)|^p dt \]

\[ = \sum_{k=0}^{\infty} \left( \frac{\lambda}{w} \right)^k \int_{ka}^{(k+1)a} |x_{\lambda}(t - ka)|^p dt = \sum_{k=0}^{\infty} \left( \frac{\lambda}{w} \right)^k \int_0^a |x_{\lambda}(t)|^p dt. \]

The convergence of the latter series implies that

\[ \left( \frac{\lambda}{w} \right)^k \to 0, \ k \to \infty, \]

which, in its turn, means that

\[ |\lambda| < |w|. \]
Thus, \( x_\lambda \) can be represented by a \( p \)-integrable on \((0, \infty)\) function \( x_\lambda(\cdot) \) of the form given by (3.8), where the corresponding \( x \in \ker T \setminus \{0\} \) is represented by
\[
x(t) := \chi_{[0,a]}(t)x_\lambda(t), \quad t \geq 0
\]
(\( \chi_\delta(\cdot) \) is the characteristic function of a set \( \delta \)).

The above proves that
\[
\sigma_p(T) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \}.
\]

Considering that \( \sigma(T) \) is a closed set in \( \mathbb{C} \) (see, e.g., [13, 14]), we infer from (3.7) and (3.10) that (3.1) holds.

Since, by [9, Lemma 2.53], the hypercyclicity of \( T \) implies the operator \( T - \lambda I \) has a dense range for all \( \lambda \in \mathbb{C} \), we infer that
\[
\sigma_r(T) = \emptyset
\]
(cf. [16, Proposition 4.1], [17, Lemma 1]), and hence, in view of (3.1) and (3.10), we conclude that
\[
\sigma_c(T) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \}.
\]

Thus, (3.2) holds as well.

\[\square\]

4. Unbounded Weighted Translations in \( L_p(0, \infty) \)

**Lemma 4.1** (Closedness of Powers).

In the (real or complex) space \( L_p(0, \infty) \) \((1 \leq p < \infty)\), for the weighted left translation
\[
(T_{w,a}x)(t) := w^t x(t + a), \quad t \geq 0,
\]
with \( w > 1, a > 0, \) and domain
\[D(T_{w,a}) := \left\{ x \in L_p(0, \infty) \middle| \int_0^\infty |w^t x(t + a)|^p dt < \infty \right\},\]
each power \( T_{w,a}^n \) \((n \in \mathbb{N})\) is a densely defined unbounded closed linear operator.

**Proof.** Let \( 1 \leq p < \infty, w > 1, a > 0, \) and \( n \in \mathbb{N} \) be arbitrary and, for the simplicity of notation, let \( T := T_{w,a}. \)

The linearity of \( T \) is obvious and implies that for \( T^n. \)

Inductively,
\[
(T^n x)(t) = w^t w^{t+a} \cdots w^{t+(n-1)a} x(t + na) = w^{nt + \frac{(n-1)na}{2}} x(t + na), \quad t \geq 0,
\]
and
\[
D(T^n) = \left\{ x \in L_p(0, \infty) \middle| \int_0^\infty |w^{nt + \frac{(n-1)na}{2}} x(t + na)|^p dt < \infty \right\}.
\]

By the denseness in \( L_p(0, \infty) \) \((1 \leq p < \infty)\) of the subspace
\[
Y := \bigcup_{m=1}^\infty \ker T^m,
\]
ON LINEAR CHAOS IN FUNCTION SPACES

(4.4) \[ \ker T^m = \{ f \in L_p(0, \infty) \mid f(t) = 0, \; t > ma \}, \; m \in \mathbb{N}, \]

of the equivalence classes represented by \( p \)-integrable on \((0, \infty)\) eventually zero functions and the inclusion

(4.5) \[ Y \subset C^\infty(T) := \bigcap_{m=1}^{\infty} D(T^m), \]

which follows from (4.2), we infer that the operator \( T^m \) is densely defined.

The unboundedness of \( T^n \) follows from the fact that, for the equivalence classes \( e_m \in L_p(0, \infty), \; m \in \mathbb{N}, \)

we have:

\[ e_m(t) := \chi_{[m, m+1]}(t), \; m \in \mathbb{N}, \; t \geq 0, \]

we have:

\[ e_m \in D(T^n). \; \|e_m\|_p = 1, \; m \in \mathbb{N}, \]

and, for all \( m \in \mathbb{N} \) sufficiently large so that \( m \geq na \), in view of \( w > 1 \),

\[ \|T^n e_m\|_p = \left[ \int_0^{\infty} \left| w^{nt + \frac{(n-1)na}{2}} e_m(t + na) \right|^p dt \right]^{1/p} \]

\[ \geq u^{n(m-\frac{na}{2}) + \frac{(n-1)na}{2}} \rightarrow \infty, \; m \rightarrow \infty. \]

Let a sequence \((x_m)_{m \in \mathbb{N}}\) in \( L_p(0, \infty) \) be such that

\[ D(T^n) \ni x_m \rightarrow x \in L_p(0, \infty), \; m \rightarrow \infty, \]

and

\[ T^n x_m \rightarrow y \in L_p(0, \infty), \; m \rightarrow \infty. \]

The sequences \((x_m(\cdot))_{m \in \mathbb{N}}\) and \((T^n x_m)(\cdot))_{m \in \mathbb{N}}\) of the \( p \)-integrable on \((0, \infty)\) representatives of the corresponding equivalence classes converging in \( p \)-norm on \((0, \infty)\), also converge in the Lebesgue measure \( \lambda_1 \) on \((0, \infty)\), and hence, by the Riesz theorem (see, e.g., [15]), there exist subsequences \((x_{m(k)}(\cdot))_{k \in \mathbb{N}}\) and \((T^n x_{m(k)})(\cdot))_{k \in \mathbb{N}}\) convergent a.e. on \((0, \infty)\) relative to \( \lambda_1 \), i.e.,

(4.6) \[ x_{m(k)}(t) \rightarrow x(t) \text{ on } (0, \infty) \; \text{ (mod } \lambda_1) \]

and

(4.7) \[ (T^n x_{m(k)})(t) \rightarrow y(t) \text{ on } (0, \infty) \; \text{ (mod } \lambda_1). \]

By (4.6),

\[ (T^n x_{m(k)})(t) = w^{nt + \frac{(n-1)na}{2}} x_{m(k)}(t + na) \]

\[ \rightarrow w^{nt + \frac{(n-1)na}{2}} x(t + na) \text{ on } (0, \infty) \; \text{ (mod } \lambda_1), \]

which by (4.7), in view of the completeness of the Lebesgue measure (see, e.g., [15]), implies that

\[ w^{nt + \frac{(n-1)na}{2}} x(t + na) = y(t) \; \text{ (mod } \lambda_1), \]
and hence,
\[ x \in D(T^n) \quad \text{and} \quad T^n x = y. \]

By the **Sequential Characterization of Closed Linear Operators** (see, e.g., [14]) the operator \( T^n \) is closed. \( \square \)

**Theorem 4.1** (Unbounded Weighted Translations in \( L_p(0, \infty) \)).

In the (real or complex) space \( L_p(0, \infty) \) (\( 1 \leq p < \infty \)), the weighted left translation
\[ (T_{w,a}x)(t) := w^t x(t + a), \quad t \geq 0, \]
with \( w > 1, \ a > 0 \), and domain
\[ D(T_{w,a}) := \left\{ x \in L_p(0, \infty) \left| \int_0^{\infty} |w^t x(t + a)|^p \, dt < \infty \right. \right\} \]
is a chaotic unbounded linear operator.

Furthermore, provided the underlying space is complex,
\[ \sigma(T_{w,a}) = \sigma_p(T_{w,a}) = \mathbb{C}. \]

**Proof.** Let \( 1 \leq p < \infty, \ w > 1, \) and \( a > 0 \) be arbitrary and, for the simplicity of notation, let \( T := T_{w,a}. \)

For the dense in \( L_p(0, \infty) \) subspace \( Y \) of the equivalence classes represented by \( p \)-integrable eventually zero functions (see (4.3) and (4.4)), we have inclusion (4.5).

The mapping
\[ Y \ni x \mapsto Sx \in Y, \]
where the equivalence class \( Sx \) is represented by
\[ (Sx)(t) := \begin{cases} w^{-(t-a)}x(t-a), & t > a, \\ 0, & \text{otherwise}, \end{cases} \]
is well defined since the function \( (Sx)(\cdot) \) is eventually zero and, in view of \( w > 1, \)
\[ \int_0^{\infty} |(Sx)(t)|^p \, dt = \int_a^{\infty} \left| w^{-(t-a)p} x(t-a) \right|^p \, dt = \int_0^{\infty} w^{-tp} |x(t)|^p \, dt \]
\[ \leq \int_0^{\infty} |x(t)|^p \, dt < \infty. \]

As is easily seen,
\[ \forall x \in Y : \ T S x = x. \]

Let \( x \in Y \), represented by a \( p \)-integrable on \((0, \infty)\) eventually zero function \( x(\cdot) \), be arbitrary. Then
\[ \exists M \in \mathbb{N} : \supp x := \{ t \in (0, \infty) \mid x(t) \neq 0 \} \subseteq [0, Ma]. \]

By (4.1),
\[ \forall n \geq M : \ T^n x = 0, \]
and hence,
\[ T^n x \to 0, \ n \to \infty. \]
Based on (4.9), inductively,

\[
(S^n x)(t) = \begin{cases} 
0, & 0 \leq t < na, \\
w^{t-2a}w^{t-3a} \ldots w^{t-na}x(t-na), & t \geq na, 
\end{cases} 
\]

\[
= \begin{cases} 
0, & 0 \leq t < na, \\
w^{-nt+\frac{n(n+1)\alpha}{2}}x(t-na), & t \geq na, 
\end{cases} 
\]

In view of \(w > 1\), we have:

\[
\|S^n x\|_p = \left[ \int_0^\infty |(S^n x)(t)|^p \, dt \right]^{1/p} = \left[ \int_{na}^\infty \left| w^{-nt+\frac{n(n+1)\alpha}{2}}x(t-na) \right|^p \, dt \right]^{1/p} 
\]

\[
\leq w^{-n\alpha + \frac{n(n+1)\alpha}{2}} \left[ \int_{na}^\infty |x(t-na)|^p \, dt \right]^{1/p} = w^{-\frac{n(n-1)\alpha}{2}}\|x\|_p, \quad x \in Y, n \in \mathbb{N}. 
\]

Whence, since \(w > 1\) and \(a > 0\), we deduce that

\[
\forall x \in Y: \lim_{n \to \infty} \|S^n x\|_p^{1/n} = 0, 
\]

or equivalently,

\[
(4.12) \quad \forall x \in Y, \forall \alpha \in (0, 1) \exists c = c(x, \alpha) > 0 \forall n \in \mathbb{N}: \|S^n x\|_p \leq c\alpha^n\|x\|_p, 
\]

which implies

\[
\forall x \in Y: S^n x \to 0, \quad n \to \infty. 
\]

From the above and the fact that, by the **Closedness of Powers Lemma** (Lemma 4.1), each power \(T^n (n \in \mathbb{N})\) is a closed operator, by the **Sufficient Condition for Hypercyclicity** (Theorem 2.2), we infer that the operator \(T\) is hypercyclic.

To prove that \(T\) has a dense set of periodic points, let us first show that each \(N \in \mathbb{N}\) is a period for \(T\).

Let \(N \in \mathbb{N}\) and

\[
(4.13) \quad x \in \ker T^N \setminus \{0\} \subseteq Y \setminus \{0\}, 
\]

where

\[
\ker T^N = \{f \in L_p(0, \infty) \mid f(t) = 0, \quad t > Na\}, 
\]

be arbitrary.

By estimate (4.12)

\[
(4.14) \quad x_N := \sum_{k=0}^{\infty} S^{kN} x \in L_p(0, \infty) 
\]

is well defined and, in view of (4.11), is represented by the \(p\)-integrable on \((0, \infty)\) function

\[
x_N(t) := w^{-kNt+\frac{kN(kN+1)\alpha}{2}}x(t-kNa), \quad t \in D_k := [kNa, (k+1)Na), \quad k \in \mathbb{Z}_+. 
\]
Since, in view of (4.13) and (4.10),
\[ \sum_{k=0}^{\infty} T^N S^k x = \sum_{k=1}^{\infty} S^{(k-1)} x = x_N, \]
by the closedness of the operator $T^N$, we infer that
\[ x_N \in D(T^N) \quad \text{and} \quad T^N x_N = x_N, \]
(see, e.g., [14]), and hence, $x_N$ is an $N$-periodic point for $T$.
Suppose that $x \in Y$ is an arbitrary equivalence class represented by a $p$-integrable on $(0, \infty)$ eventually zero function $x(\cdot)$. Then
\[ \exists M \in \mathbb{N} : x(t) = 0, \quad t > Ma. \]

Then, for an arbitrary period $N \geq M$, (4.13) holds and there exists an $N$-periodic point $x_N$ for the operator $T$ defined based on $x$ by (4.14). By estimate (4.12),
\[ \|x_N - x\| = c \sum_{k=1}^{\infty} \|S^k x\|_p \leq c \sum_{k=1}^{\infty} (\alpha^N)^k \|x\|_p \]
\[ = c \frac{\alpha^N}{1 - \alpha^N} \|x\|_p \to 0, \quad N \to \infty. \]

Whence, in view of the denseness of $Y$ in $L_p(0, \infty)$, we infer that the set $\text{Per}(T)$ of periodic points of $T$ is dense in $L_p(0, \infty)$ as well, and hence, the operator $T$ is chaotic.

Now, assuming that the space $L_p(0, \infty)$ is complex, let us prove (4.8).
Let $\lambda \in \mathbb{C}$ and
\[ x \in \ker T \setminus \{0\} \subseteq Y \setminus \{0\}, \]
where
\[ \ker T = \{ f \in L_p(0, \infty) \mid f(t) = 0, \quad t > a \}. \]
By estimate (4.12), for
\[ \alpha := (|\lambda| + 1)^{-1} \in (0, 1), \]
we have:
\[ \exists c = c(x, \alpha) > 0 \forall k \in \mathbb{N} : \|\lambda^k S^k x\|_p \leq |\lambda|^k c \alpha^k \|x\|_p = c(|\lambda| \alpha)^k \|x\|_p, \]
where $0 \leq |\lambda|^k \alpha^k = |\lambda|(|\lambda| + 1)^{-k} < 1$.
By estimate (4.16),
\[ x_\lambda := \sum_{k=0}^{\infty} \lambda^k S^k x \in L_p(0, \infty) \]
is well defined and, in view of (4.11), is represented by the $p$-integrable on $(0, \infty)$ function
\[ x_\lambda(t) := \lambda^k w^{kt} \frac{\lambda^{(k+1)a}}{\lambda^{a}} x(t - ka), \quad t \in [ka, (k + 1)a), \quad k \in \mathbb{Z}_+, \quad (0^0 := 1). \]
Since
\[ \|x_\lambda\|^p = \int_0^\infty |x_\lambda(t)|^p \, dt \geq \int_0^a |x_\lambda(t)|^p \, dt = \int_0^a |x(t)|^p \, dt = \int_0^\infty |x(t)|^p \, dt = \|x\|^p > 0, \]
we infer that \( x_\lambda \neq 0 \).

Further, since, in view of (4.15) and (4.10),
\[ \sum_{k=0}^\infty T(\lambda^k S^k x) = \lambda \sum_{k=1}^\infty \lambda^{k-1} S^{k-1} = \lambda x_\lambda, \]
by the *closedness* of the operator \( T \), we conclude that
\[ x_\lambda \in D(T) \quad \text{and} \quad Tx_\lambda = \lambda x_\lambda, \]
(see, e.g., [14]).

Thus, \( \lambda \in \sigma_p(T) \) and \( x_\lambda \) is an eigenvector of \( T \) associated with \( \lambda \), which proves (4.8).

\[ \square \]

5. **Bounded Weighted Translations on \( C_0[0, \infty) \)**

In [3, Theorem 2.3], it is shown that, on the (real or complex) space \( C_0[0, \infty) \), the bounded linear weighted left translation operator
\[ (T_{w,a})x(t) := wx(t + a), \quad t \geq 0, \]
with \( |w| > 1 \) and \( a > 0 \) is chaotic and
\[ \{ \lambda \in \mathbb{C} : 0 < |\lambda| < |w| \} \subseteq \sigma_p(T) \]
based on the simple fact that, for each \( \lambda \in \mathbb{C} \), \( \text{Re} \, \lambda < 0 \), the equation
\[ T_{w,a}x = we^{a\lambda}x \]
is satisfied by the function
\[ x(t) := e^{\lambda t}, \quad t \geq 0. \]

It is also stated (without proof) that one can show that
\[ \sigma_p(T_{w,a}) = \{ \lambda \in \mathbb{C} : |\lambda| < |w| \}. \]

Here, we completely describe the spectrum of such operators.

**Proposition 5.1** (Spectrum).

On the complex space \( C_0[0, \infty) \), for the bounded linear weighted left translation operator
\[ (Tx)(t) := wx(t + a), \quad t \geq 0, \]
where \( w \in \mathbb{C} \) with \( |w| > 1 \) and \( a > 0 \),
\[ \sigma(T) = \{ \lambda \in \mathbb{C} : |\lambda| \leq |w| \} \]
with
\[ \sigma_p(T_{w,a}) = \{ \lambda \in \mathbb{C} : |\lambda| < |w| \} \quad \text{and} \quad \sigma_c(T_{w,a}) = \{ \lambda \in \mathbb{C} : |\lambda| = |w| \}. \]
Proof. Let \( w \in \mathbb{C} \) with \( w > 1 \) and \( a > 0 \) be arbitrary and, for the simplicity of notation, let \( T := T_{w,a} \).

Since \( T = wB \), where 
\[
(Bx)(t) := x(t + a), \quad x \in C_0[0, \infty), \quad t \geq 0,
\]
is a left translation with \( \|B\| = 1 \), and hence,
\[
\|T\| = \|w\| \|B\| = \|w\|,
\]
by Gelfand’s Spectral Radius Theorem \( [14] \),
\[
\sigma(T) \subseteq \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \}.
\]

Let \( \lambda \in \mathbb{C} \) with \( |\lambda| < |w| \) and a nonzero \( x \in C[0, a] \), with
\[
x(a) = \frac{\lambda}{w} x(0)
\]
be arbitrary. E.g., for \( 0 < |\lambda| < |w| \),
\[
y(t) := e^{ct}, \quad t \in [0, a],
\]
with \( c := \frac{1}{a} \ln \frac{\lambda}{w} = \frac{1}{a} \left( \ln \left| \frac{\lambda}{w} \right| + i \text{Im} \frac{\lambda}{w} \right) \) (\( i \) is the imaginary unit).

Then, as is readily verified,
\[
x_\lambda(t) := \left( \frac{\lambda}{w} \right)^k x(t - ka), \quad t \in [ka, (k + 1)a), k \in \mathbb{Z}_+ , \quad (0^0 := 1)
\]
is a nonzero function continuous on \([0, \infty)\).

Since, in view of \( |\lambda/w| < 1 \), for any \( k \in \mathbb{Z}_+ \),
\[
\max_{ka \leq t \leq (k+1)a} |x_\lambda(t)| = \max_{ka \leq t \leq (k+1)a} \left| \left( \frac{\lambda}{w} \right)^k x(t - ka) \right| = \left| \frac{\lambda}{w} \right|^k \max_{0 \leq t \leq a} |x(t)| \to 0, \quad k \to \infty,
\]
we infer that \( x_\lambda \in C_0[0, \infty) \setminus \{0\} \).

Also,
\[
(T x_\lambda)(t) = wx(t + a) = w \left( \frac{\lambda}{w} \right)^k x(t + a - ka)
\]
\[
= \lambda \left( \frac{\lambda}{w} \right)^{k-1} x(t - (k - 1)a), \quad t \in [(k - 1)a, ka), k \in \mathbb{N},
\]
which implies that
\[
(T x_\lambda) = \lambda x_\lambda.
\]

Thus, \( \lambda \in \sigma_p(T) \).

Conversely, let \( \lambda \in \sigma_p(T) \) be an arbitrary eigenvalue for \( T \) with an associated eigenvector \( x_\lambda \in C_0[0, \infty) \setminus \{0\} \). Then, for
\[
x_k(t) := x_\lambda(t), \quad t \in [ka, (k + 1)a), k \in \mathbb{Z}_+ ,
\]
by (5.5), we have:
\[
\lambda x_{k-1}(t) = wx_k(t + a), \quad t \in [ka, (k + 1)a), k \in \mathbb{N}.
\]
Whence,
\[ x_k(t) = \left( \frac{\lambda}{w} \right)^k x_\lambda(t - ka), \quad t \in [ka, (k+1)a), \quad k \in \mathbb{Z}_+, \]
which, in view of \( x_\lambda \neq 0 \), implies that
\[ 0 < \max_{0 \leq t \leq a} |x_\lambda(t)| \leq \sup_{t \geq 0} |x_\lambda(t)| < \infty. \]
Since
\[ \lim_{t \to \infty} x_\lambda(t) = 0 \]
implies
\[ \left| \frac{\lambda}{w} \right|^k \max_{0 \leq t \leq a} |x_\lambda(t)| \to 0, \quad k \to \infty, \]
which, in its turn, means that
\[ |\lambda| < |w|. \]
Thus, \( x_\lambda \) is of the form given by (5.4), where \( x \) is the restriction to \([0, a]\) of \( x_\lambda \).

The above proves that
\[ (5.6) \quad \sigma_p(T) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \}. \]

Considering that \( \sigma(T) \) is a closed set in \( \mathbb{C} \) (see, e.g., [13, 14]), we infer from (5.3) and (5.6) that (5.1) holds.

Since, by [9, Lemma 2.53], the hypercyclicity of \( T \) implies the operator \( T - \lambda I \) has a dense range for all \( \lambda \in \mathbb{C} \), we infer that
\[ \sigma_c(T) = \emptyset \]
(cf. [16, Proposition 4.1], [17, Lemma 1]), and hence, in view of (5.1) and (5.6), we conclude that
\[ \sigma_r(T) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \}. \]
Thus, (5.2) holds as well. \( \square \)

6. Unbounded Weighted Translations in \( C_0[0, \infty) \)

**Lemma 6.1** (Closedness of Powers).
In the (real or complex) space \( (C_0[0, \infty), \| \cdot \|_\infty) \), for the weighted left translation
\[ (T_{w,a}x)(t) := w^t x(t + a), \quad t \geq 0, \]
with \( w > 1, a > 0 \), and domain
\[ D(T_{w,a}) := \left\{ x \in C_0[0, \infty) \mid \lim_{t \to \infty} w^t x(t + a) = 0 \right\}, \]
each power \( T_{w,a}^n \) \( (n \in \mathbb{N}) \) is a densely defined unbounded closed linear operator.

**Proof.** Let \( w > 1, a > 0 \), and \( n \in \mathbb{N} \) be arbitrary and, for the simplicity of notation, let \( T := T_{w,a} \).

The linearity of \( T \) is obvious and implies that for \( T^n \).
Inductively,
\[(6.1) \quad (T^n x)(t) = w^t w^{t+a} \ldots w^{t+(n-1)a} x(t + na) = w^{nt + \frac{(n-1)na}{2}} x(t + na), \quad t \geq 0,
\]
and
\[(6.2) \quad D(T^n) = \left\{ x \in C_0[0, \infty) \left| \lim_{t \to \infty} u^{nt + \frac{(n-1)na}{2}} x(t + na) = 0 \right. \right\}
\]
(cf. (4.1) and (4.2)).

By the denseness in $C_0[0, \infty)$ $(1 \leq p < \infty)$ of the subspace
\[(6.3) \quad Y := \bigcup_{m=1}^{\infty} \ker T^m,
\]
where
\[(6.4) \quad \ker T^m = \{ f \in C_0[0, \infty) \mid f(t) = 0, \quad t \geq ma \}, \quad m \in \mathbb{N},
\]
of the equivalence classes represented by $p$-integrable on $(0, \infty)$ eventually zero functions and the inclusion
\[(6.5) \quad Y \subset C^\infty(T) := \bigcap_{m=1}^{\infty} D(T^m),
\]
which follows from (6.2), we infer that the operator $T^n$ is densely defined.

The unboundedness of $T^n$ follows from the fact that, for
\[e_m(t) := \begin{cases} 
1, & 0 \leq t < ma, \\
\frac{1}{w^{-t-ma}2}, & t \geq ma,
\end{cases} \quad m \in \mathbb{N},
\]
we have:
\[e_n \in D(T), \quad ||e_n||_\infty = 1, \quad m \in \mathbb{N},
\]
and, for all $m \geq n$, in view of $w > 1$,
\[||T^n e_m||_\infty = \sup_{t \geq 0} \left| w^{nt + \frac{(n-1)na}{2}} e_m(t + na) \right| \geq w^{nt + \frac{(n-1)na}{2}} e_m(t + na) \big|_{t=ma-na} \geq w^{n(ma-na) + \frac{(n-1)na}{2}} \to \infty, \quad m \to \infty.
\]

Let a sequence $(x_m)_{m \in \mathbb{N}}$ in $C_0[0, \infty)$ be such that
\[D(T^n) \ni x_m \to x \in C_0[0, \infty), \quad m \to \infty,
\]
and
\[T^n x_m \to y \in C_0[0, \infty), \quad m \to \infty.
\]

Then, for each $t \geq 0$,
\[(6.6) \quad x_m(t) \to x(t) \quad \text{and} \quad (T^n x_m)(t) \to y(t), \quad m \to \infty.
\]

By (6.6), for each $t \geq 0$,
\[\quad (T^n x_m)(t) = w^{nt + \frac{(n-1)na}{2}} x_m(t + ma) \to w^{nt + \frac{(n-1)na}{2}} x(t + na), \quad m \to \infty,
\]
and
\[w^{nt + \frac{(n-1)na}{2}} x(t + na) = y(t), \quad t \geq 0,
\]
which implies
\[ x \in D(T^n) \quad \text{and} \quad T^n x = y. \]

Thus, by the Sequential Characterization of Closed Linear Operators (see, e.g., [14])
the operator \( T^n \) is closed. \( \square \)

**Theorem 6.1** (Unbounded Weighted Translations in \( C_0(0, \infty) \)).

In the (real or complex) space \( (C_0(0, \infty), \| \cdot \|_\infty) \), the weighted left translation
\[
(T_{w,a}x)(t) := w^t x(t + a), \quad t \geq 0,
\]
with \( w > 1 \), \( a > 0 \), and domain
\[
D(T_{w,a}) := \{ x \in C_0[0, \infty) \mid \lim_{t \to \infty} w^t x(t + a) = 0 \}
\]
is a chaotic unbounded linear operator.

Furthermore, provided the underlying space is complex,
\[
(6.7) \quad \sigma(T_{w,a}) = \sigma_p(T_{w,a}) = \mathbb{C}.
\]

**Proof.** Let \( w > 1 \) and \( a > 0 \) be arbitrary and, for the simplicity of notation, let \( T := T_{w,a} \).

For the dense in \( C_0[0, \infty) \) subspace \( Y \) of eventually zero functions (see (6.3) and (6.4)), we have inclusion (6.5).

The mapping
\[
Y \ni x \mapsto Sx \in Y,
\]
where
\[
(6.8) \quad (Sx)(t) := \begin{cases} \frac{x(0)}{a} t, & 0 \leq t < a, \\ w^{-(t-a)} x(t - a), & t \geq a, \end{cases}
\]
is well defined since the function \( Sx \) is eventually zero and, as is easily seen,
\[
(6.9) \quad \forall x \in Y : \ T S x = x.
\]

Let \( x \in Y \) be arbitrary. Then
\[
\exists M \in \mathbb{N} : \ \text{supp } x := \{ t \in [0, \infty) \mid x(t) \neq 0 \} \subseteq [0, Ma].
\]

By (6.1),
\[
\forall n \geq M : \ T^n x = 0,
\]
and hence,
\[
T^n x \to 0, \ n \to \infty.
\]
Based on (6.8), inductively,

\[
(S^n x)(t)
= \begin{cases}
  0, & 0 \leq t < (n-1)a, \\
  w^{-(t-a)} \cdots w^{-(t-(n-1)a)} x^{(0)}(t-(n-1)a), & (n-1)a \leq t < na, \\
  0, & 0 \leq t < (n-1)a, \\
  w^{-t+\frac{n(n+1)a}{2}} x(t-na), & t \geq na,
\end{cases}
\]

In view of \(w > 1\),

\[
\|S^n x\|_\infty = \sup_{t \geq 0} |(S^n x)(t)| \leq w^{-n-na+\frac{n(n+1)a}{2}} \|x\|_\infty = w^{-\frac{n(n-1)a}{2}} \|x\|_\infty, \quad x \in Y, n \in \mathbb{N}.
\]

Whence, since \(w > 1\) and \(a > 0\), we deduce that

\[
\forall x \in Y : \lim_{n \to \infty} \|S^n x\|_\infty^{1/n} = 0,
\]

or equivalently,

\[
(6.11) \quad \forall x \in Y, \forall \alpha \in (0, 1) \exists c = c(x, \alpha) > 0 \forall n \in \mathbb{N} : \|S^n x\|_\infty \leq c\alpha^n \|x\|_\infty,
\]

which implies

\[
\forall x \in Y : S^n x \to 0, \quad n \to \infty.
\]

From the above and the fact that, by the \textit{Closedness of Powers Lemma} (Lemma 6.1), each power \(T^n \ (n \in \mathbb{N})\) is a \textit{closed operator}, by the \textit{Sufficient Condition for Hypercyclicity} (Theorem 2.2), we infer that the operator \(T\) is \textit{hypercyclic}.

Based on estimate (6.11), proving that \(T\) has a dense set of periodic points, and hence, is \textit{chaotic} and that (6.7) holds is identical to proving the same parts in Theorem 4.1. 

\(\Box\)

\section{Concluding Remarks}

The foregoing results are consistent with the recent findings of [16]. According to the latter, under the premises of Theorem 3.1, Theorem 4.1, [3, Theorem 2.3], or Theorem 6.1, not only is the operator \(T_{w,a}\) \textit{chaotic} but also its every power \(T^n_{w,a} \ (n \in \mathbb{N})\) and, furthermore,

\[
\dim \ker (T^n_{w,a} - \lambda I) = \dim \ker T^n_{w,a} = \dim \{f \in X \mid f(t) = 0, \ t > na\},
\]

where \(X := L_p(0, \infty) \ (1 \leq p < \infty)\) or \(X := C_0[0, \infty)\), holds in Theorem 3.1 and Proposition 5.1 for all \(n \in \mathbb{N}\) and \(\lambda \in \mathbb{C}\) with \(|\lambda| < |w|^n\) and in Theorems 4.1 and 6.1 for all \(n \in \mathbb{N}\) and \(\lambda \in \mathbb{C}\), i.e., all eigenvalues of \(T^n_{w,a}\) are of the same geometric multiplicity.
ON LINEAR CHAOS IN FUNCTION SPACES

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