A new derivation of symmetry energy from nuclei beyond the 
\( \beta \)-stability line

V.M. Kolomietz and A.I. Sanzhur

Institute for Nuclear Research, 03680 Kiev, Ukraine

Abstract

We suggest the procedure of direct derivation of the symmetry energy from the shift of neutron-proton chemical potentials \( \Delta \lambda = \lambda_n - \lambda_p \) for nuclei beyond the beta-stability line. We observe the presence of anomalous strong (about 15\%) shell oscillations at the symmetry energy coefficient \( b_{\text{sym}} \). Our results do not confirm the existence of exceptionally large values of the symmetry energy coefficient at mass number \( A \approx 100 \) which was earlier reported in Ref.\[2\]. Using the fitting procedure, we have evaluated the volume, \( b_{\text{sym,vol}} \), and surface, \( b_{\text{sym,surf}} \), contributions to the symmetry energy. We have estimated the experimental value of surface-to-volume ratio as \( r_{S/V} = |b_{\text{sym,surf}}|/b_{\text{sym,vol}} \approx 1.7 \) for the fitting interval \( A \geq 50 \).

PACS numbers: 21.10.Dr, 21.60.-n, 21.60.Ev
1. The nuclear $\beta$-stability line is derived by the balance of both the isotopic symmetry, $E_{\text{sym}}$, and the Coulomb, $E_C$, energies. However the extraction of $E_{\text{sym}}$ and $E_C$ from the nuclear binding energy is not a simple problem because of its complicated dependency on the mass number $A$ in finite nuclei [1]. The standard procedure of extraction of the symmetry energy from a fit of mass formula to the experimental binding energies [2] is not free from ambiguities and does not allow one to separate the symmetry energy into the volume, surface and curvature contributions directly.

Moreover the symmetry energy $E_{\text{sym}}$ is usually derived on the $\beta$-stability line and some special efforts have to be applied to extend it beyond the ground state of nuclei [3]. On the other hand, new information about nuclear masses in a wide region of the stability valley can be used for straightforward derivation of the $A$-dependence of energies $E_{\text{sym}}$ and $E_C$.

In the present work, we suggest a non-standard procedure of extraction of the symmetry and Coulomb energies from the experimental data using the dependence of the isospin shift of neutron-proton chemical potentials $\Delta \lambda(X) = \lambda_n - \lambda_p$ on the asymmetry parameter $X = (N - Z)/(N + Z)$ for nuclei beyond the beta-stability line. This procedure allows one to represent the results for the $A$-dependence of energies $E_{\text{sym}}$ and $E_C$ in a transparent way, which can be easily used for the extraction of the smooth volume and surface contributions as well as the shell structure.

2. Considering the asymmetric nuclei with a small asymmetry parameter $X = (N - Z)/A \ll 1$ and assuming the leptodermous property, the total energy per nucleon $E/A$ can be represented in the following form of $A, X$-expansion

\[ E/A \equiv e_A = e_0(A) + b_{\text{sym}}(A) X^2 + E_C(X)/A, \tag{1} \]

where $e_0(A)$ includes both the bulk and the surface energies, $b_{\text{sym}}(A)$ is the symmetry energy, $E_C(X)$ is the total Coulomb energy

\[ E_C(X) = \frac{3}{20} \frac{A e^2}{R_C} (1 - X)^2 \tag{2} \]

and $R_C$ is the Coulomb radius of the nucleus.

The beta-stability line $X = X^*(A)$ can be directly derived from Eq. (1) using the condition

\[ \left. \frac{\partial E/A}{\partial X} \right|_A = 0 \quad \Rightarrow \quad X^*(A) = \frac{e_c^*(A)}{b_{\text{sym}}(A) + e_c^*(A)}, \tag{3} \]
where
\[ e_C(A) = 0.15Ae^2/R_C \]
Along the beta-stability line, the binding energy per particle is then given by
\[ E^*/A = e^*_0(A) + b^*_{\text{sym}}(A) X^{*2} + E_C(X^*)/A, \] (4)
where the upper index "*" indicates that the corresponding quantity is determined by the variational conditions \(^3\) taken for fixed \(A\) and \(X = X^*\) on the beta-stability line. For any given value of \(A\), the binding energy can be extended beyond the beta-stability line as
\[ E/A = E^*/A + b^*_{\text{sym}}(A)(X - X^*)^2 + \Delta E_C(X)/A, \] (5)
where \(\Delta E_C(X) = E_C(X) - E_C(X^*)\). The symmetry energy \(b^*_{\text{sym}}(A)\) contains the \(A\)-independent bulk term, \(b^*_{\text{sym, vol}}\), and the \(A\)-dependent surface contribution, \(b^*_{\text{sym, surf}} A^{-1/3}\),
\[ b^*_{\text{sym}}(A) = b^*_{\text{sym, vol}} + b^*_{\text{sym, surf}} A^{-1/3}. \] (6)
In general, the surface symmetry energy \(b^*_{\text{sym, surf}} A^{-1/3}\) includes also the high order curvature correction \(\sim A^{-2/3} \) \(^4\).

Using Eq. (5), one can establish an important relation for the chemical potential \(\lambda_q\) (\(q = n\) for a neutron and \(q = p\) for a proton) beyond the beta-stability line. Namely, for the fixed \(A\), we obtain the following result from Eqs. (1) and (4)
\[ \Delta \lambda(X) = \lambda_n - \lambda_p = \left| \frac{\partial E}{\partial N} \right|_Z - \left| \frac{\partial E}{\partial Z} \right|_N = 2 \left| \frac{\partial(E/A)}{\partial X} \right|_A = 4 \left[ b^*_{\text{sym}}(A) + e^*_C(A) \right] (X - X^*), \] (7)
where
\[ \lambda_n = \left( \frac{\partial E}{\partial N} \right)_Z, \quad \lambda_p = \left( \frac{\partial E}{\partial Z} \right)_N. \] (8)

On the beta-stability line, it follows Eq. (7) that \(\Delta \lambda(X)_{X=X^*} = 0\), as it should be from the definition of the beta-stability line. We point out that for finite nuclei, the condition \(\Delta \lambda = 0\) on the beta-stability line is not necessary fulfilled explicitly because of the discrete spectrum of the single particle levels for both the neutrons and the protons near Fermi surface.

3. The quantity \(\partial(E/A)/\partial X\) in Eq. (7) can be evaluated within the accuracy of \(\sim 1/A^2\) using the finite differences which are based on the experimental values of the binding energy per nucleon \(B(N, Z) = -E(N, Z)/A\). Namely,
\[ \left. \frac{\partial(E/A)}{\partial X} \right|_A = \frac{A}{4} \left[ B(N - 1, Z + 1) - B(N + 1, Z - 1) \right]. \] (9)
Since the difference (9) is taken for $\Delta Z = -\Delta N = 2$, the pairing effects do not affect the resulting accuracy.

From the binding energy tables [1] we have obtained sets of values for $\Delta \lambda(X, A = \text{const})$ covering mass numbers from 8 to 238. Each set contains 3 to 11 points $(X, \Delta \lambda/4)$. In Fig. 1, the typical sets of values are plotted for $A = 100, 120$ and 160. As seen from Fig. 1, the positions of symbols determined by (9) can be reproduced quite well by the linear dependence on $X$. This allows one to extract values of quantities $b_{\text{sym}}^*(A)$, $e_{C}^*(A)$ and $X^*$ for a given mass number $A$, see Eq. (7). The linear functions $\Delta \lambda(X)$ from the Eq. (7) obtained by the best fit to the experimental data are shown by solid lines in Fig. 1.

Fig. 1. The difference $\Delta \lambda$ between neutron and proton chemical potentials as a function of the asymmetry parameter $X$ for nuclei with mass numbers $A = 100, 120$ and 160. The experimental values are presented by the symbols. Solid lines show the result for $\Delta \lambda/4$ using Eq. (7) with values $b_{\text{sym}}^*$ and $e_{C}^*(A)$ obtained from the least square fit.
In agreement with the Eq. (7), the slopes of straight lines in Fig. 1 allow us to derive
the quantity $b_{\text{sym}}^*(A) + e_C^*(A)$. From the beta-stability condition $\Delta \lambda(X) = 0$ one can also
derive the asymmetry parameter $X^*(A)$. Finally, using the Eq. (3), we obtain the symmetry
energy coefficient $b_{\text{sym}}^*(A)$ and Coulomb energy parameter $e_C^*(A)$. The corresponding results
are shown in Figs. 2, 3 and 4. Note that the error bars in Figs. 2, 3 and 4 represent the
standard error interval obtained for $b_{\text{sym}}^*$, $e_C^*(A)$ and $X^*$ from the least square fit for the
concerning set $\Delta \lambda(X, A = \text{const})$. These errors reflect the systematic deviation of $\Delta \lambda(X)$
from the linear dependence. Experimental errors in the nuclear masses are about on order
of magnitude less than that shown by the error bars. The error bars, however, are small
enough to see the non-monotonic behavior (the shell structure) of the plots versus mass
number.

![Figure 2](image_url)

**Fig. 2.** Asymmetry parameter $X^*(A)$ versus the mass number $A$. Filled circles – experimental
data, dashed lines – calculations using Skyrme forces SkM and SLy230b, see Ref. (4). Solid
lines – $X^*(A) = 0.4A/(A+200)$ (thick), and $X^*(A) = e_C^*(A)/ [b_{\text{sym}}^*(A) + e_C^*(A)]$ (see (3)) with
$e_C^*(A) = 0.17A^{2/3}$, $b_{\text{sym}}^*(A) = 26.5 - 25.6A^{-1/3}$ (thin). Open circles – beta-stability line obtained
from the periodic system of elements.
In Fig. 2, we have plotted the value of $X^*(A)$ (solid dots) which was obtained from Eq. (7) and Fig. 1. The dashed and dotted lines in Fig. 2 were derived from the extended Thomas-Fermi approximation for SkM and SLy230b Skyrme forces, respectively, see Ref. [4]. The solid (thick) line in Fig. 2 was obtained using the empirical formula [5]

$$X^*(A) = \frac{0.4 A}{A + 200}. \quad (10)$$

The ”experimental” curve $X^*(A)$ in Fig. 2 shows the non-monotonic (sawtooth) shape as a function of the mass number $A$. This behavior is the consequence of shell structure of single particle levels near Fermi surface for both the neutrons and the protons. Because of this shell structure, the Fermi levels for protons and neutrons can coincide by chance only. For light nuclei, this significantly affects the beta-stability line, deflecting it from the empirical one for stable nuclei (open circles in Fig. 2). In agreement with Eq. (3), the smooth behavior of $X^*(A)$ is achieved by a fit of the symmetry and the Coulomb energy coefficients. Thin solid line in Fig. 2 is obtained from Eq. (3) with $e^*_C(A) = 0.17A^{2/3}$ and $b^*_\text{sym}(A) = 26.5 - 25.6A^{-1/3}$.

The $A$-dependence of the Coulomb energy coefficient $e^*_C(A)$ is shown in Fig. 3. Assuming $R_C \propto A^{1/3}$, one obtains that $e^*_C(A) \propto A^{2/3}$. Using this $A$-dependence of $e^*_C(A)$, we have the best fit to the experimental values by using $e^*_C(A) = 0.17A^{2/3}$. The corresponding result of the fit is shown in Fig. 3 as the solid line. The deviation of the Coulomb energy coefficient from the smooth $A$-dependence is mainly due to the shell oscillations of the Coulomb radius $R_C$.

The dependency of the symmetry energy coefficient $b^*_\text{sym}$ on the mass number $A$ obtained from the experimental nuclear masses using Eqs. (7) and (9) is shown in Fig. 4 as solid circles. This dependence shows the strong shell oscillations with the amplitude of about 15%. For the purpose of comparison, one could recall that shell effects contribute about 1% to the nuclear mass. In this paper, we have performed the fit of experimental data for $b^*_\text{sym}$ to the leptodermous-like functional form of Eq. (6). To extract the expansion coefficients $b^*_\text{sym,vol}$ and $b^*_\text{sym,surf}$ from the fit we have used the data with $A \geq 12$ for which the justified leptodermous expansion. Assuming $b^*_\text{sym}(A)$ given by Eq. (6) as the basic dependency of the symmetry energy coefficient on the mass number, we have obtained $b^*_\text{sym,vol} = 26.5 \text{ MeV}$ and $b^*_\text{sym,surf} = -25.6 \text{ MeV}$ with the surface-to-volume ratio $r_{s/v} = |b^*_\text{sym,surf}|/b^*_\text{sym,vol} \approx 1$. The corresponding function $b_{\text{sym}}(A)$ is plotted as the solid line in Fig. 4. The calculated values of
the coefficients \( b_{\text{sym,vol}}^* \) and \( b_{\text{sym,surf}}^* \) are in a quite good agreement with the phenomenological ones derived from the Weizsäcker mass formula \([6]\). The dashed and dotted lines in Fig. 4 show the results obtained from the extended Thomas-Fermi approximation using Skyrme forces SkM and SLy230b, see Ref. \([4]\).

One should pay attention to the sources of the uncertainty in the obtained values. Firstly, the shell effects in \( b_{\text{sym}}^* \) considerably reduce the accuracy of the fit procedure. Secondly, the range of mass numbers covered by the experimental data does not allow one to determine \( b_{\text{sym,vol}}^* \) unambiguously, since one cannot neglect the surface term in \([6]\) even for the large masses (\( A \sim 240 \)). Additional uncertainty is because of the small curvature corrections in the surface symmetry coefficient \( b_{\text{sym,surf}}^* \). Due to the above, one can obtain quite different symmetry energy coefficients \( b_{\text{sym,vol}}^* \) and \( b_{\text{sym,surf}}^* \) with approximately the same quality of the fit taking the different intervals of mass numbers \( A \) for the fitting procedure. In particular,
we have obtained $b_{\text{sym,vol}}^* = 32.5$ MeV, $b_{\text{sym,surf}}^* = -56.3$ MeV and $r_{S/V} \approx 1.7$ from the fit for $A \geq 50$. Earlier, a similar feature of the extraction of the volume and the surface terms at the symmetry energy was noted in Ref. [3] by fitting the binding energies of nuclei. Some numerical results are summarized in the Table 1. Note that in theoretical calculations, the value of surface-to-volume ratio $r_{S/V}$ varies strongly within the interval $1.6 \leq r_{S/V} \leq 2.8$, see Refs. [7, 8, 9, 10].

![Graph](image)

Fig. 4. The symmetry energy coefficient $b_{\text{sym}}$ as a function of mass number $A$. Symbols correspond to experimental data. Solid line shows $b_{\text{sym}}^*(A) = 26.5 - 25.6A^{-1/3}$ obtained from the fit to experimental values using Eq. (6). Dashed and dotted lines present the calculations using Skyrme forces SkM and SLy230b, respectively (see [4]).

Table 1. Symmetry energy coefficients $b_{\text{sym,vol}}^*$, $b_{\text{sym,surf}}^*$ and the surface-to-volume ratio $r_{S/V}$ obtained from the least square fit of symmetry energy (6) to $b_{\text{sym}}^*$ from Fig. 4 for different fitting intervals: $A \geq 50$ and $A \geq 12$.

|       | $b_{\text{sym,vol}}^*$ (MeV) | $b_{\text{sym,surf}}^*$ (MeV) | $r_{S/V}$ |
|-------|-------------------------------|-------------------------------|------------|
| $A \geq 50$ | 32.5                          | -56.3                         | 1.73       |
| $A \geq 12$ | 26.5                          | -25.6                         | 1.03       |
4. We have established the relation (7) between the nuclear symmetry energy and the isospin shift of chemical potential $\Delta \lambda = \lambda_n - \lambda_p$ that is beyond the beta-stability line. This relation allowed us to evaluate the symmetry energy independently of the standard derivation from the mass formula. Moreover this evaluation performed for different mass number $A$, has been used to derive the ”experimental” values of the $A$-dependent volume, surface and curvature terms for the isospin symmetry energy.

The presence of shell structure in the single particle levels for neutrons and protons leads to the characteristic oscillations in $A$-dependencies of the beta-stability line, the symmetry energy and the Coulomb energy (nuclear Coulomb radius). We point out the presence of strong amplitude oscillations of the symmetry energy $b_{\text{sym}}^*$ which significantly exceed the corresponding shell effects in the binding energy. We do not observe the existence of exceptionally large values of the symmetry energy coefficient $b_{\text{sym}}^*$ for the mass number $A \approx 100$ which was earlier reported in Ref. [2].

Our approach allowed us to estimate the experimental values of the volume, $b_{\text{sym},\text{vol}}^*$ and the surface, $b_{\text{sym},\text{surf}}^*$, contributions to the symmetry energy. The result of the fitting procedure for $b_{\text{sym},\text{vol}}^*$ and $b_{\text{sym},\text{surf}}^*$ depends significantly on the intervals of the mass numbers $A$ applied to the fitting procedure. By fitting all of the available information for $A \geq 12$, we have obtained $b_{\text{sym},\text{vol}}^* = 26.5$ MeV and the surface contribution $b_{\text{sym},\text{surf}}^* = -25.6$ MeV with the surface-to-volume ratio $r_{S/V} \approx 1$ which is well below the corresponding theoretical results obtained within the extended Thomas-Fermi approximation (ETFA) with the effective Skyrme forces in the present work as well as in quantum Skyrme-Hartree-Fock (SHF) approach [10]. We also point out that the change of the intervals in mass numbers $A$ for the fitting procedure leads to a significant difference in the surface contribution $b_{\text{sym},\text{surf}}^*$ to the symmetry energy, see rows 2 and 3 in the Table 1. A better agreement with theoretical results is obtained for the fitting interval with larger masses $A \geq 50$ where the leptodermous expansion is more justified. In this case, the surface-to-volume ratio is given by $r_{S/V} \approx 1.7$ which is close to the theoretical result from Ref. [10].

[1] G. Audi, A.H. Wapstra and C. Thibault, Nucl. Phys. A 729, 337 (2003).

[2] J. Jänecke, T.W. O’Donnell and V.I. Goldanskii, Nucl. Phys. A 728, 23 (2003).
[3] Akira Ono, P. Danielewicz, W. A. Friedman, W. G. Lynch and M. B. Tsang, Phys. Rev. C 70, 041604(R) (2004).

[4] V.M. Kolomietz and A.I. Sanzhur, Eur. Phys. J. A 38, 345 (2008).

[5] A.E.S. Green and N.A. Engler, Phys. Rev. 91, 40 (1953).

[6] A. Bohr and B. R. Mottelson, Nuclear Structure (W. A. Benjamin, New York, 1969), Vol. 1.

[7] E. Lipparini and S. Stringari, Phys. Lett. B112, 421 (1982).

[8] P. Möller, J.R. Nix, W.D. Myers and W.J. Swiatecki, At. Data Nucl. Data Tables 59, 185 (1995).

[9] P. Danielewicz, Nucl. Phys. A 727, 233 (2003).

[10] W. Satuła, R.A. Wyss and M. Rafalski, Phys. Rev. C 74, 011301(R) (2006).