Asymptotics of the number of partitions into p-cores and some trigonometric sums.

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1. Introduction.

A p-core is a partition that has no hook numbers divisible by p. These partitions occur in the representation theory of the symmetric group in characteristic p.

Let \( a_p(n) \) denote the number of partitions of \( n \) into p-cores. Then we have the following generating function (Garvan [7], p.449)

\[
\sum_{n=0}^{\infty} a_p(n)x^n = \prod_{j=1}^{\infty} \frac{(1 - x^{pj})^p}{(1 - x^j)}
\]

Garvan also gives the following asymptotic formula for \( a_p(n) \) if \( p \) is a prime \( \geq 5 \)

\[
a_p(n - \frac{p^2 - 1}{24}) = K_p \sum_{d|n} \left( \frac{d}{p} \right) \left( \frac{n}{d} \right)^{(p-3)/2} + O(n^{(p-3)/4+\varepsilon})
\]

where \( \left( \frac{d}{p} \right) \) is the Legendre symbol and \( K_p \) is a constant depending only on \( p \). Implicitly in Garvan’s paper one has \( K_p = 1/c_p \) where \( c_p \) seems to be an integer. This is proved here using results from [2] and [7].

Using the rather obscure methods described in [1] we will find another asymptotic formula for \( a_p(n) \). Deleting some, usually small terms, it is proved that we get the same formula as Garvan. For this we have to use that

\[
\frac{\eta(\tau)p}{\eta(p\tau)}
\]

is a modular form for the group \( \Gamma_0(p) \) with character \( \left( \frac{a}{p} \right) \) (Ogg [9], p.28).

As a byproduct we obtain some trigonometric sums which turn out to be integer valued. Here is a sample

\[
\frac{\sqrt{p}}{2^{r+1}} \sum_{j=1}^{(p-1)/2} \cot(r \left( \frac{j^2 \pi}{p} \right)) = -(-1)^{(r+1)/2} \left( \frac{-1}{p} \right) \frac{p^{r+1}}{r+1} \sum_{j=1}^{(p-1)/2} B_{r+1}(\text{frac}(\frac{j^2}{p}))
\]

are integers. Here \( B_n(x) \) is the Bernoulli polynomial and \( \text{frac}(x) = x - [x] \) is the fractional part of \( x \).

2. An asymptotic formula for \( a_p(n) \).

Let

\[
F(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-1}
\]
be the generating function for \( p(n) \), the number of partitions of \( n \). Then
\[
f(x) = \prod_{j=1}^{\infty} \frac{1 - x^{p_j}}{1 - x} = \frac{F(x)}{F(x)^p}
\]

We want to study \( f(x) \) near the point
\[
x = \exp(2\pi i h / k)
\]
where \( (h, k) = 1 \). But it is well known that (see Apostol [5], p.106)
\[
F(\exp(2\pi i h / k - t)) = \exp(\pi is(h, k) \left( \frac{kt}{2\pi} \right)^{1/2} \exp(\frac{\pi^2 t}{6k^2} - \frac{t}{24}) F(\exp(\frac{2\pi i H}{k} - \frac{4\pi^2}{k^2 t}))
\]
where
\[
H h \equiv -1 \mod k
\]
and
\[
s(h, k) = \sum_{j=1}^{k-1} ((j/k))((jh/k))
\]
is the classical Dedekind sum. Here \( ((x)) = x - \lfloor x \rfloor - 1/2 \) if \( x \notin \mathbb{Z} \) and = 0 if \( x \in \mathbb{Z} \). It follows that
\[
f(\exp(2\pi i h / k - t)) = \exp(\pi i (s(h, k) - ps(h, k))
\]
(*)
\[
(\frac{2\pi}{k})^{(p-1)/2} p^{-p/2} t^{-\frac{(p-1)}{2}} \exp\left(\frac{p^2 - 1}{24} t\right) \frac{F(\exp(2\pi i H / k - 4\pi^2 / k^2 t))}{F(\exp(2\pi i B / k - 4\pi^2 / k^2 p t))}
\]
where
\[
B ph \equiv -1 \mod k \text{ if } (k, p) = 0
\]
If \( (k, p) \neq 1 \) we put \( B = 0 \). Define
\[
A_p(k, n) = \sum_{(h,k)=1} \exp(\pi i (s(h, k) - ps(h, k)) - 2\pi ih n / k)
\]
Neglecting all analytic difficulties we compute \( a_p(n) \) as the n-th Fourier coefficient of \( f(\exp(iy)) \)
\[
a_p(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\exp(iy)) \exp(-i\nu y) dy
\]
Put \( y = 2\pi h / k + \varphi \) and sum over all \( h = 1, 2, \ldots k - 1 \) with \( (h, k) = 1 \) (we also replace the interval \(( -\pi, \pi )\) with \(( -\infty, \infty )\) in order to get a Fourier
Let us also assume that (*) is valid when we put $t = -i\varphi$. We replace the last factor in (*) by 1. Then the contribution from the singular points $\exp(2\pi i h/k)$ with $(h, k) = 1$ will be

$$\Phi_k(n) \approx \left(\frac{2\pi}{k}\right)^{(p-1)/2} p^{-p/2} A_p(k, n) \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\varphi)^{-(p-1)/2} \exp\left( - (n + (p^2 - 1)/24) i\varphi \right) d\varphi$$

(the integral exists in the distributional sense). Summing over $k$ we obtain

$$a_p(n) \approx \left(\frac{2\pi}{k}\right)^{(p-1)/2} p^{-p/2} A_p(k, n) \frac{1}{(p - 3)/2!) \sum_{k=1}^{\infty} A_p(k)$$

which looks rather different from Garvan’s formula (to begin with, it is an infinite sum!).

Making some numerical computations we found the following

1. The formula gives about half of the digits correctly. E.g.

$$a_{17}(1000) = 1829\ 06764\ 82504$$

while the approximation gives

$$1829\ 06768\ 71721$$

2. The $A_p(k, n)$ are integers if $(k, p) = 1$. Computing many values of $A_p(k, n)$ I finally came up with

**Conjecture 1.** If $p \geq 5$ is prime and $(k, p) = 1$ then

$$A_p(k, n) = \left(\frac{k}{p}\right) c_k(n + \frac{p^2 - 1}{24})$$

where

$$c_k(n) = \sum_{(h,k)=1} \exp(2\pi i h/k)$$

is the Ramanujan sum (known to be an integer). This, however, follows from the more precise

**Conjecture 2.** If $p \geq 5$ is prime and $(k, p) = 1$ then

$$ps(ph, k) - s(h, k) = \frac{(p^2 - 1)h}{12k} + \text{integer}$$

where the integer is even if and only if $(\frac{k}{p}) = 1$.
We will prove Conjecture 2 later. Let us first see what we can get from our asymptotic formula from Conjecture 1. Since we know nothing about \( A_p(k,n) \) when \( k \equiv 0 \mod p \), we throw away those terms. Then we obtain

\[
a_p(n - \frac{p^2 - 1}{24}) \approx \frac{(2\pi)^{(p-1)/2}p^{-p/2}}{(p-3)/2)!} \sum_{k=1, p \nmid k} A_p(k - \frac{p^2 - 1}{24}) \frac{n^{(p-3)/2}}{k^{(p-1)/2}} \]

\[
= \frac{(2\pi)^{(p-1)/2}p^{-p/2}}{(p-3)/2)!} \sum_{k=1, p \nmid k} \left( \frac{k}{p} \right) c_k(n) \frac{n^{(p-3)/2}}{k^{(p-1)/2}}
\]

Ramanujan ([10]) computed the sum

\[
\sum_{k=1}^{\infty} \frac{c_k(n)}{n^{1+s}}
\]

but the factor \( \left( \frac{2}{p} \right) \) causes some difficulties.

**Proposition 2.1.** We have the formula

\[
\sum_{k=1}^{\infty} \frac{c_k(n)}{n^{1+s}} = \frac{p^{1+s} \sum_{d|n} \left( \frac{2}{p} \right) d^{-s}}{\sum_{j=1}^{p-1} \left( \frac{2}{j} \right) \zeta(1+s, j/p)}
\]

where

\[
\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}
\]

is Hurwitz’s \( \zeta \)-function.

**Proof:** First we note that

\[
c_k(n) = \sum_{d|(n,k)} d \mu(k/d)
\]

where \( \mu \) is the Möbius function. It follows

\[
\sum_{k=1}^{\infty} \frac{c_k(n)}{n^{1+s}} = \sum_{k=1}^{\infty} \frac{c_k(n)}{n^{1+s}} \sum_{d|(n,k)} d \mu(k/d)
\]

\[
= \sum_{d|n} \frac{1}{d^{1+s}} \sum_{j=1}^{\infty} \frac{\mu(j)}{j^{1+s}}
\]

But \( \left( \frac{2}{p} \right) \) and \( \frac{1}{j^{1+s}} \) are both completely multiplicative functions of \( j \) and we get

\[
\sum_{j=1}^{\infty} \mu(j) \frac{\left( \frac{j}{p} \right)}{j^{1+s}} \frac{1}{\zeta(1+s, j/p)} = \frac{p^{1+s}}{\sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \frac{1}{\zeta(1+s, j/p)}}
\]

and this completes the proof.
Returning to the asymptotic formula we obtain

\[ a_p(n) \approx K_p \sum_{d|n} \left( \frac{d}{p} \right) (n/d)^{(p-3)/2} \]

which agrees with Garvan's formula with

\[ K_p = \frac{(2\pi)^{(p-1)/2}}{\sqrt{\pi((p-3)/2)!} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \zeta((p-1)/2, j/p)} \]

**Proposition 2.2.** We have \( K_p = 1/c_p \) where

(i) \[ c_p = \frac{\sqrt{p}((p-3)/2)!}{(2\pi)^{(p-1)/2}} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \zeta((p-1)/2, j/p) \]

(ii) \[ c_p = \frac{1}{2} \left( \frac{-2}{p} \right) p^{(p-1)/2} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \zeta(-(p-3)/2, j/p) \]

(iii) \[ c_p = (-1)^{(p-3)/2} \frac{\sqrt{p}}{2(p+1)/2} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \cot((p-3)/2)(\pi j/p) \]

(iv) \[ c_p = -\left( \frac{-2}{p} \right) p^{(p-1)/2} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) B_{(p-1)/2}(j/p) \]

is a positive integer (\( p \geq 5 \) prime). If \( p \equiv 3 \mod 4 \) we also have the formulas

(v) \[ c_p = \frac{\sqrt{p}}{2(p+1)/2} \sum_{j=1}^{p-1} \cot((p-3)/2)(\frac{\pi j^2}{p}) \]

(vi) \[ c_p = -\left( \frac{-2}{p} \right) p^{(p-1)/2} \sum_{j=1}^{p-1} B_{(p-1)/2}(\frac{\sqrt{p}j^2}{p}) \]

**Table of \( c_p \)**

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Proof: These formulas are consequences of more general identities. We will use the Finite Fourier Transformation (FFT) ( \( k > 1 \) a fixed integer)

\[
\hat{f}(\mu/k) = \sum_{j=0}^{k-1} f(j/k) \exp(-2\pi i j \mu/k)
\]

We make a little table of FFT for our purposes

| \( f \) | \( \hat{f} \) |
|-------|-------|
| \( \zeta(s, x) \) | \( k^s l(s, 1 - x) \) |
| \( \zeta \) | \( (-i)^{(k-1)/2} \sqrt{k(\zeta)} \) if \( k \) is prime |
| \( B_n(x) \) | \( kn(-2ki)^{-n} \cot^{(n-1)}(\pi x) \) |

where

\[
l(s, x) = \sum_{n=1}^{\infty} \frac{\exp(2\pi i x)}{n^s}
\]

is the periodic \( \zeta \)-function. There seems to be no reference to the first and third formula. Therefore they are proved in an Appendix. The second formula is in Apostol [4] p.195.

Define a scalar product

\[
\langle f, g \rangle = \sum_{j=0}^{k-1} f(j/k) \overline{g(j/k)}
\]

Then we have Parseval's formula

\[
\langle \hat{f}, \hat{g} \rangle = k \langle f, g \rangle
\]

Taking

\[
f(j/p) = \binom{j}{p}
\]

and

\[
g(j/p) = \zeta(s, j/p)
\]
we get
\[
\sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \zeta(s, j/p) = \frac{1}{p} (-i)^{(p-1)/2} \sqrt{p^r} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \zeta(s, 1 - j/p)
\]

In particular, let \( s = r + 1 \) be a positive integer:
\[
U = \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \zeta(r + 1, j/p) = (-i)^{(p-1)/2} \sqrt{p^r} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \sum_{n=1}^{\infty} \frac{\exp(2\pi i nj/p)}{n^{r+1}}
\]

**Case 1:** \( p \equiv 1 \mod 4 \)
Then \( \left( \frac{-1}{p} \right) = 1 \) and \((-i)^{(p-1)/2} = 1\). Hence
\[
U = \sqrt{p^r} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \sum_{n=1}^{\infty} \frac{\cos(2\pi nj/p)}{n^{r+1}}
\]
\[
= \sqrt{p^r} \frac{(2\pi)^{r+1}(-1)^{(r+3)/2}}{2(r+1)!} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) B_{r+1}(j/p)
\]
if \( r \) is odd (see Apostol [3], p.267). But \( r = (p - 3)/2 \) is odd so we get
\[
K_p = -\left( \frac{2}{p} \right) \frac{(-1)^{(p-3)/2}}{p^{-1/2} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) B_{(p-1)/2}(j/p)}
\]

**Case 2:** \( p \equiv 3 \mod 4 \)
Then \( \left( \frac{-1}{p} \right) = -1 \) so \( \left( \frac{-i}{p} \right) = -\left( \frac{i}{p} \right) \) and \((-i)^{(p-1)/2} = -i\). Hence
\[
U = -i \sqrt{p^r} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \sum_{n=1}^{\infty} \frac{i \sin(2\pi nj/p)}{n^{r+1}}
\]
\[
= \sqrt{p^r} \frac{(2\pi)^{r+1}(-1)^{(r+2)/2}}{2(r+1)!} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) B_{r+1}(j/p)
\]
if \( r \) is even. In particular \( r = (p - 3)/2 \) we get
\[
K_p = -\left( \frac{-2}{p} \right) \frac{-1}{p^{-1/2} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) B_{(p-1)/2}(j/p)}
\]
(one has to check that the sign is correct). The rest of the proof will follow from the results in the next section.

3. Some trigonometric sums.
Proposition 3.1. Define for a prime $p \geq 5$

$$S(r, p) = -\left(\frac{-1}{p}\right)\sqrt{p}\frac{p^{r-1}}{2^{r+2}}\sum_{j=1}^{p-1} \left(\frac{j}{p}\right) \cot^{(r)}(\pi j/p)$$

and

$$T(r, p) = (-1)^{(r-1)/2} \frac{p^{r+1}}{2(r+1)} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) B_{r+1}(j/p)$$

Then for $(p, r + 1) = 1$ both $S(r, p)$ and $T(r, p)$ are nonnegative integers and equal.

**Proof:** By symmetry one observes that $S(r, p) = T(r, p) = 0$ unless

$$p \equiv 1 \mod 4 \text{ and } r \text{ odd}$$

or

$$p \equiv 3 \mod 4 \text{ and } r \text{ even}$$

Using Parseval’s formula we obtain

$$T(r, p) = (-1)^{(r-1)/2} \frac{p^{r+1}}{2(r+1)} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) B_{r+1}(j/p)$$

$$= (-1)^{(r-1)/2} \frac{p^{r+1}}{2(r+1)} \sum_{j=1}^{p-1} (-i)^{(p-1)/2} \sqrt{p} \left(\frac{j}{p}\right) \frac{p(r+1)}{2} \cot^{(r)}(\pi j/p)$$

$$= -\left(\frac{-1}{p}\right)\sqrt{p}\frac{p^{r+1}}{2^{r+2}}\sum_{j=1}^{p-1} \left(\frac{j}{p}\right) \cot^{(r)}(\pi j/p) = S(r, p)$$

The sign is easily checked in the two cases mentioned in the beginning of the proof.

That $T(r, p)$ is an integer follows from the following facts

(i) $k^n(B_n(j/k) - B_n)$ is an integer (Almkvist-Meurman [2]). Note that

$$\sum_{j=1}^{p-1} \left(\frac{j}{p}\right) B_n = 0$$

(ii) If $(k, n) = 1$ then this integer is divisible by $k$ (Dokshitzer [6], p.3)

(iii) $$\sum_{j=1}^{p-1} \left(\frac{j}{p}\right) B_{r+1}(j/p) = 2 \sum_{j=1}^{(p-1)/2} \left(\frac{j}{p}\right) B_{r+1}(j/p)$$

in both cases.

It remains to show that, say $S(r, p)$, is nonnegative. The idea of proof is to show that the first term $\left|\cot^{(r)}(\pi/p)\right|$ dominates the sum.

(a) $\left|\cot^{(r)}(x)\right|$ is decreasing in the interval $(0, \pi/2)$
This follows from that there is a polynomial \( f_r(t) \) of degree \( r + 1 \) such that \( \cot(r)(x) = f_r(cot(x)) \) and all the coefficients of \( f_r \) are nonnegative. Indeed
\[
f_1(t) = 1 + t^2
\]
and
\[
f_{r+1}(t) = (1 + t^2)f_r'(t)
\]
(b) \( r = 1. \) Then \( \left| \cot(1)(x) \right| = \frac{1}{\sin^2(x)} \) and we have
\[
\frac{1}{\sin^2(\pi/p)} \geq \frac{p^2}{\pi^2}
\]
We will show that
\[
\sum_{j=2}^{(p-1)/2} \frac{1}{\sin^2((\pi j)/p)} < \frac{p^2}{\pi^2}
\]
if \( p \) is large enough. One has the inequality (see [3], p.45)
\[
\frac{1}{\sin(x)} \leq \frac{1}{x} + \frac{x}{3} \text{ if } 0 < x \leq \frac{\pi}{2}
\]
Hence
\[
\sum_{j=2}^{(p-1)/2} \frac{1}{\sin^2((\pi j)/p)} \leq \frac{p^2}{\pi^2} \sum_{j=2}^{(p-1)/2} \left\{ \frac{1}{j^2} + \frac{2\pi^2}{3p^2} + \frac{\pi^4}{9p^4} \right\}
\]
\[
\leq \frac{p^2}{\pi^2} \left\{ \frac{\pi^2}{6} - 1 + \frac{\pi^2}{3p} + \frac{\pi^4}{216p} \right\} < \frac{p^2}{\pi^2}
\]
if \( p > 10. \) For smaller \( p \) it is easily checked that \( S(1, p) \geq 0. \)
(c) \( r \geq 2 \) We have the expansion
\[
\cot(x) = \frac{1}{x} - \frac{2}{\pi} \sum_{j=1}^{\infty} \zeta(2j) \left( \frac{x}{\pi} \right)^{2j-1}
\]
and hence
\[
\cot(r)(x) = \frac{(-1)^r}{x^{r+1}} \left\{ 1 - 2(-1)^r \sum_{2j \geq (r+1)} \zeta(2j) \binom{2j-1}{r} \left( \frac{x}{\pi} \right)^{2j} \right\}
\]
We also have the sum
\[
\sum_{2j \geq r+1} \binom{2j-1}{r} x^{2j} = \frac{x^{r+1}}{2} \left\{ \frac{1}{(1-x)^{r+1}} - (-1)^r \frac{1}{(1+x)^{r+1}} \right\}
\]
and hence
\[
\left| \cot(r)(\pi/p) \right| \geq \frac{r^lp^{r+1}}{\pi^{r+1}} \left\{ 1 - 2\zeta(r+1) \sum_{2j \geq r+1} \binom{2j-1}{r} p^{-2j} \right\}
\]
In the same way we obtain
\[
\left| \cot^{(r)}(2\pi/p) \right| \leq \frac{p^{r+1}}{(2\pi)^{r+1}} \left\{ 1 + \zeta(r+1) \left( \frac{2}{p-2} \right)^{r+1} + \frac{2}{p+2} \right\} 
\]
Furthermore
\[
\sum_{j=3}^{(p-1)/2} \left| \cot^{(r)}(\pi j/p) \right| \leq \int_{2\pi/p}^{(p-1)/p} \cot^{(r)}(x) dx \leq \left| \cot^{(r-1)}(2\pi/p) \right| 
\]
and finally
\[
\sum_{j=1}^{(p-1)/2} \cot^{(r)}(\pi j/p) \geq \frac{p^{r+1}}{(2\pi)^{r+1}} \left\{ 1 - \zeta(r+1) \left( \frac{1}{(p-1)^{r+1}} + \frac{1}{(p+1)^{r+1}} - \frac{1}{p^{r+1}} \right) - \zeta(r) \left( \frac{2}{p-2} \right)^{r+1} + \frac{2}{p+2} \right\} > 0 
\]
if \( p \geq 5 \) (since \( r \geq 2 \)).

**Remark 3.1:** Making computer experiments it looks like
(a) \( T(r,p) \) is divisible by \( p \) if not
\[
r \equiv \frac{p-3}{2} \mod (p-1)
\]
(b) The smallest \( r \) for which \( T(r,p) \) is not an integer is \( r = (\frac{p}{2}) - 1 \).

**End of the proof of Proposition 2.2**

In the case \( p \equiv 3 \mod 4 \) we have \( \left( \frac{p-2}{p} \right) = -\left( \frac{2}{p} \right) \) and formulas (v) and (vi) follow from the definition of \( \left( \frac{2}{p} \right) \).

**Remark 3.2.** Noting that for \( x \notin \mathbb{Z} \) we have \( B_1(frac(x)) = (x) \) we define
\[
G(p) = T(0,p) = \sum_{j=1}^{p-1} ((j^2/p))
\]
Then \( G(p) \) is an integer, nonzero if \( p \equiv 3 \mod 4 \). Then computations suggest that \( G(p) = -1 \) if and only if
\[
p = 7, 11, 19, 43, 67, 163
\]
These primes are recognized as those with class number one for \( \mathbb{Q}(\sqrt{-p}) \).
Indeed, \( -G(p) \) is nothing but Dirichlet’s formula for the class number
\[
h(-p) = -\frac{1}{p} \sum_{j=1}^{p-1} \left( \frac{j}{p} \right)
\]

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which, since \( p \equiv 3 \mod 4 \), can be written

\[
h(-p) = -\sum_{j=1}^{p-1} \left( \frac{j^2}{p} \right)
\]

or after use of Parseval’s formula

\[
h(-p) = \frac{1}{\sqrt{p}} \sum_{j=1}^{(p-1)/2} \cot\left( \frac{\pi j^2}{p} \right)
\]

**Proof of Conjecture 2.**

We will actually prove a little more:

**Theorem 3.3.** Let

\[
f(x) = \prod_{n=1}^{\infty} \frac{(1 - x^{pn})^p}{1 - x^n}
\]

Then for \( (h, k) = (p, k) = 1 \) we have \( (p \geq 5 \text{ prime}) \)

\[
f(\exp(2\pi i h/k - t)) = (\frac{2\pi}{k})^{(p-1)/2} \left( \frac{k}{p} \right) \sum p/2 \left( \frac{p-1}{24} \right) - \frac{\pi i (p^2 - 1) h}{12k} \right) H(\exp \left\{ \frac{2\pi i B}{k} - \frac{4\pi^2}{k^2 p} \right\})
\]

where

\[
Bph \equiv -1 \mod k
\]

and

\[
H(x) = \frac{\eta(x)^p}{\eta(x^p)} = \prod_{n=1}^{\infty} \frac{(1 - x^n)^p}{1 - x^{pn}}
\]

**Proof:** We have

\[
f(x) = x^{-(p^2-1)/24} \frac{\eta(x^p)^p}{\eta(x)} = x^{-(p^2-1)/24} h(x)
\]

where

\[
h(x) = \frac{\eta(px)^p}{\eta(x)} \quad \text{if} \quad x = \exp(2\pi it)
\]

Here

\[
\eta(x) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n)
\]

is the Dedekind function. Put \( \tau = -\frac{1}{p^2} \). Then

\[
h(\tau) = \frac{\eta(-1/\tau')^p}{\eta(-1/p^2\tau')} = \frac{1}{\sqrt{p}} \left( \frac{\tau'}{i} \right)^{(p-1)/2} H(\tau')
\]

where

\[
H(\tau') = \frac{\eta(\tau')^p}{\eta(p^2 \tau')}
\]
But by Ogg [8], p.28  \( H \) is a modular form of weight \( (p - 1)/2 \) for the group

\[
\Gamma_0(p) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \epsilon \text{SL}(2, \mathbb{Z}); \ c \equiv 0 \mod p \right\}
\]

with character \( \left( \frac{d}{p} \right) \). This means that

\[
H\left( \frac{\alpha \tau' + b}{c \tau' + d} \right) = \left( \frac{d}{p} \right) (c \tau' + d)^{(p - 1)/2} H(\tau')
\]

if \( c \equiv 0 \mod p \). Choose

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 1 + B ph & B \\ k & k \end{array} \right) \epsilon \Gamma_0(p)
\]

where

\[
B ph \equiv -1 \mod k
\]

If we take

\[
\tau = \frac{h}{k} + \frac{2\pi t}{pk^2 t}
\]

and

\[
\frac{\tau'}{i(c \tau' + d)} = \frac{2\pi}{pk t}
\]

Hence

\[
h(\tau) = \frac{1}{\sqrt{p}} \frac{2\pi}{pk t} (p - 1)/2 \left( \frac{d}{p} \right) H(\exp \left\{ \frac{2\pi i B}{k} - \frac{4\pi^2}{k^2 pt} \right\})
\]

and the Theorem is proved

Conjecture 2 follows by comparing the limit of this formula and (*) when \( t \to 0^+ \)

**Remark.** Attempts to find a better approximation to \( a_p(n) \) like what is done in [1] failed. The convergence is too slow.

**Appendix:** Some Finite Fourier Transformations.

1. \( f(x) = B_r(x) \)

We have the generating function

\[
\frac{te^{xt}}{e^t - 1} = \sum_{r=0}^{\infty} B_r(x) \frac{t^r}{r!}
\]

Putting \( x = j/k \) and summing over \( j \) we get

\[
\frac{t}{e^t - 1} \sum_{j=0}^{k-1} \exp\left( \frac{j t}{k} - \frac{2\pi i j \mu}{k} \right) = \sum_{r=0}^{\infty} \hat{B}_r(\mu/k) \frac{t^r}{r!}
\]

Now

\[
\frac{1}{e^{\alpha t} - 1} = -\frac{1}{2} - i \frac{1}{2} \cot(\alpha/2)
\]
so the left hand side is
\[
\frac{t}{\exp(t/k - 2\pi i \mu/k) - 1} = t \left\{ -\frac{1}{2} + \frac{i}{2} \cot(\pi \mu/k + it/2k) \right\}
\]
\[= -\frac{1}{2} + k \sum_{r=1}^{\infty} \cot(r^{-1}(\pi \mu/k) t/r! \left(\frac{i}{2k}\right)^r / \]

Comparing the coefficients of \( t^r \) we find
\[
\hat{B}_r(\mu/k) = kr(\frac{i}{2k})^r \cot(r^{-1}(\pi \mu/k))
\]

2. \( f(x) = l(s, x) \)
We have
\[
\hat{l}(s, \mu/k) = \sum_{j=0}^{k-1} l(s, j/k) \exp(-2\pi ij \mu/k)
\]
\[
= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{j=0}^{k-1} \exp(2\pi ij(n - \mu)/k) = k \sum_{n \equiv \mu \mod k} \frac{1}{n^s}
\]
\[= k^{1-s} \sum_{m=0}^{\infty} \frac{1}{(m + \mu/k)^s} = k^{1-s} \zeta(s, \mu/k)
\]
Since
\[
\hat{f}(x) = kf(1 - x)
\]
we have
\[
\hat{\zeta}(s, \mu/k) = k^s l(s, 1 - \mu/k)
\]

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