Adaptive Stabilization of a Fractional-Order System with Unknown Disturbance and Nonlinear Input via a Backstepping Control Technique

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Abstract: In this paper, a new backstepping-based adaptive stabilization of a fractional-order system with unknown parameters is investigated. We assume that the controlled system is perturbed by external disturbance, the bound of external disturbance to be unknown in advance. Moreover, the effects of sector and dead-zone nonlinear inputs both are taken into account. A fractional-order auxiliary system is established to generate the necessary signals for compensation the nonlinear inputs. Meantime, in order to deal with these unknown parameters, some fractional-order adaption laws are given. The frequency-distributed model is used so that the indirect Lyapunov theory is available in designing controllers. Finally, simulation results are presented to verify the effectiveness and robustness of the proposed control strategy.

Keywords: adaptive stabilization; fractional-order system; sector and dead-zone nonlinear inputs; backstepping method

1. Introduction

Fractional-order calculus derives from the end of 17th century; it is particularly suitable for describing the viscoelastic system [1], and the memory and hereditary properties of various materials and processes. Symmetries play an important role in dynamics of fractional-order systems, and some results have been reported about the fractional symmetry; for instance, Frederico and Torres [2] studied fractional Noether symmetry and gave a definition of a fractional conserved quantity. Zhang [3,4] investigated Noether symmetry and conserved quantity for the classical fractional Birkhoffian system and the fractional Birkhoffian system with time delay. Song [5] researched Noether quasi-symmetry and perturbation to Noether quasi-symmetry. Now, studying fractional-order systems has became an active research area. In particular, control and stabilization of the fractional-order systems have attracted much attention from various scientific fields. It has been proven that applying fractional-order controllers to fractional-order system can obtain a better control effect than integer-order controllers, such as fractional-order PID control [6], fractional-order sliding mode control [7], fractional fuzzy control [8], fractional-order finite-time control [9], and so on.

The backstepping method is a recursive approach for controller design; through designing virtual controllers and partial Lyapunov functions step by step, a common Lyapunov function of the whole system can be deduced from the above operations. This method can guarantee the global stability, tracking, and transient performance of nonlinear systems [10]. In view of the excellent performance of backstepping, an increasing number of researchers have focused on this potential problem. Many studies for the backstepping-based control and synchronization of fractional-order chaotic system have been reported. For example, Luo [11] researched the robust control and synchronization of a fractional-order system by adding one power integrator. Shukla [12,13] realized the stabilization
and synchronization of fractional-order chaotic systems by using backstepping method. Wei [14,15] investigated the stability of fractional-order nonlinear systems via the adaptive backstepping technique.

However, all approaches in the aforementioned works are only focused on the linear and direct application of control inputs. In practice, input nonlinearity is often encountered in various systems and can be a cause of instability. Thus, it is obvious that the effects of input nonlinearity must be taken into account when analyzing and implementing a control scheme. Recently, Sheng [16] and Ha [17] considered the impacts of input saturation in the stabilization or synchronization of fractional-order systems. To the best of our knowledge, there is little information available in the literature about the stabilization of fractional-order systems with more complicated nonlinear inputs. Meanwhile, applying the backstepping approaches to a fractional-order system with unknown bounded external disturbances is also rare.

Motivated by the above discussions, it is still very challenging and essential to research the stabilization of fractional-order systems with complex nonlinear input by using adaptive backstepping techniques. Sector and dead-zone nonlinear inputs are more complicated than saturated input in characteristics, thus, in this paper, on the basis of symmetry principles, the effects of sector and dead-zone nonlinear inputs both are considered. For compensation of the nonlinear inputs, a fractional-order auxiliary system is constructed to generate necessary signals. For efficient handling of a system with unknown parameters, some new adaptive estimation rules are proposed. The frequency-distributed model is used to establish an indirect Lyapunov function to demonstrate the stability and design a virtual controller for every subsystem. Through the design of a virtual controller, a comprehensive actual controller and unknown disturbance estimation rule are determined.

To sum up, our approach makes the following contributions: (i) The backstepping-based stabilization of fractional-order systems with unknown bounded external disturbance is researched; (ii) The effects of two kinds of nonlinear inputs with complicated characteristic are considered; (iii) A fractional order auxiliary system is constructed to generate virtual signals for compensation of the sector and dead-zone nonlinear inputs; (iv) Based on the frequency-distributed model, indirect Lyapunov theory is used to design controllers and some adaptive estimation laws are established.

The remaining part of this paper is organized as follows: Section 2 introduces the relevant definitions, frequency-distributed model and strict feedback system. Main results are presented in Section 3. Some numerical simulations are provided in Section 4 to show the effectiveness of the proposed method. Finally, conclusions are given in Section 5.

2. Preliminaries and System Description

**Definition 1** (see [18]). The Caputo's fractional derivative of order $\alpha$ of function $f(t)$ is

$$D^\alpha_t f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \left( t - \tau \right)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & m - 1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases}$$

where $m$ is the smallest integer number, larger than $\alpha$. In the rest of this paper, we will use $D^\alpha_t$ instead of $D^\alpha_t f(t)$.

**Lemma 1** (see [19–21]). The fractional order system $D^\alpha_t x(t) = f(t)$ with $0 < \alpha < 1$, $x(t) \in \mathbb{R}$ and $f(t) \in \mathbb{R}$, can be regarded as the following linear continuous frequency distributed model

$$\begin{cases} \frac{\partial z(\omega, t)}{\partial t} = -\omega z(\omega, t) + f(t) \\ x(t) = \int_0^\infty \mu_\alpha(\omega) z(\omega, t) d\omega \end{cases}$$

where $\mu_\alpha(\omega) = \frac{\sin(\alpha \pi)}{\alpha \pi} \omega^\alpha$ and $z(\omega, t) \in \mathbb{R}$. In (2), $x(t)$ is a pseudo state variable, while $z(\omega, t)$ is an infinite dimension distributed state variable and $x(t)$ is only one weighted sum of the variable $z(\omega, t)$.
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Strict feedback system [22] can be used to express different real world systems, which can be described as follows

\[
\begin{align*}
D^a x_1 &= g_1(x_1, t)x_2 + \theta_1^T F_1(x_1, t) + f_1(x_1, t) \\
D^a x_2 &= g_2(x_1, x_2, t)x_3 + \theta_2^T F_2(x_1, x_2, t) + f_2(x_1, x_2, t) \\
& \quad \vdots \\
D^a x_{n-1} &= g_{n-1}(x_1, x_2, ..., x_{n-1}, t)x_n + \theta_{n-1}^T F_{n-1}(x_1, x_2, ..., x_{n-1}, t) + f_{n-1}(x_1, x_2, ..., x_{n-1}, t) \\
D^a x_n &= g_n(x_1, x_2, ..., x_n, t)u + \theta_n^T F_n(x_1, x_2, ..., x_n, t) + f_n(x_1, x_2, ..., x_n, t)
\end{align*}
\]

where \( \theta_i \) is the system parameters vector of the \( i \)-th state equation, \( g_i(\cdot), F_i(\cdot), f_i(\cdot) \) for \( i = 1, 2, ..., n \) are known, smooth nonlinear functions. When consider the effects of external disturbance \( \dot{d}(t) \) and the sector and dead-zone nonlinear inputs \( \phi(u(t)) \), meantime, the system parameters vector \( \theta_i \) is unknown, \( g_1(\cdot), g_2(\cdot), ..., g_n(\cdot) \) are constants, then the system can be rewritten as

\[
\begin{align*}
D^a x_1 &= k_1 x_2 + \theta_1^T F_1(x_1, t) + f_1(x_1, t) + \dot{d}(t) \\
D^a x_2 &= k_2 x_3 + \theta_2^T F_2(x_1, x_2, t) + f_2(x_1, x_2, t) + \dot{d}(t) \\
& \quad \vdots \\
D^a x_{n-1} &= k_{n-1} x_n + \theta_{n-1}^T F_{n-1}(x_1, x_2, ..., x_{n-1}, t) + f_{n-1}(x_1, x_2, ..., x_{n-1}, t) + \dot{d}(t) \\
D^a x_n &= k_n \phi(u(t)) + \theta_n^T F_n(x_1, x_2, ..., x_n, t) + f_n(x_1, x_2, ..., x_n, t) + \dot{d}(t)
\end{align*}
\]

where \( u(t) \) is the comprehensive actual controller to be designed later. \( \phi(u(t)) \) is a nonlinear input function, if it is continuous inside a sector \([\delta_1, \delta_2] \) and satisfying the following equation

\[
\delta_1 u^2(t) \leq \phi(u(t)) \leq \delta_2 u^2(t)
\]

in (5), \( \delta_2 > \delta_1 > 0 \), then \( \phi(u(t)) \) is called sector nonlinear input. A typical and symmetrical sector nonlinear function is shown in Figure 1.

\[
\phi(u(t))
\]

\[
\delta_1 u^2(t) \leq \phi(u(t)) \leq \delta_2 u^2(t)
\]

Figure 1. Sector nonlinear function \( \phi(u(t)) \) for the input \( u(t) \).
The dead-zone nonlinear function is described as follows

\[
\phi(u(t)) = \begin{cases} 
(u(t) - u_+)\phi_+(u(t)), & u(t) > u_+ \\
0, & u_- \leq u(t) \leq u_+ \\
(u(t) - u_-)\phi_-(u(t)), & u(t) < u_-
\end{cases}
\]  

where \(\phi_+(\cdot)\) and \(\phi_-(\cdot)\) are nonlinear functions of \(u(t)\), \(u_+\) and \(u_-\) are given constants. Besides, outside of the dead-band, the nonlinear input \(\phi(u(t))\) has gain reduction tolerances \(\beta_{+2}, \beta_{+1}, \beta_{-1}\) and \(\beta_{-2}\), which satisfy the following symmetric property

\[
\begin{cases} 
\beta_{+2}(u(t) - u_+)^2 \geq (u(t) - u_+)\phi(u(t)) \geq \beta_{+1}(u(t) - u_+)^2, & u(t) > u_+ \\
0, & u_- \leq u(t) \leq u_+ \\
\beta_{-2}(u(t) - u_-)^2 \geq (u(t) - u_-)\phi(u(t)) \geq \beta_{-1}(u(t) - u_-)^2, & u(t) < u_- 
\end{cases}
\]

in which, \(\beta_{+2}, \beta_{+1}, \beta_{-1}\) and \(\beta_{-2}\) are positive constants. A sample dead-zone nonlinear function is displayed in Figure 2.

![Figure 2. Dead-zone nonlinear function \(\phi(u(t))\) for the input \(u(t)\).](image)

before introducing our approach, we firstly give an assumption.

**Assumption 1.** Suppose there exist an unknown constant \(\gamma > 0\) such that the external disturbance satisfies \(|d(t)| \leq \gamma\).

Next, the main control strategy will be introduced in detail, the frequency distributed model is used so that the indirect Lyapunov theory can be applied to demonstrate the feasibility of the proposed control scheme [23–28]. In simulation, the numerical method [29] is used to solve the fractional order equations, two examples about fractional-order Chua system and fractional-order Rossler’s [30] system are given to verify the correctness of the presented control method.
3. Main Results

This section introduces the adaptive backstepping-based stabilization of fractional-order system with nonlinear input and external disturbance, to deduce the actual controller, transformation variables should be assigned firstly as

\[\begin{align*}
\dot{\xi}_1 &= x_1 - s_1 \\
\dot{\xi}_i &= x_i - \theta_{i-1} - s_i, \quad i = 2, ..., n.
\end{align*}\]

where \(s_j\) for \(j = 1, 2, ..., n\) is the virtual signal generated by the following auxiliary fractional-order system to compensate the sector and dead-zone nonlinear inputs

\[
\begin{align*}
D^\sigma s_i &= k_i s_{i+1} - (m_i s_i + l_i |s_i|^{\sigma} \text{sgn}(s_i)), \quad i = 1, 2, ..., n - 1. \\
D^\sigma s_n &= p \phi(u(t)) - (m_n s_n + l_n |s_n|^{\sigma} \text{sgn}(s_n))
\end{align*}
\]

in which, \(0 < \sigma < 1, m_i > 0, l_i > 0, m_n > 0, l_n > 0, p < k_n\). \(\theta_j(j = 1, 2, ..., n - 1)\) is virtual controller and can be designed as

\[
\begin{align*}
\theta_1 &= \frac{1}{k_1} (-c_1 \xi_1 - \hat{\delta}_1^T F_1 - f_1 - m_1 s_1 - l_1 |s_1|^{\sigma} \text{sgn}(s_1) - \hat{\gamma} \text{sgn}(\xi_1)) \\
\theta_i &= \frac{1}{k_i} (-c_i \xi_i - k_{i-1} \xi_{i-1} - \hat{\delta}_i^T F_i - f_i + D^\sigma \dot{\theta}_{i-1} - m_i s_i - l_i |s_i|^{\sigma} \text{sgn}(s_i) - \hat{\gamma} \text{sgn}(\xi_i)), \quad i = 2, 3, ..., n - 1.
\end{align*}
\]

where \(c_1 > 0, c_i > 0\). \(F_i, f_i\) is the abbreviation of \(F_i(\cdot), f_i(\cdot)\). \(\hat{\theta}_i\) is the estimation of unknown parameters vector \(\theta_i\). \(\hat{\gamma}\) is the estimation of \(\gamma\). Denote \(\hat{\delta}_i = \hat{\theta}_i - \theta_i\) and \(\hat{\gamma} = \hat{\gamma} - \gamma\) as the estimation errors, the parameters update laws are chosen as

\[
\begin{align*}
D^\sigma \hat{\delta}_i &= D^\sigma \hat{\theta}_i - D^\sigma \theta_i = D^\sigma \hat{\theta}_i = F_i, \\
D^\sigma \hat{\gamma} &= D^\sigma \hat{\gamma} - D^\sigma \gamma = D^\sigma \hat{\gamma} = \eta \sum_{j=i}^n |\xi_i|, \quad \eta > 0
\end{align*}
\]

**Theorem 1.** Consider the system (4) with sector nonlinear input, the controller which leads to asymptotic stabilization of system is given below

\[
\begin{align*}
u(t) &= -\rho (k_n - p)^{-1} e(t) \text{sgn}(s_n), \quad \rho = \delta_1^{-1} \\
e(t) &= c_n |\xi_n| + |k_n| |\xi_{n-1}| + |\theta_n|^T |F_n| + |f_n| + |\hat{\gamma}| + |D^\sigma \theta_{n-1}| + |m_n s_n + l_n |s_n|^{\sigma} \text{sgn}(s_n)|
\end{align*}
\]

**Proof.** Step 1: The first new subsystem can be obtained according to Equations (4) and (8)

\[
\begin{align*}
D^\sigma \xi_1 &= D^\sigma x_1 - D^\sigma s_1 \\
&= k_1 (\xi_2 + \theta_1 + s_2) + \theta_1^T F_1 + f_1 + d(t) - k_1 s_2 + m_1 s_1 + l_1 |s_1|^{\sigma} \text{sgn}(s_1)
\end{align*}
\]

according to Lemma 1, the new subsystem (13) and parameters adaptation laws (11) can be transformed into the frequency distributed model, that is
\[
\frac{\partial z_1(\omega, t)}{\partial t} = -\omega z_1(\omega, t) + k_1(\xi_2 + \theta_1) + \theta_1^T F_1 + f_1 + d(t) + m_1 s_1 + l_1 |s_1| \text{sgn}(s_1)
\]
\[
\xi_1 = \int_0^\infty \mu_a(\omega) z_1(\omega, t) d\omega
\]
\[
\frac{\partial z_{\tilde{\theta}_1}(\omega, t)}{\partial t} = -\omega z_{\tilde{\theta}_1}(\omega, t) + F_1 \xi_1
\]
\[
\tilde{\theta}_1 = \int_0^\infty \mu_a(\omega) z_{\tilde{\theta}_1}(\omega, t) d\omega
\]
\[
\frac{\partial z_{\tilde{\gamma}}(\omega, t)}{\partial t} = -\omega z_{\tilde{\gamma}}(\omega, t) + \eta \sum_{i=1}^n |\xi_i|
\]
\[
\tilde{\gamma} = \int_0^\infty \mu_a(\omega) z_{\tilde{\gamma}}(\omega, t) d\omega
\]

in Equation (14), \(z_1(\omega, t), z_{\tilde{\theta}_1}(\omega, t)\) and \(z_{\tilde{\gamma}}(\omega, t)\) are infinite dimension distributed state variables, while \(\xi_1, \tilde{\theta}_1\) and \(\tilde{\gamma}\) are the weighted sum of the variables \(z_1(\omega, t), z_{\tilde{\theta}_1}(\omega, t)\) and \(z_{\tilde{\gamma}}(\omega, t)\), respectively. Selecting the Lyapunov function as

\[
V_1(t) = \frac{1}{2} \int_0^\infty \mu_a(\omega) z_1^2(\omega, t) d\omega + \frac{1}{2} \int_0^\infty \mu_a(\omega) z_{\tilde{\theta}_1}^T(\omega, t) z_{\tilde{\theta}_1}(\omega, t) d\omega + \frac{1}{2\eta} \int_0^\infty \mu_a(\omega) z_{\tilde{\gamma}}^2(\omega, t) d\omega
\]

taking the derivative of \(V_1(t)\), it yields

\[
\dot{V}_1(t) = \int_0^\infty \mu_a(\omega) z_1(\omega, t) \frac{\partial z_1(\omega, t)}{\partial t} d\omega + \int_0^\infty \mu_a(\omega) z_{\tilde{\theta}_1}(\omega, t) \frac{\partial z_{\tilde{\theta}_1}(\omega, t)}{\partial t} d\omega - \frac{1}{\eta} \int_0^\infty \mu_a(\omega) z_{\tilde{\gamma}}(\omega, t) \frac{\partial z_{\tilde{\gamma}}(\omega, t)}{\partial t} d\omega + \int_0^\infty \mu_a(\omega) |\xi_1| d\omega + \frac{1}{\eta} \int_0^\infty \mu_a(\omega) z_{\tilde{\theta}_1}(\omega, t) \left( -\omega z_{\tilde{\theta}_1}(\omega, t) + F_1 \xi_1 \right) d\omega
\]

\[
\dot{V}_1(t) = -\int_0^\infty \omega \mu_a(\omega) z_1^2(\omega, t) d\omega + \frac{1}{\eta} \int_0^\infty \mu_a(\omega) |\xi_1| d\omega + \int_0^\infty \omega \mu_a(\omega) z_{\tilde{\theta}_1}^T(\omega, t) z_{\tilde{\theta}_1}(\omega, t) d\omega + \frac{1}{\eta} \int_0^\infty \omega \mu_a(\omega) |\xi_1| d\omega + \frac{1}{\eta} \int_0^\infty \mu_a(\omega) z_{\tilde{\gamma}}^2(\omega, t) d\omega + \frac{1}{\eta} \int_0^\infty \mu_a(\omega) |\xi_1| d\omega
\]

substituting \(\tilde{\theta}_1\) into Equation (16), and according to Assumption 1, we have

\[
\dot{V}_1(t) \leq -\int_0^\infty \omega \mu_a(\omega) z_1^2(\omega, t) d\omega - \int_0^\infty \omega \mu_a(\omega) z_{\tilde{\theta}_1}^T(\omega, t) z_{\tilde{\theta}_1}(\omega, t) d\omega - \frac{1}{\eta} \int_0^\infty \omega \mu_a(\omega) z_{\tilde{\gamma}}^2(\omega, t) d\omega + k_2 \xi_2 \xi_2 - c_1 \xi_2^2 + \frac{1}{\eta} \tilde{\gamma}(\eta \sum_{i=1}^n |\xi_i| - \eta |\xi_1|)
\]

if \(\xi_2 = 0\), the adaption law \(D^a \hat{\gamma} = \eta \sum_{i=1}^n |\xi_i|\) reduces to \(D^a \hat{\gamma} = \eta |\xi_1|\), then \(\dot{V}_1(t) < 0\), that is \(\xi_1, \tilde{\theta}_1, \tilde{\gamma}\) are all asymptotically converge to zero.
Step 2: The Second new subsystem about $\xi_2$ can be construct as

$$
D^\alpha \tilde{\xi}_2 = D^\alpha x_2 - D^\alpha \theta_1 - D^\alpha s_2 \\
= k_2(\xi_3 + \theta_2 + s_3) + \theta_2^T F_2 + f_2 + d(t) - D^\alpha \theta_1 - k_2s_3 + m_2s_2 + l_2|s_2|^\sigma \text{sgn}(s_2)
$$

(18)

the corresponding frequency distributed model is

$$
\frac{\partial z_2(\omega,t)}{\partial t} = -\omega z_2(\omega,t) + k_2(\xi_3 + \theta_2) + \theta_2^T F_2 + f_2 + d(t) - D^\alpha \theta_1 + m_2s_2 + l_2|s_2|^\sigma \text{sgn}(s_2)
$$

$$
\tilde{\xi}_2 = \int_0^\infty \mu_\alpha(\omega)z_2(\omega,t)d\omega
$$

$$
\tilde{\theta}_2 = \int_0^\infty \mu_\alpha(\omega)z_{\theta_2}(\omega,t)d\omega
$$

(19)

in Equation (19), $z_2(\omega,t)$ and $z_{\theta_2}(\omega,t)$ are infinite dimension distributed state variables, while $\tilde{\xi}_2$ and $\tilde{\theta}_2$ are the weighted sum of the variables $z_2(\omega,t)$ and $z_{\theta_2}(\omega,t)$, respectively. In this step to ensure $\tilde{\xi}_2$ converge to 0 as time tends to infinity. Selecting the Lyapunov function as

$$
V_2(t) = V_1(t) + \frac{1}{2} \int_0^\infty \mu_\alpha(\omega)z_2^T(\omega,t)d\omega + \frac{1}{2} \int_0^\infty \mu_\alpha(\omega)z_{\theta_2}^T(\omega,t)\tilde{z}_{\theta_2}(\omega,t)d\omega
$$

(20)

thus its derivative can be described as

$$
\dot{V}_2(t) \leq -\sum_{i=1}^2 \int_0^\infty \omega \mu_\alpha(\omega)z_{\xi_2}^2(\omega,t)d\omega - \sum_{i=1}^2 \int_0^\infty \omega \mu_\alpha(\omega)z_{\theta_2}^2(\omega,t)d\omega - \frac{1}{\eta} \int_0^\infty \omega \mu_\alpha(\omega)z_{\theta_2}^2(\omega,t)d\omega \\
+ k_1\xi_1\xi_2 - c_1\xi_2^2 + \frac{1}{\eta}\tilde{\gamma}(\eta \sum_{i=1}^n |\xi_i| - \eta |\xi|) - \int_0^\infty \omega \mu_\alpha(\omega)z_{\theta_2}^2(\omega,t)d\omega + \tilde{\xi}_2(\xi_3 + \theta_2) + \theta_2^T F_2 \\
+ f_2 + d(t) - D^\alpha \theta_1 + m_2s_2 + l_2|s_2|^\sigma \text{sgn}(s_2) - \int_0^\infty \omega \mu_\alpha(\omega)z_{\theta_2}^2(\omega,t)d\omega + \tilde{\theta}_2^T F_2 \tilde{\theta}_2
$$

(21)

substituting $\tilde{\theta}_2$ from Equation (10) to Equation (21), and according to Assumption 1, it yields

$$
\dot{V}_2(t) \leq -\sum_{i=1}^2 \int_0^\infty \omega \mu_\alpha(\omega)z_{\xi_2}^2(\omega,t)d\omega - \sum_{i=1}^2 \int_0^\infty \omega \mu_\alpha(\omega)z_{\theta_2}^2(\omega,t)d\omega - \frac{1}{\eta} \int_0^\infty \omega \mu_\alpha(\omega)z_{\theta_2}^2(\omega,t)d\omega \\
- c_1\xi_2^2 + k_2\xi_2s_3 - c_2\xi_2^2 + \frac{1}{\eta}\tilde{\gamma}(\eta \sum_{i=1}^n |\xi_i| - \eta \sum_{i=1}^2 |\xi_i|)
$$

(22)

similar to step 1, if $\xi_3 = 0$, the adaption law $D^\alpha \tilde{\gamma} = \eta \sum_{i=1}^n |\xi_i|$ reduces to $D^\alpha \tilde{\gamma} = \eta \sum_{i=1}^2 |\xi_i|$, then $V_2(t) < 0$, that is $\tilde{\xi}_2$ will be ensured to converge to zero asymptotically.

Step i ($i = 3, ..., n - 1$): We continue to investigate the i-th new subsystem with transformation variable $\xi_i$, that is

$$
D^\alpha \xi_i = D^\alpha x_i - D^\alpha \theta_{i-1} - D^\alpha s_i \\
= k_i(\xi_{i+1} + \theta_i + s_{i+1}) + \theta_i^T F_i + f_i + d(t) - D^\alpha \theta_{i-1} - k_is_i + m_is_i + l_i|s_i|^\sigma \text{sgn}(s_i)
$$

(23)
the frequency distributed model can be described as
\[
\frac{\partial z_i(\omega, t)}{\partial t} = -\omega z_i(\omega, t) + k_i(\xi_{i+1} + \theta_i) + \theta_i^T F_i + f_i + d(t) - D^a \theta_{i-1} + m_i s_i + l_i |s_i|^{\sigma} \text{sgn}(s_i)
\]
\[
\xi_i = \int_0^\infty \mu_a(\omega) z_i(\omega, t) d\omega
\]
\[
\frac{\partial \theta_i(\omega, t)}{\partial t} = -\omega \theta_i(\omega, t) + F_i \xi_i
\]
\[
\theta_i = \int_0^\infty \mu_a(\omega) \theta_i(\omega, t) d\omega
\]  
(24)

in Equation (24), \(z_i(\omega, t)\) and \(\theta_i(\omega, t)\) are infinite dimension distributed state variables, while \(\xi_i\) and \(\theta_i\) are the weighted sum of the variables \(z_i(\omega, t)\) and \(\theta_i(\omega, t)\), respectively. This step is to verify the stability of system (23) with the following Lyapunov function
\[
V_i(t) = V_{i-1}(t) + \frac{1}{2} \int_0^\infty \mu_a(\omega) z_i^2(\omega, t) d\omega + \frac{1}{2} \int_0^\infty \mu_a(\omega) \theta_i^2(\omega, t) d\omega
\]  
(25)

taking the derivative of \(V_i(t)\), one has
\[
\dot{V}_i(t) = \dot{V}_{i-1}(t) - \int_0^\infty \omega \mu_a(\omega) z_i^2(\omega, t) d\omega + \xi_i(k_i(\xi_{i+1} + \theta_i) + \theta_i^T F_i + f_i + d(t) - D^a \theta_{i-1}
\]
\[
+ m_i s_i + l_i |s_i|^{\sigma} \text{sgn}(s_i)) - \int_0^\infty \omega \mu_a(\omega) \theta_i^2(\omega, t) d\omega + \theta_i^T F_i \xi_i
\]  
\[
\leq - \sum_{j=1}^i \int_0^\infty \omega \mu_a(\omega) z_j^2(\omega, t) d\omega - \sum_{j=1}^i \int_0^\infty \omega \mu_a(\omega) \theta_j^2(\omega, t) d\omega
\]
\[
- \frac{1}{\eta} \int_0^\infty \omega \mu_a(\omega) z_i^2(\omega, t) d\omega - \sum_{j=1}^{i-1} c_j \xi_j^2 + k_i \xi_i \xi_{i-1} \xi_i + \frac{1}{\eta} \gamma (\eta \sum_{j=1}^{n} |\xi_j| - \eta \sum_{j=1}^{i-1} |\xi_j|)
\]
\[
+ \xi_i(k_i(\xi_{i+1} + \theta_i) + \theta_i^T F_i + f_i + d(t) - D^a \theta_{i-1} + m_i s_i + l_i |s_i|^{\sigma} \text{sgn}(s_i)) + \theta_i^T F_i \xi_i
\]  
(26)

introducing the virtual controller \(\bar{\theta}_i\) from Equation (10) into Equation (26), we obtain
\[
\dot{V}_i(t) \leq - \sum_{j=1}^i \int_0^\infty \omega \mu_a(\omega) z_j^2(\omega, t) d\omega - \sum_{j=1}^i \int_0^\infty \omega \mu_a(\omega) \theta_j^2(\omega, t) d\omega
\]
\[
- \frac{1}{\eta} \int_0^\infty \omega \mu_a(\omega) z_i^2(\omega, t) d\omega - \sum_{j=1}^{i-1} c_j \xi_j^2 + k_i \xi_i \xi_{i+1} \xi_i + \frac{1}{\eta} \gamma (\eta \sum_{j=1}^{n} |\xi_j| - \eta \sum_{j=1}^{i} |\xi_j|)
\]  
(27)

that is, when \(\xi_{i+1} = 0\), then the estimation law \(D^a \hat{\xi} = \eta \sum_{j=1}^{n} |\xi_j|\) reduce to \(D^a \hat{\xi} = \eta \sum_{j=1}^{i} |\xi_j|\), so, \(\dot{V}_i(t) < 0\), \(\xi_i\) is asymptotically converge to zero, which returns to step i-1.

Step n: In the last step, the actual controller is designed. Similar to the above steps, the last subsystem with transformation variable \(\xi_n\) is determined as
\[
D^a \xi_n = D^a x_n - D^a \bar{\theta}_{n-1} - D^a s_n
\]
\[
= (k_n - \varphi(u(t))) + \theta_n^T F_n + f_n + d(t) - D^a \bar{\theta}_{n-1} + m_n s_n + l_n |s_n|^{\sigma} \text{sgn}(s_n)
\]  
(28)

the corresponding frequency distributed model is
\[
\frac{\partial \varphi_n(\omega,t)}{\partial t} = -\omega \varphi_n(\omega,t) + (k_n - p) \phi(u(t)) + \theta_n^c F_n + f_n + d(t) - D^s \theta_n - 1 + m_n s_n + l_n |s_n|^2 \text{sgn}(s_n)
\]
\[
\frac{\partial \tilde{\varphi}_n(\omega,t)}{\partial t} = -\omega \tilde{\varphi}_n(\omega,t) + F_n \tilde{\varphi}_n
\]
\[
\frac{\partial \theta_n}{\partial t} = \frac{\mu_n(\omega)}{\partial t} \tilde{\varphi}_n(\omega,t) d\omega
\]

(29)

In Equation (29), \(z_n(\omega,t)\) and \(\tilde{z}_n(\omega,t)\) are infinite dimension distributed state variables, while \(\varphi_n\) and \(\tilde{\varphi}_n\) are the weighted sum of the variables \(z_n(\omega,t)\) and \(\tilde{z}_n(\omega,t)\), respectively. The overall Lyapunov function is constructed as
\[
V_n(t) = V_{n-1}(t) + \frac{1}{2} \int_0^\infty \mu_n(\omega) z_n^2(\omega,t) d\omega + \frac{1}{2} \int_0^\infty \mu_n(\omega) \tilde{z}_n^2(\omega,t) d\omega
\]

(30)

According to the previous inequality, the derivative of Equation (30) is
\[
\dot{V}_n(t) = \dot{V}_{n-1}(t) - \int_0^\infty \omega t \mu_n(\omega) z_n^2(\omega,t) d\omega + \varphi_n((k_n - p) \phi(u(t)) + \theta_n^c F_n + f_n + d(t) - D^s \theta_n - 1 + m_n s_n + l_n |s_n|^2 \text{sgn}(s_n)) \int_0^\infty \omega \mu_n(\omega) z_n^2(\omega,t) d\omega + \theta_n^c F_n \varphi_n
\]

\[
\dot{V}_n(t) \leq -\sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) z_n^2(\omega,t) d\omega - \sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) \tilde{z}_n^2(\omega,t) d\omega
\]

\[
\dot{V}_n(t) \leq -\sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) z_n^2(\omega,t) d\omega - \sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) \tilde{z}_n^2(\omega,t) d\omega
\]

(31)

According to Equations (5) and (12), we know that
\[
u(t) \phi(u(t)) = -\rho (k_n - p)^{-1} e(t) \text{sgn}(\varphi_n) \phi(u(t)) \geq \delta_1 \rho^2 (k_n - p)^{-2} e^2(t) \text{sgn}^2(\varphi_n)
\]

(32)

Since \(\rho = \delta_1^{-1}, k_n - p > 0, e(t) > 0\), according to above inequality, we have
\[
-\text{sgn}(\varphi_n) \phi(u(t)) \geq (k_n - p)^{-1} e(t) \text{sgn}^2(\varphi_n)
\]

(33)

Multiply both sides of inequality (33) by \((k_n - p)|\varphi_n|\), and according to \(\text{sgn}^2(\varphi_n) = 1, |\varphi_n| \text{sgn}(\varphi_n) = \varphi_n\), we obtain
\[
(k_n - p) \varphi_n \phi(u(t)) \leq -e(t) |\varphi_n|
\]

(34)

Substituting Equation (34) into Equation (31), and according to the second equation of Equation (12), one has
\[
\dot{V}_n(t) \leq -\sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) z_n^2(\omega,t) d\omega - \sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) \tilde{z}_n^2(\omega,t) d\omega
\]

\[
\dot{V}_n(t) \leq -\sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) z_n^2(\omega,t) d\omega - \sum_{n=1}^\infty \int_0^\infty \omega t \mu_n(\omega) \tilde{z}_n^2(\omega,t) d\omega
\]

(35)

Because of \(\dot{V}_n(t) < 0\), so the controlled system (4) with sector nonlinear input is asymptotically stable, thus the proof is completed. \(\square\)
Theorem 2. Consider the system (4) with dead-zone nonlinear input, the controller which leads to asymptotic stabilization of system is given below

\[
    u(t) = \begin{cases} 
        -\rho(k_n - p)^{-1}e(t)\text{sgn}(\xi_n) + u_- , & \xi_n > 0 \\
        0 , & \xi_n = 0 \\
        -\rho(k_n - p)^{-1}e(t)\text{sgn}(\xi_n) + u_+ , & \xi_n < 0 
    \end{cases} 
\]

where \( \rho = \beta^{-1}, \beta^{-1} = \min\{\beta_-1, \beta_{+1}\} \), and

\[
    e(t) = c_n|\xi_n| + |k_{n-1}|\xi_{n-1} + [\dot{\theta}_n]^{T}F_n + |f_n| + |D^\alpha\theta_{n-1}| + |\gamma| + |m_ns_n + l_n|\text{sgn}(s_n) | 
\]

Proof. Step 1 to i (i = 2, ..., n-1): Similar to the Proof of Theorem 1, for the concision, here is omitted, only deduce the final step.

Step n: The overall Lyapunov function is selected as

\[
    V_n(t) = V_{n-1}(t) + \frac{1}{2} \int_0^\infty \mu_n(\omega)z_n^2(\omega, t)d\omega + \frac{1}{2} \int_0^\infty \mu_n(\omega)z_{\hat{\theta}}^2(\omega, t)\dot{z}_{\hat{\theta}}(\omega, t)d\omega 
\]

Taking the derivative of \( V_n(t) \), and using the preceding derivative \( V_{n-1}(t) \), according to Assumption 1, it yields

\[
    V_n(t) \leq -\sum_{j=1}^n \int_0^\infty \omega \mu_n(\omega)z_n^2(\omega, t)d\omega - \sum_{j=1}^n \int_0^\infty \omega \mu_n(\omega)z_{\hat{\theta}}^2(\omega, t)\dot{z}_{\hat{\theta}}(\omega, t)d\omega \\
    -\frac{1}{2} \int_0^\infty \omega \mu_n(\omega)z_n^2(\omega, t)d\omega - \sum_{j=1}^n \int_0^\infty \omega \mu_n(\omega)z_{\hat{\theta}}^2(\omega, t)\dot{z}_{\hat{\theta}}(\omega, t)d\omega \\
    + |f_n| \xi_n + |D^\alpha\theta_{n-1}||\xi_n| + |m_ns_n + l_n|\text{sgn}(s_n) \leq \beta \rho^2(k_n - p)^{-2}e^2(t)\text{sgn}(\xi_n) 
\]

if \( \xi_n > 0 \), and surveying Equation (36), it is clear that \( u(t) < u_- \), then according to Equations (6) and (7), we have

\[
    (u(t) - u_-)\phi(u(t)) = -\rho(k_n - p)^{-1}e(t)\text{sgn}(\xi_n)\phi(u(t)) \\
    \geq -\beta \rho^2(k_n - p)^{-2}e^2(t)\text{sgn}(\xi_n) \\
    \geq -\beta \rho^2(k_n - p)^{-2}e^2(t)\text{sgn}(\xi_n) 
\]

since \( \rho = \beta^{-1} > 0, e(t) > 0, k_n - p > 0 \), according to the above inequality, we have

\[
    -\text{sgn}(\xi_n)\phi(u(t)) \geq (k_n - p)^{-1}e(t)\text{sgn}(\xi_n) 
\]

multiply both sides of inequality (41) by \( (k_n - p)|\xi_n| \), and according to \( \text{sgn}(\xi_n) = 1, (k_n)|\text{sgn}(\xi_n)| = \xi_n, \) we get

\[
    (k_n - p)|\xi_n|\phi(u(t)) \leq -e(t)|\xi_n| 
\]

on the basis of symmetry principles, when \( \xi_n < 0 \), through similar derivation, the inequality (42) still holds. Substituting the inequality (42) into (39), we have

\[
    V_n(t) \leq -\sum_{j=1}^n \int_0^\infty \omega \mu_n(\omega)z_n^2(\omega, t)d\omega - \sum_{j=1}^n \int_0^\infty \omega \mu_n(\omega)z_{\hat{\theta}}^2(\omega, t)\dot{z}_{\hat{\theta}}(\omega, t)d\omega \\
    -\frac{1}{2} \int_0^\infty \omega \mu_n(\omega)z_n^2(\omega, t)d\omega - \sum_{j=1}^n \int_0^\infty \omega \mu_n(\omega)z_{\hat{\theta}}^2(\omega, t)\dot{z}_{\hat{\theta}}(\omega, t)d\omega \\
    + |f_n||\xi_n| + |D^\alpha\theta_{n-1}||\xi_n| + |m_ns_n + l_n|\text{sgn}(s_n) \leq -\beta \rho^2(k_n - p)^{-2}e^2(t)\text{sgn}(\xi_n) 
\]
that is \( \dot{V}_n(t) < 0 \), so the adaptive stabilization of the controlled system (4) with dead-zone nonlinear input is realized, therefore the proof is completed. \( \square \)

4. Simulation Results

In this section, two simulation examples are given to demonstrate the effectiveness and feasibility of the proposed control strategy.

4.1. Adaptive Stabilization of Fractional-Order Chua System with Sector Nonlinear Input

The dynamics of fractional-order Chua system can be described as

\[
\begin{align*}
D^\alpha x &= a(y - x - f_1(x)) \\
D^\alpha y &= x - y + z \\
D^\alpha z &= -by
\end{align*}
\]

where \( \alpha = 0.97, f_1(x) = h_1x + \frac{1}{2}(h_0 - h_1)(|x + 1| - |x - 1|) \) is the nonlinear part of system, \( a = 15.6, b = 25, h_0 = -1.1428, h_1 = -0.714 \). This system is not strict feedback form, select \( x_1 = z, x_2 = y, x_3 = x \), considering external disturbance \( d(t) = 0.02\cos t \), the above system is transformed to the following strict feedback form

\[
\begin{align*}
D^\alpha x_1 &= -bx_2 + d(t) \\
D^\alpha x_2 &= x_3 + x_1 - x_2 + d(t) \\
D^\alpha x_3 &= \phi(u(t)) + a(x_2 - x_3 - h_1x_3 - \frac{1}{2}(h_0 - h_1)(|x_3 + 1| - |x_3 - 1|)) + d(t)
\end{align*}
\]

(45)

according to the standard form (4), we known that \( k_1 = -b, k_2 = 1, k_3 = 1 \), and the unknown parameter vector \( \theta_3 = a, F_3 = x_2 - x_3 - h_1x_3 - \frac{1}{2}(h_0 - h_1)(|x_3 + 1| - |x_3 - 1|) \). The sector nonlinear input is given by

\[
\phi(u(t)) = (0.85 + 0.21\sin(u(t)))u(t)
\]

(46)

obviously, \( \delta_1 = 0.64, \rho = 1/0.64 \), because of \( k_3 = 1 \), so, in the auxiliary system (9), we can choose \( p = 0, \sigma = 0.5, m_1 = m_2 = m_3 = 5, l_1 = l_2 = l_3 = 1 \), that is

\[
\begin{align*}
D^\alpha s_1 &= -bs_2 - (5s_1 + |s_1|^{0.5}\text{sgn}(s_1)) \\
D^\alpha s_2 &= s_3 - (5s_2 + |s_2|^{0.5}\text{sgn}(s_2)) \\
D^\alpha s_3 &= -(5s_3 + |s_3|^{0.5}\text{sgn}(s_3))
\end{align*}
\]

(47)

in this simulation, \( c_1 = c_2 = c_3 = 10, \eta = 5 \), the initial values are selected as \( x_1(0) = -0.5, x_2(0) = 0.1, x_3(0) = 0.6, \xi_1(0) = \xi_2(0) = \xi_3(0) = 0.3, \theta_3(0) = 0, \gamma(0) = 0, s_1(0) = s_2(0) = s_3(0) = 0.1 \). When the actual controller \( u(t) \) is activated, the time response of the transformation variables are shown in Figure 3, it is obvious that all variables trajectories are asymptotically tend to zero, which implies that under the control of the proposed control strategy, the adaptive stabilization of the controlled system with sector nonlinear input is realized.
4.2. Adaptive Stabilization of Fractional-Order Rossler’s System with Dead-Zone Nonlinear Input

The fractional-order Rossler’s system can be expressed as follows

\[ D^\alpha x = -y - z \]
\[ D^\alpha y = x + q_1 y \]
\[ D^\alpha z = q_2 + z(x - q_3) \] (48)

where \( \alpha = 0.92, q_1 = 0.5, q_2 = 0.2, q_3 = 10 \), this system can be transformed into strict feedback form by selecting \( x_1 = y, x_2 = x, x_3 = z \), that is

\[ D^\alpha x_1 = x_2 + q_1 x_1 + d(t) \]
\[ D^\alpha x_2 = -x_3 - x_1 + d(t) \]
\[ D^\alpha x_3 = \phi(u(t)) + q_2 - q_3 x_3 + x_2 x_3 + d(t) \] (49)

according to the standard form (4), we known that \( k_1 = 1, k_2 = -1, k_3 = 1 \), and the unknown parameter vectors \( \theta_1 = q_1, \theta_3 = [q_2, q_3]^T, F_1 = x_1, F_3 = [1, -x_3]^T \). \( d(t) = 0.5 \sin t \cos t \) is external disturbance. The dead-zone nonlinear input is given by

\[ \phi(u(t)) = \begin{cases} 
(u(t) - 2)(1 - 0.2 \sin(u(t))), & u(t) > 2 \\
0, & -4 < u(t) < 2 \\
(u(t) + 4)(0.9 - 0.3 \cos(u(t))), & u(t) < -4 
\end{cases} \] (50)
correspondingly, parameters $\beta_{+1} = 0.8, \beta_{-1} = 0.6$, so, $\beta = \min\{\beta_{+1}, \beta_{-1}\} = 0.6, \rho = \beta^{-1} = 1/0.6$, due to $k_3 = 1$, so, in the auxiliary system (9), we can choose $p = 0.2, \sigma = 0.5, m_1 = m_2 = m_3 = 4, l_1 = l_2 = l_3 = 2$. that is

$$D^\alpha s_1 = s_2 - (4s_1 + 2|s_1|^{0.5}\text{sgn}(s_1))$$
$$D^\alpha s_2 = -s_3 - (4s_2 + 2|s_2|^{0.5}\text{sgn}(s_2))$$
$$D^\alpha s_3 = 0.2\phi(u(t)) - (4s_3 + 2|s_3|^{0.5}\text{sgn}(s_3))$$

(51)

In this simulation, $c_1 = c_2 = c_3 = 5, \eta = 2$, the initial values are selected as $x_1(0) = 0.5, x_2(0) = 0.1, x_3(0) = 0.3, \dot{\theta}_1(0) = 0.1, \dot{\theta}_3(0) = [0.1, 0.1]^T, \dot{\gamma}(0) = 0.1, \xi_1(0) = \xi_2(0) = \xi_3(0) = 0.3, s_1(0) = s_2(0) = s_3(0) = 0.1$. When the actual controller $u(t)$ in (36) is activated, the time response of the transformation variables are shown in Figure 4, all variables trajectories asymptotically converge to zero. The above simulation results demonstrated the feasibility of the proposed method for fractional-order system with dead-zone nonlinear input.

**Remark 1.** The theorems firstly realized the backstepping-based stabilization of the fractional-order system with sector and dead-zone nonlinear inputs, if $\alpha = 1$, these methods change into the integer-order case, if $\alpha$ is not a fixed constant, the above theorems will be common solutions to incommensurate fractional-order systems. The indirect Lyapunov theory can ensure more degrees of freedom in designing the parameter estimation laws and can obtain better control performance.

5. Conclusions

In this paper, a backstepping-based adaptive stabilization of a fractional-order system is researched. The system is perturbed by unknown bounded external disturbance, and system parameters are unknown in advance. The effects of sector and dead-zone nonlinear inputs both
are considered in design the controller. For dealing with nonlinear input, a fractional-order auxiliary system is constructed to provide virtual signals. A frequency-distributed model is adopted so that indirect Lyapunov stability theory is used to demonstrate the stability of new subsystems with transformation variables. Through design, the virtual controllers are used to deduce an overall actual controller. Simulation results demonstrated the feasibility and effectiveness of the proposed control scheme.

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