PROJECTIVE REPRESENTATIONS OF REAL REDUCTIVE LIE GROUPS
AND THE GRADIENT MAP

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Abstract. Let $G$ be a connected semisimple noncompact real Lie group and let $\rho: G \rightarrow \text{SL}(V)$ be a representation on a finite dimensional vector space $V$ over $\mathbb{R}$, with $\rho(G)$ closed in $\text{SL}(V)$. Identifying $G$ with $\rho(G)$, we assume there exists a $K$-invariant scalar product $g$ such that $G = K \exp(p)$, where $K = \text{SO}(V, g) \cap G$, $p = \text{Sym}_0(V, g) \cap g$ and $g$ denotes the Lie algebra of $G$. Here $\text{Sym}_0(V, g)$ denotes the set of symmetric endomorphisms with trace zero. Using the $G$-gradient map techniques we analyze the natural projective representation of $G$ on $\mathbb{P}(V)$.

1. Introduction

Let $U$ be a compact connected semisimple Lie group and let $U^\mathbb{C}$ be its complexification [1]. Let $\tau: U^\mathbb{C} \rightarrow \text{SL}(V)$ be an irreducible holomorphic representation. By the classical Borel-Weyl Theorem, the representation $\tau$ is completely determined by the unique compact orbit of the $U^\mathbb{C}$-action on $\mathbb{P}(V)$ which is also the unique complex orbit of $U$. The vector $v_{\text{max}} \in V$ such that $x_0 = [v_{\text{max}}] \in \mathbb{P}(V)$ satisfies $U^\mathbb{C} \cdot x_0$ is compact, is the maximal weight vector [25].

The $U$-action on $\mathbb{P}(V)$ is Hamiltonian and so there exists a momentum map $\mu: \mathbb{P}(V) \rightarrow u^*$, where $u^*$ is the dual of the Lie algebra of $U$. If $T \subset U$ is a maximal torus with Lie algebra $t$, then $\mu_t := \mu \circ i^*$, where $i^*$ is the dual of the natural inclusion $i: t \hookrightarrow u$, is the momentum map for the $T$-action on $\mathbb{P}(V)$ [34]. With respect to the $T^\mathbb{C}$-action on $V$, there exist a finite set $\Delta(V, t^\mathbb{C}) \subset (t^\mathbb{C})^*$ called weights of $V$, so that

$$V = \bigoplus_{\lambda \in \Delta(V, t^\mathbb{C})} V_{\lambda},$$

where $V_{\lambda} = \{v \in V: \tau(H)v = \lambda(H)v, \text{ for any } H \in t^\mathbb{C}\}$. It is well-known that $v_{\text{max}} \in V_{\lambda_{\text{max}}}$ and $\dim V_{\lambda_{\text{max}}} = 1$. The functional $\lambda_{\text{max}}$ is called highest weight and $V$ is uniquely determined by $\lambda_{\text{max}}$ [31]. These data are determined by the momentum map. Indeed, $\bigcup_{\lambda \in \Delta(V, t^\mathbb{C})} \mathbb{P}(V_{\lambda})$ is the fixed point set for the $T^\mathbb{C}$-action on $\mathbb{P}(V)$ and $\mu_t(\mathbb{P}(V_{\lambda})) = \lambda_{\text{max}}$. By the Atiyah-Guillemin-Sternberg convexity Theorem [2, 22], the set of the extreme points of the polytope $\mu_t(\mathbb{P}(V))$ are weights of $V$. This means that the momentum map of the $T$-action on $\mathbb{P}(V)$ encodes roots and associated structures of the irreducible representation $\tau$. Moreover, the unique compact orbit of the $U^\mathbb{C}$-action on $\mathbb{P}(V)$ achieves the maximum of the norm square momentum map. Indeed, if $x \in \mathbb{P}(V)$

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satisfies $\| \mu(x) \| = \text{Max}_{y \in \mathbb{P}(V)} \| \mu(y) \|$, then $U^C \cdot x$ is compact and it is a $U$-orbit \cite{21,28}. In this paper, we analyze projective representations of real reductive Lie groups. The main tool is the gradient map.

Let $\rho : G \rightarrow \text{GL}(V)$ be a representation on a finite dimensional real vector space. We identify $G$ with $\rho(G) \subset \text{GL}(V)$ and we assume that $G$ is closed and it is closed under transpose. This means there exists a scalar product $\langle \cdot, \cdot \rangle$ on $V$ such that $G = K \exp(\mathfrak{p})$, where $K = G \cap O(V)$ and $\mathfrak{p} = \mathfrak{g} \cap \text{Sym}(V)$. Here $O(V)$ denotes the orthogonal group with respect to $\langle \cdot, \cdot \rangle$, $\text{Sym}(V)$ the set of symmetric endomorphisms of $V$ and $\mathfrak{g}$ the Lie algebra of $G$. By a standard theorem the existence of $\langle \cdot, \cdot \rangle$ is proved for a large class of linear representations of a real reductive algebraic groups \cite{32,39}. The scalar product $\langle \cdot, \cdot \rangle$ defines a Kempf-Ness function for the $G$-action on $V$ which is the basic tool to study certain geometric properties of linear actions of reductive real algebraic groups \cite{33,39}. Note that $G$ is self-adjoint since $G$ is invariant with respect to the map $g \mapsto g^T$, where $g^T$ is the transpose of $g$ with respect to $\langle \cdot, \cdot \rangle$. We recall that a classical Theorem of Mostow \cite{38} claims that any real reductive algebraic subgroup $G$ of $\text{GL}(n, \mathbb{R})$ is conjugate to a self-adjoint subgroup of $\text{GL}(n, \mathbb{R})$. In this paper we always assume that $G$ is a real noncompact semisimple linear Lie group and both the representations $\rho : G \rightarrow \text{SL}(V)$ and $\rho : G \rightarrow \text{SL}(V^C)$ are irreducible. If $\rho : G \rightarrow \text{SL}(V^C)$ is reducible, then there exists a linear complex structure $J$ on $V$ such that

$$\rho(G) \subset \{ A \in \text{SL}(V) : AJ = JA \},$$

\cite{17}. Therefore $(V, J)$ is a complex vector space and the $G$-action on $V$ preserves $J$. This case has been extensively studied in \cite{10}. We may also assume, up to conjugate, that $V = \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the canonical scalar product. This means that $G$ is a compatible subgroup of $\text{SL}(n, \mathbb{R})$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of $G$ induced by the Cartan decomposition of $\text{SL}(n, \mathbb{R})$. $G$ is a compatible subgroup of $\text{SL}(n, \mathbb{C})$ as well. Since $\mathfrak{k} \cap i\mathfrak{p} = \{0\}$, by \cite{26} Proposition 3.3, the Zariski closure of $G$ in $\text{SL}(n, \mathbb{C})$ is given by $U^C$, where $U$ is the connected, compact semisimple Lie subgroup of $\text{SU}(n)$ whose Lie algebra is given by $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. In particular, $G$ is a compatible real form of $U^C$ and the $U^C$-action on $\mathbb{C}^n$ is irreducible as well.

Our aim is to investigate the natural irreducible projective representation $\rho : G \rightarrow \text{PGL}(\mathbb{C}^n)$. In the sequel we also denote by $\rho : G \rightarrow \text{PGL}(\mathbb{C}^n)$ and by $\rho^C : U^C \rightarrow \text{PGL}(\mathbb{C}^n)$. Let $\langle \cdot, \cdot \rangle'$ be an $\text{Ad}(\text{SU}(n))$-invariant scalar product on $\mathfrak{su}(n)$. The $U$-action on $\mathbb{P}(\mathbb{C}^n)$ is Hamiltonian with momentum map

$$\mu : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathfrak{u}, \quad \mu = \pi_u \circ \Phi,$$

where $\Phi(z) = -\frac{1}{2} \left( \frac{z z^*}{\| z \|^2} - \frac{1}{n} \text{Id}_n \right)$ is the momentum map of the $\text{SU}(n)$-action on $\mathbb{P}(\mathbb{C}^n)$ and $\pi_u$ is the orthogonal projection of $\mathfrak{su}(u)$ onto $\mathfrak{u}$ \cite{34}. The $G$-gradient map is defined as follows.

We also denote by $\langle \cdot, \cdot \rangle'$ the $\text{Ad}(\text{SU}(n))$-invariant scalar product on $i\mathfrak{su}(n)$ requiring that $i$ is an isometry of $\mathfrak{su}(n)$ onto $i\mathfrak{su}(n)$. Since $G$ is compatible, we have the subspace $\mathfrak{p} \subset i\mathfrak{u}$. The $G$-gradient map $\mu_p : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathfrak{p}$ is the orthogonal projection of $i\mu$ onto $\mathfrak{p}$. In other world we
require
\[\langle \mu_p(z), \beta \rangle' = \langle i\mu(z), \beta \rangle' = \langle \mu(z), -i\beta \rangle',\]
for any \(\beta \in \mathfrak{p}\). If \(\mathfrak{a} \subset \mathfrak{p}\) is an Abelian subalgebra, then
\[\mu_\mathfrak{a} : \mathbb{P}(\mathbb{C}^n) \to \mathfrak{a}, \quad \mu_\mathfrak{a} := \pi_\mathfrak{a} \circ \mu_p,\]
where \(\pi_\mathfrak{a}\) is the orthogonal projection of \(\mathfrak{p}\) onto \(\mathfrak{a}\), is the \(A = \exp(\mathfrak{a})\)-gradient map. We usually restrict both \(\mu_\mathfrak{a}\) and \(\mu_p\) on \(\mathbb{P}(\mathbb{R}^n)\). By Borel-Weyl Theorem there exists a unique compact orbit \(\mathcal{O}'\) of the \(U^C\)-action on \(\mathbb{P}(\mathbb{C}^n)\). By a Theorem of Wolf [44], the \(G\)-action on \(\mathcal{O}'\) admits a unique compact orbit. We prove the following result.

**Theorem 1.** The set \(\mathcal{O} = \mathbb{P}(\mathbb{R}^n) \cap \mathcal{O}'\) is the unique compact \(G\)-orbit contained in \(\mathcal{O}'\). Moreover, \(\mathcal{O}\) is a Lagrangian submanifold of \(\mathcal{O}'\) and the fixed point set of an anti-holomorphic involutive isometry of \(\mathcal{O}'\) induced by a complex conjugation of \(\mathbb{P}(\mathbb{C}^n)\). In particular \(\mathcal{O}\) is a totally geodesic submanifold of \(\mathcal{O}'\).

The vice-versa holds as well.

Let \(\rho : U^C \to \text{SL}(n, \mathbb{C})\) be an irreducible holomorphic representation of a semisimple complex Lie group. Let \(G\) be a noncompact real form of \(U^C\). Let \(\mathcal{O}'\) be the unique compact orbit of the \(U^C\)-action on \(\mathbb{P}(\mathbb{C}^n)\) and let \(\mathcal{O}\) the unique compact \(G\)-orbit in \(\mathcal{O}'\).

**Theorem 2.** If there exists an anti-holomorphic involution \(T\) of \(\mathbb{P}(\mathbb{C}^n)\) preserving \(\mathcal{O}'\) and such that \(\mathcal{O}\) is contained in the fixed point set of \(T\), then there exists a real subspace \(V \subset \mathbb{C}^n\) such that \(G\) acts irreducibly on \(V\) and \(V^C = \mathbb{C}^n\).

Given \(\beta \in \mathfrak{p}\), we consider the parabolic subgroup
\[G^{\beta^+} := \{g \in G : \lim_{t \to -\infty} \exp(t\beta)g\exp(-t\beta) \text{ exists} \}\]
of \(G\) and \(\mu^{\beta}_p : \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}\), where \(\mu^{\beta}_p(z) = \langle \mu_p(z), \beta \rangle'.\) Since \(\beta \in \text{Sym}_0(n)\), \(\beta\) can be diagonalize. Let \(\lambda_1 > \cdots > \lambda_k\) be the eigenvalues of \(\beta\). We denote by \(V_1, \ldots, V_k\) the corresponding eigenspaces. In view of the orthogonal decompositions \(\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k\), we get
\[\mu^{\beta}_p([x_1 + \cdots + x_k]) = \frac{\lambda_1 \|x_1\|^2 + \cdots + \lambda_k \|x_k\|^2}{\|x_1\|^2 + \cdots + \|x_k\|^2}.
\]
In particular \(\text{Max}_{\mathbb{P}(\mathbb{R}^n)}(\beta) = \{p \in \mathbb{P}(\mathbb{R}^n) : \mu^{\beta}_p(p) = \max_{z \in \mathbb{P}(\mathbb{R}^n)} \mu^{\beta}_p \} = \mathbb{P}(V_1)\). By Proposition [44], we have \(\text{Max}_\mathcal{O}(\beta) \subset \text{Max}_{\mathbb{P}(\mathbb{R}^n)}(\beta)\) and \(\text{Max}_\mathcal{O}(\beta) \subset \text{Max}_{\mathcal{O}'}(\beta)\). We point out that \(\text{Max}_\mathcal{O}(\beta)\) is the unique compact orbit of the \((U^C)^{\beta^+}\)-action on \(\mathbb{P}(\mathbb{C}^n)\) [44] Corollary 1.0.1].

**Theorem 3.** In the above setting, the following results hold true:

a) \(V_1\) is the unique subspace of \(\mathbb{R}^n\) such that \(G^{\beta^+}\) acts irreducibly on it;

b) \(\text{Max}_\mathcal{O}(\beta)\) is connected and it coincides with the unique compact orbit of the \((U^C)^{\beta^+}\)-action on \(\mathcal{O}\). \(\text{Max}_\mathcal{O}(\beta)\) completely characterizes \(V_1\);

c) \(\text{Max}_\mathcal{O}(\beta)\) is a Lagrangian submanifold of \(\text{Max}_\mathcal{O'}(\beta)\) and the fixed point set of an anti-holomorphic involutive isometry of \(\text{Max}_\mathcal{O'}(\beta)\). In particular, \(\text{Max}_\mathcal{O}(\beta)\) is a totally geodesic submanifold of \(\text{Max}_\mathcal{O'}(\beta)\).
A compact orbit of the $G^{3+}$-action on $O'$ is contained in a compact orbit of $G$ due to the fact that $G = KG^{3+}$ [15], see also Proposition [23]. Since $G$ has a unique compact orbit on $O'$, it follows that Max$_O(\beta)$ is the unique compact orbit of the $G^{3+}$-action on $O'$ as well.

Let $E = \text{conv}(\mu_p(O))$. Since $O$ is a compact $G$-orbit, it is a $K$-orbit [28]. Therefore, keeping in mind that the gradient map is $K$-equivariant, $E$ is the convex hull of a $K$-orbit in $p$ and so it is a polar orbitope [6, 10]. In particular, any face of $E$ is exposed [6].

Let $F$ be a face of $E$. By Lemma [12] there exists a chain of faces

$$F = F_0 \subset F_1 \subset \cdots \subset F_k \subset E.$$ 

Since any face is exposed, there exists $\beta_0, \beta_1, \ldots, \beta_k \in p$ such that

$$F_1 = \text{Max}_E(\beta_i) := \{z \in E : \langle z, \beta_i \rangle = \max_{y \in E} \langle y, \beta_i \rangle \}$$

Applying Theorem [3] we get the following result.

**Proposition 4.** Given a chain of faces $F = F_0 \subset F_1 \subset \cdots \subset F_k \subset E$, there exist two chains of submanifolds

$$\text{Max}_O(\beta_0) \subset \text{Max}_O(\beta_1) \subset \cdots \subset \text{Max}_O(\beta_k) \subset \mathbb{P}(\mathbb{C}^n)$$

$$\text{Max}_O(\beta_0) \subset \text{Max}_O(\beta_1) \subset \cdots \subset \text{Max}_O(\beta_k) \subset \mathbb{P}(\mathbb{R}^n)$$

such that the vertical inclusions are Lagrangian and totally geodesic immersions.

In [6], the authors proved that the face structure of $E$, up to $K$-equivalence, is completely determined by the face structure of $P = E \cap a = \mu_a(O)$, where $a \subset p$ is a maximal Abelian subalgebra, up to $W = N_k(a) = \{k \in K : \text{Ad}(k)(a) = a\}$-equivalence. We recall that $W$ is called the Weyl group and it acts isometrically on $a$ as a finite group [35]. By a Theorem of Kostant [37], $P$ is the convex hull of a Weyl group orbit and so it is a polytope [42]. By Proposition [15], $\mu_a(\mathbb{P}(\mathbb{R}^n)) = \mu_a(O)$. The centralizer of $a$ in $g$ is given by

$$\mathfrak{z}(a) = m \oplus a,$$

where $m = \mathfrak{z}(a) \cap \mathfrak{k}$. If $a' \subset m$ is a maximal Abelian subalgebra of $m$, then $a' + ia \subset u = \mathfrak{t} + i\mathfrak{p}$ is a maximal Abelian subalgebra of $u$ and so $(a' + ia)^C \subset g^C = u^C$ is a Cartan subalgebra. Given $a, a'$, and $\Pi \subset \Delta(g, a)$ be a basis one can choose a basis of $(a + ia')^*$ adapted to $\Pi$ and $(ia')^*$, see [24, p.51–52], [31, p.272–273]). Let $\tilde{\mu}_\rho$ the highest weight of $g^C$ with respect to the partial ordering determined $\tilde{\Delta}$. Let $x_o = [v_\rho]$, where $v_\rho$ is any highest weight vector. It is well-known that

$$\mu : \mathbb{P}(\mathbb{C}^n) \longrightarrow a' \oplus ia, \quad \langle \mu(x_o), \xi \rangle = \tilde{\mu}_\rho(\xi),$$

see [5] that has opposite sign convection for $\mu$, and [3]. By Proposition [39] one can choose $v_\rho \in \mathbb{R}^n$. Moreover, $G \cdot x_o = O$ and so

$$\langle \mu_a(x_o), \xi \rangle = (i\tilde{\mu}_\rho)(\xi).$$

$(i\tilde{\mu}_\rho)|_a$ is the highest weight of $g$ with respect to the induced order on $a^*$. In the sequel we denote by $\mu_\rho = (i\tilde{\mu}_\rho)|_a$ and also its dual in $a$ with respect to the scalar product $\langle \cdot, \cdot \rangle$. 

Proposition 5. \( P = \text{conv}(\mathcal{W} \cdot \mu_\rho) \). In particular the weights of \( \rho \) are contained in the convex hull of the Weyl group orbit through the highest weight \( \mu_\rho \).

Let \( \beta \in \mathfrak{p} \). Then
\[
F_\beta(\mathcal{E}) = \{ p \in \mathcal{E} : \langle p, \beta \rangle = \max_{y \in \mathcal{E}} \langle y, \beta \rangle \},
\]
is a face of \( \mathcal{E} \) and any face of \( \mathcal{E} \) is given by \( F_\beta(\mathcal{E}) \) for some \( \beta \in \mathfrak{p} \). Since \( \mu_\rho(\mathcal{O}) \) is a \( K \)-orbit, it is a fundamental fact that the action of \( K \) on \( \mu_\rho(\mathcal{O}) \) extends to an action of \( G \) see [29] Lemma 5). The set of extreme points of \( F_\beta(\mathcal{E}) \), that we denote by \( \text{ext}(F_\beta(\mathcal{E})) \), is contained in \( \mu_\rho(\mathcal{O}) \). If \( V_1 \) is the eigenspace of \( \beta \) relative to the maximal eigenvalue, then \( \mathbb{P}(V_1) = \text{Max}_{\mathbb{P}(\mathfrak{r}^\mathfrak{r}^\mathfrak{c})}(\beta) = \mu_\rho^{-1}(F_\beta(\mathcal{E})) \).

We prove that
\[
\{ g \in G : gV_1 = V_1 \} = Q(F_\beta(\mathcal{E})) = \{ h \in G : \text{hext} \ F_\beta(\mathcal{E}) = \text{ext} \ F_\beta(\mathcal{E}) \},
\]
is a parabolic subgroup of \( G \) which it contains \( G^{\beta^+} \). Moreover, the \( Q(F_\beta(\mathcal{E})) \)-action on \( V_1 \) is irreducibly and \( \text{Max}_{\mathcal{O}}(\beta) \) is the unique compact orbit of the \( Q(F_\beta(\mathcal{E})) \)-action on \( \mathcal{O} \).

The group \( K \) acts on the set of faces of \( \mathcal{E} \). Up to this \( K \)-action, a face of \( \mathcal{E} \) is described in terms of root data [6]. The main tool is the notion of \( \mu_\rho \)-connected subset of the set of positive roots \( \Pi \) with respect to a fixed order. This notion was introduced by Satake in the study of the boundary components of the Satake compactifications of a symmetric space of noncompact type [41].

A subset \( I \subset \Pi \) is \( \mu_\rho \)-connected if \( I \cup \{ \mu_\rho \} \) is connected, i.e., it is not the union of subsets orthogonal with respect to the Killing form. We denote by \( I' \) the collection of all simple roots orthogonal to \( \{ \mu_\rho \} \cup I \). The set \( J := I \cup I' \) is called the \( \mu_\rho \)-saturation of \( I \). Given \( I \), we consider the standard parabolic subalgebras \( \mathfrak{q}_I \) and \( \mathfrak{q}_J \), respectively, see Section 8 and [15, 41]. We denote by \( Q_I \), respectively \( Q_J \), the parabolic subgroup of \( G \) with Lie algebra \( \mathfrak{q}_I \), respectively with Lie algebra \( \mathfrak{q}_J \). If \( Q_I = G^{\beta^+} \) then \( Q_J = Q(F_\beta(\mathcal{E})) \) [9] and both \( Q_I \) and \( Q_J \) act irreducibly on the eigenspace of \( \beta \) associated to the maximum eigenvalue, that we denote by \( W_I \). Satake proved that \( W_I \) can be defined in terms of root data [15, Lemma I.4.25, p. 69] and the \( Q_J \)-action on \( W_I \) is completely determined by the \( Q_I \)-action on \( W_I \).

Let \( Q \) be a parabolic subgroup and let \( W \) be the unique subspace of \( V \) such that \( Q \) acts irreducibly on it. We show there exists \( k \in K \) such that \( kW = W_I \) and
\[
Q_I \subseteq kQk^{-1} \subseteq Q_J.
\]
This means that up to the \( K \)-action, boundary components of \( \mathcal{E} \) completely describe the irreducible representations of the parabolic subgroups of \( G \) induced by \( \rho : G \rightarrow \text{SL}(n, \mathbb{R}) \). We also prove that irreducible representations of parabolic subgroups of \( G \) induced by \( \rho : G \rightarrow \text{SL}(n, \mathbb{C}) \) are the complexification of the irreducible representations of parabolic subgroups of \( G \) induced by \( \rho : G \rightarrow \text{SL}(n, \mathbb{R}) \). Hence, we get the following result.

Theorem 6. The face structure of \( \mathcal{E} \), up to \( K \)-equivalence, describes the irreducible representations of the parabolic subgroups of \( G \) induced by \( \rho : G \rightarrow \text{SL}(n, \mathbb{R}) \) and so induced also by \( \rho : G \rightarrow \text{SL}(n, \mathbb{C}) \).
In the sequel we always refer to Section I.1 Real Parabolic subgroups and Section II.3.

Let \( I \subset \Pi \) be a \( \mu_\rho \)-connected subset. By Theorem 60 see [6], \( Q_I \cdot \mu_\rho \) is the set of extreme points of a face of \( \mathcal{E} \) that we denote by \( F_I \). Although the \( G \)-gradient map is not \( G \)-equivariant, by Proposition 50 we get

\[
\mu_\rho(Q_I \cdot x_o) = Q_I \cdot \mu_\rho.
\]

Let \( a_I := \bigcap_{a \in I} \ker \alpha \) and let \( a' \) be the orthogonal complement of \( a_I \) in \( a \). Then \( q_I = n_I \oplus a_I \oplus m_I \), where \( m_I = \mathfrak{z}(a) \oplus a' \oplus \bigoplus_{\alpha \in I} \mathfrak{g}_\alpha \) is the Lie algebra of a Levi factor of \( Q_I \), that we denote by \( M_I \) which is not connected in general. We recall that \( g_\alpha := \{ v \in g : [H, v] = \alpha(H)v, \forall H \in a \} \). \( M_I \) is compatible and \( K_I = K \cap Q_I \) is a maximal compact subgroup of \( M_I \). The Abelian subalgebra \( a' \) is a maximal Abelian subalgebra of \( m_I \cap \mathfrak{p} \). Let \( \mathcal{W}_I = N_{K_I}(a_I) \). \( \mathcal{W}_I \) is the subgroup of \( W \) generated by the root reflections defined by the element of \( I \). We split \( \mu_\rho = y_0 + y_1 \), where \( y_0 \in a_I \) and \( y_1 \in a' \).

**Theorem 7.** The map \( I \mapsto \operatorname{conv}(\mathcal{W}_I \cdot \mu_\rho) \) induces a bijection between the \( \mu_\rho \)-connected subset of \( \Pi \) and the faces of \( P \) up to the Weyl-group action. Moreover,

\[
\mu_\rho(Q_I \cdot x_o) = \operatorname{conv}(\mathcal{W}_I \cdot \mu_\rho) = y_0 + \operatorname{conv}(\mathcal{W}_I \cdot y_1).
\]

The description of the faces of \( P \) is proved in [6], see [36, Theorem 6.2]. The above statement is quoted from Casselman [16, Theorem 3.1], where it is proved in the more general context of arbitrary finite Coxeter groups. Casselman also pointed out that the result is already implicit in the papers [11] and [13]. Our proof uses the description of the faces of \( P \) given in [6] and the techniques of the \( G \)-gradient map.

We also investigate the norm square gradient map and the norm square momentum map.

Let

\[
\nu_p : \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}, \quad p \mapsto \frac{1}{2} \| \mu_\rho(p) \|^2,
\]

denote the norm square gradient map and let

\[
\nu_u : \mathbb{P}(\mathbb{C}^n) \to \mathbb{R}, \quad p \mapsto \frac{1}{2} \| \mu(p) \|^2,
\]

denote the norm square momentum map. The gradient of \( \nu_p \) with respect to the Riemannian metric induced by the Fubini-Study metric on \( \mathbb{P}(\mathbb{C}^n) \) is given by

\[
\mathfrak{grad} \nu_p(x) = \mu(x)_{\mathbb{P}(\mathbb{C}^n)},
\]

where \( \mu(x)_{\mathbb{P}(\mathbb{C}^n)} := \frac{d}{dt} \Big|_{t=0} \exp(t \mu_p(x))x \) [28]. The gradient of \( \nu_u \) is given by

\[
\mu(x)_{\mathbb{P}(\mathbb{C}^n)} := \frac{d}{dt} \Big|_{t=0} \exp(t \mu(x))x \] [31]. We point out that \( x \in \mathbb{P}(\mathbb{R}^n) \) is a critical point of \( \nu_p \) if and only if \( x \in \mathbb{P}(\mathbb{R}^n) \) is a critical point of \( \nu_u \), see also [32], and

\[
\operatorname{Max}_{x \in \mathbb{P}(\mathbb{R}^n)} \nu_p = \operatorname{Max}_{x \in \mathbb{P}(\mathbb{C}^n)} \nu_u, \quad \operatorname{Inf}_{x \in \mathbb{P}(\mathbb{R}^n)} \nu_p = \operatorname{Inf}_{x \in \mathbb{P}(\mathbb{C}^n)} \nu_u.
\]

The negative gradient flow line of \( \nu_p \) through \( x_0 \in \mathbb{P}(\mathbb{R}^n) \) is the solution of the differential equation

\[
\begin{cases}
\dot{x}(t) = -\beta_{\mathbb{P}(\mathbb{R}^n)}(x(t)), & t \in \mathbb{R} \\
x(0) = x_0.
\end{cases}
\]
It is defined for any \( t \in \mathbb{R} \) and the limit
\[
x_\infty := \lim_{t \to \infty} x(t)
\]
exist \cite[Theorem 3.3]{12}, see also \cite{18} for the complex case. Applying results proved in \cite[Theorem 4.7]{12}, we get the following results.

**Proposition 8.** Let \( x_0 \in \mathbb{P}(\mathbb{R}^n) \). Then
\[
\| \mu_p(x_\infty) \| = \inf_{g \in G} \| \mu_p(g x_0) \|
\]
\[
= \inf_{g \in G^c} \| \mu(g x_0) \|.
\]
Let \( x_0 \in \mathbb{P}(\mathbb{C}^n) \) be such that \( U^c \cdot x_0 \cap \mathbb{P}(\mathbb{R}^n) \neq \emptyset \). Then
\[
\inf_{g \in U^c} \| \mu(g x_0) \| = \inf_{y \in U^c \cdot x_0 \cap \mathbb{P}(\mathbb{R}^n)} \| \mu_p(y) \|.
\]

We also investigate the stratification of \( \mathbb{P}(\mathbb{R}^n) \) with respect to \( \nu_p \).

**Proposition 9.** The norm square gradient map \( \nu_p \) has a unique open stratum which is the minimal stratum. This stratum is open, dense and it is given by the intersection of \( \mathbb{P}(\mathbb{R}^n) \) with the minimal stratum of \( \nu_u \).

In principle there could be many different open stratum for the norm square gradient map but we do not know any example. Any such example would imply that the nonAbelian convexity Theorem fails \cite{30}.

**2. Preliminaries**

**2.1. Convex geometry.** It is useful to recall a few definitions and results regarding convex sets. The reader may refer for instance to \cite{42} for more details.

Let \( V \) be a real vector space with a scalar product \( \langle \cdot, \cdot \rangle \) and let \( E \subset V \) be a compact convex subset. The relative interior of \( E \), denoted \( \text{relint} E \), is the interior of \( E \) in its affine hull. A face \( F \) of \( E \) is a convex subset \( F \subset E \) with the following property: if \( x, y \in E \) and \( \text{relint}[x, y] \cap F \neq \emptyset \), then \( [x, y] \subset F \). The extreme points of \( E \) are the points \( x \in E \) such that \( \{x\} \) is a face. We denote by \( \text{ext} E \) the set of the extreme points of \( E \). By a Theorem of Minkowski \cite[Corollary 1.4.5 p.19]{42}, \( E \) is the convex hull of its extremal points. Since \( E \) is compact the faces are closed \cite[p. 62]{42}. A face distinct from \( E \) and \( \emptyset \) will be called a proper face. The support function of \( E \) is the function \( h_E : V \to \mathbb{R}, h_E(u) = \max_{x \in E} \langle x, u \rangle \). If \( u \neq 0 \), the hyperplane \( H(E, u) := \{x \in E : \langle x, u \rangle = h_E(u)\} \) is called the supporting hyperplane of \( E \) for \( u \). The set
\[
F_u(E) := E \cap H(E, u)
\]
is a face and it is called the exposed face of \( E \) defined by \( u \). In general not all faces of a convex subset are exposed. A simple example is given by the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are non-exposed faces

**Lemma 11** (\cite[Lemma 3]{7}). If \( F \) is a face of a convex set \( E \), then \( \text{ext} F = F \cap \text{ext} E \).
Lemma 12 ([7, Lemma 8]). If $E$ is a compact convex set and $F \subset E$ is a face, then there is a chain of faces $F_0 = F \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$ which is maximal, in the sense that for any $i$ there is no face of $E$ strictly contained between $F_{i-1}$ and $F_i$.

Lemma 13 ([7, Prop. 5]). If $F \subset E$ is an exposed face, the set $C_F := \{ u \in V : F = F_u(E) \}$ is a convex cone. If $K$ is a compact subgroup of $O(V)$ that preserves both $E$ and $F$, then $C_F$ contains a fixed point of $K$.

We denote by $C^K_F$ the elements of $C_F$ fixed by a compact group $K$. The faces of a convex compact set give a stratification of $E$. The following result is well-known and a proof is given in [42, p. 62]

Theorem 14. If $E$ is a compact convex set and $F_1, F_2$ are distinct faces of $E$, then $\text{relint } F_1 \cap \text{relint } F_2 = \emptyset$. If $G$ is a nonempty convex subset of $E$ which is open in its affine hull, then $G \subset \text{relint } F$ for some face $F$ of $E$. Therefore $E$ is the disjoint union of the relative interiors of its faces.

The following result is probably well-known.

Proposition 15. Let $C_1 \subseteq C_2$ be two compact convex subsets of $V$. Assume that for any $\beta \in V$ we have

$$\max_{y \in C_1} \langle y, \beta \rangle = \max_{y \in C_2} \langle y, \beta \rangle.$$

Then $C_1 = C_2$.

Proof. We may assume without loss of generality that the affine hull of $C_2$ is $V$. Assume by contradiction that $C_1 \subsetneq C_2$. Since $C_1$ and $C_2$ are both compact, it follows that there exists $p \in \partial C_1$ such that $p \in C_2$. Since every face of a compact convex set is contained in an exposed face [42], there exists $\beta \in V$ such that

$$\max_{y \in C_1} \langle y, \beta \rangle = \langle p, \beta \rangle.$$

This means the linear function $x \mapsto \langle x, \beta \rangle$ restricted on $C_2$ achieves its maximum at an interior point which is a contradiction. □

Let $E$ be a $K$-invariant convex body of $\mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra of $\mathfrak{p}$ and let $W := \{ \text{Ad}(k) : k \in K \text{ and } \text{Ad}(k)(\mathfrak{a}) = \mathfrak{a} \}$. $W$ is called Weyl group and it acts on $\mathfrak{a}$ as a finite group [35]. Let $P = E \cap \mathfrak{a}$ and let $\pi_\mathfrak{a} : \mathfrak{p} \rightarrow \mathfrak{a}$ be the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$. The following result is proved in [19], see also [36].

Theorem 16. $P$ is a $W$-invariant convex body of $\mathfrak{a}$ satisfying $\pi_\mathfrak{a}(E) = P$ and $E = KP$. Hence if $E_1, E_2 \subset \mathfrak{p}$ are two $K$-invariant convex bodies of $\mathfrak{p}$ then $E_1 \cap \mathfrak{a} = E_2 \cap \mathfrak{a}$ if and only if $E_1 = E_2$.

The $W$-action on $P$ induces an action on the faces of $P$. Similarly $K$ acts on the set of faces of $E$. Denote these sets by $\mathcal{F}(P)$ respectively by $\mathcal{F}(E)$. Let $\sigma$ be a face of $P$. Then its affine hull is given by $x_o + W$, where $W \subset \mathfrak{p}$ is a subspace. We denote by $\sigma^\perp$ the orthogonal of $W$. The following result is proved in [8].
Theorem 17. The map $\mathcal{F}(P) \rightarrow \mathcal{F}(E), \sigma \mapsto K^{\sigma^\perp} \cdot \sigma$ is well-defined and induces a bijection between $\mathcal{F}(P)/\mathcal{W}$ and $\mathcal{F}(E)/K$. Moreover, $\sigma$ is an exposed face if and only if $K^{\sigma^\perp} \cdot \sigma$ does.

3. Compatible subgroups and parabolic subgroups

In the sequel we always refer to [15, 24, 28, 35].

Let $U$ be compact connected Lie group. Let $U^C$ be its universal complexification which is a linear reductive complex algebraic group [1]. We denote by $\theta$ both the conjugation map $\theta : u^C \rightarrow u^C$ and the corresponding group isomorphism $\theta : U^C \rightarrow U^C$. Let $f : U \times iu \rightarrow U^C$ be the diffeomorphism $f(g, \xi) = g \exp \xi$. Let $G \subset U^C$ be a closed subgroup. Set $K := G \cap U$ and $\mathfrak{p} := g \cap iu$. We say that $G$ is compatible if $f(K \times \mathfrak{p}) = G$. The restriction of $f$ to $K \times \mathfrak{p}$ is then a diffeomorphism onto $G$. Hence $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ is the familiar Cartan decomposition and so $K$ is a maximal compact subgroup of $G$. Note that $G$ has finitely many connected components. Since $U$ can be embedded in $\text{GL}(N, \mathbb{C})$ for some $N$, and any such embedding induces a closed embedding of $U^C$, any compatible subgroup is a closed linear group. Moreover $\mathfrak{g}$ is a real reductive Lie algebra, hence $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Denote by $G_{ss}$ the analytic subgroup tangent to $[\mathfrak{g}, \mathfrak{g}]$. Then $G_{ss}$ is closed and $G^o = Z(G)^o \cdot G_{ss}$ [35, p. 442], where $G^o$, respectively $Z(G)^o$, denotes the connected component of $G$, respectively of $Z(G)$, containing $e$. The following lemma is well-known. A proof is given in [39, pag.584].

Lemma 18.

a) If $G \subset U^C$ is a compatible subgroup, and $H \subset G$ is closed and $\theta$-invariant, then $H$ is compatible if and only if $H$ has only finitely many connected components.

b) If $G \subset U^C$ is a connected compatible subgroup, then $G_{ss}$ is compatible.

c) If $G \subset U^C$ is a compatible subgroup, and $E \subset \mathfrak{p}$ is any subset, then

$$G^E = \{g \in G : \text{Ad}(g)(z) = z, \forall z \in E\}$$

is compatible. Indeed, $G^E = K^E \exp(p^E)$, where $K^E = G^E \cap K$ and $p^E = \{v \in \mathfrak{p} : [v, E] = 0\}$.

A subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is parabolic if $\mathfrak{q}^C$ is a parabolic subalgebra of $\mathfrak{g}^C$. One way to describe the parabolic subalgebras of $\mathfrak{g}$ is by means of restricted roots. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal subalgebra, let $\Delta(\mathfrak{g}, \mathfrak{a})$ be the (restricted) roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, let $\mathfrak{g}_\lambda$ denote the root space corresponding to $\lambda$ and let $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m} = \mathfrak{z}(\mathfrak{a}) = \mathfrak{z}(\mathfrak{a}) \cap \mathfrak{t}$. We denote by $\mathfrak{z}(\mathfrak{a}) = \{x \in \mathfrak{g} : [x, \mathfrak{a}] = 0\}$. Let $\Pi \subset \Delta(\mathfrak{g}, \mathfrak{a})$ be a base and let $\Delta_+$ be the set of positive roots. If $I \subset \Pi$, set $\Delta_I := \text{span}(I) \cap \Delta$. Then

$$(19) \quad \mathfrak{q}_I := \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta_I \cup \Delta_+} \mathfrak{g}_\lambda$$

is a parabolic subalgebra. Conversely, if $\mathfrak{q} \subset \mathfrak{g}$ is a parabolic subalgebra, then there are a maximal subalgebra $\mathfrak{a} \subset \mathfrak{p}$ contained in $\mathfrak{q}$, a base $\Pi \subset \Delta(\mathfrak{g}, \mathfrak{a})$ and a subset $I \subset \Pi$ such that
q = q_I. We can further introduce

\[ a_I := \bigcap_{\lambda \in I} \ker \lambda \quad a^I := a_I^\perp \]

(20)

\[ n_I = \bigoplus_{\lambda \in \Delta_- \setminus \Delta_I} g_{\lambda} \quad m_I := m \oplus a^I \oplus \bigoplus_{\lambda \in \Delta_I} g_{\lambda}. \]

Then \( q_I = m_I \oplus a_I \oplus n_I \). Since \( \theta g_{\lambda} = g_{-\lambda} \), it follows that \( q_I \cap \theta q_I = a_I \oplus m_I \). This latter Lie algebra coincides with the centralizer of \( a_I \) in \( g \). It is a Levi factor of \( q_I \) and

\[ a_I = z(q_I \cap \theta q_I) \cap p. \]

(21)

If we denote by \( \Delta_- \) the set of negative root, then \( n_I^- = \bigoplus_{\lambda \in \Delta_- \setminus \Delta_I} g_{\lambda} \) is a subalgebra. It follows from standard commutation relations that \( z(a_I) \) normalizes \( n_I \) and \( n_I^- \) and the centralizer of \( a^I \) in either is reduced to zero. Then, keeping in mind \( g = n_I^- \oplus q_I \), \( q_I \) is self-normalizing.

**Definition 22.** A subgroup \( Q \) of \( G \) is called parabolic if it is the normalizer of a parabolic subalgebra in \( g \).

The normalizer of \( q_I \) is the standard parabolic subalgebra \( Q_I \). Let \( R_I \) and let \( A_I \) be the unique connected Lie subgroups of \( G \) with Lie algebra equals to \( n_I \) and \( a_I \) respectively. \( R_I \) is the unipotent radical of \( Q_I \). The group \( Q_I \) is the semidirect product of \( R_I \) and of \( Z(A_I) \), i.e., the centralizer of \( A_I = \exp(a_I) \) in \( G \). Moreover, \( Z(A_I) = A_I \times M_I \), where \( M_I \) is a closed Lie group whose Lie algebra is \( m_I \). It is not connected in general but it is compatible. Since \( M_I \) is stable with respect to the Cartan involution, \( K_I = M_I \cap K \) is maximal compact in \( M_I \). It is also maximal compact in \( Q_I \) and the quotient

\[ X_I = M_I/K_I = Q_I/K_IA_IN_I \]

is a symmetric space of noncompact type for \( M_I \). Finally, as a consequence of the Iwasawa decomposition \( G = NAK \), where \( N = \exp(n) \), \( n = n_0 \), and \( NA \subset Q_I \), we get the following result

**Proposition 23.** \( G = KQ_I \).

Another way to describe parabolic subalgebras of \( g \) is the following. If \( \beta \in p \), the endomorphism \( \text{ad}\beta \in \text{End} g \) is diagonalizable over \( \mathbb{R} \). Denote by \( V_{\lambda}(\text{ad}\beta) \) the eigenspace of \( \text{ad}\beta \) corresponding to the eigenvalue \( \lambda \). Set

\[ g^{\beta^+} := \bigoplus_{\lambda \geq 0} V_{\lambda}(\text{ad}\beta). \]

The following result is proved in [6].

**Lemma 24.** For any \( \beta \) in \( p \), \( g^{\beta^+} \) is a parabolic subalgebra of \( g \). If \( q \subset g \) is a parabolic subalgebra, there is some vector \( \beta \in p \) such that \( q = g^{\beta^+} \). The set of all such vectors is an open convex cone in \( z(q \cap \theta q) \cap p \).
A parabolic subgroup of $G$ is a subgroup of the form $Q = N_G(q)$ where $q$ is a parabolic subalgebra of $g$. Equivalently, a parabolic subgroup of $G$ is a subgroup of the form $Q \cap G$ where $Q$ is a parabolic subgroup of $G^\C$ and $\tilde{q}$ is the complexification of a subspace $q \subset g$.

If $\beta \in p$ set
\[ G^{\beta^+} := \{ g \in G : \lim_{t \to -\infty} \exp(t\beta)g \exp(-t\beta) \text{ exists} \} \]
\[ R^{\beta^+} := \{ g \in G : \lim_{t \to -\infty} \exp(t\beta)g \exp(-t\beta) = e \} \]
\[ v^{\beta^+} := \bigoplus_{\lambda \geq 0} V_{\lambda}(\text{ad}\beta). \]

The following result characterizes completely the parabolic subgroups of $G$. The result is classical and a proof is given in [6].

**Lemma 25.** $G^{\beta^+}$ is a parabolic subgroup of $G$ with Lie algebra $g^{\beta^+}$ and it is the semidirect product of $G^\beta$ with $R^{\beta^+}$. Moreover, $G^{\beta}$ is a Levi factor, $R^{\beta^+}$ is connected and it is the unipotent radical of $G^{\beta^+}$. Finally, every parabolic subgroup of $G$ equals $G^{\beta^+}$ for some $\beta \in p$.

### 3.1. Basic properties of the gradient map

Let $(Z, \omega)$ be a Kähler manifold. Assume that $U^\C$ acts holomorphically on $Z$, that $U$ preserves $\omega$ and that there is a momentum map $\mu : Z \to u$. If $\xi \in u$ we denote by $\xi \omega$ the induced vector field on $Z$, i.e., $\xi \omega (p) = \frac{d}{dt} \big|_{t=0} \exp(t\xi)p$, and we let $\mu^\xi \in C^\infty(Z)$ be the function $\mu^\xi(z) := \langle \mu(z), \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is an $\text{Ad}(U)$-invariant scalar product on $u$. That $\mu$ is the momentum map means that it is $U$-equivariant and that $d\mu^\xi = i_{\xi\omega} \omega$.

Let $G \subset U^\C$ be compatible. If $z \in Z$, let $\mu_p (z) \in p$ denote $-i$ times the component of $\mu(z)$ in the direction of $i\xi$. In other words, if we also denote by $\langle \cdot, \cdot \rangle$ the $\text{Ad}(U)$-invariant scalar product on $iu$ requiring the multiplication by $i$ is an isometry of $u$ onto $iu$, then $\langle \mu_p(z), \beta \rangle = (i\mu(z), \beta) = \langle \mu(z), -i\beta \rangle$ for any $\beta \in p$, defines the $G$-gradient map
\[ \mu_p : Z \to p. \]

Let $\mu_p^\beta \in C^\infty(Z)$ be the function $\mu_p^\beta(z) = \langle \mu_p(z), \beta \rangle = \mu^{-i\beta}(z)$. Let $\langle \cdot, \cdot \rangle$ be the Kähler metric associated to $\omega$, i.e. $\langle v, w \rangle = \omega(v, Jw)$. Then $\beta \omega$ is the gradient of $\mu_p^\beta$. If $M \subset Z$ is a locally closed $G$-invariant submanifold, then $\beta_M$ is the gradient of $\mu_p^\beta|_M$ with respect to the induced Riemannian structure on $M$. From now on we always assume that $M$ is compact and connected.

**Theorem 26.** [Slice Theorem [28] Thm. 3.1] If $x \in M$ and $\mu_p(x) = 0$, there are a $G_x$-invariant decomposition $T_x M = g \cdot x \oplus W$, open $G_x$-invariant subsets $S \subset W$, $\Omega \subset M$ and a $G$-equivariant diffeomorphism $\Psi : G \times G_x S \to \Omega$, such that $0 \in S$, $x \in \Omega$ and $\Psi([e, 0]) = x$.

Here $G \times G_x S$ denotes the associated bundle with principal bundle $G \to G/G_x$.

**Corollary 27.** If $x \in M$ and $\mu_p(x) = \beta$, there are a $G^\beta$-invariant decomposition $T_x M = g^\beta \cdot x \oplus W$, open $G^\beta$-invariant subsets $S \subset W$, $\Omega \subset M$ and a $G^\beta$-equivariant diffeomorphism $\Psi : G^\beta \times G_x S \to \Omega$, such that $0 \in S$, $x \in \Omega$ and $\Psi([e, 0]) = x$.

This follows applying the previous theorem to the action of $G^\beta$ with the gradient map $\tilde{\mu}_u^\beta := \mu_u^\beta - i\beta$, where $\mu_u^\beta$ denotes the projection of $\mu$ onto $u^\beta$. See [28] p.169 and [43] for more details.
If $\beta \in \mathfrak{p}$, then $\beta_M$ is a vector field on $M$, i.e., a section of $TM$. For $x \in M$, the differential is a map $T_xM \to T_{\beta_M(x)}(TM)$. If $\beta_M(x) = 0$, there is a canonical splitting $T_{\beta_M(x)}(TM) = T_xM \oplus T_xM$. Accordingly $d\beta_M(x)$ splits into a horizontal and a vertical part. The horizontal part is the identity map. We denote the vertical part by $d\beta_M(x)$. It belongs to $\text{End}(T_xM)$. Let $\{\varphi_t = \exp(t\beta)\}$ be the flow of $\beta_M$. There is a corresponding flow on $TM$. Since $\varphi_t(x) = x$, the flow on $TM$ preserves $T_xM$ and there it is given by $d\varphi_t(x) \in \text{GL}(T_xM)$. Thus we get a linear $\mathbb{R}$-action on $T_xM$ with infinitesimal generator $d\beta_M(x)$.

**Corollary 28.** If $\beta \in \mathfrak{p}$ and $x \in M$ is a critical point of $\mu^\beta_p$, then there are open invariant neighborhoods $S \subset T_xM$ and $\Omega \subset M$ and an $\mathbb{R}$-equivariant diffeomorphism $\Psi : S \to \Omega$, such that $0 \in S$, $x \in \Omega$, $\Psi(0) = x$. Here $t \in \mathbb{R}$ acts as $d\varphi_t(x)$ on $S$ and as $\varphi_t$ on $\Omega$.

**Proof.** The subgroup $H := \exp(\mathbb{R}\beta)$ is compatible. It is enough to apply the previous corollary to the $H$-action at $x$. \hfill $\square$

Let $x \in \text{Crit}(\mu^\beta_p) = \{y \in M : \beta_M(y) = 0\}$. Let $D^2\mu^\beta_p(x)$ denote the Hessian, which is a symmetric operator on $T_xM$ such that

$$(D^2\mu^\beta_p(x)v, v) = \frac{d^2}{dt^2}(\mu^\beta_p \circ \gamma)(0)$$

where $\gamma$ is a smooth curve, $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Denote by $V_-$ (respectively $V_+$) the sum of the eigenspaces of the Hessian of $\mu^\beta_p$ corresponding to negative (resp. positive) eigenvalues. Denote by $V_0$ the kernel. Since the Hessian is symmetric we get an orthogonal decomposition

$$(29) \quad T_xM = V_- \oplus V_0 \oplus V_+.$$

Let $\alpha : G \to M$ be the orbit map: $\alpha(g) := gx$. The differential $d\alpha_e$ is the map $\xi \mapsto \xi_M(x)$. The following result is well-known. A proof is given in [6].

**Proposition 30.** If $\beta \in \mathfrak{p}$ and $x \in \text{Crit}(\mu^\beta_p)$ then

$$D^2\mu^\beta_p(x) = d\beta_M(x).$$

Moreover $d\alpha_e(e^{\beta_\pm}) \subset V_\pm$ and $d\alpha_e(e^\beta) \subset V_0$. If $M$ is $G$-homogeneous these are equalities.

**Corollary 31.** For every $\beta \in \mathfrak{p}$, $\mu^\beta_p$ is a Morse-Bott function.

**Proof.** Corollary [28] implies that $	ext{Crit}(\mu^\beta_p)$ is a smooth submanifold. Since $T_x\text{Crit}(\mu^\beta_p) = V_0$ for $x \in \text{Crit}(\mu^\beta_p)$, the first statement of Proposition [30] shows that the Hessian is nondegenerate in the normal directions. \hfill $\square$

**Corollary 32.** If $M$ is $G$-homogeneous then $G^\beta$-orbits are open and closed in $\text{Crit}\mu^\beta_p$.

**Proof.** Since $T_x\text{Crit} \mu^\beta_p = T_xG^\beta \cdot x$ for $x \in \text{Crit} \mu^\beta_p$, the result follows. \hfill $\square$

Let $c_1 > \cdots > c_r$ be the critical values of $\mu^\beta_p$. The corresponding level sets of $\mu^\beta_p$, $C_i := (\mu^\beta_p)^{-1}(c_i)$ are submanifolds which are union of components of $\text{Crit}(\mu^\beta_p)$. The function $\mu^\beta_p$ defines
a gradient flow generated by its gradient which is given by $\beta_M$. By Theorem 26 it follows that for any $x \in M$ the limit:

$$\varphi_\infty(x) := \lim_{t \to +\infty} \exp(t\beta)x,$$

exists. Let us denote by $W_i^\beta$ the unstable manifold of the critical component $C_i$ for the gradient flow of $\mu_p^\beta$:

$$W_i^\beta := \{ x \in M : \varphi_\infty(x) \in C_i \}.$$

Applying Theorem 26 we have the following well-known decomposition of $M$ into unstable manifolds with respect to $\mu_p^\beta$.

**Theorem 34.** In the above assumption, we have

$$M = \bigsqcup_{i=1}^r W_i^\beta,$$

and for any $i$ the map:

$$(\varphi_\infty)|_{W_i} : W_i^\beta \to C_i,$$

is a smooth fibration with fibres diffeomorphic to $\mathbb{R}^{l_i}$ where $l_i$ is the index (of negativity) of the critical submanifold $C_i$.

Let $\beta \in \mathfrak{p}$ and let $\text{Max}_M(\beta) := \{ x \in M : \mu_p^\beta(x) = \max_{y \in M} \mu_p^\beta \}$. By Proposition 30 $\text{Max}_M(\beta)$ is a smooth, possibly disconnected, locally closed submanifold of $M$. The following result is proved in [10, Lemma 30 and Proposition 31].

**Proposition 36.** $\text{Max}_M(\beta)$ is $G^\beta+$-invariant. Moreover, $R^\beta+$ acts trivially on $\text{Max}_M(\beta)$ and the $G^\beta+$-action on $\text{Max}_M(\beta)$ admits a compact orbit which is also a $K^\beta$-orbit.

Using an $\text{Ad}(K)$-invariant inner product of $\mathfrak{p}$, we define $\nu_p(z) := \frac{1}{2} \| \mu_p(z) \|^2$. The function $\nu_p$ is $K$-invariant and it is called the norm square function. In [28] (see Corollary 6.11 and Corollary 6.12 p. 21) the following result is proved.

**Proposition 37.** Let $x \in M$. Then:

- if $\nu_p$ restricted to $G \cdot x$ has a local maximum at $x$, then $G \cdot x = K \cdot x$
- if $G \cdot x$ is compact, then $G \cdot x = K \cdot x$

A strategy to analyzing the $G$-action on $M$ is to view $\nu_p$ as generalized Morse function. In [28] the authors proved the existence of a smooth $G$-invariant stratification of $M$ and they studied its properties. If $\beta \in \mathfrak{p}$ is a critical value, then we may associate a stratum $S_\beta$ which is a $G$-invariant locally closed submanifold of $M$. The stratification Theorem proves that $S_\beta$ only depends upon $K \cdot \beta$ which is called critical orbit. Indeed, $S_\beta = S_{\beta'}$ if and only if $K \cdot \beta = K \cdot \beta'$ and

$$M = \bigsqcup_{\beta} S_\beta,$$

where $\beta$ runs through a complete set of representative $K$-orbits in the set of critical value.
Let \( a \subset p \) be an Abelian subalgebra. The \( A = \exp(a) \)-gradient map is given by \( \pi_a \circ \mu_p \), where \( \pi_a : p \rightarrow a \) denotes the orthogonal projection of \( p \) onto \( a \). If \( M \) is connected and compact, applying the convexity Theorem along \( A \)-orbits [2, 30], then \( \mu_a(M^A) \) is a finite set and \( \mu_a(M) \subseteq \text{conv}(\mu_a(M^A)) \), where \( M^A = \{ p \in M : A \cdot p = p \} \) [3 Proposition 3.1]. In particular \( \text{conv}(\mu_a(M)) = \text{conv}(\mu_a(M^A)) \). Therefore, if \( \mu_a(M) \) is convex then \( \mu_a(M) \) is the convex hull of \( \mu_a(M^A) \) and so a polytope [42].

Let \( E \) denote the convex hull of \( \mu_p(M) \) and let \( a \subset p \) be a maximal Abelian subalgebra. Since \( E \cap a = \pi_a(E) = \text{conv}(\mu_a(M)) \), applying Theorem [16] we get \( E = K \text{conv}(\mu_a(M)) \).

4. Projective representations of real reductive Lie groups

Let \( G \) be a connected real semisimple noncompact Lie group and let \( \rho : G \rightarrow \text{SL}(V) \) be an irreducible representation on a finite dimensional real vector space \( V \). We identify \( G \) with \( \rho(G) \subset \text{GL}(V) \) and we assume that \( G \) is closed and there exists \( K \)-invariant scalar product \( g \) on \( V \) such that \( G = K \exp(p) \), where \( K \subset \text{SO}(V, g) \), \( p = \text{Sym}_0(V, g) \cap g \) and \( g \) denotes the Lie algebra of \( G \). Here, \( \text{Sym}_0(V, g) \) denotes the set of symmetric endomorphisms with trace zero. Roughly speaking, if we identify \( V \) with \( \mathbb{R}^n \), then \( G \) is a closed and compatible subgroup of \( \text{SL}(n, \mathbb{R}) \). Hence \( G = K \exp(p) \), where \( K := G \cap \text{SO}(n) \) and \( p \subset \text{Sym}_0(n) \). In particular, \( K \) is a maximal compact subgroup of \( G \) and \( g = \mathfrak{k} \oplus \mathfrak{p} \) is the Cartan decomposition of \( G \), where \( \mathfrak{k} \) is the Lie algebra of \( K \), induced by the Cartan decomposition of \( \text{SL}(n, \mathbb{R}) \). Now, \( G \subset \text{SL}(n, \mathbb{C}) \) is compatible as well and \( \mathfrak{k} \cap i\mathfrak{p} = \{0\} \). By [26] Proposition 2 p. 4, the Zariski closure of \( G \) in \( \text{SL}(n, \mathbb{C}) \) is given by \( UC \), where \( U \) is the compact connected semisimple Lie group with Lie algebra \( u = \mathfrak{k} \oplus i\mathfrak{p} \). In particular \( G \) is a compatible real form of \( UC \). In the sequel we always assume that both the \( G \)-action on \( \mathbb{R}^n \) and the \( G \)-action on \( \mathbb{C}^n \) are irreducible. Hence the \( UC \)-action on \( \mathbb{C}^n \) is irreducible as well. We are interested to the natural projective representation of \( G \) on \( \mathbb{P}(\mathbb{R}^n) \). The \( G \)-action on \( \mathbb{P}(\mathbb{R}^n) \) admits a \( G \)-gradient map.

Let \( B : \mathfrak{sl}(n, \mathbb{C}) \times \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathbb{R} \) be the symmetric bilinear form given by

\[
B(X, Y) = \text{Re}(\text{Tr}(XY)).
\]

It is an \( \text{Ad}(\mathfrak{sl}(n, \mathbb{C})) \)-invariant and nondegenerate bilinear form. Indeed, the splitting \( \mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n) \) is \( B \)-orthogonal, \( B \) is negative definite on \( \mathfrak{su}(n) \) and positive definite on \( i\mathfrak{su}(n) \). We define \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{sl}(n, \mathbb{C}) \) as follows: \( \langle \cdot, \cdot \rangle = -B(\cdot, \cdot) \) on \( \mathfrak{su}(n) \); \( \langle \cdot, \cdot \rangle = B(\cdot, \cdot) \) on \( i\mathfrak{su}(n) \); \( \langle \mathfrak{su}(n), i\mathfrak{su}(n) \rangle = 0 \). \( \langle \cdot, \cdot \rangle \) is \( \text{Ad}(\mathfrak{SU}(n)) \)-invariant scalar product on \( \mathfrak{sl}(n, \mathbb{C}) \) such that the multiplication by \( i \) defines an isometry of \( \mathfrak{su}(n) \) onto \( i\mathfrak{su}(n) \). The \( U \)-action on \( \mathbb{P}(\mathbb{C}^n) \) is Hamiltonian with momentum map

\[
\mu : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathfrak{u}, \quad \mu = \pi_u \circ \Phi,
\]

where \( \Phi(z) = -\frac{i}{2} \left( \frac{zz^*}{||z||^2} - \frac{1}{n} \mathbb{I}_n \right) \) is the momentum map of the \( \text{SU}(n) \)-action on \( \mathbb{P}(\mathbb{C}^n) \) and \( \pi_u \) is the orthogonal projection of \( \mathfrak{su}(u) \) onto \( u \) [34]. The momentum map satisfies the following conditions:

- for any \( z \in \mathbb{P}(\mathbb{C}^n) \) and any \( g \in U \), we have \( \mu(gz) = \text{Ad}(g)(\mu(z)) \);
• If $\xi \in U$, we denote the induced vector field by $\xi_{\mathbb{P}(\mathbb{C}^n)}(p) = \frac{d}{dt} \big|_{t=0} \exp(t\xi) p$. Let $\mu^\xi \in C^\infty(Z)$ be the function $\mu^\xi(z) := \langle \mu(z), \xi \rangle$. Then $d\mu^\xi = i\xi_{\mathbb{P}(\mathbb{C}^n)} \omega$.

Let $z \in \mathbb{P}(\mathbb{C}^n)$. Then $\mu_p(z) \in p$ denotes the component of $i\mu(z)$ in the direction of $p$. In other words we require that $\langle \mu_p(z), \beta \rangle = \langle i\mu(z), \beta \rangle = \langle \mu(z), -i\beta \rangle$ for any $\beta \in p$. We have thus defined the $G$-gradient map

$$\mu_p : \mathbb{P}(\mathbb{C}^n) \rightarrow p,$$

which satisfies the following conditions:

a) $\mu_p(kz) = \text{Ad}(k)(\mu_p(z))$, for any $k \in K$ and for any $z \in \mathbb{P}(\mathbb{C}^n)$;

b) for any $\beta \in p$, let $\mu_p^\beta \in C^\infty(\mathbb{P}(\mathbb{C}^n))$ be the function $\mu_p^\beta(z) = \langle \mu_p(z), \beta \rangle = \mu^{-i\beta}(z)$. Then the gradient of $\mu_p^\beta$, with respect to the Riemannian metric induced by the Kähler structure, is given by $\beta_p \omega$.

If $X \subset \mathbb{P}(\mathbb{C}^n)$ is a connected $G$-stable real submanifold of $\mathbb{P}(\mathbb{C}^n)$, we consider $\mu_p$ as a $K$-equivariant mapping $\mu_p : X \rightarrow p$ such that for any $\beta \in p$, the gradient of the smooth function $\mu_p^\beta$ is given by $\beta_X$, where the gradient is computed with respect to the induced Riemannian metric on $X$.

**Lemma 38.** For any $z \in \mathbb{P}(\mathbb{R}^n)$ we have $\mu_p(z) = i\mu(z)$

**Proof.** Let $z \in \mathbb{P}(\mathbb{R}^n)$. Then $\Phi(z) = -\frac{i}{2} \left( \frac{z z^T}{\|z\|^2} - \frac{1}{n} \text{Id}_n \right) \in i\text{Sym}_0(n)$. Since $\langle \mathfrak{so}(n), i\text{Sym}_0(n) \rangle = 0$, it follows that $\mu(z)$ is the orthogonal projection of $\Phi$ onto $i\mathfrak{p}$ and so the result follows. \qed

**Proposition 39.** There exists $v \in \mathbb{P}(\mathbb{R}^n)$ such that $U^\mathbb{C} \cdot v$ is the unique compact orbit of the $U^\mathbb{C}$-action on $\mathbb{P}(\mathbb{C}^n)$. Moreover, $G \cdot v$ is the unique compact orbit of the $G$-action on $U^\mathbb{C} \cdot v$. It is also a $K$-orbit and a Lagrangian submanifold of $U^\mathbb{C} \cdot v$.

**Proof.** Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra. The centralizer of $\mathfrak{a}$ is $\mathfrak{g}$ is compatible and is given by $\mathfrak{z}_0(\mathfrak{a}) = \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{a}$.

Let $\mathfrak{b}$ be a maximal Abelian subalgebra of $\mathfrak{z}(\mathfrak{a})$ and let $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$. Then $\mathfrak{h}^\mathbb{C} = (\mathfrak{b} \oplus i\mathfrak{a}) \oplus i(\mathfrak{b} \oplus i\mathfrak{a})$ is a maximal Abelian subalgebra of $\mathfrak{u}^\mathbb{C}$. Since the $\mathfrak{a}$-action on $\mathbb{C}^n$ is the complexification of the $\mathfrak{a}$-action on $\mathbb{R}^n$, it follows that the eigenspaces of the $\mathfrak{a}$-action on $\mathbb{C}^n$ are the complexification of the eigenspaces of the $\mathfrak{a}$-action on $\mathbb{R}^n$. Since $U^\mathbb{C}$ acts irreducible on $\mathbb{C}^n$, the eigenspaces corresponding to the weight space is one dimensional. If $z$ is a nonzero vector belonging to the weight space, then it is an eigenvector for the $\mathfrak{a}$-action on $\mathbb{C}^n$. This implies $\pi(z) \in \mathbb{P}(\mathbb{R}^n)$, where $\pi : \mathbb{C}^n - \{0\} \rightarrow \mathbb{P}(\mathbb{C}^n)$ is the natural projection. We denote $v = \pi(z)$. By Borel-Weyl Theorem $U^\mathbb{C} \cdot v$ is the unique compact orbit of the $U^\mathbb{C}$-action on $\mathbb{P}(\mathbb{C}^n)$ which coincides with the unique complex orbit of $U_{[25]}$. Since $\mathbb{P}(\mathbb{R}^n)$ is Lagrangian and $G$ is a real form of $U^\mathbb{C}$, by [9, Proposition 9] it follows that $G \cdot v$ is compact as well and Lagrangian. By Proposition 37 $G \cdot v$ is a $K$-orbit. Finally, applying a Theorem of Wolf [44], see also [29], it follows that $G \cdot v$ is the unique compact $G$-orbit in $U^\mathbb{C} \cdot v$. \qed
In the sequel we always denote by $O'$ the unique compact orbit of the $U^C$-action on $\mathbb{P}(\mathbb{C}^n)$ and by $O$ the unique compact orbit of the $G$-action on $O'$.

**Corollary 40.** $O' \cap \mathbb{P}(\mathbb{R}^n) = O$.

*Proof.* By Proposition 37, $O'$ is a $U$-orbit and the set where the norm square momentum map achieves its maximum. Now, if $z \in \mathbb{P}(\mathbb{C}^n)$, then

$$\| \mu(z) \|^2 = \| \mu_{\mathfrak{k}}(z) \|^2 + \| \mu_{\mathfrak{p}}(z) \|^2,$$

where $\mu_{\mathfrak{k}}$ is the momentum map of the $K$-action on $\mathbb{P}(\mathbb{C}^n)$. Since $O' = U \cdot v$ and $v \in \mathbb{P}(\mathbb{R}^n)$, keeping in mind Lemma 38, $v$ achieves the maximum of the norm square gradient map as well. Moreover, if $z \in U \cdot v \cap \mathbb{P}(\mathbb{R}^n)$ then

$$\| \mu(z) \| = \| \mu(v) \| = \| \mu_{\mathfrak{p}}(z) \|,$$

and so $z$ achieves the maximum of the norm square gradient map. By Proposition 37, $G \cdot z$ is compact and it is contained in $O'$. By Proposition 39, $G$ has a unique compact orbit on $O'$ concluding the proof. $\square$

The map

$$T : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathbb{P}(\mathbb{C}^n), \quad [x] \mapsto [\overline{x}],$$

is an anti-holomorphic isometric involution. The Lie algebra $\mathfrak{u} = \mathfrak{k} \oplus i \mathfrak{p}$ is invariant with respect to the matrix conjugation induced by $T$. Hence, keeping in mind that the exponential map of $U$ is surjective, it follows $\mathcal{U} = U$. Therefore, keeping in mind that $v \in \mathbb{P}(\mathbb{R}^n)$, we get

$$O' = U^C \cdot v = U \cdot v = \overline{U} \cdot v.$$

This implies that complex conjugation $T$ induces an anti-holomorphic isometry on $O'$ whose fixed point set is given by

$$O' \cap \mathbb{P}(\mathbb{R}^n) = O.$$

Since the fixed point set of an isometry is a totally geodesic submanifold, we get the following result.

**Theorem 42.** $O$ is the fixed point set of an anti-holomorphic involutive isometry of $O'$. In particular $O$ is a totally geodesic submanifold of $O'$.

We claim that the vice-versa holds.

Let $\rho : U^C \rightarrow \text{SL}(n, \mathbb{C})$ be an holomorphic irreducible representation of a semisimple complex Lie group. Let $G$ be a noncompact real form of $U^C$. Assume there exists an anti-holomorphic involution $T$ of $\mathbb{P}(\mathbb{C}^n)$ preserving $O'$ and such that $O$ is contained in the fixed point set of $T$. Now, the application $T$ is induced by an anti-linear map $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $T^2 = \text{Id}_n$. Let $V = \text{Ker}(T - \text{Id}_n)$. Since $T \circ J = -J \circ T$, it follows that $\mathbb{C}^n = V \oplus J(V)$, and so the fixed point set of $T$ restricted on $O'$ is given by $O' \cap \mathbb{P}(V)$. This implies that $O \subseteq \mathbb{P}(V)$. Hence $G$ preserves a real subspace $W$ of $V$ and so $W^C$ is preserved by $U^C$. This implies that $W = V$ and $V^C = \mathbb{C}^n$. Summing up, we have proved the following result.
Theorem 43. In the above assumption, there exists a real subspace $V \subset \mathbb{C}^n$ such that $G$ acts irreducibly on $V$ and $V^C = \mathbb{C}^n$.

Let $\beta \in \mathfrak{p}$ and let $\mu^\beta_p : \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}$. Let $\lambda_1 > \cdots > \lambda_k$ be the eigenvalues of $\beta$. We denote by $V_1, \ldots, V_k$ the corresponding eigenspaces. In view of the orthogonal decomposition $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$, $\mu^\beta_p$ is given by

$$
\mu^\beta_p([x_1 + \cdots + x_k]) = \frac{\lambda_1 \| x_1 \|^2 + \cdots + \lambda_k \| x_k \|^2}{\| x_1 \|^2 + \cdots + \| x_k \|^2}.
$$

Therefore $\operatorname{Max}_{\mathbb{P}(\mathbb{R}^n)}(\beta) = \mathbb{P}(V_1)$. The gradient flow of $\mu^\beta_p$ is given by

$$
\mathbb{R} \times \mathbb{P}(\mathbb{R}^n) \to \mathbb{P}(\mathbb{R}^n), \quad (t, [x_1 + \cdots + x_k]) \mapsto [\exp(t \lambda_1 x_1 + \cdots + t \lambda_k x_k)].
$$

Hence the critical points of $\mu^\beta_p$ are given by $\mathbb{P}(V_1) \cup \cdots \cup \mathbb{P}(V_k)$ and the corresponding unstable manifolds are given by:

$$
W^\beta_1 = \mathbb{P}(\mathbb{R}^n) - \mathbb{P}(V_2 \oplus \cdots \oplus V_k),
$$

$$
W^\beta_2 = \mathbb{P}(V_2 \oplus \cdots \oplus V_k) - \mathbb{P}(V_3 \oplus \cdots \oplus V_k),
$$

$$
\vdots
$$

$$
W^\beta_{k-1} = \mathbb{P}(V_{k-1} \oplus V_k) - \mathbb{P}(V_k),
$$

$$
W^\beta_k = \mathbb{P}(V_k).
$$

Lemma 44. Let $M$ be a closed $G$-stable submanifold of $\mathbb{P}(\mathbb{R}^n)$ and let $\beta \in \mathfrak{p}$. Then

$$
\max_{x \in M} \mu^\beta_p = \max_{x \in \mathbb{P}(\mathbb{R}^n)} \mu^\beta_p = \max_{x \in \mathbb{P}(\mathbb{C}^n)} \mu^\beta_p = \max_{x \in \mathbb{C}^n} \mu^\beta_p.
$$

Proof. If $W^\beta_1 \cap M = \emptyset$, then $M \subset \mathbb{P}(V_2 \oplus \cdots \oplus V_k)$ and so there exists a proper $G$-stable subspace of $\mathbb{R}^n$. A contradiction.

Let $x \in W^\beta_1 \cap M$. Then

$$
\lim_{t \to +\infty} \exp(t \beta)x \in M \cap V_1,
$$

proving $\max_{x \in M} \mu^\beta_p = \max_{x \in \mathbb{P}(\mathbb{R}^n)} \mu^\beta_p$.

Since $\beta$ is a real matrix, we get the following orthogonal splitting $\mathbb{C}^n = V_1^C \oplus \cdots \oplus V_k^C$, with respect to the canonical Hermitian scalar product, of eigenspaces of $\beta$. In particular, the function $\mu^\beta_p : \mathbb{P}(\mathbb{C}^n) \to \mathbb{R}$, is given by

$$
\mu^\beta_p([x_1 + \cdots + x_k]) = \frac{\lambda_1 \| x_1 \|^2 + \cdots + \lambda_k \| x_k \|^2}{\| x_1 \|^2 + \cdots + \| x_k \|^2}
$$

and so $\mathbb{P}(V_1^C) \cup \cdots \cup \mathbb{P}(V_k^C)$ are the critical points of $\mu^\beta_p$. The corresponding unstable manifolds are given by:

$$
W^\beta_1 = \mathbb{P}(\mathbb{C}^n) - \mathbb{P}(V_2^C \oplus \cdots \oplus V_k^C),
$$

$$
W^\beta_2 = \mathbb{P}(V_2^C \oplus \cdots \oplus V_k^C) - \mathbb{P}(V_3^C \oplus \cdots \oplus V_k^C),
$$

$$
\vdots
$$

$$
W^\beta_{k-1} = \mathbb{P}(V_{k-1}^C \oplus V_k^C) - \mathbb{P}(V_k^C),
$$

$$
W^\beta_k = \mathbb{P}(V_k^C).
$$
This implies \( \max_{x \in \mathbb{P}(\mathbb{C}^n)} \mu^\beta_x = \lambda_1 \).

Let \( \mathcal{O}' \) be the unique compact \( U^\mathbb{C} \)-orbit in \( \mathbb{P}(\mathbb{C}^n) \). Since \( G \) acts irreducibly on \( \mathbb{C}^n \), it follows that the unstable manifold \( W^\beta_1 \) of the Morse-Bott function \( \mu^\beta : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathbb{R} \) intersects \( \mathcal{O}' \). As before, one has
\[
\Max_{\mathcal{O}'}(\beta) = \Max_{\mathbb{P}(\mathbb{C}^n)}(\beta) \cap \mathcal{O}',
\]
concluding the proof. \( \Box \)

**Proposition 45.** Let \( a \subset \mathfrak{p} \) be an Abelian subalgebra. Then
\[
\mu_a(\mathcal{O}) = \mu_a(\mathbb{P}(\mathbb{R}^n)) = \mu_a(\mathbb{P}(\mathbb{C}^n)) = \mu_a(\mathcal{O}').
\]

**Proof.** Since \( \mathcal{O} \) is a \( K \)-orbit, keeping in mind that \( \mu_a \) is \( K \)-equivariant and \( \mu_a = \pi_a \circ \mu_p \), by a Theorem of Kostant [37] it follows that \( \mu_a(\mathcal{O}) \) is a polytope. \( \mu_a(\mathbb{P}(\mathbb{R}^n)) \) is a polytope as well [30] and by Lemma 44 we get
\[
\Max_{\mathcal{O}}(\beta) = \Max_{\mathbb{P}(\mathbb{C}^n)}(\beta) \cap \mathcal{O},
\]
for any \( \beta \in \mathfrak{p} \). Applying Proposition 15 we have \( \mu_a(\mathcal{O}) = \mu_a(\mathbb{P}(\mathbb{R}^n)) \).

Lemma 44 also proves
\[
\Max_{\mathbb{P}(\mathbb{C}^n)}(\beta) = \Max_{\mathbb{P}(\mathbb{R}^n)}(\beta) = \Max_{\mathcal{O}'}(\beta),
\]
for any \( \beta \in \mathfrak{p} \). Applying again Proposition 15 we get \( \mu_a(\mathbb{P}(\mathbb{C}^n)) = \mu_a(\mathbb{P}(\mathbb{R}^n)) = \mu_a(\mathcal{O}') \), concluding the proof. \( \Box \)

Let \( v \in \mathbb{P}(\mathbb{R}^n) \) be such that \( G \cdot v = \mathcal{O} \) and \( \mathcal{O}' = U^\mathbb{C} \cdot v \).

**Proposition 46.** The restricted \( G \)-gradient map
\[
\mu_p : K \cdot v \rightarrow K \cdot \mu_p(v),
\]
is a diffeomorphism.

**Proof.** \( U^\mathbb{C} \cdot v \) is compact, and \( U^\mathbb{C} \cdot v = U \cdot v \). Then \( U_v = U^\mu(v) \) and so the centralizer of \( \mu(v) \) [23]. By Lemma 38 we have \( \mu(v) = i \mu(v) \). Therefore, keeping in mind that \( K_v = U_v \cap K \), we have \( K_v = K^{\mu_\mu(v)} \). Since \( \mu_p \) is \( K \)-equivariant the result is proved. \( \Box \)

Let \( \beta \in \mathfrak{p} \) and let \( \lambda_1 > \cdots > \lambda_k \) be the eigenvalues of \( \beta \) acting on \( \mathbb{R}^n \). We denote by \( W \) the eigenspace corresponding to \( \lambda_1 \).

**Theorem 47.** The following results hold true:

a) \( W \) is the unique subspace of \( \mathbb{R}^n \) such that \( G^{\beta^+} \) acts irreducibly on it;

b) \( \Max_{\mathcal{O}}(\beta) \) is connected and it coincides with the unique compact orbit of the \( G^{\beta^+} \)-action on \( \mathcal{O}' \). \( \Max_{\mathcal{O}}(\beta) \) completely characterizes \( W \);

c) \( \Max_{\mathcal{O}}(\beta) \) is contained in the unique compact orbit of \( (U^\mathbb{C})^{\beta^+} \)-action on \( \mathbb{P}(\mathbb{C}^n) \) given by \( \Max_{\mathcal{O}}(\beta) \). Moreover, \( \Max_{\mathcal{O}}(\beta) \) is a Lagrangian submanifold and the fixed point set of an anti-holomorphic involutive isometry of \( \Max_{\mathcal{O}}(\beta) \), and so it is totally geodesic.
Proof. Let $\mathcal{O} = K \cdot v$. By [6] Theorem 1.2] it follows that $\text{Max}_{K^\beta}(\beta)$ is a $(K^\beta)\mathcal{O}$-orbit and so it is connected. By Proposition [36], keeping in mind that $\mu$ is $K$-equivariant, we have $\text{Max}_\mathcal{O}(\beta)$ is connected. By Proposition [36], $R^{\beta+}$ acts trivially on $\text{Max}_\mathcal{O}(\beta)$. Applying [11] Theorem 1, keeping also in mind Proposition [36], it follows that $G^{\beta+}$ has a unique compact orbit in $\mathcal{O}$, which is a $(K^\beta)^\mathcal{O}$-orbit, and this orbit coincides with $\text{Max}_\mathcal{O}(\beta)$. By Lemma [11] $\text{Max}_\mathcal{O}(\beta) \subset \text{Max}_\mathcal{O}(\beta)$ which is the unique compact orbit of the $(U^C)^{\beta+}$-action on $\mathbb{P}(\mathbb{C}^n)$ [11]. Hence the first part of (b) and (c) are proved.

By Proposition [36] $G^{\beta+}$ preserves $\mathbb{P}(W)$ and $R^{\beta+}$ acts trivially on $\mathbb{P}(W)$. Since $G^\beta$ is reductive, $W$ splits as $G^\beta$-invariant, and so as $G^{\beta+}$-invariant, subspaces [35].

Let $L \subseteq W$ be a $G^{\beta+}$-invariant subspace. Since $\mathbb{P}(W^C) = \text{Max}_{\mathbb{P}(\mathbb{C}^n)}(\beta)$, it follows that $W^C$ is the unique subspace of $\mathbb{C}^n$ such that $(U^C)^{\beta+}$ acts irreducibly on it [10] Theorem 2. By Proposition [36] the unipotent part of $(U^C)^{\beta+}$ acts trivially on $\mathbb{P}(W^C)$. Hence, keeping in mind that $(\mathfrak{g}^\beta)^C = (\mathfrak{u}^C)^\beta$, it follows that $L^C$ is a $(U^C)^{\beta+}$-invariant subspace as well. This implies $L^C = W^C$ and so $L = W$. This proves (a) and (b).

The $(U^C)^{\beta}$-action on $W^C$ is irreducible and the unipotent part of the parabolic subgroup $(U^C)^{\beta+}$ acts trivially on $\mathbb{P}(W^C)$. By the Schur Lemma [35] the center of $(U^C)^{\beta}$ acts trivially on $\mathbb{P}(W^C)$. This implies that the center of $G^\beta$ acts trivially on $\mathbb{P}(W^C)$, and so on $\mathbb{P}(W)$.

Let $G^{\beta}_{ss}$ be the connected subgroup of $G^\beta$ whose Lie algebra is $[\mathfrak{g}^\beta, \mathfrak{g}^\beta]$. Let $(U^C)^{\beta}_{ss}$ be the connected subgroup of $(U^C)^{\beta}$ whose Lie algebra is $[(\mathfrak{u}^C)^\beta, (\mathfrak{u}^C)^\beta]$. $(G^{\beta})_{ss}$ is semisimple, closed [35] and a real form of the closed complex semisimple Lie group $(U^C)^{\beta}_{ss}$. By the above discussion both the $G^{\beta}_{ss}$-action on $W$ and the $(U^C)^{\beta}_{ss}$-action on $W^C$ are irreducible. The unique compact orbit of the $G^{\beta+}$-action on $\mathcal{O}$ is a compact $G^{\beta}_{ss}$-orbit. The unique compact orbit of the $(U^C)^{\beta+}$-action on $\mathbb{P}(\mathbb{C}^n)$ is a compact $(U^C)^{\beta}_{ss}$-orbit. By Proposition [39] and Theorem [42] $\text{Max}_\mathcal{O}(\beta)$ is a Lagrangian submanifold and the fixed point set of an anti-holomorphic involution of $\text{Max}_\mathcal{O}(\beta)$, concluding the proof. \qed

Corollary 48. Let $\beta \in \mathfrak{p}$ and let $W$ be the eigenspace corresponding to the maximum eigenvalue of $\beta$ acting on $\mathbb{R}^n$. Let $z \in \mathbb{P}(\mathbb{C}^n)$ be such that $(U^C)^{\beta+} \cdot z$ is compact. Then $(U^C)^{\beta+} \cdot z \cap \mathbb{P}(W)$ is the unique compact orbit of the $(U^C)^{\beta+}$-action on $\mathcal{O}$.

Proof. By the proof of the above Theorem, the unique compact $(U^C)^{\beta+}$-orbit on $\mathcal{O}$ is a compact $(U^C)^{\beta}_{ss}$-orbit contained in the unique compact orbit of the $(U^C)^{\beta}_{ss}$-action on $\mathbb{P}(W^C)$. The result follows by Corollary [40]. \qed

Corollary 49. Let $Q$ be a parabolic subgroup of $G$ and let $L \subset \mathbb{C}^n$ be a $Q$-invariant complex subspace. If the $Q$-action on $L$ is irreducible, then there exists an invariant $Q$-subspace $W$ of $\mathbb{R}^n$ such that the $Q$-action on $W$ is irreducible and $L = W^C$.

Proof. Let $\mathfrak{q}$ be the Lie algebra of $Q$. Then $\mathfrak{q}^C$ is a parabolic subalgebra of $\mathfrak{g}^C$ [15]. Let $\tilde{Q}$ denote the parabolic subgroup of $U^C$ whose Lie algebra is $\mathfrak{q}^C$. The $\tilde{Q}$-action on $\mathbb{P}(\mathbb{C}^n)$ has a unique compact orbit and it is contained in $\mathcal{O}'$. Since the unipotent part of $\tilde{Q}$ normalizes $\tilde{Q}$ and the $\tilde{Q}$ acts irreducibly on $L$, by Engel Theorem the unipotent part acts trivially on $\mathbb{P}(L)$. Hence $\tilde{Q}$ has a unique compact orbit contained in $\mathbb{P}(L)$ that we denote by $\mathcal{O}(\tilde{Q})$. \qed
Let $\beta \in \mathfrak{p}$ be such that $Q = G^{\beta +}$. By the above discussion $R^{\beta +}$ acts trivially on $\mathbb{P}(L)$. By Proposition 34, $(G^{\beta})^0$ has a compact orbit on $\mathcal{O}(\tilde{Q})$ which is a $(K^{\beta})^0$-orbit. Since $G^{\beta}$ has a finite number of connected components and any connected component intersect $K^{\beta}$, it follows that $G^{\beta}$ has a compact orbit on $\mathcal{O}(\tilde{Q})$. By Theorem 47, it is the unique compact $G^{\beta +}$-orbit in $\mathcal{O}$. This orbit is contained in $\mathbb{P}(R^n)$. Hence its real span coincides with $\mathbb{P}(W)$, where $W$ is the unique $G^{\beta +}$-invariant subspace of $R^n$ such that $G^{\beta +}$ acts irreducibly on it. Therefore $W^C$ is contained in $L^C$ and it is $\tilde{Q}$-invariant. This implies $L = W^C$.

Let $v \in \mathbb{P}(R^n)$ be such that $U^C \cdot v = O'$ and $G \cdot v = O$. Let $\beta \in \mathfrak{p}$. By Theorem 47, $\text{Max}_\mathcal{O}(\beta)$ is the unique compact $G^{\beta +}$-orbit contained in $\mathcal{O}$. Now,

$$\mu_\mathfrak{p}(\text{Max}_\mathcal{O}(\beta)) = \text{Max}_{K^\beta \mu_\mathfrak{p}(v)}(\beta).$$

By [6, Corollary 3.1 p. 593 and Proposition 3.9 p. 599], $\text{Max}_{K^\beta \mu_\mathfrak{p}(v)}(\beta)$ is the unique compact orbit of the $G^{\beta +}$-action on $K \cdot \mu_\mathfrak{p}(v)$. By Proposition 46 and Theorem 47, we get the following result

**Proposition 50.** Let $z \in \mathcal{O}$. Then $G^{\beta +} \cdot z$ is compact if and only if $G^{\beta +} \cdot \mu_\mathfrak{p}(z)$ is compact and

$$\mu_\mathfrak{p}(G^{\beta +} \cdot z) = G^{\beta +} \cdot \mu_\mathfrak{p}(z)$$

We conclude this section showing how the gradient flow of the Morse Bott function $\mu_\mathfrak{p}^\beta$ determines the unique compact orbit of the $G^{\beta +}$-action on $\mathcal{O}$.

Let $W \subset R^n$ be a real subspace. We denote by

$$\pi_W : R^n \longrightarrow W,$$

the orthogonal projection and by

$$\hat{\pi}_W : \mathbb{P}(V) - \mathbb{P}(W^\perp) \longrightarrow \mathbb{P}(V), \quad \hat{\pi}_W([v]) = [\pi_W(v)],$$

its projectivization. Let $\beta \in \mathfrak{p}$ and let $W \subset R^n$ be the eigenspace corresponding to the maximum eigenvalue of $\beta$. Since $\mathbb{P}(W) = \text{Max}_{R^n}(\beta)$, the domain of the map $\hat{\pi}_W$ is the unstable manifold of the maximum of $\mu_\mathfrak{p}^\beta$. Moreover, $\hat{\pi}_W$ coincides with the gradient flow $\varphi_\infty$. Indeed, let $\lambda_1 > \cdots > \lambda_k$ be the eigenvalues of $\beta$. Let $V_2, \ldots, V_k$ be the eigenspaces associated to $\lambda_2, \ldots, \lambda_k$. Then

$$\lim_{t \to +\infty} \exp(t\beta)[x_1 + x_2 + \cdots + x_k] = \lim_{t \to +\infty} [e^{t\lambda_1}x_1 + e^{t\lambda_2}x_2 + \cdots + e^{t\lambda_k}x_k]
\begin{align*}
&= \lim_{t \to +\infty} [x_1 + e^{t(\lambda_2-\lambda_1)}x_2 + \cdots + e^{t(\lambda_k-\lambda_1)}x_k] \\
&= [x_1] \\
&= \hat{\pi}_W(x).
\end{align*}$$

By Theorem 34, an unstable manifold flows into the corresponding critical set. Hence $\hat{\pi}_W(\mathcal{O}) = \text{Max}_\mathcal{O}(\beta)$.

Let $W^C \subset C^n$. The $(U^C)^{\beta +}$-action on $W^C$ is irreducible and

$$\mathbb{P}(W^C) = \text{Max}_{\mathbb{P}(C^n)}(\beta).$$
The projection of the orthogonal projection onto $W \subset \mathbb{C}^n$ with respect to the canonical Hermitian product, i.e.

$$\hat{\pi}_{W \subset \mathbb{C}} : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathbb{P}(V), \quad \hat{\pi}_{W \subset \mathbb{C}}([v]) = [\pi_{W \subset \mathbb{C}}(v)],$$

is the gradient flow of $\mu_\beta$ restricted to the unstable manifold corresponding to the maximum. Therefore $\hat{\pi}_{W \subset \mathbb{C}}(O') = \text{Max}_\mathcal{O}(\beta)$ and $\hat{\pi}_{W \subset \mathbb{C}}(O) = \text{Max}_\mathcal{O}(\beta)$. Summing up, we have proved the following result.

**Proposition 51.** Let $W \subset \mathbb{R}^n$ be the eigenspace corresponding to the maximum eigenvalue of $\beta$. Then both $\hat{\pi}_{W \subset \mathbb{C}}(O)$ and $\hat{\pi}_{W \subset \mathbb{C}}(O')$ are the unique compact orbit of the $G^{\beta^+}$-action on $\mathcal{O}$ and $\hat{\pi}_{W \subset \mathbb{C}}(O)$ is the unique compact orbit of the $(U^{\beta})^{\mathbb{C}^+}$-action on $\mathbb{P}(\mathbb{C}^n)$.

5. $\rho$-CONNECTED SUBSETS AND THE GRADIENT MAP

Let $\mathcal{E} = \text{conv}(\mu(\mathbb{P}(\mathbb{R}^n)))$. This highly symmetric $K$-invariant convex body has been extensively studied in [6, 8], see also [36, 40] amongst many others. Let $a \subset \mathfrak{p}$ be a maximal Abelian subalgebra. Since $P = \pi_a(\mathcal{E}) = \mu_a(\mathbb{P}(\mathbb{R}^n)) = \mu_a(\mathcal{O})$, applying Theorem 16 it follows that $\mathcal{E}$ coincides with the convex hull of $\mu_\beta(\mathcal{O})$ and so the convex hull of a $K$-orbit. This means that $\mathcal{E}$ is a polar orbitope. By Theorem 17, the face structure of $\mathcal{E}$, up to $K$-equivalence, is completely determined by the face structure of $P$ and any face of $\mathcal{E}$ is exposed. By a recent result, $\mathcal{E}$ is also a spectrahedron [36].

Let $F$ be a face of $\mathcal{E}$. By Lemma 12 there exists a chain of faces

$$F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq \mathcal{E}.$$ 

Since any face is exposed, there exists $\beta_0, \beta_1, \ldots, \beta_k \in \mathfrak{p}$ such that

$$F_i = \text{max}_{\mathcal{E}}(\beta_i) := \{ z \in \mathcal{E} : \langle z, \beta_i \rangle = \text{max}_{\beta \in \mathcal{E}}(\beta, \beta_i) \}$$

Then

$$\mu_\beta^{-1}(F_i) = \text{Max}_{\mathbb{P}(\mathbb{R}^n)}(\beta_i) = \mathbb{P}(W_i),$$

where $W_i$ is the eigenspace corresponding to the maximum eigenvalue of $\beta_i$. Moreover,

$$\text{Max}_\mathcal{O}(\beta_i) = \text{Max}_{\mathbb{P}(\mathbb{R}^n)}(\beta_i) \cap \mathcal{O} \subseteq \text{Max}_{\mathcal{O}^+}(\beta_i),$$

and

$$\mu_\beta(\text{Max}_\mathcal{O}(\beta_i)) = \text{ext } F_i.$$ 

Since $\mathbb{P}(W_i)$ is completely determined by $\text{Max}_\mathcal{O}(\beta_i)$, applying Theorem 47 we get the following result.

**Proposition 52.** Given a chain of faces $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq \mathcal{E}$, there exist two chains of submanifolds

$$\text{Max}_{\mathcal{O}^+}(\beta_0) \not\subseteq \mathbb{P}(\mathcal{O}^+) \quad \text{Max}_{\mathcal{O}^+}(\beta_1) \not\subseteq \mathbb{P}(\mathcal{O}^+) \quad \cdots \quad \text{Max}_{\mathcal{O}^+}(\beta_k) \not\subseteq \mathbb{P}(\mathcal{O}^+)$$

such that the vertical inclusions are Lagrangian and totally geodesic immersions.
Let \( a \subset p \) be a maximal Abelian subalgebra. Then

\[
\mathfrak{z}(a) = m \oplus a,
\]

where \( m = \mathfrak{z}(a) \cap \mathfrak{k} \). If \( a' \subset m \) is a maximal Abelian subalgebra of \( m \), then \( a' + ia \subset u = \mathfrak{t} + ip \) is a maximal Abelian subalgebra of \( u \) and so \((a' + ia)C \subset uC = u^C \) is a Cartan subalgebra. Given \( a, a' \), and \( \Pi \subset \Delta(g, a) \) be a basis one can choose a basis of \((a + ia')^* \) adapted to \( \Pi \) and \((ia')^* \).

Indeed, it is possible to define a set of simple roots of \( \hat{\Delta} \) such that the projection of a subset of the simple roots of \( \hat{\Delta} \) onto \( a^* \) is equal to \( \Pi \) (see \[24, p.51 - 52]; \[31, p.272 - 273]\)). In particular the Borel subalgebra of \( g^C \) is contained in \( q^C = (m + a + n)^C \).

Let \( \tilde{\mu}_\rho \) the highest weight of \( g^C \) with respect to the partial ordering determined \( \hat{\Delta} \). Let \( x_o = [v_\rho] \), where \( v_\rho \) is any highest weight vector. It is well-known that

\[
\mu : \mathbb{P}(\mathbb{C}^n) \rightarrow a' \oplus ia, \quad \langle \mu(x_o), \xi \rangle = \tilde{\mu}_\rho(\xi),
\]

see \[3\] that has opposite sign convection for \( \mu \), and \[3\]. By Proposition \[39\] one can choose \( v_\rho \in \mathbb{R}^n \). Hence \( G \cdot x_o = \mathcal{O} \), and so

\[
\langle \mu_a(x_o), \xi \rangle = (i\tilde{\mu}_\rho)(\xi).
\]

\((i\tilde{\mu}_\rho)|_a\) is the highest weight of \( g \) with respect the induced order on \( a^* \). From now on, we denote by \( \mu_\rho = (i\tilde{\mu}_\rho)|_a \) and also the vector in \( a \), which represent \((i\tilde{\mu}_\rho)|_a \) with respect to the scalar product \( \langle \cdot, \cdot \rangle \). Since \( \mu_\rho \) belongs to the positive Weyl chamber defined by the order chosen, we get the following result.

**Proposition 53.** The image of the gradient map \( \mu_\rho : \mathbb{P}(\mathbb{R}^n) \rightarrow a \), is the convex hull of the Weyl group orbit of the highest weight \( \mu_\rho \). Hence the weights of \( \rho \) are contained in \( \text{conv}(W \cdot \mu_\rho) \).

**Proof.** Denote the weights of \( \rho \) by \( \mu_1, \ldots, \mu_p \in a^* \). It is easy to check that

\[
\mu_a(\mathbb{P}(\mathbb{R}^n)) = \text{conv}(v_1, \ldots, v_p),
\]

where \( v_1, \ldots, v_p \in a \) satisfy

\[
\langle v_i, H \rangle = \mu_i(H),
\]

for any \( H \in a \) an for \( i = 1, \ldots, n \). By Proposition \[35\] we have

\[
\mu_a(\mathbb{P}(\mathbb{R}^n)) = \mu_a(\mathcal{O}).
\]

Since \( \mu_\rho \) lies in the positive Weyl chamber, it follows that \( \mu_a(\mathcal{O}) \cap a_+ = \mu_a(x_o) \). Applying the Kostant’s convexity Theorem \[37\], we have

\[
\mu_a(\mathbb{P}(\mathbb{R}^n)) = \mu_a(\mathcal{O}) = \text{conv}(W \cdot \mu_\rho),
\]

concluding the proof. \( \square \)

Let \( W \subset \mathbb{R}^n \). We say that \( W \) is \( \rho \)-admissible if \( Q(W) := \{ g \in G : gW = W \} \) is a parabolic subgroup of \( G \) and the \( Q(W) \)-action on \( W \) is irreducible. If \( Q(W) = G^{3+} \) then \( \mathbb{P}(W) \) is the linear span of the unique compact \( Q(W) \)-orbit in \( \mathcal{O} \) given by \( \text{Max}_\mathcal{O}(\beta) \) and \( \mu_\rho(\text{Max}_\mathcal{O}(\beta)) = \text{ext} \mathcal{F}_\beta(\xi) = F_W \).
Lemma 54. Let $W, W' \subseteq V$ be $\rho$-connected subspaces. Then $W \subseteq W'$ if and only if $F_W \subseteq F'_{W'}$. Moreover $W = W'$ if and only if relint $F_W = \text{relint } F_{W'}$.

Proof. If $W \subseteq W'$, then $\mathbb{P}(W) \cap \mathcal{O} \subseteq \mathbb{P}(W') \cap \mathcal{O}$, and so
\[
\text{ext } F_W = \mu_p(\mathbb{P}(W) \cap \mathcal{O}) \subseteq \mu_p(\mathbb{P}(W') \cap \mathcal{O}) \subseteq \text{ext } F_{W'}.
\]
This implies $F_W \subseteq F_{W'}$ since any face is the convex hull of its extremal points. Vice-versa, if $F_W \subseteq F_{W'}$, then
\[
\mathbb{P}(W) = \mu_p^{-1}(F_W) \subseteq \mu_p^{-1}(F_{W'}) = \mathbb{P}(W').
\]
By Theorem 14, $F_W = F_{W'}$ if and only if relint $F_W = \text{relint } F_{W'}$, concluding the proof. \qed

Since $\mu_p(\mathcal{O})$ is a $K$-orbit, it is a fundamental fact that the action of $K$ on $\mu_p(\mathcal{O})$ extends to an action of $G$ see [29, Prop. 6]. The set of extreme points of $F_W$, that we denote by $\text{ext } F_W$, is contained in $\mu_p(\mathcal{O})$. We define
\[
Q_{F_W} := \{h \in G : \text{ext } F_W = \text{ext } F_W\}.
\]

Proposition 55. $Q_{F_W} = Q(W)$. Moreover, if $\beta \in C^{H_F}_{F_W}$, then $G^{\beta +} = Q(W)$. Hence $Q(W)$ only depends on the face $F_W$.

Proof. Let $\beta \in C^{H_F}_{F_W}$. Then $\mu_p^{-1}(F_\beta(\mathcal{E})) = \mathbb{P}(W)$ and $W(G^{\beta +}) = W$. This implies $G^{\beta +} \subseteq Q(W)$.

Since $G^{\beta +} = Q_{F_W}$ [3, Proposition 3.8, p.598], it follows that $Q_{F_W} \subseteq Q(W)$.

Let $g \in Q(W)$. By Proposition 23, $g = kp$ for some $k \in K$ and some $p \in Q_{F_W}$. Then $gW = W$ implies $kW = W$ and so, keeping in mind that $\text{Max}_\mathcal{O}(\beta) = P(W) \cap \mathcal{O}$, $k$ preserves $\text{Max}_\mathcal{O}(\beta)$. By the $K$-equivariance of the $G$-gradient map $\mu_p$, we get $k \text{ext } F_W = \text{ext } F_W$ and so $k \in H_F \subseteq Q_{F_W}$. This proves $Q(W) \subseteq Q_{F_W}$, concluding the proof. \qed

Set $\mathcal{H}(\rho) = \{(W, Q(W)) : W \text{ is } \rho - \text{connected subspace of } \mathbb{R}^n\}$. The following Lemma is easy to prove.

Lemma 56. Let $W$ be a $\rho$-connected subspace of $V$ and let $g \in G$. Then

- a) $gW$ is a $\tau$-connected subspace;
- b) there exists $k \in K$ such that $gW = kW$;
- c) $Q(gW) = gQ(W)g^{-1} = kQ(W)k^{-1}$

$K$ acts on $\mathcal{H}(\tau)$ as follows:
\[
k(W, Q(W)) := (kW, kQ(W)k^{-1}).
\]

As in [10], one can prove the following result.

Proposition 57. Let $\mathcal{F}(\mathcal{E})$ denote the set of the faces of $\mathcal{E}$. Then the map
\[
\mathcal{F} : \mathcal{H}(\tau) \rightarrow \mathcal{F}(\mathcal{E}), \quad (W, Q(W)) \mapsto F_W,
\]
is $K$-equivariant and bijective.

In the sequel we always refer to [15, Section I.1 Real Parabolic subgroups].
Definition 58. A subset $I \subset \Pi$ is $\mu_\rho$-connected if $I \cup \{\mu_\rho\}$ is connected, i.e., it is not the union of subsets orthogonal with respect to the Killing form.

Connected components of $I$ are defined as usual. The $I$ is $\mu_\rho$-connected subset if and only if any connected component of $I$ contains at least one element $\alpha$ which is not orthogonal to $\mu_\rho$.

Definition 59. If $I \subset \Pi$ is $\mu_\rho$-connected, denote by $I'$ the collection of all simple roots orthogonal to $\{\mu_\rho\} \cup I$. The set $J := I \cup I'$ is called the $\mu_\rho$-saturation of $I$.

In [6], see also [36], the following theorem is proved.

Theorem 60.

a) Let $I \subset \Pi$ be a $\mu_\rho$-connected subset and let $J$ be its $\mu_\rho$-saturation. Then $Q_I \cdot \mu_\rho = Q_J \cdot \mu_\rho$ and $F := \text{conv}(Q_I \cdot \mu_\rho)$ is a face of $\mathcal{E}$. Moreover, $F = F_\beta(\mathcal{E}) = \text{conv}(K_\beta \cdot \mu_\rho)$, where $\beta \in \mathfrak{a}$ satisfies $Q_I = G^\beta \cdot ;$

b) let $\beta' \in \mathfrak{a}$ be such that $Q_J = G^{\beta'} \cdot ;$. Then $F = F_{\beta'}(\mathcal{E}) = \text{conv}(K_{\beta'} \cdot \mu_\rho)$.

c) Any face of $\mathcal{E}$ is conjugate to one of the faces constructed in (a).

Let $Q$ be a parabolic subgroup. We denote by $\mathcal{O}(Q)$ the unique closed orbit of $Q$ in $\mathcal{O}$. By Theorem 47, $\mu_p(\mathcal{O}(Q)) = \text{ext} F$ for some face $F$ of $\mathcal{O}$. Moreover, $\mu_p^{-1}(F) = W$ is the unique $\rho$-connected subspace such that $Q$ acts irreducibly on $W$, $F = F_W$ and $Q \subset Q(W)$.

Let $I \subset \Pi$ be a $\mu_\rho$-connected subset and let $F_I$ be the face of $\mathcal{E}$ such that $\text{ext} F_I = Q_I \cdot \mu_\rho$. We also denote by $W_I = \mu_p^{-1}(F_I)$ be the unique subspace of $\mathbb{R}^n$ such that $Q_I$ acts irreducible.

Proposition 61. If $Q$ is a parabolic subgroup of $G$, then there exists $\mu_\rho$-connected subset $I \subset \Pi$ and $k \in K$ such that $W_I = kW$ and $Q_I \subseteq kQk^{-1} \subseteq Q_J$.

Proof. Let $W$ be the unique subspace of $V$ be such that $Q$ acts irreducibly on $W$. By Theorem 60 there exists $k_1 \in K$ such that $W = k_1 W_I$ and $Q \subset k_1 Q_I k_1^{-1}$. Let $\mathcal{O}(W)$ denote the unique compact $Q$-orbit contained in $\mathcal{O}$. Since $k_1 \mathcal{O}(Q) = \mathcal{O}(Q_I)$, keeping in mind that $\mu_p(\mathcal{O}(Q)) = \text{ext} F_W$ and $\mu_p(\mathcal{O}(Q_I)) = \text{ext} F_I$ and $\mu_p$ is $K$-equivariant, it follows

$$\text{ext} F_I = k_1^{-1} \text{ext} F_W.$$  

By Proposition 3.5], there exists $k_2 \in K$ such that $\mathfrak{a} \subset k_2 Qk_2^{-1}$ and $(k_2 \text{ext} F_W) \cap \mathfrak{a}$ is a face of $P$. By the main result proved in [6], $\mathcal{F}(P) / W \cong \mathcal{F}(\mathcal{E}) / K$. Hence, there exists $\theta \in N_K(\mathfrak{a})$ such that

$$\text{ext} F_I \cap \mathfrak{a} = (\theta k_2 \text{ext} F_W) \cap \mathfrak{a}.$$  

By Theorem 1.1], we have $\text{ext} F_I = \theta k_2 \text{ext} F_W$.

Set $k = \theta k_2$. Since $\mathfrak{a} \subset kQk^{-1}$, there exists a $J$ subset of $\Pi$ such that $Q_J = kQk^{-1}$. By Theorem 60 and Proposition 50 we have

$$Q_J \cdot \mu_\rho = Q_I \cdot \mu_\rho = \text{ext} F_I.$$
Let $\tilde{I}$ denote the maximal $\{\mu_\rho\}$-connected subset contained in $\tilde{J}$. By Theorem 60 $Q_{\tilde{I}} \cdot \mu_\rho$ is a face and $Q_E \cdot \mu_\rho = Q_{\tilde{I}} \cdot \mu_\rho$, where $E$ is the saturation of $\tilde{I}$. Since $\tilde{I} \subseteq \tilde{J} \subseteq E$, it follows

$$Q_{\tilde{I}} \cdot \mu_\rho = Q_{\tilde{J}} \cdot \mu_\rho = Q_E \cdot \mu_\rho.$$ 

On the other hand $Q_{\tilde{I}} \cdot \mu_\rho = Q_{\tilde{J}} \cdot \mu_\rho$ and so, by Theorem 60 $Q_E = Q_{\tilde{J}}$. This implies $E = J$, $\tilde{I} = I$ due to the fact that $I$, respectively $\tilde{I}$, is the maximal $\{\mu_\rho\}$-connected subspace of $J$, and

$$Q_I \subset Q_{\tilde{J}} \subset Q_J,$$

where $I \subseteq \tilde{J} \subseteq J$. \qed

Let $I' = \tilde{J} - \{I\}$. The set $I'$ is perpendicular to $I$ and so the Langlands decomposition of $Q_{\tilde{J}}$ can be written as

$$Q_{\tilde{J}} = N_I A_I M_I M_{I'},$$

see [15]. By [15] Lemma I.4.25, p. 69, $M_{I'}$ acts trivially on $W_I$. Hence, the $Q_{\tilde{J}}$-action on $W_I$ is completely determined by the $Q_I$-action on $W_I$. Hence, keeping in mind Corollary 49 we have proved the following result.

**Theorem 62.** The face structure of $E$, up to $K$-equivalence, describes the irreducible representations of parabolic subgroups of $G$ induced by $\rho : G \to \text{SL}(n, \mathbb{R})$ and so the irreducible representations of parabolic subgroups of $G$ induced by $\rho : G \to \text{SL}(n, \mathbb{C})$.

Let $I \subset \Pi$ be a $\mu_\rho$-connected subset and let $F_I$ the face of $E$ be such that $\text{ext} F_I = Q_I \cdot \mu_\rho$. Let $a_I := \bigcap_{\alpha \in I} \text{Ker} \alpha$ and let $a^I$ be the orthogonal complement of $a_I$ in $a$. Then $q_I = n_I \oplus a^I \oplus m_I$, where $m_I = \mathfrak{z}(a) \oplus a^I \bigoplus_{\alpha \in I} g_\alpha$ is the Lie algebra of a Levi factor of $Q_I$, that we denote by $M_I$ which is not connected in general. We recall that $g_\alpha := \{v \in g : [H, v] = \alpha(H)v, \forall H \in a\}$. $M_I$ is compatible and $K_I = K \cap Q_I$ is its maximal compact subgroup. The Abelian subalgebra $a^I$ is a maximal Abelian subalgebra of $m_I \cap \mathfrak{p}$. Let $W_I = N_{K_I}(a^I)$. $W_I$ is the subgroup of the Weyl group generated by the the root reflections induced by the elements of $I$. We split $\mu_\rho = y_0 + y_1$, where $y_0 \in a_I$ and $y_1 \in a^I$. By Proposition 3.5 in [4], we have

$$Q_I \cdot \mu_\rho = K_I \cdot \mu_\rho = y_0 + K_I \cdot y_1.$$ 

By the Kostant convexity Theorem [37],

$$\pi_a(Q_I \cdot \mu_\rho) = y_0 + \text{conv}(W_I \cdot y_1).$$ 

By Proposition 50 $Q_I \cdot x_o$ is the unique compact $Q_I$-orbit on $O$ and $\mu_\rho(Q_I \cdot x_o) = Q_I \cdot \mu_\rho$. Therefore

$$\mu_a(Q_I \cdot x_o) = y_0 + \text{conv}(W_I \cdot y_1).$$ 

Summing up, keeping in mind Theorem 60 we get a result proved by Casselman [16] using the $G$-gradient map.
Theorem 63. The map \( I \mapsto \text{conv}(W_I \cdot \mu_\rho) \) induces a bijection between the \( \mu_\rho \)-connected subset of \( \Pi \) and the faces of \( P \) up to the Weyl-group action. Moreover,
\[
\mu_\rho(Q_I \cdot x_0) = \text{conv}(W_I \cdot \mu_\rho) = y_0 + \text{conv}(W_I \cdot y_1),
\]
where \( y_0 \in a_I, \ y_1 \in a' \) and \( \mu_\rho = y_0 + y_1 \).

6. Norm square gradient map

Let \( \mu_p : \mathbb{P}(\mathbb{R}^n) \to p \) be the \( G \)-gradient map and let \( \nu_p := \frac{1}{2} \| \mu_p(x) \|^2 \) denote the norm square gradient map. Let \( \nu_u(p) := \frac{1}{2} \| \mu(p) \|^2 \) denote the norm square momentum map. The gradient of \( \nu_p \), respectively \( \nu_u \), is given by \( \text{grad} \nu_p(x) = \mu(x)_{\mathbb{P}(\mathbb{R}^n)} \), respectively \( \text{grad} \nu_u(y) = \mu(x)_{\mathbb{P}(\mathbb{C}^n)} \) [28]. Lemma 38 implies \( x \in \mathbb{P}(\mathbb{R}^n) \) is a critical point of \( \nu_p \) if and only if \( x \in \mathbb{P}(\mathbb{R}^n) \) is a critical point of \( \nu_u \), see also 32. Proposition 39 implies
\[
\max_{x \in \mathbb{P}(\mathbb{R}^n)} \nu_p = \max_{x \in \mathbb{P}(\mathbb{C}^n)} \nu_u,
\]
and so the maximum of \( \nu_u \) is achieved on \( \mathbb{P}(\mathbb{R}^n) \). We shall investigate the strata associated to the critical \( K \)-orbits of \( \nu_p \). The negative gradient flow line of \( \nu_p \) through \( x_0 \in \mathbb{P}(\mathbb{R}^n) \) is the solution of the differential equation
\[
\begin{cases}
\dot{x}(t) = -\beta_{\mathbb{P}(\mathbb{R}^n)}(x(t)), \quad t \in \mathbb{R} \\
x(0) = x_0.
\end{cases}
\]
The Lojasiewicz gradient inequality holds and so the flow is defined for any \( t \in \mathbb{R} \) and the limit
\[
x_\infty := \lim_{t \to +\infty} x(t)
\]
exist [12 Theorem 3.3], see also [22] for the \( U^\mathbb{C} \)-action on \( \mathbb{P}(\mathbb{C}^n) \) and the corresponding momentum map. Moreover,
\[
\| \mu_p(x_\infty) \| = \inf_{g \in G} \| \mu_p(gx_0) \|
\]
and the \( K \)-orbit of \( x_\infty \) depends only on the \( G \)-orbit of \( x_0 \) [12 Theorem 4.7], see also [32].

By Lemma 38 \( \mu_p = i\mu \) on \( \mathbb{P}(\mathbb{R}^n) \). Then the negative gradient flow line of \( \nu_p \) through \( x_0 \) coincides with the negative gradient flow of \( \nu_u \). Hence, given \( x_0 \in \mathbb{P}(\mathbb{R}^n) \), we get
\[
\| \mu_p(x_\infty) \| = \inf_{g \in G} \| \mu_p(gx_0) \|
= \inf_{g \in U^\mathbb{C}} \| \mu_p(gx_0) \|.
\]
Let \( x \in \mathbb{P}(\mathbb{C}^n) \). Applying the negative gradient flow of the norm square momentum map, the orbit \( U^\mathbb{C} \cdot x \) flows on the \( U \)-orbit throughout \( x_\infty = \lim_{t \to +\infty} x(t) \). If \( U^\mathbb{C} \cdot x \cap \mathbb{P}(\mathbb{R}^n) \neq \emptyset \) then \( U \cdot x_\infty \cap \mathbb{P}(\mathbb{R}^n) \neq \emptyset \). Let \( y \in U \cdot x_\infty \cap \mathbb{P}(\mathbb{R}^n) \). Then, keeping in mind that the norm square momentum map is \( U \)-invariant, we have
\[
\| \mu(x_\infty) \| = \| \mu(y) \| = \| \mu_p(y) \|,
\]
and so
\[
\inf_{g \in U^\mathbb{C}} \| \mu(gx) \| = \inf_{g \in U^\mathbb{C} \cdot x \cap \mathbb{P}(\mathbb{R}^n)} \| \mu_p(y) \|.
\]
Summing up, we have proved the following result.
Proposition 64. Let $x_0 \in \mathbb{P}(\mathbb{R}^n)$. Then
\[
\| \mu_p(x) \| = \inf_{g \in G} \| \mu_p(gx) \| = \inf_{g \in G \cdot C} \| \mu_p(gx) \|.
\]
Let $x \in \mathbb{P}(\mathbb{C}^n)$ be such that $U^C \cdot x \cap \mathbb{P}(\mathbb{R}^n) \neq \emptyset$. Then
\[
\inf_{g \in U^C} \| \mu(gx) \| = \inf_{y \in U \cdot x \cap \mathbb{P}(\mathbb{R}^n)} \| \mu_p(y) \|.
\]

Let $K \cdot \beta$ be a critical orbit of the norm square gradient map. By Lemma 38, $U \cdot i\beta$ is a critical orbit of the norm square momentum map. Moreover, by the above discussion, the stratum associated to $K \cdot \beta$ is contained in the stratum associated to $U \cdot i\beta$. We recall that the stratum associated to $U \cdot i\beta$ is a smooth analytic subset of $\mathbb{P}(\mathbb{C}^n)$. Assume that the stratum associated to $K \cdot \beta$ is open. Since $\mathbb{P}(\mathbb{R}^n)$ is a Lagrangian submanifold of $\mathbb{P}(\mathbb{C}^n)$, it follows that the stratum associated to $U \cdot i\beta$ is open as well. It is well-known that there exists a unique open stratum of $\nu_u$ in $\mathbb{P}(\mathbb{C}^n)$. It is the minimal stratum. Therefore there exists a unique open stratum for $\nu_p$ which is the minimal stratum. In particular,
\[
\inf_{x \in \mathbb{P}(\mathbb{R}^n)} \nu_p = \inf_{x \in \mathbb{P}(\mathbb{C}^n)} \nu_u.
\]
Since the other strata of the norm square gradient map has codimension at least one, the minimal strata is dense. Summing up, we have proved the following result.

Proposition 65. The norm square gradient map $\nu_p$ has a unique open stratum, which is the minimal stratum. This stratum is open, dense and is given by the intersection of $\mathbb{P}(\mathbb{R}^n)$ and the minimal stratum of $\nu_u$.

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