A De Bruijn–Erdős theorem for 1-2 metric spaces

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Abstract

A special case of a combinatorial theorem of De Bruijn and Erdős asserts that every noncollinear set of \( n \) points in the plane determines at least \( n \) distinct lines. Chen and Chvátal suggested a possible generalization of this assertion in metric spaces with appropriately defined lines. We prove this generalization in all metric spaces where each nonzero distance equals 1 or 2.

It is well known that

(i) every noncollinear set of \( n \) points in the plane determines at least \( n \) distinct lines.

As noted by Erdős [5], theorem (i) is a corollary of the Sylvester–Gallai theorem (asserting that, for every noncollinear set \( S \) of finitely many points in the plane, some line goes through precisely two points of \( S \)); it is also a special case of a combinatorial theorem proved later by De Bruijn and Erdős [4].

Chen and Chvátal [2] suggested that theorem (i) might generalize in the framework of metric spaces. In a Euclidean space, line \( \overline{uv} \) is characterized as

\[
\overline{uv} = \{ p : \text{dist}(p, u) + \text{dist}(u, v) = \text{dist}(p, v) \text{ or } \text{dist}(u, p) + \text{dist}(p, v) = \text{dist}(u, v) \text{ or } \text{dist}(u, v) + \text{dist}(v, p) = \text{dist}(u, p) \},
\]

where \( \text{dist} \) is the Euclidean metric; in an arbitrary metric space \( (S, \text{dist}) \), the same relation may be taken for the definition of the line. With this definition of lines in metric spaces, Chen and Chvátal asked:

(ii) True or false? Every metric space on \( n \) points, where \( n \geq 2 \), either has at least \( n \) distinct lines or else has a line that consists of all \( n \) points.
Let us say that a metric space on \( n \) points has the *De Bruijn - Erdős property* if it either has at least \( n \) distinct lines or else has a line that consists of all \( n \) points: now we may state \( \text{(ii)} \) by asking whether or not all metric spaces on at least 2 points have the De Bruijn - Erdős property. A survey of results related to this question appears in \[1\].

By a 1-2 *metric space*, we mean a metric space where each nonzero distance is 1 or 2. Chiniforooshan and Chvátal \[3\] proved that

\( \text{(iii) every 1-2 metric space on } n \text{ points has } \Omega(n^{4/3}) \text{ distinct lines and this bound is tight.} \)

This result states that all sufficiently large 1-2 metric spaces have a property far stronger than the De Bruijn - Erdős property, but it does not imply that all 1-2 metric spaces on at least 2 points have the De Bruijn - Erdős property.

The purpose of the present note is to remove this blemish.

**Theorem 1.** All 1-2 metric spaces on at least 2 points have the De Bruijn - Erdős property.

The rest of this note is devoted to a proof of Theorem 1. A key notion in the proof, one borrowed from \[3\], is the notion of *twins* in a 1-2 metric space: these are points \( u, v \) such that \( \text{dist}(u, v) = 2 \) and \( \text{dist}(u, w) = \text{dist}(v, w) \) for all points \( w \) distinct from both \( u \) and \( v \). Use of this notion in counting lines is pointed out in the following claim (also borrowed from \[3\]), whose proof is straightforward.

**Claim 1.** If \( u_1, u_2, u_3, u_4 \) are four distinct points in a 1-2 metric space, then

- if \( \text{dist}(u_1, u_2) \neq \text{dist}(u_3, u_4) \), then \( u_1u_2 \neq u_3u_4 \),
- if \( \text{dist}(u_1, u_2) = \text{dist}(u_2, u_3) = 2 \), then \( u_1u_2 \neq u_2u_3 \),
- if \( \text{dist}(u_1, u_2) = \text{dist}(u_2, u_3) = 1 \) and \( u_1, u_3 \) are not twins, then \( u_1u_2 \neq u_2u_3 \).

By a *critical 1-2 metric space*, we shall mean a smallest counterexample to Theorem 1 in a sequence of claims, we shall gradually prove the nonexistence of a critical 1-2 metric space. We shall say that a line in a metric space is *universal* if, and only if, it consists of all points of the space.

**Claim 2.** For every pair \( u,v \) of twins in a critical 1-2 metric space, there is a third point \( w \) in this space such that \( \text{dist}(u, w) = \text{dist}(v, w) = 2 \) and \( \text{dist}(x, y) = 1 \) whenever \( x \in \{u, v, w\} \), \( y \notin \{u, v, w\} \).

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Proof. Let $S$ denote the space we are dealing with. Since $S$ is critical, $S$ does not have the De Bruijn - Erdős property and $S \setminus u$ has the De Bruijn - Erdős property. We will derive the existence of $w$ from these two facts.

The assumption that $u, v$ are twins implies that

(a) if $x, y$ are distinct points in $S \setminus \{u, v\}$, then the line $xy$ in $S$ contains either both $u, v$ or neither of $u, v$;

(b) if $w \in S \setminus u$ and $dist(w, v) = 1$, then the line $wv$ in $S$ (and the line $wu$ in $S$) contains both $u, v$;

(c) if $w \in S \setminus u$ and $dist(w, v) = 2$, then the line line $wv$ in $S$ contains $v$ and not $u$ and the line $wu$ in $S$ contains $u$ and not $v$.

Since $S$ does not have the De Bruijn - Erdős property, we have $uv \neq S$; since $u$ and $v$ are twins, it follows that

(d) there is a $w$ in $S \setminus u$ such that $dist(w, v) = 2$.

From (a), (b), (c), (d), we conclude that

(e) the number of lines in $S$ exceeds the number of lines in $S \setminus u$.

Since $S$ does not have the De Bruijn - Erdős property, the number of lines in $S$ is less than $|S|$, and so (e) implies that the number of lines in $S \setminus u$ is less than $|S \setminus u|$; since $S \setminus u$ has the De Bruijn - Erdős property, it follows that

(f) $S \setminus u$ has a universal line.

Since $S$ does not have the De Bruijn - Erdős property,

(g) $S$ has no universal line.

Facts (a), (f), and (g) together imply that some line $wv$ in $S \setminus u$ is universal. Now (b) and (g) together imply that $dist(w, v) = 2$; since $u, v$ are twins, it follows that $dist(u, v) = 2$ and $dist(w, u) = 2$. Since $wv$ is a universal line in $S \setminus u$, we have $dist(w, y) = dist(v, y) = 1$ whenever $y \not\in \{u, v, w\}$; since $u, v$ are twins, it follows that $dist(u, y) = 1$ whenever $y \not\in \{u, v, w\}$.

Claim 3. No critical 1-2 metric space contains a pair of twins.

Proof. Assume the contrary: some critical 1-2 metric space $S$ contains a pair of twins. We will show that $S$ has at least $|S|$ lines, contradicting the assumption that $S$ does not have the De Bruijn - Erdős property. For this purpose, consider the largest set $\{T_1, T_2, \ldots, T_k\}$ of pairwise disjoint three-point subsets of $S$ such that $dist(u, v) = 2$ whenever $u, v$ are distinct points in the same $T_i$ and such that $dist(u, x) = 1$ whenever $u \in T_i$, $x \not\in T_i$ for some $i$. Since $S$ contains a pair of twins, Claim $2$ guarantees that $k \geq 1$; we will derive the existence of $|S|$ lines in $S$ from this fact.
Let \( L_1 \) denote the set of all lines \( \overline{uv} \) such that \( u, v \) are distinct points in the same \( T_i \). If \( \overline{uv} \in L_1 \), then \( \overline{uv} = S \setminus w \), where \( \{u, v, w\} = T_i \) for some \( i \); it follows that

(a) \( L_1 \) consists of the \( 3k \) sets \( S \setminus w \) with \( w \) ranging through \( \cup_{i=1}^{k} T_i \).

Next, choose a point \( r \) in \( T_1 \) and let \( L_2 \) denote the set of all lines \( \overline{rx} \) such that \( x \in S \setminus \cup_{i=1}^{k} T_i \). Claim 2 and maximality of \( k \) together guarantee that \( S \) contains no pair \( x, y \) of twins such that \( x, y \in S \setminus \cup_{i=1}^{k} T_i \). This fact and Claim 1 together imply that

(b) \( |L_2| = |S| - 3k \).

Finally, note that each line in \( L_2 \) includes all points of \( T_1 \) and no points of \( T_2 \). This observation and (a) together imply that \( L_1 \cap L_2 = \emptyset \), and so \( |L_1 \cup L_2| = |S| \) by (a) and (b).

Each 1-2 metric space can be thought of as a complete graph with each edge \( uv \) labeled by \( \text{dist}(u, v) \). Given edges \( uv, xy \) of this complete graph, let us write \( uv \approx xy \) to mean that \( \overline{uv} = \overline{xy} \). The following fact is a direct consequence of Claim 4 combined with Claim 3.

**Claim 4.** Each equivalence class of the equivalence relation \( \approx \) in a critical 1-2 metric space is a set of pairwise disjoint edges with identical labels or else a (not necessarily proper) subset of a cycle of length four with alternating labels.

**Claim 5.** The size of each equivalence class of the equivalence relation \( \approx \) in a critical 1-2 metric space on \( n \) points is at most \( \max\{\frac{(n-1)}{2}, 4\} \).

**Proof.** This is a direct corollary of Claim 4 combined with the observation that an equivalence class of \( n/2 \) pairwise disjoint edges defines a universal line.

**Claim 6.** Every critical 1-2 metric space has at most 7 points.

**Proof.** Consider an arbitrary critical 1-2 metric space and let \( n \) denote the number of its points. Since this space does not have the De Bruijn - Erdős property, it has fewer than \( n \) lines, and so its equivalence relation \( \approx \) partitions the \( n(n-1)/2 \) edges of its complete graph into at most \( n - 1 \) classes. Since the largest of these classes has size at least \( n/2 \), Claim 5 implies that \( n/2 \leq \max\{\frac{(n-1)}{2}, 4\} \), and so \( n \leq 8 \). If \( n = 8 \), then the 28 edges of the complete
Claim 7. No critical 1-2 metric space has 7 points.

Proof. Consider an arbitrary critical 1-2 metric space on 7 points. Since this space does not have the De Bruijn - Erdős property, it has fewer than 7 lines, and so its equivalence relation ∼ partitions the 21 edges of its complete graph into at most 6 classes. By Claim 4, each of these classes has size at most 4, and so at least three of them have size precisely 4; by Claim 4, each of these three classes is a cycle of length four. Let $G_1, G_2, G_3$ denote these three subgraphs of the complete graph on seven vertices.

Since $G_1, G_2, G_3$ are pairwise edge-disjoint, every two of them share at most two vertices; since their union has only seven vertices, some two of them share at least two vertices; we may assume (after a permutation of subscripts if necessary) that $G_1$ and $G_2$ share precisely two vertices. Let us name these two vertices $u, v$. Since $G_1$ and $G_2$ are edge-disjoint, we may assume (after a switch of subscripts if necessary) that vertices $u, v$ are adjacent in $G_1$ and nonadjacent in $G_2$.

Next, we may name $w, x$ the remaining two vertices in $G_1$ in such a way that the four edges of $G_1$ are $uv, vw, wx, ux$; we may name $y, z$ the remaining two vertices in $G_2$ in such a way that the four edges of $G_2$ are $uy, uz, vz, vy$. Since the labels on the edges of $G_2$ alternate, we may assume (after switching $y$ and $z$ if necessary) that $\text{dist}(u, y) = 1$, $\text{dist}(u, z) = 2$, $\text{dist}(v, z) = 1$, $\text{dist}(v, y) = 2$. Since $uy = vy$, we have $u \in vy$; since $\text{dist}(v, y) = 2$, it follows that $\text{dist}(u, v) = 1$. In turn, since the labels on the edges of $G_1$ alternate, we have $\text{dist}(v, w) = 2$, $\text{dist}(w, x) = 1$, $\text{dist}(u, x) = 2$.

Now $\text{dist}(y, u) + \text{dist}(u, v) = \text{dist}(y, v)$, and so $y \in uv$; since $uv \approx vw$, it follows that $y \in vw$. But this is impossible, since $\text{dist}(v, w) = 2$ and $\text{dist}(v, y) = 2$. \qed

Claim 8. Every critical 1-2 metric space on 5 or 6 points contains points
u, v, w, x, y such that
\begin{align*}
\text{dist}(u, w) &= \text{dist}(u, x) = \text{dist}(v, w) = \text{dist}(v, x) = 1, \\
\text{dist}(u, v) &= \text{dist}(w, x) = 2, \\
\text{dist}(u, y) &\neq \text{dist}(v, y), \text{dist}(w, y) &\neq \text{dist}(x, y).
\end{align*}

Proof. Consider an arbitrary critical 1-2 metric space on $n$ points such that $n = 5$ or $n = 6$. Since this space does not have the De Bruijn - Erdős property, it has fewer than $n$ lines, and so its equivalence relation $\approx$ partitions the $n(n - 1)/2$ edges of its complete graph into at most $n - 1$ classes. Since the largest of these classes has size at least 3, Claim 4 and the absence of a universal line together imply that there are points $u, v, w, x, y$ such that
\begin{align*}
\text{dist}(u, v) &= 2, \text{dist}(v, w) = 1, \text{dist}(w, x) = 2 \quad \text{and} \quad \overline{uv} = \overline{vw} = \overline{wx}
\end{align*}
or else
\begin{align*}
\text{dist}(v, w) &= 1, \text{dist}(w, x) = 2, \text{dist}(u, x) = 1 \quad \text{and} \quad \overline{vw} = \overline{wx} = \overline{ux}.
\end{align*}
In both cases, equality of the three lines implies that
\begin{align*}
\text{dist}(u, w) &= \text{dist}(u, x) = \text{dist}(v, w) = \text{dist}(v, x) = 1, \\
\text{dist}(u, v) &= \text{dist}(w, x) = 2.
\end{align*}
Since $w, x$ are not twins, there is a point $y$ distinct from both of them and such that $\text{dist}(w, y) \neq \text{dist}(x, y)$; we will complete the proof by showing that $\text{dist}(u, y) \neq \text{dist}(v, y)$.

To do this, assume the contrary: $\text{dist}(u, y) = \text{dist}(v, y)$. Since $y \not\in \overline{wx}$ and $\overline{vw} = \overline{wx}$, we have $y \not\in \overline{vw}$, and so $\text{dist}(v, y) = \text{dist}(w, y)$. Now $\text{dist}(u, y) \neq \text{dist}(x, y)$, and so $y \in \overline{ux}$, since $y \not\in \overline{wx}$, we cannot have $\overline{uv} = \overline{wx} = \overline{ux}$, and so we must have $\overline{uv} = \overline{vw} = \overline{wx}$. In particular, $y \not\in \overline{uv}$; since $\text{dist}(u, y) = \text{dist}(v, y)$, we conclude that
\begin{align*}
\text{dist}(u, y) = \text{dist}(v, y) = \text{dist}(w, y) = 2, \text{dist}(x, y) = 1.
\end{align*}
Since $u, v$ are not twins, there is a point $z$ distinct from both of them and such that $\text{dist}(u, z) \neq \text{dist}(v, z)$; it follows that $\text{dist}(x, z)$ is distinct from one of $\text{dist}(u, z), \text{dist}(v, z)$, and so $z$ belongs to one of the lines $\overline{ux}, \overline{vx}$. But then this line is universal, a contradiction. \qed
Claim 9. No critical 1-2 metric space has 5 or 6 points.

Proof. Consider an arbitrary critical 1-2 metric space on \( n \) points such that \( n = 5 \) or \( n = 6 \) and let \( u, v, w, x, y \) be as in Claim \( \mathbb{N} \). We may assume (after a cyclic shift of \( u, w, v, x \) if necessary) that
\[
\begin{align*}
\text{dist}(u, w) &= \text{dist}(u, x) = \text{dist}(v, w) = \text{dist}(v, x) = 1, \\
\text{dist}(u, v) &= \text{dist}(w, x) = 2, \\
\text{dist}(u, y) &= \text{dist}(w, y) = 1, \; \text{dist}(v, y) = \text{dist}(x, y) = 2.
\end{align*}
\]
Since
\[
\overline{ux} \supseteq \{u, v, w, x, y\} \quad \text{and} \quad \overline{vw} \supseteq \{u, v, w, x, y\},
\]
absence of a universal line implies that \( n = 6 \) and that the sixth point of our space lies outside the lines \( \overline{ux} \) and \( \overline{vw} \). Let \( z \) denote this sixth point. Since \( z \not\in \overline{ux}, \; z \not\in \overline{vw} \), we have \( \text{dist}(u, z) = \text{dist}(x, z), \; \text{dist}(v, z) = \text{dist}(w, z) \), and so symmetry allows us to distinguish between three cases:
\begin{itemize}
  \item \( \text{dist}(u, z) = \text{dist}(x, z) = 1, \; \text{dist}(v, z) = \text{dist}(w, z) = 1 \),
  \item \( \text{dist}(u, z) = \text{dist}(x, z) = 1, \; \text{dist}(v, z) = \text{dist}(w, z) = 2 \),
  \item \( \text{dist}(u, z) = \text{dist}(x, z) = 2, \; \text{dist}(v, z) = \text{dist}(w, z) = 2 \).
\end{itemize}
Each of these three cases comprises two metric spaces, one with \( \text{dist}(y, z) = 1 \) and the other with \( \text{dist}(y, z) = 2 \). Altogether, there are six metric spaces on six points to inspect; each of them has at least six lines. \( \Box \)

Claim 10. Every metric space on 2, 3, or 4 points has the De Bruijn - Erdős property.

Proof. Consider an arbitrary critical 1-2 metric space on \( n \) points. If each of its lines has precisely 2 points or if one of its lines has precisely \( n \) points, then this space has the De Bruijn - Erdős property; otherwise one of its lines has precisely 3 points and \( n = 4 \). Let \( T \) denote the 3-point line and let \( w \) denote the fourth point of the space. If there are distinct \( x, y \) in \( T \) such that \( \overline{wx} = \overline{wy} \), then \( \overline{xy} \) is a universal line; else the three lines \( \overline{wx} \) with \( x \) ranging through \( T \) are pairwise distinct 2-point lines. \( \Box \)

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