Energy conservation in the 3D Euler equations on \( T^2 \times \mathbb{R}_+ \)

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Abstract

The aim of this paper is to prove energy conservation for the incompressible Euler equations in a domain with boundary. We work in the domain \( T^2 \times \mathbb{R}_+ \), where the boundary is both flat and has finite measure.

However, first we study the equations on domains without boundary (the whole space \( \mathbb{R}^3 \), the torus \( T^3 \), and the hybrid space \( T^2 \times \mathbb{R} \)). We make use of some of the arguments of Duchon & Robert (Nonlinearity 13 (2000) 249–255) to prove energy conservation under the assumption that \( u \in L^3(0, T; L^3(\mathbb{R}^3)) \) and one of the two integral conditions

\[
\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int_{\mathbb{R}^3} |u(x + y) - u(x)|^3 \, dx \, dt = 0
\]

or

\[
\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^3}{|x - y|^{4+\delta}} \, dx \, dy < \infty, \quad \delta > 0,
\]

the second of which is equivalent to requiring \( u \in L^3(0, T; W^{\alpha, 3}(\mathbb{R}^3)) \) for some \( \alpha > 1/3 \).

We then use the first of these two conditions to prove energy conservation for a weak solution \( u \) on \( D_+ := T^2 \times \mathbb{R}_+ \): we extend \( u \) a solution defined on the whole of \( T^2 \times \mathbb{R} \) and then use the condition on this domain to prove energy conservation for a weak solution \( u \in L^3(0, T; L^3(D_+)) \) that satisfies

\[
\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int_{T^2} \int_{|y|} \infty |u(t, x + y) - u(t, x)|^3 \, dx_3 \, dx_1 \, dx_2 \, dt = 0,
\]

and certain continuity conditions near the boundary \( \partial D_+ = \{x_3 = 0\} \).
1 Introduction

Energy conservation for solutions of the incompressible Euler equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0 \quad \nabla \cdot u = 0$$

has long been a topic of interest. While for sufficiently smooth solutions $u$ a standard integration-by-parts argument shows that energy is conserved ($\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$ for every $t \geq 0$) for weak solutions $u \in L^\infty(0,T;L^2) \cap L^3(0,T;L^3)$ we do not have the regularity needed to perform these operations. Onsager (1949) conjectured that weak solutions to the Euler equations satisfying a Hölder continuity condition of order greater than one third should conserve energy.

The study of energy conservation for this system has so far been carried out on domains without boundary, either the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3$. In this paper we aim to treat the question on the domain $\mathbb{T}^2 \times \mathbb{R}_+$, which involves a flat boundary with finite measure.

The first proof of energy conservation for weak solutions was given by Eyink (1994) on the torus, assuming that the solution satisfies $u(\cdot, t) \in C^\alpha_*$ for $\alpha > 1/3$ with a uniform bound for $t \in [0,T]$. A definition of the space $C^\alpha_*$ equivalent to that of Eyink’s is as follows: expand $u$ as the Fourier series

$$u = \sum_{k \in \mathbb{Z}^3} \hat{u}_k e^{ik \cdot x},$$

imposing conditions to ensure that $u$ is real ($\hat{u}_k = \overline{\hat{u}_{-k}}$) and is divergence free ($k \cdot \hat{u}_k = 0$); then $u \in C^\alpha_*(\mathbb{T}^3)$ if

$$\sum_{k \in \mathbb{Z}^3} |k|^\alpha |\hat{u}_k| < \infty.$$  

Requiring $u \in C^\alpha_*$ with $\alpha > 1/3$ is a stronger condition than the one-third Hölder continuity conjectured by Onsager.

Subsequently Constantin, E, & Titi (1994) gave a short proof of energy conservation, in the framework of Besov spaces (but still on the torus), under the weaker assumption that

$$u \in L^3(0,T;B^{\alpha}_{3,\infty}) \quad \text{with} \quad \alpha > 1/3.$$  

(1)

As $C^\alpha \subset B^{\alpha}_{3,\infty}$ this proves Onsager’s Conjecture. Here $B^s_{p,r}$ denotes a Besov space as defined in Bahouri et al. (2011) and Lemarié-Rieusset (2002).

Duchon & Robert (2000) showed that solutions satisfying a weaker regularity condition still conserve energy. They derived a local energy equation that contains a term $D(u)$
representing the dissipation or production of energy caused by the lack of smoothness of \( u \); this term can be seen as a local version of Onsager’s original statistically averaged description of energy dissipation. They showed that if \( u \) satisfies

\[
\int |u(t, x + \xi) - u(t, x)|^3 \, dx \leq C(t)|\xi|\sigma(|\xi|),
\]

(2)

where \( \sigma(a) \to 0 \) as \( a \to 0 \) and \( C \in L^1(0, T) \), then \( \|D(u)\|_{L^1(0,T,L^3(\mathbb{T}^3))} = 0 \) and hence the kinetic energy is conserved. The condition in (2) is weaker than (1). A detailed review examining this and further work relating to Onsager’s conjecture is given by Eyink & Sreenivasan (2006).

More recently energy conservation was shown by Cheskidov et al. (2008) when \( u \) lies in the space \( L^3(0, T; B^{1/3}_{3,\infty}(\mathbb{N})) \), where \( B^{1/3}_{3,c(\mathbb{N})} \) is a subspace of \( B^{1/3}_{3,\infty} \). In fact Cheskidov et al. (2008) showed that energy conservation holds for solutions satisfying the still weaker condition

\[
\lim_{q \to \infty} \int_0^T 2^q\|\Delta_q u\|_{L^3}^3 \, dt = 0,
\]

where \( \Delta_q \) performs a smooth restriction of \( u \) into Fourier modes of order \( 2^q \). In a follow-up paper Shvydkoy (2009) (see also Shvydkoy, 2010) states that this condition is equivalent to

\[
\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int |u(x + y) - u(x)|^3 \, dx \, dt = 0,
\]

(3)

and proves a local energy balance under this condition. We observe that condition (2) has similar form to (3), yet explicitly separates the limit and the integrability in time. This makes (3) less restrictive.

In this paper we use an approach similar to that of Shvydkoy (2009), but rather than basing our argument on the approach of Constantin, E, & Titi (1994) we adopt some of the ideas from Duchon & Robert (2000) and give a direct proof that energy conservation follows on the whole domain (this simplifies matters since the pressure no longer plays a role) under the condition that

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \varphi_\varepsilon(\xi) \cdot (v(x + \xi) - v(x))|v(x + \xi) - v(x)|^2 \, d\xi \, dx \to 0
\]

as \( \varepsilon \to 0 \), where \( \varphi \) is an even mollifier.

Given this condition it is relatively simple to show energy conservation under the assumption (3), which we do in Theorem 9, and under the alternative condition

\[
\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^3}{|x - y|^{4+\delta}} \, dx \, dy < \infty, \quad \delta > 0,
\]

(4)
which is equivalent to requiring \( u \in L^3(0, T; W^{\alpha,3}(\mathbb{R}^3)) \) for some \( \alpha > 1/3 \) (Theorem 10).

For the most significant new contribution of this paper we use condition (3) to analyse energy conservation in the domain \( D_+ := \mathbb{T}^2 \times \mathbb{R}_+ \). We show that if \((u, p)\) is a weak solution on \( D_+ \) then \((u_R, p)\) is a weak solution on \( D_- \), where \( u_R \) is an appropriately ‘reflected’ version of \( u \), and that \( u + u_R \) is a weak solution on \( D := \mathbb{T}^2 \times \mathbb{R} \). It follows that energy is conserved for \( u_E \) under condition (3); from here we deduce energy conservation for \( u \) under the condition

\[
\lim_{|y| \to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t, x + y) - u(t, x)|^3 \, dx_3 \, dx_1 \, dx_2 \, dt = 0,
\]

and additional assumptions near \( \partial D_+ \): we assume that \( u \) is continuous at \( \mathbb{T}^2 \times \{0\} \), for almost every \( t \) and \( u \in L^3(0, T; L^\infty(\mathbb{T}^2 \times [0, \delta])) \) for some \( \delta > 0 \), see Theorem 17.

## 2 Energy conservation without boundaries

In this first section we treat the incompressible Euler equations on a domain without boundaries: \( \mathbb{R}^3 \), \( \mathbb{T}^3 \), or one of the hybrid domains \( \mathbb{T} \times \mathbb{R}^2 \) or \( \mathbb{T}^2 \times \mathbb{R} \). We write \( D \) in what follows to denote any one these domains, being careful to highlight any differences required in the definitions/arguments required to deal with the periodic or hybrid cases.

### 2.1 Weak solutions of the Euler equations

For vector-valued functions \( f, g \) and matrix-valued functions \( F, G \) we use the notation

\[
\langle f, g \rangle = \int_D f_i(x) g_i(x) \, dx \quad \text{and} \quad \langle F : G \rangle = \int_D F_{ij}(x) G_{ij}(x) \, dx,
\]

employing Einstein’s summation convention (sum over repeated indices).

We use the notation \( \mathcal{D}(D) \) to denote the collection of \( C^\infty \) functions with compact support in \( D \), and \( \mathcal{S}(D) \) for the collection of all \( C^\infty \) functions with Schwartz-like decay in the unbounded directions of \( D \), e.g. for \( \mathbb{T}^2 \times \mathbb{R} \) we require

\[
\sup_{x \in \mathbb{T}^2 \times \mathbb{R}} |\partial^n \phi||x_3|^k \leq \infty,
\]
for all $\alpha, k \geq 0$ where $\alpha$ is a multi-index over all the spatial variables $(x_1, x_2, x_3)$. Note that in periodic directions the requirement of ‘compact support’ is trivially satisfied. The spaces $\mathcal{D}_\sigma(D)$ and $\mathcal{S}_\sigma(D)$ consist of all divergence-free elements of the $\mathcal{D}(D)$ or $\mathcal{S}(D)$.

We denote by $H_\sigma(D)$ the closure of $\mathcal{D}_\sigma(D)$ in the norm of $L^2(D)$; this coincides with the closure of $\mathcal{S}_\sigma(D)$ in the same norm.

Elements of $H_\sigma(D)$ are divergence free in the sense of distributions, i.e.

$$\langle u, \nabla \phi \rangle = 0 \quad \text{for all} \quad \phi \in \mathcal{D}(D); \quad (5)$$

but in fact this equality holds for all $\phi \in \mathcal{S}(D)$, and even for all $\phi \in H^1(D)$: indeed, since $\mathcal{S}_\sigma(D)$ is dense in $H_\sigma(D)$, for any $u \in H_\sigma(D)$ we can find $(u_n) \in \mathcal{S}_\sigma(D)$ such that $u_n \to u$ in $H^1(D)$, and then for any $\phi \in H^1(D)$ we have

$$\langle u, \nabla \phi \rangle = \lim_{n \to \infty} \langle u_n, \nabla \phi \rangle = \lim_{n \to \infty} \langle \nabla \cdot u_n, \phi \rangle = 0$$

(cf. Lemma 2.11 in Robinson et al., 2016, for example).

In a slight abuse of notation we denote by $C_w([0, T]; H_\sigma)$ the collection of all functions $u: [0, T] \to H_\sigma(D)$ that are weakly continuous into $L^2$, i.e.

$$t \mapsto \langle u(t), \phi \rangle \quad (6)$$

is continuous for every $\phi \in L^2(D)$. Note that $C_w([0, T]; H_\sigma) \subset L^\infty(0, T; H_\sigma)$.

We take as our space-time test functions the elements of

$$S^T_\sigma := \{ \psi \in C^\infty(D \times [0, T]) : \psi(\cdot, t) \in \mathcal{S}_\sigma(D) \text{ for all } t \in [0, T] \}.$$ 

We choose these functions to take values in $\mathcal{S}_\sigma$ (rather than in $\mathcal{D}_\sigma$) since the property of compact support is not preserved by the Helmholtz decomoposition, whereas such a decompostion respects Schwartz-like decay.

**Lemma 1.** Any $\psi \in \mathcal{S}$ can be decomposed as $\psi = \phi + \nabla \chi$, where $\phi \in \mathcal{S}_\sigma$ and $\chi \in \mathcal{S}$, and

$$\| \psi \|_{H^s} + \| \nabla \chi \|_{H^s} \leq C_s \| \psi \|_{H^s} \quad (7)$$

for each $s \geq 0$.

**Proof.** (Cf. Theorem 2.6 and Exercise 5.2 in Robinson, Rodrigo, & Sadowksi, 2016.) Since $\psi \in \mathcal{S}$ we can write $\psi$ as a hybrid Fourier series/inverse Fourier transform, using Fourier
series in the periodic directions and the Fourier transform in the unbounded directions. For example, in the case $D = T^2 \times \mathbb{R}$ we have

$$\phi(x) = \int_{-\infty}^{\infty} \sum_{(k_1, k_2) \in \mathbb{Z}^2} \left( I - \frac{k \otimes k}{|k|^2} \right) \hat{u}(k) e^{ik \cdot x} \, dk_3,$$

and

$$\chi(x) = \int_{-\infty}^{\infty} \sum_{(k_1, k_2) \in \mathbb{Z}^2} \frac{k \cdot \hat{u}(k)}{|k|^2} e^{ik \cdot x} \, dk_3;$$

in the fully periodic case we omit the $k \otimes k/|k|^2$ term when $k = 0$. It is easy to check that these functions have the stated properties. \qed

Assuming that $u$ is a smooth solution of the Euler equations

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0 \quad \nabla \cdot u = 0$$

if we multiply by an element of $S^T_{\sigma}$ and integrate by parts in space and time then we obtain (2) below; the pressure term vanishes since there are no boundaries and $\psi$ is divergence free. Requiring only (2) to hold we obtain our definition of a weak solution.

**Definition 2 (Weak Solution).** We say that $u \in C_w([0, T]; H_{\sigma})$ is a weak solution of the Euler equations on $[0, T]$, arising from the initial condition $u(0) \in H_{\sigma}$, if

$$\langle u(t), \psi(t) \rangle - \langle u(0), \psi(0) \rangle - \int_0^t \langle u(\tau), \partial_t \psi(\tau) \rangle \, d\tau = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \psi(\tau) \rangle \, d\tau$$

for every $t \in [0, T]$ and any $\psi \in S^T_{\sigma}$.

We note here that replacing $S^T_{\sigma}$ by $D^T_{\sigma}$ leads to an equivalent definition (via a simpler version of the argument of Lemma 3, below).

Throughout the paper we let $\phi$ be an even scalar function in $C^\infty_c(B(0, 1))$ with $\int_{\mathbb{R}^3} \phi = 1$ and for any $\varepsilon > 0$ we set $\phi_{\varepsilon}(x) = \varepsilon^{-3}\phi(x/\varepsilon)$. Then for any function $f$ we define the mollification of $f$ as $f_{\varepsilon} := \phi_{\varepsilon} \ast f$ where $\ast$ denotes convolution. Thus

$$f_{\varepsilon}(x) = \phi_{\varepsilon} \ast f(x) := \int_{\mathbb{R}^3} \phi_{\varepsilon}(x-y)f(y) \, dy = \int_{B(0, \varepsilon)} \phi_{\varepsilon}(y)f(x-y) \, dy.$$
In the periodic directions we extend \( f \) by periodicity in this integration. We insist that \( \varphi \) is even since this ensures that the operation of mollification satisfies the ‘symmetry property’

\[
\langle \varphi_\varepsilon \ast u, v \rangle = \langle u, \varphi_\varepsilon \ast v \rangle.
\] (8)

Our aim in the next section is to show the validity of the following two equalities that follow from the definition of a weak solution in (2). The first is

\[
\langle u(t), u_\varepsilon(t) \rangle - \langle u(0), u_\varepsilon(0) \rangle - \int_0^t \langle u(\tau), \partial_t u_\varepsilon(\tau) \rangle \, d\tau = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla u_\varepsilon(\tau) \rangle \, d\tau; \quad (9)
\]

this amounts to using \( u_\varepsilon \), a mollification of the solution \( u \), as a test function in (2): we need to show that there is sufficient time regularity to do this, which we do in Section 2.2. The second is

\[
\int_0^t \langle \partial_t u_\varepsilon(\tau), u(\tau) \rangle \, d\tau = - \int_0^t \langle \nabla \cdot [u(\tau) \otimes u(\tau)]_\varepsilon, u(\tau) \rangle \, d\tau.
\] (10)

One could see this heuristically as a “mollification of the equation” tested with \( u \); we will show that this can be done in a rigorous way in Section 2.3. We can then add these equations and take the limit as \( \varepsilon \to 0 \) to obtain the equation for conservation (or otherwise) of energy (Section 2.4).

### 2.2 Using \( u_\varepsilon \) as a test function

We will show that if \( u \) is a weak solution then in fact (2) holds for a larger class of test functions with less time regularity. We denote by \( C^{0,1}([0,T];H_\sigma) \) the space of Lipschitz functions from \([0,T]\) into \( H_\sigma \).

**Lemma 3.** If \( u \) is a weak solution of the Euler equations in the sense of Definition 2 then (2) holds for every \( \psi \in L_\sigma \), where

\[
L_\sigma := L^1(0,T;H^3) \cap C^{0,1}([0,T];H_\sigma).
\]

**Proof.** For a fixed \( u \) we can write (2) as \( E(\psi) = 0 \) for every \( \psi \in S_\sigma^T \), where

\[
E(\psi) := \langle u(t), \psi(t) \rangle - \langle u(0), \psi(0) \rangle - \int_0^t \langle u(\tau), \partial_t \psi(\tau) \rangle \, d\tau
\]

\[
- \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \psi(\tau) \rangle \, d\tau.
\]

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Since $E$ is linear in $\psi$, and $S^{T}_{\sigma}$ is dense in $L_{\sigma}$ with respect to the norm
\[
\|\psi\|_{L^{1}(0,T;H^{3})} + \|\psi\|_{C^{0,1}([0,T];L^{2})},
\]
to complete the proof it suffices to show that $\psi \mapsto E(\psi)$ is bounded in this norm. We proceed term-by-term:
\[
|\langle u(t), \psi(t) \rangle - \langle u(0), \psi(0) \rangle| \leq 2\|u\|_{L^\infty(0,T;L^{2})}\|\psi\|_{L^\infty(0,T;L^{2})},
\]
using the fact that $u \in C^{w}([0,T];H_{3})$; and
\[
\left| \int_{0}^{t} \langle u(\tau), \frac{\partial}{\partial \tau} \psi(\tau) \rangle \, d\tau \right| \leq T\|u\|_{L^{\infty}(0,T;L^{2})}\|\psi\|_{C^{0,1}([0,T];L^{2})}; \quad \text{and}
\]
\[
\left| \int_{0}^{t} \langle u(\tau) \otimes u(\tau) : \nabla \psi(\tau) \rangle \, d\tau \right| \leq T\|u\|^{2}_{L^{\infty}(0,T;L^{2})}\|\nabla \psi\|_{L^{1}(0,T;L^{\infty})}.
\]
It follows that
\[
|E(\psi)| \leq C\|u\|_{L^{\infty}(0,T;L^{2})}\|\psi\|_{C^{0,1}([0,T];L^{2})} + C\|u\|^{2}_{L^{\infty}(0,T;L^{2})}\|\psi\|_{L^{1}(0,T;H^{3})}
\]
and so we obtain the desired result.

We now study the time regularity of $u$ when paired with a sufficiently smooth function that is not necessarily divergence free.

**Lemma 4.** If $u$ is a weak solution then
\[
|\langle u(t) - u(s), \phi \rangle| \leq C|t - s| \quad \text{for all } \phi \in H^{3}(\mathbb{R}^{3}),
\]
where $C$ depends only on $\|u\|_{L^\infty(0,T;L^{2})}$ and $\|\phi\|_{H^{3}}$.

**Proof.** We use Lemma 1 to decompose $\phi \in \mathcal{S}(D)$ as $\phi = \eta + \nabla \sigma$, where $\eta \in \mathcal{S}_{\sigma}(D)$, $\sigma \in \mathcal{S}(D)$, and
\[
\|\nabla \eta\|_{L^{\infty}} \leq \|\nabla \eta\|_{H^{2}} \leq \|\eta\|_{H^{3}} \leq C\|\phi\|_{H^{3}},
\]
using (7) and the fact that $H^{2}(D) \subset L^{\infty}(D)$. Since $u(t)$ is incompressible for every $t \in [0,T]$, we have
\[
\langle u(t) - u(s), \phi \rangle = \langle u(t) - u(s), \eta + \nabla \sigma \rangle = \langle u(t) - u(s), \eta \rangle.
\]
Since $\eta \in \mathcal{S}_{\sigma}$ and $\partial_{t} \eta = 0$ it follows from the definition of a weak solution at times $t$ and $s$ that
\[
\langle u(t) - u(s), \phi \rangle = \int_{s}^{t} \langle u(\tau) \otimes u(\tau) : \nabla \eta \rangle \, d\tau
\]
and hence
\[
|\langle u(t) - u(s), \phi \rangle| \leq \|u\|^2_{L^\infty(0,T;L^2)} \|\nabla \eta\|_{L^\infty}|t-s| \leq C\|u\|^2_{L^\infty(0,T;L^2)} \|\phi\|_{H^3}|t-s|,
\]
which gives (11) for all \( \phi \in S \). Now simply observe that \( S(D) \) is dense in \( H^3(D) \) to obtain (11) as stated.

A striking corollary of this weak continuity in time is that a mollification in space only yields a function that is Lipschitz continuous in time.

**Corollary 5.** We have \( u_\varepsilon \in L_\sigma \) for any \( \varepsilon > 0 \); in particular the function \( u_\varepsilon(x,\cdot) \) is Lipschitz continuous in \( t \) as a function into \( L^2(D) \):
\[
\|u_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,s)\|_{L^2} \leq C_\varepsilon \|u\|^2_{L^\infty(0,T;L^2)}|t-s|.
\]

**Proof.** Take \( f \in L^2(D) \) with \( \|f\|_{L^2(D)} = 1 \), and let \( \phi = f_\varepsilon \). Then \( \phi \in H^3(D) \), and using the symmetry property (8) we have
\[
\langle u(t) - u(s), \phi \rangle = \langle u(t) - u(s), f_\varepsilon \rangle
= \langle (u_\varepsilon(t) - u_\varepsilon(s)), f \rangle.
\]
Since we have \( \|\phi\|_{H^3} \leq C_\varepsilon \|f\|_{L^2} = C_\varepsilon \) it follows from Lemma 4 that
\[
|\langle u_\varepsilon(t) - u_\varepsilon(s), f \rangle| \leq C_\varepsilon \|u\|^2_{L^\infty(0,T;L^2)}|t-s|.
\]
Since this holds for every \( f \in L^2(D) \) with \( \|f\|_{L^2(D)} = 1 \) we obtain the inequality (12) and \( u_\varepsilon \in C^{0,1}([0,T];L^2) \).

As mollification commutes with differentiation it follows that \( u_\varepsilon \) is divergence free. Finally, since \( u \in L^\infty(0,T;L^2) \), we observe that \( u_\varepsilon \in L^\infty(0,T;H^3) \) and
\[
\|u_\varepsilon\|_{L^1(0,T;H^3)} \leq T\|u\|_{L^\infty(0,T;H^3)}
\]
as \([0,T]\) is bounded.

Since \( u_\varepsilon \in L_\sigma \) it follows from Lemma 3 that we can use \( u_\varepsilon \) as a test function the definition of a weak solution and obtain
\[
\langle u(t), u_\varepsilon(t) \rangle - \langle u(0), u_\varepsilon(0) \rangle - \int_0^t \langle u(\tau), \partial_t u_\varepsilon(\tau) \rangle \, d\tau = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla u_\varepsilon(\tau) \rangle \, d\tau;
\]
we have validated equation (9), the first of the two equalities we need.
2.3 ‘Mollifying the equation’

We will now derive (10). The trick is to test with a mollified test function and move the mollification from the test function onto the terms involving \( u \); all terms are then smooth enough to allow for an integration by parts.

**Lemma 6.** If \( u \) is a weak solution then

\[
\int_0^t \langle \partial_t u_\varepsilon, \phi \rangle \, d\tau = -\int_0^t \langle \nabla \cdot [u \otimes u]_\varepsilon, \phi \rangle \, d\tau
\]

for every \( t \in [0, T] \) and any \( \phi \in \mathcal{S}_T^\sigma \).

**Proof.** Take \( \phi \in \mathcal{S}_T^\sigma \), and use \( \psi := \varphi_\varepsilon \ast \phi \) as the test function in the weak formulation (2). Then

\[
\langle u(t), (\varphi_\varepsilon \ast \phi)(t) \rangle - \langle u(0), (\varphi_\varepsilon \ast \phi)(0) \rangle - \int_0^t \langle u(\tau), \partial_t [\varphi_\varepsilon \ast \phi](\tau) \rangle \, d\tau
\]

\[
= \int_0^t \langle u(\tau) \otimes u(\tau), \varphi_\varepsilon \ast \nabla \phi(\tau) \rangle \, d\tau.
\]

Since the fact that we have chosen \( \varphi \) to be even implies that \( \langle \varphi_\varepsilon \ast u, v \rangle = \langle u, \varphi_\varepsilon \ast v \rangle \) (see (8)) we can move the derivatives and mollification onto the terms involving \( u \). We will do this in detail for the term on the right-hand side, since it is the most complicated; the other terms follow similarly. We obtain

\[
\int_0^t \langle u(\tau) \otimes u(\tau), \nabla [\varphi_\varepsilon \ast \phi](\tau) \rangle \, d\tau = \int_0^t \langle u(\tau) \otimes u(\tau), \varphi_\varepsilon \ast \nabla \phi(\tau) \rangle \, d\tau
\]

\[
= \int_0^t \langle [u(\tau) \otimes u(\tau)]_\varepsilon, \nabla \phi(\tau) \rangle \, d\tau = \int_0^t \langle \nabla \cdot [u(\tau) \otimes u(\tau)]_\varepsilon, \phi(\tau) \rangle \, d\tau.
\]

This implies that

\[
\langle u_\varepsilon(t), \phi(t) \rangle - \langle u_\varepsilon(0), \phi(0) \rangle - \int_0^t \langle u_\varepsilon(\tau), \partial_t \phi(\tau) \rangle \, d\tau
\]

\[
= \int_0^t \langle \nabla \cdot [u(\tau) \otimes u(\tau)]_\varepsilon, \phi(\tau) \rangle \, d\tau.
\]

Since \( u_\varepsilon \) and \( \phi \) are both absolutely continuous in time, the integration-by-parts formula

\[
\langle u_\varepsilon(t), \phi(t) \rangle - \langle u_\varepsilon(0), \phi(0) \rangle - \int_0^t \langle u_\varepsilon(\tau), \partial_t \phi(\tau) \rangle \, d\tau = \int_0^t \langle \partial_t u_\varepsilon(\tau), \phi(\tau) \rangle \, d\tau
\]

finishes the proof. \( \Box \)
We now show that (13) holds for a much larger class of functions than \( \phi \in S^T \).

**Lemma 7.** If \( u \) is a weak solution and in addition \( u \in L^3(0,T; L^3) \) then (13) holds for any \( \phi \in L^3(0,T; L^3) \cap C_w(0,T; H_\sigma) \).

(Recall that we use \( C_w(0,T; H_\sigma) \) to denote \( H_\sigma \)-valued functions that are weakly continuous into \( L^2 \).)

**Proof.** First we will obtain from (13) an equation that holds for all test functions \( \psi \) from the space \( S(D \times [0,T]) \), not just for \( \psi \in S^T \). For this we will use the Leray projection \( P \), the projection onto divergence-free vector fields. Since for any \( \psi \in S(D \times [0,T]) \) we have \( P \psi \in S^T \), it follows from (13) that

\[
\int_0^t \langle \partial_t u_\varepsilon + \nabla \cdot (u \otimes u)_\varepsilon, P \psi \rangle \, d\tau = 0.
\]

Since \( P \) is symmetric (\( \langle P g, f \rangle = \langle g, P f \rangle \)) and \( P \partial_t u_\varepsilon = \partial_t u_\varepsilon \) (since \( P \) commutes with derivatives and \( u_\varepsilon \) is incompressible) we obtain

\[
\int_0^t \langle \partial_t u_\varepsilon + P(\nabla \cdot (u \otimes u)_\varepsilon), \psi \rangle \, d\tau = 0 \quad \text{for every} \quad \psi \in S(D \times [0,T]).
\]

Since \( u_\varepsilon \) is Lipschitz in time (as a function from \([0,T]\) into \( H_\sigma\) its time derivative \( \partial_t u_\varepsilon \) exists almost everywhere (see Theorem 5.5.4 in Albiac & Kalton (2016), for example) and is integrable; we can therefore deduce using the Fundamental Lemma of the Calculus of Variations (\( u \in L^2(\Omega) \) with \( \int_\Omega u = 0 \) for all \( \psi \in C^\infty_c(\Omega) \) implies that \( u = 0 \) almost everywhere in \( \Omega \), see e.g. Lemma 3.2.3 in Jost & Li-Jost (1998)) that for almost every \((x,t) \in D \times [0,T]\)

\[
\partial_t u_\varepsilon + P(\nabla \cdot (u \otimes u)_\varepsilon) = 0.
\]

Observing that \( P \nabla \cdot (u \otimes u)_\varepsilon \in L^{3/2}(0,T; L^{3/2}) \) and that \( \partial_t u_\varepsilon \) has the same integrability since \( \partial_t u_\varepsilon = -P \nabla \cdot (u \otimes u)_\varepsilon \), we can now multiply this equality by any choice of function \( \phi \in L^3(0,T; L^3) \cap C_w(0,T; H_\sigma) \) and integrate:

\[
\int_0^t \langle \partial_t u_\varepsilon, \phi \rangle \, d\tau = -\int_0^t \langle P \nabla \cdot (u \otimes u)_\varepsilon, \phi \rangle \, d\tau
\]

\[
= -\int_0^t \langle \nabla \cdot (u \otimes u)_\varepsilon, P \phi \rangle \, d\tau = -\int_0^t \langle \nabla \cdot (u \otimes u)_\varepsilon, \phi \rangle \, d\tau,
\]

where we have used the fact that \( P \phi = \phi \) since \( \phi(t) \in H_\sigma \) for every \( t \in [0,T] \).
Note that the condition on $u \in L^3(0,T; L^3)$ is stronger than necessary for the proof but since Theorem 9 will need this condition the above result will suffice for our purposes.

We can now use $u$ as a test function in (13) and thereby obtain equation (10), the second of the equalities we need.

### 2.4 Energy Conservation

We can now add equations (9) and (10) to obtain

$$
\langle u(t), u_\varepsilon(t) \rangle - \langle u(0), u_\varepsilon(0) \rangle = 
\int_0^t \langle u(\tau) \otimes u(\tau) : \nabla u_\varepsilon(\tau) \rangle - \langle \nabla \cdot [u(\tau) \otimes u(\tau)]_\varepsilon, u(\tau) \rangle \, d\tau. 
$$

(14)

In order to proceed we will need the following identity. We note that its validity is entirely independent of the Euler equations, but relies crucially on the fact that $\varphi$ is even.

**Lemma 8.** Suppose that $v \in L^3 \cap H_\sigma$ and define

$$
J_\varepsilon(v) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \varphi_\varepsilon(\xi) \cdot (v(x + \xi) - v(x)) v(x + \xi - v(x))^2 \, d\xi \, dx.
$$

Then

$$
\frac{1}{2} J_\varepsilon(v) = \langle \nabla \cdot [v(\tau) \otimes v(\tau)]_\varepsilon, v(\tau) \rangle - \langle v(\tau) \otimes v(\tau) : \nabla v_\varepsilon(\tau) \rangle.
$$

**Proof.** We have

$$
J_\varepsilon(v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(\xi) (v_i(x + \xi) - v_i(x)) (v_j(x + \xi) - v_j(x)) (v_j(x + \xi) - v_j(x)) \, d\xi \, dx.
$$

Expanding the expression for $J_\varepsilon(v)$ yields

$$
\int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(\xi) v_i(x + \xi) v_j(x + \xi) v_j(x + \xi) \, d\xi - \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(\xi) v_i(x) v_j(x) v_j(x) \, d\xi \right. \\
+ \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(\xi) v_i(x + \xi) v_j(x) v_j(x) \, d\xi - \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(\xi) v_i(x) v_j(x + \xi) v_j(x + \xi) \, d\xi \\
+ 2 \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon v_i(x + \xi) v_j(x) v_j(x) \, d\xi - \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon v_i(x + \xi) v_j(x + \xi) v_j(x) \, d\xi \right\} \, dx.
$$
Note that the second term is zero since \( \varphi_\varepsilon \) has compact support, and the third term is zero since \( v \) is incompressible. For the fourth term we can change variable and set \( \eta = x + \xi \) to obtain

\[
- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(\eta - x)v_i(x)v_j(\eta)v_j(\eta) \, d\eta \, dx.
\]

As \( \partial_i \varphi_\varepsilon \) is an odd function we have

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(x - \eta)v_i(x)v_j(\eta)v_j(\eta) \, d\eta \, dx,
\]

which becomes

\[
\int_{\mathbb{R}^3} v_i(x) \partial_{x_i} \left[ \int_{\mathbb{R}^3} \varphi_\varepsilon(x - \eta)v_j(\eta)v_j(\eta) \, d\eta \right] \, dx = \int_{\mathbb{R}^3} v_i(x) \partial_i ([v_jv_j]_\varepsilon) \, dx = 0,
\]

where again the term becomes zero as we use the incompressibility of \( v \). A similar calculation for the first term gives

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \partial_i \varphi_\varepsilon(\xi)v_i(x + \xi)v_j(x + \xi)v_j(x + \xi) \, d\xi \, dx = \int_{\mathbb{R}^3} \partial_i ([v_iv_jv_j]_\varepsilon)(x) \, dx = 0,
\]

using periodicity. For the final two terms similar calculations yield

\[
2 \int_{\mathbb{R}^3} [v_j \partial_i(v_jv_j)_\varepsilon - v_jv_i \partial_i(v_jv_j)_\varepsilon] \, dx = 2 \left[ \langle \nabla \cdot [v \otimes v]_\varepsilon, v \rangle - \langle v \otimes v : \nabla v_\varepsilon \rangle \right]
\]

and the result follows. \( \square \)

Note that here again the assumption that \( v \in L^3 \) is stronger than needed but will hold when we use the result in Theorem 9.

We now want to look at the limit as \( \varepsilon \to 0 \) and see what condition on the solution is needed for the right hand side of (14) to converge to zero.

Let \( t_1, t_2 \in [0, T] \) with \( t_1 < t_2 \). We can set \( t = t_1 \) in (14) and also set \( t = t_2 \) in (14) and then take the difference of these two equations to obtain

\[
\langle u(t_2), u_\varepsilon(t_2) \rangle - \langle u(t_1), u_\varepsilon(t_1) \rangle
\]

\[
= \int_{t_1}^{t_2} \langle u(\tau) \otimes u(\tau) : \nabla u_\varepsilon(\tau) \rangle - \langle \nabla \cdot [u(\tau) \otimes u(\tau)]_\varepsilon, u(\tau) \rangle \, d\tau
\]

\[
= -\frac{1}{2} \int_{t_1}^{t_2} J_\varepsilon(u) \, dt.
\]

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Therefore talking the limit as $\varepsilon \to 0$, since $u \in C_w([0,T]; H_\sigma)$ we obtain
\[
\|u(t_2)\|_{L^2} - \|u(t_1)\|_{L^2} = -\frac{1}{2} \lim_{\varepsilon \to 0} \int_{t_1}^{t_2} J_\varepsilon(u) \, dt.
\]
Hence any condition on $u$ that guarantees that
\[
\lim_{\varepsilon \to 0} \int_{t_1}^{t_2} J_\varepsilon(u) \, dt \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (15)
\]
ensures energy conservation. We give two such conditions in the next section.

3 Two spatial conditions for energy conservation in the absence of boundaries

First we provide another proof (cf. Shvydkoy, 2009) of energy conservation under condition (3).

**Theorem 9.** If $u \in L^3(0,T; L^3(D))$ is a weak solution of the Euler equations that satisfies
\[
\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int_D |u(t,x + y) - u(t,x)|^3 \, dx \, dt = 0 \quad (16)
\]
then energy is conserved on $[0,T]$.

**Proof.** We take $t_1, t_2$ with $0 \leq t_1 \leq t_2 \leq T$, and consider the integral of $|J_\varepsilon(u)|$ over $[t_1, t_2]$; our aim is to show that this is zero in the limit as $\varepsilon \to 0$. We start by noticing that
\[
\int_{t_1}^{t_2} |J_\varepsilon(u)| \, dt \leq \int_{t_1}^{t_2} \int_D \frac{1}{\varepsilon^4} \left| \nabla \varphi \left( \frac{\xi}{\varepsilon} \right) \right| \left| u(x + \xi) - u(x) \right| \, dx \, dt.
\]
We can then change variables $\xi = \eta \varepsilon$ and obtain,
\[
\int_{t_1}^{t_2} |J_\varepsilon(u)| \, dt \leq \int_{t_1}^{t_2} \int_D \frac{1}{\varepsilon} \left| \nabla \varphi (\eta) \right| \left| u(x + \varepsilon \eta) - u(x) \right| \, dx \, dt.
\]
Using Fubini’s Theorem we can exchange the order of the integrals:
\[
\int_{t_1}^{t_2} |J_\varepsilon(u)| \, dt \leq \int_{\mathbb{R}^3} \int_{t_1}^{t_2} \frac{\left| u(x + \varepsilon \eta) - u(x) \right|^3}{|\varepsilon \eta|} \, dx \, dt \, |\eta| \left| \nabla \varphi (\eta) \right| \, d\eta.
\]
Taking limits as $\varepsilon$ goes to zero
\[
\lim_{\varepsilon \to 0} \int_{t_1}^{t_2} |J_\varepsilon(u)| \, dt \leq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \int_{t_1}^{t_2} \int_D \frac{|u(x + \varepsilon \eta) - u(x)|^3}{|\varepsilon \eta|} \, dx \, d\eta \, d\varepsilon \, \|\nabla \varphi(\eta)\| \, d\eta.
\]
We are finished if we can exchange the outer integral and limit. This can be done using the Dominated Convergence Theorem. To do this we define the non-negative function,
\[
f(y) = \frac{1}{|y|} \int_{t_1}^{t_2} \int_D |u(x + y) - u(x)|^3 \, dx \, dt.
\]
By assumption $\lim_{|y| \to 0} f(y) = 0$, thus for any $\varepsilon > 0$, we have $\sup_{y \in B_0(\varepsilon)} f(y) \leq K$ for some $K = K(\varepsilon)$. Further, $\text{supp}(\varphi)$ is compact. Combining these facts we obtain a dominating integrable function
\[
g(\eta) := K|\eta|\|\nabla \varphi(\eta)\|,
\]
and the result follows.

We now show how the general condition in (15) allows for a simple proof of energy conservation when $u \in L^3(0, T; W^{\alpha,3}(\mathbb{R}^3))$ for any $\alpha > 1/3$. The use of condition (17) to characterise this space is due independently to Aronszajn, Gagliardo, and Slobodeckij, see Di Nezza, Palatucci, & Valdinoci (2012), for example.

**Theorem 10.** If $u$ is a weak solution of the Euler equations on the whole space that satisfies $u \in L^3(0, T; W^{\alpha,3}(\mathbb{R}^3))$ for some $\alpha > 1/3$, i.e. if $u \in L^3(0, T; L^3(\mathbb{R}^3))$ and
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^3}{|x - y|^{3+3\alpha}} \, dx \, dy < \infty, \tag{17}
\]
then energy is conserved.

**Proof.** First observe that for $\alpha > 1/3$ the space $W^{\alpha,3}$ has a factor $|x-y|^{4+\delta}$ in the denominator of (17), where $\delta = 3\alpha - 1 > 0$.

As in the previous proof, our starting point is that
\[
\int_{t_1}^{t_2} |J_\varepsilon(u)| \, dt \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left\|\nabla \varphi\left(\frac{\xi}{\varepsilon}\right)\right\| |u(x + \xi) - u(x)|^3 \, d\xi \, dx \, dt.
\]
We can write
\[ \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \frac{1}{\varepsilon^4} |\nabla \varphi \left( \frac{y-x}{\varepsilon} \right)| |u(y) - u(x)|^3 \, d\xi \, dx \, dt \]
\[ = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \frac{1}{\varepsilon^4} |\nabla \varphi \left( \frac{y-x}{\varepsilon} \right)| |u(y) - u(x)|^3 \, d\xi \, dx \, dt \]
\[ = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \frac{|y-x|^{4+\delta}}{\varepsilon^4} |\nabla \varphi \left( \frac{y-x}{\varepsilon} \right)| |u(y) - u(x)|^3 \, d\xi \, dx \, dt \]
\[ \leq cK\varepsilon^\delta \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \frac{|y-x|^{4+\delta}}{\varepsilon^4} \, dy \, dx \, dt = c\varepsilon^\delta, \]

since \( \|\nabla\varphi\|_{L^\infty} \leq K\varphi \) and the integrand is only non-zero within the support of \( \varphi \), i.e. where \( |y-x| \leq 2\varepsilon \). Energy conservation now follows. \( \square \)

4 Energy Balance on \( \mathbb{T}^2 \times \mathbb{R}_+ \)

In this section we first give a definition of what it means to have a weak solution on the domain \( D_+ := \mathbb{T}^2 \times \mathbb{R}_+ \), where \( \mathbb{R}_+ = [0, \infty) \), that is suitable for our purposes. We then show that such a solution \( u \) can be extended to a weak solution \( u_E \) on the boundary-free domain \( D := \mathbb{T}^2 \times \mathbb{R} \). (Note that in this section we reserve \( D \) for this particular domain.)

4.1 Weak solutions of the Euler equations on \( D_+ := \mathbb{T}^2 \times \mathbb{R}_+ \)

We define \( S(D_+) \) and \( S_\sigma(D_+) \) by restricting functions in \( S(\mathbb{T}^2 \times \mathbb{R}) \) and \( S_\sigma(\mathbb{T}^2 \times \mathbb{R}) \) to \( D_+ \); this means that we have Schwartz-like decay in the unbounded direction, and that the functions have a smooth restriction to the boundary.

We define \( H_\sigma(D_+) \) to be the completion of \( D_\sigma(D_+) \) in the norm of \( L^2(D_+) \); this is equivalent to the completion of
\[ S_{n,\sigma}(D_+) := \{ \phi \in S(D_+) : \nabla \cdot \phi = 0 \text{ and } \phi_3 = 0 \text{ on } \partial D_+ \} \]
in the same norm. Functions in \( H_\sigma(D_+) \) are weakly divergence free in that they satisfy
\[ \langle u, \nabla \phi \rangle = 0 \quad \text{for every} \quad \phi \in H^1(D_+); \quad (18) \]
that this holds for every $\phi \in H^1(D_+)$ and not only for $\phi \in D(D_+)$ (proved exactly as in
Section 2.1) will be useful in what follows.

As before, in a slight abuse of notation we denote by $C_w([0,T]; H_\sigma(D_+))$ the collection
of all functions $u : [0,T] \to H_\sigma(D_+)$ that are weakly continuous into $L^2(D_+)$ i.e.

\[ t \mapsto \langle u(t), \phi \rangle_{D_+} \tag{19} \]

is continuous for every $\phi \in L^2(D_+)$. We define

\[ S^T_\sigma(D_+) := \{ \psi \in C^\infty(D_+ \times [0,T]) : \psi(\cdot, t) \in S_\sigma(D_+) \text{ for every } t \in [0,T] \}, \]

which will be our space of test functions; note that these functions are smooth and incom-
pressible, but there is no restriction on their values on $\partial D_+$.

To obtain a weak formulation of the equations on $D_+$ we consider first a smooth solution $u$ with pressure $p$ that satisfies the Euler equations

\[
\begin{cases}
\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 & \text{in } D_+ \\
\nabla \cdot u = 0 & \text{in } D_+ \\
u \cdot n = 0 & \text{on } \partial D_+,
\end{cases}
\]

where $n$ is the normal to $\partial D_+$, so that the third equation is in fact $u_3 = 0$ on $\partial D_+$. We can now multiply the first line by a test function $\phi \in S^T_\sigma$ and integrate over space and time to give

\[
\int_0^t \langle \partial_t u + \nabla \cdot (u \otimes u) + \nabla p, \phi \rangle_{D_+} d\tau = 0.
\]

We can now integrate by parts and obtain

\[
\langle u(t), \phi(t) \rangle_{D_+} - \langle u(0), \phi(0) \rangle_{D_+} - \int_0^t \langle u, \partial_t \phi \rangle_{D_+} d\tau - \int_0^t \langle u \otimes u : \nabla \phi \rangle_{D_+} d\tau - \int_0^t \langle u_3, \phi \rangle_{\partial D_+ \times [0,t]} d\tau + \int_0^t \langle p, \nabla \cdot \phi \rangle_{D_+} d\tau + \langle p, \phi \cdot n \rangle_{\partial D_+ \times [0,t]} = 0.
\]

We notice that as $u_3 = 0$ on $\partial D_+$ and $\nabla \cdot \phi = 0$ in $D_+$ the two terms involving these expressions vanish and we have

\[
\langle u(t), \phi(t) \rangle_{D_+} - \langle u(0), \phi(0) \rangle_{D_+} - \int_0^t \langle u, \partial_t \phi \rangle_{D_+} d\tau - \int_0^t \langle u \otimes u : \nabla \phi \rangle_{D_+} d\tau + \langle p, \phi \cdot n \rangle_{\partial D_+ \times [0,t]} = 0.
\]
Since we have not restricted the values of \( \phi \) on \( \partial D_+ \) we have a contribution from the boundary, namely
\[
\langle p, \phi_3 \rangle_{\partial D_+ \times [0,t]}.
\]
We therefore require \( p \in \mathcal{D}'(\partial D_+ \times [0,T]) \) in our definition of a weak solution.

**Definition 11 (Weak Solution on \( D_+ \)).** A weak solution of the Euler equations on \( D_+ \times [0,T] \) is a pair \((u,p)\), where \( u \in C_w([0,T]; H^s(D_+)) \) and \( p \in \mathcal{D}'(\partial D_+ \times [0,T]) \) such that
\[
\langle u(t), \phi(t) \rangle_{D_+} - \langle u(0), \phi(0) \rangle_{D_+} - \int_0^t \langle u(\tau), \partial_t \phi(\tau) \rangle_{D_+} \, d\tau
\]
\[
= \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \phi(\tau) \rangle_{D_+} \, d\tau - \langle p, \phi \cdot n \rangle_{\partial D_+ \times [0,t]}, \tag{20}
\]
for every \( t \in [0,T] \) and for every \( \phi \in S_T^T(D_+) \).

Note that in the final term, \( \phi \cdot n = -\phi_3 \).

### 4.2 Half plane reflection map

We introduce an extension \( u_E \) that takes a weak solution \( u \) defined in \( D_+ \) to one defined on the whole of \( D \). Essentially we extend ‘by reflection’, with appropriate sign changes to ensure that \( u_R \), the ‘reflection’ of \( u \), is a weak solution on \( D_- := \mathbb{T}^2 \times \mathbb{R}_- \). We can then show that \( u_E := u + u_R \) is a weak solution on the whole of \( D \) (in the sense of Definition 2).

Given a vector-valued function \( f : D_\pm \to \mathbb{R}^3 \) we define \( f_R : D_\pm \to \mathbb{R}^3 \) by
\[
f_R(x, y, z) := \begin{pmatrix}
    f_1(x, y, -z) \\
    f_2(x, y, -z) \\
    -f_3(x, y, -z)
\end{pmatrix}
\]
extending \( f \) and \( f_R \) by zero beyond their natural domain of definition, we set
\[
f_E(x, y, z) := \begin{cases}
    f(x, y, z) + f_R(x, y, z) & z \neq 0 \\
    \frac{1}{2}(f(x, y, z) + f_R(x, y, z)) = (f_1(x, y, 0), f_2(x, y, 0), 0) & z = 0.
\end{cases}
\]
Clearly \( f_E = f + f_R \) almost everywhere.

**Lemma 12.** If \( u \in H_s(D_+) \) then \( u_R \in H_s(D_-) \) and \( u_E \in H_s(D) \).
Proof. The only claim that requires proof is that $u_E$ remains weakly divergence-free, despite possible issues near $x_3 = 0$. However, given any $\phi \in \mathcal{D}(D)$ we can write $\phi = \phi_+ + \phi_-$, where $\phi_{\pm} := \phi|_{D_\pm} \in H^1(D_\pm)$; we can therefore use (18) to write

$$\langle u_E, \nabla \phi \rangle = \langle u, \nabla \phi_+ \rangle + \langle u_R, \nabla \phi_- \rangle = 0$$

and $u_E$ is weakly divergence-free as claimed.

Now we will show that, with an appropriate choice of the pressure, $u_R$ is a weak solution of the Euler equations in the lower half space $D_-$. Note that we do not need to extend the pressure distribution $p$.

Theorem 13. If $(u, p)$ is a weak solution to the Euler equations on $D_+$ then $(u_R, p)$ is a weak solution in $D_-$, i.e.

$$\langle u_R(t), \phi(t) \rangle_{D_-} - \langle u_R(0), \phi(0) \rangle_{D_-} - \int_0^t \langle u_R(\tau), \partial_t \phi(\tau) \rangle_{D_-} \, d\tau$$

$$= \int_0^t \langle u_R(\tau) \otimes u_R(\tau) : \nabla \phi(\tau) \rangle_{D_-} \, d\tau - \langle p, \phi \cdot n \rangle_{\partial D_- \times [0, t]}$$

for every $t \in [0, T]$ and for every $\phi \in \mathcal{S}^T_\sigma(D_-)$.

Note that now in the final term we have $\phi \cdot n = \phi_3$.

Proof. Notice first that any $\phi \in \mathcal{S}^T_\sigma(D_-)$ can be written as $\psi_R$, where $\psi = \phi_R \in \mathcal{S}^T_\sigma(D_+)$. Now, the change of variables $(x_1, x_2, x_3) \to (y_1, y_2, -y_3)$ in the linear term yields

$$\langle u_R, \psi_R \rangle_{D_-} = \langle u, \psi \rangle_{D_+}.$$

For the nonlinear term one can check case-by-case, with the same change of variables, that

$$\int_{D_-} [(u_R)_i (u_R)_j \partial_j (\psi_R)_i](x) \, dx = \int_{D_+} [u_i u_j \partial_j \psi_i](y) \, dy.$$

Finally for the pressure term we have

$$\langle p, \psi \cdot n \rangle_{\partial D_+} = \langle p, \psi_3 \rangle = -\langle p, \phi_3 \rangle = \langle p, \phi \cdot n \rangle_{\partial D_-},$$

since $\psi_3(x, y, 0) = -\phi_3(x, y, 0)$. 

\[\]
By adding (20) and (21) it follows that $u_E$ is a weak solution of $D$.

**Corollary 14.** The extension $u_E$ is a weak solution of the Euler equations on $D$ in the sense of Definition 2.

**Proof.** For $\zeta \in \mathcal{S}_T^T$ we can use $\zeta|_{D_+}$ as a test function in (20) and $\zeta|_{D_-}$ in (21) and add the two equations to obtain

$$\langle u_E(t), \zeta(t) \rangle_D - \langle u_E(0), \zeta(0) \rangle_D - \int_0^t \langle u_E(\tau), \partial_\tau \zeta(\tau) \rangle_D \, d\tau = \int_0^t \langle u_E(\tau) \otimes u_E(\tau) : \nabla \zeta(\tau) \rangle_D \, d\tau,$$

where the pressure terms have cancelled due to the opposite signs of the normal in the two domains; but this is now the definition of a weak solution of the Euler equations in $D$. \qed

Since $u_E$ is a weak solution of the incompressible Euler equations on $D$, Corollary 9 guarantees that if $u_E \in L^3(0,T;L^3(D))$ and

$$\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int_D |u_E(t,x+y) - u_E(t,x)|^3 \, dx \, dt = 0 \quad (22)$$

then $u_E$ conserves energy on $D \times [t_1,t_2]$. Due to the definition of $u_E$ this implies that

$$\|u_E(t_2)\|_{L^2(D)}^2 - \|u_E(t_1)\|_{L^2(D)}^2 = 2\|u(t_2)\|_{L^2(D_+)}^2 - 2\|u(t_1)\|_{L^2(D_+)}^2 = 0,$$

i.e. we obtain energy conservation for $u$. We now find conditions on $u$ alone (rather than $u_E = u + u_R$) that guarantee that (22) is satisfied.

## 5 Energy Conservation on $D_+$

Here we will prove our main result in Theorem 17: energy conservation on $D_+$ under certain assumptions on the weak solution $u$. The two bulk conditions we need for $u$ to conserve energy are similar to the conditions needed for Corollary 9 where we had no boundary. We will impose two extra conditions to deal with the presence of the boundary: firstly, that there exists a $\delta > 0$ such that $u \in L^3(0,T;L^\infty(T^2 \times [0,\delta)))$; secondly that $u(\cdot,t)$ is continuous at the boundary for almost every $t$.

We make some preliminary definitions and observations concerning the kind of continuity we require at $\partial D_+$. 
Definition 15 (Continuity at a Subset). We say that a function $f$ defined on $\Omega$ is $C_\Gamma$, for $\Gamma \subset \Omega$, if for all $x \in \Gamma$ and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$y \in \Omega \text{ and } |y - x| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$ 

If $\Gamma$ is a compact subset of $\Omega$ then $f \in C_\Gamma$ is in fact uniformly continuous at the subset, in the following sense.

Lemma 16. If $f$ is $C_\Gamma$ and $\Gamma$ is compact then for all $\varepsilon > 0$ there exist $\delta > 0$ such that for all $x \in \Gamma$

$$y \in \Omega \text{ and } |y - x| < \delta \quad \Rightarrow \quad |f(y) - f(x)| < \varepsilon;$$

in particular, there exists a function $w: [0, \infty) \to [0, \infty)$ with $w(0) = 0$ and continuous at $0$, such that

$$|f(x + z) - f(x)| < w(|z|)$$

whenever $x \in \Gamma$ and $x + z \in \Omega$.

Proof. Fix $\varepsilon > 0$ and take a sequence $y_n \in \Omega$ and $x_n \in \Gamma$ and assume that $|y_n - z_n| < \frac{1}{n}$ but $|f(y_n) - f(x_n)| \geq \varepsilon$. However, we know that $\Gamma$ is compact and so there exists subsequences $y_{n_j} \to x$ and $x_{n_j} \to x$; by applying continuity at a subset for $f$ we have $f(y_{n_j}) \to f(x)$ and $f(x_{n_j}) \to f(x)$, a contradiction.

We can now provide conditions on $u$ to ensure energy conservation.

Theorem 17. Let $u \in L^3(0,T;L^3(D_+))$ be a weak solution of the Euler equations that satisfies $u \in L^3(0,T;L^\infty(\mathbb{T}^2 \times [0,\delta]))$ for some $\delta > 0$, $u(\cdot,t) \in C_{\partial D_+}$ for almost every $t$, and

$$\lim_{|y| \to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \int_{|y|}^\infty |u(t,x+y) - u(t,x)|^3 \, dx \, dy \, dt = 0; \quad (23)$$

then $u$ conserves energy on $D_+ \times [t_1,t_2]$.

Proof. We can split

$$\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int_D |u_E(t,x+y) - u_E(t,x)|^3 \, dx \, dt = 0$$

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up into three sub-integrals over the regions $A := \{x|x_3 > |y|\}$, $B := \{x|x_3 < -|y|\}$ and $C := \{x||x_3| \leq |y|\}$. We have

$$|u_E(t, x + y) - u_E(t, x)|^3 \leq \left( [\mathbb{I}_A(x) + \mathbb{I}_B(x) + \mathbb{I}_C(x)]|u_E(x + y) - u_E(x)| \right)^3$$

$$= [\mathbb{I}_A(x) + \mathbb{I}_B(x) + \mathbb{I}_C(x)]|u_E(x + y) - u_E(x)|^3.$$  

For $\int_A$ we see that since $x_3 > 0$ and $x_3 + y_3 > 0$ then $u_E$ is in fact $u$, thus after integrating and taking the limit it goes to zero by (23). For $\int_B$ a very similar argument (but with the change of variables $x_3 \mapsto -z_3$) gives the same outcome.

We are left with $\int_C$: we need to show that

$$\lim_{|y| \to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \int_{-\|y\|}^{\|y\|} \int_{-\|y\|}^{\|y\|} |u_E(t, x + y) - u_E(t, x)|^3 \, dx_3 \, dx_2 \, dx_1 \, dt = 0.$$  

We have assumed that $u \in L^3(0, T; L^\infty(\mathbb{T}^2 \times [0, \delta]))$ and so

$$u_E \in L^3(0, T; L^\infty(\mathbb{T}^2 \times (-\delta, \delta))).$$

Then, since for all $|y| < \delta$ we have

$$\frac{1}{|y|} \int_{\mathbb{T}^2} \int_{-\|y\|}^{\|y\|} |u_E(t, x + y) - u_E(t, x)|^3 \, dx_3 \, dx_2 \, dx_1 \leq C \sup_{x \in \mathbb{T}^2 \times [0, \delta]} |u(t)|^3,$$  

we can then move the limit inside the time integral using the Dominated Convergence Theorem, and it suffices to show that

$$\lim_{|y| \to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \int_{-\|y\|}^{\|y\|} |u_E(t, x + y) - u_E(t, x)|^3 \, dx_3 \, dx_2 \, dx_1 = 0$$

for almost every $t \in (t_1, t_2)$.

As, $u(\cdot, t) \in C_{BD+}$ and $u \cdot n = 0$ on the boundary, the boundary values are the same for $u$ and $u_R$ and so $u_E(\cdot, t) \in C_{\{z = 0\}}$.

Now fix $t$ and let $x' \in \{z = 0\}$; then

$$|u_E(t, x' + z + y) - u_E(t, x' + z)| \leq |u_E(t, x' + z + y) - u_E(t, x') + u_E(t, x') - u_E(t, x' + z)|$$

$$\quad \leq w(t, |y + z|) + w(t, |z|) \leq 2w(t, 2|y|)$$

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and thus
\[
\frac{1}{|y|} \int_{\mathbb{T}^2} \int_{|y|}^{1/|y|} |u_E(t, x + y) - u_E(t, x)|^3 \, dx_3 \, dx_2 \, dx_1 \leq C \frac{1}{|y|} \int_{\mathbb{T}^2} \int_{|y|}^{1/|y|} |w(t, 2|y|)|^3 \, dx_3 \, dx_2 \, dx_1 \leq C \frac{1}{|y|^2} \int_{\mathbb{T}^2} |y|^2 \, |w(t, 2|y|)|^3 \to 0
\]
as $|y| \to 0$, which is what we required. \hfill \square

We note that the continuity and boundary assumptions required in the theorem could be combined into $L^3(0, T; C(\mathbb{T}^2 \times [0, \delta]))$ for some $\delta > 0$, or $L^3(0, T; C^\alpha(\mathbb{T}^2 \times [0, \delta]))$ for some $\alpha > 0$. In fact all the conditions for this theorem are satisfied by a weak solution $u$ that satisfies
\[
|u(x, t) - u(y, t)| \leq C f(x_3) |x - y|^\alpha
\]
for $\alpha > \frac{1}{3}$ and $f \in L^3(0, \infty)$.

6 Conclusion

Assuming the simple integral condition
\[
\lim_{|y| \to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t, x + y) - u(t, x)|^3 \, dx_3 \, dx_1 \, dx_2 \, dt = 0,
\]
which is similar to the weakest condition known on $\mathbb{R}^3$ or $\mathbb{T}^3$, and appropriate continuity at the boundary we have proved energy conservation of the incompressible Euler equations with a flat boundary of finite area. Further, our methods do not depend on the dimension so analogues hold in $\mathbb{T}^{d-1} \times \mathbb{R}_+$ for $d \geq 2$.

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