Bose–Einstein Condensation with Optimal Rate for Trapped Bosons in the Gross–Pitaevskii Regime

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Abstract
We consider a Bose gas consisting of \(N\) particles in \(\mathbb{R}^3\), trapped by an external field and interacting through a two-body potential with scattering length of order \(N^{-1}\). We prove that low energy states exhibit complete Bose–Einstein condensation with optimal rate, generalizing previous work in Boccato et al. (Commun Math Phys 359(3):975–1026, 2018; 376:1311–1395, 2020), restricted to translation invariant systems. This extends recent results in Nam et al. (Preprint, 2001. arXiv:2001.04364), removing the smallness assumption on the size of the scattering length.

Keywords  Bose-Einstein condensation · Interacting bosons · Gross-Pitaevskii regime

1 Introduction and Main Results
We consider a system of \(N \in \mathbb{N}\) bosons trapped by an external potential in the Gross–Pitaevskii regime; the particles interact through a repulsive two-body potential with scattering length of order \(N^{-1}\). The Hamilton operator has the form

\[
H_N = \sum_{j=1}^{N} \left[-\Delta x_j + V_{\text{ext}}(x_j)\right] + \sum_{1 \leq i < j \leq N} N^2 V(N(x_i - x_j)) \quad (1.1)
\]

and it acts on a dense subspace of the Hilbert space \(L^2_t(\mathbb{R}^{3N})\), the subspace of \(L^2(\mathbb{R}^{3N})\) consisting of functions that are symmetric with respect to permutations of the \(N\) particles. The confining potential \(V_{\text{ext}} \in L^\infty_{\text{loc}}(\mathbb{R}^3)\) diverge to infinity, as \(|x| \to \infty\) (more precise conditions on \(V_{\text{ext}}\) will be introduced later on). Furthermore, we assume
$V \in L^3(\mathbb{R}^3)$ to be pointwise non-negative, spherically symmetric and compactly supported (but our results could easily be extended to potentials decaying sufficiently fast at infinity). Notice that under these assumptions, $H_N$ is essentially self-adjoint in $(C_c^\infty(\mathbb{R}^3))^\otimes N$.

The scattering length of $V$ is defined through the zero-energy scattering equation

$$
\left[ -\Delta + \frac{1}{2} V(x) \right] f(x) = 0
$$

with the boundary condition $f(x) \to 1$, as $|x| \to \infty$. For $|x|$ large enough (outside the support of $V$) we find

$$
f(x) = 1 - \frac{a_0}{|x|}
$$

where the constant $a_0 > 0$ is known as the scattering length of $V$. A simple computation shows that

$$
8\pi a_0 = \int_{\mathbb{R}^3} V(x) f(x) dx.
$$

Moreover, by scaling, (1.2) implies that

$$
\left[ -\Delta + \frac{1}{2} N^2 V(Nx) \right] f(Nx) = 0
$$

so that the scattering length of $N^2 V(N \cdot)$ is given by $a_0/N$.

From [16], it is known that the ground state energy $E_N$ of the Hamilton operator (1.1) satisfies

$$
\lim_{N \to \infty} \frac{E_N}{N} = \inf_{\psi \in H^1(\mathbb{R}^3) : \|\psi\|_2 = 1} \mathcal{E}_{GP}(\psi),
$$

where $\mathcal{E}_{GP}$ denotes the Gross-Pitaevskii energy functional

$$
\mathcal{E}_{GP}(\psi) = \int_{\mathbb{R}^3} \left( |\nabla \psi(x)|^2 + V_{\text{ext}}(x) |\psi(x)|^2 + 4\pi a_0 |\psi(x)|^4 \right) dx.
$$

For the rest of the paper, we will lighten the notation and write $\int$ instead of $\int_{\mathbb{R}^3}$. Furthermore, we will write $\| \cdot \|$ for the $L^2$-norm and indicate other $L^p$-norms by a suitable subscript. The Gross-Pitaevskii functional $\mathcal{E}_{GP}$ admits a unique normalized, strictly positive minimizer $\varphi \in L^2(\mathbb{R}^3)$. It satisfies the Euler-Lagrange equation

$$
-\Delta \varphi + V_{\text{ext}} \varphi + 8\pi a_0 |\varphi|^2 \varphi = \varepsilon_{GP} \varphi
$$

with the Lagrange multiplier $\varepsilon_{GP} := \mathcal{E}_{GP}(\varphi) + 4\pi a_0 \|\varphi\|_4^4$. As first shown in [14], the ground state of (1.1) exhibits complete Bose–Einstein condensation in the state $\varphi$. 

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More precisely, if $\gamma_N^{(1)} = \text{tr}_{2,\ldots,N} |\psi_N\rangle \langle \psi_N|$ denotes the one-particle reduced density associated with the ground state of (1.1), then

$$\lim_{N \to \infty} \langle \varphi, \gamma_N^{(1)} \varphi \rangle = 1. \quad (1.7)$$

This implies that, in the ground state of (1.1), the fraction of particles in the state $\varphi$ approaches one, as $N \to \infty$. The convergence in (1.7) was later extended in [15, 19] to any sequence $\psi_N$ of approximate ground states, satisfying

$$\lim_{N \to \infty} \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle = E_{GP}(\varphi).$$

For translation invariant systems (particles trapped in the box $\Lambda = [0; 1]^3$, with periodic boundary conditions), (1.4) (stating, in this case, that $E_N/N \to 4\pi a_0$) and (1.7) (establishing Bose–Einstein condensation in the zero-momentum mode $\varphi(x) = 1$ for all $x \in \Lambda$) have been proved in [2, 5, 12] to hold with the optimal rate of convergence. This result was recently generalized in [18] (extending the approach of [8, 10]) to trapped systems described by the Hamilton operator (1.1), under the assumption of sufficiently small scattering length $a_0$.

Our goal in this paper is to obtain optimal bounds on the rate of convergence in (1.4) and (1.7), with no restriction on the size of the scattering length. To reach this goal, we will impose, throughout the rest of this paper, the following conditions:

1. $V \in L^3(\mathbb{R}^3)$, $V(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$, $V$ spherically symmetric, $\text{supp}(V)$ compact,
2. $V_{\text{ext}} \in C^1(\mathbb{R}^3; \mathbb{R})$, $V_{\text{ext}}(x) \to \infty$ as $|x| \to \infty$, $\exists C > 0 \forall x, y \in \mathbb{R}^3 : V_{\text{ext}}(x + y) \leq C(V_{\text{ext}}(x) + C)(V_{\text{ext}}(y) + C)$, $\nabla V_{\text{ext}}$ has at most exponential growth as $|x| \to \infty$. \quad (1.8)

Note in particular that all polynomials in $x^2$ with positive leading coefficient satisfy condition (2) in (1.8). The assumptions that $V_{\text{ext}}(x + y) \leq C(V_{\text{ext}}(x) + C)(V_{\text{ext}}(y) + C)$ and that $\nabla V_{\text{ext}}$ grows at most exponentially allow for certain simplifications of our analysis and are needed for technical reasons only.

**Theorem 1.1** Assume (1.8) and let $E_N$ denote the ground state energy of (1.1). Then, there exists a constant $C > 0$ such that

$$E_N \geq N E_{GP}(\varphi) - C \quad (1.9)$$

and

$$H_N \geq N E_{GP}(\varphi) + C^{-1} \sum_{i=1}^{N} (1 - |\varphi \rangle \langle \varphi |_i) - C. \quad (1.10)$$
In particular, if \( \psi_N \in L^2_s(\mathbb{R}^{3N}) \) with \( \|\psi_N\| = 1 \) is a sequence of approximate ground states such that

\[
\langle \psi_N, H_N \psi_N \rangle \leq N E_{GP}(\varphi) + \zeta,
\]

for \( \zeta > 0 \), then the reduced density \( \gamma^{(1)}_N \) associated with \( \psi_N \) satisfies

\[
1 - \langle \varphi, \gamma^{(1)}_N \varphi \rangle \leq C \left( \frac{C + \zeta}{N} \right).
\]

**Remark** Our techniques could also be used to prove an upper bound for \( E_N \) matching (1.9), implying that \( |E_N - N E_{GP}(\varphi)| \leq C \). We do not show it, because it would require some non-trivial additional work (this is a consequence of our choice, leading to some technical simplifications, to work on the Fock space \( \mathcal{F} \leq N \) rather than on \( \mathcal{F} \leq N \perp \varphi \), where we impose orthogonality to \( \varphi \); we will explain this point in the next section) and because it is already established in [18] (the upper bound there does not require restrictions on the size of the potential).

**Remark** We apply Theorem 1.1 in [7] to determine the low-energy spectrum of (1.1) and to establish the validity of the predictions of Bogoliubov theory, extending recent results obtained in [3, 4] for the translation invariant setting. We remark that the main result of [7] has been proved independently in [17] which also implies the lower bound (1.10) (see both [7] and [17] for a detailed comparison of the methods).

We conclude this introduction with a brief outline of the paper. In Sect. 2, we introduce the Fock space setting in which we work and we give a heuristic outline of the proof of Theorem 1.1. As in [5], our proof requires in particular a quadratic and a cubic renormalization. The related technical main results, Proposition 3.4 and Proposition 3.7 are summarized in Sect. 3. Since their proofs are rather involved, we first use these results in Sect. 4 to prove Theorem 1.1. Afterwards, Sect. 5 and Sect. 6 contain detailed proofs of Proposition 3.4 and Proposition 3.7, respectively. In the Appendix A, we collect basic results about the minimizer \( \varphi \) of (1.5).

### 2 Fock Space Setting and Outline of Proof of Theorem 1.1

We introduce the bosonic Fock space

\[
\mathcal{F} = \bigoplus_{n \geq 0} L^2_s(\mathbb{R}^{3n}) = \bigoplus_{n \geq 0} L^2(\mathbb{R}^3)^{\otimes s n}.
\]

On \( \mathcal{F} \), we consider creation and annihilation operators, satisfying the canonical commutation relations

\[
[a(g), a^*(h)] = (g, h), \quad [a(g), a(h)] = [a^*(g), a^*(h)] = 0
\]

(2.1)
for all $g, h \in L^2(\mathbb{R}^3)$. We also introduce position and momentum-space operator-valued distributions $a_x, a_x^*$ and $\hat{a}_p, \hat{a}_p^*$, for $x, p \in \mathbb{R}^3$, so that

$$a(f) = \int \hat{f}(x) a_x \, dx = \int \hat{f}(p) \hat{a}_p \, dp, \quad a^*(f) = \int f(x) a_x^* \, dx = \int \hat{f}(p) \hat{a}_p^* \, dp.$$ 

In terms of these operator-valued distributions, the number of particles operator $N$, defined by $(N/\Psi)(n) = n/\Psi(n)$ for every $\Psi \in \mathcal{F}$, takes the form

$$N = \int a_x^* a_x \, dx = \int \hat{a}_p^* \hat{a}_p \, dp.$$ 

More generally, given an operator $A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ with kernel $A(x; y)$, we define its second quantization $d\Gamma(A)$ acting in $\mathcal{F}$ through

$$d\Gamma(A) = \int dx dy \, A(x; y) a_x^* a_x.$$ 

In particular, $N = d\Gamma(1)$ is the second quantization of the identity operator.

It is simple to check that creation and annihilation operators are bounded, with respect to $N^{1/2}$. In fact, we find that

$$\|a(f)\Psi\| \leq \|f\| \|N^{1/2}\Psi\|, \quad \|a^*(f)\Psi\| \leq \|f\| \|(N + 1)^{1/2}\Psi\|. \quad (2.2)$$

To describe excitations of the Bose–Einstein condensate, we also introduce the truncated Fock spaces

$$\mathcal{F} \leq N = \bigoplus_{n=0}^{N} L^2(\mathbb{R}^3)^{\otimes n}, \quad \text{and} \quad \mathcal{F} \leq N \perp \varphi = \bigoplus_{n=0}^{N} L^2(\mathbb{R}^3)^{\otimes n}$$

defined over $L^2(\mathbb{R}^3)$ and, respectively, over $L^2(\mathbb{R}^3) \perp \varphi$, the orthogonal complement of the condensate wave function $\varphi$ in $L^2(\mathbb{R}^3)$ ($\varphi$ is the unique minimizer of (1.5)). Since they do not preserve the number of particles, creation and annihilation operators are not well-defined on $\mathcal{F} \leq N$ and $\mathcal{F} \leq N \perp \varphi$ (but notice that products of a creation and an annihilation operators are well-defined). They are replaced by modified creation and annihilation operators

$$b^*(f) = a^*(f) \sqrt{N - N'}, \quad b(f) = \sqrt{N - N'} a(f).$$

For every $f \in L^2(\mathbb{R}^3)$, $b(f)$ and $b^*(f)$ map $\mathcal{F} \leq N$ to $\mathcal{F} \leq N$. If moreover $f \perp \varphi$, we also find $b(f), b^*(f) : \mathcal{F} \leq N \perp \varphi \to \mathcal{F} \leq N \perp \varphi$. From (2.2) we have

$$\|b(f)\Psi\| \leq \|f\| \|N^{1/2}\Psi\|, \quad \|b^*(f)\Psi\| \leq \|f\| \|(N + 1)^{1/2}\Psi\|. \quad (2.2)$$
Also here, it is convenient to introduce operator-valued distributions \( b_x, b_x^* \), for any \( x \in \mathbb{R}^3 \), and, in momentum space \( \hat{b}_p, \hat{b}_p^* \), for any \( p \in \mathbb{R}^3 \). They satisfy the commutation relations (focusing here on position space operators)

\[
[b_x, b_y^*] = \left( 1 - \frac{\alpha}{N} \right) \delta(x - y) - \frac{1}{N} \alpha x \cdot a_x, \quad [b_x, b_y] = [b_x^*, b_y^*] = 0 \quad (2.3)
\]

and

\[
[b_x, a_y^* a_z] = \delta(x - y)b_z, \quad [b_x^*, a_y^* a_z] = -\delta(x - z)b_y. \quad (2.4)
\]

In particular, it follows that \([b_x, N] = b_x\) and \([b_x^*, N] = -b_x^*\).

We factor out the Bose–Einstein condensate applying a unitary map \( U_N : L_2^2(\mathbb{R}^{3N}) \to \mathcal{F}^N_{\perp \varphi} \), first introduced in [13]. To define \( U_N \), we observe that any \( \psi_N \in L_2^2(\mathbb{R}^{3N}) \) can be uniquely decomposed as

\[
\psi_N = \alpha_0 \varphi^{\otimes N} + \alpha_1 \otimes_s \varphi^{\otimes (N - 1)} + \cdots + \alpha_N
\]

with \( \alpha_j \in L_2^2(\mathbb{R}^3)^{\otimes s} \) for every \( j = 0, \ldots, N \). Thus, we can set \( U_N \psi_N = \{ \alpha_0, \alpha_1, \ldots, \alpha_N \} \in \mathcal{F}^N_{\perp \varphi} \). It is then possible to show that \( U_N \) is unitary; see [13] for details.

With \( U_N \), we can define the excitation Hamiltonian \( \mathcal{L}_N = U_N H_N U_N^* \), acting on a dense subspace of \( \mathcal{F}^N_{\perp \varphi} \). The action of \( U_N \) on creation and annihilation operators is given by

\[
U_N a^*(\varphi) a(\varphi) U_N^* = N - N',
\]

\[
U_N a^*(f) a(\varphi) U_N^* = a^*(f) \sqrt{N - N'} = \sqrt{N} b^*(f),
\]

\[
U_N a^*(\varphi) a(g) U_N^* = \sqrt{N - N'} a(g) = \sqrt{N} b(g),
\]

\[
U_N a^*(f) a(g) U_N^* = a^*(f) a(g)
\]

for all \( f, g \in L_2^2(\mathbb{R}^3) \), where \( N \) denotes the number of particles operator in \( \mathcal{F}^N_{\perp \varphi} \).

Writing \( H_N \) in second quantized form and using (2.5), we proceed as in [13, Sect. 4] and find that

\[
\mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(1)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)},
\]

where, in the sense of quadratic forms on \( \mathcal{F}^N_{\perp \varphi} \), we have that

\[
\mathcal{L}_N^{(0)} = \langle \varphi, \left[ -\Delta + V_{\text{ext}} + \frac{1}{2} (N^3 V(N\cdot) \cdot |\varphi|^2) \right] \varphi \rangle (N - N')
\]

\[
- \frac{1}{2} \langle \varphi, (N^3 V(N\cdot) \cdot |\varphi|^2) \varphi \rangle (N + 1) (1 - N'/N),
\]

\[
\mathcal{L}_N^{(1)} = \sqrt{N} \beta \left( (N^3 V(N\cdot) \cdot |\varphi|^2 - 8\pi a_0 |\varphi|^2) \varphi \right)
\]
- \frac{\mathcal{N} + 1}{\sqrt{\mathcal{N}}} b \left( \left( N^3 V(N) \ast |\varphi|^2 \right) \varphi \right) + \text{h.c.},

\mathcal{L}_N^{(2)} = \int dx \left( \nabla_x a_x^* \nabla_x a_x + V_{\text{ext}}(x) a_x^* a_x \right)
+ \int dxdy N^3 V(N(x-y)) |\varphi(y)|^2 \left( b_x^* b_x - \frac{1}{\mathcal{N}} a_x^* a_x \right)
+ \int dxdy N^3 V(N(x-y)) \varphi(x) \varphi(y) \left( b_x^* b_y - \frac{1}{\mathcal{N}} a_x^* a_y \right)
+ \frac{1}{2} \int dxdy N^3 V(N(x-y)) \varphi(y) \varphi(x) \left( b_x^* b_y + \text{h.c.} \right),

\mathcal{L}_N^{(3)} = \int dxdy N^{5/2} V(N(x-y)) \varphi(y) \left( b_x^* a_x^* a_x + \text{h.c.} \right),

\mathcal{L}_N^{(4)} = \frac{1}{2} \int dxdy N^2 V(N(x-y)) a_x^* a_x^* a_x a_x. \tag{2.6}

While \( \mathcal{L}_N \) maps \( \mathcal{F}_\perp^{-N} \) to itself, the operators \( \mathcal{L}_N^{(j)}, j \in \{0, 1, 2, 3, 4\} \) are also well defined on \( \mathcal{F}^{-N} \). In the following, it will be technically convenient to identify the \( \mathcal{L}_N^{(j)} \), through Eq. (2.6), as acting in \( \mathcal{F}^{-N} \) and in this case we will denote them by \( \widetilde{\mathcal{L}}_N^{(j)} \). Moreover, we define

\[ \widetilde{\mathcal{L}}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(1)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} \tag{2.7} \]

as a self-adjoint operator acting on a dense subspace in \( \mathcal{F}^{-N} \). In particular, if \( \Gamma(q) \) is the orthogonal projection from \( \mathcal{F}^{-N} \) onto \( \mathcal{F}_\perp^{-N} \), defined by \( (\Gamma(q) \Psi)^{(n)} = q^{\otimes n} \Psi^{(n)} \) for every \( n \in \{0, 1, \ldots, N\} \), with \( q = 1 - \langle \varphi | \varphi \rangle \), then \( \mathcal{L}_N = \Gamma(q) \widetilde{\mathcal{L}}_N \Gamma(q) \) in \( \mathcal{F}_\perp^{-N} \).

Let us conclude this section with a heuristic outline of the proof of Theorem 1.1. Translated to the setting in \( \mathcal{F}_\perp^{-N} \), Theorem 1.1 follows if we prove the coercivity bound

\[ \mathcal{L}_N \geq N \mathcal{E}_{\text{GP}}(\varphi) + cN - C \tag{2.8} \]

for \( c, C > 0 \), independent of \( N \). For technical convenience, we will first show an analogous bound for \( \widetilde{\mathcal{L}}_N \), on \( \mathcal{F}^{-N} \), which immediately implies (2.8) as well. Obtaining such a bound from the representation (2.6) is, however, not obvious as simple trial states like the vacuum \( \Omega \in \mathcal{F}_\perp^{-N} \) do not yield the right energy. Note indeed that for \( N \gg 1 \) we have

\[ \langle \Omega, \widetilde{\mathcal{L}}_N \Omega \rangle \approx N \int \left[ |\nabla \varphi(x)|^2 + V_{\text{ext}}(x) |\varphi(x)|^2 + \frac{1}{2} \tilde{V}(0) |\varphi(x)|^4 \right] dx \tag{2.9} \]

which is off by a contribution of order \( N \), compared with the ground state energy \( E_N = N \mathcal{E}_{\text{GP}}(\varphi) + \mathcal{O}(1) \). The reason for the discrepancy is that, through the map \( U_N \), we expand \( H_N \) around the energy of the uncorrelated condensate wave function.
$\varphi^{\otimes N} \in L^2_s(\mathbb{R}^{3N})$. In the Gross–Pitaevskii regime, however, it is well-known that short scale correlations among particles play a crucial role; in particular, they affect the ground state energy to leading order.

Instead of deriving (2.8) directly from the representation (2.6), our strategy is therefore to extract first the missing correlation energies by conjugating $\tilde{\mathcal{L}}_N$ through suitable unitary operators. The goal of the conjugations is to obtain a new excitation Hamiltonian which enables us to study the fluctuations around the correct energy $N \mathcal{E}_{\text{GP}}(\varphi)$.

Our choice for the renormalization procedure generalizes [2, 5] for translation invariant systems. We start (following an approach first developed in the dynamical setting in [1, 6]) by conjugating $\tilde{\mathcal{L}}_N$ through a generalized Bogoliubov transformation of the form $e^B : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}$, where

$$B = \frac{1}{2} \int dxdy \eta(x; y)\varphi(x)\varphi(y)b^*_x b^*_y - \text{h.c.}$$

for some $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Choosing $\eta(x; y) = -(1 - f(N(x - y))\varphi(x)\varphi(y)$, with $f$ solving the zero-energy scattering equation (1.2), and approximating heuristically

$$e^{-B} \mathcal{L}_N e^B = \mathcal{L}_N + [\mathcal{L}_N, B] + \int_0^1 ds \int_0^s dt \, e^{-tB}[[\mathcal{L}_N, B], B] e^{tB}$$

one observes that the large linear term $\mathcal{L}_N^{(1)}$ in (2.6) is cancelled (when combined with contributions from $[\mathcal{L}_N^{(3)}; B]$) and that the singular off-diagonal quadratic term (proportional to $b^*_x b^*_y + \text{h.c.}$) disappears (when combined with contributions from $[\mathcal{L}_N^{(2)}; B]$ and $[\mathcal{L}_N^{(4)}; B]$ that reconstruct (1.2)). At the same time, the constant contribution in $\mathcal{L}_N^{(0)}$ is renormalized to $N \mathcal{E}_{\text{GP}}(\varphi) + O(1)$. Here, it is important to notice that, while conjugation with $e^B$ changes energy expectations by order $N$ (which is needed, to correct (2.9)), it only creates a bounded number of excitations (the number of excitations is controlled by $\|\eta\|_2^2 \simeq 1$, their energy by $\|\eta\|_H^2 \simeq N$, with our choice of $\eta$).

To make this heuristic approach rigorous, we need to control the error terms in the Taylor series of $e^{-B} \tilde{\mathcal{L}}_N e^B$. To this end, instead of choosing as above the kernel $\eta = -(1 - f(N(x - y))\varphi(x)\varphi(y)$, we follow the strategy of [2, 5] and replace the solution $f$ of the zero-energy scattering equation (1.2) by the solution $f_\ell$ of a related Neumann problem in the ball $B_\ell(0) \subset \mathbb{R}^3$, with a low momentum cutoff on the scale $\ell^{-\alpha}$, for some $\alpha > 0$ (see Sect. 3 for the details). Choosing $\ell > 0$ small, this guarantees that we can expand $e^{-B} \tilde{\mathcal{L}}_N e^B$ into a convergent commutator series and that the errors are negligible, when aiming at a lower bound of the form (2.8). We remark that one can also work directly with $f(N.)$ as correlation factor with a large distance cutoff in position space. Conceptually, this is similar to our approach, but working with $f(N.)$ directly, the regularity assumptions on $V \in L^3(\mathbb{R}^3)$ can be relaxed a bit further to $V \in L^1(\mathbb{R}^3)$, see [17] for the details.

The main properties of the renormalized Hamiltonian $\mathcal{G}_N = e^{-B} \tilde{\mathcal{L}}_N e^B$ are described in Proposition 3.4. This proposition shows that, up to negligible errors,
\( G_N \) has a similar representation like \( \tilde{L}_N \) in (2.6), but with regularized linear and off-diagonal quadratic contributions and with constant term matching \( N \mathcal{E}_{\text{GP}}(\varphi) \), up to errors that are bounded, uniformly in \( N \). Although simple trial states like \( \Omega_1 \in F \subseteq N \) now yield the correct leading order energy, this is still not enough to deduce the lower bound (2.8). The reason is that cubic and quartic contributions to \( G_N \) are essentially still given by the operators \( \tilde{L}_N^{(3)} \) and \( \tilde{L}_N^{(4)} \), defined in (2.6). Observe that, by Cauchy–Schwarz and by the positivity of \( L_N^{(4)} \), we could estimate

\[
\tilde{L}_N^{(3)} + \tilde{L}_N^{(4)} \geq \frac{1}{2} \tilde{L}_N^{(4)} - 2 \tilde{V}(0) N' \geq -2 \tilde{V}(0) N'.
\]  

(2.11)

At this point it would be quite easy to conclude (2.8), if we had an additional smallness assumption on the size of the potential \( V \), as in [2, 12, 18] (in [18], a relatively simple proof of (2.8) was obtained for small \( V \), adapting ideas from [8]). Without conditions on the size of \( V \), on the other hand, it is not clear how to control the r.h.s. of (2.11) and we have to proceed differently.

To solve this problem, we conjugate \( G_N \) once more through a unitary operator of the form \( e^{A} : F \subseteq N \to F \subseteq N \), where

\[
A = \frac{1}{\sqrt{N}} \int dx dy dz \tilde{\eta}(x; y; z)(b_x^* a_y^* a_z - \text{h.c.}).
\]

In analogy with the first step, if we choose \( \tilde{\eta}(x; y; z) = -(1 - f_\ell(N(x-y)))\varphi(y) \delta(z-x) \), naive Taylor expansion shows that the cubic term \( \tilde{L}_N^{(3)} \) is cancelled out (when combined with contributions from \([L_N^{(2)}]A \) and \([L_N^{(4)}]A \)). To control various error terms arising in the expansion of \( e^{-A} G_N e^{A} \), we need to localize the correlation factor \( 1 - f_\ell(N.) \) once again to high momenta, while \( \delta(z-x) \) is smeared out and replaced by a Gaussian with Fourier transform decaying for large momenta on the scale \( \ell^{-\beta} \) (for some \( 0 < \beta < \alpha \)). With these technical modifications, the cubic conjugation produces a new Hamiltonian, with a regularized cubic term that can be controlled by \( N^{3/2}/N^{1/2} \). This is summarized in Proposition 3.7. At this point, we can proceed as in [5], using localization in the number of excitations (as originally introduced in [13]) to conclude the proof of the coercivity bound (2.8).

While our general strategy follows closely [2, 5], extending the error bounds from these works to the trapped setting is non-trivial and requires a careful analysis. Notice in particular that the minimizer \( \varphi \) of (1.5) is constant in the translation invariant setting. In this case, a large part of the analysis can be carried out conveniently in Fourier space. In the trapped setting, on the other hand, we do not have translation invariance and we need to use the regularity properties of \( \varphi \) to generalize various estimates from [2, 5].

### 3 Quadratic and Cubic Renormalizations

In our first renormalization step, we will conjugate \( \mathcal{L}_N \) with a generalized Bogoliubov transformation. To define the kernel of the quadratic phase, we consider the ground state solution of the Neumann problem.
\[
\left[ -\Delta + \frac{1}{2} V \right] f_\ell = \lambda_\ell f_\ell
\]  
(3.1)

on the ball \(|x| \leq N\ell\), for some \(0 < \ell < 1\). For simplicity, we omit here the \(N\)-dependence in the notation for \(f_\ell\) and for \(\lambda_\ell\). By radial symmetry of the interaction \(V\), \(f_\ell\) is radially symmetric and we normalize it such that \(f_\ell(x) = 1\) if \(|x| = N\ell\). By scaling, \(f_\ell(N.\) solves

\[
\left[ -\Delta + \frac{N^2}{2} V(Nx) \right] f_\ell(Nx) = N^2 \lambda_\ell f_\ell(Nx)
\]

on the ball where \(|x| \leq \ell\). Later in our analysis, we will choose the parameter \(\ell > 0\) to be sufficiently small, but it will always be of order one, independent of \(N\). We then extend \(f_\ell(N.\) to \(\mathbb{R}^3\), by setting \(f_{N,\ell}(x) = f_\ell(Nx)\) if \(|x| \leq \ell\) and \(f_{N,\ell}(x) = 1\) for \(x \in \mathbb{R}^3\) with \(|x| > \ell\). Thus, \(f_{N,\ell}(N.\) solves the equation

\[
\left( -\Delta + \frac{N^2}{2} V(N.\right) f_{N,\ell} = N^2 \lambda_\ell f_{N,\ell} \chi_\ell,
\]  
(3.2)

where we set

\[
\chi_\ell(x) = \begin{cases} 
1 & : |x| \leq \ell, \\
0 & : |x| > \ell.
\end{cases}
\]  
(3.3)

Finally, we denote by \(w_\ell\) the function \(w_\ell = 1 - f_\ell\). Notice that by scaling, \(w_\ell(N.\) has compact support in \(B_\ell(0)\), for all \(N \in \mathbb{N}\) sufficiently large. Defining the Fourier transform of \(w_\ell\) through

\[
\hat{w}_\ell(p) = \int dx \, w_\ell(x) e^{-2\pi ipx},
\]

we see that \(w_\ell(N.\) has Fourier transform

\[
\int dx \, w_\ell(Nx)e^{-2\pi ipx} = \frac{1}{N^3} \hat{w}_\ell(p/N)
\]

and we recall that (3.2) implies that

\[
-4\pi^2 p^2 \hat{w}_\ell(p/N) + \frac{N^2}{2}(\hat{V}(. / N) * \hat{f}_{N,\ell})(p) = N^5 \lambda_\ell (\hat{\chi}_\ell * \hat{f}_{N,\ell})(p).
\]

The next lemma collects important properties of \(f_\ell, w_\ell\) and the Neumann eigenvalue \(\lambda_\ell\).

**Lemma 3.1** Let \(V \in L^3(\mathbb{R}^3)\) be non-negative, compactly supported and spherically symmetric. Fix \(\ell > 0\) and let \(f_\ell\) denote the solution of (3.1).
(i) We have that
\[ \lambda_\ell = \frac{3a_0}{(\ell N)^3} \left( 1 + O\left( \frac{a_0}{\ell N} \right) \right) \tag{3.4} \]

(ii) We have \(0 \leq f_\ell, w_\ell \leq 1\) and there exists a constant \(C > 0\) such that
\[ \left| \int_{\mathbb{R}^3} V(x) f_\ell(x) dx - 8\pi a_0 \right| \leq \frac{C a_0}{\ell N} \tag{3.5} \]
for all \(\ell \in (0; 1), N \in \mathbb{N}\).

(iii) There exists a constant \(C > 0\) such that
\[ w_\ell(x) \leq \frac{C}{|x| + 1} \quad \text{and} \quad |\nabla w_\ell(x)| \leq \frac{C}{|x|^2 + 1} \tag{3.6} \]
for all \(x \in \mathbb{R}^3, \ell \in (0; 1)\) and \(N \in \mathbb{N}\) large enough. Moreover,
\[ \left| \frac{1}{(N \ell)^2} \int_{\mathbb{R}^3} w_\ell(x) dx - \frac{2}{5} \pi a_0 \right| \leq \frac{C a_0^2}{N \ell} \tag{3.7} \]
for all \(\ell \in (0; 1)\) and \(N \in \mathbb{N}\) large enough.

(iv) There exists a constant \(C > 0\) such that
\[ |\hat{w}_\ell(p)| \leq \frac{C}{|p|^2} \tag{3.8} \]
for all \(p \in \mathbb{R}^3, \ell \in (0; 1)\) and \(N \in \mathbb{N}\) large enough.

**Proof** This has already been proved in \([4, \text{Appendix B}]\), based on \([9, \text{Lemma A.1}]\) and \([5, \text{Lemma 4.1}]\).

Let us now define the correlation kernel for the generalized Bogoliubov transformation that we will be using. We denote by \(G : \mathbb{R}^3 \rightarrow \mathbb{R}\) the rescaled function
\[ G(x) = -N w_\ell(Nx) \tag{3.9} \]
which has compact support in \(B_\ell(0)\) and which, by (3.7), satisfies for all \(p \in \mathbb{R}^3\)
\[ |\hat{G}(p)| \leq \frac{C}{|p|^2}. \tag{3.10} \]
For fixed \(\alpha > 0\), we denote by \(\chi_H\) the characteristic function of \(P_H = \{ p \in \mathbb{R}^3 : |p| \geq \ell^{-\alpha} \}\), i.e.
\[ \chi_H(p) = \begin{cases} 1 & : |p| \geq \ell^{-\alpha}, \\ 0 & : |x| < \ell^{-\alpha}, \end{cases} \tag{3.10} \]
and we denote by $\tilde{\chi}_H$ its inverse Fourier transform. Finally, we define $\eta_H \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$\eta_H(x, y) = (G * \tilde{\chi}_H)(x - y)\varphi(x)\varphi(y). \quad (3.11)$$

The next lemma summarizes basic properties of the kernel $\eta_H \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$.

**Lemma 3.2** Assume (1.8), let $\ell \in (0; 1)$ and let $\alpha > 0$. Set $\eta_{H,x}(y) = \eta_H(x; y)$ for $x \in \mathbb{R}^3$. Then, there exists $C > 0$, uniform in $N$, $\ell \in (0; 1)$ and $x \in \mathbb{R}^3$, such that

$$\|\eta_H\| \leq C \ell^{\alpha/2}, \quad \|\eta_{H,x}\| \leq C \ell^{\alpha/2} |\varphi(x)|, \quad \|\nabla_1 \eta_H\|, \|\nabla_2 \eta_H\| \leq C \sqrt{N}. \quad (3.12)$$

Furthermore, identifying $\eta_H(x; y)$ with the kernel of a Hilbert-Schmidt operator on $L^2(\mathbb{R}^3)$ and $\eta^{(n)}_H(x; y)$ with the kernel of its $n$th power, we have for $n \geq 2$ and $x, y \in \mathbb{R}^3$ that

$$|\eta^{(n)}_H(x; y)| \leq \|\eta_{H,x}\| \cdot \|\nabla \eta_H\| \cdot \|\nabla \eta_H\|^{n-2} \leq C \ell^{\alpha} \|\eta_H\|^{n-2} |\varphi(x)| \cdot |\varphi(y)| \quad (3.13)$$

and, for all $N$ sufficiently large, that

$$|\eta_H(x; y)| \leq C N |\varphi(x)||\varphi(y)| \leq C N. \quad (3.14)$$

**Proof** By Theorem A.1, $\|\varphi\|_\infty \leq C < \infty$ so that together with Eq. (3.9), we find that

$$\|\eta_{H,x}\|^2 = \int dy \ |(G * \tilde{\chi}_H)(x - y)|^2 \cdot |\varphi(x)|^2 \cdot |\varphi(y)|^2$$

$$\leq \|\varphi\|_\infty^2 \cdot |\varphi(x)|^2 \cdot \|G * \tilde{\chi}_H\|^2 \leq C |\varphi(x)|^2 \int_{|p| \geq \ell^{-\alpha}} dp \ |\hat{G}(p)|^2$$

$$\leq C |\varphi(x)|^2 \int_{|p| \geq \ell^{-\alpha}} dp \ \frac{1}{|p|^4} = C \ell^{\alpha} |\varphi(x)|^2.$$

This concludes the first two bounds in (3.12), once we integrate the right hand side of the last estimate. To bound the gradient, we proceed similarly and find for $i = 1, 2$ that

$$\|\nabla_i \eta_H\|^2 \leq \int dx dy \ |\nabla(G * \tilde{\chi}_H)(x - y)|^2 \cdot |\varphi(x)|^2 \cdot |\varphi(y)|^2$$

$$+ \int dx dy \ |(G * \tilde{\chi}_H)(x - y)|^2 \cdot |\nabla \varphi(x)|^2 \cdot |\varphi(y)|^2$$

$$\leq C \|G * \tilde{\chi}_H\|^2 \|H^1\| \leq C \|G\|^2 \|H^1\| \leq C \int dx \ \frac{N^4}{(|N_x|^2 + 1)^2} = CN.$$

Notice that we used the pointwise estimate (3.6) in the second to last step. The estimates in (3.13) follow directly from (3.12) and Cauchy–Schwarz. Finally, (3.14) follows from
\[ \| \varphi \|_\infty \leq C, \| G \|_\infty \leq N \text{ as well as } \| G \ast \tilde{\chi}_{H^c} \|_\infty \leq \| G \|_1 \| \tilde{\chi}_{H^c} \|_\infty \leq C \ell^{1-3\alpha/2} \leq CN \text{ for } N \text{ large enough. Here, } \tilde{\chi}_{H^c} \text{ denotes the inverse Fourier transform of the characteristic function on } P_H = \mathbb{R}^3 \setminus \{ p \in \mathbb{R}^3 : |p| < \ell^{-\alpha} \}. \]  

With the kernel \( \eta_H \), we consider the quadratic expression

\[
B = \frac{1}{2} \int dxdy \eta_H(x; y) \left[ b_x^* b_y^* - b_x b_y \right] \tag{3.15}
\]

and the unitary operator \( e^B : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N} \) (since \( \eta_H \) is real, the operator \( B \) is antisymmetric). Because of its similarity with a Bogoliubov transformation (which would have \( a_x, a_y \), instead of the modified fields \( b_x, b_y \) in (3.15)), we call \( e^B \) a generalized Bogoliubov transformation. Notice that we do not project \( \eta_H \) into the orthogonal complement of the condensate wave function \( \varphi \). As a consequence, \( B \) and \( e^B \) do not map \( \mathcal{F}^{\leq N}_{\perp \varphi} \) into itself. This is not a problem for us, because we perform our analysis on the larger space \( \mathcal{F}^{\leq N} \) and only at the end (in Sect. 4) we switch back to the right space. An important property of the unitary operator \( e^B \) is that it preserves the number of particles, up to corrections of order one. The following lemma was proved in [6, Lemma 3.1]; it is based on the observation that \( \| \eta_H \| \leq C \ell^{\alpha/2} \leq C \).

**Lemma 3.3** Let \( B \) be the antisymmetric operator defined in (3.15). For every \( n \in \mathbb{Z} \) there exists a constant \( C > 0 \) such that

\[ e^{-B}(N + 1)^n e^B \leq C (N + 1)^n \]

as an operator inequality on \( \mathcal{F}^{\leq N} \).

Other important properties of the generalized Bogoliubov transformation \( e^B \) will be discussed at the beginning of Sect. 5 (in particular, we will show there that, on states with few excitations, \( e^B \) acts like a standard Bogoliubov transformation, up to small errors).

With \( \tilde{L}_N \) from (2.7), we can now define the quadratically renormalized excitation Hamiltonian

\[
\mathcal{G}_N = e^{-B} \tilde{L}_N e^B. \tag{3.16}
\]

The next proposition summarizes important properties of \( \mathcal{G}_N \). Before stating it, let us introduce the notation

\[
\mathcal{K} = \int dx\ a_x^*(\Delta_x) a_x, \quad \mathcal{V}_N = \frac{1}{2} \int dxdy\ N^2 V(N(x - y)) a_x^* a_x^* a_x a_x, \\
\mathcal{V}_{\text{ext}} = \int dx\ V_{\text{ext}}(x) a_x^* a_x, \quad \mathcal{H}_N = \mathcal{K} + \mathcal{V}_{\text{ext}} + \mathcal{V}_N
\]
Proposition 3.4 Assume (1.8) and let \( \eta_H \) be defined as in (3.11), with an \( \alpha > 3 \). Let
\[
\mathcal{G}_N^{\text{eff}} = N\mathcal{E}_{GP}(\varphi) - \varepsilon_{GP}N + 4\pi a_0\|\varphi\|^2N^2/N + \mathcal{H}_N + 2\vec{V}(0)\int dx|\varphi(x)|^2b_x^*b_x \\
+ 4\pi a_0\int dx\,dy\,\chi_{H^c}(x - y)\varphi(x)\varphi(y)(b_x^*b_y^* + \text{h.c.}) \\
+ \frac{1}{\sqrt{N}}\int dx\,dy\,N^3V(N(x - y))\varphi(x)(b_x^*a_y^*a_x + \text{h.c.})
\]
where \( \varepsilon_{GP} \) has been introduced in (1.6) and \( \chi_{H^c} \) denotes the characteristic function of the set \( \{p \in \mathbb{R}^3 : |p| \leq \ell^{-\alpha}\} \). Then we have \( \mathcal{G}_N = \mathcal{G}_N^{\text{eff}} + \mathcal{E}_{\mathcal{G}_N} \), where we can bound
\[
\pm \mathcal{E}_{\mathcal{G}_N} \lesssim C\ell^{(\alpha-3)/2}(N + K + V_N) + C\ell^{-5\alpha/2}N^{-1}(N + 1)^2 + C\ell^{-4\alpha}
\]
for a constant \( C \), independent of \( N \in \mathbb{N} \) and \( \ell \in (0; 1) \).

Remark In (3.15), we could have projected the kernel \( \eta_H \) orthogonally to \( \varphi \). In this way, we could have defined \( \mathcal{G}_N \) as an operator on \( \mathcal{F}^{\leq N}_\varphi \) (conjugating \( \mathcal{L}_N \), rather than \( \mathcal{L}_N^{\text{eff}} \), with \( e^B \)). Also with this definition of \( \mathcal{G}_N \), we could have proven bounds similar to those in Proposition 3.4, at the expense of a longer proof (this procedure would also give an upper bound for the ground state energy, as in the remark after Theorem 1.1).

Let us now turn to the definition of the cubic operator for our second renormalization. If \( \chi_H \) denotes the characteristic function of the set \( \{p \in \mathbb{R}^3 : |p| \geq \ell^{-\alpha}\} \) and \( G \) is defined as in (3.8), we define the kernel
\[
v_H(x; y) = (G \ast \chi_H)(x - y)\varphi(y).
\]
For our analysis below, it will also be useful to introduce
\[
\tilde{k}(x; y) = -N\ell(\ell^2(N(x - y))\varphi(y)
\]
so that in particular \( v_H(x; y) = \tilde{k}(x; y) - (G \ast \chi_H)(x - y)\varphi(y) \). The next lemma summarizes important properties of \( v_H \) and its relation to \( \tilde{k} \).

Lemma 3.5 Assume (1.8) and let \( \ell \in (0; 1) \). Then \( v_H \) satisfies
\[
\|v_H\| \lesssim C\ell^{\alpha/2}, \quad \|v_H,x\| \lesssim C\ell^{\alpha/2}, \quad \|v_H,y\| \lesssim C\ell^{\alpha/2}|\varphi(y)|
\]
for all \( x, y \in \mathbb{R}^3 \) and for all \( N \) sufficiently large. Moreover, denoting by \( \hat{v}_H(p; q) = \hat{G}(p)\hat{\chi}_H(p)\hat{\varphi}(p + q) \) the Fourier transform of \( v_H \) as a function in \( L^2(\mathbb{R}^3 \times \mathbb{R}^3) \), we have for all \( p, q \in \mathbb{R}^3 \) that
\[
\|\hat{v}_H,p\| \lesssim \frac{C}{|p|^2}\chi_H(p), \quad \|\hat{v}_H,q\| \lesssim C\ell^{\alpha/2}.
\]
Proof Using (3.9) we find that

$$\|\nu_H\| \leq \|G \ast \tilde{\chi}_H\| \|\varphi\| \leq C\|\hat{G} \chi_H\| \leq C \ell^{\alpha/2}$$

as well as

$$\|\nu_{H,x}\| \leq \|\varphi\| \|G \ast \tilde{\chi}_H\| \leq C \ell^{\alpha/2}, \quad \|\nu_{H,y}\| = \|\varphi(y)\| \|G \ast \tilde{\chi}_H\| \leq C \ell^{\alpha/2} \|\varphi(y)\|.$$ 

Finally, (3.20) follows from $\hat{\nu}_H(p; q) = \hat{G}(p) \chi_H(p) \hat{\varphi}(p + q)$ and the estimate (3.9).

Apart from $\chi_H$, we will also need a second cutoff, localizing on small momenta (we are constructing the cubic operator (3.23), involving three particles; one particle should have small momentum, the other two large momenta, similarly to [5, Eq. (5.1)] in the translation invariant case). Here, it is convenient to use the Gaussian function

$$g_L(p) = e^{-(\ell \beta) p^2}$$

for an exponent $0 < \beta < \alpha$ ($g_L$ localises on momenta $|p| \lesssim \ell^{-\beta} \ll \ell^{-\alpha}$). Notice that the inverse Fourier transform of $g_L$ is given by

$$\hat{g}_L(x) = (\sqrt{\pi \ell^{-\beta}})^3 e^{-(\pi \ell^{-\beta} x)^2}.$$ 

In particular, it satisfies

$$\|\hat{g}_L\|_1 = 1, \quad \|\hat{g}_L\| = C \ell^{-\frac{3\beta}{2}}.$$ (3.22)

With $\nu_H$ defined as in (3.18) and $g_L$ from (3.21), we introduce the operator

$$A = \frac{1}{\sqrt{\mathcal{N}}} \int dx dy dz \nu_H(x; y) g_L(x - z)(b_x^* a_y^* a_z - \text{h.c.}).$$ (3.23)

Since $A$ is anti-symmetric, $e^A : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}$ is a unitary map. An important observation is that conjugation with $e^A$ only increases the number of particles by a constant of order one, independent of $N$ (this result is similar to Lemma 3.3, for the action of the generalized Bogoliubov transform $e^B$).

Lemma 3.6 Assume (1.8), let $\ell \in (0; 1)$, $t \in [-1; 1]$, $\alpha > 0$ and let $k \in \mathbb{Z}$. Then, there exists a constant $C = C(k) > 0$ such that in the sense of forms on $\mathcal{F}^{\leq N}$, we have

$$e^{-t A}(\mathcal{N} + 1)^k e^{t A} \leq C(\mathcal{N} + 1)^k.$$ (3.24)

Proof The proof is based on a Gronwall argument. Given $\xi \in \mathcal{F}^{\leq N}$, we define $f_\xi$ by $f_\xi(s) = \langle \xi, e^{-s A}(\mathcal{N} + 1)^k e^{s A} \xi \rangle$. Taking its derivative yields

$$\partial_s f_\xi(s) = 2\text{Re} \langle \xi, e^{-s A}[(\mathcal{N} + 1)^k, A]e^{s A} \xi \rangle.$$
and it is straightforward to verify that
\[
[(\mathcal{N} + 1)^k, A] = \frac{1}{\sqrt{N}} \int dx dy dz \nu_H(x; y) \hat{g}_L(x - z) b_x^* a_y a_z ((\mathcal{N} + 2)^k - (\mathcal{N} + 1)^k) + \text{h.c.}
\]

By the mean value theorem, there exists a function \( \Theta : \mathbb{N} \to (0; 1) \) such that
\[
(\mathcal{N} + 2)^k - (\mathcal{N} + 1)^k = k(\mathcal{N} + \Theta(\mathcal{N}) + 1)^{k-1}.
\]
Thus, together with Cauchy–Schwarz, (3.19) and (3.22), we obtain that
\[
|\partial_s f_\xi (s)| \leq 2k \sqrt{N} \int dx dy dz |\nu_H(x; y)| |\hat{g}_L(x - z)| a_x a_y (\mathcal{N} + 2)^{k-1} e^{iA \xi} \parallel a_x (\mathcal{N} + 2)^{k-1} e^{iA \xi} \parallel
\]
\[
\leq C \sqrt{N} \left( \int dx dy dz |\nu_{H,x}|^2 |\hat{g}_L(x - z)| |a_x (\mathcal{N} + 1)^{k-1} e^{iA \xi} \parallel \right)^{1/2}
\]
\[
\times \left( \int dx dy dz |\hat{g}_L(x - z)| |a_x a_y (\mathcal{N} + 1)^{k-1} e^{iA \xi} \parallel \right)^{1/2}
\]
\[
\leq C \ell^{\alpha/2} \sqrt{N} \parallel (\mathcal{N} + 1)^{k/2} e^{iA \xi} \parallel (\mathcal{N} + 1)^{k/2} e^{iA \xi} \parallel \leq C \parallel (\mathcal{N} + 1)^{k/2} e^{iA \xi} \parallel^2 = C f_\xi (s).
\]
Since \( C \) is independent of \( \xi \in \mathcal{F} \subseteq \mathbb{N} \), the claim follows from Gronwall’s inequality. □

Now, recalling the definition of \( G_\mathcal{N}^{\text{eff}} \) in (3.17), let us define the cubically renormalized excitation Hamiltonian \( J_\mathcal{N} \) through
\[
J_\mathcal{N} = e^{-A} G_\mathcal{N}^{\text{eff}} e^A.
\]

Proposition 3.7 Assume (1.8), let \( \ell \in (0; 1) \) and fix \( 2\beta > \alpha > 7\beta/5 \) as well as \( \alpha > 4 \). Moreover, assume that \( N \in \mathbb{N} \) is sufficiently large and that \( \ell \in (0; 1) \) is sufficiently small (but fixed, independently of \( N \)). Then, there exist \( \kappa > 0 \) and a constant \( C > 0 \), independent of \( N \) and \( \ell \), such that
\[
J_\mathcal{N} \geq \frac{N \mathcal{E}_\text{GP}(\varphi)}{2} + \frac{1}{2} d \Gamma (-\Delta x + V_{\text{ext}}(x) + 8\pi a_0 |\varphi(x)|^2 - \varepsilon_{\text{GP}}) \]
\[
- C \ell^{\kappa} N - C \ell^{-4\alpha} (\mathcal{N} + 1)^2 / N - C \ell^{-2\beta}.
\]

4 Proof of Theorem 1.1

It is enough to prove that there exist constants \( c, C > 0 \), independent of \( N \in \mathbb{N} \), such that for all sufficiently large \( N \), we have in the sense of forms in \( \mathcal{F}_\perp \) that
\[
\mathcal{L}_N \geq N \mathcal{E}_\text{GP}(\varphi) + cN - C.
\]
To prove (4.1), we first localize the operator $\widetilde{L}_N$ defined in (2.7), based on an argument from [13] (see, in particular, [13, Proposition 6.1]). To this end, let $\delta \in (0; 1)$ (it will be determined below) and let $0 \leq f, g \leq 1$ be two smooth, real-valued functions such that $f^2 + g^2 = 1$, $f(x) = 1$ for $|x| \leq 1/2$ as well as $f(x) = 0$ for $|x| \geq 1$. With the notation $f_{\delta N} = f(N/\delta N)$, $g_{\delta N} = g(N/\delta N)$, we observe that

$$\widetilde{L}_N = f_{\delta N} \widetilde{L}_N f_{\delta N} + g_{\delta N} \widetilde{L}_N g_{\delta N} + \frac{1}{2}([f_{\delta N}, [f_{\delta N}, \widetilde{L}_N]] + [g_{\delta N}, [g_{\delta N}, \widetilde{L}_N]]).$$

Thus, in the sense of forms on $\mathcal{F}^{\leq N}$, we have

$$\mathcal{L}_N = \Gamma(q) f_{\delta N} \widetilde{L}_N f_{\delta N} \Gamma(q) + \Gamma(q) g_{\delta N} \widetilde{L}_N g_{\delta N} \Gamma(q) + \frac{1}{2} \Gamma(q) ([f_{\delta N}, [f_{\delta N}, \widetilde{L}_N]] + [g_{\delta N}, [g_{\delta N}, \widetilde{L}_N]]) \Gamma(q). \quad (4.2)$$

Recall here that $q = 1 - \langle \phi \rangle \langle \phi \rangle$ and that $\Gamma(q)$ is the orthogonal projection from $\mathcal{F}^{\leq N}$ onto $\mathcal{F}_1^{\leq N}$. The second line in (4.2) contains error terms and they can be controlled as follows. Observing that

$$[f_{\delta N}, [f_{\delta N}, \widetilde{L}_N]] = \sqrt{N} b^* \left( \left( N^3 V(N) \ast |\varphi|^2 - 8\pi a_0 |\varphi|^2 \right) \varphi \right) \left( f_{\delta N}(N + 1) - f_{\delta N}(N) \right)^2$$

$$- b^* \left( \left( N^3 V(N) \ast |\varphi|^2 \right) \frac{N + 1}{\sqrt{N}} \left( f_{\delta N}(N + 1) - f_{\delta N}(N) \right)^2 \right. \left. + \frac{1}{2} \int dxdy N^3 V(N(x - y)) \varphi(y) \varphi(x) b^*_x b^*_y \left( f_{\delta N}(N + 2) - f_{\delta N}(N) \right)^2 \right. \left. + \int dxdy N^{5/2} V(N(x - y)) \varphi(y) b^*_x a^*_y a_x \left( f_{\delta N}(N + 1) - f_{\delta N}(N) \right)^2 \right. + \text{h.c.},$$

and similarly for $[g_{\delta N}, [g_{\delta N}, \widetilde{L}_N]]$, a straight-forward application of Cauchy–Schwarz, $\|\varphi\|_\infty \leq C$, $\|N^3 V(N) \ast |\varphi|^2\|_\infty \leq C$ and the mean value theorem implies that

$$[f_{\delta N}, [f_{\delta N}, \widetilde{L}_N]] + [g_{\delta N}, [g_{\delta N}, \widetilde{L}_N]] \geq -C \left( \|f'\|_\infty^2 + \|g'\|_\infty^2 \right) \frac{(\nu_N N + 1)}{\delta^2 N^2}.$$

Hence, we find the lower bound

$$\mathcal{L}_N \geq \Gamma(q) f_{\delta N} \widetilde{L}_N f_{\delta N} \Gamma(q) + \Gamma(q) g_{\delta N} \widetilde{L}_N g_{\delta N} \Gamma(q) - C \Gamma(q) \frac{(\nu_N N + 1)}{\delta^2 N^2} \Gamma(q) - C. \quad (4.3)$$

In the next step, we control the first contribution on the r.h.s. in (4.3) through Propositions 3.4 and 3.7. We define $G_N$ as in (3.16), $J_N$ as in (3.25) and we choose the parameters $\alpha, \beta > 0$ such that $2\beta > \alpha > 7\beta/5$ and $\alpha > 5$. Moreover, we assume in the following that $N$ is sufficiently large and that $\ell \in (0; 1)$ is sufficiently small.
These assumptions ensure that the conditions in Propositions 3.4 and 3.7 are satisfied. Then, by Proposition 3.4 and Lemma 3.3, we find that
\[ f_{\delta N} \tilde{L}_N f_{\delta N} = f_{\delta N} e^B G_{\text{eff}} N e^{-B} f_{\delta N} + f_{\delta N} e^B \mathcal{J}_N e^{-B} f_{\delta N} \]
\[ \geq N \mathcal{E}_{\text{GP}}(\varphi) f_{\delta N}^2 + f_{\delta N} e^B \left( G_{\text{eff}} - N \mathcal{E}_{\text{GP}}(\varphi) \right) e^{-B} f_{\delta N} \]
\[ - C \ell^{\kappa_1} f_{\delta N} e^B (N + K + V_N) e^{-B} f_{\delta N} \]
\[ - C \ell^{2\kappa_2} N^{-1} (N + 1)^2 f_{\delta N}^2 - C \ell^{-\kappa_2} \]
\[ \geq N \mathcal{E}_{\text{GP}}(\varphi) f_{\delta N}^2 + f_{\delta N} e^B \left( G_{\text{eff}} - N \mathcal{E}_{\text{GP}}(\varphi) \right) e^{-B} f_{\delta N} \]
\[ - C \ell^{\kappa_1} f_{\delta N} e^B (N + K + V_N) e^{-B} f_{\delta N} - C \ell^{-\kappa_2} N f_{\delta N}^2 - C \ell^{-\kappa_2} \]
(4.4)

for suitable $\kappa_1, \kappa_2 > 0$. Before we can continue and apply Proposition 3.7, we need to bound the error proportional to $(N + K + V_N)$ in terms of $G_{\text{eff}}$, as defined in (3.17). To this end we observe that, since $\chi_{H^\ell}(p) \geq 0$,
\[ \int dxdy \tilde{\chi}_{H^\ell}(x - y) \varphi(x) \varphi(y) (b_x^* b_y^* + b_x b_y) \geq 0. \]

Denoting by $R$ the operator with kernel $R(x; y) = \tilde{\chi}_{H^\ell}(x - y) \varphi(x) \varphi(y)$, and observing that, with (A.2), $\|R\|_{\text{op}} \leq \sup x \int |R(x; y)| dy \leq C$, uniformly in $\ell \in (0; 1)$, we conclude that
\[ \int dxdy \tilde{\chi}_{H^\ell}(x - y) \varphi(x) \varphi(y) (b_x^* b_y^* + \text{h.c.}) \geq -2d \Gamma(R) - \tilde{\chi}_{H^\ell}(0) \]
\[ \geq -\mathcal{N} - C \ell^{-3\alpha}. \]

With this bound (and with Cauchy–Schwarz), (3.17) then easily implies that
\[ G_{\text{eff}} - N \mathcal{E}_{\text{GP}}(\varphi) \geq \frac{1}{2} \mathcal{N} - C \mathcal{N} - C \ell^{-3\alpha}. \]

Hence, plugging the last bound into (4.4), we obtain
\[ f_{\delta N} \tilde{L}_N f_{\delta N} \geq N \mathcal{E}_{\text{GP}}(\varphi) f_{\delta N}^2 + (1 - C \ell^{\kappa_1}) f_{\delta N} e^B \left( G_{\text{eff}} - N \mathcal{E}_{\text{GP}}(\varphi) \right) e^{-B} f_{\delta N} \]
\[ - C (\ell^{\kappa_1} + \ell^{-\kappa_2} \mathcal{N}) f_{\delta N}^2 - C \ell^{-\kappa_2}. \]
(4.5)

Now, we apply Proposition 3.7, Lemma 3.3 and Lemma 3.6 which yields similarly as above
\[ f_{\delta N} e^B \left( G_{\text{eff}} - N \mathcal{E}_{\text{GP}}(\varphi) \right) e^{-B} f_{\delta N} \]
\[ = f_{\delta N} e^A \left( \mathcal{J}_N - N \mathcal{E}_{\text{GP}}(\varphi) \right) e^{-A} e^{-B} f_{\delta N} \]
\[ \geq \frac{1}{2} f_{\delta N} e^B e^A d\Gamma(-\Delta_x + V_{\text{ext}}(x) + 8\pi a_0 |\varphi(x)|^2 - \varepsilon_{GP}) e^{-A} e^{-B} f_{\delta N} \]
\[ - C (\ell^{\kappa_1} + \ell^{-\kappa_2} \mathcal{N}) f_{\delta N}^2 - C \ell^{-\kappa_2}. \]

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From (1.6) and under the assumptions (1.8), the operator $h_{GP} = -\Delta + V_{\text{ext}}(x) + 8\pi a_0|\varphi(x)|^2 - \varepsilon_{GP}$ has the simple eigenvalue 0 at the bottom of its spectrum (with eigenvector $\varphi$) and a positive gap $\lambda_1 > 0$ on top of it; see, for example, [20, Theorem XIII.47]. Applying once more Lemmas 3.3 and 3.6, we therefore find that

$$f_{\delta N}e^B\left(G_N^{\text{eff}} - N\mathcal{E}_{GP}(\varphi)\right)e^{-B}f_{\delta N} \geq \frac{\lambda_1}{2}f_{\delta N}e^B e^A(N - a^*(\varphi)a(\varphi))e^{-A}e^{-B}f_{\delta N} - C(\ell^{\kappa_1} + \ell^{-\kappa_2}\delta)Nf_{\delta N}^2 - C\ell^{-\kappa_2}$$

(4.6)

With the commutators

$$\left[a^*(\varphi)a(\varphi), A\right] = \frac{1}{\sqrt{N}}\int dxdydz\, v_H(x; y)\bar{g}_L(x - z)\varphi(x)b^*(\varphi)a_x^za_z + \frac{1}{\sqrt{N}}\int dxdydz\, v_H(x; y)\bar{g}_L(x - z)\varphi(y)b^*a^*(\varphi)a_z - \frac{1}{\sqrt{N}}\int dxdydz\, v_H(x; y)\bar{g}_L(x - z)\varphi(z)b^*a^*_xa(\varphi) + \text{h.c.}$$

$$\left[a^*(\varphi)a(\varphi), B\right] = \int dxdy\, \eta_H(x; y)\varphi(y)b^*b^*(\varphi) + \text{h.c.}$$

and with Lemmas 3.5 and 3.6, we find

$$-e^B e^A a^*(\varphi)a(\varphi)e^{-A}e^{-B} \geq -a^*(\varphi)a(\varphi) - C\ell^{a/2}(N + 1).$$

Thus, inserting in (4.6) and then in (4.5), and using again Lemmas 3.3 and 3.6, we arrive at

$$f_{\delta N}\bar{L}_N f_{\delta N} \geq N\mathcal{E}_{GP}(\varphi)f_{\delta N}^2 + \bar{c}(1 - C\ell^{\kappa_1})Nf_{\delta N}^2 - \frac{\lambda_1}{2}(1 - C\ell^{\kappa_1})a^*(\varphi)a(\varphi)f_{\delta N}^2 - C(\ell^{\kappa_1} + \ell^{-\kappa_2}\delta)Nf_{\delta N}^2 - C\ell^{-\kappa_2},$$

where the constants $\bar{c}, C > 0$ are independent of $\ell \in (0; 1), \delta \in (0; 1)$ and $N \in \mathbb{N}$.

We now set $\delta = \ell^{2\kappa_2}$ and choose $\ell$ sufficiently small, so that the last bound implies

$$f_{\delta N}\bar{L}_N f_{\delta N} \geq N\mathcal{E}_{GP}(\varphi)f_{\delta N}^2 + c_1Nf_{\delta N}^2 - \frac{\lambda_1}{4}a^*(\varphi)a(\varphi)f_{\delta N}^2 - C.$$  

(4.7)

Here, the positive constant $c_1 > 0$ is independent of $\ell \in (0; 1), \delta \in (0; 1)$ and $N \in \mathbb{N}$. For the rest of the proof, we fix this choice of $\delta = \ell^{2\kappa_2}$ and this (sufficiently small) value of $\ell \in (0; 1)$ so that (4.7) holds true. Then, substituting (4.7) into (4.3), we get

$$\mathcal{L}_N \geq N\mathcal{E}_{GP}(\varphi)f_{\delta N}^2\Gamma(q) + c_1Nf_{\delta N}^2\Gamma(q) + g_{\delta N}\Gamma(q)\bar{L}_N\Gamma(q)g_{\delta N} - C\Gamma(q)\frac{\mathcal{V}_N}{\delta^2N^2}\Gamma(q) - C$$
in the sense of forms in $\mathcal{F}_{\perp \varphi}^{\leq N}$. Notice that we used that $\Gamma(q) a^*(\varphi) a(\varphi) \Gamma(q) = 0$ and $[\Gamma(q), N] = 0$. Since $\Gamma(q) \mathcal{L}_N \Gamma(q) = \mathcal{L}_N$ in the sense of forms in $\mathcal{F}_{\perp \varphi}^{\leq N}$ and since $\Gamma(q) |_{\mathcal{F}_{\perp \varphi}^{\leq N}} = \text{id}_{\mathcal{F}_{\perp \varphi}^{\leq N}}$, the previous bound translates to

$$\mathcal{L}_N \geq \mathcal{E}_{\text{GP}}(\varphi) f_{\delta N}^2 + c_1 N f_{\delta N}^2 + g_{\delta N} \mathcal{L}_N g_{\delta N} - C \frac{\mathcal{V}_N}{\delta^2 N^2} - C$$

(4.8)

in the sense of forms in $\mathcal{F}_{\perp \varphi}^{\leq N}$.

Next, we notice that the contribution proportional to $g_{\delta N} \mathcal{L}_N g_{\delta N}$ in (4.8) can be controlled as explained in [5, Eq. (6.5) to (6.8)], using the results from [15, 19], i.e. (1.7). This argument shows that there exists some constant $c_2 > 0$ such that for all sufficiently large $N \in \mathbb{N}$

$$g_{\delta N} \mathcal{L}_N g_{\delta N} \geq \mathcal{E}_{\text{GP}}(\varphi) g_{\delta N}^2 + \left[ \inf_{\xi \in \mathcal{F}_{\perp \varphi}^{\leq N} \cap \mathcal{F}_{\perp \varphi}^{\leq \delta N/2} \| \xi \| = 1} \left( \frac{1}{N} (\xi, \mathcal{L}_N \xi) - \mathcal{E}_{\text{GP}}(\varphi) \right) \right] N g_{\delta N}^2$$

(4.9)

where $\mathcal{F}_{\perp \varphi}^{\leq \delta N/2} = \{ \xi \in \mathcal{F}_{\perp \varphi} : \chi(N \geq \delta N/2) \xi = \xi \}$. Hence, plugging (4.9) into (4.8), we conclude that

$$\mathcal{L}_N \geq \mathcal{E}_{\text{GP}}(\varphi) + c_1 N f_{\delta N}^2 + c_2 N g_{\delta N}^2 - C \frac{\mathcal{V}_N}{\delta^2 N^2} - C$$

$$\geq \mathcal{E}_{\text{GP}}(\varphi) + \tilde{c} N - C \frac{\mathcal{V}_N}{\delta^2 N^2} - C$$

in the sense of forms in $\mathcal{F}_{\perp \varphi}^{\leq N}$, where $\tilde{c} = \min(c_1, c_2) > 0$.

Finally, to control the error that contains the potential energy $\mathcal{V}_N$, we recall the definitions (2.6) and a simple computation involving Cauchy–Schwarz shows that

$$\mathcal{L}_N \geq \frac{1}{2} \mathcal{H}_N - C N \geq \frac{1}{2} \mathcal{V}_N - C N$$

for some constant $C > 0$, independent of $N \in \mathbb{N}$. Therefore, we find

$$\mathcal{L}_N \geq \mathcal{E}_{\text{GP}}(\varphi) + \tilde{c} N - C \frac{\mathcal{L}_N}{\delta^2 N^2} - C,$$

so that, by choosing $N \in \mathbb{N}$ sufficiently large, we have proved that

$$\mathcal{L}_N \geq \mathcal{E}_{\text{GP}}(\varphi) + c N - C$$

for some constant $c > 0$ that is independent of $N$. \qed
5 Analysis of $\mathcal{G}_N$

In this section we analyse the operator $\mathcal{G}_N = e^{-B} \tilde{L}_N e^B$, as defined in (3.16). To compute the action of the generalized Bogoliubov transform $e^B$ on $\tilde{L}_N$, we are going to compare it with the action of a standard Bogoliubov transformation. Interpreting (3.11) as the integral kernel of a Hilbert–Schmidt operator on $L^2(\mathbb{R}^3)$, we define

$$\sinh_{\eta_H} = \sum_{j=0}^{\infty} \frac{\eta_H^{(2j+1)}}{(2j+1)!}, \quad \cosh_{\eta_H} = \sum_{j=0}^{\infty} \frac{\eta_H^{(2j)}}{(2j)!}.$$  

In addition, we define the Hilbert–Schmidt operators $p_{\eta_H}$ and $r_{\eta_H}$ by

$$p_{\eta_H} = \sinh_{\eta_H} - \eta_H = \sum_{j=1}^{\infty} \frac{\eta_H^{(2j+1)}}{(2j+1)!}, \quad r_{\eta_H} = \cosh_{\eta_H} - \text{id} = \sum_{j=1}^{\infty} \frac{\eta_H^{(2j)}}{(2j)!}. \quad (5.1)$$

Using (3.12) one obtains for $\ell \in (0; 1)$ small enough

$$\|p_{\eta_H}\|, \|r_{\eta_H}\| \leq C \ell^\alpha, \quad |p_{\eta_H}(x, y)|, |r_{\eta_H}(x, y)| \leq C \ell^\alpha \varphi(x) \varphi(y). \quad (5.2)$$

The following lemma, whose proof is an adaptation of the translation invariant case [5, Lemma 3.4], shows that, on states with few excitations, $e^B$ acts approximately like a standard Bogoliubov transformation.

**Lemma 5.1** Let $n \in \mathbb{Z}$, and let $f \in L^2(\mathbb{R}^3)$. Let $d_{\eta_H}(f)$ as well as $d_{\eta_H,x}$ be defined as

$$e^{-B} b(f)e^B = b(\cosh_{\eta_H}(f)) + b^*(\sinh_{\eta_H}(\overline{f})), \quad (5.3)$$

respectively

$$e^{-B} b_x e^B = b(\cosh_{\eta_H,x}) + b^*(\sinh_{\eta_H,x}) + d_{\eta_{H,x}}. \quad (5.4)$$

Then, there exists a constant $C > 0$ such that

$$\|(\mathcal{N} + 1)^{n/2} d_{\eta_H}(f)\| \leq \frac{C \ell^\alpha/2}{N} \|f\|\|(\mathcal{N} + 1)^{(n+3)/2}\|\xi\|, (5.5)$$

and such that, for all $x \in \mathbb{R}^3$, we have that

$$\|(\mathcal{N} + 1)^{n/2} d_{\eta_{H,x}}\| \leq \frac{C \ell^\alpha/2}{N} \|a_x(\mathcal{N} + 1)^{(n+2)/2}\|\|\eta_{H,x}\|\|(\mathcal{N} + 1)^{(n+3)/2}\|\xi\|. \quad (5.6)$$
Furthermore, if we set $d_{\eta H,x} = d_{\eta H,x} + (\mathcal{N}/\mathcal{N})b^*(\eta H,x)$, it holds true that

$$
\|(\mathcal{N} + 1)^{\alpha/2}a_yd_{\eta H,x}\xi\|
\leq \frac{C}{\mathcal{N}^2}
\left[\|\eta H,x\|\|\eta H,y\|\|(\mathcal{N} + 1)^{(n+6)/2}\xi\| + \ell^{\alpha/2}\|\eta H(x,y)\|\|(\mathcal{N} + 1)^{(n+4)/2}\xi\|
+ \ell^{\alpha/2}\|a_x(\mathcal{N} + 1)^{(n+5)/2}\xi\| + \ell^{\alpha/2}\|a_y(\mathcal{N} + 1)^{(n+4)/2}\xi\|
+ \ell^{\alpha}\|a_xa_y(\mathcal{N} + 1)^{(n+4)/2}\xi\|\right]
$$

(5.7)

and, finally, we have that

$$
\|(\mathcal{N} + 1)^{\alpha/2}d_{\eta H,x}d_{\eta H,y}\xi\|
\leq \frac{C}{\mathcal{N}^2}
\left[\|\eta H,x\|\|\eta H,y\|\|(\mathcal{N} + 1)^{(n+6)/2}\xi\| + \ell^{\alpha/2}\|\eta H(x,y)\|\|(\mathcal{N} + 1)^{(n+4)/2}\xi\|
+ \ell^{\alpha/2}\|a_x(\mathcal{N} + 1)^{(n+5)/2}\xi\| + \ell^{\alpha/2}\|a_y(\mathcal{N} + 1)^{(n+4)/2}\xi\|
+ \ell^{\alpha}\|a_xa_y(\mathcal{N} + 1)^{(n+4)/2}\xi\|\right].
$$

(5.8)

From the decomposition (2.7), we can write

$$
\mathcal{G}_N = \mathcal{G}_N^{(0)} + \mathcal{G}_N^{(1)} + \mathcal{G}_N^{(2)} + \mathcal{G}_N^{(3)} + \mathcal{G}_N^{(4)},
$$

where, for $j \in \{0, 1, 2, 3, 4\}$, we set

$$
\mathcal{G}_N^{(j)} = e^{-B}\mathcal{L}_N^{(j)} e^B.
$$

In the following subsections, we will analyze the main contributions $\mathcal{G}_N^{(j)}$ separately and, in Sect. 5.6, we combine these results to conclude Proposition 3.4.

### 5.1 Analysis of $\mathcal{G}_N^{(0)}$

From (2.6), we recall that

$$
\mathcal{L}_N^{(0)} = \langle \varphi, \left[-\Delta + V_{\text{ext}} + \frac{1}{2}(N^3V(N\cdot)^*|\varphi|^2)\right]\varphi\rangle(N - \mathcal{N})
- \frac{1}{2}\langle \varphi, (N^3V(N\cdot)^*|\varphi|^2)\varphi\rangle(N + 1)(1 - \mathcal{N}/N).
$$

(5.9)

**Proposition 5.2** There exists a constant $C > 0$ such that

$$
\mathcal{G}_N^{(0)} = \langle \varphi, \left[-\Delta + V_{\text{ext}} + \frac{1}{2}(N^3V(N\cdot)^*|\varphi|^2)\right]\varphi\rangle(N - \mathcal{N})
- \frac{1}{2}\langle \varphi, (N^3V(N\cdot)^*|\varphi|^2)\varphi\rangle(N + 1)(1 - \mathcal{N}/N) + \mathcal{E}_{N,\ell}^{(0)}.
$$
where the self-adjoint operator $E_{N,\ell}^{(0)}$ satisfies

$$\pm E_{N,\ell}^{(0)} \leq C \ell^{\alpha/2} (N + 1)$$

for all $\alpha > 0$ and $\ell \in (0; 1)$.

**Proof** We start with the observation that

$$e^{-B}N e^B - N = \int_0^1 ds \left( \int dx dy \eta_H(x; y) e^{-sB} b_x^* b_y^* e^{sB} \right) + \text{h.c.}$$

This implies together with Lemma 3.3 and Cauchy–Schwarz that

$$\pm (e^{-B}N e^B - N) \leq C \ell^{\alpha/2} (N + 1).$$

Similarly, it is straightforward to prove that

$$\pm (e^{-B}N^2 e^B - N^2) \leq C \ell^{\alpha/2} (N + 1)^2.$$

If we use these two observations together with the bounds

$$|\langle \phi, (-\Delta + V_{\text{ext}}) \phi \rangle| \leq |E_{\text{GP}}(\phi)| + 4\pi a_0 \|\phi\|_4^4$$

$$\leq C, \quad \langle \phi, (N^3 V(N) \ast |\phi|^2) \phi \rangle | \leq \|V\|_1 \|\phi\|_\infty^2 \leq C,$$

the proposition follows directly from the definition of $\tilde{L}_N^{(0)}$ in Eq. (5.9). \qed

### 5.2 Analysis of $G_N^{(1)}$

From (2.6), we recall that

$$\tilde{L}_N^{(1)} = \sqrt{N} b \left( \left( N^3 V(N) \ast |\phi|^2 - 8\pi a_0 |\phi|^2 \right) \phi \right)$$

$$- \frac{N + 1}{\sqrt{N}} b \left( \left( N^3 V(N) \ast |\phi|^2 \right) \phi \right) + \text{h.c.} \quad (5.10)$$

For the statement of the next proposition, let us define $h_N \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by

$$h_N = (N^3 (V \xi_\ell)(N) \ast |\phi|^2) \phi. \quad (5.11)$$

**Proposition 5.3** There exists a constant $C > 0$ such that

$$G_N^{(1)} = \left[ \sqrt{N} b (\cosh_{\eta_H} (h_N)) + \sqrt{N} b^* (\sinh_{\eta_H} (h_N)) \right] + \tilde{L}_N^{(1)}.$$

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where the self-adjoint operator $\mathcal{E}_{N,\ell}^{(1)}$ satisfies
\[
\pm \mathcal{E}_{N,\ell}^{(1)} \leq C \ell^{\alpha/2} (N + 1) + C \ell^{-\alpha/2} N^{-1} (N + 1)^2 + C \ell^{-1}.
\]
for all $\alpha > 0$ and $\ell \in (0; 1)$.

**Proof** First of all, we notice that a simple application of Cauchy–Schwarz and the fact that \(\| (N^3 V (N \cdot) \ast |\varphi|^2 ) \varphi \|_2 \leq C\) imply that
\[
\pm \left( \frac{N + 1}{\sqrt{N}} b \left( (N^3 V (N \cdot) \ast |\varphi|^2 ) \varphi \right) + \text{h.c.} \right)
\leq C \ell^{\alpha/2} (N + 1) + C \ell^{-\alpha/2} N^{-1} (N + 1)^2.
\]
By Lemma 3.3, we therefore obtain that
\[
\pm e^{-B} \left( \frac{N + 1}{\sqrt{N}} b \left( (N^3 V (N \cdot) \ast |\varphi|^2 ) \varphi \right) + \text{h.c.} \right) e^B
\leq C \ell^{\alpha/2} (N + 1) + C \ell^{-\alpha/2} N^{-1} (N + 1)^2.
\]
This controls the conjugation of the second contribution to $\tilde{\mathcal{L}}_{N}^{(1)}$ in (5.10). To deal with the first term on the right hand side of (5.10), we first observe that
\[
(N^3 V (N \cdot) \ast |\varphi|^2 ) \varphi = h_N + \left( N^3 (V_{f\ell} (N \cdot) \ast |\varphi|^2 - 8\pi a_0 |\varphi|^2 \right) \varphi
\]
and we find with Lemma 3.1 (ii) that
\[
\| N^3 (V_{f\ell} (N \cdot) \ast |\varphi|^2 - 8\pi a_0 |\varphi|^2 \|_{\infty}
\leq \sup_{x \in \mathbb{R}^3} \int dy (V_{f\ell} (y) |\varphi|^2 (x - y/N) - |\varphi|^2 (x)) + \frac{C}{\ell N}
\leq C \| \nabla \varphi \|_{\infty} \| \varphi \| \| V \| N^{-1} + C \ell^{-1} N^{-1} \leq C \ell^{-1} N^{-1}.
\]
Note that $\| \nabla \varphi \|_{\infty} \leq C$ by Appendix A. By Cauchy–Schwarz, this implies that
\[
\pm e^{-B} \left( \sqrt{N} b \left( (N^3 (V_{f\ell} (N \cdot) \ast |\varphi|^2 - 8\pi a_0 |\varphi|^2 \right) \varphi \right) + \text{h.c.} \right) e^B \leq C \ell^{-1}
\]
and it only remains to control
\[
e^{-B} \sqrt{N} b (h_N) e^B + \text{h.c.}
= \sqrt{N} b (\cosh_{\eta_H} (h_N)) + \sqrt{N} b^* (\sinh_{\eta_H} (h_N)) + \sqrt{N} d_{\eta_H} (h_N) + \text{h.c.}
\]
However, by Eq. (5.5) from Lemma 5.1, we know that for all $\xi \in \mathcal{F}^{\leq N}$ we have that

$$\|\langle \xi, \sqrt{N} d_{jH} (h_N) \xi \rangle\| \leq C \ell^{\alpha/2} N^{-1/2} \| h_N \| \| (N + 1)^{1/2} \xi \| \| (N + 1) \xi \|,$$

and hence, collecting the previous estimates, we conclude the proposition. 

**5.3 Analysis of $G^{(2)}_N$**

From (2.6), we recall that $\tilde{L}_N^{(2)} = K + V_{\text{ext}} + \tilde{L}_N^{(2, V)}$, where

$$K = \int dx \nabla_x a_x^* \nabla_x a_x, \quad V_{\text{ext}} = \int dx V_{\text{ext}}(x) a_x^* a_x$$

$$\tilde{L}_N^{(2, V)} = \int dx \left( N^3 V(N.x) |\varphi|^2 \right)(x) \left( b_x^* b_x - \frac{1}{N} a_x^* a_x \right)$$

$$+ \int dx dy N^3 V(N(x - y)) \varphi(x) \varphi(y) \left( b_x^* b_y - \frac{1}{N} a_x^* a_y \right)$$

$$+ \frac{1}{2} \int dx dy N^3 V(N(x - y)) \varphi(y) \varphi(x) \left( b_x^* b_y^* + \text{h.c.} \right).$$

In the following, we will analyze the contributions $e^{-B K} e^B$, $e^{-B V_{\text{ext}} e^B}$ and $e^{-B \tilde{L}_N^{(2, V)} e^B}$ separately. The analysis of the kinetic energy is the most involved so let us first treat the contributions $e^{-B V_{\text{ext}} e^B}$ and $e^{-B \tilde{L}_N^{(2, V)} e^B}$.

**Proposition 5.4** There exists a constant $C > 0$ such that

$$e^{-B V_{\text{ext}} e^B} = V_{\text{ext}} + \mathcal{E}_{N, \ell}^{(\text{ext})},$$

where the self-adjoint operator $\mathcal{E}_{N, \ell}^{(\text{ext})}$ satisfies

$$\pm \mathcal{E}_{N, \ell}^{(\text{ext})} \leq C \ell^{\alpha/2} (N + 1).$$

for all $\alpha > 0$ and $\ell \in (0; 1)$.

**Proof** We start with the observation that

$$e^{-B V_{\text{ext}} e^B} - V_{\text{ext}} = \frac{1}{2} \int_0^1 ds \left( \int dx V_{\text{ext}}(x) e^{-s B b_x^* b_x} (\eta_{H, x}) e^{s B} \right) + \text{h.c.}$$

In the appendix we show that the assumptions (2) in (1.8) imply that the external potential $V_{\text{ext}}(x)$ has at most exponential growth as $|x| \to \infty$ while, by (A.2), the minimizer $\varphi(x)$ has exponential decay as $|x| \to \infty$ with arbitrary rate. This implies
in particular that $\int dx \, V_{\text{ext}}^2(x) |\varphi(x)|^2 \leq C$. As a consequence, Cauchy–Schwarz, Lemmas 3.3 and 3.2 imply that

$$\left| \int dx \, V_{\text{ext}}(x) \left\langle \xi, e^{-sB} b_x^* b_x^* (\eta_{H,x}) e^{sB} \xi \right\rangle \right| \leq \left( \int dx \, V_{\text{ext}}^2(x) \|\eta_{H,x}\|^2 \right)^{1/2} \left( \int dx \, \|a_x e^{sB} \xi\|^2 \right)^{1/2} \leq C_\ell^{\alpha/2} \left( \int dx \, V_{\text{ext}}^2(x) |\varphi(x)|^2 \right)^{1/2} \left( \langle \xi, (N + 1)\xi \rangle \leq C_\ell^{\alpha/2} \langle \xi, (N + 1)\xi \rangle \right)$$

uniformly in $s \in [0; 1]$, which proves the claim. \hfill \square

**Proposition 5.5** There exists a constant $C > 0$ such that

$$e^{-B} \tilde{E}_{N,\ell}^{(2,V)} e^B = \sum_{j=1}^5 F_j,$$

where the self-adjoint operator $E_{N,\ell}^{(2,V)}$ satisfies

$$\pm E_{N,\ell}^{(2,V)} \leq C_\ell^{\alpha/2} (N + V_N + 1).$$

for all $\alpha > 0$, $\ell \in (0; 1)$ and $N \in \mathbb{N}$ sufficiently large.

**Proof** We split $e^{-B} \tilde{E}_{N,\ell}^{(2,V)} e^B = \sum_{j=1}^5 F_j$, setting

$$F_1 := \int dx \left(N^3 V(N_x) * |\varphi|^2\right)(x) e^{-B} b_x^* b_x e^B,$$

$$F_2 := \int dx dy N^3 V(N(x - y)) \varphi(x) \varphi(y) e^{-B} b_y^* b_y e^B,$$

$$F_3 := -\frac{1}{N} \int dx \left(N^3 V(N_x) * |\varphi|^2\right)(x) e^{-B} a_x^* a_x e^B,$$

$$F_4 := -\frac{1}{N} \int dx dy N^3 V(N(x - y)) \varphi(x) \varphi(y) e^{-B} a_y^* a_y e^B,$$

$$F_5 := \frac{1}{2} \int dx dy N^3 V(N(x - y)) \varphi(x) \varphi(y) e^{-B} (b_x^* b_y^* + b_x b_y) e^B.$$
To estimate these terms, we use the decomposition (5.4) and the bounds in Lemma 5.1. With Lemma 3.3 and Cauchy–Schwarz, we obtain

\[
\pm \left( F_1 - \int dx \left( N^3 V(N) \ast |\varphi|^2 \right) (x) b_x^* b_x \right) \leq C \ell^{\alpha/2} (N+1),
\]

\[
\pm \left( F_2 - \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) b_y^* b_y \right) \leq C \ell^{\alpha/2} (N+1)
\]\n
(5.13)

and, similarly, that

\[
\pm F_3, \pm F_4 \leq N^{-1} \|V\|_1 \|\varphi\|^2_{\infty} (N+1) \leq C \ell^{\alpha/2} (N+1)
\]\n
(5.14)

for all \( N \) sufficiently large.

Hence, let us focus on the analysis of \( F_5 \). We split it into \( F_5 = F_{51} + F_{52} + F_{53} \), where

\[
F_{51} := \frac{1}{2} \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) \\
\times \left( b(\cosh_{\eta_H,x}) + b^*(\sinh_{\eta_H,x}) \right) \left( b(\cosh_{\eta_H,y}) + b^*(\sinh_{\eta_H,y}) \right) + \text{h.c.},
\]

\[
F_{52} := \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) \left[ d_{\eta_H,x} b(\cosh_{\eta_H,y}) + d_{\eta_H,y}^* b^*(\sinh_{\eta_H,y}) \right] \\
+ b(\cosh_{\eta_H,x}) d_{\eta_H,y} + b^*(\sinh_{\eta_H,x}) d_{\eta_H,y} + \text{h.c.},
\]

\[
F_{53} := \frac{1}{2} \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) d_{\eta_H,x} d_{\eta_H,y} + \text{h.c.}
\]\n
(5.15)

and consider the different contributions separately. We use \( \sinh_{\eta_H} = \eta_H + p_{\eta_H} \) and \( \cosh_{\eta_H} = 1 + \tau_{\eta_H} \) and the estimates in Lemma 5.1 to rewrite \( F_{51} \) as

\[
F_{51} = \frac{1}{2} \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) \left( b_x^* b_y + b_x b_y^* \right) \\
+ \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) \eta_H(x; y) \left( \frac{N - N'}{N} \right) + \mathcal{E}^y_{51}
\]\n
(5.16)

with

\[
\pm \mathcal{E}^y_{51} \leq C \ell^{\alpha/2} (N+1).
\]\n
(5.17)

Let us switch to \( F_{52} \), defined in (5.15). We write \( F_{52} = F_{521} + F_{522} + F_{523} + F_{524} \) with

\[
F_{521} := \frac{1}{2} \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) d_{\eta_H,x} b(\cosh_{\eta_H,y}) + \text{h.c.},
\]

\[
F_{522} := \frac{1}{2} \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) d_{\eta_H,x} b^*(\sinh_{\eta_H,y}) + \text{h.c.},
\]
\[ F_{523} := \frac{1}{2} \int dxdy \, N^3 V(N(x - y)) \varphi(x) \varphi(y) b^*(\sinh_{\eta_{H,x}})d_{\eta_{H,y}} + \text{h.c.}, \]
\[ F_{524} := \frac{1}{2} \int dxdy \, N^3 V(N(x - y)) \varphi(x) \varphi(y) b(\cosh_{\eta_{H,x}})d_{\eta_{H,y}} + \text{h.c.} \]

Now, by applying (5.6), we find that
\[
|\langle \xi, d_{\eta_{H,x}} b(\cosh_{\eta_{H,y}}) \xi \rangle| 
\leq C \ell^{\alpha/2} \|(N + 1)^{1/2} \xi \| \left( N^{-1/2} a_x a_y \xi \| + |\varphi(x)||a_y \xi| \right) 
\]
\[ + C \ell^{\alpha/2} \|r_{\eta_{H,y}}\| (N + 1)^{1/2} \xi \| \left( \|a_x \xi\| + |\varphi(x)||N + 1\|^1/2 \xi \right) \]

and, similarly, that
\[
|\langle \xi, d_{\eta_{H,x}} b^*(\sinh_{\eta_{H,y}}) \xi \rangle| 
\leq C \ell^{\alpha/2} \|(N + 1)^{1/2} \xi \| \left( N^{-1/2} |\sinh_{\eta_{H}}(x; y)| \cdot (N + 1)^{1/2} \xi \right) 
\]
\[ + C \ell^{\alpha/2} \|\sinh_{\eta_{H,y}}\| (N + 1)^{1/2} \xi \| \left( \|a_x \xi\| + |\varphi(x)||N + 1\|^1/2 \xi \right) \]

Together with the fact that
\[
\sup_{x, y \in \mathbb{R}^3} N^{-1} |\sinh_{\eta_{H}}(x; y)| \leq N^{-1} \left( |\eta_{H}(x; y)| + C \right) \leq C
\]
by (3.14), this yields together with the estimates in Lemma 5.1
\[
\pm F_{521} \leq C \ell^{\alpha/2}(N + \mathcal{V}_{N} + 1), \quad \pm F_{522} \leq C \ell^{\alpha/2}(N + 1). \tag{5.18}
\]

A similar (but simpler) argument involving (5.6) also shows that
\[
\pm F_{523} \leq C \ell^{\alpha/2}(N + 1). \tag{5.19}
\]

Next, let us switch to \( F_{524} \), defined above. Here, we first compute
\[
\begin{align*}
&b_x N b^*(\eta_{H,y}) = b^*(\eta_{H,y}) b_x (N + 1) \\
&\quad + \eta_{H}(x; y) (1 - \mathcal{N}/N) (N + 1) - a^*(\eta_{H,y}) a_x (N + 1)/N .
\end{align*}
\]

If we recall the notation \( \bar{d}_{\eta_{H,y}} = d_{\eta_{H,y}} + (\mathcal{N}/N) b^*(\eta_{H,y}) \), we therefore obtain
\[
F_{524} = - \int dxdy \, N^3 V(N(x - y)) \varphi(x) \varphi(y) \eta_{H}(x; y) \left( \frac{N - \mathcal{N}}{N} \right) \left( \frac{\mathcal{N} + 1}{N} \right) + \mathcal{E}^{(V)}_{524}
\]
where
\[
\mathcal{E}^{(V)}_{524} = \frac{1}{2} \int dxdy \, N^3 V(N(x - y)) \varphi(x) \varphi(y) \left[ b_x \bar{d}_{\eta_{H,y}} + b(r_{\eta_{H,x}}) d_{\eta_{H,y}} \right]
\]
\[-N^{-1}b^*(\eta_{H,Y})b_x(N+1) + N^{-2}a^*(\eta_{H,Y})a_x(N+1)\] + h.c.

Now, using (5.6) and (5.7) and proceeding as in the previous steps, we find that

\[\pm \epsilon^{(V)}_{S24} \leq C \ell^{\alpha/2}(N + \mathcal{V}_N + 1). \quad (5.20)\]

Collecting the previous bounds, this controls the contribution $F_{S2}$, defined in (5.15).

Finally, to control the last contribution $F_{S3}$, defined in (5.15), we use (5.8) and the fact that

\[N^{-1}|\eta_H(x; y)| \leq C\]

so that

\[\pm F_{S3} \leq C \ell^{\alpha/2}(N + \mathcal{V}_N + 1). \quad (5.21)\]

In summary, if we combine (5.12), (5.13), (5.14), (5.16), (5.17), (5.18), (5.19), (5.20) and (5.21), we obtain the claim. \qed

Finally, we study the kinetic energy. We start with two auxiliary lemmas.

Lemma 5.6 Assume (1.8), let $\ell \in (0; 1)$ be sufficiently small and let $\alpha > 0$. Then, there exists $C > 0$, independent of $N$ and $\ell$, such that for all $n \geq 2$ and $j \in \{1, 2\}$

\[\|\nabla_j \eta_H^{(n)}\|, \|\Delta_j \eta_H^{(n)}\| \leq C \ell^{\alpha/2} \|\eta_H\|^{n-2}. \quad (5.22)\]

As a consequence, we have that

\[\|\nabla_j p_{\eta_H}\|, \|\Delta_j p_{\eta_H}\|, \|\nabla_j r_{\eta_H}\|, \|\Delta_j r_{\eta_H}\| \leq C \ell^{\alpha/2}. \quad (5.23)\]

Proof Notice first of all that (5.23) follows indeed directly from (5.22) and the definition (5.1). To prove (5.22), let us first consider

\[\|\nabla_1 \eta_H^{(n)}\|\]. Since $\eta_H$ is symmetric, this takes also care of the case $j = 2$. Moreover, note that it suffices to consider the case $n = 2$. By the triangle inequality, we then have that

\[\|\Delta_1 \eta_H^{(2)}\| \leq \left(\int dx dy (\Delta G \ast \bar{\chi}_H)(x-y)\varphi(x)\varphi(y)\eta_H(y; z) \right)^2 \right)^{1/2}

\[+ \left(2 \int dx dy (\nabla G \ast \bar{\chi}_H)(x-y) \cdot \nabla \varphi(x)\varphi(y)\eta_H(y; z) \right)^2 \right)^{1/2}

\[+ \left(\int dx dy (G \ast \bar{\chi}_H)(x-y)\Delta \varphi(x)\varphi(y)\eta_H(y; z) \right)^2 \right)^{1/2}

=: T_1 + T_2 + T_3.

Let us start with $T_1$. After switching to Fourier space, using the bound (A.5) and using (3.9) that $|\hat{\Delta G}(p)| = |p|^2|\hat{G}(p)| \leq C$, we get with Young’s inequality

\[T_1^2 = \int dr_1 dr_2 ds_1 ds_2 \hat{G}(r_1)\chi_H(r_1)\hat{G}(r_2)\chi_H(r_2)|\varphi|^2(s_1 + s_2)|\varphi|^2(r_1 - s_1)

\]
\[ \times |\varphi|^2(r_2 - s_2) |\varphi|^2(r_1 + r_2) \Delta \tilde{G}(s_1) \chi_H(s_1) \Delta \tilde{G}(s_2) \chi_H(s_2) \times C \int_{|r_1|,|r_2|,|s_1|,|s_2| \geq \ell^{-a}} \frac{dr_1 dr_2 ds_1 ds_2}{|r_1|^2 |r_2|^2 (1 + |s_1 + s_2|^2)^3 (1 + |r_1 - s_1|)^3} \times \frac{1}{(1 + |r_2 - s_2|)^3 (1 + |r_1 + r_2|)^3} \leq C \int_{|r_1|,|r_2| \geq \ell^{-a}} \frac{dr_1 dr_2}{|r_1|^2 |r_2|^2 (1 + |r_1 + r_2|)^6} \leq C \ell^a. \]

For the second term \( T_2 \), we use that \(|\nabla \varphi^2(p)| \leq C (1 + |p|)^{-2}\) by Lemma A.2 and that \(|\nabla \tilde{G}(p)| \leq \ell^a C\) for \(|p| \geq \ell^{-a}\), which yields by similar computation \( T_2^2 \leq C \ell^{2a} \).

Similarly one deals with \( T_3 \) and the bounds on \( \|\nabla \eta_H^{(n)}\| \) can be proved analogously. \( \Box \)

**Lemma 5.7** Assume (1.8), let \( \ell \in (0; 1) \) be sufficiently small and let \( \alpha > 0 \). Then, there exists a constant \( C > 0 \) such that

\[ \pm \int dx \ a^*(\nabla_x \eta_{H,x}) a(\nabla_x \eta_{H,x}) \leq C \ell^{a/2}(N + 1). \]

Moreover, recalling (5.3), (5.4) and \( \tilde{d}_{\eta,x} = d_{\eta,x} + (\tilde{N}/N)b^*(\eta_x) \), we have that

\[ \left( \int dx \ ||\sqrt{\tilde{N}}(N + 1)^{-1/2} \nabla_x d_{\eta H,x} \xi \|^2 \right)^{1/2} \leq C \| (\tilde{N} + \mathcal{K} + 1)^{1/2} \xi \|, \]

\[ \left( \int dx \ ||\sqrt{\tilde{N}}(N + 1)^{-1/2} \nabla_x d_{\eta H,x} \xi \|^2 \right)^{1/2} \leq C \ell^{a/2} \| (\tilde{N} + \mathcal{K} + 1)^{1/2} \xi \| \]

and that

\[ \left( \int dx \ ||\sqrt{\tilde{N}}(N + 1)^{-1/2} d_{\eta H} (\nabla_x \eta_{H,x}) \xi \|^2 \right)^{1/2} \leq C \ell^{a/2} \| (N + 1)^{1/2} \xi \|, \]

\[ \left( \int dx \ ||\sqrt{\tilde{N}}(N + 1)^{-1/2} d_{\eta H}^* (\nabla_x \eta_{H,x}) \xi \|^2 \right)^{1/2} \leq C \ell^{a/2} \| (N + 1)^{1/2} \xi \| \]

for all \( \xi \in \mathcal{F}^{\leq N} \) and all \( s \in [0; 1] \).

**Proof** We first note that, using arguments very similar to those in the proof of the previous Lemma 5.6, it is simple to show that

\[ \int dydz \left| \int dx \ \nabla_x \eta_H(y; x) \cdot \nabla_x \eta_H(z; x) \right|^2 \leq C \ell^a. \]
Hence, Cauchy–Schwarz implies
\[
\pm \left( \int dx \ a^*(\nabla_x \eta_{H,x}) a(\nabla_x \eta_{H,x}) \right)
= \pm \left( \int dydz \left[ \int dx \ \nabla_x \eta_{H}(y; x) \nabla_x \eta_H(z; x) \right] a^*_y a_z \right) \leq C \ell^\alpha (N + 1).
\]
This proves the first bound (5.24). To prove the bounds (5.25) and (5.26), we proceed similar as in the proof of Lemma 5.1 (which can be found in [5, Lemma 3.4]) and use Lemma 3.2, Lemma 5.6 as well as the bound (5.24); we skip the details. \qed

We are now ready to analyze the kinetic energy.

**Proposition 5.8** There exists a constant \( C > 0 \) such that
\[
e^{-B} K e^B = K + \int dx \left[ b(\nabla_x \eta_{H,x}) \nabla_x b_x + \text{h.c.} \right]
+ \| \nabla_1 \eta_H \|^2 (1 - N'/N)(1 - N'/N - 1/N) + \mathcal{E}_{N, \ell}^{(K)}
\]
where the self-adjoint operator \( \mathcal{E}_{N, \ell}^{(K)} \) satisfies
\[
\pm \mathcal{E}_{N, \ell}^{(K)} \leq C \ell^{(\alpha - 3)/2} (N + K + V_N + 1) + C \ell^{-5\alpha/2} N^{-1} (N + 1)^2
\]
for all \( \alpha > 3 \), \( \ell \in (0; 1) \) and all \( N \) large enough.

**Proof** Using a first order Taylor expansion, (2.4) and the identity (5.3), we have that
\[
e^{-B} K e^B - K
= \left( \int_0^1 ds \int dx \left[ b(\cosh_s \eta_{H,x}) + b^*(\sinh_s \eta_{H,x}) \right] \nabla_x b_x + \text{h.c.} \right)
+ \left[ \nabla_x b(\cosh_s \eta_{H,x}) + \nabla_x b^*(\sinh_s \eta_{H,x}) \right] + \text{h.c.}
+ \left( \int_0^1 ds \int dx \left[ b(\cosh_s \eta_{H,x}) + b^*(\sinh_s \eta_{H,x}) \right] \nabla_x d_{\eta H,x}
+ d_{\eta H,x} \left( \nabla_x b(\cosh_s \eta_{H,x}) + \nabla_x b^*(\sinh_s \eta_{H,x}) \right) + \text{h.c.} \right)
\]
=: G_1 + G_2 + G_3.
(5.27)

Let us start to analyze \( G_1 \). Integrating by parts and using (5.2), Lemma 5.6 as well as the bound (5.24) we conclude that
\[
G_1 = \int dx \left[ b(\nabla_x \eta_{H,x}) \nabla_x b_x + \text{h.c.} \right] + \| \nabla_1 \eta_H \|^2 (1 - N'/N) + \mathcal{E}_1,
(5.28)
where the error $\mathcal{E}_1$ satisfies

$$\pm \mathcal{E}_1 \leq C \ell^{\alpha/2}(N + 1).$$

Next we extract the relevant terms from $G_2$, defined in (5.27). We split $G_2$ into

$$G_2 = \int_0^1 ds \int dx \left[ b^* (\sinh s \eta (\nabla_x \eta, x)) \nabla_x d s \eta H, x + \text{h.c.} \right]$$

$$+ \int_0^1 ds \int dx \left[ d s \eta H (\nabla_x \eta, x) \nabla_x b (\cosh s \eta H, x) + \text{h.c.} \right]$$

$$+ \int_0^1 ds \int dx \left[ b (\cosh s \eta H (\nabla_x \eta, x)) \nabla_x d s \eta H, x + \text{h.c.} \right]$$

$$+ \int_0^1 ds \int dx \left[ d s \eta H (\nabla_x \eta, x) \nabla_x b^* (\sinh s \eta H, x) + \text{h.c.} \right]$$

$$=: G_{21} + G_{22} + G_{23} + G_{24}.$$ 

From Lemma 5.6, Lemma 5.7 and $\| \sinh \eta (\nabla_x \eta, x) \| \leq C \| \nabla_x \eta^{(2)} \|$, we easily find that

$$\pm G_{21} \leq C \ell^{\alpha/2} (N + K + 1), \quad \pm G_{22} \leq C \ell^{\alpha/2} (N + K + 1),$$

so let us continue with the analysis of $G_{23}$. We split it into

$$G_{23} = \int_0^1 ds \int dx \left[ b (r s \eta (-\Delta_x \eta, x)) d s \eta H, x + \text{h.c.} \right]$$

$$+ \int_0^1 ds \int dx \left[ b (\nabla_x \eta H, x) \nabla_x \tilde{d} s \eta H, x + \text{h.c.} \right]$$

$$- \int_0^1 ds \int dx \left[ b (\nabla_x \eta H, x) (N / N) b^* (\nabla_x \eta H, x) + \text{h.c.} \right]$$

$$=: G_{231} + G_{232} + G_{233}$$

and, similarly as above, it is simple to see that

$$\pm G_{231} \leq C \ell^{\alpha/2} (N + 1).$$

To control the contribution $G_{232}$, we use that

$$-\Delta_x \eta H (y; x) = -\Delta G (y - x) \varphi (x) \varphi (y) + 2 \nabla G (y - x) \nabla \varphi (x) \varphi (y)$$

$$- G (y - x) \Delta \varphi (x) \varphi (y) + \Delta_x \left[ (G \ast \bar{\chi}_H) (y - x) \varphi (x) \varphi (y) \right]$$
so that integrating by parts implies

\[
G_{232} = \int_0^1 ds \int dx dy \left( (-\Delta G)(y - x)\varphi(x)\varphi(y)b_y\tilde{d}_{s\eta,y} + \text{h.c.} \right) - \int_0^1 ds \int dx dy \left( G(y - x)\Delta\varphi(x)\varphi(y)b_y\tilde{d}_{s\eta,y} + \text{h.c.} \right) + \int_0^1 ds \int dx dy \left( G(y - x)\nabla\varphi(x)\varphi(y)b_y\nabla_x\tilde{d}_{s\eta,y} + \text{h.c.} \right) + \int_0^1 ds \int dx dy \left( \Delta_x \left( (G * \tilde{\chi}_{H^c})(y - x)\varphi(x)\varphi(y) \right)b_y\tilde{d}_{s\eta,y} + \text{h.c.} \right) =: G_{2321} + G_{2322} + G_{2323} + G_{2324}.
\]

Using the scattering equation (3.2) and recalling (3.3), we have that

\[
G_{2321} = -\int_0^1 ds \int dx dy \left[ \frac{1}{2}N^3 V(N(y - x)) - N^3 \lambda f_x(N(y - x)) \right] \times \chi_x(x - y)\varphi(x)\varphi(y)b_y\tilde{d}_{s\eta,y}.
\]

With Lemmas 3.1 and 5.1, we conclude, proceeding in the usual way,

\[
|G_{2321}| \leq C \ell^{2\alpha/2} \int dx dy \left[ N^3 V(N(x - y)) + C \ell^{-3}\chi_x(x - y) \right] \\
\times |\varphi(x)||\varphi(y)||\eta_x (N + 1)^{1/2}\xi| \\
\times N^{-1} \left[ \ell^{\alpha/2} ||\eta_x (N + 1)^{1/2}\xi| + ||\eta_x (N + 1)^{1/2}\xi| + \ell^{2\alpha/2}||a_x\xi|| \\
+ \ell^{\alpha/2} ||a_y(N + 1)\xi| + ||a_xa_y(N + 1)^{1/2}\xi|| \right] \\
\leq C \ell^{(\alpha - 3)/2}(\xi, (N + 1)\xi) + C \ell^{4\alpha/2}(\xi, \beta_N\xi).
\]

Using once more Lemmas 5.1 and 5.7, we also find that

\[
|G_{2322}| + |G_{2323}| \\
\leq C \int_0^1 ds \|(N + 1)^{1/2}\xi\| \left( \int dx ||\nabla_x\tilde{d}_{s\eta,y}||^2 \right)^{1/2} + \left( \int dx ||\tilde{d}_{s\eta,y}||^2 \right)^{1/2} \\
\leq C \ell^{\alpha/2}(\xi, (N + K + 1)\xi)
\]

and, since

\[
|\Delta_x [(G * \tilde{\chi}_{H^c})(x - y)\varphi(x)\varphi(y)| \leq C \ell^{-3\alpha}||\varphi(x)|| + ||\nabla_x\varphi(x)|| + ||\Delta_x\varphi(x)|| \varphi(y)|,
\]

we have furthermore by Cauchy–Schwarz and (5.7) that

\[
|G_{2324}| \leq C \ell^{-5\alpha/2} \frac{N}{N^2}(\xi, (N + 1)^2\xi).
\]
In summary, this proves that
\[ \pm G_{232} \leq C \ell^{(\alpha - 3)/2} (N + 1) + C \ell^{\alpha/2} (K + V_N) + C \ell^{-5\alpha/2} N^{-1} (N + 1)^2. \]

and since
\[ G_{233} = -\| \nabla_1 \eta_H \|^2 \frac{\mathcal{N} + 1}{N} \frac{N - \mathcal{N}}{N} - \int dx \ a^*(x) (\nabla_x \eta_H, x) a(x) \frac{\mathcal{N} + 1}{N} \frac{N - \mathcal{N}}{N}, \]

we easily deduce, with Lemma 5.7, that
\[ \pm \left( G_{23} + \| \nabla_1 \eta_H \|^2 \frac{\mathcal{N} + 1}{N} \frac{N - \mathcal{N}}{N} \right) \leq C \ell^{(\alpha - 3)/2} (N + 1) + C \ell^{\alpha/2} (K + V_N) + C \ell^{-5\alpha/2} N^{-1} (N + 1)^2. \]

With very similar arguments, one can show that
\[ \pm G_{24} \leq C \ell^{\alpha/2} (N + 1) \]

so that, in summary, we have
\[ G_2 = -\| \nabla_1 \eta_H \|^2 \frac{\mathcal{N} + 1}{N} \frac{N - \mathcal{N}}{N} + \mathcal{E}_2 \]

for an error \( \mathcal{E}_2 \) that satisfies
\[ \pm \mathcal{E}_2 \leq C \ell^{(\alpha - 3)/2} (N + 1) + C \ell^{\alpha/2} (K + V_N) + C \ell^{-5\alpha/2} N^{-1} (N + 1)^2. \]

Going back to (5.27), we finally use once more Lemmas 5.1 and 5.7 to deduce
\[ \pm G_3 \leq C \ell^{\alpha/2} (\mathcal{N} + K + 1). \]

Hence, collecting (5.28), (5.29) and (5.30) proves the proposition.

5.4 Analysis of \( G_N^{(3)} \)

From (2.6), we recall that
\[ \tilde{I}_N^{(3)} = \int dx dy N^{5/2} V(N(x - y))\varphi(y)(h^*_x a_y^* a_x + \text{h.c.}). \]

Let us also recall the definition of \( h_N = (N^3(V w_\ell)(N.) * |\varphi|^2)\varphi \) from (5.11).
Proposition 5.9 There exists a constant $C > 0$ such that

$$G_N^{(3)} = \int dxdy N^{5/2} V(N(x - y))\phi(y) (b_x^* a_y a_x + \text{h.c.})$$

$$- \left[ \sqrt{N} b^*( \cosh_{\eta_H} (h_N)) + \sqrt{N} b^* ( \sinh_{\eta_H} (h_N)) + \text{h.c.} \right] + \mathcal{E}_N^{(3)}$$

where the self-adjoint operator $\mathcal{E}_N^{(3)}$ satisfies

$$\pm \mathcal{E}_N^{(3)} \leq C \ell \alpha \ell / (N + V_N + 1) + CN^{-1/2}(N + 1)^{3/2} + C \ell^{-\alpha}.$$

for all $\alpha > 0$, $\ell \in (0; 1)$ and $N$ large enough.

Proof We use the identity

$$e^{-B} a_y^* a_x e^B = a_y^* a_x + \int_0^1 ds e^{-sB}[b(\eta_{H,y}) b_x + b_y^* b^*(\eta_{H,x})] e^{sB}$$

to split $G_N^{(3)}$ into $G_N^{(3)} = J_1 + J_2 + J_3 + \text{h.c.}$, where

$$J_1 := \int dxdy N^{5/2} V(N(x - y))\phi(y) e^{-B} b_x^* e^B a_y^* a_x,$$

$$J_2 := \int dxdy N^{5/2} V(N(x - y))\phi(y) e^{-B} b_y^* e^B \int_0^1 ds e^{-sB} b(\eta_{H,y}) b_x e^{sB},$$

$$J_3 := \int dxdy N^{5/2} V(N(x - y))\phi(y) e^{-B} b_x^* e^B \int_0^1 ds e^{-sB} b_y^* b^*(\eta_{H,x}) e^{sB}.$$ (5.31)

We start with the analysis of $J_1$. Using (5.3), we have that

$$J_1 = \int dxdy N^{5/2} V(N(x - y))\phi(y) b_x^* a_y^* a_x$$

$$+ \int dxdy N^{5/2} V(N(x - y))\phi(y) [b^*(r_{\eta_{H,x}}) + b(p_{\eta_{H,x}})] a_y^* a_x$$

$$+ \int dxdy N^{5/2} V(N(x - y))\phi(y) d_{\eta_{H,x}}^* a_y^* a_x$$

$$+ \int dxdy N^{5/2} V(N(x - y))\phi(y) b(\eta_{H,x}) a_y^* a_x$$

$$=: \int dxdy N^{5/2} V(N(x - y))\phi(y) b_x^* a_y^* a_x + J_{11} + J_{12} + J_{13}. \quad (5.32)$$

First, it is simple to see that

$$\pm J_{11} \leq C \ell \alpha (N + 1)$$
and, by (5.2) and the bound (5.7) from Lemma 5.1, we also find that
\[
|\langle \xi, J_{12} \xi \rangle| \leq \int dx dy N^{5/2} V(N(x - y)) |\varphi(y)||a_y \tilde{a}_{\eta_H,x} \xi ||a_x \xi |
\]
\[
+ \int dx dy N^{5/2} V(N(x - y)) |\varphi(y)||a_y (N^{3/4}/N) a^*(\eta_H,x) \xi ||N^{1/4} a_x \xi |
\]
\[
\leq C \ell\alpha^2 \|N + N_V + 1\|^{1/2} \xi || + CN^{-1/2} (N + 1)^{3/4} \xi ||.
\]
Note that we used $|\eta_H(x; y)| \leq CN$ for all $N$ large enough, by (3.14).

Going back to (5.32) and recalling the definition (5.11), we finally see that

\[
J_{13} = \int dx dy N^{5/2} V(N(x - y))\varphi(y)\left[ \eta_H(y; x)b_x + a^*_y a_x b(\eta_H,x) \right]
\]
\[
= -\sqrt{N}b(h_N) - \int dx dy N^{5/2} V(N(x - y))(G * \tilde{\chi}_{H^c})(x - y)\varphi^2(y)\varphi(x)b_x
\]
\[
+ \int dx dy N^{5/2} V(N(x - y))\varphi(y)a^*_y a_x b(\eta_H,x),
\]
where $\tilde{\chi}_{H^c}$ denotes the inverse Fourier transform of the characteristic function of the set $\{p \in \mathbb{R}^3 : |p| \leq \ell^{-\alpha}\}$. Using that $\|G * \tilde{\chi}_{H^c}\|_{\infty} \leq C \ell^{-\alpha}$, by (3.9), we deduce that

\[
\pm (J_{13} + \sqrt{N}b(h_N)) \leq C \ell^{-\alpha} + C \ell^{\alpha/2}(N + 1)
\]
and hence, if we collect the previous estimates, we have proved that

\[
\pm \left( J_1 - \int dx dy N^{5/2} V(N(x - y))\varphi(y)b^*_x a^*_y a_x + \sqrt{N}b(h_N) \right)
\]
\[
\leq C \ell\alpha^2 (N + N_V + 1) + C \ell^{-\alpha} N^{-1/2}(N + 1)^{3/2} + C \ell^{-\alpha}.
\]

Next, we bound $J_2$, defined in (5.31). We apply as usual the identity (5.3), Lemma 5.1 and Cauchy–Schwarz to estimate

\[
|\langle \xi, J_{22} \xi \rangle| \leq \int dx dy N^{5/2} V(N(x - y)) |\varphi(y)||\eta_{H,y}|
\]
\[
\times \int_0^1 ds \|b(\cosh \eta_{H,x}) \xi + b^*(\sinh \eta_{H,x}) \xi \||N + 1||^{1/2} b_x e^{sB} \xi ||
\]
\[
+ \int dx dy N^{5/2} V(N(x - y)) |\varphi(y)||\eta_{H,y}|
\]
\[
\times \int_0^1 ds \|d_{\eta_{H,x}} \xi \||N + 1||^{1/2} b_x e^{sB} \xi ||
\]
\[
\leq C \ell\alpha^2 ||N + 1||^{1/2} \xi ||^2
\]
\[
+ C \ell^\alpha \int dx dy N^{5/2} V(N(x - y))
\]
To control the last term $J_{33}$, on the other hand, we first rewrite it as

$$J_{33} = \int dxdy N^{5/2} V(N(x - y))\varphi(y)e^{-B}b_x^*e^{-B}b_y^*(\eta_{H,x})e^{sB}.$$ 

Finally, let us analyze the contribution $J_3$, defined in (5.31). We split this contribution into $J_3 = J_{31} + J_{32} + J_{33}$, where

$$J_{31} = \int dxdy N^{5/2} V(N(x - y))\varphi(y)e^{-B}b_x^*e^{-B}b_y^* \times \int_0^1 ds \left( e^{-sB}b_x^*e^{-B} - b_x^*\right)e^{-sB}b^*(\eta_{H,x})e^{sB},$$

$$J_{32} = \int dxdy N^{5/2} V(N(x - y))\varphi(y)\left( e^{-B}b_x^*e^{-B} - b(\eta_{H,x})\right)b_y^* \times \int_0^1 ds e^{-sB}b^*(\eta_{H,x})e^{sB},$$

$$J_{33} = \int dxdy N^{5/2} V(N(x - y))\varphi(y)b(\eta_{H,x})b_y^* \int_0^1 ds e^{-sB}b^*(\eta_{H,x})e^{sB}.$$

To control the error terms $J_{31}$ and $J_{32}$, we proceed as before to bound

$$|\langle \xi, J_{31}\xi \rangle| \leq \int dxdy N^{5/2} V(N(x - y))|\varphi(y)|\|b_xe^{B}\xi\| \times \int_0^1 ds \left\| \left[ b^*(\tau_{\eta_{H,x},y}) + b(\sinh_{\eta_{H,x},y})\right]e^{-sB}b^*(\eta_{H,x})e^{sB}\xi \right\| + C\ell^{a/2} \int dxdy N^{5/2} V(N(x - y))|\varphi(x)|\|(N + 1)^{1/2}\xi\| \times \int_0^1 ds \left\| d_{\eta_{H,x}}\left[ b(\cosh_{\eta_{H,x}}) + b^*(\sinh_{\eta_{H,x}}) + d_{\eta_{H,x}}\right]\xi \right\| \leq C\ell^{a/2} (\xi, (N + \nu_N + 1)\xi)$$

as well as

$$|\langle \xi, J_{32}\xi \rangle| \leq \int dxdy N^{5/2} V(N(x - y))|\varphi(y)|\|b_xe^{B}\xi\| \leq C\ell^{a/2} \int dxdy N^{5/2} V(N(x - y))|\varphi(x)|\|(N + 1)^{1/2}\xi\| \times \|b_y\left[ b(\cosh_{\eta_{H,x}}) + b^*(\cosh_{\eta_{H,x}}) + d_{\eta_{H,x}} - \frac{N}{\nu_N}b^*(\eta_{H,x})\right]\xi \right\| \leq C\ell^{a/2} (\xi, (N + \nu_N + 1)\xi).$$

Here we used in the in the last step (5.7) to bound the term with $\overline{d}$ and further used $b_xb^*(\eta_{H,x}) = a^*(\eta_{H,x})a_x \left( 1 - \frac{N}{\nu_N} + \eta_H(x, y) \left( 1 - \frac{N}{\nu_N} \right) \right)$ and $\|\eta_H\|_\infty \leq CN$. To control the last term $J_{33}$, on the other hand, we first rewrite it as
\[
J_{33} = -\sqrt{N} \int dx dy \, N^3(Vw_L)(N(x-y))|\varphi(y)|^2\varphi(x) \\
\times \int_0^1 ds \left[ b^*(\cosh_{\eta \|H} (\eta_{H,x})) + b(\sinh_{\eta \|H} (\eta_{H,x})) \right] \\
- \int dx dy \, N^{5/2} V(N(x-y))(G \ast \tilde{\chi}_{H^c})(x-y)|\varphi(y)|^2\varphi(x) \\
\times \int_0^1 ds \left[ b^*(\cosh_{\eta \|H} (\eta_{H,x})) + b(\sinh_{\eta \|H} (\eta_{H,x})) \right] \\
+ \int dx dy \, N^{5/2} V(N(x-y))\varphi(y)\eta_{H^c}(x; y) \int_0^1 ds d_s^* (\eta_{H,x}) \\
+ \int dx dy \, N^{5/2} V(N(x-y))\varphi(y)a_s^*a(\eta_{H,x})(1-N/N) \int_0^1 ds e^{-sB} b^*(\eta_{H,x})e^{sB},
\]

where \( \tilde{\chi}_{H^c} \) denotes the inverse Fourier transform of the characteristic function of the set \( \{ p \in \mathbb{R}^3 : |p| < \ell^{-\alpha} \} \). With very similar arguments as before, we find that

\[
\pm \left( J_{33} + \sqrt{N} \int dx dy \, N^3(Vw_L)(N(x-y))|\varphi(y)|^2\varphi(x) \\
\times \int_0^1 ds \left[ b^*(\cosh_{\eta \|H} (\eta_{H,x})) + b(\sinh_{\eta \|H} (\eta_{H,x})) \right] \right) \\
\leq C \ell^{\alpha/2}(N+1) + C \ell^{-\alpha}.
\]

Finally, if we recall the definition (5.11), we observe that

\[
\int dx dy \, N^3(Vw_L)(N(x-y))|\varphi(y)|^2\varphi(x) \\
\times \int_0^1 ds \left[ b^*(\cosh_{\eta \|H} (\eta_{H,x})) + b(\sinh_{\eta \|H} (\eta_{H,x})) \right] \\
= \int dx dy \, N^3(Vw_L)(N(x-y))|\varphi(y)|^2\varphi(x) \left[ b^* (\sinh_{\eta_{H,x}}) + b(r_{\eta_{H,x}}) \right] \\
= \left[ b(\cosh_{\eta_H}(h_N)) + b^*(\sinh_{\eta_H}(h_N)) \right] - b(h_N).
\]

Hence, the previous bounds together with (5.33) and (5.34) prove the proposition. \( \square \)

### 5.5 Analysis of \( \mathcal{G}_N^{(4)} \)

From (2.6), we recall that

\[
\widetilde{\mathcal{L}}_N^{(4)} = \frac{1}{2} \int dx dy \, N^2 V(N(x-y))a_x^*a_y^*a_ya_x.
\]
For the analysis of $G_N^{(4)} = e^{-B\mathcal{L}_N^{(4)} e^B}$, we will use the following Lemma which is a straightforward consequence of Lemmas 3.2, 5.1 and the decomposition (5.3); we omit its proof.

**Lemma 5.10** Assume \((1.8)\) and let $\ell \in (0; 1)$ be sufficiently small. Then, there exists a constant $C > 0$ such that

$$
\| (N + 1)^{n/2} e^{-sB} b_x b_y e^{sB} \xi \|
\leq C \ell^\alpha |\varphi(x)||\varphi(y)| (N + 1)^{(n+2)/2} \xi \|
+ C \ell^\alpha |\varphi(x)||a_x(N + 1)^{(n+1)/2} \xi \|
+ ||a_x a_y(N + 1)^{n/2} \xi \|
$$

for all $\xi \in \mathcal{F}^{<N}$ and all $s \in [0; 1]$.

**Proposition 5.11** There exists a constant $C > 0$ such that

$$
G_N^{(4)} = V_N - \frac{1}{2} \int dx dy N^3 (V w_\ell)(N(x - y)) \varphi(x) \varphi(y)(b_x b_y + b_x^* b_y^*)
$$

$$
+ \frac{N}{2} \int dx dy N^3 V(N(x - y)) w_\ell^2 (N(x - y)) |\varphi(x)|^2 |\varphi(y)|^2
$$

$$
\times (1 - N/N)(1 - \Delta N - 1/N) + \mathcal{E}_{N, \ell}^{(4)},
$$

where the self-adjoint operator $\mathcal{E}_{N, \ell}^{(4)}$ satisfies

$$
\pm \mathcal{E}_{N, \ell}^{(4)} \leq C \ell^\alpha(N + V_N + 1) + C \ell^{-\alpha}
$$

for all $\alpha > 0$ and $\ell \in (0; 1)$.

**Proof** Using the identity

$$
[a_x^* a_y^* a_y a_x, b_u b_v] = -\left(\delta(x - u)a_y^* a_u + \delta(x - v)a_y a_u + \delta(x - u)\delta(y - v)
$$

$$
+ \delta(x - v)\delta(u - y) + \delta(y - u) a_y^* a_v + \delta(y - v) a_y a_u \right)b_x b_y,
$$

we have that

$$
G_N^{(4)} = e^{-B\mathcal{L}_N^{(4)} e^B}
$$

$$
= V_N + \frac{1}{2} \int_0^1 ds \int dx dy N^2 V(N(x - y)) e^{-sB[a_x^* a_y^* a_y a_x, B]} e^{sB}
$$

$$
= V_N + \frac{1}{2} \int_0^1 ds \int dx dy N^2 V(N(x - y)) \eta_H(x; y) \left(e^{-sB} b_x b_y e^{sB} + \text{h.c.} \right)
$$

$$
+ \int_0^1 ds \int dx dy N^2 V(N(x - y)) \left(e^{-sB} a_y^* a(\eta_{H,x})b_x b_y e^{sB} + \text{h.c.} \right).
$$
For the conjugation of the quartic term, we use furthermore that
\[
e^{-sB}a^*_yu^*B = a^*_yu + \int_0^s d\tau e^{-\tau B}[a^*_yu, B]e^{\tau B}
\]
so that
\[
\begin{align*}
G_N^{(4)} &= V_N + \frac{1}{2} \int_0^1 ds \int dx dy N^2 V(N(x - y))\eta_H(x; y) \left( e^{-sB}b_xb_y e^{sB} + \text{h.c.} \right) \\
&\quad + \int_0^1 ds \int dx dy N^2 V(N(x - y)) \left( a^*_ya(\eta_H,x) e^{-sB}b_xb_y e^{sB} + \text{h.c.} \right) \\
&\quad + \int_0^1 ds \int_0^s d\tau \int dx dy N^2 V(N(x - y)) \times \left( e^{-\tau B}b(\eta_{H,y})b(\eta_{H,x}) e^{\tau B} e^{-sB}b_xb_y e^{sB} + \text{h.c.} \right) \\
&\quad + \int_0^1 ds \int_0^s d\tau \int dx dy N^2 V(N(x - y)) \times \left( e^{-\tau B}b^*b^*(\eta_{H,x}) e^{\tau B} e^{-sB}b_xb_y e^{sB} + \text{h.c.} \right) \\
&=: V_N + W_1 + W_2 + W_3 + W_4.
\end{align*}
\]
Combining as usual the bounds from Lemma 3.2 together with Lemma 5.10 and Cauchy–Schwarz, a tedious, but simple analysis as in the proof of [5, Proposition 7.6], shows that
\[
\pm W_2 \leq C\ell^{\alpha/2}(\mathcal{N} + V_N + 1), \quad \pm W_3 \leq C\ell^{\alpha}(\mathcal{N} + V_N + 1), \\
\pm W_4 \leq C\ell^{\alpha}(\mathcal{N} + V_N + 1).
\]
We omit the details and focus on the only relevant term \( W_1 \) which can be written as
\[
W_1 = \frac{1}{2} \int_0^1 ds \int dx dy N^2 V(N(x - y))\eta_H(x; y) \left( b(\cosh s\eta_{H,x}) + b^*(\sinh s\eta_{H,x}) + d s\eta_{H,x} \right) \\
\times \left( b(\cosh s\eta_{H,y}) + b^*(\sinh s\eta_{H,y}) + d s\eta_{H,y} \right) + \text{h.c.}
\]
\[
\begin{align*}
&= \frac{1}{2} \int_0^1 ds \int dx dy N^2 V(N(x - y))\eta_H(x; y) \left( b(\cosh s\eta_{H,x})b(\cosh s\eta_{H,y}) + \text{h.c.} \right) \\
&\quad + \frac{1}{2} \int_0^1 ds \int dx dy N^2 V(N(x - y))\eta_H(x; y) \left( b(\cosh s\eta_{H,x})b^*(\sinh s\eta_{H,y}) + \text{h.c.} \right) \\
&\quad + \frac{1}{2} \int_0^1 ds \int dx dy N^2 V(N(x - y))\eta_H(x; y) \left( b(\cosh s\eta_{H,x})d s\eta_{H,y} + \text{h.c.} \right) + \mathcal{E}_1^{(4)} \\
&=: W_{11} + W_{12} + W_{13} + \mathcal{E}_1^{(4)}.
\end{align*}
\]
where

\[
\mathcal{E}_1^{(4)} = \frac{1}{2} \int_0^1 ds \int dx dy \, N^2 V(N(x - y)) \eta_H(x; y) (b^*(\sinh_{s\eta_H,x}) + d_{s\eta_H,x}) \\
\times (b(\cosh_{s\eta_H,y}) + b^*(\sinh_{s\eta_H,y}) + d_{s\eta_H,y}) + \text{h.c.}
\]

Using (5.6), (5.7) and \(N^{-1}|\eta_H(x; y)| \leq C|\varphi(x)||\varphi(y)|\) by (3.14), we get

\[
|\langle \xi, \mathcal{E}_1^{(4)} \xi \rangle| \leq C \xi^{\alpha/2} \int_0^1 ds \int dx dy \, N^2 V(N(x - y)) |\eta_H(x; y)| ||(N + 1)^{\frac{1}{2}} \xi|| \\
\times ||(N + 1)^{-1/2} b(\cosh_{s\eta_H,y}) + b^*(\sinh_{s\eta_H,y}) + d_{s\eta_H,y}) \xi|| \\
+ N^{-1} ||(N + 1)^{1/2} a_x (b(\cosh_{s\eta_H,y}) + b^*(\sinh_{s\eta_H,y}) + d_{s\eta_H,y}) \xi|| \\
\leq C \xi^{\alpha/2} \langle \xi, (N + \mathcal{V}_N + 1) \xi \rangle.
\]

Next, let us analyze the contributions \(W_{11}, W_{12}\) and \(W_{13}\). We write

\[
W_{11} = \frac{1}{2} \int dx dy \, N^2 V(N(x - y)) \eta_H(x; y) (b_x b_y + b_x^* b_y^*) + \mathcal{E}_{11}^{(4)}
\]

for an error \(\mathcal{E}_{11}^{(4)}\) that satisfies

\[
|\langle \xi, \mathcal{E}_{11}^{(4)} \xi \rangle| \leq C \int_0^1 ds \int dx dy \, N^2 V(N(x - y)) |\eta_H(x; y)| ||(N + 1)^{\frac{1}{2}} \xi|| \\
\times ||(N + 1)^{-1/2} a_x b(p_{s\eta_H,y}) + b(p_{s\eta_H,x}) b_y + b(p_{s\eta_H,x}) b(p_{s\eta_H,y}) \xi|| \\
\leq C \xi^{\alpha/2} \langle \xi, (N + 1) \xi \rangle.
\]

Similarly, we have that

\[
W_{12} = \frac{1}{2} \int dx dy \, N^2 V(N(x - y)) \eta_H(x; y)^2 (1 - N/N) + \mathcal{E}_{12}^{(4)},
\]

where

\[
\mathcal{E}_{12}^{(4)} = \int_0^1 ds \int dx dy \, N^2 V(N(x - y)) \eta_H(x; y)(1 - N/N) \\
\times [a^*(\sinh_{s\eta_H,y}) a(\cosh_{s\eta_H,x}) + p_{s\eta_H}(x; y) + (p_{s\eta_H,x} \sinh_{s\eta_H,y}) + \text{h.c.}]
\]

and thus \(\pm \mathcal{E}_{12}^{(4)} \leq C \xi^{\alpha/2}(N + 1)\).

Finally, we have that

\[
W_{13} = -\frac{1}{2} \int dx dy \, N^2 V(N(x - y)) \eta_H(x; y)^2 \left(1 - \frac{N}{N}\right) \frac{N + 1}{N} + \mathcal{E}_{13}^{(4)}
\]

(5.35)
where, by the bound (5.7), it is simple to see that

\[ \langle \xi, \mathcal{E}_{13}^{(4)} \xi \rangle \leq C \int_0^1 ds \int dx dy N^2 V(N(x - y))|\eta_H(x; y)| \cdot \| (N + 1)^{1/2} \| \times \left[ \|(N + 1)^{-1/2} a^* (s \eta_H, y) a_x \xi \| + \ell^{\alpha/2} |\varphi(x)| \| d \eta_H, y \xi \| \right] \\
\leq C \ell^{\alpha/2} (\xi, (N + V_N + 1)\xi). \]

In summary, the analysis from above proves that

\[ \mathcal{G}_N^{(4)} = V_N + \frac{1}{2} \int dx dy N^2 V(N(x - y))\eta_H(x; y) (b_x b_y + b_x^* b_y^*) \]
\[ + \frac{1}{2} \int dx dy N^2 V(N(x - y))\eta_H(x; y)^2 (1 - N/N)(1 - N/N - 1/N) \]
\[ + \tilde{\mathcal{E}}_{N, \ell}^{(4)}, \]

for an error \( \tilde{\mathcal{E}}_{N, \ell}^{(4)} \) satisfies

\[ \pm \tilde{\mathcal{E}}_{N, \ell}^{(4)} \leq C \ell^{\alpha/2} (N + V_N + 1). \]

Replacing finally \( \eta_H(x; y) \) by \( G(x - y)\varphi(x)\varphi(y) = -N w_\ell (N(x - y))\varphi(x)\varphi(y) \) in the first two contributions on the right hand side of the last equation for \( \mathcal{G}_N^{(4)} \), we conclude the proposition, using that \( \| G \ast \tilde{\chi}_H \|_\infty \leq C \ell^{-\alpha} \) and \( N^{-1} |\eta_H(x; y)| \leq C \), by (3.14).

5.6 Proof of Proposition 3.4

Collecting the results from the previous subsections, we are now ready to prove Proposition 3.4. Since the proof is similar to the proof of [6, Theorem 4.4] and [5, Prop. 4.2], we explain the main steps only.

Proof of Proposition 3.4 Let us collect the results of Propositions 5.2, 5.3, 5.4, 5.5, 5.8, 5.9 and 5.11, noting that there is a cancellation between the linear main contributions from \( \mathcal{G}_N^{(1)} \) in Proposition 5.3 with those of \( \mathcal{G}_N^{(3)} \) in Proposition 5.9. We find that

\[ \mathcal{G}_N = \langle \varphi, [ -\Delta + V_{\text{ext}} + \frac{1}{2} (N^3 V(N \cdot) \ast |\varphi|^2)] \varphi \rangle (N - N') - \frac{1}{2} \langle \varphi, (N^3 V(N \cdot) \ast |\varphi|^2) \varphi \rangle (N + 1)(1 - N'/N) \]
\[ + \int dx dy N^3 V(N(x - y))\varphi(x)\varphi(y)\eta_H(x; y)(1 - N'/N)(1 - N'/N - 1/N) \]
\[ + \int dx dy (-\Delta_x \eta_H(x; y)) \eta_H(x; y)(1 - N'/N)(1 - N'/N - 1/N) \]
\[ \heartsuit \text{ Springer} \]
\[ + \frac{N}{2} \int dxdy N^3 V(N(x - y))w_\delta^2(N(x - y))|\varphi(x)|^2 |\varphi(y)|^2 \]
\[ \times (1 - \mathcal{N}/N)(1 - \mathcal{N}/N - 1/N) \]
\[ + \int dx (N^3 V(N.)) |\varphi|^2)(x)b_x^* b_x + \int dxdy N^3 V(N(x - y))\varphi(x)\varphi(y)b_y^* b_y \]
\[ + \frac{1}{2} \int dxdy(N^3(Vf_\ell)(N(x - y))\varphi(x)\varphi(y) - \Delta_x \eta_H(x; y)) (b_y^* b_y + b_x b_x) \]
\[ + \int dxdy N^5/2 V(N(x - y))\varphi(y)(b_x^* a_x^* a_x + \text{h.c.}) \]
\[ + \mathcal{K} + \mathcal{V}_{\text{ext}} + \mathcal{V}_N + \mathcal{E}_{\mathcal{G}N}^{(1)} \]
\[ (5.36) \]

where the error \( \mathcal{E}_{\mathcal{G}N}^{(1)} \) satisfies the estimate
\[ \pm \mathcal{E}_{\mathcal{G}N}^{(1)} \leq C \ell^{(\alpha - 3)/2}(\mathcal{N} + \mathcal{K} + \mathcal{V}_{\text{ext}} + 1) + C \ell^{-5\alpha/2} N^{-1}(\mathcal{N} + 1)^2 + C \ell^{-\alpha} . \]

To prove Proposition 3.4, we need to simplify \( \mathcal{G}_N - \mathcal{E}_{\mathcal{G}N}^{(1)} \) further. We start with the terms on the first six lines of (5.36). Recalling that
\[ - \Delta_x \eta_H(x; y) = - \Delta G(x - y)\varphi(x)\varphi(y) - 2\nabla G(x - y)\nabla \varphi(x)\varphi(y) \]
\[ - G(x - y)\Delta \varphi(x)\varphi(y) + \Delta_x [(G * \tilde{\varphi}_{H^\ell})(x - y)\varphi(x)\varphi(y)] \]

an application of the scattering equation (3.2) together with the bounds (3.4), (3.6) from Lemma 3.1 as well as the pointwise bounds \( \|G * \tilde{\varphi}_{H^\ell}\|_\infty \leq C \ell^{-\alpha} \) and
\[ \left| \nabla_x [(G * \tilde{\varphi}_{H^\ell})(x - y)\varphi(x)\varphi(y)] \right| \leq C \ell^{-2\alpha} |\varphi(y)| [ |\varphi(x)| + |\nabla \varphi(x)| + |\Delta \varphi(x)| ] , \]
\[ \left| \Delta_x [(G * \tilde{\varphi}_{H^\ell})(x - y)\varphi(x)\varphi(y)] \right| \leq C \ell^{-3\alpha} |\varphi(y)| [ |\varphi(x)| + |\nabla \varphi(x)| + |\Delta \varphi(x)| ] \]

shows that
\[ \langle \varphi, [- \Delta + V_{\text{ext}} + \frac{1}{2}(N^3 V(N.)) |\varphi|^2] \rangle (N - \mathcal{N}) \]
\[ - \frac{1}{2} \langle \varphi, (N^3 V(N.)) |\varphi|^2 \rangle (N + 1)(1 - \mathcal{N}/N) \]
\[ + \int dxdy N^3 V(N(x - y))\varphi(x)\varphi(y)\eta_H(x; y)(1 - \mathcal{N}/N)(1 - \mathcal{N}/N - 1/N) \]
\[ + \int dxdy (\Delta_x \eta_H(x; y))\eta_H(x; y)(1 - \mathcal{N}/N)(1 - \mathcal{N}/N - 1/N) \]
\[ + \frac{N}{2} \int dxdy N^3 V(N(x - y))w_\delta^2(N(x - y))|\varphi(x)|^2 |\varphi(y)|^2 \]
\[ \times (1 - \mathcal{N}/N)(1 - \mathcal{N}/N - 1/N) \]
\[ = \langle \varphi, [- \Delta + V_{\text{ext}} + \frac{1}{2}(N^3 (Vf_\ell)(N.)) |\varphi|^2] \rangle (N - \mathcal{N}) \]
\[ - \frac{1}{2} \langle \varphi, (N^3 (Vf_\ell)(N.)) |\varphi|^2 \rangle \mathcal{N}(1 - \mathcal{N}/N) + \mathcal{E}_{\mathcal{G}N}^{(2)} , \]
for an error that satisfies $\pm \overline{\epsilon}_{G_N}^{(2)} \leq C (\ell (\alpha - 3)/2 + \ell^{-3})$. Using the Gross–Pitaevskii equation (A.1), the bound (3.5) from Lemma 3.1, a simple application of the mean value theorem shows furthermore that

$$\left[ \varphi, \left[ -\Delta + V_{\text{ext}} + \frac{1}{2} (N^3 (V f_\ell)(N \cdot) * |\varphi|^2) \right] \varphi \right] (N - N')$$

$$- \frac{1}{2} \left[ \varphi, (N^3 (V f_\ell)(N \cdot) * |\varphi|^2) \varphi \right] N (1 - N/N')$$

$$= N \mathcal{E}_{GP} (\varphi) - \epsilon_{GP} N' + 4 \pi a_0 \| \varphi \|^4 N^2 / N + \overline{\epsilon}_{G_N}^{(3)} ,$$

up to an error that satisfies $\pm \overline{\epsilon}_{G_N}^{(3)} \leq C \ell^{-1}$. This shows that

$$\mathcal{G}_N = N \mathcal{E}_{GP} (\varphi) - \epsilon_{GP} N' + 4 \pi a_0 \| \varphi \|^4 N^2 / N$$

$$+ \int dx (N^3 V(N \cdot) * |\varphi|^2)(x) b_x^* b_x + \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) b_x^* b_y$$

$$+ \frac{1}{2} \int dxdy (N^3 (V f_\ell)(N \cdot)(N(x - y)) \varphi(x) \varphi(y) - \Delta_x \eta_H(x; y) (b_x^* b_y^* + b_x b_y))$$

$$+ \int dxdy N^{5/2} V(N(x - y)) \varphi(x) \varphi(y) (b_x^* a_y^* a_x + \text{h.c.})$$

$$+ \mathcal{K} + \mathcal{V}_{\text{ext}} + \mathcal{V}_N + \overline{\mathcal{\epsilon}}_{G_N}^{(1)} + \overline{\mathcal{\epsilon}}_{G_N}^{(2)} + \overline{\mathcal{\epsilon}}_{G_N}^{(3)} .$$

Now, let us simplify the quadratic contributions on the right hand side of the last equation. First of all, another application of the mean value theorem shows that

$$\pm \left( \int dx (N^3 V(N \cdot) * |\varphi|^2)(x) b_x^* b_x - \hat{V}(0) \int |\varphi(x)|^2 b_x^* b_x$$

$$+ \int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) b_x^* b_x - \hat{V}(0) \int |\varphi(x)|^2 b_x^* b_x \right)$$

$$\leq CN^{-1} (N + \mathcal{K} + 1) .$$

Here we used for the second line that $|\hat{V}(p/N) - \hat{V}(0)| \leq C |p| / N$ and that

$$\int dxdy N^3 V(N(x - y)) \varphi(x) \varphi(y) b_x^* b_x$$

$$= \int dx \hat{V}(0) \varphi(x)^2 b_x^* b_x + \int dp \left( \hat{V}(p/N) - \hat{V}(0) \right) \hat{a}^* (\hat{\varphi}_p) \left( 1 - \frac{N}{N'} \right) \hat{a}(\hat{\varphi}_p) ,$$

where $\hat{a}(\hat{\varphi}_p) = \int dy e^{2\pi i px} \varphi(x) a_x$. The previous bound follows from

$$\int |p|^2 \| \hat{a}(\hat{\varphi}_p) \|_2^2 \leq 2 \int |\nabla \varphi(x)|^2 \| a_x \xi \|_2^2 + 2 \int \varphi(x)^2 \| \nabla a_x \xi \|_2^2$$

$$\leq C \|(N + \mathcal{K})^{1/2} \xi \|_2^2$$

(5.37)
because \( \| \varphi \|_\infty, \| \nabla \varphi \|_\infty \leq C \) by (A.4).

This controls the diagonal terms. For the non-diagonal term, we use once more the scattering equation (3.2), the simple identity
\[
\| \varphi \|_\infty \leq |a_0| \|
\]
because Bose–Einstein Condensation with Optimal Rate... Page 45 of 71

12

as above to deduce that
\[
\pi - 1 \leq \sum_{\mathbf{k} \neq 0} \left| \sum_{\mathbf{q}} \frac{1}{\mathbf{k} + \mathbf{q}} \right|^2.
\]

This controls the diagonal terms. For the non-diagonal term, we use once more
\[
\| \varphi \|_\infty \leq |a_0| \|
\]
and therefore
\[
\pi - 1 \leq \sum_{\mathbf{k} \neq 0} \left| \sum_{\mathbf{q}} \frac{1}{\mathbf{k} + \mathbf{q}} \right|^2.
\]

Finally, switching to Fourier space, we see from Eq. (3.5) that for all \( p \in \mathbb{R}^3 \)
\[
\| (V_f \ell)(p/N) - 8\pi a_0 \|
\]
\[
\leq \int dx (V_f \ell)(x) \cdot \left| e^{2\pi i x p/N} - 1 \right| + \left| \int dx (V_f \ell)(x) - 8\pi a_0 \right| \quad (5.38)
\]
and therefore
\[
\| (V_f \ell)(p/N) - 8\pi a_0 \|
\]
\[
\leq C N^{-1} \| p \| + C N^{-1} \ell^{-1}
\]
and therefore
\[
\pi - 1 \leq \sum_{\mathbf{k} \neq 0} \left| \sum_{\mathbf{q}} \frac{1}{\mathbf{k} + \mathbf{q}} \right|^2.
\]

Hence, in summary, we conclude that
\[
G_N = N \varepsilon_{GP}(\varphi) - \varepsilon_{GP} \mathcal{N} + 4\pi a_0 \| \varphi \|_4^4 N^2 / N
\]
\[
+ \int dx a_s^* \left[ - \Delta_x + V_{ext}(x) \right] a_x + 2 \hat{V}(0) \int dx |\varphi(x)|^2 b_x^* b_x
\]
\[
+ 4\pi a_0 \int dxdy \hat{\chi}_{\ell}(x - y) \varphi(x) \varphi(y) (b_x b_y + b_x^* b_y^*)
\]
\[
+ \int dxdy N^{5/2} V(N(x - y)) \varphi(y) (b_x^* a_y a_x + h.c.)
\]
\[
+ \frac{1}{2} \int dxdy N^2 V(N(x - y)) a_s^* a_s^* a_x a_y + \tilde{\mathcal{G}}^{(1)}_N + \tilde{\mathcal{G}}^{(2)}_N + \tilde{\mathcal{G}}^{(3)}_N + \tilde{\mathcal{G}}^{(4)}_N,
\]
where the error $\mathcal{E}_N : = \tilde{\mathcal{E}}_N^{(1)} + \tilde{\mathcal{E}}_N^{(2)} + \tilde{\mathcal{E}}_N^{(3)} + \tilde{\mathcal{E}}_N^{(4)}$ satisfies

$$
\pm \mathcal{E}_N \leq C \left( \ell^{(\alpha-3)/2} + \ell^{(3\alpha-4)/2} \right) (N + K + V_N + 1) + C \ell^{-5\alpha/2} N^{-1} (N + 1)^2 \\
+ C \left( \ell^{(\alpha-3)/2} + \ell^{-4\alpha} + \ell^{-3\alpha-1} + \ell^{-1} \right).
$$

Choosing $\alpha > 3$, this concludes the proof of Proposition 3.4. \hfill \Box

## 6 Analysis of $\mathcal{J}_N$

The goal of this section is to show Proposition 3.7 for the excitation Hamiltonian $\mathcal{J}_N = e^{-A} \mathcal{G}_{N}^{\text{eff}} e^{A}$, where $\mathcal{G}_{N}^{\text{eff}}$ has been introduced in (3.17) and can be decomposed as

$$
\mathcal{G}_{N}^{\text{eff}} = \mathcal{D}_N + \mathcal{Q}_N + \mathcal{C}_N + \mathcal{H}_N 
$$

with $\mathcal{H}_N = K + V_{\text{ext}} + V_N$ and where

$$
\mathcal{D}_N = N E_{\text{GP}}(\varphi) - E_{\text{GP}} N + 4\pi a_0 \| \varphi \|_4^4 N^2 / N,
$$

$$
\mathcal{Q}_N = 2 \tilde{\mathcal{V}}(0) \int dx |\varphi(x)|^2 b^*_x b_x + 4\pi a_0 \int dx dy \tilde{\chi}_{H^*} (x - y) \varphi(x) \varphi(y) (b^*_x b^*_y + \text{h.c.}),
$$

$$
\mathcal{C}_N = \frac{1}{\sqrt{N}} \int dx dy N^3 V(N(x - y)) \varphi(x) [b^*_x a^*_x a_x + \text{h.c.}].
$$

(6.2)

In the next subsections, we will study the action of the unitary operator $e^{A}$ on these terms, where we recall from (3.23) that

$$
A = \frac{1}{\sqrt{N}} \int dx dy dz v_H(x; y) \tilde{g}_L(x - z) (b^*_x a^*_y a_z - \text{h.c.})
$$

with $v_H$ and $g_L$ as defined in (3.18) and (3.21), respectively, with parameters $\alpha > \beta > 0$.

First, however, we need to establish some a priori bounds controlling the growth of the expectation of kinetic and potential energies, generated by $A$.

### 6.1 Preliminary Estimates

First of all, with the next lemma we control the growth of the expectation of the external potential.

**Lemma 6.1** Assume (1.8) and let $\alpha > \beta > 0$. Then there exists $C > 0$ such that for all $\xi \in \mathcal{F}_+^{\leq N}$, $t \in [0; 1]$, $\ell \in (0; 1)$ and $N \in \mathbb{N}$ large enough,

$$
\left| \langle \xi, e^{-tA} V_{\text{ext}} e^{tA} \xi \rangle - \langle \xi, V_{\text{ext}} \xi \rangle \right| \leq C \ell^{\alpha/2} |\xi, [V_{\text{ext}} + N + C] \xi|.
$$

(6.3)
Proof The l.h.s. in (6.3) is invariant under an energy shift of $V_{\text{ext}}$ so we can assume without loss of generality that $V_{\text{ext}} \geq 0$. In other words, (6.3) follows if we prove it for $\tilde{V}_{\text{ext}} = \int_{\mathbb{R}^3} V_{\text{ext}}(x)a_x^*a_x$ where $V_{\text{ext}} = V_{\text{ext}}(x) + C \geq 0$ for some sufficiently large $C > 0$. Here, we use that $V_{\text{ext}} \geq -C$ for some $C > 0$, under our assumptions (1.8). We compute

$$[\tilde{V}_{\text{ext}}, A] = \frac{1}{\sqrt{N}} \int dx dy dz \, [\tilde{V}_{\text{ext}}(x) + \tilde{V}_{\text{ext}}(y) - \tilde{V}_{\text{ext}}(z)]$$

$$v_H(x; y)\tilde{g}_L(x - z)b_x^*a_y^*a_z + h.c.$$ 

Using (3.22) and the fact that, by (A.3), $\|\tilde{V}_{\text{ext}}\|_{\infty} \leq C$ we get

$$\left| \frac{1}{\sqrt{N}} \int dx dy dz \, \tilde{V}_{\text{ext}}(y) v_H(x; y) \tilde{g}_L(x - z) \langle \xi, b_x^*a_y^*a_z\xi \rangle \right|$$

$$\leq \frac{C}{\sqrt{N}} \int dx dy dz \, |(G * \tilde{\chi}_H)(x - y)| \tilde{g}_L(x - z) \|a_x a_y \xi \| \|a_z \xi \|$$

$$\leq C \ell^\frac{\tilde{\beta}}{2} \|(N + 1)\frac{1}{2} \xi \|^2.$$ 

Furthermore, recalling the assumption $V_{\text{ext}}(x + y) \leq C(V_{\text{ext}}(x) + C)(V_{\text{ext}}(y) + C)$ from (1.8), which trivially implies that $\tilde{V}_{\text{ext}}(x + y) \leq C(\tilde{V}_{\text{ext}}(x) + C)(\tilde{V}_{\text{ext}}(y) + C)$ as well, and using (3.19) as well as (3.22), we find that

$$\left| \frac{1}{\sqrt{N}} \int dx dy dz \, \tilde{V}_{\text{ext}}(x) v_H(x; y) \tilde{g}_L(x - z) \langle \xi, b_x^*a_y^*a_z\xi \rangle \right|$$

$$\leq \frac{1}{\sqrt{N}} \left( \int dx dy dz \, \tilde{V}_{\text{ext}}(x) \tilde{g}_L(x - z) \|v_H(x; y)\|^2 \|a_z \xi \|^2 \right)^{\frac{1}{2}}$$

$$\times \left( \int dx dy dz \, \tilde{V}_{\text{ext}}(x) \tilde{g}_L(x - z) \|a_x a_y \xi \|^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{C \ell^\frac{\tilde{\beta}}{2}}{\sqrt{N}} \left( \int dx dz \, \tilde{V}_{\text{ext}}(x + z) \tilde{g}_L(x) \|a_z \xi \|^2 \right)^{\frac{1}{2}} \|\tilde{V}_{\text{ext}}(N + 1)\frac{1}{2} \xi \|$$

$$\leq \frac{C \ell^\frac{\tilde{\beta}}{2}}{\sqrt{N}} \left( \|\tilde{V}_{\text{ext}}\|_{\ell^1} + \|(N + 1)\frac{1}{2} \xi \| \right) \|\tilde{V}_{\text{ext}}(N + 1)\frac{1}{2} \xi \| \leq C \ell^\frac{\tilde{\beta}}{2} \langle \xi, (\tilde{V}_{\text{ext}} + N + 1)\xi \rangle.$$

Notice that we used in the last step the bound $\|V_{\text{ext}} \tilde{g}_L\|_{\ell^1} \leq C$, which follows from the fact that $V_{\text{ext}}$ grows at most exponentially (see Appendix A) and the explicit formula $\tilde{g}_L(x) = (\sqrt{\pi} \ell^{\tilde{\beta}})^3 e^{-(\pi \ell^{-\tilde{\beta}})^2x^2}$. Similarly, we get

$$\left| \frac{1}{\sqrt{N}} \int dx dy dz \, \tilde{V}_{\text{ext}}(z) v_H(x; y) \tilde{g}_L(x - z) \langle \xi, b_x^*a_y^*a_z\xi \rangle \right|$$

$$\leq C \ell^\frac{\tilde{\beta}}{2} \langle \xi, (\tilde{V}_{\text{ext}} + N + 1)\xi \rangle.$$
Thus, we have

$$\pm[\tilde{V}_{\text{ext}}, A] \leq C \ell_\xi^g (\tilde{V}_{\text{ext}} + \mathcal{N} + 1).$$

With (3.24), Eq. (6.3) follows now by Gronwall’s lemma, applied to the function $f(t) = \langle \xi, e^{-tA\tilde{V}_{\text{ext}}e^{tA}\xi} \rangle$ and inserting $\tilde{V}_{\text{ext}} = V_{\text{ext}} + C$. □

Next, we need to control the growth of the kinetic and potential energy. To estimate contributions arising from the kinetic energy, we will often need to switch to momentum space. We will use the formal notation $\hat{a}_p = a(\hat{e}^{2\pi i p \cdot x})$ and $\hat{a}_p^* = a^*(\hat{e}^{2\pi i p \cdot x})$ to indicate creation and annihilation operators in momentum space. For $f \in L^2(\mathbb{R}^3)$ (interpreted as a function of momentum), we set $\hat{a}(f) = \int \hat{f}(p) \hat{a}_p$ and similarly for $\hat{a}^*(f)$. It is useful to keep in mind that

$$\int dy \, e^{2\pi i p \cdot y} \varphi(y) a_y = \hat{a}(\hat{\varphi}_p)$$

where $\hat{\varphi}_p(q) := \hat{\varphi}(p - q)$.

We will often encounter operators as in (6.5), with $\varphi$ the minimizer (or the square of the minimizer) of the Gross–Pitaevskii energy functional. Switching to position space, we can bound

$$\int dp \| \hat{a}(\hat{\varphi}_p) \xi \|^2 = \int dx |\varphi(x)|^2 \| a_x \xi \|^2 \leq \| \varphi \|_\infty^2 \| \mathcal{N}^{1/2} \xi \|^2 \leq C \| \mathcal{N}^{1/2} \xi \|^2$$

and, as already discussed in (5.37),

$$\int dp \, p^2 \| \hat{a}(\hat{\varphi}_p) \xi \|^2 \leq C \| (\mathcal{K} + \mathcal{N})^{1/2} \xi \|^2.$$  (6.7)

In the following, it will moreover be useful to introduce, for any $\theta > 0$, the notation

$$\mathcal{K}_\theta = \int_{|p| \leq \theta} p^2 \hat{a}_p^* \hat{a}_p$$

for the kinetic energy of particles having momentum below $\theta$.

**Lemma 6.2** Let $\alpha > 4$, $0 < \beta < \alpha$. Then we can write

$$[\mathcal{K}, A] = T_2 + S_1 + S_2 + \delta \mathcal{K}$$

where

$$T_2 = -\frac{2}{\sqrt{N}} \int dx dy dz \, \nabla_x H(x, y) \nabla_x \tilde{g}_L(x - z) [b^*_x a^*_y a_z + \text{h.c.}],$$

$$S_1 = -\frac{1}{\sqrt{N}} \int dx dy dz \, \mathcal{N}^2(Vf) (\mathcal{N}(x - y)) \varphi(y) \tilde{g}_L(x - z) [b^*_x a^*_y a_z + \text{h.c.}],$$

$$\delta \mathcal{K} = 0.$$
\[ S_2 = \frac{1}{\sqrt{N}} \int dxdydz \left[ (N^3(Vf_\ell)(N\cdot)) \ast \tilde{\chi}_H \right](x - y) \psi(y) \tilde{g}_L(x - z)[b_x^*a_y^*a_z + h.c.] \]

and where
\[
|\langle \xi, \delta_k \xi \rangle| \leq C \ell^{(\alpha - 4)/2} \langle \xi, N\xi \rangle + C \ell^{\alpha/2} \| K^{1/2} \xi \| \| (N + 1)^{1/2}\xi \|.
\]

Moreover, we find
\[
\pm T_2 \leq C \ell^{\alpha/2} K, \quad \pm S_1 \leq C(N + N^2), \quad \pm S_2 \leq CK + C \ell^{-\alpha}(N + 1),
\]

(6.10)

and thus
\[
\pm [K, A] \leq C(K + N) + C \ell^{-\alpha}(N + 1).
\]

(6.11)

Moreover, if in addition \( \alpha \leq 2\beta \), then for every \( \varepsilon \in (0; \alpha - \beta) \), there exists a constant \( C > 0 \) such that for all \( \ell > 0 \) sufficiently small and for \( N \) sufficiently large we have
\[
[K, A] = -\frac{1}{\sqrt{N}} \int dxdydz N^3(Vf_\ell)(x)(N(x - y)) \psi(y) \tilde{g}_L(x - z)[b_x^*a_y^*a_z + h.c.]
\]
\[
+ \frac{1}{\sqrt{N}} \int dxdydz 8\pi a_0 \tilde{\chi}_H(x - y) \tilde{g}_L(x - z) \psi(y)[b_x^*a_y^*a_z + h.c.] + \tilde{\delta}_K
\]

(6.12)

where
\[
|\langle \xi, \delta_k \xi \rangle| \leq C \ell^{(\alpha - 4)/2} \| (N + 1)^{1/2}\xi \|^2 + C \ell^{\alpha/2} \| (K + N)+ \| (K_\ell - \beta + N + 1)^{1/2}\xi \|.
\]

(6.13)

**Proof** With the commutation relations (2.1), (2.4) and integration by parts, we obtain
\[
K, A = \frac{1}{\sqrt{N}} \int dxdydz \nu_H(x; y)(x)(N - z)[b_x^*a_y^*a_z + h.c.]
\]
\[
\times [-\Delta_x a_x^*a_y^*a_z - b_x^*\Delta_x a_y^*a_z + b_x^*a_y^*\Delta_x a_z + h.c.]
\]
\[
= -\frac{1}{\sqrt{N}} \int dxdydz [\Delta_x \nu_H(x; y) + \Delta_y \nu_H(x; y)] \tilde{g}_L(x - z)[b_x^*a_y^*a_z + h.c.]
\]
\[
- \frac{2}{\sqrt{N}} \int dxdydz \nabla_x \nu_H(x; y)(x) \nabla_x \tilde{g}_L(xz)[b_x^*a_y^*a_z + h.c.]
\]
\[
= T_1 + T_2.
\]
We used here the identity $-\Delta_x \tilde{g}_L(x-z) + \Delta_z \tilde{g}_L(x-z) = 0$. We rewrite $T_1$ as

$$T_1 = -\frac{2}{\sqrt{N}} \int dxdydz \, (\Delta G * \tilde{\chi}_H)(x-y)\varphi(y)\tilde{g}_L(x-z)[b_x^*a_y^*a_z + h.c.]$$

$$+ \frac{1}{\sqrt{N}} \int dxdydz \, (G * \tilde{\chi}_H)(x-y) \Delta \varphi(y)\tilde{g}_L(x-z)[b_x^*a_y^*a_z + h.c.]$$

$$+ \frac{2}{\sqrt{N}} \int dxdydz \, (G * \tilde{\chi}_H)(x-y) \nabla \varphi(y)\tilde{g}_L(x-z)[b_x^*\nabla_ya_y^*a_z + h.c.]$$

$$=: T_{11} + T_{12} + T_{13}.$$

Using Young’s inequality and (3.22) we obtain

$$|\langle \xi, T_{12}\xi \rangle| \leq \frac{C}{\sqrt{N}} \|G * \tilde{\chi}_H\| \int dxdydz \, \tilde{g}_L(x-z)\|a_x(N+1)^{\frac{1}{2}}\xi\|\|a_z\xi\|$$

$$\leq C\ell^{\frac{q}{2}} \|(N+1)^{\frac{1}{2}}\xi\|^2$$

and

$$|\langle \xi, T_{13}\xi \rangle| \leq \frac{C}{\sqrt{N}} \|G * \tilde{\chi}_H\| \int dxdydz \, \tilde{g}_L(x-z)\|\mathcal{K}^{\frac{1}{2}}a_x\xi\|\|a_z\xi\|$$

$$\leq C\ell^{\frac{q}{2}} \|(N+1)^{\frac{1}{2}}\xi\|\|\mathcal{K}^{\frac{1}{2}}\xi\|.$$

We are left with $T_{11}$. For this term we use the scattering equation (3.2) and get

$$T_{11} = -\frac{1}{\sqrt{N}} \int dxdydz \, N^3(Vf_{\ell})(N(x-y))\varphi(y)\tilde{g}_L(x-z)[b_x^*a_y^*a_z + h.c.]$$

$$+ \frac{1}{\sqrt{N}} \int dxdydz \, [(N^3(Vf_{\ell})(N\cdot)) * \tilde{\chi}_{H^*}](x-y)\varphi(y)\tilde{g}_L(x-z)[b_x^*a_y^*a_z + h.c.]$$

$$+ \frac{2}{\sqrt{N}} \int dxdydz \, [N^3\lambda_{\ell}(f_{\ell}(N\cdot)\chi_{\ell}) * \tilde{\chi}_H](x-y)\varphi(y)\tilde{g}_L(x-z)[b_x^*a_y^*a_z + h.c.]$$

$$=: S_1 + S_2 + S_3.$$ An explicit calculation shows that $|\tilde{\chi}_{\ell}(p)| = \ell^3|\tilde{\chi}_{\ell}(\ell p)| \leq C\ell|p|^{-2}$. With (3.9), we find $\|(f_{\ell}(N\cdot)\chi_{\ell}) * \tilde{\chi}_H\| \leq \ell^{1+a/2}$ (for $N$ large enough). From (3.4), (3.22), we obtain

$$|\langle \xi, S_3\xi \rangle| \leq \frac{C}{\sqrt{N}} N^3\lambda_{\ell}\|(f_{\ell}(N\cdot)\chi_{\ell}) * \tilde{\chi}_H\| \int dxdydz \, \tilde{g}_L(x-z)\|a_x(N+1)^{\frac{1}{2}}\xi\|\|a_z\xi\|$$

$$\leq C\ell^{\frac{q}{2}} \|(N+1)^{\frac{1}{2}}\xi\|^2.$$

This proves (6.9). To show (6.10), we observe that, integrating by parts,
\[ T_2 = -\frac{2}{\sqrt{N}} \int dx dy dz \left( G \ast \tilde{\chi}_H \right)(x - y) \nabla \varphi(y) \tilde{g}_{L}(x - z) [b^*_x a_y \nabla_z a_z + h.c.] \]

\[ - \frac{2}{\sqrt{N}} \int dx dy dz \left( G \ast \tilde{\chi}_H \right)(x - y) \varphi(y) \tilde{g}_{L}(x - z) [b^*_x \nabla_y a^*_x \nabla_z a_z + h.c.] \]

With \( \| G \ast \tilde{\chi}_H \| = \| \hat{G} \chi_H \| \leq \ell^{\alpha/2} \) and (3.22) we get

\[ |\langle \xi, T_2 \xi \rangle| = \frac{C \ell^{\frac{\alpha}{2}}}{\sqrt{N}} \left| \int dx dz |\tilde{g}_{L}(x - z)| ||N^\frac{1}{2} a_x \xi || ||\nabla_z a_z \xi || \right| \]

\[ + \frac{C \ell^{\frac{\alpha}{2}}}{\sqrt{N}} \left| \int dx dz |\tilde{g}_{L}(x - z)| ||K^\frac{1}{2}_x a_x \xi || ||\nabla_z a_z \xi || \right| \]

\[ \leq \frac{C \ell^{\frac{\alpha}{2}}}{\sqrt{N}} \| \tilde{g}_{L} \|_1 ||K^\frac{1}{2}_x (N + 1)^\frac{1}{2} \xi || ||K^\frac{1}{2}_x \xi || \leq C \ell^{\alpha/2} \| K^{1/2} \xi \| ^2. \]

By (3.22), we have

\[ |\langle \xi, S_1 \xi \rangle| \leq \frac{2}{\sqrt{N}} \int dx dy dz N^3 V(N(x - y)) \varphi(y) \tilde{g}_{L}(x - z) ||a_x a_y \xi || ||a_z \xi || \]

\[ \leq C \left( \int dx dy dz N^2 V(N(x - y)) \tilde{g}_{L}(x - z) ||a_x a_y \xi || ^2 \right)^{1/2} \]

\[ \times \left( \int dx dy dz N^3 V(N(x - y)) \tilde{g}_{L}(x - z) ||a_z \xi || ^2 \right)^{1/2} \]

\[ \leq C \| V^\frac{1}{2}_N \xi \| ||(N + 1)^\frac{1}{2} \xi ||. \]

For \( S_2 \) we change to momentum space to get, using (6.7)

\[ |\langle \xi, S_2 \xi \rangle| = \frac{1}{\sqrt{N}} \int dp dq \left| \hat{V}(f_c)(p/N) \chi_H(p) g_{L}(q) \hat{a}(\hat{\phi}_p) \hat{b}_{-p} \xi, \hat{a}_q \xi \right| \]

\[ \leq C \left( \int \left| p^2 \| \hat{a}(\hat{\phi}_p) \hat{b}_{-p} \xi \| ^2 \right| \int \left| p \| \hat{a}_q \xi \| ^2 \right| \right)^{1/2} \]

\[ \leq C \ell^{-\alpha/2} \| (K + N)^{1/2} \xi \| ||(N + 1)^{1/2} \xi ||. \]

This concludes the proof of (6.10) and thus of (6.11).

The arguments to obtain the refined decomposition (6.12) are similar. Here, we prove that \( T_2 \) can be bounded as in (6.13) and that \( S_2 \) corresponds to the second term on the r.h.s. of (6.12), up to small corrections. Switching to momentum space, we get

\[ T_2 = -\frac{2}{\sqrt{N}} \int dp dq \cdot q \hat{G}(p) \chi_H(p) g_{L}(q) [\hat{b}_{p+q}^* \hat{a}(\hat{\phi}_{-p}) \hat{a}_q + h.c.]. \]
We estimate

\[ |\langle \xi, T_2 \xi \rangle| \leq \frac{C}{\sqrt{N}} \int dp dq \ |p||q| \hat{G}(p) \chi_H(p)g_L(q) \| \hat{a}_{p+q} \hat{a}(\hat{\varphi}_p) \xi \| \| \hat{a}_q \xi \| \]

\[ \leq \frac{C}{\sqrt{N}} \left[ \int dp dq \ |p|^2 \| \hat{a}_{p+q} \hat{a}(\hat{\varphi}_p) \xi \|^2 \right]^{1/2} \times \left[ \int dp dq \ |\hat{G}(p)|^2 \chi_H(p)g_L(q)^2 |q|^2 \| \hat{a}_q \xi \|^2 \right]^{1/2} \]

\[ \leq C \ell^2 \| (\mathcal{K} + \mathcal{N}) \frac{1}{2} \xi \| \left[ \| \mathcal{K} \frac{1}{2} \ell - \beta - \epsilon \xi \| + \ell^{3\alpha} \| \mathcal{N} \frac{1}{2} \xi \| \right]. \]

where in the second integral we separated \(|q| < \ell - \beta - \epsilon\) (where we can use \(\mathcal{K} \ell - \beta - \epsilon\)) and \(|q| > \ell - \beta - \epsilon\) (where \(g_L\) is effective). Now we consider \(S_2\), as defined in (6.10). Switching to Fourier space and subtracting the second term on the r.h.s. of (6.12), we can bound

\[ \left| \langle \xi, S_2 \xi \rangle - \frac{8\pi a_0}{\sqrt{N}} \int dxdydz \chi_H(x-y)\hat{g}_L(x-z)\varphi(y) \langle \xi, [b_x^* a_y^* a_z + \text{h.c.}] \xi \rangle \right| \]

\[ \leq \frac{1}{\sqrt{N}} \int dp dq \left| \tilde{V}_{f\ell}(q/N) - 8\pi a_0 |\chi_H(c)(q)g_L(p)| \| \hat{a}_{p-q} \hat{a}(\hat{\varphi}_q) \xi \| \| \hat{a}_p \xi \| \right| \]

\[ \leq \frac{C}{N^{3/2}} \int dp dq (\ell^{-1} + |q|) |\chi_H(c)(q)g_L(p)| \| \hat{a}_{p-q} \hat{a}(\hat{\varphi}_q) \xi \| \| \hat{a}_p \xi \| \]

\[ \leq C \ell^{3\alpha} \| (\mathcal{N} + 1)^{1/2} \xi \|^2 \]

where we used (5.38) and we chose \(N\) large enough to obtain the desired decay in \(\ell\). This concludes the proof of (6.12). \( \square \)

**Lemma 6.3** Let \(\alpha \geq \beta > 0\) and \(\epsilon \in (0, \alpha - \beta)\). Then there exists \(C > 0\) such that for \(\ell \in (0; 1)\) sufficiently small we have

\[ [\mathcal{V}_N, A] = \frac{1}{\sqrt{N}} \int dxdydz v_H(x, y)\hat{g}_L(x-z)N^2V(N(x-y))[b_x^* a_y^* a_z + \text{h.c.}] + \delta_V, \]

(6.15)

where

\[ \pm \delta_V \leq C \ell^{(\alpha - \beta)} [\mathcal{K} \ell - \beta - \epsilon + \mathcal{V}_N + \mathcal{N} + 1]. \]

Estimating the term on the r.h.s. of (6.15), we conclude that

\[ \pm [\mathcal{V}_N, A] \leq C [\mathcal{K} \ell - \beta - \epsilon + \mathcal{V}_N + \mathcal{N} + 1]. \]

(6.16)
Proof} With the commutation relations (2.1), (2.4), we find

\[
[\mathcal{V}_N, A] = -\frac{1}{\sqrt{N}} \int dx dy dz du v_H(x; y) \tilde{g}_L(x-z) N^2 V(N(u-z))[b_ua^*_a a^*_u a_u + h.c.]
\]
\[
+ \frac{1}{\sqrt{N}} \int dx dy dz du v_H(x; y) \tilde{g}_L(x-z) N^2 V(N(u-y))[b^*_u a^*_a a_u a_u + h.c.]
\]
\[
+ \frac{1}{\sqrt{N}} \int dx dy dz du v_H(x; y) \tilde{g}_L(x-z) N^2 V(N(x-u))[b^*_u a^*_a a_u a_u + h.c.]
\]
\[
+ \frac{1}{\sqrt{N}} \int dx dy dz du v_H(x; y) \tilde{g}_L(x-z) N^2 V(N(x-y))[b^*_u a^*_a a_u + h.c.]
\]
\[
= \sum_{j=1}^4 V_j.
\]

Switching partially to momentum space and using (6.6), (6.7), we have that

\[
|\langle \xi, V_1 \xi \rangle| \leq \frac{C}{\sqrt{N}} \int dp dq dz du N^2 V(N(u-z))|\tilde{G}(p)| \chi_H(p) g_L(q)
\times ||a_u \hat{a}_{p+q} \hat{a}_{\hat{\psi}_p} \xi|| ||a_u \xi||
\leq \frac{C \ell^{3\alpha/2}}{\sqrt{N}} \left( \int dp dq \ p^2 ||\hat{a}_{p+q} \hat{a}_{\hat{\psi}_p} \xi||^2 \right)^{1/2} \left( \int dq ||g_L(q)||^2 \right)^{1/2} ||\mathcal{V}_N^{1/2} \xi||
\leq C \ell^{3(\alpha-\beta)/2} \langle \xi, (\mathcal{K} + \mathcal{V}_N + N) \xi \rangle.
\]

We have \( \|v_{H,x} \| \leq \| G \ast \tilde{\chi}_H \| \cdot \| \varphi \| \leq C \ell^{\frac{\alpha}{2}} \). Thus, we get using (3.22)

\[
|\langle \xi, V_2 \xi \rangle| \leq \frac{C \ell^{\frac{\alpha}{2}}}{\sqrt{N}} ||\tilde{g}_L|| \int dz du N^2 V(N(u-z)) ||a_u \xi|| \cdot ||a_u a_u \xi||
\leq \frac{C \ell^{\frac{\alpha}{2}}}{N} ||\mathcal{V}_N^{1/2} \xi|| \cdot ||(N + 1)^{1/2} \xi||.
\]

Writing \( v_H(x, y) = (G \ast \tilde{\chi}_H)(x-y) \varphi(y) \) and expanding \( (G \ast \tilde{\chi}_H) \) and also \( \tilde{g}_L \) in Fourier space we can estimate

\[
|\langle \xi, V_3 \xi \rangle| \leq \frac{1}{\sqrt{N}} \left[ \int d\sigma dp dq dy N^2 V(N(u-y)) \frac{g_L(p)}{p^2} ||a_u a_v \hat{a}_{\sigma+p} \xi||^2 \right]^{1/2}
\times \left[ \int d\sigma dp dq dy |\tilde{G}(\sigma) \chi_H(\sigma)|^2 N^2 V(N(u-y)) g_L(p) \ p^2 ||a_u \hat{a}_{\hat{\sigma} \ p} \xi||^2 \right]^{1/2}
\leq C \ell^{(\alpha-\beta)/2} ||\mathcal{V}_N^{1/2} \xi|| \cdot \|K_{\ell^{\beta-\epsilon}} + N + 1 \|^{1/2} \xi||,
\]
where, to bound the parenthesis in the second line, we divided the \( p \)-integral in the two domains \(|p| < \ell^{-\beta-\epsilon}\) (where we can use the operator \( K_{\ell^{\beta-\epsilon}} \)) and \(|p| \geq \ell^{-\beta-\epsilon}\), where we use the estimate \( \sup |p| > \ell^{-\beta-\epsilon} \ p^2 e^{-p^2/2} < C \). The term \( V_3 \) can be bounded similarly. This proves (6.15). To show (6.16), we use \(|v_H(x; y)| \leq CN \varphi(y), (3.22)\)
and (3.24) to estimate
\[
|⟨ξ, V_4ξ⟩| \leq C \int dxdydz N^2 V(N(x - y))g_L(x - z)\varphi(y)\|a_xa_yξ\| \cdot \|a_zξ\|
\leq C\|g_L\|_{1,2} \cdot \|V_2^1ξ\| \left( \int dxdydz N^3 V(N(x - y))g_L(x - z)\|a_zξ\|^2 \right)^{\frac{1}{2}}
\leq C\|V_2^1ξ\| \cdot \|(N + 1)^\frac{1}{2}ξ\|.
\]

\[\square\]

Combining the last three lemmas, we obtain a bound for the growth of the Hamilton operator \(H_N = \mathcal{K} + \mathcal{V}_{\text{ext}} + \mathcal{V}_N\).

**Lemma 6.4** Assume (1.8). Let \(\alpha > \beta > 0\) and \(\alpha \geq 4\). There exists \(C > 0\) such that for all \(t \in [0; 1]\), all \(\ell \in (0; 1)\) and all \(N \in \mathbb{N}\) large enough
\[e^{-t\mathcal{A}}\mathcal{H}_N e^{t\mathcal{A}} \leq C\mathcal{H}_N + C\ell^{-\alpha}(N + 1).\]  

**Proof** We define \(f_\xi(s) := ⟨ξ, e^{-s\mathcal{A}}[\mathcal{K} + \mathcal{V}_N]\mathcal{A}^sξ⟩\). Then
\[
\partial_s f_\xi(s) = ⟨ξ, e^{-s\mathcal{A}}[\mathcal{K} + \mathcal{V}_N, \mathcal{A}]e^{s\mathcal{A}}ξ⟩.
\]

Inserting the bounds (6.11), (6.16) (with \(K_{\ell^{-\beta-\varepsilon}} \leq \mathcal{K}\)) and applying (3.24), we obtain
\[
\partial_s f_\xi(s) \leq Cf_\xi(s) + C\ell^{-\alpha}⟨ξ, e^{-s\mathcal{A}}(N + 1)\mathcal{A}^sξ⟩
\leq Cf_\xi(s) + C\ell^{-\alpha}⟨ξ, e^{-s\mathcal{A}}(N + 1)\mathcal{A}^sξ⟩ \leq Cf_\xi(s) + C\ell^{-\alpha}⟨ξ, (N + 1)ξ⟩
\]
for all \(s \in [0; 1]\). With Gronwall’s Lemma and using (6.3) to estimate the growth of the external potential, we arrive at (6.17). \[\square\]

The estimate (6.17) is still not optimal (because of the large factor \(\ell^{-\alpha}\) in front of \((N + 1)\)). To improve this bound, we first have to study the growth of the operator \(K_\theta\), defined in (6.8) measuring the kinetic energy of particles with momenta smaller than \(\theta < \ell^{-\alpha} - \ell^{-\beta}\).

**Lemma 6.5** Assume (1.8). Let \(0 < \beta < \alpha \leq 2\beta\), \(\alpha \geq 4, \varepsilon \in (0; \alpha - \beta)\). Then there exists a constant \(C\) such that for all \(s \in [0; 1]\) all \(\ell > 0\) small enough and all \(\ell^{-\beta-\varepsilon} \leq \theta < \ell^{-\alpha} - \ell^{-\beta-\varepsilon}\), we have
\[e^{-s\mathcal{A}}K_\theta e^{s\mathcal{A}} \leq CK_\theta + C\ell^{2(\alpha - \beta - \varepsilon)}(\mathcal{H}_N + N + 1).\]  

**Proof** In momentum space, we have
\[
A = \frac{1}{\sqrt{N}} \int du dv dw \tilde{v}_H(v; w)g_L(u)[\hat{b}_{u+v}^*\hat{a}_w\hat{u} - h.c.]
= \frac{1}{\sqrt{N}} \int du dv dw \tilde{G}(v)\chi_H(v)\tilde{\varphi}(v + w)g_L(u)[\hat{b}_{u+v}^*\hat{a}_w\hat{u} - h.c.].
\]  

\[\square\]
Thus, we find

$$\left[K_{\theta}, A\right] = -\frac{1}{\sqrt{N}} \int_{|u| \leq \theta} dudvdw \ \tilde{\nu}_H(v; w)g_L(u)|u|^2[\hat{b}_{u+v}^{*}\hat{a}_{u} + h.c.]$$

$$+ \frac{1}{\sqrt{N}} \int_{|u+v| \leq \theta} dudvdw \ \tilde{\nu}_H(v; w)g_L(u)|u+v|^2[\hat{b}_{u+v}^{*}\hat{a}_{u} + h.c.]$$

$$+ \frac{1}{\sqrt{N}} \int_{|w| \leq \theta} dudvdw \ \tilde{\nu}_H(v; w)g_L(u)|w|^2[\hat{b}_{u+w}^{*}\hat{a}_{u} + h.c.]$$

$$=: A_1 + A_2 + A_3.$$ 

Let \( \varepsilon \in (0; \alpha - \beta) \). For \(|u| \leq \ell^{\beta-\varepsilon} \) and \(|v| \geq \ell^{-\alpha} \) we have (since \( \alpha > \beta + \varepsilon \)) \(|u+v| \geq \ell^{-\alpha}/2 \), for \( \ell \) small enough. Therefore, by (3.20), (A.5), we find

$$\|\xi, A_1\xi\| \leq \frac{C}{\sqrt{N}} \int_{|u| \leq \theta, |v| \geq \ell^{-\alpha}} dudvdw \ \frac{1}{|v|^2} |\tilde{\nu}_0(v+w)|g_L(u)|u|^2 \|\hat{a}_u\hat{a}_{u+v}\| \|\hat{a}_v\xi\|$$

$$\leq \frac{C}{\sqrt{N}} \ell^\alpha \left[ \int dudvdw \ \frac{1}{(1+|v|)^{1-\varepsilon}} |\tilde{\nu}_0(v+w)|^{1-\varepsilon} (u+v)^2 \|\hat{a}_u\hat{a}_{u+v}\| \right]^{1/2}$$

$$\left[ \int_{|u| \leq \theta} dudvdw \ \frac{1}{|v|^2} |\tilde{\nu}_0(v+w)|^{1+\varepsilon} g_L(u) |u|^4 \|\hat{a}_v\xi\| \right]^{1/2}$$

$$\leq \frac{C}{\sqrt{N}} \ell^{(3-\varepsilon)\alpha/2} \left[ \int dudvdw \ \frac{1}{(1+|v|)^{1-\varepsilon}} |\tilde{\nu}_0(v+w)|^{1-\varepsilon} \|K_{1/2}^{1/2}\hat{a}_v\xi\| \right]^{1/2}$$

$$\left[ \int_{|u| \leq \theta} dudvdw \ |g_L(u)|^4 \|\hat{a}_v\xi\|^2 \right]^{1/2}$$

$$\leq C \ell^{3\alpha} \|K_{1/2}^{1/2}\xi\| \|N + 1\|^{1/2} \|\xi\| + C \ell^{(3\alpha-2\beta-2\varepsilon)/2} \|K_{1/2}^{1/2}\xi\| \|K_{1/2}^{1/2}\xi\|,$$

where, in the last step, we separated the \( u \)-integral in the second parenthesis between \(|u| > \ell^{\beta-\varepsilon}/2 \) (where we control \(|u|^4 \) with \( g_L(u) \)) and \(|u| < \ell^{\beta-\varepsilon}/2 \) (where we use \( K_{\theta} \)).

Now we turn to \( A_3 \). We bound

$$\|\xi, A_3\xi\| \leq \frac{C}{\sqrt{N}} \theta^{2\alpha} \int_{|u| \leq \theta, |v| \geq \ell^{-\alpha}} dudvdw \ |\tilde{\nu}(v+w)| g_L(u) \frac{g_L(u)}{|u|} |u| \|\hat{a}_u\hat{a}_{u+v}\xi\| \|\hat{a}_v\xi\|$$

$$\leq \frac{C}{\sqrt{N}} \theta^{2\alpha} \left[ \int dudvdw \ \frac{g_L(u)}{|u|^2} |w|^2 \|\hat{a}_u\hat{a}_{u+v}\xi\| \right]^{1/2}$$

$$\left[ \int dudvdw \ \frac{1}{|w|^2} |\tilde{\nu}(v+w)|^2 g_L(u) |u|^2 \|\hat{a}_v\xi\|^2 \right]^{1/2}$$

$$\leq C \ell^{3\alpha} \|K_{\theta}^{1/2}\xi\| \|N^{1/2}\xi\| + C \ell^{5/2} \ell^{2\alpha+\beta+\varepsilon} \|K_{\theta}^{1/2}\xi\|^2$$

$$\leq C \ell^{3\alpha} \|K_{\theta}^{1/2}\xi\| \|N^{1/2}\xi\| + C \ell^{(2\beta+2\varepsilon-\alpha)/2} \|K_{\theta}^{1/2}\xi\|^2,$$

where we divided the \( u \)-integral in the third line an the integral over \(|u| > \ell^{\beta-\varepsilon} \) (here we can control factors of \( u \) and extract arbitrary polynomial decay in \( \ell \) from \( g_L \)) and
an integral over $|u| \leq \ell^{-\beta - \varepsilon}$ (here we used the fact that $|v + w| \geq |v| - |w| \geq \ell^{-\beta}$ and the bound (A.5) for the decay of $\tilde{\phi}$).

We are left with $A_2$. Observing that $|v| - |u + v| \geq \ell^{-\alpha - \ell^{-\beta - \varepsilon}}$ if $|u + v| \leq \theta \leq \ell^{-\alpha} - \ell^{-\beta - \varepsilon}$ and $|v| \geq \ell^{-\alpha}$, we can extract arbitrary polynomial decay in $\ell$ from $g_L$. Thus, we easily get

$$|\langle \xi, A_2 \xi \rangle| \leq C \ell^{3\alpha} \| (\mathcal{N} + 1)^{1/2} \xi \|^2.$$  \hfill (6.22)

From (6.20), (6.21), (6.22) and the assumption $\beta < \alpha \leq 2\beta$, we conclude that

$$|\langle \xi, [\mathcal{K}_\theta, A] \xi \rangle| \leq C \| \mathcal{K}_\theta^{1/2} \xi \|^2 + C \ell^{3\alpha - 2\beta - 2\varepsilon} \| (\mathcal{K} + \mathcal{N} + 1)^{1/2} \xi \|^2.$$

Defining $f_\xi(s) = \langle \xi, e^{-sA_2} \mathcal{K}_\theta e^{sA} \xi \rangle$, we conclude that

$$|\partial_s f_\xi(s)| \leq Cf_\xi(s) + C \ell^{3\alpha - 2\beta - 2\varepsilon} \| (\mathcal{K} + \mathcal{N} + 1)^{1/2} e^{sA} \xi \|^2.$$

Estimating $\mathcal{K} \leq \mathcal{H}_N$ and applying (6.17) and Lemma 3.6, Gronwall leads us to (6.18). \hfill $\Box$

Lemma 6.5 can be used to improve the estimate (6.17) on the growth of the Hamiltonian. We begin with the potential energy operator $V_N$.

**Lemma 6.6** Assume (1.8). Let $0 < \beta < \alpha \leq 2\beta, \alpha \geq 4$. Then there exists $C > 0$ such that for $\ell > 0$ small enough and all $s \in [0; 1]$, we have

$$e^{-sA_2} V_N e^{sA} \leq C (\mathcal{H}_N + \mathcal{N} + 1).$$  \hfill (6.23)

**Proof** Eq. (6.23) follows from (6.16), using (3.24), (6.18) and Gronwall’s lemma. \hfill $\Box$

We conclude this subsection with an improvement on the growth of the kinetic energy operator $\mathcal{K}$.

**Lemma 6.7** Assume (1.8). Let $0 < \beta < \alpha \leq 2\beta, \alpha \geq 4, 0 < \varepsilon < \alpha - \beta$. Then there exists $C > 0$ such that for $\ell \in (0; 1)$ small enough, we have for all $s \in [0; 1]$

$$e^{-sA_2} \mathcal{K} e^{sA} \leq C \ell^{-(\alpha/2 + \varepsilon)} (\mathcal{H}_N + C \mathcal{N} + C).$$  \hfill (6.24)

**Proof** From (6.9) and (6.14), we find that $[\mathcal{K}, A] = S_2 + \delta'_K$, with

$$\pm \delta'_K \leq C \ell^{\alpha/2} \mathcal{K} + C (\mathcal{N} + \mathcal{N} + 1)$$

and

$$S_2 = \frac{1}{\sqrt{N}} \int dx dy dz \left[ (N^3 (V f_\ell)(N \cdot)) * \tilde{\chi}_{H^c} \right] (x - y) \varphi(y) g_L(x - z) [b_x^a a_x^a a_z + h.c.]$$
Switching to Fourier space, we have

$$|\langle \xi, S_2 \xi \rangle| \leq \frac{C}{\sqrt{N}} \int_{|p| \leq \ell^{-\alpha}} dp dq \, g_L(q) \|\hat{a}(\hat{\varphi}_p)\hat{a}_q - \hat{\xi}\| \|\hat{a}_q \xi\|$$

$$\leq C \ell^{3\alpha} \|\mathcal{N}^{1/2} \xi\|^2 + \frac{C}{\sqrt{N}} \int_{|p| \leq \ell^{-\alpha}, |q| < \ell^{-\beta-\varepsilon}} dp dq \, g_L(q) \|\hat{a}(\hat{\varphi}_p)\hat{a}_q - \hat{\xi}\| \|\hat{a}_q \xi\|,$$

where we used $g_L$ to extract decay in $\ell$ for the region $|q| > \ell^{\beta-\varepsilon}$. To control the last integral, we split the region where $|q - p| < \ell^{-\alpha} - \ell^{-\beta-\varepsilon} =: \theta$ and the second with $\theta \leq |q - p| < \ell^{-\alpha} + \ell^{-\beta-\varepsilon}$. We obtain

$$|\langle \xi, S_2 \xi \rangle| \leq C \ell^{3\alpha} \|\mathcal{N}^{1/2} \xi\|^2$$

$$+ \frac{C}{\sqrt{N}} \left[ \int_{|q - p| \leq \theta} dp dq (q - p)^2 \|\hat{a}(\hat{\varphi}_p)\hat{a}_q - \hat{\xi}\|^2 \right]^{1/2}$$

$$\times \left[ \int_{|q - p| \leq \theta} dp dq (q - p)^2 \|\hat{a}_q \xi\|^2 \right]^{1/2}$$

$$+ \frac{C}{\sqrt{N}} \left[ \int dp dq (q - p)^2 \|\hat{a}(\hat{\varphi}_p)\hat{a}_q - \hat{\xi}\|^2 \right]^{1/2}$$

$$\times \left[ \int_{\theta \leq |q - p| \leq \ell^{-\alpha} + \ell^{-\beta-\varepsilon}} dp dq (q - p)^2 \|\hat{a}_q \xi\|^2 \right]^{1/2}$$

$$\leq C \ell^{3\alpha} \|\mathcal{N}^{1/2} \xi\|^2 + C \ell^{-\alpha/2} \|\mathcal{K}^{1/2} \xi\| \|\mathcal{N}^{1/2} \xi\| + C \ell^{-(\beta+\varepsilon)/2} \|\mathcal{K}^{1/2} \xi\| \|\mathcal{N}^{1/2} \xi\|.$$

We conclude that

$$\left| \frac{d}{ds} \langle \xi, e^{sA} K e^{-sA} \xi \rangle \right| \leq C \ell^{\alpha/2} \langle \xi, e^{-sA} K e^{sA} \xi \rangle + C \ell^{-\alpha/2} \langle \xi, e^{-sA}(K_{\theta} + \mathcal{V}_N + \mathcal{N} + 1) e^{sA} \xi \rangle$$

$$+ C \ell^{-(\beta+\varepsilon)/2} \|\mathcal{K}^{1/2} e^{sA} \xi\| \|\mathcal{N} + 1\|^{1/2} \|e^{sA} \xi\|.$$

With (3.24), (6.18), (6.23), and with (6.17) (estimating $\mathcal{K} \leq \mathcal{H}_N + C \mathcal{N}$), we obtain

$$\left| \frac{d}{ds} \langle \xi, e^{sA} K e^{-sA} \xi \rangle \right| \leq C \ell^{-(\alpha+\beta+\varepsilon)/2} \langle \xi, (\mathcal{H}_N + C \mathcal{N} + C) \xi \rangle.$$

Integrating over $s$ yields (6.24).

\[\square\]

### 6.2 Analysis of $e^{-A} \mathcal{D}_N e^A$

In this section we study the contribution arising from the operator $\mathcal{D}_N$, defined in (6.2).
Lemma 6.8 There is $C > 0$ s.t. for all $F \in \mathcal{L}^\infty(\mathbb{R}^3)$, $\alpha, \beta > 0$, $\ell \in (0; 1)$, we have

$$
\left| \int du \, F(u) \langle \xi_1, \left( e^{-A} a_u^* a_u e^A - a_u^* a_u \right) \xi_2 \rangle \right| \leq C \ell^{ \frac{\alpha}{2} } \| F \|_{\mathcal{L}^\infty} \| (N + 1)^{\frac{1}{2}} \xi_1 \| \| (N + 1)^{\frac{1}{2}} \xi_2 \|,
$$

(6.25)

**Proof** With (2.4), we find

$$
\int du \, F(u) \left( e^{-A} a_u^* a_u e^A - a_u^* a_u \right)
= \frac{1}{\sqrt{N}} \int_0^1 ds \int dx dy dz \, (F(x) + F(y) - F(z)) \nu_H(x; y) \tilde{g}_L(x - z)
\times \left[ e^{-sA} b_x^* a_y^* a_z e^{sA} + h.c. \right].
$$

Using (3.22) and (3.24) we obtain

$$
\left| \frac{1}{\sqrt{N}} \int_0^1 ds \int dx dy dz \, (F(x) + F(y)
- F(z)) \nu_H(x; y) \tilde{g}_L(x - z) \langle \xi_1, e^{-sA} b_x^* a_y^* a_z e^{sA} \xi_2 \rangle \right|
\leq \frac{3\| F \|_{\mathcal{L}^\infty}}{\sqrt{N}} \int_0^1 ds \int dx dy dz \, |\nu_H(x; y)| |\tilde{g}_L(x - z)| |a_x a_y e^{sA} \xi_1| \|a_z e^{sA} \xi_2|.
$$

Interchanging $\xi_1$ and $\xi_2$ yields the same estimate for the hermitian conjugate and implies the first estimate in (6.25). Proceeding similarly, we obtain also the second bound.

Using Lemma 6.8, we can easily control the action of $e^A$ on the operator $\mathcal{D}_N$. The proof of the next lemma follows very closely the proof of [5, Proposition 8.7].

Lemma 6.9 Let $\alpha, \beta > 0$. Then there exists $C > 0$ such that for all $\ell \in (0; 1)$ holds

$$
e^{-A} \mathcal{D}_N e^A = \mathcal{D}_N + \delta \mathcal{D}_N,
$$

(6.26)

where

$$
\pm \delta \mathcal{D}_N \leq C \ell^{ \frac{\alpha}{2} } (N + 1).
$$

6.3 Analysis of $e^{-A} \mathcal{Q}_N e^A$

In this section we study the contribution to $\mathcal{J}_N$ arising from the operator $\mathcal{Q}_N$, defined in (6.2).
Lemma 6.10 Let \( 0 < \beta < \alpha \leq 2\beta, \alpha \geq 4 \) and \( \varepsilon \in (0; \alpha - \beta) \). Then there exists \( C > 0 \) such that for \( \ell \in (0; 1) \) small enough and \( N \) large enough, we have

\[
e^{-A} Q_N e^A = Q_N + \delta Q_N, \tag{6.27}
\]

where

\[
\pm \delta Q_N \leq C \left[ \ell^{(\alpha - \beta - \varepsilon)/2} + \ell^{(5\alpha - 7\beta - 2\varepsilon)/4} \right] (H_N + CN + 1).
\]

Proof The first contribution to \( Q_N \) in (6.2) can be handled with (6.25); it produces an error term bounded by \( C\ell^{\frac{a}{2}} (N + 1) \). As for the second contribution to \( Q_N \), we compute, with the commutation relations (2.3), (2.4),

\[
4\pi a_0 \int d\nu d\nu \tilde{\chi}_{H^c} (u - v) \varphi(u) \varphi(v)[b_n^a b_n^a, A]
\]

\[
= -\frac{8\pi a_0}{\sqrt{N}} \int d\nu d\nu d\nu d\nu \tilde{\chi}_{H^c} (u - v) \varphi(u) \varphi(v) \nu_H(x; y) \tilde{g}_L(x - z) b_z^x b_z^x b_z^a
\]

\[
+ \frac{8\pi a_0}{\sqrt{N}} \int d\nu d\nu d\nu d\nu \tilde{\chi}_{H^c} (u - y) \varphi(u) \varphi(y) \nu_H(x; y) \tilde{g}_L(x - z) b_z^y b_z^y b_z^a
\]

\[
+ \frac{8\pi a_0}{\sqrt{N}} \int d\nu d\nu d\nu d\nu \tilde{\chi}_{H^c} (u - x) \varphi(u) \varphi(x) \nu_H(x; y) \tilde{g}_L(x - z) b_z^x b_z^x a_y
\]

\[
+ \frac{8\pi a_0}{\sqrt{N}} \int d\nu d\nu d\nu d\nu \tilde{\chi}_{H^c} (u - x) \varphi(x) \varphi(y) \nu_H(x; y) \tilde{g}_L(x - z) b_z^y b_z^y (1 - \frac{N + 1}{N})
\]

\[
- \frac{8\pi a_0}{N \sqrt{N}} \int d\nu d\nu d\nu d\nu \tilde{\chi}_{H^c} (u - v) \varphi(u) \varphi(v) \nu_H(x; y) \tilde{g}_L(x - z) b_z^x b_z^x a_y a_x a_y
\]

\[
- \frac{16\pi a_0}{N \sqrt{N}} \int d\nu d\nu d\nu d\nu \tilde{\chi}_{H^c} (u - v) \varphi(u) \varphi(v) \nu_H(x; y) \tilde{g}_L(x - z) b_z^x b_z^x a_y a_x
\]

\[
=: \sum_{j=1}^6 \tau_j.
\]

Switching to momentum space we get, with the notation introduced in (6.5),

\[
|\langle \xi, \tau_1 \xi \rangle| \leq \frac{C}{\sqrt{N}} \int d\rho d\sigma d\tau \chi_{H^c} (p) \hat{G}(s) \chi_{H^c} (s) g_L(t) |\hat{\varphi}(p + \tau)|
\]

\[
\times \| (N + 1)^{-\frac{1}{2}} \hat{a}(\hat{\varphi}_p) \hat{a}(\hat{\varphi}_{s+t}) \hat{s}_{s+t} \| (N + 1)^{\frac{1}{2}} \xi
\]

\[
\leq C \ell^{\frac{3(\alpha - \beta)}{2}} \| (K + N)^{\frac{1}{2}} \xi \| (N + 1)^{\frac{1}{2}} \xi,
\]

where we used the bound (6.7), \( \| g_L \| \leq C \ell^{-3\beta/2} \) and \( \int |s| \geq \ell^{-\alpha} ds \| s \|^{-6} \leq C \ell^{3\alpha} \).

Similarly,

\[
|\langle \xi, \tau_2 \xi \rangle|, |\langle \xi, \tau_3 \xi \rangle| \leq C \ell^{\frac{3(\alpha - \beta)}{2}} \| (K + N)^{\frac{1}{2}} \xi \| (N + 1)^{\frac{1}{2}} \xi.
\]

\[
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\]
Moreover, we can easily bound
\[ |\langle \xi, \tau_4 \xi \rangle|, |\langle \xi, \tau_6 \xi \rangle| \leq C \ell^{3a} \|(N + 1)^{1/2} \xi\|^2 \] (6.30)
for \( N \in \mathbb{N} \) large enough (here, we can use the small factors \( N^{-1/2} \) and \( N^{-3/2} \) to gain arbitrary decay in \( \ell \)). As for \( \tau_5 \), switching to momentum space we estimate
\[
|\langle \xi, \tau_s \xi \rangle| \leq \frac{C}{N^{3/2}} \int dpdsdt \chi_H^*(p) \hat{G}(s) \chi_H(s) g_L(t) \|\hat{a}_s \hat{a}(\hat{\varphi}_p) \hat{a}(\hat{\varphi}_{-p}) \xi\| \|\hat{a}_{s+r} \hat{a}(\hat{\varphi}_s) \xi\| \\
\leq \frac{C}{N} \left[ \int_{|s| \geq \ell^{-a}} dpdsdt \frac{p^2}{|x|^{6}} \|\hat{a}_s \hat{a}(\hat{\varphi}_p) \xi\| \right]^{1/2} \\
\times \left[ \int_{|p| \leq \ell^{-a}} dpdsdt \frac{s^2}{p^2} \|\hat{a}_{s+r} \hat{a}(\hat{\varphi}_s) \xi\| \right]^{1/2} \\
\leq C \ell^{\alpha} \|(K + N)^{1/2} \xi\|^2,
\] (6.31)
where we used \(|\hat{G}(s)| \leq C/\ell^2\) and the bound (6.7). Combining (6.28), (6.29), (6.30) and (6.31) we conclude that
\[
|\langle \xi, [Q_N, A] \xi \rangle| \leq C \ell^{\alpha} \|(K + N)^{1/2} \xi\|^2 + C \ell^{3(\alpha - \beta)/2} \|(K + N)^{1/2} \xi\| \|(N + 1)^{1/2} \xi\|.
\]
With (3.24),(6.24), we obtain
\[
|\langle \xi, e^{-A} Q_N e^A \xi \rangle - \langle \xi, Q_N \xi \rangle| \leq \int_0^1 |\langle \xi, e^{-sA} [Q_N, A] e^{sA} \xi \rangle| ds \\
\leq C \left[ \ell^{(\alpha - \beta - \epsilon)/2} + \ell^{(5\alpha - 7\beta - 2\epsilon)/4} \right] \|(\mathcal{H}_N + C N + 1) \xi\|
\]
\[ \square \]

### 6.4 Contributions from \( e^{-A} C_N e^A \)

To control the action of \( e^A \) on the cubic term \( C_N \) in (6.2), we need precise estimates for the commutator \([C_N, A]\).

**Lemma 6.11** Let \( 0 < \beta < \alpha \) and \( \epsilon \in (0; \alpha - \beta) \). Then there exists a constant \( C > 0 \) such that for all \( \ell \in (0; 1) \) sufficiently small and for \( N \) sufficiently large we have
\[
[C_N, A] = \frac{1}{N} \int dx dy dz N^3 V(N(x - y))v_H(x; y)\tilde{\psi}(y)\tilde{g}_L(x - z)\left[a_x^* a_y + h.c.\right] \left(1 - \frac{N}{N}\right) \\
+ \frac{1}{N} \int dx dy dz N^3 V(N(x - y))v_H(x; y)\tilde{\psi}(x)\tilde{g}_L(x - z)\left[a_x^* a_y + h.c.\right] \left(1 - \frac{N}{N}\right) \\
+ \delta_{C_N},
\] (6.32)
where
\[
|\langle \xi, \delta_{C_N} \xi \rangle| \leq C \ell^{\frac{3(\alpha - \beta)}{2}} \|(K + N)^{1/2} \xi\| \|N^{1/2} \xi\| + C \ell^{\frac{(\alpha - \beta)}{2}} \|(K_{\ell - \beta - \epsilon} + V_N + N)^{1/2} \xi\|^2.
\]
\textbf{Proof} A long but straightforward computation using (2.3), (2.4) shows that

\begin{align*}
&= \frac{1}{N} \int dxdydz \ N^3 V(N(x - y)) \nu_H(x; y) \varphi(y) \hat{g}_L(x - z) [a_x^* a_y + h.c.] \left(1 - \frac{N}{N}\right) \\
&+ \frac{1}{N} \int dxdydz \ N^3 V(N(x - y)) \nu_H(x; y) \varphi(x) \hat{g}_L(x - z) [a_x^* a_x + h.c.] \left(1 - \frac{N}{N}\right) \\
&+ \sum_{j=1}^{12} C_j,
\end{align*}

where the error terms \( \{C_1, \ldots, C_{12}\} \) are listed and bounded below.

We begin with

\begin{align*}
C_1 &= -\frac{1}{N} \int dxdydzdv \ N^3 V(N(z - v)) \nu_H(x; y) \hat{g}_L(x - z) b_x^* b_y^* a_v^* a_z
\end{align*}

which can be bounded, switching to momentum space, by

\begin{align*}
|\langle \xi, C_1 \xi \rangle| &\leq \frac{1}{N} \int dpdsdt |\hat{V}(p/N)||\hat{G}(s)\chi_H(s)g_L(t)|\hat{a}_{s+t} \hat{a}(\hat{\varphi}_{s-p}) \hat{a}_{-p}\xi| \|\hat{a}(\hat{\varphi}_{t-p})\xi\| \\
&\leq \frac{C}{N} \left[ \int dpdsdt \ s^2 \|\hat{a}_{s+t} \hat{a}(\hat{\varphi}_{s-p}) \hat{a}_{-p}\xi\|^2 \right]^{1/2} \\
&\quad \times \left[ \int_{|s|>\ell^{-a}} dpdsdt \ |s|^{-6} g_L(t)^2 \|\hat{a}(\hat{\varphi}_{t-p})\xi\|^2 \right]^{1/2} \\
&\leq C \ell^{3(\alpha - \beta)/2} \|\mathcal{K} + \mathcal{N}\|^{1/2} \|\mathcal{N}^{1/2}\| \xi\|\xi\|.
\end{align*}

Also for the term

\begin{align*}
C_2 &= \frac{1}{N} \int dxdydzdv \ N^3 V(N(y - v)) \nu_H(x; y) \hat{g}_L(x - z) \left(1 - \frac{N}{N}\right) a_x^* a_y^* a_x a_y
\end{align*}

we switch to momentum space. We find

\begin{align*}
|\langle \xi, C_2 \xi \rangle| &\leq \frac{1}{N} \int dpdsdt |\hat{V}(p/N)||\hat{G}(s)\chi_H(s)g_L(t)|\hat{a}_{-t} \hat{a}_{-p}\xi| \|\hat{a}(\hat{\varphi}_{s-p}) \hat{a}_{s-t}\xi\| \\
&\leq C \ell^{3\alpha} \|\mathcal{N} + 1\|^{1/2} \|\xi\|^2 \\
&\quad + \frac{C}{N} \left[ \int_{|s|>\ell^{-a},|t|<\ell^{-b}\epsilon} dpdsdt \ |s|^{-4} t^2 \|\hat{a}_{-t} \hat{a}_{-p}\xi\|^2 \right]^{1/2} \\
&\quad \times \left[ \int dpdsdt \ \frac{g_L(t)^2}{t^2} \|\hat{a}(\hat{\varphi}_{s-p}) \hat{a}_{s-t}\xi\|^2 \right]^{1/2} \\
&\leq C \ell^{3\alpha} \|\mathcal{N}^{1/2}\| \|\xi\|^2 + C \ell^{(\alpha - \beta)/2} \|\mathcal{K}^{1/2}_{\ell^{-b}\epsilon}\xi\| \|\mathcal{N}^{1/2}\|,
\end{align*}
where the first term on the r.h.s. arises from $|t| > \ell^{-\beta - \varepsilon}$, where we can use $g_L(t)$ to extract (arbitrary polynomial) decay in $\ell$. The term
\[
C_3 = \frac{1}{N} \int dx dy dz dv \, N^3 V(N(x - v)) \varphi(x) \nu_H(x; y) \tilde{g}_L(x - z) \left( 1 - \frac{N}{N} \right) a_x^* a_u^* a_y a_x
\]
can be handled similarly to $C_2$. We find
\[
|\langle \xi, C_3 \xi \rangle| \leq C \ell^{3\alpha} \|\mathcal{N}^{1/2} \xi\|^2 + C \ell^{(\alpha - \beta)/2} \|K^{1/2} \xi\| \|\mathcal{N}^{1/2} \xi\|.
\]
On the other hand, to estimate
\[
C_4 = -\frac{1}{N^2} \int dx dy dz du N^3 V(N(u - v)) \varphi(u) \nu_H(x; y) \tilde{g}_L(x - z) a_u^* a_x^* a_y a_u,
\]
we switch only partially to Fourier space (keeping $V$ in position space). We obtain
\[
|\langle \xi, C_4 \xi \rangle| \leq \int du dv ds dt N^2 V(N(u - v)) \hat{G}(s) \|\hat{\chi}_H(s)g_L(t)\| \|\hat{a}_u \hat{a}_v \hat{a}_x \hat{a}_u \xi\| \|\hat{a}_u \hat{a}_v \hat{a}_x \hat{a}_u \xi\|
\leq \frac{C}{N^{3/2}} \left[ \int_{|s| > \ell^{-\alpha}} du dv ds dt N^2 V(N(u - v)) |s|^{-4} \|\hat{a}_u \hat{a}_v \hat{a}_x \hat{a}_u \xi\|^2 \right]^{1/2}
\times \left[ \int du dv ds dt N^2 V(N(u - v)) \|\hat{a}_u \hat{a}_v \hat{a}_x \hat{a}_u \xi\|^2 \right]^{1/2}
\leq C \ell^{\alpha/2} \|\mathcal{N}^{1/2} \xi\| \|\mathcal{N}^{1/2} \xi\|.
\]
The terms
\[
C_5 = -\frac{1}{N^2} \int dx dy dz du N^3 V(N(u - x)) \varphi(u) \nu_H(x; y) \tilde{g}_L(x - z) a_u^* a_x^* a_y a_u
\]
\[
C_6 = -\frac{1}{N^2} \int dx dy dz du N^3 V(N(u - y)) \varphi(u) \nu_H(x; y) \tilde{g}_L(x - z) a_u^* a_x^* a_y a_u
\]
can be bounded using the factor $N^{-2}$ to gain arbitrary decay in $\ell$. We easily find
\[
\pm C_5, \pm C_6 \leq C \ell^{3\alpha} (N + 1).
\]
As for
\[
C_7 = \frac{1}{N} \int dx dy dz dv N^3 V(N(y - v)) \varphi(y) \nu_H(x; y) \tilde{g}_L(x - z) b_y^* b_v^* a_x^* a_z,
\]
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we stay in position space and estimate

\[
|\langle \xi, C_7 \xi \rangle| \leq \int dxdydzdv \, N^2 V(N(y-v))|\varphi(y)||v_H(x; y)||\tilde{g}_L(x-z)||a_y a_v a_z \xi||a_z \xi||
\leq C \left[ \int dxdydzdv \, N^2 V(N(y-v))\tilde{g}_L(x-z)||a_y a_v a_z \xi||^2 \right]^{1/2}
\times \left[ \int dxdydzdv \, N^2 V(N(y-v))|v_H(x; y)|^2 \tilde{g}_L(x-z)||a_z \xi||^2 \right]^{1/2}
\leq C \ell^{1/2} \|\mathcal{V}_{N}^{1/2} \xi\| \|\Lambda^{1/2} \xi\|,
\]

using that (3.22) and (3.19). We can proceed very similarly to bound

\[
C_8 = \frac{1}{N} \int dxdydzdv \, N^3 V(N(x-v))\varphi(x) v_H(x; y) \tilde{g}_L(x-z)b^*_z b^*_v a_y a_z.
\]

We find

\[
|\langle \xi, C_8 \xi \rangle| \leq C \ell^{1/2} \|\mathcal{V}_{N}^{1/2} \xi\| \|\Lambda^{1/2} \xi\|.
\]

Also

\[
C_9 = -\frac{1}{N} \int dxdydzdv \, N^3 V(N(z-v))\varphi(z) v_H(x; y) \tilde{g}_L(x-z)b^*_z a_v b_x b_y
\]

can be handled analogously, estimating

\[
|\langle \xi, C_9 \xi \rangle| \leq \frac{C}{\sqrt{N}} \left[ \int dxdydzdv \, N^2 V(N(z-v))|v_H(x; y)|^2 \tilde{g}_L(x-z)||a_z a_v \xi||^2 \right]^{1/2}
\times \left[ \int dxdydzdv \, N^2 V(N(z-v))\tilde{g}_L(x-z)||a_x a_y \xi||^2 \right]^{1/2}
\leq C \ell^{1/2} \|\mathcal{V}_{N}^{1/2} \xi\| \|\Lambda^{1/2} \xi\|.
\]

To bound

\[
C_{10} = -\frac{1}{N} \int dxdydzdu \, N^3 V(N(u-z))\varphi(u) v_H(x; y) \tilde{g}_L(x-z)b^*_u b^*_x a_y a_u,
\]
we switch partially to momentum space:

\[
|\langle \xi, C_{10} \xi \rangle| \leq \int du \, dz \, ds \, dt \, N^2 V(N(u - z)) |\hat{G}(s)| \chi_H(s) g_L(t) \|a_u \hat{a} \hat{\phi}_{\eta} \hat{a}_{s+1} \xi \| \|a_u \xi \|
\]

\[
\leq \frac{C}{N} \left[ \int du \, ds \, st \|a_u \hat{a} \hat{\phi}_{\eta} \hat{a}_{s+1} \xi \|^2 \right]^{1/2}
\]

\[
\times \left[ \int_{|s| > \ell - \alpha} du \, ds \, dt \, \|a_u \xi \|^2 \right]^{1/2}
\]

\[
\leq C \ell^{3(\alpha - \beta)/2} \| (K + N)^{1/2} \xi \| \|N^{1/2} \xi\|.
\]

Also to estimate

\[
C_{11} = \frac{1}{N} \int dx \, dy \, dz \, du \, N^3 V(N(u - y)) \phi(u) v_H(x; y) \tilde{g}_L(x - z) b_u^* a_z^* b_x a_u,
\]

we switch partially to momentum space. We find

\[
|\langle \xi, C_{11} \xi \rangle| \leq \int du \, dy \, dz \, ds \, dt \, N^2 V(N(u - y)) |\hat{G}(s)| \chi_H(s) g_L(t) \|a_u \hat{a}_{s-1} \xi \| \|a_u \hat{a}_{s-1} \xi \|
\]

\[
\leq C \ell^{3\alpha} \|N^{1/2} \xi\|^2 + \frac{C}{N} \left[ \int_{|s| < \ell - \beta - \epsilon} du \, ds \, ds \, t^2 \|a_u \hat{a}_{s-1} \xi\|^2 \right]^{1/2}
\]

\[
\times \left[ \int_{|s| > \ell - \alpha} du \, ds \, \|a_u \hat{a}_{s-1} \xi\|^2 \right]^{1/2}
\]

\[
\leq C \ell^{3\alpha} \|N^{1/2} \xi\|^2 + C \ell^{(\alpha - \beta)/2} \|K^{1/2} \ell^{-\beta - \epsilon} \xi\| \|N^{1/2} \xi\|,
\]

where the first term on the r.h.s. arises from the region $|t| > \ell - \beta - \epsilon$, where we can use $g_L$ to extract decay in $\ell$. As for the term

\[
C_{12} = \frac{1}{N} \int dx \, dy \, dz \, du \, N^3 V(N(u - x)) \phi(u) v_H(x; y) \tilde{g}_L(x - z) b_u^* a_z^* b_y a_u,
\]

it can be bounded similarly as $C_{11}$. We find

\[
|\langle \xi, C_{12} \xi \rangle| \leq C \ell^{3\alpha} \|N^{1/2} \xi\|^2 + C \ell^{(\alpha - \beta)/2} \|K^{1/2} \ell^{-\beta - \epsilon} \xi\| \|N^{1/2} \xi\|.
\]

\[\square\]

### 6.5 Proof of Proposition 3.7

Recalling (6.1) and applying (6.3), (6.26) and (6.27), we obtain

\[
\mathcal{J}_N = \mathcal{D}_N + \mathcal{Q}_N + \mathcal{C}_N + \mathcal{H}_N + \int_0^1 ds \, e^{-sA} [C_N + K + V_N, A] e^{sA} + \delta_1, \quad (6.33)
\]
where

\[ \pm \delta_1 \leq C \left[ \ell^{(\alpha-\beta-\epsilon)/2} + \ell^{(5\alpha-7\beta-2\epsilon)/4} \right] (H_N + CN + 1) . \]

Combining (6.15) with the first term on the r.h.s. of (6.12) (and recalling the definition of \( C_N \) in (6.2)), we obtain

\[
[K + V_N, A] = -C_N + \frac{1}{\sqrt{N}} \int dxdydz N^3 V(N(x-y)) \varphi(y)(1 - g_L)(x - z)[b_x^* a_y^* a_z + h.c.] \\
- \frac{1}{\sqrt{N}} \int dxdydz (G \ast \tilde{\chi}_{H^c})(x - y) \varphi(y) \tilde{g}_L(x - z) N^2 V(N(x - y)) [b_x^* a_y^* a_z + h.c.] \\
+ \frac{8\pi a_0}{\sqrt{N}} \int dxdydz \tilde{\chi}_{H^c}(x - y) \tilde{g}_L(x - z) \varphi(y) [b_x^* a_y^* a_z + h.c.] + \delta_2, \tag{6.34}
\]

where

\[
|\langle \xi, \delta_2 \xi \rangle| \leq C \left[ \ell^{(\alpha-4)/2} + \ell^{(\alpha-\beta)/2} \right] \left\| (K_{\ell} - \beta - \epsilon) + N + V_N + 1 \right\|^{1/2} \xi \bigg\|^2 \\
+ C \ell^{\alpha/2} \left\| K_{\ell}^{1/2} \xi \right\| \left( (K_{\ell} - \beta - \epsilon) + N + 1 \right)^{1/2} \xi \bigg\|. 
\]

Next, we observe that the second term on the r.h.s. of (6.34) can be bounded, using that \( 0 \leq 1 - g_L(p) \leq \min(1, \ell^{2\beta} p^2) \), by

\[
\left| \frac{1}{\sqrt{N}} \int dxdydz N^3 V(N(x-y)) \varphi(y)(1 - g_L)(x - z) \langle \xi, b_x^* a_y^* a_z \xi \rangle \right| \\
\leq \left\| V_N^{1/2} \xi \right\| \left( \int dx \left\| a((1 - g_L)_x \xi \right\|^2 \right)^{1/2} \\
= \left\| V_N^{1/2} \xi \right\| \left( \int dp |1 - g_L(p)|^2 \right)^{1/2} \left\| \hat{a}_p \xi \right\|^2 \leq C \ell^{\beta} \left\| V_N^{1/2} \xi \right\| \left\| K_{\ell}^{1/2} \xi \right\| . 
\]

As for the third term on the r.h.s. of (6.34), we can first extract \( \| G \ast \tilde{\chi}_{H^c} \|_{\infty} \leq \| G \| \| \chi_{H^c} \| \leq C \ell^{-\alpha} \) and then use the factor \( N^{-1/2} \) to gain arbitrary decay in \( \ell \). We obtain

\[
\left| \frac{1}{\sqrt{N}} \int dxdydz (G \ast \tilde{\chi}_{H^c})(x - y) \times \varphi(y) \tilde{g}_L(xz) N^2 V(N(x - y)) \langle \xi, [b_x^* a_y^* a_z + h.c.] \xi \rangle \right| \\
\leq C \ell^{3\alpha} \left\| V_N^{1/2} \xi \right\| \left( N + 1 \right)^{1/2} \xi \bigg\|. 
\]
Finally, the fourth term on the r.h.s. of (6.34) can be bounded by
\[
\left| \frac{8\pi a_0}{\sqrt{N}} \int dxdydz \tilde{\chi}_F(x,y) \tilde{g}_L(x-z) \varphi(y) \langle \xi, [a_x^* a_y + h.c.] \xi \rangle \right| \leq C \sqrt{N} \left[ \int dxdydz \tilde{g}_L(x-z) \|a_x a_y \|^2 \right]^{1/2} \times \left[ \int dxdydz |\tilde{\chi}_F(x-y)|^2 \tilde{g}_L(x-z) \|a_x \xi \|^2 \right]^{1/2} \leq C \frac{\ell^{-3\alpha/2}}{\sqrt{N}} \|N \xi\| \|N^{1/2} \xi\|.
\]

Thus
\[
[K + V_N, A] = -C_N + \delta_3
\]
with
\[
|\langle \xi, \delta_3 \xi \rangle| \leq C \left[ \ell^{(\alpha-4)/2} + \ell^{(\alpha-\beta)/2} \right] \| (K_{\ell-\beta-\varepsilon} + N + V_N + 1)^{1/2} \xi \|^2 + C \left[ \ell^{\alpha/2} + \ell^\beta \right] \| K^{1/2} \xi \| \| (K_{\ell-\beta-\varepsilon} + N + V_N + 1)^{1/2} \xi \| + C \ell^{-4\alpha} \|N \xi\|^2 / N.
\]

Inserting into (6.33) and using (3.24), (6.18), (6.23), (6.24), we obtain
\[
\mathcal{J}_N = D_N + Q_N + C_N + \mathcal{H}_N - \int_0^1 ds e^{-sA}C_Ne^{sA} + \int_0^1 ds e^{-sA}[C_N, A]e^{sA} + \delta_4
\]
\[
= D_N + Q_N + \mathcal{H}_N + \int_0^1 ds e^{-sA}[C_N, A]e^{sA} + \delta_4 \quad (6.35)
\]
with
\[
\pm \delta_4 \leq C \left[ \ell^{(\alpha-4)/2} + \ell^{(\alpha-\beta-\varepsilon)/2} + \ell^{(5\alpha-7\beta-2\varepsilon)/4} + \ell^{(3\beta-\alpha-\varepsilon)/4} \right] \times (\mathcal{H}_N + CN' + 1) + C \ell^{-4\alpha} N^2 / N.
\]

We computed the commutator $[C_N, A]$ in (6.32). To deal with the two main contributions on the r.h.s. of (6.32), we switch to momentum space. For the first term, we find
\[
\frac{1}{N} \int dxdydz N^3 V(N(x-y))v_H(x, y, \varphi(y)) \tilde{g}_L(x-z) \|a_x a_y + h.c.\| \left( 1 - \frac{N}{N} \right)
\]
\[
= \int dp \left[ \frac{1}{N} \int_{|q| > \ell^{-\alpha}} dq \tilde{V}((p - q)/N) \tilde{G}(q) \right] g_L(p) \left[ \hat{a}_p \hat{\varphi}^2 \right] \left( 1 - \frac{N}{N} \right) + h.c.].
\]
\[
\quad (6.36)
\]

With (3.5), we can estimate
\[
\left| \frac{1}{N} \int_{|q| > \ell^{-\alpha}} dq \tilde{V}((p - q)/N) \tilde{G}(q) - (8\pi a_0 - \tilde{V}(0)) \right| \leq C(|p| + \ell^{-2\alpha})/N.
\]
Since moreover (using $0 \leq 1 - g_L(p) \leq \min(1, \ell/2, p^2)$)
\[ \int dp \left| 1 - g_L(p) \right| |\langle \xi, \hat{a}_p^* \hat{a} (\varphi_p) (1 - \mathcal{N}/N) \xi \rangle | \]
\[ \leq \int dp |p|^\beta \left| \hat{a}_p^* \hat{a} (\varphi_p^2) \xi \right| \leq \ell^\beta \| \mathcal{K}^{1/2} \xi \| \| \mathcal{N}^{1/2} \xi \| , \]
we conclude from (6.36), switching back to position space, that
\[ \frac{1}{N} \int dx dy dz N^3 V(N(x-y)) v_H(x; y) \varphi(y) \tilde{g}_L(x-z) [a_x^* a_y + h.c.] \left( 1 - \frac{\mathcal{N}}{N} \right) \]
\[ = (8\pi a_0 - \hat{V}(0)) \int dp \left[ \hat{a}_p^* \hat{a} (\varphi_p^2) + h.c. \right] + \delta_5 \]
\[ = 2(8\pi a_0 - \hat{V}(0)) \int dx \varphi^2(x) a_x^* a_x + \delta_5, \]
where
\[ |\langle \xi, \delta_5 \xi \rangle| \leq C \ell^\beta \| \mathcal{K}^{1/2} \xi \| \| \mathcal{N}^{1/2} \xi \| + C \| \mathcal{N} \| ^2 / N. \]
Similarly, we can also handle the second term on the r.h.s. of (6.32). We conclude that
\[ [C_N, A] = 4(8\pi a_0 - \hat{V}(0)) \int dx \varphi^2(x) a_x^* a_x + \delta_6 \]
with
\[ |\langle \xi, \delta_6 \xi \rangle| \leq C (\ell^\beta + \ell^{3(\alpha - \beta)/2}) \| (\mathcal{K} + \mathcal{N})^{1/2} \xi \| \| \mathcal{N}^{1/2} \xi \| + C \| \mathcal{N} \| ^2 / N. \]
Inserting in (6.35), we find, with (6.25), (3.24), (6.18), (6.23), (6.24),
\[ J_N = D_N + Q_N + \mathcal{H}_N + 2(8\pi a_0 - \hat{V}(0)) \int dx \varphi^2(x) a_x^* a_x + \delta_7 \]
with
\[ \pm \delta_7 \leq C \left[ \ell^{(\alpha - 4)/2} + \ell^{(\alpha - \beta - \varepsilon)/4} + \ell^{(5\alpha - 7\beta - 2\varepsilon)/4} + \ell^{(3\beta - \alpha - \varepsilon)/4} \right] \times \| \mathcal{H}_N + C \mathcal{N} + 1 \| + C \ell^{-4\alpha} \mathcal{N}^2 / N. \]
Under the assumption $\alpha > 4$, $7\beta / 5 < \alpha < 2\beta$, we can find $\varepsilon > 0$ small enough so that $\kappa = \min((\alpha - 4)/2, (\alpha - \beta - \varepsilon)/4, (5\alpha - 7\beta - 2\varepsilon)/4, (3\beta - \alpha - \varepsilon)/4) > 0$. Inserting $D_N, Q_N$ as in (6.2), we arrive therefore at
\[ J_N \geq N \mathcal{E}_{GP} (\varphi) - \varepsilon_{GP} N + 4\pi a_0 \int dx dy \tilde{\chi}_H^c (x-y) \varphi(x) \varphi(y) [b_x b_y + b_x^* b_y^*] \]
\[ + 16\pi a_0 \int dx \varphi^2(x) a_x^* a_x + (1 - C \ell^\kappa) \mathcal{H}_N - C \ell^\kappa (\mathcal{N} + 1) - C \ell^{-4\alpha} (\mathcal{N} + 1)^2 / N. \]
Next, we observe that
\[ 0 \leq 4\pi a_0 \int dx dy \tilde{\chi}_{H^c}(x-y)\varphi(x)\varphi(y)[b_x + b_x^*][b_y + b_y^*]. \]
This implies that
\[
4\pi a_0 \int dx dy \tilde{\chi}_{H^c}(x-y)\varphi(x)\varphi(y)[b_x b_y + b_x^*b_y^*] \\
\geq -8\pi a_0 \int dx dy \tilde{\chi}_{H^c}(x-y)\varphi(x)\varphi(y)b_x b_y \\
- 4\pi a_0 \int dx dy \tilde{\chi}_{H^c}(x-y)\varphi(x)\varphi(y)[b_x, b_y^*] \\
\geq -8\pi a_0 \int dx dy \tilde{\chi}_{H^c}(x-y)\varphi(x)\varphi(y)a_x^*a_y - C\ell^{-3a} - C N^2 / N,
\]
where in the last step we used the commutation relations (2.3) (and we replaced \(b_x^*, b_y\) by \(a_x^*, a_y\)). Since, switching to momentum space,
\[
\int dx dy \tilde{\chi}_{H}(x-y)\varphi(x)\varphi(y)(\xi, a_x^* a_y \xi) \\
= \int dp \chi_{H}(p)\|\hat{a}(\hat{\varphi}_p)\xi\|^2 \\
\leq \ell^{2a} \int dp p^2 \|\hat{a}(\hat{\varphi}_p)\xi\|^2 \leq \ell^{2a} \|(K + N)^{1/2}\xi\|^2,
\]
with (6.7), we conclude that
\[
4\pi a_0 \int dx dy \tilde{\chi}_{H^c}(x-y)\varphi(x)\varphi(y)[b_x b_y + b_x^*b_y^*] \\
\geq -8\pi a_0 \int dx \varphi(x)^2a_x^*a_x - C\ell^{-3a} - C N^2 / N - C\ell^{2a}(K + N).
\]
Inserting in (6.37), we arrive at
\[
J_N \geq NE_{GP}(\varphi) + (1 - C\ell^k)d\Gamma(-\Delta + V_{ext} + 8\pi a_0)\varphi(x)^2 - \varepsilon_{GP} \\
- C\ell^k N - C\ell^{-3a} - C\ell^{-4a}N^2 / N,
\]
which implies (3.26), if \(\ell > 0\) is small enough.

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Declarations

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Appendix A: Properties of the Gross–Pitaevskii Functional

In this appendix we collect several well-known results about the Gross–Pitaevskii functional $\mathcal{E}_{GP}$, defined in equation (1.5). Let us recall that $\mathcal{E}_{GP} : D_{GP} \to \mathbb{R}$ is given by

$$
\mathcal{E}_{GP}(\psi) = \int_{\mathbb{R}^3} \left( |\nabla \psi(x)|^2 + V_{\text{ext}}(x)|\psi(x)|^2 + 4\pi a_0|\psi(x)|^4 \right) \, dx
$$

with domain

$$
D_{GP} = \{ \psi \in H^1(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) : V_{\text{ext}}|\psi|^2 \in L^1(\mathbb{R}^3) \}.
$$

Recall, moreover, assumption $(2)$ in Eq. (1.8) on the external potential $V_{\text{ext}}$. The following was proved in [16, Theorems 2.1, 2.5 & Lemma A.6].

**Theorem A.1**  There exists a minimizer $\psi \in D_{GP}$ with $\|\psi\|_2 = 1$ such that

$$
\inf_{\psi \in D_{GP} : \|\psi\|_2 = 1} \mathcal{E}_{GP}(\psi) = \mathcal{E}_{GP}(\psi).
$$

The minimizer $\psi$ is unique up to a complex phase, which can be chosen so that $\psi$ is strictly positive. Furthermore, the minimizer $\psi$ solves the Gross–Pitaevskii equation

$$
-\Delta \psi + V_{\text{ext}} \psi + 8\pi a_0|\psi|^2 \psi = \varepsilon_{GP} \psi,
$$

(A.1)

with $\varepsilon_{GP}$ given by

$$
\varepsilon_{GP} = \mathcal{E}_{GP}(\psi) + 4\pi a_0\|\psi\|_4^4.
$$

Moreover, $\psi \in L^\infty(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ and for every $\nu > 0$ there exists $C_\nu$ (which only depends on $\nu$ and $a_0$) such that for all $x \in \mathbb{R}^3$ it holds true that

$$
|\psi(x)| \leq C_\nu e^{-\nu|x|}.
$$

(A.2)

We denote by $\psi$ in the following the unique, strictly positive minimizer of $\mathcal{E}_{GP}$, subject to the constraint $\|\psi\|_2 = 1$. In addition to Theorem A.1, we need to collect a few additional facts about the regularity of $\psi$. Before we do so, notice that the assumption

$$
V_{\text{ext}}(x + y) \leq C(V_{\text{ext}}(x) + C)(V_{\text{ext}}(y) + C)
$$

implies that $V_{\text{ext}}$ has at most exponential growth, as $|x| \to \infty$. Indeed, by (1.8), we find $R > 0$ such that $V_{\text{ext}}(x) > 0$ for all $|x| \geq R$. Let $\tilde{C}$ be the maximum of $V_{\text{ext}}$ in the ball of radius $2R$
around the origin. For $|x| \geq R$, we pick $n \in \mathbb{N}$ such that $nR \leq |x| < (n+1)R$ and obtain

$$|V_{\text{ext}}(x)| \leq C^n (V_{\text{ext}}(x/n) + C)^n \leq (C(\tilde{C} + C))^n \leq (C(\tilde{C} + C)) |x|/R.$$ 

Hence, $V_{\text{ext}}$ grows at most exponentially. In particular, by (A.2), this implies that

$$\|V_{\text{ext}}\|_{\infty} \leq C.$$  \hspace{1cm} (A.3)

**Lemma A.2** Let $V_{\text{ext}}$ satisfy the assumptions in (1.8). Then $\varphi \in H^2(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$ and for every $\nu > 0$ there exists $C_\nu > 0$ such that for every $x \in \mathbb{R}^3$ we have

$$|\nabla \varphi(x)|, |\Delta \varphi(x)| \leq C_\nu e^{-\nu |x|}. \hspace{1cm} (A.4)$$

Moreover, if $\hat{\varphi}$ denotes the Fourier transform of $\varphi$, we have for all $p \in \mathbb{R}^3$ that

$$|\hat{\varphi}(p)|, |\hat{\varphi}^2(p)| \leq \frac{C}{(1 + |p|)^{\gamma}}. \hspace{1cm} (A.5)$$

**Proof** By the previous Theorem A.1, the Gross–Pitaevskii equation (A.1), Eq. (A.2) and the fact that $V_{\text{ext}}, \nabla V_{\text{ext}}$ grow at most exponentially (by the assumptions (1.8) and the previous remark), we obtain the exponential decay of $\Delta \varphi$. Moreover, since $\varphi, V_{\text{ext}} \in C^1(\mathbb{R}^3)$, we get the local Hölder continuity of $\Delta \varphi$. Elliptic regularity then implies that $\varphi \in C^2(\mathbb{R}^3, \mathbb{R})$.

Next, by [11, Theorem 3.9], if $u \in C^2(B_2(y))$ solves $\Delta u = f$, then there exists a constant $C > 0$, independent of $y \in \mathbb{R}^3$, such that

$$\|\nabla u\|_{L^\infty(B_1(y))} \leq C(\|u\|_{L^\infty(B_2(y))} + \|f\|_{L^\infty(B_2(y))}).$$

Here, $B_1(y)$ and $B_2(y)$ denote the open balls of radius one and two, respectively, centered at $y \in \mathbb{R}^3$. Applying this last bound to $\varphi$ and using the exponential decay of $\varphi$, $\Delta \varphi$ implies that also $\nabla \varphi$ has exponential decay.

Finally, let us prove the decay estimate (A.5). By Eq. (A.1), we get

$$\nabla \Delta \varphi = -\nabla (\varphi V_{\text{ext}}) + \varphi G P \nabla \varphi + 8\pi a_0 \nabla \varphi^3.$$

Since, on the one hand, $\varphi, \nabla \varphi$ have exponential decay with arbitrary rate while $V_{\text{ext}}, \nabla V_{\text{ext}}$ grow at most exponentially by assumption (1.8), we conclude that $\nabla \Delta \varphi \in L^1(\mathbb{R}^3)$. This implies the estimate (A.5) by switching to Fourier space. \hfill \Box

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