Sure-Success Quantum Algorithms on Weight Decision Problem

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Conditions on sure-success decidability of weights of Boolean functions are presented for a given number of generalized Grover iterations. It is shown that the decidability problem reduces to a system of algebraic equations of a single variable. For problems that require a large number of iterations, it is observed that the iteration number of sure-success quantum algorithms scale as the square root of the iteration number of the corresponding classical probabilistic algorithms. It is also demonstrated that for a few iterations, quantum algorithms can be more efficient than this.

PACS numbers: 03.67.Ac
Keywords: Quantum computation, Weight Decision Problem, Grover Iteration, Quantum Searching

I. INTRODUCTION

Quantum algorithms have been proved to be exponentially faster than their classical counterparts in problems such as Deutsch’s problem[1, 2], Simon’s problem[3] and super-polynomially faster in order finding[4]. All of these problems have some simplifying features and the quantum algorithms which solve these problems make use of such features while exploiting quantum parallelism and, in some cases, entanglement. If there is no such simplifying feature, quantum algorithms do not perform this well. Nevertheless, as Grover showed, one can obtain at least a quadratic speedup[5, 6] for searching a single item in an unstructured database. Unfortunately this small-scale speedup is shown to be the upper limit of quantum database search algorithms[7–9]; however wide range of applications compensate for this.

Several generalizations and variations of Grover’s algorithm is explored up to now. In Ref. 8, the problem of searching for several items, instead of a single item, is studied. In this case, the algorithm gives one of the solutions randomly at the output. In Ref. 6, it is shown that using an arbitrary unitary (instead of the Hadamard transform which were used in the original Grover algorithm) does not change the \( O(\sqrt{N}) \) run-time as long as it is used consistently. If there is an inner structure to be exploited, this unitary can be chosen accordingly to obtain quadratic speedup for the new search space. Biham et al. further generalized the algorithm such that the initial state[10] and the phase inversion angle[11] are arbitrary and obtained similar results. Grover’s original algorithm is not deterministic, but with a few tweaks it is possible to obtain solutions with zero error probability[12, 13]. It does not monotonically converge to a solution (i.e., if you run it too much, it misses the target) but it can also be altered to become a fixed point algorithm[14].

An important problem related to database search is counting the number of roots of a given Boolean function \( f \). In this problem, the aim is to find the number of inputs \( x \) that gives \( f(x) = 0 \) (or equivalently the number of the roots of \( f(x) = 1 \) is sought). Brassard et al. gave a practical method to accomplish this task approximately, by employing Grover iteration as well as quantum Fourier transform[15]. Another specific problem in which Grover search iteration is used is the weight analysis of Boolean functions. The ratio of the number of solutions of \( f(x) = 1 \), to the number of all possible inputs is called the weight of a function. Weight analysis of Boolean functions has proved to be useful in areas such as cryptanalysis[16], coding theory[17], fault-tolerant circuit design[18], and for built-in self-testing circuits[19]. Weight analysis using Grover algorithm is studied by Braunstein et al. and Choi and Braunstein in a series of papers[20, 21]. They first solve the problem with a restriction where they only consider two “symmetric” weights (i.e., the function is known to have a weight either equal to \( \rho_1 \) or \( \rho_2 = 1 - \rho_1 \)). Then they generalize it to the asymmetric case where the restriction on the weights is now removed. This task can also be accomplished by using quantum counting[16]. However, in that case, one needs to introduce quantum Fourier transform and quantum counting is still slightly slower than the weight decision algorithm.

In this contribution, the weight decision problem is studied by an alternative approach. The main motivation is to see which weights could be distinguished by a given number of function evaluations, especially in the regime where only a few evaluations are required. As the algorithm devised by Braunstein et al. requires at least 3 function
evaluations, the approach used in this article covers an unexplored territory. The organization of the article is as follows. Section III starts with a general Grover iteration which consists of unitaries which are more general than the Hadamard transform of the original algorithm. The basic definitions are given and an essential theorem is presented in this section. The first problem tackled in Section III is the decision problem of a zero weight and a non-zero weight. After that, the decision problem of two non-zero weights by a few iterations is discussed. Exact solutions are given for 1 and 2 iterations. For 3 or more iterations, the associated equations become too complicated and thus we are forced to present numerical solutions for a few cases. A brief comparison of classical and quantum efficiencies and a comparison to the algorithms of Braunstein et al. are also included in this section. Finally, brief concluding remarks are given in section IV.

II. PRELIMINARIES AND THE CONSTRUCTION OF THE PROBLEM

Let us first define the problem. We are given a Boolean function \( f \) of \( N \) possible inputs \( f: \{0, 1, \ldots, N-1\} \rightarrow \{0, 1\} \). We somehow know that the function has either the weight \( \rho = r/N \) or the weight \( \rho' = r'/N \). In other words, the total number of inputs \( x \) for which \( f(x) = 1 \) is either \( r \) or \( r' \). Our job is to determine which one is the case. Note that by “weight decision problem” we imply the general case which is usually “asymmetric” as in [21], that is to say, \( r + r' \) is not necessarily equal to \( N \).

The evaluation is implemented in a quantum computer as a black box. The function evaluator, upon reading the input register (which is an \( N \)-level system), adds the value of the function on the result qubit. Denoting the unitary transformation of the function evaluation by \( U_f \) we have

\[
U_f |x\rangle_I \otimes |b\rangle_R = |x\rangle_I \otimes |b \oplus f(x)\rangle_R ,
\]

where \( I \) denotes the \( N \)-level input register and \( R \) denotes the result qubit. Using the basis \( |\pm\rangle_R = (|0\rangle_R \pm |1\rangle_R) / \sqrt{2} \), the action of \( U_f \) can also be expressed as

\[
U_f |x\pm\rangle_{IR} = (\pm 1)^{f(x)} |x\pm\rangle_{IR} .
\]

Therefore, when \( f(x) = 1 \), the overall phase of the state \( |x-\rangle \) is rotated by \( \pi \) radians. The phases of the states \( |x+\rangle \) remain unchanged irrespective of the function \( f \).

Let \( A \) be an ancilla system. Let \( |\beta_1\rangle, \ldots, |\beta_n\rangle \) be a set of orthonormal vectors in the state space of the composite system \( AIR \). Consider the following unitary operator \( S \) that acts on the composite system

\[
S \equiv 1 - 2 \sum_{i=1}^{n} |\beta_i\rangle \langle \beta_i| .
\]

This operator is essentially an inversion operation in an \( n \)-dimensional subspace of the Hilbert space, namely the subspace spanned by \( \{|\beta_i\rangle\} \). The successive application of \( U_f \) and \( S \) constitute a single iteration step \( Q_f = -S(I_A \otimes U_f) \). To express the effect of multiple iterations of \( Q_f \) on an arbitrary initial state, it is convenient to first define an \( n \times n \) matrix \( C \), which will be called as the “cosine matrix”, as

\[
C_{ij} \equiv \langle \beta_i| (I_A \otimes U_f) |\beta_j\rangle .
\]

As \( U_f \) is unitary with real eigenvalues of \( \pm 1 \), it is also hermitian. This implies that the cosine matrix \( C \) is hermitian and all of its eigenvalues are in the \([-1, +1]\) interval. In that case, we can think of \( C \) as the cosine of an angle matrix \( \Theta \), i.e., \( C = \cos \Theta \). Next, we define \( n \times n \) matrices \( R^{(m)} \) for all integer values \( m \) by

\[
R^{(m)} = \frac{\sin(m\Theta)}{\sin(\Theta)} ,
\]

which is actually a polynomial function of \( C \). Note that all of these matrices depend on the function \( f \). When necessary, this dependence will be shown by \( C(f), \Theta(f) \) and \( R^{(m)}(f) \). But, the function will not be shown explicitly when there can be no confusion.

We can express our fundamental result in terms of the \( R^{(m)} \) matrices as follows.

\textbf{Theorem 1.} If the initial state is one of \( |\beta_i\rangle \), then, after \( m \) iterations of \( Q_f = -S(I_A \otimes U_f) \), the final state is

\[
Q_f^m |\beta_i\rangle = \sum_{j=1}^{n} \left( |\beta_j\rangle R_{ji}^{(m+1)} - U_f |\beta_j\rangle R_{ji}^{(m)} \right) .
\]
The proof is rather straightforward. One first verifies Eq. (6) for $m = 0$ and $m = 1$. After that, showing that the matrices $R^{(m)}$ satisfy the following recurrence relation

$$R^{(m+2)} - 2CR^{(m+1)} - R^{(m)} = 0,$$

and using these in $Q_f^{m+1} |\beta_i\rangle$ completes the proof of the theorem. The details are left to the reader. A slightly more complicated, but still simple expression can be found for $Q_f^m |\psi\rangle$ for any arbitrary initial state $|\psi\rangle_{AIR}$. However, that general case will not be needed in this contribution.

The problem is as follows. We prepare the initial state of the input register $I$, result qubit $R$ and ancilla $A$ in the state

$$|\beta\rangle = \sum_{i=1}^{n} c_i |\beta_i\rangle$$

(8)

where the amplitudes $c_i$ will be determined later. After that, the evaluation of the unknown function $f$ and the unitary $S$ are alternatingly applied $m$ times. In other words, $Q_f^m$ is applied on the composite system. The final state of the composite system $AIR$ is

$$|\Phi_f\rangle \equiv Q_f^m |\beta\rangle.$$

(9)

Finally, a measurement is carried out on the composite system $AIR$ for determining the weight of the function $f$.

If this final measurement enables us to measure the correct weight of the function $f$ deterministically (with probability 1), then, it is necessary that all final states corresponding to functions with different weights are orthogonal. In other words, if $f$ and $g$ are any two functions that could be possibly computed by the black box device, we should have $\langle \Phi_f | \Phi_g \rangle = 0$ whenever $f$ and $g$ have different weights. In this contribution, only the case where the function computed by the black box device has either the weight $\rho$ or $\rho'$ will be considered. Therefore, if the correct weight of the unknown function could be determined after $m$ iterations, then the set of final states $|\Phi_f\rangle$ for functions with weight $\rho$ and the corresponding set for functions with weight $\rho'$ should be in orthogonal subspaces.

Undoubtedly, how the final measurement is carried out is also important from the computation point of view. However, in this contribution, the primary concern is the possibility of distinguishing functions with different weights and not how the steps of the algorithm can be implemented. Moreover, the length of the algorithm will be measured with the number of evaluations of the unknown function $f$, presumably because this is very costly. Hence, it is assumed that the preparation of the initial state $|\beta\rangle$, implementation of the unitary $S$ and the final measurement requires a much smaller number of computation steps than carrying out $U_f$. For this reason, we are inclined to find only the smallest iteration number $m$ that is necessary for distinguishing two given weights $\rho$ and $\rho'$.

For the weight decision problem, it appears that the following choice for the $|\beta_i\rangle$ is sufficient.

$$|\beta_i\rangle = |\beta_i\rangle_{AIR} \equiv |\alpha_i\rangle_A \otimes \left( \sqrt{\mu_i} \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x-\rangle_{IR} + \sqrt{1-\mu_i} |0+\rangle_{IR} \right)$$

(10)

Here, $\mu_i$ are real parameters between 0 and 1 and the ancilla states $|\alpha_i\rangle_A$ are normalized and mutually orthogonal to each other (i.e., $\langle \alpha_i | \alpha_j \rangle = \delta_{ij}$), so that $|\beta_i\rangle_{AIR}$ are also normalized and mutually orthogonal to each other. In that case, the cosine matrix $C$ is diagonal. If the unknown function $f$ has weight $\rho$, then the ith diagonal entry of $C$ is

$$C_{ii} = \cos \theta_i(f) = \langle \beta_i | I_A \otimes U_f | \beta_i \rangle = 1 - 2\rho \mu_i$$

(11)

The angle eigenvalues are therefore bounded in the interval

$$0 \leq \theta_i(f) \leq \arccos(1-2\rho)$$

(12)

The matrices $R^{(m)}$ are also diagonal in this case and their ith diagonal entry are given by

$$R^{(m)}_{ii} = \frac{\sin m\theta_i(f)}{\sin \theta_i(f)}.$$

(13)

### III. RESULTS

#### A. Distinguishing zero weight functions from non-zero weight functions

First, the case where one of the weights is identically zero, $\rho' = 0$ is investigated. In this case, the decidability of distinguishing a function $f$ with weight $\rho = \frac{r}{N}$ and the zero function $z$ (which is defined as $z(x) = 0$ for all $x$)
is studied. In other words, we are given a function $f$ and we are told that either the function $f$ has weight $\rho$ (but otherwise arbitrary) or it is the zero function. We are asked to determine if $f$ is the zero function or not with a minimum possible number of function evaluations. Classically, the deterministic algorithms require $N(1 - \rho) + 1$ evaluations in the worst case although a single evaluation is sufficient if we are lucky. It is of some interest to see how quantum algorithms perform for this problem.

In this case, $U_z = I$ and $Q_z |\beta_i\rangle = |\beta_i\rangle$, thus leading to $Q^m_z |\beta_i\rangle = |\beta_i\rangle$. If all functions with weight $\rho$ could be distinguished from $z$ after $m$ function evaluations, then we should have $\langle \Phi_z | \Phi_f \rangle = 0$ for all functions $f$ with weight $\rho$.

$$
0 = \langle \Phi_z | \Phi_f \rangle = \langle \beta | Q^m_f | \beta \rangle
$$

$$
= \sum_{ij} a_i^* \langle \beta_j | Q^m_f | \beta_j \rangle a_j
$$

$$
= \sum_{ij} a_i^* \left( R^{(m+1)}(f) - C(f) R^{(m)}(f) \right)_{ij} a_j
$$

$$
= \sum_{ij} a_i^* (\cos m \Theta(f))_{ij} a_j .
$$

For any possible iteration number $m$, the equation above can be satisfied with $n = 1$ (i.e., $S$ is an inversion in a one-dimensional subspace spanned by $|\beta_1\rangle$). In this case, $\Theta(f)$ is a $1 \times 1$ matrix, which we may denote by the value $\theta_1(f)$. We should therefore have $\cos m \theta_1(f) = 0$ for all functions $f$ with weight $\rho$. If the iteration number $m$ is the minimum possible value, then $\theta_1(f)$ should be independent of $f$ (the definition of $|\beta_1\rangle$ in the form in Eq. (10) is consistent with this) and should be given by $\theta_1(f) = \pi/2m$. The minimum iteration number is, therefore, the smallest integer $m$ where we can find a number $\mu_1$ in $[0, 1]$ interval such that

$$
\cos \frac{\pi}{2m} = 1 - 2\rho \mu_1 .
$$

It is then straightforward to show that the smallest iteration number is given by

$$
m_{\text{min}}(\rho) = \left[ \frac{\pi}{2 \arccos(1 - 2\rho)} \right] ,
$$

where $[y]$ denotes the smallest integer greater than or equal to $y$.

| $m$ | $\rho_{\text{min}}(m)$ |
|-----|---------------------|
| 1   | 0.5$^*$             |
| 2   | 0.15                |
| 3   | 0.067               |
| 4   | 0.038               |
| 5   | 0.024               |
| 10  | 0.0062              |

$^*$A special variation of Deutsch-Jozsa algorithm

TABLE I: Minimum weights of functions, which can be distinguished from the zero function $z$ by the only $m$ function evaluations.

One can also ask the reverse question: which weights $\rho$ can be distinguished by $m$ iterations? In that case, the condition on the weights is found to be

$$
\rho \geq \rho_{\text{min}}(m) = \frac{1}{2} \left( 1 - \cos \frac{\pi}{2m} \right) .
$$

For a few small $m$ values, the threshold weights $\rho_{\text{min}}$ are tabulated in Table I.

It appears that, with a single function evaluation (i.e., $m = 1$) any function $f$ with weight $\rho$ can be distinguished from the zero function $z$ provided that $\rho \geq 1/2$. This special case corresponds to a variation of the Deutsch-Jozsa problem [2]. In the Deutsch-Jozsa problem, one needs to distinguish functions $f$ with weight $\rho = 1/2$, from the constant functions $z$ and $u$ where $u = z \oplus 1$ (i.e., $u(x) = 1$ for all $x$).

If the weight $\rho$ is smaller than $1/2$, more than one function evaluations are necessary. As it will be discussed below, when the weights to be distinguished $\rho$ and $\rho'$ are closer to each other, more function evaluations are needed to identify the weight correctly. This is also the case for the current problem: when $\rho \ll 1$, one needs $m \sim \pi/4\sqrt{\rho}$ function evaluations in order to distinguish the weight $\rho$ from the weight $\rho' = 0$. 

B. Distinguishing two non-zero weights

Now, consider the problem of identifying the weight of the function when both of the possible weights $\rho$ and $\rho'$ are non-zero. For simplicity, let us consider the choices in Eq. (10) so that the angle matrices $\Theta$ are diagonal. Let $f$ and $g$ be two functions with respective weights $\rho$ and $\rho'$. The inner product of the final states is given by

$$\langle \Phi_f | \Phi_g \rangle = \sum_{i=1}^{n} |c_i|^2 \left( R_{ii}^{(m+1)}(f)R_{ii}^{(m+1)}(g) - R_{ii}^{(m+1)}(f)R_{ii}^{(m)}(g) \right) \cos \theta_i(g)$$

$$- R_{ii}^{(m)}(f)R_{ii}^{(m+1)}(g) \cos \theta_i(f) + R_{ii}^{(m)}(f)R_{ii}^{(m)}(g) \cos \theta_i(f \oplus g) = 0 .$$

(21)

Consider the last term inside the sum of Eq. (21). The matrix element can be evaluated as

$$\cos \theta_i(f \oplus g) = \langle \beta_i | I_A \otimes U_f \oplus g | \beta_i \rangle = 1 - \mu_2 \frac{2t}{N} ,$$

(22)

where $t$ is the number of inputs which make $f \oplus g$ one, i.e., $t/N$ is the weight of the function $f \oplus g$. For all possibilities for $f$ and $g$, $t$ can take on the values $t = r - r', r - r' + 2, \ldots, r + r' - 2, r + r'$. As Eq. (21) is linear in $t$, the condition in Eq. (21) is reducible to two independent equations

$$\sum_{i=1}^{n} |c_i|^2 A_i = 0$$

$$\sum_{i=1}^{n} |c_i|^2 B_i = 0$$

subject to the condition

$$\sum_{i=1}^{n} |c_i|^2 = 1 ,$$

(23)

(24)

where $A_i$ and $B_i$ are given as follows:

$$A_i = \cos(m \theta_{if}) \cos(m \theta_{ig}) + \frac{\sin(m \theta_{if}) \sin(m \theta_{ig})}{\sin \theta_{if} \sin \theta_{ig}} \left( 1 - \cos(\theta_{if}) \cos(\theta_{ig}) \right)$$

(25)

$$B_i = \frac{\sin(m \theta_{if}) \sin(m \theta_{ig})}{\sin \theta_{if} \sin \theta_{ig}} \left( 2 - \cos \theta_{if} - \cos \theta_{ig} \right) .$$

(26)

Therefore we have to find $n$ tuples $(A_i, B_i)$, which are just points on the 2-dimensional plane such that $\sum_{i=1}^{n} |c_i|^2 (A_i, B_i) = (0, 0)$. Geometrically, this means that the origin $(0, 0)$ is in the convex hull of the set of points $\{(A_i, B_i)\}$.

Note that, with the choice in Eq. (10), $A_i$ and $B_i$ depend only on the fixed parameters $m$, $\rho$ and $\rho'$; and the adjustable parameter $\mu_i$. Keeping the dependence on the fixed parameters as implicit, and showing the dependence on $\mu_i$ explicitly we can write $A_i = A(\mu_i)$ and $B_i = B(\mu_i)$. Therefore, the set of points $(A_i, B_i)$ lie on a continuous curve $(A(\mu), B(\mu))$. The weights can be distinguished, therefore, if the origin, $(0, 0)$, is inside the convex hull of the whole curve $(A(\mu), B(\mu))$ for $0 \leq \mu \leq 1$. In such a case, the problem can be solved with $n = 2$, i.e., one needs to find only two points on the curve $(A(\mu), B(\mu))$ such that the line joining them passes from the origin. The associated values of $\mu_1$ and $\mu_2$ enables us to find $|\beta_1\rangle$ and $|\beta_2\rangle$.

Therefore, the distinguishability problem of two weights $\rho$ and $\rho'$ with $m$ function evaluations reduces to a problem in convex analysis: determining whether a point lies within the convex hull of a curve. Both coordinates of the curve $(A(\mu), B(\mu))$ are actually polynomial functions of $\mu$,

$$A(\mu) = T_m(y)T_m(y') + U_{m-1}(y)U_{m-1}(y')(1 - yy')$$

(27)

$$B(\mu) = 2(\rho + \rho')U_{m-1}(y)U_{m-1}(y')$$

(28)

where $T_m$ and $U_m$ denote the Chebyshev polynomials that are defined by $T_m(t) = \cos(m \arccos(t))$ and $U_{m-1}(t) = \frac{1}{m} \frac{d}{dt} T_m(t)$ and $y = 1 - 2 \rho \mu$, $y' = 1 - 2 \rho' \mu$. However, determining whether the origin lies in the convex hull of the curve is a complicated problem which can only be solved numerically in most cases.
The case with $m = 1$ is the simplest. In that case, $A(\mu) = 1$ for all $\mu$ and therefore the origin can never be in the convex hull of the associated curve. This implies that, it is not possible to distinguish two weights which are both different from 0 or 1 by a single function evaluation.

The curve for the case $m = 2$, $\rho = 0.95$ and $\rho' = 0.45$ is shown in Fig. 1. In this example, we can see that the parametric curve have $(A, B) = (1, 0)$ at $\mu = 0$, which turns out to be the case for all situations. However, where the curve ends is nontrivial. If it cuts the horizontal axis again in the negative part of the axis, a convex combination that gives the origin is easily achieved. This simplification covers most of the solution space and can be utilized for a quick analysis. For a complete analysis one has to find compact inequalities for $\rho$ and $\rho'$. In order to understand the general case, $m = 2$ case is studied first.

1. The case where $m = 2$

For $m = 2$, $A(\mu)$ and $B(\mu)$ becomes

$$A^{(2)}(\mu) = 1 - 8\mu^2(\rho - \rho')^2$$
$$B^{(2)}(\mu) = 2\mu(\rho + \rho')(1 - 2\mu\rho)(1 - 2\mu\rho')$$

where we introduced superscripts to emphasize $m = 2$. Now we require that there exists two points $(A^{(2)}(\mu_1), B^{(2)}(\mu_1))$ and $(A^{(2)}(\mu_2), B^{(2)}(\mu_2))$ such that

$$\sum_{i=1,2} |c_i|^2 \begin{pmatrix} A^{(2)}(\mu_i) \\ B^{(2)}(\mu_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

To do this, we must find two values, $\mu_1$ and $\mu_2$, satisfying

$$\frac{B^{(2)}(\mu_1)}{A^{(2)}(\mu_1)} = \frac{B^{(2)}(\mu_2)}{A^{(2)}(\mu_2)}$$
where $B^{(2)}(\mu_1)$ and $B^{(2)}(\mu_2)$ (and, in parallel, $A^{(2)}(\mu_1)$ and $A^{(2)}(\mu_2)$) have opposite signs. It turns out that if there
is a solution to Eq. (32), it can be realized with a single variable $\mu_1$, while $\mu_2$ is set to 1. Hence, we get

\[ B^{(2)}(\mu)A^{(2)}(\mu)B^{(2)}(1) = (K\mu^2 + L\mu + M)(\mu - 1) = 0 \]  

(33)

where

\[ K = 4\rho\rho' \left(1 - 8(\rho - \rho')^2\right) \]
\[ L = 8(\rho - \rho')^2 + 4\rho\rho' - 2(\rho + \rho') \]
\[ M = (2\rho - 1)(2\rho' - 1) \].

(34, 35, 36)

Let $\Delta \equiv L^2 - 4KM$ be the discriminant of this quadratic equation. Suppose that $\rho > \rho'$. All of the following conditions have to be satisfied for being able to distinguish these two weights: $\rho/\rho' > 1 + 1/\sqrt{2}$, $\rho > 1/2$, $\rho - \rho' > 1/2\sqrt{2}$, and $\Delta > 0$. The conditions for the case $\rho < \rho'$ can be obtained simply by using $\rho - \rho'$ symmetry. These conditions are found analytically using Eq. (33) and by inspecting all possible orderings of the zeros and poles of $B^{(2)}(\mu)/A^{(2)}(\mu)$. However, even for $m = 2$, the equation for the conditions on $\rho$ and $\rho'$ requires solving a quadratic equation. Calculations for bigger $m$ are even more forbidding and therefore numerical computations had to be utilized instead. Numerical results for $m = 2$ case is pictured in Fig. 2 among further results from $m = 3$ to $m = 8$.

2. General case

Basically the same procedure can be followed for $m \geq 2$. Solving Eq. (32) would be more and more challenging analytically with increasing $m$. Fortunately this problem is suitable for numerical analysis. Note that solving this problem directly, without applying Theorem 1 is also possible, however in that case we would have needed to optimize $|\beta_i|$’s. In this formulation, there is only one parameter, $\mu$, to be optimized and this is a clear advantage.

In Fig. 2, numerical results showing decidability of weights with several iterations ($m = 2$ to $m = 8$) are plotted. It can be seen in the figure that, with each run, more weight combinations can be distinguished. However the rate of addition of distinguishable weight combinations decreases rapidly with each $m$.

Notice that for any number of iterations, the area corresponding to distinguishable weights is symmetric with respect to $\rho = \rho'$ line since we start with the same initial states and we take the inner product of the final states. On the other hand we may expect one more symmetry. The problem of distinguishing $\rho$ and $\rho'$ should be no different than distinguishing $1 - \rho$ and $1 - \rho'$ because flipping function outputs does not change the complexity of the problem, thus the figure should also have been symmetric with respect to the other diagonal. However possible advantageous output flips does not come out naturally as solutions in our formalism. This is also a strong indication for the non-optimality of the solutions provided in this contribution.

A quick but rough analysis can be made for comparing the efficiencies of classical and quantum algorithms. Classically, with a non-probabilistic algorithm we need to make

\[ m_{cl, det} \approx N(1 - |\rho - \rho'|) \]  

(37)

queries in the worst case. However we can do better with a probabilistic algorithm. In such an algorithm, the weight can be estimated by computing the function $f$ for $s$ random inputs. To be able to distinguish two weights $\rho$ and $\rho'$, the variance of the estimate should be smaller than $|\rho - \rho'|$. For large $N$, this process can be approximated as a binomial process, which leads to a value of

\[ m_{cl, prb} \approx \frac{4\rho(1 - \rho)}{|\rho - \rho'|^2} \]  

(38)

evaluations. In the quantum case, a quick estimate of minimum evaluation number can be obtained as follows: since the $(A, B)$ curve always starts at the point $(1,0)$, we can only look if this curve intersects with the negative $A$ axis. For large $m$, this approximation is good enough for estimating the order of the run-time of the algorithm. Thus we look for the roots of $B$ and see if the value of $A$ can be negative at these roots. As can be seen from Eq. (29), either $m\theta_1$ or $m\theta_2$ is an integer multiple of $\pi$ at the roots of $B$. The value of $A$ can be negative for these cases only if $|m\theta_1 - m\theta_2| > \frac{\pi}{4}$. Since we are dealing with weights that are close to each other, we can linearize the expression and finally obtain

\[ m_{quant} \approx \frac{2\sqrt{\rho(1 - \rho)}}{|\rho - \rho'|}. \]  

(39)
FIG. 2: Solutions for $m = 2$ to $m = 8$. Darker areas correspond to higher $m$ values and lighter areas correspond to lower $m$ values. Lightest region is for $m = 2$. The disc-like black region on the diagonal corresponds to weight pairs that have no solutions with $m \leq 8$ iterations.

A comparison of Equations (38) and (39) indicate that a quadratic speedup is obtained by using Grover iterations for the weight distinguishability problem. This result is compatible with the bounds given for the Grover algorithm [7–9, 15] and Choi and Braunstein’s algorithms [20, 21].

A comparison of the approaches given in [20, 21] and the approach in this article is also in order. Even though both are sure-success and achieve a square-root speedup, the algorithm given in Refs. [20 and 21] needs $m' \approx (\pi / 2)/|\rho - \rho'|$ iterations in the limit where $\rho$ approaches $\rho'$. For moderate values of the weights, that number is only slightly bigger than this article’s result of Eq. (39). However, their algorithm specifies a complete algorithm that solves the problem exactly, while the study presented in this letter lacks such a clearly constructed algorithm. A special case where our approach becomes useful is those weights where only $m = 1$ or $m = 2$ function evaluations are necessary for distinguishability. On the other hand in Refs. [20 and 21], at least three evaluations are needed.

IV. CONCLUSIONS

The sure-success weight decision problem of Boolean functions using generalized Grover iterations is discussed. Specifically, the pairs of weights that can be distinguished by $m = 1$ or $m = 2$ function evaluations are analyzed in detail. The decidability problem is reduced to a problem of determining if a point lies in the convex hull of a curve, where the curve is defined by polynomial functions whose order increases with the increasing number of iterations $m$. As a result, only for cases with very small $m$ one can obtain analytical expressions. For $m \geq 3$, it becomes necessary to follow the numerical approach.

This analysis may be compared with Braunstein and Choi’s works [20, 21] and with quantum counting [15]. Braunstein and Choi have shown that their algorithm is 4 times faster than quantum counting [21]. Being in parallel with their result and the known bounds in the literature, we also show that quadratic speedup is obtained for large values of the number of iterations.
Acknowledgments

K.U. acknowledges the support of the Scientific and Technical Research Council of Turkey (TÜBİTAK). S.T. also acknowledges the support of TÜBİTAK through project 110T335.

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