Separability and entanglement in $2 \times N$ composite quantum systems

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(February 4, 2022)

We show that all density operators of $2 \times N$-dimensional quantum systems that remain invariant after partial transposition with respect to the first system are separable. Based on this criterion, we derive a sufficient separability condition for general density operators in such quantum systems. We also give a simple proof of the separability criterion in $2 \times 2$-dimensional systems [A. Peres, Phys. Rev. Lett 77, 1413 (1996)].

03.65.Bz, 03.65.Ca, 89.70.+c

Perhaps, entanglement is the most intriguing property of Quantum Mechanics. It arises when two or more systems are non-separable states. One may wonder whether this class of non-separable states exists in other dimensions. In this Letter we show that this is not the case for $2 \times N$ systems. More specifically, we show that if $\rho^{TA} = \rho$ then necessarily $\rho$ must be separable. Based on this fact, we give a sufficient condition for separability in these systems.

The separability criterion introduced by Peres [4] states that for $M = 2$ and $N = 2, 3$ the density operator $\rho$ describing the state of systems is separable iff $\rho^{TA} \geq 0$, i.e. if its partial transpose is a positive operator. Surprisingly enough, this criterion is not valid for higher values of $N$ or $M$. In particular, for $N = M = 3$ (i.e. two-three-level systems), Bennett et al. [8] have found families of non-separable states fulfilling $\rho^{TA} = \rho \geq 0$. These states are related to the existence of sets of product states that cannot be extended with other product states to form a basis, and play a crucial role in the understanding of concepts like non-locality [3] or EPR paradox without entanglement [4]. One may wonder whether this class of non-separable states exists in other dimensions.

Let us consider the $M \times N$ case, where the Hilbert spaces $\mathcal{H}^M$ and $\mathcal{H}^N$ corresponding to systems $A$ and $B$ have dimension $M$ and $N$, respectively. Given an operator $X$ and an orthonormal basis $|m\rangle_A$ ($n = 1, \ldots, M$) for the first system, the partial transpose of $X$ with respect to system $A$ in that basis is defined as

$$X^{TA} = \sum_{n,m=1}^{M} A\langle n|\rho|m\rangle_A |m\rangle_A \langle n|$$

for any pair of vectors $|e_1\rangle, |e_2\rangle \in \mathcal{H}^2$; this property follows directly from the definition of partial transposition [3].

In order to carry out our analysis we need to recall a lemma and a corollary introduced in Ref. [12]. For their proofs we refer the reader to this reference.

$$A\langle e_1|\rho|e_2\rangle_A = A\langle e_2|^{\dagger}\rho^{TA}|e_1\rangle_A$$

$$A\langle e_1|\rho|e_2\rangle_A = A\langle e_2|^{\dagger}\rho^{TA}|e_1\rangle_A$$
Lemma 1: If \(|e, g\rangle \in \mathcal{R}(\rho)\) then \(\rho_1 \equiv \rho - \langle(e, g)\rho^{-1}(e, g)\rangle - |e, g\rangle\langle e, g|\) is positive and \(\rho^{-1}(e, g) \in K(\rho_1)\).

Corollary 1: Let \(|e, g\rangle \in \mathcal{R}(\rho), |e^*, g\rangle \in \mathcal{R}(\rho T^a), \lambda_1 \equiv \langle(e, g)\rho^{-1}(e, g)\rangle - 1\) and \(\lambda_2 \equiv \langle(e^*, g)(\rho T^a)^{-1}(e^*, g)\rangle - 1\).

Then \(\rho_1 = \lambda_1 |e, g\rangle\langle e, g| + \lambda_2 |e^*, g\rangle\langle e^*, g|\) and \(r(\rho_1) = r(\rho) - 1\) if \(\lambda_1 \leq \lambda_2 \leq \lambda_1\).

We will now consider some results that imply that if there is a product vector in the kernel of \(\rho\), then we can reduce the dimensionality of the second system.

Lemma 2: \(|e, f\rangle \in K(\rho)\) if \(|e^*, f\rangle \in K(\rho T^a)\).

\begin{proof}
If \(|e, f\rangle \in K(\rho)\) we have \(0 = \langle e, f|\rho|e, f\rangle = \langle e^*, f|\rho T^a|e^*, f\rangle\). Given that \(\rho T^a \geq 0\) this implies that \(\rho T^a |e^*, f\rangle = 0\). Similarly, we have that if \(|e^*, f\rangle \in K(\rho T^a)\) then \(\rho(e, f) = 0\).
\end{proof}

Lemma 3: If \(|e, f\rangle \in K(\rho)\), then one of the following statements is true: (i) \(|e, h\rangle \in K(\rho)\) and \(|e^*, f\rangle \in K(\rho T^a)\); (ii) There exists a non-zero \(|g\rangle \in \mathcal{F}_N\) such that \(\rho(e, f) = |e, g\rangle\) and \(\rho T^a |e^*, f\rangle = |e^*, g\rangle\).

\begin{proof}
Using Lemma 2 we have that for all \(|h\rangle \in \mathcal{F}_N\) we can write \(0 = \langle e^*, h|\rho T^a|e^*, f\rangle = \langle e, h|\rho e, f\rangle\) and therefore either \(\rho(e, f) = 0\) or \(\rho(e, f) = |e, g_1\rangle\) for some \(|g_1\rangle \in \mathcal{F}_N\). In a similar way we have that either \(\rho T^a |e^*, f\rangle = 0\) or \(\rho T^a |e^*, f\rangle = |e^*, g_2\rangle\) for some \(|g_2\rangle \in \mathcal{F}_N\). Using Lemma 2 we conclude that there are two possibilities: (i) \(\rho(e, f) = 0\) and \(\rho T^a |e^*, f\rangle = 0\), which coincides with the first statement of the Lemma; (ii) \(\rho(e, f) = |e, g_1\rangle\) and \(\rho T^a |e^*, f\rangle = |e^*, g_2\rangle\) for some \(|g_1\rangle, |g_2\rangle \in \mathcal{F}_N\). We have \(|g_2\rangle = |e^*|\rho T^a|e^*, f\rangle = |e^*|\rho^a|e, f\rangle = |g_1\rangle\) which corresponds to the second statement.
\end{proof}

Lemma 4: If \(|e, f\rangle \in K(\rho)\) then we can write \(\rho = \rho_1 + \rho_2\), where \(\rho_1\) is separable, \(\rho_1 = \rho T^a \geq 0\)

\begin{proof}
According to Lemma 3, we can have two situations: (i) \(|e, f\rangle, |e^*, f\rangle \in K(\rho)\) and therefore \(\rho\) is already supported on \(\mathcal{F}^2 \otimes \mathcal{F}_N^{-1}\); (ii) \(\rho^{-1}|e, f\rangle = |e, f\rangle\) and \(\rho T^a |e^*, f\rangle = |e^*, f\rangle\).

We define

\[\rho_1 = \rho - \frac{1}{\langle e, g|\rho^{-1}|e, g\rangle}\langle e, g|\rho^{-1}|e, g\rangle = \rho - \frac{1}{\langle g, f|\rho^{-1}|g, f\rangle}\langle g, f|\rho^{-1}|g, f\rangle\]

then \(\rho T^a = \rho T^a - \langle(e^*, g)(\rho T^a)^{-1}(e^*, g)\rangle - 1\). According to Lemma 1, \(\rho_1, \rho T^a \geq 0\). Moreover, \(\rho_1|e, f\rangle = \rho e, f\rangle - \langle(g, f)(\rho T^a)^{-1}(g, f)\rangle\langle e, g|\rho^{-1}|e, g\rangle = 0\) and \(\rho_1|e, f\rangle = \rho(e, f) = 0\). Thus \(|e, f\rangle, |e^*, f\rangle \in K(\rho_1)\) and therefore \(\rho_1\) is supported on \(\mathcal{F}^2 \otimes \mathcal{F}_N^{-1}\). Note that in this case \(r(\rho_1) = r(\rho) - 1\).
\end{proof}

This powerful lemma simply states that if there is a product vector in the kernel of \(\rho\), then we can reduce the dimensionality of the second system keeping positive the partial transposed operator. On the other hand, Corollary 1 allows to reduce the rank of \(\rho\), i.e. to increase the dimension of the kernel of \(\rho\), if we find appropriate product vectors in its range. Thus, the idea is to find out whether there are product vectors in the kernel of \(\rho\). If there are, we reduce the dimensionality. If not, then we try to find product vectors in the range until we make the kernel “sufficiently large” to include a product vector in it. Thus, the important question is to find out when we can ensure that there are product vectors in the range or in the kernel. The following lemma states that we can always find product vectors in any subspace of dimension \(\geq N\).

Lemma 5: Any subspace \(H \subset \mathcal{F}^2 \otimes \mathcal{F}_N^N\) with \(\dim(H) = M > N\) contains at least one product vector. For \(M > N\) it contains at least one product vector \(|e, f\rangle\).

\begin{proof}
Let us denote by \(|\Psi_i\rangle, i = 1, \ldots, 2N-M\) an orthonormal basis in the orthogonal complement of \(H\). We write

\[|\Psi_i\rangle = \sum_{k=1}^N A_{i,k}|0, k\rangle + B_{i,k}|1, k\rangle,\]

and \(A^1A + B^1B = 1\). We look for product vectors in \(H\) of the form \(|\Phi\rangle = (a|0\rangle + |1\rangle) \otimes \sum f_k|k\rangle\), i.e.

\[(aA + B)f = 0.\]

If \(M > N\) we have more variables than equations, and therefore there is a solution for all \(a\), and in particular for \(a\) real. For \(M = N\) we have that there is always a solution since \(\det(aA + B)\) is a polynomial of degree \(N\) in \(a\).

From the results that we have obtained so far, we have the following

Theorem 1: If \(r(\rho) = N\), and \(K(\rho)\) does not contain any pair \(|e, f\rangle, |e^*, f\rangle|\) i.e. \(R(\rho)\) is not supported on \(\mathcal{F}^2 \otimes \mathcal{F}_N^{-1}\), then \(\rho\) is separable.

\begin{proof}
We use induction. For \(N = 1\) the statement is true. Let us assume that it is true for \(N - 1\). If \(r(\rho) = N\), then \(K(\rho)\) has dimension \(N\), and according to Lemma 5 it contains a product vector \(|e, f\rangle\). Lemma 3 tells us that \(\rho(e, f) = |e, g\rangle\), with \(|g\rangle \neq 0\) and from Lemma 4 we obtain \(\rho = \rho_1 + \rho_2\), where \(\rho_1 \geq 0\) supported on \(\mathcal{F}^2 \otimes \mathcal{F}_N^{-1}\) and \(r(\rho_1) = N - 1\). Now we show that \(\rho_1\) cannot be supported on \(\mathcal{F}^2 \otimes \mathcal{F}_N^{-2}\), which would complete the proof. If it were, there would be two vectors \(|e, h\rangle, |e, h\rangle \in K(\rho_1)\).

Using that \(\rho|e, h\rangle = c|e, h\rangle\) we would have \(\rho(e, h) = 0\) and \(\rho(e, h) = c|e, h\rangle\) where \(c\) is a constant. If we define \(|f'\rangle = c|f\rangle - \langle h|\rangle\) we have that \(|e, f'\rangle, |e, f'\rangle \in K(\rho_1)\), contrary to our assumption.
\end{proof}

Now, we turn to the case \(\rho = \rho T^a\), and therefore specialize our previous results to this case. In particular, we formulate a result similar to Lemma 4.

Lemma 4b: If \(\rho = \rho T^a\) and \(|e, f\rangle \in K(\rho)\) then we can write \(\rho = \rho_{N-1} + \rho_s\), where \(\rho_s\) is separable, \(\rho_{N-1}\) is supported on \(\mathcal{F}^2 \otimes \mathcal{F}_N^{-1}\) and \(\rho_{N-1} = \rho_{N-1}\).

\begin{proof}
First, according to Lemma 2 we have that \(|e^*, f\rangle \in K(\rho T^a) = K(\rho)\) and therefore we obtain that...
\(|e, f\rangle \in K(\rho)\) with \(|e_i\rangle = |e\rangle + |e^*\rangle|/i\). Now, using the vector \(|e_i\rangle\) instead of \(|e\rangle\) we can derive the results of Lemmas 3 and 4a but with real vectors \(|e_i\rangle\). Thus, according to \([\text{Lemma 5}]\) we have \(r_1^{TA} = r^{TA} - ((g|f)^{-1}|\tilde{e}^*_i, g\rangle|\tilde{e}^*_i, g\rangle = r_1.\square\)

We are not at the position of proving the main result of this paper:

**Theorem 2:** If \(\rho = r^{TA}\) then \(\rho\) is separable.

**Proof:** We prove it by induction. First, in the case \(2 \times 1\) (i.e. \(N = 1\)) it is obviously true. Now, let us assume that it is valid for the case \((2 \times (N-1))\) and let us prove that then it is also valid for the case \(2 \times N\). In order to do that, we will show that any density operator \(\rho\) can be decomposed as \(\rho = \rho_1 + \rho_s\) where \(\rho_s\) is separable and \(\rho_1 = r_1^{TA}\) is supported on \(\mathcal{G}^2 \otimes I^{N-1}\) and therefore is also separable according to our assumption. We consider two cases: (i) \(r(\rho) \leq N\): we have \(\dim[K(\rho)] \geq N\); using Lemma 5, there is a product vector in \(K(\rho)\), so that according to Lemma 4b we have obtained the desired decomposition. (ii) \(r(\rho) > N\): according to Lemma 5 there is a product vector \(|e_i, g\rangle\) with \(|e_i\rangle = |e^*_i\rangle\). Using Corollary 1 we can use this product vector to reduce the rank of \(\rho\). But since \(|e_i, g\rangle = |e^*_i, g\rangle\) we have that the resulting operator is also equal to its partial transpose. We can proceed in this way until \(r(\rho) = N\) which corresponds to the case (i), and therefore we complete the proof.\square

Note that we could have chosen a different basis for the partial transposition. Taking into account that for any symmetric unitary operator we can always find a basis in which partial transposition is related to the original one by that such operator, we have

**Corollary 2:** If \(\rho = (U_A \otimes 1)\rho^T(U_A \otimes 1)\) for some unitary operator \(U_A = U_A^{TA}\) then \(\rho\) is separable \([\text{Lemma 6}]\).

Theorem 2 suggests that if \(\rho\) is not very different from \(\rho^T_2\), then it should also be separable. Indeed, one can construct a powerful sufficient separability condition based on that theorem. We will first introduce a lemma that generalizes Lemma 1 and that is useful to determine whether the difference of two positive operators is positive. We will use the operator norm which is defined as usual \(\|A\| = \max \|A|x\|\) with \(\|x\| = 1\).

**Lemma 6:** Given two hermitian operators \(X, Y \geq 0, X - Y \geq 0\) iff \(Y\) is supported on \(R(X)\) and \(\|X^{1/2}X^{-1/2}\| \leq 1\).

**Proof:** If \(Y\) is supported on \(R(X)\) we have that \(\exists|\phi\rangle \in R(Y), X|\phi\rangle\) and therefore \(\langle \phi|Y|\phi\rangle = \langle \phi|X|\phi\rangle < 0\). On the contrary, if \(Y\) is supported on \(R(X)\) we have that \(X - Y \geq 0\) iff \(\forall|\phi\rangle \in R(X)\) we have \(\langle \phi|Y|\phi\rangle/\langle \phi|X|\phi\rangle \leq 1\). Denoting by \(|\psi\rangle = X^{1/2}|\phi\rangle\) we obtain that \(X - Y \geq 0\) iff

\[
1 \geq \max_{|\psi\rangle} \frac{|\langle \psi|X^{-1/2}YX^{-1/2}|\psi\rangle|^2}{|\langle \psi|\psi\rangle|^2} = \|X^{1/2}X^{-1/2}\|^2. \square (7)
\]

In order to give the sufficient condition for separability, we have to introduce some definitions. In \(2 \times N\) we can always write \(\rho\) as

\[
\rho = \frac{\rho + \rho^{TA}}{2} + \frac{\rho - \rho^{TA}}{2} = \rho_s + \sigma_g^A \otimes B, \quad (8)
\]

where \(2\rho_s = \rho + \rho^{TA}, \sigma_g^A = i(|0\rangle_A|1\rangle_A - |1\rangle_A|0\rangle_A),\) and \(2B = 2B^I = tr_A[\sigma_g^A(\rho - \rho^{TA})]\). This operator \(B\) can be decomposed as

\[
B = \sum_{i=1}^{K} \lambda_i |v_i\rangle\langle v_i| . \quad (9)
\]

In particular, one of such is the spectral decomposition. Given one of such decomposition \(\{\lambda_i, |v_i\rangle\}_{i=1}^{K}\) and a set of real numbers \(\{a_i\}_{i=1}^{K}\) we define the operator

\[
C(a, \lambda, v) = \sum_{i=1}^{K} |\lambda_i| |a_i^2| |0\rangle\langle 0| + a_i^{-2} |1\rangle\langle 1| \otimes |v_i\rangle\langle v_i| , \quad (10)
\]

which is obviously positive. We have:

**Theorem 3:** Given a decomposition of \(B\) \(\{\lambda_i, |v_i\rangle\}_{i=1}^{K}\) and a set of real numbers \(\{a_i\}_{i=1}^{K}\), if \(\|C^{1/2}(a, \lambda, v)\rho_s^{-1/2}\| \leq 1\), then \(\rho\) is separable.

**Proof:** We define \(\rho_s = \rho_s - C(a, \lambda, v) = \tilde{\rho}_s^{TA} \geq 0\) according to Lemma 6. Using Theorem 2 we have that \(\tilde{\rho}_s\) is separable. Let \(|w_i\rangle = a_i|0\rangle - ia_i^{-1}\langle \text{sign}(\lambda_i)|1\rangle\). Then, it is easy to check that

\[
\rho = \tilde{\rho}_s + \sum_{i=1}^{K} |\lambda_i| |w_i, v_i\rangle\langle w_i, v_i|. \quad (11)
\]

which shows that \(\rho\) is separable.\square

Thus, we can show that a density operator is positive if we can find a decomposition of \(B\) and a set of real numbers that fulfill certain conditions. In particular we can take the spectral decomposition of \(B\) and \(a_i = 1\). Using the fact that \(\|AB\| \leq \|A\||B\|\) one can easily prove the following:

**Corollary 3:** If \(\rho + \rho^{TA}\) is of full rank and \(\|\rho + \rho^{TA}\|^{-1}\|\rho - \rho^{TA}\| \leq 1\), then \(\rho\) is separable. This corollary implies that if \(\rho\) is full rank and is very close to \(\rho^{TA}\) then it is separable.

The results introduced in the first part of this Letter also allow to prove Peres criterion \([\text{Lemma 5}]\) for separability in the \(2 \times 2\) case. In particular, in the following we will show that in these systems if \(\rho^{TA} \geq 0\) then \(\rho\) is separable. Note that the converse can be easily proved \([\text{Lemma 5}]\).

**Theorem 4:** (Peres, \([10]\)). If \(\rho^{TA} \geq 0\) are operators supported on \(\mathcal{G}^2 \otimes \mathcal{G}^2\), then \(\rho\) is separable.

**Proof:** We just have to consider the case \(\dim[K(\rho)] = \dim[K(\rho^{TA})] = 1\) in which both \(K(\rho)\) and \(K(\rho^{TA})\) contain no product vector. The reason is that: (i) if \(K(\rho)\) or \(K(\rho^{TA})\) contain a product vector then using Lemma 4a we can reduce the problem to the \(2 \times 1\) case, and therefore \(\rho\) is separable; (ii) If \(\dim[K(\rho)] = 2\) then according to Lemma 5 there is a product vector in \(K(\rho)\), so that we are back in (i) and therefore \(\rho\) is separable; (iii) similarly, if \(\dim[K(\rho^{TA})] = 2\) we conclude that \(\rho^{TA}\) and therefore \(\rho\)
are separable; (iv) If \( \dim K(\rho^{T_A}) = 0 \) and \( \dim K(\rho) < 2 \) then according to Lemma 5 \( R(\rho) \) contains at least one product vector \( |e, f\rangle \) and obviously \( |e^*, f\rangle \in R(\rho^{T_A}) \); we can apply Corollary 1 and subtract product vectors from \( \rho \) until \( \dim K(\rho) = 2 \) in which case \( \rho \) is separable, or \( \dim K(\rho) = \dim [K(\rho^{T_A})] = 1 \); (v) Similarly If \( \dim K(\rho) = 0 \) and \( \dim K(\rho^{T_A}) < 2 \) we arrive at the same conclusion. Thus, let us assume that \( \rho|\Psi_1\rangle = 0 \) and \( \rho^{T_A}|\Psi_2\rangle = 0 \), where \( |\Psi_1,\Psi_2\rangle \) are of the form \( |\hat{f}, \hat{g}\rangle \) and they are not product vectors. We will first show that these vectors can always be written as

\[
|\Psi_1\rangle \propto |e, f\rangle - |\hat{e}, g\rangle, \quad (12a)
\]
\[
|\Psi_2\rangle \propto |e^*, f\rangle, \quad (12b)
\]

where both \( \{|\hat{f}, \hat{g}\rangle \} \) are linearly independent since otherwise \( \{|\Psi_1\rangle, |\Psi_2\rangle\} \) would be a product vector. Then we will show that either \( \rho = \rho^{T_A} \) or \( |h\rangle = k^2|g\rangle \) where \( k^2 > 0 \). In the first case, according to Theorem 2, \( \rho \) is separable. In the second case, we have that if we define \( |a\rangle = |e\rangle + k|\hat{e}\rangle \) and \( |b\rangle = (\hat{f}|\hat{g}\rangle |f\rangle - k|g\rangle |\hat{g}\rangle \rangle \) then the vector \( |a, b\rangle \in R(\rho) \) and \( |a^*, b\rangle \in R(\rho^{T_A}) \), since they are orthogonal to \( |\Psi_{1,2}\rangle \) respectively. Thus, according to Corollary 1 we can subtract this product vector and increase the dimension of either \( K(\rho) \) or \( K(\rho^{T_A}) \) to 2, which as we showed in (ii) and (iii) implies that \( \rho \) is separable.

We show how to obtain the decomposition \([12]\). We look for two unnormalized vector \( |e\rangle = 0|0\rangle + |1\rangle \) and \( |\hat{f}\rangle = f_0|0\rangle + f_1|1\rangle \) such that \( \langle e, f|\Psi_1\rangle = \langle e^*, f|\Psi_2\rangle = 0 \). We obtain two linear equations for \( f_0 \) and \( f_1 \) of the form \( C(\alpha)\hat{f} = 0 \), where \( C \) is a matrix that depends linearly on \( \alpha \) and on \( \alpha^* \) \([4]\). These equations always have a solution for some given \( \alpha \) since the condition \( \det[C(\alpha)] = 0 \) is a second order equation for \( \alpha \).

We finally show that either \( \rho = \rho^{T_A} \) or \( |h\rangle = k^2|g\rangle \) where \( k^2 > 0 \). First, we show that

\[
\langle f|\rho|f\rangle = \langle g|\rho|h\rangle = \langle h|\rho|g\rangle. \quad (13)
\]

To obtain that we use the fact that the vectors \( |\Psi\rangle \) are in the kernel of \( \rho \) or \( \rho^{T_A} \), and therefore \( \langle e, f|\rho|f\rangle = \langle e, \hat{g}|\rho|g\rangle \) and \( \rho^{T_A}|e^*, h\rangle = \rho^{T_A}|\hat{e}, f\rangle \). Using \([8]\) we can write the second equation as \( \langle e|\rho|h\rangle = \langle \hat{e}|\rho|f\rangle \). Using these equations it is easy to obtain that \( \langle e, f|\rho|f\rangle = \langle e, g|\rho|h\rangle \) and \( \langle \hat{e}, \hat{f}|\rho|f\rangle = \langle \hat{e}, g|\rho|h\rangle \) which automatically proves the statement \([13]\). Now, given that \( |f\rangle \) and \( |g\rangle \) are linearly independent vectors, we can write \( |h\rangle = \alpha|f\rangle + \beta|g\rangle \) with \( \beta \neq 0 \). If \( \alpha = 0 \), then using \([13]\) we have that \( \beta \equiv k^2 > 0 \) is real and positive. If \( \alpha \neq 0 \) we have that both \( \langle g|\rho|f\rangle \) and \( \langle f|\rho|g\rangle \) can be expressed as linear combinations of \( \langle f|\rho|f\rangle \) and \( \langle g|\rho|g\rangle \). These two last operators are hermitian and they are equal to their transposes in some given basis (if we write \( \langle f|\rho|f\rangle \propto 1 + \vec{n}_f \sigma \) and \( \langle g|\rho|g\rangle \propto 1 + \vec{n}_g \sigma \) then we just have to use the basis such that both \( \vec{n}_f \) and \( \vec{n}_g \) lie in the \( x - z \) plane). Thus, \( g|\rho|f\rangle \) and \( f|\rho|g\rangle \) are also equal to their transposes in that basis. Since \( |f\rangle \) and \( |g\rangle \) are linearly independent we immediately arrive to the conclusion that if we consider the partial transpose in that basis we have that \( \rho = \rho^{T_A} \), which completes the proof. \( \square \)

Summarizing, we have demonstrated that all density operators on \( 2 \times N \) that remain invariant after partial transposition with respect to the first system are separable. Using this fact we have constructed a sufficient separability condition for such systems. We have also given a relatively simple proof of Peres criterion based on these results.

This work has been supported by Deutsche Forschungsgemeinschaft (SFB 407), the Österreichischer Fonds zur Fördierung der wissenschaftlichen Forschung (SFB F15), and the European TMR network ERB-FMRX-CT96-0087. J. I. C. thanks the University of Hannover for hospitality. We thank A. Sanpera and G. Vidal for fruitful discussions.

[1] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[2] See, for example, A. Peres, Quantum Theory: concepts and methods, (Kluwer Academic Publishers, Dordrecht, 1993).
[3] See, for example, D. P. DiVincenzo, Science 270, 255 (1995).
[4] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[5] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A223, 8 (1996).
[6] E. Stromer, Acta Math. 110, 233 (1963); S. L. Woronowicz, Rep. Math. Phys. 10, 165 (1976).
[7] P. Horodecki, Phys. Lett. A323, 333 (1997); M. Horodecki, P. Horodecki, R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).
[8] C. H. Bennett et al., quant-ph/9808038; B. Terhal, quant-ph/9810003.
[9] C. H. Bennett et al, quant-ph/9804053.
[10] R. Horodecki, M. Horodecki, and P. Horodecki, quant-ph/9811009.
[11] There exist very few sufficient conditions for separability in systems other than \( 2 \times 2 \) and \( 2 \times 3 \): G. Vidal and R. Tarrach, quant-ph/9806074; S. Braunstein et al., quant-ph/9811018.
[12] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. 80, 2261 (1998); A. Sanpera, R. Tarrach, and G. Vidal, Phys. Rev. A58, 826 (1998).
[13] This corollary can be generalized using arbitrary nonsingular matrices instead of unitary ones.
[14] Note that if coefficients depend on \( \alpha^* \) as well it might happen that the second order polynomial equation does not have solutions. This is the reason why we cannot make use of Lemma 5 directly to show that there exists a vector \( |a, b\rangle \in R(\rho) \) and \( |a^*, b\rangle \in R(\rho^{T_A}) \).