Supersymmetry (SUSY) in quantum mechanics is extended from square-integrable states to those satisfying the outgoing-wave boundary condition, in a Klein–Gordon formulation. This boundary condition allows both the usual normal modes and quasinormal modes with complex eigenvalues $\omega$. The simple generalization leads to three features: the counting of eigenstates under SUSY becomes more systematic; the linear-space structure of outgoing waves (nontrivially different from the usual Hilbert space of square-integrable states) is preserved by SUSY; and multiple states at the same frequency (not allowed for normal modes) are also preserved. The existence or otherwise of SUSY partners is furthermore relevant to the question of inversion: are open systems uniquely determined by their complex outgoing-wave spectra?

### I. INTRODUCTION

#### A. Outline

Supersymmetry (SUSY) in quantum field theory relates bosons to fermions. Its analog in quantum mechanics is interesting in its own right, and relates two (typically one-dimensional) Hamiltonians $H$ and $\tilde{H}$ with the same spectrum of normal modes (NMs), or the same spectrum apart from one state. The classical limit relates a one-parameter family of Hamiltonians $H(\xi)$.

This paper generalizes SUSY in quantum mechanics to open systems and in particular to their quasinormal modes (QNMs), which are eigenfunctions satisfying the outgoing-wave condition (OWC) at infinity. The generalization itself is straightforward, and we focus on the novel features that ensue. After a brief introduction to open systems and QNMs in the rest of this Section, this paper presents three main results.

First, in the conventional discussion for the Schrödinger equation, spectrum preservation can be conveniently expressed in terms of the number of NMs $n(E)$ for $H$ and $\tilde{n}(E)$ for $\tilde{H}$ at the same energy $E$. Namely that $n(E) = \tilde{n}(E)$ for all real $E$ except a privileged value $E_0$ (usually chosen to be $E_0 = 0$). At this value, the difference is given by the Witten index $\Delta$.

$$\Delta(E_0) = n(E_0) - \tilde{n}(E_0)$$  \hspace{1cm} (1.1)

which can be $+1$, $0$, or $-1$. For open systems with the OWC, time-reversal invariance is broken, and it is appropriate to consider a Klein–Gordon equation (KGE) instead — in effect replacing $\partial_t \rightarrow \partial_t^2$ or $E \rightarrow \omega^2$ in the time-independent equation, and distinguishing $+\omega$ from $-\omega$ (since reversing the two converts an outgoing to an incoming wave). Section I B shows that the equality of spectrum for the KGE extends to the complex $\omega$-plane except at the two privileged frequencies $\pm\Omega = \pm\sqrt{E_0}$, namely

$$n(\omega) = \tilde{n}(\omega), \quad \omega \neq \pm\Omega.$$  \hspace{1cm} (1.2)

At $\pm\Omega$, one needs to consider

$$\Delta(\pm\Omega) = n(\pm\Omega) - \tilde{n}(\pm\Omega).$$  \hspace{1cm} (1.3)

These are again $+1$, 0, or $-1$, but with $\Delta(\Omega) = -\Delta(-\Omega)$ determined by the asymptotic behavior of the SUSY generator, to be defined below. In other words, under $H \mapsto \tilde{H}$, if a state is removed (added) at $\Omega$, then a state is added (removed) at $-\Omega$. This relationship, which also applies to conservative systems and NMs provided we take the KGE point of view, sharpens the information provided by (1.1).

Secondly, SUSY preserves norms and inner products. However, for outgoing waves the usual norms and inner products are not useful. For example, an outgoing wave of frequency $\omega$ goes as $\exp[\pm i\omega(x - t)]$ at spatial infinity. With $\text{Im} \omega < 0$ for QNMs (see Section II B), the exponential growth in $|x|$ renders the wavefunction not normalizable in the usual sense. A generalized norm for QNMs was first introduced by Zeldovich many years ago, and shown to be useful for time-independent perturbation theory (of the complex eigenvalues). An associated generalized inner product can also be defined. The time-evolution operator turns out to be symmetric under this product (the analog of self-adjoint).
shows that SUSY preserves these generalized norms and inner products — a pleasant surprise, since their construction is totally unrelated to SUSY.

Thirdly, for non-conservative systems, there is no guarantee that the Hamiltonian can be completely diagonalized; in general the best that one can do is to decompose it into Jordan blocks (JBs) \[10\]. Each block \(j\), say of size \(M_j \times M_j\), is associated with an eigenfrequency \(\omega_j\), with \(M_j = 1\) being the usual case of a QNM. Section \[V\] shows that SUSY preserves the JB structure: except for \(\omega_j = \pm \Omega\), a block of size \(M_j\) maps to a block also of size \(M_j\) at the same frequency \(\omega_j\). In fact, if we generalize the definition of \(n(\omega_j)\) to be the order \(M_j\) of the block, then the relationship between \(n(\omega)\) and \(\tilde{n}(\omega)\) [cf. (1.2) and (1.3)] remains valid even for \(n, \tilde{n} > 1\).

Examples are given in Section \[V\] and a discussion is presented in Section \[VI\] including a sketch of some issues for potentials with tails — which can lead to situations with negative \(n\) and \(\tilde{n}\) that are nevertheless accommodated in the same formalism.

### B. Quasinormal modes

In open systems, waves are not confined, but can escape: acoustic waves from a musical instrument, electromagnetic waves from a laser, and linearized gravitational waves from a Schwarzschild background (to infinity and into the horizon). These systems are often described (e.g., in the case of gravitational waves \[11\]) by the KGE

\[
\left[\partial_t^2 - \partial_x^2 + V(x)\right] \phi(x,t) = 0 \quad ,
\]

or (e.g., in the case of optics \[12\]) by the closely related equation

\[
\left[\rho(x)\partial_t^2 - \partial_x^2\right] \phi(x,t) = 0 \quad .
\]

This paper deals only with the KGE, both because it is readily related to the Schrödinger equation in terms of which SUSY is usually formulated \[4\], and also because it (unlike the wave equation) admits NMs which are interesting in the present discussion.

Except for Section \[VII\] we shall assume that \(V(x)\) [or \(\rho(x) - 1\) in the case of the wave equation \(1.3\)] has finite support on the interval \(I = [-a, a]\), which is natural for describing a system of limited extent, surrounded by a trivial medium such as vacuum.

We assume that the loss is only due to the boundary conditions. In particular, the potential \(V\) is real. Absorption may be described by a complex \(V\), but causality then requires dispersion; the necessary generalization \[13\] will not be discussed here.

Among the solutions of (1.4), we consider the space \(\Gamma\) of states satisfying the OWC. We leave the time-domain definition to Section \[IV\]; in the frequency domain, a function is in \(\Gamma\) if:

\[
\phi(x) \sim e^{\pm i\omega |x|} \quad , \quad |x| \to \infty \quad .
\]

Because \(V\) is trivial outside \(I\), the asymptotic conditions \(1.4\) can be stated at \(x = \pm a\) instead:

\[
\frac{\phi'(x)}{\phi(x)} = \pm i\omega \quad , \quad x = \pm a \quad .
\]

The imposition of two boundary conditions in (1.7) forces the eigenvalues to be discrete, and these fall into two classes. First, there could be bound states or NMs \[14\]; these must (from the Schrödinger point of view) have \(E = \omega^2 < 0\), and hence \(\omega\) is purely imaginary. Since bound-state wavefunctions vanish at infinity, (1.6) dictates that \(\Im\omega > 0\). Second, there could be QNMs with complex eigenvalues \(\omega^2\). Because these waves escape, \(\phi\) decreases, so \(\Im\omega < 0\) \[15\]. Provided \(\Re\omega \neq 0\), they occur in pairs: \(\omega_j = -\omega'_j\), as is readily shown by conjugating the defining equation and boundary conditions. Those with \(\Re\omega = 0\) need not be paired; these zero modes \[16\] will be of particular importance below.

Consider the potential shown by the broken line in Figure 1(a); its NMs and QNMs are shown in Figure 1(b). In this example, there is one NM (triangle) and a sequence of QNMs (crosses), including a zero mode. In contrast to the Schrödinger formulation, the use of the KGE and the introduction of \(\Gamma\) allows NMs and QNMs to be discussed together — and at least in this example the latter manifestly carry much richer information.

Even though QNM eigenfunctions are not square-integrable and do not form a conventional Hilbert space, they are useful for analyzing outgoing waves. Importantly, the complex QNM frequencies are often directly observable: e.g., the central frequency and width of an optical line observed from a laser cavity, or the rates of repetition and decay of a gravitational-wave signal that may within the next decade be detected by instruments such as LIGO \[17\]. In fact, the spectrum is often so rich they are useful for analyzing outgoing waves. Importantly, the complex QNM frequencies are often directly observable: e.g., the central frequency and width of an optical line observed from a laser cavity, or the rates of repetition and decay of a gravitational-wave signal that may within the next decade be detected by instruments such as LIGO \[17\]. In fact, the spectrum is often so rich they are useful for analyzing outgoing waves. Importantly, the complex QNM frequencies are often directly observable: e.g., the central frequency and width of an optical line observed from a laser cavity, or the rates of repetition and decay of a gravitational-wave signal that may within the next decade be detected by instruments such as LIGO \[17\].

In cases where the \(\phi_j\)’s are not complete, it may still be possible to characterize the remainder, which could be, for example, a power law in \(t\) for long times \[20\]. Moreover, when the eigenstates are complete, one can set up a formalism that closely parallels the conservative case (see Refs. \[12\] and Section \[III\]). One can even second-quantize using these eigenstates as a basis (e.g., to discuss thermal effects and atom–field interactions in an optical cavity) \[23\]. These developments have been reviewed \[24\]. We shall see that much of the mathematical structure is preserved under SUSY.
II. FORMALISM

A. Supersymmetric quantum mechanics

In this paper we consider SUSY in the one-dimensional KGE (1.4), and especially in the corresponding eigen-value problem

\[ H \phi_j(x) = \omega_j^2 \phi_j(x) \quad , \quad (2.1) \]

where

\[ H = -\partial_x^2 + V(x) \quad . \quad (2.2) \]

The boundary conditions will be specified later. In so far as the interest centers on the time-independent problem (2.1) and the spectrum, the Schrödinger equation, to which reference is usually made (3,4), is included if we simply relabel \( \omega^2 \mapsto E \).

If there exists another system

\[ \tilde{H} = -\partial_x^2 + \tilde{V}(x) \quad , \quad (2.3) \]

with the same spectrum (or the same spectrum apart from one state), and moreover if the states in the two systems are related by

\[ \tilde{\phi}(x) = A \phi(x) \quad , \quad (2.4) \]

where

\[ A = \partial_x + W(x) \quad , \quad -A^\dagger = \partial_x - W(x) \quad , \quad (2.5) \]

then the two systems are said to be SUSY partners. In particular, if \( \phi_j(x) \) is an eigenfunction of \( H \), then \( \tilde{\phi}_j(x) \) (provided it does not vanish) is an eigenfunction of \( \tilde{H} \) with the same eigenvalue. Normalization is deferred to Section II.

In order for (2.4) to preserve the spectrum, one needs

\[ AH = \tilde{H} A \quad , \quad (2.6) \]

from which it follows that

\[ V(x) = W(x)^2 - W'(x) + \Omega^2 \quad , \quad (2.7) \]

\[ \tilde{V}(x) = W(x)^2 + W'(x) + \Omega^2 \quad , \quad (2.8) \]

with \( W(x) \) (called the SUSY potential) as in (2.3) and for some constant \( \Omega^2 \). Since both \( V \) and \( \tilde{V} \) have to be real, \( W \) and \( \Omega^2 \) are also real. Moreover, the Hamiltonians can be represented as

\[ H = A^\dagger A + \Omega^2 \quad , \quad (2.9) \]

\[ \tilde{H} = AA^\dagger + \Omega^2 \quad . \quad (2.10) \]

The two partner systems can be put into one linear space by introducing Pauli spinors, with \( H \) and \( \tilde{H} \) associated with \( 1 \pm \sigma_z \) and \( A, A^\dagger \) associated with \( \sigma_x \).

Upon reversing the sign of \( W \), (a) \( V \leftrightarrow \tilde{V} \) [cf. (2.7)], and (b) \( A \leftrightarrow -A^\dagger \) [cf. (2.5)]; thus the mapping from \( H \) back to \( H \) is (up to a sign) achieved by \( A^\dagger \). Note however that the mapping is the “inverse” only in a loose sense: \( A^\dagger A \) is not the identity but \( H - \Omega^2 \) [cf. (2.8)].

We may regard (2.7) as a Riccati equation for \( W \) in terms of the given \( V \). For \( |x| > a \), both \( V \) and \( \tilde{V} \) vanish, so \( W^2 = -\Omega^2 \). The first-order Riccati equation for \( W \) can satisfy two boundary conditions (at \( x = \pm a \)) only at special values of \( \Omega^2 \); this condition becomes familiar if we define a generator \( \Phi(x) \) by

\[ W(x) = -\frac{\Phi'(x)}{\Phi(x)} \quad . \quad (2.9) \]

Then (2.7) implies

\[ H\Phi(x) = \Omega^2 \Phi(x) \quad . \quad (2.10) \]

B. Boundary conditions

All the above may be regarded as a review of the familiar SUSY formalism for the Schrödinger equation \([3,4]\) if we put \( E = \omega^2 \), \( E_0 = \Omega^2 \) and in particular shift \( V(x) \) so that \( E_0 = 0 \). Conventionally the discussion refers to wavefunctions which vanish at infinity (or, more precisely, are square-integrable). Here we consider all eigenfunctions in \( \Gamma \), including both NMs and QNMs, with the former in the upper and the latter in the lower half of the frequency plane.

We should check immediately that \( \phi \in \Gamma \) implies \( \tilde{\phi} \in \Gamma \). For \( x > a \), if \( \phi(x) = Ce^{i\omega x} \) then \( \tilde{\phi}(x) = A\phi(x) = (i\omega + W_+)Ce^{i\omega x} \), where

\[ W(x=\pm a) = W_\pm \quad (2.11) \]

are the constant values for \( x > a \) and \( x < -a \) respectively. Thus \( \phi \) and \( \tilde{\phi} \) always satisfy the same type of boundary conditions, and the number of eigenstates in \( \Gamma \) is preserved under SUSY: \( n(\omega) = \tilde{n}(\omega) \) [cf. (1.2)] — except when \( A \) or \( A^\dagger \) destroys a state, to be discussed below.

C. Generator

The various SUSY transformations are related, in a one-to-one manner, to solutions of (2.10) for the generator \( \Phi \). First, suppose \( \Omega^2 > 0 \), so that \( \Omega \) is real. Then outside \( I \), \( \Phi \) is oscillatory: either complex (e.g., \( e^{i\Omega x} \)), inadmissible since it leads to a complex \( W \); or real (e.g., \( \sin \Omega x \)), inadmissible since its nodes lead to singularities in \( W \). Thus, \( \Omega^2 < 0 \), and we denote \( K \equiv |\Omega| \).

At each spatial extreme (\( |x| > a \)), \( \Phi \) is in general a sum of increasing and decreasing functions, i.e.,
\[ \Phi(x) = ce^{K|x|} + de^{-K|x|} \]  
(2.12)

If both \(c, d \neq 0\) (to be called the mixed type), then the logarithmic derivative is (e.g., for \(x > 0\))

\[ W(x) = -K + \frac{2iK}{c}e^{-2Kx} + \cdots \]  
(2.13)

so that \(\tilde{V} = V + 2W'\) acquires an exponential tail. (In the special case \(\Omega = 0\), the tail is not exponential but asymptotically inverse-square.) Thus, if we insist that \(\tilde{V}\) also has finite support, the mixed type is not allowed and at each extreme \(\Phi\) must be either purely decreasing (\(\Phi \propto e^{-K|x|}\), denoted as D) or purely increasing (\(\Phi \propto e^{K|x|}\), denoted as I). Outside I, the logarithmic derivative is then exactly \(\pm K\), so \(W' = 0\), implying \(\tilde{V} = 0\). When both extremes are considered together, \(\Phi\) must be one of three types, conveniently labelled with the parameter

\[ \chi = \frac{1}{2}[\text{sign}(W_+) - \text{sign}(W_-)] \]  
(2.14)

where \(\chi = +1, -1, 0\) respectively for the DD, II, and DI/ID cases (in obvious notation). In the DI/ID case, the generator is purely incoming from one extreme and purely outgoing to the other, hence is a total-transmission mode (TTM). The relaxation to allow exponential tails for \(V\) and/or \(\tilde{V}\) will be briefly mentioned in Section 3.1.

The generator is annihilated by SUSY: \(A\Phi = 0\), trivially from (2.3) and (2.4). Furthermore, since reversing the sign of \(W\) interchanges the partners [cf. below (2.5)], in view of (2.9) the transformation from \(\tilde{H}\) back to \(H\) is generated by \(\tilde{\Phi} = \Phi^{-1}\); this is guaranteed to be an eigenfunction of \(H\), also with eigenvalue \(\Omega^2\). (Despite the notation, \(\tilde{\Phi}\) is not the SUSY partner of \(\Phi\): \(\tilde{\Phi} \neq A\Phi\)\).

The boundary conditions for \(\tilde{\Phi} = \Phi^{-1}\) interchange D and I, so the reverse transformation is characterized by \(\tilde{\chi} = -\chi\).

These observations allow a simple statement of the changes in the number of states when \(H \mapsto \tilde{H}\). If \(\chi = 1\) [cf. below (2.14)], an NM \(\tilde{\Phi}\) is destroyed at \(\tilde{\Omega} = iK\) \([\Delta(iK) = 1]\) and a QNM \(\tilde{\Phi}\) is created at \(\tilde{\Omega} = -iK\) \([\Delta(-iK) = 1]\). If \(\chi = -1\), a QNM \(\tilde{\Phi}\) is destroyed at \(\tilde{\Omega} = iK\) \([\Delta(-iK) = 1]\) and an NM \(\tilde{\Phi}\) is created at \(\tilde{\Omega} = -iK\) \([\Delta(iK) = 1]\). If \(\chi = 0\), no eigenstates of \(\tilde{\Gamma}\) are created or destroyed \([\Delta(iK) = \Delta(-iK) = 0]\), since \(\Phi\) and \(\tilde{\Phi}\) are TTM's rather than NMs or QNMs. Thus, all three cases satisfy

\[ \Delta(iK) = -\Delta(-iK) = \chi \]  
(2.15)

where we emphasize the convention \(\Omega = \pm iK\) with \(K > 0\). The cases \(\chi = \pm 1\) lead to Hamiltonians whose spectra in \(\Gamma\) differ by one state (said to be essentially isospectral), whereas the case \(\chi = 0\) leads to Hamiltonians whose spectra in \(\Gamma\) are identical (said to be strictly isospectral).

These remarks provide a more complete picture of the mapping of eigenstates in \(\Gamma\) under SUSY: states do not simply appear/disappear, but are mapped to the mirror point in the complex plane.

Not all NMs, QNMs or TTM's are eligible as the generator. First, \(\Omega^2\) must be real, which restricts QNMs to zero modes. Secondly, \(\tilde{\Phi}\) cannot have nodes, or else \(W\) would acquire singularities. In the case of NMs, this restricts \(\Phi\) to the ground state. For QNMs, nodes are not required by general theorems. At least for repulsive potentials, each eigenfunction can have at most one node or antinode; thus, for a symmetric repulsive \(V\), even-parity eigenfunctions can have no nodes. There is consequently much more freedom in choosing a QNM (as opposed to an NM) as the generator. Some general statements concerning nodes in QNMs are given in Appendix A.

III. ORTHONORMALITY

A. Orthonormality for NMs

For conservative systems, SUSY preserves orthonormality. There are two issues: orthogonality is preserved because the transformed NMs are eigenvectors of the self-adjoint operator \(H\); and normalization is preserved if the transformation is changed to

\[ \phi_j \mapsto \tilde{\phi}_j = N_j \tilde{\phi}_j = N_j A\phi_j \]  
(3.1)

with

\[ N_j^{-2} = \frac{\langle \tilde{\phi}_j | \tilde{\phi}_j \rangle}{\langle \phi_j | \phi_j \rangle} \]  
(3.2)

Eq. (3.1) applies to each eigenstate \(j\) other than the generator \(\Phi\) itself, namely the ground state. It is readily shown that \(N_j^{-2} = \omega_j^2 - \Omega^2\), a result that can also be read off as a special case of the derivation below for states in \(\Gamma\). Since \(\omega_j^2 - \Omega^2\) is the eigenvalue of \(A^\dagger A\) [see (2.8)], (3.1) can be written in the operator form

\[ \phi \mapsto \tilde{\phi} = A (A^\dagger A)^{-1/2} \phi \]  
(3.3)

valid for any state \(\phi\), not just frequency eigenfunctions. (This and similar formulas below are restricted to the subspace orthogonal to \(\Phi\).) This makes it formally easy to verify the preservation of inner products:

\[ \langle \tilde{\psi} | \tilde{\phi} \rangle = \langle \psi | (A^\dagger A)^{-1/2} A^\dagger A (A^\dagger A)^{-1/2} | \phi \rangle \]

\[ = \langle \psi | \phi \rangle \]  
(3.4)

However, when operating on a general wavefunction \(\phi\), the factor \((A^\dagger A)^{-1/2}\) can only be evaluated by projecting \(\phi\) onto the eigenfunctions, and scaling each component by \(N_j\). Thus, in practice, the significant result is the evaluation of this factor. We now generalize these concepts to states in \(\Gamma\), in particular QNMs.
B. Normalization and inner product for QNMs

It is necessary to digress and review the concepts of orthogonality and normalization for QNMs. The central issue is that with the OWC, \( H \) is not self-adjoint in the usual sense, and different QNMs are not orthogonal under the usual inner product. Likewise, the norm \( \int |\phi_j|^2 \, dx \) is divergent, since the wavefunction grows exponentially at infinity.

An appropriate normalizing factor for QNMs was first introduced by Zeldovich \[7\], and later generalized and applied to other situations \[3\], including models of linearized waves propagating on a Schwarzschild background \[25\]:

\[
(\phi_j, \phi_j) = 2\omega_j \int_{-a}^{a} \phi_j(x)^2 \, dx \\
+ i [\phi_j(-a)^2 + \phi_j(a)^2] .
\]  

(3.5)

This expression goes as \( \phi_j^2 \) rather than \( |\phi_j|^2 \), and is in general not real. The limits of the integral and the surface terms can be shifted from \( \pm a \) to any \( b_x \), where \( \pm b_x > a \), without affecting the value of (3.4). This definition also applies to NMs: the surface terms vanish if we take \( b_x \to \pm \infty \), recovering the conventional norm apart from a factor of \( 2\omega_j \). In the QNM case, (3.5) is the correct normalizing factor in the sense that, e.g., under a perturbation \( V \to V + \Delta V \), the complex QNM eigenvalues change by

\[
\Delta(\omega_j^2) = \int \frac{\phi_j(x)^2 \Delta V(x) \, dx}{(\phi_j, \phi_j)} .
\]  

(3.6)

Since one no longer has positivity, there is the possibility that \( (\phi_j, \phi_j) = 0 \). This exceptional case can be separately taken care of \[26\], and some interesting aspects are dealt with in Section IV.

To go beyond the normalizing factor and discuss an analog of orthogonality, one has to first regard each state with in Section IV. An appropriate bilinear map \[22\]

\[
(\psi, \phi) = i \left\{ \int_{-a}^{a} \left[ \psi^1(x) \phi^2(x) + \psi^2(x) \phi^1(x) \right] \, dx \\
+ \left[ \psi^1(-a) \phi^1(-a) + \psi^1(a) \phi^1(a) \right] \right\} ,
\]  

(3.8)

to take the place of the usual inner product. For an eigenfunction, \( (\phi_j, \phi_j) \) agrees with (3.5). The dynamics can be written in the first-order form \( i \partial_t \phi = \mathcal{H} \phi \), with

\[
\mathcal{H} = i \begin{pmatrix} 0 & 1 \\ -\omega_j^2 - V & 0 \end{pmatrix} .
\]  

(3.9)

Importantly, \( \mathcal{H} \) is symmetric:

\[
(\mathcal{H}\psi, \phi) = (\psi, \mathcal{H}\phi) ,
\]  

(3.10)

in the proof of which the surface terms generated upon integration by parts exactly cancel against those in (3.8). The relation (3.10) is the analog of self-adjointness, and leads to the usual proof that for two eigenvectors,

\[
(\phi_k, \phi_j) = 0
\]  

whenever \( \omega_k \neq \omega_j \). Provided that \( (\phi_j, \phi_j) \neq 0 \) \[20\], one can normalize these eigenfunctions in the usual way, i.e., by requiring (3.11) to be \( 2\omega_j \delta_{kj} \) in general [cf. (3.5) for this factor]. We henceforth refer to this property as orthonormality [and to (3.11) alone as orthogonality]. It also follows trivially that, provided this orthonormal system is complete (which is the case under fairly broad assumptions; see Section IV), time evolution is given by

\[
\phi(t) = \sum_j a_j \phi_j e^{-i\omega_j t} ,
\]  

(3.12)

generalizing (3.8) to two components, and

\[
a_j = \frac{(\phi_j, \phi(t=0))}{(\phi_j, \phi_j)} .
\]  

(3.13)

The preservation of orthonormality under SUSY should therefore be sought in terms of the bilinear map (3.8).

C. Normalized SUSY transformation for QNMs

We first present a derivation of orthonormality that does not explicitly require the two-component formalism. With orthogonality already guaranteed by (3.11), it remains to compute the normalizing factor

\[
(\tilde{\phi}_j, \tilde{\phi}_j) = 2\omega_j \int_{-a}^{a} \left[ (\partial_x + W)\phi_j \right]^2 \, dx \\
+ i \left[ \tilde{\phi}_j(-a)^2 + \tilde{\phi}_j(a)^2 \right] .
\]  

(3.14)

Integrate by parts to convert \( (\partial_x \phi)^2 \) to \( -(\partial_x^2 \phi) \phi \) plus a surface term, express the second derivative in terms of \( V - \omega_j^2 \) by means of the eigenvalue equation, and write the potential as \( V = W^2 - \Omega^2 + \Omega^2 \). Then, apart from a term \( \propto \omega_j^2 - \Omega^2 \), the integrand becomes a total derivative \( \partial_x (W\phi^2) \). Using \( W(\pm a)^2 = -\Omega^2 \) and \( \partial_x \phi_j(\pm a) = \pm i\omega_j \phi_j(\pm a) \) then leads to
\[
\frac{\partial_j^* - \partial_j}{(\partial_j^*, \partial_j)} = \omega_j^2 - \Omega^2.
\] (3.15)

Incidentally, the conservative case (nodal conditions at the ends of the interval [4]) is recovered by simply dropping all surface terms.

Since the ratio \(\frac{3.15}{\text{3.13}}\) is the eigenvalue of \(A^\dagger A\), we can again write the normalized transformation for each eigenfunction as \(3.3\).

**D. SUSY for two-component form**

For eigenstates, the two components are trivially related, but in order to perform SUSY transformations on general wavefunctions in \(\Gamma\) (e.g., given a time-dependent state, to find its partner at all times), the second component must be considered explicitly.

Since SUSY must commute with time evolution and \(\phi^2 = \partial_t \phi^3\), both components must transform in the same way. Thus, the (unnormalized) SUSY transformation on two-component vectors is \(A = \text{diag} (A, A)\), which satisfies

\[
(A\psi, \phi) = (\psi, A^\dagger \phi)\ ,
\] (3.16)

where \(A^\dagger \equiv \text{diag} (A^\dagger, A^\dagger)\). In deriving \(\text{3.15}\), one has to integrate by parts: \(\partial_x\) changes sign so that \(A\) turns into \(A^\dagger\). The surface terms are seen to work out by using, e.g., \(\phi^2(a) = \partial_t \phi^3(a) = -\partial_x \phi^3(a)\), and the known values of \(W_{\pm}\). Note that \(A^\dagger A = (H - \Omega^2)\mathbb{1}\), \(\mathbb{A}A^\dagger = (H - \Omega^2)\mathbb{1}\), i.e., the products do not relate to the two-component \(\mathcal{H}\).

With \(\text{3.16}\), it is straightforward to show that the normalized SUSY transformation

\[
\phi \mapsto \tilde{\phi} = A (A^\dagger A)^{-1/2} \phi\ ,
\] (3.17)

defined on the subspace orthogonal to \(\Phi\), preserves the bilinear map, in a manner that exactly parallels \(\text{3.4}\). In the exceptional case of SUSYs that generate a doubled state (see Section \(\text{IV}\)), the subspace has to exclude the two states on which \(H - \Omega^2\) vanishes.

The linear-space structure for open systems (e.g., the replacement of inner products by bilinear maps) has an intrinsic geometric meaning for all outgoing states, not just QNMs [27]. It is therefore pleasing that this structure is preserved by SUSY, a superficially unrelated concept.

**IV. JORDAN BLOCKS**

A key concept in SUSY is the preservation of the spectrum. However, dissipative systems (such as waves satisfying the OWC) admit a spectral property not found for conservative systems. In terms of the Wronskian \(J(\omega)\) to be defined below, this is exhibited as an \(M\)th-order zero \((M > 1)\). Such a multiple zero emerges naturally when \(M\) QNM eigenvalues coalesce as system parameters are tuned, so that \(M - 1\) eigenvectors are “lost” [28] and must be replaced by other degrees of freedom. Thus, the Hamiltonian cannot be written as a diagonal matrix in the (biorthogonal) basis of eigenstates, but can only be decomposed into (Jordan) blocks \([10]\) of size \(M \times M\). When this happens, \((\phi_j, \phi_j)\) will vanish for some \(j\), invalidating the formalism in Section \(\text{III}\) [see, e.g., \(\text{3.6}\) and \(\text{3.13}\)]. These issues have been discussed in detail with reference to waves in open systems [21].

The simplest example of a JB (with \(M = 2\)) is a harmonic oscillator going through critical damping. The eigenvalues \(\omega_{\pm} = \pm \omega_R - i\tau\) coalesce when \(\omega_R \to 0\). With one eigenvalue lost, the dynamics is not given by a sum of eigenfunctions with time dependence \(\exp(-i\omega_{\pm} t)\), but by only one such eigenfunction, plus another term whose time dependence carries a prefactor \(t\).

Our purpose in this Section is to establish that SUSY maps a JB in \(H\) into a JB in \(\tilde{H}\), preserving the order \(M\) except when the eigenvalue coincides with \(\pm \Omega\).

**A. Wronskian**

In this subsection we introduce the Wronskian \(J(\omega)\), define JBs in terms of its multiple zeros and describe the mapping of JBs under SUSY by a relation between \(J(\omega)\) and its counterpart \(\tilde{J}(\omega)\).

In the original system \(H\), define solutions of the wave equation \(f(\omega, x)\) and \(g(\omega, x)\) satisfying the boundary conditions \(f(\omega, -a) = 1\), \(f'(\omega, -a) = -i\omega\), \(g(\omega, a) = 1\), \(g'(\omega, a) = i\omega\), where \(\frac{\partial}{\partial x}\). The function values are arbitrary normalizations, while the derivatives impose the OWC on the left and right respectively. An eigenstate \(\phi_j\) in \(\Gamma\) satisfies the boundary condition on both the left (as for \(f\)) and the right (as for \(g\)): \(\phi_j \propto f(\omega_j, x) \times g(\omega_j, x)\). Thus, the zeros of the (position-independent) Wronskian

\[
J(\omega) = f'(\omega, x)g(\omega, x) - f(\omega, x)g'(\omega, x)\quad (4.1)
\]

identify the eigenvalues in \(\Gamma\). It can be shown [see \(4.1\)] below that \(\phi_j < \omega_j < \omega_j^0\), so an \(M\)th-order zero of \(J(\omega) > 1\) corresponds to the generalized norm being zero, and is precisely the JB phenomenon that we wish to investigate.

It is natural to generalize the definition of \(n(\omega)\) to be the order of the zero, viz.

\[
n(\omega) = \frac{1}{2\pi i} \oint \frac{dJ(\omega')/d\omega'}{J(\omega')} d\omega'\quad (4.2)
\]
on a contour of winding number \(+1\) enclosing \(\omega\). This definition makes it clear that the total number of states (but not necessarily of eigenstates) within a contour is...
preserved under continuous changes of the system parameters. We note for future reference (see Section VI.D) that poles of $J$ (which can only occur if $V$ does not have finite support) count as negative values of $n$.

Now consider the analogous construction in the partner system $\hat{H}$, obtained for example by using an NM $\Phi$ of $H$ as the generator, i.e., for $\chi = 1$. By our convention, $\Phi$ is associated with a frequency $\Omega = iK$, and $W_\pm = \mp i\Omega$. The SUSY transformation gives

$$
\hat{f}(\omega, x) = (\partial_x + W)f(\omega, x)
$$

leading to the Wronskian

$$
\hat{J}_n(\omega) = \hat{f}(\omega, x)\hat{g}(\omega, x) - \hat{f}(\omega, x)\hat{g}'(\omega, x).
$$

When (1.4) is written out using (1.3), some terms cancel by using $J' = 0$, the second derivatives can be eliminated by the defining equation, and $V$ is expressed in terms of $W$ and $\Omega^2$. Some arithmetic then leads to

$$
\hat{J}^u(\omega) = (\omega^2 - \Omega^2)J(\omega).
$$

This Wronskian is however normalized (as indicated by the superscript), since only $C\hat{f}$ and $D\hat{g}$ satisfy the normalization conventions at $-a$ and $+a$ respectively, where $C = -D = i(\omega - \Omega)^{-1}$. Thus the normalized Wronskian $\hat{J}(\omega) = C\hat{D}\hat{J}(\omega)$ is

$$
\hat{J}(\omega) = \omega + \Omega \overline{\omega} - \Omega J(\omega).
$$

The central result (1.6) neatly summarizes the correspondence between the two spectra for $\chi = 1$. [Similar formulas for the other cases can all be consolidated by changing $\Omega \to iK$ in (1.6), and will not be separately discussed.] (a) For $\omega \neq \pm \Omega$, the spectra of $\hat{H}$ and $\hat{H}$ are the same: a JB of order $M$ in $H$ maps to a JB also of order $M$ in $\hat{H}$ at the same frequency. This should be no surprise since up to the point of coalescence (for $M \geq 2$), the eigenvalues of the two systems are guaranteed (cf. Section VII.B) to be in one-to-one correspondence; in a sense, the result here merely demonstrates that the limit is not singular. (b) Moreover, at the special frequencies $\pm \Omega$, $M$ states at $\Omega$ (of which only one is an eigenstate) are mapped into $M-1$ states at $\Omega$ plus one state at $-\Omega$. In other words, we recover (2.15) even for J Bs, i.e., even when $n, \tilde{n} > 1$. Anticipating the possibility of poles in $J$ (cf. Section VII.D), we note that (1.6) implies that such poles are also preserved under SUSY, and would be accommodated by (1.2), (1.3), and (2.15) with negative $n, \tilde{n}$.

Incidentally, (2.15) on the change in normalization under SUSY follows simply from (1.3), since the bilinear map is related to the Wronskian by

$$
(f(\omega_j), g(\omega_j)) = -\left[\frac{dJ(\omega)}{d\omega}\right]_{\omega_j}.
$$

### B. Doubling of states by SUSY

We have seen that (say for $\chi = 1$) one has $\hat{n}(-i\Omega) - n(-i\Omega) = 1$. Where this increases from 0 to 1, the situation is straightforward — a QNM is created. When the increase is from 1 to 2, the situation is more subtle and merits a detailed examination. [The general case where this increases from $M$ to $M+1$ ($M \geq 1$) will not be shown.]

Consider a system $H$ with an NM $\Phi$ at $\Omega = iK$ and accidentally also a QNM $\Psi_j$ at $-\Omega' \approx -\Omega$. If $\Omega' \neq \Omega$, there are two corresponding QNMs in the $H$-system, namely $\Phi = \Phi^{-1}$ at $-\Omega$ and $\Psi_j = A\Psi_j$ at $-\Omega'$. Now tune the parameters of $H$ so that $\Omega' \to \Omega$; in the limit we must have $\Psi_j \approx \Phi$ [26,28], as is readily verified. The proportionality constant can be evaluated by

$$
\hat{\Psi}_j = \hat{\Phi}(\omega)\frac{\hat{\Psi}_j(-a)}{\hat{\Phi}(-a)} = 2i\Omega(\hat{\Psi}_j(-a)\hat{\Phi}(-a))
$$

The agreement of these two expressions can also be seen without invoking SUSY. One notes that $\Phi$ and $\Psi_j$, being eigenfunctions of $\mathcal{H}$ with distinct eigenvalues, are orthogonal. In the bilinear map, the integral term vanishes because the frequencies are opposite, leaving only the surface terms. Thus one finds $0 = (\Psi_j, \Phi) = i[(\hat{\Psi}_j(-a)\hat{\Phi}(-a) + \hat{\Phi}_j(a)\hat{\Phi}(a))]$.

Now the frequency $-\Omega$ in $\hat{H}$ must be associated with a doubled state. This can be seen in two ways: (a) until the limit $\Omega' = \Omega$, there are two distinct states; (b) from the key relation (1.6), $\hat{J}$ has a second-order zero. With the two QNMs collapsed into one, there has to be another basis vector, to which we now turn.

Using the normalization of $f$, one has $\hat{\Psi}_j(x) = \Psi_j(-a)f(-\Omega, x)$, implying $\hat{\Phi}_j(x) = \Psi_j(-a)f(-\Omega, x)$. [Analogous formulas for $\Phi$ and $\hat{\Phi}$ follow from (4.3).] The double zero of $\hat{J}(\omega)$ at $\omega = -\Omega$ means that $\hat{f}(\omega, x)$ satisfies the OWC at $x = a$ not only at $\omega = -\Omega$, but also to first order away from the zero. This makes it plausible that, in the QNM expansion, $\partial_x f(\omega, x)\big|_{-\Omega}$ takes the place of the “missing” eigenfunction when $\Phi$ and $\Psi_j$ coincide, which has been confirmed in detail [26]. One thus defines, for an arbitrary $\alpha$, a pair of functions $\hat{\Psi}_{j, \alpha}$, where $n = 0, 1$ is an intra-block index:

$$
\hat{\Psi}_{j, 0}(x) = \hat{\Psi}_j(x)
$$

$$
\hat{\Psi}_{j, 1}(x) = \Psi_j(-a)\partial_x \hat{f}(\omega, x)|_{-\Omega} + \alpha \hat{\Psi}_j(x),
$$

and the second function satisfies...
Using this, one verifies that an outgoing solution is given by \( \tilde{\Psi}_{j,1}(t) \equiv (\tilde{\Psi}_{j,1} - it\tilde{\Psi}_{j,0})e^{itH} \). This time dependence shows that the associated second component should be \( \tilde{\Psi}_{j,2} = -i(-\Omega\tilde{\Psi}_{j,1} + \tilde{\Psi}_{j,0}) \). The prefactor \( t \) and its counterpart if any in the \( H \)-system will be further explored below.

Next consider the bilinear map and normalization for these functions. For a double zero one always has \((\tilde{\Psi}_{j,0}, \tilde{\Psi}_{j,0}) = 0;\) cf. [4.7] [a choice of \( \alpha \) in [4.9] such that also \((\tilde{\Psi}_{j,1}, \tilde{\Psi}_{j,1}) = 0 \) would be useful in wavefunction expansions]. The JB is normalized by one overall factor, the bilinear map between the two basis states [21]:

\[
\frac{(\tilde{\Psi}_{j,1}, \tilde{\Psi}_{j,0})}{(\tilde{\Psi}_{j,1}, \tilde{\Psi}_{j,1})} = \left[ \frac{-\frac{4J^0}{\omega}J^0(\omega)/d\omega^2}{-dJ(\omega)/d\omega} \right] \equiv -2\Omega ,
\]

(4.11)

where in the numerator we have used the result analogous to [4.7] for a double zero.

Finally the reverse transform generated by \(-A^\dagger [31]\) satisfies the following properties. (a) \( A^\dagger \tilde{\Psi}_{j,0} \propto A^\dagger \Phi = 0 \). (b) Hence in \( A^\dagger \tilde{\Psi}_{j,1}(t) \), the term \( \propto te^{itH} \) is annihilated. (c) The remaining term in \( A^\dagger \tilde{\Psi}_{j,1} \) is \( e^\Omega \), readily seen by observing that

\[
(H - \Omega^2)(A^\dagger \tilde{\Psi}_{j,1}) = A^\dagger (H - \Omega^2)\tilde{\Psi}_{j,1} = A^\dagger (-2\Omega\tilde{\Psi}_{j,0}) = 0 ,
\]

(4.12)

so that \( A^\dagger \tilde{\Psi}_{j,1} \) is an eigenfunction of \( H \) with eigenvalue \( \Omega^2 \). In particular, the time dependence is \( e^{itH} \) without any prefactor \( t \). A straightforward computation shows that \( c = -2\Omega \).

Since \( A^\dagger A \) is not the identity, two different states in the \( H \)-system can be associated with \( \tilde{\Psi}_{j,1} \). The first is \( A^\dagger \tilde{\Psi}_{j,1} \) as discussed above. The second is the SUSY pre-image of \( \tilde{\Psi}_{j,1} \) under \( A \), which is readily found by noticing that [22]

\[
\tilde{\Psi}_{j,1}(x) = \Psi_j(-a) \times \partial_x[\partial_x + W]f(\omega, x)]_{-\Omega} = A\Psi_j(-a)\partial_xf(\omega, x)]_{-\Omega} .
\]

(4.13)

However, \( \Psi_j(-a)\partial_xf(\omega, x)]_{-\Omega} \) is not outgoing, since \( J(\omega) \) only has a first-order zero at \( -\Omega \), and consequently \( f(\omega, x) \) satisfies the OWC at \( x = +a \) only at \( -\Omega \), but not to first order away from it.

These remarks completely solve the puzzle related to the prefactor \( t \) in the time evolution in the \( H \)-system. Namely, of the two corresponding wavefunctions of \( H \), one has an exponential time dependence, while the other is not in \( \Gamma \).

V. EXAMPLES

The formalism developed in this paper is general, in that given \( V(x) \) with finite support, SUSY partners can be constructed whenever there are nodeless generator candidates. Nevertheless, some simple examples will suffice for an illustration.

Let \( V \) be a square barrier of height \( V_0 \) on \( I \); without loss of generality henceforth \( \alpha = 1 \). In the even sector there are two zero modes for small values of \( V_0 \); e.g., for \( V_0 = 0.16 \) they occur at \( \omega_1 = -0.181i, \omega_2 = -2.500i \). The wavefunctions are \( \phi_j(x) = \cosh \alpha_j x \) within \( I \) (with \( \alpha_1 = 0.242, \alpha_2 = 2.506 \), and a real exponential for \( |x| > \alpha \); clearly each \( \phi_j \) has no nodes.

This example already illustrates, as exceptional, the existence of several nodeless zero modes — any one of which can be used as the generator \( \Phi \). We choose the state at \( \Omega = \omega_1 \); since \( \Phi \) is purely increasing at both extremes, \( \chi = -1 \). Figure 1(a) shows \( V \) (solid line) and \( V \) (dotted line). Figure 1(b) shows the spectra in the complex \( \omega \)-plane: most eigenvalues are common (crosses); one QNM exists only in \( H \) (circle), while one NM exists only in \( \tilde{H} \) (triangle) — the two systems are essentially isospectral. Another essentially isospectral partner can be constructed using the state at \( \omega_2 \) as the generator. The reverse transformations are characterized by \( \chi = 1 \).

As \( V_0 \) increases, the two zero modes move closer together and merge at \( V_0 = 0.291 \), forming a JB in \( H \) (with \( M = 2 \)). This example illustrates the general results of Section IV (with \( H \) and \( \tilde{H} \) interchanged).

As another example, consider a symmetric multi-step square barrier, with \( V(x) \) being \(-10 \) for \( |x| < 0.1 \), \( 1 \) for \( 0.1 < |x| < 1 \) and zero for \( |x| > 1 \) [solid line in Figure 2(a)]. In this example, there is a TTM at \( \Omega = -0.990i \) [square in Figure 2(b)] — in fact a purity doublet with one propagating from the left (TTM\(_L\)) and one propagating from the right (TTM\(_R\)). We choose the former as the generator for a \( \chi = 0 \) transformation. The partner potential \( \tilde{V} \) is shown by the broken line in Figure 2(a). Since the generator is not symmetric, neither is \( \tilde{V} \). The states in \( \Gamma \) are exactly preserved [crosses in Figure 2(b)].

In this example it is interesting to consider not just the states in \( \Gamma \), but also TTM\(_s\) (see also Section VII). By arguments similar to those in Section IV, one TTM\(_L\) \( \Phi \) is destroyed and one TTM\(_R\) \( \tilde{\Phi} = \Phi^{-1} \) is created at the same frequency \( \Omega \) [square in Figure 2(b)]. However, in this example, because \( V \) is symmetric, there is also a TTM\(_R\) \( \Psi(x) = \Phi(-x) \) at \( \Omega \), and its partner \( \tilde{\Psi} = A\Phi \) is again a TTM\(_R\) in the \( H \)-system. Thus, in the \( \tilde{H} \)-system, there is a doubled TTM\(_R\) state at \( \Omega \). The situation can again be analysed in terms of the double zero of a Wronskian \( J_R(\omega) \), but in this case \( J_R \) and \( \tilde{J}_R \) refer to TTM\(_R\)'s. For example, \( J_R \) is now defined in terms of a function \( f \) satisfying the outgoing-wave condition at \( x = -a \) and
a function \( g \) satisfying the incoming-wave condition at \( x = a \). The obvious adaptation of the discussion in Section IV will not be given.

VI. DISCUSSION

A. Summary

In this paper, we have extended the usual discussion of SUSY as a relation between NMs of partner systems to include the QNMs as well. By viewing all these together in the space \( \Gamma \), a more complete picture emerges. For example, in the usual discussion for NMs only, essentially isospectral transformations are said to lose or gain one state; now we see that (if \( \chi = \pm 1 \)) when an NM appears (disappears), a corresponding QNM disappears (appears).

Furthermore, we have shown that the nontrivial linear-space structure for QNMs and any possible JBs are preserved by SUSY — the latter being a feature not found in conservative systems.

QNMs differ from NMs in two further regards. First, they have complex frequencies; nevertheless, even with twice as many constraints, matching the spectra turns out to be not any more difficult. Second, QNMs need not have an increasing number of nodes, and it is often possible to find several nodeless QNMs which generate distinct SUSY transformations — whereas the analogous operation for NMs would restrict the generator to the nodeless ground state.

These wider perspectives are gained only because attention is paid to the Klein–Gordon rather than the nodeless ground state.

Two further important properties are also preserved. (a) If \( V \) has a singularity say at \( x = \pm a \) (e.g., a step), then \( \tilde{V} \) will have the same type of singularity, but with opposite sign, as can be seen from (2.3) by noticing that the most singular part is \( W' \). (b) If \( V \) has finite support, then provided \( \Phi \) is not of the mixed type, \( \tilde{V} \) would likewise have no tail. These two properties are precisely the conditions for the eigenstates in \( \Gamma \) to be complete [2].

Thus, SUSY maps a complete basis to a complete basis (if, for the \( \chi = \pm 1 \) cases, \( \Phi \rightarrow \tilde{\Phi} \) is included as well).

B. Inversion

This work partially answers the question of QNM inversion. It is well known [3] that on a finite interval, two sets of real NM frequencies uniquely determine the potential \( V \). Does one set of complex eigenfrequencies in \( \Gamma \) also uniquely determine \( V \)? The answer is negative: there can be strictly isospectral potentials if a TTM with purely imaginary frequency exists and can be used as a \( \chi = 0 \) generator; Figure 2 provides one such example. However, the following scenario is not yet ruled out. If we consider a half-line problem \( 0 < x < \infty \) (say corresponding to the radial variable in a 3-d system), imposing a nodal condition at \( x = 0 \) and the OWC for \( x > a \), can one set of QNM frequencies uniquely determine the potential? SUSY transformations (2.4) do not directly resolve this possibility — for which there is some numerical evidence [3] — since these one-sided systems do not feature an analog of TTM's with which one could construct strictly isospectral partners. Moreover, by (2.7) and (2.9) the nodal condition maps a regular \( V \) to \( V \sim 2/x^2 \) for \( x \rightarrow 0^+ \) (generalizing the well-known result that SUSY increases the angular momentum by one unit in the hydrogen atom). It would therefore be interesting to see if an enlarged class of transformations can address this question.

C. Total-transmission modes

The present paper refers in the main to states in \( \Gamma \), i.e., states that satisfy the OWC at both extremes. One could also develop the same formalism for TTM's; see, e.g., the end of Section IV. Note that we here consider TTM's as states on which SUSY acts, rather than as (\( \chi = 0 \)) generators. To be more specific, the SUSY transformation acts on the space \( \Gamma_L \) of TTM's

\[
\phi(x) \sim e^{i\omega x}, \quad |x| \rightarrow \infty,
\]

or (equivalently, under \( \omega \rightarrow -\omega \)) the space \( \Gamma_R \) of TTM's

\[
\phi(x) \sim e^{-i\omega x}, \quad |x| \rightarrow \infty.
\]

One significant difference is that a \( \chi = \pm 1 \) transformation preserves the eigenvalues in these spaces, whereas a \( \chi = 0 \) transformation may shift one state.

D. Tails

So far we have only considered potentials with finite support. We end with an outline of some issues that arise for long-range potentials.

The most obvious difference is that if \( V \) and/or \( \tilde{V} \) are allowed to have tails that decay exponentially or slower in \( |x| \), then the mixed-type SUSY transformation is allowed, and there is a continuous choice of generators.

However, there are a range of more subtle issues. First, the very definition of an outgoing wave requires care. At the numerical level, special treatment is needed to ensure convergence [3], but there are matters of principle as well when the OWC \( \phi(\omega, x) \sim e^{i\omega|x|} \) is imposed only as \( |x| \rightarrow \infty \). In that limit, the condition as stated becomes
vacuous for $\text{Im} \, \omega < 0$, since admixture of an (exponentially smaller) incoming solution would not alter this behavior. Rather, it is necessary to define the OWC in the upper half-plane in $\omega$ and analytically continue to the lower half-plane. A wave $\phi(\omega, x)$ is incoming if $\phi(-\omega, x)$ is outgoing; this is equivalent to saying that incoming waves are defined first in the lower and continued to the upper half-plane. The necessity for these procedures makes it possible that (at certain singular points $\omega$) a wavefunction can be both incoming and outgoing.

Singularity in the one-sided functions \[ \text{can only occur because of the need to integrate the defining equation over an infinite range. Those values of } \omega \text{ for which these functions (say the left function } f) \text{ are singular are said to be anomalous; if the potential is not oscillating, anomalous points can only occur on the imaginary axis} \] precisely where possible SUSY generators are to be found. The case of an exponential tail $V(x) \sim V_1 e^{-\lambda |x|}$ is of particular interest because the anomalous points (at $\omega_m = -im\lambda/2, m = 1, 2, \ldots$) can be studied by the Born approximation, which in this case is equivalent to a power-series expansion in $z = e^{-\lambda |x|}$. More generally, if

$$V(x) = \sum_{k=1}^{\infty} V_k e^{-k\lambda |x|}, \quad (6.3)$$

then for particular choices of the coefficients $V_k$, one (or more) of the generically anomalous points $\omega_m$ may turn out to be regular — a situation we refer to as miraculous. The anomalous and miraculous properties for the one-sided functions $f$ and $g$ are inherited by their Wronskian $J$, which is central to the formalism. These concepts have been discussed in relation to a particular application \[.\]

As far as SUSY is concerned, we only make one remark: such singularities lead to poles in $J(\omega)$, thus are associated with negative values of $n(\omega)$, and are related in SUSY partners by \[.\] The Pöschl–Teller potential $V(x) = V \sech^2(x/b)$ \[.\] illustrates many interesting features, including the existence of double poles of $J(\omega)$ which can merge with two zeros and lead to “nothing” — a potential that has total transmission at all energies \[.\]

Anomalous points are exceptional (in the case of exponential tails being a discrete set of measure zero in the $\omega$-plane) and miraculous cancellation of singularities doubly exceptional. One might therefore think that these are not important. Surprisingly however, the problem of linearized gravitational waves propagating on a black-hole background \[.\] is precisely miraculous in this sense, at the so-called algebraically special frequency $\omega$ \[.\] A generator at $\Omega$ leads to a SUSY transformation that exactly relates the axial and polar sectors, which are therefore isospectral in $\Gamma$. Among the more intriguing results is the following \[.\] in the polar sector there is a mode at $\Omega$ that is simultaneously a QNM and a TTM [i.e., at radial infinity it is purely outgoing but into the event horizon it is both outgoing and incoming], while no modes exist in the axial sector.

The subtle and perhaps counter-intuitive nature of these concepts demands a separate and rigorous examination, to which the foregoing is meant only as a preview — and as further illustration of the utility of SUSY in dealing with waves in open systems.

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APPENDIX A: NODES IN QNM WAVEFUNCTIONS

Nodeless eigenstates play a special role in SUSY: they are candidates for the generator $\Phi$. It is therefore useful to highlight the differences between NMs and QNMs in this regard, especially to contrast with the well-known property that there can be only one nodeless NM.

First of all, we show that for QNMs with $\text{Re} \, \omega \neq 0$, there can be at most one node or antinode. This is not surprising: since the eigenvalue is complex, the wavefunction has a changing phase, and it would be “unlikely” that the real and imaginary parts (or their derivatives) would vanish together. To prove this formally, take the Schrödinger point of view, so that the eigenvalue is $E = \omega^2$ with a nonzero imaginary part. Now consider a time-dependent QNM and suppose that it has nodes or antinodes at two points $x_1, x_2$. At these two points, the current

$$J = i [\phi^*(\partial_x \phi) - (\partial_x \phi^*)\phi] \quad (A1)$$

vanishes. Then, flux conservation implies that the total probability in the interval $[x_1, x_2]$ is constant in time. Yet the wavefunction is either growing or decaying, since $\text{Im} \, E \neq 0$, which is therefore a contradiction.

From the perspective of SUSY it is unfortunate that the above proof excludes the crucial imaginary axis. However, on that axis the statement remains valid for repulsive potentials, or more generally for potentials which are so weakly attractive that $V - \omega^2$ is positive definite \[.\] Namely, let $\phi$ be a solution with two (anti)nodes. By taking the real or imaginary part, we may assume $\phi$ to be real. Now between two nodes $\phi$ would have an extremum, i.e. $\phi'' \phi < 0$ which is incompatible with the KGE. Similarly, an antinode can only be a global maximum or minimum, precluding the presence of any other nodes or antinodes.

Thus, except for the imaginary axis in the case of attractive potentials, QNMs can have at most one node or
antinode. For symmetric potentials, in the even sector $x = 0$ is already an antinode, so there can be no nodes anywhere.

For zero modes, i.e., QNMs with $\Re \omega = 0$, nodes are more “likely”: the eigenvalue is real and the wavefunction has a constant phase (say purely real), so a node requires only one condition, rather than two. Nevertheless, in contrast to the conservative case, the proof that there can be only one nodeless eigenstate can be bypassed.

The interlacing nodal structure of NM eigenfunctions follows from well-established Sturm–Liouville theory. For the present purpose, we do not need the full apparatus. Consider, for simplicity, a finite interval $[-a,a]$ and suppose there are two distinct nodeless eigenfunctions $\phi_1, \phi_2$, both real. Then they can both be chosen to be positive, which violates the orthogonality condition for NMs

$$\int_{-a}^{a} \phi_1(x)\phi_2(x) \, dx = 0 \ . \quad (A2)$$

We can attempt to transplant the argument to QNMs. For zero modes, the wavefunctions can again be chosen to be real, and if they are nodeless, positive definite. However, the analog of $(A2)$ for two eigenfunctions with eigenvalues $\omega_j = -i\gamma_j$ is

$$-(\gamma_1 + \gamma_2) \int_{-a}^{a} \phi_1(x)\phi_2(x) \, dx + [\phi_1(-a)\phi_2(-a) + \phi_1(a)\phi_2(a)] = 0 \ . \quad (A3)$$

Note in particular the signs of the two terms. With $\gamma_j > 0$, this condition does not preclude both eigenfunctions from being positive definite.

Thus, we can make three remarks. (a) For NMs, there can be only one nodeless state. (b) For QNMs with $\Re \omega = 0$, there could be more than one state with no node. (c) For QNMs with $\Re \omega \neq 0$, or with $\Re \omega = 0$ but $V - \omega^2$ positive definite, each eigenfunction can have at most one node or antinode, and for symmetric potentials, every eigenfunction is nodeless.

Case (b) in particular opens up the possibility of multiple SUSY transformations.

G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics* (Springer, Berlin, 1996) and references therein.

[1] J. Wess, B. Zumino, Nucl. Phys. B 70, 39 (1974). J. Wess, B. Zumino, Nucl. Phys. B 78, 1 (1974).

[2] Also see, e.g., S. Ferrara, *Supersymmetry*, Vol. I & II (North Holland and World Scientific, 1987) and S. Weinberg, *The Quantum Theory of Fields*, Vol. III, Supersymmetry (Cambridge University Press, 2000).

[3] E. Witten, Nucl. Phys. B 188, 513 (1981). E. Witten, Nucl. Phys. B 202, 253 (1982).

[4] See, e.g., F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).
[21] E. S. C. Ching, P. T. Leung, W. M. Suen and K. Young, Phys. Rev. Lett. 74, 2414 (1995).
E. S. C. Ching, P. T. Leung, W. M. Suen and K. Young, Phys. Rev. D 52, 2118 (1995).
[22] P. T. Leung, W. M. Suen, C. P. Sun and K. Young, Phys. Rev. E 57, 6101 (1998).
[23] P. T. Leung, W. M. Suen, C. P. Sun and K. Young, Phys. Rev. E 58, 6101 (1998).
[24] K. C. Ho, P. T. Leung, A. Maassen van den Brink and K. Young, Phys. Rev. E 58, 2965 (1998).
[25] K. C. Ho, P. T. Leung, A. Maassen van den Brink and K. Young, Phys. Rev. E 59, 044034 (1999).
[26] A. Maassen van den Brink and K. Young, “Jordan blocks and generalized biorthogonal bases: realizations in open wave systems”, Preprint (1998) [math-ph/9905019].
[27] A. Maassen van den Brink and K. Young, “Factor-space construction of outgoing waves”, Preprint (1999).
[28] For a one-dimensional system with \( V(|x| > a) = 0 \), starting at one end (say \( x = -a \)) and using the known logarithmic derivative \( \phi'/\phi \) appropriate to the OWC and an arbitrary normalization, one gets a unique wavefunction. Thus there can be at most one eigenstate in \( \Gamma \) at any \( \omega \); there can be no degeneracies.
[29] Note the difference in notation from Ref. [37], where \( \tilde{f} \) and \( \tilde{g} \) denote normalized functions.
[30] This relation also yields a counterpart of (3.15) for the (Q)NM corresponding to the generator \( \Phi \) itself, \( \langle \Phi, \Phi \rangle = J(-\Omega)/[2\Omega \Phi(-a) \Phi(a)] \).
[31] If the double pole is split along the imaginary axis, there are two distinct SUSY transforms generated by \( \Phi \) and \( \Psi_j \) respectively, which become identical when the poles merge. Thus, in the spirit of the JB approach to the field expansion, one might try to find the “missing” SUSY map generated by \( \Psi_j \). However, no meaningful transformation emerges. This difference with the case of the field expansion can be understood by realizing that, when the poles split perpendicular to the imaginary axis upon changing the sign of the splitting perturbation, the SUSY transforms cease to exist altogether.
[32] The pre-image is unique only up to adding a multiple of \( f(\Omega, x) \); but this does not alter the conclusion that none of these pre-images are in \( \Gamma \).
[33] V. Bargmann, Phys. Rev. 75, 301 (1949); Rev. Mod. Phys. 21, 488 (1949).
G. Borg, Acta Math. 78, 1 (1946).
N. Levinson, Math. Tidsskr. B 25, 24 (1949).
F. J. Dyson, in Essays in Honor of Valentine Bargmann, E. H. Lieb, B. Simon and A. S. Wightman, eds. (Princeton University Press, N. J., 1976).
V. Barcilon, in Inverse Eigenvalue Problems (Lecture Notes in Mathematics, Vol. 1225), A. Dold and B. Eckmann, eds. (Springer Verlag, Berlin, 1986).
[34] W. S. Lee, Thesis, The Chinese University of Hong Kong (1998).
[35] P. T. Leung, Y. T. Liu, C. Y. Tam and K. Young, Phys. Lett. A 247, 253 (1998).
[36] In this case the normalization convention is that \( f \sim 1 \cdot e^{-i\omega x} \) as \( x \to -\infty \) and \( g \sim 1 \cdot e^{i\omega x} \) as \( x \to \infty \).
Fig. 1(a): A square-barrier potential $V$ (solid line) and its SUSY partner potential $\tilde{V}$ (broken line). Both potentials are symmetric and only the $x > 0$ part is shown. The SUSY transformation is constructed by using the state at $\Omega = \omega_1 = -0.18i$ [circle in Fig. 1(b)] as the generator.

Fig. 1(b): The complex $\omega$-plane showing the QNMs common to both potentials (crosses); the mode present only in $V$ (circle), which corresponds to the generator $\Phi$; and the mode present only in $\tilde{V}$ (triangle), which corresponds to $\tilde{\Phi} = \Phi^{-1}$.

Fig. 2(a): A multi-step potential $V$ (solid line) and its SUSY partner potential $\tilde{V}$ (broken line). The SUSY transformation is constructed by using the TTM$_L$ at $\Omega = -0.990i$ [square in Fig. 2(b)] as the generator.

Fig. 2(b): The complex $\omega$-plane showing the NM and QNMs common to both potentials (crosses). The square indicates a TTM$_L$ and a TTM$_R$ in $V$, and a doubled TTM$_R$ in $\tilde{V}$. 
FIGURE CAPTIONS

Fig. 1
(a) A square-barrier potential $V$ (solid line) and its SUSY partner potential $\tilde{V}$ (broken line). Both potentials are symmetric and only the $x > 0$ part is shown. The SUSY transformation is constructed by using the state at $\Omega = \omega_1 = -0.181i$ [circle in Fig. 1(b)] as the generator.
(b) The complex $\omega$-plane showing the QNMs common to both potentials (crosses); the mode present only in $V$ (circle), which corresponds to the generator $\Phi$; and the mode present only in $\tilde{V}$ (triangle), which corresponds to $\tilde{\Phi} = \Phi^{-1}$.

Fig. 2
(a) A multi-step potential $V$ (solid line) and its SUSY partner potential $\tilde{V}$ (broken line). The SUSY transformation is constructed by using the $\text{TTM}_{\text{L}}$ at $\Omega = -0.990i$ [square in Fig. 2(b)] as the generator.
(b) The complex $\omega$-plane showing the NM and QNMs common to both potentials (crosses). The square indicates a $\text{TTM}_{\text{L}}$ and a $\text{TTM}_{\text{R}}$ in $V$, and a doubled $\text{TTM}_{\text{R}}$ in $\tilde{V}$. 