Momentum space topological invariants for the 4D relativistic vacua with mass gap

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Abstract

Topological invariants for the 4D gapped system are discussed with application to the quantum vacua of relativistic quantum fields. Expression $\tilde{N}_3$ for the 4D systems with mass gap defined in [13] is considered. It is demonstrated that $\tilde{N}_3$ remains the topological invariant when the interacting theory in deep ultraviolet is effectively massless. We also consider the 5D systems and demonstrate how 4D invariants emerge as a result of the dimensional reduction. In particular, the new 4D invariant $\tilde{N}_5$ is suggested. The index theorem is proved that defines the number of massless fermions $n_F$ in the intermediate vacuum, which exists at the transition line between the massive vacua with different values of $\tilde{N}_5$. Namely, $2n_F$ is equal to the jump $\Delta \tilde{N}_5$ across the transition. The jump $\Delta \tilde{N}_3$ at the transition determines the number of only those massless fermions, which live near the hypersurface $\omega = 0$. The considered invariants are calculated for the lattice model with Wilson fermions.

1 Introduction

The momentum space topology is becoming the main tool for the investigation of the robust properties of ground states (vacua) of condensed matter systems (see review papers [1, 2, 3]). It revealed variety of vacuum structures with nontrivial topological invariants, which is now commonly known as topological materials. These invariants protect gapless fermions in bulk, or on the surface of the fully gapped topological materials, or inside the core of topological defects in these systems, such as strings, vortices, monopoles, domain walls, solitons, etc. The gaplessness (masslessness) of fermionic species
is not sensitive to the details of the microscopic physics: irrespective of the
deformation of the parameters of the microscopic theory, the energy spec-
trum of fermions remains strictly gapless. Though the physics of topological
materials started 30 years ago with the pioneering work by Nielsen and Ni-
nomiyia [4], who demonstrated that the masslessness of elementary particles
can be protected by the momentum space topology, the investigation of the
topologically nontrivial vacua in relativistic quantum field theories is still
in its infancy, with a rather few papers exploiting topological invariants in
momentum space (see e.g. [5, 6, 7, 8, 9, 10, 11, 12]).

Here we are interested in the topologically nontrivial relativistic vacua
with massive fermions. The 4D vacua with mass gap may exist in vari-
ous phases, which are described by different values of topological invariants.
Recently such an invariant was considered in [13]. This invariant \( \tilde{N}_3 \) is deter-
mined on the \( \omega = 0 \) hypersurface of 4D momentum space, which represents
the 3D Brillouin zone. Such invariant is not sufficient for the full classification
of the topological states of the lattice models, whose momentum-frequency
(\( \omega, p \)) space represents the 4D Brillouin zone. Here we introduce another
topological invariant \( \tilde{N}_5 \), which is appropriate for the lattice models (see
Eq.(20)). The invariants \( \tilde{N}_3 \) and \( \tilde{N}_5 \) are then applied to particular lattice
model with Wilson fermions (see e.g. Ref. [14]). Table 2 demonstrates the
values of these invariants in different ranges of mass parameter \( m \), in which
the vacuum represents different phases of topological insulator.

There are critical values of mass parameter in the Wilson model \( m = 0, -2, -4, -6 \) and \( -8 \), at which the topological quantum phase transitions oc-
cur between the insulators with different \( \tilde{N}_3 \) or/and \( \tilde{N}_5 \) (more on topological
phase transitions, at which the topological charge of the vacuum changes
while the symmetry does not, see [15]). At these values of \( m \) the vacuum
states are gapless and they represent analogues of topological semimetals:
similar to the four-dimensional graphene discussed in Ref. [11] they contain
massless Dirac fermions. There is an analogue of index theorem, which re-
lates the number of these massless fermions with the jump of the topological
invariant across the transition (\( \Delta \tilde{N}_3 \) or \( \Delta \tilde{N}_5 \)). The total algebraic number
of gapless fermions at the transition point is \( n_F = \Delta \tilde{N}_5/2 \), and according
to Table 2 the states with critical \( m = 0, -2, -4, -6 \) and \( -8 \) have corre-
spondingly 1, 4, 6, 4, 1 massless species. The jump \( \Delta \tilde{N}_3 \) also determines
the number of massless fermions in the intermediate states, but only those of
them which live near the hypersurface \( \omega = 0 \), i.e. \( n_F(\omega = 0) = \Delta \tilde{N}_3/2 \) [13].

The intermediate states have correspondingly 1, 3, 3, 1, 0 of such massless
species.

Since the invariants are expressed in terms of the Green’s function, they are applicable to the interacting systems as well [9]. We prove that the relation between the number of gapless fermions and the jump in the topological charge remains valid within the interacting theory for the wide class of 4D models. (Recent discussion on topological invariants for interacting condensed matter systems in terms of Green’s function and some peculiarities in the bulk-boundary correspondence there see in Refs. [16, 17].)

The paper is organized as follows. In Section 2 we consider topological invariant $\tilde{N}_3$. In Section 3 we consider the 5D massless systems. In Section 4 we derive topological invariants in 4D via the dimensional reduction of the 5D constructions. In Section 5 we suggest the method of calculation of the invariants $\tilde{N}_5$ and $\tilde{N}_3$ and formulate the index theorem. In Section 6 we apply the suggested constructions to the lattice model with Wilson fermions. In Section 7 we end with the conclusions.

2 Topological invariant $\tilde{N}_3$

2.1 Green functions

Let us consider for the definiteness $K$ flavors of Dirac fermions $\chi^A, A = 1, ..., K$ coupled to some gauge field $A$, that, in turn, may be coupled to the other unknown fields. Most of the conclusions, however, may be extended to a more general case. The fermion Green function in Euclidean space has the form:

$$-iG^B_A(x) = \frac{1}{Z} \int D\bar{\chi}DA \exp(-S_G[A])$$

$$- \int d^4 x \bar{\chi}((\partial_i + iA_i)\gamma^i + m)\chi^B(x)$$

$$= \frac{i}{Z} \int D\bar{\chi}DA \exp(-S_G[A]) \det [i(\partial_i + iA_i)\gamma^i + im]$$

$$\{[i(\partial_i + iA_i)\gamma^i + im]^{-1}\}^B_A$$

Here $S_G$ is some unknown effective action. We use Euclidean $\gamma$ - matrices such that $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. It will also be implied that a certain Lorentz invariant gauge is fixed. When the interactions are turned off, we have

$$G(\omega, p)^B_A = \int d^4 x G^B_A(0, x)e^{i\omega x^4 + i(px)} = \frac{\delta^B_A}{p_4 - \omega}, p_4 = \omega$$

(2)
When the interactions are turned on, we suppose that Lorentz invariance and parity are not broken. We shall also use the function \( Q = -i \gamma^5 G \) that is Hermitian as follows from (11). It can be diagonalized with respect to the flavor index via Unitary transformations of the fermion field. Later we shall imply that the basis is chosen, in which \( Q \) (and \( G \)) is diagonal in flavor index. Symmetry fixes the form of the Green function

\[
G(\omega, p)^B_A = Z^C_A(p^2) \left[ \frac{1}{p_i \gamma^i - im(p^2)} \right]^B_C
\]

Here \( Z(p^2) \) is the wave function renormalization function, while \( m(p^2) \) is the effective mass term. Both \( Z \) and \( m \) are diagonal matrices in indices \( A, B \). Below we omit fermion indices \( A, B \) in most of the expressions.

In addition we shall also consider more general case, when the Green function has the form

\[
G_B^A = \frac{Z[p^2]}{g^i[p] \gamma^i - im[p]} \delta_B^A, \quad i = 1, 2, 3, 4
\]

Here \( g^i[p] \) is a certain function of the 4- momentum \( p \) that is again diagonal in flavor indices. We shall also denote \( g_5^i[p] = m[p] \). The Green function of this form appears, in particular, in the lattice regularization considered in Section 6 when Lorentz symmetry is broken due to the regularization. It is worth mentioning that (11) is not the general form of the Green function in the general case when Lorentz symmetry is broken.

As the matrices \( g^A_B(p), Z^A_B(p), m^A_B(p) \) are all diagonal we further deal with them formally as with numbers.

### 2.2 When \( \tilde{N}_3 \) is the topological invariant?

In [13] the following expression has been considered:

\[
\tilde{N}_3 = \frac{1}{24 i \pi^2} \text{Tr} \gamma^5 \gamma^4 \int_{\omega=0} G^{-1} dG \wedge dG^{-1} \wedge dG
\]

Here integration is performed along the hypersurface \( \omega = 0 \) of momentum space. This is the topological invariant protected by symmetry, i.e. it is invariant under perturbations only when the Green’s function at \( \omega = 0 \) (or the effective Hamiltonian) commutes with the matrix \( \gamma^5 \gamma^4 \) (for topology protected by symmetry see Refs. [9] [3] and recent paper [18]). For the noninteracting fermions with positive masses we have \( \tilde{N}_3 \) equal to the number of
flavors of Dirac fermions. If the interactions are introduced, the Green function is changed: $G \rightarrow G + \delta G$. If the interactions do not violate the condition $[G(0, \mathbf{p}), \gamma^5 \gamma^4] = 0$, then (5) transforms as follows:

$$
\delta \tilde{N}_3 = \frac{1}{24i \pi^2} \int_{\omega=0} \text{tr} \left( \gamma^5 \gamma^4 \{ \delta G \} dG^{-1} + G dG^{-1} \wedge G dG^{-1} \right)
$$

$$
= \frac{1}{24i \pi^2} \int_{\omega=0} \text{tr} \left( \gamma^5 \gamma^4 \{ -G \delta G^{-1} \} dG^{-1} + G dG^{-1} \wedge G dG^{-1} \right)
$$

$$
= -\frac{1}{24i \pi^2} \int_{\omega=0} \text{tr} \left( \gamma^5 \gamma^4 \{ \delta G^{-1} \} dG^{-1} \wedge dG \right)
$$

$$
= -\frac{1}{24i \pi^2} \int_{\omega=0} d \text{tr} \left( \gamma^5 \gamma^4 \{ \delta G^{-1} \} \right) dG^{-1} \wedge dG
$$

$$
= -\frac{1}{24i \pi^2} \int_{\partial \Sigma} \text{tr} \left( \gamma^5 \gamma^4 \{ \delta G^{-1} \} \right) dG^{-1} \wedge dG
$$

(6)

Here $\partial \Sigma$ is the boundary of the hypersurface $\omega = 0$ in momentum space. That’s why $\tilde{N}_3$ is the topological invariant if

$$
\int_{\partial \Sigma} \text{tr} \left( \gamma^5 \gamma^4 \{ \delta G^{-1} \} \right) dG^{-1} \wedge dG = 0
$$

(7)

When the Green function is of the form (3), we consider the surface $\partial \Sigma$ such that $p^2 = \text{const} \rightarrow \infty$ and denote $n^i = p^i / |\mathbf{p}|$. Then Eq. (7) is equivalent to

$$
\frac{\delta m}{|\mathbf{p}| (1 + \frac{m^2}{p^2})^2} \rightarrow 0, \quad |\mathbf{p}| \rightarrow \infty
$$

(8)

Thus, we came to the following

**Lemma**

In the 4D system expression (3) is the topological invariant if $[G(0, \mathbf{p}), \gamma^5 \gamma^4] = 0$ and either momentum space is closed or on its boundary Eq. (7) holds. If the Green function is of the form of Eq. (3) then the latter condition is reduced to Eq. (8).

### 2.3 The asymptotic free models

It is natural to suppose, that in the realistic theory in deep ultraviolet at $p^2 \rightarrow \infty$ the system becomes effectively massless and chiral symmetric. We imply that the model is asymptotic free and the fermions are coupled to the
nonabelian gauge field. For a recent numerical investigation of the fermion propagator in such models see, for example, [19]. Numerical investigations confirm that \( \frac{m(p^2)}{p^2} \to 0 \) while \( Z(p^2) \to Z_\infty \) at \( p^2 \to \infty \). Therefore, (8) is fulfilled. Then \( \delta \tilde{N}_3 = 0 \) and \( \tilde{N}_3 = \text{Sp} 1 = K \) (the trace here is over the flavor indices) is equal to the number flavors of Dirac fermions until the phase transition is encountered. It is worth mentioning that in the case when \( Z(p^2) \to Z_\infty \) and \( m(p^2) \to \text{const} \ p^2 \) at \( p^2 \to \infty \) we also have \( \delta \tilde{N}_3 = 0 \) (see [13]).

### 3 Topological invariants for 5D massless fermions

#### 3.1 Topological invariant in 5D at \( \omega = 0 \)

We consider the system with massless fermions. Let us introduce the following expression:

\[
\mathcal{N}'_3 = \frac{1}{24\pi^2} \text{Tr} \gamma^4 \int_{\omega=0} \mathcal{G}^{-1} d\mathcal{G} \wedge d\mathcal{G}^{-1} \wedge d\mathcal{G} \quad (9)
\]

Here integral is over the 3D hypersurface that encloses the Fermi point in momentum space in the \( \omega = 0 \) subspace. Expression (9) is invariant under small perturbations of the Green function if \( \{ \delta \mathcal{G}(0, p), \gamma^4 \} = 0 \). That's why, the Fermi point in 5D is protected by the given topological invariant if the interactions are such that \( \{ \delta \mathcal{G}(0, p), \gamma^4 \} = 0 \). For example, if the Green function is of the form:

\[
\mathcal{G}(\omega, p) = \frac{Z(p^2)}{\omega^4 + p_a \gamma^a}, \quad a = 1, 2, 3, 5 \quad (10)
\]

then we obtain \( \mathcal{N}'_3 = 2 \text{Sp} 1 \) (the number of 5D Weil fermions).

#### 3.2 Topological invariant in 5D space \((\omega, p)\)

Below we generalize the construction of the topological invariant \( \mathcal{N}'_3 \) considered above. The resulting construction uses the Green function defined on the surface that encloses the Fermi point in \( \omega - p \) space. Suppose that the 5D Green function has the form

\[
\mathcal{G}(\omega, p) = \frac{Z(p^2)}{g_\alpha[p] \gamma^\alpha}, \quad a = 1, 2, 3, 4, 5, \quad p_4 = \omega \quad (11)
\]
with real functions $g^a[p]$.

Let us define the function in momentum space

$$H = \frac{G}{\sqrt{\frac{1}{4} \text{Tr} \ G^+ G}} \quad (12)$$

Here the trace is over the Dirac indices only. We can express $H$ as

$$H = \hat{g}^a \gamma^a, \quad \hat{g}^a = \frac{g^a}{|g|}, \quad |g| = \sqrt{g_a g_a}, \quad a = 1, 2, 3, 4, 5.$$ \now

Now let us consider the following integral over the closed 4D surface $\Sigma$ in momentum space such that $G$ does not have poles on $\Sigma$:

$$N_4 = \frac{3}{16 \pi^2 4!} \text{Tr} \int_{\Sigma} \mathcal{H} d\mathcal{H} \wedge d\mathcal{H} \wedge d\mathcal{H} \wedge d\mathcal{H} = \frac{3}{4 \pi^2 4!} \epsilon_{abcde} \int_{\Sigma} \hat{g}^a \hat{g}^b \wedge \hat{g}^c \wedge \hat{g}^d \wedge \hat{g}^e \quad (13)$$

The given expression for the invariant is reduced to (9) for the Green function of the form $\frac{Z[p^2]}{p^2 m^2}$. Without interactions $G$ has the pole at $p = \omega = 0$ that corresponds to the Fermi point. In this case $N_4 = 2 \text{Sp} 1$ for any surface that encloses the pole.

4 Dimensional reduction of the 5D constructions

4.1 Dimensional reduction of $N'_3$

Let us consider the 4D system with the Green function of the form $\frac{Z[p^2]}{p^2 m^2}$. We have

$$Q = \frac{Z[p^2]}{ig^l[p] \gamma^l \gamma^5 + m[p] \gamma^5} = -i \gamma^5 G \quad (14)$$

Next, we define new $\gamma$-matrices $\Gamma^5 = \gamma^5, \Gamma^i = i \gamma^i \gamma^5$ that satisfy $\{\Gamma^a, \Gamma^b\} = 2 \delta^{ab}, a = 1, 2, 3, 4, 5$. We also denote $g_5[p] = m[p]$. Then $Q$ has the form

$$Q = \frac{Z[p^2]}{g^0[p] \Gamma_a}, \quad a = 1, 2, 3, 4, 5 \quad (15)$$

Now we can treat $Q$ as a 5D Green function defined on the hypersurface $p_5 = g_5[p_1, p_2, p_3, p_4]$. We may consider, for example, the following analogue
of expression (9):

\[ \tilde{N}_3' = \frac{1}{24\pi^2} \text{Tr} \Gamma^4 \int_{\omega=0} Q^{-1} dQ \wedge dQ^{-1} \wedge dQ = \frac{1}{24i\pi^2} \text{Tr} \gamma^5 \gamma^4 \int_{\omega=0} G^{-1} dG \wedge dG^{-1} \wedge dG \equiv \tilde{N}_3 \]  

(16)

Here integral is over the closed 3D hypersurface in the slice \( \omega = 0 \) of 4D momentum space. One can see that we come to the invariant \( \tilde{N}_3 \) discussed in Section 2. This equivalence gives the way to calculate \( \tilde{N}_3 \). For example, let us consider the case of noninteracting massive fermions. We consider the closed surface \( \Sigma \) composed of the two pieces \( \Sigma_1 : p_5 = m \) and \( \Sigma_2 : p_5 = -m \) connected at infinity by the additional piece of surface. This surface encloses the pole of \( Q \) in 5D momentum space. Then we calculate expression (9) for this surface \( N_3' = 2 \text{Sp} 1 \). Therefore, \( \tilde{N}_3(m) - \tilde{N}_3(-m) = 2 \text{Sp} 1 \) while \( \tilde{N}_3(m) = -\tilde{N}_3(-m) \). As a result \( \tilde{N}_3(m) = \text{Sp} \text{sign} m \).

4.2 Dimensional reduction of \( N_4 \)

Let us define the function in momentum space

\[ H' = \frac{Q}{\sqrt{\frac{1}{4} \text{Tr} (Q)^2}} = \hat{g}_a \Gamma^a, \quad \hat{g} = \sqrt{\hat{g}^a \hat{g}^a} \]  

(17)

(Remind that here \( \hat{g}^a \) is the matrix diagonal in flavor indices.)

Now let us consider the following integral over 4D momentum space:

\[ N_4' = \frac{3}{16\pi^2 4!} \text{Tr} \int H' dH' \wedge dH' \wedge dH' \wedge dH' \]  

\[ = \frac{3}{4\pi^2 4!} \epsilon_{abcd} \int \hat{g}^a d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d \wedge d\hat{g}^e \]  

(18)

If momentum space \( M \) is closed (for example, has the form of torus or 4D - sphere), then expression (18) is the topological invariant that measures the degree of mapping \( \hat{g} : M \rightarrow S^4 \otimes S^4 \ldots \otimes S^4 \). For the calculation of \( N_4' \) in open momentum space see discussion below in Section 5.1.

4.3 Dimensional reduction of \( N_5 \)

Now let us consider the other way to derive the topological invariant for the 4D gapped systems starting from a 5D construction. Let us consider the
Euclidean Green’s function on the 4D lattice $G$ as the inverse Hamiltonian in 4D momentum space and introduce the 5D Green’s function:

$$G^{-1}(p_5, p_4, p) = p_5 \gamma^5 + G^{-1}(p_4, p) = (ip_5 + Q^{-1}(p_4, p))(-i\gamma^5).$$

(19)

Then one can introduce the topological invariant as the 5-form:

$$\mathcal{N}_5 = \frac{1}{2\pi^3 5!} \text{Tr} \int G dG^{-1} \wedge G dG^{-1} \wedge G dG^{-1} \wedge G dG^{-1},$$

(20)

where the integration is over the Brillouin zone in 4D momentum space $(p_4, p)$ and over the whole $p_5$ axis. In this form the invariant is applicable also to the interacting case. Similar to Section 2 we obtain that expression (20) is the topological invariant if the variation of the Green function $\delta G$ satisfies

$$\int_{\partial[\mathcal{M} \otimes R]} \text{tr} \left( [\delta G^{-1}] G dG^{-1} \wedge dG \wedge dG^{-1} \wedge dG \right) = 0, \quad p_5^2 \to \infty$$

(21)

Remarkably, here $G$ may be almost arbitrary. In particular, if $\mathcal{M}$ is compact closed space, then (21) is satisfied for any $G$ that does not have poles. If $\mathcal{M}$ is noncompact, then (21) is satisfied for the Green function of the form (4) when the system is massless in ultraviolet.

In terms of the 4D Green function expression (20) is

$$\tilde{\mathcal{N}}_5 = \frac{1}{2\pi^3 4!} \int_{-\infty}^{\infty} dp_5 \text{Tr} \int_{\mathcal{M}} \frac{1}{p_5 \gamma^5 + G^{-1} \gamma^5} \frac{1}{p_5 \gamma^5 + G^{-1} \gamma^5} \frac{1}{p_5 \gamma^5 + G^{-1} \gamma^5} \frac{1}{p_5 \gamma^5 + G^{-1} \gamma^5} \frac{1}{p_5 \gamma^5 + G^{-1} \gamma^5} dG^{-1}$$

(22)

In the case, when the 4D Green’s function has the form of Eq. (4):

$$G = \frac{Z[p]}{g_5(p) \gamma^5 - ig_5(p)},$$

we denote $Q = \frac{1}{f_b(p) \Gamma_b}$, $b = 1, 2, 3, 4, 5$, and $f^a = \frac{\hat{g}^a}{Z}$. Then

$$\tilde{\mathcal{N}}_5 = \frac{1}{2\pi^3 4!} \text{Tr} \int_{\mathcal{M}} \int \frac{idp_5}{(p_5^2 + f^2)^5} (f \Gamma - ip_5) \wedge (-df \Gamma (f^2 + p_5^2) + (f \Gamma - ip_5) df^2)$$

$$\wedge df \Gamma \wedge (f^2 + p_5^2) + (f \Gamma - ip_5) df^2) \wedge df \Gamma$$

(23)

The integral over $p_5$ is equal to the residue of the corresponding pole at $p_5 = if$ that can easily be calculated. As a result invariant (23) after the integration over $p_5$ is reduced to the degree of mapping of the 4D Brillouin zone to the 4D sphere of unit vector:

$$\tilde{\mathcal{N}}_5 = \frac{3}{4\pi^2 4!} \epsilon_{abcde} \int \hat{g}^a \hat{d} \hat{g}^b \wedge \hat{d} \hat{g}^c \wedge \hat{d} \hat{g}^d \wedge \hat{d} \hat{g}^e \equiv \mathcal{N}_5'$$

(24)

We proved here the following
Theorem

Eq. (22) defines the topological invariant for the gapped 4D system with momentum space $\mathcal{M}$ if Eq. (21) holds. This requirement is satisfied, in particular, for the system with compact closed $\mathcal{M}$ and for the system with the Green function of the form (3) that is massless in ultraviolet.

For the system with the Green function of the form (4) expression for the topological invariant Eq. (22) is reduced to Eq. (24).

The 5-form topological invariant (20) has been discussed in [9, 23, 21]. In particular it is responsible for the topological stability of the 3+1 chiral fermions emerging in the core of the domain wall separating topologically different vacua in 4+1 systems (see Sec. 22.2.4 in [9]) and fermion zero modes on vortices and strings [21]. The topological invariant for the general $2n+1$ insulating relativistic vacua and the bound chiral fermion zero modes emerging there have been considered in [7, 8, 12].

4.4 Topological invariant $\tilde{N}_4$

In addition to the invariant $\tilde{N}_5$ let us also consider a different construction that coincides with $\tilde{N}_5$ for the case of free fermions

$$\tilde{N}_4 = \frac{1}{48\pi^2} \text{Tr} \gamma^5 \int_\mathcal{M} dG^{-1} \wedge dG \wedge dG^{-1} \wedge dG$$

(25)

Here the integration is over the whole 4D space $\mathcal{M}$. The expression in this integral is the full derivative. Therefore, we have:

$$\tilde{N}_4 = \frac{1}{48\pi^2} \text{Tr} \gamma^5 \int_\mathcal{M} dG^{-1} \wedge dG \wedge dG^{-1} \wedge dG$$

$$= \frac{1}{48\pi^2} \text{Tr} \gamma^5 \int_{\partial\mathcal{M}} G^{-1} dG \wedge dG^{-1} \wedge dG$$

(26)

Here the integral is over the 3D hypersurface $\partial\mathcal{M}$. The last equation is identical (up to the factor $1/2$) to that of for the invariant $N_3$ for massless fermions. Therefore, Eq. (25) defines the topological invariant if the Green function anticommutes with $\gamma^5$ on the boundary of momentum space.

For the noninteracting fermions

$$G(\omega, \mathbf{p}) \rightarrow \frac{1}{p_i \gamma^i}, \quad p^2 \rightarrow \infty$$

(27)
Therefore, $\tilde{N}_4 = \text{Sp} \, 1$ (the number of Dirac fermions). The value of the given invariant coincides with $\tilde{N}_5$ in this case. However, it will be shown in Section 6 that in the other nontrivial cases like the lattice regularized models the two invariants give different values. In particular, $\tilde{N}_4 = 0$ while $\tilde{N}_5$ may be different from zero for the systems with closed momentum space $\mathcal{M}$.

When the interactions are turned on, but in the deep ultraviolet at $p^2 \to \infty$ the system becomes effectively massless and chiral symmetric, the $n_G(p)$ anticommutes with $\gamma^5$ at $p^2 \to \infty$. As a result, the value of $\tilde{N}_4$ remains equal to the number of Dirac fermions until a phase transition is encountered.

### 5 Topological invariant $\tilde{N}_5$

#### 5.1 A way to calculate $\tilde{N}_5$

In this subsection we consider the Green function of the form (4). Let us introduce the following parametrization

$$\hat{g}_5 = \cos 2\alpha, \quad \hat{g}_a = k_a \sin 2\alpha$$

(28)

$k$ may be undefined at the points of momentum space $\mathcal{M}$, where $\hat{g}_a = 0, a = 0, 2, 3, 4$. In nondegenerate case this occurs on points $p_i, i = 1, ..., \infty$. We denote the small vicinity of the regions where $k$ is undefined by $\Omega = \Omega(p_0) + \Omega(p_1) + \ldots$. Then the expression for $\tilde{N}_5$ can be rewritten as follows:

$$\tilde{N}_5 = \frac{3}{\pi^2 4!} \epsilon_{abcd} \int_{\mathcal{M} - \Omega} \sin^3 2\alpha d\alpha \wedge k^a \, dk^b \wedge dk^c \wedge dk^d$$

$$= -\frac{1}{\pi^2 4!} \epsilon_{abcd} \int_{\partial \mathcal{M} - \partial \Omega} (3\cos 2\alpha - \cos^3 2\alpha) k^a \, dk^b \wedge dk^c \wedge dk^d$$

$$= -\frac{1}{\pi^2 4!} \epsilon_{abcd} \int_{\partial \mathcal{M} - \partial \Omega} (3\hat{g}_5 - \hat{g}_5^3) k^a \, dk^b \wedge dk^c \wedge dk^d, \quad k^a = \frac{g_a}{\sqrt{g^a g^a}}$$

(29)

For the convenience we denote all boundary of momentum space (that may be consisted of several pieces) by $\partial \mathcal{M} = -\partial \Omega(p_\infty)$. This allows to rewrite the last expression as:

$$\tilde{N}_5 = \frac{1}{\pi^2 4!} \sum_{i=0,1,\ldots,\infty} \epsilon_{abcd} \int_{\partial \Omega(p_i)} (3\hat{g}_5 - \hat{g}_5^3) k^a \, dk^b \wedge dk^c \wedge dk^d$$

(30)

Let us consider the case when $\hat{g}_5 = 0$ on $\partial \mathcal{M}$. In particular, if the boundary of momentum space is placed at infinity and the system is effectively
massless in ultraviolet, then \( \hat{g}^5[p] = 0 \) at \(|p| \to \infty\). We also take into account that \( \hat{g}_5 = \pm 1 \) at \( p_i, i \neq \infty \). That’s why we obtain:

\[
\mathcal{N}_5 = \frac{2}{\pi^2 4!} \sum_{i=0, 1, \ldots} \epsilon_{abcd} \int_{\partial \Omega(p_i)} \text{sign} (\hat{g}_5) \, k^a \, dk^b \wedge dk^c \wedge dk^d \tag{31}
\]

Here the sum over \( p_i \) does not include \( p_\infty \).

In the case when there is only one zero of \( \hat{g}_a \) (we call this point \( p_0 \)) we also obtain:

\[
\tilde{N}_5 = \frac{2}{\pi^2 4!} \epsilon_{abcd} \int_{\partial \Omega(p_0)} \text{sign} (\hat{g}_5) \, k^a \, dk^b \wedge dk^c \wedge dk^d
\]

Suppose that the effective mass \( m[p] \) does not change sign anywhere. Then in this case \( \tilde{N}_5 \) is equal to the degree of mapping

\[
k^a = \frac{g^a(p)}{\sqrt{[g^1(p)]^2 + [g^2(p)]^2 + [g^3(p)]^2 + [g^4(p)]^2}} : S^3(|p| \to \infty) \to S^3 \otimes \ldots \otimes S^3
\]

Here we proved the following

\textit{Theorem}

In the system with the Green function of the form (4) the topological invariant \( \tilde{N}_5 \) is given by the sum in Eq. (31) if \( \hat{g}_5[p] = 0 \) on the boundary of momentum space.

When the Lorentz symmetry is not broken at \(|p| \to \infty\), and \( g^a[p] \sim p^a \), we have \( \tilde{N}_5 = Sp \text{sign} m \), where \( m \) is the mass. That’s why for the free fermions with positive masses \( \tilde{N}_5 \) is equal to the number of the flavors of Dirac fermions.

The situation, when several points \( p_i \) appear within the Brillouine zone will be considered later when we shall calculate \( \tilde{N}_5 \) for the lattice fermions in Section 6.

It is worth mentioning that the method suggested above could be easily extended to the calculation of \( \tilde{N}_3 \).
5.2 Index theorem

In this subsection we prove the following

**Theorem**

Suppose that the 4D system depends on parameter $\beta$ and there is a phase transition at $\beta_c$ with changing of $\tilde{N}_5$. We denote $\tilde{N}_5 = n_+$ for $\beta > \beta_c$ and $\tilde{N}_5 = n_-$ for $\beta < \beta_c$. In addition we require that $\mathcal{G}$ does not have zeros and does not contain $\gamma_5$ matrix (i.e. $\text{Tr} \mathcal{G}^{-1} \gamma_5 = 0$). Momentum space $\mathcal{M}$ of the 4D model is supposed to be either compact and closed or open. In the latter case we need that $\mathcal{G}$ does not depend on $\beta$ on $\partial \mathcal{M}$. Then at $\beta = \beta_c$ there are $n_f = (n_+ - n_-)/2$ flavors of massless Dirac fermions.

Let us consider 6D space $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_6 = \beta$. 4D subspace $p_1, p_2, p_3, p_4$ has the form of the 4D Brillouin zone $\mathcal{M}$ while $p_5$ axis and $p_6$ axis are straight lines. We then consider the 5D hypersurfaces $\Sigma_+$ at $p_6 = \beta_+ > \beta_c$ and $\Sigma_-$ at $p_6 = \beta_- < \beta_c$. These surfaces, in turn, contain the Brillouin zones of the initial model at $p_5 = 0$ for $\beta = \beta_+, \beta_-$ respectively. We connect $\Sigma_{\pm}$ at $p_5 = \pm \infty$ and at the boundary of $\mathcal{M}$ (this is possible because $\mathcal{G}$ does not depend on $\beta$ at $\partial \mathcal{M}$). The resulting hypersurface $\Sigma$ is closed in 6D space.

Let us construct the 6D Green function

$$G^{-1}(p_6, p_5, p_4, p) = p_5 \gamma_5 + G^{-1}(\beta, p_4, p), \quad p_6 = \beta$$  \hspace{1cm} (34)

Then we consider the invariant

$$N_5|_\Sigma = \frac{1}{2\pi^3 5! i} \text{Tr} \int_\Sigma GdG^{-1} \wedge GdG^{-1} \wedge GdG^{-1} \wedge GdG^{-1} \wedge GdG^{-1}, \quad (35)$$

We may neglect in this integral the region placed at infinite $p_5$. That’s why

$$N_5|_\Sigma = \tilde{N}_5(\beta_+) - \tilde{N}_5(\beta_-) = n_+ - n_- \quad (36)$$

On the other hand, we may deform $\Sigma$ in such a way that it is not already placed at infinity. $N_5|_\Sigma = n_+ - n_-$ means that the 6D system described by Green function (34) has $(n_+ - n_-)/2$ massless Dirac fermions.

$\mathcal{G}$ does not contain $\gamma_5$ matrix. Therefore, there are no poles of $G$ at nonzero $p_5$. That’s why we may deform $\Sigma$ in such a way that on this hypersurface $p_5 = \pm \epsilon$ with arbitrary small $\epsilon$. Also we make $\beta_{\pm}$ infinitely close to $\beta_c$. As a result the 6D model with Green function (34) and $(n_+ - n_-)/2$
flavors of massless Dirac fermions is dimensionally reduced to the 4D system with \( \beta = \beta_c \). There are \((n_+ - n_-)/2\) poles in (34) at \( p_0 = \beta_c, p_5 = 0 \). We conclude, therefore, that there are \((n_+ - n_-)/2\) poles in \( G(\beta_c, p_4, p) \).

The example of the 4D system with compact momentum space that satisfies the given theorem is given by the model with Wilson fermions considered in Section 6. The example of the system with noncompact momentum space is given by the free continuum fermions (in this case mass plays the role of parameter \( \beta \)).

It is worth mentioning that a similar theorem can be proved that relates the jump in \( \tilde{N}_3 \) with the number of massless fermions that exist in the \( \omega = 0 \) hypersurface of the Brillouin zone within the system at the interface between two gapped states. (Illustration of this theorem see also in Section 6).

6  Momentum space topology of lattice regularized model

6.1  \( \tilde{N}_3 \) for free Wilson fermions

In this section we consider the case of the single flavor of lattice Dirac fermions. In lattice regularization the periodic boundary conditions are used in space direction and antiperiodic boundary conditions are used in the imaginary time direction. The momenta to be considered, therefore, also belong to a lattice:

\[
p_a = \frac{2\pi K_a}{N_a}, \quad p_4 = \frac{2\pi K_4 + \pi}{N_t}, \quad K_a, K_4 \in Z, \quad a = 1, 2, 3
\]  \( (37) \)

Here \( N_a, N_t \) are the lattice sizes in \( x, y, z \), and imaginary time directions, correspondingly. However, for the lattice of infinite volume we recover continuous values of momenta that belong to the 4D torus.

Let us consider the simplest regularization of fermion action called Wilson fermions. In the absence of interactions the Green function has the form:

\[
G = \left(\sum_a \Gamma_a \sin p_a - i(m + \sum_a (1 - \cos p_a))\right)^{-1}.
\]

\[
= \frac{\sum_a \Gamma_a \sin p_a + i(m + \sum_a (1 - \cos p_a))}{\sum_a \sin^2 p_a + (m + \sum_a (1 - \cos p_a))^2}, \quad a = 1, 2, 3, 4
\]  \( (38) \)
\[ m = Ma \]

Table 1: The spectrum of the system with free Wilson fermions. In the first column the values of \( m \) are specified. In the other columns masses of the doublers are listed. Everywhere it is implied that \( |Ma| << 1 \), where \( a \) is the lattice spacing. Expression \( v^x \) means \( x \) states with masses equal to \( v \).

| \( m = Ma \) | \( M^1 \) | \( (2/a + M)^4 \) | \( (4/a + M)^b \) | \( (6/a + M)^4 \) | \( (8/a + M)^1 \) |
|-------------|--------|----------------|----------------|----------------|----------------|
| \( m = -2 + Ma \) | \( (-2/a + M)^1 \) | \( M^4 \) | \( (2/a + M)^b \) | \( (4/a + M)^4 \) | \( (6/a + M)^1 \) |
| \( m = -4 + Ma \) | \( (-4/a + M)^1 \) | \( (2/a + M)^4 \) | \( (2/a + M)^4 \) | \( (4/a + M)^1 \) |
| \( m = -6 + Ma \) | \( (-6/a + M)^1 \) | \( (-2/a + M)^b \) | \( (4/a + M)^4 \) | \( (2/a + M)^1 \) |
| \( m = -2 + Ma \) | \( (-8/a + M)^1 \) | \( (-6/a + M)^4 \) | \( (2/a + M)^1 \) | \( (2/a + M)^4 \) |

Table 2: The values of topological invariants \( \tilde{N}_3 \) and \( \tilde{N}_5 \) for free Wilson fermions.

| \( m \) | \( \tilde{N}_3 \) | \( \tilde{N}_5 \) |
|--------|-------------|-------------|
| \( m > 0 \) | 0 | 0 |
| \( -2 < m < 0 \) | -2 | -2 |
| \( -4 < m < -2 \) | 4 | 6 |
| \( -6 < m < -4 \) | -2 | -6 |
| \( -8 < m < -6 \) | 0 | 2 |
| \( m < -8 \) | 0 | 0 |

In Table 1 we represent the spectrum of the model for different values of \( m \). The diagonal values in this table represent physical massive states while the off-diagonal elements of the table represent unphysical doublers with masses that tend to infinity in the limit \( a \to 0 \). Due to the periodical boundary conditions \( \tilde{N}_3 \) remains the topological invariant on the torus. In Table 2 we represent the values of \( \tilde{N}_3 \) versus the values of \( m \). Here \( \tilde{N}_3 = \sum_{i=0}^{3} (-1)^i C_3^i \frac{m+2i}{|m+2i|} \), where the sum is over the fermion doublers in 3D, \( m+2i \) is the mass of the \( i \)-th doubler while \( C_3^i \) is its degeneracy (for the derivation see [8]). It is worth mentioning that on the given lattice \( \tilde{N}_4 = 0 \). The value of \( \tilde{N}_4 \) may be different from zero if \( G \) has poles or zeros somewhere in 4D momentum space.

Let us notice that the values of the considered topological invariants on the lattice with periodical boundary conditions contradict to the continuum result. This is due to the fermion doublers that give essential contributions in spite of their unphysically large masses.
6.2 $\tilde{N}_5$ for free Wilson fermions

As follows from Table 2, the invariant $\tilde{N}_3$ does not resolve between the fully gapped states in the region $-8 < m < -6$ and in the region $m < -8$, which are separated by the gapless state at $m = -8$ and thus should be topologically different. The reason for that is that the invariant $\tilde{N}_3$ is determined only at $\omega = 0$, while at $m = -8$ the only gapless state has nonzero $\omega$ (that is $\cos p_4 = -1$). Since at $m = -8$ there are no gapless fermions with $\omega = 0$, there is no jump in $\tilde{N}_3$ across this transition. However, the topological invariant, which is determined for all $\omega$, may differentiate between the two states with $\tilde{N}_3 = 0$. The invariant $\tilde{N}_5$ plays such a role, and one has the index theorem (see Section 5.2), which states that the total number $n_F$ of gapless fermions emerging at the critical values of mass $m$ is determined by the jump in $\tilde{N}_5$. While the jump $\Delta \tilde{N}_3$ across the transition determines the number of those gapless fermions which live in the vicinity of zero frequency, $n_F(\omega = 0)$. The resulting index theorems read:

$$n_F = \frac{1}{2} \Delta \tilde{N}_5, \quad n_F(\omega = 0) = \frac{1}{2} \Delta \tilde{N}_3. \quad (39)$$

According to Section 5.2 this equation is valid for the interacting systems too. In some cases, however, when momentum space has a complicated form, one must take into account not only poles of the Green’s function of massless fermions but also zeroes [16, 22].

For the lattice model with Wilson fermions we use the method described in Section 5.1 in order to calculate the invariant $\tilde{N}_5$. One obtains that the zeros of $\hat{g}^a$ appear at the positions of the fermion doublers. The corresponding values of sign $\hat{g}_5$ coincide with the signs of the masses of the doublers. The positions of doublers are $p_{n_i} = (\pi n_1, \pi n_2, \pi n_3, \pi n_4)$, $n_i = 0, 1$. We also have $\partial_b \hat{g}^a(p_{n_i}) \sim (-1)^n (\frac{1}{b m_{n_i}})$, where $m_{n_i}$ is the mass of the doubler. That’s why, using expression (31) we obtain:

$$\tilde{N}_5 = N'_4 = \sum_{k=0}^4 (-1)^k C_4^k \frac{m + 2k}{|m + 2k|}, \quad (40)$$

where the sum is over the fermion doublers in 4D, $m + 2k$ is the mass of the $k$-th doubler while $C_4^k$ is its degeneracy. This is similar to the result obtained in [8] for the different space-time dimension. The values of the invariant $\tilde{N}_5$ as a function of bare mass $m$ are also represented in Table 2.
The index theorem Eq.(39) gives at the transition points \( m = 0, -2, -4, -6, -8 \) the number of gapless fermions correspondingly \( n_F = 1, 4, 6, 4, 1 \) and \( n_F(\omega = 0) = 1, 3, 3, 1, 0 \).

### 6.3 Turning on interactions

When the interaction of Wilson fermions with the lattice gauge field \( U = e^{iA} \) defined on links is turned on, we have (in coordinate space):

\[
G(x, y) = i \frac{1}{Z} \int DU \exp\left(-S_G[U]\right) \text{Det}(D[U, m])D^{-1}_{x,y}[U, m]
\]

where \( S_G \) is the gauge field action while

\[
D_{x,y}[U, m] = -\frac{1}{2} \sum_i [(1 + \gamma^i)\delta_{x+e_i,y}U_{x+e_i,y} + (1 - \gamma^i)\delta_{x-e_i,y}U_{x-e_i,y}] + (m + 4)\delta_{xy}
\]

Here \( e_i \) is the unity vector in the \( i \)-th direction. We expect that the Green function in momentum space has the form [19]:

\[
G = \frac{Z[p]}{\sum_a \Gamma_a g_a[p] - im[p]}
\]

with unknown functions \( Z[p], g_a[p] \), and \( m[p] \). That’s why the Green function for the interacting Wilson fermions has the form (41) as well as the Green function for the noninteracting fermions.

Free Wilson fermions exist in several phases for different values of \( m \). These phases are marked by the values of \( \tilde{N}_5 \). When the interactions are turned on, the value of \( \tilde{N}_5 \) remains equal to its initial number until the phase transition is encountered. This is related to the fact that zeros of \( g_a[p] \) cannot disappear without changing the phase structure of the model. These zeros, in turn, are related to the value of \( \tilde{N}_5 \) according to (22). It is worth mentioning that in order to consider interactions with nonabelian gauge fields we must consider several flavors of the fermions as it was done in the previous sections. All results obtained in the present section may be generalized to this case in a straightforward way.
In the present paper we discuss the topological invariants in momentum space of the 4D models with the mass gap. In particular, the expression for $\tilde{N}_3$ introduced in [13] is considered. We find the conditions under which $\tilde{N}_3$ is the topological invariant. We also introduce new functional $\tilde{N}_5$ that is represented as an integral over the whole 4D momentum space. $\tilde{N}_5$ is obtained via the dimensional reduction of the 5D construction. In the form of Eq. (22) this functional appears to be the topological invariant for the wide class of models. In particular, it is the invariant for the asymptotic free gauge theories and for the models with compact closed momentum space. In addition, we prove the index theorem that defines the number of massless fermions $n_F$ in the intermediate vacuum, which exists at the transition line between the massive vacua with different values of $\tilde{N}_5$.

In order to illustrate the properties of the mentioned constructions we consider the lattice model with Wilson fermions. We found that the vacuum states of lattice models with fully gapped fermions (insulating vacua) in 4D space-time are characterized by two topological invariants, $\tilde{N}_3$ and $\tilde{N}_5$. Due to the index theorem mentioned above, they are responsible, in particular, for the number of gapless fermions which appear at the topological transition between the massive states with different topological charges. The vacuum in the intermediate state represents the topological semimetal. According to Eq. (39) the jump in $\tilde{N}_5$ determines the total number of massless fermions in the intermediate state, while the jump in $\tilde{N}_3$ gives the number of those massless fermions, which have $\omega = 0$.

The simple pattern described above may have a certain connection to the unification of fundamental interactions. Namely, masses of all elementary particles are extremely small compared to the Planck scale, which may indicate that we live in the vicinity of the critical line of quantum phase transition like the transition between the two gapped states in the model with Wilson fermions. There is a special reason why nature may choose the vicinity of the transition line: the massless fermions in the intermediate vacuum states are able to accommodate more entropy than the gapped states [13].

It would be interesting to apply the analysis similar to the presented here to the consideration of the other lattice models with more complicated structure of momentum space (for example, to the overlap fermions [26]).

Also it would be interesting to consider lattice models that describe vacua with broken symmetry, such as superconducting vacuum in Ref. [25], tech-
nicolor models with spontaneously broken chiral symmetry etc. However, such an analysis may necessarily involve higher Green functions because the two-point functions only do not describe appearance of the condensates completely.

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