Throughput-Improving Control of Highways Facing Stochastic Perturbations

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Abstract

In this article, we study the problem of controlling a highway segment facing stochastic perturbations, such as recurrent incidents and moving bottlenecks. To model traffic flow under perturbations, we use the cell-transmission model with Markovian capacities. The control inputs are: (i) the inflows that are sent to various on-ramps to the highway (for managing traffic demand), and (ii) the priority levels assigned to the on-ramp traffic relative to the mainline traffic (for allocating highway capacity). The objective is to maximize the throughput while ensuring that on-ramp queues remain bounded in the long-run. We develop a computational approach to solving this stability-constrained, throughput-maximization problem. Firstly, we use the classical drift condition in stability analysis of Markov processes to derive a sufficient condition for boundedness of on-ramp queues. Secondly, we show that our control design problem can be formulated as a mixed integer program with linear or bilinear constraints, depending on the complexity of Lyapunov function involved in the stability condition. Finally, for specific types of capacity perturbations, we derive intuitive criteria for managing demand and/or selecting priority levels. These criteria suggest that inflows and priority levels should be determined simultaneously such that traffic queues are placed at locations that discharge queues fast. We illustrate the performance benefits of these criteria through a computational study of a segment on Interstate 210 in California, USA.

Index terms: Traffic control, Piecewise-deterministic Markov processes, Stability analysis.

1 Introduction

Advances in smart transportation provide system operators with opportunities for improving highway system performance. Technologies such as dynamic tolling and traveler information systems can enable demand management by influencing travelers’ route choices [7]. Embedded sensors and dynamic traffic signal controllers enable ramp metering, which can be viewed as a way of allocating limited highway capacity between on-ramp and mainline [32]. Although numerous efforts have been devoted to design of performance-improving demand management and ramp metering strategies, the existing approaches typically assume deterministic highway

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capacities [5][11]. In this article, we are concerned with situations in which the highway capacity is subject to recurrent perturbations [1][16][18].

We consider a control-theoretic approach to jointly determining control inputs for demand management and capacity allocation in a highway segment facing stochastic capacity perturbations via the following formulation:

\[
\begin{align*}
\text{max} & \quad \text{throughput} \\
\text{s.t.} & \quad \text{every on-ramp queue is stable}, \\
& \quad \text{practical constraints on control inputs}.
\end{align*}
\]

We define *throughput* as the time-average rate at which the highway discharges traffic through the off-ramps. The control inputs are (i) the amount of *inflow* that is admitted at each entrance, and (ii) the *priority* of each on-ramp with respect to the mainline. For demand management, we focus on determining the inflow that leads to good performance. For capacity allocation, we focus on zero-one (non-fractional) priority between on-ramps and mainline. One can interpret *on-ramp priority* as the configuration where the system operator does not intervene an on-ramp, and *mainline priority* as where the on-ramp discharge is restricted (or *metered*) to avoid mainline congestion [10]. Our formulation seeks to determine a static control scheme in that the inflow and priority only depend on system parameters and not on state measurements, which is common in current practice [19]. In addition, the stability constraint ensures that every on-ramp queue is bounded on average, and the practical constraints ensure that the control inputs are selected from a practically reasonable domain. This paper does not consider realization of a particular inflow or parameterization of ramp metering algorithms to implement a particular priority. We refer readers to [2][7][22] for mechanisms that implement demand management and [17][20][26] for commonly used forms of ramp controllers.

Prior literature has indeed considered related control system design but mainly for deterministic settings. Papageorgiou [27] provided an overview of various traffic control schemes to reduce travel time and improve throughput. In terms of performance optimization over finite time horizons, results associated with minimization of travel time [30], weighted sum of travel time and throughput [10], or generic cost functions [4][28] have been reported. In terms of long-time performance, researchers have been focusing on the stability/convergence of traffic states [5][11]. Importantly, Varaiya [32] pointed out the need to jointly consider demand management and capacity allocation to attain higher long-run throughput.

In this article, we develop a systematic approach to solving the problem (P0) in a stochastic setting. In our previous work [13], we showed that capacity perturbations may destabilize on-ramp queues that are stable in the deterministic, nominal/average setting. The main reason is that capacity perturbations can induce traffic queues and bottlenecks that do not exist in the nominal setting and are thus not accounted for by nominally efficient controllers. This article extends the performance analysis in [13] to solve the control design problem.

To model the traffic flow dynamics under perturbations, we adopt the *stochastic switching cell transmission model* (SS-CTM) introduced in [13][15]; see Section 2. This model extends the well-known CTM [6] by introducing two features: stochastically varying capacity of certain cells and infinite-sized buffers representing the on-ramps. The state variables of the SS-CTM include traffic densities in the mainline cells and queues at the on-ramps. In this model, cell capacities stochastically evolve according to a Markov chain. Our model is also related to a class of models capturing the intrinsic randomness in traffic flow dynamics [31][34], and stochastic models for demand perturbation reported in the literature [23][29].
A major challenge of formulating the max-throughput problem is to derive the stability constraint in $[P_0]$. In Section 3, we use the classical drift condition in stability analysis of Markov processes [3, 9, 24] and properties of the CTM dynamics [11, 13] to establish a stability condition. However, application of standard stability results to our setting is not straightforward, since those results require construction of a Lyapunov function for the SS-CTM dynamics that "drifts downwards" in expectation everywhere over the continuous state space; this essentially involves solving a non-linear and non-convex optimization. To address this challenge, we extend the approach in [13] and utilize properties of the controlled traffic dynamics to show that it suffices to verify the "drift condition" only over a finite set of states.

We formulate $[P_0]$ either as a mixed integer bilinear or linear program (MIBLP/MILP), depending on whether the Lyapunov function parameters are a priori specified or solved as part of the decision variables; see Section 4. Furthermore, we characterize the structure of a class of optimal traffic control inputs under particular assumptions. For isolated merges, we show that the optimal solution prioritizes the on-ramp if it has a smaller capacity-to-inflow margin than the mainline. We call this structure the margin criterion (Propositions 2–3). An intuitive interpretation of this criterion is that, the traffic queue induced by capacity perturbations should be placed at a location (either mainline or on-ramp) that discharges the queue faster; the speed of queue discharge is quantified by the capacity-to-inflow margin. For more general settings, this criterion is implicit in the optimal solution to the MIBLP/MILP formulation.

In Section 5, we present a computational study for the margin criterion via simulation of a stretch of the State Route 134 (SR134)/Interstate 210 East-Bound (I210-EB) with capacity perturbations. The simulation results show that, compared to the classical scheme of mainline priority, the margin criterion reduces perturbation-induced throughput loss during peak hours, the longest on-ramp queue size, and the total delay (both on the mainline and at the on-ramps).

To sum up, the main contributions of this article include:
1. an easily checkable sufficient condition (Theorem 1) for stability of the on-ramp queues,
2. a mixed integer bilinear/linear program (MIBLP/MILP) formulation of the max-throughput problem based on the stability condition,
3. characterization of optimal control inputs (margin criterion) under particular assumptions (Propositions 2–3), and
4. evaluation of the effectiveness of the margin criterion via simulation.

2 Highway Traffic Model with Stochastic Capacities

We consider the SS-CTM, which models a highway segment as a compartmental system of $K$ mainline cells with on-ramp buffers and off-ramp exits, as shown in Figure 1. This model is based on [11, 13] with the exception that in this paper we also explicitly consider the on-ramp queues. For ease of presentation, we assume that every cell has a unit length of 1 km. The other parameters of traffic dynamics include the free-flow speed $\alpha_k$ (km/hr), the nominal capacity $F_k$ (veh/hr), the congestion wave speed $\beta_k$ (km/hr), and the jam (maximal) density $n^\text{max}_k$ (veh/km), where $k$ is the cell index. The $k$ buffer has a saturation rate $R_k$ (veh/hr).

Let $x = (q, n)$ denote the continuous state of the highway, where $q = [q_1 \, q_2 \, \cdots \, q_K]^T$ is the vector of lengths of the on-ramp queues and $n = [n_1 \, n_2 \, \cdots \, n_K]^T$ is the vector of traffic densities in mainline cells. For simplicity, we assume that every buffer has an infinite size; thus, the set of permissible queue lengths is $Q = \mathbb{R}_\geq 0^K$. The set of traffic densities is $\mathcal{N} := \prod_{k=1}^K [0, n^\text{max}_k]$. By including on-ramp queues as state variables, we can explicitly track the queuing induced by
capacity perturbations.

In our model, a control input is denoted by $u = (v, w)$, where $v = [v_1 \ v_2 \ \cdots \ v_K]^T$ denotes the vector of inflows (i.e., the demands that are admitted into the system) at the on-ramps, and $w = [w_1 \ w_2 \ \cdots \ w_K]^T$ denotes the vector of priorities assigned to the on-ramp traffic (with respect to the mainline traffic). Thus, each cell-buffer pair has two control inputs. The first input is $v_k \in [0, d_k]$, the inflow into mainline from on-ramp $k$; here $d_k$ denotes the exogenously given (fixed) demand parameter at the $k$th on-ramp (see Figure 1). Also, we denote $d = [d_1 \ d_2 \ \cdots \ d_K]^T$. We assume that any non-admitted demand is permanently rejected from the system and not redistributed to other locations. The second control input $w_k$ denotes the priority of inflow from buffer $k$ with respect to the mainline. Specifically, $w_k = 1$ (resp. $w_k = 0$) means that the inflow from the $k$th on-ramp (resp. mainline) is prioritized over the flow from the mainline (resp. $k$th on-ramp).

We adopt a Markovian capacity model for a class of capacity perturbations, previously introduced in [13,15,16]. This model considers that the cell capacities stochastically vary over time according to a finite-state Markov process over a set of modes denoted by $I$. The inter-mode transition rates are $\{\nu_{ij}: i, j \in I\}$. Every mode $i$ is associated with a vector of cell capacities $F(i) = [F_1(i) \ \cdots \ F_K(i)]^T$. For ease of presentation, we assume that $F_k(i) \in \{F_k, F_k - \Delta_k\}$ for each $k$ and for all $i$; that is, the capacity of the $k$th cell can only switch between two values $F_k$ and $F_k - \Delta_k$, where $\Delta_k \geq 0$ characterizes the intensity of capacity perturbation. The structure of this Markov chain depends on the particular type of capacity perturbation. Let $I(t)$ be the mode at time $t$. We assume that the Markov chain governing the the mode transition process $\{I(t): t \geq 0\}$ is ergodic and associated with a unique (row) vector of steady-state probabilities $p = [p_0 \ p_1 \ \cdots \ p_m]$ such that

$$\sum_{j \in I} \nu_{ij} p_i = \sum_{j \in I} \nu_{ji} p_j \quad \forall i \in I, \quad |p| = 1, \quad p \geq 0. \quad (1)$$

This assumption is practically reasonable, since we focus on capacity perturbations that recurrently occur and terminate. Furthermore, we exclude self-transitions by setting $\lambda_{ii} = 0$ for all $i \in I$, since such transitions do not provide significant modeling advantage for our purpose.

We use $(i, x) = (i, q, n)$ to denote state variables and $(I(t), X(t)) = (I(t), Q(t), N(t))$ to denote (hybrid) stochastic process. Following this convention, we let $Q_k(t)$ be the queue length in the $k$th buffer and $N_k(t)$ denote the traffic density in the $k$th cell at time $t$, and thus $Q(t) = [Q_1(t) \ \cdots \ Q_K(t)]^T \in Q$ and $N(t) = [N_1(t) \ \cdots \ N_K(t)]^T \in N$, where $Q = [0, \infty]^K$ and $N = [n_{\min}, \infty]^K$. The former and latter denote (hyrbid) stochastic process. Following this convention, we let $Q_k(t)$ be the queue length in the $k$th buffer and $N_k(t)$ denote the traffic density in the $k$th cell at time $t$, and thus $Q(t) = [Q_1(t) \ \cdots \ Q_K(t)]^T \in Q$ and $N(t) = [N_1(t) \ \cdots \ N_K(t)]^T \in N$, where $Q = [0, \infty]^K$ and $N = [n_{\min}, \infty]^K$.

\footnote{Instead of $v_k \in [0, d_k]$ for each $k$, we can impose the constraint $0 \leq \sum_{k=1}^K v_k \leq \sum_{k=1}^K d_k$ to allow for redirection of demand.}

\footnote{In the context of finite-state Markov chains, a state is recurrent if the expected time of two consecutive visits of this state is finite.}
\[ \prod_{k=1}^{K} [0, n_k^{\max}]. \] For a fixed control input \( u = (v, w) \in [0, d] \times \{0, 1\}^K \), the stochastic dynamics of the mode \( I(t) \), the on-ramp queues \( Q(t) \), and traffic densities \( N(t) \) can be written as follows:

\[
\Pr \left\{ I(t + \delta) = j | I(t) = i, I(s), s < t \right\} = \nu_{ij} \delta + o(\delta) \quad i, j \in \mathcal{I} : j \neq i, \tag{2a}
\]

\[
\dot{Q}(t) = G(I(t), X(t), u), \tag{2b}
\]

\[
\dot{N}(t) = H(I(t), X(t), u), \tag{2c}
\]

where \( G : \mathcal{I} \times (\mathcal{Q} \times \mathcal{N}) \times ([0, d] \times \{0, 1\}^K) \to \mathbb{R}^K \) and \( H : \mathcal{I} \times (\mathcal{Q} \times \mathcal{N}) \times ([0, d] \times \{0, 1\}^K) \to \mathbb{R}^K \) are vector fields governing the dynamics of on-ramp queues and cell traffic densities, which are described below.

For each cell \( k \), the sending flow \( S_k \) and the receiving flow \( T_k \) can be written as follows:

\[
S_k(i, x) := \min \{ \alpha_k n_k, F_k(i) \} \quad k = 1, 2, \ldots, K, \tag{3a}
\]

\[
T_k(x) := \beta_k (n_k^{\max} - n_k) \quad k = 1, 2, \ldots, K, \tag{3b}
\]

where \( S_k \) is the traffic flow that cell \( k \) can discharge downstream and \( T_k \) is the traffic flow from upstream that cell \( k \) can accept. For \( k = 1, \ldots, K - 1 \), let \( \rho_k \in (0, 1] \) denote the fixed mainline ratio, i.e. the fraction of traffic from cell \( k \) entering cell \( k + 1 \); the remaining traffic flow leaves the highway at the \( k \)th off-ramp. Since the mainline ends at cell \( K \), we have \( \rho_K = 0 \). In addition, we define

\[
\rho_k^k := 1 \quad k = 1, \ldots, K, \tag{4a}
\]

\[
\rho_k^{k_2} := \prod_{h=k_1}^{k_2-1} \rho_h \quad 1 \leq k_1 \leq k_2 - 1, \quad k_2 = 2, \ldots, K. \tag{4b}
\]

Note that \( \rho_k^{k_2} \) can be viewed as the fraction of the flow out of cell \( k_1 \) that eventually goes through cell \( k_2 \).

We assume that every on-ramp is a fluid queueing system with an infinite buffer size. That is, the sending flow from the \( k \)th buffer is given by

\[
D_k(x, u) = \left\{ \begin{array}{ll}
\min \{v_k, R_k\} & q_k = 0, \\
R_k & \text{o.w.}
\end{array} \right. \quad k = 1, \ldots, K.
\]

We refer readers to [25] for more information on fluid queueing systems.

The flow \( r_k \) (resp. \( f_k \)) discharged by the \( k \)th buffer (resp. cell) is defined as follows:

\[
r_k(i, x, u) := \min \left\{ D_k(x, u), \left( T_k(x) - (1 - w_k)S_k(i, x) \right)_+ \right\} \quad k = 1, \ldots, K, \tag{5a}
\]

\[
f_k(i, x, u) = \min \left\{ S_k(i, x), \left( T_{k+1}(x) - w_k D_{k+1}(x, u) \right)_+ \right\} \quad k = 1, \ldots, K - 1, \tag{5b}
\]

\[
f_K(i, x, u) = S_K(i, x). \tag{5c}
\]

where \((\cdot)_+\) stands for the positive part. In the above, we make the standard assumption that the \( K \)th cell is not constrained from downstream [11]. One can see from (5a)–(5c) that the priority \( w_k \) determines whether the available receiving flow \( T_k \) is first allocated to the on-ramp \((w_k = 1)\) or to the mainline \((w_k = 0)\).

We say that cell \( k \) is experiencing spillback at time \( t \) if \( S_k(t) < f_k(t) \), i.e. if the sending flow from cell \( k \) is strictly less than the actual flow. Furthermore, we say a buffer (resp. cell) \( k \) to be
congested at time $t$ if $Q_k(t) > 0$ (resp. $N_k(t) \geq F_k/\alpha_k$, where $F_k/\alpha_k$ is the critical density \cite{11}). Finally, we say that buffer $k$ is a bottleneck at time $t$ if buffer $k$ is congested but cell $k$ is not, and that cell $k$ is a bottleneck at time $t$ if cell $k$ is congested but cell $k+1$ is not.

**Remark 1** Our notion of bottlenecks can be viewed as a time-varying extension of the static notion considered in \cite{11}, which considers a setting with constant demand and constant capacities. In our model, a particular congestion pattern can recurrently occur and disappear with the occurrence and clearance of capacity perturbations.

Then, the vector fields $G$ and $H$ in (2b)–(2c) follow from mass conservation:

$$G_k(i, x, u) := v_k - r_k(i, x, u) \quad k = 1, \ldots, K,$$

$$H_1(i, x, u) := r_1(i, x, u) - f_1(i, x, u),$$

$$H_k(i, x, u) := \rho_{k-1}f_{k-1}(i, x, u) + r_k(i, x, u) - f_k(i, x, u) \quad k = 2, \ldots, K. \quad (6c)$$

One can show that $Q(t)$ and $N(t)$ are always continuous in time \cite{13, 14}. Also note that $G$ and $H$ are non-linear in $q, n$.

For a given control input $u = (v, w)$, we say that the SS-CTM is stable if the limiting time-average on-ramp queues are bounded; i.e., there exists $Z < \infty$ such that for each initial condition $(i, q, n) \in \mathcal{I} \times \mathcal{Q} \times \mathcal{N}$,

$$\limsup_{t \to \infty} \frac{1}{t} \int_{s=0}^{t} \mathbb{E}(|Q(s)|) \, ds \leq Z. \quad (7)$$

We say that a demand vector $d$ is feasible if there exists a stabilizing control input $u = (v, w)$ such that $v = d$, and infeasible otherwise.

Since (7) involves the computation of the limiting time-average of the expected queue length, it is not easy to obtain stability directly from the definition. Instead, we focus on deriving a sufficient condition for stability of the SS-CTM in the form

$$C(u, \theta) \leq 0 \quad (8)$$

where $C$ is a vector-valued function. Note that $C$ may contain auxiliary decision variables $\theta$ apart from the control inputs. Thus, the set of $u$ satisfying $C(u, \theta) \leq 0$ is a subset of stabilizing control inputs.

Next, for a given control input $u = (v, w)$, throughput is defined as the time-average off-ramp flows discharged by the highway:

$$J(u) := \lim_{t \to \infty} \frac{1}{t} \int_{s=0}^{t} \sum_{k=1}^{K} (1 - \rho_k)f_k(I(s), (Q(s), N(s)), u) \, ds. \quad (9)$$

Direct computation of the above limit involves integration of traffic flows, which evolve according to non-linear stochastic dynamics (2c)–(2b), and is thus not easy. However, we note that, if the on-ramp queues are stable in the sense of (7), then, by mass conservation, the time-average flow out of a cell becomes equal to the total inflow into the cell almost surely (a.s.):

$$\lim_{t \to \infty} \frac{1}{t} \int_{s=0}^{t} f_k(I(s), Q(s), N(s)) \, ds = \sum_{h=1}^{k} \rho_k v_h \quad a.s. \quad k = 1, \ldots, K.$$
Since $\sum_{h=k}^{K}(1-\rho_h)\rho_h^k = 1$, we can rewrite the throughput as

$$J_{a.s.} = \sum_{k=1}^{K} \sum_{h=1}^{h=k} (1-\rho_k)\rho_h^k v_{h} = \sum_{k=1}^{K} \left( \sum_{h=k}^{K} (1-\rho_h)\rho_h^k \right) v_k = \sum_{k=1}^{K} v_k. \quad (9a)$$

Then, instead of the original formulation (P0), we consider the following problem:

$$\max_{u} J = \sum_{k=1}^{K} v_k \quad (P)$$

s.t. $C(u, \theta) \leq 0$

$$u \in [0, d] \times \{0, 1\}^K.$$ 

Since $C \leq 0$ is a sufficient condition, every $u$ that is feasible w.r.t. (P) is also feasible w.r.t. (P0), and the optimal objective value of (P) is a lower bound for that of (P0).

3 Stability analysis

In this section, we develop a sufficient condition for the boundedness of on-ramp queues that can be expressed as (8), and discuss practical insights given by our analysis. In Section 3.1, we give the expression for an invariant set of the SS-CTM (Proposition 1), and discuss its implication for identification of recurrent congestion patterns and bottleneck locations. In Section 3.2, we derive the stability condition (Theorem 1) based on the invariant set. In addition to being constraints in the max-throughput problem, these results themselves lead to useful insights for traffic control, which we demonstrate via Examples 1–2.

3.1 Invariant set

Following [3], a set $M \subseteq Q \times N$ is a globally attracting and positively invariant set for the continuous state $x = (q, n)$ if

$$\forall (I(0), X(0)) \in I \times (Q \times N), \forall \epsilon > 0, \exists T \geq 0,$$

$$\forall t \geq T, \min_{x \in M} \| (X(t)) - x \|_2 \leq \epsilon,$$

$$\forall (I(0), X(0)) \in I \times M, \forall t \geq 0, \ (X(t)) \in M. \quad (10a)$$

Intuitively, $M$ is a set of states such that, for all initial conditions $(i, q, n) \in I \times Q \times N$, the process $(I(t), Q(t), N(t))$ eventually enters and does not leave $M$. For convenience, we henceforth refer to any set satisfying (10a)–(10b) simply as an invariant set.

For a given control input $u \in [0, d] \times \{0, 1\}^K$, we explicitly construct an invariant set of the continuous state $x = (q, n)$ as follows:

$$M(u) := \bigcup_{q \in \prod_{k=1}^{K} [q_k(u), \bar{q}_k(u)]} \left\{ \{q\} \times \prod_{k=1}^{K} [\pi_k(q, u), \bar{\pi}_k(u)] \right\}, \quad (11)$$

where

$$q_k(u) := \begin{cases} \infty & \text{if } v_k > \mathcal{R}_k \\ 0 & \text{o.w.} \end{cases} \quad k = 1, \ldots, K. \quad (12a)$$
Note that all the above quantities depend on the control input $u$. Since we consider fixed control inputs in this section, for notational convenience, we simply write $\nu_i$, $\bar{\nu}_k$, $\bar{n}_k(q)$, and $\bar{n}_k$. In addition, $M(u)$ only involves the mode-specific capacities $F(i)$ (see (12b), (12c)), but is independent of the transition rates $\nu_{ij}$. Furthermore, we can partition $M$ into two subsets:

$$M_0(u) := \{(q,n) \in M(u) : q = 0\},$$

$$M_1(u) := \{(q,n) \in M(u) : q \neq 0\}.$$  

(13a)

(13b)

Note that at most one of these sets could be empty.

**Proposition 1 (Invariant set)** For a given control input $u \in [0,d] \times \{0,1\}^K$, the set $M(u)$ as defined in (11) is globally attracting and positively invariant. Furthermore, the SS-CTM is stable if $M_1(u) = \emptyset$, and is unstable if $M_0(u) = \emptyset$.

The proof of this proposition is in Appendix A.1 With this result, for stability analysis, we can restrict our attention to the evolution of the continuous states over the invariant set $M$ instead of the entire continuous state space $Q \times N$.

**Remark 2** $M(u)$ is a generalization of the invariant set proposed in [13], which does not consider the impact of on-ramp queues $q_2, \ldots, q_K$.

For each control input $u = (v,w) \in [0,d] \times \{0,1\}^K$, we also define a finite set of states $\mathcal{V}(u) \subset Q \times N$:

$$\mathcal{V}(u) := \bigcup_{q \in \prod_{k=1}^{K} \{q_k, \bar{q}_k\}} \left( \{q\} \times \prod_{k=1}^{K} \{\bar{n}_k(q), \bar{n}_k\} \right).$$

(14)

Furthermore, define

$$\mathcal{V}_0(u) := \{(q,n) \in \mathcal{V}(u) : q = 0\},$$

$$\mathcal{V}_1(u) := \{(q,n) \in \mathcal{V}(u) : q \neq 0\}.$$  

(15a)

(15b)

Note that $\mathcal{V}(u)$, $\mathcal{V}_0(u)$, and $\mathcal{V}_1(u)$ are the vertices of $M(u)$, $M_0(u)$, and $M_1(u)$, respectively. We also refer to $\mathcal{V}(u)$ as the set of *critical states*, since they represent typical congestion patterns.
that can recurrently happen due to capacity perturbations. By looking at the critical states, we can identify the bottlenecks that capacity perturbations can recurrently induce. We emphasize here that a bottleneck is not always a location of capacity perturbation. Due to spillback, even after a perturbation clears, traffic can still be “stuck” in an upstream cell/buffer, which we call an induced bottleneck. The critical states clearly show where these induced bottlenecks are.

Example 1 We consider a two-cell segment of Interstate 210 with an incident hotspot, as shown in Figure 2. Calibrated parameters are listed in Table 1. Furthermore, we assume the mode transition rates to be

\[ \lambda = 1[\text{hr}^{-1}], \quad \mu = 2[\text{hr}^{-1}], \]

where \( \lambda \) (resp. \( \mu \)) is the occurrence (resp. clearance) rate of capacity perturbations. The control input \( v = [7000 ~ 1000]^T \), \( w = [1 ~ 1]^T \) is used in this example. Figure 3 illustrates the invariant set as well as the critical states.
The critical states imply the following. First, cell 2 may be recurrently congested, which means that cell 2 is a bottleneck (see Figure 4(a)). This situation is captured by the critical state \(v_0,3\) and \(v_0,4\) in Figure 3(a) and \(v_{1,3}\) and \(v_{1,4}\) in Figure 3(b). For example, at state \(v_0,3\), we have \(q_1 > 0, q_2 = 0, n_1 \geq F_1/\alpha_1\), and \(n_2 \geq F_2/\alpha_2\). Second, it recurrently happens that cell 1 is congested but cell 2 is not; that is, although cell 1 does not directly experience capacity perturbations, this cell is an induced bottleneck. This situation is captured by the critical state \(v_{1,1}\) and \(v_{1,2}\) in Figure 3(b) and illustrated in Figure 4(b). Third, if buffer 1 is congested, cell 1 is necessarily congested. That is, buffer 1 is never a bottleneck, and the situation in Figure 4(c) does not happen recurrently.

### 3.2 Derivation of stability condition

Based on Proposition 1, we now develop a sufficient condition for stability of the SS-CTM with a given control input. To state the sufficient condition, let us define \(e\) to be a \(K\)-dimensional vector of 1’s, and define a constant matrix \(D = (d_{kh}) \in \mathbb{R}^{K \times K}\) such that

\[
d_{kh} = \begin{cases} 
1 & \text{if } k = h = 1 \text{ or } h = k - 1, \\
0 & \text{o.w.} 
\end{cases}
\]

Then, we can state the sufficient condition as follows.

**Theorem 1 (Stability condition)** Consider a \(K\)-cell SS-CTM with a set of modes \(\mathcal{I}\) and with a demand vector \(d \in \mathbb{R}_{\geq 0}^K\). For a given control input \(u = (v, w) \in [0, d] \times \{0, 1\}^K\), the SS-CTM is stable in the sense of (7) if there exist a symmetric \(K \times K\) matrix \(A\) satisfying

\[
a_{kh} \geq a_{k+1,h} \quad k = 1, \ldots, K - 1, \quad h = 1, \ldots, K, 
\]

\[
a_{K,K} > 0
\]

and a set of \(K\)-dimensional vectors \(\{b^{(i)}; i \in \mathcal{I}\}\), which jointly verify the following set of inequalities linear in \(A\) and \(b^{(i)}\):

\[
A\left(DG(i, x, u) + H(i, x, u)\right) + \sum_{j \in \mathcal{I}} \nu_{ij} \left(b^{(j)} - b^{(i)}\right) \leq -e \quad \forall (i, x) \in \mathcal{I} \times V_1(u).
\]

The proof of Theorem 1 is in Appendix A.2.

Theorem 1 specializes a more general result on the stability of continuous-time Markov processes, called the Foster-Lyapunov criterion [24]. To conclude stability, the generic result in
We require the matrix $A$ i.e. the set
\[ \{ \text{chosen Lyapunov function is negative in expectation for those states far away from the "origin",} \]
that is specifically tailored to the SS-CTM:
\[
V(i, x) := \frac{1}{2}(Dq + n)^T A(Dq + n) + (b^{(i)})^T(Dq + n).
\] (19)

We require the matrix $A$ in the above to satisfy \[17a\] to \[17b\] to ensure that
1. $V$ is norm-like, i.e. $\lim_{\|x\| \to \infty} V(i, x) = \infty$ for each $i \in I$, and
2. $V$ decreases as traffic moves downstream through the highway, i.e. for each $i \in I$, each $k \in \{1, \ldots, K - 1\}$, each $\delta > 0$, and each $(q, n) \in Q \times N$, we have
\[
\frac{\partial V(i, q', n')}{\partial n'_i} \bigg|_{q' = q, n' = n} \geq \frac{\partial V(i, q', n')}{\partial n'_2} \bigg|_{q' = q, n' = n} \geq \cdots \geq \frac{\partial V(i, q', n')}{\partial n'_K} \bigg|_{q' = q, n' = n} > 0.
\]

Recall from Figure 1 that both buffer $k$ and cell $(k - 1)$ are immediately upstream of cell $k$; thus, the traffic out of buffer $k$ and the traffic out of cell $k - 1$ merge with each other and cannot be further distinguished in our model. This feature of the SS-CTM is captured by the Lyapunov function which equally penalizes $q_k$ and $n_{k-1}$ for $k = 2, \ldots, K$ thanks to the structure of $D$. Also note that the vector $b^{(i)}$ depends on the mode $i$ while the matrix $A$ does not; thus, the second term in \[19\] is intended to capture the impact of mode transitions. Finally, we do not need to restrict the range of $b^{(i)}$, since only the differences between them are relevant; one can always set $b^{(i)} \geq 0$ to ensure that $V \geq 0$.

To formally state the Foster-Lyapunov criterion, we recall that the evolution of the process \((I(t), X(t)) = (I(t), Q(t), N(t))\) is captured by the \textit{infinitesimal generator} of the traffic flow dynamics \[13\]. Since $I(t)$ is a Markov chain and since $Q(t)$ and $N(t)$ are always continuous in $t$, this process is \textit{right-continuous with left limits (RCLL)}. Hence, by \[3\] Proposition 2.1], for a given control input $u = (v, w)$, the infinitesimal generator can be written as an operator $\mathcal{L}$ as follows:
\[
\mathcal{L}V(i, x) = G^T(i, x, u) \nabla_q V(i, x) + H^T(i, x, u) \nabla_n V(i, x) + \sum_{j \in I} \nu_{ij} \left( V(j, x) - V(i, x) \right)
\] (20)

where $\nabla_q V$ (resp. $\nabla_n V$) is the gradient of $V$ with respect to $q$ (resp. $n$). With the above definition, we can state the generic result as follows:

\textbf{Theorem (Foster Lyapunov criterion \[24\]). For an RCLL Markov process with state $\xi$ and invariant set $\Xi$, if there exist a norm-like function $V : \Xi \to \mathbb{R}_{\geq 0}$, a function $g : \Xi \to \mathbb{R}_{\geq 0}$, and constants $\epsilon > 0$ and $Z < \infty$ such that}
\[
\mathcal{L}V(\xi) \leq -\epsilon g(\xi) + Z \quad \forall \xi \in \Xi,
\] (21)
\text{then, for each initial condition $\xi \in \Xi$,}
\[
\limsup_{t \to \infty} \frac{1}{t} \int_{s=0}^{t} E[g(\xi(s))] ds \leq Z / \epsilon.
\] (22)

Essentially, the drift condition involves checking that the time derivative of an appropriately chosen Lyapunov function is negative in expectation for those states far away from the “origin”, i.e. the set $\{\xi \in \Xi : g(\xi) \leq Z / \epsilon\}$. In the context of our problem, the “origin” is the set $\mathcal{M}_0(u)$.\]
With the Lyapunov function as defined in (19), verifying the drift condition reduces to checking the feasibility of the system of linear inequalities (18). More importantly, we only require checking feasibility of this system for the finite number of states in the set $\mathcal{V}(u)$. Note that straightforward use of the generic result in [24] for the SS-CTM would require checking the feasibility of (18) everywhere over $\mathcal{Q} \times \mathcal{N}$, which essentially requires maximizing the left-hand side of (18) over $\mathcal{Q} \times \mathcal{N}$; this maximization is a non-linear and non-convex optimization problem. Fortunately, it turns out that it suffices to verify the drift condition (21) over the finite set of critical states $\mathcal{V}(u)$ instead of the invariant set $\mathcal{M}(u)$, as (18) indicates. The key is to show that $\mathcal{V}(u)$ contains the states where the expected time derivative of the Lyapunov function attains its maximum, i.e. where the rate of traffic discharge is minimal. Verifying the drift condition is straightforward over $\mathcal{V}_0(u)$, but requires more work over $\mathcal{V}_1(u)$; see Appendix A.2.

When applying Theorem 1, one can consider $A$ in (18) as an unknown to be solved. Alternatively, to simplify the analysis, $A$ can also be explicitly constructed using insights about the SS-CTM dynamics. A simple construction is given by

$$a_{kh} = \gamma,$$

(23)

where $\gamma$ is any constant in $[1, \infty)$. That is, the Lyapunov function $V$ as defined in (19) penalizes the on-ramp queues and mainline traffic densities equally. An alternative construction is given by

$$a_{kh} = \gamma(K - k + 1)(K - h + 1),$$

(24)

where $\gamma$ is any constant in $[1, \infty)$. This construction accounts for the spatial distribution of traffic: the Lyapunov function will strictly decrease as traffic is discharged downstream.

Example 2 (stability, stationary) Consider the two-cell highway with a stationary incident hotspot, as described in Example 1. For ease of presentation, we only consider on-ramp priority, i.e. $w = [1 1]^T$, and only vary $v = [v_1 v_2]^T$ over the set $[0, \infty)^2$ (assuming infinite demands). In this article, solutions of inequalities are obtained using YALMIP [21]. The results are illustrated in Figure 5 and the nomenclature of various regions is in Table 3.

![Figure 5: Stability of the SS-CTM with an incident hotspot and with various control inputs $[v_1 v_2]^T$.](image)
Table 2: Stability of various regions in Figure 5.

| Region | variable $a_{kh}$ | $a_{kh}$ from (23) | $a_{kh}$ from (24) |
|--------|------------------|------------------|------------------|
| 1      | Stable           | Stable           | Stable           |
| 2      | Stable           | Stable           | Unknown          |
| 3      | Stable           | Unknown          | Stable           |
| 4      | Stable           | Unknown          | Unknown          |
| 5      | Unknown          | Unknown          | Unknown          |
| 6      | Unstable         | Unstable         | Unstable         |

In Figure 5, the union of regions 1–4 is the set of inflows $v = [v_1 \ v_2]$ that satisfy Theorem 1 (with a variable $A$) and are thus stabilizing. In addition, every $[v_1 \ v_2]^T$ in region 6 exceeds the average capacity, and is thus destabilizing; see [13, Theorem 1] for a formal statement of this necessary condition for stability. Finally, there is a gap (region 5) between the sufficient condition and the necessary condition; for control inputs in that region, our stability condition does not give a conclusive answer. Furthermore, the unions of regions 1–2 is the set of stabilizing inflows resulting from an $A$ matrix specified by (23). Compared with a variable $A$, a matrix $A$ specified by (23) only increase the unknown region by a small size (union of regions 3 and 4). Hence, (23) gives an adequate proxy to a variable $A$ for this example.

4 Reformulation and analysis of max-throughput problem

Thanks to Theorem 1, we can formulate the max-throughput problem as follows:

$$\begin{align*}
\max \quad & J = \sum_{k=1}^{K} v_k \\
\text{s.t.} \quad & A \left( G(i, x, u) + DH(i, x, u) \right) + \sum_{j \in \mathcal{I}} \nu_{ij} \left( b^{(j)} - b^{(i)} \right) \leq -e \quad \forall (i, x) \in \mathcal{I} \times \mathcal{V}_1(u), \\
& u \in [0, d] \times \{0, 1\}^K.
\end{align*}$$

The decision variables of (P1) include the control input $u = (v, w)$, and Lyapunov function parameters $A$ and $b^{(i)}$. Since $G$ and $H$ as well as the critical states $\mathcal{V}(u)$ are non-linear in $u$, even with a given matrix $A$, (25a) is non-linear in the $u$. In Section 4.1. we address this challenge by reformulating (P1) such that:

1. When $A$ is considered as a variable, the reformulation is a mixed integer bilinear program (MIBLP) with a linear objective function and bilinear constraints;

2. When $A$ is chosen to be fixed, the reformulation is a mixed integer linear program (MILP). In Section 4.2 we study the structure of optimal solutions to the max-throughput problem for isolated merges, and derive useful insights for traffic control design, mainly for fixed $A$ matrices.

4.1 Reformulation

The main techniques involved in the reformulation include (i) substitution of $G$ and $H$ with a new set of variables,

$$\left\{ j_{k,y,z}^{(i)}, \bar{r}_{k,y,z}^{(i)} : k = 1, \ldots, K, \quad i \in \mathcal{I}, \quad y \in \{0, 1\}^K \setminus \{0\}^K, \quad z \in \{0, 1\}^K \right\}.$$
and (ii) elimination of the cross-product term \( v_k w_k \) (as appears in (12e)) using the big-\( M \) method. One can interpret \( \tilde{r}^{(i)}_{k,y,z} \) and \( \tilde{r}^{(i)}_{k,y,z} \) as the cell/buffer flow evaluated at the critical states. In addition, \( y \) and \( z \) are essentially indices for the critical states: \( y_k = 0 \) (resp. \( y_k = 1 \)) corresponds to \( q_k = 0 \) (resp. \( q_k = \infty \) or \( q > 0 \)), and \( z_k = 0 \) (resp. \( z_k = 1 \)) corresponds to \( n_k = \bar{n}_k(q) \) (resp. \( n_k = \bar{n}_k \)).

We reformulate (P1) as follows:

\[
\max J = \sum_{k=1}^{K} v_k \tag{P2}
\]

s.t. \( \forall i \in I, \forall y \in \{0,1\}^K \setminus \{0\}^K, \forall z \in \{0,1\}^K, \)

\[
a_{h,1} v_1 + \sum_{k=2}^{K} a_{h,k-1} v_k - \sum_{k=1}^{K-1} (a_{h,k} - a_{h,k+1})(\rho_k \tilde{r}^{(i)}_{k,y,z} + \tilde{r}^{(i)}_{k+1,y,z}) - a_{h,K} \tilde{r}^{(i)}_{K,y,z} + \sum_{j \in I} \nu_{ij} (b_{ij}^{(i)})
\]

\[
- b^{(i)}_h \leq -1 + \sum_{k=1}^{K} M_1 y_k \xi_k \quad h = 1, \ldots, K,
\]

\[
\tilde{r}^{(i)}_{k,y,z} \leq v_k \quad \text{if} \quad y_k = 0, \; k = 1, \ldots, K,
\]

\[
\tilde{r}^{(i)}_{k,y,z} \leq R_k \quad k = 1, \ldots, K,
\]

\[
\tilde{r}^{(i)}_{k,y,z} \leq \rho_k \left( \min_{\ell} F_{h}(i) \right) + \sum_{\ell=h+1}^{k-1} \rho_{\ell} v_{\ell} + \tilde{r}^{(i)}_{k,y,z}
\]

\[
\text{if} \; z_k = 0, \; h = 1, \ldots, k - 1, \; k = 1, \ldots, K,
\]

\[
\tilde{r}^{(i)}_{k,y,z} \leq \sum_{\ell=1}^{k-1} \rho_{\ell} v_{\ell} + \tilde{r}^{(i)}_{k,y,z} \quad z_k = 0, \; k = 1, \ldots, K,
\]

\[
\tilde{r}^{(i)}_{k,y,z} \leq F_{k}(i) \quad k = 1, \ldots, K,
\]

\[
\tilde{r}^{(i)}_{k,y,z} \leq \tilde{r}^{(i)}_{k+1,y,z} - \tilde{r}^{(i)}_{k+1,y,z} \quad \text{if} \; z_{k+1} = 1, \; k = 1, \ldots, K - 1,
\]

\[
v_k \leq \tilde{r}^{(i)}_{k,y,z} - \rho_{k-1} \tilde{r}^{(i)}_{k-1,y,z} + M_2 w_k + M_2 (1 - \xi_k) \quad k = 2, \ldots, K,
\]

\[
v_k \leq \tilde{r}^{(i)}_{k,y,z} + M_2 (1 - w_k) + M_2 (1 - \xi_k) \quad k = 2, \ldots, K,
\]

\[
a_{k,h} \geq a_{k+1,h} \quad k = 1, \ldots, K - 1, \; a_{K,h} \geq 1 \quad h = 1, \ldots, K,
\]

\[
v \in [0, d], \; w \in \{0,1\}^K, \; \xi \in \{0,1\}^K.
\]

In summary, the decision variables in (P2) are \( v_k, w_k, \tilde{r}^{(i)}_{k,y,z}, \tilde{r}^{(i)}_{k,y,z}, \xi_k, a_{k,h}, \) and \( b^{(i)}_h \). Note that \( y_k, z_k \) are not decision variables; instead, they serve as indices intended for a compact representation of multiple inequalities. If \( A \) is variable, then (P2) is an MIBLP with a linear objective function and bilinear constraints. If \( A \) is given, then (P2) is an MILP.

Next, we interpret the constraints (26a)-(26k). Constraints (26a)-(26i) are imposed for each \( i \in I, \) each \( y \in \{0,1\}^K \setminus \{0\}^K, \) and each \( z \in \{0,1\}^K \). (26a) is a reformulation of (25a), where

\[q_k = \infty \text{ or } \bar{q}_k = 0.\]
the vector fields $G$ and $H$ are replaced by critical-state flows $\tilde{f}_{k,y,z}^{(i)}$ and $\tilde{r}_{k,y,z}^{(i)}$, and the matrix product is expanded for every row of $A$. The right-hand side of (26a) includes a big-$M$ term, which means that this constraint is active if and only if $y_k w_k = 1$ for some $k$. The auxiliary binary variable $\xi_k$ results from (12f), on which we elaborate below.

(26b)–(26c) result from (5a), i.e. the definition of the buffer-discharged flow. (26d)–(26g) result from (5c) and (12b)–(12d), i.e. the expressions for the flows and for the boundaries of the invariant set $M$. Specifically, (26d)–(26e) result from $n_k$; recall that $z_k = 0$ corresponds to $n = n_k$. (26d) is the capacity constraint. (26e) is associated with the receiving flow constraint, which is only active if $z_k + 1 = 1$, i.e. if $n_k + 1 = \bar{n}_k + 1$. (26f)–(26i) are associated with the expression $\bar{q}_k$ in (12f). (26i) results from (17a); this constraint is not needed if $A$ is given. (26j) indicates the set of admissible control inputs $u = (v, w)$ and the auxiliary decision variables $\xi_k$. The auxiliary decision variables $\tilde{f}_{k,y,z}^{(i)}$ and $\tilde{r}_{k,y,z}^{(i)}$ are naturally constrained by (26b)–(26i) and do not need explicit range constraints.

### 4.2 Structure of optimal solution: a single merge

Because of the coupling between demand management and capacity allocation, analytical solution to $\text{P2}$ for a general $K$-cell highway is not easy, even with a fixed $A$ matrix. However, we can analytically characterize the optimal solution for a single merge, i.e. a two-cell highway section (as in Figure 2); this is a typical setting for highway control design [20, 26]. Furthermore, we focus on two practically relevant perturbations models, viz. a stationary incident hotspot and moving bottlenecks. For ease of presentation, we do not consider the impact of off-ramps and assume $\rho_1 = 1$. Furthermore, we assume that $R_1 = F_1$.

Specifically, we show that the structure of the optimal control depends on whether the given demand is feasible or infeasible, i.e. whether there exists a stabilizing control input $u = (v, w)$ such that $v = d$:

1. For a feasible demand, an on-ramp is prioritized over the mainline if the on-ramp has a smaller capacity-to-demand margin than the mainline.
2. For an infeasible demand, the inflow and priority are coupled and must be simultaneously determined. Optimal solutions are such that high inflow is sent to a location with high priority, and high priority is assigned to a location with high inflow (or more precisely, a small capacity-to-inflow margin).

The structure of the optimal solutions in the above two cases are consistent, in that it is always optimal to avoid queuing at a location where queue is discharged slowly; the speed of discharge is quantified by the capacity-to-inflow margin. Hence, we call the structure of the optimal solutions the **margin criterion**. Our subsequent theoretical analysis only considers either feasible or infinite demand, and excludes the case of finite but infeasible demands. However, we will discuss finite but infeasible demands in the examples.

---

4. By “active”, we mean that the constraint is imposed (instead of that the constraint is binding); by “inactive”, we mean that the constraint is essentially a dummy one that does not affect the optimal solution under any circumstances.
4.2.1 Stationary incident hotspot

Consider the two-cell highway with an incident hotspot. Let \( a_{kh} \) be as defined in (23). Thus, the Lyapunov function equally penalizes traffic queue/density everywhere and does not account for spatial distribution of traffic. This construction of \( A \) is suitable for this setting, since the incident hotspot is the only bottleneck and thus the total number of vehicles in the highway is more important than how the vehicles are spatially distributed. The following result characterizes the optimal solutions to (P2).

**Proposition 2 (Optimal solution, stationary)** Consider a two-cell highway with a stationary incident hotspot. Suppose that \( \rho_1 = 1 \) and \( F_1 = R_1 \), and let \( a_{kh} \) be as defined in (23).

1. If the demand vector \( d = [d_1 \ d_2]^T \) is feasible w.r.t. the formulation (P2), then \( ([d_1 \ d_2]^T, [1 \ 1]^T) \) is an optimal solution to (P2) if
   \[
   F_1 - d_1 \geq R_2 - d_2, \tag{27}
   \]
   and \( ([d_1 \ d_2]^T, [1 \ 0]^T) \) is an optimal solution if
   \[
   F_1 - d_1 \leq R_2 - d_2. \tag{28}
   \]

2. If \( d_1 = d_2 = \infty \), denote \( \bar{F}_2 = p_0 F_2 + p_1 (F_2 - \Delta_2) \) and define
   \[
   \mathcal{V}_1^* := \left\{ v \in [0, d] : v_1 + v_2 = \bar{F}_2, \ R_2 - v_2 \geq F_2 - \bar{F}_2, \ v_1 < \min \{F_1, F_2 - \Delta_2\} \right\}, \tag{29a}
   \]
   \[
   \mathcal{V}_2^* := \left\{ v \in [0, d] : v_1 + v_2 = \bar{F}_2, \ F_1 - v_1 \geq F_2 - \bar{F}_2, \ v_2 < \min \{R_2, F_2 - \Delta_2\} \right\}. \tag{29b}
   \]

Then, every \((v, w)\) in the set \( \mathcal{V}_1^* \times \{[0 \ 0]^T\} \) \( \cup \) \( \mathcal{V}_2^* \times \{[1 \ 1]^T\} \) is an optimal solution to (P2).

The proof of Proposition 2 is in Appendix A.3.

Part (1) of the above proposition states that, for a feasible demand, if the on-ramp has a capacity-to-demand margin \( (R_2 - d_2) \) that is larger than the margin \( (F_1 - d_1) \) of the mainline, then the optimal solution assigns low priority to the on-ramp. Part (2) states that, for an infinite demand, \( v \) and \( w \) are coupled and can be analytically solved. Note that Proposition 2 can be extended to the case where the demand is finite but infeasible with straightforward modification. In fact, this result can be mechanically extended to a general \( K \)-cell highway, although the extension is notationally heavy.

We can use the two-cell highway in Example 1 to illustrate some insights from Propositions 2 and visualize the structure of the max-throughput problem. Figure 6 shows the feasible set; the meaning of each region is listed in Table 3. Note that the gap (region 4) between the stable regions and the unstable region is very small.

4.2.2 Moving bottlenecks

Consider the two-cell highway section as shown in Figure 2. Suppose that moving bottlenecks randomly arrives at and moves through the highway section. That is, the highway has three modes \( \{0, 1, 2\} \), and the mode-specific capacities are

\[
F(0) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad F(1) = \begin{bmatrix} F_1 - \Delta_1 \\ F_2 \end{bmatrix}, \quad F(2) = \begin{bmatrix} F_1 \\ F_2 - \Delta_2 \end{bmatrix},
\]
Figure 6: Stability of various control schemes ($v, w$). The margin criterion and a set of optimal solutions (green/red line segments) are also indicated.

Table 3: Stability of various regions in Figure 6.

| Reg. | Margin criterion | OR priority | ML priority |
|------|------------------|-------------|-------------|
| 1    | Stable           | Stable      | Stable      |
| 2    | Stable           | Stable      | Unknown     |
| 3    | Stable           | Unknown     | Stable      |
| 4    | Unknown          | Unknown     | Stable      |
| 5    | Unstable         | Unstable    | Unstable    |

and the inter-mode transition rates are

$$\nu_{01} = \lambda, \ \nu_{12} = \mu, \ \nu_{20} = \mu.$$  

Let $a_{kh}$ be as defined in (24). Thus, the Lyapunov function penalizes congestion in cell 1 more than that in cell 2. This construction of $A$ is suitable for this setting, since the bottleneck is moving. Congestion in cell 1 is more costly than that in cell 2, since The following result characterizes the optimal solutions to (P2) in this setting.

**Proposition 3 (Optimal solution, moving)** Consider a highway of two-cells with moving bottlenecks. Suppose that $\rho_1 = 1$ and $R_1 = F_1$. Let $\alpha$ be as defined in (24).

1. Suppose that the demand vector $d = [d_1 \ d_2]^T$ is feasible w.r.t. (P2). Then, an optimal solution is $([d_1 \ d_2]^T, [1 \ 1]^T)$ if

$$R_2 - d_2 \leq F_1 - \Delta_1 - d_1,$$

and an optimal solution is $([d_1 \ d_2]^T, [1 \ 0]^T)$ if

$$R_2 - d_2 \geq F_1 - d_1.$$  

2. Suppose that (i) $d_1 = d_2 = \infty$, (ii) $F_1 + R_2 \geq F_2$, and (iii) $F_1 - \Delta_1 + R_2 \leq F_2 - \Delta_2/2$.

Then, an optimal solution to (P2) is
where \( p_0, p_1, p_2 \) are the steady-state probabilities given by (1).

The proof of Proposition 3 is similar to that of Proposition 2.

Part (1) essentially states that, if the \( k \)th on-ramp has a capacity-to-demand margin \((R_2 - d_2)\) smaller (resp. larger) than that of the mainline \((F_1 - \Delta_1 - d_1)\) (resp. \((F_1 - d_1)\)) under the influence of the moving bottleneck, then the optimal solution assigns high (resp. low) priority to the on-ramp. If the margins do not fall in the above regions, this result does not provide a conclusive characterization; however, we can still obtain the optimal solution by solving the MILP (P2).

For an infinite demand, as in Part (2), we made the practically relevant assumptions (ii)–(iii) for ease of presentation; these assumptions usually hold for highway merges [12].

In this setting, we provide one (analytical) optimal solution.

To illustrate the insights from Proposition 3, consider again the I210 section in Figure 2 but with moving bottlenecks. Figure 7 shows the feasible set (given by Theorem 1) as well as the optimal solution (given by Proposition 3); the meaning of each region is listed in Table 4. Note that the optimal solution \( v^* \) of the MILP is close to the optimal solution \( v^\dagger \) of the MIBLP.

### 5 Case study: a full-day simulation of SR-134 East/I-210 East

In this section, we consider a 33.2-km stretch of SR-134 East/I-210 East in Los Angeles County as a test case for the margin criterion for ramp control. This highway stretch consists of 6.3 km of SR-134 East from postmile 9.46 to postmile 13.36 and 26.9 km of I-210 East from postmile 25 to postmile 41.7. There are 28 on-ramps and 25 off-ramps. We consider the on-ramp flows
Table 4: Stability of various regions in Figure 7.

| Reg. | MIBLP | MILP | OR priority | ML priority |
|------|-------|------|-------------|-------------|
| 1    | Stable| Stable| Stable      | Stable      |
| 2    | Stable| Stable| Stable      | Unknown     |
| 3    | Stable| Stable| Unknown     | Stable      |
| 4    | Stable| Unknown| Unknown   | Unknown     |
| 5    | Unknown| Unknown| Unknown    | Unknown     |
| 6    | Unstable| Unstable| Unknown   | Unstable    |

measured on a full day. Particularly, we focus on two scenarios of capacity perturbations, viz. a stationary incident hotspot and moving bottlenecks.

The model was built using data for the corresponding segments of the SR-134 East and I-210 East for Monday, October 13, 2014. This was one of the days when most vehicle detectors on mainline, on-ramps, and off-ramps of SR-134 East and I-210 East were intact, and hence the PeMS data are reliable. There are a morning peak hour and an evening peak hour. Traffic flow parameters were calibrated using Performance Measurement System (PeMS, see [33]) data following the methodology [8]. The Markovian capacity model can be calibrated following [16]. However, in this section, we consider two demonstrative capacity models: one for incident hotspots and one for moving bottlenecks. More information about the simulation tool is available in [12].

5.1 Stationary incident hotspot

We consider a stationary incident hotspot near North Azusa Avenue. The capacity of the hotspot switches between 100% (mode 0) and 75% (mode 1) of its nominal capacity, and spends equal time in both modes. That is, the time average capacity is 87.5% of the nominal capacity. The traffic flow and capacity parameters are the data in the max-throughput problem. One way to implement our design approach is to consider the whole stretch as a single highway and solve for the optimal inflow and priority at all locations in a coordinated manner. However, such a

![Figure 8: Traffic density contour for the highway with moving bottlenecks.](a) Mainline priority. (b) Margin criterion.)
coordinated ramp metering capability is not yet widely available [20]. Hence, we use an alternative, distributed implementation: we independently consider each cell and apply the margin criterion (Proposition 2). We run three simulations in this setting, viz. on-ramp priority (i.e. the uncontrolled baseline), mainline priority, and margin criterion presented in Proposition 2.

The resulting traffic density contour plot from a typical simulation sample is show in Figures 8(a)–8(b). Compared to the baseline, mainline priority (the classical traffic control scheme [11]) reduces delay by 62%. Compared to mainline priority, the margin criterion further reduces delay by 3%. More importantly, the margin criterion reduces the longest on-ramp queue by 39%. Figure 9 shows that although the peak values for vehicle counts are the same for both control strategies, margin criterion results in faster discharge of traffic.

5.2 Moving bottlenecks

We now consider moving bottlenecks randomly arriving at the highway, following a methodology analogous to that for incidents. We consider an arrival rate of moving bottlenecks of 12 per hour. When the moving bottleneck is in a cell, then the cell’s capacity is reduced by the capacity of one lane. The expected time $\mu_k$ that a moving bottleneck spends in a cell is given by $\mu_k = \frac{l_k}{\alpha_k}$, where $l_k$ is the length of the cell and $\alpha_k$ is the free-flow speed. Once again, we consider the

![Figure 9: Margin criterion accelerates discharge of traffic during peak hours.](image)

![Figure 10: Traffic density contour for the highway with moving bottlenecks.](image)
control strategies in the previous subsection, viz. on-ramp priority, mainline priority, and margin criterion (as described in Proposition 3). The resulting traffic density contour plots are given by Figures 10(a)–10(b). Compared to the baseline, mainline priority reduces delay by 16%. Compared to mainline priority, the margin criterion further reduces delay by 25%.

6 Concluding remarks

In this article, we considered the maximization of the throughput of a perturbation-prone highway section, under the constraint that every on-ramp should remain bounded on average. We developed a sufficient condition (Theorem 1) for bounded on-ramp queues, and formulate the max-throughput problem as either an MIBLP or an MILP. In addition, we characterized the structure of the optimal solutions to the max-throughput problem, which we summarized as the “margin criterion” (Propositions 2–3).

This work can be extended to a network setting with connection to upstream/downstream arterial roads. To estimate the likelihood for on-ramp queues to propagate to upstream roads, one can study properties of higher-order moments of the on-ramp queues. In addition, congestion on downstream roads may restrict the off-ramp discharges, and thus affect the design of demand management and capacity allocation strategies.

A Appendices

A.1 Proof of Proposition 1

Consider a given control input $u = (v, w)$. One can adapt the proof of [13, Proposition 1] and show that $\mathcal{M} = [0, \infty]^K \times \prod_{k=1}^K [\underline{u}_k(0), \bar{u}_k]$ is invariant. Next, we refine $\mathcal{M}$ and eventually obtain the invariance of $\mathcal{M}$ in three steps.

Step (1). For the $k$-th buffer, if $v_k > R_k$, then for any initial condition $(i, x, u) \in I \times \mathcal{M}$, we have $\lim_{t \to \infty} Q_k(t) = \infty$. Hence, the set $[0, \infty]^{k-1} \times \{\infty\} \times [0, \infty]^{K-k} \times \prod_{k=1}^K [\underline{u}_k(0), \bar{u}_k]$ is attracting and invariant. Repeating the above argument for each $k$, we conclude that the set $\mathcal{M}_1 = \prod_{k=1}^K [\underline{q}_k, \infty] \times \prod_{k=1}^K [\underline{u}_k(0), \bar{u}_k]$ is attracting and invariant.

Step (2). For the $k$-th buffer, if $v_k < R_k$ and $v_k < \beta_k (n_k^{\text{max}} - \bar{n}_k) - (1 - w_k)\rho_{k-1} \min\{\alpha_{k-1}\bar{n}_{k-1}, F_{k-1}\}$ (and thus $\bar{q}_k = 0$), we have

\[
G_k(i, x, u) = v_k - r(i, x, u) \\ \leq v_k - \min\{1_{\{q=0\}}v_k + 1_{\{q>0\}}R_k, \beta_k (n_k^{\text{max}} - \bar{n}_k) - (1 - w_k)\rho_{k-1} \min\{\alpha_{k-1}\bar{n}_{k-1}, F_{k-1}\}\} < 0
\]

Hence, the set $[0, \infty]^{k-1} \times \{0\} \times [0, \infty]^{K-k} \times \prod_{k=1}^K [\underline{u}_k(0), \bar{u}_k]$ is attracting and invariant. Repeating the above argument for each $k$, we conclude that the following set $\mathcal{M}_2 = \prod_{k=1}^K [0, \bar{q}_k] \times \prod_{k=1}^K [\underline{u}_k(0), \bar{u}_k]$ is attracting and invariant. Then, we conclude that $\mathcal{M}_1 \cap \mathcal{M}_2$ is attracting and invariant.

Step (3i). For each $k$ and each $h \leq k$, define

$$
\mathcal{M}_k^h := \left( \bigcup_{q \in (\mathcal{M}_1 \cap \mathcal{M}_2): q_k = 0} \{q\} \times \prod_{k=1}^K [\underline{u}_k(0), \bar{u}_k]\right)
$$

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Step (1) 

Let 

\[ M \]

\[ \forall \]

The rest of this proof verifies the drift condition over 

and obtain that 

\[ M \]

\[ h \]

By analogous arguments, for every 

\[ n \]

\[ q \]

\[ i \]

Suppose that there exist a matrix 

\[ A \]

2 Proof of Theorem 1

Step (3ii): 

Since 

\[ G \]

\[ T \]

\[ k \]

As defined in (11). By (2b)–(2c) and (20), for a given control input 

\[ u = (v, w) \], we have

\[ \mathcal{L} V = \left((DG + H)^T A + \sum_{i \in I} \nu_{ij} \left(b^{(j)} - b^{(i)}\right)^T\right)(Dq + n) + (b^{(i)})^T(DG + H). \] (32)

Since \( G_k \) and \( H_k \) are bounded, we can define a constant \( W := \left(\max_{i,k} |b^{(i)}_k|\right) \sum_{k=1}^K (R_k + \rho_k F_k) \) and obtain that

\[ (b^{(i)})^T(DG + H) \leq \left(\max_{i,x} |b^{(i)}_k|\right) \left(\max_{i,x} |DG + H|\right) \]

\[ \leq \left(\max_{i,x} |b^{(i)}_k|\right) \left(\max_{i,x} (v_1 - f_1 + \sum_{k=2}^K (v_k - r_k + \rho_{k-1} f_{k-1} + r_k - f_k))\right) \]

\[ \leq \left(\max_{i,k} |b^{(i)}_k|\right) \left(\sum_{k=1}^K (R_k + \rho_k F_k)\right) = W \quad \forall (i, x) \in I \times (Q \times N). \] (33)

The rest of this proof verifies the drift condition over \( M_0 \) and \( M_1 \) as defined in (13a)–(13b):

Step (1): 

For each \( (q, n) \in M_0 \), since \( M_0 \) is a bounded set, there exists \( Z_0 < \infty \) such that 

\[ \forall (i, x) \in I \times M_0 \]

\[ \left((DG + H)^T A + \sum_{j \in I} \nu_{ij} \left(b^{(j)} - b^{(i)}\right)^T\right)(Dq + n) \leq Z_0. \]
Hence, we can obtain from (33) and the above that
\[
\mathcal{L}V \leq Z_0 + W \quad \forall (i, x) \in \mathcal{I} \times \mathcal{M}_0.
\] (34)

**Step (2):** For each \((q,n) \in \mathcal{M}_1\) and for \(h = 1, \ldots, K\), note that

\[
(DG + H)^T \begin{bmatrix} a_{h,1} & \cdots & a_{h,K} \end{bmatrix}^T = a_{h,1}v_1 + \sum_{k=2}^{K-1} a_{h,k-1}v_k
\]

\[
- \sum_{k=1}^{K} \left( (a_{h,k} - \rho_k a_{h,k+1})f_k + (a_{h,k} - a_{h,k+1})r_{k+1} \right) - a_{h,K}f_K
\]

\[
\leq a_{h,1}v_1 + \sum_{k=2}^{K} a_{h,k-1}v_k - \sum_{k=1}^{K-1} (a_{h,k} - a_{h,k+1})\left( \rho_k f_k + r_{k+1} \right) - a_{h,K}f_K.
\]

(17a) ensures that \(a_{h,k} - a_{h,k+1} \geq 0\) for \(k = 1, \ldots, K - 1\). For each \(q \neq 0\), \(\rho_{k-1} f_{k-1} + r_k\) is concave in \(n\) over the box \(\prod_{k=1}^{K} [\bar{\nu}_k(q), \bar{\nu}_k]\) (see [13], Proof of Theorem 2 for details). Therefore, \(\rho_{k-1} f_{k-1} + r_k\) attains its minimum at one of the vertices of the box. Then, we have

\[
a_{k,1}v_1 + \sum_{k=2}^{K} a_{k,k-1}v_k - \min_{(q,n) \in \mathcal{V}_1} \left( \sum_{k=1}^{K-1} (a_{k,k} - a_{k,k+1})(\rho_k f_k + r_{k+1}(i, x)) + a_{k,K}f_K \right)
\]

\[
+ \sum_{j \in \mathcal{I}} \nu_{ij}(b_{k,j}^{(i)} - b_{k,i}^{(j)}) \leq -1 \quad \forall (i, x) \in \mathcal{I} \times \mathcal{M}_1.
\]

Thus, we can obtain from the above together with (32) and (33) that

\[
\mathcal{L}V \leq -c^T(Dq + n) + W \quad \forall (i, x) \in \mathcal{I} \times \mathcal{M}_1.
\]

Recalling the definition of \(D\) in (16), we have \(\mathcal{L}V \leq -|q| + W, \quad \forall (i, x) \in \mathcal{I} \times \mathcal{M}_1\). Finally, let \(Z := Z_0 + W\), we can obtain from the above and (34) that

\[
\mathcal{L}V(i, x) \leq -|q| + Z \quad \forall (i, x) \in \mathcal{I} \times \mathcal{M}.
\]

**A.3 Proof of Proposition 2**

**Part (1).** Consider a given \(u = (v, w) \in [0, d] \times \{0, 1\}^2\). To obtain the conclusion, it suffices to show the following: (i) if \((v, w) = ([d_1, d_2]^T, [1 0]^T)\) satisfies (26a)–(26k) and if (27) holds, then \((v, w') = ([d_1, d_2]^T, [1 1]^T)\) also satisfies (26a)–(26k); (ii) if \((v, w') = ([d_1, d_2]^T, [1 1]^T)\) satisfies (26a)–(26k) and if (28) holds, then \((v, w) = ([d_1, d_2]^T, [1 0]^T)\) also satisfies (26a)–(26k). Here we only prove (i); (ii) can be analogously proved.

Suppose that \((v, w) = ([d_1, d_2]^T, [1 0]^T)\) satisfies (26a)–(26k) and that (27) holds. If \(v_1 + v_2 \leq F_2 - \Delta_2\), then \(\bar{q}_1 = \bar{q}_2 = 0\) and \(\mathcal{M}\) is bounded regardless of \(w_2\), and the proof is straightforward.
If $v_1 + v_2 > F_2 - \Delta_2$ and if $v_1 \leq F_2 - \Delta_2$, then, with $w_2 = 0$, (26a)–(26k) are equivalent to the following:

$$d_1 + d_2 - p_0(d_1 + R_2) - p_1 \min\{d_1 + R_2, F_2 - \Delta_2\} \leq -\delta.$$  (35)

With $w_2 = 1$, (26a)–(26k) are implied by the following:

$$d_1 + d_2 - p_0 \min\{F_1 + d_2, d_1 + R_2\} - p_1 \min\{F_1 + d_2, d_1 + R_2, F_2 - \Delta_2\} \leq -\delta.$$  (36)

If (27) holds, then (35) implies (36). Hence, $(v, w') = ([d_1 \ d_2]^T, [1 \ 1]^T)$ also satisfies (26a)–(26k).

Part (2). We only prove the optimality of those solutions in $\mathcal{V}_1 \times \{[1 \ 0]^T\}; \mathcal{V}_2 \times \{[1 \ 1]^T\}$ can be analogously treated. On the one hand, consider a $v^* \in \mathcal{V}_1^*$ and $w^* = [1 \ 0]^T$. Then, we have

$$v_1^* + v_2^* - p_0 \min\{v_1^* + R_2, F_2\} - p_1 \{v_1^* + R_2, F_2 - \Delta_2\} \leq \bar{F}_2 - p_0 F_2 - p_1 (F_2 - \Delta_2) = 0.$$  (37)

Since $v_1 < F_2 - \Delta_2$, we obtain from (12f) that $\bar{q}_1 = 0$. Thus, (37) ensures (26a)–(26k) (see the proof of Proposition 2). Hence, $(v, w)$ is a feasible solution to (P2). On the other hand, $v_1^* + v_2^* \leq \bar{F}_2$ is a necessary condition for stability, and $\bar{F}_2$ is the maximum throughput that can be achieved (regardless of the formulation).

In conclusion, $(v^*, w^*)$ is an optimal solution to (P2).

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References

[1] Melike Baykal-Gürsoy, Weihua Xiao, and Kaan Özbay. Modeling traffic flow interrupted by incidents. European Journal of Operational Research, 195(1):127–138, 2009.

[2] Moshe Ben-Akiva and Michel Bierlaire. Discrete choice methods and their applications to short term travel decisions. In Handbook of Transportation Science, pages 5–33. Springer, 1999.

[3] Michel Benaïm, Stéphane Le Borgne, Florent Malrieu, and Pierre-André Zitt. Qualitative properties of certain piecewise deterministic Markov processes. In Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, volume 51-3, pages 1040–1075. Institut Henri Poincaré, 2015.

[4] Giacomo Como, Enrico Lovisari, and Ketan Savla. Convexity and robustness of dynamic traffic assignment and freeway network control. Transportation Research Part B: Methodological, 91:446–465, 2016.
[5] Samuel Coogan and Murat Arcak. Stability of traffic flow networks with a polytree topology. *Automatica*, 66:246–253, 2016.

[6] Carlos F Daganzo. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. *Transportation Research Part B: Methodological*, 28(4):269–287, 1994.

[7] André de Palma and Robin Lindsey. Traffic congestion pricing methodologies and technologies. *Transportation Research Part C: Emerging Technologies*, 19(6):1377–1399, 2011.

[8] Gunes Dervisoglu, Gabriel Gomes, Jaimyoung Kwon, Roberto Horowitz, and Pravin Varaiya. Automatic calibration of the fundamental diagram and empirical observations on capacity. In *Transportation Research Board 88th Annual Meeting*, number 09-3159, 2009.

[9] Robert G Gallager. *Stochastic Processes: Theory for Applications*. Cambridge University Press, 2013.

[10] Gabriel Gomes and Roberto Horowitz. Optimal freeway ramp metering using the asymmetric cell transmission model. *Transportation Research Part C: Emerging Technologies*, 14(4):244–262, 2006.

[11] Gabriel Gomes, Roberto Horowitz, Alexandr A Kurzhanskiy, Pravin Varaiya, and Jaimyoung Kwon. Behavior of the cell transmission model and effectiveness of ramp metering. *Transportation Research Part C: Emerging Technologies*, 16(4):485–513, 2008.

[12] Roberto Horowitz, Alexander A. Kurzhanskiy, and Matthew Wright. HOT lane simulation tools. Technical report, Institute of Transportation Studies, University of California, Berkeley, CA, 2016.

[13] Li Jin and Saurabh Amin. Analysis of a stochastic switched model of freeway traffic incidents. *IEEE Transactions on Automatic Control*. to appear.

[14] Li Jin and Saurabh Amin. Stability of fluid queueing systems with parallel servers and stochastic capacities. *IEEE Transactions on Automatic Control*. to appear.

[15] Li Jin and Saurabh Amin. A piecewise-deterministic Markov model of freeway accidents. In *Decision and Control (CDC)*, 2014 *IEEE 53rd Annual Conference on*. IEEE, 2014.

[16] Li Jin and Saurabh Amin. Calibration of a macroscopic traffic flow model with stochastic saturation rates. In *Transportation Research Board 96th Annual Meeting*, number 17-02496, 2017.

[17] Pushkin Kachroo and Kaan Özbay. *Feedback Ramp Metering in Intelligent Transportation Systems*. Springer Science & Business Media, 2011.

[18] Victor L Knoop. *Road Incidents and Network Dynamics: Effects on driving behaviour and traffic congestion*. PhD thesis, Technische Universiteit Delft, 2009.

[19] Alex A Kurzhanskiy and Pravin Varaiya. Traffic management: An outlook. *Economics of Transportation*, 4(3):135–146, 2015.
[20] Alexandr A Kurzhanskiy and Pravin Varaiya. Active traffic management on road networks: A macroscopic approach. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 368(1928):4607–4626, 2010.

[21] Johan Löfberg. YALMIP: A toolbox for modeling and optimization in MATLAB. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, pages 284–289. IEEE, 2004.

[22] Samer M Madanat, CY Yang, and Ying-Ming Yen. Analysis of stated route diversion intentions under advanced traveler information systems using latent variable modeling. *Transportation Research Record*, (1485), 1995.

[23] Negar Mehr and Roberto Horowitz. Probabilistic freeway ramp metering. In *ASME 2016 Dynamic Systems and Control Conference*, pages V002T31A006–V002T31A006. American Society of Mechanical Engineers, 2016.

[24] Sean P Meyn and Richard L Tweedie. Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, pages 518–548, 1993.

[25] Gordon F. Newell. *Applications of Queueing Theory*, volume 4. Springer Science & Business Media, 2013.

[26] Markos Papageorgiou, Habib Hadj-Salem, and Jean-Marc Blosseville. ALINEA: A local feedback control law for on-ramp metering. *Transportation Research Record*, (1320), 1991.

[27] Markos Papageorgiou and Apostolos Kotsialos. Freeway ramp metering: An overview. *IEEE transactions on intelligent transportation systems*, 3(4):271–281, 2002.

[28] Jack Reilly, Samitha Samaranayake, Maria Laura Delle Monache, Walid Krichene, Paola Goatin, and Alexandre M Bayen. Adjoint-based optimization on a network of discretized scalar conservation laws with applications to coordinated ramp metering. *Journal of optimization theory and applications*, 167(2):733–760, 2015.

[29] Sadra Sadraddini and Calin Belta. A provably correct mpc approach to safety control of urban traffic networks. In *American Control Conference (ACC), 2016*, pages 1679–1684. IEEE, 2016.

[30] Marius Schmitt and John Lygeros. An exact convex relaxation of the freeway network control problem with controlled merging junctions. *Transportation Research Part B: Methodological*, 114:1–25, 2018.

[31] A Sumalee, RX Zhong, TL Pan, and WY Szeto. Stochastic cell transmission model (SCTM): A stochastic dynamic traffic model for traffic state surveillance and assignment. *Transportation Research Part B: Methodological*, 45(3):507–533, 2011.

[32] Pravin Varaiya. Congestion, ramp metering and tolls. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 366(1872):1921–1930, 2008.

[33] Pravin Varaiya. *Freeway Performance Measurement System (PeMS), PeMS 9.0*. California PATH Program, Institute of Transportation Studies, University of California at Berkeley, 2009.
[34] Ren Xin Zhong, Agachai Sumalee, Tian Lu Pan, and William HK Lam. Optimal and robust strategies for freeway traffic management under demand and supply uncertainties: an overview and general theory. *Transportmetrica A: Transport Science*, 10(10):849–877, 2014.