Tackling the Trefoils

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ABSTRACT

The classical trefoil is famous for having a three-colouring which distinguishes it from the unknot. The three-colouring is also notorious for not distinguishing the right handed from the left handed trefoil. However with a bit of tweaking the three colours can also be used for this task. What lies behind the method is a new operation on biracks called doubling which converts the 3-colour quandle into a biquandle. Colouring with this biquandle distinguishes the right handed from the left handed trefoil. Equivalently it defines an element of the homology of the quandle or biquandle classifying space.

Keywords: 3-colouring, virtual knots, trefoil, quandles, racks, biquandles and their homology.

1 Introduction

There have been many proofs that the right and left handed trefoils are distinct. The earliest proof that the trefoil is inequivalent to its mirror image is due to Max Dehn, who proved it by tracking the longitude in the fundamental group. There is a good exposition of this and reference to Max Dehn in the book [S]. Perhaps the simplest proof is via the Kauffman bracket polynomial, starting the construction from scratch and using the Reidemeister moves. The next simplest to that is using the third Vassiliev invariant.

What is interesting about these and other solutions to different problems is how they drive forward the search and invention of new mathematical methods and engines. In this case it is the algebraic topology related to the classifying space of a quandle or biquandle. Indeed quandles and racks were invented in the search for classical knot invariants whilst biquandles turned out to be very useful for virtual knot theory.

Pictured is a classical trefoil (both right and left handed versions), with a 3-colouring using the colours red (r), green (g), and blue (b) which shows that they are non-trivial,

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but fails to distinguish right from left.

However the next picture shows a different 3-colouring using pairs of colours; red/green (rg), blue/red (br), green/green, (gg), red/red (rr) and red/blue (rb) which does distinguish the right trefoil from the left.

In this paper it will be shown why this is the case and how it relates to homology of quandles and biquandles.

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2 Colouring by pairs

The edges of a diagram are coloured with pairs of the three colours, red, green and blue. The pairs of colours can change on the overcrossing as well as the undercrossing. Moreover the order is important. For example (gr) means that red is on the left and green is on the right as one traverses the diagram in the given direction.

Not all pairs of colours are allowed at a crossing. The possibilities are indicated in the following diagram.
The letters $abc$ are a permutation of the colours $rgb$. There are 27 possible colourings of the positive crossings. They are labelled $abc$, $aba$, $abb$, $aab$ and $aaa$, for reasons which will become clear later.

The 27 possible colourings of negative crossings are shown on the right. They can be obtained by reflecting the positive crossings in a vertical line.

If the diagram contains a virtual crossing then the pairs cross unaltered as follows.

The letters $ijkl$ are combinations of the colours $rgb$ possibly with repetition.

3 Reidemeister moves

We now see how the colouring by pairs is affected during a Reidemeister move.

Suppose an edge is coloured $ab$ and a curl (monogon) is introduced by an expansive Reidemeister move of type I. Then a crossing of type $\pm abb$ is introduced. If the edge is coloured $aa$ then a crossing of type $\pm aaa$ is introduced. Conversely only crossings of type $\pm abb$ or $\pm aaa$ can be the vertices in a curl and the curl can be removed if desired. Recall that $abc$ is a permutation of the original three colours.

A similar description applies to a virtual Reidemeister move of type I.
A positive crossing can be cancelled with a negative crossing by a Reidemeister move of type II since any pair of colours is matched by a unique pair of colours with an opposite crossing either on the sides or top/bottom.

On the other hand an expansive Reidemeister move of type II presents difficulties since the form of the pairs have to match the limited number of colourings of the crossings. However without virtual crossings this does not present a problem as we shall now see. Even with the presence of virtual crossings there are situations where expansive Reidemeister moves of type II can take place.

Let $D$ be a classical knot diagram (or irreducible link) and orient all the regions anti-clockwise. An edge, $e$, on the boundary of a region $F$ may or may not be oriented coherently with $F$. Suppose the edge is oriented coherently and is coloured by the pair $ab$, where $a \neq b$ then colour the region $F$ by $c$. If the edge is not oriented coherently and is coloured by the pair $ab$, colour the region by $a$. If $e$ is coloured by $aa$ then colour $F$ by $a$.

**Theorem 3.1** With the notation above, the colour of the region $F$ is independent of the boundary edge $e$ chosen.

**Proof** Let $e_1$ and $e_2$ be two edges on the boundary of $F$. By induction we may as well assume that they have a crossing in common. The result follows after a consideration of all the possibilities. □

There are 45 colouring possibilities for the Reidemeister III move and we shall not consider all of them since we shall see the general pattern later in the paper. However to get an idea look at the following pair coloured Reidemeister III move.

4 Crossing Invariants

To get an invariant we sum up the crossing points of the diagram with sign. So for example the right hand trefoil gives $+rbg + rrb + rgr$. The left hand trefoil gives the negative, $-rbg - rrb - rgr$.

In order to obtain an invariant we look at how this sum changes under the Reidemeister moves. During a Reidemeister move of type I, crossing points of the form $abb$ and $aaa$
can be eliminated or constructed at will so these will be put equal to zero. For reasons which will be clear later also put $aab = 0$.

Reidemeister moves of type II confirm the inverse of a crossing.

Reidemeister moves of type III imply relations amongst the crossings. For example, the Reidemeister move of type III illustrated above implies the relation $abc + aac + bba = aab + ccc + abc$, confirming the roles of $aac$, $bba$, $aab$ and $ccc$ as zeros.

It turns out that the element $abc + aca$ is of order 3 in the resulting abelian group. This shows that the two trefoils are distinct.

5 Two sided knot diagrams

In an oriented classical knot diagram each edge has a left and right adjacent region. Moreover these two regions are distinct. This may not happen with a virtual knot diagram. If we think of a virtual knot as a knot on a surface, then a meridian on a torus divides the surface into one region. Alternatively consider the virtual trefoil in the following figure.

![Virtual Trefoil](image)

We now look at conditions which allow a diagram to be two sided. Since over and undercrossings are irrelevant to this problem we may consider only flat virtual knots. They are represented by an oriented 4-valent graph in a surface which is the image in general position of an immersed circle or a number of circles. Each crossing is a vertex and the edges inherit an orientation, so that at each vertex they enter and leave on opposite sides. Two such diagrams are equivalent if they are related by a sequence of homeomorphisms, local Reidemeister type moves and surgeries disjoint from the image. A diagram is minimal if the surface in which it lies has minimal genus. The genus of a virtual knot is this minimal genus. So classical knots have genus zero.

By a result of Kuperberg, [Kuper], two minimal diagrams representing the same class are related by a sequence of homeomorphisms and local Reidemeister type moves. Note that if a diagram is minimal and connected then the regions are open discs.

A virtual knot, $K$, flat or otherwise, is called 2-sided if it has a diagram so that any
edge is on the boundary of exactly 2 regions. It is said to be irreducible if at every crossing there are exactly 4 regions. A minimal irreducible 2-sided diagram is called cellular.

A cellular diagram is the 1-skeleton of a regular cellulation of the surface. We say that a cellular diagram is 2-colourable if the regions can be coloured, chess board fashion, with 2 colours. This is a generalisation of a more general colouring with a biquandle which is defined later in the paper. For example on a torus take a knot with 2 components, a meridian and a longitude. If we double each component then we have a cellular diagram whose regions can be 2-coloured. If we triple the meridian and longitude we have a cellular diagram which cannot be 2-coloured.

Cellular diagrams which are 2-colourable are called atoms, see [B].

We can decide which cellular diagram can be 2-coloured by looking at the orientation of the edges. Colour the edges alternately black and white along the diagram. This is always possible if the diagram represents a knot with one component but may not be possible for knots with more than one component. If an edge is coloured white keep its natural orientation. If it is black, reverse its orientation. Then each crossing is a source, sink or saddle.

![Source, sink and saddle](image)

The property of being alternately oriented is easily seen to be invariant under the Reidemeister moves and so is a knot invariant.

If the knot has one component then there is a convenient way of seeing the crossing type by means of the chord diagram. A crossing is an odd crossing if its chord crosses an odd number of other chords. Otherwise it is even. Odd crossings correspond to sources or sinks; even crossings to saddles.

If all crossings are saddles then the alternately oriented diagram is called good. This is similar to the treatment in [IKK, YM] where it is called a magnetic graph.

**Theorem 5.2** A cellular knot diagram has a 2-colouring if and only if it is a good alternately oriented diagram

**Proof** If a cellular knot diagram has a 2-colouring and if the underlying surface is
oriented, then orient the boundary of each black cell according to the orientation of the surface. This defines a good alternately oriented diagram. Conversely a good alternately oriented minimal diagram defines a 2-colouring of the cells by the reverse process. □

![Good alternatively oriented trefoil with 2-colouring](image)

**Theorem 5.3** Let $D$ be an alternatively oriented virtual diagram. Then

$$\#\text{sinks} = \#\text{sources}.$$ 

**Proof** Consider the chord diagram of $D$. This consists of $n$ circles corresponding to the $n$ components and two types of chords. The interior chords join points on the same circle and correspond to self crossings. The exterior chords join different circles and correspond to where they cross. Note that a necessary and sufficient condition for the diagram to be alternately oriented is that there are an even number of exterior chords attached to each circle.

We can simplify the diagram without changing the conclusion of the result by interchanging end points of adjacent internal and external chords. In this manner all the external chord end points can be grouped together as can those of the internal chords.

Consider the internal chords. Define cycles as follows. The vertices are the chords and the edges are the pieces of the circles at the end points travelling anticlockwise to the
adjacent chord end. Around each cycle the edges are oriented by arrows in one of two ways. It is at a change of orientation that a sink or source is generated. If the arrows converge at the vertex then it is a sink. Otherwise it is a source. A simple counting argument shows that there is an equal number of sinks and sources.

A similar argument now works for the exterior chords.

Let $K$ be a knot which has an alternately oriented diagram and let $\chi(K)$ be the minimum number of sinks (sources) for any diagram representing $K$. Note that $\chi = 0$ is a necessary and sufficient condition for $K$ to be represented by a 2-coloured diagram.

The virtual trefoil shown earlier has one source and one sink. So in this case $\chi = 1$.

I am indebted to Andrew Bartholomew for the following example of a virtual knot with a good alternate orientation.

The usual representation of virtual knot diagrams means that this corresponds to a diagram on an orientable surface of genus 2. However if we thicken the diagram then it lies on a thickened 4-valent graph with 4 vertices and 8 edges which is the image of an immersed ribbon. The diagram is 2-sided and the surface has genus 1 and 4 boundary components. Of course, to show that this is the minimal genus we would have to show that it is not classical.

6 Racks, biquandles etc:
We will slightly extend the definition of a birack and a biquandle to suit our needs later in the paper. The more usual notation and definitions can be found in [FJK] or [BaF] where a list of small biracks etc can be found.

Let $X$ be a set; the colouring or labelling set. Let $Y$ be a subset of $X^2$. A function $S : Y \rightarrow Y$ defines two binary operations by the formula

$$S(a, b) = (b^a, a_b)$$

The two binary operations,

$$(a, b) \rightarrow a^b \text{ and } (a, b) \rightarrow a_b$$

are called up and down, respectively. In previous treatments, $Y = X^2$, so the binary operations are defined for all $a, b \in X$. This extension of the definition means that $b^a$ and $a_b$ are only defined if $(a, b) \in Y$ and will give us the flexibility needed later. However whenever we write $b^a$ and $a_b$ we will always assume that the operations are defined.

The convenience of the exponential and suffix notation is that brackets can be inserted in an obvious fashion and so are not needed, For example

$$a^{bc} = (a^b)^c, \ a^b c = a^{(b,c)}, \ a^b c = (a^b)_c, \text{ etc.}$$

On the other hand expressions such as $a^c_b$ are ambiguous and are not used.

Think of $b$ as acting on $a$ in both cases. We want these actions to be invertible. So there are binary operations

$$(a, b) \rightarrow a^{b-1} \text{ and } (a, b) \rightarrow a_{b-1}$$

satisfying

$$a^{bb-1} = a^{b-1b} = a \text{ and } a_{bb-1} = a_{b-1b} = a$$

So each suitable element $a$ of $X$ defines two permutations given by $x \rightarrow x^a$ and $x \rightarrow x_\alpha$.

It is convenient at this stage to introduce the function $F$ defined by $F(a, b^a) = (b, a_b)$. In [FJK] this was called the sideways map, $S^+_\alpha$.

After these preliminaries we list the three axioms, B1-3 needed to define a biquandle.

**B1:** $F$ is invertible and preserves the diagonal, $\{(a, a)|a \in X\}$.

**B2:** $S$ is invertible. So there is a function $\overline{S} : X^2 \rightarrow X^2$ such that $S\overline{S} = \overline{S}S = id$
B3: Let $S_1 = S \times id$ and $S_2 = id \times S$ then $S_1 S_2 S_1 = S_2 S_1 S_2$

These axioms are a consequence of the three Reidemeister moves. If only B2 and B3 are satisfied then we have a birack.

The function $F$ satisfies

$$F(a, a) = (a^{a^{-1}}, a^{a^{-1}}).$$

Axiom B1 implies that for all $a \in X$ there is a unique $x \in X$ such that $a^x = x, a = x_a$ and there is a unique $y \in X$ such that $a_y = y, a = a^y$. We have $x = a_{a^{-1}}$, $y = a^{a^{-1}}$ and so B1 is equivalent to $a^{a_{a^{-1}}} = a_{a^{-1}}$ and $a^{a_{a^{-1}}} = a^{a_{a^{-1}}}.$

In [Stan] it is shown that only one half of B1 is necessary.

An amusing implication of B1 is that the infinite tower and the infinite well

$$x = a^{a^{a^{\ldots}}}, \quad y = a_{a^{a^{\ldots}}}$$

make sense.

Using B2 write $S(a, b) = (b, a^b)$. This defines two binary operations

$$(a, b) \to a^b \quad \text{and} \quad (a, b) \to a_b$$

In terms of the previous operations $a^b$ acts down on $b$ as $a^{-1}$ and $a_b$ acts up on $b$ as $a^{-1}$. So all these new operations are defined by the two initial up and down operations.

B3 is sometimes called the set theoretic Yang-Baxter equation. If we follow the progress of the triple $(a, b, c)$ through the two sides of the equation and swap variables we arrive at three relations true for all $a, b, c \in X$.

$$a^{c_b b^c} = a^{b_c}, \quad a^{b_{c a}} = a^{b_{c a}}, \quad a_{c b c} = a_{b c}$$

If the down operation is trivial, so $a_b = a$ for all $a, b \in X$, then a biquandle becomes a quandle. Symmetrically, this is also the case if the up operation is trivial. A birack with trivial down (up) operation is a rack.

7 Examples

For many examples of racks and quandles, see [FR].

The simplest quandle, with both up and down operations trivial, is the twist. In this case $S(a, b) = (b, a)$. If $X$ has $n$ elements denote the twist by $I_n$.  

10
The **3-colour** quandle is the reflection set of transpositions in the symmetric group $S_3$. The quandle operation is conjugation. It is notated $Q^3_3$ in [BaF]. It is related to the three colouring given in the introduction. Its double is the colouring by pairs.

The **black and white** biquandle is the smallest biquandle which is not a rack. It has 2 elements $\{b, w\}$ and each permutation is the transposition $(bw)$. It is notated $BQ^2_1$ in [BaF] and is an important example related to 2-colouring.

If a biquandle has the property that the up operation (or the down operation) on its own defines a quandle then it is called a **quandle-related** biquandle associated to that quandle. In a similar fashion we can define a **rack related** birack.

Our generalisation of biracks given earlier allows us to define the **double** of a birack. Let $X$ be a birack and let $Z$ be the set of pairs of pairs $\{(a, b), (c, d)|a, b, c, d \in X\}$. Then $Y$ is the subset

$$Y = \{(a^b, c), (a, b)|a, b, c, \in X\}.$$  

The doubled operations are

$$(a, b)(a^b, c) = (a^{ca}, b^c) \quad \text{and} \quad (a^b, c)(a, b) = (a, cb)$$

Doubling converts racks into biracks and quandles into biquandles.

For example consider the double of the 3-colour quandle. This is related to the colouring of pairs on the right and left trefoil.

A biquandle is said to be **linear** if it is determined by a $2 \times 2$ matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D$ are elements of an associative ring related by certain equations, see for example [BuF, BuF2, FT]. The elements $A, B, C, D$ satisfy

$$\mathcal{F} : A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A$$

and $C, D$ are defined by

$$C = A^{-1}B^{-1}A(1 - A), \quad D = 1 - A^{-1}B^{-1}AB.$$  

The only example of a commutative linear biquandle is given by

$$a^b = \lambda a + (1 - \lambda \mu)b, \quad ab = \mu a$$

where $a, b \in X$, a $\mathbb{Z}[\lambda \pm 1, \mu \pm 1]$-module [Swa]. This is called the **Alexander** biquandle and is denoted by $A_{\lambda \mu}(X)$. If $\mu = 1$ then resulting quandle is called the **Burau** quandle and is denoted by $B_\lambda(X)$. If $\mu = \lambda = 1$ then we have the twist.
There are many examples of linear non-commutative biquandles. For example let $\mathbb{H}$ denote the quaternion algebra with standard generators $1, i, j, k$. If $X$ is a left $\mathbb{H}$-module then

\[ a^b = ja + (1 + i)b, \quad a_b = -ja + (1 + i)b \]

is a biquandle. This is called the Budapest biquandle and is just one of a huge family of linear non-commutative biquandles described in \[F, BF, BuF, BuF2, FT\].

8 Colouring a virtual knot by biracks

Let $D$ be a diagram either on a surface $\Sigma$ or in the plane with virtual crossings and let $X$ be a birack. An **edge colouring** of $D$ by $X$ is an assignment of each edge to an element of $X$, its **colour**, such that at each crossing, positive, negative and virtual, the colours, $b, c, b^c, c_b \in X$ satisfy the conditions illustrated in the following figure.

```
  c
 b
  \downarrow
  c_b
  \uparrow
  b^c
```

\begin{center}
Edge colouring
\end{center}

For example the first figure of this paper shows the trefoil coloured by the 3-colouring quandle. The second figure shows the trefoil coloured by the double of the 3-colouring. An alternatively oriented diagram has been edge coloured by the black and white biquandle.

In a **whole-colouring** of $D$ by $X$ the regions of the diagram are also labelled so that at any edge the left and right are labelled as follows.

```
  ab
  \downarrow
  a^b
```

The edge is now labelled by the pair $ab$, where (perversely) the region on the right is labelled $a$ and the region on the left is labelled $a^b$. The residual labelling of the edge is by $b$. Of course if $a \neq a^b$ then the diagram has to be 2-sided.

Note that any colouring of a region transmits around the diagram and determines the colouring of all the other regions. there is also a dual colouring in which the action of the edge colouring on neighbouring regions is by the down operation.
At crossings the whole-colouring looks as follows.

\[
\begin{array}{cccc}
\text{abc} & \text{−abc} \\
\begin{array}{c}
a^b \\
ab \\
a \\
ac_b \\
a^b c \\
a^b c \\
a \\
ac_b \\
a \\
ab \\
ac_b \\
ab \\
ac_b \\
a \\
\end{array} & \begin{array}{c}
a^b c \\
a \\
ac_b \\
ab \\
a^b c \\
a^b c \\
\end{array} & \begin{array}{c}
ac_b \\
ab \\
a \\
ac_b \\
ab \\
ac_b \\
a \\
\end{array} & \begin{array}{c}
dc \\
d_c \\
d \\
d_c \\
d c \\
d c \\
\end{array}
\end{array}
\]

Whole colouring

In the example at the beginning of the two kinds of trefoil, the colouring is now by the double of the 3-colour quandle.

An edge of a 2-sided diagram coloured by a birack can be converted into an edge coloured by the double.

To determine whether a virtual knot diagram is 2-sided let us suppose that the edges are coloured alternatively by \{b, w\}, that is by \(BQ^2\). If this is possible it can be done in two ways if the knot has one component. At any crossing the colours are cyclically ordered \(b, b, w, w\). Now try and colour the regions. At a saddle the edges are now coloured by the pairs \(bb, bb, ww, ww\) or \(bw, bw, wb, wb\) in cyclic order. However at a sink say, the pairs are \(ww, wb, bb, bw\) in cyclic order.

9 The homology of racks and biracks

Unusually in the history of mathematics, the discovery of the homology and classifying space of a rack can be precisely dated to 2 April 1990, see [history].

Let \(X\) be a birack. Associated with each \(n\)-tuple \((x_1, \ldots, x_n)\) of elements of \(X\) is a cube of dimension \(n\). Each cube is canonically identified with the standard \(n\)-cube and they fit together to make a classifying space, see [FRSd].

The homology of this space can be calculated as follows. Let \(C_n^{\text{BR}}(X)\) be the free abelian group generated by the \(n\)-tuples \((x_1, \ldots, x_n)\). Define a homomorphism \(\partial_n : C_n^{\text{BR}}(X) \to C_{n-1}(X)\) by

\[
\partial_n(x_1, x_2, \ldots, x_n) = \sum_{i=2}^n (-1)^i [(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - (x_1^{x_i}, x_2^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}^{x_i}, \ldots, x_n^{x_i})]
\]

for \(n \geq 2\) and \(\partial_n = 0\) for \(n \leq 1\). Then \(C_*^{\text{BR}}(X) = \{C_n^{\text{BR}}(X), \partial_n\}\) is a chain complex and consequently has homology groups with any coefficient groups.
For example consider the black and white biquandle \( BQ_1^2 = \{b, w\} \). The operations are \( b^x = w, w^x = b \) and \( b_x = w, w_x = b \) for any \( x \). The 1-cells 1 and 2 are cycles.

For 2-cells (squares), \( \partial bb = \partial ww = 0 \) and \( \partial bw = 2(w - b) = -\partial wb \). So \( H_1^{BR} = \mathbb{Z} \oplus \mathbb{Z}_2 \).

For 3-cells (cubes), \( \partial bbb = \partial bbw = \partial bwb = \partial wbb = bb - ww \). So \( H_2^{BR} = \mathbb{Z} \oplus \mathbb{Z} \).

Assume that \( X \) is now a rack and replace \( BR \) by \( R \). Let \( C_n^D(X) \) be the subset of \( C_n^R(X) \) generated by \( n \)-tuples \( (x_1, \ldots, x_n) \) with \( x_i = x_{i+1} \) for some \( i \in \{1, \ldots, n-1\} \) if \( n \geq 2 \); otherwise let \( C_n^D(X) = 0 \). If \( X \) is a quandle, then \( \partial_n(C_n^D(X)) \subset C_{n-1}^D(X) \) and \( C_n^Q(X) = C_n^Q(X) \) is a sub-complex of \( C_n^R(X) \). Put \( C_n^Q(X) = \{C_n^Q(X), \partial_n\} \), where \( \partial_n \) is the induced homomorphism. We shall follow standard practise and denote all boundary maps by \( \partial_n \).

For an abelian group \( G \), define the chain and cochain complexes:

\[
C_*(\mathbb{Z}; G) = C_*(\mathbb{Z}) \otimes G, \quad \partial = \partial \otimes \text{id}; \quad C_*^W(X; G) = \text{Hom}(C_*^W(X), G), \quad \delta = \text{Hom}(\partial, \text{id})
\]

where \( W = D, R \) or \( Q \).

The \( n \)-th rack/degenerate/quandle homology groups and the \( n \)-th rack/degenerate/quandle cohomology groups can now be defined in the usual way where \( R = \text{rack} \), \( D = \text{degenerate} \) and \( Q = \text{quandle} \), see [CJKS], [NP], [FRSe].

For example if \( X = Q^3_3 \), the 3-colouring quandle then \( H_2^Q = 0 \) and \( H_3^Q = \mathbb{Z}_3 \) generated by the cycle \( abc + aca \), [NP] [RS].

**10 How coloured knots determines homology and homotopy classes**

Consider a cellular diagram coloured by a biquandle \( X \). Suppose initially that it is edge coloured. Then this defines a map of the underlying surface of the diagram into the classifying space of the biquandle as follows. The diagram defines a cellulation of the surface but we consider the dual cell complex. Around each crossing is a square which is mapped to the named 2-cube in the classifying space. The edges of the square define a normal bundle of the diagram arcs. Each fibre being labelled by the label of the arc. This defines the map on the fibres to the corresponding 1-cube. The points of the regions outside are mapped to the base point. Reidemeister moves correspond to a homotopy of the map. For more details see [FRSa-b].

For the 3-coloured trefoil the map defines a non-trivial element of \( \pi_2 \) showing that the trefoil is indeed non-trivial.\(^1\)

\(^1\) This is somewhat self referential as having a non-trivial 3-colouring implies a non-zero element of \( \pi_2 \).
For a whole colouring the construction is similar but defines a 3-cycle. Looking at the trefoils coloured by pairs, since $H_3 = \mathbb{Z}_3$ for the 3-colouring space and since the resulting homology class is $+1$ for the right handed trefoil and $-1$ for the left handed trefoil we see that they are distinct.

Whether a diagram can be alternately oriented depends on its class in $H_2$ of the black and white biquandle. 2-sidedness or chessboard colouring depends on its element in $H_3$.

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