On the \((h, q)\)-zeta function associated with \((h, q)\)-Bernoulli numbers and polynomials

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Abstract: In this paper we consider the following \((h, q)\)-zeta function:

\[
\zeta_q(s, x | h) = \sum_{n=0}^{\infty} \frac{q^{hn+x}}{[n + x]_q^s} + \frac{(q-1)(h-1)}{1-s} \sum_{n=0}^{\infty} \frac{q^{(h-1)n}}{[n + x]_q^{s-1}},
\]

where \(x \neq 0, -1, -2, \ldots\), \(s \in \mathbb{C} \setminus \{1\}\) and \(h \in \mathbb{C}\). Finally, we lead to a useful integral representation for the \((h, q)\)-zeta functions and give the functional equation associated with \((h, q)\)-Bernoulli numbers and polynomials.

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1. Introduction

When one talks of \(q\)-extension, \(q\) is considered in many ways such as an indeterminate, a complex number \(q \in \mathbb{C}\), or \(p\)-adic number \(q \in \mathbb{C}_p\). Throughout this paper we assume that \(q \in \mathbb{C}\) with \(|q| < 1\). We use the notation of \(q\)-number as

\[
[x]_q = \frac{1-q^x}{1-q}.
\]

Note that \(\lim_{q \to 1}[x]_q = x\), (see [6]).

The Bernoulli polynomials in \(\mathbb{C}\) are defined by the formula

\[
e^t \frac{e^{xt}}{e^t - 1} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
\]

with the usual convention of replacing \(B(x)\) by \(B_n(x)\), (see [1-14]). In the special case, \(x = 0, B_n(0) = B_n\) are called the \(n\)-th ordinary Bernoulli numbers, (see [1-10]). The Bernoulli numbers are used to express special values of Riemann zeta function, which is defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s \in \mathbb{C}, (\text{see[7]}).
\]

That is, \(\zeta(2m) = \frac{(2\pi)^{2m}(-1)^{m-1}B_{2m}}{2(2m)!}\), for \(m \in \mathbb{N}\), and \(\zeta(1 - 2m) = -\frac{B_{2m}}{2m}\), (see [7-14]).

First, we consider \((h, q)\)-Bernoulli numbers and polynomials as the \(q\)-extension of Bernoulli
numbers and polynomials. From those numbers and polynomials, we derive some \((h, q)\)-zeta functions as the \(q\)-extension of Riemann zeta function. That is, the purpose of this paper is to study the following \((h, q)\)-zeta function:

\[
\zeta_{q}(s, x \mid h) = \sum_{n=0}^{\infty} \frac{q^{hn+x}}{[n+x]_q^s} + \frac{(q - 1)(h - 1)}{1 - s} \sum_{n=0}^{\infty} \frac{q^{(h-1)n}}{[n+x]_q^{s-1}},
\]

where \(x \neq 0, -1, -2, \cdots\), and \(h \in \mathbb{C}\), \(s \in \mathbb{C} \setminus \{1\}\).

Finally, we derive to a useful integral representation for the \((h, q)\)-zeta functions and give the functional equation associated with \((h, q)\)-Bernoulli numbers and polynomials. Recently, several authors have studied the \(q\)-zeta functions and the \(q\)-Bernoulli numbers (see [1-14]). Our \(q\)-extensions of Bernoulli numbers and polynomials in this paper are different from the \(q\)-extension of Bernoulli numbers and polynomials which are treated by several authors in previous papers.

### 2. \((h, q)\)-Bernoulli numbers and polynomials associated with \((h, q)\)-zeta functions

For \(h \in \mathbb{C}\), let us consider \((h, q)\)-Bernoulli polynomials as follows:

\[
F_q(t, x \mid h) = -t \sum_{m=0}^{\infty} q^{hm+x} e^{[x+m]_q t} + (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} e^{[x+m]_q t} = \sum_{n=0}^{\infty} \beta_{n,q} h^n (x)\frac{t^n}{n!}. \tag{1}
\]

From (1), we note that

\[
F_q(t, x \mid h) = e^{\left(\frac{1}{1-q}\right)t} \sum_{l=0}^{\infty} (-1)^l q^l x^{l+h-1} \frac{1}{[l+h-1]_q} \frac{1}{1-q} \frac{t^l}{l!}. \tag{2}
\]

By (1) and (2), we get

\[
\beta_{n,q} h^n (x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) (-1)^l q^l x^{l+h-1} \frac{1}{[l+h-1]_q}. \tag{3}
\]

In the special case, \(x = 0\), \(\beta_{n,q} h^n (0) = \beta_{n,q} h^n\) are called the \(n\)-th \((h, q)\)-Bernoulli numbers.

In (3), it is easy to show that

\[
\beta_{n,q} h^n (x) = ([x]_q + q^x \beta_{h}^n)^n, \tag{4}
\]

where we use the usual convention about replacing \((\beta^h)^n = \beta_{n,q}^h\).

Note that

\[
\lim_{q \to 1} F_q(t, x \mid h) = \frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.
\]
Let $F_q(t \mid h) = \sum_{n=0}^{\infty} \beta_{n,q}^h \frac{t^n}{n!}$ be the generating function of $(h,q)$-Bernoulli numbers. Then we get

$$F_q(t \mid h) = -t \sum_{m=0}^{\infty} q^{h} m_{q} e^{[m]q} t + (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m_{q}} e^{[m]q} t. \quad (5)$$

From (1), (4), and (5), we can derive the following difference equation:

$$F_q(t, x \mid h) = e^{[x]q} F_q(q x t \mid h) = -t \sum_{n=0}^{\infty} q^{h n + x} e^{[n]q} t + (h - 1)(1 - q) \sum_{n=0}^{\infty} q^{(h-1)n} e^{[n]q} t.$$  

Therefore, we obtain the following proposition.

**Proposition 1.** For $h \in \mathbb{C}$, we have

$$\beta_{n,q}^h(x) = \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l x} \frac{l + h - 1}{[l + h - 1]_{q}} = \sum_{l=0}^{n} \binom{n}{l} q^{l x} \beta_{l,q}^h(x)^{n-l}.$$  

Note that

$$\left( \frac{d}{dt} \right)^k F_q(t, x \mid h) \bigg|_{t=0} = -k \sum_{n=0}^{\infty} q^{h n + x} [x + n]_{q}^{k-1} + (h - 1)(1 - q) \sum_{n=0}^{\infty} q^{(h-1)n} [x + n]_{q}^{k}.$$  

Thus, we obtain the following corollary.

**Corollary 2.** For $k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we have

$$\beta_{k,q}^h(x) = -k \sum_{n=0}^{\infty} q^{h n + x} [x + n]_{q}^{k-1} + (h - 1)(1 - q) \sum_{n=0}^{\infty} q^{(h-1)n} [x + n]_{q}^{k},$$  

and

$$\beta_{k,q}^h = -k \sum_{n=0}^{\infty} q^{h n} [n]_{q}^{k-1} + (h - 1)(1 - q) \sum_{n=0}^{\infty} q^{(h-1)n} [n]_{q}^{k}.$$  

It is easy to show that

$$q^{(h-1)n} F_q(t, n \mid h) - F_q(t \mid h) = t \sum_{l=0}^{n-1} q^{h l} e^{[l]q} t - (h - 1)(1 - q) \sum_{l=0}^{n-1} q^{(h-1)l} e^{[l]q} t.$$  

Thus, we have

$$\beta_{0,q}^h = \frac{h - 1}{[h - 1]_{q}}, \quad \text{and} \quad q^{h-1} \beta_{n,q}^h(1) - \beta_{n,q}^h = \delta_{1n}, \quad (6)$$
where \( \delta_{1n} \) is kronecker symbol.

Therefore, we obtain the following theorem.

**Theorem 3.** For \( h \in \mathbb{C}, n \in \mathbb{N} \) and \( m \in \mathbb{Z}_+ \), we have

\[
q^{(h-1)n} \beta_{m,q}^h(n) - \beta_{m,q}^h = m \sum_{l=0}^{n-1} q^{hl} [l]_q^{m-1} - (h - 1)(1 - q) \sum_{l=0}^{n-1} q^{(h-1)l} [l]_q^m.
\]

In the special case, \( n = 1 \), we have

\[
\beta_{0,q}^h = \frac{h - 1}{[h - 1]_q}, \text{ and } q^{h-1} \beta_{n,q}^h(1) - \beta_{n,q}^h = \delta_{1n},
\]

where \( \delta_{1n} \) is kronecker symbol.

Now, we consider the following integral representation in complex plane.

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q(-t, x | h) dt = \sum_{n=0}^\infty \frac{q^{hn+x}}{[x+n]_q^s} + \frac{\Gamma(s-1)}{\Gamma(s)} (h - 1)(1 - q) \sum_{n=0}^\infty \frac{q^{(h-1)n}}{[x+n]_q^{s-1}} - \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{m+s-2} dt \right) \frac{(-1)^m \beta_{m,q}^h(x)}{m!}.
\] (7)

where \( x \neq 0, -1, -2, \ldots, h \in \mathbb{C} \), and \( s \in \mathbb{C} \setminus \{1\} \).

By the definition of \((h, q)\)-Bernoulli polynomials, we see that

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q(-t, x | h) dt = \sum_{m=0}^\infty \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{m+s-2} dt \right) \frac{(-1)^m \beta_{m,q}^h(x)}{m!}.
\] (8)

In the special case, \( s = 1 - k (k \in \mathbb{N}) \), we see form (7) and (8) and the basic theory of complex analysis including Laurent series that

\[
\sum_{n=0}^\infty q^{hn+x}[x+n]_q^{k-1} - \frac{(h - 1)(1 - q)}{k} \sum_{n=0}^\infty [x+n]_q^k q^{(h-1)n} = -\frac{\beta_{k,q}^h(x)}{k}.
\] (9)

In the viewpoint of (7), we can define the following Hurwitz’s type \((h, q)\)-zeta function:

**Definition 4.** For \( h \in \mathbb{C}, s \in \mathbb{C} \setminus \{1\} \), and \( x \neq 0, -1, -2, \ldots, \), define

\[
\zeta_q(s, x | h) = \sum_{n=0}^\infty \frac{q^{hn+x}}{[x+n]_q^s} + \frac{(h - 1)(1 - q)}{s - 1} \sum_{n=0}^\infty \frac{q^{(h-1)n}}{[x+n]_q^{s-1}}.
\]

Note that \( \lim_{q \to 1} \zeta_q(s, x | h) = \zeta(s, x) \), where \( \zeta(s, x) = \sum_{n=0}^\infty \frac{1}{(n + x)^s} \) is called Hurwitz’s zeta function.
Remark. Note that $\zeta_q(s, x \mid h)$ has only simple pole at $s = 1$ and $\zeta_q(s, x \mid h)$ is meromorphic function except for $s = 1$ in complex $s$-plane.

By (9) and Definition 4, we obtain the following theorem.

**Theorem 5.** For $k \in \mathbb{N}$, we have

$$\zeta_q(1 - k, x \mid h) = -\frac{\beta_{k,q}^h(x)}{k}.$$ 

In the special case, $x = 1$, we see that

$$\zeta_q(s, 1 \mid h) = \sum_{n=0}^{\infty} \frac{q^{hn+1}}{[n+1]^s_q} + \frac{(h-1)(1-q)}{s-1} \sum_{n=0}^{\infty} \frac{q^{(h-1)n}}{[n+1]^{s-1}_q}.$$

$$= \sum_{n=1}^{\infty} \frac{q^{hn+1-h}}{[n]^s_q} + \frac{(h-1)(1-q)}{s-1} \sum_{n=1}^{\infty} \frac{q^{(h-1)n+1-h}}{[n]^{s-1}_q}.$$

$$= q^{-(h-1)} \left( \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s} + \frac{(h-1)(1-q)}{s-1} \sum_{n=1}^{\infty} \frac{q^{(h-1)n}}{[n]_q^{s-1}} \right).$$

Now, we define the $(h,q)$-zeta function as follows:

**Definition 6.** For $s \in \mathbb{C} \setminus \{1\}$, and $h \in \mathbb{C}$, define

$$\zeta_q(s \mid h) = \sum_{n=1}^{\infty} \frac{q^{hn}}{[n]^s_q} + \frac{(h-1)(1-q)}{s-1} \sum_{n=1}^{\infty} \frac{q^{(h-1)n}}{[n]^{s-1}_q}.$$

Note that $\zeta_q(s \mid h) = q^{h-1} \zeta_q(s, 1 \mid h)$.

For $k \in \mathbb{N}$ with $k > 1$, we have

$$\zeta_q(1 - k \mid h) = q^{h-1} \zeta_q(1 - k, 1 \mid h) = -\frac{q^{h-1}\beta_{k,q}^h(1)}{k}. \quad (10)$$

By (6), (10), and Corollary 2, we obtain the following corollary.

**Corollary 7.** For $k \in \mathbb{N}$, we have

$$\zeta_q(1 - k \mid h) = -\frac{\beta_{k,q}^h}{k}.$$

Let $\chi$ be the Dirichlet character with conductor $f \in \mathbb{N}$. Then we define the generalized $(h,q)$-Bernoulli polynomials attached to $\chi$ as follows:

$$\beta_{n,\chi,q}^h(x) = \left[f\right]_q^{n-1} \sum_{a=0}^{f-1} \chi(a)q^{(h-1)a} \beta_{n,q}^h \left(\frac{x + a}{f}\right). \quad (11)$$
Note that
\[
\lim_{q \to 1} \beta_{n,\chi,q}^h(x) = f^{n-1} \sum_{a=0}^{f-1} \chi(a) \beta_n \left( \frac{x+a}{f} \right) = B_{n,\chi}(x),
\]
where \(B_{n,\chi}(x)\) are the \(n\)-th generalized ordinary Bernoulli polynomials attached to \(\chi\). In the special case, \(x = 0\), \(\beta_{n,\chi,q}^h(0) = \beta_{n,\chi,q}^h\) are called the \(n\)-th generalized \((h, q)\)-Bernoulli numbers attached to \(\chi\).

From (3) and (11), we note that
\[
\begin{align*}
\beta_{n,\chi,q}^h(x) &= f^{n-1} \sum_{a=0}^{f-1} \chi(a) q^{(h-1)a}  
\left( \frac{1}{1-q} \right) \sum_{l=0}^{n} \left( \begin{array}{c} n \cr l \end{array} \right) (-1)^l q^{(x+a)} \frac{l + h - 1}{[f(l + h - 1)]_q}.
\end{align*}
\]
By (12), we easily get
\[
\beta_{n,\chi,q}^h(x) = (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m)[x + m]_q^k - n \sum_{m=0}^{\infty} q^{hm+x} \chi(m)[x + m]_q^{n-1}.
\]
Therefore, we obtain the following theorem.

**Theorem 8.** For \(h \in \mathbb{C}\) and \(n \in \mathbb{Z}_+\), we have
\[
\beta_{n,\chi,q}^h(x) = (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m)[x + m]_q^k - n \sum_{m=0}^{\infty} q^{hm+x} \chi(m)[x + m]_q^{n-1}.
\]

Let \(F_{\chi,q}(t, x | h) = \sum_{n=0}^{\infty} \beta_{n,\chi,q}^h(x) \frac{t^n}{n!}\). Then we see that
\[
F_{\chi,q}(t, x | h) = (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m)e^{[x+m]_q t} - t \sum_{m=0}^{\infty} q^{hm+x} \chi(m)e^{[x+m]_q t}.
\]
Therefore, we obtain the following generating function:

**Proposition 9.** Let \(F_{\chi,q}(t, x | h) = \sum_{n=0}^{\infty} \beta_{n,\chi,q}^h(x) \frac{t^n}{n!}\). Then we have
\[
F_{\chi,q}(t, x | h) = (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m)e^{[x+m]_q t} - t \sum_{m=0}^{\infty} q^{hm+x} \chi(m)e^{[x+m]_q t}.
\]

Let \(F_{\chi,q}(t | h) = \sum_{n=0}^{\infty} \beta_{n,\chi,q}^h \frac{t^n}{n!}\). Then we also get
\[
F_{\chi,q}(t | h) = (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m)e^{[m]_q t} - t \sum_{m=0}^{\infty} q^{hm} \chi(m)e^{[m]_q t}.
\]
From (14) and the definition of \(\beta_{n,\chi,q}^h\), we can derive the following functional equation:

\[
\beta_{n,\chi,q}^h = \frac{d^n}{dt^n} F_{\chi,q}(t | h) \bigg|_{t=0} = (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m)[m]_q^n - n \sum_{m=0}^{\infty} q^{hm} \chi(m)[m]_q^{n-1}.
\]
By (14) and (15), we obtain the following corollary:

**Corollary 10.** For $h \in \mathbb{C}, n \in \mathbb{Z}_+$, we have

$$
\beta_{n,\chi,q}^h = (h - 1)(1 - q) \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m) [m]_q^n - n \sum_{m=0}^{\infty} q^m \chi(m) [m]_q^{n-1}.
$$

For $s \in \mathbb{C} \setminus \{1\}, h \in \mathbb{C}$, and $x \neq 0, -1, -2, \cdots$, we consider complex integral as follows:

$$
\frac{1}{\Gamma(s)} \int_0^\infty F_{\chi,q}(-t, x \mid h) t^{s-2} dt = \frac{(h - 1)(1 - q)}{s - 1} \sum_{m=0}^{\infty} \frac{q^{(h-1)m} \chi(m)}{[x + m]_q^{s-1}} + \sum_{m=0}^{\infty} \frac{q^m x \chi(m)}{[x + m]_q^s}.
$$

From (16), we can define Hurwitz’s type $(h,q)$-L-function as follows:

**Definition 11.** For $s \in \mathbb{C} \setminus \{1\}, h \in \mathbb{C}$, and $x \neq 0, -1, -2, \cdots$, define

$$
L_q^h(s, \chi \mid x) = \frac{(h - 1)(1 - q)}{s - 1} \sum_{m=0}^{\infty} \frac{q^{(h-1)m} \chi(m)}{[x + m]_q^{s-1}} + \sum_{m=0}^{\infty} \frac{q^m x \chi(m)}{[x + m]_q^s}.
$$

By the definition of the generating function for the generalized $(h,q)$-Bernoulli polynomials attached to $\chi$, we get

$$
\frac{1}{\Gamma(s)} \int_0^\infty F_{\chi,q}(-t, x \mid h) t^{s-2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n \beta_{n,\chi,q}^h(x)}{n!} \frac{1}{\Gamma(s)} \int_0^\infty t^{n+s-2} dt.
$$

We see from (16) and (17) and the basic theory of complex analysis including Laurent series that

$$
- \frac{(h - 1)(1 - q)}{k} \sum_{m=0}^{\infty} q^{(h-1)m} \chi(m) [x + m]^k + \sum_{m=0}^{\infty} q^m x \chi(m) [x + m]_q^{k-1} = - \frac{\beta_{k,\chi,q}^h(x)}{k}, \text{ for } k \in \mathbb{N}.
$$

From (18) and Definition 2, we obtain the following functional equation.

**Theorem 12.** For $k \in \mathbb{N}$, we have

$$
L_q^h(1-k, \chi \mid x) = - \frac{\beta_{k,\chi,q}^h(x)}{k}.
$$

Let $\chi$ be non-trivial Dirichlet character with conductor $f \in \mathbb{N}$. Then we can also consider Dirichlet’s type $(h,q)$-L-function as follows:

$$
L_q^h(s, \chi) = \frac{(h - 1)(1 - q)}{s - 1} \sum_{m=1}^{\infty} \frac{q^{(h-1)m} \chi(m)}{[m]_q^{s-1}} + \sum_{m=1}^{\infty} \frac{q^m \chi(m)}{[m]_q^s},
$$

(19)
for $s \in \mathbb{C} \setminus \{1\}, h \in \mathbb{C}$.

By Corollary 10 and (19), we get

$$L^h_q(1-k, \chi) = -\frac{\beta_k^h, \chi, q}{k}, \text{ for } k \in \mathbb{N}.$$  

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