ON EULER-RAMANUJAN FORMULA, L-FUNCTIONS AND MINIMAL SURFACES

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Abstract. In this paper, we rewrite two forms of an Euler-Ramanujan identity in terms of certain L-functions and derive functional equation of the latter. We also use the Weierstrass-Enneper representation of minimal surfaces to obtain some identities involving these L-functions and one complex parameter. We also include some diagrams pertaining to the Euler-Ramanujan identities.

1. Rewriting a form of the Euler-Ramanujan identity in terms of L-functions

We have Euler-Ramanujan’s identity, \[8\], Example (1) page 38, where \(X, A\) are complex, \(A\) is not an odd multiple of \(\pi/2\):

\[
\frac{\cos (X + A)}{\cos A} = \prod_{k=1}^{\infty} \left(1 - \frac{X}{(k - \frac{1}{2} \pi) - A}\right) \left(1 + \frac{X}{(k - \frac{1}{2} \pi) + A}\right). \tag{1}
\]

We take complex logarithm on both sides, to get:

\[
\log \left(\frac{\cos (X + A)}{\cos A}\right) = \sum_{k=1}^{\infty} \log \left\{\left(1 - \frac{X}{(k - \frac{1}{2} \pi) - A}\right) \left(1 + \frac{X}{(k - \frac{1}{2} \pi) + A}\right)\right\}, \tag{2}
\]

or

\[
\log \left(\frac{\cos (X + A)}{\cos A}\right) = \sum_{k=1}^{\infty} \log \left\{\left(\frac{(k - \frac{1}{2} \pi) - (X + A)}{(k - \frac{1}{2} \pi) - A}\right) \left(\frac{(k - \frac{1}{2} \pi) + (X + A)}{(k - \frac{1}{2} \pi) + A}\right)\right\}. \tag{3}
\]

Let \(c_k = (k - \frac{1}{2})\pi\). Replace \(X + A\) by \(y\) and \(A\) by \(x\) in the identity \[8\], where \(x\) (not an odd multiple of \(\pi\)) and \(y\) are real parameters, we have,

\[
\log \left(\frac{\cos y}{\cos x}\right) = \sum_{k=1}^{\infty} \log \left(\frac{c_k - y}{c_k - x}\right) \left(\frac{c_k + y}{c_k + x}\right) \tag{4}
\]

or

\[
\log \left(\frac{\cos y}{\cos x}\right) = \sum_{k=1}^{\infty} \log \left(\frac{c_k^2 - y^2}{c_k^2 - x^2}\right), \tag{5}
\]

or

\[
\log \left(\frac{\cos y}{\cos x}\right) = \sum_{k=1}^{\infty} \left[\log(1 + \frac{x^2}{c_k^2 - x^2}) + \log(1 - \frac{y^2}{c_k^2})\right] \tag{6}
\]

Let us take the special case when \(\left|\frac{x^2}{c_k^2 - x^2}\right| < 1\) and \(\left|\frac{y^2}{c_k^2}\right| < 1\).
Then,

\[
\log \left( \frac{\cos y}{\cos x} \right) = \sum_{k=1}^{\infty} \left[ \log(1 + \frac{x^2}{c_k^2 - x^2}) + \log(1 - \frac{y^2}{c_k^2}) \right]
\]

\[
= \sum_{k=1}^{\infty} \left[ \frac{x^2}{c_k^2 - x^2} - \frac{x^4}{2(c_k^2 - x^2)^2} + \frac{x^6}{3(c_k^2 - x^2)^3} - \ldots \right] + \left[ -\frac{y^2}{c_k^2} - \frac{y^4}{2c_k^4} - \frac{y^6}{3c_k^6} - \ldots \right]
\]

\[
= \sum_{k=1}^{\infty} \left[ L_k(1, \frac{x^2}{c_k^2 - x^2}) - M_k(1, \frac{y^2}{c_k^2}) \right]
\]

where \( L_k(s,a) = \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \frac{\frac{x^{2n}}{(c_k^2 - x^2)^n}}{n^s} \) is an \( L \)-function with a real parameter \( a = \frac{x^2}{c_k^2 - x^2} \) and an integer \( k \), \( M_k(s,b) = \sum_{n=1}^{\infty} \frac{\frac{y^{2n}}{(c_k^2)^n}}{n^s} \) is an \( L \)-function with a real parameter \( b = \frac{y^2}{c_k^2} \) and an integer \( k \) and \( L_k(1, \frac{x^2}{c_k^2 - x^2}) \) and \( M_k(1, \frac{y^2}{c_k^2}) \) are these \( L \)-functions evaluated at \( s = 1 \).

We note that \( |\frac{x^2}{c_k^2 - x^2}| < 1 \) and \( |\frac{y^2}{c_k^2}| < 1 \) for all \( k \) is equivalent to \( -\frac{\pi}{2\sqrt{2}} < x < \frac{\pi}{2\sqrt{2}} \) and \( -\sqrt{\pi/2} < y < \sqrt{\pi/2} \).

Thus we have the following proposition:

**Proposition 1.1.** \( \log \left( \frac{\cos y}{\cos x} \right) = \sum_{k=1}^{\infty} \left[ L_k(1, \frac{x^2}{c_k^2 - x^2}) - M_k(1, \frac{y^2}{c_k^2}) \right] \) for \( -\frac{\pi}{2\sqrt{2}} < x < \frac{\pi}{2\sqrt{2}} \) and \( -\sqrt{\pi/2} < y < \sqrt{\pi/2} \).

In [2], using the Weierstrass-Enneper representation of minimal surfaces we derived the following way of writing the equation \( z = \log \left( \frac{\cos y}{\cos x} \right) \) (Scherk’s minimal surface) in parametric form (in terms of a complex parameter \( \zeta \)),

\[
\begin{align*}
x(\zeta, \bar{\zeta}) &= 2 \text{Re} \tan^{-1}(\zeta), \\
y(\zeta, \bar{\zeta}) &= -\text{Im} \log \left( \frac{1 + \zeta}{1 - \zeta} \right), \\
z(\zeta, \bar{\zeta}) &= \text{Re} \log \left( \frac{1 + \zeta^2}{1 - \zeta^2} \right).
\end{align*}
\]

This parametrization fails precisely at \( \zeta = \pm 1, \pm i \). Using the fact that \( \log Z = \ln|Z| + i\theta = \ln|Z| + i\tan^{-1} \left( \frac{\text{Im}Z}{\text{Re}Z} \right) \) where \( Z = |Z|e^{i\theta} \), for \( Z \) any complex number, one can easily check that if we use the above parametrization in (see [2])

\[
z = \log \left( \frac{\cos y}{\cos x} \right)
\]

and then use the identity [9], for \( \zeta \neq \pm 1, \pm i \), and belonging to a small domain in \( \mathbb{C} \) and the expression (7) in terms of \( L \)-functions look as follows:
\[ \text{Re } \log \left( \frac{1 + \zeta^2}{1 - \zeta^2} \right) \]

\[ = \sum_{k=1}^{\infty} \left[ \log(1 + \frac{(2\text{Re } \tan^{-1}(\zeta))^2}{c_k^2 - (2\text{Re } \tan^{-1}(\zeta))^2}) + \log(1 - \frac{(\text{Im } \log \left( \frac{1+\zeta}{1-\zeta} \right))^2}{c_k^2}) \right] \]

\[ = \sum_{k=1}^{\infty} \left[ L_k(1, \frac{(2\text{Re } \tan^{-1}(\zeta))^2}{c_k^2 - (2\text{Re } \tan^{-1}(\zeta))^2}) - M_k(1, \frac{(\text{Im } \log \left( \frac{1+\zeta}{1-\zeta} \right))^2}{c_k^2}) \right]. \]

The condition that \( \frac{x^2}{c_k^2 - x^2} < 1 \) and \( \frac{y^2}{c_k^2} < 1 \) for all \( k \) is satisfied if \( |\zeta| < \frac{1}{2} \). This can be seen as follows. For \( \zeta \in \mathbb{C} \) such that \( |\zeta| < \frac{1}{2} \), we have

\[-2 < 2 \frac{|\zeta| \cos \theta}{1 - |\zeta|^2} < 2, \]

which implies

\[-\tan \left( \frac{\pi}{2\sqrt{2}} \right) < 2 \frac{|\zeta| \cos \theta}{1 - |\zeta|^2} < \tan \left( \frac{\pi}{2\sqrt{2}} \right). \]

In other words,

\[-\tan \left( \frac{\pi}{2\sqrt{2}} \right) < \frac{\zeta + \bar{\zeta}}{1 - |\zeta|^2} < \tan \left( \frac{\pi}{2\sqrt{2}} \right) \]

and hence

\[-\frac{\pi}{2\sqrt{2}} < \tan^{-1}(\zeta) + \tan^{-1}(\bar{\zeta}) < \frac{\pi}{2\sqrt{2}}. \]

Thus

\[-\frac{\pi}{2\sqrt{2}} < 2\text{Re } \tan^{-1}(\zeta) < \frac{\pi}{2\sqrt{2}}. \]

This implies \( \frac{x^2}{c_k^2 - x^2} < 1 \) where \( x = 2\text{Re } \tan^{-1}(\zeta) \). From this it follows that \( \frac{x^2}{c_k^2 - x^2} < 1 \) for all \( k, \frac{y^2}{c_k^2} < 1 \) does not give any additional constraint on \( \zeta \).

Thus, we have the following proposition:

**Proposition 1.2.** For \( \zeta \in \mathbb{C} \) such that \( |\zeta| < \frac{1}{2} \),

\[ \text{Re } \log \left( \frac{1 + \zeta^2}{1 - \zeta^2} \right) = \sum_{k=1}^{\infty} \left[ L_k(1, \frac{(2\text{Re } \tan^{-1}(\zeta))^2}{c_k^2 - (2\text{Re } \tan^{-1}(\zeta))^2}) - M_k(1, \frac{(\text{Im } \log \left( \frac{1+\zeta}{1-\zeta} \right))^2}{c_k^2}) \right]. \]

2. Minimal surfaces of translation and Ramanujan’s identities

In this section, we will consider two of the very first classical examples of minimal surfaces in 3-dimensional Euclidean space known as Scherk’s surface, [4], and the helicoid, [6, 7]. In fact, we consider a one parameter family of Scherk’s type surface. We will see later they also happens to be examples of translation surfaces and show that all these surfaces will have a \( L \)-function decomposition. Then we consider the case of helicoid separately. We start with some required definitions and examples.

**Definition 2.1.** A surface \( S \) in \( \mathbb{R}^3 \) is said to be minimal if it at every point it has zero mean curvature.
**Catenoild** [5][7], is one of the very first classical examples of minimal surfaces (other than Helicoid and Scherk’s surface).

**Example 2.2.** A parametrization for a catenoid is given by

\[ X(u, v) = (\cosh v \cos u, \cosh v \sin u, v). \]

It can also be seen as the graph of the function \( \cosh^{-1}(\sqrt{x^2 + y^2}) \) and we can also write it in an another parametric form \( (x, y, \cosh^{-1}(\sqrt{x^2 + y^2})) \).

**Definition 2.3.** A surface \( S \) in \( \mathbb{R}^3 \) is called a translation surface if \( S \) (locally) can be expressed as \( X(u, v) = \alpha(u) + \beta(v) \), where \( \alpha \) and \( \beta \) are regular curves in \( \mathbb{R}^3 \) and \( X \) is a parametrization of \( S \).

**Example 2.4.** Any plane is trivially a translation surface. For instance, \( X(u, v) = (u, v, au + bv + c) = (u, 0, au) + (0, v, bv + c) = \alpha(u) + \beta(v) \) is a parametrization of plane.

We consider a one parameter family of minimal translation surfaces (also known as Scherk’s type surface) (see [5])

\[ X_\theta(u, v) = (u + v \cos \theta, v \sin \theta, \log \cosh u), \quad (8) \]

where \( \theta \in \mathbb{R} \). Indeed, \( X_\theta(u, v) = \alpha(u) + \beta_\theta(v) \), where \( \alpha(u) = (u, 0, -\log \cosh u) \) and \( \beta_\theta(v) = (v \cos \theta, v \sin \theta, \log \cosh v) \).

**Remark 2.5.** Observe that \( \theta = 0 \) corresponds to a plane in the family (8) and \( \theta = \frac{\pi}{2} \) corresponds to the classical Scherk’s surface, i.e., \( X_\frac{\pi}{2}(u, v) = (u, v, \log \cosh u) \), discussed in the previous section.

The family \( (\theta) \) of minimal translation surfaces [5] gives us a family of Euler-Ramanujan’s identities. For \( \theta \neq \pm \frac{(2n+1)\pi}{2} \), the corresponding identity we call it a twisted Euler-Ramanujan’s identity. Indeed, put \( u + v \cos \theta = x \) and \( v \sin \theta = y \), then using the Euler-Ramanujan’s identity [1], we obtain

\[
\frac{\cos(\frac{\theta}{\sin \theta})}{\cos(x - y \cot \theta)} = \prod_{k=1}^{\infty} \left( \frac{c_k^2 - \frac{y^2}{\sin^2 \theta}}{c_k^2 - (x - y \cot \theta)^2} \right), \quad (9)
\]

where \( x - y \cot \theta \) is not an odd multiple of \( \frac{\pi}{2} \). Taking log on both the sides, we get

\[
\log \left( \frac{\cos(\frac{\theta}{\sin \theta})}{\cos(x - y \cot \theta)} \right) = \sum_{k=1}^{\infty} \log \left( \frac{c_k^2 - \frac{y^2}{\sin^2 \theta}}{c_k^2 - (x - y \cot \theta)^2} \right). \quad (10)
\]

When we take \( \theta \) as an odd multiple of \( \frac{\pi}{2} \) in (10), we get back (4). Following the same idea as applied to (4), we can write an \( L \)-function decomposition for (10), which is as follows:

\[
\log \left( \frac{\cos(\frac{\theta}{\sin \theta})}{\cos(x - y \cot \theta)} \right) = \sum_{k=1}^{\infty} [L_k(1, (x - y \cot \theta)^2 - c_k^2 (x - y \cot \theta)^2)] - M_k(1, \left( \frac{y}{c_k \sin \theta} \right)^2) \quad (11)
\]

where \( L_k(1, (x - y \cot \theta)^2 - c_k^2 (x - y \cot \theta)^2) \) is an \( L \)-function with a real parameter \( a = \frac{(x - y \cot \theta)^2}{c_k^2 - (x - y \cot \theta)^2} \) for a fixed \( \theta \) and an integer \( k \), \( M_k(s, -(\frac{y}{c_k \sin \theta})^2) \), \( (\theta\text{-fixed}) \), is an \( L \)-function with a real parameter \( b = -(\frac{y}{c_k \sin \theta})^2 \) and an integer \( k \) as before. \( L_k(1, (x - y \cot \theta)^2 - c_k^2 (x - y \cot \theta)^2) \) and \( M_k(1, -(\frac{y}{c_k \sin \theta})^2) \) are these \( L \)-functions evaluated at \( s = 1 \). We can also compute
Weierstrass-Enneper type representation (which is described in terms of a complex parameter) for the family of Scherk’s type surfaces \([8]\) given by

\[
x(\zeta, \bar{\zeta}) = 2\operatorname{Re}(\tan^{-1}(\zeta)) - \operatorname{Im} \log \left(\frac{1 + \zeta}{1 - \bar{\zeta}}\right) \cos \theta,
\]

\[
y(\zeta, \bar{\zeta}) = -\operatorname{Im} \log \left(\frac{1 + \zeta}{1 - \bar{\zeta}}\right) \sin \theta,
\]

\[
z(\zeta, \bar{\zeta}) = \operatorname{Re} \log \left(\frac{1 + \zeta^2}{1 - \bar{\zeta}^2}\right).
\]

This family also has a non-parametric representation given by

\[
z = \log \left(\frac{\cos \left(\frac{y}{x} \sin \theta\right)}{\cos \left(x - y \cot \theta\right)}\right).
\]

By combining (11), (12), (13), and (14) we obtain the result as in Proposition 1.2.

For \(\zeta \in D\),

\[
\operatorname{Re} \log \left(\frac{1 + \zeta^2}{1 - \bar{\zeta}^2}\right) = \sum_{k=1}^{\infty} [L_k(1, \frac{2\operatorname{Re}(\tan^{-1}(\zeta))^2}{c_k^2}) - M_k(1, \frac{(\operatorname{Im} \log \left(\frac{1 + \zeta}{1 - \bar{\zeta}}\right))^2}{c_k^2})].
\]

Observe that the expression (16) is independent of \(\theta\). In fact, this is the common \(L\)-function decomposition for the \(\theta\)-family which is quite evident once we look at their Weierstrass-Enneper representation.

Now, let us consider the non-parametric representation of helicoid which is given by (see Osserman)

\[
z = \tan^{-1} \frac{y}{x}
\]

and also recall the identity

\[
\tan^{-1} \omega = \frac{i}{2} \left(\log (1 - i \omega) - \log (1 + i \omega)\right)
\]

where \(\omega\) is a complex number. Now if we put \(\omega = \frac{y}{x}\) in the above identity, we get

\[
\tan^{-1} \frac{y}{x} = \frac{i}{2} \left\{\log \left(1 - i \frac{y}{x}\right) - \log \left(1 + i \frac{y}{x}\right)\right\}
\]

The above expression (19) helps us to write helicoid as a sum of two \(L\)-functions evaluated at a specific value. When \(y < |x|\) we can write equation (19) as

\[
\tan^{-1} \frac{y}{x} = \frac{i}{2} \left\{\sum_{k=1}^{\infty} (-1)^{2k-1} \frac{i^k}{k} \left(\frac{y}{x}\right)^k - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{i^k}{k} \left(\frac{y}{x}\right)^k\right\}
\]

\[
= \frac{1}{2} \left\{\sum_{k=1}^{\infty} \frac{i^{k+1}}{k} \left(\frac{y}{x}\right)^k + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{i^k}{k} \left(\frac{y}{x}\right)^k\right\}
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} \{(-1)^k - 1\} \frac{i^{k+1}}{k} \left(\frac{y}{x}\right)^k
\]

\[
= -\frac{1}{2} \left[ L(1, \frac{y}{x}) - M(1, -\frac{y}{x}) \right]
\]
where \( L(s, \frac{y}{x}) = \sum_{k=1}^{\infty} \frac{e^{2\pi k}}{k^s} \) is an \( L \)-function and \( M(s, -\frac{y}{x}) = \sum_{k=1}^{\infty} \frac{e^{2\pi k}}{k^s} \) are \( L \)-functions which are in turn evaluated at \( s = 1 \). Thus we have the following proposition:

**Proposition 2.6.** When \( |y| < |x| \), \( \tan^{-1} \frac{y}{x} = \frac{1}{2} \left[ L(1, \frac{y}{x}) - M(1, -\frac{y}{x}) \right] \).

The Weierstrass-Enneper representation of the helicoid in terms of the complex parameter is given by (see [2])

\[
x(\zeta, \bar{\zeta}) = -\frac{1}{2} \text{Im} \left( \frac{\zeta + \frac{1}{\zeta}}{L(1, \frac{\text{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\text{Im}\left(\zeta + \frac{1}{\zeta}\right)})} - M \left( 1, -\frac{\text{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\text{Im}\left(\zeta + \frac{1}{\zeta}\right)} \right) \right),
\]

\[
y(\zeta, \bar{\zeta}) = \frac{1}{2} \text{Re} \left( \frac{\zeta - \frac{1}{\zeta}}{L(1, \frac{\text{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\text{Im}\left(\zeta + \frac{1}{\zeta}\right)})} - M \left( 1, -\frac{\text{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\text{Im}\left(\zeta + \frac{1}{\zeta}\right)} \right) \right),
\]

\[
z(\zeta, \bar{\zeta}) = -\frac{\pi}{2} + \text{Im}(\log \zeta).
\]

The condition \( |y| < |x| \) translates to \( |\text{Re}(\zeta - \frac{1}{\zeta})| < |\text{Im}(\zeta + \frac{1}{\zeta})| \) which is satisfied if \( |\zeta| < 1 \).

By combining (20), (24), (25), and (26), we get

**Proposition 2.7.** For \( \zeta \in \mathbb{C} \), such that \( |\zeta| < 1 \)

\[
-\frac{\pi}{2} + \text{Im}(\log \zeta) = \frac{-1}{2} \left\{ L \left( 1, \frac{\text{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\text{Im}\left(\zeta + \frac{1}{\zeta}\right)} \right) - M \left( 1, -\frac{\text{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\text{Im}\left(\zeta + \frac{1}{\zeta}\right)} \right) \right\}.
\]

Next we look at another identity. For \( X \) and \( A \) real, we have (see entry 11 in [8])

\[
\tan^{-1}(\tanh X \cot A) = \tan^{-1} \left( \frac{X}{A} \right) + \sum_{k=1}^{\infty} \left( \tan^{-1} \left( \frac{X}{k\pi + A} \right) - \tan^{-1} \left( \frac{X}{k\pi + A} \right) \right).
\]

When \( A = \frac{\pi}{2} \) the identity [28] reduces to the following identity

\[
\tan^{-1} \left( \frac{2X}{\pi} \right) = \sum_{k=1}^{\infty} \left( \tan^{-1} \left( \frac{X}{c_k} \right) - \tan^{-1} \left( \frac{X}{d_k} \right) \right),
\]

where \( c_k = (k - \frac{1}{2})\pi \) and \( d_k = (k + \frac{1}{2})\pi \). Next we put \( \frac{2X}{\pi} = \frac{y}{x} \) to obtain

\[
\tan^{-1} \left( \frac{y}{x} \right) = \sum_{k=1}^{\infty} \left( \tan^{-1} \left( \frac{eky}{x} \right) - \tan^{-1} \left( \frac{fky}{x} \right) \right),
\]

where \( e_k = \frac{\pi}{2c_k} \) and \( f_k = \frac{\pi}{2d_k} \).

Thus we have

**Proposition 2.8.**

\[
L(1, \frac{y}{x}) - M(1, -\frac{y}{x}) = \sum_{k=1}^{\infty} \left[ L_k(1, \frac{eky}{x}) - M_k(1, -\frac{eky}{x}) \right] - \left[ L_k(1, \frac{fky}{x}) - M_k(1, -\frac{fky}{x}) \right],
\]

where \( L_k(1, \frac{eky}{x}) = L(1, \frac{eky}{x}) \) and \( M_k(1, -\frac{eky}{x}) = M(1, -\frac{eky}{x}) \).
Remark: Our objects of interest here happen to be minimal surfaces of translation. In past, several authors have shown a good amount of interest in knowing what are all minimal translation surfaces in \( \mathbb{R}^3 \). The very first result in this direction is due to Dillen et al. [3], also see [4]. Recently, Lopez and Hasanis, [5] gave a complete classification of minimal translation surfaces in \( \mathbb{R}^3 \). More precisely, they proved that apart from the plane and the minimal surfaces of Scherk type, any other minimal translation surface can be described as

\[ X(s, t) = \alpha(s) + \alpha(t), \]

where \( \alpha \) is a space curve (a curve in \( \mathbb{R}^3 \)). In fact, they have given a method to construct explicit examples of such surfaces. For instance, a parametrization of helicoid is given by

\[ X(u, v) = (\cos u \cos v, \sin u \cos v, u). \]

Next, we take two circular helices in \( \mathbb{R}^3 \), given by \( \alpha(s) = (\cos s, \sin s, s/2) \) and \( \beta(t) = (\cos t, \sin t, t/2) \) and now if we consider a change of parameters \( (s, t) \to (u + v, u - v) \) then we get

\[ \alpha(u + v) + \alpha(u - v) = X(u, v). \]

It would be interesting to see if number-theoretical identities like the E-R identities are available for all minimal surfaces of translation.

3. Properties of the L-functions

We consider the general form of the L-functions defined in the previous section

\[ L_k(s, a) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s}, \quad \text{where } a = \frac{x^2}{c_k - x^2}, 0 < a < 1, \ s \in \mathbb{C} \]

\[ M_k(s, a) = \sum_{n=1}^{\infty} (-1)^{n} \frac{a^n}{n^s}, \quad \text{where } a = \frac{y^2}{c_k}, 0 < a < 1, \ s \in \mathbb{C}. \]

3.1. Convergence of the series. We show that the series is convergent for all \( s \in \mathbb{C} \) by showing its absolute convergence.

\[ L_k(s, a) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s}, 0 < a < 1, \ s = \sigma + ib, \ \sigma, b \in \mathbb{R}. \]

Now, \( |n^s| = |n^\sigma||n^b| = n^\sigma |e^{ib \ln n}| = n^\sigma. \) The absolute series then is

\[ \sum_{n=1}^{\infty} \frac{|a|^n}{n^\sigma} = \sum_{n=1}^{\infty} \frac{1}{|c|^n n^\sigma}, \ a = 1/c, c > 1. \]

Then the ratio test gives

\[ \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|c|^n n^\sigma}{|c|^{n+1} (n + 1)^\sigma} = \lim_{n \to \infty} \frac{1}{|c| (1 + 1/n)^\sigma} = \frac{1}{c} < 1. \]

Thus \( L_k(s, a) \) is convergent on \( 0 < a < 1, \) for each \( k. \) Similar is the case for \( M_k(s, a). \)
3.2. Functional equation. In this section, we follow a technique of Hardy, expounded in [1].

Our general Dirichlet series is \( \gamma(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s} \), where, \( 0 < a < 1 \), \( s \in \mathbb{C} \).

Then, defining \( F(x) = \sum_{n \leq x} (-1)^{n-1} = \frac{1 - (-1)^m}{2} \), \( m < x < m + 1 \), we have,

\[
\gamma(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s} \\
= \sum_{n=1}^{\infty} (F(n) - F(n - 1)) \frac{a^n}{n^s} \\
= \sum_{n=1}^{\infty} F(n) \frac{a^n}{n^s} - \sum_{n=1}^{\infty} F(n) \frac{a^{n+1}}{(n+1)^s} - F(0) \cdot a \\
= \sum_{n=1}^{\infty} F(n) \int_n^{n+1} \left( \frac{a^x \ln a}{x^s} - \frac{sa^x}{x^{s+1}} \right) dx \\
= \ln a \sum_{n=1}^{\infty} \int_n^{n+1} \frac{a^x F(x)}{x^s} - s \sum_{n=1}^{\infty} \int_n^{n+1} \frac{a^x F(x)}{x^{s+1}} dx \\
= \ln a \int_1^{\infty} \frac{a^x F(x)}{x^s} - s \int_1^{\infty} \frac{a^x F(x)}{x^{s+1}} dx.
\]

Let \( f(x) = \ln a \int_1^{\infty} \frac{a^x F(x)}{x^s} \\
= \ln a \int_1^{\infty} \frac{a^x (F(x) - 1/2)}{x^s} dx + \frac{\ln a}{2} \int_1^{\infty} \frac{a^x}{x^s} dx \\
= I_1(s) + I_2(s) \) (say), and

\[
g(x) = s \int_1^{\infty} \frac{a^x F(x)}{x^{s+1}} dx \\
= s \int_1^{\infty} \frac{a^x (F(x) - 1/2)}{x^{s+1}} dx + \frac{s}{2} \int_1^{\infty} \frac{a^x}{x^{s+1}} dx \\
= I_3(s) + I_4(s) \) (say).
\]

The integrals can all be seen to be convergent for all \( s \).

\( F(x) - \frac{1}{2} \) is bounded, say by \( M \). Then, taking \( a = \frac{1}{c} \), \( c > 1 \) we can see that \( \int_1^{\infty} |\frac{a^x (F(x) - 1/2)}{x^s}| dx = \int_1^{\infty} \left| \frac{F(x) - 1/2}{c^n x^n} \right| dx \leq M \int_1^{\infty} \frac{1}{x^s} dx \) \( (s > 0) \), which is convergent. \( I_1 \) and similarly \( I_2 \) are thus absolutely convergent since \( c^x \) is exponential growth and \( x^s \) has only polynomial growth. \( I_3 \) and \( I_4 \) can similarly be concluded to be convergent.

We try to evaluate \( I_4 \) first. Let \( x \ln a = u \Rightarrow du = \ln a \, dx \). Then \( u \) ranges from \( \ln a \) to \( -\infty \) as \( x \) ranges from 1 to \( \infty \).
Then,

\[
I_4(s) = \frac{s}{2} \int_1^\infty \frac{a^x}{x^{s+1}} dx
= \frac{s}{2} \int_1^\infty x^{-s-1} e^{x \ln a} dx
= \frac{s}{2 \ln a} \int_{\ln a}^{\infty} (\frac{u}{\ln a})^{s-1} e^u du
= \frac{s(\ln a)^s}{2} \int_{\ln a}^\infty (-1)^{s-1} u^{-s-1} e^{-u} du
= \frac{s}{2}(\ln a)^s \Gamma(-s, -\ln a)
\]

Similar to \(I_4(s)\), \(I_2(s)\) will then be

\[
\frac{-(\ln a)^s}{2} \Gamma(-s - 1, -\ln a),
\]

where \(\Gamma\) is the incomplete gamma function.

We now calculate \(I_3\). Since the function \(F(x)\) is piecewise continuous of period 2, we calculate its Fourier expansion. Then,

\[
F(x) = \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}),
\]

where,

\[
a_0 = \frac{1}{L} \int_0^{2L} F(x) \cos \frac{n\pi x}{L} dx = \int_0^L F(x) dx = \int_1^2 dx = 1
\]

\[
a_m = \frac{1}{L} \int_0^{2L} F(x) \cos \frac{m\pi x}{L} dx = 0
\]

\[
b_n = \frac{1}{L} \int_0^{2L} F(x) \sin \frac{m\pi x}{L} dx = \frac{2}{\pi} for n odd.
\]

Therefore, \(F(x) = \frac{1}{2} + \sum_{k=0}^\infty -\frac{2 \sin(2k\pi x + \pi x)}{(2k + 1)\pi}\).

Substituting \(F(x) - \frac{1}{2}\) by its Fourier expansion, we have,

\[
I_3(s) = s \int_1^\infty \frac{a^x (F(x) - 1/2)}{x^{s+1}} dx
= \frac{-2s}{\pi} \sum_{k=0}^\infty \frac{1}{2k + 1} \int_0^\infty a^x \sin(2k\pi x + \pi x) \frac{x^{s+1}}{x^{s+1}} dx
= \frac{-2s}{\pi} \sum_{k=0}^\infty \frac{1}{2k + 1} \int_0^\infty a^t \sin(t) \frac{t^{s+1}}{(\pi(2k + 1)\pi)^{s+1}} dt (Taking 2k\pi x + \pi x = t)
= \frac{-2s}{\pi} \sum_{k=0}^\infty \frac{\pi^s}{(2k + 1)^{1+s}} \int_0^\infty a^t \sin(t) \frac{t^{s+1}}{t^{s+1}} dt.
\]

We now try to evaluate \(I_{5k}(s) = \int_0^\infty \frac{a^t \sin(t)}{t^{s+1}} dt\).

Let \(\text{Re}(s) < 0\).

\(I_{5k}\) can be evaluated in terms of a complex gamma function as follows:
\[ I_{5k}(s) = \int_0^\infty t^{-s-1}a^{\pi/(2k+1)} \sin(t) = \int_0^\infty t^{-s-1}a^{\pi/(2k+1)}(e^{it} - e^{-it})/2i \]
\[ = \int_0^\infty t^{-s-1}(e^{Ct} - e^{-Ct})/2i \]

where \( C_{\pm} = \frac{\ln(a)}{\pi(2k+1)} \pm i \).

Let \( u = C_+ t \) and \( w = C_- t \). Then making change to these complex variables and multiplying by \((-1)^{-s-1}\) when needed, we get

\[ I_{5k}(s) = \frac{1}{2iC_+} \int_{\gamma_1} u^{-s-1}e^{-u}du - \frac{1}{2iC_-} \int_{\gamma_2} w^{-s-1}e^{-w}dw \]

where \( \gamma_1 \) is from 0 to \( C_+ \infty \) along the line \( y = \frac{(2k+1)\pi}{\ln(a)} x \) and \( \gamma_2 \) is from 0 to \( C_- \infty \) along the line \( y = -\frac{(2k+1)\pi}{\ln(a)} x \). Take contours shown in the diagrams below (with \( \gamma_1 = e \) in the first diagram and \( \gamma_2 = e \) in the second diagram) and let the radius of the circular arc grow bigger. It can be shown that the first integral evaluates to \( \int_0^\infty u^{-s-1}e^{-u}du = -\Gamma(-s) \) and so does the second one, if \( \text{Re}(s) < 0 \). Here \( \Gamma(-s) \equiv \int_0^\infty u^{-s-1}e^{-u}du \) and has poles along non-positive integers for \( \text{Re}(s) < 0 \).

It is easy to check that \( I_{5k}(s) = -\Gamma(-s)/(\ln(a) (2k+1)\pi)^2 + 1) \).

\[ I_1(s) = \frac{\ln(a)}{s-1} I_3(s-1) = \frac{2\ln(a)}{\pi}\sum_{k=0}^{\infty} \frac{\pi^{-s-1}}{(2k+1)^{2-s}((\ln(a)/(2k+1)\pi)^2+1)}\Gamma(-s+1) \]

**Figure 1.** Contour for the first integral
Thus

\[
\gamma(s) = I_1(s) + I_2(s) + I_3(s) + I_4(s) \\
= 2\ln(a) \sum_{k=0}^{\infty} \frac{\pi^{s-1}}{(2k+1)^{2s} \left( \left( \frac{\ln(a)}{(2k+1)\pi} \right)^2 + 1 \right)} \Gamma(-s+1) \\
+ \frac{(-\ln(a))^s}{2} \Gamma(-(s-1), -\ln a) \\
+ \frac{2s}{\pi} \sum_{k=0}^{\infty} (2k+1)^{1-s} \left( \left( \frac{\ln(a)}{(2k+1)\pi} \right)^2 + 1 \right) \Gamma(-s) \\
+ \frac{s}{2} (-\ln a)^s \Gamma(-s, -\ln a)
\]

Let \( N_k = (2k+1) \) and \( A = -\ln(a) \). Let us restrict ourselves to \( e^{-\pi} < a < 1 \) such that \( 0 < \frac{A}{N_k} < 1 \) for all \( k \). Let \( \tilde{\zeta}(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} \).

Notice that

\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2s} \left( \left( \frac{\ln(a)}{(2k+1)\pi} \right)^2 + 1 \right)} = \sum_{k=0}^{\infty} \frac{1}{N_k^{2-s} \left( \frac{A^2}{N_k^2} + 1 \right)} \\
= \sum_{k=0}^{\infty} \frac{1}{N_k^{2-s}} \left[ 1 - \frac{A^2}{N_k^2} + \frac{A^4}{N_k^4} - \frac{A^6}{N_k^6} + \ldots \right] \\
= \tilde{\zeta}(2-s) - A^2 \tilde{\zeta}(4-s) + \ldots \\
= \sum_{n=0}^{\infty} \tilde{\zeta}((2n+2-s)(-1)^n A^{2n}.
\]
and
\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{1-s}} \left( \frac{\ln(a)}{(2k+1)\pi} \right)^2 + 1) = \sum_{k=0}^{\infty} \frac{1}{N_k^{1-s}} \left( \frac{A^2}{N_k^2} + 1 \right) \\
= \sum_{k=0}^{\infty} \frac{1}{N_k^{1-s}} \left[ 1 - \frac{A^2}{N_k^2} + \frac{A^6}{N_k^6} - \frac{A^8}{N_k^8} \ldots \right] \\
= \zeta(1-s) - A^2 \zeta(3-s) + \ldots \\
= \sum_{n=0}^{\infty} \zeta(2n+1-s)(-1)^{n} A^{2n}.
\]

Thus, the following proposition gives us a functional relationship between the L-function and \(\zeta\) function.

**Proposition 3.1.** Let \(A = -\frac{\ln(a)}{\pi}\) and \(\tilde{\zeta}(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s}\). For \(\text{Re}(s) < 0\) and \(e^{-\pi} < a < 1\), we have the following functional equation.

\[
\gamma(s) = \frac{2\ln(a)\pi^{s-1}\Gamma(-s+1)}{\pi} \sum_{n=0}^{\infty} \zeta((2n+2-s)(-1)^n A^{2n}} \\
- \frac{(-\ln(a))^s}{2} \Gamma(-(s-1), -\ln a) \\
+ \frac{2s\pi^n}{\pi} \Gamma(-s) \sum_{n=0}^{\infty} \zeta((2n+1-s)(-1)^n A^{2n}} \\
+ \frac{s}{2}(-\ln a)^s \Gamma(-s, -\ln a)
\]

The infinite sums converge since \(0 < A < 1\).

### 3.3. Essential singularity at \(\infty\)

We consider the limit of our function as \(s \to \infty\) along two directions. \(L_k(s,a) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s}\), where \(a = \frac{x^2}{z^2 + x^2}, 0 < a < 1, s = \sigma + it \in \mathbb{C}\).

When \(\text{Im}(s)\) is fixed and \(\text{Re}(s) \to \infty\):

\[
|\sum_{n=2}^{\infty} (-1)^{n-1} \frac{a^n}{n^s}| \leq \sum_{n=2}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=2}^{\infty} \frac{1}{n^\sigma - c} = \sum_{n=2}^{\infty} \frac{1}{n^{\sigma-c}} \leq \frac{1}{2^{\sigma-c}} \sum_{n=2}^{\infty} \frac{1}{n^c} \to 0 \text{ as } \sigma \to \infty
\]

Therefore, \(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s} \to a\) as \(\sigma \to \infty\).

When \(\text{Re}(s)\) is fixed and \(\text{Im}(s) \to \infty\):

If possible, let \(L_k\left(\frac{1}{2} + it, a\right) \to a\) as \(t \to \infty\). Then given \(\epsilon > 0, \exists t_0 \in \mathbb{R}\) such
that
\[
\left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^{\frac{1}{2}} + it} - a \right| \leq \epsilon \quad \forall \ t \geq t_0
\]
or
\[
\left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^{\frac{1}{2}} + it} - \left| a \right| \right| \leq \epsilon \quad \forall \ t \geq t_0
\]
or
\[
\left| a \right| - \epsilon \leq \left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^{\frac{1}{2}} + it} \right| \quad \forall \ t \geq t_0
\]
or
\[
\left| a \right| - \epsilon \leq \zeta (\frac{1}{2})
\]
or
\[
\left| a \right| \leq -1.46... + \epsilon
\]
which is not true for sufficiently small \( \epsilon \).

Thus we arrive at a contradiction. The limit of \( L_k(s, a) \) as \( s \to \infty \) does not exist and the series has an essential singularity at infinity.

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