Rigidity of Lagrangian embeddings into symplectic tori and K3 surfaces

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Abstract

A Kähler-type form is a symplectic form compatible with an integrable complex structure. Let $M$ be either a torus or a K3-surface equipped with a Kähler-type form. We show that the homology class of any Maslov-zero Lagrangian torus in $M$ has to be non-zero and primitive. This extends previous results of Abouzaid-Smith (for tori) and Sheridan-Smith (for K3-surfaces) who proved it for particular Kähler-type forms on $M$. In the K3 case our proof uses dynamical properties of the action of the diffeomorphism group of $M$ on the space of the Kähler-type forms. These properties are obtained using Shah’s arithmetic version of Ratner’s orbit closure theorem.

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1 Main results

Recall that Kähler structure on a smooth manifold $M$ is a pair $(\omega, J)$, where $\omega$ is a symplectic form on $M$ and $J$ is an integrable complex structure on $M$ compatible with $\omega$, that is, satisfying $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) > 0$ for any non-zero tangent vectors $v, w$. (We view complex structures as tensors on $M$ – that is, as integrable almost complex structures; in particular, we only consider complex structures with the prefixed underlying smooth structure on $M$.) A symplectic form, or a complex structure, on $M$ is said to be of Kähler type, if it appears in some Kähler structure.

In this paper we will be mostly concerned with $M$ which is either an even-dimensional torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ or a smooth manifold (of real dimension 4) underlying a complex K3 surface.

Let us note the following:

- The Kähler-type symplectic forms on $\mathbb{T}^{2n}$ are exactly the ones that can be mapped by an orientation-preserving diffeomorphism of $\mathbb{T}^{2n}$ to a linear symplectic form – see e.g. [EV, Prop. 6.1]. (A linear symplectic form on $\mathbb{T}^{2n}$ is a form whose lift to $\mathbb{R}^{2n}$ has constant coefficients). We will always use the orientation on $\mathbb{T}^{2n}$ induced by the standard orientation on $\mathbb{R}^{2n} = \mathbb{C}^n$ and consider only the Kähler-type symplectic forms on $\mathbb{T}^{2n}$ inducing this orientation.

- All smooth manifolds $M$ underlying a complex K3 surface are diffeomorphic. They all are compact and connected; any complex structure on such an $M$ is of Kähler-type and its first Chern class is zero. In fact, all complex structures and Kähler-type symplectic forms on such an $M$ appear as parts of hyperkähler structures – see Proposition 5.1 below. All complex structures on such an $M$ define the same orientation on $M$ so that $b_+(M) = 3$ and $b_-(M) = 19$ – see e.g. [Bea2] (reversing the orientation would yield $b_+ = 19$ and $b_- = 3$). Further on, we will always equip a K3 surface with this standard orientation. Alternatively, to fix the orientation, one can use the fact that all complex K3 surfaces are deformation equivalent and, in particular, oriented diffeomorphic – see e.g. [Bea2].

This is the main application of the techniques developed in this paper.

**Theorem 1.1:**

Assume $M$, $\dim_{\mathbb{R}} M = 2n$, is either an even-dimensional torus or a smooth manifold underlying a K3 surface. Let $\omega$ be a Kähler-type symplectic
Then for any Maslov-zero Lagrangian torus $L \subset (M, \omega)$ the homology class $[L] \in H_n(M; \mathbb{Z})$ is non-zero and primitive.

For the proof see Section 7.

In the torus case the result of Theorem 1.1 was previously proved by Abouzaid and Smith [AS, Cor. 1.6 and Cor. 9.2] for one particular Kähler-type symplectic form on $M = \mathbb{T}^{2n}$ – namely, for the standard Darboux form. Their proof of $[L] \neq 0$ relies on a previous deep result of Fukaya and works similarly for any linear (or, equivalently, Kähler-type) symplectic form on $\mathbb{T}^{2n}$ – we recall their argument in the proof of Theorem 1.1.

In the K3 case the result of Theorem 1.1 was previously proved by Sheridan and Smith [SS, Thm. 1.3] for some Kähler-type symplectic forms on $M$. Their proof uses deep methods of homological mirror symmetry.

Here we extend the results of Abouzaid-Smith (in the torus case) and Sheridan-Smith (in the K3 case) to any Kähler-type symplectic form on $M$. Our result in the K3 case answers a question of Seidel (see [SS, Sect. 1.1, Quest. 5]).

Our proof of Theorem 1.1 reduces the case of an arbitrary Kähler-type symplectic form on $M$ to particular cases where symplectic rigidity results can be applied. Namely, if $(M, \omega)$ admits a Maslov-zero Lagrangian torus $L$ whose homology class is zero or non-primitive, then so does $(M, \omega')$ for any symplectic form $\omega'$ sufficiently close to $\omega$ and satisfying $[\omega'|_L] = 0$ (this is an easy application of Moser’s method). We show that such $\omega'$ can be chosen to lie in the $\text{Diff}^+(M)$-orbit of a Kähler-type symplectic form for which the absence of such Lagrangian tori is proved by symplectic rigidity methods. (Here and further on $\text{Diff}^+(M)$ stands for the group of orientation-preserving diffeomorphisms of $M$.) Since the set of the symplectic forms on $M$ admitting such a Lagrangian torus is $\text{Diff}^+(M)$-invariant, this yields a contradiction that proves Theorem 1.1.

In the torus case the application of symplectic rigidity results follows closely the proof of Abouzaid-Smith in [AS] and in the K3 case we use the work of Sheridan and Smith [SS].

The dynamical result on the $\text{Diff}^+(M)$-orbits needed to prove $[L] \neq 0$ in the K3 case can be deduced relatively easily from [EV], and the main effort in this paper is to strengthen it in order to prove the primitivity of $[L]$.

Remark 1.2:

The existence of Maslov-zero Lagrangian tori is well-known for certain Kähler-type symplectic forms on tori and K3 surfaces. For instance,

- For the standard Darboux form $dp \wedge dq$ on a torus, the meridian La-
grangian torus \( p = \text{const} \) is Maslov-zero.

- K3 surfaces with certain Kähler forms admit special Lagrangian tori \([\text{IL}]\) (also see \([\text{SYZ}]\)). These special Lagrangian tori are Maslov-zero.

At the same time there exist Kähler-type symplectic forms on tori and K3 surfaces that do not admit any Lagrangian submanifolds with a non-zero homology class (for instance, because of obvious homological obstructions), and, in particular, by \textbf{Theorem 1.1} no Maslov-zero Lagrangian tori.

\textbf{Remark 1.3:} Using the Kodaira-Spencer stability theorem \([\text{KoS}]\) and Torelli theorems for even-dimensional tori (see e.g. \([\text{BHPV}]\) Ch. 1, Thm. 14.2) and K3 surfaces (see \([\text{BR}]\), cf. \([\text{Bea2}]\) p.96) one can show that the Kähler-type symplectic forms on these manifolds form an open subset of the space of all symplectic forms (with respect to the \(C^\infty\)-topology).

It is an open question whether all symplectic forms on \(T^{2n}\) and K3 surfaces are of Kähler type. In \([\text{D2}]\) Donaldson outlined how one can try to prove that the answer to the question is \textit{positive}. On other hand, in view of \textbf{Theorem 1.1} a possible strategy of proving that the answer to the question is \textit{negative} would be to construct a symplectic form on \(T^{2n}\), or on a K3 surface, that admits a Maslov-zero Lagrangian torus whose integral homology class is zero or non-primitive.

Let us now describe the dynamical results at the heart of the proof of \textbf{Theorem 1.1} in more detail.

First, let us consider the case of \(T^{2n}\).

Consider the linear symplectic forms of volume 1 on \(T^{2n}\) and denote by \(\mathcal{L}\) the space of their lifts to \(\mathbb{R}^{2n}\). The topology on \(\mathcal{L}\) is induced by the standard topology on the space of bilinear forms on \(\mathbb{R}^{2n}\).

Let \(l \subset \mathbb{R}^{2n}\) be a vector subspace of (real) dimension \(n\) which is spanned over \(\mathbb{R}\) by vectors in \(\mathbb{Z}^{2n} \subset \mathbb{R}^{2n}\) – such a vector subspace is called \textit{rational}. Define \(\mathcal{L}_l \subset \mathcal{L}\) as

\[
\mathcal{L}_l := \{ \tilde{\omega} \in \mathcal{L} \mid \tilde{\omega}|_l \equiv 0 \}.
\]

Define a group \(G \subset SL(2n, \mathbb{R})\) by

\[
G := \{ g \in SL(2n, \mathbb{R}) \mid g|_l = Id \}.
\]

The group \(G\) acts on \(\mathcal{L}_l\). Let

\[
G_A := G \cap SL(2n, \mathbb{Z}).
\]
We say that $n$-dimensional rational subspaces $l, l' \subset \mathbb{R}^{2n}$ are complementary if $(l \cap \mathbb{Z}^{2n}) \oplus (l' \cap \mathbb{Z}^{2n}) = \mathbb{Z}^{2n}$ (in particular, this implies that $l \oplus l' = \mathbb{R}^{2n}$).

Given complementary $n$-dimensional rational subspaces $l, l' \subset \mathbb{R}^{2n}$, we say that a symplectic form $\tilde{\omega} \in L$ is $(l, l')$-Lagrangian split if $l$ and $l'$ are Lagrangian with respect to $\tilde{\omega}$, that is, $\tilde{\omega} \in L_l \cap L_{l'}$.

Proposition 1.4:

The union of the $G_{\Lambda}$-orbits of the $(l, l')$-Lagrangian split forms is dense in $L_l$.

For the proof see Section 2.

Now let $M$ be either $T^4$ or a smooth manifold underlying a K3 surface. Denote by $S(M)$ the space of Kähler-type symplectic forms $\omega$ on $M$ of total volume 1, meaning that $\int_M \omega^n = 1$, $2n = \dim_R M$. We equip $S(M)$ with the $C^\infty$-topology.

Let $(\cdot, \cdot) : H^2(M; \mathbb{R}) \times H^2(M; \mathbb{R}) \to \mathbb{R}$, $(x, y) := \langle x \cup y, [M] \rangle$, be the intersection product.

Further on, the positive/unit/isotropic vectors in $H^2(M; \mathbb{R})$, as well the orthogonal complement $x^\perp$ of $x \in H^2(M; \mathbb{R})$, are all taken with respect to the bilinear symmetric form $(\cdot, \cdot)$.

Definition 1.5:

We say that a positive $y \in H^2(M; \mathbb{R})$ is orthoisotropically irrational if

(A) There exists an isotropic $u \in H^2(M; \mathbb{Z})$ such that $y \in u^\perp$.

(B) For any $u \in H^2(M; \mathbb{Z})$ such that $y \in u^\perp$ one has $\text{Span}_\mathbb{R} \{u, y\} \cap H^2(M; \mathbb{Z}) = \text{Span}_\mathbb{Z} \{u\}$. (In particular, $y$ is a not a multiple of an integral cohomology class).

Remark 1.6:

The set $\mathcal{G}$ of orthoisotropically irrational classes belongs to the union $\bigcup_{u \in H^2(M; \mathbb{Z}), (u, u) = 0} u^\perp \cup \bigcup_{v \in H^2(M; \mathbb{Z}) \cap u^\perp, \text{dim Span} \{u, v\} = 2} \text{Span} \{u, v\}$.
Let \( P \subset H^2(M; \mathbb{R}) \) be the set of unit vectors:
\[
P := \{ y \in H^2(M; \mathbb{R}) \mid (y, y) = 1 \}.
\]
For \( u \in H^2(M; \mathbb{Z}) \) define a subspace \( S_{u\perp}(M) \subset S(M) \) by
\[
S_{u\perp}(M) := \{ \omega \in S(M) \mid [\omega] \in u^\perp \cap P \}.
\]
The group \( \text{Diff}^+(M) \) acts naturally on \( S(M) \).

**Theorem 1.7:**
Assume that \( \omega_0 \in S(M) \) so that the cohomology class \([\omega_0] \in H^2(M; \mathbb{R})\) is orthoisotropically irrational.

Then for any isotropic \( u \in H^2(M; \mathbb{Z}) \) the intersection of the \( \text{Diff}^+(M) \)-orbit of \( \omega_0 \) with \( S_{u\perp}(M) \) is dense in \( S_{u\perp}(M) \). In particular, it is dense in \( S(M) \) (for \( u = 0 \)).

For the proof of **Theorem 1.7** see Section 6.1. It follows ideas similar to [EV] and going back to [Ver] and is based on Shah’s arithmetic version [Sha] of the famous Ratner’s orbit closure theorem [Rat].

Let us compare **Theorem 1.7** to the results in [EV].

In the case \( M = \mathbb{T}^4 \) it was proved in [EV] that for any Kähler-type symplectic form \( \omega_0 \in S(M) \), such that \([\omega_0] \in H^2(M; \mathbb{R})\) is not a real multiple of a rational cohomology class, the \( \text{Diff}^+(M) \)-orbit of \( \omega_0 \) is dense in \( S(M) \).

In the K3 case it was proved in [EV] that for any Kähler-type symplectic form \( \omega_0 \in S(M) \), such that \([\omega_0] \in H^2(M; \mathbb{R})\) is not a real multiple of a rational cohomology class, the \( \text{Diff}^+(M) \)-orbit of \( \omega_0 \) is dense in the connected component of \( \omega_0 \) in the space of Kähler-type symplectic forms lying in \( S(M) \).

Clearly, if \([\omega_0] \) is orthoisotropically irrational, then it is not a real multiple of a rational cohomology class. Thus, **Theorem 1.7** strengthens the results in [EV]: it shows that for any \( \omega_0 \in S(M) \) such that \([\omega_0] \) is orthoisotropically irrational the \( \text{Diff}^+(M) \)-orbit of \( \omega_0 \) is not only dense \( S(M) \) (which follows from [EV]) but also that its intersection with \( S_{u\perp}(M) \) is dense in \( S_{u\perp}(M) \) for any isotropic \( u \in H^2(M; \mathbb{Z}) \).

Let us now state a proposition which is used in the proof of the results above and may be of independent interest.

**Proposition 1.8:**

\[1\]In [EV] we used the term “hyperkähler type” for a symplectic form which can be obtained as a Kähler form of some hyperkähler structure. Using Moser’s lemma and Calabi-Yau theorem, it is easy to see that on a K3 surface any symplectic form of Kähler-type is, in fact, of hyperkähler type (Section 5).
Let $M^{2n}$ be either $T^{2n}$ or a smooth manifold underlying a complex K3 surface.

Then any two Kähler-type symplectic forms on $M$ (compatible with the orientation of $M$) can be mapped into each other by a diffeomorphism of $M$ acting trivially on homology.

For the proof see Section 6.1.

2 Linear symplectic forms on tori – proof of Proposition 1.4

Let us prove Proposition 1.4.

Since $l$ and $l'$ are complementary, one can choose a basis $\mathcal{B}$ of $\mathbb{R}^{2n}$ formed by vectors in $\mathbb{Z}^{2n}$ so that the first $n$ basic vectors lie in $l$ and the last $n$ ones lie in $l'$.

Let $M_n(\mathbb{R})$ (respectively $M_n(\mathbb{Z})$) denote the spaces of $n \times n$-matrices with real (respectively integral) coefficients.

With respect to the basis $\mathcal{B}$:

- The matrices of the elements of $G$ are exactly the matrices of the form

$$
\begin{pmatrix}
I_n & B \\
0 & A
\end{pmatrix},
$$

where $A \in \text{SL}(n, \mathbb{R})$, $B \in M_n(\mathbb{R})$.

- The matrices of the elements of $G_\Lambda$ are exactly the matrices of the form

(2.1)

with integral coefficients.

- The matrices of the forms in $\mathcal{L}_l$ are exactly the matrices of the form

$$
\begin{pmatrix}
0 & -C^t \\
C & D
\end{pmatrix},
$$

where $C \in \text{SL}(n, \mathbb{R})$ and $D$ is skew-symmetric.

- The matrices of the $(l, l')$-Lagrangian split forms are exactly the matrices of the form

$$
\begin{pmatrix}
0 & -C^t \\
C & 0
\end{pmatrix},
$$

where $C \in \text{SL}(n, \mathbb{R})$.

We will identify the elements of $G$ and the linear symplectic forms with the corresponding matrices.

Define $\mathcal{X} \subset \text{SL}(n, \mathbb{R})$ by

$$
\mathcal{X} := \{ \, C \in \text{SL}(n, \mathbb{R}) \mid \text{the entries of } C^{-1} \text{ are linearly independent over } \mathbb{Q} \, \}.
$$
The set $\mathcal{X}$ satisfies the following properties:

(I) $\mathcal{X}$ is dense in $SL(n, \mathbb{R})$.

(II) For each $C \in \mathcal{X}$ the projection of the set $\mathbb{R}C^{-1}$ to the torus $T := M_n(\mathbb{R})/M_n(\mathbb{Z})$ is dense in $T$.

Property (II) implies the following claim:

(III) For each $C \in \mathcal{X}$ the set $\{CB - (CB)^t, B \in M_n(\mathbb{Z})\}$ is dense in the space of skew-symmetric $n \times n$-matrices.

Indeed, by (II), for each skew-symmetric $n \times n$-matrix $D$ the matrix $C^{-1}D$ can be approximated by matrices of the form $tC^{-1} + B$, $t \in \mathbb{R}$, $B \in M_n(\mathbb{Z})$. Accordingly, $D$ can be approximated by matrices $tId + CB$, $t \in \mathbb{R}$, $B \in M_n(\mathbb{Z})$. Since any matrix can be uniquely represented as a sum of a symmetric and a skew-symmetric matrices, this means that $D$ can be approximated by the skew-symmetric components of the matrices $CB$, $B \in M_n(\mathbb{Z})$ – that is, by the matrices $CB - (CB)^t$, $B \in M_n(\mathbb{Z})$, which proves (III).

Define $X \subset \mathcal{L}_l$ as the set of the $(l,l')$-Lagrangian split forms represented by the matrices

$$
\begin{pmatrix}
0 & -C^t \\
C & 0
\end{pmatrix}, C \in \mathcal{X}.
$$

In order to prove the proposition it suffices to show that the union of the $G_\Lambda$-orbits of the forms in $X$ is dense in $\mathcal{L}_l$ – let us prove this claim.

The action of the element

$$
\begin{pmatrix}
I_n & B \\
0 & I_n
\end{pmatrix} \in G
$$
on a form

$$
\begin{pmatrix}
0 & -C^t \\
C & 0
\end{pmatrix} \in X
$$
yields the form

$$
\begin{pmatrix}
I_n & B \\
0 & I_n
\end{pmatrix}^t
\begin{pmatrix}
0 & -C^t \\
C & 0
\end{pmatrix}
\begin{pmatrix}
I_n & B \\
0 & I_n
\end{pmatrix} =
$$

$$
= \begin{pmatrix}I_n & 0 \\ B^t & I_n\end{pmatrix}
\begin{pmatrix}0 & -C^t \\ C & 0\end{pmatrix}
\begin{pmatrix}I_n & B \\ 0 & I_n\end{pmatrix}
= \begin{pmatrix}0 & -C^t \\ C & CB - B^tC^t\end{pmatrix}.
$$

By (I) and (III) above, the set of the matrices

$$
\begin{pmatrix}0 & -C^t \\ C & CB - B^tC^t\end{pmatrix}, C \in \mathcal{X}, B \in M_n(\mathbb{Z}),
$$
is dense in the set of the matrices

$$
\begin{pmatrix}0 & -C^t \\ C & D\end{pmatrix}, C \in SL(n, \mathbb{R}), D \text{ is skew-symmetric}.
$$
In other words, any form in $L_l$ can be approximated by the image of a form

$$\begin{pmatrix} 0 & -C' \\ C & 0 \end{pmatrix} \in X$$

under the action of

$$\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \in G_A$$

for appropriate $C \in X$ and $B \in M_n(\mathbb{Z})$. This proves the claim and the proposition. ■

3 Lattices

Let us start with recalling a few generalities on lattices and Ratner’s orbit closure theorem.

3.1 Quadratic lattices

A quadratic vector space is a finite-dimensional real vector space equipped with an $\mathbb{R}$-valued symmetric product $(\cdot, \cdot)$, bilinear over $\mathbb{R}$.

A quadratic (or integral) lattice is a free finite-rank abelian group equipped with a $\mathbb{Z}$-valued symmetric product $(\cdot, \cdot)$, bilinear over $\mathbb{Z}$.

Given a quadratic vector space $V$, a quadratic lattice in $V$ is a discrete subgroup $\Lambda \subset V$ which together with the restriction of the product from $V$ forms a quadratic lattice (that is, the restriction of the product to $\Lambda$ is $\mathbb{Z}$-valued).

Given a quadratic lattice $\Lambda$, the $\mathbb{Z}$-valued product on $\Lambda$ extends to an $\mathbb{R}$-valued bilinear symmetric product on the real vector space $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, that will be also denoted by $(\cdot, \cdot)$, and $\Lambda$ becomes a quadratic lattice in $V$.

Further on in Section 3.1, $V$ will stand for $\Lambda \otimes_{\mathbb{R}} \mathbb{R}$ and we will view the elements of $\Lambda$ as vectors in $V$.

A quadratic lattice $\Lambda$ is called unimodular if the matrix of $(\cdot, \cdot)$ with respect to a basis of $\Lambda$ (over $\mathbb{Z}$) has determinant $\pm 1$. It is called even if the number $(x, x)$ is even for all $x \in \Lambda$.

The signature of $\Lambda$ is defined as the signature of the symmetric bilinear form $(\cdot, \cdot)$ on $\Lambda \otimes_{\mathbb{R}} \mathbb{R}$.

Example 3.1:
The quadratic (or integral) hyperbolic plane $U$ is a quadratic lattice of signature $(1,1)$: $U := \text{Span}_\mathbb{Z}\{x,y\}$, $(x,x) = (y,y) = 0$, $(x,y) = (y,x) = 1$. This quadratic lattice is even and unimodular.

The notions of a quadratic sublattice and the direct sum of quadratic lattices are defined in an obvious way.

A vector $x \in \Lambda$ is called primitive if it is not an integral multiple of another element in $\Lambda$.

A vector $x \in V$ is called isotropic if $(x,x) = 0$, positive if $(x,x) > 0$, and unit if $(x,x) = 1$.

The notation $\perp$ will be used for orthogonal complements in $V$ with respect to $(\cdot,\cdot)$.

Denote by $x^\perp$ the set of unit vectors in $x^\perp$:

$$x^\perp := \{ y \in x^\perp \mid (y,y) = 1 \}.$$ 

**Proposition 3.2:**

Assume $\Lambda$ is a unimodular quadratic lattice and $u \in \Lambda$ is a non-zero vector.

Then the following claims hold:

(A) If $u$ is primitive, then $(u, \Lambda) = \mathbb{Z}$.

(B) Assume $\Lambda$ is even and of signature $(p,q)$, and $u$ is isotropic. Then

(B1) $u^\perp \cap \Lambda = \text{Span}_\mathbb{Z}\{u\} \oplus \Lambda'$, where $\Lambda'$ is a unimodular integral lattice in $u^\perp$ of signature $(p-1,q-1)$;

(B2) For any $x \in u^\perp \cap \Lambda$, such that $x \notin \text{Span}_\mathbb{Z}\{u\}$, the rank of the group $(\text{Span}_\mathbb{R}\{u,x\})^\perp \cap \Lambda$ is $\text{rk} \Lambda - 2$.

**Proof of Proposition 3.2.**

Part (A) is an elementary algebraic exercise based on the fact that the ideal in $\mathbb{Z}$ generated by a finite collection of integers is a principal ideal generated by their g.c.d.

Let us prove part (B). Without loss of generality, we may assume that $u$ is primitive.

By part (A), there exists a vector $s \in \Lambda$ such that $(u,s) = 1$. Then, since $\Lambda$ is even, $(s,s)$ is even and $z := s - (s,s)u/2$ is an isotropic element of $\Lambda$ that spans together with $u$ an integral hyperbolic plane $U$. Since $U$ is a unimodular sublattice of $\Lambda$, we can write $\Lambda = U \oplus (U^\perp \cap \Lambda)$ (see e.g. [MH] Ch.1, Lem.3.1)). The signature of $U^\perp$ is $(p-1,q-1)$. Setting $\Lambda' := U^\perp \cap \Lambda$ finishes the proof of (B1).

Claim (B2) follows immediately from the following general claim: if $\Lambda_1$ is a sublattice of $\Lambda$, then $\Lambda_1^\perp := (\Lambda_1 \otimes_\mathbb{Z} \mathbb{Q})^\perp \cap \Lambda$ has rank $\text{rk} \Lambda - \text{rk} \Lambda_1$. (The
claim follows from the observation that any vector in $(\Lambda_1 \otimes_\mathbb{Z} \mathbb{Q})^\perp$ has an integral multiple that lies in $\Lambda_1^\perp$. ■

Denote by $SO(V)$ the group of linear automorphisms of $V$, with determinant 1, preserving the bilinear form. Let $SO^+(V)$ be the identity component of $SO(V)$, and let $SO^+(\Lambda)$ be the subgroup of $SO^+(V)$ formed by the elements of $SO^+(V)$ preserving $\Lambda$.

Definition 3.3:
We say that a quadratic lattice $\Lambda$ is isotropically transitive if $SO^+(\Lambda)$ acts transitively on the set of primitive isotropic vectors in $\Lambda$.

Example 3.4:
Assume that $\Lambda = U \oplus U \oplus \Lambda'$, where $\Lambda'$ is an even unimodular quadratic lattice of signature $(p,q)$, $p,q \in \mathbb{N}$.

Then $\Lambda$ is isotropically transitive by [GHS, Prop.3.3(i)] (part (A) of Proposition 3.2 allows to apply the latter result).

Remark 3.5:
We will apply this result to the set of isotropic cohomology classes on a K3 surface (Section 4.3). We will get that, since the cohomology classes of Lagrangian tori are isotropic, the mapping class group acts transitively on the lines spanned by these classes in $H^2(M;\mathbb{Q})$.

Definition 3.6:
Let $y \in V$ be a positive vector.

Given a non-zero isotropic $u \in \Lambda$, we say that $y$ is $u$-orthoirrational if $y \in u^\perp$ and $y$ does not lie in a two-dimensional plane spanned by $u$ and a vector in $\Lambda \cap u^\perp$.

We say that $y$ is orthoisotropically irrational if if it lies in $u^\perp$ for some non-zero isotropic $u \in \Lambda$ and is $u$-orthoirrational with respect to any non-zero isotropic $u \in y^\perp \cap \Lambda$. (This matches Definition 1.5).

3.2 Lattices in Lie groups and Shah’s version of Ratner’s orbit closure theorem
Let $G$ be a Lie group.

The group $G$ admits a left-invariant (respectively, right-invariant) measure, called the left (respectively, right) Haar measure, which is defined uniquely, up to a constant factor – this measure is defined by a left-invariant (respectively, right-invariant) differential volume form on $G$. 

− 11 −
Assume $\Gamma \subset G$ is a discrete subgroup. The restriction of the right Haar measure on $G$ to the fundamental domain of the right action of $\Gamma$ on $G$ induces a measure $\mu_{G/\Gamma}$ on the space $G/\Gamma$ (this measure can be defined by a differential volume form on $G/\Gamma$).

The subgroup $\Gamma$ is called a **Lie lattice** (in $G$) if $\mu_{G/\Gamma}(G/\Gamma) < +\infty$.

Recall that $g \in G$ is called **unipotent**, if $g = e^h$ for an $ad$-nilpotent element $h$ of the Lie algebra of $G$. A **unipotent one-parameter subgroup of $G$** is a subgroup of the form $\{e^{th}\}_{t \in \mathbb{R}}$ for an $ad$-nilpotent element $h$ of the Lie algebra of $G$. We say that $G$ is **generated by unipotents** if it is multiplicatively generated by unipotent elements.

A **linear** (real) Lie group is a Lie subgroup of $SL(V)$ for a finite-dimensional real vector space $V$.

Let $V$ be a finite-dimensional real vector space, $m = \dim_{\mathbb{R}} V$, and let $\Lambda \subset V$ be a free discrete subgroup of $V$ of rank $m$. A basis of $V$ is called **integral**, if it is a basis of $\Lambda$ over $\mathbb{Z}$.

Let $G \subset SL(V)$ be a linear Lie group. Let

$$G_\Lambda := \{ g \in G \mid g(\Lambda) = \Lambda \}.$$

A Lie lattice $\Gamma$ in $G$ is called **arithmetic** if $\Gamma \cap G_\Lambda$ has a finite index in $\Gamma$ and $G_\Lambda$.

A **$\mathbb{Q}$-character** on $G$ is a homomorphism $G \to \mathbb{R}_{>0}$ which is defined by algebraic equations with rational coefficients on the entries of the real $m \times m$-matrices representing the elements of $G$ with respect to an integral basis of $V$.

We say that $G$ is an **algebraic group** (respectively, an **algebraic $\mathbb{Q}$-group**) if for an integral basis of $V$ the $m \times m$-matrices of the elements of $G$ with respect to the basis form a Lie subgroup of $SL(m, \mathbb{R})$ defined by algebraic equations with real (respectively, rational) coefficients on the entries of the matrices.

**Proposition 3.7:**

Assume $p, q \in \mathbb{Z}_{>0}$ and $p + q > 2$. Then the group $SO^+(p, q)$ (the identity component of $SO(p, q)$) is generated by its algebraic one-parameter unipotent subgroups.

**Proof:**

The group $SL(2, \mathbb{R})$ contains a non-trivial algebraic one-parameter unipotent subgroup

$$U := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}.$$
There is an isomorphism between $SL(2, \mathbb{R})/\{\pm 1\}$ and $SO^+(1, 2)$ defined
by the following homomorphism $\mu : SL(2, \mathbb{R}) \to SO^+(1, 2)$ with the kernel
$\{\pm 1\}$: identify $\mathbb{R}^3$ with the space $V$ of real trace-free $2 \times 2$-matrices equipped
with the bilinear symmetric form $\langle A, B \rangle \to tr(AB)$, $A, B \in V$, of signature
$(1, 2)$; for each $C \in SL(2, \mathbb{R})$ define $\mu(C)$ as the automorphism of $V$ given
by $A \mapsto CAC^{-1}$, $A \in V$. The homomorphism $\mu$ is a polynomial map from
$SL(2, \mathbb{R})$ to $SO(1, 2)$. Hence, $\mu(U)$ is a non-trivial algebraic one-parameter
unipotent subgroup of $SO^+(1, 2)$. This implies that for all $p, q \in \mathbb{Z}_{>0}$ and $p + q > 2$ the group $SO^+(p, q)$ also contains non-trivial algebraic one-parameter
unipotent subgroups.

Any conjugate of an algebraic one-parameter unipotent subgroup of a
real Lie group is again an algebraic one-parameter unipotent subgroup of
the same Lie group. Therefore the subgroups of $SO^+(p, q)$ and of $SL(2, \mathbb{R})$
gen by their algebraic one-parameter unipotent subgroups are normal
and non-discrete. Unless $p = q = 2$, the group $SO^+(p, q)$ is a simple Lie
group and so is $SL(2, \mathbb{R})$. Therefore these groups do not contain non-discrete
proper normal subgroups \cite{Rago}. Consequently, they are generated by their
algebraic one-parameter unipotent subgroups.

Let us consider the remaining case $p = q = 2$. We have $SO^+(2, 2) =
(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\{\pm 1\}$. If $U$ is an arbitrary algebraic one-parameter
unipotent subgroup of $SL(2, \mathbb{R})$, then $Id \times U$ and $U \times Id$ are algebraic one-parame-
ter unipotent subgroups of $SO^+(2, 2) = (SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\{\pm 1\}$. Consequently, since $SL(2, \mathbb{R})$ is generated by its algebraic one-parameter
unipotent subgroups, so is $SO^+(2, 2)$.

This shows that $SO^+(p, q)$ is generated by algebraic one-parameter uni-
potent subgroups for all $p, q \in \mathbb{Z}_{>0}$ and $p + q > 2$. $\blacksquare$

The following theorem belongs to A.Borel and Harish-Chandra.

**Claim 3.8:** (Borel-Harish-Chandra theorem, \cite{BH} Thm. 9.4)
Let $G \subset SL(V)$ be an algebraic $\mathbb{Q}$-group. Then $G_A$ is a Lie lattice in $G$
(or, equivalently, $G$ admits an arithmetic lattice) if and only if $G$ does not
admit non-trivial $\mathbb{Q}$-characters. $\blacksquare$

**Proposition 3.9:**
Let $G' \subset SL(V)$ be an algebraic $\mathbb{Q}$-group and $G$ its identity compo-
nent (in the Lie group topology). Assume $G$ does not admit non-trivial
$\mathbb{Q}$-characters.

Then $G_A$ is a Lie lattice in $G$.

**Proof:**
Since $G$ does not admit non-trivial $\mathbb{Q}$-characters, $G'$ does not admit them either. Therefore, by Borel and Harish-Chandra theorem (Claim 3.8), $G'_\Lambda$ is a Lie lattice in $G'$.

The identity component $G$ is a normal subgroup of $G'$ and its index is finite, since, by Whitney’s theorem [Wh], the real algebraic subvariety of a Euclidean space has finitely many connected components. This implies that $G'_\Lambda$ has a finite index in $G'_\Lambda$. Since the indices $|G' : G|$ and $|G'_{\Lambda} : G_{\Lambda}|$ are finite and $G'_\Lambda$ is a Lie lattice in $G'$, one easily gets that $G_{\Lambda}$ is a Lie lattice in $G$.

The following fundamental theorem belongs to M.Ratner.

Claim 3.10: (Ratner’s orbit closure theorem, [Rat])

Let $G$ be a connected Lie group, $H \subset G$ its Lie subgroup generated by unipotents and $\Gamma \subset G$ a Lie lattice. Then for any $g \in G$ the closure $\Gamma gH$ of the double class is obtained as

$$\Gamma gH = \Gamma gS$$

for some closed Lie subgroup $S$, $H \subset S \subset G$. In particular, if $H$ is a closed Lie subgroup, then the closure of the orbit $\Gamma \cdot gH$ of $gH$ in $G/H$ is $\Gamma (gSg^{-1}) \cdot gH$. ■

A combination of Ratner’s orbit closure theorem (Claim 3.10) with a result of Shah [Sha, Proposition 3.2]) yields a more precise description of the group $S$ in the case where $G$ is the identity component of a linear algebraic $\mathbb{Q}$-group and $\Gamma \subset G$ is an arithmetic Lie lattice. We will state the result for $g = e$, since this is exactly what we are going to use in our proof.

Claim 3.11: ([Sha, Proposition 3.2], cf. [KSS, Proposition 3.3.7])

Let $G$ be the identity component of a linear algebraic $\mathbb{Q}$-group $G'$, and $\Gamma \subset G$ an arithmetic Lie lattice. Let $H \subset G$ be a closed Lie subgroup generated by algebraic unipotent one-parameter subgroups of $G$ contained in $H$. Let $x := eH \in G/H$, where $e \in G$ is the identity of $G$.

Then the closure of the orbit $\Gamma \cdot x$ in $G/H$ is $\Gamma S \cdot x$, where $S \subset G$ is the identity component of the smallest algebraic $\mathbb{Q}$-subgroup of $G'$ containing $H$. ■

Remark 3.12:

This result is stated in [KSS, Proposition 3.3.7] under the assumption that $G$ has no $\mathbb{Q}$-characters. In fact, in view of Borel and Harish-Chandra theorem (Claim 3.8), this assumption is redundant.
4 The $\mathbb{T}^4$ and $K3$ case – application of Shah’s version of Ratner’s orbit closure theorem

The goal of this section is to prove Theorem 1.7.

In this section we will assume that $\Lambda$ is an even unimodular lattice of signature $(p, q)$, $p, q \geq 3$. Let $V := \Lambda \otimes \mathbb{Z} \mathbb{R}$. The symmetric bilinear form on $\Lambda$ and $V$ will be denoted as before by $(\cdot, \cdot)$.

The group $SO(V) \subset SL(V)$ is a linear algebraic $\mathbb{Q}$-group and $SO^+(V)$ is its identity component. (Note, however, that $SO^+(V)$ itself is not an algebraic subgroup – its Zariski closure is $SO(V)$ [Sat, p.3]).

Let $u \in \Lambda$ be a primitive isotropic non-zero vector.

Denote the stabilizers of $u$ in $SO(V)$ and $SO^+(V)$ respectively by $G'$ and $G$:

$$G' := \{ g \in SO(V) \mid gu = u \}.$$  

$$G := \{ g \in SO^+(V) \mid gu = u \}.$$  

Each of these sets is the intersection of two Lie subgroups of the group $SL(V)$: the stabilizer of $u$ in $SL(V)$ and $SO(V)$, or, respectively, $SO^+(V)$. Since the intersection of two Lie subgroups is a Lie subgroup, $G'$ is a Lie group and $G$ is its closed Lie subgroup.

Since $u \in \Lambda$, the group $G'$ is a linear algebraic $\mathbb{Q}$-group.

The group $G'$ acts on $u^\perp$ and preserves $u^\perp, 1$.

In this case

$$G_{\Lambda} = G \cap SO^+(\Lambda).$$

Let $y \in u^\perp, 1$ be a $u$-orthoirrational unit vector.

In order to prove Theorem 1.7 which will be proved using Claim 3.11 we need the following key proposition

**Proposition 4.1:**

The $G_{\Lambda}$-orbit of $y$ is dense in $u^\perp, 1$, for any $u$-orthoirrational vector $y \in u^\perp$.

4.1 Density of $G_{\Lambda}$-orbits in $u^\perp, 1$

**Preparations for the proof of Proposition 4.1**

For the proof of Proposition 4.1 we first need a number of preparations.
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Set

\[ k := p + q - 1. \]

We have \( \dim \mathbb{R} u^\perp = k \).

The restriction map \( \phi \mapsto \phi|_{u^\perp} \) provides a canonical isomorphism between \( G \) and the identity component of the group of isometries of \( u^\perp \) fixing \( u \). Indeed, any \((k - 1)\)-dimensional vector subspace \( Z \) of \( u^\perp \) transversal to \( \text{Span}_\mathbb{R}\{u\} \) determines a unique vector \( v_Z \) such that \( v_Z \perp Z \), \((v_Z, v_Z) = 1\), \((v_Z, u) = 1\). It is easy to see that \( v_Z \notin u^\perp \). An isometry of \( u^\perp \) fixing \( u \) sends \( Z \) to another \((k - 1)\)-dimensional vector subspace \( Z' \) of \( u^\perp \) transversal to \( \text{Span}_\mathbb{R}\{u\} \) and extends uniquely to an element of \( G \) sending \( v_Z \) to \( v_{Z'} \).

This provides an inverse to the restriction map above and shows that it is an isomorphism. Thus, we can describe elements of \( G \) in terms of their restriction to \( u^\perp \).

Denote by \( H \) the stabilizer of \( y \) in \( G \):

\[ H := \{ g \in G \mid gy = y \}. \]

This is the intersection of two Lie subgroups of the group \( SL(V) \) of isomorphisms of \( V \): the stabilizer of \( y \) in \( SL(V) \) and \( G \). Since the intersection of two Lie subgroups is a Lie subgroup, \( H \) is a Lie group.

Define

\[ W := u^\perp/\text{Span}_\mathbb{R}\{u\}. \]

The bilinear form \((\cdot, \cdot)\) on \( V \) induces a non-degenerate bilinear form on \( W \) of signature \((p - 1, q - 1)\). Denote by \( SO(W) \) the group of isomorphisms of \( W \) preserving the latter bilinear form and by \( SO^+(W) \) its connected component of the identity.

Let \( y^\perp_{u^\perp} \) be the orthogonal complement of \( y \) in \( u^\perp \). Define

\[ W' := y^\perp_{u^\perp}/\text{Span}_\mathbb{R}\{u\}. \]

The bilinear form \((\cdot, \cdot)\) on \( u^\perp \) induces a non-degenerate bilinear form on \( W' \) of signature \((p - 2, q - 1)\). Denote by \( SO(W') \) the group of isomorphisms of \( W' \) preserving the latter bilinear form and by \( SO^+(W') \) its connected component of the identity.

We will consider bases \( \mathcal{B} = \{w_1, \ldots, w_{k-1}, u\} \) of \( u^\perp \) with \( u \) being the last vector of the basis. Such a basis will be called adapted.

Since the restriction of \((\cdot, \cdot)\) to \( \text{Span}_\mathbb{R}\{w_1, \ldots, w_{k-1}\} \) is a non-degenerate bilinear form of signature \((p - 1, q - 1)\), the vectors \( w_1, \ldots, w_{k-1} \) in an adapted basis can be assumed to form an orthonormal basis of their span. (A basis is said to be orthonormal with respect to an indefinite symmetric bilinear form if it is orthogonal and the square of each basic vector is \( \pm 1 \)). Such an adapted basis will be called an adapted orthonormal basis.
On the other hand, part (B1) of Proposition 3.2 implies that $\Lambda \cap u^\perp$ is a lattice in $u^\perp$ and therefore one can choose an adapted basis $B' = \{w_1, \ldots, w_{k-1}, u\}$ so that the vectors $w_1, \ldots, w_{k-1}$ all lie in $\Lambda \cap u^\perp$. Such a basis will be called an adapted integral basis.

Further on let $B$ denote an adapted orthonormal basis $B$ of $u^\perp$ of the form $B = \{w_1, \ldots, w_{k-2}, y, u\}$, with $w_{k-1} = y$.

There is a natural homomorphism $\Phi : G \to SO^+(W)$. It is surjective and admits a right inverse. Indeed, any element $\psi \in SO^+(W)$ induces a unique isometry of the $\text{Span}_\mathbb{R}\{w_1, \ldots, w_{k-2}, y\}$ that extends uniquely to an isometry of $u^\perp$ fixing $u$. Denote this isometry by $\Psi_B(\psi)$. One easily checks that such a $\Psi_B : SO^+(W) \to G$ is a right inverse of $\Phi$.

The kernel of $\Phi$ is a Lie subgroup of $G$ that will be denoted by $U$:

$$U := \ker \Phi.$$

Clearly,

Any $g \in G$ can be written in a unique way as

$$g = \phi \psi, \ \phi \in \Psi_B \left( SO^+(W) \right), \ \psi \in U. \quad (4.1)$$

There is also a canonical homomorphism $\Phi' : H \to SO^+(W')$. It is surjective as well, and admits a right inverse. Indeed, any element $\psi' \in SO^+(W')$ induces a unique isometry of the $\text{Span}_\mathbb{R}\{w_1, \ldots, w_{k-2}\}$ that extends uniquely to an isometry of $u^\perp$ fixing $u$. Denote this isometry by $\Psi'_B(\psi')$. One easily checks that $\Psi'_B : SO^+(W') \to H$ is a right inverse of $\Phi'$.

The kernel of the homomorphism $\Phi' : H \to SO^+(W')$ is a Lie subgroup of $H$ that will be denoted by $U'$:

$$U' := \ker \Phi'.$$

Clearly,

Any $h \in H$ can be written in a unique way as

$$h = \phi' \psi', \ \phi' \in \Psi'_B \left( SO^+(W') \right), \ \psi' \in U'. \quad (4.2)$$

The elements of $G$ are represented, with respect to $B$, by $k \times k$-matrices of the form

$$\begin{pmatrix}
A & 0 \\
\vdots & \\
0 & 1
\end{pmatrix} \quad (4.3)$$
(Here and further on the asterisks are arbitrary real numbers). The upper-left \((k - 1) \times (k - 1)\)-block \(A\) is a matrix in \(SO^+(p - 1, q - 1)\).

The elements of \(U\) are represented, with respect to \(B\), by matrices of the form:

\[
\begin{pmatrix}
I_{k-1} & 0 \\
\vdots & \vdots \\
0 & 0 \\
* \cdots * & 1
\end{pmatrix}
\] (4.4)

The elements of the Lie algebra of \(U\) can be then identified with the space of matrices of the form

\[
\begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
* \cdots * & 0
\end{pmatrix}
\]

The elements of \(H\) are represented, with respect to \(B\), by matrices of the form

\[
\begin{pmatrix}
A' & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
0 \ldots 0 & 1 & 0 \\
* \cdots * & 0 & 1
\end{pmatrix}
\] (4.5)

The upper-left \((k - 2) \times (k - 2)\)-block \(A'\) is a matrix in \(SO^+(p - 2, q - 1)\).

The elements of \(U'\) are represented, with respect to \(B\), by matrices of the form:

\[
\begin{pmatrix}
I_{k-2} & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
0 \ldots 0 & 1 & 0 \\
* \cdots * & 0 & 1
\end{pmatrix}
\]

The elements of the Lie algebra of \(U'\) can be then identified with the space of matrices of the form

\[
\begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 \ldots 0 & 0 & 0 \\
* \cdots * & 0 & 0
\end{pmatrix}
\] (4.6)
Remark 4.2:
The group $G$ is a (maximal) parabolic subgroup of $SO^+(V)$ and the decomposition $G = U \rtimes SO^+(W)$ is the Levi decomposition of $G$.

**Lemmas needed for the proof of Proposition 4.1:**

Now we state and prove a number of lemmas that will allow us to verify that [Claim 3.11](#) is applicable in our setting and then to apply it.

**Lemma 4.3:**
The group $G$ is the identity component of $G'$.

**Proof of Lemma 4.3.**
It follows from (4.3) that $G$ is connected. Since $G$ is the intersection of $G' \subset SO(V)$ with the identity component $SO^+(V)$ of $SO(V)$, this implies that $G$ is the identity component of $G'$.

**Lemma 4.4:**
The group $H$ is generated by its algebraic one-parameter unipotent subgroups.

**Proof of Lemma 4.4.**
The matrices of the form (4.6) are nilpotent and therefore the corresponding elements of the Lie algebra of $U'$ are $ad$-nilpotent. Moreover, it is easy to see that each element of $U'$ lies in an algebraic one-parameter unipotent subgroup of $U'$.

By [Proposition 3.7](#), since $p, q \geq 3$, the group $SO^+(p-2, q-1)$ is generated by its algebraic one-parameter unipotent subgroups. Since the isomorphism $SO^+(W') \cong SO^+(p-2, q-1)$ is a polynomial map, it sends algebraic one-parameter unipotent subgroups into algebraic one-parameter unipotent subgroups. Hence, $SO^+(W')$ is generated by its algebraic one-parameter unipotent subgroups. Since $U'$ and $SO^+(W')$ are generated by their algebraic one-parameter unipotent subgroups, so is $H$, because of (4.2).

This proves the lemma.

**Lemma 4.5:**
There is a unique connected Lie subgroup $K \subset G$ satisfying

$$H \subsetneq K \subsetneq G.$$
It is formed by all the elements of $G$ preserving both $y_u$ and send $y$ to vectors of the form $y + cu$, $c \in \mathbb{R}$.

**Proof of Lemma 4.5.**

Let us first prove that if a connected Lie group $K$ satisfies $H \subsetneq K \subsetneq G$, then it is formed by all the elements of $G$ that preserve both $y_u$ and $\text{Span}_\mathbb{R}\{y, u\}$.

Indeed, assume that $K$ is a connected Lie group such that $H \subsetneq K \subsetneq G$. As before, we work with the adapted orthonormal basis $B$ and use the identifications $W = \text{Span}_\mathbb{R}\{w_1, \ldots, w_{k-2}, y\}$ and $W' = \text{Span}_\mathbb{R}\{w_1, \ldots, w_{k-2}\}$ - these identifications preserve the bilinear forms and induce isomorphisms

$$
\varsigma_B : SO^+(W) \to SO^+(p - 1, q - 1), \quad \varsigma'_B : SO^+(W') \to SO^+(p - 2, q - 1).
$$

In terms of the matrix representations (4.3), (4.5) and the identifications $\varsigma_B, \varsigma'_B$ above, $\Phi$ can be viewed as a surjective homomorphism $\Phi : G \to SO^+(p - 1, q - 1)$ sending each matrix (4.3) to its upper-left $(k - 1) \times (k - 1)$-block, while $\Phi'$ can be viewed as a surjective homomorphism $\Phi' : H \to SO^+(p - 2, q - 1) \subset SO^+(p - 1, q - 1)$ sending each matrix (4.5) to its upper-left $(k - 2) \times (k - 2)$-block, so that $\Phi' = \Phi|_H$.

Consequently, $SO^+(p - 2, q - 1) \subset \Phi(K) \subset SO^+(p - 1, q - 1)$. Since $K$ is connected, so is $\Phi(K)$. By [EV, Lem. 9.9], this implies that either $\Phi(K) = SO^+(p - 2, q - 1)$ or $\Phi(K) = SO^+(p - 1, q - 1)$.

If $\Phi(K) = SO^+(p - 1, q - 1)$, then, in view of (4.1) and $H \subsetneq K \subsetneq G$, the matrices representing the elements of $K$ with respect to $B$ have to be exactly the matrices of the form

$$
\begin{pmatrix}
A & 0 \\
0 & 1
\end{pmatrix}
$$

where $A \in SO^+(p - 1, q - 1)$. However, this collection of matrices is not a group. This means that $\Phi(K) \neq SO^+(p - 1, q - 1)$.

Thus, $\Phi(K) = SO^+(p - 2, q - 1)$. In this case, in view of (4.1) and $H \subsetneq K \subsetneq G$, we get that the matrices representing the elements of $K$ with respect to $B$ are exactly the matrices of the form

$$
\begin{pmatrix}
A' & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\cdots & \cdots & \cdots \\
0 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix}
$$
where $A' \in SO^+(p-2, q-1)$. Such a collection of matrices does form a 
connected Lie group. It is exactly the group formed by all the elements of $G$ that preserve both $y_{u_1}^\perp = \text{Span}\{w_1, \ldots, w_{k-2}, u\}$ and sending $y$ to vectors 
of the form $y + cu$, $c \in \mathbb{R}$.

This finishes the proof of the lemma. ■

**Lemma 4.6:**

The group $G$ acts transitively on the set of vectors of $u_\perp$ of the same 
positive length (recall that the length is measured with respect to the bilinear 
form $(\cdot, \cdot)$).

**Proof of [Lemma 4.6]**

Assume $y' \in u_\perp$, $(y', y') = (y, y) =: r > 0$. We want to show that there 
is an element of $G$ mapping $y$ to $y'$.

Consider a subspace $Z \subset u_\perp$, $\dim \mathbb{R} Z = \dim \mathbb{R} u_\perp - 1 = k - 1$, that is 
transversal to $u$ and contains $y, y'$. The restriction of $(\cdot, \cdot)$ to $Z$ is a non-
degenerate bilinear form of signature $(p-1, q-1)$. The group $SO(Z)$ of 
isometries of $Z$ is isomorphic to $SO(p-1, q-1)$. In particular, this implies

that the identity component $SO^+(Z)$ of $SO(Z)$ acts transitively on the set 
of vectors of length $r$ in $Z$. An isometry of $Z$ lying in $SO^+(Z)$ and sending 
y to $y'$ can be extended to an isometry of $u_\perp$ fixing $u$ – that is, to an element 
of $G$.

This proves the lemma. ■

Assume now that $B' = \{w_1, \ldots, w_{k-1}, u\}$ is an adapted integral 

basis of $u_\perp$.

**Lemma 4.7:**

The group $G$ has no non-trivial $\mathbb{Q}$-characters.

**Proof of [Lemma 4.7]**

Let $\rho : U \rightarrow \mathbb{R}^{k-1}$ be an isomorphism sending each element of $U$, repre-
sented, with respect to $B'$, by a matrix

$$
\begin{pmatrix}
I_{k-1} & 0 \\
\vdots & \\
0 & 1
\end{pmatrix}
$$

to $(a_1, \ldots, a_{k-1})$.

Assume that $\chi$ is a $\mathbb{Q}$-character on $G$. Then $\chi|_{\psi(SO^+(W))} \equiv 1$ – indeed, 
$SO^+(p, q)$ is linear, connected and semi-simple, hence coincides with its com-
mutator subgroup and therefore does not admit any non-trivial characters.
At the same time $\chi|_U$ is a $\mathbb{Q}$-character on $U$ and consequently $f := \chi|_U \circ \rho^{-1}$ is a $\mathbb{Q}$-character on $\mathbb{R}^{k-1}$ — that is, $f$ is a polynomial with rational coefficients on $\mathbb{R}^{k-1}$, with values in $\mathbb{R}_{>0}$, satisfying $f(a + b) = f(a)f(b)$ for all $a, b \in \mathbb{R}^{k-1}$. This readily implies that $f \equiv 1$ and consequently $\chi|_U \equiv 1$. Since $\chi|_U \equiv 1$ and $\chi|_{\psi(SO^+(W))} \equiv 1$, we get $\chi \equiv 1$ because of (4.1). Thus, $G$ does not admit non-trivial $\mathbb{Q}$-characters. This proves the lemma.

**Lemma 4.8:**

The group $G_A$ is an arithmetic Lie lattice in $G$.

**Proof of Lemma 4.8**

Follows immediately from Lemma 4.3, Lemma 4.7 and Proposition 3.9.

**Lemma 4.9:**

Consider the group $H$, written in matrix form in (4.5), and $K$, defined in Lemma 4.5. Then $H$ and $K$ are not identity components of $\mathbb{Q}$-subgroups of $G'$.

**Proof of Lemma 4.9**

Set

$$n := p + q = k + 1.$$ 

Without loss of generality, we may assume that $V = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$, $u = (1, 0, \ldots, 0) \in \mathbb{Z}^n$, and $(\cdot, \cdot)$ is a bilinear symmetric form of signature $(p, q)$ on $\mathbb{R}^n$ with an invertible integral matrix.

Clearly, $G'$ is a $\mathbb{Q}$-subgroup.

Let us prove the claim about $H$. Assume, by contradiction, that $H$ is the identity component of a $\mathbb{Q}$-subgroup $S$ of $G'$.

Then the Lie algebras Lie($S$) and Lie($H$) of $S$ and $H$ coincide: Lie($S$) = Lie($H$). Since $S$ is the common zero level set of a finite number of polynomials (in the entries of matrices in $G'$) with rational coefficients, its Lie algebra Lie($S$) (that is, the tangent space of $S$ at the identity) is the common zero level set of a finite number of linear functions with rational coefficients, hence a Lie subalgebra of the Lie algebra Lie($G'$) of $G'$ admitting a basis $A_1, \ldots, A_k$ formed by matrices with rational coefficients.

Since $H$ is the stabilizer of $u$ and $y$, the Lie algebra $\mathfrak{h}$ of $H$ is the space of all matrices $h \in \mathfrak{so}(V)$ such that $h(y) = h(u) = 0$. This gives $\mathfrak{h}(V) = \langle u, y \rangle^\perp$. If $\mathfrak{h}$ is rational, its image $\mathfrak{h}(V)$ is also rational, implying that $\langle u, y \rangle^\perp$ is also rational. This is impossible, because $y$ is $u$-ortho-irrational. Thus, $G$ does not admit non-trivial $\mathbb{Q}$-characters. This proves that $H$ is not the identity component of a $\mathbb{Q}$-subgroup.
Let us prove the claim in the case of $K$. Assume, by contradiction, that $K$ is the identity component of a $\mathbb{Q}$-subgroup $S$ of $G'$. Similarly to the previous case we get that the Lie algebra $\text{Lie}(S) = \text{Lie}(K)$ admits a basis $A_1, \ldots, A_k$ formed by matrices with rational coefficients. By definition, $K$ is the subgroup of $G'$ preserving $u$ and mapping $y$ to $y + cu$, for $c \in \mathbb{R}$. The Lie algebra $\mathfrak{k}$ of $K$ is an algebra of all $h \in \mathfrak{so}(V)$ such that $h(u) = 0$ and $h(y) = cu$. Therefore, $\mathfrak{k}$ surjectively maps $V$ to $y^\perp$; it cannot be rational, because $y^\perp$ is irrational.

This finishes the proof in the case of $K$, as well as the proof of the lemma.

---

**Final steps:**

Finally, we are ready to prove Proposition 4.1.

**Proof of Proposition 4.1.**

We would like to deduce the wanted result from Claim 3.11. Let us check that Claim 3.11 is applicable here.

Indeed, $G'$ is a linear algebraic $\mathbb{Q}$-group and $G$ is its identity component – see Lemma 4.3.

The group $G_A$ is an arithmetic Lie lattice in $G$ – see Lemma 4.8.

The Lie subgroup $H \subset G$ is closed. It is generated by its algebraic one-parameter unipotent subgroups – see Lemma 4.4.

Thus, Claim 3.11 is applicable and yields that the closure of the orbit $G_A \cdot y$ in $G/H$ is $G_A S \cdot y$, where $S \subset G$ is the identity component of the smallest algebraic $\mathbb{Q}$-subgroup of $G'$ containing $H$.

By Lemma 4.5, the identity component of $S$ is either $H$, or $K$, or $G$ (the latter case, of course, means that $S = G$). Since $y \in u^\perp$ is $u$-orthoirrational, Lemma 4.9 implies that the groups $H$ and $K$ are not identity components of $\mathbb{Q}$-subgroups of $G'$.

Thus, $S = G$ and the closure of the orbit $G_A \cdot y$ in $G/H$ is

$$G_A G \cdot y = Gy = G/H.$$ 

In other words, the orbit $G_A \cdot y$ is dense in $G/H$.

On the other hand, by Lemma 4.6, $G$ acts transitively on $u^{1,1}$ and therefore $G/H = u^{1,1}$. Thus, the orbit $G_A \cdot y$ is dense in $u^{1,1}$, which finishes the proof of the proposition.
4.2 The density of $SO^+(\Lambda)$-orbits

We are now ready to deduce the following corollary of Proposition 4.1.

Theorem 4.10:
Assume that $\Lambda$ is an even unimodular lattice of signature $(p,q)$, $p,q \geq 3$, which is isotropically transitive. Let $v \in V$ be an orthoisotropically irrational unit vector.

Then for any non-zero isotropic vector $u \in \Lambda$ the intersection of the $SO^+(\Lambda)$-orbit of $v$ with $u^{\perp,1}$ is dense in $u^{\perp,1}$.

Proof of Theorem 4.10:
Since $v \in V$ is orthoisotropically irrational, there exists a primitive isotropic $x \in \Lambda$ such that $v \in x^{\perp}$ and $v$ is $x$-orthoirrational.

Let $u \in \Lambda$ be a non-zero isotropic vector. Without loss of generality, we may assume that $u$ is primitive. Since $\Lambda$ is isotropically transitive, there exists $\phi \in SO^+(\Lambda)$ mapping $x$ into $u$ and, accordingly, $x^{\perp}$ into $u^{\perp}$. The vector $\phi(v)$ then lies in $u^{\perp,1}$ and is $u$-orthoirrational. Applying Proposition 4.1 to $\phi(v) \in u^{\perp,1}$ we obtain that the needed result.

4.3 Density of the $\text{Diff}^+(M)$-orbits in cohomology

Let $M$ be either $T^4$ or a smooth manifold underlying a complex K3 surface.

The goal of this section is to formulate certain density properties of $\text{Diff}^+(M)$-orbits in $H^2(M;\mathbb{R})$. We will then use these properties in Section 6.2 to prove Theorem 1.7 about density properties of the corresponding $\text{Diff}^+(M)$-orbits in $S(M)$.

Theorem 4.11:
Assume $y \in P = \{ x \in H^2(M;\mathbb{R}) \mid (x,x) = 1 \}$ is orthoisotropically irrational.

Then for any isotropic $u \in H^2(M;\mathbb{Z})$ the intersection of the $\text{Diff}^+(M)$-orbit of $y$ with $u^{\perp} \cap P$ is dense in $u^{\perp} \cap P$. In particular, it is dense in $P$ (for $u = 0$).

Proof of Theorem 4.11:
We are going to reduce the claim to Theorem 4.10.

In the case of $M = T^4$ we identify $H^2(T^4;\mathbb{R})$, with the intersection product on it, with the quadratic space $V := (\wedge^2 \mathbb{R}^4)^*$ of bilinear forms on $\mathbb{R}^4$ (with the appropriate bilinear symmetric form on $V$), and $H^2(T^4;\mathbb{Z})$ with the appropriate quadratic lattice $\Lambda \subset V$. Since the $\text{Diff}^+(T^4)$-action on...
$H^2(\mathbb{T}^4; \mathbb{R})$ preserves $H^2(\mathbb{T}^4; \mathbb{Z})$, it induces a homomorphism from $\text{Diff}^+(\mathbb{T}^4)$ to the group $SO^+(\Lambda)$. The image of this homomorphism can be identified with $SL(4, \mathbb{Z})$ that acts naturally on $V = (\bigwedge^2 \mathbb{R}^4)^*$, as the restriction of the natural $SL(4, \mathbb{R})$-action on $V = (\bigwedge^2 \mathbb{R}^4)^*$. The quadratic lattice $\Lambda = H^2(\mathbb{T}^4; \mathbb{R})$ is isomorphic to $U \oplus U \oplus U – \text{in particular, it is even, unimodular and of signature (3,3)}$. It is also isotropically transitive, by Example 3.4.

The group $SO^+(V)$ is then isomorphic to $SL(4, \mathbb{R})/\pm 1$ and the isomorphism sends $SO^+(\Lambda) \subset SO^+(V)$ to $SL(4, \mathbb{Z})$. The orbits of the $\text{Diff}^+(\mathbb{T}^4)$-action on $H^2(\mathbb{T}^4; \mathbb{R})$ are identified with the orbits of the action of $SO^+(\Lambda) = SL(4, \mathbb{Z})$ on $V = (\bigwedge^2 \mathbb{R}^4)^*$.

In the case where $M$ is a smooth manifold underlying a complex K3 surface the group $\Lambda = H^2(M; \mathbb{Z})$, equipped with the intersection form, is a quadratic lattice isomorphic to $U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8)$ (see e.g. [Bea2] for the definition of the integral lattice $E_8$ and for the proof). In particular, it is an even unimodular quadratic lattice of signature $(3,19)$ in $V := H^2(M; \mathbb{R})$. It is also isotropically transitive, by Example 3.4.

Since the $\text{Diff}^+(M)$-action on $H^2(M; \mathbb{R})$ preserves $H^2(M; \mathbb{Z})$ and the intersection form, it induces a homomorphism $\text{Diff}^+(M) \to SO(\Lambda)$. The image of this homomorphism is $SO^+(\Lambda)$ – see [Borc], [DI]. Thus, the orbits of the $\text{Diff}^+(M)$-action on $H^2(M; \mathbb{R})$ are the orbits of the natural action of $SO^+(\Lambda)$ on $V$.

Summing up, we see that if $M = \mathbb{T}^4$ or a smooth manifold underlying a complex K3 surface, it is enough to prove that for any isotropic $u \in H^2(M; \mathbb{Z})$ the intersection of the $SO^+(\Lambda)$-orbit of $y$ with $u^\perp \cap P$ is dense in $u^\perp \cap P$.

In the case $u = 0$ the density of the $\text{Diff}^+(M)$-orbit of $y$ in $u^\perp \cap P = P$ was previously proved in [EV, Thm. 9.8]. (Note that since the unit vector $y$ is orthoisotropically irrational, it is not proportional to an integral cohomology class, and thus the result of [EV, Thm. 9.8] applies).

Therefore, we only need to prove the theorem in the case $u \neq 0$. In this case, the result follows immediately from the discussion above and Theorem 4.10.

5 Kähler-type symplectic forms on K3 surfaces

In this section we discuss Kähler-type symplectic forms on K3 surfaces. This will be used further on in Section 6.2 to prove Theorem 1.7.

Let $M$ be a K3 surface equipped with the standard orientation.

Denote by $Q : H^2(M; \mathbb{C}) \times H^2(M; \mathbb{C}) \to \mathbb{C}$ the Hermitian intersection
form:

\[ Q(a, b) := \int_M a \cup \bar{b}. \]

Recall that a hyperkähler structure on \( M \) is a collection

\[ (\omega_1, \omega_2, \omega_3, I_1, I_2, I_3) \]

of three complex structures \( I_1, I_2, I_3 \), compatible with the orientation and satisfying the quaternionic relations, and three symplectic forms \( \omega_1, \omega_2, \omega_3 \), compatible, respectively, with \( I_1, I_2, I_3 \), so that the three Riemannian metrics \( \omega_i(\cdot, I_i \cdot) \), \( i = 1, 2, 3 \), coincide. A symplectic form, or a complex structure, on \( M \) is said to be of hyperkähler type, if it appears in some hyperkähler structure.

**Proposition 5.1:**

Let \( M \) be a K3 surface equipped with the standard orientation.

Then:
- Any complex structure on \( M \) (compatible with the orientation) is of hyperkähler type (and, in particular, of Kähler type).
- A symplectic form on \( M \) (compatible with the orientation) is of Kähler-type if and only if it is of hyperkähler type.

**Proof of Proposition 5.1.**

Let \( J \) be a complex structure on \( M \).

By a theorem of Siu [Siu] (that corrected a previous result of Todorov [T]), \( J \) is of Kähler type. (Alternatively, this can be deduced from later results of Buchdahl and Lamari – see [Bu], [L] – who showed that any complex surface with an even first Betti number admits a Kähler structure).

A theorem of Friedman-Morgan [FM, p.495, Cor. 3.5], based on Donaldson’s results on gauge theory, says that any complex surface diffeomorphic to a K3 surface is itself K3 (alternatively, the latter claim can be deduced from the results of Taubes on Seiberg-Witten invariants – see e.g. [Sal, Example 3.13]). Hence, \( c_1(M, J) = 0 \) and \((M, J)\) is a complex K3 surface.

Let \( \omega \) be a Kähler form on \((M, J)\). Then, by Yau’s theorem (formerly the Calabi conjecture) [Yau], since \( c_1(M, J) = 0 \), the complex structure \( J \) belongs to a Kähler structure \((\omega', J)\) on \( M \) for which the corresponding Hermitian metric \( \omega'(\cdot, J \cdot) + \sqrt{-1} \omega'(\cdot, \cdot) \) is Ricci-flat, and moreover \( [\omega'] = [\omega] \).

Consequently, \( J \) and \( \omega' \) are of hyperkähler type – see [Bea1], cf. [Bes, Thm. 6.40]. Since \( \omega \) and \( \omega' \) are cohomologous symplectic forms compatible with the same \( J \), by a theorem of Moser [Mos], they can be identified by an isotopy of \( M \). Therefore, since \( \omega' \) is of hyperkähler type, so is \( \omega \).
Let $J$ be a complex structure on $M$. By Proposition 5.1, $J$ is of Kähler type and therefore induces a Hodge decomposition of $H^2(M; \mathbb{C})$:

$$H^2(M; \mathbb{C}) = H^{2,0}(M, J) \oplus H^{1,1}(M, J) \oplus H^{0,2}(M, J).$$

Moreover, $\dim_{\mathbb{R}} H^{2,0}(M, J) = 1$ (see e.g. [Bea2]) and the space $H^{2,0}(M, J)$ completely determines the full Hodge decomposition of $H^2(M; \mathbb{C})$:

$$H^0,2(M, J) = H^{2,0}(M, J)$$

and $H^{1,1}(M, J)$ is the orthogonal complement of $H^{2,0}(M, J) \oplus H^{0,2}(M, J)$ with respect to $Q$.

## 6 The $\mathbb{T}^4$ and K3 case – density of the $\text{Diff}^+(M)$-orbits in the space of forms

The goal of this section is to prove Theorem 1.7.

### 6.1 Symplectic Kähler-type Teichmüller space and the action of the mapping class group on it

Let $M^{2n}$ be either $\mathbb{T}^{2n}$ or a smooth manifold underlying a complex K3 surface. Let $\text{Diff}_0(M)$ be the connected component of the identity in the group $\text{Diff}^+(M)$ of orientation-preserving diffeomorphisms of $M$.

Define the **symplectic Kähler-type Teichmüller space** $T(M)$ by

$$T(M) := S(M)/\text{Diff}_0(M).$$

Equip $T(M)$ with the quotient topology. For $\omega \in S(M)$ denote the corresponding element of $T(M)$ by $\{\omega\}$.

Define the **symplectic period map** $\text{Per} : T(M) \to H^2(M; \mathbb{R})$ by

$$\text{Per}(\{\omega\}) := [\omega].$$

Using Moser’s stability theorem for symplectic structures, it is not hard to obtain that $T(M)$ is a finite-dimensional manifold (possibly non-Hausdorff) and $\text{Per}$ is a local diffeomorphism ([Mos], [FH, Proposition 3.1]).

The image of $\text{Per}$ is exactly $P = \{x \in H^2(M; \mathbb{R}) \mid x^n = 1\}$. Indeed, the inclusion $\text{Im} \text{Per} \subseteq P$ is obvious. The inclusion $P \subseteq \text{Im} \text{Per}$ in the torus case

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1. Note that although the proof of the existence of the Hodge decomposition involves the whole Kähler structure of which $J$ is a part, one can show – see e.g. [Voi, Vol.1, Prop. 6.11] – that, in fact, the Hodge decomposition depends only on $J$. 

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is easy: any point in $P$ is the cohomology class of a linear symplectic form on the torus; such a symplectic form is of Kähler type. In the K3 case the equality $P = \text{Im} \text{Per}$ follows, for instance, from [AV, Thm. 5.1], where it is proven for all compact hyperkähler manifolds of maximal holonomy.

Define the mapping class group of $M$ by

$$\Gamma := \text{Diff}^+(M) / \text{Diff}_0(M).$$

The group $\text{Diff}^+(M)$ (and, consequently, the mapping class group $\Gamma$) acts naturally on $T(M)$ and on $H^2(M; \mathbb{R})$. The period map $\text{Per}$ is equivariant with respect to these actions.

We will now prove Proposition 1.8. For convenience, we restate it here.

**Proposition 6.1:** (=Proposition 1.8)

Let $M^{2n}$ be either $T^{2n}$ or a smooth manifold underlying a complex K3 surface.

Then any two Kähler-type symplectic forms on $M$ (compatible with the orientation of $M$) can be mapped into each other by a diffeomorphism of $M$ acting trivially on homology.

**Proof of Proposition 6.1** (=Proposition 1.8).

In the case of $M = T^{2n}$ this is proved in [EV, Erratum] (it follows from the fact that any Kähler-type symplectic form on $T^{2n}$ can be identified with a linear form by a diffeomorphism acting as the identity on $H^*(T^{2n})$).

Assume that $M$ is a smooth manifold underlying a complex K3 surface. Let $\omega', \omega'' \in S_K(M)$ so that $[\omega'] = [\omega''].$

By Proposition 5.1 $\omega', \omega''$ can be included in hyperkähler structures

$$(\omega' =: \omega'_1, \omega'_2, \omega'_3, I'_1, I'_2, I'_3), (\omega'' =: \omega''_1, \omega''_2, \omega''_3, I''_1, I''_2, I''_3).$$

One can assume without loss of generality that $\text{Span}_\mathbb{R}\{[\omega'_1], [\omega'_2], [\omega'_3]\} = \text{Span}_\mathbb{R}\{[\omega''_1], [\omega''_2], [\omega''_3]\}$ – see [AV] Thm. 4.9 and the proof of Thm. 5.1.

Denote

$$W := \text{Span}_\mathbb{R}\{[\omega'_1], [\omega'_2], [\omega'_3]\} = \text{Span}_\mathbb{R}\{[\omega''_1], [\omega''_2], [\omega''_3]\}.$$  

The orthogonal complement of $[\omega']$ in $W$ determines $H^{2,0}(M, I'_1)$ and, similarly, the orthogonal complement of $[\omega'']$ in $W$ determines $H^{2,0}(M, I''_1)$ – see [AV] the proof of Thm. 4.9. Since $[\omega'] = [\omega'']$, we get that

$$H^{2,0}(M, I'_1) = H^{2,0}(M, I''_1).$$

In view of the discussion at the end of Section 5 this means that the Hodge decompositions of $H^2(M; \mathbb{C})$ induced by $I'_1$ and $I''_1$ coincide. Therefore,
by the global Torelli theorem for K3 surfaces (see [PS-S], [BR], [Siu], cf. [Ben2] p.96), there exists a biholomorphism \((M, I'_1) \to (M, I''_2)\) which acts as identity on \(H^\ast(M)\) and, consequently, preserves the orientation. Such a biholomorphism maps \(\omega'_1\) to a symplectic form \(\eta\) on \(M\) compatible with \(I'_1\) and cohomologous to \(\omega'_1\). It follows then from the theorem of Moser [Mos] that there exists an isotopy of \(M\) mapping \(\eta\) to \(\omega'_1\). Therefore, there exists a diffeomorphism of \(M\) acting trivially on the homology and mapping \(\omega''_1 = \omega'_1\) to \(\omega' = \omega'_1\).

**Proposition 6.2:**

Let \(M^{2n}\) be either \(T^{2n}\) or a smooth manifold underlying a complex K3 surface.

Then the mapping class group \(\Gamma\) (and, consequently, \(\text{Diff}^+(M)\)) acts transitively on the set of connected components of \(T(M)\).

**Proof of Proposition 6.2.**

Let \(T'(M)\) and \(T''(M)\) be two connected components of \(T(M)\). Since \(\text{Per}: T'(M) \to P\) and \(\text{Per}: T''(M) \to P\) are diffeomorphisms, there exist \(\omega', \omega'' \in S_K(M)\) such that \(\{\omega'\} \in T'(M), \{\omega''\} \in T''(M), [\omega'] = [\omega'']\).

By Proposition 6.1, there exists an element of \(\text{Diff}^+(M)\) mapping \(\omega''\) to \(\omega'\) and hence an element of \(\Gamma\) mapping \(\{\omega''\}\) to \(\{\omega'\}\).

This finishes the proof of the proposition.

**6.2 The proof of Theorem 1.7**

Now we can finally prove Theorem 1.7. We restate it here for convenience.

**Theorem 6.3:** (= Theorem 1.7)

Assume that \(\omega_0 \in S(M)\) so that the cohomology class \([\omega_0] \in H^2(M; \mathbb{R})\) is orthoisotropically irrational.

Then for any isotropic \(u \in H^2(M; \mathbb{Z})\) the intersection of the \(\text{Diff}^+(M)\)-orbit of \(\omega_0\) with \(S_{u \bot}(M)\) is dense in \(S_{u \bot}(M)\). In particular, it is dense in \(S(M)\) (for \(u = 0\)).

**Proof of Theorem 6.3** (= Theorem 1.7).

The map \(\text{Per}: T(M) \to P\) is a surjective local diffeomorphism. Consequently, its restriction to the set \(\{\omega\} | \omega \in S_{u \bot}\) is a local diffeomorphism between this set and \(a \bot \subset P\). The union of the domains and the union of the
targets of these local diffeomorphisms for different $a$ are $\text{Diff}^+(M)$-invariant and the local diffeomorphisms are $\text{Diff}^+(M)$-equivariant.

Now the theorem follows from Proposition 6.2 and Theorem 4.1.
Further on we will need the following easy lemma.

**Lemma 7.2:**
Let $R$ be the set of rational subspaces of $\mathbb{R}^m$ of dimension $k > 0$. Then $SL(m, \mathbb{Z})$ acts transitively on $R$.

**Proof:**
Let $N \subset \mathbb{Z}^m$ be a rational subspace of $\mathbb{R}^m$ of dimension $k > 0$. Then one can choose a basis $x_1, \ldots, x_k$ of the finite-rank free group $N \cap \mathbb{Z}^m$ which is a basis of $N \cap \mathbb{Q}^m$ over $\mathbb{Q}$ (because any vector in $N \cap \mathbb{Z}^m$ has an integral multiple that lies in $N \cap \mathbb{Q}^m$) and hence also a basis of $N$ over $\mathbb{R}$.

Since $SL(m, \mathbb{Z})$ acts transitively on the set of oriented bases of $\mathbb{Z}^m$, it would suffice to show that the collection $x_1, \ldots, x_k$ can be extended to a basis of $\mathbb{Z}^m$. Since $N$ is primitive, the quotient $\mathbb{Z}^m/(N \cap \mathbb{Z}^m)$ has no torsion, meaning that it is a finite rank free abelian group. Choose a basis $\bar{y}_1, \ldots, \bar{y}_{m-k}$ of $\mathbb{Z}^m/(N \cap \mathbb{Z}^m)$ and lift these elements to $y_1, \ldots, y_{m-k} \in \mathbb{Z}^m$.

Then $x_1, \ldots, x_k, y_1, \ldots, y_{m-k}$ is a basis of $\mathbb{Z}^m$.

This finishes the proof of the lemma. ■

Now we are ready to prove **Theorem 1.1** We restate it here for convenience.

**Theorem 7.3:** (=**Theorem 1.1**)
Assume $M$, $\dim_\mathbb{R} M = 2n$, is either an even-dimensional torus or a smooth manifold underlying a K3 surface. Let $\omega$ be a Kähler-type symplectic form on $M$.

Then for any Maslov-zero Lagrangian torus $L \subset (M, \omega)$ the homology class $[L] \in H_n(M; \mathbb{Z})$ is non-zero and primitive.

**Proof of Theorem 7.3** (=**Theorem 1.1**) — the torus case.
Let $x_1, \ldots, x_{2n}$ be the standard coordinates on $\mathbb{R}^{2n}$.

Let $\omega$ be a Kähler-type symplectic form on $\mathbb{T}^{2n}$ and let $L \subset (\mathbb{T}^{2n}, \omega)$ be a Maslov-zero Lagrangian torus $L \subset (\mathbb{T}^{2n}, \omega)$.

We need to prove that the homology class $[L] \in H_n(\mathbb{T}^{2n}; \mathbb{Z})$ is non-zero and primitive.

Since any Kähler-type symplectic form on $\mathbb{T}^{2n}$ can be mapped into a linear symplectic form by a diffeomorphism of $\mathbb{T}^{2n}$ (see [Ev]) and since the required property of $L$ is invariant under the rescaling of $\omega$ by a constant factor, we may assume without loss of generality that $\omega$ is a linear symplectic form of total volume 1. Let $\bar{\omega} \in \mathcal{L}$ be the lift of $\omega$ to $\mathbb{R}^{2n}$. 

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Let us identify $\pi_1(\mathbb{T}^{2n}) \cong H_1(\mathbb{T}^{2n}) \cong \mathbb{Z}^{2n} \subset \mathbb{R}^{2n} \cong H_1(\mathbb{T}^{2n};\mathbb{R})$ (recall that $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$). Set
\[
Z := \text{Im} \left( \pi_1(L) \to \pi_1(\mathbb{T}^{2n}) \right) \subset \pi_1(\mathbb{T}^{2n}) \cong \\
\cong \text{Im} \left( H_1(L) \to H_1(\mathbb{T}^{2n}) \right) \subset H_1(\mathbb{T}^{2n}) \cong \mathbb{Z}^{2n},
\]
\[
l := \text{Im} \left( H_1(L;\mathbb{R}) \to H_1(\mathbb{T}^{2n};\mathbb{R}) \right) \subset H_1(\mathbb{T}^{2n};\mathbb{R}) \cong \mathbb{R}^{2n}.
\]
Let $k := \dim l$.

Then $Z \subset l \cap \mathbb{Z}^{2n}$ and $l$ is the smallest rational subspace of $\mathbb{R}^{2n}$ containing $Z$. Moreover, $l$ is isotropic with respect to $\tilde{\omega}$, implying that $0 \leq k \leq n$. In fact, $[L] \neq 0$ if and only if $\dim l = n$.

Denote
\[
L_l := \frac{l}{l \cap \mathbb{Z}^{2n}}.
\]
It is a $k$-dimensional torus.

Let $x_1, \ldots, x_{2n}$ be the standard coordinates on $\mathbb{R}^{2n}$. Since $SL(2n,\mathbb{Z})$ acts transitively on the set of integral subspaces of $\mathbb{R}^{2n}$ of dimension $k$ ([Lemma 7.2]) and maps each linear symplectic form on $\mathbb{T}^{2n}$ to a linear symplectic form of the same total volume, we may assume, without loss of generality, that $l$ is either $\{0\}$, if $k = 0$, or the $(x_1, \ldots, x_k)$-coordinate plane, if $1 \leq k \leq n$. Let $l'$ be the $(x_{k+1}, \ldots, x_{2n})$-coordinate plane.

Let us show that $[L] \neq 0$. The proof follows the argument in [AS] for the case of $\mathbb{T}^4$ and the standard Darboux symplectic form on $\mathbb{T}^4$.

Namely, assume, by contradiction that $[L] = 0$, or, equivalently, $k < n$. The natural projection
\[
L_l \times l' \to L_l \times \frac{l'}{l' \cap \mathbb{Z}^{2n}} = \mathbb{T}^{2n}
\]
is a covering. Consider the symplectic form $\tilde{\omega}$ on $L_l \times l'$ which is the lift of $\omega$. Since a lift of a Maslov-zero Lagrangian to a covering symplectic space is again a Maslov-zero Lagrangian, the torus $L$ lifts to a Maslov-zero Lagrangian torus $\hat{L} \subset (L_l \times l', \tilde{\omega})$. It is also easy to see that the symplectic manifold $(L_l \times l', \tilde{\omega})$ is convex at infinity in the sense of [EG].

On the other hand, since $l$ is isotropic and $\dim l < n < \dim l'$, we have $l^\perp \cap l' \neq \{0\}$, where $l^\perp$ is the $\omega$-orthogonal complement of $l$. This implies that there exists a Hamiltonian on $L_l \times l'$, which is a linear function of $x_{k+1}, \ldots, x_{2n}$, whose constant Hamiltonian vector field is parallel to $l'$. Consequently, any compact subset of $L_l \times l'$ – and, in particular, $\hat{L}$ – is displaceable by a Hamiltonian isotopy. However, Maslov-zero Lagrangian torus cannot be displaceable by a theorem of Fukaya [Fu Thm. 12.2].

This leads to a contradiction and shows that $[L] \neq 0$. 

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v. 1.11, March 31, 2022
Let us now show that $[L]$ is primitive.

As we have shown, $[L] \neq 0$. Consequently, $\pi_1(L)$ is identified with $Z$ and $l$ and $l'$ are the $(x_1, \ldots, x_n)$ and $(x_{n+1}, \ldots, x_{2n})$-coordinate planes. In particular, $l$ and $l'$ are complementary $n$-dimensional rational subspaces. Let $q$ be the index of $Z$ in $l \cap \mathbb{Z}^{2n}$. The homology class $[L]$ is primitive if and only if $q = 1$.

Let us first prove the primitiveness of $[L]$ in the case where $\omega$ is $(l, l')$-Lagrangian split. The proof in this case follows the argument in [AS] for the case of $\mathbb{T}^4$ and the standard Darboux symplectic form on $\mathbb{T}^4$.

Consider the linear Lagrangian torus $L_q := l/(l \cap \mathbb{Z}^{2n}) \subset (\mathbb{T}^{2n}, \omega)$, and the covering map $L_q \times l' \to L_q \times l'/(l' \cap \mathbb{Z}^{2n}) = \mathbb{T}^{2n}$. Also consider the $q$-fold covering map $l/Z \to L_q = l/(l \cap \mathbb{Z}^{2n})$ – its direct product with the identity is a $q$-fold covering map $l/Z \times l' \to L_q \times l'$. Let $\pi : l/Z \times l' \to \mathbb{T}^{2n}$ be the composition of the covering maps $l/Z \times l' \to L_q \times l'$ and $L_q \times l' \to \mathbb{T}^{2n}$. Let $\tilde{\omega}$ be a symplectic form on $l/Z \times l'$ which is the lift of $\omega$ under $\pi$. It is easy to see that the symplectic manifold $(l/Z \times l', \tilde{\omega})$ is convex at infinity in the sense of [EG].

There are $q$ disjoint lifts of $L$ to $l/Z \times l'$ and they all are Maslov-zero Lagrangian tori in $(l/Z \times l', \tilde{\omega})$ mapped into each other by diffeomorphisms of $l/Z \times l'$ induced by parallel translations in $l/Z$. Since $\omega$ is $(l, l')$-Lagrangian split, such parallel translations are Hamiltonian symplectomorphisms of $(l/Z \times l', \tilde{\omega})$ generated by Hamiltonians that are linear functions of the coordinates $(x_{n+1}, \ldots, x_{2n})$ on $l/Z \times l'$. Thus, $q > 1$ would have implied that there exists a displaceable Maslov-zero Lagrangian torus in $(l/Z \times l', \tilde{\omega})$, in contradiction to the same theorem of Fukaya [Fu, Thm. 12.2] that we have already used above.

This proves the primitiveness of $[L]$ in the case where $\omega$ is $(l, l')$-Lagrangian split.

Now let $\omega$ be an arbitrary linear symplectic form on $\mathbb{T}^{2n}$ of total volume 1.

Assume, by contradiction, that $[L]$ is not primitive. It follows from [Proposition 7.1] that for any linear symplectic form $\omega'$ of total volume 1, which is sufficiently close to $\omega$ and satisfies $[\omega'|_L] = 0$ (for linear symplectic forms $\omega$ this holds if and only if $\omega'|_l \equiv 0$), the symplectic manifold $(\mathbb{T}^{2n}, \omega')$ admits a Maslov-zero Lagrangian torus whose homology class is not primitive. By [Proposition 1.4], one can find such an $\omega'$ which is the image of an $(l, l')$-Lagrangian split form $\omega''$ under a diffeomorphism of $\mathbb{T}^{2n}$ given by an element of $SL(2n, \mathbb{Z})$ acting trivially on $l$. Hence, $(\mathbb{T}^{2n}, \omega'')$ too admits a Maslov-zero Lagrangian torus whose homology class is not primitive, in contradiction to the primitiveness in the $(l, l')$-Lagrangian split case that we have proved above.

This finishes the proof of the theorem.
Proof of \textbf{Theorem 7.3} (= \textbf{Theorem 1.1}) – the K3 case.

Let $M$ be a smooth manifold underlying a smooth K3 surface.

It follows from [SS, Thm. 1.3] that for a certain Kähler-type complex structure $J$ on $M$ (a so-called “mirror quartic”) there exists a symplectic form $\omega_0 \in S(M)$, compatible with $J$, such that $(M, \omega)$ does not admit a Maslov-zero Lagrangian torus in the zero or a non-primitive homology class. In the terminology of [SS], such an $\omega_0$ can be chosen to be “ambient irrational”. In view of [SS, Example 3.10], the latter property of $\omega_0$ implies that the quadratic lattice $[\omega_0]^\perp \cap H^2(M; \mathbb{Z})$ is isomorphic to $U \oplus U \oplus U \oplus \langle 4 \rangle$. (Here $\langle 4 \rangle$ is the quadratic lattice formed by $\mathbb{Z}$ equipped with the symmetric bilinear form $(\cdot, \cdot)$ defined by $(1, 1) = 4$).

We claim that $[\omega_0]$ is orthoisotropically irrational.

Indeed, first, $[\omega_0]^\perp$ contains a non-zero primitive isotropic element $u \in H^2(M; \mathbb{Z})$ (from a $U$ summand), hence $[\omega_0] \in u^\perp$.

Second, let us show that for each such $u$ we have

$$\text{Span}_R\{u, [\omega_0]\} \cap H^2(M; \mathbb{Z}) = \text{Span}_Z\{u\}.$$  

Assume by contradiction that this is false. Then $[\omega_0] = \kappa_1 u + \kappa_2 x$ for some $\kappa_1, \kappa_2 \in \mathbb{R}$ and $x \in u^\perp \cap H^2(M; \mathbb{Z})$. Since $H^2(M; \mathbb{Z})$ together with the intersection pairing is an even unimodular quadratic lattice (see part (B2) of \textbf{Proposition 3.2}), the rank of $(\text{Span}_R\{u, x\})^\perp \cap H^2(M; \mathbb{Z})$ equals $rk H^2(M; \mathbb{Z}) - 2 = 20$. On the other hand, $(\text{Span}_R\{u, x\})^\perp \cap H^2(M; \mathbb{Z})$ is contained in $[\omega_0]^\perp \cap H^2(M; \mathbb{Z})$, whose rank is 7, yielding a contradiction. Thus, $\text{Span}_Z\{u, [\omega_0]\} \cap H^2(M; \mathbb{Z}) = \text{Span}_Z\{u\}$ and the claim holds.

Since, as we have shown, $[\omega_0]$ is orthoisotropically irrational, \textbf{Theorem 1.7} implies that the $\text{Diff}^+(M)$-orbit of $\omega_0$ is dense in $S_{u^\perp}$ for each isotropic $u \in H^2(M; \mathbb{Z})$.

Now for each such $u$ denote by $\mathcal{L}_u$ the (possibly empty) set of $\omega \in S$ such that $(M, \omega)$ admits a Maslov-zero Lagrangian torus in the homology class Poincaré-dual to $u$. Clearly, $\mathcal{L}_u \subset S_{u^\perp}$. Moreover, it follows from \textbf{Proposition 7.1} that $\mathcal{L}_u$ is open in $S_{u^\perp}$.

The claim of the theorem is equivalent to showing that $\mathcal{L}_u$ is empty if $u$ is zero or non-primitive.

Assume, by contradiction, that $\mathcal{L}_u \neq \emptyset$ for such a $u$. Hence, by the density property above, $\text{Diff}^+(M)$-orbit of $\omega_0$ intersects $\mathcal{L}_u$ – that is, there exists $\omega$ in the $\text{Diff}^+(M)$-orbit of $\omega_0$ such that $(M, \omega)$ admits a Maslov-zero Lagrangian torus in the zero or a non-primitive homology class. Then, by the $\text{Diff}^+(M)$-invariance, so does $(M, \omega_0)$, which yields a contradiction with the choice of $\omega_0$.

This finishes the proof of the theorem. \blacksquare
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