Data-driven distributed MPC of dynamically coupled linear systems

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Abstract: In this paper, we present a data-driven distributed model predictive control (MPC) scheme to stabilise the origin of dynamically coupled discrete-time linear systems subject to decoupled input constraints. The local optimisation problems solved by the subsystems rely on a distributed adaptation of the Fundamental Lemma by Willems et al., allowing to parametrise system trajectories using only measured input-output data without explicit model knowledge. For the local predictions, the subsystems rely on communicated assumed trajectories of neighbours. Each subsystem guarantees a small deviation from these trajectories via a consistency constraint. We provide a theoretical analysis of the resulting non-iterative distributed MPC scheme, including proofs of recursive feasibility and (practical) stability. Finally, the approach is successfully applied to a numerical example.

1. INTRODUCTION

Model predictive control (MPC) is a modern control method with a well-researched theoretical foundation, see e.g. Rawlings et al. (2020); Grüne and Pannek (2017). Advantages of MPC are the possibility to explicitly consider constraints on the controlled system and the incorporation of a performance objective. If the system to be controlled is large-scale or consists of many interconnected but otherwise discernible subsystems, a central controller might not be desirable or even computationally infeasible. This is alleviated by distributed MPC (DMPC) where a local MPC is designed for each subsystem and the respective control input is computed based on locally available information. Various different setups for DMPC exist which are suitable for different classes of systems, interconnection structures, and communication topologies using an iterative or non-iterative scheme. See e.g. the surveys Christofides et al. (2013); Müller and Allgöwer (2017) or the collection Maestre and Negenborn (2014) for a selection and classification of various schemes.

Typically, a model derived from first principles is used in MPC to predict the behaviour of the controlled system. In practice, however, it might be difficult to derive such a model in detail, but obtaining input-output data is comparatively simple. Recently, for this reason, there has been a push to use only input-output data and no explicit model knowledge to design direct data-driven controllers. Many results, including our approach, rely on Willems’ Fundamental Lemma Willems et al. (2005), which states that all trajectories of a linear time-varying (LTI) system can be constructed from one persistently exciting trajectory.

Based on this, MPC schemes can be designed using only input-output data and no explicit model knowledge Yang and Li (2015); Coulson et al. (2019), even admitting closed-loop guarantees on stability and robustness Berberich et al. (2021a). Further references may be found in the survey Markovsky and Li (2015); Coulson et al. (2019), even admitting closed-loop guarantees on stability and robustness. Recently, extensions to data-driven distributed MPC for dynamically coupled systems with stability guarantees Allibhoy and Cortés (2021); Alonso et al. (2021) have been made, which require state measurements and state coupling. Both employ iterative distributed optimisation, requiring a multitude of communication at each time step.

In this paper, we propose a non-iterative data-driven distributed MPC (D3MPC) scheme for dynamically coupled LTI systems. The local MPC optimisation problems are based on only local input-output data and are solved in parallel assuming that the neighbours follow a previously communicated trajectory. Therefore, the scheme scales well with the total number of subsystems. Communication is kept to a minimum since in each time step only one trajectory needs to be shared. We show that the proposed scheme practically asymptotically stabilises the origin of the system while complying with input constraints. The main tool is the inclusion of a consistency constraint in the optimisation problems, which provides that the subsystems do not deviate too much from their communicated trajectories. The idea is based on Dunbar (2007) which uses a similar consistency constraint in a model-based setting for continuous-time systems in a stabilising dual-mode DMPC scheme.

2. PRELIMINARIES

We denote the natural numbers containing 0 with $\mathbb{N}_0$. We write $A \succ 0$ ($A \succeq 0$) if $A = A^\top$ is positive (semi-)definite.
The smallest eigenvalue of a matrix $A = A^\top$ is denoted by $\lambda_{\min}(A)$. Given another matrix $B = B^\top$, $\lambda_{\min}(A, B) = \min\{\lambda_{\min}(A), \lambda_{\min}(B)\}$. For a set of vectors $v_1, \ldots, v_n$ we denote the stacked vector $v = [v_1^\top \ldots v_n^\top]^\top = \col_{n\times 1}(v_i)$. For a sequence $\{x_k\}_{k=0}^{N-1}$, we define the Hankel matrix

$$H_L(x) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{N-L} \\ x_1 & x_2 & \cdots & x_{N-L+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{L-1} & \cdots & \cdots & x_{N-L} \end{bmatrix}.$$  

Moreover, we introduce the notation $x_{[a,b]} = \col_{b-a}(x_k)$. For a (block-)diagonal matrix with $n$ (block-)diagonal elements $A$, we write $\diag_{n\times 1}(A_i)$. The set of integers in the interval $[a, b]$ with $a \leq b$ is denoted by $[a, b]$. We make use of class-$K$, $\mathcal{K}_\infty$ and -$\mathcal{KL}$ comparison functions and refer to Kellett (2014) for their definition. The cardinality of a set $S$ is denoted by $|S|$. With * we denote parts that can be inferred by symmetry.

Definition 1. We say that a sequence $\{x_k\}_{k=0}^{N-1}$ with $x_k \in \mathbb{R}^n$ is persistently exciting of order $L$ if $\rank(H_L(x)) = nL$.

3. DATA-DRIVEN DISTRIBUTED MPC

3.1 Distributed system representation

Our objective is to stabilise a group of $M \geq 2$ dynamically coupled subsystems about the origin. The dynamic coupling between the subsystems is given by a directed graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, M\}$ is the set of nodes corresponding to subsystems and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of all directed edges between nodes in the graph. If the output $y_j$ of a subsystem $j$ appears in the dynamic equation of subsystem $i$, and $i \neq j$, then subsystem $j$ is neighbour of subsystem $i$ and part of the set of neighbours $\mathcal{N}_i = \{j \in \mathcal{V} \mid j \text{ is a neighbour of } i\}$. Then, $\mathcal{E} = \{(j, i) \in \mathcal{V} \times \mathcal{V} \mid j \in \mathcal{N}_i\}$. It is assumed that communication along the directed edges of $G$ is possible.

We consider the following input-output dynamics for subsystem $i \in \mathcal{V}$:

$$x_{i,t+1} = A_i x_{i,t} + B_i u_{i,t} + \sum_{j \in \mathcal{N}_i} B_{ij} y_{j,t}$$

$$y_{i,t} = C_i x_{i,t} + D_{i} u_{i,t}$$

for $i = 1, \ldots, M$, where $x_{i,t} \in \mathbb{R}^n$ is the state, $u_{i,t} \in \mathbb{R}^m$ is the input, and $y_{i,t} \in \mathbb{R}^p$ is the output of system $i$, all at time $t \in \mathbb{N}_0$. The input is subject to constraints $u_{i,t} \in \mathcal{U}_i \subseteq \mathbb{R}^m$ with compact $\mathcal{U}_i$. We abbreviate $y_{i,t} = \col_{1\times \mathcal{E}_i}(y_{i,t})$. The dimensions of every subsystems’ input and output are assumed to be the same, for notational simplicity and without loss of generality.

For each individual subsystem $i$, the matrices $A_i$, $B_i$, $C_i$, and $D_i$ are unknown, but a data set

$$D_i := \{(i,k)_{i=1}^{N-1} : y_{i,k} \in \mathcal{N}_i \}$$

(2) that satisfies the dynamics (1) is available.

Assumption 2.  (1) For each $i \in \mathcal{V}$, the pair $(A_i, B_i)$ is controllable, where

$$B_i = [B_{i1} \ B_{i2} \ \cdots \ B_{ij} \mathcal{N}_i]$$

(3) For each $i \in \mathcal{V}$, the sequence

$$[y_{i,0}^-, y_{i,1}^-, \ldots, y_{i,N-1}^-]$$

is persistently exciting of order $L + n$.

(3) $(A, C)$ is observable, where $C = \diag_{M\times 1}(C_{ii})$ and

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & \ddots & \vdots \\ A_{M1} & \cdots & A_{MM} \end{bmatrix}$$

with $A_{ij} = 0$ if $j \notin \mathcal{N}_i$ and $A_{ij} = B_{ij} C_{ij}$ if $j \in \mathcal{N}_i$.

The following lemma is a consequence of Willems’ Fundamental Lemma Willems et al. (2005) with the neighbours’ outputs interpreted as additional inputs. It allows the characterisation of any input-output trajectory of (1) based on suitable local data $D_i$ and will be the basis of our D3M MPC scheme.

Lemma 3. Suppose Assumption 2 holds. Then, for any $i \in \mathcal{V}$, \{u_k, y_k\}_{k=0}^{t-1}, \{y_k\}_{k=0}^{t-2} \text{ is a trajectory of (1) if and only if there exists } \alpha_t \in \mathbb{R}^{N_L+1} \text{ such that}

$$\begin{bmatrix} H_L(u_{[0,N-1]}^t) \\ H_L(y_{[0,N-2]}^t) \\ H_L(y_{[0,N-1]}^t) \end{bmatrix} \alpha_t = \begin{bmatrix} u_{[0,L-1]}^t \\ y_{[0,L-2]}^t \\ y_{[0,L-1]}^t \end{bmatrix}.$$  

(4) Since we only rely on input-output measurement, it is useful to define the extended state

$$\xi_t = \begin{bmatrix} u_{[t-n,t-1]}^t \\ y_{[t-n,t-1]}^t \end{bmatrix}.$$  

(5)

The system

$$\begin{aligned}
\xi_{t+1} &= \tilde{A}_i \xi_t + \tilde{B}_i u_t + \tilde{B}_{i-1} y_{i-1}^t \\
y_t &= \tilde{C}_i \xi_t + \tilde{D}_i u_t,
\end{aligned}$$

(6a)

with suitable matrices $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{B}_{i-1}$, $\tilde{C}_i$ and $\tilde{D}_i$, then has the same input-output behaviour as (1), see, e.g. Goodwin and Sin (2014), (Koch et al., 2021, Lemma 2). In particular, (6a) looks like (8) with unknown $G_{i,n,1}$, $G_{i,1}$, $S_{i,n,1}$, $S_1$, $F_{i,n,1}$, $F_{i,1}$ (cf. Berberich et al. (2021b)). Note that (4) contains only $y_{[0,L-2]}^t$ since there is no feed-through from $y_{L-1}^t$ to $y_{L-1}^t$ in (1b), and thus $y_{L-1}^t$ has no effect on $y_{L-1}^t$. However, we included $y_{L-1}^t$ in (5) since then $\tilde{B}_{i-1}$ in (6a) is known. We exploit this in the data-driven design of terminal conditions in Section 3.3. We also define $T_{i,i} = [0 \ldots 0]$ with $y_{i,i}^t = T_{i,i} \xi_{i+1}$. Similarly, a global extended state system

$$\begin{aligned}
\xi_{t+1} &= \tilde{A}_i \xi_t + \tilde{B}_i u_t \\
y_t &= \tilde{C}_i \xi_t + \tilde{D}_i u_t,
\end{aligned}$$

(7a)

can be defined with $\xi_t = \col_{1\times \mathcal{E}_i}(\xi_t)$, $u_t = \col_{1\times \mathcal{E}_i}(u_t)$ and $y_t = \col_{1\times \mathcal{E}_i}(y_t)$ and suitable matrices $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$.

3.2 Data-driven distributed MPC optimisation problem

Each subsystem locally solves at time step $t$ the following optimisation problem given the current measurements $\{u_k, y_k\}_{k=t-n}^{t}$ and communicated trajectories $y_{[t-n+1,L]}^t(t-1)$ of the neighbours.

$$\min_{\alpha} \sum_{k=0}^{L-1} \|y_k^t(t)\|_{Q_k}^2 + \|u_k^t(t)\|_{R_k}^2 + V_1^t(\xi_k^t(t))$$

(9a) subject to...
whereas (9c) fixes the initial condition. The predicted extended output behaviour of the system over the prediction horizon

\[
\begin{bmatrix}
H_{L+n}(\theta_{[0,N-1]})
H_{L+n-1}(y_{[0,N-2]}^i)
H_{L+n}(\hat{y}_{[0,N-1]}^i)
\end{bmatrix}
\begin{bmatrix}
u_{t-n-1}^i
\dot{y}_{t-n-1}^i
y_{t-n-1}^i
\end{bmatrix}
= \begin{bmatrix}
0 & I & \ldots & 0 & \ldots & 0
0 & 0 & 0 & I & \ldots & 0
0 & 0 & 0 & 0 & \ldots & I
\end{bmatrix}
\begin{bmatrix}
u_{t-n-L-1}^i(t)
y_{t-n-L-1}^i(t)
y_{t-n-L-1}^i(t-1)
\end{bmatrix}
\]

(9b)

with a constant \(\Omega_i \geq 0\), and momentarily defined trajectories \(\hat{u}_{[0,L-1]}^i(t)\) and \(\hat{y}_{[0,L-1]}^i(t)\) as well as a terminal cost \(V_f^i\) and a set \(X_f^i\). The optimal solution of (9) is denoted by \(\alpha_t^i(t)\) with corresponding optimal input sequence \(u_{t-n-L-1}^i(t)\) and corresponding predicted output sequence \(y_{t-n-L-1}^i(t)\). Constraint (9b) is based on Lemma 3 and used to predict the input-output behaviour of the system over the prediction horizon \(L\), whereas (9c) fixes the initial condition. The predicted extended state (9h) is confined to a terminal set through (9e) which is tightened using a factor \(\theta_i > 0\) to provide recursive feasibility. Each subsystem assumes that the outputs of their neighbours follow a communicated trajectory (cf. (9b)). The neighbours, however, will in general deviate from this trajectory. To ensure recursive feasibility despite this discrepancy, the consistency constraints (9f) and (9g) are included. It forces the predicted input and output to stay as close to what has been communicated as the in 3.4 defined feasible trajectories \(\hat{u}_{[0,L-1]}^i(t)\) and \(\hat{y}_{[0,L-1]}^i(t)\) can. If \(U_i\) and \(X_f^i(\theta,\epsilon)\) are ellipsoidal or polytopic, (9) is a convex quadratically constraint quadratic program, which can be solved efficiently.

3.3 Terminal ingredients

Assumption 4. For every subsystem \(i\), there exist a matrix \(P_i \succ 0\) defining \(V_f^i(\xi^i) = \|\xi\|^2_{P_i}\), \(\eta_i, \epsilon_i > 0\) and a terminal control law \(\kappa_i : X_f^i(\epsilon_i) \to U_i\) such that

\[
V_f^i(\xi^i+) - V_f^i(\xi^i) \leq -\eta_i\|\xi^i\|^2 - \|y_i^i\|^2_{Q_i} - \|\kappa_i(\xi^i)\|^2_{R_i}
\]

(10)

\[
\kappa_i(\xi^i) \in U_i
\]

(11)

This assumption implies a decrease in \(V_f^i\) inside \(X_f^i(\epsilon_i)\) for every subsystem despite the influence of the neighbours’ outputs. Implicitly, the dynamic coupling needs to be sufficiently weak.

We comment on how the method in Berberich et al. (2021b) could be adapted to compute distributed terminal ingredients as in Assumption 4 from input-output data, if the coupling is sufficiently weak. A general data-driven method to compute these terminal ingredients is left open for future research. We rewrite (6) into

\[
\begin{bmatrix}
\xi_{i+1}^\top
y_{i+1}^i
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_i & B_{w_i} & B_{w_i}
I & 0 & 0
0 & I & 0
\end{bmatrix}
\begin{bmatrix}
\xi_i^\top
u_{i}^l
y_{i}^i
\end{bmatrix}
\]

(12a)

\[
u_{i}^l = \Delta_i z_{i+1}^l
\]

(12b)

where (12b) contains all unknown elements in (6), i.e. \(\Delta_i = \begin{bmatrix} G_{i,n} \ldots G_{i,1} \ldots S_{i,n} \ldots S_{i,1} \ldots F_{i,n} \ldots F_{i,1}\end{bmatrix}\). We define the matrices

\[
\Xi_i = \begin{bmatrix} \xi_{i}^d & \xi_{i}^d & \ldots & \xi_{i}^d_{N-1}\end{bmatrix},
\Xi_i^\top = \begin{bmatrix} \xi_{i}^d_{N-1} \xi_{i}^d_{N-2} \ldots \xi_{i}^d\end{bmatrix},
\]

\[
U_i = \begin{bmatrix} u_{i,n}^d \ldots u_{i,1}^d \ldots u_{i,n}^d_{N-1}\end{bmatrix},
Y_{i} = \begin{bmatrix} y_{i,n}^d \ldots y_{i,n}^d_{N-1}\end{bmatrix},
\]

and \(Z_i = [\Xi_i^\top \ U_i^\top] \) with the extended state \(\xi_{i}^d\) based on the available input-output data \(D_i\). We further define \(M_i = \Xi_i^\top - \bar{A}_i \Xi_i - B_{w_i} U_i - B_{y_i} Y_{i}\) and the matrix

\[
P_{\Delta_i} = [x]^\top \begin{bmatrix} -Z_i Z_i^\top & Z_i M_i^\top B_{w_i}^\top
B_{w_i}^\top M_i Z_i & 0 & I
\end{bmatrix}.
\]

Lemma 5. (cf. (Berberich et al., 2021b, Proposition 10)). Factorise \(T_i \bar{Q}_i T_i^\top R_i = Q_i \Lambda_i Q_i^\top R_i^\top R_i^\top\). Suppose there exist \(X_1 \succ 0, \Gamma_i \succ 0, M_i, \tau_i \geq 0, \gamma_i > 0\) such that

\[
\text{trace}(\Gamma_i) < \gamma_i^2, \Gamma_i I X_1 \Gamma_i I X_1 \succ 0,
\]

and
where Algorithm 1 based on the updated neighbours’ trajectories $t$.

Define $P_{\Delta t} = X_t^{-1} - T_{\Delta t}^T \bar{Q} T_{\Delta t}$ and $K_t = M_t X_t^{-1}$. Then, there exists $\tilde{\eta}_t > 0$ such that

$$\|\bar{\xi}_t\|^2_{\Delta t} - \|\xi_t\|^2_{\Delta t} \leq -\tilde{\eta}_t \|\bar{\xi}_t\|^2_{\Delta t} - \|y_t\|_{\Delta t}^2 - \|\kappa_t(\bar{\xi}_t)\|^2_{\Delta t},$$

(13)

where $\bar{\xi}_t = (A_t + B_t \bar{K}_t) \xi_t$ and $\kappa_t(\xi_t) = K_t \xi_t$.

This result is a slight extension of (Berberich et al., 2021b, Proposition 10) and we refer to Berberich et al. (2021b) for the proof. The existence of the additional parameter $\tilde{\eta}_t$ is guaranteed by strictness of the LMIs in (Berberich et al., 2021b, Proposition 10). Note that (13) is similar to (10), except that no coupling is considered. Thus, the neglected dynamic coupling needs to be sufficiently weak for (13) to imply (10). Furthermore, if a bound on the neighbours’ outputs is known, they can be interpreted as bounded noise and it is straightforward to adapt the method in Berberich et al. (2021b) to design a robust feedback law, which would alleviate this issue. Feasibility of the conditions in Lemma 5 requires a potentially restrictive condition on the dimensions of subsystem (1), as further discussed in Berberich et al. (2021b).

### 3.4 Distributed data-driven MPC scheme

The following assumption bypasses the difficult task of constructing an initially feasible candidate in (9).

**Assumption 6.** At time $t = 0$, for all subsystems $i \in \mathcal{V}$, there exists $\hat{\alpha}(0)$ with corresponding $\hat{y}_{[-n,L-1]}(0)$ such that it is a feasible candidate in (9) for $y_{[-n+1,L]}(-1) = \hat{y}_{[-n,L-1]}(0)$.

In addition, each subsystem knows $\hat{\alpha}(0)$.

Since the prediction relies on communicated trajectories of the neighbours, the first communicated output trajectory is $\hat{y}_{[-n,L-1]}(0)$ from Assumption 6.

After solving (9) at $t = 1$, the predicted output trajectory of subsystem $i$, $y_{[0,L-1]}(t-1)$, is available for communication after a one-step extension as described momentarily. However, in general, it does not correspond to a feasible candidate at time $t$ because of (9b), which now depends on the updated $y_{[-n+1,L]}(t-1)$, whereas $y_{[0,L-1]}(t-1)$ was computed based on $y_{[-n+1,L]}(t-2)$. Instead, a feasible candidate at time $t \in \mathbb{N}$ can be constructed using

$$\hat{u}_{[-n,L-2]}^i(t) = u_{[-n+1,L-1]}^i(t-1),$$

$$\hat{u}_{[-L+1]}^i(t) = \kappa_i(\hat{\xi}_{[-L+1]}^i(t)),$$

(14a)

(14b)

where

$$\hat{\xi}_{[-L+1]}^i(t) = \begin{bmatrix} \hat{u}_{[-n+1,L-2]}^i(t) \\ y_{[-n+1,L-1]}^i(t-1) \\ y_{[-n,L-1]}^i(t-2) \end{bmatrix},$$

(15)

as will be shown momentarily. The corresponding candidate $\hat{\alpha}(t)$ and output trajectory $\hat{y}_{[-n,L-1]}(t-1)$ are computed with Algorithm 1 based on the updated neighbours’ trajectories $y_{[-n+1,L]}(t-1)$ and updated output measurement $y_{[-n,L-1]}(t-1)$.

### Algorithm 1. Data-driven simulation (subsystem $i$)

**Input:**

- Data $D_i$, where $[u_{d,[0,N-1]}^i, y_{[0,N-2]}^i]$ is persistently exciting of order $L + 2n$ and $(A_i, B_i)$ is controllable.

- Initial condition $\{u_k^i, y_k^i\}_{k=-n}^{-1}$

- New input and neighbours’ output data $\{u_k^i\}_{k=0}^{L-1}$, $\{y_k^i\}_{k=-n}^{L-2}$

**Procedure:**

1. Compute $\alpha^i$ satisfying

$$\begin{bmatrix} H_{L+n}[u_{[0,N-1]}^i] \\ H_{L+n-1}[y_{[0,N-2]}^i] \\ H_n[y_{[-n,L-1]}^i] \end{bmatrix} \alpha^i = \begin{bmatrix} u_{[-n,L-1]}^i \\ y_{[-n,L-2]}^i \\ y_{[-n-1]}^i \end{bmatrix}$$

(16)

2. Compute $y_{[0,L-1]}^i = H_L[y_{[-n,L-1]}^i] \alpha^i$.

**Output:** Resulting simulated output trajectory $\{y_k^i\}_{k=0}^{L-1}$.

The D3MPC scheme is stated in Algorithm 2. Note that each subsystem can solve its MPC problem in parallel in Step 1) of Algorithm 2. Hence, the complexity of the scheme increases only with the number of neighbours each subsystem has, but not with the total number of subsystems. In addition, since communication is necessary only once in each time step, the communication overhead is kept to a minimum. In Step 4) of Algorithm 2, we compute an extension of the optimal output trajectory by one step, which is denoted by $\hat{y}_{L}^i(t)$ with a slight abuse of notation.

### Algorithm 2. Data-driven distributed MPC scheme

**Input for all $i \in \mathcal{V}$:**

- Data $D_i$, where $[u_{d,[0,N-1]}^i, y_{[0,N-2]}^i]$ is persistently exciting of order $L + 2n$ and $(A_i, B_i)$ is controllable.

- Initial measurement $\{u_k^i, y_k^i\}_{k=-n}^{-1}$

- Initially communicated trajectories $y_{[-n+1,L]}(-1)$.

**Procedure:** For all $i \in \mathcal{V}$:

1. Solve the local MPC problem (9).

2. Apply $u_k = u_k^0(t)$.

3. Measure $y_k^i(t)$.

4. Compute $u_k^L(t) = \kappa_i(\xi_L^i(t))$. Then, compute $y_L^i(t)$ using Algorithm 1 with initial condition $u_{[-n,1]}^i(t)$, $y_{[-n+1,1]}^i(t-1)$, $y_{[-n-1]}^i(t)$ and new input and neighbours’ output trajectories $u_{[0,L]}^i(t), y_{[-n+1,L]}^i(t-1)$.

5. Transmit $y_{[-n+1,L]}^i(t)$.

6. Receive $y_{[-n+1,L]}^i(t)$ from neighbours $N_i$.

7. Set $t = t + 1$.

8. Compute $\hat{u}_{[-n,L-1]}^i(t)$ according to (14) as well as $\hat{y}_{[-n,L-1]}^i(t)$ using Algorithm 1 with initial condition $\hat{u}_{[-n,1]}^i(t), y_{[-n+1,0]}^i(t-1), y_{[-n-1]}^i(t)$ and new input and neighbours’ output trajectories $\hat{u}_{[0,L]}^i(t), y_{[-n+1,L]}^i(t-1)$, $y_{[-n,L-1]}^i(t-1)$. 


In this section, we prove that the origin is practically asymptotically stabilised if Algorithm 2 is applied to the system (7). An important requirement to this end is that in each time step each subsystem is able to solve the MPC optimisation problem (9), i.e. (9) is recursively feasible, which in particular implies that the input constraints are not violated.

4.1 Recursive feasibility

The following assumption is crucial to prove recursive feasibility of (9) and captures the central idea of the scheme. If all subsystems stay close to their communicated trajectory, the unexpected influence on neighbours is sufficiently bounded.

Assumption 7. There exist \( \sigma'_i, \hat{\sigma}_i \in \mathcal{K} \) such that \( \| \xi^{\ast}_i(t) - \xi^+_i(t) \|_{P_i} \leq \hat{\sigma}_i(\Omega_i) \) and \( \| y^+_i(t) - y^{\ast}_i(t) \|_{Q_i} \leq \sigma'_i(\Omega_i) \) for all \( t \in \mathbb{N}_0 \) and \( k \in [0, L-2] \) if Algorithm 2 is used, with \( \xi^+_i(t) \) as in (15) and where

\[
\xi^+_i(t) = \begin{bmatrix}
  u^+_i(t) \\
  y^+_i(t) \\
  y^{\ast}_i(t)
\end{bmatrix}.
\]

As stated in Dunbar (2007), it is to be expected that this requires sufficiently weak dynamic coupling. Although we conjecture that Assumption 7 can be shown to hold (cf. Dunbar, 2007, Lemma 3), it is not straightforward to compute \( \hat{\sigma}_i \) and \( \sigma'_i \) only based on the available input-output data. This is beyond the scope of this paper, yet we want to highlight two possibilities. It may be possible to extract this information from the Hankel matrix of each subsystem in Lemma 3, since it contains the dynamic coupling. Alternatively, one may be able to use bounds on the norms of \( A_i \) and \( B_i \), \( i \in \mathcal{N} \), in (1) to construct \( \sigma_i \) and \( \sigma'_i \). Using a similar approach as in Wildhagen et al. (2021), the latter can be estimated using input-output data.

We now show that \( u_{[0,L-1]}(t) \) from (14) indeed results in a feasible candidate in (9) and the MPC problem is recursively feasible.

Theorem 8. Let Assumptions 2–7 hold. In particular, from Assumption 6, let the MPC problem (9) be feasible for all \( i \in \mathcal{V} \) at time \( t = 0 \). Then, the MPC problem (9) is also feasible for all \( i \in \mathcal{V} \) and \( t \in \mathbb{N} \), if \( \sigma_i(\Omega_i) \leq (1 - \sqrt{\delta_i}) \sqrt{\epsilon_i} \) and \( \theta_i \geq 1 - \eta_i \lambda_{\max}(P_i)^{-1} \) for all \( i \in \mathcal{V} \) with \( \epsilon_i, \eta_i, \) and \( P_i \) from Assumption 4.

Proof. Let \( i \in \mathcal{V} \). From Assumption 6, a solution \( \alpha_i^{\ast}(t) \) with corresponding \( u^{\ast}_{[n,n+1,L-1]}(0) \) and \( y^{\ast}_{[n,n+1,L-1]}(t) \) to (9) exists. Assume for induction that (9) is feasible at time \( t \). Consider the candidate input trajectory \( u_{[n,n+1,L-1]}(t+1) \) from (14) together with the updated output data of the neighbours \( y^{\ast}_{[n,n+1,L]}(t) \) which, if applied in (1a), yields \( y^+_i(t) \). By Lemma 3, there exists a corresponding \( \alpha'_i(t+1) \) such that (9b) is satisfied. By definition of \( u^+_i(t+1) \) and \( y^{\ast}_i(t+1) \) the constraint (9c) holds. From feasibility of \( u^{\ast}_{[n,n+1,L-1]}(t) \) it follows that \( u_{[n,n+1,L-1]}(t+1) \) is \( \hat{U}_i \). Clearly (9d) and (9e) hold. It remains to be shown that (9d) for \( k = L - 1 \) and (9e) are satisfied. Feasibility of (9) at time \( t \) and Assumption 7 yield

\[
\| \hat{\xi}^+_i(t+1) \|_{P_i} \leq \| \xi^+_i(t) \|_{P_i} + \| \hat{\xi}^+_i(t+1) - \xi^+_i(t) \|_{P_i} \leq \sqrt{\theta_i \epsilon_i} + \hat{\sigma}_i(\Omega_i) \leq \sqrt{\epsilon_i},
\]

since \( \hat{\sigma}_i(\Omega_i) \leq (1 - \sqrt{\delta_i}) \sqrt{\epsilon_i} \) with \( \theta_i \in (0, 1) \). Hence, \( \hat{\xi}^+_i(t+1) \in \mathcal{X}_i^i(\epsilon_i) \) and by Assumption 4, (9d) is satisfied for \( k = L - 1 \) and for all \( i \in \mathcal{V} \). Then, again from Assumption 4,

\[
\| \hat{\xi}^+_i(t+1) \|_{P_i} \leq \| \hat{\xi}^+_i(t+1) - \xi^+_i(t+1) \|_{P_i} + \eta_i \| \hat{\xi}^+_i(t+1) \|_{P_i} \leq \theta_i \| \hat{\xi}^+_i(t+1) \|_{P_i} \leq \theta_i \epsilon_i.
\]

since \( \theta_i \geq 1 - \eta_i \lambda_{\max}(P_i)^{-1} \).

4.2 Practical stability

In order to show practical stability, we define

\[
V^*_i = \sum_{i=1}^{M} \sum_{k=0}^{L-1} \| y^+_i(t) \|_{Q_i}^2 + \| u^+_i(t) \|_{R_i}^2 + \hat{V}^*_i(\xi^+_i(t))
\]

and assume the following.

Assumption 9. There exists \( c_{ab} > 0 \) such that \( V^*_i \leq c_{ab} \| \xi^*_i \|^2 \) holds for all \( t \in \mathbb{N}_0 \).

This assumption can be shown to hold if, e.g. \( \mathcal{U}_i \) are compact polytopes for all \( i \in \mathcal{V} \) and the cost of the initially feasible candidate from Assumption 6 admits a quadratic upper bound.

The following theorem establishes practical stability of the origin of the global extended system (7). It shows a trade-off between giving each subsystem a larger margin of freedom, i.e. choosing \( \Omega_i \) larger, compared to a tight stabilisation of the origin, i.e. choosing \( \Omega_i \) smaller.

Theorem 10. Let Assumptions 2–7 hold. If the D3MPC scheme in Algorithm 2 is applied, then the origin is input-to-state stable (cf. Limon et al. (2006); Jiang and Wang (2001)) with respect to \( \Omega = \max_i \Omega_i \) for the resulting closed-loop system. That is, there exist \( \beta \in K\mathcal{L} \), \( \delta \in K \) and \( \Omega_{\max} \) such that for all \( \Omega \leq \Omega_{\max} \)

\[
\| \xi^*_i \| \leq \beta(\| \xi^*_0 \|, t) + \delta(\Omega)
\]

holds for the closed-loop solution of (7).

Proof. Since \( \alpha^+_i(t+1) \) with \( u^{\ast}_{[n,n+1,L-1]}(t+1) \) and \( y^{\ast}_{[n,n+1,L-1]}(t+1) \) is a feasible choice in (9) at time \( t+1 \) (see Theorem 8) and \( \hat{u}^+_i(t+1) = u^{\ast}_{k+1}(t) \) for \( k \in [0,L-2] \).
\[ V^*_{t+1} \leq \sum_{i=1}^{M} \left( \sum_{k=0}^{L-1} \| \hat{g}_k^i(t+1) \|_{Q_i} + \| \hat{a}_k^i(t+1) \|_{R_i} \right) \]
\[ + V^*_f(\xi^*_i(t+1)) \]
\[ = \sum_{i=1}^{M} \sum_{k=0}^{L-2} \| \hat{g}_k^i(t+1) \|_{Q_i}^2 - \| y^*_{k+1}^i(t) \|_{Q_i}^2 \]
\[ + \sum_{i=0}^{M} V^*_f(\xi^*_i(t+1)) - V^*_f(\xi^*_i(t)) \]
\[ - \left( \sum_{i=0}^{M} \| y^*_{k}^i \|_{Q_i}^2 + \| u^*_t \|_{R_i}^2 \right) + V^*. \]

Then, since \( \xi^*_i(t+1) \) for all \( i \in \mathcal{V} \) as shown in the proof of Theorem 8, and by definition of \( \hat{u}_{L-1}^i(t+1) \) in (14b), from Assumption 4 it follows that

\[ V^*_{t+1} - V^*_t \leq -\| y^*_t \|_{Q}^2 - \| u^*_t \|_{R}^2 \]
\[ + \sum_{i=0}^{M} \| y^*_{k}^i \|_{Q_i}^2 - \| y^*_{k+1}^i(t) \|_{Q_i}^2 \]
\[ + \sum_{i=0}^{M} V^*_f(\xi^*_i(t+1)) - V^*_f(\xi^*_i(t)) \]
\[ \leq -\| y^*_t \|_{Q}^2 - \| u^*_t \|_{R}^2 + \sum_{i=0}^{M} \frac{L-1}{2} \sigma'(\Omega_i) + 2\sqrt{\sigma'(\Omega_i)} \]
\[ + \sum_{i=0}^{M} \frac{\sigma(\Omega_i)}{\sqrt{\Omega_i}} + 2V^*_t \sqrt{\sigma'(\Omega_i)} \]
\[ + \sum_{i=0}^{M} \frac{(L-1)\sigma(\Omega_i)}{\sqrt{\Omega_i}}. \]

As in the proof of (Berberich et al., 2021b, Theorem 8), since \((A,C)\) is observable by Assumption 2, \((\tilde{A}, \tilde{C})\) is detectable and (7) admits an input-output-to-state stability Lyapunov function \( W(\xi) = \| \xi \|_{P_W}^2 \) satisfying

\[ W(\tilde{A}\xi + \tilde{B}u) - W(\xi) \leq -\frac{1}{2}\| \xi \|_{P_W}^2 + \sum_{i=0}^{M} \frac{L-1}{2} \sigma(\sigma'(\Omega_i)) \]
\[ + \frac{\sigma(\sigma'(\Omega_i))}{\sqrt{\Omega_i}} \]
\[ \leq -\| y \|_{Q}^2 - \| u \|_{R}^2 + \sum_{i=0}^{M} \frac{L-1}{2} \sigma(\sigma'(\Omega_i)) \]
\[ + \frac{\sigma(\sigma'(\Omega_i))}{\sqrt{\Omega_i}}. \]

We consider the same example as in Alonso et al. (2021): a system comprising a chain of 64 subsystems with dynamics

\[ x_{i+1} = \left[ \begin{array}{c} \frac{1}{m} \frac{\beta}{m} \frac{\beta}{m} - 0.2 \frac{\beta}{m} \\ 0.2 \frac{\beta}{m} - 0.2 \frac{\beta}{m} \end{array} \right] x_{i} + \left[ \begin{array}{c} 0 \end{array} \right] u_{i} + \sum_{j \in \mathcal{N}_i} \left[ \begin{array}{c} 0 \end{array} \right] y_{ij}, \]

\[ y_{ij} = \left[ \begin{array}{c} \frac{\beta}{m} \frac{\beta}{m} \frac{\beta}{m} \end{array} \right] x_{ij}. \]

The parameters are set to \( m_i = 1, d_i = 0.75, k_{ij} = 1.25 \) and \( k_i = \sum_{j \in \mathcal{N}_i} k_{ij} \) for all \( i \in \mathcal{V} \) and \( j \in \mathcal{N}_i \). We consider input-output data of each subsystem of length \( N = 100 \). We choose \( L = 5, Q_i = R_i = I, U_i = [-2, 2] \) and design terminal ingredients as discussed with \( c_i = 10^{-5} \) for all \( i \in \mathcal{V} \). For the consistency constraint, we choose \( \Omega_i = 0.01 \) for all \( i \in \mathcal{V} \). We use a suboptimal feasible trajectory as the initially feasible trajectory. The LMI.s in Lemma 5 were solved using YALMIP Löfberg (2004) and MOSEK MOSEK ApS (2020), whereas MOSEK was used for (9). We observe that the states converge to a neighbourhood of the origin if the proposed D3MPC scheme is applied, as expected from Theorem 10. To illustrate this, we plot the closed-loop state evolution of subsystems \( i \in [1,5] \) in Figure 1.
We have proposed a direct data-driven distributed MPC scheme for a group of dynamically coupled LTI systems. Each local MPC optimisation problem uses only past measured input-output data for the prediction, without any prior system identification step. We showed that if the dynamic coupling is sufficiently weak, the data-driven distributed MPC scheme is recursively feasible and practically stabilises the origin of the global system. The main mechanism are so-called consistency constraints, i.e. keeping close to a previously communicated trajectory, based on the model-based approach in Dunbar (2007).

This enables a non-iterative parallel distributed MPC scheme with minimal communication. The complexity of the scheme does not increase with the total number of subsystems, but only with the number of neighbours of each subsystem. Future research will investigate deriving suitable bounds for the consistency based on input-output data, as well as data-driven design of terminal ingredients that take the dynamic coupling into account.

6. CONCLUSION

We have proposed a direct data-driven distributed MPC scheme for a group of dynamically coupled LTI systems. Each local MPC optimisation problem uses only past measured input-output data for the prediction, without any prior system identification step. We showed that if the dynamic coupling is sufficiently weak, the data-driven distributed MPC scheme is recursively feasible and practically stabilises the origin of the global system. The main mechanism are so-called consistency constraints, i.e. keeping close to a previously communicated trajectory, based on the model-based approach in Dunbar (2007). This enables a non-iterative parallel distributed MPC scheme with minimal communication. The complexity of the scheme does not increase with the total number of subsystems, but only with the number of neighbours of each subsystem. Future research will investigate deriving suitable bounds for the consistency based on input-output data, as well as data-driven design of terminal ingredients that take the dynamic coupling into account.

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