Dynamical Properties of Gaussian Thermostats

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Abstract

In this work we show that the set of Kupka-Smale Gaussian thermostats on a compact manifold is generic. A Gaussian thermostat is Kupka-Smale if the closed orbits are hyperbolic and the heteroclinic intersections are transversal.

We also show a dichotomy between robust transitivity and existence of arbitrary number of attractors or repellers orbits. The main tools are the concept of transitions adapted to the conformally symplectic context and a perturbative theorem which is a version of the Franks lemma for Gaussian thermostats.

Finally we provide some conditions in terms of geometrical invariants for an invariant set of a Gaussian thermostat to have dominated splitting. From that we conclude some dynamical properties for the surface case.

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1 Introduction

Gaussian thermostats or isokinetic dynamics were introduced by Hoover [15] who considered a class of mechanical dynamical systems with forcing and thermostatting term based on the Gauss least Constraint Principle for nonholonomic constraints.

Let $(M, g)$ be a Riemannian manifold and let $E : M \rightarrow TM$ be a vector field in $M$. We call Gaussian thermostat the flow $\phi : TM \rightarrow TM$ in which an orbit $\eta(t) = (x(t), v(t))$ satisfies the equation

$$\begin{cases}
\dot{x} = v \\
\nabla_v v = E - \frac{g(E,v)}{g(v,v)}v
\end{cases}$$

where $\nabla$ is the Riemannian connection of $(M, g)$.

The thermostat is the parcel $-\frac{g(E,v)}{g(v,v)}v$ of the second equation. It subtracts the component of $E$ which is not parallel to the orbit tangent vector $v$, changing only the direction of $v$ and keeping its modulus constant.

Gaussian thermostats have been proposed as models for systems out of equilibrium in statistical mechanics as it is discussed in the papers by Gallavotti and Ruelle [1], [2], [3]. Gauss least Constraint Principle is particularly useful in describing the motion of constrained systems. No fundamental rules or variational principles are available for such constraints and it is not true that the work performed by the constraints should be a minimum. Such flows require special methods to compensate for the natural dissipation of work into heat. For simplicity, it is convenient to remove the heat in such a way that the nonequilibrium state is a steady one. By steady we mean some state variables are held constant and for the Gaussian thermostat we maintain the kinetic energy constant.

As an example of Gaussian thermostat provided in [30], let $N$ be a Riemannian manifold and consider $M = S^1 \times N$ equipped with the product metric. Let $E$ a vector field tangent to $S^1$ with constant modulus $|E|$. For $v \in SM$, write $v = v_0 + v_1$ the decomposition of $v$ in a component parallel to $S^1$ and $N$, respectively. The equation of the Gaussian thermostat on the component $v_0$ is $v_0 = |E|(1 - v_0^2)$ and the orbits of the Gaussian thermostat are parallel lines tangent to $S^1$ or have this component equal as $v_0(t) = \tanh(|E|t + \lambda_0)$. This Gaussian thermostat

![Figure 1: Typical orbit over $S^1 \times N$.](image-url)
has a global attractor $A$ and a global repeller $R$, both normally hyperbolic. They are given by $A = \{ \xi = \xi_0 + \xi_1 \in SM | \xi_0 = E \}$ and $R = \{ \xi = \xi_0 + \xi_1 \in SM | \xi_0 = -E \}$.

If now we consider $N = S^1$ and change the vector field $E$ to make an irrational angle with $N$ then we still have two invariant sets, both normally hyperbolic and the dynamics on each one is conjugated to a linear irrational flow on $S^1 \times S^1$. We call each torus with these properties an irrational torus.

The Hamiltonian formulation of the Gaussian thermostat was obtained by Dettmann and Morris [12]. This also connects us with the conformally symplectic dynamics [18]. A Gaussian Thermostat can be an example of a conformally Hamiltonian system. These systems are more general than Hamiltonian and symplectic systems but are closely related to them. They are determined by a non-degenerate 2-form $\Omega$ on the phase space and a function $H$, called again a Hamiltonian. The form $\Omega$ is not assumed to be closed but $d\Omega = \gamma \wedge \Omega$ for some closed 1-form $\gamma$. This condition guarantees, at least locally, the form $\Omega$ can be multiplied by a nonzero function to give a symplectic structure. The skew-ortogonality of tangent vectors is preserved under multiplication of the form by any nonzero function, hence the name conformally symplectic structure.

A geometrical perspective for Gaussian thermostats was introduced by Wojtkowski. He shows that the Gaussian thermostat coincide with W-flow, a modification of the geodesic flow on a Weyl manifold and, with this approach, he establish some connections between negative curvature of the Weyl structure and the hyperbolicity of W-flows [30]. Latter, in [23] he proposed that Gaussian thermostats are geodesic flows of a special metric connections which is non-symmetric but has isometric parallel transport.

In the present work we approach Gaussian thermostats from a dynamical system point of view and we discuss some of their generic properties as Kupka-Smale; after that, we propose dynamical dichotomies adapting to the Gaussian thermostats context, the results in [8]. Moreover, extending results in [30] we provides sufficient conditions for weak form of hyperbolicity based on the curvature.

For the first part of this work we restrict ourselves to a subset of the external forces $E$. More precisely, we fixed a Riemmanian metric $g$ and we consider perturbations only by the action of an external force. Following that, we define $\mathcal{X}_g(M) \subset \mathcal{X}^\infty(M)$ as the set of vector fields $E$ such that the 1-form $\gamma(.) = g(E,.)$ is closed.

We say that a Gaussian thermostat $(M, g, E)$ on a compact manifold $M$ is Kupka-Smale if it satisfies:

(i) all closed orbits are hyperbolic,
(ii) all heteroclinic intersections are transversal.

And we also say a property $P$ is generic in $\mathcal{X}_g(M)$ if there exists a residual subsets in $\mathcal{X}_g(M)$ which satisfies $P$.

The first result in this work is

**Theorem A.** The set of vector fields in $\mathcal{X}_g(M)$ that induce Kupka-Smale Gaussian thermostats is generic in $\mathcal{X}_g(M)$.

We also prove a dichotomy between robust transitivity and the existence of an arbitrary number of attractor or repeller orbits for the Gaussian thermostat.

Given a flow $\phi$ and a hyperbolic periodic saddle orbit $\eta$, we define the homoclinic class $H(\eta, \phi)$ as the closure of the set of transversal intersection of the unstable and stable manifolds of $\eta$.

$$H(\eta, \phi) = \overline{W^u_\phi(\eta) \cap W^s_\phi(\eta)}.$$
Let $M$ be a differential manifold with $\dim(M) = 2n + 1$, let $F : M \to TM$ be a vector field such that $F(x) \neq 0$ for all $x \in M$ and $\phi$ the associated flow. We denote $\hat{T}M$ the quotient of the tangent bundle $TM$ by the vector field $F$. Let $\pi : TM \to T\hat{M}$ be the canonical projection of $v \in T_x\hat{M}$ in $\hat{T}_xM$. The composition $A^\varepsilon_x = \pi \circ d\phi^\varepsilon_x : \hat{T}_xM \to \hat{T}_{\phi^\varepsilon(x)}M$ defines the transversal derivative cocycle associated to $\phi$ or the linear Poincaré application. We say that an invariant set $\Lambda$ has dominated splitting (on the linear Poincaré application) if $\hat{T}_\theta M = N^c_\theta \oplus N^u_\theta$ is an invariant decomposition of the transversal derivative cocycle $\hat{T}^t$ and there exists constants $\lambda, C > 0$ such that for $t > 0$ and for all $\theta \in \Lambda$

$$||T^t|_{N^c_\theta}|| \leq Ce^{-\lambda t}m(T^t|_{N^u_\theta})$$

where $m(L) = \min\{|Lv| : ||v|| = 1\}$.

We say that an invariant set $\Lambda$ is hyperbolic, if the linear Poincaré flow has a decomposition $\hat{T}_\theta M = N^s_\theta \oplus N^u_\theta$ and any vector in $N^s$ is uniformly contracted by forward iterates and any vector in $N^u$ is uniformly contracted for backward iterates.

In both examples described before, the attracting and repelling normally hyperbolic torus exhibits a dominated splitting and they are not hyperbolic.

In the next theorem we proved a dichotomy in the $C^1$-category for homoclinic classes:

**Theorem B.** Let $E \in \mathcal{X}_g(M)$ and $\eta$ be a hyperbolic saddle periodic orbit for the Gaussian thermostat $(M, g, E)$ which flow is $\phi$.

One of the alternatives is true:

(i) the homoclinic class $H(\eta, \phi)$ has a dominated splitting,

(ii) given a small neighborhood $V$ of $H(\eta, \phi) \subset V$, $k \in \mathbb{N}$, and $\varepsilon > 0$, exists $\tilde{E} \in \mathcal{X}_g(M)$ with $d(\tilde{E}, E)_C < \varepsilon$ such that the Gaussian thermostat $(M, g, \tilde{E})$ has $k$ attractor or repeller periodic orbits in $V$.

For the case of surfaces, it is possible to get the following theorem which is a local version of theorem 5.2 in [30].

**Theorem B’.** Let $M$ be a Riemannian surface. Let $E \in \mathcal{X}_g(M)$ and $\eta$ be a hyperbolic saddle periodic orbit for the Gaussian thermostat $(M, g, E)$ which flow is $\phi$.

One of the alternatives is true:

(i) generically, the homoclinic class $H(\eta, \phi)$ is hyperbolic,

(ii) given a small neighborhood $V$ of $H(\eta, \phi) \subset V$, $k \in \mathbb{N}$, and $\varepsilon > 0$, exists $\tilde{E} \in \mathcal{X}_g(M)$ with $d(\tilde{E}, E)_C < \varepsilon$ such that the Gaussian thermostat $(M, g, \tilde{E})$ has $k$ attractor or repeller periodic orbits in $V$.

The main tools for the proof of Theorem B are the concept of transitions adapted to the conformally symplectic context and the perturbative theorem, stated hereafter.

**Theorem C.** Given $E \in \mathcal{X}_g(M) \cap \mathcal{X}_T(M)$, $4 \leq r \leq \infty$, $\theta$ a point in a periodic orbit $\eta$ of a Gaussian thermostat $(M, g, E)$, and $T : \hat{T}_\theta SM \to \hat{T}_\theta SM$ the transversal derivative cocycle along $\eta$.

Given $\varepsilon > 0$ there exists $\delta > 0$ such that for any conformally symplectic cocycle $L$ such that $||L - T|| < \delta$ then there exists $\tilde{E} \in \mathcal{X}_g(M)$ with $d(\tilde{E}, E)_C < \varepsilon$ which defines a Gaussian thermostat $(M, g, \tilde{E})$ such that $\eta$ is an orbit of it and the transversal derivative cocycle associated to $\theta$ is $L$. 

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In [14], Franks proved that given a $C^1$ diffeomorphism $f : M \to M$ over a Riemannian manifold $(M,g)$ and $\varepsilon > 0$, if we take a periodic point $x \in M$, we can perform a $C^1$ small perturbation $g$ of $f$ such that $g^n(x) = f^n(x)$, $n \in \mathbb{Z}$, and $dg^n$ is any isomorphism $\varepsilon$-close of $df^n$, for $n \in \mathbb{Z}$. This result is known as Franks lemma. The key point in that lemma is since is only required that $g$ is $C^1$-close to $f$, the support of the perturbation can be done arbitrary small in such a way that the perturbation preserves the trajectory.

In our context, a perturbation on the Gaussian thermostat is a perturbation on the vector field $E$. This means that perturbations are not local, in fact, a perturbation on the vector field implies a perturbation on a cylinder on the tangent space where the Gaussian thermostat flow is defined. Therefore, even the support of the perturbation of the vector field $E$ can be done arbitrary small, this produces a perturbation on the Gaussian thermostat that is not localized.

For geodesic flows, a similar problem appears, since geodesic flow perturbations are metric perturbations and these perturbations are not local as was described above. In fact, to perturb a metric on a neighborhood of a closed geodesic means that it is performed a perturbation in a cylinder in the tangent space where the geodesic flow is defined. However, and overcoming such difficulty, in [10] Contreras proved a version of the Franks lemma for the geodesic flow. A similar version for the particular case of Magnetic flows have been proved in [21].

As we said, for a Gaussian thermostat the situation is the same as the geodesic flow. In this case, we keep the metric unperturbed and we perturb the vector field $E$, and this implies a perturbation over a cylinder in the tangent space where the flow is defined.

To perform Franks lemma for Gaussian thermostats we adapt the ideas of Contreras in [10] keeping in mind that the metric is untouched and any perturbation is done on the external vector field.

We complete this work relating some geometrical properties with dynamical ones for the Gaussian thermostat in the following theorems.

**Theorem D.** Let $\Lambda \subset SM$ be an invariant subset of a $C^2$ Kupka-Smale Gaussian thermostat flow with non-positive sectional curvature of the Weyl structure and for any $\theta = (x,v) \in \Lambda$ we have $\gamma(v) > 0$. Then $\Lambda$ has dominated splitting.

Using the results in [6] for non-singular three dimensional flows which are based on the theorems in [24], we can conclude the following.

**Theorem D'.** Let $M$ be a 2-dimensional manifold and $\Lambda \subset SM$ be an invariant subset of a $C^2$ Kupka-Smale Gaussian thermostat flow with non-positive curvature of the Weyl structure and for any $\theta = (x,v) \in \Lambda$ we have $\gamma(v) > 0$. Then $\Lambda = \hat{\Lambda} \cup T$, where $\hat{\Lambda}$ is hyperbolic and $T$ is a finite union of normally hyperbolic irrational tori.

We want to point out that for Gaussian thermostat both type of sets listed in the conclusion of theorem D’ can exist. In the beginning of the present section we show that there are examples of normally hyperbolic invariant torus. Also, given a metric with negative curvature (and so the geodesic flow is Anosov) it is possible to chose an external field $E$ such that the Gaussian thermostat is a Derived of Anosov and in particular has an attracting hyperbolic set.

We start our work in the next section in which we make an introduction to conformally symplectic dynamics and its properties. Then, we show the conformally Hamiltonian approach to the Gaussian thermostats. We also construct some perturbations used along the work and good coordinates to work with. The remaining is dedicated to the proofs of the theorems. The first one, which says the set of vector fields in $\mathcal{X}_g(M)$ which defines Kupka-Smale Gaussian thermostats is generic in $\mathcal{X}_g(M)$. The second corresponds to the dichotomy between dominated splitting and the existence of attractors/repellers on the neighborhood of homoclinical classes for Gaussian thermostats. Then we prove the third theorem, the perturbative theorem, which
was used as a tool for the proof of the previous one. Finally, in the last section we prove the relationship between geometrical and dynamical properties.

2 Conformally symplectic dynamics

In this section, we introduce the conformally symplectic dynamics. This systems are more general than the symplectic ones and deeply related to these. A conformally symplectic system is determined by a manifold $M$ and a 2-form $\Omega$ over $M$, non degenerated and which satisfies $d\Omega = \gamma \wedge \Omega$ with $\gamma$ a closed 1-form. The closeness of $\gamma$ guarantees that, locally, $\Omega$ can be multiplied by a non null function to obtain a symplectic structure.

Let $\Omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$ the canonical symplectic 2-form in $\mathbb{R}^{2n}$.

**Proposition 1 (18).** Given a invertible linear application $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, the following are equivalent:

1. $\Omega_0(Su, Sv) = \mu \Omega_0(u, v)$ for a scalar $\mu > 0$ and $u, v \in \mathbb{R}^{2n}$,
2. $\Omega_0(Su, Sv) = 0$ if and only if $\Omega_0(u, v) = 0$,
3. Lagrangian subspaces are invariant by $S$.

**Definition 1.** The set of matrices that satisfy the properties of the proposition form a subgroup of $GL(\mathbb{R}^{2n})$, it is called a conformally symplectic group and is denoted by $CS(\mathbb{R}^{2n})$.

**Definition 2.** Given $\mathcal{B} = \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ a basis for $\mathbb{R}^{2n}$. We say that $\mathcal{B}$ a canonical conformally symplectic basis if $\Omega(e_i, e_j) = \delta_{i,\mod(n)}$.

This basis coincides with the symplectic basis for $\mathbb{R}^{2n}$ and, given a conformally symplectic application, is always possible to find a conformally symplectic basis associated to this application.

**Proposition 2.** If $\lambda$ is an eigenvalue of $S \in CS(\mathbb{R}^{2n})$ then there exists $\mu \in \mathbb{R}$ such that $\frac{\mu}{\lambda}$ also is a eigenvalue of $S$.

Let $X$ be a metric space and $f : X \to X$ be a continuous application. A continuous application $A : X \to CS(\mathbb{R}^{2n})$ is a a conformally symplectic cocycle if $x$ is a periodic trajectory of $f$ (i.e. $f^k(x) = x$) then the matrix $A_k = \prod_{i=0}^{k-1} A f^i(x)$ verifies proposition. Furthermore, if all the eigenvalues has modulus different than one we say that $A_k$ is hyperbolic and the periodic trajectory is hyperbolic. The next results extends proposition to the Lyapunov exponent for conformally symplectic cocycle. Let $X$ be a measurable space with probability measure $m$ and $f : X \to X$ be an ergodic application. Let $A : X \to CS(\mathbb{R}^{2n})$ be a measurable application and consider the associated cocycle which is called a measurable conformally symplectic cocycle. From Oselecc's theorem,

**Theorem 1 (18).** If a measurable conformally symplectic cocycle $A(x), x \in X$, satisfies the integrability condition, i.e., $\int_X \log_+ \|A(x)\| \, dm(x) < +\infty$, then the Lyapunov exponents $\lambda_1 < \ldots < \lambda_s$ and the flag

$\{0\} = V_0 \subset \ldots \subset V_{s-1} \subset V_s = \mathbb{R}^{2n}$

flag are well defined. Moreover:

1. $\lambda_k + \lambda_{s-k+1} = b$ with $b = \int_X \log |det A(x)| \, dm(x)$;
2. the multiplicity of $\lambda_k$ and $\lambda_{s-k+1}$ are equal for $k = 1, 2, \ldots, s$;
3. $V_{s-k}$ is the orthogonal complement of $V_s$ with respect to $\Omega_0$. 

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Conformally symplectic cocycles are going to be redefined and revisited in section 5.1 in a more general frameworks. A particular case of a conformally derivative cocycle is the transversal derivative cocycle associated to the flow of Gaussian thermostat.

Let \( \Omega \) a 2-form defined on the quotient \( \hat{TM}, \Omega_x : \hat{TM} \times \hat{TM} \to \mathbb{R} \), with \( x \in M \) such that

(i) \( \Omega \) is non degenerated,

(ii) there exists a closed 1-form \( \gamma \) defined on the quotient \( \hat{TM}, \gamma_x : \hat{TM} \to \mathbb{R} \), such that \( d\Omega = \gamma \wedge \Omega \),

(iii) there exists \( \beta : M \to \mathbb{R} \) such that

\[
(\pi \circ \phi_t^x)^*\Omega = e^{\int_0^t \beta(\phi_s^x)ds} \Omega, \quad \forall x \in M, \quad t \in \mathbb{R}.
\]

The pair \((M, \Omega)\) is called a conformally symplectic manifold.

**Definition 3.** Let \((M, \Omega)\) be a conformally symplectic manifold. A \( C^\infty \) function \( H : M \to \mathbb{R} \) is called a Hamiltonian and the conformally Hamiltonian vector field is the uniquely defined vector field \( F \) provided by the relation \( \Omega(., F) = dH \).

Restricted to \( \hat{TM} \), the 2-form \( \Omega \) is conformally symplectic with \( \beta(t) = \gamma(F(\phi_t^x)) \) and the Hamiltonian \( H \) is a first integral for the flow.

Given an orbit segment \( \eta \) of the conformally symplectic flow \( \phi \) then there exist local coordinates such that the linear cocycle satisfies

\[
(A^t)^*JA^t = e^{\int_0^t \beta(\phi_s^x)ds} J,
\]

where \( J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \) is an \( 2n \times 2n \) matrix and \( I \) the \( n \times n \) identity. This cocycle has symmetry of the Lyapunov spectrum.

A matrix \( Y \) is called *infinitesimally conformally symplectic* if

\[
Y^*J + JY = vJ.
\]

This matrices can be written as

\[
Y = \begin{pmatrix} \beta & \gamma \\ \alpha & vI - \beta^* \end{pmatrix}
\]

with \( \alpha \) and \( \gamma \) symmetric. The tangent space of a matrix \( X \in CS(\mathbb{R}^{2n}) \) at the energy level \( v \) satisfies

\[
T^v_XCS(\mathbb{R}^n) = XT_1CS(\mathbb{R}^n)
\]

with \( T_1CS(\mathbb{R}^n) \) the tangent space of the identity.

Given a Gaussian thermostat \((M, g, E)\), a \( \varepsilon \)-perturbation of \((M, g, E)\) is a \( \varepsilon \)-perturbation of \( E \), i.e., it is the Gaussian thermostat defined by \((M, g, \tilde{E})\) with \( d(\tilde{E}, E) < \varepsilon \) in the adequate topology.

### 3 The Gaussian thermostat as a conformally symplectic Hamiltonian flow

Given \((M, g, E)\) a Gaussian thermostat restricted to the energy level \( c \). The conformally Hamiltonian structure of this Gaussian thermostat is defined as follows.

Let
(i) The $C^\infty$ function $H : TM \to \mathbb{R}$, the Hamiltonian, and defined by $H(x,v) = \frac{1}{2}g(v,v)$.
(ii) The tautological 1-form in $TM$ induced by $g$: $\kappa = h^{-1}(\kappa^*)$.
(iii) The canonical 2-form in $TM$ induced by $g$: $\omega = -d\kappa$.
(iv) The 1-form $\gamma$ defined in $TM$ by $\gamma = g(E,\cdot)$.
(v) The non-degenerated 2-form $\Omega = \omega - \frac{1}{4}\gamma \wedge \kappa$.

Then, restricted to an energy level $c$, the conformally Hamiltonian flow $(M,H,\Omega)$ associated to the vector field $E \in \mathcal{X}_g(M)$ coincides with the Gaussian thermostat $(M,g,E)$.

If $E \in \mathcal{X}_g(M)$ then $\Omega$ is conformally symplectic and the conformally symplectic structure gives us the following results.

**Definition 4.** A closed orbit $\eta$ is called a prime orbit if it is not an iterate of a closed orbit of a smaller period.

**Proposition 3.** Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R}$. Given $E \in \mathcal{X}_g(M)$ and the Gaussian thermostat $(M,g,E)$ with conformally Hamiltonian structure $(M,H,\Omega)$ such that $\gamma = \langle E,\cdot \rangle$ and $\Omega = \omega - \gamma \wedge \kappa$. Suppose the Gaussian thermostat has a prime closed orbit $\eta$ and the eigenvalues of the linear Poincaré application in $p \in \eta$ satisfies $\lambda_i \lambda_{i+n} = e^{\beta}$. Then it is possible to realize a surgery to obtain a Gaussian thermostat $(M,g,\tilde{E})$ such that $\|\tilde{E} - E\|_{C^0} < \frac{\alpha}{\text{period}(\eta)} + \varepsilon$, the Gaussian thermostat $(M,g,\tilde{E})$ has $\eta$ as an orbit and the eigenvalues of the linear Poincaré application in $p \in \eta$ satisfies $\lambda_i \lambda_{i+n} = e^{\beta + \alpha}$.

**Proof.** Consider $\mu$ Dirac function supported in $\eta$. By the theorem $\boxdot$,

$$\beta = \int_M \gamma(E(x)) d\mu(x)$$

which $\beta = \lambda_1 \lambda_{n+1} = \lambda_2 \lambda_{n+2} = \cdots = \lambda_n \lambda_{2n}$. Write $E = E_0 \oplus E_1$ where $E_0$ is the component of $E$ parallel to $\dot{\eta}$ and $E_1$ is the orthogonal component.

Let

- $W \subset M$ be a tubular neighborhood of $c = \pi \circ \eta$,
- $\tau > 0$ such that $m\tau = \text{period}(\eta)$ with $m \in \mathbb{N}$ and $\tau < r_{\text{inj}}$ with $r_{\text{inj}}$ is the injectivity radius of $M$. Consider, for $0 \leq k < m$, $\eta_k(t) = \eta((t+k)\tau)$ with $t \in [0,1]$. We call $c_k$ the projection of $\eta_k$ in $M$, $c_k = \pi \circ \eta_k$.

The segments $c_k$ can intersect transversally. Suppose for $c_0$ the set of intersection points of $c_k$ with $c_0$ is $\mathcal{F}_0 = \{p_1, \ldots, p_l\}$. Note that $l < m$.

Consider the sequence $t_0 = 0, \ldots, t_i, \ldots, t_{l+1} = 1$ where $t_i$ is such that $c(t_i) = p_i$ for each $p_i \in \mathcal{F}_0$. Let $h_i : [0,1] \to \mathbb{R}$ bump functions satisfying $\int_{t_i}^{t_{i+1}} h_i(s) ds = 1$ and $h_i(t_i) = h_i(t_{i+1}) = 0$ for $i = 0, \ldots, l$. Also consider $V \subset M$ a neighborhood of $c_0$ and a local chart $\phi(t,x)$ of $M$ such that $\phi(t,0) = c_0$. Finally, let $g_i : V_i \to \mathbb{R}$ a bump function such that $\text{supp}(g_i) \subset V$ and $g_i(t,0) = 1$.

We locally change de vector field $E$ as $\tilde{E}(\phi(t,x)) = \frac{\alpha}{m(t+1)} h_i(t) g_i(t,x) \frac{E_0(\phi(t,x))}{\|E_0\|} + E(\phi(t,x))$ and repeat this procedure to $c_k$, $k = 1, \ldots, m$. Consider the conformally Hamiltonian flow defined by $(M,H,\tilde{\omega})$ such that $\tilde{\gamma} = \langle E,\cdot \rangle$ and $\tilde{\Omega} = \omega + \tilde{\gamma} \wedge \kappa$. 

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Thus η is a periodic orbit associated to the flow \( \tilde{\phi} \) and, moreover,

\[
\tilde{b} = \int_{M^e} \tilde{\gamma}(p(x))d\mu(x) = \int_{M^e} \langle \tilde{E}(\pi(x)), F(x) \rangle d\mu(x) = \sum_{k=1}^{m} \int_{0}^{\tau} \langle \tilde{E}(\pi(t)), F(\pi(\eta(t))) \rangle dt = \beta + \alpha
\]

Proposition 4. Fix \( \varepsilon > 0 \). Let \( E \in \mathcal{Z}_g(M) \) and the Gaussian thermostat \( (M, g, E) \) with conformally symplectic Hamiltonian structure \((M, H, \Omega)\) such that \( \gamma = \langle E, \cdot \rangle \) and \( \Omega = \omega - \gamma \wedge \kappa \). Suppose that the Gaussian thermostat has a closed orbit \( \eta \) with an eigenvalue of the linear Poincaré application in \( p \in \eta \) satisfying

\[
\lambda_i \lambda_{i+n} = e^\beta.
\]

Then there exists a surgery \((M, g, \tilde{E})\) of \((M, g, E)\) such that \( \| \tilde{E} - E \|_{C^\infty} < \varepsilon \), the Gaussian thermostat \((M, g, \tilde{E})\) has \( \eta \) as an orbit and, furthermore, the eigenvalue of the linear Poincaré application in \( p \in \eta \) satisfies

\[
\lambda_i \lambda_{i+n} > e^\beta.
\]

Proof. It is enough to consider the last proof with \( h \) of class \( C^\infty \) such that \( h(p_i) = 0 \), \( h(x) \neq 0 \) for \( x \in [0, \tau] \setminus \{ p_i \}_{i=1,\ldots,t} \) and \( \| h \|_{C^\infty} < \varepsilon \).

Now, we introduce local coordinates useful to this work.

3.1 Fermi coordinates and Jacobi equation

Given \((M, g)\) a Riemannian manifold of dimension \( n + 1 \) and \( E \in \mathcal{Z}^r(M) \), we denote

(i) \( \pi : SM \to M \) the unitary bundle,
(ii) \( \phi^t : SM \to SM \) the Gaussian thermostat flow \((M, g, E)\),
(iii) \( \exp_{\theta} \) the exponential application in \( v \in T_{\theta}SM \) at \( \theta \in SM \).

Let \( \eta : [0, \tau] \to SM \) be an orbit segment and consider \( c(t) = \pi \circ \eta(t) \) the projection of \( \eta \) in \( M \). We assume \( \tau < r_{inj} \), where \( r_{inj} \) is the injectivity radius of \((M, g)\). Let \( \{ e_0 = c'(0), e_1, \ldots, e_{n-1} \} \) be a basis of \( T_{c(0)}M \) and consider \( e_i(t) \), \( i = 1, \ldots, n - 1 \), the parallel transport of \( e_i \) along \( c \) with respect to the Riemannian connexion \( \nabla \).

Consider \( \psi : [0, \tau] \times \mathbb{R}^n \to M \) given by

\[
\psi(t, x) = \exp_{c(t)} \sum_{i=1}^{n-1} x_i e_i(t)
\]

This application is a diffeomorphism on a neighborhood \( V \subset [0, \tau] \times \{0, \ldots, 0\} \) then it defines a local coordinated system in a neighborhood of \( \eta \). Furthermore, \( c(t) = \exp_{c(t)} 0 = \psi(t, 0, \ldots, 0) \) and considering the canonical parametrization \( \tilde{\psi} \) de \( TM \), we have

\[
\eta(t) = (c(t), \dot{c}(t)) = \tilde{\psi}( t, 0, \ldots, 0, 1, 0, \ldots, 0).
\]

Using the identity \( \frac{d}{dt}(d\phi_t) = (dX \circ \phi_t) \, d\phi_t \) with \( X = \frac{d}{dt} \phi_t \), we obtain the differential equation to the linearization of the conformally Hamiltonian flow over the orbit \( \eta(t) \), which we call Jacobi equation in \( TM \) and are summarize in the next theorem:
Theorem 2. Let $\eta : (-\varepsilon, \varepsilon) \rightarrow SM$ a Gaussian thermostat orbit segment. Then the linear Poincaré cocycle $\hat{T}$ along $\eta$ satisfies the Jacobi equation which is written in Fermi coordinates as

$$\frac{d}{dt} \hat{\mathcal{T}} \bigg|_{(t,x=0)} = \left\{ \begin{bmatrix} 0 & I \\ \hat{K} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \hat{B} & \hat{C} \end{bmatrix} \right\} \hat{T},$$

where

$$\hat{K} = (-K_{ij})_{ij} \quad i,j \in \{1, \ldots, n\},$$

$$\hat{B} = \left( \frac{\partial E_i}{\partial x_j} \right)_{ij} \quad i,j \in \{1, \ldots, n\},$$

$$\hat{C} = -\frac{E_0}{c}I = \sigma I \text{ with } I \text{ the } n \times n \text{ identity.}$$

4 Proof of theorem A

In this section, we show that the set of vector fields which define a Kupka-Smale Gaussian thermostat is generic in $\mathcal{X}_g(M)$. Before rigorously stating the theorem, we need the following definitions.

Definition 5. We say a Gaussian thermostat $(M,g,E)$ over a compact manifold $M$ is Kupka-Smale if it satisfies:

(i) The closed orbits are hyperbolic,

(ii) The heteroclinic intersections are transversal.

Theorem A. The Kupka-Smale property is generic in $\mathcal{X}_g(M)$.

The vector field $E$ defines the 1-form $\gamma_E(\cdot) = \langle E, \cdot \rangle$. Consider $\text{Per}(E)$ as the set of periodic orbits of $(M,g,E)$ and $\eta \in \text{Per}(E)$ a prime periodic orbit with period $L$. We define the application $\beta_E : \text{Per}(E) \rightarrow \mathbb{R}$ by $\beta_E(\eta) = \int_0^L \gamma_E(\dot{\eta}(s))ds$. We can also denote $\beta_E(\theta)$ by $\beta_E(\eta)$ where $\theta$ is a point of $\eta$. We split the main theorem in two parts, i.e., in two lemmas:

Definition 6. The subset $G_1 \subset \mathcal{X}_g(M)$ of vector fields which defines Gaussian thermostats such that if $\eta$ is a periodic orbit then

(i) the transversal cocycle associated is hyperbolic,

(ii) $\beta_E(\eta) \neq 0$.

Lemma 1. The subset $G_1 \subset \mathcal{X}_g(M)$ is residual in $\mathcal{X}_g(M)$.

Definition 7. Let $G_2 \subset G_1$ such that if $\eta_i, \eta_j \in \text{Per}(E)$ and $W^u(\eta_i) \cap W^s(\eta_j) \neq \emptyset$ then this intersection is transversal.

Lemma 2. The set $G_2$ is residual in $\mathcal{X}_g(M)$.

Fixed a coordinate system along the orbit $\eta$ such that for $\tau > 0$ the segment $\eta([0,\tau])$ is contained on the image of this coordinated system, we consider the application $\psi : [0,\tau] \times \mathbb{R}^n \rightarrow \mathbb{R}$ of class $C^\infty$ such that $\int_0^\tau \psi(t,0)dt = 1$ and $\psi$ has support on a neighborhood of $[0,\tau] \times \{0\}$ contained in the image of that coordinate system. Furthermore we consider, $\pi_V(\theta) : T_\theta SM \rightarrow H(\theta)$ the projection on the vertical space, and $\pi_H(\theta) : T_\theta SM \rightarrow V(\theta)$ the projection on the horizontal space.
4.1 Auxiliary results

We state a transversality theorem due to Abraham. Let $\mathcal{A}$ be a Baire topological space, $M$ and $N$ manifolds satisfying the second axiom of enumerability and with finite dimension, $K \subset M$ a subset, $V \subset N$ a submanifold and an application

$$F : \mathcal{A} \rightarrow C^1(M, N)$$
$$a \mapsto F_a.$$

If the map

$$ev_F : \mathcal{A} \times TM \rightarrow TN$$
$$(a, v) \mapsto DF_a v$$

is continuous, we call $F$ a $C^1$-pseudorepresentation.

If $F$ is a $C^1$-pseudorepresentation and there exists a dense subset $D \subset \mathcal{A}$ such that for $a \in D$ there exists an open set $B_a$ in a separable Banach space, $\psi_a : B_a \rightarrow \mathcal{A}$ continuous and $a' \in B_a$ such that

(i) $\psi_a(a') = a$,

(ii) $ev_F\psi_a : B_a \times M \rightarrow N$, is $C^r$ transversal to $V$ in $a' \times K$,

then we say $F$ is $C^r$-pseudotransversal to $V$ in $K$.

**Theorem 3** (Abraham’s Transversality theorem). Suppose $F : \mathcal{A} \rightarrow C^1(M, N)$ is $C^r$-pseudo transversal to $V$ in $K$ with

$$r \geq \max(1, 1 + \dim M - \text{codim} V).$$

Let $R = \{a \in \mathcal{A} : F(a) \cap V \neq \emptyset \} = \{a \in \mathcal{A} : F(a) \text{ is transversal to } V \text{ in points of } K\}$.

If $K = M$ then $R$ is residual in $A$. If $V$ is a closed submanifold and $K \subset M$ is compact then $R$ is open and dense in $A$.

**Lemma 3.** Let $T$ be a positive number and $\eta$ be a hyperbolic periodic orbit of the Gaussian thermostat $(M, g, E)$ with $E \in \mathcal{H}_g(M)$ and such that $\beta_E(\eta) \neq 0$. Then there exist neighborhoods $U \subset SM$ of $\eta$ and $U \subset \mathcal{H}_g(M)$ of $E$ such that

(i) all $\tilde{E} \in U$ has a periodic orbit $\eta_{\tilde{E}} \subset U$ and all orbit of $(M, g, \tilde{E})$, different from $\eta_{\tilde{E}}$ passing by $U$ has period $T$,

(ii) the orbit $\eta_{\tilde{E}}$ depends continuously on $\tilde{E}$,

(iii) $\beta_{\eta_{\tilde{E}}}(\eta_{\tilde{E}}) \neq 0$.

**Proof.** Let $\Sigma$ be a transversal section over $\theta \in \eta$ and consider $P_{\tilde{E}} : \Sigma \rightarrow \Sigma$ the Poincaré application associated to $E$ in $\theta$. Let $L$ be the period of $\eta$ and $n$ a positive integer such that $nL > 2T$. The Poincaré application depends continuously of the Gaussian thermostat and therefore, for small enough $V \subset \Sigma$, the application $(P_{\tilde{E}})^n$ is defined in $V$ for all $E \in U$.

The point $\theta$ is a hyperbolic fixed point for $P_{\tilde{E}}$, then there exists for possibly smaller $U$ and $V$, a continuous application $\rho : U \rightarrow V$ which associate each $E \in U$ to the unique fixed point $\rho(E)$ of $P_{\tilde{E}}$ in $V$ and $\rho(E)$ is a hyperbolic point.

By the Hartman-Grobman theorem and continuous dependency of the Poincaré application, there exist a neighborhood $\tilde{V} \subset V$ of $\theta$ and a neighborhood $U$ of $E$ such that for $E \in U$ then $(P_{\tilde{E}})^k(\theta) \in V$, $k = 1, \ldots, n$, and every closed orbit of $E \in U$ different from $\eta_{\tilde{E}}$ has period larger than $T$.

Furthermore, for possibly smaller $\tilde{V}$, we have $\beta_{\eta_{\tilde{E}}}(\eta_{\tilde{E}}) \neq 0$. To finish the proof, let $U = \cup_{t \in [0, L+\varepsilon]} \phi^t(\tilde{V})$ for $\varepsilon$ small enough.
Lemma 4. Let \( E \in TG(M, g) \) and \( K \subset SM \) be a compact subset such that all closed orbits of \( K \) have period larger than \( T \). Then there exists a neighborhood \( U \subset \mathcal{X}_g(M) \) of \( E \) such that if \( \tilde{E} \in U \) then the closed orbits of \( \tilde{E} \) passing by \( K \) have period larger than \( T \).

Proof. Let \( \theta \in K \). The orbit \( \theta \) is regular or has period \( > T \) then there exists \( \varepsilon > 0 \) and a neighborhood \( U_\theta \) of \( \theta \) such that for \( \tilde{\theta} \in U_\theta \) we have \( \phi^t(\tilde{\theta}) \notin U_\theta \) for \( t \in [\varepsilon, T + \varepsilon] \).

Furthermore, the flow depends continuously on \( E \) and there exists a neighborhood \( U_\theta \subset \mathcal{X}_g(M) \) such that the same property holds for all Gaussian thermostats in \( U_\theta \).

Consider an open cover of \( K \) formed by \( U_\theta \), \( \{ U_\theta \}_{\theta \in K} \), and let a finite subcover \( \{ U_{\theta_i} \}_{i=1,...,k} \). The neighborhood \( U \subset \mathcal{X}_g(M) \) which satisfies the properties of the lemma is \( U = \cap_{i=1}^k U_{\theta_i} \). \( \square \)

The next two results are about isotropic subspaces.

Definition 8. Let \( \theta \in SM \) and a subspace \( U \subset T_\theta SM \). We call \( U \) isotropic when \( \Omega(u, v) = 0 \) for \( u, v \in U \).

Lemma 5. Let \( \theta \in \hat{T}_\theta SM \) and an isotropic subspace \( Q \in S(\theta) = \hat{T}_\theta SM \) then the set \( t \in \mathbb{R} \) such that \( D\phi_t Q \cap V(\phi_t(\theta)) \neq \{0\} \) is discrete.

Proof. First let us show show if \( Q \) is an isotropic subspace of \( T_\theta SM \) such that \( E \cap V(\theta) \neq \{0\} \) then there exists a neighborhood \( W \) of \( t = 0 \) such that \( D\phi_t Q \cap V(\phi_t(\theta)) = \{0\} \) for all \( t \in W \setminus \{0\} \).

Let \( \pi_{H(\theta)} : \hat{T}_\theta SM \rightarrow H(\theta) \) be the orthogonal projection over the horizontal bundle. There exists an isometry \( J_\theta : \hat{T}_\theta SM \rightarrow \hat{T}_\theta SM \) such that \( J_\theta^2 = -I \), \( J_\theta V(\theta) = H(\theta) \), \( J_\theta H(\theta) = V(\theta) \).

The subspaces \( \pi_{H}(Q) \) and \( J_\theta(Q \cap V(\theta)) \) are orthogonal. In fact, let \( x \in \pi_{H}(Q) \). We can write \( x = y - z \) where \( y \in Q \) and \( z \in Q \cap V(\theta) \). If \( Jw \in J_\theta(Q \cap V(\theta)) \) then \( \Omega(y, w) = 0 \) because \( Q \) is isotropic and, therefore, \( (\gamma \wedge \kappa)(y, w) = 0 \) because \( w \) is a vertical vector. So \( g(y, Jw) = 0 \). We also have \( g(z, Jw) = 0 \) as \( z \in Q \cap V(\theta) \). Then \( g(x, Jw) = 0 \).

Let \( \{ h_1, ..., h_k \} \) be a basis of \( \pi_{H(\theta)}(Q) \). If \( t \) is sufficiently small then there exists a set of linearly independent vectors \( \{ h_1(t), ..., h_k(t) \} \subset \pi_{H(\theta)}(D\phi_t Q) \).

Let \( \{ w_1, ..., w_l \} \) be a basis of \( Q \cap V(\theta) \) and the Jacobi fields \( J_1, ..., J_l \) with \( J(0) = 0 \) and \( J(0) = \pi_{V(\theta)} w_i \), for \( i = 1, ..., l \). Define, for \( t > 0 \), the vector fields \( W_i(t) = -\frac{1}{\|J_i(t)\|} J_i(t) \) and thus

\[
\lim_{t \to 0} W_i(t) = \lim_{t \to 0} \frac{J_i(t)}{\|J_i(t)\|} = -\pi_{V(\theta)} w_i = Jw_i.
\]

So \( \pi_{H(\theta)}(D\phi_t w_i) = J_i(t) \neq 0 \) when \( t > 0 \). Thus \( \{ h_1(t), ..., h_k(t), w_1(t), ..., w_l(t) \} \) is included in \( \pi_{H(\theta)}(D\phi_t Q) \). When \( t \) is small this set is close to \( \{ h_1, ..., h_k, J_\theta w_1, ..., J_\theta w_l \} \) then it is a linearly independent set.

Finally, we conclude for \( t \) sufficiently small \( \dim(\pi_{H(\theta)}(D\phi_t Q)) = \dim(Q) \) and then \( D\phi_t Q \cap V(\phi_t(\theta)) = \{0\} \). \( \square \)

Corollary 1. Let \( \theta \in SM \) and \( Q \in \hat{T}_\theta SM \) be an isotropic subspace. Then the set \( t \in \mathbb{R} \) such that \( \dim(\pi_{H}(D\phi_t Q)) \neq \dim(Q) \) is discrete.

After previous corolary, we say that isotropic spaces of Gaussian thermostat deviate from the vertical.

Finally, a technical lemma about symmetric applications.

Lemma 6. Let \( U \) and \( Q \) be subspaces of \( \mathbb{R}^n \) such that \( \dim(U) = \dim(Q) \). Then there exists a symmetric application \( \tilde{B} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
\pi_Q(U) = Q
\]

where \( \pi_Q \) is the orthogonal projection over \( Q \).
Proof. Consider the following decomposition of $\mathbb{R}^n$:

$$U \cap Q \oplus U \cap Q^\perp \oplus U^\perp \cap Q \oplus U^\perp \cap Q^\perp.$$ 

Suppose $\dim(U \cap Q) = l$ and take $\{w_1, \ldots, w_l\}$ a basis of $\dim(U \cap Q)$. Define on this subspaces $\tilde{B}(w_i) = w_i$, for $i = 1, \ldots, l$, the identity application.

Suppose $\dim(U \cap Q^\perp) = k$ then $\dim(U^\perp \cap Q) = k$. Let $\{u_1, \ldots, u_k\}$ be a basis of $\dim(U \cap Q^\perp)$ and $\{v_1, \ldots, v_k\}$ a basis of $U^\perp \cap Q$. Define $\tilde{B}(u_i) = v_i$, for $i = 1, \ldots, k$. On the subspace $U^\perp \cap Q$ define the application $\tilde{B}$ to make it symmetric.

Suppose $\dim(U^\perp \cap Q^\perp) = m$ and $\{z_1, \ldots, z_m\}$ is a basis of $U^\perp \cap Q^\perp$. Define $\tilde{B}(z_i) = 0$, for $i = 1, \ldots, m$. So we have that the application $\tilde{B}$ is completely defined.

4.2 Genericity of hyperbolic closed orbits

In this section we prove lemma 1. Let

- $\eta$ be a periodic orbit of period $L$ of the Gaussian thermostat $(M, g, E) \in \mathcal{X}_g(M)$,
- $\theta \in \eta$ be a point of this orbit,
- $\Sigma$ be a transversal section in $\theta \in \eta$,
- $T : \hat{T}_\theta SM \longrightarrow \hat{T}_\theta SM$ be the transversal derivative cocycle associated to $\theta$.

By the implicit function theorem, there exist $O \subset \mathcal{X}_g(M)$ a neighborhood of $E$ and $U \subset SM$ a neighborhood of $\theta$ such that the Poincaré application

$$P : O \times \Sigma \cap U \longrightarrow \Sigma,$$

is well defined as their iterates. Let

- For $i = 1, \ldots, k$
  $$\rho_i : O \longrightarrow C^\infty(\Sigma \cap U, \Sigma \times \Sigma),$$
  $$E \longmapsto \theta \mapsto (\theta, P^\rho_i E \theta),$$

- $W = \{(\theta, \theta) : \theta \in \Sigma\}$,
- $V \subset \nabla \subset U$ a neighborhood of $\theta$ with $\nabla$ compact,
- $R_0 = O$, $R_j = \{E \in O : \rho_i \upharpoonright_{\Sigma \cap \nabla} W \text{ para } i = 1, \ldots, j\}$,
- $S_0 = O$, $S_j = \{E \in O : \text{fixed points of } \rho_i \text{ are hyperbolic and } \beta_E(\theta) \neq 0 \text{ to } i = 1, \ldots, j\}$.

From theorem 2, we can fix Fermi coordinates along $\eta$ and the application $T$ is written as $T = e^A$ with $A$ of the form

$$A = \begin{bmatrix} 0 & I \\ S & \lambda I \end{bmatrix}$$

wherein

- $S = \int_0^L \dot{K}(t) + \dot{B}(t) dt$ is a symmetric matrix such that
  $$\dot{K}(t) = (-K_{ij})_{ij}(t) \quad i, j \in \{1, \ldots, n\},$$
  $$\dot{B}(t) = \left(\frac{\partial E_i}{\partial x_j}\right)_{ij}(t) \quad i, j \in \{1, \ldots, n\},$$
\[ \lambda = - \int_0^L E_0(t)dt = - \int_0^L \gamma_E(v(t))dt = - \int_0^L (E, v)(t)dt \text{ (at the level } c = 1), \]

- \( I \) is the identity matrix.

**Lemma 7.** If \( \rho_i(\theta) \in W, \rho_j(\theta) \notin W, j < i \) and \( E \in \mathcal{X}_d(M) \) then there exists a \( \varepsilon \)-perturbation \( \tilde{E} \in \mathcal{X}_d(M) \) of \( E \) such that \( \theta \) is a hyperbolic periodic orbit and \( \beta(\theta) \neq 0 \).

**Proof.** Let \( \varepsilon \) be a \( \varepsilon \)-perturbation \( \tilde{E}^1 \) of the Gaussian thermostat \( E \) such that \( \tilde{E}^1 \) preserves the periodic orbit \( \eta \) and the eigenvalues of \( T \) have modulus different from 1.

In Fermi coordinates, the matrix \( T_\theta = e^A \) has an eigenvalue with modulus 1 if and only if the matrix \( A \) has an eigenvalue with real part equals to zero. If the eigenvalue is complex, we apply proposition 4.4 with \( \alpha = \varepsilon > 0 \).

In addition, \( \det(A) = (-1)^n \det(S) \) is equal to the product of its eigenvalues. Thus the application \( A \) has real eigenvalues with modulus equal to 0 if and only if \( \det(S) = 0 \). We denote \( S^k, k \in \{1, \ldots, n\} \) the matrix \((n-k) \times (n-k)\) constructed from \( S \) removing the first \( k \) lines and \( k \) columns.

Let \( k_0 \) the smaller \( k \) such that \( \det(S^k) \neq 0 \). If \( \det(S^k) = 0 \) for \( k = 1, \ldots, n \) take \( k_0 = n \).

Consider \( \tau > 0 \) such that \( \eta([0, \tau]) \) is contained in a coordinate Fermi neighborhood.

Take \( \lambda_1 < \varepsilon \tau \). The perturbation on the Gaussian thermostat is constructed from a perturbation \( \tilde{E}^1 \) of the vector field \( E \):

\[
\begin{align*}
\tilde{E}_0^1(t, x) &= E_0 \\
\tilde{E}_1^1(t, x) &= E_1 + \psi(t, x) \frac{\lambda_1}{\tau} x_1 \\
& \quad \vdots \\
\tilde{E}_{k_0}^1(t, x) &= E_{k_0} + \psi(t, x) \frac{\lambda_1}{\tau} x_{k_0} \\
\tilde{E}_{k_0+1}^1(t, x) &= E_{k_0+1} \\
& \quad \vdots \\
\tilde{E}_n^1(t, x) &= E_n
\end{align*}
\]

We can write the perturbation matrix \( \tilde{S} \) as

\[ \tilde{S} = \tilde{K} + \tilde{B} + \varepsilon D = S + \varepsilon D, \]

wherein \( D \) is a diagonal matrix such that if \( i \leq k_0 \) then \( D_{ii} = 1 \) and if \( i > k_0 \) then \( D_{ii} = 0 \).

Calculating the determinant of \( \tilde{S} \) by cofactors in terms of the first row, we have

- If \( k_0 < n \) then \( \det(\tilde{S}) > \lambda_1^{k_0} \det(S^{k_0}) \neq 0 \),
- If \( k_0 = n \) then \( \det(\tilde{S}) > \lambda_1^n \neq 0 \).

If \( \beta_{\tilde{E}^1}(\eta) \neq 0 \) then there is nothing to do. Otherwise, take \( 0 < \lambda_2 < \varepsilon \tau \) and \( \tilde{E}^2 \) defined by

\[
\begin{align*}
\tilde{E}_0^2 &= \tilde{E}_0^1 + \psi(t, x) \frac{\lambda_2}{\tau}  \\
\tilde{E}_i^2 &= \tilde{E}_i^1 \text{ para } i = 1, \ldots, (n-1)
\end{align*}
\]

Consider the application \( \beta_{\tilde{E}^2} \) applied to \( \eta \) which is also an orbit of the Gaussian thermostat \((M, g, \tilde{E}^2)\), then

\[ \beta_{\tilde{E}^2}(\eta) = \int_0^\tau \gamma_{\tilde{E}^2}(\tilde{\eta}(s)) ds = \lambda_2 > 0. \]

**Lemma 8.** Suppose \( S_{j-1} \) is open and dense in \( \mathcal{O} \). Then \( \rho_j \) is a pseudo \( C^r \)-representation transversal to \( W \) in \( \Sigma \cap \overline{V} \).
Moreover, due to the symmetric application such that $s$ is open and dense in $O$ is open and dense in $O$ is open and dense in $O$ is open and dense in $O$ is open and dense in $O$. We show that given an integer $\rho_j(\theta, t, x)$ is an open and dense subset. Once $G_1 = \cap_{T \geq 1} G_1(T)$ then $G_1$ is residual.

**Proof.** With $B = TG(M, g)$ we denote the Banach space of perturbations and with $S_j \cap TG(M, g)$ the dense set provided from the definition of pseudo transversality. We need to prove $\rho_j$ is $C^\epsilon$-transversal to $W$ in $E \times \Sigma \cap V$ for $E \in S_j$.

If $\rho_i(E, \theta) \in W$ for $i < j$ and $\rho_j(E, \theta) \in W$ then $\theta$ is a periodic hyperbolic point, $\rho_j(E) \cap \theta W$, and $\rho_j \cap (E, \theta) W$.

If $\rho_i(E, \theta) \notin W$ for $i < j$, $\rho_j(E, \theta) \in W$ and $D\rho_j(E, \theta) \cap T_0W = \{0\}$ then we have the transversality.

If $D\rho_j(E, \theta) \cap TW \neq \{0\}$ then let $u \in T_0SM$ such that $D\rho_i(u) = (u, D\phi^i u = u)$. Moreover, the projection of an eigenvalue of $D\phi^i$ in $T_0SM$ is an eigenvalue of the linear Poincaré application.

We can take $u$ with nonzero horizontal component. In fact, by lemma 5 the subspace generated by $u$, denoted by $U$, is isotropic and then the intersection with the vertical bundle occurs in isolated points along the orbit of $\theta$. It is enough to consider the point $\theta'$ in $\eta \in \theta$ such that $U \cap V(\theta') = \{0\}$. We keep calling $\theta'$ of $\theta$.

We consider $Q \subset T_0SM$ the orthogonal subspace of $u$. Fix Fermi coordinates along the orbit segment $\eta : [0, \tau] \to SM$ (see $r_{inj}$) in such way that $\eta(0) = \theta$ and $\eta(\tau) = \theta$.

Consider the path $c : (-\varepsilon, \varepsilon) \to CS(\mathbb{R}^{2n})$ such that $c(s) = e^{A+B}\theta$ wherein $B = \left[\begin{array}{cc} 0 & 0 \\ \tilde{B} & 0 \end{array}\right]$ and $\tilde{B} : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric.

Note that $c(0) = d\phi^\varepsilon_\theta$ and this path is the derivative cocycle of the Gaussian thermostat $(M, g, \tilde{E})$ wherein $\tilde{E}$ is given by

$$
\begin{align*}
\tilde{E}_0^s(t, x) &= E \\
\tilde{E}_s^i(t, x) &= E_i + \frac{\varepsilon}{\tau} \sum_{j=1}^n \psi(t, x) \tilde{B}_{ij} x_j \quad \text{for} \quad i = 1, \ldots, n
\end{align*}
$$

We are looking for an application $B$ such that for $s \neq 0$ we have

$$
\pi_Q e^{(A+B)} U \neq 0.
$$

Calculating the differential of the path in $s = 0$

$$
\frac{d}{ds} \pi_Q e^{(A+B)} U \bigg|_{s=0} = \pi_Q (i e^{iA}) BU
$$

this differential must be non trivial and, for this, it is enough to show $BU \neq U$ and $dim(BU) = dim(U)$ once $U$ is an eigenspace of the isomorphism $e^A$.

Let $Q' \subset Q$ such that $Q' \subset V(\theta)$. This choice is possible because $U \cap V(\theta) = \{0\}$. Let $\tilde{B}$ be the symmetric application such that $\tilde{B} \pi_{\{U\}} U = Q'$ whose existence is guaranteed by lemma 6. Moreover, due to $U \cap V(\theta) = \{0\}$, we have $1 = dim(Q' = BU) = dim(U)$.

**Lemma 9.** There exists a $\varepsilon$-perturbation $\tilde{E} \in \mathcal{X}_0(M)$ of $E$ such that the Poincaré application $P$ of the Gaussian thermostat $(M, g, \tilde{E})$ has only a finite number of fixed points, all hyperbolic, and if $\eta$ is an orbit of the flow associated to a fixed point then $\beta_{\tilde{E}}(\eta) \neq 0$.

**Proof.** The proof is done performing an inductive argument: By lemma 8 if $S_{j-1}$ is open and dense in $O$ then $\rho_j$ is a pseudo $C^\epsilon$-representation transversal to $W$ in $\Sigma \cap V$. By theorem 3 $R_j$ is open and dense in $O$. By lemma 7 if $\rho_i(\theta) \in W$ and $\rho_j(\theta) \notin W$, $j \leq i$, then there exists a $\varepsilon$-perturbation $\tilde{E}$ of $E$ such that $\theta$ is a hyperbolic periodic orbit. Therefore if $R_j$ is open and dense in $O$ then $S_j$ is open and dense in $O$.

We are going to prove lemma 11

**Proof of lemma 11** We show that given an integer $T > 0$ then the set of vector fields $G_1(T) \subset \mathcal{X}_0(M)$ which defines Gaussian thermostats whose orbits of period smaller than $T$ are hyperbolic is an open and dense set. Once $G_1 = \cap_{T \geq 1} G_1(T)$ then $G_1$ is residual.
\(G_1(T)\) is open in \(\mathcal{X}_g(M)\). Let \(E \in G_1(M)\). The Gaussian thermostat \((M, g, E)\) has only a finite number of orbits of period \(\leq T\).

Let \(\theta \in SM\). We have two possibilities:

(i) The point \(\theta\) is contained in a regular orbit or a periodic orbit of period \(> T\). By the tubular flow theorem, there exists a neighborhood \(U_\theta\) of \(\theta\) in \(SM\) such that every orbit of \((M, g, E)\) which intersects \(U_\theta\) has period \(> T\). By lemma 4 there exists a neighborhood \(N_\theta \subset \mathcal{X}_g(M)\) of \(E\) such that the orbit of a Gaussian thermostat in \(N_\theta\) which passes through \(U_\theta\) has periodic orbit of period \(> T\).

(ii) The point \(\theta\) is contained in a periodic orbit of period \(\leq T\) and \(\beta_E(\theta) \neq 0\). By lemma 3 there exist neighborhoods \(U_\theta \subset SM\) and \(N_\theta \subset \mathcal{X}_g(M)\) such that if \(E \in N_\theta\) has a unique closed hyperbolic orbit \(\theta_E\) in \(U_\theta\) which is hyperbolic then \(\beta_E(\theta_E) \neq 0\) and all others periodic orbits intersecting \(U_\theta\) have period \(> T\).

Let \(\{U_\theta, \theta \in SM\}\) an open cover of \(SM\). Take a finite subcover \(U_1, \ldots, U_k\) and consider \(N_1, \ldots, N_k\) from this cover. The vector fields in \(N = N_1 \cap \cdots \cap N_k\) define Gaussian thermostats with all periodic orbits \(\theta_E\) close to periodic orbits of \((M, g, E)\) with period \(\leq T\) also have period \(\leq T\). Furthermore, such orbits \(\theta_E\) are all hyperbolic and \(\beta(\theta_E) \neq 0\).

\(G_1(T)\) is dense in \(\mathcal{X}_g(M)\). Consider the set \(\Gamma(T) = \{\theta \in SM : O(\theta)\) is closed with period \(\leq T\}\). This set is compact. In fact, let \(\theta_n\) be a sequence in \(\Gamma(T)\) such that \(\theta_n \to \theta\). If the orbit of \(\theta\) has period \(\leq T\) then \(\theta \in \Gamma(T)\). If the orbit of \(\theta\) is regular or closed with period greater than \(T\) then, by the tubular flow theorem, there exists a neighborhood \(U_\theta\) of \(\theta\) such that a closed orbit in \(\overline{U_\theta}\) has period greater than \(T\). Contradiction.

We will show there exists \(\tilde{E}\) arbitrarily close to \(E\) such that \(\tilde{E} \in G_1(T)\).

Let \(\{W_\theta\}_{\theta \in \Gamma}\) be an open cover of \(\Gamma\) where \(W_\theta\) is a neighborhood of the orbit \(\theta\) and let \(\{W_{\theta_i}\}_{i=1, \ldots, k}\) be a finite subcover. Consider \(W = \cup_{i=1}^k W_{\theta_i}\) and the compact subset \(K = SM \setminus W\). Every periodic orbit in \(K\) has period greater than \(T\). By lemma 4 there exists an open set \(N \subset \mathcal{X}_g(M)\) such that every closed orbit in \(K\) of a Gaussian thermostat in \(N\) has period greater than \(T\). Consider the Poincaré application \(P_t\) associated to \(\theta_1\). By lemma 3 there exists a \(\varepsilon\)-close Gaussian thermostat such that the application \(P_t\) has only hyperbolic periodic points and for this points \(\beta \neq 0\). Applying this lemma for \(l = 1, \ldots, k\), we conclude the result.

\[\square\]

4.3 Transversality of invariant manifolds

To finish the proof of theorem A, it remains to prove lemma 2. Consider \(T > 0\). Let \(E \in G_1(T)\) and \(\eta_1, \ldots, \eta_l\) be the periodic orbits of \((M, g, E)\) of period smaller and equal than \(T\). For each \(\eta_i\), we take compact neighborhoods \(W^s_{\eta_i}(\eta_i, E)\) and \(W^u_{\eta_i}(\eta_i, E)\) of \(\eta_i\) in \(W^s(\eta_i, E)\) and \(W^u(\eta_i, E)\), respectively, such that the boundaries of \(W^s_{\eta_i}(\eta_i, E)\) and \(W^u_{\eta_i}(\eta_i, E)\) are fundamental domains. Let \(\Sigma^s_i\) be a submanifold of \(SM\) of codimension 1 transversal to the flow direction which intersection with \(W^s(\eta_i, E)\) is \(\partial W^s_{\eta_i}(\eta_i, E)\). For \(\tilde{E}\) in a neighborhood \(N^s\) of \(E\) where the flow is transversal to \(\Sigma^s_i\), we take a neighborhood \(W^s_{\eta_i}(\tilde{\eta}_i, \tilde{E})\) such that \(\tilde{\eta}_i\) is a continuation of \(\eta_i\) which boundary is the intersection of \(W^s(\tilde{\eta}_i, \tilde{E})\) with \(\Sigma^s_i\). We perform the same construction for \(W^u_{\eta_i}(\eta_i, E)\) to obtain the set \(N^u\) and set \(N = N^s \cap N^u\). For each positive integer \(n\), define \(W^s_n(\eta_i, E) = \phi_n(W^s(\eta_i, E))\) and \(W^u_n(\eta_i, E) = \phi_n(W^u(\eta_i, E))\).

Let \(G^2_1(T) \subset G_1(T)\) such that if \(E \in G^2_1(T)\) then \(W^s_n(\eta_i, E)\) is transversal to \(W^u_n(\eta_j, E)\) for all closed orbits \(\eta_i\) and \(\eta_j\) of period \(\leq T\) of \(E\). To prove lemma 2 it is enough to show the following lemma.
Lemma 10. Let $E \in G_2(T)$ and $N$ be a neighborhood of $E$ as above. Then for all $n \in \mathbb{N}$, the set $G_2^n(T)$ is open and dense in $N$.

Proof. We denote by $G_2^{n,i,j}(T)$ the set of Gaussian thermostats $E \in N$ such that if $\eta_1, \ldots, \eta_k$ are hyperbolic closed orbits of $(M, g, E)$ with period smaller and equal than $T$ then the invariant manifolds $W^s_n(\eta_i, E)$ and $W^u_n(\eta_j, E)$ are transversal.

The set $G_2^n$ satisfies

$$G_2^n(T) = \cap_{i,j=1}^k G_2^{n,i,j}(T).$$

Therefore it is enough to show that $G_2^{n,i,j}(T)$ is open and dense.

$G_2^{n,i,j}(T)$ is open in $\mathcal{F}_g(M)$. Let $E \in G_2^{n,i,j}(T)$. Once $W^s_n(\eta_i, E)$ and $W^u_n(\eta_j, E)$ are transversal and the applications $E \rightarrow W^s_n(\eta_i, E)$ and $E \rightarrow W^u_n(\eta_j, E)$ are continuous, it follows that there exists a neighborhood $N_{ij} \subset N$ of $E$ such that for all $E \in N_{ij}$ the manifolds $W^s_n(\eta_i, E)$ and $W^u_n(\eta_j, E)$ are transversal.

$G_2^{n,i,j}(T)$ is dense in $\mathcal{F}_g(M)$. Consider the compact set $K = W^s_n(\eta_i, E) \cap W^u_n(\eta_j, E)$. For $\theta \in K$ we take $A_\theta$ a neighborhood of $\theta$ with only one connected component in $K$. We consider $\{A_\theta\}_{\theta \in K}$ a finite subcover of an open cover $\{A_\theta\}_{\theta \in K}$ of $K$.

There exists a neighborhood $\tilde{N} \subset N$ such that if $E \in \tilde{N}$ then $W^s_n(\eta_i, \tilde{E}) \cap W^u_n(\eta_j, \tilde{E}) \subset \bigcup_{l=1,\ldots,k} A_{\theta_l}$. Consider $\tilde{N}^l \subset \tilde{N}$ whose Gaussian thermostat $(M, g, E)$ in that set verifies that $W^s_n(\eta_i, E)$ is traversal to $W^u_n(\eta_j, E)$ in $A_{\theta_l}$. We show that $\tilde{N}^l$ is dense in $\tilde{N}$. Consider $T^*W^s_nW(\theta) = T^*gSM$ the stable and unstable manifolds tangent space, respectively, restricted to the derivative cocycle on the point $\theta \in SM$. Now, we fix Fermi coordinates along the orbit segment $W : [0, \tau] \rightarrow SM$ ($\tau < r_{\text{inv}}$) such that $W(0) = \tilde{\theta}$ and $W(\tau) = \theta$.

Consider the path $c : (-\varepsilon, \varepsilon) \rightarrow CS(\mathbb{R}^{2n})$ such that $c(s) = e^{A^sB}$ where $B = \begin{bmatrix} 0 & 0 \\ \tilde{B} & 0 \end{bmatrix}$ and $\tilde{B} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is symmetric. Note that $c(0) = T : T^*gSM \rightarrow T^*gSM$ is the transversal derivative cocycle of $(M, g, E)$ and this path represents the transversal derivative cocycle of the Gaussian thermostats $(M, g, E)$ where $\tilde{E}$ is given by

$$\begin{cases} \tilde{E}_0(t, x) = E \\ \tilde{E}_i(t, x) = E_i + \sum_{j=1}^n \psi(t, x)\tilde{B}_{ij}x_j \end{cases} \text{ to } i = 1, \ldots, n.$$ 

Consider $U = T^*W^s(\theta) \cap T^*W^u(\tilde{\theta})$, $Q = (T^*W^s(\theta) + T^*W^u(\theta))^\perp$ and $\dim(U) = \dim(Q) = k$.

Let $\pi_Q$ be the orthogonal projection on the subspace $Q$. Observe that

$$\pi_Qe^{A^sB}U = 0$$

and to finish the proof we look for an application $B$ such that for $s \neq 0$ we have

$$\pi_Qe^{A^sB}U \neq 0.$$

Considering

$$\frac{d}{ds} \pi_Qe^{A^sB}U \bigg|_{s=0} = \pi_Qe^{A}BU$$

we require that

$$BU = \pi_V(\theta)(BU) \subset \pi_V(\theta)(e^{-\delta}Q)$$

and $\dim(U) = \dim(B\pi_H(\theta)U) \leq \pi_V(\theta)(e^{-\delta}Q)$ where $\pi_V(\theta)$ is the orthogonal projection on the vertical subspace and $\pi_H(\theta)$ is the orthogonal projection on the horizontal subspace.

From the fact that $\beta(\eta_i) \neq 0$ and $\beta(\eta_j) \neq 0$ follows that $T^*W^s(\theta)$ and $T^*W^u(\tilde{\theta})$ are isotropic and the lemma [5] gives us $\dim(\pi_H(\theta)U) = \dim(\pi_V(\theta)Q) = k$. Finally, the existence of an application $\tilde{B}$ such that $\tilde{B}(\pi_H(\theta)U) = Q$ is guaranteed by lemma [5].

\qed
5 Proof of theorems B and B’

The proof of theorem B is an adaptation to the context of conformally symplectic applications of the work in [8]. In the proof, we use the perturbative theorem from section 6.

The idea of the proof goes as follows. Let \( \Sigma \) be a transversal section in \( \theta \in \eta \) and \( f : \Sigma \rightarrow \Sigma \) be the Poincaré application. Consider \( \pi : T_\theta SM \rightarrow T_\theta SM \) the projection of \( T_\theta SM \) over \( \hat{T}_\theta SM \). Suppose \( \pi \circ df_\theta \) has 2\(n\) invariant subspaces \( E_1, \ldots, E_{2n} \) on the homoclinic class \( H(\theta, f) \).

Suppose also \( H(\theta, f) \) has \( 2n-1 \) periodic points \( \theta_1, \ldots, \theta_{2n-1} \) such that \( \theta_1 \) has complex eigenvalue related to the subspaces \( E_i \) and \( E_{i+1} \) (we say in this case that there is a sequence of periodic orbits with concatenated complex eigenvalues). Then by a perturbation \( \tilde{f} \) of \( f \) we can make \( \pi \circ df_{\tilde{\theta}_i}^{\text{per}(\theta_i)}(E_i) = \pi \circ df_{\theta_i}^{\text{per}(\theta_i)}(E_{i+1}) \) and \( \pi \circ df_{\tilde{\theta}_i}^{\text{per}(\theta_i)}(E_{i+1}) = \pi \circ df_{\theta_i}^{\text{per}(\theta_i)}(E_i) \) where \( \text{per}(\theta_i) \) is the period of \( \theta_i \). Using the notion of transitions, it is possible to create a point \( \theta \) whose orbit has some iterates near to each \( \theta_i \) and then the differential of \( f \) on \( \theta \) inherits some properties of the differential of \( f \) on \( \theta_i \). Then it mixes all the subspaces \( E_k \) creating the sink or the source of the theorem. However, if in a robust way the subspaces \( E_i \) have not a sequence of concatenated complex eigenvalues, it is impossible to create the complex eigenvalues and we have dominated splitting. The perturbative theorem from section 6 permit us to show an homoclinic class has transitions and also to translate the dynamical problem to a linear systems problem. The proof is explained in details in the following subsections: in subsection 5.1 we makes the translation from diffeomorphisms problems to linear systems language; in subsections 5.2, 5.3 and 5.4 we show how to create either a sink or repeller by perturbation of a linear system if there are a collection of periodic orbits with concatenated complex eigenvalues; in subsection 5.5 we we provide a sufficient condition to get a dominated splitting and we show that if there are not a collection of periodic orbits with complex eigenvalues then the sufficient condition holds.

The entire proof of theorem B can be done in the symplectic category, so together with the work of Contreras in [10] it is valid the following geodesic flow version of theorem B:

**Theorem B’ (symplectic version of theorem B).** Given \((M, g)\) Riemannian manifold with geodesic flow \( \phi \) and \( \eta \) a saddle hyperbolic periodic orbit. For a neighborhood \( V \) of \( \eta \), one of this alternatives is valid:

(i) the homoclinic class \( H(\eta, \phi) \subset V \) has a dominated splitting decomposition

(ii) given a neighborhood \( U \) of \( H(\eta, \phi) \subset V \) and \( k \in \mathbb{N} \), there exists a geodesic flow \((\tilde{M}, \tilde{g})\) arbitrarily close to \((M, g)\) with \( k \) islands arbitrarily close to \( \eta \) which its orbit is contained in \( U \).

The proof of theorem B’ follows from the thesis of theorem B and from the fact that for \( C^2 \) three dimensional Kupka-Smale vector fields, if the closure of saddle periodic orbits has a dominated splitting then it is hyperbolic. The statement is proved in [6], which extends to three dimensional flows initially proved for surfaces diffeomorphisms in [24]. Observe that we can apply the result in [6] since we already proved that generically, Gaussian thermostat are Kupka-Smale.

5.1 Conformally symplectic linear system and transitions

In this section we make the translation from diffeomorphisms problems to linear systems language.

**Definition 9.** Let \( \Sigma \) be a topological space, \( f \) an homeomorphism defined in \( \Sigma \), a locally trivial fiber bundle \( \pi : \mathcal{E} \rightarrow \Sigma \) over \( \Sigma \) such that the fiber dimension \( \mathcal{E}_x \) for \( x \in \Sigma \) is constant over \( \Sigma \), and an application \( A : \mathcal{E} \rightarrow \mathcal{E} \) such that \( A_x = A(x,.) : \mathcal{E}_x \rightarrow \mathcal{E}_{f(x)} \) is a conformally symplectic linear isomorphism in the following sense. There are coordinates such that \( A^*\omega = \mu \omega \) where \( \omega \) is
a the canonical symplectic 2-form $\omega$ and $\mu : \Sigma \to \mathbb{R}$ is a positive function. The application $A$ also satisfies $\|A_x\| < \infty$ for all $x \in \Sigma$. We call $(\Sigma, f, \mathcal{E}, A, \omega, \mu)$ a conformally symplectic linear system or a conformally symplectic linear cocycle over $f$. When there is no ambiguity, we denote the linear system $(\Sigma, f, \mathcal{E}, A, \mu)$ only by $A$.

The norm of the application $A$, is defined as $\|A\| = \max\{\sup_{x \in \Sigma}\|A_x\|, \sup_{x \in \Sigma}\|A_x^{-1}\|\}$.

**Lemma 11.** Let $(\Sigma, f, \mathcal{E}, A, \mu)$ and $(\Sigma, f, \mathcal{E}, B, \mu)$ be conformally symplectic linear systems. Then $(\Sigma, f, \mathcal{E}, A \circ B, \mu)$ satisfies $\|A \circ B\| \leq \|A\|\|B\|$.

We denote $CS(\Sigma, f, \mathcal{E})$ the set of conformally symplectic linear systems $(\Sigma, f, \mathcal{E}, A, \mu)$ for all $\mu$ and equip $CS(\Sigma, f, \mathcal{E})$ with the distance $d(A, B) = \max\{\|A - B\|, \|A^{-1} - B^{-1}\|\}$ for $A, B \in CS(\Sigma, f, \mathcal{E})$.

**Lemma 12.** Given $K > 0, \varepsilon_1 > 0$ and a conformally symplectic linear system $(\Sigma, f, \mathcal{E}, A, \mu)$ such that $\|A\| < K$ then for all symplectic $\varepsilon_1$-perturbation of the identity $(\Sigma, Id_{\Sigma}, \mathcal{E}, E, 1)$ the composition $E \circ A$ and $A \circ E$ are $\varepsilon$-perturbation of $A$ with $\varepsilon = \varepsilon_1 K$. Furthermore, for all $\varepsilon$-perturbation $\tilde{A}$ of $A$ there exists a symplectic $\varepsilon_1$-perturbation of the identity $(\Sigma, Id_{\Sigma}, \mathcal{E}, E, 1)$ such that $\tilde{A} = E \circ A$.

**Proof.** The application $E \circ A$ satisfies $\|E \circ A - A\| = \|(E - Id_{\Sigma}) \circ A\| \leq \|E - Id_{\Sigma}\| \|A\| < \varepsilon_1$. Define $E = \tilde{A} \circ A^{-1}$, the $E \circ A = \tilde{A} \circ E \circ A^{-1}$ and $E^* \omega = (\tilde{A} \circ A^{-1})^* \omega = (A^{-1})^* (\omega \circ \tilde{A}) = (A^{-1})^* (\omega) = \mu \omega$. Analogously, $A \circ E$ satisfies $\|A \circ E - A\| = \|A \circ (E - Id_{\Sigma})\| \leq \|A\| \|E - Id_{\Sigma}\| < \varepsilon$ and $E^* \omega = (E - Id_{\Sigma})^* \omega = (E^* \omega = (A-1)^* \omega = (A^{-1})^* (\mu \omega) = \mu \omega$.

**Definition 10.** The conformally symplectic linear system $(\Sigma, f, \mathcal{E}, A, \mu)$ is called a conformally symplectic periodic linear system if all $x \in \Sigma$ is a periodic point of $f$. We denote $p(x)$ the period of $x$.

**Definition 11.** The conformally symplectic linear system $(\Sigma, f, \mathcal{E}, A, \mu)$ is called a conformally symplectic continuous linear system if the fiber structure varies continuously and $A : \mathcal{E} \to \mathcal{E}$ is continuous.

**Definition 12.** The conformally symplectic linear system $(\Sigma, f, \mathcal{E}, A, \mu)$ is called a conformally symplectic matrix system if $\mathcal{E} = \Sigma \times \mathbb{R}^n$, $\mathbb{R}^n$ is equipped with the euclidean canonical metric and there exists a positive function $\mu : \Sigma \to \mathbb{R}$ such that $A^T J A = \mu J$. We denote the conformally symplectic matrix system by $(\Sigma, f, A, \mu)$.

Given a set $\mathcal{A}$, a word with letters in $\mathcal{A}$ is a finite sequence of elements of $\mathcal{A}$. The size of this word is its number of letters. The set of words admits a natural semi group structure: the product of $[a] = (a_1, \ldots, a_n)$ by $[b] = (b_1, \ldots, b_n)$ is $[a][b] = (a_1, \ldots, a_n, b_1, \ldots, b_n)$. We say a word $[a]$ is not a power if $[a] \neq [b]^k$ for all word $[b]$ with $k > 1$.

If $(\Sigma, f, A, \mu)$ is a conformally symplectic matrix system then for $x \in \Sigma$ we denote the word $A(f^{p(x)-1}(x)), \ldots, A(x))$ by $[M]_A(x)$. The matrix $M_A(x) = A(f^{p(x)-1}) \circ \cdots \circ A(x)$ is the product of letters of $[M]_A(x)$.

**Definition 13.** Given $\varepsilon > 0$, a conformally symplectic periodic linear system $(\Sigma, f, \mathcal{E}, A, \mu)$ admits $\varepsilon$-transitions if:

(i) for all finite family of points $x_1, \ldots, x_n = x_1 \in \Sigma$ there exists a coordinate system over $\mathcal{E}$ such that we can consider this system as a conformally symplectic matrix system $(\Sigma, f, A, \omega, \mu)$ and
(ii) For all \((i, j) \in \{1, \ldots, n\}^2\) there exists a finite word \([t^{i, j}]\) of matrices in \(CS(n, \mathbb{R})\) such that the word

\[
[W(t, a)] = [t^{i_1, \ldots, i_m}] [M_A(x_{i_m})] \ldots [M_A(x_{i_{m-1}})] [M_A(x_{i_1})]^a m \ldots [t^{j_1, \ldots, j_n}] [M_A(x_{j_n})]^a n
\]

with \(t = (i_1, \ldots, i_m) \in \{1, \ldots, n\}^m\), \(a = (a_1, \ldots, a_m) \in \mathbb{N}^m\) and the word \((x_{i_1}, a_{i_1}), \ldots, (x_{i_m}, a_{i_m})\) with letters in \(\Sigma \times \mathbb{N}\) is not a power.

Then there exists \(x = x(t, a) \in \Sigma\) such that

(ii.1) the size of \([W(t, a)]\) is \(\text{per}(x)\),

(ii.2) The word \([M_A(x)]\) is \(\varepsilon\)-close to \([W(t, a)]\) and there exists a \(\varepsilon\)-perturbation \(\tilde{A}\) of \(A\) such that \([M]_{\tilde{A}}(x) = [W(t, a)]\),

(ii.3) We can choose \(x\) such that the distance from the orbit of \(x\) and \(x_i, i \in \{1, \ldots, n-1\}\), is bounded by a function \(\alpha_i\) which tends to zero when \(a_i\) tends to infinity.

Given \(t\) and \(a\) as above, we denote \([t^{i, j}]\) an \(\varepsilon\)-transiton from \(x_1\) to \(x_j\) and we call an \(\varepsilon\)-transitons matrix to the product \(T_{i,j}\) of letters composing \([t^{i, j}]\).

Definition 14. A conformally symplectic periodic linear system admits transitions if for any \(\varepsilon > 0\) the system admits \(\varepsilon\)-transitions.

Below, the dictionary between a conformally symplectic linear system and the tangent bundle dynamics of the Gaussian thermostat over an homoclinic class:

Lemma 13. Given a hyperbolic periodic saddle point \(p\) of index \(k\) (the dimension of the stable subbundle). The differential \(df\) of the Poincaré application \(f\) induces a continuous linear system with transitions on the set \(\Sigma\) of hyperbolic saddles in \(H(p, f)\) of index \(k\) which are homoclinically related to \(p\).

Proof. Fix \(\varepsilon > 0\) and a finite family \(x_1, \ldots, x_n \in \Sigma\). The points \(x_i\) are homoclinically related to \(p\) then there exists a transitive compact hyperbolic subset \(K \subset H(p, f)\) which contains all points \(x_i\). It is possible to cover \(K\) by a Markov partition composed by rectangles \(R_k\). For each \(x \in K\) we define conformally symplectic coordinates \(\phi_x: U_x \to M\). It is possible to refine the Markov partition such that \(R_k\) is contained in a open set \(U_x\) and for \(x\) and \(y\) at the same rectangle \(\|A(x) - A(y)\| < \varepsilon\) and \(\|A(x)^{-1} - A(y)^{-1}\| < \varepsilon\). For each rectangle \(R_k\) we write \(df\) on the conformally symplectic coordinates and consider the associated matrix system \((K, f, A, \mu)\). The transition from \(x_1\) to \(x_j\) is obtained by the property of Markov partition and the conformally symplectic perturbative theorem.

On what follows it is showed that a property valid on a point of a conformally symplectic linear systems which admits transitions is extended over a dense set. Next lemma is only true to the case of strict conformally symplectic case, i.e., \(\mu\) is not identically 1 since for \(\mu \equiv 1\) it is impossible to create contractions or dilations:

Lemma 14 (Spreading property). Let \((\Sigma, f, \mathcal{E}, A, \mu)\) be a conformally symplectic periodic linear system with transitions. Fix \(\varepsilon > \varepsilon_0 > 0\), assume there exists a \(\varepsilon_0\)-perturbation \(B\) of \(A\) and \(x \in \Sigma\) such that \([M]_B\) is a dilation (or a contraction). Then there exists an \(f\)-invariant dense set \(\Sigma_C \subset \Sigma\) and a \(\varepsilon\)-perturbation \(C\) of \(A\) such that for all \(y \in \Sigma\), \([M]_C(y)\) is a dilation (or a contraction).

Proof. Let \(\Sigma_B \subset \Sigma\) a dense subset and \(\varepsilon_1 = \varepsilon - \varepsilon_0\).

Fix \(\delta > 0\) and consider \(z \in \Sigma\), the family of points \(x_1 = z \in \Sigma_B, x_2 = x \in x_3 = z\) and the word

\[
[W] = [t^{1,2}] [M_A^n(x)][t^{2,1}] [M_A^{n(z, \delta)}(z)]
\]
the linear system \((\Sigma, f, \mathcal{E}, A, \mu)\) admit transitions so there exists \(z_n \in \Sigma\) such that \(d(z_n, z) < \delta\) and a \(\varepsilon_1\)-perturbation \(\tilde{A}\) of \(A\) such that \([M]_{\tilde{A}}(z_n) = [W]\).

Consider a \(\varepsilon_0\)-perturbation \(C\) of \(\tilde{A}\) defined along the orbit of \(z_n\) by

\[
[M]_{C}(z_n) = [t^{1,2}] [M]^{C}_{B}(x) [t^{2,1}] [M]^{\varepsilon, \delta}_{A}(z).
\]

With \(n\) big enough, we have \([M]_{C}(z_n)\) a dilation or a contraction and \(\Sigma_C\) is defined as the union of \(z_n\).

**Definition 15.** Given a conformally symplectic linear system \((\Sigma, f, \mathcal{E}, A, \mu)\). A \(\varepsilon\)-perturbation \(\tilde{A}\) of \(A\) is a conformally symplectic linear system \((\Sigma, f, \mathcal{E}, \tilde{A}, \mu)\) such that \(d(\tilde{A}, A) < \varepsilon\) and which preserves the same conformally symplectic structure \(\mu\), i.e., \(\tilde{A}^* \omega = \mu \omega\).

**Lemma 15.** Fix \(\varepsilon > 0\). Any \(\varepsilon\)-perturbation of a matrix \(A \in CS(n, \mathbb{R})\) can be written as a composition of \(A\) with a symplectic matrix \(\varepsilon_1\)-close to the identity \(E\) and \(\varepsilon = \varepsilon_1\|A\|\). The compositions of \(A\) with \(E\) a symplectic matrix \(\varepsilon_1\)-close to the identity, \(A \circ E\) and \(E \circ A\), are \(\varepsilon\)-perturbations of \(A\).

**Proof.** Let \(\tilde{A}\) be an \(\varepsilon\)-perturbation of \(A\) and the symplectic matrix \(E = \tilde{A} \circ A^{-1}\) then \(E \circ A = \tilde{A}\) and \(\|E \circ A - A\| < \varepsilon\).

Let \(E \in S(n, \mathbb{R})\) be a symplectic matrix \(\varepsilon_1\)-close to the identity then \(E \circ A\) and \(A \circ E\) are \(\varepsilon\)-perturbations of \(A\).

**Definition 16.** A conformally symplectic periodic linear system \((\Sigma, f, \mathcal{E}, B, \mu)\) of dimension \(2n\) is diagonalizable if for all \(x \in \Sigma\) the matrix \(M_B(x)\) has real positive eigenvalues with multiplicity 1. We denote \(\lambda_1(x) \leq \cdots \leq \lambda_n(x) \leq \lambda_{\pi}(x) \leq \cdots \leq \lambda_{\tau}(x)\) the eigenvalues of \(M_B(x)\) such that \(\lambda_i(x)\lambda_j(x) = \mu(x)\) for \(i = 1, \ldots, n\). We also denote \(E_i(x)\) the eigenspace corresponding to \(\lambda_i(x)\) and \(E_\pi(x)\) the eigenspace corresponding to \(\lambda_{\pi}(x)\). Then \(\mathcal{E} = (\oplus_i E_i) \oplus (\oplus_\pi E_\pi)\) is an invariant decomposition of \(B\).

**Lemma 16.** Let \((\Sigma, f, \mathcal{E}, A, \mu)\) be a conformally symplectic periodic linear system with transitions. Then for \(\varepsilon > 0\) there exists diagonalizable \(\varepsilon\)-perturbation \(B\) of \(A\) defined on a dense set \(\tilde{\Sigma} \subset \Sigma\).

**Proof.** Given a matrix \(A \in CS(n, \mathbb{R})\), there exists an arbitrarily small perturbation \(B \in CS(n, \mathbb{R})\) of \(A\) such that \(B\) has all its eigenvalues are positive real numbers and with multiplicity 1.

This follow from [17] which constructs a symplectic change of coordinates \(T\) such that

\[T^{-1}AT = \begin{bmatrix}
A_{11} & 0 & 0 & A_{12} & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & A_{n1} & 0 & 0 & A_n2 \\
A_{13} & 0 & 0 & A_{14} & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & A_{n3} & 0 & 0 & A_{n4}
\end{bmatrix}\]

where the submatrices \(\begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix}\) are canonical blocks analogous to the Jordan blocks. \(\square\)

### 5.2 Restriction and decompositions

An invariant subbundle \(F\) is a collection of subspaces \(F_x \subset \mathcal{E}_x\) such that \(\dim(F(x)) = c \ \forall x \in \Sigma\) and \(A(F(x)) = F(f(x))\). A \(A\)-splitting \(E = F \oplus G\) is given by invariant subbundles such that \(\mathcal{E}_x = E(x) \oplus G(x)\), \(\forall x \in \Sigma\).
Definition 17. Let $(\Sigma, f, E, A, \mu)$ be a conformally symplectic linear system and a $A$-splitting $E = F \oplus G$. We call this splitting a dominated splitting, and we denote $F \prec G$, if there exists $l \in \mathbb{N}$ such that all $x \in \Sigma$:
\[
\|A^l(x)|_F\| \|A^{-l}(f^l(x))|_G\| < \frac{1}{2}.
\]

Definition 18. Let $(\Sigma, f, E, A, \mu)$ be a conformally symplectic linear system and a $A$-splitting $E = \bigoplus_{i=1}^{k} E_i$ into $k$ invariant subbundles $E_1, \ldots, E_k$. We say two subspaces $E_m, E_n$ in $k$ invariant subbundles have dominated splitting associated to the subspaces $E_m$ and $E_n$ if there exists $l \in \mathbb{N}$ such that for all $x \in \Sigma$ the restriction of $A$ in each subspace satisfies:
\[
\|A^l(x)|_{E_m}\| \|A^{-l}(f^l(x))|_{E_n}\| < \frac{1}{2}.
\]
To emphasize $l$, we say $F \oplus G$ is a $l$-dominated splitting and we write $F \prec_l G$.

Differently from the general case, treaded in [8], we need to preserve the conformally symplectic structure and thus we can not make the restriction and quotient of a matrix in such a way that the perturbation on the restriction has no influence on the quotient and vice versa. We replace the concept of restriction and quotient only by the concept of restriction: let $F_1, \ldots, F_k$ be $k$ invariant subspaces such that $E = \bigoplus F_i$, we can write $A$ as:
\[
A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_k \end{pmatrix}
\]
such that $A_i(F_i) \subset F_i$.

The next lemma corresponds to lemma 4.4 in [8] and it uses strongly the symplectic conformal structure.

Lemma 17. Given $K > 0$, $l \in \mathbb{N}$ and a linear system $(\Sigma, f, E, A, \mu)$ bounded by $K$ with invariant decomposition $E = F \oplus \bigoplus_{i=1}^{k} F_i$ then there exists $L$ such that it holds:
(i) if $E \prec_l F$ and $E \prec_l G$ then $E \prec_L (F \oplus G)$.
(ii) if $F \prec_l G$ and $E \prec_l G$ then $(E \oplus F) \prec_L G$.

Proof. Using the conformally symplectic basis we can write the matrices $A$ as:
\[
A = \begin{pmatrix} A_E & 0 & 0 \\ 0 & A_F & 0 \\ 0 & 0 & A_G \end{pmatrix}
\]
Observe that the entry of the second line and third column is zero, differently to the proof of lemma 4.4 in [8] where that entry corresponds to a non-zero matrix $B(x)$. To conclude, the proof follows as in lemma 4.4 in [8] but now using that $B(x) = 0$.

Definition 19. Consider a $f$-invariant set $\Sigma' \subset \Sigma$ and the restriction of $E$ over $\Sigma'$. The conformally symplectic linear system induced by $A$, $(\Sigma', f|_{\Sigma'}, E_{\Sigma'})$, is called a subsystem induced by $A$ in $\Sigma'$.

The next lemma correspond to lemma 1.4 in [8] adapted to the context of conformally symplectic linear systems. The statement in [8] holds for any linear systems so the proof of next lemmas follows as a corollary.
Lemma 18. Let $(\Sigma, f, E, A, \mu)$ be a conformally symplectic continuous linear system such that there exists an invariant dense set $\Sigma_1 \subset \Sigma$ which subsystem admits $l$-dominated splitting, then $(\Sigma, f, E, A, \mu)$ admits an $l$-dominated splitting. More generally, suppose there exists a sequence of subsystems $(\Sigma, f, E, A_k, \mu)$ converging to $(\Sigma, f, E, A, \mu)$ such that for all $k$ there exists a dense invariant subset $\Sigma_k \subset \Sigma$ where $A_k$ admits $l$-dominated splitting. Then $A$ admits $l$-dominated splitting on all $\Sigma$. Finally, a dominated splitting over a conformally symplectic continuous linear systems is continuous.

Lemma 19. Let $(\Sigma, f, E, A, \mu)$ be a conformally symplectic periodic linear system with $\varepsilon$-transitions and with dominated splitting $E_1 < \cdots < E_n$. Fix $\varepsilon_0 > \varepsilon$. Then given two points $x_i$ and $x_j \in \Sigma$, $k \in \{1, \ldots, m\}$ and $[i^j]$ $\varepsilon$-transition between $p_i$ and $p_j$. Then there exists $\varepsilon_0$-transition $[i^j]$ such that $\tilde{T}_{ji}$ takes $E_k(x_i)$ in $E_k(x_j)$.

Proof. The application $T_{ji}$ satisfies $\text{angle}(\tilde{T}_{ji}(E_k(x_i)), E_k(x_j)) < \varepsilon$. Consider a perturbation of $\tilde{T}_{ii}$ such that $\tilde{T}_{ji}(E_k(x_i))$ does not have components in $E_m(x_i)$, $m > k$. Due to the dominated splitting, the image of $E_k(x_i)$ by $M^n_j(x_j)\tilde{T}_{ji}$ is arbitrarily close to $E_k(x_j)$ when $n$ is big enough. By a perturbation of $M_A$ with a small rotation we have that $M^n_A(x_j)\tilde{T}_{ji}(E_k(x_i)) = E_k(x_j)$. \qed

The next lemma needs the definition of the rank of an eigenvalue.

Definition 20. Let $M \in GL(n, \mathbb{R})$ such that $M$ has a complex eigenvalue $\lambda$. We say that $\lambda$ has rank $(i, i + 1)$ if there exists an $M$-invariant splitting of $\mathbb{R}^n$, $F \oplus G \oplus H$, such that:

(i) the eigenvalues $\sigma$ of $F$ satisfies $|\sigma| < |\lambda|,
(ii) the eigenvalues $\sigma$ of $H$ satisfies $|\sigma| > |\lambda|,
(iii) dim(F) = i - 1$ and $dim(F) + dim(G) + dim(H) = n,
(iv) the plane $G$ is the eigenspace of $\lambda$.

Lemma 20. Fix $\varepsilon > 0$. Let $(\Sigma, f, E, A, \mu)$ be a conformally symplectic linear system and $p \in \Sigma$ a periodic point with complex eigenvalue of rank $(i, i + 1)$ associated to the subspace $E_i \oplus E_{i+1}$. Then there exists $\varepsilon$-perturbation $\tilde{A}$ of $A$ and $m \in \mathbb{N}$ such that $[M]^m_{\tilde{A}}(p)(E_{i+1}) = E_i$.

Proof. We perform a $\frac{\varepsilon}{2}$-perturbation $\tilde{A}$ of $A$ such that the complex eigenvalue of rank $(i, i + 1)$ of $\tilde{A}$ has irrational entries. Then there exists $m$ such that the angle between $[M]^m_{\tilde{A}}(p)(E_{i+1})$ and $E_i$ is equal to $\alpha < \frac{\varepsilon}{2}$. Composing $\tilde{A}$ with a rotation by $\alpha$ (which is symplectic), we conclude the result. \qed

Lemma 21. Given a vector $v$ and two conformally symplectic matrices $T$ and $M$ such that the vector $w$ is the eigenvector associated to the largest eigenvalue in modulus of $M$, then there exist $n$ and a $\varepsilon$-perturbation $\tilde{B}$ of $B = M^n \circ T$ such that $\tilde{B}(v) = w$.

Proof. If $v$ is an invariant vector, we perform a symplectic perturbation to make $v$ not invariant. Hereafter, we iterate $M$ to make the angle between $Bv = M^n \circ T(v)$ and $w$ equal to $\alpha < \varepsilon$. We compose $B$ with a rotation by $\alpha$ and then we have the result. \qed

5.3 Bidimensional case

The proof of the next two results follows from an argument of R. Mañé in [19] which is also valid in the conformally symplectic case.

Proposition 5. Given $K > 0$ and $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that for any two dimensional conformally symplectic linear system $(\Sigma, f, E, A, \mu)$ with norm bounded by $K$ and matrices $M_A(x)$ preserving orientation, one of the following possibilities holds:
(i) $A$ admits $l$-dominated splitting.

(ii) There exists $\varepsilon$-perturbation $\tilde{A}$ of $A$ and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ has a complex eigenvalue.

We denote with $R^\theta : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation by the angle $\theta$.

**Lemma 22.** For all $\alpha > 0$ and all matrix $M \in GL_+(2,\mathbb{R})$ with two eigenspaces $E_1$ and $E_2$ whose angle is less than $\alpha$ then there exists $s \in [-1,1]$ such that $R^{s\alpha} \circ M$ has a complex eigenvalue.

### 5.4 Generic Dichotomy

To prove the main theorem, it suffices to proof the analogous result for conformally symplectic linear systems, once we reduced the problem to this case.

**Proposition 6.** For $K > 0$, $n > 0$ and $\varepsilon > 0$ there exists $l > 0$ such that a conformally symplectic periodic linear system $(\Sigma, f, E, A, \mu)$ bounded by $K$ and with transitions satisfying one of the next possibilities:

(i) $A$ admits a $l$-dominated splitting.

(ii) There exists a $\varepsilon$-perturbation $\tilde{A}$ of $A$ and a point $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is an homothety.

The proof is divided into two steps: the first one says if we can not create complex eigenvalues of rank $(i, i+1)$ for $i \in \{1, \ldots, 2n-1\}$ then we have dominated splitting and the second one says if we can create such eigenvalues then we can perturb the original linear system with transition to get homotheties.

**Proposition 7.** Fix $\varepsilon > 0$, $n \in \mathbb{N}$, and $K > 0$. Let $(\Sigma, f, E, A, \mu)$ be a conformally symplectic continuous linear system with transitions, of dimension $2n$ bounded by $K$ such that all $\varepsilon$-perturbation of $A$ does not admit complex eigenvalues of rank $(i, i+1)$ for $i = 1, \ldots, 2n-1$. Then there exists $l \in \mathbb{N}$ such that $(\Sigma, f, E, A, \omega, \mu)$ admits $l$-dominated splitting $E = F \oplus G$, $F \prec_l G$, with $\dim(F) = i$

The proof of previous proposition is given in subsection 5.5

**Proposition 8.** Fix $\varepsilon > \varepsilon_0 > 0$. Let $(\Sigma, f, E, A, \mu)$ be a conformally symplectic periodic linear system with transitions. Suppose for all $i = 1, \ldots, 2n-1$ there exists $\varepsilon_0$-perturbation of $A$ with complex eigenvalues of rank $(i, i+1)$. Then there exists $\varepsilon$-perturbation $\tilde{A}$ of $A$ and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is a homothety.

### 5.5 Sufficient condition for $l$-dominated splitting

In this section, we prove the proposition using the next three lemmas which assure us that if a conformally symplectic periodic diagonalizable linear system with transitions has no complex eigenvalues of rank $(i, i+1)$ then there exists dominated splitting $E \prec F$ such that $\dim(E) = i$. In lemma 23 it is proved that provided two invariant subspaces, $E_j$ and $E_{k+1}$, if they do not have a $l$-dominated splitting then it is possible to create a complex eigenvalue of rank $(i, i+1)$ for $i \in \{j, \ldots, k\}$. The idea of the proof is to restrict the application $A$ to the subspaces $E_j$ and $E_{k+1}$ and to use the bidimensional results observing that all the perturbation are performed in the conformally symplectic space. This means that when we perturb the eigenvalues associated to the eigenvectors $\lambda_j$ and $\lambda_{k+1}$ then this perturbation affects all the subspaces $E_j$, $E_{k+1}$, $E_j$, and $E_{k+1}$. In lemma 24 we prove that if we do not have complex eigenvalue of rank $(i, i+1)$ then there exists $L_1$ such that there is a $L_1$-dominated restricted to the sum of subspaces $E_{k+1}$. Finally,
lemma \text{23} says that there exists \( L_2 \) such that the \( L_2 \)-dominated splitting is still valid for the sum of \( E_k \). With this three lemmas, the proof of proposition \text{7} follows from lemmas \text{16} and \text{18}.

For the proof we will need the rotation \( R_{i,j}^\theta : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} 

\begin{equation*}
R_{i,j}^\theta = \begin{pmatrix}
i & j \\
\begin{array}{ccccccccc}
1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & \cos(\theta) & 0 & \cdots & 0 & -\sin(\theta) & \cdots & 0 \\
\vdots \\
0 & \cdots & \sin(\theta) & 0 & \cdots & 0 & \cos(\theta) & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array}
\end{pmatrix}
\end{equation*}

\text{Lemma 23.} Let \((\Sigma, f, \mathcal{E}, B, \mu)\) be a conformally symplectic diagonalizable linear system of dimension \( 2n \) and bounded by \( K > 0 \). Given \( \varepsilon > 0 \) and \( 1 \leq i \leq 2n - 1 \), there exists \( l \in \mathbb{N} \) such that one of the alternatives is valid:

(i) There exists \( \varepsilon \)-perturbation of \( B \) with complex eigenvalue of rank \((i, i + 1)\),

(ii) \( E_j \prec_l E_{k+1} \) for all \( j \leq i \leq k \leq 2n - 1 \).

\text{Proof.} Recall that \( \lambda_i \lambda_{n+i} = \mu \) for \( i = 1, \ldots, n \) and \( \tilde{\lambda}_i = \lambda_{(i+n-1)(\text{mod} 2n)+1} \). Let \( l \) be the constant of domination given by proposition \text{5}. If \( E_j \prec_l E_{k+1} \) for all \( j \leq i \leq k \leq 2n - 1 \) then it is done. Moreover, by the proposition \text{22} the angle between \( E_j \) and \( E_{k+1} \) is less than \( \alpha \) and we can perturb \( B \) to obtain a complex eigenvalues associated to the subspaces \( E_j \) and \( E_k \) using the rotation \( R_{j,k}^\alpha \). In order to preserve the conformally symplectic structure, we define the isotopy

\[ \tilde{B}_t = R_{j,k+1}^\alpha \circ R_{j,k+1}^\alpha \circ B. \]

By continuity, there exists \( t_0 \in [0, 1] \) such that one of this alternatives is valid:

- \( \lambda_i^f = \lambda_{i+1} \leq \lambda_{k+1}^f \),
- \( \lambda_i^f \leq \lambda_i = \lambda_{k+1}^f \),
- \( \lambda_i < \lambda_i^f = \lambda_{k+1}^f < \lambda_{i+1} \),
- \( \lambda_{k+1}^f = \lambda_{i+1} \leq \lambda_{j+1}^f \),
- \( \lambda_{k+1}^f \leq \lambda_i = \lambda_{j+1}^f \),
- \( \lambda_i < \lambda_{k+1}^f = \lambda_{j+1}^f < \lambda_{i+1} \).

The last three possibilities are related to the cases \( j \leq n, k + 1 > n \) e \( j + n \neq k + 1 \). In all cases a small perturbation produces a complex eigenvalue of rank \((i, i + 1)\). Considering the difference between the general case and the conformally symplectic case we can create besides the eigenvalue of rank \((i, i + 1)\) associated to the subspaces \( E_j e E_{k+1} \), a complex eigenvalue associated to the subspaces \( E_j e E_{k+1} \)

\[ \begin{array}{cc}
\lambda_1 & \lambda_{i+1} \\
\lambda_2 & \lambda_3 \\
\lambda_4 & \lambda_5 \\
\lambda_6 & \lambda_7 \\
\lambda_8 & \lambda_9 \\
\lambda_{2n} & \\
\hline
\end{array} \]

\[ \begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\hline
A & B \\
\end{array} \]

\[ \begin{array}{c}
\lambda_1 \quad \lambda_2 \quad \lambda_3 \\
\hline
i & i+1 \\
\end{array} \]

\text{Lemma 24.} Let \((\Sigma, f, \mathcal{E}, B, \mu)\) be a conformally symplectic diagonalizable linear system of dimension \( 2n \) and bounded by \( K > 0 \). Given \( \varepsilon > 0 \) there exists \( l_0 \in \mathbb{N} \) such that for \( 1 \leq i \leq 2n - 1 \) is valid one of the alternatives below:
(i) $B$ admits $\varepsilon$-perturbation with complex eigenvalue of rank $(i, i + 1)$,
(ii) for all $j \leq i$, $E_j \prec \oplus_{i+1}^{2n} E_k$.

Proof. The proof is an inductively application of the lemmas $17$ and $23$. \hfill $\Box$

Lemma 25. Let $(\Sigma, f, E, B, \mu)$ be a conformally symplectic diagonalizable linear system of dimension $2n$ and bounded by $K > 0$. Given $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that for $1 \leq i \leq 2n - 1$ is valid one of the alternatives below:

(i) $B$ admits $\varepsilon$-perturbation with complex eigenvalue of rank $(i, i + 1)$,
(ii) $\oplus_{i}^{1} E_k \prec \oplus_{i+1}^{2n} E_k$.

Proof. The proof is an inductively application of the lemmas $17$ and $24$. \hfill $\Box$

6 Proof of Theorem C (Franks’ Lemma)

In $[14]$, Franks proved that given a $C^1$ diffeomorphism $f : M \to M$ over a Riemannian manifold $(M, g)$ and $\varepsilon > 0$, if we take a periodic point $x \in M$, we can perform a $C^1$ small perturbation $g$ of $f$ such that $g^n(x) = f^n(x)$, $n \in \mathbb{Z}$, and $dg^n$ is any isomorphism $\varepsilon$-close of $df^n$, for $n \in \mathbb{Z}$. This result is known as Franks lemma. The key point in that lemma is since is only required that $g$ is $C^1$-close to $f$, the support of the perturbation can be done arbitrary small in such a way that the perturbation preserves the trajectory.

For geodesic flows, an analogous result requires a different technique from that used by Franks, since geodesic flow perturbations are metric perturbations and these perturbations are not local as was described above. In fact, to perturb a metric on a neighborhood of a closed geodesic means that it is performed a perturbation in a cylinder in the tangent space where the geodesic flow is defined. However, and overcoming such difficulty, in $[10]$, Contreras proved a version of the Franks lemma to the geodesic flow.

For a Gaussian thermostat, the situation is the same as the geodesic flow. In this case, we perturb the vector field $E$ and similarly, a perturbation of $E$ in a neighborhood of the manifold implies a perturbation over a cylinder in the tangent space where the flow is defined. In this section, we prove a version of the Franks lemma for Gaussian thermostats adapting the ideas of Contreras in $[10]$.

Let $(M, g)$ be a Riemannian manifold of dimension $n + 1$. We denote with $\mathcal{X}^k(M)$ the space of vector fields over $M$ of class $C^k$ and with $\mathcal{R}^l(M)$ the space of Riemannian metrics over $M$ with $C^l$ topology.

Let $(M, g, E)$ be the Gaussian thermostat defined over $(M, g)$ with $E \in \mathcal{X}_g(M) \cap \mathcal{X}^r(M)$ and $\eta$ a closed orbit of $(M, g, E)$. Our goal is to find a perturbation of $E$ which has $\eta$ as a closed orbit and, moreover, the transversal derivative cocycle associated to a point $p$ in $\eta$ (i.e., its linear Poincaré application) is any conformally symplectic application close to the original one. We consider on $\mathcal{X}^r(M)$ the topology defined by the open sets $B(E, r) = \{ \bar{E} \in \mathcal{X}^r(M) | \max \{ \| \bar{E} - E \|_{C^1} < r \} \}$.

The main theorem of this section is:

**Theorem C.** Let $E \in \mathcal{X}_g(M) \cap \mathcal{X}^r(M)$, $4 \leq r \leq \infty$, a periodic orbit $\eta$ of the Gaussian thermostat $(M, g, E)$, a point $\theta$ in $\eta$ and $T : T_{\theta}SM \to T_{\theta}SM$ its transversal derivative cocycle.

Fixed $\varepsilon > 0$ there exists $\delta > 0$ such that given a conformal linear map $L \| L - T \| < \delta$ then there exists $\bar{E} \in \mathcal{X}_g(M)$ with $d(\bar{E}, E)_{C^1} < \varepsilon$ which defines the Gaussian thermostat $(M, g, \bar{E})$ and such that it has $\eta$ as an orbit and its transversal derivative cocycle at $\theta$ is $L$. 

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The idea of the proof goes as follows. Consider \((M, g, E)\) the Gaussian thermostat associated to the vector field \(E \in \mathcal{X}_g(M) \cap \mathcal{X}^r(M)\). Let \(\eta\) a periodic orbit of \((M, g, E)\), \(p \in M\) a point of \(\eta\) and \(T : T_pM \rightarrow T_pM\) the linear Poincaré application associated to \(p\). We consider the application
\[
S : \mathcal{X}_g(M) \cap \mathcal{X}^r(M) \rightarrow CS(n)
\]
whose domains are the vector fields of class \(C^r\) which defines a closed 1-form \(\gamma\) and the image are the conformally symplectic matrices such that \(S(E) = T\).

We consider subsets in \(\mathcal{X}_g(M)\) satisfying the following:

\(\mathcal{G}_1\) For every orbit segment of size \(\frac{r_{inj}}{2}\), where \(r_{inj}\) is the injectivity radius of \((M, g)\), has at least one point such that the matrix \(K_g + B_g\) has all distinct eigenvalues, where \(K\) is the curvature matrix and \([B]_{ij} = \frac{\partial E^j}{\partial x^i}\) is the derivative matrix of \(E \in \mathcal{X}_g(M) \cap \mathcal{X}^1(M)\),

\(\mathcal{G}_2\) Provided a finite segment of orbit of the Gaussian thermostat, we consider the set of vector fields \(C^1\)–close to \(E\) such that they coincides with \(E\) on the self intersections of the given finite segment of orbit.

Let \(\mathcal{U}\) be a neighborhood of \(E\), we prove that the image by \(S\) of \(\mathcal{U} \cap \mathcal{G}_1 \cap \mathcal{G}_2\) contains a ball of radius \(\delta > 0\). This means that any \(\delta\)-perturbation of the transversal derivative cocycle \(T\) can be performed by a perturbation in \(\mathcal{U} \cap \mathcal{G}_1 \cap \mathcal{G}_2\).

We do this by partitioning the orbit \(\eta\) in some segments \(\eta_i, i = 1, \ldots, n\). For each segment, we consider the transversal derivative linear cocycle on the extremal points and we define the applications \(S_i : \mathcal{X}_g(M) \cap \mathcal{X}^r(M) \rightarrow CS(M)\) which satisfies \(S_i(E) = T_i\) where \(T_i : T_{\eta^0_i}M \rightarrow T_{\eta^1_i}M\) is transversal derivative linear cocycle on the extremal points \(\eta^0_i\) and \(\eta^1_i\) of \(\eta_i\).

Using the Jacobi equation for Gaussian thermostats, we show \(\|dS_E\| > d > 0\) and this implies the result.

We consider \(\pi : TM \rightarrow M\) the canonical projection. We denote \(S(n)\) the symmetric \(n \times n\) matrices, \(S^*(n)\) the symmetric \(n \times n\) matrices with null diagonal and \(O(n)\) the orthogonal \(n \times n\) matrices.

Before we prove the theorem we need some preliminary:

### 6.1 Generic condition

Given \(g \in \mathcal{R}^2(M)\), we define

- \(K_g : SM \rightarrow S(n)_{\mathcal{O}(n)}\)

\[K_g(\theta)_{ij} = -(R_g(\theta, e_i(\theta), e_j(\pi(\theta)))\]

- \(B_g : SM \rightarrow S(n)_{\mathcal{O}(n)}\)

\[B_g(\theta)_{ij} = \frac{\partial E^i}{\partial x^j}(\pi(\theta)),\]

where \(\{\theta, e_1 = e_1(0), \ldots, e_n = e_n(0)\}\) is an orthonormal basis of \(T_{\phi^E_0(\theta)}M\) and \(\{\theta(t), e_1(t), \ldots, e_n(t)\}\) is the parallel transport of this basis along \(c(t) = \pi \circ \phi^E_t\left(\frac{\theta}{\|\theta\|_g}\right)\) according to the Riemannian connection associated to the metric \(g\). Let us remember the definition of \(\mathcal{G}_1\):

\[\mathcal{G}_1 = \{E \in \mathcal{X}_g(M) \cap \mathcal{X}^1(M)|\]

\[
every \text{orbit segment of size } \frac{r_{inj}}{2} \text{ has at least one point}
\]

\[
such \text{that the matrix } K_g + B_g \text{ has all distinct eigenvalues}\}\}

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The following theorem states that $\mathcal{G}_1$ is generic.

**Theorem 4.** The set $\mathcal{G}_1$ is open in $\mathcal{X}_g(M) \cap \mathcal{X}^1(M)$ and $\mathcal{G}_1 \cap \mathcal{X}^\infty(M)$ is dense in $\mathcal{X}_g(M)$.

The proof only differs from the proof in [10] by one aspect: now we need the eigenvalues of $K_g + B_g$ to be distinct instead of the eigenvalues of $K_g$. Since the proof is similar, we leave it to the reader.

### 6.2 Perturbative theorem

We begin this section with the definition of a prime orbit:

**Definition 21.** Let $\mathcal{F}_0$ be an orbit segment and $\mathcal{F}_0$ the set of orbit segments which intersect $c_0$. Let $W_0 \subset W$ be a tubular neighborhood of $c = \pi \circ \eta$, where $W$ is open in $\mathcal{X}^\tau(M)$. Given a segment $c_0$ the set of orbit segments which intersect it is noted $\mathcal{F}_0 = \{c_1, \ldots, c_l\}$.

![Diagram](image)

For each segment $\eta_k$, we define the application $S_k : \mathcal{X}_g(M) \cap \mathcal{X}^\tau(M) \rightarrow CS(n)$ by

$$S_k(E) = T_k^E,$$

where $T_k^E$ is the transversal linear cocycle between $\eta_k(0)$ and $\eta_k(1)$.

It can happen that the segments $c_k$ transversally intersect on $M$. Given a segment $c_0$ the set of segments which intersect it is noted $\mathcal{F}_0 = \{c_1, \ldots, c_l\}$.

For the segment $\eta_0$, after we take $\mathcal{F}_0$ and $W_0$ we also have the set $\mathcal{G}_2(\eta_0, W_0, \mathcal{F}_0)$ and we apply the following result, which is the perturbation theorem on an orbit segment:

**Theorem 5.** Let $E \in \mathcal{G}_1 \cap \mathcal{X}_g(M) \cap \mathcal{X}^\tau(M)$, $4 \leq \tau \leq \infty$, and $\eta_0$ an orbit segment associated to the Gaussian thermostat $(M, g, E)$. Given a neighborhood $U \in \mathcal{X}_g(M) \cap \mathcal{X}^1(M)$, there exists $\delta_0 = \delta_0(g, E, U) > 0$ such that given $W_0$ and $\mathcal{F}_0$ as above then the image of $U \cap \mathcal{G}_1 \cap \mathcal{G}_2(\eta_0, W_0, \mathcal{F}_0)$ by the application $S_0$ contains a ball of radius $\delta_0$ centered on $S_0(E)$.

For $c_1$, we take $\mathcal{F}_1$, $W_1$, and $\mathcal{G}_2(c_1, W_1, \mathcal{F}_1)$. Hereafter, we apply again the above theorem to obtain the image of a neighborhood of the vector field $E$ by $S_1$ which contains a ball of radius $\delta_1$ centered at $S_1(E)$, $B_{\delta_1}(S_1(E))$. We repeat this procedure for $\eta_2, \ldots, \eta_m$.

The sets $\mathcal{G}_2(c_i, W_i, \mathcal{F}_i)$, $i \in \{0, \ldots, m\}$, guarantee us there is no interference between one perturbation and the next one and, moreover, they are contained in the following set.
Definition 22. We define $\mathcal{G}_2(\eta, E, W)$ the set of vector field $\tilde{E} \in \mathcal{X}^\infty(M)$ such that

$$\text{supp}(\|\tilde{E} - E\|) \subset W.$$ 

To finish the proof, we consider the application

$$S = (S_0, \ldots, S_m) : \mathcal{X}_g(M) \cap \mathcal{X}^\tau(M) \rightarrow CS(n) \times \cdots \times CS(n).$$

The image of $U \cap \mathcal{G}_1 \cap \mathcal{G}_2(\eta, E, W)$ contains the product $B_{\delta_0}(S_0(E)) \times \cdots \times B_{\delta_m}(S_m(E))$. This result, translated to the transversal linear cocycle language means that a perturbation on the transversal linear cocycle contained on a ball centered at $T$ and radius $\min_{i \in \{0, \ldots, m\}} \{\delta_i\}$ can be realized with a vector field $\tilde{E}$ in $U \cap \mathcal{G}_1 \cap \mathcal{G}_2(\eta, E, W)$.

6.3 Perturbative theorem along an orbit segment. Proof of theorem 5

Now we prove theorem 5, which is the intermediate step for the main result of this section. First we show how to perturb the vector field $E$. We take Fermi coordinates along $\eta$ and consider the application $\alpha : [0, \tau] \times \mathbb{R}^n \rightarrow S(n)$ of class $C^\infty$ with support in a neighborhood of $[0, \tau] \times \{0\}$ and $\lambda : [0, \tau] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\lambda(t, 0) \equiv \frac{\lambda}{2}$ for $t \in [0, \tau]$, $\lambda > 0$ and has its support in a neighborhood of $[0, \tau] \times \{0\}$ with $\tau < \tau_{\text{inj}}$.

Let $\eta$ be an orbit of the Gaussian thermostat, $\eta(t) = (u(t), v(t))$, and consider

$$\begin{align*}
\tilde{E}_0 &= E_0 + \lambda(t, x) \\
\tilde{E}_i &= E_i + \sum_j \alpha_{ij} x_j \quad \text{para } i = 1, \ldots, n.
\end{align*}$$

The orbit $\eta$ of $(M, g, E)$ is also an orbit of $(M, g, \tilde{E})$. In fact,

$$\begin{align*}
\dot{u} &= v \\
\frac{\partial}{\partial t} v &= \tilde{E} - \text{Proj}_v \tilde{E} = E - \text{Proj}_v E.
\end{align*}$$

We call the intersection of the set of the perturbations of $E$ given as above with $E \in \mathcal{G}_2(\eta_0, W_0, \mathcal{F}_0)$ as $\mathcal{G}_2$. For the proof, we also need some special functions:

$\phi_\varepsilon$: Given $\varepsilon > 0$, let $\phi_\varepsilon : \mathbb{R}^n \rightarrow [0, 1]$ be a function of class $C^\infty$ such that $\phi_\varepsilon(x) = 1$ if $x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^n$ and $\phi_\varepsilon(x) = 0$ if $x \notin [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^n$. We also require that there is a fixed $k$ such that

$$\|\phi_\varepsilon(x) x^* P(t) x\| \leq k\|P\|_{C^0} + \varepsilon k\|P\|_{C^1} + \varepsilon^2 k\|P\|_{C^2},$$

where $P$ is a function that is defined in proposition 3 and depends on the next two functions defined below.
\( \bar{h} \): The function \( \bar{h} : [0, 1] \to [0, 1] \) is of class \( C^\infty \) with support far from the intersection points, i.e., \( \text{supp}(\bar{h}) \cap (\pi \circ \eta)^{-1}V_i = \emptyset \) where \( V_i \) is a neighborhood of \( F_i \) and such that
\[
\int_0^1 (1 - \bar{h}(s)) \, ds < \rho.
\]

\( \delta \): The function \( \delta : [0, 1] \to [0, +\infty] \) is of class \( C^\infty \) such that \( \delta(s) = 0 \) if \( |s - \tau| \geq \lambda \) and \( \int_0^1 \delta(s) \, ds = 1 \).

To prove theorem 5 we need to compute \( d_0 F\zeta (\sigma = 0) \) but for completeness we compute \( d_\sigma F\zeta \) at any value of \( \sigma \).

**Proposition 9.** Let \( F : S(n)^3 \times S^*(n) \times \mathbb{R} \to CS(n) \) given by \( F(\sigma, \lambda) = d\phi_1^E = X_1 \), where \( (\sigma, \lambda) \in S(n)^3 \times S^*(n) \times \mathbb{R} \) and

(i) \( \sigma = (a, b, c; d) \in S(n)^3 \times S^*(n) \),
(ii) \( P(t) = \bar{h}(t)[a \delta(t) + b \delta'(t) + c \delta''(t) + d \delta'''(t)] \),
(iii) \( \alpha(t, x) = P(t)\phi_{\epsilon}(x)x_j \),
(iv) \( \tilde{E}(t, x)_0 = (1 + (\lambda \bar{h}(t)\delta(t) - 1)\phi_{\epsilon}(x))E_0 \),
(v) \( \tilde{E}(t, x)_i = E_i + \sum_j \alpha_{ij}x_j \).

Then there exists \( k \) such that
\[
\|d_{(\sigma, \lambda)} F\zeta\| \geq k\|\zeta\| \quad \forall \zeta = (a, b, c; d; e) \in S(n)^3 \times S^*(n) \times \mathbb{R}.
\]

*Proof.* Consider the path in \( \mathcal{X}_g(M) \) given by,
\[
\gamma : s \to (\sigma, \lambda) + s\zeta,
\]
and the Jacobi equation for \( \gamma(s) \) along \( \eta \):
\[
\ddot{T}_s = \mathbb{A}_s T_s,
\]
where
\[
\mathbb{A}_s = \begin{bmatrix} 0 & I \\ K_s + B_s & C_s \end{bmatrix}
\]
\[
K_s = K,
B_s = B + sP(t),
C_s = (1 + s\lambda)C.
\]

Differentiating this equation with respect to \( s \), we have a differential equation for \( Z_t = \frac{dT_s(t)}{ds} \bigg|_{s=0} \)
\[
\dot{Z} = \mathbb{A} Z + \mathbb{B} X,
\]
where
\[
\mathbb{A} = \begin{bmatrix} 0 & I \\ K + B & C \end{bmatrix},
\]
\[
\mathbb{B} = \begin{bmatrix} P(t) & 0 \\ 0 & \lambda C \end{bmatrix}.
\]
We know $Z_1 = d_{(\sigma, \lambda)} S \cdot \zeta$. This follows from the definition of $Z$:

$$Z(1) = \left. \frac{d}{dr} T^r(1) \right|_{r=0} = \left. \frac{d}{dr} T^{(\sigma, \lambda) + r \zeta}(1) \right|_{r=0} = \left. \frac{d}{dr} (F((\sigma, \lambda) + r \zeta)) \right|_{r=0}.$$

If we write $Z_t = T_t Y_t$ then $T Y = T$ (motivation: $T X CS(\mathbb{R}^n) = T(T_t CS(\mathbb{R}^n)))$. Moreover, $T_{\lambda}(0) \equiv I$ then $Z(0) = 0$ and $Y(0) = 0$. Thus,

$$Y(t) = \int_0^t (T^s)^{-1} \mathbb{B}^s T^s ds$$

$$\mathbb{B}(X) = \left[ \begin{array}{cc} 0 & 0 \\ P(t) & \lambda C \end{array} \right] = \tilde{h}(t) \{ \delta(t) \tilde{A} + \delta'(t) \tilde{B} + \delta''(t) \tilde{C} + \delta'''(t) \tilde{D} + \delta(t) \tilde{F} \}$$

$$\tilde{A} = \left[ \begin{array}{cc} 0 & 0 \\ a & 0 \end{array} \right], \tilde{B} = \left[ \begin{array}{cc} 0 & 0 \\ b & 0 \end{array} \right], \tilde{C} = \left[ \begin{array}{cc} 0 & 0 \\ c & 0 \end{array} \right], \tilde{D} = \left[ \begin{array}{cc} 0 & 0 \\ d & 0 \end{array} \right],$$

$$\tilde{F} = \lambda \sigma \left[ \begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right]$$

Integrating by parts $Y(1)$, we have:

$$\int_0^1 T^{-1}_s \delta'(s) \tilde{B} T_s ds = \int_0^1 \delta(s) T^{-1}_s [\tilde{A} \tilde{B} - \tilde{B} \tilde{A}] T_s ds$$

$$= \int_0^1 \delta(s) T^{-1}_s \left[ \begin{array}{cc} b & 0 \\ \sigma b & -b \end{array} \right] T_s ds.$$

$$\int_0^1 T^{-1}_s \delta''(s) \tilde{C} T_s ds = \int_0^1 \delta'(s) T^{-1}_s \left[ \begin{array}{cc} c & 0 \\ \sigma c & -c \end{array} \right] T_s ds$$

$$= \int_0^1 \delta(s) T^{-1}_s \left( H \left[ \begin{array}{cc} c & 0 \\ \sigma c & -c \end{array} \right] + \left[ \begin{array}{cc} c & 0 \\ \sigma c & -c \end{array} \right] H \right) T_s ds$$

$$= \int_0^1 \delta(s) T^{-1}_s \left[ \begin{array}{cc} 0 & -2c \\ Kc + cK & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & 0 \\ Bc + cB & 0 \end{array} \right] + \left[ \begin{array}{cc} \sigma c & 0 \\ \sigma^2 c & -\sigma c \end{array} \right] T_s ds.$$

$$\int_0^1 T^{-1}_s \delta'''(s) \tilde{D} T_s ds = \int_0^1 \delta'(s) T^{-1}_s \left[ \begin{array}{cc} 0 & -2d \\ Kd + dK & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & 0 \\ Bd + dB & 0 \end{array} \right] + \left[ \begin{array}{cc} \sigma d & 0 \\ \sigma^2 d & -\sigma d \end{array} \right] T_s ds$$

$$= \int_0^1 \delta(s) T^{-1}_s \left( \left[ \begin{array}{cc} Kd + 3dK & 0 \\ 0 & -3Kd - dK \end{array} \right] + \left[ \begin{array}{cc} Bd + 3dB & 0 \\ 2\lambda((K + B)d + d(K + B)) & -3Bd - dB \end{array} \right] + \left[ \begin{array}{cc} \sigma^2 d & 0 \\ \sigma^3 d & -\sigma^2 d \end{array} \right] T_s ds.ight.$$
Lemma 26. Let \(d\) determined by the following diagram is commutative.

\[
\begin{aligned}
\alpha &= a + \sigma b + (K + B)c + c(K + B) + \sigma^2 c + 2\sigma \{(K + B)d + d(K + B)\} + \sigma^3 d, \\
\beta &= b + \sigma c + (K + B)d + 3d(K + B) + \sigma^2 d, \\
\gamma &= -2c, \\
v &= \lambda \sigma.
\end{aligned}
\]

The matrix \(\beta\) decomposes itself in \(\beta = \beta_{sim} + \beta_{asim}\) where \(\beta_{sim}\) is symmetric and \(\beta_{asim} = d(K + B) - (K + B)d\) is antisymmetric. The following lemma, proved in [10], shows that \(\beta_{asim}\) is determined by \(d\). Moreover, \(\beta_{sim}, \alpha,\) and \(\gamma\) are determined by \(b, a,\) and \(c,\) respectively.

Lemma 27. Let \(W\) a symmetric matrix and consider \(L_W : S^*(n) \to AS(n)\) given by \(L_W(d) = Wd - dW\). Suppose the eigenvalues \(\lambda_i\) of \(W\) are all distinct. Then for all \(f \in AS(n)\) there exists \(d \in S^*(n)\) such that \(L_W(d) = f\) and

\[
\|f\| \leq \frac{\|d\|}{\min_{i \neq j} |\lambda_i - \lambda_j|}.
\]

To finish the argument we will need the following lemma which shows there exists \(k\) such that \(\|Z_1\| \geq k\|\zeta\|\) and whose proof can be found in [10].

Lemma 27. Let \(\mathcal{N}\) a connected Riemannian manifold with dimension \(m\) and \(F : \mathbb{R}^m \to \mathcal{N}\) a smooth application such that

\[
|d_x F(v)| \geq a > 0 \quad \forall (x, v) \in T \mathbb{R}^m \quad \text{com} \quad |v| = 1 \quad |x| \leq r
\]

then for all \(0 < b < ar\),

\[
\{ \omega \in \mathcal{N} | d(\omega, F(0)) < b \} \subseteq F(\{ x \in \mathbb{R}^m | |x| < \frac{b}{a} \}).
\]

We will show the image of \(U_0\) by \(S\) contains a ball in \(CS(n)\) centered at \(S(g)\) and radius \(r = r(g, U)\). Consider the application \(G : \mathbb{R}^{2n(n+1)} \to \mathcal{X}^{-r}(M)\) where \(G(\sigma, \lambda) = (\hat{E})\). The following diagram is commutative

\[
\begin{array}{ccc}
B(0, k_5^{-1}r) \subset \mathbb{R}^{2n(n+1)} & \xrightarrow{G} & \mathcal{X}^{-r}(M) \\
& \xrightarrow{F} & CS(n) \\
& \xrightarrow{S} &
\end{array}
\]

The proposition [2] and the lemma [27] show that \(B(S(E), r) \subset F(B(0, k_5^{-1}r))\) but we need \(B(S(E), r) \subset S(U_0)\). To obtain this, it is enough to show \(G(B(0, k_5^{-1}r)) \subset U_0\). In fact, \(\|\hat{E} - E\|_{C^1} < \varepsilon\) therefore \(\hat{E} \in V_0 \subset U_0\) and we have the result.

7 The Gaussian thermostat as a Weyl flow. Proof of theorems D and D’

In this section we describe the linear connection \(\hat{\nabla}_X Y\) introduced in [30] and defined as

\[
\hat{\nabla}_X Y = \nabla_X Y - (X, Y)E + \gamma(Y)X + \gamma(X)Y.
\]
Proposition 10. The linear connection $\hat{\nabla}$ is symmetric and it is non compatible with the metric.

Proof. Symmetry

$$
\hat{T}(X,Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X,Y]
= \nabla_X Y - \langle X,Y \rangle E + \gamma(Y)X
+ \gamma(X)Y - \nabla_Y X + \langle X,Y \rangle E - \gamma(X)Y - \gamma(Y)X - \nabla_X Y + \nabla_Y X
= 0.
$$

Non compatibility with the metric

$$
Z\langle X,Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle
= \langle \hat{\nabla}_Z X + \langle Z, X \rangle E - \gamma(Z)X, Y \rangle
+ \langle X, \hat{\nabla}_Z Y + \langle Z, Y \rangle E - \gamma(Z)Y \rangle
= -2\gamma(Z)\langle X,Y \rangle + \langle \hat{\nabla}_Z X,Y \rangle + \langle X, \hat{\nabla}_Z Y \rangle.
$$

The reparametrization by arc length of the geodesic flow of $\hat{\nabla}$ is called Weyl flow. The following proposition shows the Weyl flow restricted to $SM$ is a Gaussian thermostat.

**Proposition 11.** The Weyl flow is the Gaussian thermostat $(M,g,E)$ when we consider the restriction of it to $SM$.

Proof. Let $u$ a geodesic of $\hat{\nabla}$ and $\psi : \mathbb{R} \to \mathbb{R}$ the reparametrization by arc length of $u$ such that $\eta(t) = u(\psi(t)) \quad e \quad u(s) = \eta(\psi^{-1}(s)).$

Let $\frac{dw}{ds} = w$ and $\frac{dt}{ds} = v$ then

$$
w = \frac{dt}{ds} = v\|w\|,
$$

$$
\frac{d\|w\|}{ds} = -\gamma(w)\|w\|,
$$

$$
\frac{d\|w\|}{dt} = -\gamma(w)\frac{\|w\|}{\|w\|} = -\gamma(w),
$$

and

$$
0 = \hat{\nabla}_w w
= \nabla_w w + 2\gamma(w)w - \|w\|^2 E
= \nabla_v \frac{dt}{ds} + 2\gamma \left( \frac{dt}{ds} \right) v - \left( \frac{dt}{ds} \right)^2 v - \left( \frac{dt}{ds} \right)^2 \|v\|^2 E
$$

$$
= \frac{dt}{ds} v \left( \frac{dt}{ds} \right) v + \left( \frac{dt}{ds} \right)^2 \nabla_v v + 2 \left( \frac{dt}{ds} \right) \gamma(v) v - \left( \frac{dt}{ds} \right)^2 \|v\|^2 E
$$

$$
= -\left( \frac{dt}{ds} \right)^2 \gamma(v) v + \left( \frac{dt}{ds} \right)^2 \nabla_v v + 2 \left( \frac{dt}{ds} \right) \gamma(v) v - \left( \frac{dt}{ds} \right)^2 \|v\|^2 E.
$$

In terms of the Riemannian connection, the equation can be written as

$$
\begin{cases}
\dot{x} = v \\
\dot{v} = E - \gamma(v) v.
\end{cases}
$$

This means $\eta$ is an orbit of $(M,g,E)$ when $\langle v,v \rangle = 1$. 

\[\square\]
We define the curvature tensor associated to $\hat{\nabla}$ as
\[
\hat{R}(X,Y) = \hat{\nabla}_X \hat{\nabla}_Y - \hat{\nabla}_Y \hat{\nabla}_X + \hat{\nabla}_{[X,Y]},
\]
and the sectional curvature
\[
\hat{K}(X,Y) = \langle \hat{R}(X,Y)X, Y \rangle.
\]

Let $f(s,u) = \pi \circ \phi^t(z(u))$ be a family of geodesic of the connection $\hat{\nabla}$ parametrized by $u$. Then the Jacobi field for the geodesic flow $\hat{\nabla}$ is given by,
\[
w(s) = \frac{\partial}{\partial s} f(s,u), \\
J(s) = \frac{\partial}{\partial u} f(s,u), \\
\dot{J}(s) = \hat{\nabla}_{w(s)} J(s), \\
\ddot{J}(s) = -\hat{R}(w, J) w.
\]

Consider now the reparametrization of that family by arclength $\tilde{f}(t,u) = f(\psi(t,u),u)$ where $\psi(t,u)$ is the reparametrization by arc length of $\phi_t(z(u))$. The Jacobi field $\hat{J} = \frac{\partial}{\partial u} \tilde{f}$ satisfies
\[
\dot{\hat{J}} = \frac{\partial \psi}{\partial u} w + J, \\
\ddot{\hat{J}} = v(\frac{\partial \psi}{\partial u}) w + \frac{1}{\|w\|} J + \frac{1}{\|w\|^2} \dot{J}, \\
\dddot{\hat{J}} = v(v(\frac{\partial \psi}{\partial u})) w + \frac{1}{\|w\|} \gamma(v) \dot{J} - \frac{1}{\|w\|^2} \hat{R}(w, J) w.
\]

Therefore, over $\hat{T}_bSM$, we have the Jacobi field for the Weyl flow
\[
\dddot{\hat{J}} = \gamma(v) \dot{J} - \frac{1}{\|w\|^2} \hat{R}(w, J) w.
\]

### 7.1 Proof of geometrical theorem (Theorems D and D’)

The aim now is to relate sectional curvature with the dynamics of a Gaussian thermostat. We start with some results proved in [30].

**Theorem 6 (30).** If the sectional curvature of the Weyl structure is negative everywhere then the W-flow has dominated splitting, with exponential growth/decay of volumes.

In particular, for the case of surfaces follows that

**Theorem 7 (30).** For surfaces, if the curvature of the Weyl structure is negative everywhere then the W-flow is a transitive Anosov flow.
In theorem D and D’ the hypothesis of negative (sectional) curvature are relaxed.

Proof of theorem D: Consider the function $\mathcal{L} : \hat{T}SM \rightarrow \mathbb{R}$ defined as

$$\mathcal{L}(\xi) = \langle \xi_h, \xi_v \rangle,$$

where $\xi_h$ and $\xi_v$ are the horizontal and vertical components of $\xi$, respectively.

We can write $\xi \in \hat{T}SM$ as $\xi = (\xi_h, \xi_v)$ where $\xi_h = \hat{J}(t)$ and $\xi_v = \dot{\hat{J}}(t)$, where $\hat{J}$ is a jocobi field. So the jocobi equation for the Weyl flow over the quotient $\hat{T}SM$ give us

$$\frac{d}{dt}\mathcal{L}(\xi) = \langle \xi_v, \xi_v \rangle + \gamma(v)\langle \xi_h, \xi_v \rangle - \langle \xi_h, \hat{R}(v, \xi_h)v \rangle.$$

Fix $k > 0$. Defining the positive cone $C^+_k(\pi(\xi)) = \{\xi \in \hat{T}_{\pi(\xi)}SM | \mathcal{L}(\xi) \geq k\}$. This defines a cone field over $SM$. The application $\mathcal{L}$ is continuous and monotonically increasing over $C^+_k$ then we can conclude that positive cones are sent in the interior of positive cones by $T^t$.

Analogously, using the reversibility of the flow we get negative cones $C^-_k(\pi(\xi)) = \{\xi \in \hat{T}_{\pi(\xi)}SM | \mathcal{L}(\xi) \leq -k\}$ are sent in the interior of negative cones by $T^{-t}$. This caracterizes dominated splitting for the flow.

Using the following result, the proof of theorem D’ holds.

Theorem 8. ([6]) Let $M$ be a three dimensional closed manifold and $\Lambda$ be a non-singular compact invariant set for $X \in \mathcal{X}^2(M)$ with a dominated splitting such that all periodic trajectory points are hyperbolic saddles, then $\Lambda = \tilde{\Lambda} \cup T$, where $\tilde{\Lambda}$ is hyperbolic and $T$ is a finite union of irrational tori.

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