Optical Poynting singularities of propagating and evanescent vector Bessel beams

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For propagating and evanescent vector Bessel beams, we study the singularities of the Poynting vector (Poynting singularities), at which the energy flux density turns to zero. Poynting singularities include all the phase singularities and some of polarization ones (L- and C-points). We reveal the existence conditions and positions of singularities, which are located at cylindrical surfaces around the beam axis. We mark the special case of the evanescent Bessel beam in the form of cylindrical standing wave, that is singular at any spatial point.

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I. INTRODUCTION

We define the optical Poynting singularity as an isolated point, line or surface at which the Poynting vector of electromagnetic beam vanishes [1]. This singularity is understood in the sense of indeterminate direction of the Poynting vector [2]. In Refs. [2, 3, 4, 5, 6] I. Mokhun et al use another definition of the Poynting vector singularity, according to which transversal component vanishes. This definition can be preferable, because Poynting vector does not achieve exact zero at any point of the field [3]. It is evident that Mokhun’s definition of the singularity includes our definition. In other words, we are to study only for a part of singular points which appear in terms of transverse component singularity.

The most well-known type of singularity is the phase one. It arises, when the points of zero Poynting vector are at the same time the points, where electric or magnetic field (or both) is zero, so that the field phase cannot be defined [7]. It seems to be the typical singularity for scalar beams and finds a use for super-resolution [8, 9, 10], or construction and propagation of optical vortices [8, 11]. Another well-known type of singularity is the polarization singularity (see theoretical [12, 13, 14, 15] and experimental [16] results), when the state of polarization (in general, elliptical) is not fully determined. These singularities include the points of linear polarization, or L-points (rotation direction of polarization is not defined), and the points of circular polarization, or C-points (polarization azimuth is not defined). At L- and C-points the Poynting vector is not zero in general. In some special situations, the energy flux density may vanish owing to the certain directions of the vectors of electric and magnetic fields. It can happen at the points of linear or circular polarization. However, it is not the case in general, because vanishing of the Poynting vector due to polarization (polarization induced Poynting singularity) can be at the points of elliptical polarization, too. Finally, it should be noted that the fundamental connection between polarization and Poynting singularities was noted in Refs. [2, 3]. Both singularities do not necessarily coincide, but go together. Such a relation can be applied for estimation of the magnitude of the angular momentum in some spatial region using field polarization characteristics.

In Ref. [1], the connection between Poynting singularities and singular points of the dynamic systems has been developed. The main definitions and theoretical methods related to arbitrary electromagnetic fields have been introduced therein. In the present paper we study the singularities of vector Bessel beams. Bessel beams have been studied for several decades due to the properties of non-diffraction, self-reconstruction, and angular momentum transfer (see, e.g., [18, 19, 20] and reviews [21, 22] and references therein). In the current paper we consider the exact solutions of the Maxwell equations called vector Bessel beams. The fields of vector Bessel beams are characterized by some polarization distribution, while the intensity of the beam has the form depicted in Fig. 1. A conventional Bessel beam follows from the vectorial one in the paraxial approximation. Some peculiarities connected with the vector nature of electromagnetic Bessel beams were discussed in the papers [23, 24, 25, 26]. The number of applications which use both propagating and evanescent Bessel beams (such as super-resolution lenses mentioned above) grows from year to year. That is why it seems to be very important to study the regularities of spatial structure of complex electromagnetic fields in details. Singular peculiarities of electromagnetic beams (especially vortices) can be used as well, for example, in optical tweezers technique [27], detection of astronomical objects [28], quantum cryptography [29], contrast enhancement in microscopy [30], etc.

The special discussion is necessary on how to create a vector Bessel beam. The field with such intensity and po-
larization distributions arises in the core of the ordinary circular fibers. If the core radius is great, the intensity distribution contains a lot of minima and maxima. The field getting out the fiber can be considered as the vector Bessel beam. The transverse wavenumber \( q \) of such a beam is defined by the dispersion equation.

The paper consists of introduction, two sections and conclusion. Section 2 is devoted to propagating vector Bessel beams: we give the basic information about their origin from Maxwell’s equations and consider the conditions of singularity generation. Section 3 deals with the evanescent beams which may arise under the conditions of total internal reflection and attenuate when propagating. Both sections 2 and 3 include the discussion of electromagnetic field properties at singular points (in particular, the state of polarization) and the identification of the singularity type.

II. SINGULARITIES OF PROPAGATING BESSEL BEAMS

The \( m \)-th order Bessel beam solution of Maxwell’s equations can be written in cylindrical coordinates \((r, \varphi, z)\) as

\[
\begin{pmatrix}
E(r, t) \\
H(r, t)
\end{pmatrix} = \begin{pmatrix}
E(r, \phi) \\
H(r, \phi)
\end{pmatrix} \exp(i \beta z + i m \varphi - i \omega t),
\]

where longitudinal field components (along beam’s axis) satisfy the Bessel equation

\[
\frac{d^2}{dr^2} \begin{pmatrix} E_z \\ H_z \end{pmatrix} + \frac{1}{r} \frac{d}{dr} \begin{pmatrix} E_z \\ H_z \end{pmatrix} + \left(q^2 - \frac{m^2}{r^2}\right) \begin{pmatrix} E_z \\ H_z \end{pmatrix} = 0.
\]

Here \( \omega \) is the circular frequency of electromagnetic wave, \( \beta \) is the longitudinal wavenumber, \( q = \sqrt{k^2 \varepsilon - \beta^2} \) is the transverse (radial) wavenumber, \( m \) is an integer number. Assuming finite-valued solution of equation (2), it can be written as follows:

\[
H_z = c_1 J_m(qr), \quad E_z = c_2 J_m(qr),
\]

where \( c_1 \) and \( c_2 \) are arbitrary complex numbers. Using these expressions for the longitudinal components, one can derive the other components of the fields [23, 24]:

\[
\begin{align*}
E_r &= -\frac{m \mu k}{q^2 r} c_1 J_m(qr) + i \frac{\beta}{2q} c_2 (J_{m-1}(qr) - J_{m+1}(qr)), \\
E_\varphi &= -i \frac{\mu k}{2q} c_1 (J_{m-1}(qr) - J_{m+1}(qr)) - \frac{m \beta}{q^2 r} c_2 J_m(qr), \\
H_r &= \frac{\beta}{2q} c_1 (J_{m-1}(qr) - J_{m+1}(qr)) + \frac{m \varepsilon k}{q^2 r} c_2 J_m(qr), \\
H_\varphi &= -\frac{m \beta}{q^2 r} c_1 J_m(qr) + i \frac{\varepsilon k}{2q} c_2 (J_{m-1}(qr) - J_{m+1}(qr)).
\end{align*}
\]

where \( k = \omega/c \) is the wavenumber in vacuum, \( c \) is the speed of light. The meaning of the parameters \( c_1 \) and \( c_2 \) becomes clear, if we turn one of them to zero. If \( c_1 = 0 \), then \( H_\varphi = 0 \) and the general expressions (3) and (4) are reduced to the fields of TM-polarized beam. Otherwise (\( c_2 = 0 \)) TE-polarized Bessel beam is obtained. It is very important that the amplitudes \( c_1 \) and \( c_2 \) are complex numbers which describe all possible vector Bessel beams of the order \( m \). In contrast to commonly used scalar beams, one of the most important properties of these vector (electromagnetic) beams is polarization.

The information to be discussed below is connected with the energy flux density (Poynting vector) of the beam considered. By substituting field components (3) and (4) into \( \mathbf{S} = (c/8\pi)\text{Re}(\mathbf{E} \times \mathbf{H}^*) \), the quantity \( \mathbf{S} \) can be derived as

\[
\begin{align*}
\mathbf{S}_r &= c_1^2 J_m(qr)^2 \left[ J_{m-1}(qr) - J_{m+1}(qr) \right] + \frac{m \mu k}{q^2 r} c_1 J_m(qr) c_2 (J_{m-1}(qr) - J_{m+1}(qr)), \\
\mathbf{S}_\varphi &= -i c_2 J_m(qr) \left[ J_{m-1}(qr) - J_{m+1}(qr) \right] + i \frac{\beta}{2q} c_1 J_m(qr) c_2 (J_{m-1}(qr) - J_{m+1}(qr)), \\
\mathbf{S}_z &= -\frac{m \varepsilon k}{q^2 r} c_2 J_m(qr) c_1 J_m(qr) + \frac{m \beta}{2q} c_2 J_m(qr) c_1 J_m(qr).
\end{align*}
\]
\[
S = \frac{c}{8\pi} \left( \frac{k\beta}{q^2} (\mu |c_1|^2 + \varepsilon |c_2|^2)(J'_m^2 + \frac{m^2}{q^2 r^2} J_m^2) - \frac{2m}{q^2 r} (\beta^2 + k^2 \varepsilon \mu) \text{Im}(c_1 c_2^*) J'_m J_m \right) e_z \\
+ \frac{c}{8\pi} \left( \frac{mk}{q^2 r} (\mu |c_1|^2 + \varepsilon |c_2|^2) J_m^2 - \frac{2\beta}{q} \text{Im}(c_1 c_2^*) J'_m J_m \right) e_\phi,
\]

(5)

where the derivative is calculated with respect to the entire function argument, i.e. \( J'_m(qr) = dJ_m/dr(qr) \). This expression describes the dependence of Poynting vector on radial coordinate \( r \) for vector Bessel beam with any particular value of \( m \) which is a discrete parameter. This means that if the condition \( m = 0 \) is taken (zeroth order beam), all terms with \( m \) in numerator vanish and there is no any ambiguities (and discontinuities) at the point \( r = 0 \).

Two situations, \( m = 0 \) and \( m \neq 0 \), are different from the mathematical point of view. Therefore, we will consider them separately. In the case \( m = 0 \), the expression for Poynting vector \( \mathbf{S} \) is reduced to

\[
S_r(r) = 0, \\
S_\phi(r) = -\frac{c\beta}{4\pi q} \text{Im}(c_1 c_2^*) J_0(qr) J_1(qr), \\
S_z(r) = \frac{ck\beta}{8\sigma q^2} (\mu |c_1|^2 + \varepsilon |c_2|^2) J_1^2(qr).
\]

(6)

It is seen that, though the beam as a whole is directed along \( z \)-axis, the Poynting vector is not. This is due to vector nature of the beam considered which is a general solution of Maxwell’s equations. Taking only TE- or TM-component \((c_1 \text{ or } c_2 \text{ is null})\) and proceeding to paraxial limit, one can obtain the usual (scalar) result with the only \( S_z \) component. Note, that in Eqs. \((6)\) the first expression stands for the property of diffractionless of the Bessel beam, while the second one corresponds to angular momentum transfer by the beam. Finally, it is worth to stress that the straight physical sense of energy has not the Poynting vector itself, but the integral of it over the whole cross section of the beam. This integral is always directed along propagation of the beam as a whole (say, \( z \)-axis). But the Poynting vector of a complex beam (which consists of enormous number of plane waves) at a certain point can be directed in various ways (including opposite direction, \(-z\)) \([22, 33]\).

The position of an optical Poynting singularity \( \mathbf{r}_0 \) can be found from equation \( \mathbf{S}(\mathbf{r}_0) = 0 \) or, in the case of propagating vector Bessel beam,

\[
S_\phi(\mathbf{r}_0) = 0, \quad S_z(\mathbf{r}_0) = 0.
\]

(7)

The couple of equations \((7)\) is satisfied, when

\[
f(\mathbf{r}_0) = J_1(qr_0) = 0.
\]

(8)

The positions of singularities \( \mathbf{r}_0 \) are defined by the single beam parameter \( q \). For paraxial beams (transverse wavenumber \( q \) is small compared with the wavenumber), the radial coordinates \( \mathbf{r}_0 \) are much greater than that for non-paraxial vector beams. It is important that the singularities exist for any \( c_1 \) and \( c_2 \). This feature is emphasized in contrast to what will be discussed below for \( m \neq 0 \).

If \( m \neq 0 \), then using the formulae

\[
J'_m(qr) = \frac{1}{2} (J_{m-1}(qr) - J_{m+1}(qr)), \\
J_m = \frac{qr}{2m} (J_{m-1} + J_{m+1}),
\]

(9)

the Poynting vector components \((6)\) can be represented in the form

\[
S_r(r) = 0, \\
S_\phi(r) = \frac{cr}{32m\pi} \left[ k(\mu |c_1|^2 + \varepsilon |c_2|^2)(J_{m-1} + J_{m+1})^2 + 2\beta \text{Im}(c_1 c_2^*) (J_{m-1}^2 - J_{m+1}^2) \right], \\
S_z(r) = \frac{c}{8\pi} \left[ \frac{k\beta}{2q} (\mu |c_1|^2 + \varepsilon |c_2|^2) (J_{m-1}^2 + J_{m+1}^2) - \frac{\beta^2 + k^2 \varepsilon \mu}{2q} \text{Im}(c_1 c_2^*) (J_{m-1}^2 - J_{m+1}^2) \right].
\]

(10)

It leads to the two systems of equations (with upper and lower signs):

\[
f(\mathbf{r}_0) = (-\beta \pm k\sqrt{\varepsilon \mu}) J_{m-1}(\mathbf{r}_0) + (\beta \pm k\sqrt{\varepsilon \mu}) J_{m+1}(\mathbf{r}_0) = 0,
\]

(11)

\[
\mu |c_1|^2 + \varepsilon |c_2|^2 \mp 2\sqrt{\varepsilon \mu} \text{Im}(c_1 c_2^*) = 0.
\]

(12)

The solutions of equation \((11)\) are \( \mathbf{r}_0 \) at different \( \beta \geq 0 \) and \( m \). Positions of singular points are determined by the
only coordinate \( r_0 \) in three-dimensional space. Therefore, the singularities are located on the cylindrical surfaces of certain radius. Reduced to one dimension, the surface can be considered as the point in the radial direction. The quantity of these surfaces is infinite due to infinite number of solutions of equation (11). The choice of sign in equation (11) is connected with the choice of sign in relation (12) which defines the coefficients \( c_1 \) and \( c_2 \) and, hence, the Bessel beam itself. Equation (12) can be represented as quadratic form due to the common representation of complex numbers \( c_1 = a_1 e^{\phi_1} \) and \( c_2 = a_2 e^{\phi_2} \) (\( a_1 \) and \( a_2 \) are real and positive):

\[
\mu a_1^2 + \varepsilon a_2^2 + 2\sqrt{\varepsilon \mu a_1 a_2} \sin \Delta \phi = 0, \tag{13}
\]

where \( \Delta \phi = \varphi_1 - \varphi_2 \). Expressing the phase difference \( \Delta \phi \), we have

\[
\sin \Delta \phi = \pm \frac{\mu a_1^2 + \varepsilon a_2^2}{2\sqrt{\varepsilon \mu a_1 a_2}}. \tag{14}
\]

Taking into account that for \( a_1, a_2 \geq 0 \) the inequality \( \mu a_1^2 + \varepsilon a_2^2 \geq 2\sqrt{\varepsilon \mu a_1 a_2} \) holds, it should be stated that \( |\sin \Delta \phi| \geq 1 \). Owing to the limitation on value of sine equation (14) can be reduced to \( \sin \Delta \phi = \pm 1 \), so that

\[
a_1 = \sqrt{\frac{\mu}{\varepsilon}} a_2, \quad \Delta \phi = \pm \frac{\pi}{2} + 2\pi p, \tag{15}
\]

where \( p \) is integer. Thus, we conclude that vector Bessel beam can contain the singularities in beam’s cross-section, only if it is constructed of TE- and TM-components oscillating out of phase and possessing amplitudes matched by the wave impedance \( \sqrt{\mu/\varepsilon} \). The simplest case of the relation (15) (for propagation in vacuum) gives \( c_1 = 1, c_2 = -i \) and \( c_1 = 1, c_2 = i \) for upper-sign and lower-sign equations (15), respectively.

As examples, we consider vector Bessel beams of the orders \( m = 0 \) and \( m = 1 \). In figure 2(a) we graphically solve equation (8) for \( m = 0 \) and equation (11) for \( m = 1 \). The zero crossings of the function \( f(r) \) (left-hand sides of equations (8) and (11)) in the upper figures correspond to the vertical arrows in the bottom figures. In figure 2(b) the most of Poynting singularities for the amplitudes \( c_2 = i \) and \( c_2 = -i \) appear together. The single exception is the first singular point for \( c_2 = i \). Spatial distance between a couple of Poynting singularities diminishes with increasing the radial coordinate \( r \), because the components of the energy flux densities are similar for great \( r \). In figure 2(a), \( f(r) \)-curves for \( c_2 = i \) and \( c_2 = -i \) (as well as for any amplitudes \( c_1 \) and \( c_2 \)) coincide.

Since the radial component of the Poynting vector \( S_r \) is absent (see equation (10)), the direction of vector \( S \) in the plane \( (\varphi, z) \) can be specified by the angle \( \psi(r) = \arctan[S_\varphi(r)/S_z(r)] \).

Figure 3(a) shows the change of the angle \( \psi(r) \) with radial coordinate. Its value oscillates between \(-\pi\) and \( \pi \), the function \( \psi \) being discontinuous. The function jumps from \( \pi \) to \(-\pi \). For \( m = 0 \), the positions of jumps \( r_j \) coincide with that of singularities. In fact, at the singularities the expression for \( \psi(r) \) contains an ambiguity of the 0/0 type, which can be unwound as

\[
\psi(r_0) = \arctan S'_\varphi(r_0)/S'_z(r_0), \tag{16}
\]

where \( S'_\varphi(r_0) \) and \( S'_z(r_0) \) are the derivatives of the Poynting vector components at the singular point \( r_0 \). For \( m = 0 \) we get

\[
S'_\varphi(r_0) = -\frac{c}{4\pi} \beta \text{Im}(c_1 c_2^*), \quad S'_z(r_0) = 0,
\]

so that \( \psi(r_0) \) equals \( \pi \), i.e. singularity points coincide with points of jumps \( r_j \). At the singularity only one of components of Poynting vector (namely, \( S_\varphi \)) changes the sign. Such a situation can occur, if \( S_z \) is tangent to the
fluctuation deforms the beam form, so that it becomes

where

imperfections during their realistic generation. So, if we

They are very attractive due to the existence of the

vector Bessel beams are the centrally symmetric beams.

another value of

increases, whereas the angle

the singularity, then

It is evident that after adding

s

by substituting the coefficients

and

Thus, the singularities considered are due to the spe-

Further we consider the electric and magnetic fields at

the singular surfaces. None of them vanish as it would

be in the case of phase singularity. In the case

the azimuthal components vanish and both electric and

magnetic fields are directed along

while

and

are arbitrary complex numbers, i.e.

Such longitudinal fields are linearly polarized and can be

treated as polarization L-singularities.

For the rest of

one can derive the fields at singular points from

and

We may note that the magnetic and electric field

strengths are connected as

By substituting the coefficients

and

we represent the relation

in the form

The inset depicts the polarization ellipses at the points of solid curve marked by vertical arrows. Parameters:

FIG. 3: Angle \( \psi \) vs. radial coordinate \( r \) for (a) \( m = 0 \) and (b) \( m = 1 \). Vertical dashed lines mark the radii of singularities. Parameters: \( \beta = 0.5k, \varepsilon = 1, \mu = 1, c_1 = 1, c_2 = -i \).

FIG. 4: Polarization parameter \( \gamma \) versus radial coordinate for (a) \( m = 0, c_2 = -i \), (b) \( m = 1 \). The inset depicts the polarization ellipses at the points of solid curve marked by vertical arrows. Parameters: \( c_1 = 1, \beta = 0.5k, \varepsilon = 1, \mu = 1 \).

It should be noted that considered solutions for the

vector Bessel beams are the centrally symmetric beams.

They are very attractive due to the existence of the

closed-form expressions for the fields. On the other hand,

the studied beams are very specific, because of different

imperfections during their realistic generation. So, if we

introduce a small electric field \( \mathbf{e}(r) \) as a fluctuation,

the field can be written as \( \mathbf{E} = \mathbf{E}_B + \mathbf{e} \) and \( \mathbf{H} = \mathbf{H}_B + \mathbf{h} \),

where \( \mathbf{E}_B \) and \( \mathbf{H}_B \) are the electric and magnetic fields of

the ideal vector Bessel beam, and \( \mathbf{h} = 1/(ik\mu) \nabla \times \mathbf{e} \). The fluctuation deforms the beam form, so that it becomes

asymmetric. Then the Poynting vector takes the form

It is evident that after adding \( \mathbf{s} \) the singular point \( r_0 \) will

be destroyed. Since the fluctuation field is an arbitrary

one, it is quite impossible to meet the condition \( \mathbf{S}(r) = 0 \)

now (only for some specific fields \( \mathbf{e} \) it can be done). How-

ever, as the field \( \mathbf{e}(r) \) is small, the singularities appear to

be approximately situated at \( r_0 \), however \( \mathbf{S}(r) = 0 \) is not

satisfied exactly at any point.
The state of polarization can be studied by means of parameter $\gamma$:

$$\gamma = \frac{|E|^2}{|E|^2}.$$  \hspace{1cm} (21)

The introduced parameter characterizes the polarization of an electromagnetic beam using the complex electric field vector. The same analysis can be performed for the polarization of the magnetic field. All the types of polarization can be described with $\gamma$. So, circular polarization corresponds to $\gamma = 0$, elliptical one — to $0 < \gamma < 1$, linear polarization — to $\gamma = 1$. In figure 4 the singularities are located at the maxima of curves $\gamma(r)$. That is why the electromagnetic fields at the singular surfaces are linearly polarized ($\gamma = 1$). The proof of $\gamma = 1$ can be fulfilled analytically: the value $\gamma$ for the fields (19) equals $|c_1^2|/|c_2|^2 = 1$. The inset of the figure 4(b) clearly demonstrates the evolution of the forms of polarization ellipses. Starting with circular polarization at beam’s axis ($r = 0$), it changes step-by-step through elliptical to linear one at singularity $r_0$. It should be noted that circular polarization appears only at beam’s axis. Hence the axis is the place of C-points. Moreover, at $r = 0$ the fields can be equal to zero (e.g., for $m = 2$), i.e. the phase singularity exists at this point. So, the beam axis simultaneously corresponds to phase, polarization, and Poynting singularities. In this situation, the Poynting singularity is caused by the field vanishing, but not by the special polarization. However, the other Poynting singularities of the vector Bessel beam are the polarization induced ones. Sometimes, they can be identified as L-points according to the classification of polarization singularities. In our example, the Poynting singularities correspond to the L-points. In general, such is not the case, because the definitions of both singularities are not connected directly. For $m = 0$ the beam axis $r_0 = 0$ is also the singular line (see figure 4(b)), the electric and magnetic fields being non-zero ones and the Poynting singularity being caused by the linear field polarization. The link between polarization and Poynting singularities for arbitrary electromagnetic fields is discussed in Refs. [3, 6].

### III. POYNTING SINGULARITIES FOR EVANESCENT BEAMS

When a vector Bessel beam falls onto the interface between two media, the total internal reflection can occur. The Bessel beam that penetrates the interface exponentially decays, when moving from the boundary. Such a wave is called the evanescent (inhomogeneous) one. Thus, the evanescent Bessel beam corresponds to the situation when $q > k\sqrt{\varepsilon\mu}$. Applying replacement $\beta \rightarrow i\beta' = i\sqrt{q^2 - k^2}\varepsilon\mu$ to the fields (3) and (4) we find the Poynting vector components [26]:

$$S_r = -\frac{c\beta' m}{4\pi q^2} \Im(c_1 c_2^*) J_m^2(q r) e^{-2\beta' z},$$

$$S_\phi = \frac{c k m}{8\pi q^4 r} (\mu|c_1|^2 + \varepsilon|c_2|^2) J_m^2(q r) e^{-2\beta' z},$$

$$S_z = -\frac{c m (\beta'^2 + k^2\varepsilon\mu)}{4\pi q^4 r^2} \Im(c_1 c_2^*) J_m(q r) J'_m(q r) e^{-2\beta' z}.$$  \hspace{1cm} (22)

In contrast to the propagating beam (see equation (5)), the evanescent Bessel beam has the radial component $S_r$, i.e. it is diffracted. As in previous section, one can mark out two cases, $m = 0$ and $m \neq 0$.

For $m = 0$, it is turned out that the Poynting vector $S$ vanishes for each radial coordinate $r$. However, neither electric nor magnetic field is equal to zero. The fields are expressed as

$$E(r) = e^{-\beta' z} \left( \frac{\beta'}{q} c_2 J_1(q r) e_r + i \frac{\mu k}{q} c_1 J_1(q r) e_\phi + c_2 J_0(q r) e_z \right),$$

$$H(r) = e^{-\beta' z} \left( \frac{\beta'}{q} c_1 J_1(q r) e_r - i \frac{\varepsilon k}{q} c_2 J_1(q r) e_\phi + c_1 J_0(q r) e_z \right).$$  \hspace{1cm} (23)

We have obtained the very exciting situation. There exist non-zero cylindrically symmetric solutions of the Maxwell equations, for which the energy is not transferred (the energy flux is entirely zero). Such a situation is similar to the case of the plane-wave standing waves. Being the Beltrami-fields, these standing waves do not transfer the energy, too [33].

Now the singularities are not located at certain cylindrical surfaces. They occupy the whole three-dimensional space. So, the singularity can be not only the isolated point, or line, or surface, but the space itself. In general, the polarization of electromagnetic field [23] is elliptical. Exactly such a field polarization provides the complete vanishing of the energy flux. At the same time, polarization is linearly or circularly polarized at some points. For example, L-points arise, when the condition $J_1(q r) = 0$ holds true. It is obvious, that Poynting singularities include the polarization singularities, which are situated at the cylindrical surfaces.

For $m \neq 0$, the Poynting vector components [22] can
be rewritten using the properties of Bessel functions \(9\):

\[
S_r = -\frac{c\beta r}{16\pi m} \text{Im}(c_1 c_2^*)(J_{m-1} + J_{m+1})^2 e^{-2\beta' z},
\]
\[
S_\phi = \frac{e\beta r}{32\pi m} (|c_1|^2 + \varepsilon |c_2|^2) (J_{m-1} + J_{m+1})^2 e^{-2\beta' z},
\]
\[
S_z = -\frac{e(\beta'^2 + k^2 \varepsilon \mu)}{16\pi q^2} \text{Im}(c_1 c_2^*)(J_{m-1}^2 - J_{m+1}^2) e^{-2\beta' z}.
\]

The singularity position \(r_0\) is determined from the equation

\[
f(r_0) = J_{m-1}(qr_0) + J_{m+1}(qr_0) = 0.
\]

The above equation holds for any complex coefficients \(c_1\) and \(c_2\). This means that any evanescent Bessel beam has singular points or, rather, cylindrical surfaces. In figure 5(a) the left-hand side of equation (25) is shown. For \(m > 1\) the Poynting singularity at beam’s axis arises. It is caused by zero fields and coincides with the phase singularity as for the propagating electromagnetic Bessel beams.

As to the electric and magnetic fields at singular points, they take the form

\[
E(r) = e^{i\beta' - \beta' z} J_{m+1}(qr_0) \left( \frac{\beta'}{q} c_2 e_r + i \frac{\mu k}{q} c_1 e_\phi \right),
\]
\[
H(r) = e^{i\beta' - \beta' z} J_{m+1}(qr_0) \left( \frac{\beta'}{q} c_1 e_r - i \frac{\varepsilon k}{q} c_2 e_\phi \right).
\]

It is important that the fields satisfy the relation

\[
H = \alpha_1 E + \alpha_2 E^*,
\]

where \(\alpha_1\) and \(\alpha_2\) are real and complex numbers, respectively. The numbers are equal to

\[
\alpha_1 = \frac{\mu |c_1|^2 - \varepsilon |c_2|^2}{2\mu \text{Re}(c_1 c_2^*)}, \quad \alpha_2 = \frac{\mu \beta'^2 + \varepsilon |c_2|^2}{2\mu \text{Re}(c_1 c_2^*)}.
\]

Relation between the electric and magnetic fields (27) is necessary to turn the Poynting vector to zero. Generally, the field link (27) corresponds to the elliptical polarization. The state of polarization can be described by the parameter \(\gamma\) (see its definition (21)). For evanescent Bessel beams, we will express it in the closed form as

\[
\gamma = \frac{\beta'^2 \varepsilon c_2^2 - \mu^2 k^2 c_1^2}{\mu^2 k^2 |c_1|^2 + \beta'^2 |c_2|^2} = \sqrt{1 - \frac{2\mu^2 k^2 \beta'^2 |c_1|^2 |c_2|^2 + \text{Re}(c_1^2 c_2^2)}{\left(\mu^2 k^2 |c_1|^2 + \beta'^2 |c_2|^2\right)^2}}.
\]

Figure 5(b) demonstrates three possible polarizations at the singularity point \(r_0\) indicated by the vertical arrows. The linear polarization (L-point) is achieved for \(\gamma = 1\), i.e. for

\[
|c_1|^2 |c_2|^2 + \text{Re}(c_1^2 c_2^2) = 0.
\]

Using the above introduced representation of the complex amplitudes \(c_1 = a_1 e^{i\varphi_1}\) and \(c_2 = a_2 e^{i\varphi_2}\) one can easily obtain the phase difference

\[
\Delta \varphi \equiv \varphi_1 - \varphi_2 = \frac{\pi}{2} + \pi p, \quad p = 0, \pm 1, \pm 2, \ldots
\]

In figure 5(b) this condition is fulfilled for the amplitudes \(c_1 = 1\) and \(c_2 = -1\).

Circulary polarized fields at singularities (C-points) are described by \(\gamma = 0\). From equation (28) it follows the necessary condition to obtain C-point at the Poynting singularity:

\[
\beta'^2 c_2^2 - \mu^2 k^2 c_1^2 = 0.
\]

By substituting the amplitudes, we get

\[
\left(\frac{a_2}{a_1}\right)^2 = \left(\frac{\mu k}{\beta'}\right)^2 e^{2i\Delta \varphi}.
\]

Since \(a_1\) and \(a_2\) are real, the condition \(\exp(2i\Delta \varphi) = 1\)
should be valid. Hence,
\[ \Delta \varphi = \pi p, \quad p = 0, \pm 1, \pm 2 \ldots \] (33)
and the relation between coefficients \( c_1 \) and \( c_2 \) should satisfy the expression
\[ \frac{c_2}{c_1} = \pm \frac{\mu k}{\beta}. \] (34)

Such C-singularities are shown in figure 5(b) as that corresponding to the curve with amplitudes \( c_1 = 1 \) and \( c_2 = 2 \).

If neither equation (30) nor (34) holds, the polarization of the field at the singular point is elliptical. For example, the elliptically polarized field with \( \gamma = 0.6 \) appears for \( c_1 = c_2 = 1 \) (see figure 5(b)). We can be convinced again that the Poynting singularity for the vector Bessel beams can include the polarization singularities as particular cases. Of course, the polarization singularities can be generated at the points different from the Poynting singularities, too.

**IV. CONCLUSION**

Summing up, vector Bessel beams (solutions of Maxwell’s equations in cylindrical coordinates) demonstrate the occurrence of non-phase optical singularities, namely Poynting singularities, which are situated at cylindrical surfaces in three-dimensional space. To create the singularity of the Poynting vector, TE and TM components of propagating Bessel beams defined by the complex amplitudes \( c_1 \) and \( c_2 \) should be out of phase and be matched with the wave impedance. Such a strong restriction on the amplitudes arises only for propagating waves. Any evanescent Bessel beam is appeared to be singular. Moreover, the evanescent beam with \( n = 0 \) is singular at any spatial point. This case describes a sort of cylindrical standing wave. The points of Poynting singularities always include the phase singularities as particular case (at Bessel beam’s axis). However this is not the case for L- and C-points (polarization singularities). The manifold of the Poynting singularities and manifold of L- and C-points can intersect, however, it is only the coincidence. Since vector cylindrical solutions of the Maxwell equations describe the electromagnetic modes of circular fibers, the theory of Poynting singularities can be applied for the fields of guiding structures.

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