Generalized Cross Curvature Flow

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Abstract—In this paper, for a given compact 3-manifold with an initial Riemannian metric and a symmetric tensor, we establish the short-time existence and uniqueness theorem for extension of cross curvature flow. We give an example of this flow on manifolds.

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1. INTRODUCTION

The cross curvature flow (XCF) on 3-manifolds were first introduced by B. Chow and R. S. Hamilton [6] as

$$\frac{\partial g}{\partial t} = -2\epsilon h, \quad g(0) = g_0,$$

where $h$ is cross curvature tensor, $\epsilon = \pm 1$ is the sectional curvature sign of the metric $g_0$. They obtained several monotonicity formula to show that for an initial metric with negative sectional curvature the XCF exists for short-time and converge to a hyperbolic metric after an appropriate normalizative of it. Then, J. A. Buckland [2], studied its short-time existence on closed 3-manifolds. Moreover, Cao et al. [3, 4], investigated the asymptotic behavior of the cross curvature flow on locally homogeneous three-manifolds. Several examples of solutions to XCF have been obtained by L. Ma and D. Chen [14] for warped product metrics on 2-torus and 2-sphere bundle over the circle. Also, in [11, 13] have been obtained some results about XCF.

The Ricci flow [12] is the most successful example for deforming a Riemannian structure via a geometric evolution equation and it is defined as

$$\frac{\partial g}{\partial t} = -2\text{Ric}, \quad g(0) = g_0,$$

where $\text{Ric}$ denotes the Ricci curvature and this flow is a natural analogue of the heat equation for metrics. The existence solution of the Ricci flow on closed Riemannian manifolds was studied by Hamilton [12] and DeTurck [9].

Another geometric flow is the Ricci–Bourguignon flow which is a generalization of the Ricci flow and it is defined as follows

$$\frac{\partial g}{\partial t} = -2\text{Ric} + 2\rho R g = -2(\text{Ric} - \rho R g), \quad g(0) = g_0,$$

where $R$ is the scalar curvature of $g$ and $\rho$ is a real constant. The Ricci–Bourguignon flow was introduced by Bourguignon [1] for the first time in 1981. Short-time existence and uniqueness for the solution of the Ricci–Bourguignon flow for $\rho < \frac{1}{2(n - 1)}$ on $[0, T)$ have been shown by Catino et al. in [5]. When $\rho = 0$, the Ricci–Bourguignon flow is the Ricci flow.

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Motivated by the above works, in this article we consider a 3-dimensional compact Riemannian manifold \( M \) whose metric \( g = g(t) \) is evolving according to the flow equation
\[
\frac{\partial g}{\partial t} = -2h + 2\rho Rg, \quad g(0) = g_0,
\]  
(1)
where \( h \) is the curvature tensor and \( \epsilon = \pm 1 \) is the sectional curvature sign of the metric \( g_0 \). We establish the short-time existence of the solution to (1) by using more elementary method than [8] on 3-dimensional compact Riemannian manifold \( M \) and give some evolution equations of curvatures. This flow called hyperbolic cross curvature flow and for simplicity, we will denote it by ECCF in short.

2. THE CROSS CURVATURE TENSOR

Let \((M, g)\) be a 3-dimensional Riemannian manifold with positive sectional curvatures. The Einstein tensor is \( E_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} \). Raising the indices, define \( P^{ij} = R^{ij} - \frac{1}{2}Rg^{ij} \). Let \( V_{ij} \) be the inverse of \( P^{ij} \). The cross curvature tensor is given by
\[
h_{ij} = \left( \frac{\det D^{kl}}{\det g^{kl}} \right) V_{ij}.
\]
Let denote the volume form by \( h \) and raise indices by \( \mu_{ijk} = g^{ip}g^{jq}g^{kr}\mu_{pqr} \) and normalize such that \( \mu_{123} = \mu^{123} = 1 \). Therefore
\[
\mu_{ijkkl} \delta_{jk} = \delta^{l}_{j} \delta^{m}_{k} - \delta^{m}_{j} \delta^{l}_{k},
\]
and we can rewrite \( h_{ij} \) as \( h_{ij} = \frac{1}{2}R_{ijkl}^{pqr}R_{kjsr}^{il} \), or equivalently as \( h_{ij} = \frac{1}{2}P^{rs}R_{rsij} \), where \( P^{ij} = \frac{1}{4}\mu_{ijkl}R_{ijkl} \) is the Einstein tensor. If we choose orthonormal basis so that the eigenvalues of the Einstein tensor are \( a = R_{2323}, b = R_{1313}, \) and \( c = R_{1212} \), then the eigenvalues of \( h_{ij} \) are \( bc, ac, \) and \( ab \) and the eigenvalues of \( R_{ij} \) are \( b+c, a+c, \) and \( a+b \). Since \( R_{ij} \) \((i \neq j)\) are sectional curvatures, so if manifold has positive sectional curvatures, then both \( P_{ij} \) and \( h_{ij} \) are positive definite.

3. SHORT-TIME EXISTENCE AND UNIQUENESS FOR THE ECCF

In this section we investigate the short-time existence and uniqueness of the solution to the ECCF.

**Theorem.** Let \((M, g_0)\) be a compact 3-dimensional Riemannian manifold with positive sectional curvature. Then there exists a positive constant \( T \) such that the evolution equation
\[
\frac{\partial g}{\partial t} = -2h + 2\rho Rg, \quad g(0) = g_0,
\]
(3)
has a unique solution metric \( g(t) \) on \([0, T)\) for \( \rho < \frac{n(n + 1)}{4} \).

**Proof.** We first compute the linearized operator \( DL_{g_0} \tilde{g}_{ij} = \frac{d}{d\tau}|_{\tau=0}L(\tilde{g}_0 + \tau \tilde{g}) \) of the operator \( L = -2h + 2\rho Rg \) at a metric \( g_0 \), where \( \tilde{g}_{ij} \) denotes a variation in the metric. The curvature tensor and the scalar curvature have the following linearizations [15]
\[
D(R_{ijkl})_{g_0}(\tilde{g}) = \frac{1}{2} \left( \nabla_{i} \nabla_{j} \tilde{g}_{kl} - \nabla_{k} \nabla_{l} \tilde{g}_{ij} - \nabla_{k} \nabla_{l} \tilde{g}_{il} + \nabla_{k} \nabla_{l} \tilde{g}_{jk} - R_{ijkl}^{pq}R_{pqil} - R_{ijkl}^{pq}R_{pqjk} + \tilde{g}_{jm}R_{iklm} \right),
\]
\[
DR_{g_0}(\tilde{g}) = -\Delta(\tau g_0) + \nabla^{i} \nabla^{j} \tilde{g}_{ij} - R_{ij}^{pq} \tilde{g}_{ij}^{pq},
\]
Fix a point \( x \in M \), and consider normal coordinates at \( x \). We can write the above operators at \( x \),
\[
D(R_{ijkl})_{g_0}(\tilde{g}) = \frac{1}{2} \left( \frac{\partial^2 \tilde{g}_{ij}}{\partial x^{i} \partial x^{j}} - \frac{\partial^2 \tilde{g}_{kl}}{\partial x^{k} \partial x^{l}} + \frac{\partial^2 \tilde{g}_{il}}{\partial x^{k} \partial x^{l}} - \frac{\partial^2 \tilde{g}_{ij}}{\partial x^{k} \partial x^{l}} \right) + \text{lower order terms},
\]
\[
DR_{g_0}(\tilde{g}) = -\frac{\partial^2 (\tau g_0 \tilde{g})}{\partial x^{i} \partial x^{j}} + \frac{\partial^2 \tilde{g}_{ij}}{\partial x^{i} \partial x^{j}} + \text{lower order terms}.
\]
Hence, we get
\[ D(P^{ij})_{g_0}(\tilde{g}) = \frac{1}{4} \mu^{ipq} \mu^{jrs} D(R_{pqrs})_{g_0}(\tilde{g}) + \text{lower order terms}. \]

Therefore, using (2) we obtain
\[ D(h_{ij})_{g_0}(\tilde{g}) = P^{kl}(g_0)D(R_{ijkl})_{g_0}(\tilde{g}) + \text{lower order terms} \]
\[ = \frac{1}{2} P^{kl}(g_0) \left( \frac{\partial^2 \tilde{g}_{kj}}{\partial x^i \partial x^l} - \frac{\partial^2 \tilde{g}_{kl}}{\partial x^i \partial x^j} + \frac{\partial^2 \tilde{g}_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 \tilde{g}_{ij}}{\partial x^k \partial x^l} \right) + \text{lower order terms}, \]

and
\[ DL_{g_0} \tilde{g}_{ij} = -P^{kl}(g_0) \left( \frac{\partial^2 \tilde{g}_{kj}}{\partial x^i \partial x^l} - \frac{\partial^2 \tilde{g}_{kl}}{\partial x^i \partial x^j} + \frac{\partial^2 \tilde{g}_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 \tilde{g}_{ij}}{\partial x^k \partial x^l} \right) + 2\rho \left( \frac{\partial^2 (tr_{g_0} \tilde{g})}{\partial x^k \partial x^k} + \frac{\partial^2 g_{ij}}{\partial x^i \partial x^j} \right) + \text{lower order terms}. \]

Now, we obtain the symbol of the linear differential operator \( DL_{g_0} \tilde{g}_{ij} \) in the direction of an arbitrary cotangent vector \( \xi \) by replacing each derivative \( \frac{\partial}{\partial x^a} \) appearing in the higher order terms with \( \xi_a \):
\[ \sigma(DL_{g_0})(\xi) \tilde{g}_{ij} = -P^{kl}(g_0) (\xi_i \xi_k \tilde{g}_{kj} - \xi_i \xi_j \tilde{g}_{ki} + \xi_k \xi_j \tilde{g}_{il} - \xi_k \xi_l \tilde{g}_{ij}) + 2\rho \left( -\xi_k \xi_l (tr_{g_0} \tilde{g}) + \xi_i \xi_k \tilde{g}_{kl} \right) (g_0)_{ij}. \]

Since the principal symbol is homogeneous, we may assume that \( \xi = (1, 0, \ldots, 0) \) satisfies \( \xi_1 = 1 \) and \( \xi_i = 0 \) for \( i \neq 1 \). A simple computation shows that
\[ \sigma(DL_{g_0})(\xi) \tilde{g}_{ij} = P^{11}(g_0) \tilde{g}_{ij} + P^{lk}(g_0) (\delta_{il} \delta_{j1} \tilde{g}_{kl} - \delta_{i1} \delta_{k1} \tilde{g}_{lj} - \delta_{l1} \delta_{j1} \tilde{g}_{ik}) + 2\rho \left( -\delta^{ij} (tr_{g_0} \tilde{g}) + \delta_{ij} \tilde{g}_{11} \right). \]

In coordinate system we have \((\tilde{g}_{11}, \tilde{g}_{12}, \tilde{g}_{13}, \tilde{g}_{22}, \tilde{g}_{23}, \tilde{g}_{33}, \tilde{g}_{23})\) and we deduce that
\[ \sigma(DL_{g_0})(\xi) = \begin{pmatrix} 0 & 0 & P^{22}(g_0) - 2\rho & P^{33}(g_0) - 2\rho & 2P^{23}(g_0) \\ 0 & 0 & -P^{12}(g_0) & 0 & -P^{13}(g_0) \\ 0 & 0 & P^{11}(g_0) - 2\rho & -2\rho & 0 \\ 0 & 0 & -2\rho & P^{11}(g_0) - 2\rho & 0 \\ 0 & 0 & 0 & 0 & P^{11}(g_0) \end{pmatrix}. \]

The eigenvalues of this matrix are 0 with multiplicity 3, \( P^{11}(g_0) \) with multiplicity 2 and \( P^{11}(g_0) - 4\rho \) with multiplicity 1. The sectional curvature is assumed to be positive, so the system (3) is weakly parabolic. In order to eradicate the zero eigenvalues, we apply the so-called DeTurk’s trick \cite{9} to show that the flow (3) is equivalent to a strictly parabolic initial-value problem, under the action of diffeomorphisms group of \( M \). Let \( V \) be DeTurk’s vector field defined by
\[ V^i(g) = -g_0^{ij} g^{kl} \nabla_k \left( \frac{1}{2} tr_{g_0} g_{jl} - (g_0)_{jl} \right), \]
then
\[ (L_V g)_{ij} = \nabla_j V_i + \nabla_i V_j = g^{kl} \left( \frac{\partial^2 g_{ki}}{\partial x^l \partial x^j} - \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{kj}}{\partial x^i \partial x^l} \right) + \text{lower order terms}. \]

With the same notation used above, the principal symbol of operator \( L_V \) is given by
\[ \sigma(DL_V)_{g_0}(\xi) \tilde{g}_{ij} = \delta_{i1} \delta_{j1} tr_{g_0}(\tilde{g}) - \delta_{i1} \tilde{g}_{j1} - \delta_{j1} \tilde{g}_{i1}. \]

Now we consider the initial-value problem
\[ \frac{\partial g}{\partial t} = -2h + 2\rho Rg + L_V g, \quad g(0) = g_0. \]
The computations above show that the linearized of operator $L - L_V$ has principal symbol in the direction $\xi$ is given by

$$
\sigma(DL_{g_0})(\xi) = \begin{pmatrix}
1 & 0 & 0 & P^{22}(g_0) - 2\rho - 1 & P^{33}(g_0) - 2\rho - 1 & 2P^{23}(g_0) \\
0 & 1 & 0 & -P^{12}(g_0) & 0 & -P^{13}(g_0) \\
0 & 0 & 1 & 0 & -P^{13}(g_0) & -P^{12}(g_0) \\
0 & 0 & 0 & P^{11}(g_0) - 2\rho & -2\rho & 0 \\
0 & 0 & 0 & -2\rho & P^{11}(g_0) - 2\rho & 0 \\
0 & 0 & 0 & 0 & 0 & P^{11}(g_0) \\
\end{pmatrix}.
$$

This matrix has 3 eigenvalues equal to 1, 2 eigenvalues equal to $P^{11}(g_0)$, and one eigenvalue equal to $P^{11}(g_0) - 4\rho$. Therefore the flow (4) is a strictly parabolic system and a sufficient condition for the short-time existence of a solution (4) is that $\rho < \frac{P^{11}(g_0)}{4}$, hence a unique solution exists for a short-time for (3) by standard parabolic theory.

In what follows, we will show how to go from a solution of the (4) back to a solution for (3). Define a one-parameter family of maps $\phi_t : M \to M$ by

$$\frac{\partial}{\partial t} \phi_t(x) = -V(\phi_t(x), t), \quad \phi_0 = Id_M.$$ 

From [7] the maps $\phi_t$ exist and are diffeomorphisms as long as the solution $g(t)$ exists. The same way as the Ricci flow [7, 9, 15], if $g(t)$ be a solution to (4) then $\tilde{g}(t) = \phi_t^* g(t)$ is a solution to (3). Now, assume we have a solution $\tilde{g}(t)$ to (3). Let $\psi_t$ be the solution to the harmonic map flow

$$\frac{\partial \psi_t}{\partial t} = \Delta_{\tilde{g}(t)} \psi_t, \quad \psi_0 = Id_M,$n

where $\tilde{g}$ is any reference metric and by [10] the maps $\psi_t$ exist and are diffeomorphisms. By direct computation, we see that $g(t) = (\psi_t)_* \tilde{g}(t)$ is a solution for (4). For uniqueness, if we have two solutions $\tilde{g}_1(t)$ and $\tilde{g}_2(t)$ of (3) with the same initial data, then using above method, we get two solutions $g_1(t)$ and $g_2(t)$ of (4) with the same initial data. By uniqueness of the solution to (4), we conclude that $g_1(t) = g_2(t)$ and hence $\tilde{g}_1(t) = \tilde{g}_2(t)$.

**Remark.** Let $(M, g_0)$ be a compact 3-dimensional Riemannian manifold with negative sectional curvature. Similarly to the proof of Theorem 1, we can show that there exists a positive constant $T$ such that the evolution equation

$$\frac{\partial g}{\partial t} = 2h + 2\rho Rg, \quad g(0) = g_0,$n

has a unique solution metric $g(t)$ on $[0, T)$ for $\rho < -\frac{P^{11}(g_0)}{2}$.

4. **EXAMPLE**

Let $(M, g(0))$ be a closed 3-dimensional Riemannian manifold, the initial metric $g(0)$ is Einstein metric that is for some constant $\lambda$ it satisfies $R_{ij}(0) = \lambda g_{ij}(0)$. Since, the initial metric is Einstein for some constant $\lambda$, let $g_{ij}(t, x) = (c(t)) g_{ij}(0)$. By the definition of the Ricci tensor we obtain

$$R_{ij}(t) = R_{ij}(0) = \lambda g_{ij}(0), \quad R(t) = \frac{3\lambda}{c(t)}.$$ 

Hence,

$$P^{ij}(t) = \frac{2c^2(t) - 3\lambda^2}{2\lambda c^2(t)} g^{ij}(0), \quad V_{ij}(t) = \frac{2\lambda c^2(t)}{2c^2(t) - 3\lambda^2} g_{ij}(0), \quad h_{ij} = \left(\frac{2c^2(t) - 3\lambda^2}{2\lambda c^2(t)}\right)^2 c^{-3} g_{ij}(0).$$

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\[-2h_{ij} + 2 \rho R g_{ij} = \left(-2 \left(\frac{2c^2(t) - 3 \lambda^2}{2 \lambda c^2(t)} \right)^2 \left(c^{-3}(t) + 6 \rho a \right) \right) g_{ij}(0) .\]

In the present situation, the equation (3) becomes
\[
\frac{\partial (c(t) g_{ij}(0))}{\partial t} = \left(-2 \left(\frac{2c^2(t) - 3 \lambda^2}{2 \lambda c^2(t)} \right)^2 c^{-3}(t) + 6 \rho a \right) g_{ij}(0),
\]
this gives an ODE of first order as follows
\[
\frac{dc(t)}{dt} = \left(-2 \left(\frac{2c^2(t) - 3 \lambda^2}{2 \lambda c^2(t)} \right)^2 c^{-3}(t) + 6 \rho a \right) , \quad c(0) = 1 .
\]

Therefore the solution of (3) flow remains Einstein.

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