THE WHITNEY DUALS OF A GRADED POSET

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Abstract. We introduce the notion of a Whitney dual of a graded poset. Two posets are Whitney duals to each other if (the absolute value of) their Whitney numbers of the first and second kind are interchanged between the two posets. We define new types of edge and chain-edge labelings which we call Whitney labelings. We prove that every graded poset with a Whitney labeling has a Whitney dual. Moreover, we show how to explicitly construct a Whitney dual using a technique involving quotient posets.

As applications of our main theorem, we show that geometric lattices, the lattice of non-crossing partitions, the poset of weighted partitions studied by González D’León-Wachs, and most of the $R^*_S$-labelable posets studied by Simion-Stanley all have Whitney duals. Our technique gives a combinatorial description of a Whitney dual of the noncrossing partition lattice in terms of a family of noncrossing Dyck paths. Our method also provides an explanation of the Whitney duality between the poset of weighted partitions and a poset of rooted forests studied by Reiner and Sagan. An integral part of this explanation is a new chain-edge labeling for the poset of weighted partitions which we show is a Whitney labeling.

Finally, we show that a graded poset with a Whitney labeling admits a local action of the 0-Hecke algebra of type $A$ on its set of maximal chains. The characteristic of the associated representation is Ehrenborg’s flag quasisymmetric function. The existence of this action implies, using a result of McNamara, that when the maximal intervals of the constructed Whitney duals are bowtie-free, they are also snellable. In the case where these maximal intervals are lattices, they are supersolvable.

Keywords: graded posets, Whitney numbers, Whitney duality, edge labelings, chain-edge labelings, quotient posets, noncrossing partitions, weighted partitions, rooted forests, 0-Hecke algebra actions, flag quasisymmetric function.

Contents

1. Introduction 2
2. Basic Examples of posets with and without Whitney Duals 6
3. Whitney labelings and quotient posets 8
3.1. Edge labelings 8
3.2. Constructing Whitney Duals 10
3.3. Chain-edge labelings 18
4. The Whitney dual $Q_{\lambda}(P)$ 21
4.1. The Möbius function of $Q_{\lambda}(P)$ 21
4.2. Another description of $Q_{\lambda}(P)$ 22
5. Examples of posets with Whitney labelings 23
5.1. Geometric lattices 23
1. Introduction

All partially ordered sets (or posets) considered here will be finite, graded, and contain a minimum element (denoted by $\hat{0}$). We assume familiarity with poset and poset topology terminology and notation. For background on posets the reader should visit [34, Chapter 3] and [35].

Throughout the paper, $P$ will denote a finite graded poset with $\hat{0}$ and $\rho$ will denote its rank function. The M"obius function of a poset $P$ is defined recursively for pairs $x < y$ in $P$ by

$$
\mu(x, y) = \begin{cases} 
1 & \text{if } x = y, \\
- \sum_{x \leq z < y} \mu(x, z) & \text{if } x \neq y.
\end{cases}
$$

We illustrate, with two examples, how to calculate the values $\mu(\hat{0}, x)$ of the M"obius function. These two examples will be of a particular relevance throughout this article.

**Example 1.1.** Let $\Pi_n$ denote the poset whose underlying set is formed by the partitions of the set $[n] := \{1, 2, \ldots, n\}$ with order relation given for $\pi, \pi' \in \Pi_n$ by $\pi \leq \pi'$ if every block of $\pi$ is contained in some block of $\pi'$. Equivalently, the cover relation $\pi \prec \pi'$ is defined whenever $\pi'$ is obtained from $\pi$ when exactly two blocks of $\pi$ are merged to form a single block in $\pi'$ while the remaining blocks of $\pi$ and $\pi'$ are the same. We say that the partitions are ordered by refinement and we call $\Pi_n$ the partition lattice (since this poset has additional structure, that of a lattice). In Figure 1 we illustrate the values $\mu(\hat{0}, x)$ of the M"obius function for every $x \in \Pi_3$.

**Example 1.2.** Let $T$ be a tree with vertices labeled by distinct integers. We call the smallest vertex of $T$ the root. We say $T$ is an increasing tree if the sequence of vertex labels read along any path starting at the root of $T$ is increasing. An increasing spanning forest is a collection of increasing trees whose vertex labels form a partition of $[n]$. The word “spanning” here indicates that these forests are spanning forests of the complete graph. For more information about increasing spanning forests see [13]. We use $\mathcal{ISF}_n$ to denote the set of increasing spanning forests on $[n]$. A partial order on $\mathcal{ISF}_n$ is defined...
by \( F_1 \preceq F_2 \) if exactly two trees in \( F_1 \) are replaced by the tree in \( F_2 \) that is obtained after joining their roots with an edge. Note that the root of the resulting tree is the smaller label among the roots of the two joined trees. See Figure 1 for the Hasse diagram of \( ISF_3 \) together with the Möbius values \( \mu(\hat{0}, x) \) for every \( x \in ISF_3 \).

Two important invariants that we can associate to a graded poset \( P \) are its Whitney numbers of the first and second kind. The \( k \)-th Whitney number of the first kind, \( w_k(P) \), is defined by

\[
    w_k(P) = \sum_{\rho(x) = k} \mu(\hat{0}, x),
\]

and the \( k \)-th Whitney number of the second kind, \( W_k(P) \), is defined by

\[
    W_k(P) = |\{ x \in P \mid \rho(x) = k \}|.
\]

The Whitney numbers of a graded poset play an important role in many areas of combinatorics. For example, they appear as coefficients of the chromatic polynomial of a finite graph [24]. Stanley [30] showed they can be used to count the number of acyclic orientations of a graph. When the poset is the intersection lattice of a real hyperplane arrangement, Zaslavsky [37] showed that its Whitney numbers can be used to count the number of bounded and unbounded regions. For complex hyperplane arrangements the Whitney numbers can be used to compute the dimensions of the Orlik-Solomon algebra that is isomorphic to the Whitney cohomology of the intersection lattice [21]. Very recently, a long standing conjecture of Heron [14], Rota [25] and Welsh [36] concerning the log-concavity of the Whitney numbers of the first kind of geometric lattices was settled by Adiprasito, Huh, and Katz [1] using ideas coming from Hodge theory.

Now consider the previous two examples in Figure 1. When we compute the Whitney numbers of \( \Pi_3 \) and \( ISF_3 \) and we list them side by side (see Table 1), we notice a curious coincidence: (up to sign) their Whitney numbers of the first and second kind are interchanged. This surprising phenomenon was initially noticed by the authors of [11] for a
different, but closely related, pair of posets: the poset $\Pi^w_n$ of weighted partitions and the poset $\mathcal{SF}_n$ of rooted spanning forests on $[n]$ studied by Reiner in [22] and Sagan in [26]. It turns out that this phenomenon occurs for many other pairs of posets, in particular, the authors announced in [10] that for every geometric lattice there exist another poset with their Whitney numbers interchanged. This seemingly common phenomenon motivates the following definition.

**Definition 1.3.** Let $P$ and $Q$ be graded posets. We say that $P$ and $Q$ are **Whitney Duals** if for all $k \geq 0$ we have that

\[ |w_k(P)| = W_k(Q) \text{ and } |w_k(Q)| = W_k(P). \]

According to this definition, $\Pi_3$ and $\mathcal{ISF}_3$ are Whitney duals. In fact, this is true in general for $\Pi_n$ and $\mathcal{ISF}_n$ for $n \geq 1$ (see [10]).

Our investigation on Whitney duals is driven by the following two questions that we address to different extents in the present work:

**Question 1.4.** When does a graded poset $P$ has a Whitney dual?

**Question 1.5.** Is there a method to construct a Whitney dual for a graded poset $P$? Perhaps under certain assumptions about $P$.

In this article we advance towards an answer to the first question as we provide an answer to the second question. To do this, we use poset topology tools including edge labeling, chain-edge labelings, and quotient posets.

To state our main theorem, we now briefly review a few basic ideas concerning edge and chain-edge labelings. Recall that the **Hasse diagram of** $P$ is the directed graph on $P$ with directed edges the covering relations $x < y$ in $P$, i.e., $x < y$ where there is no $z \in P$ satisfying $x < z < y$. Also recall that an **edge labeling or E-labeling** of $P$ is a map $\lambda : \mathcal{E}(P) \to \Lambda$ where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of $P$ and $\Lambda$ is some other poset of labels. An edge labeling is said to be an **ER-labeling** if in every interval $[x, y]$ of $P$ there is a unique saturated or unrefinable chain

\[ c : (x = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = y) \]

that is increasing, i.e., such that

\[ \lambda(x_0 < x_1) < \lambda(x_1 < x_2) < \cdots < \lambda(x_{\ell-1} < x_\ell). \]

The concept of an edge labeling was generalized by Björner and Wachs to a labeling of pairs formed by maximal chains and edges of the Hasse diagram known as **chain-edge labelings** or

| $k$ | $w_k(\Pi_n)$ | $W_k(\Pi_n)$ | $w_k(\mathcal{ISF}_n)$ | $W_k(\mathcal{ISF}_n)$ |
|-----|-------------|-------------|-----------------|----------------|
| 0   | 1           | 1           | 1               | 1              |
| 1   | -3          | 3           | -3              | 3              |
| 2   | 2           | 1           | 1               | 2              |

**Table 1.** Whitney numbers of the first and second kind for $\Pi_3$ for $\mathcal{ISF}_3$. 

C-labelings. The generalization of an ER-labeling is then known as a CR-labeling. Chain-edge labelings are more involved and technical in their definition than edge labelings. Thus, as a presentation strategy across this article we choose to first discuss the definitions and constructions in the context of edge labelings and then explain how these constructions generalize to the context of chain-edge labelings. The reader should visit [2, 5, 34, 35] for information on edge and edge-chain labelings.

In Section 3 we define new types of edge and chain-edge labelings that we call Whitney labelings or W-labelings, the name coming from the fact that they provide sufficient conditions for the construction of Whitney duals of graded posets. A crucial property of Whitney labelings is the rank two switching property. We say that an edge labeling \( \lambda \) has the rank two switching property if for every maximal chain \( c : (\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k) \) that has an increasing step \( \lambda(x_{i-1} \prec x_i) < \lambda(x_i \prec x_{i+1}) \) at rank \( i \) there is a unique maximal chain \( c' : (\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{i-1} \prec x_i' \prec x_{i+1} \prec \cdots \prec x_{k-1} \prec x_k) \), whose labels are the same as the ones from \( c \) except for \( \lambda(x_{i-1} \prec x_i') = \lambda(x_i \prec x_{i+1}) \) and \( \lambda(x_i' \prec x_{i+1}) = \lambda(x_{i-1} \prec x_i) \). For chain-edge labelings we require an additional condition for the rank two switching property, namely that the choice of the \( x_i' \) is consistent among maximal chains that coincide in the bottom \( d \) elements when \( d > i \).

A Whitney labeling of \( P \) is an ER or CR-labeling with the rank two switching property and with the property that in every interval \([x, y]\) (a rooted interval in the case of CR-labelings) of \( P \) each ascent-free maximal chain is determined uniquely by its sequence of labels from bottom to top. We will call Whitney labelings respectively EW or CW-labelings depending if the underlying labeling is an ER or a CR-labeling.

The main result of this paper is the following:

**Theorem 1.6.** Let \( P \) be a poset with a Whitney labeling \( \lambda \). Then \( P \) has a Whitney dual. Moreover, using \( \lambda \) we can explicitly construct a Whitney dual \( Q_\lambda(P) \) of \( P \).

We actually prove a more general result. We define a more general kind of labeling that we call a generalized Whitney labeling. Its definition relies on a set of more general but, at the same time, more technical conditions that also imply the existence of a Whitney dual. However, all the examples that we know of so far satisfy the nicer definition that we gave above.

Theorem 1.6 guarantees for every Whitney labeling \( \lambda \) of a graded poset \( P \) a Whitney dual \( Q_\lambda(P) \). In general, there is no reason to expect that this construction is independent of \( \lambda \) and in fact in Section 5.3 we show that for the poset of weighted partitions \( \Pi_n^w \) there are at least two non-isomorphic Whitney duals. The two Whitney duals that we find are of the form \( Q_\lambda(P) \) for two different Whitney labelings. One of the labelings is an ER-labeling that was already introduced in [11]. We prove here that this labeling is also an EW-labeling. The second labeling is a new CR-labeling that we also show is a CW-labeling. The ascent-free chains of the new CW-labeling are indexed by rooted forests of \([n]\) and the construction of the poset \( Q_\lambda(P) \) provides an explanation of the Whitney
duality between the weighted partition poset, $\Pi_w^n$, and the poset of rooted spanning forest, $\mathcal{SF}_n$. This duality can be seen from the work of González D'León - Wachs in [11] and the work of Sagan in [26] after comparing the Whitney numbers of $\Pi_w^n$ and $\mathcal{SF}_n$. Our construction gives an explanation for this duality, which was our initial motivation for the current project.

**Theorem 1.7.** For all $n \geq 1$, there is a CW-labeling $\lambda_C$ of $\Pi_w^n$ such that $Q_{\lambda_C}(\Pi_w^n)$ and $\mathcal{SF}_n$ are isomorphic posets. In particular $\Pi_w^n$ and $\mathcal{SF}_n$ are Whitney duals.

The rest of the paper is organized as follows. In Section 2 we consider basic notions and examples of Whitney duals. In Section 3 we introduce the notion of Whitney labeling and we give a construction of a Whitney dual using a Whitney labeling and quotient posets.

In Section 4 we give a characterization of the Möbius values of $Q_{\lambda}(P)$. We also provide a simpler description of the posets $Q_{\lambda}(P)$ when $\lambda$ is a Whitney labeling (in the strict sense of the definition, i.e., not a generalized one).

In Section 5 we provide a series of examples of posets for which Theorem 1.6 applies. These include all geometric lattices, the poset of weighted partitions $\Pi_w^n$ and the R*S-labelable posets studied by Simion and Stanley [28]. This last family includes the lattice $\mathcal{NC}_n$ of noncrossing partitions of $[n]$, the posets of shuffles studied by Greene [12] and also the noncrossing partition lattices of types B and D studied by Reiner [23] (see also [15]). With these examples we illustrate both the existence of EW and CW-labelings. For the particular cases of $\mathcal{NC}_n$ and $\Pi_w^n$ we also give explicit combinatorial descriptions of their Whitney duals $Q_{\lambda}(P)$.

In Section 6 we show that a Whitney labeling induces a local action of the 0-Hecke algebra $H_n(0)$ of type A on the set of maximal chains of $P$ and hence a representation $\chi_P$ of $H_n(0)$ on the space spanned by the maximal chains of $P$. This action can be transported to a local action on the set of maximal chains of its Whitney dual $Q_{\lambda}(P)$ and hence also induces a representation $\chi_{Q_{\lambda}(P)}$. In particular, we prove the following theorem.

**Theorem 1.8.** Let $P$ be a poset with a (generalized) CW-labeling $\lambda$. Then

$$\text{ch} \left( \chi_{Q_{\lambda}(P)} \right) = \text{ch} \left( \chi_P \right) = F_P(x) = \omega F_{Q_{\lambda}}(x),$$

where $F_P(x)$ is Ehrenborg’s flag quasisymmetric function of the graded poset $P$, $\text{ch}(V)$ indicates the quasisymmetric characteristic of the $H_n(0)$-representation $V$ and $\omega$ is the classical involution in the ring of quasisymmetric functions that maps Gessel’s fundamental quasisymmetric function $L_{S,n}$ indexed by a set $S$ to the one indexed by its complement $[n - 1] \setminus S$.

Theorem 1.8 implies that $\chi_{Q_{\lambda}}(P)$ is induced by a “good action” in the sense of [28]. The results in [18] then imply that the bowtie-free maximal intervals of the posets $Q_{\lambda}(P)$ are snellable (see [18]) and the maximal intervals of $Q_{\lambda}(P)$ that are lattices are supersolvable.

2. Basic Examples of posets with and without Whitney Duals

We start by a discussion on some basic examples of posets which have and do not have Whitney duals.
A graded poset $P$ is **Eulerian** if $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ for all $x \leq y$ in $P$ (see Figure 2a for an example). Other examples of Eulerian posets include the face lattice of a convex polytope and the (strong) Bruhat order. For more information on Eulerian posets see the survey article [32]. From the definition of Eulerian poset, it is immediate that we have $|w_k(P)| = W_k(P)$ for all $k$. Therefore, every Eulerian poset has a Whitney dual, namely itself. It is natural to ask if all posets which are their own Whitney dual are Eulerian. The poset in Figure 2b shows that this is not the case \(^1\). This leads to the currently open question of whether there is a natural characterization of self-Whitney dual posets.

Not every ranked poset has a Whitney dual. For example, consider the three element chain $C$ in Figure 2c. We have that $w_2(C) = 0$ and $W_2(C) = 1$. If $Q$ was a Whitney dual of $C$, then $|w_2(Q)| = 1$ and $W_2(Q) = 0$, which is clearly impossible. This illustrates the fact that a poset $P$ with $|w_k(P)| = 0$ for some $k$ smaller than the rank of the poset cannot have a Whitney dual.

We also remark that there are more complicated reasons that can prevent a poset from having a Whitney dual. As an example, consider the poset in Figure 2d. Suppose that this poset had a Whitney dual, $Q$. Then the absolute value of the sequence of Whitney numbers of the first kind for $Q$ would be $1, 3, 2, 1$. Moreover, $Q$ would necessarily have a unique maximal element ($\hat{1}$). This would imply that the sum of the Whitney numbers of

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\(^1\)We thank Cyrus Hettle for pointing out this example to the authors.
the first kind would be 0. Now it must be the case that $w_0(Q) = 1$ and $w_2(Q) = -3$. As a result we should be able to assign a sign to 2 and 1 so that $1 - 3 \pm 2 \pm 1 = 0$, which is impossible.

One might wonder if Whitney duals are unique. By considering the posets in Figures 2a and 2b, one can see this is not the case. These posets are both self Whitney dual and Whitney dual to each other. In fact, a poset that is not self-Whitney dual can have multiple non-isomorphic Whitney duals. We show in Section 5.3 that $\Pi_n^w$ is an example of such a poset.

Now that we have seen some basic examples of posets with and without Whitney duals, we turn our attention to an approach on constructing Whitney duals.

3. Whitney labelings and quotient posets

The idea of edge labelings or E-labelings is pervasive in the poset literature. The concept of an ER-labeling (originally called an R-labeling) was introduced by Stanley (see [29]) to study the Möbius function of rank selected subposets of a graded poset. Björner [2] extended this notion by adding a lexicographic condition. Such labelings are called EL-labelings. The extra condition on an EL-labeling implies the shellability of the order (simplicial) complex of the poset and hence has stronger topological consequences that allow for the determination of its homotopy type. Björner and Wachs [4] extended the theory of lexicographic shellability to certain type of labelings called chain-edge labelings or C-labelings (see also [5, 6, 7]). This new concept of a CR-labeling (or CL-labeling when a lexicographic condition is considered) provided a more flexible description that helped in the determination of the Möbius numbers of posets that were not known to be included in the family of ER-labelable posets. For example, C-labelings can be used to understand the Möbius numbers of the Bruhat order of a Coxeter group. An ER-labeling is a special case of a CR-labeling and hence chain-edge labelings are in principle more general than edge labelings. In fact it is not known whether a poset that has a CR-labeling also has an ER-labeling. In this section we show how edge and chain-edge labelings can be used to construct Whitney duals. We will first describe all constructions using only the concept of an E-labeling since the presentation and proofs will be more clear and pleasant to the reader. The use of C-labelings require certain technical details, like the concept of a rooted interval, that might obscure the relevant ideas involved. However, we hope that the reader will find that the bottom-to-top nature of our constructions are compatible with C-labelings.

In the last subsection we discuss how all the theory developed for E-labelings continue to hold for C-labelings. We describe the important modifications to the definitions and properties involved when transferring from the context of E-labelings to C-labelings.

3.1. Edge labelings. We now discuss edge labelings and their relation with Whitney numbers. First, let us recall some basic facts about edge labelings. For complete treatments on the topic, see [2, 5, 34]. Let $P$ be a poset, and let $\mathcal{E}(P)$ be the set of edges of the Hasse diagram of $P$. Moreover, let $\Lambda$ be an arbitrary fixed poset that will be considered as the poset of labels. An edge labeling of $P$ is a map $\lambda : \mathcal{E}(P) \to \Lambda$. 

Let $P$ be a poset with edge labeling $\lambda$. To every saturated chain (also known as an unrefinable chain)

$$c : (x = x_0 \prec x_1 \prec \cdots \prec x_{\ell-1} \prec x_\ell = y)$$

we can associate a corresponding word of labels

$$\lambda(c) = \lambda(x_0 \prec x_1)\lambda(x_1 \prec x_2) \cdots \lambda(x_{\ell-1} \prec x_\ell).$$

We say that $c$ is increasing if its word of labels $\lambda(c)$ is strictly increasing, that is, $c$ is increasing if

$$\lambda(x_0 \prec x_1) < \lambda(x_1 \prec x_2) < \cdots < \lambda(x_{\ell-1} \prec x_\ell).$$

We say that $c$ is ascent-free if its word of labels $\lambda(c)$ has no ascents, i.e. $\lambda(x_{i-1} \prec x_i) \neq \lambda(x_i \prec x_{i+1})$, for all $i = 1, \ldots, \ell - 1$. Clearly, there are chains that are neither increasing nor ascent-free.

**Definition 3.1.** An edge labeling is an ER-labeling if in each closed interval $[x, y]$ of $P$, there is a unique increasing maximal chain. By analogy, we say that an edge labeling is an ER$^*$-labeling if in each closed interval $[x, y]$ of $P$, there is a unique ascent-free maximal chain.

We note that in [34], ER-labelings are referred to as R-labelings and have a slightly different definition. First, it is assumed that the labels are totally ordered. Also, increasing refers to weakly increasing and ascent-free is replaced with strictly decreasing. An almost identical proof as the one of Theorem 3.14.2 in [34] gives the following.

**Theorem 3.2** (c.f. Theorem 3.14.2 in [34]). Let $P$ be a graded poset with an ER-labeling (ER$^*$-labeling). Then

$$\mu(x, y) = (-1)^{\rho(y) - \rho(x)}|\{c \mid c \text{ is an ascent-free (increasing) maximal chain in } [x, y]\}|.$$ 

Let us now consider examples of ER and ER$^*$ labelings. In both examples the labels will come from the set $[n] \times [n]$ and we assume that this set is ordered lexicographically using the natural order on $[n]$ as integers.

**Example 3.3.** Let $\lambda : \mathcal{E}(\Pi_n) \to [n] \times [n]$ be the edge labeling defined on the partition lattice $\Pi_n$ by $\lambda(\pi \prec \sigma) = (i, j)$ where $i < j$ and $i$ and $j$ are the minimum elements of the two blocks of $\pi$ that were merged to obtain $\sigma$. This edge labeling is an ER-labeling and is a special case of Björner’s minimum labeling for geometric lattices described in [3]. In Figure 3 the labeling $\lambda$ of $\Pi_3$ is depicted.

**Example 3.4.** Let $\lambda^* : \mathcal{E}(\mathcal{ISF}_n) \to [n] \times [n]$ be the edge labeling defined on $\mathcal{ISF}_n$ by setting $\lambda^*(F_1 \prec F_2)$ to be the unique edge in $F_2$ that is not in $F_1$. It was proved in [10, Proposition 3] that this edge labeling is an ER$^*$-labeling. In Figure 3, the labeling $\lambda^*$ of $\mathcal{ISF}_3$ is depicted.

The reader may have noticed how similar are the labelings of $\Pi_3$ and $\mathcal{ISF}_3$. This is no coincidence and as we will see later, the labeling of $\mathcal{ISF}_3$ can be obtained from the labeling of $\Pi_3$. 
Figure 3. Example of ER and ER∗-labelings on Π₃ and ISF₃

| λ is an   | \[w_k(P)\]                              | \[W_k(P)\]                              |
|-----------|----------------------------------------|----------------------------------------|
| ER-labeling | # (ascent-free sat. chains of length \(k\) starting at \(\hat{0}\)) | # (increasing sat. chains of length \(k\) starting at \(\hat{0}\)) |
| ER∗-labeling | # (increasing sat. chains of length \(k\) starting at \(\hat{0}\)) | # (ascent-free sat. chains of length \(k\) starting at \(\hat{0}\)) |

Table 2. Proposition 3.5

Using the definition of Whitney numbers, Definition 3.1, and Theorem 3.2, we can describe the Whitney numbers of a poset using an ER-labeling (ER∗-labeling) by the enumeration of saturated chains as follows.

**Proposition 3.5.** Let \(P\) be a graded poset with an ER-labeling (ER∗-labeling). Then \([w_k(P)]\) is the number of ascent-free (increasing) saturated chains starting at \(\hat{0}\) of length \(k\). Moreover, \([W_k(P)]\) is the number of increasing (ascent-free) saturated chains starting at \(\hat{0}\) of length \(k\).

In Table 2 we summarize in the conclusions of Proposition 3.5 to highlight the importance of this proposition and the fact that ER and ER∗-labelings switch the role of increasing and ascent-free saturated chains. This fact will be used later in a construction of Whitney duals.

3.2. Constructing Whitney Duals. In the following we use edge labelings with the following property.

**Definition 3.6.** Let λ be an edge labeling on \(P\). We say that λ has the rank two switching property if for every maximal chain

\[c : (\hat{0} = x_0 < x_1 < \cdots < x_{k-1} < x_k)\]
that has an increasing step $\lambda(x_{i-1} < x_i) < \lambda(x_i < x_{i+1})$ at rank $i$ there is a unique maximal chain

$$c': (\emptyset = x_0 < x_1 < \cdots < x_{i-1} < x'_i < x_{i+1} < \cdots < x_{k-1} < x_k),$$

whose labels are the same as the ones from $c$ except for $\lambda(x_{i-1} < x'_i) = \lambda(x_i < x_{i+1})$ and $\lambda(x'_i < x_{i+1}) = \lambda(x_{i-1} < x_i)$.

**Remark 3.7.** For E-labelings, there is a simpler way to describe the rank two switching property. We can say that $\lambda$ has the rank two switching property provided that for every interval $[x, y]$ with $\rho(y) - \rho(x) = 2$, if $\lambda_1 \lambda_2$ is the word of labels of the unique increasing maximal chain in the interval, then there exists a unique maximal chain in $[x, y]$ whose word of labels is $\lambda_2 \lambda_1$. We choose to give the seemingly more complicated definition because it closely resembles the condition for C-labelings that we provide later.

In Figure 3, one can see that the labeling of $\Pi_3$ given in Example 3.3 has the rank two switching property. Indeed, the increasing chain in the unique rank two interval of $\Pi_3$ is labeled by $(1, 2)(1, 3)$ and there is a unique chain labeled by $(1, 3)(1, 2)$. In fact, $\Pi_n$ has the rank two switching property for all $n \geq 1$. One can verify this using the fact that there are only two types of rank two interval in $\Pi_n$. Each interval is isomorphic to $\Pi_3$ or to a boolean algebra of rank two.

**Definition 3.8.** Let $P$ be a graded poset and let $\lambda$ be an ER-labeling of $P$ with the rank two switching property. Denote $\mathcal{M}_{[x,y]}$ the set of maximal chains in $[x, y] \subseteq P$ and let $c : (x = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = y) \in \mathcal{M}_{[x,y]}$ be a saturated chain having an ascent $\lambda(x_{i-1} < x_i) < \lambda(x_i < x_{i+1})$ at rank $i$. By the rank two switching property there is an element $\tilde{x}_i \in [x_{i-1}, x_{i+1}]$ such that $\lambda(x_{i-1} < \tilde{x}_i) = \lambda(x_i < x_{i+1})$ and $\lambda(x_{i-1} < x_i) = \lambda(\tilde{x}_i < x_{i+1})$.

We say that the chain $c \setminus \{x_i\} \cup \{\tilde{x}_i\}$, that is obtained from $c$ after removing $x_i$ and adding $\tilde{x}_i$, was obtained by a *quadratic exchange at rank* $i$. We will use the notation $U_i(c) = c'$ if $c'$ is obtained from $c$ by applying a quadratic exchange at rank level $i$ and whenever $c$ does not have an ascent at rank $i$ we define $U_i(c) = c$.

For every interval $[x, y]$ in $P$ we can define a labeled directed graph $G_{[x,y]}$ whose vertex set is $\mathcal{M}_{[x,y]}$ and where there is a labeled directed edge $c_1 \xrightarrow{U_i} c_2$ if $c_2$ is obtained from $c_1$ by a quadratic exchange at rank $i$. We define also $S(c) := \{\lambda(x_{i-1} < x_i) \mid i \in [\ell]\}$ the multiset of labels of the chain $c$. Note that a quadratic exchange leaves the multiset of labels invariant, i.e., $S(c_1) = S(c_2)$ whenever $c_1$ and $c_2$ are related by a sequence of quadratic exchanges. Thus in general $G_{[x,y]}$ is not a connected graph, for example two chains in $\mathcal{M}_{[x,y]}$ with different multisets of labels belong to different connected components of $G_{[x,y]}$.

We call a pair $(i < j)$ such that $\lambda(x_{i-1} < x_i) < \lambda(x_{j-1} < x_j)$ a *label inversion* of $c$. Note that a quadratic exchange reduces the number of label inversions of $c$ and, since the directed edges in $G_{[x,y]}$ are given by quadratic exchanges, this implies that $G_{[x,y]}$ does not contain directed cycles.

By our construction ascent-free chains in $\mathcal{M}_{[x,y]}$ are precisely the vertices of $G_{[x,y]}$ that have outdegree 0, also known as *sinks*. Indeed, quadratic exchanges can only happen at
an ascending step of a saturated chain. In particular, by repeatedly applying quadratic exchanges we get the following lemma.

**Lemma 3.9.** For every chain \( c \in M_{[x,y]} \) there exists at least one ascent-free maximal chain \( c' \in M_{[x,y]} \) such that \( c \) and \( c' \) belong to the same connected component of \( G_{[x,y]} \). Any such chain \( c' \) satisfies \( S(c') = S(c) \).

Lemma 3.9 tells us that each connected component of \( G_{[x,y]} \) has at least one vertex that is a sink, however this sink vertex is not necessarily unique. Ideally, for our construction, we would like to have a unique ascent-free chain \( c \in M_{[x,y]} \) in each connected component of \( G_{[x,y]} \). This would imply that the connected components of \( G_{[x,y]} \) are indexed by ascent-free chains in \( M_{[x,y]} \). To guarantee this condition, we will use a classical result in graph theory and in the study of term rewriting systems, known either as the Diamond Lemma or Newman’s Lemma (see [19, 17]).

We say that a directed graph \( G \) is confluent if for every pair of vertices \( x \) and \( y \) in the same connected component of \( G \) there are directed walks \( x \sim u \) (a sequence of directed edges \( x \rightarrow z \rightarrow \cdots \rightarrow u \)) and \( y \sim u \) that meet at a common vertex \( u \) of \( G \). Confluence has the following easy but interesting consequence.

**Lemma 3.10.** If \( G \) is a confluent directed graph that does not contain infinite directed walks \( x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \) (or cycles) then every connected component of \( G \) contains a unique sink vertex.

We say that \( G \) is locally confluent if for every pair of directed edges \( v \rightarrow x \) and \( v \rightarrow y \) there are directed walks \( x \sim u \) and \( y \sim u \) that meet at a common vertex \( u \) of \( G \).

**Lemma 3.11** (Newman’s Lemma c.f. [19, 17]). A directed graph \( G \) without infinite directed walks is confluent if and only if it is locally confluent.

Lemma 3.11 simplifies the procedure of checking confluence by restricting to local confluence which is an easier condition to test, especially given that \( G_{[x,y]} \) is a finite graph without cycles. In \( G_{[x,y]} \) the local confluency condition can be tested by considering two labeled directed edges \( c \xleftarrow{U_i} c_1 \) and \( c \xleftarrow{U_j} c_2 \). Note that since \( U_i(c) \) produces a unique chain by the rank two switching property, we may always assume \( i \neq j \). If \( |i - j| > 1 \) then we can always obtain a chain \( c' \) either as \( c \xleftarrow{U_i} c_1 \xleftarrow{U_j} c' \) or \( c \xleftarrow{U_j} c_2 \xleftarrow{U_i} c' \) since the elements involved in the ascending steps are not adjacent to each other. In other words, if \( |i - j| > 1 \), then \( U_i U_j(c) = U_j U_i(c) \) for all chains \( c \) with ascents at \( i \) and \( j \). When \( j = i + 1 \) we say that \( c \) has a double-ascent or that \( c \) has a critical condition at rank \( i \), i.e., for \( c : (x = x_0 < x_1 < \cdots < x_{\ell-1} < x_{\ell} = y) \) we have that
\[
\lambda(x_{i-1} < x_i) < \lambda(x_i < x_{i+1}) < \lambda(x_{i+1} < x_{i+2}).
\]

**Definition 3.12.** Let \( P \) be a graded poset and let \( \lambda \) be an ER-labeling of \( P \) with the rank two switching property. For every saturated chain \( c \in M_{[x,y]} \) with a critical condition at rank \( i \) we can obtain saturated chains \( c' \) and \( c'' \) by removing the ascents at ranks \( i \) and \( i + 1 \) by a sequence of exchanges \( c \xleftarrow{U_i^\lambda} c_1 \xleftarrow{U_i^\lambda} c_2 \xleftarrow{U_i} c' \) and \( c \xleftarrow{U_i^\lambda} c_3 \xleftarrow{U_i} c_4 \xleftarrow{U_i^\lambda} c'' \). We say
that \( \lambda \) satisfies the braid relation if for every such \( c \in \mathcal{M}_{[x,y]} \) we have that \( c' = c'' \). In other words, we have \( U_i U_{i+1} U_i(c) = U_{i+1} U_i U_{i+1}(c) \) for chains \( c \) which have a critical condition at rank \( i \).

Note that by the discussion above, if the ER-labeling \( \lambda \) satisfies the braid relation then the graph \( G_{[x,y]} \) is locally confluent, hence we have the following corollary.

**Corollary 3.13.** Let \( P \) be a graded poset and let \( \lambda \) be an ER-labeling of \( P \) satisfying

- the rank two switching property, and
- the braid relation.

Then for every interval \([x, y]\) in \( P \), we have that each connected component of \( G_{[x,y]} \) has a unique sink, i.e., a unique ascent-free saturated chain.

3.2.1. **Quotient posets.** We now turn our attention to quotient posets, which is the other main tool we use for the construction of Whitney duals. We begin with a definition.

**Definition 3.14.** Let \( P \) be a graded poset and let \( \sim \) be an equivalence relation on \( P \) such that if \( x \sim y \), then \( \rho(x) = \rho(y) \). We define the quotient poset \( P/ \sim \) to be the set of equivalence classes ordered by \( X \leq Y \) if and only if there exists \( x \in X \), \( y \in Y \) and \( z_1, z_2, \ldots, z_k \in P \) such that

\[
(3.1) \quad x = z_0 \leq z_1 \sim z_2 \leq \cdots \leq z_{n-1} \sim z_k \leq z_{k+1} = y.
\]

For any element \( x \in P \) we will denote by \([x]\) its corresponding equivalence class in \( P/ \sim \).

The next proposition follows from Definition 3.14.

**Proposition 3.15.** Let \( P \) be a graded poset and let \( \sim \) be an equivalence relation on \( P \) such that if \( x \sim y \), then \( \rho(x) = \rho(y) \). Then we have the following.

1. \( P/ \sim \) is a poset.
2. For \( X, Y \in P/ \sim \), \( X \preceq Y \) if and only if \( x \preceq y \) for some \( x \in X \) and \( y \in Y \).
3. \( P/ \sim \) is graded and for \( X \in P/ \sim \), we have \( \rho(X) = \rho(x) \) for all \( x \in X \).

**Proof.** Part (2) can be easily verified from the definitions and part (3) is a consequence of parts (1) and (2) since they imply that the function \( \rho(X) = \rho(x) \), for an arbitrary \( x \in X \), is a well-defined rank function. We show that \( P/ \sim \) together with the relation \( \leq \) satisfies the three properties of a poset.

1. (Reflexive) This is clear.
2. (Antisymmetric) Suppose that \( X \leq Y \) and \( Y \leq X \). Since elements in each equivalence class have the same rank, then for any elements \( x \in X \) and \( y \in Y \) we have that \( X \leq Y \) implies \( \rho(x) \leq \rho(y) \). Similarly, \( Y \leq X \) implies \( \rho(y) \leq \rho(x) \) and hence \( \rho(x) = \rho(y) \). Thus in (3.1) none of the inequalities can be strict, implying \( x \sim y \) and hence \( X = Y \).
3. (Transitive) Suppose that \( X \leq Y \) and \( Y \leq Z \). Then by the definition of \( \leq \) in (3.1), \( X \leq Y \) implies that there exist \( x \in X \), \( y \in Y \) and \( u_1, \ldots, u_k \) such that

\[
(3.2) \quad x = u_0 \leq u_1 \sim u_2 \leq \cdots \leq u_{k-1} \sim u_k \leq u_{k+1} = y.
\]
Also \( Y \leq Z \) implies that there are \( y' \in Y, z \in Z \) and \( w_1, \ldots, w_s \) such that

\[
y' = w_0 \leq w_1 \sim w_2 \leq \cdots \leq w_{n-1} \sim w_n \leq w_{n+1} = z.
\]

Since both \( y \) and \( y' \) are in \( Y \), we have that \( y \sim y' \). This together with (3.2) and (3.3) imply that \( X \leq Z \).

The poset \( P/ \sim \) with the relation defined above is called the *quotient poset* of \( P \) by the relation \( \sim \).

**Definition 3.16.** Given a poset \( P \), let \( C(P) \) denote the poset whose elements are saturated chains of \( P \) starting at \( \hat{0} \) ordered by inclusion. We call \( C(P) \) the *chain poset* of \( P \). Figure 4 depicts \( \Pi_3 \) and \( C(\Pi_3) \). If \( c \in C(P) \), we write \( e(c) \) for the maximal element of \( c \), i.e., the element of \( P \) where \( c \) terminates. Suppose that \( \lambda \) is an ER-labeling of \( P \) with the rank two switching property. Let \( \sim_\lambda \) be the equivalence relation on \( C(P) \) defined by \( c_1 \sim_\lambda c_2 \) whenever, \( c_1 \) and \( c_2 \) are in the same connected component of \( G_{[0, e(c_1)]} \). We will use \( Q_\lambda(P) \) to denote \( C(P)/ \sim_\lambda \).

Note that by the nature of the quadratic exchanges, for every \( X \in Q_\lambda(P) \) and \( c, c' \in X \) we have that \( e(c) = e(c') \). Thus, we can also define \( e(X) = e(c) \) for any \( c \in X \).

**Example 3.17.** Consider again \( \Pi_3 \) and its chain poset \( C(\Pi_3) \) shown in Figure 4b. Every element of \( C(\Pi_3) \) is in its own equivalence class except for the chains \( 1/2/3 \leq 12/3 \leq 123 \) and \( 1/2/3 \leq 13/2 \leq 123 \) since \( 1/2/3 \leq 12/3 \leq 123 \) is the only directed edge in \( G_{[1/2/3, 123]} \). Taking the quotient to obtain \( Q_\lambda(\Pi_3) \), we get the poset in Figure 4c, where we have identified the equivalence classes by the underlying set of labels on the chains. By comparing with Figure 3, one can observe that \( \mathcal{LSF}_3 \) and \( Q_\lambda(\Pi_3) \) are isomorphic.

Note that by the definition of \( \sim_\lambda \), if \( \lambda \) satisfies the conditions of Corollary 3.13 then each equivalence class \( X \in Q_\lambda(P) \) contains a unique ascent-free maximal chain in \( [\hat{0}, e(X)] \). In fact this is a correspondence between ascent-free saturated chains starting at \( \hat{0} \) in \( P \) of length \( k \) and equivalence classes in \( Q_\lambda(P) \) of rank \( k \). Hence, using Proposition 3.5 and the
definition of Whitney numbers of the second kind in Equation 1.3 we are able to conclude at this point that \(|w_k(P)| = W_k(Q_\lambda(P))\).

We now turn our attention to the task of satisfying the other half of Definition 1.3, that is, we would like to have in addition that \(W_k(P) = |w_k(Q_\lambda(P))|\). This will allow us to conclude that \(Q_\lambda(P)\) is a Whitney Dual of \(P\). Our strategy will be to define an edge labeling \(\lambda^*\) on \(Q_\lambda(P)\) that under certain conditions is an \(ER^*\)-labeling. We will then show that the saturated chains from \(0\) in \(Q_\lambda(P)\) under the newly defined \(ER^*\)-labeling and the ones in \(P\) under the labeling \(\lambda\) are in a label-preserving bijection. This together with Proposition 3.5 imply that \(P\) and \(Q_\lambda(P)\) are Whitney duals.

To define this labeling recall that, by definition, \(c_1 \sim_\lambda c_2\) implies \(S(c_1) = S(c_2)\). In light of this, we will use \(S(X)\) to denote the multiset of labels in any chain in \(X\). Moreover, if \(X \preceq Y\) in \(Q_\lambda(P)\) then there exists a unique element in \(S(Y) \setminus S(X)\). Define an edge labeling \(\lambda^*\) on \(Q_\lambda(P)\) by

\[
\lambda^*(X \preceq Y) = S(Y) \setminus S(X).
\]

This edge labeling for \(Q_\lambda(P)\) appears in Figure 4c. We will now consider a pair of structural lemmas that will be useful in the following discussion.

**Lemma 3.18.** Let \(X_1 \preceq X_2 \preceq \cdots \preceq X_k\) be a saturated chain in \(Q_\lambda(P)\).

(a) We have that \(e(X_1) \prec e(X_2) \prec \cdots \prec e(X_k)\) is a saturated chain in \(P\).

(b) For any chain \(c \in X_i\) we have that \(c \cup \{e(X_2), \ldots, e(X_i)\} \in X_i\) for all \(i = 1, \ldots, k\).

(c) If \(X_1 = X'_1 \preceq X'_2 \preceq \cdots \preceq X'_k\) is another saturated chain with \(e(X_i) = e(X'_i)\) for all \(i\), then \(X_i = X'_i\) for all \(i\).

**Proof.** First we show (a) holds. Consider the cover \(X_{i-1} \preceq X_i\). By Proposition 3.15, there exists a \(c \in X_{i-1}\) and \(d \in X_i\) with \(c \prec d\) in \(C(P)\). By the definition of the poset \(C(P)\) we have that \(d = c \cup \{e(X_i)\}\) and so \(e(X_{i-1}) = e(c) \prec e(d) = e(X_i)\). Thus, \(e(X_1) \prec e(X_2) \prec \cdots \prec e(X_k)\) is a saturated chain in \([e(X_1), e(X_k)]\).

To show (b), we use induction on \(k\). The case when \(k = 1\) is trivial. Now suppose \(k \geq 2\). By induction, for any \(c \in X_1\) we have \(c \cup \{e(X_2), \ldots, e(X_i)\} \in X_i\) for \(i \leq k - 1\). We want to show that \(c \cup \{e(X_2), \ldots, e(X_{k-1}), e(X_k)\} \in X_k\). First, note that by part (a), \(c \cup \{e(X_2), \ldots, e(X_k)\} \} is a saturated chain in \(P\). Now since \(X_{k-1} \preceq X_k\), Proposition 3.15 implies that there exists \(d \in X_{k-1}\) and \(d' \in X_k\) with \(d' = d \cup \{e(X_k)\}\). Since \(c \cup \{e(X_2), \ldots, e(X_{k-1})\}\) and \(d\) both belong to \(X_{k-1}\), we know these two chains are related by a sequence of quadratic moves. It follows that \(c \cup \{e(X_2), \ldots, e(X_{k-1}), e(X_k)\}\) and \(d'\) are related by the exact same moves and so are in the same equivalence class. Thus, \(c \cup \{e(X_2), \ldots, e(X_{k-1}), e(X_k)\}\) is in \(X_k\).

Note that part (c) follows directly from part (b).

**Lemma 3.19.** Let \(C : (X_1 \preceq X_2 \preceq \cdots \preceq X_k)\) be a saturated chain in \(Q_\lambda(P)\) and let \(c : (e(X_1) \prec e(X_2) \prec \cdots \prec e(X_k))\) be the corresponding saturated chain in \(P\). The words of labels of these two saturated chains under their respective labelings are equal, i.e., \(\lambda^*(C) = \lambda(c)\).

**Proof.** By Equation (3.4), \(\lambda^*(X_{i-1} \preceq X_i) = S(X_i) \setminus S(X_{i-1}) = S(d') \setminus S(d) = \lambda(e(X_{i-1}) \prec e(X_i))\), where \(d \in X_{i-1}\) and \(d' \in X_i\) are such that \(d \preceq d'\).
To prove that the labeling $\lambda^*$ given in Equation (3.4) is an ER*-labeling of $Q_\lambda(P)$ we will need the additional following condition.

**Definition 3.20.** Let $\lambda$ be an ER-labeling on $P$ with the rank two switching property. We say that $\lambda$ is *cancellative* if for every $z < x < y$ in $P$, $c \in M_{[z,y]}$ and $c_1, c_2 \in M_{[x,y]}$ we have that

$$c \cup c_1 \sim_\lambda c \cup c_2 \implies c_1 \sim_\lambda c_2.$$

**Proposition 3.21.** Let $P$ be a graded poset and let $\lambda$ be an ER-labeling of $P$ satisfying

- the rank two switching property,
- the braid relation, and
- the cancellative property.

Then the labeling $\lambda^*$ of $Q_\lambda(P)$ given by Equation (3.4) is an ER* labeling.

**Proof.** Let $[X, Y]$ be an interval in $Q_\lambda(P)$. We will show that there is a unique ascent-free maximal chain in $[X, Y]$.

The interval $[X, Y]$ contains an ascent-free maximal chain. Pick any $c \in X$ and let $X = X_1 < X_2 < \cdots < X_k = Y$ be a maximal chain in $[X, Y]$. By Lemma 3.18 parts (a) and (b), $(e(X_1) < e(X_2) < \cdots < e(X_k))$ is a saturated chain in $[e(X), e(Y)]$ and $c \cup \{e(X_2), e(X_3), \ldots, e(X_k)\} \in Y$. The chain $(e(X_1) < e(X_2) < \cdots < e(X_k))$ may not be ascent-free in $[e(X), e(Y)]$, but it is related to one by a sequence of quadratic exchanges. Suppose the related ascent-free chain is $e(X) = x'_1 < x'_2 < \cdots < x'_k = e(Y)$. For each $1 \leq i \leq k$, let $X'_i$ be the equivalence class containing $c \cup \{x'_2, x'_3, \ldots, x'_i\}$. Then it must be the case that $X'_k = Y$ since we are using quadratic exchanges on $c \cup \{e(X_2), e(X_3), \ldots, e(X_k)\} \in Y$. In $C(P)$ we have that $c \not< c \cup \{x'_2, x'_3, \ldots, x'_k\}$ and hence, by the quotient poset definition of $Q_\lambda(P)$ we have that $X = X'_1 < X'_2 < \cdots < X'_k = Y$ is a maximal chain in $[X, Y]$. Moreover, by Lemma 3.19, the labels along this chain are the same as along $x'_1 < x'_2 < \cdots < x'_k$. It follows that $[X, Y]$ has an ascent-free maximal chain.

The ascent-free maximal chain found above is unique. Suppose this was not the case and that $X = X_1 < X_2 < \cdots < X_k = Y$ and $X = X'_1 < X'_2 < \cdots < X'_k = Y$ are both ascent-free maximal chains in $[X, Y]$. Pick any $c \in X$. Then by Lemma 3.18 part (b), $c \cup \{e(X_2), e(X_3), \ldots, e(X_k)\}$ and $c \cup \{e(X'_2), e(X'_3), \ldots, e(X'_k)\}$ are both chains in $Y$. By the cancellative property $c \cup \{e(X_2), e(X_3), \ldots, e(X_k)\} \sim c \cup \{e(X'_2), e(X'_3), \ldots, e(X'_k)\}$ implies that $e(X_1) < e(X_2) < \cdots < e(X_k)$ and $e(X'_1) < e(X'_2) < \cdots < e(X'_k)$ are related by quadratic exchanges. It follows that both of these chains are in the same connected component of $G_{[e(X), e(Y)]}$ and by Lemma 3.19, they are both ascent-free. However, Corollary 3.13, asserts that there is a unique ascent-free maximal chain in each connected component implying that $e(X_i) = e(X'_i)$ for all $i$. Applying Lemma 3.18 part (c) we conclude that $X_i = X'_i$ for all $i$. It follows that there is a unique ascent-free maximal chain in each interval and so $\lambda^*$ is an ER* labeling.

Before stating the main theorem of this section let us discuss a very important class of labelings that satisfy the conditions of Proposition 3.21.

**Theorem 3.22.** Let $\lambda$ be an ER-labeling satisfying
Remark 3.23

Definition 3.24. Let \( \lambda : \mathcal{E}(P) \to \Lambda \) be an ER-labeling of \( P \). We say \( \lambda \) is an EW-labeling if it satisfies

- the rank two switching property, and
- in each interval each ascent-free maximal chain has a unique word of labels;

then \( \lambda \) satisfies

- the rank two switching property,
- the braid relation, and
- the cancellative property.

Proof. We need to check that \( \lambda \) satisfies the braid relation and the cancellative property.

The braid relation is satisfied by \( \lambda \). Indeed, let \( c \) be a saturated chain that has a critical condition at rank \( i \). We obtain saturated chains \( c' \) and \( c'' \) by removing the ascents at ranks \( i \) and \( i+1 \) by a sequence of exchanges \( c \xleftarrow{U_{i+1}} c_1 \xleftarrow{U_i} c_2 \xleftarrow{U_{i+1}} c' \) and \( c \xleftarrow{U_{i+1}} c_3 \xleftarrow{U_i} c_4 \xleftarrow{U_{i+1}} c'' \). By the definition of a quadratic exchange we have that \( c' \) and \( c'' \), when restricted to the interval between ranks \( i-1 \) and \( i+2 \), have the same ascent-free word of labels and hence \( c' = c'' \).

The cancellative property is satisfied by \( \lambda \). We are going to prove, using induction on the value of \( \rho(x) - \rho(z) \), that for every \( z < x < y \) in \( P \), \( c \in \mathcal{M}_{[z,x]} \) and \( d, d' \in \mathcal{M}_{[x,y]} \), we have that \( c \cup d \sim_{\Lambda} c \cup d' \) implies \( d \sim_{\Lambda} d' \).

When \( \rho(x) - \rho(z) = 1 \) we have \( c : (z < x) \) and without loss of generality we can assume, perhaps after applying enough quadratic exchanges, that \( d \) and \( d' \) are ascent-free. Hence the only possible ascents must happen in the step \( z < x < d'_1 \) for \( c \cup d \) and in the step \( z < x < d'_1 \) for \( c \cup d' \). Note that then quadratic exchanges on \( c \cup d \) can only shuffle the label \( \lambda(z < x) \) across the word \( \lambda(d) \) and quadratic exchanges on \( c \cup d' \) can only shuffle \( \lambda(z < x) \) across the word \( \lambda(d') \). Because we have \( c \cup d \sim_{\Lambda} c \cup d' \) and the braid relation, Corollary 3.13 implies that the ascent-free word of labels obtained after all the quadratic exchanges have been applied to both \( c \cup d \) and \( c \cup d' \) is the same. Hence we originally had \( \lambda(d) = \lambda(d') \). Uniqueness of the ascent-free word of labels implies then \( d = d' \).

Now consider the case when \( \rho(x) - \rho(z) = k > 1 \) and we have chains \( c \in \mathcal{M}_{[z,x]} \) and \( d, d' \in \mathcal{M}_{[x,y]} \) such that \( c \cup d \sim_{\Lambda} c \cup d' \). Note first that \( z < c_{k-1} < y \), and we will consider instead the saturated chains \( c' := c \setminus \{x\} \in \mathcal{M}_{[z,c_{k-1}]} \), \( m := \{c_{k-1}\} \cup d \in \mathcal{M}_{[c_{k-1},y]} \) and \( m' := \{c_{k-1}\} \cup d' \in \mathcal{M}_{[c_{k-1},y]} \). We have that \( \rho(c_{k-1}) = \rho(z) = \rho(x) - \rho(z) - 1 \) and \( c' \cup m = c \cup d \sim_{\Lambda} c \cup d' = c' \cup m' \), hence by induction we conclude that \( m \sim_{\Lambda} m' \). Now if we consider \( c_{k-1} < x < y \), the argument above says that \( \{c_{k-1} < x\} \cup d = m \sim_{\Lambda} m' = \{c_{k-1} < x\} \cup d' \) in \( \mathcal{M}_{[c_{k-1},y]} \) and since \( \rho(x) - \rho(c_{k-1}) = 1 \) we are back in the base case that implies \( d \sim_{\Lambda} d' \). \( \square \)

Remark 3.23. Note that in Theorem 3.22 we can replace the second condition by the stronger requirement that maximal chains have unique word of labels.

We are now in a position to provide names to the type of edge labelings that allow us to construct Whitney duals. We call these labelings EW-labelings, where the letter “W” comes from the fact that they provide sufficient conditions to construct Whitney duals.

**Definition 3.24.** Let \( \lambda : \mathcal{E}(P) \to \Lambda \) be an ER-labeling of \( P \). We say \( \lambda \) is an EW-labeling if it satisfies
• the rank two switching property, and
• in each interval each maximal chain has a unique word of labels.

**Definition 3.25.** Let \( \lambda : \mathcal{E}(P) \to \Lambda \) be an ER-labeling of \( P \). We say \( \lambda \) is a *generalized EW-labeling* if it satisfies

• the rank two switching property,
• the braid relation, and
• the cancellative property.

The following proposition provides an insight into the construction process that generates \( Q_\lambda(P) \) from \( P \) using \( \lambda \). According to Proposition 3.26, we can think of \( Q_\lambda(P) \) as a poset that is obtained from \( P \) by “pulling apart” saturated chains from \( \hat{0} \).

**Proposition 3.26.** Let \( \lambda \) be a generalized EW-labeling of \( P \). There is a label preserving bijection from the set of saturated chains from \( \{\hat{0}\} \) of length \( k \) in \( Q_\lambda(P) \) and the set of saturated chains from \( \hat{0} \) of length \( k \) in \( P \). In particular, there is a label preserving bijection \( M_{Q_\lambda(P)} \to M_P \) between maximal chains.

**Proof.** Fix \( k \) and let \( S_{Q,k} \) be the set of saturated chains from \( \{\hat{0}\} \) of length \( k \) in \( Q_\lambda(P) \) and \( S_{P,k} \) be the set of saturated chains from \( \hat{0} \) of length \( k \) in \( P \). Let \( \varphi : S_{Q,k} \to S_{P,k} \) be defined by

\[
\varphi([\hat{0}] = X_0 \prec X_1 \prec \cdots \prec X_k) = (\hat{0} = e(X_0) \prec e(X_1) \prec \cdots \prec e(X_k)).
\]

By Lemma 3.18 part (a) we know \( \varphi \) is well-defined and by Lemma 3.19 we know that \( \varphi \) preserves the word of labels. By Lemma 3.18 part (c), \( \varphi \) is injective.

Finally, we show that \( \varphi \) is surjective. Let \( \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k \in S_{P,k} \). Let \( c_i = \{x_0, x_1, \ldots, x_i\} \), then by definition \( c_0 \prec c_1 \prec \cdots \prec c_k \) is in \( C(P) \). By the definition of a quotient poset, \( [c_0] \prec [c_1] \prec \cdots \prec [c_k] \) is in \( S_{Q,k} \) and it is a preimage of \( \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k \). \( \square \)

Definition 1.3 together with Propositions 3.5, 3.21 and 3.26 imply our main theorem, that we are ready to state in the language of Definitions 3.24 and 3.25.

**Theorem 3.27.** Let \( P \) be a graded poset with a generalized EW-labeling \( \lambda \). Then \( Q_\lambda(P) \) is a Whitney dual of \( P \).

**Remark 3.28.** In [10] the authors defined the related concept of an \( \overline{\text{EW}} \)-labeling. The reason for the use of an overline in that article is that the conditions on those labelings are more restrictive but imply the conditions of Definition 3.24. While the definition of an \( \overline{\text{EW}} \)-labeling greatly simplifies the proofs of the theorems we have presented here, there are posets with EW-labelings, but no known \( \overline{\text{EW}} \)-labeling. See Section 5.3 for some examples.

3.3. **Chain-edge labelings.** We show in this subsection that the definitions and constructions given for EW-labelings also extend to the generality of chain-edge labelings with the same consequences with respect to Whitney duality.

**Definition 3.29.** Let \( \mathcal{ME}(P) \) denote the set of pairs \((m, e)\) where \( m \) is a maximal chain in \( P \) and \( e \) is an edge in the Hasse diagram of \( m \). A *chain-edge labeling* or *C-labeling* of \( P \) is a map \( \lambda : \mathcal{ME}(P) \to \Lambda \), where \( \Lambda \) is some poset of labels, satisfying the condition
that whenever two maximal chains coincide along the bottom \(d\) edges then their labels also coincide on these \(d\) edges.

**Definition 3.30.** A rooted interval \([x, y]_r\) in \(P\) is a pair \(([x, y], r)\) where \([x, y]\) is an interval in \(P\) and \(r\) is a saturated chain from \(0\) to \(x\).

The rationale behind Definition 3.30 is that in a C-labeling, when we want to restrict to a smaller interval \([x, y]\) in \(P\), the labels depend on the initial saturated chain \(r\) from \(0\) to \(x\).

**Definition 3.31.** A C-labeling \(\lambda\) of \(P\) naturally induces a C-labeling \(\lambda_r\) in a rooted interval \([x, y]_r\) by letting the labels of a maximal chain \(c\) of \([x, y]\) be the ones corresponding to the maximal chain \(r \cup c\) in \([0, y]\). A C-labeling is a CR-labeling if in every rooted interval \([x, y]_r\) there is a unique increasing maximal chain.

It was shown by Björner and Wachs [6] that a CR-labeling on a poset \(P\) has the same implications with respect to Möbius numbers as described in Theorem 3.2 in the case of an ER-labeling. Hence we can describe the Whitney numbers of a poset with a CR-labeling by the enumeration of saturated chains in the same way than Proposition 3.5. We have that \(|w_k(P)|\) is the number of as ascent-free saturated chains starting at \(\hat{0}\) of length \(k\) and \(|W_k(P)|\) is the number of increasing saturated chains starting at \(\hat{0}\) of length \(k\) as before.

**Definition 3.32.** We say a C-labeling has the **rank two switching property** if for every maximal chain of the form

\[
m : (0 = m_0 < m_1 < \cdots < m_k)
\]

with an ascending step \(\lambda(m, m_{i-1} < m_i) < \lambda(m, m_i < m_{i+1})\) at some rank \(i < k\) there is a unique element \(m'_i \neq m_i\) such that the chains \(m\) and

\[
m' : (0 = m_0 < m_1 < \cdots < m_{i-1} < m'_i < m_{i+1} < \cdots < m_k)
\]

have the same word of labels except at rank \(i\) where \(\lambda(m', m_{i-1} < m'_i) = \lambda(m, m_i < m_{i+1})\) and \(\lambda(m', m'_i < m_{i+1}) = \lambda(m, m_{i-1} < m_i)\). Moreover, the rank two switching property requires a consistency condition, that for any other maximal chain \(\hat{m}\) that coincides with \(m\) in the first \(i+2\) elements \((\hat{m}_j = m_j\) whenever \(j \leq i+1\)) the choice of the unique element also coincides, i.e., \(\hat{m}'_i = m'_i\).

**Remark 3.33.** Note that in Definition 3.32 there is an additional consistency condition that is not present in Definition 3.6. This condition guarantees that the restriction of the rank two switching property for intervals of the form \([0, y]\) is well-defined when \(y\) any element of \(P\) which is not necessarily maximal.

In the situation of Definition 3.32 we say that the chain \(m'\), is obtained from \(m\) by a **quadratic exchange at rank** \(i\) and will use the notation \(m' = U_i(m)\). If \(m\) does not have an ascent at rank \(i\) we define \(U_i(m) = m\).

As in the discussion after Definition 3.8, we define graphs \(G_{[x, y]_r}\) given by quadratic exchanges but this time the elements are maximal chains in a rooted interval \([x, y]_r\). To check confluency in \(G_{[x, y]_r}\) we invite the reader to verify that by the consistency condition...
of the rank two switching property for C-labelings in Definition 3.32, if a maximal chain \( c \) has ascents at ranks \( i \) and \( j \) with \( |i - j| > 1 \) it is necessarily true that \( U_i U_j(c) = U_j U_i(c) \).

We say that \( c : (\hat{0} = x_0 \ll x_1 \ll \cdots \ll x_{\ell - 1} \ll x_\ell = y) \) has a double-ascent or a critical condition at rank \( i \) if

\[
\lambda(c, x_{i-1} \ll x_i) < \lambda(c, x_i \ll x_{i+1}) < \lambda(c, x_{i+1} \ll x_{i+2}).
\]

For every saturated chain \( c \) from \( \hat{0} \) with a critical condition at rank \( i \) we say that \( \lambda \) satisfies the braid relation if we have \( U_i U_{i+1} U_i(c) = U_{i+1} U_i U_{i+1}(c) \).

When we have a CR-labeling \( \lambda \) satisfying the rank two switching property and the braid relation, we ensure local confluency in \( G_{[x,y]} \), and hence, by Lemma 3.11, the conclusion of Corollary 3.13 holds. Therefore each connected component of \( G_{[x,y]} \) has a unique sink, i.e., a unique ascent-free saturated chain. We use the exact same definitions of chain poset \( C(P) \) and quotient poset \( Q_\lambda(P) \) given in Definition 3.16; and also give the same definition of the edge labeling \( \lambda^* \) on \( Q_\lambda(P) \) of Equation 3.4, that is,

\[
\lambda^*(X < Y) = S(Y) \setminus S(X).
\]

Note that \( \lambda^* \) is actually an E-labeling on \( Q_\lambda(P) \), i.e. does not depend on maximal chains, even though the labeling \( \lambda \) of \( P \) is a C-labeling. We would like to conclude that \( \lambda^* \) is also an ER*-labeling in this case. To do this, we show that the lemmas and propositions for E-labelings that appeared in the previous subsection still hold in the C-labeling scenario.

It is straightforward to verify that Lemma 3.18 is still valid in our new setting, but we need a C-labeling analogue (Lemma 3.34 below) of Lemma 3.19 to be able to produce an analogue of Proposition 3.26.

**Lemma 3.34.** Let \( D : (X_1 \ll X_2 \ll \cdots \ll X_k) \) be a saturated chain in \( Q_\lambda(P) \) and let \( d : (e(X_1) \ll e(X_2) \ll \cdots \ll e(X_k)) \) be the corresponding saturated chain in \( P \). Let \( c \in X_1 \) and let \( \lambda_c \) be the induced labeling coming from \( \lambda \) on the rooted interval \([e(X_1), e(X_k)] \). Then the words of labels of \( D \) and \( d \) under their respective labelings are equal, i.e., \( \lambda^*(D) = \lambda_c(d) \).

**Proof.** By Lemma 3.18 we have that for all \( i \leq k \), \( c \cup \{e(X_1), e(X_2), \cdots, e(X_i)\} \in X_i \). Then by Equation (3.4), we have that

\[
\lambda^*(X_{i-1} \ll X_i) = S(X_i) \setminus S(X_{i-1})
\]

\[
= S(c \cup \{e(X_1), e(X_2), \cdots, e(X_i)\}) \setminus S(c \cup \{e(X_1), e(X_2), \cdots, e(X_{i-1})\})
\]

\[
= \lambda(c \cup d, e(X_{i-1}) \ll e(X_i)). \quad \square
\]

**Definition 3.35.** Let \( \lambda \) be a CR-labeling on \( P \) with the rank two switching property. We say that \( \lambda \) is cancellative if for every \( z < x < y \) in \( P \), \( r \in M_{[0,z]} \), \( c \in M_{[z,x]} \) and \( c_1, c_2 \in M_{[x,y]} \) we have that

\[
c \cup c_1 \sim_{\lambda_r} c \cup c_2 \text{ implies } c_1 \sim_{\lambda_{c_1, c_2}} c_2.
\]

The reader can verify that the proof of Proposition 3.21 is still valid when \( \lambda \) is a CR-labeling with the corresponding properties, replacing whenever necessary intervals in \( P \) by rooted intervals to take into account the labeling. We then have that a CR-labeling \( \lambda \) of
$P$ with the rank two switching property, the braid relation and the cancellative property induces the ER$^*$ labeling $\lambda^*$ of $Q_\lambda(P)$ given by Equation (3.4).

**Definition 3.36.** A *generalized CW-labeling* $\lambda : \mathcal{ME}(P) \to \Lambda$ is a CR-labeling that satisfies

- the rank two switching property,
- the braid relation, and
- the cancellative property.

We say $\lambda$ is a *CW-labeling* if it satisfies

- the rank two switching property, and
- in each rooted interval each ascent-free maximal chain has a unique word of labels.

As with EW-labelings, it turns out in this scenario that every CW-labeling is a generalized CW-labeling but we do not know if the converse is true.

We are now ready to state the main theorem for (generalized) CW-labelings that follows from Definition 1.3 together with the C-labeling analogues of Propositions 3.5, 3.26, and 3.21.

**Theorem 3.37.** Let $P$ be a graded poset with a (generalized) CW-labeling $\lambda$. Then $Q_\lambda(P)$ is a Whitney dual of $P$.

### 4. The Whitney dual $Q_\lambda(P)$

In this section we first give a formula for the Möbius function of $Q_\lambda(P)$ given that $\lambda$ is a generalized CW-labeling. In the second part of the section we provide a somewhat simpler description of $Q_\lambda(P)$ given that $\lambda$ is a CW-labeling. This characterization only applies to CW-labelings in the strict sense of the definition, so our description does not include all generalized CW-labelings.

#### 4.1. The Möbius function of $Q_\lambda(P)$

We can characterize the Möbius numbers of $Q_\lambda(P)$ using the fact that $\lambda^*$ of Equation 3.4 is an ER$^*$-labeling. An interesting fact is that $Q_\lambda(P)$ belongs to the famous family of posets whose Möbius numbers are 0 or $\pm 1$. Hersh and Mészáros in [16] have defined a family of edge labelings, that they coined SB-labelings, and that allowed them to conclude that a lattice with such a labeling has Möbius numbers 0 or $\pm 1$. Their result partially answers a question posed by Björner and Greene on why posets with these Möbius values are plentiful in combinatorics. It is still an open problem to find a characterization of when the posets $Q_\lambda(P)$ are lattices. The family of posets $Q_\lambda(P)$ provide a plethora of examples of posets whose Möbius numbers are 0 or $\pm 1$.

Fix $x \in P$ and let $X^1, X^2, \ldots, X^n$ be the different elements of $Q_\lambda(P)$ such that for all $i$, $e(X^i) = x$. Since there is exactly one increasing maximal chain in $[0, x]$, Definition 3.16 implies that there is exactly one $X^i$ that contains this maximal increasing chain and all other $X^j$ for $j \neq i$ do not contain any increasing maximal chain. As we see in the next proposition it is exactly this class which has a nonzero Möbius value $\mu([[0], X])$ in $Q_\lambda(P)$. 

Proposition 4.1. Let $\lambda$ be a generalized CW-labeling of $P$; $X, Y \in Q_\lambda(P)$ and $c \in X$. Then in $Q_\lambda(P)$ we have

$$
\mu([X, Y]) = \begin{cases} 
(-1)^{\rho(Y) - \rho(X)} & \text{if } Y \text{ contains } c \cup d, \text{ where } d \text{ is the unique increasing maximal chain in } M_{[e(X), e(Y)]c}, \\
0 & \text{otherwise}.
\end{cases}
$$

Proof. Since $\lambda$ is a CR-labeling, there exists a unique maximal chain $d$ in $[e(X), e(Y)]_c$ which is increasing. If $Y \in Q_\lambda(P)$ is the class that contains $c \cup d$ then, by the definition of $Q_\lambda(P)$ and Equation (3.4), there is an increasing saturated chain which terminates at $Y$ say $X = X_1 \prec X_2 \prec \cdots \prec X_k = Y$. Now suppose that there was another increasing chain $X = X'_1 \prec X'_2 \prec \cdots \prec X'_k$ in $Q_\lambda(P)$ with $e(X'_k) = e(Y)$. Then by Lemma 3.34 there would be a corresponding maximal chain in $[e(X), e(Y)]_c$ which is increasing. Since there is a unique increasing maximal chain in $[e(X), e(Y)]_c$, we know that $e(X_i) = e(X'_i)$ for all $i$. But (the CW-labeling version of) Lemma 3.18 part (c) implies that $X_i = X'_i$ which is impossible. Since (the CW-labeling version of) Proposition 3.21 asserts that $\lambda'$ given in Equation (3.4) is an ER*-labeling of $Q_\lambda(P)$ Theorem 3.2 gives the desired result. \qed

Remark 4.2. Note that in the EW-version of Proposition 4.1 the chain $c \in X$ is irrelevant.

4.2. Another description of $Q_\lambda(P)$. Let $\Lambda$ be a poset and let $w$ be a word with letters in the alphabet $\Lambda$. Assume that whenever $w_i < w_{i+1}$ we are allowed to do exchanges on $w$ of the form

$$
w_1w_2 \cdots w_{i-1}w_iw_{i+1}w_{i+2} \cdots w_n \xrightarrow{i} w_1w_2 \cdots w_{i-1}w_{i+1}w_iw_{i+2} \cdots w_n.
$$

It is not hard to check that this type of exchange produces a locally confluent relation and after using Newman’s Lemma 3.11 we can conclude that there is a unique ascent-free word $w'$ that is related to $w$ in this manner. We define $\text{sort}(w) := w'$ to be this unique ascent-free word. For example, if $\Lambda = \mathbb{Z}$, $w = 85324$ then $\text{sort}(w) = 85432$.

Definition 4.3. Let $P$ be a poset with a CW-labeling $\lambda$. Let $R_\lambda(P)$ be the poset whose elements are pairs $(x, w)$ where $x \in P$ and $w$ is the word of labels of an ascent-free saturated chain $c \in M_{[0, x]}$ (note that by the definition of CW-labeling $w$ uniquely determines $c$); and such that $(x, w) \leq (y, u)$ whenever $x \leq y$ and $u = \text{sort}(w\lambda(c, x \leq y))$ (uv here means the concatenation of the words $u$ and $v$).

Example 4.4. If we consider the EW-labeling of $\Pi_3$ given in Example 3.3, we obtain the poset $R_\lambda(\Pi_3)$ depicted in Figure 5. Comparing $R_\lambda(\Pi_3)$ and $Q_\lambda(\Pi_3)$ given in Figure 4c, we can observe directly that these two posets are isomorphic.

Theorem 4.5. If $\lambda$ is a CW-labeling of $P$, then $R_\lambda(P) \cong Q_\lambda(P)$.

Proof. Let $\varphi : R_\lambda(P) \rightarrow Q_\lambda(P)$ be given by $\varphi((x, w)) = [c]$ where $c \in M_{[0, x]}$ is the unique ascent-free chain determined by $w$. Using the fact that when $\lambda$ is a CW-labeling each equivalence class in $Q_\lambda(P)$ contains a unique ascent-free saturated chain, one can see that $\varphi$ is a well-defined bijection. We want to see that $\varphi$ and $\varphi^{-1}$ are poset maps.
We now show that \( \varphi \) is an order-preserving map. Since we are working with finite posets it is enough to show that \( \varphi \) preserves cover relations. Suppose that \((x, w) \prec (y, u)\) in \( R_{\lambda}(P) \). Let \( c \) and \( d \) be the unique saturated chains from \( \hat{0} \) determined by \( w \) and \( u \), respectively. Then \( \varphi((x, w)) = [c] \) and \( \varphi((y, u)) = [d] \). Let \( d' = c \cup \{y\} \). Since \( d' \) is a saturated chain from \( \hat{0} \) to \( y \) with word of labels \( w\lambda(c \cup \{y\}, x \prec y) \), we can use the rank two switching property to see that \( d' \) is equivalent to an ascent-free chain with labels \( \text{sort}(w\lambda(c, x \prec y)) = u \). Since \( u \) determines a unique saturated chain we have that \( d' \sim d \). Moreover, \( c \prec d' \) in \( C(P) \), so we have \( \varphi((x, w)) = [c] \prec [d] = \varphi((y, u)) \).

Now we show \( \varphi^{-1} \) is also an order-preserving map. To see why, suppose that \( X \prec Y \) in \( Q_{\lambda}(P) \); and let \( x = e(X) \), \( y = e(Y) \), \( \varphi^{-1}(X) = (x, w) \) and \( \varphi^{-1}(Y) = (y, u) \). By the definition of \( Q_{\lambda}(P) \) and Proposition 3.15, there are chains \( c \in X \) and \( d' \in Y \) such that \( d' = c \cup \{y\} \) and hence \( x \prec y \). We may assume, without loss of generality, that \( c \) is the ascent-free chain in \( X \) with word \( w \) (otherwise, apply quadratic exchanges until you obtain an ascent-free chain in \( X \)). Note that \( w\lambda(d', x \prec y) \) is the word of labels of \( d' \). Let \( d \) be the ascent-free chain in \( Y \) with word of labels \( u \). Since \( d \sim d' \) we also have that the word of labels of \( d \) is \( \text{sort}(w\lambda(c, x \prec y)) \) and so \( u = \text{sort}(w\lambda(c, x \prec y)) \). We obtain then that \( \varphi^{-1}(X) = (x, w) \preceq (y, u) = \varphi^{-1}(Y) \) as desired. \( \square \)

The new characterization of \( Q_{\lambda}(P) \) that was given in Theorem 4.5 can be helpful providing combinatorial descriptions of Whitney duals (see Section 5.2).

5. Examples of posets with Whitney labelings

In this section we give several examples of posets with Whitney labelings. By Theorem 3.37, this implies that these posets also have Whitney duals.

5.1. Geometric lattices. In [31] Stanley introduced an edge labeling for geometric lattices that is an ER-labeling (In fact, as shown by Björner in [3] it is an EL-labeling). We give the definition below and then show it is also an EW-labeling.
Definition 5.1. Let $L$ be a geometric lattice with set of atoms $A(L)$. Fix a total order on $A(L)$. Now define $\lambda: \mathcal{E}(L) \to A(L)$ by setting $\lambda(x < y) = a$ where $a$ is the smallest atom such that $x \vee a = y$. We call $\lambda$ a minimum labeling of $L$. Note that this labeling can be different for different total orders on $A(L)$.

Example 5.2. The labeling $\lambda$ of $\Pi_n$ in Example 3.3 is a minimum labeling. Here we associate each atom of $\Pi_n$ with the ordered pair $(i, j)$ where $ij$ is the unique nontrivial block in the atom and such that $i < j$. We then order the atoms lexicographically.

Proposition 5.3. For any geometric lattice $L$ a minimum labeling of $L$ is an EW-labeling.

Proof. It was shown in [31] that a minimum labeling is an ER-labeling. Also, for any interval $[x, y]$ the labels along any maximal chain uniquely determine the chain since one can read off the elements of the chain by taking joins of $x$ with the labels along the chain. Thus it suffices to show that a minimum labeling has the rank two switching property.

Let $\lambda$ be a minimum labeling of $L$, let $[x, y]$ be an interval of rank two and suppose that $ij$ is the word of labels of the increasing chain, $x < x \vee i < x \vee i \vee j = y$. Since $L$ is geometric and $j$ is an atom not underneath $x$, $x < x \vee j < y$. Observe that $\lambda(x < x \vee j) = j$, since if this was not the case, this would imply $\lambda(x \vee i < y) < j$ which is a contradiction. Moreover, $i$ is not below $x \vee j$ and $i$ is below $y$. Since there is a unique increasing chain in $[x, y]$, $i$ is the smallest atom that appears as a label in $[x, y]$. It follows that $\lambda(x \vee j < y) = i$. We conclude that the chain $x < x \vee j < y$ has the word of labels $ji$. Since joins are unique, there is only one chain in $[x, y]$ with word of labels $ji$ and thus any minimum labeling satisfies the rank two switching property. \qed

We have the following theorem as a corollary.

Theorem 5.4 ([10]). Every geometric lattice has a Whitney dual.

Remark 5.5. For the poset $\Pi_n$ the authors proved in [10] that $\mathcal{ISF}_n$ is the Whitney dual corresponding to the minimal labeling of Example 5.2.

5.2. The noncrossing partition lattice. We say a partition $\pi = B_1/B_2/\cdots/B_k$ of $[n]$ is noncrossing if there are no $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ for some $i \neq j$. For example, $124/35/67$ is not a noncrossing partition since $2 < 3 < 4 < 5$ and $\{2, 4\}$ and $\{3, 5\}$ are in two different blocks, but $127/36/45$ is noncrossing. The noncrossing partition lattice, denoted $\mathcal{NC}_n$, is the set of noncrossing partitions of $[n]$ ordered by refinement.

As the name suggest, $\mathcal{NC}_n$ is a lattice and has many nice combinatorial properties (see Simions survey article [27] for more information). $\mathcal{NC}_n$ is an induced subposet of $\Pi_n$, but it is not a sublattice of $\Pi_n$. Figure 6 depicts $\mathcal{NC}_4$.

In [33], Stanley found a beautiful connection between $\mathcal{NC}_n$ and a set of combinatorial objects known as parking functions. A parking function of $n$ is a sequence of $n$ positive integers $(p_1, p_2, \ldots, p_n)$ with the property that when it is rearranged in a weakly increasing order $p_{i_1} < p_{i_2} < \cdots < p_{i_n}$, then $p_{i_j} \leq j$ for all $j$. An edge labeling of $\mathcal{NC}_n$ is given in [33] with the property that the words of labels along the maximal chains are exactly the parking functions of $n - 1$. We will show that this edge labeling is in fact an EW-labeling,
establishing that $\mathcal{NC}_n$ has a Whitney dual. To describe the labeling of Stanley, first note that just as in the partition lattice $\Pi_n$, the cover relation is given by merging two blocks together. Suppose that $\sigma$ is obtained from $\pi$ by merging $B_i$ and $B_j$, where $\min B_i < \min B_j$, then define

$$\lambda_{\mathcal{NC}}(\pi \lessdot \sigma) = \max\{a \in B_i \mid a < \min B_j\}. \quad (5.1)$$

See Figure 6 to see this labeling for $\mathcal{NC}_4$.

The work in [33] uses a slightly different definition of ER-labeling. There, an ER-labeling is defined as a labeling such that each interval has a unique weakly increasing maximal chain. It is not hard to see that the labeling in (5.1) does not fit this definition. However, we have chosen to define an ER-labeling as a labeling where each maximal interval has a unique strictly increasing maximal chain. Under this definition, one can check that $\lambda_{\mathcal{NC}}$ is indeed an ER-labeling.

In [33] this ER-labeling is used to prove that there is a local $\mathfrak{S}_n$-action on the maximal chains of $\mathcal{NC}_{n+1}$. This action is local in the sense that if a transposition of the form $(i, i+1)$ acts on a maximal chain it only changes the chain in at most the element at rank $i$. Suppose that $[x, y]$ is an interval of rank two in $\mathcal{NC}_n$ and such that $\rho(x) = i - 1$ and $\rho(y) = i + 1$. Then for a maximal chain $c$ in $[x, y]$,

$$(i, i+1)c = \begin{cases} c' & \text{if } c \text{ has a strict ascent or strict descent in } [x, y], \\ c & \text{otherwise,} \end{cases}$$

where $c'$ is the unique maximal chain $[x, y]$ with the same label set as $c$ which reverses the labels in $c$. In other words, the action switches strict ascents and strict descents and leaves equal labels fixed. Note that this local action of $\mathfrak{S}_{n-1}$ coincides with the action on the set of parking functions where the transposition $(i, i+1)$ permutes the letters $i$ and $i+1$. 

Figure 6. Edge labeling of $\mathcal{NC}_4$. For clarity, the edge labels are represented by line patterns. The (red) solid lines represent the label 1, the (blue) dashed lines represent 2, and the (black) dotted lines represent 3.
of a parking function. Under this action, there is exactly one weakly decreasing parking function in each orbit. The fact that this action exists immediately implies that Stanley’s labeling of $\mathcal{NC}_n$ has the rank two switching property. This, together with the fact that the maximal chains are in one-to-one correspondence with parking functions (which are all different) implies that the labeling $\lambda_{\mathcal{NC}}$ in (5.1) satisfies the conditions of Definition 3.24, and so is an EW-labeling. Hence by Theorem 1.6 we conclude that $\mathcal{NC}_n$ has a Whitney dual.

**Theorem 5.6.** The labeling $\lambda_{\mathcal{NC}}$ is an EW-labeling of $\mathcal{NC}_n$. Hence $Q_{\lambda}(\mathcal{NC}_n)$ is a Whitney dual of $\mathcal{NC}_n$.

We will now use Theorem 4.5 to provide a more familiar combinatorial description of the Whitney dual $Q_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n)$ of $\mathcal{NC}_n$.

Recall that a Dyck path of order $n$ is a lattice path from $(0,0)$ to $(n,n)$ that never goes below the line $y = x$ and only takes steps in the directions of the vectors $(1,0)$ (East) and $(0,1)$ (North). We will consider Dyck paths $D$ that come with a special labeling. Given an increasing sequence $b_1 < b_2 < \cdots < b_{n+1}$ of positive integers, we label the point $(i-1,0)$ of $D$ by $b_i$ (see Figure 7a). In Figure 7b we illustrate two labeled Dyck paths.

We now define a process of “merging” two labeled Dyck paths $D_1$ and $D_2$ to obtain a new labeled Dyck path $D$. Suppose that $D_1$ and $D_2$ have disjoint and noncrossing label sets $B = \{b_1, b_2, \ldots, b_j\}$ and $C = \{c_1, c_2, \ldots, c_k\}$, where both sets are written in increasing order and $b_1 < c_1$. Since the sets are noncrossing then there exists an $i$ such that $b_i < c_1 < c_2 < \cdots < c_k < b_{i+1}$ (where we use the convention $b_{j+1} = \infty$). Then, the new lattice path $D$ will be a path from $(0,0)$ to $(j+k,j+k)$ whose labels along the bottom row are $b_1, b_2, \ldots, b_i, c_1, c_2, \ldots, c_k, b_{i+1}, \ldots, b_j$. From left to right until we reach the vertical line labeled $b_i$, $D$ looks exactly the same as $D_1$. In the line labeled $b_i$ in $D$ we add all the north steps that $D_1$ had originally at $b_i$ plus one additional north step followed by an additional east step from the line labeled $b_i$ to the line labeled $c_1$. Then we glue $D_2$ where we left off in the line labeled $c_1$. After we finish gluing $D_2$, we glue the remaining part of $D_1$ that goes from the line labeled $b_i$ to the line labeled $b_j$. As an example, suppose we wish to obtain a labeled Dyck path $D$ by merging the two labeled Dyck paths $D_1$ and $D_2$ in Figure 7b on label sets $\{1,3,6,7\}$ and $\{2,3,4\}$ respectively. We start by creating a grid from $(0,0)$ to $(6,6)$ and label the bottom row with the (ordered) union of the two labeled sets (see Figure 8a). Since $1$ is the largest element in $\{1,3,5,6,7\}$ smaller than all
the elements of \( \{2, 3, 4\} \), we add in \( D \) a new north step at the line labeled 1 and add a new east step afterwards between lines labeled 1 and 2. Since at the line labeled 1, \( D_1 \) had two north steps, \( D \) will have now 3 north steps (see Figure 8b). Next, glue \( D_2 \) where we left off (see Figure 8c) and then the remaining part of \( D_1 \) to obtain \( D \) (see Figure 8d).

In order to verify that the resulting labeled lattice path is also a labeled Dyck path (that is, it has the same number of north and east steps and is always above the diagonal), we rely on an equivalent definition of a Dyck path. A ballot sequence of length \( 2n \) is a string \( s_1 s_2 \cdots s_{2n} \) with the same number of 1’s and 0’s and such that for every \( i \in [2n] \) the subword \( s_1 s_2 \cdots s_i \) has at least as many 1’s as 0’s. It is well-known that a lattice path that takes only north and east steps is a Dyck path if and only if the sequence obtained associating to each north step a 1 and to each east step a 0 is a ballot sequence. Relying on this equivalent definition, we see that in the resulting path \( D \) the number of north steps and east steps is equal and the construction never breaks the property that every preamble in \( D \) contains at least as many north steps as east steps. Hence \( D \) is a well-defined labeled Dyck path.

Let \( \mathcal{NC}\text{Dyck}_n \) be the set whose objects are collections of labeled Dyck paths such that their underlying sets of labels form a noncrossing partition of \([n]\). We provide \( \mathcal{NC}\text{Dyck}_n \) with a partial order by defining for \( F, F' \in \mathcal{NC}\text{Dyck}_n \) the cover relation \( F \triangleleft F' \) whenever \( F' \) can be obtained from \( F \) by merging exactly two of the labeled Dyck paths in \( F \). Note here that each labeled Dyck path can be represented by its set of labels together with an exponent for each label. The exponent of an element \( i \) being the number of north steps in the vertical line labeled \( i \) in its Dyck path. This notation extends to the elements in \( \mathcal{NC}\text{Dyck}_n \). For example, we can denote the collection of Dyck paths in Figure 7b by \( 1^2 5^0 6^1 7^0 / 2^1 3^1 4^0 \). In Figure 9 we illustrate \( \mathcal{NC}\text{Dyck}_4 \).

**Theorem 5.7.** For all \( n \geq 1 \), \( Q_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n) \cong \mathcal{NC}\text{Dyck}_n \).

**Proof.** Theorem 4.5 characterizes the poset \( Q_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n) \) as being isomorphic to the poset \( R_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n) \) whose elements are pairs \((\pi, w)\) where \( \pi \in \mathcal{NC}_n \) and \( w \) is the word of labels of an ascent-free chain in \([\hat{0}, \pi]\). We show that \( R_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n) \cong \mathcal{NC}\text{Dyck}_n \).
The bijection assigns to a parking function with $k_i$ occurrences of the label $i$ the Dyck path with $k_i$ north steps on the line $x = i - 1$. In our notation, the pairs $(4, w) \in R_{\lambda_{NC}}(\mathcal{NC}_4)$ can be represented as $1^{2}2^{3}4^{0}, 1^{2}2^{1}3^{0}4^{0}, 1^{2}2^{2}3^{0}4^{0}, 1^{2}2^{0}3^{1}4^{0}$ and $1^{2}2^{1}3^{1}4^{0}$. Now, it is not hard to see that any interval of the form $[0, B_1/B_2/\cdots/B_k]$ is isomorphic to the product of smaller noncrossing partition lattices $\mathcal{NC}_{B_1} \times \mathcal{NC}_{B_2} \times \cdots \times \mathcal{NC}_{B_k}$, where $\mathcal{NC}_{B_j}$ is the lattice of noncrossing partitions of $B_j \subset [n]$. Moreover, the labels in any cover relation in $[0, B_1/B_2/\cdots/B_k]$ depend only on the two blocks being merged. So any ascent-free maximal chain can be represented as a noncrossing partition where each of the blocks $B_j$ have been decorated with exponents representing an ascent-free maximal chain in $\mathcal{NC}_{B_j}$.

Note that words of labels on maximal chains of $\mathcal{NC}_{B_j}$ are “parking functions” on $B_j$, that is, if $B_j = \{b_1 < b_2 < \cdots < b_l\}$ then in the word of labels of a maximal chain the number of occurrence of the letter $b_i$ is greater or equal to $i$ (an equivalent definition of a parking function). For example, the chain in $[0, 1457/23/6/89]$ with word of labels $(8, 4, 2, 1)$ is represented by $1^{1}4^{2}5^{0}7^{0}/2^{1}3^{0}/6^{0}/8^{1}9^{0}$. Since $\lambda_{NC}$ is an EW-labeling, the cover relation $(\pi, w) < (\pi', w')$ in $R_{\lambda_{NC}}(\mathcal{NC}_n)$ is completely determined by the cover relation $\pi < \pi'$. Hence $(\pi', w')$ is obtained from $(\pi, w)$ by merging two blocks $B_i$ and $B_j$ of $\pi$ such that $\min B_i < \min B_j$ and $w' = \text{sort}(w)$ where $p = \lambda_{NC}(\pi < \pi') = \max\{a \in B_i \mid a < \min B_j\}$. The reader can note that this is equivalent to the definition of merging labeled Dyck paths. Indeed, in our notation this amounts to merging the weighted blocks on the sets $B_i$ and $B_j$ and increasing the exponent of $p \in B_i$ by one. For example, if in $1^{1}4^{2}5^{0}7^{0}/2^{1}3^{0}/6^{0}/8^{1}9^{0}$ we merge the blocks with sets $\{1, 4, 5, 7\}$ and $\{2, 3\}$ we get $1^{2}2^{1}3^{0}4^{2}5^{0}7^{0}/6^{0}/8^{1}9^{0}$. If we further merge the blocks with sets $\{1, 2, 3, 4, 5, 7\}$ and $\{6\}$ we get $1^{2}2^{1}3^{0}4^{2}5^{1}6^{0}7^{0}/8^{1}9^{0}$. We then have that $R_{\lambda_{NC}}(\mathcal{NC}_n)$ is indeed isomorphic to the poset $\mathcal{NCDyck}_n$. 

It is interesting to note the well-known fact that the M"{o}bius function value of $\mathcal{NC}_n$ is (up to a sign) the Catalan number $C_{n-1}$. This information is recovered here since $\mathcal{NCDyck}_n$.
is a Whitney dual of $\mathcal{NC}_n$ and its maximal elements are Dyck paths of order $n - 1$ which are Catalan objects.

5.3. The weighted partition poset and the poset of rooted spanning forests. Here we discuss the original example that motivated our work. This example was noticed by González D’León and Wachs in [11] while studying a poset of partitions where each block has a weight that is a natural number. This poset, known as the poset of weighted partitions and denoted $\Pi_n^w$, was originally introduced by Dotsenko and Khoroshkin in [8] and is related to the study of the operad of Lie algebras with two compatible brackets. The authors of [11] realized that the Whitney numbers of the first and second kind were switched with respect to those of the poset of rooted spanning forests $\mathcal{SF}_n$ on $[n]$ studied by Reiner [22] and Sagan [26]. Since the two pairs of Whitney numbers were already computed, by direct comparison, we can conclude that $\Pi_n^w$ and $\mathcal{SF}_n$ are Whitney duals. In this section we use the theory developed in Section 3 to give a different proof of this fact.

A weighted partition $B_1^{v_1}/B_2^{v_2}/\cdots/B_t^{v_t}$ of $[n]$ is a partition $B_1/B_2/\cdots/B_t$ of $[n]$ such that each block $B_i$ is assigned a weight $v_i \in \{0, 1, 2, \ldots, |B_i| - 1\}$. The poset of weighted partitions $\Pi_n^w$ is the set of weighted partitions of $[n]$ with order relation given by

$$A_1^{w_1}/A_2^{w_2}/\cdots/A_s^{w_s} \leq B_1^{v_1}/B_2^{v_2}/\cdots/B_t^{v_t}$$

if the following conditions hold:

- $A_1/A_2/\cdots/A_s \leq B_1/B_2/\cdots/B_t$ in $\Pi_n$
- if $B_k = A_i \cup A_{i2} \cup \cdots \cup A_{it}$ then $v_k - (w_i + w_{i2} + \cdots + w_{it}) \in \{0, 1, \ldots, l - 1\}$.

Equivalently, we can define the covering relation by

$$A_1^{w_1}/A_2^{w_2}/\cdots/A_s^{w_s} \prec B_1^{v_1}/B_2^{v_2}/\cdots/B_{s-1}^{v_{s-1}}$$

if the following conditions hold:

- $A_1/A_2/\cdots/A_s \prec B_1/B_2/\cdots/B_{s-1}$ in $\Pi_n$
- if $B_k = A_i \cup A_j$, where $i \neq j$, then $v_k - (w_i + w_j) \in \{0, 1\}$
- if $B_k = A_i$ then $v_k = w_i$.

![Figure 10. Weighted partition poset for $n = 3$](image-url)
The poset $\Pi_n^w$ has a minimum element
$$\hat{0} := 1^0/2^0/\ldots/n^0$$
and $n$ maximal elements
$$[n]^0, [n]^1, \ldots, [n]^{n-1}.$$ 
Note that for all $i = 0, 1, \ldots, n-1$, the maximal intervals $[\hat{0}, [n]^i]$ and $[\hat{0}, [n]^{n-1-i}]$ are isomorphic to each other, and the two maximal intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are isomorphic to $\Pi_n$. See Figure 10 for the example of $\Pi_3^w$.

A rooted spanning forest on $[n]$ is a spanning forest of the complete graph on vertex set $[n]$ such that in every connected component there is a unique specially marked vertex, called the root. Let $\mathcal{SF}_n$ be the set of rooted spanning forests on $[n]$. For $F \in \mathcal{SF}_n$, let $E(F)$ denote the edge set of $F$ and $R(F)$ be the set of roots in $F$. The set $\mathcal{SF}_n$ has the structure of a poset with order relation given by $F_1 \leq F_2$ whenever
$$E(F_1) \subseteq E(F_2) \text{ and } R(F_2) \subseteq R(F_1).$$
Equivalently, the cover relation $F_1 \lessdot F_2$ occurs if $F_2$ is obtained from $F_1$ by adding a new edge $\{x, y\} \in E(F_2)$ such that $x, y \in R(F_1)$ and by choosing either $x$ or $y$ as the new root of its component. Note that $R(F_2)$ is either $R(F_1) \setminus \{x\}$ or $R(F_1) \setminus \{y\}$. See Figure 11 for the example of $\mathcal{SF}_3$. For more on this poset see [22, 26].

Let $F \in \mathcal{SF}_n$ and $x, y$ be vertices in $F$, we say that $x$ is the parent of $y$ (and $y$ the child of $x$) if $\{x, y\}$ is an edge of $F$ that belongs to the unique path from $y$ to the root of their component in $F$. We denote $x = p(y)$ whenever $x$ is the parent of $y$. We say that $x$ is an ancestor of $y$ (y is a descendant of $x$) if $x$ is in the unique path from $y$ to the root of their connected component in $F$. To every forest $F \in \mathcal{SF}_n$, we can associate an element of $\Pi_n^w$,

$$\pi(F) := \{V(T)^{w_T} \mid T \text{ a tree in } F\},$$

where $w_T$ is the number of descents in $T$, i.e., edges $\{x, p(x)\}$ in $T$ where $p(x) > x$. As an example, if $F$ is the forest formed by the trees $T_1$ to $T_6$ of Figure 13, then $w_{T_1} = 0$, $w_{T_2} = 0$, $w_{T_3} = 0$, $w_{T_4} = 3$, $w_{T_5} = 1$, $w_{T_6} = 1$ and thus $\pi(F) = 168911^{14^3}/23^0/4^0/5^0/1012^1/713^1$. 

\[\text{Figure 11. } \mathcal{SF}_3 \text{ (here the roots are marked with red squares)}\]
(where we have underlined the two digit numbers). The following propositions were proved in [11] and [26].

**Proposition 5.8** ([11]). For all \( x \in \Pi_n^w \),
\[
\mu([\hat{0}, x]) = (-1)^{\rho(x)}|\{ F \in SF_n \mid \pi(F) = x \}|.
\]
In particular,
\[
\mu([\hat{0}, [n]^i]) = (-1)^{n-i}|\mathcal{T}_{n,i}|,
\]
where \( \mathcal{T}_{n,i} \) is the set of rooted trees with \( i \) descents.

**Proposition 5.9** ([26]). For all \( F \in SF_n \),
\[
\mu([\hat{0}, F]) = \begin{cases} 
(-1)^{\rho(F)} & \text{if } F \text{ is a forest in which every nonroot vertex is a leaf} \\
0 & \text{otherwise}
\end{cases}
\]

Note that the map \( \pi : SF_n \to \Pi_n^w \) defined in Equation (5.2) is a bijection when we restrict the domain to the set of forests in which every one of its nonroot vertices are leaves. Furthermore, note that a forest \( F \in SF_n \) and its associated weighted partition \( \pi(F) \in \Pi_n^w \) have the same rank in their respective posets. We obtain the following theorem as a corollary of the previous two propositions.

**Theorem 5.10** (González D’León - Wachs, personal communication). We have that
\[
w_k(\Pi_n^w) = |\{ F \in SF_n \mid \rho(F) = n - k \}| = W_k(SF_n)
\]
\[
w_k(SF) = |\{ x \in \Pi_n^w \mid \rho(x) = n - k \}| = W_k(\Pi_n^w).
\]
Hence, the posets \( \Pi_n^w \) and \( SF_n \) are Whitney duals.

5.3.1. A CW-labeling of \( \Pi_n^w \). In the following we will use the structure of the poset \( SF_n \) and a surjective map from the poset of saturated chains \( C(\Pi_n^w) \) to \( SF_n \) to give a CW-labeling \( \lambda_C \) for \( \Pi_n^w \). Theorem 3.37 will then imply that \( Q_{\lambda_C}(\Pi_n^w) \) is a Whitney dual of \( \Pi_n^w \). We will also show that \( Q_{\lambda_C}(\Pi_n^w) \simeq SF_n \) providing a different proof of Theorem 5.10.

We first discuss some structural properties of \( SF_n \). For a rooted tree \( T \) on vertex set \( V(T) \), let \( r(T) \) denote the root of \( T \); and for any pair of vertices \( v, w \in V(T) \) define the distance \( d(v, w) \) to be the number of edges in the unique path between \( v \) and \( w \). Note that by the definition of the cover relations in \( SF_n \), a saturated chain \( \hat{0} = F_0 < F_1 < \cdots < F_k = F \) in \( SF_n \) can be seen as a step-by-step instruction set on how to build the forest \( F \). Here in the \( i \)-th step, exactly two trees \( T_{i,1} \) and \( T_{i,2} \) of \( F_{i-1} \) are combined by adding the edge \( \{r(T_{i,1}), r(T_{i,2})\} \) to get a new tree \( T'_i \) in \( F_i \) whose root \( r(T'_i) \in \{r(T_{i,1}), r(T_{i,2})\} \). We call this process merging the trees \( T_{i,1} \) and \( T_{i,2} \) by the roots \( r(T_{i,1}) \) and \( r(T_{i,2}) \). At \( F_0 = \hat{0} \) every element of \( [n] \) is a root \( (R(F) = [n]) \), but at each step \( i \) there is exactly one element \( v_i \in \{r(T_{i,1}), r(T_{i,2})\} \) that stops being a root, i.e., \( v_i \in R(F_{i-1}) \) and \( v_i \notin R(F_i) \). This process defines an ordered listing \( v_1, v_2, \ldots, v_k \) of the non-root vertices of \( F \).

We can also consider the converse situation: let \( v_1, v_2, \ldots, v_k \) be an ordered listing of the non-root vertices of \( F \) and let \( \hat{0} = F_0, F_1, \cdots, F_k = F \) be the sequence defined by obtaining \( F_i \) from \( F_{i-1} \) by adding the edge \( \{v_i, p(v_i)\} \) (where \( p(v_i) \) is the parent of \( v_i \) in \( F \)
and letting \( R(F_i) = R(F_{i-1}) \setminus \{v_i\} \). It is clear that the rank \( \rho(F_i) = i \) in \( SF_n \), but it is not clear if the set \( \hat{0} = F_0, F_1, \ldots, F_k = F \) forms a saturated chain in \( SF_n \) since a cover relation in \( SF_n \) happens **exactly** when the two trees in \( F_{i-1} \) are merged using an edge between their roots. The following lemma characterizes which sequences \( v_1, v_2, \ldots, v_k \) give a valid saturated chain \( \hat{0} = F_0 < F_1 < \cdots < F_k = F \) in \( SF_n \).

An ordered listing \( v_1, v_2, \ldots, v_k \) of some subset of \( V(F) \) is said to be a linear extension if whenever \( v_i \) is a descendant of \( v_j \) in \( F \) then we have that \( i < j \).

**Lemma 5.11.** The sequence \( \hat{0} = F_0, F_1, \ldots, F_k = F \) in which \( F_i \) is obtained from \( F_{i-1} \) by adding the edge \( \{v_i, p(v_i)\} \) and setting \( R(F_i) = R(F_{i-1}) \setminus \{v_i\} \) is a saturated chain in \( SF_n \) if and only if the ordered listing \( v_1, v_2, \ldots, v_k \) of the non-root vertices of \( F \) is a linear extension.

**Proof.** First let \( \hat{0} = F_0 < F_1 < \cdots < F_k = F \) be a saturated chain in \( SF_n \) and suppose that the associated ordered listing \( v_1, v_2, \ldots, v_k \) is not a linear extension. Then there are non-root vertices \( v_i \) and \( v_j \) in \( F \) such that \( v_i \) is a descendant of \( v_j \) and \( j < i \). We choose \( v_i \) and \( v_j \) such that \( d(v_i, v_j) \) is minimal and we claim that in this case it must be that \( v_j = p(v_i) \). Otherwise, there is a vertex \( v_l \) that is a descendant of \( v_j \) and an ancestor of \( v_i \). If \( l < j \) then the pair \( (v_i, v_l) \) satisfies the condition above with \( d(v_i, v_l) < d(v_i, v_j) \) and if \( l > j \) then the pair \( (v_i, v_j) \) satisfies the condition above with \( d(v_i, v_j) < d(v_i, v_j) \). Now, if \( v_j = p(v_i) \) and \( j < i \) this implies that in the step \( i \) between \( F_{i-1} \) and \( F_i \) we added the edge \( \{v_i, v_j\} \) but \( v_j \notin R(F_{i-1}) \) since \( v_j \) has been already removed from the set of roots in step \( j \). This implies that \( F_i \) does not cover \( F_{i-1} \), that is a contradiction. We conclude that \( v_1, v_2, \ldots, v_k \) is a linear extension.

On the other hand, let the ordered listing \( v_1, v_2, \ldots, v_k \) be a linear extension of the non-root vertices of \( F \) and let \( \hat{0} = F_0, F_1, \ldots, F_k = F \) in which \( F_i \) is defined as stated in the lemma. Note that \( F_{k-1} \) is the forest obtained from \( F \) by removing the edge \( \{v_k, p(v_k)\} \) and is such that \( R(F_{k-1}) = R(F) \cup \{v_k\} \). We also clearly have that \( v_1, v_2, \ldots, v_{k-1} \) is a linear extension of the non-root vertices of \( F_{k-1} \). Since we know that the trivial sequence \( F_0 = \hat{0} \) is a saturated chain in \( SF_n \) we assume by induction that \( \hat{0} = F_0 < F_1 < \cdots < F_{k-1} \) is also one. Note that \( p(v_k) \neq v_j \) for \( j = 1, \ldots, k \) since the ordered listing is a linear extension and so \( p(v_k) \) cannot appear before \( v_k \). But this implies that both \( v_k \) and \( p(v_k) \) are in \( R(F_{k-1}) \) and that \( F_k \) is obtained from \( F_{k-1} \) by adding the edge \( \{v_k, p(v_k)\} \) such that \( R(F_k) = R(F_{k-1}) \setminus \{v_k\} \). Thus \( F_{k-1} \prec F_k \) and so \( \hat{0} = F_0 < F_1 < \cdots < F_k = F \) is a saturated chain in \( SF_n \).

We define the **cost** of a rooted tree \( T \) as

\[
\gamma(T) = \sum_{v \in V(T)} d(v, r(T)).
\]

See Figure 12 for an example.

Recall that \( C(\Pi_n^w) \) is the poset of saturated chains from \( \hat{0} \) in \( \Pi_n^w \). For a chain

\[
c : (\hat{0} = x_0 < x_1 < \cdots < x_k) \in C(\Pi_n^w)
\]
let \( c_i \) be the subchain of \( c \) consisting of its bottom \( i + 1 \) elements, i.e.,

\[
\begin{align*}
c_i : (x_0 < x_1 < \cdots < x_i).
\end{align*}
\]

We recursively define a map \( \mathcal{F} : C(\Pi_n^w) \to SF_n \) as follows:

1. Let \( \mathcal{F}(0_{\Pi_n^w}) = 0_{SF_n} \), the rooted forest on \([n]\) with no edges. Note that \( \pi(0_{SF_n}) = 0_{\Pi_n^w} \) where the function \( \pi \) is defined in Equation (5.2).

2. Let \( c : (\hat{0} = x_0 < x_1 < \cdots < x_k) \in C(\Pi_n^w) \). The cover relation

\[
\begin{align*}
x_{k-1} := A_1^{w_1}/A_2^{w_2}/ \cdots /A_s^{w_s} \lesssim B_1^{w_1}/B_2^{w_2}/ \cdots /B_{s+1}^{w_{s+1}} := x_k
\end{align*}
\]

in \( \Pi_n^w \) is such that exactly two weighted blocks \( A_i^{w_i} \) and \( A_j^{w_j} \) of \( x_{k-1} \) are merged into a weighted block \( B_m^{w_m} \) of \( x_k \) where \( B_m = A_i \cup A_j \) and \( v_m = (w_i + w_j) \in \{0, 1\} \). We assume that we have recursively defined \( \mathcal{F}(c_{k-1}) \) as a rooted spanning forest with the property that \( \pi(\mathcal{F}(c_{k-1})) = x_{k-1} \). We then define \( \mathcal{F}(c) \) to be the rooted spanning forest obtained from \( \mathcal{F}(c_{k-1}) \) by connecting the two trees \( T_i \) and \( T_j \) with vertex sets \( V(T_i) = A_i \) and \( V(T_j) = A_j \) using the edge \( \{r(T_i), r(T_j)\} \) and choosing the root of the new tree to be \( \min\{r(T_i), r(T_j)\} \) if \( v_m = (w_i + w_j) = 0 \) or \( \max\{r(T_i), r(T_j)\} \) if \( v_m = (w_i + w_j) = 1 \).

Note that by construction and equation (5.2), we have that \( \pi(\mathcal{F}(c)) = x_k \) and so the map \( \mathcal{F} \) is inductively well-defined. Moreover, it has the property that for every \( c \in C(\Pi_n^w) \) we have that \( \pi(\mathcal{F}(c)) = e(c) \) (where, just as before, \( e(c) \) is maximum element of \( c \)). Indeed, the blocks of \( x_k \) are the same as the ones of \( x_{k-1} \) except for \( A_i^{w_i} \) and \( A_j^{w_j} \) that now form the block \( (A_i \cup A_j)^{w_m} \). It is not hard to see that \( \mathcal{F} \) is sending cover relations in \( C(\Pi_n^w) \) to cover relations in \( SF_n \) and so \( \mathcal{F} \) is a rank preserving poset map.

**Lemma 5.12.** Let \( \mathcal{F} : C(\Pi_n^w) \to C(SF_n) \) be defined for \( c : (\hat{0} = x_0 < x_1 < \cdots < x_k) \in C(\Pi_n^w) \) by

\[
\mathcal{F}(c) = \mathcal{F}(c_0) \lesssim \mathcal{F}(c_1) \lesssim \cdots \lesssim \mathcal{F}(c_k),
\]

and \( \overline{\pi} : C(SF_n) \to C(\Pi_n^w) \) defined for \( F : (\hat{0} = F_0 \lesssim F_1 \lesssim \cdots \lesssim F_k) \in C(SF_n) \) by

\[
\overline{\pi}(F) = (\pi(F_0) \lesssim \pi(F_1) \lesssim \cdots \lesssim \pi(F_k)).
\]

Then we have that these maps are well-defined and that \( \overline{\pi} \circ \mathcal{F} = id_{C(\Pi_n^w)} \) and \( \mathcal{F} \circ \overline{\pi} = id_{C(SF_n)} \). Hence, \( \mathcal{F} \) and \( \overline{\pi} \) define an isomorphism \( C(\Pi_n^w) \simeq C(SF_n) \).

**Figure 12.** A tree \( T \) with cost \( \gamma(T) = 7 \)
Proof. Note that if $F < F'$, we have that $F'$ is obtained from $F$ by merging two trees $T_1$ and $T_2$ to get a tree $T'$ of $F'$ such that $r(T') = r(T_1)$. Hence we have that $\pi(F)$ and $\pi(F')$ are almost identical except that they differ in two weighted blocks $V(T_1)^{w_{T_1}}$ and $V(T_2)^{w_{T_2}}$ of $\pi(F)$ and one weighted block $(V(T_1) \cup V(T_2))^{w_{T_1}+w_{T_2}+\chi(r(T_1) > r(T_2))}$ of $\pi(F')$ (where $\chi(A) = 1$ if the statement $A$ is satisfied and 0 otherwise). This means exactly that $\pi(F) < \pi(F')$ so $\tilde{\pi}$ is a well-defined map that preserves cover relations and hence is order-preserving. The comments preceding this lemma also imply that $\tilde{\mathfrak{F}}$ is a well-defined order-preserving map.

By the recursive definition of $\mathfrak{F}$ it follows that $\tilde{\pi} \circ \tilde{\mathfrak{F}} = id_{C(\Pi_n^w)}$. The reader can verify using induction on the length $k$ of a chain $F : (0 = F_0 < F_1 < \cdots < F_k) \in C(S\mathcal{F}_n)$ that $\tilde{\mathfrak{F}} \circ \tilde{\pi} = id_{C(S\mathcal{F}_n)}$. Hence we obtain the desired isomorphism $C(\Pi_n^w) \simeq C(S\mathcal{F}_n)$.

Remark 5.13. Note that Lemma 5.12 implies that the information encoded in the saturated chain $c : (0 = x_0 < x_1 < \cdots < x_k) \in C(\Pi_n^w)$ can be recovered uniquely from the saturated chain

$$\tilde{\mathfrak{F}}(c) = \tilde{\mathfrak{F}}(c_0) < \tilde{\mathfrak{F}}(c_1) < \cdots < \tilde{\mathfrak{F}}(c_k) \in C(S\mathcal{F}_n).$$

We will use this fact in the construction of a C-labeling of $\Pi_n^w$.

We define first an E-labeling $\lambda_{S\mathcal{F}} : \mathcal{E}(S\mathcal{F}_n) \to \Lambda$ as follows: Let $F < F'$ be such that $T_i$ and $T_j$ are the trees of $F$ that have been merged to get a tree $T'$ of $F'$ and assume without loss of generality that $r(T_i) < r(T_j)$. We define

$$\lambda_{S\mathcal{F}}(F < F') = \begin{cases} (-\gamma(T_j), r(T_i), r(T_j)) & \text{if } r(T') = r(T_i) \\ (-\gamma(T_i), r(T_j), r(T_i)) & \text{if } r(T') = r(T_j), \end{cases}$$

and we define $\Lambda$ to be the poset $\mathbb{Z}^3$ with lexicographic order.

We will define now a C-labeling $\lambda_C : \mathcal{M}\mathcal{E}(\Pi_n^w) \to \Lambda$ as follows: Let $c : (0 = x_0 < x_1 < \cdots < x_{\rho(\Pi_n^w)})$ be a maximal chain of $\Pi_n^w$. We define

$$\lambda_C(c, x_{i-1} < x_i) = \lambda_{S\mathcal{F}}(\tilde{\mathfrak{F}}(c_{i-1}) < \tilde{\mathfrak{F}}(c_i)).$$

The bottom to top construction that we have used to define $\lambda_C$, i.e., using the information on the saturated chains from $0$ to $x_{i-1}$, guarantees that this labeling is a C-labeling of $\Pi_n^w$. Indeed, for any maximal chain $\tilde{c}$ that coincides with $c$ in the bottom $d$ elements we have that $\tilde{\mathfrak{F}}(\tilde{c}_i) = \tilde{\mathfrak{F}}(c_i)$ for $i = 0, 1, \ldots, d - 1$ and so $\tilde{c}$ shares the same labels with $c$ along the first $d - 1$ edges. We will prove that $\lambda_C$ is in fact a CW-labeling.

Note that the definition of $\lambda_C$ says that for a chain $c : (0 = x_0 < x_1 < \cdots < x_k)$, the label $\lambda_C(c, x_{i-1} < x_i)$ only depends on the forests $\tilde{\mathfrak{F}}(c_{i-1})$ and $\tilde{\mathfrak{F}}(c_i)$. Hence the label sequence

$$\lambda_C(c, x_i \leq x_{i+1}), \lambda_C(c, x_{i+1} \leq x_{i+2}), \ldots, \lambda_C(c, x_{j-1} \leq x_j)$$

only depends on the sequence $\tilde{\mathfrak{F}}(c_i) < \tilde{\mathfrak{F}}(c_{i+1}) < \cdots < \tilde{\mathfrak{F}}(c_j)$.

**Lemma 5.14.** Let the sequence $F_i < F_{i+1} < \cdots < F_j$ be a saturated chain in $S\mathcal{F}_n$ such that the following holds:
• The forest $F_{i+1}$ is obtained from $F_i$ by merging two trees $T_1$ and $T_2$ of $F_i$ to obtain a new tree $T'$ in $F_{i+1}$ with $r(T') = r(T_1)$, and
• the forest $F_j$ is obtained by merging $T'$ with another tree $T_3$ to obtain a new tree $T''$ with root $r(T'') = r(T_3) \neq r(T') = r(T_1)$.

Then

$$\lambda_{SF}(F_i \preceq F_{i+1}) > \lambda_{SF}(F_j \preceq F_j).$$

**Proof.** Under these assumptions we have that

$$\lambda_{SF}(F_i \preceq F_{i+1}) = (-\gamma(T_2), r(T_1), r(T_2)) > (-\gamma(T'), r(T_3), r(T')) = \lambda_{SF}(F_j \preceq F_j),$$

since $T_2$ is a proper subtree of $T'$ and so $\gamma(T') > \gamma(T_2)$. \qed

**Proposition 5.15.** $\lambda_C : \mathcal{ME}(\Pi'_n) \to \Lambda$ is a CR-labeling.

**Proof.** To show that $\lambda_C$ is a CR-labeling we have to show that in each rooted interval $[x, y]$ there is a unique increasing chain. Another way to describe this is to say that for any saturated chain $c : (\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k = x)$ and $y \in \Pi'_n$ such that $x < y$ there is a unique saturated chain

$$\hat{c} : (\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k \prec x_{\rho(y)} = y)$$

such that

$$\lambda_C(\hat{c}, x_k \preceq x_{k+1}) < \lambda_C(\hat{c}, x_{k+1} \preceq x_{k+2}) < \cdots < \lambda_C(\hat{c}, x_{\rho(y)-1} \preceq x_{\rho(y)}). \quad (5.6)$$

Note that by the comments that precede Lemma 5.14 this label sequence only depends on the saturated chain $\hat{\mathcal{F}}(\hat{c}_k) \preceq \hat{\mathcal{F}}(\hat{c}_{k+1}) \preceq \cdots \preceq \hat{\mathcal{F}}(\hat{c}_{\rho(y)})$ in $\mathcal{SF}_n$. Hence if we are able to determine that there is a unique saturated chain $\hat{\mathcal{F}}(\hat{c}_k) = F_k \preceq F_{k+1} \preceq \cdots \preceq F_{\rho(y)}$ in $\mathcal{SF}_n$ with $\pi(F_{\rho(y)}) = y$ that has a sequence of labels that is increasing, then as a consequence of Lemma 5.12, the chain

$$\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k = x = \pi(F_k) \prec \pi(F_{k+1}) \prec \cdots \prec \pi(F_{\rho(y)}) = y$$

is the unique saturated chain $\hat{c}$ with the desired property.

Lemma 5.14 implies that for $\hat{c}$ to satisfy the increasing condition in equation (5.6) from steps $k+1$ to $\rho(y)$ we can only merge trees in a way that after a root $r(T_1)$ has been chosen in a step between $k+1$ and $\rho(y)$ the same root has to continue being a root in all consecutive steps. Now, recall from the recursive definition of the map $\hat{\mathcal{F}}$ that the process of selecting a root depends on the value of $u \in \{0, 1\}$ where $A_1^{w_1}$ and $A_2^{w_2}$ are the blocks of $x_{s-1}$ that are merged to obtain the block $(A_1 \cup A_2)^{w_1+w_2+u}$ of $x_s$. By the definition of the order relation in $\Pi'_n$, we have that each block in $y$ is of the form $(A_1 \cup A_2 \cup \cdots \cup A_l)^{w_1+w_2+\cdots+w_l+v}$ where $v \in \{0, 1, \ldots, l - 1\}$ and $A_1^{w_1}, A_2^{w_2}, \ldots, A_l^{w_l}$ are blocks of $x_k$. Assume that in $\hat{\mathcal{F}}(c_k) = F_k$ the trees corresponding to these weighted blocks are $T_1, T_2, \ldots, T_l$ and without loss of generality assume that the indexing is such that $r(T_1) < r(T_2) < \cdots < r(T_l)$. The reader can easily check that there is a unique tree $T'$ obtained by merging the $l$ trees by the roots step by step selecting at each step the same root $r(T_j)$ such that exactly $v$ of the other $l - 1$ roots are smaller than $r(T_j)$. In fact this tree is the one where $j = v + 1$. All
labels that come from the step-by-step merging process that creates $T'$ are then of the form $(-\gamma(T_i), r(T_{v+1}), r(T_j))$ for $i \in \{1, \ldots, l\} \setminus \{v + 1\}$. If there are two trees $T_i$ and $T_j$ for $i, j \in \{1, \ldots, l\} \setminus \{v + 1\}$ such that $\gamma(T_i) < \gamma(T_j)$ then

$$(-\gamma(T_j), r(T_{v+1}), r(T_j)) < (-\gamma(T_i), r(T_{v+1}), r(T_i)),$$

and if $\gamma(T_i) = \gamma(T_j)$ but $r(T_i) < r(T_j)$ then

$$(-\gamma(T_i), r(T_{v+1}), r(T_i)) < (-\gamma(T_j), r(T_{v+1}), r(T_j)).$$

Hence there is a unique increasing way of constructing $T'$ by attaching the roots $r(T_i)$ to the selected root $r(T_{v+1})$ by going first in reverse order of $\gamma(T_i)$ and then in order of $r(T_i)$.

As an example of the argument above, consider the forest formed by the trees $T_1$ to $T_6$ of Figure 13 with $r(T_1) < r(T_2) < r(T_3) < r(T_4) < r(T_5) < r(T_6)$. Suppose that we want to find an increasing maximal chain in the rooted interval

$$[16891114^3/23^0/4^0/5^0/10^1/12^1/713^1, 123456789101112131417]_c$$

where $c$ is a saturated chain from $\hat{0}$ to $16891114^3/23^0/4^0/5^0/10^1/12^1/713^1$ such that $\mathcal{F}(c)$ is the forest $\{T_1, T_2, T_3, T_4, T_5, T_6\}$. Since $7 - w_{T_1} - w_{T_2} - w_{T_3} - w_{T_4} - w_{T_5} - w_{T_6} = 2$, the unique increasing chain $\bar{c}$ in this rooted interval produces a tree $T'$ whose root $r(T') = r(T_3)$ (since the edges $\{r(T_3), r(T_1)\}$ and $\{r(T_3), r(T_2)\}$ will create exactly the 2 additional descents in $T'$). Since $\gamma(T_4) > \gamma(T_1) = \gamma(T_5) = \gamma(T_6) > \gamma(T_2)$ the new steps in $\bar{c}$ consist in adding first the edge $\{r(T_3), r(T_4)\}$ then the edges $\{r(T_3), r(T_1)\}$, $\{r(T_3), r(T_5)\}$, $\{r(T_3), r(T_6)\}$ in increasing order of their roots and finally the edge $\{r(T_3), r(T_2)\}$. This will give the sequence of labels

$$(-7, 5, 6) < (-1, 5, 2) < (-1, 5, 12) < (-1, 5, 13) < (0, 5, 4).$$

Finally, the sequence $\bar{F}(\bar{c}_k) = F_k \ll F_{k+1} \ll \cdots \ll F_{\rho(y)}$ describes a process of merging the trees in $F_k$ until we obtain the various trees in $F_{\rho(y)}$. By the discussion above we have that each tree $T' \in F_{\rho(y)}$ uniquely determines the subsequence of steps to build it given that the labels $(-\gamma(T_i), r(T_{v+1}), r(T_i))$ are required to satisfy equation (5.6). Furthermore, since these labels are all distinct and belong to the total order $\Lambda = \mathbb{Z}^3$, there is a unique way

![Figure 13. Example of the unique tree that can be constructed along an increasing maximal chain in a rooted interval $[x, y]_c$ of $\Pi_n^w$. Here the edges have been distinguished using red for descending edges and blue for ascending edges.](image-url)
of organizing the labels (the unique increasing shuffle of all the increasing subsequences of labels for all trees) for the various trees $T'$ in an increasing order, hence there is a unique sequence $\mathfrak{f}(\tilde{c}_k) = F_k \leq F_{k+1} \leq \cdots \leq F_{\rho(y)}$ with $\pi(F_{\rho(y)}) = y$, that satisfy the increasing condition in equation (5.6).

\textbf{Lemma 5.16.} The labeling $\lambda_C$ satisfies the rank two switching property. Moreover for every maximal chain $c$ in $\Pi_n^w$ and $i = 1, \ldots, \rho(\Pi_n^w)$ we have that $\mathfrak{f}(c) = \mathfrak{f}(U_i(c))$, where $U_i(c)$ is the unique saturated chain obtained after possibly applying a quadratic exchange to $c$ at rank $i$.

\textbf{Proof.} Any rank two increasing sequence

$$\lambda_C(c, x_{k-1} < x_k) < \lambda_C(c, x_k < x_{k+1})$$

is determined by a sequence $\mathfrak{f}(c_{k-1}) < \mathfrak{f}(c_k) < \mathfrak{f}(c_{k+1})$ of forests. In any such sequence we have two possible cases.

\textbf{Case I:} We start with trees $T_1, T_2, T_3, T_4$ of $\mathfrak{f}(c_{k-1})$ and obtain trees $T'$ and $T''$ of $\mathfrak{f}(c_{k+1})$ such that $T'$ is obtained by merging $T_1$ and $T_2$ with $r(T') = r(T_1)$; and $T''$ is obtained by merging $T_3$ and $T_4$ with $r(T'') = r(T_3)$. In this case, the label sequence is $(-\gamma(T_2), r(T_1), r(T_2)) < (-\gamma(T_4), r(T_3), r(T_4))$ and there is another forest $\mathfrak{f}(c_{k-1}) < F < \mathfrak{f}(c_{k+1})$ obtained by merging first $T_3$ and $T_4$ and then merging $T_1$ and $T_2$ in $F$ to get $\mathfrak{f}(c_{k+1})$. Let $\tilde{x}_k = \pi(F)$ and observe (by considering the definition of the cover relations in $\Pi_n^w$) that $\pi(\mathfrak{f}(c_{k-1})) < \pi(\mathfrak{f}(c_k))$ and $\pi(\mathfrak{f}(c_{k+1}))$. Then the chain $\tilde{c} = c \cup \{\tilde{x}_k\} \{x_k\}$ satisfies that $\lambda_C(c, x_{k-1} < \tilde{x}_k) = (\gamma(T_1), r(T_1), r(T_2)) = \lambda_C(c, x_k < x_{k+1})$ and $\lambda_C(c, \tilde{x}_k < x_{k+1}) = (\gamma(T_2), r(T_1), r(T_2)) = \lambda_C(c, x_k < x_{k+1})$.

\textbf{Case II:} In this case we start with trees $T_1, T_2, T_3$ of $\mathfrak{f}(c_{k-1})$ and obtain a tree $T''$ in $\mathfrak{f}(c_{k+1})$ by first merging $T_1$ and $T_2$ to get $T'$ with $r(T') = r(T_1)$ in $\mathfrak{f}(c_k)$ and then merge $T'$ and $T_3$ in $\mathfrak{f}(c_k)$ to get $T''$. Note that by Lemma 5.14, it is necessarily true that $r(T'') = r(T') = r(T_1)$, otherwise the labels would not be increasing. In this case, the label sequence is $(-\gamma(T_2), r(T_1), r(T_2)) < (-\gamma(T_3), r(T_1), r(T_3))$. Again here there is another forest $\mathfrak{f}(c_{k-1}) < F < \mathfrak{f}(c_{k+1})$ obtained by merging first $T_1$ and $T_3$ to get a tree $T'$ with $r(T') = r(T_1)$ in $F$ and then merging $T'$ and $T_2$ in $F$ to get again the same $T''$ in $\mathfrak{f}(c_{k+1})$. We let $\tilde{x}_k = \pi(F)$ and observe here again that $\pi(\mathfrak{f}(c_{k-1})) < \tilde{x}_k < \pi(\mathfrak{f}(c_{k+1}))$. Then $\tilde{c} = c \cup \{\tilde{x}_k\} \{x_k\}$ satisfies that $\lambda_C(c, x_{k-1} < \tilde{x}_k) = (\gamma(T_3), r(T_1), r(T_3)) = \lambda_C(c, x_k < x_{k+1})$ and $\lambda_C(c, \tilde{x}_k < x_{k+1}) = (\gamma(T_2), r(T_1), r(T_2)) = \lambda_C(c, x_k < x_{k+1})$.

Note that in the two cases above the choice of the element $\tilde{x}_k$ is unique. A label of the form $(-\gamma(T_2), r(T_1), r(T_2))$ determines exactly that the trees of $\mathfrak{f}(c_{k-1})$ with roots $r(T_1)$ and $r(T_2)$ are being merged into a tree $T'$ of $\mathfrak{f}(c_k)$ with root $r(T') = r(T_1)$. Hence Lemma 5.12 implies the uniqueness of the resulting saturated chain in $\Pi_n^w$. Note that a common feature of the two cases above is that after the quadratic exchange we have $\mathfrak{f}(\tilde{c}_{k+1}) = \mathfrak{f}(c_{k+1})$. Hence applying a quadratic exchange at level $k$ does not affect the sequence

$$\mathfrak{f}(c_{k-1}) < \mathfrak{f}(\tilde{c}_{k+2}) < \cdots < \mathfrak{f}(\tilde{c}_{\rho(\Pi_n^w)}),$$
nor the labels \( \lambda_C(c, x_{i-1} \leq x_i) \) for \( i = k + 2, \ldots, \rho(\Pi^w_n) \) since they depend only on this sequence. In particular we have \( \widetilde{\mathcal{F}}(c) = \mathcal{F}(c) = \mathcal{F}(U_t(c)) \). Also, the choice of the element \( x_k \) in the two cases above only depends on \( \mathcal{F}(c_{k+1}) \) and so it has to be the same for any other chain that coincides with \( c \) in the bottom \( k + 2 \) elements. We conclude then that \( \lambda_C \) satisfies the rank two switching property. \( \square \)

We note here that whenever \( c \in C(\Pi^w_n) \), we have \( \widetilde{\mathcal{F}}(c) = (\mathcal{F}(c_0) \prec \mathcal{F}(c_1) \prec \cdots \prec \mathcal{F}(c_k)) \) in \( C(\mathcal{SF}_n) \). Since the sequence of ordered pairs \( (r(T_{i,1}), r(T_{i,2})) \) of roots that are being merged at each step \( i \) (selecting \( r(T_{i,1}) \) as the new root) provides enough information to reconstruct the element of \( C(\mathcal{SF}_n) \), by Lemma 5.12, it also provides enough information to reconstruct the elements in \( C(\Pi^w_n) \). Hence the sequences of labels given by \( \lambda_C \) uniquely determine elements in \( C(\Pi^w_n) \). This also implies that the same is true for all maximal chains in rooted intervals \([x, y]_c\), or otherwise we can extend two maximal chains \( c_1 \) and \( c_2 \) with the same word of labels to \( c \cup c_1 \) and \( c \cup c_2 \) with the same property. This together with Proposition 5.15, Lemma 5.16 and Theorem 3.37 imply the following corollary.

**Corollary 5.17.** \( \lambda_C \) is a CW-labeling and hence \( Q_{\lambda_C}(\Pi^w_n) \) is a Whitney dual of \( \Pi^w_n \)

Note here that Lemma 5.16 also implies that the map \( \mathcal{F} : C(\Pi^w_n) \to \mathcal{SF}_n \) has the property that if \( c \) and \( c' \) are such that \( [c] = [c'] \) in \( Q_{\lambda_C}(\Pi^w_n) \) then \( \mathcal{F}(c) = \mathcal{F}(c') \). Hence we obtain a well-defined map \( \widetilde{\mathcal{F}} : Q_{\lambda_C}(\Pi^w_n) \to \mathcal{SF}_n \) given by \( \widetilde{\mathcal{F}}(X) = \mathcal{F}(c) \) for any \( c \in X \).

**Theorem 5.18.** The map \( \widetilde{\mathcal{F}} : Q_{\lambda_C}(\Pi^w_n) \to \mathcal{SF}_n \) is a poset isomorphism.

*Proof.* To be able to prove that this is a poset isomorphism we should show that \( \mathcal{F} \) is a bijection and that \( \mathcal{F}^{-1} \) are both poset (order preserving) maps.

**\( \mathcal{F} \) is a bijection:** To prove this we will show that for any \( F \in \mathcal{SF}_n \) there exist a unique ascent-free chain \( c \in C(\Pi^w_n) \) such that \( \mathcal{F}(c) = F \). The conclusion then follows from the fact that for every \( X \in Q_{\lambda_C}(\Pi^w_n) \) there is a unique ascent-free chain \( c \in C(\Pi^w_n) \) such that \( X = [c] \).

Recall from Lemma 5.11 that a sequence \( 0 = F_0 \ll F_1 \ll \cdots \ll F_k = F \) is a saturated chain in \( \mathcal{SF}_n \) if and only if the associated ordered listing \( v_1, v_2, \ldots, v_k \) of the non-root vertices of \( F \) is a linear extension. Call \( T(v_i) \) the induced rooted subtree of \( F \) formed by all descendants of \( v_i \) (including itself) and recall that \( p(v_i) \) is the the parent of \( v_i \) with respect to \( F \). Since \( v_1, v_2, \ldots, v_k \) is a linear extension, every \( v_j \in T(v_i) \) satisfies \( j < i \). This implies that every saturated chain \( 0 = F_0 \ll F_1 \ll \cdots \ll F_k = F \) has associated labels of the form \( \lambda_{\mathcal{SF}}(F_{i-1} \ll F_i) = (-\gamma(T(v_i)), p(v_i), v_i) \). Hence all saturated chains \( 0 = F_0 \ll F_1 \ll \cdots \ll F_k = F \) in \([0, F]\) for a given \( F \in \mathcal{SF}_n \) have the same set of labels

\[
L(F) := \{ (-\gamma(T(v_i)), p(v_i), v_i) \mid i \in [k] \}.
\]

The labels in \( L(F) \) are clearly all different and come from a totally ordered set, so there is a unique ascent-free way to order them

\[
(-\gamma(T(v_{i_1})), p(v_{i_1}), v_{i_1}) > (-\gamma(T(v_{i_2})), p(v_{i_2}), v_{i_2}) > \cdots > (-\gamma(T(v_{i_k})), p(v_{i_k}), v_{i_k}).
\]

Let \( v_{i_1}, v_{i_2}, \ldots, v_{i_k} \) be the ordered listing that we obtain in this way. We want to check that this sequence is also a linear extension which, by Lemma 5.11, implies that it has
an associated saturated chain from \( \hat{0} \) to \( F \). Indeed, if this ordered listing is not a linear extension then there are vertices \( v_i \) and \( v_j \) such that \( v_i \) is an ancestor of \( v_j \) and \( i < j \). But then we have that \( -\gamma(T(v_i)), p(v_i), v_i) < (\gamma(T(v_j)), p(v_j), v_j) \) since \( \gamma(T(v_i)) > \gamma(T(v_j)) \), a contradiction. Hence the sequence \( v_1, v_2, \ldots, v_k \) is a linear extension that gives a valid chain \( \hat{0} = F_0 < F_1 < \cdots < F_k = F \) in \( SF_n \) and so by Lemma 5.12 \( c : (\hat{0} = \pi(F_0) < \pi(F_1) < \cdots < \pi(F_k) = \pi(F)) \) is the unique ascent-free chain with \( \bar{\mathfrak{g}}(c) = F \).

\[ \overline{\mathfrak{g}} \text{ and } \overline{\mathfrak{g}}^{-1} \text{ are poset maps:} \] The fact that \( \overline{\mathfrak{g}} \) is order preserving implies, using Definition 3.14, Proposition 3.15 and the well-definedness of \( \overline{\mathfrak{g}} \), that \( \overline{\mathfrak{g}} \) is also order preserving. Now, if we have \( F < F' \) in \( SF_n \), given that \( \overline{\mathfrak{g}} \) is surjective, there is a chain \( c \) such that \( F = \overline{\mathfrak{g}}(c) \) and the recursive definition of \( \overline{\mathfrak{g}} \) implies that \( F' = \overline{\mathfrak{g}}(c \cup \pi(F')) \). But \( c < (c \cup \pi(F')) \) in \( C(\Pi_n^w) \) and so \( \overline{\mathfrak{g}}^{-1}(F) = [c] < [(c \cup \pi(F'))] = \overline{\mathfrak{g}}^{-1}(F') \) in \( Q_{\lambda_C}(\Pi_n^w) \). Since all posets are finite this implies \( \overline{\mathfrak{g}}^{-1} \) is order preserving.

\[ \text{Remark 5.19.} \] As we mentioned at the beginning of this subsection, Theorem 5.18 provides a new proof of Theorem 5.10 as a corollary.

5.3.2. A different Whitney dual for \( \Pi_n^w \). In [11], González D’León and Wachs gave an ER-labeling (that is in fact an EL-labeling) for \( \Pi_n^w \), quite different from the CR-labeling constructed above.

The map \( \lambda_E : \mathcal{E}(\Pi_n^w) \to \Lambda_n \) was defined as follows: let \( x < y \) in \( \Pi_n^w \) so that \( y \) is obtained from \( x \) by merging two blocks \( A^{w_A} \) and \( B^{w_B} \) into a new block \( (A \cup B)^{w_A + w_B + u} \), where \( u \in \{0, 1\} \) and where we assume without loss of generality that \( \min A < \min B \). We define

\[ \lambda_E(x < y) = (\min A, \min B)^u. \]

Here \( \Lambda_n \) is defined as follows: for each \( a \in [n] \), let \( \Gamma_a := \{(a, b)^u : a < b \leq n + 1, u \in \{0, 1\}\} \). We partially order \( \Gamma_a \) by letting \( (a, b)^u \leq (a, c)^v \) if \( b \leq c \) and \( u \leq v \). Note that \( \Gamma_a \) is isomorphic to the direct product of the chain \( a + 1 < a + 2 < \cdots < n + 1 \) and the chain \( 0 < 1 \). Now define \( \Lambda_n \) to be the ordinal sum \( \Lambda_n := \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n \). See Figure 14 for an example.

The labeling \( \lambda_E \) has the property that when restricted to the intervals \( [\hat{0}, [n]^0] \) and \( [\hat{0}, [n]^{n-1}] \), which are both isomorphic to \( \Pi_n \), it reduces to the minimal labeling of \( \Pi_n \) in \( [3, 31] \).

\[ \text{Theorem 5.20} \] ([11] Theorem 3.2). \( \lambda_E \) is an ER-labeling.

\[ \text{Theorem 5.21.} \] \( \lambda_E \) is an EW-labeling and hence \( Q_{\lambda_E}(\Pi_n^w) \) is a Whitney dual of \( \Pi_n^w \).

\[ \text{Proof.} \] Note that the information contained in the labels \( (\min A, \min B)^u \) is enough to recover any saturated chain from \( \hat{0} \). Hence the sequence of labels in each interval uniquely determines a chain. To show that \( \lambda_E \) is an EW-labeling we are left to show that it satisfies the rank two switching property. The rank two intervals \( [x, y] \) in \( \Pi_n^w \) are of three possible different types (see Figure 15), we will show that in each of these types the rank two switching property is satisfied.

\[ \text{Type I:} \] Two pairs of distinct blocks \( \{A^{w_A}, B^{w_B}\} \) and \( \{C^{w_C}, D^{w_D}\} \) of \( x \) are merged to get \( y \). Assume without loss of generality that \( \min A < \min B, \min A < \min C \) and
Type III: Three distinct blocks 

\[
\{ (1.3)^6, (1.2)^1, (1.2)^1 \} \quad \{ (1.3)^7, (1.2)^1, (1.2)^1 \} \quad \{ (1.3)^8, (1.2)^1, (1.2)^1 \} \quad \{ (1.3)^9, (1.2)^1, (1.2)^1 \}
\]

Then adding either 0 or 1 to the total weight of the resulting block. In this case the interval has a unique increasing chain with labels

\[
\lambda_E(x < z_1) = (\min A, \min B)^u < (\min A, \min C)^u = \lambda_E(z_1 < y),
\]

and there is a unique chain with labels

\[
\lambda_E(x < z_2) = (\min A, \min C)^u > (\min A, \min B)^u = \lambda_E(z_2 < y).
\]

Type III: Three distinct blocks 

\[
\{ (1.3)^6, (1.2)^1, (1.2)^1 \} \quad \{ (1.3)^7, (1.2)^1, (1.2)^1 \} \quad \{ (1.3)^8, (1.2)^1, (1.2)^1 \} \quad \{ (1.3)^9, (1.2)^1, (1.2)^1 \}
\]

Then adding either 0 or 1 to the total weight of the resulting block. In this case the interval has a unique increasing chain with labels

\[
\lambda_E(x < z_1) = (\min A, \min B)^0 < (\min A, \min C)^1 = \lambda_E(z_1 < y),
\]

and there is a unique chain with labels

\[
\lambda_E(x < z_2) = (\min A, \min C)^1 > (\min A, \min B)^0 = \lambda_E(z_2 < y).
\]

Thus, \( \lambda_E \) has the rank two switching property. \( \square \)
Now that we know, by Theorem 5.21, that $\lambda_E$ is an EW-labeling of $\Pi_n^w$, we can use Theorem 4.5 to describe $Q_{\lambda_E}(\Pi_n^w)$. We leave the general characterization of $Q_{\lambda_E}(\Pi_n^w)$ for a future article, but we explicitly compute the example of $Q_{\lambda_E}(\Pi_3^w)$ in Figure 16. An interesting fact here is that $Q_{\lambda_E}(\Pi_3^w)$ and $Q_{\lambda_C}(\Pi_3^w) \simeq SF_3$ are evidently not isomorphic (see Figures 11 and 16).

**Theorem 5.22.** There exist a poset $P$ and two CW-labelings $\lambda_1$ and $\lambda_2$ of $P$ such that the posets $Q_{\lambda_1}(P)$ and $Q_{\lambda_2}(P)$ are not isomorphic.

5.4. **R*S-labelable posets.** In [28] Simion and Stanley introduced the notion of an R*S-labeling as a tool to study local actions of the symmetric group on maximal chains of a
poset. In this subsection, we show that the $R^*S$ labelings that respect the consistency condition of the rank two switching property are CW-labelings. Simion and Stanley [28] showed the posets of shuffles $W_{M,N}$ introduced by Greene [12] have $R^*S$ labelings and Hersh [15] showed that the noncrossing partition lattices $NC^{B}_n$ and $NC^{D}_n$ of types B and D introduced by Reiner [23] also have such labelings. As noted in [28], Stanley’s parking function labeling of the noncrossing partition lattice $NC^A_n$ of type A described in a previous subsection is an $R^*S$ labeling. All of these are examples of $R^*S$ labelings that are CW-labelings, hence we have as a corollary that these posets have Whitney duals.

We start with the definition of an $S$-labeling of a poset given in [28]. We present it here in a slightly different language than how was originally stated in [28] to highlight the connection with CW-labelings.

**Definition 5.23** ([28]). Let $P$ be a graded poset of rank $n$ with a $\hat{0}$ and $\hat{1}$. Let $\lambda$ be a C-labeling such that the labels are totally ordered. We say $\lambda$ is an $S$-labeling if

1. For each maximal chain $c = (\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1})$ and for each $1 \leq i \leq n - 1$ such that $\lambda(c, x_{i-1} < x_i) \neq \lambda(c, x_i < x_{i+1})$, there exists a unique maximal chain $c' = (\hat{0} = x_0 < x_1 < \cdots < x_{i-1} < x'_i < x_{i+1} < \cdots < x_n = \hat{1})$ such that $c$ and $c'$ have the same sequence of labels except that $\lambda(c, x_{i-1} < x_i) = \lambda(c', x'_i < x_{i+1})$ and $\lambda(c, x_i < x_{i+1}) = \lambda(c', x_{i-1} < x'_i)$

2. $\lambda$ is one-to-one on maximal chains. That is, two different maximal chains must have different sequences of labels from bottom to top.

Considering the condition (1) of an S-labeling in Definition 5.23, one can see that if the choice of the element $x'_i$ is consistent among all maximal chains that coincide in the first $i + 2$ edges then the S-labeling also satisfies the rank two switching property of Definition 3.32. We will call S-labelings satisfying this additional condition consistent. In that case, we note that condition (1) on S-labelings is in fact stronger than the rank two switching property. For example, the labeling of $\Pi_3$ given in Figure 3 has the rank two switching property, but it is not an S-labeling since the chain with label sequence $(2, 3), (1, 3)$ cannot be switched with anything as it should be in the case of an S-labeling. Also, note that condition (2) of Definition 5.23 is also stronger than the simpler requirement of Definition 3.24 that ascent-free chains are one-to-one. Hence any S-labeling satisfying the consistency condition and which is also a CR-labeling, is a CW-labeling.

In [28] Simion and Stanley refer to C-labelings that are also CR-labelings as $R^*$-labelings. Moreover, any labeling which is both an $R^*$-labeling and an S-labeling is called an $R^*S$-labeling. We warn the reader of a possible source of confusion since in this paper we have used the term ER$^*$ to mean a different type of labeling. Using Theorem 3.37, we have the following.

**Theorem 5.24.** A consistent $R^*S$-labeling is a CW-labeling. Consequently, every poset with a consistent $R^*S$-labeling has a Whitney dual.

**Remark 5.25.** The consistency condition is automatically satisfied for E-labelings and so it only needs to be checked when the underlying labeling is a C-labeling.
The reader can check in [28] that the labeling for the poset of shuffles $W_{M,N}$ is an example of a consistent $R^\ast S$-labeling and hence a CW-labeling. In [33] and [15] edge labelings of the noncrossing partition lattices of type A, B and D are given. These are all examples of $R^\ast S$-labelings and hence they are also EW-labelings. We then have the following corollary.

**Corollary 5.26.** The poset of shuffles $W_{M,N}$ and the noncrossing partition lattices of type A, B and D all have Whitney duals.

6. $H_n(0)$-actions and Whitney labelings

In this section we describe an action of the 0-Hecke algebra on the maximal chains of a poset $P$ with a generalized CW-labeling $\lambda$. We will also see that the same action can be associated to the Whitney dual $Q_\lambda(P)$ constructed in Section 3. The characteristic of this action is Ehrenborg’s flag quasisymmetric function in the case of $P$ and is Ehrenborg’s flag quasisymmetric function with $\omega$ applied in the case of $Q_\lambda(P)$. The techniques we describe here closely follow the work of McNamara in [18] who studied actions of this kind on posets with EL-labelings in which the word of labels in every chain is a permutation of $S_n$, also known as $S_n$ EL-shellable or snellable posets.

6.1. **An action of the 0-Hecke algebra.** Suppose that $P$ is a graded poset of rank $n$. Moreover, suppose that $\lambda$ is a generalized CW-labeling of $P$. Recall that $\mathcal{M}_P$ is the set of maximal chains of $P$. Define maps $U_1, U_2, \ldots, U_{n-1} : \mathcal{M}_P \to \mathcal{M}_P$ such that for $c : (0 = x_0 < x_1 < \cdots < x_n)$

$$ U_i(c) = \begin{cases} c' & \text{if } \lambda(c, x_{i-1} < x_i) < \lambda(c, x_i < x_{i+1}) , \\ c & \text{otherwise} , \end{cases} $$

where $c'$ the unique maximal chain of $P$ obtained by applying a quadratic exchange at rank $i$. As an example, consider the maximal chain $c : (1/2/3/4 < 13/2/4 < 123/4 < 1234)$ in $\mathcal{NC}_4$ with the parking function labeling (see Figure 6). Since there is no ascent at rank 1, $U_1(c) = c$. However, there is an ascent at rank 2, and $U_2(c) = 1/2/3/4 < 13/2/4 < 134/2 < 1234$.

We note that in [18], where the labelings are snellings, the maps $U_i$ are similar except that instead of exchanging ascents by descents, they exchange descents by ascents.

**Proposition 6.1.** The maps $U_1, U_2, \ldots, U_{n-1}$ have the following properties.

1. For all $c \in \mathcal{M}_P$, $U_i(c)$ and $c$ are the same except possibly at rank $i$.
2. $U_i^2 = U_i$ for all $i$.
3. $U_i U_j = U_j U_i$ for all $i, j$ such that $|i - j| > 1$.
4. $U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}$ for all $i$.

**Proof.** The first three properties are immediate from the definition of $U_i$ and Definition 3.32 of the rank two switching property for C-labelings. The last property is a consequence of the braid relation in a generalized CW-labeling when there is a critical condition at rank $i$ and is easily verified when there is no critical condition at rank $i$. 

□
The 0-Hecke algebra of type $A$ is defined by abstract generators satisfying the same relations of those in Proposition 6.1. Thus the properties described in the proposition imply that there is an action of the generators of the 0-Hecke algebra $H_n(0)$ on the set $\mathcal{M}_P$. This action is said to be local since the chains $U_i(c)$ and $c$ are the same except possibly at rank $i$. Moreover, this action gives rise to a representation of the 0-Hecke algebra on the space $\mathbb{C}\mathcal{M}_P$ linearly spanned by $\mathcal{M}_P$.

It turns out that the characteristic of this action is a well-known quasisymmetric function. Before we look at this characteristic, we need to review some material on quasisymmetric functions.

### 6.2. Ehrenborg’s flag quasisymmetric function.

In [9], Ehrenborg introduced a formal power series now known as Ehrenborg’s flag quasisymmetric function. Given a graded poset $P$ with a $\hat{0}$ and $\hat{1}$, it is defined by

$$F_P(x_1, x_2, \ldots) = F_P(x) = \sum_{\hat{0}=t_1 \leq t_2 \leq \cdots \leq t_{k-1} < t_k = \hat{1}} x_1^{r_{k(t_0,t_1)}} x_2^{r_{k(t_1,t_2)}} \cdots x_k^{r_{k(t_{k-1},t_k)}}$$

where the sum is over all multichains from $\hat{0}$ to $\hat{1}$ where $\hat{1}$ appears exactly once. As the name suggests, $F_P(x)$ belongs to the ring of quasisymmetric functions. That is, for each sequence $n_1, n_2, \ldots, n_k$ the monomial $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ has the same coefficient as $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. In addition to being a quasisymmetric function, $F_P(x)$ also keeps track of the flag $f$-vector and the flag $h$-vector of $P$ as we describe next.

Let $P$ be a graded poset with a $\hat{0}$ and $\hat{1}$. For $S \subseteq [n-1]$ define

$$\alpha_P(S) = |\{\hat{0} < x_1 < x_2 < \cdots < x_{|S|} < \hat{1} \mid \{\rho(x_1), \rho(x_2), \ldots, \rho(x_{|S|})\} = S\}|.$$

In other words, $\alpha_P(S)$ is the number of chains from $\hat{0}$ to $\hat{1}$ which use elements whose rank set is $S$. The function given by $\alpha_P : 2^{[n-1]} \to \mathbb{Z}$ is called the flag $f$-vector of $P$. We also define

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S\setminus T|} \alpha_P(T).$$

The function $\beta_P : 2^{[n-1]} \to \mathbb{Z}$ is called the flag $h$-vector of $P$. The reason for the names flag $f$-vector and flag $h$-vector is that they refine the classical $f$-vector and $h$-vector of the order complex of $P$. See [34][§3.13] for more details.

When $P$ has a $\hat{0}$ and $\hat{1}$, there is a nice relationship between $F_P(x)$ and $\beta_P(S)$. Indeed, it is well-known that if $P$ has rank $n$, then

$$F_P(x) = \sum_{S \subseteq [n-1]} \beta_P(S) L_{S,n}(x)$$

where $L_{S,n}$ is Gessel’s fundamental quasisymmetric function defined by

$$L_{S,n}(x) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \atop i_j < i_{j+1} \text{ if } j \in S} x_{i_1} x_{i_2} \cdots x_{i_n}.$$
The original definition of \( F_P(x) \) requires that \( P \) have a \( \hat{1} \), however we would like to extend this to more general posets. In order to do this, we consider a slight generalization of \( F_P(x) \) to deal with posets with a single minimal element \( \hat{0} \), but possibly with multiple maximal elements. Let \( P \) be a graded poset with a \( \hat{0} \), then we define

\[
F_P(x) = \sum_m F_{[\hat{0},m]}(x)
\]

where the sum is over all maximal elements \( m \) of \( P \). Note that in the case that \( P \) has a \( \hat{1} \), this is just Ehrenborg’s classical definition. Since intervals always have a \( \hat{0} \) and a \( \hat{1} \), we have that

\[
F_P(x) = \sum_m \left( \sum_{S \subseteq [n-1]} \beta_{[\hat{0},m]}(S) L_{S,n} \right).
\]

Now suppose that \( \lambda \) is a CR-labeling of \( P \) and that \( P \) has a \( \hat{0} \) and a \( \hat{1} \). Recall that \( \mathcal{M}_P \) denotes the set of maximal chains in \( P \). For \( c : (x_0 < x_1 < x_2 < \cdots < x_n) \in \mathcal{M}_P \), the descent set of \( c \) is defined to be

\[
D(c) = \{ i \mid \lambda(c, x_{i-1}, x_i) \neq \lambda(c, x_i, x_{i+1}) \}.
\]

It was shown by Stanley [34][c.f. Theorem 3.14.2] for ER-labelings and of Björner and Wachs [7] for CR-labelings that \( \beta_S(P) \) is the number of maximal chains with descent set \( S \). A simple modification of Stanley’s proof of the combinatorial description of the numbers \( \beta_P(S) \), shows that if \( \lambda \) is an ER*-labeling of a poset \( P \) with \( \hat{0} \) and \( \hat{1} \), then \( \beta_P(S) = \{ c \in \mathcal{M}_P : \lambda(c) = S \} \) for any \( S \subseteq [n-1] \).

**Example 6.2.** We compute \( F_P(x) \) for \( P = \Pi_3 \). As one can see in Figure 3, \( \Pi_3 \) has three maximal chains. Under the ER-labeling of Example 3.3, one of the maximal chains is increasing and the other two are ascent-free. It follows that \( \beta_{\Pi_3}^*(\{0\}) = 1 \) and \( \beta_{\Pi_3}^*(\{1\}) = 2 \). Thus,

\[
F_{\Pi_3}(x) = L_{0,2}(x) + 2L_{\{1\},2}(x).
\]

**Example 6.3.** We consider the example of \( F_P(x) \) when \( P = ISF_3 \). As one can see in Figure 3, there are two maximal intervals with three maximal chains altogether. Moreover, using the ER*-labeling of Example 3.4 we can compute \( \beta_{p,m}(S) \) with the number of maximal chains with strict ascent set given by \( S \).

If \( F_1 \) is the increasing spanning forest with edge set \{\{(1,2), (1,3)\}\} and \( F_2 \) is the one with \{\{(1,2), (2,3)\}\}, we see that

\[
\beta_{[\hat{0},F_1]}(\emptyset) = 1, \beta_{[\hat{0},F_1]}(\{1\}) = 1, \beta_{[\hat{0},F_2]}(\emptyset) = 1, \text{and } \beta_{[\hat{0},F_2]}(\{1\}) = 0.
\]

Therefore

\[
F_{ISF_3}(x) = 2L_{0,2}(x) + L_{\{1\},2}(x).
\]

**Example 6.4.** We compute \( F_P(x) \) for \( P = NC_4 \). Recall from Section 5.2 that \( NC_4 \) has an ER-labeling where the labels on the set \( \mathcal{M}_{NC_4} \) of maximal chains of \( NC_4 \) correspond to parking functions of length 3, see Figure 6. So \( NC_4 \) has 16 maximal chains with label words given by \((1,1,1)\), the three permutations of each \((1,1,2)\) \((1,1,3)\) and \((1,2,2)\); and
the six permutations of $(1, 2, 3)$. Considering the descent sets of each of these sequences we can compute that $\beta_{\mathcal{NC}_4}(\emptyset) = 1$, $\beta_{\mathcal{NC}_4}({1}) = 5$, $\beta_{\mathcal{NC}_4}({2}) = 5$ and $\beta_{\mathcal{NC}_4}({1, 2}) = 5$. Thus,

$$F_{\mathcal{NC}_4}(x) = L_{\emptyset,3}(x) + 5L_{\{1\},3}(x) + 5L_{\{2\},3}(x) + 5L_{\{1,2\},3}(x).$$

The quasisymmetric function $F_{\mathcal{NC}_n}(x)$ is in fact symmetric. Stanley [33] showed that $\omega(F_{\mathcal{NC}_n}(x))$ is Haiman’s Parking Function Symmetric Function of $n$, where $\omega$ is the involution on the ring of quasisymmetric functions given by $\omega(L_{S,n}) = L_{S^c,n}$ where $S^c$ is the complement of $S$ in $[n - 1]$.

**Example 6.5.** Now consider $F_{\mathcal{P}}(x)$ when $P = \mathcal{NC}_{\text{Dyck}}_4 \cong Q_{\lambda}(\mathcal{NC}_4)$ together with its inherited ER*-labeling from $\mathcal{NC}_4$ in Section 5.2. Using Proposition 3.26 we know that the maximal chains are in a label-preserving bijective correspondence with the ones of $\mathcal{NC}_4$, so they are labeled by parking functions as well. One can show then that $F_{\mathcal{NC}_{\text{Dyck}}_4}(x) = 5L_{\emptyset,3}(x) + 5L_{\{1\},3}(x) + 5L_{\{2\},3}(x) + 5L_{\{1,2\},3}(x)$.

The reader may have noticed that the quasisymmetric functions above are very closely related. Our examples show that

$$F_{\mathcal{NSF}_3}(x) = \omega(F_{\mathcal{ISF}_3}(x)) \text{ and } F_{\mathcal{NC}_4}(x) = \omega(F_{\mathcal{NC}_{\text{Dyck}}_4}(x)).$$

This is no coincidence as we now see.

**Theorem 6.6.** Let $\lambda$ be a generalized CW-labeling of $P$. Then

$$F_{Q_{\lambda}(P)}(x) = \omega(F_{\mathcal{P}}(x)).$$

**Proof.** Recall that since $\lambda$ is a CR-labeling, $\beta_{\mathcal{P}}(S)$ counts the number of maximal chains with descent set $S$. Similarly since $\lambda^*$ is an ER*-labeling of $Q_{\lambda}(P)$, $\beta_{Q_{\lambda}(P)}(S)$ is the number of maximal chains with ascent set $A(c) := \{i \mid \lambda(c, x_{i-1}, x_i) < \lambda(c, x_i, x_{i+1})\} = S$. The CW-analogue of Proposition 3.26 implies that there is a bijection between maximal chains in $P$ and $Q_{\lambda}(P)$ which preserves labels. It follows that for each $S \subseteq [n - 1]$,

$$\sum_{m \in P} \beta_{[\emptyset, m]}(S^c) = \sum_{m' \in Q_{\lambda}(P)} \beta_{[\emptyset, m']}\beta_{[\emptyset, m']}(S)$$
where each sum is over maximal elements of $P$ and $Q_\lambda(P)$ respectively. Therefore

$$\omega F_{Q_\lambda(P)}(x) = \omega \left( \sum_{m' \in Q_\lambda(P)} F_{[\hat{0}, m']}([x]) \right)$$

$$= \omega \left( \sum_{m' \in Q_\lambda(P)} \left( \sum_{S \subseteq [n-1]} \beta_{[\hat{0}, m']}(S) L_{S,n} \right) \right)$$

$$= \omega \left( \sum_{S \subseteq [n-1]} \left( \sum_{m' \in Q_\lambda(P)} \beta_{[\hat{0}, m']}(S) L_{S,n} \right) \right)$$

$$= \sum_{S \subseteq [n-1]} \left( \sum_{m \in P} \beta_{[\hat{0}, m]}(S^c) L_{S,n} \right)$$

$$= \sum_{S \subseteq [n-1]} \left( \sum_{m \in P} \beta_{[\hat{0}, m]}(S^c) \omega(L_{S,n}) \right)$$

Thus we have proved the desired result. □

6.3. The characteristic of the action. In [20], Norton investigated the representation theory of $H_n(0)$. It is known that there are $2^{n-1}$ irreducible representations, all of them one-dimensional and hence they can be indexed by subsets of $[n-1]$. With this indexing, we have that if $U_i$ is one of the generators of $H_n(0)$ then the representation $\psi_S$ is given by

$$\psi_S(U_i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise}. \end{cases}$$

Hence, the character of the action is given by

$$\chi_S(U_{i_1}U_{i_2}\cdots U_{i_k}) = \begin{cases} 1 & \text{if } i_1, i_2, \ldots, i_k \in S, \\ 0 & \text{otherwise}. \end{cases}$$

The (quasisymmetric) characteristic of the character $\chi_S$ is defined by

$$ch(\chi_S) = L_{S,n}$$
where, as before, $L_{S,n}$ is Gessel’s fundamental quasisymmetric function. We will use $\chi_P$ to denote the character of the defining representation of a $H_n(0)$-action on $P$.

**Theorem 6.7.** Let $P$ be a graded poset of rank $n$ with a generalized CW-labeling. The local $H_n(0)$-action previously described is such that

$$ch(\chi_P) = F_P(x).$$

We note that the proof we present is almost identical to the one in [18, Proposition 4.1].

**Proof.** Let $[L_{S,n}]f(x)$ denote the coefficient of $L_{S,n}$ in the expansion of the quasisymmetric function $f(x)$ in the fundamental basis. We will show for any subset $S$ of $[n-1]$,

$$[L_{S,n}]ch(\chi_P) = [L_{S,n}]F_p(x).$$

As we saw earlier, the coefficient in $F_p(x)$ is

$$\sum_m \beta_{[0,m]}(S).$$

Thus, it suffices to show

$$[L_{S,n}]ch(\chi_P) = \sum_m \beta_{[0,m]}(S).$$

Now let $J \subseteq [n - 1]$ and let $\{i_1, i_2, \ldots, i_k\}$ be a multiset of $J$ where each element of $J$ appears at least once. For $c \in \mathcal{M}_P$, if $U_i(c) \neq c$, then $c$ has an ascent at $i$. It follows that $U_{i_1}U_{i_2} \cdots U_{i_k}(c) = c$ if and only if $c$ has descent set containing $J$. Therefore,

$$\chi_P(U_{i_1}U_{i_2} \cdots U_{i_k}) = \#\{c \in \mathcal{M}_P \mid D(c) \supseteq J\}$$

$$= \sum_{S \supseteq J} \#\{c \in \mathcal{M}_P \mid D(c) = S\}$$

$$= \sum_m \sum_{S \supseteq J} \#\{c \in \mathcal{M}_{[0,m]} \mid D(c) = S\}$$

$$= \sum_m \left(\sum_{S \supseteq J} \beta_{[0,m]}(S)\right)$$

$$= \sum_{S \subseteq [n-1]} \left(\sum_m \beta_{[0,m]}(S)\right) \chi_S(U_{i_1}U_{i_2} \cdots U_{i_k})$$

It follows that

$$[L_{S,n}]ch(\chi_P) = [L_{S,n}]ch\left(\sum_{S \subseteq [n-1]} \left(\sum_m \beta_{[0,m]}(S)\right) \chi_S\right) = \sum_m \beta_{[0,m]}(S)$$

which completes the proof. \qed
The analogue of Proposition 3.26 for CW-labelings implies that there is a bijection between maximal chains of $P$ and $Q_\lambda(P)$ which preserves labels. It follows that the local $H_n(0)$-action on $CM(P)$ can be also transported to a $H_n(0)$-action on $CM(Q_\lambda)$. It turns out that this action on the maximal chains of $Q_\lambda(P)$ is local.

**Lemma 6.8.** Let $P$ be a graded poset with a generalized CW-labeling $\lambda$. The 0-Hecke algebra action on $Q_\lambda(P)$ is local.

**Proof.** We must show that if we apply $U_i$ to any maximal chain $d$ of $Q_\lambda(P)$, the chain we get agrees with $d$ everywhere except possible at rank $i$. Suppose that $d$ and $d'$ are maximal chains in $Q_\lambda(P)$ such that $U_i(d) = d'$ and $d \neq d'$. Let $c$ and $c'$ be respectively the preimages of these chains under the label preserving bijection between $M_{Q_\lambda(P)}$ and $M_P$ described in the generalized CW-labeling version of Proposition 3.26. Then $U_i(c) = c'$ and $c \neq c'$. Since the action on $P$ is local, we can write $c : (0 = x_0 < x_1 < \cdots < x_{i-1} < x_i < x_{i+1} < \cdots < x_n)$ and $c' : (0 = x_0 < x_1 < \cdots < x_{i-1} < x'_i < x_{i+1} < \cdots < x_n)$. Denote $c_k$ the subchain formed by the smallest $k + 1$ elements of $c$ and $c'_k$ the one formed by the smallest $k + 1$ elements of $c'$. We have that $c_j = c'_j$ for all $0 \leq j \leq i - 1$ and $c_i \neq c'_i$, but $c_j$ and $c'_j$ are equivalent for $j \geq i + 1$ (since one chain is obtained from the other after applying a quadratic exchange at rank $i$). Thus the chains $[c_0] < [c_1] < \cdots < [c]$ and $[c'_0] < [c'_1] < \cdots < [c']$ in $Q_\lambda(P)$ agree everywhere except at rank $i$. Moreover, by the (generalized CW-labeling versions of) Lemma 3.18 and the proof of Proposition 3.26 these chains are exactly $d$ and $d'$. We conclude that the action on $Q_\lambda(P)$ is local. \qed

**Proposition 6.9.** Let $P$ be a poset with a generalized CW-labeling $\lambda$. For any maximal interval $I$ in $Q_\lambda(P)$,

$$\text{ch}(\chi_I) = \omega(F_I(x))$$

**Proof.** First note that since the action on $Q_\lambda(P)$ is local, the action only permutes maximal chains within maximal intervals of $Q_\lambda(P)$. Also, just as with $P$, $U_i(c) \neq c'$ if and only if $c$ has ascent at rank level $i$. Finally, note that since the labeling on $Q_\lambda(P)$ (and hence on $I$) is an ER*-labeling, we have that $\beta_i(S) = \{c \in M_I \mid D(c) = S^c\}$ for any $S \subseteq [n - 1]$. With this in mind, one can check that a slight modification of the proof of Theorem 6.7 gives the result. \qed

**Remark 6.10.** Note that $F_{Q_\lambda(P)}(x) = \sum F_I(x)$ and that $\chi_P = \chi_{Q_\lambda(P)} = \sum \chi_I$, where the sums are over maximal intervals $I$ of $Q_\lambda(P)$. Hence we obtain Theorem 1.8 as a corollary of Theorems 6.6 and 6.7; and Proposition 6.9.

A poset $P$ is called bowtie-free if there does not exist distinct $a, b, c, d \in P$ with $c \ll a$, $d \ll a$, $a \ll b$, and $d \ll b$. In [18], McNamara showed that a bowtie-free poset $P$ with a $0$ and a $1$ has a local $H_n(0)$-action with the property that the characteristic of this action is $\omega(F_P(x))$ if and only if $P$ is snellable. Additionally, he showed that if $P$ is a lattice, then $P$ is supersolvable. Proposition 6.9 then implies the following corollary.

**Corollary 6.11.** Let $P$ be a poset with a generalized CW-labeling $\lambda$. If $I$ is a maximal interval of $Q_\lambda(P)$ and is bowtie free, then $I$ is snellable. Moreover, if $I$ is a lattice, then $I$ is supersolvable.
7. OPEN QUESTIONS AND FURTHER WORK

In this Section we leave a few open questions that are motivated by the present work. In Theorem 1.6 we showed that posets that have Whitney labelings also have Whitney duals. It is reasonably to expect that there are posets without Whitney labelings that have Whitney duals. Indeed, the poset \( ISF_3 \) has \( \Pi_3 \) as a Whitney dual. However \( ISF_3 \) cannot have a Whitney labeling since in one of the maximal intervals the rank two switching property cannot be satisfied, see Figure 1. We would like to know if there is a general characterization of graded posets that have Whitney duals that completely answers Question 1.4. Additionally, we would like to know if there are other different and insightful methods of constructing Whitney duals, we propose the following question.

**Question 7.1.** Is there a systematic way to construct a Whitney dual of \( P \) without the use of labelings?

In the context of Whitney labelings we have provided two definitions: Whitney labelings and generalized Whitney labelings. Although the conditions of a generalized Whitney labeling are the ones we use in the proofs, all our examples satisfy the, a priori stronger, requirements of Whitney labelings.

**Question 7.2.** Are the families of Whitney labelable graded posets and of generalized Whitney labelable graded posets equal?

Of main interest is also to better understand the structure of the posets \( Q_\lambda(P) \) that are constructed using Whitney labelings \( \lambda \) of \( P \). We know from Theorem 5.22 that these posets are strongly dependent on \( \lambda \) and in Theorem 4.5 we provide a different description of its poset structure.

**Question 7.3.** Is there a nice way of characterizing all the posets that are of the form \( Q_\lambda(P) \) for some poset \( P \) and some Whitney labeling \( \lambda \)?

In light of Corollary 6.11 determining the structural properties of the posets \( Q_\lambda(P) \) also becomes relevant.

**Question 7.4.** Are all \( Q_\lambda(P) \) lattices? If this is not the case, are all of them bowtie-free?

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