The only syzygy-free solution is Lagrange’s.

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Abstract. A syzygy in the three-body problem is a collinear instant. We prove that with the exception of Lagrange’s solution every solution to the zero angular momentum Newtonian three-body problem suffers syzygies. The proof works for all mass ratios.

We consider the Newtonian three-body problem with zero angular momentum and negative energy. Masses are positive, but arbitrary. A ‘syzygy’ means an eclipse: an instant at which the three masses are collinear.

Theorem 0.1. Every solution admits a syzygy except one: the Lagrange homothety solution.

Solutions are defined over their maximal interval of existence and analytically continued through binary collisions a la Levi-Civita [2]. Binary collisions counts as syzygies. A solution cannot be extended past a finite time $t = b$ if and only if as $t \to b$ the three positions of the three bodies tend to the same point. In other words, a solution fails to exist past a certain time if and only if it ends in triple collision at that time. (See [11] or [10])

The Lagrange homothety solution [1] begins and ends in triple collision. At every other instant of its existence the masses form an equilateral triangle. This triangle evolves by homothety (scaling). Half-way through its evolution the three bodies are instantaneously at rest, forming an equilateral triangle whose size is determined by the value of the negative energy.

In [6] I proved theorem 0.1 upon imposing two additional hypotheses on solutions: that they are bounded, and that they do not end in triple collision. The contribution of the present paper is to dispense with these hypotheses.

I first dispense with the hypothesis on collision, keeping the boundedness hypothesis. Again, in [6] I proved that bounded solutions which do not end in collision have syzygies. The same proof, plus invariance of the equations and zero angular momentum condition under time reversal proves existence of syzygies for solutions which are bounded and do not begin in triple collision. All that remains of the bounded solutions are those, excluding Lagrange, which begin and end in triple collision. The proof for these solutions will follow the same qualitative lines as [6]. According to Moeckel [1989] (see the corollary at the top of p. 53) there are, for generic mass ratios, an infinite number of these finite-interval solutions bi-asymptotic to triple collision.

Moeckel, Chenciner and others have pointed out that dispensing with the boundedness hypothesis on solutions ought to be easy. In unbounded negative energy solutions two of the masses must form a bound pair with the third mass far away for long periods of time. During these long times the bound pair moves according to a differential equation which is a slight, but time-dependent perturbation of the Kepler equation and so the pair should spin about each other frequently crossing the line joining their center of mass to the distant mass, and thus making syzygies. However I was unable to turn this idea into a proof. The difficulties include the existence of oscillatory unbounded solutions, and the difficulty of establishing syzygies for systems looking like highly eccentric nearly Keplerian orbits subject to small time-dependent perturbations concentrated along the semi-major axis of
the orbits. Instead, I use the methods of [6]. The bulk of this paper is devoted to proving the existence of infinitely many syzygies for unbounded solutions with zero angular momentum. I expect a more skilled analyst could get a more direct proof based on the Kepler idea, and valid for unbound negative energy solutions with nonzero angular momentum.

**Motivation.** I have been trying for some time to establish a symbolical dynamical description for the zero angular momentum three-body. The symbols are to be the syzygies, marked as 1, 2, 3, depending on which mass crosses between the other two. See [7], [8]. I have successfully established a complete symbolical dynamical description if I allow myself to change the potential from the Newtonian $1/r$ potential to the $1/r^2$ potential and if I take the three masses to be equal. Theorem [1.1] is a first step toward the more interesting Newtonian case. The theorem asserts that, with one exception, every solution has syzygies, and hence a symbol sequence.

**Proof.**

We continue to use the methods of [6] where we introduced the “height” variable $z$ on the three-body configuration space minus triple collisions. The crucial properties of this $z$ are

$$-1 \leq z \leq 1$$

(0.1)

$$|z| = 1 \iff \text{equilateral}$$

(0.2)

$$z = 0 \iff \text{syzygy}$$

(0.3)

and that along any solution

$$d(f\dot{z})/dt = -qz, \quad f > 0, q \geq 0$$

(0.4)

where $f$ is a smooth function on shape space, $q$ is a smooth function on the tangent space to shape space,

$$q = 0 \iff \text{tangent to Lagrange homothety}.$$ 

(0.5)

and

$$f \to \infty \iff \text{unbounded}.$$ 

(0.6)

We recall that a solution is called bounded if all the distances $r_{ij}$ between the pairs $i, j$ of masses are bounded functions of time. Thus the solution is unbounded if

$$\limsup r_{ij}(t) = +\infty$$

for some mass pair $i, j$.

**The bounded case.** Let $x$ be a solution as per the theorem, and suppose it to be bounded. Thus $x$ is a bounded zero-angular momentum negative energy solution to the three-body problem besides the Lagrange solution. We may suppose it is not a collinear solution since every instant of a collinear solution is a syzygy. Reflecting a solution about a line effects the transformation $z \to -z$, and a time reflection $t \to -t$ effects the transformation $\dot{z} \to -\dot{z}$. Using these symmetries and time translation, we may assume at some initial time $t = -\epsilon$ we have $z > 0$ and $\dot{z} \leq 0$. Because the solution is not the Lagrange homothety solution, $z$ cannot be identically 1 and $q$ must be positive along the solution (see (0.5)). It follows from (0.4) that $(d/dt)(f\dot{z}) < 0$. In particular $\dot{z} = 0$ identically is impossible. Upon translating time forward slightly from $-\epsilon$ to 0 we will have $\dot{z} < 0$. Now we have
z(0) > 0 and \( \dot{z}(0) < 0 \). We must prove that at some finite time \( t = b \) later we have \( z(b) = 0 \).

According to \( \text{(0.4)} \) \( f \dot{z} \) is strictly decreasing as long as \( z > 0 \). Since \( f \) is positive, the derivative \( \dot{z} \) must remain negative over any interval \([0, b)\) of time during which \( z(t) > 0 \). Thus \( z(t) \) is monotonically decreasing over every interval of time \([0, b)\) for which the solution exists and for which \( z(t) > 0 \). The solution cannot fail to exist in such interval, because the only way it can terminate itself is by ending in triple collision. But non-collinear solutions which end in triple collision must asymptote to the Lagrange solution \[5\] implying \( z \to 1 \) or \( z \to -1 \) which we have excluded. Hence either \( b < \infty \) and \( z(b) = 0 \), in which case we have our syzygy, or \( b = \infty \) and the solution stays in the upper hemisphere \( z > 0 \) for all positive time. We invoke the hypothesis that the solution is bounded to exclude the second possibility.

So, suppose that \( z > 0 \) on \([0, \infty)\), that \( \dot{z}(0) < 0 \) and that the motion is bounded. According to the bound \( \text{(0.6)} \) the function \( 1/f \) is bounded away from zero along our solution, so that \( 1/f > k \) on \([0, \infty)\) for some positive constant \( k \). Now \( \dot{z}(0) \) is negative by assumption, and \( f(0) \) is positive so that \( f(0) \dot{z}(0) = -a < 0 \) is negative. According to the differential equation \( \text{(0.4)} \) \( f \dot{z} \) is monotonically decreasing so that

\[
\dot{z}(t) \dot{z}(t) < -a. \quad \text{Then } \dot{z} = -\frac{1}{f}\dot{z} < -ka. \quad \text{But then}
\]

\[
z(t) = z(0) + \int_0^t \dot{z} \, dt < z(0) - kat.
\]

which violates the positivity of \( z \) as soon as \( t > z(0)/ka \). This contradiction shows that in fact \( z \) has a zero before time \( t = z(0)/ka \).

**The unbounded case.**

There are two types of unbounded solutions, escape and oscillatory. A solution is an escape solution if \( \lim r_{ij}(t) = +\infty \) for some pair \( ij \). It is an oscillatory solution if for some pair \( \limsup r_{ij} = \infty \) while for every pair \( \liminf r_{ij} < \infty \). The existence of oscillatory type unbounded solutions was established by Sitnikov \[6\]. Our proof deals simultaneously with both types.

The function \( \max_i r_{ij} \) is a measure of the size of the configuration. Another equivalent measure is \( R \) where \( R^2 = I \) is the moment of inertia: \( I = \Sigma m_i m_j r_{ij}^2 / \Sigma m_i \).

The \( m_i \) are the values of the masses. Then

\[
c \max_i r_{ij} < R < C \max_i r_{ij}.
\]

where here and throughout \( c, C \) denote positive constants depending only on the masses and occasionally on the energy. The precise values of these constants will not be important. It follows that a motion is an escape motion if \( \lim_{t \to \infty} R(t) = +\infty \) and it is oscillatory if \( \limsup R(t) = +\infty \) while \( \liminf R(t) < \infty \). The function \( f \) of the basic equation \( \text{(0.4)} \) is related to \( R^2 \) by

\[
f = R^2 \lambda^2
\]

where \( c \leq \lambda \leq C \). Relation \( \text{(0.6)} \) follows from this expression for \( f \).

Let \( x \) be an unbounded solution. Being unbounded, for any \( R_0 > 0 \) there is a time \( t \) such that \( R(t) \geq R_0 \). Rewrite the \( z \) equation \( \text{(0.4)} \) by introducing a new

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\[1\] See Table 1 there, the entry \( \text{dim}(\text{St}(R)) \) with \( R = C^* \). The linearization at \( C^* \) for collinear motion also has \( \text{dim}(\text{St}(R)) = 1 \), showing that the stable manifold ingoing to a collinear triple collision \( C^* \) lies within the collinear submanifold.

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time variable $\sigma$:

$$f \frac{dz}{dt} = \frac{dz}{d\sigma},$$

so that

$$\frac{dt}{d\sigma} = f$$

The differential equation for $z$ becomes:

$$\frac{d^2 z}{d\sigma^2} = -qz$$

which is a harmonic oscillator: (0.8)

of variable frequency $\omega$. If $\omega$ were to be a constant $\omega_0$ then this would be the equation of a linear oscillator and the zeros of any solution would be spaced equally at ($\sigma$-) intervals of length $\pi/\omega_0$. Returning to our case, from standard Sturm-Liouville theory it follows that if $\omega^2 > \omega_0^2$ then within each of these intervals of length $\pi/\omega_0$ the function $z(\sigma)$ has a zero. Let $\ell$ be the length of an interval of $\sigma$ during which $R \geq R_0$ and suppose that $\omega \geq \omega_0$ during this interval. In the $\sigma$ variable escape to infinity takes finite time so the lengths $\ell$ will be finite. If we can show that for $R_0$ sufficiently large $\ell > \pi/\omega_0$ then we will know there is an oscillation during this interval. We will establish below the asymptotics:

$$\ell \omega_0 \geq CR_0, R_0 \to \infty$$

It follows that there are many syzygies during the interval $\ell$ (at least $CR_0/\pi$ syzygies).

The following two estimates yield (0.9)

(0.10) $\omega \geq CR_0^2$

(0.11) $\ell \geq C/R_0$  

**Proving estimate (0.10) the $\omega$ bound.**

Let

$$r = \min_{i \neq j} r_{ij}$$

be the minimum distance. Fix the total energy $H$ to be negative and write $h = -H > 0$. Then, as is well-known, there exists a constant $c$ depending only on the masses such that the minimum intermass distance $r$ satisfies

(0.12) $r \leq c/|H|.$

See equation (1.3) of the appendix for a proof.

The total energy is given

$$H = (K/2) - U$$

where $K \geq 0$ is the potential energy and

$$U = \Sigma m_i m_j / r_{ij}.$$  

is the negative of the potential energy. Because our solution has negative energy and $R$ is large along intervals of the solution of it, we know that along any one such 'long' interval that one of the distances, say $r_{12}$ is much smaller than the other two and these other two are of order $R$:

(0.13) $r_{12} = r \leq C, r_{13}, r_{23} \geq CR.$

(See the appendix , equations (1.6, 1.8) for proofs.) Introduce the spherical coordinates $(R, \theta, \phi)$ used in [4], and the squared distance variables

$$s_k = r_{ij}^2$$
for $i, j, k$ any permutation of 123. These systems of coordinates are related by $s_k = R^2 \lambda (1 - \gamma_k(\theta) \cos(\phi))$ where $\gamma_k = \cos(\theta - \theta_k)$, and $\theta = \theta_k$, $\phi = 0$ describes the location of the binary collision ray $r_{ij} = 0$. See [3] eq. (4.3.14). (The angles $\theta_k - \theta_j$ between the collision rays depend on the masses.)

The function $q$ satisfies:

$$ q = \text{positive} + \frac{-4 \cos(\phi) \partial U}{\sin(\phi)} \partial \phi. $$

It follows that the bound (0.10) will follow from the bound

$$ \frac{-4 \cos(\phi) \partial U}{\sin(\phi)} \partial \phi \geq CR^2 $$

valid for all $R$ large enough, together with the defining relations (0.8) and (0.7).

We proceed to establish the bound (0.14). We have

$$ \frac{\partial U}{\partial \phi} = -\sum \frac{m_i m_j}{r_{ij}^3} \frac{\partial s_k}{\partial \phi} $$

and

$$ \frac{\partial s_k}{\partial \phi} = -R^2 \lambda \gamma_k \sin(\phi) + R^2 \frac{\partial \lambda}{\partial \phi} (1 - \gamma_k(\theta) \cos(\phi)) $$

$$ = \sin(\phi) R^2 \lambda \gamma_k + \Lambda s_k $$

where $\Lambda = \frac{\partial \log \lambda}{\partial \phi}$. The function $\lambda$ and hence log $\lambda$ are even functions on the shape sphere, where ‘even’ and ‘odd’ refer to behaviour under the reflection $\phi, \theta \mapsto -\phi, \theta$ about the equator. It follows that $\Lambda$ is an odd function and so vanishes on the equator. Thus

$$ \Lambda = \sin(\phi) W(\phi, \theta) $$

where $W$ is a smooth function on the sphere. In particular $W$ is uniformly bounded. Now we have

$$ \frac{-1}{\sin(\phi)} \frac{1}{r_{ij}^3} \frac{\partial s_k}{\partial \phi} = \frac{1}{r_{ij}^3} [R^2 \lambda \gamma_k(\theta) - W s_k] $$

Since $\gamma_3 = 1$ at collision and since the point in the shape sphere representing our triangle is arbitrarily close to this same collision point for $R$ big (because $r/R << 1$), we have that $\gamma_3$ is as close to 1 as we like along our solution interval, by taking $R$ large along the interval. Using this fact, and the bound (0.12) we have

$$ \frac{-1}{\sin(\phi)} \frac{1}{r_{ij}^3} \frac{\partial s_k}{\partial \phi} = \frac{1}{r_{ij}^3} [R^2 C - C] $$

$$ \geq \frac{CR^2}{r_{ij}^3} $$

And for the other two distances:

$$ | \frac{-1}{\sin(\phi)} \frac{1}{r_{ij}^3} \frac{\partial s_2}{\partial \phi} |, | \frac{-1}{\sin(\phi)} \frac{1}{r_{ij}^3} \frac{\partial s_1}{\partial \phi} | \leq C/R $$

Thus:

$$ -4 \frac{\cos(\phi) \partial U}{\sin(\phi)} \partial \phi \geq CR^2 $$

as claimed.

**Proving the estimate on $\ell$, the bound (0.11).**

We will be using the length $\rho = ||\xi||$ of the long Jacobi vector $\xi$ as a measure of escape. This vector connects the 12 center of mass to the distant mass 3. We have

$$ R^2 = a \rho^2 + b r_{12}^2 $$

where $a, b$ depend only on the masses. (See (1.4) of the appendix). It follows from equation (0.15) and the bound $r \leq c/|H|$ on $r = r_{12}$ that $R$ and $\rho$ are related by

$$ c_a \rho \leq R \leq C_a \rho, $$

...
where the constants $c_a, C_a$ depend only on the masses. (These constants can be taken arbitrarily close to the constant $1/\sqrt{a}$ by taking $R_0$ sufficiently large and $R > R_0$.)

The desired length bound follows from the following assertion

**Proposition 0.2.** Let $\rho(t)$ be the length at time $t$ of the long Jacobi vector for a future-unbounded negative energy solution. Then there exists a constant $c_3$ such that for all $\rho_0$ sufficiently large there exists a $\rho_* \geq \rho_0$ and two times $t_1 < t_*$ such that $\rho(t_1) = \rho_*$ while $\rho(t_*) = 2\rho_*$ and $\rho$ is monotonically increasing over the interval $t_1 < t < t_2$ with the derivative bound

\[
|\dot{\rho}(t)| \leq c_3.
\]

If the solution is oscillatory, then we can take $t_*$ such that $\dot{\rho}(t_*) = 0$ and continuing further, find $t_3 > t_*$ such that $\rho(t_3) = \rho_*$, and $\rho$ decreases monotonically over $[t_*, t_3]$ with the bound (0.18) in place.

In the oscillatory case, the constant $c_3$ can be taken as small as we like. In the escape case the limit $\nu_\infty := \lim_{t \to \infty} \dot{\rho}(t)$ exists and we can take for $c_3$ any constant greater than $\nu_\infty$. (See the (0.19) of the appendix regarding this limit.)

We show how the bounds of the proposition imply the desired bound (0.11) on $\ell$. We measure the length $\ell$ of the domain of the arc of solution guaranteed by the proposition

\[
\ell = \int d\sigma = \int \frac{d\alpha}{R^2} dt \\
\geq K \int_{t_1}^{t_*} \frac{d\sigma}{d\rho} d\rho = K \int_{t_1}^{t_*} \frac{\rho}{\rho^2} d\rho \\
\geq \frac{K}{c_3} \int_{t_1}^{t_*} \rho \frac{1}{\rho_*} d\rho \\
\geq \frac{K}{c_3} \int_{t_1}^{t_*} \rho \frac{1}{\rho_*} d\rho
\]

which is the desired bound.

0.1. **Proving Proposition 0.2.** The proof divides into two cases, escape and oscillatory. Both cases rely on the inequality:

\[
-c_-/\rho^2 < \dot{\rho} < -c_+/\rho^2,
\]

valid for $\rho > \rho_0$ with $\rho_0$ large enough. As usual, the constants $c_- > c_+ > 0$ depend only on the masses. By taking $\rho_0$ arbitrarily large, we can make $c_-, c_+$ arbitrarily close to each other and to the total mass. See the appendix, lemma 1.1 for the proof of (0.19).

**Case 1: Escape.** Say that $\rho(t) \to \infty$ with $t$. According to (0.19) its speed $\dot{\rho}$ decreases with increasing $\rho$. For $t$ sufficiently large, we have $\dot{\rho}(t) > 0$, for otherwise we would have arbitrarily large times at which $t$ would turn back around and $\rho$ would decrease, contradicting escape. (See the comparison lemma immediately below, and the appendix for more details.) It follows that $\dot{\rho}(t)$ is monotonically decreasing with increasing $t$ and so tends to a limit $\nu_\infty \geq 0$. Given any $\epsilon > 0$,
choose \( t_* \) large enough so that \( 0 < \dot{\rho} \leq \nu_{\infty} + \epsilon \) while \( \rho(t_*) := \rho_* > \rho_0 \). Then for all \( t > t_* \) we have \( \rho(t) \leq \rho_* \) while \( 0 < \dot{\rho}(t) \leq \nu_{\infty} + \epsilon \). Thus \( \rho \) travels between \( \rho_* \) and \( \infty \) all the while satisfying \( |\dot{\rho}| \leq c_3 = \nu_{\infty} + \epsilon \).

Case 2: Oscillatory. We use inequality (0.19) in conjunction with:

**Lemma 0.3.** [Comparison Lemma]. Consider three scalar differential equations
\[
\ddot{x}_- = F_-(x_-), \quad \ddot{x} = F(x, t), \quad \ddot{x}_+ = F_+(x_+) \quad \text{with } C^1 \text{ right hand sides satisfying}
\]
\[
F_-(x) < F(x, t) < F_+(x) < 0 \quad \text{for } x > x_c, \quad x_c \text{ a fixed constant. Suppose that}
\]
\[
F_-(x) \text{ and } F_+(x) \text{ are monotone increasing for } x > x_c \text{ Let } x_-(t), x(t), x_+(t) \text{ be the}
\]
solutions to their respective differential equation sharing ‘at rest’ initial conditions at time 0: \( x_-(0) = x_1(0) = x_+(0) := x_* > x_c, \dot{x}_-(0) = \dot{x}(0) = \dot{x}_+(0) = 0 \). Then,
for all times \( t \) such that \( x_-(t) \geq x_c \) we have

1. \( x_- < x(t) < x_+ \) with equality only at \( t = 0 \), and
2. \( \frac{dx_-}{dt} < \frac{dx}{dt} < \frac{dx_+}{dt} \) for \( t > 0 \) and \( \frac{dx_-}{dt} > \frac{dx}{dt} > \frac{dx_+}{dt} \) for \( t < 0 \)

See Figure 1.

We prove the lemma below. Assuming the lemma we continue with the proof of the proposition in the oscillatory case. Let \( \rho_0 \) large be chosen so that the estimate of (0.19) is in force for \( \rho > \rho_0 \), with the constants \( c_+, c_- \) sufficiently close to each other. How close is detailed in the next paragraph. Since the solution is oscillatory, given any \( \rho_* > 0 \) we can find times \( t_* \) arbitrarily large such that

\[ \rho(t_*) := 2\rho_* \]

and

\[ \dot{\rho}(t_*) = 0. \]

Since \( \rho_* \) is arbitrarily large, we may suppose that with \( \frac{1}{2}\rho_* \geq \rho_0 \). The comparison lemma sandwiches \( \rho \) between the solutions \( \rho_+, \rho_- \) to the ‘bounding’ differential equations: \( \ddot{\rho}_\pm = -c_\pm/\rho_\pm^2 \) of (0.19) which share initial conditions with \( \rho \) at \( t = t_* \). See figure 1. Thus

\[ \rho_-(t) < \rho(t) < \rho_+(t); t \in I \]

where \( I \) is an interval containing \( t_* \) such that for \( t \in I \) the bound \( \rho(t) > \rho_0 \) needed to obtain (0.19) is in force.

We can describe the comparison solutions \( \rho_{\pm} \) in sufficient detail by using the scaling symmetry of Kepler’s equation. Let \( \phi(t) \) be the solution to the model Kepler equation \( \ddot{\phi} = -1/\phi^2 \) with initial condition \( \phi(0) = 1, \dot{\phi}(0) = 0 \). Then

\[ \rho_+(t) = \lambda \phi(\lambda^{-3/2} \sqrt{c_+(t-t_*))} \]

and

\[ \rho_-(t) = \lambda \phi(\lambda^{-3/2} \sqrt{c_-(t-t_*))} \]

where we take

\[ \lambda = 2\rho_* \]
to guarantee agreement of initial conditions at \( t = t_* \). By taking \( \rho_0 \) sufficiently large we can make \( c_- \) arbitrarily close to \( c_+ \). Consequently, for \( \rho_0 \) large enough we will have that \( 1/4 \leq \phi(\sqrt{\alpha \tau_j}) \) where \( \tau_j > 0 \) is time such that \( \phi(\sqrt{\alpha \tau_j}) = 1/2 \).

Now the times \( \tau \) and \( t \) for the scaled solutions are related by \( \tau = \lambda^{-3/2}(t - t_*) \). It follows at the time \( t_2 \) corresponding to \( \tau \) we have

\[
\rho_+(t_2) = \rho_+
\]
and \( \rho_0 < \rho_+ / 2 < \rho_-(t_f) < \rho(t_f) \leq \rho_+ \). Over the time interval \([0, \tau_j]\) the uniform derivative bound \(-k < \sqrt{\alpha - \frac{\lambda}{\rho^2}}(\sqrt{\alpha - \tau}) < 0 \) holds. Under the scaling and translation symmetry used to make \( \rho_{\pm} \) we find that velocities transform by \( \nu(t) = \frac{1}{\sqrt{\lambda}} \frac{d\phi}{dt}(\lambda^{-3/2}(t - t_*)) \). Consequently \( k/\sqrt{\lambda} < \rho_- < 0 \) during the time interval \([t_*, t_f]\). By the comparison lemma then

\[
-k/\sqrt{2\rho_0} < \dot{\rho} < 0
\]

over this same time interval. Now \( \rho(t_f) \) may be less than \( \rho_* \) but \( \rho \) is monotonically decreasing. So take for \( t_3 \) the unique time in the interval \([t_*, t_f]\) such that \( \rho(t_3) = \rho_* \). This completes the argument in the oscillating case for the decreasing interval \([t_*, t_3]\) of \( \rho \). The argument for the increasing arc \([t_2, t_*]\) of \( \rho \) is the time reversal (about \( t_* \)) of this argument.

QED (for the proposition.)

**Proof of Lemma.**

Proof of lemma. (1) follows from (2) by integration. We will just prove (2) in the \( - \) case, i.e. the inequality \( dx_- / dt < dx / dt \) for \( t > 0 \). The argument in the other cases is identical. Looking at the Taylor expansions of \( x_-, x \) at \( t = 0 \) we see that the inequality holds in a small right-hand neighborhood of 0, say \((0, \delta)\). Now proceed by contradiction. If the inequality fails before \( x_- \) reaches \( x_0 \), then there is a \( t \) with \( dx_- (t)/dt \geq dx(t)/dt \). Let \( t_* \) be the first such \( t > 0 \) such that \( dx_- (t)/dt = dx(t)/dt \). We have \( t_* > \delta \). By integration, \( \int_0^{t_*} dx_- (x(t))dt = \int_0^{t_*} dx (x(t))dt \). These two integrals are equal. But \( F(x(t), t) = F(x_-(t)) \). And in the interval \([0, t_*]\) we have \( dx_- / dt < dx / dt \), and so, by integration \( x_- (t) < x(t) \). Then \( F_-(x(t)) > F_-(x_-(t)) \) by the monotonicity of \( F_- \). So \( \int_0^{t_*} F(x(t), t)dt > \int_0^{t_*} F_-(x_-(t))dt \), contradicting the equality of the two integrals.

QED

**Remark.** Differential inequalities involving \( \rho \) are much better behaved at large \( R \) (and hence \( \rho \)) than those involving \( R \). The \( R \) differential equation is the Lagrange-Jacobi identity \( 2 \frac{d}{dt} RR = 4H + 2U \) and yields a huge second derivative for \( R \) when \( r = r_{12} \) is sufficiently small. Thus \( R \) can oscillates wildly, despite the fact that the bound (0.16) is in force.

**Discussion. Open questions.** Could the theorem hold for arbitrary energy \( H \) and angular momentum \( J \)? No. It does not hold for \( H > 0 \) and \( J = 0 \). For the direct method of the calculus of variations yields action-minimizing hyperbolic escape orbits which leave triple collision and tend to any desired noncollinear point of the shape sphere in infinite time. The reflection argument (see, eg. [11]) shows that these minimizers never become collinear. The theorem might hold for \( H = J = 0 \) but I suspect not. In this case there is a manifold of parabolic escape orbits whose shapes tend to Lagrange. I would guess some of these have no syzygies, but this is just a guess. For \( H < 0 \), and general \( J \neq 0 \) the theorem is false, at least for mass
ratios in which one mass dominates. For in this case, the near-circular Lagrange solutions are KAM stable, and so are surrounded by a nearby cloud of KAM tori on which the solutions stay near Lagrange, and hence away from \( z = 0 \) for all time. It is possible that for some values of \( H < 0, J \neq 0 \) and some values of the mass ratios that the theorem continues to hold. If the Dziobek constants \( J^2 H \) and mass ratios are such that the Lagrange solution is unstable (which is the case for nearly equal masses and \( J^2 H \) being a value near that which supports the circular Lagrange solution) then there is some chance for the theorem to hold.

According to the theorem, all solutions bi-asymptotic to triple collision except for the Lagrange solution have syzygies. This number is necessarily finite. What numbers are possible? Is any finite number of syzygies achieved? Is any finite syzygy sequence realized? Write \( m(x) \) for the time interval on which the solution \( x \) is defined. (Thus \( m(x) = +\infty \) for all solutions except those bi-asymptotic to triple collision.) Is it true that \( m(x) \) is minimized (among all solutions \( x \) with \( J = 0 \) and \( H < 0 \) fixed) by the Lagrange solution?

1. Appendix: Bounds near \( \infty \) for negative energy.

We suppose the total energy \( H = K/2 - U \) to be negative and write \( H = -|H| \).

Then
\[
(1.1) \quad U \geq |H|
\]
Write \( r = \min\{r_{ij} : i \neq j\} \) for the minimum of the intermass distances. Then there is a constant \( c \) depending only on the masses such that
\[
(1.2) \quad c/r \geq U.
\]
(For instance, if the masses are all equal to \( m \) then \( c = 3m^2 \).) It follows that
\[
(1.3) \quad c/|H| \geq r.
\]

Let us suppose that 12 realize the minimum distance:
\[
r = r_{12}.
\]

Associated to the decomposition 12; 3 we have Jacobi vectors and their lengths:
\[
\zeta = x_1 - x_2, \quad |\zeta| = r
\]
\[
\xi = x_3 - x_{cm}^{12}, \quad |\xi| = \rho.
\]
Here
\[
x_{cm}^{12} := (m_1 x_1 + m_2 x_2)/(m_1 + m_2)
\]
\[
:= \mu_1 x_1 + \mu_2 x_2
\]
is the 12 center of mass and
\[
\mu_1 = m_1/(m_1 + m_2), \quad \mu_2 = m_2/(m_1 + m_2).
\]

One computes
\[
(1.4) \quad R^2 = \alpha_1 r^2 + \alpha_3 \rho^2
\]
where
\[
(1.5) \quad \alpha_1 = m_1 m_2/(m_1 + m_2), \quad \alpha_3 = (m_1 + m_2) m_3/(m_1 + m_2 + m_3).
\]
from which it follows that 
\[ (1.6) \quad c_a \rho \leq R \leq C_a \rho \]

and 
\[ (1.7) \quad U \geq C/\rho \]

Set \( \hat{U} = RU \). Combine (1.1), (1.2), (1.7) with (1.6) to get that 
\[ C\rho/r \geq \hat{U} \geq R|H| \]

or 
\[ \frac{C}{R|H|} \geq r/\rho \]

which asserts that by making \( R \) or \( \rho \) large we can make the ratio \( r/\rho \) as small as we wish. We view \( r/\rho \) as a perturbation parameter. From the last inequality it follows that for every \( \epsilon > 0 \) there is a \( \rho_0 \) (or \( R_0 \)) sufficiently large such that \( \rho \geq \rho_0 \) \((R \geq R_0)\) implies that \( r/\rho < \epsilon \). In what follows, let \( \epsilon \) small be given, and suppose we have chosen the corresponding \( \rho_0 \) (or \( R_0 \)) be taken so as to guarantee \( r/\rho < \epsilon \). And let \( c, C, .. \) denote constants depending only on this \( \rho_0 \), the masses, the total energy, and, in a moment, the total angular momentum.

We can express the other distances in terms of \( \xi, \zeta \):
\[ (1.8) \quad r_{13} = \|\xi - \mu_1 \zeta\| ; \quad r_{23} = \|\xi + \mu_2 \zeta\| \]

where \( \mu_i \) are the reduced masses described above. Note that \( r_{13} = \rho + O(\epsilon) \) and \( r_{23} = \rho + O(\epsilon) \).

We have 
\[ H = H_{12} + H_3 + g \quad (5a) \]

where 
\[ H_{12} = \frac{1}{2} \alpha_1 \|\dot{\xi}\|^2 - \beta_1/r, \quad (5b) \]
\[ H_3 = \frac{1}{2} \alpha_3 \|\dot{\xi}\|^2 - \beta_3/\rho, \quad (5c) \]

where \( \alpha_1, \alpha_2 \) are given in eq. (1.5) and where 
\[ \beta_1 = m_1 m_2, \beta_3 = (m_1 + m_2)m_3 \]

and where the “error term” \( g \) is given by
\[ g = \frac{(m_1 + m_2)m_3}{\|\xi\|^2} - \frac{m_1 m_3}{\|\xi - \mu_1 \zeta\|^2} - \frac{m_2 m_3}{\|\xi + \mu_2 \zeta\|^2} \langle \xi, \zeta \rangle + O\left(\frac{1}{\rho^4}\right) \]

Note that 
\[ (1.9) \quad |g| \leq C\epsilon/\rho. \]

We will also need bounds for the gradients of \( g \):
\[ g_\xi = c\zeta/\rho^3 + O(\rho^{-4}) \]
so that 
\[ (1.10) \quad |g_\xi| \leq C\epsilon/\rho^2 \]

If we set \( g = 0 \) then \( H \) becomes the Hamiltonian for two uncoupled Kepler systems. The next lemma describes some details of the asymptotics of this decoupling as \( \rho \to \infty \). Introduce 
\[ J_{12} = \mu_1 \zeta \wedge \dot{\xi}, \]
the angular momentum of the 12 system, and the radial and transverse separation velocities $\nu, V^\perp$ by

$$\dot{\nu} = \nu \dot{\xi} + V^\perp ; \quad \dot{\rho} = \nu$$

where $\dot{\xi} = \xi/\rho$ is the unit vector in the $\xi$ direction and where $V^\perp$ is orthogonal to $\xi$.

**Lemma 1.1.** Consider any solution to the three-body problem along which $\rho(t) \geq \rho_0$ with $\rho_0$ as above. There exists a positive constant $c$, depending only on the total energy $H$ and total angular momentum $J$, the masses $m_i$ and $\rho_0$ such that

- (a) $|J_{12}| \leq c$
- (b) $\|V^\perp\| \leq c/\rho$
- (c) $|\hat{\rho} + M/\rho^2| \leq c/\rho^3$ where $M = m_1 + m_2 + m_3$ is the total mass.

Similar bounds hold for the time derivatives of $H_{12}, H_3, J_3, \xi$, and the 12 and 3 Laplace (or Runge-Lenz) vectors.

**Proof of Estimate (a):** We show that $\|J_{12}\| \leq \beta_1^2/|H| + O(1/\rho_0)$. We have $J_{12} = \alpha_1 \zeta \wedge \zeta$, $\|\zeta \wedge \zeta\|^2 \leq \|\zeta\|^2 \|\zeta\|^2$ and $\|\zeta\|^2 = r^2$. It follows that $|J_{12}| \leq \alpha_1 r^2 \|\zeta\|^2$.

Set $H' = H_3 + g$. Note that $-H' \leq \beta_3/r + \epsilon/\rho \leq c/\rho_0$, and that $H_{12} = H - H'$. If it follows that $\text{But } H_{12} \leq -|H| + c/\rho_0$. Use the formula for $H_{12}$ to rewrite this inequality as

$$\alpha_1 \|\dot{\zeta}\|^2 \leq -2|H| + 2c/\rho_0 + 2\beta_1/r$$

Multiply through by $r^2$ to get

$$|J_{12}| \leq \alpha_1^2 r^2 \|\dot{\zeta}\|^2 \leq [-2|H| + 2c/\rho_0]r^2 + 2\beta_1 r$$

The right hand side is a quadratic function of $r$ with negative quadratic term. The maximum value of this quadratic function is $(\frac{1}{2})(2\beta_1)^2/(2|H| - 2c/\rho_0)) = \beta_1^2/|H| + O(1/\rho_0)$. Thus $|J_{12}| \leq \beta_1^2/|H| + O(1/\rho_0)$.

**Proof of estimate (b).** We have $\alpha_3 \xi \wedge \xi = \alpha_3 \xi \wedge V^\perp$, $|\xi \wedge V^\perp| = \rho|V^\perp|$ and $\alpha_3 \xi \wedge \xi = J - J_{12}$. Thus

$$\alpha_3 \rho|V^\perp| = |J - J_{12}| \leq |J| + |J_{12}|$$

Now use estimate (a).

**Proof of estimate (c).** We have $\bar{\rho} := \langle \xi, \dot{\xi} \rangle$, so that

(1.11)

$$\bar{\rho} = \langle \frac{d}{dt} \hat{\xi}, \dot{\xi} \rangle + \langle \hat{\xi}, \ddot{\xi} \rangle.$$

We compute that $\langle \frac{d}{dt} \hat{\xi}, \dot{\xi} \rangle = -\frac{\dot{\xi}^2}{\rho} + \frac{\Omega^2 \zeta}{\rho} = \|V^\perp\|^2/\rho$ so that by estimate (b):

(1.12)

$$|\langle \frac{d}{dt} \hat{\xi}, \dot{\xi} \rangle| \leq c/\rho^3$$

Now use Newton’s equations for $\xi$

$$\alpha_3 \ddot{\xi} = U_\xi = -\beta_3 \xi/\rho^3 + g_\xi.$$

which yields

$$\ddot{\xi} = -M \xi/\rho^3 + \frac{1}{\alpha_3} g_\xi.$$
because $\beta_3/\alpha_3 = M$. Thus

\begin{equation}
\langle \dot{\xi}, \ddot{\xi} \rangle = -\frac{M}{\rho^2} + \frac{1}{\alpha_3} \langle \dot{\xi}, g_\xi \rangle.
\end{equation}

Using the estimate (1.10) on the gradient $g_\xi$ of $g$, with equations (1.11), (1.12), and (1.13) we get the desired result $|\ddot{\rho} + M/\rho^2| \leq c/\rho^3$.

QED

As a general reference for some of the inequalities appearing here, and many others, see Marchal, esp. pp. 327-7, equations (885)-(894).

References

[1] J-L. Lagrange, [1772], Essai sur le Problème des Trois Corps. Prix de l’Académie Royale des Sciences de Paris, tome IX, in volume 6 of œuvres (page 292).
[2] T. Levi-Civita [1921], “Sur La Régularisation du Problème des Trois Corps”, Acta Math., v. 42, 99-144; especially, p. 105, eq. (12), (1921).
[3] C. Marchal The Three-body Problem, Elsevier, 1990.
[4] A. Chenciner A. and R. Montgomery A remarkable periodic solution of the three-body problem in the case of equal masses, Annals of Math., 152, pp. 881-901 (2000)
[5] R. Moeckel, “Chaotic Dynamics Near Triple Collision” Arch. for Rat. Mechanics, v. 107, no. 1, 37-69, 1989.
[6] R. Montgomery, “Infinitely many syzygies”, Archives for Rational Mechanics and Analysis, v. 164, 311-340, 2002.
[7] R. Montgomery, “Fitting hyperbolic pants to a three-body problem”. Ergodic Theory Dynam. Systems v. 25 no. 3, 921–947, 2005,
[8] R. Montgomery, “The N-body problem, the braid group, and action-minimizing periodic orbits”, Nonlinearity, vol. 11, no. 2, 363-376, 1998.
[9] K.A. Sitnikov, “The existence of oscillatory motions in the three-body problem”, Sov. Phys. Dokl., v. 5, 647-650, 1961.
[10] D. Saari, Collisions, Rings, and Other Newtonian N-Body Problems, CBMS Regional Conference Series in Mathematics, no. 104, AMS, 2005.
[11] K. Sundman, “Memoire sur Le Probleme des Trois Corps”, Acta Math., v. 36, 105-179, (1912).