Tropicalisation for topologists

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Abstract

We consider the problem of translating notions from classical topology to tropical language. We consider the tropical projective and Grassmannian spaces. We give a fairly easy classification of the projective gadget whereas the Grassmannians seems rather more difficult. We also consider the notion of the tropical matrices, and define a variant of tropical orthogonal matrices. We completely determine $GL(R_\geq, n)$ and $O(n)_\text{trop}$. We also give some results on the idempotent elements of the tropical matrix algebra $M_{n,n}(R_\geq)$. Such a notion will be important in the related bundle theory. We note that tropical phenomena have been studies by algebraic geometers, and our work here may overlap with them and there is no claim on the originality of these notes, neither no claim on them being unoriginal.

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1 Motivation, Basics

The \( \min \)-algebra, \( \mathbb{N}^\infty = \mathbb{N} \sqcup \{+\infty\} \) together with the binary operation of taking minimum is our main motivation.

Definition 1.1. By a tropical space, we mean a pointed set \((T, \infty)\) together with a minimum-function \( \min = \min_T : T \times T \to T \) satisfying

\[
\min(t, \infty) = \min(\infty, t) = t \text{ for all } t \in T,
\]

and \((t, t) = t \) for all \( t \in T \).

Notice that under this operation, \( T \) turns into a monoid, whose each element is idempotent. The tropical space \( T \) is called commutative if \( \min(t, t') = \min(t', t) \) for all \( t, t' \in T \). Next, we make it clear what we mean by a tropical module, tropical algebra, etc. Let \( R \) be a commutative ring. By a left tropical \( R \)-module \( T \) we mean a tropical set \( T \) of which has the structure of a left \( R \)-module, and the \( R \)-action fixes the \( \infty \), i.e. \( r\infty = \infty \). Similarly, we can define a tropical algebra \( T \) to be a tropical space \( T \) together with a multiplication which is compatible with the tropical structure.

Notice that we may consider the category of these objects, say category of tropical \( R \)-modules together with the forgetful functor

\[
\text{RMod}_{\text{trop}} \to \text{RMod}.
\]

Next, we show how it is possible to construct examples of these spaces. This of course reduces to find reasonable ways of defining the \( \min \)-function.

Note 1.2. We note that it is possible to have a dual tropical structure given by taking \( \max \)imum of two given elements. We will use this in some of our important examples.

1.1 Examples

1.1.1 Tropicalising normed spaces

Let us write \( \mathbb{R}_\geq \) for the half line of nonnegative real numbers. Let \((M, | |)\) be a space with a norm function \( | | : M \to \mathbb{R}_\geq \). Notice that any norm gives rise to a metric, and any metric space \((M, d)\) with a chosen point \( 0 \in M \) can be turned into a normed space by setting \(|v| = d(v, 0)\) for any \( v \in M \). Having a norm define \( m : M \to \mathbb{R}_\geq \) by

\[
m(a, b) = \min(|a|, |b|)
\]

where the right hand side is the minimum function of the real line. Notice that if we choose \( M = \mathbb{R} \) with the usual the metric on \( \mathbb{R} \), then \( m(a, b) = \min(a, b) \). Now, we may define \( \min = \min_M : M \times M \to M \) by \( \min(a, a) = a \) and

\[
\min(a, b) = \begin{cases} 
a & \text{if } m(a, b) = |a|, 
b & \text{if } m(a, b) = |b|. 
\end{cases}
\]
For a given normed space $M$, we define tropicalisation of $M$ by $M^{\text{trop}} = M \sqcup \{\infty\}$ with generalised $\text{min}$-function satisfying

$$\text{min}(m, \infty) = \text{min}(\infty, m) \text{ for all } m \in M$$

and $\text{min}(\infty, \infty) = \infty$. Hence we obtain, a tropicalisation functor

$$Trop : \text{Normed-Space} \longrightarrow \text{Trop-Space}.$$ 

Notice that category of normed spaces contains the category of finite dimensional vector spaces inside it.

The fact that the tropicalisation of $M$ depends on the norm, or say the metric, shows that this can be used as an invariant of metric spaces, and not necessarily as an invariant of topological spaces. The reason is that it is possible to choose different metrics, yielding the same topology on a space.

**Note 1.3.** We note that the tropicalisation functor in fact is defined from the category of totally ordered sets into the category of tropical sets, as the notion of order naturally tells how to choose smaller element among two given ones.

### 1.1.2 Tropicalising Modules with projections

Notice that on the real line one has

$$\text{min}(a, b) = \frac{a}{2} + \frac{b}{2} - \frac{1}{2}|a - b|.$$ 

We use this as analogy to define $\text{min}$ on a class of modules. Suppose we have a ring $R$ which includes $\frac{1}{2}$, say $R = \mathbb{Z}[\frac{1}{2}], \mathbb{Q}, \mathbb{R}, \mathbb{C}$, etc. Let $M$ be an $R$-module together with a projection, i.e. a mapping $\tau : M \rightarrow M$ such that $\tau^2 = \tau$. We then define

$$\text{min}^\tau_M(a, b) = \frac{1}{2}(a + b - \tau(a - b)).$$

This then defines a $\text{min}$-function on $M$. Note that one may restrict to choose $\tau$ as an automorphism of $R$-modules. But for the moment we shall not put this restriction. We can consider the tropicalisation of $M^{\text{trop}}_\tau = M \sqcup \{\infty\}$ with generalised $\text{min}$-function defined as

$$\text{min}_{M^{\text{trop}}_\tau}(m, \infty) = \text{min}_{M^{\text{trop}}_\tau}(\infty, m) = m \text{ for all } m \in M,$$

and $\text{min}_{M^{\text{trop}}_\tau}(\infty, \infty) = \infty$. This then defines the tropicalisation functor

$$Trop : \text{Rmod}^{\text{pro}} \longrightarrow \text{Trop-Space}$$

where the left hand side is the category of $R$-modules with projections. If we give structure of a monoid to $M^{\text{trop}}_\tau$ by $\text{min}_{M^{\text{trop}}_\tau}$ and define the multiplication on $M^{\text{trop}}_\tau$ by generalising the addition operation of $M$, then we obtain an example of a tropical algebra. More precisely, we define $\oplus, \odot$ by

$$m \oplus m' = \text{min}_{M^{\text{trop}}_\tau}(m, m') \text{ for all } m, m' \in M^{\text{trop}}_\tau,$$

$$m \odot m' = \begin{cases} m + m' & \text{if } m, m' \in M \\ \infty & \text{if } m = \infty, \text{ or } m' = \infty. \end{cases}$$
Notice that here $\odot$ is commutative. We usually will drop $\odot$ from our notation. This then gives structure of a semi-ring to $(M^\tau_{\text{trop}}, \oplus, \otimes)$. Observe that the inclusion

$$M \to M^\tau_{\text{trop}}$$

acts like the exponential map as it sends addition to multiplication. If we choose $\tau$ to be an automorphism, then we can give structure of an $R$-module to to $M^\tau_{\text{trop}}$ by requiring that the action fixes $\infty$, and on the other points of $M^\tau_{\text{trop}} \setminus \{\infty\}$ acts in the same way as on $M$. This then makes the inclusion map $M \to M^\tau_{\text{trop}}$ into a map of $R$-modules.

Finally, we note that it is possible to define a “dual” tropical structure on $M$. For motivation, notice that in $\mathbb{R}$, we have

$$\max(a, b) = \frac{1}{2}(a + b + |b - a|).$$

One then may fix $\tau : M \to M$ with $\tau^2 = 1$ and define

$$\max^\tau_M(a, b) = \frac{1}{2}(a + b + \tau(a - b)).$$

It is then possible to perform as we did with the $\min$-function.

## 2 Tropical Euclidean and Grassmannian Spaces

I like to approach these topics with hat of a topologist on. These are quite familiar objects, and it will be interesting how they look like geometrically.

In topology, there are different approaches to define Grassmannian spaces. Let us to work with $\mathbb{R}$ for a moment. We consider $\mathbb{R}$ with its usual addition and multiplication. The Grassmannian $G_k(\mathbb{R}^{n+k})$ may be viewed as the space of $k$-dimensional subspaces of $\mathbb{R}^{n+k}$ which can be described as quotient of another space, namely the Steifel space $V_k(\mathbb{R}^{n+k})$ of $k$-frames in $\mathbb{R}^{n+k}$ together the quotient map

$$V_k(\mathbb{R}^{n+k}) \to G_k(\mathbb{R}^{n+k}).$$

Another way of defining $G_k(\mathbb{R}^{n+k})$ is to consider it as the quotient

$$\frac{O(n+k)}{O(n) \times O(k)}$$

where $O(n)$ is the orthogonal group, the group of isometries of $\mathbb{R}^n$.

The key element, in any of these approaches, is the type of algebraic structure of $\mathbb{R}^n$ is a “set with addition” together with its structure as an $\mathbb{R}$-module. On the other hand, the standard $\mathbb{R}$-module structure on $\mathbb{R}^n$ is obtained by using the multiplication operation $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

We consider the tropicalisation of the real line and determine a model for it, which is more familiar to a topologist. We then consider the ways that one can define the multiplication operation on the tropical line, have granted that the addition is given by the tropical structure.
2.1 A Model for $(\mathbb{R}_{\text{trop}})^n$

To begin with, notice that $\mathbb{R}_{\text{trop}}$ as a set is in a one to one correspondence with $\mathbb{R}_{\geq}$ where by the latter we mean the nonnegative real numbers. We fix such a correspondence as following. First, notice that there is a homeomorphism of topological spaces $f : \mathbb{R}_{>} \to \mathbb{R}$ defined by

$$f(x) = -\ln x$$

where $\mathbb{R}_{>}$ stands for the open half line of positive real numbers. We extend this to $f : \mathbb{R}_{\geq} \to \mathbb{R}_{\text{trop}}$ by setting

$$f(0) = \infty.$$ 

This then enables us to get a one to one correspondence

$$f^{\times n} = (f, \ldots, f) : \mathbb{R}^n_{\geq} \to (\mathbb{R}_{\text{trop}})^n$$

where $X^n$ denotes $n$-fold Cartesian product of $X$ with itself. Note that it is quite straightforward to see that the category of tropical spaces is closed under the Cartesian product.

**Note 2.1.** For a topologist the object $\mathbb{R}^n_{\geq}$ is a familiar one. This is the space that is used as the model space to define *manifolds with corners*, where

$$\partial \mathbb{R}^n_{\geq} = \mathbb{R}^n_{>} - \text{interior}(\mathbb{R}^n_{\geq})$$

corresponds to different types of (potential) singularities that such a manifold can have, whereas the interior corresponds to the smooth parts of such manifolds.

**Remark 2.2.** Notice that the set $\mathbb{R}_{\text{trop}}$ has two components, hence the set $(\mathbb{R}_{\text{trop}})^n$ will have $2^n$ components. We may write this as

$$(\mathbb{R}_{\text{trop}})^n = \coprod_{0 \leq k \leq n} (\mathbb{R}^k_{\geq})^{\binom{n}{k}},$$

where by $(\mathbb{R}^k_{\geq})^{\binom{n}{k}}$ we mean $(\binom{n}{k})$-fold $\bigoplus$-product of $\mathbb{R}^k$ with itself. We can make this more precise. For instance, in $(\mathbb{R}_{\text{trop}})^2$ we have one copy of $\mathbb{R}^2$, two copies of $\mathbb{R}$ and one copy of $\mathbb{R}^0$ in the following way: $\mathbb{R}^2$ corresponds to itself, one copy of $\mathbb{R}$ corresponds to $\{\infty\} \times \mathbb{R}$, another copy of $\mathbb{R}$ corresponds to $\mathbb{R} \times \{\infty\}$, and $\mathbb{R}^0$ corresponds to $(\infty, \infty)$. Under the correspondence of $(\mathbb{R}_{\text{trop}})^2$ with $\mathbb{R}^2_{\geq}$ we see that $\mathbb{R}^2$ corresponds to the interior of $\mathbb{R}^2_{\geq}$, $\mathbb{R} \times \{\infty\}$ corresponds to the $x$-axis without the origin, $\{\infty\} \times \mathbb{R}$ corresponds to the $y$-axis, and $(\infty, \infty)$ corresponds to the origin.

Our next objective, is to define an action on $\mathbb{R}_{\text{trop}}$ and make sense of $k$-dimensional subspaces of $(\mathbb{R}_{\text{trop}})^{n+k}$. Here, we have to choose which object we like to act on $\mathbb{R}_{\text{trop}}$. We may choose $\mathbb{R}$ to acts on $\mathbb{R}_{\text{trop}}$, or we may choose $\mathbb{R}_{\text{trop}}$ to act on $\mathbb{R}_{\text{trop}}$.

### 2.1.1 A Tropical Structure on $\mathbb{R}^n_{\geq}$

We consider the “dual” tropical structure on $\mathbb{R}_{\geq}$ by using the taking the maximum of two given values. This operation has a min element $0 \in \mathbb{R}_{\geq}$ as its minimum. This then induces a tropical structure on $\mathbb{R}^n_{\geq}$ given

$$v \oplus v' = (\max(v_1, v'_1), \ldots, \max(v_n, v'_n))$$
where \( v, v' \in \mathbb{R}^n_\geq \). In this case \( 0 \in \mathbb{R}^n_\geq \) correspond to the minimum element where its image under \( f^{\times n} : \mathbb{R}^n_\geq \to (\mathbb{R}^{\text{trop}})^n \) corresponds to maximum element, i.e. \( \infty \). Notice that \( g \) and \( f \) both are decreasing functions, hence they respect the tropical structure. Observe that the positive real line \( \mathbb{R}_\geq \) is a group under multiplication. In fact \( \mathbb{R}_\geq \) is the tropicalisation of \( \mathbb{R}_\geq \) viewed as a group under multiplication. This then enables us define the multiplication \( \mathbb{R}_\geq \times \mathbb{R}_\geq \to \mathbb{R}_\geq \) to be the usual multiplication of two real numbers, i.e.

\[
(r, s) \mapsto rs.
\]

It is straightforward to see that \( f \) and \( g \) become maps of tropical (semi)rings.

Our next goal is to make it clear what we mean by the action of real numbers on these spaces.

2.1.2 Flows on \( \mathbb{R}^n_\geq \)

The one to one correspondence

\[
f^{\times n} : \mathbb{R}^n_\geq \to (\mathbb{R}^{\text{trop}})^n
\]

motivates us, and provides us with a tool, to define an action of \( \mathbb{R}_\geq \) on \( \mathbb{R}^n_\geq \). Let us write \( g \) for the inverse of \( f \), i.e. \( g : \mathbb{R}^{\text{trop}} \to \mathbb{R}_\geq \) is given by

\[
g(x) = e^{-x} \quad g(\infty) = 0.
\]

Recall from previous section that we have tropicalisation of any module with projection. In this case, \( \mathbb{R}^{\text{trop}} \) is the same as tropicalisation of \( \mathbb{R} \) with \( \tau \) given by the norm function. This then shows that it is possible to have a multiplication \( \circ : \mathbb{R}^{\text{trop}} \times \mathbb{R}^{\text{trop}} \to \mathbb{R}^{\text{trop}} \) given by

\[
t \circ t' = \begin{cases} t + t' & \text{if } t, t' \in \mathbb{R}, \\
\infty & \text{if } t = \infty, \text{ or } t' = \infty.
\end{cases}
\]

We use this to define the action of the additive group \( \mathbb{R} \) on \( \mathbb{R}^{\text{trop}} \) as \( \mathbb{R} \times \mathbb{R}^{\text{trop}} \to \mathbb{R}^{\text{trop}} \) given by

\[
(r, t) \mapsto r \circ t.
\]

We use \( g \) to define an action of \( \mathbb{R} \times \mathbb{R}_\geq \to \mathbb{R}_\geq \) by requiring that \( g \) respects this action, i.e. \( g \) has to be a map of \( \mathbb{R} \)-modules. This induces the action \( \mathbb{R} \times \mathbb{R}_\geq \to \mathbb{R}_\geq \) given by

\[
(r, v) \mapsto e^{-r}v.
\]

Notice that here we consider the action of \( (\mathbb{R}, +) \) on \( \mathbb{R}_\geq \). It is again clear that the mappings \( f, g \) respect these actions. Notice that it is then quite clear that how to define the corresponding actions of \( \mathbb{R} \) on \( (\mathbb{R}^{\text{trop}})^n \) and \( \mathbb{R}^n_\geq \) respectively. This is done by defining the action component-wise. We investigate this in the next part, where we look at analogous of \( k \)-planes in these spaces.
Note 2.3. Finally we explain the title for this section. The word flow, for a differential topologist, reminds the action of the real numbers under addition on a (smooth) manifold. The above is only one particular flow that we use, and let call it the “standard flow” on \( \mathbb{R}_\geq \).

It is possible to consider a more general setting. Notice that we have defined the mappings \( f \) and \( g \) in a way that they respect the tropical (semi)ring structures. Hence, having any action of the additive group \((\mathbb{R}, +)\) on \( \mathbb{R}_\geq \) will determine a corresponding action on \( \mathbb{R}^{trop} \), hence on \( \mathbb{R}^n_\geq \) and \( (\mathbb{R}^{trop})^n \) respectively. Note that in the case of the standard flow the point \( 0 \in \mathbb{R}^n_\geq \) corresponds to a singular point of the flow.

2.2 Tropical Projective Spaces

The study of tropical projective spaces, i.e. the space of lines in the tropical Euclidean spaces, seems to be the first natural step in attempt to understand the tropical Grassmannian spaces. It turns out that, using our model for \( (\mathbb{R}^{trop})^n \), it is an easy task to identify \( \mathbb{P}^n_{\text{trop}} \) as a set where \( \mathbb{P}^n_{\text{trop}} \) is the set of all lines in \( (\mathbb{R}^{trop})^{n+1} \) which “pass” through the origin. We state the result.

**Lemma 2.4.** There is a one to one correspondence

\[ \mathbb{P}^n_{\text{trop}} \rightarrow \Delta^n \]

where \( \Delta^n \) is the standard \( n \)-simplex, i.e.

\[ \Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_\geq : x_i \geq 0, \sum x_i = 1\}. \]

This is quite straightforward to see, when we use \( \mathbb{R}^{n+1}_\geq \) as our model for \( (\mathbb{R}^{trop})^{n+1} \). For instance, let \( n = 1 \). Then we need to identify all lines in \( \mathbb{R}^{n+1}_\geq \). Notice that in \( \mathbb{R}^{n+1}_\geq \) any point together with the “origin” determines a line. Let \( (a, b) \in \mathbb{R}^2_\geq \). Then the line passing through this point and “reaching” the origin is determined by the orbit of this point under the action of real line, i.e. all points \( e^{-r}(a, b) = (e^{-r}a, e^{-r}b) \) where \( r \in \mathbb{R} \). Notice that \( e^{-r} \) is nothing but a positive real number. Hence, the orbit of \( (a, b) \) is the line passing \( (a, b) \) and the origin. However, notice that this will never reach the origin, and the origin will be a limit point for this line when \( r \) tends to \( +\infty \). This then shows that the set \( \{(a, b) : a > 0, b > 0, a + b = 1\} \) is in one to one correspondence with these line. We need to identify the lines that correspond to the cases with \( a = 0 \) and \( b = 0 \). The \( x \)-axis and the \( y \)-axis then give these two end points. Notice that this look like two point compactification of the real line.

For cases \( n > 2 \) a similar approach gives the result. We only note that the boundary of the simplex will correspond to the lines on \( \partial \mathbb{R}^{n+1}_\geq \) where as its interior points correspond to interior\( (\mathbb{R}^{n+1}_\geq) \).

**Remark 2.5.** The tropical projective space does not seem to be a tropical space, however its correspondence an standard simplex may be an evidence that it is a kind of a variety(?!).
2.3 Subspaces in Tropical Euclidean Spaces

We like to look at the tropical version of the Grassmannian spaces. We note that there
is some work on this from an algebraic-geometric point of view such as

David Speyer, Bernd Sturmfels *The tropical Grassmannian* Adv. Geom. Vol.4 No.3
pp389–411, 2004

But I am not aware of the contents of this work, so I don’t make any comment.
We choose to work with our model $\mathbb{R}^{n+k}_{\geq}$ to study the Grassmannian objects. The
Grassmannian space $G_k(\mathbb{R}^{n+k})$ is the set of $k$-planes in $\mathbb{R}^{n+k}$. We take this approach
and look at the $k$-planes in $\mathbb{R}^{n+k}_{\geq}$.

2.3.1 Linear independence in the tropical sense

In the Euclidean space, it is quite straightforward to identify the $k$-dimensional sub-
spaces of $\mathbb{R}^{n+k}$, i.e. the space spanned by $k$ linear independent vector \{\(v_1, \ldots, v_k : v_i \in \mathbb{R}^{n+k}\)\}.

In tropical space ($\mathbb{R}^{\text{trop}})^{n+k}$ two vectors \(t_1, \ldots, t_n, (t'_1, \ldots, t'_n)\) are linearly depen-
dent in the tropical sense if and only if there exist \(r \in \mathbb{R}\) such that
\[
t_1 + r = t'_1, \ldots, t_n + r = t'_n.
\]

However, our model is much easier to use. More precisely, two vectors \(t_1, \ldots, t_n, (t'_1, \ldots, t'_n) \in \mathbb{R}^{n+k}_{\geq}\) are linearly dependent if there is a real number \(r\) such that
\[
(t_1, \ldots, t_n) = e^{-r}(t'_1, \ldots, t'_n)
\]
i.e.
\[
t_1 = e^{-r}t'_1, \ldots, t_n = e^{-r}t'_n.
\]
This tells that two vectors in $\mathbb{R}^{n+k}_{\geq}$ are linear independent if one is not multiple of
the other one, similar to the notion of the linear independent in the Euclidean sense.
However, the notion of linear combination is quite different in two cases.
Next, we have two identify what is meant by the space spanned by $k$ linear independent
vectors $v_1, \ldots, v_k \in \mathbb{R}^{n+k}_{\geq}$ in the tropical. The addition in $\mathbb{R}^{n+k}_{\geq}$ defined in previous
sections shows that spanning in tropical sense is given by
\[
\text{Span}^{\text{trop}}\{v_1, \ldots, v_k\} = \text{interior}(\text{Cone}\{v_1, \ldots, v_k\})
\]
where \{\(v_1, \ldots, v_k\)\} is an arbitrary set of $k$ vectors in $\mathbb{R}^{n+k}_{\geq}$ and the cone \text{Cone}\{\(v_1, \ldots, v_k\)\} is the cone taken in usual Euclidean sense. For instance, for $k = 2$ the cone on two
vectors is the area between the two lines determined by two vectors. By a $k$-subspace
$C \subset \mathbb{R}^{n+k}_{\geq}$ we mean a cone which is span of $k$ independent vectors, i.e.
\[
C = \text{Span}^{\text{trop}}\{v_1, \ldots, v_k\}
\]
where \{\(v_1, \ldots, v_k\)\} is a linearly independent set.
Notice that there is not a precise notion of dimension here. The reason is that not
every point in \( \mathbb{R}^{n+k} \) is a linear combination of finite number of vectors. The reason for this lies in the way that the tropical addition, and scalar multiplication on \( \mathbb{R}^{n+k} \) are defined. The following observation provides us with a framework to look at this.

**Lemma 2.6.** Let \( u_1, \ldots, u_n \in \mathbb{R}^n \) denote the standard Euclidean basis elements, i.e. \( u_1 = (1, 0, \ldots, 0), \ldots, u_n = (0, \ldots, 0, 1) \). We then have

\[
v \in \text{interior}(\mathbb{R}_n^+) \iff v \in \text{Span}^{\text{trop}}\{u_1, \ldots, u_n\},
\]

where \( \text{interior}(\mathbb{R}_n^+) = \mathbb{R}_n^+ - \partial \mathbb{R}_n^+ \). The space \( \partial \mathbb{R}_n^+ \) is characterised by

\[
(x_1, \ldots, x_n) \in \partial \mathbb{R}_n^+ \iff x_t = 0 \text{ for some } 1 \leq t \leq n.
\]

**Remark 2.7.** We note that according to this lemma, the space \( \mathbb{R}_n^+ \) is not finitely generated as an \((\mathbb{R}, +)\)-set. The reason for this is that we don’t have \( \infty \) in \( \mathbb{R} \). Later on, we will consider the action of \( \mathbb{R}^{\text{trop}} \) on this set, where \( \mathbb{R}_n^+ \) becomes an \( \mathbb{R}^{\text{trop}} \)-module.

Notice that a vector \( v \) is in \( \text{interior}(\mathbb{R}_n^+) \) if all of its component, written in the usual Euclidean basis, are positive. Observe that in particular, if we choose any vector, \( u_i \) such as in the above lemma, then under the correspondence \( \mathbb{R}_n^+ \to (\mathbb{R}^{\text{trop}})^n \) we can see that

\[
\text{Span}^{\text{trop}}\{u_i\} \simeq \mathbb{R}.
\]

Moreover, let us write

\[
\text{Span}^{\text{trop}}\{\hat{u}_i\} \simeq \{\infty\}.
\]

The above lemma together with the notation that just introduced our allows us to formally rewrite the decomposition of \( \mathbb{R}_n^+ \) corresponding to the one given by Remark 2.2. The result reads as following.

**Corollary 2.8.** The space \( \mathbb{R}_n^+ \) has the following decomposition as

\[
\text{Span}^{\text{trop}}\{u_1, \ldots, u_n\} \bigcup \\
\text{Span}^{\text{trop}}\{u_1, \ldots, u_{n-1}, \hat{u}_n\} \bigcup \cdots \bigcup \text{Span}^{\text{trop}}\{\hat{u}_1, u_2, \ldots, u_n\} \bigcup \\
\text{Span}^{\text{trop}}\{u_1, \ldots, u_{n-2}, \hat{u}_{n-1}, \hat{u}_n\} \bigcup \cdots \bigcup \text{Span}^{\text{trop}}\{\hat{u}_1, \hat{u}_2, u_3, \ldots, u_n\} \bigcup \\
\bigcup \cdots \\
\text{Span}^{\text{trop}}\{u_1, \hat{u}_2, \ldots, \hat{u}_n\} \bigcup \cdots \bigcup \text{Span}^{\text{trop}}\{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{n-1}, u_n\} \\
\bigcup \{(0, 0, \ldots, 0)\}.
\]

Here \( \hat{u}_i \) means that the vector \( u_i \) is not in the set. Moreover, under the correspondence

\[
\mathbb{R}_n^+ \to (\mathbb{R}^{\text{trop}})^n
\]

the space \( \text{Span}^{\text{trop}}\{u_1, \ldots, u_{i-1}, \hat{u}_i, \ldots, \hat{u}_j, u_{j+1} \ldots u_n\} \) maps to

\[
\text{Span}^{\text{trop}}\{u_1\} \times \cdots \times \text{Span}^{\text{trop}}\{u_{i-1}\} \times \text{Span}^{\text{trop}}\{\hat{u}_i\} \cdots \times \\
\text{Span}^{\text{trop}}\{\hat{u}_j\} \times \text{Span}^{\text{trop}}\{u_{j+1}\} \times \cdots \times \text{Span}^{\text{trop}}\{u_n\}
\]

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which is the same as
\[ \mathbb{R} \times \cdots \times \mathbb{R} \times \{\infty\} \times \cdots \times \{\infty\} \times \mathbb{R} \times \cdots \times \mathbb{R} \]

where for each \( u_k \) in the spanning set we obtain a copy of \( \mathbb{R} \) at kth position, and for each \( \tilde{u}_i \) we obtain a copy of \( \{\infty\} \) at the ith position.

Finally, we have a little observation which will be important later on when we consider the generalised tropical Grassmannians.

**Lemma 2.9.** Let \( v_1, \ldots, v_k \in \mathbb{R}^n \) be linearly independent with \( k > 1 \). Let \( v \in \text{Span}^{\text{trop}}\{v_{\alpha_1}, \ldots, v_{\alpha_j}\} \) where \( 1 \leq \alpha_1, \ldots, \alpha_j \leq k \) and \( j < k \). Then \( v \notin \text{Span}^{\text{trop}}\{v_1, \ldots, v_k\} \).

In particular, \( v_i \notin \text{Span}^{\text{trop}}\{v_{\alpha_1}, \ldots, v_{\alpha_j}\} \).

This is easy to see, as if \( v \in \text{Span}^{\text{trop}}\{v_{\alpha_1}, \ldots, v_{\alpha_j}\} \) then \( v \) has to belong to boundary of the \( k \)-subspace determined by \( v_1, \ldots, v_k \). The result then follows from Corollary 2.7.

### 2.3.2 Relating \( G_k(\mathbb{R}_n) \) to configuration spaces

We like relate the Grassmannian space \( G_k(\mathbb{R}_n) \) to some configuration spaces. The mapping fails to be an isomorphism. But it at least provides a tool which presumably will help to analyse, and calculate more sophisticated algebraic invariants of these spaces.

We start by recalling the analogous construction in homotopy. Let us fix an arbitrary basis for \( \mathbb{R}^n \). Let \( V \subset \mathbb{R}^n \) be a \( k \)-dimensional subspace. We then can choose a basis for it, say \( \{v_1, \ldots, v_k\} \). The fact that there are linearly independent means that they give rise to \( k \) distinct lines, hence defines a mapping

\[ G_k(\mathbb{R}^n) \to F(P^{n-1}, k), \]

where given any set \( X \) we define the set of configuration of \( n \) point in \( X \) as

\[ F(X, n) = \{(x_1, \ldots, x_n) : x_i \in X, i \neq j \implies x_i \neq x_j\}. \]

The above mapping fails to be a homeomorphism as it is possible to choose \( k \) distinct points in \( P^{n-1} \), or say \( k \) distinct vectors, which are not necessarily linearly independent.

We write \( G_k(\mathbb{R}_n) \) for the set of all \( k \)-subspaces of \( \mathbb{R}_n \). Assume that we have a \( k \)-subspace \( C = \text{Span}^{\text{trop}}\{v_1, \ldots, v_k\} \subset \mathbb{R}_n \). Let \( l_1, \ldots, l_k \) be \( k \) distinct lines determined by \( v_1, \ldots, v_k \) respectively. This determines a mapping

\[ G_k(\mathbb{R}_n) \to F(\mathbb{P}_n^{n-1}, k). \]

Notice that \( \mathbb{P}_n^{n-1} \) is the same as \( \Delta_n^{n-1} \). The fact that \( v_1, \ldots, v_k \) are linearly independent in the tropical sense, implies that non of these vectors falls into the cone generated by the other ones. If we use \( v_1, \ldots, v_k \) to denote those \( k \) points in \( \Delta_n^{n-1} \) that correspond to these lines, we obtain a convex set, where here convex means convex as a subset of \( \mathbb{R}^n \) with its usual metric.

On the other hand, if we choose any convex set in \( \Delta_n^{n-1} \) with \( k \) vertices we obtain \( k \) vector in \( \mathbb{R}_n \) which are independent in the tropical sense. This completes the proof of the following observation.
Theorem 2.10. There is an isomorphism of sets
\[ G_k(\mathbb{R}^n_{\geq}) \rightarrow F(\Delta^{n-1}, k)^{\text{convex}} \]
where \( F(\Delta^{n-1}, k)^{\text{convex}} \) refers to a subset of \( F(\Delta^{n-1}, k) \) whose points are in one to one correspondence with convex subset of \( \Delta^{n-1} \) with \( k \) vertices.

2.4 \( \mathbb{R}^n_{\geq} \) as an \( \mathbb{R}^{\text{trop}} \)-module

Recall from previous sections that \( \mathbb{R} \times \mathbb{R}^n_{\geq} \rightarrow \mathbb{R}^n_{\geq} \) gives \( \mathbb{R}^n_{\geq} \) structure of an \((\mathbb{R}, +)\)-set. This action is not compatible when we consider the field of real lines, with its usual addition and multiplication. However, it is possible to obtain structure of an \( \mathbb{R}^{\text{trop}} \)-module on \( \mathbb{R}^n_{\geq} \).

First, define \( \mathbb{R}^{\text{trop}} \times \mathbb{R}^n_{\geq} \rightarrow \mathbb{R}^n_{\geq} \) by
\[
(r, t) \mapsto e^{-r}t, \\
(+\infty, t) \mapsto 0.
\]

Recall that \( \mathbb{R}^{\text{trop}} \) has a (semi)ring structure when regarded as \((\mathbb{R} \cup \{+\infty\}, \min, +)\) whereas \( \mathbb{R}^n_{\geq} \) has the tropical structure when regarded as \((\mathbb{R}^n_{\geq}, \max, \cdot)\) where \( \cdot \) denotes the usual product. We then define the action \( \mathbb{R}^{\text{trop}} \times \mathbb{R}^n_{\geq} \rightarrow \mathbb{R}^n_{\geq} \) to be the component-wise action, i.e.
\[
(r, (t_1, \ldots, t_n)) \mapsto (e^{-r}t_1, \ldots, e^{-r}t_n), \\
(+\infty, (t_1, \ldots, t_n)) \mapsto (0, \ldots, 0).
\]

This definition is very similar to the previous one. It does not change the notion of the linear independence. However, there is slight difference in the notion of span. We write \( \text{Span}^{\text{Trop}} \) to distinguish it from \( \text{Span}^{\text{trop}} \).

Lemma 2.11. Suppose \( v_1, \ldots, v_k \in \mathbb{R}^n_{\geq} \). Then
\[
\text{Span}^{\text{Trop}}\{v_1, \ldots, v_k\} = \text{Cone}\{v_1, \ldots, v_k\}.
\]

Hence, a slight change in the ground set acting on \( \mathbb{R}^n_{\geq} \), i.e. replacing \( \mathbb{R} \) with \( \mathbb{R}^{\text{trop}} \) has the effect that it adds the limit points of a cone to it. As a corollary \( \mathbb{R}^n_{\geq} \) becomes finitely generated over \( \mathbb{R}^{\text{trop}} \). We have the following.

Corollary 2.12. Suppose \( u_1, \ldots, u_n \) denote the usual basis elements for \( \mathbb{R}^n_{\geq} \). We then have
\[
\mathbb{R}^n_{\geq} = \text{Span}^{\text{Trop}}\{u_1, \ldots, u_n\}.
\]

Here, any point on \( \partial \mathbb{R}^n_{\geq} \) will have tropical coordinates which are formed of real numbers, and \(+\infty\). Moreover, \( \{u_1, \ldots, u_n\} \) is the only set of vectors satisfying this property, i.e. if there is any set of vectors \( \{v_1, \ldots, v_n\} \) such that
\[
\mathbb{R}^n_{\geq} = \text{Span}^{\text{Trop}}\{v_1, \ldots, v_n\},
\]
then each \( v_i \) will be a re-scaling of of \( u_j \) for unique \( 1 \leq j \leq n \).
As an example, consider $\mathbb{R}^2_{\geq}$ and consider the point $(1, 0)$ which is on its boundary. The Corollary 2.7 implies that it cannot be written as any linear combination of two vectors in $\mathbb{R}^2_{\geq}$ when regarded as an $\mathbb{R}$-set. However, as an $\mathbb{R}_{\text{trop}}$-module, we may write

$$(1, 0) = e^0(1, 0) + \infty(0, 1)$$

i.e. as a vector in $\mathbb{R}^2_{\geq}$ the point $(1, 0)$ may be written as the column vector

$$\begin{bmatrix} 1 \\ +\infty \end{bmatrix}.$$ 

2.4.1 The space $G^\text{Trop}_k(\mathbb{R}^n_{\geq})$

Likewise the space $G^k(\mathbb{R}^n_{\geq})$ we define $G^\text{Trop}_k(\mathbb{R}^n_{\geq})$ to be the set of all $k$-subspaces in $\mathbb{R}^n_{\geq}$ when regarded as an $\mathbb{R}_{\text{trop}}$-module. We say $C \subseteq \mathbb{R}^n_{\geq}$ is a $k$-subspace if there are $k$ linearly independent vector $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq}$ such that

$$C = \text{Span}^\text{Trop}\{v_1, \ldots, v_k\}.$$ 

We may refer to $G^\text{Trop}_k(\mathbb{R}^n_{\geq})$ as the generalised tropical Grassmannian space. Notice that there is a one to one correspondence

$$G_k(\mathbb{R}^n_{\geq}) \rightarrow G^\text{Trop}_k(\mathbb{R}^n_{\geq})$$

given by

$$C \mapsto \overline{C}$$

where $\overline{C}$ denotes the closure of $C$; i.e. $\overline{C} = C \cup \partial C$. The inverse mapping

$$G^\text{Trop}_k(\mathbb{R}^n_{\geq}) \rightarrow G^k(\mathbb{R}^n_{\geq})$$

given by

$$C \mapsto \text{interior}C = C - \partial C.$$ 

Accordingly we obtain the following description of $G^\text{Trop}_k(\mathbb{R}^n_{\geq})$.

**Theorem 2.13.** There is a one to one correspondence

$$G^\text{Trop}_k(\mathbb{R}^n_{\geq}) \rightarrow F(\Delta^{n-1}, k)^{\text{convex}}.$$ 

**Remark 2.14.** Before proceeding further, we like to draw the reader’s attention to an essential difference between $G^k(\mathbb{R}^n_{\geq})$ and $G^\text{Trop}_k(\mathbb{R}^n_{\geq})$ in one hand and their Euclidean analogous $G^k(\mathbb{R}^n)$ on the other hand. In the Euclidean space $\mathbb{R}^n$ any set of $n$-linearly independent set will span $\mathbb{R}^n$, however in $\mathbb{R}^n_{\geq}$ either as a $(\mathbb{R}, +)$ or as an $\mathbb{R}_{\text{trop}}$-module the only option for such a choice is provided by the standard basis. Although, according to Lemma 2.6 as an $(\mathbb{R}, +)$-set this it is not possible to generate all of $\mathbb{R}^n_{\geq}$ in tropical sense.

For instance, consider $\mathbb{R}^3_{\geq}$ and let $C \in G^k(\mathbb{R}^3_{\geq})$ be defined as

$$C = \text{Span}^\text{trop}\{(1, 0, 0), (0, 1, 0), (1, 1, 2)\}.$$
This is a 3-subspace in $\mathbb{R}_3^n$, and yet it is not equal to $\mathbb{R}_3^n$. We note that all of those three vectors defining $C$ are linearly independent. We can also consider to $\overline{C} \in G_3^{\text{trop}}(\mathbb{R}_3^n)$ where $C \neq \mathbb{R}_3^n$.

A consequence of this is that we may choose another vector $v \in \mathbb{R}_3^n$ which does not belong to $C$, i.e the set

$$\{(1,0,0),(0,1,0),(1,1,2),v\}$$

is a linearly independent set in the tropical sense. This determines a cone formed by 4 vectors which can not be generated by any 3 vectors. Hence we obtain a 4-subspace in $\mathbb{R}_3^n$ giving rise to an element of $G_4(\mathbb{R}_3^n)$.

In general, we may choose $k > n$ when we consider $G_k(\mathbb{R}_3^n)$. In fact $k$ can be any arbitrary number.

It is quite interesting to see how a $k$-subspace in $\mathbb{R}_n^n$ maps under the tropical isomorphism

$$\mathbb{R}_n^n \to (\mathbb{R}^{\text{trop}})^n.$$  

Recall from previous sections that $(\mathbb{R}^{\text{trop}})^n$ is disjoint union of $2^n$ copies of $\mathbb{R}^m$ with $0 \leq m \leq n$. Now assume that $C \in G_k(\mathbb{R}_n^n)$, i.e.

$$C = \text{interior}(\text{Cone}\{v_1, \ldots, v_k\})$$

where $v_1, \ldots, v_k \in \mathbb{R}_n^n$ are linearly independent(in the tropical sense). Notice that in this case $C$ is given by the interior of the cone, which implies that $C \subset \text{interior}(\mathbb{R}_n^n)$. We now that interior($\mathbb{R}_n^n$) maps to $\mathbb{R}_n^n \subset (\mathbb{R}^{\text{trop}})^n$. Moreover, note that the mapping $f : \mathbb{R}_n^n \to (\mathbb{R}^{\text{trop}})^n$ is continuous when restricted to interior($\mathbb{R}_n^n$). This implies that $C$ also maps into $\mathbb{R}_n^n \subset (\mathbb{R}^{\text{trop}})^n$.

Next, we consider $C \in G_k^{\text{trop}}(\mathbb{R}_n^n)$ and its image under the $f : \mathbb{R}_n^n \to (\mathbb{R}^{\text{trop}})^n$.

Notice that if $C \in G_k^{\text{trop}}(\mathbb{R}_n^n)$ then it is a closed cone, where by being closed we mean closed as a set in $\mathbb{R}_n^n$, viewed as a topological space with its topology inherited from the standard topology on $\mathbb{R}^n$.

If $C \subset \text{interior}(\mathbb{R}_n^n)$ then according to the previous case, all of $C$ maps to only one component of $(\mathbb{R}^{\text{trop}})^n$, namely to $\mathbb{R}^n$.

Another possibility is that $C \cap \partial \mathbb{R}_n^n \neq \phi$. In this case then the image of $C$ under $f : \mathbb{R}_n^n \to (\mathbb{R}^{\text{trop}})^n$ will land in more than one factor of $(\mathbb{R}^{\text{trop}})^n$ viewed as a disjoint union. The following provides us with an example.

**Example 2.15.** Consider the space $\mathbb{R}_3^n$, together with vectors $v_1 = (1,0,1)$ and $v_2 = (0,1,1)$. This determines

$$C = \text{Span}^{\text{trop}}\{v_1, v_2\}$$

as an element of $G_2^{\text{trop}}(\mathbb{R}_3^n)$. Let

$$L_1 = \{(t,0,t) : t > 0\}, \ L_2 = \{(0,t,t) : t > 0\},$$

i.e. $L_i \cup \{(0,0,0)\}$ is the line determined by $v_i$. It is then clear that

$$\partial C = L_1 \cup L_2.$$
Under the correspondence $f : \mathbb{R}^3_\geq \rightarrow (\mathbb{R}^{\text{trop}})^3$ we have

$$f(L_1) \subset \mathbb{R} \times \{\infty\} \times \mathbb{R}$$
$$f(L_2) \subset \{\infty\} \times \mathbb{R} \times \mathbb{R}$$
$$f(0,0,0) = (\infty, \infty, \infty).$$

The image of $C$ under $f$ is an example of a 2-subspace in $(\mathbb{R}^{\text{trop}})^3$.

In general, suppose $C \in G_{k}^{\text{Trop}}(\mathbb{R}^n_\geq)$ with $C = \text{Span}_{\text{Trop}}\{v_1, \ldots, v_k\}$. If $C \cap \partial \mathbb{R}^n_\geq \neq \emptyset$ then there are $\alpha_i \in \{1, \ldots, k\}$ with $v_{\alpha_i} \in \partial \mathbb{R}^n_\geq$. Recall from Corollary 2.8 that $\mathbb{R}^n_\geq$ has a decomposition as

$$\text{Span}_{\text{Trop}}\{u_1, \ldots, u_n\} \sqcup \text{Span}_{\text{Trop}}\{u_1, \ldots, u_{n-1}, \hat{u}_n\} \sqcup \cdots \sqcup \text{Span}_{\text{Trop}}\{\hat{u}_1, u_2, \ldots, u_n\} \sqcup \text{Span}_{\text{Trop}}\{u_1, \ldots, u_{n-2}, \hat{u}_{n-1}, \hat{u}_n\} \sqcup \cdots \sqcup \text{Span}_{\text{Trop}}\{\hat{u}_1, \hat{u}_2, u_3, \ldots, u_n\} \sqcup \text{Span}_{\text{Trop}}\{u_1, \hat{u}_2, \ldots, \hat{u}_n\} \sqcup \cdots \sqcup \text{Span}_{\text{Trop}}\{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{n-1}, u_n\} \sqcup \{(0,0,\ldots,0)\}.$$  

In this decomposition the first factor, i.e. $\text{Span}_{\text{Trop}}\{u_1, \ldots, u_n\}$ corresponds to $\text{interior}(\mathbb{R}^n_\geq)$, and the other factors correspond to $\partial \mathbb{R}^n_\geq$. Hence, each of $v_{\alpha_i}$ will belong to one, and only one, of the factors in the above decomposition. For instance, assume

$$v_{\beta_1}, v_{\beta_2}, \ldots, v_{\beta_j} \in \text{Span}_{\text{Trop}}\{u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_n\} =: S.$$  

Notice that apart from $\{(0,0,\ldots,0)\}$, any other factor in the above decomposition is an open set, when viewed as a subspace of $\mathbb{R}^n$ with its usual topology. This then implies that

$$\text{Span}_{\text{Trop}}\{v_{\beta_1}, v_{\beta_2}, \ldots, v_{\beta_j}\} \subset \text{Span}_{\text{Trop}}\{u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_n\}.$$  

Applying Corollary 2.8 shows that $\text{Span}_{\text{Trop}}\{v_{\beta_1}, v_{\beta_2}, \ldots, v_{\beta_j}\}$ maps into the factor of $(\mathbb{R}^{\text{trop}})^n$ corresponding to $S$. This then completely determines where $C \cap \partial \mathbb{R}^n_\geq$ maps under the tropical isomorphism $f : \mathbb{R}^n_\geq \rightarrow (\mathbb{R}^{\text{trop}})^n$. This concludes our notes on the generalised Grassmannian tropical spaces.

### 2.4.2 k-subspaces of $(\mathbb{R}^{\text{trop}})^n$.

We like to analyse the those subspaces of $(\mathbb{R}^{\text{trop}})^n$ which are in the images of $G_{k}^{\text{Trop}}(\mathbb{R}^n_\geq)$ and map to more than one factor in the disjoint union decomposition for $(\mathbb{R}^{\text{trop}})^n$.

We define what is meant by a k-subspace in $(\mathbb{R}^{\text{trop}})^n$. Recall that an example of a 2-subspace was given the previous section.
In order to proceed, we need to fix an order on the disjoint union decomposition for \((\mathbb{R}^{trop})^n\). Let \(I = (i_1, \ldots, i_n)\) with each of its entries belonging to \(\{0, 1\}\). By the set \((\mathbb{R}^{trop})^I\) we mean a product of copies of the real line \(\mathbb{R}\), and the singleton \(\{\infty\}\) in the following way: if \(i_j = 1\) then we have a copy of \(\mathbb{R}\), and if \(i_j = 0\) then we have a copy of \(\{\infty\}\). For example, for \(n = 2\), we have \((\mathbb{R}^{trop})^2_{(1,0)} = \mathbb{R} \times \{\infty\}\). It is then clear that
\[
(\mathbb{R}^{trop})^n = \bigcup_{I \in \{0,1\}^n} (\mathbb{R}^{trop})^I.
\]
Moreover, let \(|I| = \sum 2^{i_j} + 1\). We then will refer to \((\mathbb{R}^{trop})^I\) as the \(|I|\)-th factor of \((\mathbb{R}^{trop})^n\). This also induces an order on these sequences (really the binary expansion of positive natural numbers) by
\[
I > J \iff |I| > |J|.
\]
This is the same as the the lexicographic order on the sequences \(I\) and induces an order on the factors of \((\mathbb{R}^{trop})^n\) as following
\[
(\mathbb{R}^{trop})^n_I \leq (\mathbb{R}^{trop})^n_J \iff I \leq J.
\]
Next, let \(k > 0\) and let \(k = (k_1, \ldots, k_{2^n}) \in \mathbb{Z}^{2^n}\) be a sequence of nonnegative integers, with the most left nonzero entry equal to \(k\). Consider a collection of spaces \(K = \{K_i : 1 \leq i \leq 2^n\}\) where \(K_i\) is a subspace of the \(i\)-th factor of \((\mathbb{R}^{trop})^n\) with
\[
K_i = \text{Span}^{trop}\{v^i_1, \ldots, v^i_{k_i}\}
\]
and \(\{v^i_1, \ldots, v^i_{k_i}\}\) being a linear independent set in the tropical sense. Moreover, we set the span of the empty set to be the empty set. We call \(K\) as \(k\)-subspace, if there is a \(k\)-subspace \(C \in G^k_{\mathbb{Trop}}(\mathbb{R}^n)\) such that \(f(C)\) maps into \((\mathbb{R}^{trop})^n\) with its image in different factors of \((\mathbb{R}^{trop})^n\) being given by \(K_i\)'s.

**Remark 2.16.** It is possible to give a more explicit account of the above calculations. In order to do this, we need to label different components of \(\mathbb{R}^n\). It is similar to what we did in above. Let \(I\) be a sequence of length \(n\) whose entries are either 1 or 0. Let \(u_1, \ldots, u_n\) denote the usual Euclidean basis for \(\mathbb{R}^n\). Let \((\mathbb{R}^n)_I \subset \mathbb{R}^n\) be given by a product of copies of the open real half line \(\mathbb{R}_+\) and the singleton \(\{0\}\), where for \(i_j = 1\) we have a copy of \(\mathbb{R}_+\) at \(j\)th position, and for \(i_j = 0\) we have a copy of \(\{0\}\) at the \(j\)th position. For example, in the case of \(n = 2\) we have \((\mathbb{R}^2)_{(1,0)} = \mathbb{R}_+ \times \{0\}\) which is the \(x\)-axis without \(\{(0,0)\}\). We refer to \((\mathbb{R}^n)_I\) as the \(|I|\)-th component of \(\mathbb{R}^n\). Notice that \((\mathbb{R}^n)_I = \text{Span}^{trop}\{e_1, \ldots, e_n\}\) where \(e_j = u_j\) if \(i_j = 1\) and \(e_j = u_j\) if \(i_j = 0\). Hence, according to Corollary 2.8 we have
\[
\mathbb{R}^n = \bigcup_{I \in \{0,1\}^n} (\mathbb{R}^n)_I.
\]
It is now evident that the \(|I|\)-th component of \((\mathbb{R}^n)\) maps homeomorphically onto the \(|I|\)-th factor of \((\mathbb{R}^{trop})^n\).

Now assume that \(C \in G^k_{\mathbb{Trop}}(\mathbb{R}^n)\) with \(C \cap \partial \mathbb{R}^n \neq \phi\). Let \(C_I = C \cap (\mathbb{R}^n)_I\) be the \(|I|\)-th face of \(C\). Then \(C_I\) maps homeomorphically into the \(|I|\)-th factor of \((\mathbb{R}^{trop})^n\) under the tropical isomorphism
\[
\mathbb{R}^n \rightarrow (\mathbb{R}^{trop})^n.
\]
### 2.5 Tropical Matrices, Tropical Orthogonal Matrices

An alternative description of Grassmannian spaces is provided by viewing them as the orbit space of two orthogonal groups acting on another one. This is our main motivation study “tropical matrices”.

Let $M_{m,n}(\mathbb{R}_>)$ be the set of all $m \times n$ matrix whose entries are elements in $\mathbb{R}_>$. This set admits structure of a semi-ring induced by the addition and multiplication in $\mathbb{R}_>$. More precisely, for $A \in M_{m,n}(\mathbb{R}_>)$ let $A_{ij}$ denote the $(i, j)$ entry of $A$. Then for $A, B \in M_{m,n}(\mathbb{R}_>)$ we define the addition, denoted with $\oplus$, by

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \max(A_{ij}, B_{ij}).$$

If $A \in M_{m,p}(\mathbb{R}_>)$ and $B \in M_{p,n}(\mathbb{R}_>)$, then the multiplication, denoted with $\odot$, is defined by

$$(A \odot B)_{ij} = \bigoplus_{k=1}^p A_{ik} \odot B_{kj} = \max(A_{ik}B_{kj} : 1 \leq k \leq p).$$

Under these operations, the identity element for $\oplus$ is the zero matrix, whereas the identity matrix for $\odot$ is the identity matrix. Moreover, notice that $M_{m,n}(\mathbb{R}_>)$ is an $\mathbb{R}_{\text{trop}}$-module.

One may ask whether any matrix in $M_{n,n}(\mathbb{R}_>)$ is invertible. The answer is positive, and an example is provided by the set of all diagonal matrices whose diagonal does not have any nonzero element. This is a “universal example” of such matrices as the following theorem confirms, by giving a complete classification of all such matrices.

**Theorem 2.17.** Let $A \in M_{n,n}(\mathbb{R}_>)$ be a matrix with right $\odot$-inverse with, i.e. there exists $B \in M_{n,n}(\mathbb{R}_>)$ such that $A \odot B = I$. Then there exists $\sigma \in \Sigma_n$ and a diagonal matrix $D = (D_1, \ldots, D_n) \in M_{n,n}(\mathbb{R}_>)$ whose diagonal entries are nonzero, and

$$A = (D_{\sigma^{-1}(1)}, \ldots, D_{\sigma^{-1}(n)}).$$

Here $\Sigma_n$ is the permutation group on $n$ letters. Moreover, the matrix $B$ is determined by

$$B_{ij} = \begin{cases} 
0 & \text{if } A_{ji} = 0 \\
\frac{1}{A_{ji}} & \text{if } A_{ji} \neq 0.
\end{cases}$$

Notice that according to this theorem any matrix with right inverse also has a left inverse and they are the same. This is again straightforward to see this once we observe that

$$(A \odot B)_{ii} = \max(A_{ik}B_{ki} : 1 \geq k \geq n) = 1,$$

$$(A \odot B)_{ij} = \max(A_{ik}B_{kj} : 1 \geq k \geq n) = 0 \text{ for } i \neq j.$$  

For instance, let $i = 1$. The fact that $(A \odot B)_{11} = 1$ implies that there exists $k$ such that $A_{1k} = \frac{1}{B_{11}} \neq 0$. Combining $A_{1k} \neq 0$ together with $(A \odot B)_{ij} = 0$ for $i \neq j$ implies that $B_{kj} = 0$ for $j \neq i$. Applying this for each row of $A \odot B$ completes the proof.

Next, we consider the problem of determining tropical orthogonal matrices. Recall that in the Euclidean case, the orthogonal matrix $O(n)$ is an $n \times n$ matrix whose
each column viewed has a vector in $\mathbb{R}^n$ has unit norm, i.e. lies on the $(n - 1)$-sphere $S^{n-1} = \{ x \in \mathbb{R}^n : ||x|| = 1 \}$, and is perpendicular to all other columns. We may define the tropical inner product $(\cdot, \cdot)_{\text{trop}} : \mathbb{R}^n_\geq \times \mathbb{R}^n_\geq \to \mathbb{R}_{\text{trop}}$ by

$$
\langle v, w \rangle_{\text{trop}} = \oplus_{i=1}^n v_i \circ w_i = \max(v_i w_i : 1 \geq i \geq n)
$$

with $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. In particular, we have the tropical norm on $\mathbb{R}^n_\geq$ given by

$$
||x||_{\text{trop}} = (\max(x_i^2 : 1 \leq i \leq n))^{1/2}.
$$

In particular, the tropical circle is given by

$$
S^1_{\text{trop}} = \{ x \in \mathbb{R}^2_\geq : ||x||_{\text{trop}} = 1 \}
= \{ (1, x_2) \in \mathbb{R}^2 : x_2 \leq 1 \} \cup \{ (x_1, 1) : x_1 \leq 1 \}.
$$

More generally, the tropical $n$-sphere is given by

$$
S^n_{\text{trop}} = \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_\geq : ||x||_{\text{trop}} = 1 \}
= \bigcup_{i=1}^{n+1} \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_\geq : x_i = 1, j \neq i \implies x_j \leq 1 \}.
$$

Moreover, these spaces admit a tropical structure.

**Lemma 2.18.** The tropical sphere $S^n_{\text{trop}}$ together with the maximum operation, inherited from $\mathbb{R}^{n+1}_\geq$, is a tropical space with the identity element for this operation given by $(1, 1, \ldots, 1)$.

It is quite tempting to see what the analogous of the orthogonal group will be. It is quite easy to determine form of such matrices. The reason is provided with the following lemma.

**Lemma 2.19.** Let $A = (A_1, \ldots, A_n) \in M_{n,n}(\mathbb{R}_\geq)$ where $A_i$ denotes the $i$-th column of $A$. Suppose $||A_i||_{\text{trop}} = 1$ and $(A_i, A_j)_{\text{trop}} = 0$ for $i \neq j$. Then $A$ has the same columns as the identity matrix.

Let $O(n)_{\text{trop}}$ be the set of all matrices identifies by the above lemma, i.e. set of all tropical orthogonal matrices.

**Lemma 2.20.** Let $\Sigma_n$ denote the permutation group on $n$ letters. Let the action $\Sigma_n \times M_{m,n}(\mathbb{R}_\geq) \to M_{m,n}(\mathbb{R}_\geq)$ be given by

$$(\sigma, (A_1, \ldots, A_n)) \mapsto (A_{\sigma^{-1}(1)}, \ldots, A_{\sigma^{-1}(n)}),$$

where $A = (A_1, \ldots, A_n)$ is an arbitrary $m \times n$ matrix written in a column form. Then $O(n)_{\text{trop}}$ is given by the orbit of the identity matrix under the action $\Sigma_n \times M_{n,n}(\mathbb{R}_\geq) \to M_{n,n}(\mathbb{R}_\geq)$.  

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Notice that there is an inclusion, in fact a map of monoids,
\[ O(n) \to GL(\mathbb{R}_\geq, n). \]

The above lemma tells us that the action of \( O(n) \times M_{n,n}(\mathbb{R}_\geq) \to M_{n,n}(\mathbb{R}_\geq) \) given by the tropical matrix multiplication, will be only the permutation of the rows of a given matrix. Let us write \( GL(\mathbb{R}_\geq, n) \) for the set of all tropical \( n \times n \) invertible matrices. For \( A \in GL(\mathbb{R}_\geq, n+k) \) we write
\[ A = \begin{pmatrix} A_{nn} & B \\ C & A_{kk} \end{pmatrix} \]
where \( A_{nn} \) is the \( n \times n \) and \( A_{kk} \) is a \( k \times k \) block. This allows one to define the action of \( O(n)^{\text{trop}} \times O(k)^{\text{trop}} \) on \( GL(\mathbb{R}_\geq, n+k) \). One then will guess that there should be a one to one correspondence
\[ \frac{GL(\mathbb{R}_\geq, n+k)}{O(n)^{\text{trop}} \times O(k)^{\text{trop}}} \to G_k(\mathbb{R}^{n+k}). \]

Finally, notice that \( O(n)^{\text{trop}} \) is a monoid under the tropical matrix multiplication.

### 2.5.1 Idempotents

Suppose \( M \) is an arbitrary monoid. An element \( a \in M \) is idempotent if \( a^2 = a \). We consider to the problem of determining the idempotent in the monoid \( M_{n,n}(\mathbb{R}_\geq) \). The result reads as following.

**Lemma 2.21.** Let \( A \) be an idempotent \( n \times n \) matrix entries from \( \mathbb{R}_\geq \). Then \( A \) satisfies the following conditions
\[ A_{ii} \leq 1 \]
\[ A_{ik}A_{ki} \leq \min(A_{ii}, A_{kk}) \text{ if } i \neq k. \]

The proof is straightforward once we consider the diagonal elements. The equation \( A \odot A = A \) implies that
\[ \max(A_{ik}A_{ki} : 1 \leq k \leq n) = A_{ii}. \]
For instance, this implies that \( A_{ii}^2 \leq A_{ii} \) which means that \( A_{ii} \leq 1 \). The other inequality is obtained in a similar fashion, by comparing the equations for \( A_{ii} \) and \( A_{kk} \).

### 2.5.2 Stablisation

We consider the idea of the infinite dimensional orthogonal tropical matrices. Notice that for any \( m, n \) there is a mapping \( M_{m,n}(\mathbb{R}_\geq) \to M_{m+1,n+1}(\mathbb{R}_\geq) \) given by
\[ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \]
In particular, this induces a mapping \( O(n)^{\text{trop}} \to O(n+1)^{\text{trop}} \). We then define the analogous of the infinite orthogonal group by
\[ O^{\text{trop}} = \text{colim } O(n)^{\text{trop}}. \]
This object inherits a monoid structure induces from the monoid structure on the finite dimensional tropical orthogonal monoids. One then hopes that this will give a characterisation of the infinite dimensional Grassmannians.

2.5.3 Comments on Tropical Bundle Theory

The algebraic topology of fibre bundles with a given topological space $F$ as the fibre, is understood in terms of the classifying space of the groups of automorphisms of $F$. By analogy one may consider to fibre bundle theory of surjections $E \rightarrow B$ whose fibres are given by copies of the tropical space $\mathbb{R}_n^{\geq}$. This then makes it quite reasonable to consider the classifying space $BO(n)^{trop}$ of the tropical orthogonal monoids $O(n)^{trop}$ where these are monoids under the tropical matrix multiplication. The classifying space functor is defined for monoid (in fact for topological monoids which carry a weaker structure). Hence, one may observe that the $\mathbb{R}_n^{\geq}$-bundles are classified in terms of mapping into $BO(n)^{trop}$.

The interest in such theory, and the theory of associated characteristic classes, seems to come from the theory of singularities. We note that a canonical example for a tropical bundle will be “tangent bundle” of a tropical manifold. This then motivates one to claim that the classification of these singularities might be done in terms of the characteristic classes of the associated tangent bundle.

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