The flow around a macroscopical body by a colloid solution and the drag crisis

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Abstract

The motion of colloids in the flow field of a viscous liquid is investigated. The small colloid size compared to the macroscopical scale of the flow allow to calculate their velocity relative to that of the liquid. If the inner colloid density is larger than the density of the liquid the flow field has the domains where the colloid velocity is close to the liquid velocity. But in the domains with a strong braking of the liquid velocity the colloids are accelerated relative to the liquid. This effect is used for the qualitative explanation of the drag reduction in the flow around macroscopical bodies and in the pipes.

More than 60 years ago [1] it was discovered that a small concentration of polymers in liquid solution essentially decrease the drag in pipes. This effect is used in the oil transportation. There is a lot of theoretical and experimental publications on this subject. However there is no accepted qualitative interpretation of the physical origin for the observed drag reduction. Rather detailed paper [2] use a complicate theory of the polymer deformation and its dependence on the inner strain but does not state its connection with the liquid flow. The recent work [3] shows a bad agreement of the performed experiment with existing theories. The large amount of publications is devoted to the rheological properties of the concentrated polymer solutions see e.g. the review [4], or the book [5]. We shall not discuss this complicate subject suggesting that in the dilute polymer solutions the main problem is connected with the interaction of the separate polymer and the flow field of the liquid. The more subtle effects of the polymer deformation may be important for more exact quantitative description.

In this note we consider first the more investigated problem of the flow around an immobile macroscopical body by Newtonian viscous liquid and its modification due to the dilute solution of comparatively large polymer molecules. The description of the large polymer having thousands of the connected links is well developed (see e.g.[7] or [8]). The equilibrium state is represented by a coil having on average a spherical form with the radius \( l = \sqrt{\frac{Na^2}{6}} \). Where \( N \) is the number of the links, and \( a \) is their length. The molecular weight of such a coil is much more then the molecular weight of the solvent. Therefore it is possible to neglect the Brownian motion because the polymer thermal velocity is small.
to the considering flow velocities. To simplify the problem further we shall treat the polymer as a large spherical colloid weakly compressible. The largest scale of the motion is given by the size of the macroscopic body $L$ which is much more then the average distance between the nearest colloids $c^{-1/3}$ where $c$ is the small volume concentration of the colloids. This distance is large compare to the size of the colloid

$$L \gg c^{-1/3} \gg l$$

(1)

1 The motion of the colloid relative to the liquid

At the low colloid concentration it is possible to use the linear approximation (see e.g [9]). Let us consider one colloid in the flow field. At the large distances compare to the colloid size $l$ the liquid flow can be considered as uniform. The equations of the motion have the form of the standard equations for an incompressible viscous liquid. We assume that the colloid is incompressible also. At the colloid boundary the liquid velocities and that of the colloid surface are equal and the tensor of the momentum transfer is continuous.

The macroscopic motion connected with the large scale $L$ is a kind of an external force acting on the separate colloid and we can use the well known result (see e.g. [6]) for the calculation of the force acting on the body (colloid) immersed in the liquid. If the body is moved with the liquid this force is equal to $\rho V_0 \frac{dV}{dt}$, where $\rho$ is the density of the liquid, $V_0 = \frac{4\pi l^3}{3}$ is the volume of the colloid. But really one must take into account the relative motion and add $-m_{i,k} \frac{d}{dt}(w^P_i - v^l_i)$, where $m_{i,k}$ is the tensor of the associated masses. The relative motion gives also Stokes friction force $-6\pi \eta l (w^P_i - v^l_i)$. Summing the various contributions one get the force acting on the colloid

$$\rho V_0 \frac{dV}{dt} = \rho V_0 \frac{dV}{dt} - m_{i,k} \frac{d}{dt}(w^P_i - v^l_i) - 6\pi \eta l (w^P_i - v^l_i)$$

(2)

Using Stokes law suggests that the time of the "viscous" relaxation $\frac{\rho V_0}{\eta}$ is small compare to the "hydrodynamical" time $L/U$, where $U$ is the velocity of the
Stokes force is proportional to the first power of the colloid size and therefore the terms with the accelerations are comparatively small. Therefore the zero approximation gives \( w_i^p = v_i^l \) where \( v_i^l \) is the local fluid velocity at a small distance \( l \) from the colloid.

The next approximation can be obtained by introducing the zero approximation in (2) that gives

\[
w_i^p - v_i^l = -(\rho^p - \rho^l) \frac{2l^2}{9\eta} \frac{dv_i^l}{dt} \tag{3}
\]

We shall use this approximation for the relative velocity of the colloid. This local "microscopic" motion gives the contribution to the averaged equations of motion on the distances of the order \( c^{-1/3} \). On the contrary neglecting this contribution the colloids have no effect on the averaged equations of motion.

In Stokes approximation the force acting on unit area of the spherical colloids surface is constant \( F_i = -\frac{3}{2l^2}(w_i^p - v_i^l) \) (see e.g. [6]). Therefore the colloid deformations are absent. In order to find the deformations one needs to consider Oseen corrections connected with the nonlinear inertial terms. The procedure to find the corrections in Reynolds number

\[
Re = \frac{|\vec{u}^p - \vec{v}^l|}{\eta \rho^l} \tag{4}
\]

made in [10, 11] can be found in [6]. It is necessary to investigate the solution of the equation

\[
(\vec{u}^p \nabla)\sigma^l = -\frac{1}{\rho^p} \vec{\nabla}p + \nu \Delta \sigma^l \tag{5}
\]

where \( \nu = \frac{\eta}{\rho^l} \) is the kinematic viscosity, \( \vec{u} \) is the relative velocity equal to its constant value far from the coil. In the external flow field there are two domains: the near one at \( r \ll l/Re \) and the far at \( r \gg l \), overlapping at \( l/Re \gg r \gg l \). In the near domain the starting approximation coincides with Stokes solution, in the far domain the starting is Oseen approximation \( \vec{u} = const \). The sewing of the appropriate solutions in the overlapping region gives the corrections to Stokes solution \( v^l(1) \)

\[
v_i^{(2)} = \frac{3Re}{8} v_i^{(1)} + \frac{3Re}{32} \left(1 - \frac{1}{r'}\right) \left(2 + \frac{1}{r'} + \frac{1}{(r')^2}\right) \left(1 - 3\cos^2 \vartheta\right) \tag{6}
\]

\[
v_i^{(2)} = \frac{3Re}{8} v_i^{(1)} + \frac{3Re}{32} \left(1 - \frac{1}{r'}\right) \left(4 + \frac{1}{r'} + \frac{1}{(r')^2} + \frac{2}{(r')^2}\right) \sin \vartheta \cos \vartheta \tag{7}
\]

Here the spherical coordinates with the polar axis along the relative velocity are used and the non dimensional quantities are introduced: \( r' \) in the units of the colloid radius \( l \) and the velocities in the units of the relative velocity \( u \).

The calculations are simplified at small velocities and give at the colloid surface the pressure

\[
p^{(2)} = -(1 - 3\cos^2 \vartheta) \frac{3}{8l} \eta Re |\vec{u}^l - \vec{w}^p| \tag{8}
\]

and the tangential strain

\[
\sigma_{r\vartheta} = \frac{3\eta Re}{8l} |\vec{u}^l - \vec{w}^p| \sin \vartheta \cos \vartheta \tag{9}
\]
At the picture (1) it is shown some section of the colloid and the points on its surface with the maximal stresses and their directions. There is the compression along the relative velocity direction and the elongation along the meridians with zeros at the poles and the equator. These stresses produce the deformation of the colloid but the calculation require a definite model for the connection of deformations and stresses inside the colloid. There is the general statement belonging to Maxwell that the rapid (short time) stresses correspond to the elasticity theory but the slow (long time) stresses correspond to the viscous liquid. For the case of colloid solutions at high Reynolds numbers the colloids have the velocity close to the velocity of the solvent that gives the short time action on the separate colloid in the domain with the large acceleration and most probable is the validity of the elasticity theory. As a result the colloid is compressed in moving direction and is elongated in the perpendicular directions. That gives the increased colloid cross section and it moves in the direction with the largest drag. Therefore the motion instability is possible. However this question has not direct relation to the drug reduction at the flow around a macroscopic body.

2 The flow around a macroscopic body.

The averaged hydrodynamical equations for the incompressible liquid with the solution of weakly compressible colloids at the small concentration contain the equation

\[ \text{div} \vec{v} = 0 \] (10)

and the conservation law of the colloid number

\[ \frac{\partial c}{\partial t} + \text{div}(c \vec{w}p) = 0 \] (11)

The colloid velocity is close to that of the liquid \( \vec{w}p \approx \vec{v} \). Therefore it follows from the last two equations that \( c = \text{const} \) is the solution.

The last equation is the conservation law of the total momentum for the system of the liquid and colloids

\[ \frac{\partial}{\partial t} \left[ < \rho > v'_i + cm(w'_p - v'_i) \right] = - \frac{\partial}{\partial x_k} \left[ \Pi_{ik} + cm(w'_p - v'_i)v'_k \right] \] (12)

where \( < \rho > = \rho' + cm, \Pi_{ik} = < \rho' > v'_iv'_k + p\delta_{ik} - \eta \frac{\partial v'_i}{\partial x_k} \) and \( m = \frac{4\pi l^3}{3} \rho_p \).

Stokes drag is absent due to the conservation of the total momentum.

We consider first the absence of the colloids. The general picture of the liquid flow around an immobile macroscopic body is well investigated. For large Reynolds number \( Re = \frac{UL}{\nu} \) at the far domain on the distances large compare to the boundary layer thickness the flow is vortex free. At the boundary layer approximate external surface the normal to the body velocity is almost zero. The boundary layer begins in the forward critical point with a finite thickness. Moving in the flow direction one get the slow increase of the tangent velocity and the boundary layer thickness till the maximum of the tangent velocity is achieved. After that the tangent velocity is lowering and the laminar layer become unstable. Its thickness has a sharp increase with a sharp decrease of the tangent velocity in the same region. This phenomenon is known as "the
tearing off" the boundary layer from the body surface and the creation of the turbulent "stagnation" domain for the flow adjoining to the second critical point as it is shown on fig. 2.

If one has the colloid solution in the liquid then the large laminar part of the boundary layer has a small liquid acceleration and the colloid velocity is quite close to the liquid velocity. Therefore they do not affect on the average flow according to (9). However in the region where the boundary layer is "tearing off" one has the large acceleration ("braking") at large Reynolds number. That gives the strong colloid stream in the initial flow direction if the colloid density is larger then that of the liquid according to (3).

The drag at large Reynolds numbers in the colloid absence is given ([6]) by formula

\[ F = C(Re) \frac{1}{2} U^2 \rho l S \]

where \( S \) is the body cross section, \( U \) is the body velocity relative to the liquid, \( C(Re) \) is a constant depending on Reynolds number only. From the experiment it is known that at the large Reynolds number \( Re* \) the function \( C(Re) \) essentially changes its form giving a sharp drag reduction at comparatively small increase of Reynolds number \( \delta Re* \ll Re* \) known as "a drag crisis". This phenomenon is defined as the moving the" tearing off" domain down the flow and the strong decrease of the "effective" body cross section.

According to (12) the influence of the colloids in the domain for "tearing off" is quite close to the enlargement of Reynolds number because the effective momentum stream is increased

\[ \Pi_{ik}^f = \langle \rho > v_i v_k + p \delta_{ik} - \eta \frac{\partial v_i}{\partial x_k} + \frac{\partial v_i}{\partial x_k} + \frac{cm2 l^2 \partial v_i}{9 \eta} \partial x_k v_k (\rho p - \rho l) \] (13)

The colloid contribution can be written as \( \delta \Pi_{ik} = \rho l v_i \delta v_k \) where

\[ \delta v_i = cV_0 \left( \frac{\rho p}{\rho l} \right)^2 (1 - \frac{\rho l}{\rho p}) \frac{2 l^2 \delta v_i}{9 \eta} \frac{\partial v_i}{\partial x_n} \]

The contribution to the integral can be estimated as

\[ \int \frac{\delta v_i}{\nu} dx \sim cV_0 \left( \frac{\rho p}{\rho l} \right)^2 (1 - \frac{\rho l}{\rho p}) \frac{2 l^2 |\nu|^2}{9 \nu^2} \]

This expression gives the estimate of the colloid contribution to Reynolds number. If this quantity is equal to the experimental \( \delta Re* > 0 \) then one get the drug reduction like the drug crisis without the polymer solution. The condition of the drug reduction by the polymer solution define the col concentration and
the sign of the difference $\rho^p - \rho^l > 0$. This condition can be rewritten in the form

$$\delta \text{Re}^* = cV_0 \left( \frac{\rho^l}{\rho^p} \right)^2 \left( 1 - \frac{\rho^l}{\rho^p} \right) \frac{2l^2}{9L^2} (\text{Re}^*)^2$$

By suggestion $cV_0 \ll 1$ therefore the requirement $\delta \text{Re}^* \frac{l^2}{(\text{Re}^*)^2} \ll l^2$ for the colloid size must be fulfilled. This consideration does not give the difficult numerical estimate.

The details of the turbulent flows in a long pipes at large Reynolds numbers are not so well investigated. Partially phenomenological Karman-Prandtl theory is based (as was shown by L.D.Landau) on the dimensional analysis and the experimental determination of some empirical constants. The main statement is the existence of the central area in the pipe cross section with a weak dependence of the mean velocity on the radius (fig.3 the shaded area) and the viscous sublayer close to the wall of the tube with the linear decrease of the liquid velocity to zero at the wall. The velocity derivative in the viscous sublayer is large compare to that in the central area.

If the colloids are present the flow in the central domain is weakly affected because the mean colloid velocities are very close to that of the liquid. However near the transition boundary colloids can achieve an essential velocity according to (3) if $\rho^p \neq \rho^l$. Therefore the central domain with the large mean velocity must increase due to the colloids motion. The numerical estimate is difficult because one must know the mean value of the acceleration $\frac{dv}{dt}$ for the turbulent flow in the transition region, and the unknown empirical constants defining the central and side domains.

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