Global embeddings and hydrodynamic properties of Kerr black hole

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The global embedding Minkowski space structures are investigated to yield (2+4) and (3+4) dimensionalities outside and inside the event horizon of the Kerr black hole on the equatorial plane, respectively. The accretion onto the Kerr black hole is also studied to yield the general relativistic equation for the steady state axisymmetric massive particles and massless photons. The bound orbits of the massive particles and photons around the Kerr black hole are also investigated to calculate the harmonic motion frequency in the radial direction and angular frequency in the azimuthal one of the massive particles, and the impact parameter of the photons, respectively. The hydrodynamic aspects of the rotating Kerr black hole are discussed to yield the relations among the number density, pressure and internal energy density of the massive particles and photons on the rotating Kerr curved manifold.

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I. INTRODUCTION

The Kerr black hole [1] is recently involved in many influential studies such as an accretion disk [2], a particle accelerator problem [3, 4], numerical relativity [5] and a hidden conformal symmetry [6, 7] in general relativity community. A familiar feature of exact solutions to the field equations of general relativity is the presence of singularities. There have been lots of progresses in global embeddings of various black holes with coordinate singularities [8–16].

As novel ways of removing the coordinate singularities, the higher dimensional global flat embeddings of the black hole solutions are subjects of great interest both to mathematicians and to physicists. It has been well known in differential geometry that four dimensional Schwarzschild metric [17] is not embedded in $\mathbb{R}^5$ [8]. Recently, (5+1) dimensional global embedding Minkowski space (GEMS) structure for the Schwarzschild black hole has been obtained [9] to investigate a thermal Hawking effect on a curved manifold [18] associated with an Unruh effect [19] in the higher dimensional spacetime. The global flat embeddings inside and outside of event horizons of Schwarzschild and Reissner-Nordström black holes have been constructed and on these overall patches of the curved manifolds four accelerations and Hawking temperatures have been evaluated by introducing relevant Killing vectors [13]. Recently, the GEMS scheme has been applied to stationary motions in spherically symmetric spacetime [14], and the Banados-Teitelboim-Zanelli black hole [20] has been embedded in (3+2) dimensions.

In this paper, exploiting the timelike Killing field and the axial Killing one of the Kerr black hole, we investigate hydrodynamic properties of the perfect fluid of the massive particles and massless photons spiraling toward the Kerr black hole along a conical surface. On the equatorial plane of the Kerr black hole, we obtain the GEMS structures which have the (2+4) and (3+4) dimensionalities outside and inside the event horizon of the rotating black hole, respectively. Next, we study the bound orbits of the massive particles and photons around the Kerr black hole, to calculate their physical properties such as their harmonic motion frequency and angular frequency in the radial and azimuthal directions, respectively. The hydrodynamic aspects of the rotating Kerr black hole are also discussed to yield the relations among the number density, pressure and internal energy density of the massive particles and photons around the rotating Kerr black hole.

This paper is organized as follows. In section II we introduce the Kerr black hole metric to study the hydrodynamics of the massive particles and photons, and in section III we investigate the GEMS natures of the Kerr black hole horizons and also we study the bound states of the steady state axisymmetric flow of the perfect fluid of the massive particles and photons. Section IV includes summary and discussions.

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II. HYDRODYNAMICS OF AXISYMMETRIC KERR BLACK HOLE

In order to introduce the rotating Kerr black hole embedded in an asymptotically flat spacetime, we start with the Kerr metric \[1, 21\]
\[
ds^2 = - \frac{\Delta}{\Sigma} dt^2 - \frac{4M a r \sin^2 \theta}{\Sigma} dt d\phi + \frac{A \sin^2 \theta}{\Sigma} d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2
\] (2.1)

where
\[
\Delta = r^2 - 2Mr + a^2, \\
\Sigma = r^2 + a^2 \cos^2 \theta, \\
A = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta,
\] (2.2)

and \(M\) and \(a\) (\(= J/M\) \(0 \leq a \leq M\)) are the mass and angular momentum per unit mass of the black hole, respectively.

Here one can easily see that in the limit of \(a = 0\) the above Kerr metric reduces to the Schwarzschild case. We consider the four velocity given by
\[
u_a = \frac{dx^a}{d\tau}
\]
where one can choose \(\tau\) to be the proper time (affine parameter) for timelike (null) geodesics. From the equations of motion of a massive particle and/or a photon around the Kerr black hole, the massive particles initially at rest and the photons with an initial velocity \(u^r_\infty \approx 1\) respectively, at infinity spiral toward the black hole along a conical surface of constant \(\theta = \theta_\infty\) where \(\theta_\infty\) is the polar angle at infinity. For a fluid of massive particles and photons which approach supersonically to the black hole, one may take the approximation to simplify the hydrodynamical equations, \(u^\theta \approx d\theta/d\tau \approx 0\).

Now, the fundamental equations of relativistic fluid dynamics can be obtained from the conservation of particle number and energy-momentum fluxes. In order to derive an equation for the conservation of particle numbers one can use the particle flux four-vector \(n u^a\) to yield
\[
\nabla_a (n u^a) = 0
\] (2.3)

where \(n\) is the proper number density of particles measured in the rest frame of the fluid of massive particles and photons and \(\nabla_a\) is the covariant derivative in the Kerr curved manifold of interest and \(g = \det g_{ab}\). For steady state axisymmetric flow of the perfect fluid of the massive particles and photons, the conservation of energy-momentum fluxes is similarly described by the Einstein equation as below
\[
\nabla_b T^b_a = 0
\] (2.4)

where the stress-energy tensor \(T^{ab}\) for perfect fluid is given by
\[
T^{ab} = \rho u^a u^b + P (g^{ab} + u^a u^b),
\] (2.5)

with \(\rho\) and \(P\) being the proper energy densities and pressures of the massive particles or photons, respectively.

For the steady state axisymmetric flow of the perfect fluid of the massive particles and photons, the equations (2.3) and (2.4) yield
\[
4\pi r^2 n u^r \left(1 + \frac{a^2}{r^2} \cos^2 \theta\right) \sin \theta = C_0,
\] (2.6)

\[
(P + \rho) u^r r^2 \left(1 + \frac{a^2}{r^2} \cos^2 \theta\right) \sin \theta = C_t,
\] (2.7)

where \(C_0\) is the accretion rate of the massive particles or photons, and \(C_i\) \((i = t, \phi)\) are the other constants of the motion which can be evaluated at infinity to yield the ratio \(C_\phi/C_t = u_\phi/u_t = 0\). Combining the equations (2.6) and (2.7), one can derive the relations \(^1\)

\(^1\) The timelike case of the massive particles corresponding to \(\kappa = 1\) in (2.8) appeared in [22], and now we include the null case of the photons with \(\kappa = 0\). Here, \(\kappa\) is defined in (2.9). Additionally, the \(\sin \theta\) factors in (2.6) and (2.7) are missing in the corresponding equations in [22].
where \( n_{\infty}, P_{\infty} \) and \( \rho_{\infty} \) are the particle number density, pressure and internal energy density of the fluid of the massive particles or photons at infinity, respectively. Here we have introduced a new parameter \( \kappa \) defined as

\[
\kappa = -g_{\alpha\beta}u^\alpha u^\beta = \begin{cases} 
1 & \text{for timelike geodesics} \\
0 & \text{for null geodesics.} 
\end{cases}
\]

Next, using the projection operators in (2.4) one can obtain the general relativistic equation on the direction perpendicular to the four-velocity [21]

\[
(P + \rho)u^b \nabla_b u_a + (g_{ab} + u_a u_b) \nabla^b P = 0
\]

(2.10)

from which, after some algebra, we obtain the radial component of the above equation for the steady state axisymmetric accretion of the massive particles and photons on the rotating Kerr black hole of mass \( M \)

\[
\frac{1}{2} \frac{d}{dr}(u^r u^r) + \frac{M(1 - a^2 r^{-2} \cos^2 \theta)}{r^2(1 + a^2 r^{-2} \cos^2 \theta)^2} + \frac{Ma^2 \sin^2 \theta}{r^4(1 + a^2 r^{-2} \cos^2 \theta)(1 - 2M r^{-1} + a^2 r^{-2})} \left( \frac{r}{M + A} \right) u^r u^r + \frac{Ma^2 \sin^2 \theta}{r^4(1 + a^2 r^{-2} \cos^2 \theta)^2 A} + \frac{1}{P + \rho} \left( u^r u^r + \frac{1 - 2M r^{-1} + a^2 r^{-2}}{1 + a^2 r^{-2} \cos^2 \theta} \right) \frac{dP}{dr} = 0,
\]

(2.11)

where \( A \) is defined in (2.2) and, for the first time up to our knowledge, we have introduced the function \( B \), which is a characteristic for the rotating Kerr black hole

\[
B = (r^2 - a^2 \cos^2 \theta) \Delta - 4Mr^3.
\]

(2.12)

The equation (2.11) is one of our main results which, for the vanishing \( a \) limit, reduces to the spherically symmetric Schwarzschild black hole result [23]

\[
\frac{1}{2} \frac{d}{dr}(u^r u^r) + \frac{M}{r^2} + \frac{1}{P + \rho} \left( u^r u^r + \frac{1 - 2M}{r} \right) \frac{dP}{dr} = 0.
\]

(2.13)

We observe that, in (2.11) as in the other rotating Kerr black hole equations (2.6), (2.7) and (2.8), the axisymmetric gravitational terms due to the angular momentum of the black hole vary as \( a^2/r^2 \leq M^2/r^2 \), while the terms independent of the black hole rotation vary as \( M/r \).

### III. Global Embeddings and Bound Orbits around Kerr Black Hole

In the Kerr metric (2.1), where \( t \) and \( \phi \) are cyclic, namely \( \partial_t g_{ab} = \partial_\phi g_{ab} = 0 \), we have two Killing vector fields \( \xi^a_i \) \( (i = t, \phi) \) satisfying the Killing equations

\[
\mathcal{L}_{\xi} g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0.
\]

(3.1)

Using the above Killing equations and geodesic ones, one can readily see that \( u^c \nabla_c (g_{ab} \xi^a_i u^b) = 0 \) \( (i = t, \phi) \) to produce the constants of motion \( \epsilon \) and \( l \) corresponding to the Killing vector fields,

\[
\begin{align*}
    g_{ab} \xi^a_i u^b &= - \left( 1 - \frac{2Mr}{\Sigma} \right) u^t - \frac{2Mar \sin^2 \theta}{\Sigma} u^\phi = -\epsilon, \\
    g_{ab} \xi^a_\phi u^b &= \frac{2Mar \sin^2 \theta}{\Sigma} u^t + \frac{A \sin^2 \theta}{\Sigma} u^\phi = l,
\end{align*}
\]

(3.2)

where \( \epsilon \) and \( l \) are the conserved energy per unit mass and angular momentum per unit mass, for the massive particles. For the photons, we note that \( \hbar \epsilon \) and \( \hbar l \) are the total energy and the angular momentum of the photons, respectively.
In order to investigate the global embedding structure of the Kerr black hole, we consider the equatorial plane on $\theta = \pi/2$ where the Kerr metric (2.1) reduces to the following three-metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)^{-1} dr^2 - \frac{4Ma}{r} dtd\phi + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) d\phi^2. \quad (3.3)$$

The event horizon of this Kerr metric is located at $r = r_+$ with

$$r_\pm = M \pm (M^2 - a^2)^{1/2} \quad (3.4)$$

which are given by the pole of $g_{rr}$ of the Kerr metric.

After some tedious algebra, for the Kerr black hole in the region $r \geq r_+$ and $r \leq r_-$ we obtain the (2+4) global embedding Minkowski space (GEMS) structure

$$ds^2 = -(dz^0)^2 - (dz^1)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 \quad (3.5)$$

with the coordinate transformations

$$
egin{align*}
  z^0 &= t \quad (3.6) \\
  z^1 &= k^{-1} \left(\frac{2M}{r}\right)^{1/2} \cosh k(t - a\phi), \\
  z^2 &= k^{-1} \left(\frac{2M}{r}\right)^{1/2} \sinh k(t - a\phi), \\
  z^3 &= (r^2 + a^2)^{1/2} \cos \phi, \\
  z^4 &= (r^2 + a^2)^{1/2} \sin \phi, \\
  z^5 &= \int dr \left(\frac{k^{-2}M}{2r^3} + \frac{2Mr^3}{(r^2 + a^2)(r - r_+)(r - r_-)}\right)^{1/2}, \quad (3.7)
\end{align*}
$$

where $k$ is given by

$$k = \frac{r_+ - r_-}{2r^2_+}. \quad (3.8)$$

Next, in order to investigate the region $r_- \leq r \leq r_+ (r_+ \leq 2M)$ we rewrite the Kerr black hole three-metric as

$$ds^2 = \left(\frac{2M}{r} - 1\right) dt^2 - \left(-1 + \frac{2M}{r} - \frac{a^2}{r^2}\right)^{-1} dr^2 - \frac{4Ma}{r} dtd\phi + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) d\phi^2. \quad (3.9)$$

in terms of the positive definite functions $-g_{tt}$ and $-g_{rr}$ inside the event horizon $r_+$ to yield the (3+4) GEMS structure

$$ds^2 = -(dz^0)^2 - (dz^1)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 - (dz^6)^2 \quad (3.10)$$

with the coordinate transformation

$$
egin{align*}
  z^5 &= k^{-1} \left(\frac{2M}{r}\right)^{1/2}, \\
  z^6 &= \int dr \left(\frac{2Mr^3}{(r^2 + a^2)(r - r_+)(r - r_-)}\right)^{1/2}, \quad (3.11)
\end{align*}
$$

where $(z^0, z^1, z^2, z^3, z^4)$ are the same as those in (3.7). Here one notes that the GEMS in the region $r_- \leq r \leq r_+$ has one more timelike dimensionality. In fact, the $z^5$ coordinate transformation in (3.7) split into positive definite part and negative definite one in the region $r_- \leq r \leq r_+$ to yield $(z^5, z^6)$ in (3.11).

Now, we proceed to calculate the radial equation for the particle on the equatorial plane whose metric is given by (3.3) to yield

$$\frac{1}{2} u^r u^r + V(r) = 0, \quad (3.12)$$
where the effective potential is given by [21]

$$V(r) = -\frac{\kappa M}{r} + \frac{l^2}{2r^2} + \frac{1}{2}(\kappa - \epsilon^2) \left(1 + \frac{a^2}{r^2}\right) - \frac{M}{r^3}(l - a\epsilon)^2.$$  \hfill (3.13)

Here one notes that the first and second terms are the Newtonian and centrifugal barrier terms respectively and the others are the general relativistic corrections, including the black hole rotating effects with the parameter $a$. One also notes for the massive particle that if $\epsilon < 1$ the orbit of the particle is bound so that it can not reach $r = \infty$, while if $\epsilon > 1$ the orbit is unbound except for a measure-zero set of unstable orbits [24].

The effective potential (3.13) now should fulfill the condition

$$\frac{dV}{dr}(r = r_{s,us}) = 0,$$  \hfill (3.14)

to yield the radius of the stable and unstable bound orbits on the equatorial plane for a given $\epsilon$ and $l$

$$r_{s,us} = \frac{1}{2M}[l^2 + (1 - \epsilon^2)a^2 \pm D^{1/2}],$$  \hfill (3.15)

where the upper (lower) sign refers to stable (unstable) orbit and

$$D = [l^2 + (1 - \epsilon^2)a^2 - 2\sqrt{3}M(l - a\epsilon)][l^2 + (1 - \epsilon^2)a^2 + 2\sqrt{3}M(l - a\epsilon)].$$  \hfill (3.16)

In order to guarantee the positive value of $D$, $l$ should satisfy the constraints that $l \geq \sqrt{3}M + (\sqrt{3}M + a - a\epsilon)^{1/2}(\sqrt{3}M - a - a\epsilon)^{1/2}$ for $a(1 + \epsilon) \leq \sqrt{3}M$ and $l \geq -\sqrt{3}M + (\sqrt{3}M + a + a\epsilon)^{1/2}(\sqrt{3}M - a + a\epsilon)^{1/2}$ for $a(1 + \epsilon) \geq \sqrt{3}M$, respectively. In the limit with $a = 0$, one readily notes that (3.15) reduces to the well known Schwarzschild case for $l \geq 2\sqrt{3}M$ [21]

$$r_s = \frac{1}{2M}[l^2 + (l^4 - 12M^2a^2)^{1/2}].$$  \hfill (3.17)

The energy per unit mass of the particle in the circular orbit of the radius $r = r_s$ is just the value of the effective potential $V$ at that radius

$$V(r = r_s) = 0,$$  \hfill (3.18)

which, together with (3.14), yields the energy $\epsilon_c$ per unit mass for a circular orbit and the angular momentum $l_c$ per unit mass whose explicit forms are given in [25]. The angular frequency in the azimuthal direction for the zero mode of the circular orbit with $\epsilon_c$ and $l_c$ at the stable bound orbit of $r = r_s$ is then given by

$$\omega_\phi = \frac{1}{r_s^2 + a^2 - 2Mr_s} \left[\frac{2Ma}{r_s} \epsilon_c + \left(1 - \frac{2M}{r_s}\right) l_c\right].$$  \hfill (3.19)

On the other hand, in the radial direction we have the harmonic motion frequency of the massive particle of the form

$$\omega_r = \frac{1}{r_s^2} \left[2Mr_s - (l_c^2 + (1 - \epsilon_c^2)a^2)\right]^{1/2}.$$  \hfill (3.20)

Here one notes that, since there exists some difference between $\omega_\phi$ and $\omega_r$ in general, the massive particle performs the precession motion with the frequency

$$\omega_p = \omega_\phi - \omega_r.$$  \hfill (3.21)

Next, we study the photons following the null geodesics spiraling toward the rotating Kerr black holes on the equatorial plane, where the potential (3.13) reduces to

$$V(r) = \frac{l^2}{2r^2} - \frac{1}{2}a^2 \left(1 + \frac{a^2}{r^2}\right) - \frac{M}{r^3}(l - a\epsilon)^2,$$  \hfill (3.22)

from which, exploiting the condition (3.14), we obtain the radius of the unstable bound orbit on the equatorial plane for a given $\epsilon$ and $l$

$$r_0 = \frac{3M(l - a\epsilon)^2}{l^2 - \epsilon^2a^2},$$  \hfill (3.23)
which shows that, in the strong gravitational region, the propagation direction of the light changes following the above unstable orbit. Inserting $r = r_0$ into both (3.14) and (3.18), one can readily show that $r_0$ satisfies the following identity

$$r_0^{3/2} - 3M r_0^{1/2} + 2aM^{3/2} = 0,$$

(3.24)

which, in the $a = 0$ limit, reduces to the well known Schwarzschild black hole result, $r_0 = 3M$. Here, the real solution of (3.24) is given by

$$r_0 = 2 \left[1 + \cos \left(\frac{2}{3} \cos^{-1}(-a)\right)\right] n^2.$$

(3.25)

We reemphasize that, since the orbit of the photon is unstable, we do not have a circular trajectory of the photon, and instead we have a bending of the light in the strong gravitational field originated from the rotating Kerr black hole of mass $M$. In fact, the impact parameter defined by $l/\epsilon$ is evaluated in our case to produce

$$\frac{l^2}{\epsilon^2} = a^2 + 3r_0^2,$$

(3.26)

which also in the vanishing $a$ limit reduces to the Schwarzschild black hole result, $l^2/\epsilon^2 = 27M^2$.

Now, we have couple of comments to address. If the massive particles in the perfect fluid described in the previous section happen to be trapped in the circular orbit at $r = r_s$, (2.8) and (2.11) become

$$\frac{(P + \rho)^2}{n^2} \frac{1 - 2Mr_s^{-1} + a^2 r_s^{-2}}{1 + a^2 r_s^{-2} + 2Ma^2 r_s^{-3}} = \frac{(P_\infty + \rho_\infty)^2}{n_\infty^2},$$

(3.27)

and

$$\frac{M}{r_s^2} \frac{Ma^2(r_s^2 + a^2 - 6Mr_s)}{r_s^2(r_s^2 + a^2 + 2Ma^2 r_s^{-1})} \frac{1 - 2Mr_s^{-1} + a^2 r_s^{-2}}{P + \rho} \frac{dP}{dr} = 0.$$

(3.28)

In the limit $r_s \gg M$, from (3.19) and (3.20) one can obtain the closed bound orbit with $\omega_r = \omega_\phi = M^{1/2}r^{-3/2}$ to yield $\omega_p = 0$. One notes that the radiation in the perfect fluid around the stable circular orbit of the rotating Kerr black hole plays a major role in producing the circular ring with sensible luminosity around the black hole.

IV. CONCLUSIONS

In the context of the general relativistic description of the rotating Kerr black hole, we have investigated the steady state axisymmetric flow of the perfect fluid of the massive particles and photons. The hydrodynamic aspects around the rotating Kerr black hole have been discussed to yield the relations among the number density, pressure and internal energy density of the massive particles and photons on the rotating Kerr curved manifold. Moreover, we have constructed the GEMS with the (2+4) and (3+4) dimensionalities outside and inside the event horizon of the Kerr black hole on the equatorial plane, respectively. In the neighborhood close to the circular equilibrium orbit of the massive particle, the particle has been shown to perform the small harmonic oscillation. Together with the azimuthal zero mode, this vibrational mode in the radial direction has been able to generate the general relativistic precession in the vicinity of the black hole. The photon bound orbit has been also discussed in terms of the impact parameter of the photon trajectory.

In the realistic astrophysical situation, we assume that the supernova SN1987A is decaying into the rotating Kerr black hole as in the case of SN 1979C [26] and the particles consist of the hydrogen and helium gas almost fully ionized to perform the thermal vibration, as well as the gravitational one with $\omega_r$, through the mechanism such as electron-ion and electron-electron bremsstrahlungs and Compton heating. The gravitational mode constrained by the fluid dynamic equations can generate the radio wave envelope, especially in the vicinity of the black hole, of the high frequency thermal radiation, which play a major role in producing the circular ring with sensible luminosity around the black hole [27].

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