Tree amplitudes and color decomposition in broken SU(2)

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ABSTRACT: We propose a color decomposition for general tree amplitudes in a SU(2) gauge theory which is spontaneously broken via the Higgs mechanism. Working in the unitary gauge, we construct color-ordered amplitudes by explicitly presenting a set of color-ordered Feynman rules. Those primitive amplitudes are gauge-invariant, and they preserve perturbative unitarity in the high-energy limit. Serving as building blocks of color-dressed tree amplitudes, they allow for efficient evaluation of tree-level scattering amplitudes involving gauge bosons and the Higgs boson via the Berends-Giele recursion relations for color-ordered currents. We demonstrate the efficiency of this computational scheme by calculating on-shell amplitudes for scattering of five, six and nine W-bosons in the limit of vanishing Weinberg angle.

KEYWORDS: color decomposition, broken gauge symmetry, tree amplitudes
1 Introduction

Interactions of electroweak gauge bosons at high energies probe into the very nature of electroweak symmetry breaking. Such interactions can, eventually, be studied at CERN Large Hadron Collider (LHC). However, detailed investigations of the electroweak sector at high energies require the development of efficient techniques to calculate amplitudes for scattering processes with electroweak gauge bosons both at tree- and the one-loop level. It has long been known that broken electroweak gauge invariance makes such perturbative computations formidable.

Indeed, in standard renormalizable gauges the presence of non decoupling Goldstone bosons quickly leads to an explosion of the number of Feynman diagrams. On the other hand, in the unitary gauge large cancellations among the longitudinal parts \( p_\mu p_\nu / m_W^2 \) of vector boson propagators would occur, leading to severe numerical stability issues. Partly because of this, our knowledge of multi vector boson scattering is quite limited. Tree-level results for \( \gamma\gamma \rightarrow W^+W^- ZZ \) and \( \gamma\gamma \rightarrow W^+W^-W^+W^- \) were computed in [1] using an optimized gauge choice. Beyond the tree-level the situation is even worse: to the best of our knowledge, only the simplest case of \( VV \rightarrow VV \) scattering has been studied [2–5].

In recent years, we have witnessed enormous progress in developing computational techniques for scattering amplitudes in massless gauge theories, both regular and supersymmetric (for a recent review, see e.g. the special issue [6] and the review [7]). However, these techniques were mainly developed within QCD-like theories and must then be generalized in order for them to cope with non colored particle and with massive vector bosons. A first step in this direction was made in [8], where processes with up to two external vector
bosons were considered and in [9], where multi-photon tree-level amplitudes were studied. Another step in this direction was made in [10], where it was shown how to generalize the CSW construction (see [6]) in order to deal with a broken gauge theory.

A large number of on-shell computational techniques in massless gauge theories – both at tree- and the one-loop level – are based on the idea of color ordering. In that approach, scattering amplitudes are represented by sums of products of color factors and color-stripped objects – the so-called color-ordered amplitudes. As an example, a useful color decomposition of $n$-gluon scattering amplitudes in a gauge theory with the group $SU(N_c)$ reads [11, 12]

$$A_{n}^{\text{tree}} (1, 2, \cdots, n) = \sum_{\sigma \in S_n / Z_n} \text{Tr} (T^{a_\sigma(1)} T^{a_\sigma(2)} \cdots T^{a_\sigma(n)}) A_{n}^{\text{tree}} (\sigma(1), \sigma(2), \cdots, \sigma(n)), \quad (1.1)$$

other useful decompositions were presented in [13], [14] (see the review [7] for details). Here $A_{n}^{\text{tree}}$ are color-ordered or primitive amplitudes, which only depend on the momenta and polarizations of external gluons.

The primitive amplitudes $A_{n}^{\text{tree}}$ have many attractive properties (see e.g. [15]). Each primitive amplitude receives contribution only from planar diagrams with external legs arranged in the corresponding order. These color-ordered diagrams can be computed by introducing a set of color-ordered Feynman rules from which the color degrees of freedom are removed. The color-stripped primitive amplitudes are gauge-invariant and, in this sense, physical. Moreover, kinematic singularities of tree amplitudes are closely related to their on-shell constructibility, as reflected by the BCFW on-shell recursion relation [16, 17]. Compared with the full color-dressed amplitude, primitive amplitudes, being color-ordered, have simpler structure of kinematic singularities; for this reason, they can be thought of as basic objects for studying analytic properties of scattering amplitudes.

We would like to define and work with color-ordered amplitudes to describe interactions of electroweak gauge bosons. However, in a theory where gauge invariance is broken, it is not clear how to do that. There are multiple reasons for that, from vacuum having preferred direction in the “color” space, to the existence of color-neutral “Higgs particle” in the spectrum, which makes the concept of color ordering ambiguous. One option is to give up on the idea of color ordering and to generalize existing algorithms for calculating scattering amplitudes to make them applicable to color-dressed quantities. This program has been successfully carried out to address computation of high-multiplicity processes with electroweak gauge bosons [18] and gluons [19], both at tree-level and beyond.

In this paper we investigate if the concept of color ordering can be used to describe scattering of massive gauge bosons, in spite of the caveats pointed above. We focus on a model with the $SU(2)$ gauge group which is completely broken by the Higgs mechanism. We explain how to define color-ordered amplitudes in this model and show that those amplitudes satisfy the electroweak Ward identity and respect perturbative unitarity bound. We present explicit results for scattering amplitudes of five, six and nine $W$-bosons, by computing them in the unitary gauge using color-ordered currents that satisfy Berends-Giele recursion.
The paper is organized as follows. In Section 2 we describe our model and mention some problems with arranging the color decomposition of scattering amplitudes. In Section 3 we derive color-ordered Feynman rules and explain how color-ordered amplitudes are constructed. In Section 4 we prove that color-ordered currents satisfy electroweak Ward identity. In Section 5 we present our conclusions. Some results, including color-ordered Feynman rules and discussion of numerical computation of five- and six- and nine-\(W\) scattering amplitudes are relegated to the Appendix.

2 SU(2) gauge theory, Higgs mechanism and the color decomposition

We consider a SU(2) gauge theory which is broken by the Higgs mechanism. In such a theory, three gauge fields \(W^a\) are labelled by color indices \(a = 1, 2, 3\). The gauge field part of the Lagrangian reads

\[
\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \varepsilon^{abc} W^b_\mu W^c_\nu, \tag{2.1}
\]

where \(\varepsilon^{abc}\) is the Levi-Civita tensor. We use SU(2) Lie algebra generators \(T^a = \sigma^a / \sqrt{2}\), where \(\sigma^{1,2,3}\) are the Pauli matrices. The orthogonality and commutation relations read

\[
\text{Tr} \left( T^a T^b \right) = \delta^{ab}, \quad \left[ T^a, T^b \right] = i \sqrt{2} \varepsilon^{abc} T^c. \tag{2.2}
\]

While the above relations generalize to an arbitrary SU(\(N\)) group, generators of the SU(2) group enjoy an anti-commutation relation

\[
\left\{ T^a, T^b \right\} = 1 \delta^{ab}, \tag{2.3}
\]

that will play an important role in our construction. A completeness relation of generators in the fundamental representation is useful for dealing with color algebra. In the case of SU(2), it reads

\[
(T^a)_{ij} (T^a)_{kl} = \delta_{il} \delta_{kj} - \frac{1}{2} \delta_{ij} \delta_{kl}. \tag{2.4}
\]

We break the gauge symmetry in the Standard Model-like way; to this end, we introduce a scalar SU(2) doublet \(\Phi\) and give it a non-vanishing vacuum expectation value \(\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}\). In general, we parameterize the SU(2) doublet in terms of four real scalar

\[
\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -i (\phi^1 - i \phi^2) \\ v + H + i \phi^3 \end{pmatrix}. \tag{2.5}
\]

We identify \(H\) with the physical Higgs boson; the three fields \(\phi^a, a = 1, 2, 3\) are Goldstone degrees of freedom; they are absorbed by gauge fields \(W^a\), as they acquire equal masses \(m_W = g v / 2\) and obtain longitudinal modes. The Higgs part of the Lagrangian reads

\[
\mathcal{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) + \mu^2 \Phi^\dagger \Phi - \lambda \left( \Phi^\dagger \Phi \right)^2, \tag{2.6}
\]
where the covariant derivative in the fundamental representation is given by $D_{\mu} = \partial_{\mu} - ig W_{\mu}^a T^a / \sqrt{2}$. This broken gauge theory can be quantized in a standard way by introducing a gauge-fixing term

$$L_{g.f.} = -\frac{1}{2\xi} \left( \partial_{\mu} W_{\mu}^a - \xi \frac{q}{2} \phi^a \right)^2,$$

and the ghost Lagrangian

$$L_{\text{ghost}} = \bar{u}^a \left[ -\delta^{ab} \partial^2 + g \varepsilon^{abc} \xi \phi^c - \xi \frac{g^2 v}{4} \delta_{ab} - \xi \frac{g^2 v^2}{4} \delta_{ab} \right] u^b.$$

The unphysical degrees of freedom, i.e. the Goldstone fields and the ghost fields, all have masses $\sqrt{\xi} m_W$. In the limit $\xi \to \infty$, which is usually referred to as the unitarity gauge, the unphysical degrees of freedom decouple from the theory. In such a gauge, intermediate states appearing in any physical scattering amplitude, i.e. gauge bosons $W^a$ and the Higgs boson $H$, are physical degrees of freedom. In particular, the unitary gauge propagator for the gauge field $W^a$ is

$$D_{a\mu\nu} = -i \frac{\delta^{ab}}{p^2 - m_W^2} \left( g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{m_W^2} \right).$$

The unitary gauge deals with only physical degrees of freedom and, for this reason, is particularly suitable for unitarity-related tools such as on-shell recursion relations and unitarity cuts. We also point out that in this particular model, a global $SU(2)$ symmetry survives as the particle content nicely fits into its various representations, even though the locally gauged $SU(2)$ symmetry is broken. This observation will help us to construct the color decomposition in what follows.

We also mention that in this paper we restrict our discussion to self-interaction of gauge bosons and their interactions with the Higgs boson. The self-interaction of the Higgs bosons, which can be traced back to the scalar potential Eq. (2.6), is not necessary to describe consistent interaction pattern in the gauge sector. Hence, the mass of the Higgs boson $m_H$ can be viewed as a free input parameter. Its value is not important for ensuring the gauge invariance of the theory, although it determines the value of scattering amplitudes in high-energy scattering and, therefore, controls perturbative unitarity.

It is well-known that the description of multi-particle scattering – even at tree level – becomes very difficult within conventional Feynman-diagrammatic approach. This is especially true for gauge field theories where the number of Feynman diagrams grows factorially when the number of external particles increases, see Table 1. In the unitary gauge, where the number of Feynman diagrams is greatly reduced due to the absence of Goldstone bosons and ghosts, severe cancellations occur between individual diagrams as longitudinal structures in propagators of gauge bosons introduce bad scaling behaviors in the high energy limit. The color decomposition that we introduce in this Section reduces full amplitudes to simpler objects, which can be computed in the recursive fashion, thereby keeping growth of Feynman diagrams in check and avoiding large numerical cancellations at intermediate steps.

A color decomposition for the $n$-gluon scattering is shown in Eq.(1.1). We remind the reader that this color-decomposition is achieved by rewriting the structure constant.
| # of external gauge bosons | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------------------|---|---|---|---|---|---|---|
| # of color-dressed diagrams | 1 | 7 | 55 | 730 | 11410 | 226765 | 5230225 |

Table 1. Number of color-dressed tree diagrams for multi-\(W\) scattering in the broken \(SU(2)\) model, as generated by automation package Qgraf\[20\]. Intermediate Higgs bosons contribute a large number of additional diagrams.

\(f^{abc}\) – which enters Feynman rules in case of gluodynamics – through a difference of traces of products of \(SU(N_c)\) generators in the fundamental representation and then using the completeness relations to combine various traces.

We would like to repeat the same procedure in the broken gauge theory; the immediate obstacle that we face is that – in addition to the structure constants of the \(SU(2)\) group that control self-interactions of the gauge bosons, there are symmetric structure constants \(\delta^{ab}\) in the coupling of the gauge bosons to the Higgs boson. We can deal with the anti-symmetric structure constants \(\sim \epsilon^{abc}\) in the standard way by writing

\[
\epsilon^{abc} = -\frac{i}{\sqrt{2}} \left( \text{Tr}(T^a T^b T^c) - \text{Tr}(T^a T^c T^b) \right).
\]  

(2.10)

To deal with the symmetric structure constants \(\delta^{ab}\), we use the fact that for the \(SU(2)\) gauge group, they can be written as anti-commutators of Lie algebra generators, Eq.\((2.3)\).

We can employ this representation for \(\delta^{ab}\) to insert it in relevant places inside traces created by the repeated use of Eq.\((2.10)\) and the completeness relation Eq.\((2.4)\). We conclude that a general tree \(W\)-boson scattering amplitude can be written as a linear combination of kinematic structures multiplied by traces of products of \(SU(2)\) generators in the fundamental representation

\[
A_{\text{tree}}^{\text{tree}}(1_W, 2_W, \ldots, n_W) = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{\sigma(1)} \cdots T^{\sigma(n)}) A_{\text{tree}}^{\text{tree}}(\sigma(1), \sigma(2), \ldots, \sigma(n)).
\]  

(2.11)

We note that primitive amplitudes in the above formula are defined to contain the gauge coupling constant in the appropriate power.

To make use of the full power of color decomposition, it is important to understand how color-ordered amplitudes can be computed. In case of pure gluodynamics, a powerful way to compute ordered amplitudes is based on Berends-Giele recursion relations \[11\]. If we want to apply a similar technique to compute scattering amplitudes in a broken gauge theory, we face the following problem: because of the existence of \(WWH\) vertex, iterations of Berends-Giele currents for electroweak gauge bosons must involve the Higgs boson currents. However, since Higgs bosons are color-neutral, we face an immediate question of how to incorporate the neutral particles into the color-ordering scheme. A similar issue arises if we think about using tree color-ordered amplitudes as building blocks in one-loop computations. In this case, unitarity cuts clearly produce tree amplitudes with intermediate (multiple) Higgs particles and we need to understand how to define “color-ordered” amplitudes with Higgs particles and electroweak gauge bosons.
We require that color-ordered amplitudes receive contribution only from planar color-stripped diagrams with particular ordering of all physical external particles. Besides, we require that these ordered amplitudes satisfy electroweak Ward identity, in a similar way as the color-dressed amplitudes do. This last feature – that we will loosely refer to as “gauge invariance of scattering amplitudes” – is important for enabling applications of these color-ordered objects to one-loop computations. It turns out that for broken $SU(2)$ such color-stripped objects do exist. In the following Sections we construct them explicitly.

3 Constructing physical primitive amplitudes

We begin by addressing the color-neutrality of the Higgs boson. To deal with this issue, we extend the gauge group from $SU(2)$ to $U(2)$, by introducing the abelian $U(1)$ generator $T^0 = 1/\sqrt{2}$. We can now consider the completeness relation in an $U(2)$ theory, by adding the $U(1)$ generator to Eq.\((2.4)\). For definiteness, we will label operators of $SU(2)$ with $a, b, c$, while generators of $U(2)$ will be labeled with $\hat{a}$, etc. The tilded indices run from 0 to 3 while the untilded ones run from 1 to 3. For the $U(2)$ group, we still have the commutation relation

$$[T^\hat{a}, T^\hat{b}] = i\sqrt{2} f^{\hat{a}\hat{b}\hat{c}} T^\hat{c},$$

(3.1)

where $f^{\hat{a}\hat{b}\hat{c}}$ vanishes if any of the indices is zero and $f^{\hat{a}\hat{b}\hat{c}} = \varepsilon^{\hat{a}\hat{b}\hat{c}}$ otherwise. In addition, the simplified completeness relation is valid

$$(T^\hat{a})_{ij} (T^\hat{a})_{kl} = \delta_{il} \delta_{kj}.$$  

(3.2)

On the other hand, the anticommutation relation Eq.\((2.3)\) requires care since it becomes invalid for a generic choice of $U(2)$ generators.

We can now extend the particle content of the theory by promoting the gauge bosons and the Higgs boson to full $U(2)$ multiplets. This implies that we introduce the Higgs triplet $H^a$, $a = 1, 2, 3$, in addition to the regular $SU(2)$ Higgs boson that (in this notation) is denoted as $H^0$, and the $U(1)$ gauge boson $W^0$. The interactions between these particles are controlled by $U(2)$ Feynman rules. In the gauge boson sector, we obtain those rules by writing

$$\varepsilon^{abc} \rightarrow \varepsilon^{\hat{a}\hat{b}\hat{c}} = -\frac{i}{\sqrt{2}} \left[ \text{Tr} \left(T^{\hat{a}} T^{\hat{b}} T^{\hat{c}}\right) - \text{Tr} \left(T^{\hat{b}} T^{\hat{c}} T^{\hat{a}}\right) \right].$$

It is then obvious that with this extension of the Feynman rules, the $U(1)$ gauge bosons completely decouple from the gauge sector of the theory although it is useful to have them, to prove the color decomposition in a straightforward way.

In the Higgs sector, we need to extend the interactions between $W$-bosons and the Higgs boson. Again, we want to make this extension in such a way, that the decoupling of unphysical particles is obvious. Recall that, eventually, we are interested in computing multi-$W$ and multi-Higgs scattering amplitudes where all external states are taken to be
physical. To this end, we write $W^{\tilde{a}} W^{\tilde{b}} H^{\tilde{c}}$ vertex as

$$\hat{t}, \nu \quad \quad \hat{c} = \frac{i g m_W g^{\mu \nu}}{\sqrt{2}} \left( \text{Tr} \left( T^{\tilde{a}} T^{\tilde{b}} T^{\tilde{c}} \right) + \text{Tr} \left( T^{\tilde{b}} T^{\tilde{a}} T^{\tilde{c}} \right) \right). \quad (3.4)$$

It is easy to understand that this equation leads to decoupling of the interaction between unphysical Higgses $H^a$, $a = 1, 2, 3$ and physical gauge bosons. Indeed, in this case $\text{Tr}(T^{a} T^{b} T^{c}) \sim \epsilon^{abc}$, so the sum of the two traces vanishes. The non-vanishing contribution requires that either $\tilde{a} = \tilde{b} = \tilde{c} = 0$, which gives an interaction of a physical $H$ with two unphysical $W$-bosons or that one of $\tilde{a}, \tilde{b}, \tilde{c}$ is zero and the other two are not. This latter case contains an interaction of a physical $W$ with unphysical Higgs and unphysical $W$, as well as the interaction of a physical Higgs with two physical $W$-bosons. An important feature of the above vertex is that unphysical particles always appear in pairs; this will be a crucial element for understanding their decoupling from physical amplitudes.

Similarly, the $WWHH$ vertex can be generalized in the following way

$$\hat{b}, \nu \quad \quad \quad \quad \hat{c} = \frac{i g^2}{4} g^{\mu \nu} \left( \text{Tr} \left( T^{\tilde{a}} T^{\tilde{b}} T^{\tilde{d}} \right) + \text{Tr} \left( T^{\tilde{b}} T^{\tilde{a}} T^{\tilde{d}} \right) \right) \quad (3.5)$$

The right hand side produces a variety of vertices that involve both physical and unphysical particles; again, the unphysical particles appear in pairs.

By repeated use of the completeness relation for Lie algebra generators Eq. (3.2), we combine individual traces into traces of products of $T$-matrices that correspond to color-states of all external particles, including the Higgs boson. The relevant color-stripped Feynman rules are given in Appendix A. We emphasize that the $U(2)$ Feynman rules imply that unphysical particles can be produced in pairs only; therefore, if external particles are physical, the unphysical particles automatically decouple from full tree amplitude, in spite of contributing to color-ordered ones. This decoupling is identical to how ghosts in QCD or super-partners in supersymmetric QCD do not contribute to scattering amplitudes of regular quarks and gluons at tree level. We therefore conclude that scattering amplitudes can be represented in the following way

$$A_n^{\text{tree}} (1X, 2X, \ldots, nX) = \sum_{\sigma \in S_n/Z_n} \text{Tr} \left( T^{\tilde{a}(1)} T^{\tilde{a}(2)} \ldots T^{\tilde{a}(n)} \right) A_n^{\text{tree}} (\sigma(1), \ldots, \sigma(n)), \quad (3.6)$$

where $X^{\tilde{a}}$ is a generic notation for the Higgs boson and $W$ bosons, and color-ordered amplitudes $A_n$ are obtained from the color-ordered Feynman rules.
As we pointed out already, we would like to construct the color-ordered amplitudes that satisfy electroweak Ward identity that connects matrix elements of “gauge” and “Goldstone” currents

\[ \frac{k^\mu}{m_W} \cdot k \cdot \frac{\mu}{m_W} \cdot = -i \cdot \cdots \cdot \phi \quad \cdot (3.7) \]

This relation between matrix elements of the two currents should be valid in any gauge, including the unitary one.

For the purpose of computing color-ordered amplitudes, we can define the color-stripped currents, where all external particles – including the Higgs bosons – are ordered, in full analogy with QCD. Similar to ordered amplitudes, these currents can be computed as sums of all color-ordered diagrams, using a set of color-ordered Feynman rules. The \( n \)-particle partial amplitude is obtained by computing the scalar product of a \((n-1)\)-point current with the polarization vector of the \( n \)-th particle and taking the on-shell limit

\[ A_{\text{tree}}^n (1, 2, \cdots, n-1, n) = \lim_{k_1^2 \to m_W^2} \epsilon_\mu (k_1) W_\mu^\nu (2, \cdots, n-1, n) . \quad (3.8) \]

We envision that ordered amplitudes can, eventually, be used in one-loop computations based on generalized unitarity [7]. To enable computations in the unitary gauge, it is crucial that ordered amplitudes satisfy the electroweak Ward identity, in the sense of Eq.(3.7) since this relation allows, formally, to start a calculation in the Feynman gauge and then argue that, after taking the unitarity cuts, contributions of unphysical \( W \)-polarization and contribution of the Goldstone boson cancel out exactly, leaving out the unitary gauge result. Therefore, we require that Eq.(3.7) holds for color-ordered currents

\[ \frac{k^\mu}{m_W} W_\mu^\nu (2, \cdots, n-1, n) = -iG (2, \cdots, n-1, n) . \quad (3.9) \]

Finally, we require that partial amplitudes are perturbative-unitary. By this we mean that amplitudes for gauge-boson scattering approach a constant in the limit of infinitely large center-of-mass energy. Explicitly,

\[ A_{\text{tree}}^n (1, 2, \cdots, n) = \text{constant} + \mathcal{O} \left( \frac{m_W^2}{s_{ij}} \frac{m_H^2}{s_{ij}} \right) , \quad s_{ij} \to \infty , \quad (3.10) \]

where 1, 2, \cdots, \( n \) can be gauge bosons of any polarization or Higgs bosons, and \( s_{ij} = (k_i + k_j)^2 \) where \( i, j \) represents any two of the external particles. Empirically, we find that, after enforcing gauge invariance, perturbative unitarity works out automatically.

Since we plan to use unitary gauge for the computation of color-ordered amplitudes, we do not need to discuss interaction vertices where Goldstone bosons appear. However,\footnote{We do not include the external propagator of \( n \)-th particle into the definition of the current.}
we need such vertices to check the electroweak Ward identity. Having in mind the unitary
gauge, we only require a vertex with a single Goldstone boson $\phi HW$. Indeed, vertices with
larger number of Goldstone bosons lead to diagrams where Goldstone bosons appear as
internal particles; such diagrams decouple in the unitary gauge, due to the infinitely large
mass of the Goldstone boson. We will take

$$p_1^\mu = \frac{g}{2\sqrt{2}} (p_1 - p_2)^\mu$$

as the color-stripped Feynman rule for the interaction of the Goldstone boson with physical
degrees of freedom and check if this is sufficient to maintain gauge invariance.

We are now in position to start checking the electroweak Ward identity for color-
ordered amplitudes. We begin with the ordered amplitude that describes scattering of four
$W$-bosons $0 \to W_1(p_1) + W_2(p_2) + W_3(p_3) + W_4(p_4)$. The corresponding diagrams are
shown in Fig. 1. The vertices that contribute to the description of the $W$-boson scattering
all follow from the color-stripped version of vertices that naturally arise in the unitary
gauge, see Eqs. (3.4). The right hand side of the Ward identity Eq. (3.9) is even simpler and
receives contributions from the color-ordered vertex shown in Eq. (3.11).

Because of the symmetry of the problem, we only have to check the Ward identity with
respect to the momentum of one gauge boson. We choose the gauge boson with momentum
$p_1$ for this purpose. We write the scattering amplitude as

$$M = \epsilon_{1\mu} \left( J_s^\mu + J_t^\mu + J_{4W}^\mu + J_{s,H}^\mu + J_{t,H}^\mu \right),$$

where the currents are defined in Fig. 1. We begin by considering all diagrams without
the intermediate Higgs boson. We consider first the $s$–channel diagram that contributes
and to \( J_s^\mu \). We obtain

\[
J_s^\mu = \left( \frac{ig}{\sqrt{2}} \right)^2 [(p_1 - p_2)^\alpha \epsilon_2^\mu + (p_2 - q)^\mu \epsilon_2^\alpha - 2p_1 \cdot \epsilon_2 g^\alpha \eta^\mu] \frac{i}{q^2 - m_W^2} \\
\times \left( -g_{\alpha\beta} + \frac{q_\alpha q_\beta}{m_W^2} \right) [(p_3 - p_4)^3 \epsilon_3 \cdot \epsilon_4 + 2p_4 \cdot \epsilon_3 \epsilon_4^\beta - 2p_3 \cdot \epsilon_4 \epsilon_3^\beta],
\]

(3.13)

with \( q = -p_1 - p_2 \). To contract \( J_s \) with \( p_1 \), we note that

\[
p_{1,\mu} \cdot [(p_1 - p_2)^\alpha \epsilon_2^\mu + (p_2 - q)^\mu \epsilon_2^\alpha - 2p_1 \cdot \epsilon_2 g^\alpha \eta^\mu] = q^\alpha p_1 \cdot \epsilon_2 + (q^2 - m_W^2) \epsilon_2^\alpha.
\]

(3.14)

and

\[
[q^\alpha p_1 \cdot \epsilon_2 + (q^2 - m_W^2) \epsilon_2^\alpha] \left( -g_{\alpha\beta} + \frac{q_\alpha q_\beta}{m_W^2} \right) = -(q^2 - m_W^2) \epsilon_{2,\beta}.
\]

(3.15)

Putting everything together we obtain

\[
p_1 \cdot J_s = \frac{ig^2}{2} [(p_3 - p_4) \cdot \epsilon_2 (\epsilon_3 \cdot \epsilon_4) + 2p_4 \cdot \epsilon_3 (\epsilon_2 \cdot \epsilon_4) - 2p_3 \cdot \epsilon_4 (\epsilon_2 \cdot \epsilon_3)].
\]

(3.16)

Contribution of the \( t \)-channel diagram \( J_t^\mu \) is easily obtained from the \( J_s^\mu \) by interchanging 2 and 4. The result is

\[
p_1 \cdot J_t = \frac{ig^2}{2} [(p_3 - p_2) \cdot \epsilon_4 (\epsilon_2 \cdot \epsilon_3) + 2p_2 \cdot \epsilon_3 (\epsilon_2 \cdot \epsilon_4) - 2p_3 \cdot \epsilon_2 (\epsilon_3 \cdot \epsilon_4)].
\]

(3.17)

The \( 4W \)-vertex gives a contribution

\[
p_1 \cdot J_{4W} = ig^2 \left[ p_1 \cdot \epsilon_3 (\epsilon_2 \cdot \epsilon_4) - \frac{1}{2} p_1 \cdot \epsilon_2 (\epsilon_3 \cdot \epsilon_4) - \frac{1}{2} p_1 \cdot \epsilon_4 (\epsilon_2 \cdot \epsilon_3) \right].
\]

(3.18)

Putting together the “pure-gauge” contributions, we find

\[
p_1 \cdot (J_s + J_t + J_{4W}) = \frac{ig^2}{2} [\epsilon_2 \cdot \epsilon_3 (-2p_3 + p_3 - p_2 - p_1) \cdot \epsilon_4 \\
+ \epsilon_2 \cdot \epsilon_4 (2p_4 + 2p_2 + 2p_1) \cdot \epsilon_3 + \epsilon_3 \cdot \epsilon_4 (p_3 - p_4 - 2p_3 - p_1) \cdot \epsilon_2] \\
- \frac{ig^2}{2} [(\epsilon_2 \cdot \epsilon_3) p_4 \cdot \epsilon_4 - 2(\epsilon_2 \cdot \epsilon_4) p_3 \cdot \epsilon_3 + (\epsilon_3 \cdot \epsilon_4) p_2 \cdot \epsilon_2] \\
= 0.
\]

(3.19)

We now consider diagrams with the intermediate Higgs boson. In the \( s \)-channel we have

\[
p_1 \cdot J_{s,H} = \left[ \frac{ig}{\sqrt{2}} m_W p_1 \cdot \epsilon_2 \right] \frac{i}{q^2 - m_H^2} \left[ \frac{ig}{\sqrt{2}} m_W \epsilon_3 \cdot \epsilon_4 \right].
\]

(3.20)

with \( q = -p_1 - p_2 \). If we use the physical condition \( p_2 \cdot \epsilon_2 = 0 \), we can write \( p_1 \cdot \epsilon_2 = -(q - p_1) \cdot \epsilon_2/2 \) and

\[
p_1 \cdot J_{s,H} = -im_W \left[ \frac{q}{2\sqrt{2}} (q - p_1) \cdot \epsilon_2 \right] \frac{i}{q^2 - m_H^2} \left[ \frac{iq}{\sqrt{2}} m_W \epsilon_3 \cdot \epsilon_4 \right].
\]

(3.21)
Because of the Feynman rule shown in Eq. (3.11), this result is in exactly $-i m_W J_{s,\phi}$. The same result clearly holds also for $J_{t,H}$ and $J_{t,\phi}$. Hence, we conclude that in case of $W$-scattering, the Ward identity holds and it works out in the following way: it holds diagram by diagram for Higgs exchanges while the sum of diagrams that only involve gauge bosons is transverse on its own in the unitarity gauge. This result is summarized in Fig. 8: the four-$W$ ordered amplitude satisfies the relevant Ward identity, with the need for new interaction vertices. We note that a tight relation between Higgs exchanges and pure gauge scattering diagrams comes from the requirement that color-ordered amplitudes are unitary. We have checked that perturbative unitarity holds for 4$W$ scattering.

As the next step, we consider the $0 \rightarrow W_1(p_1) + W_2(p_2) + W_3(p_3) + H_4(p_4)$ ordered scattering amplitude. This is no longer a fully symmetric case, and we have to check three different Ward identities separately, one for each $W$ leg. However, given the fact that ordered amplitudes are cyclic-symmetric, only two cases are independent. The relevant currents are shown in Fig 2.

We begin by checking the Ward identity with respect to the $W_1$ leg. We write the scattering amplitude as

$$\mathcal{M} = \epsilon_{1\mu} \left( \tilde{J}_s^\mu + \tilde{J}_t^\mu \right).$$

Using partial results presented in the discussion of four-$W$ scattering amplitude, it is straightforward to compute the $s$–channel contribution

$$p_1 \cdot \tilde{J}_s = \left[ \frac{ig}{\sqrt{2}}(m_W^2 - q^2) \epsilon_2^\beta \right] \frac{i}{q^2 - m_W^2} \left[ \frac{ig}{\sqrt{2}} m_W \epsilon_{3,\beta} \right] = \frac{ig^2}{2} m_W \epsilon_2 \cdot \epsilon_3. \quad (3.23)$$

For the $t$–channel contribution we can write

$$p_1 \cdot \tilde{J}_t = \left[ \frac{ig}{\sqrt{2}} m_W p_1^\gamma \right] \frac{i}{q^2 - m_W^2} \left( -g_{\alpha \beta} + \frac{q_{\alpha} q_{\beta}}{m_W^2} \right) V_3^\beta (-q, p_2, p_3), \quad (3.24)$$

where $V_3$ is the all-outgoing three-boson vertex. Writing $p_1 = -(q + p_4 - p_1)/2$, we obtain

$$p_1 \cdot \tilde{J}_t = -im_W \left[ \frac{g}{2\sqrt{2}} (p_4 - p_1)^a \right] \frac{i}{q^2 - m_W^2} \left( -g_{\alpha \beta} + \frac{q_{\alpha} q_{\beta}}{m_W^2} \right) V_3^\beta (-q, p_2, p_3)$$

$$-im_W \left[ \frac{g}{2\sqrt{2}} q^\gamma \right] \frac{i}{q^2 - m_W^2} \left( -g_{\alpha \beta} + \frac{q_{\alpha} q_{\beta}}{m_W^2} \right) V_3^\beta (-q, p_2, p_3). \quad (3.25)$$
Since \( q \cdot V_3(-q, p_2, p_3) = 0 \), the second line vanishes and the first term coincides with 
\(-imW\tilde{J}_{t,\phi}\). Hence, if we put everything together, we obtain the violation of the Ward identity – the divergence of the \( s \)-channel contribution does not match any term on the right hand side of the Ward identity. Explicitly, we obtain

\[
p_1 \cdot W(2, 3, 4) = -imW G(2, 3, 4) + \frac{ig^2}{2} m_W \epsilon_2 \cdot \epsilon_3. \tag{3.26}
\]

At the color-dressed level the offending term cancels between \( s \)- and \( u \)-channel contributions but, if we want ordered amplitudes to satisfy the Ward identity, we need to introduce additional vertices. The simplest one to introduce to enforce the Ward identity is a local \( \phi WWH \) vertex. We can take the corresponding color-ordered vertices to be

\[
\begin{align*}
\mu & \nu \\
\text{ } & = g^2 \frac{1}{4} g^{\mu\nu},
\end{align*}
\]

\[
\begin{align*}
\mu & \nu \\
\text{ } & = -g^2 \frac{1}{4} g^{\mu\nu}.
\end{align*}
\tag{3.27}
\]

The color-dressed version of this vertex is constructed in such a way that, when the sum over all colors is taken, this vertex vanishes. This is important for ensuring that this vertex does not contribute to color-dressed amplitudes.

We are now in position to check the Ward identity for the \( W_2 \) boson. We write the amplitude as

\[
\mathcal{M} = \epsilon_{2\mu} \left( \tilde{J}_s^\mu + \tilde{J}_t^\mu \right). \tag{3.28}
\]

We start with the \( s \)-channel current. We can use the result in Eq. (3.23), after \( 1 \leftrightarrow 2 \) flip. The flip gives a minus sign and we obtain

\[
p_2 \cdot J_s = -\frac{ig^2}{2} m_W \epsilon_1 \cdot \epsilon_3. \tag{3.29}
\]

The same is true for the \( t \)-channel diagram. We just have to take the result above and exchange \( 1 \) with \( 3 \). In this case the \( VVV \) vertex picks up a minus sign, while the \( VVH \) vertex is unchanged. We are left then with an overall minus sign, with the result

\[
p_2 \cdot J_t = +\frac{ig^2}{2} m_W \epsilon_3 \cdot \epsilon_1. \tag{3.30}
\]

We see that the Ward identity is satisfied without additional vertices. Hence, we must forbid the four-particle ordered vertex \( W\phi WH \), in spite of the existence of the ordered vertex \( \phi WWH \).

Finally, we have to consider amplitudes with two \( W \)-bosons and two Higgs bosons, \( 0 \rightarrow W(p_1) + W(p_2) + H(p_3) + H(p_4) \). Because of symmetry, we only have to check the Ward identity with respect to one of the vector bosons; we choose the \( W_1 \). The relevant currents are shown in Fig. 3. We write the amplitude as

\[
\mathcal{M} = \epsilon_{1\mu} \left( \tilde{J}_s^\mu + \tilde{J}_t^\mu + \tilde{J}_{2W2H}^\mu \right). \tag{3.31}
\]
We start with the $t$–channel contribution

\[ p_1 \cdot J_t = -im_W \left[ \frac{g}{2\sqrt{2}} (p_4 - p_1) \right] \frac{i}{q^2 - m_W^2} \left( -g_{\alpha\beta} + \frac{q_\alpha q_\beta}{m_W^2} \right) \left[ \frac{ig}{\sqrt{2}} m_W \epsilon_2^\alpha \right] \]

\[ -im_W \left[ \frac{g}{2\sqrt{2}} q^\alpha \right] \frac{i}{q^2 - m_W^2} \left( -g_{\alpha\beta} + \frac{q_\alpha q_\beta}{m_W^2} \right) \left[ \frac{ig}{\sqrt{2}} m_W \epsilon_2^\beta \right]. \tag{3.32} \]

The first line cancels with the single term on the right-hand side of the Ward identity, while the second one gives

\[ -im_W \left[ \frac{g}{2\sqrt{2}} q^\alpha \right] \frac{i}{q^2 - m_W^2} \left( -g_{\alpha\beta} + \frac{q_\alpha q_\beta}{m_W^2} \right) \left[ \frac{ig}{\sqrt{2}} m_W \epsilon_2^\beta \right] = ig^2 \frac{4}{4} q \cdot \epsilon_2. \tag{3.33} \]

The $WWHH$ vertex gives a contribution

\[ p_1 \cdot J_{W2H} = ig^2 \frac{4}{4} p_1 \cdot \epsilon_2, \tag{3.34} \]

so that – once we put everything together – we find a mismatch in the Ward identity

\[ p_1 \cdot W(2, 3, 4) = -im_W G(2, 3, 4) + ig^2 \frac{4}{4} (p_1 + q) \cdot \epsilon_2. \tag{3.35} \]

To fix the last term, we modify the $W(2, 3, 4)$ current by making use of the fact that the $s$-channel current mediated by the exchange of the $W$-boson is propagator-free, when contracted with $p_1$. Since we need $HH$ final state, we introduce the ordered $WHH$ interaction vertex

\[ \mu, \tilde{a} \cdots \frac{g}{2} (p_1 - p_2)^\mu. \tag{3.36} \]

To ensure that this vertex does not contribute to color-dressed amplitudes with external physical particles, we assign the $f^{\tilde{a}\tilde{b}\tilde{c}}$ color factor to it. As a result, unphysical particles are produced in pairs by this vertex and, therefore, do not contribute to amplitudes with physical external particles. Note, however, that the new $WHH$ interaction vertex leads to additional contributions to $WWW \nu H$ amplitude that we already considered and found to satisfy the Ward identity without it. Therefore, we have to make sure that the addition
of the new vertex does not destroy the Ward identity. A simple computation shows that the Ward identity for $WWWH$ amplitude remains valid even after the addition of diagrams with new vertices. The full Ward identities with additional couplings are shown in Appendix B.

In fact, the color-ordered Feynman rules that appeared in the discussion of four-particle scattering amplitudes are sufficient to define tree-level color-ordered amplitudes for arbitrary multiplicities of external states. Given the color-ordered Feynman rules, partial amplitude $A_{\text{tree}}^{n}(1, \cdots, n)$ can be computed in an extremely efficient way using the Berends-Giele recursion relations for off-shell currents. The Berends-Giele recursion relations for the Higgs boson current, the gauge boson current, and for the Goldstone boson current are shown in Figs. 5, 6, 7. These color-ordered currents satisfy coupled recursion relations, which resemble the recursion relations in QCD [11] but are more complicated.

Working in an unitary gauge, we checked numerically that partial amplitudes satisfy the Ward identity Eq. (3.9) and are unitary Eq. (3.10) for up to 9 external particles. We have also checked that full amplitude obtained by computing color-ordered amplitudes and assembling them into a full amplitude using Eq. (3.6) agrees with the full scattering amplitude computed with Feynman diagrams.

The left pane in Fig. 4 shows the dependence of the ordered amplitude for 9-$W$ scattering as a function of the collision energy. It is clear from that Figure that the ordered amplitude approaches the constant limit at high-energy, consistent with perturbative unitarity. We have checked that this behavior is typical for other ordered multi-particle amplitudes. Note that zeroes of the ordered amplitude also exist, but we checked that they do not correspond to zeroes of the full amplitude. In the next Section, we prove that the color-ordered currents that can be constructed using color-ordered Feynman rules satisfy electroweak Ward identity Eq. (3.9).

The right pane in Fig. 4 shows the time required to compute a single color-ordered amplitude. We have checked that – for $n$-point amplitude the time scales like $n^4.4$ under double precision, roughly independent of the type and polarizations of external particles. This time scaling is similar to what has been achieved in computations of pure gluon

Figure 4. Left pane: a typical color-ordered amplitude as a function of the center-of-mass energy clearly shows behavior consistent with perturbative unitarity. Right pane: time required to compute a typical partial amplitude for $W$-scattering under double or quadruple precision.
amplitudes, see e.g. Ref. [21]. The roughly $n^4$ scaling can be understood as follows: (1) to calculate $(n - 1)$-point currents one needs to calculate 1-point, 2-point,...,$(n - 2)$-point currents, requiring $O(n)$ recursions; (2) there are $O(n) (n - 1)$-point ordered currents to calculate; (3) to calculate each $(n - 1)$-point ordered current via recursion relation, the maximum number of ways to split is $O(n^2)$, corresponding to 4-point vertices. One would expect improved scaling as $n^3$ if 4-point vertices are traded for 3-point vertices by introducing auxiliary fields. Similar conclusion has also been achieved in the study of color-dressed recursions [19]. We also compared the time required for calculating comparable processes using our recursive code and MadGraph5 [22]; the results of the comparison are shown in Table 2. We found the recursive approach much more efficient than the traditional Feynman diagrammatic approach, especially for large number of external gauge bosons.

### Table 2.

| Process       | MadGraph5 Time | Recursive Time | Ratio |
|---------------|----------------|----------------|-------|
| $WW\rightarrow 4W$ | 0.026 s        | $WW\rightarrow 4W$ | 0.006 s | 4.3   |
| $WW\rightarrow 4W + Z$ | 6.66 s         | $WW\rightarrow 5W$ | 0.072 s | 92.5  |

Efficiency comparison between the recursive method and MadGraph5. Computation was performed on the same computer, in double precision. We studied comparable, but not identical, processes with the same number of external particles and similar number of Feynman diagrams. The MadGraph calculation refers to full weak-boson scattering in the Standard Model, while the recursive computation refers to the broken $SU(2)$ model that we consider in this paper.

4 Proof of electroweak Ward identity for arbitrary multiplicity

In this Section we present the recursive proof of the Ward identity Eq.(3.9) for ordered tree amplitudes of arbitrary multiplicity, generalizing the discussion in Ref.[11] to the case of currents in broken gauge theory. To facilitate the proof, we introduce some compact notations. We consider $n$-point off-shell currents, where the $n$ on-shell physical particles have outgoing momenta $k_i^\mu$, $i = 1, 2, \cdots, n$ and are physically ordered. We use $\tilde{W}_a, \tilde{H}_a$ and $\tilde{G}_a$ to denote off-shell currents coupled to gauge bosons $W$, the Higgs boson $H$ and the Goldstone $G$, respectively. Note that for the sake of compactness, we do not display Lorentz index of the gauge current. We also choose not to multiply the currents by an off-shell propagator.

The subscript in currents $\tilde{W}_a(\tilde{H}_a)$, $a = 1, 2, 3, \cdots$, is used to indicate how many currents the original current has been divided into, and their relative ordering; summation over all possible partitions is implicitly assumed. For example a term $\tilde{H}_1 (\tilde{W}_2 \cdot \tilde{W}_3)$ is the shorthand notation for

$$
\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \tilde{H}_1 (i, i) \left( \tilde{W}_1 (i + 1, j) \cdot \tilde{W}_3 (j + 1, n) \right),
$$

(4.1)
and a term $\tilde{H}_1 \tilde{H}_2 \left( p_2 \cdot \tilde{W}_3 \right) \tilde{H}_4$ is the shorthand notation for

$$\sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \tilde{H} \left( 1, i \right) \tilde{H} \left( i + 1, j \right) \left( p \left( i + 1, j \right) \cdot \tilde{W} \left( j + 1, k \right) \right) \tilde{H} \left( k + 1, n \right).$$

(4.2)

where the momentum sum $p(i, j)$ is defined as

$$p \left( i, j \right) = \sum_{m=i}^{j} k_m.$$

(4.3)

We also denote by $W_a$ and $H_a$ the currents where the propagator of the off-shell leg is multiplied in (since Goldstone boson only appears as external current in the unitary gauge,
there is no need to define such an object also for $G$). These currents read

$$W_a = \frac{-i}{p_a^2 - m_W^2} \left( \tilde{W}_a - \frac{p_a \cdot \tilde{W}_a}{m_W^2} p_a \right), \quad H_a = \frac{i}{p_a^2 - m_H^2} \tilde{H}_a. \quad (4.4)$$

We want to show that

$$p_a \cdot \tilde{W}_a = -i m_W \tilde{G}_a, \quad p_a \cdot W_a = \frac{1}{m_W} \tilde{G}_a. \quad (4.5)$$

We prove the Ward identity by induction. The induction starts with one-particle gauge current, which is simply the polarization vector of the relevant particle. Note that a single-particle Goldstone current vanishes since all external particles are physical. The Ward identity for single-particle currents then follows trivially.

Furthermore, we introduce a useful notation to describe three- and four-point gauge vertices

$$W_1 (-2p_1 - p_2) \cdot W_2 + W_2 (p_1 + 2p_2) \cdot W_1 + (p_1 - p_2) (W_1 \cdot W_2) = [W_1, W_2], \quad (4.6)$$

$$2W_2 (W_1 \cdot W_3) - W_1 (W_2 \cdot W_3) - W_3 (W_1 \cdot W_2) = \{W_1, W_2, W_3\}. \quad (4.7)$$

Using this notation, Berends-Giele recursion relations can be written in a compact way. For example, the recursion relation for the gauge current reads

$$\tilde{W} = \frac{ig}{\sqrt{2}} [W_1, W_2] + \frac{ig^2}{2} \{W_1, W_2, W_3\} + \cdots. \quad (4.8)$$

The advantage of this notation is that recursion relations can be re-inserted into the right-hand-side of the recursion relation in a compact way. For example

$$\tilde{H}_1 (\tilde{W}_2 \cdot \tilde{W}_3) = \frac{ig}{\sqrt{2}} \tilde{H}_1 [W_2, W_3] \cdot \tilde{W}_4 + \frac{ig^2}{2} \tilde{H}_1 \{W_2, W_3, W_4\} \cdot \tilde{W}_5 + \cdots, \quad (4.9)$$
where in the right hand side we wrote the recursion relation for $\tilde{W}_2$ and re-named the currents following the convention that they are numbered according to their clock-wise appearance.

We will prove by induction that the Ward identity

$$p \cdot \tilde{W} - (-im_W)\tilde{G} = 0 \quad (4.10)$$

holds for any multiplicity of the external particles. First, we consider the gauge current $\tilde{W}$ and use the recursion relation for it. The contribution of the three-$W$ vertex gives

$$\frac{ig}{\sqrt{2}} (p_1 + p_2) \cdot [W_1, W_2] = \frac{ig}{\sqrt{2}} \left( (p_1^2 - m_W^2) - (p_2^2 - m_W^2) \right) W_1 \cdot W_2$$

$$- \frac{ig}{\sqrt{2} m_W} \left( \tilde{G}_1 (p_1 \cdot W_2) - \tilde{G}_2 (p_2 \cdot W_1) \right) = \frac{g}{\sqrt{2}} \left( \tilde{W}_1 \cdot W_2 - W_1 \cdot \tilde{W}_2 \right). \quad (4.11)$$

The four-$W$ vertex evaluates to

$$\frac{ig^2}{2} (p_1 + p_2 + p_3) \cdot \{W_1, W_2, W_3\} = \frac{ig^2}{2} \left\{ W_1 \cdot [W_2, W_3] - [W_1, W_2] \cdot W_3 \right\}. \quad (4.12)$$

If we insert the recursion relation for $\tilde{W}_0$ into Eq.(4.11) and combine the resulting expressions for three- and four-gluon vertex contributions in Eq.(4.10), we observe that pure-gauge contributions cancel out. The remaining contributions to $p \cdot \tilde{W}$ necessarily contain the Higgs boson current. To investigate those terms, we write

$$p \cdot \tilde{W} = A_1 + A_2, \quad (4.13)$$

where $A_1$ is the sum of Higgs-dependent terms shown as ellipses in Eq.(4.8) and $A_2$ is the sum of Higgs-dependent terms that arise when the recursion relation for $\tilde{W}_{1,2}$ is inserted into Eq.(4.11). Those terms read

$$A_1 = \frac{-ig}{2\sqrt{2}} \left( (p_1^2 - m_H^2) - (p_2^2 - m_H^2) \right) H_1 H_2$$

$$+ \frac{ig}{\sqrt{2} m_W} \left( H_1 (p_1 \cdot W_2) + H_2 (p_2 \cdot W_1) \right) + \frac{ig}{\sqrt{2}} \left( \tilde{G}_2 H_1 + \tilde{G}_1 H_2 \right) \quad (4.14)$$

$$+ \frac{ig^2}{4 m_W} \left( \tilde{G}_1 H_2 H_3 + H_1 H_2 \tilde{G}_3 \right) + \frac{ig^2}{4} \left( W_1 \cdot (p_2 + p_3) H_2 H_3 + H_1 H_2 (p_1 + p_2) \cdot W_3 \right);$$

$$A_2 = \frac{ig^2}{4} (-H_2 H_3 (p_2 - p_3) \cdot W_1 + H_1 H_2 (p_1 - p_2) \cdot W_3)$$

$$+ \frac{ig^2}{4 m_W} (-H_3 (W_1 \cdot W_2) + H_1 (W_2 \cdot W_3))$$

$$+ \frac{ig^3}{4 \sqrt{2}} \left( - (W_1 \cdot W_2) H_3 H_4 + H_1 H_2 (W_3 \cdot W_4) \right). \quad (4.15)$$

Note that the first term in Eq.(4.14) is $\frac{a^2}{2\sqrt{2}} \left( \tilde{H}_1 H_2 - H_1 \tilde{H}_2 \right)$. To simplify it, we insert recursion relation to eliminate $H_2$. The appearance of Goldstone currents $\tilde{G}_a$ is the consequence of applying Ward identity to gauge currents of lower multiplicity. Finally we need
the recursion relation for the Goldstone boson current; it reads

$$im_W \tilde{G} = -\frac{ig}{\sqrt{2}} m_W (H_2 (p_2 \cdot W_1) + H_1 (p_1 \cdot W_2))$$

$$- \frac{ig}{2\sqrt{2}} (\tilde{G}_1 H_2 + H_1 \tilde{G}_2) + \frac{ig^2}{4} m_W ((W_1 \cdot W_2) H_3 - H_1 (W_2 \cdot W_3)).$$  (4.16)

Similar to the previous discussion, terms with $\tilde{G}_{1,2}$ currents arise because of electroweak Ward identity. After collecting all terms, we finally arrive at the following equation

$$p \cdot \tilde{W} - (-im_W)\tilde{G} = \frac{ig}{2\sqrt{2}} \left( \tilde{G}_1 H_2 + H_1 \tilde{G}_2 \right) + \frac{ig^2}{8\sqrt{2}} (H_1 H_2 (W_3 \cdot W_4) - (W_1 \cdot W_2) H_3 H_4)$$

$$+ \frac{ig^2}{8m_W} \left( \tilde{G}_1 H_2 H_3 + H_1 H_2 \tilde{G}_3 + 2H_1 \tilde{G}_2 H_3 \right)$$

$$+ \frac{ig^2}{4} (H_2 H_3 (p_2 \cdot W_1) + H_1 H_2 (p_2 \cdot W_3))$$

$$+ H_1 H_3 (p_1 \cdot W_2) + H_1 H_3 (p_3 \cdot W_2)).$$  (4.17)

Eliminating $\tilde{G}_a$ in the first term by inserting recursion relation, and applying the Ward identity for lower-multiplicity currents, we find that all terms cancel out exactly. This proves the assertion that the color-ordered Feynman rules that we constructed allow us to define color-ordered currents that satisfy electroweak Ward identity.

5 Conclusion

In this paper, we studied gauge boson scattering amplitudes in $SU(2)$ gauge theory, spontaneously broken by the Higgs mechanism. We constructed color-ordered scattering amplitudes that satisfy electroweak Ward identities and respect perturbative unitarity. Those color-ordered amplitudes are peculiar in that both external gauge bosons – that carry color – and the Higgs boson – that is neutral – are physically ordered. We present explicitly a set of color-ordered Feynman rules, which lead to coupled Berends-Giele recursion relations for color-ordered currents. Similar to QCD, these color-ordered currents can be used to efficiently compute tree color-ordered amplitudes. We presented a proof of gauge invariance for off-shell currents of arbitrary multiplicity. Full color-dressed tree-level scattering amplitudes can be constructed from color-ordered amplitudes via the color decomposition in terms of traces of products of group generators in the fundamental representation of $SU(2)$.

Our decomposition is restricted to $SU(2)$, due to the relative simple group structure. For a gauge theory of broken $SU(N)$ with a variety of different breaking schemes, the surviving global symmetry can be very different. As the result, the usefulness of color decomposition and color-ordered amplitudes in that case can be questioned. In particular, in realistic electroweak models, gauge bosons acquire different masses and have different couplings to the Higgs sector, so there is a lack of symmetries to make use of.

Another issue is the generalization of our decomposition to the one-loop level. The central question is how to construct gauge-invariant color-stripped objects that properly
reflect the cut structure of the full amplitude and how to assemble those objects into the full one-loop amplitude. The unitarity gauge, which deals with only the physical degrees of freedom in intermediate states, is clearly ideal for the on-shell methods. However, as we emphasized several times, our construction of color-stripped amplitudes introduces unphysical fields and unphysical vertices whose contribution to the entire amplitude cancels out once the sum over colors is taken. It is unclear to us at the moment if this can be also arranged at the one-loop level. This remains an interesting open question for the future.

Acknowledgments

K.M. would like to thank Zoltan Kunszt for useful discussions. This research is supported by the NSF under grants PHY-0855365 and by the start-up funds provided by the Johns Hopkins University. L.D. is supported by the Rowland Research Fellowship awarded by the Department of Physics and Astronomy of the Johns Hopkins University.
A Relevant Feynman rules

We list all non-vanishing color-ordered Feynman rules in the unitary gauge, which give
gauge invariant and unitary partial amplitudes. Gauge bosons, the Higgs boson, and
Goldstone bosons are represented by wavy lines, dashed lines and dotted lines, respectively. External legs are ordered clockwise and all momenta are outgoing.

\[ k_1 \quad \mu \quad k_2 \quad \nu \quad k_3 \quad \rho \]

\[ k_1 \quad \mu \quad k_2 \quad \nu \quad k_3 \quad \rho \]

\[ \rho = ig \frac{\sqrt{2}}{2} [g_{\mu \nu} (k_1 - k_2)^\rho + g_{\nu \rho} (k_2 - k_3)^\mu + g_{\rho \mu} (k_3 - k_1)^\nu] \]

\[ \rho = ig^2 \left[ g_{\mu \sigma} g_{\nu \rho} - \frac{1}{2} g_{\mu \nu} g_{\rho \sigma} - \frac{1}{2} g_{\mu \rho} g_{\nu \sigma} \right] \]

\[ \rho = ig^2 \frac{\sqrt{2}}{4} m_W g_{\mu \nu} \]

\[ \rho = g^2 \frac{4}{4} g_{\mu \nu} \]

\[ \rho = \frac{g^3}{4} g_{\mu \nu} \]

\[ \rho = g \frac{2 \sqrt{2}}{4} (p_1 - p_2)^\mu \]

\[ \rho = \frac{g}{2 \sqrt{2}} (p_1 - p_2)^\mu \]

\[ \rho = \frac{g}{2 \sqrt{2}} (p_1 - p_2)^\mu \]

We also present below a set of color-dressed Feynman rules for the extended particle content, which is useful in the proof of color decomposition.
\[ \rho, \bar{c} = -g \varepsilon^{\bar{a}b\bar{c}} [g^\mu\nu (k_1 - k_2)^\rho + g^\nu\rho (k_2 - k_3)^\mu + g^\rho\mu (k_3 - k_1)^\nu] \]

\[ \rho, \bar{c} = -ig^2 \left[ \varepsilon^{\bar{a}b\bar{c}} \varepsilon^{\bar{d}\bar{e}} \left( g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + \varepsilon^{\bar{a}d\bar{e}} \varepsilon^{\bar{c}\bar{b}} \left( g^{\mu\rho} g^{\sigma\nu} - g^{\mu\nu} g^{\rho\sigma} \right) \right] \]

\[ \bar{c} = \frac{ig}{\sqrt{2}} m_W g^{\mu\nu} \left\{ \text{Tr} \left( T^{\bar{a}} T^{\bar{b}} T^{\bar{c}} \right) + \text{Tr} \left( T^{\bar{b}} T^{\bar{a}} T^{\bar{c}} \right) \right\} \]

\[ \bar{c} = \frac{ig}{\sqrt{2}} g^{\mu\nu} \left\{ \left( \text{Tr} \left( T^{\bar{a}} T^{\bar{b}} T^{\bar{c}} \right) + \text{Tr} \left( T^{\bar{b}} T^{\bar{a}} T^{\bar{c}} \right) \right) \times \left( \text{Tr} \left( T^{\bar{d}} T^{\bar{e}} T^{\bar{c}} \right) + \text{Tr} \left( T^{\bar{d}} T^{\bar{e}} T^{\bar{c}} \right) \right) \right\} \]

\[ \bar{c} = -\frac{g}{2\sqrt{2}} (p_1 - p_2)^\mu \left\{ \text{Tr} \left( T^{\bar{a}} T^{\bar{b}} T^{\bar{c}} \right) + \text{Tr} \left( T^{\bar{b}} T^{\bar{a}} T^{\bar{c}} \right) \right\} \]

\[ \bar{c} = -i \frac{g}{2\sqrt{2}} (p_1 - p_2)^\mu \left\{ \text{Tr} \left( T^{\bar{a}} T^{\bar{b}} T^{\bar{c}} \right) - \text{Tr} \left( T^{\bar{b}} T^{\bar{a}} T^{\bar{c}} \right) \right\} \]

\[ \bar{c} = -\frac{g^2}{4} g^{\mu\nu} \left\{ \left( \text{Tr} \left( T^{\bar{a}} T^{\bar{b}} T^{\bar{c}} \right) + \text{Tr} \left( T^{\bar{b}} T^{\bar{a}} T^{\bar{c}} \right) \right) \times \left( \text{Tr} \left( T^{\bar{d}} T^{\bar{e}} T^{\bar{c}} \right) - \text{Tr} \left( T^{\bar{d}} T^{\bar{e}} T^{\bar{c}} \right) \right) \right\} \]
B The Ward identity

\[ i \frac{1}{m_W} + + + + = + + \]

Figure 8. Gauge invariance for $WW \rightarrow WW$ color-ordered amplitude.

\[ i \frac{2}{m_W} + + + + = + + \]

Figure 9. Gauge invariance for $WW \rightarrow WH$ partial amplitude.

\[ i \frac{1}{m_W} + + + + = \]

Figure 10. Gauge invariance for $WW \rightarrow HH$ partial amplitude.
C Numerical results for amplitudes

For future reference, we present numerical results for multi-W scattering amplitudes for a typical phase space point. We choose \( m_W = 80 \text{ GeV} \) and \( m_H = 114 \text{ GeV} \) and set the gauge coupling constant to \( g = 0.1 \). All momenta are given in GeV.

**Scattering of five W bosons**

We first list all color-ordered primitive amplitudes (there are \( 4! = 24 \) of them) for \( W(k_1)W(k_2) \rightarrow W(k_3)W(k_4)W(k_5) \) scattering. In the center of mass frame, the momenta of gauge bosons are

\[
\begin{align*}
  k_1^\mu &= (400.0, 0.0, 0.0, 0.0) \\ 
  k_2^\mu &= (400.0, 0.0, 0.0, -391.91835884531) \\ 
  k_3^\mu &= (141.60000091791, -21.221509298529, -65.313093963515, -94.521995112027) \\ 
  k_4^\mu &= (272.59123471245, -47.332020563805, -145.67298975201, -210.81992583272) \\ 
  k_5^\mu &= (385.80876436964, 68.553529862334, 210.98608371552, 305.3419209475)
\end{align*}
\]

\[(C.1)\]

The first benchmark configuration has all longitudinal polarizations \( \{h_i\} = \{L, L, L, L, L\} \)\(^2\)

\[
\begin{align*}
  \epsilon_1^\mu &= (4.8989794855664, 0.0, 0.0, 5.0) \\ 
  \epsilon_2^\mu &= (4.8989794855664, 0.0, 0.0, -5.0) \\ 
  \epsilon_3^\mu &= (1.460451515266, -0.32149505634144, -0.98946010522866, -1.4319600795855) \\ 
  \epsilon_4^\mu &= (3.2573470139166, -0.61890348724549, -1.9047691951593, -2.7566367487485) \\ 
  \epsilon_5^\mu &= (4.7177921654432, 0.87595769515527, 2.6959207494118, 3.9015730603830)
\end{align*}
\]

\[(C.2)\]

\(^2\)To avoid confusion, throughout this paper we do not take complex conjugate when multiplying polarization vectors, regardless of whether the external particle is incoming or outgoing.
Color-ordered primitive amplitudes are evaluated to be

\[
\begin{align*}
A^5_{\text{tree}}(1, 2, 3, 4, 5) &= 8.082744225626406 \times 10^{-6} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 2, 3, 5, 4) &= -9.052099561980900 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 2, 4, 3, 5) &= 2.485269458065873 \times 10^{-5} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 2, 4, 5, 3) &= 3.859247304761025 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 2, 5, 3, 4) &= 6.760721099941013 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 2, 5, 4, 3) &= -1.55210097888805 \times 10^{-6} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 3, 2, 4, 5) &= -7.534154547657562 \times 10^{-6} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 3, 2, 5, 4) &= 3.909646009037787 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 3, 4, 2, 5) &= 2.741897891315819 \times 10^{-5} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 3, 4, 5, 2) &= 1.55210097890106 \times 10^{-6} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 3, 5, 2, 4) &= 7.476469442186263 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 3, 5, 4, 2) &= -3.85924730476688 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 4, 2, 3, 5) &= -1.052143197490972 \times 10^{-5} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 4, 2, 5, 3) &= -7.476469442214452 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 4, 3, 2, 5) &= 6.344716114626181 \times 10^{-6} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 4, 3, 5, 2) &= -6.760721099878129 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 4, 5, 2, 3) &= -3.909646009018272 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 4, 5, 3, 2) &= 9.052099561980900 \times 10^{-7} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 5, 2, 3, 4) &= -6.344716114630626 \times 10^{-6} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 5, 2, 4, 3) &= -2.741897891315862 \times 10^{-5} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 5, 3, 2, 4) &= 1.052143197490939 \times 10^{-5} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 5, 3, 4, 2) &= -2.485269458066134 \times 10^{-5} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 5, 4, 2, 3) &= 7.534154547657562 \times 10^{-6} \text{ GeV}^{-1} \\
A^5_{\text{tree}}(1, 5, 4, 3, 2) &= -8.082744225625105 \times 10^{-6} \text{ GeV}^{-1}
\end{align*}
\]

The second benchmark configuration has both transverse and longitudinal polarizations \(\{h_i\} = \{+,-,+, L, L\}\). The polarization vectors are

\[
\begin{align*}
\epsilon^u_1 &= (0.0, -0.70710678118655, 0.70710678118655i, 0.0) \\
\epsilon^u_2 &= (0.0, 0.70710678118655, -0.70710678118655i, 0.0) \\
\epsilon^u_3 &= (0.0, 0.17677668390637 - 0.67249851605560i, \\
& \quad 0.54406272447999 + 0.21850799963164i, -0.41562694313348i) \\
\epsilon^u_4 &= (3.2573470139166, -0.61890348724549, -1.9047891951593, -2.7566367487485) \\
\epsilon^u_5 &= (4.7177921654432, 0.87595769515527, 2.6959207494118, 3.9015730603830)
\end{align*}
\]

(C.3)
Color-ordered primitive amplitudes are evaluated to be

\[ A^\text{tree}_5(1, 2, 3, 4, 5) = (-1.587502368520595E - 1.153387893203229i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 2, 3, 5, 4) = (-5.687601324129297 - 4.13228391887344i) \times 10^{-8} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 2, 4, 3, 5) = (-1.715712547908148 - 1.2465580533715343i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 2, 4, 5, 3) = (-1.881276398229126 - 1.366827202600273i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 2, 5, 3, 4) = (2.773037381457798 + 2.014729430705663i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 2, 5, 4, 3) = (4.71624158970851 + 3.42654981746144i) \times 10^{-8} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 3, 2, 4, 5) = (2.433008233863460 + 1.767683813673294i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 3, 2, 5, 4) = (1.085894764699578 + 0.7889486653585891i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 3, 4, 2, 5) = (-1.188112309463586 - 0.863214052886585i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 3, 4, 5, 2) = (-4.716241589779813 - 3.426549817461366i) \times 10^{-8} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 3, 5, 2, 4) = (2.529171126417367 + 1.83755023927348i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 3, 5, 4, 2) = (1.881276398229151 + 1.366827202600311i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 4, 2, 3, 5) = (-2.800106605902566 - 2.034396371916651i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 4, 2, 5, 3) = (-2.529171126417509 - 1.83755023927342i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 4, 3, 2, 5) = (1.31208508984693 + 0.953285542320996i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 4, 3, 5, 2) = (-2.733037381457778 - 2.014729430705651i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 4, 5, 2, 3) = (-1.085894764699594 - 0.7889486653588198i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 4, 5, 3, 2) = (5.687601324147984 + 4.13228391887015i) \times 10^{-8} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 5, 2, 3, 4) = (-1.31208508984696 - 0.953285542320850i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 5, 2, 4, 3) = (1.188112309463583 + 0.863214052886511i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 5, 3, 2, 4) = (2.800106605902975 + 2.034396371916417i) \times 10^{-7} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 5, 3, 4, 2) = (1.715712547908162 + 1.246538033715348i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 5, 4, 2, 3) = (-2.433008233863598 - 1.767683813673305i) \times 10^{-6} \text{ GeV}^{-1} \]
\[ A^\text{tree}_5(1, 5, 4, 3, 2) = (1.587502368520560 + 1.153387893203234i) \times 10^{-6} \text{ GeV}^{-1} \]

(C.5)

We point out that the results exhibit reflection symmetry of the primitive amplitudes for general \(W\)-Higgs multi-particle scattering

\[ A^\text{tree}_n(1, 2, \cdots, n - 1, n) = (-1)^m A^\text{tree}_n(n, n - 1, \cdots, 2, 1) \]

(C.6)

where \(m\) is the number of external gauge bosons \(W\).
Scattering of six $W$ bosons

Next we present numerical results for full amplitude of 6-$W$ scattering, i.e. $W(k_1)W(k_2) \rightarrow W(k_3)W(k_4)W(k_5)W(k_6)$. Their momenta in center of mass frame are

\[
k_1^\mu = (400.0, 0.0, 0.0, 391.91835884531)
\]

\[
k_2^\mu = (400.0, 0.0, 0.0, -391.91835884531)
\]

\[
k_3^\mu = (137.60000085831, -20.334885027467, -62.5843444866943, -90.572912422541)
\]

\[
k_4^\mu = (111.42091925759, -14.086567220910, -43.353998793403, -62.742494856025)
\]

\[
k_5^\mu = (185.38050614865, -30.374957508527, -93.484512622753, -135.29205414942)
\]

\[
k_6^\mu = (365.59857373546, 64.796409756903, 199.442285628310, 288.60746142799)
\]

(C.7)

The first benchmark configuration has all longitudinal polarizations $\{h_i\} = \{L, L, L, L, L, L\}$

\[
A_6^{\text{tree}}(1^1_L, 2^1_L, 3^2_L, 4^2_L, 5^3_L, 6^3_L) = 1.745869319633557 \times 10^{-8} \, \text{GeV}^{-2}
\]

(C.9)

To evaluate full amplitude we also have to specify the color of the gauge bosons $\{a_i, i = 1, 2, \cdots, 6\}$. We choose $\{a_i\} = \{1, 1, 2, 2, 3, 3\}$. Then the full color-dressed amplitude is found to be

\[
A_6^{\text{tree}}(1^1_L, 2^1_L, 3^2_L, 4^2_L, 5^3_L, 6^3_L) = 1.745869319633557 \times 10^{-8} \, \text{GeV}^{-2}
\]

(C.10)

The second benchmark configuration has both transverse and longitudinal polarizations $\{h_i\} = \{+, -, +, L, L, L\}$

\[
A_6^{\text{tree}}(1^1_+, 2^1_-, 3^2_L, 4^2_L, 5^3_L, 6^3_L) = \left( -5.08936118528462 - 4.002476058956751i \right) \times 10^{-9} \, \text{GeV}^{-2}
\]

(C.11)
These results are cross-checked using the conventional Feynman diagrammatic method. In that case we calculate and sum over all 730 tree diagrams for 6-W scattering. In WScattering of nine W bosons

Finally, we present numerical results for full amplitude of 9-W scattering, i.e. W(k₁)W(k₂) → W(k₃)W(k₄)W(k₅)W(k₆)W(k₇)W(k₈)W(k₉). Their momenta in center of mass frame are

\[
\begin{align*}
k_{1}^{\mu} &= (400.0, 0.0, 0.0, 391.91835884531) \\
k_{2}^{\mu} &= (400.0, 0.0, 0.0, -391.91835884531) \\
k_{3}^{\mu} &= (116.00000053644, -15.257392539990, -46.957428832438, -67.957427664543) \\
k_{4}^{\mu} &= (100.88056163819, -11.162669509069, -34.355166367926, -49.719262561900) \\
k_{5}^{\mu} &= (89.9433196606884, -7.4665293771818, -22.979616008133, -33.256411849213) \\
k_{6}^{\mu} &= (82.900106361594, -3.9479230294227, -12.150458487855, -17.584308261977) \\
k_{7}^{\mu} &= (80.019253013145, -0.31881335310116, -0.98120667078859, -1.4200155973621) \\
k_{8}^{\mu} &= (83.380380566575, -4.2685834850712, -13.13734956107, -19.012550975304) \\
k_{9}^{\mu} &= (246.87637827718, 42.421911293836, 130.56122632325, 188.94997691030) \quad (C.12)
\end{align*}
\]

The first benchmark configuration has all longitudinal polarizations \(\{h_i\} = \{L, L, L, L, L, L, L, L, L\}\)

\[
\begin{align*}
e_{1}^{i} &= (4.8989794855664, 0.0, 0.0, 5.000000000000) \\
e_{2}^{i} &= (4.8989794855664, 0.0, 0.0, -5.000000000000) \\
e_{3}^{i} &= (1.050000092600, -0.26337165583558, -0.81057466096947, -1.1730746392889) \\
e_{4}^{i} &= (0.76820485919648, -0.22904379687417, -0.70492436784595, -1.0201761026526) \\
e_{5}^{i} &= (0.51383982516229, -0.20421138712643, -0.62849806430746, -0.90957091996858) \\
e_{6}^{i} &= (0.27169250621209, -0.18822015672784, -0.57928211463475, -0.83834493032251) \\
e_{7}^{i} &= (0.021940447742366, -0.18167933678739, -0.5591553950179, -0.80921168905572) \\
e_{8}^{i} &= (0.29376006988772, -0.18931059303850, -0.58263813273021, -0.84320180521184) \\
e_{9}^{i} &= (2.9194377174610, 0.56051931235228, 1.7251011698101, 2.4965898022169) \quad (C.13)
\end{align*}
\]

For color degree of freedom, we choose \(\{a_i\} = \{1, 1, 1, 2, 2, 2, 2, 3, 3, 3\}\). The full amplitude evaluates to

\[
A_{9}^{\text{tree}}(1_{L}^{1}, 2_{L}^{1}, 3_{L}^{1}, 4_{L}^{1}, 5_{L}^{2}, 6_{L}^{2}, 7_{L}^{3}, 8_{L}^{3}, 9_{L}^{2}) = -i8.941784390273400 \times 10^{-18} \text{ GeV}^{-5} \quad (C.14)
\]
The second benchmark configuration has both transverse and longitudinal polarizations \( \{ h_i \} = \{ +, -, +, - , L, L, L, L \} \)

\[
\begin{align*}
\epsilon_1^i &= (0.0, -0.70710678118655, 0.70710678118655i, 0.0) \\
\epsilon_2^i &= (0.0, 0.70710678118655, -0.70710678118655i, 0.0) \\
\epsilon_3^i &= (0.0, 0.17677668390637 - 0.67249851605560i, \\
& \quad 0.54406272447999 + 0.21850799963164i, -0.41562694313348i) \\
\epsilon_4^i &= (0.0, 0.17677668390637 + 0.67249851605560i, \\
& \quad 0.54406272447999 - 0.21850799963164i, -0.41562694313348i) \\
\epsilon_5^i &= (0.0, 0.17677668390637 - 0.67249851605560i, \\
& \quad 0.54406272447999 + 0.21850799963164i, -0.41562694313348i) \\
\epsilon_6^i &= (0.27169250621209, -0.18822015672784, -0.57928211463475, -0.83834493032251) \\
\epsilon_7^i &= (0.021940447742366, -0.18167933678739, -0.55915153950179, -0.80921168905572) \\
\epsilon_8^i &= (0.29376006988772, -0.18931059303850, -0.58263813273021, -0.84320180521184) \\
\epsilon_9^i &= (2.9194377174610, 0.56051931235228, 1.7251011698101, 2.4965898022169)
\end{align*}
\]

Again we choose \( \{ a_i \} = \{ 1, 1, 1, 2, 2, 3, 3 \} \). The full amplitude evaluates to

\[
\mathcal{A}_{9}^{\text{tree}} \left( 1^1_+, 2^1_-, 3^1_+, 4^2_-, 5^2_+, 6^2_L, 7^3_L, 8^3_L, 9^2_L \right) \\
= (-1.621175976214353 + 2.231357479296454i) \times 10^{-17} \, \text{GeV}^{-5}
\]

By contrast, numerical implementation based on Feynman diagrams is too inefficient to yield a result for 9-W scattering in a sensible amount of time.
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