Cosmological perturbations and observational constraints on nonlocal massive gravity

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Nonlocal massive gravity can provide an interesting explanation for the late-time cosmic acceleration, with a dark energy equation of state \(w_{\text{DE}}\) smaller than \(-1\) in the past. We derive the equations of linear cosmological perturbations to confront such models with the observations of large-scale structures. The effective gravitational coupling to nonrelativistic matter associated with galaxy clusterings is close to Newton’s gravitational constant \(G\) for a mass scale \(m\) slightly smaller than today’s Hubble parameter \(H_0\). Taking into account the background expansion history as well as the evolution of matter perturbations \(\delta_m\), we test for these models with Type Ia Supernovae (SNIa) from Union 2.1, the cosmic microwave background (CMB) measurements from Planck, a collection of baryon acoustic oscillations (BAO), and the growth rate data of \(\delta_c\). Using a higher value of \(H_0\) derived from its direct measurement \((H_0 \gtrsim 70 \text{ km s}^{-1} \text{ Mpc}^{-1})\) the data strongly support the nonlocal massive gravity model \((-1.1 \lesssim w_{\text{DE}} \lesssim -1.04\) in the past) over the \(\Lambda\)CDM model \((w_{\text{DE}} = -1)\), whereas for a lower prior \((67 \text{ km s}^{-1} \text{ Mpc}^{-1} \lesssim H_0 \lesssim 70 \text{ km s}^{-1} \text{ Mpc}^{-1})\) the two models are statistically comparable.

I. INTRODUCTION

Modified gravitational theories have received much attention in connection to the dark energy problem \([1]\). In particular, the recent observational constraints derived from Planck and other data show that the dark energy equation of state \(w_{\text{DE}}\) smaller than \(-1\) is favored \([2]\). This may imply the infrared modification of gravity from general relativity (GR), because the models in the framework of GR — such as quintessence \([3]\) and k-essence \([4]\) — generally predict \(w_{\text{DE}}\) larger than \(-1\).

So far many dark energy models based on the large-distance modification of gravity have been proposed — including the Dvali-Gabadadze-Porrati (DGP) model \([5]\), \(f(R)\) gravity \([6]\), Brans-Dicke theories \([7]\), and Galileons \([8]\). In the DGP model the cosmic acceleration can be realized by the gravitational leakage to the fifth dimension, but it suffers from the incompatibility with observations \([9]\) as well as the ghost problem \([10]\). In \(f(R)\) gravity and Brans-Dicke theories it is possible to construct viable dark energy models at the expense of designing scalar potentials to be compatible with both cosmological and local gravity constraints \([11]\). In covariant Galileons there exists a tracker solution along which \(w_{\text{DE}}\) evolves from \(-2\) (matter era) to \(-1\) (de Sitter era) \([12]\), but only the late-time tracking solutions are allowed from the joint data analysis of SNIa, CMB, and BAO \([13]\).

Besides the theories mentioned above, massive gravity has recently received significant attention due to the possibility of the late-time cosmic acceleration with a mass scale \(m\) of the order of today’s Hubble parameter \(H_0\). In the original Fierz-Pauli theory \([14]\) there exists a so-called van Dam-Veltman-Zakharov (vDVZ) discontinuity \([15]\) with which the linearized GR cannot be recovered in the \(m \rightarrow 0\) limit. In the presence of nonlinear interactions the problem of the vDVZ discontinuity can be cured \([16]\), but an instability mode called the Boulware-Deser (BD) ghost appears due to nonlinearities \([17]\).

De Rham, Gabadadze and Tolley (dRGT) constructed a massive gravity theory \([18]\) in which the BD ghost is absent. In addition to acausality of the theory \([19]\) and the requirement of an external reference metric, there are some problems when the dRGT theory is applied to the cosmology. On the homogenous and isotropic cosmological background, it was shown that at least one ghost exists among five propagating degrees of freedom \([20]\). The self-accelerating solutions in the dRGT theory are also unstable against scalar and vector perturbations \([21]\). The possible way out of these problems is to break the homogeneity or the isotropy of the Universe \([22, 23]\) or to introduce other degrees of freedom \([24, 25]\).

An alternative approach to massive gravity was recently suggested by Jaccard et al. \([27]\), who introduced nonlocal terms to obtain fully covariant equations of motion without referring to any external reference metric (see also Refs. \([28, 30]\) for related works). This theory — dubbed non-local massive gravity (NLMG) — respects causality and

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reduces to a massless one without the vDVZ discontinuity in the $m \to 0$ limit. The covariant equations of motion are given by

$$G_{\mu\nu} - m^2(\Box_{\text{ret}}^{-1}G_{\mu\nu})^T = 8\pi GT_{\mu\nu},$$  \hspace{1cm} (1.1)$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor, $G$ is the gravitational constant, $\Box_{\text{ret}}^{-1}$ is the inverse of d’Alembertian operator computed with the retarded Green’s function, and the superscript $T$ represents the extraction of the transverse part.

The background cosmological dynamics based on Eq. (1.1) was studied in Ref. [37]. There is a rapidly growing scalar mode responsible for the late-time cosmic acceleration, in which case the dark energy equation of state evolves from $w_{\text{DE}} = -1.725$ (matter era) to $w_{\text{DE}} = -1.506$ (accelerated era). Since the Planck data combined with the SnIa and WMAP polarization data placed the bound $w_{\text{DE}} = -1.13^{+0.13}_{-0.11}$ (95% C.L.) for constant $w_{\text{DE}}$ [2], the NLMG model (1.1) is in tension with the current observations of CMB and SnIa. In order to avoid the rapid growth of the scalar mode, we also require that the mass $m$ is much smaller than $H_0$.

Alternatively, Maggiore [38] proposed a model given by the field equation

$$G_{\mu\nu} - \frac{1}{3}m^2(g_{\mu\nu}\Box_{\text{ret}}^{-1}R)^T = 8\pi GT_{\mu\nu},$$  \hspace{1cm} (1.2)$$

where $g_{\mu\nu}$ is the metric tensor and $R$ is the Ricci scalar. In this case the strong instability of a scalar mode present in the theory (1.1) is avoided, so that the dark energy equation of state does not significantly deviate from $-1$ ($w_{\text{DE}} \approx -1.1$ in the deep matter era). The model has a predictive power due to the presence of a single parameter $m$ alone. For today’s dark energy density parameter $\Omega_{\text{DE}}^{(0)} \simeq 0.68$, the mass $m$ is fixed to be $m \simeq 0.67H_0$ [38, 39]. It was also shown that the general relativistic behavior can be recovered inside the solar system [10].

The retarded operator $\Box_{\text{ret}}^{-1}$ mentioned above is required for causality. However, the variation of some nonlocal action involving the inverse d’Alembertian operator $\Box_{\text{ret}}^{-1}$ (such as $\phi \Box_{\text{ret}}^{-1}\phi$ with $\phi$ being a scalar) usually symmetrizes the Green’s function [41]. Hence the field equations of motion involving the retarded operator $\Box_{\text{ret}}^{-1}$ do not follow from a variational principle from some nonlocal action. Such a retarded nonlocal operator will only emerge when some classical or quantum averaging prescription is performed in a more fundamental local quantum field theory (QFT). In this sense, the field equations of motion (1.1) and (1.2) should be considered only as effective classical equations that do not have a direct link to the action of a nonlocal QFT [41]. This is also the approach we follow in our paper.

Furthermore, if we use a quadratic action of gravitational waves associated with the perturbation equation of the theory (1.2) by simply replacing $\Box_{\text{ret}}^{-1}$ with $\Box^{-1}$, the resulting propagator apparently involves a ghostlike massive scalar [38]. Foffa et al. [11] showed that this apparent ghost is not a propagating degree of freedom and in the $m \to 0$ limit it smoothly approaches a nonradiative degree of freedom of GR. This implies that we should regard Eq. (1.2) as an effective classical equation of motion rather than promoting it directly to a full QFT (which typically involves some classical or quantum averaging). The issue of quantization—including ghosts—would be addressed in an underlying fundamental theory with a possible ultraviolet completion.

In this paper we study the cosmology (at the classical level) and observational constraints on the NLMG models. In Sec. II the background equations of motion are derived for general models including (1.1) and (1.2). We then discuss the evolution of $w_{\text{DE}}$ as well as the mass scale $m$ constrained from the background cosmology. In Sec. III we obtain the full equations of linear cosmological perturbations for the NLMG model (1.2). We also discuss the behavior of perturbations for the subhorizon modes relevant to large-scale structures. In Sec. IV we confront the NLMG model (1.2) with the latest observations of SnIa, CMB, BAO, and redshift-space distortions. Section V is devoted to the conclusions.

II. BACKGROUND EQUATIONS OF MOTION

We start with the following equations of motion of the NLMG models\footnote{Following Ref. [27] we use the metric signature ($-$, $+$, $+$, $+$) in this paper. Note that the metric signature used in Ref. [37] is ($+$, $-$, $-$, $-$).}

$$G_{\mu\nu} - m^2(\Box_{\text{ret}}^{-1}G_{\mu\nu})^T = 8\pi GT_{\mu\nu},$$  \hspace{1cm} (2.1)$$

where $a_1$ and $a_2$ are constants, $R_{\mu\nu}$ is the Ricci tensor, $\Box_{\text{ret}}^{-1}$ is the inverse of the d’Alembertian operator computed by using the retarded Green’s function due to causality [34]. The model (1.1) corresponds to $a_1 = 1$ and $a_2 = -1/2$, whereas the model (1.2) is characterized by $a_1 = 0$ and $a_2 = 1/3$. \hspace{1cm}
We now introduce a tensor $S_{\mu\nu}$ obeying the relation

$$\square S_{\mu\nu} = a_1 R_{\mu\nu} + a_2 g_{\mu\nu} R.$$

(2.2)

In order to respect the continuity equation $\nabla^\mu T_{\mu\nu} = 0$ of matter in Eq. (2.1), we pick up the transverse part $S^T_{\mu\nu}$ of the symmetric tensor $S_{\mu\nu}$ satisfying $\nabla^\mu S^T_{\mu\nu} = 0$, that is

$$G_{\mu\nu} - m^2 S^T_{\mu\nu} = 8\pi G T_{\mu\nu}.$$ 

(2.3)

The tensor $S_{\mu\nu}$ can be decomposed as

$$S_{\mu\nu} = S^T_{\mu\nu} + (\nabla_\mu S_\nu + \nabla_\nu S_\mu)/2.$$ 

(2.4)

Let us consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime described by the metric $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, where $t$ is the cosmic time. On this background the vector $S_\mu$ has a time component $S_0$ alone, so that

$$(S^0_0)^T = u \dot{S}_0, \quad (S^i_i)^T = v + 3HS_0,$$

(2.5)

where $u \equiv S^0_0$ and $v \equiv S^i_i$, $H \equiv \dot{a}/a$, and a dot represents a derivative with respect to $t$. For the energy-momentum tensor $T_{\mu\nu}$, we take into account a perfect fluid obeying the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0,$$

(2.6)

where $\rho$ is the energy density and $P$ is the pressure of the fluid. From Eq. (2.1) we obtain the following equations of motion

$$3H^2 + m^2(u + \dot{S}_0) = 8\pi G \rho,$$

(2.7)

$$2\dot{H} + 3H^2 + \frac{m^2}{3}(v + 3HS_0) = -8\pi GP.$$ 

(2.8)

From the (00) and (ii) components of Eq. (2.2) it follows that

$$\ddot{u} + 3H\dot{u} - 6H^2u + 2H^2v = -3(a_1 + 4a_2)H^2 - 3(a_1 + 2a_2)\dot{H},$$

(2.9)

$$\ddot{v} + 3H\dot{v} - 2H^2v + 6H^2u = -9(a_1 + 4a_2)H^2 - 3(a_1 + 6a_2)\dot{H}.$$ 

(2.10)

The divergence of Eq. (2.4) gives $2\nabla^\nu S_{\mu\nu} = \nabla^\nu(\nabla_\mu S_\nu + \nabla_\nu S_\mu)$. The $\nu = 0$ component of this equation reads

$$\ddot{S}_0 + 3H\dot{S}_0 - 3H^2S_0 = -\left(\ddot{u} + 3Hu - Hv\right).$$ 

(2.11)

In order to simplify the analysis, we define

$$U \equiv u + v, \quad V \equiv u - v/3,$$

(2.12)

by which $u = (U + 3V)/4$ and $v = (3/4)(U - V)$. The field $U$ corresponds to the trace part of the tensor $S^\nu_\nu$, whereas the field $V$ characterizes the difference between the time and spatial diagonal components of $S^\nu_\nu$. On the right-hand side of Eqs. (2.7) and (2.8) we take into account the contribution of nonrelativistic matter (density $\rho_m$, pressure $P_m = 0$) and radiation (density $\rho_r$, pressure $P_r = \rho_r/3$). We can write Eqs. (2.7) and (2.8) in the following forms

$$3H^2 = 8\pi G(\rho_m + \rho_r + \rho_{DE}),$$

(2.13)

$$2\dot{H} + 3H^2 = -8\pi G(P_r + P_{DE}),$$ 

(2.14)

where

$$\rho_{DE} = \frac{m^2}{32\pi G}(4\zeta X - 4X' - U - 3V), \quad P_{DE} = \frac{m^2}{32\pi G}(U - V + 4X),$$ 

(2.15)

and

$$X = HS_0, \quad \zeta = \frac{H'}{H}.$$ 

(2.16)
A prime represents a derivative with respect to $N = \ln(a/a_i)$, where $a_i$ is the initial scale factor. The dark energy equation of state is then given by

$$w_{\text{DE}} = \frac{P_{\text{DE}}}{\rho_{\text{DE}}} = \frac{U - V + 4X}{4\zeta X - 4X' - U - 3V},$$

(2.17)

whereas the effective equation of state is $w_{\text{eff}} = -1 - \frac{2}{3}\zeta$. From Eqs. (2.9)–(2.11) it follows that

$$U'' + (3 + \zeta)U' = -6(a_1 + 4a_2)(2 + \zeta),$$

(2.18)

$$V'' + (3 + \zeta)V' - 8V = -2a_1\zeta,$$

(2.19)

$$X'' + (3 - \zeta)X' - (3 + 3\zeta + \zeta')X = -\frac{1}{4}(U' + 3V' + 12V).$$

(2.20)

From Eq. (2.14) the functions $\zeta$ and $\zeta'$ obey

$$\zeta = -\frac{3}{2} - \frac{m^2}{8H^2}(U - V + 4X) - \frac{1}{2}\Omega_r,$$

(2.21)

$$\zeta' = 2\Omega_r - 3\zeta - 2\zeta^2 - \frac{m^2}{8H^2}(U' - V' + 4X'),$$

(2.22)

where we used the fact that the radiation density parameter $\Omega_r = 8\pi G\rho_r/(3H^2)$ satisfies

$$\Omega_r' = -(4 + 2\zeta)\Omega_r.$$ \hfill (2.23)

From Eq. (2.13) the matter density parameter $\Omega_m = 8\pi G\rho_m/(3H^2)$ is known to be

$$\Omega_m = 1 - \Omega_r - \Omega_{\text{DE}}, \quad \text{where} \quad \Omega_{\text{DE}} = \frac{m^2}{12H^2}(4\zeta X - 4X' - U - 3V).$$

(2.24)

The density parameter of radiation today (corresponding to $a = 1$) is given by

$$\Omega_r^{(0)} = \Omega_r^{(0)}(1 + 0.2271N_{\text{eff}}),$$

(2.25)

where $\Omega_r^{(0)}$ is the photon density parameter and $N_{\text{eff}}$ is the relativistic degrees of freedom. We adopt the standard values $\Omega_r^{(0)} = 2.469 \times 10^{-5} h^{-2}$ and $N_{\text{eff}} = 3.04$, where $H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$. So, given that $\Omega_r^{(0)}$ is fixed from the CMB, the only free parameter is $\Omega_{\text{DE}}^{(0)}$. From Eq. (2.24) the mass ratio $m/H_0$ is calculated as

$$\frac{m}{H_0} = \sqrt{\frac{12(1 - \Omega_m - \Omega_r)}{4\zeta X - 4X' - U - 3V}}_{a=1}.$$ \hfill (2.26)
We integrate the dynamical equations of motion (2.18)-(2.23) with the initial conditions
\[ U(t_i) = \dot{U}(t_i) = V(t_i) = \dot{V}(t_i) = X(t_i) = \dot{X}(t_i) = 0, \tag{2.27} \]
where \( t_i \) corresponds to the time at the deep radiation era. The initial conditions can be fixed as above once the definition of the retarded inverse d’Alembertian \( \Box^{-1}_\text{ret} \) based on the retarded Green function is given. Note that this issue was discussed in detail in Ref. [39]. The initial time \( t_i \) (taken at the deep radiation era) is regarded at the epoch when the effective description of our classical equations of motion becomes appropriate. The initial conditions (2.27) correspond to the minimal case in which the influence of the nonlocal corrections to general relativity is negligible at \( t = t_i \). For example, the field \( U \) is proportional to \( \Box^{-1}_\text{ret} R = - \int_{t_i}^t dt' a^{-3(t')} \int_{t_i}^{t'} dt'' a^3(t'') R(t'') \) and hence \( U(t_i) = \dot{U}(t_i) = 0 \) [39]. For given \( m \), we numerically solve the background equations of motion (2.18)-(2.23) and evaluate (2.28) to check the consistency of the solutions.

From Eq. (2.19) it is clear that the field \( V \) is unstable for \( a_1 \neq 0 \) due to the presence of the \( -8V \) term in Eq. (2.19) and the term \(-2a_1 \zeta \) on the right-hand side that acts as a source term if \( a_1 \) is not equal to zero. Even if all the fields \( U, V, X \) and their first time derivatives are 0, we cannot avoid this instability. However, for \( a_1 = 0 \), the field \( V \) is decoupled from the dynamical system because it can stay at \( V = 0 \). In other words, for the equation of motion (1.2) corresponding to the case \( a_1 = 0 \), the degree of freedom associated with the field \( V \) does not appear. In fact, for the theory with \( a_1 = 1 \) and \( a_2 = -1/2 \) [27], this instability leads to the rapid growth of \( V \) and \( X \), by which the dark energy equation of state evolves as \( w_{DE} \simeq -1.791 \) (radiation era), \( w_{DE} \simeq -1.725 \) (matter era), and \( w_{DE} \simeq -1.506 \) (accelerated era) [37]. Unless \( a_1 \) is very close to 0, the same property also holds for the theories with \( a_1 \neq 0 \).

In Fig. 1 we plot the evolution of \( w_{DE} \) and \( w_{\text{eff}} \) for \( a_1 = 0.01 \) and \( a_2 = 1/3 \) with today’s matter density parameter \( \Omega_m(0) = 0.323 \). As in the theory with \( a_1 = 1 \) and \( a_2 = -1/2 \), the dark energy equation of state during the matter-dominated epoch is around \( w_{DE} \simeq -1.725 \) and it finally approaches the value \(-1.506 \). In this case the mass \( m \) is much smaller than \( H_0 \), which is required to avoid the early dominance of dark energy [37]. The evolution of \( w_{DE} \) shown in Fig. 1 is incompatible with the joint data analysis of CMB, SNIa, and BAO, due to the large deviation from \(-1 \). This conclusion holds not only from the WMAP data combined with the SNIa and BAO measurements [13, 14] but also from the Planck data combined with the SNIa and WMAP polarization measurements [2].

When \( a_1 = 0 \) the rhs of Eq. (2.19) vanishes, so that the field \( V \) does not contribute to the cosmological dynamics for the initial conditions given in (2.27). In the deep radiation era (\( \zeta \simeq -2 \) and \( \zeta' \simeq 0 \)) the field \( U \) is almost frozen, but it starts to evolve once \( \zeta \) deviates from \(-2 \). During the matter-dominated epoch (\( \zeta \simeq -3/2 \) and \( \zeta' \simeq 0 \)), integrations of Eqs. (2.18) and (2.20) read
\[ U \simeq -8a_2 N + c_1, \tag{2.28} \]
\[ X \simeq \frac{4}{3}a_2 + c_2 e^{-9-(\sqrt{37}/2)N/4}, \tag{2.29} \]
where \( c_1 \) and \( c_2 \) are integration constants, and we neglected the decaying-mode solutions. If the field value \(|U|\) at the radiation-matter equality (identified as \( N = 0 \)) is much smaller than 1, it follows that \(|c_1| \ll 1 \). On using the solution
the dark energy density \( \rho_{\text{DE}} \) in the matter era is given by

\[
\rho_{\text{DE}} \simeq \frac{m^2}{4\pi G}a_2 \left( N - 1 - \frac{c_1}{8a_2} \right),
\]

(2.30)

where the term \( c_1/(8a_2) \) is much smaller than 1 for \( |a_2| = O(1) \). In the regime \( N \gtrsim 1 \), \( \rho_{\text{DE}} \) is positive for \( a_2 > 0 \).

This condition is assumed in the following discussion.

Neglecting the second term on the rhs of Eq. (2.20), the dark energy equation of state (2.17) in the deep matter era reads

\[
w_{\text{DE}} \simeq -\frac{1 - 16r}{1 - 24r}, \quad r \equiv -\frac{a_2}{3H}.
\]

(2.32)

In the regime \( N \gtrsim 1 \) we have \( U \simeq -8a_2N \) and \( r \simeq 1/(24N) \) for \( a_2 = O(1) \), so that \( w_{\text{DE}} \simeq -[1 - 2/(3N)]/(1 - 1/N) \). Hence the dark energy equation of state (2.32) is in the range \(-1.11 < w_{\text{DE}} < -1 \) for \( N > 4 \). At late times \( (N \gg 1) \) the terms \( m^2/H^2 \) grow in Eqs. (2.21) and (2.22), so Eq. (2.32) starts to lose its validity.

In the left panel of Fig. 2 the evolution of \( w_{\text{DE}} \) and \( w_{\text{DE}}^\text{eff} \) is plotted for \( a_1 = 0, a_2 = 1/3 \), and \( \Omega_m^0 = 0.323 \). The value \( a_2 = 1/3 \) was chosen to match with the one in Ref. [38], but its precise value does not matter provided \( a_2 = O(1) \). As estimated above, the dark energy equation of state exhibits mild growth around \(-1.1 < w_{\text{DE}} < -1.04 \) by today \((-4 < \ln a < 0) \). This mild variation of \( w_{\text{DE}} \) is followed by more rapid growth toward \( w_{\text{DE}} \approx -0.5 \) in the future. In this case the mass is found to be \( m/H_0 \approx 0.67 \), which shows good agreement with the one derived in Ref. [38]. In the right panel of Fig. 2 we also plot the mass ratio \( m/H_0 \) versus \( \Omega_m^0 \) in the range \( \Omega_m^0 \in [0, 1] \). For larger \( \Omega_m^0 \) the mass \( m \) gets smaller, but it is typically of the order of \( m/H_0 = O(0.1) \).

## III. COSMOLOGICAL PERTURBATIONS

In this section we shall derive the equations of motion for linear cosmological perturbations for the NLMG model (2.21) with \( a_1 = 0 \). From Eq. (2.22) we have \( \Box S_\mu = a_2 \delta_\mu^\nu R \) and hence

\[
\Box \bar{U} = 4a_2R,
\]

(3.1)

\[
S_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\bar{U},
\]

(3.2)

where \( \bar{U} \equiv S_0^0 + S_i^i \). On the flat FLRW background the field \( \bar{U} \) is identical to \( U \) introduced in Eq. (2.12), with \( V = 0 \). From Eqs. (3.1) and (3.2) the field equations of motion (2.3) read

\[
G_{\mu\nu} - m^2 \left[ \frac{1}{4}g_{\mu\nu}\bar{U} - \frac{1}{2}(\nabla_\mu S_\nu + \nabla_\nu S_\mu) \right] = 8\pi GT_{\mu\nu}.
\]

(3.3)

Taking the covariant derivative of Eq. (2.4), we obtain

\[
\nabla_\mu \bar{U} = 2(\Box S_\mu + \nabla_\nu S_\mu) .
\]

(3.4)

We decompose the field \( \bar{U} \) into the background component \( U(t) \) and the perturbation \( \delta U(t, x) \), as

\[
\bar{U} = U(t) + \delta U(t, x).
\]

(3.5)

The time component of the vector \( S_\mu \) can be also decomposed as \( S_0 = \bar{S}_0(t) + \delta S_0(t, x) \), where we omit the bar in the following for simplicity. The spatial component of \( S_\mu \) can be written as \( S_i = S_i^T + \partial_i \delta S \), where \( \delta S \) is a scalar and \( S_i^T \) is a transverse vector satisfying \( \partial_i S_i^T = 0 \). Since we are interested in only scalar perturbations, we do not consider the contribution of vector perturbations \( S_i^T \). Then, the four-vector \( S_\mu \) can be expressed as

\[
S_\mu = (S_0 + \delta S_0, \partial_\mu \delta S).
\]

(3.6)

In order to derive the full perturbation equations of motion, we need to expand Eqs. (3.1), (3.3), and (3.5) up to first order in perturbations. In doing so, we consider scalar metric perturbations \( \Phi \) and \( \Psi \) described by the following metric in longitudinal gauge [32]:

\[
ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j ,
\]

(3.7)
for which the perturbations of the Ricci scalar $R$ and the Einstein tensor $G_{\mu\nu}$, etc. can be computed. Since our interest is the evolution of perturbations during the matter era, we take into account a non-relativistic perfect fluid characterized by the energy-momentum tensor:

$$T_0^0 = -(\rho_m + \delta \rho_m), \quad T_i^0 = -\rho_m v_{m,i}, \quad T_j^i = 0 \quad (i,j = 1,2,3),$$

where $\delta \rho_m$ is the density perturbation and $v_m$ is the velocity potential.

### A. Perturbation equations

The perturbation $\delta T^{\mu\nu}$ of the matter energy-momentum tensor $T^{\mu\nu}$ obeys the continuity equation,

$$\delta T^{\mu\nu}_{;\mu} = 0.$$  \hfill (3.9)

From the $\nu = 0$ and $\nu = i$ components of Eq. (3.9), we obtain the following equations in Fourier space, respectively:

$$\dot{\delta \rho_m} + 3H \delta \rho_m - 3\rho_m \dot{\Phi} + \frac{k^2}{a^2}\rho_m v_m = 0,$$  \hfill (3.10)

$$\dot{v}_m = \Phi,$$  \hfill (3.11)

where $k$ is a comoving wave number. We introduce the gauge-invariant density contrast

$$\delta_m = \frac{\delta \rho_m}{\rho_m} + 3H v_m.$$  \hfill (3.12)

Taking the time derivative of Eq. (3.10) and using Eq. (3.11), the density contrast satisfies

$$\ddot{\delta m} + 2H \dot{\delta m} + \frac{k^2}{a^2} \delta m = 3\dot{B} + 6H\dot{B},$$  \hfill (3.13)

where $B \equiv \Psi + Hv_m$.

From the (00), (0i), (ij) $[i \neq j]$, and the trace of the (ii) parts of the perturbed version of Eq. (3.3), we obtain the following equations of motion in Fourier space respectively:

$$\frac{2k^2}{a^2}\dot{\Psi} + 6H \left(\dot{\Psi} + H\Phi\right) - m^2 \left(\frac{1}{4} \delta U + \delta S_0 - 2\dot{S}_0 \Phi - S_0 \dot{\Phi}\right) = -8\pi G\delta \rho_m,$$  \hfill (3.14)

$$2 \left(\dot{\Psi} + H\Phi\right) + \frac{m^2}{2} \left(\delta S + \delta S_0 - 2\dot{S}_0 \Phi - 2H\delta S\right) = 8\pi G\rho_m v_m,$$  \hfill (3.15)

$$\Psi - \Phi + m^2 \delta S = 0,$$  \hfill (3.16)

$$6\dot{\Psi} + 6H \left(\dot{\Phi} + 3\Psi\right) + 6 \left(3H^2 + 2\dot{H}\right) \Phi - 2\frac{k^2}{a^2} \left(\Phi - \Psi\right) - m^2 \left[\frac{3}{4} \delta U + \frac{k^2}{a^2} \delta S + 3H\delta S_0 - 3\dot{S}_0 \left(\Psi + 2H\Phi\right)\right] = 0.$$  \hfill (3.17)

From Eq. (3.17) it follows that

$$\delta U + 3H\delta U + \frac{k^2}{a^2} \delta U - 2\Phi \left(\dot{U} + 3H\dot{U}\right) - \left(\Phi + 3\Psi\right) \dot{U} = 8a_2 \left[3 \left(\Psi + 4H\dot{\Psi} + H\Phi\right) + 6 \left(2H^2 + \dot{H}\right) \Phi + \frac{k^2}{a^2} \left(2\Psi - \Phi\right)\right].$$  \hfill (3.18)

The $\nu = 0$ and $\nu = i$ components of Eq. (3.14) read

$$\dot{\delta U} = -4 \left[\delta S_0 + 3H\delta S_0 - S_0 \Phi - 2\dot{S}_0 \Phi - 3\dot{S}_0 \left(\Phi + \Psi + 2H\Phi\right) - 3H^2 \delta S_0\right]$$

$$+ 12H S_0 \left(\dot{\Phi} - \dot{\Psi} - 2H\Phi\right) - 2\frac{k^2}{a^2} \left(\delta S_0 + \dot{\delta S} - 4H\delta S - 2S_0 \Phi\right),$$  \hfill (3.19)

$$\delta U = -2 \left[\ddot{\delta S} + 3H\dot{\delta S} + 2\frac{k^2}{a^2} \delta S - 2 \left(H + 3H^2\right) \delta S + \dot{\delta S}_0 + 5H\delta S_0 - 2S_0 \left(\Phi + \Psi + 4H\Phi\right) - 4\dot{S}_0 \Phi\right].$$  \hfill (3.20)

The evolution of the density contrast $\delta m$ is known by solving Eqs. (3.10) and (3.11) and (3.14)—(3.20) for given $k$. In the $m \to 0$ limit, all the mass-dependent terms involving the perturbations $\delta U, \delta S_0$, and $\delta S$ in Eqs. (3.14)—(3.17)
vanish to recover the general relativistic behavior. When \( m \neq 0 \) the evolution of the gravitational potentials \( \Phi \) and \( \Psi \) is subject to change, which affects the growth of \( \delta_m \) through Eq. (3.13). Eliminating the terms \( \Psi + H\Phi \) from Eqs. (3.14) and (3.15), we obtain

\[
\frac{k^2}{a^2} \Psi - \frac{m^2}{8} \delta F = -4\pi G \rho_m \delta_m ,
\]

where

\[
\delta F = \delta U + 4\delta S_0 + 6H\delta S_0 + 6H\delta S - 12H^2\delta S - 4S_0\Phi - 4 \left( 2\dot{S}_0 + 3HS_0 \right) \Phi .
\]

In the \( m \to 0 \) limit, Eq. (3.21) reduces to the standard Poisson equation \( (k^2/a^2)\Psi = -4\pi G \rho_m \delta_m \). In GR we have \( \delta S = 0 \) and \( \Psi = \Phi \), so that the third term on the left-hand side (lhs) of Eq. (3.13) reads \( (k^2/a^2)\Psi = -4\pi G \rho_m \delta_m \). This term works as a driving force for the growth of \( \delta_m \) with the gravitational coupling characterized by \( G \). In the presence of the mass term \( m \), there is a modification to the gravitational constant \( G \).

**B. Evolution of perturbations on subhorizon scales**

Let us consider the perturbations relevant to the linear regime of galaxy clusterings. This corresponds to the wave numbers \( 0.01 \, h \, \text{Mpc}^{-1} \lesssim k \lesssim 0.1 \, h \, \text{Mpc}^{-1} \) \( [46] \), i.e.,

\[
30 \lesssim k/H_0 \lesssim 300 .
\]

In the redshift range where the redshift distortions of galaxies have been measured \( (z \equiv 1/a - 1 \lesssim 2 ) \), the modes \( [3.23] \) are deep inside the Hubble radius \( (k/a \gg H) \). Under a subhorizon approximation we can ignore some of the terms in the perturbation equations \( (3.14) - (3.20) \) (see e.g., Refs. \[47–49\]). We also note that the gravitational potentials \( \Phi \) and \( \Psi \) are nearly constant during the deep matter era and they start to vary after the onset of the cosmic acceleration, so that \( |\Phi| \lesssim |H\Phi| \) and \( |\Psi| \lesssim |H\Psi| \) by today.

Under the subhorizon approximation the dominant contributions to Eq. (3.18) should be the terms including \( k^2/a^2 \), and hence

\[
\delta U \simeq 8a_2 (2\Psi - \Phi) = 8a_2 \left( \Phi - 2m^2 \delta S \right) ,
\]

where in the second equality we used Eq. (3.16). For the validity of this approximation we also require that \( (k^2/a^2)|\delta U| \gg |\Phi H\dot{U}| \), which can be interpreted as \( k^2/(aH)^2 \gg |U| \) for \( a_2 = 1/3 \) and \( |U| \lesssim |H\dot{U}| \). Since today’s value of \( U \) is of the order of 10, the condition \( k^2/(aH)^2 \gg |U| \) is satisfied for the wave numbers \( [3.24] \). We also note that Eq. (3.18) does not contain a large mass term exceeding \( k/a \), so the oscillating mode induced by the second derivative \( \delta U \) can be neglected relative to the mode \( [3.24] \).

We recall that the mass scale \( m \) is slightly smaller than \( H_0 \), in which case \( k^2/a^2 \) is much larger than \( m^2 \) for the subhorizon modes \( [3.24] \). From Eqs. (3.14) and (3.20) we can estimate the orders of the subhorizon perturbations \( \delta S \) and \( \delta S_0 \), as

\[
|\delta S| \approx \frac{a^2}{k^2} |\Phi| , \quad |\delta S_0| \approx \frac{a^2 H}{k^2} |\Phi| .
\]

From Eq. (3.10) it follows that

\[
\left| \frac{\Psi}{\Phi} - 1 \right| = O(\epsilon_k) , \quad \text{where} \quad \epsilon_k \equiv \frac{(ma)^2}{k^2} .
\]

Since \( \epsilon_k \ll 1 \), the difference between \( \Psi \) and \( \Phi \) is small. On using Eq. (3.25), the perturbation \( \delta F \) in Eq. (3.22) is approximately given by

\[
\delta F \simeq \delta U - 4S_0 \dot{\Phi} - 4 \left( 2\dot{S}_0 + 3HS_0 \right) \Phi .
\]

The term \( X = HS_0 \) grows to the order close to \( O(1) \) by today. Since \( \delta U = O(\Phi) \) from Eq. (3.24), the perturbation \( |\delta F| \) is of the order of \( |\Phi| \). From Eqs. (3.21) and (3.26) the third term on the lhs of Eq. (3.13) can be estimated as

\[
\frac{k^2}{a^2} \dot{\Phi} \simeq -4\pi G \rho_m \delta_m ,
\]
FIG. 3: The evolution of $f \sigma_8(z)$ for the NLMG model with $a_1 = 0$ and $a_2 = 1/3$, compared to the ΛCDM model in the redshift range $z \in [0, 2]$. The two lines correspond to $k = 30H_0$ (solid black line) and the ΛCDM model (dashed red line) for $\Omega_m^{(0)} = 0.3$ and $\sigma_{8,0} = 0.8$. We plot $f \sigma_8$ in this redshift range as this is where the current data exist.

where the difference between the effective gravitational coupling $G_{\text{eff}}$ and the gravitational constant $G$ is

$$\left| \frac{G_{\text{eff}}}{G} - 1 \right| = O(\epsilon_k).$$

(3.29)

For the modes (3.23) the parameter $\epsilon_k$ is in the range $5 \times 10^{-6} \lesssim \epsilon_k \lesssim 5 \times 10^{-4}$, so $G_{\text{eff}}$ is very close to $G$. The rhs of Eq. (3.13) can be negligible relative to its lhs, and hence

$$\ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi G_{\text{eff}} \rho_m \delta_m \approx 0.$$  

(3.30)

Numerically we have solved the full perturbation equations (3.10)–(3.11) and (3.14)–(3.20) for the initial conditions $\delta U(t_i) = \delta S_0(t_i) = \delta S_0(t_i) = \delta S(t_i) = 0$. In spite of the presence of the second derivative $\delta U$ in Eq. (3.18), the term $(k^2/a^2)\delta U$ soon starts to balance with the term $8a^2(k^2/a^2)(2\Psi - \Phi)$ on the rhs of Eq. (3.18). After that, the solutions can be well described by the analytic estimation given above. Numerically we also confirmed the accuracy of Eq. (3.29) and found that in practice $G_{\text{eff}}/G \approx 1$ to better than 0.05% precision for the wave numbers in the range (3.23). This suggests that, apart from the difference of the background evolution, it is difficult to distinguish the NLMG model from the ΛCDM model for the perturbations relevant to large-scale structures. Therefore, in what follows and the likelihood analysis in Sec. IV we shall solve Eq. (3.30) together with the background equations of motion by setting $G_{\text{eff}} = G$.

The observations of redshift space distortions can place bounds on the quantity $f \sigma_8$, where $f \equiv \dot{\delta}_m/(H \delta_m)$ characterizes the growth rate of matter perturbations and $\sigma_8$ is the rms amplitude of $\delta_m$ at the comoving $8h^{-1}$ Mpc scale [46]. In Fig. 3 we plot $f \sigma_8$ versus the redshift $z$ for the NLMG model with $a_1 = 0$ and $a_2 = 1/3$ as well as for the ΛCDM model. Today’s values of $\Omega_m$ and $\sigma_8$ are chosen to be $\Omega_m^{(0)} = 0.3$ and $\sigma_{8,0} = 0.8$, respectively. The solid black curve in Fig. 3 corresponds to the wave number $k = 30H_0$, but we confirmed that the evolution of $f \sigma_8$ for $k > 30H_0$ is similar to that for $k = 30H_0$.

Since $G_{\text{eff}}$ is very close to $G$ for the modes (3.23), the growth rate of matter perturbations in the NLMG model is similar to that in the ΛCDM model. The main reason of the small difference seen in Fig. 3 is that the background evolution of $w_{\text{DE}}$ is different. This is similar to what happens for the constant $w_{\text{DE}}$ models in the framework of GR [51]. For a practical purpose, the evolution of $f \sigma_8$ in the NLMG model can be known in good accuracy by solving Eq. (3.30) with $G_{\text{eff}} = G$. 

\[ \left| \frac{G_{\text{eff}}}{G} - 1 \right| = O(\epsilon_k). \]
IV. OBSERVATIONAL CONSTRAINTS

In this section we will confront the NLMG model (2.1) characterized by \( a_1 = 0 \) and \( a_2 = 1/3 \) with the latest cosmological data and study whether they can be distinguished from the \( \Lambda \)CDM model.

A. The data

In order to constrain the NLMG model, we use the same numerical code\(^2\) and the same data of SNIa, BAO and growth rate as those in Refs. \[52\, 53\], so we refer the readers to the aforementioned references for detail. We also employ the correlation matrix of the Planck CMB shift parameters \( (l_n, R, z_a) \) presented in Ref. \[54\]. These three parameters are related to the background quantities such as \( \Omega_m^{(0)}, \Omega_b^{(0)}, \) and \( h \). They can efficiently summarize the CMB information on dark energy in a model-independent way \[55\].

Compared to Refs. \[52\, 53\], there is a difference in the analysis of the growth-rate data. Instead of using the well-known \( \gamma(z) \) parametrization and modeling the growth rate as \( f(z) = \Omega_m^{(z)} \), we directly fit the numerical solution of the perturbation equations. Regarding the data of the growth rate given in Table I of Ref. \[52\], they are based on the WiggleZ, SDSS, 2dF, PSCz, VVDS, 6dF, 2MASS and BOSS galaxy surveys. The data themselves are given in terms of \( f(z)\sigma_8(z) \). It should be stressed that the main benefit of using \( f(z)\sigma_8(z) \), instead of just \( f(z) \), is that the former is directly related to the power spectrum of peculiar velocities of galaxies.

B. Fitting method and model comparisons

As mentioned in the previous section, the mass \( m \) is known from Eq. (2.20) and it is not a free parameter of the theory. For the case \( a_1 = 0 \) it is known as a function of the matter density parameter \( \Omega_m^{(0)} \) and this is explicitly shown in the right panel of Fig. 2. In the general case, regarding the value of \( a_2 \), it can be fixed to \( 1/3 \) in accordance with the literature and the mass scale \( m \) can as well be computed in this case. However, its precise value does not matter because we can always absorb the coefficient \( a_2 \) into \( m \).

The mass scale \( m \) is determined by the demand that the system of the background equations of motion is consistent, i.e., the value initially used for the solution has to be the same as the one derived from Eq. (2.20). Since the results depend on \( \Omega_m^{(0)} \), we implement an iterative algorithm in which the value of \( m \) is found for each value of \( \Omega_m^{(0)} \) via Eq. (2.20) to check the consistency of Eqs. (2.18)—(2.23). These values are also saved and used later on to simplify and speed up the fitting procedure. Therefore, the final set of parameters employed in the minimization is \( (\Omega_m^{(0)}, \Omega_b^{(0)}h^2, \sigma_{8,0}) \), where \( \Omega_b^{(0)} \) is today’s baryon density parameter. This situation is analogous to what happens in the \( \Lambda \)CDM model.

We compute the total chi square

\[
\chi^2 = \chi^2_{\text{SNIa}} + \chi^2_{\text{BAO}} + \chi^2_{\text{CMB}} + \chi^2_{\text{growth}},
\]

where each term on the rhs is derived by fitting with the SNIa, BAO, CMB, and growth-rate data, respectively, along the line of Refs. \[52\, 53\]. The best-fit corresponds to model parameters for which \( \chi^2 \) takes a minimum value \( \chi^2_{\text{min}} \). We will also consider the same parameters \( (\Omega_m^{(0)}, \Omega_b^{(0)}h^2, \sigma_{8,0}) \) in the \( \Lambda \)CDM model and evaluate the total chi square for the comparison with the NLMG model.

Following Ref. \[54\], we discuss the effect of the \( H_0 \) prior on the results. The Planck team essentially pinned down the parameter \( \Omega_m^{(0)}h^2 \) to very high precision, so changing \( h \) also affects \( \Omega_m^{(0)} \), since \( \delta(\Omega_m^{(0)}h^2) \approx 0 \) or equivalently \( \delta \ln \Omega_m^{(0)} \approx -2\delta \ln h \). The latter implies that, while fitting the data, increasing \( h \) forces \( \Omega_m^{(0)} \) to lower values and vice versa. However, due to the degeneracies in the CMB+BAO data, a lower value of \( \Omega_m^{(0)} \) implies a more negative dark energy equation of state, i.e. \( w_{\text{DE}} < -1 \). In simple terms, increasing \( h \) reduces \( \Omega_m^{(0)} \) and forces \( w_{\text{DE}} \) to more negative values and vice versa. Therefore, the choice of \( h \) that we make is important in the rest of the analysis especially since, as mentioned before, the NLMG model has a corresponding equation of state \( w_{\text{DE}} \) between \(-1.1\) and \(-1.04 \). In order to accommodate the cases with different values of \( H_0 \), we will test some priors on \( h \): (i) the Planck best fit: \( h = 0.673 \) \[2\], (ii) the best-fit \( h = 0.738 \) derived by the direct measurement of \( H_0 \) \[56\], and (iii) other four values of \( h \) ranging 0.673 < \( h \) < 0.738.

\(^2\) General minimization and MCMC cosmological codes can be found freely available \[50\].
We will also consider the Akaike information criterion (AIC) \[^{[57]}\] as in Refs. \[^{[52, 53]}\]. The AIC is defined, for the case of Gaussian errors, as

\[
\text{AIC} = \chi^2_{bf} + 2\ell ,
\]

where \(\ell\) is the number of free parameters. A smaller value of the AIC indicates a better fit to the data. In order to effectively compare two different models, we need to estimate the differences \(\Delta \text{AIC}_{1,2} = AIC_1 - AIC_2\) for the two models 1 and 2. Since \(\ell = 3\) in both the NLMG and the \(\Lambda\)CDM models, the difference of AIC between the two models is actually equivalent to that of \(\chi^2_{bf}\) between them. The larger the value of \(\Delta \text{AIC}\), the higher the evidence against the model with a larger value of AIC, with a difference \(\Delta \text{AIC} \gtrsim 2\) indicating weak evidence and \(\Delta \text{AIC} \gtrsim 6\) indicating a stronger evidence in favor of the model with smaller AIC, while a value \(\lesssim 2\) indicates consistency among the two comparison models. The AIC penalizes models with more parameters, but it should be stressed that these numbers are provided only as a rule of thumb and they should be used with caution \[^{[58]}\].

| Model       | \(\Omega_m^{(0)}\) | \(\Omega_b h^2\) | \(\sigma_{8,0}\) | \(\chi^2_{bf}\) | AIC | \(\Delta \text{AIC}\) |
|-------------|-------------------|-----------------|----------------|----------------|-----|-----------------|
| \(h = 0.673\) |                  |                 |               |                |     |                 |
| \(\Lambda\text{CDM}\) | 0.328 ± 0.002      | 0.0234 ± 0.0002 | 0.735 ± 0.019 | 583.470       | 589.470 | 0               |
| \(\text{NLMG}\)   | 0.334 ± 0.002      | 0.0223 ± 0.0002 | 0.726 ± 0.019 | 585.570       | 591.570 | 2.100          |
| \(h = 0.738\) |                  |                 |               |                |     |                 |
| \(\Lambda\text{CDM}\) | 0.252 ± 0.002      | 0.0249 ± 0.0002 | 0.789 ± 0.021 | 599.620       | 605.620 | 10.011         |
| \(\text{NLMG}\)   | 0.257 ± 0.002      | 0.0245 ± 0.0002 | 0.775 ± 0.020 | 589.609       | 595.609 | 0               |

**TABLE I:** Statistical results of the overall likelihood analysis: The first column indicates the model, while the second, third, and fourth columns provide the \(\Omega_m^{(0)}\), \(\Omega_b h^2\), and \(\sigma_{8,0}\) best-fit values. The last three columns present the goodness-of-fit statistics \((\chi^2_{bf}, \text{AIC}, \Delta \text{AIC}_{bf, j} = AIC_i - AIC_j)\). All the error estimates come from the inverse of the Fisher matrix. The upper part of Table shows the values for the Planck prior \(h = 0.673\), while the lower half the corresponding values for the Riess et al. prior \(h = 0.738\).
FIG. 4: The 1, 2 and 3σ contour plots for the ΛCDM (left) and the NLMG (right) for \( a_1 = 0 \) and \( a_2 = 1/3 \) in the \((\Omega_m^{(0)}, \sigma_{8,0})\) parameter space. The top row shows the contours for \( h = 0.673 \) and the bottom row for \( h = 0.738 \).

The ΛCDM models. As a function of \( h \), \( \chi^2_{bf} \) has a minimum around \( h_1 = 0.703 \) in the NLMG and \( h_2 = 0.697 \) in the ΛCDM. We adopt the following quadratic functions expanded around \( h_1 \) and \( h_2 \) respectively:

\[
\chi^2_{bf,NLMG}(h) = \chi^2_{bf,NLMG}(h_1) + \frac{1}{2} \partial^2_{h_1} \chi^2_{bf,NLMG}(h_1)(h - h_1)^2 + \ldots,
\]

\[
\chi^2_{bf,\Lambda CDM}(h) = \chi^2_{bf,\Lambda CDM}(h_2) + \frac{1}{2} \partial^2_{h_2} \chi^2_{bf,\Lambda CDM}(h_2)(h - h_2)^2 + \ldots,
\]

(4.3)

where we used the fact that at the minimum the first derivative is zero. Taking the difference, we obtain

\[
\Delta \chi^2_{bf}(h) \equiv \chi^2_{bf,NLMG}(h) - \chi^2_{bf,\Lambda CDM}(h) = b_1 + b_2 h + b_3 h^2 + \ldots,
\]

(4.4)

where the constants \((b_1, b_2, b_3)\) are related to the various terms of Eqs. (4.3). As we see in Fig. 6 this fit shows good agreement with \( \chi^2_{bf} \) derived for some discrete values of \( h \). For \( h > 0.686 \), \( \chi^2_{bf} \) in the NLMG is smaller than that in the ΛCDM. In particular, for \( h > 0.70 \), the NLMG is favored over the ΛCDM according to the AIC.
When we consider the general case with a small but nonvanishing \( a_1 \), the instability of the field \( V \) leads to the dark energy equation of state much smaller than \(-1\). We have carried out the joint data analysis based on the SnIa, CMB, and BAO data for the NLMG model with \( a_1 = 0.01 \) and \( a_2 = 1/3 \) and for the \( \Lambda \)CDM model. For \( h = 0.738 \) we find that the bestfits correspond to \( \chi^2_{bf} = 727.802 \) in the NLMG and \( \chi^2_{bf} = 591.528 \) in the \( \Lambda \)CDM, respectively. The difference of AIC from the model \( a_1 = 0 \) and \( a_2 = 1/3 \) is \( |\Delta \text{AIC}| \sim 140 \), so the models with \( a_1 \neq 0 \) are significantly disfavored from the data.

V. CONCLUSIONS

In this paper we studied cosmological perturbations and observational constraints on the NLMG model. Our analysis of the background cosmology covers the two models given by Eqs. (1.1) and (1.2). We dealt with (2.1) as an effective classical equation of motion for discussing the cosmology relevant to dark energy. The issues of ghosts and ultraviolet completion should be addressed in a more fundamental theory with a Lagrangian implementing quantum
and classical averaging.

We derived the background equations of motion from Eq. (2.1) on the flat FLRW background. For the models with \( a_1 \neq 0 \) there is an instability for the field \( V \) induced by the \( -SV \) term on the lhs of Eq. (2.19). In this case the mass \( m \) is required to be much smaller than \( H_0 \) to avoid the early onset of the cosmic acceleration. Since the dark energy equation of state significantly deviates from \( -1 \), the models with \( a_1 \neq 0 \) are strongly disfavored from the joint analysis of the SNIa, CMB, and BAO data.

For the models with \( a_1 = 0 \) the rhs of Eq. (2.19) vanishes, so that the field \( V \) does not grow for the appropriate initial conditions \( \phi(0) \). In order for the dark energy density \( \rho_{DE} \) to be positive, we found that the parameter \( a_2 \) has to satisfy the condition \( a_2 > 0 \). The dark energy equation of state \( w_{DE} \) in the deep matter era can be estimated as Eq. (2.32), which shows good agreement with the numerically integrated solution \( (-1.1 < w_{DE} < -1.04 \) for \( -4 < \ln a < 0 \)).

Expanding the field equations of motion and the metric to first order in perturbations about the flat FLRW background, we derived the full equations of cosmological perturbations for the NLMG model (1.2). The behavior of perturbations is also estimated for the modes relevant to galaxy clusterings. We found that the effective gravitational coupling \( G_{eff} \) is very close to the gravitational constant \( G \) for subhorizon perturbations characterized by the wave numbers \((k)\). Therefore, the evolution of \( f\sigma_8 \) is similar to that in the \( \Lambda \)CDM model (see Fig. 3). In this sense, the current growth-rate measurement alone is not able to distinguish between the NLMG and the \( \Lambda \)CDM models.

We compared the NLMG model (1.2) against the latest cosmological observations, including SNIa, BAO, CMB, and redshift space distortions. Since the mass \( m \) is not a free parameter, we developed an iterative algorithm to compute \( m \) for each value of \( \Omega_m^0 \) via Eq. (2.20) and checked the consistency of Eqs. (2.18)–(2.29). The mass \( m \) in the NLMG plays a similar role to the cosmological constant \( \Lambda \) in the \( \Lambda \)CDM, such that the number of free parameters in the two models is the same. For the sake of comparison we considered the same parameters \((\Omega_m^0, \Omega_b^0 h^2, \sigma_8, \Lambda)\) in both models.

The observational constraints on the NLMG model mainly come from the background expansion history rather than the growth history. The dark energy equation of state varies slowly from the deep matter era to today around \( -1.1 \lesssim w_{DE} \lesssim -1.04 \). Since \( w_{DE} \) is approximately constant in the past, the situation is quite similar to the case of the constant \( w_{DE} \) models studied in Ref. [54]. The likelihood results depend on the value of the \( H_0 \) prior due to the degeneracies of the CMB parameters.

We computed the chi squares in both the NLMG and the \( \Lambda \)CDM models for several different values of \( H_0 \) ranging from the Planck best fit \( h = 0.673 \) to the Riess et al. best fit \( h = 0.738 \). The results of our analysis are presented in Table I and Figs. 4–6. For \( 0.67 < h < 0.70 \) the AIC shows that the NLMG and the \( \Lambda \)CDM models are statistically comparable, but for \( h \geq 0.70 \) the NLMG model is strongly favored over the \( \Lambda \)CDM model. We hope that future observations will pin down the values of \( h \) to exquisite accuracy, clarifying whether the NLMG model is really preferred to the models with \( w_{DE} \geq -1 \).
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