In the classical multiple linear regression model

\[ y = X\beta + \varepsilon, \tag{1} \]

where \( y \) is an \( n \times 1 \) vector of observations, \( X = [1, x_1, \ldots, x_{p-1}] \) a known \( n \times p \) design matrix with \( 2 < p < n \), \( \beta = (\beta_0, \beta_1, \ldots, \beta_{p-1})^T \) an unknown \( p \times 1 \) vector of regression parameters, and \( \varepsilon \) an \( n \times 1 \) vector of random errors with mean zero and variance \( \sigma^2 I \). The (ordinary) least squares estimator for \( \beta \) is \( \hat{\beta} = (X^TX)^{-1}X^Ty \).

Suppose the first \( q \) variables \( x_1, x_2, \ldots, x_q \) are strongly correlated \( (2 \leq q < p) \). Then, model (1) has a multicollinearity problem. Detailed discussions about this problem may be found in, for example, Draper and Smith (1998), Belsley, Kuh and Welsch (2004), and Montgomery, Peck and Vining (2012). The most well-known consequence of the problem is that it leads to large variances for the least squares estimators \( \hat{\beta}_1, \ldots, \hat{\beta}_{p-1} \).
\hat{\beta}_2, \ldots, \hat{\beta}_q. Consequently, individual parameters \beta_1, \beta_2, \ldots, \beta_q cannot be accurately estimated.

In this note, we study the estimation of linear combinations of \beta_1, \beta_2, \ldots, \beta_q,

\[ w_1\beta_1 + w_2\beta_2 + \cdots + w_q\beta_q, \]  

(3)

where \( \mathbf{w} = (w_1, w_2, \ldots, w_q)^T \) is any \( q \times 1 \) vector satisfying \( \sum_{i=1}^q |w_i| = 1 \). Although none of the underlying parameters can be accurately estimated, surprisingly there are linear combinations of the form (3) that can be extremely accurately estimated. Tsao (2019) found such linear combinations for a special uniform model with a uniform correlation structure, but the uniform correlation condition is restrictive. In this note, we look for such linear combinations without assuming any parametric correlation structure. Our main results are (i) when variables \( \mathbf{x}_i \) in model (1) are standardized variables, an average of \( \beta_1, \beta_2, \ldots, \beta_q \) can be highly accurately estimated and (ii) when the variables are not standardized variables, a variability weighted average can be highly accurately estimated. We call these averages “group effects” of the strongly correlated variables. We also briefly discuss the applications of these group effects in the least squares regression.

2 Main results

We first study relevant limit properties of the correlation matrix of the strongly correlated variables \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_q \). We then apply these properties to find linear combinations of the form (3) that can be accurately estimated.

2.1 Limit properties of the correlation matrix of strongly correlated variables

Let \( \mathbf{R} \) be the full rank correlation matrix of \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_q \),

\[
\mathbf{R} = \begin{bmatrix}
1 & r_{12} & \cdots & r_{1q} \\
r_{21} & 1 & \cdots & r_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
r_{q1} & r_{q2} & \cdots & 1
\end{bmatrix}_{q \times q},
\]  

(4)
where \( r_{ij} = \text{corr}(x_i, x_j) \). Since \( x_1, x_2, \ldots, x_q \) are strongly correlated, all \( |r_{ij}| \) are close to +1 but \( r_{ij} \) may be close to either +1 or −1. For convenience, we assume that all \( r_{ij} \) are positive so that they are all close to +1. To see there is no loss of generality by making this assumption, there are \( 2^q \) sign configurations of \( \pm x_1, \pm x_2, \ldots, \pm x_q \) that we may use to build a linear model. Models based on different configurations are equivalent in that they differ only in the signs of some parameters. A linear combination of model parameters under one configuration can be easily converted to one under another, so we may choose any configuration to look for linear combinations that can be accurately estimated. Let \( \text{sgn}(r_{1j}) \) be the sign of \( r_{1j} = \text{corr}(x_1, x_j) \). Since all \( |r_{ij}| \) are close to +1, configuration

\[
x_1, \text{sgn}(r_{12})x_2, \ldots, \text{sgn}(r_{1q})x_q
\]

(5)
satisfies that all pairwise correlations are positive; see Theorem 3.1 in Tsao (2019). So making the assumption amounts to choosing configuration (5) which does not affect the generality. We call (5) an all positive correlations arrangement of \( x_1, x_2, \ldots, x_q \), and will illustrate this arrangement through a numerical example in Section 2.3.

An immediate benefit of making the assumption is that we can now use \( r_M = \min \{r_{ij}\} \) to measure the level of multicollinearity and to formulate the question of interest. To see this, since all \( r_{ij} \) satisfy \( 0 < r_M \leq r_{ij} < 1 \), when \( r_M \) goes to 1 all \( r_{ij} \) will approach 1 which makes the multicollinearity problem worse. In this sense we say that an increase in \( r_M \) represents an increase in the level of multicollinearity. Our question of interest can now be formulated as that of finding linear combinations of the form (3) that can still be accurately estimated when \( r_M \) approaches its upper bound 1. To find an answer to this question, we first study the limit properties of \( R \) and \( R^{-1} \) when \( r_M \) goes to 1.

Since \( R \) is positive definite, it has \( q \) positive eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q > 0 \). Let \( v_1, v_2, \ldots, v_q \) be their corresponding orthonormal eigenvectors, respectively, and let \( 1_q = (1, 1, \ldots, 1)^T \) be the \( q \times 1 \) vector whose elements are all 1. We have

**Lemma 2.1** Correlation matrix \( R \) satisfies

(i) \( \lambda_1 \to q \) and \( \lambda_i \to 0 \) for \( i = 2, 3, \ldots, q \) as \( r_M \to 1 \); and

(ii) \( v_1 \to \frac{1}{\sqrt{q}} 1_q \) as \( r_M \to 1 \).
Proof. Let $A$ be the $q \times q$ matrix whose elements are all 1. Then, $A$ has two distinct eigenvalues, $\lambda_1^A = q$ and $\lambda_2^A = 0$. Eigenvalue $\lambda_1^A$ has multiplicity 1 and $\lambda_2^A$ has multiplicity $(q - 1)$. The unique orthonormal eigenvector of $\lambda_1^A$ is $\frac{1}{\sqrt{q}}1_q$. Here, for simplicity, we have ignored the other eigenvector of $\lambda_1^A$, $\frac{-1}{\sqrt{q}}1_q$, which differs only in sign from $\frac{1}{\sqrt{q}}1_q$.

Let $P = [p_{ij}]$ be a perturbation matrix of $A$ defined by

$$P = A - R.$$  \hfill (6)

Then, $P$ is real and symmetric and $p_{ij} = 1 - r_{ij}$. Since $p_{ij}^2 \rightarrow 0$ when $r_M \rightarrow 1$, thus $\|P\|_2 \rightarrow 0$ when $r_M \rightarrow 1$. It follows that $\lambda_1 \rightarrow \lambda_1^A = q$ and $\lambda_i \rightarrow \lambda_2^A = 0$ for $i = 2, 3, \ldots, q$ as $r_M \rightarrow 1$ (Horn and Johnson, 1985). Also, by $\lambda_1 \rightarrow q$ and $r_{ij} \rightarrow 1$, $v_1 \rightarrow \frac{1}{\sqrt{q}}1_q$. □

Lemma 2.2 The inverse matrix $R^{-1}$ satisfies

(i) $1_q^T R^{-1} 1_q > 1$; and

(ii) $1_q^T R^{-1} 1_q \rightarrow 1$ as $r_M \rightarrow 1$.

Proof. Since $R$ is positive definite, $R^{-1}$ is also positive definite. Let $\lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_q'$ be the eigenvalues of $R^{-1}$. Then, $\lambda_i' = \lambda_{q-i+1}^{-1}$ and its eigenvector is $v_i' = v_{q-i+1}$ for $i = 1, 2, \ldots, q$. In particular, $\lambda_q' = \lambda_1^{-1}$ and $v_q' = v_1$. Denote by $\gamma(R^{-1}, x)$ the Rayleigh-Ritz ratio of matrix $R^{-1}$. We have $\gamma(R^{-1}, x) \geq \lambda_q'$ (Horn and Johnson, 1985), so

$$\gamma(R^{-1}, x) = \frac{x^T R^{-1} x}{x^T x} \geq \frac{1}{\lambda_1}.$$  \hfill (7)

Since all $\lambda_i > 0$ and $\text{trace}(R) = \sum_{i=1}^q \lambda_i = q$, we have $0 < \lambda_1 < q$. Setting $x = 1_q$ and noting that $1_q^T 1_q = q$, the above inequality and $0 < \lambda_1 < q$ imply that

$$1_q^T R^{-1} 1_q \geq \frac{q}{\lambda_1} > 1,$$

which proves (i). To show (ii), note that

$$1_q^T R^{-1} 1_q = q \times \gamma(R^{-1}, 1_q) = q \times \gamma(R^{-1}, \frac{1}{\sqrt{q}}1_q).$$  \hfill (7)
Further, \( \gamma(R^{-1}, v_q') = \lambda_q' = 1/\lambda_1 \) and by Lemma 2.1 \( \lambda_1 \to q \) as \( r_M \to 1 \), thus \( \gamma(R^{-1}, v_q') \to 1/q \). Also, by Lemma 2.1 \( v_q' = v_1 \to \frac{1}{\sqrt{q}1_q} \) as \( r_M \to 1 \). By the continuity of \( \gamma(R^{-1}, x) \), we have \( \gamma(R^{-1}, \frac{1}{\sqrt{q}}1_q) \to 1/q \). This and (7) imply that \( 1^T_q R^{-1} 1_q \to 1 \). □

### 2.2 Average group effect of strongly correlated variables is estimable

We first consider the average group effect of a standardized version of model (1) and then a variability weighted average effect of the original model (1). For convenience, let \( X_1 = [x_1, x_2, \ldots, x_q] \) and \( X_2 = [x_{q+1}, x_{q+2}, \ldots, x_{p-1}] \). Also, let \( \beta_1 = (\beta_1, \beta_2, \ldots, \beta_q)^T \), \( \beta_2 = (\beta_{q+1}, \beta_{q+2}, \ldots, \beta_{p-1})^T \) and write model (1) as

\[
y = \beta_0 1_n + X_1 \beta_1 + X_2 \beta_2 + \varepsilon, \tag{8}
\]

where \( 1_n \) is the \( n \times 1 \) vector whose elements are all 1. Variables in \( X_1 \) are assumed to be strongly correlated in the sense that (i) their correlation coefficients are large in absolute value and (ii) they are at most weakly correlated with variables in \( X_2 \).

Without loss of generality, we assume variables in \( X_1 \) are in an all positive correlations arrangement so that their correlation matrix is \( R \) in (1) where all elements are close to +1.

Let \( x_i = (x_{i1}, x_{i2}, \ldots, x_{im})^T \), \( \bar{x}_i = \frac{1}{n} \sum_{j=1}^{n} x_{ji} \) and \( s_i^2 = \sum_{j=1}^{n} (x_{ji} - \bar{x}_i)^2 \). We call

\[
x_i' = \frac{x_i - \bar{x}_i 1_n}{s_i} \tag{9}
\]

the standardized variable which has mean zero and length one. Let \( y = (y_1, y_2, \ldots, y_n)^T \), \( \bar{y} = \frac{1}{n} \sum_{j=1}^{n} y_j \) and \( y' = y - \bar{y} \). We can write (8) as

\[
y' = X'_1 \beta'_1 + X'_2 \beta'_2 + \varepsilon, \tag{10}
\]

where \( X'_1 = [x'_1, x'_2, \ldots, x'_q] \), \( X'_2 = [x'_{q+1}, x'_{q+2}, \ldots, x'_{p-1}] \), \( \beta'_1 = (\beta'_1, \beta'_2, \ldots, \beta'_q)^T \), and \( \beta'_2 = (\beta'_{q+1}, \beta'_{q+2}, \ldots, \beta'_{p-1})^T \). We call model (10) the standardized model. The relationship between parameters in model (10) and those in the original model (8) is

\[
\beta_0 = \bar{y} - \sum_{i=1}^{p-1} \bar{x}_i \beta'_i / s_i \quad \text{and} \quad \beta_i = \beta'_i / s_i \quad \text{for} \quad i = 1, 2, \ldots, p - 1. \tag{11}
\]
Let \( X' = [X'_1, X'_2] \). Then, \( X'^T X' = [r_{ij}] \in \mathbb{R}^{(p-1) \times (p-1)} \) is the correlation matrix for variables in models (8) and (10) where \( r_{ij} = \text{corr}(x'_i, x'_j) = \text{corr}(x_i, x_j) \). Partition this correlation matrix as follows:

\[
X'^T X' = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in \mathbb{R}^{(p-1) \times (p-1)},
\]

where \( R_{11} = \mathbf{R} \in \mathbb{R}^{q \times q} \) is the correlation matrix of the strongly correlated variables in \( X'_1 \) or \( X_1 \). By (12),

\[
[X'^T X']^{-1} = \begin{bmatrix}
[R_{11} - R_{12}R_{22}^{-1}R_{21}]^{-1} & R_{11}^{-1}R_{12}[R_{21}R_{11}^{-1}R_{12} - R_{22}]^{-1} \\
[R_{21}R_{11}^{-1}R_{12} - R_{22}]^{-1}R_{21}R_{11}^{-1} & [R_{22} - R_{21}R_{11}^{-1}R_{12}]^{-1}
\end{bmatrix}.
\]

Define the average group effect of the \( q \) strongly correlated variables in \( X'_1 \) as

\[
\xi_a = \frac{1}{q}(\hat{\beta}'_1 + \hat{\beta}'_2 + \cdots + \hat{\beta}'_q).
\]

Let \( \hat{\beta}' = (\hat{\beta}'_1, \hat{\beta}'_2, \ldots, \hat{\beta}'_{p-1})^T \) be the least squares estimator for \( \beta' = (\beta'_1^T, \beta'_2^T)^T \). The minimum variance unbiased linear estimator for \( \xi'_a \) is

\[
\hat{\xi}_a = \frac{1}{q}(\hat{\beta}'_1 + \hat{\beta}'_2 + \cdots + \hat{\beta}'_q).
\]

Although none of the \( \beta'_i \) in (14) is accurately estimated by \( \hat{\beta}'_i \) in (15), the following theorem shows the average of \( \beta'_i \) is accurately estimated by the average of \( \hat{\beta}'_i \).

**Theorem 2.1** For the group of strongly correlated variables \( X'_1 \) in (10),

(i) if they are uncorrelated with variables in \( X'_2 \), then (i1) \( \text{var}(\hat{\xi}_a) > \sigma^2/q^2 \) and (i2) \( \text{var}(\hat{\xi}_a) \to \sigma^2/q^2 \) as \( r_M \to 1 \);

(ii) if they are weakly correlated with variables in \( X'_2 \) so that elements of \( R_{12}R_{22}^{-1}R_{21} \) are all very small, then \( \text{var}(\hat{\xi}_a) \) will be approximately \( \sigma^2/q^2 \) as \( r_M \to 1 \).

When variables in \( X'_1 \) are uncorrelated with those in \( X'_2 \), result (i1) gives a lower bound on \( \text{var}(\hat{\xi}_a) \), and (i2) shows \( \text{var}(\hat{\xi}_a) \) approaches this lower bound as \( r_M \) approaches its extreme value of 1. Thus \( \xi_a \) is more accurately estimated by \( \hat{\xi}_a \) at higher levels of multicollinearity. Also, when variables in \( X'_1 \) are orthogonal, there is no multicolinearity and \( \text{var}(\hat{\xi}_a) \) is exactly \( \sigma^2/q \). But when they are highly correlated,
(i2) implies \( \text{var}(\hat{\xi}_a) \) is approximately the lower bound \( \sigma^2/q^2 \), so \( \xi_a \) is approximately \( q \) times more accurately estimated (by \( \hat{\xi}_a \)) than when the variables are orthogonal.

We now prove Theorem 2.1.

**Proof.** For any constant vector \( \mathbf{c} \in \mathbb{R}^{p-1} \), we have

\[
\text{var}(\mathbf{c}^T \hat{\mathbf{\beta}}') = \sigma^2 \mathbf{c}^T [\mathbf{X}'^T \mathbf{X}']^{-1} \mathbf{c}. \tag{16}
\]

Let \( \mathbf{c}_a = (1/q, \ldots, 1/q, 0, \ldots, 0)^T \) where the first \( q \) elements are 1/q and the remaining elements are 0. Then, \( \xi_a = \mathbf{c}_a^T \mathbf{\beta}' \) and \( \hat{\xi}_a = \mathbf{c}_a^T \hat{\mathbf{\beta}}' \). By (13) and (16),

\[
\text{var}(\hat{\xi}_a) = \text{var}(\mathbf{c}_a^T \hat{\mathbf{\beta}}') = \frac{\sigma^2}{q^2} \mathbf{1}_q^T [\mathbf{R}_{11} - \mathbf{R}_{12} \mathbf{R}^{-1}_{22} \mathbf{R}_{21}]^{-1} \mathbf{1}_q. \tag{17}
\]

To show (i), when variables in \( \mathbf{X}_1' \) are uncorrelated with variables in \( \mathbf{X}_2' \), all elements of \( \mathbf{R}_{12} = \mathbf{R}^T_{21} \) are zero. By (17),

\[
\text{var}(\hat{\xi}_a) = \frac{\sigma^2}{q^2} \mathbf{1}_q^T \mathbf{R}^{-1}_{11} \mathbf{1}_q = \frac{\sigma^2}{q^2} \mathbf{1}_q^T \mathbf{R}^{-1} \mathbf{1}_q. \tag{18}
\]

Applying Lemma 2.2 to the right-hand side of (18), we obtain (i1) and (i2).

To show (ii), note that the matrix \([\mathbf{R}_{11} - \mathbf{R}_{12} \mathbf{R}^{-1}_{22} \mathbf{R}_{21}]\) in (17) is a real symmetric matrix. When \( r_M \to 1 \), \( \mathbf{R}_{11} = \mathbf{R} \) will approach matrix \( \mathbf{A} \) in (6). Under the condition that elements of \( \mathbf{R}_{12} \mathbf{R}^{-1}_{22} \mathbf{R}_{21} \) are small, we can again define a perturbation matrix \( \mathbf{P}' \) where

\[
\mathbf{P}' = \mathbf{A} - [\mathbf{R}_{11} - \mathbf{R}_{12} \mathbf{R}^{-1}_{22} \mathbf{R}_{21}]
\]

like what we did in (6). If we are to let all elements of \( \mathbf{R}_{12} \mathbf{R}^{-1}_{22} \mathbf{R}_{21} \) go to zero and \( r_M \) go to 1, then we can repeat the same analysis we have performed above to show that \([\mathbf{R}_{11} - \mathbf{R}_{12} \mathbf{R}^{-1}_{22} \mathbf{R}_{21}]\) has all the properties of \( \mathbf{R} \) given in Lemmas 2.1 and 2.2. It follows that \( \text{var}(\hat{\xi}_a) \) in (17) will approach \( \sigma^2/q^2 \) as \( r_M \to 1 \). So when elements of \( \mathbf{R}_{12} \mathbf{R}^{-1}_{22} \mathbf{R}_{21} \) are very small, which they would be when variables in \( \mathbf{X}_1' \) and \( \mathbf{X}_2' \) are weakly correlated, \( \text{var}(\hat{\xi}_a) \) is approximately \( \sigma^2/q^2 \) as \( r_M \to 1 \).

Theorem 2.1 does not cover the case where variables in \( \mathbf{X}_1' \) are fairly strongly correlated with variables in \( \mathbf{X}_2' \). We are not interested in this case as it weakens the notion of \( \mathbf{X}_1' \) being a (stand-alone) group of strongly correlated variables which renders its group effects not meaningful. Returning now to the original model (8)
where $\beta_1, \beta_2, \ldots, \beta_q$ are that of the strongly correlated variables in $X_1$. Let $w^* = (w^*_1, w^*_2, \ldots, w^*_q)^T$ where

$$w^*_i = \frac{s_i}{\sum_{j=1}^{q} s_j},$$  \hspace{1cm} (19)$$
for $i = 1, 2, \ldots, q$. We call the following weighted average

$$\xi_W = w^*_1 \beta_1 + w^*_2 \beta_2 + \cdots + w^*_p \beta_q,$$
the variability weighted average (group) effect of the strongly correlated variables in $X_1$. Let $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_{p-1})^T$ be the least squares estimator for $\beta = (\beta_1^T, \beta_2^T)^T$. The minimum variance unbiased linear estimator for $\xi_W$ is

$$\hat{\xi}_W = w^*_1 \hat{\beta}_1 + w^*_2 \hat{\beta}_2 + \cdots + w^*_p \hat{\beta}_q.$$  

Although variances of individual $\hat{\beta}_i$ will go to infinity when $r_M \to 1$, the following corollary shows that the variance of $\hat{\xi}_W$ will not.

**Corollary 2.1** For the group of strongly correlated variables $X_1$ in (8), if they are uncorrelated or weakly correlated with variables in $X_2$, then

$$\text{var}(\hat{\xi}_W) \approx \frac{\sigma^2}{(\sum_{i=1}^{q} s_i)^2} \quad \text{as } r_M \to 1.$$  

Corollary 2.1 can be proved by using Theorem 2.1 and (11). To summarize, when the level of multicollinearity $r_M$ is high (close to 1), the variances of estimators for the two average group effects $\xi_a$ and $\xi_W$ are near their respective lower bounds, in general comparable to or substantially smaller than the error variance $\sigma^2$. In this sense, we say that $\xi_a$ and $\xi_W$ are estimable. In contrast, variances of the estimators for individual parameters $\beta_i$ and $\beta'_i$ are much larger than the error variance, so $\beta_i$ and $\beta'_i$ are not estimable.

### 2.3 Hald cement data example

The Hald cement data has been widely used to illustrate the impact of multicollinearity. See, for example, Draper and Smith (1998). The data set contains 13 observations with 4 predictor variables $x_1, x_2, x_3, x_4$ and a response $y$:

$$y = \text{heat evolved in calories per gram of cement};$$
\[ x_1 = \text{amount of tricalcium aluminate}; \]
\[ x_2 = \text{amount of tricalcium silicate}; \]
\[ x_3 = \text{amount of tetracalcium alumino ferrite}; \]
\[ x_4 = \text{amount of dicalcium silicate}. \]

Here, we use this data set to illustrate several points of this section. We obtained this data set from the R package “ridge” by Moritz and Cule (2018).

We first illustrate the all positive correlations arrangement of a group of strongly correlated variables. In R display 1 below, the correlation matrix on the left is that of the four predictor variables in the original Hald data set. It shows that there are two strongly correlated groups \( \{x_1, x_3\} \) and \( \{x_2, x_4\} \) but correlation coefficients of both groups are negative. So \( \{x_1, -x_3\} \) and \( \{x_2, -x_4\} \) are the all positive correlations arrangements for these two groups. For convenience, we rename and rearrange signs of the variables so that “\( x_1 = x_1 \)” (that is, “new \( x_1 \) equals the old \( x_1 \)”), “\( x_2 = -x_3 \)”, “\( x_3 = x_2 \)” and “\( x_4 = -x_4 \)”. The correlation matrix of the renamed variables is on the right side of R display 1. The two strongly correlated groups are now \( \{x_1, x_2\} \) and \( \{x_3, x_4\} \), both with positive correlations, and there are no strong correlations between variables from different groups.

R display 1: Correlations of original Hald data (left) and renamed data (right)

\[
\begin{array}{cccc}
X1 & X2 & X3 & X4 \\
X1 & 1.0000 & 0.2285 & -0.8241 & -0.2454 \\
X2 & 0.2286 & 1.0000 & -0.1392 & -0.9729 \\
X3 & -0.8241 & -0.1392 & 1.0000 & 0.0295 \\
X4 & -0.2454 & -0.9729 & 0.0295 & 1.0000 \\
\end{array}
\]

\[
\begin{array}{cccc}
X1 & X2 & X3 & X4 \\
X1 & 1.0000 & 0.8241 & 0.2286 & 0.2454 \\
X2 & 0.8241 & 1.0000 & 0.1392 & 0.0295 \\
X3 & 0.2286 & 0.1392 & 1.0000 & 0.9729 \\
X4 & 0.2454 & 0.0295 & 0.9729 & 1.0000 \\
\end{array}
\]

The following illustrations are concerned with the standardized model (10) based on the renamed variables, so the matrix \( X'^T X' \) in (12) is just the correlation matrix on the right of the above R display. Matrix \( R_{11} \) in the proof of Theorem 2.1 is the upper-left quarter of this correlation matrix and \( R_{22} \) is the lower-right quarter. A key argument in the proof of (ii) of Theorem 2.1 is that elements of \( R_{12} R_{22}^{-1} R_{21} \) are small when variables in \( X'_1 \) are weakly correlated with variables in \( X'_2 \). R display 2 below shows both \( R_{12} R_{22}^{-1} R_{21} \) (on the left) and \( R_{21} R_{11}^{-1} R_{12} \). Their elements are indeed small relative to that of \( R_{11} \) and \( R_{22} \).
The four parameters are poorly estimated with large standard errors due to multicollinearity generated by the two groups of strongly correlated variables. The two average group effects, on the hand, are very accurately estimated. The estimated error variance is \( \hat{\sigma}^2 = 2.306^2 \). So the (estimated) lower bound for the standard errors of the two group effects from Theorem 2.1 is \( \hat{\sigma}/2 = 1.153 \). We see from R display 3 that the standard errors of the two estimated group effects are not too far above the lower bound.

### 3 Concluding remarks

The average group effect \( \xi_a \) has the interpretation as the expected change in response \( y' \) when variables in \( X'_1 \) all increase by \((1/q)\)th of a unit. As such, it represents a group impact or a group effect on the response variable. In the Hald cement data analysis above, for example, when both (the renamed) \( x_1 \) and \( x_2 \) increase by half a unit, the response variable \( y' \) is expected to increase by 14.673 units. All linear combinations of the form (3) are group effects with similar interpretations. The variability weighted average effect \( \xi_W \) is the expected change in the original response variable \( y \) when all original variables in \( X_1 \) change by the amount \( w^* \).
For the uniform model in Tsao (2019), other estimable group effects are found in a neighborhood of $\xi_a$. Based on numerical examples that we have examined, this is also true for a general standardized model (11), so $\xi_a$ serves as a location around which other estimable group effects of variables in $X'_1$ can be found. For example, in the Hald cement data example above, group 1 effect $0.45\hat{\beta}_1 + 0.55\hat{\beta}_2$ is in the neighborhood of the average group effect in that the weight vector $(0.45, 0.55)^T$ which defines this effect is close to the weight vector $(0.5, 0.5)^T$ of the average group effect. The estimated value is $0.45\hat{\beta}_1 + 0.55\hat{\beta}_2 = 12.979$ with a standard error of 2.568. So it is also accurately estimated, although not as accurately as the average group effect. In general, a group effect farther away from the average group effect will be less accurately estimated. Individual parameters of the variables are special group effects at the maximum distance away from the average group effect. They cannot be accurately estimated. The variability weighted average effect is also the location around which other estimable effects of variables in $X_1$ may be found.

The average group effect (14) and the variability weighted average group effect (20) are only defined for strongly correlated predictor variables in an all positive correlations arrangement. If the original variables are not in such an arrangement, then the average of their parameters will not be accurately estimated but all linear combinations of their parameters corresponding to estimable group effects under an all positive correlations arrangement of these variables can be accurately estimated. Finally, we briefly discuss how estimable group effects could be used in the least squares regression. Traditionally, to handle multicollinearity due to strongly correlated predictor variables, the least squares regression is often abandoned in favor of alternatives such as ridge regression and principle component regression. But these alternatives are more complicated in implementation and interpretation. Although individual parameters of these variables cannot be accurately estimated by the least squares regression, estimable group effects can be. Indeed, with strong correlations among variables these group effects are more meaningful than individual parameters as we cannot speak of the impact of a variable in a strongly correlated group in isolation. As such, the group effects provide an alternative means for studying strongly correlated predictor variables to the individual parameters. When we focus on such group effects, the least squares regression works perfectly fine; multicollinearity gen-
erated by these variables is not a problem but a source of useful information as it enables important group effects to be very accurately estimated.

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