Three-flavoured neutrino oscillations and the Leggett–Garg inequality

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Abstract Three-flavoured neutrino oscillations are investigated in the light of the Leggett–Garg inequality (LGI). The results obtained are: (a) The maximum violation of the LGI is 2.17036 for neutrino path length $L_1 = 140.15$ km and $\Delta L = 1255.7$ km. (b) The presence of the mixing angle $\theta_{13}$ enhances the maximum violation of LGI by 4.6%. (c) The currently known mass hierarchy parameter $\alpha = 0.0305$ increases the maximum violation of LGI by 3.7%. (d) The presence of a CP-violating phase parameter enhances the maximum violation of LGI by 0.24%, thus providing an alternative indicator of CP violation in three-flavoured neutrino oscillations. The outline of an experimental proposal is suggested whereby the findings of this investigation may be verified.

1 Introduction

The Leggett–Garg inequality (LGI) [1] is useful to test the quantumness of a system through successive measurement outcomes at different times on the same system. In a previous work [2] we showed that two-state neutral kaon oscillations and two-state neutrino oscillations are quantum phenomena by demonstrating that the LGI is violated in both cases.

Note that the kaon and neutrino cases comprised two different kinds of two-state quantum systems. Oscillations between $K^0$ and $\bar{K}^0$ states indicate a decaying two-state oscillating quantum system. On the other hand, neutrino oscillations between the two-flavour eigenstates $\nu_e$ and $\nu_\mu$ signify a conservative two-state quantum system. In [2] for a decaying kaon system, the maximum violation of LGI in the presence of CP violation is when the correlator $C = 2.36463$ (defined below in Sect. 2), while in the absence of CP violation the LGI violation is maximum when $C = 2.36448$. This is significantly smaller than the Tsirelson bound for the LGI in two-state system given by $C_{\text{Tsirelson}} = 2\sqrt{2} = 2.82843$. In the case of conservative two-flavour neutrino oscillations the maximum violation of LGI is when $C = 2.76000$. Similar work has also been done in two-state neutrino oscillations [3]. There the authors have demonstrated how oscillation phenomena can be used to test for violations of the classical bound by performing measurements on an ensemble of neutrinos at distinct energies.

The existence of neutrino mass has been a subject of keen interest over the last 50 years [4–6]. In 2001 the third generation of neutrinos (tau neutrino) was discovered by the DONUT collaboration [7]. Exhaustive details regarding various aspects of neutrino masses and oscillations can be found in [8,9] and the references therein. The next investigation, therefore, logically should be the LGI in the scenario of three-flavoured neutrino oscillations, both without and with CP violation. This is what we set out to accomplish in the present work. The effect of CP violation for three-flavoured neutrino oscillations may stimulate further investigations in this area. We also consider matter interactions with the neutrino. Here we have analysed the LGI in the context of two small parameters, viz. the sine of the mixing angle $\theta_{13}$, $\sin \theta_{13} << 1$ and the mass hierarchy parameter $\alpha << 1$. Note that the mixing angles are Eulerian angles relating the set $(\nu_e, \nu_\mu, \nu_\tau)$ to the mass eigenstates $(\nu_1, \nu_2, \nu_3)$ in the relevant space as shown in Fig. 1, while $\alpha \equiv \frac{\Delta m^2_{31}}{\Delta m^2_{21}}$ with $\Delta m^2_{31} \equiv m^2_3 - m^2_1$, $\Delta m^2_{21} \equiv m^2_2 - m^2_1$, $\Delta m^2_{12} \equiv m^2_2 - m^2_1$, where $m_i$, $i = 1, 2, 3$, denotes the mass of the $i$th species of neutrino.

In Sect. 2, we give a brief introduction of LGI. In Sect. 3 we discuss the three-flavoured neutrino oscillations. In Sect. 4 the LGI is evaluated and analysed. In Sect. 5 an outline is given of how one can actually experimentally verify the LGI in three-flavoured neutrino oscillations. Section 6 summarises our results. Appendix is in Sect. 7.
2 Leggett–Garg inequality

Bell’s inequality (BI) [10] is based on the assumption of local realism—an intrinsic property of classical physics. Violation of local realism signifies quantum phenomena. BI is a testable algebraic inequality constructed from certain combinations of correlation functions for the outcomes of an observable quantity measurement on two spatially separated systems. BI is violated by quantum physics in the presence of quantum entanglement between two spatially separated systems and implies that the quantum world is non-local [11–16]. Leggett and Garg [1] constructed another algebraic inequality based on the assumption of macrorealism in terms of time separated correlation functions corresponding to the successive measurement outcomes at different times on a single system.

The assumptions underlying the LGI [1] are macroscopic realism (MR) and noninvasive measurability (NIM). MR means that a macroscopic system during its time evolution, is (at any instant time) in a definite one of the available states. NIM means it is possible in principle to determine which of the states the system is in, without affecting the states themselves or the system’s subsequent dynamics. These two aspects together constitute macrorealism.

Consider a two-state system and an observable quantity $Q(t)$ such that whenever $Q(t)$ is measured it takes values +1 or −1 depending on whether it is in state 1 or state 2, respectively. Next consider a collection of runs starting from identical initial conditions such that in the first set of runs $Q$ is measured at times $t_1$ and $t_2$; in the second at $t_2$ and $t_3$; in the third at $t_3$ and $t_4$; in the fourth at $t_1$ and $t_4$ ($t_1 < t_2 < t_3 < t_4$). From such measurements it is straightforward to determine the temporal correlation function $C_{ij} = \langle Q(t_1)Q(t_2) \rangle$. Any physical system obeying the assumptions of a macrorealistic theory will then give the Leggett–Garg inequality [1]:

$$C ≡ C_{12} + C_{23} + C_{34} - C_{14} ≤ 2.$$  \hspace{1cm} (1)

A wide range of quantum systems violate the upper bound of the LGI. This allows one to use the LGI to probe quantum mechanics (QM) in the macroscopic regime [17–34]. A detailed review on LGI is given in [35].

The Legget–Garg Inequality involves the time parameter whereas the relevant probabilities (given below in Sect. 3 onwards) are expressed in terms of the base line length parameter $L$. But $L = ct$, $c$ is the velocity of light. So $t$ is automatically present. Now the correlations in time are transcribed into correlations in length.

Consider an $n$-state system. As before, measurements of a macroscopic property $Q$ can yield only two values $±1$, i.e. $Q$ is a dichotomic variable. If some states (say $k$ states where $k < n$) take the value $+1$ then all the remaining $n - k$ states will take the value $-1$. This is no problem because states with the same value of $Q$ may be considered as microscopically distinct states with the same macroscopic property $Q$. MR and NIM then imply that the system has a definite value of $Q$ at all times and this value is independent of previous measurements on the system. Therefore, the bound for Eq. (1) in macrorealistic theories remains the same.

We now consider LGI in the three-flavoured neutrino oscillations.

3 Three-flavoured neutrino oscillations

During propagation neutrinos undergo oscillations between the three-flavoured eigenstates $\nu_e$, $\nu_\mu$ and $\nu_\tau$. Consider the standard parameterisation of the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix $U$ that mixes the three neutrino flavour states [36,37]:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}e^{i\delta_{CP}} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}e^{i\delta_{CP}} & c_{13} \\ s_{12}s_{23} - c_{12}s_{13}e^{i\delta_{CP}} & -s_{12}s_{23} - c_{12}s_{13}e^{i\delta_{CP}} & c_{13} \end{pmatrix}$$

(2)

where $\theta_{ij}$ are the mixing angles, $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ and $\delta_{CP}$ is the Dirac-type CP-violating phase. If $P_{\alpha\beta} \equiv P(\nu_\alpha \rightarrow \nu_\beta)$ is the transition probability from one neutrino flavour $\alpha$ to another flavour $\beta$, then in general the functional dependence of $P_{\alpha\beta}$ is

$$P_{\alpha\beta} = P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, E, L, V(x)).$$

(3)

where $\alpha, \beta \equiv e, \mu, \tau$. Here $\Delta m^2_{ij} \equiv m_i^2 - m_j^2$ with $m_i$ being the mass of the $i$th species. $E$ is the neutrino energy, $L$ is the baseline length, and $V(x)$ is the matter-induced effective potential, $x \in [0, L]$ is the coordinate along the neutrino path.

$\Delta m^2_{ij} \theta_{ij}$’s and $\delta_{CP}$ are fundamental parameters and the same for all experiments. On the other hand $E$, $L$ and $V$ vary from experiment to experiment.

In [38] complete sets of series expansion formulae for neutrino oscillation probabilities in matter of constant density.
have been calculated taking into account the three flavours. We will consider the neutrino energies of the order of 1 GeV. Therefore we consider the appropriate double expansion given in [38] up to the second order in both mass hierarchy parameter $\alpha \equiv \frac{\Delta m^2_{31}}{\Delta m^2_{21}}$ and $s_{13} = \sin \theta_{13}$.

Let us start with an electron-neutrino beam at time $t = 0$, i.e. $L = 0$. After time $t$, i.e. distance $L = ct$, the probability of finding $\nu_e$, $\nu_\mu$ and $\nu_\tau$ are, respectively [38],

$$P_{\nu_e} = 1 - \alpha^2 \sin^2 2\theta_{12} \sin^2 \left( \frac{V L}{2} \right) \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} L}{4E} - 4s_{13}^2 \frac{\Delta m^2_{31} L}{4E},$$

$$P_{\nu_\mu} = \alpha^2 \sin^2 2\theta_{12} \sin^2 \left( \frac{V L}{2} \right) \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} L}{4E} + 4s_{13}^2 \frac{\Delta m^2_{31} L}{4E},$$

$$\times \left[ \frac{\Delta m^2_{31} L}{4E} \sin \left( \frac{V L}{2} \right) \sin \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} L}{4E} \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \right].$$

(4)

$$P_{\nu_\tau} = \alpha^2 \sin^2 2\theta_{12} \sin^2 \left( \frac{V L}{2} \right) \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} L}{4E} + 2s_{13} \sin 2\theta_{12} \sin 2\theta_{23}$$

$$\times \cos \left( \frac{\Delta m^2_{31} L}{4E} \sin \left( \frac{V L}{2} \right) \sin \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} L}{4E} \right).$$

(5)

$$P_{\nu_e} = \alpha^2 \sin^2 2\theta_{12} \sin^2 \left( \frac{V L}{2} \right) \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} L}{4E} + 4s_{13}^2 \frac{\Delta m^2_{31} L}{4E},$$

$$\times \cos \left( \frac{\Delta m^2_{31} L}{4E} \sin \left( \frac{V L}{2} \right) \sin \left( \frac{2E \Delta m^2_{31}}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} L}{4E} \right).$$

(6)

Similarly one can find the other eight joint probabilities:

$$P_{\nu_e,\nu_e}(L_1, L_2), P_{\nu_e,\nu_\mu}(L_1, L_2), P_{\nu_e,\nu_\tau}(L_1, L_2), P_{\nu_\mu,\nu_\mu}(L_1, L_2), P_{\nu_\mu,\nu_\tau}(L_1, L_2), P_{\nu_\tau,\nu_\tau}(L_1, L_2), P_{\nu_e,\nu_\tau}(L_1, L_2), P_{\nu_e,\nu_e}(L_1, L_2).$$

The transition probabilities required to evaluate the above joint probabilities are given in detail in [38].

4 Evaluating and analysing LGI for three flavours of neutrino

In the three-flavoured neutrino oscillations, we assume that the dichotomic observable $Q$ takes the value $+1$ when the system is found in the electron-neutrino flavour state $\nu_e$, $Q$ takes the value $-1$ if the system is found in any one of the muon neutrino $\nu_\mu$ or tau neutrino $\nu_\tau$ states. Then the correlation function $C_{12}$ can be evaluated by using all the 9 joint probabilities as

$$C_{12} = \langle Q(L_1) Q(L_2) \rangle = P_{\nu_e,\nu_e}(L_1, L_2) - P_{\nu_e,\nu_\mu}(L_1, L_2) - P_{\nu_e,\nu_\tau}(L_1, L_2)$$

$$-P_{\nu_\mu,\nu_\mu}(L_1, L_2) + P_{\nu_\mu,\nu_\tau}(L_1, L_2) + P_{\nu_\tau,\nu_\tau}(L_1, L_2)$$

$$-P_{\nu_\tau,\nu_\tau}(L_1, L_2) + P_{\nu_e,\nu_\tau}(L_1, L_2) + P_{\nu_e,\nu_e}(L_1, L_2).$$

(8)

The exact expression of $C_{12}$ is given in the appendix. An interesting point in the expression for $C_{12}$ is that for a neutrino beam with given energy the correlation $C_{12}$ shows dependence on $L_1$ as well as the spatial separation $(L_2 - L_1)$. It is also important to note that in the case of two-flavoured neutrino oscillations the correlation function depends only on the spatial separation $(L_2 - L_1)$ [2]. The other correlation functions, viz., $C_{23}, C_{34}$ and $C_{14}$ can be calculated in the same way and they exhibit similar features. Next one can evaluate the correlation function $C$ defined in Eq. (1) in order to study the maximum violation of LGI for three-flavoured neutrino oscillations. Varying the spatial separations, it is found that the maximum value of $C$ is attained essentially when all the spatial separations are taken to be same, i.e. $(L_4 - L_3) = (L_3 - L_2) = (L_2 - L_1) = \Delta L$ and the cor-
matter density is a very good approximation [40–42]. A typical value of this case is significantly smaller than the maximum value of neutrino oscillations. For the given value of \( L \), the variation of the quantity \( C \) increases \( \theta \) because it is larger than 2. So the presence of the mixing angle \( \delta \) also increases the quantumness in three-flavoured neutrino oscillations. This is logical because this means there is only one neutrino mass, so there cannot be any oscillations. However, for three-state neutrino oscillations [2], the condition \( m_1 = m_2 \) implies that the maximum value of the quantity \( C \) is greater than 2, i.e., we are still in the quantum domain. For two-state neutrino oscillations [2], the condition \( m_1 = m_2 \) implies that the maximum value of \( C \) is 2, i.e. one is in the classical domain! This is logical because this means there is only one neutrino mass, so there cannot be any oscillations. However, for three-state neutrino oscillations there are three neutrino masses and if two of them become equal then also there will exist possibility of neutrino oscillations because now there are effectively two masses. In the present case the presence of non-zero value of \( \alpha \) increases the maximum value of \( C \) as shown in Fig. 4. In Fig. 4 blue, orange, green and red color graphs correspond to the behaviour of the quantity \( C \) for values of \( \alpha = 0, 0.01, 0.0305 \) (actual experimentally measured value) and 0.06, respectively. For the present experimentally measured value of \( \alpha \) the maximum value of the quantity \( C \) increases by about 3.7%. So the presence of non-zero \( \alpha \) increases the quantumness in three-flavoured neutrino oscillations.

Next consider the effect of the CP-violating phase parameter \( \delta_{CP} \) on the maximum value of \( C \). If we ignore \( \delta_{CP} \) in the present case the presence of non-zero \( \alpha \) increases the maximum value of \( C \) as shown in Fig. 4. In Fig. 4 blue, orange, green and red color graphs correspond to the behaviour of the quantity \( C \) for values of \( \alpha = 0, 0.01, 0.0305 \) (actual experimentally measured value) and 0.06, respectively. For the present experimentally measured value of \( \alpha \) the maximum value of the quantity \( C \) increases by about 3.7%. So the presence of non-zero \( \alpha \) increases the quantumness in three-flavoured neutrino oscillations.
the expression for $C$, the maximum value of $C$ reduces to 2.16553 for $L_1 = 140.15$ km and $\Delta L = 1253.8$ km. So presence of $\delta_{CP}$ actually enhances the maximum violation of LGI by an amount 0.00483. This is a significant enhancement. Thus CP violation actually enhances the quantumness of the three-flavoured neutrino oscillations. It is worth mentioning that in the case of neutral kaon oscillations the presence of CP violation increases the maximum violation of LGI by an amount 0.00015 [2] which is a 0.008% enhancement, whereas here the increase is 0.24% i.e. a 30-fold increase. Therefore, so far as LGI is concerned, the effect of the CP violation is much more in three-flavoured neutrino oscillations compared to neutral kaon oscillations. In Fig. 5 we focus around the region where the quantity $C$ takes its maximum value both with and without CP violation. The solid curve is the behaviour of $C$ including CP violation and the dashed curve is the behaviour of $C$ without CP violation. Figure 5 tells that the presence of CP violation enhances the maximum QM violation of LGI purpose the observable quantity $Q$ has to be measured at two different times $t_1$ and $t_2$ ($t_2 > t_1$) or equivalently at two different base line lengths $L_1$ and $L_2$ where $L_2 > L_1$. As already mentioned $Q$ takes the value +1 when the system is found in the electron-neutrino flavour state. Otherwise $Q$ takes the value $-1$. So

$$C_{12} = \mathcal{P}_{++}(L_1, L_2) - \mathcal{P}_{+-}(L_1, L_2) - \mathcal{P}_{-+}(L_1, L_2) + \mathcal{P}_{--}(L_1, L_2),$$

(9)

where $\mathcal{P}_{++}(L_1, L_2) = P_{\nu_e,\nu_e}(L_1, L_2)$ is the joint probability of finding the system in the electron-neutrino flavour state at both the distances $L_1$ and $L_2$. Similar arguments hold for the other 3 joint probabilities:

$$\mathcal{P}_{--}(L_1, L_2) = P_{\nu_e,\nu_e}(L_1, L_2),$$

$$\mathcal{P}_{+-}(L_1, L_2) = P_{\nu_e,\nu_\mu}(L_1, L_2) + P_{\nu_\mu,\nu_e}(L_1, L_2),$$

$$\mathcal{P}_{-+}(L_1, L_2) = P_{\nu_\mu,\nu_\mu}(L_1, L_2) + P_{\nu_e,\nu_\mu}(L_1, L_2),$$

$$+ P_{\nu_\mu,\nu_e}(L_1, L_2) + P_{\nu_\tau,\nu_e}(L_1, L_2).$$

Note that the scripted probabilities $\mathcal{P}$ are the ones that are actually measured. These are related to the theoretically calculated unscripted probabilities as shown above. This is necessitated by the fact that here more than one state can have the same value for the dichotomic variable $Q$.

It is to be noted that to experimentally verify the maximum violation of LGI the first measurement of $Q$ at length $L_1$ must satisfy NIM. Otherwise the measurement process will destroy the state of the system and measurement of $Q$ at the later length $L_2$ will be meaningless as the state has already been disturbed. This (NIM in the first measurement) can be ensured using the negative result measurement (NRM) [43] as follows.

Let the measuring set-up be arranged so that if the probe is triggered, $Q(L_1) = +1$, while if it is not triggered, $Q(L_1) =$
−1. This ensures that while the untriggered probe provides information as regards the value of $Q$, there is no interaction occurring between the probe and the measured particle. So NIM is satisfied. Now use only the results of untriggered runs for which $Q(L_1) = −1$. Follow this by the measurement of $Q$ at $L_2$. These results can be used for determining the joint probabilities $P_{-+}(L_1, L_2)$ and $P_{--}(L_1, L_2)$. Similarly, for determining the other two joint probabilities $P_{+-}(L_1, L_2)$ and $P_{++}(L_1, L_2)$ occurring in $C_{12}$, the measuring set-up can be inverted so that a value of $Q(L_1) = +1$ triggers the probe, while for $Q(L_1) = +1$ it does not. In this way, one can determine $C_{12}$ and all the two-time correlation functions occurring in LGI by ensuring NIM through the use of the NRM procedure for the first measurement of any pair. Then one can calculate the total correlation $C$ using Eq. (1) and experimentally verify our results as regards the maximum violation of LGI in the case of three-flavoured neutrino oscillations.

6 Concluding remarks

In this work we have investigated the violation of the LGI in the case of three-flavoured neutrino oscillations. Our findings are as follows:

1. The maximum value of the correlation $C$ is 2.17036 for $L_1 = 140.15$ km and $\Delta L = 1255.7$ km.
2. The violation of the classical bound of $C$ given by LGI for three-flavoured neutrino oscillation is 8.5%. Note that in the case of two-flavoured neutrino oscillations [2] this violation was 38%. So the maximum violation of LGI in the case of three-flavoured neutrino oscillations is significantly lower than the maximum violation for the two-state neutrino oscillation.
3. If we put $\theta_{13} = 0$, the maximum value of $C$ is 2.07762 for $L_1 = 638$ km and $\Delta L = 1376.34$ km. This is much lower than (2.17036) which is obtained for the experimental value of $\theta_{13} = 8.5^\circ$. So the presence of $\theta_{13}$ enhances the maximum violation of LGI by the amount 0.09274 i.e.4.6%. Increasing $\theta_{13}$ increases the maximum value of $C$ (Fig. 3).
4. For the mass hierarchy parameter $\alpha = 0$, i.e. $m_1 = m_2$ the maximum value of $C$ is 2.09606 for $\Delta L = 1252.74$ km. Note that although now $m_1 = m_2$ the maximum bound of $C$ is greater than 2, i.e., we are still in the quantum domain. For two-state neutrino oscillations, [2] $m_1 = m_2$ implied that the maximum value of $C$ is 2, i.e. the classical domain. $\alpha = 0.0305$ increases the maximum value of the quantity $C$ by 3.7% as shown in Fig. 4.
5. If $\delta_{CP} = 0$ in the expression for $C$, the maximum value of $C$ reduces to 2.16553 for $L_1 = 140.15$ km and $\Delta L = 1253.8$ km. So the presence of a CP-violating phase parameter actually enhances the maximum violation of LGI by an amount 0.00483 which is 0.24%, a significant enhancement (Fig. 5). Compare this to the case of neutral kaon oscillations where including CP violation increased the maximum violation of LGI by 0.008% [2].

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7 Appendix

The exact expression of the correlation function $C_{12}$ is given by

$$
C_{12} = P_{\nu_e, \nu_e}(L_1, L_2) - P_{\nu_e, \nu_e}(L_1, L_2) - P_{\nu_e, \nu_e}(L_1, L_2) - P_{\nu_e, \nu_e}(L_1, L_2) + P_{\nu_e, \nu_e}(L_1, L_2) + P_{\nu_e, \nu_e}(L_1, L_2) + P_{\nu_e, \nu_e}(L_1, L_2) + P_{\nu_e, \nu_e}(L_1, L_2)
$$

\[
= \left[ 1 - \alpha^2 \sin^2 2\theta_{12} \frac{\sin^2 \left( \frac{L_1}{2} \right)}{\left( \frac{2EV}{\Delta m^2_{31}} \right)^2} \right] \\
-4\xi_{13}^2 \sin^2 \left( \frac{2EV}{\Delta m^2_{31}} \right) \left[ \left( \frac{2EV}{\Delta m^2_{31}} \right) \left( \frac{2EV}{\Delta m^2_{31}} \right) \right] \\
\times \left[ 1 - 2\alpha^2 \sin^2 2\theta_{12} \frac{\sin^2 \left( \frac{L_1}{2} - \frac{L_1}{2} \right)}{\left( \frac{2EV}{\Delta m^2_{31}} \right)^2} - 8\xi_{13}^2 \sin^2 \left( \frac{2EV}{\Delta m^2_{31}} \right) \left( \frac{2EV}{\Delta m^2_{31}} \right) \left( \frac{2EV}{\Delta m^2_{31}} \right) \right] \\
- \alpha^2 \sin^2 2\theta_{12} \frac{\sin^2 \left( \frac{L_1}{2} \right)}{\left( \frac{2EV}{\Delta m^2_{31}} \right)^2} \\
+4\xi_{13}^2 \sin^2 \left( \frac{2EV}{\Delta m^2_{31}} \right) \left( \frac{2EV}{\Delta m^2_{31}} \right) \left( \frac{2EV}{\Delta m^2_{31}} \right) \left( \frac{2EV}{\Delta m^2_{31}} \right) \\
\times \sin 2\theta_{12} \sin 2\theta_{32} \cos \left( \frac{\Delta m^2_{31} L_1}{4E} - \delta_{CP} \right) \\
\times \left[ \frac{2EV}{\Delta m^2_{31}} \sin \left( \frac{V_{L1}}{2} \right) \sin \left( \frac{2EV}{\Delta m^2_{31}} \right) \left( \frac{2EV}{\Delta m^2_{31}} \right) \right]
\]
\[
\times \left[ 2a^2 \sin^2 2\theta_{12} \frac{\tan^2 (\frac{V(L_2-L_1)}{2})}{2} \left( \frac{2EV}{\Delta m^2_{31}} \right)^2 \right.
\]
\[
+ 8s_{13}^2 \sin^2 \left( \frac{2EV}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right]
\]
\[
\times \left[ 4a^2 \sin^2 2\theta_{12} \frac{\tan^2 (\frac{V(L_2-L_1)}{2})}{2} + 4s_{13}^2 \sin^2 \left( \frac{2EV}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right]
\]
\[
+ 4a^2 \sin^2 2\theta_{12} \sin 2\theta_{23} \tan \left( \frac{V(L_1-L_2)}{2} \right) \sin \left( \frac{2EV}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} (L_1)}{4E} \right]
\]
\[
\times \left[ \cos \left( \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right) - \delta_{\text{CP}} \right]
\]
\[
- \sin \delta_{\text{CP}} \sin \left( \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right) \right]
\]
\[
- \left[ 2a^2 \sin^2 2\theta_{12} \frac{\tan^2 (\frac{V(L_2-L_1)}{2})}{2} \right.
\]
\[
+ 8s_{13}^2 \sin^2 \left( \frac{2EV}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right]
\]
\[
\times \left[ 4a^2 \sin^2 2\theta_{12} \frac{\tan^2 (\frac{V(L_2-L_1)}{2})}{2} + 4s_{13}^2 \sin^2 \left( \frac{2EV}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right]
\]
\[
+ 4a^2 \sin^2 2\theta_{12} \sin 2\theta_{23} \tan \left( \frac{V(L_1-L_2)}{2} \right) \sin \left( \frac{2EV}{\Delta m^2_{31}} - 1 \right) \frac{\Delta m^2_{31} (L_1)}{4E} \right]
\]
\[
\times \left[ \cos \left( \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right) - \delta_{\text{CP}} \right]
\]
\[
- \sin \delta_{\text{CP}} \sin \left( \frac{\Delta m^2_{31} (L_2-L_1)}{4E} \right) \right] \right] \right] - 1. \tag{10}
\]