RANDOM TOEPLITZ MATRICES: THE CONDITION NUMBER UNDER HIGH STOCHASTIC DEPENDENCE

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ABSTRACT. In this paper, we study the condition number of a random Toeplitz matrix. Since a Toeplitz matrix is a diagonal constant matrix, its rows or columns cannot be stochastically independent. This situation does not permit us to use the classic strategy to analyze its minimum singular value when all the entries of a random matrix are stochastically independent. Using Cauchy Interlacing Theorem as a decoupling technique, we can break the stochastic dependence of the structure of the Toeplitz matrix and reduce the problem to analyze the extreme singular values of a random circulant matrix. A circulant matrix is, in fact, a particular class of a Toeplitz matrix, but with a more specific structure, where it is possible to obtain explicit formulas for its eigenvalues and also for its singular values. Among our results, we show the condition number of non–symmetric random Toeplitz matrix of dimension $n$ under the existence of moment generating function of the random entries is $\kappa(\mathcal{T}_n) = O\left(\frac{1}{\varepsilon n^{\rho+1/2} \log n^{1/2}}\right)$ with probability $1 - O\left(\varepsilon^2 + \varepsilon n^{-2\rho} + n^{-1/2+o(1)}\right)$ for any $\varepsilon > 0$, $\rho \in (0, 1/4)$. Moreover, if the random entries only have the second moment, the condition number satisfies $\kappa(\mathcal{T}_n) = O\left(\frac{1}{\varepsilon n^{\rho+1/2} \log n}\right)$ with probability $1 - O\left((\varepsilon^2 + \varepsilon n^{-2\rho} + (\log n)^{-1/2})\right)$. Also, Cauchy Interlacing Theorem permits to analyze the condition number of a symmetric random Toeplitz matrix. In this case, the condition number $\kappa(\mathcal{T}_n^{sym}) = O\left(\frac{1}{\varepsilon n^{1.01} \log n^{1/2}}\right)$ with probability $1 - O\left(\varepsilon n^{-1/10} + n^{-77/300+o(1)}\right)$, when the random entries have moment generating function. Additionally, we show that the results on the random Toeplitz matrices hold for random Hankel matrices.

1. INTRODUCTION

The singularity of random matrices has been an intensely studied topic in the last years; see e.g., [3, 16, 17, 21, 22]. Recall that a square matrix is called singular if its determinant is zero. A criterium to determine a matrix is singular is to verify if its minimum singular value is zero. The singular values of a matrix carry more useful information about the properties of the matrix, inclusive if it is rectangular. For example, they play an important aspect in the celebrated Circular Law Theorem; see [3] for a systematic presentation.

The singular values of a square matrix $A$ are the eigenvalues of the matrix $\sqrt{A^T A}$, where $A^T$ denotes the transpose matrix of $A$. Therefore, the singular values are non–negative real numbers. Note the singular values can be defined when $A$ is no square since $A^T A$ is always square. The extreme singular values are related to the norm of a matrix. The operator norm of an $n$-dimensional square matrix $A$ is defined by

$$\|A\| := \max_{\|x\|_2 = 1} \|Ax\|_2,$$

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where $\| \cdot \|_2$ denotes the Euclidean norm. If $0 \leq \sigma_n \leq \sigma_{n-1} \leq \cdots \leq \sigma_1$ are the singular values of matrix $A$, we have

$$\|A\| = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1, \quad \|A^{-1}\| = \left( \min_{\|x\|_2=1} \|Ax\|_2 \right)^{-1} = \sigma_n^{-1}. $$

The last equality has only meaning when $A$ is non-singular. In the rest of this paper, we consider the following notation for the extreme singular values $\sigma_{\max} := \sigma_1$ and $\sigma_{\min} := \sigma_n$. In this context, it is known that $\sigma_{\min}$ measures the distance of a matrix $A$ to the set of singular matrices. More precisely,

$$\sigma_{\min} = \inf \{ \|E\| : A + E \text{ is singular and } E \text{ is } n \times n \text{ matrix} \}.$$ 

From the above identity, we can verify that if the minimum singular value is zero, then the matrix is singular. If $\sigma_{\min} \neq 0$, we can define the so-called condition number $\kappa(A)$ of a matrix $A$ as

$$\kappa(A) := \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}.$$ 

The condition number was independently introduced by Alan Turing (1948) and by John von Neumann and Herman Goldstine (1947) in order to study the accuracy in the solution of a linear system in the presence of finite-precision arithmetic [6]. By the definition of condition number is easy to see $\kappa \geq 1$. If $\kappa$ is very large, the corresponding matrix is said to be ill-conditioned. The logic for this terminology is that if $\kappa$ is very large, then $\sigma_{\min}$ should be small and the matrix $A$ is close to the set of singular matrices. Then, a small perturbation of $A$ can cause loss accuracy in the computed solution of the system $Ax = b$; see [8]. Thus, it is interesting to set up conditions under which $\kappa$ is close to low values and this requires the estimation from below of the minimum singular value $\sigma_{\min}$ as well as the estimation from above of the maximum singular value $\sigma_{\max}$. These are precisely the main goals of this paper for the specific class of structured random matrices which are (non-symmetric and symmetric) Toeplitz matrices.

Among the first papers on the condition number of random matrices, we have one from Demmel [8]. He assumes that $A$ is an $n$-dimensional random square matrix such that $A/\|A\|_F$ ($\| \cdot \|_F$ is the Frobenius norm) is uniformly distributed on the unit sphere. Demmel defines $\kappa_1(A) := \|A\|_F \|A^{-1}\|$ as an approximation to the condition number and shows

$$\frac{C(1 - 1/x)n^2 - 1}{x} \leq \mathbb{P} (\kappa_1(A) \geq x) \leq \sum_{k=1}^{n^2} \frac{n^2}{k} \left( \frac{2n}{x} \right)^k,$$

where $C > 0$ depends on $n$.

Other papers study the behavior of $\sigma_{\min}$ or the condition number of random matrices under either strong independency assumptions or some structure specification on their entries. For example, Rudelson and Vershynin [21] prove that if $A$ has entries which are independent and identically distributed (i.i.d. for short) in the class of sub-Gaussian random variables (r.v. for short) with variance at least 1, then for all $\varepsilon \geq 0$, $\mathbb{P} (\sigma_{\min}(A) \leq \varepsilon n^{-1/2}) \leq C \varepsilon + c^n$, for some constants $C > 0$ and $c \in (0, 1)$ depending on the sub-Gaussian r.v. Vershynin [25] proves a similar estimation for a symmetric matrix where the upper triangle part has independent and identically sub-Gaussian r.v. entries. Recently, Litvak and et al. [15] consider that $A$ is a matrix with sub-Gaussian i.i.d. entries with zero mean and unit variance. They show $\mathbb{P} (\kappa(A) \leq n/t) \leq 2\exp(-ct^2)$ for $t \geq 1$ and positive constant $c$ which depends on the sub-Gaussian r.v.

On the other hand, random matrices with structure have been analyzed, i.e., matrices whose entries follow certain disposition. For example, random triangular matrices $L_n$ with entries in the
A circulant matrix is a particular case of a Toeplitz matrix, which structure permits to give an explicit expression for its eigenvalues. It is well known that a Toeplitz matrix can be approximated by circulant matrices (for example see [24]). Circulant matrices have a lot of useful properties; see [26]. Actually, as we will see, the random circulant matrices are closely related to random polynomials. So, all our statements on the extreme singular values of a random Toeplitz matrix are direct implications from our results on random circulant matrices.

The main tools used to bring Toeplitz problem to the circulant problem are the Cauchy Interlacing Theorem and the circulant embedding, which can be considered as a decoupling technique. They permit broke the strong stochastic dependence in the Toeplitz structure into circulant structure, where we can handle the estimation of the extreme singular values. Once the problem is reduced to circulant structure, we estimate the maximum singular value by estimating the maximum modulus of a random polynomial on the unit circle. To do this, we use the so-called *Salem–Zygmund inequality*. Under the existence of the moment generating function (m.g.f. for short), we show that the maximum singular value of a random (non–symmetric or symmetric) circulant matrix is $O\left((n \log n)^{1/2}\right)$ with probability $1 - O\left(n^{-2}\right)$. In the non–symmetric case, we can relax our conditions up to the only existence of the second moment, in which case the maximum singular value is $O\left(n^{1/2} \log n\right)$ with probability $1 - O\left(\log n\right)^{-1/2}\right)$. On the other hand, using the concept of least common denominator, a tool developed to handle the so–called small ball probability problem, we obtain a lower bound for the minimum singular value. For any $\varepsilon > 0$ and $\rho \in (0, 1/4)$, we show under mild conditions (see below the condition (H)) that the minimum singular value for non–symmetric circulant matrix is at least $\varepsilon n^{-\rho}$ with probability $O\left(\frac{e^{2\pi^2}}{n^{9/2}} + \frac{1}{n^{1/2}}\right)$. In the symmetric case, we show that the minimum singular value is at least $\varepsilon n^{-0.51}$ with probability $O\left(\frac{e^{2\pi^2}}{n^{9/2}} + \frac{1}{n^{1/2} \log n + \varepsilon^{-1/2}}\right)$ for any $\varepsilon > 0$. Since a Hankel matrix can be transformed into a Toeplitz matrix as we will see later, all our
results in random Toeplitz matrices hold for random Hankel matrices.

This paper is organized as follows. In Section 2 we state the main results of this paper. The reduction of the Toeplitz problem to the circulant problem is explained in Section 3. The results of random circulant matrices are stated in Section 4. In Section 5 we prove the Salem–Zygmund inequalities for non–symmetric and symmetric cases. In Section 6 we give the proof of Theorem 4.3 about the minimum singular value of random non–symmetric circulant. In Section 7 we prove the lower bound of the minimum singular value for random symmetric circulant. In Section 8 we give the proof of our results on the condition number of a random Toeplitz matrix. The Appendix A and Appendix B contain additional material in order to provide clarity to this paper.

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2. Main results

A Toeplitz matrix $T_n$ is an $n \times n$ matrix with constant diagonals, i.e., $T_n$ has the following structure

$$T_n = \begin{bmatrix}
\xi_0 & \xi_1 & \cdots & \xi_{n-2} & \xi_{n-1} \\
\xi_{-1} & \xi_0 & \xi_1 & \cdots & \xi_{n-2} \\
\vdots & \xi_{-1} & \xi_0 & \cdots & \xi_{n-3} \\
\xi_{-n+2} & \cdots & \xi_{-2} & \xi_{-1} & \xi_0
\end{bmatrix}.$$ 

When the entries of $T_n$ are r.v., we say it is a random Toeplitz matrix. Let $\Xi := \{\xi_j : j \in \mathbb{Z}\}$ be a set of i.i.d. r.v. We assume that the random entries of $T_n$ belong to $\Xi$.

In the rest of this paper, any positive constant will be denoted by $C_0, C_1, C_2, \ldots$, which are not necessarily equal in each statement as they appear. We denote the norm of a real or complex number $z$ as $|z|$.

In the following, we state our results on the random Toeplitz matrix and their implications on random Hankel matrices.

2.1. Non–symmetric Toeplitz. Let $\xi_j \in \Xi$ for $j = -n, \ldots, n$, and we consider the respective random Toeplitz matrix $T_n$ with $2n - 1$ i.i.d. entries. For the first result on the Toeplitz matrix, we assume that the random entries have m.g.f. The existence of m.g.f. permits to use Chernoff bounding technique to estimate the $\sigma_{\max}$.

**Theorem 2.1** (Non–symmetric Toeplitz: Maximum singular value I). Suppose $\xi_0$ has zero mean and finite positive variance. If the m.g.f. of $\xi_0$ exists in an open interval around zero. Then

$$\mathbb{P}\left(\sigma_{\max}(T_n) \geq C_0 \left((2n) \log(2n)\right)^{1/2}\right) \leq \frac{C_1}{(2n)^2},$$

where the $C_0, C_1$ are positive constants depending on the distribution of $\xi_0$.

**Remark 2.2.** By the Borel–Cantelli lemma, we have from Theorem 2.1 that

$$\limsup_{n \to \infty} \frac{\sigma_{\max}(T_n)}{((2n) \log(2n))^{1/2}} \leq C_0 \quad \text{almost surely.}$$

Actually, we can relax the conditions in Theorem 2.1 up to the existence of the second moment of $\xi_0$. 

Theorem 2.3 (Non–symmetric Toeplitz: Maximum singular value II). Suppose $\xi_0$ has zero mean and $E[\xi_0^2] < \infty$ exists. Then
\[ P\left(\sigma_{\max}(T_n) \geq C_0 (2n)^{1/2} \log(2n)\right) \leq \frac{C_1}{(\log(2n))^{1/2}} , \]
where the $C_0, C_1$ are positive constants depending on the distribution of $\xi_0$.

For our result on the minimum singular value of $T_n$, we need to introduce the condition (H). We say a r.v. $\xi$ satisfies the condition (H) if
\[ \sup_{u \in \mathbb{R}} P(|\xi - u| \leq 1) \leq 1 - q \quad \text{and} \quad P(|\xi| > M) \leq q/2 \quad (H) \]
for some $M > 0$ and $q \in (0, 1)$.

The first part of the condition (H) says that a r.v. $\xi$ is not concentrated around any single value. Usually, it is referred to as Lévy concentration function, which in general is defined as follows.

Definition 2.4. The Lévy concentration function of a random vector $\xi \in \mathbb{R}^n$ is defined for any $\varepsilon \geq 0$ as
\[ L(\xi, \varepsilon) := \sup_{x \in \mathbb{R}^n} P(||x - x||_2 \leq \varepsilon) . \]

In fact, our task will be to bound $L((2n)^\rho \sigma_{\min}(T_n), \varepsilon)$ for any $\varepsilon > 0$ and $\rho \in (0, 1/4)$. To do this we use the concept of the least common denominator (LCD). LCD permits to give an upper bound of the Lévy concentration of a random sum $\sum_{i=1}^{n-1} \alpha_i Z_j$ in terms of its coefficients $\{\alpha_j\}_{j=0}^{n-1}$, assuming that $Z_j$ are i.i.d. r.v. satisfying the condition (H). In later sections, we show how to reduce $L((2n)^\rho \sigma_{\min}(T_n), \varepsilon)$ to Lévy concentration of a sum of r.v. Thus, we can prove the following statement.

Theorem 2.5 (Non–symmetric Toeplitz: minimum singular value). Suppose $\xi_0$ satisfies the condition (H). Then, for each $\rho \in (0, 1/4)$ and for any $\varepsilon > 0$ we have for all large $n$
\[ P\left(\sigma_{\min}(T_n) \leq \varepsilon(2n)^{-\rho}\right) \leq C \left(\frac{\varepsilon^2 + \varepsilon}{(2n)^{2\rho}} + \frac{1}{(2n)^{1/2-o(1)}}\right) , \]
where $C$ is a positive constant depending on the distribution of $\xi_0$.

Remark 2.6. In Theorem 2.5 as in the rest of this paper, we can take $o(1) = C (\log \log n)^{-1}$, where $C$ is a universal positive constant. This error is a consequence of Lemma 6.5 used in the proof of the equivalent statement for the minimum singular value of a random circulant matrix.

If the conditions of the previous results on $\sigma_{\max}(T_n)$ and $\sigma_{\min}(T_n)$ hold at the same time, we can bound the condition number of a random Toeplitz matrix $T_n$.

Theorem 2.7 (Non–symmetric Toeplitz: Condition number). If the conditions of Theorem 2.7 and Theorem 2.5 hold. Then, for any $\varepsilon > 0$ and $\rho \in (0, 1/4)$, the condition number $\kappa(T_n)$ of a random (non–symmetric) Toeplitz matrix $T_n$ satisfies for all large $n$
\[ P\left(\kappa(T_n) \leq \frac{C_0}{\varepsilon} n^{\rho + 1/2} (\log n)^{1/2}\right) \geq 1 - C_1 \left(\varepsilon^2 + \varepsilon\right)n^{-2\rho} + n^{-1/2 + o(1)} , \]
where $C_0, C_1$ are positive constants depending on the distribution of $\xi_0$. If the conditions of Theorem 2.3 and Theorem 2.5 hold, the condition number satisfies for all large $n$
\[ P\left(\kappa(T_n) \leq \frac{C_0}{\varepsilon} n^{\rho + 1/2} \log n\right) \geq 1 - C_1 \left(\varepsilon^2 + \varepsilon\right)n^{-2\rho} + (\log n)^{-1/2} , \]
where $C_0, C_1$ is a positive constant depending on the distribution of $\xi_0$. 


2.2. **Toeplitz symmetric.** Let \( \xi_j \in \Xi \) for \( j = 0, 1, \ldots, n \) and we consider the respective random symmetric Toeplitz matrix \( T_n^{sym} \) with \( n \) i.i.d. entries. In the following, we give upper and lower bounds for the maximum and minimum singular values, respectively, and consequently for the condition number of \( T_n^{sym} \).

**Theorem 2.8** (Symmetric Toeplitz: Maximum singular value). Suppose \( \xi_0 \) has zero mean and finite positive variance. If the m.g.f. of \( \xi_0 \) exists in an open interval around zero. Then

\[
P(\sigma_{\max}(T_n^{sym}) \geq C_0 ((2n) \log(2n))^{1/2}) \leq \frac{C_1}{(2n)^2},
\]

where the \( C_0, C_1 \) are positive constants depending on the distribution of \( \xi_0 \).

**Remark 2.9.** By the Borel–Cantelli lemma, we have from Theorem 2.8 that

\[
\limsup_{n \to \infty} \frac{\sigma_{\max}(T_n^{sym})}{(2n) \log(2n)^{1/2}} \leq C_0 \quad \text{almost surely.}
\]

**Theorem 2.10** (Symmetric Toeplitz: Minimum singular value). Suppose \( \xi_0 \) satisfies the condition \( (H) \). Then, for any \( \varepsilon > 0 \) we have for all large \( n \)

\[
P(\sigma_{\min}(T_n^{sym}) \leq \varepsilon(2n)^{-0.51}) \leq C \left( \frac{\varepsilon}{(2n)^{0.1}} + \frac{1}{(2n)^{77/300-o(1)}} \right),
\]

where \( C \) is a positive constant depending on the distribution of \( \xi_0 \).

**Theorem 2.11** (Symmetric Toeplitz: Condition number). If the conditions of Theorem 2.8 and Theorem 2.10 hold. Then, for any \( \varepsilon > 0 \) the condition number \( \kappa(T_n^{sym}) \) of a random symmetric Toeplitz matrix \( T_n^{sym} \) satisfies for all large \( n \)

\[
P\left( \kappa(T_n^{sym}) \leq \frac{C_0}{\varepsilon n^{1.01}} (\log n)^{1/2} \right) \geq 1 - C_1 \left( \varepsilon n^{-0.1} + n^{-77/300+o(1)} \right),
\]

where \( C_0, C_1 \) are positive constants depending on the distribution of \( \xi_0 \).

**Remark 2.12.** We observe that the bound for the condition number of a random symmetric Toeplitz matrix increases by a factor of \( n^{0.51-\rho} \) to respect the non–symmetric case with m.g.f. Intuitively, this is caused because we reduce the number of independent r.v. from \( 2n-1 \) to \( n \). However, the bound is not extremely large.

2.3. **Random Henkel matrix.** Let \( J := (J_{i,j}) \) be the \( n \times n \) exchange matrix, i.e., the entries of \( J \) are \( J_{i,j} = 1 \) if \( j = n - i + 1 \), and \( J_{i,j} = 0 \) if \( j \neq n - i + 1 \). Note \( J \) has the following properties:

- \( J^2 = I_n \), where \( I_n \) is the \( n \times n \) identity matrix
- \( J^T = J \)

An \( n \times n \) matrix \( H \) is called Henkel if the \( JH \) is a Toeplitz matrix. For more details in Hankel matrices see [26]. Observe

\[
\sqrt{(JH)^T(JH)} = \sqrt{H^T J^T JH} = \sqrt{H^T H}.
\]

(1)

By \( [\text{I}] \) we have that \( JH \) and \( H \) have the same singular values. Then, all results in this section hold for random (non–symmetric or symmetric) Hankel matrices with the respective assumptions.
3. FROM TOEPLITZ MATRICES TO CIRCULANT MATRICES

A circulant matrix $C_n$ is a particular case of a Toeplitz matrix of dimension $n$ where the entries are circulated row by row. A circulant matrix $C_n$ looks like

$$C_n = \begin{bmatrix}
  \xi_0 & \xi_1 & \cdots & \xi_{n-2} & \xi_{n-1} \\
  \xi_{n-1} & \xi_0 & \cdots & \xi_{n-3} & \xi_{n-2} \\
  \xi_{n-2} & \xi_{n-1} & \cdots & \xi_{n-4} & \xi_{n-3} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \xi_1 & \xi_2 & \cdots & \xi_{n-1} & \xi_0 
\end{bmatrix}.$$  

Note that a circulant matrix is defined by its first row. Let $w_n := \exp \left( \frac{i2\pi}{n} \right)$, $i^2 = -1$. It is well known any circulant matrix is diagonalized by the matrix of Fourier $F_n$, whose entries are power of $w_n$, i.e., $F_n = \frac{1}{\sqrt{n}} (w_n^{jk})_{0 \leq j, k \leq n-1}$. By a straightforward computation, it follows:

$$C_n = F_n^* \text{diag} \left( G_n(1), G_n(w_n), \ldots, G_n(w_n^{n-1}) \right) F_n,$$

where $F_n^*$ is the conjugate transpose of $F_n$, and $G_n(z) := \sum_{j=0}^{n-1} \xi_j z^j$ is a complex polynomial. This property permits to find explicit expressions for its eigenvalues. If $\lambda_0, \ldots, \lambda_{n-1}$ denote the eigenvalues of a circulant matrix, we have

$$\lambda_k = G_n(w_n^k) = \sum_{j=0}^{n-1} \xi_j w_n^{jk} \text{ for } k = 0, \ldots, n-1. \quad (2)$$

If the circulant matrix is symmetric, the expressions for eigenvalues are reduced to a linear combination of cosine values, i.e., they can be expressed as:

- $n$ odd: $\lambda_k = \lambda_{n-k}$ for $1 \leq k \leq \lfloor n/2 \rfloor$, then
  $$\lambda_0 = \xi_0 + 2 \sum_{j=1}^{\lfloor n/2 \rfloor} \xi_j, \quad \lambda_k = \xi_0 + 2 \sum_{j=1}^{\lfloor n/2 \rfloor} \xi_j \cos \left( \frac{2\pi k}{n} j \right), \quad (3)$$

- $n$ even: $\lambda_k = \lambda_{n-k}$ for $1 \leq k \leq n/2$, then
  $$\lambda_0 = \xi_0 + 2 \sum_{j=1}^{n/2-1} \xi_j, \quad \lambda_k = \xi_0 + 2 \sum_{j=1}^{n/2-1} \xi_j \cos \left( \frac{2\pi k}{n} j \right) + (-1)^k \xi_{n/2}. \quad (4)$$

In fact, a circulant with real or complex entries is a normal matrix, i.e., it satisfies the condition $C_n^* C_n = C_n C_n^*$, where $C_n^*$ denotes the conjugate transpose of $C_n$. This property implies that the extreme singular values of a circulant matrix satisfy the following relationships

$$\sigma_{\max}(C_n) = \max_{k=0, \ldots, n-1} |\lambda_k|, \quad \sigma_{\min}(C_n) = \min_{k=0, \ldots, n-1} |\lambda_k|.$$  

Hence, the condition number of a circulant matrix is

$$\kappa(C_n) = \left( \frac{\sigma_{\max}(C_n)}{\sigma_{\min}(C_n)} \right)^{-1}.$$  

3.1. Circulant embedding. Every Toeplitz matrix $T_n$ can be embedded into a circulant matrix of dimension $2n$. In fact, let $C_{2n}$ be a circulant matrix defined as

$$C_{2n} = \begin{bmatrix} T_n & B_n \\ B_n & T_n \end{bmatrix}.$$  

(5)
where
\[
\mathbf{B}_n := \begin{bmatrix}
\xi_* & \xi_{-n+1} & \cdots & \xi_{-2} & \xi_{-1} \\
\xi_{n-1} & \xi_* & \xi_{-n+1} & \cdots & \xi_{-2} \\
\vdots & \xi_{n-1} & \xi_* & \cdots & \vdots \\
\xi_2 & \cdots & \cdots & \xi_{-n+1} \\
\xi_1 & \xi_2 & \cdots & \xi_{n-1} & \xi_*
\end{bmatrix}.
\]

The variable \(\xi_*\) does not have any restrictions. Note \(\mathbf{B}_n\) is a Toeplitz matrix. If \(\mathcal{T}_n\) is symmetric, \(\mathcal{C}_{2n}\) is also symmetric. This embedded is one of the key points in the development of our arguments. To see it, we need to mention the Cauchy Interlacing Theorem, see [12, Corollary 8.6.3].

**Theorem 3.1** (Cauchy Interlacing Theorem). Let \(A = [a_1|\cdots|a_n] \in \mathbb{R}^{m \times n}\) be a column partitioning with \(m \geq n\). If \(A_r = [a_1|\cdots|a_r]\), then for \(r = 1,\ldots,n-1\)
\[
\sigma_1(A_{r+1}) \geq \sigma_1(A_r) \geq \sigma_2(A_{r+1}) \geq \cdots \geq \sigma_r(A_{r+1}) \geq \sigma_r(A_r) \geq \sigma_{r+1}(A_{r+1}).
\]

From Cauchy Interlacing Theorem and the circulant embedding of \(\mathcal{T}_n\), we obtain the key relation between the extreme singular values of \(\mathcal{T}_n\) and \(\mathcal{C}_{2n}\). Indeed, we observe that
\[
\sigma_{\max}(\mathcal{C}_{2n}) \geq \sigma_{\max}(\mathcal{T}_n) \quad \text{and} \quad \sigma_{\min}(\mathcal{T}_n) \geq \sigma_{\min}(\mathcal{C}_{2n}). \tag{6}
\]
To see it, if \(A_1\) is the matrix obtained by deleting the last column of \(\mathcal{C}_{2n}\), we have \(\sigma_{\max}(\mathcal{C}_{2n}) \geq \sigma_{\max}(A_1)\) and \(\sigma_{\min}(A_1) \geq \sigma_{\min}(\mathcal{C}_{2n})\). Now, if \(A_2\) is the matrix obtained by deleted the last column of \(A_1\), we have \(\sigma_{\max}(\mathcal{C}_{2n}) \geq \sigma_{\max}(A_1) \geq \sigma_{\max}(A_2)\) and \(\sigma_{\min}(A_2) \geq \sigma_{\min}(A_1) \geq \sigma_{\min}(\mathcal{C}_{2n})\). Thus, to do this up to
\[
\sigma_{\max}(\mathcal{C}_{2n}) \geq \sigma_{\max}\left(\begin{bmatrix} \mathcal{T}_n \\ \mathbf{B}_n \end{bmatrix}\right) \quad \text{and} \quad \sigma_{\min}\left(\begin{bmatrix} \mathcal{T}_n \\ \mathbf{B}_n \end{bmatrix}\right) \geq \sigma_{\min}(\mathcal{C}_{2n}).
\]

Since the singular values of \(A^T\) are the same as \(A\), applying the above arguments to \([\mathcal{T}_n^T \mathbf{B}_n^T]\) we get \([\mathcal{C}_{2n}^T]\). Thus, we observe that if we want to understand the behavior of the extreme singular values of a Toeplitz matrix, we need to analyze the extreme singular values of a circulant matrix. Let \(D_n, d_n\) be non–negative real numbers for \(n \in \mathbb{N}\), by the relationships in \([6]\), we have
\[
\mathbb{P}(\sigma_{\max}(\mathcal{T}_n) \geq D_n) \leq \mathbb{P}(\sigma_{\max}(\mathcal{C}_{2n}) \geq D_n), \quad \mathbb{P}(\sigma_{\min}(\mathcal{T}_n) \leq d_n) \leq \mathbb{P}(\sigma_{\min}(\mathcal{C}_{2n}) \leq d_n).
\]

Thus, the advantage of the relationships in \([6]\) is to say us one manner to **decoupling** the strong dependence in the structure of a Toeplitz matrix. To do this we do not add more information, i.e., we use in \(\mathcal{C}_{2n}\) the same r.v. than \(\mathcal{T}_n\) (adding \(\xi_*\)). For our purpose, we consider \(\xi_*\) has the same distribution of \(\xi_o \in \Xi\) and it is independent of all r.v. in \(\Xi\).

The content of the following sections will be to establish adequate values of \(D_n, d_n\). As we were mentioned before, the results on random Toeplitz matrices are direct implications of the statements on random circulant matrices.

4. **Random circulant matrices**

We assume the entries of a circulant matrix \(\mathcal{C}_n\) are r.v. in \(\Xi\). In the following, we give a lower bound and upper bound for \(\sigma_{\min}(\mathcal{C}_n)\) and \(\sigma_{\max}(\mathcal{C}_n)\), respectively, when \(\mathcal{C}_n\) is a random non–symmetric or symmetric matrix.

Since the eigenvalues of any circulant matrix \(\mathcal{C}_n\) are \(G_n(w_n^k), k = 0,\ldots,n-1\), we have
\[
\sigma_{\max}(\mathcal{C}_n) = \max_{k=0,\ldots,n-1} |A_k| \leq \max_{z \in \mathbb{C}, |z|=1} |G_n(z)|. \tag{7}
\]

We will take advantage of the relationship between the maximum singular of a circulant matrix and the maximum modulus of a complex polynomial on the unit circle as is shown in \([7]\). Since
the coefficients of \( G_n \) are the entries of the first row of \( C_n \), we have that if \( C_n \) is a random matrix, then \( G_n \) is a random polynomial with i.i.d. coefficients in \( \Xi \). Thus, in order to estimate \( \sigma_{\text{max}}(C_n) \), we can estimate the maximum modulus of a random complex polynomial on the unit circle. This problem is interesting in itself. In fact, this problem was studied during a large time; for example see [10] [14] [28]. When it is established an upper bound of the maximum modulus of a random polynomial on the unit circle, the obtained inequality is usually called Salem–Zygmund inequality [14]. The following statements give an upper bound to the maximum modulus of a random polynomial on the unit circle when its random coefficients are i.i.d. r.v. with m.g.f. or they only have the second moment.

Since we are interested in the maximum modulus of \( G_n(z) = \sum_{j=0}^{n-1} \xi_j z^j \) on the unit circle, we can consider \( W_n(x) := G_n(e^{ix}) \) for \( x \in \mathbb{T} \), where \( \mathbb{T} \) denotes the unit circle \( \mathbb{R}/(2\pi\mathbb{Z}) \). In this way, the maximum modulus of \( G_n \) on the unit circle is denoted by \( \|W_n\|_\infty \).

**Theorem 4.1** (Salem–Zygmund inequality type I). Suppose \( \xi_0 \) has zero mean and finite positive variance. If the m.g.f. of \( \xi_0 \) exists in an open interval around zero. Then

\[
\mathbb{P}\left(\|W_n\|_\infty \geq C_0 (n \log n)^{1/2}\right) \leq \frac{C_1}{n^2}
\]

where \( C_0 \) and \( C_1 \) are positive constants that only depend on the distribution of \( \xi_0 \).

From the expression (7), we deduce that if the first row of a random (non–symmetric) circulant matrix has i.i.d. r.v. with zero mean, finite positive variance, and they have m.g.f. around zero, we have

\[
\mathbb{P}\left(\sigma_{\text{max}}(C_n) \geq C_0 (n \log n)^{1/2}\right) \leq \mathbb{P}\left(\|W_n\|_\infty \geq C_0 (n \log n)^{1/2}\right) \leq \frac{C_1}{n^2}.
\]  

(8)

**Remark 4.2.** By the Borel–Cantelli lemma, we deduce from (8) that

\[
\limsup_{n \to \infty} \frac{\sigma_{\text{max}}(C_n)}{(n \log n)^{1/2}} \leq C_0 \quad \text{almost surely},
\]

and also we have an equivalent statement for the maximum modulus of a random polynomial on the unit circle,

\[
\limsup_{n \to \infty} \frac{\|W_n\|_\infty}{(n \log n)^{1/2}} \leq C_0 \quad \text{almost surely}.
\]

The conditions in Theorem 4.1 can be relaxed up to the random coefficients are i.i.d. r.v. with only zero mean and finite second moment. For this, we use the expectation of the maximum modulus of random polynomial on the unit circle. More precisely, Weber [28] shows

\[
\mathbb{E}\left(\max_{x \in \mathbb{T}} \left| \sum_{j=0}^{n-1} \xi_j e^{ijx}\right|\right) \leq C \min\left\{ (n \log(n+1)\mathbb{E}(|\xi_0|^2))^{1/2}, n\mathbb{E}|\xi_0| \right\}
\]

\[
\leq C(n \log(n+1)\mathbb{E}(|\xi_0|^2))^{1/2},
\]

(9)

where \( C \) is a universal positive constant. Hence, using the Markov inequality, we can deduce for a random (non–symmetric) circulant matrix that

\[
\mathbb{P}\left(\sigma_{\text{max}}(C_n) \geq C_0 n^{1/2} (\log n)\right) \leq \mathbb{P}\left(\|W_n\|_\infty \geq C_0 n^{1/2} (\log n)\right) \leq \frac{C_1}{C_0 n^{1/2} (\log n)} \leq \frac{C_1}{(\log n)^{1/2}},
\]

(10)
where $C_1$ is a positive constant depending on the distribution of $\xi_0$.

For the minimum singular value of a random (non-symmetric) circulant matrix $\sigma_{\min}(C_n)$, we have the following result.

**Theorem 4.3** (Non-symmetric circulant: Minimum singular value). Suppose $\xi_0$ satisfies the condition (H). Then, for any $\varepsilon > 0$ and $\rho \in (0, 1/4)$, we have for all large $n$

$$\mathbb{P}\left(\sigma_{\min}(C_n) \leq \varepsilon n^{-\rho}\right) \leq C \left(\frac{\varepsilon^2}{n^{2\rho}} + \frac{1}{n^{1/2 - o(1)}}\right),$$

where the positive constant $C$ depends on the distribution of $\xi_0$.

In the case of a random symmetric circulant, we establish equivalent results to Theorems 4.1 and 4.3. Observed, if $C_n^{\text{sym}}$ is a random symmetric circulant matrix, half of its entries in the first row are i.i.d. First, we established the corresponding Salem–Zygmund inequality for a trigonometric random polynomial where the coefficients of the terms $z^j$ and $z^{n-j}$ are equal.

**Theorem 4.4** (Salem–Zygmund type II). Suppose $\xi_0$ has zero mean and finite positive variance. Also, suppose the m.g.f. of $\xi_0$ exists in an open interval around zero. Let $W_n^{\text{sym}}(x) := \sum_{j=0}^{n-1} \xi_j e^{ijx}$ for $x \in \mathbb{T}$ with $\xi_j = \xi_{n-j} \in \Xi$ for $j = 1, \ldots, \lfloor n/2 \rfloor + a_n$, $a_n = -1$ if $n$ is even and $a_n = 0$ if $n$ is odd. If $\|W_n^{\text{sym}}\|_{\infty} := \max_{x \in \mathbb{T}} |W_n^{\text{sym}}(x)|$, then

$$\mathbb{P}\left(\|W_n^{\text{sym}}\|_{\infty} \geq C_0 (n \log n)^{1/2}\right) \leq \frac{C_1}{n^2},$$

where $C_0, C_1$ are positive constants depending on the distribution of $\xi$.

We denote the eigenvalues of a random symmetric circulant matrix $C_n^{\text{sym}}$ by $\lambda_k^{\text{sym}}$ for $k = 0, \ldots, n-1$. From Theorem 4.4 is observed

$$\mathbb{P}\left(\max_{k=0,\ldots,n-1} \left|\lambda_k^{\text{sym}}\right| \geq C_0 (n \log n)^{1/2}\right) \leq \mathbb{P}\left(\|W_n^{\text{sym}}\|_{\infty} \geq C_0 (n \log n)^{1/2}\right) \leq \frac{C_1}{n}. \quad (11)$$

Thus, $\sigma_{\max}(C_n^{\text{sym}})$ of a random symmetric circulant matrix $C_n^{\text{sym}}$ is at most $C_0 (n \log n)^{1/2}$ with probability $1 - O\left(n^{-2}\right)$.

**Remark 4.5.** If the random entries of a symmetric circulant are Gaussian, Adhikari and Saha [13] show $\limsup_{n \to \infty} \frac{\sigma_{\max}(C_n^{\text{sym}})}{\sqrt{n \log n}} \leq C_0$ almost surely, where $C_0$ is a positive constant. Actually, they mention that this result holds for sub-Gaussian r.v. But, using our result from random polynomials and Borel-Cantelli lemma, the same result holds for random variables with m.g.f. Moreover, we have

$$\limsup_{n \to \infty} \frac{\|W_n^{\text{sym}}\|_{\infty}}{\sqrt{n \log n}} \leq C_0$$

almost surely.

The next result is about the minimum singular value of a random symmetric circulant matrix.

**Theorem 4.6** (Symmetric circulant: Minimum singular value). Suppose $\xi_0$ satisfies the condition (H). Then for any $\varepsilon > 0$ and all large $n$

$$\mathbb{P}\left(\min_{0 < k \leq \lfloor n/2 \rfloor} \left|\lambda_k^{\text{sym}}\right| \leq \varepsilon n^{-0.51}\right) \leq C \left(\frac{\varepsilon}{n^{0.1}} + \frac{1}{n^{1/7/300-o(1)}}\right),$$

with $C$ a positive constant depending on the distribution of $\xi_0$.

**Remark 4.7.** About condition number of random (non-symmetric or symmetric) circulant matrix, we can establish similar results to Theorem 2.7 and Theorem 2.11. We omit them, since only is necessary to change Toeplitz by Circulant in Theorem 2.7 and Theorem 2.11. The proofs are similar to the Toeplitz case (see Section 5).
5. Proof of the Salem–Zygmund inequalities

The strategy to prove Theorem 4.1 is essential given in [2], see the proof of Theorem 1.2 therein. Here, we give the outline of the proof, but since the proof of Theorem 4.1 follows similar ideas, we include details of our arguments in Appendix A.

The existence of m.g.f. of $\xi_0$ around zero permits to obtain for any $x \in \mathbb{T}$
\[
E \left[ e^{t W_n(x)} \right] \leq e^{-\alpha^2 t^2 n/2},
\]
for some fixed positive constant $\alpha^2$ which depends on the distribution of $\xi_0$. It is possible to show that there exists an interval $I \subset \mathbb{T}$ such that $|W_n(x)| \geq \frac{1}{2} \| W_n \|_{\infty}$ for $x \in I$ and the length of $I$ is $\frac{\pi \alpha}{3n}$. Then
\[
E \left[ \exp \left( \frac{1}{2} t \| W_n \|_{\infty} \right) \right] \leq \frac{8n}{3} E \left[ \int_I \left( e^{t W_n(x)} + e^{-t W_n(x)} \right) \mu(dx) \right] \leq \frac{16n}{3} \exp \left( 3\alpha^2 t^2 n/2 \right).
\]

Finally, we use Chernoff bounding technique to obtain an upper bound for $\| W_n \|_{\infty}$ with high probability. Let $b_n$ be a positive real number for $n \in \mathbb{N}$, then
\[
P \left( \| W_n \|_{\infty} \geq b_n \right) = P \left( e^{t \| W_n \|_{\infty}} \geq e^{tb_n} \right) \leq e^{-tb_n} E \left[ e^{t \| W_n \|_{\infty}} \right].
\]
In Appendix A is shown how to select the adequate $t$ and $b_n$ such that
\[
P \left( \| W_n \|_{\infty} \geq C_0 (n \log n)^{1/2} \right) \leq \frac{C_1}{n^2}.
\]

6. Proof of Theorem 4.3

Before starting, we need to introduce little notations. The floor of a real number $x$, denoted by $\lfloor x \rfloor$, is the greatest integer $n$ such that $n \leq x$. Remember that $\| \cdot \|_2$ is the Euclidean norm in $\mathbb{R}^n$. The determinant of a square matrix is denoted by det$(\cdot)$. Let $f_n, g_n$ be two real sequences, we write $f_n = o(g_n)$ if for every $\alpha > 0$ there exists $n_0$ such that for all $n \geq n_0$ we have $|f_n| \leq \alpha |g_n|$. The target is to find a nice upper bound of the Lévy concentration of $n^\rho \sigma_{\min}(C_n)$. Rememer that the eigenvalues of a circulant matrix are $\lambda_k = G_n(e^{i2\pi k/n})$, $k = 0, \ldots, n-1$, where $G_n(z) = \sum_{j=0}^{n-1} \xi_j z^j$ and $\xi_0, \ldots, \xi_{n-1} \in \Xi$ i.i.d. r.v. If $x_k := k/n$, we have
\[
P \left( \sigma_{\min}(C_n) \leq \varepsilon n^{-\rho} \right) = P \left( n^\rho \sigma_{\min}(C_n) \leq \varepsilon \right) \leq \sum_{k=0}^{n-1} P \left( |n^\rho G_n(e^{i2\pi x_k})| \leq \varepsilon \right)
\leq L(n^\rho G_n(1), \varepsilon) + L(n^\rho G_n(-1), \varepsilon) + \sum_{k=0}^{n-1} L \left( n^\rho G_n(e^{i2\pi x_k}), \varepsilon \right). \tag{12}
\]
Note in the expression (12), $n^\rho G_n(e^{i2\pi x_k})$ is a sum of r.v. with (deterministic) real or complex coefficients for all $k = 0, \ldots, n-1$. To estimate the Lévy concentration for each of these sums, we use the least common denominator (LCD), which is defined as follows.

Definition 6.1. Let $L$ be any fixed positive number. The least common denominator (LCD) of a matrix $V \in \mathbb{R}^{m \times n}$ is defined as
\[
D(V) := \inf \left\{ \| \theta \|_2 > 0 : \theta \in \mathbb{R}^m, \text{dist} \left( V^T \theta, \mathbb{Z}^n \right) < L \sqrt{\log \left( \frac{\| V^T \theta \|_2}{L} \right)} \right\},
\]
where dist$(v, \mathbb{Z}^n)$ denotes the distance from the vector $v \in \mathbb{R}^n$ to the set $\mathbb{Z}^n$ and $\log_+(x) = \max\{\log(x), 0\}$. 
The notion of LCD used here was introduced by Rudelson and Vershynin \cite{15} in the study of the eigenvectors of random matrices with independent random entries. For a given matrix $V$, we denote by $\|V\|_\infty$ the maximum Euclidean norm of the columns of $V$. Rudelson and Vershynin show that LCD is no trivial (see Proposition 7.4 in \cite{15}).

**Proposition 6.2** (Simple lower bound for LCD). For every matrix $V$ and $L > 0$, one has

$$D(V, L) \geq \frac{1}{2\|V\|_\infty}.$$

Moreover, Rudelson and Vershynin show how to relate the Lévy concentration function with LCD (see Theorem 7.5 in \cite{15}).

**Theorem 6.3** (Small Ball Probability Inequality). If $V$ is an $m \times n$ matrix and $X \in \mathbb{R}^n$ is a random vector with i.i.d. entries such that they satisfy the condition (II). Then for every $L \geq \sqrt{m/q}$ we have

$$\mathcal{L}(V^T X, \varepsilon \sqrt{m}) \leq \frac{(CL/\sqrt{m})^m}{(\det (V V^T))^{1/2}} \left( \varepsilon + \sqrt{m/D(V)} \right)^m,$$

where $C$ depends on the distribution of the entries of $X$, and $D(V)$ is the LCD of $V$.

A special case of Theorem 6.3 is when $m = 1$. In this case, $V^T X$ represents a sum of r.v.\]

**Corollary 6.4** (Small ball probabilities for sums). Let $\xi_k$ be i.i.d. copies of $\xi$ satisfying condition (II). Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then for every $L \geq (1/q)^{1/2}$ we have

$$\mathcal{L}\left(\sum_{k=1}^{n} a_k \xi_k, \varepsilon\right) \leq \frac{C L}{\|a\|_2} \left( \varepsilon + \frac{1}{D(a, L)} \right), \quad \varepsilon \geq 0,$$

where $C$ depends on the distribution of $\xi_k$, and $D(a, L)$ is the LCD of $a$.

Now, we can proceed to give an upper bound of $\mathcal{L}(n^p G_n(e^{i2\pi x_k}), \varepsilon)$. Actually, this is shown by Barrera and Manrique in \cite{1}, see Theorem 1.6. But for the sake of clarity, we decide to include here the important parts of the used strategy.

To apply Theorem 6.3, we distinguish two cases given by the expression (12), when $n^p G_n(e^{i2\pi x_k})$ has real or complex coefficients.

**Lemma 6.5** (Real coefficients). Under the hypothesis of Theorem 6.3, we have for any $\varepsilon \geq 0$

$$\mathcal{L}(n^p G_n(1), \varepsilon) + \mathcal{L}(n^p G_n(-1), \varepsilon) \leq C_1 \left( \frac{\varepsilon}{n^{p+1/2}} + \frac{1}{n^{1/2}} \right),$$

where the positive constant $C_1$ depends on the distribution of $\xi_0$.

**Proof.** Note $n^p G_n(1) = n^p \sum_{j=0}^{n-1} \xi_j$. Write $a = (1, \ldots, 1)^T \in \mathbb{R}^n$. By Proposition 6.2 we have the LCD of $a$ is such that $D(a) \geq \frac{1}{2} n^{-p}$. Let $L \geq (1/q)^{1/2}$. By Corollary 6.4 we have

$$\mathcal{L}(n^p G_n(1), \varepsilon) \leq \frac{CL}{n^{p+1/2}} \left( \varepsilon + 2n^p \right) \leq C_1 \left( \frac{\varepsilon}{n^{p+1/2}} + \frac{1}{n^{1/2}} \right),$$

where $C_1$ is a positive constant depending on the distribution of $\xi_0$.

For $n^p G_n(-1) = n^p \sum_{j=0}^{n-1} (-1)^j \xi_j$, the proof is similar, but taking $a = (1, -1, \ldots, (-1)^n) \in \mathbb{R}^n$. \hfill \Box
For our second case with complex coefficients, we define the $2 \times n$ matrix $V_k$, $k = 0, \ldots, n-1$, as
\begin{equation}
V_k := \begin{bmatrix} 1 & \cos (2\pi x_k) & \ldots & \cos ((n-1)2\pi x_k) \\ 0 & \sin (2\pi x_k) & \ldots & \sin ((n-1)2\pi x_k) \end{bmatrix},
\end{equation}
where $x_k = \frac{k}{n}$. Let $X := [\xi_0, \ldots, \xi_{n-1}]^T \in \mathbb{R}^n$, then
\begin{equation}
V_kX = \begin{bmatrix} \sum_{j=0}^{n-1} \xi_j \cos (j2\pi x_k) \\ \sum_{j=0}^{n-1} \xi_j \sin (j2\pi x_k) \end{bmatrix}^T \in \mathbb{R}^2
\end{equation}
which implies
\begin{equation}
\|V_kX\|_2 = \left| \sum_{j=0}^{n-1} \xi_j e^{ij2\pi x_k} \right| = \left| G_n(e^{i2\pi x_k}) \right|.
\end{equation}
On the other hand, we have for all $k$
\begin{equation}
\det (V_kV_k^T) = \det \begin{bmatrix} \sum_{j=0}^{n-1} \cos^2 (j2\pi x_k) & \frac{1}{2} \sum_{j=0}^{n-1} \sin (2j2\pi x_k) \\ \frac{1}{2} \sum_{j=0}^{n-1} \sin (2j2\pi x_k) & \sum_{j=0}^{n-1} \sin^2 (j2\pi x_k) \end{bmatrix} = \frac{n^2}{4}.
\end{equation}

Before continuing, we need to introduce two auxiliary lemmas, which can be found in [2], but for the sake of clarity, we include their proves in Appendix B. The first lemma gives an upper bound for the number of integers whose greatest common denominator with $n$ is at most $1$ where the positive constant is a universal positive constant.

**Lemma 6.6.** Let $y, M \in [1, \infty)$ be fixed numbers. The cardinality of the set
\[ \{k \in [1, M] \cap \mathbb{N} : \gcd(k, M) \geq y \} \]
is at most $\frac{1}{[y]}M^{1+C(\log \log M)^{-1}}$, where $C$ is a universal positive constant.

**Lemma 6.7.** Fix $\theta \in [0, 2\pi)$ and positive $m \in \mathbb{Z}$. Let $V$ be a vector in $\mathbb{R}^m$ which entries are $V_j = r \cos (j2\pi x - \theta)$ for $j = 0, \ldots, m-1$ with positive integer $r \geq 2$ and $x = 1/m$. Then whenever $\frac{1}{2r(2\pi x)} \geq 6$.

**Lemma 6.8 (Complex coefficients).** Under the hypothesis of Theorem 4.3, we have for all large $n$
\begin{equation}
\sum_{k=0 \atop k\neq 0,n/2}^{n-1} \mathcal{L}(n^\rho G(e^{i2\pi x_k}), \varepsilon) \leq C \left( \frac{\varepsilon^2}{n^{2\rho}} \frac{1}{n^{1/2-\alpha(1)}} \right),
\end{equation}
where the positive constant $C$ depends on the distribution of $\xi_0$.

**Proof.** Remember that $x_k = \frac{k}{n}$. We need to distinguish two cases for $\gcd(k, n)$. First, we assume $\gcd(k, n) \geq n^{1/2}$, then by Lemma 6.6 the number of integers $k$ that satisfies this condition is at most $n^{1/2+o(1)}$. Note, if $V_k$ is the matrix defined by (13), then by Proposition 6.2, the LCD of $V_k$ satisfies $D(n^\rho V_k) \geq \frac{1}{2} n^{-\rho}$. Thus, using the expressions (14) and (15), by Theorem 6.3 we have
\begin{equation}
\sum_{k=0 \atop k\neq 0,n/2}^{N-1} \mathcal{L}(n^\rho G(e^{i2\pi x_k}), \varepsilon) \leq n^{1/2+o(1)} \frac{C^2 L^2}{2n^{1+2\rho}} (\varepsilon + 4n^{\rho})^2 \leq C_1 \left( \frac{\varepsilon^2}{n^{1/2+2\rho-\alpha(1)}} + \frac{1}{n^{1/2-\alpha(1)}} \right),
\end{equation}
where in the last inequality we use the fact \((a + b)^2 \leq 2a^2 + 2b^2\) and \(C_1\) is a positive constant depending on the distribution of \(\xi_0\).

Now, we assume \(\gcd(n, k) \leq n^{1/2}\). Let \(V_k\) be the matrix defined by \((13)\) and \(x = k/n\). Let \(\Theta = r[\cos(\theta), \sin(\theta)]^T \in \mathbb{R}^2\), where \(r > 0\) and \(\theta \in [0, 2\pi]\). For fixed \(r, \theta\), we have

\[
V_k^T \Theta = r[\cos(-\theta), \cos(2\pi x - \theta), \ldots, \cos(2(n-1)\pi x - \theta)]^T.
\]

(17)

Note \(\|\Theta\|_2 = r\) and \(\|V_k^T \Theta\|_2 \leq r\sqrt{n}\). Now, we need to estimate the LCD of \(V_k\). Since \(\gcd(n, k) \leq n^{1/2}\), we can apply Lemma 6.7 to \(n' = \frac{n}{\gcd(n, k)} \geq n^{1/2}\) and \(k' = \frac{k}{\gcd(n, k)}\). Taking into account the expression \((17)\), from the definition of LCD for \(n'\ V_k\) with \(n' r \leq \frac{1}{12\pi}\), by Lemma 6.7 we have

\[
\frac{1}{48} \cdot \frac{n^{1/2} - 1}{2\pi} \leq \text{dist}(n' V_k^T \Theta, \mathbb{Z}^n) < L \sqrt{\log_+ \frac{n'}{L}} \leq L \sqrt{\log_+ \frac{n}{L}},
\]

which is a contradiction for all large \(n\) since \(L\) is fixed. Thus, the LCD of \(n' V_k^T\) satisfies

\[
D(n' V_k^T) \geq r \geq \frac{n^{1/2} - 1}{48\pi}
\]

for all large \(n\). Notice that here we assume that \(r\) is a positive integer. Actually, by Proposition 6.2 we can assume that \(r \geq 1/2\). To handle \(2 > r \geq 1/2\), we observe that we can replicate the ideas in the proof of Lemma 6.7 and show \(\text{dist}(n' V_k^T \Theta, \mathbb{Z}^n) \geq C (n^{1/2} - \rho)\) for some positive constant \(C\). If \(r \geq 1\), we use \(\lceil r \rceil\) instead of \(r\) to apply Lemma 6.7. Then, by Theorem 6.3 (recall \(V_k \in \mathbb{R}^{2 \times n}\)) and expression \((15)\), we have

\[
\sum_{\substack{k = 0 \atop \alpha : k \neq 0, n/2, \gcd(k, n) \leq n^{1/2}}}^{N-1} \mathcal{L}(n' G(e^{i2\pi x_k}), \varepsilon) \leq n \cdot \frac{C^2 L^2}{2n^{1+2\rho}} \left(\varepsilon + \frac{48\pi}{n^{1/2}}\right)^2 \bigg) \leq C_2 \left(\frac{\varepsilon}{n^{2\rho}} + \frac{1}{n}\right),
\]

(18)

where the positive constant \(C_2\) depends on the distribution of \(\xi_0\). Thus, from \((10)\) and \((18)\) we have for all large \(n\)

\[
\sum_{\substack{k = 0 \atop k \neq 0, n/2}}^{n-1} \mathcal{L}(n' G(e^{i2\pi x_k}), \varepsilon) \leq C_1 \left(\frac{\varepsilon}{n^{1/2+2\rho}} + \frac{1}{n^{1/2}}\right) + C_2 \left(\frac{\varepsilon^2}{n^{2\rho}} + \frac{1}{n}\right)
\]

\[
\leq C_3 \left(\frac{\varepsilon^2}{n^{2\rho}} + \frac{1}{n^{1/2}}\right),
\]

where the positive constant \(C_3\) depends on the distribution of \(\xi_0\). \(\square\)

**Proof Theorem 4.3** By expression \((12)\) and lemmas 6.5 and 6.8 for any \(\varepsilon \geq 0\) and \(\rho \in (0, 1/4)\) we have for all large \(n\)

\[
\mathbb{P}(\sigma_{\text{min}}(C_n) \leq \varepsilon n^\rho) \leq C_1 \left(\frac{\varepsilon}{n^{\rho+1/2}} + \frac{1}{n^{1/2}}\right) + C \left(\frac{\varepsilon^2}{n^{2\rho}} + \frac{1}{n^{1/2}}\right)
\]

\[
\leq C_2 \left(\frac{\varepsilon^2 + \varepsilon}{n^{2\rho}} + \frac{1}{n^{1/2}}\right),
\]

where the positive constant \(C_2\) depends on the distribution of \(\xi_0\).
7. Proof of Theorem 4.6

In this section, we give the proof of Theorem 4.6. Again, we use LCD to give a nice upper bound of the probability of the event \( \{ \min_k |\lambda_k^{\text{sym}}| \leq \varepsilon n^{-0.51} \} \). To do this, we need to observe a useful property of the Lévy concentration of the sum of independent r.v.

**Proposition 7.1.** Let \( \varepsilon \geq 0 \). If \( X, Y \in \mathbb{R} \) are independent random variables then

\[
\mathcal{L}(X + Y, \varepsilon) \leq \min(\mathcal{L}(X, \varepsilon), \mathcal{L}(Y, \varepsilon)).
\]

**Proof.** It is immediate from Definition 2.4. \( \square \)

From Proposition 7.1 and expressions (3) and (4), we can observe for any large \( n \) that

\[
\mathbb{P} \left( \min_{0 \leq k \leq \lfloor n/2 \rfloor} |\lambda_k| \leq \varepsilon n^{-0.51} \right) \leq \mathbb{P} (|\lambda_0| \leq \varepsilon n^{-0.51}) + \sum_{k=1}^{\lfloor n/2 \rfloor} \mathbb{P} (|\lambda_k| \leq \varepsilon n^{-0.51}) \\
\leq \mathcal{L}(n^{0.51} S_0, \varepsilon) + \sum_{k=1}^{\lfloor n/2 \rfloor} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon),
\]

where

\[
S_0 := \sum_{j=1}^{\lfloor n/2 \rfloor - 1} \xi_j, \quad S_{n,k} := \sum_{j=1}^{\lfloor n/2 \rfloor - 1} \xi_j \cos \left( \frac{2\pi k j}{n} \right).
\]

Let \( v \in \mathbb{R}^{\lfloor n/2 \rfloor - 1} \) with entries \( v_j := \cos \left( \frac{2\pi k j}{n} \right) \) for \( j = 1, \ldots, \lfloor n/2 \rfloor - 1 \). From Corollary 6.4, we observe if the LCD of \( v \) is sufficiently large then \( \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) \) will be small. Hence, our problem is reduced to analyze the arithmetic structure of \( v \). For this, we establish the next lemma.

**Lemma 7.2.** Let \( n, k \) be a fixed positive integers with \( \gcd(n, k) = 1 \) and \( n > k \). Let \( v \) be a vector in \( \mathbb{R}^{\lfloor n/2 \rfloor - 1} \) whose entries are \( v_j = \cos(2\pi k j x) \) for \( j = 1, \ldots, \lfloor n/2 \rfloor - 1 \) with \( x := 1/n \). Then for all large \( n \)

\[
\text{dist} \left( \mathbb{Z}^{\lfloor n/2 \rfloor}, vr \right) \geq \frac{1}{1728\pi x},
\]

whenever \( \frac{1}{30 \cdot 2\pi x} \geq r \geq 1 \).

**Proof.** Here \( i \) is the imaginary unit. Fix \( k \) and we assume \( \gcd(k, n) = 1 \). Note \( \cos(2\pi k j x) \) is the real part of \( \exp(i2\pi k j x) \) for all \( j \). The set of points of the form \( \exp(i2\pi k j x) \) for \( j = 0, \ldots, n - 1 \) can be seen as the vertices of regular polygon \( P \) inscribed in the unit circle. The vector \( v \) considers at most half of the vertices of this regular polygon of \( n \) sides. By the pigeonhole principle, we have that there exists a quadrant \( Q \) of the plane where there are at least \( \lfloor n/2 \rfloor/4 \) vertices of \( P \) which are entries of \( v \). Note \( \lfloor n/2 \rfloor/4 \geq n/9 \) for all \( n \geq 18 \). In the following, we fix the quadrant \( Q \) obtained by the pigeonhole principle. Note that the difference between the arguments of adjacent vertices of \( P \) in \( Q \) is at most \( 3 \cdot 2\pi x \) for all \( n \geq 18 \).

Let \( J := [-1, 1] \cap Q \). Note that \( J \) is a close interval, which can be \([-1, 0]\) or \([0, 1]\). Let \( [y, y + 9 \cdot 2\pi x] \) be a closed interval in \( J \). Let \( \widehat{A} \) the arc on the unit circle in the quadrant \( Q \) such that its projection in the horizontal axis is \([y, y + 9 \cdot 2\pi x] \). If the length of \( \widehat{A} \) is \( l \), then the number of values \( \cos(2\pi j k x) \) which are in \((y, y + 9 \cdot 2\pi x)\) are at least

\[
\frac{l}{3 \cdot 2\pi x} - 2 \geq \frac{9 \cdot 2\pi x}{3 \cdot 2\pi x} - 2 = 1,
\]

since \( l \geq 9 \cdot 2\pi x \).
Let $I := \{j \in \{1, \ldots, \lfloor n/2 \rfloor - 1 \} : \cos (2\pi jkx) \in J\}$. Note $|I| \geq n/9$ for $n \geq 18$. Fix a positive integer $r \leq \frac{n}{9}$. Let $K^r$ be the set of integer $s$ with $|s| < r$ and $[\frac{s}{r}, \frac{s+1}{r}] \subset J$. Note $|K^r| \geq |r|$. We take an $s \in K^r$. For each $j \in I$ there exists at least one value
\[
\cos (2\pi jkx) \in \left(\frac{s}{r} + 9(\alpha - 1)(2\pi x), \frac{s}{r} + 9\alpha(2\pi x)\right) \subset \left[\frac{s}{r}, \frac{s+1}{r}\right],
\]
for all positive integer $\alpha \leq \frac{1}{(9 \cdot 2\pi x)}$.

Let $I^r_s \subset I$ such that $\cos (2\pi jkx) \in \left[\frac{s}{r}, \frac{s+1}{r}\right]$ for all $j \in I^r_s$. We define
\[
d_j := \min \left\{\left|\cos (2\pi jkx) - \frac{s}{r}\right|, \left|\cos (2\pi jkx) - \frac{s+1}{r}\right|\right\}.
\]
Let $L$ be the biggest integer that satisfies $(9 \cdot 2\pi x) L \leq \frac{1}{2r}$, i.e., $L = \left\lfloor \frac{1}{2r \cdot 9 \cdot 2\pi x} \right\rfloor$. Observe
\[
L \geq \frac{1}{2r \cdot 9 \cdot 2\pi x} - 1 \geq \frac{1}{2} \left(\frac{1}{2r \cdot 9 \cdot 2\pi x}\right) \quad \text{whenever} \quad 1 \geq 36r \cdot 2\pi x.
\]
Then
\[
\sigma_s^r := \sum_{j \in I^r_s} d_j \geq \sum_{\lambda = 1}^{L} 2\lambda (3 \cdot 2\pi x) = 6 \cdot 2\pi x \sum_{\lambda = 1}^{L} \lambda = 6 \cdot 2\pi x \frac{L(L+1)}{2} \geq 3 \cdot 2\pi x L^2 \geq 3 \cdot 2\pi x \left(\frac{1}{2} \cdot \frac{1}{2r \cdot 9 \cdot 2\pi x}\right)^2 = \frac{1}{4} \cdot \frac{1}{4r^2 \cdot 27 \cdot 2\pi x}
\]
Now, we take the sum of all $\sigma_s^r$ with $s \in K^r$, \[
\sum_{s \in K^r} \sigma_s^r \geq |r| \cdot \frac{1}{432} \cdot \frac{1}{r^2 \cdot 2\pi x}.
\]
By the previous analysis, we have that the distance from $v$ to $\mathbb{Z}^{\lfloor n/2 \rfloor}$ is at least
\[
r \left(\frac{1}{432} \cdot \frac{|r|}{r^2} \cdot \frac{1}{2\pi x}\right) = \frac{1}{432} \cdot \frac{|r|}{r} \cdot \frac{1}{2\pi x} \geq \frac{1}{864} \cdot \frac{1}{2\pi x},
\]
whenever $\frac{1}{36 \cdot 2\pi x} \geq r \geq 1$.

\begin{remark}
Note the condition $\gcd(n, k) = 1$ in the above Lemma \ref{lemma:7.2} can be broken. If $\gcd(n, k) = m$, we use lemma with $n' = n/m$ and $k' = k/m$.
\end{remark}

Now, as $v \in \mathbb{R}^{\lfloor n/2 \rfloor - 1}$ has entries $v_j = \cos \left(\frac{2\pi k}{n} \cdot j\right)$ for all $j$, we have $\sqrt{n/8} \leq ||v|| \leq \sqrt{n/2}$. Using Lemma \ref{lemma:7.2} we can estimate the LCD of $v$. Assume $\theta \leq \frac{1}{2\pi x}$ with $x = \frac{\gcd(n, k)}{n}$. If $\gcd(n, k) \leq n^{1/3}$, by Lemma \ref{lemma:7.2} Remark \ref{remark:7.3} and the definition of LCD for $v$ we get
\[
\frac{1}{1728\pi} n^{2/3} \leq \frac{1}{1728\pi x} \leq \text{dist} \left(\theta v, \mathbb{Z}^{\lfloor n/2 \rfloor}\right) \leq L \sqrt{\log_+ \frac{||v||}{L}} \leq L \sqrt{\log_+ \left(\frac{1}{L} n^{3/2}\right)},
\]
which is a contradiction for all large $n$ since $L$ is fixed. We can conclude that LCD of $v$ is
\[
D(v) \geq \frac{1}{72\pi} n^{2/3}.
\]
Thus, by definition of LCD and expression (20), we have
\[
D(n^{0.51}v) \geq n^{-0.51} D(v) \geq \frac{1}{72\pi} n^{2/3 - 0.51}.
\]
Notice that we have assumed $\theta \geq 1$. By Proposition \ref{proposition:5.2} we can consider that $\theta \geq 1/2$. To handle $1 > \theta \geq 1/2$, we observe that we can replicate the ideas in the proof of Lemma \ref{lemma:7.2} and show that
Thus, from Proposition 6.2, we have $\dist(\theta v, Z^{[n/2]}) \geq C(n^{2/3})$ for some positive constant $C$.

Using Corollary 6.4 and expression (21), we give an upper bound for the second sum of (19). Thus,

$$
\sum_{k=1}^{[n/2]} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) = \sum_{\gcd(n,k) \leq n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) + \sum_{\gcd(n,k) > n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) \\
\leq \frac{n}{2} \left[ \frac{C L}{n^{1.1}} \left( \varepsilon + \frac{72 \pi}{n^{2/3-0.51}} \right) \right] + \sum_{\gcd(n,k) > n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) \\
\leq C_1 \left( \frac{\varepsilon}{n^{0.1}} + \frac{1}{n^{2/3-0.4}} \right) + \sum_{\gcd(n,k) > n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon). \tag{22}
$$

By Lemma 6.6, the second term of the sum (22) has at most $2n^{2/3+O(1)}$ terms. If $n^{1/3} \leq \gcd(k, n) \leq n^{2/3}$, we have

$$
D(n^{0.51} v) \geq n^{-0.51} D(v) \geq \frac{1}{72 \pi} n^{1/3-0.51}.
$$

From Proposition 6.2, we have $D(v) \geq 1/2$. By Lemma 6.6, the number of positive integer $k$ such that $\gcd(k, n) > n^{2/3}$ is at most $2n^{1/3+O(1)}$. Then

$$
\sum_{\gcd(n,k) > n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) = \sum_{n^{2/3} \geq \gcd(n,k) > n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) + \sum_{\gcd(n,k) > n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) \\
\leq 2n^{2/3+O(1)} \left[ \frac{C L}{n^{1.1}} \left( \varepsilon + \frac{72 \pi}{n^{1/3-0.51}} \right) \right] + \sum_{\gcd(n,k) > n^{1/3}} \mathcal{L}(n^{0.51} S_{n,k}, \varepsilon) \\
\leq C_2 \left( \frac{\varepsilon}{n^{13/30-O(1)}} + \frac{1}{n^{77/300-O(1)}} \right) + 2n^{1/3+O(1)} \left[ \frac{C L}{n^{1.1}} \left( \varepsilon + 2n^{0.51} \right) \right] \\
\leq C_2 \left( \frac{\varepsilon}{n^{13/30-O(1)}} + \frac{1}{n^{77/300-O(1)}} \right) + C_3 \left( \frac{\varepsilon}{n^{23/30-O(1)}} + \frac{1}{n^{77/300-O(1)}} \right) \\
\leq C_4 \left( \frac{\varepsilon}{n^{13/30-O(1)}} + \frac{1}{n^{77/300-O(1)}} \right). \tag{23}
$$

Finally, taking $w = (1, \ldots, 1) \in \mathbb{R}^{[n/2]-1}$. Note $\|w\| = n^{0.5}$ and $D(w) \geq 1/2$ (Proposition 6.2). By Corollary 6.4, we have

$$
\mathcal{L}(n^{0.51} S_0, \varepsilon) \leq C_5 \left( \frac{\varepsilon}{n^{0.1}} + \frac{1}{n^{0.5}} \right). \tag{24}
$$

Joining the estimation (22)–(24), we get for all large $n$

$$
\mathbb{P} \left( \min_{0 \leq k \leq [n/2]} |\lambda_k^{sym}| \leq \varepsilon n^{-0.51} \right) \leq C_6 \left( \frac{\varepsilon}{n^{0.1}} + \frac{1}{n^{77/300-O(1)}} \right).
$$

\[ \square \]

8. Condition number of a random Toeplitz matrix

In this section, we give the proof of the first part of Theorem 2.7. The proofs of the second part of Theorem 2.7 and Theorem 2.11 are similar and they are omitted.
Proof of Theorem 2.7. By Theorem 2.1 and Theorem 2.5 we have for any \( \varepsilon > 0 \), \( \rho \in (0, 1/4) \), and for all large \( n \), we have
\[
P\left( \kappa (T_n) \leq \frac{C_0}{\varepsilon n^{\rho + 1/2} (\log n)^{1/2}} \right) \geq P \left( \sigma_{\max}(T_n) \leq C_0 (n \log n)^{1/2}, \sigma_{\min}^{-1}(T_n) \leq \frac{1}{\varepsilon n^\rho} \right)
\]
\[
\geq 1 - P \left( \sigma_{\max}(T_n) \geq C_0 (n \log n)^{1/2} \right) - P \left( \varepsilon n^{-\rho} \geq \sigma_{\min}(T_n) \right)
\]
\[
\geq 1 - \frac{C_1}{(2n)^2} - C_2 \left( \frac{\varepsilon^2 + \varepsilon}{(2n)^{2\rho}} + \frac{1}{n^{1/2-o(1)}} \right)
\]
\[
\geq 1 - C_2 \left( \frac{\varepsilon^2 + \varepsilon}{n^{2\rho}} + \frac{1}{n^{1/2-o(1)}} \right),
\]
for a positive constant \( C_2 \) depending on the distribution of \( \xi_0 \).

\[\square\]

**APPENDIX A. SALEM–ZYGMUND INEQUALITIES**

Here, we include the details of the proofs of Theorem 4.1 and Theorem 4.4.

We introduce two important auxiliary lemmas. The proof of these lemmas can be found in [2], see Lemma 1.1 and Claim 1 there in. The first lemma is related to the random variable with m.g.f. that sometimes is called *locally sub–Gaussian* r.v. The second lemma establishes the existence of an interval where the function of \( W_n(x) \) reaches at least half of its maximum modulus. Since the second lemma has important aspects of the proof of Theorem 4.4, we decide to include it here.

**Lemma A.1** (Locally sub–Gaussian r.v.). Let \( \xi \) be a random variable such that its m.g.f. \( M_\xi \) exists in an interval around zero. Assume that \( \mathbb{E} [\xi] = 0 \) and \( \mathbb{E} [\xi^2] = s^2 > 0 \). Then there is a \( \delta > 0 \)
\[
M_\xi(t) \leq e^{\alpha^2 t^2/2} \quad \text{for any } t \in (-\delta, \delta) \text{ and } \alpha^2 > s^2.
\]

**Lemma A.2.** There exists a random interval \( I \subset \mathbb{T} \) (Lebesgue measure) of length \( \frac{8}{3n} \) such that
\[
|W_n(x)| \geq \frac{1}{2} \|W_n\|_\infty \quad \text{for any } x \in I.
\]

**Proof.** In fact, let \( p_n(x) := \sum_{j=0}^{n-1} b_j e^{ijx} \), \( x \in \mathbb{T} \) be a trigonometric polynomial on \( \mathbb{T} \), where \( b_j \), \( j = 0, \ldots, n-1 \) are real numbers. For \( x \in \mathbb{T} \), write
\[
g_n(x) := |p_n(x)|^2 = \left( \sum_{j=0}^{n-1} b_j \cos(jx) \right)^2 + \left( \sum_{j=0}^{n-1} b_j \sin(jx) \right)^2
\]
and
\[
h_n(x) := \left( \sum_{j=0}^{n-1} j b_j \cos(jx) \right)^2 + \left( \sum_{j=0}^{n-1} j b_j \sin(jx) \right)^2.
\]
Then
\[
\|p_n\|_\infty^2 = \sup_{x \in \mathbb{T}} g_n(x) = \|g_n\|_\infty^2 \quad \text{and} \quad \|p_n'\|_\infty^2 = \sup_{x \in \mathbb{T}} h_n(x).
\]

Recall the Bernstein inequality \( \|p_n'\|_\infty \leq n \|p_n\|_\infty \) (see for instance Theorem 14.1.1, Chapter 14, page 508 in [20]). For any \( x \in \mathbb{T} \) we have
\[
|g_n'(x)| \leq 4 \|p_n\|_\infty \|p_n'\|_\infty \leq 4n \|p_n\|_\infty^2 = 4n \|g_n\|_\infty^2.
\]
(25)

Since \( g \) is continuous then there exists \( x_0 \in \mathbb{T} \) such that \( g(x_0) = \|g_n\|_\infty \). Moreover, from the Mean Value Theorem and relation (25) we get
\[
|g(x) - g(x_0)| \leq \|g_n'\|_\infty \|x - x_0\| \leq 4n \|g_n\|_\infty |x - x_0|,
\]
for any \( x \in \mathbb{T} \). Take \( I := [x_0 - \frac{3}{16n}, x_0 + \frac{3}{16n}] \subset \mathbb{T} \). Notice that the length of \( I \) is \( \frac{3}{8n} \). Moreover,
\[
|g(x) - g(x_0)| \leq \frac{3}{4} \|g_n\|_\infty, \quad \text{for any } x \in I.
\]
Since \( g(x_0) = \|g_n\|_\infty \) then from the triangle inequality we deduce \( \frac{1}{4} \|g_n\|_\infty \leq |g_n(x)| \), for any \( x \in I \).

Therefore,
\[
\frac{1}{2} \|p_n\|_\infty \leq |p_n(x)|, \quad \text{for any } x \in I.
\]

\[\Box \]

**Proof of Theorem 4.1.** By Lemma A.1 there exists a \( \delta > 0 \) such that
\[
M_\xi(t) \leq e^{\alpha^2 t^2 / 2} \quad \text{for any } t \in (-\delta, \delta), \quad \text{where } \alpha^2 > \mathbb{E} [\xi_0^2] > 0.
\]

At first, we assume \( W_n(x) = \sum_{j=0}^{n-1} \xi_j e^{ijx} \) is real. Later, we take the imaginary part, but the analysis will be the same. Note
\[
e^{\alpha^2 t^2 / 2} = \prod_{j=0}^{n-1} e^{\alpha^2 t^2 / 2} \geq \prod_{j=0}^{n-1} \mathbb{E} [e^{i\xi_j \cos(jx)}] = \mathbb{E} \left[ \prod_{j=0}^{n-1} e^{i\xi_j \cos(jx)} \right] = \mathbb{E} [e^{iW_n(x)}]
\]
for every \( t \in (-\delta, \delta) \).

From Lemma A.2 there exists a random interval \( I \subset \mathbb{T} \) of length \( \frac{3}{8n} \), such that \( W_n(x) \geq \|W_n\|_\infty \) or \( W_n(x) \geq \|W_n\|_\infty \) on \( I \). Denote by \( \mu \) the normalized Lebesgue measure on \( \mathbb{T} \). Note
\[
\exp \left( \frac{1}{2} \|W_n\|_\infty \right) = \frac{1}{\mu(I)} \int_I \exp \left( \frac{1}{2} t \|W_n\|_\infty \right) dx \leq \frac{1}{\mu(I)} \int_\mathbb{T} \left( e^{tW_n(x)} + e^{-tW_n(x)} \right) dx.
\]

Then, for every \( t \in (-\delta, \delta) \) we have
\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \|W_n\|_\infty \right) \right] \leq \frac{8n}{3} \mathbb{E} \left[ \int_I \left( e^{tW_n(x)} + e^{-tW_n(x)} \right) \mu(dx) \right] 
\leq \frac{8n}{3} \mathbb{E} \left[ \int_\mathbb{T} \left( e^{tW_n(x)} + e^{-tW_n(x)} \right) \mu(dx) \right] 
\leq \frac{16n}{3} \exp \left( 3\alpha^2 t^2 n / 2 \right).
\]

From the previous expression we have
\[
\mathbb{E} \left[ \exp \left\{ \frac{t}{2} \left( \|W_n\|_\infty - 3\alpha^2 t n - \frac{2}{t} \log \left( \frac{16n}{3} \right) \right) \right\} \right] \leq \frac{1}{t}
\]
for all \( l > 0 \) and every \( t \in (-\delta, \delta) \). Taking \( l_n = cn^2 \) with \( c = 3/16 \), the inequality \( \left| \log \left( \frac{4\pi n^2}{c^3} \right) \right| < \delta^2 \)
for all large \( n \). Let \( t_n = \left( \frac{\log(4\pi n^2)}{c^3} \right)^{1/2} \), thus
\[
\mathbb{P} \left( \|W_n\|_\infty \geq 5 \left( \alpha^2 n \log \left( 16 \frac{n}{3} \cdot l_n \right) \right)^{1/2} \right) \leq \frac{1}{l_n} \quad \text{for all large } n.
\]

As \( e^{ijx} = \cos(jx) + i \sin(jx) \), for all large \( n \) se have
\[
\mathbb{P} \left( \|\text{Re}(W_n)\|_\infty \geq 5 \left( \alpha^2 n \log \left( \frac{16}{3} cn^3 \right) \right)^{1/2} \right) \leq \frac{1}{cn^2}
\]
and
\[
\mathbb{P} \left( \|\text{Im}(W_n)\|_\infty \geq 5 \left( \alpha^2 n \log \left( \frac{16}{3} cn^3 \right) \right)^{1/2} \right) \leq \frac{1}{cn^2}.
\]
From the above, we have for some suitable positive constants $C_0, C_1$ depending on the distribution of $\xi_0$ that
\[
P \left( \|W_n\|_\infty \geq C_0 (n \log n)^{1/2} \right) \leq \frac{C_1}{n^2}, \quad \text{for all large } n.
\]

The proof of Theorem 4.4 is similar to the proof of Theorem 4.1 but it is needed to handle the condition $\xi_j = \xi_{n-j}$.

**Proof of Theorem 4.4.** Note Lemma A.2 holds for the conditions of $W_{n \text{sym}}$. The arguments are the same as in the proof of Theorem 4.1 up to the analysis of m.g.f. of $W_{n \text{sym}}(x)$, where it is necessary to observe the following. We take the real part of $W_{n \text{sym}}(x) = \sum_{j=0}^{n-1} \xi_j e^{ijx}$. Assume that $n$ is even, the other case is similar. Then
\[
e^{3\alpha^2 t^2 n/2} \geq e^{\alpha^2 t^2 / 2} \prod_{j=1}^{n/2-1} e^{2\alpha^2 t^2} \geq \mathbb{E} \left[ e^{t\xi_0} \prod_{j=1}^{n/2-1} \mathbb{E} \left[ \exp (t\xi_j \cos(jx) + t\xi_{n-j} \cos((n-j)x)) \right] \right]
\]
\[
= \mathbb{E} \left[ e^{t\xi_0} \prod_{j=1}^{n/2-1} e^{t\xi_j \cos(jx)} \right] = \mathbb{E} \left[ e^{\text{Re}(W_{n \text{sym}}(x))} \right]
\]
for every $2t \in (-\delta, \delta)$. The next arguments as the same as the proof of Theorem 4.1 with the only difference of taking $2t \in (-\delta, \delta)$. And then for some suitable positive constants $C_0, C_1$ depending on the distribution of $\xi_0$ we have
\[
P \left( \|W_{n \text{sym}}\|_\infty \geq C_0 (n \log n)^{1/2} \right) \leq \frac{C_1}{n^2}, \quad \text{for all large } n.
\]

**Appendix B. Arithmetics properties**

**B.1. Proof of Lemma 6.6.** Denote by $T$ the Euler totient function. Observe that
\[
\sum_{k: \gcd(k,M) \geq y} \frac{1}{1 \leq k \leq M} \leq \sum_{d=1 \leq d \leq M} T \left( \frac{M}{d} \right).
\]
Recall that $T(s) \leq s - \sqrt{s}$ for all $s \in \mathbb{N}$. Moreover, if $d(s)$ denotes the number of divisors of $s$, then by Theorem 13.12 in [1], there exists a positive constant $C$ such that $d(s) \leq s^{C(\log \log(s))^{-1}}$. Hence
\[
\sum_{k: \gcd(k,M) \geq y} \frac{1}{1 \leq k \leq M} \leq \left( \frac{M}{y} - \sqrt{\frac{M}{y}} \right) M^{C(\log \log(M))^{-1}} \leq \frac{1}{y} M^{1+C(\log \log M)^{-1}}.
\]

**B.2. Proof of Lemma 6.7.** We define the following sequence
\[
P = \{ \exp \left( i \left( j 2\pi x - \theta \right) \right) : j = 0, \ldots, m-1 \},
\]
where $i$ is the imaginary unit. Note $P$ is a set of points on the unit circle which can be seen as vertices of a regular polygon with $m$ sides inscribed in the unit circle.

Since the arguments of points of the form $\exp \left( i \left( j 2\pi x - \theta \right) \right)$ are separated exactly by a distance $2\pi x$, the number of points $\exp \left( i \left( j 2\pi x - \theta \right) \right)$ which are in an arc on the unit circle is at least $\frac{1}{2\pi x} - 2$,
where \( l \) is the length of the arc.

Let \([y, y + 3(2\pi x)]\) be a subinterval of \([-1, 1]\) and we consider the arc \( A \) on the unit circle whose projection on the horizontal axis is \([y, y + 3(2\pi x)]\). If the length of the arc \( A \) is \( l \), then the number of values \( \cos (j2\pi x - \theta) \) which are still in \((y, y + 3(2\pi x))\) is at least \( \frac{l}{2\pi x} - 2 \geq \frac{3(2\pi x)}{2\pi x} - 2 = 1 \) since \( l \geq 3(2\pi x) \).

Let \( s \in \(-(r-1), (r-1)\) \cap \mathbb{Z} \). Note that there exists at least one value
\[
\cos (j2\pi x - \theta) \in \left[ \frac{s}{r}, \frac{s+1}{r} \right]
\]
for all positive integers \( k \leq \frac{1}{3r(2\pi x)} \).

Now, we consider all the values \( \cos (j2\pi x - \theta) \in \left[ \frac{s}{r}, \frac{s+1}{r} \right] \) and define
\[
d_j := \min \left\{ \left| \cos (j2\pi x - \theta) - \frac{s}{r} \right|, \left| \cos (j2\pi x - \theta) - \frac{s+1}{r} \right| \right\}.
\]
Let \( L \) be the biggest integer which satisfies \((3 \cdot 2\pi x)L \leq \frac{1}{2r}, \) or equivalently, \( L = \left\lfloor \frac{1}{2r(3 \cdot 2\pi x)} \right\rfloor \).

Therefore, the sum of \( d_j \) for all \( \cos (j2\pi x - \theta) \in \left[ \frac{s}{r}, \frac{s+1}{r} \right] \) is at least
\[
\sum_{\lambda=1}^{L} 2\lambda (3 \cdot 2\pi x) = 6 (2\pi x) \sum_{\lambda=1}^{L} \lambda \geq 6 (2\pi x) \frac{L^2}{2} \geq 3 (2\pi x) \left( \frac{1}{2} \cdot \frac{1}{(2r) (3 \cdot 2\pi x)} \right)^2 = \frac{1}{12} \cdot \frac{1}{(2r)^2 (2\pi x)},
\]
where the following inequality was used
\[
L \geq \frac{1}{2r (3 \cdot 2\pi x)} - 1 \geq \frac{1}{2} \cdot \frac{1}{2r (3 \cdot 2\pi x)},
\]
which holds if \( \frac{1}{2r (2\pi x)} \geq 6 \). Let \( \sigma_s \) be the sum of \( d_j \) for each interval \( \left[ \frac{s}{r}, \frac{s+1}{r} \right] \), \( s = -(r-1), \ldots, (r-1) \). Since \( r \geq 2 \), we have
\[
\sum_{s=-(r-1)}^{r-1} \sigma_s \geq (2r - 2) \left( \frac{1}{12} \cdot \frac{1}{(2r)^2 (2\pi x)} \right) \geq \frac{1}{24} \cdot \frac{1}{(2r) (2\pi x)}.
\]

From the previous analysis, we have that the distance between the vector \( V \in \mathbb{R}^m \) whose entries are \( V_j = r \cos (j2\pi x - \theta) \) for \( j = 0, \ldots, m-1 \) with \( x = 1/m \) to \( Z^m \) is at least
\[
r \left( \frac{1}{12} \cdot \frac{1}{(2r) (2\pi x)} \right) = \frac{1}{48} \cdot \frac{1}{2\pi x},
\]
verifying that expression \( \frac{1}{2r(2\pi x)} \geq 6 \) is fulfilled.

\[\square\]

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