The Development of interfaces in a Parabolic p-Laplacian type diffusion equation with weak convection

Habeeb A. Aal-Rkhais¹*, Ruba H. Qasim¹

1Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar, Nasiriyah, Iraq.

*Corresponding Author: habeebk@utq.edu.iq

Abstract. This work has the objective to analyse the initial growth of interface and structure of nonnegative weak solution for one-dimensional parabolic p-Laplacian type diffusion-convection with non-positive convection coefficient c. In this situation, the interfaces may expand, shrink or remain stationary relying on the competition between these two factors. In this paper, we concentrate on three regions to classify the behavior of local solutions near the asymptotic interface in the irregular domain. In the first and second regions, the slow diffusion dominates over the convection term with expanding interfaces under some restrictions. In the third region, the slow diffusion dominates over the convection, but the interfaces have a waiting time. In our proof, the rescaling method and blow-up techniques are applied.

1. Introduction

In this section we introduce the Cauchy problem (CP) for the parabolic p-Laplacian type diffusion equation with nonlinear advection as follow

\[ \begin{align*}
  \mathcal{L}u &\equiv u_t - \left( |u_x|^{p-2} u_x \right)_x + c \left( u^\lambda \right)_x = 0, \quad x \in \mathbb{R}, 0 < t < T \\
  u(x, 0) &= u_0(x), \quad x \in \mathbb{R}
\end{align*} \tag{1} \]

where \( p > 2, c < 0, \lambda > 0, 0 < T \leq +\infty \) and \( u_0 \) is non-negative and continuous. The equation (1) consists of a p-Laplacian type diffusion equation with a convection term. The last term \( c \left( u^\lambda \right)_x \) has a significant role for changing the behavior of the interface and local solution. This issue has gained much attention due to its rich mathematical material and numerous applications in quantum physics, plasma physics, biophysics, chemical reaction architecture (see [9, 14, 18]).

The main features of the parabolic p-Laplacian type diffusion-advection process (1) is the nonlinearity’s character. Equations of the form (1) are often referred to as equations with unusual growth conditions because the gap that is resulted from the nonlinear factors. The p-Laplacian equation can be interpreted as a generalization of equation (1) with \( c=0 \). During the last few decades, equation (1) has received a lot of attention and has been cast as a touchstone in the theory of nonlinear partial differential equations. There are many literatures on the equation (1). Let us refer to readers some significant studies [15], [21], and [11], which offer a lot of information on how to analyse p-Laplacian equations qualitatively. The solution to the CP (1), (2) may have interfaces separating regions where \( u > 0 \) from the other one where \( u = 0 \). We shall assume that

\[ u_0(x) = \Gamma(-\alpha)^{\frac{1}{\alpha}} \text{ as } x \to 0^- \quad \text{for } \Gamma > 0, \alpha > 0. \tag{3} \]

where \( \approx \) is a two-sided inequality. The behavior and direction of the interface movement depends on the competitions of these two factors, diffusion and convection. Since initial data \( u_0 \) is bounded
function and satisfies some restrictions of the parameters on its growth rate as $x \to -\infty$ that is appropriate for general theory, see [12]. Moreover, the global case is considered as
\[ u_0(x) = \Gamma(-x)^\alpha, \quad x \in \mathbb{R} \]  
(4)

It will be clearly introduced to find the self-similar solution to the CP(1), (4). Optimal growth rates conditions for the degenerated nonlinear parabolic equation were discussed in [7, 10, 19, 5] for the PME with fast and slow diffusion. Studying the qualitative behavior of the interfaces for diffusion-convection equation
\[ u_t - (u^m)_{xx} + c(u^\lambda)_x = 0, \]  
(5)

with $m > 0$ was considered in [8], [3], [6], [1]. The general theory as well as qualitative properties for the CP(5), (2), are established on non-smooth domains with compactly supported initial function, see [8, 19]. Our goal in this paper is to offer a classification of the development of asymptotic interfaces (or free boundaries) with the solutions of (1)-2 near the interfaces. Since the equation of $p$-Laplacian type diffusion-convection (1) is invariant with respect to translation, so we will study the initial growth of the interface $\eta(t) = \{x: u(x,t) > 0\}$ and $\eta(0) = 0$.

A brief summary of the contents of the article follows: section 2 shows some definitions of the weak solutions and super-solutions (or sub-solutions) of the equation (1) that are necessary for our main results. Our main results are offered in three sections: section 3 two lemmas (lemma 3.1 and lemma 3.2) are introduced to estimate the self-similar solutions in the case where the slow diffusion dominates. Also, theorem 3.1 is discussed to estimate the asymptotic local solution near interface which expands. In section 4, we shows that the case where both the diffusion and convection in balance depending on the critical value $\Gamma$. Also, the self-similar solutions and behavior of interfaces are considered in lemma 4.1 and theorem 4.1. In section 5, the waiting time interface is proved in theorem 5.1. Finally, the conclusions is offered in section 6.

![Fig.1](image_url)

Fig.1: The $(\alpha, \lambda)$-plane to classify the situation for the growth of interfaces in CP(1)-(4).

We introduce in our work The $(\alpha, \lambda)$-plane to classify the case for the development of interfaces for the local solution of CP(1)-(4) if $c < 0$ in Fig.1. The our study is concentrated to discuss the solutions and interfaces depending on the following three regions in Fig.1:

**Region(1):** $\lambda > 0, \alpha < (p - 1)/(p - 1 - \min\{\lambda, p/2\})$,

**Region(2):** $0 < \lambda < p/2, \alpha = (p - 1)/(p - 1 - \lambda)$,

**Region(3):** $p/2 \leq \lambda < p - 1, 2(p - 1)/(p - 2) \leq \alpha < p - 1/(p - 1 - \lambda)$.

2. Preliminaries

Make sure that your Equation Editor or MathType fonts, including sizes, are set up to match the text of your document.

In this work, to prove the main results we have to consider the basic idea and some definitions such as weak solutions and super-solutions (or sub-solutions) of (1):
Theorem 3.1. Let $u(x, t) \geq 0$ and $u \in C(D_1)$, where

$$D_1 = \{(x, t): \xi_0(t) < x < +\infty, 0 < t < T \leq +\infty\}$$

$c$ is in $C^2(\mathbb{R})$ and $x = \xi(t)$ are curves that divide $D_1$ into subdomains $D_i$, where for arbitrary $\delta > 0$ and $\tau_1 \in (\delta, T]$ and $\xi(t) \in C[0, T]$, which is absolutely continuous on a closed interval $[\delta, \tau_1]$. Assume that $\xi$ satisfies the following inequality

$$L\xi = \frac{\partial}{\partial x}(|\xi_x|^p - 2\xi_x) + c \left(\frac{\partial^2}{\partial x^2}\xi\right) \geq 0, (0 \leq \xi \leq \xi_1)$$

Where $\xi \in C^2(\mathbb{R})$, $\xi \in L^\infty(D_1 \cap (t \leq T))$ for any $\tau_1 \in (0, T]$ and $|\partial^2\xi_x|^p - 2\xi_x \in C(D_1)$. Also, if the boundary conditions satisfy

$$\frac{\partial}{\partial n}\xi|_{x=\xi_0(t)} \geq (\leq) u|_{x=\xi_0(t)}, \quad \frac{\partial}{\partial n}\xi|_{t=0} \geq (\leq) u|_{t=0}.$$  

Thus we have

$$\xi \geq (\leq) u \quad \text{in} \quad D_1.$$  

Assume that $c < 0$ and that $u_0$ sometimes has unbounded growth as $|x| \to +\infty$. In [24, 20], the equation (1) with $c = 0$; $p > 2$, has condition of the optimal growth situation was derived.

3. Slowly Dominating of Diffusion over Convection

Throughout this section we estimate the asymptotic local solution of the CP (1) – (3) near interface which expands. The slow diffusion is dominated over the convection factor in region(1), Fig.1. We will prove that in two different subcases in the following theorem:

Theorem 3.1. Let $p > 2$, $c < 0$, $\lambda > 0$ and $\alpha < (p - 1)/(p - 1 - \min\{\lambda, p/2\})$. Then the diffusion term dominates and the interface initially expands such that

$$\eta(t) \sim \zeta', t^{1/(p-\alpha(p-2))} \quad \text{as} \quad t \to 0^+,$$

$$\zeta' = \Gamma^{p/\alpha(p-2)} \zeta,$$  

where $\zeta' > 0$ depends on $\alpha$ and $p$. For arbitrary $p < \zeta$, $\exists \delta(p) > 0$ depends on $\Gamma, \alpha, p$ such that

$$u(x, t) \sim t^{\alpha/(p-\alpha(p-2))} S(p), \quad x = \zeta(t) = t^{1/(p-\alpha(p-2))}.$$  

The self-similar solution of (1), (4), from Lemma 3.1, has the function $S$ which is called a shape function:

$$u(x, t) \sim t^{\alpha/(p-\alpha(p-2))} S(\zeta), \quad \zeta = \Gamma^{p/\alpha(p-2)} \zeta,'$$  

Indeed, $S$ is a unique solution that satisfies the nonlinear ODE problem:

$$\left\{ \begin{array}{l}
S(-\infty) = \Gamma(-\zeta) \alpha, \\
S(\zeta) = 0, \quad \zeta \geq \zeta, \\
S(\zeta) = 0.
\end{array} \right.$$  

Its dependence on $\Gamma$ is given through the following:
\[ S(\rho) = S_0 \left( \frac{(2 - p)}{p - \alpha(p - 2)} \right) \Gamma^{\frac{p}{p - \alpha(p - 2)}}, \quad (12a) \]
\[ S_0(\rho) = w(\rho, 1), \quad \zeta'_0 = \sup\{\rho; S_0(\rho) > 0\} > 0. \quad (12b) \]

A solution \( w \) satisfies the CP(1), (4) with the convection coefficient \( c = 0 \). The explicit formula (7) and (10) represents the asymptotic interfaces and local solutions along the curves \( x = \zeta_0(t) \) matching to CP(1)-(2) with \( c = 0 \). The following lemma which was proved in [5], that is very significant in proving of lemma 3.2.

**Lemma 3.1.** If the restrictions \( p > 2 \) and \( 0 < \alpha < p/(p - 2) \) satisfy, then the CP(2), (4) has the self-similar solution (10) with the shape function \( S \) satisfies (12). Also, the solution to the CP (1), (2) satisfies (7) – (9) if \( u_0 \) satisfies (3).

**Lemma 3.2.** Let \( u \) be a solution to the CP(1) – (2) and \( u_0 \) satisfies (3). If one condition of the following
(a) \( 0 < \lambda \leq p/2, 0 < \alpha < (p - 1)/(p - 1 - \lambda) \).
(b) \( p/2 \leq \lambda \leq p - 1, 0 < \alpha < 2(p - 1)/(p - 2) \).
(c) \( p - 1 \leq \lambda, 0 < \alpha < 2(p - 1)/(p - 2) \).

is valid. Then (9) holds.

**Proof of Lemma 3.2.** Let the condition of the cases (a) and (b) with \( c < 0 \) be valid. Assume that the initial condition (3) is satisfied. For \( \epsilon > 0 \) which is arbitrary sufficiently small value, and there exists \( x_\epsilon < 0 \), then
\[ (\Gamma - \epsilon)(-x)_\epsilon^p \leq u_\epsilon(x) \leq (\Gamma + \epsilon)(-x)_\epsilon^p, \quad x_\epsilon \leq x < +\infty, \quad (13) \]
Since the existence and uniqueness of the CP(1), (1.2) with \( u_\epsilon = (\Gamma \pm \epsilon)(-x)_\epsilon^p, T = +\infty \) hold, it follows that \( u_\epsilon(x,t) \) (resp. \( u_{-\epsilon}(x,t) \) be solutions to the CP (1), (2) with \((\Gamma + \epsilon)(-x)_\epsilon^p \) (resp.\((\Gamma - \epsilon)(-x)_\epsilon^p \)). Also, because of the continuity of the solutions to the CP (1) – (2), \( 3\sigma = \sigma(\epsilon) > 0 \) such that
\[ u_{-\epsilon}(x_\epsilon, t) \leq u_\epsilon(x_\epsilon, t) \leq u_\epsilon(x_\epsilon, t), \quad \text{for} \quad 0 \leq t \leq \sigma. \quad (14) \]
From a comparison principle and (13) – (14), we have that
\[ u_{-\epsilon} \leq u_\epsilon, \quad x_\epsilon \leq x < \infty, 0 \leq t \leq \sigma. \quad (15) \]
Let us consider
\[ u_k^{\pm}(x,t) = k^{\frac{1}{\alpha}}(k^{-\frac{1}{\alpha}}x^\frac{1}{\alpha}), \quad (16) \]
\[ u_k^\pm(x,t) \text{ solves the following CP:} \]
\[ u_t - (\left|u_x \right|p - 2u)u_x + c \left( u_x \right)^2 = 0, \quad (17a) \]
\[ u_0(x,0) = (\Gamma \pm \epsilon)(-x)_\epsilon^p. \quad (17b) \]
The solution of CP(17) is existed and unique. Since \( \alpha(p - 1 - \lambda) - (p - 1 - \lambda) < 0 \) it is easy to prove that
\[ \lim_{k \to +\infty} u_k^{\pm}(x,t) = v_{\pm}(x,t) ; \quad t > 0, x \in \mathbb{R}. \quad (18) \]
According to the lemma 3.1, \( v_{\pm} \) is the solution of the CP(1), (2), with \( u_0 = (\Gamma \pm \epsilon)(-x)_\epsilon^p, T = +\infty \) and \( c = 0 \). Now, we take \( x = \zeta_0(t) = \rho t^{1/\alpha(p - 2)}, \) where \( \rho < \zeta_0 \), such that
\[ x_t(t, \tau) = S(\rho, \Gamma \pm \epsilon) \tau^{\alpha/(\alpha(p - 2))}, t \geq 0. \quad (19) \]
and from (18) it follow that
\[ \lim_{k \to +\infty} u_k^{\pm}(k^{-1/\alpha}x^\frac{1}{\alpha}), k^{-\frac{1}{\alpha}(p - \alpha(p - 2))/\alpha} = S(\rho, \Gamma \pm \epsilon) \tau^{\alpha/(\alpha(p - 2))}, t \geq 0. \quad (20) \]
If we take \( \tau = k^{-\frac{1}{\alpha}(p - \alpha(p - 2))/\alpha} \) then (19) implies
\[ u_k^{\pm}(x, \tau) \sim S(\rho, \Gamma \pm \epsilon) \tau^{\alpha/(\alpha(p - 2))} \] as \( \tau \to 0^+ \) \quad (21)
then (10) follows from (15), (20).

Now consider the case (c). Suppose that \( u_{\pm} \) solves the Dirichlet problem (DP)
\[ u_t - (\left|u_x \right|p - 2u)u_x + c \left( u_x \right)^2 = 0, \quad \text{for} \quad 0 < |x| < \sigma. \quad (22a) \]
\[ u(x,0) = (\Gamma \pm \epsilon)(-x)_\epsilon^p, \quad 0 < |x| < \sigma. \quad (22b) \]
\[ u(x, t) = (\Gamma \pm \epsilon)(-x)_\epsilon^p, \quad 0 \leq t \leq \sigma. \quad (22c) \]
The rescaling function \( u_k^{\pm}(x,t) \), satisfies the following DP
\[ \psi_t - (|\psi|^{p-2} \psi)_x + \frac{c k}{\alpha} (\psi^\lambda)_x = 0 \text{ in } D^k \]
\[ \psi \left( - k^2 x_\epsilon, t \right) = 0, \psi \left( k^2 x_\epsilon, t \right) = k(\Gamma \pm \epsilon)(-x)^\alpha, 0 \leq t \leq \frac{\sigma k}{a} \]
\[ \psi(x, 0) = (\Gamma \pm \epsilon)(-x)^\alpha, |x| < \frac{1}{\sigma k}|x_\epsilon|, \]

where
\[ D^k = \{ (x, t); \ |x| < \frac{1}{\sigma k}|x_\epsilon|, 0 < t \leq \frac{\sigma k}{a} \} \]

Then \( \exists \sigma > 0 \) that is not influenced by \( k \). The DPs (22a) \(-\) (22c) and (23a) \(-\) (23c) have a unique solution. Keeping in mind, property of finite speed of propagation, it can be chosen that \( \sigma > 0 \), depending on \( \epsilon \), such that
\[ \psi(-x_\epsilon, t) = 0, \quad 0 \leq t \leq \sigma \]

By using comparison theorem (see lemma 2.1), from (13), (14) and (15), (24) follows for \( |x| \leq |x_\epsilon|, 0 \leq t \leq \sigma \). Let us now prove that the sequence \( \{ \psi^{+\epsilon}_k \} \) is convergent by assuming a function
\[ \psi = (\Gamma + 1)(1 + x^2)^{\alpha/2} e^t, x \in \mathbb{R}, 0 \leq t \leq \sigma. \]

Then we have
\[ I_k \psi \equiv \psi(t) - (|\psi|^{p-2} \psi)_x + \frac{c k}{\alpha} (\psi^\lambda)_x \equiv (\Gamma + 1)(1 + x^2)^{\alpha/2} \text{ in } D^k \]
\[ \psi(x, t) = (\Gamma \pm \epsilon)(-x)^\alpha, \psi(-x_\epsilon, t) = 0, \quad 0 \leq t \leq \sigma. \]

\[ T = 1 - \mathcal{H}(x) + R, \text{ where} \]
\[ \mathcal{H}(x) = \alpha^{p-1}(\Gamma + 1)^{p-2} e^{t(p-2)(p-1)}(1 + x^2)^{\frac{a(p-2) - 2(p-1)}{2}} x^{p-2} \]
\[ \times (1 + (\alpha - 2)(1 + x^2)^{-1} x^2), \]

Since \( \mathcal{H} \) is continuous and for \( |x| < \frac{1}{\sigma k}|x_\epsilon|, \)
\[ \mathcal{H} \approx 0 \left( \frac{k}{\alpha} \right)^{p-1} (\Gamma + 1)^{\lambda-1} e^{(\lambda-1)t} \]
\[ \alpha(\lambda-1)-2 \]
\[ x, \]

and hence
\[ I_k \psi = (\Gamma + 1)(1 + x^2)^{\alpha/2} e^t \geq \psi \quad \text{in } D^k \]
\[ D^k_0 = D^k \cap \{ (x, t); 0 < t \leq \sigma \}, \text{ such that} \]
\[ R = 0 \left( \frac{k}{\alpha} \right)^{p-1} \text{ uniformly on } D^k_0 \text{ as } k \to \infty. \]

Thus, for \( 0 < \epsilon < 1 \), we get
\[ \psi_\epsilon(0, 0) \geq \psi_\epsilon^{+\epsilon}(x, 0) \text{ on } |x| < \frac{1}{\sigma k}|x_\epsilon|, \]
\[ \psi_\epsilon \left( \pm k^2 x_\epsilon, t \right) \geq \psi_\epsilon^{+\epsilon} \left( \pm k^2 x_\epsilon, t \right), \quad 0 \leq t \leq \sigma. \]

Since \( \exists k_0 = k_0(\alpha, \gamma) \) thus \( \forall k \geq k_0 \) and the comparison principle, lemma 2.1 implies
\[ \psi_\epsilon^{+\epsilon}(x, t) \leq \psi(x, t) \text{ in } D^k_0. \]

Assume that \( B \subset \mathcal{P} \) and \( \mathcal{P} = \{ (x, t) \in \mathbb{R}, 0 \leq t \leq t_0 \}, \) where \( B \) is an arbitrary compact set. If \( k \geq k_0, \)
with so large value \( k_0 \) such that \( B \subset D^k_0 \). Inequality (27) implies that the \( \{ \psi^{+\epsilon}_k \} \) is uniformly bounded in \( B \). They are uniformly Hölder continuous and \( \exists \nu \psi \) such that the subsequence \( \psi^{+\epsilon}_k \) is convergent
\[ \lim_{k \to +\infty} \psi^{+\epsilon}_k = \nu \psi, \text{ on } \mathcal{P}. \]

Proof of Theorem 3.1. By assumption of the theorem, let the restriction of \( \alpha \) and \( \lambda \). Since the formula
\[ (7) \text{ comes from lemma 3.1. The lower bound of the interface satisfies as follows} \]
\[ \zeta_n \leq \lim_{t \to 0^+} \inf t \psi^{+\epsilon}_{t(2(p-2)-p}. \]
On the other hand, to get the upper bound of the interface: we assume that $\epsilon > 0$ is arbitrary sufficiently small and $u_\epsilon$ solves the CP(1), (4) with $c = 0$. Let us replace $\Gamma$ by $\Gamma + \epsilon$. Same as given before, the first inequality of (14) and the second one of (13). That will be considered. Now, Let us prove that $\bar{u}_\epsilon$ satisfies a supersolution of (1) with $c < 0$, then we have

$$L\bar{u}_\epsilon = c(\bar{u}_\epsilon^2)_x.$$  

We want to prove $c(\bar{u}_\epsilon^2)_x \geq 0$, and since $c < 0$, so we will prove $(\bar{u}_\epsilon^2)_x \leq 0$: then we will use regularization to solve (1, 1) with $c = 0$, and $u_\epsilon(x, t) = (\Gamma + \epsilon)(-\chi)_\epsilon^\ast$.

We will prove $(\bar{u}_\epsilon)_x \leq 0$. Let $w = u_\epsilon$ and define

$$\bar{u}_\epsilon(x, t) = \max\left(0, w\left(xe^{ct}, \frac{1}{ct} \left(e^{ct} - 1\right)\right)\right) = \max(0, w(\zeta, \tau)).$$

$$L\bar{u}_\epsilon = e^{ct}Lw + cw(\zeta + \lambda \bar{u}_\epsilon^{\lambda - 1}e^{ct}) = cw(\zeta + \lambda \bar{u}_\epsilon^{\lambda - 1}e^{ct}) \geq 0,$$

where,

$$Lw = w_t - \left(\left|w_t\right|^{p-2}w_t\right)_\zeta = 0.$$  

From here since $c < 0$, and $\zeta + \lambda \bar{u}_\epsilon^{\lambda - 1}e^{ct} > 0$. It follows that $w_\zeta < 0$, and since $w$ is a classical solution and if we make $[x_\zeta]$ and $\sigma$ so small, maximum principle can be used to get that $\bar{u}_\epsilon < 1$ in $\mathcal{D}$, and so $(\bar{u}_\epsilon)_x \leq 0$. So it must be that $(\bar{u}_\epsilon)_x \leq 0$. So we obtain that

$$\bar{u}_\epsilon \geq 0$$  

$\mathcal{D} = \{(x, t): \, x \geq x_\epsilon; \, 0 < t < \rho\}.$

From a comparison principle and (13), (14), the right-side inequality of (15) holds. Thus

$$\frac{1}{t^{\alpha(p-2)-p}}\eta(t) \leq (\Gamma + \epsilon)^{2-p}\frac{w_\rho}{\zeta}, \quad 0 \leq t < \sigma,$$

it implies that

$$\lim_{t \to 0^+} \sup_{\eta(t)} t^{\frac{1}{\alpha(p-2)-p}} \leq \zeta, \quad 0 \leq t < \sigma. \tag{30}$$

Therefore, by (29) and (30), $\eta(t)\sim_t \zeta, t^{\frac{p-\alpha(p-2)}{p-1}}, \, t \to 0^+$ is valid. 

### 4. Diffusion and Convection in Balance

In this section, we identify the range of parameters when the $p$-Laplacian type diffusion and convection are in balance and it will be presented in the following theorem as shown in Fig. 1 (region(2)). We present the case ($\Gamma > \Gamma$), that shows the same behavior of local solution and expanding interface as in theorem 3.1.

**Theorem 4.1.** Let $p > 2, 0 < \lambda < p/2, \alpha = p - 1/(p - 1 - \lambda)$, and

$$\Gamma_* = \left(\frac{(-c)^{1/(p-1)}(p - 1 - \lambda)(p - 1)}{p - 1 - \lambda}\right). \tag{31}$$

If $u_0$ satisfies (3) then the interface expands accordingly as $\Gamma > \Gamma_*$ and

$$\eta(t) \sim_t \zeta, t^{\frac{p-1-\lambda}{p}}, \, \text{as} \, t \to 0^+ \tag{32}$$

where if $\Gamma > \Gamma_*$, then $\zeta_* > 0$, and there exists $S(p)$ > 0 for arbitrary $p < \zeta_*$ such that

$$u(x, t) \sim_t S(p)t^{\frac{p-1}{p}}, \, \text{as} \, t \to 0^+, \tag{33}$$

where, $S(p) = (p - 1 - \lambda) - (p - 1)(p - 2)$ and $x = \zeta(p, t) = pt^{1/\alpha(p-2)}$.

In theorem 4.1, we considered the local solution in region (2), Fig. 1. Let $u_0$ be satisfied (1.4). The situations that will be discuss expanding interface if $\Gamma > \Gamma_*$ and waiting time if $\Gamma = \Gamma_*$. Then, a stationary solution to the CP(1), (4), is proved if $\Gamma = \Gamma_*$. On the other hand, if $\Gamma > \Gamma_*$ then the solution to CP(1), (4) has the following formula

$$u(x, t) = S_1(t)^{\frac{p-1}{\theta}}, \zeta = x t^{\frac{(p-1-\lambda)}{\theta}}, \tag{34}$$

$$\eta(t) = \zeta, t^{\frac{p-1-\lambda}{\theta}}, \, 0 \leq t < +\infty. \tag{35}$$

It is an expanding interface, $S_1(0) = 0$ (see Lemma 4.1), and
\[ \Gamma_2 t^{p-1} \frac{\theta}{\Gamma} (\xi_2 - \xi_0) \frac{\theta}{\Gamma} \leq u \leq \Gamma_1 t^{p-1} \frac{\theta}{\Gamma} (\xi_1 - \xi_0) \frac{\theta}{\Gamma} \]  
(36)

Where
\[ \alpha_0 = (p - 1)/(p - 2) \text{ if } \lambda > 1; \quad \alpha_0 = (p - 1)/(p - 1 - \lambda) \text{ if } \lambda < 1, \]
(37)

which indicates
\[ \xi_2 \leq \xi \leq \xi_1. \]
(38)

We observe that the right-hand side of (36) (resp. (38)) relays to the self-similar formula (34) (resp. (35)).

**Lemma 4.1.** Let \( u(x, t) \) be a function that satisfies the CP(1), (4) with \( 0 < \lambda < p/2 \) and \( \alpha = (p - 1)/(p - 1 - \lambda) \). Then \( u(x, t) \) satisfies
\[ u(x, t) = \delta(\xi) t^{p-1} \frac{\theta}{\Gamma}, \quad \xi = xt^{p-1-\lambda} \frac{\theta}{\Gamma}, \]
(39)

that is a self-similar solution. If \( \Gamma > \lambda \), then \( u_1(0) = A_1 \); where \( A_1 = A_1(\lambda, \Gamma, c) \) is a positive. If \( u_0 \) satisfies (3) with \( c > 0; \Gamma > \lambda \), then \( u \) satisfies
\[ u(0, t) = A_1 t^{(p-1)/(p-1-\lambda)} \frac{\theta}{\Gamma}. \]
(40)

**Proof of Lemma 4.1.** Let us consider a rescaling function
\[ u_k(x, t) = ku \left( \frac{\theta}{\Gamma} \frac{\lambda + p - 1}{k^{p-1}} x, k^{p-1}t \right), \quad k > 0, \]
(41)

satisfies (1), (4). There exists a unique global solution to (1), (4) under the condition of this lemma. Therefore, we have
\[ u(x, t) = ku \left( \frac{\theta}{\Gamma} \frac{\lambda + p - 1}{k^{p-1}} x, k^{p-1}t \right), \quad k > 0. \]
(42)

If we choose \( k = t^{(p-1)/\theta} \) then (43) implies (34) for \( u \) with \( S(\xi) = u(\xi, 1) \). Indeed, the shape function \( S \) is a unique nonnegative differentiable solution of the BVP
\[ \begin{cases} (\delta' | p - 2 \delta')' - \alpha \frac{1}{p - \alpha} \delta'(\xi) - \alpha \frac{p - \alpha}{p - \alpha} S - c(\delta^\lambda)' = 0, \xi < \xi_*, \\ S(-\infty) \sim \Gamma(\xi)^\alpha, \quad S(\xi_*) = 0, \quad S(+\infty) \equiv 0, \quad \xi \geq \xi_*, \end{cases} \]
(43)

where \( \xi_* > 0 \) such that \( f \) satisfies (43). Also, \( S \) equals to 0 for \( \xi \geq \xi_* \) and it is positive and smooth for \( \xi < \xi_* \). Then, (35) holds.

Now assume that \( u_0 \) satisfies (3). Same as the proof of lemma 3.1, from (3), the inequalities (13) and (14) are satisfied. Then the existence and uniqueness of CP(1) – (2) with \( u_0 = (\Gamma \pm \epsilon)(-x)^\alpha \), hold. As described previously, (14) follows from (13). Now if we take
\[ u_k^{\pm}(x, t) = ku_0 \left( \frac{\theta}{\Gamma} \frac{\lambda + p - 1}{k^{p-1}} x, k^{p-1}t \right), \quad k > 0. \]
(44)

thus \( u_k^{\pm}(x, t) \) solves the CP
\[ \begin{align*} u_t - ((u)_x | p - 1 (u)_x)_x + c((u)_x^\lambda) & \equiv 0, x \in \mathbb{R}, t > 0, \\ u(x, 1) = (\Gamma \pm \epsilon)(-x)^{p(1)/(p - 1 - \lambda)}, & \quad x \in \mathbb{R}. \end{align*} \]
(45a)
\[ (45b) \]

The CP(45a), (45b) has a unique solution. By using the comparison principle, we have
\[ \lim_{k \to +\infty} u_k^{\pm}(x, t) = v_\pm; \]
(46)

where \( v_\pm \) solves the CP(1) – (2), \( c = 0 \); and \( u_0 = (\Gamma \pm \epsilon)(-x)^{p(1)/(p - 1 - \lambda)} \). Thus, \( v_\pm \) satisfies (34). Let us take \( x = \xi_*(t) \), for \( \rho < \xi_* \), and \( \rho \) is an arbitrary fixed number then (46) becomes
\[ \lim_{k \to +\infty} ku_0 \left( \xi_*(t), k^{p-1} t \right) = \delta(\rho, \Gamma \pm \epsilon) t^{(p-1)/(\Gamma \pm \epsilon)}, \quad t \geq 0. \]
(47)

If we take \( \tau = k^{p-1} t \), then (47) implies
\[ u_k^{\pm}(\xi_*(\tau), \tau) \sim \delta(\rho, \Gamma) t^{(p-1)/(\Gamma \pm \epsilon)} \quad \text{as} \quad t \to 0^+. \]
(48)

From (48) and (15), for the arbitrary value \( \epsilon > 0 \), the formula (41) follows.
Proof of theorem 4.1. Let us assume the initial condition (4). The CP(1),(4) has a unique weak solution. We get the self-similar solution (34) from Lemma 4.1. To concentrate on the case \( \Gamma > \Gamma_* \), we assume a function
\[
\mathcal{H}(x,t) = \frac{p-1-\lambda}{\sigma} S_1(\zeta), \quad \zeta = xt^\frac{1}{\sigma},
\]
then
\[
L_\mathcal{H} = \frac{p(y-1)+1}{\sigma} L^0 S_1, \tag{50a}
\]
\[
L^0 S_1 = \frac{p-1}{\sigma} f_1(\zeta) - \frac{p-1-\lambda}{\sigma} \zeta S'_1(\zeta) - (S'_1(\zeta)|S'_1(\zeta)|^{p-2} + c\left(S^2_1(\zeta)\right)' \tag{50b}
\]
Let us choose \( S_1 \) such that \( S_1(\zeta) = \Gamma_0(\zeta_0 - \zeta)^+, 0 < \zeta < +\infty \) where the constants \( \Gamma_0, \zeta_0, \alpha_0 \) are positive. Let us Take \( \alpha_0 = (p-1)/(p-1-\lambda) \), from (50b), we have
\[
L^0 S_1 = \frac{-c\lambda(p-1)}{\sigma} \Gamma_0^\lambda(\zeta_0 - \zeta)^+ \left\{ 1 - \left( \frac{\Gamma_0}{\Gamma_1} \right)^{p-1-\lambda} \right\} + \frac{p-1-\lambda}{(-c\lambda) \zeta_0(\zeta_0 - \zeta)^+} \left( \frac{(p-1)(1-\lambda)}{p-1-\lambda} \right)
\]
To prove the upper estimation, we choose \( \Gamma_0 = \Gamma_1, \zeta_0 = \zeta_1 \). If \( \lambda > 1 \), then
\[
L^0 S_1 \geq \frac{-c\lambda(p-1)}{\sigma} \Gamma_1^\lambda(\zeta_1 - \zeta)^+ \left\{ 1 - \left( \frac{\Gamma_1}{\Gamma_1} \right)^{p-1-\lambda} \right\} = 0,
\]
where \( \Gamma_1 > \Gamma_* \), and
\[
\Gamma_1 = \Gamma_* \left( \frac{p-1-\lambda}{(-c\lambda) \zeta_1} \right)^{\frac{1}{p-1-\lambda}} \left( \frac{2(p-1-\lambda)p}{p-1-\lambda} \right)
\]
While if \( \lambda < 1 \), and for \( 0 \leq \zeta \leq \xi_1 \), then
\[
L^0 S_1 \geq \frac{-c\lambda(p-1)}{\sigma} \Gamma_1^\lambda(\zeta_1 - \zeta)^+ \left\{ 1 - \left( \frac{\Gamma_1}{\Gamma_1} \right)^{p-1-\lambda} \right\} = 0,
\]
where \( \Gamma_1 = \Gamma_* \). From (50a) it follows that
\[
L_\mathcal{H} \geq 0 \text{ for } 0 < x < \zeta_2 t^\frac{p-1-\lambda}{\sigma} \text{, } t \in (0, +\infty),
\]
\[
L_\mathcal{H} = 0 \text{ for } x > \zeta_2 t^\frac{p-1-\lambda}{\sigma} \text{, } t \in (0, +\infty).
\]
Then \( \mathcal{H} \) is the supersolution of (1) in \( U = \{(x,t): x > 0, t > 0\} \), by lemma 2.1. Also, since
\[
\mathcal{H}(x,0) = u(x,0) = 0 \text{ for } 0 \leq x < +\infty,
\]
\[
\mathcal{H}(0,t) = u(0,t) \text{ for } 0 \leq t < +\infty.
\]
the right-hand side of (38) follows. Let \( \alpha_0 = (p-1)/(p-1-\lambda) \) then we will prove the lower estimation by choosing \( \Gamma_0 = \Gamma_2, \zeta_0 = \zeta_2 \). If \( \lambda < 1 \) then from (4.22) and for \( 0 \leq \zeta \leq \xi_2 \), we derive by using the same calculation above. Also, from (50a) it follows that
\[
L_\mathcal{H} \leq 0 \text{ for } 0 < x < \zeta_2 t^\frac{p-1-\lambda}{\sigma} \text{, } t \in (0, +\infty),
\]
\[
L_\mathcal{H} = 0 \text{ for } x > \zeta_2 t^\frac{p-1-\lambda}{\sigma} \text{, } t \in (0, +\infty).
\]
Lemma 2.1 implies that \( \mathcal{H} \) is a subsolution of (1) in \( U \), the left-hand side of (36) follows. Let \( \alpha_0 = (p-1)/(p-2) \) and then we will prove the lower estimation by choosing \( \lambda > 1, \) for \( 0 \leq \zeta \leq \xi_2 \) we have
\[
L^0 S_1 \leq \frac{p-1}{\sigma} \Gamma_2(\zeta_2 - \zeta)^+ \left\{ \zeta_2 t^p - \Gamma_2^p - \frac{\theta(p-1)^p}{(p-2)^p(p-1)} \right\} = 0,
\]
where \( \Gamma_2 > \Gamma_* \), and
\[
\Gamma_2 = \xi_2 \frac{\Gamma_1}{\Gamma_1} - \tau - \tau_2 = \frac{\Gamma_1}{\Gamma_1} - \tau - \tau_2
\]

which again implies (54). As before, from lemma 2.1, (53), and the comparison theorem, the left-hand side of (36) follows.

5. Diffusion dominates with waiting time interface

**Theorem 5.1.** Let \( 2(p-1)/(p-2) \leq \alpha < (p-1)/(p-1-\lambda) \) and \( p/2 \leq \lambda < p-1 \), then the diffusion dominates term with waiting time interface.

**Proof of theorem 5.1.** Consider the upper estimate \( \hat{u} (x, t) = (\Gamma + \epsilon)/(\alpha) \) which represents supersolution of \( u \). Also, Let consider the lower estimate

\[
\hat{u} (x, t) = (\Gamma - \epsilon)/(\alpha) \]

where \( \tau \) and \( \lambda \) are positive constants. since \( \alpha < (p-1)/(p-1-\lambda) \), we get

\[
\hat{u} (x, t) \geq (C - \epsilon)/(\alpha) \text{ and } \exists x_c < 0, \text{ such that } x_c < x < 0, \text{ then by continuity there exist } \sigma_0 > 0 \text{ such that}
\]

\[
\hat{u} (x_e, t) \geq (\Gamma - \epsilon)/(\alpha) \text{ where } \sigma_0 > 0.
\]

By the calculating, we get

\[
\text{L} \hat{u}_e \leq (\Gamma - \epsilon)/(\alpha) \]

\[
(\Gamma - \epsilon)/(\alpha) \leq (\Gamma - \epsilon)/(\alpha) \]

Then applying the comparison theorem, and Lemma 2.1 implies that \( \hat{u}_e \) is the lower estimate of \( u \). Therefore, the estimation \( \hat{u}_e \leq u \leq \hat{u}_e \) is satisfied.

6. Conclusion

The self-similar solutions to local weak solutions for the model of parabolic p-Laplacian diffusion-convection processes with non-positive convection coefficient are estimated. Also, Classification of the behavior of interfaces and local solutions near the interfaces in non-smooth domain is presented on three regions. It is evidence that on that regions, the slow diffusion is either increasing or having a waiting time. The interfaces may expand or remain stationary relying on the competition between the diffusion and convection. Significant methods are used such as, rescaling method and blow up techniques. As a result, the conclusion that comes through this work is that model can be applied to quantum physics, biophysics, plasma physics, chemical reaction design and so on.

**References.**

[1] Aal-Rkhais, H. A., Hashoosh, A. E. 2018 *Asymptotic Behavior of Solutions to the Nonlinear Fokker-Planck Equation with Absorption.* In Jour of Adv Research in Dynamical& Control Systems: (Vol. 10, 10-Special Issue).

[2] Abdulla, U. G. 2000 *Reaction–diffusion in irregular domains.* Journal of Differential Equations, 164(2), 321-354.

[3] Abdulla, U. G., & Aal-Rkhais, H. A. 2019, July. *Development of the Interfaces for the Nonlinear Reaction-Diffusion equation with Convection.* In IOP Conference Series: Materials Science and Engineering (Vol. 571, No. 1, p. 012012). IOP Publishing.

[4] Abdulla, U. G., & Jeli, R. 2016 *Evolution of Interfaces for the Nonlinear Parabolic p-Laplacian Type Reaction-Diffusion Equations.* arXiv preprint arXiv:1605.07279.

[5] Abdulla, U. G., & King, J. R. 2000. *Interface development and local solutions to reaction-diffusion equations.* SIAM Journal on Mathematical Analysis, 32(2), 235-260.
[6] Abed, B. N., Majeed, S. J., & Aal-Rkhais, H. A. 2020, July. Qualitative Analysis and Traveling wave Solutions for the Nonlinear Convection Equations with Absorption. In Journal of Physics: Conference Series (Vol. 1591, No. 1, p. 012052). IOP Publishing.

[7] Alvarez, L. A., Maytal, J., & Shinnar, S. 1986 Idiopathic external hydrocephalus: natural history and relationship to benign familial macrocephaly. Pediatrics, 77(6), 901-907.

[8] Alvarez, L., Diaz, J. I., & Kersner, R. 1988 On the initial growth of the interfaces in nonlinear diffusion-convection processes. In Nonlinear Diffusion Equations and Their Equilibrium States I (pp. 1-20). Springer, New York, NY.

[9] Antontsev, S. N., Díaz, J. I., Shmarev, S., & Kassab, A. J. 2002 Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics. Progress in Nonlinear Differential Equations and Their Applications, Vol 48. Appl. Mech. Rev., 55(4), B74-B75.

[10] Avarez, L., & Diaz, J. I. 1993 On the initial growth of interfaces in reaction diffusion equations with strong absorption. Proceedings of the Royal Society of Edinburgh A, 123, 803-817.

[11] Boccardo, L., & Gallouet, T. 1996 Summability of the solutions of nonlinear elliptic equations with right hand side measures. Journal of Convex Analysis, 3, 361-366.

[12] de Pablo, A., & Sanchez, A. 2000. Global travelling waves in reaction–convection–diffusion equations. Journal of Differential Equations, 165(2), 377-413.

[13] Di Benedetto, E. 1986 On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 13(3), 487-535.

[14] Diaz, J. I. 1985 Nonlinear partial differential equations and free boundaries. Elliptic Equations. Research Notes in Math., 1, 106.

[15] DiBenedetto, E. 1993 Degenerate Parabolic Equations, Springer Verlag, Series Universitext, New York.

[16] DiBenedetto, E., & Herrero, M. A. 1989 On the Cauchy problem and initial traces for a degenerate parabolic equation. Transactions of the American Mathematical Society, 314(1), 187-224.

[17] Esteban, J. R., & Vázquez, J. L. 1986 On the equation of turbulent filtration in one-dimensional porous media. Nonlinear Analysis: Theory, Methods & Applications, 10(11), 1303-1325.

[18] Ettwein, F., & Růžička, M. 2003 Existence of strong solutions for electrorheological fluids in two dimensions: steady Dirichlet problem. In Geometric analysis and nonlinear partial differential equations (pp. 591-602). Springer, Berlin, Heidelberg.

[19] Gladkov, A. L. 1995 The Cauchy problem in classes of increasing functions for the equation of filtration with convection. Sbornik: Mathematics, 186(6), 803-825.

[20] Ishige, K. 1996 On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation. SIAM Journal on Mathematical Analysis, 27(5), 1235-1260.

[21] Kačur, J. 1990 On a solution of degenerate elliptic-parabolic systems in Orlicz-Sobolev spaces II. Mathematische Zeitschrift, 203(1), 569-579.

[22] Kalashnikov, A. S. 1978 On a nonlinear equation appearing in the theory of non-stationary filtration. Trudy. Sem. Petrovsk, 4, 137-146. (in Russian)

[23] Kalashnikov, A. S. 1982 On the propagation of perturbations in the first boundary value problem of a doubly-nonlinear degenerate parabolic equation, Trud. Semin. I.G. Pertovski, 8, 128-134. (in Russian)

[24] Kalashnikov, A. S. 1987 Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations. Russian Mathematical Surveys, 42(2), 169-222.