Quantum properties of supersymmetric theories regularized by higher covariant derivatives

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Abstract. We investigate quantum corrections in $\mathcal{N}=1$ non-Abelian supersymmetric gauge theories, regularized by higher covariant derivatives. In particular, by the help of the Slavnov–Taylor identities we prove that the vertices with two ghost legs and one leg of the quantum gauge superfield are finite in all orders. This non-renormalization theorem is confirmed by an explicit one-loop calculation. By the help of this theorem we rewrite the exact NSVZ $\beta$-function in the form of the relation between the $\beta$-function and the anomalous dimensions of the matter superfields, of the quantum gauge superfield, and of the Faddeev–Popov ghosts. Such a relation has simple qualitative interpretation and allows suggesting a prescription producing the NSVZ scheme in all loops for the theories regularized by higher derivatives. This prescription is verified by the explicit three-loop calculation for the terms quartic in the Yukawa couplings.

1. Introduction

It is well known that the ultraviolet behaviour of supersymmetric theories is better than in the non-supersymmetric case. In particular, in $\mathcal{N}=1$ supersymmetric theories the superpotential has no divergent quantum corrections [1] due to the non-renormalization theorem. The NSVZ $\beta$-function [2, 3, 4, 5] can be also considered as a non-renormalization theorem. In particular, it is closely related to the $\mathcal{N}=2$ non-renormalization theorem [6, 7, 8], see, e.g., [9, 10, 11]. Certainly, the good ultraviolet behaviour of supersymmetric theories appears due to a wide symmetry of a theory. Therefore, to reveal attractive features of supersymmetric theories, one should make manifest as many symmetries as possible [12]. From this point of view, the dimensional regularization is not convenient, because it breaks supersymmetry. Its modification called the dimensional reduction [13] is better, but due to mathematical inconsistency [14] can lead to supersymmetry breaking in higher loops [15, 16]. Unlike the dimensional technique, the higher covariant derivative regularization [17, 18, 19] is a mathematically consistent method, which can be formulated in a manifestly supersymmetric way [20, 21]. It is also possible to formulate this regularization in the $\mathcal{N}=2$ harmonic superspace [11], so that $\mathcal{N}=2$ supersymmetry remains a manifest symmetry at all stages of calculating quantum corrections.

Application of the higher derivative regularization to $\mathcal{N}=1$ supersymmetric theories revealed some interesting features of quantum corrections. For example, it was noted that the integrals giving the $\beta$-function of supersymmetric gauge theories are integrals of total derivatives [22] and even double total derivatives [23]. A lot of calculations in various supersymmetric theories confirm that this factorization seems to be a general feature of supersymmetric theories, see, e.g., [24, 25, 26, 27]. For $\mathcal{N}=1$ supersymmetric electrodynamics (SQED) it has been proven...
in all orders of the perturbations theory by two different methods [28, 29]. Such a structure of loop integrals is closely related to the NSVZ \( \beta \)-function. In particular, with the higher derivative regularization the renormalization group (RG) functions of \( N = 1 \) SQED defined in terms of the bare coupling constant satisfy the NSVZ relation in all loops independently of the renormalization prescription. This allows constructing the all-loop prescription giving the NSVZ scheme for the RG functions standardly defined in terms of the renormalized coupling constant [33, 34, 35], while with the dimensional technique such a scheme should be tuned in every order of the perturbation theory [36, 37, 38]. However, investigation of the non-Abelian case is much more complicated and is a subject of this paper.

2. The higher covariant derivative regularization for \( N = 1 \) supersymmetric theories

We consider the massless \( N = 1 \) SYM theory described by the action

\[
S = \frac{1}{2e_0^2} \text{Re} \text{tr} \int d^4x d^2\theta W^a W_a + \frac{1}{4} \int d^4x d^4\theta \phi^* i^j j^j \phi_j + \left( \frac{1}{6} \int d^4x d^2\theta \lambda^{ij \kappa} \phi_i \phi_j \phi_k + \text{c.c.} \right),
\]

assuming the invariance under the gauge transformations. In the supersymmetric case the quantum-background splitting is performed by making the substitution \( e^{2\Omega} \rightarrow e^{\Omega^+} e^{2\Omega} \). The background superfield \( V \) is then defined as \( e^{2\Omega} = e^{\Omega^+} e^\Omega \). The theory is regularized by adding the gauge invariant higher derivative term

\[
S_\Lambda = \frac{1}{2e_0^2} \text{Re} \text{tr} \int d^4x d^2\theta e^{\Omega} W^a e^{-\Omega} \left[ R \left( -\frac{\nabla^2 \nabla^2}{16\Lambda^2} \right) - 1 \right]_{\text{Adj}} e^{\Omega} W^a e^{-\Omega} e^{-\Omega} + \frac{1}{4} \int d^4x d^4\theta \phi^* e^{\Omega^+} e^{\Omega^+} \left[ F \left( -\frac{\nabla^2 \nabla^2}{16\Lambda^2} \right) - 1 \right] e^{\Omega} e^{\Omega} \phi,
\]

where the functions \( R(x) \) and \( F(x) \) rapidly increase at infinity and are equal to 1 at \( x = 0 \). The gauge can be fixed by adding the term invariant under the background gauge transformations,

\[
S_{gf} = \frac{1}{e_0^2} \text{tr} \int d^4x d^4\theta \left( 16\xi_0 f^+ \left[ e^{\Omega^+} K^{-1} \left( -\frac{\nabla^2 \nabla^2}{16\Lambda^2} \right) e^\Omega \right]_{\text{Adj}} f + e^{\Omega} f e^{-\Omega} \nabla^2 V + e^{-\Omega} f^+ e^{\Omega^+} \nabla^2 V \right),
\]

where \( K(0) = 1 \) and \( K(x) \) also rapidly grows at infinity. The actions for the Faddeev–Popov and Nielsen–Kallosh ghosts have the form

\[
S_{FP} = \frac{1}{e_0^2} \text{tr} \int d^4x d^4\theta \left( e^{\Omega} e^{-\Omega} + e^{-\Omega^+} e^{\Omega^+} \right) \times \left\{ \left( \frac{V}{1 - e^{2\Omega}} \right)_{\text{Adj}} (e^{-\Omega^+} e^{\Omega^+}) + \left( \frac{V}{1 - e^{-2\Omega}} \right)_{\text{Adj}} (e^{\Omega} e^{-\Omega}) \right\};
\]

\[
S_{NK} = \frac{1}{2e_0^2} \text{tr} \int d^4x d^4\theta b^+ \left[ e^{\Omega^+} K \left( -\frac{\nabla^2 \nabla^2}{16\Lambda^2} \right) e^\Omega \right]_{\text{Adj}} b.
\]

The Pauli–Villars determinants should also be inserted into the generating functional to cancel the remaining one-loop divergences [19], see Ref. [27] for the discussion of details in the supersymmetric case.

1 The same arguments allow to reveal similar features of some other theories, [30, 31, 32].
The total gauge fixed theory is invariant under the BRST transformations

\[ \delta V = -\varepsilon \left( \frac{V}{1 - e^{2V}} \right) A_{ij} \left( e^{-\Omega^+} c^+ e^{\Omega^+} \right) + \left( \frac{V}{1 - e^{-2V}} \right) A_{ij} \left( e^{\Omega} c e^{-\Omega} \right) \]; \\
\[ \delta \bar{c} = \varepsilon \bar{D}^2 (e^{-2V} f^+ e^{2V}) ; \quad \delta \bar{c}^+ = \varepsilon D^2 (e^{2V} f e^{-2V}) ; \quad \delta c = \varepsilon e^2 ; \quad \delta c^+ = \varepsilon (e^+)^2 , \quad (5) \]

where \( \varepsilon \) is an anticommuting scalar parameter which is independent of the coordinates. The superfields \( f, b, \text{ and } \Omega \) are not transformed.

The renormalization constants are defined by the equations

\[ \frac{1}{\alpha_0} = Z_\alpha ; \quad V = Z_V Z_{\alpha}^{-1/2} V_R ; \quad \bar{c} c = Z_{c} Z_{\alpha}^{-1} \bar{c}_R c_R ; \quad \phi_i = (\sqrt{Z_\alpha}) i^j (\phi_R)_{ij} . \quad (6) \]

In our notation \( R \) denotes renormalized superfields, \( \alpha \) and \( \lambda \) are the renormalized coupling constant and the Yukawa couplings, respectively. The renormalization constants for the Yukawa couplings and for the other parameters can be expressed in terms of the renormalization constants defined in Eq. (6).

### 3. Finiteness of the triple gauge-ghost vertices

First, we prove that four three-point vertices (\( \bar{c} V c, \bar{c}^+ V c, \bar{c} V c^+, \text{ and } \bar{c}^+ V c^+ \)) with two ghost legs and a single leg of the quantum gauge superfield are finite in all orders [39]. All these vertices have the same renormalization constant \( Z_{\alpha}^{1/2} Z_c Z_V \), so that the above statement can be rewritten as²

\[ \frac{d}{d \ln \Lambda} (Z_{\alpha}^{-1/2} Z_c Z_V) = 0 . \quad (7) \]

Below we will demonstrate that in the general \( \xi \) gauge the two-point ghost Green function is divergent. This implies that all Green functions of the structure \( \bar{c} V^n c \) are also divergent for all \( n \neq 1 \).

To prove finiteness of the \( \bar{c} V c \)-vertices, we use the the Slavnov–Taylor identities for the three-point functions, which can be obtained using the standard methods [40, 41]. Taking into account that

\[ \frac{\delta^2 \Gamma}{\delta c^2 \delta c_x} = - \frac{D^2_c D^2_v}{16} G_c \delta^{xy} \delta_{AB} ; \quad \frac{\delta}{\delta c_x} \langle \delta V^B \rangle = - \varepsilon \cdot \frac{1}{4} G_c D^2 \delta^{xy} \delta_{AB} , \quad (8) \]

it is possible to derive the identity

\[ \varepsilon \cdot G_c (\partial^2 / \Lambda^2) \bar{D}^2_v \delta^{xy} \delta_{AB} - \varepsilon \cdot G_c (\partial^2 / \Lambda^2) \bar{D}^2_v \delta^{xy} \delta_{AB} = \frac{1}{2} G_c (\partial^2 / \Lambda^2) \bar{D}^2_v \delta^{xy} \delta_{AB} \langle \delta c^B \rangle = 0 . \quad (9) \]

Explicit expression for the Green function corresponding to the triple ghost-gauge vertex in this equation can be derived using dimensional and chirality considerations.

² In the one-loop approximation this has first been noted in the paper [27].
\[
\frac{\delta^3 \Gamma}{\delta \epsilon_x \delta V_y \delta \epsilon_c} = \frac{-ie_0}{16} f^{ABC} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \left( f(p,q) \partial^2 \Pi_{1/2} \right.
\]
\[
- F_{\mu}(p,q)(\gamma^\mu)_y^b \bar{D}^b D_b + F(p,q) \right) _y \left( \partial^2 \delta_{xg}(p+q) \bar{D}_y^2 \delta_{y2}(q) \right),
\]
where \( \partial^2 \Pi_{1/2} \) is the supersymmetric projection operator and \( \delta_{xg}(p) \equiv \delta^4(\theta_x - \theta_y) e^{ip_x(x^\mu - y^\mu)} \).

Introducing the chiral source \( \mathcal{J} \) and the source term
\[
- \frac{e_0}{2} \int d^4 x d^2 \theta f^{ABC} \mathcal{J}^A c^B c^C + \text{c.c.,}
\]
from dimensional and chirality considerations we obtain
\[
\frac{\delta^2}{\delta \epsilon_c \delta C} \left( \delta_{\epsilon_y} \right) = -ie \frac{\delta^3 \Gamma}{\delta \epsilon_x \delta \epsilon_c \delta \mathcal{J}_y} = - \frac{ie_0 e}{4} f^{BCD} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} H(p,q) \bar{D}_y^2 \delta_{xg}(q+D_w \delta_{y2}(q),
\]
where \( H(p,q) \) is the dimensionless function and, by construction, \( H(p,q) = H(p,-q-p) \).

Using explicit expressions for the Green functions, the Slavnov–Taylor identity (9) can be rewritten in the form
\[
G_c(q) F(q,p) + G_c(p) F(p,q) = 2G_c(q+p) H(-q-p). \tag{13}
\]

The function \( H(p,q) \) is finite. Really, it is contributed by diagrams in which one leg corresponds to the chiral source \( \mathcal{J} \) and two other legs correspond to chiral ghost superfields \( c \). Each of these diagrams contains the expression
\[
\int d^4 y d^2 \theta_y \mathcal{J}_y^A \cdot \bar{D}_y^2 \delta_{y1} \cdot \bar{D}_y^2 \delta_{y2} = -2 \int d^4 y d^4 \theta y \mathcal{J}_y^A \cdot \bar{D}_y^2 \delta_{y1} \cdot \bar{D}_y^2 \delta_{y2}. \tag{14}
\]
From this equation we see that all expressions for the considered superdiagrams can be presented as integrals over the total superspace, which include integration over
\[
\int d^4 \theta = -\frac{1}{2} \int d^2 \theta D^2 + \text{ total derivatives in the coordinate space.} \tag{15}
\]

Consequently, two left spinor derivatives act to the chiral external legs, so that the non-vanishing result can be obtained only if two right spinor derivatives also act to the external legs. This implies that the result should be proportional to, at least, second degree of the external momenta and is finite in the ultraviolet region. Thus, the function \( H(p,q) \) appears to be UV finite.

Multiplying the Slavnov–Taylor identity (9) to the renormalization constant \( Z_c \) and differentiating the result with respect to \( \ln \Lambda \), due to finiteness of \( (G_c)_R \) and \( H \) in the limit \( \Lambda \to \infty \) we obtain
\[
\left( (G_c)_R(q) \frac{d}{d \ln \Lambda} F(q,p) + (G_c)_R(p) \frac{d}{d \ln \Lambda} F(p,q) \right) \bigg|_{\Lambda \to \infty} = 0. \tag{16}
\]
For \( p = -q \) this equation gives
\[
\frac{d}{d \ln \Lambda} F(-q,q) \bigg|_{\Lambda \to \infty} = 0. \tag{17}
\]
Therefore, the corresponding renormalization constant is finite,
\[
\frac{d}{d \ln \Lambda} (Z_{\alpha}^{-1/2} Z_c Z_V) = 0, \tag{18}
\]
so that the function \( F(p,q) \) is also finite. This implies that all other three-point ghost-gauge vertices are finite.

4. One-loop verification of the finiteness of the \( V \bar{c}c \)-vertices

In the one-loop approximation the two-point ghost Green function is given by two superdiagrams presented in Fig. 1. The result for these diagrams (in the Euclidean space after the Wick rotation) can be written as

\[
G_c(p) = 1 + e_0^2 C_2 \int \frac{d^4 k}{(2\pi)^4} \left( \frac{\xi_0}{K_k} - \frac{1}{R_k} \right) \left( -\frac{1}{6k^4} + \frac{1}{2k^2(k+p)^2} - \frac{p^2}{2k^4(k+p)^2} \right) + O(e_0^4, \epsilon_0^2 \lambda_0^2), \tag{19}
\]

where \( R_k \equiv R(k^2/\Lambda) \) and \( K_k \equiv K(k^2/\Lambda^2) \). We see that this expression is divergent in the ultraviolet region (in the limit \( \Lambda \to \infty \)). The anomalous dimension of the Faddeev–Popov ghosts obtained from Eq. (19) is

\[
\gamma_c(\alpha_0, \lambda_0) = \left. \frac{d \ln G_c}{d \ln \Lambda} \right|_{p=0; \alpha, \lambda=\text{const}} = -\frac{\alpha_0 C_2(1 - \xi_0)}{6\pi} + O(\alpha_0^3, \alpha_0^2 \lambda_0^2). \tag{20}
\]

Thus, in the general \( \xi \)-gauge the two-point Green function of the Faddeev–Popov ghosts is divergent starting from the one-loop approximation.

\[
G_c(p) = 1 + e_0^2 C_2 \int \frac{d^4 k}{(2\pi)^4} \left( \frac{\xi_0}{K_k} - \frac{1}{R_k} \right) \left( -\frac{1}{6k^4} + \frac{1}{2k^2(k+p)^2} - \frac{p^2}{2k^4(k+p)^2} \right) + O(e_0^4, \epsilon_0^2 \lambda_0^2), \tag{19}
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\]

Thus, in the general \( \xi \)-gauge the two-point Green function of the Faddeev–Popov ghosts is divergent starting from the one-loop approximation.
The three-point gauge-ghost Green functions in the one-loop approximation are contributed by the superdiagrams presented in Fig. 2. Their sum has been found in [39]. It is convenient to write the results in the form

\[
\frac{i\varepsilon_0}{4} f_{ABC} \int d^4\theta \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} e^{A}(\theta, p + q) \left( f(p, q) \partial^2 \Pi_{1/2} V^B(\theta, -p) + F_{\mu}(p, q)(\gamma^\mu)_a^b D_b \bar{D}^a V^B(\theta, -p) + F(p, q) V^B(\theta, -p) \right) c^C(\theta, -q); \tag{21}
\]

\[
\frac{i\varepsilon_0}{4} f_{ABC} \int d^4\theta \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} e^{A}(\theta, p + q) \tilde{F}(p, q) V^B(\theta, -p) c^C(\theta, -q). \tag{22}
\]

Then, in the one-loop approximation the functions \(F\) and \(\tilde{F}\) are given by the following expressions:

\[
F(p, q) = 1 + \frac{\varepsilon_0^2 C_2}{4} \int \frac{d^4k}{(2\pi)^4} \left\{ -\frac{(q + p)^2}{K_k k^2(k + p)^2(k - q)^2} - \frac{\varepsilon_0 p^2}{K_k k^2(k + q)^2(k + q + p)^2} + \frac{\varepsilon_0 q^2}{K_k k^2(k + p)(k + q)^2} + \left( \frac{\varepsilon_0}{K_k} - \frac{1}{R_k} \right) - \frac{2(q + p)^2}{k^2(k + q + p)^2} + \frac{2}{k^2(k + q + p)^2} \right\} + O(\alpha_0^2, \alpha_0 \lambda_0^3). \tag{23}
\]

\[
\tilde{F}(p, q) = 1 - \frac{\varepsilon_0^2 C_2}{4} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{p^2}{R_k k^2(k + q)^2(k + q + p)^2} + \frac{\varepsilon_0 (q + p)^2}{K_k k^2(k - p)^2(k + q)^2} + \frac{\varepsilon_0 q^2}{K_k k^2(k + p)(k + q + p)^2} + \left( \frac{\varepsilon_0}{K_k} - \frac{1}{R_k} \right) \right\} + O(\alpha_0^2, \alpha_0 \lambda_0^3). \tag{24}
\]

![Figure 3. One-loop contribution to the function \(H(p, q)\).](image)

We see that they are finite in the ultraviolet region. Much more complicated one-loop expression for the functions \(f(p, q)\) and \(F_\mu(p, q)\) are presented in [39]. However, in the one-loop approximation their finiteness follows from simple dimensional considerations.

The finiteness of the function (23) and (24) confirms the general statement about the finiteness of the triple vertices with two ghost legs and one leg of the quantum gauge superfield. Note that the important ingredient of the general proof is the finiteness of the function \(H\) defined in Eq. (12). The one-loop contribution to this function is given by the diagram presented in Fig. 3 and can be written as
\[ H(p, q) = 1 - \frac{e_0^2 C_2}{4} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{p^2}{R_k k^2 (k + q)^2 (k + p)^2} + \frac{(q + p)^2}{k^4 (k + q + p)^2} \left( \frac{\xi_k}{K_k} - \frac{1}{R_k} \right) \right\} + O(e_0^4, e_0^2 \lambda_0^2). \]  

From this expression we see that the function \( H \) is really finite in the ultraviolet region and is really quadratic in the external momenta \( p \) and \( q \).

5. Finiteness of the triple ghost-gauge vertices and the NSVZ \( \beta \)-function

According to \([2, 3, 4, 5]\), in \( \mathcal{N} = 1 \) supersymmetric theories the \( \beta \)-function is related to the anomalous dimension of the matter superfields by the equation

\[ \beta(\alpha) = - \frac{\alpha^2 (3C_2 - T(R) + C(R)^i j \gamma_j^i (\alpha)/r)}{2\pi (1 - C_2\alpha/2\pi)}, \]  

which is called the exact NSVZ \( \beta \)-function. There are a lot of various arguments supporting correctness of this relation, based on instantons, anomalies, etc. For \( \mathcal{N} = 1 \) SQED the exact NSVZ \( \beta \)-function has been constructed in \([42, 43]\). In this simplest case the NSVZ relation was also obtained by summing supergraphs in the case of using the higher derivative regularization \([28, 29]\). Generalization of this result to the case of using the dimensional reduction is a complicated and so far unsolved problem \([44, 45]\).

Qualitatively, the existence of the relation between the \( \beta \)-function and the anomalous dimension of the matter superfields in the Abelian case can be explained as follows \([23]\). Let us consider a supergraph without external lines. If two external legs of the gauge superfield \( V \) are attached to this graph by all possible ways, then one obtains diagrams contributing to the \( \beta \)-function. For the other hand, cutting matter lines in the original graph gives diagrams contributing to the anomalous dimension of the matter superfields. The NSVZ equation relates these contributions if the theory is regularized by higher derivatives and the RG functions are defined in terms of the bare coupling constant.

In the non-Abelian case which is considered here the RG functions are defined in terms of the bare couplings by the equations

\[
\begin{align*}
\beta(\alpha_0, \lambda_0) &\equiv \frac{d\alpha_0}{d\ln \Lambda}; \\
(\gamma_i^j)(\alpha_0, \lambda_0) &\equiv -\frac{d\ln(Z_{\gamma}^i j (\alpha, \lambda, \Lambda/\mu))}{d\ln \Lambda}; \\
\gamma_V(\alpha_0, \lambda_0) &\equiv -\frac{d\ln(Z_V(\alpha, \lambda, \Lambda/\mu))}{d\ln \Lambda}; \\
\gamma_c(\alpha_0, \lambda_0) &\equiv -\frac{d\ln(Z_c(\alpha, \lambda, \Lambda/\mu))}{d\ln \Lambda},
\end{align*}
\]  

where the differentiation is made at fixed values of the couplings \( \alpha \) and \( \lambda^{ijk} \). It is well-known (see, e.g., \([33, 35]\)) that these RG functions are scheme independent at a fixed regularization, but depend on a regularization.

Let us equivalently present the NSVZ relation for the RG functions (27) in the form

\[ \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = - \frac{3C_2 - T(R) + C(R)^i j (\gamma_j^i (\alpha_0, \lambda_0))/r}{2\pi} + \frac{C_2}{2\pi} \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0}, \]  

and express the \( \beta \)-function in the right hand side in terms of the renormalization constant \( Z_\alpha \) using the equation
\[
\beta(\alpha_0, \lambda_0) = \frac{d\alpha_0(\alpha, \lambda, \Lambda/\mu)}{d\ln \Lambda}\bigg|_{\alpha, \lambda = \text{const}} = -\alpha_0 \frac{d\ln Z_\alpha}{d\ln \Lambda}\bigg|_{\alpha, \lambda = \text{const}}. \tag{29}
\]

Taking into account Eq. (18), this expression can be rewritten in terms of the Faddeev–Popov ghost anomalous dimension \(\gamma_c\) and the quantum gauge superfield anomalous dimension \(\gamma_V\) defined in terms of the bare couplings,

\[
\beta(\alpha_0, \lambda_0) = -2\alpha_0 \frac{d\ln(Z_c Z_V)}{d\ln \Lambda}\bigg|_{\alpha, \lambda = \text{const}} = 2\alpha_0 \left(\gamma_c(\alpha_0, \lambda_0) + \gamma_V(\alpha_0, \lambda_0)\right). \tag{30}
\]

Substituting this expression into the right hand side of Eq. (28) we present the NSVZ \(\beta\)-function in a different form,

\[
\frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi} \left(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0, \lambda_0) - 2C_2\gamma_V(\alpha_0, \lambda_0) + C(R)^{ij}_{jk}(\alpha_0, \lambda_0)/r\right). \tag{31}
\]

It is interesting to note that in this form of the NSVZ equation the matter superfields and the Faddeev–Popov ghosts similarly contribute to the right hand side. Moreover, Eq. (31) admits the same simple graphical interpretation as in the Abelian case. Really, if we consider a supergraph without external lines, then attaching two external legs of the background gauge superfield produces a set of diagrams contributing to the \(\beta\)-function, while cutting internal lines gives superdiagrams contributing to the anomalous dimensions of the Faddeev–Popov ghosts, of the quantum gauge superfield, and of the matter superfields, see Fig. 4.

![Figure 4](image_url)

**Figure 4.** This picture illustrates why the NSVZ \(\beta\)-function appears in the non-Abelian case.

### 6. NSVZ scheme in the non-Abelian case

In the Abelian case the RG functions defined in terms of the bare coupling constant satisfy the NSVZ relation independently of the renormalization prescription if the theory is regularized by higher covariant derivatives. However, if the RG functions are defined in the standard way, in terms of the renormalized coupling constant, they satisfy the NSVZ relation only in a certain subtraction scheme, which is called the NSVZ scheme. In the non-Abelian case the RG functions are defined in terms of the renormalized couplings by the equations

\[
\tilde{\beta}(\alpha, \lambda) \equiv \frac{d\alpha}{d\ln \mu}; \quad (\tilde{\gamma}_\phi)^{ij}_{jk}(\alpha, \lambda) \equiv \frac{d\ln(Z_\phi)^{ij}_{jk}(\alpha_0, \lambda_0, \Lambda/\mu)}{d\ln \mu}; \\
\tilde{\gamma}_V(\alpha, \lambda) \equiv \frac{d\ln Z_V(\alpha_0, \lambda_0, \Lambda/\mu)}{d\ln \mu}; \quad \tilde{\gamma}_c(\alpha, \lambda) \equiv \frac{d\ln Z_c(\alpha_0, \lambda_0, \Lambda/\mu)}{d\ln \mu}, \tag{32}
\]

where the differentiation is made at fixed values of \(\alpha_0\) and \(\lambda^{ijk}_0\). These RG functions are scheme and regularization dependent and satisfy the NSVZ relation only for a special renormalization
prescription. The general equations describing the scheme dependence of the NSVZ relation have been derived in [46, 35].

Let us assume that, similarly to the Abelian case, the RG functions defined in terms of the bare coupling constant satisfy the NSVZ relation (31) with the higher covariant derivative regularization. This is reasonable, because qualitative way of its derivation looks exactly as in the Abelian case. Similarly to [33, 34, 35] it is possible to prove that in the non-Abelian case the RG functions defined in terms of the bare couplings and the ones defined in terms of the renormalized couplings coincide if the renormalization constants satisfy the conditions

$$Z_\alpha(\alpha, \lambda, x_0) = 1; \quad (Z_\phi)_{ij}(\alpha, \lambda, x_0) = \delta_{ij}; \quad Z_c(\alpha, \lambda, x_0) = 1,$$

where $x_0$ is a certain fixed value of $\ln \Lambda/\mu$. Therefore, under the assumption that the NSVZ relation is valid for the RG functions defined in terms of the bare couplings with the higher derivative regularization, we see that the boundary conditions (33) give the NSVZ scheme. It is easy to see that for $x_0 = 0$ in this scheme only powers of $\ln \Lambda/\mu$ are included into the renormalization constants. This looks like the minimal subtractions (which are usually used within the dimensional technique), so that the NSVZ scheme constructed according to the above prescription (for $x_0 = 0$) can be called HDMS, i.e. the higher derivative regularization supplemented by minimal subtractions. Certainly, it is also necessary to require that the renormalization constants also satisfy the condition

$$Z_V = Z_\alpha^{1/2}Z_c^{-1},$$

under which the NSVZ relations (31) and (26) are equivalent.

Thus, the conditions (33) and (34) possibly give the NSVZ scheme with the higher covariant derivative regularization for non-Abelian gauge theories.

7. The NSVZ relation for the three-loop terms quartic in the Yukawa couplings

To verify the above results we consider the three-loop terms quartic in the Yukawa couplings [47]. They correspond to the graphs presented in Fig. 5. (In the three-loop approximation only these graphs give non-trivial contributions of the considered structure to the $\beta$-function.)

![Figure 5. The terms in the NSVZ relation which are investigated here are obtained from these two graphs.](attachment:image.png)

Attaching two external lines of the background gauge superfield we obtain a large number of two- and three-loop the diagrams contributing to the $\beta$-function, which are presented in Fig. 6. The corresponding diagrams for the anomalous dimension are obtained by cutting internal lines in the graphs presented in Fig. 5. They are shown in Fig. 7.

The calculation of the diagrams presented in Fig. 6 gives the following result for the corresponding part of the $\beta$-function defined in terms of the bare couplings:

$$\frac{\Delta \beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{2\pi}{\tau} C(R)^j_i \frac{d}{d \ln \Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \lambda_{ijmn}^{\lambda_0} \lambda_{ijmn}^{\alpha_0} \frac{\partial}{\partial \eta_{ij}} \frac{\partial}{\partial \eta^m} \left( k^2 F_k q^2 F_q (q + k)^2 F_{q+k} \right).$$
Figure 6. These superdiagrams give a part of the $\beta$-function corresponding to the supergraphs presented in Fig. 5.

$$+rac{4\pi}{r}C(R)^i j \frac{d}{d\ln \Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \left( \lambda_0^{ab} \lambda_0^{*ab} \lambda_0^{cd} \lambda_0^{*cd} \left( \frac{\partial}{\partial k^\mu} \frac{\partial}{\partial k^\nu} - \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\nu} \right) + 2\lambda_0^{iab} \lambda_0^{*jab} \times \left( \lambda_0^{cde} \lambda_0^{*cde} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\nu} \right) \right) \frac{1}{k^2 F_k^2 q^2 F_q (q + k)^2 F_{q+k}^2 l^2 F_l^2 (l + k)^2 F_{l+k}^2} = -\frac{1}{2\pi r} C(R)^i j \Delta \gamma^\phi(\lambda_0)^j i,$$ (35)

where $\Delta \gamma^\phi(\lambda_0)^j i$ is the part of the anomalous dimension (defined in terms of the bare couplings) corresponding to the diagrams presented in Fig. 7. From Eq. (35) we see that the considered contribution to the $\beta$-function is given by integrals of double total derivatives. This confirms the suggestion that it is a general feature of quantum corrections in supersymmetric theories. Moreover, according to Eq. (35) the NSVZ relation is satisfied for terms of the considered structure.

For the simplest regulator $F(k^2/\Lambda^2) = 1 + k^2/\Lambda^2$ it is possible to calculate all loop integrals. In this case

$$\Delta \gamma^\phi(\alpha_0, \lambda_0)^j i = \frac{1}{4\pi^2} \lambda_0^{iab} \lambda_0^{*jab} - \frac{1}{16\pi^4} \lambda_0^{iab} \lambda_0^{*jab} \lambda_0^{cde} \lambda_0^{*cde};$$

$$\frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi} \left( 3C_2 - T(R) \right) - \frac{1}{2\pi r} C(R)^i j \Delta \gamma^\phi(\lambda_0)^j i + O(\alpha_0) + O(\lambda_0^6).$$ (36)

Integrating the RG equations (27) one can find the renormalization constants,
(ln Zφ)j = \frac{1}{2\pi} \lambda_{ij} \lambda^*_{0j} (ln \frac{\Lambda}{\mu} + g_1) + \frac{1}{2\pi} \lambda_{ij} \lambda^*_{0j} \chi_{kcd} \Lambda^*_{0j} (ln \frac{\Lambda}{\mu} + 2g_1 ln \frac{\Lambda}{\mu} + 2g_2^2 - g_2) + O(\alpha_0) + O(\lambda_0); (37)

where g_1, b_1 etc. are arbitrary finite constants. Fixing them one fixes the subtraction scheme in the considered approximation. From these renormalization constants we construct the RG functions defined in terms of the renormalized couplings according to the prescription (32),

\frac{\beta}{\alpha^2} = -\frac{1}{2\pi} (3C_2 - T(R)) + \frac{1}{2\pi} \frac{C(R)_j}{\delta_j^2} \left(-\frac{1}{4\pi^2} \lambda_{ij} \chi_{l} + \frac{1}{16\pi^4} \lambda_{ij} \chi_{kcd} \Lambda^*_{0j} (b_2 - g_1) + \frac{1}{16\pi^2} \lambda_{ij} \chi_{l} \chi_{kcd} \Lambda^*_{0j} (1 + 2b_2 - 2g_1) \right) + O(\alpha) + O(\Lambda^6). (40)

Thus, the considered part of this \beta-function is scheme-dependent due to the presence of the finite constants b_2 and g_1. Their values can be fixed by imposing the conditions

Z_\phi(\alpha, \lambda, x_0) = J_i = \delta_i; \quad \alpha = \alpha_0 = 1, (41)

from which we obtain g_1 = b_1 = b_2 = -x_0, so that b_2 - g_1 = 0. (For x_0 = 0 all finite constants vanish and we obtain the HDMS scheme.) It is easy to verify that for b_2 - g_1 = 0 the NSVZ relation (31) is really valid for the RG functions defined in terms of the renormalized couplings,

\frac{\beta}{\alpha^2} = -\frac{1}{2\pi} (3C_2 - T(R)) + \frac{1}{2\pi} \frac{C(R)_j}{\delta_j^2} \left(-\frac{1}{4\pi^2} \lambda_{ij} \chi_{l} + \frac{1}{16\pi^4} \lambda_{ij} \chi_{kcd} \Lambda^*_{0j} \right) + O(\alpha) + O(\Lambda^6). (42)

In particular, we see that the NSVZ relation appears to be valid with the higher covariant derivative regularization supplemented by minimal subtractions.

8. Conclusion

We have demonstrated that for N = 1 SYM the three-point vertices with two ghost legs and a single leg of the quantum gauge superfield are finite. Consequently, the renormalization constants can be chosen so that Z_\alpha^{1/2} Z_\gamma Z_c = 1. This equation allows to rewrite the NSVZ \beta-function in a new form, in terms of the anomalous dimensions of the quantum gauge superfield, of the Faddeev–Popov ghosts, and of the matter superfields. This new form of the NSVZ equation has a simple qualitative interpretation, which is similar to the Abelian case. Under the assumption that the RG functions defined in terms of the bare couplings satisfy this equation in the case of using the higher covariant derivative regularization, the NSVZ scheme in the non-Abelian
The case appears to be the minimal subtraction scheme (HDMS), in which only powers of $\ln \Lambda/\mu$ are included into the renormalization constants. The non-trivial three-loop calculation for the terms quartic in the Yukawa couplings confirms the above proposal and this prescription for constructing the NSVZ scheme.

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