Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions

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Abstract. The symmetry algebra of asymptotically flat spacetimes at null infinity in three dimensions is the semi-direct sum of the infinitesimal diffeomorphisms on the circle with an abelian ideal of supertranslations. The associated charge algebra is shown to admit a non trivial classical central extension of Virasoro type closely related to that of the anti-de Sitter case.

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Central charge for $bms_3$

1. Introduction

Classical central extensions in the Poisson bracket algebra representation of asymptotic symmetries in general relativity have been originally discovered in the example of three dimensional, asymptotically anti-de Sitter spacetimes at spatial infinity [1]. In this case, the Poisson algebra of surface charges was shown to consist of two copies of the Virasoro algebra. This fact has been relevant in the context of the AdS$_3$/CFT$_2$ correspondence [2] and used to give a microscopical derivation of the Bekenstein-Hawking entropy for black holes with near horizon geometry that is locally AdS$_3$ [3]. The analysis of the asymptotic charge algebra has subsequently been performed in the context of asymptotically de Sitter spacetimes at timelike infinity [4] with results very similar to those obtained in the anti-de Sitter case.

For asymptotically flat spacetimes, the appropriate boundary from a conformal point of view is null infinity [5]. The asymptotic symmetry algebra has been derived a long time ago in four dimensions [6, 7, 8] and more recently by conformal methods [9] also in three dimensions [10].

The purpose of this letter is to complete the picture for classical central charges in three dimensions. We begin by computing the symmetry algebra $bms_n$ of asymptotically flat spacetimes at null infinity in $n$ dimensions, i.e., the $n$-dimensional analog of the four dimensional Bondi-Metzner-Sachs algebra, by solving the Killing equations to leading order.

In four dimensions, we make the obvious observation that the asymptotic symmetry algebra can be larger than the one originally discussed in [8] if the conformal transformations of the 2-sphere are not required to be globally well-defined. In three dimensions, we recover the known results [10]: $bms_3$ is the semi-direct sum of the infinitesimal diffeomorphisms on the circle with the abelian ideal of supertranslations.

In three dimensions, we then derive the space of allowed metrics by requiring (i) that $bms_3$ be the symmetry algebra for all allowed metrics, (ii) that the asymptotic symmetries leave the space of allowed metrics invariant, (iii) that the associated charges be linear and finite. As a new result, the associated Poisson algebra of charges is shown to be centrally extended. A non trivial central charge of Virasoro type with value $c = \frac{3}{4}$ appears between the Poisson brackets of the charges of the two summands. To conclude our analysis we point out that the centrally extended asymptotic charge algebras in flat and anti-de Sitter spacetimes are related in the same way than their exact counterparts [11]. Related recent work on holography in asymptotically flat spacetimes can be found for example in [12].

2. Surface charges

For pure Einstein gravity with a possibly non-vanishing cosmological constant $\Lambda$, we associate to a vector $\xi$ the following $n - 2$ forms in spacetime:

$$k_\xi[h, g] = -\delta_h k^K_\xi[g] - \xi \cdot \Theta[h, g] - k^S_{\xi g}[h, g],$$

where $\delta_h g_{\mu\nu} = h_{\mu\nu}$, $(d^{n-2}x)_{\mu_1...\mu_p} \equiv \frac{1}{p!(n-p)!} \epsilon_{\mu_1...\mu_n} dx_{\mu_1+1}...dx_{\mu_n}$. The first two terms of (2.1) are expressed in the form derived using covariant phase space methods [13, 14].

$$k^K_\xi[g] = \frac{\sqrt{-g}}{16\pi G} (D^\mu \xi^\nu - D^\nu \xi^\mu)(d^{n-2}x)_{\mu\nu},$$

$$\Theta[h, g] = \frac{\sqrt{-g}}{16\pi G} (g^{\alpha\alpha} D^\beta h_{\alpha\beta} - g^{\alpha\beta} D^\mu h_{\alpha\beta})(d^{n-1}x)_{\mu}.$$  

The supplementary term

$$k^S_{\xi g}[h, g] = \frac{\sqrt{-g}}{16\pi G} \frac{1}{2} g^{\alpha\alpha} h_{\alpha\beta} (D^\beta \xi^\nu + D^\nu \xi^\beta) - (\mu \leftrightarrow \nu))(d^{n-2}x)_{\mu\nu},$$

vanishes for exact Killing vectors of $g$, but not necessarily for asymptotic ones. Expression (2.1) has been derived using cohomological methods in [14] and coincides with the one originally derived in the context of four dimensional asymptotically anti-de Sitter spacetimes [16].
By construction, the $n - 2$ form $k_\xi$ is closed, $dk_\xi = 0$, when $h$ satisfies the equations of motion, for the equations linearized around $g$ and when $\xi$ is a Killing vector of $g$. Under these conditions, its integral

$$ \oint_S k_\xi[h, g] $$

over a closed $n - 2$ dimensional surface only depends on the homology class of this surface. In the context of linearized gravity around a solution $g$, one can furthermore prove

- for $n \geq 3$, that all $n - 2$ forms that are closed for all solutions $h$ of the linearized theory are given, on solutions $h$, by $k_\xi$ for some Killing vector $\xi$, up to $d$ exact $n - 2$ forms which do not contribute to the integrals [15] [17] [18],

- if $\xi_1, \xi_2$ are Killing vectors of the solution $g$ and $h$ is a solution to the Einstein equations linearized around $g$, then so is $\mathcal{L}_\xi h$ and [19]

$$ \{\delta Q_{\xi_1}, \delta Q_{\xi_2}\}[h, g] := -\delta Q_{\xi_2}[\mathcal{L}_\xi, h, g] = \delta Q_{[\xi_1, \xi_2]}[h, g]. $$

- on solutions and for Killing vectors $\xi$ of $g$ with $S$ chosen in the $t$ constant hyperplane, $\delta Q_{\xi}[h, g]$ coincides with the values computed in the context of the Hamiltonian formalism [20].

The question is then how to use these exact results to get a theory of asymptotic symmetries and charges in full general relativity for some chosen boundary surface $S^\infty$ and background solution $\bar{g}$. This involves two related aspects: finding the boundary conditions on the fields and the asymptotic symmetry algebra. One possibility is to impose boundary conditions such that, at the boundary, the theory of charges for suitably defined asymptotic Killing vectors is controlled by the linearized theory.

Even though these generalizations enlarge the space of allowed metrics, we do not expect them to affect the conclusions concerning the central charge below. This is the reason why we loosely refer to this situation as “asymptotic linearity” (cf. [19] for details). In that case, one simply has

$$ \oint_{S^\infty} k_\xi[h, \bar{g}] = \oint_{S^\infty} k_\xi[h, \bar{g}] + N_\xi[\bar{g}]. $$

While this approach trivially guarantees integrability of the charges, it requires strong fall-off conditions that can be too restrictive and eliminate solutions of interest. Indeed, these fall-off conditions impose that the individual terms in the charges are finite, while in general it may be necessary to allow for cancellation of infinite contributions from different terms and for non linear corrections to the charges (see e.g. [21] [22]). Even though these generalizations enlarge the space of allowed metrics, we do not expect them to affect the conclusions concerning the central charge below. This is the reason why we restrict ourselves at this stage to the linear approach.

In the asymptotic linear case, the Poisson bracket of the charges is given by $\{Q_{\xi_1}, Q_{\xi_2}\}[g^f, \bar{g}] = -\delta_{\xi_1}^f Q_{\xi_2}[g^f, \bar{g}]$, with $\delta_{\xi} g_{\mu \nu}^f = \mathcal{L}_\xi g_{\mu \nu}^f$ and

$$ \{Q_{[\xi_1, \xi_2]}, g^f, \bar{g}\} = Q_{[\xi_1, \xi_2]}[g^f, \bar{g}] + \mathcal{K}_{\xi_1, \xi_2}[\bar{g}] = N_{[\xi_1, \xi_2]}[\bar{g}] = 0. $$

It then follows that the expression

$$ \mathcal{K}_{\xi_1, \xi_2}[\bar{g}] = \oint_{S^\infty} k_{\xi_1}[\mathcal{L}_{\xi_2} \bar{g}, \bar{g}] $$

defines a 2-cocycle on the Lie algebra of asymptotic Killing vector fields $\xi$, 

$$ \mathcal{K}_{\xi_1, \xi_2}[\bar{g}] = -\mathcal{K}_{\xi_2, \xi_1}[\bar{g}], \quad \mathcal{K}_{[\xi_1, \xi_2], \xi_3}[\bar{g}] + \mathcal{K}_{[\xi_2, \xi_3], \xi_1}[\bar{g}] + \mathcal{K}_{[\xi_3, \xi_1], \xi_2}[\bar{g}] = 0. $$
For Einstein gravity, the explicit formula for the central charge follows from \( bms \) and is given by
\[
\mathcal{K}_{\xi,\xi'}[\bar{g}] = \frac{1}{16\pi G} \int_{\partial S^2} (\sigma^\nu) \mu_\nu \sqrt{-\sigma} \left( -2 \bar{D}_\sigma \xi^\sigma \bar{D}^\nu \xi'_\nu + 2 \bar{D}_\sigma \xi'^\sigma \bar{D}^\nu \xi^\nu + 4 \bar{D}_\sigma \xi^\nu \bar{D}^\nu \xi'^\sigma \right)
\]
\[
+ \frac{8a}{2 - n} \bar{\xi}^\nu \xi'_\nu - 2 \bar{R}^\mu_\nu \sigma_\rho \xi'^\rho + (\bar{D}^\sigma \xi'^\nu + \bar{D}^\nu \xi'^\sigma)(\bar{D}^\mu \xi_\sigma + \bar{D}^\sigma \xi^\mu)
\]
(2.12)
Note that the central charge vanishes if either \( \xi \) or \( \xi' \) is an exact Killing vector of \( \bar{g} \). Because (2.11) can be proved for all vector fields \( \xi_1, \xi_2 \) and solutions \( g \) of Einstein’s equations, it follows from the definition of the Poisson bracket that if either \( \xi \) or \( \xi' \) is in addition an exact Killing vector of \( g' \) then \( Q_{\xi,\xi'}[g', \bar{g}] = 0 \) by choosing the normalization to vanish.

3. The \( bms_n \) algebra

Introducing the retarded time \( u = t - r \), the luminosity distance \( r \) and angles \( \theta^A \) on the \( n - 2 \) sphere by \( x^1 = r \cos \theta^1, x^A = r \sin \theta^1 \cdots \sin \theta^{A-1} \cos \theta^A \), for \( A = 2, \ldots, n - 2 \), and \( x^{n-1} = r \sin \theta^1 \cdots \sin \theta^{n-2} \), the Minkowski metric is given by
\[
ds^2 = -du^2 - 2dudr + r^2 \sum_{A=1}^{n-2} s_A(d\theta^A)^2,
\]
(3.1)
where \( s_1 = 1, s_A = \sin^2 \theta^1 \cdots \sin^2 \theta^{A-1} \) for \( 2 \leq A \leq n - 2 \). The (future) null boundary is defined by \( r = \text{constant} \to \infty \) with \( u, \theta^A \) held fixed.

We require asymptotic Killing vectors to satisfy the Killing equation to leading order. They have the form \( \xi^\mu = \chi^\mu(\theta^A) + o(\chi^\mu) \) for some fall-offs \( \chi^\mu(r) \) to be determined. Here, round brackets around a single index mean that the summation convention is suspended. For such vectors, \( \mathcal{L}_\xi \bar{g}_{\mu\nu} = O(\rho_{\mu\nu}) \). Solving the Killing equation to leading order means finding the highest orders \( \chi^\mu(r) \) in \( r \) such that equation
\[
\mathcal{L}_\xi \bar{g}_{\mu\nu} = o(\rho_{\mu\nu}),
\]
(3.2)
admits non-vanishing \( \chi^\mu(u, \theta) \) as solutions. After a straightforward computation (summarized in Appendix A), one finds
\[
\xi^\mu = T(\theta^A) + u\partial_1 Y^1(\theta^A) + o(r^0), \quad \xi^\nu = -r\partial_1 Y^1(\theta^A) + o(r),
\]
(3.3)
where \( T(\theta^A) \) is an arbitrary function on the \( n - 2 \) sphere, and \( Y^A(\theta^A) \) are the components of the conformal Killing vectors on the \( n - 2 \) sphere. These asymptotic Killing vectors form a sub-algebra of the Lie algebra of vector fields and the bracket induced by the Lie bracket \( [\xi, \xi'] \) is determined by
\[
\begin{align*}
\dot{T} &= Y^B \partial_B Y^A - Y'^B \partial_B Y'^A, \\
\dot{Y}^A &= Y^B \partial_B Y^A - Y'^B \partial_B Y'^A.
\end{align*}
\]
(3.4)
(3.5)
It follows that asymptotic Killing vectors with \( T = 0 = Y^A \) form an ideal in the algebra of asymptotic Killing vectors. The quotient algebra is defined to be \( bms_n \). It is the semi-direct sum of the conformal Killing vectors \( Y^A \) of Euclidean \( n - 2 \) dimensional space with an abelian ideal of so-called infinitesimal supertranslations. Note that the exact Killing vectors of \( \bar{g}, \xi_\mu = a_\mu + b_{\mu\nu} x^\nu \) give rise to
\[
Y^A_E = \frac{1}{s(A)} (b_{[i|j]} + b_{[ij]} \frac{2}{r}) \frac{1}{r} \partial_x^i, \quad T_E = -[a_0 + a_i \frac{x^i}{r}],
\]
(3.6)
and belong to \( bms_n \), so that \( \text{iso}(n - 1, 1) \) is a subalgebra of \( bms_n \).

\( \dagger \) This expression differs from the one derived in [19] by an overall sign because we have changed the sign convention for the charges and also by the fact that we use here the Misner-Thorne-Wheeler convention for the Riemann tensor. See also [23] for similar expressions.
In order to make contact with conformal methods, we just note that if $\bar{g}_{\mu\nu} = r^{-2}g_{\mu\nu}$ is the metric induced at the boundary $r$ constant,
\[
ds^2 = -\frac{1}{r^2} du^2 + \sum_{A=1}^{n-2} s_A(d\theta^A)^2.
\]
(3.7)

one can easily verify that $\mathfrak{bms}_3$ is isomorphic to the Lie algebra of conformal Killing vectors of the boundary metric (3.7), in the limit $r \to \infty$.

For $n > 4$, the asymptotic algebra contains the infinitesimal supertranslations parameterized by $T(\theta^A)$ and the $n(n-1)/2$ dimensional conformal algebra of Euclidean space $\mathfrak{so}(n-1,1)$ in $n-2$ dimensions, isomorphic to the Lorentz algebra in $n$ dimensions.

In four dimensions, the conformal algebra of the 2-sphere is infinite-dimensional and contains the Lorentz algebra $\mathfrak{so}(3,1)$ as a subalgebra. It would of course be interesting to analyze whether central extensions arise in the charge algebra representation of $\mathfrak{bms}_4$, but we will not do so here. Note that in the original discussion [5], the transformations were required to be well-defined on the 2-sphere and $\mathfrak{bms}_4$ was restricted to the semi-direct sum of $\mathfrak{so}(3,1)$ with the infinitesimal supertranslations. In this case, there are no non trivial central extensions, see e.g. [24].

In three dimensions, the conformal Killing equation on the circle imposes no restrictions on the function $Y(\theta)$. Therefore, $\mathfrak{bms}_3$ is characterized by 2 arbitrary functions $T(\theta), Y(\theta)$ on the circle. These functions can be Fourier analyzed by defining $P_n = \xi(T = \exp(in\theta), Y = 0)$ and $J_n = \xi(T = 0, Y = \exp(in\theta))$. In terms of these generators, the commutation relations of $\mathfrak{bms}_3$ become
\[
i[J_m, J_n] = (m - n)J_{m+n}, \quad i[P_m, P_n] = 0, \quad i[J_m, P_n] = (m - n)P_{m+n}.
\]
(3.8)

In other words, the 6 dimensional Poincaré algebra $\mathfrak{iso}(2,1)$ of 3 dimensional Minkowski spacetime is enhanced to the semi-direct sum of the infinitesimal diffeomorphisms on the circle with the infinitesimal supertranslations.

4. Charge algebra representation of $\mathfrak{bms}_3$

In order to determine the Poisson algebra representation of $\mathfrak{bms}_3$, we need to specify the boundary conditions on the metric, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} = \chi_{\mu\nu}(r)$ are fall-offs to be determined, and also subleading terms in the asymptotic Killing vectors.

If we want the asymptotic symmetry algebra to be the same for all allowed metrics, we need to require that solving the Killing equation to leading order for $g$ in place of $\bar{g}$ will lead to the $\xi^\mu$ given in (3.3). We will also need $\mathcal{L}_\xi g_{\mu\nu} = O(\chi_{\mu\nu})$ so that the asymptotic symmetry $\mathcal{L}_\xi$ leaves the space of allowed metrics invariant. These conditions are satisfied for metric deviations of the form
\[
h_{uu} = O(1), \quad h_{ur} = O(r^{-1}), \quad h_{u\theta} = O(1),
\]
\[
h_{rr} = O(r^{-2}), \quad h_{r\theta} = O(1), \quad h_{\theta\theta} = O(r),
\]
and asymptotic Killing vectors defined by
\[
\xi^u = T(\theta) + u\partial_\theta Y(\theta) + O(r^{-1}), \quad \xi^r = -r\partial_\theta Y(\theta) + O(r^0),
\]
\[
\xi^\theta = Y(\theta) + \frac{n}{r}\partial_\theta \bar{Y}(\theta) + \frac{1}{r}f^\theta_{sub}(\theta) + O(r^{-2}),
\]
(4.2)

where $f^\theta_{sub}(\theta)$ is an arbitrary function. In addition, the associated charges are finite and linear in the sense of condition [27] for metric deviations of the form
\[
h_{uu} = O(1), \quad h_{ur} = O(r^{-1}), \quad h_{u\theta} = O(1),
\]
\[
h_{rr} = O(r^{-2}), \quad h_{r\theta} = h_1(\theta) + O(r^{-1}), \quad h_{\theta\theta} = h_2(\theta)r + O(r^0).
\]
(4.3)

These boundary conditions contain for example the metric
\[
ds^2 = -(1 - 4m)^2 du^2 - 2udu + 8J(1 - 4m)du \theta \frac{8J}{1 - 4m} d\theta d\theta + (r^2 - 16J^2) d\theta^2.
\]
(4.4)
which describes a spinning particle in Minkowski spacetime [25]. The space of allowed metrics also contains the dimensional reduction of the Einstein-Rosen waves from four to three dimensions [26], for which the metric deviation at infinity in a suitable coordinate system is given by \( h_{uu} = O(1) \), \( h_{ur} = O(r^{-1}) \), the others zero.

The charges (2.8) reduce to

\[
Q_\xi[g^f, g] = \frac{1}{16\pi G} \int_0^{2\pi} d\theta \left( h_{uu} T + (2h_{u\theta} - u\partial_\theta h_{uu} + 2\partial_\theta h_{.hh} + \partial_\theta h_{h1} + \partial_\theta h_{h2} + r\partial_\theta h_{r\theta} - r\partial_\theta h_{ur}) Y \right),
\]

where the charge of the background has been set to zero. These charges obviously vanish for trivial asymptotic Killing vectors. One can also verify that when the metric deviations satisfy the linearized Einstein equations these charges are conserved, i.e., independent of \( u \). Finally, we note that \( \int_0^{2\pi} d\theta k_\xi^S[h, g] = 0 \) for \( \xi, h \) satisfying (4.2) respectively (4.3) so that the charges agree with those defined in [13].

The expression (4.3) allows us to compute the central extension of the Poisson algebra representation of \( \mathfrak{bms}_3 \) by replacing \( h_{\mu\nu} \) by \( \mathcal{L}^\xi \mathcal{g}_{\mu\nu} \) with \( \xi \) given in (3.3). The result is

\[
\mathcal{K}_{\xi, \xi'} = \frac{1}{8\pi G} \int_0^{2\pi} d\theta \left[ \partial_\theta Y^\theta (\partial_\theta Y^\theta T') - \partial_\theta Y^\theta (\partial_\theta Y^\theta T) \right].
\]

Using algebra (3.5), cocyle condition (2.11) can be explicitly checked. In terms of the generators \( Q_{\mathcal{P}_n} = \mathcal{P}_n, Q_{\mathcal{J}_n} = \mathcal{J}_n \), we get the centrally extended algebra

\[
i \{ \mathcal{J}_m, \mathcal{J}_n \} = (m-n)\mathcal{J}_{m+n}, \quad i \{ \mathcal{P}_m, \mathcal{P}_n \} = 0, \\
i \{ \mathcal{J}_m, \mathcal{P}_n \} = (m-n)\mathcal{P}_{m+n} + \frac{1}{4G} m(m^2 - 1) \delta_{n+m,0}.
\]

(4.7)

It can easily be shown to be non-trivial in the sense that it cannot be absorbed into a redefinition of the generators. Only the commutators of generators involving either \( \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_{-1} \) or \( \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1} \) corresponding to the exact Killing vectors of the Poincaré algebra \( \mathfrak{iso}(2,1) \) are free of central extensions.

The algebra (4.14) has many features in common with the anti-de Sitter case: it has the same number of generators, and a Virasoro type central charge. In fact, these algebras are related in the same way than their exact counterparts [11]: if one introduces the negative cosmological constant \( \Lambda = -\frac{1}{l^2} \) and considers

\[
i \{ \mathcal{J}_m, \mathcal{J}_n \} = (m-n)\mathcal{J}_{m+n}, \quad i \{ \mathcal{P}_m, \mathcal{P}_n \} = \frac{1}{l^2} (m-n)\mathcal{J}_{m+n}, \quad i \{ \mathcal{J}_m, \mathcal{P}_n \} = (m-n)\mathcal{P}_{m+n},
\]

(4.8)

the \( \mathfrak{bms}_3 \) algebra (3.3) corresponds to the case \( l \rightarrow \infty \). For finite \( l \), the charges \( \mathcal{L}_m^\pm \) corresponding to the generators \( L_\pm^m = \frac{1}{l^2} (lP_m \pm J_m^-) \) form the standard two copies of the Virasoro algebra,

\[
i \{ \mathcal{L}_m^-, \mathcal{L}_n^- \} = (m-n)\mathcal{L}_{m+n}^-, \quad \frac{c}{12} m(m^2 - 1) \delta_{n+m,0}, \quad \{ \mathcal{L}_m^-, \mathcal{L}_n^+ \} = 0,
\]

(4.9)

where \( c = \frac{2l}{l^2} \) is the central charge for the anti-de Sitter case.

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Appendix A. Explicit computation of the \( \mathfrak{bms}_n \) algebra

Introducing the notation \( \hat{\xi}^u = U(u, \theta^A), \hat{\xi}^r = R(u, \theta^A), \hat{\xi}^A = Y^A(u, \theta^B) \), the \( rr \)-component of equation (3.2) reduces to

\[
-2U\partial_r \chi^u + \partial_r o(\chi^u) = o(\rho_{rr}).
\]

(A.1)
The last two equations allow one to identify in \( n \) dimensions with metric \( g_{\mu\nu} \), \( R \equiv \nabla^2 \). The \( ur \)-component of equation (3.2) then gives
\[
- \partial_u U + o(r^0) - \partial_r \chi' R + \partial_r o(r^r) = o(\rho_{ur}).
\]
This leads to \( \chi' = r \), \( R + \partial_u U = 0 \) and \( \rho_{ur} = r^0 \). The \( uu \)-component of equation (3.2) reduces to
\[
-2r \partial_u R + o(r) = o(\rho_{uu}).
\]
It imposes \( \partial_u R = 0 \) and gives \( \rho_{uu} = r \). From the \( ra \) component,
\[
- \partial_a U + o(r^0) + \partial_r \chi A r^A r^2 s_A + r^2 \partial_r o(\chi^A) = o(\rho_r A),
\]
we get \( \chi^A = r^0 \) and \( \rho_r A = r \). The \( uA \)-component of equation (3.2) is
\[
 r^2 \partial_u Y^A + r^2 o(r^0) + r \partial_A R + o(r^1) = o(\rho_{uA}),
\]
implying \( \partial_u Y^A = 0 \), and \( \rho_{uA} = r^2 \). Finally, the \( AA \) and \( AB \) with \( A \neq B \) components of equation (3.2) are given by
\[
2 r^2 R s_A + 2 r^2 \partial(A) Y^A s_{(A)} + 2 Y^C C s_A + o(r^2) = o(\rho_{AA}),
\]
\[
 r^2 \partial_A Y^A s_{(A)} + r^2 \partial_A Y^B s_{(B)} + o(r^2) = o(\rho_{AB}).
\]
One finds the following conditions
\[
\partial_u Y^A = 0, \quad R + \partial_u Y^A + \sum_{C < A} Y_C \cot \theta^C = 0, \quad \partial_B Y^A s_{(A)} + \partial_A Y^B s_{(B)} = 0,
\]
with \( \rho_{AA} = r^2 = \rho_{AB} \). The constraints imposed by (3.2) on \( U, R \) and \( Y^A \) are summarized by
\[
 R = -\partial_1 Y^1, \quad \partial_a U = \partial_1 Y^1, \quad \partial_u \partial_1 U = 0, \quad \partial_u Y^A = 0,
\]
\[
 \partial_1 Y^1 = \partial(A) Y^A + \sum_{B < A} \cot \theta^B Y^B, \quad \forall A,
\]
\[
 \partial_A Y^B s_{(B)} + \partial_B Y^A s_{(A)} = 0, \quad A \neq B, \quad A, B = 1, \ldots, n - 2.
\]
The last two equations allow one to identify \( Y^A(\theta^B) \) with the conformal Killing vectors of the sphere in \( n - 2 \) dimensions with metric \( g_{AB}^{(n-2)} = \delta_{AB}s_A \).

References

[1] J. D. Brown and M. Henneaux, “Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity,” Commun. Math. Phys. 104 (1986) 207.
[2] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183–386, hep-th/9905111.
[3] A. Strominger, “Black hole entropy from near-horizon microstates,” JHEP 02 (1998) 009, hep-th/9712251.
[4] A. Strominger, “The dS/CFT correspondence,” JHEP 10 (2001) 034, hep-th/0106113.
[5] E. Witten, “Quantum gravity in de Sitter space,” hep-th/0106109.
[6] H. Bondi, M. G. van der Burg, and A. W. Metzner, “Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems,” Proc. Roy. Soc. Lond. A 269 (1962) 21.
[7] R. K. Sachs, “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times,” Proc. Roy. Soc. Lond. A 270 (1962) 103.
[8] R. K. Sachs, “Asymptotic symmetries in gravitational theory,” Phys. Rev. 128 (1962) 2851.
[9] R. Penrose, “Asymptotic properties of fields and space-times,” Phys. Rev. Lett. 10 (1963) 66–68.; R. Penrose, “Zero rest mass fields including gravitation: Asymptotic behavior,” Proc. Roy. Soc. Lond. A 284 (1965) 159.; R. Geroch, “Asymptotic structure of space-time,” in Symposium on the asymptotic structure of space-time eds., Plenum: NY.
Central charge for $bms_3$.

[10] A. Ashtekar, J. Bicak, and B. G. Schmidt, “Asymptotic structure of symmetry reduced general relativity,” Phys. Rev. D55 (1997) 669–686. [gr-qc/9608042]

[11] E. Witten, “(2+1)-dimensional gravity as an exactly soluble system,” Nucl. Phys. B311 (1988) 46.

[12] J. de Boer and S. N. Solodukhin, “A holographic reduction of minkowski space-time,” Nucl. Phys. B665 (2003) 545–593. [hep-th/0303006]; G. Arcioni and C. Dappiaggi, “Exploring the holographic principle in asymptotically flat spacetimes via the BMS group,” Nucl. Phys. B674 (2003) 553–592. [hep-th/0306142]; G. Arcioni and C. Dappiaggi, “Holography in asymptotically flat space-times and the BMS group,” Class. Quant. Grav. 21 (2004) 5655. [hep-th/0312186]; C. Dappiaggi, V. Moretti, and N. Pinamonti, “Rigorous steps towards holography in asymptotically flat spacetimes,” Rev. Math. Phys. B665 (2003) 545–593. [hep-th/0306142]; G. Arcioni and C. Dappiaggi, “Holography in asymptotically flat spacetimes and the BMS group,” Class. Quant. Grav. 21 (2004) 5655. [hep-th/0312186]; D. Astefanesei and E. Radu, “Quasilocal formalism and black ring thermodynamics,” Phys. Rev. D73 (2006) 044014. [hep-th/0509144]; R. B. Mann and D. Marolf, “Holographic renormalization of asymptotically flat spacetimes,” Class. Quant. Grav. 23 (2006) 2927–2950. [hep-th/0511096]; D. Astefanesei, R. B. Mann, and C. Stelea, “Note on counterterms in asymptotically flat spacetimes,” hep-th/0608037.

[13] V. Iyer and R. M. Wald, “Some properties of Noether charge and a proposal for dynamical black hole entropy,” Phys. Rev. D50 (1994) 846–864. [gr-qc/9403028]

[14] R. M. Wald and A. Zoupas, “A general definition of conserved quantities in general relativity and other theories of gravity,” Phys. Rev. D61 (2000) 084027. [gr-qc/9911095]

[15] G. Barnich, F. Brandt, and M. Henneaux, “Local BRST cohomology in the antifield formalism. I. General theorems,” Commun. Math. Phys. 174 (1995) 57–92. [hep-th/9405109]

[16] L. F. Abbott and S. Deser, “Stability of gravity with a cosmological constant,” Nucl. Phys. B195 (1982) 76.

[17] I. M. Anderson and C. G. Torre, “Asymptotic conservation laws in field theory,” Phys. Rev. Lett. 77 (1996) 4109–4113. [hep-th/9608008]

[18] G. Barnich, S. Leclercq, and P. Spindel, “Classification of surface charges for a spin 2 field on a curved background solution,” [gr-qc/0404006]

[19] G. Barnich and F. Brandt, “Covariant theory of asymptotic symmetries, conservation laws and central charges,” Nucl. Phys. B633 (2002) 3–82. [hep-th/0111246]

[20] T. Regge and C. Teitelboim, “Role of surface integrals in the Hamiltonian formulation of general relativity,” Ann. Phys. 88 (1974) 286.

[21] M. Henneaux, C. Martínez, R. Troncoso, and J. Zanelli, “Black holes and asymptotics of 2+1 gravity coupled to a scalar field,” Phys. Rev. D65 (2002) 104007. [hep-th/0201170]

[22] M. Henneaux, C. Martínez, R. Troncoso, and J. Zanelli, “Asymptotically anti-de Sitter spacetimes and scalar fields with a logarithmic branch,” [hep-th/0404236]

[23] J.-I. Koga, “Asymptotic symmetries on killing horizons,” Phys. Rev. D64 (2001) 124012. [gr-qc/0107096]; S. Silva, “Black hole entropy and thermodynamics from symmetries,” Class. Quant. Grav. 19 (2002) 3947–3962. [hep-th/0204179]; M. Blagojevic and M. Vasilic, “On the classical central charge,” Class. Quant. Grav. 22 (2005) 3891–3910. [hep-th/0410111]

[24] P. McCarthy, “Lifting of projective representations of the Bondi-Metzner-Sachs group,” Proc. R. Soc. London A358 (1978) 141–171.

[25] S. Deser, R. Jackiw, and G. ’t Hooft, “Three-dimensional Einstein gravity: Dynamics of flat space,” Ann. Phys. 152 (1984) 220.

[26] A. Ashtekar, J. Bicak, and B. G. Schmidt, “Behavior of Einstein-Rosen waves at null infinity,” Phys. Rev. D55 (1997) 687–694. [gr-qc/9608041]