Numerical simulation of the electron capture process in a magnetar interior

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Abstract

In a superhigh magnetic field, direct Urca reactions can proceed for an arbitrary proton concentration. Since only the electrons with high energy $E > Q$, $Q$ is the threshold energy of inverse $\beta$-decay) at large Landau levels can be captured, we introduce the Landau level effect coefficient $q$ and the effective electron capture rate $\Gamma_{\text{eff}}$. By using $\Gamma_{\text{eff}}$, the values of $L_X$ and $L_\nu$ are calculated, where and $L_\nu$, $L_X$ are the average neutrino luminosity of Anomalous X-ray Pulsars (AXPs) and the average X-ray luminosity of AXPs, respectively. The complete process of electron capture inside a magnetar is simulated numerically.

Keywords Magnetar Superhigh magnetic fields Electron capture rate

1 Introduction

Anomalous X-ray Pulsars (AXPs) and soft gamma-ray repeaters (SGRs) are a small group of peculiar neutron stars (NSs) that are currently believed to have superhigh magnetic fields $B \sim 10^{14} - 10^{15}$ G, and are hence identified as magnetar candidates (Duncan & Thompson 1992; Kouveliotou et al 1998, 1999; Paczynski 1992; Thompson & Duncan 1993, 1996). Based on the convective dynamo mechanism, some works have focused on the origin of the internal magnetic fields of magnetars (Thompson & Duncan 1995, 1996). Magnetars are considered to be NSs and might reasonably be expected to have very small radii $R \sim 10^6$ cm, although measured masses of these stars are about $1.4M_\odot$, where $M_\odot$ denotes the solar mass $\sim 2 \times 10^{33}$ g.

In a common magnetar, the typical internal temperature is $\sim 10^8$ K as estimated using the standard cooling mechanism (Bachcall & Wolf 1965; Yakovlev et al. 2001, 2000) and its intrinsic superhigh magnetic fields may be produced by the induced magnetic moments of the $^3P_2$ Cooper pairs in an anisotropic neutron superfluid at a moderate lower interior temperatures ($\sim 2.38 \times 10^8$ K)(Peng & Lai 2007, Peng & Tong 2007). Observations show that some SGRs and AXPs have a thermal type X-ray flux, with the magnitude of average X-ray luminosity $L_X \sim 1.0 \times 10^{34} - 5 \times 10^{36}$ erg s$^{-1}$. A NS can be treated as a system of magnetic dipoles ($B = B_p$, where $B_p$ is the polar magnetic field strength) because of the existence of a $^3P_2$ neutron superfluid in its interior. In the presence of a background magnetic field, the dipoles tend to become aligned in the same direction, which would give rise to a phase transition from paramagnetism to ferromagnetism in the interior of a NS if the temperature drops below a critical temperature. Superhigh magnetic fields of magnetars may originate from this phase transition and the maximum field strength is about $(3.0 - 4.0) \times 10^{15}$ G (Peng & Tong 2007).

When $B \gg B_{\text{cr}}$, direct Urca reactions occur rapidly for an arbitrary proton concentration due to the fact that strong magnetic fields can alter matter compo-
sitions and increase phase space for protons, with a resulting increase in $Y_e$, where $Y_e$ is the mean electron number per baryon and $B_{cr}$ is the quantum critical magnetic field (Lai & Shapiro 1991). After entering the neutrino cooling epoch, the direct Urca reaction is the simplest neutrino emission process (Gamov & Schoenberg 1941; Pethick 1992). In the interior of a magnetar, where matter is assumed to be totally transparent to neutrinos and antineutrinos, the simple decay of neutrons and successive electron capture take place simultaneously, as required by charge neutrality. In order to provide a detailed view of the actual evolutionary scenario of a magnetar, we should take into consideration the effective electron capture rate $\Gamma_{\text{eff}}$ (effective number of electrons captured by one proton per second) due to the existence of Landau levels of charged particles.

The incorrect notion that $E_F(e)$ decreases with increasing $B$ in an intense field ($B \gg B_{cr}$) has been universally adopted for a long time. This misunderstanding has arisen because the solution of the non-relativistic electron cyclotron motion equation $\hbar \omega_B$ is wrongly (or unsuitably) applied to calculate the energy state density in a relativistic degenerate electron gas. In addition, in some textbooks on statistical physics, the torus located between the $n$-th Landau level and the $(n+1)$-th Landau level is ascribed to the $n$-th Landau level when calculating the statistical weight in momentum space. If so, energy (or momentum) will change continuously in the direction perpendicular to the magnetic field, which is contradictory to the quantization of energy (see Gao et al. 2010 for further details). To the contrary, our point of view is that in the case of an intense magnetic field, the stronger the magnetic field, the higher the Fermi energy of electrons released from the magnetic field energy. The possible interpretations of this viewpoint are also given in (Gao et al. 2010).

This paper is organized as follows: in § 2 the values of $\Gamma$ and $\langle E_n \rangle$ are calculated; with regard to the relationships between the magnetic fields and Landau levels, we put forward our points of view and hypotheses in § 3.1; in § 3.2 we calculate the magnitude of $q$ and give an appropriate initial temperature $T_0$; in § 3.3 a numerical simulation of the whole process of electron capture inside a magnetar is presented; discussions and conclusions are given in Section 4; and the formula for the electron Fermi energy is deduced briefly in Appendix.

2 The calculations of $\Gamma$ and $\langle E_n \rangle$

As the core density increases, the high electron Fermi energy drives electron capture by nuclei and free protons, which reduces $Y_e$ and decreases the contribution of the degenerate electrons to the total pressure supporting the core against gravitation collapse. In this paper, we focus on non-relativistic, degenerate nuclear matter and ultra-relativistic, degenerate electrons under $\beta$-equilibrium implying the following relationship among chemical potentials (called the Fermi energies $E_F$) of the particles: $\mu_p + \mu_e = \mu_n$, where the neutrino chemical potential is ignored. We assume that at zero-temperature the NS is $\beta$-stable, but at non-zero temperature ($kT \ll E_F(i), i = n, p, e, k = 1.38 \times 10^{-16}$ erg K$^{-1}$ is the Boltzmann constant), reactions $e^- + p \rightarrow n + \nu_e$ and $n \rightarrow e^- + p + \nu_e$ proceed near the Fermi energies $E_F(i)$ of the participating particles. In the case of gentle $\rho_0 \leq \rho \leq 2\rho_0$, electrons are relativistic, neutrons and protons are non-relativistic, and the following expressions are hold approximately: $p_{F}(n)/2m_n = 60(\rho/\rho_0)^{\frac{5}{2}}$ MeV, $p_{F}(p)/2m_p = 1.9(\rho/\rho_0)^{\frac{5}{2}}$ MeV, $(m_n - m_p)c^2 = 1.29$ MeV, where $\rho_0 = 2.8 \times 10^{14}$ g cm$^{-3}$ is the standard nuclear density (see Chapter 11 of Shapiro et al. 1983).

For the purpose of convenient calculation, we set $\rho = \rho_0$ in our model, yielding $Q = E_{cm}(n) - E_F(p) = (m_n - m_p)c^2 + (p_{F}(n)/2m_n - p_{F}(p)/2m_p) = 59.4$ MeV. However, $E_n > 0$ and the minimum of $Q$ is not less than the minimum Fermi kinetic energy of the outgoing neutrons $p_{F}(n)/2m_n = 60$ MeV, otherwise the process of $e^- + p \rightarrow n + \nu_e$ will not occur. Solving Eq.(9) in Appendix gives the expression $E_F(e) = 40(B/B_{cr})^{\frac{5}{2}}$ MeV when $B \gg B_{cr}$ (Peng & Tong 2007, 2008), where we assign $\rho = \rho_0$ and $Y_e \sim (0.08-0.11)$ (see Appendix). In this paper, the range of $B$ is assumed to be $(0.22346-3.0) \times 10^{15}$ G corresponding to $E_F(e) \sim (60 - 114.85)$ MeV, where $0.22346 \times 10^{15}$ G is the minimum strength of a superhigh magnetic field denoted as $B_l$. When $B$ drops below $B_l$, the direct Urca process is quenched everywhere in the magnetar interior. The electron capture rates can be calculated by the following equation:

$$\Gamma = \frac{2\pi}{\hbar} \frac{G_F^2 C_{m}^2}{(2\pi)^3 \hbar^3 e^6} \langle E_n \rangle^2 I$$

$$I = \int_{E_F(e)}^{E_F(p)} \frac{1}{(E^2 - m_e^2 c^4)^{\frac{5}{2}}} E_e \langle E_n - 60 \rangle^2 \frac{1}{e^{\frac{E_{\nu} - E_F(e)}{kT}}} + \frac{1}{e^{\frac{E_e - E_{\nu}}{kT}}} dE_e,$$  (1)

where $m_ec^2 = 0.511$ MeV, $T = 1.0 \times 10^8$ K, and other terms appearing in Eq. (1) have already been defined in Chapter 18 of (Shapiro et al. 1983). For convenience, we use the symbol $A$ to represent $\frac{2\pi}{\hbar} \frac{G_F^2 C_{m}^2}{(2\pi)^3 \hbar^3 e^6} \approx 0.018$ (MeV)$^{-5}$ s$^{-1}$. If we want to determine $L_X$ and the average kinetic energy of the outgoing neutrons $\langle E_{\nu} \rangle$, the average kinetic energy of the outgoing neutrinos $\langle E_{\nu} \rangle$ must firstly be calculated. The expression for $\langle E_{\nu} \rangle$
is of the form:

\[ \langle E_n \rangle = \int \frac{E F(e)}{Q} S(E - Q)^3 E_n (E_n^2 - m_e^2 c^4)^{1/2} dE_n / I, \]  

where \( I = \int \frac{E F(e)}{Q} S(E - Q)^2 E_n (E_n^2 - m_e^2 c^4)^{1/2} dE_n \). By employing energy conservation via \( E_{\nu'} = E_{\nu} - Q \), \( \langle E_n \rangle \) can be calculated by the equation \( \langle E_n \rangle = E F(e) - \langle E_{\nu'} \rangle - 1.29 \text{ MeV} \). The calculation results are shown below in tabular form.

It can be seen from Table 1 that \( \Gamma \) decreases with the decreasing \( E F(e) \), and is reduced to zero when \( E F(e) = Q \). The rate of the electron capture reaction varies clearly with magnetic field strength: when \( B \gg B_1 \), \( E F(e) \gg Q = 60 \text{ MeV} \), the reaction will happen very quickly and the \( ^3P_2 \) Cooper pairs will be destroyed immediately by the outgoing neutrons in this process, which causes anisotropy in the neutron superfluid and causes the induced magnetic field to disappear; when the magnetic field weakens, the reaction rate becomes smaller; the reaction will end if the field decreases below \( B_{cr} \). By colliding with the neutrons produced in the process \( n + (n \uparrow n \downarrow) \rightarrow n + n + n \), the kinetic energy of the outgoing neutrons will be transformed into thermal energy and then transformed into radiation energy via soft X-ray and \( \gamma \)-ray emission \( \text{(Peng & Tong 2005, 2009).} \)

However, the values of \( \Gamma \) in Table 1 are too large to represent the full extent of electron capture in the interior of a magnetar. We believe that the calculated values of \( \Gamma \) from Eq.(1) are not the actual values of \( \Gamma \) and need to be modified. The same should be true of the neutron decay rate \( (1/927 \text{ s), calculated by Eq.(11.4.1) of (Shapiro et al. 1983).} \) To explain this phenomenon, careful analysis found that when Eq.(1) is used to calculate \( \Gamma \), we simply assume that electrons and protons are free when magnetic field effects are too weak to be taken into consideration. In other words, only when \( B = 0 \) can Eq.(1) be used, so the values of \( \Gamma \) in Table 1 are simply the values of the electron capture rate onto free protons. When Eq.(11.4.1) of \( \text{(Shapiro et al. 1983) is used to calculate neutron decay rate, we make a simple assumption: pure neutron decay in a vacuum and strong interaction effects combined with the magnetic field effects are ignored. Suppose the values of \( \Gamma \) in Table 1 were not modified. If reaction } e^- + p \rightarrow n + \nu_e \text{ proceeded at the minimum rate of \( \Gamma \) in Table 1, the value of } \nu \text{ calculated by Eq.(13) in § 3 would be far larger than the observed value } (\sim 1.0 \times 10^{34}, 5 \times 10^{36} \text{ erg s}^{-1}, \text{see § 1. Further, electrons and protons would be depleted within a few minutes, which is not consistent with the actual circumstances in a neutron star interior. Instead, } Y_e \text{ decreases insignificantly in the whole process of electron capture, and the mean value of } Y_e \text{ of a neutron star is } \sim 0.05 \text{ and is comparatively stable, which implies that the electron capture reaction proceeds very slowly due to the existence of Landau levels. In accordance with the Pauli exclusion principle, electrons are situated in disparate Landau levels from the lowest energy state (the ground energy state) up to the highest energy state (the Fermi energy state), individually with the overwhelming majority occupying } n = 0, 1 \text{ Landau levels. Only the electrons occupying large Landau levels with high energy } (E > Q) \text{ are allowed to participate in the direct Urca process (Yakovlev et al. 2001). In other words, whether an electron could be captured depends not only on the electron’s energy } E_e \text{ but also on the number of the Landau level it occupies. In Eq.(1), } E_e \text{ must be treated as a continuous function, otherwise } \Gamma \text{ cannot be calculated by using discrete Landau levels. If } E_e \text{ is treated as continuous, as in the free-field case, we must modify } \Gamma \text{ by introducing } q \text{ considering that the symmetry is broken in the momentum space caused by the superhigh magnetic fields. In the interior of an NS, different forms of magnetic field (weakly quantizing field, strongly quantizing field and non-quantizing field) could exist simultaneously. The properties and distributions of these fields, are described in more detail in (Gao et al. 2010). Here, we focus on the magnetar interior, where the weakly quantizing strong magnetic fields permit direct Urca processes. The details will be given in § 3.} \]

3 Numerical simulating the direct Urca process in magnetar interior

This section is composed of three subsections. For each subsection we present different methods and considerations.

3.1 Our points of view and hypotheses

For extremely strong magnetic fields, the cyclotron energy of an electron becomes comparable to its rest-mass energy, and the transverse motion electron becomes relativistic. We can define a critical magnetic field (often called the relativistic magnetic field) \( B_{cr} \) by the relation \( h \omega = m_e c^2 \) which gives \( B_{cr} = m_e^2 c^3 / (e h) = 4.14 \times 10^{13} \text{ G.} \) In the case of \( B \geq B_{cr} \), solving the relativistic Dirac Equation for electrons gives the electron energy levels \( \text{(Canuto & Ventura 1977).} \)

\[ E = [m_e^2 c^4 (1 + \nu (2B / B_{cr}^2)) + p^2 c^2]^{1/2}, \]  

where \( B \) is directed along the \( z \)-axis, quantum number \( \nu \) is given by \( \nu = n + 1/2 + \sigma, n = 0, 1, 2, \cdots \) is the Landau
level number, and spin \( \sigma = \pm \frac{1}{2}, p_z \) is the z-component of the electron/momentum. Combining \( B_{cr} = m_e^2 c^3/\varepsilon h \) with \( \mu_e = e/2m_e c \), we obtain

\[
E_\sigma^2(p_z, B, n, \sigma) = m_e^2 c^4 + p_z^2 c^2 + (2n + 1 + \sigma)2m_e c^2 \mu_e B,
\]

with \( \mu_e \sim 0.927 \times 10^{-20} \) erg G\(^{-1}\) being the magnetic moment of an electron. From Eq.(4), for electrons in a given Landau level, \( E_\sigma \) increases with \( p_z c \); if the value of \( p_z c \) is invariable, \( E_\sigma \) increases with increasing Landau level number \( n \). In terms of the relation between the magnetic fields and Landau levels, our points of view and hypotheses are as follows:

1. Firstly, for electrons (or protons) in an intense magnetic field, the maximum Landau level number, \( n_m \), decreases with increasing magnetic field strength \( B \). However, it is very difficult to calculate exactly \( n_m \) occupied by a homogeneous gas of cold electrons (or protons) in a given field, and so we can only estimate it. Considering this limitation, we define a quantity \( q_e \), the ratio of the electron number in higher Landau levels to that in all Landau levels, written as

\[
q_e = \frac{1}{N_{tot}} (N(n) + N(n+1) + \cdots) ,
\]

where \( n \) denotes the number of the lowest Landau level (not the ground Landau level), below which electrons cannot be captured.

2. Secondly, in the core of a NS, charge neutrality gives \( n_e = n_p \). Therefore, whenever the magnetic field significantly affects the electrons, it also affects the protons, thus the values of \( n_m \) for electrons and protons are essentially the same in a given magnetic field [Lai \\& Shapiro 1991]. Similarly, we define the quantity \( q_p \) as the ratio of the proton number in higher Landau levels to that in all Landau levels, whose expression is the same as that of \( q_e \). We firstly introduce the Landau level effect coefficient \( q = q_e q_p \), and the effective electron capture rate \( \Gamma_{eff} \) is then defined as:

\[
\Gamma_{eff} = q \Gamma = q_e q_p .
\]

3. Finally, in the vicinity of the Fermi surface, the electrons with the same energy \( E \) could come from different Landau levels because the electrons are degenerate. However, the electrons occupying lower Landau levels cannot be captured even if their energies are higher than the threshold reaction energy; for higher Landau levels, there still exist some electrons with lower energies \( E (E < Q) \) that are not captured.

In order to determine the order of magnitude of the Landau level effect coefficient \( q \) and to estimate the maximum Landau level number \( n_m \) as accurately as possible, we should firstly rewrite Eq.(4), as demonstrated in § 3.2. we must modify \( \Gamma \) by introducing \( q \) considering that the symmetry is broken in the momentum space caused by the superhigh magnetic fields.

### 3.2 The evaluations of \( q \) and \( T_0 \)

When \( B \gg B_{cr}, p_z c \gg p_z c \) and \( p_z c \gg m_e c^2 = 0.511 \) MeV, Eq.(4) can be well approximated as

\[
E(n) \approx p_z c \cdot (1 + \frac{1}{2} \frac{m_e c^2}{p_z c})^2 + (2n+1+\sigma) \frac{m_e c^2}{p_z c} \mu_e B .
\]

Degenerate electron gas is distributed exponentially [Peng \\& Tong 2007], so we define

\[
q_e(n) = \frac{N(n)}{N(0)} = \exp\{-\frac{E(n) - E(0)}{kT}\}
\]

where \( N(n) \) denotes the number of electrons in the \( n \)-th Landau level. When \( B \sim 10^{15} \) G, \( T \sim 10^8 \) K, \( 2m_e c^2 \mu_e B \sim 10^{-5}, 2m_e c^2 \mu_e B/(p_F(z)c) \sim 10^{-6}, 2m_e c^2 \mu_e B/(p_F(z)c) \sim 10 \). From these evaluations, it is clear that \( N(n) \gg N(n+1) \gg N(n+2) \cdots \), so \( q_{n+e}(n) \ll q_{n+1}(n) \ll q_{n}(n) \ll 1 \). According to the Pauli exclusion principle, \( N_{tot} = n_e \), here \( N_{tot} \) and \( n_e \) are electron state density, electron number density, respectively. Suppose \( N(0) \sim 0.9 n_e \) in the ground state Landau level, then \( N(0)/N_{tot} \sim 0.9 \). Since \( n_e \sim 10^{46} \) cm\(^{-3}\) [Shapiro et al. 1983] and \( N(n) \geq 1 \), \( n_m \) is estimated to be \( \sim \) several or \( \sim 10 \). It is important to note that, in a non-relativistic weak field, the electron cyclotron energy is \( \hbar \omega_B = \hbar c B/(m_e c^3) = 11.5 B_{12} \) KeV, the maximum Landau level number \( n_m \sim E_\sigma/(\hbar \omega_B) \sim 10^2 \) or higher, where \( B_{12} \) is magnetic field in units of 10\(^{12} \) G; also, in the case of a weakly quantizing relativistic strong magnetic field (\( B \approx 10^{14} \sim 10^{15} \) G), the solution of non-relativistic electron cyclotron motion equation \( \hbar \omega_B \) is no longer suitable, but if this equation is used, the rest mass of an electron \( m_e \) must be replaced by its effective mass \( m_e^* \), which is far larger than the former after taking into account the effect of relativity. In this latter case \( n_m \) could be estimated to be \( \sim 10 \) or higher, rather than 0 or 1, which shows our evaluations are reasonable. From the above discussion, we obtain the following approximate relationship:

\[
q(e) \approx \frac{1}{N_{tot}} N(0) q_n(e) = 0.9 q_p(e) .
\]
Thus the electron number in the lowest Landau level can be accurately approximated by

$$N_n \approx 0.9n_e \exp \left\{ -\frac{2nm_e c^2 \mu_e B}{p_F(z)c \cdot kT} \right\}. \quad (10)$$

If we want to determine $q_n$, the value of $p_F(z)c$ should be calculated first. Inserting $p_c c = p_F(z)c$ and $E(e) = E_F(e)$ into Eq.(4) gives

$$[40 \left( \frac{B}{B_{cr}} \right)^{\frac{1}{2}}]_2^{\frac{2}{3}} \approx m_e^2 c^4 + p_F^2(z) c^2 + (2n + 1 + \sigma) 2 m_e c^2 \mu_e B,$$

with $p_F(z)$ the highest momentum along the magnetic field. Next, a way of calculating the value of $p_F(z)c$ in Eq.(6) is introduced as follows. For example, in the case of $B = 3.0 \times 10^{15}$ G and $n = 5$, firstly, we calculate the values of $p_F(z)c$ corresponding to $\sigma = 1$ and $\sigma = -1$, respectively, by using Eq.(6), then calculate the mean value of $p_F(z)c$ $\approx 113.96$ MeV. From the analysis above, the effective electron capture $\Gamma_{\text{eff}}$ can be expressed as:

$$\Gamma_{\text{eff}} = q\Gamma = q(e)q(p)\Gamma = 0.9\exp \left\{ -\frac{2nm_e c^2 \mu_e B}{p_F(z)c \cdot kT} \right\} 2\Gamma. \quad (12)$$

The steady X-ray emission ($10^{34} \sim 10^{36}$ erg s$^{-1}$) could be from the magnetar interior (Peng & Tong 2007). We assume that all protons take part in the process of electron capture, and that all the kinetic energy of the outgoing neutrons is converted and radiated in the form of thermal energy, then $L_X$ can be expressed as

$$L = \Gamma_{\text{eff}} n_p V(3p_2)(E_n), \quad (13)$$

where $\langle E_n \rangle \geq 60$ MeV, otherwise the reaction ceases; $V(3p_2)$ denotes the volume of $3p_2$ anisotropic neutron superfluid, $V(3p_2) = \frac{4}{3} \pi R_3^3$ cm$^3$, $R_3 = 10^5$ cm, $\pi = 3.14159$, and $n_p = n_e = 9.6 \times 10^{45}$ cm$^{-3}$ setting $\rho = \rho_0$. We also gain an approximate expression of $T$ from Eqs.(12-13)

$$T = \frac{2nm_e c^2 \mu_e B}{p_F(z)c \cdot \ln \left[\frac{0.9}{L_X n_e \frac{2}{3} \pi R_3^3 \langle E_n \rangle} \right]^{1/2}} \quad (14)$$

In the initial stage of electron capture process, the initial X-ray luminosity $L_{X0}$ could be higher than the generally observed values $1 \times (10^{34} \sim 5 \times 10^{36}$ erg s$^{-1}$), so it is reasonable to assume here that $L_{X0} = 9 \times 10^{46}$ erg s$^{-1}$ and $B_0 = 3.0 \times 10^{15}$ G. In order to estimate the order of magnitude of $q$ inside a neutron star, an appropriate initial temperature $T_0$ needs to be determined. Solving Eq.(10) gives the values of the possible initial temperatures corresponding to $n = 1, 2, 3, 4, 5, \cdots$, respectively. The calculation results are listed in Table 2.

According to neutron star cooling theory, the typical magnetar internal temperature is about $3 \times 10^8$ K (Yakovlev et al. 2001). When $B \sim 1.0 \times 10^{15}$ G, $T \sim 10^8$ K, but $T < T_C(3p_2)$, the critical temperature of $3p_2$ a neutron superfluid, $T < T_C(3p_2) = \Delta_{\text{max}}(3p_2)/2k \approx 2.78 \times 10^8$ K, where $\Delta_{\text{max}}(3p_2) \sim 0.05$ MeV is the energy gap maximum of $3p_2$ (Peng & Lu 2007, Peng & Tong 2009). From Table 2, an appropriate initial temperature ($\sim 2.6646 \times 10^8$ K) is selected by considering that the calculated values for $n = 1, 2$ are too low, whereas the calculated values for $n = 4, 5, 6$ are too high (higher than $T_C(3p_2)$). We must stress that this initial temperature is selected arbitrarily for a particular case and may be different for other magnetars. From Table 2, once $L_{X0}$ has been calculated, $q$ is determined ($\sim 1.9543 \times 10^{-18}$) but has little effect on $n$ and $T$. It is worth noting that the calculated values of temperature are the possible values (or the expected values) of the initial temperature, not the variable range of temperature. In the vicinity of the Fermi surface, the electrons with the same energy could come from different Landau levels because the electrons are degenerate. However, not all the electrons near the Fermi surface can be captured, since electrons with energy $(E_n > Q = 60$ MeV) cannot be captured if their Landau level number is too small ($n < 3$); similarly, some electrons occupying Landau levels ($n \geq 3$) are not captured because of their lower energy ($E < Q$). We next define the effective captured electron number of the $n$-th Landau level $N_{\text{eff}}(n)$,

$$N_{\text{eff}}(n) = N(n) - N_{E=Q}(n), \quad (15)$$

where $N_{E=Q}(n)$ denotes the number of electrons with energy $E$ ($E_n < Q$) in the $n$-th Landau level. Accordingly, the expression for $q_n$ should be modified:

$$q_n = \exp \left\{ -\frac{2nm_e c^2 \mu_e B}{p_F(z)c \cdot kT} \right\} - \exp \left\{ -\frac{2nm_e c^2 \mu_e B}{p_{E=Q}(z)c \cdot kT} \right\}. \quad (16)$$

If $B = 3.0 \times 10^{15}$ G, $T = 2.6646 \times 10^8$ K, $n = 3$, $E = 60$ MeV, from Eq.(7), we get $p_{E=Q}(z)c = 59.102$ MeV, $\exp \left\{ -\frac{2nm_e c^2 \mu_e B}{p_{E=Q}(z)c \cdot kT} \right\} = 9.141 \times 10^{-18}$. Further, when $E_F(e) = 114.85$ MeV, then $p_{F}(z)c = 114.31$ MeV, $\exp \left\{ -\frac{2nm_e c^2 \mu_e B}{p_F(z)c \cdot kT} \right\} = 1.549 \times 10^{-9}$. In a given Landau level, similar to a mushroom cloud, the electron number increases exponentially with increasing $p(z)c$, so $q_n(e) \approx \exp \left\{ -\frac{2nm_e c^2 \mu_e B}{p_{E=Q}(z)c \cdot kT} \right\}$, which illustrates our approximate ways above are reasonable.
Table 1 The calculated values of $\Gamma$, $(E_n)$ and $(E_n)\textsuperscript{a}$.

| B (G)   | $E_p$ (eV) | $\Gamma$ (s$^{-1}$) | $(E_n)$ (MeV) | $(E_n)\textsuperscript{a}$ (MeV) |
|--------|------------|---------------------|---------------|-------------------------------|
| 3.0×10\textsuperscript{15} | 114.850 | 1.022×10\textsuperscript{7} | 43.20 | 70.36 |
| 2.8×10\textsuperscript{15} | 112.886 | 8.892×10\textsuperscript{6} | 41.66 | 69.94 |
| 2.5×10\textsuperscript{15} | 109.733 | 7.043×10\textsuperscript{6} | 39.10 | 69.34 |
| 2.0×10\textsuperscript{15} | 103.779 | 4.367×10\textsuperscript{6} | 34.31 | 68.18 |
| 1.5×10\textsuperscript{15} | 96.577 | 2.255×10\textsuperscript{6} | 28.54 | 66.75 |
| 1.0×10\textsuperscript{15} | 87.267 | 7.892×10\textsuperscript{5} | 21.12 | 64.86 |
| 6.0×10\textsuperscript{14} | 76.805 | 1.501×10\textsuperscript{5} | 12.88 | 62.64 |
| 3.7×10\textsuperscript{14} | 68.805 | 1.3696 | 6.13 | 60.66 |
| 3.5×10\textsuperscript{14} | 67.123 | 9.2448 | 5.40 | 60.43 |
| 2.5×10\textsuperscript{14} | 61.707 | 1.1193 | 0.40 | ~60  |
| 2.25×10\textsuperscript{14} | 60.103 | 0.02375 | 0.075 | ~60 |

$\textsuperscript{a}(E_n)$ is calculated by using the relation of $(E_n)=E_p\langle e\rangle \cdot (E_\nu)-1.29$ MeV.

Table 2 The possible values of the initial temperature

| n | $p\nu(z)c$ | $T^a$ | $q^b$ | $\Gamma\text{eff}^c$ |
|---|------------|-------|-------|---------------------|
|   | (MeV)     | (K)   | $(\times 10^{-18})$ | $(\times 10^{-11}\text{ s}^{-1})$ |
| 1 | 114.62    | 8.8579×10\textsuperscript{7} | 1.9543 | 1.9881 |
| 2 | 114.46    | 1.77407×10\textsuperscript{8} | 1.9543 | 1.9881 |
| 3 | 114.31    | 2.66459×10\textsuperscript{8} | 1.9543 | 1.9881 |
| 4 | 114.15    | 3.55777×10\textsuperscript{8} | 1.9543 | 1.9881 |
| 5 | 113.99    | 4.45345×10\textsuperscript{8} | 1.9543 | 1.9881 |
| 6 | 113.84    | 5.35118×10\textsuperscript{8} | 1.9543 | 1.9881 |

$^aT=\frac{2\pi m_e c^2 u_B}{p\nu(z)c}\ln[0.9(n_t + R_3(E_n))^{1/3}]$

$^bq=\frac{L_X}{n_0 V(\nu F_2)(b)}$

$^c\Gamma\text{eff} = \frac{L_X}{n_0 V(\nu F_2)(b)}$. $(E_n)\sim 70.36$ MeV, assuming that the initial X-ray luminosity $L_X=9\times10^{36}$ erg s$^{-1}$ and the initial magnetic field strength $B_0=3.0\times10^{15}$ G.

Table 3 The details of numerical simulating magnetar cooling

| $T$ | $q$ | $\Gamma\text{eff}$ | $L_X$ |
|-----|-----|-------------------|-------|
| $\times 10^8$ K | $(\times 10^{-18})$ | (s$^{-1}$) | (erg s$^{-1}$) |
| 2.5303 | 1.9582 | 1.7412×10$^{-11}$ | 7.835×10$^{36}$ |
| 2.5302 | 1.9551 | 1.7385×10$^{-11}$ | 7.821×10$^{36}$ |
| 2.5301$^a$ | 1.9519$^a$ | 1.7356×10$^{-11a}$ | 7.810×10$^{36a}$ |
| 2.5300 | 1.9488 | 1.7330×10$^{-11}$ | 7.798×10$^{36}$ |

$^a$The possible values of related quantities in physics

3.3 Numerical simulation of a complete electron capture process

In this section, a simple way of simulating magnetar cooling and magnetic field decay is introduced briefly. For an example, in order to determine the temperature corresponding to $B=2.8\times10^{15}$ G, by combining $q_4 = [0.9 \exp(-4m_e c^2 u_B B)]^2$ with $p\nu(z)c=112.37$ MeV, $T$ is decreased by step $\Delta T=0.0001\times10^8$ K from an initial temperature $T_0$. The expected value of $T$ is reached when the value of $q$ is just below 1.9543×10$^{-18}$. Once $T$ is determined, the relevant values of $L_X$, $q$ and $\Gamma\text{eff}$ can be calculated easily. The details of numerical simulations are shown in Table 3, where $(E_n)=69.94$ MeV, $n=3$, $\Gamma=8.892\times10^6$ s$^{-1}$. By using the same method, the values of $q$, $\Gamma\text{eff}$, $T$ and $L_{rmX}$ in different stages of the electron capture process are calculated and listed. The details of numerical simulation of the whole process of electron capture inside magnetar are shown in Table 4.

From the simulations above, we infer that, once the value of $L_X$ is given, $q$ decreases insignificantly and can be treated as a constant which could be explained...
as follows: on the one hand, in a magnetar interior, the electrons are extremely relativistic and degenerate \((E_\nu \gg m_e c^2, E_F(e) \gg kT)\), such that when \(T\) falls, electron transition between Landau levels is not permitted because the electrons can be treated as having a zero-temperature approximation; on the other hand, the processes of electron capture and \(\beta\)-decay occur at the same time (as required by electrical neutrality), so when \(B\) decays, the depleted protons and electrons are recruited for many times leading to only a small decrease in the value of \(Y_e\) . In order to validate our speculations further, we can assume \(L_{\nu\nu} = 1.0 \times 10^{36} \text{ erg s}^{-1}\), and simulate the whole process of electron capture numerically in the same way. The details are shown in Table 5.

In reality, the observed values of \(L_X\) are different for different magnetars. However, for most magnetars, \(B \sim 10^{14} \sim 10^{15} \text{ G}, T \sim 10^7 \sim 10^8 \text{ K}\), so the observed X-ray luminosity \(\sim 10^{34} \sim 10^{36} \text{ erg s}^{-1}\) which can be explained by the calculations above. The values of the average neutrino luminosity of AXP, \(L_{\nu\nu}\), are determined as follows:

\[
L_{\nu\nu} = \Gamma_{\nu\nu} n_p V (3 P_2) (E_{\nu\nu}).
\]  

(17)

Clearly, our approach to calculating \(L_{\nu\nu}\) is completely different from previous methods. For instance, the neutrino emissivity \(Q_{\nu\nu}\) can be calculated by Eq.(14) of \cite{Baiko & Yakovlev} as follows:

\[
Q = Q_0^0 \times F_{\nu B}^C,
\]

\[
Q_0^0 = \frac{457\pi G^2 (1 + 3g_2^2)}{10080} m_n^* m_p^* \mu_T T_0^6,
\]

(18)

where \(T_0 = T/10^9 \text{ K}\), \(m_n^*\) and \(m_p^*\) are nucleon effective masses in dense matter, \(G = G_F \cos \theta_C\), \(G_F = 1.436 \times 10^{-49} \text{ erg cm}^3\) is the Fermi weak coupling constant, \(\theta_C \approx 13^\circ\) is the Cabibbo angle, \(g_A = 1.261\) is the axial-vector coupling constant, \(Q_0^0\) is the field-free emissivity, the factor \(R_{\nu B}^C\) describes the effect of the magnetic field. In Eq.(18), \(E_{\nu\nu} \sim kT_0\) and \(B\) can be as high as \(B \geq 10^{18} \text{ G}\), but is not discussed (here the range of \(B\) is \(0 \sim 3 \times 10^{16} \text{ G}\)). Thus the value of \(E_{\nu\nu} (E_{\nu\nu} \sim 0.086 \text{ MeV}, \text{when} \ T \sim 10^9 \text{ K})\) is far less than those of Table 1 in § 2, and the range of \(B\) is not consistent with that of our model. We therefore ask what is the essential difference between \(E_{\nu\nu}\) of the direct Urca reaction and that of the modified Urca reaction (in the process of a modified Urca reaction, \(E_{\nu\nu} \sim kT_0\)). It is universally acknowledged that the neutrino emissivity of the modified Urca reaction is about six orders of magnitude smaller than that of the direct Urca reaction. May this be because of the increase of the number of particles participating in the direct Urca reaction while the average kinetic energy of the outgoing neutrinos \(\langle E_{\nu\nu}\rangle\) is invariant in both cases? Ultrastrong magnetic fields can generate a noticeable magnetic broadening of the direct Urca process threshold and the thresholds of other reactions in a magnetar interior \cite{Yakovlev et al. 2001}. However, the modified Urca process are negligible if the direct Urca process is allowed. What is more, it is easy to imagine that the neutrino flux comes from the thermal energy in the magnetar interior, rather than from the free energy of the superhigh magnetic field, if \(E_{\nu\nu} \sim kT_0\) is used in the direct Urca reaction. There exist many different cooling mechanisms, including neutrino emission, inside a NS. In the direct Urca reaction, \(\langle E_{\nu\nu}\rangle\) and \(L_{\nu\nu}\) are ultimately determined by \(B\), and are weak functions of \(T\), which is only equivalent to background temperature and decreases with decreasing \(B\). The \(\beta\)-decay and related reactions in strong

### Table 4 Numerically simulating the whole process of electron capture inside magnetar.

| \(B\) (G) | \(\nu(e)c\) | \(\Gamma_{\nu\nu}\) | \(\nu_{\nu}^{b}\) |
|----------|-------------|-----------------|-------------|
| 3.0 \times 10^{14} | 114.31 | 2.6646 \times 10^{8} | 1.9543 \times 10^{-18} | 1.4881 \times 10^{-11} |
| 2.8 \times 10^{15} | 112.37 | 2.5310 \times 10^{8} | 1.9519 \times 10^{-18} | 1.7536 \times 10^{-12} |
| 2.5 \times 10^{15} | 109.26 | 2.3233 \times 10^{8} | 1.9513 \times 10^{-18} | 1.3743 \times 10^{-11} |
| 2.0 \times 10^{15} | 103.38 | 1.9643 \times 10^{8} | 1.9491 \times 10^{-18} | 8.5117 \times 10^{-12} |
| 1.5 \times 10^{15} | 96.15 | 1.5840 \times 10^{8} | 1.9489 \times 10^{-18} | 4.3948 \times 10^{-12} |
| 1.0 \times 10^{15} | 87.03 | 1.1666 \times 10^{8} | 1.9448 \times 10^{-18} | 1.5348 \times 10^{-12} |
| 6.0 \times 10^{14} | 76.64 | 7.9485 \times 10^{7} | 1.9446 \times 10^{-18} | 2.9188 \times 10^{-13} |
| 3.7 \times 10^{14} | 67.95 | 5.5284 \times 10^{7} | 1.9441 \times 10^{-18} | 2.6626 \times 10^{-14} |
| 3.5 \times 10^{14} | 66.01 | 5.3028 \times 10^{7} | 1.9423 \times 10^{-18} | 1.7956 \times 10^{-14} |
| 2.5 \times 10^{14} | 61.62 | 4.1190 \times 10^{7} | 1.9417 \times 10^{-18} | 2.1733 \times 10^{-16} |
| 2.25 \times 10^{14} | 60.02 | 3.8059 \times 10^{7} | 1.9412 \times 10^{-18} | 4.6104 \times 10^{-20} |

\(a q = [0.9 \exp\{-\frac{2 m_e c^2 \mu \beta B}{\nu(e)c}\}]^2\)

\(b L_{\nu\nu} = \Gamma_{\nu\nu} n_p V (3 P_2)(E_{\nu\nu})\)
magnetic fields have been investigated since the late 1960s (e.g., Canuto & Chiu 1971; Debades et al. 1998; Lai & Shapiro 1991) and references therein). Despite this previous research, existing analysis is approximate and many assumptions are invoked consequently, because our computational methods are completely new approaches, that need to be validated empirically. By using Eq. (11), schematic diagrams of the neutrino luminosity $L_{\nu}$ as a function of magnetic field strength $B$ are shown in Figure 1.

From Table 1 and Tables 4-5, we find that the neutrino luminosity $L_{\nu}$ is also a weak function of the background temperature $T$. The schematic diagrams of the neutrino luminosity $L_{\nu}$ versus the background temperature $T$ are plotted in Figure 2.

Figure 2 illustrates that the neutrino emission decreases with falling temperature. The relationship between magnetic field strength $B$ and the background temperature $T$ is shown in Figure 3, based on the data in Tables 4-5.

Figure 3 shows that the direct Urca reaction proceeds as long as the magnetic field $B$ is high than the critical value $B_{cr}$ corresponding to the minimum value of the background temperature $T$ in the core, and when $T$ falls further with decreasing $B$ to below the critical value, the neutrino luminosity and direct Urca process are quenched everywhere in the core. It should be noted that the surface temperature is controlled by crustal physics, and is independent of the evolution of the core. In these three figures, $B$ is assumed to be in the range $2.24 \times 10^{14}$ G - $3.0 \times 10^{15}$ G. Note that when $B \leq B_{c}$, the direct Urca processes cease, while the modified Urca processes still occur, producing the weaker X-ray and neutrino flux.

### Table 5

| $B$ (G) | $\rho B (\rho G)$ | $T$ (K) | $\xi$ (g cm$^{-3}$) | $\Gamma_{\text{eff}}$ (s$^{-1}$) | $L_{\nu}$ (erg s$^{-1}$) |
|---------|-------------------|---------|---------------------|-----------------------------|-----------------------------|
| $3.0 \times 10^{15}$ | $1.14 \times 10^{11}$ | $2.5 \times 10^{11}$ | $2.16 \times 10^{11}$ | $2.69 \times 10^{11}$ | $1.0 \times 10^{11}$ |
| $2.8 \times 10^{15}$ | $1.23 \times 10^{11}$ | $2.7 \times 10^{11}$ | $2.16 \times 10^{11}$ | $1.95 \times 10^{11}$ | $8.6 \times 10^{10}$ |
| $2.5 \times 10^{15}$ | $1.29 \times 10^{11}$ | $2.7 \times 10^{11}$ | $2.15 \times 10^{11}$ | $1.52 \times 10^{11}$ | $6.7 \times 10^{10}$ |
| $2.0 \times 10^{15}$ | $1.38 \times 10^{11}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $9.4 \times 10^{10}$ | $4.3 \times 10^{10}$ |
| $1.5 \times 10^{15}$ | $1.52 \times 10^{11}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $4.9 \times 10^{10}$ | $2.1 \times 10^{10}$ |
| $1.0 \times 10^{15}$ | $7.0 \times 10^{10}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $7.0 \times 10^{10}$ | $8.4 \times 10^{10}$ |
| $6.0 \times 10^{14}$ | $7.6 \times 10^{10}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $3.2 \times 10^{10}$ | $1.3 \times 10^{10}$ |
| $3.7 \times 10^{14}$ | $6.7 \times 10^{10}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $2.9 \times 10^{10}$ | $1.1 \times 10^{10}$ |
| $3.5 \times 10^{14}$ | $6.6 \times 10^{10}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $1.9 \times 10^{10}$ | $7.7 \times 10^{9}$ |
| $2.5 \times 10^{14}$ | $6.1 \times 10^{10}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $2.4 \times 10^{10}$ | $9.3 \times 10^{9}$ |
| $2.25 \times 10^{14}$ | $6.0 \times 10^{10}$ | $2.16 \times 10^{11}$ | $2.16 \times 10^{11}$ | $5.1 \times 10^{9}$ | $1.9 \times 10^{9}$ |

$^{a} \xi = [0.9 \exp\{-2 \mu_{B}^2 \sigma^2 B \}]^{2}$

$^{b} L = \Gamma_{\text{eff}} n_{\nu} V^{3} P_{2}(E_{0})$
4 Discussions and Conclusions

In this paper, we introduce an approximate method for investigating the effects of Landau levels on the evolution of a superhigh magnetic field, and also numerically simulate the process of magnetar cooling and magnetic field decay. The main conclusions are as follows:

1. The effect of a superhigh magnetic field would be to speed up the cooling of magnetar. When $B$ decays, the values of $\Gamma$, $T$ and $Y_e$ decrease, but $\Delta Y_e$ could be very small.

2. In the magnetar interior, the $^3P_2$ Cooper pairs will be destroyed quickly by the outgoing neutrons via the process of electron capture, so the induced magnetic field will disappear.

3. The abnormal X-ray flux $L_X$ and neutrino flux $L_\nu$ come from the free energy of the superhigh magnetic field, not from the thermal energy in the core of magnetar, and are all ultimately determined by the magnetic field strength.

Finally, we are hopeful that our assumptions and numerical simulations can be combined with observations in the future, to provide a deeper understanding of the nature of the superhigh magnetic fields of magnetars.

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Appendix

A The calculation of $E_{F}(e)$ in the presence of magnetic field

By summing over electron energy states (per unit volume) in a 6-dimension phase space, we can express $N_{\text{pha}}$ as follows

$$N_{\text{pha}} = \frac{2\pi}{h^{3}} \int dp_{z} n_{m}(p_{z}, \sigma, B^{*}) \sum_{n=0}^{\infty} \sum g_{n}$$

$$\int \delta\left(\frac{p_{\perp}}{m_{e}c} - [(2n + 1 + \sigma)B^{*}]^{\frac{1}{2}}\right) dp_{\perp}$$

in which the Dirac $\delta$-function and the relation $2\mu_{e}B_{\text{cr}}/m_{e}c^{2} = 1$ are used, $B^{*} = B/B_{\text{cr}}$ is a non-dimensional magnetic field, $p_{\perp} = m_{e}c\sqrt{(2n + 1 + \sigma)B^{*}}$ denotes electron momentum perpendicular to the magnetic field, $2\mu_{e}B_{\text{cr}}/m_{e}c^{2} = 1$. For $n = 0$, the spin is antiparallel to $B$, the spin quantum number $\sigma = -1$, so the ground state Landau level is non-degenerate; whereas at higher levels $n \geq 1$ are doubly degenerate, and the spin quantum number $\sigma = \pm 1$. Therefore the spin degeneracy $g_{n} = 1$ for $n = 0$ and $g_{n} = 2$ for $n \geq 1$, then Eq.(A1) can be rewritten

$$N_{\text{pha}} = 2\pi\left(\frac{m_{e}c}{h}\right)^{3} \int d(p_{z}/m_{e}c) n_{m}(p_{z}, \sigma, B^{*})$$

$$\int \delta\left(\frac{p_{\perp}}{m_{e}c} - (2nB^{*})^{\frac{1}{2}}\right) dp_{\perp}$$

$$n_{m}(p_{z}, \sigma, B^{*})$$

+ $\sum_{n=1}^{\infty} \int \delta\left(\frac{p_{\perp}}{m_{e}c} - (2(n + 1)B^{*})^{\frac{1}{2}}\right) dp_{\perp}$

$$\int d(p_{\perp}/m_{e}c) n_{m}(p_{\perp}, B^{*})$$

(A2)

The maximum Landau level number $n_{m}$ is the upper limit of the summation over $n$ in Eq.(A2), which is uniquely determined by the condition $(p_{F}(z)c)^{2} \geq 0$ (Lai & Shapiro 1991). The expression for $n_{m}$ is

$$n_{m}(p_{z}, B^{*}, \sigma = -1) = \text{Int}\left[\frac{1}{2B^{*}}[\left(\frac{E_{F}}{m_{e}c^{2}}\right)^{2} - 1 - \left(\frac{p_{z}}{m_{e}c}\right)^{2}]\right]$$

(A3)

$$n_{m}(p_{z}, B^{*}, \sigma = 1) = \text{Int}\left[\frac{1}{2B^{*}}[\left(\frac{E_{F}}{m_{e}c^{2}}\right)^{2} - 1 - \left(\frac{p_{z}}{m_{e}c}\right)^{2}] - 1\right]$$

(A4)

where Int$[x]$ denotes an integer value of the argument $x$. After a complicated process, Eq.(A6) may now be rewritten

$$N_{\text{pha}} = 6\pi\sqrt{2B^{*}}\left(\frac{m_{e}c}{h}\right)^{3} \int \left[\frac{E_{F}}{m_{e}c^{2}}\right]^{\frac{1}{2}} n_{m}(p_{z}, B^{*})$$

$$d\left(\frac{p_{z}}{m_{e}c}\right) - 2\pi\left(\frac{m_{e}c}{h}\right)^{3} \sqrt{2B^{*}}\left(\frac{E_{F}}{m_{e}c^{2}}\right)$$

$$= 6\pi\sqrt{2B^{*}}\left(\frac{m_{e}c}{h}\right)^{3} \left(\frac{1}{2B^{*}}\right)^{\frac{1}{2}} \int \left[\frac{E_{F}}{m_{e}c^{2}}\right]^{\frac{1}{2}} \left[\left(\frac{E_{F}}{m_{e}c^{2}}\right)^{2} - 1 - \left(\frac{p_{z}}{m_{e}c}\right)^{2}\right]^{\frac{1}{2}}$$

$$d\left(\frac{p_{z}}{m_{e}c}\right) - 2\pi\left(\frac{E_{F}}{m_{e}c^{2}}\right)\left(\frac{m_{e}c}{h}\right)^{3} \sqrt{2B^{*}}$$

$$= \frac{3\pi}{B^{*}}\left(\frac{m_{e}c}{h}\right)^{3} \int \left[\frac{E_{F}}{m_{e}c^{2}}\right]^{\frac{1}{2}} \left[\left(\frac{E_{F}}{m_{e}c^{2}}\right)^{2} - 1 - \left(\frac{p_{z}}{m_{e}c}\right)^{2}\right]^{\frac{1}{2}}$$

$$d\left(\frac{p_{z}}{m_{e}c}\right) - 2\pi\left(\frac{E_{F}}{m_{e}c^{2}}\right)\left(\frac{m_{e}c}{h}\right)^{3} \sqrt{2B^{*}}$$

(A5)
In order to deduce the formula for $E_F(e)$, we firstly introduce two non-dimensional variables $\chi$ and $\gamma_e$, which are defined as $\chi = \frac{p_z}{m_e c} = \frac{p_z c}{E_F m_e}$ and $\gamma_e = \frac{E_F}{m_e c^2}$, respectively, then Eq.(A5) can be rewritten as

$$N_{\text{ph}} = \frac{3\pi}{B^*} \left(\frac{m_e c}{\hbar}\right)^3 \left(\gamma_e\right)^4 \int_0^1 \left(1 - \frac{1}{\gamma_e^2} - \chi^2\right)^{\frac{3}{2}} d\chi$$

$$-2\pi \gamma_e \left(\frac{m_e c}{\hbar}\right)^3 \sqrt{2B^*}$$

(A6)

The electron number density is determined by

$$n_e = N_A \rho Y_e$$

(A7)

where $N_A = 6.02 \times 10^{23}$ is the Avogadro constant. For a given nucleus with proton number $Z$ and nucleon number $A$, the relation $Y_e = Z/A$ always holds. Combining Eq.(A5) with Eq.(A6), we find

$$\frac{3\pi}{B^*} \left(\frac{m_e c}{\hbar}\right)^3 \left(\gamma_e\right)^4 \int_0^1 \left(1 - \frac{1}{\gamma_e^2} - \chi^2\right)^{\frac{3}{2}} d\chi$$

$$-2\pi \gamma_e \left(\frac{m_e c}{\hbar}\right)^3 \sqrt{2B^*} = N_A \rho Y_e$$

(A8)

In the case of field-free, for reactions $e^- + p \rightarrow n + \nu_e$ and $n \rightarrow p + e^- + \nu_e^-$ to take place, there exists the following inequality among the Fermi momenta of the proton($p_F$), the electron($k_F$) and the neutron ($q_F$): $p_F + k_F \geq q_F$ required by momentum conservation near the Fermi surface. Together with the charge neutrality condition, the above inequality brings about the threshold for proton concentration $Y_p = n_p / (n_p + n_n) \geq \frac{1}{10} = 0.11$, this means that, in the field-free case, direct Urca reactions are strongly suppressed by Pauli blocking in the neutron-rich nuclear matter formed only by protons, neutrons and electrons. However, in a magnetic field $B \gg B_{cr}$, direct Urca reactions are open for an arbitrary proton concentration ($Y_e \leq 0.11$) due to the fact that strong magnetic field can alter matter compositions and increase phase pace for protons which leads to the increase of $Y_e$ [Lai & Shapiro 1991]. Calculations indicate that $E_F(e)$ is $(39.3 \sim 42.2)(B/B_{cr})^{1/4}$ MeV corresponding to $Y_e \sim 0.08 - 0.11$ at a given nuclear density $2.8 \times 10^{14}$ g cm$^{-3}$. We assume that direct Urca reactions must occur in the core of neutron star. According to the calculation above, in the range of allowable error ($\leq 5\%$) we gain an approximate relationship between $E_F(e)$ and $B$, which can be expressed as $E_F(e) = 40(B/B_{cr})^{1/4}$ MeV when $\rho = 2.8 \times 10^{14}$ g cm$^{-3}$ and $Y_e$ is $\sim 0.08 - 0.11$.

(Cited from the paper: ‘Neutron Star Magnetic Field and The Electron Fermi Energy’ Authors: Gao Z. F., Wang N., Yuan J.P., et.al., 2010, Prepared)