PERTURBING ORDINARY DIFFERENTIAL EQUATIONS TO GENERATE RESONANT AND REPEATED ROOT SOLUTIONS

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Abstract. In the study of ordinary differential equations (ODEs) of the form \( \hat{L}[y(x)] = f(x) \), where \( \hat{L} \) is a linear differential operator, two related phenomena can arise: resonance, where \( f(x) \propto u(x) \) and \( \hat{L}[u(x)] = 0 \), and repeated roots, where \( f(x) = 0 \) and \( \hat{L} = \hat{D}^n \) for \( n \geq 2 \). We illustrate a method to generate exact solutions to these problems by taking a known homogeneous solution \( u(x) \), introducing a parameter \( \epsilon \) such that \( u(x) \to u(x; \epsilon) \), and Taylor expanding \( u(x; \epsilon) \) about \( \epsilon = 0 \). The coefficients of this expansion \( \frac{\partial^k u}{\partial \epsilon^k} \bigg|_{\epsilon=0} \) yield the desired resonant or repeated root solutions to the ODE. This approach, whenever it can be applied, is more insightful and less tedious than standard methods such as reduction of order or variation of parameters. While the ideas can be introduced at the undergraduate level, we could not find any elementary or advanced text that illustrates these ideas with appropriate generality.

Key words. ordinary differential equations, ODEs, resonance, repeated roots

AMS subject classifications. 34A05, 34B05, 34B30

1. Introduction. Introductory courses on ODEs often focus on illustrating methods of solution for problems in the natural sciences and engineering. One common topic is resonance, which for the typical case of a mechanical or electrical oscillator corresponds to forcing the system at its natural frequency. The unwanted oscillations that occurred when the Millennium Bridge, which crosses over the Thames River in London, was opened to foot traffic in June 2000 serves a modern realization [14]. Mathematically, this response occurs because the governing ODE is forced with one of its homogeneous solutions.

Of course, modern computer algebra systems such as Mathematica and Maple provide immediate, often analytical, solutions to these problems. However, because standard algorithms are utilized, it is not uncommon that the results are cumbersome and not insightful. For the special case of resonance, we show here using ideas grounded in perturbation theory and analytic continuation that the resonant solution can be obtained simply by constructing an appropriate Taylor series. This method yields simpler, more transparent functional forms for the resonant solution that have an obvious relationship to the homogeneous solution. We will also see that the same ideas are applicable when constructing linearly independent solutions of equations with “repeated roots”.

To illustrate the method, we first provide some general background on linear ODEs to orient the reader on the scope of the problems we seek to solve. We then proceed to go through a series of examples that explicitly demonstrate how the method is used. To conclude, we provide a general derivation that summarizes the method and reveals its underlying structure. Because this approach seems particularly flexible, practically requires only knowledge of Taylor series, and in non-elementary problems produces much simpler solutions than mathematical software packages, we believe it

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will be of interest. These ideas can be taught in a course at the undergraduate level, although we are not aware of any book on ODEs at any level that illustrates the approach nor emphasizes its generality.

2. Background. In the study of \(n\)th-order linear ODEs of the form

\[
\hat{L}[y(x)] = f(x), \quad \text{where} \quad \hat{L} = a_0(x) + \sum_{j=1}^{n} a_j(x) \frac{d^j}{dx^j},
\]

solutions may be represented as \(y(x) = \sum_{j=1}^{n} c_j u_j(x) + u_p(x)\). Here, \(u_j(x)\) are the \(n\) linearly independent homogeneous solutions such that \(\hat{L}[u_j(x)] = 0\) and \(u_p(x)\) is a particular solution such that \(\hat{L}[u_p(x)] = f(x)\). The constants \(\{c_j\}\) are determined by auxiliary data. Two common special cases arise when seeking the solution set \(\{u_j(x), u_p(x)\}\): resonance and repeated roots.

The first, resonance, occurs when \(f(x) \propto u_j(x)\), i.e., when the forcing function includes any term proportional to one of the homogeneous solutions. Therefore, positing a particular solution of the form \(u_p(x) \propto f(x)\), an intuitive ansatz for a linear system, will automatically fail since \(\hat{L}[f(x)] = 0\).

The second is the case of repeated roots, which concerns the homogeneous problem \(f(x) = 0\) whenever the linear operator can be factored as \(L = \prod_{k=1}^{m} D_k\), where all of the \(D_k\) commute. In this case it is necessary to solve the lower order sub-problems \(D_k[y(x)] = 0\). The difficulty arises when one of the operators, say \(D_1\), is repeated \(r\) times, and so solving the sub-problems can produce only \(n-r-1\) linearly independent solutions.

Both these cases are systematically resolved by reduction of order\(^1\), which is a standard topic in introductory courses [4, sec. 3.4, 3.8], [12, sec. 2.8, 2.10], [10, sec. 6.7], [8, sec. 3.4.4, 3.6.2]. Substituting a solution of the form \(y(x) = u(x)g(x)\), where \(u(x)\) is an already obtained homogeneous solution, into the ODE results in a reduced-order ODE for \(g(x)\) that is easier to solve and will generate the remaining resonant or repeated root solutions. In practice, the challenge with applying this method is that it only reduces the order of the ODE by one, and so it typically fails to be useful for ODEs that are higher than second order. Also, for non-elementary ODEs, the method may only guarantee complicated integral representations, which are difficult to put into a simpler form. Furthermore, even when the method is successful, it is a tedious calculation and the connection between the resonant or repeated root solution and the known homogeneous solution \(u_j(x)\) is not obvious.

Here, we highlight what we believe is a better, or at least more insightful, way to obtain resonant or repeated root solutions to linear ODEs. For the case of resonance, the idea is to introduce a small parameter \(\epsilon\) in the forcing function so that \(f(x) \rightarrow f(x; \epsilon)\) such that \(\lim_{\epsilon \rightarrow 0} f(x; \epsilon) = f(x)\). It will then be possible to construct the particular solution by positing \(u_p(x; \epsilon) \propto f(x; \epsilon)\). Taylor expanding \(u_p(x; \epsilon)\) in \(\epsilon\) and setting \(\epsilon = 0\) will generate the desired resonant solution. In the examples, we will show that repeated root solutions may be constructed in the exact same way. In the end, we will learn that the obtained resonant or repeated root solution is simply an analytic continuation of the known homogeneous solution in \(\epsilon\). This description is formal, but the method will become concrete in the following step-by-step examples.

The rudiments of this idea have been discussed in advanced texts [3, pgs. 11-12], [16, pgs. 17-18], but only in the context of repeated roots for constant coefficient

\(^1\)Variation of parameters is the most general method to solve all nonhomogeneous linear ODEs, but it is just a logical extension of reduction of order.
or equidimensional ODEs, where reduction to an algebraic characteristic equation is possible. As we will see, the method as presented here is more general and useful in a variety of non-elementary problems, where no algebraic characteristic equation exists globally. Makarov et. al [13] give a more general presentation of the idea, so it is certainly known in the Russian mathematical literature, but even there the discussion is limited to the case of resonance with no mention of repeated root solutions.

While our method only applies to problems involving resonance or repeated roots, such problems occur often enough in applied mathematics and the physical sciences that we believe it is worth highlighting. Furthermore, as previously noted, this approach is not presented at all in introductory texts on differential equations to the best of our knowledge, but is certainly comprehensible at the undergraduate level and is just as useful as reduction of order. Whenever one encounters resonance or a repeated root, we suggest that the method we now illustrate be the method of choice.

3. Examples. In this section we give seven concrete examples, in order of increasing complexity. We suggest reading them in order, as the ideas naturally build off each other. The first example of resonance in a constant coefficient 2nd-order ODE is discussed in many introductory texts using reduction of order [4, 12, 10, 8].

3.1. Constant coefficient equation at resonance. Consider the constant coefficient ODE

$$\frac{d^2y}{dx^2} + y = \sin x. \quad (3.1)$$

The homogeneous solutions are $u_1(x) = \sin x$ and $u_2(x) = \cos x$. We observe that the forcing function $f(x) = \sin x$ is linearly dependent on $u_1(x)$, in fact it is $u_1(x)$, and so we have resonance. Therefore, naively guessing $u_p(x) = A \sin x$, where $A$ is a to be determined coefficient, will fail as it would render the left-hand side of equation (3.1) equal to zero. The strategy is to then introduce a parameter $\epsilon$ to equation (3.1) such that

$$\frac{d^2y}{dx^2} + y = \sin ((1 + \epsilon)x), \quad (3.2)$$

where we are interested in the limit $\epsilon \to 0$. Now, substituting $u_p(x) = A \sin ((1 + \epsilon)x)$ provides a particular solution so long as $A = \frac{1}{1-(1+\epsilon)^2}$. Hence, the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x - \frac{\sin (x + \epsilon x)}{2\epsilon + \epsilon^2}. \quad (3.3)$$

Constructing a Taylor expansion around $\epsilon = 0$, and neglecting terms of $O(\epsilon^2)$ or higher, gives

$$y(x) = \left( c_1 - \frac{1}{2\epsilon} \right) \sin x + c_2 \cos x - \frac{x \cos x}{2}. \quad (3.4)$$

Because we have complete freedom in choosing $c_1$, we can make the transformation $c_1 = \tilde{c}_1 - 1/2\epsilon$, which removes the divergence as $\epsilon \to 0$. Thus, we have obtained the general solution

$$y(x) = \tilde{c}_1 \sin x + c_2 \cos x - \frac{x \cos x}{2}. \quad (3.5)$$

If $x$ represents time, the $x \cos x$ behavior is the signature of the ever growing oscillations of a conservative oscillator at resonance.
Let us take stock of the solution approach. The first step is to introduce a parameter $\epsilon$ that allows us to guess a particular solution $u_p(x; \epsilon) \propto f(x; \epsilon)$, where the limit $\epsilon \to 0$ corresponds to the problem we wish to solve. This limit will be singular, but Taylor expanding $u_p(x; \epsilon)$ to linear order in $\epsilon$ allows grouping the singular part with a homogeneous solution. Relabeling a free constant removes the singularity and gives the desired $\epsilon = 0$ solution.

### 3.2. Repeated roots of the equidimensional equation.

Consider the equidimensional ODE in the form

\[(3.6) \quad x^2 \frac{d^2 y}{dx^2} + (1 - 2b) x \frac{dy}{dx} + b^2 y = 0,\]

where $b$ is a given constant. Carefully staring at this equation, we observe that the linear operator can be factored and the ODE can be rewritten as

\[(3.7) \quad \left( x \frac{d}{dx} - b \right)^2 y = 0,\]

and thus this equation has a repeated root. We construct the homogeneous solutions $\{u_1(x), u_2(x)\}$ by solving the sub-problems $(x \frac{d}{dx} - b) u_1 = 0$ and $(x \frac{d}{dx} - b) u_2 = u_1$. By inspection we observe that both $u_1(x)$ and $u_2(x)$ are indeed solutions to equation (3.7). Integration gives $u_1(x) = x^b$ and we are left to solve

\[(3.8) \quad (x \frac{d}{dx} - b) u_2 = x^b.\]

We are now in the same position as example 3.1, since the homogeneous solution of equation (3.8) is exactly the forcing term $x^b$, and so we have resonance. Thus, we learn that resonance and repeated roots are effectively the same feature. Writing $b \to b + \epsilon$ gives

\[(3.9) \quad (x \frac{d}{dx} - b) u_2 = x^{b+\epsilon}.\]

The equation and solution of interest correspond to the limit $\epsilon \to 0$. Substituting $u_2(x) = Ax^{b+\epsilon}$ as the particular solution is successful if $A = 1/\epsilon$, and therefore the solution to equation (3.7) can be written

\[(3.10) \quad y(x) = c_1 x^b + c_2 x^{b+\epsilon}.\]

Constructing the Taylor expansion for $0 < \epsilon \ll 1$, we have $x^\epsilon = 1 + \epsilon \log x + O(\epsilon^2)$, therefore

\[(3.11) \quad y(x) = \left( c_1 + \frac{c_2}{\epsilon} \right) x^b + c_2 x^b \log x.\]

Relabeling the constant $c_1 = \tilde{c}_1 - c_2/\epsilon$ removes the divergence and gives the desired solution

\[(3.12) \quad y(x) = \tilde{c}_1 x^b + c_2 x^b \log x.\]

Of course, in this particular example, equation (3.8) could have been solved more directly via an integrating factor. However, that approach only works for first-order ODEs, whereas our method easily generalizes to higher-order ODEs, as illustrated in example 3.4.
3.3. Airy’s equation at resonance. So far, we have only looked at ODEs that may be reduced to algebraic characteristic equations, the solutions to which yield elementary functions. Furthermore, we have yet to consider a well-posed initial (IVP) or boundary (BVP) value problem. To increase the complexity and demonstrate the generality of our method, consider the forced Airy BVP

\[
\frac{1}{x} \frac{d^2 y}{dx^2} - y = \text{Ai}(x) \tag{3.13a}
\]

\[
y(0) = 1 \tag{3.13b}
\]

\[
y(x \to \infty) \to 0 , \tag{3.13c}
\]

where \( \text{Ai}(x) \) is the Airy function of the first kind. Similar equations come up when studying the Schrödinger equation for a particle in a linear potential [9]. The homogeneous solutions are the two linearly independent Airy functions of the first and second kind, respectively, \( u_1(x) = \text{Ai}(x) \) and \( u_2(x) = \text{Bi}(x) \), and thus equation (3.13a) corresponds to resonance since the forcing function is proportional to a homogeneous solution.

Following the same steps as in the first two examples, we introduce a parameter \( \epsilon \) so that

\[
\frac{1}{x} \frac{d^2 y}{dx^2} - y = \text{Ai}((1 + \epsilon)x) , \tag{3.14}
\]

where we will let \( \epsilon \to 0 \). Proposing a particular solution \( u_p(x) = C \text{Ai}((1 + \epsilon)x) \) is successful if \( C = \frac{1}{(1 + \epsilon)^3} \), where we have used \( \frac{d^2}{dx^2} [\text{Ai}(cx)] = c^3 \text{Ai}(cx) \). Therefore, the solution to (3.14) is

\[
y(x) = c_1 \text{Ai}(x) + c_2 \text{Bi}(x) + \frac{\text{Ai}((1 + \epsilon)x)}{(1 + \epsilon)^3 - 1} . \tag{3.15}
\]

Constructing the Taylor expansion \( \text{Ai}(x + \epsilon x) = \text{Ai}(x) + \epsilon x \text{Ai}'(x) + O(\epsilon^2) \), where \( \frac{d}{dx} = ' \), and retaining terms only up to first order in \( \epsilon \) gives

\[
y(x) = \left( c_1 + \frac{1}{3\epsilon} \right) \text{Ai}(x) + c_2 \text{Bi}(x) + \frac{\epsilon x \text{Ai}'(x)}{3} . \tag{3.16}
\]

We can now apply the boundary data. The condition (3.13c) forces \( c_2 = 0 \) since \( \text{Bi}(x \to \infty) \to \infty \). The condition (3.13b) results in \( 1 = (c_1 + \frac{1}{\epsilon}) \text{Ai}(0) \Rightarrow c_1 + \frac{1}{\epsilon} = \frac{2\pi^{3/6}}{\Gamma(1/3)} \), where we have used \( \text{Ai}(0) = \frac{\Gamma(1/3)}{2\pi^{3/2}} \) (see Appendix 5.1 for a derivation). Thus we see that the divergence at \( \epsilon = 0 \) goes away naturally when we consider a well-posed BVP\(^2\), and the final solution is

\[
y(x) = \frac{2\pi^{3/6}}{\Gamma(1/3)} \text{Ai}(x) + \frac{\epsilon x \text{Ai}'(x)}{3} . \tag{3.17}
\]

For problems involving special functions, it is often useful to utilize Mathematica as a first check on solvability. Entering the appropriate commands results in

\[
\text{sol} = \text{DSolve}\left[\frac{1}{x} y''[x] - y[x] == \text{AiryAi}[x], y[x], x\right] \to 1/6 (\pi \text{AiryAi}[x] \text{AiryAiPrime}[x] \text{AiryBi}[x] - 2\pi x \text{AiryAiPrime}[x])^2
\]

\[
\text{AiryBi}[x] - \pi \text{AiryAi}[x]^2 \text{AiryBiPrime}[x] + 2\pi x \text{AiryAi}[x] \text{AiryAiPrime}[x] \text{AiryBiPrime}[x] + \text{AiryAi}[x] c[1] + \text{AiryBi}[x] c[2] ,
\]

\(^2\)We thank Dionisios Margetis for emphasizing this point to us.
which is rather disastrous compared to our solution! Because the structure of Mathematica’s solution involves products of \( \text{Ai}(x) \) and \( \text{Bi}(x) \), it is clear that reduction of order is the algorithm underpinning this solution. Yet, we are guaranteed that the particular solution is unique up to added multiples of the homogeneous solutions (the Fredholm alternative). Indeed, when we apply Mathematica’s brute-force simplification algorithm to this solution we find

\[
\text{sol2} = \text{FullSimplify[sol]}
\]

\[
y(x) \rightarrow 1/3x \text{AiryAiPrime}[x] + \text{AiryAi}[x](-1/6 + c[1]) + \text{AiryBi}[x]c[2],
\]

which is precisely our general solution (3.16) with a relabeling of a constant.

Using our method, one can now appreciate how to obtain the simplified solution directly from the ODE. There is no need to generate the complicated solution via reduction of order and use esoteric properties of Airy functions to simplify it further.

### 3.4. Repeated roots of a fourth-order Bessel-like equation.

Our method easily generalizes to higher-order ODEs. Consider the fourth-order equation

\[
(3.20) \quad \left( \frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} - k^2 \right)^2 y = 0,
\]

which arises in the study of hydrodynamic stability [7]. Here \( k \) is a given constant. Equation (3.20) has a repeated root, so we construct the homogeneous solutions by solving the sub-problems \( \left( \frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} - k^2 \right) u_1 = 0 \) and \( \left( \frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} - k^2 \right) u_2 = u_1 \). Substituting \( u_1(x) = x^2 \zeta(x) \) into the first sub-problem furnishes \( x^2 \frac{d^2 \zeta}{dx^2} + x \frac{d \zeta}{dx} - (1 + k^2 x^2) \zeta = 0 \), for which the solutions are the modified Bessel functions \( \zeta(x) = c_1 I_1(kx) + c_2 K_1(kx) \). Therefore, \( u_1(x) = c_1 x I_1(kx) + c_2 x K_1(kx) \) and the second sub-problem becomes

\[
(3.21) \quad \left( \frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} - k^2 \right) u_2 = c_1 x I_1(kx) + c_2 x K_1(kx).
\]

Equation (3.21) displays resonance, as the forcing functions are the homogeneous solutions of the differential operator. Therefore, we introduce \( \epsilon \) so that

\[
(3.22) \quad \left( \frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} - k^2 \right) u_2 = c_1 x I_1((k + \epsilon)x) + c_2 x K_1((k + \epsilon)x)
\]

and guess the particular solution \( u_p(x) = A x I_1((k + \epsilon)x) + B x K_1((k + \epsilon)x) \). Substituting this guess works only if we choose \( A = \frac{c_1}{(k + \epsilon)^2 - k^2} \) and \( B = \frac{c_2}{(k + \epsilon)^2 - k^2} \). Therefore, the solution to equation (3.20) is

\[
(3.23) \quad y(x) = c_1 x I_1(kx) + c_2 x K_1(kx) + c_3 \frac{x I_1((k + \epsilon)x)}{(k + \epsilon)^2 - k^2} + c_4 \frac{x K_1((k + \epsilon)x)}{(k + \epsilon)^2 - k^2},
\]

where we have renamed some integration constants. We now construct the Taylor expansions

\[
(3.24a) \quad I_1(kx + \epsilon x) = I_1(kx) + \epsilon (I_1(kx)/k + x I_2(kx)) + O(\epsilon^2) \\
(3.24b) \quad K_1(kx + \epsilon x) = K_1(kx) + \epsilon (K_1(kx)/k - x K_2(kx)) + O(\epsilon^2),
\]
where we have made use of the formulas \( I'_1(z) = I_1(z)/z + I_2(z) \) and \( K'_1(z) = K_1(z)/z - K_2(z) \). Combining (3.23-3.24) and keeping only terms to linear order in \( \epsilon \) gives, after some rearrangement,

\[
(3.25a) \quad y(x) = \left( c_1 + \frac{c_3}{2k} + \frac{c_4}{2k^2} \right) xI_1(kx) + \left( c_2 + \frac{c_4}{2k} + \frac{c_4}{2k^2} \right) xK_1(kx) + \frac{c_3}{2k} x^2 I_2(kx) - \frac{c_4}{2k} x^2 K_2(kx).
\]

Relabeling all constants to remove the \( \epsilon \to 0 \) divergence and to clean up the final solution results in

\[
(3.26) \quad y(x) = \tilde{c}_1 x I_1(kx) + \tilde{c}_2 x K_1(kx) + \tilde{c}_3 x^2 I_2(kx) + \tilde{c}_4 x^2 K_2(kx).
\]

Calling Mathematica again for comparison, we find

\[
\text{inner} = D[y[x], \{x, 2\}] - (1/x)*D[y[x], x] - k^2*y[x]
\]
\[
\text{sol} = DSolve[D[\text{inner}, \{x, 2\}] - (1/x)*D[\text{inner}, x] - k^2*\text{inner} == 0, y[x], x]
\]
\[
y[x] \rightarrow x^2 BesselI[2, ikx] c[1] + x^2 BesselY[2, -ikx] c[2] + i/8 (ikx BesselJ[0, ikx] BesselJ[2, ikx] BesselY[1, -ikx]) + \kappa x^3 BesselJ[0, ikx] BesselJ[1, ikx] BesselY[2, -ikx] c[3] + i/8 (ikx BesselJ[2, ikx] BesselY[0, -ikx] BesselY[1, -ikx]) + \kappa x^3 BesselJ[1, ikx] BesselY[0, -ikx] BesselY[2, -ikx] c[4],
\]

which is not pleasant. Applying the brute force simplification algorithm gives

\[
\text{sol2} = \text{FullSimplify[} \text{sol]}\]
\[
y[x] \rightarrow -x^2/4(4BesselI[2, kx] c[1] - 4 BesselY[2, -ikx] c[2] + BesselI[0, kx] c[3] + BesselY[0, -ikx] c[4]),
\]

Using the relationship \( Y_\beta(i\pi x) \propto K_\mu(z) \) between Bessel and modified Bessel functions as well as the recurrence relations \( J_0(z) = 3J_1(z)/z + J_2(z) \) and \( K_0(z) = 3K_1(z)/z + K_2(z) \) converts Mathematica’s simplified solution into our solution (3.26).

### 3.5. Repeated roots of Bessel’s equation

So far, we have solved problems that Mathematica could manage, albeit more clumsily. Let us now tackle a problem that Mathematica fails to solve directly, namely

\[
(3.29) \quad \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \right)^3 y = 0.
\]

Sixth-order equations do have applications, for example when studying the interactions between fluids and elastic media [1]. In this case, we construct the homogeneous solutions by solving the sub-problems \( \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \right)^2 u_1 = 0 \) and \( \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \right)^2 u_2 = u_1 \). To solve the first sub-problem for \( u_1(x) \), we break things down further as before and solve the problems \( \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \right) v_1 = 0 \) and \( \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \right) v_2 = v_1 \) so that \( u_1(x) = \text{span}\{v_1(x), v_2(x)\} \).

The solutions for \( v_1(x) \) are the Bessel functions \( v_1(x) = c_1 J_0(x) + c_2 Y_0(x) \). Anticipating resonance in the equation for \( v_2(x) \), we introduce \( \epsilon \) and write

\[
(3.30) \quad \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \right) v_2 = c_1 J_0((1+\epsilon)x) + c_2 Y_0((1+\epsilon)x).
\]
Positing the particular solution \( u_\epsilon = AJ_\epsilon ((1 + \epsilon)x) + BY_\epsilon (1 + \epsilon) \), works only if we choose 
\[ A = \frac{c_1}{1-1+1+\epsilon^2} \]
and 
\[ B = \frac{c_2}{1-1+1+\epsilon^2}. \]
Therefore, at this point the solution for \( u_1(x) \) is
\[ u_1(x) = c_1 J_\epsilon(x) + c_2 Y_\epsilon(x) + c_3 J_\epsilon ((1 + \epsilon)x) \]
\[ + c_4 Y_\epsilon ((1 + \epsilon)x) \]
where we have relabelled some constants. Taylor expanding \( J_\epsilon(x) + O(\epsilon^2) \) and \( Y_\epsilon(x + \epsilon) = Y_\epsilon(x) - \epsilon x J_1(x) + O(\epsilon^2) \), substituting them into the equation (3.31), and keeping terms only up to first order in \( \epsilon \) gives
\[ u_1(x) = \left( c_1 - \frac{c_3}{2\epsilon} \right) J_\epsilon(x) + \left( c_2 - \frac{c_4}{2\epsilon} \right) Y_\epsilon(x) + \frac{c_3}{2} x J_1(x) + \frac{c_4}{2} x Y_1(x). \]
Relabelling constants as usual gives the final solution for \( u_1 \)
\[ u_1(x) = \hat{c}_1 J_\epsilon(x) + \hat{c}_2 Y_\epsilon(x) + \hat{c}_3 x J_1(x) + \hat{c}_4 x Y_1(x). \]
We can now proceed to solve the sub-problem for \( u_2(x) \). We have
\[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \] 
\( u_2 = \hat{c}_1 J_\epsilon(x) + \hat{c}_2 Y_\epsilon(x) + \hat{c}_3 x J_1(x) + \hat{c}_4 x Y_1(x). \)
As usual, we proceed by inserting the parameter \( \epsilon \) such that
\[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 \]
\[ u_2 = \hat{c}_1 J_\epsilon((1 + \epsilon)x) + \hat{c}_2 Y_\epsilon((1 + \epsilon)x) \]
\[ + \hat{c}_3 x J_1((1 + \epsilon)x) + \hat{c}_4 x Y_1((1 + \epsilon)x) \]
and propose a particular solution \( u_\epsilon = AJ_\epsilon((1 + \epsilon)x) + BY_\epsilon((1 + \epsilon)x) + C x J_1((1 + \epsilon)x) + D x Y_1((1 + \epsilon)x) \). Substituting this anzats, we find after some simplification and rearrangement that
\[ A = \frac{c_1 - 2(1 + \epsilon) C}{1 - (1 + \epsilon)^2} \]
\[ B = \frac{\hat{c}_2 - 2(1 + \epsilon) D}{1 - (1 + \epsilon)^2} \]
\[ C = \frac{\hat{c}_3}{1 - (1 + \epsilon)^2} \]
\[ D = \frac{\hat{c}_4}{1 - (1 + \epsilon)^2}. \]
The full solution to the original problem (3.29) is \( y = \text{span}\{u_1, u_2\} \). Combining our results from equations (3.33) and (3.36) gives
\[ y(x) = \hat{c}_1 J_\epsilon(x) + \hat{c}_2 Y_\epsilon(x) + \hat{c}_3 x J_1(x) + \hat{c}_4 x Y_1(x) \]
\[ + \frac{A x J_1((1 + \epsilon)x)}{1 - (1 + \epsilon)^2} + \frac{B x Y_1((1 + \epsilon)x)}{1 - (1 + \epsilon)^2}. \]
The simplest way to proceed is to set \( \epsilon = 0 \) in the terms involving \( A \) and \( B \), because we have freedom to make the transformations \( \hat{c}_1 = c_1 - A \) and \( \hat{c}_2 = c_2 - B \) to remove divergence issues. Thus we have
\[ y(x) = \hat{c}_1 J_\epsilon(x) + \hat{c}_2 Y_\epsilon(x) + \hat{c}_3 x J_1(x) + \hat{c}_4 x Y_1(x) \]
\[ - \frac{x J_1((1 + \epsilon)x)}{2\epsilon + \epsilon^2} - \frac{x Y_1((1 + \epsilon)x)}{2\epsilon + \epsilon^2}. \]
We can now perform the Taylor expansions

\[ J_1(x + c\epsilon) = J_1(x) + \epsilon (J_1(x) - xJ_2(x)) + O(\epsilon^2) \]
\[ Y_1(x + c\epsilon) = Y_1(x) + \epsilon (Y_1(x) - xY_2(x)) + O(\epsilon^2), \]

where we have used the formulas \( J_1'(z) = J_1(z)/z - J_2(z) \) and \( Y_1'(z) = Y_1(z)/z - Y_2(z). \) Substituting these expansions in equation (3.38) while retaining terms only up to linear order in \( \epsilon \) furnishes

\[ y(x) = \hat{c}_1 J_0(x) + \hat{c}_2 Y_0(x) + \left( \hat{c}_3 - \frac{c_5}{2\epsilon} - \frac{c_5}{2} \right) x J_1(x) + \left( \hat{c}_4 - \frac{c_6}{2\epsilon} - \frac{c_6}{2} \right) x Y_1(x) + \frac{c_5}{2} x^2 J_2(x) + \frac{c_6}{2} x^2 Y_2(x). \]

Relabelling constants to clean up the solution and remove divergences finally gives all six linearly independent solutions to equation (3.29):

\[ y(x) = \hat{c}_1 J_0(x) + \hat{c}_2 Y_0(x) + \hat{c}_3 x J_1(x) + \hat{c}_4 x Y_1(x) + \hat{c}_5 x^2 J_2(x) + \hat{c}_6 x^2 Y_2(x). \]

While we did all this work to show the explicit steps, the general structure of our method should now be clear. Problems of the form \( \hat{D}^n[y(x)] = 0 \) can be broken down into \( n - 1 \) resonance problems, the solutions to which can be generated from the function \( u(x) \) where \( \hat{D}[u(x)] = 0 \) by constructing the Taylor expansion

\[ u(x; \epsilon) = u(x) + \frac{\partial u}{\partial \epsilon} \bigg|_{\epsilon=0} \epsilon + \frac{\partial^2 u}{\partial \epsilon^2} \bigg|_{\epsilon=0} \frac{\epsilon^2}{2} + \cdots + \frac{\partial^{n-1} u}{\partial \epsilon^{n-1}} \bigg|_{\epsilon=0} \frac{\epsilon^{n-1}}{(n-1)!}. \]

Therefore the function \( \frac{\partial^n u}{\partial \epsilon^n} \bigg|_{\epsilon=0} \) gives the \( k^{th} \) repeated root of \( \hat{D}^n[y(x)] = 0. \) In this example, \( u(x; \epsilon) = \hat{J}_0((1 + \epsilon)x) + \hat{Y}_0((1 + \epsilon)x), \) and applying the formula (3.42) furnishes the delightful formula

\[ u_{(k)}(x) = b_1 x^k J_k(x) + b_2 x^k Y_k(x) \]

for the \( k^{th} \) repeated root of Bessel’s equation.

### 3.6. Legendre’s equation at resonance

We now turn to an example where the manner in which we introduce the parameter \( \epsilon \) is quite different. We consider the following Legendre BVP at resonance

\[ \frac{d}{dx} \left[ (1 - x^2) \frac{dy}{dx} \right] + n(n + 1) y = P_n(x) \]
\[ y(1) = 1 \]
\[ y(x) \text{ is finite on } x \in (-1, 1), \]

where \( P_n(x) \) is the Legendre polynomial of the first kind of order \( n \) for \( n \in \mathbb{Z}. \) In this case, the homogeneous solutions are \( u_1(x) = P_n(x) \) and \( u_2(x) = Q_n(x), \) which are the Legendre functions of the first and second kind, respectively.

Clearly, we have resonance, but if we try our usual strategy of perturbing the forcing function as \( P_n((1 + \epsilon)x) \) and utilize the ansatz \( u_\epsilon(x) = AP_n((1 + \epsilon)x), \) it will fail because of the structure of the \( (1 - x^2) \) term in equation (3.44a). Instead, for equations of this type it is more useful to perturb the order of the forcing function, in this case the order of the Legendre polynomial \( n. \) We have

\[ \frac{d}{dx} \left[ (1 - x^2) \frac{dy}{dx} \right] + n(n + 1) y = P_{n+\epsilon}(x), \]
where we interpret $P_{n+\epsilon}(x)$ as a homogeneous solution to equation (3.44a) for $n \to n + \epsilon \in \mathbb{R}$, which can be represented by a hypergeometric function [5, eq. 15.9.7], although we will not make use of this directly. Positing the ansatz $u_p(x) = AP_{n+\epsilon}(x)$ works and we find $A = \frac{1}{n(n+1)-(n+\epsilon)(n+\epsilon+1)}$, and so the general solution is

$$y(x) = c_1 P_n(x) + c_2 Q_n(x) - \frac{P_{n+\epsilon}(x)}{(2n+1)\epsilon + \epsilon^2}.$$  

(3.46)

We now Taylor expand the particular solution in the usual way, except now we are expanding in the order of the Legendre polynomial. Keeping terms only up to $O(\epsilon)$ we obtain

$$-u_p(x) = \frac{P_n(x)}{(2n+1)\epsilon} + \frac{1}{2n+1} \frac{\partial P_n(x)}{\partial n}.$$ 

(3.47)

We now have to evaluate the derivative $\frac{\partial P_n(x)}{\partial n}$, which we do by thinking of $n$ as a continuous variable, then evaluating the result only at integer $n$. There are many ways to do this, for example in terms of the aforementioned hypergeometric function, or by using an integral representation valid for arbitrary $n$ [15, eq. 4.1]. There is actually a simpler way of computing $\frac{\partial P_n(x)}{\partial n}$, which was discovered by Jolliffe [11]. His method leads to the remarkable formula

$$\frac{\partial P_n(x)}{\partial n} := P_{n,1}(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n \log \left(\frac{x+1}{2}\right)] - P_n(x) \log \left(\frac{x+1}{2}\right),$$

(3.48)

which is interesting in that it is almost the standard Rodrigues formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$ apart from logarithmic corrections. Equipped with equation (3.48), the general solution becomes

$$y(x) = \left(c_1 - \frac{1}{(2n+1)\epsilon}\right) P_n(x) + c_2 Q_n(x) - \frac{P_{n,1}(x)}{2n+1}.$$ 

(3.49)

Equation (3.49) has the same structure as the solution derived by Backhouse [2, eqs. 13-14] in his study of the resonant Legendre equation, where he too remarked on the appearance of logarithmic terms.

We can now apply the boundary data. Condition (3.44c) forces $c_2 = 0$ while condition (3.44b) gives $1 = \left(c_1 - \frac{1}{(2n+1)\epsilon}\right) P_n(1) - \frac{P_{n,1}(1)}{2n+1}$. Using the fact that $P_n(1) = 1$, one can see that $P_{n,1}(1) = 0$ due to the logarithm terms, hence $c_1 = 1 + \frac{1}{(2n+1)\epsilon}$ and the solution to the BVP (3.44) is

$$y(x) = P_n(x) - \frac{P_{n,1}(x)}{2n+1},$$

(3.50)

where $P_{n,1}(x)$ is given by equation (3.48).

3.7. Repeated roots of the Hermite equation. As our final example we now consider repeated roots of Hermite’s equation

$$\left(\frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2n\right)^2 y = 0$$

(3.51)

for $n \in \mathbb{Z}$. As usual, we seek to solve the sub-problems $\left(\frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2n\right) u_2 = u_1$. The solutions to the first sub-problem are the Hermite
functions \( u_1(x) = c_1 H_n(x) + c_2 G_n(x) \), where \( H_n(x) \) is the Hermite polynomial of order \( n \) and \( G_n(x) = H_n(x) \int^x dx' e^{x'^2}/H_n^2(x') \). Therefore, the second sub-problem we need to solve is

\[
\left( \frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2n \right) u_2 = c_1 H_n(x) + c_2 G_n(x),
\]

which is a resonance problem. As in the previous example, we perturb the order of the Hermite functions and construct the perturbed problem

\[
\left( \frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2n \right) u_2 = c_1 H_{n+\epsilon}(x) + c_2 G_{n+\epsilon}(x),
\]

where we interpret \( H_{n+\epsilon}(x) \) and \( G_{n+\epsilon}(x) \) as a homogeneous solutions to equation (3.51) for \( n+\epsilon \in \mathbb{R} \). As usual, we propose the form \( u_p(x) = AH_{n+\epsilon}(x) + BG_{n+\epsilon}(x) \) and substitute into equation (3.53). We find this works as long as we choose \( A = -c_1/2\epsilon \) and \( B = -c_2/2\epsilon \). Hence the general solution so far is

\[
y(x) = c_1 H_n(x) + c_2 G_n(x) + c_3 \frac{H_{n+\epsilon}(x)}{\epsilon} + c_4 \frac{G_{n+\epsilon}(x)}{\epsilon},
\]

where we have relabeled some integration constants. To finish, we construct the Taylor expansions \( H_{n+\epsilon}(x) = H_n(x) + \frac{\partial H_n(x)}{\partial n} \epsilon + O(\epsilon^2) \) and \( G_{n+\epsilon}(x) = G_n(x) + \frac{\partial G_n(x)}{\partial n} \epsilon + O(\epsilon^2) \), where the task now becomes computing \( \frac{\partial H_n(x)}{\partial n} \), from which \( \frac{\partial G_n(x)}{\partial n} \) follows by invoking the chain rule. Unfortunately this derivative appears less often in practice than the Legendre case, so there are no simple formulas commonly used. Thus we give an answer in terms of the confluent hypergeometric function \( _1F_1 \) [1],

\[
\frac{\partial H_n(x)}{\partial n} := H_{n,1}(x) = x \frac{d}{d\xi} \left[ _1F_1 \left( \frac{3}{2}, 1; x^2 \right) \right] \bigg|_{\xi = \frac{1}{2n}}.
\]

Using this definition and the chain rule we also find

\[
\frac{\partial G_n(x)}{\partial n} := G_{n,1}(x) = H_{n,1}(x) \int^x dx' \frac{e^{x'^2}}{H_n^2(x')} - 2H_n(x) \int^x dx' \frac{e^{x'^2} H_n(x')}{H_n^2(x')},
\]

With everything known, we substitute these functions into their respective Taylor expansions, group terms of order \( \epsilon^{-1} \) with the homogeneous solutions, relabel integration constants, and set \( \epsilon = 0 \). The final result is

\[
y(x) = \hat{c}_1 H_n(x) + \hat{c}_2 G_n(x) + c_3 H_{n,1}(x) + c_4 G_{n,1}(x).
\]

4. General structure and final remarks. In this paper we highlighted how to construct resonant or repeated root solutions to ODEs via analytic continuation of an already known homogeneous solution in an introduced parameter \( \epsilon \). For the practitioner who comes across these problems in their work, we hope these examples have offered a concrete guide on how to find such solutions. To conclude, we now give a more general derivation that encompasses all our examples.

We are interested in finding a particular solution \( u_p(x) \) of the resonant ODE \( \hat{L}[u_p(x)] = u(x) \), where \( \hat{L}[u(x)] = 0 \). Suppose that \( \hat{L} \) can be written as \( \hat{L} = \hat{M} - \lambda \),

\[3\]There is no “Hermite function of the 2nd kind” commonly cited in the literature, so we give an integral representation here that can be derived using reduction of order (see Appendix 5.2).
where \( \lambda \) is either a parameter in the original problem, or it can be introduced to the problem and set to unity at the end. The homogeneous solution will therefore depend on this parameter, and we denote this dependence by \( u = u(x; \lambda) \). Therefore we seek to solve

\[
\left( \hat{M} - \lambda \right) [u_p(x)] = u(x; \lambda).
\]

We now perturb this problem by introducing a small parameter \( \epsilon \)

\[
\left( \hat{M} - \lambda \right) [u_p(x)] = u(x; \lambda + \epsilon)
\]

and propose that the particular solution has the form \( u_p(x) = Au(x; \lambda + \epsilon) \). Since \( \hat{M}[u(x; \lambda + \epsilon)] = (\lambda + \epsilon)u(x; \lambda + \epsilon) \), this ansatz satisfies equation (4.2) if we choose \( A = 1/\epsilon \), hence \( u_p(x) = u(x; \lambda + \epsilon)/\epsilon \). We now Taylor expand \( u_p(x) \) around \( \epsilon = 0 \) to obtain

\[
u_p(x) = \frac{u(x)}{\epsilon} + \frac{\partial u}{\partial \lambda} + O(\epsilon).
\]

The \( u(x)/\epsilon \) term is proportional to the homogeneous solution \( u(x) \), so we are free to remove it by lumping it with the integration constant associated with \( u(x) \). This removes the \( 1/\epsilon \) divergence and allows us to set \( \epsilon = 0 \), resulting in

\[
u_p(x) = \frac{\partial u}{\partial \lambda}.
\]

With this general derivation, we see that the resonant solution to an ODE is always given by the derivative with respect to the eigenvalue \( \lambda \) of the governing differential operator \( \hat{L} \). We note that equation (4.4) has also been derived by Makarov et al. [13]. See Table 4.1 for a guide on how to apply this equation for the differential operators discussed in our examples.

From example 3.5, we showed that the repeated root problem \( \hat{D}^k[y(x)] = 0 \) is equivalent to \( k - 1 \) resonance problems. Thus in the context of this derivation, the \( k \)th repeated root solution is given by

\[
u_{(k)}(x) = \frac{\partial^k u}{\partial \lambda^k}
\]

where \( \hat{D}[u(x)] = 0 \).

While all the problems presented here may be solved using reduction of order, the benefit of using the method presented here should be apparent. There is minimal algebra and no integration of a reduced-order ODE is required. It is only necessary to compute a Taylor series. In this way, the connection between the homogeneous solutions and resonant or repeated root solutions becomes clear. We hope that this note motivates instructors to present this approach along side reduction of order in their differential equation courses. These problems do arise in practice, and we believe this method is helpful in exercising thought processes useful in applied mathematics, as well as producing the most elegant solution.

\[\text{Note that } \frac{\partial u}{\partial \lambda} = \frac{\partial u}{\partial \epsilon} \bigg|_{\epsilon=0} \text{ identically, so this is consistent with our previous notation up until now.} \]
Table 4.1

Construction of the resonant particular solution \( u_p(x) \) from its associated homogeneous solution \( u(x) \) using equation (4.4). The calculation is done using the chain rule \( \frac{du}{d\lambda} = \frac{du}{d\mu} \frac{d\mu}{d\lambda} \). To generate higher order repeated root solutions, simply take higher derivatives with respect to \( \lambda \) as per equation (4.5).

| \( L \) | \( \lambda \) | Homogeneous Solution \( u(x) \) | Resonant Solution \( u_p(x) \) |
|---|---|---|---|
| \( \frac{d^2}{dx^2} + \mu^2 \) | \( -\mu^2 \) | \( \sin \mu x, \cos \mu x \) | \( -\frac{x \cos \mu x}{2\mu}, -\frac{x \sin \mu x}{2\mu} \) |
| \( x \frac{d}{dx} - \mu \) | \( \mu \) | | \( x^k \) |
| \( \frac{1}{x} \frac{d^2}{dx^2} - \mu^2 \) | \( \mu^2 \) | \( \text{Ai}(\mu x), \text{Bi}(\mu x) \) | \( \frac{x \text{Ai}(\mu x)}{3\mu^2}, \frac{x \text{Bi}(\mu x)}{3\mu^2} \) |
| \( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \mu^2 \) | \( -\mu^2 \) | \( J_0(\mu x), Y_0(\mu x) \) | \( \frac{x J_1(\mu x)}{2\mu}, \frac{x Y_1(\mu x)}{2\mu} \) |
| \( \frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \right) + \mu(\mu + 1) \) | \( -\mu(\mu + 1) \) | \( P_\mu(x), Q_\mu(x) \) | \( -\frac{P_{\mu+1}(x)}{2\mu+1}, -\frac{Q_{\mu+1}(x)}{2\mu+1} \) |
| \( \frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2\mu \) | \( -2\mu \) | \( H_\mu(x), G_\mu(x) \) | \( -\frac{H_{\mu+1}(x)}{2}, -\frac{G_{\mu+1}(x)}{2} \) |

5. Appendix.

5.1. Calculation of \( \text{Ai}(0) \). We start with the well-known integral representation \( \text{Ai}(x) = \frac{1}{\pi} \int_0^\infty dt \cos \left( \frac{t^3}{3} + xt \right) \). Our method of approach in evaluating \( \text{Ai}(0) \) will be standard contour integration, so instead we consider the complex integral

\[
I(\alpha) := \int_0^\infty dw \ e^{i\alpha w^3} = \frac{1}{\alpha^{1/3}} \int_0^\infty dz \ e^{iz^3}.
\]

One can see that \( \text{Re} \left[ I(1/3) \right] = \pi \text{Ai}(0) \), and so our task becomes evaluating \( I \). Because \( e^{iz^3} \) is entire, \( \oint_C dz \ e^{iz^3} = 0 \) by Cauchy’s theorem. We choose the contour \( C \) as show in Fig. 5.1, where we will take \( R \to \infty \). Writing out the integrals we have

\[
\lim_{R \to \infty} \left[ \int_0^R dx \ e^{ix^3} + \int_0^{\pi/6} \Re e^{i\theta} d\theta \ e^{iR^3 e^{i3\theta}} + \int_{\gamma} e^{iR \pi/6} dr \ e^{-r^3} \right] = 0,
\]

where \( I \) is the integral we wish to compute.
We can prove that $J$ vanishes in the $R \to \infty$ limit by the following argument:

\begin{align}
|J| &= \int_0^{\pi/6} |Re^{i\theta}e^{iR^3e^{i3\theta}}| \\
&\leq R \int_0^{\pi/6} |d\theta e^{-R^3\sin 3\theta}| \\
&\leq R \int_0^{\pi/6} \theta e^{-R^3\theta/\pi} \\
&= \frac{\pi}{6R^2} \left( 1 - e^{-R^3} \right) \to 0 \text{ as } R \to \infty.
\end{align}

In the first step, we get rid of all pure unimodular phases. In the next step, we use the bound $\sin 3\theta > 6\theta/\pi$ on $\theta \in [0, \pi/6]$, which allows us to evaluate the integral and prove that it vanishes. Hence, $I = -K$, which we can evaluate in terms of the Gamma function as follows:

\begin{align}
I &= -K \\
&= e^{i\pi/6} \int_0^\infty dr e^{-r^3} \\
&= \frac{e^{i\pi/6}}{3} \int_0^\infty du \, u^{-2/3} e^{-u} \\
&= \frac{e^{i\pi/6}}{3} \Gamma(1/3).
\end{align}

Here we have just made the change of variables $u = r^3$ and utilized the definition of the Gamma function.

Hence we have $\pi \operatorname{Ai}(0) = \Re\{I(1/3)\} = 3^{1/3} \Re\{I\} = 3^{1/3} \cos(\pi/6)\Gamma(1/3) = 3^{5/6} \Gamma(1/3)/6$.

Simplifying a bit gives the final result $\operatorname{Ai}(0) = \frac{\Gamma(1/3)}{2^{5/3}\pi}$.  

5.2. Calculation of the Hermite function of the 2nd kind $G_n(x)$. Let $v_2(x) = H_n(x)f(x)$ be the 2nd linearly independent homogeneous solution to Her-
mite’s equation \( v''_2 - 2xv'_2 + 2nv_2 = 0 \), where \( f(x) \) is to be determined. Substituting this ansatz into Hermite’s equation and simplifying a bit results in

\[
H_n f'' + 2H_n' f' - 2x f'H_n + (H''_n - 2xH'_n + 2nH_n) f = 0
\]

which is a first order ODE for \( f' \). We can divide equation (5.5) through by \( f'H_n \) and then rearrange it into the form

\[
\frac{d}{dx} \log f'H_n^2 = 2x,
\]

which can be integrated to give \( f'(x) = e^{x^2}/H_n^2(x) \) (neglecting the integration constant). Another integration produces \( f(x) = \int e^{x^2}/H_n^2(x) \), and thus our desired result is

\[
v_2(x) = H_n(x) \int e^{x^2}/H_n^2(x).
\]

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REFERENCES

[1] Hermite Differential Equation. From MathWorld–A Wolfram Web Resource., https://mathworld.wolfram.com/HermiteDifferentialEquation.html. Weisstein, Eric W.
[2] N. Backhouse, Resonant equations and special functions, Journal of computational and applied mathematics, 133 (2001), pp. 163–169.
[3] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory, Springer Science & Business Media, 2013.
[4] W. E. Boyce, R. C. DiPrima, and D. B. Meade, Elementary Differential Equations, John Wiley & Sons, 2017.
[5] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15, http://dlmf.nist.gov/. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
[6] C. Duprat, J. M. Aristoff, and H. A. Stone, Dynamics of elastocapillary rise, Journal of Fluid Mechanics, 679 (2011), pp. 641–654.
[7] S. L. Goren, The instability of an annular thread of fluid, Journal of Fluid Mechanics, 12 (1962), pp. 309–319.
[8] M. D. Greenberg, Advanced Engineering Mathematics, Prentice-Hall, 1988.
[9] D. J. Griffiths and D. F. Schroeter, Introduction to Quantum Mechanics, Cambridge University Press, 2018.
[10] A. Jeffrey, Advanced Engineering Mathematics, Academic Press, 1st ed., 2001.
[11] A. Jolliffe, A form for \( \frac{1}{\sqrt{p_n(\mu)}} \), where \( p_n(\mu) \) is the Legendre polynomial of degree \( n \), Mess. Math, 49 (1919), pp. 125–127.
[12] E. Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, 9th edition, international edition ed., 2005.
[13] V. L. Makarov and T. Arazmyradov, The construction of particular solutions of resonance differential equations, Differentsial’nye Uravneniya, 14 (1978), pp. 1255–1261.
[14] S. H. Strogatz, D. M. Abrams, A. McRobie, B. Eckhardt, and E. Ott, Crowd synchrony on the millennium bridge, Nature, 438 (2005), pp. 43–44.
[15] R. Szmytkowski, *On the derivative of the Legendre function of the first kind with respect to its degree*, Journal of Physics A: Mathematical and General, 39 (2006), p. 15147.

[16] R. B. White, *Asymptotic Analysis of Differential Equations*, World Scientific, 2010.