GENERIC NONDEGENERACY FOR SOLUTIONS OF THE ALLEN-CAHN EQUATION UNDER A VOLUME CONSTRAINT IN CLOSED MANIFOLDS

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Abstract. Let $M^n$ be a connected closed smooth manifold with $n \geq 2$. We adapt the techniques in [MP09] and [GM11] to prove the generic nondegeneracy for solutions of the Van der Waals-Allen-Cahn-Hilliard equation under a volume constraint in $M$.

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1. Introduction and main result

Let $(M^n, g)$ be a connected closed smooth Riemannian manifold, where $n \geq 2$. Let $W: \mathbb{R} \to \mathbb{R}$ be a function of class $C^2$. Fix $\nu, \epsilon > 0$. A pair $(u, \lambda) \in H_g(M) \times \mathbb{R}$ is a solution for the Van der Waals-Allen-Cahn Hilliard equation under volume constraint $\nu$ when

$$(P_{W,\nu,\epsilon,g}) \quad \begin{cases} -\epsilon^2 \Delta_g u + W'(u) = \lambda \\ \int_M u d\mu_g = \nu \end{cases},$$

where $\mu_g$ is a measure induced by $g$ defined on the Borel subsets of $M$ and $H_g(M)$ is a convenient Sobolev space of functions defined in section 2.

In [BNAP20], the authors establish lower bounds on the number of solutions for $(P_{W,\nu,\epsilon,g})$ in function of topological invariants of $M$ for sufficiently small $\nu, \epsilon > 0$ and under specific hypotheses on the potential function $W$. In particular: if $(P_{W,\nu,\epsilon,g})$ only admits nondegenerate solutions, then Morse theory may be applied to prove that it admits at least $P_M(1)$ solutions, where $P_M(t)$ is the Poincaré polynomial of $M$.

Our main result is that under suitable growth conditions for $W'$ and $W''$, this is indeed the case generically with respect to $(\epsilon, g) \in [0, \infty] \times \mathcal{M}^k$, where $1 \leq k < \infty$ and $\mathcal{M}^k$ is the space of Riemannian metrics of class $C^k$ on $M$:

**Theorem 1.1.** Fix $g_0 \in \mathcal{M}^k$. Suppose that 1 and 2 hold. Then

$$D_{W,\nu}^* = \left\{ (\epsilon, g) \in [0, \infty] \times \mathcal{M}^k : \text{any solution } (u, \lambda) \in H_{g_0}(M) \times \mathbb{R} \text{ for } (P_{W,\epsilon,\nu,g}) \text{ is nondegenerate} \right\}$$

is an open dense subset of $[0, \infty] \times \mathcal{M}^k$.

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This result is obtained by the application of an abstract transversality theorem through an appropriate adaptation of the techniques in [MP09] and [GM11] to the context of this article.

More precisely, we say that a solution $(u, \lambda) \in H_g(M) \times \mathbb{R}$ for $(P_{W,\nu,\epsilon,g})$ is nondegenerate when the only pair $(v, \Lambda) \in H_g(M) \times \mathbb{R}$ which solves the linearized problem

$$
\begin{aligned}
-\epsilon^2 \Delta_g v + W''(u)v &= \Lambda \\
\int_M v \, d\mu_g &= 0
\end{aligned}
$$

is the trivial one $(v, \Lambda) = (0, 0)$.

In fact, this notion coincides with the Morse theoretic notion of a nondegenerate critical point for the functional $J_{W,\epsilon,g}: H_g(M) \times \mathbb{R} \to \mathbb{R}$ given by

$$
J_{W,\epsilon,g}(u, \lambda) = \int_M \frac{\epsilon^2}{2} g(\nabla u, \nabla u) + W(u) - \lambda u \, d\mu_g - \lambda \nu.
$$

Indeed, $J_{W,\epsilon,g}$ is a functional of class $C^2$ for which $(v, \Lambda)$ is a solution for $(Q_{W,\epsilon,g,u})$ if, and only if, $\int_M v \, d\mu_g = 0$ and $(v, \Lambda) \in \text{ker Hess}(J_{W,\epsilon,g})(u, \lambda)$. Therefore, $(u, \lambda)$ is a nondegenerate critical point of $J_{W,\epsilon,g}$ such that $\int_M u \, d\mu_g = \nu$.

For Differential Geometry, interest for the Van der Waals-Allen-Cahn-Hilliard equation under a volume constraint is justified by the results of [PR03], where Pacard and Ritoré showed that one can approach constant mean curvature hypersurfaces by the nodal sets of critical points for $J_{W,\epsilon,g,\lambda}$ as $\epsilon \to 0^+$. If we consider critical points without the volume constraint, these sets approach a minimal hypersurface.

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2. Preliminaries

Basic constructions. Fix $1 \leq k < \infty$. Denote by $S_k^k$ the Banach space of symmetric 2-covectors on $M$ of class $C^k$. The space $\mathcal{M}^k$ of Riemannian metrics on $M$ of class $C^k$ is an open convex cone in $S_k^k$.

Consider any $(\epsilon, g) \in ]0, \infty[ \times \mathcal{M}^k$. $(\epsilon, g)$ induces the following inner products on $C^\infty(M)$:

$$
\langle u, v \rangle_g = \int_M g(\nabla u, \nabla v) + uv \, d\mu_g;
$$

$$
E_{\epsilon,g}(u, v) := \int_M \epsilon^2 g(\nabla u, \nabla v) + uv \, d\mu_g.
$$

$H_g(M)$, $H_{\epsilon,g}(M)$ are, respectively, the Hilbert spaces endowed with $\langle \cdot, \cdot \rangle_g$, $E_{\epsilon,g}$ obtained as completions of $C^\infty(M)$. Similarly: given $1 \leq q < \infty$, $L^q_g(M)$ is the Banach space obtained as completion of $C^\infty(M)$ with respect to the norm

$$
\|u\|_{q,g} := \left( \int_M |u|^q \, d\mu_g \right)^{1/q}.
$$

One may check that the norms induced by $\langle \cdot, \cdot \rangle_g$, $E_{\epsilon,g}$ on $C^\infty(M)$ are equivalent. In particular, this implies $H_g(M) = H_{\epsilon,g}(M)$ as sets and that the canonical inclusion $H_g(M) \to H_{\epsilon,g}(M)$ is an isomorphism of Banach spaces. The same holds for
the canonical inclusion $H_{p'}(M) \to H_p(M)$ for any $g' \in \mathcal{M}_k$. For details, we refer the reader to [Heb00, Proposition 2.2].

**Considered setting.** Suppose that

\[(1) \quad \exists K_1 > 0 \forall t \in \mathbb{R}, \ |W'(t)| \leq K_1(1 + |t|^{p-1});\]
\[(2) \quad \exists K_2 > 0 \forall t \in \mathbb{R}, \ |W''(t)| \leq K_2(1 + |t|^{p-2});\]

for a certain $p \in [2, p_n]$, where $p_n = \infty$ for $n = 2$, $p_n = (2n)/(n - 2)$ for $n \geq 3$.

Fix $g_0 \in \mathcal{M}_k$. Consider any $(\epsilon, g) \in [0, \infty] \times \mathcal{M}_k$. Due to the Kondrakov theorem, the canonical inclusion $i_{\epsilon,g} : H_{\epsilon,g}(M) \to L_p^g(M)$ is a compact operator. Set $p' := p/(p - 1)$. We define $A_{\epsilon,g}$ as the adjoint of $i_{\epsilon,g}$ while considering the canonical Banach space isomorphisms $(L_p^g(M))' \simeq L_p^g(M)$ and $H_{\epsilon,g}(M) \simeq (H_{\epsilon,g}(M))'$:

**Definition 2.1.** $A_{\epsilon,g} = i_{\epsilon,g}^* : L_p^g(M) \to H_{\epsilon,g}(M)$.

**Remark 2.2.** $A_{\epsilon,g}$ is a compact self-adjoint operator and $E_{\epsilon,g}(A_{\epsilon,g}u, v) = \int_M uv \, d\mu_g$ for any $u, v \in H_g(M)$.

For details on lemmas 2.3 and 2.4 we refer the reader to [MP09, Lemmas 2.1, 2.3].

**Lemma 2.3.** $E : [0, \infty] \times \mathcal{M}_k \to \text{Bil}(H_{g_0}(M))$ is a map of class $C^1$, where $E(\epsilon, g) := E_{\epsilon,g}$. In particular,

$$dE_{(\epsilon,g)}[\eta, h](u, v) = 2c \eta \int_M g(\nabla u, \nabla v) \, d\mu_g + c^2 \int_M b_{g,h}(\nabla u, \nabla v) \, d\mu_g +$$
$$+ \frac{1}{2} \int_M (\text{tr}_g h) uv \, d\mu_g,$$

where $b_{g,h}$ is a symmetric $2$-covector on $M$ of class $C^k$ given locally by

$$(b_{g,h})_{ij} = (\text{tr}_g h) g^{ij}/2 - g^{ij} h_{kl} g^{kl}.$$

**Lemma 2.4.** $A : [0, \infty] \times \mathcal{M}_k \to B \left( L_p^{g_0}(M), H_{g_0}(M) \right)$ is a map of class $C^1$, where $A(\epsilon, g) := A_{\epsilon,g}$. In particular,

$$dE_{(\epsilon,g)}[\eta, h](A_{\epsilon,g}u, v) + E_{\epsilon,g}(dA_{\epsilon,g})[\eta, h]u, v) = \frac{1}{2} \int_M (\text{tr}_g h) uv \, d\mu_g.$$

$W' : \mathbb{R} \to \mathbb{R}$ is a function of class $C^1$ with suitable growth conditions, so $H_{g_0}(M) \ni u \mapsto W'(u) \in L_p^{g_0}(M)$ is a Nemytskii operator of class $C^1$. For details on this argument, we recommend the reference [Kav93]. This implies:

**Lemma 2.5.** The Nemytskii operator $B_W : H_{g_0}(M) \times \mathbb{R} \to L_p^{g_0}(M)$ given by $B_W(u, \lambda) = \lambda + u - W'(u)$ is a map of class $C^1$. In particular,

$$d(B_W)(u, \lambda)[v, \Lambda] = \Lambda + v - vW''(u).$$

In the next definition, we identify the space of constant real-valued functions on $M$ with $\mathbb{R}$:

**Definition 2.6.** Let $F_W : [0, \infty] \times \mathcal{M}_k \times (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R} \to H_{g_0}(M) \times \mathbb{R}$ be given by

$$F_W(\epsilon, g, u, \lambda) = \left( u - A_{\epsilon,g} \circ B_W(u, \lambda), \int_M u \, d\mu_g \right).$$
Using remark 2.2, we can prove that the set of solutions \((u, \lambda) \in (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R}\) for \(P_{W, \nu, e, g}\) is a level-set of \(F_W\):

**Remark 2.7.** \((u, \lambda) \in H_{g_0}(M) \times \mathbb{R}\) is a solution for \(P_{W, \nu, e, g}\) if, and only if, 
\(F_W(\epsilon, g, u, \lambda) = (0, \nu)\).

**Lemma 2.8.** \(F_W : [0, \infty] \times \mathcal{M}^k \times (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R} \to H_{g_0}(M) \times \mathbb{R}\) is a map of class \(C^1\). In particular, 
\[
d(F_W)_{(\epsilon, g, u, \lambda)}[\eta, h, v, \Lambda] = 
\left( v - A_{\epsilon, g} \circ d(B_W)_{(u, \lambda)}[v, \Lambda] - dA_{(\epsilon, g)}[\eta, h] \circ B_W(u, \lambda), \right.
\left. \int_M \frac{1}{2} (\text{tr}_g h) u + v \, d\mu_g \right)
\]

3. PROOF OF MAIN RESULT

Consider the following abstract transversality theorem:

**Theorem [HHL05, Theorem 5.4]** Let \(X, Y, Z\) be real Banach spaces and \(U, V\) be respective open subsets of \(X, Y\). Let \(F : V \times U \to Z\) be a map of class \(C^m\), where \(m \geq 1\). Let \(z_0 \in \text{im } F\). Suppose that

1. Given \(y \in V\), \(F(y, \cdot) : x \mapsto F(x, y)\) is a Fredholm map of index \(l < m\), i.e., \(dF(y, \cdot)_x : X \to Z\) is a Fredholm operator of index \(l\) for any \(x \in U\);
2. \(z_0\) is a regular value of \(F\), i.e., \(dF_{(y_0, x_0)} : Y \times X \to Z\) is surjective for any \((y_0, x_0) \in F^{-1}(z_0)\);
3. Let \(l : F^{-1}(z_0) \to Y \times X\) be the canonical embedding and \(\pi_Y : Y \times X \to Y\) be the projection of the first coordinate. Then \(\pi_Y \circ l : F^{-1}(z_0) \to Y\) is \(\sigma\)-proper, i.e., \(F^{-1}(z_0) = \bigcup_{s=1}^\infty C_s\), where given \(s = 1, 2, \ldots\), \(C_s\) is a closed subset of \(F^{-1}(z_0)\) and \(\pi_Y \circ l|_{C_s}\) is proper.

Then the set \(\{y \in V : z_0\) is a regular value of \(F(y, \cdot)\}\) is an open dense subset of \(V\).

The first step to prove our main result is the lemma that follows, in which we restrict ourselves to nonconstant solutions:

**Lemma 3.2.** Fix \(g_0 \in \mathcal{M}^k\). Suppose that [1] and [2] hold. Then
\[
\mathcal{D}_{W, \nu} = \{ (\epsilon, g) \in [0, \infty] \times \mathcal{M}^k : \text{any solution } (u, \lambda) \in (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R} \text{ for } P_{W, \nu, e, g} \text{ is nondegenerate} \}
\]
is an open dense subset of \([0, \infty] \times \mathcal{M}^k\).

Its proof consists of a direct application of the abstract transversality theorem. Specifically, we consider \(X = Z = H_{g_0}(M) \times \mathbb{R}\), \(Y = V = [0, \infty] \times \mathcal{S}^k\), \(U = (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R}\), \(F = F_W\) and \(z_0 = (0, \nu)\). We verify that its hypotheses hold in section 4.

After analysing the constant solutions for \(P_{W, \nu, e, g}\), we refine lemma 3.2 to prove our main result:

**Proof of Theorem 1.1.** Let \((\epsilon, g) \in [0, \infty] \times \mathcal{M}^k\) and \(\mathcal{U}\) be a neighborhood of \((\epsilon, g)\) in \([0, \infty] \times \mathcal{M}^k\).

\(\mathcal{D}_{W, \nu}^\ast \cap \mathcal{U}\) is not empty. Indeed, let \((\tau, \mathcal{F}) \in \mathcal{D}_{W, \nu}^\ast \cap \mathcal{U}\). If \((P_{W, \tau, \mathcal{F}, W})\) does not admit constant solutions, then \((\tau, \mathcal{F}) \in \mathcal{D}_{W, \nu}^\ast \cap \mathcal{U}\). Otherwise, the volume constraint
shows that the unique constant solution is \( \nu/\mu_\tau(M) \). This is a degenerate solution if, and only if, \( (Q_{W,\epsilon,g,u}) \) admits a nontrivial solution. This only happens when there exists \( j = 1, 2, \ldots \) such that

\[
\tau^2 = -\frac{W''(\nu/\mu_\tau(M))}{\alpha_j(\mathcal{g})},
\]

where \( \mathcal{E}_\tau = \{ \alpha_j(\mathcal{g}) : j = 1, 2, \ldots \} \) is the set of nonzero eigenvalues of \(-\Delta_\tau\). \( \mathcal{E}_\tau \) is a discrete subset of \([0, \infty[\), so there exists \( \hat{\epsilon} > 0 \) such that \((\hat{\epsilon}, \mathcal{g}) \in \mathcal{D}_{W,\nu} \cap \mathcal{U} \) and \((3)\). This implies \((\hat{\epsilon}, \mathcal{g}) \in \mathcal{D}_{W,\nu} \).

\( \mathcal{D}_{W,\nu} \) is an open subset of \([0, \infty[\times M^k \). Indeed, let \((\hat{\epsilon}, \hat{g}) \in \mathcal{D}_{W,\nu} \). If \((P_{W,\nu,\hat{\epsilon},\hat{g}}) \) does not admit constant solutions, the result is a corollary of lemma 3.2. Otherwise, note that \( M^k \ni g \mapsto W''(\nu/\mu_\tau(M)) = \mathcal{E}_\tau \) and \( M^k \ni g \mapsto \alpha_j(g) \in \mathbb{R} \) are continuous maps for any positive integer \( j \), so \((\hat{\epsilon}, \hat{g}) \) admits a neighborhood \( \mathcal{V} \) in \([0, \infty[\times M^k \) in which the constant solutions are nondegenerate. To conclude, \( \mathcal{V} \cap \mathcal{D}_{W,\nu} \) is a neighborhood of \((\epsilon, g)\) in \([0, \infty[\times M^k \) for which the respective Allen-Cahn equation does not admit degenerate solutions. \( \square \)

4. Technical steps

For a pair \((\epsilon, g) \in [0, \infty[ \times M^k \), let \( F_{W,\epsilon,g} : (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R} \to H_{g_0} \times \mathbb{R} \) be given by \( F_{W,\epsilon,g}(u,\lambda) = F_W(\epsilon, g, u, \lambda) \). We adopt similar notation when fixing other variables.

In lemma 4.1, we shall verify that the first hypothesis of theorem 3.1 holds. With that objective in mind, consider the following preliminary result:

**Lemma 4.1.** Let \( g \in M^k, C_g : H_{g_0}(M) \to \mathbb{R} \) be given by \( C_g(v) = \int_M v \, d\mu_g \) and \( T_g : H_{g_0}(M) \times \mathbb{R} \to H_{g_0}(M) \times \mathbb{R} \) be given by \( T_g(v, \Lambda) = (v, C_g(v)) \). Then \( T_g \) is a Fredholm operator of index 0.

**Proof.** \( C_g \) is a linear functional, so \( \text{codim } \ker C_g = 1 \) in \( H_{g_0}(M) \). This implies

\[
\text{codim } T_g(\ker C_g \times \mathbb{R}) = 2
\]

in \( H_{g_0}(M) \times \mathbb{R} \).

\[
T_g(\ker C_g \times \mathbb{R}) \cap T_g(\mathbb{R}(1,0)) = 0,
\]

so

\[
\text{codim } [T_g(\ker C_g \times \mathbb{R}) + T_g(\mathbb{R}(1,0))] = \text{codim } T_g(\ker C_g \times \mathbb{R}) - 1 = 1.
\]

\[
H_{g_0}(M) \times \mathbb{R} = (\ker C_g \times \mathbb{R}) \oplus (\mathbb{R}(1,0)),
\]

so \( \text{codim } \text{im } T_g = 1 \). \( \ker T_g = \{0\} \times \mathbb{R} \), so \( T_g \) is a Fredholm operator of index 0. \( \square \)

**Lemma 4.2.** Given \((\epsilon, g) \in [0, \infty[ \times M^k, F_{W,\epsilon,g} \) is a Fredholm map of index 0.

**Proof.** Fix \((u, \lambda) \in H_{g_0}(M) \times \mathbb{R} \) and let \( K_{W,\epsilon,g,u,\lambda} : H_{g_0}(M) \times \mathbb{R} \to H_{g_0}(M) \times \mathbb{R} \) be given by

\[
K_{W,\epsilon,g,u,\lambda}(v, \Lambda) = \left( A_{\epsilon,g} \circ d (B_{W})(u, \lambda), [v, \Lambda], 0 \right).
\]

\[
d (F_{W,\epsilon,g})(u,\lambda) = T_g - K_{W,\epsilon,g,u,\lambda},
\]

where \( T_g \) was defined in lemma 4.1. Therefore, it suffices to prove that \( K_{W,\epsilon,g,u,\lambda} \) is a compact operator to conclude that \( d (F_{W,\epsilon,g})(u,\lambda) \) is a Fredholm operator with index 0. This is indeed the case, because \( A_{\epsilon,g} \) is a compact operator. \( \square \)
Let us examine the second hypothesis of the abstract transversality theorem. Let \((\epsilon, g, u, \lambda) \in F_{W}^{-1}(0, \nu)\). To conclude that \(d \langle F_{W}\rangle_{(\epsilon, g, u, \lambda)}\) is surjective, it suffices to show that

\[
(4) \quad \left[ \operatorname{im} d \langle F_{W, \epsilon, g}\rangle_{(\epsilon, g, u, \lambda)} \right]_{\perp} \subset \operatorname{im} d \langle F_{W, \epsilon, u, \lambda}\rangle_{g},
\]

which we shall prove in lemma 4.3.

The following defines an inner product on \(H_{g_{0}}(M) \times \mathbb{R}\):

\[
\langle (u_{1}, t_{1}), (u_{2}, t_{2}) \rangle_{\epsilon, g} = E_{\epsilon, g} (u_{1}, u_{2}) + t_{1} t_{2}.
\]

This allows us to establish the characterization:

Remark 4.3. Let \((\epsilon, g) \in [0, \infty[ \times M^{k}\) and \((u, \lambda), (v, \Lambda) \in H_{g_{0}}(M) \times \mathbb{R}\). Then

\[
(v, \Lambda) \in \left[ \operatorname{im} d \langle F_{W, \epsilon, g}\rangle_{(\epsilon, g, u, \lambda)} \right]_{\perp} \text{ if, and only if, } (v, -\Lambda) \text{ is a solution for } \langle Q_{W, \epsilon, g, u} \rangle.
\]

We use this characterization to prove inclusion (4):

Lemma 4.4. Let \((\epsilon, g, u, \lambda) \in F_{W}^{-1}(0, \nu)\). Let \((v, -\Lambda) \in H_{g_{0}}(M) \times \mathbb{R}\) be a solution for \(\langle Q_{W, \epsilon, g, u} \rangle\). If

\[
\langle d \langle F_{W, \epsilon, u, \lambda}\rangle_{g} [\mu], (v, \Lambda) \rangle_{\epsilon, g} = 0
\]

for all \(h \in S^{k}\), then \((v, \Lambda) = (0, 0)\).

Proof. Due to lemmas 2.3, 2.4 and 2.8 the equation on the statement is rewritten

\[
(5) \quad \int_{M} \epsilon^{2} b_{g, h} \langle \nabla u, \nabla v \rangle + \frac{\langle \operatorname{tr} g \rangle}{2} \left[ \langle W'(u) - \lambda \rangle v + \Lambda u \right] d\mu_{g} = 0,
\]

where we recall that \(b_{g, h}\) is a symmetric \(2\)-covector on \(M\) of class \(C^{k}\) given locally by

\[
(b_{g, h})_{ij} = \left( \operatorname{tr} g \right) / 2 - g^{\alpha} h_{\alpha \beta} g^{\beta}.
\]

An argument with normal coordinates centered at arbitrary \(x \in M\) which considers specific perturbations of \(g\) proves that

\[
(6) \quad g \langle \nabla u, \nabla v \rangle = b_{g, h} \langle \nabla u, \nabla v \rangle = 0 \in L_{g_{0}}^{1} (M)
\]

for any \(h \in S^{k}\). For details, see [GMT11] Lemma 12.

Taking \(h = \varphi g\) for arbitrary \(\varphi \in C^{\infty}(M)\) shows that (5) and (6) imply

\[
(7) \quad \langle W'(u) - \lambda \rangle v + \Lambda u = 0 \in L_{g_{0}}^{1} (M).
\]

On one hand: integrating the equation above (7) yields

\[
(8) \quad \int_{M} (W'(u) - \lambda) v d\mu_{g} = -\Lambda v.
\]

On the other hand: taking into account (6) and the fact that \(u\) is a weak solution for \(-\epsilon^{2} \Delta_{g} u + W'(u) = \lambda\),

\[
\int_{M} W'(u) v d\mu_{g} = \int_{M} \lambda v d\mu_{g},
\]

\(\nu \neq 0\), so the last equation and (8) imply \(\Lambda = 0\). Due to (7), \(\Lambda = 0\) implies

\[
(W'(u) - \lambda) v = 0 \in L_{g_{0}}^{1} (M).
\]

If \(\lambda - W'(u) \equiv 0\), then \(u\) is a weak solution for \(-\epsilon^{2} \Delta_{g} u = 0\) which only happens with a constant \(u\). We do not consider constant solutions, so \(\lambda - W'(u)\) does not
vanish identically. Due to proposition 4.3, \( u, v \) are functions of class \( C^1 \). Therefore, 
\( \lambda - W'(u) \) is a continuous function which does not vanish identically.

In particular, \( v \) vanishes in a nonempty open subset of \( M \). In this context, we can use strong unique continuation ([PRS08 Theorem A.5]) in problem \( \{Q_{W,\epsilon,g,u}\} \) to conclude that \( v = 0 \in H_{g_0}(M) \).

**Proposition 4.5.** Fix \( (\epsilon, g) \in ]0, \infty[ \times \mathcal{M}^k \) and \( \alpha \in ]0, 1[ \). If \( (u, \lambda) \in H_{g_0}(M) \times \mathbb{R} \) is a solution for \( \{P_{W,\epsilon,g,u}\} \), then \( u \in C^{1,\alpha}(M) \). If it also holds that \( (v, \Lambda) \in H_{g_0}(M) \times \mathbb{R} \) is a solution for \( \{Q_{W,\epsilon,g,u}\} \), then \( v \in C^{1,\alpha}(M) \).

**Proof.** Regularity is a local problem, so we fix a coordinate system \( (\epsilon > q < 1) \). Arguing as in [JOS10 Theorem 12.2.2], one may show that given \( q > 1 \), \( W'((\bar{u}) \in L^q(\rho(\Omega)) \) implies \( u \in H^{2,q}(\rho(\Omega)) \). To conclude, we use the Sobolev Embedding Theorem.

The third hypothesis is proved analogously as [GM11 Lemma 11]:

**Lemma 4.6.** \( \pi_Y \circ \iota : F^{-1}_{W,\epsilon}(0, \nu) \to Y \) is \( \sigma \)-proper, where \( \pi_Y \) and \( \iota \) are defined in theorem 3.1.

**Proof.** Given \( s = 1, 2, \ldots \); let
\[
C_s = \left[ \frac{1}{s}, s \right] \times \mathcal{B}_s \times I(0, s) \setminus B(\mathbb{R}, 1/s) \times [-s, s], \cap F^{-1}_{W,\epsilon}(0, \nu),
\]
where \( \mathcal{B}_s, I(0, s) \) are respective open balls in \( \mathcal{S}^k, H_{g_0}(M) \) centered at 0 with radius \( s \) and 
\[
B(\mathbb{R}, 1/s) = \left\{ u \in H_{g_0}(M) : \inf_{v \in \mathbb{R}} \|u + v\|_{H_{g_0}} < 1/s \right\}.
\]

Fix a positive integer \( s \). Let us prove that \( \pi_Y \circ \iota |_{C_s} \) is a proper map. Let \( \{((\epsilon_n, g_n, u_n, \lambda_n)) \}_{n} \subset C_s \) be a sequence such that \( \lim_{n} g_n = g \in \mathcal{M}^k, \lim_{n} \epsilon_n = \epsilon \in [1/s, s] \) and given \( n \), \( (u_n, \lambda_n) \) is a solution for \( \{P_{W,\epsilon_n,g_n,u_n}\} \).

We claim that \( (u_n, \lambda_n) \) has a convergent subsequence. Due to the Kondrakov theorem, the canonical inclusion \( i_{\epsilon, g, t} : H_{\epsilon, g_0}(M) \to L^t_{g_0}(M) \) is a compact operator for any \( t \in [2, p^*_n] \); so \( (u_n) \) converges in \( L^t_{g_0}(M) \) up to subsequence to a certain \( u \in L^t_{g_0}(M) \). \( (\lambda_n) \) is bounded, so it converges up to a subsequence to a certain \( \lambda \in [-s, s] \). Arguing as in lemma 4.2 we see that \( \lim_{n} A_{\epsilon, g} \circ B_W(u_n, \lambda_n) = A_{\epsilon, g} \circ B_W(u, \lambda) \). We can use the Mean Value Inequality and lemma 2.4 to prove that, in fact, \( \lim_{n} A_{\epsilon, g} \circ B_W(u_n, \lambda_n) = A_{\epsilon, g} \circ B_W(u, \lambda) \).

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