FULLY NON-LINEAR DEGENERATE ELLIPTIC EQUATIONS IN COMPLEX GEOMETRY

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ABSTRACT. We derive an a priori real Hessian estimate for solutions of a large family of geometric fully non-linear elliptic equations on compact Hermitian manifolds, which is independent of a lower bound for the right-hand side function. This improves on the estimates of Székelyhidi [58] and additionally applies to elliptic equations with a degenerate right-hand side. As an application, we establish the optimal C^{1,1} regularity of envelopes of (θ, m)-subharmonic functions on compact Hermitian manifolds.

1. INTRODUCTION

Suppose that (X^n, ω) is a compact Hermitian manifold of complex dimension n without boundary. Let g denote the Riemannian metric corresponding to ω. We will additionally fix a real (1,1)-form χ_0 on X. For any \( u \in C^2(X) \), we will look at the form:

\[ \chi := \chi_0 + i\partial\bar{\partial}u, \]

and its associated Hermitian endomorphism \( A \) on \( T^{1,0}X \):

\[ A_{p\bar{q}} := g^{p\bar{q}} \chi_0. \]

Following the set-up of Székelyhidi [58], we will be interested in solving elliptic equations of the form:

\[ F(A) := f(\lambda_1, \ldots, \lambda_n) = h, \]

where here \( h \in C^\infty(X) \) is fixed and \( f : \Gamma \to \mathbb{R} \) is a concave symmetric function of the eigenvalues of \( A \) (which we denote by \( \lambda_1, \ldots, \lambda_n \), defined on the convex, open, symmetric cone \( \Gamma \subset \mathbb{R}^n \). We refer the reader to [58] or the beginning of Section 2 for a complete description of the necessary assumptions we will need on \( f \) and \( \Gamma \).

Equations of the form (1.1) were first studied by Caffarelli, Nirenberg, and Spruck, who solved the Dirichlet problem for domains in \( \mathbb{R}^n \), in their pioneering paper [9]. Their work was subsequently generalized to compact Riemannian manifolds by Guan [35], who also introduced the idea of using the existence of certain subsolutions to derive a priori \( C^2 \) estimates for (1.1). The complex case was recently studied by Székelyhidi [58], using what he

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calls $C$-subsolutions, which generalize the subsolutions of \cite{[35]}. In particular, he shows in \cite{[58]} that the existence of a $C$-subsolution implies a priori $C^\infty$-estimates for solutions to \eqref{1.1}.

The importance of \eqref{1.1} comes from the fact that it is general enough to simultaneously cover many natural geometric PDEs. The most well-known example is the complex Monge-Ampère equation, famously solved by Yau \cite{[65]} when $(M, \omega)$ is Kähler in his resolution of the Calabi conjecture \cite{[10]}. For general Hermitian $(M, \omega)$, the complex Monge-Ampère equation was solved by Tosatti-Weinkove \cite{[61]}, building on several earlier works (see for instance Cherrier \cite{[13]}, Hanani \cite{[38]}, Guan-Li \cite{[36]}, Tosatti-Weinkove \cite{[60]}, Zhang-Zhang \cite{[67]}, etc.).

Another well-known example is the complex Hessian equation. When $(M, \omega)$ is Kähler, Dinew-Ko\l dziej \cite{[25]} proved a Liouville type theorem for $(\omega, m)$-subharmonic functions in $\mathbb{C}^n$, which, when combined with the estimate of Hou-Ma-Wu \cite{[40]}, solved the complex Hessian equation. When $(M, \omega)$ is Hermitian, the complex Hessian equation was solved by Székelyhidi \cite{[58]} and Zhang \cite{[66]}, independently.

A third example is the complex Hessian quotient equation (these are equations of the form $f = (\sigma_m/\sigma_\ell)^{1/(m-\ell)}$, for $\ell < m$ and with $\sigma_m$ being the $m$th-symmetric function), which was solved by Székelyhidi \cite{[58]} when the right-hand side is constant (assuming the existence of a $C$-subsolution – see below). Previous special cases of this equation had been investigated by Song-Weinkove \cite{[53]}, in connection with stable points of the $J$-flow of Donaldson \cite{[27]} and Chen \cite{[12]}, and also by Fang-Lai-Ma \cite{[30]}. When the right-hand side is not constant, analogous results were obtained by Sun \cite{[54, 55, 56]} (see also Li \cite{[47]}, Guan-Sun \cite{[37]}).

The last example of a geometric equation that falls under the purview of \eqref{1.1} is the Monge-Ampère equation for $(n-1)$-plurisubharmonic functions (see \cite{[58], Section 7}), introduced by Fu-Wang-Wu \cite{[31]} as a generalization of the Monge-Ampère equation. This equation can be written as:

\begin{equation}
\left(\omega_h + \frac{1}{n-1}((\Delta \omega)u - \sqrt{-1} \partial \bar{\partial} u)\right)^n = e^h \omega^n
\end{equation}

where $\omega_h$ and $\omega$ are Hermitian metrics. When $\omega$ is a Kähler metric with non-negative orthogonal bisectional curvature, Fu-Wang-Wu \cite{[32]} solved \eqref{1.2}. When $\omega$ is a general Kähler metric, \eqref{1.2} was solved by Tosatti-Weinkove in \cite{[62]}. They later relaxed this to only requiring that $\omega$ be a Hermitian metric in \cite{[63]}.

Equation \eqref{1.1} is in general fully non-linear, and as such, strict ellipticity is not guaranteed without some assumption on the right-hand-side function $h$ – the necessary condition obtained in \cite{[58]} is to require that

$$\sup_{\Omega} f < h < \sup_{\Gamma} f,$$
where here we write:

$$\sup_{\partial \Gamma} f := \sup_{\lambda' \in \partial \Gamma} \limsup_{\lambda \to \lambda'} f(\lambda).$$

However, a number of geometric applications require that one consider the “degenerate” elliptic case when $$\sup_{\partial \Gamma} f = \inf_X h$$ – the most well-known such situation occurs when one studies the space of Kähler metrics on a compact Kähler manifold $$X$$. In [48], Mabuchi introduced a Riemannian metric on this indefinite dimensional space. Semmes [52] and Donaldson [28] independently showed that the geodesic equation in [48] can be rewritten as a Dirichlet problem for the homogeneous Monge-Ampère equation on $$X$$ cross annuli $$A \subset \mathbb{C}$$. Such geodesics have played a large role in many recent developments in Kähler geometry, and so understanding their regularity was an important question, resolved by Chu-Tosatti-Weinkove [17], who showed that these geodesics are $$C^{1,1}$$ regular. Their result builds on an extensive body of previous work, including that of Chen [11], Berman [5], and Donaldson [28], and is known to be optimal thanks to examples of Lempert-Vivas [44], Darvas-Lempert [22], and Darvas [21]. Later, Chu-McCleerey [16] generalized this to $$C^{1,1}$$-regularity of geodesics between Kähler metrics on singular Kähler varieties (on the smooth locus of the variety).

Another important application requiring degenerate right-hand sides is the study of the envelope:

$$P(b) := \sup \{ v \in \text{PSH}(X, \omega) \mid v \leq b \}$$

for a smooth obstacle function $$b \in C^\infty(X)$$ (here PSH($$X, \omega$$) is the set of all $$\omega$$-plurisubharmonic functions). It was shown by Tosatti [59] and Chu-Zhou [20] that $$P(b) \in C^{1,1}(X)$$. To see the connection, note that envelope $$P(b)$$ satisfies a free-boundary problem involving the contact set $$K := \{ P(b) = b \}$$:

$$F_{\text{MA}}(\omega_{P(b)}) = \chi_K \left( \frac{\omega_{b}^n}{\omega^n} \right)^{1/n},$$

where here we write $$\omega_b := \omega + i\partial\bar\partial b$$ and $$F_{\text{MA}}(A) := (\lambda_1 \cdots \lambda_n)^{1/n}$$ for the Monge-Ampère operator. Berman-Demailly [7, Corollary 2.5] show that Equation (1.3) follows if $$P(b)$$ has bounded Laplacian (the proof is in fact already basically contained in [44]), which was later shown to be true by Berman [6] when $$b \in C^\infty(X)$$. Berman’s result was later strengthened by Darvas-Rubenstein [23] to requiring that $$b$$ have only bounded Laplacian, and this was subsequentially used by Di Nezza-Trapani [24] to show (1.3) for more singular envelopes as well.

To deal with degenerate $$h$$, the most common method is to approximate $$h$$ by a family of non-degenerate $$h_i$$. One then shows that solutions to $$F(A) = h_i$$ satisfy a priori estimates independent of $$\inf_X h$$, so they pass to the limiting (degenerate) solution. In this paper, we will derive such an estimate for the real Hessian of $$u$$ (which we mean to be the Hessian of $$u$$...
with respect to the Levi-Civita connection induced by $g$), which allows us to apply this technique to the degenerate case of Equation (1.1).

As in [58], our estimates will depend on the existence of a $C$-subsolution to (1.1) – in Section 2, we show that the alternate definition given on [58, p.345] extends easily to the degenerate setting, so the notion of $C$-subsolution is still well-defined. Moreover, it will be clear from the definition that if $u$ is a $C$-subsolution to (1.1), then there exists a $\sigma_0 > 0$, depending only $(X, \omega)$, $\Gamma$, $f$, $\chi_0$, $u$ and $h$, such that $u$ is a $C$-subsolution to the non-degenerate equation:

$$F(A) = h + 2\sigma_0.$$

In Section 3, we show that $\sigma_0$ can be used to make the estimates independent of $\inf_X h$. Moreover (see Remark 2.5), it is easy to see that $\sigma_0$ can always be chosen to be 1 for many geometric PDEs (including the complex Monge-Ampère, the complex Hessian equations and the Monge-Ampère equation for $(n-1)$-plurisubharmonic functions), so that the estimates for these equations also do not depend on $\sigma_0$.

With this background, we can now state our main result:

**Theorem 1.1.** Let $u$ be a smooth solution of (1.1) with $\sup_X u = -1$. Suppose that $u$ is a $C$-subsolution of (1.1) and $\sup_{\partial \Gamma} f < h < \sup_{\Gamma} f$, then there exists a constant $C$ depending only on $(X, \omega)$, $\chi_0$, $u$, $\sigma_0$, $\sup_X h$, $\sup_X |\partial h|_g$, and a lower bound of $\nabla^2 u$ such that

$$\sup_X |u| + \sup_X |\partial u|_g + \sup_X |\nabla^2 u|_g \leq C,$$

where $\nabla$ is the Levi-Civita connection of $g$.

As mentioned, our main contribution is the real Hessian estimate. If $\partial X \neq \emptyset$, our argument gives an interior bound on the real Hessian (see Remark 4.7). As mentioned previously, there exist examples of smooth boundary data for the Dirichlet problem for the homogenous complex Monge-Ampère equation such that no $C^2$ solution exists. This implies that one should not expect a degenerate solution to (1.1) to be better than $C^{1,1}$ if $h$ is degenerate, meaning that our estimate should be optimal (although, to the best of the authors’ knowledge, a concrete example on a closed manifold appears to be unknown – c.f. [50, 26] for related examples).

To prove Theorem 1.1, our approach builds off previous such estimates for the Monge-Ampère equation – in particular, we proceed by trying to control the largest eigenvalue of (a perturbed version of) $\nabla^2 u$ using the maximum principle (cf. [19]). The main task is to control a third order term of the form:

$$\sum_i F^\tilde{\iota} |u_{V_1 V_2 i}|^2 / \lambda_i^2,$$

where $V_1$ is the unit eigenvector corresponding to $\lambda_1$ and $F^\tilde{\iota}$ is the $\tilde{\iota}$-derivative of $F$ ($F(A) = \log(\lambda_1 \cdot \ldots \cdot \lambda_n)$). Previous arguments for the
Monge-Ampère equation \((19, 15)\) heavily utilize special properties of the corresponding \(F\). More precisely, for the Monge-Ampère equation, both \(F_{i}\) and \(F_{ijkl}\) have simple expressions in terms of the solution metric – in the general context of Equation \((1.1)\), this is no longer true however, so we need to apply new techniques to control \((1.4)\).

Our solution starts with the term:

\[
\sum_{i \neq q} \frac{(F_{ij} - F_{ij}) |\chi_{i} \cdot \nabla_{V}|^{2}}{\lambda_{1}(\chi_{i} - \chi_{q})},
\]

which is non-negative thanks to the concavity of \(F\). In order to effectively use this term, our first contribution is to finesse a technique of Hou-Ma-Wu \([40]\) (which is in turn based on ideas developed for the real Hessian equation by Chou-Wang \([14]\)), to control the individual summands in \((1.4)\) based on the relative sizes of the corresponding \(F_{ij}\). This extracts more non-negative terms than previous versions of this technique (cf. Hou-Ma-Wu \([40]\), Székelyhidi \([58]\), Chu-Tosatti-Weinkove \([19]\), etc.), allowing us to control most of the terms in \((1.4)\).

Even then, \((1.5)\) is not sufficient to control all of \((1.4)\), as it is possible that \(F_{ij} - F_{ij}\) might be quite small. In this case however, we show that the corresponding non-positive terms are also quite small, picking up an extra factor of \(\lambda_{-2}\) compared to the other summands in \((1.4)\). Thus, we show that these terms can be controlled by the addition of an extra quantity to the maximum principle depending on \(\nabla^{2}u\), similar to \([15]\). As a remark, we point out that the extra term in \([15]\) depends on \(\partial \bar{u}\), and that the argument there also depends heavily on the structure of \(F_{MA}\). In contrast, our new extra term depends on \(\nabla^{2}u\), making the argument significantly more delicate.

A direct consequence of Theorem \(1.1\) is the \(C^{1,1}\)-estimate and the existence of \(C^{1,1}\) solutions in the degenerate case, provided there exists an approximating family of solutions to the non-degenerate equations:

**Theorem 1.2.** Suppose that \(h \in C^{2}(X)\) satisfies \(\sup_{\partial \Gamma} f \leq h < \sup_{\Gamma} f\), and that \(h_{\varepsilon} \in C^{2}(X)\) satisfy \(\sup_{\partial \Gamma} f \leq h_{\varepsilon} < \sup_{\Gamma} f\) and \(h_{\varepsilon} \to h\) in \(C^{2}\). Let \(u\) be a \(C\)-subsolution to the degenerate equation

\[
F(A(u)) = h(x),
\]

and let \(u_{\varepsilon}\) be smooth solutions to the non-degenerate equations:

\[
F(A(u_{\varepsilon})) = h_{\varepsilon}(x)
\]

for all \(0 < \varepsilon\) sufficiently small. Then there exists a constant \(C\), independent of \(\varepsilon\) and depending only on \((X, \omega), \chi_{0}, u, \sigma_{0}, \sup X h, \sup X |\partial h|_{g}\), and a lower bound of \(\nabla^{2} h\) such that

\[
\sup_{X} |u_{\varepsilon}| + \sup_{X} |\partial u_{\varepsilon}|_{g} + \sup_{X} |\nabla^{2} u_{\varepsilon}|_{g} \leq C,
\]

where \(\nabla\) is the Levi-Civita connection of \(g\).
In particular, up to extracting a subsequence, the \(u_\varepsilon\) converge to a \(u \in C^{1,1}(X)\) solving the degenerate version of equation (1.1):

\[
f^*(\lambda(u)) = h(x),
\]

where here \(f^*\) is the upper semi-continuous extension of \(f\) to \(\Gamma\).

As discussed in [58], the \textit{a priori} estimates in Theorem 1.1 cannot be used to show the existence of solutions to (1.1) (e.g. using the continuity method) without further assumptions to guarantee the existence of appropriate \(C\)-subsolutions along the continuity path. As such, it is necessary to assume the existence of the solutions \(u_\varepsilon\) in Theorem 1.2. It is quite non-trivial to come up with natural conditions guaranteeing the existence of \(C\)-subsolutions – see, for instance, the well-known conjectures of Lejmi-Székelyhidi [43] and Székelyhidi [58] for the complex Hessian quotient equation. For the degenerate complex Hessian equation however, the zero function is automatically a \(C\)-subsolution, which allows us to solve degenerate equations without further assumptions:

**Corollary 1.3.** Let \((X, \omega)\) be a compact Hermitian manifold and \(1 \leq m \leq n\). Let \(\theta\) be a (strictly) \(m\)-positive form on \(X\) and write \(\text{mSH}(M, \theta)\) for the set of all \((\theta, m)\)-subharmonic functions on \(X\) with respect to \(\omega\) (see Section 5 for definitions). Then for any non-negative function \(h\) on \(X\) such that \(\int_X h\omega^n > 0\) and \(h^{\frac{1}{m}} \in C^2(X)\), there exists a pair \((u, c) \in C^{1,1}(X) \times \mathbb{R}_+\) such that

\[
\begin{align*}
(\theta + \sqrt{-1}\partial\bar{\partial} u)^m \wedge \omega^{n-m} &= c h\omega^n, \\
u \in \text{mSH}(X, \theta), \quad \sup_X u &= -1.
\end{align*}
\]

Lastly, in Section 6, we give a geometric application of our result, by showing how to adapt the estimates in Theorem 1.1 to get regularity of envelopes of \(m\)-subharmonic functions, generalizing the above results for the Monge-Ampère equation.

**Theorem 1.4.** Let \((X, \omega)\) be a compact Hermitian manifold and \(\theta\) a (strictly) \(m\)-positive form. If \(b \in C^{1,1}(X)\), we define the envelope:

\[
P_{m,\theta}(b) := \sup\{v \in \text{mSH}(X, \theta) \mid v \leq b\}.
\]

Then \(P_{m,\theta}(b) \in C^{1,1}(X)\). In particular, \(P_{m,\theta}(b)\) solves:

\[
(\theta + i\partial\bar{\partial} P_{m,\theta}(b))^m \wedge \omega^{n-m} = \chi_K \theta^m_b \wedge \omega^{n-m},
\]

where \(\theta_b = \theta + i\partial\bar{\partial} b\) and \(K = \{P_{m,\theta}(b) = b\}\) is the contact set.

We now outline the contents of the rest of the paper. Section 2 recalls some necessary background material, largely along the lines of [58]. In section 3, we show that the \textit{a priori} estimates of [58] depend only on the background data in Theorem 1.1. We prove our main result, Theorem 1.1, in Section 4. In Section 5, we prove Theorem 1.2 and show how to get Corollary 1.3.
Finally, in Section 6, we apply our estimates to $P_m(b)$ and prove Theorem 1.4.

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2. Background and Notation

Recall that $(X^n, \omega)$ is a compact Hermitian manifold, and we write $g$ for the Riemannian metric corresponding to $\omega$. We will also let $\nabla$ denote the Levi-Civita connection induced by $g$.

We begin by recalling all the necessary assumptions required on Equation (1.1). First, we shall suppose that $\Gamma \subseteq \mathbb{R}^n$ is an open, symmetric, convex cone with vertex at the origin. Additionally, we will require that $\Gamma$ contain the positive orthant:

$$\Gamma_n \subseteq \Gamma,$$

$$\Gamma_n = \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_i > 0 \text{ for each } i\}.$$

Now, we will assume that $f : \Gamma \to \mathbb{R}$ is a smooth function on $\Gamma$ such that

(i) $f$ is concave,
(ii) $f_i = \frac{\partial f}{\partial \lambda_i} > 0$ for each $i$.
(iii) For any $\sigma < \sup_{\partial \Gamma} f$ and $\lambda \in \Gamma$, we have $\lim_{t \to \infty} f(t\lambda) > \sigma$.

Let $\lambda(u) \in \mathbb{R}^n$ be the function taking $u$ to the un-ordered $n$-tuple of eigenvalues of $A$, the Hermitian endomorphism on $T_{1,0}X$ defined in the introduction. Since both $\Gamma$ and $f$ are symmetric, we have that $f(\lambda(u))$ is well-defined and smooth on $X$. We are then interested in the equation:

(2.1) \[
\begin{cases}
  f(\lambda(u)) = h(x) \in C^{\infty}(X) \\
  \sup_{\partial \Gamma} f < h < \sup_{\Gamma} f.
\end{cases}
\]

We say (2.1) is a non-degenerate equation (we shall also refer to its solutions as being non-degenerate). We will also be interested in the degenerate version of equation (2.1), which is when $\sup_{\partial \Gamma} f = \inf_X h$; more precisely, when we say that (2.1) is degenerate, we actually refer to the equation:

(2.2) \[
\begin{cases}
  f^*(\lambda(u)) = h(x) \in C^{\infty}(X) \\
  \sup_{\partial \Gamma} f \leq h < \sup_{\Gamma} f,
\end{cases}
\]

where $f^*$ is the upper semi-continuous extension of $f$ to $\overline{\Gamma}$. We will also abuse terminology slightly further, and say that solutions to (2.2) are degenerate solutions to equation (2.1). Note that a degenerate solution $u$ will in general only have $\lambda(u) \in \overline{\Gamma}$.

For any $\sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f)$, we write

$$\Gamma^\sigma = \{\lambda \in \Gamma \mid f(\lambda) > \sigma\}.$$
It is easy to check that $\Gamma^\sigma$ is a convex open set, and that the level set $f^{-1}(\sigma) = \partial \Gamma^\sigma$ is a smooth hypersurface in $\mathbb{R}^n$ which doesn’t intersect the boundary of $\Gamma$.

We now recall the definition of $C$-subsolutions for non-degenerate equations, as given by Székelyhidi [58, Definition 1].

**Definition 2.1.** A function $u$ on $X$ is said to be a $C$-subsolution of the non-degenerate equation $F(A) = h$, if at each $x \in X$, the set

$$(\mu(x) + \Gamma_n) \cap \partial \Gamma^b(x)$$

is bounded, where

$$\mu(x) = \lambda(g^{\sigma} \nabla_{\sigma}(x)), \quad \chi = \chi_0 + \sqrt{-1}\partial \overline{\partial} u.$$

In [58], Székelyhidi also proved the following equivalent characterization of $C$-subsolutions. Following Trudinger [64], define

$$\tilde{\Gamma} = \{ \mu \in \mathbb{R}^n \mid \text{there exists } t > 0 \text{ such that } \mu + te_i \in \Gamma \text{ for all } i \}.$$  

Note that $\tilde{\Gamma}$ is also an open, symmetric, convex cone. For each $\mu \in \tilde{\Gamma}$, we define

$$f_{\infty,i}(\mu) = \lim_{t \to \infty} f(\mu + te_i), \quad f_{\infty,\min}(\mu) = \min_{1 \leq i \leq n} f_{\infty,i}(\mu),$$

where here $e_i \in \mathbb{R}^n$ is the $i^{th}$ standard basis vector. Note that, from the concavity of $f$, we have that $f_{\infty,\min}$ is concave and is thus either everywhere finite on $\tilde{\Gamma}$, or uniformly $+\infty$ (cf. [64]).

**Proposition 2.2.** (Székelyhidi, [58, p.345]) The smooth function $u$ is a $C$-subsolution to the non-degenerate equation $F(A) = h$ if and only if:

$$\mu(x) \in \tilde{\Gamma} \text{ and } f_{\infty,\min}(\mu) > h(x), \text{ for any } x \in X.$$  

**Proof.** We first check the forward implication. Suppose that $x \in X$, $t > 0$, and $i \in \{1, \ldots, n\}$ are such that $\mu(x) + te_i \notin \Gamma$. Since $\Gamma$ is an open convex cone containing $\Gamma_n$, there must exist some $s_0 > 0$ such that $\mu(x) + te_i + s_01 \in \partial \Gamma$. Further, our assumptions on $f$ imply that:

$$\lim_{s \to \infty} f(\mu(x) + te_i + s1) = \sup_{\Gamma} f.$$  

Since $\sup_{\partial \Gamma} f < h < \sup_{\Gamma} f$, it follows that there exists some $s_1 > s_0 > 0$ with:

$$f(\mu(x) + te_i + s_11) = h(x).$$

Thus:

$$\mu(x) + te_i + s_11 \in (\mu(x) + \Gamma_n) \cap \partial \Gamma^b(x).$$

So if $u$ is a $C$-subsolution, it follows that, for any $i$, the set of all $t$ such that $\mu(x) + te_i \notin \Gamma$ is bounded; which is to say, $\mu(x) \in \tilde{\Gamma}$.

We check the second requirement in a similar manner – let $\mu(x) \in \tilde{\Gamma}$ be such that:

$$f_{\infty,i}(\mu(x)) = \lim_{t \to \infty} f(\mu(x) + te_i) \leq h(x),$$

where
for some \( i \in \{1, \ldots, n\} \). Since \( f \) is strictly increasing in the \( \mathbf{e}_i \)-direction, we have:

\[
f(\mu(x) + t\mathbf{e}_i) < h(x) \quad \text{for all } t \text{ sufficiently large.}
\]

But as we just argued, this implies there is an \( s_1 > 0 \) such that:

\[
m(x) + t\mathbf{e}_i + s_1 \mathbf{1} \in (\mu(x) + \Gamma_n) \cap \partial \Gamma^h(x)
\]

for each \( t \) sufficiently large. Clearly,

\[
\lim_{t \to \infty} |\mu(x) + t\mathbf{e}_i + s_1 \mathbf{1}| \geq \lim_{t \to \infty} (t - |\mu(x)|) = +\infty.
\]

Thus, if \( u \) is a \( C \)-subsolution, then we cannot have \( f_{\infty, i}(\mu(x)) \leq h(x) \) for any \( i \). This concludes the forward direction.

Conversely, suppose that \( \mu(x) \in \tilde{\Gamma} \) and \( f_{\infty, \min}(\mu(x)) > h(x) \) but \( (\mu(x) + \Gamma_n) \cap \partial \Gamma^h(x) \) is unbounded. Then there exists an unbounded sequence \( \{v_j\} \in (\mu(x) + \Gamma_n) \cap \partial \Gamma^h(x) \) — if we write \( v_j = (v_{j1}, \ldots, v_{jn}) \), then after taking a subsequence, we can assume that:

\[
\lim_{j \to \infty} v_{j0}^* = +\infty,
\]

for at least one index \( i_0 \). But then:

\[
f_{\infty, \min}(\mu(x)) \leq f_{\infty, i_0}(\mu(x)) \leq \lim_{j \to \infty} f(\mu(x) + v_j) = h(x),
\]

contradicting with \( f_{\infty, \min}(\mu(x)) > h(x) \). \( \square \)

We will take this alternate characterization as our definition of a \( C \)-subsolution for degenerate equations:

**Definition 2.3.** We say that a function \( u \) on \( X \) is a \( C \)-subsolution to the degenerate equation \((2.2)\) if at each \( x \in X \), we have:

\[
\mu(x) \in \tilde{\Gamma}, \quad f_{\infty, \min}(\mu) > h(x).
\]

The advantage of using this characterization of \( C \)-subsolutions for degenerate equations is that it is easily seen to be preserved under small perturbations of \( h \). As mentioned above, \( f_{\infty, \min} \) is either \( \equiv +\infty \), or everywhere finite and concave on the open cone \( \tilde{\Gamma} \); it follows that, in this second case, \( f_{\infty, \min}(\mu(x)) \) is continuous on \( X \). Thus, we immediately get:

**Proposition 2.4.** If \( u \) is a \( C \)-subsolution to the non-degenerate equation \((2.1)\) or the degenerate equation \((2.2)\), then there exists a constant \( \sigma_0 > 0 \), depending only on \( (X, \omega), \Gamma, f, \chi_0, \mu \) and \( h \), such that \( u \) is a \( C \)-subsolution to the non-degenerate equation:

\[
F(A) = h + 2\sigma_0.
\]

Consequently, if \( \tilde{h} \in C^\infty(X) \) satisfies \( \sup_{\tilde{\Gamma}} f \leq \tilde{h} < \sup_{\Gamma} f \) and \( |\tilde{h} - h| < 2\sigma_0 \), then \( u \) is also a \( C \)-subsolution to \( F(A) = \tilde{h} \).

Given this, we will often refer to a \( C \)-subsolution to \((2.2)\) as a \( C \)-subsolution to \((2.1)\), leaving context to dictate if the equation is degenerate or not.
Remark 2.5. For many geometric PDEs (including the complex Monge-Ampère equation, the complex Hessian equation and the Monge-Ampère equation for \((n-1)\)-plurisubharmonic functions), it is easy to see that \(\sup_{\Gamma} f = +\infty\) and \(f_{\infty, \min} \equiv +\infty\). This implies that one can always choose \(\sigma_0 = 1\) in this case, i.e. it can be taken to be a universal constant independent of \(u\) and \(h\).

3. Estimates for the Complex Hessian

In this section, we will show that the \textit{a priori} estimates of [58] hold in the degenerate setting. Specifically, we claim:

**Theorem 3.1.** Let \(u\) be a smooth solution of (1.1) with \(\sup_X u = -1\). Suppose that \(u\) is a \(C\)-subsolution of (1.1), then there exists a constant \(C\) depending only on \((X, \omega), \Gamma, f, \chi_0, u, \sup_X h, \sup_X |\partial h|_g\) and on a lower bound of \(\sqrt{-1} \partial \bar{\partial} h\) such that

\[
\sup_X |u| + \sup_X |\partial u|_g + \sup_X |\partial^2 u|_g \leq C.
\]

In [58], Székelyhidi established that

\[
\sup_X |u| \leq C \quad \text{and} \quad \sup_X |\partial \bar{\partial} u|_g \leq C \sup_X |\partial u|_g^2 + C
\]

for some constant \(C\) depending only on \((X, \omega), \Gamma, f, \chi_0, u, \sup_X h, \sup_X |\partial h|_g\), a lower bound of \(\sqrt{-1} \partial \bar{\partial} h\), and on four geometric constants, \(\kappa, \tau, R, \delta\), which might depend on \(\inf_X h\). He also establishes the \(C^1\) estimate using a blow-up argument, which can be seen to depend only on \(C, (X, \omega), \Gamma, f, \chi_0\), and \(\sup_X h\). Thus, to prove Theorem 3.1 it will be enough to check that the constants \(\kappa, \tau, R, \delta\) can be made independent of \(\inf_X h\).

3.1. The constants \(\kappa\) and \(\tau\): We begin by showing that, if one is more careful with the geometry of the \(C\)-subsolution, the constant \(\kappa\) in [58 Proposition 5] (which is a refinement of [35 Theorem 2.16]) can be made quite explicit:

**Lemma 3.2.** Suppose that \(\mu \in \overline{\Gamma}, \sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f)\) are such that:

\[
\sigma + \sigma_0 < \sup_{\Gamma} f, \quad \mu - 2\delta \mathbf{1} \in \overline{\Gamma},
\]

and

\[
(\mu - 2\delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^{\sigma + \sigma_0} \subseteq B_R(0),
\]

for some constants \(\delta, R, \sigma_0 > 0\). Then there exists a constant \(\kappa > 0\), depending only on \(\mu, \delta, R, n\) and \(\Gamma\) such that, for any \(\lambda \in \partial \Gamma^\sigma\), we either have:

1. \(\sum_i f_i(\lambda)(\mu_i - \lambda_i) > \kappa \mathcal{F}(\lambda)\), or
2. \(f_i(\lambda) > \kappa \mathcal{F}(\lambda)\) for all \(i = 1, \ldots, n\),

where here \(f_i := \frac{\partial f}{\partial \lambda_i}\) and \(\mathcal{F}(\lambda) = \sum_i f_i(\lambda)\).
Proof. We start with a claim:

Claim: Suppose that \( \lambda' \in (\mu - 2\delta \mathbf{1} + \overline{\Gamma_n}) \cap (\mathbb{R}^n \setminus \overline{\Gamma^\sigma + \sigma_0}) \). Then for any non-zero \( v \in \overline{\Gamma_n} \), there exists a \( t > 0 \) such that \( \lambda' + tv \in \Gamma \) and \( f(\lambda' + tv) \geq \sigma + \sigma_0 \).

Proof of Claim: That \( \lambda' + tv \in \Gamma \) for any non-zero \( v \in \overline{\Gamma_n} \) and \( t \) sufficiently large follows immediately from the fact that \( \mu - 2\delta \mathbf{1} \in \overline{\Gamma} \).

Suppose then for the sake of a contradiction that the other conclusion does not hold, i.e. that there exists a non-zero \( v \in \overline{\Gamma_n} \) such that \( f(\lambda' + tv) < \sigma + \sigma_0 \) for all \( t > 0 \) sufficiently large. By property (ii) of \( f \), for each \( t \) there exists a \( w_t \in \Gamma_n \) such that \( f(\lambda' + tv + w_t) = \sigma + \sigma_0 \), i.e. \( \lambda' + tv + w_t \in \partial \Gamma^\sigma + \sigma_0 \).

Since \( tv + w_t \in \Gamma_n \), it follows that:

\[
\lambda' + tv + w_t \in (\mu - 2\delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^\sigma + \sigma_0 \quad \text{for each } t.
\]

By (3.1), the above set is bounded though, so we have:

\[
|\lambda' + tv + w_t| \leq R \quad \text{for each } t,
\]

which is clearly absurd, as \( |tv + w_t| \to \infty \) as \( t \to \infty \).

We now begin the proof proper. Consider the set:

\[
A_{\delta} := (\mu - \delta \mathbf{1} + \overline{\Gamma_n}) \cap (\mathbb{R}^n \setminus \overline{\Gamma^\sigma + \sigma_0}).
\]

Suppose for now that \( A_{\delta} \) is non-empty – we will deal with the case \( A_{\delta} = \emptyset \) at the end of the proof. The assumption that \( A_{\delta} \neq \emptyset \) is clearly equivalent to assuming that \( \mu - \delta \mathbf{1} \in \mathbb{R}^n \setminus \overline{\Gamma^\sigma + \sigma_0} \).

For each \( v \in A_{\delta} \), we define the associated convex cone:

\[
C_v := \{ w \in \mathbb{R}^n \mid v + tw \in (\mu - 2\delta \mathbf{1} + \overline{\Gamma_n}) \cap \overline{\Gamma^\sigma + \sigma_0} \text{ for some } t > 0 \}.
\]

Applying the above claim twice (once with \( \lambda' = \mu - \delta \mathbf{1} \) and once with \( \lambda' = \mu - 2\delta \mathbf{1} \)), we see that

- \( \overline{\Gamma_n} \setminus \{0\} \subset C_{\mu - \delta \mathbf{1}} \), and
- for each index \( i = 1, \ldots, n \), set:

\[
T_i := \{ t > 0 \mid \mu - 2\delta \mathbf{1} + te_i \in \overline{\Gamma^\sigma + \sigma_0} \}
\]

is non-empty. Set \( t_i := \min T_i \).

Let \( 1_i := \sum_{j \neq i} e_j \). The second point implies that we can actually find a larger cone than \( \overline{\Gamma_n} \setminus \{0\} \) inside \( C_{\mu - \delta \mathbf{1}} \) by rewriting:

\[
\mu - \delta \mathbf{1} + (t_i - \delta)e_i - \delta 1_i = \mu - 2\delta \mathbf{1} + t_i e_i \in (\mu - 2\delta \mathbf{1} + \overline{\Gamma_n}) \cap \overline{\Gamma^\sigma + \sigma_0}.
\]

By definition, we thus get:

\[
(t_i - \delta)e_i - \delta 1_i \in C_{\mu - \delta \mathbf{1}}.
\]

It follows now that if \( w = (w_1, \ldots, w_n) \in C_{\mu - \delta \mathbf{1}}^* \) (the dual cone of \( C_{\mu - \delta \mathbf{1}} \)) then we must have:

\[
\langle w, (t_i - \delta)e_i - \delta 1_i \rangle \geq 0.
\]
Since $\Gamma_n \subset C_{\mu - \delta 1}$, then $C_{\mu - \delta 1}^* \subset \Gamma_n$. Each component of $w$ must be positive, so we have:

$$t_i w_i \geq (t_i - \delta) w_i \geq \delta \sum_{j \neq i} w_j.$$  

Squaring both sides and dropping cross-terms gives:

$$t_i^2 w_i^2 \geq \delta^2 \left( \sum_{j \neq i} w_j \right)^2 \geq \delta^2 \sum_{j \neq i} w_j^2 = \delta^2 (|w|^2 - w_i^2),$$

implying

$$w_i^2 \geq \frac{\delta^2}{t_i^2} |w|^2.$$  

It is clear that $\mu - 2\delta 1 + t_i e_i \in \partial \Gamma^\sigma + \sigma_0$. By (3.1), $|\mu - 2\delta 1 + t_i e_i| \leq R$, so:

$$t_i \leq R + |\mu - 2\delta 1| \leq C(\mu, \delta, R, n).$$

It thus follows that:

$$w_i \geq \frac{1}{C} |w|$$

for any vector $w \in C_{\mu - \delta 1}^*$ and some constant $C$ independent of $\sigma$ and $\sigma_0$.

We now show the same result holds for any $v \in A_\delta$, i.e.,

$$w_i \geq \frac{1}{C} |w| \text{ for any } w \in C_v^*.$$  

This will follow immediately if we can show:

$$C_{\mu - \delta 1} \subseteq C_v.$$  

This is quite clear however— suppose that $w \in C_{\mu - \delta 1}$, so that there exists $t > 0$ with:

$$\mu - \delta 1 + tw \in (\mu - 2\delta 1 + \Gamma_n) \cap \Gamma^\sigma + \sigma_0.$$  

Since $v \in A_\delta$, then there exists a $w_2 \in \Gamma_n$ with $\mu - \delta 1 + w_2 = v$. It follows then that:

$$\sigma + \sigma_0 \leq f(\mu - \delta 1 + tw) \leq f(v + tw),$$

so $w \in C_v$ by definition.

We may conclude as follows (we now drop the assumption that $A_\delta$ is non-empty). Let $\lambda \in \partial \Gamma^\sigma$ and define $T_\lambda$ to be the tangent plane to $\partial \Gamma^\sigma$ at $\lambda$ with inward normal vector

$$n_\lambda = (f_1(\lambda), \ldots, f_n(\lambda)).$$

Define $H_\lambda$ to be the upper-half space:

$$H_\lambda = \{ w \in \mathbb{R}^n \mid (w - \lambda) \cdot n_\lambda \geq 0 \}.$$  

Note that $\Gamma^\sigma + \sigma_0 \subseteq \Gamma^\sigma \subseteq H_\lambda$, by concavity of $f$. We now have two cases:

**Case 1:** $T_\lambda \cap A_\delta \neq \emptyset$.  


In this case, choose some \( v \in T_\lambda \cap A_\delta \). By definition, the cone \( v + C_v \) has cross section \( (\mu - 2\delta 1 + \Gamma_\sigma) \cap \partial \Gamma_{\sigma + \sigma_0} \), so clearly
\[
v + C_v \subset H_\lambda.
\]
It follows then that \( n_\lambda \in C_v^* \) (as \( (v - \lambda) \cdot n_\lambda = 0 \)), so we get
\[
f_i(\lambda) \geq \frac{1}{C} |\nabla f|.
\]
Combining this with \( |\nabla f| \geq \frac{1}{\sqrt{n}} F > 0 \) (cf. [58, (19)]), we see that (2) holds.

**Case 2:** \( T_\lambda \cap A_\delta = \emptyset \).

There are two possibilities: either \( A_\delta = \emptyset \) or \( A_\delta \neq \emptyset \). As mentioned previously, \( A_\delta = \emptyset \) if and only if \( \mu - \delta 1 \in \Gamma_{\sigma + \sigma_0} \subset H_\lambda \). If alternatively \( A_\delta \neq \emptyset \), then \( A_\delta \) lies on one side of \( T_\lambda \), as \( A_\delta \cap T_\lambda = \emptyset \). Thanks to the claim, \( A_\delta \cap \Gamma_{\sigma + \sigma_0} = \emptyset \), so we must have \( A_\delta \subset H_\lambda \).

Thus, in both cases we have \( \mu - \delta 1 \in H_\lambda \). It then follows that
\[
(\mu - \delta 1 - \lambda) \cdot n_\lambda \geq 0,
\]
which implies
\[
\sum_i f_i(\lambda) (\mu_i - \lambda_i) \geq \delta F(\lambda),
\]
finishing the proof.

The next lemma is essentially the same as [58, Lemma 9].

**Lemma 3.3.** For any \( \sup_{\partial \Gamma} f < \sigma \leq \Lambda < \sup_{\Gamma} f \), there exists a constant \( \tau > 0 \) depending only on \( \Lambda, f \) and \( \Gamma \) such that
\[
F(\lambda) > \tau \quad \text{for any } \lambda \in \partial \Gamma_{\sigma}.
\]

**Proof.** Take \( \sigma' \in (\Lambda, \sup_{\Gamma} f) \). By [58, Lemma 9], there exists a constant \( N \) depending only on \( \Lambda, f \), and \( \Gamma \) such that
\[
\Gamma + N 1 \subset \Gamma_{\sigma'}.
\]
Using the concavity of \( f \), for any \( \lambda \in \partial \Gamma_{\sigma} \),
\[
\sigma' < f(\lambda + N 1) \leq f(\lambda) + N \sum_i f_i(\lambda) = \sigma + NF(\lambda).
\]
This implies
\[
F(\lambda) \geq \frac{\sigma' - \sigma}{N} \geq \frac{\sigma' - \Lambda}{N}.
\]
\[\square\]
3.2. The constants $\delta$ and $R$. We now show the existence of appropriate $\delta$ and $R$:

**Lemma 3.4.** Let $h$ be a smooth function on $X$ such that $\sup_{\partial F} f < h \leq \Lambda < \sup_X f$ and $\sup_X |\partial h|_g \leq \Lambda$. Suppose that $\underline{u}$ is a $C$-subsolution of $F(A) = h + 2\sigma_0$. Then there exist constants $\delta$ and $R$, depending only on $(X, \omega)$, $\Gamma$, $f$, $\chi_0$, $\underline{u}$, $\sigma_0$, and $\Lambda$, such that for any $x \in X$,

$$(\mu(x) - 2\delta 1 + \Gamma_n) \cap \partial h^{h(x)} + \sigma_0 \subset B_R(0).$$

**Proof.** We argue by contradiction. Suppose that there exist sequences $\{h_i\} \subset C^\infty(X)$ and $\{x_i\} \subset X$ such that:

(a) $\sup_{\partial F} f < h_i \leq \Lambda$ and $\sup_X |\partial h_i|_g \leq \Lambda$,

(b) $\underline{u}$ is a $C$-subsolution of $F(A) = h_i + 2\sigma_0$, and

(c) there exists $v_i = \sum_{j=1}^n v_i^j e_j \in \Gamma_n$ with $|v_i| > i$ such that $\mu(x_i) - 2i^{-1}1 + v_i \in \Gamma$ and

$$f(\mu(x_i) - 2i^{-1}1 + v_i) = h_i(x_i) + \sigma_0.$$  

After passing to a subsequence, we may assume that $x_i \rightarrow x_\infty \in X$. Fix an $\varepsilon > 0$ so that $\mu(x_\infty) - \varepsilon 1 \in \Gamma$ (which we can do since $\Gamma$ is open). Then for all $i$ large enough, we have:

$$\mu(x_i) - 2i^{-1}1 \in \mu(x_\infty) - \varepsilon 1 + \Gamma_n.$$  

Since $v_i \in \Gamma_n$ and $|v_i| > i$, after passing to an additional subsequence, we can assume that $v_i^j \rightarrow \infty$ for some fixed index $j \in \{1, \ldots, n\}$. It follows then that:

$$f_{\infty, \min}(\mu(x_\infty) - \varepsilon 1) \leq f_{\infty, j}(\mu(x_\infty) - \varepsilon 1) = \lim_{i \rightarrow \infty} f(\mu(x_i) - \varepsilon 1 + v_i^j e_j)$$

$$\leq \lim_{i \rightarrow \infty} f(\mu(x_i) - 2i^{-1}1 + v_i) = \lim_{i \rightarrow \infty} h_i(x_i) + \sigma_0.$$  

By the Arzelà-Ascoli theorem, after passing to a subsequence once more, we can assume that the $h_i$ either converge uniformly to some $h_\infty \in C(X)$ or that $\sup_X h_i \searrow -\infty$, as the $h_i$ are uniformly bounded above by $\Lambda$. In the second case, we conclude that $f_{\infty, \min}(\mu(x_\infty) - \varepsilon 1) = -\infty$, which violates previously stated properties of $f_{\infty, \min}$ (see the paragraph above Proposition 2.2). Thus, we must have that $h_i \xrightarrow{C_0} h_\infty \in C(X)$, so that:

$$f_{\infty, \min}(\mu(x_\infty) - \varepsilon 1) \leq h_\infty(x_\infty) + \sigma_0.$$  

Letting $\varepsilon \rightarrow 0$ gives:

$$f_{\infty, \min}(\mu(x_\infty)) \leq h_\infty(x_\infty) + \sigma_0.$$  

But since $\underline{u}$ is a $C$-subsolution to $F(A) = h_i + 2\sigma_0$ for each $i$, we must have:

$$f_{\infty, \min}(\mu(x_\infty)) > h_i(x_\infty) + 2\sigma_0.$$  

Letting $i \rightarrow \infty$ gives

$$f_{\infty, \min}(\mu(x_\infty)) \geq h_\infty(x_\infty) + 2\sigma_0,$$

which is a contradiction, as desired. \qed
Substituting the above lemmas into the proof of \[35\], Theorem 2.18 (see also \[58\], Proposition 6), we then obtain the following proposition:

**Proposition 3.5.** Let \(u\) be a smooth solution of (1.1) and suppose that \(u\) is a \(C\)-subsolution of (1.1). Set

\[
\chi = \chi_0 + \sqrt{-1} \partial\bar{\partial}u.
\]

Then there exist positive constants \(\kappa\) and \(\tau\) depending only on \((X, \omega), \Gamma, f, \chi_0, u, \sigma_0, \sup_X h, \sup_X |\partial h|_g\) such that at each \(x \in X\), both:

1. either \(\sum_i F_{ij}(\chi_{ij} - \chi_{ij}) > \kappa F\), or \(F_{ii} > \kappa F\) for all \(i\); and
2. \(F \geq \tau\),

where \(F_{ij}\) is the first derivative of \(F\) and \(F = \sum_i F_{ii}\).

### 4. Proof of Theorem 1.1

To prove Theorem 1.1 it suffices to establish the following real Hessian estimate:

**Theorem 4.1.** Let \(u\) be a smooth solution of (1.1) with \(\sup_X u = -1\). Suppose that \(u\) is a \(C\)-subsolution of (1.1), then there exists a constant \(C\) depending only on \((X, \omega), \sup_X |u|, \sup_X |\partial u|_g, \sup_X |\partial \partial u|_g, \chi_0, u, \sigma_0, \sup_X h, \sup_X |\partial h|_g\) and on a lower bound of \(\nabla^2 h\) such that

\[
\sup_X |\nabla^2 u|_g \leq C.
\]

First, we assume without loss of generality that \(\chi_0 = 0\). Otherwise, we replace \(\chi_0\) by \(\chi\). Let \(\lambda_1(\nabla^2 u) \geq \lambda_2(\nabla^2 u) \geq \cdots \geq \lambda_{2n}(\nabla^2 u)\) be the eigenvalues of real Hessian \(\nabla^2 u\) with respect to the Riemannian metric \(g\). Under the assumptions of \(\Gamma\), we see that

\[
\Gamma \subset \Gamma_1 = \{(v_1, \cdots, v_n) \in \mathbb{R}^n \mid \sum_i v_i > 0\}.
\]

As such, we have \(g(\nabla^2 \chi_{ij}) > 0\), which implies (cf. (4.2))

\[
\sum_{\alpha=1}^{2n} \lambda_{\alpha}(\nabla^2 u) = \Delta u > -C.
\]

This shows

\[
|\nabla^2 u|_g = \left(\sum_{\alpha=1}^{2n} (\lambda_{\alpha}(\nabla^2 u))^2 \right)^{\frac{1}{2}} \leq C \lambda_1(\nabla^2 u) + C.
\]

Hence, it suffices to show that \(\lambda_1(\nabla^2 u)\) is uniformly bounded from above.

We define

\[
L := \sup_X |\nabla^2 u|_g + 1, \quad \rho := \nabla^2 u + Lg.
\]
Clearly, $\rho$ is positive everywhere and $|\rho|_g^2 \leq 4L^2$. We consider the following quantity

$$Q = \log \lambda_1(\nabla^2 u) + \xi(|\rho|_g^2) + \eta(|\partial u|_g^2) + e^{-Au},$$

where

$$\xi(s) = -\frac{1}{3} \log (5L^2 - s), \quad \eta(s) = -\frac{1}{3} \log \left(1 + \sup_X |\partial u|_g^2 - s\right)$$

and $A \gg 1$ is a large constant to be determined later. It is very easy to check:

$$\xi'' = 3(\xi')^2, \quad \eta'' = 3(\eta')^2.$$

We will show that $Q$ is uniformly bounded above using the maximum principle. We first show that this is sufficient to conclude that $\lambda_1(\nabla^2 u)$ is also uniformly bounded. Suppose then that $Q \leq C$, and let $y \in X$ be a point at which $\lambda_1(\nabla^2 u)$ attains its supremum. It follows from the above that:

$$\lambda_1(\nabla^2 u)(y) = \sup_X \lambda_1(\nabla^2 u) \geq \frac{1}{C_1} L - C_2,$$

for uniform constants $C_1, C_2$. Then, since $|u|$ and $|\partial u|_g^2$ are uniformly bounded, at $y$, we have:

$$C \geq \log \lambda_1 - \frac{1}{3} \log(5L^2 - |\rho|_g^2) \geq \log (L - C_1 C_2) - \frac{1}{3} \log(L^2) - \log C,$$

as $|\rho|_g^2 \geq 0$. This shows

$$C \geq \frac{1}{3} \log L - \log C$$

so that $L \leq C$. By definition, $\sup_X \lambda_1(\nabla^2 u) \leq L$, so the theorem will follow.

We thus bound $Q$ from above. Let $x_0$ be a maximum point of $Q$. We choose real coordinates $\{x^\alpha\}_{\alpha=1}^{2n}$ near $x_0$ such that at $x_0$,

$$g_{\alpha\beta} := g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = \delta_{\alpha\beta}, \quad \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 0, \quad \text{for any } \alpha, \beta, \gamma = 1, 2, \ldots, 2n.$$

and

$$J \left(\frac{\partial}{\partial x^{2i-1}}\right) = \frac{\partial}{\partial x^{2i}}, \quad \text{for any } i = 1, 2, \ldots, n,$$

where $J$ denotes the complex structure of $(X, \omega)$. Define at $x_0$

$$e_i = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x^{2i-1}} - \sqrt{-1} \frac{\partial}{\partial x^{2i}}\right), \quad \text{for any } i = 1, 2, \ldots, n.$$

Thus $g_{ij} = g(e_i, \overline{e_j}) = \delta_{ij}$ and $\{e_i\}_{i=1}^n$ forms a frame of $(1, 0)$ vectors at $x_0$. After rotating coordinates, we further assume that

$$\chi_{ij} = \chi_i \delta_{ij}, \quad \chi_1 \geq \chi_2 \geq \cdots \geq \chi_n, \quad \text{at } x_0.$$

We then extend $\{e_i\}_{i=1}^n$ smoothly to a $g$-unitary frame of $(1, 0)$ vectors near $x_0$. 
At \( x_0 \), let \( V_\alpha \) be the \( g \)-unit eigenvector of \( \lambda_\alpha \) and denote its components by \( \{ V_\beta^\alpha \}_{\beta=1}^{2n} \). Extend \( V_1, V_2, \ldots, V_{2n} \) to be vector fields in a neighborhood of \( x_0 \) by taking the components to be constant.

When \( \lambda_1(\nabla^2 u)(x_0) = \lambda_2(\nabla^2 u)(x_0) \), the quantity \( Q \) might not be smooth at \( x_0 \). To avoid this situation, we apply a standard perturbation argument (see e.g., [58, 19]). Define

\[
B = B_{\alpha\beta} dx^\alpha \otimes dx^\beta := (\delta_{\alpha\beta} - V_1^\alpha V_1^\beta) dx^\alpha \otimes dx^\beta
\]

and

\[
\Phi = \Phi^\alpha_{\beta} \frac{\partial}{\partial x^\alpha} \otimes dx^\beta := g^{\alpha\gamma} (\nabla_{\gamma} \nabla_\beta u - B_{\gamma\beta}) \frac{\partial}{\partial x^\alpha} \otimes dx^\beta,
\]

where

\[
\nabla_\gamma \nabla_\beta u = \nabla^2 u \left( \frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\beta} \right).
\]

Write \( \lambda_\alpha(\Phi) \) for the eigenvalues of \( \Phi \) with respect to \( g \). It is clear that \( \lambda_1(\Phi)(x_0) > \lambda_2(\Phi)(x_0) \), which implies \( \lambda_1(\Phi) \) is smooth at \( x_0 \). So we consider the modified quantity near \( x_0 \):

\[
\hat{Q} = \log \lambda_1(\Phi) + \xi(|\rho|_g^2) + \eta(|\partial u|_g^2) + e^{-A u}.
\]

Since \( \lambda_1(\Phi)(x_0) = \lambda_1(\nabla^2 u)(x_0) \) and \( \lambda_1(\nabla^2 u) \geq \lambda_1(\Phi) \) near \( x_0 \), we see that \( Q \) still achieves its maximum at \( x_0 \). Note that the vector fields \( V_1, V_2, \ldots, V_{2n} \) are still eigenvectors for \( \Phi \) at \( x_0 \), with eigenvalues \( \lambda_1(\Phi) > \lambda_2(\Phi) \geq \cdots \geq \lambda_{2n}(\Phi) \). In the following argument, we use \( \lambda_\alpha \) to denote \( \lambda_\alpha(\Phi) \) for convenience.

Let \( F^{ij} \) and \( F^{ij,kl} \) be the first and second derivatives of \( F \). Then at \( x_0 \), we have (see e.g. [21, 43, 57])

\[
F^{ij} = \delta_{ij} f_i
\]

and

\[
F^{ij,kl} = f_{i\rho} \delta_{ij} \delta_{pq} + \frac{f_i - f_j}{\chi_{ii} - \chi_{jj}} (1 - \delta_{ij}) \delta_{iq} \delta_{jp}.
\]

Note that the second term is interpreted as a limit if \( \chi_{ii} = \chi_{jj} \). Recalling that \( \chi_{11} \geq \chi_{22} \geq \cdots \geq \chi_{nn} \), we have (see e.g. [29, 57])

\[
F^{1\eta} \leq F^{2\eta} \leq \cdots \leq F^{n\eta}.
\]

We always use subscripts to denote covariant differentiation with respect to the Levi-Civita connection. Since \( (X, \omega) \) may be not Kähler, the Levi-Civita connection will in general not be compatible with the complex structure. Thus, there exists a tensor field \( \Theta \) depending only on \( (X, \omega) \) such that (cf. [15, Section 2])

\[
(\overline{\partial \partial} u)(e_i, \overline{e}_j) = u_{\overline{\partial}} e_i + \Theta_{ij}^k u_k + \overline{\Theta}_{ij}^\overline{k} u^\overline{k},
\]

with

\[
\overline{\Theta}_{ij}^k = \Theta_{ij}^k \text{ and } \overline{\Theta}_{ij}^\overline{k} = \Theta_{ij}^\overline{k}.
\]
We emphasize that $u_\rho$ represents $(\nabla u)(e_i, e_j)$, not $(\partial \bar{\partial} u)(e_i, e_j)$. In particular,
\begin{equation}
(\partial \bar{\partial} u)(e_i, e_j) = u_\rho + 2\text{Re}(\Theta_{\rho}^{k} u_k).
\end{equation}
The following commutation formula for covariant derivatives will be used many times:
\begin{equation}
(\nabla^p u)_{\alpha\beta} - (\nabla^p u)_{\beta\alpha} = (\nabla^p u) * \text{Rm},
\end{equation}
where $*$ means contraction and $\text{Rm}$ denotes the curvature tensor of $(X, g)$.

4.1. **Some calculations.** By the maximum principle, at $x_0$, we have
\[0 \geq F^\tilde{\tau}_i \tilde{Q}_{\tilde{\tau}},\]
\begin{equation}
F^\tilde{\tau}_i \tilde{Q}_{\tilde{\tau}} = \frac{F^\tilde{\tau}_i (\lambda_1)_{\tilde{\tau}}}{\lambda_1} - \frac{F^\tilde{\tau}_i (|\lambda_1|)_{\tilde{\tau}}^2}{\lambda_1^2} + \xi' F^\tilde{\tau}_i (|\rho|_{g}^2)_{\tilde{\tau}} + \xi'' F^\tilde{\tau}_i (|\partial u_{g}|_{\tilde{\tau}}^2) + \eta' F^\tilde{\tau}_i (|\partial u_{g}|_{\tilde{\tau}}^2) + \eta'' F^\tilde{\tau}_i (|\partial u_{g}|_{\tilde{\tau}}^2) + A^2 e^{-Au} F^\tilde{\tau}_i |u|^2 - Ae^{-Au} F^\tilde{\tau}_i u_{\tilde{\tau}}.
\end{equation}
The task of this subsection is to obtain a lower bound of $F^\tilde{\tau}_i \tilde{Q}_{\tilde{\tau}}$.

**Lemma 4.2.** At $x_0$, we have
\begin{equation}
F^\tilde{\tau}_i u_{\tilde{\tau}} = -F^\tilde{\tau}_i (\chi_{\tilde{\tau}} - \chi_{\tilde{\tau}}) - 2F^\tilde{\tau}_i \text{Re}(\Theta_{\rho}^{k} u_k),
\end{equation}
\begin{equation}
F^\tilde{\tau}_i (|\partial u_{g}|_{\tilde{\tau}}^2) = \sum_p F^\tilde{\tau}_i (|u_{ip}|^2 + |u_{p\tilde{\tau}}|^2) - CF
- 2 \sum_p F^\tilde{\tau}_i \text{Re}(\Theta_{\rho}^{k} u_k u_{p} + \Theta_{\rho}^{k} u_k u_{p}),
\end{equation}
\begin{equation}
F^\tilde{\tau}_i (\lambda_1)_{\tilde{\tau}} \geq 2 \sum_{\alpha > 1} F^\tilde{\tau}_i |u_{\tilde{\tau} V_{\alpha}|}^2 - F^\tilde{\tau}_i \text{Re}(\Theta_{\rho}^{k} u_{\tilde{\tau} V_{\alpha} k} - C\lambda_1），
\end{equation}
\begin{equation}
F^\tilde{\tau}_i (|\rho|_{g}^2)_{\tilde{\tau}} \geq 2 \sum_{\alpha, \beta} F^\tilde{\tau}_i |u_{\alpha \beta}|^2 - 4 \sum_{\alpha, \beta} F^\tilde{\tau}_i \rho_{\alpha \beta} \text{Re}(\Theta_{\rho}^{k} u_{\alpha \beta k}) - C L^2 F.
\end{equation}

**Proof.** Using $\chi = \chi_0 + \sqrt{-1} \partial \bar{\partial} u$ and (4.3), we have
\[\chi_{\tilde{\tau}} = (\chi_0)_{\tilde{\tau}} + u_{\tilde{\tau}} + 2\text{Re}(\Theta_{\rho}^{k} u_k).
\]
For (4.5), we compute
\[F^\tilde{\tau}_i u_{\tilde{\tau}} = F^\tilde{\tau}_i \left(\chi_{\tilde{\tau}} - (\chi_0)_{\tilde{\tau}} - 2\text{Re}(\Theta_{\rho}^{k} u_k)\right).
\]
Recalling that $u = 0$, so $\chi = \chi_0$. It then follows that
\[F^\tilde{\tau}_i u_{\tilde{\tau}} = -F^\tilde{\tau}_i (\chi_{\tilde{\tau}} - \chi_{\tilde{\tau}}) - 2F^\tilde{\tau}_i \text{Re}(\Theta_{\rho}^{k} u_k).
\]
For (4.6), we compute

\[ F_{i}^{\tau}(\partial_{\tau}^{2})_{\tau} = \sum_{p} F_{i}^{\tau}(\partial_{\tau}u_{\tau})^{2} + \sum_{p} F_{i}^{\tau}(u_{\tau}^{2}) + \sum_{p} F_{i}^{\tau}(u_{\tau}^{2} + u_{\tau}^{2}u_{\tau}^{2}) \]

\[ \geq \sum_{p} F_{i}^{\tau}(\partial_{\tau}^{2})_{\tau} + 2\text{Re}(F_{i}^{\tau}(u_{\tau}^{2} + u_{\tau}^{2}u_{\tau}^{2})) - CF \]

\[ = \sum_{p} F_{i}^{\tau}(\partial_{\tau}^{2})_{\tau} + 2\text{Re}(F_{i}^{\tau}(u_{\tau}^{2} + O(1))) - CF \]

\[ \geq \sum_{p} F_{i}^{\tau}(\partial_{\tau}^{2})_{\tau} + 2\text{Re}(F_{i}^{\tau}(u_{\tau}^{2} + O(1))) - CF. \]

Applying \( \nabla \) to (1.1),

\[ F_{i}^{\tau}\chi_{i} = h_{\tau}, \]

so

\[ F_{i}^{\tau}u_{\tau} = -F_{i}^{\tau}(\Theta_{i}u_{k\tau} + \Theta_{i}u_{k\tau}) + h_{\tau} + O(F). \]

It then follows that

\[ 2\text{Re}(F_{i}^{\tau}u_{\tau}u_{\tau}) \geq -2\sum_{p} F_{i}^{\tau}\text{Re}(\Theta_{i}u_{k\tau}u_{\tau} + \Theta_{i}u_{k\tau}u_{\tau}) - CF. \]

Thus,

\[ F_{i}^{\tau}(\partial_{\tau}^{2})_{\tau} \geq \sum_{p} F_{i}^{\tau}(\partial_{\tau}^{2})_{\tau} + 2\sum_{p} F_{i}^{\tau}\text{Re}(\Theta_{i}u_{k\tau}u_{\tau} + \Theta_{i}u_{k\tau}u_{\tau}) - CF. \]

For (4.7), we recall the following formulas for differentiating the largest eigenvalue of a matrix (see e.g. [19, Lemma 5.2]):

\[ \lambda_{1}^{\alpha} := \frac{\partial \lambda_{1}}{\partial \Phi_{\beta}} = V_{1}^{\alpha}V_{1}^{\beta}, \]

\[ \lambda_{1}^{\alpha,\gamma} := \frac{\partial^{2} \lambda_{1}}{\partial \Phi_{\beta} \partial \Phi_{\delta}} = \sum_{\mu > 1} V_{1}^{\alpha}V_{1}^{\beta}V_{1}^{\gamma}V_{1}^{\delta} + V_{1}^{\alpha}V_{1}^{\gamma}V_{1}^{\gamma}V_{1}^{\delta} \]

Since \( \Phi_{i}^{\alpha} = g^{\alpha\gamma}(u_{\gamma\beta} - B_{\gamma\beta}) \) and \( B_{\alpha\beta} = 0 \) at \( x_{0} \), then

\[ (\Phi_{i}^{\alpha})_{i} = u_{\alpha\beta} - B_{\alpha\beta} = u_{\alpha\beta}. \]
Applying \( \nabla V_1 \nabla V_1 \) to (4.8),

\[
F^{\tilde{\alpha}}(\lambda_1)_{\tilde{\alpha}} = F^{\tilde{\alpha}} \chi_{\tilde{\alpha} \lambda_1 \chi_{\tilde{\alpha}}} = h_{\lambda_1},
\]

so

\[
F^{\tilde{\alpha}} u_{\lambda_1 V_1} = -F^{\tilde{\alpha} j \tilde{\alpha} q} \chi_{\tilde{\alpha} j} \chi_{\tilde{\alpha}} V_1 = -2 F^{\tilde{\alpha}} \text{Re}(\Theta_{\tilde{\alpha}}^{h_{\lambda_1}} u_{k V_1}) + h_{\lambda_1} - C \lambda_1 \mathcal{F}.
\]

Thus,

\[
F^{\tilde{\alpha}}(\lambda_1)_{\tilde{\alpha}} \geq 2 \sum_{\alpha > 1} \frac{F^{\tilde{\alpha}}|u_{\lambda_1 V_1}|^2}{\lambda_1 - \lambda_\alpha} - 2 F^{\tilde{\alpha}} \text{Re}(\Theta_{\tilde{\alpha}}^{h_{\lambda_1}} u_{k V_1}) - C \lambda_1 \mathcal{F}.
\]

For (4.8), recalling that \( \rho = \nabla^2 u + Lg \), we have

\[
\rho_{\alpha \beta i} = u_{\alpha \beta i}, \quad \rho_{\alpha \beta \tilde{\alpha}} = u_{\alpha \beta \tilde{\alpha}}.
\]

We compute

\[
F^{\tilde{\alpha}}(|\rho_{\tilde{\alpha}}|^2)_{\tilde{\alpha}} = 2 \sum_{\alpha, \beta} F^{\tilde{\alpha}}|u_{\alpha \beta i}|^2 + 2 \sum_{\alpha, \beta} F^{\tilde{\alpha}} \rho_{\alpha \beta u_{\alpha \beta i}}
\]

\[
= 2 \sum_{\alpha, \beta} F^{\tilde{\alpha}} (|u_{\alpha \beta}|^2 + O(1)) + 2 \sum_{\alpha, \beta} F^{\tilde{\alpha}} \rho_{\alpha \beta} \left(u_{\alpha \beta} + O(1)\right)
\]

\[
\geq 2 \sum_{\alpha, \beta} F^{\tilde{\alpha}}|u_{\alpha \beta}|^2 + 2 \sum_{\alpha, \beta} F^{\tilde{\alpha}} \rho_{\alpha \beta} u_{\alpha \beta} - CL^2 \mathcal{F}.
\]

Applying \( \nabla \beta \nabla \alpha \) to (1.1),

\[
F^{\tilde{\alpha}} \chi_{\tilde{\alpha} \alpha} + F^{\tilde{\alpha} j \tilde{\alpha} q} \chi_{\tilde{\alpha} j} \chi_{\tilde{\alpha}} = h_{\alpha \beta},
\]

then

\[
F^{\tilde{\alpha}} u_{\alpha \beta i} = -F^{\tilde{\alpha} j \tilde{\alpha} q} \chi_{\tilde{\alpha} j} \chi_{\tilde{\alpha}} - 2 F^{\tilde{\alpha}} \text{Re}(\Theta_{\tilde{\alpha}}^{h_{\lambda_1}} u_{k \alpha \beta}) + h_{\alpha \beta} + O(\lambda_1 \mathcal{F})
\]

\[
\geq -F^{\tilde{\alpha} j \tilde{\alpha} q} \chi_{\tilde{\alpha} j} \chi_{\tilde{\alpha}} - 2 F^{\tilde{\alpha}} \text{Re}(\Theta_{\tilde{\alpha}}^{h_{\lambda_1}} u_{\lambda_1 \beta}) + h_{\alpha \beta} - C L \mathcal{F}.
\]
which implies
\[
2 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} u_{i \alpha \beta} \\
\geq -2 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} \chi_{\bar{\alpha}} \chi_{\bar{\beta}} - 4 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} \Re(\Theta_{\bar{\alpha}} u_{\alpha \beta k}) + 2 \sum_{\alpha, \beta} \rho_{\alpha \beta} h_{\alpha \beta} - CL^2 F
\]
\[
\geq -2 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} \chi_{\bar{\alpha}} \chi_{\bar{\beta}} - 4 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} \Re(\Theta_{\bar{\alpha}} u_{\alpha \beta k}) - CL^2 F.
\]

Here we used \( \rho > 0 \), so constant \( C \) does not depend on the upper bound of \( \nabla^2 h \). Using the concavity of \( F \) and \( \rho > 0 \),
\[
-2 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} \chi_{\bar{\alpha}} \chi_{\bar{\beta}} \geq 0.
\]
This shows
\[
2 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} u_{i \alpha \beta} \geq -4 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} \Re(\Theta_{\bar{\alpha}} u_{\alpha \beta k}) - CL^2 F,
\]
and so
\[
F_{\alpha \beta} (|\rho|_2^2)_{\alpha \beta} \geq 2 \sum_{\alpha, \beta} F_{\alpha \beta} |u_{i \alpha \beta}|^2 - 4 \sum_{\alpha, \beta} F_{\alpha \beta} \rho_{\alpha \beta} \Re(\Theta_{\bar{\alpha}} u_{\alpha \beta k}) - CL^2 F.
\]

Lemma 4.3. At \( x_0 \), we have
\[
0 \geq 2 \sum_{\alpha > 1} \frac{F_{\alpha \alpha} |u_{V_{\alpha \alpha}}|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + \sum_{i \neq q} \frac{(F_{\bar{\alpha} \bar{\beta}} - F_{\alpha \beta})|u_{\bar{\alpha} \bar{\beta}}|^2}{\lambda_1 (\lambda_1 - \lambda_{\bar{\alpha} \bar{\beta}})} + \sum_{\alpha, \beta} \frac{F_{\alpha \beta} |u_{i \alpha \beta}|^2}{C_A \lambda_1^2}
\]
\[
- \frac{F_{\alpha \alpha} |u_{V_{\alpha \alpha}}|^2}{\lambda_1^2} + \frac{1}{C} \sum_{\alpha, \beta} \sum_{p} F_{\alpha \beta} \left(|u_{i \alpha}|^2 + |u_{i \beta}|^2\right)
\]
\[
+ \xi'' \sum_{\alpha, \beta} F_{\alpha \beta} (|\rho|_g^2)^2 + \eta'' F_{\alpha \beta} (|\partial u_{i \alpha}|^2)^2
\]
\[
+ A^2 e^{-A u} F_{\alpha \beta} |u_{i \alpha}|^2 + Ae^{-A u} \sum_{i} F_{\alpha \beta} (\chi_{\bar{i}} - \chi_{\bar{\alpha}}) - CF,
\]
where \( C_A \) denotes a uniform constant depending on \( A \).
Proof. Combining (4.11) and Lemma 4.2

\[0 \geq 2 \sum_{\alpha > 1} \frac{F^{b} |u_{V_{1}V_{\alpha}}|^2}{\lambda_{1}(\lambda_{1} - \lambda_{\alpha})} - \frac{F^{b} \chi_{i} \chi_{j} V_{i} \chi_{j} V_{i}}{\lambda_{1}} - \frac{F^{b} |(\lambda_{1})|^2}{\lambda_{1}^2} \]

\[+ 2\xi' \sum_{\alpha, \beta} F^{b} |u_{\alpha \beta}|^2 + \xi'' \sum_{\alpha, \beta} F^{b} |(|\rho|_{\beta})|^2| \]

\[+ \eta' \sum_{\alpha, \beta} F^{b} (|u_{\alpha \beta}|^2 + |u_{\beta \pi}|^2) + \eta'' F^{b} |(|\vartheta|_{i}^2)|^2 \]

\[+ A^2 e^{-A u} F^{b} |u_{i}|^2 + A e^{-A u} F^{b} (\chi_{i} - \chi_{\pi}) - (C + C \eta' + CL^2 \xi') \mathcal{F} \]

\[- 2F^{b} \text{Re}(\Theta^{k}_{i} \overline{u_{V_{1}V_{i}}}) - 4 \xi' \sum_{\alpha, \beta} F^{b} \rho_{\alpha \beta} \text{Re}(\Theta^{k}_{i} \overline{u_{\alpha \beta}}) \]

\[- 2 \eta' \sum_{\alpha, \beta} F^{b} \text{Re} \left( \Theta^{k}_{i} \overline{u_{V_{1}V_{i}}} + \overline{\Theta^{b}_{i}} \overline{u_{\beta \pi}} u_{p} \right) + 2A e^{-A u} F^{b} \text{Re}(\Theta^{b}_{i} u_{i}) \]

Thanks to (4.11) and the concavity of \(f\),

\[- \frac{F^{b} \chi_{i} \chi_{j} V_{i}}{\lambda_{1}} \leq - \sum_{i, q} f_{i q} \chi_{i} \chi_{j} V_{i} \chi_{j} V_{i} \lambda_{1} - \sum_{i, q} \frac{(F^{b} - F^{b} \chi_{i} \chi_{j} V_{i} \chi_{j} V_{i})}{\lambda_{1}(\chi_{i} - \chi_{q})} \]

\[\geq \sum_{i, q} \frac{(F^{b} - F^{b} \chi_{i} \chi_{j} V_{i} \chi_{j} V_{i})}{\lambda_{1}(\chi_{i} - \chi_{q})} \]

Using (4.9) and \(B_{\alpha \beta} = 0\) at \(x_0\), we see that

\[(\lambda_{1})_{\alpha} = \lambda_{1}^{\alpha \beta} (\phi_{\beta})_{\alpha} = V_{1}^{\alpha} V_{1}^{\beta} (u_{\alpha \beta} - B_{\alpha \beta}) = u_{V_{1}V_{i}}, \]

and so

\[- F^{b} |(\lambda_{1})_{\alpha}|^2 \leq - F^{b} |u_{V_{1}V_{i}}|^2 \]

By the definition of \(\eta, \xi\) and \(\hat{Q}\), we have

\[\frac{1}{C} \leq \eta' \leq C, \quad \frac{1}{18 L^2} \leq \xi' \leq \frac{1}{3 L^2}, \quad \frac{L}{C A} \leq \lambda_{1} \leq L. \]

From \(\hat{Q}_{k} = 0\), we obtain \(-2F^{b} \text{Re}(\Theta^{k}_{i} \overline{Q}_{k}) = 0\). Combining this with \(\Theta^{k}_{i} = \Theta^{b}_{i}\), we have that the last two lines of (4.11) are zero. Substituting the above inequalities into (4.11) gives (4.10).

4.2. Third order terms. In this subsection, we deal with the third order terms in (4.10). The bad (non-positive) third order term is

\[B := \frac{F^{b} |u_{V_{1}V_{i}}|^2}{\lambda_{1}^2}. \]
To control $B$, we define the set
$$S = \{ i \in \{ 1, 2, \ldots, n-1 \} \mid F^i < A^{-2} e^{2Au(x_0)} F^{i+1+\epsilon}. \}.$$ 

Note that $S$ may be empty. Let $i_0$ be the maximal element of $S$. If $S = \emptyset$, then set $i_0 = 0$. Using $i_0$, we define another set
$$I = \{ i \mid i > i_0 \}.$$ 

Clearly, $I$ is not empty since $n \in I$. Roughly speaking, if $i \in I$, then $F^i$ is comparable with $F^{n\bar{m}}$. If $i \not\in I$, then $F^i \ll F^{n\bar{m}}$. Specifically, note that if $i \in I$, then $i \not\in S$, so:
$$F^i \geq A^{-2} e^{2Au} F^{i+1+\epsilon}.$$ 

Since $i + 1, i + 2, \ldots, n \in I$ also, we get:
$$F^i \geq A^{-4} e^{4Au} F^{i+2+\epsilon} \geq \ldots \geq A^{-2(n-i)} e^{2(n-i)Au} F^{n\bar{m}} \geq A^{-2n} e^{2nAu} F^{n\bar{m}}$$
as $A \geq 1$ and $u \leq -1$. As we always have $F^i \ll F^{n\bar{m}}$, it follows that, for those $i \in I$, $F^i$ is comparable to $F^{n\bar{m}}$ by a uniform factor of $A^{-2n} e^{2nAu}$ (as $\| u \|_{L^\infty}$ is under control).

We now decompose the bad term $B$ into three terms based on $I$
$$B = \sum_{i \not\in I} \frac{F^i |u_{i1} V_{i1}|^2}{\lambda_i^2} + 2\varepsilon \sum_{i \not\in I} \frac{F^i |u_{i1} V_{i1}|^2}{\lambda_i^2} + (1 - 2\varepsilon) \sum_{i \not\in I} \frac{F^i |u_{i1} V_{i1}|^2}{\lambda_i^2} \equiv B_1 + B_2 + B_3,$$
where $\varepsilon \in (0, \frac{1}{2})$ is a constant to be determined later.

- **The terms $B_1$ and $B_2$.** Since $x_0$ is the maximum point of $\hat{Q}$, then $Q_i(x_0) = 0$. Combining this with $(\lambda_i) = u_{i1} V_{i1}$ (see (4.12)), we obtain
$$0 = \hat{Q}_i = \frac{(\lambda_i) i}{\lambda_1} + \xi' \cdot (|\rho|^2 g_i) + \eta' \cdot (|\partial u|^2 g_i) - A e^{-Au} u_i$$

(4.14)
$$= u_{i1} V_{i1} + \xi' \cdot (|\rho|^2 g_i) + \eta' \cdot (|\partial u|^2 g_i) - A e^{-Au} u_i.$$ 

We use (4.14) to deal with $B_1$ and $B_2$ as follows.

**Lemma 4.4.** At $x_0$, we have
$$B_1 + B_2 \leq \xi'' F^i \leq 6 \varepsilon A^2 e^{-2Au} F^i \leq 2 + C \mathcal{F}.$$ 

**Proof.** By (4.14.1) and the Cauchy-Schwarz inequality, we see that
$$B_1 = \sum_{i \not\in I} \frac{F^i \xi' \cdot (|\rho|^2 g_i) + \eta' \cdot (|\partial u|^2 g_i) - A e^{-Au} u_i}{2}$$

$$\leq 3(\xi')^2 \sum_{i \not\in I} F^i (|\rho|^2 g_i) + 3(\eta')^2 \sum_{i \not\in I} F^i (|\partial u|^2 g_i) + 3A^2 e^{-2Au} \sum_{i \not\in I} F^i u_i^2$$

$$\leq 3(\xi')^2 \sum_{i \not\in I} F^i (|\rho|^2 g_i) + 3(\eta')^2 \sum_{i \not\in I} F^i (|\partial u|^2 g_i) + CA^2 e^{-2Au} \sum_{i \not\in I} F^i.$$
For \( i \notin I \), by definition of \( I \), we have \( i \in S \) and so

\[
F^n_i < A^{-2}e^{2Au}F^i + 1 \leq A^{-2}e^{2Au}F,
\]

which implies

\[
B_1 \leq 3(\xi')^2 \sum_{i \notin I} F^n_i |(\rho|_i^2)|^2 + 3(\eta')^2 \sum_{i \notin I} F^n_i |(\partial u|_g)_i|^2 + C_F.
\]

On the other hand,

\[
B_2 \leq 6\varepsilon(\xi')^2 \sum_{i \in I} F^n_i |(\rho|_i^2)|^2 + 6\varepsilon(\eta')^2 \sum_{i \in I} F^n_i |(\partial u|_g)_i|^2 + 6\varepsilon A^2 e^{-2Au} \sum_{i \in I} F^n_i |u_i|^2
\]

\[
\leq 6\varepsilon(\xi')^2 \sum_{i \in I} F^n_i |(\rho|_i^2)|^2 + 6\varepsilon(\eta')^2 \sum_{i \in I} F^n_i |(\partial u|_g)_i|^2 + 6\varepsilon A^2 e^{-2Au} F^n_i |u_i|^2.
\]

Combining the above inequalities and using \( \varepsilon \in (0, \frac{1}{2}) \), we obtain

\[
B_1 + B_2 \leq 3(\xi'')^2 F^n_i |(\rho|_i^2)|^2 + 3(\eta'')^2 F^n_i |(\partial u|_g)_i|^2 + 6\varepsilon A^2 e^{-2Au} F^n_i |u_i|^2 + C_F.
\]

Then Lemma 4.4 follows from \( \xi'' = 3(\xi')^2 \) and \( \eta'' = 3(\eta')^2 \).

- The term \( B_3 \). In \( 4.11 \), the good (non-negative) third order terms are

\[
G_1 + G_2 + G_3 := 2 \sum_{\alpha > 1} F^n_i |u_1 V_\alpha|_i^2 + \left( F^n_i - F^n_{\eta} \right) |\chi_{\eta} V_1|_i^2 + \sum_{\alpha, \beta} \frac{F^n_i |u_{\alpha \beta}|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)}
\]

We will use \( G_1, G_2 \) and \( G_3 \) to control \( B_3 \). Define

\[
W = \frac{1}{\sqrt{2}} (V_1 - \sqrt{-1} J V_1), \quad W_i = \sum_{q} \nu_q e_q, \quad J V_1 = \sum_{\alpha > 1} \mu_\alpha V_\alpha,
\]

where we used \( V_1 \) is orthogonal to \( J V_1 \). At \( x_0 \), since \( V_1 \) and \( e_q \) are \( g \)-unit, then

\[
\sum_{q} |\nu_q|^2 = 1, \quad \sum_{\alpha > 1} \mu_\alpha^2 = 1.
\]

Lemma 4.5. At \( x_0 \), we have

\[
|\nu_q| \leq \frac{C_A}{\lambda_1}, \quad \text{for } q \in I,
\]

where \( C_A \) is a uniform constant depending on \( A \).

Proof. By \( 4.11 \) and the Cauchy-Schwarz inequality,

\[
B \leq 3(\xi')^2 F^n_i |(\rho|_i^2)|^2 + 3(\eta')^2 F^n_i |(\partial u|_g)_i|^2 + C A^2 e^{-2Au} F.
\]

Substituting this into \( 4.11 \) and dropping non-negative terms \( G_i \) \( (i = 1, 2, 3) \), we see that

\[
0 \geq (\xi'' - 3(\xi')^2) \sum_{\alpha, \beta} F^n_i |(\rho|_i^2)|^2 + (\eta'' - 3(\eta')^2) F^n_i |(\partial u|_g)_i|^2 + \frac{1}{C} \sum_{p} F^n_i (|u_{ip}|^2 + |u_{p}||^2) - C A^2 e^{-2Au} F.
\]
Thanks to $\xi'' = 3(\xi')^2$ and $\eta'' = 3(\eta')^2$,
\[
\sum_p F_i^p \left( |u_ip|^2 + |u_{ip}|^2 \right) \leq C_A F.
\]

For $i \in I$, using (4.13), we have
\[
F_i^p \geq A^{-2n} c^{2n} A u F_n^p,
\]
which implies
\[
\sum_{i \in I} \sum_p \left( |u_ip|^2 + |u_{ip}|^2 \right) \leq C_A.
\]

Since $e_i = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^2_i} - \sqrt{-1} \frac{\partial}{\partial x_i} \right)$ and $I = \{ i_0 + 1, \ldots, n \}$, then
\[
(4.15) \quad \sum_{\alpha = 2i_0 + 1}^{2n} \sum_{\beta = 1}^{2n} |u_{\alpha\beta}|^2 \leq C_A.
\]

Recalling that $V_1$ is the eigenvector of $\Phi$ corresponding to $\lambda_1$, we have
\[
V_1^\alpha = \frac{1}{\lambda_1} \sum_{\beta = 1}^{2n} \Phi_\beta^\alpha V_1^\beta.
\]

Combining this with $\Phi_\beta^\alpha = g^\alpha_\gamma (u_{\gamma\beta} - B_{\gamma\beta})$ and (4.15),
\[
|V_1^\alpha| \leq \frac{1}{\lambda_1} \left( \sum_{\beta = 1}^{2n} |u_{\alpha\beta}| + C \right) \leq \frac{C_A}{\lambda_1}, \quad \text{for } \alpha = 2i_0 + 1, \ldots, 2n.
\]

It then follows that
\[
|\nu_q| \leq |V_1^{2q-1}| + |V_1^{2q}| \leq \frac{C_A}{\lambda_1}, \quad \text{for } q \in I.
\]

We can now deal with the term $B_3$. By the definitions of $W_1, \nu_q$ and $\mu_\alpha$, we compute
\[
u_{V_1 V_1} = - \sqrt{-1} u_{V_1 V_1} + \sqrt{2} u_{V_1 \tilde{\eta}_1}.
\]
By (4.12),
\[
\chi_{\tilde{\eta}_1} = (\chi_0)_{\tilde{\eta}_1} + u_{\tilde{\eta}_1} + \Theta_{\tilde{\eta}_1} u_{\tilde{\eta}_1} + \Theta_{\tilde{\eta}_1}^{\tilde{X}} u_{\tilde{\eta}_1},
\]
we have
\[
u_{\tilde{\eta}_1 V_1} = \chi_{\tilde{\eta}_1} V_1 + O(\lambda_1).
Thus,
\[ u_{V_1 V_1} = - \sqrt{-1} \sum_{\alpha > 1} \mu_{\alpha} u_{V_1 V_{\alpha}} + \sqrt{2} \sum_{q \notin I} v_q \chi_{\sigma_q} V_1 + \sqrt{2} \sum_{q \in I} v_q u_{V_1} + O(\lambda_1). \]

Using the Cauchy-Schwarz inequality and Lemma 4.5, we decompose \( B_3 \) into several parts:
\[
B_3 = (1 - 2\varepsilon) \sum_{i \in I} \frac{F_i^\alpha |u_{V_1 V_{i\alpha}}|^2}{\lambda_i^2} \leq (1 - \varepsilon) \sum_{i \in I} \frac{F_i^\alpha}{\lambda_i^2} \left( \sum_{\alpha > 1} \mu_{\alpha} u_{V_1 V_{\alpha}} \right)^2 + \frac{C}{\varepsilon} \sum_{i \in I} \frac{F_i^\alpha |\chi_{\sigma_i} V_1|^2}{\lambda_i^2} + \frac{CF}{\varepsilon} \]
\[
= : B_{31} + B_{32} + B_{33} + \frac{CF}{\varepsilon}.
\]

**Lemma 4.6.** At \( x_0 \), we have
\[ B_3 \leq G_1 + G_2 + G_3 + \frac{CF}{\varepsilon}, \]
assuming without loss of generality that \( \lambda_1 \geq \frac{C}{\varepsilon} \) for some uniform constant \( C_A \) depending on \( A \).

**Proof.** We use \( G_i \) to control \( B_{3i} \) for \( i = 1, 2, 3 \). For \( B_{31} \), we compute
\[
B_{31} \leq (1 - \varepsilon) \sum_{i \in I} \frac{F_i^\alpha}{\lambda_i^2} \left( \sum_{\alpha > 1} (\lambda_1 - \lambda_\alpha) \mu_{\alpha} \right) \left( \sum_{\alpha > 1} \mu_{\alpha} u_{V_1 V_{\alpha}} \right)^2 = \left( 1 - \varepsilon \right) \sum_{i \in I} \frac{F_i^\alpha}{\lambda_i^2} \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha} \right) \left( \sum_{\alpha > 1} \mu_{\alpha} u_{V_1 V_{\alpha}} \right)^2,
\]
where we used \( \sum_{\alpha > 1} \mu_{\alpha}^2 = 1 \). On the other hand, since \( \sup_X |\partial u| \) and \( \sup_X |\partial \overline{\partial} u| \) are under control,
\[
-C \leq \chi(W_1, W_1) = \chi_0(W_1, W_1) + (\sqrt{-1} \partial \overline{\partial} u)(W_1, W_1)
\]
\[
\leq \nabla^2 u(W_1, W_1) + C = \frac{1}{2} (u_{V_1 V_1} + u_{V_1 J V_1} + C = \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \mu_{\alpha}^2 \lambda_\alpha \right) + C,
\]
which implies
\[
\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \leq 2\lambda_1 + C \leq 2(1 + \varepsilon)\lambda_1,
\]
provided that \( \lambda_1 \geq \frac{C}{\varepsilon} \). It then follows that
\[
B_{31} \leq (1 - \varepsilon^2) \sum_{i \in I} \sum_{\alpha > 1} \frac{F_i^\alpha |u_{V_1 V_{i\alpha}}|^2}{\lambda_i^2 (\lambda_1 - \lambda_\alpha)} \leq G_1.
\]
For $B_{32}$, we compute
\[
G_2 \geq \sum_{i \in I, q \notin I} \frac{(F_i - F_q) |\chi_i \sigma \nu_1|^2}{\lambda_1 (\chi_q \sigma \nu_1 - \chi_i \sigma \nu_1)} \geq \sum_{i \in I, q \notin I} \frac{(F_i - F_q) |\chi_i \sigma \nu_1|^2}{C \lambda_1}.
\]
For $i \in I$ and $q \notin I$, by definition of $I$,
\[
F_q \leq F_{i0} \leq A^{-2}e^{2Au} F_{i0 + 130 + 1} \leq A^{-2}e^{2Au} F_i.
\]
Since $A \gg 1$ and $u \leq -1$:
\[
F_i - F_q \geq \frac{F_i}{2}.
\]
Thus,
\[
B_{32} = \frac{C}{\epsilon} \sum_{i \in I, q \notin I} \frac{F_i |\chi_i \sigma \nu_1|^2}{\lambda_1^2} \leq \frac{C}{\epsilon \lambda_1} \sum_{i \in I, q \notin I} \frac{(F_i - F_q) |\chi_i \sigma \nu_1|^2}{\lambda_1} \leq G_2,
\]
as long as $\lambda_1 \geq \frac{C}{\epsilon}$. For $B_{33}$, we compute
\[
B_{33} \leq \frac{C_A}{\epsilon \lambda_1^2} \sum_{\alpha, \beta} \frac{F_i |u_{i0\beta}|^2}{\lambda_1^2} \leq \sum_{\alpha, \beta} \frac{F_i |u_{i0\beta}|^2}{C_A \lambda_1^2} = G_3,
\]
provided that $\lambda_1 \geq \frac{C_A}{\epsilon}$.\]

4.3. End of the Proof. We now finish the proof of Theorem 4.1

Proof. Combining Lemma 4.3 4.4 and 4.6 we obtain
\[
0 \geq \frac{1}{C_0} \sum_i F_i (|u_{i0}|^2 + |u_i|^2) + A e^{-Au} \sum_i F_i (\chi_i - \chi_i) + (A^2 e^{-Au} - 6e^{-2Au}) F_i |u_i|^2 - \frac{C_0 F}{\epsilon},
\]
for a uniform constant $C_0$. Choose
\[
A = \frac{6C_0 + 1}{\kappa}, \quad \epsilon = \frac{e^{Au(x_0)}}{6}.
\]
According to Proposition 3.3 there are then two possible cases:

Case 1: $\sum_i F_i (\chi_i - \chi_i) \geq \kappa F$.

Combining this with our choice of $A$ and $\epsilon$ implies that:
\[
\sum_i F_i (|u_{i0}|^2 + |u_i|^2) + F \leq 0,
\]
which is a contradiction.

Case 2: $F_i \geq \kappa F$ for all $i$.\]
In this case, we obtain
\[
\sum \mathcal{F}^2 (|u_{ip}|^2 + |u_{ip}|^2) \leq CF.
\]
It then follows that \( \sum_{i,p} (|u_{ip}|^2 + |u_{ip}|^2) \leq C \), which implies \( \lambda_1(x_0) \leq C \). □

Remark 4.7. If \( \partial X \neq \emptyset \), then the above argument shows that
\[
\sup_X |\nabla^2 u|_g \leq \sup_{\partial X} |\nabla^2 u|_g + C,
\]
where \( C \) depends only on \((X, \omega), \sup_X |u|, \sup_X |\partial u|_g, \sup_X |\partial \bar{u}|_g, \chi_0, \mathcal{W}, \sigma_0, \sup_X h, \sup_X |\partial h|_g \) and on a lower bound of \( \nabla^2 h \).

5. Degenerate Equations

In this short section, we give the proofs of Theorem 1.2 and Corollary 1.3, which follow immediately from Theorem 1.1.

Theorem 5.1. (Theorem 1.2) Suppose that \( h \in C^2(X) \) satisfies
\[
\sup_{\partial \Gamma} f \leq h < \sup_{\Gamma} f,
\]
and that \( h_\varepsilon \in C^2(X) \) satisfy \( \sup_{\partial \Gamma} f < h_\varepsilon < \sup_{\Gamma} f \) and \( h_\varepsilon \to h \) in \( C^2 \). Let \( \mathcal{u} \) be a \( C \)-subsolution to the degenerate equation
\[
F(A(u)) = h(x),
\]
and let \( u_\varepsilon \) be smooth solutions to the non-degenerate equations:
\[
F(A(u_\varepsilon)) = h_\varepsilon(x)
\]
for all \( 0 < \varepsilon \) sufficiently small. Then there exists a constant \( C \), independent of \( \varepsilon \) and depending only on \((X, \omega), \chi_0, \mathcal{W}, \sigma_0, \sup_X h, \sup_X |\partial h|_g \), and a lower bound of \( \nabla^2 h \) such that
\[
\sup_X |u_\varepsilon| + \sup_X |\partial u_\varepsilon|_g + \sup_X |\nabla^2 u_\varepsilon|_g \leq C,
\]
where \( \nabla \) is the Levi-Civita connection of \( g \).

In particular, up to extracting a subsequence, the \( u_\varepsilon \) converge to a \( u \in C^{1,1}(X) \) solving the degenerate version of equation (1.1):
\[
f^*(\Lambda(u)) = h(x),
\]
where here \( f^* \) is the upper semi-continuous extension of \( f \) to \( \Gamma \).

Proof. By Proposition 2.4, we have that \( \mathcal{u} \) is a \( C \)-subsolution to \( F(A(u_\varepsilon)) = h_\varepsilon(x) \) for all \( \varepsilon > 0 \) sufficiently small. Thus, by Theorem 1.1 and the \( C^2 \) convergence of the \( h_\varepsilon \), we have the required \( a \ priori \) estimate on the \( u_\varepsilon \). The existence of \( u \) follows then immediately from the Arzelà-Ascoli theorem. □

Before we prove Corollary 1.3, we briefly recall the necessary definitions. Fix an integer \( m \) with \( 1 \leq m \leq n \). Let \( U \) be a domain in \( \mathbb{C}^n \) and \( \omega \) be a Hermitian metric on \( \Omega \). Let \( \Gamma_m(U) \) be the set of all \( m \)-positive smooth \((1,1)\)-forms on \( \Omega \) with respect to \( \omega \) (we will always measure positivity using the fixed reference form \( \omega \)).
**Definition 5.2.** ([35] Definition 2.10) Suppose that $\theta \in \Gamma_m(U)$. We say an upper semi-continuous function $v : U \to [\mathbb{R}, +\infty)$ is $(\theta, m)$-subharmonic on $U$ if $v \in L^1_{\text{loc}}(U)$ and

1. $v + \rho$ is $\omega$-subharmonic for $\rho$ solving $i\partial\bar{\partial}\rho \wedge \omega^{n-1} = \theta \wedge \omega^{n-1}$ and
2. for any $\gamma_1, \ldots, \gamma_{m-1} \in \Gamma_m(U)$:

$$
(\theta + \sqrt{-1}\partial\bar{\partial}v) \wedge \gamma_1 \wedge \ldots \wedge \gamma_{m-1} \wedge \omega^{n-m} \geq 0
$$

in the sense of distributions.

We write $\text{mSH}(U, \theta)$ be the set of all $(\theta, m)$-subharmonic functions on $U$.

For a compact Hermitian manifold $(X, \omega)$, the definition is similar. Let $A^{1,1}(X)$ be the space of smooth real $(1, 1)$-forms on $X$. For any $\alpha \in A^{1,1}(X)$, write

$$
\sigma_m(\alpha) = \left(\begin{array}{c} n \\ m \end{array}\right) \frac{\alpha^m \wedge \omega^{n-m}}{\omega^n}.
$$

Then we define

$$
\Gamma_m(X, \omega) = \{ \theta \in A^{1,1}(X) \mid \sigma_l(\theta) > 0 \text{ for } l = 1, 2, \ldots, m \},
$$

to be the cone of (strictly) $m$-positive forms on $X$. As $\omega$ is fixed, we will often write $\Gamma_m(X)$ instead of $\Gamma_m(X, \omega)$.

**Definition 5.3.** Let $(X, \omega)$ be a compact Hermitian manifold and $\theta \in \Gamma_m(X)$. We say an upper semi-continuous function $v : X \to [\mathbb{R}, +\infty)$ is $(\theta, m)$-subharmonic on $X$ if $v \in L^1(X)$ and $v|_U \in \text{mSH}(U, \theta)$ for any local coordinate system $U$. Write $\text{mSH}(X, \theta)$ for the set of all $(\theta, m)$-subharmonic functions on $X$.

**Corollary 5.4.** ([Corollary 1.3]) Let $(X, \omega)$ be a compact Hermitian manifold and $\theta \in \Gamma_m(X)$ for some $1 \leq m \leq n$. Then for any non-negative function $h$ on $X$ such that $\int_X h\omega^n > 0$ and $h^\frac{1}{m} \in C^2(X)$, there exists a pair $(u, c) \in C^{1,1}(X) \times \mathbb{R}_+$ such that

$$
\begin{cases}
(\theta + \sqrt{-1}\partial\bar{\partial}u)^m \wedge \omega^{n-m} = c h^n, \\
u \in \text{mSH}(X, \theta), \quad \sup_X u = -1.
\end{cases}
$$

**Proof.** Write $h_i = h + i^{-1}$. Since it is well known that $\sigma_m^\frac{1}{m}$ falls under the frame work of [1,1], all we need to do to apply Theorem 1.2 is show that there exist pairs $(u_i, c_i) \in C^\infty(X) \times \mathbb{R}_+$ solving:

$$
\begin{cases}
(\theta + \sqrt{-1}\partial\bar{\partial}u_i)^m \wedge \omega^{n-m} = c_i h_i \omega^n, \\
u_i \in \text{mSH}(X, \theta), \quad \sup_X u_i = -1.
\end{cases}
$$

and such that the $c_i$ are also uniformly bounded in $i$:

$$
\frac{1}{C} \leq c_i \leq C.
$$

The existence of such pair follows from [58] Proposition 21] or [60] Theorem 1.1]. The boundedness of the $c_i$ follows from [42] Lemma 3.13]. For the
reader’s convenience, we repeat the short proof here. The lower bound for
the $c_i$ follows immediately from applying the maximum principle at the mini-
mum of $u_i$. Then, weak compactness of sup-normalized $(\theta, m)$-subharmonic
functions implies that the $L^1$-norm of the $u_i$ is also controlled:
$$\|u_i\|_{L^1} \leq C.$$ 
By Maclaurin’s inequality and Stokes’ formula, we have
$$\int_X (c_i h_i)^{1/2} \omega^n \leq C \int_X \sigma_1 (\theta + \sqrt{-1} \partial \bar{\partial} u_i) \omega^n \leq C + C\|u_i\|_{L^1} \leq C.$$ 
Combining this with $\int_X h_i^{1/2} \omega^n \geq C^{-1}$, we obtain
$$c_i \leq C,$$
as desired. Thus, after possibly taking a subsequence, we get that $c_i \to c > 0$. Since the $(c_i h_i)^{1/2}$ are non-degenerate and converge to $(c h)^{1/2}$ in $C^2$, we can thus apply Theorem 1.2 to conclude the existence of $u$. □

6. $C^{1,1}$ REGULARITY OF $(m, \theta)$-ENVELOPES

In this section, we prove Theorem 1.4.

**Theorem 6.1.** (Theorem 1.4) Let $(X, \omega)$ be a compact Hermitian manifold
and $\theta \in \Gamma_m(X)$. If $h \in C^{1,1}(X)$, we define the envelope:
$$P_{m,\theta}(h) := \sup \{ v \in m\text{SH}(X, \theta) \mid v \leq h \}.$$ 
Then $P_{m,\theta}(h) \in C^{1,1}(X)$. In particular, $P_{m,\theta}(h)$ solves:
$$\log \sigma_m (\theta + \sqrt{-1} \partial \bar{\partial} u) = \frac{1}{\epsilon} (u - h),$$
where $\epsilon \in (0, 1)$ and $h \in C^\infty(X)$. We derive some estimates for (6.2) that are essentially the same as those in Theorem 1.1.

**Lemma 6.2.** Let $u$ be a smooth solution of (6.2). Then there exists a constant $C$ depending only on $\|h\|_{C^2}$, $\theta$ and $(X, \omega)$ such that
(1) $\sup_X (u - h) \leq C \epsilon$, $\inf_X u \geq -C$.
(2) $\sup_X |\partial u|_g + \sup_X |\partial \bar{\partial} u|_g \leq C$.
(3) $\sup_X |\nabla^2 u|_g \leq C$. 

Proof. For (1), let $x_0$ be the maximum point of $(u - h)$. At $x_0$, we have
\[ u - h \leq \varepsilon \log \sigma_m(\theta + \sqrt{-1} \partial \bar{\partial} h) \leq C \varepsilon, \]
which implies $\sup_X (u - h) \leq C \varepsilon$. Let $y_0$ be the minimum point of $u$. At $y_0$, we have
\[ u - h \geq \varepsilon \log \sigma_m(\theta) \geq -C, \]
which implies $\inf_X u \geq -C$.

(2) and (3) can be proved by applying arguments similar to [58, Theorem 2] and Theorem 1.1, respectively, with only minor modifications to deal with the dependency of the right hand side on the unknown function $u$. The modifications are standard adaptations from the Monge-Ampère case [3, 65], and are well known – for the reader’s convenience, we give a sketch of them in the context of proving (3). In this setting,
\[ f = \log \sigma_m, \quad u = 0, \quad X = \chi_0 = \theta. \]
Note that $\log \sigma_m$ satisfies all the requisite assumptions of Theorem 1.1.

We define a similar maximum principle quantity:
\[ Q = \log \lambda_1(\nabla^2 u) + \xi(|\rho|^2) + \eta(|\partial u|^2) + e^{-A(u-B)}. \]
where $B$ is a constant such that $\sup_X (u - B) \leq -1$. We also modify the definitions of $\xi$ and $\eta$ slightly:
\[ \xi(s) = -\frac{1}{3} \log \left( 100n^2 L^2 - s \right), \quad L = \sup_X |\nabla^2 u|_g + 1, \]
\[ \eta(s) = -\frac{1}{3} \log \left( 1 + 4 \sup_X |\partial h|^2 + \sup_X |\partial u|^2 - s \right). \]
Let $x_0$ be the maximum point of $Q$, and $\hat{Q}$ be the corresponding perturbed quantity. Near $x_0$, we choose the same coordinates $\{ x^\alpha \}_{\alpha=1}^{2n}$ and local frame $\{ e_i \}_{i=1}^n$ as in the proof of Theorem 4.1. Equation (6.2) can be written as
\[ (6.3) \quad F(A) = \frac{1}{\varepsilon}(u - h), \quad A_j^i = g^{ij} \chi_0 \chi_j. \]
As mentioned, the only difference between (1.1) and (6.3) is the right hand side. To apply the maximum principle at $x_0$, we need to establish some inequalities. We begin by showing that the conclusion of Lemma 4.3 still holds for (6.3).

The following calculations are very similar to Lemma 4.2. It is clear that
\[ (6.4) \quad F\bar{\nabla} u_{\bar{\alpha}} = -F\bar{\nabla}(\Delta u - \chi_0) - 2F\bar{\nabla} \text{Re}(\Theta^2 u). \]
Applying $\nabla_{\bar{\alpha}}$ to equation (6.3),
\[ F\bar{\nabla} \chi_0 \chi_{\bar{j}} = \frac{1}{\varepsilon}(u_{\bar{j}} - h_{\bar{j}}), \]
which implies

\[ 2 \text{Re} \left( F^{\bar{u}_p u_p} \right) \geq -2 \sum_p F^{\bar{u}_p} \text{Re} \left( \Theta^{\ell, \bar{u}_p u_p} + \Theta^{\ell, u_p \bar{u}_p} \right) + \frac{2}{\varepsilon} (|\partial u|^2 - |\partial u| |\partial h|) - C F. \]

Combining this with the Cauchy-Schwarz inequality,

\[
F^{\bar{u}_p} (|\partial u|_g) \geq \sum_p F^{\bar{u}_p} (|u_p|^2 + |u_p|^2) + 2 \text{Re} \left( F^{\bar{u}_p u_p} \right) - C F
\]

\[
(6.5)
\]

\[
\geq \sum_p F^{\bar{u}_p} (|u_p|^2 + |u_p|^2) - \frac{1}{\varepsilon} |\partial h|^2 - C F
\]

\[
- 2 \sum_p F^{\bar{u}_p} \text{Re} \left( \Theta^{\ell, u_p \bar{u}_p} + \Theta^{\ell, u_p \bar{u}_p} \right).
\]

Applying \( \nabla V_1 \nabla V_1 \) to (6.3), we get

\[
F^{\bar{u}_p} \chi^{\ell, V_1 V_1} + F^{\bar{u}_p} \chi^{\ell, V_1 V_1} = \frac{1}{\varepsilon} (\lambda_1 - h V_1 V_1) \geq \frac{1}{2\varepsilon} \lambda_1,
\]

where we assume without loss of generality that \( \lambda_1 - h V_1 V_1 \geq \frac{1}{2} \lambda_1 \). Thus,

\[
F^{\bar{u}_p} \chi^{\ell, V_1 V_1} \geq -F^{\bar{u}_p} \chi^{\ell, V_1 V_1} - 2 F^{\bar{u}_p} \text{Re} \left( \Theta^{\ell, u_p \ell V_1 V_1} \right) + \frac{1}{2\varepsilon} \lambda_1 - C \lambda_1 F,
\]

and so

\[
F^{\bar{u}_p} (\lambda_1) \geq 2 \sum_{\alpha > 1} \frac{F^{\bar{u}_p} |u_{\ell, V_1 \ell V_1}|^2}{\lambda_1 - \lambda_{\alpha}} + F^{\bar{u}_p} u_{\ell, V_1 V_1} - C \lambda_1 F
\]

\[
\geq 2 \sum_{\alpha > 1} \frac{F^{\bar{u}_p} |u_{\ell, V_1 \ell V_1}|^2}{\lambda_1 - \lambda_{\alpha}} - F^{\bar{u}_p} \chi^{\ell, V_1 V_1} - 2 F^{\bar{u}_p} \text{Re} \left( \Theta^{\ell, u_p \ell V_1 \ell V_1} \right) + \frac{1}{2\varepsilon} \lambda_1 - C \lambda_1 F.
\]

(6.6)

Applying \( \nabla \beta \nabla \alpha \) to (6.3) gives,

\[
F^{\bar{u}_p} \chi^{\ell, \alpha \beta} + F^{\bar{u}_p} \chi^{\ell, \alpha \beta} = \frac{1}{\varepsilon} (u_{\alpha \beta} - h_{\alpha \beta}),
\]
which implies
\[
2 \sum_{\alpha, \beta} F_{\bar{\alpha}} \rho_{\alpha \beta} u_{\bar{\ni} \alpha \beta} \geq -2 \sum_{\alpha, \beta} F_{\bar{\alpha}} \rho_{\alpha \beta} \chi_{\bar{\gamma} \alpha \gamma_{\bar{\beta}}} - 4 \sum_{\alpha, \beta} F_{\bar{\alpha}} \rho_{\alpha \beta} \text{Re}(\Theta_{\bar{\alpha}}^{k} u_{\alpha \beta k})
\]
\[+ \sum_{\alpha, \beta} \rho_{\alpha \beta} (\rho_{\alpha \beta} - h_{\alpha \beta}) - CL^2 \mathcal{F}
\]
\[
\geq -4 \sum_{\alpha, \beta} F_{\bar{\alpha}} \rho_{\alpha \beta} \text{Re}(\Theta_{\bar{\alpha}}^{k} u_{\alpha \beta k}) - 24 \frac{n^2}{\varepsilon} L^2 - CL^2 \mathcal{F},
\]
where we assume without loss of generality that \( L \geq \sup X |\nabla^2 h|_g \). It then follows that
\[
F_{\bar{\gamma}} \left( |\rho|^2 \right)_{\bar{\gamma}} \geq 2 \sum_{\alpha, \beta} F_{\bar{\alpha}} |u_{\alpha \beta}|^2 + 2 \sum_{\alpha, \beta} F_{\bar{\alpha}} \rho_{\alpha \beta} u_{\bar{\ni} \alpha \beta} - CL^2 \mathcal{F}
\]
\[
\geq 2 \sum_{\alpha, \beta} F_{\bar{\alpha}} |u_{\alpha \beta}|^2 - 4 F_{\bar{\alpha}} \rho_{\alpha \beta} \text{Re}(\Theta_{\bar{\alpha}}^{k} u_{\alpha \beta k}) - \frac{24 n^2}{\varepsilon} L^2 - CL^2 \mathcal{F}.
\]

We can now combine (6.4), (6.5), (6.6), and (6.7) and apply the same argument as in Lemma 4.3 to obtain:
\[
0 \geq \text{(right hand side of (4.10))} + \frac{1}{\varepsilon} \left( \frac{1}{2} - 24 n^2 L^2 \xi' - \eta' |\partial h|_g^2 \right).
\]
But by the definition of \( \xi \) and \( \eta \), we have
\[
\xi' \leq \frac{1}{100 n^2 L^2}, \quad \eta' \leq \frac{1}{4 |\partial h|_g^2 + 1},
\]
so
\[
\frac{1}{\varepsilon} \left( \frac{1}{2} - n^2 L^2 \xi' - \eta' |\partial h|_g^2 \right) \geq 0.
\]
Hence, the conclusion of Lemma 4.3 still applies for equation (6.3).

We can now continue the argument as in Section 4 until we get to equation (4.16) – in our new setting, this becomes the slightly modified inequality:
\[
0 \geq \frac{1}{C_0} \sum_p F_{\bar{\gamma}} \left( |u_{\bar{\gamma} p}|^2 + |u_{\bar{\gamma}}|^2 \right) + A e^{-A(u-B)} F_{\bar{\gamma}} (\chi_{\bar{\gamma} \tau} - \chi_{\bar{\gamma}})
\]
\[+ (A^2 e^{-A(u-B)} - 6 \varepsilon' A^2 e^{-2A(u-B)}) F_{\bar{\gamma}} |u_{\bar{\gamma}}|^2 - \frac{C_0 \mathcal{F}}{\varepsilon'} - C_0,
\]
where we have replaced the \( \varepsilon \) in (4.16) with \( \varepsilon' \), to avoid confusion with the \( \varepsilon \) already being used in this section. We now need to apply Proposition 3.5.
to finish. Since the right hand side of the equation (6.3) depends on \( \varepsilon \), we cannot apply Proposition 3.5 directly, as we need the real Hessian estimate to be independent of \( \varepsilon \). Instead, note that there exists \( \kappa > 0 \) depending only on \( (X, \omega) \) and \( \theta \) such that

\[ \theta - \kappa \omega \in \Gamma_m(X). \]

By \( \chi = \theta \) and Gårding’s inequality,

\[ F^\overline{\nabla}(\chi_{\nabla} - \chi_{\nabla}) = F^\overline{\nabla}\theta_{\nabla} - m = F^\overline{\nabla}(\theta_{\nabla} - \kappa g_{\nabla} + \kappa g_{\nabla}) - m \geq \kappa F - m. \]

Choosing then \( A = \frac{6C_0 + 1}{\kappa}, \varepsilon' = \frac{e^{A(u(x_0) - B)}}{6} \), it follows that

\[ \sum_p F^\overline{\nabla} \left( |u_{ip}|^2 + |u_{p\nabla}|^2 \right) + F \leq C. \]

Thanks to [40, Lemma 2.2 (2)], \( F \leq C \) implies \( F^\overline{\nabla} \geq \frac{1}{\kappa} \). We thus get:

\[ \sum_p \left( |u_{ip}|^2 + |u_{p\nabla}|^2 \right) \leq C, \]

which shows that \( \lambda \leq C \), as required. \( \square \)

6.2. \((\theta, m)\)-subharmonic functions. We now recall some basic facts about \((\theta, m)\)-subharmonic functions. In what follows, we will write:

\[ P_\theta(h) := P_{m, \theta}(h), \]

for simplicity.

**Lemma 6.3.** Let \( v \in \text{mSH}(X, \theta) \) and \( p \in X \). Suppose that \( U \) is a coordinate system centered at \( p \). If \( \varphi \in C^2(U) \) satisfies

\[ \varphi(p) = v(p), \quad \varphi \geq v \quad \text{on} \quad U; \]

then for any \( \gamma_1, \ldots, \gamma_{m-1} \in \Gamma_m(U) \),

\[ (\varphi + \sqrt{-1} \partial \overline{\partial} \varphi) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \geq 0 \quad \text{at} \quad p. \]

In particular, \( \sigma_m(\theta + \sqrt{-1} \partial \overline{\partial} \varphi) \geq 0 \quad \text{at} \quad p. \)

**Proof.** We follow the argument of [51, Lemma 7]. Since \( \gamma_1, \ldots, \gamma_{k-1} \in \Gamma_m(U) \), we have that

\[ \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \]

is a positive \((n-1, n-1)\)-form. By [49] (4.8), there exists a Hermitian metric \( \beta \) on \( U \) such that

\[ \beta^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m}. \]

After shrinking \( U \), we can find a bounded \( f \in C^\infty(U) \) solving the linear equation

\[ \theta \wedge \beta^{n-1} = \sqrt{-1} \partial \overline{\partial} f \wedge \beta^{n-1}. \]
Combining this with \( v \in mSH(X, \theta) \), we obtain
\[
\sqrt{-1} \partial \bar{\partial} (f + v) \wedge \beta^{n-1} = (\theta + \sqrt{-1} \partial \bar{\partial} v) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \geq 0 \quad \text{on } U
\]
in the sense of distributions. Clearly, \( f + \varphi \in C^2(U) \) satisfies
\[
(f + \varphi)(p) = (f + v)(p), \quad f + \varphi \geq f + v \quad \text{on } U.
\]
By [39, Theorem 9.2 and 9.3], we have
\[
\sqrt{-1} \partial \bar{\partial} (f + \varphi) \wedge \beta^{n-1} \geq 0 \quad \text{at } p.
\]
It then follows that
\[
(\theta + \sqrt{-1} \partial \bar{\partial} \varphi) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \geq 0 \quad \text{at } p,
\]
as desired.

The last claim, that \( \sigma_m(\theta + \sqrt{-1} \partial \bar{\partial} \varphi) \geq 0 \) at \( p \), now follows from a simple application of Gårding’s inequality. \( \square \)

Let \( \varepsilon \in (0, 1) \). By Lemma 6.2 and the continuity method in [58], there exists a smooth function \( u_\varepsilon \in mSH(X, \theta) \) solving (6.2):
\[
\begin{align*}
\log \sigma_m(\theta + \sqrt{-1} \partial \bar{\partial} u_\varepsilon) &= \frac{1}{\varepsilon}(u_\varepsilon - h), \\
\theta + \sqrt{-1} \partial \bar{\partial} u_\varepsilon &\in \Gamma_m(X).
\end{align*}
\]

**Lemma 6.4.** Let \( 0 < \delta < 1 \) and \( \theta \in \Gamma_m(X) \). Then for any \( v \in mSH(X, (1 - \delta)\theta) \) such that \( v \leq h \), we have
\[
v \leq u_\varepsilon + \delta \quad \text{on } X,
\]
for all \( \varepsilon > 0 \) sufficiently small. In particular, \( P_{(1-\delta)\theta}(h) \leq u_\varepsilon + \delta \).

**Proof.** The argument is extracted from [46, Theorem 3.2]. Let \( p \) be the minimum point of the lower semi-continuous function \( u_\varepsilon - v \), and set
\[
B = (u_\varepsilon - v)(p) = \min_{X} (u_\varepsilon - v).
\]
It suffices to prove \( B \geq -\delta \). Pick a coordinate system \( U \subset X \) centered at \( p \), and set
\[
\varphi = u_\varepsilon - B \in C^\infty(X).
\]
It is clear that
\[
\varphi(p) = v(p), \quad \varphi \geq v \quad \text{on } U.
\]
By Lemma 6.3, we have
\[
\sigma_m((1 - \delta)\theta + \sqrt{-1} \partial \bar{\partial} \varphi) \geq 0 \quad \text{at } p,
\]
which implies
\[
\sigma_m(\theta + \sqrt{-1} \partial \bar{\partial} u_\varepsilon) \geq \delta^m \sigma_m(\theta) > 0 \quad \text{at } p.
\]
Using \( v \leq h \) and 6.3, we get:
\[
v(p) - \delta \leq h(p) + \varepsilon \log(\delta^m \sigma_m(\theta)) \leq u_\varepsilon(p)
\]
as long as \( \varepsilon \) is sufficiently small, depending only on \( \theta \) and \( \delta \). Rearranging yields \( B \geq -\delta \), as desired. \( \square \)
6.3. **Proof of Theorem 6.1.** We now prove Theorem 6.1.

**Proof of Theorem 6.1.** We begin by showing that it suffices to prove the theorem for $h \in C^\infty(X)$. To see this, suppose that $h \in C^{1,1}(X)$ and pick a sequence of smooth functions $h_i$ such that

$$\lim_{i \to \infty} \| h_i - h \|_{C^0} = 0, \quad \| h_i \|_{C^2} \leq C,$$

where $C$ is independent of $i$. By the definition of the envelope, we have

$$\| P_\theta(h_i) - P_\theta(h) \|_{L^\infty} \leq \| h_i - h \|_{C^0},$$

which implies

$$\lim_{i \to \infty} \| P_\theta(h_i) - P_\theta(h) \|_{L^\infty} = 0.$$

Thus, the theorem will follow if we can prove that $\| P_\theta(h_i) \|_{C^2} \leq C$ for the smooth functions $h_i$.

Suppose then that $h \in C^\infty(X)$. As already noted, by Lemma 6.2 and the continuity method in [58], there exists smooth functions $u_\varepsilon \in \mbox{mSH}(X, \theta)$ solving

$$\log \sigma_m(\theta + \sqrt{-1} \partial \overline{\partial} u_\varepsilon) = \frac{1}{\varepsilon}(u_\varepsilon - h),$$

for any $\varepsilon \in (0, 1)$. We claim

$$\lim_{\varepsilon \to 0} \| u_\varepsilon - P_\theta(h) \|_{C^0} = 0. \tag{6.9}$$

Given this, the theorem follows immediately from Lemma 6.2 and the Arzelà-Ascoli theorem.

To prove (6.9), by (1) of Lemma 6.2, we have

$$u_\varepsilon - C\varepsilon \leq h.$$

Since $u_\varepsilon - C\varepsilon \in \mbox{mSH}(X, \theta)$ also, we thus have:

$$u_\varepsilon - C\varepsilon \leq P_\theta(h). \tag{6.10}$$

Combining this with Lemma 6.4 now gives the two-sided bound:

$$P_{(1-\delta)\theta}(h) - \delta \leq u_\varepsilon \leq P_\theta(h) + C\varepsilon,$$

for all $0 < \varepsilon$ sufficiently small. By (2) of Lemma 6.2, any subsequence $\varepsilon_i \to 0$ has a further subsequence (which we will also denote by $\varepsilon_i$), such that $u_{\varepsilon_i} \to u_0 \in C^0(X)$ uniformly as $i \to \infty$. Taking $i \to \infty$ in the above bound gives:

$$P_{(1-\delta)\theta}(h) - \delta \leq u_0 \leq P_\theta(h).$$

But it is clear from the definition that $\lim_{\delta \to 0} P_{(1-\delta)\theta}(h) = P_\theta(h)$, so we can now let $\delta \to 0$ to get:

$$u_0 = P_\theta(h).$$

Thus, any $C^0$-convergent subsequence of $\{ u_\varepsilon \}$ converges to $P_\theta(h)$, so we conclude that $u_\varepsilon \overset{C^0}{\to} P_\theta(h)$, as claimed.

Finally, we show (6.1) – the proof is the same as in the Monge-Ampère case, but we reproduce the details for the readers convenience (see [59]).
First, we show that $(\theta + i\partial\bar{\partial}u)(h)^m \wedge \omega^{n-m}$ vanishes outside the contact set $K := \{P_\theta(h) = h\}$ – this is classical when $X$ is a Kähler manifold, and the same proof can be adapted to the Hermitian case [33, Theorem 4.1]. Here however, we give a different proof, utilizing the estimates we have already show for the $u_\varepsilon$. Note that, by equation (6.3), we have that $(\theta + i\partial\bar{\partial}u_\varepsilon)^m \wedge \omega^{n-m} \to 0$ on the open set $X \setminus K$. Thus, we only need to show that the measures $(\theta + i\partial\bar{\partial}u_\varepsilon)^m \wedge \omega^{n-m}$ converge to $(\theta + i\partial\bar{\partial}P_\theta(h))^m \wedge \omega^{n-m}$.

By the uniform convergence of the $u_\varepsilon$ combined with the uniform lower bound for $i\partial\bar{\partial}u_\varepsilon$, there exists a constant $C$ independent of $\varepsilon$ such that $u_\varepsilon, P_\theta(h) \in \text{PSH}(X, C\omega)$. By [11, Proposition 1.2], we have that

$$(C\omega + i\partial\bar{\partial}u_\varepsilon)^i \to (C\omega + i\partial\bar{\partial}P_\theta(h))^i,$$

for each $1 \leq i \leq n$. Writing then:

$$(\theta + i\partial\bar{\partial}u_\varepsilon)^m \wedge \omega^{n-m} = (\theta - C\omega + C\omega + i\partial\bar{\partial}u_\varepsilon)^m \wedge \omega^{n-m} = \sum_{i=0}^{m} \binom{m}{i} (C\omega + i\partial\bar{\partial}u_\varepsilon)^i \wedge (\theta - C\omega)^{m-i} \wedge \omega^{n-m},$$

we immediately see

$$(\theta + i\partial\bar{\partial}u_\varepsilon)^m \wedge \omega^{n-m} \to (\theta + i\partial\bar{\partial}P_\theta(h))^m \wedge \omega^{n-m},$$
as needed.

We now deal with the measure on the contact set. Since $P_\theta(h) - h \in C^1(X)$, we have that $\nabla(P_\theta(h) - h) = 0$ on the closed set $K$. By the first half of the proof, $P_\theta(h) \in C^{1,1}(X)$, so we have that $\nabla_i(P_\theta(h) - h)$ is actually Lipschitz for any $i = 1, \ldots, 2n$ (working in a local coordinate chart, with the $i$ being real indices). It follows that:

$$\nabla\nabla_i(P_\theta(h) - h) = 0$$

almost everywhere on $\{\nabla_i(P_\theta(h) - h) = 0\} \supset K$ (see [11, Theorem 3.2.6], for instance). Therefore, we have $\nabla^2 P_\theta(h) = \nabla^2 h$ a.e. on $K$, and so $\theta + i\partial\bar{\partial}P_\theta(h) = \theta + i\partial\bar{\partial}h$ on $K$, as the complex derivatives are just complex linear combinations of the real ones. This establishes [41].

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