PRE-CALABI-YAU ALGEBRAS AND TOPOLOGICAL QUANTUM FIELD THEORIES

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Abstract. We introduce a notion generalizing Calabi-Yau structures on A-infinity algebras and categories, which we call pre-Calabi-Yau structures. This notion does not need either one of the finiteness conditions (smoothness or compactness) which are required for Calabi-Yau structures to exist. In terms of noncommutative geometry, a pre-CY structure is as a polyvector field satisfying an integrability condition with respect to a noncommutative analogue of the Schouten-Nijenhuis bracket. We show that a pre-CY structure defines an action of a certain PROP of chains on decorated Riemann surfaces. In the language of the cobordism perspective on TQFTs, this gives a partially defined extended 2-dimensional TQFT, whose 2-dimensional cobordisms are generated only by handles of index one. We present some examples of pre-CY structures appearing naturally in geometric and topological contexts.

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1. Introduction

The complex of Hochschild chains $C^*_e(A)$ of a compact Calabi-Yau algebra $A$ has the structure of an algebra over a differential graded (dg) PROP, which encodes the topology of certain moduli spaces of topological surfaces. This structure has been studied from many perspectives [KS06; Cos07; Cos05; WW16], and is given by a collection of operations $C^*_e(A)^{\otimes m} \to C^*_e(A)^{\otimes n}$ for $m \geq 1, n \geq 0$; the spaces of such operations are parametrized by chains on the moduli space $\mathcal{M}_{g,m,n}$ of Riemann surfaces, decorated with $m \geq 1$ incoming and $n \geq 0$ outgoing marked points and choices of a real tangent direction on each marked point.

Over the past decade, there have been various developments in the study of Calabi-Yau structures, so we recapitulate the general outline of this theory, in the context of $A_\infty$-algebras. An $A_\infty$-algebra $(A, \mu = \sum_{i \geq 1} \mu^i)$ is called compact if its cohomology $H^*(A, \mu^1)$ is finite dimensional. A Calabi-Yau structure of dimension $d$ on a compact $A_\infty$-algebra $A$ is a class $\omega \in \text{Hom}_k(CC^*_e(A), k[-d])$, that is, in the dual of the cyclic complex of $A$, whose projection to $\text{Hom}(C^*_e(A), k[-d]) \cong R\text{Hom}_{A\text{-mod}}(A[d], A^\vee)$ defines a quasi-isomorphism of $A$-bimodules between a shift of the diagonal bimodule $A[d]$ and the linear dual $A^\vee$. This has been called a compact, proper or right Calabi-Yau structure in the existing literature.

The action of the PROP of chains on moduli spaces of decorated surfaces has been interpreted by Lurie in the context of the cobordism hypothesis; using a certain unfolding construction [Lur09], one shows that a compact CY structure produces a ‘non-compact’ fully extended 2d TQFT, whose 2-dimensional cobordisms are required to have at least one input. In the case of cyclic $A_\infty$-algebras, that is, finite-dimensional $A_\infty$-algebras with a cyclically compatible pairing, one can concretely write down formulas for these operations.

When $A$ is homologically smooth [KS06], there is a dual notion of Calabi-Yau structure, which has appeared in the literature by the name of smooth or left Calabi-Yau structure. It is given by a class $\omega \in CC^-_e(A)[-d]$ in the negative cyclic homology of $A$, whose projection to $C^*_e(A)[-d] \cong R\text{Hom}_{A\text{-mod}}(A'[d], A[-d])$ gives a quasi-isomorphism between the left or bimodule dual $A' = R\text{Hom}_{A\text{-mod}}(A, A^e)$ and the shifted diagonal bimodule $A[d]$. The idea that one should consider such quasi-isomorphisms appeared initially in [Gin06], where it was not required that this quasi-isomorphism come from a class negative cyclic homology; this weaker has been called a ‘weak CY structure’ or ‘Ginzburg CY structure’ in the literature.
Each one of these types of structures, compact and smooth CY, requires a certain finiteness condition in order to exist: a compact CY structure on $A$ can only exist if the cohomology of $(A, \mu^1)$ is finite-dimensional, and a smooth CY structure, only if $A$ is homologically smooth, that is, if the diagonal bimodule $A$ has a finite-length resolution. In many applications, it is necessary to go beyond the finite case, for instance in string topology, certain types of Fukaya categories and categories of coherent sheaves appearing in homological mirror symmetry.

The purpose of this paper is to describe as explicitly as possible a type of structure that generalizes compact and smooth CY structures. We call this a pre-Calabi-Yau structure; it can exist on an $A_\infty$-category $A$ that is neither compact or smooth, and it defines a certain type of partial 2d TQFT structure.

**Theorem 1.** The Hochschild chain complex $C_*(A)$ of a pre-CY category $A$ of dimension $d$ has the action of the PROP $Q^d$, containing operations $C_*(A)^{\otimes m} \to C_*(A)^{\otimes n}$ for $m \geq 1, n \geq 1$, that is, with at least one input and at least one output. The PROPS $Q^d$ calculate the cohomologies of the moduli spaces $\mathcal{M}_{g, \vec{m}, \vec{n}}$ with coefficients in powers $L^d$ of a certain rank one local system.

Moreover, there is an open-closed extension of this PROP (as a colored PROP) acting on both $C_*(A)$ and the morphism spaces $A(X,Y)$ of the $A_\infty$-category $A$.

Let us now describe this structure, first for an $A_\infty$-algebra. For any graded vector space $A$, we define its space of ‘order $k$ higher Hochschild cochains’

$$C^*_m(A) = \prod_{n_1, \ldots, n_k \geq 0} \text{Hom}(\bigotimes_{i=1}^k A[1]^{\otimes n_i}, A^{\otimes k})$$

Each order $k$ higher cochain $\phi$ can be visualized as a vertex with $k$ outgoing arrows, and $n_i$ incoming arrows on each angle, as in the figure above. For any choice of integer $d$, we define the space of ‘order $k$ dimension $d$ cyclic Hochschild cochains’

$$C^*_{(k,d)}(A) := (C^*_m(A))^{(\mathbb{Z}_k,d)}(d-2)(k-1)$$

where $(\mathbb{Z}_k,d)$ denotes invariants under a certain action of the cyclic group rotating the vertex, with signs depending on the parity of $d$. We endow the space

$$C^*_{[d]}(A) := \prod_k C^*_{(k,d)}(A)$$

with a binary operation we call the necklace product $\circ_{\text{neck}}$. Its associated necklace bracket $[-,-]_{\text{neck}}$ gives $C^*_{[d]}(A)[1]$ the structure of a dg Lie algebra.

**Definition 1.** A pre-Calabi-Yau structure of dimension $d$ on the graded vector space $A$ is a solution $m = \sum_k m_k$ to the Maurer-Cartan equation $m \circ_{\text{neck}} m = 0$, of degree one in the dg Lie algebra $C^*_{[d]}(A)[1]$.
Recall that an $A_\infty$-structure is the data of a module structure over chains on the topological operad of Stasheff associahedra; each cell on this chain complex is labeled by a corolla, a disk with many inputs and one output. A pre-CY structure is a collection of maps giving a module structure over a bigger structure, a dioperad $MC_{n_1,\ldots,n_k}$ of chains on spaces of multicorollas, that is, disks with many inputs and many outputs.

Another description can be given in the language of noncommutative geometry developed in [KS06]. In that setting, an $A_\infty$-structure on $A$ is equivalent to a solution of a certain Maurer-Cartan problem in the dg Lie algebra given by the (shifted) Hochschild cochains $C^*(A)[1]$, with Lie bracket given by the Gerstenhaber bracket $[-,-]_G$; this is the data of a homological vector field on the noncommutative pointed scheme associated to $A$. In this interpretation, the space $C^*_d(A)$ is the space of shifted polyvector fields on the noncommutative space associated to $A$, and the necklace bracket is the analog of the Schouten-Nijenhuis bracket. The $k=1$ component of the Maurer-Cartan equation shows that a pre-CY algebra is also an $A_\infty$-algebra; a pre-CY structure extending a given $A_\infty$ structure is a system of integrable polyvector fields extending the corresponding vector field.

When $A$ is finite-dimensional, a cyclic $A_\infty$-structure on $A$ easily gives a pre-CY structure of a particularly simple form. We generalize this to a more homotopically-invariant statement: in the compact case, we demonstrate how to construct a pre-CY structure on a minimal model of a noncommutative Lagrangian inside of a noncommutative symplectic space. This gives a close relation between pre-CY structures (on minimal models) and relative notions of Calabi-Yau structures as introduced in [BD18; BD19].

Pre-CY structures have a well-behaved deformation theory, extending the deformation theory of $A_\infty$-structures. In Section 4, we describe how to calculate the relevant obstructions for smooth or compact $A_\infty$ categories, using the diagonal bimodule and its duals. This allows us to produce pre-CY structures on some examples of geometric and topological interest, which we discuss in Section 5.

The proof of Theorem 1 relies on a new cell complex describing the spaces $M_{g,n,\vec{m},\vec{n}}$. This cell complex is obtained by using a modified form of uniformization of surfaces by Strebel differentials; we use quadratic differentials with higher order poles. We rely on the description of moduli spaces of meromorphic differentials given by [GW16; GW19], itself a generalization of the Hubbard-Masur theorem [HM79]. The use of higher-order poles allows us to easily describe an open-closed extension of this PROP.

The classical theory of Strebel differential is related to the topology of moduli spaces of Riemann surfaces as explained in [Kon92]. A Riemann surface with a Strebel differential determines the data of a metric ribbon graph; this gives a cell decomposition of the corresponding moduli space into cells labeled by topological types of ribbon graphs. In our case, we will obtain instead acyclic marked ribbon quivers, i.e., ribbon graphs endowed with directed edges in an acyclic manner, and with some markings. Each such ribbon quiver gives a cell in the open-closed moduli space; this cell decomposition is similar to, but finer than, the cell decomposition given by ‘black-and-white graphs’ appearing in e.g. [WW16; Ega15]. The combinatorial data of the quiver gives the action of this colored open-closed PROP on the morphism spaces of a pre-CY category $A$ and its Hochschild complex $C_*(A)$;
this action reduces to the black-and-white graph PROP action discussed in op.cit. when $A$ is a cyclic $A_{\infty}$-algebra.

Let us briefly explain how this work fits in the cobordism perspective on TQFTs. As we already mentioned, that a (compact) CY structure leads to an action of a PROP of noncompact surfaces has been interpreted by Lurie to be the characterization of a certain type of fully extended oriented 2d TQFT. Namely, there is a notion of Calabi-Yau object in any symmetric monoidal $(\infty, 2)$-category $\mathcal{A}$, such that the data of such an object in $\mathcal{A}$ is equivalent to the data of a ‘non-compact’ fully extended oriented 2d TQFT $Z : \text{Bord}^\text{nc}_2 \to \mathcal{A}$. Such a TQFT only assigns values to cobordisms given by surfaces with $\geq 1$ incoming boundary components, which form the $(\infty, 2)$-category of ‘noncompact’ 2d cobordisms $\text{Bord}^\text{nc}_2$. The precise relation between the spaces $\mathcal{M}_{g,N,\vec{m},\vec{n}}$ and the $\text{Bord}^\text{nc}_2$ is given by a certain ‘unfolding construction’ [Lur09].

Using this construction allows one to understand Theorem 1 in this cobordism framework.

**Corollary 2.** Let $\mathcal{C}$ be a ‘good’ symmetric monoidal $(\infty, 2)$-category (in the sense of [Lur09]), linear over $\mathbb{Q}$. Then a pre-CY object in $\mathcal{C}$, that is, an object of $\mathcal{C}$ with an appropriate action of the multi-corolla dioperad, determines a middle-index oriented 2d TQFT valued in $\mathcal{C}$: a symmetric monoidal functor $Z : \text{Bord}^\text{mid}_2 \to \mathcal{C}$ from the $(\infty, 2)$-category of 2d cobordisms generated by handles of index one.

This result can be seen as a conceptual explanation of why, for example, compact and smooth CY structures can produce pre-CY structures, essentially by restriction of the TQFT structure to middle-index cobordisms. We postpone these discussions about the relations between smooth CY structures and pre-CY structures to [KTV].

### 1.1. Relation to existing literature.

Both this paper and [KTV] are updated and expanded versions of unfinished preprints that were written by the first and third named authors in 2013, and which have since then circulated among the community in their unfinished form, being cited by other articles and lectures such as [IK20; IKV21; Yeu18; FH19; Sei20].

Due to this state of affairs, some results and definitions here have also been discussed elsewhere in the literature, often citing that unfinished version of this paper. Throughout the text, we have made an attempt to be thorough in referring to those articles, and to be clear about what results in this expanded version are new. Let us give a non-extensive overview of related work. In the finite-dimensional case, pre-CY algebras have been defined and studied before by another name: some time between 2013 and now we learned from T. Tradler and M. Zeinalian that they had made an equivalent definition in [TZ07] where this structure was called a $V_{\infty}$-algebra. Moreover, P. Seidel also informed us that in [Sei12] he gave the same definition (Definition 3.5), calling them boundary algebras. In the infinite dimensional case, some relations between pre-CY structures and symplectic/Poisson geometry appear in [Yeu18; IK20].

Tradler-Zeinalian also describe the proof of an analogous result to our Theorem 1, describing a PROP they call $D\mathcal{G}_\infty$ (for directed graphs); this turns out to be the closed part of our PROP, and in the finite-dimensional case their proof is equivalent to ours. On the other hand, the uniformization of open-closed moduli space by meromorphic Strebel differentials is new, and gives an answer the question posed in Remark 5.6 of [TZ07].
Another place where a similar PROP appears is in the work of Wahl-Westerland [WW16] on black-and-white graphs, which are similar objects to ours but without directions along the edges. The moduli spaces we describe here are homotopic to the ones proved by [Ega15] to give classifying spaces for open-closed cobordisms. Our quiver structure gives a finer stratification; the induced action on $A$ and $C_*(A)$ agrees with their description in the case where $A$ is a cyclic $A_\infty$-algebra, but the finer stratification allows us to generalize away from the finite-dimensional case.

This model of black-and-white graphs, acting on a cyclic $A_\infty$-algebra, has appeared in the recent papers [CCT20; CT20], where the authors refer to its use in the context of categorical Gromov-Witten invariants. We believe that the formalism of this paper can be used to extend their formalism to calculate invariants of categories that are not cyclic $A_\infty$.

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2. Background material

Throughout this paper, we work over a fixed field $k$ of characteristic zero. We will denote by Vect the symmetric monoidal categories of $\mathbb{Z}$-graded vector spaces, with monoidal structure given by the tensor product $\otimes$. We will use cohomological grading, and denote by $[1]$ the shift functor acting on objects of Vect as $(V[1])^n = V^{n+1}$. For any homogeneous element $a \in V$, we write $\deg(a)$ for its degree and denote by $\bar{a} = \deg(a) - 1$ its degree in $V[1]$. In fact, all our statements hold equally for the $\mathbb{Z}/2$-graded case, and we will simply write 'graded vector space'.

In many definitions of certain graded vector spaces (of Hochschild complexes, spaces associated to the PROPs etc.) we must take products over a sequence filtered by length. In all of those cases we will simply write a product $\prod \{...\}$, but more precisely one should take the completion of the direct sum $\bigoplus \{...\}$ with respect to the filtration by length.

2.1. $A$-infinity algebras and categories. Let us denote by $\mathcal{A}$ the data of a set of objects $\text{Ob}(\mathcal{A})$ and for any two objects $X, Y \in \text{Ob}(\mathcal{A})$, a graded vector space $\mathcal{A}(X, Y)$.

The space of Hochschild cochains on $\mathcal{A}$ of length $n$ is the graded vector space

$$C^*(\mathcal{A})^n := \prod_{X_0, \ldots, X_n \in \text{Ob}(\mathcal{A})} \text{Hom}(\mathcal{A}(X_0, X_1)[1] \otimes \cdots \otimes \mathcal{A}(X_{n-1}, X_n)[1], \mathcal{A}(X_0, X_n))$$

Definition 2. The space of Hochschild cochains of $\mathcal{A}$ is the graded vector space given by

$$C^*(\mathcal{A}) = \prod_{n \geq 0} C^*(\mathcal{A})^n$$
Example. If \( \mathcal{A} \) has a single object \( X \) with a graded vector space \( \mathcal{A}(X, X) = A \) concentrated in degree zero, the complex \( C^*(\mathcal{A}) \) is given by \( \text{Hom}(A^\otimes n, A) \) in degree \( +n \).

We endow the space \( C^*(\mathcal{A}) \) with the Gerstenhaber product \( \circ \), a non-associative operation defined by

\[
    f \circ g(a_1, \ldots, a_n) = \sum_{i,j} (-1)^{\#} f(a_1, \ldots, a_i, g(a_{i+1}, \ldots), a_j, \ldots, a_n)
\]

where \( \# = \bar{g} \sum_{k=1}^i \bar{a}_k \), with \( \bar{g} = \deg(g) - 1 \). This gives a product of degree \(-1\), that is, a morphism of graded vector spaces

\[
    C^*(\mathcal{A}) \otimes C^*(\mathcal{A}) \to C^*(\mathcal{A})[-1].
\]

Definition 3. The Gerstenhaber bracket \([-, -]\) is the binary operation on \( C^*(\mathcal{A}) \) defined by

\[
    [f, g] = f \circ g - (-1)^{\bar{f}\bar{g}} g \circ f
\]

The Gerstenhaber bracket endows the shifted space of Hochschild cochains \( C^*(\mathcal{A})[1] \) with the structure of a graded Lie algebra.

Definition 4. An \( A_\infty \) structure on \( \mathcal{A} \) is an element \( \mu \in C^2(\mathcal{A}) \) satisfying \( \mu \circ \mu = 0 \), with vanishing length zero component \( \mu^0 = 0 \).

We will often refer to \( \mathcal{A} \) as an \( A_\infty \)-category while leaving the \( A_\infty \) structure \( \mu \) implicit. For any \( n \geq 1 \), we denote by \( \mu^n \in C^2(\mathcal{A})^n \) the length \( n \) component of such a structure; this is the data of maps

\[
    \mu^n : \mathcal{A}(X_0, X_1)[1] \otimes \cdots \otimes \mathcal{A}(X_{n-1}, X_n)[1] \to \mathcal{A}(X_0, X_n)
\]

for any \((n + 1)\)-tuple of objects \( X_i \).

Example. In the single-object case, each \( \mu^n \) is a map \( A^\otimes n \to A \) of degree \( 2 - n \), the collection of such maps satisfying the equation

\[
    \sum_{r+s+t=n} (-1)^{\#} \mu^{r+1+t}(a_1, \ldots, a_r, \mu^s(a_{r+1}, \ldots, a_{r+s}), a_{s+1}, \ldots, a_n) = 0
\]

where \( \# = \bar{a}_1 + \cdots + \bar{a}_r \). That is, \( A \) is an \( A_\infty \)-algebra.

Note that the equation for the first component is \( \mu^1 \circ \mu^1 = 0 \), so \( \mu^1 \) is a differential. If instead we drop the requirement that \( \mu^0 = 0 \), we have a curved \( A_\infty \)-category, in which case \( \mu^1 \) does not necessarily square to zero. Throughout this paper we will only deal with the non-curved case.

If \( \mu^n = 0 \) for all \( n \geq 3 \), \( \mathcal{A} \) is a dg (differential graded) category with set of objects \( \text{Ob}(\mathcal{A}) \) and morphism spaces \( \mathcal{A}(X, Y) \), with differential and composition maps given by

\[
    df := \mu^1(f), \quad f \cdot g := (-1)^{\deg(f)} \mu^2(f, g).
\]

An \( A_\infty \)-structure determines an action of certain dg operad, the \( A_\infty \)-operad, given by chains on the topological operad of Stasheff associahedra; the algebraic definition above comes from a cell decomposition of those associahedra whose cells are labeled by rooted planar trees.
2.1.1. Geometric interpretation. We now recall a noncommutative geometry perspective of $A_{\infty}$-algebras [KS06]. In this interpretation, the Gerstenhaber bracket corresponds to the bracket of vector fields on a noncommutative space, and an $A_{\infty}$-structure is an integrable vector field.

Consider a graded vector space $V$ with dual space $V^\vee$. Let

$$TV^\vee := \prod_{k \geq 0} (V^\otimes k)^\vee \supset \bigoplus_{k \geq 0} (V^\otimes k)^\otimes k$$

be the completed tensor algebra of $V^\vee$, where $(V^\otimes k)^\vee$ is equipped with the topology of the dual linear space and $TV^\vee$ with the product topology. We note that

$$TV^\vee \cong (\oplus_{k \geq 0} V^\otimes k)^\vee \cong \prod_{k \geq 0} (V^\otimes k)^\otimes k$$

where $\hat{\otimes}$ is the completed tensor product.

Let $T_+ V^\vee := \prod_{k \geq 1} (V^\otimes k)^\vee$ be the maximal ideal. A derivation $\delta : T_+ V^\vee \to T_+ V^\vee$ of degree $n$ is a continuous linear map of degree $n$ satisfying the equation

$$\delta(a_1 a_2) = \delta(a_1) a_2 + (-1)^{n a_1} a_1 \delta(a_2).$$

It follows from linearity and the Leibniz rule that a derivation is determined by its restriction to $V^\vee$, that is, by the collection of maps $\delta_k : V^\vee \to (V^\otimes k)^\vee$, for $k \geq 1$. We think of $\delta = \sum_{k \geq 1} \delta_k$ as a Taylor expansion for the derivation.

Suppose that for each $k$, $\delta_k$ is the dual of a map $\mu_k : V^\otimes k \to V$ (this is automatically true for $A$ finite-dimensional, but not necessarily in general). These maps are components of the dual map $\mu : T_+ V \to T_+ V$ which is a coderivation on the coalgebra $T_+ V$.

Let $\text{Der}(T_+ V^\vee)$ denote the graded vector space of continuous derivations of the algebra $T_+ V^\vee$. The commutator $[-,-]$ gives $\text{Der}(T_+ V^\vee)$ the structure of a graded Lie algebra. We now set $V := A[1]$, where $A$ is a graded vector space; there is then a correspondence between collections of maps $\mu_k : A[1]^\otimes k \to A[1]$ and derivations of $T_+ (A[1])^\vee$ which are duals of coderivations. The Lie bracket on $\text{Der}(T_+ (A[1])^\vee)$, restricted to the duals of coderivations of $T_+ (A[1])$, is equal to the Gerstenhaber bracket on $C^*(A)[1]$.

As a consequence, if $\mu : T_+ V \to T_+ V$ is a coderivation of degree one such that its dual derivation satisfies $[\delta, \delta] = 0$, then $\mu$ defines an $A_{\infty}$ structure on $A$. In that case $\delta$ is also called a homological vector field. This interpretation also gives a natural way to define morphisms in the category of $A_{\infty}$-algebras.

**Definition 5.** A morphism of $A_{\infty}$ algebras $g : (A, \mu_A) \to (B, \mu_B)$ is a morphism of tensor coalgebras $g : T_+ (A[1]) \to T_+ (B[1])$ commuting with the respective coderivations.

Unraveling this definition, we have a collection of maps $g^n : A[1]^\otimes n \to B[1]$ satisfying a family of equations, given by

$$\sum_{l+t+1 = k} g^k (\text{id}^\otimes l \otimes \mu_A^s \otimes \text{id}^\otimes t) = \sum_{i_1 + \cdots + i_r = n} \mu_B^r (g^{i_1} \otimes g^{i_2} \otimes \cdots \otimes g^{i_r}).$$

for all $n \geq 1$ and $n = l + s + t$.

**Definition 6.** The morphism $g : (A, \mu_A) \to (B, \mu_B)$ is called a quasi-isomorphism of $A_{\infty}$ algebras if the chain map $g_1 : (A, \mu_A^1) \to (B, \mu_B^1)$ is a quasi-isomorphism of complexes.
The following ‘homological inverse function’ theorem holds for $A_\infty$-algebras, upon considering minimal models for them.

**Proposition 3.** [KS06, Prop.3.2.3] If $g : (A, \mu_A) \to (B, \mu_B)$ is a quasi-isomorphism of $A_\infty$ algebras then it has a (non-canonical) inverse $g'$.

### 2.1.2. Unitality

Let us now discuss some notions of unitality for $A_\infty$-algebras and categories.

**Definition 7.** An $A_\infty$-category $(A, \mu)$ is cohomologically unital if $H = H^*(A, \mu)$ has an identity morphism in $H(X, X)$ for every object $X$ (in this case, $H$ is an ordinary category). It is strictly unital if for every object $X$ there is an element $1_X \in A(X, X)$ such that

$$\mu^2(1_X, a) = (-1)^{\deg(a)} \mu^2(a, 1_X) = a$$

and $\mu^2(\ldots, 1_X, \ldots) = 0$.

Given any $A_\infty$-category $(A, \mu)$, one can always adjoin an unit $1_X$ to the automorphism space of each object $X$ to get a strictly unital $A_\infty$-category $A^\dagger$. That is, by setting

$$A^\dagger(X, X) := A(X, X) \oplus k1_X$$

for every object $X$. The structure maps are extended to make $1_X$ a strict unit.

**Proposition 4.** If $A$ is cohomologically unital, then the canonical map $A \to A^\dagger$ is a quasi-equivalence.

See e.g. [Lef02, Sec.3.2] for a precise proof.

### 2.1.3. Modules and bimodules

Throughout this paper we will make use of modules and bimodules over $A_\infty$-categories. There are several detailed expositions of this theory, for example [Gan13; She20]. We will skip most details but recall some relevant concepts of their theory, together with geometric interpretations in the case of $A_\infty$-algebras.

**Definition 8.** A right module $M$ over an $A_\infty$-category $(A, \mu)$ is a graded vector space $M(X)$ for every object $X \in A$, along with maps of degree one

$$\mu_M^{1,n}(X_1, \ldots, X_n) : M(X_1) \otimes A(X_1, X_2)[1] \otimes \cdots \otimes A(X_{n-1}, X_n)[1] \to M(X_n)$$

for $n \geq 1$, such that the system of equations defining $A_\infty$ algebras holds if we replace $\mu^i$ by $\mu_M^i$ every time the argument starts by an element in $M$.

For an $A_\infty$-algebra $A$, a system of maps $\mu_M^i$ making $M$ an $A$-module is equivalent to the data of $\mu_M$ of degree $+1$ on $M \otimes T(A[1])$ which makes it a dg comodule over the coalgebra $T(A[1])$. Dualizing the factors of $A$, this gives a degree one derivation $\delta_M \in \text{Der}(M \otimes T(A[1]^{\vee}))$, satisfying the Leibniz rule with respect to the homological vector field $\delta_A$ on $\text{Spec}(A)$, and $[\delta_M, \delta_M] = 0$.

**Definition 9.** A $(A, B)$-bimodule $M$ over a pair of $A_\infty$-categories $(A, \mu_A)$ and $(B, \mu_B)$ is a graded vector space $M(X, X')$ for every pair of object $X \in B, X' \in A$, along with maps of degree one

$$\mu_M^{r,s} : B(X_1, X_2)[1] \otimes \cdots \otimes B(X_{r-1}, X_r)[1] \otimes M(X_r, X'_1) \otimes A(X'_1, X'_2)[1] \otimes \cdots \otimes A(X'_{s-1}, X'_s)[1] \to M(X_1, X'_s)$$

for $r, s \geq 0$ and tuples of objects $X_i, X'_i$, such that the system of equations defining $A_\infty$ algebras holds if we appropriately replace $\mu^i$ by $\mu^i_A, \mu_M^i$, or $\mu^i_B$. 
In geometric terms, the bimodule structure over a pair $A, B$ of $A_\infty$-algebras is given by a differential of degree +1 on the graded vector space $T_+(A[1]) \otimes M \otimes T_+(B[1])$ whose dual derivation satisfies the Leibniz rule with respect to the derivations $\delta_A^\op$ and $\delta_B$. The $A_\infty$-relations imply that $\mu_M^{0,1,0}$ is a differential and that $\mu_M^{1,1,0}$ and $\mu_M^{1,1,1}$ descend to module actions on the level of cohomology. If $A$ and $B$ are homologically unital, we can define:

**Definition 10.** The $(A, B)$-bimodule $M$ is **homologically unital** if for any objects $X \in A, X' \in B$, any choice of homological units $e_X$ and $e_{X'}$ act as the identity at the level of cohomology.

From now on, we will work only with homologically unital $A_\infty$-categories and homologically unital modules over them. We now present some facts about $A_\infty$-bimodules whose proof can be found in e.g. [Gan13].

**Proposition 5.** Given any pair $(A, B)$ of $A_\infty$-categories, there is a dg category $A \text{-Mod}\-B$ whose objects are $(A, B)$-bimodules. There is an operation of tensor product over $B$ given by a dg functor $- \otimes_B - : A \text{-Mod}\-B \times B \text{-Mod}\-C \to A \text{-Mod}\-C$ and a two-sided tensor functor given by a dg functor $- \otimes_{A, B} - : A \text{-Mod}\-B \times B \text{-Mod}\-A \to \text{Vect}$

Both of these maps are functorial, in the sense that $A_\infty$-morphisms on the factors induce $A_\infty$-morphisms on the product, sending quasi-isomorphisms to quasi-isomorphisms.

For $A_\infty$-algebras, a morphism $M \to N$ of $(A, B)$-bimodules is a map of dg $T_+(A[1])^{op} \otimes T_+(B[1])$-comodules whose dual derivation intertwines the homological vector fields $\delta_A$ and $\delta_B$.

**Remark.** The reason for having these hands-on separate definitions for modules and bimodules is to avoid discussing the tensor product of $A_\infty$-algebras/categories, since there is a lack of functorial choices for such a product; see [KS06].

**Proposition 6.** For any $A_\infty$-category $A$, there is an object $A_\Delta \in A \text{-Mod}\-A$, called its diagonal bimodule, such that there are quasi-isomorphisms $M \otimes_A A_\Delta \cong M$ and $A_\Delta \otimes_A N \cong N$ for any right $A$-module $M$ and left $A$-module $N$.

2.1.4. Hochschild co/homology. Let $(A, \mu)$ be an $A_\infty$ category.

**Definition 11.** The **Hochschild cochain complex** is the graded vector space $C^*(A)$ endowed with the differential $d := [\mu, -]$ of degree +1; its cohomology $HH^*(A) := H^*(C^*(A), d)$ is called the Hochschild cohomology of the $A_\infty$ category $A$.

Note that by definition, $[\mu, \mu] = 0$ so $d^2 = 0$ and we have a class $[\mu] \in HH^2(A)$. We define the space of Hochschild chains $C_*(A)$ as the graded vector space

$$C_n(A) := \prod_{X_0, \ldots, X_n \in \text{Ob}(A)} A(X_0, X_1) \otimes \cdots \otimes A(X_{n-1}, X_n) \otimes A(X_n, X_0)[n]$$
Definition 12. The Hochschild chain complex is the graded vector space $C_\ast(A)$ endowed with a differential $b$ of degree +1 defined on generators as follows

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i,j} (-1)^{\#_1} \mu^{i+j}(a_{j+1}, \ldots, a_0, \ldots, a_i) \otimes a_{i+1} \otimes \cdots \otimes a_j$$

$$+ \sum_{i,j} (-1)^{\#_2} a_0 \otimes \cdots \otimes \mu^{j-i}(a_{i+1}, \ldots, a_j) \otimes \cdots \otimes a_n,$$

where $\#_1 = (\bar{a}_1 + \cdots + \bar{a}_j)(\bar{a}_{j+1} + \cdots + \bar{a}_n)$ and $\#_2 = \bar{a}_0 + \cdots + \bar{a}_i$. This complex calculates the Hochschild homology $HH_\ast(A) := H^\ast(C_\ast(A), b)$ of the $A_\infty$ category $A$.

Remark. Note that even though we denote Hochschild homology with a subscript, we are still using the cohomological grading convention for it; the differential $b$ has degree +1.

2.2. Graphical calculus for $A$-infinity categories and bimodules. Anyone who tries to do calculations with $A_\infty$-structures will eventually encounter the annoying appearance of signs everywhere. We now present a unified graphical calculus describing $A_\infty$-categories and -bimodules, which allows one to write the relevant formulas with a minimum of explicit signs. The resulting sign conventions are in conformity with the signs appearing in e.g. [Sei08; Gan13]. For simplicity we work with an $A_\infty$-algebra $A$, but the procedure is easily extended to categories.

2.2.1. Representation for Hochschild cochains. Let $\phi \in C^\ast(A)$ denote a Hochschild cochain of $A$. We interpret $\phi = \sum_n \phi^n$ as a collection of maps

$$\phi^n : A[1] \otimes^n \rightarrow A[1]$$

(note the shift in the target) of degree $\tilde{\phi} = \deg(\phi) - 1$ and denote it graphically by the diagram

One can then compose such vertices into larger diagrams, e.g.

An ordering on the graph $\Gamma$ is a linear extension of the partial ordering induced by the directions along the arrows. In the diagram above, we have a partial ordering

$$\phi_1 > \phi_3, \quad \phi_2 > \phi_3$$

and we can choose for instance the ordering

$$(\phi_3 \phi_2 \phi_1), \text{ meaning } \phi_3 < \phi_2 < \phi_1$$

which we suggested in the figure above by drawing $\phi_1$ above $\phi_2$. 

Each diagram is embedded in a disc, with a marked point on the boundary corresponding to the outgoing arrow at the bottom. To the ordered diagram above we assign a Hochschild cochain \( (\Gamma, (\phi_3 \phi_2 \phi_1)) \) whose value on \( a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A[1] \otimes^n \) is calculated as follows.

1. We write the elements \( a_i \) as incoming arrows around the circle, clockwise starting from the bottom. We then choose some way of connecting these arrows to the vertices while respecting the cyclic order without crossings. For example, given an element \( a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5 \in A[1] \otimes^5 \), one such diagram is

\[
\phi_1 \phi_2 \phi_3 \phi_4 \phi_5
\]

2. We write the ordering and the element in \( A[1] \otimes^n \) next to each other:

\[
(\phi_3 \phi_2 \phi_1)(a_1 a_2 a_3 a_4 a_5)
\]

3. We permute the \( a_i \) such that \( \phi_1 \) precedes its inputs immediately, recording the Koszul sign:

\[
(\phi_3 \phi_2 \phi_1)(a_1 a_2 a_3 a_4 a_5) = (-1)^{\bar{\phi}_1(\bar{a}_2 + \bar{a}_3)}(\phi_3 \phi_2 \phi_1)(a_2 a_3 a_1 a_4 a_5)
\]

4. We evaluate \( \phi_1(a_2, a_3) \) and write the result in the place occupied by \( \phi_1 \) and its inputs:

\[
(-1)^{\bar{\phi}_1(\bar{a}_2 + \bar{a}_3)}(\phi_3 \phi_2)(\phi_1(a_2, a_3) a_1 a_4 a_5)
\]

5. Repeat steps (3) and (4) until we are left with an element of \( A[1] \), which we then interpret as an element of \( A \). In the case above, this result is

\[
(-1)^{\bar{\phi}_1 \bar{a}_1 + \bar{\phi}_2 (\bar{a}_1 + \bar{a}_2 + \bar{a}_3) + \bar{\phi}_1 \bar{\phi}_2}\phi_3(a_1, \phi_1(a_2, a_3), \phi_2(a_4), a_5)
\]

In the sign, \( \bar{\phi} \) as usual denotes the degree of \( \phi \) as a map from copies of \( A[1] \) to copies of \( A[1] \); that is, \( \bar{\phi} = \deg(\phi) - 1 \).

In the example above, we could have chosen the ordering \( (\phi_3 \phi_1 \phi_2) \) instead, switching \( \phi_1 \) and \( \phi_2 \). The exponent in the sign then would have been

\[
\bar{\phi}_1 \bar{a}_1 + \bar{\phi}_2 (\bar{a}_1 + \bar{a}_2 + \bar{a}_3)
\]

which differs by \( \bar{\phi}_1 \bar{\phi}_2 \) from the one above. One can check that this is a general fact: maps determined by any two orderings differ by a minus sign with exponent given by \( \sum_{(ij)} \bar{\phi}_i \bar{\phi}_j \), where we sum over transpositions \( (ij) \) in the permutation between the orderings.
This reproduces the signs appearing in $A_\infty$-relations. For example, the Gerstenhaber product and bracket simply become

$$\phi \circ \psi = \psi \phi$$

and

$$[\phi, \psi] = -(-1)^{\bar{\phi} \bar{\psi}} \phi \psi - \psi \phi$$

and the $A_\infty$-structure equations are just

$$\mu = 0,$$

with deg($\mu$) = $\bar{\mu} + 1 = 2$

and $\mu^0 = 0$.

2.2.2. Hochschild chains. It is also possible to describe Hochschild chains by modifying the graphical calculus above: instead of a disc, a Hochschild chain travels along a cylinder with a distinguished point on each boundary. We squash the cylinder into an annulus for ease of representation, with the input end around the outside. For example, the identity map on Hochschild chains is

$$\begin{array}{c}
\bullet \\
\downarrow
\end{array}$$

We then send a chain $a_0 \otimes a_1 \otimes \cdots \otimes a_p$ from the outside, with $a_0$ along the arrow leaving the marked point $\bullet$ and the $a_i$ around in clockwise order, i.e.

$$\begin{array}{c}
a_0 \\
\downarrow \\
a_1 \\
\downarrow \\
\vdots \\
\downarrow \\
a_p
\end{array}$$

Given a directed tree embedded in this cylinder, we can analogously define the action of diagrams of Hochschild cochains, as we did before in the disk. First we pick an ordering of the vertices as above and evaluate

$$(\phi_N \ldots \phi_1)(a_0 a_1 \ldots a_p)$$

with the same signs rules as above, and at the end, we permute the resulting outputs to put them in clockwise order starting from the marking $\bullet$.

The natural operations on Hochschild chains can be described very simply. For example, the Hochschild differential corresponding to some $A_\infty$-structure $\mu \in$
\[ C^2(A) \] is

\[ b = \]

and the cap product giving an action of \( C^*(A) \) and \( C_*(A) \), is

\[ \]

In the case where \( A \) has a strict unit \( e_A \), we can also describe the Connes differential

\[ B = \]

where \( 1 \in C^0(A) \) denotes the constant Hochschild cochain that evaluates \( \mathbb{k} \to e_A \) and \( T_+(A[1]) \to 0 \). This graphical calculus already appears in [KS06]; the prescription above allows one to write the signs systematically.

2.2.3. Morphisms and bimodules. We now describe the graphical calculus for bimodules. Let \( (A, \mu_A) \) and \( (B, \mu_B) \) be two \( A_\infty \)-algebras. We interpret an \((A,B)\)-bimodule \( M \) as a ‘boundary condition’ (bold line) between planar regions labeled by \( B \) (white) and \( A \) (shaded), together with a ‘boundary point operator’ \( \mu_M \) which sits on the bold line, as follows:

Note that the bold arrow representing an \((A,B)\)-bimodule \( M \) has \( A \) to its left (as seen by someone walking along the arrow) and \( B \) to its right. On some element \((a_1, \ldots, a_s, b_1, \ldots, b_r) \in T(A[1]) \otimes T(B[1])\), we evaluate the vertex by inserting \( b_i \) elements on the left and \( a_i \) elements on the right, clockwise, and \( m \in M \) on the top. We require that \( \deg(\mu_M) = 1 \), seen as a map

\[ B[1]^{\otimes r} \otimes M \otimes A[1]^{\otimes s} \to M \]

In order to properly use the sign convention, we need to specify how to do steps (2) and (3), i.e. how to assign degrees for the arrows and for the regions in the
presence of a bimodule line. The prescription is essentially the same, except that we count elements \( m \in M \) according to their degree \( \text{deg}(m) \) in \( M \).

With this convention in mind, we can express the \( A_\infty \) structure equation for bimodules as

\[
\mu_B M + \mu M M + \mu M A = 0
\]

The dg category \( A \text{-Mod-} B \) of \((A,B)\)-bimodules has objects given by boundary conditions with a \( \mu \) vertex as above, that is, a pair of a graded vector space \( M \) together with a set of maps \( \mu_M : B[1]^{\otimes r} \otimes M \otimes A[1]^{\otimes s} \to M \). Morphisms between bimodules are given by ‘boundary condition changing operators’: a morphism \( F : M \to N \) is represented by vertex between a \( M \)-line and an \( N \)-line, and the differential on the space of morphisms is given by \( F \mapsto [\mu, F] \), where \([\mu, F]\) is the vertex given by the sum of diagrams

Left- and right-modules over some \( A_\infty \)-category can be analogously described by setting one of the sides to be the rank one \( A_\infty \)-algebra given by the ground field \( k \) in degree zero.

2.2.4. Tensor products of bimodules. We can also use this graphical calculus to describe the tensor product of bimodules. Let \((A,\mu_A),(B,\mu_B),(C,\mu_C)\) be three \( A_\infty \)-algebras, and \( M \in A \text{-Mod-} B, N \in B \text{-Mod-} C \) two bimodules. We have the following graphical ‘definition’ of \( M \otimes_B N \):

\[
\begin{array}{c}
M \otimes_B N \\
\downarrow \\
\text{shaded area}
\end{array}
= 
\begin{array}{c}
N \\
\downarrow \\
M
\end{array}
\]

where the shaded area is where \( B[1] \)-lines travel. That is, as a graded vector space it is \( M \otimes T(B[1]) \otimes N \), with structure map \( \mu = \mu_{M \otimes_B N} \) given by the expression:

\[
M \otimes_B N = \mu_N + \mu_B + \mu_M
\]
Given two $A_{\infty}$-algebras $A, B$, we can tensor two bimodules $M \in A\text{-Mod}\text{-}B$ and $N \in B\text{-Mod}\text{-}A$ simultaneously over $A$ and $B$, and get a differential graded vector space $M \otimes_{A\otimes B} N$, the two-sided tensor product. As a graded vector space this is given by $M \otimes T(B[1]) \otimes N \otimes T(A[1])$. We can express this as traveling along a cylinder (or annulus) with two marked lines along which elements of $M$ and $N$ travel. The differential $d_{M \otimes_{A \otimes B} N}$ is then given by the diagram:

seen as an operation from $M \otimes_{A \otimes B} N$ (outside the annulus) to itself (inside the annulus) of degree +1.

2.3. Calabi-Yau structures.

2.3.1. The diagonal bimodule and its duals. Recall that given any $A_{\infty}$-algebra $A$ there is a canonical object of $A\text{-Mod}\text{-}A$, its diagonal bimodule $A_\Delta$. As a graded vector space, it is equal to $A$, and its structure maps are produced from the structure maps of $A$ with sign changes to account for shifts.

Given graded vector spaces $V_0, V_1, \ldots, V_n$, consider the graded vector space $\text{Hom}(V_1 \otimes \cdots \otimes V_n, V_0)$. Shifting one of the factors by $[-1]$, say $V_i$, gives an isomorphism

$$\text{Hom}(V_1 \otimes \cdots \otimes V_i[-1] \otimes \cdots \otimes V_n, V_0) \cong \text{Hom}(V_1 \otimes \cdots \otimes V_n, V_0)$$

sending $\phi$ to $\phi'$ given by

$$\phi'(v_1, \ldots, v_n) = (-1)^{\#} \phi(v_1, \ldots, v_n)$$

where $\# = \sum_{j=1}^i \deg(v_j)$; recall also that in the category of dg vector spaces, the differential on the shift $d_{V[1]}$ is given by $-d_V$.

The structure map of the bimodule $A_\Delta$, that is, a morphism $\mu_{A_\Delta} : T(A[1]) \otimes A \otimes T(A[1]) \to A$ given by the structure map $\mu_A : T(A[1]) \to A[1]$ of the $A_{\infty}$-algebra, but with the appropriate shift coming from the considerations above:

$$\mu_{A_\Delta}(a_1, \ldots, a_n, a, a'_1, \ldots, a'_m) = (-1)^{\#} \mu_A(a_1, \ldots, a_n, a, a'_1, \ldots, a'_m),$$

with $\# = \bar{a}_1 + \cdots + \bar{a}_n + 1$, and on the right-hand side $a$ is seen as an element of $A[1]$.

We now describe two other canonical objects in the category $A\text{-Mod}\text{-}A$, the linear dual bimodule $A_\Delta^\vee$ and the inverse dualizing bimodule $A_\Delta'$.

The linear dual bimodule $A_\Delta^\vee$, as a graded vector space, is given by the dual $\text{Hom}_k(A, k)$, and it has structure map $\mu_{A_\Delta^\vee} : T(A[1]) \otimes A^\vee \otimes T(A[1]) \to A^\vee$ given by

$$\mu_{A_\Delta^\vee}(a_1, \ldots, a_m, a^\vee, a'_1, \ldots, a'_n)(a) = (-1)^{\#} a^\vee(\mu_A(a'_1, \ldots, a'_n, a, a_1, \ldots, a_m))$$

where $\# = (\bar{a}_1 + \cdots + \bar{a}_m + \deg(a^\vee)) \cdot (\bar{a}'_1 + \cdots + \bar{a}'_n + \deg(a)) + \bar{a}'_1 + \cdots + \bar{a}'_n$. 

---

**Diagram:**

- Diagram showing the structure of the tensor product and the differential.
- Two circles with marked lines indicating the travel of elements.
- Annotations explaining the differential and structure maps.
- Notation for $\mu_{A_\Delta}$, $\mu_{A_\Delta^\vee}$, and $\mu_B$. 

---

**Additional Notes:**

- The notation $\otimes$ represents the tensor product.
- $\Delta$ and $\Delta'$ denote the diagonal and dualizing bimodules, respectively.
- $\mu$ represents the structure map.
- $\text{Hom}$ denotes the homomorphism space.
- Degree notation is used to account for shifts in the differential.

---

**Mathematical Symbols:**

- $A_{\infty}$-algebras
- $A\text{-Mod}\text{-}B$
- $T(B[1])$
- $T(A[1])$
- $\mu_{A_\Delta}$
- $\mu_{A_\Delta^\vee}$
- $\mu_B$
- $\text{Hom}_k(A, k)$
- $\Delta$
- $\Delta'$
- $\otimes$
- $\Delta$
Using the two-sided tensor product of bimodules, we can extend the canonical evaluation \( A \otimes A^\vee \to k \) to an evaluation morphism

\[
ev_A : A^\vee_\Delta \otimes_{A,A} A_\Delta \to k
\]

which satisfies the following graphical equation in \( \text{Hom}_k(A^\vee_\Delta \otimes_{A,A} A_\Delta, k) \):

\[
\begin{align*}
A^\vee_\Delta & \quad \mu_{A_\Delta} & \quad A^\vee_\Delta \\
\mu_{A_\Delta} & \quad \ev_A & \quad A^\vee_\Delta \\
A_\Delta & \quad \ev_A & \quad A_\Delta
\end{align*}
\]

We use the linear dual bimodule when \( A \) is compact, that is, when \( H^*(A, \mu_A) \) is finite-rank as a vector space. Let \( M \) be any compact object of \( A\text{-Mod}-A \), and consider some morphism \( F \in \text{Hom}_{A,A}(M, A^\vee_\Delta) \). Let \( \hat{F} \) be the map of graded vector spaces \( M \otimes_{A,A} A_\Delta \to k \) given by

\[
\hat{F}(a_1, \ldots, a_k, m, a'_1, \ldots, a'_l, a) = \ev(F(a_1, \ldots, a_k, m, a'_1, \ldots, a'_l), a)
\]

Recall that \( M \otimes_{A,A} A_\Delta \) is a model for the Hochschild chain complex \( C_* (A, M) \) of \( A \) with coefficients in the bimodule \( M \). We can express the relevant tensor-hom adjunction using bimodules as follows.

**Lemma 7.** When \( A \) and \( M \) are compact, the map \( \text{Hom}_{A,A}(M, A^\vee_\Delta) \to \text{Hom}_k(M \otimes_{A,A} A_\Delta, k) \) given by \( F \mapsto \hat{F} \) is a quasi-isomorphism.

Graphically, the lemma above says that the following local replacement

\[
\begin{align*}
M & \quad M & \quad M \\
\hat{F} & \quad \hat{F} & \quad \ev
\end{align*}
\]

of a part of some larger diagram induces an quasi-isomorphism.

We also have the inverse dualizing bimodule \( A^{1}_\Delta \), defined following Ginzburg’s definition in [Gin06] for the dg case. We recall its definition in the case of an \( A_\infty \)-algebra; the \( A_\infty \)-category case is analogous, and the relevant formulas can be found in [Gan13]. As a graded vector space, \( A^{1}_\Delta \) is given by

\[
A^{1}_\Delta = \text{Hom}_k(T(A[1]) \otimes A \otimes T(A[1]), A \otimes A)
\]
with structure maps given schematically by

\[
\mu^{r|0} = \begin{array}{ccc}
A \otimes A & \xrightarrow{\sim} & A \\
\mu & \otimes & 1 & \otimes & 0
\end{array}, \quad \mu^{0|s} = \begin{array}{ccc}
A \otimes A & \xrightarrow{\sim} & A \\
\mu & \otimes & 1 & \otimes & s
\end{array}
\]

and \(\mu^{r|s} = 0\) when \(r, s > 0\). \(^1\) This is an \(A_\infty\)-analog of the definition of the bimodule dual \(M^! = \text{Hom}_{A^e}(M, A^e)\) over an associative algebra \(A\); and can also be defined by this formula using the formalism of \(n\)-modules over an \(A_\infty\)-algebra.

Recall that an \(A_\infty\)-algebra \(A\) is (homologically) smooth \([KS06]\) if the diagonal bimodule \(A_{\Delta}\) is perfect as a \((A,A)\)-bimodule, i.e. if it is quasi-isomorphic to a direct summand of a finite extension of copies of the bimodule \(A \otimes A\).

**Lemma 8.** \([Gan13]\) If \(A\) is homologically smooth and \(M\) is perfect, then there is a natural quasi-isomorphism

\[
A_{\Delta}^! \otimes_{A,A} M \xrightarrow{\sim} \text{Hom}_{A,A}(A_{\Delta}, M)
\]

for any \((A,A)\)-bimodule \(M\).

For \(M = A_{\Delta}\), we pick an inverse of the quasi-isomorphism above to obtain a distinguished element \(\text{ev}^!\) in the graded vector space \(A_{\Delta}^! \otimes_{A,A} A_{\Delta}\). We then have a map \(\text{Hom}_{A,A}(A_{\Delta}, M) \to A_{\Delta}^! \otimes_{A,A} A_{\Delta}\) given graphically by:

which gives a quasi-inverse to the map in the Lemma above.

### 2.3.2. Compact and smooth Calabi-Yau structures.

We now recall two notions of Calabi-Yau structures on \(A_\infty\)-categories.

**Definition 13.** If \(A\) is compact, a dual cycle \(\theta : C_\ast(A) \to \mathbb{k}[-d]\) is a weak compact Calabi-Yau structure of dimension \(d\) on \(A\) if it maps to an isomorphism of bimodules \(A_{\Delta} \to A_{\Delta}^\vee\) under the map of Lemma 7.

If \(A\) is smooth, a cycle \(\omega : \mathbb{k}[d] \to C_\ast(A)\) is a weak smooth Calabi-Yau structure of dimension \(d\) on \(A\) if it maps to an isomorphism of bimodules \(A_{\Delta}^! \to A_{\Delta}\) under the inverse to the map in Lemma 8.

\(^1\)Note we can analogously define the bimodule dual of any bimodule \(M\) by substituting \(M\) on top instead of \(A\).
See e.g. [Gin06; KS06; BD19; Gan19].

There is a canonical $S^1$-action on the Hochschild complex, whose homotopy fixed points are modeled by the negative cyclic complex $CC^-(A)$ and whose homotopy orbits are modeled by the (positive) cyclic complex $CC^*(A)$. As a consequence there are canonical maps

$$CC^-(A) \to C^*(A), \quad C^*(A) \to CC^*(A).$$

**Definition 14.** A (strong) compact Calabi-Yau structure is a lift of a weak compact Calabi-Yau structure to a dual class in (positive) cyclic homology

$$CC^*(A) \to \mathbb{k}[-d],$$

and a (strong) smooth Calabi-Yau structure is a lift of a weak smooth Calabi-Yau structure to a class in negative cyclic homology

$$\mathbb{k}[d] \to CC^-(A).$$

**2.4. Cyclic A-infinity structures.** There is another notion of Calabi-Yau structure on an $A_\infty$-category, which is closely related with the definition of compact Calabi-Yau structure given above; this was defined in [KS06] under the name of $A_\infty$-algebra/category with nondegenerate scalar product.

**Definition 15.** A cyclic $A_\infty$-structure of degree $d$ on an $A_\infty$-category $A$ is a collection of (chain-level) nondegenerate $\mathbb{k}$-linear pairings

$$\langle -, - \rangle : \mathcal{A}(X,Y) \otimes \mathcal{A}(Y,X) \to \mathbb{k}[-d]$$

for any objects $X, Y$ of $A$, such that

$$(\mu^n(a_1, \ldots, a_n), a_{n+1}) = (-1)^{\bar{a}_1 + 1} \langle a_1, \mu^n(a_2, \ldots, a_n, a_0, \ldots, a_{n+1}) \rangle$$

for any collection of objects $X_1, \ldots, X_n, X_{n+1} = X_0$ and morphisms $a_i \in \text{hom}(X_i, X_{i+1})$.

Note that nondegeneracy of the pairing implies that $A$ is finite-dimensional. A cyclic $A_\infty$-structure should be seen a strictification of a compact CY structure, in the sense that given any compact CY structure on a compact $A_\infty$-algebra, one can find a quasi-isomorphic finite-dimensional cyclic $A_\infty$-algebra; we discuss more about this equivalence in Section 3.3.4, see also [Gan19].

**2.4.1. The necklace bracket for cyclic A-infinity algebras.** Recall that the data of an $A_\infty$-structure $\mu$ on $A$ is given by a solution to a Maurer-Cartan equation on the space $C^*(A)$ of its Hochschild cochains. We now explain an analogous description of cyclic $A_\infty$-structures.

Recall that the space of Hochschild chains of $A$ is defined as $C_*(A) = \prod_{s \geq 1} C_s(A)^s$, where

$$C_s(A)^s = \prod_{X_0, \ldots, X_s \in \text{Ob}(A)} \mathcal{A}(X_0, X_1)[1] \otimes \cdots \otimes \mathcal{A}(X_{s-1}, X_s)[1] \otimes \mathcal{A}(X_s, X_0)[1][-1]$$

This space has a $\mathbb{Z}_s$ action rotating the factors of $\mathcal{A}(\ldots)[1]$ with a Koszul sign; let us denote by $CC_*(A)^s = (C_*(A)^s)^{2s}$, and $CC_*(A) = \bigoplus_{s \geq 1} CC^s(A)^s$.

**Definition 16.** Let $A$ have a pairing $\langle -, - \rangle$ of degree $d$. The necklace bracket of $(A, \langle -, - \rangle)$ is the map

$$[-,-]_{\text{necc}} : CC^s(A)[1] \otimes CC^s(A)[1] \to CC^s(A)[1]$$
defined by summing up over pairings, using $\langle,\rangle$, of two cyclic words in elements of $A$, in all possible ways. Explicitly,

$$[(a_0 \otimes \cdots \otimes a_r)_{\text{cyc}}, (b_0 \otimes \cdots \otimes b_s)_{\text{cyc}}]_{\text{nec}} = \sum_{i,j} \langle a_i, b_j \rangle (a_0 \otimes \cdots \otimes a_{i-1} \otimes b_{j+1} \otimes \cdots \otimes b_s \otimes b_0 \otimes \cdots \otimes b_{j-1} \otimes a_{i+1} \otimes \cdots \otimes a_r)_{\text{cyc}},$$

where $(-)_{\text{cyc}}$ denotes we take the sum over all cyclic permutations.

We then have the following characterization of a cyclic $A_\infty$-structure.

Lemma 9. If $(-,-)$ is non-degenerate, there is an equivalence between the data of a cyclic $A_\infty$-structure on $A$ and the data of a solution $\omega \in CC^*(A)$ of homogeneous degree 2 of the Maurer-Cartan equation $[\omega,\omega]_{\text{nec}} = 0$.

Proof. The equivalence between the $A_\infty$-structure maps and the solution $\omega$ is given by dualizing the inputs:

$$\mu^s \rightarrow (\langle,\rangle) \quad \text{and} \quad \omega^s \rightarrow \langle,\rangle$$

i.e., using the pairing on the first $s$ of the $s+1$ outgoing legs of $\omega^s$. The fact that such maps $\mu^s$ satisfy the $A_\infty$-relations and are compatible with the pairing follows from the non-degeneracy of the pairing and the cyclic symmetry of $\omega$. □

3. PRE-CALABI-YAU ALGEBRAS AND CATEGORIES

We now present the main definition of this paper. A pre-CY structure on an $A_\infty$-algebra, or more generally on an $A_\infty$-category, is an extension of its $A_\infty$-structure given by a solution to a Maurer-Cartan equation on a certain dg Lie algebra containing the Hochschild cochains as a subcomplex. This equation is defined by a necklace bracket, generalizing the case of cyclic $A_\infty$-structures we saw in Section 2.4 to algebras/categories that are not finite-dimensional.

3.1. Higher Hochschild cochains. For any integer $\ell \geq 1$, and any $\ell$-tuple of non-negative integers $n_1, \ldots, n_\ell$, we denote by $\{X^i_j\}$ a collection of $\sum(n_i + 1)$ objects of $A$, indexed by $i = 1, \ldots, \ell$ and $j = 0, \ldots, n_i$.

We then sum over all such collections to define a graded vector space

$$C^*_\ell(A; n_1, \ldots, n_\ell) := \prod_{\{X^i_j\}} \text{Hom} \left( \bigotimes_{i=1}^\ell (A(X^i_0, X^i_1)[1] \otimes \cdots \otimes A(X^i_{n_i-1}, X^i_{n_i})[1]), \bigotimes_{i=1}^\ell A(X^i_0, X^i_{n_i-1}) \right)$$

Note the indices in the target of the Hom above. To make sense of this when $i = 0$, we set $X^{-1}_j = X^{\ell-1}_j$.

Example. Let us give an example to demonstrate how to organize the objects $X^i_j$. Consider the case $\ell = 2$ with $n_1 = 2, n_2 = 1$. We then pick five objects $X^1_0, X^1_1, X^2_1, X^1_2, X^2_2$, and organize them around the circle in clockwise fashion.
We can depict an element \( \phi \in C^{*}_{(3)} \) as a vertex inside of this circle which takes in elements of \( A(X^0_0, X^1_1), \ldots \) and then outputs an element of \( A(X^2_0, X^1_2) \otimes A(X^0_0, X^2_1) \).

\[
\begin{array}{c}
A(X^2_0, X^1_2) \\
A(X^1_1, X^2_1) \\
A(X^0_0, X^1_1) \\
A(X^0_0, X^2_1)
\end{array}
\xrightarrow{X^2_0} X^2_0 \\
\xrightarrow{X^1_1} \otimes \\
\xrightarrow{X^0_0} A(X^2_0, X^1_2)
\]

The objects \( X^i_j \) label regions around the vertex, and each arrow with an \( X \) region to its right and a \( Y \) region to its left carries an element of \( A(X, Y) \).

**Definition 17.** For any integer \( \ell \geq 1 \), the space of \( \ell \)-higher Hochschild cochains is the graded vector space

\[
C^{*}_{(\ell)}(A) := \prod_{n_1, \ldots, n_\ell} C^{*}_{\ell}(A; n_1, \ldots, n_\ell)
\]

Note that when \( \ell = 1 \) this is the usual space of Hochschild cochains, and when \( A = A \) is some \( A_\infty \)-algebra we have

\[
C^{*}_{(\ell)}(A) = \prod_{n_1, \ldots, n_k} \text{Hom}_k \left( A[1] \otimes n_1 \otimes \cdots \otimes A[1] \otimes n_k, A \otimes k \right).
\]

**Remark.** There is another concept of generalized Hochschild invariants which sometimes appears with this same name, described by [Pir00] under the name of ‘higher order Hochschild homology’. This is a distinct notion from our definitions.

3.1.1. **Graphical representation of higher Hochschild cochains.** We now extend the graphical calculus for (ordinary) Hochschild cochains to include higher Hochschild cochains. We visualize an element \( \phi \in C^{*}_{(k)}(A) \) as a vertex drawn on a plane, with \( k \) arrows coming out of the vertex. As an example, a cochain \( \phi \in C^{*}_{(3)}(A) \) is drawn as a vertex:

\[
\begin{array}{c}
\phi
\end{array}
\]

with the white arrow marking the first factor of the (output) tensor product, and the other factors are read from the outgoing arrows in *clockwise* direction. The vertex above represents a collection of maps

\[
\phi^{(n_1, n_2, n_3)} : A[1] \otimes n_2 \otimes A[1] \otimes n_2 \otimes A[1] \otimes n_3 \to A[1] \otimes A[1] \otimes A[1]
\]

(note the shifts on the outputs) for all choices of \( n_i \geq 0 \). We visualize the input factors as arrows incoming into the vertex, starting next to the white arrow, and the outputs as arrows going out of the vertex, starting on the white arrow, all clockwise. For example, if the cochain \( \phi \) evaluates

\[
\phi^{(2,1,3)}(a^1_1, a^2_2; a^1_2, a^2_2, a^3_3) = (b_1, b_2, b_3),
\]
we visualize this operation as

This notation extends by linear combination to inputs/outputs that are general tensors, and also to $A_{\infty}$-categories; in that case we must label the regions between the arrows by objects of the category.

We now describe how to evaluate a directed tree diagram composed of vertices like the one above, extending the procedure from Section 2.2. For example, for higher cochains

$$\phi \in C_{(3)}^*(A), \quad \psi \in C_{(2)}^*(A), \quad \lambda \in C_{(3)}^*(A)$$

we would like to interpret a diagram such as

as giving an element in $C_{(6)}^*(A)$. The white arrow to the right denotes that we want to read the output of this diagram starting clockwise from there.

We again must order our diagram, by choosing an ordering of the vertices compatible with the partial order given by the arrows. For example, on the diagram above we can choose the ordering $(\lambda \psi \phi)$ (in ascending order as before).

We interpret the diagram above as giving maps

$$A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_e} \rightarrow A^{\otimes 6}$$

as follows:

1. We draw incoming lines around the circle for each of the factors in the source, starting right after the marked white arrow tip, then connect them to the internal vertices of the tree in all possible ways without crossing. For
example, here is one such diagram:

(2) We then write the vertices in their order next to the inputs:

\[(\lambda \psi \phi)(a_1^1 a_2^1 \ldots a_6^1)\]

(3) Permute the entries with Koszul signs until the inputs of the last vertex are
next to it, in order:

\[-1\# (\lambda \psi \phi)(a_4^1 a_5^1 a_5^2 a_1^1 \ldots a_6^1)\]

for the appropriate sign \#.

(4) Evaluate the vertex and then write its outputs, also in order. Note that in
general the output is not a simple tensor, but instead a combination, such
as

\[\phi(\emptyset; a_4^1, a_5^1, a_5^2 a_1^1 \ldots a_6^1) = b_1 \otimes b_2 \otimes b_3 + b_1' \otimes b_2' \otimes b_3'\]

so we sum over all terms. So we have

\[-1\# ((\lambda \psi)(b_1 b_2 b_3 a_1^1 \ldots a_6^1) + (\lambda \psi)(b_1' b_2' b_3' a_1^1 \ldots a_6^1) + \ldots)\]

(5) Repeat steps (3) and (4) until all vertices are gone.

Note that in the drawing above we marked the rightmost outgoing arrow with
a white arrowhead; this indicates the first factor of \(A\) in the output \(A^{\otimes 6}\). After
the steps above we ended up with 6 elements of \(A[1]\); so we permute them with
Koszul sign to match the outgoing order, going clockwise and starting from the
white arrow, and then applying an overall shift to read the output in \(A^{\otimes 6}\).

By construction, the description above matches what we already established for
ordinary Hochschild cochains, when all vertices have exactly one outgoing arrow.

3.1.2. Cyclic actions. We now describe cyclic actions on spaces of higher Hochschild
cochains. Recall that an element \(\phi \in C^*_{(k)}(A)\) is a collection of maps given by

\[\phi(a_1^1, \ldots, a_{n_1}^1; \ldots; a_k^1, \ldots, a_{n_k}^1) = b_1 \otimes \ldots b_k + b_1' \otimes \ldots b_k' + \ldots\]

We now choose an integer \(d\) and write \(t\) for the generator 1 in the cyclic group \(\mathbb{Z}_k\)
of order \(k\).

**Definition 18.** The action of dimension \(d\) of the cyclic group \(\mathbb{Z}_k\) on \(C^*_{(k)}(A)\) is given by

\[(t \phi)(a_2^2, \ldots, a_{n_2}^2; \ldots; a_1^k, \ldots, a_{n_k}^k; a_1^1, \ldots, a_{n_1}^1) = (-1)^{\#_a}(-1)^{(d^{-1})(k-1)}((-1)^{\#_b}b_2 \otimes b_3 \otimes b_3 + \ldots)\]

where \(\#_a = (\bar{a}_1^1 + \ldots + \bar{a}_1^{n_1})(\bar{a}_2^2 + \ldots + \bar{a}_{n_2}^2)\) and \(\#_b = \bar{b}_1(\bar{b}_2 + \ldots + \bar{b}_k)\) are the
Koszul signs for permuting the factors of \(a\) and \(b\), seen as elements of \(A[1]\). That
is, the action of dimension $d$ has an extra sign of $(d - 1)(k - 1)$ compared to the usual Koszul sign.

We denote the action of dimension $d$ by $(\mathbb{Z}_k, d)$. It only depends on the parity of $d$.

**Definition 19.** The space of cyclic $k$-cochains of dimension $d$ on $A$ is defined as

$$C_{(k, d)}^*(A) := (C_{(k)}^*(A))^{(\mathbb{Z}_k, d)}[(d - 2)(k - 1)]$$

that is, the $(d - 2)(k - 1)$ shift of the higher Hochschild cochains that are invariant under the action of dimension $d$.

We assemble all these spaces into the tangent complex

$$C_{(d)}^*(A) := \prod_{k \geq 1} C_{(k, d)}^*(A)$$

Now that we have all these different complexes, related by shifts, let us define some notation that will simplify the calculation of signs later.

**Definition 20.** Given a cochain $\phi \in C_{(k, d)}^*(A)$, we denote by

- $\deg(\phi)$ its degree in $C_{(k)}^*(A)$, i.e. as a map from copies of $A[1]$ to copies of $A$.
- $\tilde{\phi}$ its degree as a map from copies of $A[1]$ to copies of $A$.
- $|\phi|$ its degree in $C_{(k, d)}^*(A)$, or equally, in $C_{(d)}^*(A)$.

These degrees are related by $\deg(\phi) = \tilde{\phi} + k = |\phi| + (d - 2)(k - 1)$.

3.1.3. The necklace bracket. We now use the graphical notation to define some operations on higher cyclic cochains. Let $\phi \in C_{(k, d)}^*(A), \psi \in C_{(\ell, d)}^*(A)$ be two higher cyclic cochains of dimension $d$ on $A$.

**Definition 21.** The necklace product $\phi \circ_{\text{necc}} \psi$ is the element of $C_{(k + \ell - 1)}^*(A)$ given by the following expression:

$$\phi \circ_{\text{necc}} \psi = \sum_{n=1}^{k} (-1)^{r_n} \phi \circ \psi$$

where the sign exponents are given by

$$r_n = (\ell - 1)(d - 1)(k - n + |\phi| + 1), \quad s_m = (k - 1)(d - 1)(|\phi| + 1) + (\ell - 1)(d - 1)n$$

Intuitively, the necklace product $\phi \circ \psi$ is given by placing $\psi$ in all possible ways around $\phi$ (making a ‘necklace’) connecting $\psi \rightarrow \phi$, summing over all possibilities with appropriate signs.

**Example.** Let us discuss the cases where $k$ or $\ell = 1$. When $k = 1$, i.e., $\phi$ is an ordinary Hochschild chain, we sum over putting $\phi$ along all the $\ell$ outgoing arrows of $\psi$, always with the same sign $(-1)^{(\ell-1)(d-1)(|\phi|+1)}$. 
When \(\ell = 1\), then we get a sum over all ways of putting \(\psi\) in the regions around \(\phi\), all with sign +1. Finally, when \(k = \ell = 1\) this is just the ordinary Gerstenhaber product.

Note that the necklace product has degree \(-1\), since it involves interpreting an output (element in \(\mathcal{A}(X,Y)\)) as an input (element in \(\mathcal{A}(X,Y)[1]\)).

**Definition 22.** The **necklace bracket** of dimension \(d\) is the map

\[
[-,-]_{\text{nec}} : C^*_\ell(\mathcal{A})[1] \otimes C^*_\ell(\mathcal{A})[1] \to C^*_{k+\ell-1}(\mathcal{A})[1]
\]

defined by \([\phi,\psi]_{\text{nec}} = \phi \circ \psi - (-1)^{(|\phi|-1)(|\psi|-1)} \psi \circ \phi\).

When restricted to ordinary Hochschild cochains \(C^*(\mathcal{A})\), this gives the usual notion of Gerstenhaber bracket.

The following proposition could be proven by doing an explicit computation of the signs involved. However, there is a more conceptual way of organizing the signs which we will discuss in Section 6.1.4, so we postpone the proof until then.

**Proposition 10.** The map defined above does land in \(C^*_{k+\ell-1}(\mathcal{A})[1]\), that is, its image satisfies the appropriate cyclic invariance under \(\mathbb{Z}_{k+\ell-1}\), and also gives \(C^*_d(\mathcal{A})[1]\) the structure of a dg Lie algebra.

### 3.2. Pre-Calabi-Yau structures

Using the necklace bracket we now come to the main definition of this paper.

**Definition 23.** A **pre-Calabi-Yau structure** of dimension \(d\) on \(\mathcal{A}\) is an element

\[
m = \sum_{k=1}^{\infty} m(k) \in C^*_d(\mathcal{A})
\]

of degree \(|m| = 2\) (that is, of degree 1 in the dg Lie algebra \(C^*_d(\mathcal{A})[1]\)) solving the Maurer-Cartan equation \(m \circ m = 0\).

We will say that \(\mathcal{A}\) is a pre-CY category to mean that there exists a pre-CY structure \(m\) as above. Restricting the equation \(m \circ m = 0\) to the component \(C^*_{(1)}(\mathcal{A})[1] = C^*(\mathcal{A})[1]\) gives the equation \(m_{(1)} \circ m_{(1)} = 0\), whose solution \(\mu = m_{(1)}\) is an \(A_\infty\) structure on \(\mathcal{A}\).

For concreteness, let us repeat the definition above in more detail, for the case of an \(A_\infty\)-algebra \(A\). The data of a pre-CY structure of dimension \(d\) on \(A\) is then a collection of maps

\[
m^{n_1,\ldots,n_k}_{(k)} : A[1] \otimes A[1] \otimes \cdots \otimes A[1] \to A^\otimes k
\]

of degree \(dk - d - 2k + 4\), cyclically invariant or anti-invariant (depending on the parity of \((k-1)(d-1)\)), satisfying

\[
\sum_{k+\ell=n+1} m_{(k)} \circ m_{(\ell)} = 0
\]

for every \(n \geq 1\).
3.2.1. Unitality. Recall from Proposition 4 that if \((\mathcal{A}, \mu)\) is a nonunital \(A_\infty\)-category, one can adjoin an unit \(1_X\) to the endomorphism space of each object \(X\) to get a strictly unital \(A_\infty\)-category \(\mathcal{A}^+\), which is moreover quasi-equivalent to \(\mathcal{A}\) when \(\mathcal{A}\) is homologically unital.

**Definition 24.** A pre-CY category \((\mathcal{A}, m = \sum_{k \geq 1} m_{(k)})\) is called strictly unital if and only if, for every object \(X\) of \(\mathcal{A}\), there is an element \(1_X \in \text{hom}_\mathcal{A}(X, X)\) such that

\[
m_{(1)}^2(1_X, a) = a, \quad (-1)^k m_{(1)}^2(b, 1_X) = b \quad \text{for all } a \in \mathcal{A}(X, Y), b \in \mathcal{A}(Y, X) \text{ and any object } Y,
\]

and every higher structure map \(m_{(k)}, k \geq 2\), evaluates to zero on any sequence containing \(1_X\).

Note that by the definition above a strictly unital pre-CY category is also a strictly unital \(A_\infty\)-category. The following lemma follows directly from the definitions.

**Proposition 11.** Let \((\mathcal{A}, \mu = m_{(1)})\) be an \(A_\infty\)-category, not necessarily strictly unital, and \((\mathcal{A}^+, \mu^+)\) its strictly unital augmentation. Then any pre-CY structure \(m = \{m_{(k)}\}\) of dimension \(d\) on \(\mathcal{A}\) extends to a pre-CY structure \(m^+ = \{m^+_{(k)}\}\) of dimension \(d\) on \(\mathcal{A}^+\), given by setting

\[
m_{(1)}^+ = \mu^+, \quad m_{(k)}^+|_{\mathcal{A}} = m_{(k)}, \forall k \geq 2
\]

and \(m_{(k \geq 2)}\) evaluated on any sequence containing a unit \(1_X\) gives zero.

3.2.2. The category of pre-CY algebras. \(A_\infty\)-algebras over \(k\) form a category, with morphisms given by \(A_\infty\)-functors

\[
f = \{f^n\}, f^n : A[1] \otimes^n \rightarrow B[1]
\]

and differential \(d : f \mapsto (\mu_B \circ f - f \circ \mu_A)\).

This can be easily extended to pre-CY algebras. Let \((A, m), (B, n)\) be two pre-CY algebras of dimension \(d\) over \(k\). Let us say a pre-morphism from \(A\) to \(B\) is a collection of maps \(f = \{f_{(k)}^{n_1, \ldots, n_k}\}\), for all \(k \geq 1\) and \(n_i \geq 1\), where

\[
f_{(k)}^{n_1, \ldots, n_k} \in \text{Hom}_k(A[1] \otimes^{n_1} \cdots A[1] \otimes^{n_k}, B[1] \otimes^k ((Z/k, d)[(d - 1)(k - 1)])
\]

where as before \((Z/k, d)\) indicates the dimension \(d\) action of the cyclic group.

We now define the composition of two such pre-morphisms, using the graphical calculus defined in Section 3.1.1. The pre-morphism \(g \circ_{\text{pre}} f\) has components given by sums over diagrams of the form

![Diagram](image)

over all ways of drawing any number \(N\) of \(f_{(k)}^{i}, 1 \leq i \leq N\) vertices around a \(g_{(k)}\) vertex. The number of outgoing arrows of such a diagram is given by \(k + \sum_{i} -N\).
Calculating the degrees, we see that the degree of such a diagram, as a map between factors of $A[1]$ and $B[1]$ is
\[
\sum_{i=1}^{N} (d-1)(n_i-1) + (d-1)(k-1) = (d-1)(k + \sum \ell_i - N - 1)
\]
so the collection of such diagrams also defines a pre-morphism.

Comparing to the definitions in Section 3.1, we see that an element of $C^*_n(A)$ defines a pre-morphism $A \to A$, with a shift by the degree of the element. That way, a pre-CY structure $n = \{n_i(\ell_i)\}$ on $B$ can be seen as a morphism of degree one, and we can extend the composition map $\circ$ above to its components.

**Definition 25.** The morphisms $(A,m) \to (B,n)$ in the category of pre-CY algebras over $k$ are given by pre-morphisms $f : A \to B$ satisfying the equation
\[
f \circ m = n \circ f_{\text{pre}}
\]
in the space $\text{Hom}_k(A[1]^{\otimes n_1} \otimes \ldots A[1]^{\otimes n_k} , B[1]^{\otimes k})[(d-1)(k-1) + 1]$.

Drawing the appropriate diagrams proves that the composition $\circ$ preserves solutions of the equation above, so the definition above does give a category.

We note that restricting attention to the structure maps $m_{(1)}$ gives a definition of morphism that agrees with the definition of $A_\infty$-morphism, so we have the following result.

**Proposition 12.** The functor $(A,m) \mapsto (A,\mu = m_{(1)})$ gives a functor from the category of pre-CY algebras of any dimension $d$ over $k$ to the category of $A_\infty$-algebras over $k$.

Finally we note that the two results above also generalize immediately to the setting of pre-CY categories (i.e. with multiple objects), in the same way that functors between $A_\infty$-categories are defined.

3.3. **Pre-CY algebras in noncommutative geometry.** We argue now that pre-CY algebras should be seen as giving a notion of Lagrangian subspaces inside a noncommutative symplectic space.

3.3.1. **The finite-dimensional case.** We start with the case where $A$ is finite-dimensional as a graded vector space. The following result means that in this case the data of a pre-CY structure can be rephrased in terms of cyclic CY structures.

**Proposition 13.** Let $A$ be a finite-dimensional graded vector space. Then the data of a pre-Calabi Yau structure of dimension $d$ on $A$ is equivalent to the data of a cyclic Calabi-Yau structure of dimension $d - 1$ on the space $A \oplus A^*[1-d]$, such that the subspace $A$ is an $A_\infty$-subalgebra.

**Proof.** Recall that by definition a cyclic CY structure of dimension $d-1$ on a graded vector space $B$ is a pair $(\mu_B, \langle \cdot, \cdot \rangle)$ of an $A_\infty$-structure on $B$ and a nondegenerate pairing
\[
\langle \cdot, \cdot \rangle : B \otimes B \to k[1-d]
\]
such that the tensor $\mu_B^0(-,\ldots,-)$ is (graded) invariant under the cyclic action.
Let $m$ be a pre-CY structure of degree $d$ on $A$. We now produce an $A_\infty$-structure $\mu$ on $B = A \oplus A^\vee[1 - d]$; by definition this is the data of maps

$$\mu^N : ((A \oplus A^\vee[1 - d])[1])^\otimes N \to (A \oplus A^\vee[1 - d])[2]$$

We produce all these maps from $m$ by dualizing the appropriate map $m_{1,\ldots,n_k}^{n_1,\ldots,n_k}$. Let us be explicit: consider the component

$$m_{1,\ldots,n_k}^{n_1,\ldots,n_k} : A[1]^\otimes n_1 \otimes \cdots \otimes A[1]^\otimes n_k \to A^\otimes k$$

We then produce components of $\mu$ from it, in the following way. For simplicity we denote

$$m_{1,\ldots,n_k}^{n_1,\ldots,n_k}(a_1,\ldots,a_{n_k}^k) = b_0 \otimes \cdots \otimes b_{k-1}$$

and regard all factors as living in $A[1]$.

(1) We make a map

$$A[1]^\otimes n_1 \otimes (A^\vee[2 - d]) \otimes A[1]^\otimes n_2 \otimes \cdots \otimes A^\vee[2 - d] \otimes A[1]^\otimes n_k \to A$$

which on $(a_1^1,\ldots,a_{n_1}^1,c_1,\ldots,c_{k-1},\ldots,a_{n_k}^k)$ first permutes all the $c_i$ factors (elements of $A^\vee[2 - d]$) to the end, evaluates $m_{1,\ldots,n_k}^{n_1,\ldots,n_k}$ on the $a_i$ factors, then permutes the outputs $b_i$ to pair $b_i$ with $c_i$, giving the result

$$\langle b_1,c_1 \rangle \cdots \langle b_{k-1}c_{k-1} \rangle b_0$$

with the Koszul sign coming from all the permutations.

(2) If $n_k \geq 1$, we also make a map

$$A[1]^\otimes n_1 \otimes (A^\vee[2 - d]) \otimes A[1]^\otimes n_2 \otimes \cdots \otimes A^\vee[2 - d] \otimes A[1]^\otimes n_k \to A^\vee[1 - d]$$

in the same way as above, but dualizing the last incoming factor of $A[1]$ instead of the first outgoing factor of $A$.

One then has to check that the resulting structure is cyclic with respect to the canonical pairing of degree $d - 1$ on $B$, and that it satisfies the $A_\infty$-relations, by performing a computation of the signs. Cyclicality follows from the cyclic invariance of the $m$ maps, and the $A_\infty$-relations follow from the necklace Maurer-Cartan equation. Recall that in the definition of the necklace product there are two sums; these correspond respectively to the terms in the $A_\infty$-relation for $\mu$ given by types (1) and (2) above.

**Remark.** It is instructive to consider the differential $\mu^1 : A \oplus A^\vee[1 - d] \to (A \oplus A^\vee[1 - d])[1]$; the component $A \to A^\vee[-d]$ (or equivalently, a pairing $A \otimes A[d] \to k$) is identically zero because of the condition that $A$ be an $A_\infty$-subalgebra. The component $A^\vee[-d] \to A$ may be nonzero; in terms of the pre-CY structure this is the copairing on $A$ given by the component $m_{(2)}^{0,0} : k \to A \otimes A[d]$.

Another way of stating Proposition 13 is to say that a finite-dimensional pre-CY algebra is a Lagrangian $A_\infty$-subalgebra. As remarked in the Introduction, the definition of pre-CY structure already appeared in the work of Tradler and Zeinalian [TZ07] (under the name of $V_\infty$-algebras), and in the work of Seidel [Sei12; Sei10] under the name of boundary algebras; these definitions do not apply to infinite-dimensional algebras. In the case where $A$ is not finite-dimensional, but still compact (that is, $H^\ast A$ is finite dimensional), we will relate this notion to cyclic $A_\infty$-structures in Section 3.3.4.

We have yet another simple relation between pre-CY structures and cyclic CY structures:
Proposition 14. A cyclic CY structure of dimension $d$ on a finite-dimensional graded vector space $A$ also defines a pre-CY structure of dimension $d$ on $A$.

Proof. We simply set $m(1) = \mu_A$ to be the $A_\infty$-structure on $A$, $m(2) = m^{0,0}_{(2)} : k \to A \otimes A[d]$ to be the inverse of the pairing coming from the cyclic CY structure, and $m_{(k \geq 3)} = 0$. The cyclic relation for $\mu$ then implies that $m \circ m = 0$, as desired. □

As usual, the Lemma above also holds in the setting with multiple objects.

3.3.2. Cyclic forms. The result of Proposition 13 is not compatible with quasi-equivalences of $A_\infty$-structures; for applications it will be useful to relax the cyclicity condition so that it is only required to hold up to homotopy. Since we will make use of $A_\infty$-minimal models, in this section we will restrict our attention to $A_\infty$-algebras, which is the setting where the theory of minimal models is more readily available in the literature.

These results are more naturally understood in the language of noncommutative formal manifolds, as developed in [KS06]. Recall that the data of a nonunital $A_\infty$-algebra $A$ is the same as the data of noncommutative formal pointed dg-manifold $(X, x_0)$ with a homological vector field $Q$ and an isomorphism $A[1] \cong T_{x_0}X$.

The space of functions $\mathcal{O}(X)$ on $X$ is then identified with the tensor algebra $T(A[1])$; its space of cyclic $0$-forms is then

$$\Omega^0_{\text{cyc}}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{\text{top}},$$

where $[,]_{\text{top}}$ is the topological completion of the algebraic commutator. Roughly, if $\{x_i\}$ are coordinates on $A[1]$, $\Omega^0_{\text{cyc}}(X)$ is composed of cyclic formal series $f(\{x_i\})$ on free variables $\{x_i\}$.

The functions on the odd tangent bundle $\mathcal{O}(T[1]X)$ are then formal series on free variables $\{x_i, dx_i\}$, with $\deg(dx_i) = \deg(x_i) + 1$. The spaces of cyclic nc differential forms $\Omega^m_{\text{cyc}}(X)$ are given by the decomposition

$$\Omega^0_{\text{cyc}}(T[1]X) = \prod_{m \geq 0} \Omega^m_{\text{cyc}}(X)$$

into spaces spanned by expressions with exactly $m$ variables of the type $dx_i$.

We have two distinct differentials acting on cyclic nc differential forms: the cyclic de Rham differential $d_{\text{cyc}}$ and the Lie derivative $\text{Lie}_Q$ with respect to the homological vector field $Q$. We denote by $\Omega^m_{\text{cyc}}(X)$ the closed forms with respect to $d_{\text{cyc}}$; the Lie derivative descends to this subcomplex so we can consider the complexes $(\Omega^m_{\text{cyc}}(X), \text{Lie}_Q)$.

We can express the action of the Lie differential graphically by relating to the definitions of Section 2 in terms of bimodules. Translating the definitions, we have an isomorphism

$$\Omega^m_{\text{cyc}}(X) = \left(\left(\left(\cdots \left(\mathcal{A}_\Delta \otimes_A \mathcal{A}_\Delta \otimes_A \cdots \otimes_A \mathcal{A}_\Delta\right)^x\right)^x\right)^x\right)^{z_m}.$$
that is, a element $\omega \in \Omega^k_{\text{cyc}}(X)$ can be seen as a vertex receiving $m$ cyclically ordered $A_\Delta$ arrows and any $k$ numbers of $A[1]$ arrows between them, for example:

![Diagram](image)

where the bold arrows label the $A_\Delta$ arrows. The Lie derivative $\text{Lie}_Q$ is then given by ‘circling’ this vertex with a vertex corresponding to the $A_\infty$-structure maps $\mu_A$ and $\mu_{A_\Delta}$, as in Section 2.2.4.

### 3.3.3. Symplectic structures and minimal models.

From now on we assume the $A_\infty$ algebra $A$ is homologically unital and compact. Recall from Definition 14 that a compact CY structure on $A$ is a morphism of complexes

$$\omega : CC_*(A) \to k[-d]$$

satisfying a nondegeneracy condition.

With the above assumptions on $A$, we have quasi-isomorphisms between the following three complexes, all of which calculate the cyclic cohomology $HC_*(A)$:

1. The dual of the ‘cyclic Cuntz-Quillen complex’ [KS06]

$$CC^*_{\text{mod}}(A) = ((C_*(A))^\vee[u^{-1}], b^* + u^{-1}B^*)$$

The dual of the canonical map $HH_*(A) \to HC_*(A)$ is realized by the map $CC^*_{\text{mod}}(A) \to (C_*(A))^\vee$ sending $u^{-1} \mapsto 0$, and a class

$$\omega = \sum_{n \geq 0} \omega_n u^{-n} \in CC^*(A)$$

represents a compact CY structure if when $\omega_0$ induces a quasi-isomorphism $A \to A^\vee[-d]$.

2. The complex of closed cyclic nc 2-forms $(\Omega^2_{\text{cyc}}(X), \text{Lie}_Q)$ defined above. Recall that elements of this space are cyclically symmetric combinations of expressions of the form $f(x_1, \ldots)dx_i g(x_1, \ldots)dx_j$ where $f, g$ are formal power series in the free variables $x_i$. Recall that basis one-forms $dx_i$ give functions on the shifted tangent space $T_{x_0}X[-1] \cong A$; a class $\omega \in \Omega^2_{\text{cyc}}(X)$ is a compact CY structure if its evaluation at zero $\omega|_0$ induces a quasi-isomorphism $A \to A^\vee[-d]$.

3. Finally, we have the complex $(\Omega^0_{\text{cyc}}(X)/k, \text{Lie}_Q)$ of cyclic nc 0-forms modulo constants. Taking the length one part gives a map $\Omega^0_{\text{cyc}}(X)/k \to (A/[A, A])^\vee$ which we compose with the natural map $(A/[A, A])^\vee \to (\text{Sym}^2 A)^\vee$ given by $\phi \mapsto \phi(\mu_2(-, -))$. A class $\omega \in \Omega^0_{\text{cyc}}(X)/k$ gives a compact CY structure if its image under this map induces a quasi-isomorphism $A \to A^\vee[-d]$.

We now recall the theory of $A_\infty$-minimal models. An $A_\infty$-algebra $(A_0, \mu_0)$ is minimal if the differential $\mu_0^1$ is zero; if there is an $A_\infty$ quasi-isomorphism $(A_0, \mu_0) \to (A, \mu)$, we say that $A_0$ is a minimal model of $A$. 
One can prove that any $A_\infty$-algebra has a minimal model $A_0$, which as a vector space is $H^*(A, \mu^1)$. Moreover, the structure maps on $A_0$ can be algorithmically constructed by the procedure known as homological perturbation. Given a section $i : A_0 \to A$ of the projection $\pi : A \to A_0$, together with a homotopy $H : A \to A[-1]$, satisfying

$$\text{id}_A - i \circ \pi = \mu^1 \circ H - H \circ \mu^1,$$

there is a minimal $A_\infty$-structure $\mu_{A_0} = \{\mu^{k \geq 2}_i\}$ and a quasi-isomorphism $i_A = \{i_A^{k \geq 1}: A_0 \to A\}$, extending $i_A^0 = i$. These maps can be constructed from $i, \pi, H$ by an appropriate sum over tree diagrams with those maps along the edges.

In geometric terms, the quasi-isomorphism $A_0 \simeq A$ corresponds to performing a change of coordinates around the base point $x_0 \in X$ in the corresponding formal noncommutative manifold. The induced action of $i_A^*$ on $(\Omega^2_{\text{cyc}}(X), \text{Lie}_Q)$ is the transformation of forms induced by that change of coordinates. The following result says that one can always find such a change of coordinates which makes a given nondegenerate cyclic nc 2-form $\omega$ constant.

**Proposition 15.** [KS06] Let $(X, x_0)$ be a formal noncommutative pointed dg manifold with dim$(H^*(T_{x_0}X)) < \infty$. Then any class $\omega \in H^*(\Omega^2_{\text{cyc}}(X), \text{Lie}_Q)$ which is nondegenerate (i.e., a compact CY structure on $A = T_{x_0}X[-1]$) gives a constant nondegenerate class $\omega_0$ on a minimal model (i.e., a cyclic CY structure on $A_0$).

**3.3.4. A noncommutative Lagrangian neighborhood theorem.** We now come to the main result of this Subsection, an extension of the proposition above which can be seen as a noncommutative version of the Lagrangian neighborhood theorem.

**Theorem 16.** Let $(A, \mu_A)$ be a (not necessarily unital) compact $A_\infty$-algebra, and $A_0 \to A$ a minimal model of $A$. Then $A_0$ has a pre-CY structure of dimension $d$ if and only if there is a tuple $((B, \mu_B), f, \omega_A, \omega_B)$ where

1. $(B, \mu_B)$ is an $A_\infty$-algebra, $f = \{f^n\}$ is an $A_\infty$-morphism $f : A \to B,$
2. $\omega_A, \omega_B$ are elements of $\Omega^2_{\text{cyc}}(A), \Omega^2_{\text{cyc}}(B)$ such that
   $$\text{Lie}_{Q_B} \omega_B = 0,$$
3. $\omega_B$ is a compact Calabi-Yau structure on $B$; inducing a symplectic form $\omega_{B,0}$ on $H^*(B, m_B)$,
4. The image $f^1(H^*(A, \mu_A^1))$ is Lagrangian in $H^*(B, \mu_B^1)$ with respect to $\omega_{B,0}$, and
5. The quadratic form $\omega_{A,0}$ on $H^*(A, \mu_A^1)$ is a perfect pairing when restricted to $\ker(f^1) \otimes \ker(f^1)$.

Moreover, if $A$ is homologically unital then $(B, f)$ is also homologically unital and $f$ is a morphism of homologically unital $A_\infty$-algebras.

Note that the equations on the Lie derivatives of $\omega_A, \omega_B$ above are equivalent to the statement that the element

$$\omega = (\omega_A, \omega_B) \in \text{Cone} \left(f^*: (\Omega^2_{\text{cyc}}(B), \text{Lie}_{Q_B}) \to (\Omega^2_{\text{cyc}}(A), \text{Lie}_{Q_A})\right)$$

of the cone on cyclic nc 2-forms is closed.

**Proof.** Let us prove the easy direction first. We pick $B_0 = A_0 \oplus A_0^\vee[1-d]$; by Proposition 13 the pre-CY structure of dimension $d$ on $A_0$ gives a cyclic CY structure of dimension $(d-1)$ on $B_0$, or equivalently a constant symplectic form $\omega_0$ giving a perfect pairing $B_0 \otimes B_0 \to k[1-d]$, and a compatible $A_\infty$-structure $\mu_{B_0}$ such that $A_0$
is an $A_\infty$-subalgebra. We then pick a quasi-inverse $s = \sum s^n$ to the $A_\infty$-morphism $i_A : A_0 \to A$, and declare $f$ to be the $A_\infty$ composition $j \circ s$, where $j = j^1$ is the inclusion of $A_0$. By assumption, $j^*(\omega_0) = 0$, so the tuple $((B_0, \mu_{B_0}), f, \omega_A = 0, \omega_0)$ gives the desired structure.

For the other direction, we must produce a pre-CY structure on $A_0$ from a tuple satisfying conditions (1)–(5). We start by noting that these conditions are invariant under quasi-isomorphism, so we take $A$ and $B$ to be already minimal, and use an automorphism of the minimal $A_\infty$-algebra $B$ to make $\omega_B$ constant. We then have a morphism of graded vector spaces $f^1 : A \to B$. Let us denote $K = \ker(f^1)$ and $L = f^1(A)$; we then have the short exact sequence of graded vector spaces

$$K \xrightarrow{i} A \xrightarrow{f^1} L \subset B$$

We now split the proof into four steps:

**Step 1:** We find a non-minimal $A_\infty$-algebra $D$ quasi-isomorphic to $B$ such that $f^1$ lifts to an injective map $A \to D$. We construct this explicitly as follows. The quadratic form $\omega_{A,0}$ defines maps

$$A \to A^\vee[-d], \quad K \xrightarrow{\sim} K^\vee[-d],$$

the latter being an isomorphism by condition (5). We use these maps to define a projection $\pi : A \to K$ given by

$$A \to A^\vee[-d] \xrightarrow{\text{res}} K^\vee[-d] \to K$$

such that $\pi \circ i = \text{id}_K$. This gives a section $s : L \to A$ and a splitting

$$K \xrightarrow{s} A \xrightarrow{i} L \subset B$$

We also pick any complementary subspace to the Lagrangian $L$; this gives us a decomposition $B \cong L \oplus L^\vee[1-d]$.

We now define a nonminimal $A_\infty$-algebra

$$D := B \oplus K \oplus K^\vee[1-d] = L \oplus L^\vee[1-d] \oplus K \oplus K^\vee[1-d]$$

with a differential given by $m_D^1 : K^\vee[1-d] \to K[1]$ given by $\omega_{A,0}$, and zero on $B \oplus K$. The higher structure maps are given by $\mu_D^n = \mu_B^n$ on $B$ and zero on $K \oplus K^\vee[1-d]$; one checks promptly that these maps satisfy the $A_\infty$-relations. By definition, the inclusion $B \to D$ is a quasi-isomorphism of $A_\infty$-algebras.

We now define a constant 2-form $\omega_D : D \otimes D \to k[1-d]$ on $D$ by using $\omega_B$ on the subspace $B \oplus B$ plus the standard pairing between $K$ and $K^\vee$. We can pick the sign for the differential $\mu_D^1$ to satisfy the following relation for any $x, y \in K^\vee[1-d]$

$$\omega_A(\mu_D^1(x), \mu_D^1(y)) = -\omega_D(\mu_D^1(x), y) = (-1)^x \omega_D(x, \mu_D^1(y))$$

Together with cyclicity of $\omega_B$ with respect to $\mu_B$, we then get

$$\omega_D(\mu_D(a_1, \ldots, a_n), a_{n+1}) - (-1)^{a_1} \omega_D(a_1, \mu_D(a_2, \ldots a_{n+1})) = 0$$
for any \( n \), establishing cyclicity of \( \omega_D \). Using the graphical calculus for signs from Section 2.2 we can concisely express this relation as

\[
\text{Lie}_D \omega = \mu - \mu = 0
\]

We now define an \( A_\infty \)-morphism \( g : A \to D \) such that \( g^1 \) is injective, explicitly as follows. The first map \( g^1 \) embeds \( A \) as \( L \oplus K \), in other words \( g \) is equal to \( f \) plus a correction in \( K \):

\[
g^1 : A \to B \oplus K \oplus K^\vee[1 - d]
\]

\[
a \mapsto \langle f^1(a), \pi(a), 0 \rangle
\]

and the higher maps are equal to \( f \) but with a correction in \( K^\vee[1 - d] \):

\[
g^n : A \otimes^n \to B \oplus K \oplus K^\vee[1 - d]
\]

\[
\bar{a} \mapsto \langle f^n(\bar{a}), 0, (\mu_D)^{-1}(\mu_A^n(\bar{a})) \rangle
\]

where \( (\mu_D)^{-1} : K[1] \to K^\vee[1 - d] \) is the inverse of the differential. Using the \( A_\infty \)-relations for \( A \), one can check that this indeed defines a morphism of \( A_\infty \)-algebras \( g : A \to D \).

**Step 2:** Now that we have an \( A_\infty \)-embedding, we further calculate an \( A_\infty \)-automorphism \( t : (D, \mu_D) \to (D, \nu) \) such that the composition \( h = t \circ g : (A, \mu_A) \to (D, \nu) \) satisfies the property that \( \text{Im}(h^n) \subseteq \text{Im}(h^1) \) for all \( n \). Note that the \( A_\infty \)-structure \( \nu \) is different by quasi-isomorphic to \( \mu_D \); also we will require \( t^1 = \text{id}_D \).

We must now calculate the higher maps \( t^n \) and the structure maps \( \nu^n \). The maps \( t^n, n \geq 2 \) are only nonzero on elements of \( L \oplus K \), and map to elements of \( L^\vee[1 - d] \oplus K^\vee[1 - d] \). We define them inductively; let us suppose that we have defined them up to \( t^{n-1} \).

Now for \( n \), first we define a map \( \tau^n : A \otimes^n \to L \oplus L^\vee[1 - d] \oplus K^\vee[1 - d] \) given by

\[
\tau^n = - \sum_{k=2}^{n-1} \sum_{n_1, \ldots, n_k \text{ such that } \sum_i n_i = n} g^{n_1} \cdots g^{n_k}
\]

We then define \( t^n \) to be zero on any sequence with factors in \( L^\vee[1 - d] \) and \( K^\vee[1 - d] \), and on powers of the subspace \( L \oplus K \) to be given by

\[
\tau = \pi_{L^\vee[1 - d] \oplus K^\vee[1 - d]} \tau^n
\]

that is, using the identification \( A = L \oplus K \) on the source, applying \( \tau \) and projecting to \( L^\vee[1 - d] \oplus K^\vee[1 - d] \). This definition implies that the all the maps of the composition \( h = t \circ g \) have image in \( L \).
We use the sequence of maps $t^n$ above to calculate the new $A_\infty$-structure $\nu$, also inductively, by requiring that $t$ be an $A_\infty$-morphism. Explicitly, the following formula (with Koszul signs coming from the graphical calculus) gives an $A_\infty$-structure $\nu$:

\[
\nu^n = \sum_{k=2}^{n-1} \sum_{n_1, \ldots, n_k} \sum_{i_1, \ldots, i_k} \sum_{n_{i_1}=n} \sum_{n_{i_2}=n} \cdots \sum_{n_{i_k}=n} t^{n_1} \cdots t^{n_k} + \sum_{k=1}^{n-1} \sum_{0 \leq i \leq n-k} t^{n_1} \cdots t^{n_i} \sum_{n_{i+1}=n} \sum_{n_{i+2}=n} \cdots \sum_{n_{i+k}=n} t^{n_1} \cdots t^{n_i} \sum_{n_{i+k+1}=n} \cdots \sum_{n_{n+1}=n},
\]

where the terms in the second sum have signs coming from the graphical calculus, namely $(-1)^{a_1 + \cdots + a_i}$ for the term shown.

**Step 3:** We now calculate that with the definitions above, the constant 2-form $\omega_D$ is indeed cyclic for $\nu$; this is a consequence of the relation $f^* \omega_B = \text{Lie}_A \omega_A$.

We first make an auxiliary calculation using the transformation $t$: we calculate the following identity for all elements $a_1, \ldots, a_{n+1}$ in the image of $h^1 : A \to D$:

\[
t^* \omega_D(a_1, \ldots, a_{n+1}) = d_1 \cdots d_n d_{n+1} + d_1 d_2 \cdots d_{n+1} = - \sum_{k=2}^{n-1} \sum_{i=0}^{n-k} (-1)^\# \sum_{n_{i+1}=n} \sum_{n_{i+2}=n} \cdots \sum_{n_{i+k}=n} \sum_{n_{i+k+1}=n} \cdots \sum_{n_{n+1}=n},
\]

where $\# = \deg(a_1) + a_2 + \cdots + a_i + 1$, which equals $\sum_{j=1}^i a_j$ when $i \geq 1$ and 1 when $i = 0$. Note also that

\[
t^* \omega_D(\ldots, x, \ldots) = 0
\]
on a sequence of length $\geq 3$ containing an element $x \in L^\vee [1-d] \oplus K^\vee [1-d]$, since all the $t^{n \geq 2}$ vanishes on those elements.

By the bounds on the last sum, there are no terms when $n = 1, 2$; also for every $n$ the sum only depends only on the nonconstant part of $\omega_A$ (i.e. $\omega_A$ vertices with $\geq 3$ incoming arrows). The calculation above follows from the definition of $t^n$ and the relation between $\omega_A$ and $\omega_B$.

We then calculate

\[
d_1 \cdots d_n d_{n+1} - d_1 d_2 \cdots d_{n+1}
\]
by using the inductive definition of $\nu^n$ in terms of $t^{k<n}, \nu^{k<n}$ and plugging in the equation for $t^*\omega_D$: this reduces the expression above to similar expressions for $k < n$, down to the base cases $k = 1, 2$ which can be calculated explicitly to be zero. Therefore $\omega_D$ is cyclic for $\nu$.

**Step 4:** Now we have a cyclic $A_\infty$-algebra $(D, \nu)$ with an $A_\infty$-map $h : A \to D$ all of whose components land on the Lagrangian subspace $L \oplus K$. It remains to prove that the structure maps $\nu$ preserve this subspace; we can express this as

$$\omega_D(\nu^n(h^1(-), \ldots, h^1(-), h^1(-))) = 0$$

This can also be proved by induction: the $A_\infty$-relation between $h$ and $\nu$ implies that the expression above can be reduced to similar expressions for $\nu^{k<n}$. The base case with $n = 1$ then follows by assumption, since $\nu^1 = \mu^1$ lands in $K$, a subset of the Lagrangian subspace. By Proposition 13, this is equivalent to a pre-CY structure of dimension $d$ on the minimal $A_\infty$-algebra $A$.

As for the last part of the statement, regarding unitarity of the minimal model, it is a general fact that units in cyclic $A_\infty$-categories can always be strictified by a cyclic $A_\infty$-quasi-isomorphism; see [Dav21, Prop.4.8].

**Remark.** Recall that when $A$ is homologically unital we have a quasi-isomorphism

$$(\Omega^2_{\text{cyc}}(A), \text{Lie}_{Q_A}) \cong (\Omega^0_{\text{cyc}}(A)/k, \text{Lie}_{Q_A})$$

so Theorem 16 can be rephrased in terms of cyclic 0-forms with no constant term, with an entirely analogous statement.

3.3.5. Pre-CY structures as noncommutative integrable polyvector fields. We continue in the setting of a compact $A_\infty$-algebra $(A, \mu)$. Recall that an $A_\infty$-structure is a homological vector field $Q$ on the pointed formal dg manifold $X$ corresponding to $A$. The extension of this $A_\infty$-algebra to a pre-CY algebra $(A, m)$ with $m_{(1)} = \mu$ should be seen as an extension of $Q$ to a polyvector field on $X$ satisfying an integrability condition expressed by the vanishing of a noncommutative Schouten-Nijenhuis bracket.

For that, we go to a minimal model $A_0$ of $A$; that is, to a particular coordinate system around $x_0$.

**Definition 26.** The shifted degree $(2 - d)$-cotangent bundle $\Pi T^*[2 - d]X$ is the pointed formal dg manifold corresponding to the graded vector space $A_0 \oplus A_0^*[1 - d]$.

From Theorem 16, we get a cyclic CY structure on $A_0 \oplus A_0[1 - d]$, with $A_0$ as an $A_\infty$-subalgebra. Thus we get a homological vector field $Q'$ on $\Pi T^*[2 - d]X$ which preserves the zero section $X$, restricting to $Q$ on it.

The vector field $Q'$ is Hamiltonian with respect to the constant two-form $\omega_0$ given by the standard pairing, that is, $\text{Lie}_{Q'} \omega = 0$. As in [KS06], there is a cyclic function $H \in \Omega^0_{\text{cyc}}(\Pi T^*[2 - d]X)$ satisfying $i_{Q'} \omega = dH$.

In analogy with the commutative world, note that the space of polyvector fields is identified with the space of functions on the shifted cotangent bundle. Thus, we set the space of noncommutative $(2 - d)$-shifted polyvector fields to be given by this latter space of cyclic functions $\Omega^0_{\text{cyc}}(\Pi T^*[2 - d]X)$. One sees that the Poisson bracket $\{,\}$ on this space coming from $\omega_0$ is the analogue of the Schouten-Nijenhuis bracket acting on polyvector fields; which gives the following characterization.
Lemma 17. The data of a pre-CY structure of dimension $d$ on a compact $A_\infty$-algebra $A$ is a polyvector field $H$ on the degree $d$ cotangent bundle $T^*|2-d|X$, satisfying the Maurer-Cartan equation $\{H, H\} = 0$.

4. Deformation theory of pre-Calabi-Yau structures

Each infinitesimal deformation problem of algebraic structures such as $A_\infty$ and pre-CY structures is governed by some type of Maurer-Cartan equation in an appropriate dg Lie algebra.

For $A_\infty$-structures on $A$, that Lie algebra is the shifted Hochschild cochains $C^*\langle 1 \rangle(A)$ with differential given by the Gerstenhaber bracket $[\mu, -]_G$ with the $A_\infty$-structure maps. More abstractly, there is a formal derived stack $\text{Def}_{A_\infty}(A)$ over $k$ parametrizing $A_\infty$-structures on $A$ whose derived tangent complex at a given point $\mu$ is calculated by the Hochschild cohomology $HH^*(A, \mu)$.

There is a similar description for the deformation theory of pre-CY structures, which we now present, together with some methods to compute the relevant deformation spaces in the case of smooth or compact $A_\infty$-categories.

4.1. Higher Hochschild invariants. Let $(A, \mu)$ be an $A_\infty$-category. The $A_\infty$ structure $\mu$ is an element of $C^*_\langle 1 \rangle(A)$, and therefore $[\mu, -]_{\text{nec}}$ defines a map $C^*_{\langle \ell \rangle}(A) \to C^*_{\langle \ell \rangle}(A)[1]$ for every $\ell$, which squares to zero as a consequence of Proposition 10.

Definition 27. The $\ell$th higher Hochschild cohomology of the $A_\infty$-category $(A, \mu)$ is the graded vector space

$$HH^*_\langle \ell \rangle(A) := H^*(C^*_{\langle \ell \rangle}(A), [\mu, -]_{\text{nec}}).$$

This definition agrees with the usual Hochschild cohomology of $A_\infty$-categories when $\ell = 1$.

For fixed $\ell \geq 1, d \in \mathbb{Z}$, recall the space of higher cyclic cochains from Definition 19. Taking the necklace bracket with the $A_\infty$-structure $\mu$ preserves $d$ and cyclic invariance, so we define:

Definition 28. The $(\ell, d)$-higher cyclic cohomology of the $A_\infty$-category $(A, \mu)$ is the graded vector space

$$HC^*_{\langle \ell, d \rangle}(A) := H^*(C^*_{\langle \ell, d \rangle}(A), [\mu, -]_{\text{nec}}).$$

Recall that we introduced a shift depending on $d$ between the gradings for higher cyclic cochains and higher Hochschild cochains: an element of degree $n$ in $C^*_{\langle \ell, d \rangle}(A)$ is an element of degree $n + (d-2)(\ell - 1)$ in $C^*_{\langle \ell \rangle}(A)$, cyclically invariant or anti-invariant depending on the parity of $(d-1)(\ell - 1)$.

Proposition 18. If for some $n \geq 0, k \geq 1$ we have $HH^n_{\langle k \rangle}(A) = 0$, then $HC^{(n-(d-2)(\ell-1))}_{\langle \ell, d \rangle}(A) = 0$.

Proof. Suppose that we have a cochain $\phi \in C^{n-(d-2)(\ell-1)}_{\langle \ell, d \rangle}(A)$; this is a cyclically invariant/anti-invariant element of $C^*_\langle \ell \rangle(A)$, which by assumption is exact, i.e. $\phi = [\mu, \psi]_{\text{nec}}$ for some $\psi$ which might not be cyclically invariant. We then take the symmetrization/antisymmetrization $\frac{1}{\ell!} \sum_{\sigma} \pm \sigma(\psi)$ over cyclic permutations $\sigma$, which now lives in higher cyclic cochains and is also a primitive of $\phi$, since the differential also has cyclic symmetry. □
Let \( m \) be a pre-CY structure of dimension \( d \) extending the \( A_\infty \)-structure \( \mu \) on \( \mathcal{A} \). The map \([m, -]_{\text{nc}}\) mixes the spaces above for different \( \ell \); it now defines a differential on the space

\[
C^*_d(\mathcal{A}) := \bigoplus_{\ell \geq 1} C^*_\ell(\mathcal{A}, [m, -]_{\text{nc}})
\]

**Definition 29.** The tangent cohomology of the pre-CY category \((\mathcal{A}, m)\) is the graded vector space

\[
H^*_d(\mathcal{A}) := H^*(C^*_d(\mathcal{A}), [m, -]_{\text{nc}}).
\]

By the general theory of deformations, if \( \mathcal{M}_{\text{preCY}} \) denotes the (derived) moduli stack of pre-CY structures on \( \mathcal{A} \), the tangent complex of \((\mathcal{A}, m)\) models the tangent space \( T_m \mathcal{M}_{\text{preCY}} \).

**4.1.1. The higher cyclic to tangent cohomology spectral sequence.** Remembering only the \( \mu = m_{(1)} \) component of a pre-CY structure gives a map

\[
\mathcal{M}_{\text{preCY}} \to \mathcal{M}_{A_\infty}
\]

which on tangent spaces at any given point \( m \) is \( C^*_d(\mathcal{A}, m) \to C^*(\mathcal{A}, \mu) \).

Consider the decreasing filtration

\[
F^k_d(\mathcal{A}) := \prod_{n \geq k} C^*_n(\mathcal{A}, [m, -]_{\text{nc}})
\]

We note that the differential \( d = [m, -]_{\text{nc}} \) preserves this filtration: more precisely, the bracket with the component \( m_{(\ell)} \) increases the number of outgoing arrows by \( \ell - 1 \), giving a map \( F^k_d(\mathcal{A}) \to F^{k+\ell-1}_d(\mathcal{A}) \) of cohomological degree one. So we have a spectral sequence associated to this filtered cochain complex, which abuts to the tangent cohomology.

The associated graded of the filtration on cochain is

\[
\text{Gr}^k C^*_n(\mathcal{A}) := F^k_d(\mathcal{A}) / F^{k+1}_d(\mathcal{A}) = C^*_n(\mathcal{A}, \mu)
\]

and the differential induced on \( \text{Gr}^k C^*_n(\mathcal{A}) \) agrees with the bracket \([\mu, -]_{\text{nc}}\) with the \( A_\infty \)-structure \( \mu = m_{(1)} \). The standard theory of filtered complexes then gives:

**Proposition 19.** Given any pre-CY structure on \( \mathcal{A} \), there is a spectral sequence \( E^{p,q}_r \), starting from the higher cyclic cohomology

\[
E^{p,q}_1 = HC^{p+q}_{(0,d)}(\mathcal{A})
\]

and converging to the tangent cohomology

\[
E^{p,q}_\infty = \text{Gr}^p H^{p+q}_d(\mathcal{A}).
\]

**4.1.2. Extending an \( A \)-infinity structure to a pre-CY structure.** We now give an interpretation of the relation between the higher cyclic and tangent cohomologies, in terms of extending a solution to the \( A_\infty \) Maurer-Cartan equation to a solution of the pre-CY Maurer-Cartan equation.

**Proposition 20.** Given an \( A_\infty \) category \((\mathcal{A}, \mu)\), for each \( k \geq 3 \), the higher cyclic cohomology in degree two \( HC^2_{(k,d)}(\mathcal{A}) \) is the group of obstructions to extending a solution of the Maurer-Cartan equation from

\[
C^*_d(\mathcal{A}) / F^k_d(\mathcal{A}) \text{ to } C^*_d(\mathcal{A}) / F^{k+1}_d(\mathcal{A}).
\]
Proof. Let us simply write $C^*$ etc. and leave $\mathcal{A}$ implicit, for conciseness. We first describe the induction step when $k = 3$. Suppose that we have a solution of the Maurer–Cartan equation modulo $F^3_{[d]}$; that is, elements $m(1), m(2)$ such that

$$(m(1) + m(2)) \circ \text{nec} \ (m(1) + m(2)) \equiv 0 \pmod{F^3_{[d]}}$$

which is equivalent to requiring $[m(1), m(2)]_{\text{nec}} = 0$. We then have

$$(m(1) + m(2)) \circ \text{nec} \ (m(1) + m(2)) = m(2) \circ \text{nec} \ m(2)$$

We see that this is $[m(1), m(2)]_{\text{nec}} = 0$. We then have

$$[m(1), m(3)]_{\text{nec}} = -m(2) \circ \text{nec} \ m(2)$$

we then have

$$(m(1) + m(2) + m(3)) \circ \text{nec} \ (m(1) + m(2) + m(3)) \equiv 0 \pmod{F^4_{[d]}(\mathcal{A})},$$

that is, an extension of our solution to $C^*_{[d]}(\mathcal{A})/F^3_{[d]}$.

In general, if we know that

$$(m(1) + \cdots + m_{k-1}) \circ \text{nec} \ (m(1) + \cdots + m_{k-1}) \equiv 0 \pmod{F^k_{[d]}}$$

then we have

$$[m(1), m(1) + \cdots + m_{k-1}] \circ \text{nec} \ (m(1) + \cdots + m_{k-1})_{\text{nec}} \equiv 0 \pmod{F^k_{[d]}}$$

We write all the terms that appear in $F^k_{[d]}/F^{k+1}_{[d]}$, giving

$$[m(1), (m(1) + \cdots + m_{k-1})]_{\text{nec}} \equiv \sum_{i=1}^{k-1} [m(1), [m(i), m_{k-i}]_{\text{nec}}] \pmod{F^{k+1}_{[d]}}$$

which after applying graded Leibniz and cancellations gives

$$[m(1), (m(1) + \cdots + m_{k-1})]_{\text{nec}} \equiv 0 \pmod{F^{k+1}_{[d]}}$$

Thus if $HC^2_{k,d} = 0$ we can find a primitive $m(k)$ of $(m(1) + \cdots + m_{k-1})$ modulo $F^{k+1}_{[d]}$ implying

$$(m(1) + \cdots + m_{k}) \circ \text{nec} \ (m(1) + \cdots + m_{k}) \equiv 0 \pmod{F^{k+1}_{[d]}}.$$

We combine the proposition above with Proposition 18 to give a sufficient condition in terms of higher Hochschild cohomology.

**Corollary 21.** If $HH^{3\ell-d-2\ell+4}_{(\ell)}(\mathcal{A}) = 0$ for every $\ell \geq 3$, then any cocycle $m(2) \in C^2_{(2,d)}(\mathcal{A})$ can be extended to a pre-CY structure on $\mathcal{A}$. 

$\square$
4.2. Calculating higher Hochschild cohomology. It becomes important therefore to compute \( HH_{i,j}(\mathcal{A}) \). Recall from Section 2.3.1 that, in the case where the category \( \mathcal{A} \) is compact and/or smooth, one can express the (ordinary) Hochschild invariants in terms of certain dual bimodules, which are related to Serre functors. Here we extend that description to higher Hochschild cohomology groups.

4.2.1. Compact \( \mathcal{A} \)-infinity categories. Let \( \mathcal{A} \) be a compact \( \mathcal{A}_\infty \)-category. Recall the two canonical objects in \( \mathcal{A} \text{-Mod}\rightarrow \mathcal{A} \) given by the diagonal bimodule \( \mathcal{A} \Delta \) and its linear dual \( \mathcal{A} \check{\vee} \). The linear dual has the property that for any perfect \( \mathcal{A} \)-bimodule \( M \), there is a quasi-isomorphism of complexes

\[
(M \otimes_{\mathcal{A}} \mathcal{A} \Delta)^\vee \cong \text{Hom}_{\mathcal{A}}(M, \mathcal{A} \check{\vee})
\]
picking \( M = \mathcal{A} \check{\vee} \), the preimage of the identity gives an element \( \text{ev}_{\mathcal{A}} \in (\mathcal{A} \check{\vee} \otimes_{\mathcal{A}} \mathcal{A} \Delta)^\vee \). In our graphical notation, we picture \( \text{ev}_{\mathcal{A}} \) as a vertex

\[
\begin{array}{c}
\mathcal{A} \check{\vee} \\
\text{ev}_{\mathcal{A}}
\end{array} \rightarrow \mathcal{A} \Delta
\]

taking two bimodule arrows and any number of \( \mathcal{A}[1] \) arrows along the top and bottom (not pictured).

Let \( \phi \in C_k^c(\mathcal{A}) \) be a kth higher Hochschild cochain. We define an element

\[
\tilde{\phi} \in \text{Hom}_{\mathcal{A}}(\mathcal{A} \Delta \otimes_{\mathcal{A}} (\mathcal{A} \check{\vee} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{A} \check{\vee}), \mathcal{A} \Delta)
\]

by evaluating the following diagram:

that is, we use the evaluation element to convert the last \( k - 1 \) outgoing elements of \( \phi \) into incoming elements of \( \mathcal{A} \check{\vee} \).

If \( \phi \) is closed under the differential on \( C_k^c(\mathcal{A}) \), that is, \( [\mu, \phi]_{\text{nc}} = 0 \), then by closedness of \( \text{ev}_{\mathcal{A}} \) and the structure equation for \( \mu_{\mathcal{A} \Delta} \) we see that the map above satisfies the structure equations to be a morphism of bimodules. We now precompose that morphism with some quasi-isomorphism

\[
(\mathcal{A} \check{\vee})^{\otimes_{\mathcal{A}} (k-1)} \cong \mathcal{A} \Delta \otimes_{\mathcal{A}} (\mathcal{A} \check{\vee})^{\otimes_{\mathcal{A}} (k-1)}
\]

to get a map which we also denote by \( \tilde{\phi} \in \text{Hom}_{\mathcal{A}}((\mathcal{A} \check{\vee})^{\otimes_{\mathcal{A}} (k-1)}, \mathcal{A} \Delta) \).
Proposition 22. When $A$ is homologically unital and compact, the map $\phi \mapsto \tilde{\phi}$ gives a quasi-isomorphism

$$C^*_k(A) \xrightarrow{\sim} \text{Hom}_{A-A}([A^\vee]_{\otimes A(k-1)}, A_\Delta)$$

for any $k \geq 1$.

Proof. Note that for $k = 1$ we get an isomorphism $CC^*(A) \cong \text{Hom}_{A-A}(A_\Delta, A_\Delta)$ which in this formalism of $A_\infty$-bimodules is proven in [Gan13, Sec.2.6]. Let us consider the case where $A$ is an $A_\infty$-algebra $A$ so we can omit the sums over tuples of objects; we will follow the same strategy used $\text{op.cit.}$ Namely, we will use length filtrations to reduce the calculation to bar complexes for associative algebras.

Consider the cone of the morphism $\Psi : CC^*_k(A) \to \text{Hom}_{A-A}(A_\Delta \otimes A(A^\vee)_{\otimes A(k-1)}, A_\Delta)$:

$$\text{Cone}(\Psi) = C^*_k(A) \oplus \text{Hom}_{A-A}(A_\Delta \otimes_A (A^\vee)_{\otimes A(k-1)}, A_\Delta)[1]$$

$$= \bigoplus_{\{n_i \geq 0\}, \sum i \leq k} \text{Hom}_{A}(A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_k}, A^{\otimes k})$$

$$\oplus \text{Hom}_{A}(A[1]^{\otimes r_0} \otimes A \otimes A[1]^{\otimes r_1} \otimes A^\vee \otimes \cdots \otimes A[1]^{\otimes r_k}, A)[1]$$

with differential given by the triangular matrix

$$d_{\text{Cone}} = \begin{pmatrix} [\mu, -] & 0 \\ \Psi & [\mu, -]_{\text{sec}} \end{pmatrix}$$

where we schematically denote $\mu$ for the appropriate combination of structure maps for $A$, $A_\Delta$ and $A^\vee$.

Consider the following decreasing filtrations:

$$F_p(C^*_k(A)) = \bigoplus_{\sum n_i \geq p} \text{Hom}_{A}(A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_k}, A^{\otimes k})$$

$$F_p(\text{Hom}_{A-A}(A_\Delta \otimes_A (A^\vee)_{\otimes A(k-1)}, A_\Delta))$$

$$= \bigoplus_{\sum r_i \geq p-1} \text{Hom}_{A}(A[1]^{\otimes r_0} \otimes A \otimes A[1]^{\otimes r_1} \otimes A^\vee \otimes \cdots \otimes A[1]^{\otimes r_k}, A)$$

The differentials on each complex separately preserve this filtration; the only maps preserving $p$-degree are the components containing only $\mu_1^A$ and $\mu_1^{A^\vee}$, and all higher $\mu^{n \geq 2}$ increase $p$-degree. Moreover $\Psi$ sends

$$F_p(C^*_k(A)) \to F_{p+1}(\text{Hom}_{A-A}(A_\Delta \otimes_A (A^\vee)_{\otimes A(k-1)}, A_\Delta))$$

because the total number of inputs does not decrease; the $p$-degree one part of $\Psi$ is just given by the component $\mu^A_{\Delta} \otimes$.

Therefore we get a filtration on $\text{Cone}(\Psi)$ which is compatible with the differential, so we have an associated spectral sequence computing its cohomology. The components of $d_{\text{Cone}}$ preserving the length are only the ones containing the differentials $\mu^3$ so the first page of this spectral sequence is given by the same complex above, but on the unital associative algebra $H = H^*(A^1)$ instead.

By compactness, $H$ is finite dimensional and $\text{ev}_H$ only has a single component given by the perfect pairing between $H$ and $H^\vee$. The only nonzero terms in the first page differential are the terms containing $\mu_H^H, \mu_H^{H^\vee}$ and $\mu_H^A \Delta$.

We can then put another filtration on this complex, now by counting the number of $H[1]$ inputs minus $r_0$, that is, by the number $\sum_i (n_i + r_i)$. The differential
also preserves this filtration so we again take the associated spectral sequence; using the pairing to shift \( k - 1 \) of the outgoing factors of \( CC^*_k(H) \) to we then see that the total complex of the first page is the sum of total bar complexes for \( H^\vee \otimes_H \cdots \otimes_H H^\vee \otimes H^{r_{k-1}} \) as a left \( H \)-module (total meaning including the last term \( \to H^\vee \otimes_H \cdots \otimes_H H^\vee \otimes H^{r_0} \) coming from the \( r_0 = 0 \) component), which is acyclic for unital algebras, so the second spectral sequence, and therefore the first also, converge to zero. \( \square \)

4.2.2. Smooth \( A \)-infinity categories. We now prove an analogous result to Proposition 22, but for (homologically) smooth, instead of compact, \( A_\infty \)-categories. Recall that \( A \) is smooth when its diagonal bimodule is perfect, and has as bimodule dual the “inverse dualizing bimodule” \( A! \) which represents Hochschild homology, i.e. for any perfect bimodule \( \mathcal{M} \) there is a quasi-isomorphism

\[
\text{Hom}_{A\otimes A}(A, \mathcal{M}) \cong A^! \otimes_{A\otimes A} \mathcal{M}
\]

which is given by composing with a canonical coevaluation element \( \text{ev}_A^! \in A_{\Delta} \otimes_{A\otimes A} A! \). We picture \( \text{ev}_A^! \) as a vertex

\[
\xymatrix{ A^* \ar[r]^{\text{ev}_A} & \Delta }$

with two outgoing bimodule arrows and any number of outgoing \( A[1] \) arrows along the top and bottom.

Let \( \phi \in C^*_k \) be a \( k \)th higher Hochschild cochain. We define an element

\[
\tilde{\phi} \in \text{Hom}_{A\otimes A}(A_{\Delta}, A\Delta \otimes_A A^! \otimes_A \cdots \otimes_A A^!_{k-1})
\]

by the following diagram:

\[
\xymatrix{ & A_{\Delta} \ar[dl]_{\mu_A} \ar[dr]^{\phi} & \\
& A^!_{\Delta} \ar[dl]_{\text{ev}_A^!} \ar[dr]^{\text{ev}_A^!} & \\
A_{\Delta} & A^! & A^! & \cdots & A^!}
\]

**Proposition 23.** When \( A \) is homologically unital and smooth, the map \( \phi \mapsto \tilde{\phi} \) gives a quasi-isomorphism

\[
C^*_k(A) \cong \text{Hom}_{A\otimes A}(A_{\Delta}, (A^!)^{\otimes k}(k-1))
\]

for any \( k \geq 1 \).

The proof of this proposition will be similar to the proof of Proposition 22, but we must first make an auxiliary definition. Let \( A \) be an \( A_\infty \)-category and \( \mathcal{M}_1, \ldots, \mathcal{M}_k \) any tuple of \( A \)-bimodules.
**Definition 30.** The $A$-bimodule $W(M_1, \ldots, M_n)$ as a graded vector space is given by

$$\prod_{\{X_i^j\}} \text{Hom}_k \left( \bigotimes_{i=1}^{k-1} A(X_i^j, X_i^{j+1}), \bigotimes_{i=1}^{k-1} M(X_i^j, X_i^{j+1-1}) \right).$$

This graded vector space gets a differential from the structure maps of $A$ and of the bimodules $M$, and an $A$-bimodule structure from the maps $\mu_{M_1}^{r|1}$ and $\mu_{M_k}^{r|1}$ of the first and last bimodule.

We can picture an element of $W(M_1, \ldots, M_n)$ as a vertex with outgoing $M$ arrows, and $n-1$ groups of incoming $A[1]$ arrows in between them. In particular, $W(M) = M$ for a single bimodule.

The following fact is a consequence of the universal property of the inverse dualizing bimodule $A!$ and of the fact that if $A$ is smooth, $A!$ is also perfect and there is a quasi-isomorphism $A!! \sim A$.

**Lemma 24.** Let $A$ be smooth, $M_1, M_2$ any two $A$-bimodules. Then the map

$$\begin{array}{ccc}
M_2 & \to & M_1 \\
\downarrow & & \downarrow \\
M_2 & \to & A! \otimes_A M_2[-1].
\end{array}$$

**Proof.** (of Proposition 23) Applying the lemma above $(k-1)$ times we can prove instead the quasi-isomorphism

$$C_{(k)}^*(A) \sim W(A \Delta, A \Delta[1], \ldots, A \Delta[1]).$$

We argue this in an entirely analogous way as in the proof of Proposition 22, by using the filtration on the cone

$$\text{Cone}(\Psi) = C_{(k)}^*(A) \oplus W(A \Delta, A \Delta[1], \ldots, A \Delta[1])^1[1].$$

given by the filtration induced by the number of $A[1]$ arrows on $C_{(k)}^*(A)$ and the number of $A[1]$ arrows plus one on $W(A \Delta, A \Delta[1], \ldots, A \Delta[1])$.

We again get a spectral sequence whose first page is given by the same cone but for the unital associative algebra $H = H^*A$. We now use the second filtration, by the total length ‘on the right’, getting a second spectral sequence whose first page has as total complex the total bar complex for $W(\Gamma \Delta, \ldots, \Gamma \Delta)$ as a left $H$-module, which is acyclic for unital $H$. □
4.3. **Relation to smooth CY structures.** Let \((A, \mu)\) be a homologically smooth and unital \(A_\infty\)-category. Let \(m\) be a pre-CY structure on \(A\) compatible with \(\mu\), that is, \(m(1) = \mu\). The next component is

\[
m(2) \in C^2_{(2,d)}(A) \subset C^{2+[(d-2)(2-1)]}_{(2)}(A) = C^d_{(2)}(A)
\]

Using the quasi-isomorphism \(C^*_d(A) \simeq \text{Hom}_{A-A}(A_\Delta, A^!d)\) we get a morphism \(\Phi \in \text{Hom}_{A-A}(A_\Delta, A^!d)\).

Recall that for \(A\) smooth there is another quasi-isomorphism \(\text{Hom}_{A-A}(A^!, A_\Delta) \simeq C_* (A)\) between the Hochschild chain complex and the inverse morphism space of bimodules. By definition, if \(\Phi\) is a quasi-isomorphism of \(A\)-bimodules, any quasi-inverse \(\Phi^{-1}\) defines a weak smooth CY structure of dimension \(d\) on \(A\). As mentioned in Section 2.3.2, an algebra \(A\) with such a structure is also known as a ‘Ginzburg CY algebra’. In [KTV] we prove the following result.

**Theorem 25.** Let \(\tilde{\omega} \in CC^*_d(A)\) be a (strong) smooth CY structure on \(A\), whose image \(\omega \in C_* (A)\) induces a quasi-isomorphism \(A^!d \simeq A_\Delta\). Then there is a pre-CY structure \(m\) on \(A\) whose component \(m(2)\) induces an inverse quasi-isomorphism; conversely, given any such pre-CY structure one can produce a (strong) smooth CY structure on \(A\).

Note that this result requires the existence of the lift \(\tilde{\omega}\) in negative cyclic homology in order to produce the pre-CY structure. On the other hand, it guarantees the existence of this lift in the following case

**Corollary 26.** If \(\omega \in CC^*_d(A)\) is a weak smooth CY structure of dimension \(d\) which has an inverse \(m(2) \in CC^*_d(A)\) such that \(\mu + m(2)\) extends to a pre-CY structure, then \(\omega\) has a lift \(\tilde{\omega} \in CC^*_d(A)\) giving a strong smooth CY structure of dimension \(d\).

For instance, if \([m(2), m(2)]_{\text{nc}} = 0\) then \(\mu + m(2)\) is already a pre-CY structure (with \(m(n \geq 3) = 0\)) so this result applies.

5. **Examples**

We now present some examples where one naturally finds pre-Calabi-Yau structures: topology of finite-dimensional manifolds with boundary and the algebraic geometry of varieties with anticanonical section.

5.1. **Finite-dimensional manifolds with boundary.** Recall from Section 3.3.3 that, given an homologically unital and compact \(A_\infty\)-algebra \(A\), there are three equivalent ways of describing compact CY structures on \(A\); here we will use the third description, namely, as classes

\[
[\omega] \in H^* (\Omega^0_{\text{cyc}}(X)/k, \text{Lie}_Q)
\]

in the complex of nc 0-forms with no constant term on the corresponding formal pointed dg manifold \(X\) with homological vector field \(Q\). We have the following application, which already appears in [Kon93]

**Proposition 27.** Let \(M\) be a compact, closed, oriented manifold of dimension \(d\). The fundamental class of \(Y\) gives a compact CY structure of dimension \(d\) on the dg algebra of de Rham forms \(B = \Omega^*(M)\) (with coefficients in \(k\)).
Proof. We regard \( B = \Omega^*(M) \) as an \( A_\infty \)-algebra with \( \mu^1 = d_{dR}, \mu^2 = \wedge, \mu^{\geq 3} = 0 \).
Integration against the fundamental class \([M]\) gives a map
\[
\omega = \int_{[Y]} : B \to k[-d]
\]
simply by assigning zero to forms of degree \(< d\). We extend this to a nc 0-form \( \omega \in \Omega^0_{cyc}(B)/k \). Stokes’ theorem implies that this form is closed under \( \text{Lie}_Q \), and Poincaré duality of \( M \) implies that the pairing
\[
\omega \circ \wedge : B \otimes B \to k[d]
\]
is nondegenerate.

In terms of minimal models, [KS06, Thm.10.2.2] then implies that

**Corollary 28.** There is a cyclic CY structure on the graded vector space \( H^*_dR(M) = H^*(B, \mu^1) \).

In other words, there is a minimal \( A_\infty \) structure on \( H^*_dR(M) \), quasi-isomorphic to the dg-algebra \( \Omega(Y) \) and cyclic with respect to the Poincaré pairing.

Paul Seidel, in private communication, made the conjecture that if a compact oriented manifold \( M \) has non-empty boundary then its cohomology \( H^*(M) \) should have a pre-CY structure. Indeed, this follows from the results of Section 3.3.4:

**Theorem 29.** Let \( M \) be a compact oriented manifold of dimension \( d \) with compact boundary \( \partial M \) then the cohomology \( H^*(M) \) of \( M \) has the structure of a pre-CY algebra of dimension \( d \).

**Proof.** Let us apply Theorem 16 for the dg-algebras \( A = \Omega(M) \) and \( B = \Omega(\partial M) \), with \( f : A \to B \) given by restriction of forms, i.e., pullback under the inclusion \( i : \partial M \to M \).
Integration against the fundamental classes \([M]\) in degree \( d \) and \([\partial M]\) in degree \( d - 1 \) give maps
\[
\int_{[M]} : A \to k[-d], \quad \int_{[\partial M]} : B \to k[1-d]
\]
which we extend to nc 0-forms \( \omega_A, \omega_B \).

As before, \( \omega_B \) is closed and gives a nondegenerate pairing. We now check the other conditions of Theorem 16. For any forms \( \alpha_i \) on \( M \) we calculate
\[
f^*\omega_B(\alpha_1, \alpha_2) = \omega_B(i^*\alpha_1, i^*\alpha_2) = \int_{[\partial M]} i^*\alpha_1 \wedge i^*\alpha_2
\]
and by Stokes’ theorem on \( M \),
\[
(Lie_Q \omega_A)(\alpha_1, \alpha_2) = \int_{[M]} (d\alpha_1 \wedge \alpha_2 + (-1)^{\deg(\alpha_2)}\alpha_1 \wedge d\alpha_2) = \int_{[\partial M]} i^*\alpha_1 \wedge i^*\alpha_2
\]
Moreover, the component of length 3 \( (Lie_Q \omega_A)(\alpha_1, \alpha_2, \alpha_3) \) vanishes by associativity of \( \wedge \), and the higher components vanish because \( \mu^{\geq 3} = 0 \). Thus we have \( f^*\omega_B = Lie_Q \omega_A \).

The calculation above also shows that \( f^1(H^*(A)) = i^*(H^*(M)) \) is Lagrangian, since \( \int_{[\partial M]} i^*\alpha_1 \wedge i^*\alpha_2 = 0 \) if both \( \alpha_1, \alpha_2 \) are closed on \( M \), and it has maximal dimension by Poincaré-Lefschetz duality.
It remains to check that $\omega_A = \int_M \omega_B$ defines a non-degenerate pairing on the kernel $K = \ker(H^*(M) \to H^*(\partial M))$. To see this, consider the long exact sequence
\[
\cdots \to H^n(M) \to H^n(\partial M) \to H^{n+1}(M, \partial M) \to \cdots
\]
Poincaré-Lefschetz duality and the compatibility $f^* \omega_B = L\iota_{\lambda\omega_A}$ implies then that the map
\[
K = \ker(H^*(M) \to H^*(\partial M)) \cong \coker(H^{*+1}(\partial M) \to H^*(M, \partial M))
\]
$\cong (\ker((H^*(M, \partial M))^\vee \to (H^{*+1}(\partial M))^\vee))^\vee \to \ker(H^{d-*}(M) \to H^{d-*}(\partial M))^\vee = K^\vee[-d]$ is nondegenerate.

5.1.1. Poincaré pairs. The result above can be applied to a slight generalization of oriented manifolds with boundary, given by the formalism of Poincaré pairs, explained in [BD19]. A Poincaré pair of dimension $d$ is a continuous map of topological spaces of finite type $f : X \to Y$, together with a class $[Y, X] \in H_d(Y, X)$, satisfying a certain nondegeneracy condition. An instance of such an object is an oriented manifold $M = Y$ with boundary $\partial M = X$.

Given any topological space of finite type $X$ and any field $k$, there is a linearization $\mathcal{L}(X)$; this is a $k$-linear dg category such that there is a (noncanonical) equivalence $\mathcal{L}(X) \simeq C_*(\Omega_{pt}X)$ to the dg algebra of chains on the based loop space, and such that there is an equivalence between $\mathcal{L}(X)$-modules and $(\infty, 1)$-local systems on $X$ valued in $k$-chain complexes.

**Theorem 30.** [BD19, Thm.5.7] A Poincaré pair of dimension $d$ determines a relative smooth Calabi-Yau structure of dimension $d$ on the functor $\mathcal{L}(X) \to \mathcal{L}(Y)$, and therefore a relative compact Calabi-Yau structure on the functor $\text{Loc}^{\text{fd}}(Y) \to \text{Loc}^{\text{fd}}(X)$.

Translating the definition of relative CY structure into the language of $A_\infty$-structures and nc forms, we see that it corresponds exactly to the structure in the assumptions of Theorem 16. Therefore, we conclude the following.

**Corollary 31.** If the dg categories $\text{Loc}^{\text{fd}}(Y)$ and $\text{Loc}^{\text{fd}}(X)$ have minimal models (as $A_\infty$-categories with $\mu^1 = 0$) then the minimal model for $\text{Loc}^{\text{fd}}(Y)$ has a pre-CY structure of dimension $d$.

**Remark.** Strictly speaking, we only proved Theorem 16 in the setting of $A_\infty$-algebras, for the sole reason that the existence of minimal models has only been proven in the algebra case. But if we add the assumption of their existence the rest of the proof is the same.

**Remark.** The results of [KTV] imply that it is also possible to show that the dg category $\mathcal{L}(Y)$ itself carries a pre-CY structure, which is moreover nondegenerate in the sense of Section 4.3.

5.2. Varieties with section of the anticanonical. Let $X$ be a quasi-compact separated scheme over $k$. We denote by $\mathcal{A} = D_{\text{perf}}(X)$ the derived category of perfect complexes on $X$. A result of Bondal and van den Bergh [BV03] is that $\mathcal{A}$ is generated under taking cones and direct sums by a single object $E$.

Setting $A_X = \text{End}(E)$, seen as a dg algebra of endomorphisms, there is a triangulated equivalence $\mathcal{A} \cong [\text{Perf}(A_X)]$ to the derived category of perfect $A_X$-modules. This choice of generator exhibits the dg category $\text{Perf}(A_X)$ as an enhancement of
the triangulated category \( \mathcal{A} \). We also have a description of the category of \( A_X \)-bimodules; there is a quasi-isomorphism

\[
D_{\text{perf}}(X \otimes X) \cong [\text{Perf}(A_X \otimes A_X^{op})]
\]
described more precisely in e.g. [Toë07]. The algebra \( A_X \) is homologically smooth when \( X \) is smooth and compact when \( X \) is compact.

We can use this enhancement to define Hochschild co/homology of \( X \) as the corresponding invariants of \( A_X \); the resulting complexes can be shown to be invariant up to quasi-isomorphism under derived equivalence, and agree with the geometric responding invariants of \( A_X \) when \( X \) described more precisely in e.g. [Toë07]. The algebra with outer and inner bimodule structure given by the identifications

\[
\text{To be more precise, we must calculate } \text{Hom}(\Delta \otimes \Delta, \Delta)
\]

We now consider the isomorphism \( \eta \) to \( \Delta \) of the support of the sheaf above is contained in the product of the diagonals we have

\[
\text{Agree with the geometric responding invariants of } A_X \text{ when } X \text{ described more precisely in e.g. [Toë07].}
\]

\[
\eta \circ \Delta \otimes \Delta = \Delta \otimes \Delta
\]

We now extend this calculation to higher Hochschild homology.

**Proposition 32.** If \( X \) is smooth of dimension \( d \), under the equivalence \( D_{\text{perf}}(X \otimes X) \cong [\text{Perf}(A_X \otimes A_X^{op})] \), the inverse dualizing bimodule \( A' \) corresponds to \( \Delta_*(\omega_X^{-1})[-d] \).

**Proof.** Roughly this follows from the adjunction (of derived functors) \( \Delta_* \circ \Delta' = \Delta^*(-) \otimes O_X \omega_X^{-1}[-d] \) and the identification of the diagonal bimodule as \( O_\Delta = \Delta_0 \). To be more precise, we must calculate \( \text{Hom}_{A_X \otimes A_X}(A_X, A_X) \) as a \( A_X \) module itself; for that we must consider four actions of the algebra \( A_X \) which translates to sheaves on \( X_1 \times X_2 \times X_3 \times X_4 \). We number the copies of \( X \) for clarity, and denote by \( \pi_i, \pi_{ij}, \Delta_i \) the appropriate projections and diagonal embeddings.

The 4-module \( A^e \) has two bimodule structures, outer and inner, and corresponds to the sheaf

\[
O_{\Delta_{12}} \otimes O_{\Delta_{34}} \in \text{Perf}(X_1 \times X_2 \times X_3 \times X_4)
\]

with outer and inner bimodule structure given by the identifications

\[
\pi_{14} \circ (O_{\Delta_{12}} \otimes O_{\Delta_{34}}) \cong O_{X_1} \otimes O_{X_4}, \quad (\pi_{23})_\ast(O_{\Delta_{12}} \otimes O_{\Delta_{34}}) \cong O_{X_2} \otimes O_{X_2}
\]

The bimodule \( A^i_X = \text{Hom}_{A_X \otimes A_X}((A_X)_{\Delta}, A_X) \) is then given by

\[
(\pi_{23})_\ast(\pi_{14} \circ (O_{\Delta_{12}} \otimes O_{\Delta_{34}}))
\]

We now consider the isomorphism \( \eta : X_2 \times X_3 \cong X_1 \times X_4 \); from the fact that the support of the sheaf above is contained in the product of the diagonals we have calculate that it is isomorphic to

\[
\eta \circ (\pi_{14})_\ast(\pi_{14} \circ (O_{\Delta_{12}} \otimes O_{\Delta_{34}})) \cong (\Delta_{23})_\ast(\omega_X^{-1})[-d]
\]

Together with Proposition 23, this implies that:

**Corollary 33.** For any \( k \geq 1 \) there is a quasi-isomorphism of complexes

\[
C^i_{(k)}(A_X) \cong \text{Hom}_{X \times X}(\Delta_0 O_X, \Delta_0 (\omega_X^{-k})[d(1-k)]).
\]
Consider now the pair of a smooth variety $X$ of dimension $d$ and a section $s$ of its anticanonical bundle $\omega_X^{-1}$. Its image under the pushforward $\Delta_*$ is an element of

$$\text{Ext}^d_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X^{-1}) \cong \mathcal{H}H^d_{(2)}(A_X)$$

by the proposition above. We define $m_{(2)} \in C_{(2,d)}(A_X)$ to be (a cocycle representative) of the symmetrization of this element in $\mathcal{H}H^d_{(2)}(A_X)$. We also denote $m_{(1)}$ to be the $A_\infty$ structure on $A_X$ (which is just a dg algebra structure in this case).

**Theorem 34.** For any smooth $X$ and anticanonical section as above, the element $m_{(1)} + m_{(2)}$ can be extended to a pre-CY structure of dimension $d$ on $A_X$.

**Proof.** We note first that since $\Delta$ is a closed immersion, $\Delta_*$ is an exact functor, and therefore for any coherent sheaf $\mathcal{F}$ (in the abelian category, in degree zero in $\text{Perf}(X)$), $\Delta_* \mathcal{F}$ is also in the abelian category of coherent sheaves on $X \times X$.

We now calculate, using the proposition above:

$$\mathcal{H}H^{d-k-d-2k+4}_{(k)}(A_Y) = \text{Ext}^{4-2k}(\Delta_* \mathcal{O}_X, \Delta_* (\omega_X^{-1} - k))$$

But the Ext groups vanish in negative degree since both objects are in the abelian category of coherent sheaves. So $\mathcal{H}H^{d-k-d-2k+4}_{(k)}(A_X) = 0$ for all $k \geq 3$ and we can apply Corollary 21.

5.2.1. Calabi-Yau spaces. A special case of the result above applies to varieties with non-vanishing section of their anticanonical sheaf $\omega_X$. An example of such a space is a Calabi-Yau variety of any dimension, open or closed; here we take the broad definition that a Calabi-Yau is just a smooth variety with trivial canonical bundle.

**Proposition 35.** For any Calabi-Yau variety $Y$ of dimension $d$, the dg category $\text{Coh}(Y)$ has a pre-CY structure of dimension $d$; moreover its component $m_{(2)}$ is nondegenerate in the sense of Section 4.3.

**Proof.** Since $\omega_Y \simeq \mathcal{O}_Y$, we pick a section trivializing its inverse $\omega_Y^{-1}$, which by Theorem 34 gives a pre-CY structure. Nondegeneracy follows from the fact that this section is nonvanishing.

In fact the result above can be extended a little further, to any Gorenstein scheme with trivial canonical bundle. Moreover, by Theorem 25, existence of this pre-CY structure implies that there is a smooth Calabi-Yau structure of dimension $d$ on $\text{Coh}(Y)$ for any such space $Y$; this has also been shown by [BD19].

6. PROPs of marked ribbon quivers

We now arrive at the proof of our main result, Theorem 1. To recall, it says that the data of a pre-Calabi-Yau structure of dimension $d$ on an $A_\infty$-algebra/category $A$ determines an action of a certain colored dg PROP $Q^d$ on the morphism spaces $\mathcal{A}(X,Y)$ for all objects $X,Y$ of $A$ and on its Hochschild chain complex $C_*(\mathcal{A})$. This structure will be related to surfaces whose boundary has ‘open strings’ (corresponding to elements of some space $\mathcal{A}(X,Y)$) and ‘closed strings’ (corresponding to Hochschild chains).

The PROPs $Q^d$ will be defined combinatorially in terms of ribbon quivers in Section 6.2, and using the graphical calculus for signs explained in Section 3.1.1 we prove that its action commutes with the relevant differentials. This proves half of Theorem 1.
We postpone to Section 7 the other half of this proof, namely, the fact that the complexes $Q^2$ compute chains on the moduli spaces of open-closed surfaces; this description relies on an extension of Strebel’s theorem to meromorphic quadratic differentials with higher-order poles, based on the description of Gupta and Wolf of the space of such objects [GW16; GW19]. This geometric description also makes manifest the fact that the composition maps of the combinatorial PROP describe the maps induced on chains by gluing of surfaces.

6.1. Marked ribbon quivers. Let us introduce the combinatorial objects that we will use to define the PROPs.

6.1.1. Ribbon quivers. For us, a ribbon graph (or fatgraph) is a finite, connected graph whose vertices are equipped with a cyclic order of the incident half-edges. We will allow vertices of any valence $\text{val}(v) \in \mathbb{Z}_+ = \{1, 2, \ldots\}$. Every ribbon graph $\Gamma$ gives rise to an oriented topological surface with boundary $\Sigma_\Gamma$ by assigning a disc to each vertex and a rectangle to each edge, and then gluing according to incidence and ribbon structure.

**Definition 31.** A acyclic ribbon quiver $\vec{\Gamma}$ is a ribbon graph $\Gamma$ together with an orientation of each edge of $\Gamma$, such that

1. the underlying quiver of $\vec{\Gamma}$ has no oriented cycles, and
2. any vertex of valence two is either a source or a sink; it cannot have one arrow in and another one out.

Given any $\vec{\Gamma}$ as above, we denote by $\text{Source}(\vec{\Gamma})$ and $\text{Sink}(\vec{\Gamma})$ the corresponding subsets of the set $V(\Gamma)$ of vertices. Any vertex that is not a source or a sink we will call a flow vertex, denoting their subset $\text{Flow}(\vec{\Gamma})$. We also denote by $\text{Source}^1(\vec{\Gamma})$ and $\text{Sink}^1(\vec{\Gamma})$ the subsets of those that have valence one.

Note that $\text{Source}^1(\vec{\Gamma}) \cup \text{Sink}^1(\vec{\Gamma})$ is all the vertices of valence one, and to each element of this set there is a well-defined boundary circle, i.e., component of $\partial \Sigma_\Gamma$ which it sits on.

**Definition 32.** A marking on an acyclic ribbon quiver $\vec{\Gamma}$ is the data of five ordered subsets of $V(\Gamma)$, labeled

$$V_x, V_{\text{open-in}}, V_{\text{open-out}}, V_o, V_1$$

all pairwise disjoint, with the following properties:

1. $V_x \subset \text{Source}^1(\vec{\Gamma})$, such that if $v \in V_x$, then no other vertices in $\text{Source}^1(\vec{\Gamma})$ sit on the same boundary component of $v$.
2. $V_{\text{open-out}} \subset \text{Sink}^1(\vec{\Gamma})$ such that if $v \in V_{\text{open-out}}$, the boundary component it sits on doesn’t have any vertex in $V_x$.
3. $V_o \subset \text{Sink}(\vec{\Gamma}) \setminus V_{\text{open-out}}$.
4. $V_1 \subset \text{Source}^1(\vec{\Gamma})$ and every vertex in $V_1$ is directly connected to a vertex in $V_o$, such that for each vertex in $V_o$ there is at most one vertex in $V_1$ connected to it.
5. $V_{\text{open-in}} = \text{Source}^1(\vec{\Gamma}) \setminus (V_x \cup V_1)$.

Together with a choice of distinguished outgoing arrow for each vertices not in $\text{Sink}(\vec{\Gamma})$, and a distinguished incoming arrow for each vertex in $V_o$. 
We will see later, in Section 7, that marked ribbon quivers label cells in some moduli space of open-closed surfaces; that is, topological surfaces whose boundary has subsets marked as incoming/outgoing ‘open strings’ (intervals) and ‘closed strings’ (parametrized circles). Each ribbon quiver is embedded in such a surface, with some vertices associated to these marked boundaries.

Figure 1. A ribbon quiver embedded in its corresponding open-closed surface $\Sigma$. The $V_x$ vertices attach to closed inputs, the $V_o$ vertices map attach to closed outputs, and the open-in/open-out vertices to open inputs/outputs. We denote the distinguished arrows by white triangles when there are multiple possible choices.

Let us paraphrase the definition above: we choose some valence one sources which are alone on their boundary component to be $\times$ vertices; on the other boundary components we can choose some valence one sinks to be open-out vertices. Every other sink is a $V_o$ vertex; to some of those we can label one of the leaves attached to it as $V_1$. Every other valence one source is then an open-in vertex. The remaining unmarked vertices are all the sources of valence $\geq 2$ and all the flow vertices, which by assumption have valence $\geq 3$.

The following marked ribbon quiver corresponds to the open-closed surface in Fig. 1:

Recall that we are free to attach a $V_1$ vertex at the $\circ$-sink, as we did in this quiver.

6.1.2. Genus and degree. Let $(\vec{\Gamma}, V_x, V_{\text{open-in}}, V_{\text{open-out}}, V_o, V_1)$ be a marked acyclic ribbon graph, which for simplicity we will just call $\vec{\Gamma}$.
**Definition 33.** The genus $g(\vec{\Gamma})$ is the genus of the closed surface $\Sigma_{\vec{\Gamma}}$. The homological $d$-degree of $\vec{\Gamma}$ depends on the choice of an integer $d$, and is given by the formula:

$$\deg_d(\vec{\Gamma}) = \sum_{v \in \text{Source} \geq 2} ((2-d) \text{out}(v)+d-4) + \sum_{v \in \text{Flow}} ((2-d) \text{out}(v)+d+\text{in}(v)-4) + \sum_{v \in V_\circ} (\text{in}(v)-1)$$

where $\text{in}(v)$ and $\text{out}(v)$ are the number of incoming and outgoing arrows of $v$.

**Remark.** Note that when $d = 0$, the graphs with degree zero are exactly the ones with only trivalent flow vertices, bivalent unmarked sources, and valence one $\circ$-sinks. One gets the other graphs by starting from such a graph and contracting edges; each contracted edge contributes $+1$ to the homological degree. Adding a $V_1$-leaf to a $V_\circ$-vertex also contributes $+1$ to the degree.

---

6.1.3. **Orientations on ribbon quivers.** In order to define the differential on the PROP of ribbon quivers, and moreover to assign its action with correct minus signs, it is necessary to introduce the notion of orientations.

Again let us fix an integer $d$, and a graph $\Gamma$. Suppose now that we assign a degree $|v| \in \mathbb{Z}$ to each vertex $v \in V(\Gamma)$, independently of $\Gamma$ itself.\footnote{This degree will be related to the degree of operation we insert at $v$, but for defining the orientations let us just describe it as an arbitrary integer.}

Consider the set $V(\Gamma) \sqcup E(\Gamma)$ of all its vertices and edges. Let us denote by $\text{Ord}(\Gamma)$ the set of orderings of $V(\Gamma) \sqcup E(\Gamma)$; an element of $\text{Ord}(\Gamma)$ is for instance a sequence $(e_1 e_2 v_1 e_3 \ldots v_n)$.

We define an action of the symmetric group $S_{|V(\Gamma)|+|E(\Gamma)|}$ on $\{\pm 1\} \times \text{Ord}(\Gamma)$ by the following rule:

- Vertices have weight $d + |v|$, edges have weight $d - 1$.
- When we commute any two elements $x, y$ in the sequences, we multiply by a factor $(-1)^{\text{weight}(x)\text{weight}(y)}$.

**Definition 34.** A $d$-orientation on a graph $\Gamma$ with vertex degrees $\{|v|\}$ is an element of the two-element set $(\mathbb{Z}/2 \times \text{Ord}(\Gamma))/S_{|V(\Gamma)|+|E(\Gamma)|}$ where we take the quotient by the action of dimension $d$ above.

Note that by definition, the notion of orientations only depends on the degrees $|v|$ up to parity. The case that will be most important to us is when all the degrees are even.

**Lemma 36.** If all the degrees $|v|$ are even, then:

- if $d$ is even, a $d$-orientation on $\Gamma$ is the same as an orientation (in the classical sense) on the vector space $\text{Span}_\mathbb{R}(E(\Gamma))$, and
- if $d$ is odd, it is an orientation on $\text{Span}_\mathbb{R}(V(\Gamma))$.

Let us now fix a ribbon quiver structure $\vec{\Gamma}$. This defines a partial order on the set of vertices, with $v > w$ if there is a path $v \to w$, and a cyclic order on the half-edges incident at each vertex. We have the following notion of compatibility:

**Definition 35.** An ordering in $\text{Ord}(\Gamma)$ is in normal form with respect to the ribbon quiver structure $\vec{\Gamma}$ if it is of the following form:

$$(e_{11}, \ldots, e_{1k_1}, v_1, \ldots, v_{n-1}, e_{n1}, \ldots, e_{nk_n}, v_n)$$
where \(v_1, v_2, \ldots, v_n\) is non-decreasing in the partial order, \((e_{i1}, \ldots, e_{ik_i})\) are the edges going out of \(v_i\) in some order compatible with the clockwise cyclic order.

The data of an ordering in normal form can be given by a linear extension of the partial order of vertices, together with marking one outgoing edge for every vertex \(v_i\), indicating the edge \(e_{i1}\) in the notation above.

**Example.** Consider the following marked ribbon quiver of genus zero with two \(\times\) sources and one \(\circ\) sink:

![Diagram of a marked ribbon quiver]

We can pick for instance the linear extension \((v_1 > v_2 > v_3 > v_4 > v_5 > v_6)\) with the marked edges indicated by the white arrows (when there is a choice). The corresponding ordering in normal form is then

\[(v_6 f v_5 e v_4 c b v_3 d v_2 a v_1)\].

6.1.4. *Action of oriented marked ribbon quivers.* Recall that in Section 3.1.1, for an \(A_\infty\)-algebra/category \(A\) we defined an action of directed trees of higher Hochschild cochains on copies of \(A[1]\). We now show that this action naturally generalizes to an action of oriented marked ribbon quivers as above.

This action is described graphically in the following way. For each \(\times\)-source, we input a Hochschild chain \(a_0 \otimes a_1 \otimes \cdots \otimes a_p\) by sending \(a_0\) along the edge and drawing \(p\)-arrows corresponding to \(a_1, \ldots, a_p\) counter-clockwise from it:

![Diagram of Hochschild cochain evaluation]

Now take \((\Gamma, (\ldots))\) a marked ribbon quiver with an ordering in normal form, such that all the sinks (i.e., vertices in \(V_\circ\)) are first, then all the \(V_1\) vertices, then all the flow vertices, then all the sources. The ordering then looks like

\[(o_1 o_2 \ldots o_n e_1 e \ldots e_N e \ldots v_1 e \ldots s_m e \ldots s_1)\]

where \(o\) are the \(V_\circ\) sinks, \(v\) are the flow vertices and \(s\) are the sources, and \(e\) just generically denotes the edges in normal form.

Now, into each flow vertex with \(out(v) = k\) we can insert a higher cyclic \(k\)-cochain, i.e. an element \(\phi \in C^{\ast}_{(k,d)}(A)\). We evaluate this on \(m\) Hochschild chains and output \(n\) Hochschild chains, in the following way:
(1) For each source $s_i$ corresponding to each Hochschild chain $a_0^i \otimes a_1^i \otimes \cdots \otimes a_p^i$, we draw the arrows coming out as above, then choose a way to connect them to the vertices embedded in the surface $\Sigma_{\Gamma}$, \textit{without crossing} each other.

(2) We write the ordering in normal form and the components of the Hochschild chains next to each other:

\[
(o_1 o_2 \cdots o_N e \cdots e v_N \cdots e \cdots v_1 e \cdots e s_m \cdots e \cdots s_1)(a_0^1 a_1^1 \cdots a_p^m)
\]

where now we regard all the $a$’s as elements of $A[1]$.

(3) We now proceed as we did in Section 3.1.1; we bring the inputs of the last vertex to the beginning of the $a$ string.

(4) We then evaluate the cochain $\phi_i$ and write in the place the output, in the given ordering of the output edges. For the sources $s$, we interpret them as just producing the corresponding Hochschild chain.

(5) We repeat steps (3) and (4) until there are only outputs vertices left. We now reorder the elements of $A[1]$ to correspond to the output vertices in the order $o_1 \cdots o_n$, beginning from the incoming marked direction of $o_i$; recall that if there is a $V_1$ vertex attached we must start from it. We then sum over all the possibilities in step (1), with the Koszul sign coming from the transpositions.

Note that all the Koszul signs are for $a$ as seen as elements of $A[1]$. For clarity, we now do an example.

\textit{Example.} We consider the graph below with the normal form ordering

\[
(o \ e_6 \ v_3 \ e_5 \ v_2 \ e_3 \ e_2 \ v_1 \ e_4 \ s_2 \ e_1 \ s_1)
\]

and input a pair of Hochschild chains $a_0 \otimes a_1 \otimes a_2$ and $b_0 \otimes b_1$ through the sources $s_1, s_2$, and put higher cyclic cochains $\phi \in C^*_C(2,d)(A)$, $\psi, \lambda \in C^*_C(1,d)(A) = C^*(A)$. Let us connect the arrows in the following way, for instance:
and then evaluate $X = (o \ e_6 \ v_3 \ e_5 \ v_2 \ e_3 \ e_2 \ v_1 \ e_4 \ s_2 \ e_1 \ s_1)(a_0 \ a_1 \ a_2 \ b_0 \ b_1)$.

The source vertices just output the chains themselves, permuting the $a$s and $b$s back and forth, so we have no sign:

$$X = (o \ e_6 \ v_3 \ e_5 \ v_2 \ e_3 \ e_2 \ v_1)(a_0 \ a_1 \ a_2 \ b_0 \ b_1)$$

We then permute to get

$$X = (-1)^{b_1(\bar{a}_0 + \bar{a}_1 + \bar{a}_2 + \bar{b}_0)}(o \ e_6 \ v_3 \ e_5 \ v_2 \ e_3 \ e_2 \ v_1)(b_1 \ a_0 \ a_1 \ a_2 \ b_0)$$

and evaluate $\phi$, suppose for instance that $\phi(b_1; a_0) = c_1 \otimes c_2$, so that

$$X = (-1)^{b_1(\bar{a}_0 + \bar{a}_1 + \bar{a}_2 + \bar{b}_0)}(o \ e_6 \ v_3 \ e_5 \ v_2 \ e_3 \ e_2)(c_1 \ c_2 \ a_1 \ a_2 \ b_0)$$

and then continuing this process until we have $(-1)^\#(a)(x \ a_2)$. We then permute $x$ and $a_2$ to read the output, since the edge $e_6$ is the marked edge going into the output $o$.

It is straightforward to check that once we choose higher cyclic cochains $\phi_i$ for each vertex, the action of a graph $\Gamma$ of degree $\deg(\Gamma)$ defines a map of graded vector spaces

$$\Phi(\Gamma) : (C_*(A))^\otimes m \to (C_*(A))^\otimes n$$

of degree $\sum_i(|\phi_i| - 2) - \deg(\Gamma)$. We have the following result, which motivates our definition of orientation:

**Proposition 37.** The dimension $d$ action of the symmetric group on edges and vertices intertwines the assignment $(\Gamma, \ldots) \to \Phi(\Gamma)$, with the sign representation on the target. Therefore, it descends to an action of the set of graphs with $d$-orientation.

*Proof.* This statement just says that a permutation between two normal forms acts by the same sign on orderings and on the resulting operator $\Phi(\Gamma)$. To see this, note that any two normal forms are related by two types of moves:

- Switching the order of two vertices (corresponding to cochains $\phi, \psi$) that are incomparable in the partial order, together with their outgoing vertices. This introduces a sign

  $$((k - 1)(d - 1) + |\phi| + d)((\ell - 1)(d - 1) + |\psi| + d)$$

  where $\phi \in C^\ell_{(k,d)}(A)$, $\psi \in C^\ell_{(\ell,d)}(A)$.

- Performing a cyclic permutation of the output edges of any given vertex $\phi$, which introduces a sign $(k - 1)(d - 1)$.

For moves of type (1), we explicitly verify that changing the order of evaluation in $\Phi(\Gamma)$ introduces the same sign

$$\tilde{\phi}\tilde{\psi} \equiv ((k - 1)(d - 1) + |\phi| + d)((\ell - 1)(d - 1) + |\psi| + d) \pmod{2}$$

(recall that $\tilde{\phi}$ is the degree as a map to factors of $A[1]$) and for moves of type (2), we get a Koszul sign, which differs from $(k - 1)(d - 1)$ exactly by the sign used in the definition of higher cyclic cochains $C^\ell_{(k,d)}(A)$.

By the proposition above, we can define the action of a graph with any ordering of its edges and vertices, even if they are not in normal form. This now allows us to explain intuitively the signs in the necklace product:
Lemma 38. The necklace product of $\phi \in C^*_k(A), \psi \in C^*_\ell(A)$ is given by the symmetrization of the result of the oriented ribbon quiver

$$\phi \circ_{\text{nec}} \psi = \begin{pmatrix} e_{n+1} & \ldots & e_{n+\ell-1} \\ e_n & & e_{n+\ell} \\ & \vdots & \vdots \\ & & e_{k+\ell-1} \\ e_1 & e_2 & \ldots & e_k & e_{k+\ell-1} \\ \psi & \phi & e \\ e_{n+1} & \ldots & e_{n+\ell-1} \end{pmatrix}, (e_1e_2\ldots e_{k+\ell-1}\phi\psi) + \text{(cyc)}$$

where by (cyc) we denote the sum over cyclic permutations of the labels $e_1, \ldots, e_{k+\ell-1}$ on the diagram.

One proves the lemma above simply by computing the signs and comparing with the previously given Definition 22. However, this definition in terms of the orientation allows us to easily prove properties about the necklace product and bracket.

Lemma 39. The necklace product lands in higher cyclic cochains for the dimension $d$ action of $\mathbb{Z}_{k+\ell-1}$ and the corresponding necklace bracket, seen as a map

$$[-,-]_{\text{nec}} : C^*_d(A)[1] \otimes C^*_d(A)[1] \to C^*_d(A)[1],$$

satisfies the graded Jacobi relation.

Proof. For the first statement, note that cyclic shift under $\mathbb{Z}_{k+\ell-1}$ does not change the set of graphs we sum over, but does change their orientations by cyclically rotating the edges $e_1 \ldots e_{k+\ell-1}$, which introduces a sign $(k-1)(d-1)$, which is exactly the sign added to the Koszul sign in the definition of higher cyclic cochain. The second statement just follows directly from the observation that all graphs appear with orientation given by the same expression. □

6.2. The PROP of marked ribbon quivers. We are now ready to define the PROPs acting on Hochschild chains of pre-CY algebras and categories. For simplicity of presentation, let us first describe the ‘closed string’ part of the PROP; for each dimension $d$ this is a (single-colored) PROP $Q^d$ which will act on Hochschild chains of pre-CY categories of dimension $d$. Later, in Section 6.4, we will explain how the general open-closed case works.

6.2.1. Generators. Let us fix any pair $m,n$ of positive integers. Consider the set $RQ(m,n)$ of all marked ribbon quivers with $|V_e| = m, |V_o| = n, V_{\text{open-in}} = V_{\text{open-out}} = \emptyset$. We will be inserting the pre-CY structure maps $m_{(k)}$ into every flow vertices of our quivers; as those have degree $|m_{(k)}| = 2$ in $C^*_d(A)$, we fix the degree of all flow vertices to be $|v| = 2$.

Definition 36. As a graded vector space, the space $Q^d_{m,n}$ is defined as

$$Q^d_{m,n} = \text{Span}_k \left( \{ (\vec{\Gamma}, \mathcal{O}) \mid \vec{\Gamma} \in RQ_{m,n}, \mathcal{O} \text{ d-orientation on } \vec{\Gamma} \} \right) / \sim,$$

where $d$-orientation was defined in Definition 34, and $\sim$ denotes the equivalence relation that sends $(\vec{\Gamma}, \mathcal{O}) \mapsto - (\vec{\Gamma}, \mathcal{O}^{\text{op}})$, with $\mathcal{O}^{\text{op}}$ the opposite orientation to $\mathcal{O}$. The generating vector $(\vec{\Gamma}, \mathcal{O})$ is placed in (cohomological) degree $-\deg(\vec{\Gamma})$. 
Because each ribbon graph has a genus $g \geq 0$, we get decompositions $RQ_{m,n} = \bigsqcup_g RQ_g(m,n)$ and $Q^d(m,n) = \prod_g Q^d_g(m,n)$.

Let us present some examples in low genus, with their respective degrees:

**Example.** Let us present some examples in genus zero, together with their (homological) degrees:

$m = 1, n = 1$

1. $\circ \xrightarrow{\times} \circ$ deg $= 0$
2. $\circ \xrightarrow{\times} \circ$ deg $= 1$
3. $\circ \xrightarrow{\times} \circ$ deg $= -d$

$m = 1, n = 2$

4. $\circ \xrightarrow{\times} \circ$ deg $= -d$
5. $\circ \xrightarrow{\times} \circ$ deg $= -d + 1$

We give the ribbon structure from the embedding into the page, and indicate the orientation by ordering all vertices (by height) and choosing the first outgoing (white arrow) when there is ambiguity. Note that graph (3) has a boundary component (dashed circle) without any incoming or outgoing leg; this will be a free boundary of the surface.

6.2.2. The differential. We now describe a differential on each of the spaces $Q^d_{m,n,g}$.

Recall from Definition 33 that the vertices contributing nonzero degree to a given marked ribbon quiver $\vec{\Gamma}$ are: (unmarked) sources with valence $> 2$, flow vertices with $\text{in} > 2$ and/or $\text{out} > 1$, and $\circ$-marked sinks with valence $> 1$.

For each such vertex $v$, draw it on the plane and draw a dashed curve separating the incident half-edges into two non-empty subsets.

**Definition 37.** A separation of the vertex $v$ is a ribbon quiver with two vertices $a,b$ obtained by splitting the vertex into two, and connecting them by an arrow $e : a \to b$, with the following conditions:

1. None of the resulting vertices have $\text{in} = \text{out} = 1$.
2. If $v$ is not a sink, then $b$ is not a sink.

For the distinguished edge at a $\circ$-sink, when we separate such a vertex, we end up with a flow vertex and a $\circ$-sink; if the previously distinguished edge now lands in the $\circ$-sink, we mark it. If not, we mark the new edge.

And as for the $\circ$-sinks with a $V_1$-vertex attached, if such a vertex has valence 2 we just declare it to have no separations; otherwise, if the resulting graph has a $V_1$-vertex attached to a flow vertex of valence 3, we delete the $V_1$-vertex and its incident edge; if it has a $V_1$-vertex attached to a flow vertex of valence $> 3$, we exclude this separation.
For example, along the dashed curve below, there are two possible separations

\[ \implies \quad \& \quad \] and

In contrast, if we separate only incoming arrows to one side, by condition (2) we only have one separation

\[ \implies \]

Finally, to exemplify the special rules at \( \circ \)-sinks, we decreed that the vertex has no separations. With more incoming edges we do have separations at vertices with 1 attached, for example,

\[ \implies \quad \& \quad \]

We now define the differential on \( Q^d_{m,n,g} \) by defining it on the basis elements given by a marked ribbon quiver \( \vec{\Gamma} \) together with some orientation. We first put the orientation into some normal form

\[ (o_1 o_2 \ldots o_n \ldots 1_1 \ldots v_N \ldots v_1 x_m \ldots x_1) \]

with all the \( \circ \) sinks \( o_i \) before all the \( V_1 \) vertices \( 1_i \), before all the flows and unmarked sources \( v_i \), before all the \( \times \) sources \( x_i \).

**Definition 38.** The differential \( \partial \) of the element above is given by

\[
\partial((\vec{\Gamma};((o_1 o_2 \ldots o_n \ldots 1_1 \ldots v_N \ldots v_1 \ldots x_m \ldots x_1))) =
\sum_{v \in V(\vec{\Gamma})} \left( \vec{\Gamma}_{(e:a \to b)}((o_1 o_2 \ldots o_n e a \ldots 1_1 \ldots v_N \ldots b \ldots v_1 \ldots x_m \ldots x_1) \right)
\]

In other words, we sum over all separations of all vertices that can be separated (that is, all vertices with nonzero degree), with the orientation such that the extra vertex \( a \) and extra edge \( e \) created are always immediately after the output vertices, with the other vertex \( b \) replacing \( e \).

As for the special case of separations involving \( V_1 \)-vertices, if we deleted a \( V_1 \)-vertex attached as

\[ \implies \]

we just produce the orientation given by deleting the vertex 1 and its incident edge; if we instead deleted

\[ \implies \]

we produce *minus* that orientation.
Remark. The rules for the separations involving $V_1$-vertices might seem arbitrary, but in our prop action we will input strict units at these vertices; the conventions in fact they follow from the relations satisfied by those strict units (Definition 7). For example, the reason for excluding separations of vertices with valence 2 and one 1-vertex attached is that when evaluating the ribbon quiver with the $A_\infty$-maps, the differential above would give a term $\mu^2(1, x) + (-1)^{\bar{x}} \mu^2(x, 1) = 0$.

Lemma 40. The differential $\partial$ squares to zero.

Proof. Any two consecutive separations can be performed in either order; resulting in the same graph but with orientations given by

$$\left(o_1 o_2 \ldots o_n e_2 a_2 e_1 a_1 \ldots 1_i \ldots v_N b_1 \ldots b_2 v_1 \ldots x_m \ldots x_1\right)$$

if we do the separation $(e_1 : a_1 \to b_1)$ and then $(e_1 : a_1 \to b_1)$, and

$$\left(o_1 o_2 \ldots o_n e_1 a_1 e_2 a_2 \ldots 1_i \ldots v_N b_1 \ldots b_2 v_1 \ldots x_m \ldots x_1\right)$$

if we do them in the opposite order. The sign difference between these is from switching a pair of an edge and vertex with another pair, giving $(-1)^{(d+d+1)(d+d+1)} = -1$. □

6.2.3. Isomorphisms between even and odd dimensions. We defined each of the complexes $\{Q^d(m,n)\}$ separately for each choice of integer $d$; changing $d$ shifts the degree assigned to each marked ribbon quivers in $Q^d(m,n)$, and also changes the signs in the differential.

Upon fixing the number of inputs $m$, the number of outputs $n$ and the genus $g$, for any two integers $d_1, d_2$ the complexes $Q^d_1(m,n)$ and $Q^d_2(m,n)$ are spanned by the same marked ribbon quivers, with an overall shift depending on those integers. Moreover, if $d_1, d_2$ have the same parity, the signs in the differentials are all the same, so we have that:

Proposition 41. If $d_1 \equiv d_2 \pmod{2}$, the complexes $Q^d_1(m,n)$ and $Q^d_2(m,n)$ are isomorphic up to shift.

It turns out that at partial form of the result above also holds between dimensions $d_1, d_2$ of different parity, but only once we restrict to ribbon quivers corresponding to surfaces without free boundary circles. We will be more precise about the relations between marked ribbon quivers and open-closed surfaces in Section 7, but for now we will say that if a free boundary circle of $\Gamma$ is a boundary component of the surface $\Sigma_\Gamma$ that has no vertex of valence one neighboring it.

Definition 39. For any $d, g, m, n$, the subcomplex of marked ribbon quivers without free boundaries

$$Q^d_{g,F=0}(m,n) \subset Q^d_g(m,n)$$

is the subcomplex spanned by the marked quivers with no free boundary circles.

Note that the differential preserves the number of free boundary circles, so it preserves the subcomplex $Q^d_{g,F=0}(m,n)$. We will now argue that the complexes $Q^d_{g,F=0}(m,n)$ is independent of the dimension $d$, up to shift; this is seen by a computation involving the signs in the differential, which we now explain.

Recall that when $d$ is even, a $d$-orientation on a marked ribbon quiver $\bar{\Gamma}$ is an ordering of the edges of $\bar{\Gamma}$ modulo the alternating group, and when $d$ is odd, it is an ordering of the vertices of $\bar{\Gamma}$ modulo the alternating group. The sets of orientations
for either the edges or the vertices are both abstractly isomorphic to $\mathbb{Z}_2$ so in order to define a bijection between orientations, it is enough to determine a single sign between a fixed ordering of edges and a fixed ordering of vertices.

We now choose a pair of those fixed orderings. Recall that the surface $\Sigma_\Gamma$ is divided by $\Gamma$ into regions; as long as there are no free boundary circles, each such region corresponds to a $\times$-vertex and is homeomorphic to a disk with a cut going from the $\times$-vertex to the boundary. The boundary of each such disk is made up of a sequence of edges and vertices, and possibly some trees attached to some of these vertices.

We now fix an ordering of the $\times$-vertices and an ordering of the $\circ$-vertices (corresponding to fixing the ordering on the inputs/outputs of the PROP operation); let that ordering be

$$(x_1, \ldots, x_m), \quad (o_1, \ldots, o_n)$$

respectively.

On each region corresponding to the source $x_i$, we start from $x_i$ along the edge incident to it, and go around the perimeter of the disk. We record the data of the edges along the boundary of this perimeter in the following way: we fill a matrix with two rows, one column at a time, such that

1. when we encounter an new vertex followed by an outgoing edge with the same orientation of our boundary walk, we write that vertex and edge as a new column, and
2. when we encounter a vertex followed by an outgoing edge with the same orientation of our boundary walk, but we already encountered that vertex before, we write a new column leaving the top entry empty.

We perform the operation above for each $\times$-vertex $x_i$ in sequence, keeping the columns as we move to the next disc.

Because the surface $\Sigma_\Gamma$ is orientable, if there are no free boundary circles we guarantee that every edge appears exactly once in this matrix; and every vertex that is not a sink ($\circ$-vertex) also appears exactly once.

Remark. Note that if, on the other hand, there are a nonzero number of free boundary circles, it is possible that the procedure above will miss some edges; namely, every edge that is part of the clockwise-oriented cycle around each free boundary circle.

Once we are done, this matrix contains each edge and vertex exactly once, except for the sink $\circ$-vertices. We now add those back in the beginning, according to their order, and define $\text{Sgn}(\text{sinks, top row})$ to be the fixed vertex orientation. As for the edges, we define $\text{Sgn}(\text{bottom row})$ to be the fixed edge orientation. Note that these depend only of the marked ribbon quiver $\vec{\Gamma}$ and on the ordering of the $\times$- and $\circ$-vertices.

We now define a sign between these orientations, in the following way:

1. For each column that has a vertex and an edge, we record the number of nonempty entries of the matrix before that column,
2. For each column that has just an edge, we record the number of nonempty entries of the matrix after that column, and
3. For each column that has just a vertex, we record zero.

We then sum over all these numbers to get an integer $S$. 

**Definition 40.** The isomorphism between the sets of vertex orientations and edge orientations is given by relating the fixed orientations we specified by

\[ \text{Sgn(top row)} = (-1)^S \text{Sgn(bottom row)}. \]

Doing this for every marked ribbon quiver \( \Gamma \) gives isomorphisms of graded vector spaces

\[ Q_{g,F=0}^{\text{even}}(m,n) \cong Q_{g,F=0}^{\text{odd}}(m,n)[s], \]

for some shift \( s \).

**Proposition 42.** The isomorphisms above intertwine the differential \( \partial \), and thus give isomorphisms of complexes between all the \( Q_{g,F=0}^{d}(m,n) \) for varying \( d \), up to shift.

**Proof.** The proof consists in checking the sign introduced by the differential \( \partial \) with respect to our fixed orientations. Recall the definition of \( \partial \): for each vertex \( v \) that is either a flow vertex or a sink we consider all separations \( e : a \rightarrow b \) at \( v \), and sum over all of those. For the orientation, starting an orientation given by inserting the new higher vertex \( a \) and the new edge \( e \) between all the sinks and the other vertices:

\[ (o_1 o_2 \ldots o_n e \ a \ldots 1_i \ldots v_N \ldots b \ldots v_1 \ldots x_m \ldots x_1) \]

Let \( \Gamma \) be our starting graph and \( \Gamma' = \Gamma_{e:a \rightarrow b} \) be one of its separations at \( v \). Starting now with the fixed orientation (we got by identifying the fixed edge and vertex orientations) for \( \Gamma \), we can permute it to some normal form (getting two signs for vertices and edges), apply the separation and then permute the vertices/edges other than \( a \) and \( e \) again to the fixed orientations (getting those same two signs for vertices and edges).

We then compare the obtained orientations with the fixed orientations of \( \Gamma' \), and calculate that the relative sign between the new edge orientation and the new vertex orientation is always +1. This involves checking each possible configuration of edge directions around the new edge \( e \), together with each possible ordering of the regions around it.

Essentially there are three distinct cases to be checked, depending on whether in the matrix the new edge and vertex \( e \) and \( a \) appear together (in the same column) or separately, and if not whether the vertex \( b \) appears right after that column or not. We now do one of those cases explicitly for the sake of clarity.

Suppose that the region of the surface \( \Sigma_\Gamma \) around \( v \), and the separation \( e : a \rightarrow b \) of \( v \), look like the following drawing:

On the left we have a piece of \( \Gamma \) and on the right a piece of \( \Gamma' \), and the roman numerals indicate in which order those three regions around \( v \) appear in our fixed orientations.

We now make the matrix of fixed orientations for \( \Gamma \):

\[
\begin{bmatrix}
\ldots & v_1 & v & \ldots & v_3 & \square & \ldots & v_5 & \ldots \\
\ldots & e_1 & e_2 & \ldots & e_3 & e_4 & \ldots & e_5 & \ldots
\end{bmatrix}
\]
and for $\Gamma'$:

\[
\begin{bmatrix}
\ldots & v_1 & a & \ldots & v_3 & b & \ldots & v_5 & e & \ldots \\
\ldots & e_1 & e_2 & \ldots & e_3 & e_4 & \ldots & e_5 & e & \ldots
\end{bmatrix}
\]

where we emphasize the vertices and edge being separated for convenience.

Using the sign prescription above gives us two signs $S_\Gamma$ and $S_{\Gamma'}$; we now calculate their difference to be equal to

\[\#(\ldots v_1) + \#(a\ldots v_3) + \#(\ldots e_1\ldots e_5) + 1\]

modulo 2, where each $\#$ term is the length of the indicated string in the matrix. We note that this is exactly the sign of the permutation that brings $a$ and $e$ to the beginning of their respective rows, and $b$ to where $v$ was in the top row for $\Gamma$.

Therefore we have that both the differential for even and odd $d$, applied to the fixed edge/vertex orientation that we identified in Definition 40, give $\Gamma'$, also in its fixed edge/vertex orientation, with the same sign. We then check other cases (with different ordering of the regions and orientations of $e_1, \ldots, e_5$) and get analogous results.

6.2.4. Compositions. Finally, we must discuss compositions; in a PROP one can compose along any number of outputs and inputs, without connectedness restrictions. We now describe how to compose marked ribbon quivers with orientations.

To describe the composition, one must consider the topology of the surface $\Sigma_\Gamma$ associated to the ribbon graph. We embed $\Gamma$ into this surface and for each $\times$ source $x_i$, we consider the region of $\Sigma_\Gamma \setminus \Gamma$ adjacent to it. We interpret this region as a disc with a cut from the boundary to the center, going along the edge incident at $x_i$.

We call the boundary of this disc the boundary cycle $B_{x_i}$ associated to the source $x_i$. Note that this boundary cycle might include the same edge of $\Gamma$ once or twice (on opposite sides), and might include a vertex of $\Gamma$ multiple times.

Note also that different $\times$-sources $x_i, x_j$ have adjacent regions that are disjoint from one another; also, even though their boundary cycles might overlap, each angle around a vertex of $\Gamma$ is at most associated to one $\times$-source.

Let $\Gamma_1$ be some graph with a $\circ$-vertex $o$ and $\Gamma_2$ with a $\times$-vertex $x$. By definition, a $\times$-vertex has only one outgoing arrow, but a $\circ$-vertex has any number $k \geq 1$ of incoming arrows, one of which is distinguished. A composition of $\Gamma_1$ and $\Gamma_2$ at the pair $(o, x)$ is given by:

- Deleting the $\circ$-vertex $o$ from $\Gamma_1$ and the $\times$-vertex $x$ from $\Gamma_2$,
- Connecting the marked arrow in $\Gamma_1$ to the arrow in $\Gamma_2$ leaving the removed vertex $\times$
- Connecting the $k - 1$ other arrows in $\Gamma_1$ to vertices in the boundary cycle of $x$, respecting the cyclic ordering.

One checks that this operation produces another marked ribbon quiver, and that it is additive on degrees. We show an example in Fig. 2. Moreover, since all the regions adjacent to the $\times$-vertices are disjoint, we can compose along multiple pairs as above, by performing the connections in each of those regions.

We must now describe how to compose orientations. Note that composition at each pair $(o_i, x_i)$ deletes the two vertices and replaces the edge $e_i$ incident at $x_i$ with an edge connecting $\Gamma_1$ to $\Gamma_2$. We define the composition of orientations at a
Figure 2. Composition of two marked ribbon quivers with one closed input and one closed output (i.e. $(m,n) = (1,1)$). The composition at the pair $(s_j,o_i)$ of an input and output is a sum over ways of distributing the three arrows incident at $o_i$ around the boundary cycle of $s_j$; the distinguished arrow (white triangle) always gets connected to the unique edge leaving $s_j$.

pair $(o_i,x_i)$ as follows:

$$(\ldots e_i x_i) \circ (o_i \ldots) = (\ldots e_i \ldots)$$

and if there are multiple pairs, we equally define

$$(\ldots e_3 x_3 e_2 x_2 e_1 x_1) \circ (o_1 o_2 o_3 \ldots) = (\ldots e_3 e_2 e_1)$$

Note that the formula above is equivariant with respect to changing the order of compositions; transposing $(e_i x_i e_j x_j) \leftrightarrow (e_j x_j e_i x_i)$ gives a $-1$ sign, $(o_i o_j) \leftrightarrow (o_j o_i)$ gives $(-1)^d$ and $(e_i e_j) \leftrightarrow (e_j e_i)$ gives $(-1)^{d+1}$.

**Definition 41.** The composition of two elements $(\Gamma_1, O_1) \in Q^d(m_1,n_1)$ and $(\Gamma_2, O_2) \in Q^d(m_2,n_2)$ at some number of pairs $(o_1,x_1), \ldots, (o_p,x_p)$ is given by the linear combination

$$(\Gamma_1, O_1) \circ ((o_i,x_i)) (\Gamma_2, O_2) = \sum_{\text{compositions } \Gamma} (\Gamma, O_2 \circ O_1)$$

where we sum over all compositions, i.e. all ways of distributing the extra incident vertices on $o_i$ around the boundary cycle of $x_i$, with the orientation given by the composition as above.

The following result follows directly from checking the axioms of a symmetric monoidal category.

**Proposition 43.** The spaces $\{Q^d(m,n)\}$ for all $m,n \geq 1$ form a dg PROP, that is, a symmetric monoidal category enriched over the category of chain complexes $\text{Ch}_k$, with object monoid given by the natural numbers.
More precisely, each $Q^d_g(m,n)$ is the space of connected operations of genus $g$ in the PROP: to get all operations (or equivalently, the morphisms spaces of the symmetric monoidal category) one must allow disconnected ribbon quivers, which means allowing tensor products of any number of spaces $Q^d_g(m,n)$.

6.3. Action of the PROP on Hochschild chains. Let $A$ be a pre-CY category of dimension $d$, with pre-CY structure $\{m_{(k)}\}$. As before we will call $m = m_{(1)}$ its $A_{\infty}$-structure. Recall that in Section 6.1.4 we describe how a marked ribbon quiver acts on Hochschild chains, once we input higher cyclic cochains into the vertices of the quiver.

Let us be more precise about which Hochschild complex we will use. We assume the category $A$ is homologically unital as an $A_{\infty}$-category; we extend it to an $A_{\infty}$-equivalent, strictly unital $A_{\infty}$-category $A^+$, with strict units $1_X \in A(X,X)$ for each $X$.

The Hochschild chain complex $C_*(A)$ has a subcomplex spanned by all terms $a_0 \otimes a_1 \otimes \cdots \otimes a_p$ where some $a_{i\neq 0} = 1_X$ for some $X$. The nonunital Hochschild chain complex $C^m_*(A)$ is then defined as the quotient of $C_*(A^+)$ by this subcomplex; as $A$ is homologically unital the composition $C_*(A) \rightarrow C_*(A^+) \rightarrow C^m_*(A)$ is a quasi-isomorphism. So for simplicity of notation we will just denote $C_*(A)$ for this non-unital complex, which allows us to work with the strict units in $A^+$.

We now input the pre-CY structure map $m_{(k)}$ into every vertex with $k$ outgoing edges, and get a map of graded vector spaces $\Phi : Q^d(m,n) \otimes C_*(A)^{\otimes m} \rightarrow C_*(A)^{\otimes n}$.

We have the Hochschild differential $b$ acting on $C_*(A)$ and the differential $\partial$ acting on $Q^d(m,n)$ as we defined using separations.

**Theorem 44.** The map $\Phi$ commutes with the differential and defines a map

$$H^*(Q^d(m,n)) \otimes HH_*(A)^{\otimes m} \rightarrow HH_*(A)^{\otimes n}.$$ 

**Proof.** It will be easier to shift the Hochschild complex, so we will instead describe an action $\Phi : Q^d(m,n)[m-n] \otimes (C_*(A)[1])^{\otimes m} \rightarrow (C_*(A)[1])^{\otimes n}$

In other words, given Hochschild chains $a^i, 1 \leq i \leq m$, we want to prove the following identity:

$$\Phi(\partial \Gamma)(a^1, \ldots, a^m) + (-1)^{\deg(\Gamma) + m-n} \Phi(\Gamma) \circ b(a^1, \ldots, a^m) = b \circ \Phi(\Gamma)(a^1, \ldots, a^m)$$

The action of a graph on a Hochschild chain is as described in Section 6.1.4; in order to prove the identity above we have to understand how to express the Hochschild differential $b$, applied before and after $\phi(\Gamma)$, in terms of modifications to $\Gamma$.

We describe the ribbon quivers corresponding to $\Phi(\Gamma) \circ b$: for the component of the Hochschild differential on the $i$th chain in $(C_*(A)[1])^{\otimes m}$, we sum over insertions of a vertex $\mu$ attached to all angles around the boundary cycle of the $i$th $\times$-source of $\Gamma$, added to an insertion of a vertex $\nu$ along the edge incident at that $\times$-source.

We orient all those graphs in the following way: if $\Gamma$ had an orientation in normal form

$$(o_1 o_2 \ldots o_n \ldots v_N \ldots v_1 e_m \ldots x_m \ldots e_1 \ldots x_1)$$
where $e_1, \ldots, e_m$ are the edges incident at the sources $x_1, \ldots, x_m$, for each modified graph we insert the new vertex $\mu$ and its outgoing edge $e$ as

$$(o_1 o_2 \ldots o_n \ldots v_N \ldots v_1 e \mu e_m \ldots e_{m} \ldots e_1 \ldots x_1)$$

between the $\times$-sources and their edges, and all the other edges and vertices.

Now we sum over all these new ribbon quivers with the orientation above. We see that the effect of each new vertex $\mu$ is to exactly precede the application of $\Phi(\Gamma)$ by an operation

$$a_0^1 \otimes a_1^1 \otimes \cdots \otimes a_{p_1}^1 \otimes \cdots \otimes a_0^m \otimes a_1^m \otimes \cdots \otimes a_{p_m}^m \mapsto a_0^1 \otimes a_1^1 \otimes \cdots \otimes a_{p_1}^1 \otimes \cdots \otimes a_i^1 \otimes \cdots \otimes \mu(a_j^i, \ldots) \otimes \cdots \otimes a_{p_1}^1 \otimes \cdots \otimes a_0^m \otimes a_1^m \otimes \cdots \otimes e_p^m$$

and the sum over such operations is the (shifted) Hochschild differential $b$ on $(C_*(A)[1])^{\otimes m}$.

Now we describe the ribbon quivers corresponding to $b \circ \Phi(\vec{\Gamma})$: for the $i$th $\circ$-vertex $o_i$, we sum over all insertions of $\mu$ around the angles of $o_i$ and also over all insertions of $\mu$ along the edges incident at $o_i$.

We orient all those graphs in a similar way, by placing the new vertex $\mu$ and its outgoing edge as follows

$$(o_1 o_2 \ldots o_n \ldots v_N \ldots v_1 e_{m} \ldots e_{m} \ldots e_1 \ldots x_1)$$

that is, right between the $\circ$-outputs and all the other elements. We sum over all these new ribbon quivers with this orientation, and observe that the effect is almost to follow the application of $\Phi(\Gamma)$ by $b$ on $(C_*(A)[1])^{\otimes n}$: what we are missing are terms where some subset of size $\geq 2$ of the $k$ incoming edges to $o_i$ themselves get input into $\mu$; these are exactly the separations of a sink in Definition 37, which appear in $\partial \Gamma$.

We now turn to the flow vertices. The Maurer-Cartan equation satisfied by the elements $\{m_{(k)}\}$ is

$$\sum_{i+j=k+1} m_{(i)} \circ_{\text{unc}} m_{(j)} = 0$$

for every $k \geq 1$. Consider now some vertex $v \in \vec{\Gamma}$, with $k$ outgoing edges and $\ell$ incoming edges. Two of the types of terms in the equation above are terms with $j = 1$ or $i = 1$; by calculating their orientation they enter in the equation above respectively as

in terms of the ribbon quiver, the first type of term corresponds to the sum of two types of modifications to the ribbon quiver: adding $\mu$ in angles around $v$, and also to separations of $v$ where one side of the dashed line in Definition 37 only
has incoming arrows. Note that the sign for all terms in the second sum is the
same, and that
\[(k - 1)(d - 1) = dk - d - k + 1 \equiv \tilde{v} + 1\]
where again \(\tilde{v} = \tilde{m}_{(k)} = dk - d + 3k - 4\) is the degree of the vertex as a map from
copies of \(A[1]\) to copies of \(A[1]\).

All the other separations of \(v\) appearing in \(\partial \Gamma\) correspond to the other terms
in the necklace equation above, with \(i, j \geq 2\). The last calculation we need is to
analyze the effect on the orientation of the operation that does not change anything
about the ribbon quiver, but commutes the order between some new vertex \(\mu\)
or \(\mu\) past some vertex \(v\) that is not attached to it. We calculate that
this has the effect of introducing a sign
\[
\tilde{\mu}\tilde{v} = \tilde{v}
\]

We are now ready to assemble all these calculations. Starting with the sum
of ribbon quivers for \(\Phi(\vec{\Gamma}) \circ b\), in sequence we ‘pass’ the \(\mu\) vertices past the flow
vertices \(v_i\), either by the necklace relation, if they are connected, or by switching
the orientation as shown above, if they are not. The sign gained is always \(\tilde{v}\). If \(v\)
has \(k\) outgoing edges and \(\ell\) incoming edges this is
\[
\tilde{v} = \tilde{m}_{(k)} \equiv dk - d - k \equiv \deg(v) + \text{out}(v) + \text{in}(v) \quad (\text{mod } 2)
\]
Therefore repeating this procedure for all the flow vertices we get a global sign
\[
\sum_{v \in \text{Flow}} (\deg(v) + \text{out}(v) + \text{in}(v))
\]
but every edge appears twice in the sum above, with the exception of the edges
connected to the \(x\)-sources and \(o\)-sinks. As each \(o\)-sink has a degree of \((\text{in}(o) - 1)\),
this global sign is \(\deg(\Gamma) + m - n\) and we have the desired identity. \(\square\)

6.4. The open-closed PROPs. We now describe a modification of the PROPs
\(Q^d\), which will act not only on Hochschild chains of a pre-CY category \(A\),
which are associated to closed strings, but also on the morphism spaces \(A(X,Y)\)
for objects \(X, Y\) of \(A\), which are associated to open strings. We already described
the graphs appearing in this PROP in Section 6.1.1; the open-in and open-out vertices
are inputs and outputs of open strings.

6.4.1. Colors and boundary type. A colored PROPs is just a particular type of
symmetric monoidal category. Given a set \(S\) of colors, a \(S\)-PROP is a strict symmetric
monoidal category whose monoid of objects is isomorphic to the free monoid
generated by \(S\).

That is, if \(Q\) is a \(S\)-PROP, for any two sequences \(\vec{c} = (c_1, \ldots, c_n)\) and \(\vec{c}' = (c'_1, \ldots, c'_m)\), there is a set of morphisms \(Q(\vec{c}, \vec{c}')\). Just from the axioms of
symmetric monoidal categories these spaces then come with appropriate actions by the
symmetric groups \(S_n, S_m\), compatible with permutations of \(\vec{c}, \vec{c}'\).

Let \(A\) be some \(A_\infty\)-category, and denote \(\text{Ob}(A)\) its set of objects. We now fix a
set of colors associated to \(A\) to be the set
\[
S_A = (\text{Ob}(A) \times \text{Ob}(A)) \sqcup \{\ast\}
\]
In other words, there is one color for each ordered pair \((X,Y)\) of objects of \(A\) and
one extra color \(\ast\). If \(A\) is an algebra, that is, has a single object \(X\), the set of colors
is the two-element set \(\{(X,X), \ast\}\) (for open and closed boundaries, respectively).
The closed PROP $Q^d$ we described previously can be seen as colored PROP where we only use the color $\ast$; in that case, the sequences $\vec{c}, \vec{c}'$ are described solely by the two positive integers $m, n$ which determine the space $Q^d(m, n)$, by specifying how many $\times$- and $\circ$-vertices were required.

For the open-closed PROP we need to describe which ribbon quivers (also with open in/outputs) are compatible with a pair $\vec{c}, \vec{c}'$.

**Definition 42.** The boundary type of $\vec{\Gamma}$ is the tuple

$$(|V_x|, |V_o|, (i, o, i, i, o, \ldots, o), \ldots, (o, i, o, \ldots, i), F)$$

of three integers, where $F$ is the number of boundary components without any marked vertices, and some finite number of cyclically ordered sequences on the symbols $i, o$, each corresponding to a boundary component with open-in and open-out vertices.

Note that by definition, the sets $V_x, V_o$ and all the $i$'s and $o$'s are also linearly ordered, corresponding to some order of open inputs and outputs; this ordering has nothing to do with the cyclic ordering in the boundary type.

**Definition 43.** A marked acyclic ribbon quiver $\vec{\Gamma}$ is compatible with the sequences of colors $\vec{c}, \vec{c}'$ if the boundary type of $\vec{\Gamma}$ satisfies these conditions:

1. $|V_x| = \text{number of } \ast \text{'s appearing in } \vec{c}$,
2. $|V_o| = \text{number of } \ast \text{'s appearing in } \vec{c}'$,
3. $|\vec{c}| - |V_x| = \text{number of } i \text{'s in the boundary type}$,
4. $|\vec{c}'| - |V_o| = \text{number of } o \text{'s in the boundary type}$,

Together with the following condition on the colors. By (3) and (4) above we have a bijection between the incoming/outgoing open colors $c_k = (X_k, Y_k)$ (pairs of objects of $A$) and the sets of $i$ and $o$; we require that along every boundary component:

1. if $i_k$ appears immediately before $i_\ell$ then $Y_k = X_\ell$,
2. if $i_k$ appears immediately before $o_\ell$ then $Y_k = Y_\ell$,
3. if $o_k$ appears immediately before $i_\ell$ then $X_k = X_\ell$, and
4. if $o_k$ appears immediately before $o_\ell$ then $X_k = Y_\ell$.

Paraphrasing the conditions above in more informal terms, once we draw the surface associated to the marked acyclic quiver $\vec{\Gamma}$, we have exactly the right numbers of open/closed colors on each side, and along each boundary component made up of open colors, we can draw them as incoming/outgoing oriented strings such that their sources and targets are compatible.

**6.4.2. Action of the open-closed PROPs.** We now use the compatibility condition above to define the desired PROPs. For any pair of colors $\vec{c}, \vec{c}'$, consider the set $RQ(\vec{c}, \vec{c}')$ of all marked ribbon quivers compatible with it.

**Definition 44.** The space $Q^d(\vec{c}, \vec{c}')$ is defined analogously to $Q^d(m, n)$ (Section 6.2), but summing over all ribbon quivers in $RQ(\vec{c}, \vec{c}')$, with orientations.

As for the action of $Q^d(\vec{c}, \vec{c}')$, we proceed the same way with the closed inputs ($\times$-vertices), and on each open input (valence one vertex in $V_{\text{open-in}}$) labeled by a color $(X, Y)$, we input the corresponding element of the hom space $A(X, Y)$; on each open output labeled by a color $(X, Y)$ (valence one vertex in $V_{\text{open-out}}$) we again read out the arrow traveling along the incident edge as an element of...
Figure 3. Boundary type along a boundary component of $\Gamma$ with two open inputs (on the left) and two open outputs (on the right). The associated boundary type has a cyclically ordered tuple $(i, i, o, o)$ which is compatible with any sequence of colors of the form $((Y, X), (X, W), (Z, W), (Y, Z))$, for any four objects $X, Y, Z, W$ of $\mathcal{A}$.

some hom space $\mathcal{A}(X, Y)$ on components where the source and target $X, Y$ of that element agree with the desired color, or as zero if they do not.

The same argument as in Theorem 44 proves the following result.

**Theorem 45.** For any pre-CY category $\mathcal{A}$ of dimension $d$, and any sequences of colors $\vec{c}, \vec{c}'$, having respectively $m, n$ instances of the color $\ast$, and open colors given by pairs of objects $(X_i, Y_i)$ and $(X'_i, Y'_i)$, there is an morphism of complexes

$$Q^d(\vec{c}, \vec{c}') \otimes (C_\ast(\mathcal{A}))^{\otimes m} \otimes \prod_i \mathcal{A}(X_i, Y_i) \rightarrow (C_\ast(\mathcal{A}))^{\otimes n} \otimes \prod_j \mathcal{A}(X'_j, Y'_j)$$

Finally, composition of ribbon quivers $\Gamma_2 \circ \Gamma_1$ along open in/outputs with compatible colors is done by erasing the pair of vertices from $V_{\text{open-out}} \subset V(\Gamma_1)$ and $V_{\text{open-in}} \subset V(\Gamma_2)$ and identifying their incident edges; this makes the collection of spaces $Q^d_{\vec{c}, \vec{c}'}$, into a $\mathcal{S}_\mathcal{A}$-colored dg PROP.

In order to define composition and the action as above, one has to define the orientation on these ribbon quivers with open in/outputs just as we have done before. This can be done by pretending that the open inputs are $\times$-vertices and the open outputs are $\circ$-vertices of valence one; and proceeding just as we did for the closed case.

**Remark.** We would like to point out a new feature that occurs when composing open inputs and outputs: the creation of free boundary circles. Let $\Gamma_1, \Gamma_2$ be graphs which contain consecutive open outputs and inputs as follows:

![Figure 4. Creation of a new free boundary component (dashed circle from open gluing along two neighboring open intervals)](image)

The composition $\Gamma_2 \circ \Gamma_1$ then will have a free boundary (i.e. without any marked sources or sinks), indicated by the dashed circle. On the other hand, all the free
boundaries that $\Gamma_1$ and $\Gamma_2$ already had will still be in $\Gamma_2 \circ \Gamma_1$; the composition map $\circ$ is superadditive on the number $F$ of free boundary circles. Because of this, in the open-closed case one cannot restrict to the case without no free boundary circles, and one does not get the isomorphisms between the even and odd dimensions that we discussed in the closed case in Section 6.2.3.

6.5. **Features of the open-closed PROPs.** We discuss some features of the PROPs $Q^d$, as well as some smaller algebraic structures that are part of it.

6.5.1. **Connes’ differential and identity maps.** When marking ribbon quivers, we allowed some $\circ$-outputs to have a valence one vertex attached to them, labeled by 1; this corresponds to inputting the cochain $1 \in C^0(A)$, for the case of an algebra, or the unit morphism $1_X \in A(X, X)$ for the appropriate object $X$, for the case of a category.

Recall that we have been denoting by $C_\ast(A)$ the ‘nonunital chain complex’ $C^\text{nu}_\ast(A)$ for Hochschild homology, that is the quotient of the usual complex $C_\ast(A^+)$ of the augmented $A_\infty$-category by the subcomplex of chains that have some strict unit $1_X$ in some place with nonzero index.

Therefore, if a certain $\circ$-vertex $o_i$ has a 1 attached to it, unless the edge connecting $o_i$ and 1 is the distinguished edge of $o_i$, the resulting output chain is zero in $C_\ast(A)$, so we will always assume that edge is the distinguished edge.

With this convention, the ribbon quiver giving Connes’ differential $B$ of cohomological degree $-1$ is given by

$$\Gamma_B = \begin{array}{c} \times \\ \circ \end{array}$$

The ribbon quiver above has (homological) degree $\deg = 1$ and genus zero, and is part of the closed PROP. The identity map on $C_\ast(A)$ is also in that same space $Q^d(1, 1)$, but with degree zero, and is given simply by

$$\Gamma_{id} = \begin{array}{c} \times \\ \circ \end{array}$$

Another simple operation is given by the disc with $k$ open inputs and one closed output at the origin:

which describes the map of degree $1 - k$ that sends a sequence

$$(a_0, a_1, \ldots, a_{k-1}) \in A(X_1, X_2) \otimes A(X_2, X_3) \otimes \cdots \otimes A(X_k, X_1)$$

to the Hochschild chain $a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \in C^\ast(A)$. 
6.5.2. The $A$-infinity operad. Consider now the sub-PROP of the open-closed PROP $Q^d$ made up of genus zero ribbon quivers without $\times$- or $\circ$-vertices (therefore without $1$-vertices), without free boundary components and with a single open output; every such ribbon quiver is a union of some directed trees, with every vertex having a single output.

Note that the degree and differential on all these ribbon quivers is independent of the integer $d$. Moreover, if we restrict to connected ribbon quivers, only composition along one output is allowed, and we just have an operad. The following proposition follows from just checking that the signs in our prescription are just the Koszul signs appearing in the $A_\infty$-relations.

**Proposition 46.** This operad of genus zero ribbon graphs with open inputs and one open outputs is equivalent to the $A_\infty$-operad, that is, chains of the operad of rooted planar trees.

6.5.3. The multicorolla dioperad. We now consider a slightly bigger sub-PROP of $Q^d$, where we still only have genus zero connected ribbon tree quivers with open in/outputs, without free boundaries, but now we allow multiple outputs.

If we still only allow composition along a single edge, the composed ribbon quiver still satisfies those conditions, and we get a dioperad (in the language of e.g. [Gan03]). This dioperad is generated by `multicorollas’ such as:

Let us denote this multicorolla operad by $MC^d$. Each space of operations of $MC^d$ splits as a sum of complexes

$$MC^d_{n_1, \ldots, n_k}$$

from $n = \sum n_i$ inputs to $k$ outputs; this space is spanned by directed trees embedded in the disk with $k$ arrows going out, and $n_i$ arrows coming in between the $i$th and the $(i + 1)$th outgoing arrows.

Later in Section 7.5.1 we will prove that the complexes $Q^d$ model chains on appropriate moduli spaces of metric ribbon quivers. In this case, each of those
spaces is given by a quotient
\[ \text{MetRT}_{n_1, \ldots, n_k}/\mathbb{Z}_k, \]
where MetRT\(_{n_1, \ldots, n_k}\) is a space of metric ribbon tree quivers with \(k\) outputs, and \(n = \sum_i n_i\) inputs (distributed as we described).

Note the quotient by the cyclic action; this is because in identifying the surfaces with \(n\) inputs as belonging to one of the spaces MetRT\(_{n_1, \ldots, n_k}\) we must be free to apply a cyclic rotation. The spaces MetRT\(_{n_1, \ldots, n_k}\) are contractible and retract to the unique cell given by the ribbon tree with a single vertex.

Thus the spaces MetRT\(_{n_1, \ldots, n_k}/\mathbb{Z}_k\) are rationally contractible; for \(k\) of characteristic zero we have that
\[ MC^d_{n_1, \ldots, n_k} \cong C_\ast(\text{MetRT}_{n_1, \ldots, n_k}/\mathbb{Z}_k, \mathcal{L}^d), \]
where \(\mathcal{L}\) is a certain \(k\)-local system, and thus we have the following characterization:

**Proposition 47.** A has a pre-CY structure of dimension \(d\) if and only if it is a module over the dioperad \(MC^d\).

Similar spaces to the these already appeared in the work of Poirier-Tradler [PT17], where the authors consider trees with a distinguished outgoing edge, meaning that they do not take the quotient of the chain complex by the cyclic action. Some related works also include [PT19; DPR15].

### 7. Meromorphic Strebel differentials and the open-closed moduli space

In this section, we turn to the theory of Strebel differentials and explain how the open-closed PROP \(Q\) that we defined in the previous section relates to certain moduli spaces of surfaces with open/closed/free boundaries.

#### 7.1. Strebel differentials

Let us briefly recall some relations between the geometry of quadratic differentials and the description of moduli spaces of Riemann surfaces.

Let us fix a compact and connected Riemann surface \(S\). A meromorphic quadratic differential \(\varphi\) on \(S\) determines a flat metric \(|\varphi|\) on the complement of its set of zeros and poles, and a measured foliation given by its horizontal foliation.

The classical work of Jenkins and Strebel [Jen57; Str84] deals with meromorphic quadratic differentials with poles of order at most two. Let us first discuss the holomorphic case. Such a differential \(\varphi\) is a (holomorphic) Strebel differential if the union of all non-closed leaves of its horizontal foliation has measure zero.

Such a differential determines a finite graph \(\Gamma_\varphi\) embedded in \(S\), consisting of the union of all the critical leaves, zeros and simple poles of \(\varphi\), and decomposes \(S \setminus (\Gamma \cup \{\text{double poles}\})\) into some number of maximal ring domains, or finite-height cylinders. Each such cylinder is foliated by the horizontal leaves of \(\varphi\), which are simple closed curves of some isotopy class \(\gamma_i\), all pairwise distinct and each not nullhomotopic.

**Theorem 48.** [Str84] Fix \(S\) (of genus \(\geq 2\)), \(n\) (isotopy classes) of simple closed curves \(\gamma_i\) as above, and \(n\) positive real numbers \(m_i\); then there is a unique (up to scale) Jenkins-Strebel differential \(\varphi\) whose ring domains are cylinders associated to \(\gamma_i\) with modulus \(m_i\).
Hubbard and Masur gave another perspective on the result above. Let $\mathcal{MF}$ denote the space of measured foliations; a holomorphic quadratic differential gives such an object by taking its horizontal foliation.

**Theorem 49.** [HM79] Any measured foliation $F \in \mathcal{MF}(S)$ is realized by a unique holomorphic quadratic differential on $S$.

This gives a homeomorphism between the space of measured foliations and the space of holomorphic quadratic differentials on $S$, both homeomorphic to $\mathbb{R}^{6g-6}$. In other words, the map $Q \rightarrow T_g$ presents the space of quadratic differentials as a fiber bundle over Teichmüller space, with fiber identified with $\mathcal{MF}(\Sigma_g)$. Strebel’s theorem for holomorphic differentials is then recovered by taking a particular measured foliation.

For differentials with double poles, the story is similar but the maximal ring domains surrounding each double pole is a infinite-height cylinder. If we now set all the heights of the finite-height cylinders to be zero, we then have the following variant of Strebel’s theorem.

**Theorem 50.** [Str84] For a fixed Riemann surface $C$ with $k$ distinct points $p_1, \ldots, p_k$, and a choice of positive real numbers $\ell_1, \ldots, \ell_k$, there is a unique Strebel differential on $C$ with double poles at $p_i$ and holomorphic on $C \setminus \{p_i\}$, such that all the maximal ring domains of $\varphi$ are half-infinite cylinders of circumference $\ell_i$ surrounding the points $p_i$.

For simplicity we refer to such objects as Strebel differentials; in particular, for such a differential the residue of $\sqrt{\varphi}$ at every double pole is real. Each Strebel differential then determines a finite metric ribbon graph $\Gamma$ embedded in $S$, given by the critical leaves of $\varphi$, to which $C \setminus \{p_i\}$ contracts.

Strebel’s uniqueness theorem can then be used to give an interpretation of moduli space of punctured curves by such graphs. The set $\mathcal{M}^{\text{comb}}_{g,k}$ of all such metric ribbon graphs with genus $g$ and $k$ boundary cycles can be given a natural topology and orbifold structure.

**Theorem 51.** [Kon92] The map $\mathcal{M}_{g,k} \times \mathbb{R}_+^k \rightarrow \mathcal{M}^{\text{comb}}_{g,k}$, given by taking the graph of critical leaves of Strebel differentials, is a homeomorphism of orbifolds.

### 7.2. Higher-order poles

Recent work of Gupta and Wolf [GW16; GW19] has described a generalization of the Hubbard-Masur theorem to meromorphic quadratic differentials with poles of arbitrary order, precisely describing the compatibility between geometric data on the surface (e.g. measured foliations) and the analytic behavior of $\varphi$.

As before, let $S$ be a Riemann surface with $k \geq 1$ points $p_1, \ldots, p_k$ and choose $k$ positive integers $n_i \geq 2$. A quadratic differential $\varphi$ with poles of order $n_i$ at $p_i$ induces a measured foliation with pole singularities on $S$; this is a measured foliation on $S \setminus \{p_i\}$ but with some specific local behavior around each $p_i$. The space of such measured foliations is denoted by $\mathcal{MF}(S, \{n_i\})$.

Around each pole, for some arbitrary choice of coordinate $z$, we have the local expressions

$$\sqrt{\varphi} = \frac{1}{z^{n/2}} \left( p(z) + z^{n/2}g(z) \right) dz$$

for $n$ even, where $p(z)$ is a polynomial of degree $(n-2)/2$, and

$$\sqrt{\varphi} = \frac{1}{z^{(n-1)/2}} \left( p(z) + z^{(n-1)/2}g(z) \right) dz$$

for $n$ odd.
for $n$ odd, where $p(z)$ is a polynomial of degree $(n - 3)/2$. In both formulas $g(z)$ is some non-vanishing holomorphic function.

The polynomials $p(z)$ are then the principal parts of $\sqrt{\varphi}$; in the even $n$ case there is one real compatibility condition between $p$ and the measured foliation determined by $\varphi$. Taking into account this condition, one calculates that at the point $p_i$, the space of compatible principal parts is a manifold of real dimension $n - 1$. When $n \geq 3$ this is homeomorphic to $\mathbb{R}^{n-2} \times S^1$, and when $n = 2$ this is homeomorphic to $\mathbb{R}_+$. One can prove then a generalization of the Hubbard-Masur theorem, which we paraphrase from [GW16; GW19].

**Theorem 52.** With $S$ and $\{p_i\}, \{n_i\}$ as above, given a measured foliation with poles $F \in \mathcal{MF}(S, \{n_i\})$ and the data of 'compatible principal parts' (as defined in op.cit.) at each $p_i$, there is a unique meromorphic quadratic differential $\varphi$ with poles of order $n_i$ at $p_i$ realizing $F$ and with the chosen principal parts, depending continuously on that data.

That is, up to the one real compatibility condition at poles of even order, one can pick the measured foliation and the principal parts independently. The continuity statement implies, in particular, that the natural map $Q(g, \{n_i\}) \to T_{g,k}$ presents the space of such meromorphic quadratic differentials as a fiber bundle with fiber homeomorphic to

$$\mathcal{MF}(S, \{n_i\}) \times \{\text{compatible principal parts}\} \cong \mathbb{R}^{6g-6+\sum_i n_i} \times \left( \prod_{i, n_i \geq 3} \mathbb{R}^{n_i-2} \times S^1 \right) \times \left( \prod_{i, n_i = 2} \mathbb{R}_+ \right)$$

For quadratic differentials with poles of orders $\leq 2$, one recovers Strebel’s theorem from this description: one chooses the zero measured foliation in $\mathcal{MF}(S, \{n_i\})$ and gets a homeomorphism $Q(\{n_i\})^{\text{Str}} \cong T_{g,k} \times \mathbb{R}^k_+$. Here $\mathbb{R}^k_+$ comes from the principal parts, which in this case are the residues of $\phi$ at the double poles. This isomorphism is moreover equivariant with respect to the action of the mapping class group $\text{Mod}(\Sigma_{g,n})$, giving the isomorphism of orbifolds $\mathcal{M}^{\text{Str}} \cong \mathcal{M}_{g,n} \times \mathbb{R}^k_+$.

We now extend this to the case of higher order poles.

**Definition 45.** (Meromorphic Strebel differentials) A meromorphic quadratic differential $\varphi$ is meromorphic Strebel if it maps to the zero measured foliation.

One can also characterize such differentials by the following properties:

1. Every leaf asymptotic to a pole of $\varphi$ of order $\geq 2$ in one direction either
   - Goes to a zero of $\varphi$ in the other direction, or
   - Goes to the same pole in the other direction, and is homotopic to the constant curve at that pole, through a homotopy of curves that are also leaves of the horizontal foliation.
2. The closure $\Gamma$ of the union of all the other critical leaves (i.e. leaf going to a zero or simple pole, and not contained in the item above) of the horizontal foliation is measure zero, and
3. The complement of the graph $\Gamma$ in $S$ is a disjoint union of some number of disks (with no cylinders).

**Remark.** In [GW16] the authors refer to these differentials as 'half-plane differentials', but do not include differentials with simple poles. Or rather, they get rid of
simple poles by taking double covers ramified at them; for our purposes we cannot
do that, and will eventually need to include simple poles. We have decided to call
this notion ‘meromorphic Strebel’ to emphasize this difference and the relation to
Strebel’s theorem.

\[ \text{Figure 5. Picture of the horizontal foliation of a meromorphic Strebel}
\text{differential on } \mathbb{C}P^1 \text{ with one pole of order 4 (inside the circle), one pole}
of order 3 (to the right) and a simple pole (white circle at the bottom).} \]

Consider the locus \( Q(g, \{ n_i \})^{\text{mStr}} \) of Riemann surfaces of genus \( g \) with meromorphic
Strebel differentials with poles of orders \( n_i \), and no simple poles, together with
the natural map \( \pi : Q(g, \{ n_i \})^{\text{mStr}} \to T_{g,k} \). As a consequence of Theorem 52 we
have

**Corollary 53.** Each fiber of \( \pi \) is identified with the space of compatible principal
parts, homeomorphic to

\[
\left( \prod_{i,n_i \geq 3} \mathbb{R}_{+}^{n_i-2} \times S^1 \right) \times \left( \prod_{i,n_i=2} \mathbb{R}_{+} \right).
\]

7.3. **A moduli space of meromorphic Strebel differentials.** We now construct a space that will give us a classifying space for certain open-closed cobordisms. Consider a compact topological surface with boundary \( (\Sigma, \partial \Sigma) \) of genus \( g \). Let us choose a partition of its boundary \( \partial \Sigma \) into the following subsets:
- Incoming closed boundaries \( C_{\text{in}} \) given by some disjoint union of circles.
- Incoming open boundaries \( O_{\text{in}} \) given by some disjoint union of (open) intervals.
- Outgoing closed boundaries \( C_{\text{out}} \) given by some disjoint union of circles.
- Outgoing open boundaries \( O_{\text{out}} \) given by some disjoint union of (open) intervals.
- Free boundaries, given by the complement of the above subsets, a disjoint
union of circles and (closed) intervals.

We now pick a linear ordering of each of the set of connected components of the first
four subsets above; i.e. we label all the incoming/outgoing open/closed boundaries
by some ordered sets.

Consider the ‘bordered mapping class group’ preserving all the open and closed
boundaries pointwise; in contrast, it can freely rotate and permute the free boundary
circles. We will simply denote the corresponding mapping class group by \( \text{Mod}^{\text{oc}}(\Sigma) \).
Let us now take a classifying space for this group, decomposed as

$$\bigsqcup_{F \text{ free boundary circles}} B \text{Mod}^{\text{oc}}(\Sigma_F),$$

where we keep open and closed boundaries fixed, but include an arbitrary number $F$ of free boundary circles. This disjoint union is a classifying space for cobordisms between $\text{O}_{\text{in}} \sqcup C_{\text{in}}$ and $\text{O}_{\text{out}} \sqcup C_{\text{out}}$, with any number of holes in the interior. We can work with each one of those spaces separately by fixing the number of free boundary circles.

We now define some data from this surface.

**Definition 46.** (Pole data) To each connected component $i$ of $\partial \Sigma$, we assign an integer $n_i$ as follows:

- If $i \in \pi_0(C_{\text{in}})$ (i.e. incoming closed), we assign $n_i = 3$, and if $i \in \pi_0(C_{\text{out}})$ (i.e. outgoing closed) we assign $n_i = 2$.
- If $i$ contains exactly $N \geq 1$ open intervals (either incoming or outgoing) we assign $n_i = N + 2$.
- Finally, if $i$ is a free boundary circle, we assign $n_i = 2$.

Consider then the space $Q(g, \{n_i\})^{\text{mStr}}$ of Riemann surfaces $S$ of genus $g$ with meromorphic Strebel differentials of pole orders $\{n_i\}$, and no simple poles. Consider one such differential. By definition, each pole $p$ of order 2 is surrounded by a ring domain that is a half-infinite cylinder, with boundary given by some circle $S^1_p$ which is the union of some critical leaves.

**Definition 47.** An *marking with $\ell$ outputs* on $\varphi \in Q(g, \{n_i\})^{\text{mStr}}$ is a choice of some subset of the double poles of $\varphi$, of size $\ell$, together with a single point in $S^1_p$ for each $p$ in that subset. We denote the space of all such objects by $Q(g, \{n_i\})^{\text{mStr}}_\ell$.

For each double pole marked as an output, we take the unique vertical geodesic going from that pole to its marked point; this defines a tangent direction in $T_p S$. Moreover, for every pole $p_i$ of order $n_i \geq 3$, we have distinguished $n_i - 2$ directions, given by the critical leaves asymptotic to $p_i$.

Therefore such a differential with marked outputs gives a point in the ‘bordered Teichmüller space’ $\mathcal{T}_{g, \vec{k}}$ of Riemann surfaces of genus $g$ with choices of some numbers of distinguished points on the boundaries. Here we use some generic subscript $\vec{k}$ to indicate all the data of the boundary; note that the free boundary circles (corresponding to unmarked double poles) do not have any distinguished points.

**Proposition 54.** There is an isomorphism of fiber bundles between

$$\pi : Q(g, \{n_i\})^{\text{mStr}}_\ell \to \mathcal{T}_{g, \vec{k}}$$

and the trivial $\mathbb{R}_+^k$-bundle over $\mathcal{T}_{g, \vec{k}}$, which is equivariant for the action of the bordered mapping class group $\text{Mod}^{\text{oc}}(\Sigma)$.

**Proof.** The existence of an isomorphism follows directly from Gupta-Wolf’s result; the only non-trivial fact is that one can construct this isomorphism equivariantly, where the trivial $\mathbb{R}_+^k$-bundle has a trivial action along the fiber. For this we must give a $\mathbb{R}_+^k$-valued invariant function on $Q(g, \{n_i\})^{\text{mStr}}_\ell$.

For $\varphi \in Q(g, \{n_i\})^{\text{mStr}}_\ell$, to each pole of order $n_i \geq 3$, resp. $= 2$ there is a punctured disk around it given by the union of $n_i - 2$ half-planes, resp. half-infinite
cylinder, whose boundary is a union of critical leaves between zeros. The lengths of such boundaries give the desired \( \text{Mod}^{\text{oc}}(\Sigma) \)-invariant function. One must check that this indeed gives an isomorphism of fiber bundles; this follows from observing that, using the description of the spaces of compatible principal parts, such a function gives on each fiber a continuous embedding \( \mathbb{R}^k \hookrightarrow \{ \text{compatible principal parts} \} \), moreover continuous with respect to variations of complex structure.

7.4. The perimeter-shrinking map. Each meromorphic Strebel differential with marked outputs we used above gives a metric ribbon graph, which in turns determines the surface and differential; one could use it to give a cell model for the moduli space above. We will now explain how to relate this model to the marked ribbon quivers we discussed in Section 6.

For that, we will need an operation that shrinks the perimeter of the half-infinite cylinders corresponding to outputs. We start with a surface \( S \) and a meromorphic Strebel differential \( \varphi_0 \) on \( S \) having a double pole at a point \( p \). The horizontal foliation and metric associated to \( \varphi_0 \) give a half-infinite cylinder surrounding \( p \), with an \( S^1 \) boundary made of a sequence of horizontal geodesics between zeros of \( \varphi \); this is a cycle in the associated metric ribbon graph \( \Gamma \).

Let us say there are \( n \) zeros on that cycle; pick one of these zeros as a starting point, and encode the data of \( \Gamma \) near this cycle as a tuple of lengths

\[
(d_1, c_1, d_2, c_2, \ldots, d_n, c_n) \in \mathbb{R}^{2n}_{\geq 0}
\]

where \( d_i \) is associated to the \( i \)th zero in the cyclic order as follows: if the order of that zero is \( \geq 2 \) (i.e. the vertex in \( \Gamma \) has valency \( \geq 4 \)) then we assign \( d_i = 0 \). If it is a simple zero, then \( d_i \) is the length of the edge in \( \Gamma \) incident there and not contained in the cycle (i.e. the edge pointing out). We define \( c_i \) to be the length of the edge between the \( i \)th and the \((i + 1)\)th zero in the cycle.

**Definition 48.** A partial perimeter-shrinking family of meromorphic Strebel differentials is a family over \([0, T)\) of Riemann surfaces and meromorphic Strebel differentials \((S_t, \varphi_t)\), such that their tuples of lengths around some cycle as above satisfy

\[
\begin{align*}
  d_i^t &= d_i^0 + \frac{t}{2}, \\
  c_i^t &= c_i^0 - t
\end{align*}
\]

while keeping all other lengths of the ribbon graph constant.

In other words, as we increase \( t \), the sides of the cycle ‘zip up’ by a distance of \( t/2 \) on each side, gluing more of the cells neighboring the cycle and reducing the circumference of the cycle by \( Nt \).

It is obvious from continuity that if none of the \( c_i^0 \) is equal to \( T \), the family extends to \( t = T \). But also if, say, \( c_1 = T \), we can extend the family with a differential over \( t = T \) that has \((n - 1)\) zeros on the cycle; as long as at least one of the \( c_i \) is bigger than \( T \), this still gives a continuous family of quadratic differentials over \([0, T]\).

We can iterate this process, shrinking the perimeter while decreasing \( n \) accordingly, until we end up with some differential where all the \( c_i \) are equal to some \( T \). Then we can complete this family by a meromorphic Strebel differential with one less double pole. This gives a continuous family valued in the locus of quadratic differentials with a bounded above number of higher-order poles.

**Definition 49.** A perimeter-shrinking family of meromorphic Strebel differentials is a sequence of completed partial perimeter-shrinking families \( \varphi_t^i, 1 \leq i \leq M \), each
over some interval \([0, T_i]\), such that
\[
\varphi_{t=T_i}^{t=0} = \varphi_{t+1}^{t=0}
\]
and such that in the last family, \(\varphi_{T_M}^{t=0}\) has one less double pole than \(\varphi_1^{t=0}\).

**Lemma 55.** Such a perimeter-shrinking family from \(\varphi\) to \(\varphi'\) gives a map of metric ribbon graphs \(\Gamma \to \Gamma'\) between the corresponding metric ribbon graphs.

**Proof.** We can construct this map by hand on each partial family, by defining it on the edges with lengths \(c_i\) and \(d_i\) by the zipping description. One can easily check that this map extends to a the endpoint of such a family. This uniquely defines the map, since the other edges of \(\Gamma\) are kept constant.

\(\square\)

It follows from the definitions that once we pick a starting meromorphic Strebel differential \(\varphi\) on some surface \(S\) and one of its double poles \(p\), there is a unique family shrinking the corresponding perimeter. At the end of such a family we have a Riemann surface \(S'\) and a differential \(\varphi'\) with one less double pole. Let \(\Gamma'\) denote the ribbon graph associated to it.

Consider now starting from a differential with marked outputs \(\varphi \in \mathcal{Q}(g, \{n_i\})_{m\text{Str}}\) as in Definition 47. Let \(p\) be a marked output (i.e., double pole of \(\varphi\)) and \(q \in S_{p}^1\) the point giving its marked direction.

**Lemma 56.** The data of a perimeter-shrinking family from \(\varphi = \varphi^0\) to \(\varphi'\) gives a distinguished point \(p'\) in \(\Gamma\), coming from the vanishing double pole, and either a marked edge of \(\Gamma\) incident at \(p'\), or a marked angle at \(p'\), coming from the marked direction \(q\).

**Proof.** Such a perimeter-shrinking family is made of a sequence of partial perimeter-shrinking families; let \(\varphi_{T_M}^{t=0} = \varphi'\). At the start of such a family we have a differential \(\varphi'' = \varphi_{T_M}^{0}\) with some number \(m\) of zeros around the relevant double pole, such that all the lengths \(c_i\) between them are equal to \(T_M\).

We take the image \(q''\) of the point \(q\) in \(\Gamma''\). Let us take the distance in \(\Gamma''\) between \(q''\) and each of the \(m\) zeros around that pole. By construction, there are only two possibilities:

1. There is a unique minimum among those distances, for some zero labeled by \(j\), or
2. There are two consecutive zeros, labeled by \(j\) and \(j+1\), which are equidistant to \(q''\).

Now we produce the markings. The point \(p'\) is defined to be the image of the circle in the graph \(\Gamma'\); this is either a simple pole (if \(m = 1\)), a regular point (if \(m = 2\)) or a zero of order \(m - 2\) (if \(m \geq 3\)). As for the direction, in case 1 above we pick the edge incident at \(p'\) corresponding to the \(j\)th zero, and in case 2 we pick the angle between the edges corresponding to \(j\) and \(j + 1\).

\(\square\)

Consider now such a differential with marked outputs \(\varphi \in \mathcal{Q}(g, \{n_i\})_{m\text{Str}}\) as in Definition 47. Let us pick any linear ordering of the \(\ell\) marked outputs. Doing the procedure above for each of the double poles in order results in a sequence \((S_i, \varphi^i, \{\text{markings}\}^i)\) where \(S^0 = S\), \(\varphi^{i+1}\) has one less double pole than \(\varphi^i\) and there are \(i\) markings (i.e. points in the ribbon graph with a distinguished incident direction or angle).
Figure 6. A perimeter-shrinking family for the double pole $p$ with a generic position of its marking $q$. The end result is a differential with one less double pole; the image of $p$ is $p'$ and its distinguished edge is the top one, as it leads to the image of $q$, denoted $q'$.

Figure 7. A perimeter-shrinking family for the double pole $p$ with a non-generic position of its marking $q$. If the images $p', q'$ coincide, we then mark the angle inside of which the image of $q$ is before the last partial perimeter shrinking.

After contracting the perimeters of all the half-infinite cylinders corresponding to the $\ell$ marked double poles, we get a metric ribbon graph $\Gamma$, with $\ell$ marked points, each with a marked direction (either incident half-edge or angle between half-edges). All of the internal edges of $\Gamma$ have finite length, but some of its leaves (corresponding to the higher-order poles) have infinite length. This data satisfies the conditions

(1) The marked points $p_i$ are all distinct.
(2) Every leaf of $\Gamma$ with finite length ends in a marked point (this comes from a simple pole after contracting perimeters of half-infinite cylinders)

Let $\Gamma$ be such a metric ribbon graph with $\ell$ marked points $p_i$, some number $e$ of marked incident half-edges and $(\ell - e)$ marked angles. Let us denote the set of such data as $\text{MetGr}_\ell$. We now make a larger set $\text{MetRG}'_\ell$ by replacing every element in $\text{MetRG}_\ell$ by the set $\mathbb{R}^e$. 
This replacement accounts for the fact that from the perimeter-shrinking map, for each time we ended up with a marked half-edge, there was a whole angular sector on which our marked point around the double pole could have been, whereas for each marked angle there was a single such point.

We now partition the set $\text{MetRG}'$ by genus and number of infinite-length leaves on each boundary component of the corresponding surface. Let $(m_i), \ 1 \leq i \leq N$ denote that tuple of numbers, with $m_i \geq 0$. We then have a partition

$$\text{MetRG}'_\ell = \bigsqcup_{g,(m_i)} \text{MetRG}'_{\ell,g,(m_i)}$$

We then make another tuple $(n_i)$, of length $N + \ell$, by setting

$$n_i = m_i + 2, \ 0 \leq i \leq N, \text{ and } n_i = 2, \ N + 1 \leq i \leq N + \ell$$

The tuple $(n_i)$ is the orders of poles before perimeter-shrinking.

**Corollary 57.** Perimeter-shrinking gives a bijection of sets

$$\mathcal{Q}(g,(n_i))_{\ell}^{\text{mStr}} \xrightarrow{\sim} \text{MetRG}'_{\ell,g,(m_i)} \times \mathbb{R}^\ell$$

Therefore we can endow $\text{MetRG}'_{\ell,g,(m_i)}$ with the action of the corresponding mapping class group $\text{Mod}_{g,(m_i)}$, acting trivially on the $\mathbb{R}^\ell$ component, and with the finest topology for which the map above is a homeomorphism, so that the quotient

$$\mathcal{M}_{\ell,g,(m_i)} := \left[\text{MetRG}'_{\ell,g,(m_i)}/\text{Mod}_{g,(m_i)}^{\text{oc}}\right]$$

is an orbifold classifying space for the open-closed mapping class group.

### 7.5. Cell decomposition by marked ribbon quivers

We now use the model discussed in the previous Subsection to produce a cell decomposition of the open-closed moduli space, and relate it to the dg PROP constructed in Section 6.

Let us paraphrase the result of the previous section. We choose a genus $g$ and a boundary type of $\Sigma$: each boundary component is either incoming closed, outgoing closed, a free circle, or a combination of incoming open intervals, outgoing open intervals and free intervals.

The number of outgoing closed boundaries is the integer $\ell \geq 1$ and the rest of the boundaries determines the tuple $\{m_i\}$. Following the perimeter-shrinking map in the previous subsection gives a space $\mathcal{M}'_{\ell,g,(m_i)}$: each point in this space is given by $(S, \varphi, \{\vec{v}_i\}_{1 \leq i \leq \ell})$ where $S$ is a Riemann surface diffeomorphic to a compactification of $\Sigma$, $\varphi$ is a meromorphic Strebel differential with critical graph $\Gamma$, and $\vec{v}_i$ are unit length tangent vectors, projecting down to points $p_i \in \Gamma$.

Each such vector $\vec{v}_i$ corresponds to one outgoing closed boundary, and encodes a distinguished direction; the quadratic differential $\varphi$ has poles of order $\geq 2$ corresponding to the other boundary components of $\Sigma$. Namely, it has poles of:

- order 2 for each free boundary circle,
- order 3 for each incoming closed circle,
- order $n + 2$ for each boundary component with $n$ open intervals (incoming and outgoing)

Each pole of order 3 has a single edge of $\Gamma$ asymptotic to it, and each pole of order $n + 2$ has $n$ such edges. The data of such a differential is encoded by the metric graph $\Gamma$. We now use the marking data $\{\vec{v}_i\}$ to give $\Gamma$ the structure of a *marked ribbon quiver* as defined in Section 6.1.1.
For that, we need to make a small modification to $\Gamma$ to properly deal with the open outputs. Recall that every open in/output corresponds to an infinite-length edge of $\Gamma$, asymptotic to a higher-order pole.

Let $|V_{\text{open-out}}|$ be the number of open outputs. We now pick a tuple of positive reals $(\lambda_1, \lambda_2, \ldots) \in (\mathbb{R}_{>0})^{|V_{\text{open-out}}|}$ and regularize $\Gamma$ by setting the length of the corresponding edge to be given by these numbers $\lambda_i$.

We now use the marking to define a height function: consider the subset $P \subset \Gamma$ consisting of all the $\ell$ points $p_i$ corresponding to the markings $\vec{v}_i$, together with all the end points of the regularized edges. That is, $P$ has one point for each output, open or closed.

**Definition 50.** The height function $h$ on the metric graph $\Gamma$ is the distance along $\Gamma$ to the set $P$. We define the ribbon quiver $\vec{\Gamma}$ to be the ribbon graph $\Gamma$ directed with the negative gradient of this height function.

The regularization procedure guarantees that the outgoing open edges always have directions pointing into their corresponding sink.

We then use the rest of the data to produce a marking on the ribbon quiver, in the sense of Section 6.1.1, as follows

- $V_x$ are the valence one sources (corresponding to poles of order 3) associated to incoming closed boundaries,
- $V_{\text{open-in}}, V_{\text{open-out}}$ are the valence one sources, resp. sinks associated to the other higher order poles,
- $V_o$ is the set of marked points $\{p_i\}$ in $\Gamma$,
- for each point in $V_o$, if after perimeter-shrinking we ended up with a marked angle, we attach an extra leaf

in that angle.

The procedure above uniquely defines a marked ribbon quiver, and it stratifies the space

$$\mathcal{M}_{e,g,(m_i)} \times (\mathbb{R}_{>0})^{|V_{\text{open-out}}|}$$

by the type of marked ribbon quiver obtained; this gives a cell decomposition of this space.

**Remark.** Note that the quiver obtained generically will have more vertices than the ribbon graph; there will be some number of valence two sources, one for each maximum of the height function occurring inside an edge of the graph.

7.5.1. **Dimensions and orientations.** We now discuss the dimensions and orientations of the cells labeled by marked ribbon quivers.

The dimension of the Teichmüller space of a genus $g$ surface with $k$ punctures is given by $6g - 6 + 2k$. We considered instead the moduli space $\mathcal{T}_{g,k}$ where each puncture is decorated by a number of tangent directions depending on the boundary type; in terms of those boundary types, we have that

$$\dim \mathcal{T}_{g,k} = 6g - 6 + 2F + 3|V_x| + 2O + |V_{\text{open-in}}| + |V_{\text{open-out}}| + 3|V_o|$$

where $F$ is the number of free boundaries (i.e. boundary components of $\Gamma$ without marked sources or sinks) and $O$ is the number of boundary components with open-in and open-out intervals. We get this formula from the dimension of the punctured
Teichmüller space since every ×-source, open-in, open-out, o-sink contributes one extra dimension (by picking a direction at the corresponding puncture).

We then had the space \( Q_{g, (n_i)}^{\text{St}}|_{V_x} \) of meromorphic Strebel differentials, before shrinking perimeters; this was a \((\mathbb{R}_{>0})^k\) bundle over \( T_{g, k} \) so its dimension is

\[
\dim T_{g, k} = 6g - 6 + 3F + 4|V_x| + 3O + |V_{\text{open-in}}| + |V_{\text{open-out}}| + 4|V_o|
\]

and after perimeter-shrinking we have the space \( \text{MetRG}_{\ell, g, (m_i)} \) and its quotient, our desired moduli space \( M = M_{g, (m_i), |V_x|} \), which by Corollary 57 has dimension

\[
\dim M = 6g - 6 + 3F + 4|V_x| + 3O + |V_{\text{open-in}}| + |V_{\text{open-out}}| + 3|V_o|
\]

Recall now the definition of the \( d \)-degree of a marked ribbon quiver, from Definition 33:

\[
\deg_d(\vec{\Gamma}) = \sum_{v \in \text{Source} \geq 2} ((2-d)\text{out}(v) + d - 4) + \sum_{v \in \text{Flow}} ((2-d)\text{out}(v) + d + \text{in}(v) - 4) + \sum_{v \in V_o} (\text{in}(v) - 1).
\]

Calculating this for \( d = 0 \) gives

\[
\deg_0(\vec{\Gamma}) = \sum_{v \in \text{Source} \geq 2} (2\text{out}(v) - 4) + \sum_{v \in \text{Flow}} (2\text{out}(v) + \text{in}(v) - 4) + \sum_{v \in V_o} (\text{in}(v) - 1),
\]

so \( \vec{\Gamma} \) has \( \deg_0(\vec{\Gamma}) = 0 \) when it has only valence 2 unmarked sources, valence 3 flow vertices and valence 1 o-vertices. Such marked ribbon quivers are exactly the ones labeling a top-dimensional cell in the decomposition of \( M' \times (\mathbb{R}_{>0})^{|V_{\text{open-out}}|} \) that we described; other marked ribbon quivers label cells with (real) codimension given by \( \deg_0 \).

**Lemma 58.** The (real) dimension of a cell in \( M' \times (\mathbb{R}_{>0})^{|V_{\text{open-out}}|} \) labeled by a marked ribbon quiver \( \vec{\Gamma} \) is given by

\[
\dim_{\mathbb{R}} C_{\vec{\Gamma}} = |V(\Gamma)| - |V_x| - |V_{\text{open-in}}| + |V_o| - |V_1| - 1.
\]

**Proof.** Recall that each point in this cell can be given by a marked ribbon quiver with a metric; however, we are not free to choose the lengths of all the edges independently, since the lengths all come from differences of values of the height function.

Given any two vertices, the length of any directed path between them is the same, given by the difference of their heights no matter the path. Conversely, fixing all height differences fixes all the edge lengths. Note that some of the vertices are at infinite height, namely the marked sources \( |V_x| \) and \( |V_{\text{open-in}}| \).

Thus the space of possible edge lengths has dimension given by the number of vertices at finite height minus one. Besides the edge lengths, recall that for every o-sink with distinguished incident edge we added a factor of \( \mathbb{R} \), and for every o-sink with distinguished angle we replaced that angle with a 1-vertex. Counting all these factors we get the formula above. \( \square \)

We are now ready to relate this space to the props \( Q^d \) constructed in Section 6.1.1. Note that here it is important that we assume \( \mathbb{Q} \subseteq \mathbb{k} \).

**Theorem 59.** The complexes \( Q_{g=0}^d \) with their differential \( \partial \) calculate the homology \( H_{-\infty}(\mathcal{M}, \mathbb{k}) \) of the corresponding classifying spaces \( \mathcal{M} \) for the open-closed mapping class groups.
Proof. We basically follow the argument for the analogous statement about the usual graph complex, see [Pen86; CV03; Kon93]. With the degree conventions we set, it will be easier to work cohomology so we use Poincaré duality; since the spaces $\mathcal{M}$ are not compact we must use an appropriate form of Poincaré duality by working in cohomology with compact supports $H^*_c(\mathcal{M}, k)$ instead.

We now construct a compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$, by using the cell decomposition by ribbon graphs, and adding cells where the length of any number of edges of finite length goes to zero or $\infty$. Each cell of $\overline{\mathcal{M}}$ is labeled by a ribbon graph with edge lengths $\in [0, +\infty]$; the (homological) boundary differential is still given by contracting some number of the finite lengths. We note now that the boundary $\partial \overline{\mathcal{M}}$ is a subcomplex of the cell complex for $\mathcal{M}$; one cannot get rid of an infinite-length edge by contracting a finite-length edge, and if in $\Gamma$ a cycle has length zero, it also has length zero in any edge contraction of $\Gamma$. We thus see that the cell complex of ribbon graphs we constructed for $\overline{\mathcal{M}}$ computes the compactly supported rational cohomology $H^*_c(\mathcal{M}, \mathbb{Q})$.

To check orientations, it will be more straightforward to deal with vertex orderings instead of edge orderings; the correspondence between those is given in Section 6.2.3. The terms of the differential not involving $1$-vertices are given by dual of the (homological) differential that sums over contractions of edges; such contractions correspond to making the difference in height $h(a) - h(b)$ between the source and target vertices go to zero. When this happens we delete one of the vertices (say $a$); recall that in the definition of $\partial$ the induced orientation was given by putting $a$ at the start, so $\partial$ is indeed the dual of the cell boundary differential.

As for the terms involving $V_1$, these can be understood by a local calculation as the $\circ$-vertices they are attached to are not allowed to collide; we check that the differential $\partial$ agrees with the required orientations. □

Example. We take the surface of genus zero with two closed inputs and one closed output; each input and output is a puncture with a distinguished tangent direction. The corresponding Teichmüller space $T_{0,3}$ is homeomorphic to $(S^1)^3$. Therefore the space $Q(0, \{3, 3, 2\})_{l=1}^{n_{\text{Str}}}$ of meromorphic Strebel differentials has (real) dimension 6, and the modified space of metric graphs $\text{MetRG}'$ has dimension 5. Therefore our moduli space has $\dim_{\mathbb{R}} \mathcal{M} = 5$, and is homeomorphic to $(S^1)^3 \times (\mathbb{R}_{\geq 0})^2$, retracting to the 3-torus. The following 1-cycles (given by sums of diagrams with some appropriate orientation) map to three 1-cycles spanning $H_1(\mathcal{M}, \mathbb{Z}) \cong \mathbb{Z}^3$:

\begin{align*}
C_1 &= \begin{array}{c}
\begin{array}{c}
\circ \\
\times
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times \\
\circ
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\circ \\
\times
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times \\
\circ
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\circ \\
\times
\end{array}
\end{array}
\end{align*}

\begin{align*}
C_2 &= \begin{array}{c}
\begin{array}{c}
\times \\
\circ
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\circ \\
\times
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times \\
\circ
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\circ \\
\times
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times \\
\circ
\end{array}
\end{array}
\end{align*}

\begin{align*}
C_3 &= \begin{array}{c}
\begin{array}{c}
\circ \\
\times
\end{array}
\end{array}
\end{align*}
7.5.2. The determinant line bundle. Finally, let us discuss the meaning of the integer \( d \) in terms of the geometry of the moduli spaces \( \mathcal{M} \).

For that we calculate the shift in the degrees for some dimension \( d \) and dimension zero. Let us take a marked ribbon quiver \( \vec{\Gamma} \) whose unmarked sources are all of valence 2; let \( N \) be the number of those sources. Calculating the degrees, we find that the difference between the degrees for \( d \) and zero is:

\[
\deg_d(\vec{\Gamma}) - \deg_0(\vec{\Gamma}) = dN
\]

**Proposition 60.** The number \( N \) is equal for all such graphs with a fixed genus and boundary type, and is given by

\[
N = 2g - 2 + |V_x| + |V_o| + O + F + |V_{\text{open-out}}| = -\chi(\Sigma_{\text{OC}}) + |V_{\text{open-out}}|
\]

where \( O \) is again the number of boundary circles with open intervals, \( F \) is the number of free boundary circles, and \( \chi(\Sigma_{\text{OC}}) \) is the Euler characteristic of the corresponding open-closed surface.

**Proof.** The first statement follows from the fact that the space \( \mathcal{M} \) is connected for each fixed genus and boundary type, and the fact that any two graphs with all unmarked sources of valence 2 are related by contracting and expanding edges.

As for the formula, it is enough to construct one such quiver; we can produce a quiver with \( 2g \) such sources from a polygon with \( 4g \) edges, one \( \text{\times} \)-vertex and one \( \text{\circ} \)-vertex. One then sees by construction that each other source in \( V_x \), each other sink and each boundary component with open-ins and -outs contributes another valence 2 unmarked source.

The lemma above implies that

\[
Q^d = H^{\dim \mathcal{M} - \ast}_c(\mathcal{M}, \mathcal{L}^\otimes d)
\]

for a line bundle \( \mathcal{L} \) over \( \mathcal{M} \), with a shift of \(-\chi(\Sigma_{\text{OC}}) + |V_{\text{open-out}}|\). This agrees with the description given by Costello in [Cos07]; the action of the open-closed PROP in degree \( d \) should be twisted by \( \mathcal{L} = \text{det} \) which is a certain determinant line bundle on the open-closed moduli space. The argument in Section 6.2.3 can thus be seen as a combinatorial proof that, in the case where there are no free boundary circles, the determinant line bundle is trivial up to shift, a claim which already appears in *op. cit.*

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