Differential games and Hamilton–Jacobi equations in the Heisenberg group

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Abstract

The purpose of this work is twofold. First we study the solutions of a Hamilton–Jacobi equation of the form

$$
\frac{\partial u}{\partial t}(t,x) + H(t,x,\nabla_H u(t,x)) = 0,
$$

where $\nabla_H u$ represents the horizontal gradient of a function $u$ defined on the Heisenberg group $\mathbb{H}$. Motivated by [12], we prove a Lipschitz continuity preserving property for $u$ with respect to the Korányi homogeneous distances $d_G$ in $\mathbb{H}$. Secondly, we are keenly interested in introducing the game theory in $\mathbb{H}$, taking into account its Sub–Riemannian structure: inspired by [9] and [1], we prove $d_G$-Lipschitz regularity results for the lower and the upper value functions of a zero game with horizontal curves as its trajectories, and we study the Hamilton–Jacobi–Isaacs equations associated to such zero game. As a consequence, we also provide a representation of the viscosity solution of the initial Hamilton–Jacobi equation.

Keywords: Heisenberg group; game theory; Hamilton–Jacobi–Isaacs equation; viscosity solution.

MSC: 35R03; 49L20; 91A25

1 Introduction

The first aim of this paper is to study the properties of the viscosity solutions of the Hamilton–Jacobi equation in the Heisenberg group $\mathbb{H}$

$$
\begin{align*}
\frac{\partial u}{\partial t}(t,x) + H(t,x,\nabla_H u(t,x)) &= 0 & \text{in } (0,T) \times \mathbb{H} \\
\quad u(0,x) &= g(x) & \text{in } \mathbb{H},
\end{align*}
$$

where $\nabla_H u$ is the horizontal gradient of the function $u$, and $H$ and $g$ are bounded functions satisfying suitable assumptions (see 3. and 4.); in particular, they are Lipschitz continuous in $x$ w.r.t. the left–invariant Korányi distance $d_G$.

This study is motivated by the work of Liu, Manfredi and Zhou [12]; however, our approach is different. It is well known that the study of the Hamilton–Jacobi equations is strictly related to the game theory; the pioneer of this approach was Isaacs [10]. Several authors like Evans, Souganidis, Bardi, Lions have connected the Isaacs theory with the notion of viscosity solution (see [9], [3] [11]). Along this line of investigation, and since we are interested in embedded the game theory in the Sub–Riemannian structure of the

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We emphasize that in assumptions 2., 3. and 4. a \(d_G\)-Lipschitz condition is considered, where the mentioned Korányi–gauge metric \(d_G\) is a natural metric in \(\mathcal{H}\) that turns out to be equivalent to the Carnot–Carathéodory metric \(d_{CC}\). If we replace these \(d_G\)-Lipschitz properties with the Euclidean Lipschitz requirements, we know that the lower \(V^-\) and the upper \(V^+\) value functions for the game (2) (see Definition 2.2) are Euclidean Lipschitz (see Remark 3.1). Our result is indeed more precise and in the spirit of proving Lipschitz preserving properties as in [12].
Theorem 1.1 (\(d_G\)-Lipschitz continuity preserving properties for \(V^-\)) Let us consider the zero game (2) with the assumptions 1., 2. and 3.. Then its lower value function \(V^-\) is bounded and uniformly Lipschitz continuous w.r.t. the metric \(d_G\), i.e. there exists a constant \(C'\) such that

\[
|V^-(t, x) - V^-(t', x')| \leq C' \left( |t - t'| + d_G(x, x') \right),
\]

for every \(t, t' \in [0, T]\) and \(x, x' \in \mathcal{H}\). A similar result holds for \(V^+\).

The second important result of the paper is the following:

Theorem 1.2 (\(V^-\) as viscosity solution) Let us consider the zero game (2) with the assumptions 1., 2. and 3.. Then, the lower value function \(V^-\) is a viscosity solution of the lower Hamilton–Jacobi–Isaacs equation

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) + H^-(t, x, \nabla_H u(t, x)) &= 0 \quad \text{for } (t, x) \in (0, T) \times \mathcal{H} \\
u(T, x) &= g(x) \quad \text{for } x \in \mathcal{H},
\end{aligned}
\]

(3)

where \(H^-\) is the lower Hamiltonian for the game (2).

The lower Hamiltonian \(H^-\) is defined as a maxmin-function (see Definition 4.2). Clearly, a similar result holds for \(V^+\). The proofs of Theorem 1.1 and Theorem 1.2 require a fine use of the horizontal curves and their properties in \(\mathcal{H}\).

A precise estimate of the \(d_G\)-Lipschitz constant for \(V^-\) in Theorem 1.1 allows us to provide a representation of the viscosity solution \(u\) for the initial problem (1) as a value function (see Theorem 5.1): as a consequence of the previous results, the mentioned \(d_G\)-Lipschitz assumptions in 3. and 4. for the functions \(H\) and \(g\) involved in (1) is inherited by \(u\), that turns out to be \(d_G\)-Lipschitz.

The paper is organized as follows: in Section 2 we introduce and connect the fundamental notions in the Heisenberg group and in the game theory; moreover, we prove some fine properties of the horizontal curves in \(\mathcal{H}\). Section 3 and Section 4 are essentially devoted to the proofs of Theorem 1.1 and Theorem 1.2 respectively. In Section 5 we study problem (1) and raise an open question related to the Hopf–Lax formula.

2 Preliminaries.

2.1 A short introduction on the Heisenberg group \(\mathcal{H}\)

The Heisenberg group \(\mathcal{H}\) is \(\mathbb{R}^3\) endowed with a non–commutative law \(\circ\): it is the Lie group whose Lie algebra \(\mathfrak{h}\) admits a stratification of step 2; in particular \(\mathfrak{h} = \mathbb{R}^3 = V_1 \oplus V_2\), with

\[
\begin{aligned}
V_1 &= \text{span}\{X_1, X_2\} \quad \text{with } X_1 = \frac{x_2}{2} \partial_{x_3} - \frac{x_3}{2} \partial_{x_2} \quad \text{and } X_2 = \partial_{x_2} + \frac{x_2}{2} \partial_{x_3}, \\
V_2 &= \text{span}\{X_3\} \quad \text{with } X_3 = \partial_{x_3}.
\end{aligned}
\]

(4)

The bracket \([\cdot, \cdot]: \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}\) is defined as \([X_1, X_2] = X_3\), and it vanishes for all the other basis vectors. For a sufficiently regular function \(f : \mathcal{H} \to \mathbb{R}\), we define the horizontal gradient \(\nabla_H f\) by

\[
\nabla_H f(x) = (X_1 f(x), X_2 f(x)) \quad \text{with } x \in \mathcal{H};
\]

for our purpose it is convenient to think \(\nabla_H f(x)\) as a vector in \(\mathbb{R}^2\). We say that such \(f\) is in \(\Gamma^1(\mathcal{H})\) if its horizontal derivatives \(X_1 f\) and \(X_2 f\) are continuous functions.
The group law is defined by the relation
\[(x_1, x_2, x_3) \circ (x'_1, x'_2, x'_3) = (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + (x_1 x'_2 - x'_1 x_2)/2).\]

Consequently, the null element is \(e = (0, 0, 0)\) and \((x_1, x_2, x_3)^{-1} = (-x_1, -x_2, -x_3)\). The dilation is a family of automorphisms given by \(\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)\), and hence the homogeneous dimension is 4.

We denote by \(\| \cdot \| \) the Euclidean norm in \(\mathbb{R}^n\) and by \(d_E\) the Euclidean distance. In contrast with Analysis in Euclidean spaces, where the Euclidean distance is the most natural choice, in the Heisenberg group several distances have been introduced for different purposes (for example, see \(d_{CC}\) below). The Korányi gauge \(\| \cdot \|_G\), defined by
\[
\| (x_1, x_2, x_3) \|_G = \left( (x_1^2 + x_2^2 + x_3^2)^{1/4} \right) \quad \forall x = (x_1, x_2, x_3) \in \mathbb{H},
\]
allows us to introduce the left invariant metric \(d_G\) via \(d_G(x, x') = \| (x')^{-1} \circ x \|_G\). This Korányi distance \(d_G\) is homogeneous, namely, it is continuous, left invariant and behaves well respect to the dilations \(\delta_\lambda\). For our purpose it is important to mention that for every compact set \(\Omega \subset \mathbb{H}\) there exists a constant \(C = C(\Omega)\) such that (see for example [5])
\[
\frac{1}{C} \| x \| \leq \| x \|_G \leq C \| x \|^{1/2} \quad \forall x \in \Omega.
\]

Given a metric \(d\) in \(\mathbb{H}\), we say that a function \(f : \mathbb{H} \rightarrow \mathbb{R}\) is Lipschitz w.r.t. the metric \(d\) (or shortly is \(d\)-Lipschitz) it there exists a constant \(C\) such that
\[
|f(x) - f(x')| \leq C d(x, x') \quad \forall x, x' \in \mathbb{H}.
\]

In order to emphasize the dependence on the distance \(d\), we stress that the function \(f(x) = \| x \|_G\) is \(d_G\)-Lipschitz, but it is not \(d_E\)-Lipschitz. On the other hand, by (6), it is clear that every \(d_E\)-Lipschitz function is \(d_G\)-Lipschitz.

### 2.2 Horizontal curves in \(\mathbb{H}\)

Let us start with this fundamental notion:

**Definition 2.1 (horizontal curve)** A horizontal curve \(x = (x_1, x_2, x_3) : [a, b] \rightarrow \mathbb{H}\), with \([a, b] \subset \mathbb{R}\), is an absolutely continuous function a.e. tangent to horizontal directions, i.e.
\[
\dot{x}(s) = \dot{x}_1(s) X_1(x(s)) + \dot{x}_2(s) X_2(x(s)) \quad \text{a.e. } s \in [a, b].
\]

Equivalently, \(x\) is horizontal if
\[
\dot{x}_3 = (x_1 \dot{x}_2 - x_2 \dot{x}_1)/2 \quad \text{a.e. in } [a, b],
\]
that is \(\dot{x} = f^\mathbb{H}(x, z)\), for some measurable function \(z : \mathbb{R} \rightarrow \mathbb{R}^2\), where \(f^\mathbb{H} : \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}^3\) is given by
\[
f^\mathbb{H}(x, z) = \begin{pmatrix} z_1 \\ z_2 \\ z_2 x_1 - z_1 x_2/2 \end{pmatrix} \quad \forall z \in \mathbb{R}^2, x \in \mathbb{H}.
\]

We say that \(x\) is a horizontal curve on \([a, b] \subset \mathbb{R}\) with horizontal velocity \(z\) and initial point \(\xi \in \mathbb{H}\) if
\[
x(t) = \xi + \int_a^t f^\mathbb{H}(x(s), z(s)) \, ds \quad \forall s \in [a, b].
\]
It is well-known that the Carnot–Carathéodory metric \( d \) is easy to see that, for a given \( z \), the curves \( x \) and \( \tilde{x} \) defined by

\[
\begin{align*}
\dot{x} &= -f^H(x, z) & \text{a.e. in } [a, b] \\
x(a) &= \xi & \text{a.e. in } [a, b]
\end{align*}
\]

are the same object, i.e. the horizontal curve on \([a, b]\) with initial point \( x(0) = \xi \) and final point \( x(1) = \xi' \). Chow’s theorem (see, for example, [7]) guarantees that \( \Gamma(\xi, \xi') \neq \emptyset \) and allows us to introduce the Carnot–Carathéodory metric \( d_{CC} \) by the formula

\[
d_{CC}(\xi, \xi') = \inf_{x \in \Gamma(\xi, \xi')} \int_0^1 \| (\dot{x}_1(s), \dot{x}_2(s)) \| \, ds.
\]

It is well-known that the Carnot–Carathéodory metric \( d_{CC} \) and the gauge metric \( d_G \) are bi-Lipschitz equivalent.

The following property is known and can be easily proved:

**Remark 2.1** Let \( F : \mathbb{H} \to \mathbb{R} \) be a function in \( \Gamma^1 \), and let \( x = (x_1, x_2, x_3) : [0, T] \to \mathbb{H} \) be a horizontal curve. Then, for a.e. \( s \in (0, T) \),

\[
\frac{dF(x(s))}{ds} = (\dot{x}_1(s), \dot{x}_2(s)) : \nabla_H F(x(s)).
\]

The next three propositions are crucial in order to prove our Lipschitz preserving property for the value functions. For every fixed \( \tau \in [0, T] \), let us introduce the set of controls at time \( \tau \) for Player I as \( \mathcal{Z}(\tau) = \{ z : [\tau, T] \to Z \subset \mathbb{R}^2 \text{, measurable} \} \).

**Proposition 2.1** Let assumption 1 be satisfied. Let \( \tau \) be fixed in \([0, T] \), and let us consider a horizontal curve \( x = (x_1, x_2, x_3) : [\tau, T] \to \mathbb{H} \) with horizontal velocity \( z = (z_1, z_2) \in \mathcal{Z}(\tau) \) and initial point \( \xi \in \mathbb{H} \), i.e.

\[
\begin{align*}
\dot{x} &= f^H(x, z) & \text{a.e. in } [\tau, T] \\
x(\tau) &= \xi
\end{align*}
\]

Then

\[
d_G(\xi, x(t)) \leq 3R_Z(t - \tau) \quad \forall t \in [\tau, T]
\]

**Proof** The assertion \([10]\) comes from

\[
\begin{align*}
| (\xi^{-1} \circ x(t))_i | & \leq | x_i(t) - \xi_i | = \int_{\tau}^t |\dot{x}_i(s)| \, ds \leq (t - \tau) R_Z, \quad i = 1, 2 \\
| (\xi^{-1} \circ x(t))_3 | & = \left| -\xi_3 + x_3(t) + \frac{1}{2} ( -\xi_1 x_2(t) + \xi_2 x_1(t) ) \right| \\
& = \frac{1}{2} \left| \int_{\tau}^t [ z_2(s)(x_1(s) - \xi_1) - z_1(s)(x_2(s) - \xi_2) ] \, ds \right| \\
& \leq \frac{1}{2} \int_{\tau}^t | (x_1(s) - \xi_1, x_2(s) - \xi_2, z_2(s)) \cdot (z_2(s), -z_1(s)) | \, ds \\
& \leq (t - \tau)^2 R_Z^2.
\]
Since \( \|\xi^{-1} \circ x(t)\|_G = d_G(\xi, x(t)) \), by (5) we have the claim.

**Proposition 2.2** Let us suppose that assumption 1. holds, and let \( x \) be a horizontal curve as in Proposition 2.1. Let \( \xi \) be fixed in \( H \) and let us consider the horizontal curve \( \tilde{x} \) on \([\tau, T]\) with horizontal velocity \( z \), i.e.

\[
\begin{cases}
\dot{x} = f_H(\tilde{x}, z) \text{ in } [\tau, T] \\
\tilde{x}(\tau) = \xi
\end{cases}
\]

(11)

Then there exists a constant \( \tilde{C} \) that depends only on \( R_Z \) and \( T \) such that

\[
d_G(\xi, x(t), \tilde{x}(t)) \leq \tilde{C} d_G(\xi, \tilde{x}) \quad \forall t \in [\tau, T].
\]

(12)

**Proof** Let us define \( \phi : [\tau, T] \to [0, \infty) \) by \( \phi(t) = d_G(\xi, \tilde{x}(t)) \). Then

\[
\phi(t) = \left\| (\tilde{x}(t))^{-1} \circ x(t) \right\|_G
\]

\[
= \left\| \left(-\tilde{x}_1(t) + x_1(t), -\tilde{x}_2(t) + x_2(t), -\tilde{x}_3(t) + x_3(t) - \frac{1}{2} (\tilde{x}_1(t)x_2(t) - \tilde{x}_2(t)x_1(t)) \right) \right\|_G
\]

\[
= \left\| \left(-\tilde{\xi}_1 - \int_{\tau}^{t} z_1(s) \, ds + \xi_1 + \int_{\tau}^{t} z_1(s) \, ds,
-\tilde{\xi}_2 - \int_{\tau}^{t} z_2(s) \, ds + \xi_2 + \int_{\tau}^{t} z_2(s) \, ds,
-\tilde{\xi}_3 - \frac{1}{2} \int_{\tau}^{t} (\tilde{x}_1(s)z_2(s) - \tilde{x}_2(s)z_1(s)) \, ds + \xi_3 + \frac{1}{2} \int_{\tau}^{t} (x_1(s)z_2(s) - x_2(s)z_1(s)) \, ds + \frac{1}{2} \left[ \left( \tilde{\xi}_1 + \int_{\tau}^{t} z_1(s) \, ds \right) \left( \tilde{\xi}_2 + \int_{\tau}^{t} z_2(s) \, ds \right) + \left( \tilde{\xi}_2 + \int_{\tau}^{t} z_2(s) \, ds \right) \left( \tilde{\xi}_1 + \int_{\tau}^{t} z_1(s) \, ds \right) \right] \right\|_G
\]

\[
= \left\| \left(\xi_1 - \tilde{\xi}_1, \xi_2 - \tilde{\xi}_2, \xi_3 - \tilde{\xi}_3 + \frac{1}{2} (\xi_1 - \tilde{\xi}_1, \xi_2 - \tilde{\xi}_2) + \int_{\tau}^{t} ((\xi_1 - \tilde{\xi}_1)z_2(s) - (\xi_2 - \tilde{\xi}_2)z_1(s)) \, ds \right) \right\|_G
\]

By (5) we have, a.e.,

\[
\frac{d \phi(t)}{dt} = \frac{\left(\tilde{x}(t))^{-1} \circ x(t)\right)_3 (\xi_1 - \tilde{\xi}_1)z_2(t) - (\xi_2 - \tilde{\xi}_2)z_1(t)}{2 (\phi(t))^3}
\]

\[
= \frac{\left|\left(\tilde{x}(t))^{-1} \circ x(t)\right)_3 (\xi_1 - \tilde{\xi}_1, \xi_2 - \tilde{\xi}_2)(z_2(t), -z_1(t))\right|}{2 (\phi(t))^2}
\]

\[
\leq \frac{1}{2} \frac{\left\| (\xi_1 - \tilde{\xi}_1, \xi_2 - \tilde{\xi}_2) \right\| \|z(t)\|}{\left\| (\xi_1 - \tilde{\xi}_1, \xi_2 - \tilde{\xi}_2) \right\|}
\]

\[
\leq \frac{R_Z}{2}
\]
Now, by the Gronwall inequality, we obtain
\[ \phi(t) \leq \phi(\tau) \exp \left( \int_{\tau}^{t} \frac{R_{Z}}{2} \, ds \right) \leq \exp \left( T \frac{R_{Z}}{2} \right) d_{G}(\xi, \tilde{\xi}) := \tilde{C} d_{G}(\xi, \tilde{\xi}). \] (13)

Let us note that the curve \( \tilde{x} \) in (11) is exactly a left translation of the first curve \( x \), i.e. \( \tilde{x}(t) = \tilde{\xi} \circ \xi^{-1} \circ x(t) \).

**Proposition 2.3** Let us suppose that assumption 1. is satisfied and let \( x \) be a horizontal curve as in Proposition 2.1. Let \( \xi \in \mathcal{H} \) and \( 0 \leq \tau \leq \tau' \leq T \) be fixed. Let \( \bar{x} \) be the horizontal curve on \([\tau', T]\) with horizontal velocity \( z|_{[\tau', T]} \in \mathcal{Z}(\tau') \) and initial point \( \tilde{\xi} \), i.e.
\[
\begin{cases}
\dot{\bar{x}} = f_{\mathcal{H}}(\bar{x}, z) \text{ in } [\tau', T] \\
\bar{x}(\tau') = \tilde{\xi}.
\end{cases}
\]

Then there exists a constant \( \tilde{C} \) that depends only on \( R_{Z} \) and \( T \) such that
\[ d_{G}(x(t), \bar{x}(t)) \leq \tilde{C} \left( d_{G}(\tilde{\xi}, \xi) + (\tau' - \tau) \right) \quad \forall t \in [\tau', T]. \]

**Proof** By Proposition 2.1, we have \( d_{G}(\xi, x(\tau')) \leq 3R_{Z}(\tau' - \tau) \). Now, it is easy to see that \( \bar{x} \) is a left translation of the curve \( x \) restricted to \([\tau', T]\): more precisely, \( \bar{x}(t) = \tilde{\xi} \circ (x(\tau'))^{-1} \circ x(t) \).

By (12) we have, for every \( t \in [\tau', T] \),
\[
d_{G}(x(t), \bar{x}(t)) \leq \tilde{C} d_{G}(x(\tau'), \bar{x}(\tau'))
\leq \tilde{C} \left( d_{G}(x(\tau'), \xi) + d_{G}(\tilde{\xi}, \xi) \right)
\leq \tilde{C} (1 + 3R_{Z}) \left( d_{G}(\tilde{\xi}, \xi) + (\tau' - \tau) \right)
\leq \tilde{C} \left( d_{G}(\tilde{\xi}, \xi) + (\tau' - \tau) \right) \quad (14)
\]

### 2.3 Differential games in \( \mathcal{H} \)

The game which we are interested in is the following (see (2))
\[
\begin{align*}
\text{Player I: } & \quad \max_{y \in \mathcal{Y}(0)} J(y, z) \\
\text{Player II: } & \quad \min_{z \in \mathcal{Z}(0)} J(y, z) \\
J(y, z) & = \int_{0}^{T} F(t, x, y, z) \, dt + g(x(T)) \\
\dot{x} & = -f_{\mathcal{H}}(x, z) \quad \text{a.e. in } [0, T] \\
x(0) & = x_{0}
\end{align*}
\]
under the assumptions 1., 2. and 3., where \( T > 0 \) and \( x_{0} \in \mathcal{H} \) are fixed.

Three comments on such zero game are required. Following the idea in (11), the dynamics we consider is horizontal: essentially, we consider a game where, for every strategy of the two players, the associated trajectory is a horizontal curve on \( \mathcal{H} \). Secondly, in the dynamics appears a minus (see (8)) whose only reason is to keep consistency with the classical case. Finally, the reader who is not expert in game theory would be surprise by the fact that the dynamics does not involve the control of the first Player. We want to reassure these readers,
because this is a classical situation, asymmetric for the two players, and it turn out to be very useful to obtain representations for the solutions of Hamilton–Jacobi equations.

Starting from this game, let us introduce the classical notions of controls and strategies for the two players (see for example [2], [3]).

For every fixed \( \tau \in [0,T] \), let us introduce the set of controls at time \( \tau \) for Player I as \( \mathcal{Y}(\tau) = \{ y : [\tau,T] \to Y \subset \mathbb{R}^2, \text{ measurable} \} \). In a similar way for the second Player, we define \( \mathcal{Z}(\tau) = \{ z : [\tau,T] \to Z \subset \mathbb{R}^2, \text{ measurable} \} \). We say that a map \( \alpha : \mathcal{Z}(\tau) \to \mathcal{Y}(\tau) \) is a nonanticipative strategy for Player I at time \( \tau \) if, for any time \( t \in [\tau,T] \) and any controls \( z, z' \in \mathcal{Z}(\tau) \), then we have \( \alpha[z] = \alpha[z'] \) a.e. in \([\tau,t]\). We denote by \( \mathcal{S}_\alpha(\tau) \) the set of such nonanticipative strategies at time \( \tau \) for Player I. In a symmetric way, we denote by \( \mathcal{S}_\beta(\tau) \) the set of nonanticipative strategies for Player II, which are the nonanticipative maps \( \beta : \mathcal{Y}(\tau) \to \mathcal{Z}(\tau) \).

**Definition 2.2 (upper and lower value functions)** Let us consider the zero game (15). The lower value function \( V^- : [0,T] \times \mathcal{H} \to \mathbb{R} \) is defined by

\[
V^-(\tau,\xi) = \inf_{\beta \in \mathcal{S}_\beta(\tau)} \sup_{y \in \mathcal{Y}(\tau)} \left\{ \int_{\tau}^{T} F(t,x,y,\beta[y]) \, dt + g(x(T)) \right\},
\]

where \( x \) is the horizontal curve on \( [\tau,T] \) with horizontal velocity \( -\beta[y] \) and initial point \( \xi \). The upper value function \( V^+ : [0,T] \times \mathcal{H} \to \mathbb{R} \) is defined by

\[
V^+(\tau,\xi) = \sup_{\alpha \in \mathcal{S}_\alpha(\tau)} \inf_{z \in \mathcal{Z}(\tau)} \left\{ \int_{\tau}^{T} F(t,x,\alpha[z],z) \, dt + g(x(T)) \right\},
\]

where \( x \) is the horizontal curve on \( [\tau,T] \) with horizontal velocity \( -z \) and initial point \( \xi \).

It is well known that in general \( V^- \leq V^+ \) and such two functions are different (see [2] for an example of game where the previous inequality is strict). We say that the game (15) admits value function \( V \) if

\[
V(\tau,\xi) = V^+(\tau,\xi) = V^-(\tau,\xi), \quad \forall (\tau,\xi) \in [0,T] \times \mathcal{H}.
\]

The following Dynamic Programming optimality condition is a classical result proved in [5]:

**Theorem 2.1** Let us consider the problem (15). Then

\[
V^-(\tau,\xi) = \inf_{\beta \in \mathcal{S}_\beta(\tau)} \sup_{y \in \mathcal{Y}(\tau)} \left\{ \int_{\tau}^{T+\sigma} F(t,x,y,\beta[y]) \, dt + V^-(\tau+\sigma, x(\tau+\sigma)) \right\}
\]

for every \( \tau, \tau + \sigma \in [0,T] \) and \( \xi \in \mathcal{H} \). A similar result holds for \( V^+ \).

### 3 Lipschitz continuity preserving properties

This section is devoted to the proof of Theorem [11]. First of all, let us remark that, if we consider the dynamics \( \dot{x} = -f^\mathcal{H}(x,z) \) in game (15), under assumption 1., it is easy to see that \( f^\mathcal{H} \) is uniformly continuous with

\[
\| f^\mathcal{H}(x,z) - f^\mathcal{H}(x',z) \| = \frac{1}{2} \| (z_2, -z_1) \cdot (x_1 - x'_1, x_2 - x'_2) \| \leq \frac{1}{2} R_Z \| x - x' \|,
\]

for all \( x = (x_1, x_2, x_3) \), \( x' = (x'_1, x'_2, x'_3) \) in \( \mathcal{H} \) and \( z = (z_1, z_2) \) in \( Z \). Now, let us replace in assumptions 2. and 3. the gauge distance \( d_G \) with the Euclidean distance \( d_E \), i.e. let us assume for a moment that
2'. \( |F(t, x, y, z)| \leq C_1 \), \( |F(t, x, y) - F(t, x', y, z)| \leq C'_1 \|x - x'| \)

3'. \( |g(x)| \leq C_2 \), \( |g(x) - g(x')| \leq C'_2 \|x - x'| \)

for some constants \( C_1, C'_1, C_2, C'_2 \) and for every \( t \in [0, T], \ x, x' \in \mathbb{H}, \ y \in Y \) and \( z \in Z \).

Theorem 3.2 in \([9]\) implies easily the following result:

**Remark 3.1** Let us consider the problem (15) with the assumptions 1., 2', and 3'. Then \( V^- \) is bounded and uniformly Lipschitz continuous w.r.t. the Euclidean distance \( d_E \), i.e.

\[
|V^-(t, x) - V^-(t', x')| \leq C(|t - t'| + \|x - x'|),
\]

for every \( t, t' \in [0, T] \) and \( x, x' \in \mathbb{H} \). Consequently, \( V^- \) is Lipschitz continuous w.r.t. the gauge distance \( d_G \). A similar result holds for \( V^+ \).

We note that our result in Theorem 1.1 is more precise under weaker assumptions, since there exists \( d_G \)-Lipschitz functions that are not \( d_E \)-Lipschitz.

**Proof of Theorem 1.1** The idea of the proof follows from Theorem 3.2 in \([9]\), but here we use all the fine properties of the horizontal curves in \( \mathbb{H} \) w.r.t. the \( d_G \)-distance proved in subsection 2.2.

Let us fix \( \tau < \tau' \in [0, T] \) and \( \xi, \xi' \in \mathbb{H} \). It is immediate to see that

\[
|V^-(\tau, \xi)| \leq C_1 T + C_2.
\]

Now let us fix \( \epsilon > 0 \). There exists \( \tilde{\beta} \in \mathcal{S}_{\tilde{\beta}}(\tau) \) such that

\[
V^-(\tau, \xi) \geq \sup_{y \in \mathcal{Y}(\tau)} \left\{ \int_{\tau}^{T} F(t, x, y, \tilde{\beta}[y]) \, dt + g(x(T)) \right\} - \epsilon. \tag{18}
\]

Fix \( y_0 \in Y \). For every \( y \in \mathcal{Y}(\tau') \), let us define \( \tilde{y} \in \mathcal{Y}(\tau) \) by

\[
\tilde{y}(t) = \begin{cases} y_0, & \text{for } t \in [\tau, \tau') \\ y(t), & \text{for } t \in [\tau', T] \end{cases}
\]

Let us define \( \tilde{\beta} \in \mathcal{S}_{\tilde{\beta}}(\tau') \) such that

\[
\tilde{\beta}[y](t) = \tilde{\beta}[\tilde{y}](t), \quad \forall y \in \mathcal{Y}(\tau'), \ t \in [\tau', T]
\]

Clearly,

\[
V^-(\tau', \xi') \leq \sup_{y \in \mathcal{Y}(\tau')} \left\{ \int_{\tau'}^{T} F(t, x, y, \tilde{\beta}[y]) \, dt + g(x(T)) \right\}.
\]

Let \( \tilde{y} \in \mathcal{Y}(\tau') \) be such that

\[
V^-(\tau', \xi') \leq \int_{\tau'}^{T} F(t, x, \tilde{y}, \tilde{\beta}[\tilde{y}]) \, dt + g(x(T)) + \epsilon. \tag{20}
\]

From (18) we get

\[
V^-(\tau, \xi) \geq \int_{\tau}^{T} F(t, x, \tilde{y}, \tilde{\beta}[\tilde{y}]) \, dt + g(x(T)) - \epsilon, \tag{21}
\]
where \( \tilde{y} \) is defined by \( \tilde{y} \) via (19) replacing \( y \) with \( \tilde{y} \). Note that the trajectories \( x \) that appear in (20) and in (21) are different functions. In particular, denoting by \( \tilde{x} \) and \( \tilde{\hat{x}} \) such trajectories in (20) and in (21) respectively, we have that \( \tilde{x} \) is a horizontal curve on \( [\tau', T] \) with horizontal velocity \( -\tilde{\beta}[\tilde{y}] \) and initial point \( \xi' \), while \( \tilde{\hat{x}} \) is a horizontal curve on \( [\tau, T] \) with horizontal velocity \( -\tilde{\beta}[\tilde{y}] \) and initial point \( \xi \). Since \( \tilde{y} = \tilde{y} \) and \( \tilde{\beta}[\tilde{y}] = \tilde{\beta}[\tilde{y}] \) on \([\tau', T] \), by Proposition 2.3 it is easy to prove that, for every \( t \in [\tau', T] \),

\[
d_G(\tilde{x}(t), \tilde{\hat{x}}(t)) \leq C \left( d_G(\xi', \xi) + (\tau' - \tau) \right).
\]

By (20) and (21), assumptions 2. and 3. we obtain

\[
V^-(\tau', \xi') - V^-(\tau, \xi) \leq \int_{\tau'}^{T} \left( F(t, x, \tilde{\hat{y}}, \tilde{\beta}[\tilde{y}]) - F(t, \tilde{x}, \tilde{\hat{y}}, \tilde{\beta}[\tilde{y}]) \right) dt + \int_{\tau}^{\tau'} F(t, \tilde{\hat{x}}, \tilde{\beta}[\tilde{y}]) dt + g(\tilde{x}(T)) - g(\tilde{x}(T)) + 2\epsilon
\]

\[
\leq C' \int_{\tau'}^{T} d_G(\tilde{x}(t), \tilde{\hat{x}}(t)) dt + (\tau' - \tau)C_1 + C_2 d_G(\tilde{x}(T), \tilde{\hat{x}}(T)) + 2\epsilon
\]

\[
\leq C' \left( d_G(\xi', \xi) + (\tau' - \tau) \right) + 2\epsilon
\]

with

\[
C' = C(C_1 T + C_2^2) + C_1.
\]

This concludes the first part of the proof.

Let \( \epsilon \) again be fixed. Then there exists \( \tilde{\beta} \in S_{\beta}(\tau') \) such that

\[
V^-(\tau', \xi') \geq \sup_{y \in \mathcal{Y}(\tau')} \left\{ \int_{\tau'}^{T} F(t, x, y, \tilde{\beta}[y]) dt + g(x(T)) \right\} - \epsilon.
\]

For every \( y \in \mathcal{Y}(\tau) \), let us define \( \tilde{y} \in \mathcal{Y}(\tau') \) by

\[
\tilde{y}(t) = y(t), \quad \forall t \in [\tau', T]
\]

Fix \( y_0 \in Y \). Let us define \( \tilde{\beta} \in S_{\beta}(\tau) \) such that, for every \( y \in \mathcal{Y}(\tau) \)

\[
\tilde{\beta}[y](t) = \begin{cases} y_0, & \text{for } t \in [\tau, \tau') \\ \tilde{\beta}[\tilde{y}](t), & \text{for } t \in [\tau', T] \end{cases}
\]

Clearly,

\[
V^-(\tau, \xi) \leq \sup_{y \in \mathcal{Y}(\tau)} \left\{ \int_{\tau}^{T} F(t, x, y, \tilde{\beta}[y]) dt + g(x(T)) \right\}.
\]

Let \( \tilde{y} \in \mathcal{Y}(\tau) \) be such that

\[
V^-(\tau, \xi) \leq \int_{\tau}^{T} F(t, x, \tilde{\hat{y}}, \tilde{\beta}[\tilde{y}]) dt + g(x(T)) + \epsilon.
\]

The inequality (24) gives

\[
V^-(\tau', \xi') \geq \int_{\tau'}^{T} F(t, x, \tilde{\hat{y}}, \tilde{\beta}[\tilde{y}]) dt + g(x(T)) - \epsilon.
\]
where \( \tilde{y} \) is defined by \( \tilde{y} \) via relation (25) replacing \( y \) with \( \tilde{y} \). Note that \( \tilde{x} \) is a horizontal curve on \([\tau,T]\) with horizontal velocity \(-\tilde{\beta}[\tilde{y}]\) and initial point \( \xi \), while \( \bar{x} \) is a horizontal curve on \([\tau',T]\) with horizontal velocity \(-\beta[y]\) and initial point \( \xi' \). Since \( \hat{y} = \tilde{y} \) and \( \hat{\beta}[\hat{y}] = \tilde{\beta}[\tilde{y}] \) on \([\tau',T]\), by Proposition 2.3 we have that, for every \( t \in [\tau',T] \),

\[
d_G(\tilde{x}(t), \bar{x}(t)) \leq \bar{C} (d_G(\xi', \xi) + (\tau' - \tau)).
\]

By (26) and (27), assumptions 2. and 3. we obtain

\[
V^-(\tau, \xi) - V^-(\tau', \xi') \leq \int_{\tau'}^{\tau} \left( F(t, \tilde{x}, \hat{y}, \tilde{\beta}[\tilde{y}]) - F(t, \bar{x}, \hat{y}, \tilde{\beta}[\tilde{y}]) \right) dt + \int_{\tau'}^{\tau} F(t, \tilde{x}, \hat{y}, \tilde{\beta}[\tilde{y}]) dt + g(\bar{x}(T)) - g(\bar{x}(T)) + 2\varepsilon
\]

\[
\leq C'_1 \int_{\tau'}^{\tau} d_G(\tilde{x}(t), \bar{x}(t)) dt + (\tau' - \tau) C_1 + C_2 d_G(\bar{x}(T), \bar{x}(T)) + 2\varepsilon
\]

\[
\leq C' (d_G(\xi', \xi) + (\tau' - \tau)) + 2\varepsilon
\] (28)

with \( C' \) as in (23). This inequality and (22) conclude the proof. \(\square\)

In the fundamental paper [15], Pansu provides a Rademacher–Stefanov type result in the Carnot group setting; in particular, he proves that every Lipschitz continuous function w.r.t. a homogeneous distance on \( II \) is differentiable almost everywhere in the horizontal directions. Hence, our previous result implies that the lower value function admits the horizontal gradient and the derivative w.r.t. \( t \), i.e.

\[
\nabla_H V^-(t, x) = \left( X_1 V^-(t, x), X_2 V^-(t, x) \right)
\]

and \( \frac{\partial V^-}{\partial t}(t, x) \),

for almost everywhere \((t, x) \in [0, T] \times II\). Therefore, \( V^- \) could be a candidate for a viscosity solution, as we will see in definition 4.1. Moreover, a more precise estimate in (22) and in (28) gives

\[
|V^-(t, x) - V^-(t', x')| \leq \bar{C}(C'_1 T + C'_2) \left( d_G(x', x) + |t' - t| \right) + C_1|t' - t|
\]

This implies that, taking into account (14), we have the following \( d_G \)-Lipschitz constant

**Remark 3.2** For a.e. \((t, x) \in [0, T] \times II\) we have

\[
\| \nabla_H V^-(t, x) \| \leq (1 + 3 R Z) e^{\frac{R Z}{2}} (C'_1 T + C'_2) : = C^2
\]

It is important to notice that \( C^2 \) does not depend on \( R_Y \).

4 **Viscosity solutions for Hamilton–Jacobi–Isaacs equation**

This section is essentially devoted to the proof of Theorem 4.1. In order to recall the notion of viscosity solution in our context (see for example [14]) we say that, given an open interval \( I \), a function \( \psi : I \times II \rightarrow \mathbb{R} \) is in \( \Gamma^1(I \times II) \) if \((t, x) \mapsto \left( \frac{\partial \psi(t,x)}{\partial t}, X_1 \psi(t, x), X_2 \psi(t, x) \right) \) is a continuous function.
Definition 4.1 (viscosity solution) Let $\mathcal{H} : [0, T] \times \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let $u : [0, T] \times \mathbb{H} \to \mathbb{R}$ be a bounded and uniformly continuous function, with $u(T, x) = g(x)$ in $\mathbb{H}$. We say that $u$ is a viscosity subsolution of the Hamilton–Jacobi equation

$$
\begin{cases}
\frac{\partial u}{\partial t}(t, x) + \mathcal{H}(t, x, \nabla_x u(t, x)) = 0 & \text{in } (0, T) \times \mathbb{H} \\
u(T, x) = g(x) & \text{in } \mathbb{H}
\end{cases}
$$

if, whenever $(t_0, x_0) \in (0, T) \times \mathbb{H}$ and $\psi$ is a test function in $\Gamma^1((0, T) \times \mathbb{H})$ touching $u$ from above at $(t_0, x_0)$, i.e.

$$u(t_0, x_0) = \psi(t_0, x_0) \quad \text{and} \quad u(t, x) \leq \psi(t, x) \quad \text{in a neighborhood of } (t_0, x_0),$$

we have

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \mathcal{H}(t_0, x_0, \nabla_x \psi(t_0, x_0)) \geq 0.$$ 

(30)

We say that $u$ is a viscosity supersolution of equation (29) if, whenever $(t_0, x_0) \in (0, T) \times \mathbb{H}$ and $\psi$ is a test function in $\Gamma^1((0, T) \times \mathbb{H})$ touching $u$ from below at $(t_0, x_0)$, i.e.

$$u(t_0, x_0) = \psi(t_0, x_0) \quad \text{and} \quad u(t, x) \geq \psi(t, x) \quad \text{in a neighborhood of } (t_0, x_0),$$

we have

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \mathcal{H}(t_0, x_0, \nabla_x \psi(t_0, x_0)) \leq 0.$$ 

(31)

A function that is both a viscosity subsolution and a viscosity supersolution is called viscosity solution.

An equivalent definition of viscosity solution (29) uses the notion of supejets (see [14], [13]). We recall that if in (29) we change the condition $u(T, x) = \psi(x)$ with an initial condition of the type

$$u(0, x) = \psi(x) \quad x \in \mathbb{H},$$

then the viscosity solution of the new problem is defined by reversing the inequalities (30) and (31).

Definition 4.2 (upper and lower Hamiltonian) Let us consider the zero game (15). We define the lower Hamiltonian $H^- : [0, T] \times \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H^-(t, x, \lambda) = \max_{y \in Y} \min_{z \in Z} \left( F(t, x, y, z) - \lambda \cdot z \right)$$

(32)

and the upper Hamiltonian $H^+ : [0, T] \times \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H^+(t, x, \lambda) = \min_{z \in Z} \max_{y \in Y} \left( F(t, x, y, z) - \lambda \cdot z \right).$$

It is easy to see that $H^- \leq H^+$. We say that the min max condition, or Isaacs’ condition, is satisfied if $H = H^+$. In this case, we define the Hamiltonian $H$ by

$$H(t, x, \lambda) = H^-(t, x, \lambda) = H^+(t, x, \lambda).$$

Let us spend few lines to make some comments on the Definition 4.2. In a classical zero game case, if we have a trajectory $x$ in $\mathbb{R}^n$, then we usually introduce a multiplier $\lambda$ with
the same dimension, i.e. $\lambda \in \mathbb{R}^n$; more precisely, if $\dot{x} = h(t, x, y, z)$ is the dynamics of the zero game, in the definition (32) of $H^-$ the function $(F + \lambda \cdot g)$ appears as argument of the max min. In our case, the trajectory $x$ is in $\mathcal{H}$ but the multiplier $\lambda$ is 2-dimensional and takes into account only the horizontal velocity of the horizontal curve $x$.

Now we are ready to prove Theorem 1.2, i.e. Theorem 4.1 Let us consider the problem (15) with the assumptions 1., 2. and 3.. Then $V^-$ is a viscosity solution of the lower Hamilton–Jacobi–Isaacs equation

$$\begin{align*}
\frac{\partial u}{\partial t}(t, x) + H^-(t, x, \nabla_H u(t, x)) &= 0 &\text{for } (t, x) \in (0, T) \times \mathcal{H} \\
u(T, x) &= g(x) &\text{for } x \in \mathcal{H}
\end{align*}$$

(33)

and $V^+$ is a viscosity solution of the upper Hamilton–Jacobi–Isaacs equation

$$\begin{align*}
\frac{\partial u}{\partial t}(t, x) + H^+(t, x, \nabla_H u(t, x)) &= 0 &\text{for } (t, x) \in (0, T) \times \mathcal{H} \\
u(T, x) &= g(x) &\text{for } x \in \mathcal{H}
\end{align*}$$

(34)

The proof of this theorem requires the following lemma:

**Lemma 4.1** Let $\psi \in \Gamma^1((0, T) \times \mathcal{H})$.

If there exists $\theta > 0$ such that

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + H^-(t_0, x_0, \nabla_H \psi(t_0, x_0)) \geq \theta,$$

(35)

then, for all sufficiently small $\tau > 0$, there exists $\bar{y} \in \mathcal{Y}(t_0)$ such that for every $\bar{\beta} \in S_{\bar{\beta}}(t_0)$ we have

$$\int_{t_0}^{t_0 + \tau} \left( \frac{\partial \psi}{\partial t}(s, \bar{x}(s)) + F(s, \bar{x}(s), \bar{y}(s), \bar{\beta}[\bar{y}](s)) - \bar{\beta}[\bar{y}](s) \cdot \nabla_H \psi(s, \bar{x}(s)) \right) ds \geq \frac{\tau \theta}{2},$$

(36)

where $\bar{x}$ is the horizontal curve on $[t_0, t_0 + \tau]$ with horizontal velocity $\bar{\beta}[\bar{y}]$ and initial point $x_0$.

If there exists $\theta > 0$ such that

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + H^-(t_0, x_0, \nabla_H \psi(t_0, x_0)) \leq -\theta,$$

(37)

then, for all sufficiently small $\tau > 0$, there exists $\bar{\beta} \in S_{\bar{\beta}}(t_0)$ such that for every $\bar{y} \in \mathcal{Y}(t_0)$ we have

$$\int_{t_0}^{t_0 + \tau} \left( \frac{\partial \psi}{\partial t}(s, \bar{x}(s)) + F(s, \bar{x}(s), \bar{y}(s), \bar{\beta}[\bar{y}](s)) - \bar{\beta}[\bar{y}](s) \cdot \nabla_H \psi(s, \bar{x}(s)) \right) ds \leq -\frac{\tau \theta}{2},$$

(38)

where $\bar{x}$ is as before.

The proof of this lemma can be found in [9] (see Lemma 4.3, where a classical gradient instead of our horizontal gradient appears) and it is based on the continuity of the function

$$(t, x) \mapsto \frac{\partial \psi}{\partial t}(t, x) + F(t, x, y, z) - z \cdot \nabla_H \psi(t, x)$$

13
and on the compactness of the control sets $Y$ and $Z$.

**Proof of Theorem 4.1** The proof follows the idea of Theorem 4.1 in [9] and uses the properties of the horizontal curves in subsection 2.2. It is obvious, by definition, that $V^-(T, x) = g(x)$, for every $x \in \mathcal{H}$. So, let us fix $(t_0, x_0) \in (0, T) \times \mathcal{H}$.

First, let $\psi \in \Gamma^1((0, T) \times \mathcal{H})$ be a test function touching $V^-$ from below at $(t_0, x_0)$, i.e.

$$V^-(t_0, x_0) = \psi(t_0, x_0) \quad \text{and} \quad V^-(t, x) \geq \psi(t, x) \quad \text{in a neighborhood of} \quad (t_0, x_0). \quad (39)$$

We have to prove that (31) holds with $\mathcal{H} = H^-$. By contradiction, let us assume that this is not true and that there exists $\theta > 0$ such that holds (35); then, (36) implies that

$$\inf_{\beta \in S(\theta)} \sup_{y \in \mathcal{Y}(t_0)} \int_{t_0}^{t_0 + \tau} \left( \frac{\partial \psi}{\partial t}(s, x) + F(s, x, y, \beta[y]) - \beta[y] \cdot \nabla_H \psi(s, x) \right) ds \geq \frac{\theta}{2} \quad (40)$$

where $x$ solves

$$\begin{cases}
\dot{x} = -f_H(x, \beta[y]) & \text{in } [t_0, T] \\
x(t_0) = x_0,
\end{cases} \quad (41)$$

Now by Theorem 2.1 we know that

$$V^-(t_0, x_0) = \inf_{\beta \in S(\theta)} \sup_{y \in \mathcal{Y}(t_0)} \left\{ \int_{t_0}^{t_0 + \tau} F(s, x, y, \beta[y]) ds + V^-(t_0 + \tau, x(t_0 + \tau)) \right\} \quad (42)$$

with $x$ as before. For every such horizontal curve $x$, (39) and Proposition 2.1 imply that, for $\tau$ small enough,

$$0 = V^-(t_0, x_0) - \psi(t_0, x_0) \leq V^-(t_0 + \tau, x(t_0 + \tau)) - \psi(t_0 + \tau, x(t_0 + \tau)) \quad (43)$$

Since $x$ is horizontal and $\psi$ is in $\Gamma^1$, Remark 2.1 and (41) imply

$$\begin{align*}
\psi(t_0 + \tau, x(t_0 + \tau)) - \psi(t_0, x_0) &= \int_{t_0}^{t_0 + \tau} \frac{d\psi(s, x(s))}{ds} ds \\
&= \int_{t_0}^{t_0 + \tau} \left( \frac{\partial \psi}{\partial t}(s, x(s)) - \beta[y](s) \cdot \nabla_H \psi(s, x(s)) \right) ds \quad (44)
\end{align*}$$

Relations (42)–(44) give

$$0 \geq \inf_{\beta \in S(\theta)} \sup_{y \in \mathcal{Y}(t_0)} \left\{ \int_{t_0}^{t_0 + \tau} F(s, x, y, \beta[y]) ds + \psi(t_0 + \tau, x(t_0 + \tau)) - \psi(t_0, x_0) \right\}$$

$$= \inf_{\beta \in S(\theta)} \sup_{y \in \mathcal{Y}(t_0)} \left\{ \int_{t_0}^{t_0 + \tau} \left( F(s, x, y, \beta[y]) + \frac{\partial \psi}{\partial t}(s, x) - \beta[y] \cdot \nabla_H \psi(s, x) \right) ds \right\}$$

This inequality contradicts (40), hence (35) is false and this concludes the first part of the proof.

Now, let $\psi \in \Gamma^1((0, T) \times \mathcal{H})$ be a test function touching $V^-$ from above at $(t_0, x_0)$, i.e.

$$V^-(t_0, x_0) = \psi(t_0, x_0) \quad \text{and} \quad V^-(t, x) \leq \psi(t, x) \quad \text{in a neighborhood of} \quad (t_0, x_0). \quad (45)$$

We have to prove that (30) holds with $\mathcal{H} = H^-$. Let us assume that this is not true and that there exists $\theta > 0$ such that (37) holds; then (38) implies that

$$\inf_{\beta \in S(\theta)} \sup_{y \in \mathcal{Y}(t_0)} \int_{t_0}^{t_0 + \tau} \left( \frac{\partial \psi}{\partial t}(s, x) + F(s, x, y, \beta[y]) - \beta[y] \cdot \nabla_H \psi(s, x) \right) ds \leq -\frac{\theta}{2} \quad (46)$$
where $x$ is as in (41). For every such horizontal curve $x$, requirement (45) and Proposition 2.1 imply that, for $\tau$ small enough,

$$
0 = V^-(t_0, x_0) - \psi(t_0, x_0) \geq V^-(t_0 + \tau, x(t_0 + \tau)) - \psi(t_0 + \tau, x(t_0 + \tau))
$$

(47)

Relations (42), (44) and (47) give

$$
0 \leq \inf_{\beta \in S_{\beta(t_0)}} \sup_{y \in Y_{\beta(t_0)}} \left\{ \int_{t_0}^{t_0+\tau} F(s, x, y, \beta[y]) ds + \psi(t_0 + \tau, x(t_0 + \tau)) - \psi(t_0, x_0) \right\}
$$

$$
= \inf_{\beta \in S_{\beta(t_0)}} \sup_{y \in Y_{\beta(t_0)}} \left\{ \int_{t_0}^{t_0+\tau} \left( F(s, x, y, \beta[y]) + \frac{\partial \psi}{\partial t}(s, x) - \beta[y] \cdot \nabla_H \psi(s, x) \right) ds \right\}.
$$

This inequality contradicts (46): hence (37) is false and this concludes the proof for $V^-$. In a similar way one proves that $V^+$ is a viscosity solution of (34).

5 Representation of solutions of Hamilton–Jacobi equations

We are now in the position to study the viscosity solution of our initial Hamilton–Jacobi problem (1), i.e.

$$
\begin{align*}
\frac{\partial u}{\partial t}(t, x) + H(t, x, \nabla_H u(t, x)) &= 0 \quad \text{for } (t, x) \in (0, T) \times I^H \\
u(0, x) &= g(x) \quad \text{for } x \in I^H
\end{align*}
$$

(48)

under assumptions 3. and 4..

Having in mind problem (48), let us introduce a zero sum game as follows: set

$$
R_Z = K, \quad R_Y = (1 + 3K)e^{TK}(D_1^T + C_2')
$$

(49)

in assumption 1., and consider the function

$$
F(t, x, y, z) = -H(T - t, x, y) + z \cdot y.
$$

(50)

Relations (49) and (50) give us a zero sum game as in (15) associated to our initial problem (48). It is clear that assumptions 1. and 4. guarantee that $F$ in (50) satisfies assumption 2. with $C_1 = D_1 + R_Z R_Y$ and $C_1' = D_1'$. Hence, Theorem 1.1 implies that the lower value function $V^-_F$ for the zero game (15), with $F$ as in (50) and $R_Y$ and $R_Z$ as in (49),

$$
V^-_F(\tau, \xi) = \inf_{\beta \in S_{\beta(\tau)}} \sup_{y \in Y_{\beta(\tau)}} \left\{ \int_{\tau}^{T} \left( -H(T - t, x, y) + \beta[y] \cdot y \right) dt + g(x(T)) \right\}
$$

$$
\text{with } x(t) = \xi - \int_{\tau}^{t} f^H(x, \beta[y]) ds,
$$

is bounded and $d_C$-Lipschitz w.r.t. $x$. Remark 3.2 gives $\|\nabla_H V^-_F(t, x)\| \leq C^s$, with

$$
C^s = (1 + 3K)e^{TK}(D_1^T + C_2').
$$
Note that $C^* = R_Y$. Moreover, by Theorem 4.1 $V^-_F$ is a viscosity solution of the lower Hamilton–Jacobi–Isaacs equation (33), i.e.

$$
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t}(t,x) + H^-(t,x,\nabla_H u(t,x)) = 0 & \text{for } (t,x) \in (0,T) \times \mathcal{I}H \\
u(T,x) = g(x) & \text{for } x \in \mathcal{I}H
\end{array} \right.
\end{aligned}
$$

(51)

where, as in (32),

$$
H^-(t,x,\lambda) = \max_{y \in \mathcal{Y}} \min_{z \in \mathcal{Z}} (-H(T-t,x,y) + z \cdot y - \lambda \cdot z),
$$

(52)

and $\mathcal{Y} = B_{\mathbb{R}^2}(0,R_Y)$, $\mathcal{Z} = B_{\mathbb{R}^2}(0,R_Z)$. Clearly, assumption 4. implies

$$
\mathcal{H}(t,x,\lambda) \leq \mathcal{H}(t,x,y) + K\|\lambda - y\|
$$

for every $y,\lambda \in \mathcal{Y}$. Hence, taking into account that $K = R_Z$, for every $t \in [0,T]$, $x \in \mathcal{I}H$ and $\lambda \in \mathcal{Y}$,

$$
\begin{aligned}
\mathcal{H}(t,x,\lambda) &= \min_{y \in \mathcal{Y}} (\mathcal{H}(t,x,y) + R_Z\|\lambda - y\|) \\
&= \min_{y \in \mathcal{Y}} \max_{z \in \mathcal{Z}} (\mathcal{H}(t,x,y) + z \cdot (\lambda - y)) \\
&= -\max_{y \in \mathcal{Y}} \min_{z \in \mathcal{Z}} (\mathcal{H}(t,x,y) - z \cdot (\lambda - y)).
\end{aligned}
$$

The equality above and (52) imply that

$$
H^-(T-t,x,\lambda) = -\mathcal{H}(t,x,\lambda) \quad \forall t \in [0,T], x \in \mathcal{I}H, \lambda \in \mathcal{Y}.
$$

(53)

The function $U$, defined by $U(t,x) = V^-_F(T-t,x)$ and (51), is a viscosity solution for

$$
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t}(t,x) - H^-(T-t,x,\nabla_H u(t,x)) = 0 & \text{for } (t,x) \in (0,T) \times \mathcal{I}H \\
u(0,x) = g(x) & \text{for } x \in \mathcal{I}H
\end{array} \right.
\end{aligned}
$$

Taking into account that $\|\nabla_H U(t,x)\| \leq C^* = R_Y$, and using (53), we finally have the following representation for the viscosity solution for the Hamilton–Jacobi equation (48).

Theorem 5.1 Let us consider problem (48) with the assumptions 1. ($R_Z$ and $R_Y$ as in (49)), 3. and 4. Then, the function $U$ defined by

$$
\begin{aligned}
U(\tau,\xi) = \inf_{\beta \in \mathcal{S}_{\beta}(T-\tau)} \sup_{y \in \mathcal{Y}(T-\tau)} \left\{ \int_{T-\tau}^{T} (-\mathcal{H}(T-t,x,y) + \beta[y] \cdot y) \, dt + g(x(T)) \right\}
\end{aligned}
$$

with $x(t) = \xi - \int_{T-\tau}^{t} f^\mathcal{H}(x,\beta[y]) \, ds$,

is $d_G$-Lipschitz and is a viscosity solution for (48).
5.1 A particular case and a question.

Let us consider the particular case $\mathcal{H} = \mathcal{H}(y)$. The previous arguments give that, under the same assumptions 1., 3. and 4., the function $U$ defined by

$$U(\tau, \xi) = \inf_{\beta \in S(0)} \sup_{y \in Y(0)} \left\{ \int_0^\tau \left( -\mathcal{H}(y) + \beta |y| \cdot y \right) \, dt + g(x(\tau)) \right\}$$

with $x(t) = \xi - \int_0^t f^H(x, \beta |y|) \, ds$,

with $R_Z$ and $R_Y$ as in (49), is a viscosity solution for

$$\begin{aligned}
\frac{\partial u}{\partial t}(t, x) + \mathcal{H}(\nabla H u(t, x)) &= 0 \quad \text{for } (t, x) \in (0, T) \times \mathbb{H} \\
u(0, x) &= g(x) \quad \text{for } x \in \mathbb{H}.
\end{aligned}$$

Theorem 4 in [14] guarantees the uniqueness of such solution:

**Proposition 5.1** In the assumptions 1., 3. and 4., the function $U$ in (54) is $d_C$-Lipschitz and is the unique viscosity solution of (55) satisfying

$$\lim_{t \to 0} \sup_{x \in \mathbb{H}} |u(x, t) - u(x, 0)| = 0.$$

It is well known that, if the Hamiltonian function depends only on the gradient, it is possible to write a Hopf–Lax formula: the first result in this line of investigation in $\mathbb{H}$ is in [14], a more general result can be found in [1]. Hence, a very interesting question is the following: if we add to the previous assumptions the following

5. the function $g : \mathbb{H} \to \mathbb{R}$ is “convex”,

is it possible to apply the ideas in [3] to obtain a Hopf–Lax formula for the function $U$ in (54)? If $g$ is convex in the $\mathbb{R}^3$ classical sense, and hence is locally $d_E$-Lipschitz, one can try to apply the same arguments of section 3 in [3] where a Jensen inequality plays a fundamental role.

But the very interesting and natural question, taking into account the Sub–Riemannian setting and assumption 4., arises if we require that $g$ is only $H$–convex, i.e., for every fixed $x \in \mathbb{H}$ and $w \in V_1$ the function $s \mapsto g(x \circ \exp(sw))$ is convex. The simplest example of a $H$–convex function, but not $\mathbb{R}^3$–convex, is $x \mapsto ||x||_G$; moreover, we recall that an $H$–convex function is $d_C$-Lipschitz (see [6] for details on the properties of these $H$–convex functions). Unfortunately with this notion of convexity, to our knowledge, there is not in the literature a Jensen-type inequality in the Heisenberg group that would allow to follow the ideas in [3] in order to obtain a Hopf–Lax formula: this is a very big obstacle. We are working on this obstacle.

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