AFFINE OPEN COVERING OF THE QUANTIZED FLAG MANIFOLDS AT ROOTS OF UNITY

TOSHIYUKI TANISAKI

Abstract. We show that the quantized flag manifold at a root of unity has natural affine open covering parametrized by the elements of the Weyl group. In particular, the quantized flag manifold turns out to be a quasi-scheme in the sense of Rosenberg [12].

1. Introduction

Let $G$ be a connected semisimple algebraic group over the complex number field $C$, and let $B, B^+$ be Borel subgroups of $G$ such that $B \cap B^+$ is a maximal torus of $G$. The homogeneous space $\mathcal{B} = G/B$ is a projective algebraic variety called the flag manifold. Let $W$ be the Weyl group of $G$. We have an affine open covering $\mathcal{B} = \bigcup_{w \in W} \mathcal{B}^w$, where $\mathcal{B}^w = wB^+B/B$. Let $\mathcal{R}$ be the homogeneous coordinate algebra of $\mathcal{B}$, and let $\mathcal{R}^w$ be the coordinate algebra of $\mathcal{B}^w$ so that

$$\mathcal{B} = \text{Proj} \mathcal{R}, \quad \mathcal{B}^w = \text{Spec} \mathcal{R}^w \quad (w \in W).$$

Let us consider the situation where $G$ is replaced by the corresponding quantum group. Let $K$ be a field equipped with $q \in K^\times$. Using the quantum group we can naturally define $q$-analogues $\mathcal{R}_{K,q}$, $\mathcal{R}^w_{K,q}$ of $\mathcal{R}$, $\mathcal{R}^w$ respectively. Here, $\mathcal{R}_{K,q}$ is a graded $K$-algebra, and $\mathcal{R}^w_{K,q}$ is a $K$-algebra. We have $\mathcal{R}_{C,1} \cong \mathcal{R}$ and $\mathcal{R}^w_{C,1} \cong \mathcal{R}^w$. A major difference compared to the ordinary case $q = 1$ is the fact that $\mathcal{R}_{K,q}$ and $\mathcal{R}^w_{K,q}$ are non-commutative in general. Hence in order to understand the “quantized flag manifold” in a geometric manner, we need the language of non-commutative algebraic geometry, which has been developed by Artin-Zhang [2], Verëvkin [15], Rosenberg [12] following Manin’s idea [10]. Using $\mathcal{R}_{K,q}$ and $\mathcal{R}^w_{K,q}$, we can define as in Rosenberg [12], Lunts-Rosenberg [7] (see also Joseph [5]) the abelian categories

$$(1.1) \quad \text{Mod}(\mathcal{O}_{\mathcal{B}_{K,q}}), \quad \text{Mod}(\mathcal{O}_{\mathcal{B}^w_{K,q}}) \quad (w \in W),$$

which are regarded as the categories of “quasi-coherent sheaves” on the virtual spaces

$$\mathcal{B}_{K,q} = \text{Proj} \mathcal{R}_{K,q}, \quad \mathcal{B}^w_{K,q} = \text{Spec} \mathcal{R}^w_{K,q} \quad (w \in W)$$

respectively, and the exact functors

$$(1.2) \quad (i^w_{K,q})^* : \text{Mod}(\mathcal{O}_{\mathcal{B}_{K,q}}) \to \text{Mod}(\mathcal{O}_{\mathcal{B}^w_{K,q}}) \quad (w \in W).$$

In order to verify that $\text{Mod}(\mathcal{O}_{\mathcal{B}_{K,q}})$ defines a quasi-scheme $\mathcal{B}_{K,q}$ in the sense of [12] we need to show the patching property

$$(1.3) \quad M \in \text{Mod}(\mathcal{O}_{\mathcal{B}_{K,q}}), \quad (i^w_{K,q})^*M = 0 \quad (\forall w \in W) \implies M = 0.$$
This holds for \( q = 1 \) since \( \mathcal{B}_{K,1} \) is isomorphic to the ordinary flag manifold over \( K \). From this we can derive the property (1.3) when \( q \) is transcendental over the prime field \( K_0 \) of \( K \) (see [7]).

The main result of this paper is (1.3) where \( q \) is a root of unity. By the aid of Lusztig’s quantum Frobenius homomorphism we can reduce its proof to the case where \( q = \pm 1 \). In the case \( q = 1 \) (1.3) is a classically known fact as mentioned above. The proof is rather involved in the case \( q = -1 \). We will construct an isomorphism \( \mathcal{R}_{K,-1} \cong \mathcal{R}_{K,1} \) of graded vector spaces. Although this isomorphism does not preserve the ring structure, it satisfies some favorable properties so that we can derive (1.3) for \( q = -1 \) from that for \( q = 1 \).

2. Quantized enveloping algebras

2.1. Let \( G \) be a connected reductive algebraic group over \( \mathbb{C} \), and let \( H \) be a maximal torus of \( G \). We denote by \((X, \Delta, Y, \Delta^\vee)\) the root datum associated to \( G \) and \( H \). Namely,

\[
X = \text{Hom}(H, \mathbb{C}^\times), \quad Y = \text{Hom}(\mathbb{C}^\times, H),
\]

and \( \Delta \) (resp. \( \Delta^\vee \)) is the set of roots (resp. coroots). The coroot corresponding to \( \alpha \in \Delta \) is denoted as \( \alpha^\vee \in \Delta^\vee \). We fix a set of simple roots \( \{\alpha_i \mid i \in I\} \) of \( \Delta \), and denote the corresponding set of positive roots by \( \Delta^+ \). The Weyl group \( W \) is the subgroup of \( GL(h^\ast) \) generated by the simple reflections \( s_i : h^\ast \to h^\ast \) (\( \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \)) for \( i \in I \). Set

\[
Q = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha \subset X, \quad Q^\vee = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha^\vee \subset Y,
\]

\[
Q^+ = \sum_{\alpha \in \Delta^+} \mathbb{Z}_{\geq 0}\alpha \subset Q, \quad X^+ = \{\lambda \in X \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ (i \in I)\} \subset X.
\]

For \( i, j \in I \) we set \( a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle \).

Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be the Lie algebras of \( G \) and \( H \) respectively. We will identify \( X \) (resp. \( Y \)) with a \( \mathbb{Z} \)-lattice of \( h^\ast \) (resp. \( h \)). We have the root space decomposition

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \left[ h, x \right] = \alpha(h)x \ (h \in \mathfrak{h})\}.
\]

For \( i \in I \) we take \( \tau_i \in \mathfrak{g}_{\alpha_i}, \bar{f}_i \in \mathfrak{g}_{-\alpha_i} \) such that \( [\tau_i, \bar{f}_i] = \alpha_i^\vee \).

We denote by \( B, B^+ \) the Borel subgroups of \( G \) with Lie algebras

\[
\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{b}^+ = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha
\]

respectively. We denote by \( N, N^+ \) the unipotent radicals of \( B, B^+ \) respectively. Their Lie algebras are given by

\[
\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha
\]

respectively.
2.2. For a Lie algebra \( \mathfrak{a} \) we denote its enveloping algebra by \( \mathcal{U}(\mathfrak{a}) \). For \( \lambda \in X \) we define a character \( \chi_\lambda : \mathcal{U}(\mathfrak{h}) \to \mathbb{C} \) by
\[
\chi_\lambda(h) = \langle \lambda, h \rangle \quad (h \in \mathfrak{h}).
\]
For \( n \in \mathbb{Z}_{\geq 0} \) set
\[
\binom{x}{n} = \frac{x(x-1) \ldots (x-n+1)}{n!} \in \mathbb{Q}[x].
\]
We denote by \( \mathcal{U}_Z(\mathfrak{h}) \) the \( \mathbb{Z} \)-subalgebra of \( \mathcal{U}(\mathfrak{h}) \) generated by the elements
\[
\binom{y}{n} \quad (y \in Y, n \in \mathbb{Z}_{\geq 0}).
\]
For \( i \in I \) and \( n \in \mathbb{Z}_{\geq 0} \) we define \( \overline{e}_i^{(n)}, \overline{f}_i^{(n)} \in \mathcal{U}(\mathfrak{g}) \) by
\[
\overline{e}_i^{(n)} = \frac{e_i^n}{n!}, \quad \overline{f}_i^{(n)} = \frac{f_i^n}{n!}.
\]
We define \( \mathbb{Z} \)-subalgebras \( \mathcal{U}_Z(\mathfrak{n}), \mathcal{U}_Z(\mathfrak{n}^+), \mathcal{U}_Z(\mathfrak{b}), \mathcal{U}_Z(\mathfrak{g}) \) of \( \mathcal{U}(\mathfrak{g}) \) by
\[
\mathcal{U}_Z(\mathfrak{n}) = \langle \overline{e}_i^{(n)} | i \in I, n \geq 0 \rangle, \quad \mathcal{U}_Z(\mathfrak{n}^+) = \langle \overline{e}_i^{(n)} | i \in I, n \geq 0 \rangle,
\]
\[
\mathcal{U}_Z(\mathfrak{b}) = \langle \mathcal{U}_Z(\mathfrak{h}), \mathcal{U}_Z(\mathfrak{n}) \rangle, \quad \mathcal{U}_Z(\mathfrak{g}) = \langle \mathcal{U}_Z(\mathfrak{h}), \mathcal{U}_Z(\mathfrak{n}), \mathcal{U}_Z(\mathfrak{n}^+) \rangle.
\]
For a commutative ring \( R \) we set
\[
\mathcal{U}_R(\mathfrak{h}) = R \otimes_{\mathbb{Z}} \mathcal{U}_Z(\mathfrak{h}), \quad \mathcal{U}_R(\mathfrak{n}) = R \otimes_{\mathbb{Z}} \mathcal{U}_Z(\mathfrak{n}), \quad \mathcal{U}_R(\mathfrak{n}^+) = R \otimes_{\mathbb{Z}} \mathcal{U}_Z(\mathfrak{n}^+),
\]
\[
\mathcal{U}_R(\mathfrak{b}) = R \otimes_{\mathbb{Z}} \mathcal{U}_Z(\mathfrak{b}), \quad \mathcal{U}_R(\mathfrak{g}) = R \otimes_{\mathbb{Z}} \mathcal{U}_Z(\mathfrak{g}).
\]
They are Hopf algebras over \( R \). Note that for \( \lambda \in X \) we have \( \chi_\lambda(\mathcal{U}_Z(\mathfrak{h})) \subset \mathbb{Z} \). Hence the character \( \chi_\lambda : \mathcal{U}(\mathfrak{h}) \to \mathbb{C} \) induces the character \( \chi_\lambda : \mathcal{U}_R(\mathfrak{h}) \to R \) of \( \mathcal{U}_R(\mathfrak{h}) \).

2.3. For an integer \( m \) we define its \( t \)-analogues \( [m]_t \) and \( \{m\}_t \) by
\[
[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}], \quad \{m\}_t = \frac{t^m - 1}{t - 1} \in \mathbb{Z}[t].
\]
For a non-negative integer \( n \) we set
\[
[n]_t! = [1]_t [2]_t \ldots [n]_t \in \mathbb{Z}[t, t^{-1}], \quad \{n\}_t! = \{1\}_t \{2\}_t \ldots \{n\}_t \in \mathbb{Z}[t].
\]
We have
\[
[m]_t = t^{-m+1} \{m\}_t!, \quad [n]_t! = t^{-n(n-1)/2} \{n\}_t!.
\]

2.4. We fix a \( W \)-invariant symmetric bilinear form
\[
(\cdot, \cdot) : \sum_{\alpha \in \Delta} \mathbb{Q} \alpha \times \sum_{\alpha \in \Delta} \mathbb{Q} \alpha \to \mathbb{Q}
\]
satisfying \( (\alpha, \alpha) \in 2\mathbb{Z} \) for any \( \alpha \in \Delta \). For \( \alpha \in \Delta \) we set \( d_\alpha = (\alpha, \alpha)/2 \), and for \( i \in I \) we set \( d_i = d_{\alpha_i} \).

Set \( F = \mathbb{Q}(q) \). The quantized enveloping algebra \( U_F(\mathfrak{g}) \) is the associative algebra over \( F \) with 1 generated by the elements
\[
k_y \quad (y \in Y), \quad e_i, \quad f_i \quad (i \in I)
\]
satisfying the relations
\begin{align*}
k_0 &= 1, \quad k_y k_y = k_{y_1 y_2} \quad (y_1, y_2 \in Y), \\
k_y e_i &= q^{(\alpha_i, y)} e_i k_y, \quad k_y f_i = q^{-(\alpha_i, y)} f_i k_y \quad (y \in Y, i \in I), \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I), \\
(1-\alpha_{ij}) \sum_{r=0}^{1} (-1)^r e_i^{(1-\alpha_{ij}-r)} e_j e_i^{(r)} &= 0 \quad (i, j \in I, i \neq j), \\
(1-\alpha_{ij}) \sum_{r=0}^{1} (-1)^r f_i^{(1-\alpha_{ij}-r)} f_j f_i^{(r)} &= 0 \quad (i, j \in I, i \neq j).
\end{align*}

Here, \( q_i = q^{k_i}, \) \( k_i = k_{d_i \alpha_i} \) for \( i \in I, \) and \( e_i^{(n)} = e_i^n/[n]_q, \) \( f_i^{(n)} = f_i^n/[n]_q \) for \( i \in I, \) \( n \in \mathbb{Z}_{\geq 0}. \)

We will use the Hopf algebra structure of \( U_F(g) \) given by
\begin{align*}
\Delta(k_y) &= k_y \otimes k_y, \\
\Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \\
\varepsilon(k_y) &= 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\
S(k_y) &= k_y^{-1}, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i
\end{align*}
for \( y \in Y, i \in I. \) We define \( F \)-subalgebras \( U_F(h), U_F(b), U_F(n), U_F(n^+) \) of \( U_F(g) \) by
\begin{align*}
U_F(h) &= \langle k_y \mid y \in Y \rangle, \quad U_F(b) = \langle k_y, f_i \mid y \in Y, i \in I \rangle, \\
U_F(n) &= \langle f_i \mid i \in I \rangle, \quad U_F(n^+) = \langle e_i \mid i \in I \rangle.
\end{align*}

Then we have
\[ U_F(h) = \bigoplus_{y \in Y} F k_y. \]

For \( \lambda \in X \) we define a character \( \chi_\lambda : U_F(h) \to F \) by
\[ \chi_\lambda(k_y) = q^{(\lambda, y)} \quad (y \in Y). \]

For \( \gamma \in Q^+ \) set
\begin{align*}
U_F(n)_{-\gamma} &= \{ u \in U_F(n) \mid k_y u k_y^{-1} = q^{-(\gamma, y)} u \quad (y \in Y) \}, \\
U_F(n^+)_{\gamma} &= \{ u \in U_F(n^+) \mid k_y u k_y^{-1} = q^{\gamma, y} u \quad (y \in Y) \}.
\end{align*}

Then we have
\[ U_F(n) = \bigoplus_{\gamma \in Q^+} U_F(n)_{-\gamma}, \quad U_F(n^+) = \bigoplus_{\gamma \in Q^+} U_F(n^+)_{\gamma}. \]

2.5. Set \( A = \mathbb{Z}[g, q^{-1}]. \) We define an \( A \)-subalgebra \( U_A(h) \) of \( U_F(h) \) by
\[ U_A(h) = \{ u \in U_F(h) \mid \chi_\lambda(u) \in A \quad (\forall \lambda \in X) \}. \]
By the definition of $U_A(h)$, the character $\chi_\lambda : U_F(h) \to F$ for $\lambda \in X$ induces an algebra homomorphism $\chi_\lambda : U_A(h) \to A$. For $y \in Y$, $n \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}$ we have

$$k_y \in U_A(h), \quad \{q^m k_y\}_q \in U_A(h),$$

where, for $n \in \mathbb{Z}_{\geq 0}$, we set

$$\left\{ x \right\}_t = \prod_{s=1}^{n} \frac{x t^{-s+1} - 1}{t^s - 1} \in (Q(t))[x].$$

The proof of the following result is easily reduced to the case where $Y$ is of rank one. Details are omitted.

**Lemma 2.1.** Let $y_1, \ldots, y_m$ be a basis of the free $\mathbb{Z}$-module $Y$.

(i) (see [3, Theorem 3.1]) $U_A(h)$ is a free $A$-module with basis

$$\prod_{a=1}^{m} \left\{ k_{y_a} \right\}_{n_a} \left( n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0} \right).$$

(ii) $U_A(h) \cap F[k_{y_1}, \ldots, k_{y_m}]$ is a free $A$-module with basis

$$\prod_{a=1}^{m} \left\{ k_{y_a} \right\}_{n_a} \left( n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0} \right).$$

(iii) The ring $U_A(h)$ is the localization of $U_A(h) \cap F[k_{y_1}, \ldots, k_{y_m}]$ with respect to the multiplicative set $\{k_{n_1 y_1 + \cdots + n_m y_m} \mid n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}\}$.

We denote by $U_A(g)$ the $A$-subalgebra of $U_F(g)$ generated by $U_A(h)$ and $e_i^{(n)}$, $f_i^{(n)}$ for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$. It is naturally a Hopf algebra over $A$. We define $A$-subalgebras $U_A(b)$, $U_A(n)$, $U_A(n^+)$ of $U_A(g)$ by

$$U_A(b) = \langle U_A(h), e_i^{(n)} \mid i \in I, n \in \mathbb{Z}_{\geq 0} \rangle, \quad U_A(n) = \langle f_i^{(n)} \mid i \in I, n \in \mathbb{Z}_{\geq 0} \rangle, \quad U_A(n^+) = \langle e_i^{(n)} \mid i \in I, n \in \mathbb{Z}_{\geq 0} \rangle.$$

We have the triangular decomposition

$$U_A(g) \cong U_A(n^+) \otimes U_A(h) \otimes U_A(n), \quad U_A(b) \cong U_F(h) \otimes U_A(n),$$

where the isomorphisms are induced by the multiplication. For $\gamma \in Q^+$ set

$$U_A(n)_{-\gamma} = U_A(n) \cap U_F(n)_{-\gamma}, \quad U_A(n^+)_{\gamma} = U_A(n^+) \cap U_F(n)_{\gamma}.$$

Then we have

$$U_A(n) = \bigoplus_{\gamma \in Q^+} U_A(n)_{-\gamma}, \quad U_A(n^+) = \bigoplus_{\gamma \in Q^+} U_A(n^+)_{\gamma}.$$

It is known that $U_A(n)_{-\gamma}$ and $U_A(n^+)_{\gamma}$ are free $A$-modules of finite rank (see [3]).
2.6. Let $R$ be a commutative ring equipped with $\zeta \in R^\times$. We set

$$U_{R,\zeta}(g) = R \otimes_A U_A(g), \quad U_{R,\zeta}(b) = R \otimes_A U_A(b), \quad U_{R,\zeta}(h) = R \otimes_A U_A(h),$$

$$U_{R,\zeta}(n) = R \otimes_A U_A(n), \quad U_{R,\zeta}(n^+) = R \otimes_A U_A(n^+),$$

where $A \to R$ is given by $q \mapsto \zeta$. Then $U_{R,\zeta}(g)$ is a Hopf algebra over $R$, and $U_{R,\zeta}(b)$, $U_{R,\zeta}(h)$, $U_{R,\zeta}(n)$, $U_{R,\zeta}(n^+)$ are naturally identified with $R$-subalgebras of $U_{R,\zeta}(g)$. Moreover, $U_{R,\zeta}(b)$, $U_{R,\zeta}(h)$ are Hopf subalgebras. We have the triangular decomposition

$$U_{R,\zeta}(g) \cong U_{R,\zeta}(n) \otimes U_{R,\zeta}(h) \otimes U_{R,\zeta}(n^+), \quad U_{R,\zeta}(b) \cong U_{R,\zeta}(n) \otimes U_{R,\zeta}(h).$$

For $\gamma \in \mathbb{Q}^+$ we set

$$U_{R,\zeta}(n)_{-\gamma} = R \otimes_A U_A(n)_{-\gamma}, \quad U_{R,\zeta}(n^+)_{\gamma} = R \otimes_A U_A(n^+)_\gamma.$$

Then we have

$$U_{R,\zeta}(n) = \bigoplus_{\gamma \in \mathbb{Q}^+} U_{R,\zeta}(n)_{-\gamma}, \quad U_{R,\zeta}(n^+) = \bigoplus_{\gamma \in \mathbb{Q}^+} U_{R,\zeta}(n^+)_{\gamma}.$$

**Lemma 2.2.** Let $y_1, \ldots, y_m$ be a basis of the free $\mathbb{Z}$-module $Y$.

(i) $U_{R,\zeta}(h)$ is a free $R$-module with basis

$$1 \otimes \prod_{a=1}^m \left\{ \frac{k_{ya}}{n_a} \right\}_q k_{ya}^{-|[na+1]/2]} (n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}).$$

(ii) The elements

$$1 \otimes \prod_{a=1}^m \left\{ \frac{k_{ya}}{n_a} \right\}_q (n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0})$$

of $U_{R,\zeta}(h)$ are linearly independent over $R$.

(iii) Denote by $U_{R,\zeta}^+(h)$ the $R$-submodule of $U_{R,\zeta}(h)$ generated by the elements in (ii). Then $U_{R,\zeta}^+(h)$ is a subring of $U_{R,\zeta}(h)$. Moreover, $U_{R,\zeta}(h)$ is a localization of $U_{R,\zeta}^+(h)$ with respect to the multiplicative set $\{k_{n_1y_1+\cdots+n_my_m} \mid n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}\}$. Hence

$$U_{R,\zeta}(h) = \bigcup_{n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}} k_{n_1y_1+\cdots+n_my_m}^{-1} U_{R,\zeta}^+(h).$$

**Proof.** Set

$L = R \otimes_A (U_A(h) \cap \mathbb{F}[k_{y_1}, \ldots, k_{y_m}])$, \quad $S = \{k_{n_1y_1+\cdots+n_my_m} \mid n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}\} \subset L$.

In view of Lemma 2.1 it is sufficient to verify that the canonical homomorphism $L \to S^{-1}L$ is injective. Hence we have only to show that the map $L \ni z \mapsto k_{y_1}^{-1}z \in L$ is injective for any $a$. This is easily reduced to the case where $Y$ is of rank one. Details are omitted.

Let $\lambda \in X$. By abuse of notation we denote by

$$\chi_\lambda : U_{R,\zeta}(h) \to R$$

the $R$-algebra homomorphism induced by $\chi_\lambda : U_F(h) \to F$.

The proof of the following fact is reduced to the rank one case. Details are omitted.
Lemma 2.3. \(\text{(i)}\) The subset \(\{\chi_\lambda \mid \lambda \in X\}\) of \(\text{Hom}_R(U_{R,\zeta}(\mathfrak{h}), R)\) is linearly independent over \(R\).

\(\text{(ii)}\) Let \(h \in U_{R,\zeta}(\mathfrak{h})\). If \(\chi_\lambda(h) = 0\) for any \(\lambda \in X\), then we have \(h = 0\).

We set \(\zeta_\alpha = \zeta_{d_\alpha} (\alpha \in \Delta), \quad \zeta_i = \zeta_{d_i} (i \in I)\).

2.7. In this subsection we assume that \(\zeta_\alpha = \pm 1\) for any \(\alpha \in \Delta\). We compare \(U_{R,\zeta}(\mathfrak{g})\) with \(U_{R,\zeta}(\mathfrak{h})\) in the following.

Note that \(\zeta_s = 1\) for some \(s \in \mathbb{Z}_{>0}\) by our assumption. For \(y \in Y\) and \(n \in \mathbb{Z}_{\geq 0}\) we set \(h(y,n) = 1 \otimes \left\{ k_{sy} \right\}_{q^s} \in U_{R,\zeta}(\mathfrak{h})\).

By Lemma 2.3 \(\text{(ii)}\) it is characterized as the element of \(U_{R,\zeta}(\mathfrak{h})\) satisfying

\[\chi_\lambda(h(y,n)) = \left( \langle \lambda, y \rangle \right)_{n} 1_R\]

for any \(\lambda \in X\). In particular, \(h(y,n)\) does not depend on the choice of \(s\). Denote by \(U'_{R,\zeta}(\mathfrak{h})\) the subalgebra of \(U'_{R,\zeta}(\mathfrak{h})\) generated by the elements \(h(y,n)\) for \(y \in Y\), \(n \in \mathbb{Z}_{\geq 0}\). By Lemma 2.3 \(\text{(ii)}\) we see easily the following

Lemma 2.4. If \(y_1, \ldots, y_m\) is a basis of the free \(\mathbb{Z}\)-module \(Y\), then

\[\prod_{a=1}^m h(y_a, n_a) \quad (n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0})\]

form a basis of the \(R\)-module \(U'_{R,\zeta}(\mathfrak{h})\).

For \(i \in I\), \(n \in \mathbb{Z}_{\geq 0}\), \(s \in \mathbb{Z}\) set

\[\left[ k_i; s \atop n \right]_{q_i} = \prod_{a=1}^n \frac{q_i^{s-a+1}k_i - q_i^{-s+a-1}k_i^{-1}}{q_i^a - q_i^{-a}} \in U_{A}(\mathfrak{h}),\]

\[t(i, n, s) = 1 \otimes q_i^{-n(n-s)}k_i \left[ k_i; s \atop n \right]_{q_i} \in U_{R,\zeta}(\mathfrak{h}).\]

Then we have

\[\chi_\lambda(t(i, n, s)) = \left( \langle \lambda, \alpha_i^\vee \rangle + s \right)_{n} 1_R\]

for any \(\lambda \in X\). Note

\[\left( x + s \atop n \right) \in \sum_{m=0}^n \mathbb{Z} \left( x \atop m \right)\]

in \(\mathbb{Q}[x]\). Hence by Lemma 2.3 \(\text{(ii)}\) we have

\[t(i, n, s) \in \sum_{m=0}^n R h(\alpha_i^\vee, m) \in U'_{R,\zeta}(\mathfrak{h}) \quad (i \in I, n \in \mathbb{Z}_{\geq 0}).\]

Take a subset \(J\) of \(I\) satisfying

\[(2.2) \quad i, j \in I, \quad a_{ij} < 0 \implies |J \cap \{i, j\}| = 1.\]
For $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$ we define $e(i, n), f(i, n) \in U_{R, \xi}(\mathfrak{g})$ by

$$e(i, n) = \begin{cases} \zeta_i^n e_{\xi i}^{(n)} & (i \in J) \\ \zeta_i^{n-1} e_{\xi i}^{(n)} & (i \notin J), \end{cases}$$

$$f(i, n) = \begin{cases} \zeta_i^n f_{\xi i}^{(n)} k_i^n & (i \in J) \\ \zeta_i^{n-1} f_{\xi i}^{(n)} & (i \notin J). \end{cases}$$

We define $U_{R, \xi}(n)$ (resp. $U_{R, \xi}(n^+)$) the $R$-subalgebra of $U_{R, \xi}(\mathfrak{g})$ generated by

$$\{ f(i, n) \mid i \in J, n \in \mathbb{Z}_{\geq 0} \} \text{ (resp. } \{ e(i, n) \mid i \in J, n \in \mathbb{Z}_{\geq 0} \}).$$

For $\gamma = \sum_{i \in I} m_i \alpha_i \in Q$ we set $\gamma^{+} = \sum_{i \in J} m_i \alpha_i \in Q$. Then we have

$$(2.3) \quad U_{R, \xi}^{+}(n) = \bigoplus_{\gamma \in Q^+} U_{R, \xi}(n)^{-\gamma} k_{\gamma}, \quad U_{R, \xi}^{+}(n^+) = \bigoplus_{\gamma \in Q^+} U_{R, \xi}(n^+) \gamma k_{\gamma-\gamma}.$$}

We define $U_{R, \xi}^{+}(\mathfrak{g})$ to be the $R$-subalgebra of $U_{R, \xi}(\mathfrak{g})$ generated by $U_{R, \xi}^{+}(\mathfrak{h}), U_{R, \xi}^{+}(\mathfrak{n}), U_{R, \xi}^{+}(n^+)$.

By well-known relations in $U_{\mathfrak{g}}(\mathfrak{f})$ we have

$$(2.4) \quad he(i, n) = \xi_{\alpha_{i}}(h) e(i, n) h, \quad hf(i, n) = \xi_{-\alpha_{i}}(h) f(i, n) h,$$

$$(2.5) \quad e(i, n) f(j, m) = f(j, m) e(i, n), \quad (i \neq j),$$

$$(2.6) \quad e(i, n) f(i, m) = \sum_{0 \leq a \leq n, m} f(i, m-a) t(i, a, 2a - m) e(i, n-a)$$

for $h \in U_{R, \xi}^{+}(\mathfrak{h}), n, m \in \mathbb{Z}_{\geq 0}, i, j \in I$. By (2.4), (2.5), (2.6) we see that the multiplication of $U_{R, \xi}^{+}(\mathfrak{g})$ induces the isomorphism

$$(2.7) \quad U_{R, \xi}^{+}(\mathfrak{g}) \cong U_{R, \xi}^{+}(\mathfrak{n}) \otimes U_{R, \xi}^{+}(\mathfrak{h}) \otimes U_{R, \xi}^{+}(n^+)$$

of $R$-modules.

**Lemma 2.5.** Recall that $\xi_{\alpha} = \pm 1$ for any $\alpha \in \Delta$. We have isomorphisms

$$U_{R}(\mathfrak{n}) \cong U_{R}^{+}(\mathfrak{n}) \quad (f_{i}^{(n)} \leftrightarrow f(i, n)), \quad U_{R}(n^+) \cong U_{R, \xi}^{+}(n^+) \quad (e_{i}^{(n)} \leftrightarrow e(i, n))$$

of $R$-algebras.

**Proof.** We may assume $\mathfrak{g}$ is simple. By considering the situation where the bilinear form $(\ , \ )$ in (2.1) is replaced by $d_{\alpha}^{-1}(\ , \ )$ for short roots $\alpha$, we may further assume that $q = q_{\alpha}$ for short roots $\alpha$. In this case $\xi = \pm 1$, and we have

$$U_{R, \xi}^{+}(n) = R \otimes \mathbb{Z} U_{Z, \xi}(\mathfrak{n}), \quad U_{R, \xi}^{+}(n^+) = R \otimes \mathbb{Z} U_{Z, \xi}(n^+).$$

Here $\xi = 1 \in \mathbb{Z}$ if $\xi = 1 \in R$, and $\xi = -1 \in \mathbb{Z}$ if $\xi = -1 \in R$. Hence we have only to show

$$U_{Z}(\mathfrak{n}) \cong U_{Z, \xi}(\mathfrak{n}), \quad U_{Z}(n^+) \cong U_{Z, \xi}(n^+).$$

By using the embeddings

$$U_{Z}(\mathfrak{n}) \subset U_{\mathfrak{Q}}^{+}(\mathfrak{n}), \quad U_{Z}(n^+) \subset U_{\mathfrak{Q}}^{+}(n^+), \quad U_{Z, \xi}(\mathfrak{n}) \subset U_{\mathfrak{Q}, \xi}(\mathfrak{n}), \quad U_{Z, \xi}(n^+) \subset U_{\mathfrak{Q}, \xi}(n^+)$$

the proof is reduced to showing

$$U_{\mathfrak{Q}}^{+}(\mathfrak{n}) \cong U_{\mathfrak{Q}, \xi}(\mathfrak{n}), \quad U_{\mathfrak{Q}}^{+}(n^+) \cong U_{\mathfrak{Q}, \xi}(n^+).$$

This is easily verified by checking the Serre type relations. \hfill \Box
In view of the relations (2.4), (2.5), (2.6) we see from (2.7), Lemma 2.5 the following.

**Proposition 2.6.** Recall that \( \zeta_\alpha = \pm 1 \) for any \( \alpha \in \Delta \). We have an isomorphism

\[
\mathcal{U}_R(\mathfrak{g}) \cong U_{R,\zeta}(\mathfrak{g})
\]

of \( R \)-algebras given by

\[
(y) \mapsto h(y, n), \quad \mathcal{e}^{(n)}(i) \mapsto e(i, n), \quad \mathcal{f}^{(n)}(i) \mapsto f(i, n)
\]

for \( y \in Y, i \in I, n \in \mathbb{Z}_{\geq 0} \) (compare [9] Proposition 33.2.3).

**Remark 2.7.** For \( i \in I \) satisfying \( \zeta_i = 1 \) we have \( k_i = 1 \) in \( U_{R,\zeta}(\mathfrak{h}) \) by Lemma 2.3. Hence if \( \zeta_i = 1 \) for any \( i \in I \), then (2.8) turns out to be a Hopf algebra isomorphism. In general (2.8) does not preserve the comultiplication.

2.8. Let \( R \) be a commutative ring equipped with \( \zeta \in R^\times \). In this subsection we assume that there exists some \( \ell \in \mathbb{Z}_{>0} \) such that \( f_\ell(\zeta) = 0 \), where \( f_\ell \) is the \( \ell \)-th cyclotomic polynomial. As in [9] we define a root datum \((\tilde{\Delta}, \tilde{\mathfrak{g}}, \tilde{\Delta}^\vee)\) as follows. Let

\[
\tilde{\Delta} = \{ \tilde{\alpha} \mid \alpha \in \Delta \} \text{ turns out to be a root system with } \{ \tilde{\alpha}_i \mid i \in I \} \text{ a set of simple roots. Moreover, } \tilde{\Delta}^\vee = \{ \tilde{\alpha}^\vee \mid \alpha \in \Delta \} \text{ is the set of coroots for the root system } \tilde{\Delta}. \text{ Set}
\]

\[
\tilde{\Delta} = \{ \tilde{\alpha} \mid \alpha \in \Delta \} \text{ turns out to be a root system with } \{ \tilde{\alpha}_i \mid i \in I \} \text{ a set of simple roots. Moreover, } \tilde{\Delta}^\vee = \{ \tilde{\alpha}^\vee \mid \alpha \in \Delta \} \text{ is the set of coroots for the root system } \tilde{\Delta}. \text{ Set}
\]

Let \( \tilde{\mathfrak{g}} \) be the connected reductive algebraic group over \( C \) with root datum \((\tilde{\Delta}, \tilde{\mathfrak{g}}, \tilde{\Delta}^\vee)\), and let \( \mathfrak{g} \) be its Lie algebra. We denote by \( U_{R,\zeta}(\tilde{\mathfrak{g}}) \) the quantized enveloping algebra over \( R \) associated to the root datum \((\tilde{\Delta}, \tilde{\mathfrak{g}}, \tilde{\Delta}^\vee)\) and the \( W \)-invariant symmetric bilinear form (2.1) on \( \sum_{\alpha \in \Delta} Q^\alpha \). We similarly define \( R \)-subalgebras \( U_{R,\zeta}(\mathfrak{h}), U_{R,\zeta}(\mathfrak{t}), U_{R,\zeta}(\mathfrak{m}), U_{R,\zeta}(\mathfrak{m}^+) \) of \( U_{R,\zeta}(\tilde{\mathfrak{g}}) \).

Note that for any \( \alpha \in \Delta \) we have \( \zeta_\alpha = \zeta^\alpha = \pm 1 \). Hence we can apply the results in the preceding subsection to \( U_{R,\zeta}(\tilde{\mathfrak{g}}) \). In particular, we have \( \overline{U}_R(\mathfrak{g}) \cong U_{R,\zeta}(\mathfrak{g}) \subset U_{R,\zeta}(\tilde{\mathfrak{g}}) \).

Following Lusztig we define the quantum Frobenius homomorphism

\[
\mathcal{F} : U_{R,\zeta}(\mathfrak{g}) \to U_{R,\zeta}(\tilde{\mathfrak{g}})
\]
as follows. By $Y \subset \mathfrak{y}$ we have an inclusion $U_{\mathfrak{F}}(\mathfrak{h}) \subset U_{\mathfrak{F}}(\mathfrak{y})$ sending $k_y$ for $y \in Y$ to $k_y$ for $y \in Y \subset \mathfrak{y}$. This induces $U_{\mathfrak{A}}(\mathfrak{h}) \subset U_{\mathfrak{A}}(\mathfrak{y})$ because $\mathfrak{y} \subset \mathfrak{x}$. Hence we obtain a natural homomorphism
\[
F_\mathfrak{h} : U_{R,\zeta}(\mathfrak{h}) \to U_{R,\zeta}(\mathfrak{y}).
\]
On the other hand by [9, Theorem 35.1.7] we have well-defined algebra homomorphisms
\[
F_n : U_{R,\zeta}(n) \to U_{R,\zeta}(\mathfrak{y}), \quad F_{n+} : U_{R,\zeta}(n^+) \to U_{R,\zeta}(\mathfrak{y}^+)
\]
satisfying
\[
F_n(f_i^{(n)}) = \begin{cases} f_i^{(n/r_i)}(r_i|n) & \text{(otherwise)} \\ 0 & \text{(otherwise)} \end{cases}, \quad F_{n+}(e_i^{(n)}) = \begin{cases} e_i^{(n/r_i)}(r_i|n) & \text{(otherwise)} \\ 0 & \text{(otherwise)} \end{cases}
\]
for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$ (see also the last paragraph of [9, 35.5.2]). The following result is proved exactly as in [9, Theorem 35.1.9] using Lemma 2.3 (ii).

**Proposition 2.8.** There exists a unique Hopf algebra homomorphism
\[
F : U_{R,\zeta}(\mathfrak{g}) \to U_{R,\zeta}(\mathfrak{y})
\]
satisfying $F|_{U_{R,\zeta}(\mathfrak{h})} = F_\mathfrak{h}$, $F|_{U_{R,\zeta}(n)} = F_n$, $F|_{U_{R,\zeta}(n^+)} = F_{n+}$.

3. Representations

3.1. Let $\mathcal{H}$ be a Hopf algebra over a commutative ring $R$. For left $\mathcal{H}$-modules $M_1$, $M_2$ we regard the left $\mathcal{H} \otimes \mathcal{H}$-module $M_1 \otimes M_2 = M_1 \otimes_R M_2$ as a left $\mathcal{H}$-module via the comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. For a left (resp. right) $\mathcal{H}$-module $M$ we regard $M^* = \text{Hom}_R(M, R)$ as a right (resp. left) $\mathcal{H}$-module by
\[
\langle m^* h, m \rangle = \langle m^*, hm \rangle, \quad \text{(resp. } \langle hm^*, m \rangle = \langle m^*, mh \rangle).\]
for $h \in \mathcal{H}$, $m \in M$, $m^* \in M^*$.

3.2. For a left (resp. right) $\mathcal{U}^e(\mathfrak{g})$-module $V$ and $\mu \in X$ we set
\[
V_\mu = \{ v \in V \mid hv = \mu(h)v \ (h \in \mathfrak{h}) \}, \quad \text{(resp. } V_\mu = \{ v \in V \mid vh = \mu(h)v \ (h \in \mathfrak{h}) \}).\]
For $\lambda \in X^+$ we define a $\mathcal{U}^e(\mathfrak{g})$-module $\mathcal{V}(\lambda)$ by
\[
\mathcal{V}(\lambda) = \mathcal{U}(\mathfrak{g})/\left( \sum_{h \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})(h - \lambda(h) + \sum_{i \in I} \mathcal{U}(\mathfrak{g})\mathfrak{e}_i + \sum_{i \in I} \mathcal{U}(\mathfrak{g})f_i^{(\lambda, n^\vee) + 1} \right).
\]
Then $\mathcal{V}(\lambda)$ is a finite-dimensional irreducible $\mathcal{U}^e(\mathfrak{g})$-module, which has the weight space decomposition $\mathcal{V}(\lambda) = \bigoplus_{\mu \in X} \mathcal{V}(\lambda)_\mu$. Set $\mathfrak{v}_\lambda = \mathfrak{I} \in \mathcal{V}(\lambda)$, so that $\mathcal{V}(\lambda)_\mu = C_{\mathfrak{v}_\lambda}$. Set $\mathcal{V}^e(\lambda) = \text{Hom}_C(\mathcal{V}(\lambda), C)$. It is a finite-dimensional irreducible right $\mathcal{U}^e(\mathfrak{g})$-module. The weight space decomposition of $\mathcal{V}(\lambda)$ gives the weight space decomposition $\mathcal{V}^e(\lambda) = \bigoplus_{\mu \in X} \mathcal{V}^e(\lambda)_\mu$ of $\mathcal{V}^e(\lambda)$, where $\mathcal{V}^e(\lambda)_\mu = (\mathcal{V}(\lambda)_\mu)^*$. We define $\mathfrak{v}_\lambda \in \mathcal{V}^e(\lambda)_\lambda$ by $\langle \mathfrak{v}_\lambda, \mathfrak{v}_\lambda \rangle = 1$.

For $\lambda \in X^+$ we define a $\mathcal{U}_Z(\mathfrak{g})$-submodule $\mathcal{Z}_\lambda$ of $\mathcal{V}(\lambda)$ and a right $\mathcal{U}_Z(\mathfrak{g})$-submodule $\mathcal{Z}_\lambda^e$ of $\mathcal{V}^e(\lambda)$ by
\[
\mathcal{Z}_\lambda = \mathcal{U}_Z(\mathfrak{g})\mathfrak{v}_\lambda, \quad \mathcal{Z}_\lambda^e = \mathfrak{v}_\lambda^e \mathcal{U}_Z(\mathfrak{g}).
\]
Then $\Delta_Z(\lambda)$ and $\Delta_Z^*(\lambda)$ are free $\mathbb{Z}$-modules satisfying
\[
C \otimes_{\mathbb{Z}} \Delta_Z(\lambda) \cong \nabla(\lambda), \quad C \otimes_{\mathbb{Z}} \Delta_Z^*(\lambda) \cong \nabla^*(\lambda).
\]
For $\lambda \in X^+$ we define a $U_Z(\mathfrak{g})$-submodule $\nabla_Z(\lambda)$ of $\nabla(\lambda)$ and a right $U_Z(\mathfrak{g})$-submodules $\nabla_Z^*(\lambda)$ of $\nabla^*(\lambda)$ by
\[
\nabla_Z(\lambda) = \{ v \in \nabla(\lambda) \mid \langle \Delta_Z(\lambda), v \rangle \subset \mathbb{Z} \} \cong \text{Hom}_{\mathbb{Z}}(\Delta_Z(\lambda), \mathbb{Z}),
\]
\[
\nabla_Z^*(\lambda) = \{ v^* \in \nabla^*(\lambda) \mid \langle v^*, \Delta_Z(\lambda) \rangle \subset \mathbb{Z} \} \cong \text{Hom}_{\mathbb{Z}}(\Delta_Z^*(\lambda), \mathbb{Z}).
\]
Then we have
\[
\Delta_Z(\lambda) \subset \nabla_Z(\lambda) \subset \nabla(\lambda), \quad \Delta_Z^*(\lambda) \subset \nabla_Z^*(\lambda) \subset \nabla^*(\lambda).
\]
Moreover, $\nabla_Z(\lambda)$ and $\nabla_Z^*(\lambda)$ are free $\mathbb{Z}$-modules satisfying
\[
C \otimes_{\mathbb{Z}} \nabla_Z(\lambda) \cong \nabla(\lambda), \quad C \otimes_{\mathbb{Z}} \nabla_Z^*(\lambda) \cong \nabla^*(\lambda).
\]
Let $R$ be a commutative ring. For a $\mathcal{U}_R(\mathfrak{g})$-module $V$ and $\mu \in X$ we set
\[
V_\mu = \{ v \in V \mid hv = \nabla_\mu(h)v \ (h \in \mathcal{U}_R(\mathfrak{h})) \}.
\]
We say that a $\mathcal{U}_R(\mathfrak{g})$-module $V$ is integrable if it has the weight space decomposition $V = \bigoplus_{\mu \in X} V_\mu$, and for any $v \in V$ and $i \in I$ we have $e_i^{(n)} v = f_i^{(n)} v = 0$ for $n \gg 0$. We denote by $\text{Mod}_{\text{int}}(\mathcal{U}_R(\mathfrak{g}))$ the category of integrable $\mathcal{U}_R(\mathfrak{g})$-modules.

Let $V$ be an integrable $\mathcal{U}_R(\mathfrak{g})$-module. For $i \in I$ we define an invertible $R$-homomorphism $T_i|_V = T_i : V \to V$ by
\[
T_i v = \sum_{a-b+c=(\lambda, \alpha_i^\vee)} (-1)^b f_i^{(a)} e_i^{(b)} f_i^{(c)} v \quad (\lambda \in X, v \in V_\lambda).
\]
For $w \in W$ we define an invertible $R$-homomorphism $T_w|_V = T_w : V \to V$ by
\[
T_w = T_{w_1} \cdots T_{w_N},
\]
where $w = s_{i_1} \cdots s_{i_N}$ is a reduced expression of $w$. It does not depend on the choice of a reduced expression. Moreover, we have $T_wV_\lambda = V_{w\lambda}$ for $\lambda \in X$. Regarding $\mathcal{U}_R(\mathfrak{g})$ as an integrable $\mathcal{U}_R(\mathfrak{g})$-module via the adjoint action we have
\[
(T_w|_V)(uv) = (T_w|_{\mathcal{U}_R(\mathfrak{g})}(u))(T_w|_V(v)) \quad (u \in \mathcal{U}_R(\mathfrak{g}), v \in V).
\]
The following is well-known.

**Lemma 3.1.** For integrable $\mathcal{U}_R(\mathfrak{g})$-modules $V_1$, $V_2$ we have
\[
T_w|_{V_1 \otimes V_2} = T_w|_{V_1} \otimes T_w|_{V_2}.
\]

For $\lambda \in X^+$ we define left $\mathcal{U}_R(\mathfrak{g})$-modules $\Delta_R(\lambda)$, $\nabla_R(\lambda)$ and right $\mathcal{U}_R(\mathfrak{g})$-modules $\Delta_R^*(\lambda)$, $\nabla_R^*(\lambda)$ as the base changes of $\Delta_Z(\lambda)$, $\nabla_Z(\lambda)$, $\Delta_Z^*(\lambda)$, $\nabla_Z^*(\lambda)$ respectively. We call $\Delta_R(\lambda)$, $\nabla_R(\lambda)$ the (left and right) Weyl modules with highest weight $\lambda$, and $\Delta_R^*(\lambda)$, $\nabla_R^*(\lambda)$ the (left and right) dual Weyl modules with highest weight $\lambda$. The left $\mathcal{U}_R(\mathfrak{g})$-modules $\Delta_R(\lambda)$, $\nabla_R(\lambda)$ are integrable.
3.3. For a left (resp. right) $U_F(\mathfrak{g})$-module $V$ and $\mu \in X$ we set

$$V_\mu = \{ v \in V \mid hv = \chi_\mu(h)v \ (h \in U_F(\mathfrak{h})) \},$$

(resp. $V_\mu = \{ v \in V \mid vh = \chi_\mu(h)v \ (h \in U_F(\mathfrak{h})) \}).$

For $\lambda \in X^+$ we define a $U_F(\mathfrak{g})$-module $V_F(\lambda)$ by

$$V_F(\lambda) = U_F(\mathfrak{g})/ \left( \sum_{h \in U_F(\mathfrak{h})} U_F(\mathfrak{g})(h - \chi_\lambda(h)) + \sum_{i \in I} U_F(\mathfrak{g}) e_i + \sum_{i \in I} U_F(\mathfrak{g}) f_i^{(\lambda, \alpha_i)} \right).$$

Then $V_F(\lambda)$ is a finite-dimensional irreducible $U_F(\mathfrak{g})$-module. Set $V_F^*(\lambda) = \text{Hom}_F(V_F(\lambda), F)$. It is a finite-dimensional irreducible right $U_F(\mathfrak{g})$-module. The weight space decomposition $V_F(\lambda) = \bigoplus_{\mu \in X} V_F(\lambda)_\mu$ of $V_F(\lambda)$ gives the weight space decomposition $V_F^*(\lambda) = \bigoplus_{\mu \in X} V_F^*(\lambda)_\mu$ of $V_F^*(\lambda)$, where $V_F^*(\lambda)_\mu = (V_F(\lambda)_\mu)^*$. Set $v_\lambda = \overline{1} \in V_F(\lambda)$. Then we have $V_F(\lambda)_\lambda = Fv_\lambda$. We define $v_\lambda^* \in V_F^*(\lambda)_\lambda$ by $\langle v_\lambda^*, v_\lambda \rangle = 1$.

3.4. For $\lambda \in X^+$ we define a $U_A(\mathfrak{g})$-submodule $\Delta_A(\lambda)$ of $V_F(\lambda)$ and a right $U_A(\mathfrak{g})$-submodule $\Delta_A^*(\lambda)$ of $V_F^*(\lambda)$ by

$$\Delta_A(\lambda) = U_A(\mathfrak{g})v_\lambda, \quad \Delta_A^*(\lambda) = v_\lambda^* U_A(\mathfrak{g}).$$

They have the weight space decomposition

$$\Delta_A(\lambda) = \bigoplus_{\mu \in X} \Delta_A(\lambda)_\mu, \quad \Delta_A^*(\lambda) = \bigoplus_{\mu \in X} \Delta_A^*(\lambda)_\mu,$$

where

$$\Delta_A(\lambda)_\mu = \{ v \in \Delta_A(\lambda) \mid hv = \chi_\mu(h)v \ (h \in U_A(\mathfrak{h})) \},$$

$$\Delta_A^*(\lambda)_\mu = \{ v \in \Delta_A^*(\lambda) \mid vh = \chi_\mu(h)v \ (h \in U_A(\mathfrak{h})) \}.$$ 

It follows from the deep theory of canonical bases that $\Delta_A(\lambda)_\mu$ and $\Delta_A^*(\lambda)_\mu$ are free $A$-modules (see [9]). In particular, $\Delta_A(\lambda)$ and $\Delta_A^*(\lambda)$ are free $A$-modules satisfying

$$F \otimes_A \Delta_A(\lambda) \cong V_F(\lambda), \quad F \otimes_A \Delta_A^*(\lambda) \cong V_F^*(\lambda).$$

We define a $U_A(\mathfrak{g})$-submodule $\nabla_A(\lambda)$ of $V_F(\lambda)$ and a right $U_A(\mathfrak{g})$-submodules $\nabla_A^*(\lambda)$ of $V_F^*(\lambda)$ by

$$\nabla_A(\lambda) = \{ v \in V_F(\lambda) \mid \langle \Delta_A^*(\lambda), v \rangle \subset A \} \cong \text{Hom}_A(\Delta_A^*(\lambda), A),$$

$$\nabla_A^*(\lambda) = \{ v^* \in V_F^*(\lambda) \mid \langle v^*, \Delta_A(\lambda) \rangle \subset A \} \cong \text{Hom}_A(\Delta_A(\lambda), A).$$

Then we have

$$\Delta_A(\lambda) \subset \nabla_A(\lambda) \subset V_F(\lambda), \quad \Delta_A^*(\lambda) \subset \nabla_A^*(\lambda) \subset V_F^*(\lambda).$$

We have the weight space decomposition

$$\nabla_A(\lambda) = \bigoplus_{\mu \in X} \nabla_A(\lambda)_\mu, \quad \nabla_A^*(\lambda) = \bigoplus_{\mu \in X} \nabla_A^*(\lambda)_\mu,$$

where

$$\nabla_A(\lambda)_\mu = \{ v \in \nabla_A(\lambda) \mid hv = \chi_\mu(h)v \ (h \in U_A(\mathfrak{h})) \} = \text{Hom}_A(\Delta_A^*(\lambda)_\mu, A),$$

$$\nabla_A^*(\lambda)_\mu = \{ v \in \nabla_A^*(\lambda) \mid vh = \chi_\mu(h)v \ (h \in U_A(\mathfrak{h})) \} = \text{Hom}_A(\Delta_A(\lambda)_\mu, A).$$
By the duality $\nabla_A(\lambda)_\mu$ and $\nabla^*_A(\lambda)_\mu$ are free $A$-modules satisfying \[
\mathbf{F} \otimes_A \nabla_A(\lambda)_\mu \cong V_\mathbf{F}(\lambda)_\mu, \quad \mathbf{F} \otimes_A \nabla^*_A(\lambda)_\mu \cong V^*_\mathbf{F}(\lambda)_\mu.
\] In particular, $\nabla_A(\lambda)$ and $\nabla^*_A(\lambda)$ are free $A$-modules satisfying \[
\mathbf{F} \otimes_A \nabla_A(\lambda) \cong V_\mathbf{F}(\lambda), \quad \mathbf{F} \otimes_A \nabla^*_A(\lambda) \cong V^*_\mathbf{F}(\lambda).
\]

3.5. Let $R$ be a commutative ring equipped with $\zeta \in R^\times$, and consider $U_{R,\zeta}(g) = R \otimes_A U_A(g)$. For a $U_{R,\zeta}(g)$-module $V$ and $\mu \in X$ we set \[
V_\mu = \{ v \in V \mid hv = \chi_\mu(h)v \ (h \in U_{R,\zeta}(h)) \}.
\] We say that a $U_{R,\zeta}(g)$-module $V$ is integrable if it has the weight space decomposition \[
V = \bigoplus_{\mu \in X} V_\mu,
\] and for any $v \in V$ and $i \in I$ we have $e_i^{(n)}v = f_i^{(n)}v = 0$ for $n \gg 0$. We denote by $\text{Mod}_m(U_{R,\zeta}(g))$ the category of integrable $U_{R,\zeta}(g)$-modules.

Let $V$ be an integrable $U_{R,\zeta}(g)$-module. Following [9] we define an invertible $R$-homomorphism $T_i|_V = T_i : V \rightarrow V$ for $i \in I$ by \[
T_i v = \sum_{a-b+c=\langle \lambda, \alpha_i^\vee \rangle} (-1)^{b+c-a} f_i^{(a)} e_i^{(b)} f_i^{(c)} v \quad (v \in V_\lambda).
\] Note $T_i = T_{i-1}$ in the notation of [9]. We will use the following fact (see [9 Proposition 5.3.4]).

**Lemma 3.2.** Let $i \in I$, and let $V_1, V_2$ be integrable $U_{R,\zeta}(g)$-modules.

(i) Assume that $v_1 \in V_1$ satisfies $e_i^{(n)}v_1 = 0$ for any $n > 0$. Then we have \[
(T_i|_{V_1} \otimes |_{V_2})(v_1 \otimes v_2) = T_i v_1 \otimes v_2
\] for any $v_2 \in V_2$.

(ii) Assume $\zeta^2 = 1$. Then we have $T_i|_{V_1} \otimes |_{V_2} = T_i|_{V_1} \otimes T_i|_{V_2}$.

For $v \in V$ we define an $R$-homomorphism $T_w|_V = T_w : V \rightarrow V$ by $T_w = T_{i_1} \ldots T_{i_N}$, where $w = s_{i_1} \ldots s_{i_N}$ is a reduced expression of $w$. It does not depend on the choice of a reduced expression. Moreover, we have $T_wV_\lambda = V_{\lambda + \mu}$ for $\lambda \in X$. By [9] there exists an automorphism $T_w$ of the $R$-algebra $U_{R,\zeta}(g)$ satisfying \[
T_wv = (T_w u)(T_w v) \quad (u \in U_{R,\zeta}(g), v \in V).
\]

**Lemma 3.3.** For a $U_{R,\zeta}(g)$-module $V$ the following conditions are equivalent.

(a) $V$ has the weight decomposition $V = \bigoplus_{\mu \in X} V_\mu$, and for any $v \in V$ the $U_{R,\zeta}(g)$-submodule $U_{R,\zeta}(g)v$ of $V$ is a finitely generated $R$-module.

(b) $V$ is an integrable $U_{R,\zeta}(g)$-module.

**Proof.** The indication (a)⇒(b) is clear from $e_i^{(n)}V_\lambda \subset V_{\lambda + n\alpha_i}$ and $f_i^{(n)}V_\lambda \subset V_{\lambda - n\alpha_i}$. Assume (b) holds. For any $v \in V$, $w \in W$, $i \in I$ we have $(T_w e_i^{(n)})v = (T_w f_i^{(n)})v = 0$ for $n \gg 0$ by (3.3). Hence we can deduce (a) using the PBW-type basis for $U_{R,\zeta}(g)$ (see [9]).

For $\lambda \in X^+$ we define left $U_{R,\zeta}(g)$-modules $\Delta_{R,\zeta}(\lambda)$, $\nabla_{R,\zeta}(\lambda)$ and right $U_{R,\zeta}(g)$-modules $\Delta_{R,\zeta}^*(\lambda)$, $\nabla_{R,\zeta}^*(\lambda)$ as the base changes of $\Delta_A(\lambda)$, $\nabla_A(\lambda)$, $\Delta_A^*(\lambda)$, $\nabla_A^*(\lambda)$ respectively. We call $\Delta_{R,\zeta}(\lambda)$, $\Delta_{R,\zeta}^*(\lambda)$ the (left and right) Weyl modules with highest
weight $λ$, and $∇_{R,ζ}(λ)$, $∇_{R,ζ}(λ)$ the (left and right) dual Weyl modules with highest weight $λ$.

3.6. In this subsection we assume $ζ_α = ±1$ for any $α ∈ Δ$. Take a subset $J$ of $I$ as in (2.2), and identify $U_R(ĝ)$ with a subalgebra of $U_{R,ζ}(ĝ)$ via Proposition 2.6. Since $U_{R,ζ}(ĝ)$ is generated by $U_R(ĝ)$ and $U_R(h)$, we see easily the following.

**Proposition 3.4.** The embedding $U_R(ĝ) ⊂ U_{R,ζ}(ĝ)$ induces the equivalence

$$\text{Mod}_{\text{int}}(U_{R,ζ}(ĝ)) ≅ \text{Mod}_{\text{int}}(U_R(ĝ))$$

of abelian categories.

**Lemma 3.5.** For $w ∈ W$ and $λ ∈ X$ there exists $ε_{w,λ} ∈ \{±1\}$ such that for any integrable $U_{R,ζ}(ĝ)$-module $V$ we have

$$T_w v = ε_{w,λ} T_w v \quad (v ∈ V_λ).$$

Here, $T_w$ (resp. $T_ν$) is defined as an operator on the integrable $U_R(ĝ)$-module (resp. $U_{R,ζ}(ĝ)$-module) $V$.

**Proof.** We may assume $w = s_i$ for $i ∈ I$. Recall

$$T_i v = ∑_{a-b+c=\langle λ,α_i^∨ \rangle} (-1)^b c^{\langle λ,α_i^∨ \rangle} f_i^n c_i e_i^n d_i v.$$

For $i ∈ J$ we have

$$c_i^{(n)} = ζ_i^{n(n-1)/2} c_i^{(n)}, \quad f_i^{(n)} = ζ_i^{n(n+1)/2} f_i^{(n)} k_i^{-n} \quad (n ∈ Z_{≥0}),$$

and hence

$$T_i v = ζ_i^{\langle λ,α_i^∨ \rangle(\langle λ,α_i^∨ \rangle-1)/2} ∑_{a-b+c=\langle λ,α_i^∨ \rangle} (-1)^b c_i^{(a)} e_i^{(b)} f_i^{(c)} v = ζ_i^{\langle λ,α_i^∨ \rangle(\langle λ,α_i^∨ \rangle-1)/2} T_i v.$$  

For $i ∉ J$ we have

$$c_i^{(n)} = ζ_i^{n(n+1)/2} c_i^{(n)} k_i^{-n}, \quad f_i^{(n)} = ζ_i^{n(n-1)/2} f_i^{(n)} \quad (n ∈ Z_{≥0}),$$

and hence

$$T_i v = ζ_i^{\langle λ,α_i^∨ \rangle(\langle λ,α_i^∨ \rangle+1)/2} ∑_{a-b+c=\langle λ,α_i^∨ \rangle} (-1)^b c_i^{(a)} e_i^{(b)} f_i^{(c)} v = ζ_i^{\langle λ,α_i^∨ \rangle(\langle λ,α_i^∨ \rangle+1)/2} T_i v. \quad \Box$$

**Proposition 3.6.** As left or right $U_R(ĝ)$-modules we have

$$Δ_{R,ζ}(λ) ≅ Δ_R(λ), \quad Δ^*_{R,ζ}(λ) ≅ Δ_R^*(λ), \quad ∇_{R,ζ}(λ) ≅ ∇_R(λ), \quad ∇^*_{R,ζ}(λ) ≅ ∇_R^*(λ)$$

for $λ ∈ X^+$.  

**Proof.** By duality we have only to show the first two isomorphisms. The proofs being similar, we only verify the first one. We may assume $ĝ$ is simple. Then as in the proof of Proposition 2.6 we may assume $q = q_α$ for short roots $α$, and $R = Z$. Using the embeddings

$$Δ_{Z,±1}(λ) ⊂ Δ_{Q,±1}(λ), \quad Δ_Z(λ) ⊂ Δ_Q(λ)$$

we have
the proof is reduced to showing $\Delta_{Q^+1}(\lambda) \cong \Sigma_Q(\lambda)$. By

$$\Delta_{Q^+1}(\lambda) \cong \overline{U}_Q(\mathfrak{g})/\left(\overline{U}_Q(\mathfrak{g})n^+ + \sum_{h \in T(h)} \overline{U}_Q(\mathfrak{g})(h - t\lambda(h)) + \sum_{i \in I} \overline{U}_Q(\mathfrak{g})f_i^{(\lambda, a_i^\nu + 1)}\right)$$

we can check that we have a surjective homomorphism $\Delta_{Q}(\lambda) \to \Delta_{Q^+1}(\lambda)$ given by $\tau_\lambda \mapsto \nu_\lambda$. We conclude that this is actually an isomorphism considering the dimensions. \hfill \Box

3.7. Let $R$ be a commutative ring equipped with $\zeta \in R^\times$. In this subsection we assume that there exists some $\ell \in \mathbb{Z}_{>0}$ such that $f_\ell(\zeta) = 0$, where $f_\ell$ is the $\ell$-th cyclotomic polynomial. Recall that we have the quantum Frobenius homomorphism $\mathcal{F} : U_{R,\zeta}(\mathfrak{g}) \to U_{R,\zeta}(\mathfrak{g})$. Applying Proposition 2.20 to $U_{R,\zeta}(\mathfrak{g})$ we obtain an embedding $\overline{U}_R(\mathfrak{g}) \subset U_{R,\zeta}(\mathfrak{g})$ of $R$-algebras (depending on the choice of a subset $J$ of $I$). To avoid the confusion we denote the left and right Weyl modules over $\overline{U}_R(\mathfrak{g})$ with highest weight $\lambda \in \mathfrak{h}$ by $\nabla_{R,\lambda}(\mathfrak{g})$, $\Delta_{R,\lambda}(\mathfrak{g})$, and the left and right dual Weyl modules over $\overline{U}_R(\mathfrak{g})$ with highest weight $\lambda \in \mathfrak{h}$ by $\nabla_{R,\lambda}^*(\mathfrak{g})$, $\Delta_{R,\lambda}^*(\mathfrak{g})$ respectively. Similarly, we denote the left and right Weyl modules over $U_{R,\zeta}(\mathfrak{g})$ with highest weight $\lambda \in \mathfrak{h}$ by $\nabla_{R,\zeta,\lambda}(\mathfrak{g})$, $\Delta_{R,\zeta,\lambda}(\mathfrak{g})$, and the left and right dual Weyl modules over $U_{R,\zeta}(\mathfrak{g})$ with highest weight $\lambda \in \mathfrak{h}$ by $\nabla_{R,\zeta,\lambda}^*(\mathfrak{g})$, $\Delta_{R,\zeta,\lambda}^*(\mathfrak{g})$ respectively. As in Proposition 3.3.6 we can identify the left or right $\overline{U}_R(\mathfrak{g})$-modules $\nabla_{R,\lambda}(\mathfrak{g})$, $\Delta_{R,\lambda}(\mathfrak{g})$, $\nabla_{R,\lambda}^*(\mathfrak{g})$, $\Delta_{R,\lambda}^*(\mathfrak{g})$ for $\lambda \in \mathfrak{h}$ with the left or right $U_{R,\zeta}(\mathfrak{g})$-modules $\nabla_{R,\zeta,\lambda}(\mathfrak{g})$, $\Delta_{R,\zeta,\lambda}(\mathfrak{g})$, $\nabla_{R,\zeta,\lambda}^*(\mathfrak{g})$, $\Delta_{R,\zeta,\lambda}^*(\mathfrak{g})$ respectively. Hence they can also be regarded as left or right $U_{R,\zeta}(\mathfrak{g})$-modules via $\mathcal{F}$. Under this identification we have homomorphisms

$$\Delta_{R,\zeta}(\lambda) \to \nabla_{R,\lambda}(\mathfrak{g}) \to \nabla_{R,\zeta,\lambda}(\mathfrak{g}) \to \nabla_{R,\zeta,\lambda}^*(\mathfrak{g}) \to \Delta_{R,\lambda}^*(\mathfrak{g})$$

of $U_{R,\zeta}(\mathfrak{g})$-modules for $\lambda \in \mathfrak{h}$. They induce isomorphisms

(3.4) $\Delta_{R,\zeta}(\lambda)_{w,\lambda} \cong \nabla_{R,\lambda}(\mathfrak{g})_{w,\lambda} \cong \Delta_{R,\zeta,\lambda}(\mathfrak{g})_{w,\lambda} \cong \Delta_{R,\zeta,\lambda}^*(\mathfrak{g})_{w,\lambda}$

(3.5) $\Delta_{R,\zeta}^*(\lambda)_{w,\lambda} \cong \nabla_{R,\lambda}^*(\mathfrak{g})_{w,\lambda} \cong \Delta_{R,\zeta,\lambda}(\mathfrak{g})_{w,\lambda} \cong \Delta_{R,\zeta,\lambda}^*(\mathfrak{g})_{w,\lambda}$

of $R$-modules for any $w \in W$.

4. DUALITY FOR HOPF ALGEBRAS

4.1. Let $\mathcal{T}$ be a Hopf algebra over a field $K$, which is commutative and cocommutative. Then the set $\text{Hom}_{\text{alg}}(\mathcal{T}, K)$ of algebra homomorphisms from $\mathcal{T}$ to $K$ is endowed with a structure of abelian group by

$$\langle \phi \psi \rangle(t) = (\phi \otimes \psi) (\Delta(t)) \quad (\phi, \psi \in \text{Hom}_{\text{alg}}(\mathcal{T}, K)).$$

For a subgroup $\mathcal{Y}$ of $\text{Hom}_{\text{alg}}(\mathcal{T}, K)$ we denote by $\text{Mod}_{\mathcal{Y}}(\mathcal{T})$ the category of finite-dimensional $\mathcal{T}$-modules $M$ with the weight space decomposition

$$M = \bigoplus_{\varphi \in \mathcal{Y}} M_{\varphi}, \quad M_{\varphi} = \{m \in M \mid tm = \varphi(t)m \ (t \in \mathcal{T})\}.$$
Assume that we are given a Hopf algebra \( H \), which contains \( T \) as a Hopf subalgebra. Note that the dual space \( H^* = \text{Hom}_K(H, K) \) is endowed with an \( H \)-bimodule structure by
\[
\langle h_1 f h_2, h \rangle = \langle f, h_2 h h_1 \rangle \quad (f \in H^*, h, h_1, h_2 \in H).
\]
By a standard argument we have the following (see for example [14]).

**Proposition 4.1.** The following conditions on \( f \in H^* \) are equivalent to each other.

(a) \( Hf \in \text{Mod}_{f,\text{irr}}(T) \),
(b) \( fH \in \text{Mod}_{f,\text{irr}}(T) \),
(c) \( HfH \in \text{Mod}_{f,\text{irr}}(T \otimes T) \),
(d) there exists a two-sided ideal \( I \) of \( H \) such that \( \langle f, I \rangle = \{0\} \) and \( H/I \in \text{Mod}_{f,\text{irr}}(T \otimes T) \).

We denote by \( H^*_{T,\text{irr}} \) the subspace of \( H^* \) consisting of \( f \in H^* \) satisfying the equivalent conditions of Proposition 4.1. Then \( H^*_{T,\text{irr}} \) turns out to be a Hopf algebra whose multiplication, unit, comultiplication, counit, antipode are induced by the transpose of the comultiplication, the counit, the multiplication, the unit, the antipode respectively of \( H \).

We denote by \( \text{Mod}_{T,\text{irr}}(H) \) the category of left \( H \)-modules which belong to \( \text{Mod}_{f,\text{irr}}(T) \) as a \( T \)-module. More generally, we denote by \( \text{Mod}_{T,\text{irr}}^l(H) \) the category of left \( H \)-modules which is a sum of submodules belonging to \( \text{Mod}_{f,\text{irr}}^l(T) \). For \( M \in \text{Mod}_{T,\text{irr}}^l(H) \) we have a homomorphism
\[
(4.1) \quad \Phi_M : M \otimes M^* \to H^*_{T,\text{irr}}
\]
of \( H \)-bimodules given by
\[
\langle \Phi_M(m \otimes m^*), h \rangle = \langle m^*, hm \rangle \quad (m \in M, m^* \in M^*, h \in H).
\]
Here, \( M \otimes M^* \) is regarded as an \( H \)-bimodule by
\[
h_1(m \otimes m^*)h_2 = h_1 m \otimes m^* h_2 \quad (m \in M, m^* \in M^*, h_1, h_2 \in H).
\]
Denote by \( \text{Mod}_{T,\text{irr}}^\text{irr}(H) \) the set of isomorphism classes of the irreducible \( H \)-modules belonging to \( \text{Mod}_{T,\text{irr}}^l(H) \). By a standard argument we have the following (see for example [14]).

**Proposition 4.2.** (i) We have
\[
H^*_{T,\text{irr}} = \sum_{M \in \text{Mod}_{T,\text{irr}}^l(H)} \text{Im}(\Phi_M) = \bigcup_{M \in \text{Mod}_{T,\text{irr}}^l(H)} \text{Im}(\Phi_M).
\]
(ii) Assume that \( \text{Mod}_{T,\text{irr}}^l(H) \) is a semisimple category and that \( \text{End}_H(M) = K \text{id} \) for any \( M \in \text{Mod}_{T,\text{irr}}^\text{irr}(H) \). Then
\[
\bigoplus_{M \in \text{Mod}_{T,\text{irr}}^l(H)} \Phi_M : \bigoplus_{M \in \text{Mod}_{T,\text{irr}}^\text{irr}(H)} M \otimes M^* \to H^*_{T,\text{irr}}
\]
is an isomorphism of \( H \)-bimodules.
4.2. For a finite subset $\Gamma$ of $\text{Mod}^\text{irr}_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$ we define a two-sided ideal $\mathcal{I}(\Gamma)$ of $\mathcal{H}$ by

$$\mathcal{I}(\Gamma) = \{ h \in \mathcal{H} \mid hM = \{0\} \ (\forall M \in \Gamma) \}.$$ 

**Lemma 4.3.** Assume that $\text{Mod}^f_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$ is a semisimple category and that $\text{End}_\mathcal{H}(M) = K\text{id}$ for any $M \in \text{Mod}^\text{irr}_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$.

(i) For any finite subset $\Gamma$ of $\text{Mod}^\text{irr}_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$ we have

$$\mathcal{H}/\mathcal{I}(\Gamma) \cong \bigoplus_{M \in \Gamma} \text{End}_K(M).$$

(ii) For $f \in \mathcal{H}^*$ we have $f \in \mathcal{H}^*_{\mathcal{T},\mathcal{Y}}$ if and only if there exists a finite subset $\Gamma$ of $\text{Mod}^\text{irr}_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$ such that $\langle f, \mathcal{I}(\Gamma) \rangle = \{0\}$.

**Proof.** (i) Note that any $M \in \Gamma$ is an irreducible $\mathcal{H}/\mathcal{I}(\Gamma)$-module. Hence by a well-known fact on finite dimensional algebras the assertion follows from

$$\dim \mathcal{H}/\mathcal{I}(\Gamma) \leq \sum_{M \in \Gamma} (\dim M)^2.$$ 

To verify this it is sufficient to show for finite subsets $\Gamma, \Gamma'$ of $\text{Mod}^\text{irr}_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$ satisfying $\Gamma' = \Gamma \cup \{ M \}$ that $\dim \mathcal{I}(\Gamma)/\mathcal{I}(\Gamma') \leq (\dim M)^2$. This follows from

$$\text{Ker}(\mathcal{I}(\Gamma) \to \text{End}_K(M)) = \mathcal{I}(\Gamma').$$

(ii) Assume $f \in \mathcal{H}^*_{\mathcal{T},\mathcal{Y}}$. For

$$\Gamma = \{ M \in \text{Mod}^\text{irr}_{\mathcal{T},\mathcal{Y}}(\mathcal{H}) \mid \text{Hom}_\mathcal{H}(M, \mathcal{H}f) \neq \{0\} \}$$

we have

$$\langle f, \mathcal{I}(\Gamma) \rangle = \langle \mathcal{I}(\Gamma)f, 1 \rangle \subset \langle \mathcal{I}(\Gamma)(\mathcal{H}f), 1 \rangle = \{0\}.$$ 

The converse is clear from (i). \qed

4.3. In general, for a coalgebra $\mathcal{C}$ we denote by $\text{Comod}(\mathcal{C})$ (resp. $\text{Comod}^f(\mathcal{C})$) the category of right $\mathcal{C}$-comodules (resp. finite dimensional right $\mathcal{C}$-comodules).

Note that for $M \in \text{Mod}^f_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$ we have a right $\mathcal{H}^*_{\mathcal{T},\mathcal{Y}}$-comodule structure $\gamma_M : M \to M \otimes \mathcal{H}^*_{\mathcal{T},\mathcal{Y}}$ given by

$$hm = \sum_a \langle f_a, h \rangle m_a \ (h \in \mathcal{H}) \implies \gamma_M(m) = \sum_a m_a \otimes f_a.$$ 

This induces functors

$$\text{Mod}^f_{\mathcal{T},\mathcal{Y}}(\mathcal{H}) \to \text{Comod}^f(\mathcal{H}^*_{\mathcal{T},\mathcal{Y}}), \quad \text{Mod}^f_{\mathcal{T},\mathcal{Y}}(\mathcal{H}) \to \text{Comod}(\mathcal{H}^*_{\mathcal{T},\mathcal{Y}}).$$

**Proposition 4.4.** The functors in (4.2) give equivalences of categories.

**Proof.** Assume that $M$ is a finite-dimensional right $\mathcal{H}^*_{\mathcal{T},\mathcal{Y}}$-comodule with respect to $\gamma : M \to M \otimes \mathcal{H}^*_{\mathcal{T},\mathcal{Y}}$. Then we can define a left $\mathcal{H}$-module structure of $M$ by

$$\gamma(m) = \sum m_a \otimes f_a \implies hm = \sum_i \langle f_a, h \rangle m_a \ (h \in \mathcal{H}).$$ 

It is easily seen that this left $\mathcal{H}$-module belongs to $\text{Mod}^f_{\mathcal{T},\mathcal{Y}}(\mathcal{H})$. Moreover, this induces the inverse to the functors in (4.2). \qed
We will sometimes identify $\text{Mod}^f_{T,Y}(\mathcal{H})$ with $\text{Comod}(\mathcal{H}^*_T)$.

5. COORDINATE ALGEBRAS OF THE QUANTIZED ALGEBRAIC GROUPS

5.1. Let $K$ be a field. We set

$$\overline{O}_K(G) = U_K(\mathfrak{g})_{U_K(\mathfrak{h}),X}.$$  

It is isomorphic as a Hopf algebra to the coordinate algebra of the reductive algebraic group $G_K$ over $K$ with the same root datum $(X, \Delta, Y, \Delta^\vee)$ as $G$.

5.2. Let $K$ be a field equipped with $\zeta \in K^*$. By Lemma 2.3 the map $X \ni \lambda \mapsto \chi_\lambda \in \text{Hom}_{\text{alg}}(\mathcal{U}_{K,\zeta}(\mathfrak{g}), K)$ is an injective group homomorphism. We will regard $X$ as a subgroup of $\text{Hom}_{\text{alg}}(\mathcal{U}_{K,\zeta}(\mathfrak{g}), K)$ in the following. Set

$$O_{K,\zeta}(B) = \mathcal{U}_{K,\zeta}(\mathfrak{b})_{\mathcal{U}_{K,\zeta}(\mathfrak{h}),X}, \quad O_{K,\zeta}(H) = \mathcal{U}_{K,\zeta}(\mathfrak{b})_{\mathcal{U}_{K,\zeta}(\mathfrak{h}),X},$$

$$O_{K,\zeta}(G) = \mathcal{U}_{K,\zeta}(\mathfrak{g})_{\mathcal{U}_{K,\zeta}(\mathfrak{h}),X}.$$  

It is easily seen that

$$O_{K,\zeta}(H) = \bigoplus_{\lambda \in X} K\chi_\lambda.$$  

We identify $\mathcal{U}_{K,\zeta}(\mathfrak{n})^* \otimes \mathcal{U}_{K,\zeta}(\mathfrak{h})^* \otimes \mathcal{U}_{K,\zeta}(\mathfrak{n}^+)^*$ with a subspace of $\mathcal{U}_{K,\zeta}(\mathfrak{g})^*$ by

$$\langle \psi \otimes \chi \otimes \varphi, yhx \rangle = \langle \psi, y \rangle \langle \chi, h \rangle \langle \varphi, x \rangle \quad (y \in \mathcal{U}_{K,\zeta}(\mathfrak{n}), h \in \mathcal{U}_{K,\zeta}(\mathfrak{h}), x \in \mathcal{U}_{K,\zeta}(\mathfrak{n}^+)).$$

Similarly, we identify $\mathcal{U}_{K,\zeta}(\mathfrak{n})^* \otimes \mathcal{U}_{K,\zeta}(\mathfrak{h})^*$ with a subspace of $\mathcal{U}_{K,\zeta}(\mathfrak{g})^*$. Set

$$\mathcal{U}_{K,\zeta}(\mathfrak{n})^* = \bigoplus_{\gamma \in Q^+} (\mathcal{U}_{K,\zeta}(\mathfrak{n})_{-\gamma})^* \subset \mathcal{U}_{K,\zeta}(\mathfrak{g})^*,$$

$$\mathcal{U}_{K,\zeta}(\mathfrak{n}^+)^* = \bigoplus_{\gamma \in Q^+} (\mathcal{U}_{K,\zeta}(\mathfrak{n}^+)_\gamma)^* \subset \mathcal{U}_{K,\zeta}(\mathfrak{n}^+)^*.$$  

Then we have

$$O_{K,\zeta}(G) \subset \mathcal{U}_{K,\zeta}(\mathfrak{n})^* \otimes O_{K,\zeta}(H) \otimes \mathcal{U}_{K,\zeta}(\mathfrak{n}^+)^* \subset \mathcal{U}_{K,\zeta}(\mathfrak{g})^*.$$  

Moreover, we have

$$O_{K,\zeta}(B) = \mathcal{U}_{K,\zeta}(\mathfrak{n})^* \otimes O_{K,\zeta}(H) \subset \mathcal{U}_{K,\zeta}(\mathfrak{g})^*,$$

and the natural homomorphism $O_{K,\zeta}(G) \to O_{K,\zeta}(B)$ is surjective (see, for example [13, Section 2.7]). Hence we have

$$O_{K,\zeta}(B) \cong O_{K,\zeta}(G)/\{f \in O_{K,\zeta}(G) \mid \langle f, U_{K,\zeta}(\mathfrak{b}) \rangle = \{0\}\}.$$  

\textbf{PROPOSITION 5.1.} Assume $\zeta^a_\alpha = 1$ for any $\alpha \in \Delta$. The surjection $\mathcal{U}_{K,\zeta}(\mathfrak{g})^* \to \mathcal{U}_{K}(\mathfrak{g})^*$ induced by the embedding $\mathcal{U}_{K}(\mathfrak{g}) \subset \mathcal{U}_{K,\zeta}(\mathfrak{g})$ given in Proposition 2.4 restricts to an isomorphism

$$O_{K,\zeta}(G) \cong \overline{O}_K(G)$$  

of $\overline{U}_K(\mathfrak{g})$-bimodules.
PROOF. By Proposition 3.4 the surjection $U_{K,\zeta}(g)^* \to \overline{U}_K(g)^*$ restricts to a surjective homomorphism $O_{K,\zeta}(G) \to \overline{O}_K(G)$ of $\overline{U}_K(g)$-bimodules. In order to prove that it is injective it is sufficient to show

$$\text{Ker}(M \otimes M^* \to \overline{U}_K(g)^*) = \text{Ker}(M \otimes M^* \to U_{K,\zeta}(g)^*)$$

for any $M \in \text{Mod}_{\text{int}}(U_{K,\zeta}(g)) = \text{Mod}_{\text{int}}(\overline{U}_K(g))$ by Proposition 4.2. Since $M \otimes M^* \to \overline{U}_K(g)^*$ is a homomorphism of $\overline{U}_K(g)$-bimodule, we have only to show

$$\text{Ker}(M_{\lambda} \otimes M^* \to \overline{U}_K(g)^*) = \text{Ker}(M_{\lambda} \otimes M^* \to U_{K,\zeta}(g)^*)$$

for any $\lambda \in X$. By $\overline{U}_K(g) \cong \overline{U}_K(n) \otimes \overline{U}_K(n^+) \otimes \overline{U}_K(h)$ we have

$$\text{Ker}(M_{\lambda} \otimes M^* \to \overline{U}_K(g)^*) = \text{Ker}(M_{\lambda} \otimes M^* \to (\overline{U}_K(n) \otimes \overline{U}_K(n^+))^*)$$.

Similarly, we have

$$\text{Ker}(M_{\lambda} \otimes M^* \to U_{K,\zeta}(g)^*) = \text{Ker}(M_{\lambda} \otimes M^* \to (U'_{K,\zeta}(n) \otimes U'_{K,\zeta}(n^+))^*)$$.

They coincide under the identification $\overline{U}_K(g) = U'_{K,\zeta}(g)$.

REMARK 5.2. The isomorphism (5.3) does not preserve the ring structure in general (see Remark 2.7).

5.3. Assume that $\zeta \in K^\times$ has the order $\ell < \infty$. Define the root datum $(\check{X}, \check{\Delta}, \check{Y}, \check{\Delta}^\circ)$ and the corresponding complex reductive group $\check{G}$ as in Section 2. The quantum Frobenius homomorphism $\mathcal{F}$ induces the injection $\mathcal{F} : U_{K,\zeta}(\check{g})^* \hookrightarrow U_{K,\zeta}(g)^*$. This restricts to an injective Hopf algebra homomorphism

(5.4)

$$O_{K,\zeta}(\check{g}) \hookrightarrow O_{K,\zeta}(g).$$

6. INDUCTION FUNCTOR

6.1. Let $\mathcal{H}, \mathcal{H}'$ be Hopf algebras over a field $K$ with invertible antipodes, and let $p : \mathcal{H} \to \mathcal{H}'$ be a surjective Hopf algebra homomorphism. We have a natural exact functor

$$\text{Res}_{\mathcal{H}'}^\mathcal{H} : \text{Comod}(\mathcal{H}) \to \text{Comod}(\mathcal{H}'),$$

where, for a $K$-module $V$ with the right $\mathcal{H}$-comodule structure $\gamma : V \to V \otimes \mathcal{H}$, we associate the right $\mathcal{H}'$-comodule given by $(1 \otimes p) \circ \gamma : V \to V \otimes \mathcal{H}'$. We can also define the induction functor

$$\text{Ind}_{\mathcal{H}'}^\mathcal{H} : \text{Comod}(\mathcal{H}') \to \text{Comod}(\mathcal{H})$$

as follows. Let $M$ be a right $\mathcal{H}'$-comodule. Regarding $\mathcal{H}$ as a right $\mathcal{H}'$-comodule via $(1 \otimes p) \circ \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}'$, the tensor product $M \otimes \mathcal{H}$ of the two right $\mathcal{H}'$-comodules is endowed with a right $\mathcal{H}'$-comodule structure $\gamma : M \otimes \mathcal{H} \to (M \otimes \mathcal{H}) \otimes \mathcal{H}'$. We set

$$\text{Ind}_{\mathcal{H}'}^\mathcal{H}(M) = \{ n \in M \otimes \mathcal{H} \mid \gamma(n) = n \otimes 1 \}.$$

Then $\text{Ind}_{\mathcal{H}'}^\mathcal{H}(M)$ is endowed with a right $\mathcal{H}$-comodule structure induced by that of $M \otimes \mathcal{H}$ given by

$$M \otimes \mathcal{H} \to (M \otimes \mathcal{H}) \otimes \mathcal{H} \quad (m \otimes h \mapsto \sum_k (m \otimes h_k') \otimes S^{-1}h_k),$$
where $\Delta(h) = \sum_k h_k \otimes h_k'$. For $V \in \text{Comod}(\mathcal{H})$, $M \in \text{Comod}(\mathcal{H}')$ we have
$$\text{Hom}(V, \text{Ind}_{\mathcal{H}'}^H(M)) \cong \text{Hom}(\text{Res}_{\mathcal{H}'}^H(V), M).$$
It follows that $\text{Ind}_{\mathcal{H}'}^H$ is left exact, and we have
$$\text{Ind}_{\mathcal{H}''}^H = \text{Ind}_{\mathcal{H}'}^H \circ \text{Ind}_{\mathcal{H}''}^H$$
for a sequence $\mathcal{H} \to \mathcal{H}' \to \mathcal{H}''$ of surjective homomorphisms of Hopf algebras with invertible antipodes. The following fact is well-known.

**Lemma 6.1.** For $M \in \text{Comod}(\mathcal{H}')$ and $V \in \text{Comod}(\mathcal{H})$ we have
$$\text{Ind}_{\mathcal{H}'}^H(\text{Res}_{\mathcal{H}'}^H(V) \otimes M) \cong V \otimes \text{Ind}_{\mathcal{H}'}^H(M), \quad \text{Ind}_{\mathcal{H}'}^H(M \otimes \text{Res}_{\mathcal{H}'}^H(V)) \cong \text{Ind}_{\mathcal{H}'}^H(M) \otimes V.$$

6.2. Assume that $K$ is a field equipped with $\zeta \in K^\times$. We consider the left exact functor
$$\text{Ind} = \text{Ind}^{\mathcal{O}_K,\zeta(G)}: \text{Comod}(\mathcal{O}_K,\zeta(G)) \to \text{Comod}(\mathcal{O}_K,\zeta(G)).$$
It is known that the abelian category $\text{Comod}(\mathcal{O}_K,\zeta(G))$ has enough injectives, and we have its right derived functors
$$R^k \text{Ind}: \text{Comod}(\mathcal{O}_K,\zeta(G)) \to \text{Comod}(\mathcal{O}_K,\zeta(G)) \quad (k \geq 0).$$

For $\lambda \in X$ we denote by $K_\lambda = K_{1,\lambda}$ the one dimensional $U_{K,\zeta}(b)$-module given by
$$h_{1,\lambda} = \chi_\lambda(h)1_\lambda \quad (h \in U_{K,\zeta}(h)), \quad z_{1,\lambda} = \varepsilon(z)1_\lambda \quad (z \in U_{K,\zeta}(n)).$$
We also denote the corresponding right $\mathcal{O}_{K,\zeta}(B)$-comodule by $K_\lambda = K_{1,\lambda}$. The following fact is standard (see [1]).

**Lemma 6.2.** For $\lambda \in X$ we have
$$\text{Ind}(K_\lambda) \cong \begin{cases} \nabla_{K,\zeta}(\lambda) & (\lambda \in X^+) \\ 0 & (\lambda \notin X^+). \end{cases}$$

The following Kempf type vanishing theorem is a consequence of the deep theory of crystal bases due to Kashiwara.

**Proposition 6.3** (Ryom-Hansen [13]). For $\lambda \in X^+$ and $k \neq 0$ we have $R^k \text{Ind}(K_\lambda) = 0$.

7. **Homogeneous coordinate algebras**

7.1. For a field $K$ we set
$$\overline{\mathcal{O}}_K(N \setminus G) = \{ f \in \mathcal{O}_K(G) \mid f z = \varepsilon(z)f \ (z \in \mathcal{O}_K(n)) \}. $$
It is a subalgebra of $\mathcal{O}_K(G)$ and a $(\mathcal{O}_K(n), \mathcal{O}_K(h))$-bimodule. Set
$$\overline{\mathcal{O}}_K(N \setminus G; \lambda) = \{ f \in \overline{\mathcal{O}}_K(N \setminus G) \mid fh = \chi_\lambda(h)f \ (h \in \mathcal{O}_K(h)) \}$$
for $\lambda \in X^+$. It is known that
$$\overline{\mathcal{O}}_K(N \setminus G) = \bigoplus_{\lambda \in X^+} \overline{\mathcal{O}}_K(N \setminus G; \lambda).$$
Moreover, we have an isomorphism
$$\nabla_K(\lambda) \cong \overline{\mathcal{O}}_K(N \setminus G; \lambda) \quad (v \leftrightarrow \Phi_{\nabla_K}(\lambda)(v \otimes \tau_\lambda))$$
of left $\mathcal{U}_K(g)$-modules. Here, the image of $v^*_\lambda \in \mathfrak{X}^+_\mathfrak{A}(\lambda)$ in $\Sigma^*_K(\lambda) = (\nabla_K(\lambda))^*$ is denoted by $v^*_\lambda$ by abuse of the notation. By

\begin{equation}
(7.4) \quad \mathcal{O}_K(N\backslash G; \lambda) \mathcal{O}_K(N\backslash G; \mu) \subset \mathcal{O}_K(N\backslash G; \lambda + \mu) \quad (\lambda, \mu \in X^+)
\end{equation}

$\mathcal{O}_K(N\backslash G)$ turns out to be a commutative $K$-algebra graded by the abelian group $X$. Under the identification \((7.3)\) we obtain a non-zero homomorphism

\begin{equation}
(7.5) \quad \nabla_K(\lambda) \otimes \nabla_K(\mu) \to \nabla_K(\lambda + \mu)
\end{equation}

of left $\mathcal{U}_K(g)$-modules corresponding to \((7.4)\). By

\begin{align*}
\text{Hom}_{\mathcal{U}_K(g)}(\nabla_K(\lambda) \otimes \nabla_K(\mu), \nabla_K(\lambda + \mu)) \\
\cong \text{Hom}_{\mathcal{U}_K(g)}^\ast(\Sigma^+_K(\lambda + \mu), \Sigma^+_K(\lambda) \otimes \Sigma^+_K(\mu)) \cong K
\end{align*}

\((7.5)\) is the unique (up to a scalar multiple) non-zero homomorphism of $\mathcal{U}_K(g)$-modules.

7.2. Let $K$ be a field equipped with $\zeta \in K^\times$. We set

\begin{equation}
(7.6) \quad O_{K,\zeta}(N\backslash G) = \{ f \in O_{K,\zeta}(G) \mid f z = \varepsilon(z) f \ (z \in U_{K,\zeta}(n)) \}.
\end{equation}

It is a subalgebra of $O_{K,\zeta}(G)$ as well as a $(U_{K,\zeta}(g), U_{K,\zeta}(h))$-bimodule. By the definition of $O_{K,\zeta}(G)$ it is easily seen that $O_{K,\zeta}(N\backslash G)$ is a direct sum of subspaces

\begin{equation}
O_{K,\zeta}(N\backslash G; \lambda) = \{ f \in O_{K,\zeta}(N\backslash G) \mid f h = \chi_\lambda(h) f \ (h \in U_{K,\zeta}(h)) \}
\end{equation}

for $\lambda \in X$. Note that we have $O_{K,\zeta}(N\backslash G; \lambda) \cong \text{Ind}(K\lambda)$ by the definition of Ind. Hence by Proposition \(6.2\) we have $O_{K,\zeta}(N\backslash G; \lambda) \neq \{0\}$ only if $\lambda \in X^+$, and hence

\begin{equation}
(7.7) \quad O_{K,\zeta}(N\backslash G) = \bigoplus_{\lambda \in X^+} O_{K,\zeta}(N\backslash G; \lambda).
\end{equation}

Moreover, we have $O_{K,\zeta}(N\backslash G; \lambda) \cong \nabla_{K,\zeta}(\lambda)$ for $\lambda \in X^+$. More precisely, we have an isomorphism

\begin{equation}
(7.8) \quad \nabla_{K,\zeta}(\lambda) \cong O_{K,\zeta}(N\backslash G; \lambda) \quad (v \leftrightarrow \Phi_{\nabla_{K,\zeta}(\lambda)} (v \otimes v^*_\lambda))
\end{equation}

of $U_{K,\zeta}(g)$-modules. Here, the image of $v^*_\lambda \in \mathfrak{X}^+_\mathfrak{A}(\lambda)$ in $\Delta^*_{K,\zeta}(\lambda) = (\nabla_{K,\zeta}(\lambda))^*$ is denoted by $v^*_\lambda$ by abuse of the notation.

By

\begin{equation}
(7.9) \quad O_{K,\zeta}(N\backslash G; \lambda) O_{K,\zeta}(N\backslash G; \mu) \subset O_{K,\zeta}(N\backslash G; \lambda + \mu) \quad (\lambda, \mu \in X^+)
\end{equation}

$O_{K,\zeta}(N\backslash G)$ turns out to be a $K$-algebra graded by the abelian group $X$. Similarly to the case of $\mathcal{O}_K(N\backslash G)$, the multiplication

\begin{equation}
O_{K,\zeta}(N\backslash G; \lambda) \otimes O_{K,\zeta}(N\backslash G; \mu) \to O_{K,\zeta}(N\backslash G; \lambda + \mu)
\end{equation}

corresponds to the unique (up to a non-zero scalar multiple) non-zero homomorphism

\begin{equation}
(7.10) \quad \nabla_{K,\zeta}(\lambda) \otimes \nabla_{K,\zeta}(\mu) \to \nabla_{K,\zeta}(\lambda + \mu)
\end{equation}

of left $U_{K,\zeta}(g)$-modules under the identification \((7.8)\).
7.3. Assume that $\zeta \in K^\times$ has the multiplicative order $\ell < \infty$. Define a subalgebra 
$O_{K,\zeta}(\sharp N \backslash G)$ of $O_{K,\zeta}(G)$ and its subspace $O_{K,\zeta}(\sharp N \backslash G; \lambda)$ for $\lambda \in \sharp X$ similarly to the case of $O_{K,\zeta}(G)$, so that
$$
O_{K,\zeta}(\sharp N \backslash G) = \bigoplus_{\lambda \in \sharp X^+} O_{K,\zeta}(\sharp N \backslash G; \lambda).
$$

It is easily seen that the embedding (5.11) induces the embedding
(7.11) $$
O_{K,\zeta}(\sharp N \backslash G) \subset O_{K,\zeta}(N \backslash G)
$$
of algebras so that $O_{K,\zeta}(\sharp N \backslash G; \lambda) \subset O_{K,\zeta}(N \backslash G; \lambda)$ for $\lambda \in \sharp X$. Note also that we have an isomorphism $O_{K,\zeta}(G) \cong \overline{O}_K(\sharp G)$ of $\overline{U}_K(\sharp g)$-bimodules by Proposition 5.1.

It induces an isomorphism
(7.12) $$
O_{K,\zeta}(\sharp N \backslash G) \cong \overline{O}_K(\sharp N \backslash G)
$$
of left $\overline{U}_K(\sharp g)$-modules.

**Proposition 7.1.** For any $\mu \in \sharp X^+$ there exists some $N \geq 0$ such that for $\lambda \in X^+$ satisfying $\langle \lambda, \alpha_i^\vee \rangle \geq N$ for any $i \in I$ we have
$$
O_{K,\zeta}(N \backslash G; \lambda)O_{K,\zeta}(\sharp N \backslash G; \mu) = O_{K,\zeta}(N \backslash G; \lambda + \mu).
$$

**Proof.** Recall
$$
O_{K,\zeta}(N \backslash G; \lambda) \cong \nabla_{K,\zeta}(\lambda) \quad (\lambda \in X^+), \quad O_{K,\zeta}(\sharp N \backslash G; \mu) \cong \sharp \nabla_{K,\zeta}(\mu) \quad (\mu \in \sharp X^+).
$$
Under this identification, the multiplication
$$
O_{K,\zeta}(N \backslash G; \lambda) \otimes O_{K,\zeta}(\sharp N \backslash G; \mu) \to O_{K,\zeta}(N \backslash G; \lambda + \mu)
$$
corresponds to a non-zero homomorphism
(7.13) $$
\nabla_{K,\zeta}(\lambda) \otimes \sharp \nabla_{K,\zeta}(\mu) \to \nabla_{K,\zeta}(\lambda + \mu)
$$
of $U_{K,\zeta}(g)$-modules, where $\sharp \nabla_{K,\zeta}(\mu)$ is regarded as a $U_{K,\zeta}(g)$-module through the quantum Frobenius homomorphism $\mathcal{F}: U_{K,\zeta}(g) \to \sharp U_{K,\zeta}(g)$. Note that (7.13) is obtained by taking the dual of the unique (up to a scalar multiple) non-zero homomorphism
$$
\Delta_{K,\zeta}(\lambda + \mu) \to \Delta_{K,\zeta}(\lambda) \otimes \sharp \Delta_{K,\zeta}(\mu)
$$
of right $U_{K,\zeta}(g)$-modules.

Assume $\langle \lambda, \alpha_i^\vee \rangle \gg 0$ for any $i \in I$. We have a filtration
$$
K_\lambda \otimes \sharp \nabla_{K,\zeta}(\mu) = M_1 \supset M_2 \supset \cdots \supset M_s \supset M_{s+1} = 0
$$
of the $U_{K,\zeta}(b)$-module $K_\lambda \otimes \sharp \nabla_{K,\zeta}(\mu)$ such that $M_j/M_{j+1} \cong K_{\lambda + \nu_j}$, where $\nu_1 = \mu, \nu_2, \ldots, \nu_s$ are the weights of $\sharp \nabla_{K,\zeta}(\mu)$ with multiplicity. From the short exact sequence
$$
0 \to M_2 \to K_\lambda \otimes \sharp \nabla_{K,\zeta}(\mu) \to K_{\lambda + \mu} \to 0
$$
we obtain an exact sequence
$$
\text{Ind}(K_\lambda \otimes \sharp \nabla_{K,\zeta}(\mu)) \to \text{Ind}(K_{\lambda + \mu}) \to R^i \text{Ind}(M_2)
$$
of $U_{K,\zeta}(g)$-modules. By Proposition 6.3 and Lemma 6.1 we have
$$
\text{Ind}(K_\lambda \otimes \sharp \nabla_{K,\zeta}(\mu)) \cong \nabla_{K,\zeta}(\lambda) \otimes \sharp \nabla_{K,\zeta}(\mu), \quad \text{Ind}(K_{\lambda + \mu}) \cong \nabla_{K,\zeta}(\lambda + \mu).
$$
By our assumption on $\lambda$ we have $\lambda + \nu_j \in X^+$ for any $j$, and hence $R^k \text{Ind}(M_j/M_{j+1}) = 0$ for any $j \geq 2$, $k \geq 1$. Therefore, we have $R^1 \text{Ind}(M_2) = 0$. We obtain a surjective homomorphism

$$\nabla_{K,\zeta}(\lambda) \otimes \nabla_{K,\zeta}(\mu) \to \nabla_{K,\zeta}(\lambda + \mu)$$

of $U_{K,\zeta}(\mathfrak{g})$-modules. Since (7.13) is the unique (up to a scalar multiple) non-zero homomorphism of $U_{K,\zeta}(\mathfrak{g})$-modules, we obtain the surjectivity of (7.13). \hfill \Box

8. QUANTIZED FLAG MANIFOLDS

8.1. We assume that $G$ is semisimple and simply-connected. Namely, we assume $Y = Q^\nu$, so that the canonical homomorphism $X \to \text{Hom}_{\mathbb{Z}}(Q^\nu, \mathbb{Z})$ is bijective. Let $K$ be a field, and let $G_K$ be the split semisimple algebraic group defined over $K$ with the same root datum $(X, \Delta, Y, \Delta^\vee)$ as $G$. We denote by $B_K$, $N_K$, $N_K^+$ the subgroups of $G_K$ similarly defined as $B$, $N$, $N^+$ respectively. Set $B_K = B_K \backslash G_K$. It is a projective algebraic variety defined over $K$, called the flag variety. Denote by $\text{Mod}(\mathcal{O}_{B_K})$ the category of quasi-coherent $\mathcal{O}_{B_K}$-modules. It can be described using $\mathcal{O}_K(N\backslash G)$ as follows. Let $\text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G))$ be the category of graded $\mathcal{O}_K(N\backslash G)$-modules. We denote by $\text{Tor}_\text{gr}(\mathcal{O}_K(N\backslash G))$ the full subcategory of $\text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G))$ consisting of $M \in \text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G))$ such that for any $m \in M$ there exists some positive integer $N$ satisfying

$$\lambda \in X, \ <\lambda, \alpha_i^\vee> \geq N \ (\forall i \in I) \implies \mathcal{O}_K(N\backslash G; \lambda)m = \{0\}.$$ 

Note that $\text{Tor}_\text{gr}(\mathcal{O}_K(N\backslash G))$ is closed under taking subquotients and extensions in $\text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G))$. Then we have

$$\text{Mod}(\mathcal{O}_{B_K}) \cong \text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G))/\text{Tor}_\text{gr}(\mathcal{O}_K(N\backslash G)) := \mathcal{P}^{-1} \text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G)),$$

where $\mathcal{P}$ consists of morphisms in $\text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G))$ whose kernel and cokernel belong to $\text{Tor}_\text{gr}(\mathcal{O}_K(N\backslash G))$, and $\mathcal{P}^{-1} \text{Mod}_\text{gr}(\mathcal{O}_K(N\backslash G))$ denotes the localization of the category so that the morphisms in $\mathcal{P}$ turn out to be isomorphisms (see [4, 13]).

For $w \in W$ and $\lambda \in X^+$ we set

$$\mathcal{T}_w \mathcal{V}_\lambda = \Phi_{\mathcal{T}_w \mathcal{V}_\lambda} \in \mathcal{O}_K(N\backslash G; \lambda) \subset \mathcal{O}_K(N\backslash G).$$

Here, the images of $\mathcal{V}_\lambda \in \nabla_{\mathbb{Z}}(\lambda)$ and $\mathcal{T}_w \mathcal{V}_\lambda \in \nabla_{\mathbb{Z}}(\lambda)$ in $\mathcal{O}_K(N\backslash G)$ and $\nabla_{\mathbb{K}}(\lambda)$ are denoted by $\mathcal{V}_\lambda$ and $\mathcal{T}_w \mathcal{V}_\lambda$ respectively. We have

$$\mathcal{V}_\lambda \mathcal{T}_w \mathcal{V}_\mu = \mathcal{T}_{\lambda + \mu} \quad (\lambda, \mu \in X^+, w \in W),$$

and hence $\mathcal{S}_w = \{\mathcal{V}_\lambda | \lambda \in X^+\}$ is a homogeneous multiplicative subset of the commutative graded ring $\mathcal{O}_K(N\backslash G)$. Therefore, the localization $(\mathcal{S}_w)^{-1} \mathcal{O}_K(N\backslash G)$ turns out to be a commutative graded ring graded by $X$. Set

$$\mathcal{O}_K(B^w) = ((\mathcal{S}_w)^{-1} \mathcal{O}_K(N\backslash G))(0).$$

This commutative ring is naturally identified with the coordinate algebra of the affine open subset $B_K^w = B_K \backslash B_K N_K^+ w^{-1}$ of $B_K$. In particular, the category $\text{Mod}(\mathcal{O}_{B_K^w})$ of quasi-coherent $\mathcal{O}_{B_K^w}$-modules is isomorphic to the category $\text{Mod}(\mathcal{O}_K(B^w))$ of $\mathcal{O}_K(B^w)$-modules. The natural exact functor

$$\mathcal{F}\mathcal{E}\mathcal{S}_w : \text{Mod}(\mathcal{O}_{B_K}) \to \text{Mod}(\mathcal{O}_{B_K^w})$$
Lemma 8.1 holds for general root data $(X, \Delta, Y, \Delta^\vee)$ without the assumption $Y = Q^\vee$. The general case easily follows from the special case.

8.2. We continue to assume $Y = Q^\vee$. Let $K$ be a field equipped with $\zeta \in K^\times$. We define an abelian category $\text{Mod}(\mathcal{O}_{B_{K, \zeta}})$ by

$$\text{Mod}(\mathcal{O}_{B_{K, \zeta}}) := \text{Mod}_{\text{gr}}(O_{K, \zeta}(N\setminus G))/\text{Tor}_{\text{gr}}(O_{K, \zeta}(N\setminus G))$$

for every $\lambda \in X^+$ such that $\langle \lambda, \alpha_i \rangle \geq 0$ for any $i \in I$. Moreover, $\mathcal{P}$ consists of morphisms in $\text{Mod}_{\text{gr}}(O_{K, \zeta}(N\setminus G))$ whose kernel and cokernel belong to $\text{Tor}_{\text{gr}}(O_{K, \zeta}(N\setminus G))$, and $\mathcal{P}^{-1}\text{Mod}_{\text{gr}}(O_{K, \zeta}(N\setminus G))$ denotes the localization of the category so that the morphisms in $\mathcal{P}$ turn out to be isomorphisms.

For $w \in W$ and $\lambda \in X^+$ we set

$$\sigma^w_\lambda = \Phi_{\nabla_{K, \zeta}(\lambda)}(T_w v_\lambda \otimes v^*_\lambda) \in O_{K, \zeta}(N\setminus G) \subset O_{K, \zeta}(G).$$

Here, the images of $v_\lambda \in \nabla_A(\lambda)$ and $v^*_\lambda \in \nabla_{K, \zeta}(\lambda)$ in $\nabla_{K, \zeta}(\lambda)$ and $\Delta^*_{K, \zeta}(\lambda) = (\nabla_{K, \zeta}(\lambda))^*$ are denoted by $v_\lambda$ and $v^*_\lambda$ respectively. We have

$$\sigma^w_\lambda \sigma^w_\mu = \sigma^w_{\lambda + \mu} \quad (\lambda, \mu \in X^+, w \in W).$$

Set $S^w_{K, \zeta} = \{ \sigma^w_\lambda \mid \lambda \in X^+ \}$ for $w \in W$. It is known that the multiplicative set $S^w_{K, \zeta}$ satisfies the left and right Ore conditions in the ring $O_{K, \zeta}(N\setminus G)$ (see [6], [14]). Hence the localization $(S^w_{K, \zeta})^{-1}O_{K, \zeta}(N\setminus G)$ is a graded ring graded by $X$. We set

$$O_{K, \zeta}(B^w) := ((S^w_{K, \zeta})^{-1}O_{K, \zeta}(N\setminus G))(0).$$

Remark 8.3. In the case $w = 1$ we can identify the $K$-algebra $O_{K, \zeta}(B^1)$ with a subalgebra of $O_{K, \zeta}(B^\dagger)$ (see [14] Proposition 4.5]). For general $w$ we do not know such an explicit description of the algebra $O_{K, \zeta}(B^w)$. In fact the ring $O_{K, \zeta}(B^w)$ does depend on the choice of $w \in W$. 

Given by the restriction of quasi-coherent $\mathcal{O}$-modules is induced by

$$\text{Mod}_{\text{gr}}(O_{K}(N\setminus G) \ni M \mapsto ((S^w_{K})^{-1}M)(0) \in \text{Mod}(O_{K}(B^w)).$$

By $B_{K} = \bigcup_{w \in W} B^K_w$ we have obviously

$$M \in \text{Mod}(\mathcal{O}_{B_{K}}), \text{res}_w(M) = 0 \quad (\forall w \in W) \implies M = 0.$$

It is easily seen that this is equivalent to the following.

**Lemma 8.1.** For any $\mu \in X^+$ we have

$$\overline{O}_{K}(N\setminus G; \lambda + \mu) = \sum_{w \in W} \overline{O}_{K}(N\setminus G; \lambda) \sigma^w_\mu$$

for $\lambda \in X^+$ such that $\langle \lambda, \alpha_i \rangle \geq 0$ for any $i \in I$. 

Remark 8.2. Lemma 8.1 holds for general root data $(X, \Delta, Y, \Delta^\vee)$ without the assumption $Y = Q^\vee$. The general case easily follows from the special case.
We define an abelian category \( \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}}) \) to be the category of left \( O_{K,\zeta}(\mathcal{B}^w) \)-modules;

\[
(8.4) \quad \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}}) := \text{Mod}(O_{K,\zeta}(\mathcal{B}^w)).
\]

Then we have a natural exact functor

\[
(8.5) \quad \text{res}_w : \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}}) \to \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}})
\]

induced by

\[
\text{Mod}_{gr}(O_{K,\zeta}(N\backslash G)) \ni M \mapsto ([\mathbb{S}_N^{k'}]^{-1}M)(0) \in \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}}).
\]

**Example 8.4.** Let \( G = SL_2(\mathbb{C}) \). Let \( \alpha \) be the unique positive root, and set \( \rho = \alpha/2 \).

Then we have \( X = \mathbb{Z}_\rho, X^+ = \mathbb{Z}_{\geq 0}\rho \). In this case it is well-known that the \( X \)-graded \( K \)-algebra \( O_{K,\zeta}(N\backslash G) \) is generated by the elements \( a_\zeta, b_\zeta \) of degree \( \rho \) satisfying the fundamental relation \( a_\zeta b_\zeta = \zeta b_\zeta a_\zeta \). We define a functor \( F : \text{Mod}_{gr}(O_{K,\zeta}(N\backslash G)) \to \text{Mod}_{gr}(O_{K,1}(N\backslash G)) \) as follows. For \( M = \bigoplus_{n \in \mathbb{Z}} M(n\rho) \in \text{Mod}_{gr}(O_{K,\zeta}(N\backslash G)) \) we set \( F(M) = M \) as an \( X \)-graded \( K \)-module and define the action of \( O_{K,1}(N\backslash G) \) on \( F(M) = M \) by

\[
a_1m = a_\zeta m, \quad b_1m = \zeta^n b_\zeta m \quad (m \in M(n\rho)).
\]

It is easily seen that this gives equivalences

\[
\text{Mod}_{gr}(O_{K,\zeta}(N\backslash G)) \cong \text{Mod}_{gr}(O_{K,1}(N\backslash G)), \quad \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}}) \cong \text{Mod}(\mathcal{O}_{B_w^K}), \quad \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}}) \cong \text{Mod}(\mathcal{O}_{B_w^K})
\]

of categories.

The main result of this paper is the following.

**Theorem 8.5.** Recall that \( Y = Q^\vee \). Assume that \( \zeta \) is transcendental over the prime field \( K_0 \) of \( K \), or the multiplicative order of \( \zeta \) is finite. Then we have

\[
(8.6) \quad M \in \text{Mod}(\mathcal{O}_{B_w^{K,\zeta}}), \quad \text{res}_w(M) = 0 \quad (\forall w \in W) \implies M = 0.
\]

8.3. We recall the notion of quasi-schemes following Rosenberg [12]. Let \( \mathfrak{A}, \mathfrak{B} \) be abelian categories. We say that a morphism \( f : X_\mathfrak{A} \to X_\mathfrak{B} \) between the corresponding virtual spaces is given if we are given an isomorphism class of a right exact functor \( f^* : \mathfrak{B} \to \mathfrak{A} \) admitting a right adjoint \( f_* : \mathfrak{A} \to \mathfrak{B} \). If moreover \( f_* \) is exact and faithful, we say that \( f \) is an affine morphism. Let \( \mathfrak{A}, \mathfrak{A}_\lambda (\lambda \in \Lambda) \) be abelian categories and assume that we are given morphisms \( f_\lambda : X_{\mathfrak{A}_\lambda} \to X_{\mathfrak{A}} \). We say that \( \{f_\lambda\}_{\lambda \in \Lambda} \) is a Zariski cover of \( X_{\mathfrak{A}} \) if each \( f_\lambda^* \) is a localization of categories in the sense of [4] and the patching property

\[
M \in \mathfrak{A}, \quad f_\lambda^* M = 0 \quad (\forall \lambda \in \Lambda) \implies M = 0
\]

is satisfied. Let \( \mathfrak{A}, \mathfrak{B} \) be abelian categories, and assume that we are given a morphism \( f : X_\mathfrak{A} \to X_\mathfrak{B} \) between the corresponding virtual spaces. Then we say that \( X_\mathfrak{A} \) is a quasi-scheme over \( X_\mathfrak{B} \) if there exists a Zariski cover \( \{f_\lambda\}_{\lambda \in \Lambda} \) of \( \mathfrak{A}_\lambda \) such that \( f_\lambda, f \circ f_\lambda \) are affine. Note that an ordinary scheme \( Y \) is regarded as a quasi-scheme \( X_{\text{Mod}(O_Y)} \).

By Theorem 8.5 we have the following.
Recall that $Y = Q^\vee$. Assume that $\zeta$ is transcendental over the prime field $K_0$ of $K$, or the multiplicative order of $\zeta$ is finite. Then (8.3), (8.4), (8.5) give a quasi-scheme $B_{K,\zeta}$ over $\text{Spec}(K)$ with affine open covering $B_{K,\zeta} = \bigcup_{w \in W} B_{K,\zeta}^w$, in the sense of Rosenberg [12].

8.4. We no longer assume $Y = Q^\vee$. Theorem 8.5 follows easily from the following Theorem applied to the special case $Y = Q^\vee$.

Theorem 8.7. Assume that $\zeta$ is transcendental over the prime field $K_0$ of $K$, or the multiplicative order of $\zeta$ is finite. For any $\mu \in X^+$ we have

\begin{equation}
O_{K,\zeta}(N \setminus G; \lambda + \mu) = \sum_{w \in W} O_{K,\zeta}(N \setminus G; \lambda) \sigma^w_{\mu}
\end{equation}

for $\lambda \in X^+$ such that $\langle \lambda, \alpha_{\gamma}^i \rangle \gg 0$ for any $i \in I$.

From Theorem 8.7 we obtain the following.

Corollary 8.8. Assume that $\zeta$ is transcendental over the prime field $K_0$ of $K$, or the multiplicative order of $\zeta$ is finite. For any $\mu \in X^+$ we have

\begin{equation}
O_{K,\zeta}(G) = \sum_{w \in W} O_{K,\zeta}(G) \sigma^w_{\mu}.
\end{equation}

Proof. We first show

\begin{equation}
O_{K,\zeta}(G) = O_{K,\zeta}(G)O_{K,\zeta}(N \setminus G; \nu)
\end{equation}

for any $\nu \in X^+$. Take $\varphi \in O_{K,\zeta}(N \setminus G; \nu)$ such that $\varepsilon(\varphi) = 1$. Then we have

\[
\sum_k (S^{-1} \varphi_k) \varphi_k = \varepsilon(\varphi) = 1,
\]

where $\Delta(\varphi) = \sum_k \varphi_k \otimes \varphi_k'$. By the definition of $O_{K,\zeta}(N \setminus G; \nu)$ we have

\[
\Delta(O_{K,\zeta}(N \setminus G; \nu)) \subset O_{K,\zeta}(N \setminus G; \nu) \otimes O_{K,\zeta}(G).
\]

Hence we have $1 \in O_{K,\zeta}(G)O_{K,\zeta}(N \setminus G; \nu)$, from which we obtain (8.8).

Now let $\mu \in X^+$. By Theorem 8.7 there exists some $\lambda \in X^+$ such that

\[ O_{K,\zeta}(N \setminus G; \lambda + \mu) = \sum_{w \in W} O_{K,\zeta}(N \setminus G; \lambda) \sigma^w_{\mu}. \]

Then we obtain

\[ O_{K,\zeta}(G) = O_{K,\zeta}(G)O_{K,\zeta}(N \setminus G; \lambda + \mu) = \sum_{w \in W} O_{K,\zeta}(G)O_{K,\zeta}(N \setminus G; \lambda) \sigma^w_{\mu} \]

\[ \subset \sum_{w \in W} O_{K,\zeta}(G) \sigma^w_{\mu}. \]

□
8.5. The remainder of this section is devoted to the proof of Theorem 8.7. We will derive it from Lemma 8.1.

We first consider the case ζ is transcendental over the prime field $K_0$ of $K$. This case was already dealt with in [5] and [7]. We include its proof here for the sake of the readers.

Set $R = K_0[q, q^{-1}]$. For $λ ∈ X^+$ we define a subspace $O_{\mathcal{R}, q}(N \setminus G; λ)$ of $\text{Hom}_{\mathcal{R}}(U_{\mathcal{R}, q}(\mathfrak{g}), \mathcal{R})$ by

$$O_{\mathcal{R}, q}(N \setminus G; λ) = \{ \Phi_{\nabla_{\mathcal{R}, q}(λ)}(v ⊗ v^*_λ) \mid v ∈ \nabla_{\mathcal{R}, q}(λ) \},$$

where

$$\langle \Phi_{\nabla_{\mathcal{R}, q}(λ)}(v ⊗ v^*_λ), u \rangle = \langle v^*_λ, uv \rangle \quad (u ∈ U_{\mathcal{R}, q}(\mathfrak{g})).$$

We set

$$O_{\mathcal{R}, q}(N \setminus G) = \bigoplus_{λ ∈ X^+} O_{\mathcal{R}, q}(N \setminus G; λ).$$

It is a ring graded by $X$. Moreover, we have

$$K ⊗_R O_{\mathcal{R}, q}(N \setminus G) ≃ O_{K, ζ}(N \setminus G)$$

with respect to $s_ζ : R → K$ ($q → ζ$), and

$$K_0 ⊗_R O_{\mathcal{R}, q}(N \setminus G) ≃ \overline{O}_{K_0}(N \setminus G)$$

with respect to $s_1 : R → K_0$ ($q → 1$). Set

$$\hat{σ}_μ = \Phi_{\nabla_{\mathcal{R}, q}(μ)}(T_w v_μ ⊗ v^*_μ) ∈ O_{\mathcal{R}, q}(N \setminus G)$$

for $μ ∈ X^+$, and consider the $\mathcal{R}$-linear map

$$F_λ : O_{\mathcal{R}, q}(N \setminus G; λ)^{\oplus W} → O_{\mathcal{R}, q}(N \setminus G; λ + μ) \quad ((f_w)_{w ∈ W} ↦ \sum_{w ∈ W} f_w \hat{σ}_μ^w)$$

between free $\mathcal{R}$-modules of finite rank. Assume $⟨λ, α_τ^\vee⟩ ≫ 0$. We see by Lemma 8.1 and Remark 8.2 that $K_0 ⊗_R F_λ$ with respect to $s_1$ is surjective. Hence $K ⊗_R F_λ$ with respect to $s_ζ$ is surjective when $ζ$ is transcendental over $K_0$. This is exactly what we need to show. The proof of Theorem 8.7 is now complete in the case $ζ$ is transcendental over the prime field $K_0$ of $K$.

8.6. We consider the case $ζ^2 = 1$ for any $α ∈ Δ$. Let $λ, μ ∈ X^+$. Let

$$\Xi : \nabla_K(λ) ⊗ \nabla_K(μ) → \nabla_K(λ + μ)$$

and

$$\Xi : \nabla_{K, ζ}(λ) ⊗ \nabla_{K, ζ}(μ) → \nabla_{K, ζ}(λ + μ)$$

be the unique (up to a non-zero scalar multiple) non-zero homomorphisms of $\overline{U}_K(\mathfrak{g})$-modules and $U_{K, ζ}(\mathfrak{g})$-modules respectively. In view of (7.3), (7.8) and Lemma 8.1 (see also Remark 8.2), we have only to show that $\Xi(\nabla_{K, ζ}(λ) ⊗ T_w v_μ)$ coincides with $\Xi(\nabla_K(λ) ⊗ T_w v_μ)$ under the identification $\nabla_{K, ζ}(λ + μ) ≃ \nabla_K(λ + μ)$ of Proposition 8.6. By Lemma 3.7.i and Lemma 3.2 (ii) we have

$$\Xi(\nabla_K(λ) ⊗ T_w v_μ) = \Xi(T_w(\nabla_K(λ) ⊗ v_μ)) = T_w \Xi(\nabla_K(λ) ⊗ v_μ),$$

$$\Xi(\nabla_{K, ζ}(λ) ⊗ T_w v_μ) = \Xi(T_w(\nabla_{K, ζ}(λ) ⊗ v_μ)) = T_w \Xi(\nabla_{K, ζ}(λ) ⊗ v_μ).$$
Hence by Lemma \ref{lem} it is sufficient to show
\begin{equation}
\Xi(\nabla_K(\lambda) \otimes \mu) = \Xi(\nabla_K(\lambda) \otimes v_\mu).
\end{equation}
Setting
\begin{equation}
\begin{array}{c}
M = \{ m \in \Delta_{K,\zeta}^*(\lambda + \mu) \mid \langle m, \Xi(\nabla_K(\lambda) \otimes v_\mu) \rangle = \{0\} \}, \\
\overline{M} = \{ \overline{m} \in \overline{\Delta}_K(\lambda + \mu) \mid \langle \overline{m}, \overline{\Xi}(\nabla_K(\lambda) \otimes \overline{\mu}) \rangle = \{0\} \},
\end{array}
\end{equation}
\begin{equation}
(8.9)
\end{equation}
is equivalent to \( \overline{M} = M \) under the identification \( \Delta_{K,\zeta}^*(\lambda + \mu) = \overline{\Delta}_K(\lambda + \mu) \). Let
\begin{equation}
\Xi^*: \Delta_{K,\zeta}^*(\lambda + \mu) \to \overline{\Delta}_K(\lambda) \otimes \overline{\Delta}_K(\mu)
\end{equation}
and
\begin{equation}
\Xi^*: \Delta_{K,\zeta}^*(\lambda + \mu) \to \Delta_{K,\zeta}^*(\lambda) \otimes \Delta_{K,\zeta}^*(\mu)
\end{equation}
be the unique (up to a scalar multiple) non-zero homomorphisms of \( \overline{U}_K(\mathfrak{g}) \)-modules and \( U_{K,\zeta}(\mathfrak{g}) \)-modules respectively. Note that any \( m \in \Delta_{K,\zeta}^*(\lambda + \mu)_{\lambda+\mu-\gamma} \) can be written in the form \( m = v_{\lambda+\mu}^* y \) for \( y \in U_{K,\zeta}(\mathfrak{n})_{-\gamma} \). For such \( m \) we have
\begin{equation}
\langle m, \Xi(\nabla_K(\lambda) \otimes v_\mu) \rangle = \langle \Xi^*(m), \nabla_{K,\zeta}(\lambda) \otimes v_\mu \rangle = \langle \Xi^*(v_{\lambda+\mu}^* y), \nabla_{K,\zeta}(\lambda) \otimes v_\mu \rangle
\end{equation}
\begin{equation}
= \langle (v_\lambda^* \otimes v_\mu^*) y, \nabla_{K,\zeta}(\lambda) \otimes v_\mu \rangle.
\end{equation}
By \( \Delta(y) \in y \otimes k_+ + \sum_{\delta \in Q^+ \setminus \{0\}} U_{K,\zeta}(\mathfrak{n}) \otimes U_{K,\zeta}(\mathfrak{h}) U_{K,\zeta}(\mathfrak{n})_{-\delta} \) we have
\begin{equation}
\langle m, \Xi(\nabla_K(\lambda) \otimes v_\mu) \rangle = \langle v_\lambda^* y, \nabla_{K,\zeta}(\lambda) \rangle.
\end{equation}
Hence \( M = v_{\lambda+\mu}^* A \) with \( A = \{ y \in U_{K,\zeta}(\mathfrak{n}) \mid v_\lambda^* y = 0 \} \). By \( 2.3 \) we can also write \( M = v_{\lambda+\mu}^* A' \) with \( A' = \{ y \in U_{K,\zeta}(\mathfrak{n}) \mid \overline{\nabla}_\lambda y = 0 \} \). On the other hand by a similar argument we have \( \overline{M} = \overline{\nabla}_{\lambda+\mu} A \) with \( \overline{A} = \{ y \in \overline{U}_K(\mathfrak{n}) \mid \overline{\nabla}_\lambda y = 0 \} \). Therefore, we obtain \( \overline{M} = M \) from Lemma \ref{lem} \( \Xi \) is now complete in the case \( \zeta_0 = 1 \) for any \( \alpha \in \Delta \).

8.7. Finally, we consider the case where the multiplicative order of \( \zeta \in K^\times \) is finite. Denote the multiplicative order of \( \zeta \) by \( \ell \), and consider the root datum \((\ell X, \ell \Delta, \ell Y, \ell \Delta^\vee)\) as in Section 2. Recall that we have an embedding
\begin{equation}
O_{K,\zeta}(\ell N \backslash \ell G) \subset O_{K,\zeta}(N \backslash G)
\end{equation}
of algebras satisfying
\begin{equation}
O_{K,\zeta}(\ell N \backslash \ell G; \lambda) \subset O_{K,\zeta}(N \backslash G; \lambda)
\end{equation}
for \( \lambda \in \ell X^+ \). Let \( \mu \in X^+ \). We can take \( \nu \in X^+ \) and \( \mu' \in \ell X^+ \) such that \( \mu + \nu = \mu' \). By the result of the preceding subsection together with \( 3.4 \), \( 3.5 \) we obtain
\begin{equation}
O_{K,\zeta}(\ell N \backslash \ell G; \lambda' + \mu') = \sum_{w \in W} O_{K,\zeta}(\ell N \backslash \ell G; \lambda') \sigma_{\mu'}^w
\end{equation}
for some $\lambda' \in \mathbb{X}^+$. Let $\xi \in \mathbb{X}^+$ such that $\langle \xi, \alpha_i \rangle \gg 0$ for any $i \in I$. Then by Proposition 7.1 we have
\[
O_{K,\zeta}(N \setminus G; \xi + \lambda' + \nu + \mu) = O_{K,\zeta}(N \setminus G; \xi + \lambda' + \mu')
\]
\[
= O_{K,\zeta}(N \setminus G; \xi)O_{K,\zeta}(N \setminus G; \lambda')\sigma_{\mu'}^w
\]
\[
\subset \sum_{w \in W} O_{K,\zeta}(N \setminus G; \xi + \lambda')\sigma_{\mu'}^w = \sum_{w \in W} O_{K,\zeta}(N \setminus G; \xi + \lambda')\sigma_{\nu}^w\sigma_{\mu}^w
\]
The proof of Theorem 8.7 is complete.

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9-3-12 JIYUGAOKA, MUNAKATA, FUKUOKA, 811-4163 JAPAN
Email address: ttanisaki@icloud.com