A realistic non-local electrical thermometer based on Coulomb coupled systems

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We propose a realistic design for Coulomb-coupled system based non-local cryogenic electrical thermometer. The operation of the proposed thermometer relies on electrostatic interaction between Coulomb coupled quantum dots, resulting in an overall change in conductance due to change in the remote reservoir temperature. The performance and regime of operation of the proposed thermometer is then theoretically investigated using density matrix formulation and compared with another recently proposed Coulomb coupled system based thermometer. It is demonstrated that the proposed design ensures a superior temperature sensitivity and robustness compared to the other design proposed in literature (Physica E, 114, 113635, 2019). At the end we investigate the regime of optimal operation and comment on the ground state configuration for optimal operation of the proposed thermometer. The design proposed in this paper can be employed to construct highly efficient non-local cryogenic thermometers.
I. INTRODUCTION

Electrical sensing of temperature and heat flow in nano-scale systems, particularly in the cryogenic regime, has been one of the biggest challenges in the modern era. Engineering devices to couple heat with electrically measurable quantities has been extremely difficult in the domain of solid-state nano-technology. In the aspect of thermally controlled electrical transport in nano-scale systems; thermoelectric engines [1–26], refrigerators [27–38], rectifiers [39–44] and transistors [45–52] have been proposed in the last decade. Recently, the provision towards non-local thermal control of electrical transport, where electrical variables between two terminals are manipulated via thermal action at a remote third terminal, has been proposed and realized experimentally [53–65].

Proposal towards non-local thermal control of electrical transport mainly includes multi-terminal devices, where current/voltage between two terminals may be manipulated via heat energy stored at one or many non-local remote reservoirs [53–65]. Non-local coupling between thermal and electrical quantities offer many distinct advantages over their local counterparts which include (but are not limited to) the provision towards an independent manipulation of electrical and lattice thermal conductance, isolation of the remote target reservoir from Joule heat dissipation, etc. Proposals towards temperature induced control of electronic flow in quantum dots [66] and quantum-point-contacts [67], due to stochastic thermal fluctuation in a capacitively coupled quantum dot, have already been put-up in literature. In addition, a lot of effort have been geared towards theoretical and experimental realization of electrical thermometers [68–82].

In this paper, we propose a non-local electrical thermometer employing Coulomb coupled systems that can be used to sense temperature from a remote target reservoir. The spatial separation of the target reservoir from the current flow track not only shields the reservoir from unnecessary Joule heating, but also offers the provision towards independent manipulation of lattice thermal conductance, such that good thermal isolation of the target reservoir can be achieved. Although electrical thermometers based on Coulomb coupled systems have already been put-up in literature [66], our proposed design offers a superior sensitivity and robustness compared to earlier proposed systems. In addition, the temperature sensitivity in our proposed system for sufficient bias voltage is independent of the average temperature of the current flow path, making such systems suitable for practical purpose.

This paper is organized as follows. In Sec. II, we elaborate the proposed system and briefly describe the transport formulation employed to analyze its performance. The detailed derivations for the transport formulation is given in Appendix A. In Sec. III, we discuss the regime of operation and performance of the proposed thermometer. This section also presents a brief discussion on the relative performance of the proposed thermometer compared to the one recently proposed in literature [66]. Finally, we conclude the paper briefly in Sec. IV.

II. PROPOSED DESIGN AND TRANSPORT FORMULATION

The proposed thermometer, schematically demonstrated in Fig. 1(a) as system-I, consists of four quantum dots $S_L$, $S_M$, $S_R$, and $S_G$. The dots $S_L$, $S_R$, and $S_G$ are electrically coupled to reservoirs $L$, $R$, and $G$ respectively, $G$ being the target reservoir whose temperature is to be accessed. The ground state energy levels of the four dots $S_L$, $S_M$, $S_R$, and $S_G$ are denoted by $\varepsilon_L$, $\varepsilon_M$, $\varepsilon_R$ and $\varepsilon_G$ respectively, where $\varepsilon_L$, $\varepsilon_M$, and $\varepsilon_R$ are aligned with each other. The dot $S_M$ is electrically coupled to both $S_L$ and $S_R$, while capacitively coupled to $S_G$. The capacitive coupling permits energy exchange between the dots $S_M$ and $S_G$ while restricting any electronic flow between them, resulting in zero net electronic current out of (into) the reservoir $G$. Coming to the provision towards practical fabrication of such a system, the considerable advancement in solid-state nano-fabrication technology has led to the realization of three and four dot systems with and without Coulomb coupling [83–88]. In addition, two specific quantum dots that are far in space may be bridged together to achieve strong capacitive coupling, without affecting the electrostatic energy of adjacent quantum dots [89–93].

As discussed earlier, the proposed system operates on the basis of Coulomb coupling between $S_M$ and $S_G$. Due to capacitive coupling between $S_G$ and $S_M$, any change in electron number in $S_M$ changes the electrostatic energy of $S_G(M)$. Maximum conductance between the reservoirs $L$ and $R$ is thus achieved when the ground state of $S_G$ is unoccupied. As demonstrated in Fig. 2, any electronic tunnelling into the ground state of $S_G$ enhances the electrostatic energy in $S_M$ and restricts electronic flow between the dots. The average conductance between $L$ and $R$ is thus dependent on the average occupancy probability of $S_G$. Any change in temperature of the reservoir $G$ changes the average ground state occupancy probability of $S_G$, and thus affects the current flow between $L$ and $R$, resulting in a temperature sensitive conductance. We will later demonstrate that the sign of sensitivity is dependent on the relative position of $\varepsilon_G$ relative to the Fermi energy.

We employ density matrix formulation to study the transport phenomena and performance of the proposed set-up (derived in Appendix A). In Fig. 1(a), we denote the reservoir-to-system coupling by $\gamma_c$ while, $t_L$ and $t_R$ represents the inter-dot tunnel coupling amplitude. The fluctuation in electrostatic energy of the entire system consisting of four
Thus, a possible state of consideration in the system may be denoted as \( | \)...

\[ U(n_{S_L}, n_{S_M}, n_{S_R}, n_{S_G}) = \sum_x U_x^{\text{self}} (n_x^{\text{tot}} - n_x^{eq})^2 + \sum_{x_1 \neq x_2} U_x^{\text{m}} (n_{x_1}^{\text{tot}} - n_{x_1}^{eq}) (n_{x_2}^{\text{tot}} - n_{x_2}^{eq}) \]  

\( n_x^{\text{tot}} \) being the total electron number, and \( U_x^{\text{self}} = \frac{q^2}{C_x^{\text{self}}} \) is the electrostatic potential energy due to self-capacitance \( C_x^{\text{self}} \) (with the surrounding leads) of the dot \( x \) (details given in Appendix A). \( U_x^{\text{m}} \) is the electrostatic energy due to Coulomb coupling between two neighbouring quantum dots, and \( n_x^{eq} \) is the overall equilibrium number of electrons present in dot \( x \) at 0K and is determined by the minimum attainable electrostatic energy of the system. 

\( n_x = n_x^{\text{tot}} - n_x^{eq} \), hence, defines the total number of electrons added in the ground state of the dot \( x \) because of stochastic thermal fluctuations from the reservoirs. Here, a minimal physics model is used to study the thermometer performance under the assumption that the fluctuation in potential due self-capacitance of the quantum dots is much more compared to the average thermal voltage \( kT/q \) or the applied bias voltage \( V \), that is \( U_x^{\text{self}} = \frac{q^2}{C_x^{\text{self}}} >> (kT, qV) \). Thus, the electron occupancy probability or transport rate via the self-capacitance induced Coulomb blocked energy state is negligibly small. The analysis of the entire system of four quantum dots may hence be completed by limiting the maximum number of electrons in the ground state of each dot to one. Hence, the entire system investigation may be limited to sixteen multi-electron states, which we denote by the electron occupation number in the ground state of each quantum dot. Thus, a possible state of consideration in the system may be denoted as \( | n_{S_L}, n_{S_M}, n_{S_R}, n_{S_G} \rangle = | n_{S_L} \rangle \otimes | n_{S_M} \rangle \otimes | n_{S_R} \rangle \otimes | n_{S_G} \rangle \), where \( n_{S_L}, n_{S_M}, n_{S_R}, n_{S_G} \in (0, 1) \), denote the number of electrons present in the ground-states of \( S_L, S_M, S_R \) and \( S_G \) respectively. We also assume that the strength of capacitive coupling between the dots, except for that between \( S_M \) and \( S_G \), are negligible, such that, for all practical purposes under consideration,
Due to capacitive coupling induced transport dependence among the ground states of $S_M$ and $S_G$, we treat this pair of dots as a sub-system ($\varsigma_M$) of the entire system of four dots. The other two sub-systems $\varsigma_L$ and $\varsigma_R$ consists of the dots $S_L$ and $S_R$ respectively. The state probability of the sub-system $\varsigma_M$ is denoted by $P_{x,y}^{\varsigma_M}$, where $x$ and $y$ represent the electron number in the ground state of dot $S_M$ and $S_G$ respectively. $P_{k}^{\varsigma_L(R)}$, on the other hand, denotes the probability of occupancy of the dot $S_{L(R)}$. Under the condition that $U_{S_M,S_G}$ is a few times greater than the ground state broadening due to system-to-reservoir coupling, the inter-dot tunnelling rates are are maximized when either $\varepsilon_M$ or $\varepsilon_M + U_{S_M,S_G}$ coincides with the energy levels $\varepsilon_L$ and $\varepsilon_R$ (See Appendix A). For simplicity, we henceforth represent $U_{S_M,S_G}$ as $U_m$. The state probabilities $P_{x,y}^{\varsigma_M}$ and $P_{k}^{\varepsilon_L(R)}$ are computed using density matrix formulation derived in Appendix A (Eqs. A59), assuming quasi-equilibrium Fermi-Dirac statistics at the reservoirs. The probability of occupancy of the reservoirs at energy $\varepsilon$ is thus given by:

$$f_\upsilon(\varepsilon) = \left(1 + \exp\left(\frac{\varepsilon - \mu_\upsilon}{kT_\upsilon}\right)\right)^{-1},$$

where $\mu_\upsilon$ and $T_\upsilon$ respectively represent the quasi-Fermi energy and temperature of the reservoir $\upsilon$. To analyze the performance of the proposed thermometer, we use a voltage-controlled model, where a bias voltage $V$ is applied, with the positive and negative terminals being connected to $R$ and $L$ respectively. On application of a bias voltage $V$, the different quasi-Fermi energy of the reservoirs under quasi-equilibrium condition may be written as $\mu_G = \mu_0$, $\mu_L = \mu_0 + qV/2$ and $\mu_R = \mu_0 - qV/2$, where $\mu_G$, $\mu_L$ and $\mu_R$ denote the quasi-Fermi energy of the reservoirs $G$, $L$ and $R$ respectively and $\mu_0$ is the equilibrium Fermi energy of the entire system under consideration. On calculation of the sub-system state probabilities, the electronic current flowing into (out-of) the left (right) reservoirs can be written as
In this section, we investigate the performance and the regime of operation of the proposed thermometer (system-I). In addition, we also conduct a performance comparison between system-I and system-II in terms of temperature sensitivity. To investigate the proposed thermometer (system-I), without loss of generality, we choose \( T_L = T_R = T_G = 10K \).

![Graph showing the variation of sensitivity with temperature](image)

**FIG. 3.** Benchmark of the maximum current of system-II with that of system-I. In case of system-I, \( \gamma_c = 10^{-5} eV \). The maximum current of system-II becomes identical with that of system-I for \( \gamma = 3.75 \times 10^{-7} eV \) (See Fig. 1). For calculating the maximum current through the systems, the ground states of the dots \( S_G \) (for system-I) and \( G_1 \) (for system-II) are assumed to be empty (See Fig. 1) and the ground states of the dots \( S_L, S_M, S_R \) (for system-I) and \( S_1 \) (for system-II) are assumed to be aligned with the equilibrium Fermi potential. The temperature of the reservoirs are assumed to be \( T_L = T_R = T_G = 10K \).

(Appendix A):

\[
I_{L(R)} = \frac{q^2}{h} \gamma_c \left\{ f_{L(R)}^{\ast}(\varepsilon_{L(R)}) f_{L(R)}(\varepsilon_{L(R)}) - f_{L(R)}^{\ast}(\varepsilon_{L(R)}) [1 - f_{L(R)}(\varepsilon_{L(R)})] \right\},
\]

where \( \gamma_c \) denotes the reservoir-to-system coupling in eV (Fig. 1.a). The thermometer sensitivity is defined as the rate of change of electronic current between \( L \) and \( R \) with the remote reservoir temperature \( T_G \). We thus define the current sensitivity as:

\[
\text{Sensitivity} (\chi) = \left( \frac{dI}{dT_G} \right),
\]

where \( I = I_L = -I_R \).

The recently proposed electrical thermometer, demonstrated in Fig. 1(b), consists of two Coulomb coupled quantum dots \( S_1 \) and \( G_1 \) [66]. \( S_1 \) is electrically coupled to the reservoirs \( L \) and \( R \) respectively, while \( G_1 \) is electrically coupled to the reservoir \( G \). The dots \( S_1 \) and \( G_1 \) are capacitively coupled with mutual charging energy \( U_m \). We will call this set-up as system-II.

### III. RESULTS

In this section, we investigate the performance and the regime of operation of the proposed thermometer (system-I). In addition, we also conduct a performance comparison between system-I and system-II in terms of temperature sensitivity. To investigate the proposed thermometer (system-I), without loss of generality, we choose \( \gamma_c = 10^{-5} eV \) and \( t_L = t_R = 0.1 \gamma_c \). Such values of the coupling parameters signify the weak coupling limit and thus restrict the electronic transport in the sequential tunnelling regime where the effects of co-tunnelling and higher order tunnelling processes can be neglected. We also choose \( T_L = T_R = 10K \). In addition, to compare the performance of our proposed thermometer (system-I) to the one demonstrated in literature (system-II), we choose \( \gamma \approx 3.75 \times 10^{-7} eV \) (Fig. 1.b), which results in identical maximum current for both the systems (See Fig. 3).

**Performance comparison between system-I and system-II:** Fig. 4(a) demonstrates the variation of sensitivity of the proposed thermometer (system-I) with the applied bias, for \( U_m = 2meV \), \( \varepsilon_L = \varepsilon_M = \varepsilon_R = \varepsilon_G = \mu_0 \), and different values of remote reservoir temperature \( T_G \). We note that sensitivity magnitude increases with increase in bias \( V \), up to the point of saturation. Such a behaviour can be expected as the maximum saturation current for quantum dot systems is achieved when the bias voltage exceeds a few \( kT_{L(R)} \). An increase and subsequent saturation in the total current marks identical behaviour for the system sensitivity. We note that for \( U_m = 2meV \), the system temperature sensitivity is zero for \( T_G = 2K \) for the entire range of applied bias. This is because under the condition \( \varepsilon_L = \varepsilon_M = \varepsilon_R = \varepsilon_G = \mu_0 \), \( \frac{d}{dT_G} f_G(\varepsilon_G + U_m) = 0 \) at \( T_G = 2K \), which results in zero temperature sensitivity. With an
FIG. 4. Variation of temperature sensitivity for (a) system-I and (b) system-II against voltage bias for various values of $T_G$ and fixed $T_{L(R)} = 10K$. The mutual Coulomb coupling energy is taken to be $U_m = 2meV \approx 2.3209\frac{kT_G}{q}$. We assume that the ground state energy levels are aligned to $\mu_0$, that is, $\epsilon_L = \epsilon_M = \epsilon_R = \epsilon_G = \mu_0$.

increase in $T_G$, $\frac{4}{dT_G} f_G(\epsilon_G + U_m)$ acquires a finite value resulting in non-zero thermometer sensitivity (Appendix B). Although not demonstrated here, a finite system sensitivity at $T_G=2K$ can be achieved for lower values of $U_m$ (shown later). Fig. 4(b) demonstrates the variation in sensitivity with applied bias for system-II. We note that our proposed thermometer (system-I) demonstrates a higher maximum sensitivity compared to the one already put up in literature (system II) [66]. This is due to the fact that in our proposed system, the dots $S_L$ and $S_R$ act as energy filters, thereby prohibiting appreciable current through the Coulomb blocked level $\epsilon_M + U_m$, resulting in a drastic current reduction when the ground state $S_G$ is occupied. We also note from Fig. 4(b) that the sensitivity of system-II varies over a range of applied bias and reduces to zero when the applied bias is increased. Such a behaviour implies that the system is prone to voltage noise. When the applied bias is gradually increased, the current through the system-II increases, which increases the sensitivity magnitude. On further increase in applied bias voltage, electrons can tunnel into (out of) the Coulomb blocked energy level $\epsilon^1_s + U_m$, which gradually reduces the temperature sensitivity. When the applied bias is sufficiently high, such that $f_L(\epsilon^1_s + U_m) \approx 1$ and $f_R(\epsilon^1_s) \approx 0$, current through the system is maximized, prohibiting any further change in current due to variation in remote reservoir temperature $T_G$. This results in zero temperature sensitivity for system-II at high values of applied bias. The sensitivity of the proposed system-I, as
FIG. 5. Variation of temperature sensitivity for (a) system-I and (b) system-II against voltage bias for various values of $T_{L(R)}$ and fixed remote reservoir $T_G = 10\,K$ and Coulomb coupling energy $U_m = 2\,meV \approx 2.3209\frac{kT_G}{q}$. We assume that the ground state energy levels are aligned to $\mu_0$, that is, $\varepsilon_L=\varepsilon_M=\varepsilon_R=\varepsilon_G=\mu_0$.

**demonstrated in Fig. 4(a) on the other hand remains constant when the bias voltage $V$ increases beyond $10\frac{kT_{L(R)}}{q}$, which makes system-I robust against noise at high values of applied bias.**

**Fig. 5 demonstrates the variation in temperature sensitivity with voltage bias for various values of $T_{L(R)}$. In particular Fig. 5(a) demonstrates the temperature sensitivity for our proposed system-I. We note that at low values of the applied voltage bias ($V$), the system sensitivity is dependent on $T_{L(R)}$. However, when the bias voltage is sufficiently increased, the sensitivity for different values of $T_{L(R)}$ saturates at the same magnitude. Thus for sufficiently high bias $V$, our proposed system-I is robust against any variation of the average current path temperature. On the other hand, for the recently proposed system-II [66], as demonstrated in Fig. 5(b), there is no value of the applied bias $V$ for which the system sensitivity is independent of $T_{L(R)}$. Thus in case of system-II, the sensitivity is dependent on the average temperature of the current path. This implies that system-II is not robust against variation in the average current path temperature. It should again be noted that the robustness of our proposed system-I against variation in the temperature of current path, at high values of applied bias $V$, should be attributed to the energy filtering effect of $S_L$ and $S_R$.**

**Fig. 6 demonstrates the variation of saturation sensitivity (at high bias) with remote reservoir temperature $T_G$**
FIG. 6. Sensitivity as a function of the remote reservoir temperature $T_G$ for various values of $U_m$ for (a) our proposed system-I and (b) recently proposed system-II [66]. The temperature of reservoirs $L$ and $R$ are fixed at $T_{L(R)} = 10K$, while the quantum dot ground states are aligned with the equilibrium Fermi energy, that is, $\varepsilon_L = \varepsilon_M = \varepsilon_R = \varepsilon_G = \mu_0$.

for different values of the Coulomb-coupling energy ($U_m$) for both system-I and system-II. In particular, Fig. 6 (a) demonstrates the variation of system-I saturation sensitivity with $T_G$ (at $V = 0.0129V$ or $\approx 15kT_{L(R)}/q$, which results in the maximum saturation sensitivity). We note that at lower target reservoir temperature $T_G$, the system sensitivity is higher for lower values of $U_m$. Clearly, for a particular $U_m$, there exists an optimal remote reservoir temperature $T_G = T_G^{opt}$ at which the system sensitivity is maximum. We also note that the optimal sensitivity at a higher temperature demands a higher value of $U_m$. This optimal $T_G$ ($T_G^{opt}$) as a function of $U_m$ can be given as (Appendix B):

$$T_G^{opt} = (2.399)^{-1} \frac{U_m}{k}$$  

(5)

From Eqn. 5, it can be noted that the value of $T_G^{opt}$ increases with increase in $U_m$, a trend which is clearly clearly reflected in Fig. 6(a). Fig. 6(b) demonstrates the maximum sensitivity of the recently proposed thermometer (system-II) [66] for different values of $U_m$ with a variation in the remote reservoir temperature. For computing the maximum sensitivity of system-II, we chose $V = 0.0032V$. It is clear that in the low temperature regime, the proposed thermome-
ter (system-I) demonstrates a much higher magnitude of sensitivity compared to the recently proposed system-I \[66\]. Thus, the proposed thermometer (system-I) demonstrates an overall superior performance compared to system-II in terms of high magnitude of temperature sensitivity and robustness against fluctuations in voltage bias and average temperature of the current path.

**Regime of operation:** Fig. 7 demonstrates the operation regime of the proposed thermometer (system-I) in terms of the ground state configurations of the quantum dots for \(U_m = 2mV \approx 2.3209kT/q\) and \(T_L=T_R=T_G=10K\). While investigating the regime of operation, the ground states of \(S_L, S_M\) and \(S_R\) are assumed to be aligned as \(\varepsilon_L = \varepsilon_M = \varepsilon_R = \varepsilon_0\). We note that the sensitivity magnitude is maximum when \(\varepsilon_G\) lies a few \(kT_G\) above the equilibrium Fermi energy. This can be explained as follows. When \(\varepsilon_G = 0\), a change in \(T_G\) results in a variation of stochastic fluctuation at the Coulomb blocked level \(\varepsilon_G + U_m\). There is, however, no variation in stochastic fluctuation of the ground state \(\varepsilon_G\). As the ground state \(\varepsilon_G\) increases above \(\mu_0\), a change in \(T_G\) affects the stochastic fluctuation of both the ground state \(\varepsilon_G\) and the Coulomb blocked level \(\varepsilon_G + U_m\), which increases the system sensitivity to \(T_G\). When \(\varepsilon_G\) is further increased above \(\mu_0\), both the function \(\frac{d\varepsilon_G}{dT_G}\) and \(\frac{dG}{dT_G}(\varepsilon_G + U_m)\) reduces causing a drop in the overall sensitivity. We thus note that the regime of optimal sensitivity, in terms of \(\varepsilon_G\) is a few \(kT_G\) above \(\mu_0\). We also note that the sensitivity, as \(\varepsilon_G\) goes below \(\mu_0\), becomes positive. This is due to the fact that as \(\varepsilon_G\) gradually goes below \(\mu_0\), an increase in \(T_G\) causes a decrease in the ground state occupancy of \(S_G\) \(\left(\frac{dG}{dT_G} < 0 \text{ when } \zeta < \mu_0\right)\), resulting in an increase in current flow with an increase in temperature. Fig. 7 also demonstrates that our proposed thermometer offers good temperature sensitivity over a wide range of \(\varepsilon_0\), which is \(-4kT_{L(G)} < \varepsilon_0 < 4kT_{L(G)}\).

The variation in sensitivity with the ground state configuration of \(S_M\) is demonstrated in Fig. 8. In this case the ground states of the dots \(S_L, S_R\) and \(S_G\) are kept fixed at \(\varepsilon_L = \varepsilon_R = \varepsilon_G = \mu_0\), while \(\varepsilon_M\) is varied to obtain the positions of optimal sensitivity. As already noted in the previous discussions, the sensitivity is negative and high when \(\varepsilon_M\) coincides with \(\varepsilon_{L(R)}\). This is expected since the current through the system is maximum when the ground states of the dots \(S_L, S_M\) and \(S_R\) are aligned with each other. The negative value of sensitivity arises from the fact that an increase temperature results in an increase in electron occupation probability in the ground state of the dot \(S_G\) and thus decreases the current through the system. A misalignment in the ground state of \(S_M\) with respect to the ground states of \(S_{L(R)}\) results in a sharp deterioration in system sensitivity. Interestingly, we also note a positive peak in sensitivity around the \(\varepsilon_M - \varepsilon_{L(R)} \approx 2.3\frac{kT}{q}\), which is equal to the value of \(U_m\) in this case. When \(\varepsilon_M + U_m = \varepsilon_{L(R)}\), the ground states \(\varepsilon_{L(R)}\) is aligned with the Coulomb blocked level \(\varepsilon_M + U_m\). Hence, an electron tunneling into the ground state of \(S_G\) increases the current flow through the system. When \(\varepsilon_G - \mu_0 = 0\), an increase in temperature increases the occupancy probability of the Coulomb blockaded level \(\varepsilon_G + U_m\) which in turn enhances the current flow. This in-turn results in a positive sensitivity peak at \(\varepsilon_M + U_m = \varepsilon_{L(R)}\). We thus note two possible configurations of the ground state of \(S_M\) which results in finite optimal temperature sensitivity of the proposed thermometer (system-I).

![Fig. 7. Colour plot depicting the performance of the proposed thermometer with variation in the ground states for \(U_m = 2mV \approx 2.3209kT/q\) and \(V = 0.0129V \approx 15kT/q\). The ground state of the dots \(S_L, S_M\) and \(S_R\) are aligned with each other as \(\varepsilon_L = \varepsilon_M = \varepsilon_R = \varepsilon_0\). The temperature of the reservoirs \(L, R\) and \(G\) are chosen as \(T_L = T_R = T_G = 10K\).](image-url)
compact, I rename the states as

\[ |0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle, |8\rangle, |9\rangle, |10\rangle, |11\rangle, |12\rangle, |13\rangle, |14\rangle, |15\rangle, |16\rangle. \]

Under this representation and the assumptions already stated in the main text, the Hamiltonian of the entire system (excluding the reservoirs) can be

\[
U(n_{S_L}, n_{S_M}, n_{S_R}, n_{S_G}) = \sum_x U^\text{self}_x (n^\text{tot}_x - n^\text{eq}_x)^2 + \sum_{x_1 \neq x_2} U^m_{x_1, x_2} (n^\text{tot}_{x_1} - n^\text{eq}_{x_1}) (n^\text{tot}_{x_2} - n^\text{eq}_{x_2}),
\]  

(A1)

where the symbols have already been defined in the main text. As stated earlier, a possible state of the system may be denoted by \([n_L, n_M, n_R, n_G]\), where \(n_L, n_M, n_R, n_G \in (0, 1)\). With a slight abuse of notations, to make the equations compact, I rename the states as \(|0, 0, 0, 0\rangle \rightarrow |1\rangle, |0, 0, 0, 1\rangle \rightarrow |2\rangle, |0, 0, 1, 0\rangle \rightarrow |3\rangle, |0, 0, 1, 1\rangle \rightarrow |4\rangle, |0, 1, 0, 0\rangle \rightarrow |5\rangle, |0, 1, 0, 1\rangle \rightarrow |6\rangle, |0, 1, 1, 0\rangle \rightarrow |7\rangle, |0, 1, 1, 1\rangle \rightarrow |8\rangle, |1, 0, 0, 0\rangle \rightarrow |9\rangle, |1, 0, 0, 1\rangle \rightarrow |10\rangle, |1, 0, 1, 0\rangle \rightarrow |11\rangle, |1, 0, 1, 1\rangle \rightarrow |12\rangle, |1, 1, 0, 0\rangle \rightarrow |13\rangle, |1, 1, 0, 1\rangle \rightarrow |14\rangle, |1, 1, 1, 0\rangle \rightarrow |15\rangle, |1, 1, 1, 1\rangle \rightarrow |16\rangle. \] 

Under this representation and the assumptions already stated in the main text, the Hamiltonian of the entire system (excluding the reservoirs) can be

IV. CONCLUSION

To conclude, in this paper, we have proposed a realistic electrical thermometer based on Coulomb coupled systems. The performance and regime of operation of the proposed thermometer was then theoretically analyzed using the density matrix formulation. It was demonstrated that the proposed thermometer demonstrates a superior performance and robustness compared to the earlier proposed thermometers \([66]\). In addition, the symmetrical design of the system allows for the possibility of electron-phonon scattering on the proposed system performance also constitutes an interesting direction. Other possible design strategies for electrical thermometers employing Coulomb-coupled quantum dots is left for future investigation. Nevertheless, the design proposed in this paper can be used to fabricate highly sensitive and robust non-local electrical thermometers.

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Appendix A: Derivation of state probabilities and current equations via density matrix formulation

The total electrostatic energy of the system shown in Fig. 1 can be given by the equation:

\[
U(n_{S_L}, n_{S_M}, n_{S_R}, n_{S_G}) = \sum_x U^\text{self}_x (n^\text{tot}_x - n^\text{eq}_x)^2 + \sum_{x_1 \neq x_2} U^m_{x_1, x_2} (n^\text{tot}_{x_1} - n^\text{eq}_{x_1}) (n^\text{tot}_{x_2} - n^\text{eq}_{x_2}),
\]  

(A1)

FIG. 8. Plot of sensitivity of the proposed thermometer with variation in the energy level \(\varepsilon_M\). The Coulomb-coupling energy is assumed to be \(U_m = 2meV (\approx 2.3209 \mu eV)\). The bias \(V\) is chosen as \(V = 0.0129V (\approx 15.2F)\). Temperature of the reservoirs \(L, R\) and \(G\) are taken as \(T_L = T_R = T_G = 10K\). The other ground state energy levels are aligned to \(\mu_0\), that is, \(\varepsilon_L = \varepsilon_R = \varepsilon_G = \mu_0\).
where $h.c.$ denotes hermitian conjugate. In the above equations, $\epsilon_s$ is the total energy of the state system in state $|x\rangle$, assuming the state $|0, 0, 0, 0\rangle$ or $|1\rangle$ to be the vacuum state. Hence, the different values of $\epsilon_s$ may be expressed as: $\epsilon_1 = 0; \epsilon_2 = \epsilon_G; \epsilon_3 = \epsilon_R; \epsilon_4 = \epsilon_R + \epsilon_E; \epsilon_5 = \epsilon_M; \epsilon_6 = \epsilon_G + \epsilon_M + U_m; \epsilon_7 = \epsilon_M + \epsilon_R; \epsilon_8 = \epsilon_G + \epsilon_M + \epsilon_R + U_m; \epsilon_9 = \epsilon_L; \epsilon_{10} = \epsilon_L + \epsilon_G; \epsilon_{11} = \epsilon_L + \epsilon_E + \epsilon_G; \epsilon_{12} = \epsilon_L + \epsilon_E + \epsilon_G + \epsilon_E + \epsilon_M + U_m; \epsilon_{13} = \epsilon_L + \epsilon_E + \epsilon_G + \epsilon_M + U_m; \epsilon_{14} = \epsilon_L + \epsilon_E + \epsilon_G + \epsilon_M + \epsilon_R; \epsilon_{15} = \epsilon_L + \epsilon_E + \epsilon_M + \epsilon_R$.

Under the assumption that the inter-dot coupling is much weaker than the system to reservoir coupling, the temporal evolution of the density matrix elements, in the regime of dominant sequential transport between the reservoir and the system can be given by a set of modified Liouville equations $[94–100]$:

\[
\frac{\partial \rho_{\eta\eta}}{\partial t} = -i[H, \rho]_{\eta\eta} - \sum_{\nu} \Gamma_{\eta\nu} \rho_{\eta\nu} + \sum_{\delta} \Gamma_{\delta\eta} \rho_{\delta\eta},
\]

\[
\frac{\partial \rho_{\eta\beta}}{\partial t} = -i[H, \rho]_{\eta\beta} - \frac{1}{2} \sum_{\nu} \left( \Gamma_{\eta\nu} + \Gamma_{\beta\nu} \right) \rho_{\eta\beta},
\]

where $\rho_{\eta\eta} = \langle \eta | \rho | \eta \rangle$ and $\rho_{\eta,\beta} = \langle \eta | \rho | \beta \rangle$, $\rho$ being the density matrix operator. In the above equation, $[A, B]$ denotes the commutator of the operators $A$ and $B$, while $\Gamma_{\eta\nu}$ denotes the rate of transition from state $\nu$ to state $\eta$ due to electronic transitions via system-to-reservoir coupling. $\Gamma_{\eta\nu}$ can be given by

\[
\Gamma_{\eta\nu} = \gamma_c \times f_\nu (\epsilon_\eta - \epsilon_\nu - \mu_\nu),
\]

where the electronic transition is driven by the reservoir $\nu$. We now write the equation for the density matrix element $\rho_{5,5}$ from Eq. (A3), assuming steady state.

\[
0 = \dot{\rho}_{5,5} = it_L \{ \rho_{5,9} - \rho_{5,3} \} + it_R \{ \rho_{5,3} - \rho_{5,5} \} + \sum_{j=6,7,13} \left( \Gamma_{5,j} \rho_{5,j} - \rho_{5,5} \Gamma_{5,j} \right)
\]

(A5)

A change in the state $|5\rangle$ of the system can occur due to external system-to-reservoir coupling or due to inter-dot tunnelling. Inter-dot tunnelling, can drive the system from state $|5\rangle$ to $|3\rangle$ or $|9\rangle$. In Eq. (A5), the first term accounts to inter-dot tunnelling between the dots $S_L$ and $S_M$, the second term accounts for the inter-dot tunnelling between dot $S_M$ and $S_R$, while the third term accounts for the change in system state due to system-to-reservoir coupling. The expression of the non-diagonal density matrix elements $\rho_{3,5} = \rho_{5,3}^*$, under steady state, can be derived from Eq. (A3) as:

\[
0 = \dot{\rho}_{3,5} = \dot{\rho}_{5,3}^* = i \rho_{3,5} \{ \epsilon_5 - \epsilon_3 \} + it_R \{ \rho_{3,3} - \rho_{5,5} \} + it_L \rho_{3,9} - \frac{1}{2} \left( \sum_{j=6,7,13} \Gamma_{5,j} + \sum_{j=1,4,11} \Gamma_{3,j} \right) \rho_{3,5}
\]

(A6)

\[
\rho_{3,5} = \rho_{5,3}^* = \frac{t_R \{ \rho_{3,3} - \rho_{5,5} \} + t_L \rho_{3,9}}{\epsilon_3 - \epsilon_5} - \frac{1}{2} \Gamma_{3,5}
\]

(A7)

\[
\Upsilon_{3,5} = \sum_{j=6,7,13} \Gamma_{5,j} + \sum_{j=1,4,11} \Gamma_{3,j}
\]

(A8)

Similarly,

\[
\rho_{9,5} = \rho_{5,9}^* = \frac{t_L \{ \rho_{9,9} - \rho_{5,5} \} + t_R \rho_{9,3}}{\epsilon_9 - \epsilon_5} - \frac{1}{2} \Gamma_{9,5}
\]

(A9)

\[
\Upsilon_{9,5} = \sum_{j=6,7,13} \Gamma_{5,j} + \sum_{j=1,10,11} \Gamma_{9,j}
\]

(A10)

The numerator in Eqns. (A7) and (A9) contains non-diagonal terms $\rho_{3,9}$ and $\rho_{9,3}$. Since, the Hamiltonian in Eq. (A2) doesn’t contain a direct tunnelling matrix element between the states $|9\rangle$ and $|3\rangle$, it is imperative to understand the
effect of this term on the electronic transport. Assuming steady state, from the non-diagonal elements of the density matrix, we get

$$0 = \dot{\rho}_{3,9} = \rho^*_{9,3} = i\rho_{3,9} \{ \epsilon_9 - \epsilon_3 \} + it_L \rho_{3,5} - it_R \rho_{5,9} - \frac{1}{2} \left\{ \sum_{j=4,7,11} \Gamma_{3,j} + \sum_{j=1,10,11} \Gamma_{9,j} \right\} \rho_{3,9} \quad (A11)$$

$$\rho_{3,9} = \rho^*_{9,3} = \frac{t_L \rho_{3,5} - t_R \rho_{5,9}}{\{ \epsilon_3 - \epsilon_9 \} - \frac{i}{2} \gamma_{3,9}}$$

$$\gamma_{3,9} = \sum_{j=4,7,11} \Gamma_{3,j} + \sum_{j=1,10,11} \Gamma_{9,j} \quad (A12)$$

The numerator in Eq. (A12), the element $t_L \rho_{3,5}$, accounts for co-tunnelling processes where simultaneous transfer of electrons take place from dot $M$ to $R$ and from dot $L$ to $M$. Similarly, the element $t_R \rho_{5,9}$ accounts for another co-tunnelling process where the simultaneous transfer of electrons take place from dot $R$ to $M$ and from dot $L$ to $L$. For simplifying further derivations, we assume $t_L = t_R = t$. The values of $\rho_{3,5}$ and $\rho_{5,9}$ in Eq. (A12) can be replaced from Eqs. (A7) and (A9) to get an expansion containing the density matrix elements weighted by $t^2$. Repeated expansion of Eq. (A12) by employing Eqn. (A7), (A9) and (A12) results in terms of higher order in $t$ and accounts for the higher order co-tunnelling processes. It can hence be noted that the terms in Eq. (A12) are at-least of second order in $t$, when expanded in terms of the diagonal density-matrix elements. However, the expansion $\rho_{3,5}$ and $\rho_{5,9}$ from Eqs. (A7) and (A9) contains terms which are first order in $t$. Under the assumption that the system-to-reservoir coupling $\gamma_c$ is much higher than the inter-dot tunnelling parameter $\gamma$, the contribution of the term $\rho_{3,9}$ in electronic transport properties can be neglected with respect to the terms $\rho_{5,3}$ and $\rho_{5,9}$. This approximation accounts to neglecting the contributions of co-tunnelling or higher order tunnelling processes and estimating the properties of the entire system via sequential transport or first order tunnelling processes. Hence Eq. (A7) can be approximated as,

$$\rho_{3,5} = \rho^*_{5,3} \approx \frac{t_R \{ \rho_{3,3} - \rho_{5,5} \}}{\{ \epsilon_3 - \epsilon_5 \} - \frac{i}{2} \gamma_{3,5}} \quad (A14)$$

$$\rho_{9,5} = \rho^*_{5,9} \approx \frac{t_L \{ \rho_{9,9} - \rho_{5,5} \}}{\{ \epsilon_9 - \epsilon_5 \} - \frac{i}{2} \gamma_{9,5}} \quad (A15)$$

Putting the values of $\rho_{3,5}$ and $\rho_{9,5}$ from Eqs. (A14) and (A15) in (A5), we get,

$$\dot{\rho}_{5,5} = \tau_{3,5} \rho_{3,3} + \tau_{9,5} \rho_{9,9} - (\tau_{5,3} + \tau_{5,9}) \rho_{5,5} + \sum_{j=6,7,13} \left\{ \Gamma_{j,5} \rho_{j,5} - \rho_{5,5} \Gamma_{5,j} \right\} \quad (A16)$$

where $\tau_{3,5} = \tau_{5,3} = \tau_{3,5} = \frac{t^2 \gamma_{3,3}}{(\epsilon_3 - \epsilon_5)^2 + \left( \frac{\gamma_{3,3}}{\epsilon_3 - \epsilon_5} \right)^2}$ and $\tau_{9,5} = \tau_{9,5} = \frac{t^2 \gamma_{9,9}}{(\epsilon_9 - \epsilon_5)^2 + \left( \frac{\gamma_{9,9}}{\epsilon_9 - \epsilon_5} \right)^2}$. It is clear that in Eq. (A16), the term $\tau_{3,5} \rho_{3,3}$ takes into account state transition from from $|3(9)\rangle$ to $|5\rangle$, due to tunnelling from the the dot $S_{R(L)}$ to the dot $S_M$, while $\tau_{5,3} \rho_{5,3}$ takes into account the transition from $|5\rangle$ to $|3(9)\rangle$ due to tunnelling from $S_M$ to $S_{L(R)}$. Thus, these terms denote the rate of inter-dot tunnelling.

Similarly, the rate equations corresponding to the other diagonal elements of the density matrix, in the limit of
negligible co-tunnelling phenomena can ultimately be derived as:

\[
\dot{p}_{1,1} = \sum_{j=2,3,9} (\Gamma_{j,1}p_{j,j} - \Gamma_{1,j}p_{1,1}) \tag{A17}
\]

\[
\dot{p}_{2,2} = \sum_{j=1,4,10} (\Gamma_{j,2}p_{j,j} - \Gamma_{2,j}p_{2,2}) \tag{A18}
\]

\[
\dot{p}_{3,3} = \tau_{3,3}p_{3,3} - \tau_{3,3}p_{3,3} + \sum_{j=1,4,11} (\Gamma_{j,3}p_{j,j} - \Gamma_{3,j}p_{3,3}) \tag{A19}
\]

\[
\dot{p}_{4,4} = \tau_{4,4}p_{4,4} - \tau_{4,4}p_{4,4} + \sum_{j=2,3,12} (\Gamma_{j,4}p_{j,j} - \Gamma_{4,j}p_{4,4}) \tag{A20}
\]

\[
\dot{p}_{5,5} = \tau_{5,5}p_{5,5} + \tau_{5,5}p_{9,9} - (\tau_{5,5} + \tau_{9,9})p_{5,5} + \sum_{j=6,7,13} \left\{ \Gamma_{j,5}p_{j,j} - \rho_{5,5}\Gamma_{5,j} \right\} \tag{A21}
\]

\[
\dot{p}_{6,6} = \tau_{4,6}p_{4,4} + \tau_{6,10}p_{10,10} - (\tau_{6,4} + \tau_{6,10})p_{6,6} + \sum_{j=5,8,14} (\Gamma_{j,6}p_{j,j} - \Gamma_{6,j}p_{6,6}) \tag{A22}
\]

\[
\dot{p}_{7,7} = \tau_{7,7}p_{11,11} - \tau_{7,7}p_{7,7} + \sum_{j=5,8,15} (\Gamma_{j,7}p_{j,j} - \Gamma_{7,j}p_{7,7}) \tag{A23}
\]

\[
\dot{p}_{8,8} = \tau_{8,8}p_{12,12} - \tau_{8,8}p_{8,8} + \sum_{j=6,7,16} (\Gamma_{j,8}p_{j,j} - \Gamma_{8,j}p_{8,8}) \tag{A24}
\]

\[
\dot{p}_{9,9} = \tau_{9,9}p_{5,5} - \tau_{9,9}p_{9,9} + \sum_{j=1,10,11} (\Gamma_{j,9}p_{j,j} - \Gamma_{9,j}p_{9,9}) \tag{A25}
\]

\[
\dot{p}_{10,10} = \tau_{6,10}p_{6,6} - \tau_{10,6}p_{10,10} + \sum_{j=2,9,12} (\Gamma_{j,10}p_{j,j} - \Gamma_{10,j}p_{10,10}) \tag{A26}
\]

\[
\dot{p}_{11,11} = \tau_{11,11}p_{7,7} + \tau_{13,11}p_{13,13} - (\tau_{11,7} + \tau_{11,13})p_{11,11} + \sum_{j=3,9,12} (\Gamma_{j,11}p_{j,j} - \Gamma_{11,j}p_{11,11}) \tag{A27}
\]

\[
\dot{p}_{12,12} = \tau_{8,12}p_{12,12} - \tau_{8,12}p_{8,8} + \tau_{14,12}p_{14,14} - (\tau_{12,8} + \tau_{12,14})p_{12,12} + \sum_{j=4,10,11} (\Gamma_{j,12}p_{j,j} - \Gamma_{12,j}p_{12,12}) \tag{A28}
\]

\[
\dot{p}_{13,13} = \tau_{11,13}p_{11,11} - \tau_{13,11}p_{13,13} + \sum_{j=5,14,15} (\Gamma_{j,13}p_{j,j} - \Gamma_{13,j}p_{13,13}) \tag{A29}
\]

\[
\dot{p}_{14,14} = \tau_{12,14}p_{12,12} - \tau_{14,12}p_{14,14} + \sum_{j=6,13,16} (\Gamma_{j,14}p_{j,j} - \Gamma_{14,j}p_{14,14}) \tag{A30}
\]

\[
\dot{p}_{15,15} = \sum_{j=7,13,16} (\Gamma_{j,15}p_{j,j} - \Gamma_{15,j}p_{15,15}) \tag{A31}
\]

\[
\dot{p}_{16,16} = \sum_{j=8,14,15} (\Gamma_{j,16}p_{j,j} - \Gamma_{16,j}p_{16,16}) \tag{A32}
\]

In the above set of Eqs. (A17)-(A32), \( \Gamma_{x,y} \) is related to the rate of state transition \(|x| \rightarrow |y|\) due to system-to-reservoir tunnelling, and is dependent on the system-to-reservoir coupling as well as the probability of occupancy of the reservoirs. \( \tau_{x,y} \), on the other hand, is related to the rate of state transition \(|x| \rightarrow |y|\) due to inter-dot tunnelling and is to be derived from the density matrix equations. The various parameters \( \Gamma_{x,y} \) can be given by the following equations, assuming statistical quasi-equilibrium at the reservoirs.

\[
\Gamma_{1,2} = \Gamma_{3,4} = \Gamma_{9,10} = \Gamma_{11,12} = \gamma_c f_G (\varepsilon_G) \tag{A33}
\]

\[
\Gamma_{5,6} = \Gamma_{7,8} = \Gamma_{13,14} = \Gamma_{15,16} = \gamma_c f_G (\varepsilon_G + U_m) \tag{A34}
\]

\[
\Gamma_{2,1} = \Gamma_{4,3} = \Gamma_{10,9} = \Gamma_{12,11} = \gamma_c \{1 - f_G (\varepsilon_G)\} \tag{A35}
\]

\[
\Gamma_{6,5} = \Gamma_{8,7} = \Gamma_{14,13} = \Gamma_{16,15} = \gamma_c \{1 - f_G (\varepsilon_G + U_m)\} \tag{A36}
\]

\[
\Gamma_{1,9} = \Gamma_{2,10} = \Gamma_{3,11} = \Gamma_{4,12} = \Gamma_{5,13} = \Gamma_{6,14} = \Gamma_{7,15} = \Gamma_{8,16} = \gamma_c f_L (\varepsilon_L) \tag{A37}
\]

\[
\Gamma_{9,1} = \Gamma_{10,2} = \Gamma_{11,3} = \Gamma_{12,4} = \Gamma_{13,5} = \Gamma_{14,6} = \Gamma_{15,7} = \Gamma_{16,8} = \gamma_c \{1 - f_L (\varepsilon_L)\} \tag{A38}
\]

\[
\Gamma_{1,3} = \Gamma_{2,4} = \Gamma_{5,7} = \Gamma_{6,8} = \Gamma_{9,11} = \Gamma_{10,12} = \Gamma_{13,15} = \Gamma_{14,16} = \gamma_c f_R (\varepsilon_R) \tag{A39}
\]

\[
\Gamma_{3,1} = \Gamma_{4,2} = \Gamma_{7,5} = \Gamma_{8,6} = \Gamma_{11,9} = \Gamma_{12,10} = \Gamma_{15,13} = \Gamma_{16,14} = \gamma_c \{1 - f_R (\varepsilon_R)\} \tag{A40}
\]
In Eqns. (A17)-(A32), for the system under consideration, the various values of $\tau_{x,y}$ were derived as:

$$
\tau_{x,y} = \frac{t^2 \gamma_{x,y}}{(\epsilon_x - \epsilon_y)^2 + \left(\frac{\gamma_{x,y}}{2}\right)^2},
$$
(A41)

where the different values of $\gamma_{x,y}$ can be expressed in terms of the following equations:

$$
\gamma_5 = \gamma_3 = \sum_{j=6,7,13} \Gamma_{5,j} + \sum_{j=1,4,11} \Gamma_{3,j} = \gamma_c \left\{1 + 2f_L(\epsilon_L) + f_G(\epsilon_G) + f_G(\epsilon_G + U_m)\right\}
$$
(A42)

$$
\gamma_{4,6} = \gamma_{6,4} = \sum_{j=2,3,12} \Gamma_{4,j} + \sum_{j=5,8,14} \Gamma_{6,j} = \gamma_c \left\{3 + 2f_L(\epsilon_L) - f_G(\epsilon_G) - f_G(\epsilon_G + U_m)\right\}
$$
(A43)

$$
\gamma_{5,9} = \gamma_{9,5} = \sum_{j=6,7,13} \Gamma_{5,j} + \sum_{j=1,10,11} \Gamma_{9,j} = \gamma_c \left\{1 + 2f_R(\epsilon_R) + f_G(\epsilon_G) + f_G(\epsilon_G + U_m)\right\}
$$
(A44)

$$
\gamma_{6,10} = \gamma_{10,6} = \sum_{j=5,8,14} \Gamma_{6,j} + \sum_{j=2,9,12} \Gamma_{10,j} = \gamma_c \left\{3 + 2f_R(\epsilon_R) - f_G(\epsilon_G) - f_G(\epsilon_G + U_m)\right\}
$$
(A45)

$$
\gamma_{7,11} = \gamma_{11,7} = \sum_{j=5,8,15} \Gamma_{7,j} + \sum_{j=3,9,12} \Gamma_{11,j} = \gamma_c \left\{3 - 2f_R(\epsilon_R) + f_G(\epsilon_G) + f_G(\epsilon_G + U_m)\right\}
$$
(A46)

$$
\gamma_{8,12} = \gamma_{12,8} = \sum_{j=6,7,16} \Gamma_{8,j} + \sum_{j=4,10,11} \Gamma_{12,j} = \gamma_c \left\{5 - 2f_R(\epsilon_R) - f_G(\epsilon_G) - f_G(\epsilon_G + U_m)\right\}
$$
(A47)

$$
\gamma_{11,13} = \gamma_{13,11} = \sum_{j=3,9,12} \Gamma_{11,j} + \sum_{j=5,14,15} \Gamma_{13,j} = \gamma_c \left\{3 - 2f_L(\epsilon_L) + f_G(\epsilon_G) + f_G(\epsilon_G + U_m)\right\}
$$
(A48)

$$
\gamma_{12,14} = \gamma_{14,12} = \sum_{j=6,7,16} \Gamma_{12,j} + \sum_{j=4,10,11} \Gamma_{14,j} = \gamma_c \left\{5 - 2f_L(\epsilon_L) - f_G(\epsilon_G) - f_G(\epsilon_G + U_m)\right\}
$$
(A49)

From Eqn. (A42)-(A49) and (A33)-(A40), the different expressions for the inter-dot tunnelling rates were derived as:

$$
\tau_{5,3} = \tau_{3,5} = \frac{t^2 \gamma_{5,3}}{(\epsilon_M - \epsilon_R)^2 + \left(\frac{\gamma_{5,3}}{2}\right)^2}
$$
(A50)

$$
\tau_{4,6} = \tau_{6,4} = \frac{t^2 \gamma_{4,6}}{(\epsilon_M + U_m - \epsilon_R)^2 + \left(\frac{\gamma_{4,6}}{2}\right)^2}
$$
(A51)

$$
\tau_{5,9} = \tau_{9,5} = \frac{t^2 \gamma_{5,9}}{(\epsilon_M - \epsilon_L)^2 + \left(\frac{\gamma_{5,9}}{2}\right)^2}
$$
(A52)

$$
\tau_{6,10} = \tau_{10,6} = \frac{t^2 \gamma_{6,10}}{(\epsilon_M + U_m - \epsilon_L)^2 + \left(\frac{\gamma_{6,10}}{2}\right)^2}
$$
(A53)

$$
\tau_{7,11} = \tau_{11,7} = \frac{t^2 \gamma_{7,11}}{(\epsilon_M - \epsilon_L)^2 + \left(\frac{\gamma_{7,11}}{2}\right)^2}
$$
(A54)

$$
\tau_{8,12} = \tau_{12,8} = \frac{t^2 \gamma_{8,12}}{(\epsilon_M + U_m - \epsilon_L)^2 + \left(\frac{\gamma_{8,12}}{2}\right)^2}
$$
(A55)

$$
\tau_{11,13} = \tau_{13,11} = \frac{t^2 \gamma_{11,13}}{(\epsilon_M - \epsilon_R)^2 + \left(\frac{\gamma_{11,13}}{2}\right)^2}
$$
(A56)

$$
\tau_{12,14} = \tau_{14,12} = \frac{t^2 \gamma_{12,14}}{(\epsilon_M + U_m - \epsilon_R)^2 + \left(\frac{\gamma_{12,14}}{2}\right)^2}
$$
(A57)

The set of Eqns. (A17)-(A32) form a set of linear equations. Once the different parameters are calculated using Eqns. (A33)-(A57), the diagonal elements of the density matrix, given by Eqns. (A17)-(A32), can be solved using
matrix inversion method, under the condition \( \sum_{j=1}^{16} \rho_{j,j} = 1 \). Once the diagonal elements of the density matrix are computed, the current \( I_{L(R)} \) through the left (right) reservoirs into the system can be given as:

\[
I_{L(R)} = \frac{e^2}{h} \gamma_c \left\{ P_0^{(L(R))} f_{L(R)}(\varepsilon_{L(R)}) - P_1^{(L(R))} (1 - f_{L(R)}(\varepsilon_{L(R)})) \right\},
\]

where \( P_j^{(L(R))} \) denotes the probability that the ground state of the dot \( S_{L(R)} \) contains \( j \) electrons. \( P_j^{(L(R))} \) can be calculated by taking the partial trace over the density matrix elements as:

\[
\begin{align*}
P_0^{(L)} &= \sum_{j=1}^{8} \rho_{j,j} \\
P_1^{(L)} &= \sum_{j=9}^{16} \rho_{j,j} \\
P_0^{(R)} &= \sum_{j=0}^{3} (\rho_{4j+1,4j+1} + \rho_{4j+2,4j+2}) \\
P_1^{(R)} &= \sum_{j=0}^{3} (\rho_{4j+3,4j+3} + \rho_{4j+4,4j+4})
\end{align*}
\]

### Appendix B: Derivation of \( T_{G}^{opt} \) (Fig. 6)

Here we provide a detailed explanation of Eqn. (5). To find the remote reservoir temperature \( T_{G} \) at which maximum sensitivity is achieved for a given value of Coulomb coupling energy \( U_m \), we need to analyze the maxima of the functions \( \frac{d f_G(\varepsilon_G)}{dT_G} \big|_{\varepsilon_G=0} \) and \( \frac{d}{dT_G} f_G(\varepsilon_G + U_m) \big|_{\varepsilon_G=0} \).

\[
f_G(\varepsilon_G) = \left( 1 + \exp \left( \frac{\varepsilon_G + U_m}{kT_G} \right) \right)^{-1}
\]

(B1)

In case of the proposed system, we need to take into account the impact of both \( \frac{d f_G(\varepsilon_G)}{dT_G} \big|_{\varepsilon_G=0} \) and \( \frac{d}{dT_G} f_G(\varepsilon_G + U_m) \big|_{\varepsilon_G=0} \) on the system performance. However, \( \frac{d f_G(\varepsilon_G)}{dT_G} \big|_{\varepsilon_G=0} = 0 \) and thus the system sensitivity magnitude is maximized when \( \frac{d}{dT_G} f_G(\varepsilon_G + U_m) \big|_{\varepsilon_G=0} \) is maximum. The derivative of the function \( f_G(\varepsilon_G + U_m) \) with respect to \( T_G \) can be expressed as follows:

\[
\frac{d}{dT_G} f_G(\varepsilon_G + U_m) = \left( \frac{\varepsilon_G + U_m}{kT_G^2} \right) \times \exp \left( \frac{\varepsilon_G + U_m}{kT_G} \right) \times \left( 1 + \exp \left( \frac{\varepsilon_G + U_m}{kT_G} \right) \right)^{-2},
\]

(B2)

For further simplification, assuming \( \left( \frac{\varepsilon_G + U_m}{k} \right) = c \), and \( \frac{1}{T_G} = x \), we can express \( \frac{d}{dT_G} f_G(\varepsilon_G + U_m) \) as:

\[
\frac{d}{dT_G} f_G(\varepsilon_G + U_m) = cx^2 \times \exp(cx) \times (1 + \exp(cx))^{-2},
\]

(B3)

On differentiating the above equation (to obtain the point of maxima), and after some algebraic manipulations, we obtain an expression as:

\[
\frac{d^2}{dT_G^2} f_G(\varepsilon_G + U_m) = 0 \Rightarrow (1 + \exp(cx)) \times (cx + 2) = 2 \times cx \times \exp(cx) \Rightarrow \frac{cx + 2}{cx - 2} = \exp(cx)
\]

(B4)

After some algebraic manipulations, we obtain the expression as:

\[
\frac{(\varepsilon_G + U_m) + 1}{(\varepsilon_G + U_m) - 1} = 0.5 \times \exp \left( \frac{\varepsilon_G + U_m}{kT_G^{opt}} \right)
\]

(B5)
We note that the above equation is satisfied for \( \frac{U_m}{kT_G} = 2.399 \). Hence, the optimal remote reservoir temperature \( T_G^{opt} \) at which a maximum magnitude of sensitivity is obtained is given by:

\[
T_G^{opt} = (2.399)^{-1} \frac{\varepsilon_G + U_m}{k}
\]  

(B6)

For \( \varepsilon_G = 0 \), the value of \( T_G^{opt} \) is hence given by \( T_G^{opt} = (2.399)^{-1} \frac{U_m}{k} \).

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