Cameron’s operator in terms of determinants and hypergeometric numbers

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Abstract

By studying Cameron’s operator in terms of determinants, two kinds of “integer” sequences of incomplete numbers were introduced. One was the sequence of restricted numbers, including $s$-step Fibonacci sequences. Another was the sequence of associated numbers, including Lamé sequences of higher order. By the classical Trudi’s formula and the inverse relation, more expressions were able to be obtained. These relations and identities can be extended to those of sequence of negative integers or rational numbers. As applications, we consider hypergeometric Bernoulli, Cauchy and Euler numbers with some modifications.

Keywords Operator · Restricted numbers · Associated numbers · Hypergeometric Bernoulli numbers · Hypergeometric Cauchy numbers · Hypergeometric Euler numbers · Hypergeometric functions · Determinant

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1 Introduction

In 1989, Cameron [3] considered the operator $A$ defined on the set of sequences of nonnegative integers as follows: for $x = \{x_n\}_{n \geq 1}$ and $z = \{z_n\}_{n \geq 1}$, set $Ax = z$, where

$$1 + \sum_{n=1}^{\infty} z_n t^n = \left(1 - \sum_{n=1}^{\infty} x_n t^n\right)^{-1}. \quad (1)$$

Suppose that $x$ enumerates a class $C$. Then, $Ax$ enumerates the class of disjoint unions of members of $C$, where the order of the “component” members of $C$ is significant. The operator $A$ also plays an important role for free associative (non-commutative) algebras. More motivations and background together with many concrete examples (in particular, in the aspects of Graph Theory) by this operator can be seen in [3].

There are many examples in Combinatorics. For example, if the sequence of the numbers $x_n$ is in arithmetic progression, the numbers $z_n$ are yielded from the three-term recurrence relation, in terms of the Cameron’s operator by (1). That is, when $x_n = (n-1)a + b \ (n \geq 1)$ then the numbers $z_n$ satisfy the recurrence relation

$$z_n = (b+2)z_{n-1} + (a-b-1)z_{n-2} \quad (n \geq 3), \quad z_2 = a+b(b+1), \quad z_1 = b, \quad (2)$$

because the identity in (1) is equal to

$$\frac{1-2t+t^2}{1-(b+2)t-(a-b-1)t^2}.$$ 

In particular, if $a = 1$ and $b = 2$, $z_n$ is yielded from the sequence

$$\{x_n\}_{n \geq 1} = 2, 3, 4, 5, 6, 7, \ldots,$$

and the number of $L$-convex polyominoes with $n$ cells, that is convex polyominoes where any two cells can be connected by a path internal to the polyomino and which has at most 1 change of direction (see also [4, 5] and [31, A003480]).

In [6], two kinds of generalized Stirling numbers of both kinds are introduced in Combinatorial interpretations. The Stirling numbers of the second kind enumerate the number of partitions of a set with $n$ elements into $k$ non-empty blocks. The Stirling numbers can be generalized using a restriction on the size of the blocks and cycles. In particular, the restricted Stirling numbers of the second kind give the number of partitions of $n$ elements into $k$ subsets, with the additional restriction that none of the blocks contain more than $m$ elements. The associated Stirling numbers of the second kind equals the number that each block contains at most $m$ elements. These numbers are called incomplete Stirling numbers of the second kind, together. The (unsigned) Stirling numbers of the first kind enumerate the number of permutations on $n$ elements with $k$ cycles. The restricted Stirling numbers of the first kind and the associated Stirling numbers of the first kind are similarly defined.
Using these restricted and associated numbers, several generalizations of Bernoulli and Cauchy numbers are introduced and studied in [20, 22–24].

In this paper, by applying the similar concept of incomplete Stirling numbers, we study more general operators. One type includes Fibonacci $s$-step numbers, satisfying the recurrence relation $F_n^{(s)} = F_{n-1}^{(s)} + F_{n-2}^{(s)} + \cdots + F_{n-s}^{(s)}$, as special cases. Another includes the numbers in Lamé sequence of higher order [11] or Fibonacci $p$-numbers [16], satisfying the recurrence relation $L_n = L_{n-1} + L_{n-s}$, as special cases. When $s=3$, it is called Narayana’s cows sequence (see, e.g., 1, 7) By the classical Trudi’s formula and the inverse relation, more expressions can be obtained. These relations and identities are not restricted to the nonnegative integer sequences, but can be extended to those of sequence of negative integers or rational numbers. As applications, we consider hypergeometric Bernoulli, Cauchy and Euler numbers with some modifications.

2 Incomplete numbers with their expressions

Given two integer\(^1\) (finite or infinite) sequences $x = \{x_1, x_2, \ldots\}$ and $z = \{z_1, z_2, \ldots\}$. We assume that $x_0 = 1$ and $z_0 = 1$ unless we specify. We shall consider a more general operator $A x = z$, where for a positive integer $m$,

$$1 + \sum_{n=1}^{\infty} z_n t^n = \left(1 - \sum_{n=1}^{m} x_n t^n\right)^{-1}. \tag{3}$$

When $m \to \infty$, the relation (3) is reduced to the relation (1). On the contrary to the associated numbers in Sect. 2.2, the numbers $z_n$ may be called restricted numbers, yielding from the sequence of the numbers $x_n$ in terms of the operator in (3). For example, in the case where $\{x_n\}_{n=1}^{m} = \{1, \ldots, 1\}$, for $m = 2$, $A\{x_n\}$ yields the Fibonacci sequence $\{z_n\} = \{F_{n+1}\}$, where $F_n = F_{n-1} + F_{n-2}$ ($n \geq 2$) with $F_1 = F_2 = 1$. For $m = 3$, $A\{x_n\}$ yields the Tribonacci sequence $\{z_n\} = \{T_{n+1}\}$, where $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ ($n \geq 3$) with $T_1 = T_2 = 1$ and $T_3 = 2$. Therefore, the combinatorial interpretation is that $z_n = T_{n+1}$ is the number of ways of writing $n$ as an ordered sum of 1’s, 2’s and 3’s. For example, $T_5 = 7$ because $n = 4$ can be expressed in 7 ways as

$$1 + 3 = 3 + 1 = 2 + 2 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$ 

Using a recurrence relation

$$z_n = \sum_{k=1}^{\min\{n,m\}} x_k z_{n-k} \quad (n \geq 1), \tag{4}$$

the number $z_n$ can be expressed in terms of the determinant [21, Lemma 1].

\(^1\) In the applications’ section, rational sequences will be considered.
Lemma 1  For integers $1 \leq n \leq m$,

$$
\begin{align*}
z_n &= \begin{bmatrix}
 x_1 & 1 & 0 \\
 -x_2 & x_1 & \\
 \vdots & \vdots & \ddots & 1 \\
(-1)^n x_{n-1} & (-1)^{n-1} x_{n-2} & \cdots & x_1 & 1 \\
(-1)^{n-1} x_n & (-1)^n x_{n-1} & \cdots & -x_2 & x_1
\end{bmatrix}.
\end{align*}
$$

(5)

For integers $n \geq m$,

$$
\begin{align*}
\begin{array}{c}
 x_1 & 1 & 0 \\
 \vdots & \vdots & \ddots \ \\
(-1)^{m-1} x_m & \vdots \\
 0 & \cdots & 0 \\
 0 & \cdots & 1
\end{array}
\begin{array}{c}
\underbrace{\cdots}_{n-m} \\
\overbrace{\cdots}^{m-n}
\end{array}
\begin{array}{c}
(-1)^{m-1} x_m \\
\cdots \\
0 \cdots 0
\end{array} \begin{array}{c}
\underbrace{\cdots}_{n-m} \\
\overbrace{\cdots}^{m-n}
\end{array}
\begin{array}{c}
0 \\
\cdots \\
-1 \\
\cdots \\
1
\end{array}
\end{align*}
$$

(6)

Remark  For integers $n \geq 3$, the $n$-th Fibonacci number $F_n$ can be expressed as

$$
\begin{align*}
F_n &= \begin{bmatrix}
 1 & 1 & 0 \\
 -1 & 1 & 1 \\
 0 & -1 & \ddots \\
 & \ddots & \ddots & 1 & 0 \\
 & & \ddots & -1 & 1 \\
 & & & 0 & 1
\end{bmatrix}.
\end{align*}
$$

([30, 365 (p. 54)]^2). For negative indices,

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2 In this book, Fibonacci numbers do not begin from 1, 1, 2, 3, . . . but from 1, 2, 3, 5, . . .
were studied in [17]. Recently, Fibonacci numbers in the form \( \pm F_{n-k+l} \) with determinants of tridiagonal matrices with 1’s on the superdiagonal and subdiagonal and alternating blocks of 1’s and -1’s of the length \( a \), where \( a \) is any positive integer and \( a \not\equiv 0 \pmod{3} \), studied in [32].

Using a combinatorial summation, the number \( z_n \) has an explicit expression [21, Theorem 1].

**Lemma 2**  For integers \( n \geq 1 \),

\[
z_n = \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} x_{i_1} \cdots x_{i_k}.
\]

1 \( \leq i_1, \ldots, i_k \leq m \)

By applying Trudi’s formula [2, 33] (see also [29, Vol.3, pp. 208–209, p. 214]), to Lemma 1, we obtain an explicit expression for numbers \( z_n \) in terms of \( x_n \) with multinomial coefficients [21, Theorem 4].

**Lemma 3**  For \( n \geq 1 \), we have

\[
z_n = \sum_{t_1 + 2t_2 + \cdots + mt_m = n} \left( \begin{array}{c} t_1 + \cdots + t_m \\ t_1, \ldots, t_m \end{array} \right) x_1^{t_1} \cdots x_m^{t_m}.
\]

By applying the inversion relation (see, e.g. [25]):
\[ \sum_{k=0}^{n} (-1)^{n-k} \alpha_k D(n-k) = 0 \quad (n \geq 1) \]

\[ \Leftrightarrow \]

\[ \alpha_n = \begin{vmatrix} D(1) & 1 \\ D(2) & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix} \quad \Leftrightarrow \quad D(n) = \begin{vmatrix} \alpha_1 & 1 \\ \alpha_2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix} \quad (7) \]

with \( \alpha_0 = D(0) = 1 \), from Lemma 1, we have the following relation [21, Theorem 4].

**Lemma 4** For \( n \geq 1 \), we have

\[
\begin{vmatrix} z_1 & 1 & 0 \\ z_2 & z_1 \\ \vdots & \ddots & \ddots & 1 & 0 \\ z_{n-1} & z_{n-2} & \cdots & z_1 \\ z_n & z_{n-1} & \cdots & z_2 & z_1 \end{vmatrix} = \begin{cases} (-1)^{n-1} x_n & (1 \leq n \leq m) \\ 0 & (n \geq m + 1) \end{cases}.
\]

Therefore, by the inversion relations with Lemmas 2 and 3, we have the following relation [21, Theorem 5].

**Lemma 5** For \( n \geq 1 \), we have

\[
\sum_{k=1}^{n} (-1)^{k-1} \sum_{i_1 + \cdots + i_k = n, i_1, \ldots, i_k \geq 1} z_{i_1} \cdots z_{i_k} = \sum_{t_1+2t_2+\cdots+nt_n=n} \left( t_1 + \cdots + t_n \right) (-1)^{t_1+\cdots+t_n-1} z_1^{t_1} \cdots z_n^{t_n} = \begin{cases} x_n & (1 \leq n \leq m) \\ 0 & (n \geq m + 1) \end{cases}.
\]

**2.1 Examples**

For Fibonacci numbers \( F_n \), we have
\[
F_n = \sum_{t_1 + 2t_2 = n - 1} \frac{(t_1 + t_2)!}{t_1!t_2!} (-1)^{n-t_1-1}
\]
\[t_1, t_2 \geq 0\]

(8)

and

\[
\begin{pmatrix}
F_2 & 1 & 0 \\
F_3 & F_2 & \\
\vdots & \vdots & \ddots & 1 & 0 \\
F_n & F_{n-1} & \cdots & F_2 & 1 \\
F_{n+1} & F_n & \cdots & F_3 & F_2
\end{pmatrix} = \begin{cases}
(-1)^{n-1} & (n = 1, 2) \\
0 & (n \geq 3)
\end{cases}
\]

(9)

For example, in (8),

\[
F_6 = \frac{3!}{112!} + \frac{4!}{31!} + \frac{5!}{510!} = 8
\]

and

\[
F_7 = \frac{3!}{013!} + \frac{4!}{212!} + \frac{5!}{411!} + \frac{6!}{610!} = 13.
\]

For Tribonacci numbers \( T_n \), we have

\[
\sum_{k=1}^{n} (-1)^{k-1} \sum_{i_1 + \cdots + i_k = n} T_{i_1+1} \cdots T_{i_k+1}
\]

\[i_1, \ldots, i_k \geq 1\]

\[t_1 + 2t_2 + \cdots + nt_n = n, t_1, \ldots, t_n\]

\[= \sum_{t_1 + 2t_2 + \cdots + nt_n = n} (-1)^{t_1+\cdots+t_n-1} T_2^{t_1} \cdots T_n^{t_n}
\]

\[= \begin{cases}
1 & (n = 1, 2, 3) \\
0 & (n \geq 4)
\end{cases}
\]

(11)

For example, for \( n = 3 \) in (10),

\[
\sum_{k=1}^{3} (-1)^{k-1} \sum_{i_1 + \cdots + i_k = n_1, \ldots, i_k \geq 1} T_{i_1+1} \cdots T_{i_k+1}
\]

\[= T_4 - 2T_2T_3 + T_2T_2T_2 = 4 - 2 \cdot 1 \cdot 2 - 1 \cdot 1 \cdot 1 = 1
\]

and in (11),
\[
\sum_{t_1+2t_2+3t_3=3} \binom{t_1+t_2+t_3}{t_1,t_2,t_3} (-1)^{t_1+t_2+t_3-1} T_2^2 T_3^3 T_4^4 \\
= \frac{3!}{3!0!0!} (-1)^{3-1} 1^3 2^0 4^0 + \frac{2!}{1!1!0!0!} (-1)^{1+1-1} 1^1 2^1 4^0 + \frac{1!}{0!1!1!1!} (-1)^{1-1} 1^0 2^0 4^1 \\
= 1 \cdot 1 + (-2) \cdot 2 + 1 \cdot 4 = 1.
\]

For \( n = 4 \) in (10),
\[
\sum_{k=1}^{4} (-1)^{k-1} \sum_{i_1 + \cdots + i_k = n} T_{i_1+1} \cdots T_{i_k+1} \\
= T_5 - (2T_2 T_4 + T_3^2) + 3T_2^2 T_3 - T_4^2 \\
= 7 - 2 \cdot 1 \cdot 4 - 2^2 + 3 \cdot 1^2 \cdot 2 - 1^4 = 0
\]
and in (11),
\[
\sum_{t_1+2t_2+3t_3+4t_4=4} \binom{t_1+t_2+t_3+t_4}{t_1,t_2,t_3,t_4} (-1)^{t_1+t_2+t_3+t_4-1} T_2^0 T_3^0 T_4^0 T_5^4 \\
= \frac{4!}{4!0!0!0!} (-1)^{4-1} 1^4 2^0 4^0 7^0 + \frac{3!}{2!1!0!0!0!} (-1)^{2+1-1} 1^2 2^1 4^0 7^0 \\
+ \frac{2!}{0!2!0!0!0!} (-1)^{2-1} 1^0 2^2 4^0 7^0 + \frac{2!}{1!0!1!1!0!} (-1)^{1+1-1} 1^1 2^0 4^1 7^0 \\
+ \frac{1!}{0!0!0!0!1!} (-1)^{1-1} 1^0 2^0 4^0 7^1 \\
= -1 + 3 \cdot 2 - 4 - 2 \cdot 4 + 7 = 0.
\]

### 2.2 Associated numbers

In [20, 22–24], some kinds of associated numbers are studied in terms of associated Stirling numbers in [6].

In general, for a positive integer \( m \), the associated operator \( \mathcal{A}_{\geq m} x = z \) is defined by the generating function

\[
1 + \sum_{n=1}^{\infty} z_n t^n = \left( 1 - \sum_{n=m}^{\infty} x_n t^n \right)^{-1}.
\]

When \( m = 1 \), the relation (12) is reduced to the relation (1). In this sense, the operator in (3) is called the restricted operator because of the restricted Stirling numbers in [6].

By a recurrence relation between two sequences
with $z_0 = 1$ and $z_1 = \cdots = z_{m-1} = 0$, or

$$z_n = x_n + \sum_{k=0}^{n-2m} x_{n+k}z_{n-m-k} \quad (n \geq 1), \quad (14)$$

we have a determinant expression for $z_n$ by means of the associated operator [21, Theorem 2].

**Lemma 6** For integers $n \geq m \geq 1$,

$$z_n = \begin{vmatrix} 0 & 1 & 0 \\ \vdots & 0 & 1 \\ 0 & \cdots & \ddots \\ (-1)^{m-1}x_m & 1 & 0 \\ \vdots & \cdots & 0 & 0 & 1 \\ (-1)^{n-1}x_n & \cdots & (-1)^{m-1}x_m & 0 & \cdots & 0 \end{vmatrix}_{n-m+1 \times m-1} \quad (15)$$

The number $z_n$ has an explicit expression by means of the associated operator [21, Theorem 3].

**Lemma 7** For integers $n \geq 1$,

$$z_n = \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} x_{i_1} \cdots x_{i_k}, \quad i_1, \ldots, i_k \geq m$$

In order to describe this special case more precisely, consider the geometric sequence of the associated numbers, $x_n = a^{n-m}b$ $(n \geq m)$ with $x_0 = 1$ and $x_1 = \cdots = x_{m-1} = 0$, where $a$ and $b$ are nonzero integers. Then, by the Cameron’s operator, we have the sequence of numbers $z_n$, satisfying

$$z_n = az_{n-1} + bz_{n-m} \quad (n \geq m) \quad \text{with} \quad z_0 = 1 \quad \text{and} \quad z_1 = \cdots = z_{m-1} = 0.$$ 

That is, the right-hand side of (12) is equal to

$$\frac{1 - at}{1 - at - bt^m}.$$

Since the number of the elements in the set \{(i_1, \ldots, i_k)|i_1 + \cdots + i_k = n, i_1, \ldots, i_k \geq m\} is equal to
\[
\begin{array}{c}
\binom{n - km + k - 1}{k - 1},
\end{array}
\]

we have

\[
z_n = \sum_{k=1}^{n} \sum_{i_1 + \ldots + i_k = n} \sum_{i_1, \ldots, i_k \geq m} a^{i_1-m} b \ldots a^{i_k-m} b
\]

\[
= \sum_{k=1}^{\left\lfloor \frac{n}{m} \right\rfloor} \sum_{i_1 + \ldots + i_k = n} a^{n-km} b^k
\]

\[
= \sum_{k=1}^{\left\lfloor \frac{n}{m} \right\rfloor} \binom{n - km + k - 1}{k - 1} a^{n-km} b^k.
\]

**Proposition 1** For integers \(a, b, n\) and \(m\) with \(a \neq 0, b \neq 0\) and \(n \geq m \geq 1\), we have

\[
z_n = \sum_{k=1}^{\left\lfloor \frac{n}{m} \right\rfloor} \binom{n - km + k - 1}{k - 1} a^{n-km} b^k.
\]

**Remark** For initial values \(z_n = a^{n-m} b (m \leq n \leq 2m - 1)\), \(z_n = a^{n-m} b + (n - 2m + 1) a^{n-2m} b^2 \quad (2m \leq n \leq 3m - 1)\) and \(z_n = a^{n-m} b + (n - 2m + 1) a^{n-2m} b^2 + \binom{n - 3m + 2}{2} a^{n-3m} b^3 \quad (3m \leq n \leq 4m - 1)\).

In particular, when \(a = b = 1\), we can get a more explicit expression.

**Corollary 1** For integers \(n\) and \(m\) with \(n \geq m \geq 1\), we have

\[
z_n = \sum_{k=1}^{\left\lfloor \frac{n}{m} \right\rfloor} \binom{n - km + k - 1}{k - 1}.
\]

**Remark** When \(m = 1\) in Corollary 1, we have

\[
\sum_{k=1}^{n} \binom{n - 1}{k - 1} = 2^{n-1}.
\]

When \(m = 2\) in Corollary 1, we have
\[
\sum_{k=1}^{[\frac{n}{k}]} \binom{n-k-1}{k-1} = F_{n-1}
\]

[28, Theorem 12.4].

If the sequence of the associated numbers \(x_n\) forms the arithmetic progression, that is, \(x_n = (n - m)a + b\) \((n \geq m)\) with \(x_0 = 1\) and \(x_1 = \cdots = x_{m-1} = 0\), where \(a\) and \(b\) are nonzero integers, then by the Cameron’s operator, for \(m \geq 3\) we have the sequence of numbers \(z_n\), satisfying

\[
z_n = 2z_{n-1} - z_{n-2} + bz_{n-m} + (a - b)z_{n-m-1} \quad (n \geq m + 1)
\]

with \(z_0 = 1\), \(z_1 = \cdots = z_{m-1} = 0\) and \(z_m = b\). That is, the right-hand side of (12) is equal to

\[
\frac{1 - 2t + t^2}{1 - 2t + t^2 - bt^m + (b - a)t^{m+1}}.
\]

When \(m = 2\), \(z_n\)’s satisfy the recurrence relation

\[
z_n = 2z_{n-1} + (b - 1)z_{n-2} + (a - b)z_{n-3} \quad (n \geq 3)
\]

with \(z_0 = 1\), \(z_1 = 0\) and \(z_2 = b\). The right-hand side of (12) is equal to

\[
\frac{1 - 2t + t^2}{1 - 2t - (b - 1)t^2 - (b - a)t^3}
\]

When \(m = 1\), \(z_n\)’s satisfy the recurrence relation (2), as the original operator mentioned in the Introduction.

By applying Trudi’s formula and the inversion formula to Lemma 6, we obtain the following expressions for associated numbers \(z_n\).

**Lemma 8** For \(n \geq 1\), we have

\[
z_n = \sum_{m_1 + (m+1)m_2 + \cdots + m_n = n} \binom{t_m + t_{m+1} + \cdots + t_m}{t_m, t_{m+1}, \ldots, t_n} x_n^{m_1} x_{m+1}^{m_2} \cdots x_n^{m_n}, \quad (16)
\]

\[
\begin{vmatrix}
1 & 0 & & & \\
& z_1 & & & \\
& & \ddots & 1 & 0 \\
& & & z_{n-1} & z_n \\
& & & & z_1
\end{vmatrix} = \begin{cases} (-1)^{n-1} x_n & (n \geq m) \\ 0 & (1 \leq n \leq m - 1) \end{cases}, \quad (17)
\]

and
\[ \sum_{k=1}^{n} (-1)^{k-1} \sum_{i_{1}+\cdots+i_{k}=n} z_{i_{1}} \cdots z_{i_{k}} \]
\[ = \sum_{i_{1}+2i_{2}+\cdots+ni_{n}=n} \left( i_{1} + \cdots + i_{n} \right) (-1)^{i_{1}+\cdots+i_{n}} z_{i_{1}} \cdots z_{i_{n}} \]
\[ = \left\{ \begin{array}{ll}
X_{n} & (n \geq m) \\
0 & (1 \leq n \leq m - 1)
\end{array} \right. \] (18)

3 Applications to hypergeometric numbers

Cameron’s operator can be applied to the sequences of fractional numbers too. Recently, incomplete Bernoulli numbers [22] are introduced and studied as one kind of generalizations of the classical Bernoulli numbers. Similarly, incomplete Cauchy numbers [23] are introduced and studied as one kind of generalizations of the classical Cauchy numbers. Both generalizations are based upon the incomplete Stirling numbers of both kinds. However, these incomplete Bernoulli and Cauchy numbers cannot be generalized as hypergeometric numbers using the hypergeometric functions.

Therefore, we introduce modified hypergeometric numbers. For integers \( N \geq 1 \), \( n \geq 0 \) and \( m \geq 1 \), define modified restricted hypergeometric Bernoulli numbers \( B_{N,n}^{*} \) by

\[ \left( \sum_{n=0}^{m} \frac{x^{n}}{(N+1)^{(n)}} \right)^{-1} = \sum_{n=0}^{\infty} B_{N,n}^{*} \frac{x^{n}}{n!}, \] (19)

where \( (x)^{(n)} := x(x+1) \cdots (x+n-1) \) \( (n \geq 1) \) is the rising factorial with \( (x)^{(0)} = 1 \). When \( N = 1 \), \( B_{n}^{*} = B_{1,n}^{*} \) are modified restricted Bernoulli numbers [25]. When \( m \rightarrow \infty \), \( B_{N,n} = B_{N,n}^{*} \) are the original hypergeometric Bernoulli numbers [9, 10, 12–15] defined by the generating function

\[ \frac{1}{1F_{1}(1;N+1;x)} = \frac{x^{N}/N!}{e^{x} - \sum_{n=0}^{N-1} x^{n}/n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^{n}}{n!}, \] (20)

where \( 1F_{1}(a; b; z) \) is the confluent hypergeometric function defined by the generating function

\[ 1F_{1}(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^{n}}{n!}. \]

When \( N = 1 \) and \( m \rightarrow \infty \), \( B_{n} = B_{1,n}^{*} \) are the classical Bernoulli numbers defined by the generating function
\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}
\]

with \(B_1 = -1/2\).

On the other hand, for integers \(N \geq 1\), \(n \geq 0\) and \(m \geq 1\), define modified associated hypergeometric Bernoulli numbers \(B_{N,n}^*\) by

\[
\left(1 + \sum_{n=0}^{\infty} \frac{x^n}{(N+1)^n}\right)^{-1} = \sum_{n=0}^{\infty} B_{N,n}^* \frac{m^n}{n!}.
\]

When \(N = 1\), \(B_{1,n}^* = B_{1,n}^* = B_{1,n}^* = m\) are modified associated Bernoulli numbers \([25]\).

When \(m = 1\), \(B_{N,n} = B_{N,n}^* = B_{N,n}^* = 1\) are the original hypergeometric Bernoulli numbers.

When \(N = 1\) and \(m = 1\), \(B_n = B_{1,n}^* = B_{1,n}^* = 1\) are the classical Bernoulli numbers.

For integers \(N \geq 1\), \(n \geq 0\) and \(m \geq 1\), define modified restricted hypergeometric Cauchy numbers \(c_{N,n}^*\) by

\[
\left(N \sum_{n=0}^{m} \frac{(-x)^n}{N+n}\right)^{-1} = \sum_{n=0}^{\infty} c_{N,n}^* \frac{m^n}{n!}.
\]

When \(N = 1\), \(c_{1,n}^* = c_{1,n}^* = c_{1,n}^* = m\) are modified restricted Cauchy numbers \([25]\). When \(m \to \infty\), \(c_{N,n} = c_{N,n}^* = c_{N,n}^* = \infty\) are the original hypergeometric Cauchy numbers \([18]\) defined by the generating function

\[
\frac{1}{2F_1(1, N; N+1; -x)} = \frac{(-1)^{N-1} x^N / N}{\log(1+t) - \sum_{n=1}^{N-1} (-1)^{n-1} x^n / n} = \sum_{n=0}^{\infty} c_{N,n} x^n / n!,
\]

where \(2F_1(a, b; c; z)\) is the Gauss hypergeometric function defined by the generating function

\[
2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)} (b)^{(n)} z^n}{(c)^{(n)} n!}.
\]

When \(N = 1\) and \(m \to \infty\), \(c_n = c_{1,n}^* = c_{1,n}^* = \infty\) are classical Cauchy numbers defined by the generating function

\[
\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} c_n x^n / n!.
\]

On the other hand, for integers \(N \geq 1\), \(n \geq 0\) and \(m \geq 1\), define modified associated hypergeometric Cauchy numbers \(c_{N,n}^*\) by

\[
\left(1 + N \sum_{n=m}^{\infty} \frac{(-x)^n}{N+n}\right)^{-1} = \sum_{n=0}^{\infty} c_{N,n}^* \frac{m^n}{n!}.
\]

When \(N = 1\), \(c_{1,n}^* = c_{1,n}^* = c_{1,n}^* = m\) are modified associated Cauchy numbers \([25]\). When
For integers \( N \geq 0, n \geq 0 \) and \( m \geq 1 \), define modified restricted hypergeometric Euler numbers \( E_{N,n,\leq m} \) by

\[
\left( \sum_{n=0}^{m} \frac{x^{2n}}{(2N+1)(2n)!} \right)^{-1} = \sum_{n=0}^{\infty} E_{N,n,\leq m} \frac{x^n}{n!}.
\]

(25)

Note that \( E_{N,n,\leq m}=0 \) when \( n \) is odd. When \( N = 0 \), \( E_{n,\leq m} = E_{0,n,\leq m} \) are modified restricted Euler numbers. When \( m \to \infty \), \( E_{N,n} = E_{N,n,\leq \infty} \) are the original hypergeometric Euler numbers [19, 27] defined by the generating function

\[
\frac{1}{1-F_2(1;N+1,(2N+1)/2;x^2/4)} = \frac{x^{2N}/(2N)!}{\cosh x - \sum_{n=0}^{N-1} x^{2n}/(2n)!} = \sum_{n=0}^{\infty} E_{N,n} \frac{x^n}{n!},
\]

(26)

where \( 1-F_2(a;b,c;z) \) is the hypergeometric function defined by the generating function

\[
1-F_2(a;b,c;z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}(c)^{(n)}}{(n)!} z^n.
\]

When \( N = 0 \) and \( m \to \infty \), \( E_n = E_{0,n,\leq \infty} \) are the classical Euler numbers defined by the generating function

\[
\frac{1}{\cosh x} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.
\]

On the other hand, for integers \( N \geq 0, n \geq 0 \) and \( m \geq 1 \), define modified associated hypergeometric Euler numbers \( E_{N,n,\geq m} \) by

\[
\left( 1 + \sum_{n=m}^{\infty} \frac{x^{2n}}{(2N+1)(2n)!} \right)^{-1} = \sum_{n=0}^{\infty} E_{N,n,\geq m} \frac{x^n}{n!}.
\]

(27)

Note that \( E_{N,n,\geq m}=0 \) when \( n \) is odd. When \( N = 0 \), \( E_{n,\geq m} = E_{0,n,\geq m} \) are modified associated Euler numbers. When \( m = 1 \), \( E_{N,n} = E_{N,n,\geq 1} \) are the original hypergeometric Euler numbers. When \( N = 0 \) and \( m = 1 \), \( E_n = E_{0,n,\geq 1} \) are the classical Euler numbers.

For integers \( N \geq 0, n \geq 0 \) and \( m \geq 1 \), define modified restricted hypergeometric Euler numbers of the second kind \( \widehat{E}_{N,n,\leq m} \) by

\[
\left( \sum_{n=0}^{m-1} \frac{x^{2n}}{(2N+2)(2n)!} \right)^{-1} = \sum_{n=0}^{\infty} \widehat{E}_{N,n,\leq m} \frac{x^n}{n!}.
\]

(28)

Note that \( \widehat{E}_{N,n,\leq m}=0 \) when \( n \) is odd. When \( N = 0 \), \( \widehat{E}_{n,\leq m} = \widehat{E}_{0,n,\leq m} \) are modified restricted Euler numbers of the second kind. When \( m \to \infty \), \( \widehat{E}_{N,n} = \widehat{E}_{N,n,\leq \infty} \) are the
original hypergeometric Euler numbers of the second kind [19, 27] defined by the generating function
\[
\frac{1}{\text{1_F}_2(1; N + 1, (2N + 3)/2; x^2/4)} = \frac{x^{2N+1}/(2N + 1)!}{\sinh x - \sum_{n=0}^{N} x^{2n+1}/(2n + 1)!}
\]
\[\quad = \sum_{n=0}^{\infty} \hat{E}_{N,n} \frac{x^n}{n!}.
\] (29)

When \(N = 0\) and \(m \to \infty\), \(\hat{E}_n = \hat{E}_{0,n} \leq \infty\) are the Euler numbers of the second kind defined by the generating function
\[
\frac{x}{\sinh x} = \sum_{n=0}^{\infty} \hat{E}_n \frac{x^n}{n!}.
\]

On the other hand, for integers \(N \geq 0\), \(n \geq 0\) and \(m \geq 1\), define modified associated hypergeometric Euler numbers of the second kind \(\hat{E}_{n,n,m}^*\) by
\[
\left(1 + \sum_{n=m}^{\infty} \frac{x^{2n}}{(2N + 2)^{2n}}\right)^{-1} = \sum_{n=0}^{\infty} \hat{E}_{n,n,m}^* \frac{x^n}{n!}.
\] (30)

Note that \(\hat{E}_{n,n,m}^* = 0\) when \(n\) is odd. When \(N = 0\), \(\hat{E}_{n,m}^* = \hat{E}_{0,n,m}^*\) are modified associated Euler numbers of the second kind. When \(m = 1\), \(\hat{E}_{n,n}^* = \hat{E}_{n,n,1}^*\) are the hypergeometric Euler numbers of the second kind. When \(N = 0\) and \(m = 1\), \(\hat{E}_n = \hat{E}_{0,n,1}^*\) are the Euler numbers of the second kind.

There are many kinds of generalizations of Bernoulli, Cauchy and Euler numbers, including poly numbers, multiple numbers, Apostol numbers and various \(q\)-Bernoulli numbers, \(p\)-adic Bernoulli numbers. However, modified restricted hypergeometric Bernoulli, Cauchy and Euler numbers and modified associated hypergeometric Bernoulli, Cauchy and Euler numbers have natural extensions in terms of determinant expressions.

Applying Lemmas 1 and 6 to Bernoulli, Cauchy and Euler numbers yields the following determinant expressions, respectively.

For convenience, put for \(n \geq 0\)
\[
\xi_n := \begin{cases} 
(-1)^n \frac{n!}{n!} & \text{if } A_{n,m}^* = B_{N,n,m}^* \text{ or } A_{n,m}^* = B_{N,n,m}^* \\
(-1)^n (2n)! & \text{if } A_{n,m}^* = C_{N,n,m}^* \text{ or } A_{n,m}^* = C_{N,n,m}^* \\
(-1)^n (2n)! & \text{if } A_{n,m}^* = E_{N,n,m}^* \text{ or } A_{n,m}^* = E_{N,n,m}^* \\
\end{cases}
\]
\[
&\text{and for } j \geq 1
\]
\[ x_j := \begin{cases} 
\frac{N!}{(N+j)!} & (N \geq 1) \\
\frac{N}{N+j} & (N \geq 1) \\
\frac{(2N)!}{(2N+2j)!} & (N \geq 0) \\
\frac{(2N+1)!}{(2N+2j+1)!} & (N \geq 0) 
\end{cases} \]

if \( A_{n,\leq m}^* = B_{N,n,\leq m}^* \) or \( A_{n,\geq m}^* = B_{N,n,\geq m}^* \)

We omit \( N \) from the notation for brevity and convenience, as the omission causes no confusion. Note that \( N \geq 1 \) for Bernoulli’s and Cauchy’s and \( N \geq 0 \) for Euler’s.

**Proposition 2** For integers \( n \) and \( m \) with \( n \geq m \geq 1 \), we have

\[
A_{n,\leq m}^* = \frac{x_1}{x_m} \begin{array}{cccc}
1 & 0 & & \\
\vdots & \ddots & \ddots & \\
0 & \ddots & \ddots & 0 \\
& & \ddots & 1 \\
0 & & \ddots & \ddots & \ddots \\
& & \ddots & 1 & 0 \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
\end{array}.
\]

**Proposition 3** For integers \( n \) and \( m \) with \( n \geq m \geq 1 \), we have

\[
A_{n,\geq m}^* = \frac{x_n}{x_m} \begin{array}{cccc}
0 & 1 & 0 & \\
\vdots & \ddots & \ddots & \\
0 & \ddots & \ddots & 0 \\
& & \ddots & 1 \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
\end{array}.
\]

**Remark** When \( n \leq m - 1 \) for \( A_{n,\leq m}^* = B_{N,n,\leq m}^* \) in Proposition 2 or when \( m = 1 \) for \( A_{n,\geq m}^* = B_{N,n,\geq m}^* \) in Proposition 3, the result is reduced to a determinant expression of hypergeometric Bernoulli numbers. In addition, when \( N = 1 \), we have a determinant expression of Bernoulli numbers [8, p. 53]. When \( n \leq m - 1 \) for \( A_{n,\leq m}^* = c_{N,n,\leq m}^* \) in Proposition 2 or when \( m = 1 \) for \( A_{n,\geq m}^* = c_{N,n,\geq m}^* \) in Proposition 3, the result is reduced to a determinant expression of hypergeometric Cauchy
numbers [26]. In addition, when \(N = 1\), we have a determinant expression of Cauchy numbers in [8, p. 50].

When \(n \leq m - 1\) for \(A_{n, \leq m}^* = E_{N,n,\leq m}^*\) in Proposition 2 or when \(m = 1\) for \(A_{n, \geq m}^* = E_{N,n, \geq m}^*\) in Proposition 3, the result is reduced to a determinant expression of hypergeometric Euler numbers [19, 27]. In addition, when \(N = 1\), we have a determinant expression of Euler numbers in (cf. [8, p. 52]).

When \(n \leq m - 1\) for \(A_{n, \leq m}^* = \widetilde{E}_{N,n,\leq m}^*\) in Proposition 2 or when \(m = 1\) for \(A_{n, \geq m}^* = \widetilde{E}_{N,n, \geq m}^*\) in Proposition 3, the result is reduced to a determinant expression of hypergeometric Euler numbers of the second kind [27]. In addition, when \(N = 1\), we have a determinant expression of Euler numbers of the second kind [19, 27].

Applying Lemma 2 yields different explicit expressions of the modified restricted hypergeometric Bernoulli, Cauchy and Euler numbers.

**Proposition 4**  For positive integers \(n\) and \(m\), we have

\[
A_{n, \leq m}^* = \zeta_n \sum_{k=1}^{n} (-1)^{n-k} \sum_{i_1 + \cdots + i_k = n} \kappa_{i_1} \cdots \kappa_{i_k},
\]

where \(1 \leq i_1, \ldots, i_k \leq m\).

There exists an alternative expression including binomial coefficients when every \(x_n\) \((n \geq 1)\) in (1) is replaced by \(-x_n\) with \(x_0 = 1\) as

\[
1 + \sum_{n=1}^{\infty} z_n t^n = \left( \sum_{n=0}^{\infty} x_n t^n \right)^{-1}.
\] (31)

Note that \(i_1, \ldots, i_k\) take value 0 too.

**Proposition 5**  For positive integers \(N\), \(n\) and \(m\), we have

\[
A_{n, \leq m}^* = \zeta_n \sum_{k=1}^{n} (-1)^{n-k} \binom{n+1}{k+1} \sum_{i_1 + \cdots + i_k = n} \kappa_{i_1} \cdots \kappa_{i_k},
\]

where \(0 \leq i_1, \ldots, i_k \leq m\).

**Proof**  The proof is based upon a more general result, given in the following lemma. The restricted case is obtained similarly. For Cauchy numbers, \(x_n\)’s are further replaced by \((-1)^n x_n\). \(\square\)

**Lemma 9**  If the sequence of \(z_n\)’s is given by (31), we have for \(n \geq 1\)
\[ z_n = \sum_{k=1}^{n} \binom{n+1}{k+1} \sum_{i_1 + \cdots + i_k = n} x_{i_1} \cdots x_{i_k}, \quad i_1, \ldots, i_k \geq 0 \]

**Proof** From the definition in (31), we have for \( n \geq 1 \)

\[
\begin{align*}
Z_n &= \frac{1}{n!} \frac{d^n}{dr^n} \left( 1 - \left( 1 + \sum_{n=1}^{\infty} x_n t^n \right)^{-1} \right) \bigg|_{r=0} \\
&= \frac{1}{n!} \frac{d^n}{dr^n} \sum_{l=0}^{\infty} \left( 1 - \sum_{n=0}^{\infty} x_n t^n \right)^l \bigg|_{r=0} \\
&= \frac{1}{n!} \frac{d^n}{dr^n} \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{l}{k} \left( -\sum_{n=0}^{\infty} x_n t^n \right)^k \bigg|_{r=0} \\
&= \sum_{k=1}^{n} (-1)^k \sum_{l=k}^{n} \binom{l}{k} \sum_{i_1 + \cdots + i_k = n} x_{i_1} \cdots x_{i_k}, \quad i_1, \ldots, i_k \geq 0
\end{align*}
\]

Notice that the term of \( k = 0 \) disappears when \( n \geq 1 \). Using the relation

\[
\sum_{l=k}^{n} \binom{l}{k} = \binom{n+1}{k+1},
\]

we get the result. \( \square \)

Applying Lemma 7 yields different explicit expressions of the modified associated hypergeometric Bernoulli numbers.

**Proposition 6** For positive integers \( n \) and \( m \), we have

\[
A_{n, \leq m}^* = \zeta_n \sum_{k=1}^{n} (-1)^n k^{n-k} \sum_{i_1 + \cdots + i_k = n} x_{i_1} \cdots x_{i_k}, \quad 1 \leq i_1, \ldots, i_k \leq m
\]

**Remark** When \( m \to \infty \) in Proposition 4 or \( m = 1 \) in Proposition 6, we have
\[
B_{N,n} = n! \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} \frac{(-N!)^k}{(N + i_1)! \cdots (N + i_k)!},
\]

\[
c_{N,n} = n! \sum_{k=1}^{n} (-1)^{n-k} \sum_{i_1 + \cdots + i_k = n} \frac{N^k}{(N + i_1) \cdots (N + i_k)}
\]

[26],

\[
E_{N,2n} = (2n)! \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} \frac{(-2N!)^k}{(2N + 2i_1)! \cdots (2N + 2i_k)!}
\]

and

\[
\tilde{E}_{N,2n} = (2n)! \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} \frac{(-2N + 1!)^k}{(2N + 2i_1 + 1)! \cdots (2N + 2i_k + 1)!}
\]

[27].

There exists an alternative expression including binomial coefficients. The results are similarly obtained using Lemma 9 in the associated case.

**Proposition 7** For positive integers \(N, n\) and \(m\), we have

\[
A_{n, \geq m}^* = \xi_n \sum_{k=1}^{n} \binom{n + 1}{k + 1} \sum_{i_1 + \cdots + i_k = n} x_{i_1} \cdots x_{i_k},
\]

\(i_1, \ldots, i_k \geq m - 1\)

**Remark** When \(m \to \infty\) in Proposition 5 or \(m = 1\) in Proposition 7, we have
\[ B_{N,n} = n! \sum_{k=1}^{n} \binom{n+1}{k+1} \sum_{i_1 + \cdots + i_k = n} \frac{(-N!)^k}{(N+i_1)! \cdots (N+i_k)!}, \]
\[ i_1, \ldots, i_k \geq 0 \]
\[ c_{N,n} = n! \sum_{k=1}^{n} (-1)^{n-k} \binom{n+1}{k+1} \sum_{i_1 + \cdots + i_k = n} \frac{N^k}{(N+i_1) \cdots (N+i_k)}, \]
\[ i_1, \ldots, i_k \geq 0 \]
\[ E_{N,2n} = (2n)! \sum_{k=1}^{n} \binom{n+1}{k+1} \sum_{i_1 + \cdots + i_k = n} \frac{(-2N!)^k}{(2N+2i_1)! \cdots (2N+2i_k)!}, \]
\[ i_1, \ldots, i_k \geq 0 \]
\[ \hat{E}_{N,2n} = (2n)! \sum_{k=1}^{n} \binom{n+1}{k+1} \sum_{i_1 + \cdots + i_k = n} \frac{(-2N+1)!^k}{(2N+2i_1+1)! \cdots (2N+2i_k+1)!}, \]

By applying Lemmas 3 and 8 (16) to Propositions 2 and 3, we obtain an explicit expression for modified incomplete Bernoulli, Cauchy and Euler numbers.

**Proposition 8**  For \( n \geq m \geq 1 \), we have

\[ A_{n,\leq m}^* = (-1)^n \xi_n \sum_{t_1+2t_2+\cdots+mt_m=n} \binom{t_1 + \cdots + t_m}{t_1, \ldots, t_m} (-1)^{t_1+\cdots+t_m} \gamma_1^1 \cdots \gamma_m^m \]

and

\[ A_{n,\geq m}^* = (-1)^n \xi_n \sum_{mt_m+(m+1)t_{m+1}+\cdots+n=n} \binom{t_m + t_{m+1} + \cdots + t_n}{t_m, t_{m+1}, \ldots, t_n} \]
\[ \times (-1)^{t_m + t_{m+1} + \cdots + t_n} \gamma_m^m \gamma_{m+1}^{m+1} \cdots \gamma_n^n. \]

By applying Lemmas 4 and 8 (17) as the result of the inversion relation to Propositions 2 and 3, respectively, we have the following.

**Proposition 9**  For \( n \geq 1 \), we have
By applying the inversion relations to Proposition 8, we also have the following.

**Proposition 10** For \( n \geq 1 \), we have

\[
\alpha_n = \sum_{t_1+2t_2+\cdots+mt_m=n} \left( t_1 + \cdots + t_m \right) \left( t_1, \ldots, t_m \right) (-1)^{n-t_1-\cdots-t_m}
\]
\[
\times \left( \frac{A_n^{*}, \leq m}{\xi_1} \right)_1 \left( \frac{A_n^{*}, \leq m}{\xi_2} \right) \cdots \left( \frac{A_n^{*}, \leq m}{\xi_m} \right)_m \quad (n \leq m)
\]
\[
= \sum_{mt_m+(m+1)t_{m+1}+\cdots+nt_n=n} \left( t_m + t_{m+1} + \cdots + t_n \right) (-1)^{n-t_m-t_{m+1}-\cdots-t_n}
\]
\[
\times \left( \frac{A_n^{*}, \geq m}{\xi_m} \right)_m \left( \frac{A_n^{*}, \geq m}{\xi_{m+1}} \right) \cdots \left( \frac{A_n^{*}, \geq m}{\xi_n} \right)_n \quad (n \geq m).
\]

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