A Wave Model of Metric Spaces

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Abstract. Let Ω be a metric space. By \( A_t \) we denote the metric neighborhood of radius \( t \) of a set \( A \subset \Omega \) and by \( \mathcal{O} \), the lattice of open sets in \( \Omega \) with partial order \( \subseteq \) and order convergence. The lattice of \( \mathcal{O} \)-valued functions of \( t \in (0, \infty) \) with pointwise partial order and convergence contains the family \( \mathcal{I}_O = \{ A(\cdot) \mid A(t) = A_t, A \in \mathcal{O} \} \). Let \( \tilde{\Omega} \) be the set of atoms of the order closure \( \overline{\mathcal{O}} \). We describe a class of spaces for which the set \( \tilde{\Omega} \) equipped with an appropriate metric is isometric to the original space \( \Omega \).

The space \( \tilde{\Omega} \) is the key element of the construction of the wave spectrum of a lower bounded symmetric operator, which was introduced in a work of one of the authors. In that work, a program for constructing a functional model of operators of the aforementioned class was laid down. The present paper is a step in the realization of this program.

Key words: metric space, lattice of open subsets, isotony, lattice-valued function, atom, wave model.

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Introduction

In the paper [1] a program for constructing a new functional model of symmetric semibounded operators was laid down. This model was named the wave model, which was motivated by its origin in inverse problems of mathematical physics. In [3] and [7] its systematic investigation was started. The key element of the wave model is the wave spectrum—a unitary invariant of a symmetric semibounded operator [1]. This is a topological space determined by the operator. As it turned out, the fundamental problem of reconstructing a Riemannian manifold from boundary spectral and dynamical data [2] can be reduced to finding the wave spectrum of the minimal Laplacian corresponding to this manifold. This spectrum, equipped with an appropriate metric, first, is determined by the inverse problem data and secondly, turns out to be isometric to the manifold to be recovered. Thus, it solves the problem.

The subject of the present paper is general properties of the wave spectrum as a set-theoretic construction and its possible structure. Here we study them separately, without connection to their origin—semibounded operators. The results of the paper, in our opinion, give reason to speak of the wave spectrum as a new and rich-in-content attribute of this important class of operators.

Relatively simple facts are stated as propositions, their proofs are not included. In this paper some inaccuracies in [1] are corrected. We are grateful to the referee for helpful comments.

1. The Lattice \( \mathcal{O} \)

Throughout the paper \((\Omega, d)\) is a complete metric space. Definitions and terminology are taken mostly from [4] and [5].

- The metric neighborhood of radius \( t \) of a set \( A \subset \Omega \) is the set
  \[ A^t := \{ x \in \Omega \mid d(x, A) < t \}, \quad t > 0; \]

we also denote \( \varnothing^t := \varnothing \). By int \( A \) we denote the set of interior points of the set \( A; \overline{\mathcal{A}} \) is the closure of \( A \) in \( \Omega \). We note that \( A^t = \overline{A^t} \) for \( t > 0 \).

Let \( B_r(x) := \{ y \in \Omega \mid d(x, y) < r \} \) and \( B_r[x] := \{ y \in \Omega \mid d(x, y) \leq r \} \) be the open and the closed ball of radius \( r > 0 \) centered at \( x \in \Omega \).
• Let $\mathcal{D}$ be the lattice of open sets in $\Omega$ with partial order $\subseteq$ and operations $G_1 \cup G_2 = G_1 \cup G_2$ and $G_1 \cap G_2 = G_1 \cap G_2$. It is complete: every family of sets $G_\alpha$ in it has the least upper bound $\bigcup_{\alpha} G_\alpha$ and the greatest lower bound $\inf \bigcap_{\alpha} G_\alpha$. In particular, the lattice $\mathcal{D}$ has least and greatest elements—the sets $\varnothing$ and $\Omega$, respectively.

In the lattice $\mathcal{D}$ one can introduce the order convergence (o-convergence) [4]. Let $\{G_\alpha\}$ be a net in $\mathcal{D}$. Then, by definition,

$$G_\alpha \xrightarrow{o} G \quad \text{if} \quad \sup_{\alpha} \inf_{\beta > \alpha} G_\beta = \inf_{\alpha} \sup_{\beta > \alpha} G_\beta = G,$$

which in our case is equivalent to

$$G_\alpha \xrightarrow{o} G \quad \text{if} \quad \bigcup_{\alpha} \bigcap_{\beta > \alpha} G_\beta = \bigcap_{\alpha} \bigcup_{\beta > \alpha} G_\beta = G.$$

A net $\{G_\alpha\}$ increases (decreases) if $\alpha \geq \beta$ implies $G_\alpha \geq G_\beta$ ($G_\alpha \leq G_\beta$). Let us note a simple fact.

**Proposition 1.** Every increasing (decreasing) net $\{G_\alpha\}$ of sets in $\mathcal{D}$ converges in order to its greatest lower bound $\inf \bigcap_{\alpha} G_\alpha$ (to its least upper bound $\bigcup_{\alpha} G_\alpha$).

Let $\mathcal{F} \subset \mathcal{D}$ be a family of open subsets. We denote the set of limits of all order-convergent nets in $\mathcal{F}$ by $\overline{\mathcal{F}}$ and refer to the operation $\mathcal{F} \mapsto \overline{\mathcal{F}}$ as order closure.

2. The Set $\overline{\mathcal{D}}$

• Consider the family $\mathcal{F} := F((0, +\infty); \mathcal{D})$ of functions on the half-line taking values in the lattice $\mathcal{D}$. It is easy to see that $\mathcal{F}$ is a complete lattice with respect to the pointwise order.

$$f \leq g \iff f(t) \subseteq g(t), \quad t > 0,$$

which determines the operations

$$(f \vee g)(t) := f(t) \cup g(t), \quad (f \wedge g)(t) := f(t) \cap g(t), \quad t > 0.$$  

The lattice has the least element $0_\mathcal{F}(\cdot) \equiv \varnothing$ and the greatest element $1_\mathcal{F}(\cdot) \equiv \Omega$. The order convergence in $\mathcal{F}$ coincides with the pointwise order convergence:

$$f_\alpha \xrightarrow{o} f \iff f_\alpha(t) \xrightarrow{o} f(t), \quad t > 0.$$

• A map $i: \mathcal{P} \rightarrow \mathcal{D}$ between two partially ordered sets is called an isotonic map (or an isotony) if it preserves the order, i.e., $p \leq q$ implies $i(p) \leq i(q)$ [4]. By the metric isotony we mean the map $I: \mathcal{D} \rightarrow \mathcal{F}$. $(IG)(t) := G^t, \quad t > 0$. Note that $I\varnothing = 0_\mathcal{F}$ according to the above definitions.

Consider the image

$$I\mathcal{D} = \{g \mid g(t) = G^t, \ G \in \mathcal{D}, \ t > 0\} \subset \mathcal{F}$$

of the whole lattice. Its order closure $\overline{I\mathcal{D}}$ in the lattice $\mathcal{F}$ is of special interest to us. Note that, generally speaking, it is not a lattice. Let us list some of the properties of its elements.

**Proposition 2.** All elements of $\overline{I\mathcal{D}}$ are increasing functions. For every $g \in \overline{I\mathcal{D}}$, there exists a decreasing net $\{G_\alpha\}$ in $\mathcal{D}$ such that $IG_\alpha \xrightarrow{o} g$ in $\mathcal{F}$.

Furthermore, $g(t) = \inf \bigcap_{\alpha} G_\alpha$ for every $t > 0$.

Indeed, the order closure $\overline{I\mathcal{D}}$ consists of those functions in $\mathcal{F}$ which are limits of at least one net in $I\mathcal{D}$. For $g \in \overline{I\mathcal{D}}$ and a net $\{G_\alpha\}$ in $\mathcal{D}$ such that $IG_\alpha \xrightarrow{o} g$ one can take $G_\alpha := \bigcup_{\beta > \alpha} G_\beta$. This is a decreasing net and, moreover, $G_\alpha = \bigcup_{\beta > \alpha} G_\beta$. The remaining assertions of the proposition are also easy to check. We add that there may be no increasing net which converges to $g$.

**Lemma 1.** Let $g \in \overline{I\mathcal{D}}$, and let $\{G_\alpha\}$ be a decreasing net in $\mathcal{D}$ such that $IG_\alpha \xrightarrow{o} g$. Then $\bigcap_{t>0} g(t) = \bigcap_{t>0} \bigcap_{\alpha} G_\alpha = \bigcap_{\alpha} G_\alpha$.  

*We note that, generally, $\overline{\mathcal{F}} \neq \overline{\mathcal{F}}$ (see [5]). Nevertheless, o-convergence determines a topology on $\mathcal{D}$, in which the families with the property $\mathcal{F} = \overline{\mathcal{F}}$ are closed by definition.
Since this means that the compactness of the closed ball that there exists a finite set \( t > 0 \), and hence \( \bigcap_{t>0} g(t) \subseteq \bigcap_{t>0} G^t_\alpha = \overline{G^t_\alpha} \). Since \( \alpha \) is arbitrary, we have
\[
\bigcap_{t>0} g(t) \subseteq \bigcap_{\alpha} \overline{G^t_\alpha}. \tag{1}
\]

On the other hand,
\[
\bigcap_{\alpha} \overline{G^t_\alpha} \subseteq \left( \bigcap_{\alpha} \overline{G^t_\alpha} \right)^t \subseteq \text{int} \left( \bigcap_{\alpha} (\overline{G^t_\alpha})^t \right) = \text{int} \bigcap_{\alpha} G^t_\alpha = g(t)
\]
for every \( t > 0 \), and hence \( \bigcap_{\alpha} \overline{G^t_\alpha} \subseteq \bigcap_{t>0} g(t) \). Together with (1) this gives the assertion of the lemma. \( \blacksquare \)

- To each function \( g \in \overline{\Omega} \) we assign the set
\[
\hat{g} := \bigcap_{t>0} g(t) \subseteq \Omega
\]
(which is always closed by Lemma 1) and call it the nucleus of the function \( g \). For every function \( g \in \overline{\Omega} \),
\[
(\hat{g})^t \subseteq g(t), \quad t > 0. \tag{2}
\]
Indeed, using Lemma 1, one has
\[
(\hat{g})^t = \left( \bigcap_{\alpha} \overline{G^t_\alpha} \right)^t \subseteq \text{int} \left( \bigcap_{\alpha} (\overline{G^t_\alpha})^t \right) = \text{int} \bigcap_{\alpha} G^t_\alpha = g(t).
\]
Relation (2) becomes trivial if the nucleus is empty. However, there is a simple condition which ensures the nonemptiness of the nucleus.

**Condition 1.** For any \( x \in \Omega \) and \( r > 0 \), the closed ball \( B_r[x] \) is compact.

**Lemma 2.** Under Condition 1, if \( g \in \overline{\Omega} \) and \( g \not\equiv 0 \), then \( \hat{g} \not= \emptyset \).

**Proof.** We make use of the following simple statement:
\[
A \cap B^t = \emptyset \iff A^t \cap B = \emptyset. \tag{3}
\]

By Proposition 2, given \( g \), there exists a decreasing net \( \{G_\alpha\} \) such that \( IG_\alpha \to g \) and \( \hat{g} = \bigcap_\alpha \overline{G^t_\alpha} \). Let \( \hat{g} = \emptyset \). For any \( x \in \Omega \) and \( t > 0 \), we have \( B_t[x] \cap (\bigcap_\alpha \overline{G^t_\alpha}) = \emptyset \). It follows from the compactness of the closed ball that there exists a finite set \( \{\alpha_1, \ldots, \alpha_n\} \) of indices such that \( B_t[x] \cap (\bigcap_{i=1}^n \overline{G^t_{\alpha_i}}) = \emptyset \). Since the set of indices is directed, there exists an upper bound \( \gamma \) of the set \( \{\alpha_1, \ldots, \alpha_n\} \), and hence \( B_t[x] \cap \overline{G^t_{\gamma}} = \emptyset \). This means that \( \{x\}^t \cap G^t_{\gamma} = \emptyset \), and this by the statement proved above is equivalent to the relation \( \{x\} \cap G^t_{\gamma} = \emptyset \), or \( x \not\in G^t_{\gamma} \). Therefore, \( x \not\in \bigcap_\alpha G^t_\alpha \) for any \( x \) and \( t \). Hence \( g(t) = \text{int} \bigcap_\alpha G^t_\alpha = \emptyset \) for every \( t > 0 \), which leads to a contradiction. \( \blacksquare \)

It is clear that under Condition 1 relation (2) is a nontrivial lower bound for the functions in \( \overline{\Omega} \).

- In what follows, we will need one more condition on the metric space \((\Omega, d)\).

**Condition 2.** For any \( x, y \in \Omega \) and \( r, s > 0 \), the relation \( B_r(x) \cap B_s(y) = \emptyset \) implies \( d(x, y) \geq r + s \).

Let us remark that the class of metric spaces which satisfy Conditions 1 and 2 contains complete locally compact spaces with intrinsic metric [6] (including Riemannian manifolds).

**Proposition 3.** Under Condition 2, for every \( A \subseteq \Omega \), one has \((A^r)^s = A^{r+s}\).

**Corollary.** Under Condition 2, for any \( x \in \Omega \) and \( r > 0 \), one has \( B_r(x) = B_{r[x]} \).

The following lemma has technical character.
Lemma 3. Let Conditions 1 and 2 hold. Then, for every decreasing net \( \{A_\alpha\} \) of sets in the space \( \Omega \),
\[
\left( \bigcap_\alpha A_\alpha \right)^t \subseteq \bigcap_\alpha A_\alpha^t \subseteq \left( \bigcap_\alpha \overline{A_\alpha} \right)^t, \quad t > 0.
\]

**Proof.** The first inclusion is always true. Let us prove the second. Let \( x \notin \left( \bigcap_\alpha A_\alpha \right)^t \). Then, by Hausdorffness, there exists an \( r > 0 \) such that \( B_r(x) \cap \left( \bigcap_\alpha A_\alpha \right)^t = \emptyset \). Hence \( B_r(x) \cap \left( \bigcap_\alpha \overline{A_\alpha} \right)^t = \emptyset \), and using formula (3) and Proposition 3, one has \( B_r(x)^t \cap \left( \bigcap_\alpha \overline{A_\alpha} \right)^t = \emptyset \); therefore, \( B_t[x] \cap \left( \bigcap_\alpha \overline{A_\alpha} \right) = \emptyset \). The closed ball is compact; hence there exists a finite set \( \{\alpha_1, \ldots, \alpha_n\} \) of indices such that \( B_t[x] \cap \bigcap_{\alpha=1}^n \overline{A_\alpha} = \emptyset \). The net \( \{A_\alpha\} \) decreases; hence, for \( \gamma = \sup\{\alpha_1, \ldots, \alpha_n\} \), we have \( B_t[x] \cap A_\gamma = \emptyset \). Consequently, \( B_t(x) \cap A_\gamma = \emptyset \), which, due to (3), is equivalent to \( x \notin A_\gamma^t \); thus, \( x \notin \bigcap_\alpha A_\alpha^t \). Therefore, \( \bigcap_\alpha A_\alpha^t \subseteq \left( \bigcap_\alpha \overline{A_\alpha} \right)^t \), which completes the proof. \( \Box \)

- Now we can obtain an upper bound for an element of \( T\Omega \) in terms of its nucleus.

Lemma 4. Let Conditions 1 and 2 hold. Then, for every function \( g \in T\Omega \),
\[
g(t) \subseteq \operatorname{int} g^t, \quad t > 0. \tag{4}
\]

**Proof.** If \( g \equiv \emptyset \), then the assertion is obvious. Let \( g \neq \emptyset \). Then by Lemma 3 one has \( \bigcap_\alpha G_\alpha^t \subseteq \left( \bigcap_\alpha \overline{G_\alpha} \right)^t = g^t \); therefore, \( g(t) = \operatorname{int} \bigcap_\alpha G_\alpha^t \subseteq \operatorname{int} g^t \). \( \Box \)

- Let us show that the set \( T\Omega \) contains a sufficient amount of functions whose nuclei are single points. We define \( b_*[x], b^*[x] \in \mathfrak{F} \) by
\[
(b_*[x])(t) = B_t(x), \quad (b^*[x])(t) = \operatorname{int} B_t[x], \quad t > 0, \tag{5}
\]
and note that \( b_*[x] \subseteq b^*[x] \).

Lemma 5. Let Condition 2 hold. Then \( b^*[x] \in T\Omega \) for every \( x \in \Omega \), and the nucleus of the function \( b^*[x] \) consists of the single point \( x \).

**Proof.** Let us prove that the net \( \{IB_\varepsilon(x)\}_{\varepsilon>0} \) converges in \( \mathfrak{F} \) to \( b^*[x] \) as \( \varepsilon \to +0 \). Since this net is decreasing, by Proposition 1 one has to check that \( \operatorname{int} \bigcap_{\varepsilon>0} (B_\varepsilon(x))^t = \operatorname{int} B_t[x] \) for every \( t > 0 \). By Proposition 3 we have \( (B_\varepsilon(x))^t = B_{t+\varepsilon}(x) = (B_t(x))^\varepsilon \). Furthermore, \( \bigcap_{\varepsilon>0} (B_\varepsilon(x))^t = \bigcap_{\varepsilon>0} (B_t(x))^\varepsilon = B_t(x) = B_t[x] \) by the corollary to Proposition 3. From this it follows that
\[
\operatorname{int} \bigcap_{\varepsilon>0} (B_\varepsilon(x))^t = \operatorname{int} B_t[x], \quad t > 0.
\]
The nucleus of the function \( b^*[x] \), which is \( \bigcap_{t>0} \operatorname{int} B_t[x] \), obviously contains the point \( x \) and cannot contain any other points, which means that it coincides with \( \{x\} \). \( \Box \)

3. The Wave Model

- Consider the equivalence relation on functions in the set \( T\Omega \) defined as follows:
\[
f_1 \sim f_2 \iff \bigcap_{t>0} f_1(t) = \bigcap_{t>0} f_2(t) \iff \hat{f}_1 = \hat{f}_2.
\]

Let \( \langle f \rangle \) be the equivalence class of the function \( f \). For the functions defined by (5), one has \( b_*[x] = b^*[x] = \{x\} \), whence \( b_*[x] \sim b^*[x] \) and \( \langle b_*[x] \rangle = \langle b^*[x] \rangle \). All representatives of the same class have the same nucleus, which allows us to refer to the latter as the *nucleus of the equivalence class*, which we denote in what follows by \( \langle \ldots \rangle \). Thus, \( \langle b_*[x] \rangle = \langle b^*[x] \rangle = \{x\} \).

On the set \( T\Omega/ \sim \) of classes we define a partial order in the following way:
\[
\langle f \rangle \leq \langle g \rangle \iff \bigcap_{t>0} f(t) \subseteq \bigcap_{t>0} g(t) \iff \hat{f} \subseteq \hat{g}.
\]
Let $\mathcal{P}$ be a partially ordered set with least element 0. An element $a \in \mathcal{P}$ is called an atom (written as $a \in \text{At}(\mathcal{P})$) if $0 < p \leq a$ implies $p = a$ [4]. Note that the lattice $\mathfrak{D}$ of open sets may contain no atoms, e.g., in the case $\Omega = \mathbb{R}^n$.

The partially ordered set $\overline{\mathcal{O}}/\sim$ has the least element $0_\sim$, and therefore one can speak of its atoms:

$$\overline{\Omega} := \text{At}(\overline{\mathcal{O}}/\sim).$$

The set $\overline{\Omega}$ is the main object of the present paper.

**Lemma 6.** Let Conditions 1 and 2 hold. Then $\overline{\Omega} = \{\langle b_\ast \rangle \mid x \in \Omega\}$.

**Proof.** Let us prove that, for every $x \in \Omega$, the class $\langle b_\ast \rangle$ is an atom. Since its nucleus consists of only one point $x$, every class which is strictly less than $\langle b_\ast \rangle$ should have empty nucleus, but then this class should be the least. Consequently, $\langle b_\ast \rangle$ is an atom. To prove the converse, let $\langle a \rangle$ be an atom. Then, for every $x \in \langle a \rangle^\ast$, the inequality $\langle b_\ast \rangle \leq \langle a \rangle$ holds. By the definition of an atom this means that $\langle a \rangle = \langle b_\ast \rangle$ and $\langle a \rangle^\ast = \{x\}$. □

Let functions $a, b \in \overline{\mathcal{O}}$ be representatives of atoms $\langle a \rangle$ and $\langle b \rangle$, respectively. We refer to the function

$$\tau((a), (b)) := 2\inf\{t \mid a(t) \cap b(t) \neq \emptyset\}$$

as the wave distance between these atoms. The choice of this name is motivated by applications where the points corresponding to atoms initiate waves which propagate in $\Omega$ with unit speed. At the moment $t = \tau((a), (b))/2$ these waves start to overlap [1]. The correctness of this definition (finiteness for every pair of atoms, independence of the choice of representatives, and the three properties of the metric) follows from the next lemma.

**Lemma 7.** Let Conditions 1 and 2 hold. Then $\tau((b_\ast \rangle, \langle b_\ast \rangle) = d(x, y)$ for every $x, y \in \Omega$.

**Proof.** For any representatives $a \in \langle b_\ast \rangle$ and $b \in \langle b_\ast \rangle$, the inclusions $B_t(x) \subseteq a(t) \subseteq \text{int} B_t[x]$ and $B_t(y) \subseteq b(t) \subseteq \text{int} B_t[y]$ hold; hence, for $t < d(x, y)/2$, the relations $a(t) \cap b(t) \subseteq \text{int} B_t[x] \cap \text{int} B_t[y] = \emptyset$ take place. For $t > d(x, y)/2$, we have $a(t) \cap b(t) \supseteq B_t(x) \cap B_t(y) \neq \emptyset$, because if $B_t(x) \cap B_t(y) = \emptyset$, then $d(x, y) \geq 2t$ by Condition 2. Therefore, $\inf\{t \mid a(t) \cap b(t) \neq \emptyset\} = d(x, y)/2$, which proves the lemma. □

We refer to the space $(\overline{\Omega}, \tau)$ as the wave model of the original space $(\Omega, d)$. Summarizing the preceding considerations, we obtain the main result of the paper.

**Theorem 1.** Under Conditions 1 and 2 the wave model $(\overline{\Omega}, \tau)$ is isometric to the original $(\Omega, d)$.

The isometry is realized by the bijection $\Omega \ni x \mapsto \langle b_\ast \rangle \in \overline{\Omega}$.

### 4. Examples and Comments

The passage to equivalence classes is very important. One can prove that in the set $\overline{\mathcal{O}}$ atoms correspond to points of the space $\Omega$. At the same time, several atoms may correspond to one point and be incomparable with each other but belong to the segment $[\langle b_\ast \rangle, \langle b_\ast \rangle] := \{g \in \mathfrak{S} \mid b_\ast \leq g \leq b_\ast \}$. We demonstrate this fact by examples.

- In $\Omega = \mathbb{R}^n$ to each $x \in \mathbb{R}^n$ there corresponds a unique atom of the set $\overline{\mathcal{O}}$: $a_x(t) = B_t(x)$, $t > 0$, $a_x = \{x\}$, so that the wave model can be constructed without factorization.

- If $\Omega = [0, 1]$ and $d(x, y) = |x - y|$, then, for every $x \in (0, 1)$, the segment $[\langle b_\ast \rangle, \langle b_\ast \rangle]$ consists of four functions (see Fig. 1); at the same time, $B_t(x)$ does not belong to the set $\overline{\mathcal{O}}$, and among
these functions two are atoms of $\mathcal{I}_\Omega$ ($a^{(1)}$ and $a^{(2)}$ in the figure). For $x \in (0, 1/2]$, the atoms are

$$a^{(1)}(t) := \begin{cases} 
B_t(x), & 0 < t < 1 - x, \ t \neq x, \\
[0, 2x), & t = x, \\
[0, 1), & t = 1 - x, \\
\Omega, & t > 1 - x, 
\end{cases}$$

$$a^{(2)}(t) := \begin{cases} 
B_t(x), & 0 < t < 1 - x, \ t \neq x, \\
(0, 2x), & t = x, \\
\Omega, & t \geq 1 - x. 
\end{cases}$$

One can take $G^{(1)}_\varepsilon = (x - \varepsilon, x)$ and $G^{(2)}_\varepsilon = (x, x + \varepsilon)$ as initial approximating sets for these atoms: as $\varepsilon \to +0$, one has $(G^{(i)}_\varepsilon)^t \to a^{(i)}(t)$ for $i = 1, 2$. In the figure the points marked with small circles are excluded. After factorization all the four functions become identical.

![Diagram](image)

Fig. 1. The elements of the segment $[[b_*[x], b^*[x]]]

- The situation in which more than one atom of the set $\mathcal{I}_\Omega$ corresponds to one point of $\Omega$ is possible for Riemannian manifolds if the boundaries of the balls $B_t(x)$ with large $t$ are allowed to have self-intersections. Furthermore, the picture becomes more complicated if $\Omega$ has boundary. Nevertheless, in this case, Conditions 1 and 2 are satisfied and the wave model is isometric to the manifold. This fact plays the key role in the problem of reconstructing a manifold from the inverse spectral and dynamical data; see [1].

- In [1, p. 303] a (non-Hausdorff) topology on $\text{At} \mathcal{I}_\Omega$ which separates the atoms of this set was introduced.

- It would be interesting to find out to which extent the wave model remains meaningful when Conditions 1 and 2 are weakened. For example, in the case of a space with discrete metric

$$d(x, y) = \begin{cases} 
1, & x \neq y, \\
0, & x = y, 
\end{cases}$$

Condition 2 is not satisfied, and we have $\tau(x, y) = 2d(x, y)$. The wave model is isometric to the original “up to homothety.”

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