The theoretical analysis of an instrument for linear and angular displacements of the steered wheel measuring

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Abstract. In the paper the theoretical analysis of the measuring instrument for determination of translation and rotation of the stub axle with the steered wheel against car body was presented. The instrument is made of nine links with elongation sensors embedded in it. One of several possible structures of instrument of this kind was presented. Basing on solution of the geometrical constraints system of equations of the device, the numerical analysis of the measurement accuracy was conducted.

1. Introduction
The analysis of the car movement parameters on the curved track is one of the fundamental issues of stability and steerability. The change of these parameters is always caused by the change of the external forces acting on the vehicle. The car comparative studies carried out so far show that even at the same kinematic extortion, realized by the change of steering angle, the change of the forces generated at the wheel-road contact patch is dependent on many factors associated with tire, suspension and steering system construction. Steering wheels are carried out against a car body through the spatial mechanisms with flexible constraints [1]. The flexibility is the reason that during a car ride along the same path, with different speeds, the real kinematic steering ratio is changing; a significant difference between the real and theoretical steering angles appears. A measurement of real steering and camber angles in experimental car studies has a significant value. The results of these measurements are used to work out the relationships between car movement parameters as well as for stability and steerability valuation [2, 3, 4].

In case of independent suspensions, wheel vertical movements caused by unevenness of the road surface cause a track change. It leads to wheels drifting and adversely affects the straight driving [5, 6]. The measurement of the position and orientation of the wheel relative to the car body is very difficult, only a few studies on this topic, mainly related to the dynamic measurement of the steering angle, can be found in the literature [2, 7, 8]. A Datron RV-4 instrument [9] allows the measurement of the position and orientation of the steering wheel relative to the car body, the measurement, however, is complicated and the measured values are not obtained directly but as a result of complex calculations. This instrument has large dimensions and considerable weight in proportion to the weight of the wheel. The durability of the instrument is being reduced under the influence of the dynamic loads generated while driving the car over the road unevenness.
2. The goal and scope of the work
The main goal of the work is an introduction of the indirect measurement method of the stub axle with steering wheel translation and rotation.

The scope of this work concerns problems of solving the kinematics of four-link suspension and the proposed instrument mechanisms as well as determination of the steering and camber angles and the characteristics of lateral displacements of the wheel centre. The analysis of the measurement accuracy were conducted.

3. Multi-link steered wheels suspension mechanism system

In Figure 1 the four-link steering wheel suspension mechanism scheme is shown. Points: B₁, B₂, B₄ and B₅ are centres of ball joints connecting links with the stub axle. Point B₃ is a centre of the ball joint connecting steering linkage with the stub axle arm. Points: A₁, A₂, A₄ and A₅ are centres of ball joints, which replace a metal-rubber joints connecting links with the car body. Point A₃ is centres of the ball joint connecting steering linkage with a rack. A lower front link, represented in the figure by link A₁B₁, is connected with anti-roll bar at the point W. A telescopic column is connected with this link at the point C. Frames {N} and {K} are associated, respectively, with the car body and the stub axle.

![Figure 1. Scheme of four-link steered wheels suspension mechanism.](image)

4. Equations of geometric constraints of the suspension mechanism

The equations of geometric constraints of the mechanism shown in Figure 1 were written in the form of 14 nonlinear algebraic equations. These equations express squares of distances between characteristic points of the suspension:

\[
\begin{align*}
\vec{r}_{A_jB_j}^T \vec{r}_{A_jB_j} &= l_j^2, & \text{for } j = 1(5), \\
\vec{r}_{B_jB_k}^T \vec{r}_{B_jB_k} &= l_{jk}^2, & \text{for } \begin{cases} j = 1 \text{ and } k = 2(5), \\ j = 2 \text{ and } k = 3(5), \\ j = 3 \text{ and } k = 4(5). \end{cases}
\end{align*}
\]

(1)

In the above system of equations given parameters are coordinate \(z_{B1}\) of the point \(B_1(x_{B1}, y_{B1}, z_{B1})\) and steering rack displacement- \(u_p\), added to coordinate \(y_{A3}\) of the point \(A_3(x_{A3}, y_{A3}+u_p, z_{A3})\). At the given parameters \(z_{B1}\) and \(u_p\), coordinates of the point \(B_j(x_{Bj}, y_{Bj}, z_{Bj})\), for \(j=(1,5)\) and \(k=(2)5\) were determined from the system (1) using the perturbation method [10]. A constructional position of the points \(B_6\) and \(B_7\) is given, therefore determination of their coordinates in the movements space of the suspension \(\{N\}\) is possible from the following systems of equations:

for point \(B_6\):

\[
\vec{r}_{B_kB_j}^T \vec{r}_{B_kB_j} = l_{kj}^2, & \text{for } \begin{cases} k = 6 \text{ and } j = 1, \\ k = 6 \text{ and } j = 3, \\ k = 6 \text{ and } j = 5, \end{cases}
\]

(2)
for point $B_7$:

$$\hat{r}_{B_kB_j}^T \hat{r}_{B_kB_j} = l_{k_j}^2, \quad \text{for} \quad \begin{cases} k = 7 \text{ and } j = 1, \\ k = 7 \text{ and } j = 2, \\ k = 7 \text{ and } j = 4. \end{cases}$$

(3)

Next, a rotation angles: $\alpha, \beta, \gamma$ of $\{K\}$ against $\{N\}$ were determined. After calculating the coordinates of points $K$ and $B_j$ ($j=(1)5$) a three vectors for $\hat{r}_{KB_j}$ can be created. For each of these vectors a matrix equation is satisfied:

$$\hat{r}_{KB_j} = A_{KN} \cdot \hat{r}_{KB_j},$$

(4)

where: $\hat{r}_{KB_j} - \text{vector in } \{K\}$, $\hat{r}_{KB_j} - \text{vector in } \{N\}$,

$$A_{KN} = A_{NK}^T = \begin{bmatrix} c\beta \cdot c\gamma & c\beta \cdot s\gamma & -s\beta \\ s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & s\alpha \cdot c\beta \\ c\alpha \cdot s\beta \cdot c\gamma + s\alpha \cdot s\gamma & c\alpha \cdot s\beta \cdot s\gamma - c\alpha \cdot c\gamma & c\alpha \cdot c\beta \end{bmatrix}. \quad \text{(5)}$$

Denoting coordinates of vectors: $\hat{r}_{KB_j} = [x_{b_j}, y_{b_j}, z_{b_j}]^T$, $\hat{r}_{KB_j} = [x_{b_j}, y_{b_j}, z_{b_j}]^T$ and assuming $j = n, m, v$ on the basis of (4) we obtain:

$$x_{bn} = (x_{nb} \cdot c\gamma + y_{nb} \cdot s\gamma)c\beta - z_{nb} \cdot s\beta,$$

$$x_{bm} = (x_{mb} \cdot c\gamma + y_{mb} \cdot s\gamma)c\beta - z_{mb} \cdot s\beta,$$

$$x_{bv} = (x_{vb} \cdot c\gamma + y_{vb} \cdot s\gamma)c\beta - z_{vb} \cdot s\beta.$$ 

(6)

From the system of equations (6) the rotation angles $\alpha$ and $\beta$ are calculated. In order to calculate the angle $\gamma$ equation (7) is used:

$$y_{bv} = (x_{vb} \cdot s\beta \cdot c\gamma)\alpha - (x_{vb} \cdot s\gamma)\alpha + (y_{vb} \cdot s\beta \cdot c\gamma)\gamma + (y_{vb} \cdot c\gamma)c\alpha + (z_{vb} \cdot c\beta)s\alpha.$$ 

(7)

Then, having rotation angles of system $\{K\}$ against system $\{N\}$, coordinates of vector $\hat{r}_{B_kB_7}$:

$$\hat{r}_{B_kB_7} = A_{KN} \cdot \hat{r}_{B_kB_7},$$

(8)

unit vector $\hat{e}_k = [e_{kx}, e_{ky}, e_{kz}]^T$, lying on the wheel rotation axis as well as steering and camber angles:

$$\delta_k = -arctg \left( \frac{e_{kx}}{e_{ky}} \right),$$

(9)

$$\gamma_k = -arcsin(e_{kz}),$$

(10)

5. Coordinates of characteristic points of the suspension mechanism

A constructional location of the mechanism in the suspension movements space $\{N\}$ is defined by the coordinates of ball joints connecting links with the car body and the stub axle [11]:

- $A_1 (144.1, 345.2, -92.2); A_2 (-229.2, 362.2, -101.7); A_3 (-99.7, 400.0, 306.2); A_4 (-69.0, 396.3, 413.5); A_5 (134.6, 428.5, 408.9); B_1 (28.7, 690.9, -98.0); B_2 (-24.4, 687.0, -131.6); B_3 (-135.7, 617.1, 286.9); B_4 (-18.1, 639.8, 388.4); B_5 (15.4, 673.3, 389.5); B_6 (0.5, 647.0, 0.1); B_7 (1.0, 747.0, 0.6).

Coordinates values were given in mm.
6. Suspension characteristics
In Figure 2 characteristics of the analysed mechanism, obtained on the basis of its kinematics solution are shown.

![Figure 2](image1)

**Figure 2.** Dependences of steering and camber angles: $\delta_k$, $\gamma_k$ respectively and relative lateral displacement of the wheel centre $\Delta K_y$ on suspension deflection- $q$ and steering rack displacement- $u_p$, determined on the basis of the suspension kinematics solution.

7. Structure and mobility of the measuring instrument mechanism

![Figure 3](image2)

**Figure 3.** Scheme of the measuring instrument for determination of the translation and rotation of the steered wheel stub axle. Points $D_i$, $i=1(3)$ are the centres of three concentric ball joints. 1- disc attached to a rim, 2- disc immobilized against a stub axle.

The mechanism of the measuring instrument, shown in Figure 3, is composed of nine links $d_{ij}$, $j=1(9)$ connected by rotary- sliding kinematic pairs $s_{ij}$, $j=1(9)$. At the points: $D_i$, $i=1(3)$, which are the ball joints centres, the links are attached to the bearing disc, which is connected to the stub axle and immobilized against it. By points: $H_j$, $j=1(9)$, centres of the ball joints connecting links with the disc connected with...
the car body were noted in the figure. Characteristic for this mechanism is that each point marked as $D_i$, $i = (1)3$ is a common centre of three concentric ball joints.

The instrument mechanism has 6 mobility degrees- using the formula from the of mechanisms and machines theory [12]:

$$R = R_t - R_p,$$

(11)

where:

$$R = 6(n - 1) - \sum_{i=1}^{5} p_i,$$

(12)

- $R$ - real mobility of the mechanism,
- $R_t$ - theoretical mobility of the mechanism,
- $R_p$ - apparent mobility of the mechanism,
- $p_i$ - kinematic pair of the $i$- class,
- $n$ - number of links creating mechanism.

With regard to the considered mechanism: $n = 20$, $p_4 = 9$, $p_5 = 18$, $p_2 = p_1 = 0$, $R_t = 24$. After subtracting the apparent degrees of mobility $R_p = 18$ from theoretical mobility, the real mobility $R = 6$; is equal to the stub axle degrees of freedom in $\{N\}$.

8. Kinematics of the measuring instrument mechanism

Centres of the ball joints: $D_i$, $i = (1)3$ belong to the wheel stub axle. So the distances of these points from points $B_j$, $j = 1(6)$ can be calculated. However, the coordinates of points: $D_i$, $i = (1)3$ in $\{N\}$ can be determined from systems of equations:

$$\mathbf{r}_{B_jD_j}^T \mathbf{r}_{B_jD_j} = l_{B_jD_j}^2,$$

for $D_1$:

$$\begin{align*}
\mathbf{r}_{B_1D_1}^T \mathbf{r}_{B_1D_1} &= (l_{B_1H_1} + s_j)^2, & \text{for } j = 1, \\
\mathbf{r}_{B_2D_2}^T \mathbf{r}_{B_2D_2} &= (l_{B_2H_2} + s_j)^2, & \text{for } j = 2, \\
\mathbf{r}_{B_3D_3}^T \mathbf{r}_{B_3D_3} &= (l_{B_3H_3} + s_j)^2, & \text{for } j = 3, \\
\mathbf{r}_{B_4D_4}^T \mathbf{r}_{B_4D_4} &= (l_{B_4H_4} + s_j)^2, & \text{for } j = 4, \\
\mathbf{r}_{B_5D_5}^T \mathbf{r}_{B_5D_5} &= (l_{B_5H_5} + s_j)^2, & \text{for } j = 5.
\end{align*}$$

(13)

Thus it becomes possible to calculate the relative elongations $s_j$ of links $d_{ij}$, $i = (1)9$, relative to their constructional distances. Elongations $s_j$ depend on given parameters $q$ and $u$: $s_j(q$ and $u_p)$.

In the practical application of the measuring instrument for determining the rotation and translation of the steered wheel stub axle it is needed to measure the coordinates of points $D_i$, $i = (1)3$ as well as $H_j$, $j = (1)9$, for constructional suspension configuration; links elongations $s_j(q$ and $u_p)$, measured by the sensors are also needed.

Determination of the steered wheel stub axle rotation and translation using a measuring instrument boils down to solving of the inverse problem. In this case, at given elongations $s_j(q$ and $u_p)$ of the links $d_{ij}$, $j = (1)9$ and constructional positions of the points $D_i$, $i = (1)3$ as well as the additional points, eg. $D_4 = B_7$ and $D_5 = B_6$, the coordinates of these points, as the functions of parameters $q$ and $u$, should be determined from the systems of equations:

for point $D_1$:

$$\mathbf{r}_{D_1H_1}^T \mathbf{r}_{D_1H_1} = (l_{D_1H_1} + s_j)^2, \quad \text{for } j = 1, j = 4,$$

(14)

for point $D_2$:

$$\mathbf{r}_{D_2H_2}^T \mathbf{r}_{D_2H_2} = (l_{D_2H_2} + s_j)^2, \quad \text{for } j = 2, j = 6,$$

(15)

for point $D_3$:

$$\mathbf{r}_{D_3H_3}^T \mathbf{r}_{D_3H_3} = (l_{D_3H_3} + s_j)^2, \quad \text{for } j = 3, j = 8,$$

(16)

The coordinates of point $D_4$ are obtained by solving the system of equations:
The coordinates of the point \( D_5 \equiv B_6 \) are calculated similarly as the point \( D_4 \). The above systems of equations were solved using triangulation method. The steering and camber angles \( \delta_d \) and \( \gamma_d \) respectively, as well as the relative changes in lateral position of the wheel \( \Delta K_{yd} \) were calculated in the same way as it is shown in paragraph 4.

9. Constructional coordinates of the characteristic points of the measuring instrument mechanism

The coordinates of centres of the joints: \( D_i \), \( i = 1 \) to 3 as well as coordinates of the points \( D_4 \), \( D_5 \) belonging to the wheel rotation axis assigned to the constructional configuration of the suspension are given below in millimetres: \( D_1 \) (50.0, 815.0, 30.0); \( D_2 \) (25.0, 835.0, -90.0); \( D_3 \) (-50.0, 855.0, 200.0); \( H_1 \) (90.0, 1165.0, -40.0); \( H_2 \) (10.0, 1164.0, -70.0); \( H_3 \) (-40.0, 1169.0, 270.0); \( H_4 \) (45.0, 1171.0, -90.0); \( H_5 \) (110.0, 1162.0, 10.0); \( H_6 \) (30.0, 1167.0, -95.0); \( H_7 \) (30.0, 1166.0, -20.0); \( H_8 \) (-85.0, 1163.0, 220.0); \( H_9 \) (-20.0, 1164.0, 320.0); \( D_4 \) (1.0, 747.0, 0.6); \( D_5 \) (0.5, 647.0, 1.1).

10. Elongations of the measuring instrument links

The following graphs show sample dependences of the elongations of links of the measuring instrument on the steering rack displacement- \( u_p \) and the suspension deflection- \( q \).

a)  

b)  

c)  

d)  

e)  

f)
11. Suspension characteristics
Suspension characteristics obtained via numerical simulation using the proposed measuring instrument are show in the Figure 4.

**Figure 4.** Sample dependences of the measuring instrument links elongations $s_j$ on suspension deflection- $q$ and steering rack displacement- $u_p$. 

![Figure 4](image)
12. Measurement uncertainty analysis

Standard uncertainty of the measurement was determined on the basis of numerical simulation using formula shown in equation (17):

$$
\Delta f = \pm \left[ \sum_{j=1}^{9} \left( \frac{\partial f}{\partial s_j} \cdot \Delta s_j \right)^2 \right]^{1/2}
$$

Figure 5. Dependences of steering and camber angles: $\delta_d$, $\gamma_d$, respectively and relative lateral displacement of the wheel centre $\Delta K_{yd}$ on suspension deflection- $q$ and steering rack displacement- $u_p$, determined on the basis of the suspension kinematics solution.

where:
- $f$ - respectively:
  - steering angle $\delta_d(s_1,\ldots,s_9)$;
  - camber angle $\gamma_d(s_1,\ldots,s_9)$;
  - relative lateral displacement of the wheel centre $\Delta K_{yd}(s_1,\ldots,s_9)$;
- $s_j$ - $j$-th displacement sensor elongation
- $\Delta s_j$ - an accuracy of $j$-th displacement sensor, $\Delta s_j = 0,02$ mm was assumed for $j=1,\ldots,9$.

The sample values of measurement uncertainty were shown below in table 1.

Table 1. The sample maximum values of the measurement uncertainty for the steering $\delta_d$ and camber $\gamma_d$ angels, as well as for the relative lateral displacement of the wheel centre $\Delta K_{yd}$.

| Maximum value of measurement uncertainty in case of the maximum left turn and constructional load of the vehicle | $\Delta \delta$ [°] | $\Delta \gamma$ [°] | $\Delta \Delta K_{yd}$ [mm] |
|---|---|---|---|
| $\pm 0,09$ | $\pm 0,03$ | $\pm 0,11$ |
| Maximum value of measurement uncertainty in case of the return of steering wheels to straight ahead position and constructional load of the vehicle | $\pm 0,10$ | $\pm 0,02$ | $\pm 0,02$ |
| Maximum value of measurement uncertainty in case of the maximum right turn and constructional load of the vehicle | $\pm 0,04$ | $\pm 0,01$ | $\pm 0,11$ |

13. Conclusions

The mechanism of the proposed instrument consists of nine links with optical elongation sensors built-in. The structure of it enables its connection to the stub axle via three joints. Each of them is a conjunction of three joints with a common centre, each with three degrees of freedom.
A four link suspension and rack- and- pinion steering system kinematics problem was solved in the paper. The basic kinematic characteristics of steering and camber angles and the relative lateral displacements of the wheel centre, as well as the links lengths changes of the measuring mechanism were determined. Changes of links lengths were taken into account in the equations of geometric constraints of the instrument, the quantities needed to make up analogous kinematic characteristics of suspension and steering system were calculated. The susceptibility of coupling measuring instrument and car’s bod was not taken into account during the presented analysis, while the ideal stiff coupling was assumed.

The analysis of kinematics of steering wheels suspensions mechanisms should be done using spatial characteristics, which are described by functions, which independent variables are the suspension deflection and steering rack displacement. Cross sections of these characteristics can be used for comparison with experimentally determined characteristics.

Basing on the results of the computer simulation and the determined measurement uncertainty, it was found, that- considering practical use- the developed device enables an accurate determination of suspension characteristics. Moreover, there is always a solution of the system of equations of the geometrical constraints of the measuring instrument mechanism, in contrast to an instrument based on a structure of the Stewart platform [13].

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