A further look at the truncated pentagonal number theorem

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Abstract. In this paper, we study the asymptotic behavior of the following function

\[ M_k(n) := (-1)^{k-1} \sum_{j=0}^{k-1} \left( p(n - j(3j + 1)/2) - p(n - j(3j + 5)/2 - 1) \right), \]

which arises from Andrews and Merca’s truncated pentagonal number theorem.

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1. Introduction

In [3], Andrews and Merca studied a truncated version of Euler’s pentagonal number theorem. The motivation of their work arises from the non-negativity of the following function:

\[ M_k(n) := (-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j (p(n - j(3j + 1)/2) - p(n - j(3j + 5)/2 - 1)), \quad (1.1) \]

where \( n \) and \( k \) are positive integers, and \( p(n) \) denotes the number of partitions of \( n \) [2]. Andrews and Merca also gave a partition-theoretic interpretation of \( M_k(n) \). Namely, it denotes the number of partitions of \( n \) in which \( k \) is the least integer that is not a part and there are more parts > \( k \) than there are < \( k \).

Their proof of the non-negativity of \( M_k(n) \) relies on a clever reformulation of the generating function of \( M_k(n) \). Namely, if we put \( M_k(0) = (-1)^{k-1} \), then

\[ M_k(q) := \sum_{n \geq 0} M_k(n) q^n = \frac{(-1)^{k-1}}{(q; q)_\infty} \sum_{j=0}^{k-1} (-1)^j q^{j(3j+1)/2} (1 - q^{2j+1}) \]

\[ = (-1)^{k-1} + \sum_{n \geq 1} q^{(2) + (k+1)n} \frac{[n-1]}{[k-1]_q}, \quad (1.3) \]

where

\[ (A: q)_n = \prod_{j=0}^{n-1} (1 - Aq^j), \]

and

\[ \left[ \begin{array}{c} A \\ B \end{array} \right]_q = \begin{cases} 0, & \text{if } B < 0 \text{ or } B > A, \\ \frac{(q)_A}{(q)_B (q)_A - q}, & \text{otherwise}. \end{cases} \]

One immediately sees that the non-negativity of \( M_k(n) \) for \( n \geq 1 \) follows from (1.3).
Now, if one fixes \( k \) and computes some values of \( M_k(n) \), one may notice that \( M_k(n) \) grows rapidly as \( n \) becomes large. This stimulates us to study the asymptotic behavior of \( M_k(n) \). In this paper, we shall show

**Theorem 1.1.** Let \( \epsilon > 0 \) be arbitrary small. Then as \( n \to \infty \), we have, for \( k \ll n^{\frac{1}{8} - \epsilon} \),

\[
M_k(n) = \frac{\pi}{12\sqrt{2}}kn^{-\frac{3}{2}}e^{\frac{2\pi}{\sqrt{3}}n^{-\frac{3}{4}}} + O \left( k^3n^{\frac{3}{2}}e^{\frac{2\pi}{\sqrt{3}}n^{-\frac{3}{4}}} \right).
\]  

**Remark 1.1.** Here the assumption \( k \ll n^{\frac{1}{8} - \epsilon} \) ensures that \( O \left( k^3n^{\frac{3}{2}}e^{\frac{2\pi}{\sqrt{3}}n^{-\frac{3}{4}}} \right) \) is indeed an error term.

Apparently, (1.4) demonstrates the positivity of \( M_k(n) \) for sufficiently large \( n \) if we fix \( k \). In fact, this asymptotic formula allows us to have a better understanding of \( M_k(n) \). The interested reader may also compare (1.4) with the celebrated asymptotic expression for \( p(n) \) due to Hardy and Ramanujan [6]

\[
p(n) \sim \frac{1}{2\sqrt{6\pi n}} e^{\frac{2\pi}{\sqrt{3}}n^{\frac{3}{2}}}.
\]

2. Proof

Throughout this section, we let \( q = e^{2\pi i \tau} \) with \( \tau = x + iy \in \mathbb{H} \) (i.e. \( y > 0 \)). We also put

\[
y = \frac{1}{2\sqrt{6n}} \quad \text{and} \quad M = \sqrt{\left( \frac{12}{12 - \pi^2} \right)^2 - 1}.
\]

Note that we may take \( M \) to be other (positive) absolute constant. However, we choose the above value for computational convenience.

2.1. Asymptotics of \( \mathcal{M}_k(q) \) near \( q = 1 \). We first estimate \( \mathcal{M}_k(q) \) near \( q = 1 \).

**Lemma 2.1.** For \( |x| \leq My \), we have, as \( n \to \infty \) (and hence \( y \to 0^+ \)),

\[
\mathcal{M}_k(q) = -2e^{\frac{2\pi}{\sqrt{3}}k\tau^2}e^{\frac{2\pi}{\sqrt{6}}} + O \left( k^3n^{-\frac{3}{2}}e^{\frac{2\pi}{\sqrt{3}}n^{-\frac{3}{4}}} \right).
\]  

**Proof.** We have

\[
1 - q^{2j+1} = 1 - e^{2(2j+1)\pi i \tau} = -2(2j + 1)\pi i \tau + \mathcal{E}_j,
\]

where

\[
|\mathcal{E}_j| = \left| e^{2(2j+1)\pi i \tau} - 1 - 2(2j + 1)\pi i \tau \right|
\]

\[
\leq e^{2|2(2j+1)\pi i \tau|} - 1 - |2(2j + 1)\pi i \tau|
\]

\[
\leq 4(2j + 1)^2|\tau|^2,
\]

since \( |2(2j + 1)\pi i \tau| < 1 \) for \( 0 \leq j \leq k - 1 \) (which is ensured by the assumption \( k \ll n^{\frac{1}{8} - \epsilon} \)) whereas \( e^x - 1 - x \leq x^2 \) when \( 0 < x < 1 \). Hence,

\[
\sum_{j=0}^{k-1} (-1)^{j} (1-q^{2j+1}) = \sum_{j=0}^{k-1} (-1)^{j} (-2(2j + 1)\pi i \tau) + \mathcal{E}
\]

\[
= (-1)^{k} 2\pi i k \tau + \mathcal{E},
\]
where
\[ |\mathcal{E}| \leq \sum_{j=0}^{k-1} 4(2j + 1)^2 |\tau|^2 \ll k^3 y^2. \]

Consequently, we have
\[ \sum_{j=0}^{k-1} (-1)^j q^{(3j+1)/2}(1 - q^{2j+1}) = (-1)^k 2\pi ik\tau + O(k^3 y^2). \quad (2.2) \]

Furthermore, we know from the modular inversion formula for Dedekind’s eta-function (cf. [7, p. 121, Proposition 14]) that
\[ \left( \frac{q}{q} \right)_{\infty} = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i}{4\tau}} \left( 1 + O \left( e^{-\frac{2\pi}{\tau}} \right) \right), \quad (2.3) \]

where the square root is taken on the principal branch, with \( z^{1/2} > 0 \) for \( z > 0 \). Hence,
\[ \left( \frac{q}{q} \right)_{\infty} = \sqrt{-i\tau} e^{-\frac{\pi i}{4\tau}} + O \left( y^2 e^{-\frac{\pi y}{4\tau}} \right) \quad (2.4) \]

Finally, (2.1) follows from (1.2), (2.2), (2.4) and the fact that
\[ \Im \left( \frac{-1}{\tau} \right) = \frac{y}{x^2 + y^2} \leq \frac{1}{y}. \]
This finishes the proof of Lemma 2.1. \( \square \)

2.2. Asymptotics of \( \mathcal{M}_k(q) \) away from \( q = 1 \). We next estimate \( \mathcal{M}_k(q) \) away from \( q = 1 \).

**Lemma 2.2.** For \( My < |x| \leq \frac{1}{2} \), we have, as \( n \to \infty \) (and hence \( y \to 0^+ \)),
\[ \mathcal{M}_k(q) \ll kn^{-\frac{1}{2}} e^{-\frac{\pi y}{2\tau}}. \quad (2.5) \]

**Proof.** We first have the following trivial bound
\[ \left| \sum_{j=0}^{k-1} (-1)^j q^{(3j+1)/2}(1 - q^{2j+1}) \right| \leq 2k. \quad (2.6) \]

On the other hand,
\[ \log \left( \frac{1}{\left( \frac{q}{q} \right)_{\infty}} \right) = - \sum_{n \geq 1} \log(1 - q^n) \]
\[ = \sum_{n \geq 1} \sum_{m \geq 1} \frac{q^{nm}}{m} \]
\[ = \sum_{m \geq 1} \frac{q^m}{m(1 - q^m)}. \]

Hence,
\[ \left| \log \left( \frac{1}{\left( \frac{q}{q} \right)_{\infty}} \right) \right| \leq \sum_{m \geq 1} \frac{|q|^m}{m|1 - q^m|} \]
\[ \leq \sum_{m \geq 1} \frac{|q|^m}{m(1 - |q|^m)} - \frac{|q|}{1 - |q|} + \frac{|q|}{|1 - q|}. \]
It follows from (2.3) that
\[
\frac{1}{(|q|; |q|)_{\infty}} = \sqrt{q} e^{\frac{\pi i}{12}} \left( 1 + O \left( e^{-\frac{\pi y}{2}} \right) \right). \tag{2.8}
\]
Furthermore, we know from the fact \(|x| > My\) that \(\cos(2\pi x) < \cos(2\pi My) \leq 1\).
Hence,
\[
|1 - q|^2 = 1 - 2e^{-2\pi y} \cos(2\pi x) + e^{-4\pi y} > 1 - 2e^{-2\pi y} \cos(2\pi My) + e^{-4\pi y}.
\]
Computing the Taylor expansion around \(y = 0\) yields
\[
\left| \frac{1}{(q; q)_{\infty}} \right| \ll \sqrt{y} \exp \left( \frac{1}{y} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1+M^2}} \right) \right) \right). \tag{2.10}
\]
Finally, (2.5) follows from (2.6) and (2.10). \(\square\)

2.3. Applying Wright’s circle method. Let \(C\) denote the circle \(q = e^{2\pi i \tau} = e^{2\pi i (x+iy)}\) where \(x \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \). Cauchy’s integral formula tells us that
\[
M_k(n) = \frac{1}{2\pi i} \int_C \frac{M_k(q)}{q^{n+1}} dq
= \frac{1}{2\pi i} \int_C \left( e^{2\pi i \tau} \right) e^{-2\pi i \tau} \, dx
= \int_{|x| \leq My} + \int_{My < |x| \leq \frac{1}{2}} =: I_1 + I_2,
\tag{2.11}
\]
where the integrands in \(I_1\) and \(I_2\) are both \(M_k \left( e^{2\pi i \tau} \right) e^{-2\pi i \tau}.
\]
We first compute \(I_1\), which contributes to the main term. Our evaluation relies on a function \(P_s(u)\) defined by Wright [8]. For fixed \(M > 0\) and \(u \in \mathbb{R}_{>0}\), let
\[
P_s(u) := \frac{1}{2\pi i} \int_{1-Mi}^{1+Mi} v^s e^{u(v+\frac{1}{2})} \, dv.
\]
Wright [8, p. 138, Lemma XVII] showed that this function can be rewritten in terms of the \(I\)-Bessel function up to an error term.

Lemma 2.3 (Wright). We have, as \(u \to \infty\),
\[
P_s(u) = I_{s-1}(2u) + O(e^u), \tag{2.12}
\]
where \(I_\ell\) denotes the usual \(I\)-Bessel function of order \(\ell\).
We also recall that the asymptotic expansion of \(I_\ell(x)\) (cf. [1, p. 377, (9.7.1)]) states that, for fixed \(\ell\), when \(|\arg x| < \frac{\pi}{2}\),
\[
I_\ell(x) \sim e^x \sqrt{\frac{2\pi x}{\sqrt{2\pi}x}} \left( 1 - \frac{4\ell^2 - 1}{8x} + \frac{(4\ell^2 - 1)(4\ell^2 - 9)}{2!(8x)^2} - \cdots \right). \tag{2.13}
\]
It follows from Lemma 2.1 that
\[
I_1 = \int_{|x| \leq M_y} e^{-2n \pi i x} \left( -2e^{\frac{\pi i}{4}} \pi k \tau^{\frac{3}{2}} e^{\frac{2n \pi i}{\sqrt{\tau}}} + O \left(k^3 n - \frac{2}{\pi} e^{\frac{2n \pi i}{\sqrt{\tau}}}ight) \right) dx.
\]

Making the change of variables \( v = -i \tau / y \) yields
\[
I_1 = \int_{1-M_v}^{1+M_i} (-iy)e^{2n \pi y v} \left( -2e^{\frac{\pi i}{4}} \pi k (iy)^{\frac{3}{2}} e^{\frac{2n \pi y v}{\sqrt{6}}} + O \left(k^3 n - \frac{2}{\pi} e^{\frac{2n \pi y v}{\sqrt{6}}}ight) \right) dv
\]
\[
= 2^{\frac{3}{2}} 3^{-\frac{7}{4}} \pi^2 k n^{-\frac{3}{4}} P_{\frac{1}{2}} \left(\frac{\pi \sqrt{n}}{\sqrt{6}}\right) + O \left(k^3 n - \frac{2}{\pi} e^{\frac{2n \pi y v}{\sqrt{6}}}ight)
\]
\[
= 2^{\frac{7}{4}} 3^{-\frac{7}{4}} \pi^2 k n^{-\frac{3}{4}} I_{\frac{1}{2}} \left(\frac{2\pi \sqrt{n}}{\sqrt{6}}\right) + O \left(k^3 n - \frac{2}{\pi} e^{\frac{2n \pi y v}{\sqrt{6}}}ight)
\]
\[
= \frac{\pi}{12\sqrt{2}} k n^{-\frac{3}{4}} e^{\frac{2n \pi y v}{\sqrt{6}}} + O \left(k^3 n - \frac{2}{\pi} e^{\frac{2n \pi y v}{\sqrt{6}}}ight).
\] (2.14)

We now evaluate \( I_2 \). It follows from Lemma 2.2 that
\[
I_2 \ll \int_{M_y < |x| \leq \frac{1}{4}} kn^{-\frac{1}{4}} e^{\frac{\pi \sqrt{n}}{2\sqrt{\pi}}} e^{\frac{\pi \sqrt{n}}{\sqrt{6}}} dx \ll kn^{-\frac{1}{4}} e^{\frac{3n \pi x}{2\sqrt{6}}}.
\] (2.15)

Consequently, we know from (2.14) and (2.15) that as \( n \to \infty \),
\[
M_k(n) = \frac{\pi}{12\sqrt{2}} k n^{-\frac{3}{4}} e^{\frac{2n \pi y v}{\sqrt{6}}} + O \left(k^3 n - \frac{2}{\pi} e^{\frac{2n \pi y v}{\sqrt{6}}}ight).
\]

This is our main result.

3. Closing remarks

There are more truncated theta series identities. Two interesting examples are due to Guo and Zeng [5]:
\[
\frac{(-q; q)^{\infty}}{(q; q)^{\infty}} \sum_{j=-k}^{k} (-1)^{j} q^{j^{2}} = 1 + (-1)^{k} \sum_{n \geq k+1} \frac{(-q; q)_{k} (-1; q)_{n-k} q^{(k+1)n}}{(q; q)_{n}} \left[\frac{n-1}{k}\right],
\] (3.1)

and
\[
\frac{(-q; q^{2})^{\infty}}{(q^{2}; q^{2})^{\infty}} \sum_{j=0}^{k-1} (-1)^{j} q^{j(2j+1)} (1 - q^{2j+1})
\]
\[
= 1 + (-1)^{k-1} \sum_{n \geq k} \frac{(-q; q^{2})_{k} (-q; q^{2})_{n-k} q^{2(k+1)n-k}}{(q^{2}; q^{2})_{n}} \left[\frac{n-1}{k-1}\right].
\] (3.2)

Let \( \overline{p}(n) \) denote the number of overpartitions of \( n \) (i.e., partitions of \( n \) where the first occurrence of each distinct part may be overlined) and let \( \text{pod}(n) \) denote the number of partitions of \( n \) wherein odd parts are not repeated. The above two identities respectively reveal the non-negativity of the following two functions (the notation of which is due to Andrews and Merca [4]) for \( n, k \geq 1 \):
\[
\overline{M}_{k}(n) := (-1)^{k} \sum_{j=-k}^{k} (-1)^{j} \overline{p}(n - j^{2}),
\] (3.3)
and

\[ MP_k(n) := (-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \left( \text{pod}(n-j(2j+1)) - \text{pod}(n-(j+1)(2j+1)) \right). \] (3.4)

In [4], to answer a question of Guo and Zeng [5, p. 702], Andrews and Merca also presented the partition-theoretic interpretations of \( M_k(n) \) and \( MP_k(n) \):

- \( M_k(n) \) denotes the number of overpartitions of \( n \) in which the first part larger than \( k \) appears at least \( k + 1 \) times;
- \( MP_k(n) \) denotes the number of partitions of \( n \) in which the first part larger than \( 2k - 1 \) is odd and appears exactly \( k \) times whereas all other odd parts appear at most once.

Using similar arguments to that in Sect. 2, we are also able to show the asymptotic behaviors of \( M_k(n) \) and \( MP_k(n) \).

**Theorem 3.1.** Let \( \epsilon > 0 \) be arbitrary small. Then as \( n \to \infty \), we have, for \( k \ll n^{1\over 12} - \epsilon \),

\[ M_k(n) = \frac{1}{8} n^{-1} e^{\pi \sqrt{n}} + O \left( k^3 n^{-7} e^{\pi \sqrt{n}} \right). \] (3.5)

**Remark 3.1.** It is interesting to point out that the main term of \( M_k(n) \) is identical to the main term in the asymptotic expression of \( \mathcal{P}(n) \).

**Theorem 3.2.** Let \( \epsilon > 0 \) be arbitrary small. Then as \( n \to \infty \), we have, for \( k \ll n^{1\over 8} - \epsilon \),

\[ MP_k(n) = \frac{\pi}{16} k n^{-{3\over 2}} e^{\pi {\sqrt{2}\over 4}} + O \left( k^3 n^{-7} e^{\pi {\sqrt{2}\over 4}} \right). \] (3.6)

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