A Simple Proof of Siegel’s Theorem Using Mellin Transform

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Abstract

In this paper, we present a simple analytic proof of Siegel’s theorem that concerns the lower bound of $L(1, \chi)$ for primitive quadratic $\chi$. Our new method compares an elementary lower bound with an analytic upper bound obtained by the inverse Mellin transform of $\Gamma(s)$.

Keywords: Analytic number theory, Dirichlet L-function, Mellin transform, Siegel’s theorem, Siegel-Walfisz theorem

I. Introduction

In 1935, Siegel introduces the function

$$f(s) = (s) L(s, \chi_1) L(s, \chi_2) L(s, \chi_1 \chi_2)$$

where $\chi_1$ and $\chi_2$ are primitive quadratic characters modulo $q_1$ and $q_2$ respectively. By exploring its algebraic properties, he shows that a very strong lower bound can be established for $L(1, \chi)$:

**Theorem 1 (Siegel).** For all $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$L(1, \chi) > C(\varepsilon) q^{-\varepsilon}$$

holds for any primitive quadratic $\chi$ modulo $q$.

Although the statement of Theorem 1 is analytic, it leads to strong conclusions in the distribution of prime numbers in arithmetic progressions. Using this result, Walfisz improved the zero-free region of $L(s, \chi)$ to obtain the prime number theorem for arithmetic progressions in the following form:

**Theorem 2 (Siegel-Walfisz).** Let $\pi(x; q, a)$ denotes the number of primes $\leq x$ that are $\equiv a$ (mod $q$). Then for all $A > 0$, there exists $C_A > 0$ such that the following estimate

$$\pi(x; q, a) = \frac{1}{\phi(q)} \int_2^x \frac{dt}{\log t} + O_A \left\{ x \exp \left( -C_A \sqrt{\log x} \right) \right\}$$

holds when $(a, q) = 1$ and $q \leq (\log x)^A$. 

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This result is very powerful in additive problems concerning primes. For instance, Vinogradov deduces from Theorem 2 that every large odd integer is a sum of three primes. Under Theorem 2, Mirsky[5] shows that every large integer is a sum of a prime and a k-free integer.

The original proof of Theorem 1 uses algebraic number theory. Later in 1949, Estermann[2] obtained a simple proof using purely analytic methods. Few decades after that, Goldfeld[3][4] gave a much more simplified analytic proof using contour integration. In this paper, we propose a new contour-integration proof of Theorem 1 based on the inverse Mellin transform of \( \Gamma(s) \). In particular, the new approach simplifies Goldfeld’s method because it uses Abelian summation instead of Cesàro summation\(^1\).

II. Analytic Lemmas

From now on, \( s = \sigma + it \) always denotes a complex number with an abscissa of \( \sigma \) and an ordinate of \( t \).

**Lemma 1** (Phragmén-Lindelöf). If \( \phi(s) \) is regular and \( O(e^{\varepsilon|t|}) \), for any \( \varepsilon > 0 \), in the strip \( \sigma_1 \leq \sigma \leq \sigma_2 \), and

\[
\phi(\sigma_1 + it) \ll |t|^{k_1}, \quad \phi(\sigma_2 + it) \ll |t|^{k_2}
\]

then \( \phi(s) \ll |t|^\max(k_1, k_2) \) holds uniformly in \( \sigma_1 \leq \sigma \leq \sigma_2 \).

*Proof.* See §5.65 of [8]. Q.E.D.

**Lemma 2.** Let \( f(s) \) be defined as in (1), then for all \( \varepsilon > 0 \). The estimate

\[
f(s) \ll \varepsilon (q_1 q_2)^{1+\varepsilon} |t|^{2+\varepsilon}
\]

holds uniformly in \( \sigma \geq 0 \).

*Proof.* It is well known that when \( \sigma \) lies in a fixed strip and \( |t| \to \infty \), \( \zeta(s) \) and \( L(s, \chi) \) satisfies the following asymptotic functional equations\(^2\):

\[
\zeta(s) \ll |t|^{1/2-\sigma}\zeta(1-s)
\]

\[
L(s, \chi) \ll (q|t|)^{1/2-\sigma}|L(1-s, \chi)|
\]

where \( \chi \) is a primitive character modulo \( q \). For. Since \( \zeta(s) \) and \( L(s, \chi) \) converge absolutely for all \( \sigma \geq 1 + \varepsilon \), we see that when \( \sigma = -\varepsilon \), (5) and (6) can be simplified into

\[
\zeta(s) \ll \varepsilon |t|^{1/2+\varepsilon}
\]

\[
L(s, \chi) \ll \varepsilon (q|t|)^{1/2+\varepsilon}
\]

Plugging (7) and (8) into (1), we see that (4) holds for \( \sigma = -\varepsilon/4 \), and finally we can apply Lemma 1 to extend this estimate to \( \sigma \geq -\varepsilon/4 \). Q.E.D.

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\(^*\)This is often known as the ternary Goldbach’s conjecture. See §25 and §26 of [1] for an account of Vinogradov’s proof.

\(^1\)A detailed description of these summation methods is accessible in §5.2 of [6]

\(^2\)See §10.1 of [6] for a full derivation.
Lemma 3. Let $\Gamma(s)$ denote the Gamma function. Then the estimate
\[ \Gamma(s) \ll |t|^{\sigma - 1/2} e^{-\pi |t|/2} \tag{9} \]
holds whenever $\sigma$ lies in a fixed interval.

Proof. By Stirling’s formula, we know that when $\sigma$ lies in a fixed interval
\[
\log \Gamma(s) = \left( s - \frac{1}{2} \right) \log s - s + O(1)
\]
\[\begin{align*}
&= \left( s - \frac{1}{2} \right) \log \left( \frac{s}{\iota t} \right) + \left( s - \frac{1}{2} \right) \log \left( 1 + \frac{\sigma}{\iota t} \right) - s + O(1) \\
&= \left( s - \frac{1}{2} \right) \log \left( \frac{s}{\iota t} \right) - \frac{s}{\iota t} + O(1)
\end{align*}\]
Taking real parts on both side, we see that as $t \to +\infty$ there is
\[\log |\Gamma(s)| = \left( \sigma - \frac{1}{2} \right) \log t - \pi t/2 + O(1) \tag{10}\]
Therefore, exponentiating on both side of (10) yields the desired result. Q.E.D.

Lemma 4. For all $y > 0$ there is
\[ e^{-y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^{-s} \Gamma(s) ds \tag{11}\]

Proof. The result follows directly by applying Mellin’s inversion formula to
\[ \Gamma(s) = \int_0^\infty y^{s-1} e^{-y} dy \tag{12}\]
Q.E.D.

Lemma 5. For $0 < \sigma < 1$, we have $\zeta(\sigma) < 0$.

Proof. By partial summation, we have
\[ \zeta(\sigma) = \frac{\sigma}{\sigma - 1} - \sigma \int_1^\infty x^{\sigma} \{x\} \, dx \leq \frac{\sigma}{\sigma - 1} \tag{13}\]
The right hand side immediately concludes the proof. Q.E.D.

Lemma 6. Let $\chi$ be nonprincipal character modulo $q$ then $L(1, \chi) \ll \log q$. 

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Proof. Using the fact that $|\chi(n)| \leq 1$, we have

$$|L(1, \chi)| \leq \left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| + \left| \sum_{n > N} \frac{\chi(n)}{n} \right|$$

$$\leq \sum_{n \leq N} \frac{1}{n} + \left[ \frac{1}{x} \sum_{N < n \leq x} \chi(n) \right] \int_{N}^{\infty} \left( \sum_{N < n \leq t} \chi(n) \right) \frac{dt}{t^2}$$

$$\leq 1 + \sum_{2 \leq n \leq N} \frac{\int_{n-1}^{n} \frac{dt}{t} + \max_{x} \sum_{N < n \leq x} \chi(n)}{\int_{N}^{\infty} \left( \frac{1}{N} + \int_{N}^{\infty} \frac{dt}{t^2} \right)}$$

$$\leq 1 + \log N + \frac{2\phi(q)}{N} \ll \log N + \frac{q}{N}$$

Setting $N = q$ gives the desired result. Q.E.D.

Lemma 7. Let $\chi$ be a quadratic character such that $L(s, \chi)$ is free of real zeros in $s > 1 - \varepsilon$. Then $L(\beta, \chi) > 0$ holds for any $1 - \varepsilon < \beta < 1$.

Proof. Since $L(s, \chi)$ is continuous in $[1 - \varepsilon, 1]$ and $L(1, \chi) > 0$, the result immediately follows. Q.E.D.

Lemma 8. For any $\varepsilon > 0$ there exists a primitive quadratic $\chi_1$ modulo $q_1$ and $1 - \varepsilon < \beta < 1$ such that $f(\beta) \leq 0$ holds for all quadratic $\chi_2$ modulo $q_2$.

Proof. On one hand, if no $\chi$ can be found such that $L(s, \chi)$ has a zero in $(1 - \varepsilon, 1)$, then it follows from Lemma 5 and Lemma 7 that $f(\beta) < 0$ for all $1 - \varepsilon < \beta < 1$.

On the other hand, if we are unable to find a quadratic primitive $\chi$ such that $L(s, \chi)$ does possess a real zero in $(1 - \varepsilon, 1)$, then let $\chi_1 = \chi$ and $\beta$ be the real zero so that $f(\beta) = 0$. Consequently for every $\varepsilon > 0$, there exists a primitive quadratic $\chi_1$ modulo $q_1$ and $1 - \varepsilon < \beta < 1$ such that $f(\beta) \leq 0$. Q.E.D.

### III. Proof of Siegel’s Theorem

Similar to Goldfeld’s method[3], our approach also studies the partial sum of $f(s)$

$$A(x, \beta) = \sum_{n \leq x} \frac{a_n}{n^\beta}$$

where $a_n$ denote the Dirichlet series coefficient of $f(s)$ and $1 - \varepsilon < \beta < 1$ It follows from literature that $a_n \geq 0$ and $a_1 = 1$, so we have $A(x, \beta) \geq 1$ when $x \geq 1$. In addition, because the exponential decay function satisfies

$$e^{-n/x} \begin{cases} \geq 0 & n > x \\ \geq e^{-1} & n \leq x \end{cases}$$

we also have

$$1 \leq A(x, \beta) \leq e \sum_{n \geq 1} \frac{a_n}{n^\beta} e^{-x/n}$$

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§This is an auxiliary result used to prove Dirichlet’s theorem. See §4.3 of [6] for a derivation

§See §21 of [1] for a detailed account
Now, we apply Lemma 4 to the exponential function in (15) so that
\[ e^{-1} \leq \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^s \Gamma(s) f(s + \beta) ds \triangleq I \] (16)

To estimate the integral, we move the path of integration to \( \sigma = -\beta \) so that it follows from Lemma 2 that
\[ I = x^{-1-\beta} \Gamma(1 - \beta) \lambda + f(\beta) + x^{-\beta} \int_{-\beta - i\infty}^{-\beta + i\infty} |\Gamma(s) f(s + \beta)| ds \] (17)
\[ = x^{-1-\beta} \Gamma(1 - \beta) \lambda + f(\beta) + O \left\{ x^{-\beta} (q_1 q_2)^{1+\epsilon} \int_0^\infty t^{2+\epsilon} |\Gamma(-\beta + it)| dt \right\} \] (18)
where \( \lambda = L(1, \chi_1) L(1, \chi_2) L(1, \chi_1 \chi_2) \) is the residue of \( f(s) \) at \( s = 1 \). Since \( \Gamma(s+1) = s \Gamma(s) \), the remaining integral will be bounded by
\[ \int_0^\infty t^{2+\epsilon} |\Gamma(-\beta + it)| dt \ll \frac{1}{1 - \beta} \int_0^\infty t^{5/2 - \beta + \epsilon} e^{-\pi t/2} dt \ll \frac{1}{1 - \beta} \]

Now, if we choose \( \beta \) and \( \chi \) according to Lemma 8, then we can ignore the \( f(\beta) \) term to simplify (18) into
\[ 1 \ll x^{-1-\beta} \lambda + x^{-\beta} q_1^{1+\epsilon} \] (19)
in which all \( \beta \) and \( q_1 \) terms in the coefficients are absorbed into \( \ll \). To simplify this even further, we set \( x^\beta = q_2^{1+\epsilon} \) for some small \( c > 0 \) so that the left hand side of (19) will still be positive even after subtracted by \( x^{-\beta} q_2^{1+\epsilon} \). To further simplify the right hand side, we apply Lemma 6 to \( \lambda \) so that for \( q_2 > q_1(\epsilon) \) there is
\[ 1 \ll x^{-1-\beta} \lambda \ll x^{-1-\beta} (\log q_1)(\log q_1 q_2)L(1, \chi_2) \ll x^\epsilon (\log q_2)L(1, \chi_2) \] (20)

Transforming this equation, we have
\[ L(1, \chi_2) \gg x^{-\epsilon} (\log q_2)^{-1} \gg q_2^{-\epsilon(1+\epsilon)/\beta} (\log q_2)^{-1} \gg q_2^{-\epsilon \frac{\log q_2}{\log q_2}} (\log q_2)^{-1} \]

Without loss of generality, we assume \( \epsilon \leq 1/2 \), so that
\[ L(1, \chi_2) \gg q_2^{-3\epsilon} (\log q_2)^{-1} \gg q_2^{-4\epsilon} \] (21)

This lower bound becomes Theorem 1 after a change of variable.

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