Quickest Change Detection of a Markov Process Across a Sensor Array

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Abstract

Recent attention in quickest change detection in the multi-sensor setting has been on the case where the densities of the observations change at the same instant at all the sensors due to the disruption. In this work, a more general scenario is considered where the change propagates across the sensors, and its propagation can be modeled as a Markov process. A centralized, Bayesian version of this problem, with a fusion center that has perfect information about the observations and a priori knowledge of the statistics of the change process, is considered. The problem of minimizing the average detection delay subject to false alarm constraints is formulated as a partially observable Markov decision process (POMDP). Insights into the structure of the optimal stopping rule are presented. In the limiting case of rare disruptions, we show that the structure of the optimal test reduces to thresholding the a posteriori probability of the hypothesis that no change has happened. We establish the asymptotic optimality (in the vanishing false alarm probability regime) of this threshold test under a certain condition on the Kullback-Leibler (K-L) divergence between the post- and the pre-change densities. In the special case of near-instantaneous change propagation across the sensors, this condition reduces to the mild condition that the K-L divergence be positive. Numerical studies show that this low-complexity threshold test results in a substantial improvement in performance over naive tests such as a single-sensor test or a test that wrongly assumes that the change propagates instantaneously.

Index Terms

Change-point problems, distributed decision-making, optimal fusion, quickest change detection, sensor networks, sequential detection.

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I. INTRODUCTION

An important application area for distributed decision-making systems is in environment surveillance and monitoring. Specific applications include: i) Intrusion detection in computer networks and security systems [2], [3], ii) monitoring cracks and damages to vital bridges and highway networks [4], iii) monitoring catastrophic faults to critical infrastructures such as water and gas pipelines, electricity connections, supply chains, etc. [5], iv) biological problems characterized by an event-driven potential including monitoring human subjects for epileptic fits, seizures, dramatic changes in physiological behavior, etc. [6], [7], v) dynamic spectrum access and allocation problems [8], vi) chemical or biological warfare agent detection systems to protect against terrorist attacks, vii) detection of the onset of an epidemic, and viii) failure detection in manufacturing systems and large machines. In all of these applications, the sensors monitoring the environment take observations that undergo a change in statistical properties in response to a disruption (change) in the environment. The goal is to detect the point of disruption (change-point) as quickly as possible, subject to false alarm constraints.

In the standard formulation of the change detection problem, studied over the last fifty years, there is a sequence of observations whose density changes at some unknown point in time and the goal is to detect the change-point as soon as possible. Two classical approaches to quickest change detection are: i) The minimax approach [9], [10], where the goal is to minimize the worst-case delay subject to a lower bound on the mean time between false alarms, and ii) The Bayesian approach [11]–[13], where the change-point is assumed to be a random variable with a density that is known a priori and the goal is to minimize the expected (average) detection delay subject to a bound on the probability of false alarm. Significant advances in both the minimax and the Bayesian theories of change detection have been made, and the reader is referred to [9]–[22] for a representative sample of the body of work in this area. The reader is also referred to [9], [16], [18], [22]–[27] for performance analyses of the standard change detection approaches in the minimax context, and [28], [29] in the Bayesian context.

Extensions of the above framework to the multi-sensor case where the information available for decision-making is distributed has also been explored [29]–[32]. In this setting, the observations are taken at a set of $L$ distributed sensors, as shown in Fig. I. The sensors may send either quantized/unquantized versions of their observations or local decisions to a fusion center, subject to communication delay, power and bandwidth constraints, where a final decision is made, based on all the sensor messages. In particular, in recent work [29]–[32], it is assumed that the statistical properties of all the sensors’ observations change at the same time. However, in many scenarios, it is more suitable to consider the case where the statistics of each sensor’s observations may change at different points in time. An application of this model is in the detection of pollutants and...
biological warfare agents, where the change process is governed by the movement of the agent through the medium under consideration. Numerous other examples, including those described earlier, can be modeled in the change process detection framework.

We consider a Bayesian version of this problem and assume that the point of disruption (that needs to be detected) is a random variable with a geometric distribution. We assume that the $L$ sensors are placed in an array or a line and they observe the change as it propagates through them. We model the inter-sensor delay with a Markov model and in particular, the focus is on the case where the inter-sensor delay is also geometric. More general inter-sensor delay models can be considered, but the case of a geometric prior has an intuitive and appealing interpretation due to the memorylessness property of the geometric random variable.

We study the centralized case, where the fusion center has complete information about the observations at all the $L$ sensors, the change process statistics, and the pre- and the post-change densities. This is applicable in scenarios where: i) the fusion center is geographically collocated with the sensors so that ample bandwidth is available for reliable communication between the sensors and the fusion center; and ii) the impact of the disruption-causing agent on the statistical dynamics of the change process and the statistical nature of the change so induced can be modeled accurately.

**Summary of Main Contributions:** The goal of the fusion center is to come up with a strategy (or a stopping rule) to declare change, subject to false alarm constraints. Towards this goal,
we first show that the problem fits the standard partially observable Markov decision process (POMDP) framework [33] with the sufficient statistics given by the \textit{a posteriori} probabilities of the state of the system conditioned on the observation process. We then establish a recursion for the sufficient statistics, which generalizes the recursion established in [32] for the case when all the sensors observe the change at the same instant.

Following the logic of [34] and [32], we then establish the optimality of a more general stopping rule for change detection. This rule takes the form of the smallest time of cross-over (intersection) of a linear functional (or hyperplane) in the space of sufficient statistics with a non-linear concave function, and generalizes the threshold test of [32]. While further analytical characterization of the optimal stopping rule is difficult in general, in the extreme scenario of a rare disruption regime, we show that the structure of this rule reduces to a simple threshold test on the \textit{a posteriori} probability that no change has happened. This low-complexity test is denoted as \( \nu_A \) (corresponding to an appropriate choice of threshold \( A \)) for simplicity.

While \( \nu_A \) is obtained as a limiting form of the optimal test, it is not clear (as yet) if it is a “good” test. Towards this goal, we show that it is asymptotically optimal (as the false alarm probability \( P_{\text{FA}} \) vanishes) under a certain condition on the Kullback-Leibler (K-L) divergence between the post- and the pre-change densities. Meeting this condition becomes more easier as change propagates more instantaneously across the sensor array, and in the extreme case of [32], this condition reduces to the mild one that the K-L divergence be positive.

The difference between the setting in this work and the setting in [32] is in the non-asymptotic, but small \( P_{\text{FA}} \) regime. Asymptotic optimality of a particular test in the setting of [32] translates to an \( L \)-fold increase in the slope of \( E_{\text{D}} \) vs. \( P_{\text{FA}} \) in the regime where the false alarm probability is small, but not vanishing (e.g., \( P_{\text{FA}} \approx 10^{-4} \) or \( 10^{-5} \)). However, if the change propagates too “slowly” across the sensor array, numerical studies indicate that not all of the \( L \) sensors’ observations may contribute to the performance of \( \nu_A \) in this regime. Nevertheless, as \( P_{\text{FA}} \to 0 \), all the \( L \) sensors are expected (in general) to contribute to the slope.

Thus, while it is not clear if \( \nu_A \) is asymptotically optimal in general, or even if all the sensors’ observations contribute to its performance in the non-asymptotic regime, numerical studies also show that it can result in substantial performance improvement over naïve tests such as the \textit{single sensor test} (where only the first sensor’s observation is used in decision-making) or the \textit{mismatched test} (where all the sensors’ observations are used in decision-making, albeit with a wrong model that change propagates instantaneously), especially in regimes of practical importance (rare disruption, and reasonably quick, but non-instantaneous change propagation across the sensors). The performance improvement possible with \( \nu_A \), in addition to its low-complexity, make it an attractive choice for many practical applications with a basis in multi-
sensor change process detection.

**Organization:** This paper is organized as follows. The change process detection problem is formally set-up in Section II. In Section III, this problem is posed in a POMDP framework and the sufficient statistics of the dynamic program (DP) are identified. Recursion for the sufficient statistics are then established. The structure of the optimal stopping rule in the general case and the rare disruption regime are illustrated in Section IV. The limiting form of the optimal test is denoted as \( \nu_A \) for simplicity. Using elementary tools from renewal theory, asymptotic optimality of \( \nu_A \) is established in Sections V–VII under certain conditions. (The main results are stated in Sec. V and they are established in detail in the appendices and in Sec. VI and VII.) A discussion of the main results and numerical studies to illustrate our results are provided in Section VIII. Concluding remarks are made in Section IX.

II. Problem Formulation

Consider a distributed system with an array of \( L \) sensors, as in Fig. 1 that observes an \( L \)-dimensional discrete-time stochastic process \( Z_k = [Z_{k,1}, \ldots, Z_{k,L}] \), where \( Z_{k,\ell} \) is the observation at the \( \ell \)-th sensor and the \( k \)-th time instant. A disruption in the sensing environment occurs at the random time instant \( \Gamma_1 \) and hence, the density of the observations at each sensor undergoes a change from the null density \( f_0 \) to the alternate density \( f_1 \).

**Change Process Model:** Previous works on quickest change detection in multi-sensor systems consider strategies to detect the change-point, \( \Gamma_1 \), when the change occurs at the same instant across all the sensors [29]–[32]. As described in the introduction, it is useful to consider more general scenarios where there exists random propagation delays in the change-point across the sensors.

In this work, we consider a change process where the change-point evolves across the sensor array. In particular, the change-point as seen by the \( \ell \)-th sensor is denoted as \( \Gamma_{\ell} \). We assume that the evolution of the change process is Markovian across the sensors. That is,

\[
P\left( \{\Gamma_{\ell_1+\ell_2+\ell_3} = m_1 + m_2 + m_3\} \mid \{\Gamma_{\ell_1+\ell_2} = m_1 + m_2\}, \{\Gamma_{\ell_1} = m_1\} \right) = P\left( \{\Gamma_{\ell_1+\ell_2+\ell_3} = m_1 + m_2 + m_3\} \mid \{\Gamma_{\ell_1+\ell_2} = m_1 + m_2\} \right)
\]

for all \( \ell_i \) and \( m_i \geq 0 \), \( i = 1, 2, 3 \). Further simplification of the analysis is possible under a joint-geometric model on \( \{\Gamma_{\ell}\} \). Under this model, the change-point \( (\Gamma_1) \) evolves as a geometric random variable with parameter \( \rho \), and inter-sensor change propagation is modeled as a geometric

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1We assume that the pre-change \( (f_0) \) and the post-change \( (f_1) \) densities exist.
random variable with parameter \( \{ \rho_{\ell-1, \ell}, \ \ell = 2, \ldots, L \} \). That is,

\[
P(\{ \Gamma_1 = m \}) = \rho (1 - \rho)^m, \ m \geq 0 \quad \text{and} \quad P(\{ \Gamma_\ell = m_1 + m_2 \mid \{ \Gamma_{\ell-1} = m_2 \} \}) = \rho_{\ell-1, \ell} (1 - \rho_{\ell-1, \ell})^{m_1}, \ m_1 \geq 0
\]

independent of \( m_2 \geq 0 \) for all \( \ell \) such that \( 2 \leq \ell \leq L \).

We will find it convenient to set \( \rho_{0,1} = \rho \) and \( \rho_{L,L+1} = 0 \) so that \( \rho_{\ell-1, \ell} \) is defined for all \( \ell = 1, \ldots, L + 1 \). This is also consistent with an equivalent \((L+2)\)-sensor system where sensor indices run through \( \{ \ell = 0, \ldots, L + 1 \} \). The hypothetical zero-th sensor models the disruption point, the first real sensor observes change with respect to the zero-th sensor with a geometric parameter \( \rho \) (and so on). The hypothetical \((L+1)\)-th sensor models an “observer at infinity” that observes change from the \( L \)-th sensor with an infinite delay on average. This is reflected by setting \( \rho_{L,L+1} = 0 \). At this point, it should be noted that [29]–[32] consider this equivalent framework explicitly by modeling \( \gamma \), the probability that the disruption took place before the observations were made. The setup in [29]–[32] can be obtained by setting:

\[
P(\{ \Gamma_0 < 0 \}) = \gamma \quad \text{and} \quad P(\{ \Gamma_0 = 0 \}) = 1 - \gamma \quad \text{for some} \ \gamma \in [0, 1].
\]

In this work, we focus on the case where \( \gamma = 0 \) with extension to the general case being straightforward.

While a joint-geometric model is consistent with the Markovian assumption as only the inter-sensory (one-step) propagation parameters are modeled, the change-points at the individual sensors themselves are not geometric. For example, it can be checked that

\[
P(\{ \Gamma_2 = m \}) = \frac{\rho \rho_{1,2}}{\rho - \rho_{1,2}} \times \left( (1 - \rho_{1,2})^m - (1 - \rho)^m \right)
\]

\[
P(\{ \Gamma_3 = m \}) = \frac{\rho \rho_{1,2} \rho_{2,3}}{(\rho - \rho_{1,2})(\rho_{1,2} - \rho_{2,3})(\rho - \rho_{2,3})} \times \left( (\rho - \rho_{1,2})(1 - \rho_{2,3})^{m+2} - (\rho - \rho_{2,3})(1 - \rho_{1,2})^{m+2} + (\rho_{1,2} - \rho_{2,3})(1 - \rho)^{m+2} \right)
\]

and so on. It should be clear from the above expressions that a joint-geometric model does not impose any constraints on \( \{ \rho_{\ell-1, \ell} \} \) except that \( \rho_{\ell-1, \ell} \in [0, 1] \).

Note that \( \rho \to 1 \) corresponds to the case where instantaneous disruption (that is, the event \( \{ \Gamma_1 = 0 \} \)) has a high probability of occurrence. On the other hand, \( \rho \to 0 \) uniformizes the change-point in the sense that the disruption is equally likely to happen at any point in time. This case where the disruption is “rare” is of significant interest in practical systems [16], [19], [29]–[32]. This is also the case where we will be able to make insightful statements.

\[^2\]“Observer at infinity” interpretations are often used in distributed decision-making and stochastic control problems [33], [34].
about the structure of the optimal stopping rule. Similarly, we can also distinguish between two extreme scenarios at sensor \( \ell \) depending on whether \( \rho_{\ell-1,\ell} \to 0 \) or \( \rho_{\ell-1,\ell} \to 1 \). The case where \( \rho_{\ell-1,\ell} \to 1 \) corresponds to instantaneous change propagation at sensor \( \ell \) and \( \{ \Gamma_\ell = \Gamma_{\ell-1} \} \) with high probability. The case where \( \rho_{\ell-1,\ell} \to 0 \) corresponds to uniformly likely propagation delay. The widely-used assumption [29], [32] of instantaneous change propagation across sensors is equivalent to assuming \( \rho_{\ell-1,\ell} = 1 \) for all \( \ell = 2, \cdots, L \).

**Observation Model:** To simplify the study, we assume that the observations (at every sensor) are independent, conditioned on the change hypothesis corresponding to that sensor, and are identically distributed pre- and post-change, respectively. That is,

\[
Z_{k,\ell} \sim \begin{cases} 
\text{i.i.d. } f_0 & \text{if } k < \Gamma_\ell, \\
\text{i.i.d. } f_1 & \text{if } k \geq \Gamma_\ell.
\end{cases}
\]

We will describe the above assumption as that corresponding to an “i.i.d. observation process.” Let \( D(f_1, f_0) \) denote the Kullback-Leibler divergence between \( f_1 \) and \( f_0 \). That is,

\[
D(f_1, f_0) = \int \log \left( \frac{f_1(x)}{f_0(x)} \right) f_1(x) dx.
\]  \hspace{1cm} (1)

We also assume that the measure described by \( f_0 \) is absolutely continuous with respect to that described by \( f_1 \). That is, if \( f_1(x) = 0 \) for some \( x \), then \( f_0(x) = 0 \). This condition ensures that \( E_{f_1} \left[ \frac{f_0(x)}{f_1(x)} \right] = 1 \).

**Performance Metrics:** We consider a centralized, Bayesian setup where a fusion center has complete knowledge of the observations from all the sensors, \( I_k \triangleq \{ Z_1, \cdots, Z_k \} \), in addition to knowledge of statistics of the change process (equivalently, \( \{ \rho_{\ell-1,\ell} \} \)) and statistics of the observation process (equivalently, \( f_0 \) and \( f_1 \)). The fusion center decides whether a change has happened or not based on the information, \( I_k \), available to it at time instant \( k \) (equivalently, it provides a stopping rule or stopping time \( \tau \)).

The two conflicting performance measures for quickest change detection are the probability of false alarm, \( P_{FA} \triangleq P(\{ \tau < \Gamma_1 \}) \), and the expected detection delay, \( E_{DD} \triangleq E[(\tau - \Gamma_1)^+] \), where \( x^+ = \max(x, 0) \). This conflict is captured by the Bayes risk, defined as,

\[
R(c) \triangleq P_{FA} + c E_{DD} = E \left[ \mathbb{I} \left( \{ \tau < \Gamma_1 \} \right) + c (\tau - \Gamma_1)^+ \right]
\]

for an appropriate choice of per-unit delay cost \( c \), where \( \mathbb{I}(\{ \cdot \}) \) is the indicator function of the event \( \{ \cdot \} \). We will be particularly interested in the regime where \( c \to 0 \). That is, a regime where

\(^3\)More general observation (correlation) models are important in practical settings. This will be the subject of future work.

\(^4\)We assume that the fusion center has knowledge of \( f_0 \) and \( f_1 \) so that it can use this information to declare that a change has happened. Relaxing this assumption is important in the context of practical applications and is the subject of current work.
minimizing $P_{FA}$ is more important than minimizing $E_{DD}$, or equivalently, the asymptotics where $P_{FA} \to 0$.

The goal of the fusion center is to determine

$$\tau_{opt} = \operatorname*{arg\,inf}_{\tau \in \Delta_\alpha} E_{DD}(\tau)$$

from the class of change-point detection procedures $\Delta_\alpha = \{\tau : P_{FA}(\tau) \leq \alpha\}$ for which the probability of false alarm does not exceed $\alpha$. In other words, the fusion center needs to come up with a strategy (a stopping rule $\tau$) to minimize the Bayes risk. Note that the strategy developed by optimizing the Bayes risk can also be used for the other classical problem formulation in change detection, that of the minimax type [32, Theorem 1], [13], [33].

### III. Dynamic Programming Framework

It is straightforward to check that [13, pp. 151-152], [32] the Bayes risk can be written as

$$R(c) = P(\{\Gamma_1 > \tau\}) + cE \left[ \sum_{k=0}^{\tau-1} P(\{\Gamma_1 \leq k\}) \right].$$

Towards solving for the optimal stopping time, we restrict attention to a finite-horizon, say the interval $[0, T]$, and proceed via a dynamic programming (DP) argument.

The state of the system at time $k$ is the vector $S_k = [S_{k,1}, \ldots , S_{k,L}]$ with $S_{k,\ell}$ denoting the state at sensor $\ell$. The state $S_{k,\ell}$ can take the value 1 (post-change), 0 (pre-change), or $t$ (terminal). The system goes to the terminal state $t$, once a change-point decision $\tau$ has been declared. The state evolves as follows:

$$S_{k,\ell} = \mathbb{1}(\{\Gamma_{\ell} \leq k\} \cap \{S_{k-1,\ell} \neq t\} \cap \{\tau \neq k\}) + t \mathbb{1}(\{S_{k-1,\ell} = t\} \cup \{\tau = k\}).$$

with $S_0 = 0$. Since $S_{k-1}$ captures the information contained in $\{\Gamma_\ell \leq j\}$ for $0 \leq j \leq k-1$ and all $\ell$, given $S_{k-1}$, $\{\Gamma_{\ell} \leq k\}$ is independent of $\{\Gamma_\ell \leq j, j \leq k-1\}$ for all $\ell$. Thus, the state evolution satisfies the Markov condition needed for dynamic programming.

The state is not observable directly, but only through the observations. The observation equation can be written as

$$Z_{k,\ell} = V^{(S_{k,\ell})}_{k,\ell} \mathbb{1}(\{S_{k,\ell} \neq t\}) + \xi \mathbb{1}(\{S_{k,\ell} = t\}), \ \ell \geq 1$$

where $V^{(0)}_{k,\ell}$ and $V^{(1)}_{k,\ell}$ are the $k$-th samples from independently generated infinite arrays of i.i.d. data according to $f_0$ and $f_1$, respectively. When the system is in the terminal state, the observations do not matter (since a change decision has already been made) and are hence denoted by a dummy random variable, $\xi$. It is clear that the observation uncertainty $(V^{(0)}_{k,\ell}, V^{(1)}_{k,\ell})$ satisfies the necessary Markov conditions for dynamic programming since they are i.i.d. in time.
Finally, the expected cost (Bayes risk) can be expressed as the expectation of an additive cost over time by defining
\[ g_k(S_k) = c \mathbb{1} \{ \{ S_{k,1} = 1 \} \} \]
and a terminal cost \( \mathbb{1} \{ \{ S_{k,1} = 0 \} \} \). Thus the problem fits the standard POMDP framework with termination [33], with the sufficient statistic (belief state) being given by
\[ P(\{ S_k = s_k \} | I_k), \]
where \( I_k = \{ Z_1, \ldots, Z_k \} \) for \( k \) such that \( S_k \neq t \), i.e., \( S_{k,\ell} \in \{ 0, 1 \} \) for each \( \ell \). Note that this sufficient statistic is described by \( 2L \) conditional probabilities, corresponding to the \( 2L \) values that \( s_k \) can take. We will next see that this sufficient statistic can be further reduced to only \( L \) independent probability parameters in the general case.

The fusion center determines \( \tau \) and hence, the minimum expected cost-to-go at time \( k \) for the above DP problem can be seen to be a function of \( I_k \). For a finite horizon \( T \), the cost-to-go function is denoted as \( \tilde{J}_T^k(I_k) \) and is of the form (see [32], [33, p. 133] for examples of similar nature):
\[
\tilde{J}_T^k(I_k) = \min \left\{ P(\{ \Gamma_1 > k \} | I_k), c P(\{ \Gamma_1 \leq k \} | I_k) + E \left[ \tilde{J}_T^{k+1}(I_{k+1}) | I_k \right] \right\}, \quad 0 \leq k < T
\]
where \( I_0 \) is the empty set. The first term in the above minimization corresponds to the cost associated with stopping at time \( k \), while the second term corresponds to the cost associated with proceeding to time \( k+1 \) without stopping. The minimum expected cost for the finite-horizon optimization problem is \( \tilde{J}_0^T(I_0) \).

**Recursion for the Sufficient Statistics:** Consider the special case where change at all the sensors happens at the same instant. In this setting, it can be shown that the random variable \( p_k \triangleq P(\{ \Gamma_1 \leq k \} | I_k) \) serves as the sufficient statistic for the above dynamic program and affords a recursion [32]. To consider the more general case, we define an \((L+1)\)-tuple of conditional probabilities, \( \{ p_{k,\ell}, \ell = 1, \ldots, L + 1 \} \):
\[ p_{k,\ell} \triangleq P(\{ \Gamma_1 \leq k, \ldots, \Gamma_{\ell-1} \leq k, \Gamma_\ell > k, \ldots, \Gamma_L > k \} | I_k). \]
The special setting of [32] is then equivalent to
\[ p_{k,L+1} = p_k, \quad p_{k,1} = 1 - p_k, \quad \text{and} \quad p_{k,\ell} = 0, \quad \ell = 2, \ldots, L. \]

\(^5\)This should not be entirely surprising since there exists a “natural” ordering on the sensors’ change-points. They can be arranged in non-decreasing order: \( \Gamma_\ell \geq \Gamma_{\ell-1} \) for all \( \ell \). The primary reason for such an ordering to exist is that we assume an array (or line) of sensors in this work. Extensions to more general (or unknown) geometries of sensors is of interest in practice.
We now show that \( p_k \triangleq [p_{k,1}, \ldots, p_{k,L+1}] \) can be obtained from \( p_{k-1} \) via a recursive approach. For this, we note that the underlying probability space \( \Omega \) in the setup can be partitioned as
\[
\Omega = \bigcup_{\ell=1}^{L+1} T_{k,\ell}
\]
\[
T_{k,\ell} \triangleq \{ \Gamma_1 \leq k, \ldots, \Gamma_{\ell-1} \leq k, \Gamma_\ell \geq k+1, \ldots, \Gamma_L \geq k+1 \}.
\]
The event where no sensor has observed the change is denoted as \( T_{k,1} \). (The test that will be proposed and studied later in the paper thresholds the \textit{a posteriori} probability of \( T_{k,1} \).) On the other hand, \( T_{k,\ell} \) (for \( \ell \geq 2 \)) corresponds to the event where the maximal index of the sensor that has observed the change before time instant \( k \) is \( \ell - 1 \). Observe that \( p_{k,\ell} \) is the probability of \( T_{k,\ell} \) conditioned on \( I_{k-1} \).

To show that \( p_{k,\ell} \) can be written in terms of \( p_{k-1} \), the observations \( Z_k \) and the prior probabilities, we partition \( T_{k,\ell} \) further as
\[
T_{k,\ell} = \bigcup_{j=1}^{\ell} U_{k,\ell,j}
\]
\[
U_{k,\ell,j} \triangleq \{ \Gamma_1 \leq k - 1, \ldots, \Gamma_{j-1} \leq k - 1, \Gamma_j = k, \ldots, \Gamma_{\ell-1} = k, \Gamma_\ell \geq k+1, \ldots, \Gamma_L \geq k+1 \}, \ 1 \leq j \leq \ell.
\]
Note that \( U_{k,\ell,j} \cap T_{k-1,j} = U_{k,\ell,j} \). Using the new partition \( \{ U_{k,\ell,j}, \ j = 1, \ldots, \ell \} \) and applying Bayes’ rule repeatedly, it can be checked that \( p_{k,\ell} \) can be written as
\[
p_{k,\ell} = \frac{\sum_{m=1}^{\ell} f(Z_k|I_{k-1}, U_{k,\ell,m}) P(U_{k,\ell,m}|I_{k-1})}{\sum_{j=1}^{L+1} \sum_{m=1}^{L+j} f(Z_k|I_{k-1}, U_{k,j,m}) P(U_{k,j,m}|I_{k-1})} \triangleq \frac{N_{\ell}}{\sum_{j=1}^{L+1} N_j}
\]
where \( f(\cdot|\cdot) \) denotes the conditional probability density function of \( Z_k \) and \( N_{\ell} \) denotes the numerator term.

From the i.i.d. assumption on the statistics of the observations, the first term within the summation for \( N_{\ell} \) can be written as:
\[
f(Z_k|I_{k-1}, U_{k,\ell,m}) = \prod_{j=1}^{\ell-1} f_1(Z_{k,j}) \prod_{j=\ell}^{L} f_0(Z_{k,j}) = \prod_{j=1}^{\ell-1} L_{k,j} \prod_{j=1}^{L} f_0(Z_{k,j})
\]
where \( L_{k,j} \triangleq \frac{f_1(Z_{k,j})}{f_0(Z_{k,j})} \) is the likelihood ratio of the two hypotheses given that \( Z_{k,j} \) is observed at the \( j \)-th sensor at the \( k \)-th instant. For the second term, observe from the definitions that
\[
P(U_{k,\ell,m}|I_{k-1}) = P(T_{k-1,m}|I_{k-1}) \frac{P(U_{k,\ell,m})}{P(T_{k-1,m})}.
\]
Thus, we have
\[ N_\ell = \left( \sum_{m=1}^{\ell} \frac{P(U_{k,\ell,m})}{P(T_{k-1,m})} \cdot p_{k-1,m} \right) \times \prod_{m=1}^{\ell-1} L_{k,m} \prod_{m=1}^{L} f_0(Z_{k,m}) \]
\[ \triangleq \left( \sum_{m=1}^{\ell} w_{k,\ell,m} p_{k-1,m} \right) \Phi_{\text{obs}}(k, \ell) \]
where the first part is a weighted sum of \( p_{k-1,m} \) with weights decided by the prior probabilities, and the second part of the evolution equation, \( \Phi_{\text{obs}}(k, \ell) \), can be viewed as that part that depends only on the observation \( Z_k \).

Many observations are in order at this stage:

- The above expansion for \( N_\ell \) can be easily explained intuitively: If the maximal sensor index observing the change by time \( k \) is \( \ell - 1 \), then the maximal sensor index observing the change by time \( k - 1 \) should be from the set \( \{0, \ldots, \ell - 1\} \).
- Using the joint-geometric model for \( \{\Gamma_\ell\} \), it can be shown that \( w_{k,\ell,m} \) is of the form:
  \[ w_{k,\ell,m} = \frac{P(U_{k,\ell,m})}{P(T_{k-1,m})} = (1 - \rho_{\ell-1,\ell}) \cdot \prod_{j=m-1}^{\ell-2} \rho_{j,j+1} \triangleq (1 - \rho_{\ell-1,\ell}) \cdot w_{m}^{\ell} \]
\[ N_\ell = \prod_{m=1}^{\ell-1} L_{k,m} \prod_{m=1}^{L} f_0(Z_{k,m}) \cdot (1 - \rho_{\ell-1,\ell}) \times \left( \sum_{m=1}^{\ell} p_{k-1,m} \cdot w_{m}^{\ell} \right) \]
with the understanding that the product term in the definition of \( w_{m}^{\ell} \) is vacuous (and is to be replaced by 1) if \( m = \ell \). It is important to note that the joint-geometric assumption renders the weights \( (w_{k,\ell,m}) \) associated with \( p_{k-1,m} \) independent of \( k \). This will be useful later in establishing convergence properties for the DP.
- It is important to note that given a fixed value of \( \ell \), \( p_{k,\ell} \) is dependent on the entire vector \( p_{k-1} \) and not on \( p_{k-1,\ell} \) alone. Thus, the recursion for \( N_\ell \) implies that \( p_k \) forms the sufficient statistic and the function \( \tilde{J}_k^T(I_k) \) can be written as a function of only \( p_k \), say \( J_k^T(p_k) \). The finite-horizon DP equations can then be rewritten as
  \[ J_k^T(p_k) = \min \left\{ p_{k,1}, c(1 - p_{k,1}) + A_k^T(p_k) \right\} \]
with
\[ A_k^T(p_k) \triangleq E[J_{k+1}^T(p_{k+1})|I_k] \]
\[ = \int \left[ J_{k+1}^T(p_{k+1}) f(Z_{k+1}|I_k) \right]_{Z_{k+1}=z} dz. \]
The previously established recursion for \( p_{k+1} \) ensures that the right-hand side is indeed a function of \( p_k \).
It is easy to check that the general framework reduces to the special case when all the change-points coincide with $\Gamma_1$ [32]. In this case, only $T_{k,1}$ and $T_{k,L+1}$ are non-empty sets with

$$T_{k,1} = \{ \Gamma_1 \geq k + 1 \}, \quad \text{and} \quad T_{k,L+1} = \{ \Gamma_1 \leq k \},$$

$$p_{k,1} = p_k, \quad p_{k,1} = 1 - p_k \quad \text{and} \quad p_{k,\ell} = 0, \quad \ell = 2, \ldots, L.$$

Furthermore, the recursion for $p_k$ reduces to

$$p_k = \frac{\mathcal{N}}{\prod_{j=1}^L f_0(Z_{k,j}) (1 - p_{k-1}) (1 - \rho) + \mathcal{N}},$$

$$\mathcal{N} = \prod_{j=1}^L f_1(Z_{k,j}) ((1 - p_{k-1}) \rho + p_{k-1})$$

which coincides with [32, eqn. (13)-(15)]. This case can also be obtained from the formula in (2) by setting $\rho_{\ell-1,\ell} = 1$ for all $\ell$ with $2 \leq \ell \leq L$.

IV. STRUCTURE OF THE OPTIMAL STOPPING RULE ($\tau_{\text{opt}}$)

The goal of this section is to study the structure of the optimal stopping rule, $\tau_{\text{opt}}$. For this, we follow the same outline as in [32], [34] (see, also [33, p. 133] for a similar example) and study the infinite-horizon version of the DP problem by letting $T \to \infty$.

**Theorem 1:** Let $p = [p_1, \ldots, p_{L+1}]$ be an element of the standard $L$-dimensional simplex $\mathcal{P}$, defined as, $\mathcal{P} \triangleq \{ p : \sum_{j=1}^{L+1} p_j = 1 \}$. The infinite-horizon cost-to-go for the DP is of the form

$$J(p) = \min \left\{ p_1, \ c(1 - p_1) + A_J(p) \right\},$$

where the function $A_J(p)$: i) is concave in $p$ over $\mathcal{P}$; ii) is bounded as $0 \leq A_J(p) \leq 1$; and iii) satisfies $A_J(p) = 0$ over the hyperplane $\{ p : p_1 = 0 \}$.

**Proof:** Before considering the infinite-horizon DP, we will study the finite-horizon version and establish some properties along the directions of [32]–[34]. A straightforward induction argument shows that if $T$ is fixed,

$$0 \leq J^T_k(p) \leq 1 \quad \text{for all} \quad 0 \leq k \leq T,$$

$$0 \leq A^T_k(p) \leq 1 \quad \text{for all} \quad 0 \leq k \leq T.$$ 

Similarly, it is easy to observe that for any $k$, $A^T_k(p)$ and $J^T_k(p)$ equal zero if $p_1 = 0$. In Appendix A, the concavity of $A^T_k(\cdot)$ and $J^T_k(\cdot)$ are established via a routine induction argument.

We now consider the infinite-horizon DP and show that it is well-defined. (That is, we remove the restriction that the stopping time is finite and let $T \to \infty$. ) Towards this end, we need to
establish that \( \lim_{T} J_{k}^{T}(\cdot) \) exists, which is done as follows: By an induction argument, we note that for any \( p \) and \( T \) fixed, we have

\[
J_{k}^{T}(p) \leq J_{k+1}^{T}(p), \quad 0 \leq k \leq T - 1.
\]

It is important to note that this conclusion critically depends on the joint-geometric assumption of the change process (in particular, the memorylessness property that results in the independence of \( w_{k,\ell,m} \) on \( k \) in (2)) and the i.i.d. nature of the observation process conditioned on the change-point.

Using a similar induction approach, observe that for any \( p \) and \( k \) fixed, \( J_{k}^{T+1}(p) \leq J_{k}^{T}(p) \). Heuristically, this can also be seen to be true because the set of stopping times increases with \( T \). Since \( J_{0}^{T}(p) \geq 0 \) for all \( k \) and \( T \), for any fixed \( k \), we can let \( T \rightarrow \infty \) and we have

\[
\lim_{T} J_{k}^{T}(p) = \inf_{T : T > k} J_{k}^{T}(p) \triangleq J_{k}^{\infty}(p).
\]

Furthermore, the memorylessness property and the i.i.d. observation process results in the invariance of \( J_{k}^{\infty}(p) \) on \( k \). This can be shown by a simple time-shift argument. Denote this common limit as \( J(p) \).

A simple dominated convergence argument [35] then shows that \( \lim_{T} A_{k}^{T}(p) \) is well-defined and independent of \( k \). If we denote this limit as \( A_{j}(p) \), we have

\[
A_{j}(p) = \int \left[ J(p) f(Z|I_{\bullet}) \right]_{Z=z} dz
\]

\[
= \int J(p) \left\{ \sum_{j=1}^{L+1} \left( (1 - \rho_{j-1,j}) \cdot \sum_{m=1}^{j} w_{j,m} \rho_{m} \right) \Phi_{obs}(\bullet, j) \right\} \bigg|_{Z=z} \] d\(z\),

where the fact that \( \Phi_{obs}(k, j) \bigg|_{Z=z} \) is independent of \( k \) is denoted as \( \Phi_{obs}(\bullet, j) \). Hence, the infinite-horizon cost-to-go can be written as

\[
J(p) = \min \left\{ p_{1}, c(1 - p_{1}) + A_{j}(p) \right\}.
\]

The structure of \( A_{j}(p) \) follows from the finite-horizon characterization by letting \( T \rightarrow \infty \).

At this stage, it is a straightforward consequence that the optimal stopping rule is of the form

\[
\tau_{opt} = \inf_{k} \left\{ p_{k,1}(1 + c) - c < A_{j}(p_{k}) \right\}.
\]

That is, a change is declared when the hyperplane on the left side is exceeded by \( A_{j}(p_{k}) \) and no change is declared, otherwise.

We will next see that this test characterization reduces to a degenerate one as \( \rho \rightarrow 0 \). To establish this degeneracy result, along the lines of [32], we now define a one-to-one and invertible
transformation \(\{q_{k,\ell}, \ell = 1, \ldots, L + 1\}\), as follows:

\[ q_{k,\ell} = \frac{p_{k,\ell}}{\rho p_{k,1}}. \]

The inverse transformation is given by:

\[ p_{k,\ell} = \frac{q_{k,\ell}}{\sum_{j=1}^{L+1} q_{k,j}}, \ell = 1, \ldots, L + 1, \]

which is equivalent to

\[ p_{k,1} = \frac{1}{1 + \rho \sum_{j=2}^{L+1} q_{k,j}} \quad \text{and} \quad p_{k,\ell} = \frac{\rho q_{k,\ell}}{1 + \rho \sum_{j=2}^{L+1} q_{k,j}}, \ell = 2, \ldots, L + 1. \]

We can write \(q_{0,\ell}\) in terms of the priors as

\[ q_{0,1} = \frac{p_{0,1}}{\rho p_{0,1}} = \frac{1}{\rho}, \]
\[ q_{0,\ell} = \frac{p_{0,\ell}}{\rho p_{0,1}} = \frac{P(\{\Gamma_1 = \cdots = \Gamma_{\ell-1} = 0, \Gamma_\ell > 0\})}{\rho P(\{\Gamma_1 > 0\})} = \frac{\prod_{j=0}^{\ell-2} \rho j+1 (1 - \rho \ell-1,\ell)}{\rho (1 - \rho)}, \ell = 2, \ldots, L + 1. \]

Note that while \(p_{k,\ell}\) are conditional probabilities of certain events and hence lie in the interval \([0, 1]\), the range of \(q_{k,\ell}\) is in general \([0, \infty)\).

It can be checked that the evolution equation can be rewritten in terms of \(q_{k,\ell}\) as

\[ q_{k,\ell} = \frac{1 - \rho \ell-1,\ell}{1 - \rho} \cdot \prod_{j=1}^{\ell-1} L_{k,j} \cdot \left(\sum_{j=1}^{\ell} q_{k-1,j} w_j^\ell\right). \] (3)

It is interesting to note from (3) that the update for \(q_{k,\ell}\) is a weighted sum of \(q_{k-1,j}, j = 1, \ldots, \ell\) with progressively increasing weight as \(j\) increases. Similarly, we can define \(J_k^T(\cdot)\) and \(A_k^T(\cdot)\) in terms of \(q_k\). Using the transformation \(\{q_{k,\ell}\}\), \(\tau_{\text{opt}}\) is seen to have the form:

\[ \tau_{\text{opt}} = \inf_k \left\{ \sum_{\ell=2}^{L+1} q_{k,\ell} > \frac{1 - A_j(q_k)}{\rho (c + A_j(q_k))} \right\}. \]

When all \(\Gamma_\ell\) coincide [32], we have

\[ q_{k,L+1} = \frac{p_k}{\rho(1 - p_k)} \triangleq q_k, \quad q_{k,1} = \frac{1}{\rho}, \quad q_{k,\ell} = 0, \ell = 2, \ldots, L. \]

It is important to note that the transformation in [32] can be generalized in more than one direction. For example, i) \(q_{k,\ell} = \frac{\sum_{j=1}^{L+1} p_{k,j}}{\rho p_{k,1}}\), ii) \(q_{k,\ell} = \frac{1 - p_{k,\ell}}{\rho p_{k,1}}\) etc. are consistent with the definition in [32]. While these definitions of \(q_{k,\ell}\) ensure that the structure of \(\tau_{\text{opt}}\) (as \(\rho \to 0\)) becomes simple, the recursion for \(q_{k,\ell}\) (and hence, an understanding of the performance of the proposed test) becomes more complicated. We believe that the definition of \(q_{k,\ell}\), as provided here, is the most natural generalization in the goal of understanding the performance of change process detection schemes.
Further, it is straightforward to check that the evolution in (3) reduces to

\[ q_{k,L+1} = \frac{\prod_{j=1}^{L} L_{k,j}}{1 - \rho} \cdot (1 + q_{k-1,L+1}), \] (4)

which is [32, eqn. 32]. Thus, the space of sufficient statistics and the optimal test reduce to a one-dimensional variable \( p_k = P(\{\Gamma_1 \leq k\} | I_k) \) or equivalently, \( q_k \) and a threshold test on \( p_k \) (or equivalently, on \( q_k \)), respectively.

In the general case, unless something more is known about the structure of \( A_J(\cdot) \) (which is possible if there is some structure on \( \{\rho_{\ell-1,\ell}\} \)), we cannot say more about \( \tau_{opt} \). Nevertheless, the following theorem establishes its structure in the practical setting of a rare disruption regime (\( \rho \to 0 \)). The limiting test thresholds the \textit{a posteriori} probability that no-change has happened (from below), and is denoted as \( \nu_A \).

**Theorem 2:** The structure of \( \tau_{opt} \) converges to a simple threshold rule in the asymptotic limit as \( \rho \to 0 \). This test is of the form:

\[
\nu_A = \begin{cases} 
\text{Stop} & \text{if } \log\left(\sum_{\ell=2}^{L+1} q_{k,\ell}\right) \geq A \\
\text{Continue} & \text{if } \log\left(\sum_{\ell=2}^{L+1} q_{k,\ell}\right) < A 
\end{cases}
\]

for an appropriate choice of threshold \( A \).

**Proof:** See Appendix B.

The test \( \nu_A \) is of low-complexity because of the following properties: i) a simple recursion formula (3) for the sufficient statistics; ii) a threshold operation for stopping; and iii) the threshold value that can be pre-computed given the \( P_{FA} \) constraint (see Prop. 3). However, it is important to note that the complexity of \( \nu_A \) is \textit{not} equivalent to that of the threshold test of [32] because the recursion for the sufficient statistics depends on \( (L + 1) \text{ a posteriori} \) probabilities, in general, in contrast to a single parameter in [32].

The fact that \( \tau_{opt} \uparrow \nu_A \) for an appropriate choice of \( A \) does not imply that \( \nu_A \) is asymptotically (as \( \rho \to 0 \) or as \( P_{FA} \to 0 \)) optimal. However, the low-complexity of this test, in addition to Theorem 2 and the fact that the structure of \( A_J(q_k) \) (and hence, \( \tau_{opt} \)) are not known suggest that it is a good candidate test for change detection across a sensor array. In fact, we will see this to be the case when we establish sufficient conditions under which \( \nu_A \) is asymptotically optimal.

**V. MAIN RESULTS ON \( \nu_A \)**

Towards this end, our main interest is in understanding the performance \( (E_{DD} \text{ vs. } P_{FA}) \) of \( \nu_A \) for any general choice of threshold \( A \). We make a few preliminary remarks before providing performance bounds for \( \nu_A \).
**Special Cases of Change Parameters:** We start by considering some special scenarios of change propagation modeling. The first scenario corresponds to the case where one (or more) of the \( \rho_{\ell-1, \ell} \) is 1. The following proposition addresses this setting.

**Proposition 1:** Consider an \( L \)-sensor system described in Sec. II, parameterized by \( \{\rho_{\ell-1, \ell}\} \), where \( \rho_{\ell', \ell+1} = 1 \) for some \( \ell' \) and \( \max_{j \neq \ell'} \rho_{j, j+1} < 1 \). This system is equivalent to an \( (L-1) \)-sensor system, parameterized by \( \{\beta_{\ell, \ell+1}\} \), where

\[
\begin{align*}
\beta_{j, j+1} &= \rho_{j, j+1}, & j &\leq \ell' - 1 \\
\beta_{j, j+1} &= \rho_{j+1, j+2}, & j &\geq \ell'
\end{align*}
\]

with the \((\ell' + 1)\)-th sensor observing (a combination of) \( Z_{k, \ell'+1} \) and \( Z_{k, \ell'+2} \) with a geometric delay parameter of \( \beta_{\ell', \ell'+1} = \rho_{\ell'+1, \ell'+2} \).

**Proof:** The proof is straightforward by studying the evolution of \( \{q_{k, \ell}\} \) for the original \( L \)-sensor system. From (3), it can be seen that \( q_{k, \ell'} = 0 \) (identically) for all \( k \) and the reduced \((L-1)\)-dimensional system discards this redundant information, while the observation corresponding to the \((\ell' + 1)\)-th sensor is carried over to the \((\ell' + 2)\)-th original sensor.

The second scenario corresponds to the case where one (or more) of the \( \rho_{\ell-1, \ell} \) is 0.

**Proposition 2:** Consider an \( L \)-sensor system, parameterized by \( \{\rho_{\ell-1, \ell}\} \), with \( \ell' \) indicating the smallest index such that \( \rho_{\ell', \ell+1} = 0 \). This system is equivalent to an \( \ell' \)-sensor system with the same parameters as that of the original system. It is as if sensors \((\ell' + 1)\) and beyond do not exist (or contribute) in the context of change detection.

**Proof:** The proof is again straightforward by considering the evolution of \( \{q_{k, \ell}\} \) in (3) and noting that \( q_{k, j}, j \geq \ell' + 2 \) are identically 0 for all \( k \).

It is useful to interpret Props. 1 and 2 via an “information flow” paradigm. If change propagation is instantaneous across a sensor (corresponding to the first case), it is as if the fusion center is oblivious to the presence of that sensor conditioned upon the previous sensors’ observations. In this setting, the detection delay corresponding to that sensor is zero, as would be expected from the fact that the geometric parameter is 1. In the second case, information flow to the fusion center (concerning change) is cut-off or blocked past the first sensor with a geometric parameter of 0. That is, the observations made by sensors \( \{\ell' + 1, \cdots, L\} \) (if any) do not contribute information to the fusion center in helping it decide whether the disruption has happened or not. Apart from these extreme cases of oblivious/blocking sensors, we can assume without any loss in generality that

\[
0 < \min_{\ell} \rho_{\ell-1, \ell} \leq \max_{\ell} \rho_{\ell-1, \ell} < 1.
\]

Continuity arguments suggest that if some \( \rho_{\ell-1, \ell} \) is small (but non-zero), it should be natural to expect that the \( \ell \)-th sensor and beyond may not “effectively” contribute any information to the
fusion center. We will interpret this observation after establishing tractable performance bounds for $\nu_A$.

**Probability of False Alarm:** We first show that letting $A \to \infty$ in $\nu_A$ corresponds to considering the regime where $P_{FA} \to 0$.

*Proposition 3:* The probability of false alarm with $\nu_A$ can be upper bounded as

$$P_{FA} \leq \frac{1}{1 + \rho \cdot \exp(A)}.$$  

That is, if $\alpha \leq 1$ and the threshold $A$ is set as $A = \log \left( \frac{1}{\rho \alpha} \right)$, then $P_{FA} \leq \alpha$.

*Proof:* The proof is elementary and follows the same argument as in [29], [36]. Note that $p_{k,1}$ and $\nu_A$ can also be written as

$$p_{k,1} = P \left( \{ \Gamma_1 > k \} \mid I_k \right)$$

$$\nu_A = \inf_k \left\{ p_{k,1} \leq \frac{1}{1 + \rho \cdot \exp(A)} \right\}.$$

Thus, we have

$$P_{FA} = P \left( \{ \nu_A < \Gamma_1 \} \right) = E [p_{\nu_A,1}] \leq \frac{1}{1 + \rho \cdot \exp(A)}.$$

**Universal Lower Bound on $E_{DD}$:** We now establish a lower bound on $E_{DD}$ for the class of stopping times $\Delta_\alpha$. That is, any stopping time $\tau$ should have an $E_{DD}$ larger than the lower bound if $P_{FA}$ is to be smaller than $\alpha$.

*Proposition 4:* Consider the class of stopping times $\Delta_\alpha = \{ \tau : P_{FA}(\tau) \leq \alpha \}$. Under the assumption that $\min_{\ell=2,\ldots,L} \rho_{\ell-1,\ell} > 0$, as $\alpha \to 0$, we have

$$\inf_{\tau \in \Delta_\alpha} E_{DD}(\tau) \geq \frac{\log \left( \frac{1}{\rho \alpha} \right) \cdot (1 + o(1))}{LD(f_1, f_0) + |\log(1 - \rho)|}.$$  

*Proof:* The proof follows on similar lines as [29, Lemma 1 and Theorem 1], but with some modifications to accommodate the change process setup. See Appendix C.

**Upper Bound on $E_{DD}$ of $\nu_A$:** We will establish an upper bound on $E_{DD}$ of $\nu_A$. Using this bound, it can be seen that $\nu_A$ meets the lower bound (proved above) for an appropriate choice of $A$, thus establishing its asymptotic optimality. The main result is as follows.

*Theorem 3:* Let $\{\rho_{\ell-1,\ell}\}$ be such that $0 < \min_{\ell} \rho_{\ell-1,\ell} \leq \max_{\ell} \rho_{\ell-1,\ell} < 1$. Further, assume that $D(f_1, f_0)$ be such that there exists some $j$ satisfying $\ell \leq j \leq L$ and

$$D(f_1, f_0) > \frac{1}{j - \ell + 1} \log \left( \frac{\sum_{\ell=0}^{j-1}(1 - \rho_{p,p+1})}{1 - \rho_{j,j+1}} \right),$$

(5)
for all $2 \leq \ell \leq L$. Then, $\nu_A$ with $A = \log \left( \frac{1}{\rho_0} \right)$ is asymptotically optimal (as $\alpha \to 0$). Furthermore, the performance of $\nu_A$ in this regime is of the form:

$$E_{DD} = \frac{\log \left( \frac{1}{\rho} \right) + |\log(P_{FA})|}{LD(f_1, f_0) + |\log (1 - \rho)|} + o(1).$$

The proof of Theorem 3 in the general case of an arbitrary number ($L$) of sensors with an arbitrary choice of $\{\rho_{\ell-1, \ell}\}$ results in cumbersome analysis. Hence, it is worthwhile considering the special case of two sensors that can be captured by just two change parameters: $\rho$ and $\rho_{1,2}$. The main idea that is necessary in tackling the general case is easily exposed in the $L = 2$ setting in Sec. VI. The general case is subsequently studied in Sec. VII.

VI. EXPECTED DETECTION DELAY: SPECIAL CASE ($L = 2$)

The main statement in the $L = 2$ case is the following result.

**Theorem 3 ($L = 2$):** The stopping time $\nu_A$ is such that $\nu_A \to \infty$ as $A \to \infty$. Further, if $D(f_1, f_0)$ satisfies

$$D(f_1, f_0) > \log (2 - \rho - \rho_{1,2}),$$

as $A \to \infty$, we also have

$$E_{DD} = E[\nu_A] \leq \frac{A}{2D(f_1, f_0) + |\log (1 - \rho)|}.$$  

We will work our way to the proof of the above statement by establishing some initial results.

**Proposition 5:** If $0 < \{\rho, \rho_{1,2}\} < 1$, we can recast $\{q_{k,\ell}\}$ as follows:

\[
q_{k,1} = \frac{1}{\rho} \\
q_{k,2} = \left( \frac{1 - \rho_{1,2}}{1 - \rho} \right)^k \cdot \left( 1 + \frac{1 - \rho_{1,2}}{1 - \rho} \right)^{\alpha_{k,2}} \cdot \prod_{m=1}^{k} L_{m,1}^{\alpha_{k,2}} \cdot \prod_{m=0}^{k-2} (1 + \zeta_{m,2}) \\
\zeta_{m,2} = \frac{1 - \rho}{(1 - \rho_{1,2}) \cdot (1 + q_{m,2}) \cdot L_{m+1,1}} \\
q_{k,3} = \frac{\rho_{1,2}}{(1 - \rho)^k} \cdot \left( 1 + \frac{1 - \rho_{1,2}}{1 - \rho} + \frac{1}{1 - \rho} \right)^{\alpha_{k,3}} \cdot \prod_{m=1}^{k} L_{m,1} L_{m,2}^{\alpha_{k,3}} \cdot \prod_{m=0}^{k-2} (1 + \zeta_{m,3}) \\
\zeta_{m,3} = \frac{\rho_{1,2} \cdot \left( 1 - \rho + (1 - \rho_{1,2}) \cdot L_{m+1,1} \cdot (1 + q_{m,2}) \right)}{L_{m+1,1} L_{m+1,2} \cdot (\rho_{1,2} + \rho_{1,2} q_{m,2} + q_{m,3})}.}
\]
Proof: We start with the recursions
\[ q_{k,2} = \frac{(1 - \rho_{1,2})}{1 - \rho} \cdot L_{k,1} \cdot (1 + q_{k-1,2}) \]
\[ q_{k,3} = \frac{L_{k,1} L_{k,2}}{1 - \rho} \cdot (\rho_{1,2} + \rho_{1,2} q_{k-1,2} + q_{k-1,3}) \cdot \]
The expression for \( q_{k,2} \) is obtained by isolating the term \((1 + q_{j-2,2})\) at every stage as \( j \) increases from 2 to \( k \). The expression for \( q_{k,3} \) is obtained by isolating the term \((\rho_{1,2} + \rho_{1,2} q_{j-2,2} + q_{j-3,2})\) at every stage as \( j \) increases.

The test \( \nu_A \) can now be rewritten as
\[ \nu_A = \inf_k \left\{ \log \left( q_{k,2} + q_{k,3} \right) > A \right\} \]
\[ = \inf_k \left\{ \log \left( \alpha_{k,2} \cdot C_1 \cdot J_2 + \alpha_{k,3} \cdot C_1 C_2 \cdot J_3 \right) > A \right\} \]
\[ = \inf_k \left\{ \log(\alpha_{k,2} \cdot C_1 \cdot J_2) + \log \left( 1 + C_2 \cdot \frac{\alpha_{k,3}}{\alpha_{k,2}} \cdot J_3 \right) > A \right\}. \]

We need the following preliminaries in the course of our analysis.

Lemma 1: Since \( q_{m,2} \geq 0 \), note that \( J_2 \) can be trivially upper bounded as
\[ J_2 \leq \prod_{m=1}^{k-1} \left( 1 + \frac{1 - \rho}{(1 - \rho_{1,2}) \cdot L_{m,1}} \right). \]

Lemma 2: If \( \{x, x_1, x_2, \cdots\} \) are i.i.d. with \( x \geq 0 \) and \( E[\log(x)] > 0 \), then
\[ \frac{1}{k} \log \left( 1 + \prod_{m=1}^{k} x_m \right) - \frac{\sum_{m=1}^{k} \log(x_m)}{k} \xrightarrow{k \to \infty} 0 \text{ a.s. and in mean.} \]

If \( \{x, x_1, x_2, \cdots\} \) are i.i.d. with \( x \geq 0 \) and \( E[\log(x)] \leq 0 \), then
\[ \frac{1}{k} \log \left( 1 + \prod_{m=1}^{k} x_m \right) \xrightarrow{k \to \infty} 0 \text{ a.s. and in mean.} \]

Note that both these conclusions are true even if \( \{x_m\} \) are not i.i.d. (or even independent) as long as the condition on the sign of \( E[\log(x)] \) can be replaced with an almost sure (and in mean) statement on the sign of \( \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \log(x_m) \) (or an appropriate variant thereof).

The following statement, commonly referred to as the Blackwell’s elementary renewal theorem [35, pp. 204-205], is needed in our proofs.

Lemma 3: Let \( x_m \) be i.i.d. positive random variables and define \( T_m \) as follows:
\[ T_m = T_{m-1} + x_m, \quad m \geq 1 \text{ and } T_0 = 0. \]
The number of renewals in \([0, t]\) is \(N_t = \inf_k \left\{ T_k > t \right\}\). Then, we have
\[
\frac{N_t}{t} \to \frac{1}{\mu} \quad \text{a.s. as } t \to \infty \quad \text{and} \quad \frac{E[N_t]}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty,
\]
where \(\mu \triangleq E[x_m] \in (0, \infty]\).

**Proof of Theorem 3 (L = 2):** We will postpone the proof of the first statement to Sec. VII when we consider the general case in Prop. 8. For the second statement, we first use the bound for \(J_2\) from Lemma 1 and the fact that \(\zeta_{m,t} \geq 0\), and thus we have
\[
\log \left( 1 + C_2 \cdot \frac{\alpha_{k,3}}{\alpha_{k,2}} \cdot \frac{J_3}{J_2} \right) \geq \log \left( 1 + C_2 \cdot \frac{\alpha_{k,3}}{\alpha_{k,2}} \cdot \frac{1}{\prod_{m=1}^{k-1} \left( 1 + \frac{1-\rho}{(1-\rho) L_{m,1}} \right)} \right)
\]
\[
\geq \log \left( 1 + \prod_{m=1}^{k} \frac{\rho_{1,2}^{1/k} \cdot L_{m,2}}{(1-\rho_{1,2}) \cdot \left( 1 + \frac{1-\rho}{(1-\rho) L_{m,1}} \right)} \right).
\]
Now, observe that
\[
E \left[ \log \left( \frac{L_{m,2}}{(1-\rho_{1,2}) \cdot \left( 1 + \frac{1-\rho}{(1-\rho) L_{m,1}} \right)} \right) \right]
\]
\[
= D(f_1, f_0) + \log \left( \frac{1}{1-\rho_{1,2}} \right) - E \left[ \log \left( 1 + \frac{1-\rho}{(1-\rho_{1,2}) L_{m,1}} \right) \right]
\]
\[
\geq D(f_1, f_0) + \log \left( \frac{1}{1-\rho_{1,2}} \right) - \log \left( 1 + E \left[ \frac{1-\rho}{(1-\rho_{1,2}) L_{m,1}} \right] \right)
\]
\[
= D(f_1, f_0) - \log (2 - \rho - \rho_{1,2}) > 0
\]
where the first equality follows since \(\rho_{1,2} > 0\) (change has to eventually happen at the second sensor to ensure that \(E[\log(L_{m,2})] = D(f_1, f_0)\), the second step follows from Jensen’s inequality and the third equality from the fact that \(E\left[\frac{1}{L_{m,1}}\right] = 1\). Using this fact in conjunction with Lemma 2 and noting that \(\rho_{1,2} > 0\), as \(k \to \infty\), we have
\[
\log(\alpha_{k,2} \cdot C_1 \cdot J_2) + \log \left( 1 + C_2 \cdot \frac{\alpha_{k,3}}{\alpha_{k,2}} \cdot \frac{J_3}{J_2} \right) \geq \log (C_1 C_2 \cdot \alpha_{k,3} \cdot J_3)
\]
\[
\geq \sum_{m=1}^{k} \log \left( \frac{\rho_{1,2}^{1/k} \cdot L_{m,1} \cdot L_{m,2}}{L_k \cdot 1-\rho} \right).
\]
The above relationship implies that \(\nu_A \leq \nu_{L,A}\) where
\[
\nu_{L,A} \triangleq \inf_k \left\{ L_k > A \right\}.
\]
Applying Lemma 3 (since the entries in the definition of $\nu_{L,A}$ are independent) and the first statement of the theorem that $\nu_A \to \infty$ as $A \to \infty$, we have

$$\frac{E[\nu_A]}{A} \leq \frac{E[\nu_{L,A}]}{A} \xrightarrow{A \to \infty} \frac{1}{2D(f_1, f_0) + |\log (1 - \rho)|}.$$  

\[\blacksquare\]

VII. EXPECTED DETECTION DELAY: GENERAL CASE ($L \geq 3$)

We now consider the general case where $L \geq 3$. The main statement here is as follows.

**Theorem 3 ($L \geq 3$):** If $D(f_1, f_0)$ is such that the condition (5) is satisfied, as $A \to \infty$, we have

$$E_{DD} = E[\nu_A] \leq \frac{A}{L D(f_1, f_0) + |\log (1 - \rho)|}.$$  

\[\blacksquare\]

As before, we will work towards the proof of this statement. For this, the following generalizations of Prop. 5 and Lemma 1 are necessary.

**Proposition 6:** We have

$$q_{k,\ell} = \alpha_{k,\ell} \cdot \prod_{j=1}^{\ell-1} \prod_{m=1}^{k} L_{m,j} \cdot \prod_{m=0}^{k-2} (1 + \zeta_{m,\ell}), \quad \ell = 2, \ldots, L + 1$$

where

$$\alpha_{k,2} = \left( \frac{1 - \rho_{1,2}}{1 - \rho} \right)^k \cdot \left( 1 + \frac{1 - \rho_{1,2}}{1 - \rho} \right)$$

$$\alpha_{k,\ell} = \left( \frac{1 - \rho_{\ell-1,\ell}}{1 - \rho} \right)^k \cdot \prod_{j=1}^{\ell-2} \rho_{j,j+1} \cdot \left( \sum_{j=0}^{\ell-1} \frac{1 - \rho_{j,j+1}}{1 - \rho} \right), \quad \ell \geq 3$$

$$\zeta_{m,\ell} = \frac{1}{(1 - \rho_{\ell-1,\ell}) \cdot \prod_{j=1}^{\ell-1} L_{m+1,j}} \cdot \sum_{j=1}^{\ell} q_{m,j} w_j^\ell C_{m+1,j,\ell}$$

$$B_{m,n,\ell} = \sum_{p=n-1}^{\ell-1} (1 - \rho_{p,p+1}) \cdot \prod_{j=1}^{p} L_{m,j}, \quad n = 1, \ldots, \ell$$

$$C_{m,n,\ell} = B_{m,n,\ell} - (1 - \rho_{\ell-1,\ell}) \cdot \prod_{j=1}^{\ell-1} L_{m,j}, \quad n = 1, \ldots, \ell.$$

**Proof:** The proof is provided in Appendix D for the sake of completeness. Also, see Appendix D for how this proposition can be reduced to the case of [32].

**Lemma 4:** The following upper bound for $\zeta_{m,\ell}$ is obvious when $\max_{\ell} \rho_{\ell-1,\ell} < 1$:

$$\zeta_{m,\ell} \leq \frac{B_{m+1,1,\ell}}{(1 - \rho_{\ell-1,\ell}) \cdot \prod_{j=1}^{\ell-1} L_{m+1,j}} = \frac{\sum_{p=0}^{\ell-2} (1 - \rho_{p,p+1}) \prod_{j=1}^{p} L_{m+1,j}}{(1 - \rho_{\ell-1,\ell}) \cdot \prod_{j=1}^{\ell-1} L_{m+1,j}}.$$  

\[\blacksquare\]
From Prop. 6, \( \nu_A \) can be conveniently rewritten as

\[
\nu_A = \inf_k \left\{ \log \left( \sum_{\ell=2}^{L+1} \alpha_{k,\ell} \cdot C_1 \cdots C_{\ell-1} \cdot J_\ell \right) > A \right\}.
\]

Unlike the setting in Sec. VI, the structure of \( \nu_A \) (as of now) is not amenable to studying \( E_{DD} \) (in further detail). This is because it has the form of log of sum of random variables (see [36] for similar difficulties in the multi-hypothesis testing problem). We alleviate this difficulty by rewriting the test statistic in terms of quantities whose asymptotics can be easily studied.

**Proposition 7:** We have the following expansion for the test statistic:

\[
\log \left( \sum_{\ell=2}^{L+1} \alpha_{k,\ell} \cdot C_1 \cdots C_{\ell-1} \cdot J_\ell \right) = \log \left( \frac{\alpha_{k,2} \cdot C_1 \cdot J_2}{\alpha_{k,2} \cdot J_2} \right) + \sum_{\ell=2}^{L} \log \left( 1 + \frac{\eta_\ell \cdot \alpha_{k,\ell+1} \cdot C_\ell \cdot J_{\ell+1}}{\alpha_{k,\ell} \cdot J_\ell} \right)
\]

where

\[
\beta_{k,\ell} = \frac{\alpha_{k,\ell+1}}{\alpha_{k,\ell}} = \left( 1 - \frac{1 - \rho_{\ell,\ell+1} - 1}{1 - \rho_{\ell,\ell+1} - 1} \right) \cdot \rho_{\ell-1,\ell} \cdot \left( 1 + \frac{1 - \rho_{\ell,\ell+1}}{\sum_{m=0}^{\ell-1} 1 - \rho_{m,m+1}} \right), \quad \ell = 2, \ldots, L
\]

\[
\eta_{\ell+1} = \frac{\eta_\ell \cdot \beta_{k,\ell} \cdot C_\ell \cdot J_{\ell+1}}{1 + \eta_\ell \cdot \beta_{k,\ell} \cdot C_\ell \cdot J_{\ell+1}}, \quad \ell = 2, \ldots, L - 1
\]

with \( \eta_2 = 1 \).

**Proof:** The proof is straightforward by using the induction principle.

The following proposition establishes the general asymptotic trend of \( \nu_A \).

**Proposition 8:** The test \( \nu_A \) is such that \( \nu_A \to \infty \) a.s. as \( A \to \infty \).

**Proof:** See Appendix D.

As we try to understand \( \nu_A \) further, it is important to note that the behavior of the decision statistic of \( \nu_A \) is determined (only) by the trends of

\[
x_\ell \triangleq \beta_{k,\ell} \cdot C_\ell \cdot \frac{J_{\ell+1}}{J_\ell}, \quad \ell = 2, \ldots, L.
\]

This is so because the asymptotics of \( \{ \eta_\ell \} \) are also primarily determined by the trends of \( \{ x_\ell \} \). We now develop the generalized version of the heuristic in Sec. VI for the upper bound of \( E_{DD} \).
Consider the case where \( L = 4 \). The second piece in the description of the test statistic (in Prop. 7) can be written as

\[
\mathcal{L} \triangleq \log (1 + \eta_2 x_2) + \log (1 + \eta_3 x_3) + \log (1 + \eta_4 x_4)
\]

where the evolution of \( \eta_\ell \) and \( x_\ell, \ell = 2, 3, 4 \) is described in Prop. 7. In the regime where \( k \to \infty \), note that if \( x_2 \to \infty \) (with high probability), then \( \eta_3 \to 1 \). On the other hand, if \( x_2 \to 0 \) (with high probability), then \( \eta_3 \to x_2 \). Thus, we can identify (and partition) eight cases as follows:

- **Case 1**: \( x_2 \to 0, x_2 x_3 \to 0, x_2 x_3 x_4 \to 0 \Rightarrow \eta_3 \to x_2, \eta_4 \to x_2 x_3 \Rightarrow \mathcal{L} \to 0 \)
- **Case 2**: \( x_2 \to 0, x_2 x_3 \to 0, x_2 x_3 x_4 \to \infty \Rightarrow \eta_3 \to x_2, \eta_4 \to x_2 x_3 \Rightarrow \mathcal{L} \to \log(x_2 x_3 x_4) \)
- **Case 3**: \( x_2 \to 0, x_2 x_3 \to \infty, x_4 \to 0 \Rightarrow \eta_3 \to x_2, \eta_4 \to 1 \Rightarrow \mathcal{L} \to \log(x_2 x_3) \)
- **Case 4**: \( x_2 \to 0, x_2 x_3 \to \infty, x_4 \to \infty \Rightarrow \eta_3 \to x_2, \eta_4 \to 1 \Rightarrow \mathcal{L} \to \log(x_2 x_3 x_4) \)
- **Case 5**: \( x_2 \to \infty, x_3 \to 0, x_3 x_4 \to 0 \Rightarrow \eta_3 \to 1, \eta_4 \to x_3 \Rightarrow \mathcal{L} \to \log(x_2) \)
- **Case 6**: \( x_2 \to \infty, x_3 \to 0, x_3 x_4 \to \infty \Rightarrow \eta_3 \to 1, \eta_4 \to x_3 \Rightarrow \mathcal{L} \to \log(x_2 x_3 x_4) \)
- **Case 7**: \( x_2 \to \infty, x_3 \to \infty, x_4 \to 0 \Rightarrow \eta_3 \to 1, \eta_4 \to 1 \Rightarrow \mathcal{L} \to \log(x_2 x_3) \)
- **Case 8**: \( x_2 \to \infty, x_3 \to \infty, x_4 \to \infty \Rightarrow \eta_3 \to 1, \eta_4 \to 1 \Rightarrow \mathcal{L} \to \log(x_2 x_3 x_4) \)

In all the eight cases, we have a universal description for \( \mathcal{L} \) (as \( k \to \infty \)) that holds with high probability:

\[
\mathcal{L} \xrightarrow{k \to \infty} \sum_{m=2}^{\ell^* - 1} \log (x_m), \quad \ell^* = \arg \min \left\{ \prod_{m=\ell}^{j} x_m \to 0 \text{ for all } j \geq \ell \right\}.
\]

If \( \ell^* = 2 \), then the above summation is replaced by 0, and if there exists no \( \ell \in \{2, 3, 4\} \) such that the above condition holds, then \( \ell^* \) is set to 5.

The following proposition provides a precise mathematical formulation of the above heuristic.

**Proposition 9**: Let the following limit be well-defined and be denoted as \( \gamma_{\ell,j} \):

\[
\gamma_{\ell,j} \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{m=1}^{k} \log \left( \frac{1 + \zeta_{m,j+1}}{1 + \zeta_{m,\ell}} \right).
\]

Define \( \ell^* \) as

\[
\ell^* \triangleq \arg \min_{\ell : 2 \leq \ell \leq L} \left\{ \Delta_{\ell,j} \leq 0 \text{ for all } j = \ell, \ldots, L \right\}
\]

where

\[
\Delta_{\ell,j} = \log \left( \frac{1 - \rho_{j,j+1}}{1 - \rho_{\ell-1,j,\ell}} \right) + (j - \ell + 1)D(f_1, f_0) + \gamma_{\ell,j}.
\]

\[
\Delta_{\ell,j} \triangleq \log \left( \frac{1 - \rho_{j,j+1}}{1 - \rho_{\ell-1,j,\ell}} \right) + (j - \ell + 1)D(f_1, f_0) + \gamma_{\ell,j}.
\]
If there exists no element in the set for the arg min operation in (6), we set \( \ell^* = L + 1 \). Then, as \( A \to \infty \) (and hence, \( k = \nu_A \to \infty \) a.s. from Prop. 8), we have

\[
\frac{1}{k} \sum_{\ell=2}^{L} \log (1 + \eta \ell x) - \frac{1}{k} \sum_{\ell=2}^{\ell^*-1} \log(x) \to 0 \text{ a.s.}
\]  

(7)

If \( \ell^* = 2 \), then the second term in the above expression is set to 0.

**Proof:** See Appendix D.

Following Props. 8 and 9 as \( A \to \infty \), \( \nu_A \) can be restated as

\[
\nu_A \to \inf_k \left\{ \sum_{m=1}^{k} \left( \log \left( \frac{1 - \rho_{1,2}}{1 - \rho} \right) + \log(L_{m,1}) + \log(1 + \zeta_{m,2}) + \sum_{\ell=2}^{\ell^*-1} \log(x) \right) > A \right\}
\]

with \( \ell^* \) defined in (6).

Observe that if the condition in Prop. 9 is satisfied, the first \( \ell^* - 1 \) sensors contribute to the slope of \( E_{DD} \) and the rest of the sensors \( \ell^*, \ldots, L \) (if any) do not contribute to the slope. It is useful to understand the conditions under which \( \ell^* = L + 1 \).

Theorem 3 provides a simple condition such that the observations from all the \( L \) sensors contribute to the slope. We are now prepared to prove it.

**Proof of Theorem 3 (\( L \geq 3 \)):** First, using Lemma 4 note that, we can bound \( \Delta_{\ell,j} \) as

\[
\Delta_{\ell,j} \geq (j - \ell + 1) D(f_1, f_0) + \log(1 - \rho_{j,j+1}) - \mathbb{E} \left[ \log \left( \sum_{p=0}^{\ell-1} \frac{1 - \rho_{p,p+1}}{1 - \rho_{\ell,i}} L_{\ell,i} \right) \right].
\]

Using Jensen’s inequality and noting that \( E_{f_1} \left[ \frac{1}{\prod_{i=p+1}^{\ell-1} L_{\ell,i}} \right] = 1 \), (5) is sufficient to ensure that for all \( \ell = 2, \ldots, L \), there exists some \( j \geq \ell \) such that \( \Delta_{\ell,j} > 0 \). It is important to realize that the above condition is necessary as well as sufficient for \( \ell^* = L + 1 \). Thus, under the assumption that (5) holds, invoking Prop. 8 as \( A \to \infty \) (that is, letting \( k = \nu_A \to \infty \) a.s. and using Prop. 9), \( \nu_A \) can be written as

\[
\nu_A \to \inf_k \left\{ \sum_{m=1}^{k} \left( \sum_{\ell=1}^{L} \log(L_{m,\ell}) + \log \left( \frac{1}{1 - \rho} \right) + \log(1 + \zeta_{m,L+1}) \right) > A \right\}.
\]

Note that since \( \zeta_{m,L+1} \geq 0 \), we have

\[
\sum_{m=1}^{k} \left( \sum_{\ell=1}^{L} \log(L_{m,\ell}) + \log \left( \frac{1}{1 - \rho} \right) + \log(1 + \zeta_{m,L+1}) \right) \geq \sum_{m=1}^{k} \left( \sum_{\ell=1}^{L} \log(L_{m,\ell}) + \log \left( \frac{1}{1 - \rho} \right) \right),
\]
and hence, \( \nu_A \leq \nu_{L,A} \) where

\[
\nu_{L,A} \triangleq \inf_k \left\{ L_k > A \right\}.
\]

Thus, we have

\[
\frac{E[\nu_A]}{A} \leq \frac{E[\nu_{L,A}]}{A} \xrightarrow{A \to \infty} \frac{1}{LD(f_1, f_0) + \log \left( \frac{1}{1-\rho} \right)}
\]

where the convergence is again due to Lemma \(3\).

VIII. DISCUSSION AND NUMERICAL RESULTS

**Discussion:** A loose sufficient condition for all the \( L \) sensors to contribute to the slope of \( \nu_A \) is that

\[
D(f_1, f_0) > \max_{\ell=1,\ldots,L-1} \min_{j \geq \ell+1} \frac{1}{j - \ell} \cdot \log \left( \frac{\sum_{p=0}^{\ell} (1 - \rho_{p,p+1})}{1 - \rho_{j,j+1}} \right) \triangleq \gamma_u.
\]

Another sufficient condition is that

\[
D(f_1, f_0) > \max_{\ell=1,\ldots,L-1} \frac{1}{L - \ell} \cdot \log \left( 1 - \rho + \sum_{j=1}^{\ell} (1 - \rho_{j,j+1}) \right).
\]

That is, if \( \rho \) is such that

\[
\rho \geq \sum_{\ell=2}^{L} (1 - \rho_{\ell,1,\ell}),
\]

then \( \gamma_u \leq 0 \) and the condition of Theorem \(3\) reduces to a mild one that the K-L divergence between \( f_1 \) and \( f_0 \) be positive. A special setting where the above condition is true (irrespective of the rarity of the disruption-point) is the regime where change propagates across the sensor array “quickly.” The case of [32] is an extreme example of this regime and Theorem \(3\) recaptures this extreme case.

In more general regimes where change propagates across the sensor array “slowly”, either the disruption-point should become less rare (independent of the choice of \( f_1 \) and \( f_0 \)) or that the densities \( f_1 \) and \( f_0 \) be sufficiently discernible (independent of the rarity of the disruption-point) so that all the \( L \) sensors can contribute to the asymptotic slope. When these conditions fail to hold, it is not clear whether the theorems are applicable, or even if all the \( L \) sensors contribute to the slope of \( E[\nu_A] \). Nevertheless, it is reasonable to conjecture that as long as \( \min_{\ell} \rho_{\ell,1,\ell} > 0 \), then all the \( L \) sensors contribute to the asymptotic slope.

However, the difference between the asymptotic and the non-asymptotic regimes need a careful revisit. Following the initial remark (Prop. \(2\)) on the extreme case of blocking sensors (where some \( \rho_{\ell-1,\ell} = 0 \)), in the more realistic case where some \( \rho_{\ell-1,\ell} \) may be small (but non-zero),
it is possible that if \( D(f_1, f_0) \) is smaller than some threshold value (determined by the change propagation parameters), not all of the \( L \) sensors may “effectively” contribute to the slope of \( E_{DD} \), at least for reasonably small, but non-asymptotic values of \( P_{FA} \). For example, see the ensuing discussion where numerical results illustrate this behavior at \( P_{FA} \) values of \( 10^{-4} \) to \( 10^{-5} \) for some choice of change propagation parameters, \emph{even} when the condition in Theorem \( \text{[3]} \) is met. When the condition in Theorem \( \text{[3]} \) is not met, such a behavior is expected to be more typical.

The final comment is on the approach pursued in this paper. While the approach pursued in Sec. \( \text{VI} \) and \( \text{VII} \) results in interesting conclusions, it is not clear if this approach is \emph{fundamental} in the sense that this is the only approach possible for characterizing \( E_{DD} \) vs. \( P_{FA} \). Furthermore, this approach assumes the existence of \( \{\gamma_{\ell,j}\} \). Even if these quantities exist and are hence, theoretically computable, such a computation is complicated by the fact that \( \{\zeta_{m,\ell}, m = 1, \cdots, k\} \) are correlated. Thus, verification of the exact condition in Prop. \( \text{[9]} \) (equivalently, computing \( \ell^* \)) has to be achieved either via Monte Carlo methods or by bounding \( \Delta_{\ell,j} \), as done here. Furthermore, correlation of \( \{\zeta_{m,\ell}\} \) and hence, \( y_m \) (see \( \text{[8]} \)) implies that statistics of \( \nu_A \) have to be obtained using non-linear renewal theoretic techniques for general (correlated) random variables \( \text{[37]} \). This is the subject of current work.

\textbf{Numerical Study I – Performance Improvement with} \( \nu_A \): Given that the structure of \( \tau_{opt} \) is not known in closed-form, we now present numerical studies to show that \( \nu_A \) results in substantial improvement in performance over both a single sensor test (which uses the observations only from the first sensor and ignores the other sensor observations) and a test that uses the observations from all the sensors but under a mismatched model (where the change-point for all the sensors is assumed to be the same), even under realistic modeling assumptions.

The first example corresponds to a two sensor system where the occurrence of change is modeled as a geometric random variable with parameter \( \rho = 0.001 \). Change propagates from the first sensor to the second with the geometric parameter \( \rho_{1,2} = 0.1 \). The pre- and post-change densities are \( \mathcal{CN}(0, 1) \) and \( \mathcal{CN}(1, 1) \), respectively so that \( D(f_1, f_0) = 0.50 \). Fig. \( \text{2} \) shows that \( \nu_A \) can result in an improvement of at least 4 units of delay at even marginally large \( P_{FA} \) values on the order of \( 10^{-3} \).

The second example corresponds to a five sensor system where \( \rho = 0.005 \). Change propagates across the array according to the following model: \( \rho_{1,2} = 0.1, \rho_{2,3} = 0.2, \rho_{3,4} = 0.5 \) and \( \rho_{4,5} = 0.7 \). The pre- and the post-change densities are \( \mathcal{CN}(0, 1) \) and \( \mathcal{CN}(0.75, 1) \) so that \( D(f_1, f_0) \approx 0.2813 \). With \( D(f_1, f_0) \) and the change parameters as above, Theorem \( \text{[3]} \) assures us that at least \( L = 2 \) sensors contribute to the \( E_{DD} \) vs. \( P_{FA} \) slope asymptotically. On the other hand, Fig. \( \text{3} \) shows that more than two sensors indeed contribute to the slope. Thus, it can be seen that
Fig. 2. False alarm vs. Expected detection delay for a $L = 2$ setting with $\rho = 0.001$ and $\rho_{1,2} = 0.1$.

Fig. 3. False alarm vs. Expected detection delay for a typical $L = 5$ setting.

Theorem 3 provides only a sufficient condition on performance bounds. It is also worth noting
the transition in slope (unlike the case in [32]) for both the mismatched test and $\nu_A$ as $P_{FA}$ decreases from moderately large values to zero, whereas the slope of the single sensor test (as expected) remains constant.

**Numerical Study II – Performance Gap Between the Tests:** We now present a second case-study with the main goal being the understanding of the relative performance of $\nu_A$ with respect to the single sensor and the mismatched tests. We again consider a $L = 2$ sensor system and we vary the change process parameters, $\rho$ and $\rho_{1,2}$, in this study. The pre- and the post-change densities are $CN(0,1)$ and $CN(1.2,1)$ so that $D(f_1, f_0) = 0.72$.

![Graphs showing false alarm vs. expected detection delay for different parameter settings.](image)

Fig. 4. False alarm vs. Expected detection delay for a $L = 2$ setting with different model parameters.

Fig. 4 and Fig. 5(b) show the performance of the three tests with varying $\rho$ parameters for a
fixed choice of $\rho_{1,2}$. We observe that the gap in performance between the single sensor test and $\nu_A$ increases as $\rho$ decreases, whereas the gap between $\nu_A$ and the mismatched test stays fairly constant. Similarly, Fig. 5 shows the performance of the three tests with varying $\rho_{1,2}$ parameters for a fixed choice of $\rho$. We observe from these plots that the gap between the mismatched test and $\nu_A$ increases as $\rho_{1,2}$ decreases, whereas the gap between the single sensor test and $\nu_A$ increases as $\rho_{1,2}$ increases.

![Graphs showing performance comparison](image)

Fig. 5. False alarm vs. Expected detection delay for a $L = 2$ setting with different model parameters.

The choice of $D(f_1, f_0) = 0.72$ is such that the sufficient condition in Theorem are satisfied, independent of the change parameters. Hence, we expect the slope of the $E_{DD}$ vs. $P_{FA}$ plot to be of the form $\frac{1}{2D(f_1,f_0)+|\log(1-\rho)|}$ asymptotically as $P_{FA} \to 0$. Nevertheless, Fig. 5(c) and (d) show
that, when both $\rho$ and $\rho_{1,2}$ are small, the slope of $\nu_A$ is only as good as (or slightly better than) the single sensor test, which is known to have a slope of the form $\frac{1}{D(f_1, f_0) + |\log(1-\rho)|}$. Thus, we see that even though our theory guarantees that both the sensors’ observations contribute in the eventual performance of $\nu_A$ asymptotically, we may not see this behavior for reasonable choices of $P_{FA}$ like $10^{-4}$. The case of observation models not meeting the conditions of Theorem 3 is expected to show this trend for even lower $P_{FA}$ values.

To summarize these observations, if $E_{DD, \nu_A}$, $E_{DD, MM}$ and $E_{DD, SS}$ denote the expected detection delays for $\nu_A$, mismatched and single sensor tests (respectively) for some fixed choice of $P_{FA}$, then

$$
E_{DD, MM} - E_{DD, \nu_A} \propto \frac{1}{\rho_{1,2}} \text{ and independent of } \rho \\
E_{DD, SS} - E_{DD, \nu_A} \propto \frac{\rho_{1,2}}{\rho}.
$$

It is interesting to note from the above equations that $\rho_{1,2}$ impacts the gap between the two tests in a contrasting way. The test $\nu_A$ is expected to result in significant performance improvement in the regime where $\rho$ is small, but $\rho_{1,2}$ is neither too small nor too large. In fact, this regime where $\nu_A$ is expected to result in significant performance improvement is the precise regime that is of importance in practical contexts. This is so because we can expect the occurrence of disruption (e.g., cracks in bridges, intrusions in networks, onset of epidemics etc.) to be a rare phenomenon. Once the disruption occurs, we expect change to propagate across the sensor array fairly quickly due to the geographical (network proximity in the case of computer networks) proximity of the other sensors, but not so quick that the extreme case of [32] is applicable. Classifying the regime of $\{\rho_{l-1, l}\}$ and $D(f_1, f_0)$ where significant performance improvement is possible with $\nu_A$ is ongoing work. It is also of interest to come up with better test structures in the regime where $\nu_A$ does not lead to a significant performance improvement.

**IX. CONCLUDING REMARKS**

We considered the centralized, Bayesian version of the change process detection problem in this work and posed it in the classical POMDP framework. This formulation of the change detection problem allows us to establish the sufficient statistics for the DP under study and a recursion for the sufficient statistics. While we obtain the broad structure of the optimal stopping rule ($\tau_{opt}$), any further insights into it are rendered infeasible by the complicated nature of the infinite-horizon cost-to-go function. Nevertheless, $\tau_{opt}$ reduces to a threshold rule (denoted in this work as $\nu_A$) in the rare disruption regime. The test $\nu_A$ possesses many attractive properties: i) it is of low-complexity; ii) it is asymptotically optimal in the vanishing false alarm probability
regime under certain mild assumptions on the K-L divergence between the post- and the pre-change densities; and iii) numerical studies suggest that it can lead to substantially improved performance over naive tests. Thus, $\nu_A$ serves as an attractive test for practical applications that can be modeled as a change process.

To the best of our knowledge, this is the first work to consider the change process detection problem in extensive detail. Thus, there exists potential for extending this work in multiple new directions. While we established the asymptotic optimality of $\nu_A$ when $D(f_1, f_0) \geq \gamma_u$, it is unclear as to what happens when $D(f_1, f_0) < \gamma_u$. In other words, is $\ell^* = L + 1$ when $D(f_1, f_0) < \gamma_u$ given that $\gamma_u > 0$? It is most likely that $\nu_A$ is asymptotically optimal even in this regime as long as $\min \rho_{\ell-1, \ell} > 0$, but establishing this result may involve some ingenious techniques. However, if $\nu_A$ is not asymptotically optimal in this regime, it is of interest to design better low-complexity stopping rules; e.g., Threshold tests on weighted sums of the a posteriori probabilities based on further study of the structure of $\tau_{opt}$ etc.

More careful asymptotic analysis of $\nu_A$ and performance gap between: i) $\nu_A$ and the mismatched test, ii) $\nu_A$ and the single sensor test, and iii) $\nu_A$ and weighted threshold tests etc. would involve tools from non-linear renewal theory [26], [29], [37] and is the subject of current attention. Such an asymptotic study could in turn drive the design of better test structures. Our numerical results also illustrate and motivate the need for non-asymptotic characterization (piece-wise linear approximations of the $E_{DD}$ vs. $P_{FA}$ curve) of the proposed tests. Unlike the case of instantaneous change propagation [29], [32], we showed that asymptotic characterizations may not kick in quickly for small $P_{FA}$ values if the change propagates too “slowly” across the sensor array. Under such circumstances, it is also of interest to revisit the precise definition of optimality of a stopping rule.

Decentralized [32], [34], censored [38], multi-channel [18] and robust [39], [40] versions of change detection are motivated by these constraints. Extensions of this work to more general observation models are important in the context of practical applications. For example, non-iid [29] and Hidden-Markov models [24] have found increased interest in biological problems determined by an event-driven potential [6], [7]. Practical applications will in turn drive the need for understanding change detection with certain specific observation models.

**APPENDIX**

A. **Completing Proof of Theorem 1**: Establishing Concavity of $A_k^T(\cdot)$ and $J_k^T(\cdot)$

We now show that $A_k^T(p_k)$ and $J_k^T(p_k)$ are concave in $p_k$. First, note that $J_T^T(p_T) = p_{T,1}$ is concave in $p_T$ because it is affine. Using the recursion for $p_T$, it is straightforward to check that

$$A_{T-1}^T(p_{T-1}) = E[J_T^T(p_T) | I_{T-1}] = p_{T-1,1} \cdot (1 - \rho).$$
Using this in the definition of \( J_{T-1}^T(p_{T-1}) \), we have

\[
J_{T-1}^T(p_{T-1}) = \begin{cases} 
  p_{T-1,1} & 0 \leq p_{T-1,1} \leq \frac{c}{c+\rho} \\
  c + p_{T-1,1}(1 - \rho - c) & \frac{c}{c+\rho} \leq p_{T-1,1} \leq 1.
\end{cases}
\]

Since both \( A_{T-1}^T(p_{T-1}) \) and \( J_{T-1}^T(p_{T-1}) \) are affine and piecewise-affine (It is important to note that the slope of the second affine part, which is \( 1 - \rho - c \), is smaller than the first (= 1) in \( p_{T-1,1} \) respectively, they are concave.

We now assume that \( J_{k+1}^T(p_{k+1}) \) is concave in \( p_{k+1} \) and show that \( A_k^T(p_k) \) is also concave in \( p_k \). For this, consider \( \lambda A_k^T(p_k^1) + (1 - \lambda) A_k^T(p_k^2) \) with \( p_k^1 \) and \( p_k^2 \) being two elements in the standard \( L \)-dimensional simplex. We have

\[
\lambda A_k^T(p_k^1) + (1 - \lambda) A_k^T(p_k^2) = \int \left[ \lambda J_{k+1}^T(p_{k+1}^1) \mu_1 + (1 - \lambda) J_{k+1}^T(p_{k+1}^2) \mu_2 \right]_{Z_{k+1} = z} dz
\]

\[
= \int \left[ \mu J_{k+1}^T(p_{k+1}^1) + (1 - \mu) J_{k+1}^T(p_{k+1}^2) \right] \times \left( \lambda \mu_1 + (1 - \lambda) \mu_2 \right)_{Z_{k+1} = z} dz
\]

where

\[
\mu_i = f(Z_{k+1} | I_k)|_{p_k = p_k^i} = \sum_{j=1}^{L+1} \left[ \left( \sum_{m=1}^{j} w_{k+1, j, m} p_{k, m}^j \right) \Phi_{obs}(k+1, j) \right], \quad i = 1, 2, \text{ and }
\]

\[
\mu = \frac{\lambda \mu_1}{\lambda \mu_1 + (1 - \lambda) \mu_2}.
\]

Using the concavity of \( J_{k+1}^T(\cdot) \), we can upper bound the above as follows:

\[
\lambda A_k^T(p_k^1) + (1 - \lambda) A_k^T(p_k^2) \leq \int \left[ J_{k+1}^T(\mu p_{k+1}^1 + (1 - \mu) p_{k+1}^2) \right] \times \left( \lambda \mu_1 + (1 - \lambda) \mu_2 \right)_{Z_{k+1} = z} dz
\]

If we define

\[
p_k^3 \triangleq \lambda p_k^1 + (1 - \lambda) p_k^2,
\]

it is straightforward to check that

\[
p_{k+1}^3 = \mu p_{k+1}^1 + (1 - \mu) p_{k+1}^2.
\]

Using these facts, we have

\[
\lambda A_k^T(p_k^1) + (1 - \lambda) A_k^T(p_k^2) \leq A_k^T(\lambda p_k^1 + (1 - \lambda) p_k^2),
\]

thus establishing the concavity of \( A_k^T(\cdot) \). The concavity of \( J_k^T(\cdot) \) follows since the minimum and sum of concave functions is concave. An inductive argument completes the proof.
B. Proof of Theorem 2

We will show that

\[ \tau_{\text{opt}} \rightarrow \begin{cases} 
  \text{Stop} & \text{if } \sum_{j=2}^{L+1} q_{k,j} \geq \frac{1}{c} \\
  \text{Continue} & \text{if } \sum_{j=2}^{L+1} q_{k,j} \leq \frac{1-h(\rho)}{c} 
\end{cases} \]

for an appropriately chosen function \( h(\rho) \) that satisfies \( \lim_{\rho \to 0} h(\rho) = 0 \). We start with the finite-horizon DP and define \( \Phi_k \) and \( \Psi_k \) as follows:

\[ \Phi_k \triangleq \frac{1}{1+\rho \sum_{j=2}^{L+1} q_{k,j}} - J^T_k(q_k), \quad 0 \leq k \leq T, \]

\[ \Psi_k \triangleq A^T_k(q_k) - \frac{1-\rho}{1+\rho \sum_{j=2}^{L+1} q_{k,j}}, \quad 0 \leq k \leq T-1. \]

The main idea behind the proof is to show that \( \Phi_k \) and \( \Psi_k \) are bounded by a function of \( \rho \) (that goes to 0 as \( \rho \to 0 \)), uniformly for all \( k \). Thus, the structure of the test in the limit as \( \rho \to 0 \) can be obtained.

Towards this goal, note from Appendix A that \( \Phi_T = \Psi_T = 0 \). Also, note that \( J^T_{T-1}(q^T_{T-1}) \) can be written as

\[ J^T_{T-1}(q^T_{T-1}) = \begin{cases} 
  \frac{1-\rho+\rho c \sum_{j=2}^{L+1} q^T_{T-1,j}}{1+\rho \sum_{j=2}^{L+1} q_{T-1,j}} & 0 \leq \sum_{j=2}^{L+1} q_{T-1,j} \leq \frac{1}{c} \\
  \frac{1}{1+\rho \sum_{j=2}^{L+1} q_{T-1,j}} & \sum_{j=2}^{L+1} q_{T-1,j} \geq \frac{1}{c}, 
\end{cases} \]

which can be equivalently written as

\[ \Phi_{T-1} = \frac{1}{1+\rho \sum_{j=2}^{L+1} q_{T-1,j}} \cdot \mathbb{I} \left( \left\{ \sum_{j=2}^{L+1} q_{T-1,j} \leq \frac{1}{c} \right\} \right). \]

Note that \( 0 \leq \Phi_{T-1} \leq \rho \) and we have

\[ 0 \leq E[\Phi_{T-1} | I_{T-2}] \triangleq -\Psi_{T-2} = \rho g_2(\rho) \text{ where} \]

\[ g_2(\rho) \triangleq E \left[ \frac{1-\rho + \rho c \sum_{j=2}^{L+1} q_{T-1,j}}{1+\rho \sum_{j=2}^{L+1} q_{T-1,j}} \cdot \mathbb{I} \left( \left\{ \sum_{j=2}^{L+1} q_{T-1,j} \leq \frac{1}{c} \right\} \right) | I_{T-2} \right]. \]

Now observe that \( X_\rho \) can be rewritten as

\[ X_\rho = \frac{1-\rho + \rho c \sum_{j=2}^{L+1} q_{T-1,j}}{1+\rho \sum_{j=2}^{L+1} q_{T-1,j}} \cdot \mathbb{I} \left( \left\{ \sum_{j=2}^{L+1} q_{T-1,j} \geq \frac{1}{c} \right\} \right). \]

Furthermore, \( X_\rho \leq 1 \) for all \( \rho \) and the set within the indicator function (above) converges to the empty set as \( \rho \downarrow 0 \). Thus, a straightforward consequence of the bounded convergence theorem
for conditional expectation [35] is that
\[
\lim_{\rho \to 0} q_2(\rho) = 0
\]
\[
\Psi_{T-2} \to 0,
\]
independent of the choice of \(T\).

Plugging the above relation in the expression for \(J_{T-2}^T(q_{T-2})\), we have
\[
J_{T-2}^T(q_{T-2}) = \min \left\{ \frac{1}{1 + \rho \sum_{j=2}^{L+1} q_{T-2,j}}, \frac{1 - \rho + \rho c \sum_{j=2}^{L+1} q_{T-2,j}}{1 + \rho \sum_{j=2}^{L+1} q_{T-2,j}} + \Psi_{T-2} \right\}
\]
\[
= \min \left\{ \frac{1}{1 + \rho \sum_{j=2}^{L+1} q_{T-2,j}}, \frac{1 - \rho}{1 + \rho} + \frac{\rho c \sum_{j=2}^{L+1} q_{T-2,j}}{1 + \rho \sum_{j=2}^{L+1} q_{T-2,j}} \right\}
\]
\[
= \frac{1}{1 + \rho \sum_{j=2}^{L+1} q_{T-2,j}} - \Phi_{T-2}
\]
\[
\Phi_{T-2} = \frac{\rho - \Psi_{T-2} - \rho (c + \Psi_{T-2}) \sum_{j=2}^{L+1} q_{T-2,j}}{1 + \rho \sum_{j=2}^{L+1} q_{T-2,j}} \right\} \cdot \left\{ \sum_{j=2}^{L+1} q_{T-2,j} \leq \frac{1}{c} \cdot \frac{1 - \Psi_{T-2}}{1 + \Psi_{T-2}} \right\}
\]
with \(0 \leq \Phi_{T-2} \leq \rho(1 + g_2(\rho))\). As before, it is straightforward to check that the set within the indicator function converges to the empty set as \(\rho \downarrow 0\) and we can write \(\Psi_{T-3}\) as
\[
\Psi_{T-3} = E[\Phi_{T-2}|I_{T-3}] = \rho g_3(\rho)
\]
\[
g_3(\rho) = E\left[ \left( \frac{1 - c \sum_{j=2}^{L+1} q_{T-2,j}}{1 + \rho \sum_{j=2}^{L+1} q_{T-2,j}} + g_2(\rho) \right) \cdot \left\{ \sum_{j=2}^{L+1} q_{T-2,j} \geq \frac{c - \rho g_2(\rho)}{c + \rho} \right\} \right]
\]
\[
\lim_{\rho \to 0} g_3(\rho) = 0 \quad \text{and} \quad \frac{\Psi_{T-3}}{\rho} \to 0.
\]

Following the same logic inductively, it can be checked that
\[
\frac{\Psi_{T-k}}{\rho} \to 0, \quad 1 \leq k \leq T,
\]
independent of the choice of \(T\). That is, we have
\[
J_k^T(q_k) = \min \left\{ \frac{1}{1 + \rho \sum_{j=2}^{L+1} q_{k,j}}, \frac{1 - \rho + \rho c \sum_{j=2}^{L+1} q_{k,j}}{1 + \rho \sum_{j=2}^{L+1} q_{k,j}} + \Psi_k \right\}
\]

Thus, the test structure reduces to stopping when
\[
\sum_{j=2}^{L+1} q_{k,j} \geq \frac{1}{c} \cdot \frac{1 - \Psi_k}{1 + \Psi_k},
\]
and using the limiting form for $\Psi_k$ as $\rho \to 0$, we have the threshold structure (as stated). The proof is complete by going from the finite-horizon DP to the infinite-horizon version as in the proof of Theorem 1. Note that while we expect the limiting test structure in the finite-horizon setting to be dependent on $T$, it is not seen to be the case in this work because $\rho = 0$ is a discontinuity point for the DP.

C. Proof of Proposition 4

We first intend to show that a version of [29, Lemma 1] holds in our case. More precisely, our goal is to show that for any $\epsilon \in (0,1)$, we have

$$\lim_{\alpha \to 0} \sup_{\tau \in \Delta, \alpha} P_k(\{k \leq \tau < k + (1 - \epsilon)L_\alpha\}) = 0,$$

where $P_k(\{\cdot\})$ denotes the probability measure when $\Gamma_1 = k$ and

$$L_\alpha \triangleq \frac{\log \left( \frac{1}{\rho^\alpha} \right)}{LD(f_1, f_0) + |\log(1 - \rho)|}.$$ 

Note that $L_\alpha \to \infty$ as $\alpha \to 0$. Following along the logic of the proof of [29, Lemma 1] here, it can be seen that

$$P_k(\{k \leq \tau < k + (1 - \epsilon)L_\alpha\}) \leq \exp \left( (1 - \epsilon^2)qL_\alpha \right) P_\infty(\{k \leq \tau < k + (1 - \epsilon)L_\alpha\}) + P_k(\{\max_{0 \leq n < (1 - \epsilon)L_\alpha} Z_{k+n} \geq (1 - \epsilon^2)qL_\alpha\}),$$

(9)

where $q \triangleq LD(f_1, f_0)$, $P_\infty(\{\cdot\})$ denotes the probability measure when no change happens, and

$$Z_{k+n}^k = \sum_{\ell=1}^L \sum_{i=\Gamma_\ell}^{k+n} \log \left( \frac{f_1(Z_i, \ell)}{f_0(Z_i, \ell)} \right)$$

with $\Gamma_1 = k$.

For the first term in (9), we have the following. With the appropriate definitions of $q$ and $L_\alpha$, and the tail probability distribution of a geometric random variable, it is again easy to check (as in the proof of Lemma 1) that for any $\tau \in \Delta_\alpha$, we have

$$\exp \left( (1 - \epsilon^2)qL_\alpha \right) P_\infty(\{k \leq \tau < k + (1 - \epsilon)L_\alpha\}) \to 0 \quad \text{as} \quad \alpha \to 0$$

for any $\epsilon \in (0,1)$ and all $k \geq 1$. For the second term in (9), we need a condition analogous to [29, eqn. (3.2)]:

$$P_k \left( \left\{ \frac{1}{M} \max_{0 \leq n < M} Z_{k+n}^k \geq (1 + \epsilon)q \right\} \right) \xrightarrow{M \to \infty} 0 \quad \text{for all} \quad \epsilon > 0 \quad \text{and} \quad k \geq 1.$$
This is trivial since the following is true:
\[
\frac{Z_{k+n}^k}{n} \xrightarrow{a.s.} LD(f_1, f_0) \quad \text{as} \quad n \to \infty
\] (10)
for all \( k \in [1, \infty) \).

The above condition follows from the following series of steps. First, note that the strong law of large numbers for i.i.d. random variables implies that:
\[
\frac{Z_{k+n}^k}{n} + \frac{1}{n} \sum_{\ell=2}^{L} \sum_{i=\Gamma_1}^{\Gamma_{\ell-1}} \log \left( \frac{f_1(Z_{i,\ell})}{f_0(Z_{i,\ell})} \right) \xrightarrow{a.s.} LD(f_1, f_0) = q \quad \text{as} \quad n \to \infty.
\]

Then, it can be easily checked that
\[
E[z_\ell] = D(f_1, f_0) \sum_{j=2}^{\ell} \frac{(1 - \rho_{j-1,j})^L}{\rho_{j-1,j}}.
\]

Since \( \min \rho_{\ell-1,\ell} > 0 \) from the statement of the proposition, we have \( E[z_\ell] \in (0, \infty) \) for all \( \ell = 2, \cdots, L \), and hence, the condition in (10) holds. Applying the condition in (10) with \( M = (1 - \epsilon)L_\alpha \) as \( \alpha \to 0 \), we have the equivalent of [29, Lemma 1].

The proposition follows by application of an equivalent version of [29, Theorem 1, eqn. (3.14)] which follows exactly as in [29].

D. Completing Proofs of Statements in Sec. VII

Proof of Prop. 6: We start from (3) and apply the recursion relationship for \( \{q_{k-1,\ell}\} \). Noting that \( w_m^j w_j^\ell = w_m^j \) for all \( j \) such that \( m \leq j \leq \ell \), we can collect the contributions of different terms and write \( \sum_{j=1}^{\ell} q_{k-1,j} w_j^\ell \) as
\[
\sum_{j=1}^{\ell} q_{k-1,j} w_j^\ell = \frac{1}{1 - \rho} \cdot \sum_{j=1}^{\ell} q_{k-2,j} w_j^\ell B_{k-1,j,\ell}
\]
where \( \{B_{k-1,j,\ell}\} \) is as defined in the statement of the proposition. Thus, we have
\[
\sum_{j=1}^{\ell} q_{k-1,j} w_j^\ell = \frac{(1 - \rho_{\ell-1,\ell}) \prod_{j=1}^{\ell-1} L_{k-1,j}}{1 - \rho} \cdot \left( \sum_{j=1}^{\ell} q_{k-2,j} w_j^\ell \right) \cdot \{1 + \zeta_{k-2,\ell}\}
\]

\[
\zeta_{k-2,\ell} = \frac{1}{(1 - \rho_{\ell-1,\ell}) \prod_{j=1}^{\ell-1} L_{k-1,j}} \cdot \frac{\sum_{j=1}^{\ell-1} q_{k-2,j} w_j^\ell C_{k-1,j,\ell}}{\sum_{j=1}^{\ell} q_{k-2,j} w_j^\ell}.
\]

Iterating the above equation, we have the conclusion in the statement of the proposition.
It is useful to reduce Prop. 6 to the case of \([32]\) when \(\rho_{\ell-1,\ell} = 1\) for all \(\ell = 2, \ldots, L\). For this, note that \(\alpha_{k,\ell}\) (and hence, \(q_{k,\ell}\)) are identically zero for all \(2 \leq \ell \leq L\). Thus, we have
\[
q_{k,L+1} = \alpha_{k,L+1} \cdot \prod_{j=1}^{L} \prod_{m=1}^{k} L_{m,j} \cdot \prod_{m=0}^{k-2} (1 + \zeta_{m,L+1}).
\]
We then have the following reductions:
\[
\alpha_{k,L+1} = \frac{1}{(1-\rho)^k} \cdot \left(1 + \frac{1}{1-\rho}\right),
\]
\[
\zeta_{m,L+1} = \frac{1}{\prod_{j=1}^{L} L_{m+1,j}} \cdot \frac{B_{m+1,1,L+1}}{1 + q_{m,L+1}}
\]
\[
B_{m+1,1,L+1} = 1 - \rho \text{ and hence},
\]
\[
q_{k,L+1} = \frac{\prod_{j=1}^{L} L_{k,j}}{1-\rho} \cdot \frac{k-1}{m=0} \left\{ \frac{1}{1+q_{m-1,L+1}} \right\} + \frac{\prod_{j=1}^{L} L_{m,j}}{1-\rho}
\]
\[
= \frac{\prod_{j=1}^{L} L_{k,j}}{1-\rho} \cdot \frac{1}{\prod_{j=1}^{L} L_{m,j}} \cdot \frac{k-1}{m=0} \left\{ 1 + \frac{\prod_{j=1}^{L} L_{m,j}(1+q_{m-1,L+1})}{1-\rho} \right\}
\]
with the initial condition that \(q_{-1,L+1} = 0\) and \(L_{0,j} = 1\) for all \(j\). It is straightforward to establish via induction that the only way in which the above recursion can hold is if \(q_{k,L+1}\) satisfies
\[
q_{k,L+1} = \frac{\prod_{j=1}^{L} L_{k,j}}{1-\rho} \cdot (1 + q_{k-1,L+1})
\]
which, as expected, is the same recursion as \([4]\).

**Proof of Prop. 8** First, note that if we can find \(\{U_k\}\) such that for all \(k\)
\[
\log \left( \sum_{\ell=2}^{L+1} \alpha_{k,\ell} \cdot C_{1} \cdots C_{\ell-1} \cdot J_{\ell} \right) \leq U_k,
\]
then \(\nu_A \geq \nu_{U,A}\) where
\[
\nu_{U,A} \triangleq \inf_k \left\{ U_k > A \right\}.
\]
We use Lemma \([4]\) to obtain the following bound and the associated \(\{U_k\}\):
\[
\sum_{\ell=2}^{L+1} \alpha_{k,\ell} \cdot C_{1} \cdots C_{\ell-1} \cdot J_{\ell} \leq \sum_{\ell=2}^{L+1} \frac{(1-\rho_{\ell-1,\ell}) \cdot \prod_{j=1}^{\ell-1} L_{k,j} \cdot D_{\ell}}{1-\rho} \cdot \frac{k-1}{m=1} \sum_{p=0}^{\ell-1} (1-\rho_{p,p+1}) \prod_{j=1}^{p} L_{m,j} \cdot \frac{k-1}{m=1} \sum_{p=0}^{L} (1-\rho_{p,p+1}) \prod_{j=1}^{p} L_{m,j}
\]
\[
\leq \frac{1}{1-\rho} \cdot \left( \sum_{\ell=2}^{L+1} D_{\ell} \cdot \prod_{j=1}^{\ell-1} L_{k,j} \right) \cdot \frac{k-1}{m=1} \sum_{p=0}^{L} (1-\rho_{p,p+1}) \prod_{j=1}^{p} L_{m,j}
\]
\[
\leq \frac{D}{1-\rho} \cdot \left( \sum_{p=1}^{L} (1-\rho_{p,p+1}) \cdot \prod_{j=1}^{p} L_{k,j} \right) \cdot \frac{k-1}{m=1} \sum_{p=0}^{L} (1-\rho_{p,p+1}) \prod_{j=1}^{p} L_{m,j}
\]
\[
\leq \frac{D}{1-\rho} \cdot \prod_{m=1}^{k} \sum_{p=0}^{L} (1-\rho_{p,p+1}) \prod_{j=1}^{p} L_{m,j}
\]
where \( D_\ell = \prod_{j=1}^{\ell-2} \rho_{j,j+1} \cdot \left( \sum_{j=0}^{\ell-1} \frac{1 - \rho_{j,j+1}}{1 - \rho} \right) \), \( D = 1 + \max_{\ell=1, \ldots, L} \frac{\ell}{1 - \rho_{\ell, \ell+1}} \). With the above bound, we have

\[
\nu_A \geq \inf_k \left\{ \sum_{m=1}^{k} \log \left( \frac{\sum_{p=0}^{L} (1 - \rho_{p,p+1}) \prod_{j=1}^{p} L_{m,j}}{1 - \rho} \right) \right\} > A + \log \left( \frac{1 - \rho}{D} \right),
\]

The conclusion follows by using Lemma 3 and noting that \( E \left[ \log \left( \sum_{p=0}^{L} \left(1 - \rho_{p,p+1}\right) \prod_{j=1}^{p} L_{m,j} \right) \right] \in (0, \infty) \).

**Proof of Prop. 9**

This proof is a formal write-up of the heuristic presented before the statement of Prop. 9. Following the definition of \( \eta_j \) and the fact that \( 0 \leq \eta_j \leq 1 \), we have

\[
\eta_j x_j \leq \prod_{m=\ell^*}^{j} x_m, \quad j \geq \ell^*.
\]

Suppose there exists an \( \ell^* \leq L \) as defined in (6), invoking Lemma 2 with the fact that \( \Delta_{\ell^*,j} \leq 0 \) for all \( j \geq \ell^* \), we have

\[
\frac{1}{k} \sum_{\ell=\ell^*}^{L} \log (1 + \eta_{\ell} x_\ell) \xrightarrow{k \to \infty} 0 \text{ a.s. and in mean}.
\]

Thus, we have

\[
\frac{1}{k} \sum_{\ell=2}^{\ell^*-1} \log (1 + \eta_{\ell} x_\ell) \xrightarrow{k \to \infty} 0 \text{ a.s. and in mean}.
\]

The main contribution to (7) is now established via induction. Since \( \eta_2 = 1 \), we can expand the sum as (modulo the a.s. and in mean convergence parts):

\[
\frac{1}{k} \sum_{\ell=2}^{\ell^*-1} \log (1 + \eta_{\ell} x_\ell) - \frac{1}{k} \log \left( 1 + \sum_{\ell=2}^{\ell^*-1} \prod_{m=2}^{\ell} x_m \right) \xrightarrow{k \to \infty} 0.
\]

If \( \ell^* = 2 \), it is clear that the proposition is true. If \( 3 \leq \ell^* \leq L + 1 \), since \( 2 < \ell^* \), by the definition of \( \ell^* \), there exists (a smallest choice) \( j_2 \geq 2 \) such that

\[
\prod_{m=2}^{j_2} x_m \xrightarrow{k \to \infty} \infty \text{ with}
\]

\[
\prod_{m=2}^{p} x_m \xrightarrow{k \to \infty} 0 \text{ or } \mathcal{O}(1) \text{ for all } 2 \leq p \leq j_2 - 1
\]

provided the set \([2, \ldots, j_2 - 1]\) is not empty. There are two possibilities: \( j_2 = \ell^* - 1 \) or \( j_2 \leq \ell^* - 2 \). (Note that \( j_2 \geq \ell^* \) results in a contradiction since it will imply \( \prod_{m=\ell^*}^{j_2} x_m \to \infty \), but we know this is not true from the definition of \( \ell^* \)). In the first case, we are done upon invoking Lemma 2. In the second case, iterating by replacing 2 with \( j_2 + 1 \) (as many times as necessary) and finally invoking Lemma 2 and noting the main contribution of the sum in (7), we arrive at the conclusion of the proposition.
REFERENCES

[1] V. Raghavan and V. V. Veeravalli, “Quickest Detection of a Change Process Across a Sensor Array,” Proc. 11th IEEE Intern. Conf. on Inform. Fusion, Cologne, Germany, pp. 1305–1312, July 2008.
[2] A. G. Tartakovsky, B. Rozovskii, R. Blazek, and H. Kim, “A Novel Approach to Detection of Intrusions in Computer Networks via Adaptive Sequential and Batch-Sequential Change-point Detection Methods,” IEEE Trans. Sig. Proc., vol. 54, no. 9, pp. 3372–3382, Sept. 2006.
[3] J. S. Baras, A. Cardenas, and V. Ramezani, “Distributed Change Detection for Worms, DDOS and Other Network Attacks,” Proc. American Cont. Conf. (ACC), Boston, MA, pp. 1008–1013, 2004.
[4] K. Mechitov, W. Kim, G. Agha, and T. Nagayama, “High-Frequency Distributed Sensing for Structure Monitoring,” Trans. of the Soc. of Instr. and Cont. Engineers, vol. E-S-1, no. 1, pp. 109–114, 2006.
[5] “National Science Foundation Workshop on Monitoring and Controlling the Nation’s Critical Infrastructures,” 2006, Final Report, Available: [Online]. http://www.ece.wisc.edu/~nowak/ci.
[6] L. A. Farwell and E. Donchin, “Talking Off the Top of Your Head: A Mental Prosthesis Utilizing Event-Related Brain Potentials,” Electroencephalography and Clinical Neurophysiology, vol. 70, pp. 510–523, 1988.
[7] R. Ratnam, J. B. M. Goense, and M. E. Nelson, “Change-point Detection in Neuronal Spike Train Activity,” Neurocomputing, vol. 52-54, pp. 849–855, 2003.
[8] H. Li, C. Li, and H. Dai, “Quickest Spectrum Sensing in Cognitive Radio,” Proc. 42nd IEEE Conf. on Inform. Systems and Sciences (CISS), 2008.
[9] G. Lorden, “Procedures for Reacting to a Change in Distribution,” Ann. Math. Statist., vol. 42, no. 6, pp. 1897–1908, Dec. 1971.
[10] M. Pollak, “Optimal Detection of a Change in Distribution,” Ann. Statist., vol. 13, no. 1, pp. 206–227, Mar. 1985.
[11] A. N. Shiryaev, “The Detection of Spontaneous Effects,” Sov. Math. Dokl., vol. 2, pp. 740–743, 1961.
[12] A. N. Shiryaev, “On Optimum Methods in Quickest Detection Problems,” Theory Prob. Appl., vol. 8, no. 1, pp. 22–46, Jan. 1963.
[13] A. N. Shiryaev, Optimal Stopping Rules, Springer-Verlag, NY, 1978.
[14] D. Siegmund, Sequential Analysis: Tests and Confidence Intervals, Springer-Verlag, NY, 1985.
[15] A. G. Tartakovsky, Sequential Methods in the Theory of Information Systems (in Russian), Radio i Svyaz’, Moscow, 1991.
[16] M. Basseville and I. V. Nikiforov, Detection of Abrupt Changes: Theory and Applications, Prentice Hall, Englewood Cliffs, NJ, 1993.
[17] T. L. Lai, “Sequential Analysis: Some Classical Problems and New Challenges (with discussion),” Stat. Sinica, vol. 11, pp. 303–408, 2001.
[18] A. G. Tartakovsky and V. V. Veeravalli, “Change-point Detection in Multi-Channel and Distributed Systems with Applications,” In Applications of Sequential Methodologies, (N. Mukhopadhyay, S. Datta and S. Chattopadhyay, Eds.), Marcel Dekker, Inc., NY, pp. 331–363, 2004.
[19] H. V. Poor and O. Hadjiliadis, Quickest Detection, Cambridge University Press, 2008.
[20] T. L. Lai, “Sequential Change-point Detection in Quality Control and Dynamical Systems,” J. Roy. Statist. Soc. Ser. B (Meth), vol. 57, no. 4, pp. 613–658, 1995.
[21] E. S. Page, “Continuous Inspection Schemes,” Biometrika, vol. 41, pp. 100–115, 1954.
[22] G. V. Moustakides, “Optimal Stopping Times for Detecting Changes in Distributions,” Ann. Statist., vol. 14, no. 4, pp. 1379–1387, Dec. 1986.
[23] M. Beibel, “Sequential Detection of Signals with Known Shape and Unknown Magnitude,” Stat. Sinica, vol. 10, pp. 715–729, 2000.
[24] C. D. Fuh, “SPRT and CUSUM in Hidden Markov Models,” Ann. Statist., vol. 31, pp. 942–977, 2003.
[25] G. Peskir and A. N. Shiryaev, “Solving the Poisson Disorder Problem,” In Advances in Finance and Stochastics, Springer: Berlin, pp. 295–312, 2002.
[26] A. G. Tartakovsky, “Extended Asymptotic Optimality of Certain Change-point Detection Procedures,” Submitted, 2003.
[27] B. Yakir, “A Note on Optimal Detection of a Change in Distribution,” Ann. Statist., vol. 25, pp. 2117–2126, 1997.
[28] T. L. Lai, “Information Bounds and Quick Detection of Parameter Changes in Stochastic Systems,” IEEE Trans. Inform. Theory, vol. 44, no. 7, pp. 2917–2929, July 1998.
[29] A. G. Tartakovsky and V. V. Veeravalli, “General Asymptotic Bayesian Theory of Quickest Change Detection,” SIAM Theory Prob. and its Appl., vol. 49, no. 3, pp. 458–497, 2005.
[30] A. G. Tartakovsky and V. V. Veeravalli, “An Efficient Sequential Procedure for Detecting Changes in Multi-Channel and Distributed Systems,” Proc. 5th IEEE Intern. Conf. on Inform. Fusion, Annapolis, MD, vol. 1, pp. 41–48, July 2002.
[31] A. G. Tartakovsky and V. V. Veeravalli, “Quickest Change Detection in Distributed Sensor Systems,” Proc. 6th IEEE Intern. Conf. on Inform. Fusion, Cairns, Australia, vol. 1, pp. 756–763, July 2003.
[32] V. V. Veeravalli, “Decentralized Quickest Change Detection,” IEEE Trans. Inform. Theory, vol. 47, no. 4, pp. 1657–1665, May 2001.
[33] D. P. Bertsekas, Dynamic Programming: Deterministic and Stochastic Models, Prentice Hall, NJ, 1987.
[34] V. V. Veeravalli, T. Basar, and H. V. Poor, “Decentralized Sequential Detection with a Fusion Center Performing the Sequential Test,” IEEE Trans. Inform. Theory, vol. 39, no. 2, pp. 433–442, Mar. 1993.
[35] R. A. Durrett, Probability: Theory and Examples, Duxbury Press, 2nd edition, 1995.
[36] C. W. Baum and V. V. Veeravalli, “A Sequential Procedure for Multi-Hypothesis Testing,” IEEE Trans. Inform. Theory, vol. 40, no. 6, pp. 1994–2007, Nov. 1994.
[37] M. Woodroofe, Nonlinear Renewal Theory in Sequential Analysis, Society for Industrial and Applied Mathematics, 1982.
[38] S. Appadwedula, V. V. Veeravalli, and D. L. Jones, “Decentralized Detection with Censoring Sensors,” IEEE Trans. Sig. Proc., vol. 56, no. 4, pp. 1362–1373, Apr. 2008.
[39] V. V. Veeravalli, T. Basar, and H. V. Poor, “Minimax Robust Decentralized Detection,” IEEE Trans. Inform. Theory, vol. 40, no. 1, pp. 35–40, Jan. 1994.
[40] P. J. Huber, Robust Statistics, Wiley Series in Probability and Statistics, Wiley-Interscience, 1981.