Small noise limit and convexity for generalized incompressible flows, Schrödinger problems, and optimal transport

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Abstract

This paper is concerned with six variational problems and their mutual connections: The quadratic Monge-Kantorovich optimal transport, the Schrödinger problem, Brenier’s relaxed model for incompressible fluids, the so-called Brödinger problem recently introduced by M. Arnaudon & al. [3], the multiphase Brenier model, and the multiphase Brödinger problem. All of them involve the minimization of a kinetic action and/or a relative entropy of some path measures with respect to the reversible Brownian motion. As the viscosity parameter $\nu \to 0$ we establish Gamma-convergence relations between the corresponding problems, and prove the convergence of the associated pressures arising from the incompressibility constraints. We also present new results on the time-convexity of the entropy for some of the dynamical interpolations. Along the way we extend previous results by H. Lavenant [30] and J-D. Benamou & al. [10].

Keywords: generalized incompressible flows; multimarginal optimal transport; Schrödinger problem; relative entropy; Gamma-convergence
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1 Introduction

Since the works of Y. Brenier in the 90’s [14, 15, 17] it is known that there is a
deep connection between generalized incompressible flows (inviscid Euler equa-
tions) and Monge-Kantorovich optimal transport. A decade later, C. Léonard,
M. Cuturi, and T. Mikami established independently a link between determin-
istic optimal transport and the so-called Schrödinger problem (also referred to
as entropic optimal transport) [37, 31, 22, 32]. The two approaches are closely
related, see e.g. [26] for a thorough discussion. Recently both variational
problems were blended into a single framework and the corresponding entropic in-
terpolation problem for incompressible viscid fluids, also known as the Brödinger
problem (contraction of Brenier and Schrödinger), was introduced in [3]. Just
as Brenier’s variational relaxed formulation leads to the incompressible Euler
(inviscid) equation, this new approach leads to some particular incompressible
motion of multiphase viscid fluids. See [3, 6] for more discussions on these
fluid-mechanical aspects.
In this paper, we will deal with six optimization problems: the Monge-Kantorovic optimal transport problem, the Schrödinger problem, Brenier’s relaxed model for incompressible fluids, the Brödinger problem, and multiphase versions of the last two problems. All of them amount to minimizing various path-measure action functionals, either an ensemble-average of the kinetic energy or a stochastic entropic version thereof, together with suitable constraints. We will introduce these problems in full details in Section 2 but let us first present them in a few words: In the optimal transport problem \( OT \) the action to be minimized is given by a deterministic average kinetic energy, and the initial and terminal distributions are prescribed. The Schrödinger problem \( Sch_\nu \) is identical, except that an entropic stochastic perturbation is now added to the kinetic energy. For Brenier’s problem, i.e. the Relaxed Euler formulation \( REu \), the functional is again the total kinetic action, but the flow must now satisfy an additional incompressibility constraint for all times. This forces in particular the initial and terminal distributions to equal the Lebesgue measure, and one is given instead the initial-to-terminal coupling of the particle trajectories. Brödinger’s problem \( Br\nu \) is identical to Brenier’s formulation, with the addition of an extra regularizing term as in \( Sch_\nu \). Finally, the Multiphase Relaxed Euler and Multiphase Brödinger problems \( MREu, MBr\nu \) are multiphase versions of \( REu \) and \( Br\nu \), respectively, where the fluid under consideration is now seen as a superposition of a whole continuum of phases, each of them having prescribed initial and final distributions.

Let us highlight that the Schrödinger problem, the Brödinger problem and the multiphase Brödinger problem feature a viscosity parameter \( \nu > 0 \) encoding the thermal, stochastic fluctuations in the system. In this work we mainly take interest in the small noise limit \( \nu \to 0 \), and we present here a complete picture in terms of Gamma-convergence for the corresponding variational problems. We fill some missing gaps in the theory, in particular we prove the convergence of the Brödinger problem towards Brenier’s relaxed formulation (including multiphase versions thereof).

We will use slightly different approaches for \( REu \) and \( Br\nu \) on the one hand, and for \( OT, Sch_\nu, MREu, \) and \( MBr\nu \) on the other hand. Indeed, for the latter four models we favor dynamical Benamou-Brenier-like formulations [8]. For the first two problems such a formulation does not exist, and we will argue directly at the level of the underlying stochastic processes and Lagrangian trajectories. This will result in more involved proofs but also in finest statements, and explains why we will study first \( REu \) and \( Br\nu \) and then the other problems.

Informal presentation of the contents We will give rigorous statements of our main results later on in Section 2.3 but for now let us briefly summarize our contributions and ideas. As already mentioned, we will be mostly concerned with the small noise limit, and the convergences below should be understood as \( \nu \to 0 \).

Our first result (Theorem 2.2) will assert the \( \Gamma \)-convergence of the Brödinger problem towards Brenier’s Relaxed formulation of Euler’s equation, \( Br\nu \to \)
REu. This improves results of [10] where the same convergence was recently proved for a discrete-time version of the problem only, but our method of proof is completely different. As already said, this will be the most involved part of the paper. As often for Gamma-convergence problems, the $\Gamma$–$\lim\inf$ part will follow from more or less standard lower semicontinuity arguments and the main challenge is rather the construction of suitable recovery sequences for the $\Gamma$–$\lim\sup$. Roughly speaking, the latter sequences will be constructed by stochastic deformation of classical trajectories: given a deterministic path-measure $P$, we will construct a stochastic perturbation $P^\nu \to P$ by superposing Brownian bridges of small variance $\nu$ centered at paths $\omega = (\omega_t)_{t \in [0,1]}$ initially charged by $P$. A careful analysis will then reveal that the resulting Brödinger stochastic action of $P^\nu$ does not exceed the Brenier’s deterministic action of $P$ by more than $\mathcal{O}(\nu)$.

Our second contribution consists in a new independent proof of convergence of the entropic Schrödinger problem towards deterministic optimal transport (Theorem 2.3), $\text{Sch}_\nu \to \text{OT}$, already known from [37, 31, 22, 19]. The main ingredient will be a quantified regularization procedure, Lemma 4.1 which is nothing but an Eulerian PDE-version of our previous Lagrangian stochastic deformation: Given a curve of probability measures $\rho = (\rho_t)_{t \in [0,1]}$ with finite kinetic energy, we construct a noisy approximation $\rho_t^\nu := \tau_{s(t(1-t))} * \rho_t$ by running the heat flow for short times ($\tau_s$ is the heat kernel at time $s > 0$), and show that its dynamical Schrödinger action deviates from the Benamou-Brenier action of the original curve by at most $\mathcal{O}(\nu)$ in order to retrieve the $\Gamma$–$\lim\sup$.

Our third result (Theorem 2.4) establishes the convergence of the Multi-phase Brödinger problem towards the multiphase Relaxed Euler formulation, $\text{MBrö}_\nu \to \text{MREu}$, and is completely new as far as we know. The proof essentially consists in superposing by linearity our previous construction of recovery sequences for single phases, using the fact that the heat flow preserves the Lebesgue measure. Since both problems include an incompressibility constraint some pressure fields arise as the corresponding Lagrange multipliers, and we prove convergence of the pressures $p^\nu \to p$ as well.

Finally, and as a byproduct of the previous analysis, our last set of contributions will focus on the time-convexity of the entropy functional along the relevant dynamical interpolations: the optimal transport and the Schrödinger problems on the one hand (Proposition 2.7), and on the other hand the multiphase Brenier and multiphase Brödinger problems (Proposition 2.8). Part of our results here are well-known [30, 33], but we either provide again a new and rather elementary proof of independent interest, or improve recent results from [30]. The argument will be again purely variational: Starting from an optimal curve $\rho$, we exploit the same previous Eulerian regularization to construct a noisy competitor $\rho^\nu$ for the optimization problem under consideration. The necessarily positive sign of the defect at first order $\mathcal{O}(\nu)$ reveals that the entropy of $\rho^\nu$ must be convex in time, and we conclude by taking $\nu \to 0$. 

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Outline. The paper is organized as follows: Section 2 contains the precise formulations of the six variational problems, as well as rigorous statements of our main results and preliminary definitions. In Section 3 we discuss the convergence \( \text{Br}\nu \to \text{REu} \) by probabilistic arguments. Section 4 contains the PDE regularization procedure, Lemma 4.1, as well as our new proof of the convergence \( \text{Sch}\nu \to \text{OT} \) of entropic towards deterministic optimal transport. In Section 5 we prove the corresponding result for incompressible multiphase flows, namely \( \text{MBr}\nu \to \text{MREu} \). We also show that the associated pressures converge. Our last Section 6 is devoted to the time-convexity of the entropy in the various models. We include in Appendix A a self-contained proof of the existence and uniqueness of solutions for \( \text{MBr}\nu \). In Section 3 we shall heavily rely on some explicit properties of the Brownian motion and bridges on the torus: in order not to interrupt the exposition we will simply give the technical statements when needed in the text, and defer the proofs to Appendix B.

2 Formulation of the problems and main statements

In this paper we carry out the whole analysis in the \( d \)-dimensional flat torus \( \mathbb{T}^d = (\mathbb{R}^d / \mathbb{Z}^d) \) for technical convenience, but some of the arguments might be adapted for convex domains \( \Omega \subset \mathbb{R}^d \) (in which case the natural boundary conditions should be the homogeneous Neumann ones, corresponding to mass conservation and reflected Brownian motion). We denote by \( d \) the geodesic distance on the torus and \( \pi : \mathbb{R}^d \to \mathbb{T}^d \) the canonical projection. The normalized Lebesgue measure on the torus is denoted by \( \text{Leb} \) (sometimes also \( dx \) or \( dy \) when no ambiguity arises). If \( X \) is a Polish space, \( \mathcal{P}(X) \) stands for the set of Borel probability measures on \( X \). If \( \rho \in \mathcal{P}(X) \) we use the notation \( d\rho \) in the integrals, i-e we write \( \int_X \varphi(x) \, d\rho(x) \) for the integral of \( \varphi \) with respect to the measure \( \rho \).

When \( \rho \in \mathcal{P}(\mathbb{T}^d) \) is absolutely continuous with respect to the Lebesgue measure \( dx \) we will often identify it with its Radon-Nikodym density, and simply write \( d\rho(x) = \rho(x) \, dx \) with a slight abuse of notations. The standard Brownian motion on the torus with diffusivity \( \nu > 0 \) and starting from \( x \in \mathbb{T}^d \) will be denoted by \( R^\nu_x \), and similarly \( R^\nu_z \) will stand for the Brownian motion on the whole space started from \( \bar{x} \in \mathbb{R}^d \). The reversible Brownian motion on the torus is obtained by choosing a uniform initial distribution,

\[
R^\nu := \int_{\mathbb{T}^d} R^\nu_x \, dx.
\]

2.1 The variational problems

First of all, let us describe precisely the six optimization problems that we will be dealing with throughout.
The quadratic optimal transport problem. Given \( \rho_0 \) and \( \rho_1 \) in \( \mathcal{P}(\mathbb{T}^d) \), the quadratic Monge-Kantorovich (square) distance is defined as:

\[
\text{d}_{\text{MK}}^2(\rho_0, \rho_1) := \inf_{\gamma} \frac{1}{2} \int d(x, y)^2 \, d\gamma(x, y),
\]

where the infimum runs over the set of admissible plans, that are by definition the measures \( \gamma \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d) \) with first marginal \( \rho_0 \) and second marginal \( \rho_1 \). This provides a distance on \( \mathcal{P}(\mathbb{T}^d) \) that metrizes the topology of narrow convergence and we refer to the classical monographs \([42, 38]\) for a detailed account on the theory of optimal transport and extended bibliography.

Note that \([1]\) is a static formulation. In order to introduce the celebrated dynamical Benamou-Brenier counterpart \([8]\), let us first recall the notion of absolutely continuous curves. Following \([2]\) Chapter 1, if \((\mathcal{X}, \mathcal{A})\) is a Polish space and \(1 \leq p < +\infty\), we say that a curve \( x : [0, T] \to \mathcal{X} \) belongs to \( AC^p([0, T]; \mathcal{X}) \) if the metric speed \( |\dot{x}(t)| := \lim_{s \to t} \frac{s(x(s), x(t))}{|s-t|} \) exists for a.e. \( t \in [0, T] \) and belongs to \( L^p(0, T) \). In the (complete) Wasserstein space \((\mathcal{P}(\mathbb{T}^d), d_{\text{MK}})\), \( AC^2 \) curves can be further characterized in terms of continuity equations and kinetic energy as:

**Theorem 2.1** \((\text{[2] Thm. 8.3.1})\). Let \( \rho = (\rho_t)_{t \in [0, 1]} \) be a curve from \([0, 1]\) to \( \mathcal{P}(\mathbb{T}^d) \). Then \( \rho \) belongs to \( AC^2([0, 1]; \mathcal{P}(\mathbb{T}^d)) \) if and only if there exists a measurable vector field \( c = c_t(x) \) belonging to \( L^2([0, 1] \times \mathbb{T}^d, dt \otimes \rho_t) \), i.e.

\[
\|c\|_{L^2([0, 1] \times \mathbb{T}^d, dt \otimes \rho_t)} := \int_0^1 \int |c_t(x)|^2 \, d\rho_t(x) \, dt < +\infty,
\]

such that the continuity equation

\[
\partial_t \rho_t + \text{div}(\rho_t c_t) = 0
\]

holds in a weak sense. In that case

\[
\int_0^1 |\dot{\rho}_t|^2 \, dt = \min_c \int_0^1 |c_t(x)|^2 \, d\rho_t(x) \, dt, \tag{2}
\]

where the metric derivative \( |\dot{\rho}_t| = \lim_{b \to 0} \frac{d_{\text{MK}}(\rho_t, \rho_t + b c_t)}{b} \) is well defined for almost all \( t \) and the minimum is taken among all such \( L^2 \) vector fields \( c \).

This allows to reformulate the optimal transport problem by saying that a curve of measures \( \rho = (\rho_t)_{t \in [0, 1]} \) is admissible for the problem \( \text{OT}(\rho_0, \rho_1) \) if it is of regularity \( AC^2([0, 1]; \mathcal{P}(\mathbb{T}^d)) \), and if it coincides with \( \rho_0 \) at time \( t = 0 \) and with \( \rho_1 \) at time \( t = 1 \). It is a solution to \( \text{OT}(\rho_0, \rho_1) \) if it minimizes the kinetic action functional defined by

\[
\mathcal{A}(\rho) := \frac{1}{2} \int_0^1 |\dot{\rho}_t|^2 \, dt \tag{3}
\]

\(^1\) We recall that narrow convergence is the weak-* convergence of measures in duality with continuous, bounded test functions.
for absolutely continuous curves, and set to $+\infty$ otherwise, on the set of all admissible curves.

The equivalence between the static formulation and the dynamical one is due to Benamou and Brenier in [8], and this formulation in terms of metric derivatives can be found for example in [38, Chapter 5]. The functional $A$ is convex, proper (the preimage of bounded sets are relatively compact) and lower semicontinuous for the topology of uniform convergence on $C([0, 1]; \mathcal{P}(\mathbb{T}^d))$ when $\mathcal{P}(\mathbb{T}^d)$ is endowed with the distance $d_{MK}$. There always exists at least one minimizer – as for the static problem – but uniqueness does not hold in general (typically due to non-uniqueness of geodesics in the torus).

The Schrödinger problem. We also formulate this problem in a dynamical Benamou-Brenier version. To do so, we first need to introduce another functional on the curves of probability measures: if $\rho$ is a curve on $\mathcal{P}(\mathbb{T}^d)$ with the additional regularity $\sqrt{\rho} \in L^2([0, 1]; H^1(\mathbb{T}^d))$ (recall that when $\rho$ has density with respect to the Lebesgue measure we abuse notations and write $d\rho_t(x) = \rho_t(x) \, dx$), the following vector field is well defined for almost all $t \in (0, 1)$ and $\rho_t$-almost all $x \in \mathbb{T}^d$:

$$\nabla \log \rho_t := 2 \frac{\nabla \sqrt{\rho_t}}{\sqrt{\rho_t}}.$$  \hspace{1cm} (4)

For such curves $\rho$, we define

$$\mathcal{F}(\rho) := \frac{1}{8} \int_0^1 \int |\nabla \log \rho_t|^2 \, d\rho_t \, dt = \frac{1}{2} \int_0^1 \int |\nabla \sqrt{\rho_t}|^2 \, dx \, dt.$$  \hspace{1cm} (4)

If $\rho$ is not regular enough we set $\mathcal{F}(\rho) := +\infty$. The quantity $\mathcal{F}(\rho)$ is nothing but the time-integral of the Fisher information

$$F(\rho) := \frac{1}{2} \int \left| \nabla \frac{1}{2} \log \rho_t \right|^2 \, d\rho_t$$

along the curve $\rho$. (We use here the customary probabilistic $\frac{1}{2}$ factor for the generator of the Brownian motion, accounting for the final $\frac{1}{8}$ factor in (4): indeed the osmotic velocity field in $\frac{1}{2} \Delta \rho = \text{div}(\rho \nabla \frac{1}{2} \log \rho) = w(\rho) = \nabla \frac{1}{2} \log \rho$, and $F(\rho) = \frac{1}{2} \int |w(\rho)|^2 \, d\rho$ is the corresponding kinetic energy.) It is strictly convex and lower semicontinuous for the topology of uniform convergence on $C([0, 1]; \mathcal{P}(\mathbb{T}^d))$.

Given $\rho_0, \rho_1$ in $\mathcal{P}(\mathbb{T}^d)$ and a diffusion parameter $\nu > 0$, we say that a curve $\rho$ is admissible for the Schrödinger problem $\text{Sch}_\nu(\rho_0, \rho_1)$ if it belongs to $AC^2([0, 1]; \mathcal{P}(\mathbb{T}^d))$, if it coincides with $\rho_0$ at time 0 and with $\rho_1$ at time 1, and if $\mathcal{F}(\rho) < +\infty$. A solution to $\text{Sch}_\nu(\rho_0, \rho_1)$ is an admissible curve that minimizes

$$\mathcal{H}_\nu(\rho) := \frac{1}{2} \int_0^1 |\dot{\rho}_t|^2 \, dt + \frac{\nu^2}{8} \int_0^1 \int |\nabla \log \rho_t|^2 \, d\rho_t \, dt = A(\rho) + \nu^2 \mathcal{F}(\rho)$$  \hspace{1cm} (5)

in the set of all admissible curves.
The Schrödinger problem dates back to articles by E. Schrödinger himself [39, 40]. Since then, it has given rise to a large amount of work, see for example [43, 24, 21, 32] and references therein. Classically, this problem is rather formulated in terms of relative entropy with respect to the Brownian motion, as we will do for the Brödinger problem. Our dynamical framework is however equivalent, as observed in numerous papers – see for instance [21, Section IV], [26, Cor. 5.8] or the introduction of [27].

The functional $H_\nu$ is strictly convex, proper, and lower semicontinuous for the topology of uniform convergence on $C([0,1]; \mathcal{P}(T^d))$. As proved for example in [32], minimizers exist if and only if the initial and final entropies are finite:

$$\int \rho_0 \log \rho_0 \, dx < +\infty \quad \text{and} \quad \int \rho_1 \log \rho_1 \, dx < +\infty,$$

and uniqueness is automatically guaranteed by strict convexity.

In most situations the structure of the minimizers is well understood, we refer once again to [32] for a survey on the following results: Typically (and usually requiring soft conditions on $\rho_0$ and $\rho_1$), the curve $\rho$ is a solution to $\text{Sch}_\nu(\rho_0, \rho_1)$ if and only if there are two nonnegative functions $f = f(x)$ and $g = g(x)$ such that for all $t \in [0,1]$

$$\rho_t = (f \ast \tau_{\nu t}) \times (g \ast \tau_{\nu(1-t)}),$$

where $(\tau_s)_{s \geq 0}$ is the heat kernel on $T^d$. Due to the endpoint constraints, $f$ and $g$ are obtained by solving the so-called Schrödinger system:

$$\begin{cases} f \times (g \ast \tau_\nu) = \rho_0, \\ (f \ast \tau_\nu) \times g = \rho_1, \end{cases}$$

a nonlinear integral system studied in great details by various authors [25, 13, 24]. Most of the known results on the Schrödinger problem strongly leverage this decomposition property of $\rho$. We stress that our variational analysis below will not exploit this structure at all, hence our approach is more robust when new constraints are added.

The Schrödinger problem is used nowadays as an entropic regularization of optimal transport. Indeed, the functional $H_\nu$ Gamma-converges to the functional $A$, as proved by Léonard in [31] and by Carlier et Al. in [19] after partial results by Mikami in [37]. (We also provide a new independent proof of this fact, see Theorem 2.3 below.) One of the advantages of this entropic optimal transport is clearly that $H_\nu$ has a unique minimizer, but most importantly that it is more tractable numerically speaking and allows for extremely fast and efficient computational strategies for optimal transport – see e.g. [22, 9].

**The Brenier model for incompressible fluids.** Brenier’s original model [14] is somehow an incompressible version of the optimal transport problem. Here the data is a plan $\gamma \in \mathcal{P}(T^d \times T^d)$ whose marginals are both the Lebesgue
measure (we say that $\gamma$ is bistochastic), corresponding to assigning the initial-to-terminal distribution of particles in the fluid. Brenier’s problem is an optimization problem in the set of generalized flows, i.e., in the set of probability measures on continuous paths on the torus $\mathcal{P}(C([0, 1]; \mathbb{T}^d))$ (This is the set of laws of continuous processes.) In this paper we choose to adopt the notation $\text{REu}$ for “Relaxed Euler” instead of $\text{Bre}$ (for “Brenier”) in order to avoid confusion with our notation $\text{Brö}$ for the Brödinger problem. A generalized flow $P \in \mathcal{P}(C([0, 1]; \mathbb{T}^d))$ is said to be admissible for Brenier’s problem $\text{REu}(\gamma)$ if:

(i) the joint law $P_{0,1}$ between the initial and final times under $P$ is $\gamma$, which means that for all test functions $\varphi$ on $\mathbb{T}^d \times \mathbb{T}^d$,
\[
\int \varphi(\omega_0, \omega_1) \, dP(\omega) = \int \varphi(x, y) \, d\gamma(x, y),
\]
(ii) $P$ is incompressible, i.e., the marginal $P_t$ of $P$ at time $t$ is the Lebesgue measure: For all test-functions $\varphi$ on $\mathbb{T}^d$ and all $t \in [0, 1]$,
\[
\int \varphi(\omega_t) \, dP(\omega) = \int \varphi(x) \, dx,
\]
(iii) the following action functional is finite:
\[
\mathcal{A}(P) := \int A(\omega) \, dP(\omega) < +\infty,
\]
where the action $A$ of a path $\omega \in C([0, 1]; \mathbb{T}^d)$ is defined by
\[
A(\omega) := \frac{1}{2} \int_0^1 |\dot{\omega}_t|^2 \, dt
\]
for absolutely continuous curves, and is otherwise set to $+\infty$. In particular an admissible $P$ should only charge absolutely continuous (in fact $AC^2$) curves.

A solution to $\text{REu}(\gamma)$ is then an admissible generalized flow which minimizes $\mathcal{A}$ in the set of all admissible generalized flows.

This model has been introduced and studied for the first time by Y. Brenier in [14] as a relaxation of the incompressible Euler equation in its variational formulation (due to V. Arnold in [4], developed in [5]). The functional $\mathcal{A}$ is affine, proper and lower semicontinuous for the topology of narrow convergence, and there always exist minimizers [14, Section 4]. Note that uniqueness does not hold in general, see e.g. [12].

The Brödinger problem. The Brödinger problem is an entropic version of the Brenier problem. Given a bistochastic $\gamma \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d)$ and a diffusivity parameter $\nu > 0$, a generalized flow $P$ is said to be admissible for the problem $\text{Brö}_\nu(\gamma)$ if conditions (i) and (ii) in Brenier’s model hold, and if (iii) is replaced by:
(iii) the relative entropy of $P$ with respect to the reversible Brownian motion $R^\nu$ with diffusivity $\nu$ is finite,

$$\mathcal{H}_\nu(P) := \nu H(P \mid R^\nu) < +\infty.$$ 

The relative entropy functional $(p, r) \mapsto H(p \mid r)$ will be properly defined in section 2.4 below. A solution to $\text{Brö}_\nu(\gamma)$ is then an admissible generalized flow $P$ that minimizes $\mathcal{H}_\nu$ in the set of all admissible generalized flows.

The Brödinger problem has been introduced and studied for the first time in [3]. The relative entropy $\mathcal{H}_\nu$ is strictly convex, proper and lower semicontinuous for the topology of narrow convergence, and it is proved in [3] that there exists a unique minimizer if and only if the following marginal entropy is finite:

$$H(\gamma \mid \text{Leb} \otimes \text{Leb}) < +\infty.$$  

As in the Schrödinger problem, uniqueness is guaranteed by strict convexity of the relative entropy.

**The multiphase Brenier model.** The Brenier and Brödinger models each have a multiphase version. These are useful when studying the pressure field (which is the Lagrange multiplier associated with the incompressibility constraint) in these generalized incompressible models. For Brenier’s model, the multiphase version was introduced by Brenier himself in [15, 16]. Following Lavenant in [30], we adopt here the point of view of traffic plans (see also [11]), which are probability measures on the set $C([0, 1]; \mathcal{P}(\mathbb{T}^d))$, i.e. elements of the set $\mathcal{P}(C([0, 1]; \mathcal{P}(\mathbb{T}^d)))$. Heuristically, each curve of measures $\rho \in C([0, 1]; \mathcal{P}(\mathbb{T}^d))$ charged by a traffic plan $P \in \mathcal{P}(C([0, 1]; \mathcal{P}(\mathbb{T}^d)))$ represents a phase in the fluid. The ideal fluid can then be seen as a (possibly continuous) superposition of all those phases. All the phases are coupled by an incompressibility constraint, and evolve so as to minimize the total kinetic action.

In this setting, the data is a probability measure $\Gamma \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d))$ with the compatibility condition

$$\int \rho_0 \, d\Gamma(\rho_0, \rho_1) = \int \rho_1 \, d\Gamma(\rho_0, \rho_1) = \text{Leb}.$$ 

We say that $\Gamma$ is *bistochastic in average*. Intuitively, $d\Gamma(\rho_0, \rho_1)$ can be thought of as the fraction of phases $\rho$ that coincide with $\rho_0$ at time 0 and with $\rho_1$ at time 1. A traffic plan $P$ is declared admissible for the problem $\text{MREu}(\Gamma)$ if:

(i) the joint law between the initial and terminal times under $P$ is $\Gamma$, which means that for all test function $\varphi$ on $\mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d)$,

$$\int \varphi(\rho_0, \rho_1) \, dP(\rho) = \int \varphi(\rho_0, \rho_1) \, d\Gamma(\rho_0, \rho_1),$$  

(ii) for all $t$, the average of the density at time $t$ under $P$ is the Lebesgue measure, which means that for all $t \in [0, 1],$

$$\int \rho_t \, dP(\rho) = \text{Leb}.$$
(iii) the following action functional is finite:

\[ \mathcal{A}(P) := \int \mathcal{A}(\rho) \, dP(\rho) < +\infty, \quad (11) \]

where \( \mathcal{A} \) is defined in (3). In particular \( P \) should only charges absolutely continuous (and in fact \( AC^2 \)) curves.

A solution to \( \text{MREu} (\Gamma) \) is then a minimizer of \( \mathcal{A} \) over all admissible traffic plans.

The functional \( \mathcal{A} \) is affine, proper and lower semicontinuous for the topology of narrow convergence, and there always exist minimizers as proved in [15]. (The proof only requires slight modifications in order to fit within our description in terms of traffic plans, see also [30, Thm. 2.12]) Analogously to the Brenier model, uniqueness does not hold in general.

Brenier’s original model can be seen as a particular case of the multiphase Brenier model. Indeed, if \( \gamma \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d) \) is bistochastic, one can canonically construct an associated \( \Gamma \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d)) \) by prescribing, for all test functions \( \varphi \) on \( \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d) \):

\[ \int \varphi(\rho_0, \rho_1) \, d\Gamma(\rho_0, \rho_1) = \int \varphi(\delta_x, \delta_y) \, d\gamma(x, y). \quad (12) \]

One can then slightly adapt [1] Section 4 and build a solution to \( \text{MREu} (\Gamma) \) from a solution to \( \text{REu} (\gamma) \), and vice versa. We also refer to [30] for an explanation on how to reformulate the works of Brenier and Ambrosio–Figalli in terms of traffic plans.

The multiphase Brödinger problem. The multiphase Brödinger problem \( \text{MBrö}_{\nu} \) is to the multiphase Brenier model what the Schrödinger problem is to the optimal transport problem, namely an entropic stochastic regularization of some sort. As for the multiphase Brenier model, we choose here an exposition in terms of traffic plans. If \( \Gamma \) is bistochastic in average and \( \nu > 0 \), a traffic plan \( P \in \mathcal{P}(C([0,1]; \mathcal{P}(\mathbb{T}^d))) \) is said to be admissible for \( \text{MBrö}_{\nu} (\Gamma) \) if points (i) (ii) (iii) in the multiphase Brenier model hold, and if in addition:

(iv) the average Fisher information

\[ \mathcal{F}(P) := \int \mathcal{F}(\rho) \, dP(\rho) \quad (13) \]

is finite. We recall that \( \mathcal{F} \) is defined in [4].

A solution to \( \text{MBrö}_{\nu} (\Gamma) \) is then a traffic plan that minimizes the functional

\[ \mathcal{H}_{\nu}(P) := \mathcal{A}(P) + \nu^2 \mathcal{F}(P) = \int \mathcal{H}_{\nu}(\rho) \, dP(\rho) \]

among all admissible traffic plans (where \( \mathcal{H}_{\nu} \) is defined in [5]). In this setting, the problem was introduced by the first author in [6] in order to prove the existence of a scalar pressure field in the Brödinger problem. The functional \( \mathcal{H}_{\nu} \)
is affine, proper and lower semicontinuous for the topology of narrow convergence. In appendix A we prove existence and uniqueness of minimizers under the entropy condition:

\[ \int \int \rho_0 \log \rho_0 \, d\Gamma(\rho_0, \rho_1) < +\infty \quad \text{and} \quad \int \int \rho_1 \log \rho_1 \, d\Gamma(\rho_0, \rho_1) < +\infty. \tag{14} \]

In fact this condition is also necessary for the existence of minimizers because any plan with \( \mathcal{H}_\nu(P) < \infty \) must have finite marginal entropies, but we omit the details (see Remark A.3 later on). However, contrarily to what happens in the Brenier model, the Brödinger problem is not a particular case of the multiphase Brödinger problem. Indeed, the construction (12) described in the Brenier case gives rise to a \( \Gamma \) that cannot satisfy the entropy condition (14) since \( H(\delta_x \mid \text{Leb}) = +\infty \) for all \( x \in \mathbb{T}^d \). There is still a link between Brö and MBrö, but we refrain from discussing further this connection and rather refer to [6, Section 9].

2.2 The pressure field in the incompressible models

As already mentioned, each of the four incompressible models (REu, MREu, Brö, MBrö) features its own Lagrange multiplier corresponding to the incompressibility constraint. This pressure field, a scalar distribution, can be interpreted as the differential of the optimal action when the prescribed density is perturbed, in accordance with the envelope theorem. In this paper we will study these pressure fields for the multiphase models only, but similar results might be developed in the Brenier model or for the Brödinger problem.

The precise definition of the pressure fields will be recalled later on in Theorem 5.4 but let us make a bit more explicit the idea. One can prove that, if \( P \) is a solution to MREu(\( \Gamma \)), then there exists a unique \( p \in \mathcal{D}'((0,1) \times \mathbb{T}^d) \) such that, for all \( Q \) satisfying the joint law condition (9) but with perturbed density \( \int \rho \, dQ(\rho) = (1 + \phi) \text{Leb} \) for small but arbitrary \( \phi \in \mathcal{D}((0,1) \times \mathbb{T}^d) \), there holds

\[ \mathcal{A}(Q) \geq \mathcal{A}(P) + \langle p, \phi \rangle. \]

As a consequence, in the perturbed problem with perturbed density \((1 + \phi) \text{Leb}\) instead of Leb, the optimal action is perturbed at first order by the quantity \( \langle p, \phi \rangle \). This was first proved in [15] for the Brenier model (the proof in the multiphase Brenier model case is a direct adaption). As proved by the first author in [6], it is still true replacing MREu(\( \Gamma \)) by MBrö_\nu(\( \Gamma \)) and \( \mathcal{A} \) by \( \mathcal{H}_\nu \).

In turn, the pressure field is a relevant object to study the dynamics of the solutions: It is a self-induced potential accelerating the particles (i.e. \( \dot{\omega}_t = -\nabla p(t, \omega_t) \) for almost all trajectories at least when the pressure is regular enough) in such a way as to preserve the incompressibility constraint. We refer to [11, Thm. 6.8] for an illustration of this fact for the Brenier model REu.
2.3 Contributions

Our first result will assert the convergence of the Brödinger problem towards the Brenier model as the diffusivity \( \nu \to 0 \). If \( \gamma \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d) \) we write \( \iota_\gamma \) for the characteristic function corresponding to the marginal constraint \( \gamma \), and \( \iota_{\text{Inc}} \) is the characteristic function corresponding to the incompressibility constraint \( \mathcal{I} \).

In other words, if \( P \) is a generalized flow,

\[
\iota_\gamma(P) := \begin{cases} 0 & \text{if } P_{0,1} = \gamma \\
+\infty & \text{else,}
\end{cases}
\]

and

\[
\iota_{\text{Inc}}(P) := \begin{cases} 0 & \text{if } P_t = \text{Leb} \text{ for all } t \\
+\infty & \text{else.}
\end{cases}
\]

Our statement reads then

**Theorem 2.2.** With the same notations as before,

1. The following \( \Gamma \)-convergence holds:

\[
\Gamma - \lim_{\nu \to 0} \{ \overline{H}_\nu + \iota_{\text{Inc}} \} = \overline{A} + \iota_{\text{Inc}}.
\]

2. If \( \gamma \) is bistochastic and satisfies \( H(\gamma | \text{Leb} \otimes \text{Leb}) < +\infty \), then

\[
\Gamma - \lim_{\nu \to 0} \{ \overline{H}_\nu + \iota_{\text{Inc}} + \iota_\gamma \} = \overline{A} + \iota_{\text{Inc}} + \iota_\gamma.
\]

This will be seen as a straightforward consequence of Theorem 3.1 below, and the result is stronger than convergence of the minimizers. Note that the second part of our statement only addresses the case of a fixed marginal law \( P_{0,1} = \gamma \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d) \), while the first part does not and will typically require suitable regularization \( \gamma^\nu \rightharpoonup \gamma \). The key step in the proof will be to build a recovery sequence \( P^\nu \) by adding a Brownian bridge (with diffusivity \( \nu \)) to any absolutely continuous curve charged by any admissible generalized flow \( P \) in the Brenier model. We will then use a Cameron-Martin formula to compute the entropy of the resulting process. We will also exploit the narrow continuity of the optimal action in Brenier’s model \( \mathcal{RE}u(\gamma) \) with respect to \( \gamma \) [7, Thm. 1] to obtain a necessary and sufficient condition for a sequence \( (\gamma^\nu)_{\nu > 0} \) of bistochastic measures to be the marginal laws of a recovery sequence \( P^\nu \rightharpoonup P \), see point 2 in Theorem 3.1 later on.

We will address next the convergence \( \text{Sch}_\nu \to \text{OT} \) of entropic towards deterministic optimal transport: For curves \( \rho \in C([0,1]; \mathcal{P}(\mathbb{T}^d)) \), and given \( \rho_0, \rho_1 \in \mathcal{P}(\mathbb{T}^d) \), we write

\[
\iota_{(\rho_0, \rho_1)}(\rho) := \begin{cases} 0 & \text{if } \rho|_{t=0} = \rho_0 \text{ and } \rho|_{t=1} = \rho_1 \\
+\infty & \text{else,}
\end{cases}
\]

for the characteristic function of the endpoints constraint. The entropy functional will be rigorously defined in section 2.3 but for now let us anticipate and write \( H(\mu) := H(\mu | \text{Leb}) \) for the entropy of \( \mu \in \mathcal{P}(\mathbb{T}^d) \) computed relatively to the Lebesgue measure on the torus. The statement then reads
Theorem 2.3. Let $\rho_0, \rho_1 \in \mathcal{P}(\mathbb{T}^d)$ such that $H(\rho_0) < +\infty$ and $H(\rho_1) < +\infty$. Then
\[
\Gamma - \lim_{\nu \to 0} \left\{ H_\nu + \iota_{(\rho_0, \rho_1)} \right\} = A + \iota_{(\rho_0, \rho_1)}
\] (15)
for the uniform topology on $C([0, 1]; \mathcal{P}(\mathbb{T}^d))$.

We stress again that the result is not new [37, 31, 22, 19], but we will give an independent proof that is elementary and new to the best of our knowledge. In particular, we will provide an explicit PDE construction of recovery sequences inspired from the previous probabilistic arguments and Brownian bridges.

Our third result is the convergence $\text{MBr}\beta_\nu \to \text{MREu}$ of the multiphase Brödinger problem towards the multiphase Brenier model:

Theorem 2.4. If $\Gamma$ is bistochastic in average with finite average marginal entropies as in (14), then
\[
\Gamma - \lim_{\nu \to 0} \left\{ H_\nu + \iota_{\text{Inc}} + \iota_{\Gamma} \right\} = A + \iota_{\text{Inc}} + \iota_{\Gamma}
\] (16)
for the narrow topology of $C([0, 1]; \mathcal{P}(\mathbb{T}^d))$.

Here we write as before $\iota_{\Gamma}(P)$ and $\iota_{\text{Inc}}(P)$ for the characteristic functions of the marginal and incompressibility constraints, [9] and [10], respectively. Again, this result is stronger than convergence of the minimizers.

Given that both problems include an incompressibility constraint, a natural question to ask is whether the associated Lagrange multipliers converge as well, i.e. whether the Brödinger pressure $p_\nu$ converges towards the Brenier pressure $p$. The answer to that question is positive, but in order to make a rigorous statement we first need to introduce the following functional space:

Definition 2.5. Let $C^{1,2}([0, 1] \times \mathbb{T}^d)$ denote the space of functions having continuous time-derivative and continuous second order space-derivatives. For such a function $f \in C^{1,2}([0, 1] \times \mathbb{T}^d)$, we say that $f \in \mathcal{G}$ if in addition:
- for all $x \in \mathbb{T}^d$, $f(0, x) = f(1, x) = 0$,
- for all $t \in [0, 1]$,
\[
\int_{\mathbb{T}^d} f(t, x) \, dx = 0.
\]
Endowed with its natural $C^{1,2}$ norm, $\mathcal{G}$ is a Banach space.

Our result is the following:

Theorem 2.6. Take $\Gamma$ bistochastic in average and satisfying [14]. For all $\nu > 0$ let $p_\nu$ be the pressure field associated with $\text{MBr}\beta_\nu(\Gamma)$, and let $p$ be the pressure field associated with $\text{MREu}(\Gamma)$ (both being defined in Theorem 5.4). Then
\[
p_\nu \rightharpoonup p \quad \text{in} \quad \mathcal{G}'
\]
for the weak-* convergence on the topological dual $\mathcal{G}'$ of $\mathcal{G}$.
Finally, we will investigate the time-convexity of the entropy $H = H(\bullet | \text{Leb})$ along solutions to (some of) the previous problems. Note that, for the one-phase problems $\text{RE}_u, \text{Brö}$, the incompressibility constraint $\rho_t \equiv \text{Leb}$ forces the entropy to be constant in time $H(\rho_t) \equiv H(\text{Leb}) = 0$, so nothing interesting can be said there. We will give a new independent proof of the convexity along the interpolations $\text{OT}, \text{Sch}_\alpha$:

**Proposition 2.7.** Let $\rho_0, \rho_1 \in \mathcal{P}(\mathbb{T}^d)$ have finite entropies $H(\rho_0), H(\rho_1) < \infty$, and let $\rho$ be a solution of $\text{OT}(\rho_0, \rho_1)$ or $\text{Sch}_\alpha(\rho_0, \rho_1)$ for fixed diffusivity $\alpha > 0$. Then $t \mapsto H(\rho_t)$ is convex.

For the optimal transport problem this is nothing but R. McCann’s celebrated *displacement convexity* [36], and the convexity for the Schrödinger problem is also known from [33]. However, we would like to stress that our unified proof is elementary, purely variational, and exploits neither prior knowledge on - nor particular structure of - the minimizers.

In the multiphase settings $\text{MRE}_u, \text{MBrö}_\alpha$, we will establish

**Proposition 2.8.** Let $\Gamma$ be bistochastic in average with finite marginal entropies as in (14), and let $P$ be any solution to $\text{MRE}_u(\Gamma)$ or $\text{MBrö}_\alpha(\Gamma)$ for $\alpha > 0$. Then the average entropy

$$t \mapsto \int H(\rho_t) \, dP(\rho)$$

is convex.

The convexity for $\text{MRE}_u$ was conjectured by Y. Brenier in [18]. This was recently proved by H. Lavenant in [30] for particular solutions only (roughly speaking, solutions with minimal entropy in some integral sense), while we stress that our statement holds for any solution. For $\text{MBrö}$ the result is completely new.

### 2.4 Notations and preliminary results

**Canonical processes.** In the Brenier model and in the Brödinger problem we are dealing with generalized flows, which are probability measures on $C([0, 1]; \mathbb{T}^d)$. We will denote by $X = (X_t)_{t \in [0, 1]}$ the canonical process on this space. Put differently, for all $t \in [0, 1]$, the random variable $X_t$ is the evaluation map at time $t$:

$$X_t : \omega \in C([0, 1]; \mathbb{T}^d) \mapsto \omega_t \in \mathbb{T}^d.$$  

Likewise, in the multiphase Brenier and Brödinger models, we are dealing with traffic plans, i.e. with probability measures on $C([0, 1]; \mathcal{P}(\mathbb{T}^d))$. We will denote by $X = (X_t)_{t \in [0, 1]}$ the canonical process on this space. For all $t \in [0, 1]$, $X_t$ is the evaluation map at time $t$:

$$X_t : \rho \in C([0, 1]; \mathcal{P}(\mathbb{T}^d)) \mapsto \rho_t \in \mathcal{P}(\mathbb{T}^d).$$
Push-forward and disintegration. If $\mathcal{X}$ and $\mathcal{Y}$ are two Polish spaces, $p$ is a Borel measure on $\mathcal{X}$, and $\Phi : \mathcal{X} \to \mathcal{Y}$ is measurable, we will denote by $\Phi_\# p$ the push-forward of $p$ by $\Phi$, i.e., the law of $\Phi$ under $p$.

When there is no ambiguity on the map $\Phi$ to be used, we simply denote by $p^\#$ the conditional law $p(\bullet | \Phi = y) \in \mathcal{P}(\mathcal{X})$. By virtue of the disintegration theorem, $p^\#$ is well defined for $\Phi_\# p$-almost every $y \in \mathcal{Y}$, and concentrates on the fiber $\Phi^{-1}(y)$. We recall that, by definition, if $\varphi$ is a test function on $\mathcal{X}$ then

$$
\int_{\mathcal{X}} \varphi(x) \, dp(x) = \int_{\mathcal{Y}} \left\{ \int_{\Phi^{-1}(y)} \varphi(x) \, dp^\#(x) \right\} \, d(\Phi_\# p)(y),
$$

(17)
or equivalently:

$$
p = \int_{\mathcal{Y}} p^\# \, d(\Phi_\# p)(y).
$$

(18)

If $P$ is the law of a process on $\mathbb{T}^d$ or $\mathbb{R}^d$ (that is, an element of $\mathcal{P}(\mathbb{C}[0,1];\mathbb{T}^d)$ or $\mathcal{P}(\mathbb{C}[0,1];\mathbb{R}^d)$), and if the map $\Phi : \mathcal{X} \to \mathcal{Y}$ is the evaluation at time $t$ defined above, we will write $P_t := X_t \# P$ for the marginal of $P$ at time $t$. Following the standard notations, $P_{0,1}$ will stand for the joint law $(X_0, X_1) \# P$, and $P^x,y$ will refer to the conditional law $P(\bullet | X_0 = x, X_1 = y)$. These laws will frequently have their diffusivity as a superscript (typically $P^\mu$). In that case, we write

$$
P^{\mu,x,y} := P^\mu(\bullet | X_0 = x, X_1 = y).
$$

With these notations, the marginal constraint (6) can be reformulated as $P_{0,1} = \gamma$ and the incompressibility (7) reads $P_t = \text{Leb}$.

Similarly, if $P$ is a traffic plan and $t \in [0,1]$, we will write $P_t := X_t \# P$ for the time-$t$ marginal, $P_{0,1}$ will stand for the joint law $(X_0, X_1) \# P$, and $P^{\rho_0, \rho_1}$ will refer to the conditional law $P(\bullet | X_0 = \rho_0, X_1 = \rho_1)$. These laws will frequently have their diffusivity as a superscript (typically $P^\mu$). In that case, we write

$$
P^{\mu,\rho_0, \rho_1} := P^\mu(\bullet | X_0 = \rho_0, X_1 = \rho_1).
$$

With these notations, (9) rewrites $P_{0,1} = \Gamma$ and the generalized incompressibility (10) reads $\int \rho \, dP_t(\rho) = \text{Leb}$.

The relative entropy. If $\mathcal{X}$ is a Polish space, $r$ is a reference positive and finite Radon measure on $\mathcal{X}$, and $p$ is a Borel probability measure on $\mathcal{X}$, the relative entropy of $p$ with respect to $r$ (also known as the Kullback-Leibler divergence) is defined by

$$
H(p | r) := \begin{cases} 
\int \log \rho \, dp = \int \rho \log \rho \, dr & \text{if } p \ll r \text{ and } \rho = \frac{dp}{dr}, \\
+ \infty & \text{else.}
\end{cases}
$$

Jensen’s inequality applied to the convex function $\eta \mapsto \eta \log \eta$ implies the lower bound $H(p | r) \geq - \log r(\mathcal{X})$ (which is 0 when $r$ is a probability measure). In
Section 3, we will need several elementary results about the relative entropy, listed here without proofs. The first one concerns the change of reference measure.

**Proposition 2.9.** Let $r$ and $p$ be as above and let $f \in L^1(X, r)$ be nonnegative and $p$-almost surely positive. Then

$$H(p \mid f \cdot r) = H(p \mid r) - \int \log f \, dp.$$  

The behaviour of the relative entropy with respect to disintegration is given by

**Proposition 2.10.** Let $X$ and $Y$ be Polish spaces, $r$ and $p$ be as above and take $\Phi : X \to Y$ a measurable map. Then with the same notations as before,

$$H(p \mid r) = H(\Phi \# p \mid \Phi \# r) + \int H(p^x \mid r^x) \, d\Phi \# p(x).$$

Finally, if $\Phi$ is one-to-one, then simultaneously pushing forward $r$ and $p$ by $\Phi$ does not change their relative entropy:

**Proposition 2.11.** Take $X$, $Y$, $r$, $p$ and $\Phi$ as in Proposition 2.10. Assume furthermore that $p \ll r$ and that there exists $\Psi : Y \to X$ such that $r$-almost surely, $\Psi \circ \Phi = \text{Id}_X$. Then

$$H(p \mid r) = H(\Phi \# p \mid \Phi \# r).$$

For probability measures on the torus $\rho \in P(T^d)$ and if no confusion arises, we simply write

$$H(\rho) := H(\rho \mid \text{Leb}) = \int_{T^d} \rho(x) \log \rho(x) \, dx$$

for the entropy computed relatively to the Lebesgue measure. (Once again, we keep the same notation $\rho$ for a measure and its density with respect to $\text{Leb}$.)

**The heat flow.** Let us denote by $\tau_s$ the heat kernel in the torus

$$\partial_s \tau_s = \frac{1}{2} \Delta \tau_s$$

at time $s > 0$, started from the initial Dirac distribution $\tau_0 = \delta$. We will use the following Gaussian estimate several times:

**Lemma 2.12.** There are two dimensional constants $k_d, K_d > 0$ such that for all $s \in (0, 1]$ and all $x, y \in T^d$,

$$\frac{k_d}{\sqrt{2\pi s}} \exp \left( -\frac{d^2(x, y)}{2s} \right) \leq \tau_s(y - x) \leq \frac{K_d}{\sqrt{2\pi s}} \exp \left( -\frac{d^2(x, y)}{2s} \right).$$  

(19)
This type of results can be obtained under general assumptions on the domain and we refer e.g. to [41, 34, 28] for this delicate topic. In the torus we have the explicit formula

$$\forall x, y \in \mathbb{T}^d, \quad \tau_s(y - x) = \frac{1}{\sqrt{2\pi s}} \sum_{l \in \mathbb{Z}^d} \exp \left( -\frac{|\bar{y} - \bar{x} + \bar{l}|^2}{2s} \right),$$

(20)

where $\bar{x}, \bar{y} \in \mathbb{R}^d$ are chosen so that $\pi(\bar{x}) = x$ and $\pi(\bar{y}) = y$. Hence in this particular case the bounds (19) could be worked out by hand. As such, the upper bound can only be valid for short times (note that we took care to assume $s \leq 1$ in our statement) and indeed we shall only use this in the limit $s \to 0$.

### 3 Convergence of Brö towards REu

The goal of this section is to prove Theorem 2.2. In fact the statement will be obtained as a consequence of the following stronger, but more technical result:

**Theorem 3.1.**

1. Let $(P^\nu)_{\nu>0}$ be a sequence of incompressible generalized flows narrowly converging to $P$. Then

$$\liminf_{\nu \to 0} H_\nu(P^\nu) \geq \mathcal{A}(P).$$

2. Let $P$ be an admissible generalized flow for $\text{REu}(\gamma)$, and $\gamma^\nu \rightharpoonup \gamma$. The following are equivalent:

   (a) there exists a sequence of generalized incompressible flows $P^\nu$ with marginals $P_{0,1}^\nu = \gamma^\nu$ that narrowly converges to $P$ and such that

$$\limsup_{\nu \to 0} H_\nu(P^\nu) \leq \mathcal{A}(P)$$

(21)

   (b) the sequence of marginals satisfies

$$\limsup_{\nu \to 0} \nu H(\gamma^\nu | R_{0,1}^\nu) \leq \frac{1}{2} \int d(x, y)^2 d\gamma(x, y).$$

(22)

**Remark 3.2.** Condition (22) exactly requires $(\gamma^\nu)_{\nu>0}$ to be a recovery sequence for the $\Gamma$-convergence

$$\Gamma - \lim_{\nu \to 0} \{\nu H(\bullet | R_{0,1}^\nu) + \iota_{\text{Bis}}\} = C_{\text{MK}} + \iota_{\text{Bis}},$$

(23)

where:

$$C_{\text{MK}}(\gamma) := \frac{1}{2} \int d(x, y)^2 d\gamma(x, y)$$

is the Monge-Kantorovich quadratic cost functional and $\iota_{\text{Bis}}(\gamma)$ is the characteristic function of the bistochasticity constraint. The $\Gamma$-convergence (23) is well
known as a particular case of results from [31] and [19], hence for a given \( \gamma \) there always exists a sequence \( \gamma^\nu \) as in [22] and in practice our Theorem 3.1 guarantees that one can always construct a recovery sequence \( P^\nu \rightharpoonup P \) in [21].

Before going into the technical details we will need a few preliminary definitions and results. In order not to interrupt the flow of the exposition, we postpone the proofs of the latter to the appendix.

Observe first that the kinetic action \( A(\omega) = \frac{1}{2} \int_0^1 |\dot{\omega}_t|^2 \, dt \) is only lower semi-continuous for the uniform topology on \( C([0,1]; \mathbb{T}^d) \), and unbounded. In order to circumvent this lack of regularity we will approximate \( A \) by difference quotients as follows: for large \( N \in \mathbb{N} \) we set \( \tau = 1/N \), \( t_n = n\tau \) for \( n = 0, \ldots, N \), and for any \( \omega \in C([0,1]; \mathbb{T}^d) \) we define

\[
A_N(\omega) := \frac{1}{2\tau} \sum_{n=0}^{N-1} d^2(\omega_{t_n}, \omega_{t_{n+1}}). \tag{24}
\]

This is a a good approximation of \( A \) in the sense that, for any fixed \( \omega \) and by classical properties of difference quotients in the one-dimensional Sobolev space \( AC^2([0,1]; \mathbb{T}^d) = H^1([0,1]; \mathbb{T}^d) \), we have pointwise convergence

\[
A_N(\omega) \to A(\omega) \quad \text{as} \quad N \to \infty
\]

(in particular \( A_N(\omega) \to +\infty \) if \( \omega \notin AC^2 \)). Note moreover that, for fixed \( N \), \( A_N \) is continuous for the uniform topology (as the sum of finitely many evaluations) and bounded, since in the torus \( d^2(\omega_{t_n}, \omega_{t_{n+1}}) \leq \text{diam}(\mathbb{T}^d)^2 < +\infty \). (In the whole space \( \mathbb{R}^d \) one should replace \( A_N \) by its truncation \( \tilde{A}_N := \min(A_N, N) \) to guarantee boundedness, and the rest of the argument below then applies \textit{mutatis mutandis}.)

Our first technical statement reads

\textbf{Lemma 3.3.} Consider \( A_N \) as defined in [24], and let \( R^\nu \) denote the reversible Brownian motion with diffusivity \( \nu \). For all \( \alpha \in (0,1) \) and \( \nu \in (0,1) \) there holds

\[
\int \exp\left( \frac{\alpha}{\nu} A_N(\omega) \right) \, dR^\nu(\omega) \leq \frac{1}{(1-\alpha)^{Nd/2}}. \tag{25}
\]

Similarly, for all \( x, y \in \mathbb{T}^d \) let \( R^\nu_{x,y} := R^\nu(\bullet | X_0 = x, X_1 = y) \) be the Brownian bridge joining \( x \) to \( y \). There is a dimensional constant \( C_d > 0 \) such that

\[
\int \exp\left( \frac{\alpha}{\nu} A_N(\omega) \right) \, dR^\nu_{x,y}(\omega) \leq \frac{C_d}{(1-\alpha)^{Nd/2}} \exp\left( \frac{\alpha}{2\nu} d^2(x, y) \right). \tag{26}
\]

\footnote{In [19], our term \( \nu H(\gamma | R^\nu_{0,1}) \) is replaced by \( C_{\text{logk}}(\gamma) + \nu H(\gamma | \text{Leb} \otimes \text{Leb}) \) and the analysis is carried out in the whole space \( \mathbb{R}^d \). In that setting one has the explicit formula \( R^\nu_{0,1} = \exp\left( -|x-y|^2 \right) / \sqrt{2\pi \nu} \, dx \otimes dy \), and expanding the logarithms in Proposition 2.9 gives exactly \( \nu H(\gamma | R^\nu_{0,1}) = \nu \log((2\pi\nu)^{d/2}) + C_{\text{logk}}(\gamma) + \nu H(\gamma | \text{Leb} \otimes \text{Leb}) \). Consequently, the corresponding optimization programs are the same. We also note that the construction given in [19] is easily adapted to the torus.}
In the whole space this would readily follow from explicit computations for Gaussian vectors. Working in the torus makes the proof slightly more intricate, and we postpone the proof to Appendix B for convenience.

We also need to define a notion of translated Brownian bridges, which will play a crucial role in our construction of the recovery sequence \((P^\nu)\). To do so, let us first denote by \(\Pi\) the projection

\[\Pi : \omega \in C([0, 1]; \mathbb{R}^d) \mapsto (t \mapsto \pi(\omega_t)) \in C([0, 1]; T^d).\]

Then, if \(\omega \in C([0, 1]; T^d)\), we write \(T_\omega\) for the translation map

\[T_\omega : \alpha \in C([0, 1]; T^d) \mapsto \omega + \alpha \in C([0, 1]; T^d).\] (27)

Let \(B^\nu := R^\nu,0,0\) be the standard Brownian bridge in \(\mathbb{R}^d\) with diffusivity \(\nu\) and joining 0 to 0. Our translated bridges are defined as follows:

**Definition 3.4.** If \(\omega \in C([0, 1]; T^d)\) and \(\nu > 0\), we set

\[B^\nu_\omega := T_\omega \# \Pi \# B^\nu.\]

Roughly speaking, \(B^\nu_\omega\) is obtained by adding the projection of the Brownian bridge to \(\omega\). Note however that the Brownian bridge in the torus is not the projection of the Brownian bridge in \(\mathbb{R}^d\), i.e., \(R^\nu,0,0 \neq \Pi \# B^\nu\). As a consequence, \(B^\nu_\omega \neq T_\omega \# R^\nu,0,0\). This alternative choice of translated bridges would have made the proof of Lemma 3.5 below more delicate.

The entropy of \(B^\nu_\omega\) with respect to the bridges of \(R^\nu\) will be computed thanks to

**Lemma 3.5.** There exists a dimensional constant \(C = C_d > 0\) such that, for all \(\nu \leq 1\) and all \(\omega \in C([0, 1]; T^d)\),

\[\nu H(B^\nu_\omega | R^\nu,0,0,\omega) \leq A(\omega) - \frac{1}{2}d^2(\omega_0, \omega_1) + C\nu.\] (28)

Once again, we postpone the proof to the appendix. Note that, in the whole space \(\mathbb{R}^d\), the corresponding result is stated in Lemma B.2 and follows from the classical Cameron-Martin formula. In that case, equality holds in (28) with \(C = 0\).

With this preliminary material at hand, let us first prove Theorem 2.2 using Theorem 3.1.

**Proof of Theorem 2.2.** Both \(\Gamma - \lim inf\) parts are a direct consequence of point 1 of Theorem 3.1 regardless of any marginal constraint.

For the \(\Gamma - \lim sup\) part in point 1, fix an admissible generalized flow \(P\) with marginals \(P_{0,1} = \gamma\). By [19, Thm. 2.7] there always exists a recovery sequence \(\gamma^\nu \rightharpoonup \gamma\) for the optimal transport problem, i.e., satisfying (22) (see Footnote 2).
Thus by Theorem 3.1 we can construct a recovery sequence $P^\nu \rightharpoonup P$ satisfying (21).

For the $\Gamma - \limsup$ part of point 2 we claim that the particular sequence $\gamma^\nu = \gamma$ satisfies (22). On this premise, Theorem 3.1 immediately provides a recovery sequence $P^\nu \rightharpoonup P$ with marginals $P^\nu_{0,1} = \gamma$ and satisfying (21) as required, hence it suffices to check our claim. To this end, observe that the density $r^\nu_{0,1}$ of $R^\nu_{0,1}$ with respect to $\text{Leb} \otimes \text{Leb}$ is

$$r^\nu_{0,1}(x,y) = \tau^\nu_{s}(y-x)$$

where as before, $(\tau^s_s)$ is the heat kernel in the torus at time $s$. By Proposition 2.9 (with $f = r^\nu_{0,1}$ and $r = \text{Leb} \otimes \text{Leb}$) we have thus

$$\nu H(\gamma | R^\nu_{0,1}) = \nu H(\gamma | \text{Leb} \otimes \text{Leb}) - \int \nu \log r^\nu_{0,1}(x,y) \, d\gamma(x,y)$$

$$= \nu H(\gamma | \text{Leb} \otimes \text{Leb}) - \int \nu \log \tau^\nu_{s}(y-x) \, d\gamma(x,y)$$

$$\xrightarrow{\nu \to 0} 0 + \frac{1}{2} \int d^2(x,y) \, d\gamma(x,y),$$

where the last line is obtained using our assumption $H(\gamma | \text{Leb} \otimes \text{Leb}) < +\infty$ as well as (19) (which implies in particular $\nu \log \tau^\nu_{s}(y-x) \to -d^2(x,y)$ uniformly on $T^d \times T^d$). Hence our claim holds and the proof is complete.

Let us now carry on with the proof of Theorem 3.1 which will go through several steps. We begin with

Proof of point 1 of Theorem 3.1. This result is weaker than the same statement without the incompressibility constraint. Hence, it is direct a consequence of the works of C. Léonard, see e.g. [31, Prop. 2.5]. For the sake of self-completeness we choose to present here an independent proof, fully leveraging the explicit structure of the reversible Brownian motion $R^\nu$ as a particular reference measure (whereas C. Léonard covers much more general settings). We will also recycle part of the argument later on in the proof of $2a \implies 2b$ in Theorem 3.1 hence we give the full details.

Consider a sequence $P^\nu \rightharpoonup P$ as in our statement. Observe first that the Legendre transform of $h(u) = u \log u$ is $h^*(v) = e^{v-1}$, in particular

$$u \log u \geq uv - e^{v-1} \quad \text{for all } u, v. \quad (29)$$

Fix any bounded and continuous function $f(\omega)$ on $C([0,1]; T^d)$. Taking $u = \frac{dP^\nu}{d\mu^\nu}(\omega), v = f(\omega)/\nu$ in the previous convexity inequality and integrating with
respect to $R^\nu$, we get
\[ \nu \mathcal{H}_\nu(P^\nu) = \nu \int \frac{dP^\nu}{dR^\nu}(\omega) \log \frac{dP^\nu}{dR^\nu}(\omega) dR^\nu(\omega) \]
\[ \geq \nu \int \frac{dP^\nu}{dR^\nu}(\omega) \frac{f(\omega)}{\nu} dR^\nu(\omega) - \nu \int \exp \left( \frac{f(\omega)}{\nu} - 1 \right) dR^\nu(\omega) \]
\[ = \int f(\omega) dP^\nu(\omega) - \nu e^{-1} \int \exp \left( \frac{f(\omega)}{\nu} \right) dR^\nu(\omega). \quad (30) \]

Ideally, one wishes to test $f = A$ (the kinetic action (8)) in this formula, and pass to the limit $\nu \to 0$ hoping that the exponential term $\nu e^{-1}$ to conclude that $\lim \inf \nu \mathcal{H}_\nu(P^\nu) \geq \int A dP$. However this is not possible since Brownian paths are nowhere differentiable hence $A(\omega) = +\infty$ for $R^\nu$-almost all paths $\omega$. Instead, we use as a surrogate the difference quotient approximation $A_N$ defined in (24).

For technical reasons let us fix a parameter $\alpha \in (0, 1)$ close to 1, and recall that for large $N$ we write $\tau = 1/N$ and $t_n = n\tau$. Taking $f = \alpha A_N$ in (30), the last integral in the r.h.s. is immediately bounded by Lemma 3.3 as
\[ \int \exp \left( \frac{\alpha A_N(\omega)}{\nu} \right) dR^\nu(\omega) \leq \frac{1}{(1 - \alpha)^{Nd/2}}. \]

As a consequence we get
\[
\lim \inf_{\nu \to 0} \nu \mathcal{H}_\nu(P^\nu) \geq \lim \inf_{\nu \to 0} \left\{ \alpha \int A_N(\omega) dP^\nu(\omega) \right\} - \lim \sup_{\nu \to 0} \left\{ \nu e^{-1} \int \exp \left( \frac{\alpha A_N(\omega)}{\nu} \right) dR^\nu(\omega) \right\} \]
\[ \geq \lim \inf_{\nu \to 0} \left\{ \alpha \int A_N(\omega) dP^\nu(\omega) \right\} - \lim \sup_{\nu \to 0} \left\{ \frac{\nu e^{-1}}{(1 - \alpha)^{Nd/2}} \right\} \]
\[ = \alpha \int A_N(\omega) dP(\omega) - 0 \]

because we assumed $P^\nu \rightharpoonup P$ (and $A_N$ is bounded continuous). Taking next $\alpha \to 1$ and then $\lim \inf$ as $N \to \infty$ with pointwise convergence $A_N(\omega) \to A(\omega)$ for all $\omega \in C([0, 1]; T^d)$, we finally get by Fatou’s lemma
\[ \lim \inf_{\nu \to 0} \nu \mathcal{H}_\nu(P^\nu) \geq 1 \cdot \lim \inf_{N \to \infty} \int A_N(\omega) dP(\omega) \geq \int A(\omega) dP(\omega) = A(P) \]
and the proof is complete. $\square$ $\square$

We proceed now with the proof of the equivalence $2a \iff 2b$ in Theorem 3.1.

In order to ease the exposition we opted for dividing the argument in two steps, one for each implication.

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Proof of Theorem 3.1. Assume that \( P^\nu \) converges to \( P \) and satisfies the \( \Gamma-\limsup \) inequality (21). Disintegrating with respect to \( (X_0, X_1) \), we have by Proposition 2.10
\[
\overline{H}_\nu(P^\nu) = \nu H(\gamma^\nu | R^\nu_{0,1}) + \nu \int H(P^\nu_{\alpha,x,y} | R^\nu_{\alpha,x,y}) \, d\gamma^\nu(x,y).
\]
Hence with our assumption (21) there holds
\[
\limsup_{\nu \to 0} \{ \nu H(\gamma^\nu | R^\nu_{0,1}) \}
\leq \limsup_{\nu \to 0} \overline{H}_\nu(P^\nu) - \liminf_{\nu \to 0} \left\{ \nu \int H(P^\nu_{\alpha,x,y} | R^\nu_{\alpha,x,y}) \, d\gamma^\nu(x,y) \right\}
\leq \mathcal{A}(P) - \liminf_{\nu \to 0} \left\{ \nu \int H(P^\nu_{\alpha,x,y} | R^\nu_{\alpha,x,y}) \, d\gamma^\nu(x,y) \right\}.
\] (31)

In order to estimate the last integral we proceed using the same strategy as in the proof of point 1 of Theorem 3.1 earlier. For large \( N \in \mathbb{N} \) and \( \tau = 1/N \) we write again \( t_n = n\tau \) for \( n = 0 \ldots N \), and consider now
\[
\hat{A}_N(\omega) := \frac{1}{2\tau} \sum_{n=0}^{N-1} d^2(\omega_{t_n}, \omega_{t_{n+1}}) - \frac{1}{2} d^2(\omega_0, \omega_1) = A_N(\omega) - \frac{1}{2} d^2(\omega_0, \omega_1).
\]
This function is continuous for the uniform topology on \( C([0,1]; \mathbb{T}^d) \), bounded, and it is as before a good approximation of
\[
\hat{A}(\omega) := \begin{cases} \frac{1}{2} \int_0^1 |\omega_t|^2 \, dt - \frac{1}{2} d^2(\omega_0, \omega_1) & \text{if } \omega \in AC^2 \\ \text{otherwise} \end{cases}
\]
as \( N \to \infty \). We fix as before a parameter \( \alpha \in (0,1) \) close to 1. Exploiting the convexity inequality (29) with \( u = \frac{dP^\nu_{\alpha,x,y}}{dR^\nu_{\alpha,x,y}}(\omega) \) and \( v = \frac{2}{\nu} \hat{A}_N(\omega) \), integrating first with respect to \( R^\nu_{\alpha,x,y} \) (note that \( \omega_0 = x \) and \( \omega_1 = y \) for \( R^\nu_{\alpha,x,y} \)-almost all \( \omega \)) and then with respect to \( \gamma^\nu \), we get
\[
\nu \int H(P^\nu_{\alpha,x,y} | R^\nu_{\alpha,x,y}) \, d\gamma^\nu(x,y) \geq \alpha \int \hat{A}_N(\omega) \, dP^\nu_{\alpha,x,y}(\omega) \, d\gamma^\nu(x,y)
- \nu e^{-1} \int \exp \left( -\frac{\alpha}{2\nu} d(x,y)^2 \right) \int \exp \left( \frac{\alpha}{2\nu} A_N(\omega) \right) \, dR^\nu_{\alpha,x,y}(\omega) \, d\gamma^\nu(x,y). \quad (32)
\]
Since by assumption \( P^\nu \overset{\star}{\to} P \), and because \( \hat{A}_N \) is bounded continuous, we see that the first term in the r.h.s
\[
\int \hat{A}_N(\omega) \, dP^\nu_{\alpha,x,y}(\omega) \, d\gamma^\nu(x,y) = \int \hat{A}_N(\omega) \, dP^\nu(\omega) \overset{\nu \to 0}{\longrightarrow} \int \hat{A}_N(\omega) \, dP(\omega),
\] (33)
thus it only remains to show that the \( \limsup \) of the exponential term in (32) goes to zero (just like in the previous proof of point 1 of Theorem 3.1). This is
where we need to use Lemma 3.3. From (32) (33) (26), we get:

\[ \liminf_{\nu \to 0} \left\{ \nu \int H(P^{\nu,x,y} | R^{\nu,x,y}) \, d\gamma^{\nu}(x, y) \right\} \geq \alpha \int \hat{A}_N(\omega) \, dP(\omega) - \limsup_{\nu \to 0} \left\{ \nu e^{-1} \int \exp \left( -\frac{\alpha}{2\nu} d^2(x, y) \right) \frac{C_d}{(1 - \alpha)^{Nd/2}} \, d\gamma^{\nu}(x, y) \right\} \]
\[ \geq \alpha \int \hat{A}_N(\omega) \, dP(\omega) - \limsup_{\nu \to 0} \left\{ \nu e^{-1} \times \frac{C_d}{(1 - \alpha)^{Nd/2}} \right\} \]
\[ = \alpha \int \hat{A}_N(\omega) \, dP(\omega). \]

Taking next \( \alpha \to 1 \) and \( \liminf \) as \( N \to \infty \) with now

\[ \hat{A}_N(\omega) = A_N(\omega) - \frac{1}{2} d^2(\omega_0, \omega_1) \]

for all \( \omega \), we conclude by Fatou’s lemma that

\[ \liminf_{\nu \to 0} \left\{ \nu \int H(P^{\nu,x,y} | R^{\nu,x,y}) \, d\gamma^{\nu}(x, y) \right\} \geq \int \left\{ A(\omega) - \frac{1}{2} d^2(\omega_0, \omega_1) \right\} \, dP(\omega) = \mathcal{A}(P) - \frac{1}{2} \int d^2(x, y) \, d\gamma(x, y). \]

Substituting into (31) finally gives (22) and concludes the proof. \( \Box \)

Let us now establish the converse implication:

**Proof of Theorem 3.1, 2b \( \Rightarrow \) 2a.** Take \((\gamma^\nu)_{\nu > 0}\) as in our statement. Since we have \( \gamma^\nu \rightharpoonup \gamma \), a closer look into the proof of [7, Thm. 1] (continuity of the optimal action in Brenier’s problem \( \mathcal{REu}(\gamma) \) with respect to the marginal \( \gamma \)) gives a sequence \( Q^\nu \) of generalized flows converging to \( P \) such that \( Q^\nu \) is admissible for \( \mathcal{REu}(\gamma^\nu) \) and

\[ \lim_{\nu \to 0} \mathcal{A}(Q^\nu) = \mathcal{A}(P). \] (34)

Let now

\[ P^\nu := \int B^\nu_\omega \, dQ^\nu(\omega), \] (35)

with \( B^\nu_\omega \) as in Definition 3.4. Roughly speaking, \( P^\nu \) is a noisy version of \( Q^\nu \), where all the paths initially charged by \( Q^\nu \) receive now an additional small Brownian perturbation.

First of all, we claim that \( P^\nu \Rightarrow P \) as \( \nu \to 0 \). Indeed, if \( \varphi \) is a test function on \( C([0, 1]; \mathbb{T}^d) \), let us check that

\[ \varphi^\nu : \omega \mapsto \int \varphi(\alpha) \, dB^\nu_\omega(\alpha) \]
converges uniformly towards $\varphi$ on the compact sets of $C([0, 1]; \mathbb{T}^d)$. Indeed if $K$ is such a compact set, take $m : \mathbb{R}_+ \to \mathbb{R}_+$ a modulus of continuity of $\varphi|_K$. Of course, $m$ can be chosen continuous and bounded with $m(0) = 0$. Then, for all $\omega \in K$,

$$
|\varphi'(\omega) - \varphi(\omega)| \leq \int |\varphi(\alpha) - \varphi(\omega)| \, dB_\nu(\alpha)
= \int |\varphi(\omega + \alpha) - \varphi(\omega)| \, dB_\nu(\alpha) \leq \int m(\|\alpha\|_{\infty}) \, dB(\alpha).
$$

Since $m$ is bounded and continuous so is the map $\alpha \mapsto m(\|\alpha\|_{\infty})$ for the uniform topology on $C([0, 1]; \mathbb{T}^d)$, and it is therefore an admissible test-function for the narrow convergence $B_\nu \Rightarrow \delta_0$ in $\mathcal{M}(C([0, 1]; \mathbb{T}^d))$. As a consequence the right-hand side converges to $\int m(\|\alpha\|_{\infty}) \, d\delta_0(\alpha) = m(0) = 0$ and therefore $\|\varphi' - \varphi\|_{\infty} \to 0$ as required. Since $Q_\nu \Rightarrow P$ we get now

$$
\int \varphi(\omega) \, dP(\omega) = \int \left( \int \varphi(\alpha) \, dB_\nu(\alpha) \right) \, dQ_\nu(\omega)
= \int \varphi(\omega) \, dQ_\nu(\omega) \underset{\nu \to 0}{\longrightarrow} \int \varphi(\omega) \, dP(\omega)
$$

as claimed.

Moreover, it is straightforward to check that the marginal constraint is satisfied, $(X_0, X_1) \# P_\nu = \gamma_\nu$. Let us check now the incompressibility. Take $\varphi$ a test function on $\mathbb{T}^d$: by Definition 3.4 of $B_\nu$ and by incompressibility of $Q_\nu$, we have for all $t \in [0, 1]$

$$
\int \varphi(\omega) \, dP(\omega) = \int \int \varphi(\alpha_t) \, dB_\nu(\alpha) \, dQ_\nu(\omega)
= \int \int \varphi(\pi_t + \omega) \, dB_\nu(\alpha) \, dQ_\nu(\omega)
= \int \int \varphi(\pi_t + \omega) \, dQ_\nu(\omega) \, dB_\nu(\alpha)
= \int \int \varphi(\pi_t + x) \, dx \, dB_\nu(\alpha)
= \int \int \varphi(x') \, dx' \, dB_\nu(\alpha) = \int \varphi(x') \, dx'.
$$

Next, let us estimate $\mathcal{H}_\nu(P_\nu) = \nu H(P_\nu | R_\nu)$. Conditioning on the endpoints, we get by Proposition 2.10

$$
H(P_\nu | R_\nu) = H(\gamma_\nu | R_0^\nu) + \int H(P_\nu | R_{x,y}^\nu) \, d\gamma_\nu(x, y). \quad (36)
$$

Moreover, conditioning $Q_\nu$, we get for $\gamma_\nu$-almost all $(x, y) \in \mathbb{T}^d \times \mathbb{T}^d$:

$$
P_{\nu,x,y} = \int B_\nu \, dQ_{\nu,x,y}(\omega).
$$
In particular for \( \gamma^\nu \)-almost all \((x, y) \in T^d \times T^d\), by Jensen’s inequality, and because \( Q^\nu_{x,y} \)-almost surely \( \omega_0 = x \) and \( \omega_1 = y \), there holds
\[
H(P^\nu_{x,y} | R^\nu_{x,y}) \leq \int H(B^\nu_\omega | R^\nu_{x,y}) \, dQ^\nu_{x,y}(\omega) = \int H(B^\nu_\omega | R^\nu_{x,y}) \, dQ^\nu_{x,y}(\omega).
\]
Substituting this inequality in formula (36), we get:
\[
H(P^\nu | R^\nu) \leq H(\gamma^\nu | R^\nu_{0,1}) + \int \left\{ A(\omega) - \frac{1}{2} d^2(\omega_0, \omega_1) + C\nu \right\} \, dQ^\nu(\omega)
\]
Multiplying by \( \nu \) and using (28), we get for \( \nu < 1 \):
\[
\nu H(P^\nu | R^\nu) \leq \nu H(\gamma^\nu | R^\nu_{0,1}) + \int \left\{ A(\omega) - \frac{1}{2} d^2(\omega_0, \omega_1) + C\nu \right\} \, dQ^\nu(\omega)
\]
where \( C \) is a dimensional constant. With our assumption (22) and by (34), we finally obtain
\[
\limsup_{\nu \to 0} \frac{\nu H(P^\nu | R^\nu)}{\nu H(\gamma^\nu | R^\nu_{0,1})} \leq \frac{1}{2} \int d(x,y)^2 \, d\gamma(x,y) + A(P) - \frac{1}{2} \int d(x,y)^2 \, d\gamma(x,y)
\]
and the proof is complete.

4 Convergence of Sch towards OT

Here we give a new proof of the convergence of entropic optimal transport towards deterministic optimal transport as the diffusivity \( \nu \to 0 \), Theorem 2.3.

We stress again that the result itself is not new [37, 31, 22, 19], but our proof only relies on elementary PDE arguments and we believe it is worth including the details for the sake of completeness.

Before going into the proof we shall need a fundamental regularization procedure (Lemma 4.1 below), to be used repeatedly in the sequel. To motivate the approach, observe that, in the previous section, the key step was the construction of a suitable recovery sequence \( P^\nu \triangledown P \) by means of Brownian bridges –
see in particular 35]. Our regularization below will simply consist in a similar construction at the PDE level.

More precisely, recall that we write \( \tau_s(x) \) for the heat kernel at time \( s > 0 \)
\[
\partial_t \tau_s = \frac{1}{2} \Delta \tau_s
\]
started from the initial Dirac distribution \( \tau_0 = \delta_0 \). For a given curve \( \rho \in C([0,1]; \mathcal{P}(\mathbb{T}^d)) \) and diffusivity parameter \( \nu > 0 \) we shall always write \( \rho^\nu \in C([0,1]; \mathcal{P}(\mathbb{T}^d)) \) for the curve defined by
\[
t \mapsto \rho^\nu_t := \rho_t \ast \tau_{\nu t(1-t)} \in \mathcal{P}(\mathbb{T}^d),
\]
(37)
where the convolution only acts in space. In other words \( \rho^\nu_t \) is defined as the solution of the heat flow at time \( s = \nu t(1-t) \) started from \( \rho_t \) at time \( s = 0 \), and in particular \( \rho^\nu \) has the same endpoints as \( \rho \)
\[
\rho^\nu_0 = \rho_0 \quad \text{and} \quad \rho^\nu_1 = \rho_1.
\]

Our regularity estimate takes the following quantitative form:

**Lemma 4.1.** For all \( \rho \in C([0,1]; \mathcal{P}(\mathbb{T}^d)) \), defining \( \rho^\nu \) as in (37), there holds
\[
\mathcal{A}(\rho^\nu) + \frac{\nu^2}{2} \int_0^1 \left( t - \frac{1}{2} \right)^2 \left| \nabla \log \rho^\nu_t \right|^2 \, dt + \nu \int_0^1 H(\rho^\nu_t) \, dt \\
\leq \mathcal{A}(\rho) + \nu \frac{H(\rho_0) + H(\rho_1)}{2}.
\]
(38)
Moreover, there exists a dimensional constant \( C = C_d > 0 \) such that,
\[
\mathcal{A}(\rho^\nu) + \frac{\nu^2}{8} \int_0^1 \left| \nabla \log \rho^\nu_t \right|^2 \, dt + \nu \int_0^1 H(\rho^\nu_t) \, dt \\
\leq \mathcal{A}(\rho) + \nu \left[ \frac{H(\rho_0) + H(\rho_1)}{2} + C \right].
\]
(39)

Similarly, for \( \alpha > 0 \) the entropic version holds as:
\[
\mathcal{H}_\alpha(\rho^\nu) + \frac{\nu^2}{2} \int_0^1 \left( t - \frac{1}{2} \right)^2 \left| \nabla \log \rho^\nu_t \right|^2 \, dt + \nu \int_0^1 H(\rho^\nu_t) \, dt \\
\leq \mathcal{H}_\alpha(\rho) + \nu \frac{H(\rho_0) + H(\rho_1)}{2},
\]
(40)
and
\[
\mathcal{H}_\alpha(\rho^\nu) + \frac{\nu^2}{8} \int_0^1 \left| \nabla \log \rho^\nu_t \right|^2 \, dt + \nu \int_0^1 H(\rho^\nu_t) \, dt \\
\leq \mathcal{H}_\alpha(\rho) + \nu \left[ \frac{H(\rho_0) + H(\rho_1)}{2} + C \right].
\]
(41)
Note that all four right-hand sides are allowed to be infinite, in which case our statement is vacuous.

Proof. Let us start with (38). We can always assume that $\rho$ has regularity $AC^2([0,T]; P(\mathbb{T}^d))$, since otherwise $A(\rho) = +\infty$ in the r.h.s. By theorem 2.1 there exists a velocity field $c \in L^2(dt \otimes \rho)$ such that $A(\rho) = \frac{1}{2} \int_0^1 \int |c_t|^2 \rho_t \, dt$. Defining the classical regularization

$$\hat{c}^\nu_t := \frac{(\rho_t c_t) * \tau_{\nu t(1-t)}}{\rho_t * \tau_{\nu t(1-t)}} = \frac{(\rho_t c_t) * \tau_{\nu t(1-t)}}{\rho_t^\nu},$$

(42) it is easy to check that

$$\partial_t \rho_t^\nu + \text{div}(\rho_t^\nu \hat{c}^\nu_t) = -\nu \left(t - \frac{1}{2}\right) \Delta \rho_t^\nu$$

at least in the sense of distributions. (This is a first reason why we defined the regularization (42) as acting on the momentum variable $m_t^\nu = \rho_t^\nu \hat{c}_t^\nu = (\rho_t c_t) * \tau_{\nu t(1-t)}$, rather than directly on the velocities.) The extra Laplacian in the right-hand side arises because the regularizing kernel $\tau_{\nu t(1-t)}$ is not fixed but depends on time. Setting moreover

$$c_t^\nu := \hat{c}^\nu_t - \nu \left(t - \frac{1}{2}\right) \nabla \log \rho_t^\nu$$

(43) and recalling that $\text{div}(\rho \nabla \log \rho) = \text{div} \left(\rho \frac{\nabla \rho}{\rho}\right) = \Delta \rho$, we have now

$$\partial_t \rho_t^\nu + \text{div}(\rho_t^\nu c_t^\nu) = 0,$$

hence by definition of the kinetic action (3) and of the metric speed (2)

$$A(\rho^\nu) \leq \frac{1}{2} \int_0^1 \int |c_t^\nu|^2 \, d\rho_t^\nu \, dt.$$  

(44)

Since $\tau_{\nu t(1-t)}$ is a probability measure and $(\rho, m) \mapsto \frac{|m|^2}{2\rho}$ is jointly convex, an immediate application of Jensen’s inequality gives

$$\frac{1}{2} \int_0^1 \int |c_t^\nu|^2 \, d\rho_t^\nu \, dt \leq \frac{1}{2} \int_0^1 \int |c_t|^2 \, d\rho_t \, dt = A(\rho)$$

(45)

(This is another reason for the particular definition of $\hat{c}^\nu$.) Gathering (44,45) and exploiting (43) to expand $|c_t^\nu|^2$, we find

$$A(\rho^\nu) + \frac{\nu^2}{2} \int_0^1 \left(t - \frac{1}{2}\right)^2 \int |\nabla \log \rho_t^\nu|^2 \, d\rho_t^\nu \, dt$$

$$\leq A(\rho) + \nu \int_0^1 \left(t - \frac{1}{2}\right) \int \nabla \log \rho_t^\nu \cdot c_t^\nu \, d\rho_t^\nu \, dt.$$  

(46)

Writing next

$$\frac{d}{dt} H(\rho_t^\nu) = \int (1 + \log \rho_t^\nu) \partial_t \rho_t^\nu = -\int (1 + \log \rho_t^\nu) \text{div}(\rho_t^\nu c_t^\nu) = \int \nabla \log \rho_t^\nu \cdot c_t^\nu \, d\rho_t^\nu \, dt$$

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we obtain after integration by parts
\[
\int_0^1 \left(t - \frac{1}{2}\right) \nabla \log \rho_t^\nu \cdot c_t^\nu \, d\rho_t^\nu \, dt = \int_0^1 \left(t - \frac{1}{2}\right) \frac{d}{dt} H(\rho_t^\nu) \, dt
\]
\[
= \frac{H(\rho_0^\nu) + H(\rho_1^\nu)}{2} - \int_0^1 H(\rho_t^\nu) \, dt
\]
\[
= \frac{H(\rho_0) + H(\rho_1)}{2} - \int_0^1 H(\rho_t^\nu) \, dt.
\]

Here we crucially used the fact that the endpoints \(\rho_0^\nu = \rho_0\) and \(\rho_1^\nu = \rho_1\) remain unchanged. Our first estimate \(38\) immediately follows by substituting this identity in \(46\).

To get \(39\), we first add \(\frac{\nu^2}{2} \int_0^1 t(1-t) \int |\nabla \log \rho_t^\nu|^2 \, d\rho_t^\nu \, dt\) to both sides of \(38\) and use the algebraic identity \(t(1-t) + (t - 1/2)^2 = 1/4\) to get
\[
\mathcal{A}(\rho^\nu) + \frac{\nu^2}{8} \int_0^1 \int |\nabla \log \rho_t^\nu|^2 \, d\rho_t^\nu \, dt + \nu \int_0^1 H(\rho_t^\nu) \, dt
\]
\[
\leq \mathcal{A}(\rho) + \frac{\nu H(\rho_0) + H(\rho_1)}{2} + \nu \int_0^1 \nu \frac{t(1-t)}{2} \int |\nabla \log \rho_t^\nu|^2 \, d\rho_t^\nu \, dt, \quad (47)
\]
and it only remains to control the last term in the right-hand side. By the celebrated Li-Yau inequality \[34\] Thm. 1.1, the Fisher information decays at a universal rate along the heat flow uniformly in the initial datum, here (with \(\tilde{\rho}_s = \tau_s \ast \tilde{\rho}_0\)):
\[
\int |\nabla \log \tilde{\rho}_s|^2 \, d\tilde{\rho}_s \leq \frac{d}{2s} \quad \forall \, \tilde{\rho}_0 \in \mathcal{P}(\mathbb{T}^d), \forall \, s > 0.
\]
Recalling that, by definition, \(\rho_t^\nu\) is the solution at time \(s = \nu t(1-t)\) of the heat flow started from \(\tilde{\rho}_0 = \rho_t\), the last term in \(47\) can thus be controlled as
\[
\nu \int_0^1 \frac{\nu t(1-t)}{2} \int |\nabla \log \rho_t^\nu|^2 \, d\rho_t^\nu \, dt \leq \nu \int_0^1 \nu \frac{t(1-t)}{2} \int |\nabla \log \rho_t^\nu|^2 \, d\rho_t^\nu \, dt \leq C_d \nu
\]
and \(39\) follows.

As for \(40\)(41), we recall that the Fisher information
\[
F(\tilde{\rho}_s) = \frac{1}{8} \int |\nabla \log \tilde{\rho}_s|^2 \, d\tilde{\rho}_s
\]
is nonincreasing along the heat flow (by the same Jensen’s inequality used in \(45\)). In our particular setting this gives \(F(\rho_t^\nu) \leq F(\rho_t)\) for all \(t \in [0,1]\). The result immediately follows by adding \(\alpha^2 F(\rho_t) \leq \alpha^2 F(\rho)\) to \(38\) and \(39\), respectively, and the proof is complete.

We are now in position of proving the convergence of \(\text{Sch}_\nu\) towards \(\text{OT}\).
Proof of Theorem 2.3: The $\Gamma$–lim inf is obvious, as $\mathcal{H}_\nu = \mathcal{A} + \nu^2 \mathcal{F} \geq \mathcal{A}$ and $\mathcal{A}$ is lower semicontinuous. Let us therefore consider the $\Gamma$–lim sup, and fix $\rho \in C([0,1]; \mathcal{P}(\mathbb{T}^d))$ with endpoints $\rho_0, \rho_1$. We can always assume that $\rho \in AC^2$, otherwise there is nothing to prove.

For $\rho \in AC^2([0,1]; \mathcal{P}(\mathbb{T}^d))$, we claim that $(\rho^\nu)_{\nu > 0}$ defined in (37) is an admissible recovery sequence. Indeed, as already discussed $\rho^\nu$ has same endpoints $\rho_0, \rho_1$ as $\rho$. Moreover from (39) we get

$$H_\nu(\rho^\nu) = A(\rho^\nu) + \nu^2 \mathcal{F}(\rho^\nu) \leq A(\rho) + \nu \left[ \frac{H(\rho_0) + H(\rho_1)}{2} + C \right],$$

and taking the lim sup gives

$$\limsup_{\nu \to 0} H_\nu(\rho^\nu) \leq A(\rho)$$

as required. \qed

Remark 4.2. From this proof it is clear that, apart from the static entropy term $\nu \frac{H(\rho_0) + H(\rho_1)}{2}$, the purely dynamical lim sup-gap for the $\Gamma$-convergence $H_\nu \to OT$ is of order at most $C\nu$ for a dimensional constant $C = C_d$.

Remark 4.3. When $H(\rho_0)$ or $H(\rho_1)$ is infinite, the $\Gamma$-convergence $[15]$ cannot hold because $H_\nu + \iota(\rho_0, \rho_1) \equiv +\infty$ for all $\nu > 0$. However, similarly to the scenario of Theorem 3.1 it is still possible to prove

$$\Gamma - \lim_{\nu \to 0} H_\nu = \mathcal{A},$$

and more precisely

$$\Gamma - \lim_{\nu \to 0} \left\{ H_\nu + \iota(\rho^\nu_0, \rho^\nu_1) \right\} = \mathcal{A} + \iota(\rho_0, \rho_1)$$

if and only if $\rho^\nu_0 \stackrel{\Gamma}{\to} \rho_0$, $\rho^\nu_1 \stackrel{\Gamma}{\to} \rho_1$ and

$$\lim_{\nu \to 0} \nu H(\rho^\nu_0) = 0 \quad \text{and} \quad \lim_{\nu \to 0} \nu H(\rho^\nu_1) = 0.$$

Such $(\rho^\nu_0)$ and $(\rho^\nu_1)$ are easy to build for instance by convolution.

5 Convergence of MBrö towards MREu

Here we prove the $\Gamma$-convergence in Theorem 2.4 as well as the convergence of the pressures associated with the incompressibility constraints, Theorem 2.6.
5.1 $\Gamma$-convergence

Proof of Theorem 2.4. Again, the $\Gamma - \lim \inf$ easily follows from the standard lower semicontinuity of $A$ together with $H_\nu = A + \nu^2 F \geq A$, and we only focus on the $\Gamma - \lim \sup$ inequality. The argument essentially consists in superposing the proof of Theorem 2.3 by linearity, i.e. integrating with respect to $P$.

More precisely: For $\nu > 0$ we define the mapping

$$\Phi^\nu : \rho \mapsto \rho^\nu$$

from $C([0,1] ; \mathcal{P}(\mathbb{T}^d))$ to itself, where the curve $\rho^\nu = (\rho^\nu_t)_{t \in [0,1]}$ is defined in (37). For any incompressible traffic plan $P$, we first claim that $P^\nu := \Phi^\nu \# P$ shares its marginals with $P$ and automatically inherits incompressibility from that of $P$. Indeed, as already observed, $\rho^\nu$ leaves the endpoints unchanged $\rho^\nu_0 = \rho_0$ and $\rho^\nu_1 = \rho_1$, hence for all test functions $\varphi$ on $\mathcal{P} \times \mathcal{P}$

$$\int \varphi(\rho_0, \rho_1) dP^\nu(\rho) = \int \varphi(\Phi^\nu(\rho)_0, \Phi^\nu(\rho)_1) dP(\rho)$$

$$= \int \varphi(\rho_0, \rho_1) dP(\rho) = \int \varphi(\rho_0, \rho_1) dP(\rho)$$

and therefore $P^\nu_0 = P_{0,1}$. In particular, the constraint $P^\nu_0 = \Gamma$ is satisfied as soon as $P_{0,1} = \Gamma$. For the incompressibility, since Leb is invariant for the heat flow $(\tau_s) \# \text{Leb} = \text{Leb}$ $\Rightarrow$ $\Phi^\nu(\text{Leb}) = \text{Leb}$, and because $\Phi^\nu$ is linear, we have

$$\int \rho_t dP^\nu(\rho) = \int \Phi^\nu(\rho)_t dP(\rho) = \Phi^\nu \left( \int \rho dP(\rho) \right)_t = \Phi^\nu(\text{Leb}) = \text{Leb}$$

for all $t$.

Taking now an admissible $P$ in $\text{MREu}(\Gamma)$, we just showed that $P^\nu$ is admissible too, and we claim that it is a suitable recovery sequence.

First, we claim that $P^\nu \overset{\ast}{\rightharpoonup} P$: according to [2, Lem. 5.2.1], it suffices to show that $\Phi^\nu$ converges uniformly towards the identity on the compact sets of $C([0,1] ; \mathcal{P}(\mathbb{T}^d))$. In fact, this convergence is even uniform (and not compactly uniform), which will follow from the estimate

$$\forall \rho \in \mathcal{P}(\mathbb{T}^d), \forall s \geq 0, \quad d_{MK}(\rho, \rho \ast \tau_s) \leq C_d \sqrt{s}$$

for some dimensional constant $C_d > 0$. To check this, we use as an admissible coupling between $\rho$ and $\rho \ast \tau_s$ the joint law of the Brownian motion starting from $\rho$ between times 0 and $s$: testing $d_{\gamma_s}(x, y) := \tau_s(y-x) \, d\rho(x) \otimes dy$ as a competitor in (1) yields

$$d_{MK}^2(\rho, \rho \ast \tau_s) \leq \frac{1}{2} \int d^2(x,y) d_{\gamma_s}(x,y)$$

$$= \frac{1}{2} \int \left( \int d^2(x,y) \tau_s(x-y) \, dy \right) d\rho(x) \leq C_d s.$$

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As a consequence,
\[
\forall \rho \in C([0,1]; \mathcal{P}(\mathbb{T}^d)), \quad \sup_{t \in [0,1]} \text{dMK}(\rho_t, \rho_t') \leq \sup_{t \in [0,1]} C_d \sqrt{\nu t (1 - t)} = C_d \sqrt{\nu},
\]
which proves the uniform convergence.

For the lim sup inequality, we can always assume that \( \mathcal{A}(P) < \infty \) hence that \( P \) only charges \( AC^2 \) curves, otherwise there is nothing to prove. We can therefore appeal to Lemma 4.1 and (39) (for \( P \)-a.e. \( \rho \)) to estimate
\[
\mathcal{H}_\nu(P) = \int \mathcal{H}_\nu(\rho) \, dP(\rho) = \int \mathcal{H}_\nu(\rho^\nu) \, dP(\rho) \leq \int \mathcal{A}(\rho) \, dP(\rho) + \nu \int \left[ \frac{H(\rho_0) + H(\rho_1)}{2} + C \right] \, dP(\rho) \leq \mathcal{A}(P) + \nu \left[ \int \frac{H(\rho_0) + H(\rho_1)}{2} \, d\Gamma(\rho_0, \rho_1) + C \right].
\]
Taking the lim sup gives the desired inequality and achieves the proof.

Remark 5.1. As in Remark 4.3, if (14) does not hold the \( \Gamma \)-convergence (16) cannot hold due to \( \mathcal{H}_\nu + \nu \Gamma \equiv +\infty \). However, it is still possible to prove
\[
\Gamma - \lim_{\nu \to 0} \mathcal{H}_\nu = \mathcal{A}
\]
by regularizing \( \Gamma \) as \( \Gamma^\nu := (\Psi_\nu, \Psi_\nu) \# \Gamma \), where
\[
\Psi_\nu : \rho \in \mathcal{P}(\mathbb{T}^d) \mapsto \rho * \tau_\nu \in \mathcal{P}(\mathbb{T}^d)
\]
and the admissible generalized flow must also be regularized correspondingly. In fact, we expect as in Remark 4.3 that
\[
\Gamma - \lim_{\nu \to 0} \left\{ \mathcal{H}_\nu + \nu \Gamma^\nu \right\} = \mathcal{A} + \nu \Gamma
\]
if and only if \( \Gamma^\nu \rightharpoonup \Gamma \) and
\[
\lim_{\nu \to 0} \nu \int H(\rho_0) \, d\Gamma^\nu(\rho_0, \rho_1) = 0 \quad \text{and} \quad \lim_{\nu \to 0} \nu \int H(\rho_1) \, d\Gamma^\nu(\rho_0, \rho_1) = 0.
\]
To prove this statement, one needs a result corresponding to [7, Thm. 1] in that setting but we did not pursue in this direction.

5.2 Convergence of the pressures

Let us now turn to the proof of Theorem 2.6. We will need the following definition of the average density of a traffic plan.

Definition 5.2. Let \( P \) be a traffic plan. Its density \( \rho^P \in C([0,1]; \mathcal{P}(\mathbb{T}^d)) \) is defined at time \( t \in [0,1] \) by:
\[
\rho^P_t := \int \rho_t \, dP(\rho).
\]
In other words, \((\rho^P_t)_{t \in [0, 1]}\) is the curve obtained by averaging all the phases at time \(t\) with respect to \(P\). The key ingredient below will rely on the proof of Theorem 8.4 in \cite{6}. There, the first author introduced the following functional space:

**Definition 5.3.** We define \(E_0\) the space of continuous functions \(f : [0, 1] \times \mathbb{T}^d \to \mathbb{R}\) satisfying:

- for all \(x \in \mathbb{T}^d\), \(f(0, x) = f(1, x) = 0\),
- for all \(t \in [0, 1]\),
  \[
  \int_{\mathbb{T}^d} f(t, x) \, dx = 0, 
  \]  
  \((50)\)
- for all \(t \in [0, 1]\), \(f(t, \bullet) \in W^{2, \infty}(\mathbb{T}^d)\) and
  \[
  \sup_{t \in [0, 1]} \|d^2 f(t, \bullet)\|_\infty < +\infty, 
  \]  
- \(f(\bullet, x) \in AC^2([0, 1])\) for all \(x \in \mathbb{T}^d\), and \(\partial_t f\), which is well defined for almost all \(t\) and all \(x\), satisfies
  \[
  \int_0^1 \|\partial_t f(t, \bullet)\|^2_\infty \, dt < +\infty. 
  \]

We endow \(E_0\) with the norm

\[
\|f\|_{E_0} := \sup_{t \in [0, 1]} \|d^2 f(t, \bullet)\|_\infty + \left( \int_0^1 \|\partial_t f(t, \bullet)\|^2_\infty \, dt \right)^{1/2}, 
\]

for which \((E_0, \|\bullet\|_{E_0})\) is a Banach space. We write \(E'_0\) for its topological dual and \(\|\bullet\|_{E'_0}\) for the dual norm. Note that \(E'_0 \subset D'(([0, 1] \times \mathbb{T}^d))\) is a subspace of distributions.

Taking into account the dependence on the diffusivity parameter \(\nu\), an easy extension of [6, Thm. 8.4] gives

**Theorem 5.4** (Existence of the pressure fields).

1. Let \(\Gamma\) be bistochastic in average. There exists a unique \(p \in E'_0\) such that, for all solutions \(P\) to MREu(\(\Gamma\)) and all traffic plans \(Q\) satisfying \((9)\) with \(\rho^Q - 1 \in E_0\), there holds

\[
A(Q) \geq A(P) + \langle p, \rho^Q - 1 \rangle_{E'_0, E_0}, 
\]

The distribution \(p\) is called the pressure field associated with MREu(\(\Gamma\)).
2. Let $\Gamma$ be bistochastic in average and satisfy (14), and let $\nu > 0$. There exists a unique $p^{\nu} \in \mathcal{E}_0$ such that, if $P^{\nu}$ is the unique solution to $\text{MBr}^{\nu}_0(\Gamma)$ and $Q$ is any traffic plan satisfying (9) with $\rho^Q - 1 \in \mathcal{E}_0$, then

$$H_\nu(Q) \geq H_\nu(P^{\nu}) + \langle p^{\nu}, \rho^Q - 1 \rangle_{\mathcal{E}_0, \mathcal{E}_0}. \quad (51)$$

Moreover, there exists a dimensional constant $C$ such that

$$\|p^{\nu}\|_{\mathcal{E}_0} \leq C(1 + \nu^2). \quad (52)$$

The distribution $p^{\nu}$ is called the pressure field associated with $\text{MBr}^{\nu}_0(\Gamma)$.

Proof. In the absence of viscosity, i.e. point 1 in our statement, the result is due to Brenier in [15] but also follows from our argument below. In the viscous setting only our estimate (52) is new compared to [6] hence we only sketch the proof for $\nu > 0$ and refer to [6] for more details.

Given a scalar function $\varphi = (\varphi(t, x))$, we say that a traffic plan $Q$ is admissible for $\text{MBr}^{\varphi}_\nu(\Gamma)$ if (9)(11) hold, the previous incompressibility (10) is replaced by

$$\rho^Q_t = (1 + \varphi(t, \cdot)) \text{Leb},$$

and the Fisher information (13) is finite. Of course, this is possible only if $1 + \varphi \geq 0$ and $\varphi$ satisfies (50), but both properties hold if $\varphi$ is chosen sufficiently small in $\mathcal{E}_0$.

Note that $\text{MBr}^\varphi_\nu$ corresponds to $\varphi \equiv 0$: essentially, as explained in the introduction, the idea of proof below will consist in relaxing the incompressibility constraint and performing a small perturbation of $\text{MBr}^0_\nu = \text{MBr}^\varphi_\nu|_{\varphi = 0}$ around $\varphi = 0$.

With the convention that $\inf_{\emptyset} \{ \ldots \} = +\infty$ and following [6], the map

$$\phi^\nu : \varphi \in \mathcal{E}_0 \rightarrow \inf_{Q \text{ admissible for } \text{MBr}^\varphi_\nu(\Gamma)} \mathcal{H}_\nu(Q)$$

is convex and lower semicontinuous (for the topology induced by the norm $\| \cdot \|_{\mathcal{E}_0}$). Hence, in order to get the existence of a subgradient $p^{\nu} \in \partial_{\varphi = 0} \phi^\nu$ satisfying (51) and (52), it suffices to show that there exists a constant $C = C_d > 0$ such that for all $\varphi \in \mathcal{E}_0$ close to 0 – say with $\|\varphi\|_{\mathcal{E}_0} \leq 1/2$ – there holds

$$\phi^\nu(\varphi) \leq C(1 + \nu^2).$$

But for such a $\varphi$, using a famous result by Dacorogna and Moser (see [6] Thm. 8.1 or [23] for the original version) and arguing as in [6], one can build from the minimizer $P^{\nu}$ in $\text{MBr}^\varphi_\nu$ a traffic plan $Q^{\nu}$ that is admissible for $\text{MBr}^\varphi_\nu(\Gamma)$, with moreover

$$\mathcal{A}(Q^{\nu}) \leq C(1 + \mathcal{A}(P^{\nu})), \quad \mathcal{F}(Q^{\nu}) \leq C(1 + \mathcal{F}(P^{\nu})).$$
and $C$ depending only on the dimension. In particular, we have

$$\phi^\nu(\varphi) \leq \mathcal{H}_\nu(Q^\nu) = \mathcal{A}(Q^\nu) + \nu^2 \mathcal{F}(Q^\nu) \leq C(1 + \nu^2) + \mathcal{H}_\nu(P^\nu)).$$

We conclude using the fact that $P^\nu$ is a minimizer, hence

$$\mathcal{H}_\nu(P^\nu) \leq \mathcal{H}_\nu(P^1) = \mathcal{A}(P^1) + \nu^2 \mathcal{F}(P^1) \leq C(1 + \nu^2).$$

(53)

(Here $P^1$ is the minimizer of $\text{MBr}^1$ with $\nu = 1$, but any other fixed admissible traffic plan would have worked as well.) We refer to [15, Thm. 1.2] and [6, Thm. 8.4] for the uniqueness part of the statement.

We are now ready to prove the convergence of the pressures:

**Proof of Theorem 2.6.** Let $P^\nu$ be the unique minimizer for $\text{MBr}^\nu(\Gamma)$. From (53) we have

$$\mathcal{A}(P^\nu) \leq \mathcal{A}(P^\nu) + \nu^2 \mathcal{F}(P^\nu) = \mathcal{H}_\nu(P^\nu) \leq C(1 + \nu^2)$$

as $\nu \to 0$, and we recall that $\mathcal{A}$ is proper for the narrow topology. Consequently, the sequence $(P^\nu)$ is tight and has at least one cluster point $P^\nu \rightharpoonup P$ (up to extraction of a discrete subsequence if needed). By Theorem 2.4 and recalling that $\Gamma$-convergence implies convergence of minimizers to minimizers, it is clear that any such cluster point $P$ is a solution to $\text{REu}(\Gamma)$.

First of all, by Theorem 5.4, the pressures $(p^\nu)_{\nu>0}$ are bounded in $\mathcal{E}'_0$ uniformly in $\nu$. But, as the functional space $\mathcal{G}$ from Definition 2.5 is continuously embedded in $\mathcal{E}_0$, $(\rho^\nu)$ is also bounded in $\mathcal{G}'$. By separability of $\mathcal{G}$ and the Banach-Alaoglu theorem there is $p^* \in \mathcal{G}'$ such that, up to extraction of a further subsequence, $p^\nu \rightharpoonup p^*$ for the weak-$*$ topology of $\mathcal{G}'$. Thus it suffices to show that $p^* = p$, and convergence of the whole sequence towards $p$ will follow by standard arguments. To do so, and by uniqueness in Theorem 5.4, it suffices to show that, for all $\varphi \in \mathcal{E}_0$ and all traffic plans $Q$ satisfying (9)(11) as well as $\rho^Q = 1 + \varphi$, there holds

$$\mathcal{A}(Q) \geq \mathcal{A}(P) + (p^* - \varphi)\mathcal{E}'_0, \mathcal{E}_0.$$ (54)

To test this inequality, take $\varphi \in \mathcal{E}_0$ and $Q$ such that $\rho^Q = 1 + \varphi$, and define

$$Q^\nu := \Phi^\nu_{\rho^Q} Q$$

as in the proof of Theorem 2.4. We recall that $\Phi^\nu$ is defined in (48) via (37).

From the subdifferential characterization (51) of $p^\nu$, we have

$$\mathcal{H}_\nu(Q^\nu) \geq \mathcal{H}_\nu(P^\nu) + (p^\nu, \rho^Q - 1)\mathcal{E}'_0, \mathcal{E}_0.$$ (55)

Repeating the exact same argument from the proof of Theorem 2.4 (construction of the recovery sequences), we have moreover

$$\mathcal{A}(Q) \geq \limsup_{\nu \to 0} \mathcal{H}_\nu(Q^\nu).$$ (56)
But by linearity of $\Phi^\nu$ with $\Phi^\nu(\text{Leb}) = \text{Leb}$ (or, abusing notations, $\Phi^\nu(1) = 1$) it is easy to check that

$$ \varphi^\nu := \rho^Q - 1 = \Phi^\nu(\varphi), $$

which by definition of $\Phi^\nu$ simply means that $\varphi^\nu(t, \cdot)$ is the solution at time $s = \nu t (1 - t)$ of the heat flow started from $\varphi(t, \cdot)$. Therefore, by standard properties of the heat flow, $\varphi^\nu \to \varphi$ in any reasonable topology, and in particular strongly in $\mathcal{E}_0$. Together with $p^\nu \rightharpoonup p$, this allows to take the limit in the product

$$ \langle p^\nu, \varphi^\nu \rangle_{\mathcal{E}'_0, \mathcal{E}_0} \to \nu \to 0 \langle p^*, \varphi \rangle_{\mathcal{E}'_0, \mathcal{E}_0}. \quad (57) $$

Moreover, by the $\Gamma - \text{lim inf}$ property in Theorem 2.4 with $P^\nu \rightharpoonup P$,

$$ \liminf_{\nu \to 0} H_{\nu}(P^\nu) \geq A(P). \quad (58) $$

We finally retrieve (54) by passing to the limit in (55) using (56), (57), and (58).

6 Time-convexity of the entropy

Using our regularization lemma 4.1, we can now give the proofs of Proposition 2.7 and Proposition 2.8. We begin with the single-phase setting:

Proof of Proposition 2.7. Let us start with OT. For small $\nu > 0$ consider the curve $\rho^\nu$ defined by (37). As already discussed the endpoints remain invariant, $\rho^\nu_0 = \rho_0, \rho^\nu_1 = \rho_1$: the curve $\rho^\nu$ is therefore an admissible competitor in the OT problem, and since $\rho$ is a minimizer we have $A(\rho) \leq A(\rho^\nu)$. Simply discarding the term $\frac{\nu^2}{8} \int \ldots \geq 0$ in (38), we have

$$ A(\rho) \leq A(\rho^\nu) \leq A(\rho) + \nu \left[ \frac{\int H(\rho_0) + H(\rho_1)}{2} - \int_0^1 H(\rho^\nu_t) \, dt \right] $$

hence

$$ \int_0^1 H(\rho^\nu_t) \, dt \leq \frac{\int H(\rho_0) + H(\rho_1)}{2} $$

for all $\nu > 0$. By standard properties of the heat flow we have moreover $\rho^\nu_t = \tau_{\nu t(1-t)} \ast \rho_t \rightharpoonup \rho_t$ for all $t \in [0,1]$ as $\nu \to 0$. Since the entropy is lower semicontinuous with respect to narrow convergence, we get by Fatou’s lemma

$$ \int_0^1 H(\rho_t) \, dt \leq \int_0^1 \liminf_{\nu \to 0} H(\rho^\nu_t) \, dt \leq \liminf_{\nu \to 0} \int H(\rho^\nu_t) \, dt \leq \frac{\int H(\rho_0) + H(\rho_1)}{2}. $$

In particular $H(\rho_t) < \infty$ for a.e. $t$, and in fact for all $t$ by narrow continuity of $\rho \in C([0,1]; \mathcal{P}(\mathbb{T}^d))$ and lower semicontinuity of $H$.

This was carried out in times $t \in [0,1]$, but $\rho$ is of course a minimizer for the optimal transport problem $\text{OT}(\rho_{t_0}, \rho_{t_1})$ for all intermediate times $t_0 \leq t_1$. 36
Since we just proved that \( \frac{1}{2} [H(\rho_{t_0}) + H(\rho_{t_1})] < +\infty \), we can repeat the exact same argument to conclude that

\[
\int_{t_0}^{t_1} H(\rho_t) \, dt \leq \frac{H(\rho_{t_0}) + H(\rho_{t_1})}{2}, \quad \forall \, t_0 \leq t_1.
\]

Since \( \rho \) is narrowly continuous and \( H \) is l.s.c. for the narrow convergence, and because \( t_0, t_1 \) can now vary arbitrarily, this implies the desired convexity.

The proof for \( \text{Sch}_\alpha(\rho_0, \rho_1) \) is identical, simply using (40) instead of (38).

For the multiphase setting the proof essentially consists in superposing the previous argument by linearity:

**Proof of Proposition 2.8.** We consider first the \( \text{MREu} \) problem. As in the proof of Theorem 2.4 earlier in section 5.1, the argument essentially consists in superposing (i.e. integrating with respect to \( P \)) the corresponding statement for a single phase, here Proposition 2.7.

More precisely: let \( P \) be a solution to \( \text{MREu} \), and consider as before the map \( \Phi : \rho \mapsto \rho^\nu \) from \( C([0, 1]; P(T^d)) \) to itself defined by (37). We already checked in the proof of Theorem 2.4 that the traffic plan \( P^\nu := \Phi^\nu \# P \) is incompressible and shares its marginals \( \Gamma \) with \( P \). Since \( P \) is a minimizer in \( \text{MREu} \) there holds

\[
\mathcal{A}(P) \leq \mathcal{A}(P^\nu) = \int \mathcal{A}(\rho) \, dP^\nu(\rho) = \int \mathcal{A}(\rho^\nu) \, dP(\rho) = \int \mathcal{A}(\rho^\nu) \, dP(\rho).
\]

Discarding \( \frac{x^2}{8} \{ ... \} \geq 0 \) in (38) we can estimate as before

\[
\mathcal{A}(\rho^\nu) \leq \mathcal{A}(\rho) + \nu \left( \frac{H(\rho_0) + H(\rho_1)}{2} - \int_0^1 H(\rho^\nu_t) \, dt \right)
\]

for \( P \)-a.e. \( \rho \), and integrating with respect to \( P \) thus gives

\[
\mathcal{A}(P) \leq \int \left\{ \mathcal{A}(\rho) + \nu \left( \frac{H(\rho_0) + H(\rho_1)}{2} - \int_0^1 H(\rho^\nu_t) \, dt \right) \right\} \, dP(\rho)
\]

\[
= \mathcal{A}(P) + \nu \left( \int \frac{H(\rho_0) + H(\rho_1)}{2} \, dP(\rho) - \int_0^1 \int H(\rho^\nu_t) \, dt \, dP(\rho) \right).
\]

Whence

\[
\int_0^1 H(\rho^\nu_t) \, dP(\rho) \, dt \leq \int \frac{H(\rho_0) + H(\rho_1)}{2} \, dP(\rho)
\]

for all \( \nu > 0 \). The right-hand side is finite since the marginal entropies \( \int \{ H(\rho_0) + H(\rho_1) \} \, dP(\rho) = \int \{ H(\rho_0) + H(\rho_1) \} \, d\Gamma(\rho_0, \rho_1) < +\infty \). Taking first \( \nu \to 0 \) and repeating next the argument in arbitrary subintervals \( [t_0, t_1] \subset [0, 1] \), the rest of
the proof is identical to the previous proof of Proposition 2.7 and we omit the
details.

For $\text{MBr} \tilde{\alpha}$ we simply use (40) instead of (38) as before, and the proof is complete.

Remark 6.1. In [30] H. Lavenant proves (a slightly weaker version of) the same
convexity by discretizing $\text{MREu}$ in time, which gives a minimization problem
over a large number $K$ of intermediate marginals at times $t = t_0, \ldots, t_K = 1$.
Performing an infinitesimal perturbation of the $k$-th optimal marginal using
the heat flow as well as the flow interchange technique from [35], one retrieves then
some convexity in the discrete time variable $k$ and finally passes to the limit
$K \to \infty$ to conclude. The technical details differ compared to our proof above,
but the main idea is somehow similar: the heat flow gives admissible competitors
in the variational problem, and tends to simultaneously diminish and convexify
the entropy. Hence if the entropy were not convex, one could construct better
competitors by running the heat flow for short times while improving convexity.
However, our regularization $\rho'(t) = \tau_\nu(t(1-t)) \ast \rho_t$ is more global, roughly speaking
because we simultaneously perturb the whole continuum of time marginals in a
unified fashion and thus we avoid any delicate time-slicing. More importantly,
our approach has a clear counterpart at the level of the underlying stochastic
processes, $\nu t(1-t)$ being of course the intrinsic scale of the Brownian bridges
$B^\nu,x,y$ involved in section 3.

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Appendices

A Existence and uniqueness for $\text{MBr} \tilde{\alpha}$

Here we establish

**Theorem A.1.** Take $\Gamma$ bistochastic in average satisfying the entropy condition [14], and let $\nu > 0$. Then $\text{MBr}_\nu(\Gamma)$ admits a unique solution.

This is perhaps not completely standard in this form due to our choice of
exposition in terms of traffic plans, and we include the details for the sake of
completeness.
Then the conditional law \( C \) measurable map, which assigns to any possible traffic plan \( \Gamma \) can either adapt the proof of [3, Cor. 5.2], or also observe that, given an admissible traffic plan. (In order to find such a traffic plan, one can either adapt the proof of Theorem 2.4 is admissible for \( MBr\nu(\Gamma) \) in particular (49) ensures that \( \mathcal{H}_\nu(P) < +\infty \).

For the uniqueness part, we first show that if \( P \) is a solution to \( MBr\nu(\Gamma) \), then the conditional law \( P^{0_\nu, 0_\nu} := P(\bullet | X_0 = \rho_0, X_1 = \rho_1) \) is a Dirac mass for \( \Gamma \)-almost all \((\rho_0, \rho_1)\). In other words, \( P \) is supported on the graph of a measurable map, which assigns to any \((\rho_0, \rho_1)\) a unique curve \( m = m[\rho_0, \rho_1] \in C([0, 1]; \mathcal{P}(\mathbb{T}^d)) \) joining \( \rho_0 \) to \( \rho_1 \). Indeed, let us define the average:

\[
m[\rho_0, \rho_1] := \int \rho \, dP^{\rho_0, \rho_1}(\rho) \in C([0, 1]; \mathcal{P}(\mathbb{T}^d)).
\]

This curve is well defined for \( \Gamma \)-almost all \((\rho_0, \rho_1)\), and we claim that \( P^{\rho_0, \rho_1} = \delta_{m[\rho_0, \rho_1]} \) for \( \Gamma \)-almost all \((\rho_0, \rho_1)\). To check this, let us define

\[
\tilde{P} := \int \delta_{m[\rho_0, \rho_1]} \, d\Gamma(\rho_0, \rho_1).
\]

Because \( m[\rho_0, \rho_1] \) has endpoints \( \rho_0, \rho_1 \) one can check that \( \tilde{P}_{0,1} = \Gamma \), and in the same spirit it is easy to see that \( \tilde{P} \) is incompressible in average (because \( P \) is). By strict convexity of \( \mathcal{H}_\nu \) and Jensen’s inequality, we have for \( \Gamma \)-almost all \((\rho_0, \rho_1)\)

\[
\mathcal{H}_\nu(m[\rho_0, \rho_1]) = \mathcal{H}_\nu\left(\int \rho \, dP^{\rho_0, \rho_1}(\rho)\right) \leq \int \mathcal{H}_\nu(\rho) \, dP^{\rho_0, \rho_1}(\rho) \quad (59)
\]

with equality if and only if \( P^{\rho_0, \rho_1} \) is a Dirac mass. To verify that equality holds as desired, let us integrate (59) with respect to \( \Gamma \): by definition of \( \tilde{P} \) on the left-hand side, and using the disintegration formula (17) with respect to \( P \) on the right-hand side, we get

\[
\mathcal{H}_\nu(\tilde{P}) = \int \mathcal{H}_\nu(\rho) \, d\tilde{P}(\rho) = \int \int \mathcal{H}_\nu(\rho) \, d\delta_{m[\rho_0, \rho_1]}(\rho) \, d\Gamma(\rho_0, \rho_1)
\]

\[
= \int \mathcal{H}_\nu(m[\rho_0, \rho_1]) \, d\Gamma(\rho_0, \rho_1) \overset{(59)}{\leq} \int \left( \int \mathcal{H}_\nu(\rho) \, dP^{\rho_0, \rho_1}(\rho) \right) \, d\Gamma(\rho_0, \rho_1)
\]

\[
= \int \mathcal{H}_\nu(\rho) \, dP(\rho) = \mathcal{H}_\nu(P).
\]

Since \( P \) is a minimizer and \( \tilde{P} \) is admissible the reverse inequality \( \mathcal{H}_\nu(P) \leq \mathcal{H}_\nu(\tilde{P}) \) holds as well, thus we must have equality in (59) for \( \Gamma \)-a.e. \((\rho_0, \rho_1)\) and therefore \( P^{\rho_0, \rho_1} = \delta_{m[\rho_0, \rho_1]} \) as claimed.
Throughout this appendix, barred quantities will live in Brownian motion and bridges on the torus, mainly Lemma 3.3 and Lemma 3.5. Properties of the Brownian motion on $\mathbb{T}^d$ to propagate injection of a pinned Brownian motion on $\mathbb{T}^d$ starting points are chosen consistently, $\bar{x}$ is affine, $\bar{P}_t := (P_1 + P_2)/2$ is a solution as well and must be supported on a graph. But, $P_1, P_2$ being themselves supported on a graph, $P_1 + P_2$ can be supported on a graph if and only if $P_1$ and $P_2$ coincide. Hence, uniqueness is proved. 

Remark A.2. From this proof it is clear that we established a slightly stronger statement, namely that any minimizer for $\text{MBr}\bar{0}(\Gamma)$ must be supported on a graph $(\rho_0, \rho_1) \mapsto \delta_{m[\rho_0, \rho_1]}$. This shows that the framework of traffic plans is not much more general than multiphase flows in the sense of Brenier’s parametric setting, i.e when the phases $p = (p^a)_a$ are labeled using the initial and final positions $a = (x, y) \in \mathbb{T}^d \times \mathbb{T}^d$ and the incompressibility reads $\int_{\mathbb{T}^d \times \mathbb{T}^d} \rho_i^a \, da = \text{Leb}$ for all $t$. Somehow we just proved that one can allow labeling on couples $(\rho_0, \rho_1) \in \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d)$ instead of $(x, y) \in \mathbb{T}^d \times \mathbb{T}^d$, but no better.

Remark A.3. In addition to being a sufficient condition as stated above in Theorem A.1, the entropy condition (14) is in fact also necessary for $\text{MBr}\bar{0}(\Gamma)$ to admit a (unique) solution. Indeed, by the classical Logarithmic Sobolev Inequality, the Fisher Information controls the entropy $H(\rho) \leq C_d F(\rho)$. Since $\mathcal{F}(P) = \int_0^1 \int F(\rho_t) \, dP(\rho) \, dt < \infty$ there exists at least a time $t_0 \in [0, 1]$ such that the average entropy $\int H(\rho_{t_0}) \, dP(\rho) \leq C \int F(\rho_{t_0}) \, dP(\rho) < \infty$. Moreover for an $AC^2$ curve the time derivative of the entropy can be computed by the chain rule $\frac{d}{dt} H(\rho_t) = \int \nabla \log \rho_t \cdot c_t \, d\rho_t$, where $c_t(x)$ corresponds to the metric speed $\bar{\rho}_t$ in Theorem 2.1. By definition of $\mathcal{H}_\nu$, any plan with finite entropy $\mathcal{H}_\nu(\nu) < \infty$ has both its metric speed $\bar{\rho}_t$ and Fisher information $F(\rho_t)$ controlled in the $L^2$ sense, hence $\frac{d}{dt} H(\bar{\rho}_t)$ is controlled in $L^1$. This $L^1$ bound allows to propagate $\int H(\rho_{t_0}) \, dP(\rho) < \infty$ to $\int H(\rho_t) \, dP(\rho) < \infty$ the whole interval $t \in [0, 1]$, in particular to $t = 0$ and $t = 1$.

B Properties of the Brownian motion on $\mathbb{T}^d$

Here we give detailed proofs of some technical results used in Section 3 for the Brownian motion and bridges on the torus, mainly Lemma 3.3 and Lemma 3.5. Throughout this appendix, barred quantities will live in $\mathbb{R}^d$, while unbarred quantities will live in the torus. Typically, we shall write $\bar{x} \in C([0, 1]; \mathbb{R}^d)$ and $\omega \in C([0, 1]; \mathbb{T}^d)$, and the canonical processes will read $\bar{X}_t : \bar{x} \mapsto \bar{\omega}_t$ and $X_t : \omega \mapsto \omega_t$. The pinned Brownian motions read accordingly $\bar{R}^\nu_x$ on the whole space and $R^\nu_x$ on the torus.

B.1 Brownian bridges on the torus

By definition, the pinned Brownian motion on $\mathbb{T}^d$ is nothing but the projection of a pinned Brownian motion on $\mathbb{R}^d$, i.e $R^\nu_x = \Pi_\bar{x} \bar{R}^\nu_x$ as soon as the starting points are chosen consistently, $x = \pi(\bar{x})$. However, the bridges on $\mathbb{T}^d$

$$R^{\nu,x,y} := R^\nu_x(\cdot | X_1 = y) = R^\nu(\cdot | X_0 = x, X_1 = y)$$
are not the projection of the bridges on \( \mathbb{R}^d \)
\[ R^\nu,\tilde{x},\tilde{y} := R^\nu_x(\bullet | X_1 = \tilde{y}). \]
The former can however be expressed as mixtures of the latter

**Lemma B.1.** Take \( x \) and \( y \) in \( \mathbb{T}^d \), and choose any lifts \( \bar{x} \) and \( \bar{y} \) in \( \mathbb{R}^d \) such that \( \pi(\bar{x}) = x \) and \( \pi(\bar{y}) = y \). Then
\[ R^\nu,x,y = \frac{1}{Z^\nu,x,y} \sum_{\bar{l} \in \mathbb{Z}^d} \exp\left(-\frac{|\bar{y} - \bar{x} + \bar{l}|^2}{2\nu}\right) \Pi_# R^\nu,\bar{x},\bar{y} + \bar{l}, \]
where \( Z^\nu,x,y := \sum_{\bar{l} \in \mathbb{Z}^d} \exp\left(-\frac{|\bar{y} - \bar{x} + \bar{l}|^2}{2\nu}\right) > 0 \) is a normalization constant.

Remark that because of (20),
\[ Z^\nu,x,y = (2\pi\nu)^{d/2} \tau^\nu(x - y). \]

**Proof.** We only give the proof for \( x = 0 \), the general case is identical by shift invariance. We first observe that the right-hand side in our statement obviously does not depend on the particular choice of the lifts, hence for \( x = 0 \) it suffices to establish equality with \( x = 0 \). Let \( R^\nu_0 \) and \( R^\nu_0 = \Pi_# R^\nu_0 \) be the Brownian motions started from the origin on \( \mathbb{R}^d \) and \( \mathbb{T}^d \), respectively, and consider \( i : \mathbb{T}^d \to \mathbb{R}^d \) a measurable right inverse of the canonical projection \( \pi : \mathbb{R}^d \to \mathbb{T}^d \). Applying (18) to disintegrate \( p = R^\nu_0 \) with respect to \( \Phi = X_1 \), we have
\[ R^\nu_0 = \int_{\mathbb{R}^d} \mathcal{T}^\nu,0,\pi \frac{1}{\sqrt{2\pi\nu}^d} \exp\left(-\frac{|y|^2}{2\nu}\right) dy. \]
Partitioning \( \mathbb{R}^d \) in cubes \( \{\bar{l} + i(\mathbb{T}^d)\}_{\bar{l} \in \mathbb{Z}^d} \) leads to
\[ R^\nu_0 = \int_{\mathbb{T}^d} \frac{1}{\sqrt{2\pi\nu}^d} \sum_{\bar{l} \in \mathbb{Z}^d} \exp\left(-\frac{|i(y) + \bar{l}|^2}{2\nu}\right) R^\nu,0,i(y) + \bar{l} dy, \]
and hence by linearity of the pushforward operation \( \Pi_# \) we get
\[ R^\nu_0 = \Pi_# R^\nu_0 = \int_{\mathbb{T}^d} \left\{ \frac{1}{\sqrt{2\pi\nu}^d} \sum_{\bar{l} \in \mathbb{Z}^d} \exp\left(-\frac{|i(y) + \bar{l}|^2}{2\nu}\right) \Pi_# R^\nu,0,i(y) + \bar{l} \right\} dy. \]
Remark next that, for all \( y \) and by definition of the inverse \( \pi \circ i(y) = y \), the integrand in this right-hand side is a measure supported on the fiber \( \{X_1 = y\} \). By uniqueness in the disintegration theorem [2 Thm. 5.3.1] one simply reads off the last equality the conditioning
\[ R^\nu,0,y = R^\nu_0(\bullet | X_1 = y) = \frac{1}{\sqrt{2\pi\nu}^d} \sum_{\bar{l} \in \mathbb{Z}^d} \exp\left(-\frac{|i(y) + \bar{l}|^2}{2\nu}\right) \Pi_# R^\nu,0,i(y) + \bar{l} \]
and the result follows. \( \square \)
B.2 Proof of Lemma 3.3

Let us start with estimate (25). Since the increments of the Brownian motion are independent and stationary we have first

\[
\int \exp \left( \alpha \frac{A_N(\omega)}{\nu} \right) \, dR^\nu(\omega) = \int \exp \left( \alpha \sum_{n=0}^{N-1} \frac{d^2(\omega_t^n, \omega_{t_n+1})}{2\nu \tau} \right) \, dR^\nu(\omega)
\]

\[
= \left[ \int \exp \left( \frac{\alpha d^2(\omega_0, \omega_\tau)}{2\nu \tau} \right) \, dR^\nu(\omega) \right]^N
\]

with \( \tau = 1/N \). Whence, by definition of the reversible Brownian motion \( R^\nu \),

\[
\int \exp \left( \alpha \frac{d^2(\omega_0, \omega_\tau)}{2\nu \tau} \right) \, dR^\nu(\omega)
\]

\[
= \frac{1}{(2\pi \nu \tau)^{d/2}} \int_{\mathbb{R}^d} \exp \left( \alpha \frac{d^2(0, \pi(\bar{y}))}{2\nu \tau} \right) \frac{\exp \left( -\frac{|\bar{y}|^2}{2\nu \tau} \right) \, d\bar{y}}{2\pi \nu \tau}) \exp \left( (\alpha - 1) \frac{|\bar{y}|^2}{2\nu \tau} \right) \, d\bar{y},
\]

where we used \( d(0, \pi(\bar{y})) \leq |\bar{y} - 0| \) in the last line. Because we were cautious enough to choose \( \alpha < 1 \) this quantity is finite, and changing variables \( \bar{z} = \sqrt{\frac{1-\alpha}{\nu \tau}} \bar{y} \) in the integral yields

\[
\int \exp \left( \alpha \frac{d^2(\omega_0, \omega_\tau)}{2\nu \tau} \right) \, dR^\nu(\omega) \leq \frac{1}{(2\pi(1-\alpha))^{d/2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|\bar{z}|^2}{2} \right) \, d\bar{z}
\]

\[
= \frac{1}{(1-\alpha)^{d/2}}. \quad (62)
\]

Gathering (61) (62) gives exactly (25).

For the conditioned version (26), choose arbitrary lifts \( \bar{x}, \bar{y} \in \mathbb{R}^d \) of \( x, y \in \mathbb{T}^d \). From Lemma B.1 we deduce

\[
\int \exp \left( \frac{\alpha A_N(\omega)}{\nu} \right) \, dR^\nu,\omega,\bar{x},\bar{y}(\omega)
\]

\[
= \frac{1}{Z_{\nu,x,y}} \sum_{\bar{l} \in \mathbb{Z}^d} \exp \left( -\frac{|\bar{y} - \bar{x} + \bar{l}|^2}{2\nu} \right) \int \exp \left( \frac{\alpha A_N \circ \Pi(\bar{z})}{\nu} \right) \, dR^\nu,\bar{x},\bar{y} + \bar{l}(\bar{z}). \quad (63)
\]

For arbitrary points \( \bar{p}, \bar{q} \in \mathbb{R}^d \) we first estimate the terms

\[
\Lambda(\bar{p}, \bar{q}) := \int \exp \left( \frac{\alpha A_N \circ \Pi(\bar{z})}{\nu} \right) \, dR^\nu,\bar{p},\bar{q}(\bar{z})
\]

\[
= \int \exp \left( \frac{\alpha}{\nu} \sum_{n=0}^{N-1} \frac{d^2(\pi(\bar{x}_n), \pi(\bar{x}_{n+1}))}{2\tau} \right) \, dR^\nu,\bar{p},\bar{q}(\bar{z})
\]
appearing in the summand of (63). Since \( d(\pi(u), \pi(v)) \leq |v - u| \) we have

\[
\Lambda(\bar{p}, \bar{q}) \leq \int \exp \left( \frac{\alpha}{2\nu T} \sum_{n=0}^{N-1} |\omega_{t_{n+1}} - \omega_{t_n}|^2 \right) d\mathcal{R}^{\bar{p}, \bar{q}}(\omega)
\]

\[
= \mathbb{E}_{\mathcal{R}^{\bar{p}, \bar{q}}} \left[ \exp \left( \frac{\alpha}{2\nu T} \sum_{n=0}^{N-1} |\bar{X}_{t_{n+1}} - \bar{X}_{t_n}|^2 \right) \right]
\]

Moreover, the law of the canonical process \( \bar{X}_t \) under the bridge \( \mathcal{R}^{\bar{p}, \bar{q}} \) is the same as the law of \( \bar{Y}_t := \bar{X}_t + (1 - t)(\bar{p} - \bar{X}_0) + t(\bar{q} - \bar{X}_1) \) under the law of the standard Brownian motion \( \mathcal{R}^0 \) started from the origin. Thus

\[
\Lambda(\bar{p}, \bar{q}) \leq \mathbb{E}_{\mathcal{R}^0} \left[ \exp \left( \frac{\alpha}{2\nu T} \sum_{n=0}^{N-1} |\bar{Y}_{t_{n+1}} - \bar{Y}_{t_n}|^2 \right) \right]
\]

\[
= \mathbb{E}_{\mathcal{R}^0} \left[ \exp \left( \frac{\alpha}{2\nu T} \sum_{n=0}^{N-1} |(\bar{X}_{t_{n+1}} - \bar{X}_{t_n}) + \tau \{ (\bar{q} - \bar{p}) - (\bar{X}_1 - \bar{X}_0) \}|^2 \right) \right]
\]

Expanding \((a + b - c)^2\) in the sum and recalling that \( N\tau = 1 \), it is easy to get:

\[
\sum_{n=0}^{N-1} |(\bar{X}_{t_{n+1}} - \bar{X}_{t_n}) + \tau (\bar{q} - \bar{p}) - \tau (\bar{X}_1 - \bar{X}_0)|^2
\]

\[
= \sum_{n=0}^{N-1} |\bar{X}_{t_{n+1}} - \bar{X}_{t_n}|^2 + \sum_{n=0}^{N-1} \tau^2 |\bar{q} - \bar{p}|^2 + \sum_{n=0}^{N-1} \tau^2 |\bar{X}_1 - \bar{X}_0|^2
\]

\[
+ 2 \sum_{n=0}^{N-1} (\bar{X}_{t_{n+1}} - \bar{X}_{t_n}) \cdot \tau (\bar{q} - \bar{p}) - \sum_{n=0}^{N-1} \tau (\bar{q} - \bar{p}) \cdot \tau (\bar{X}_1 - \bar{X}_0)
\]

\[
- 2 \sum_{n=0}^{N-1} (\bar{X}_{t_{n+1}} - \bar{X}_{t_n}) \cdot \tau (\bar{X}_1 - \bar{X}_0)
\]

\[
= \sum_{n=0}^{N-1} |\bar{X}_{t_{n+1}} - \bar{X}_{t_n}|^2 + \tau |\bar{q} - \bar{p}|^2 + \tau |\bar{X}_1 - \bar{X}_0|^2
\]

\[
+ 2\tau (\bar{X}_1 - \bar{X}_0)(\bar{q} - \bar{p}) - 2\tau (\bar{q} - \bar{p})(\bar{X}_1 - \bar{X}_0) - 2\tau |\bar{X}_1 - \bar{X}_0|^2
\]

\[
= \sum_{n=0}^{N-1} |\bar{X}_{t_{n+1}} - \bar{X}_{t_n}|^2 + \tau |\bar{q} - \bar{p}|^2 - \tau |\bar{X}_1 - \bar{X}_0|^2.
\]
As a consequence, and by independence of the Brownian increments,

\[ \Lambda(\bar{p}, \bar{q}) \leq \exp \left( \frac{2}{\nu} \| \bar{q} - \bar{p} \|^2 \right) \mathbb{E}_{\mathbb{T}_0} \left[ \exp \left( \frac{\alpha}{2\nu} \left\{ \sum_{n=0}^{N-1} |\bar{X}_{\tau_{n+1}} - \bar{X}_{\tau_n}|^2 - \tau |\bar{X}_1 - \bar{X}_0|^2 \right\} \right) \right] \]

\[ \leq \exp \left( \frac{\alpha}{2\nu} \| \bar{q} - \bar{p} \|^2 \right) \mathbb{E}_{\mathbb{T}_0} \left[ \exp \left( \frac{\alpha}{2\nu} \sum_{n=0}^{N-1} |\bar{X}_{\tau_{n+1}} - \bar{X}_{\tau_n}|^2 \right) \right] \]

\[ = \exp \left( \frac{\alpha}{2\nu} \| \bar{q} - \bar{p} \|^2 \right) \times \left( \mathbb{E}_{\mathbb{T}_0} \left[ \exp \left( \frac{\alpha}{2\nu} |\bar{X}_{\tau} - \bar{X}_0|^2 \right) \right] \right)^N \]

where the last equality follows from the same explicit computation as in (62) with \( \bar{X}_0 = 0 \) for \( \mathbb{T}_0 \)-almost all \( \omega \).

Finally, setting \( \bar{p} = \bar{x} \) and \( \bar{q} = \bar{y} + \bar{l} \) as in (63), using formulas (20)(60) and the dimensional bounds (19) on the heat kernel, we get when \( \nu \leq 1 \):

\[ \int \exp \left( \frac{\alpha}{\nu} A_N(\omega) \right) dR^{x,y}(\omega) \]

\[ \leq \frac{1}{(1 - \alpha)dN/2} \times \frac{1}{Z^{x,y}} \sum_{l \in \mathbb{Z}^d} \exp \left( - \frac{(1 - \alpha)}{2\nu} |\bar{y} - \bar{x} + \bar{l}|^2 \right) \]

\[ = \frac{1}{(1 - \alpha)dN/2} \times \tau_{\frac{1}{2\nu}}(y - x) \]

\[ \leq \frac{1}{(1 - \alpha)dN/2} \times \frac{K_d \exp \left( - \frac{(1 - \alpha)}{2\nu} d^2(x, y) \right)}{k_d \exp \left( \frac{1}{2\nu} d^2(x, y) \right)} \]

\[ \leq \frac{C_d}{(1 - \alpha)dN/2} \exp \left( \frac{\alpha}{2\nu} d^2(x, y) \right) \]

and the proof is achieved. \( \square \)

### B.3 Proof of Lemma 3.5

We first establish the corresponding result in the whole space. As in the torus in (27), we define the translation operator in \( \mathbb{R}^d \) as:

\[ \mathcal{T}_\pi : \pi \in C([0, 1]; \mathbb{R}^d) \rightarrow \pi + \pi \in C([0, 1]; \mathbb{R}^d). \]

We recall that \( \mathcal{B}^{\nu} = \mathcal{R}^{\nu,0,0} \) is the Brownian bridge of diffusivity \( \nu \) on \( \mathbb{R}^d \) joining 0 to 0. We have:

**Lemma B.2.** Let \( \pi \in AC^2([0, 1]; \mathbb{R}^d) \) and \( \nu > 0 \). Then

\[ \nu H \left( \mathcal{T}_\pi \mathcal{B}^{\nu} \mid \mathcal{R}^{\nu,0,0} \right) = \frac{1}{2} \int_0^1 |\dot{\pi}_t|^2 dt - \frac{|\pi_1 - \pi_0|^2}{2}. \]
Proof. We will rather establish the following equivalent formula: if \( \pi \in AC^2([0,1]; \mathbb{R}^d) \) satisfies \( \pi_0 = \pi_1 = 0 \), then for all \( \nu > 0 \) and \( \bar{x}, \bar{y} \in \mathbb{R}^d \),

\[
\nu H \left( T_{\pi\#} \bar{R}^{\nu, \bar{x}, \bar{y}} \mid \bar{R}^{\nu, \bar{x}, \bar{y}} \right) = \frac{1}{2} \int_0^1 |\bar{\pi}_t|^2 \, dt.
\]

(64)

If \( \bar{\xi}_t := (1-t)\bar{x}_0 + t\bar{x}_1 \), it will then suffice to apply this formula with \( \bar{\pi}_t := \bar{x}_t - \bar{\xi}_t \) and to use the identities \( T_{\bar{\pi}} = T_{\pi} \circ T_{\bar{\xi}} \) and \( T_{\xi\#} \bar{B}^{\nu} = \bar{B}^{\nu, \bar{\pi}_0, \bar{\pi}_1} \).

So let us prove (64). We fix \( \pi \in AC^2([0,1]; \mathbb{R}^d) \) with \( \pi_0 = \pi_1 = 0 \) and \( \nu > 0 \). First, by the standard Cameron-Martin formula, if \( \bar{R}^{\nu} \) is any \( \nu \) Brownian motion on \( \mathbb{R}^d \) then

\[
\nu H \left( T_{\pi\#} \bar{R}^{\nu} \mid \bar{R}^{\nu} \right) = \frac{1}{2} \int_0^1 |\bar{\pi}_t|^2 \, dt.
\]

Noticing that the marginals \( (\bar{R}^\nu)_{0,1} \) and \( (T_{\pi\#} \bar{R}^{\nu})_{0,1} \) coincide (because \( \pi_0 = \pi_1 = 0 \)), we can apply Proposition 2.10 in order to condition on the endpoints \( (X_0, X_1) \) and get:

\[
H(T_{\pi\#} \bar{R}^{\nu} \mid \bar{R}^{\nu}) = 0 + \int H \left( (T_{\pi\#} \bar{R}^{\nu})^{\bar{x}, \bar{y}} \mid \bar{R}^{\nu, \bar{x}, \bar{y}} \right) d\bar{R}^{\nu}_{0,1}(\bar{x}, \bar{y}).
\]

Gathering these two formulas and observing that \( (T_{\pi\#} \bar{R}^{\nu})^{\bar{x}, \bar{y}} = T_{\pi\#} (\bar{R}^{\nu, \bar{x}, \bar{y}}) \) we get

\[
\nu \int H(T_{\pi\#} \bar{R}^{\nu, \bar{x}, \bar{y}} \mid \bar{R}^{\nu, \bar{x}, \bar{y}}) d\bar{R}^{\nu}_{0,1}(x, y) = \frac{1}{2} \int_0^1 |\bar{\pi}_t|^2 \, dt.
\]

(65)

Finally, take \( \bar{x}, \bar{y} \in \mathbb{R}^d \) and consider the geodesic \( \bar{\xi}_t := (1-t)\bar{x} + t\bar{y} \). Then \( \bar{R}^{\nu, \bar{x}, \bar{y}} = T_{\bar{\xi}\#} \bar{B}^{\nu} \), the translations \( T_{\bar{\xi}} \) and \( T_{\bar{\pi}} \) commute, and \( T_{\bar{\xi}} \) is invertible. Whence by Proposition 2.11

\[
H(T_{\pi\#} \bar{R}^{\nu, \bar{x}, \bar{y}} \mid \bar{R}^{\nu, \bar{x}, \bar{y}}) = H(T_{\bar{\pi}\#} T_{\bar{\xi}\#} \bar{B}^{\nu} \mid T_{\bar{\pi}\#} T_{\bar{\xi}\#} \bar{B}^{\nu})
\]

\[
= H(T_{\bar{\xi}\#} T_{\bar{\pi}\#} \bar{B}^{\nu} \mid T_{\bar{\xi}\#} \bar{B}^{\nu}) = H(T_{\bar{\pi}\#} \bar{B}^{\nu} \mid \bar{B}^{\nu}).
\]

Finally exploiting (65), we get for all \( \bar{x} \) and \( \bar{y} \) in \( \mathbb{R}^d \):

\[
\nu H(T_{\pi\#} \bar{R}^{\nu, \bar{x}, \bar{y}} \mid \bar{R}^{\nu, \bar{x}, \bar{y}}) = \nu H(T_{\pi\#} \bar{B}^{\nu} \mid \bar{B}^{\nu}) = \frac{1}{2} \int_0^1 |\bar{\pi}_t|^2 \, dt
\]

and the proof is complete.

\( \square \)

In order to deduce Lemma 3.5 from Lemma B.2 we need a canonical construction of a process on \( \mathbb{R}^d \) out of a process on \( \mathbb{T}^d \). To this end, we choose \( i : \mathbb{T}^d \to \mathbb{R}^d \) a measurable right inverse of the projection \( \pi \) with bounded image. For \( \omega \in C([0,1]; \mathbb{T}^d) \) we denote by \( I(\omega) \) the unique lift of \( \omega \) starting from \( i(\omega_0) \), and \( I \) is of course a measurable right inverse of \( \Pi \). The entropy is invariant under the canonical projection in the following sense

\[
\nu H(T_{\pi\#} \bar{R}^{\nu} \mid \bar{R}^{\nu}) = \nu H(T_{\pi\#} \bar{B}^{\nu} \mid \bar{B}^{\nu}) = \frac{1}{2} \int_0^1 |\bar{\pi}_t|^2 \, dt
\]

and the proof is complete.

\( \square \)
Lemma B.3. Take $\overline{P}$ a probability measure and $\overline{R}$ a finite positive Radon measure on $C([0,1];\mathbb{R}^d)$. Suppose that $\overline{P} \ll \overline{R}$ and that, $\overline{R}$-almost surely, $X_0 = i(\pi(X_0))$. Then

$$H(\Pi_#\overline{P} | \Pi_#\overline{R}) = H(\overline{P} | \overline{R}).$$

Proof. On the set $\{X_0 = i(\pi(X_0))\}$ we have $\overline{R}$-almost surely $I \circ \Pi = \text{Id}$, and our statement is a direct consequence of Proposition 2.11.

With the above definition of the lift $I(\omega)$, observe that for all $\omega \in C([0,1];\mathbb{T}^d)$ the shifted bridge $B'_\omega$ from Definition 3.4 satisfies:

$$B'_\omega := T_\omega \# \overline{B'} = \Pi_# T_\omega(B'_\omega).$$

We are finally in position of establishing Lemma 3.5.

Proof of Lemma 3.5. For notational convenience, we denote the lift of $\omega$ by $
abla := I(\omega) \in C([0,1];\mathbb{R}^d)$.

By Lemma B.1, we have

$$R^{\nu,\omega_0,\omega_1} = \frac{1}{Z^{\nu,\omega_0,\omega_1}} \sum_{\ell \in \mathbb{Z}^d} \exp \left( -\frac{|\nabla_1 - \nabla_0 + \ell|^2}{2\nu} \right) \Pi_# R^{\nu,\nabla_0,\nabla_1 + \ell}$$

$$= \Pi_# \left( \frac{1}{Z^{\nu,\omega_0,\omega_1}} \sum_{\ell \in \mathbb{Z}^d} \exp \left( -\frac{|\nabla_1 - \nabla_0 + \ell|^2}{2\nu} \right) R^{\nu,\nabla_0,\nabla_1 + \ell} \right).$$

(The superscripts in $B^{\nu,\omega_0,\omega_1}$ do not stand for the conditioning of some $B'$, but rather emphasizes the dependence of the measure $B^{\nu,\omega_0,\omega_1}$ on the fixed endpoints $\omega_0, \omega_1 \in \mathbb{R}^d$. ) Observe that

- all the measures involved in the definition of $B^{\nu,\omega_0,\omega_1}$ are mutually singular (because $R^{\nu,\pi,\nabla_0} \perp R^{\nu,\pi,\nabla_1}$ as soon as $\nabla_0 \neq \nabla_1$),
- $T_{\nabla,#} B' \ll R^{\nu,\nabla_0,\nabla_1}$ by Lemma B.2,
- $R^{\nu,\nabla_0,\nabla_1} \ll R^{\nu,\omega_0,\omega_1}$ (because $R^{\nu,\nabla_0,\nabla_1}$ appears in the sum defining the measure $B^{\nu,\omega_0,\omega_1}$ for $\ell = 0$).

As a consequence $T_{\nabla,#} B' \ll B^{\nu,\omega_0,\omega_1}$, and the corresponding Radon-Nikodym derivative only involves the $\ell = 0$ contribution in $B^{\nu,\omega_0,\omega_1}$. Moreover by definition 3.4 of $B'_\omega = T_{\omega,#} B'$ and because $T_{\omega} \circ \Pi = \Pi \circ T_{\omega}$ one can write $B' = \Pi_# T_{\omega,#} B'$. Since $B^{\nu,\omega_0,\omega_1}$-almost surely $X_0 = i(\pi(X_0)),$
we can apply Lemma B.3

\[ H(B_{\nu} \mid R^{\nu, (\omega_{0}, \omega_{1})}) = H(\Pi_{\#} T_{\#} B^{\nu} \mid \Pi_{\#} B^{\nu, (\omega_{0}, \omega_{1})}) \\
= H(T_{\#} B^{\nu} \mid B^{\nu, (\omega_{0}, \omega_{1})}) \\
= H \left( T_{\#} B^{\nu} \bigg| \frac{1}{Z_{\nu, \omega_{0}, \omega_{1}}} \exp \left( -\frac{|\omega_{1} - \omega_{0}|^{2}}{2\nu} \right) \right) \\
= H \left( T_{\#} B^{\nu} \bigg| R^{\nu, (\omega_{0}, \omega_{1})} \right) + \frac{|\omega_{1} - \omega_{0}|^{2}}{2\nu} + \log Z^{\nu, (\omega_{0}, \omega_{1})} \]

where we used Proposition 2.9 in the last equality. We can compute the first entropy term in the right hand side using Lemma B.2 (remark that the kinetic action of \( \omega \) on the torus coincides with that of its lift \( \overline{\omega} \)) and we can estimate the last term using (60)(19), which leads to

\[ \nu H(B_{\nu} \mid R^{\nu, (\omega_{0}, \omega_{1})}) \leq \left( \frac{1}{2} \int_{0}^{1} |\dot{\omega}_{t}|^{2} dt - \frac{1}{2}|\omega_{1} - \omega_{0}|^{2} \right) \]

\[ + \nu \left( \frac{|\omega_{1} - \omega_{0}|^{2}}{2\nu} + \log K_{d} - \frac{d^{2}(\omega_{0}, \omega_{1})}{2\nu} \right) \]

\[ = \frac{1}{2} \int_{0}^{1} |\dot{\omega}_{t}|^{2} dt - \frac{d^{2}(\omega_{0}, \omega_{1})}{2\nu} + \nu \log K_{d} \]

and concludes the proof with \( C := \log K_{d} \). \( \square \)

References

[1] Luigi Ambrosio and Alessio Figalli. Geodesics in the space of measure-preserving maps and plans. *Archive for rational mechanics and analysis*, 194(2):421–462, 2009.

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer, 2006.

[3] Marc Arnaudon, Ana Bela Cruzeiro, Christian Léonard, and Jean-Claude Zambrini. An entropic interpolation problem for incompressible viscid fluids. *arXiv preprint arXiv:1704.02126*, 2017.

[4] Vladimir I. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits. In *Annales de l’institut Fourier*, volume 16, pages 319–361. Institut Fourier, 1966.

[5] Vladimir I. Arnold and Boris A. Khesin. *Topological methods in hydrodynamics*, volume 125. Springer Science & Business Media, 1999.

[6] Aymeric Baradat. On the existence of a scalar pressure field in the Bredinger problem. *arXiv preprint arXiv:1803.06299*, 2018.
[7] Aymeric Baradat. Continuous dependence of the pressure field with respect to endpoints for ideal incompressible fluids. *Calc. Var. Partial Differential Equations*, 58(1), 2019.

[8] Jean-David Benamou and Yann Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numerische Mathematik*, 84(3):375–393, 2000.

[9] Jean-David Benamou, Guillaume Carlier, Marco Cuturi, Luca Nenna, and Gabriel Peyré. Iterative Bregman projections for regularized transportation problems. *SIAM Journal on Scientific Computing*, 37(2):A1111–A1138, 2015.

[10] Jean-David Benamou, Guillaume Carlier, and Luca Nenna. Generalized incompressible flows, multi-marginal transport and Sinkhorn algorithm. *arXiv preprint arXiv:1710.08234*, 2017.

[11] Marc Bernot, Vicent Caselles, and Jean-Michel Morel. Traffic plans. *Publicacions Matemàtiques*, pages 417–451, 2005.

[12] Marc Bernot, Alessio Figalli, and Filippo Santambrogio. Generalized solutions for the Euler equations in one and two dimensions. *Journal de mathématiques pures et appliquées*, 91(2):137–155, 2009.

[13] Arne Beurling. An automorphism of product measures. *Annals of Mathematics*, pages 189–200, 1960.

[14] Yann Brenier. The least action principle and the related concept of generalized flows for incompressible perfect fluids. *J. Amer. Math. Soc.*, 2(2):225–255, 1989.

[15] Yann Brenier. The dual least action problem for an ideal, incompressible fluid. *Archive for rational mechanics and analysis*, 122(4):323–351, 1993.

[16] Yann Brenier. A homogenized model for vortex sheets. *Archive for Rational Mechanics and Analysis*, 138(4):319–353, 1997.

[17] Yann Brenier. Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Communications on pure and applied mathematics*, 52(4):411–452, 1999.

[18] Yann Brenier. Extended Monge-Kantorovich theory. In *Optimal transportation and applications*, pages 91–121. Springer, 2003.

[19] Guillaume Carlier, Vincent Duval, Gabriel Peyré, and Bernhard Schmitzer. Convergence of entropic schemes for optimal transport and gradient flows. *SIAM Journal on Mathematical Analysis*, 49(2):1385–1418, 2017.
[20] Yongxin Chen, Tryphon T Georgiou, and Michele Pavon. On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint. *Journal of Optimization Theory and Applications*, 169(2):671–691, 2016.

[21] AB Cruzeiro, Liming Wu, and JC Zambrini. Bernstein processes associated with a markov process. In *Stochastic Analysis and Mathematical Physics*, pages 41–72. Springer, 2000.

[22] Marco Cuturi. Sinkhorn distances: Lightspeed computation of optimal transport. In *Advances in neural information processing systems*, pages 2292–2300, 2013.

[23] Bernard Dacorogna and Jürgen Moser. On a partial differential equation involving the Jacobian determinant. In *Annales de l’Institut Henri Poincaré. Analyse non linéaire*, volume 7, pages 1–26. Elsevier, 1990.

[24] Hans Föllmer. Random fields and diffusion processes. In *École d’Été de Probabilités de Saint-Flour XV–XVII, 1985–87*, pages 101–203. Springer, 1988.

[25] Robert Fortet. Résolution d’un systeme d’equations de m. schrödinger. *Journal de Mathématiques Pures et Appliquées IX*, 1:83–105, 1940.

[26] Ivan Gentil, Christian Léonard, and Luigia Ripani. About the analogy between optimal transport and minimal entropy. *Ann. Fac. Sci. Toulouse Math. (6)*, 26(3):569–601, 2017.

[27] Nicola Gigli and Luca Tamanini. Benamou-Brenier and duality formulas for the entropic cost on RCD*(K, N) spaces. *arXiv preprint arXiv:1805.06325*, 2018.

[28] Alexander Grigor’yan. Estimates of heat kernels on Riemannian manifolds. *London Math. Soc. Lecture Note Ser*, 273:140–225, 1999.

[29] Benton Jamison. The markov processes of schrödinger. *Probability Theory and Related Fields*, 32(4):323–331, 1975.

[30] Hugo Lavenant. Time-convexity of the entropy in the multiphasic formulation of the incompressible Euler equation. *Calculus of Variations and Partial Differential Equations*, 56(6):170, 2017.

[31] Christian Léonard. From the Schrödinger problem to the Monge–Kantorovitch problem. *Journal of Functional Analysis*, 262(4):1879–1920, 2012.

[32] Christian Léonard. A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete Contin. Dyn. Syst. Serie A*, 34(4):1533–1574, 2014.
[33] Christian Léonard. On the convexity of the entropy along entropic interpolations. In *Measure theory in non-smooth spaces*, Partial Differ. Equ. Meas. Theory, pages 194–242. De Gruyter Open, Warsaw, 2017.

[34] Peter Li and Shing Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, 156(1):153–201, 1986.

[35] Daniel Matthes, Robert J McCann, and Giuseppe Savaré. A family of nonlinear fourth order equations of gradient flow type. *Communications in Partial Differential Equations*, 34(11):1352–1397, 2009.

[36] Robert J McCann. A convexity principle for interacting gases. *Advances in mathematics*, 128(1):153–179, 1997.

[37] Toshio Mikami. Monge’s problem with a quadratic cost by the zero-noise limit of h-path processes. *Probability theory and related fields*, 129(2):245–260, 2004.

[38] Filippo Santambrogio. Optimal transport for applied mathematicians. *Birkhäuser, NY*, 2015.

[39] Erwin Schrödinger. *Über die Umkehrung der naturgesetze*. Verlag Akademie der wissenschaften in kommission bei Walter de Gruyter u. Company, 1931.

[40] Erwin Schrödinger. Sur la théorie relativiste de l’électron et l’interprétation de la mécanique quantique. In *Annales de l’institut Henri Poincaré*, volume 2, pages 269–310, 1932.

[41] Srinivasa RS Varadhan. Diffusion processes in a small time interval. *Communications on Pure and Applied Mathematics*, 20(4):659–685, 1967.

[42] Cédric Villani. *Optimal transport: old and new*, volume 338. Springer Science & Business Media, 2008.

[43] Jean-Claude Zambrini. Variational processes and stochastic versions of mechanics. *Journal of Mathematical Physics*, 27(9):2307–2330, 1986.