Quantum Evolution of the Time-Dependent Non-Hermitian Hamiltonians: Real Phases

Mustapha Maamache\textsuperscript{a}\textsuperscript{*}, Oum Kaltoum Djeghiour\textsuperscript{a,\textsuperscript{b}\dagger}, Naima Mana\textsuperscript{a}\textsuperscript{‡} and Walid Koussa\textsuperscript{a}\textsuperscript{§}

\textsuperscript{(a)}Laboratoire de Physique Quantique et Systèmes Dynamiques, Faculté des Sciences, Université Ferhat Abbas Sétif 1, Sétif 19000, Algeria.
\textsuperscript{(b)}Département de Physique, Université de Jijel, BP 98 Ouled Aissa, 18000 Jijel, Algeria.

Abstract

Explicitly time-dependent pseudo-Hermitian (TDPH) invariants theory systems, with a time-dependent (TD) metric, is developed for a time-dependent non Hermitian (TDNH) quantum systems. We derive a simple relation between the eigenstates of this pseudo-Hermitian (PH) invariant and the solutions of the Schrodinger equation. A physical system is treated in detail: the TD Swanson model, where an explicitly TDPH invariant is derived for this system, the eigenvalues and eigenstates of the invariant are calculated explicitly.

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1 Introduction

Quantum mechanics is based on a set of axioms among them we mention: (i) the inner products of state vectors have a positive norm, (ii) the time evolution is unitary, iii) the Hamiltonian of a system must be a Hermitian operator $h \dagger = h$ in order to guarantee that its eigenvalues are real. Recently, Bender et al \cite{1, 2} found that non-Hermitian (NH) Hamiltonians satisfy these conditions and have interpreted the reality of the spectrum as being due to its $\mathcal{PT}$-symmetry which comes from the invariance of $\mathcal{PT}$-symmetric Hamiltonians under both parity and time reversal transformation $H \mathcal{PT} = \mathcal{PT} H$. The parity operator $\mathcal{P}$ and the time reversal operator $\mathcal{T}$ are defined by their action on the position operator $x$ and the momentum operator $p$ as: $\mathcal{P}: x \rightarrow -x$, $p \rightarrow -p$, $\mathcal{T}: x \rightarrow x$, $p \rightarrow -p$, $i \rightarrow -i$.

The crucial point is to redefine the inner product that enables to re-establish the consistent probabilistic interpretation of the theory. Given that in $\mathcal{PT}$ quantum mechanics, $\mathcal{P}$ and

\begin{itemize}
  \item $\text{E-mail: maamache@univ-setif.dz}$
  \item $\text{E-mail: k.djeghiourjijel@gmail.com}$
  \item $\text{E-mail: na3ima_mn@hotmail.fr}$
  \item $\text{E-mail: koussawalid@yahoo.com}$
\end{itemize}
\( \mathcal{T} \) take a role analogous to the Hermitian conjugate in ordinary quantum mechanics, a natural way to define the \( \mathcal{PT} \) inner product of two eigenfunctions of \( H \) is given by \( \langle \phi^H_m | \phi^H_n \rangle_{\mathcal{PT}} = (\mathcal{PT} | \phi^H_m \rangle \cdot | \phi^H_n \rangle = (-1)^n \delta_{mn} \) which shows that this \( \mathcal{PT} \) inner product is not always definite positive. In quantum theory the inner product in Hilbert space of state vectors has a positive norm. Positive definiteness is restored by introducing a linear operator \( C \) that takes eigenstates of the Hamiltonian that have negative norm under the \( \mathcal{PT} \) inner product and turns them into positive \[ \langle C \mathcal{PT} | \phi^H_m \rangle \cdot | \phi^H_n \rangle = -\delta_{mn} \].

It was established, in Refs. [3, 4, 5], that \( \mathcal{PT} \) symmetry is neither necessary nor sufficient for the reality of the spectrum which can be attributed to the pseudo Hermiticity of the Hamiltonian \( H \). A Hamiltonian is called quasi-Hermitian \[ \mathcal{C} \] or pseudo-Hermitian (PH) if it exists a bounded with respect to an invertible Hermitian operator \( \eta = \rho^+ \rho \) satisfies

\[
H^\dagger = \eta H \eta^{-1}.
\]  

The NH Hamiltonian \( H \) can be related to an equivalent Hermitian one by

\[
h = \rho H \rho^{-1},
\]  

showing that the eigenvalues of \( h \) and \( H \) are identical, although the relations between their eigenvectors will differ

\[
| \psi^h_n \rangle = \rho | \phi^H_n \rangle.
\]  

This, in turn, requires a redefinition of the usual inner product to

\[
\langle \phi^H_m | \phi^H_n \rangle_{\eta} = \langle \phi^H_m | \eta | \phi^H_n \rangle = \delta_{mn}.
\]  

All these efforts have been devoted to study TDNH systems. Systems with TDNH Hamiltonians operators and time-independent metrics have been studied in [7, 8], a number of conceptual difficulties have been encountered in the generalization to TD metric operators [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Recent contributions [18, 19] have advanced that it is incompatible to maintain unitary time evolution for TDNH Hamiltonians when the metric operator is explicitly TD. In other words, both Hamiltonians \( H(t) \) and \( H^\dagger(t) \) involved are related to each other as

\[
H^\dagger(t) = \eta(t) H(t) \eta^{-1}(t) + i \hbar \dot{\eta}(t) \eta^{-1}(t),
\]  

The key feature in this equation is the fact that the Hamiltonian \( H(t) \) is no longer quasi-Hermitian due to the presence of the last term and thus it generalizes the well known standard quasi-Hermiticity relation [11] in the context TDNH quantum mechanics [18, 19].

This work investigate in detail the main frames of TDNH systems ruled by the Schrödinger equation

\[
H(t) | \Phi^H(t) \rangle = i \hbar \delta_t | \Phi^H(t) \rangle,
\]  

2
where \( |\Phi^H(t)\rangle \) is related to the solution \( |\Psi^h(t)\rangle \) of the Hermitian Schrödinger equation

\[
h(t) |\Psi^h(t)\rangle = i\hbar \partial_t |\Psi^h(t)\rangle ,
\]

by a TD invertible operator \( \rho(t) \) as

\[
|\Psi^h(t)\rangle = \rho(t) |\Phi^H(t)\rangle .
\]

For this we introduce, in section 2, the pseudo invariant operator theory for the TD Schrödinger equation related with the NH Hamiltonian. Then we give the solution of the TD Schrödinger equation in terms of eigenstates of PH invariant operator \( I^{PH}(t) \) and goes on to examine how the reality of their phases can be established. In section 3, by using the Lewis-Riesenfeld method \[24\] of invariants and a TD metric, we construct a TD solutions for the generalized version of the NH Swanson Hamiltonian with TD coefficients. Section 4, concludes our work.

2 Pseudo-invariant operator method

The use of invariants theory to solve quantum systems, whose Hamiltonian is an explicit function of time, has the advantage to offer an exact solution for problems solved by the traditional TD perturbation theory \[25\]. There is a class of exact invariants for TD harmonic oscillators, both classical and quantum, that has been reported in \[26\].

The invariants method \[24\] is very simple due to the relationship between the eigenstates of the invariant operator and the solutions to the Schrödinger equation by means of the phases; in this case the problem is reduced to find the explicit form of the invariant operator and the phases.

Now we proceed to introduce and analyze the spectral properties of PH invariant operator \( I^{PH}(t) \). Particular attention is given to the special subset of quasi Hermitian operators. We start by considering a NH quantum mechanics in its most general form by studying TD Hamiltonian operators \( H(t) \) satisfying the Schrödinger equation \[6\] and where the metric operator \( \eta(t) = \rho^+(t) \rho(t) \) associated with \( H(t) \) and \( I^{PH}(t) \) is also TD.

Suppose the existence of a pseudo Hermitian, explicitly TD, non trivial invariant operator \( I^{PH}(t) \); that means, \( I^{PH}(t) \) satisfies

\[
I^{PH\dagger}(t) = \eta(t) I^{PH}(t) \eta^{-1}(t) \Leftrightarrow I^h(t) = \rho(t) I^{PH}(t) \rho^{-1}(t) = I^{h\dagger}(t),
\]

\[
\frac{\partial I^{PH}(t)}{\partial t} = \frac{i}{\hbar} [I^{PH}(t), H(t)].
\]

Thus \( I^{PH}(t) \) may be mapped to \( I^{h}(t) \), by a similarity transformation \( \rho(t) \).

It is easy to see that the action of the invariant operator in a Schrödinger state vector is also solution to the Schrödinger equation, that is

\[
H(t) \left( I^{PH}(t) |\Phi^H(t)\rangle \right) = i\hbar \partial_t \left( I^{PH}(t) |\Phi^H(t)\rangle \right),
\]

which is a valid result for any invariant operator.

Now, we generalize the Lewis-Riesenfeld theory so that it can be used to find the eigenstates \( |\phi^H_n(t)\rangle \) of \( I^{PH}(t) \)

\[
I^{PH}(t) |\phi^H_n(t)\rangle = \lambda_n |\phi^H_n(t)\rangle ,
\]

where \( |\phi^H_n(t)\rangle \) is related to the solution \( |\Psi^h(t)\rangle \) of the Hermitian Schrödinger equation

\[
h(t) |\Psi^h(t)\rangle = i\hbar \partial_t |\Psi^h(t)\rangle ,
\]

by a TD invertible operator \( \rho(t) \) as

\[
|\Psi^h(t)\rangle = \rho(t) |\Phi^H(t)\rangle .
\]
and
\[ \langle \phi^H_n(t) | \eta(t) | \phi^H_n(t) \rangle = \delta_{m,n}. \] (12)

The eigenvalues \( \lambda_n \) are also time-independent, as we can deduce in the following simple way. By differentiating Eq. (11) with respect to time, it follows that
\[ \frac{\partial F^{PH}}{\partial t} | \phi^H_n(t) \rangle + F^{PH} \frac{\partial | \phi^H_n(t) \rangle}{\partial t} = \frac{\partial \lambda_n}{\partial t} | \phi^H_n(t) \rangle + \lambda_n \frac{\partial | \phi^H_n(t) \rangle}{\partial t}, \] (13)
taking the scalar product of Eq. (13) with a state \( \langle \phi^H_n(t) | \eta(t) \rangle \) and using the left-hand side of Eq. (10), we obtain
\[ \frac{\partial \lambda_n}{\partial t} = \langle \phi^H_n(t) | \eta(t) \frac{\partial F^{PH}}{\partial t} | \phi^H_n(t) \rangle = 0. \] (14)

Since the Hermitian invariant \( I^H(t) \) and the NH one \( I^{PH}(t) \) are related by a similarity transformation (9), therefore they have the same eigenvalues. The reality of the eigenvalues \( \lambda_n \) is guaranteed, since one of the invariants involved, i.e. \( I^H(t) \), is Hermitian.

In order to investigate the connection between eigenstates of \( I^{PH}(t) \) and solutions of the Schrödinger equation,
\[ i\hbar \frac{\partial}{\partial t} | \Phi^H_n(t) \rangle = H(t) | \Phi^H_n(t) \rangle, \] (15)
we first start by projecting Eq. (13) onto \( \langle \phi^H_m(t) | \eta(t) \rangle \) and using Eq. (14), we obtain
\[ i\hbar \langle \phi^H_m(t) | \eta(t) \frac{\partial}{\partial t} | \phi^H_n(t) \rangle = \langle \phi^H_m(t) | \eta(t) H(t) | \phi^H_n(t) \rangle, \quad (m \neq n). \] (16)

The next step in the method is showing the existence of a simple and explicit rule for choosing the phases of the eigenstates of \( I^{PH}(t) \) such that these states satisfy themselves the Schrödinger equation with the only requirement the invariant does not involve time differentiation. The new eigenstates \( | \Phi^H_n(t) \rangle \) of \( I^{PH}(t) \) are
\[ | \Phi^H_n(t) \rangle = e^{i\gamma_n(t)} | \phi^H_n(t) \rangle, \] (17)
will satisfy the Schrodinger equation. This is to say, \( | \Phi^H_n(t) \rangle \) is a particular solution to the Schrodinger equation. This requirement is equivalent to the following first-order differential equation for the \( \gamma_n(t) \):
\[ \frac{d\gamma_n(t)}{dt} = \langle \phi^H_n(t) | \eta(t) \left[ i\hbar \frac{\partial}{\partial t} - H(t) \right] | \phi^H_n(t) \rangle \] (18)

In Eq. (18), the first term is parallel to a familiar non-adiabatic geometrical phase, but the second term representing effects due to a time-dependent Hamiltonian is a dynamical phase. It is the sum of these two terms that can ensure a real \( \gamma_n(t) \).

The general solutions of the Schrodinger equation for system with non-Hermitian time-dependent Hamiltonian \( H(t) \) are readily obtained as follows:
\[ | \Phi^H(t) \rangle = \sum_n C_n e^{i\gamma_n(t)} | \phi^H_n(t) \rangle, \] (19)
where the \( C_n = \langle \phi^H_n(0) | \eta(0) | \Phi^H(0) \rangle \) are time-independent coefficients.
3 Generalized time dependent non-Hermitian Swanson Hamiltonian

The first model of a NH \( \mathcal{PT} \)-symmetric Hamiltonian quadratic in position and momentum was studied by Ahmed \[27\] and made popular by Swanson \[28\] namely

\[
H = \omega (a^+ a + \frac{1}{2}) + \alpha a^2 + \beta a^+ + \ldots
\]

with \( \omega, \alpha \) and \( \beta \) real parameters, such that \( \alpha \neq \beta \) and \( \omega^2 - 4\alpha \beta > 0 \) and where \( a^+ \) and \( a \) are the usual harmonic oscillator creation and annihilation operators for unit frequency. This Hamiltonian has been studied extensively in the literature by several authors \[29, 30, 31, 32, 33, 34\].

We construct here, by employing the Lewis-Riesenfeld method of invariants, the solutions for the non-Hermitian Swanson Hamiltonian with TD coefficients \[19\]

\[
H(t) = \omega(t) (a^+ a + \frac{1}{2}) + \alpha(t) a^2 + \beta(t) a^+ + \ldots
\]

where \((\omega(t), \alpha(t), \beta(t)) \in \mathbb{C}\) are time-dependent parameters. We set \( \hbar = 1 \).

Following the previous idea, the problem is reduced to find a pseudo invariant operator. The most general invariant \( I^{PH}(t) \), for the generalized Swanson oscillator \[20\], can be written in the form

\[
I^{PH}(t) = \delta_1(t) (a^+ a + \frac{1}{2}) + \delta_2(t) a^2 + \delta_3(t) a^+ + \ldots
\]

where \( \delta_1(t), \delta_2(t), \delta_3(t) \) are time dependent real parameters. The invariant \[21\] is of course manifestly NH when \( \delta_2(t) \neq \delta_3(t) \).

Let us solve the standard quasi-Hermiticity relation \[19\] by making the following general and, for simplicity, Hermitian ansatz for a TD metric \( \rho(t) \)

\[
\rho(t) = \exp \left[ \epsilon(t) (a^+ a + \frac{1}{2}) + \mu(t) a^2 + \mu^*(t) a^+ + \ldots \right],
\]

\[
= \exp \left[ [\vartheta_+(t) K_+] \exp [\ln \vartheta_0(t) K_0] \exp [\vartheta_-(t) K_-] \right],
\]

where \( K_+ = a^+ a^2 + K_- = a^2 + K_0 = (a^+ a/2 + 1/4) \) form SU(1, 1)-algebra

\[
\begin{align*}
[K_0, K_+] &= K_+ \\
[K_0, K_-] &= -K_- \\
[K_+, K_-] &= -2K_0
\end{align*}
\]

with the TD coefficients

\[
\begin{align*}
\vartheta_+(t) &= \frac{2\mu^* \sinh \theta}{\theta \cosh \theta - \epsilon \sinh \theta} = -\Phi(t)e^{-i\varphi(t)}, \\
\vartheta_0(t) &= \left( \cosh \theta - \frac{\epsilon}{\theta} \sinh \theta \right)^{-2} = \Phi^2(t) - \chi(t), \\
\vartheta_-(t) &= \frac{2\mu \sinh \theta}{\theta \cosh \theta - \epsilon \sinh \theta} = -\Phi(t)e^{i\varphi(t)}, \\
\chi(t) &= -\frac{\cosh \theta + \frac{\epsilon}{\theta} \sinh \theta}{\cosh \theta - \frac{\epsilon}{\theta} \sinh \theta}, \quad \theta = \sqrt{\epsilon^2 - 4|\mu|^2}.
\end{align*}
\]
The key point is the construction of the Hermitian invariant operator $I^h(t) = \rho(t)I^{PH}(t)\rho^{-1}(t)$ from the NH one $I^{PH}(t)$. It follows that

$$I^h(t) = \frac{2}{\vartheta_0} \left[\left[-\delta_1 (\vartheta_- \vartheta_+ + \chi) - 2 (\delta_2 (\vartheta_+ - \delta_3 \chi \vartheta_-)] K_0 + (\delta_1 \vartheta_+ + \delta_2 \vartheta_+^2 + \delta_3 \chi^2) K_+\right].$$

The equation (25) has been derived with the help of the following relations

$$\rho(t) K_+\rho^{-1}(t) = \frac{1}{\vartheta_0} \left[-2 \vartheta_- \chi K_0 + \vartheta_+^2 K_- + \chi^2 K_+\right],$$

$$\rho(t) K_0\rho^{-1}(t) = \frac{1}{\vartheta_0} \left[- (\vartheta_- \vartheta_+ + \chi) K_0 + \vartheta_+ K_- + \chi \vartheta_+ K_+\right],$$

$$\rho(t) K_-\rho^{-1}(t) = \frac{1}{\vartheta_0} \left[-2 \vartheta_+ K_0 + K_- + \vartheta_+^2 K_+\right].$$

For $I^h(t)$ to be Hermitian ($I^h(t) = I^{h\dagger}(t)$) we require the coefficient of $K_0$ is real, and the coefficients of $K_-$ and $K_+$ are complex conjugate of one another. Using these two requirements, we have:

$$[-\delta_1 (\vartheta_- \vartheta_+ + \chi) - 2 (\delta_2 (\vartheta_+ - \delta_3 \chi \vartheta_-)] = [-\delta_1 (\vartheta_- \vartheta_+ + \chi) - 2 (\delta_2 \vartheta_- + \delta_3 \chi \vartheta_+)],$$

$$\delta_1 \vartheta_- + \delta_2 \vartheta_-^2 + \delta_3 \chi^2) = (\delta_1 \chi \vartheta_- + \delta_2 \vartheta_-^2 + \delta_3 \chi^2),$$

which correspond to

$$\delta_2 = \delta_3 \chi,$$

$$\delta_1 = -\frac{\delta_3 (\vartheta_-^2 + \chi)}{\vartheta_-} = -\frac{\delta_3 (\vartheta_-^2 + \chi)}{\vartheta_+}.$$  

From equation (28), it follows that $\vartheta_+ = \vartheta_- \equiv -\Phi(t)$ implying that the TD parameter $\mu(t)$ must be real, i.e. $\mu(t) = \mu^*(t)$. Finally the similarity transformation (22) maps the NH quadratic invariant (21) into $I^h(t)$ given by

$$I^h(t) = -\frac{2}{\vartheta_0} \left[\delta_1 \chi \Phi + \chi - 4\delta_3 \chi \Phi\right] K_0.$$  

Let $|\psi_n^h\rangle$ be the eigenstate of $K_0$ with the eigenvalue $k_n$, i.e.

$$K_0 |\psi_n^h\rangle = k_n |\psi_n^h\rangle.$$  

The eigenstates of $I^h(t)$ (29) are obviously given by

$$I^h(t) |\psi_n^h(t)\rangle = -\frac{2}{\vartheta_0} \left[\delta_1 \chi \Phi + \chi - 4\delta_3 \chi \Phi\right] k_n |\psi_n^h\rangle,$$

because of the time-dependence, the invariant $I^h(t)$ is a conserved quantity whose eigenvalues are real constants. However, without loss of generality, the factor $-\left[\delta_1 \chi \Phi + \chi - 4\delta_3 \chi \Phi\right]/\vartheta_0$
can be taken equal to 1. It follows that the eigenstates $\left| \phi_n^H(t) \right\rangle$ of $I^{PH}(t)$ can be directly deduced from the basis $\left| \psi_n^H \right\rangle$ of its Hermitian counterpart $I^h(t)$ through the similarity transformation $\left| \phi_n^H(t) \right\rangle = \rho^{-1}(t) \left| \psi_n^h \right\rangle$ with time-independent eigenvalue $k_n$.

According to the above discussion, the problem is reduced to find a PH invariant operator and the suitable real phases of its eigenfunctions to take them as the solution for the Schrödinger equation. In a first step, we will determine the real parameters $\delta_1, \delta_2, \delta_3$ so that our invariant operator $I^{PH}(t)$ (21) is PH. Imposing the quasi-Hermiticity condition (19) on $I^h(t)$, we get

$$I^{PH}(t) = \rho^+(t) I^h(t) \rho^{-1+}(t) = 2\delta_1 K_0 + 2\delta_2 K_- + 2\delta_3 K_+$$

From the above equation, the real parameters $\delta_1, \delta_2, \delta_3$ follow straightforwardly:

$$\delta_1 = -\frac{(\Phi^2 + \chi)}{\vartheta_0}, \quad \delta_2 = -\frac{\chi \Phi}{\vartheta_0}, \quad \delta_3 = -\frac{\Phi}{\vartheta_0}.\quad (33)$$

Therefore, the PH invariant operator $I^{PH}(t)$ is written in the following form

$$I^{PH}(t) = -\frac{2}{\vartheta_0} \left[ (\Phi^2 + \chi) K_0 + \chi \Phi K_- + \Phi K_+ \right].\quad (34)$$

The second step in the method is imposing for $I^{PH}(t)$ (34) the invariance condition (10) which leads to the following relations:

$$\dot{\vartheta}_0 = \frac{\vartheta_0}{\Phi} \left[ -2\Phi |\omega| \sin \varphi_\omega + |\alpha| \sin \varphi_\alpha + (2\Phi^2 + \chi) |\beta| \sin \varphi_\beta \right],\quad (35)$$

$$\dot{\Phi} = -\Phi |\omega| \sin \varphi_\omega + |\alpha| \sin \varphi_\alpha + \Phi^2 |\beta| \sin \varphi_\beta,$n

$$\chi |\beta| \cos \varphi_\beta = |\alpha| \cos \varphi_\alpha,$n

$$(\Phi^2 + \chi) |\alpha| \cos \varphi_\alpha = \chi \Phi |\omega| \cos \varphi_\omega,$n

$$\Phi |\omega| \cos \varphi_\omega = (\Phi^2 + \chi) |\beta| \cos \varphi_\beta.\quad (37)$$

here, $\varphi_\omega$, $\varphi_\alpha$ and $\varphi_\beta$ are the polar angles of $\omega$, $\alpha$, and $\beta$, respectively.

The final step consists in determining the Schrodinger solution (17) which is an eigenstate of the PH invariant (34) multiplied by a time-dependent factor (18)

$$\frac{d\gamma_n(t)}{dt} = \langle \phi_n^H(t) | \eta(t) \left[ i \frac{\partial}{\partial t} - H(t) \right] | \phi_n^H(t) \rangle$$

Using the NH Hamiltonian $H(t)$ (20) and then deriving the transformed Hamiltonian $[i\rho \dot{\rho}^{-1} - \rho H \rho^{-1}]$ through the metric operator $\rho(t)$ (22), we further identify this transformed Hamiltonian as

$$i \rho \dot{\rho}^{-1} - \rho H \rho^{-1} = 2W(t) K_0 + 2U(t) K_- + 2V(t) K_+,$$\quad (39)
where the coefficient functions are

\[ W(t) = \frac{1}{\vartheta_0} \left[ \omega (\Phi^2 + \chi) - 2\Phi (\alpha + \beta \chi) - \frac{i}{2} \left( \vartheta_0 - 2\Phi \dot{\Phi} \right) \right], \quad (40) \]

\[ U(t) = \frac{1}{\vartheta_0} \left[ \omega \Phi - \alpha - \beta \Phi^2 + \frac{i}{2} \Phi \right], \quad (41) \]

\[ V(t) = \frac{1}{\vartheta_0} \left[ \omega \chi \Phi - \alpha \Phi^2 - \beta \chi^2 + \frac{i}{2} \left( \vartheta_0 \dot{\Phi} + \Phi^2 \dot{\Phi} - \Phi \dot{\vartheta}_0 \right) \right]. \quad (42) \]

Considering Eqs. (37), these TD coefficients can be simplified as

\[ W(t) = \frac{1}{\vartheta_0} \left[ |\omega| (\Phi^2 + \chi) \cos \varphi_\omega - 4\Phi |\alpha| \cos \varphi_\alpha \right. \]

\[ -i \frac{\vartheta_0}{2\Phi} \left[ -\Phi |\omega| \sin \varphi_\omega + |\alpha| \sin \varphi_\alpha + \chi |\beta| \sin \varphi_\beta \right], \quad (43) \]

\[ U(t) = 0, \quad (44) \]

\[ V(t) = 0. \quad (45) \]

Knowing that the phase \( \gamma_n(t) \) must be real, we need to impose that the frequency \( W(t) \) is real. Then, we obtain the exact phase of the eigenstate

\[ \gamma_n(t) = k_n \int_0^t \frac{2}{\vartheta_0} \left[ |\omega| (\Phi^2 + \chi) \cos \varphi_\omega - 4\Phi |\alpha| \cos \varphi_\alpha \right] dt'. \quad (46) \]

Therefore, the solutions for the Schrödinger equation (6) are given by

\[ |\Phi^H(t)\rangle = \sum_n C_n(0) \exp \left( i k_n \int_0^t \frac{2}{\vartheta_0} \left[ |\omega| (\Phi^2 + \chi) \cos \varphi_\omega - 4\Phi |\alpha| \cos \varphi_\alpha \right] dt' \right) |\phi^H_n(t)\rangle. \quad (47) \]

The canonical representation (34), when is expressed in terms of \( x = \sqrt{\frac{1}{2}}(a + a^+) \) and \( p = \sqrt{2}(a^+ - a) \), becomes

\[ I^P^H(t) = \frac{1}{2\vartheta_0} \left\{ \left[ (\Phi - \chi) (1 - \Phi) \right] p^2 - i \Phi (\chi - 1) (px + xp) - \left[ (\Phi + \chi) (1 + \Phi) \right] x^2 \right\}, \quad (48) \]

and the eigenfunctions are given as

\[ \phi^H_n(x,t) = \sqrt{\frac{1}{n!2^n\sqrt{n!}}} \sqrt{\frac{\vartheta_0}{(\Phi - \chi)(1 - \Phi)}} \exp \left[ -\frac{1}{2} \left( \frac{\vartheta_0 + \Phi(\chi - 1)}{(\Phi - \chi)(1 - \Phi)} \right) x^2 \right] \]

\[ \times H_n \left( \left[ \frac{\vartheta_0}{(\Phi - \chi)(1 - \Phi)} \right]^{\frac{1}{2}} x \right). \quad (49) \]
Clearly the eigenvalues are $2k_n = (n + 1/2)$ and the $H_n$ are the Hermite polynomials of order $n$. These eigenfunctions are orthonormal with respect to the weight factor $\eta(t) = \exp \left[ \frac{\Phi(\chi - 1)}{(\Phi - \chi)(1 - \Phi)} x^2 \right]$. That is,

$$\int \phi_m^*H(x,t) \exp \left[ \frac{\Phi(\chi - 1)}{(\Phi - \chi)(1 - \Phi)} x^2 \right] \phi_nH(x,t) dx = \delta_{mn}. \quad (50)$$

4 Conclusion

Recently, the general framework for a description of a unitary time evolution for TDNH Hamiltonians has been stated and the use of a TD metric operator cannot ensure the unitarity of the time evolution simultaneously with the observability of the Hamiltonian [18, 19]. They adapted the method based on a TD unitary transformation of TD Hermitian Hamiltonians [35, 36] to solve the Schrödinger equation for the generalized version of the NH Swanson Hamiltonian with TD coefficients.

In this work, using quasi-Hermiticity relation (9) between a NH invariant operator $I^{PH}(t)$ and Hermitian one $I^H(t)$, we have presented an alternative approach to solve the time-evolution of quantum systems. We investigated in detail the main frames of TD systems in the framework of the Lewis and Riesenfeld method which ensures that a solution to the Schrödinger equation governed by a TDNH Hamiltonian is an eigenstate of an associated PH invariant operator $I^{PH}(t)$ with a TD global real phase factor $\gamma_n(t)$.

The properties derived here help us to understand better systems described by TDNH Hamiltonians and should play a central role in TDNH quantum mechanics. After going through these properties, we then have presented an illustrative example: the generalized Swanson Hamiltonian with TD complex coefficients.

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