Analysis of high school students’ errors in mathematical proving: the case of mathematical induction

Y Ahmadi1*, Y S Kusumah1, A Jupri1

1Study Program of Mathematics Education, School of Postgraduate Studies, Universitas Pendidikan Indonesia, Bandung, Indonesia

E-mail: yusuf.bangmedi@upi.edu

Abstract. The ability in proving mathematical statements is a pivotal skill to own. Mathematical induction as a tool for proving mathematical statements related to natural numbers requires three steps to conduct, namely the basic step, the hypothetical step and the induction step. Meanwhile, errors and mistakes can occur during the process of proving. The purpose of this study is to describe mistakes made by students when they use mathematical induction, based on Newman’s Error Analysis (NEA) stages model. The research was a qualitative descriptive study with the type of a case study. The subjects of this study were 25 eleventh grader students in a senior high school. The data were collected by giving tests and conducting interviews. The results of the research, viewed from the stages of NEA, show that the students can reach the first stage (Reading and Recoding) and the second stage (Comprehension) in doing the proving properly. Most of them, however, made errors beginning from the third stage and afterwards (Transformation, Process Skills and Encoding). It is also found that their errors are due to their poor understanding in using hypothetical step in mathematical induction.

1. Introduction

There are mathematical statements or expressions which have to be proved by using mathematical proving. Working in mathematical proving certainly requires us to have particular logical steps in order to do it in the right way [1]. But of course, we can generalize the proving methods into just two broad stages, i.e. understanding the evidence and doing it (proving the mathematical statements) [2]. To be specific in mathematical induction, the steps are (1) the initial step, (2) the hypothetical step and (3) the induction step. Mathematical induction is required in proving some of mathematical sentences involving the set of natural numbers (or the set of whole numbers, depending on the initial base number to prove), so that the mathematical sentences will satisfy for all natural numbers which are greater than the natural number tested in the initial step of the induction. Some basic mathematical knowledge like algebraic operations properties were of course required to be mastered by them (i.e. exponential operations, real number, arithmetical operations, etc.) in order to conduct the proving process in an accurate and comprehensive way.

From the eyes of teachers, mathematical proving still seems like an exclusive skill owned by a small number of excellent students [3]. This ability should be used as a tool only in learning mathematics; it is not required by all students in proving what have to be proved [4].

Some researchers describe that students’ ability in mathematical proving are quite low, and even in some indicators measured, their ability can be categorized as failed [5]. There is no enough evidence to say that students’ inductive and deductive reasoning in the mathematical proving are high [6].
Particularly, it is no wonder why mathematical proving seems so hard to be carried out by the students because mathematical proving itself is one of students’ learning obstacles, as epistemologically categorized by Sundawan [7]. The students have difficulties or errors in mathematical proving. It is so apparent that they: (1) do not know how to begin constructing proofs; (2) do not know how to use definitions, concepts or principles; and (3) do not know how to begin the construction of proof from what to be proved.

It was also found that among the reasons why students have difficulties in proving are (1) the students did not remember the procedures; (2) the students were not familiar with doing mathematical proving; and (3) the students did not understand the concepts or the problems [8]. Further, in some cases, where proving was conducted systematically by presenting symbols accurately, yet the argument presented was not clear, due to insufficient details and incomplete explanation [9].

In this research, we tried to focus our attention in finding the errors on students’ works in mathematical proving by induction, and then compare them to the stages of errors in Newman’s Error Analysis (NEA). In NEA, there are five stages of errors, namely (1) Reading and Recoding, (2) Comprehension, (3) Transformation, (4) Process Skills and (5) Encoding [10].

The research on students’ ability in mathematical proving which has been mentioned above, however, can be classified as quantitative research [5, 6]. In this study, a qualitative research was carried out to investigate how and why the errors or mistakes in mathematical proving conducted by senior high school students could occur compared to the stages of Newman’s Error Analysis.

2. Method
The case study was used in this research to provide broader and deeper opportunities in gaining data and analysing them intensively [11]. The purpose of this case study is to find out directly any mistakes made by the students in proving mathematical statements using mathematical induction. In Indonesia, this topic has been already learned by the eleventh grader students of senior high school (age 16-17 years old) in the first semester.

Table 1. Mathematical induction questions.

| No. | Type of mathematical induction | Questions |
|-----|--------------------------------|------------|
| 1   | Number series                  | Prove that the expression below is true for all \( n \geq 0 \) where \( n \) is an element of the whole number set. 
\[ 1 + 2 + 2^2 + 2^3 + 2^4 + \ldots + 2^n = 2^{n+1} - 1 \] |
| 2   | Divisibility by a real number [Type A] | Prove that \( 9^n - 1 \) is divisible by 8 for each natural number \( n \).  
[Type B] Prove that \( 6^n - 1 \) is divisible by 5 for each natural number \( n \). |
| 3   | Inequalities                    | Prove that \( 2^{n-1} \leq n! \) for all natural number \( n \).  
(The notation \( n! \), called “factorial of \( n \)”, is \( 1 \times 2 \times 3 \times \ldots \times n \).  
Example: \( 4! = 1 \times 2 \times 3 \times 4 = 24 \)) |

Based on a limited observation through Indonesian mathematics text books, the concepts in the statements to be proved using mathematical induction can be regarded into three types: number series, divisibility (by a real number and by an algebraic expression) and inequalities. Because the induction involving divisibility by an algebraic expression is considered relatively hard for high school students,
later on the test, they were not given that type of divisibility question. Instead, they were asked to prove a question involving divisibility by a real number.

In the test, the students were given three questions in the forms of mathematical statements to be proved by using mathematical induction. The first question is an expression of a number series with its general formula for the \( n^{th} \) term. Secondly, the students were asked to prove an algebraic expression which is assumedly divisible by a given particular real number. Lastly, they were asked to prove mathematical statements containing the concepts of inequalities using mathematical induction. Table 1 shows the questions used to test the students’ ability in mathematical proving using mathematical induction.

3. Results and Discussion

Figure 1 shows a student’s work in answering the question related to number series expression (question number 1). The students were asked to prove the equation (1) for all whole numbers \( n \geq 0 \).

\[
1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1
\]

We can see that the student has done it in three steps as required in the induction process but failed concluding the proof in the final step, namely the induction step. The student actually had hypothesized in the second step and used that hypothesis in the third step, and also targeted right to the expression \( 2^{k+2} - 1 \). Unfortunately, when the student reached \( 2^{k+1} + 2^{k+1} - 1 = 2(2^{k+1}) - 1 \), which was still true, it was then written in the form of \( 2^{2k+2} - 1 \), instead of \( 2^{k+2} - 1 \). This led to the wrong conclusion that stated the expression was not successfully proven. A simple algebraic mistake has been made by this student and it caused the wrong result, despite knowing the steps and the final objective in mathematical induction process. The algebraic mistake made by the student was actually one of the very basic of the power properties, i.e. \( a^m \cdot a^n = a^{m+n} \).
When this result was evaluated by comparing it to the Newman’s Error Analysis stages, it is obvious that the first and the second stage of NME have been reached by the student, i.e. Reading and Recoding stage and Comprehension stage. In the third stage (Transformation), the student might slightly do a small mistake led to a vital decision. In this case, the student could not transform and manipulate algebraic expressions accurately. This also implicated to not fulfilling the stages of Process Skills and Encoding of the NME stages.

Figure 2 shows a student’s work in the question number 2. It can be seen that the student’s work which was related to divisibility expression has a mistake in terms of using algebraic manipulation. Until the second step, the student wrote an algebraic manipulation by equalizing the expression $9^k - 1$ with $8m$. It means that it is a multiply of 8. In this case, $9^k$ can be substituted by $8m + 1$, so the expression can be written as $9 \times 9^k - 1 = 9(8m + 1) - 1 = 72m + 8$, where this is obviously

Figure 3. A student’s mistake in proving by using mathematical induction in inequalities expression.
divisible by 8. Unfortunately, the student failed to use this manipulation correctly in the induction step. Instead, the student directly stated that $9 \times 9^k - 1 = 9(8m)$, forgetting that $-1$ is a different term from $9^k$. Then, the student concluded that the third step has been confirmed. From the above description, it is apparent that, the Newman’s Error Analysis stage of Reading and Recoding, as well as the stage of Comprehension, have been fulfilled by the student. But it obviously seems that the student started getting into an error while entering the stage of Transformation, and it continued to the stages of Process Skills and Encoding.

Figure 3, meanwhile, shows a student’s work in the question number 3. It can be seen that the students has reached the hypothetical step where an assumption was made by stating $2^{k-1} \leq k!$ is true. We can see that the student wanted to prove the expression in the third step, namely for the $(k+1)^{th}$ term. But what the student did was completely not correct by any means of the third step. The student meant to add something in both sides of the expression $2^{k-1} \leq k!$, but it broke the rule of algebra. The next line in the work was the expression $2^{k-1} + 2^k \leq (k + 2^{k+1})!$, where we know that it is incorrect. Uniquely, even though it was not proven, the student wrote some words next to the induction work, stating that the inequality expression $2^{n-1} \leq n!$ has been proven true for all natural numbers $n$. This indicated that the student had been frustrated of what he or she did not know, implicating to a direct conclusion made by him/herself, not by the work performed.

4. Conclusion

Based on the results and discussions above, we can draw a conclusion of this study that students made mistakes in mathematical proving by using mathematical induction and most of them could not reach the Transformation stage (the third stage) of the Newman’s Error Analysis stages. They could read and recode (the first stage) and they even comprehend (the second stage) the given problems very well, yet they found difficulties when they have to transform (the third stage) and manipulate the expressions they had to lead them to the main purpose of the proving. This also means that simple error in comprehension like wrong algebraic understanding influenced in the result of wrong conclusion in the students’ process of proving using mathematical induction. The students did not optimize the process of mathematical induction because they did not understand how to use the hypothetical step (the second step) of mathematical induction in the induction step (the third step). What we found here confirmed the previous research that students have obstacles and difficulties in proving mathematics because they could not use and implement appropriate principles [7].

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References

[1] Mubarok M S, Pujiastuti E and Suparsih H 2018 Meningkatkan kemampuan pembuktian matematis dan rasa ingin tahu siswa kelas XI MIPA SMA Negeri 6 Semarang melalui model PBL PRISMA 1677-83
[2] Syafitri, F S 2017 Kemampuan representasi matematis dan kemampuan pembuktian matematika J. Edumath 3(1) 49-55
[3] Knuth E J 2014 Secondary school mathematics teachers’ conceptions of proof *J. for Res. in Math. Educ.* **33**(5)379-405

[4] Knuth E J 2002 Teachers’ conceptions of proof in the context of secondary school mathematics *J. Math. Teacher Educ.* **5** (The Netherlands: Kluwer Academic Publishers) pp 61-88

[5] Hartono, Jamilah and Susiaty U D 2017 Kemampuan pembuktian matematis mahasiswa melalui model pembelajaran penemuan terbimbing *Proc. of Seminar Nasional Pendidikan MIPA dan Teknologi IKIP PGRI Pontianak* pp 76-85

[6] Lestari K E 2015 Analisis kemampuan pembuktian matematis mahasiswa menggunakan pendekatan induktif-deduktif pada mata kuliah analisis real *Mendidik: J. Kajian Pend. Dan Pengajaran* **1**(2) 128-35

[7] Sundawan M D, Dewi I L K and Noto M S 2018 Kajian kesulitan belajar mahasiswa dalam kemampuan pembuktian matematis ditinjau dari aspek epistemology pada mata kuliah geometri transformasi *Inspiramatika: J. Inov. Pend. Dan Pemb. Mat.* **4**(1) 13-26

[8] Perbowo K S and Pradipta T R 2017 Pemetaan kemampuan pembuktian matematis sebagai prasyarat mata kuliah analisis riil mahasiswa pendidikan matematika *Kalamatika: J. Pend. Mat.* **2**(1) 81-90

[9] Pantaleon K V, Juniati D, Lukito A and Mandur K 2018 The written mathematical communication profile of prospective math teacher in mathematical proving *J. Phys.: Conf. Ser.* **947** 012070

[10] Alhassora N S A, Abu M S and Abdullah A H 2017 Newman error analysis on evaluating and creating thinking skills *Man in India* **97**(19) 413-27

[11] Bungin B 2003 *Analisis data penelitian kualitatif: Pemahaman filosofs dan metodologis kearah penguasaan model aplikasi* (Jakarta: Raja Grafindo Persada)