Derivative estimates of solutions of elliptic systems in narrow regions

HaiGang Li∗, YanYan Li†, Ellen ShiTing Bao‡ and Biao Yin§

1 Introduction

In this paper we establish local derivative estimates for solutions to a class of elliptic systems arising from studies of fiber-reinforced composite materials. From the structure of the composite, there are a relatively large number of fibers which are touching or nearly touching. The maximal strains can be strongly influenced by the distances between the fibers.

Stimulated by some works on damage analysis of fiber composites ([6]), there have been a number of papers, starting from [9], [15] and [16], on gradient estimates for solutions of elliptic equations and systems with piecewise smooth coefficients which are relevant in such studies. See, e.g. [1, 2, 3, 4, 5], [7, 8], [10], [12], [17], [18, 19]. Earlier studies on such and closely related issues can be found in [11], [13, 14].

In a recent paper [2], some gradient estimates were obtained concerning the conductivity problem where the conductivity is allowed to be ∞ (perfect conductor).

Theorem A. ([2]) Let B1 and B2 be two balls in $\mathbb{R}^3$ with radius R and centered at $(0, 0, \pm R \pm \frac{\epsilon}{2})$, respectively. Let $H$ be a harmonic function in $\mathbb{R}^3$ such that $H(0) = 0$. Define $u$ to be the solutions of

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \mathbb{R}^3 \setminus (B_1 \cup B_2), \\
u &= 0 \quad \text{on } \partial B_1 \cup \partial B_2, \\
\end{aligned}
\]

then

\[
\begin{aligned}
u(x) - H(x) &= O(|x|^{-1}) \quad \text{as } |x| \to +\infty.
\end{aligned}
\]

Then there exists a constant $C$ independent of $\epsilon$ such that

\[
\|\nabla (u - H)\|_{L^n(\mathbb{R}^3 \setminus (B_1 \cup B_2))} \leq C.
\]

---

*School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China. Email: hgli@bnu.edu.cn

†Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854, USA. Email: yyli@math.rutgers.edu

‡Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.

§Quantitative Advisory Services, Ernst and Young LLP, 5 Times Square, New York, NY 10036, USA.
Contrary to scalar equations, less is known about derivative estimates of solutions of systems. In this paper we extend Theorem A to general elliptic systems, including linear systems of elasticity, in all dimensions. Moreover, we allow the two balls in Theorem A to be replaced by any two smooth domains, and we establish a stronger local version.

We use \( B_r(0') = \{ x' \in \mathbb{R}^{n-1} \mid |x'| < r \} \) to denote a ball in \( \mathbb{R}^{n-1} \) centered at the origin \( 0' \) of radius \( r \). Let \( h_1 \) and \( h_2 \) be smooth functions in \( B_1(0') \) satisfying

\[
h_1(0') = h_2(0') = 0, \quad \nabla h_1(0') = \nabla h_2(0') = 0,
\]

and

\[
-\frac{\epsilon}{2} + h_2(x') < \frac{\epsilon}{2} + h_1(x'), \quad \text{for } |x'| < 1.
\]

For \( 0 < r \leq 1 \), we define

\[
\Omega_r := \left\{ x \in \mathbb{R}^n \mid -\frac{\epsilon}{2} + h_2(x') < x_n < \frac{\epsilon}{2} + h_1(x'), \; x' \in B_r(0') \right\}.
\]

Its lower and upper boundaries are, respectively,

\[
\Gamma^+_r = \left\{ x \in \mathbb{R}^n \mid x_n = \frac{\epsilon}{2} + h_1(x'), \; |x'| \leq r \right\}, \quad \Gamma^-_r = \left\{ x \in \mathbb{R}^n \mid x_n = -\frac{\epsilon}{2} + h_2(x'), \; |x'| \leq r \right\}.
\]

Let \( u = (u^1, \cdots, u^N) \) be a vector-valued function. We consider the following boundary value problems

\[
\begin{cases}
\partial_\alpha \left( A^\alpha_{ij}(x) \partial_\beta u^j + B^\alpha_{ij} u^j \right) + C^\beta_{ij} \partial_\beta u^j + D_{ij} u^i = 0 & \text{in } \Omega_1, \\
\quad u = 0 & \text{on } \Gamma^+_1 \cup \Gamma^-_1.
\end{cases}
\]  

We use the usual summation convention: \( \alpha \) and \( \beta \) are summed from 1 to \( n \), while \( i \) and \( j \) are summed from 1 to \( N \). For \( 0 < \lambda < \Lambda < \infty \), we assume that the coefficients \( A^\alpha_{ij}(x) \) are measurable and bounded,

\[
|A^\alpha_{ij}| \leq \Lambda, \tag{4}
\]

and satisfy the rather weak ellipticity condition

\[
\int_{\Omega_1} A^\alpha_{ij} \partial_\alpha \psi^i \partial_\beta \psi^j \geq \lambda \int_{\Omega_1} |\nabla \psi|^2, \quad \forall \psi \in H^1_0(\Omega_1, \mathbb{R}^N). \tag{5}
\]

Furthermore, we assume that \( A^\alpha_{ij}, B^\alpha_{ij}, C^\beta_{ij}, D_{ij}, h_1 \) and \( h_2 \) are in \( C^k(\Omega_1) \) for some \( k \geq 0 \), denote

\[
\|A\|_{C^k(\Omega_1)} + \|B\|_{C^k(\Omega_1)} + \|C\|_{C^k(\Omega_1)} + \|D\|_{C^k(\Omega_1)} \leq \beta_k,
\]

and

\[
\|h_1\|_{C^k(\Omega_1)} + \|h_2\|_{C^k(\Omega_1)} \leq \gamma_k,
\]

where \( \beta_k \) and \( \gamma_k \) are some positive constants. Hypotheses (4) and (5) are satisfied by linear systems of elasticity (see [20]).
We give local estimates of weak solutions \( u \) of (3), that is, \( u \in H^1(\Omega_1, \mathbb{R}^N) \), \( u = 0 \) on \( \Gamma^+ \cup \Gamma^- \) a.e., and satisfies
\[
\int_{\Omega_1} \left( A_{ij}^{\beta}(x) \partial_{\beta} u^i + B_{ij}^{\beta} u^i \right) \partial_{\alpha} \zeta^i - C_{ij}^{\beta} \partial_{\beta} u^i \xi^i - D_{ij} \xi^i = 0
\]
for every vector-valued function \( \zeta = (\zeta^1, \cdots, \zeta^N) \in C_c^\infty(\Omega_1, \mathbb{R}^N) \), and hence for every \( \zeta \in H^1_0(\Omega_1, \mathbb{R}^N) \).

**Theorem 1.1.** Assume the above, let \( u \in H^1(\Omega_1, \mathbb{R}^N) \) be a weak solution of (3). Then for \( k \geq 0 \), there exist constants \( 0 < \mu < 1 \) and \( C \), depending only on \( n, N, \lambda, \Lambda, k, \beta_{k+1+\frac{\mu}{2}} \) and \( \gamma_{k+1+\frac{\mu}{2}} \), such that
\[
|\nabla^k u(x)| \leq C \mu^{\frac{k+1}{2}} \|u\|_{L^2(\Omega_1)}, \quad \text{for all } x = (x', x_n) \in \Omega_{\frac{1}{2}}.
\]

In particular,
\[
\max_{-\frac{\mu}{2} + h_1(0') < x_n < -\frac{\mu}{2} + h_2(0')} |\nabla^k u(0', x_n)| \to 0, \quad \text{as } \epsilon \to 0.
\]

A consequence of Theorem 1.1 is an extension of Theorem A to all dimensions and to any smooth domains.

**Corollary 1.1.** Let \( D_1 \) and \( D_2 \) be two disjoint bounded open sets in \( \mathbb{R}^n, n \geq 2 \), with \( C^k \) boundaries for \( k = \left[ \frac{n}{2} + 1 \right] \), and \( \text{dist}(\partial D_1, \partial D_2) = \epsilon \in (0, 1) \). Let \( H \) be a harmonic function in \( \mathbb{R}^n \setminus (D_1 \cup D_2) \). Assume that \( u \) satisfies
\[
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n \setminus (D_1 \cup D_2), \\
u = 0 & \text{on } \partial D_1 \cup \partial D_2, \\
\liminf_{|x| \to \infty} |u(x) - H(x)| \leq K, & \text{for some } K > 0.
\end{cases}
\]

Then there exists a constant \( C \), depending only on \( K \), \( \|H\|_{L^\infty(\partial D_1 \cup \partial D_2)} \) and the \( C^k \) norms and diameters of \( D_1 \) and \( D_2 \) (but independent of \( \epsilon \)), such that
\[
\|\nabla (u - H)\|_{L^\infty(\mathbb{R}^n \setminus (D_1 \cup D_2))} \leq C.
\]

### 2 Proof of Theorem 1.1

In this section, we derive the \( C^k \) estimates for solutions of elliptic systems (3). In the following, we first show that the energy in \( \Omega \), decays exponentially as \( r \) tends to 0. Unless otherwise stated, we use \( C \) to denote some positive constants, whose values may vary from line to line, which depend only on \( n, N, \Lambda, \beta_0 \) and \( \gamma_2 \), but is independent of \( \epsilon \).

**Lemma 2.1.** Let \( u \in H^1(\Omega_1, \mathbb{R}^N) \) be a weak solution of (3), then there exist \( 0 < \mu_0 < 1 \) and \( C \), depending only on \( n, N, \lambda, \Lambda, \beta_0 \) and \( \gamma_2 \), such that, for any \( \sqrt{\epsilon} \leq r < \frac{1}{2} \),
\[
\int_{\Omega} |\nabla u|^2 dx \leq C(\mu_0)^\frac{1}{2} \int_{\Omega_1} |\nabla u|^2.
\]
Proof. Without loss of generality, we can assume that \( \int_{\Omega_1} |\nabla u|^2 = 1 \). For any \( 0 < t < s \leq 1 \), we introduce a cutoff function \( \eta \in C^\infty(\Omega_1) \) satisfying \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \( \Omega_t \), \( \eta = 0 \) in \( \Omega_1 \setminus \Omega_t \), and \( |\nabla \eta| \leq \frac{2}{s-t} \). Multiplying \((u\eta)^2\) on both sides of the equation in (3) and integrating by parts, we have

\[
\int_{\Omega_1} \left( A_{ij}^\alpha(x) \partial_\beta u^i + B_{ij}^\alpha u^i \right) \partial_\alpha (u^i \eta^2) - C_{ij}^\alpha \partial_\beta (u^i \eta^2) - D_{ij} u^i (u^i \eta^2) = 0.
\]

Since

\[
\int_{\Omega_1} \left( A_{ij}^\alpha(x) \partial_\beta (u^i \eta) \right) \partial_\alpha (u^i \eta^2)
\]

\[
= \int_{\Omega_1} A_{ij}^\alpha(x) \partial_\beta (u^i \eta) \partial_\alpha (u^i \eta) - \int_{\Omega_1} A_{ij}^\alpha(x) (u^i \partial_\beta \eta) \partial_\alpha (u^i \eta) + \int_{\Omega_1} B_{ij}^\alpha (u^i \eta) \partial_\alpha (u^i \eta)
\]

\[
+ \int_{\Omega_1} A_{ij}^\alpha(x) \partial_\beta (u^i \eta) \partial_\alpha (u^i \eta)
\]

\[
\eta \int_{\Omega_1} \left( \int_{\Omega_1} \left( \int_{-\frac{\delta}{2}+h_2(x')} \partial_n (u\eta)(x', x') dx_n \right)^2 \right) dx
\]

\[
\leq \int_{\Omega_1} \left( \int_{-\frac{\delta}{2}+h_2(x')} \partial_n (u\eta)(x', x') dx_n \right)^2 \right) dx
\]

\[
\leq C(\epsilon + s^2) \int_{\Omega_1} |\nabla (u\eta)|^2 dx.
\]

Taking \( 0 < \delta_0 < 1 \) such that \( C^2(\delta_0 + \delta_0^2) = \frac{4}{7} \), then we have

\[
\int_{\Omega_t} |\nabla u|^2 \eta^2 dx \leq C \int_{\Omega_t} u^2 |\nabla \eta|^2 dx, \quad \text{for } \epsilon, s < \delta_0,
\]

(9)
Again using \( u = 0 \) on \( \Gamma \), and by Hölder inequality, we have

\[
\int_{\Omega} u^2 dx \leq C(\epsilon + s^2) \int_{\Omega} |\nabla u|^2 dx.
\]  

(10)

Combining (9) and (10), we have

\[
\int_{\Omega} |\nabla u|^2 dx \leq C(\epsilon + s^2) \int_{\Omega} |\nabla u|^2 dx, \quad \text{for } s < \delta_0.
\]  

(11)

For simplicity of notation, we denote

\[
F(t) = \int_{\Omega} |\nabla u|^2 dx,
\]  

then (11) can be written as

\[
F(t) \leq C \left( \frac{\epsilon + s^2}{s - t} \right)^2 F(s).
\]  

(12)

For \( \sqrt{\epsilon} \leq t < s \leq \delta_0 \), we have the following iterative formula

\[
F(t_i) \leq \left( \frac{C_0 s^2}{s - t} \right)^2 F(t_{i+1}),
\]  

(13)

where \( C_0 \) is a fixed constant, depending only on \( n, N, \lambda, \Lambda, \beta_0 \) and \( \gamma_2 \). Let \( \delta = \min\{\frac{1}{\sqrt{\epsilon}}, \delta_0\} \) and \( t_0 = r < \delta, t_{i+1} = 2\delta(1 - \sqrt{1 - t_i/\delta}) \) if \( t_i \leq \delta \). Then

\[
\frac{C_0 t_{i+1}^2}{t_{i+1} - t_i} = \frac{1}{2},
\]  

(13)

and \( \{t_i\} \) is an increasing sequence. It is easy to see that for some \( i, t_i > \delta \). Let \( k \) be the integer satisfying \( t_k \leq \delta \) and \( t_{k+1} > \delta \). Clearly \( t_{k+1} \leq 2\delta \). Then for any \( 0 \leq i \leq k \), we have

\[
F(t_i) \leq \left( \frac{C_0 t_{i+1}^2}{t_{i+1} - t_i} \right)^2 F(t_{i+1}) = \frac{1}{4} F(t_{i+1}),
\]  

(14)

Iterating (14) \( k \)-times, we have

\[
F(t_0) \leq \left( \frac{1}{4} \right)^{k+1} F(t_{k+1}) \leq \left( \frac{1}{4} \right)^{k+1} F(2\delta) \leq \left( \frac{1}{4} \right)^{k+1}.
\]  

(15)

Now we estimate \( k \). From (13) it follows that

\[
\frac{1}{2C_0 t_i} = \frac{1}{2C_0 t_{i+1}} + \frac{1}{1 - 2C_0 t_{i+1}}, \quad \text{for } 0 \leq i \leq k.
\]  

Then summing it from \( i = 0 \) to \( i = k \), we have

\[
\frac{1}{2C_0 t_0} = \frac{1}{2C_0 t_{k+1}} + \sum_{i=1}^{k+1} \frac{1}{1 - 2C_0 t_i},
\]  

5
Since \(0 < t_i \leq \delta \leq \frac{1}{8C_0}\) for \(1 \leq i \leq k\), it follows that
\[
1 < \frac{1}{1 - 2C_0t_i} \leq \frac{4}{3}.
\]

Then
\[
k + 1 < \frac{1}{2C_0} \left( \frac{t_0}{t_0 - t_{k+1}} \right) < \frac{4}{3}(k + 1).
\]

Recalling \(t_0 = r\), and \(\delta < t_{k+1} \leq 2\delta\), we have
\[
\frac{3}{8C_0r} - 3 \leq k + 1 < \frac{1}{2C_0r} - 2.
\]

Therefore, from (15),
\[
F(r) = F(t_0) \leq \left( \frac{1}{4} \right)^{k+1} \leq \left( \frac{1}{4} \right)^{\frac{2\delta}{3}}.
\]

The proof is completed. \(\Box\)

**Proof of Theorem 1.1.** Given a point \(z = (z', z_n) \in \Omega_1\), define
\[
\widehat{\Omega}_s(z) := \left\{ x = (x', x_n) \in \Omega_1 \big| - \frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), |x' - z'| < s \right\}. \tag{16}
\]

We consider the following scaling in \(\widehat{\Omega}_{\frac{1}{2}(\epsilon+h_1(z')-h_2(z'))}(z)\),
\[
\begin{align*}
Ry' + z' &= x', \\
Ry_n - \frac{\varepsilon}{2} + h_2(z') &= x_n,
\end{align*}
\]
where \(R = (\epsilon + h_1(z') - h_2(z'))\). Denote
\[
\widehat{h}_1(y') := \frac{1}{R} \left( \varepsilon - h_2(z') + h_1(z' + Ry') \right),
\]
\[
\widehat{h}_2(y') := \frac{1}{R} \left( -h_2(z') + h_2(z' + Ry') \right).
\]

Then
\[
\widehat{h}_1(0') = 1, \quad \widehat{h}_2(0') = 0,
\]
and
\[
\widehat{h}_2(y') < \widehat{h}_1(y'), \quad |\nabla \widehat{h}_1(y')|, |\nabla \widehat{h}_2(y')| \leq C_l, \quad \text{for} \ |y'| \leq 1, \ l \geq 1.
\]

Let
\[
\widehat{u}(y', y_n) = u \left( Ry' + z', Ry_n - \frac{\varepsilon}{2} + h_2(z') \right),
\]
then \(\widehat{u}(y)\) satisfies
\[
\partial_a \left( \widehat{A}_{ij}^{\beta}(y) \partial_\beta \widehat{u}(y) + \widehat{B}_{ij}^\alpha(y) \widehat{u}(y) \right) + \widehat{C}_{ij}^{\beta}(y) \partial_\beta \widehat{u}(y) + \widehat{D}_{ij}(y) \widehat{u}(y) = 0 \quad \text{in} \ Q_1, \tag{17}
\]
where
\[ \hat{A}(y) = A\left( R_y' + z', R_y - \frac{\epsilon}{2} + h(z') \right), \quad \hat{B}(y) = RB\left( R_y' + z', R_y - \frac{\epsilon}{2} + h(z') \right), \]
\[ \hat{C}(y) = RC\left( R_y' + z', R_y - \frac{\epsilon}{2} + h(z') \right), \quad \hat{D}(y) = R^2 D\left( R_y' + z', R_y - \frac{\epsilon}{2} + h(z') \right), \]
and for \( r < 1 \),
\[ Q_r := \left\{ (y', y_n) \in \mathbb{R}^n \mid \hat{h}_2(y') < y_n < \hat{h}_1(y'), |y'| < r \right\}. \]

Using \( L^2 \) estimates for elliptic systems \([17]\) and by the Sobolev imbedding theorems, we have
\[ \max_{0 \leq y_n \leq 1} |\nabla^k \hat{u}(0', y_n)| \leq C \|\nabla \hat{u}\|_{L^2(Q_1)}, \]
where \( C \) depends only on \( n, N, \lambda, \Lambda, k, \beta_{k+\frac{4}{5}+1} \) and \( \gamma_{k+\frac{4}{5}+1} \). It follows, in view of Lemma \([2.1]\), that
\[ |\nabla^k u(z)| \leq C \left( \epsilon + |z'|^2 \right)^{1-k-\frac{4}{5}} \|\nabla u\|_{L^2(\Omega_{\epsilon, \frac{4}{5}})} \leq C \left( \epsilon + |z'|^2 \right)^{1-k-\frac{4}{5}} (\mu_0)^{-\frac{1}{5}}, \quad (18) \]
where \( \mu_0 < 1 \) was defined in Lemma \([2.1]\) and \( C \) depends only on \( n, N, \lambda, \Lambda, k, \beta_{k+\frac{4}{5}+1} \) and \( \gamma_{k+\frac{4}{5}+1} \). The proof is completed. \( \square \)

**Proof of Corollary 1.1** Without loss of generality, we assume that \( D_1 \) and \( D_2 \) are separated by the plane \( x_n = 0 \), with \( (0', \frac{\epsilon}{2}) \in \partial D_1 \) and \( (0', -\frac{\epsilon}{2}) \in \partial D_2 \). Since
\[ \Delta (u - H) = 0 \quad \text{in} \ \mathbb{R}^n \setminus \overline{D_1 \cup D_2}. \]
We have, after applying the maximum principle to \( u - H \), that
\[ |u - H| \leq \|H\|_{L^\infty(\partial D_1 \cup \partial D_2)} + K, \quad \text{in} \ \mathbb{R}^n \setminus \overline{D_1 \cup D_2}. \]
Corollary \([1.1]\) then follows from Theorem \([1.1]\). \( \square \)

**Acknowledgements.** The first author was partially supported by NSFC (11071020) (11126038), SRFDPHE (201000003120005) and Ky and Yu-Fen Fan Fund Travel Grant from the AMS. He also would like to thank the Department of Mathematics and the Center for Nonlinear Analysis at Rutgers University for the hospitality and the stimulating environment. The work of the second author was partially supported by NSF grant DMS-0701545, DMS-1065971 and DMS-1203961. The first and second author were both partially supported by Program for Changjiang Scholars and Innovative Research Team in University in China (IRT0908).
References

[1] H. Ammari, G. Ciraolo, H. Kang, H. Lee and K. Yun: Spectral analysis of the Neumann-Poincaré operator and characterization of the gradient blow-up, arXiv:1206.2074.

[2] H. Ammari, H. Dassios, H. Kang and M. Lim: Estimate for the electric field in the presence of adjacent perfectly conducting sphere, Quat. Appl. Math. 65 (2007), 339-355.

[3] H. Ammari, H. Kang and M. Lim: Gradient estimates to the conductivity problem, Math. Ann. 332 (2005), 277-286.

[4] H. Ammari, H. Kang, H. Lee, J. Lee and M. Lim: Optimal estimates for the electrical field in two dimensions, J. Math. Pure Appl. 88 (2007), 307-324.

[5] H. Ammari, H. Kang, H. Lee, M. Lim and H. Zribi: Decomposition theorems and fine estimates for electrical fields in the presence of closely located circular inclusions, J. Differential Equations 247 (2009), 2897-2912.

[6] I. Babuška, B. Anderson, P.J. Smith and K. Levin: Damage analysis of fiber composites. I. Statistical analysis on fiber scale, Comput. Methods Appl. Mech. Engrg. 172 (1999), 27-77.

[7] E.S. Bao, Y.Y. Li and B. Yin: Gradient estimates for the perfect conductivity problem, Arch. Rational Mech. Anal. 193 (2009), 195-226.

[8] E.S. Bao, Y.Y. Li and B. Yin: Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions, Comm. Partial Differential Equations 35 (2010), 1982-2006.

[9] E. Bonnetier and M. Vogelius: An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section, SIAM J. Math. Anal. 31 (2000), 651-677.

[10] E. Bonnetier and F. Triki, Pointwise bounds on the gradient and the spectrum of the Neumann-Poincaré operator: The case of 2 discs, Contemp. Math., to appear.

[11] B. Budiansky and G.F. Carrier, High shear stresses in stiff fiber composites, J. Appl. Mech., 51 (1984), 733-735.

[12] H. Kang, M. Lim and K. Yun, Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities, J. Math. Pure Appl., to appear.

[13] J.B. Keller, Stresses in narrow regions, Trans. ASME J. Appl. Mech., 60 (1993), 1054-1056.
[14] J.B. Keller, Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders, J. Appl. Phys., 3 (1963), 991-993.

[15] Y.Y. Li and L. Nirenberg: Estimates for elliptic system from composite material, Comm. Pure Appl. Math. 56 (2003), 892-925.

[16] Y.Y. Li and M. Vogelius: Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Rational Mech. Anal. 153 (2000), 91-151.

[17] M. Lim and Y. Yun: Blow-up of electric fields between closely spaced spherical perfect conductors, Comm. Part. Diff. Eqs. 34 (2009), 1287-1315.

[18] K. Yun: Estimates for electric fields blow up between closely adjacent conductors with arbitrary shape, SIAM J. Appl. Math. 67 (2007), 714-730.

[19] K. Yun: Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections, J. Math. Anal. Appl. 350 (2009), 306-312.

[20] O.A. Oleĭnik, A.S. Shamaev and G.A. Yosifian: Mathematical problems in elasticity and homogenization, Studies in Mathematics and Its Applications, 26. North-Holland, Amsterdam, (1992).