Playing with functions of positive type, classical and quantum

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Abstract

A function of positive type can be defined as a positive functional on a convolution algebra of a locally compact group. In the case where the group is abelian, by Bochner’s theorem a function of positive type is, up to normalization, the Fourier transform of a probability measure. Therefore, considering the group of translations on phase space, a suitably normalized phase-space function of positive type can be regarded as a realization of a classical state. Thus, it may be called a function of classical positive type. Replacing the ordinary convolution on phase space with the twisted convolution, one obtains a noncommutative algebra of functions whose positive functionals we may call functions of quantum positive type. In fact, by a quantum version of Bochner’s theorem, a continuous function of quantum positive type is, up to normalization, the (symplectic) Fourier transform of a Wigner quasi-probability distribution; hence, it can be regarded as a phase-space realization of a quantum state. Playing with functions of positive type—classical and quantum—one is led in a natural way to consider a class of semigroups of operators, the classical-quantum semigroups. The physical meaning of these mathematical objects is unveiled via quantization, so obtaining a class of quantum dynamical semigroups that, borrowing terminology from quantum information science, may be called classical-noise semigroups.

Keywords: function of positive type, quantum mechanics on phase space, Wigner distribution, semigroup of operators, open quantum system

1. Introduction

As is well known, theoretical physics and mathematics are inextricably intertwined sciences. A striking example of this fruitful relationship is the profound role that the theory of groups and of group representations plays in quantum mechanics and quantum field theory, and, conversely, the impulse that the former subject has received due to its interaction with the latter [1–7]. In this context, a central position—historically and conceptually—is certainly held by the phase-space approach to quantum theory, a topic that has developed into a vast and interesting research field thanks to the pioneering work of outstanding scientists like Weyl, Wigner, Groenewold and Moyal [1, 8–10], at an early stage, and to the efforts of several other researchers up to this day; see, e.g., [11–19], and references therein.

The present paper is aimed at providing a simple example of how a mathematical notion arising in the realm of (abstract) harmonic analysis—essentially, the theory of topological groups, of their representations and of suitable spaces of functions on such groups—assumes a precise meaning when considered from the point of view of classical statistical mechanics or of phase-space quantum mechanics, and may give rise to new insights and applications.

A fundamental concept in harmonic analysis is that of function of positive type on a locally compact group [20, 21]. In the case where the group is abelian, by a classical theorem of Bochner [22, 23] such a function can be regarded as the Fourier transform of a probability measure—a so-called characteristic function—up to normalization. In classical statistical mechanics, probability measures play the role of physical states. Thus, it turns out that one can realize such a
state, alternatively, by a function of positive type on phase space. The latter realization is often more convenient to deal with, since it relies on ordinary C*-valued functions, rather than on quite abstract objects like measures.

In the standard formulation of quantum mechanics, the physical states are realized as density operators, namely, as normalized, positive trace class operators. In this language, there is no direct link with the notion of function of positive type. However, from an algebraic point of view, states are positive (linear) functionals on the *-algebra of observables (the C*-algebra of bounded operators) [24, 25]. This is analogous to the classical case, where the complex (Radon) measures [26] on phase space provide a realization of the bounded functionals on the commutative algebra of continuous complex functions vanishing at infinity. In this setting, the states are the normalized positive functionals—namely, the probability measures—while the observables are the real functions (the selfadjoint part of the *-algebra of observables). Alternatively, as already mentioned, one can realize the physical states as normalized functions of positive type on phase space, that can be regarded as functionals on the Banach space of integrable functions, which is an algebra with respect to convolution.

The analogy between the classical and the quantum case becomes much deeper, however, by considering the phase-space formulation of quantum mechanics. In this approach, states are realized as quasi-probability distributions (the Wigner functions) and the (symplectic) Fourier transform of these distributions—the quantum characteristic functions—can also be regarded as suitably normalized positive functionals on a certain noncommutative *-algebra of square integrable functions. We call (all) the positive functionals on this algebra functions of quantum positive type. The relation between the intrinsic characterization of functions of quantum positive type and the property of being a quantum characteristic function is a quantum counterpart of Bochner’s theorem [27–29] (also see [30, 31]).

Beside these remarkable analogies between the classical and the quantum case, there are a few noteworthy differences. Whereas in the classical setting function spaces are automatically continuous, in the quantum setting continuity is precisely the condition that allows one to single out, up to normalization, the quantum characteristic functions among all functions of quantum positive type. Moreover, in the classical case functions of positive type are normalized in the sense of functionals. In the quantum case, a different kind of normalization criterion (for the continuous functions of quantum positive type) has to be applied for selecting the physical states and, for these states, normalization in the sense of functionals has to be interpreted as the square root of purity [32]. The two normalization criteria coincide precisely for pure states.

The framework of functions of positive type can also be regarded as a common arena where classical and quantum states can be put all together in the same game. Indeed, probability measures on phase space form a semigroup, with respect to convolution, and, accordingly, functions of positive type form a semigroup with respect to point-wise multiplication. In this context, natural objects are a convolution semigroup of probability measures and, associated with this, a multiplication semigroup of (normalized) positive definite functions. On the other hand, one can show that the point-wise product of a function of positive type by a continuous function of quantum positive type is again a function of the latter type. These ingredients allow one to obtain a semigroup of operators—a so-called classical-quantum semigroup [33]—acting in the Banach space generated by the continuous functions of quantum positive type, out of a multiplication semigroup of positive definite functions. At this point, it is natural to wonder whether such a semigroup of operators, which arises when playing with functions of positive type, classical and quantum, has any physical meaning.

By quantizing a classical-quantum semigroup, one obtains a new semigroup of operators acting in the Banach space of trace class operators. This semigroup of operators turns out to be a quantum dynamical semigroup, namely, it describes the evolution of an open quantum system [34, 35]. The quantized counterpart of a classical-quantum semigroup belongs, in particular, to the class of twirling semigroups [36–38], and has a precise role in the context of quantum information science. Because of this role, we will call it a classical-noise semigroup.

The paper is organized as follows. In section 2, we introduce the notions of function of classical (section 2.1) and quantum (section 2.2) positive type, and we discuss their physical meaning. Next, in section 3, we show that with every multiplication semigroup of functions of positive type is associated a semigroup of operators, i.e., a classical-quantum semigroup. Via quantization, one finds out that a classical-quantum semigroup is mapped into a quantum dynamical semigroup; see section 4. Finally, in section 5, a few conclusions are drawn, and possible extensions of the results presented in the paper are briefly outlined.

2. Functions of positive type: classical and quantum

In this section, we will establish some basic facts concerning functions of positive type on phase space and their quantum counterpart, and discuss the relevance of these functions in classical (statistical) mechanics and in quantum mechanics. Usually, we will feel free to mention and use well known results from functional analysis and abstract harmonic analysis that can be found in standard references; see, e.g., [20–22] for functions of positive type, [21, 22, 26] for Fourier analysis on abelian groups and complex measures, [25, 39] for *-algebras and [11, 14, 18, 19] for Wigner distributions, the associated quantum characteristic functions and the twisted convolution. Explicit reference to the literature will be made for nonstandard facts only.

2.1. The classical case

Given a locally compact group G (precisely, we will assume that it is a locally compact, second countable, Hausdorff
topological group), let $L^1(G)$ be the Banach space of $\mathbb{C}$-valued functions on $G$, integrable w.r.t. the left Haar measure $\nu_G$. As is well known, $L^1(G)$—endowed with the convolution product $(\cdot)\otimes(\cdot)$,

$$
(\varphi_1 \otimes \varphi_2)(g) = \int_G \varphi_1(h)\varphi_2(h^{-1}g) \, d\nu_G(h),
$$

and with the involution $1 : \varphi \mapsto \varphi^*$,

$$
\varphi^*(g) = \Delta_G \left( g^{-1} \right) \overline{\varphi \left( g^{-1} \right)},
$$

with $\Delta_G$ denoting the modular function on $G$—becomes a Banach $*$-algebra $(L^1(G), \otimes, 1)$.

**Definition 1.** A positive, bounded linear functional on the Banach $*$-algebra $(L^1(G), \otimes, 1)$, realized as a function in the Banach space of $\nu_G$-essentially bounded functions $L^\infty(G)$, is called a function of positive type on $G$. Namely, a function $\chi$ on $G$ is said to be of positive type if it belongs to $L^\infty(G)$ and

$$
\int_G \chi(g)(\varphi^* \otimes \varphi)(g) \, d\nu_G(g) \geq 0,
$$

for all $\varphi \in L^1(G)$.

**Remark 1.** By our previous assumptions, $G$ is $\sigma$-compact and $\nu_G$ is $\sigma$-finite, so the Banach space $L^\infty(G)$ of essentially bounded functions on $G$, defined in the usual way [26], can be identified with the dual space of $L^1(G)$, and certain technicalities regarding the (re-)definition of $L^\infty(G)$—see, e.g., section 2.3 of [21]—are not necessary in this case.

A function of positive type $\chi \in L^\infty(G)$ agrees $\nu_G$-almost everywhere with a (bounded) continuous function$^1$ and

$$
\|\chi\|_\infty := \sup_{g \in G} |\chi(g)| = \chi(e),
$$

with $e$ denoting the identity in $G$ and $\chi(e)$ the value at $e$ of the ‘continuous version’ of $\chi$.

For a bounded continuous function $\chi : G \to \mathbb{C}$ the following facts are equivalent:

- (P1) $\chi$ is of positive type;
- (P2) $\chi$ satisfies condition (3), for all $\varphi \in C_c(G)$ (the linear space of continuous $\mathbb{C}$-valued functions on $G$, with compact support);
- (P3) $\chi$ satisfies the condition

$$
\int_G \int_G \chi \left( g^{-1}h \right) \overline{\varphi(h)} \varphi(h) \, d\nu_G(g) \, d\nu_G(h) \geq 0,
$$

for all $\varphi \in C_c(G)$;
- (P4) $\chi$ is a positive definite function, i.e.,

$$
\sum_{j,k} \chi \left( g_j^{-1} g_k \right) c_j c_k \geq 0,
$$

for every finite set $\{g_1, \ldots, g_m\} \subset G$ and arbitrary complex numbers $c_1, \ldots, c_m$.

Note that the equivalence between the second and the third point above is obtained by a simple change of variables in the integrals, and relation (6) can be regarded as a discretized version of (5).

Let us now focus on the case where $G$ is abelian (in particular, a vector group). We will denote by $\hat{G}$ the Pontryagin dual of $G$, namely, the group of all irreducible (one-dimensional) representations of $G$, the unitary characters, endowed with a suitable topology. E.g., $\mathbb{R}$ can be identified with $\mathbb{R}$ itself, $\mathbb{Z}$ with the circle group $\mathbb{T}$ (and, by Pontryagin duality, $\mathbb{T} = \mathbb{Z}$), and the cyclic group $\mathbb{Z}(k)$ of integers mod $k$ is selfdual, like $\mathbb{R}$. We will denote by $\text{CM}(\hat{G})$ the Banach space of complex Radon measures on $\hat{G}$. In this case, according to Bochner’s theorem, we can add a further item to the previous list of equivalent properties:

- (P5) $\chi$ is the Fourier (i.e., Fourier-Stieljes) transform of a positive measure $\mu \in \text{CM}(\hat{G})$.

In the case where the positive measure $\mu \in \text{CM}(\hat{G})$ is a probability measure—$\mu(\hat{G}) = 1$—this normalization condition translates into $\chi(0) = 1$ (with 0 denoting the identity in the abelian group $G$) and, taking into account relation (4), the latter condition is nothing but the normalization of $\chi$ regarded as a functional on $L^1(G)$. In the context of probability theory, $\chi$ is usually called the characteristic function associated with $\mu$ [23].

If $G$ is the group of translations on the $(n + n)$-dimensional phase space—i.e., the vector group $\mathbb{R}^n \times \mathbb{R}^n$—we can obviously identify $\hat{G}$ with $G$ itself, and use the symplectic Fourier transform [11], instead of the ordinary one.

The physical meaning and relevance of functions of positive type become evident as soon as one considers that probability measures on phase space play the role of physical states in classical statistical mechanics; see, e.g., [25]. Indeed, from the algebraic point of view a classical state is a normalized positive functional on the $C^*$-algebra of (classical) observables. Taking into account Gelfand theory of commutative $C^*$-algebras [39], according to which such an algebra is isomorphic to an algebra of continuous functions vanishing at infinity, a natural choice for the algebra of classical observables on phase space is $C_0(\mathbb{R}^n \times \mathbb{R}^n)$, the Banach space of continuous $C$-valued functions on $\mathbb{R}^n \times \mathbb{R}^n$ vanishing at infinity, which is a $C^*$-algebra if endowed with the point-wise product (the true observables forming the selfadjoint part $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ of the whole algebra). The dual space of $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ is $\text{CM}(\mathbb{R}^n \times \mathbb{R}^n)$, the space of complex Radon measures on $\mathbb{R}^n \times \mathbb{R}^n$, and the states on the $C^*$-algebra $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ are precisely the probability measures on $\mathbb{R}^n \times \mathbb{R}^n$. Clearly, the expectation value of an observable

$$
\langle f \rangle_{\mu} = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(q, p) \, d\mu(q, p).
$$

Unfortunately, probability measures are in general rather abstract objects to deal with, so that it is often useful to replace a classical state $\mu \in \text{CM}(\mathbb{R}^n \times \mathbb{R}^n)$ with its symplectic
Fourier transform $\tilde{\mu}$,

$$\tilde{\mu}(q, p) := \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\omega((q, p), (q', p'))} \, d\mu(q', p').$$

(8)

— where $\omega$ is the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\omega(q, p, (q', p')) := (q, p)居住 \Omega (q', p') = q \cdot q' - p \cdot p', \quad \Omega \equiv (0_n, I_n).$$

(9)

— which is a continuous function of positive type on $\mathbb{R}^n \times \mathbb{R}^n$.

E.g., the Dirac measure $\delta_{q_0, p_0}$ concentrated at $(q_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, a pure (i.e., extremal) classical state, is mapped into the function $(q, p) \mapsto e^{i((q, p) - (q_0, p_0))}$; a Gaussian measure $\gamma$,

$$d\gamma(q, p) = \pi^{-n/2} \det(\Omega)^{-1/2} \exp(-(q - q_0, p - p_0)^T \times M^{-1}(q - q_0, p - p_0)) \, dq \, dp,$$

(10)

— where $M$ is a positive definite, symmetric, $2n \times 2n$ real matrix, which (with a slight abuse with respect to the conventional normalization) will be called the covariance matrix—is mapped into the function of positive type $\tilde{\gamma}$ of the form

$$\tilde{\gamma}(q, p) = \exp(-(q, p)^T \times M(q, p))/4 + i(q, p)^T \times \Omega(q_0, p_0), \quad \Omega = -\Omega \times \Omega.$$  

(11)

Here $\Omega$ is the skew-symmetric $2n \times 2n$ matrix associated with the standard symplectic form (9).

Moreover, as is well known, the (symplectic) Fourier transform maps $L^1(\mathbb{R}^n \times \mathbb{R}^n)$, continuously and injectively, into a dense linear subspace $FL^1(\mathbb{R}^n \times \mathbb{R}^n)$ of $C_0(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\varphi \mapsto \tilde{\varphi}, \quad \tilde{\varphi}(q, p) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi(q', p') e^{i((q - q') \cdot (p - p') \cdot \Omega)} \, dq' \, dp'.$$

(12)

Then, by Fubini’s theorem, for every observable $f \in FL^1(\mathbb{R}^n \times \mathbb{R}^n)$ the expectation value $\langle f \rangle_\mu$ can be expressed in the following form:

$$\langle f \rangle_\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi(q, p) \chi(q, p) \, dq \, dq' \, dp = (\varphi, \chi),$$

with $\tilde{\varphi} = f$ and $\tilde{\chi} = \tilde{\mu}$.

(13)

Here $\langle \varphi, \chi \rangle$ should be regarded as the pairing between $\varphi \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and $\chi \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (a function of positive type), where $\varphi$ and $\chi$ represent an observable and a state, respectively, in analogy with the pairing (7).

**Remark 2.** The condition that $f \in FL^1(\mathbb{R}^n \times \mathbb{R}^n)$ be an observable—$f = \tilde{f}$—corresponds to the condition that $\varphi$ be selfadjoint: $\varphi = \varphi^*$; namely, that $\varphi(q, p) = \varphi(-q, -p)$.

**Remark 3.** The pairing between $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ and $CM(\mathbb{R}^n \times \mathbb{R}^n)$ can be completely restored considering that the C*-algebra $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ is isomorphic, via the Fourier transform, to the group $C^*(\mathbb{R}^n \times \mathbb{R}^n)$ (the universal C*-completion of $L^1(\mathbb{R}^n \times \mathbb{R}^n)$), and the dual of the Banach space $C^0(\mathbb{R}^n \times \mathbb{R}^n)$ can be identified with the subspace $L^p_\mu \equiv L^p(\mathbb{R}^n \times \mathbb{R}^n)$ of $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ consisting of all complex linear superpositions of continuous functions of positive type on $\mathbb{R}^n \times \mathbb{R}^n$; see [40].

We can summarize the picture outlined above as follows:

- The notion of function of positive type is group-theoretical and can be defined for any locally compact group, giving rise to various equivalent characterizations. In the case of an abelian group, a function of positive type can be characterized, up to normalization, as the Fourier transform of a probability measure.
- In the standard approach to classical statistical mechanics, the physical states are realized as probability measures on phase space, while the observables form the selfadjoint part of the $C^*$-algebra $C_0(\mathbb{R}^n \times \mathbb{R}^n)$.
- Alternatively, one can undertake a characteristic function approach by replacing probability measures with the corresponding normalized functions of positive type (characteristic functions). In this approach, the standard observables are (densely) replaced with the functions belonging to the selfadjoint part of the Banach $*$-algebra $L^1(\mathbb{R}^n \times \mathbb{R}^n)$.
- Whereas the dual space of $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ is $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the positive functionals on $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ are precisely the (continuous) functions of positive type on phase space. Therefore, in the characteristic function approach $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ may be thought of as the algebra of observables tout court (instead of the group C*-algebra $C^*(\mathbb{R}^n \times \mathbb{R}^n)$).

Continuous functions of positive type form a convex cone $P_\mu \equiv P(\mathbb{R}^n \times \mathbb{R}^n)$ in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and characteristic functions—i.e., normalized continuous functions of positive type—form a convex subset $P_0$ of $P_\mu$. Since, as it will be shown in section 2.2, there is a quantum analogue of functions of positive type on $\mathbb{R}^n \times \mathbb{R}^n$, the elements of $P_0$ will be sometimes referred to as functions of classical positive type.

**2.2. The quantum case**

In the standard formulation of quantum mechanics normal states\(^2\) are realized as density operators, namely, as normalized, positive trace class operators in a certain Hilbert space. Within this formalism, there is no direct connection between a physical state and anything analogous to the notion of function of positive type. On the other hand, a profound link of this kind does emerge if one adopts a phase-space approach to

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\(^2\) Recall that among all states (normalized positive functionals) on the $C^*$-algebra of observables, the normal states—that are usually considered as the physically relevant ones—are characterized by the property of being $\sigma$-additive (or completely additive), see [24, 25]. This property is the direct analogue of the $\sigma$-additivity of a measure. Of course, in the finite-dimensional case, which is often considered in the study of entanglement and in applications to quantum information science [41], all states are normal.
quantum mechanics. Whereas various phase-space formalisms have been proposed in the literature, see e.g. [18, 42–49], the prototype is certainly the remarkable approach developed by Weyl, Wigner, Groenewold and Moyal [1, 8–10]—which includes a suitable quantization–dequantization scheme and the notion of star product of functions—approach that may be called the WWGM formulation of quantum mechanics.

In the WWGM formulation, a pure state \( \rho = |\psi\rangle\langle\psi| \), \( \psi \in L^2(\mathbb{R}^n) \) (\( \|\psi\| = 1 \)), is replaced with a function \( q_\rho(q, p) := \frac{1}{(2\pi)^n} \int \mathbb{R}^n e^{-i\psi(q - \frac{x}{2})}\psi(q + \frac{x}{2}) d^nx \),

\[
q_\rho(q, p) := \frac{1}{(2\pi)^n} \int \mathbb{R}^n e^{-i\psi(q - \frac{x}{2})}\psi(q + \frac{x}{2}) d^nx,
\]

(14)

where we have set \( \hbar = 1 \). This definition, by means of the spectral decomposition of a positive trace class operator, extends in a natural way to any density operator in \( L^2(\mathbb{R}^n) \) and then, by taking (finite) linear superpositions, to every trace class operator. The phase-space functions associated with (normal) states are usually called Wigner functions.

By this construction, one obtains a separable complex Banach space of functions that will be denoted by \( LW_\nu \) (the image via the Wigner map, see section 4 infra, of the Banach space of trace class operators in \( L^2(\mathbb{R}^n) \)). This linear space contains a convex cone \( W_\nu \), formed by those functions that are associated with positive trace class operators in \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \), and \( W_\nu \) contains the convex set \( W_\nu \) formed by the Wigner functions. Within \( W_\nu \), the Wigner functions are precisely those functions satisfying the normalization condition

\[
\lim_{r \to +\infty} \int_{\|p\| < r} q_\rho(q, p)d^np = \text{tr}(\hat{\rho}) = 1,
\]

(15)

see [50], where \( q \in W_\nu \) is the function canonically associated with a certain state \( \hat{\rho} \).

A Wigner function \( q \) is real but, in general, it is not a genuine probability distribution, since it may assume negative values; moreover, it is square integrable but, in general, not integrable [50], fact that is taken into account in the lhs of (15). Nevertheless, one can express the expectation value of an observable \( \hat{A} \) in the state \( \hat{\rho} \) as \( \langle \hat{A} \rangle_\rho = \text{tr}(\hat{A}\hat{\rho}) \) as an integral on phase space,

\[
\langle \hat{A} \rangle_\rho = \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha(q, p)q_\rho(q, p) d^npd^np,
\]

(16)

where \( \alpha \) is a real function associated with the operator \( \hat{A} \). Formula (15) and the normalization condition (16) explain why the Wigner function \( q \) is often called a quasi-probability distribution.

Remark 4. As an example of a class of genuine probability distributions on phase space—hence, representing classical states—which also represent quantum states, it is worth mentioning that, according to the classical Hudson–Littlejohn characterization [51, 52], the Wigner functions of pure states which assume non-negative values only are precisely those probability distributions associated with Gaussian measures (of the general form (10)) whose covariance matrix \( \Sigma \)—which is, by definition, a positive definite symmetric matrix—is also symplectic, i.e., \( \Sigma^T \Sigma = \Sigma \Sigma^T = \Omega \).

Remark 5. Actually, not all the observables can be realized as ordinary functions in the WWGM formulation. In general, they form a suitable class of distributions (generalized functions), see [53]. This fact, however, will not play any role in the following.

As in the classical setting, also in the quantum case one can replace a Wigner quasi-probability distribution with its symplectic Fourier transform. Precisely, taking into account the fact that a Wigner function is square integrable (but, in general, not integrable), one should use the symplectic Fourier-Plancherel operator \( \hat{F}_{qp} \), which is the selfadjoint unitary operator in \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \) determined by

\[
\hat{F}_{qp}f(q, p) := \frac{1}{(2\pi)^n} \int \mathbb{R}^n f(q', p)e^{i(q'q - p)p'} dq'dp',
\]

(17)

for all \( f \in L^1(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^n \times \mathbb{R}^n) \). Then, the space \( LW_\nu \) is mapped onto a dense subspace \( Q_\nu \) of \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \).

\[
Q_\nu := \hat{F}_{qp}LW_\nu,
\]

(18)

and the convex cone \( W_\nu \subset LW_\nu \) is mapped onto a convex cone \( Q_\nu \subset Q_\nu \). Each function in \( Q_\nu \) agrees almost everywhere (w.r.t. Lebesgue measure) with a continuous function. Actually, it will often be convenient, in the following, to regard \( Q_\nu \) as a linear space of continuous functions.

By analogy with the classical case, we may call a function \( \tilde{q} \) in \( Q_\nu \), defined by

\[
\tilde{q} := (2\pi)^n\hat{F}_{qp}q, \quad q \in W_\nu,
\]

(19)

the quantum characteristic function associated with the quasi-probability distribution \( q \). The factor \( (2\pi)^n \) appearing in (19) is chosen in such a way that the quantum characteristic functions, similarly to the classical case, are those functions in \( Q_\nu \) satisfying the normalization condition

\[
\tilde{q}(0) = 1,
\]

(20)

with \( 0 \equiv (0, 0) \) denoting the origin in \( \mathbb{R}^n \times \mathbb{R}^n \). Thus, the quantum characteristic functions form a convex subset \( Q_\nu = (2\pi)^n\hat{F}_{qp}W_\nu \) of \( Q_\nu \).

A natural task in the WWGM approach is to provide a fully self-consistent formulation of quantum theory, task that obviously includes the problem of characterizing intrinsically the convex set of physical states \( W_\nu \) or, equivalently, the convex set \( Q_\nu \) of quantum characteristic functions. The analysis of this problem leads in a natural way to the notion of function of quantum positive type.

As in the classical setting, we first consider a *-algebra of functions, and then define the functions of positive type as suitable functionals on this algebra. To this aim, we recall that the Hilbert space \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \) becomes a *-algebra—precisely, a \( \mathbb{H}^* \)-algebra [19, 54]—once endowed with the twisted...
convolution () ⊗ ()

\[
(A_1 \otimes A_2)(q, p) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A_1(q', p') d^q d^p p',
\]

\[
A_2(q - q', p - p') e^{i(p' - p) \cdot q} d^q d^p p',
\]

(21)

Note that involution J is formally identical to that arising in the classical case, but it acts in a different space. An element \( A_1, A_2 \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \), and with the involution: \( A \mapsto A^\dagger \),

\[
A^\dagger(q, p) = A(-q, -p), \quad A \in L^2(\mathbb{R}^n \times \mathbb{R}^n).
\]

(22)

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(22)

Note that involution J is formally identical to that arising in the classical case, but it acts in a different space. An element \( A_1, A_2 \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \), and with the involution: \( A \mapsto A^\dagger \),

\[
A^\dagger(q, p) = A(-q, -p), \quad A \in L^2(\mathbb{R}^n \times \mathbb{R}^n).
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\]

(22)
precisely, it agrees almost everywhere with a continuous function. Thus, requiring continuity may be regarded as a way of removing the ambiguity in the choice of an item among all functions implementing the same functional on $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ (namely, among all functions giving rise to the same element of $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$). On the other hand, in the quantum case, continuity must be imposed in order to single out, up to normalization, precisely the quantum characteristic functions.

**Remark 8.** Since $Q_n = \hat{P}_n W_n$, taking into account remark 4 and the fact that the normalization condition satisfied by both classical and quantum characteristic functions is of the form (20), it is clear that the intersection $P_n \cap Q_n$ is not empty and contains, in particular, every phase-space function $\gamma$ of the type (11) such that $\hat{\gamma}$ is a symplectic, positive definite symmetric matrix. This is actually the general form of a quantum characteristic function of a pure state which is also a classical characteristic function. Note, however, that such a function represents a pure state in the quantum setting only (the elements of $\hat{P}_n$ associated with pure classical states are the functions of the form $(q, p) \mapsto e^{i(qp - p\eta - \eta^2)}$).

Comparing relations (3), (5) and (6) with (23), (27) and (28), respectively, one should note something more than a formal similarity. In the latter inequalities, a major role is played by the function

$$\left(\mathbb{R}^n \times \mathbb{R}^n\right) \times \left(\mathbb{R}^n \times \mathbb{R}^n\right) \ni (z, z') \mapsto e^{iw(z; z')/2}, \quad (30)$$

which is a nontrivial multiplier for the group $\mathbb{R}^n \times \mathbb{R}^n$—more specifically, a non-exact multiplier [5]—i.e., the multiplier associated with the Weyl system and with the integrated form of canonical commutation relations [55], whereas in inequalities (3), (5) and (6) the trivial multiplier is implicitly involved. This is quite an impressive fact: the symplectic structure that plays such a fundamental role in classical mechanics emerges in a spontaneous way in the quantum setting, e.g., in condition (27) satisfied by quantum characteristic functions. Also note that, while an ordinary function of positive type on the vector group $\mathbb{R}^d$ is defined for every $k = 1, 2, \ldots$, the notion of function of positive type emerging in the quantum setting entails the symplectic form $\omega$—a non-degenerate skew-symmetric bilinear form—and hence in this case $k$ must be an even number.

### 3. Playing with functions of positive type

The convolution $\mu_1 \otimes \mu_2$ of a pair of positive measures $\mu_1, \mu_2 \in \text{CM}(G)$ on a locally compact group $G$—

$$\int_G \varphi(g) \, d\mu_1 \otimes \mu_2 (g) = \int_G \int_G \varphi(gh) \, d\mu_1(g) d\mu_2(h), \quad \varphi \in C_c(G)$$

— is a positive measure in $\text{CM}(G)$ too [21]; in particular, a probability measure if $\mu_1$ and $\mu_2$ are both normalized. Endowed with convolution the set $\text{PM}(G)$ of probability measures on $G$ becomes a semigroup. The identity of this semigroup is given by the Dirac measure at the identity of $G$. If $G$ is abelian, to the convolution of probability measures corresponds, via the Fourier-(Stieltjes) transform, the point-wise multiplication of characteristic functions [22]. By Bochner’s theorem, it follows that the point-wise product $\chi_1 \chi_2$ of two continuous functions of positive type $\chi_1$ and $\chi_2$ on $G$ is again a continuous function of positive type, and clearly point-wise multiplication preserves normalization too.

Therefore—considering now the case where $G = \mathbb{R}^n \times \mathbb{R}^n$—endowed with the point-wise product of functions the set $\hat{P}_n$ of normalized functions of classical positive type is a semigroup, the identity being the function $\chi \equiv 1$. The Fourier transform determines an algebraic isomorphisms between the semigroups $\text{PM}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\hat{P}_n$, which becomes a topological isomorphism if $\text{PM}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\hat{P}_n$ are endowed with the weak topology and the topology of uniform convergence on compact sets, respectively (Lévy’s continuity theorem, see [23]). The latter topology turns out to coincide with the topology induced on $\hat{P}_n$ by the weak*-topology of $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (regarded as the dual of $L^1(\mathbb{R}^n \times \mathbb{R}^n)$), see [56, 57], which is the natural one in this context.

What happens with the point-wise multiplication of a function of classical positive type by a continuous function of quantum positive type?

**Theorem 1.** The point-wise product $\chi Q$ of a function $\chi \in P_n$ by a function $Q \in Q_n$ belongs to $Q_n$; in particular, it belongs to $Q_n$ if $\chi$ and $Q$ are normalized.

**Proof.** First observe that condition (6), for $G = \mathbb{R}^n \times \mathbb{R}^n$, becomes

$$\sum_{j,k} (z_k - z_j) c_j c_k \geq 0, \quad z \equiv (q, p). \quad (32)$$

By Schur’s product theorem [58], the Hadamard product (i.e., the entrywise product) of two positive (semi-definite) matrices is positive too. Applying this result to conditions (32) and (28), that for continuous functions on $\mathbb{R}^n \times \mathbb{R}^n$ are equivalent to the property of being of positive type—classical and quantum, respectively—one readily proves the result.

The previous result allows us to play with the point-wise product of functions of positive type (classical or quantum). Consider then a multiplication semigroup of functions of positive type, i.e., a set $\{\chi_t\}_{t \in \mathbb{R}^+}$ of normalized continuous functions of positive type on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\chi_t \chi_s = \chi_{t+s}, \quad t, s \geq 0, \quad \chi_0 \equiv 1, \quad (33)$$

where $\chi_t \chi_s$ is a point-wise product. We also require that the homomorphism

$$\mathbb{R}^+ \ni t \mapsto \chi_t \in \hat{P}_n \quad (34)$$

be continuous (w.r.t. the topology of uniform convergence on compact sets on $\hat{P}_n$). Such semigroups can be classified. In fact, the symplectic Fourier transform of a multiplication
semigroup of functions of positive type on \( \mathbb{R}^a \times \mathbb{R}^b \) is a (continuous) convolution semigroup of probability measures, and convolution semigroups on Lie groups admit a well known characterization associated with the Lévy–Kintchine formula [36, 59].

Given a multiplication semigroup of functions of positive type \( \{ \chi_t \}_{t \in \mathbb{R}^+} \subset \mathcal{C}_a \), since \( \chi_t \) is a bounded continuous function, we can define a bounded operator \( \hat{\mathcal{C}}_t \) in \( L^2(\mathbb{R}^a \times \mathbb{R}^b) \):

\[
(\hat{\mathcal{C}}_t f)(q, p) = \chi_t(q, p) f(q, p), \quad f \in L^2(\mathbb{R}^a \times \mathbb{R}^b).
\]  

(35)

The set \( \{ \hat{\mathcal{C}}_t \}_{t \in \mathbb{R}^+} \) is a semigroup of operators [60], namely,

1. \( \hat{\mathcal{C}}_t \hat{\mathcal{C}}_s = \hat{\mathcal{C}}_{t+s} \), \( t, s \geq 0 \) (one-parameter semigroup property);
2. \( \hat{\mathcal{C}}_0 = I \) (where \( I \) is the identity operator).

**Remark 9.** Note that, since \( |\chi_t(q, p)| \leq \chi_t(0) = 1 \), \( \{ \hat{\mathcal{C}}_t \}_{t \in \mathbb{R}^+} \) is a contraction semigroup, i.e., \( \| \hat{\mathcal{C}}_t \| \leq 1 \).

**Remark 10.** One may define an analogous contraction semigroup replacing the Hilbert space space \( L^2(\mathbb{R}^a \times \mathbb{R}^b) \) with the Banach space \( L^1(\mathbb{R}^a \times \mathbb{R}^b) \). The Fourier transform intertwines this further semigroup of operators with a probability semigroup in \( \text{Co}(\mathbb{R}^a \times \mathbb{R}^b) \), describing (under a certain assumption) a drift-diffusion process [36–38]. We leave the details to the reader.

It is now natural to consider the restriction of the semigroup of operators \( \{ \hat{\mathcal{C}}_t \}_{t \in \mathbb{R}^+} \) to a linear subspace of \( L^2(\mathbb{R}^a \times \mathbb{R}^b) \). Indeed, as argued in section 2, by complex linear superpositions one can extend the convex cone \( \mathcal{Q}_a \) of functions of quantum positive type on \( \mathbb{R}^a \times \mathbb{R}^b \) to a dense subspace \( LQ_a \) of \( L^2(\mathbb{R}^a \times \mathbb{R}^b) \). A semigroup of operators \( \{ \hat{\mathcal{C}}_t \}_{t \in \mathbb{R}^+} \) in \( LQ_a \) is then defined as follows. Since, according to theorem 1, the point-wise product of a continuous function of classical positive type by a continuous function quantum positive type is a function of the latter type, we can (consistently) set

\[
(\mathcal{C}, \mathcal{Q})(q, p) = \chi_t(q, p) \mathcal{Q}(q, p), \quad \mathcal{Q} \in LQ_a.
\]  

(36)

where, with a slight abuse w.r.t. our previous notation, \( \mathcal{Q} \) here denotes a linear superposition of four functions of quantum positive type:

\[
\mathcal{Q} = Q_1 - Q_2 + i(Q_3 - Q_4), \quad Q_1, \ldots, Q_4 \in Q_a.
\]  

(37)

It is clear, moreover, that we have:

\[
\mathcal{C}_t Q_a \subset Q_a, \quad \mathcal{C}_t \hat{Q}_a \subset \hat{Q}_a.
\]  

(38)

We will call the semigroups of operators \( \{ \hat{\mathcal{C}}_t \}_{t \in \mathbb{R}^+} \) and \( \{ \mathcal{C}_t \}_{t \in \mathbb{R}^+} \) a classical-quantum semigroup and a proper classical-quantum semigroup, respectively. Up to this point, the introduction of these semigroups of operators may be regarded as a mere mathematical divertissement, based on the properties of functions of positive type. It turns out, however, that they do have a precise physical interpretation which can now be unveiled by means of suitable quantization–dequantization maps.

4. **Introducing quantization–dequantization maps into the game**

The procedure of associating with a quantum-mechanical operator (a state or an observable) a function on phase space—a Wigner function or a quantum characteristic function—may be thought of as the application of a suitable dequantization map, which can be considered as the reverse arrow of a quantization map transforming functions into operators. The definition of a star product [18, 19, 61]—the twisted convolution, in the case where quantum characteristic functions are involved—allows one to achieve a self-consistent formulation of quantum mechanics in terms of phase-space functions and, as discussed in section 2, a fundamental step of this construction is the notion of function of quantum positive type. In order to illustrate the meaning of this notion in terms of Hilbert space operators it is worth reconsidering the dequantization map in a slightly more formal setting, which also allows us to highlight its group-theoretical content.

A basic ingredient in the definition of a dequantization map is a *square integrable* (in general, projective) representation \( U \) of a locally compact group \( G \) in a Hilbert space \( H \); see [14, 62, 63], and references therein. Here, \( H \) should be though of as the Hilbert space of a quantum-mechanical system and \( G \) as a symmetry group. Denoting by \( B_2(H) \) the Hilbert space of Hilbert–Schmidt operators in \( H \), the representation \( U \) allows one to define a dequantization map

\[
\mathcal{D} : B_2(H) \to L^2(G),
\]  

(39)

see [14, 18, 19], which is a (linear) *isometry*. In the case where the group \( G \) is unimodular (e.g., an abelian or a compact group) and we consider, in particular, a trace class operator \( \hat{\rho} \) in \( B_2(H) \), the function \( \mathcal{D} \hat{\rho} \) associated with \( \hat{\rho} \) is of the form

\[
(\mathcal{D} \hat{\rho})(g) = d_U^{-1} \text{tr} \left( U(g)^* \hat{\rho} \right),
\]  

(40)

where \( d_U > 0 \) is a normalization constant depending on \( U \) and on the choice of the Haar measure on \( G \). The range \( \text{Ran}(\mathcal{D}) \) of the dequantization map is a closed subspace of \( L^2(G) \). The quantization map associated with \( U \) is the adjoint of the dequantization map, i.e., it is the partial isometry \( \mathcal{D} \) defined by

\[
\mathcal{D} = \mathcal{D}^* : L^2(G) \to B_2(H).
\]  

(41)

Clearly, we have that \( \text{Ker} (\mathcal{D}) = \text{Ran}(\mathcal{D}) \perp \), and in general this kernel is not trivial. Also for the quantization map one can provide explicit formulae; see, e.g., [19]. At this point, the star product

\[
(\cdot) \star (\cdot) : L^2(G) \times L^2(G) \to L^2(G)
\]  

(42)

associated with \( U \) is defined by

\[
f_1 \star f_2 := \mathcal{D} \left( (\mathcal{D} f_1)(\mathcal{D} f_2) \right),
\]  

(43)

with \( (\mathcal{D} f_1)(\mathcal{D} f_2) \) denoting the ordinary product (composition) of the operators \( \mathcal{D} f_1 \) and \( \mathcal{D} f_2 \). Therefore, for a pair of functions living in the range of the dequantization map the star product is nothing but the dequantized version of the product of
operators. Explicit formulae for the star products can be found in [19].

Let us consider the case where G is the group of translations on phase space $\mathbb{R}^n \times \mathbb{R}^n$ [18, 19]. In this case, $\mathcal{H} = L^2(\mathbb{R}^n)$, $L^2(G) = L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is given by $d^m dq dp$; (C) (the Haar measure is normalized in such a way that in (40) $d_U = 1$) and the representation $U$ has to be identified with the Weyl system [19, 55], i.e.,

$$U(q, p) = \exp \left(i(p \cdot \hat{\varrho} - q \cdot \hat{\rho})\right),$$

where $\hat{\varrho} = \{\hat{\varrho}_1, \ldots, \hat{\varrho}_n\}, \hat{\rho} = \{\hat{\rho}_1, \ldots, \hat{\rho}_n\}$, with $\varrho_j, \rho_j$ denoting the $j$th coordinate position and momentum operators in $L^2(\mathbb{R}^n)$. The Weyl system is a (square integrable) projective representation,

$$U(q + \varrho, p + \rho) = m(q, p; \varrho, \rho)U(q, p)U(q, \rho),$$

where the multiplier $m$ is given by

$$m(q, p; \varrho, \rho) := \exp \left(\frac{i}{2}(q \cdot \rho - p \cdot \varrho)\right).$$

It can be shown that for every density operator $\hat{\rho}$ in $L^2(\mathbb{R}^n)$, denoting with $\varrho$ the associated Wigner distribution, the function $(\mathcal{D}_\varrho(p, q) = \text{tr}(U(q, p)^* \hat{\rho}))$ coincides with the quantum characteristic function $\varrho$ (defined by (19), see [14, 18, 19]). Therefore, since $P_{\varrho} = P_{\varrho}$, we have:

$$\varrho = (2\pi)^{-n}P_{\varrho} \mathcal{D}_\varrho;$$

namely, quantization–dequantization à la Weyl–Wigner is obtained by composing the maps $\mathcal{D}$ and $\mathcal{D}$ with the symplectic Fourier-Plancherel operator.

One can prove, moreover, the following facts [18, 19]:

1. In this case, $\text{Ran}(\mathcal{D}) = L^2(\mathbb{R}^n \times \mathbb{R}^n)$, hence $\mathcal{D}$ and $\mathcal{D}$ are unitary operators.
2. The quantization map $\mathcal{D}$ intertwines the involution $J : \mathcal{A} \mapsto \mathcal{A}$ in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, see (22), with the standard involution $\hat{\mathcal{A}} = (\mathcal{A})^*: \mathcal{A} \mapsto \mathcal{A}$ in $B_2(L^2(\mathbb{R}^n))$:

$$J \mathcal{A} = (\mathcal{D} \mathcal{A})^*.$$

3. The star product defined by (43) coincides with the twisted convolution $(\cdot) \mathcal{D} (\cdot)$, see (21).

From the previous facts and the results outlined in section 2.2 one draws the following further conclusions:

4. The quantum positivity condition (23), or (24), expressed in terms of operators, becomes the condition that $\mathcal{B} \in B_2(L^2(\mathbb{R}^n))$ is such that

$$\text{tr}(\mathcal{B} \hat{\mathcal{A}}^\dagger \mathcal{A}) \geq 0, \ \forall \hat{\mathcal{A}} \in B_2(L^2(\mathbb{R}^n)).$$

This condition is equivalent to the fact that $\mathcal{B} \geq 0$.

5. Denoting by $B_1(L^2(\mathbb{R}^n))$ the Banach space of trace class operators in $L^2(\mathbb{R}^n)$, and by $B_2(L^2(\mathbb{R}^n))^+$ the positive trace class operators, we have that

$$LQ = \mathcal{D} B_1 \left(L^2(\mathbb{R}^n)\right), \ \ Q = \mathcal{D} B_1 \left(L^2(\mathbb{R}^n)\right)^+. \ (50)$$

Also note that, for every $\hat{\rho} \in B_1(L^2(\mathbb{R}^n))^+$, $\|\mathcal{D}_\varrho\|_{\infty} = (\mathcal{D}_\varrho)(0) = \text{tr}(\varrho)$.

6. Therefore, the quantization map $\mathcal{D} = D^*$ transforms bijectively a function of quantum positive type into a positive Hilbert–Schmidt operator, a continuous function of quantum positive type into a positive trace class operator and a normalized continuous function of quantum positive type into a density operator. By relation (48), a function of quantum positive type $Q$ is selfadjoint: $Q = Q^*$.

7. Denoting by $\| \cdot \|_2$ the norm of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, with the normalization of the Haar measure fixed above, for every density operator $\hat{\rho}$ in $L^2(\mathbb{R}^n)$ we have that

$$\|\mathcal{D}_\varrho\|_2 = \sqrt{\text{tr}(\varrho^2)} = \sqrt{(\mathcal{D}_\varrho) \mathcal{D}_\varrho(0)} \leq 1; \ (51)$$

i.e., as anticipated in section 2.2 (see (29)), the norm of a normalized continuous function of quantum positive type, regarded as a functional in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, corresponds to the square root of the purity of the associated state.

We are finally ready to unveil the true nature of a classical-quantum semigroup, see (35) and (36). To this aim, first observe that associated with the Weyl system $U$ we have a (strongly continuous) isometric representation $U \mapsto U$ of $\mathbb{R}^n \times \mathbb{R}^n$ acting in the Banach space $B_1(L^2(\mathbb{R}^n))$, i.e.,

$$U \mapsto U(q, p) : B_1 \left(L^2(\mathbb{R}^n)\right) \ni \hat{\rho} \mapsto U(q, p)\hat{\rho} \times U(q, p)^* \in B_1 \left(L^2(\mathbb{R}^n)\right), \ (52)$$

which is the standard symmetry action of $\mathbb{R}^n \times \mathbb{R}^n$ on trace class operators. Now, given a convolution semigroup $\{\mu_t\}_{t \in \mathbb{R}^+}$ of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$, a semigroup of operators $\{\mu_t(U \mapsto U)\}_{t \in \mathbb{R}^+}$ in $B_1(L^2(\mathbb{R}^n))$ is defined by setting

$$\mu_t(U \mapsto U) \hat{\rho} := \int_{\mathbb{R}^n \times \mathbb{R}^n} U \mapsto U(q, p)\hat{\rho} \ d\mu_t(q, p). \ (53)$$

The integral of a vector-valued function on the rhs of (53) can be understood as a Bochner integral [37]. It can be shown that the semigroup of operators $\{\mu_t[U \mapsto U]\}_{t \in \mathbb{R}^+}$, a so-called twirling semigroup, is actually a quantum dynamical semigroup (a completely positive, trace-preserving semigroup of operators in $B_1(L^2(\mathbb{R}^n))$); see [36–38, 64].

**Theorem 2.** Let $\{\xi_t\}_{t \in \mathbb{R}^+}$ be the multiplication semigroup of functions of positive type associated, via the symplectic Fourier-Stieltjes transform, with the convolution semigroup
\[ \{ \mu_t \}_{t \in \mathbb{R}^+}, \text{ i.e.,} \]
\[ \chi_t(q, p) = \int_{\mathbb{R}^d} e^{i(q\cdot p' - p\cdot q')} \, d\mu_t(q', p'), \quad (54) \]

and let \( \{ \mathcal{E}_t \}_{t \in \mathbb{R}^+} \) be the proper classical-quantum semigroup associated with \( \{ \chi_t \}_{t \in \mathbb{R}^+} \). Then, the quantization map \( \mathcal{J} \) intertwines the semigroup of operators \( \{ \mathcal{E}_t \}_{t \in \mathbb{R}^+} \) with the quantum dynamical semigroup \( \{ \mu_t[U \cup U] \}_{t \in \mathbb{R}^+} \); namely, for every \( Q \in \mathcal{L}_Q \)
\[ \mathcal{J}_t Q = \mu_t[U \cup U] \mathcal{J}_t Q, \quad t \geq 0. \quad (55) \]

**Proof.** The crucial observation is that \( \mathcal{J} = \mathcal{J}_t \) intertwines the representation \( U \cup U \) with the representation \( T \) of \( \mathbb{R}^d \times \mathbb{R}^d \) in \( \mathcal{L}_Q \), defined by
\[ (T(q, p)Q)(\tilde{q}, \tilde{p}) = e^{-i(q \cdot p - p \cdot q)} Q(q, p), \quad (56) \]

see [19]. Relation (55) then follows easily. \( \square \)

**Remark 11.** It is clear that, by theorem 1, for every normalized function of positive type \( \chi \in \mathcal{P} \)—the symplectic Fourier transform of a probability measure \( \mu \) on \( \mathbb{R}^d \times \mathbb{R}^d \)—the map \( \mathcal{E}_\chi : \mathcal{L}_Q \to \mathcal{L}_Q \), \( \mathcal{E}_\chi Q(q, p) = \chi(q, p) Q(q, p) \), defines a bounded operator, a ‘classical-quantum map’, such that \( \mathcal{E}_\chi \mathcal{Q}_t \subset \mathcal{Q}_t \) and \( \mathcal{E}_\chi \mathcal{Q}_t \subset \mathcal{Q}_t \), whether or not \( \chi \) is a member of a multiplication semigroup of functions of positive type. The quantization map \( \mathcal{J} \) intertwines \( \mathcal{E}_\chi \) with a quantum dynamical map (a quantum channel)—a so-called twirling (super-)operator [36]—\( \mu[U \cup U] \) in \( B_1(\mathcal{L}_Q^2) \), which is of the form (53) for \( \mu = \mu_t \).

We conclude observing that the isometric representation \( U \cup U \) in \( B_1(\mathcal{L}_Q^2) \) can be extended in a natural way to a unitary representation in \( B_2(\mathcal{L}_Q^2) \). This extension gives rise to a new semigroup of operators in the Hilbert space \( B_2(\mathcal{L}_Q^2) \), whose definition is completely analogous to (53). This semigroup of operators is unitarily equivalent, via the dequantization map, to the classical-quantum semigroup \( \{ \mathcal{E}_t \}_{t \in \mathbb{R}^+} \) in \( \mathcal{L}_Q^2(\mathbb{R}^d \times \mathbb{R}^d) \). We leave the details to the reader.

### 5. Conclusions and perspectives

The WWGM phase-space formulation unveils remarkable analogies of quantum mechanics with classical statistical mechanics. In this approach, as noted in section 2, the intrinsic characterization of phase-space functions corresponding to density operators is not only an interesting problem per se, but also a fundamental issue of self-consistency. It is worth mentioning, in this regard, that in addition to the previously cited classical papers [27–31]—where necessary and sufficient conditions for a phase-space function to be a quantum characteristic function (the Fourier transform of a Wigner function) are derived—it is possible to obtain criteria holding directly for Wigner functions as well; see the recent paper [65].

Here, following the path traced in [27–31], we have adopted a group-theoretical approach based on functions of positive type for the following reasons: first, this approach fully highlights the analogies, and a few noteworthy differences, between the classical and the quantum setting; second, it allows us to achieve a very simple characterization of a remarkable class of quantum dynamical semigroups, see (36), which is the main aim of the present contribution; third, it is suitable for generalizations associated with square integrable group representations. Let us briefly analyze these points.

Continuous functions of positive type on phase space play a central role both in classical statistical mechanics and in the WWGM formulation of quantum mechanics, due to their relation with probability measures and quasi-probability distributions, respectively. In the classical setting, they can be defined as positive functionals on the group algebra \( L(\mathbb{R}^d \times \mathbb{R}^d), \{\cdot, \cdot\} \), and continuity can be regarded as a byproduct of this definition. In the quantum setting, continuous functions of (quantum) positive type are embedded in the positive functionals on the twisted convolution \( \mathcal{H} \)-algebra \( \{L(\mathbb{R}^d \times \mathbb{R}^d), \{\cdot, \cdot\}, J\} \). This corresponds, via quantization, to the embedding of the positive trace class operators in the positive Hilbert–Schmidt operators. We have therefore found it natural to consider these functionals, implemented by square integrable functions, as a quantum analogue of functions of (classical) positive type. The functions representing quantum states are characterized in this framework precisely by the property of being continuous and by a suitable normalization condition, which in general differs from their normalization as functionals. Note that, following [21]—where (standard) functions of positive type are discussed in detail—we have made a precise distinction between functions of positive type and positive definite functions, both in the classical and in the quantum setting. The two notions coincide if the functions involved are continuous.

Analyzing the fundamental properties of functions of classical and quantum positive type it is also natural to consider a class of semigroup of operators, the classical-quantum semigroups. We have recently introduced these semigroups of operators in [33], with the aim of highlighting their relations with other classes of semigroups of operators and the role that they play in quantum information science. As shown in section 4, by quantizing a classical-quantum semigroup one obtains a quantum dynamical semigroup belonging to the class of twirling semigroups. Every twirling semigroup is generated by a pair formed by a projective representation \( U \) of a locally compact group and by a convolution semigroup \( \{\mu_t\}_{t \in \mathbb{R}^+} \) of probability measures on that group [36–38]. In the case where \( U \) is a Weyl system, and \( \{\mu_t\}_{t \in \mathbb{R}^+} \) is a Gaussian convolution semigroup [36, 38, 59], the associated twirling semigroup consists of certain quantum dynamical maps called classical-noise channels in the context of quantum information, see [66] and references therein. Therefore, borrowing this terminology we can conclude that, under a suitable assumption, by quantizing a classical-quantum semigroup one obtains a classical-noise semigroup. In this regard, even though this should be clear from the discussion in section 4, it
is worth stressing that by ‘quantization’ here we do not mean passing from a classical to a quantum setting, but switching from a phase-space to the standard Hilbert space formulation of quantum mechanics.

We believe that the group-theoretical approach undertaken in this paper may pave the way to extend the results obtained for the group of translations on phase space to other groups. First, one may consider a suitable generalization of the notion of function of quantum positive type based on group-theoretical star products [19]. This notion would provide a natural characterization for those functions representing quantum states in the general quantization–dequantization scheme outlined in section 4, which relies on square integrable representations. Having such a tool at one’s disposal, it would be then natural to consider a generalization of the notion of classical-quantum semigroup (or map). The straightforward case is a group of the form $\mathbb{A} \times \hat{\mathbb{A}}$, where $\mathbb{A}$ is a locally compact abelian group and $\hat{\mathbb{A}}$ its dual. Phase spaces of this kind—e.g., related with cyclic groups or finite fields—and the associated phase-space representations of quantum states (with applications to quantum information science, quantum-state tomography, teleportation etc) are of current interest; see [67–73] and references therein.

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