INFINITE LADDERS INDUCED BY PREPROJECTIVE ALGEBRAS

NAN GAO AND CHRYSOSTOMOS PSAROUDAKIS

Abstract. In this paper we characterize when a recollement of compactly generated triangulated categories admits a ladder of some height going either upwards or downwards. As an application, we show that the derived category of the preprojective algebra of Dynkin type $A_n$ admits a periodic infinite ladder, where the one outer term in the recollement is the derived category of a differential graded algebra.

CONTENTS

1. Introduction 1
2. Recollements and adjoint functors 2
3. Compactly generated triangulated categories and ladders 5
4. Infinite ladders and preprojective algebras 12
References 18

1. Introduction

Ladders of triangulated categories were introduced by Beilinson, Ginzburg and Schechtman [5] in their attempt to formalize the equivalence between derived categories of graded modules over symmetric and exterior algebras, known now as Koszul duality. A ladder is a recollement of triangulated categories $R_{tr}(U, T, V)$ together with a (possible infinite) sequence of triangle functors going upwards or downwards such that any three consecutive rows form a recollement of triangulated categories. Recall that a recollement $R_{tr}(U, T, V)$ is a short exact sequence of triangulated categories $0 \rightarrow U \rightarrow T \rightarrow V \rightarrow 0$ such that the inclusion functor $i: U \rightarrow T$ as well as the quotient functor $e: T \rightarrow V$ have a left and a right adjoint functor. Recollements at the level of triangulated categories were introduced by Beilinson, Bernstein and Deligne in their fundamental work on perverse sheaves [4]. In our days, recollements of triangulated categories play an important role in various contexts since the main idea is to encode information for $T$ from the possibly simpler outer terms $U$ and $V$.

In the context of representation theory of finite dimensional algebras, ladders of derived categories have recently attracted a lot of attention due to their connection with several homological and $K$-theoretic invariants, see [1,9,17,18,22]. Another important aspect of ladders of derived categories of rings is the relation to the problem of lifting or restricting recollements to different levels of derived categories, see [1, Theorem III]. Moreover, by [1, Section 5] the height of the ladder measures how far is an algebra from being derived simple [35], i.e. its derived category is not the middle term of a non-trivial recollement with outer terms derived categories of algebras. For example, indecomposable symmetric and selfinjective algebras are derived simple at the level of bounded derived categories, see [26] and [9] respectively. A standard example of ladders of recollements arises from the derived category of a triangular matrix ring [1, Example 3.4]. Moreover, in case that the underlying rings are Gorenstein algebras, it was proved in [36] that the derived category of certain triangular matrix algebras admits an infinite ladder.

In light of the above applications of a ladder, it is natural to search for more examples. The main aim of this paper is to present a new example of a recollement situation which admits an infinite ladder. More precisely, consider the preprojective algebra $\Pi_n(Q)$ in the sense of Gelfand and Ponomarev [15], where $Q$
is a quiver of Dynkin type $A_n$. For a finite dimensional algebra $\Lambda$ over a field $k$ denote by $\Pi_n(\Lambda, Q)$ the algebra $\Lambda \otimes_k \Pi_n(Q)$. The main result of this paper is the following.

**Main Theorem.** The derived category of $\Pi_n(\Lambda, Q)$ admits an infinite ladder $L_{\Pi}(D(\Gamma), D(\Pi_n(\Lambda, Q)), D(\Lambda))$ of period four, where $\Gamma$ is a dg algebra, and this ladder restricts to bounded as well as to perfect complexes.

The key ingredients of the proof are: (a) a characterization of when a recollement of compactly generated triangulated categories admits a ladder of some height, and (b) a careful analysis of the recollement situation of $\Pi_n(\Lambda, Q)$ at the level of module categories.

The Main Theorem also shows that the preprojective algebra $\Pi_n(Q)$ admits a periodic infinite ladder. However, since $\Pi_n(Q)$ is a finite dimensional selfinjective algebra it follows from [10] that the bounded derived category of $\Pi_n(Q)$ is derived simple. Hence, this fact combined with our main result shows that we cannot expect to obtain a better recollement situation for $\Pi_n(Q)$ than the one we have in our Main Theorem. Another interesting aspect is that the periodicity of $L_{\Pi}(D(\Gamma), D(\Pi_n(\Lambda, Q)), D(\Lambda))$ is not a consequence of the existence of a Serre functor (compared to [1]).

The structure of the paper is as follows. In Section 2 we collect preliminary notions and results on recollements of triangulated categories and existence of adjoint functors. In Section 3 we provide necessary and sufficient conditions for a recollement of compactly generated triangulated categories to admit a ladder of some height going either upwards or downwards. This characterization is proved in Theorem 3.6 and Theorem 3.7. Note that the results of this section generalize, and are inspired by, related results of Angeleri H"ulgen-K"onig-Liu-Yang [1], formulated in the setting of derived categories of module categories over finite dimensional algebras. In Section 4 we prove the main theorem of this paper. This section is divided into two subsections. In the first subsection we provide sufficient conditions for a recollement of derived categories of dg algebras to restrict to a recollement between the full subcategories of dg modules with finite dimensional total cohomology. This is proved in Proposition 4.3 and is used in the proof of the Main Theorem. In the second subsection, after recalling the description of the module category of $\Pi_n(\Lambda, Q)$, we show in Proposition 4.4 that there is a recollement of module categories where the left term is the module category of $\Pi_{n-1}(\Lambda, Q)$ and the right term is the module category of $\Lambda$. In Theorem 4.5 we prove the main result of this paper as stated above. We close this section with a remark about the the ladder that we obtain for $\Pi_n(\Lambda, Q)$ in connection to the Nakayama functor and we also discuss the relation of $\Pi_2(\Lambda, Q)$ with certain Morita rings.

## 2. Recollements and adjoint functors

We start this section with the notion of a recollement due to Beilinson-Bernstein-Deligne [4].

**Definition 2.1.** A recollement of triangulated categories, denoted by $R_{\Pi}(U, T, V)$, is a diagram

\[
\begin{array}{ccc}
\Xi & \xrightarrow{i} & \Theta\\
\downarrow q & & \downarrow e\\
\Pi & \xrightarrow{i} & \Gamma
\end{array}
\]

of triangulated categories and triangle functors satisfying the following conditions:

1. $(q, i, p)$ and $(l, e, r)$ are adjoint triples.
2. The functors $i$, $l$, and $r$ are fully faithful.
3. $\text{Im } i = \text{Ker } e$.

We write $\text{Ker } e$ for the kernel of the functor $e$ and $\text{Im } i$ for the essential image of $i$. Also, we usually write $i(U)$, $l(V)$ and $r(V)$ for the essential image of the functor $i$, $l$ and $r$ respectively. Note that Definition 2.1 comes with a pair of triangles arising from the units and counits of the adjoint pairs. In particular, it is easy to see that for any object $X$ in $\Theta$ we have triangles $ip(X) \to X \to re(X) \to ip(X)[1]$ and $le(X) \to X \to iq(X) \to le(X)[1]$ in $\Theta$.

It is well known that recollements of triangulated categories correspond bijectively to torsion, torsion-free triples, TTF-triples for short, in triangulated categories (see [4, Section 1.4.4], [6, Remark 2.14] and [32, Section 4.2]). We explain how this bijection works since it is used later.

Let $(X, Y, Z)$ be a TTF-triple in a triangulated $\mathcal{T}$, i.e. $(X, Y)$ and $(Y, Z)$ are torsion pairs ([6, Definition 2.1, Chapter I]) and $X$, $Y$ and $Z$ are triangulated categories. Note that the latter notion is also known as stable t-structures, see [29]. Then we have the adjoint pairs $(i_X, R_X)$, $(L_Y, i_Y)$, $(i_Y, R_Y)$ and $(L_Z, i_Z)$, as
indicated in the following diagrams:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{i_X} & \mathcal{Y} \\
\downarrow & & \downarrow \\
R_X & \xrightarrow{i_Y} & R_Y
\end{array}
\quad (1)
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{i_Y} & \mathcal{Z} \\
\downarrow & & \downarrow \\
R_Y & \xrightarrow{i_Z} & R_Z
\end{array}
\quad (2)
\]

where \(i_X\), \(i_Y\) and \(i_Z\) are the inclusion functors. Associated with the TTF-triple \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) there is a recollement of triangulated categories as follows:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{i_X} & \mathcal{Y} \\
\downarrow & & \downarrow \\
L_Y & \xrightarrow{i_Y} & L_Z
\end{array}
\quad (2.2)
\]

Clearly \(\text{Im} i_Y = \text{Ker} R_X\) and the functors \(i_X, i_Y\) are fully faithful. Thus, it remains to prove that \(i_Z L_Z i_X\) is the right adjoint of \(R_X\), i.e. there is a natural isomorphism \(\text{Hom}_X(R_X(T), X) \cong \text{Hom}_Y(T, i_Z L_Z i_X(X))\) for every \(T \in \mathcal{Y}\) and \(X \in \mathcal{X}\). Note that from the adjoint triple \((i_X, R_X, i_Z L_Z i_X)\) it follows that the functor \(i_Z L_Z i_X\) is also fully faithful. First, we have \(\text{Hom}_X(R_X(T), X) \cong \text{Hom}_Y(i_Z L_Z i_X(T), i_Z i_X(X))\). From the diagram (1) we have the triangle \(i_X R_X(T) \rightarrow T \rightarrow i_Y L_Y(T) \rightarrow i_X R_X(T)\).[1]. Applying the functor \(\text{Hom}_Y(\_ , i_Z L_Z i_X(X))\) and using that \(\text{Hom}_Y(i_Y(Y), i_Z i_X(Z)) = 0\) we get the isomorphism

\[
\text{Hom}_Y(T, i_Z L_Z i_X(X)) \cong \text{Hom}_Y(i_Z R_X(T), i_Z L_Z i_X(X))
\]

Next, from diagram (2) we have the triangle \(i_Y R_Y i_X(X) \rightarrow i_X(X) \rightarrow i_Z L_Z i_X(X) \rightarrow i_Y R_Y i_X(X)\).[1]. Applying the functor \(\text{Hom}_X(i_Z R_X(T), \_ )\) and using that \(\text{Hom}_X(i_X(X), i_Y(Y)) = 0\) we get the isomorphism

\[
\text{Hom}_X(i_Z R_X(T), i_X(X)) \cong \text{Hom}_X(i_Z R_X(T), i_Z L_Z i_X(X)).
\]

The above isomorphisms show that \((R_X, i_Z L_Z i_X)\) is a recollement of triangulated categories. Conversely, if \(R_{\sigma}(U, \mathcal{Y}, \mathcal{V})\) is a recollement of triangulated categories it is easy to check that the triple \((I(V), i(\mathcal{Y}), r(\mathcal{V}))\) is a TTF-triple in \(\mathcal{T}\).

The base of a recollement \(R_{\sigma}(U, \mathcal{Y}, \mathcal{V})\) is the short exact sequence of triangulated categories \(0 \rightarrow U \rightarrow \mathcal{Y} \rightarrow \mathcal{V} \rightarrow 0\). The following useful result shows that given a short exact sequence as above, it suffices to have adjoints (left and right) either for the inclusion functor \(i: U \rightarrow \mathcal{Y}\) or for the quotient functor \(e: \mathcal{Y} \rightarrow \mathcal{V}\) to obtain a recollement situation \(R_{\sigma}(U, \mathcal{Y}, \mathcal{V})\).

**Lemma 2.2.** ([11, Theorem 1.1], [12, Theorem 2.1]) Let \(0 \rightarrow U \xrightarrow{i} \mathcal{Y} \xrightarrow{e} \mathcal{V} \rightarrow 0\) be an exact sequence of triangulated categories. Then the following hold.

(i) The functor \(i: U \rightarrow \mathcal{Y}\) admits a left adjoint functor \(q: \mathcal{Y} \rightarrow U\) if and only if the functor \(e: \mathcal{Y} \rightarrow \mathcal{V}\) admits a left adjoint functor \(l: \mathcal{V} \rightarrow \mathcal{Y}\).

(ii) The functor \(i: U \rightarrow \mathcal{Y}\) admits a right adjoint functor \(p: \mathcal{Y} \rightarrow U\) if and only if the functor \(e: \mathcal{Y} \rightarrow \mathcal{V}\) admits a right adjoint functor \(r: \mathcal{V} \rightarrow \mathcal{Y}\).

In this case, the functor \(l\) (respectively, the functor \(r\)) is fully faithful.

Recall that a full subcategory \(\mathcal{X}\) of an additive category \(\mathcal{A}\) is called contravariantly finite if for any object \(A\) in \(\mathcal{A}\) there is a morphism \(f: X_A \rightarrow A\) in \(\mathcal{A}\) with \(X_A\) in \(\mathcal{X}\) such that the map \(\text{Hom}_{\mathcal{A}}(X_A, f): \text{Hom}_{\mathcal{A}}(X', X_A) \rightarrow \text{Hom}_{\mathcal{A}}(X', A)\) is surjective for every object \(X'\) in \(\mathcal{X}\). Dually we define a subcategory to be covariantly finite and if it is both covariantly and contravariantly finite then it is called factorially finite. Also, if \(\mathcal{X}\) is a class of objects in \(\mathcal{A}\), then we denote by \(\overleftarrow{\mathcal{X}} = \{ A \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(\mathcal{X}, A) = 0 \}\) the right orthogonal subcategory of \(\mathcal{X}\) and by \(\overrightarrow{\mathcal{X}} = \{ A \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(A, \mathcal{X}) = 0 \}\) the left orthogonal subcategory of \(\mathcal{X}\).

Due to our discussion in Section 3 about ladders of recollements, it is useful to know when the inclusion functor of a triangulated subcategory \(U\) in a triangulated category \(\mathcal{T}\) admits a left or right adjoint. In the following result we provide necessary and sufficient conditions for the inclusion functor \(i: U \rightarrow \mathcal{T}\) to have a right adjoint. The dual statement for left adjoints is left to the reader. Note that this result is due to Keller-Vossieck [24], see also [6, Chapter I, Proposition 2.3], [7, Lemma 3.1] and [?, Proposition 1.4], but we provide a short proof.

**Lemma 2.3.** Let \(\mathcal{T}\) be a triangulated category with a triangulated subcategory \(U\). The following statements are equivalent.
(i) The inclusion functor \( i: \mathcal{U} \to \mathcal{T} \) has a right adjoint, i.e. there is a functor \( p: \mathcal{T} \to \mathcal{U} \) such that \((i, p)\) is an adjoint pair.

(ii) The subcategory \( \mathcal{U} \) is contravariantly finite in \( \mathcal{T} \) and for every object \( X \) in \( \mathcal{T} \) there is a triangle \( U_X \to X \to U'_X \to U_X[1] \) in \( \mathcal{T} \) such that the map \( U_X \to X \) is a right \( \mathcal{U} \)-approximation of \( X \) in \( \mathcal{T} \) and \( U'_X \) lies in \( \mathcal{U}^\perp \).

(iii) \((\mathcal{U}, \mathcal{U}^\perp)\) is a stable t-structure in \( \mathcal{T} \).

Proof. (i) \(\implies\) (ii): Let \( X \) be an object in \( \mathcal{T} \). Using the adjunction isomorphism of the pair \((i, p)\), it follows that the counit map \( p(X) = ip(X) \to X \) is a right \( \mathcal{U} \)-approximation of \( X \) in \( \mathcal{T} \). We denote by \( U_X \) the object \( p(X) \). Consider now the triangle \((\ast): U_X \to X \to U'_X \to U_X[1]\) in \( \mathcal{T} \) and let \( U' \) be an object in \( \mathcal{U} \). Applying the functor \( \text{Hom}_\mathcal{T}(U', \_\_\_\_) \) to \((\ast)\) we get the following long exact sequence:

\[
(U', U_X) \xrightarrow{\cong} (U', X) \xrightarrow{u} (U', U'_X) \xrightarrow{u} (U', U_X[1]) \xrightarrow{\cong} (U', X[1])
\]

Note that the above isomorphisms follow from the adjunction isomorphism of the adjoint pair \((i, p)\) together with \( i \) being the inclusion functor. This shows that \( \text{Hom}_\mathcal{T}(U', U'_X) = 0 \) and therefore \( U'_X \) lies in \( \mathcal{U}^\perp \).

(ii) \(\implies\) (i): By assumption there is a right \( \mathcal{U} \)-approximation \( U_X \to X \) for every \( X \) in \( \mathcal{T} \). We claim that the assignment \( X \mapsto (p(X) := U_X \text{ induces a functor } \mathcal{T} \to \mathcal{U} \text{ which is a right adjoint of the inclusion functor } i \). We first show that the above assignment gives a well defined functor.

Let \( g: Y \to X \) be a morphism in \( \mathcal{T} \) and consider a right \( \mathcal{U} \)-approximation \( U_Y \to Y \) of \( Y \) in \( \mathcal{T} \) such that \( U'_Y \) lies in \( \mathcal{U}^\perp \). Since \( \text{Hom}_\mathcal{T}(U_Y, U'_X) = 0 \) and \( \text{Hom}_\mathcal{T}(U_Y, U'_X[-1]) = 0 \), we obtain the following commutative diagram:

\[
\begin{array}{ccc}
U_Y & \xrightarrow{h} & U'_Y \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
U_X & \xrightarrow{\eta} & U_X[1] \\
\end{array}
\]

where the morphism \( h \) is unique. We now show that the right \( \mathcal{U} \)-approximation \( U_X \to X \) is the unique up to isomorphism right \( \mathcal{U} \)-approximation of \( X \) in \( \mathcal{T} \). Let \( V_X \to X \) be another right \( \mathcal{U} \)-approximation of \( X \) in \( \mathcal{T} \). Then we have the following commutative diagram and \( h' \circ h = \text{Id}_{U_X} \):

\[
\begin{array}{ccc}
U_X & \xrightarrow{h} & V_X \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
X & \xrightarrow{\text{id}} & X \\
\end{array}
\]

We infer that the map \( h \) is an isomorphism.

Finally, let \( U' \) be an object in \( \mathcal{U} \). Applying the functor \( \text{Hom}_\mathcal{T}(U', \_\_\_\_) \) to the given triangle \( U_X \to X \to U'_X \to U_X[1] \), we get the isomorphism \( \text{Hom}_\mathcal{U}(U', U_X) \cong \text{Hom}_\mathcal{T}(U', X) \). This means that \((i, p)\) is an adjoint pair.

(ii) \(\iff\) (iii): This implication (ii) \(\implies\) (iii) is clear. Conversely, assume that \((\mathcal{U}, \mathcal{U}^\perp)\) is a stable t-structure. Then for any object \( X \in \mathcal{T} \) there is a triangle \( U_X \to X \to U'_X \to U_X[1] \) with \( U_X \in \mathcal{U} \) and \( U'_X \in \mathcal{U}^\perp \). Let \( U \) be an object of \( \mathcal{U} \). Applying the functor \( \text{Hom}_\mathcal{T}(U, \_\_\_\_) \) to this triangle, we get that the induced morphism \( \text{Hom}_\mathcal{T}(U, U_X) \to \text{Hom}_\mathcal{T}(U, X) \) is an isomorphism. Clearly, the map \( U_X \to X \) is a right \( \mathcal{U} \)-approximation.

\(\square\)

The next result shows that in some cases we can lift adjoint functors from abelian categories to derived categories. Its proof is standard, see for instance [28]. This is used in the proof of the Main Theorem in Section 4.

**Lemma 2.4.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories such that \( D(\mathcal{A}) \) and \( D(\mathcal{B}) \) exists. Assume that there is an adjoint pair of exact functors \((F, G)\), i.e. \( F: \mathcal{A} \to \mathcal{B} \) and \( G: \mathcal{B} \to \mathcal{A} \), between \( \mathcal{A} \) and \( \mathcal{B} \). Then there is an adjoint pair \((F', G')\) between the unbounded derived categories of \( \mathcal{A} \) and \( \mathcal{B} \) which restricts also to the bounded derived categories. In particular, if \( G: \mathcal{B} \to \mathcal{A} \) is fully faithful, then the induced functor \( G: D(\mathcal{B}) \to D(\mathcal{A}) \) is fully faithful.
3. COMPACTLY GENERATED TRIANGULATED CATEGORIES AND LADDERS

Our aim in this section is to characterize when a recollement of compactly generated triangulated categories admits a ladder of some height. We first recall the notion of a ladder due to Beilinson-Ginzburg-Schechtman \cite[Section 1.5]{[5]}, see also \cite[Section 3]{[1]}.

**Definition 3.1.** A ladder $L_{tr}(U, T, V)$ is a finite or infinite diagram of triangulated categories and triangle functors:

\[
\begin{array}{cccc}
& q_1 & & \\
& & l & \\
& & i & e \downarrow \\
& & r & \\
& p_1 & & \\
& & l & \\
& & i & \\
& & q & \\
& & & \\
\end{array}
\]

such that any three consecutive rows form a recollement of triangulated categories. Multiple occurrence of the same recollement is allowed. The height of the ladder $L_{tr}(U, T, V)$ is the number of recollements contained in it (counted with multiplicities). A ladder is periodic, if there exists a positive integer $n$ such that the $n$-th recollement going upwards (respectively, going downwards) in $L_{tr}(U, T, V)$ is equivalent to the recollement $R_{tr}(U, T, V)$ which is considered to be a ladder of height one. The minimal such positive integer $n$ is the period of the ladder.

We now describe a method to built a ladder of triangulated categories. Let $R_{tr}(U, T, V)$ be a recollement as in (2.1). Since the functors $l$ and $r$ are fully faithful, a first natural step is to consider when the inclusion functor $l(V) \rightarrow T$ has a left adjoint or when the inclusion functor $r(V) \rightarrow T$ has a right adjoint. If this is the case, then pictorially we get the following diagram where $(l^1, l)$ and $(r, r^1)$ are adjoint pairs:

\[
\begin{array}{cccc}
& q & & \\
& i & l & \\
& & e & \\
& & i & p \downarrow \\
& & r & \\
& & e & \\
& p & & \\
& & l & \\
& & i & \\
& & q & \\
& & & \\
\end{array}
\]

The second step is to analyze the right part of the above diagram. In particular, we have the adjoint triples $(l^1, l, e)$ and $(e, r, r^1)$ where $l$ and $r$ are fully faithful. Then we obtain the next two recollements:

\[
\begin{array}{cccc}
& q & & \\
& i & l & \\
& & e & \\
& & i & p \downarrow \\
& & r & \\
& & e & \\
& p & & \\
& & l & \\
& & i & \\
& & q & \\
& & & \\
\end{array}
\]

Summarizing so far we have the following ladder of the recollement $R_{tr}(U, T, V)$:

\[
\begin{array}{cccc}
& q & & \\
& i & l & \\
& & e & \\
& & i & p \downarrow \\
& & r & \\
& & e & \\
& p & & \\
& & l & \\
& & i & \\
& & q & \\
& & & \\
\end{array}
\]

From the adjoint triples $(q^1, q, i)$, $(i, p, p^1)$ and since the functor $i$ is fully faithful we get that the functors $q^1$ and $p^1$ are fully faithful. Then, if the inclusion functor $q^1(U) \rightarrow T$ has a left adjoint, say $q^2$, or the
inclusion functor $p^1(U) \to \mathcal{T}$ has a right adjoint, say $p^2$, we can continue similarly as above and obtain the following ladder of the recollement $R_r(U, \mathcal{T}, V)$:

![Diagram](image)

The functor $l$ being fully faithful implies that $l^2$ is fully faithful and similarly for the functor $r^2$. Then we continue as above by examining if the inclusion functor $l^2(V) \to \mathcal{T}$, respectively $r^2(V) \to \mathcal{T}$, has a left adjoint, respectively a right adjoint. It should be noted that the key results of this construction are Lemma 2.2 and Lemma 2.3.

In this section the triangulated categories involved in a recollement situation are assumed to be compactly generated. Let $\mathcal{T}$ be a triangulated category with small coproducts. An object $X$ in $\mathcal{T}$ is called compact if the functor $\text{Hom}(X, -) : \mathcal{T} \to \mathcal{Ab}$ preserves coproducts. The compact objects in $\mathcal{T}$ form a thick triangulated subcategory which we denote by $\mathcal{T}^c$. Then $\mathcal{T}$ is compactly generated if $\mathcal{T}^c$ is skeletally small and the vanishing $\text{Hom}(X, Y) = 0$ for all $X \in \mathcal{T}^c$ implies that $Y = 0$. In other words, $\mathcal{T}$ is compactly generated if $\mathcal{T}$ admits a set of compact generators.

In the sequel we need the following useful results which are consequences of Brown representability for compactly generated triangulated categories.

**Lemma 3.2.** ([30, Theorem 4.1, Theorem 5.1]) Let $F : S \to \mathcal{T}$ be a triangle functor between compactly generated triangulated categories. Assume that the functor $F$ has a right adjoint $G : \mathcal{T} \to S$. Then the following are equivalent:

(i) The functor $G$ preserves coproducts.

(ii) There is an adjoint triple $(F, G, H)$.

(iii) The functor $F$ preserves compact objects.

**Lemma 3.3.** ([30, Lemma 3.2]) Let $\mathcal{T}$ be a compactly generated triangulated category. Let $S$ be a full coproduct-closed triangulated subcategory containing a set of compact generators of $\mathcal{T}$. Then $S = \mathcal{T}$.

Let $R_r(U, \mathcal{T}, V)$ be a recollement of triangulated categories. We call $R_r(U, \mathcal{T}, V)$ a recollement of compactly generated triangulated categories if the triangulated categories $U$, $\mathcal{T}$ and $V$ are compactly generated.

Given a recollement $R_r(U, \mathcal{T}, V)$ and triangulated subcategories $X$, $Y$ and $Z$ in $U$, $\mathcal{T}$ and $V$ respectively, we say that $R_r(U, \mathcal{T}, V)$ restricts to an upper recollement $R_r(X, Y, Z)$ if there is a half recollement:

![Diagram](image)

This means precisely that $0 \to X \to Y \to Z \to 0$ is an exact sequence of triangulated categories which admits left adjoints. Note that the above functors are exactly the triangle functors of $R_r(U, \mathcal{T}, V)$.

We also say that $R_r(U, \mathcal{T}, V)$ induces an upper recollement $R_r(Z, Y, X)$ if there is a half recollement:

![Diagram](image)

for some triangle functors $l^1$ and $q^1$.

We are now ready to characterize when a recollement of compactly generated triangulated categories admits a ladder of height two going downwards. This result generalizes [1, Proposition 3.2, Lemma 4.3].

**Proposition 3.4.** Let $R_r(U, \mathcal{T}, V)$ be a recollement of compactly generated triangulated categories. The following are equivalent.

(i) There is a ladder of height two going downwards.
(ii) There is a functor \( p^1 : \mathcal{U} \to \mathcal{I} \) such that \( (p, p^1) \) is an adjoint pair. In particular, the subcategory \( p^1(\mathcal{U}) \) is covariantly finite in \( \mathcal{I} \) and the pair \( (\perp, p^1(\mathcal{U}), p^1(\mathcal{U})) \) is a stable t-structure in \( \mathcal{I} \).

(iii) There is a functor \( r^1 : \mathcal{I} \to \mathcal{V} \) such that \( (r, r^1) \) is an adjoint pair.

(iv) The functor \( i : \mathcal{U} \to \mathcal{I} \) preserves compact objects.

(v) The functor \( i \circ e : \mathcal{I} \to \mathcal{I} \) preserves compact objects.

(vi) The functor \( e : \mathcal{I} \to \mathcal{V} \) preserves compact objects.

(vii) The subcategory \( r(\mathcal{V}) \) is contravariantly finite in \( \mathcal{I} \) and the pair \( (r(\mathcal{V}), r(\mathcal{V}) \perp) \) is a stable t-structure in \( \mathcal{I} \). In particular, \( r(\mathcal{V}) \) is a functorially finite subcategory in \( \mathcal{I} \).

(viii) The recollement \( R_\mathcal{U}(\mathcal{U}, \mathcal{I}, \mathcal{V}) \) restricts to an upper recollement \( R_\mathcal{V}(\mathcal{U}^c, \mathcal{I}^c, \mathcal{V}^c) \). In particular, \( i(\mathcal{U}^c) \) is covariantly finite in \( \mathcal{I}^c \) and the pair \( (\perp(i(\mathcal{U}^c)), i(\mathcal{U}^c)) \) is a stable t-structure in \( \mathcal{I}^c \).

**Proof.** The implication (i) \( \implies \) (ii) follows from Definition 3.1. The second part of statement (ii) follows from the dual of Lemma 2.3. The equivalence (ii) \( \iff \) (iii) follows from Lemma 2.2.

(iii) \( \implies \) (i): Since we have the adjoint triple \( (e, r, r^1) \) and \( r \) is fully faithful, we have the recollement:

This implies that we have the following ladder for \( R_\mathcal{U}(\mathcal{U}, \mathcal{I}, \mathcal{V}) \) of height two going downwards:

(ii) \( \implies \) (iv): By assumption we have the adjoint triple \( (i, p, p^1) \). Then from Lemma 3.2 the functor \( i : \mathcal{U} \to \mathcal{I} \) preserves compact objects.

(iv) \( \implies \) (v): Let \( X \) be an object in \( \mathcal{I}^c \). From Lemma 3.2 the functor \( q \) preserves compact objects. Thus, by assumption the object \( i q(X) \) belongs to \( \mathcal{I}^c \) and then from the canonical triangle

\[
\text{le}(X) \to X \to \text{iq}(X) \to \text{le}(X)[1]
\]

we obtain that the object \( \text{le}(X) \) lies in \( \mathcal{I}^c \).

(v) \( \implies \) (vi): Let \( X \) be a compact object in \( \mathcal{I} \) and \( \{V_i \mid i \in I\} \) a set of objects in \( \mathcal{V} \). Then the commutativity of the following diagram:

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{V}(\text{le}(X), \bigsqcup_{i \in I} V_i) & \xrightarrow{\text{t.f.l.}} & \bigsqcup_{i \in I} \text{Hom}_\mathcal{V}(\text{le}(X), V_i) \\
\text{Hom}_\mathcal{I}(\text{le}(X), \bigsqcup_{i \in I} V_i)) & \cong & \text{Hom}_\mathcal{I}(\text{le}(X), \bigsqcup_{i \in I} l(V_i)) \\
\end{array}
\]

shows that the object \( e(X) \) is compact in \( \mathcal{V} \).

(vi) \( \implies \) (iii): This implication follows from Lemma 3.2.

(iii) \( \iff \) (vii): Since the functor \( r \) has a right adjoint \( r^1 \) if and only if the inclusion functor \( r(\mathcal{V}) \to \mathcal{I} \) has a right adjoint \( \mathcal{I} \to r(\mathcal{V}) \), the result follows from Lemma 2.3. From the dual of this lemma, the subcategory \( r(\mathcal{V}) \) is always covariantly finite in \( \mathcal{I} \) since the inclusion functor \( r(\mathcal{V}) \to \mathcal{I} \) has the quotient functor \( e \) as a left adjoint. We infer that the subcategory \( r(\mathcal{V}) \) is functorially finite in \( \mathcal{I} \).

So far we have proved that the first seven conditions are equivalent. We now show that if one of these holds then (viii) holds. Since we have the adjoint triples \( (l, e, r) \) and \( (e, r, r^1) \), it follows from Lemma 3.2 that the triangle functors \( l \) and \( e \) preserve compact objects. Moreover, the functor \( l : \mathcal{V}^c \to \mathcal{I}^c \) is fully faithful since the functor \( l : \mathcal{V} \to \mathcal{I} \) is fully faithful. This implies that the functor \( e : \mathcal{I}^c \to \mathcal{V}^c \) is essentially surjective since we have the adjoint pair \( (l, e) \). In particular, by Lemma 2.2 we have the following diagram:

\[
\begin{array}{ccc}
\text{Ker } e & \xrightarrow{\text{inc}} & \mathcal{I}^c \\
\text{inc} & & i \\
\mathcal{I} & & e \\
\mathcal{V}^c & & \\
\end{array}
\]
which is a half recollement. This means that \( 0 \rightarrow \text{Ker } e \rightarrow T^c \rightarrow V^c \rightarrow 0 \) is an exact sequence of triangulated categories, i.e. \( T^c/\text{Ker } e \cong V^c \), and we have the adjoint pairs \( (q^1, \text{inc}) \) and \( (l, e) \). It suffices to show that \( \text{Ker } e \) is triangle equivalent to \( T^c \). From Lemma 3.2 the triangle functor \( i: U \rightarrow T \) preserves compact objects since we have the adjoint triple \( (i, p, p^1) \). Thus, we get a triangle functor \( i: U^c \rightarrow T^c \) which is fully faithful since the functor \( i: U \rightarrow T \) is fully faithful. Moreover, the composition of functors \( U^c \rightarrow T^c \rightarrow V^c \) is zero since the composition \( e \circ i: U \rightarrow V \) is zero, i.e. \( i(U^c) \subseteq \text{Ker } e \). Let \( Y \) be an object in \( T^c \) such that \( e(Y) = 0 \). Then there is an object \( U \) in \( U \) such that \( i(U) = Y \). We claim that \( U \) lies in \( U^c \). Let \( \{ U_i \mid i \in J \} \) be a set of objects in \( U \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}(U, \bigoplus_{i \in J} U_i) & \xrightarrow{\text{def}} & \bigoplus_{i \in J} \text{Hom}(U, U_i) \\
\text{Hom}(i(U), \bigoplus_{i \in J} U_i) & \xrightarrow{i \text{ left adjoint}} & \text{Hom}(i(U), \bigoplus_{i \in J} i(U_i)) \quad \text{def} & \bigoplus_{i \in J} \text{Hom}(i(U), i(U_i))
\end{array}
\]

and therefore the object \( U \) is compact in \( U \). Hence \( i(U^c) = \text{Ker } e \). Finally, the second part of statement (viii) follows from the dual of Lemma 2.3.

Conversely, assume that (viii) holds. Then from Lemma 3.2 and since the functor \( e \) preserves compact objects we obtain the adjoint triple \( (e, r, r^1) \). Thus (iii) holds.

We continue with the dual result of Proposition 3.4, that is, when a recollement of compactly generated triangulated categories admits a ladder of height two going upwards. Note that there is some difference between Proposition 3.4 and Proposition 3.5. In particular, compare the half recollements that we obtain at the level of compact objects.

**Proposition 3.5.** Let \( R(U, T, V) \) be a recollement of compactly generated triangulated categories. The following are equivalent.

(i) There is a ladder of height two going upwards.

(ii) There is a functor \( q^1: U \rightarrow T \) such that \( (q^1, q) \) is an adjoint pair. In particular, the functor \( q^1 \) preserves compact objects, the subcategory \( q^1(U) \) is contravariantly finite in \( T \) and the pair \( (q^1(U), q^1(U)^{-1}) \) is a stable \( t \)-structure in \( T \).

(iii) There is a functor \( l^1: T \rightarrow V \) such that \( (l^1, l) \) is an adjoint pair. In particular, the functor \( l^1 \) preserves compact objects.

(iv) The subcategory \( l(V) \) is covariantly finite in \( T \) and the pair \( (l^1(V), l(V)) \) is a stable \( t \)-structure in \( T \). In particular, \( l(V) \) is a functorially finite subcategory in \( T \).

(v) The recollement \( R(U, T, V) \) induces an upper recollement \( R(V^c, T^c, U^c) \).

**Proof.** The implication (i) \( \Rightarrow \) (ii) follows from Definition 3.1. The second part of statement (ii) follows from Lemma 3.2 since we have the adjoint triple \( (q^1, q, i) \), and the rest follows from Lemma 2.3. The equivalence (ii) \( \iff \) (iii) follows from Lemma 2.2.

(iii) \( \Rightarrow \) (i): Since we have the adjoint triple \( (l^1, l, e) \) and \( l \) is fully faithful, we obtain the recollement:

and therefore we have the following ladder for \( R(U, T, V) \) of height two going upwards:

The equivalence of (iii) and (iv) follows from the dual of Lemma 2.3. We show that if one of the four equivalent statements hold, then (v) holds. In particular, we prove that the following diagram

\[ (3.5) \]
is a half recollement. Since we have the adjoint triples \((I^1, l, e), (l, e, r), (q^1, q, i)\) and \((q, i, p)\), it follows from Lemma 3.2 that the triangle functors \(I^1, l, q^1\) and \(q\) preserve compact objects. Hence, we get the functors between the compact subcategories as indicated already in the above diagram. Clearly the functors \(l, q\) and \(q^1\) are fully faithful. It remains to show that \(\text{Im} l = \text{Ker} q\). This proof follows in the same way as the proof of the corresponding part in Proposition 3.4, the details are left to the reader.

Conversely, assume that (v) holds. From the recollement \(R_c(U, T, V)\), the functors \(l\) and \(q\) preserve compact objects. In particular, our assumption means that we have the recollement (3.5) and we want to show that the adjunctions \((I^1, l)\) and \((q^1, q)\) extend from the categories of compact objects to the whole triangulated categories. More precisely, we show that \((I^1, l)\) is an adjoint pair between \(V\) and \(T\), i.e. statement (iii) holds. Let \(A\) be an object in \(\mathcal{T}\) and consider the full subcategory of \(V\):

\[
\mathcal{A}M = \{ Y \in V \mid f_{AX}: \text{Hom}_V(I^1(A), Y[k]), \forall k \in \mathbb{Z} \}
\]

It is easy to check that \(\mathcal{A}M\) is a triangulated subcategory of \(V\) and by assumption \(V^c\) is contained in \(\mathcal{A}M\). Let \((Y_i)_{i \in I}\) be a family of objects in \(\mathcal{A}M\). From the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_V(I^1(A), Y_i) & \xrightarrow{\begin{smallmatrix} f_{AX} \end{smallmatrix}} & \text{Hom}_A(I_1 Y_i) \\
\begin{smallmatrix} I_{i \in I} \text{Hom}(I^1(A), Y_i) \end{smallmatrix} & \xrightarrow{\begin{smallmatrix} \cong \end{smallmatrix}} & \begin{smallmatrix} \text{Hom}_A(I_1 (Y_i)) \end{smallmatrix} \\
\end{array}
\]

it follows that the map \(f_{AX}: \text{Hom}_V(I^1(A), Y_i)\) is an isomorphism and therefore the triangulated subcategory \(\mathcal{A}M\) is closed under coproducts. Then, from Lemma 3.3 we obtain that \(\ast\): \(\mathcal{A}M = V\) for every compact object \(A\) in \(T\). This means that the map \(f_{AX}\) is an isomorphism for every \(A\) in \(T\) and \(Y\) in \(V\). On the other hand, for an arbitrary but fixed \(Y\) in \(V\) consider the following subcategory of \(T\):

\[
\mathcal{M}_Y = \{ X \in T \mid f_{XY}: \text{Hom}_V(I^1(X[k]), Y) \xrightarrow{\cong} \text{Hom}_T(X[k], Y[k]), \forall k \in \mathbb{Z} \}
\]

It follows easily as above that \(\mathcal{M}_Y\) is a coproduct-closed triangulated subcategory of \(T\) and by the relation \((\ast)\) we deduce that \(T^c \subseteq \mathcal{M}_Y\). Then, Lemma 3.3 implies that \(\mathcal{M}_Y = T\) for all \(Y\) in \(V\). We infer that \((I^1, l)\) forms an adjoint pair between the triangulated categories \(V\) and \(T\).

For a ladder of a recollement \(R_c(U, T, V)\), as in diagram (3.1), we fix the following notation: \(p := p^0, p^{-1} := i, p^{-2} := q, r := r^0, r^{-1} := e\), and similarly for the functors going upwards. Note that height one means the recollement \(R_c(U, T, V)\) that we start with. We are now ready to characterize when a recollement of compactly generated triangulated categories admits a ladder of height \(n\) going downwards. Parts of the following result generalizes [1, Theorem 4.4]. For simplicity in the presentation of the next result we fix a positive odd integer \(n \geq 3\). The case that \(n\) is even is treated in a similar way.

**Theorem 3.6.** Let \(R_c(U, T, V)\) be a recollement of compactly generated triangulated categories and \(n \geq 3\) a positive odd integer. The following statements are equivalent.

(i) The recollement \(R_c(U, T, V)\) admits a ladder of height \(n\) going downwards.

(ii) There are sequences of functors \(i, p^1, p^3, \ldots, p^{n-3}; U \rightarrow T\) and \(p, p^2, \ldots, p^{n-1}; T \rightarrow U\) which preserve compact objects and the pairs \((p, p^1), \ldots, (p^{n-2}, p^{n-1})\) are adjoint pairs.

(iii) There are sequence of functors \(r, r^1, r^3, \ldots, r^{n-3}; T \rightarrow V\) and \(r, r^2, r^4, \ldots, r^{n-3}; V \rightarrow T\) which preserve compact objects and the pairs \((r, r^1), \ldots, (r^{n-2}, r^{n-1})\) are adjoint pairs.

(iv) The recollement \(R_c(U, T, V)\) restricts to a recollement \(R_c(U^c, T^c, V^c)\) which admits a ladder of height \(n-2\) going downwards.

(v) There is a sequence of triangulated subcategories \(l(V), i(U), r(V), p^1(U), r^2(U), \ldots, r^{n-3}(V), p^{n-2}(U), r^{n-1}(V)\) in \(T\) such that

\[
((l(V), i(U), r(V)), (i(U), r(V), p^1(U)), \ldots, (r^{n-3}(V), p^{n-2}(U), r^{n-1}(V))
\]

are TTF-triples in \(T\).

(vi) There is a sequence of TTF-triples in \(T^c\) of the form: \((l(V) \cap T^c, i(U) \cap T^c, r(V) \cap T^c), (i(U) \cap T^c, r(V) \cap T^c, p^1(U) \cap T^c), \ldots, (r^{n-3}(V) \cap T^c, p^{n-2}(U) \cap T^c, r^{n-1}(V) \cap T^c)\).

**Proof.** For simplicity we prove the result for \(n = 3\). We refer once more to diagram (3.1) for the notation of the involved functors. The case where \(R_c(U, T, V)\) has a ladder of height \(n \geq 5\) is treated similarly.
(i) $\iff$ (ii) $\iff$ (iii): Assume that $R_\nu(U, T, V)$ admits a ladder of height three going downwards. Then we have the adjoint triples $(i, p, p^1)$, $(p, p^3, p^2)$ and $(e, r, r^1)$, $(r, r^1, r^2)$. Hence from Lemma 3.2 we get the desired properties for the functors $i$, $p$ and $e$, $r$. The converse directions follow also from Lemma 3.2.

(iv) $\implies$ (i): Restricting the recollement to compact objects means precisely that the functors $i$, $p$ and $e$, $r$ preserve compact objects. Then Lemma 3.2 provides us with the extra adjoints such that $R_\nu(U, T, V)$ admits a ladder of height three going downwards.

(i) $\implies$ (iv): Since the recollement $R_\nu(U, T, V)$ admits a ladder of height three going downwards, we have the adjoint triples: $(i, e, r)$, $(e, r, r^1)$ and $(r, r^1, r^2)$. From Lemma 3.2 the triangle functors $i$, $e$ and $r$ preserves compact objects, thus they give rise to the following adjoint triple:

![Diagram](image)

Clearly the functor $e: T^c \to U^c$ is essentially surjective since the functor $l: U^c \to T^c$ is fully faithful. Moreover, from the adjoint triple $(l, e, r)$ we also get that the functor $r: U^c \to T^c$ is also fully faithful (or just because $r: U \to T$ is fully faithful). It remains to show that $\text{Ker} e$ is equivalent to $U^c$. This follows as in the last part of the proof of Proposition 3.4. We infer that $R_\nu(U^c, T^c, V^c)$ is a recollement and in this case the height is one.

(i) $\implies$ (v): By assumption we have the following three recollements of triangulated categories:

![Diagram](image)

From the bijection between recollements of triangulated categories and TTF-triples, we obtain that $(i(V), i(U), r(V))$, $(i(U), r(V), p^1(U))$ and $(r(V), p^1(U), r^2(V))$ are TTF-triples in $T$.

(v) $\implies$ (i): From assumption we have the torsion pairs $(i(V), i(U))$, $(i(U), r(V))$, $(r(V), p^1(U))$ and $(p^1(U), r^2(V))$ in $T$. In particular, each of these torsion pairs gives rise to the following diagrams (for the notation see the text before diagram (2.2)):

\[ (i(V), i(U)): \quad i(V) \xrightarrow{i(U)} T \xrightarrow{i(U)} i(U) \quad (1) \quad (i(U), r(V)): \quad i(U) \xrightarrow{i(U)} T \xrightarrow{i(U)} r(V) \quad (2) \]

\[ (r(V), p^1(U)): \quad r(V) \xrightarrow{i(U)} T \xrightarrow{i(U)} p^1(U) \quad (3) \quad (p^1(U), r^2(V)): \quad p^1(U) \xrightarrow{i(U)} T \xrightarrow{i(U)} r^2(V) \quad (4) \]

We now show how from the above diagrams we obtain a recollement $R_\nu(U, T, V)$ of ladder three going downwards. From the diagrams (1) and (2) the inclusion functor $i_{i(U)}: i(U) \to T$ has a left and right adjoint. Then from Lemma 2.2 we obtain the recollement $R_\nu(i(U), T, (T/i(U)))$. In particular, from [6, Proposition 2.6 (vi), Chapter I] and Lemma 2.2 we obtain the following recollement of triangulated categories:

![Diagram](image)

On the other hand, from diagrams (2) and (3) the inclusion functor $i_{r(V)}: r(V) \to T$ has a left and right adjoint. Using again [6, Proposition 2.6, Chapter I] and Lemma 2.2, but now for the torsion pair

![Diagram](image)
Let functor \( R_{\mathcal{V}}(V) \) have a right adjoint, say \( R^1_{\mathcal{V}} \) . Hence, we have the adjoint triple \((i_{\mathcal{V}}, R_{\mathcal{V}}, R^1_{\mathcal{V}})\). From Lemma 2.2 we infer that the recollement (3.6) admits a ladder of height three going downwards. 

(iv) \( \Rightarrow \) (vi): Since \( R_{\mathcal{V}}(\mathcal{U}, \mathcal{T}, \mathcal{V}) \) restricts to \( R_{\mathcal{V}}(\mathcal{U}^c, \mathcal{T}^c, \mathcal{V}^c) \), we get the TTF-triple \((l(\mathcal{V}^c), i(\mathcal{U}^c), r(\mathcal{V}))\) in \( \mathcal{T}^c \). We claim that \( l(\mathcal{V}^c) = l(\mathcal{V}) \cap \mathcal{T}^c, i(\mathcal{U}^c) = i(\mathcal{U}) \cap \mathcal{T}^c \) and \( r(\mathcal{V}) = r(\mathcal{V}) \cap \mathcal{T}^c \). We first show that \( r(\mathcal{V}^c) = r(\mathcal{V}) \cap \mathcal{T}^c \). Let \( r(X) \) be an object in \( r(\mathcal{V}) \), i.e., \( X \) lies in \( \mathcal{V}^c \). Since the functor \( r \) preserves compact objects, the object \( r(X) \) belongs to \( r(\mathcal{V}) \cap \mathcal{T}^c \). Conversely, if we take an object \( Y \) in \( r(\mathcal{V}) \cap \mathcal{T}^c \), then \( Y = r(Y') \) for some \( Y' \) in \( \mathcal{V} \) and \( r(Y') \) is compact. Since the commutative diagram \( \mathcal{T} \rightarrow \mathcal{V} \) preserves compact objects, we get that the object \( Y' \) lies in \( \mathcal{V}^c \). Hence the object \( r(Y') \) lies in \( r(\mathcal{V}^c) \). Similarly we show that \( l(\mathcal{V}^c) = l(\mathcal{V}) \cap \mathcal{T}^c \) and \( l(\mathcal{U}^c) = i(\mathcal{U}) \cap \mathcal{T}^c \). We infer that \((l(\mathcal{V}) \cap \mathcal{T}^c), i(\mathcal{U}) \cap \mathcal{T}^c, r(\mathcal{V}) \cap \mathcal{T}^c)\) is a TTF-triple in \( \mathcal{T}^c \).

(vi) \( \Rightarrow \) (iv): Assuming that \((l(\mathcal{V}) \cap \mathcal{T}^c), i(\mathcal{U}) \cap \mathcal{T}^c, r(\mathcal{V}) \cap \mathcal{T}^c)\) is a TTF-triple in \( \mathcal{T}^c \), we show that the recollement \( R_{\mathcal{V}}(\mathcal{U}, \mathcal{T}, \mathcal{V}) \) restricts to a recollement \( R_{\mathcal{V}}(\mathcal{U}^c, \mathcal{T}^c, \mathcal{V}^c) \). Equivalently, we show that \((l(\mathcal{V}^c), i(\mathcal{U}^c), r(\mathcal{V}))\) is a TTF-triple in \( \mathcal{T}^c \). It suffices to prove that \( l(\mathcal{V}) \cap \mathcal{T}^c \cong l(\mathcal{V}^c) \). Clearly, if \( Y \) lies in \( l(\mathcal{V}^c) \), that is, \( Y = l(\mathcal{V}) \) for some \( V \) in \( \mathcal{V} \), then \( l(V) \) is a compact object in \( \mathcal{T} \) since the functor \( l \) preserves compact objects. Thus, \( l(\mathcal{V}^c) \subseteq l(\mathcal{V}) \cap \mathcal{T}^c \). On the other hand, let \( X \) be an object in \( l(\mathcal{V}) \cap \mathcal{T}^c \). This means that \( X \) is of the form \( l(V) \) for some \( V \) in \( \mathcal{V} \) and we claim that \( V \) lies in \( \mathcal{V}^c \). Indeed, the commutativity of the diagram

\[
\begin{align*}
\text{Hom}_\mathcal{V}(V, \prod_{i \in I} V_i) & \cong \prod_{i \in I} \text{Hom}_\mathcal{V}(V, V_i) \\
\text{Hom}_\mathcal{T}(l(V), \prod_{i \in I} l(V_i)) & \cong \prod_{i \in I} \text{Hom}_\mathcal{T}(l(V), l(V_i)) \cong l(V)^c
\end{align*}
\]

implies that \( V \) is compact. Hence, \( l(\mathcal{V}) \cap \mathcal{T}^c \subseteq l(\mathcal{V}^c) \) and this completes the proof.

We continue with a characterisation of when a recollement of compactly generated triangulated categories admits a ladder of height \( n \) going upwards. The proof is analogous with the proof of Theorem 3.6 using now Proposition 3.5 (and its proof), so it is left to the reader. However, we remark that this result is not exactly the dual of Theorem 3.6, for instance compare statements (iv).

**Theorem 3.7.** Let \( R_{\mathcal{V}}(\mathcal{U}, \mathcal{T}, \mathcal{V}) \) be a recollement of compactly generated triangulated categories and \( n \geq 3 \) a positive odd integer. The following statements are equivalent.

(i) The recollement \( R_{\mathcal{V}}(\mathcal{U}, \mathcal{T}, \mathcal{V}) \) admits a ladder of height \( n \) going upwards.

(ii) There are sequences of functors \( q^1, q^3, \ldots, q^{n-2} : \mathcal{U} \rightarrow \mathcal{T} \) and \( q^1, q^3, \ldots, q^{n-2} : \mathcal{T} \rightarrow \mathcal{U} \) such that \( (q^{n-1}, q^{n-2}, \ldots, q^1, q) \) are adjoint pairs. In particular, the functors \( q^1, q^3, \ldots, q^{n-2} : \mathcal{U} \rightarrow \mathcal{T} \) and \( q^1, q^3, \ldots, q^{n-2} : \mathcal{T} \rightarrow \mathcal{U} \) preserve compact objects.

(iii) There are sequences of functors \( l^1, l^3, \ldots, l^{n-2} : \mathcal{V} \rightarrow \mathcal{T} \) and \( l^1, l^3, \ldots, l^{n-2} : \mathcal{T} \rightarrow \mathcal{V} \) such that \( (l^{n-1}, l^{n-2}, \ldots, l^1, l) \) are adjoint pairs. In particular, the functors \( l^1, l^3, \ldots, l^{n-2} : \mathcal{T} \rightarrow \mathcal{V} \) and \( l^1, l^3, \ldots, l^{n-2} : \mathcal{V} \rightarrow \mathcal{T} \) preserve compact objects.

(iv) The recollement \( R_{\mathcal{V}}(\mathcal{U}, \mathcal{T}, \mathcal{V}) \) induces a recollement \( R_{\mathcal{V}}(\mathcal{U}^c, \mathcal{T}^c, \mathcal{V}^c) \) which admits a ladder of height \( n - 2 \) going upwards.

(v) There is a sequence of triangulated subcategories \( l^{n-1}(\mathcal{V}), q^{n-2}(\mathcal{U}), \ldots, l^2(\mathcal{V}), q^1(\mathcal{U}), l(\mathcal{V}), i(\mathcal{U}), r(\mathcal{V}) \) in \( \mathcal{T} \) such that

\[
(l^{n-1}(\mathcal{V}), q^{n-2}(\mathcal{U}), l^{n-3}(\mathcal{V})), \ldots, (l^2(\mathcal{V}), q^1(\mathcal{U}), l(\mathcal{V})), (l(\mathcal{V}), i(\mathcal{U}), r(\mathcal{V}))
\]

are TTF-triples in \( \mathcal{T} \).

(vi) There is a sequence of TTF-triples in \( \mathcal{T} \) of the form: \( (l^{n-1}(\mathcal{V}) \cap \mathcal{T}^c, q^{n-2}(\mathcal{U}) \cap \mathcal{T}^c, l^{n-3}(\mathcal{V}) \cap \mathcal{T}^c), \ldots, (q^1(\mathcal{U}) \cap \mathcal{T}^c, l^2(\mathcal{V}) \cap \mathcal{T}^c, q^1(\mathcal{U}) \cap \mathcal{T}^c), (l^2(\mathcal{V}) \cap \mathcal{T}^c, q^1(\mathcal{U}) \cap \mathcal{T}^c, l(\mathcal{V}) \cap \mathcal{T}^c) \).

As a consequence of Theorem 3.6 and Theorem 3.7 we obtain the following bijections (up to equivalence).
Corollary 3.8. For a positive integer \( n \geq 1 \) we have the following bijections:

\[
\left\{ \begin{array}{l}
R_n(U, T, V) \text{ admits a ladder of height } n \text{ going downwards} \\
R_n(U, T, V) \text{ admits a ladder of height } n \text{ going upwards}
\end{array} \right. \leftrightarrow \left\{ \begin{array}{l}
X_1, \ldots, X_{n+2} \text{ of subcat } \leq_T \{ (X_1, X_2, X_3), (X_2, X_3, X_4), \ldots \\
(X_n, X_{n+1}, X_{n+2}) : \text{TTF-triples in } T
\end{array} \right. \\
\left\{ \begin{array}{l}
X_1, \ldots, X_{n+2} \text{ of subcat } \leq_T \{ (X_{n+2}, X_{n+1}, X_n), (X_{n+1}, X_n, X_{n-1}), \ldots \\
(X_3, X_2, X_1) : \text{TTF-triples in } T
\end{array} \right.
\]

Note that in [8] and [19] the authors have also considered sequences of triangulated subcategories such that each two form a stable t-structure. We close this section with a remark on the description of the subcategories giving rise to a ladder.

Remark 3.9. Let \( T \) be a compactly generated triangulated category and fix a generating set \( S \) of compact generators in \( T \). Assume that there is a hereditary torsion pair \( (X, Y) \) of finite type in \( T \) [6], that is, \( (X, Y) \) is a stable t-structure and \( Y \) is closed under coproducts. Then from [6, Proposition 1.1, Chapter IV] it follows that there is a TTF-triple \( (X, Y, Z) \) in \( T \) where \( Z = Y^\perp \). Moreover, the triangulated category \( Y \) is compactly generated by the set \( L_Y(S) \). Hence, we obtain the recollement \( R_n(Y, T, X) \), see diagram (2.2).

From [6, Lemma 1.2, Chapter III] the functor \( R_X : T \to X \) preserves coproducts. Then in [6, Proposition 1.11, Chapter IV] the authors provide necessary and sufficient conditions for the hereditary torsion pair \( (X, Y) \) in \( T \) to induce a torsion pair in \( T^c \). However, these equivalent conditions characterize exactly when the recollement diagram (2.2) admits a ladder of height two going downwards. Moreover, in this case the authors show that the torsion pair \( (X, Y) \) is compactly generated by the set of object \( R_X(S) \) and \( X^c = X \cap T^c \). This means precisely that \( X = X^\perp = \{ R_X(S)[n] | n \geq 0 \} \), see [6, Definition 2.4, Chapter III]. Hence, in this way we obtain a description of the sequence of subcategories in Corollary 3.8 that provide us the ladder of a recollement of compactly generated triangulated categories.

4. Infinite Ladders and Preprojective Algebras

In this section we show that the derived category of the preprojective algebra \( \Pi_n(\Lambda, Q) \) admits an infinite ladder of period four which restricts to \( D^b(\text{mod}) \) and \( K^b(\text{proj}) \). This section is divided into two subsection and the main result is proved in the second one. We start by recalling some basic facts about differential graded algebras from [21,23].

4.1. Differential graded algebras. Let \( k \) be a field. A differential graded algebra, dg algebra for short, is a \( \mathbb{Z} \)-graded \( k \)-algebra \( A = \oplus_{n \in \mathbb{Z}} A^n \) endowed with a differential \( d_A \) of degree one such that the graded Leibniz rule: \( d(ab) = d(a)b + (-1)^n ad(b) \) for all \( a \in A^n \) and \( b \in A^m \), holds. A dg (left) \( A \)-module \( X \) is a \( \mathbb{Z} \)-graded (left) \( A \)-module \( X = \oplus_{n \in \mathbb{Z}} X^n \) endowed with a differential \( d_X \) of degree one such that \( d_X(ax) = ad_X(x) + (-1)^n d_A(a)x \) for all \( x \in X^n \) and \( a \in A \). A morphism \( f: X \to Y \) of dg \( A \)-modules is a morphism of the underlying graded \( A \)-modules which is homogeneous of degree zero and commutes with the differential. Then the category of all left dg \( A \)-modules is an abelian category and is denoted by \( \mathcal{C}(A) \). A morphism \( f \) in \( \mathcal{C}(A) \) is a quasi-isomorphism if it induces isomorphisms in homology. Moreover, a morphism \( f: X \to Y \) of dg \( A \)-modules is null-homotopic if there is a morphism of dg \( A \)-modules \( h: X \to Y \) of degree minus one such that \( f = hd_X + dyh \). Then the homotopy category \( \mathcal{H}(A) \) has the same objects with \( \mathcal{C}(A) \) and its morphism space consists of the equivalences classes of morphisms in \( \mathcal{C}(A) \) modulo the null-homotopic ones. The homotopy category \( \mathcal{H}(A) \) is triangulated and then the derived category \( \mathcal{D}(A) \) is the localization of \( \mathcal{H}(A) \) with respect to the quasi-isomorphisms.

Let \( A \) be a dg algebra. Recall from [21, Theorem 5.3] that the full subcategory of compact objects of \( \mathcal{D}(A) \) coincides with the category \( \mathcal{P}(A) \) of perfect \( \mathcal{D}(A) \)-modules. The latter is the smallest full triangulated subcategory of \( \mathcal{D}(A) \) containing \( A \) and closed under finite coproducts and direct summands. We denote by \( \mathcal{D}_A(A) \) the full subcategory of \( \mathcal{D}(A) \) consisting of \( \mathcal{D}(A) \)-modules whose total cohomology is finite dimensional, i.e. \( \mathcal{D}_A(A) = \{ X \in \mathcal{D}(A) | \oplus_{n \in \mathbb{Z}} \mathcal{H}^n(X) \text{ is finite dimensional} \} \).

To proceed, we need the following auxiliary results.

Lemma 4.1. \( \mathcal{D}_A(A) = \{ X \in \mathcal{D}(A) | \oplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(A)}(P, X[n]) \text{ is finite dimensional for any } P \in \mathcal{P}(A) \} \).
Proof. Since we have the isomorphisms $\text{Hom}_{D(A)}(A, X[i]) \cong \text{Hom}_{H(A)}(A, X[i]) \cong H^0 \text{Hom}(A, X[i]) \cong H^0(X[i]) = H^0(X)$ and $\text{per}(A)$ is, by definition, closed under direct summands, shifts and extensions, the desired description of $D_{\text{id}}(A)$ follows immediately. □

Lemma 4.2. Let $A$ and $B$ be two dg algebras. Assume that there is an adjoint pair $(F, G)$ between the derived categories $D(A)$ and $D(B)$ such that the functor $F$ restricts to $F: \text{per}(A) \rightarrow \text{per}(B)$. Then the functor $G$ restricts to $G: D_{\text{id}}(B) \rightarrow D_{\text{id}}(A)$.

Proof. Let $X$ be an object in $D_{\text{id}}(B)$ and let $P$ an object in $\text{per}(A)$. Since there is an isomorphism $\text{Hom}_{D(A)}(P, G(X)[n]) \cong \text{Hom}_{D(B)}(F(P), X[n])$, for any $n \in \mathbb{Z}$, and $F(P)$ lies in $\text{per}(B)$, it follows from Lemma 4.1 that the hom space $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D(A)}(P, G(X)[n])$ is finite dimensional. We infer from Lemma 4.1 that the object $G$ lies in $D_{\text{id}}(A)$. □

In the next result we provide a sufficient condition for a recolllement of derived categories of dg algebras to restrict to $D_{\text{id}}$. Compare this with [1, Theorem 4.6].

Proposition 4.3. Assume that there is a recolllement of derived categories of dg algebras:

Consider the following conditions:

(i) The recollement $R_{\text{id}}(D(S), D(R), D(T))$ restricts to a recollement

(ii) The functor $l$ restricts to $D_{\text{id}}$ and $i(S)$ lies in $\text{per}(R)$.

(iii) The functor $q$ restricts to $D_{\text{id}}$ and $e(R)$ lies in $\text{per}(T)$.

Then (ii) $\implies$ (i) and (iii) $\implies$ (i). If $p$ preserves coproducts, then all three conditions are equivalent.

Proof. (ii) $\implies$ (i): By Lemma 3.2 we know that the functors $q$ and $l$ restrict to $\text{per}(R)$ and $\text{per}(T)$, respectively. Then by Lemma 4.2 it follows that the functors $i$ and $e$ restrict to $D_{\text{id}}$. Since $i(S)$ lies in $\text{per}(R)$, i.e. the functor $i$ preserves compact objects, Lemma 3.2 implies that there is an adjoint triple $(i, p, p^*)$ and therefore an adjoint triple $(e, r, r^*)$ by Lemma 2.2. Then the functors $i$ and $e$ restrict to $\text{per}$ and by Lemma 4.2 again we get that the functors $p$ and $r$ restrict to $D_{\text{id}}$. Note that the functor $l$ restricts to $D_{\text{id}}$ by assumption. It remains to show that $q$ restricts to $D_{\text{id}}$. Let $X$ be an object in $D_{\text{id}}(R)$. From the canonical triangle

and since $le(X)$ lies in $D_{\text{id}}(R)$, we get that $iq(X)$ lies in $D_{\text{id}}(R)$. Since the functor $i$ is fully faithful, there are isomorphisms for all $n \in \mathbb{Z}$ and any $P \in \text{per}(S)$:

$\text{Hom}_{D(S)}(lP, q(X)[n]) \cong \text{Hom}_{D(R)}(i(P), iq(X)[n])$

This implies that the space $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D(S)}(lP, q(X)[n])$ is finite dimensional and thus Lemma 4.1 shows that $q(X)$ lies in $D_{\text{id}}(S)$. We infer that $R_{\text{id}}(D(S), D(R), D(T))$ is a recollement.

(i) $\implies$ (ii): We only need to check that $i(S)$ belongs to $\text{per}(R)$. Since the functor $p$ preserves coproducts, Lemma 3.2 implies that $i(S)$ lies in $\text{per}(R)$.

The implications (iii) $\implies$ (i) and (i) $\implies$ (iii) follow similarly as above. □

4.2 Preprojective algebras. Let $k$ be an algebraically closed field and $Q$ a finite quiver. Denote by $Q$ the double quiver of $Q$ which is obtained from $Q$ by adding for each arrow $a \in Q_1$ an arrow $a^*$ in the opposite direction. Then the preprojective algebra [15] is defined as $\Pi_n(Q) := kQ/(c)$ where $kQ$ is the path algebra of $Q$ over $k$ and $(c)$ is the two-sided ideal generated by $c = \Sigma_{a \in Q_1} (a^*a - aa^*)$. It is known that if $Q$ is Dynkin, then $\Pi_n(Q)$ is a finite-dimensional selfinjective algebra.

Let $\Lambda$ be a finite dimensional $k$-algebra and denote by $\Pi_n(\Lambda, Q)$ the algebra $\Lambda \otimes_k \Pi_n(Q)$. From now on we assume that the quiver $Q$ is Dynkin of type $\Lambda_n$. This implies that $\Pi_n(\Lambda, Q)$ is a finite dimensional $k$-algebra and is called the path algebra of $\Pi_n(Q)$ over $\Lambda$, see [27, subsection 2B]. Note that $\Pi_n(\Lambda, Q)$ can
be realized as a preprojective algebra of Dynkin species, we refer to [25] for more details. The module category of $\Pi_n(\Lambda, Q)$ has objects representations of $\Pi_n(Q)$ over $\Lambda$, see [2], [27, Lemma 2.1], [25, Proposition 4.12]. More precisely, for $n = 2$ and $n = 3$ we have the following descriptions:

$$\text{Mod-}\Pi_2(\Lambda, Q) = \{ \begin{array}{c} X \xrightarrow{\eta} Y \end{array} \mid g \circ f = 0, \ f \circ g = 0 \ \text{and} \ \ X, Y, Z \in \text{Mod-}\Lambda \}$$

$$\text{Mod-}\Pi_3(\Lambda, Q) = \{ \begin{array}{c} X \xrightarrow{g_1} \begin{array}{c} f_1 \end{array} \xrightarrow{g_2} X_3 \xrightarrow{\cdots} X_n-1 \xrightarrow{g_n-1} X_n \end{array} \mid g_1 \circ f_1 = 0 = f_2 \circ g_2, \ f_1 \circ g_1 = g_2 \circ f_2$$

and $X, Y, Z \in \text{Mod-}\Lambda$$

and more generally we have

$$\text{Mod-}\Pi_n(\Lambda, Q) = \{ \begin{array}{c} X_1 \xrightarrow{g_1} \begin{array}{c} f_1 \end{array} \xrightarrow{g_2} X_3 \xrightarrow{\cdots} X_{n-1} \xrightarrow{g_n-1} X_n \end{array} \mid g_1 \circ f_1 = 0 = f_2 \circ g_2, \ f_1 \circ g_1 = g_2 \circ f_2$$

for all $1 \leq i \leq n \leq 2$ and $X_1, \ldots, X_n \in \text{Mod-}\Lambda$.

If we restrict to finitely generated $\Lambda$-modules, we get the module category $\text{mod-}\Pi_n(\Lambda, Q)$. An object of $\text{Mod-}\Pi_n(\Lambda, Q)$ is denoted by $(X_1, \ldots, X_n, f_1, g_1, \ldots, f_{n-1}, g_{n-1})$.

We define the following functors:

(i) The functor $T_1: \text{Mod-}\Lambda \to \text{Mod-}\Pi_n(\Lambda, Q)$ is given by

$$X \xrightarrow{0} X \xrightarrow{0} \begin{array}{c} 1 \end{array} \xrightarrow{0} \begin{array}{c} 1 \end{array} \xrightarrow{0} \begin{array}{c} 1 \end{array} \xrightarrow{0} X$$

for a $\Lambda$-module $X$, and given a morphism $a: X \to X'$ in $\text{Mod-}\Lambda$ then $T_1(a) = (a, a, \ldots, a)$.

(ii) The functor $T_2: \text{Mod-}\Lambda \to \text{Mod-}\Pi_n(\Lambda, Q)$ is given by

$$X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \begin{array}{c} 1 \end{array} \xrightarrow{1} \begin{array}{c} 1 \end{array} \xrightarrow{1} X$$

for a $\Lambda$-module $X$, and given a morphism $a: X \to X'$ in $\text{Mod-}\Lambda$ then $T_2(a) = (a, a, \ldots, a)$.

(iii) The functor $U_1: \text{Mod-}\Pi_n(\Lambda, Q) \to \text{Mod-}\Lambda$ is given by

$$U_1(X_1, \ldots, X_n, f_1, g_1, \ldots, f_{n-1}, g_{n-1}) = X_1$$

on objects, and for a morphism $(a_1, \ldots, a_n)$ in $\text{Mod-}\Pi_n(\Lambda, Q)$ we have $U_1(a_1, \ldots, a_n) = a_1$.

(iv) The functor $U_2: \text{Mod-}\Pi_n(\Lambda, Q) \to \text{Mod-}\Lambda$ is given by

$$U_2(X_1, \ldots, X_n, f_1, g_1, \ldots, f_{n-1}, g_{n-1}) = X_n$$

on objects, and for a morphism $(a_1, \ldots, a_n)$ in $\text{Mod-}\Pi_n(\Lambda, Q)$ we have $U_2(a_1, \ldots, a_n) = a_n$.

(v) The functor $Z_1: \text{Mod-}\Pi_{n-1}(\Lambda, Q) \to \text{Mod-}\Pi_n(\Lambda, Q)$ is defined by

$$Z_1(X_1, \ldots, X_{n-1}, f_1, g_1, \ldots, f_{n-2}, g_{n-2}) = (X_1, \ldots, X_{n-1}, 0, f_1, g_1, \ldots, f_{n-2}, g_{n-2}, 0, 0)$$

on objects, and for a morphism $(a_1, \ldots, a_{n-1})$ in $\text{Mod-}\Pi_2(\Lambda, Q)$ we have $Z_2(a_1, \ldots, a_{n-1}) = (a_1, \ldots, a_{n-1}, 0)$.

(vi) The functor $Z_2: \text{Mod-}\Pi_{n-1}(\Lambda, Q) \to \text{Mod-}\Pi_n(\Lambda, Q)$ is defined by

$$Z_2(X_1, \ldots, X_{n-1}, f_1, g_1, \ldots, f_{n-2}, g_{n-2}) = (0, X_1, \ldots, X_{n-1}, 0, 0, f_1, g_1, \ldots, f_{n-2}, g_{n-2})$$

on objects, and for a morphism $(a_1, \ldots, a_{n-1})$ in $\text{Mod-}\Pi_{n-1}(\Lambda, Q)$ we have $Z_2(a_1, \ldots, a_{n-1}) = (0, a_1, \ldots, a_{n-1})$.

In the next result we show that $\text{Mod-}\Pi_n(\Lambda, Q)$ admits a recollement of module categories. The definition of a recollement of module categories is completely analogous to Definition 2.1. In the abelian case, by definition we have that only the middle functors are exact. For more on recollements of abelian categories see [13,33].
Proposition 4.4. Let $\Lambda$ be a finite dimensional algebra. Then the algebra $\Pi_n(\Lambda, Q)$ admits the following equivalent recollements of module categories:

\[
\begin{array}{ccc}
\text{Mod-}\Pi_{n-1}(\Lambda, Q) & \xrightarrow{z_1} & \text{Mod-}\Pi_n(\Lambda, Q) \\
\downarrow & & \downarrow \\
\text{Mod-}\Lambda & \xrightarrow{u_1} & \text{Mod-}\Lambda
\end{array}
\] (4.1)

and

\[
\begin{array}{ccc}
\text{Mod-}\Pi_{n-1}(\Lambda, Q) & \xrightarrow{z_2} & \text{Mod-}\Pi_n(\Lambda, Q) \\
\downarrow & & \downarrow \\
\text{Mod-}\Lambda & \xleftarrow{u_2} & \text{Mod-}\Lambda
\end{array}
\] (4.2)

Proof. We first show that (4.1) is a recollement. From [33, Remark 2.3] it suffices to show that $(T_1, U_1, T_2)$ is an adjoint triple with $T_1$ (or $U_1$) fully faithful and that the kernel $\ker U_1$ is the module category $\text{Mod-}\Pi_{n-1}(\Lambda, Q)$. Let $(X'_1, \ldots, X'_{n}, f_1, g_1, \ldots, f_{n-1}, g_{n-1})$ be an object in $\text{Mod-}\Pi_n(\Lambda, Q)$ and $X$ be an object in $\text{Mod-}\Lambda$. Consider a morphism $(a_1, \ldots, a_n) : T_1(X) \rightarrow (X'_1, \ldots, X'_{n}, f_1, g_1, \ldots, f_{n-1}, g_{n-1})$. Then, it is easy to observe that $a_2 = f_1 \circ a_1$ and $a_{i+1} = f_i \circ \cdots \circ f_1 \circ a_1$ for all $2 \leq i \leq n-1$. The assignment $(a_1, f_1 \circ a_1, \ldots, f_{n-1} \circ \cdots \circ f_1 \circ a_1) \mapsto a_1$ implies that there is a natural isomorphism between $\text{Hom}_n(\Lambda, Q)(T_1(X), (X'_1, \ldots, X'_{n}, f_1, g_1, \ldots, f_{n-1}, g_{n-1}))$ and $\text{Hom}_n(\Lambda, Q)(X, X')$, proving that $(T_1, U_1)$ is an adjoint pair. Similarly, if $(a_1, \ldots, a_n) : (X'_1, \ldots, X'_{n}, f_1, g_1, \ldots, f_{n-1}, g_{n-1}) \rightarrow T_2(X)$ is a morphism in $\text{Mod-}\Pi_n(\Lambda, Q)$, then we get that $a_2 = a_1 \circ g_1$ and $a_{i+1} = a_1 \circ g_1 \circ \cdots \circ g_1$ for all $2 \leq i \leq n-1$. This shows that the assignment $(a_1, a_1 \circ g_1, \ldots, a_1 \circ g_1 \circ \cdots \circ g_{n-1}) \mapsto a_1$ induces a natural isomorphism $\text{Hom}_n(\Lambda, Q)((X'_1, \ldots, X'_{n}, f_1, g_1, \ldots, f_{n-1}, g_{n-1}), T_2(X)) \cong \text{Hom}_n(X, X')$. Hence, $(U_1, T_2)$ is an adjoint pair. Clearly, the functor $T_1$ is fully faithful and the adjoint triple $(T_1, U_1, T_2)$ implies that $T_2$ is also fully faithful. Moreover, the functor $T_2$ is fully faithful and the kernel $\ker U_1$ consists of all objects $(X_1, \ldots, X_n, f_1, g_1, \ldots, f_{n-1}, g_{n-1})$ such that $U_1(X_1, \ldots, X_n, f_1, g_1, \ldots, f_{n-1}, g_{n-1}) = 0$, that is, $X_1 = 0$. This implies that the maps $f_1$ and $g_1$ are also zero. Hence, the kernel $\ker U_1$ consists of all objects of the form $(0, X_2, \ldots, X_n, 0, 0, f_2, g_2, \ldots, f_{n-1}, g_{n-1})$ and this subcategory is exactly $Z_2(\text{Mod-}\Pi_{n-1}(\Lambda, Q))$. We infer that $(\text{Mod-}\Pi_{n-1}(\Lambda, Q), \text{Mod-}\Pi_n(\Lambda, Q), \text{Mod-}\Lambda)$ is a recollement and similarly we show that (4.2) is a recollement as well.

Finally, we show that the recollements (4.1) and (4.2) are equivalent. Let $(X_1, \ldots, X_n, f_1, g_1, \ldots, f_{n-1}, g_{n-1})$ be an object in $\text{Mod-}\Pi_n(\Lambda, Q)$. We define the endofunctor $\mathcal{F} : \text{Mod-}\Pi_n(\Lambda, Q) \rightarrow \text{Mod-}\Pi_n(\Lambda, Q)$ by $\mathcal{F}(X_1, \ldots, X_n, f_1, g_1, \ldots, f_{n-1}, g_{n-1}) = (X_n, \ldots, X_1, g_1, f_1, \ldots, f_{n-1}, g_{n-1})$ on objects, and given a morphism $(a_1, \ldots, a_n)$ then $\mathcal{F}(a_1, \ldots, a_n) = (a_n, \ldots, a_1)$. It follows easily that the functor $\mathcal{F}$ is an equivalence of categories. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Mod-}\Pi_n(\Lambda, Q) & \xrightarrow{u_1} & \text{Mod-}\Lambda \\
\downarrow \cong & & \downarrow \text{id}_{\text{Mod-}\Lambda} \\
\text{Mod-}\Pi_n(\Lambda, Q) & \xleftarrow{u_2} & \text{Mod-}\Lambda
\end{array}
\]

It implies that the functors $U_2 \mathcal{F}$ and $U_1$ are naturally isomorphic. Thus, from [34, Definition 4.1, Lemma 4.2] we obtain that the recollements (4.1) and (4.2) of $\text{Mod-}\Pi_n(\Lambda, Q)$ are equivalent.

We are now ready to state and prove the main result of this paper.

Theorem 4.5. Let $\Lambda$ be a finite dimensional $k$-algebra over a field $k$. For the algebra $\Pi_n(\Lambda)$ the following statements hold:

(i) There is an infinite ladder $R_\Gamma(D(\Gamma), D(\Pi_n(\Lambda, Q)), D(\Lambda))$ of period four, where $\Gamma$ is a dg algebra such that $H^0(\Gamma) \cong \Pi_{n-1}(\Lambda, Q)$.

(ii) There is an infinite ladder $R_\Gamma(D\pi(\Gamma), D^b(\mod-\Pi_n(\Lambda, Q)), D^b(\mod-\Lambda))$ of period four.

(iii) There is an infinite ladder $R_\Gamma(\per(\Gamma), K^b(\proj\Pi_n(\Lambda, Q)), K^b(\proj\Lambda))$ of period four.

Proof. (i) Consider the idempotent element $e_1 \in \Pi_n(\Lambda, Q)$ and the right $\Pi_n(\Lambda, Q)$-module $e_1\Pi_n(\Lambda, Q)$. Clearly, the module $e_1\Pi_n(\Lambda, Q)$ is finitely generated projective. Then from [3, Proposition 5.2], see also
we obtain the following recollement of triangulated categories:

\[
\begin{array}{c}
\text{D}(\Gamma) \\
\downarrow i \\
\text{D}(\Pi_n(\Lambda, Q)) \\
\downarrow e \\
\text{D}(\text{End}(e_1\Pi_n(\Lambda, Q))) \\
\end{array}
\]

(4.3)

where \(\Gamma\) is a dg algebra such that \(\mathbb{H}^0(\Gamma) \cong \Pi_{n-1}(\Lambda, Q)\) and \(e = e_1\Pi_n(\Lambda) \cong e_1\Pi_n(\Lambda)\). We now explain the right part of the above recollement. First, note that the endomorphism algebra \(\text{End}(e_1\Pi_n(\Lambda, Q))\) is isomorphic to \(\Lambda\). Since \(e_1\Pi_n(\Lambda, Q)\) is projective as a right \(\Pi_n(\Lambda, Q)\)-module, the functor \(e\) is the derived functor of the exact functor \(e_1(-) : \text{Mod-}\Pi_n(\Lambda, Q) \rightarrow \text{Mod-}\Lambda\) which is left multiplication with the idempotent element \(e_1\). The latter functor coincides with the functor \(U_1 : \text{Mod-}\Pi_n(\Lambda, Q) \rightarrow \text{Mod-}\Lambda\), see Proposition 4.4.

Then, from Lemma 2.4 and Proposition 4.4 we get the following recollement:

\[
\begin{array}{c}
\text{D}(\Gamma) \\
\downarrow i \\
\text{D}(\Pi_n(\Lambda, Q)) \\
\downarrow u_1 \\
\text{D}(\Lambda) \\
\end{array}
\]

(4.4)

The triangle functors \(T_1, U_1\) and \(T_2\) appeared in (4.4) are the derived functors of the underlying exact functors at the level of module categories. From the recollement (4.2), we have the adjoint pair \((U_2, T_1)\) between \(\text{Mod-}\Pi_n(\Lambda, Q)\) and \(\text{Mod-}\Lambda\). Since both \(U_2\) and \(T_1\) are exact functors, Lemma 2.4 yields an adjoint pair, still denoted by \((U_2, T_1)\), between the derived categories \(\text{D}(\Pi_n(\Lambda, Q))\) and \(\text{D}(\Lambda)\). Then, from Lemma 2.2 there is a triangle functor \(q^1 : \text{D}(\Gamma) \rightarrow \text{D}(\Pi_n(\Lambda, Q))\) such that \((q^1, q)\) is an adjoint pair. This implies that the following diagram is a recollement of triangulated categories:

\[
\begin{array}{c}
\text{D}(\Lambda) \\
\downarrow T_1 \\
\text{D}(\Pi_n(\Lambda, Q)) \\
\downarrow q^1 \\
\text{D}(\Gamma) \\
\end{array}
\]

(4.5)

So far, the above diagram shows that the recollement (4.4) admits a ladder of height two going upwards. From the recollement of abelian categories (4.2), we also have the adjoint pair \((T_2, U_2)\). Then we get an induced adjoint pair at the level of derived categories still denoted by \((T_2, U_2)\). Hence, there is a sequence of triangle functors \(T_1, U_1, T_2, U_2, T_1\) such that any two consecutive functors form an adjoint pair between \(\text{D}(\Pi_n(\Lambda, Q))\) and \(\text{D}(\Lambda)\). Then Theorem 3.7 yields an infinite ladder for (4.4) of period four going upwards. Recall that the recollement (4.4) is considered to be a ladder of height one, so period four means that the fifth recollement that we obtain is the recollement (4.4). The same method and Theorem 3.6 gives an infinite ladder of period four going downwards. We infer that \(\text{D}(\Pi_n(\Lambda, Q))\) admits an infinite ladder of period four as follows:

(ii) & (iii) Since we have the adjoint triple \((U_2, T_1, U_1)\), Lemma 3.2 implies that the functor \(U_2\) preserves compact objects, i.e. it restricts to the category \(\text{per}\). Then from Lemma 4.2 the functor \(T_1\) restricts to \(\text{D}_a\). Since \((i, p, p')\) is an adjoint triple, it follows from Lemma 3.2 that \(i(\Gamma)\) lies in \(\text{per}(\Pi_n(\Lambda, Q))\). Then from Proposition 4.3 the recollement (4.4) restricts to \(\text{R}_{(0)}(\text{D}_a(\Gamma), \text{D}_a(\Pi_n(\Lambda, Q))), \text{D}_a(\Lambda))\). Since \(\Lambda\) and \(\Pi_n(\Lambda, Q)\) are finite dimensional algebras (recall that \(Q\) is Dynkin of type \(A_n\)), so they are considered as dg algebras concentrated in degree zero, we have triangle equivalences \(\text{D}_a(\Pi_n(\Lambda, Q)) \cong \text{D}^b(\text{mod-}\Pi_n(\Lambda))\) and
\( D_{\text{id}}(\Lambda) \cong D^b(\text{mod-}\Lambda) \). Then we obtain the recollement \( R_\nu (D_{\text{id}}(\Gamma), D^b(\text{mod-}\Pi_n(\Lambda, Q)), D^b(\text{mod-}\Lambda)) \). As in part (i) we get an infinite ladder of period four for \( D^b(\text{mod-}\Pi_n(\Lambda, Q)) \) using now the bounded version of Lemma 2.4. Finally, since all the involved functors in (4.6) fit into an adjoint triple, Lemma 3.2 implies that they restrict to compact objects. Then part (iii) follows from Theorem 3.6 and Theorem 3.7.

We now explain that the ladder of Theorem 4.5 is in fact a consequence of the Nakayama functor. Compare this with ladders of recollements arising from algebras of finite global dimension treated in [1].

**Remark 4.6.** We keep the notation and assumptions as in Theorem 4.5. For simplicity we consider the case \( n = 2 \) and suppose that \( \Lambda \) is a selfinjective algebra. We denote by \( D : \text{mod-}\Lambda \rightarrow \text{mod-}\Lambda^\text{op} \) the usual duality, see [2]. Consider the Nakayama functor \( \nu = D\Pi_2(\Lambda, Q) \oplus \Pi_2(\Lambda, Q) \rightarrow \text{mod-}\Pi_2(\Lambda, Q) \rightarrow \text{mod-}\Pi_2(\Lambda, Q) \). Then its right adjoint \( \nu^{-1} = \text{Hom}_{\Pi_2(\Lambda, Q)}(D\Pi_2(\Lambda, Q), -) \). We claim that there is an isomorphism \( U_2 \cong \nu^{-1} \circ U_1 \circ \nu \) and \( (\nu^{-1} \circ U_1 \circ \nu, T_1) \) is an adjoint pair.

We now show the fist claim. Since \( \Lambda \) is a selfinjective algebra, it follows that the algebra \( \Pi_2(\Lambda, Q) \) is also selfinjective. Since the functors \( \nu \) and \( \nu^{-1} \) are exact, it suffices to prove the desired isomorphism for projective modules. Using the decrion of \( \Pi_2(\Lambda, Q) \) as a Morita ring, see Remark 4.8 below, we have from [16, Proposition 3.1] that the indecomposable projective modules are of the form \( T_1(P) = (P, P, \text{Id}_P, 0) \) or \( T_2(P) = (P, P, 0, \text{Id}_P) \), where \( P \) is an indecomposable projective \( \Lambda \)-module. We also refer to [27, Proposition 2.4] for the general case. Then we compute that \( \nu^{-1} \circ U_1 \circ \nu(F, P, \text{Id}_P, 0) = \nu^{-1}(U_1(\nu(P), \nu(P), 0, \text{Id}_\nu(P))) = \nu^{-1}(\nu(P)) \cong P = U_2(P, P, \text{Id}_P, 0) \) and similarly for the projective \( T_2(P) \). For the second claim, it suffices to check the desired adjunction isomorphism for projective modules. Take a projective \( \Pi_2(\Lambda, Q) \)-module \( E = (E_1, E_2, f_1, g_1) \) and a \( \Lambda \)-module \( X \). Since \( \text{Hom}_\Lambda^{\text{op}}(\nu^{-1} \circ U_1 \circ \nu, E, X) = \text{Hom}_\Lambda(E_2, X) \cong \text{Hom}_\Pi_2(\Lambda, Q)(E, T_1(X)) \) it follows that \( (\nu^{-1} \circ U_1 \circ \nu, T_1) \) is an adjoint pair.

In case that \( \Lambda \) is a selfinjective algebra (or when we consider just the preprojective algebra \( \Pi_n(\Lambda, Q) \)), the above considerations show that from the adjoint pair \( (T_1, U_1) \) between the module categories we can produce the adjoint pair \( (\nu^{-1} \circ U_1 \circ \nu, T_1) \). The main idea of the proof of Theorem 4.5 is to lift the underlying exact functors of (4.1) and (4.2) to derived categories. The key property is that by Proposition 4.4 we have the adjoint triples \( (T_1, U_1, T_2) \) and \( (T_2, U_2, T_1) \) at the level of module categories. Thus when \( \Lambda \) is selfinjective, this sequence of adjoints can be interpreted as we explained above via the Nakayama functor.

**Remark 4.7.** Let \( \Lambda \) be a finite dimensional selfinjective \( k \)-algebra over a field \( k \). From the recollections (4.1) and (4.2), we have the adjoint triples \( (U_2, T_1, U_1) \) and \( (U_1, T_2, U_2) \) between \( \text{mod-}\Pi_n(\Lambda, Q) \) and \( \text{mod-}\Lambda \). Since \( U_2, T_1, U_1 \) and \( T_2 \) are exact functors, it follows that \( U_2, T_1, U_1 \) and \( T_2 \) preserve projective modules. This implies that we get adjoint triples between the stable categories \( \text{mod-}\Pi_n(\Lambda, Q) \) and \( \text{mod-}\Lambda \), still denoted by \( (U_2, T_1, U_1) \) and \( (U_1, T_2, U_2) \). Hence, there is an infinite sequence of exact functors between the triangulated categories \( \text{mod-}\Pi_n(\Lambda, Q) \) and \( \text{mod-}\Lambda \) going upwards and downwards such that any two consecutive functors are adjoint pairs.

![Diagram](image)

Then the kernel \( \text{Ker} U_1 \) is a triangulated subcategory of \( \text{mod-}\Pi_n(\Lambda, Q) \) and as in Theorem 4.5 we obtain a periodic infinite ladder \( L_\nu(\text{Ker} U_1, \text{mod-}\Pi_n(\Lambda, Q), \text{mod-}\Lambda) \). Note that \( \text{Ker} U_1 \) is not \( \text{mod-}\Pi_{n-1}(\Lambda, Q) \) since the functor \( Z_2 : \text{mod-}\Pi_{n-1}(\Lambda, Q) \rightarrow \text{mod-}\Pi_n(\Lambda, Q) \) doesn’t not preserve projectives.

We close this section with the following remark where we explain the relation of preprojective algebras with Morita rings.
Remark 4.8. Let $\Lambda$ be a finite dimensional algebra and consider the Morita ring

$$\Delta_{(0,0)} = \begin{pmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{pmatrix}$$

The addition of elements of $\Delta_{(0,0)}$ is componentwise and multiplication is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \cdot \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + nb' & an' + nb' \\ ma' + bm' & bb' \end{pmatrix}$$

We refer to [16] for more details on Morita rings. The module category of this class of Morita rings was investigated further in [14] in connection with aspects of Gorenstein homological algebra. In particular, the module category $\text{Mod-}\Delta_{(0,0)}$ is equivalent to the double morphism category $\text{DMor}(\text{Mod-}\Lambda)$ of $\text{Mod-}\Lambda$, introduced in [14, subsection 2.2]. The latter category is exactly the module category $\text{Mod-}\Pi_2(\Lambda, Q)$ described in the beginning of subsection 4.2. The same conclusion holds if we consider categories of finitely generated modules. Note that the Morita ring $\Delta_{(0,0)}$ is isomorphic to the algebra $\Lambda \otimes_k \left( \begin{smallmatrix} k & k \\ k & k \end{smallmatrix} \right)_{(0,0)}$, where the Morita ring $\left( \begin{smallmatrix} k & k \\ k & k \end{smallmatrix} \right)_{(0,0)}$ is the preprojective algebra $\Pi_2(Q)$ over the Dynkin quiver $Q$ of type $A_2$.

Hence, from Theorem 4.5 and [14, Example 2.7] the derived category $D(\Delta_{(0,0)})$ of the double morphism category admits a periodic infinite ladder $R_\Gamma(D(\Gamma), D(\Delta_{(0,0)}), D(\Lambda))$, where $\Gamma$ is a dg algebra such that $H^0(\Gamma) \cong \Lambda$, and this ladder restricts to bounded as well as to perfect complexes.

References

[1] L. Angeleri H"ugel, S. K"ong, Q.H. Liu and D. Yang, Ladders and simplicity of derived module categories, J. Algebra 472 (2017), 15–66.
[2] M. Auslander, I. Reiten and S. Smalo, Representation Theory of Artin Algebras, Cambridge University Press, (1995).
[3] S. Bazzoni and A. Pavarin, Recollements from partial tilting complexes, J. Algebra 388 (2013), 338–363.
[4] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux Pervers, (French) [Perverse sheaves], Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Asterisque 100 Soc. Math. France, Paris, (1982).
[5] A. Beilinson, V. Ginzburg and V. Schechtman, Koszul duality, J. Geom. Phys. 5 (1998), no. 3, 317–350.
[6] A. Beligiannis and I. Reiten, Homological and homotopical aspects of torsion theories, Mem. Amer. Math. Soc. 188 (2007), no. 883, viii+207 pp.
[7] A. I. Bondal, Representations of associative algebras and coherent sheaves, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 25–44; translation in Math. USSR-Izv. 34 (1990), no. 1, 23–42.
[8] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors, and reconstructions, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337; translation in Math. USSR-Izv. 35 (1990), no. 3, 519–541.
[9] H.X. Chen and C.C. Xi, Recollements of derived categories II: Algebraic $K$-theory, arXiv:1212.1879.
[10] Y.P. Chen and S. Könenk, Recollements of self-injective algebras, and classification of self-injective diagram algebras, preprint.
[11] E. Cline, B. Parshall and L. Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew. Math. 391 (1988) 85–99.
[12] E. Cline, B. Parshall and L. Scott, Algebraic stratification in representation categories, J. Algebra 117 (1988) 504–521.
[13] V. Franjou and T. Pirashvili, Comparison of abelian categories recollements, Documenta Math. 9 (2004), 41–56.
[14] N. Gao and C. Psaroudakis, Gorenstein homological aspects of monomorphism categories via Morita rings, to appear in Algebr. Represent. Theory. DOI 10.1007/s10468-016-9652-1.
[15] I. M. Gel’fand and V. A. Ponomarev, Model algebras and representations of graphs, Funktsional. Anal. i Prilozhen. 13 (1979), 1–12.
[16] E.L. Green and C. Psaroudakis, On Artin algebras arising from Morita contexts, Algebr. Represent. Theory 17 (2014), no. 5, 1485–1525.
[17] Y. Han and Y. Qin, Reducing homological conjectures by $n$-recollements, Algebr. Represent. Theory 19 (2016), no. 2, 377–395.
[18] D. Happel, Reduction techniques for homological conjectures, Tsukuba J. Math. 17, (1993), no. 1, 115–130.
[19] O. Iyama, K. Kato and J.-I. Miyachi, Polygon of recollements and $N$-complexes, arXiv:1603.06056.
[20] M. Kalck and D. Yang, Relative singularity categories I: Auslander resolutions, Adv. Math. 301, 973–1021 (2016).
[21] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63–102.
[22] B. Keller, Invariance and localization for cyclic homology of DG algebras, J. Pure Appl. Algebra 123 (1998), 223–273.
[23] B. Keller, On differential graded categories, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zurich, 2006.
[24] B. Keller and D. Vossieck, Sous les catégories dérivées, C. R. Acad. Sci. Paris 305 (1987), 225-228.
[25] J. Kulshammer, Prospects of algebra I: Basic properties, arXiv:1608.01934.
[26] Q. Liu and D. Yang, Blocks of group algebras are derived simple, Math. Z. 272 (2012), 913–920.
[27] X. H. Luo and P. Zhang, Monic representations and Gorenstein-projective modules, Pacific J. Math. 264 (2013), no. 1, 163–194.
19

[28] G. Maltsiniotis, Le théorème de Quillen, d’adjonction des foncteurs dérivés, revisité, C. R. Acad. Sci. Paris, Ser. I 344, pp. 549–552 (2007).

[29] J. Miyachi, Localization of triangulated categories and derived categories, J. Algebra 141 (1991), 463–483.

[30] A. Neeman, The Grothendieck duality theorem via Bousfield’s techniques and Brown representability, J. Amer. Math. Soc. 9 (1996), no. 1, 205–236.

[31] A. Neeman, Some adjoints in homotopy categories, Ann. of Math. (2) 171 (2010), no. 3, 2143–2155.

[32] P. Nicolás, On torsion torsionfree triples, Phd thesis, Murcia, (2007).

[33] C. Psaroudakis, Homological Theory of Recollements of Abelian Categories, J. Algebra 398 (2014), 63–110.

[34] C. Psaroudakis and J. Vitória, Recollements of Module Categories, Appl. Categ. Structures 22 (2014), no. 4, 579–593.

[35] A. Wiedemann, On stratifications of derived module categories, Canad. Math. Bull. 34 (1991), no. 2, 275–280.

[36] P. Zhang, Y. Zhang, G. D. Zhou, and L. Zhu, Unbounded ladders induced by Gorenstein algebras, arXiv:1507.07333.

Nan Gao, Department of Mathematics, Shanghai University, Shanghai 200444, PR China
E-mail address: nangao@shu.edu.cn

Chrysostomos Psaroudakis, Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway
E-mail address: chrysostomos.psaroudakis@math.ntnu.no