Tableau-based procedure for deciding satisfiability in the full coalitional multiagent epistemic logic

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Abstract

We study the multiagent epistemic logic CMAEL(CD) with operators for common and distributed knowledge for all coalitions of agents. We introduce Hintikka structures for this logic and prove that satisfiability in such structures is equivalent to satisfiability in standard models. Using this result, we design an incremental tableau based decision procedure for testing satisfiability in CMAEL(CD).

1 Introduction

Over the last two decades, multiagent epistemic logics have been found to be a useful tool for a variety of applications in computer science and AI ([1], [8]), the main among them being design, specification, and verification of distributed protocols ([6], [5], [2]). In this paper, we consider the full coalitional multiagent epistemic logic, involving modal operators for individual knowledge for each agent, as well as operators for common and distributed knowledge among any (non-empty) coalition of agents; we call that logic CMAEL(CD)2.

Most of the multiagent epistemic logics studied so far only cover fragments of CMAEL(CD); e.g., the logic considered in [2] contains, besides the individual knowledge modalities, the operator of distributed knowledge only for the whole set of agents in the language, while [9] extends that system with common knowledge operator for the whole set of agents. As far as we know, no provably complete deductive system or a decision procedure has been developed so far for CMAEL(CD), although [1] propose (without proof) an axiomatic system which is presumed to be complete for this logic.

One of the major issues in applying multiagent epistemic logics to design of distributed systems is the development of algorithms for constructive checking

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1The notion of agents used in this paper is abstract; for example, agents can be thought of as components of a distributed system.
2Abbreviated from Coalitional MultiAgent Epistemic Logic with Common and Distributed knowledge.
of formulae of those logics for satisfiability, i.e., checking if a formula is satisfiable and, if so, constructing a model for it. The main purpose of this paper is to develop a tableau-based algorithm for the constructive satisfiability problem for $\text{CMAEL}(\text{CD})$. In the recent precursor ([4]) to the present paper, we have developed such an algorithm for the multiagent epistemic logic with operators of individual knowledge, as well as common and distributed knowledge for the set of all agents. In the present paper, we extend the results of [4] to $\text{MAEL}(\text{CD})$. The main challenge in such an extension lies in handling the operators of distributed knowledge parameterized by coalitions of agents. Thus, even though the style of the tableaux presented here is similar to the ones from [4], the proof of correctness of the procedure required more involved model-theoretic techniques building on those used in [2]. Consequently, the present paper substantially focuses on overcoming challenges raised by the presence in the language of coalitional distributed knowledge modalities.

The satisfiability-checking algorithms of both [4] and the present paper are based on the incremental tableaux in the style first proposed in [10] and adapted recently to logics of strategic ability in multiagent systems in [3]. Besides our conviction that this approach to building decision procedures for logics of multiagent systems is practically most optimal, the uniformity of method and style of these tableaux is deliberate, as it reflects our intention to eventually integrate them into a tableau-based decision procedure for comprehensive logical systems for reasoning about knowledge, time, and strategic abilities of agents and coalitions in multiagent systems.

The structure of the paper is as follows: Section 2 presents the syntax and semantics of $\text{CMAEL}(\text{CD})$; in Sections 3 and 4 we introduce Hintikka structures for this logic and prove that satisfiability in Hintikka structures is equivalent to satisfiability in models. Then, in Sections 5 and 6 we develop the tableau procedure for testing satisfiability of $\text{CMAEL}(\text{CD})$-formulae and sketch proofs of its soundness, completeness, and termination, and briefly estimate its complexity. We illustrate our tableau procedure with two examples in the Appendix.

2 Syntax and semantics of the logic $\text{CMAEL}(\text{CD})$

The language $\mathcal{L}$ of $\text{CMAEL}(\text{CD})$ contains a (finite or countable) set $\mathcal{AP}$ of atomic propositions, whose arbitrary members we typically denote by $p, q, r, \ldots$; a finite, non-empty set $\Sigma$ of names of agents, whose arbitrary members we typically denote by $a, b, \ldots$ and whose subsets, called coalitions, we typically denote by $A, B, \ldots$ (possibly with decorations); a sufficient repertoire of the Boolean connectives; and, for every non-empty coalition $A$, the modal operators $D_A$ (“it is distributed knowledge among $A$ that . . .”) and $C_A$ (“it is common knowledge among $A$ that . . .”). The formulae of $\mathcal{L}$ are thus defined as follows:

$$\varphi ::= p \mid \neg \varphi \mid (\varphi_1 \land \varphi_2) \mid D_A \varphi \mid C_A \varphi,$$

where $p$ ranges over $\mathcal{AP}$ and $A$ ranges over non-empty subsets of $\Sigma$; the set of all such subsets will henceforth be denoted by $\mathcal{P}^+(\Sigma)$. The other Boolean
connectives can be defined as usual. We denote formulae of $\mathcal{L}$ by $\varphi, \psi, \chi, \ldots$ (possibly with decorations) and omit parentheses in formulae whenever it does not result in ambiguity. We write $\varphi \in \mathcal{L}$ to mean that $\varphi$ is a formula of $\mathcal{L}$.

The distributed knowledge operator $D_A \varphi$ intuitively means that a “super-agent”, somebody who knows everything that any of the agents in $A$ knows, can obtain $\varphi$ as a logical consequence of his knowledge. For example, if agent $a$ knows $\psi$ and agent $b$ knows $\psi \rightarrow \chi$, then $D_{\langle a,b \rangle} \chi$ is true even though neither $a$ nor $b$ knows $\chi$. The operators of individual knowledge $K_a \varphi$ (“agent $a$ knows that $\varphi$”), for $a \in \Sigma$, can then be defined as $D_{\langle a \rangle} \varphi$, henceforth written $D_a \varphi$.

The common knowledge operator $C_A \varphi$ means that $\varphi$ is “public knowledge” among $A$, i.e., that every agent in $A$ knows $\varphi$, and knows that every agent in $A$ knows $\varphi$, etc. For example, it is common knowledge among drivers that green light means “go” and red light means “stop”. Formulae of the form $\neg C_A \varphi$ are referred to as (epistemic) eventualities, for the reasons given later on.

Formulae of $\mathcal{L}$ are interpreted over coalitioni multiagent epistemic models. In this paper, we also need the auxiliary notions of coalitioni multiagent epistemic structures and frames, which we now define.

**Definition 2.1** A coalitioni multiagent epistemic structure (CMAES, for short) is a tuple $\mathfrak{S} = (\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)})$, where

1. $\Sigma$ is a finite, non-empty set of agents;
2. $S \neq \emptyset$ is a set of states;
3. for every $A \in \mathcal{P}^+(\Sigma)$, $R^D_A$ is a binary relation on $S$;
4. for every $A \in \mathcal{P}^+(\Sigma)$, $R^C_A$ is the reflexive, transitive closure of $\bigcup_{A' \subseteq A} R^D_{A'}$.

**Definition 2.2** A coalitioni multiagent epistemic frame (CMAEF) is a CMAES $\mathfrak{F} = (\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)})$, where each $R^D_A$ is an equivalence relation satisfying the following condition:

$$\tag{†} \quad R^D_A = \bigcap_{a \in A} R^D_a$$

If condition $(\dagger)$ above is replaced by the following, weaker, one:

$$\tag{††} \quad R^D_B \subseteq R^D_A \quad \text{whenever} \quad B \subseteq A,$$

then $\mathfrak{F}$ is a coalitioni multiagent epistemic pseudo-frame (pseudo-CMAEF).

Note that in every (pseudo-)CMAEF $R^D_A \subseteq \bigcap_{a \in A} R^D_a$, and hence $\bigcup_{A' \subseteq A} R^D_{A'} = \bigcup_{a \in A} R^D_a$. Thus, condition $(4)$ of Definition 2.1 is equivalent to requiring that, in (pseudo-)CMAEFs, $R^C_A$ is the transitive closure of $\bigcup_{a \in A} R^D_a$, for every $A \in \mathcal{P}^+(\Sigma)$. Moreover, each $R^C_A$ in a (pseudo-)CMAEF is an equivalence relation.

**Definition 2.3** A coalitioni multiagent epistemic model (CMAEM) is a tuple $\mathcal{M} = (\mathfrak{F}, \mathcal{AP}, L)$, where $\mathfrak{F}$ is a CMAEF, $\mathcal{AP}$ is a set of atomic propositions, and $L : S \rightarrow \mathcal{P}(\mathcal{AP})$ is a labeling function, assigning to every state $s$ the set $L(s)$ of atomic propositions true at $s$. If $\mathfrak{F}$ is a pseudo-CMAEF, then $\mathcal{M} = (\mathfrak{F}, \mathcal{AP}, L)$ is a multiagent coalitioni pseudo-model (pseudo-CMAEM).
The satisfaction relation between (pseudo-)CMAEMs, states, and formulae is defined in the standard way. In particular,

- \( M, s \models D_A \varphi \) iff \( (s, t) \in R^D_A \) implies \( M, t \models \varphi \);

- \( M, s \models C_A \varphi \) iff \( (s, t) \in R^C_A \) implies \( M, t \models \varphi \).

**Definition 2.4** Given a (pseudo-)CMAEM \( M \) and \( \varphi \in L \), we say that \( \varphi \) is satisfiable in \( M \) if \( M, s \models \varphi \) holds for some \( s \in M \) and say that \( \varphi \) is valid in \( M \) if \( M, s \models \varphi \) holds for every \( s \in M \). Satisfiability and validity in a class of (pseudo-)models are defined accordingly.

The truth condition for the operator \( C_A \) can be re-stated in terms of reachability. Let \( \mathcal{F} \) be a (pseudo-)CMAEF with state space \( S \) and let \( s, t \in S \). We say that \( t \) is \( A \)-reachable from \( s \) if either \( s = t \) or, for some \( n \geq 1 \), there exists a sequence \( s = s_0, s_1, \ldots, s_{n-1}, s_n = t \) of elements of \( S \) such that, for every \( 0 \leq i < n \), there exists \( a \in A \) such that \( (s_i, s_i+1) \in R^D_a \). It is then easy to see that the following truth condition for \( C_A \) is equivalent to the one given above:

- \( M, s \models C_A \varphi \) iff \( M, t \models \varphi \) whenever \( t \) is \( A \)-reachable from \( s \).

Note also, that if \( \Sigma = \{ a \} \), then \( D_a \varphi \leftrightarrow C_a \varphi \) is valid for all \( \varphi \in L \). Thus, the single-agent case is trivialized and, therefore, we assume throughout the remainder of the paper that \( \Sigma \) contains at least 2 agents.

### 3 Hintikka structures

Despite our ultimate interest in satisfiability of finite sets of formulae in CMAEMs, the tableaux we present check for the existence of a more general kind of semantic structure for \( \Theta \) than a model, namely a **Hintikka structure**. In this section, we show that Hintikka structures satisfy the same sets of formulae as pseudo-CMAEMs; in the next section, we show that CMAEMs satisfy the same sets of formulae as pseudo-CMAEMs. Consequently, testing for satisfiability in a Hintikka structure can replace testing for satisfiability in a CMAEM. In the following discussion, for brevity, we only consider single formulae; the extension to finite sets of formulae is straightforward.

The most fundamental difference between (pseudo-)models and Hintikka structures is that while the former specify the truth value of every formula of \( L \) at each state, the latter only do so for the formulae relevant to the evaluation of a fixed formula \( \theta \). Another important difference is that the accessibility relations in (pseudo-)models must satisfy the explicitly stated conditions of Definition 2.2, while in Hintikka structures conditions are only imposed on the labels of the states in such a way that every Hintikka structure generates, through the construction of Lemma 3.5 below, a pseudo-CMAEM so that the “truth” of formulae in the labels is preserved in the resultant pseudo-model. To define Hintikka structures, we need the following auxiliary notion.
Definition 3.1 A set $\Delta \subseteq \mathcal{L}$ is fully expanded if it satisfies the following conditions:

- if $\neg \phi \in \Delta$, then $\phi \in \Delta$;
- if $\phi \land \psi \in \Delta$, then $\phi \in \Delta$ and $\psi \in \Delta$;
- if $\neg (\phi \land \psi) \in \Delta$, then $\neg \phi \in \Delta$ or $\neg \psi \in \Delta$;
- if $D_A \phi \in \Delta$, then $D_{A'} \phi \in \Delta$ for every $A'$ such that $A \subseteq A' \subseteq \Sigma$;
- if $C_A \phi \in \Delta$, then $\phi \in \Delta$;
- if $C_A \phi \in \Delta$, then $D_a (\phi \land C_A \phi) \in \Delta$ for every $a \in A$;
- if $\neg C_A \phi \in \Delta$, then $\neg D_a (\phi \land C_A \phi) \in \Delta$ for some $a \in A$;
- if $\neg D_A \neg D_B \phi \in \Delta$, then $D_{(A \land B)} \phi \in \Delta$.

Definition 3.2 A coalitional multi-agent epistemic Hintikka structure (CMAEHS for short) is a tuple $(\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, H)$ such that $(\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)})$ is a CMAES, and $H$ is a labeling of the elements of $S$ with sets of formulae of $\mathcal{L}$ satisfying the following constraints:

- H1 if $\neg \phi \in H(s)$, then $\phi \notin H(s)$, for every $s \in S$;
- H2 $H(s)$ is fully expanded, for every $s \in S$;
- H3 if $\neg D_A \phi \in H(s)$, then there exists $t \in S$ such that $(s, t) \in R^D_A$ and $\neg \phi \in H(t)$;
- H4 if $(s, t) \in R^D_A$, then $D_{A'} \phi \in H(s)$ iff $D_{A'} \phi \in H(t)$, for every $A' \subseteq A$;
- H5 if $\neg C_A \phi \in H(s)$, then there exists $t \in S$ such that $(s, t) \in R^C_A$ and $\neg \phi \in H(t)$.

Definition 3.3 Let $\theta \in \mathcal{L}$, $\Theta \subseteq \mathcal{L}$, and $\mathcal{H}$ be a CMAEHS with state space $S$. We say that $\mathcal{H}$ is a CMAEHS for $\theta$, or that $\theta$ is satisfiable in $\mathcal{H}$, if $\theta \in H(s)$ for some $s \in S$; we say that $\Theta$ is satisfiable in $\mathcal{H}$ if $\Theta \subseteq H(s)$.

We now prove that $\theta \in \mathcal{L}$ is satisfiable in a pseudo-CMAEM iff there exists a CMAEHS for $\theta$. Given a pseudo-CMAEM $\mathcal{M} = (\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, \mathcal{AP}, L)$, we define the extended labeling function $L^+: S \mapsto \mathcal{P}(\mathcal{L})$ on $\mathcal{M}$ as follows: $L^+(s) = \{ \phi \mid M, s \models \phi \}$. Then, it is routine to check the following.

Lemma 3.4 Let $\mathcal{M} = (\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, \mathcal{AP}, L)$ be a pseudo-CMAEM satisfying $\theta$ and let $L^+$ be the extended labeling on $\mathcal{M}$. Then, $(\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, L^+)$ is a CMAEHS for $\theta$.

Now, we argue in the opposite direction.
Lemma 3.5 Let $\theta \in \mathcal{L}$ be such that there exists a CMAEHS for $\theta$. Then, $\theta$ is satisfiable in a pseudo-CMAEM.

Proof. Let $\mathcal{H} = (\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, H)$ be a CMAEHS for $\theta$. We construct a pseudo-CMAEM $\mathcal{M}'$ satisfying $\theta$ out of $\mathcal{H}$ as follows.

First, for every $A \in \mathcal{P}^+(\Sigma)$, let $R^D_A$ be the reflexive, symmetric, and transitive closure of $\bigcup_{a \in A} R^D_a$ and let $R^C_A$ be the transitive closure of $\bigcup_{a \in A} R^C_a$. Notice that $R^D_A \subseteq R^D_A$ and $R^C_A \subseteq R^C_A$ for every $A \in \mathcal{P}^+(\Sigma)$. Second, let $L(s) = H(s) \cap \mathcal{A}P$, for every $s \in S$. It is then easy to check that $\mathcal{M}' = (\Sigma, S, \{R^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, \mathcal{A}P, L)$ is a pseudo-CMAEM.

To complete the proof of the lemma, we show, by induction on the structure of $\chi \in \mathcal{L}$ that, for every $s \in S$ and every $\chi \in \mathcal{L}$, the following hold:

(i) $\chi \in H(s)$ implies $\mathcal{M}', s \Vdash \chi$;

(ii) $\neg \chi \in H(s)$ implies $\mathcal{M}', s \Vdash \neg \chi$.

Let $\chi$ be some $p \in \mathcal{A}P$. Then, $p \in H(s)$ implies $p \in L(s)$ and, thus, $\mathcal{M}', s \Vdash p$; if, on the other hand, $\neg p \in H(s)$, then due to (H1), $p \notin H(s)$ and thus $p \notin L(s)$; hence, $\mathcal{M}', s \Vdash \neg p$.

Assume that the claim holds for all subformulae of $\chi$; then, we have to prove that it holds for $\chi$, as well.

Suppose that $\chi$ is $\neg \varphi$. If $\neg \varphi \in H(s)$, then the inductive hypothesis immediately gives us $\mathcal{M}', s \Vdash \neg \varphi$; if, on the other hand, $\neg \varphi \notin H(s)$, then by virtue of (H2), $\varphi \in H(s)$ and hence, by inductive hypothesis, $\mathcal{M}', s \Vdash \varphi$ and thus $\mathcal{M}', s \Vdash \neg \varphi$.

The case of $\chi = \varphi \land \psi$ is straightforward, using (H2).

Suppose that $\chi$ is $\mathcal{A}D_A \varphi$. Assume, first, that $\mathcal{A}D_A \varphi \in H(s)$. In view of the inductive hypothesis, it suffices to show that $(s, t) \in R^D_A$ implies $t \in H(t)$. So, assume that $(s, t) \in R^D_A$. There are two cases to consider. If $s = t$, then the conclusion immediately follows from (H2). If, on the other hand, $s \neq t$, then there exists an undirected path from $s$ to $t$ along the relations $R^D_A$, where each $A'$ is a superset of $A$. Then, in view of (H4), $\mathcal{A}D_A \varphi \in H(t)$; hence, by (H2), $\varphi \in H(t)$, as desired.

Assume, next, that $\neg \mathcal{A}D_A \varphi \in H(s)$. In view of the inductive hypothesis, it suffices to show that there exist $t \in S$ such that $(s, t) \in R^D_A$ and $\neg \varphi \in H(t)$. By (H3), there exists $t \in S$ such that $(s, t) \in R^D_A$ and $\neg \varphi \in H(t)$. As $R^D_A \subseteq R^D_A$, the desired conclusion follows.

Suppose now that $\chi$ is $\mathcal{A}C_A \varphi$. Assume that $\mathcal{A}C_A \varphi \in H(s)$. In view of the inductive hypothesis, it suffices to show that if $t$ is $A$-reachable from $s$ in $\mathcal{M}'$, then $\varphi \in H(t)$. So, assume that either $s = t$ or, for some $n \geq 1$, there exists a sequence of states $s = s_0, s_1, \ldots, s_{n-1}, s_n = t$ such that, for every $0 \leq i < n$, there exists $a_i \in \Sigma$ such that $(s_i, s_{i+1}) \in R^D_A$. In the former case, the desired conclusion follows from (H2). In the latter case, we can show by induction on $0 \leq i < n$ that $\mathcal{A}C_{a_i} (\varphi \land \mathcal{A}C_{a_i} \varphi) \in H(s_i)$. Then, $\mathcal{A}C_{a_{n-1}} (\varphi \land \mathcal{A}C_{a_{n-1}} \varphi) \in H(s_{n-1})$, and thus, in view of (H3) and (H2), $\varphi \in H(t)$.

Assume, on the other hand, that $\neg \mathcal{A}C_A \varphi \notin H(s)$. Then, the desired conclusion follows from (H6), the fact that $R^C_A \subseteq R^C_A$, and the inductive hypothesis.
Theorem 3.6 Let \( \theta \in \mathcal{L} \). Then, \( \theta \) is satisfiable in a pseudo-CMAEM iff there exists a CMAEHS for \( \theta \).

**Proof.** Immediately follows from Lemmas 3.4 and 3.5. \( \square \)

### 4 Equivalence of CMAEMs and pseudo-CMAEMs

In the present section, we prove that pseudo-CMAEMs and CMAEMs satisfy the same sets of formulae. The right-to-left direction is immediate, as every CMAEM is a pseudo-CMAEM. For the left-to-right direction, we use a modification of the construction from [2, appendix A1] to show that if \( \theta \in \mathcal{L} \) is satisfiable in a pseudo-CMAEM, then it is satisfiable in a “tree-like” pseudo-CMAEM that actually turns out to be a bona-fide CMAEM.

**Definition 4.1** Let \( \mathcal{M} = (\Sigma, S, \{\mathcal{R}^A_D\}_{A \in \mathcal{P}^+(\Sigma)}, \{\mathcal{R}^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, \mathcal{AP}, L) \) be a (pseudo-) CMAEM and let \( s,t \in S \). A maximal path from \( s \) to \( t \) in \( \mathcal{M} \) is a sequence \( s = s_0, A_0, s_1, \ldots, s_{n-1}, A_{n-1}, s_n = t \) such that, for every \( 0 \leq i < n \), \((s_i, s_{i+1}) \in \mathcal{R}^D_A \); but \((s_i, s_{i+1}) \in \mathcal{R}^D_B \) does not hold for any \( B \) with \( A_i \subset B \subset \Sigma \).

A segment \( \rho' \) of a maximal path \( \rho \) starting and ending with a state is a sub-path of \( \rho \).

**Definition 4.2** Let \( \rho = s_0, A_0, \ldots, A_{n-1}, s_n \) be a maximal path in \( \mathcal{M} \). The reduction of \( \rho \) is obtained by, first, replacing in \( \rho \) every longest sub-path \( s_p, A_p, s_{p+1}, \ldots, A_{p+q-1}, s_{p+q} \) such that \( s_p = s_{p+1} = \ldots = s_{p+q} \) with \( s_p \) (i.e., removing loops) and, then, by replacing in the resultant path every longest sub-path \( s_j, A_j, s_{j+1}, \ldots, A_{j+m-1}, s_{j+m} \) such that \( A_j = A_{j+1} = \ldots = A_{j+m-1} \) with \( s_j, A_j, s_{j+m} \) (i.e., collapsing multiple consecutive transitions along the same relation with a single transition). A maximal path is reduced if it equals its own reduction.

**Definition 4.3** A (pseudo-)CMAEM \( \mathcal{M} \) is tree-like if, for every \( s,t \in \mathcal{M} \), there exists at most one reduced maximal path from \( s \) to \( t \).

**Lemma 4.4** If \( \theta \in \mathcal{L} \) is satisfiable in a pseudo-CMAEM, then it is satisfiable in a (tree-like) CMAEM.

**Proof.** Suppose that \( \theta \) is satisfied in a pseudo-CMAEM \( \mathcal{M} = (\Sigma, S, \{\mathcal{R}^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{\mathcal{R}^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, \mathcal{AP}, L) \) at state \( s \). To build a tree-like CMAEM satisfying \( \theta \), we use a slight modification of the standard technique of tree-unraveling. The only difference between our construction and the standard tree-unraveling is that the state space of our tree model is made up of all maximal paths in \( \mathcal{M} \) rather than all paths whatsoever.

Let \( \mathcal{M}' = (\Sigma, S', \{\mathcal{R}^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{\mathcal{R}^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, \mathcal{AP}, L') \) be the submodel of \( \mathcal{M} \) generated by \( s \). Then, \( \mathcal{M}', s \models \theta \) since \( \mathcal{M} \) and \( \mathcal{M}' \) are locally bisimilar at \( s \).
Next, we unravel $\mathcal{M}'$ into a model $\mathcal{M}'' = (\Sigma, S'', \{R'^D_A\}_{A \in \mathcal{P}^+(\Sigma)}, \{R'^C_A\}_{A \in \mathcal{P}^+(\Sigma)}, A\mathcal{P}, L'')$ as follows. First, call a maximal path $\rho$ in $\mathcal{M}'$ an $s$-max-path if the first component of $\rho$ is $s$. Denote the last element of $\rho$ by $l(\rho)$.

Notice that $s$ by itself is an $s$-max-path. Now, let $S''$ be the set of all $s$-max-paths in $\mathcal{M}'$. For every $A \in \mathcal{P}^+(\Sigma)$, let $R'^D_A$ be $\{(\rho, \rho') \mid \rho, \rho' \in S'' \text{and } \rho = A, l(\rho')\}$ and let, furthermore, $R'^D_A$ to be the reflexive, symmetric, and transitive closure of $R'^D_A$. Notice that $(\rho, \rho') \in R'^D_A$ holds iff one of the paths $\rho$ and $\rho'$ extends the other by a sequence of $A$-steps. Therefore, two different states in $S''$ can only be connected by $R'^D_A$ for at most one maximal coalition $A$.

Further, we stipulate the following downwards closure condition: whenever $(\rho, \tau) \in R'^D_A$ and $B \subseteq A$, then $(\rho, \tau) \in R'^D_B$. The relations $R'^D_A$ are then defined as in any CMAEF. To complete the definition of $\mathcal{M}''$, we put $L''(\rho) = L'(l(\rho))$, for every $\rho \in S''$.

It is clear from the construction that $\mathcal{M}''$ is a pseudo-CMAEM. We will now show that it actually is a (tree-like) CMAEM and that it satisfies $\theta$. To prove the first part of the claim, we need some extra terminology.

We call a maximal path $\rho_1, A_1, \rho_2, \ldots, A_{n-1}, \rho_n$ in $\mathcal{M}''$ primitive if, for every $0 \leq i < n$, either $(\rho_i, \rho_{i+1}) \in R'^D_{A_i}$ or $(\rho_{i+1}, \rho_i) \in R'^D_{A_i}$. A primitive path $\rho_1, A_1, \rho_2, \ldots, A_{n-1}, \rho_n$ is non-redundant if there is no $0 \leq i < n$ such that $\rho_i = \rho_{i+2}$ and $A_i = A_{i+1}$. Intuitively, in a non-redundant path we never go from a state $\rho$ (forward or backward) along a relation and then immediately back to $\rho$ along the same relation. Since the relations $R'^D_A$ are edges of a tree, it immediately follows that:

(‡) for every pair of states $\rho, \tau \in S''$, there exists at most one non-redundant primitive path from $\rho$ to $\tau$.

Lastly, we call a primitive path $\rho_1, A, \rho_2, \ldots, A, \rho_n$ an $A$-primitive path.

We will now show that maximal reduced paths in $\mathcal{M}''$ stand in one-to-one correspondence with non-redundant primitive paths. It will then follow from (‡) that maximal reduced paths between any two states of $\mathcal{M}''$ are unique, and thus $\mathcal{M}''$ is tree-like, as claimed. Let $P = \rho_1, A_1, \ldots, A_{n-1}, \rho_n$, where $\rho_1 = \rho$ and $\rho_n = \tau$, be a maximal reduced path from $\rho$ to $\tau$ in $\mathcal{M}''$. Since $(\rho_1, \rho_{i+1}) \in R'^D_{A_i}$, there exists a non-redundant $A_i$-primitive path from $\rho_i$ to $\rho_{i+1}$, in view of (‡) is unique. Let us obtain a path $P'$ from $\rho$ to $\tau$ by replacing in $\rho$ every link $(\rho_i, A_i, \rho_{i+1})$ by the corresponding non-redundant $A_i$-primitive path from $\rho_i$ to $\rho_{i+1}$. Call $P'$ an expansion of $P$. In view of (‡), every path has a unique expansion. Now, it is easy to see that $P$ is a reduction of $P'$. Since the reduction of a given path is unique, too, it follows that there exists a one-to-one correspondence between maximal reduced paths and non-redundant primitive paths in $\mathcal{M}''$.

Next, we prove that $\bigcap_{a \in A} R'^D_a \text{ for every } A \in \mathcal{P}^+(\Sigma)$, and therefore that $\mathcal{M}''$ is a CMAEM. The left to right inclusion is immediate from the construction (namely, because of the downward saturation condition we imposed on epistemic relations). For the opposite direction, assume that $(s, t) \in R'^D_a$ holds for every $a \in A$. Then, for every $a \in A$, there exists a path, and therefore a maximal reduced path, from $s$ to $t$ along relations $R'^D_{A'}$ such that $a \in A'$. As $\mathcal{M}''$ is tree-like, there is only one maximal reduced path from $s$ to $t$. Therefore,
the relations \( R^D_A \) linking \( s \) to \( t \) along this path are such that \( A \subseteq A' \) for every \( A' \). Then, by the downwards closure condition, there is a path from \( s \) to \( t \) along the relation \( R''_A \) and, hence, \( (s,t) \in R''_A \), as desired.

Finally, it remains to prove that \( M'' \) satisfies \( \theta \). First, notice that \( (\rho,\tau) \in R''_A \) iff there exists an \( A \)-primitive path from \( \rho \) to \( \tau \); hence, as every \( R'_A \) is an equivalence relation, if \( (\rho,\tau) \in R''_A \), then \( (l(\rho),l(\tau)) \in R'_A \). It is now easy to check that the relation \( Z = \{ (\rho,l(\rho) \mid \rho \in S'' \} \) is a bisimulation between \( M'' \) and \( M' \). Since \( (s,l(s)) \in Z \), it follows that \( M'',s \models \theta \), and we are done. \( \square \)

**Theorem 4.5** Let \( \theta \in \mathcal{L} \). Then, \( \theta \) is satisfiable in a CMAEM iff there exists a Hintikka structure for \( \theta \).

**Proof.** Immediate from Theorem 3.6 and Lemma 4.4. \( \square \)

### 5 Tableaux for CMAEL(CD)
#### 5.1 Basic ideas and overview of the tableau procedure

The tableau procedure for testing a formula \( \theta \in \mathcal{L} \) for satisfiability is an attempt to construct a non-empty graph \( T^\theta \) (called tableau) representing all possible CMAEHSs for \( \theta \). The philosophy underlying our tableau algorithm is essentially the same as the one underpinning the tableau procedure for LTL from [10], recently adapted to Alternating-time logic ATL in [3] and to multi-agent epistemic logic with operators of common and distributed knowledge for the whole set of agents in [4]. To make the present paper self-contained, we first outline the basic idea behind the tableau algorithm for CMAEL(CD), following [3] and [4]. The details of the tableaux presented here, obviously, are specific to CMAEL(CD).

Usually, tableaux check for satisfiability by decomposing the input formula into “semantically simpler” formulae. In the classical propositional case, “semantically simpler” implies “smaller”, thus ensuring the termination of the procedure. Another feature of the tableaux for classical propositional logic is that the decomposition into simpler formulae results in a simple tree, representing an exhaustive search for a model—or, to be more precise, a Hintikka set (the classical analogue of Hintikka structures)—for the input formula \( \theta \). If at least one leaf of the tree produces a Hintikka set for \( \theta \), the search has succeeded and \( \theta \) is pronounced satisfiable.

These two defining features of the classical tableau method do not directly apply to logics containing fixed point operators, such as \( C_A \). First, decomposing of a formula \( C_A \varphi \) produces the formulae of the form \( D_0(\varphi \land C_A \varphi) \), which are not exactly “semantically simpler”; rather, the unfolding of the monotone operator whose fixed point is \( C_A \varphi \) is effected. Hence, we cannot take termination for granted and need to put a mechanism in place that would guarantee it—in our tableaux, this mechanism consists in the use of prestates, whose role
is to ensure the finiteness, and hence termination, of the construction. Second, in the classical case, the only reason why the tableau might fail to produce a Hintikka set for the input formula is that every attempt to build such a set results in a collection of formulae containing a *patent inconsistency* (a pair of formulae \( \phi, \neg \phi \)). In the case of \textbf{CMAEL(CD)}, there are other such reasons, as the tableau is meant to represent CMAEHSs, which are more complicated structures than classical Hintikka sets. One additional possible reason for a failure of a node of the tableau to be satisfiable has to do with eventualities: the presence of an eventuality \( \neg \text{C}_A \varphi \) in the label of a state \( s \) of a CMAEHS requires that there is an \( A \)-path from \( s \) to a state \( t \) whose label contains \( \neg \varphi \). The analogue of this condition in the tableau is called *realization of eventualities*. Thus, all eventualities in a tableau should be realized in order for the tableau to be “good”, i.e. to eventually produce a Hintikka structure. The third possible reason for existence of “bad” nodes in the tableau has to do with successor nodes—it may so happen that some of the successors of a node \( s \) whose satisfiability is necessary for the satisfaction of \( s \) itself are unsatisfiable; a “good” tableau should not contain such “bad” nodes.

The tableau procedure consists of three major phases: *construction*, *prestate elimination*, and *state elimination*. During the construction phase, we produce a directed graph \( \mathcal{P}^\theta \)—called the *pretableau* for \( \theta \)—whose set of nodes properly contains the set of nodes of the tableau \( \mathcal{T}^\theta \) we are building. Nodes of \( \mathcal{P}^\theta \) are sets of formulae, some of which, called *states*, are meant to represent states of a Hintikka structure, while others, called *prestates*, play an auxiliary, technical role in the construction of \( \mathcal{P}^\theta \). During the prestate elimination phase, we create a smaller graph \( \mathcal{T}^\theta_0 \) out of \( \mathcal{P}^\theta \), called the *initial tableau for \( \theta \)*, by eliminating all the prestates of \( \mathcal{P}^\theta \) and adjusting its edges—prestates have already fulfilled their role and can be discharged. Finally, during the state elimination phase, we remove from \( \mathcal{T}^\theta_0 \) all the states, if any, that cannot be satisfied in any CMAEHS, for one of the three above-mentioned reasons. The elimination procedure results in a (possibly empty) subgraph \( \mathcal{T}^\theta \) of \( \mathcal{T}^\theta_0 \), called the *final tableau for \( \theta \)*. If some state \( \Delta \) of \( \mathcal{T}^\theta \) contains \( \theta \), we declare \( \theta \) satisfiable; otherwise, we declare it unsatisfiable.

The reader is referred to the examples given in the Appendix, to trace all phases of the construction of the tableau.

### 5.2 Construction phase

At this phase, we build the pretableau \( \mathcal{P}^\theta \)—a directed graph whose nodes are sets of formulae, coming in two varieties: *states* and *prestates*. States are meant to represent states of CMAEHSs the tableau attempts to construct, while prestates are “embryo states”, expanded into states in the course of the construction. Formally, states are fully expanded (recall Definition 3.1), while prestates do not have to be so. Moreover, \( \mathcal{P}^\theta \) contains two types of edge. As already mentioned, a tableau attempt to produce a compact representation of all possible CMAEHSs for the input formula; in this attempt, it organizes an exhaustive search for such CMAEHSs. One type of edge, depicted by unmarked
double arrows $\Rightarrow$, represents the expansion of the tableau as a search tree. That exhaustive search considers all possible alternatives, which arise when expanding prestates into (fully expanded) states by branching in the disjunctive cases. Thus, when we draw a double arrow from a prestate $\Gamma$ to states $\Delta$ and $\Delta'$ (depicted as $\Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta'$, respectively), this intuitively means that, in any CMAEHS, a state satisfying $\Gamma$ has to satisfy at least one of $\Delta$ and $\Delta'$. Our first construction rule, (SR), prescribes how to create states from prestates.

Given a set $\Gamma \subseteq \mathcal{L}$, we say that $\Delta$ is a \textit{minimal fully expanded extension of} $\Gamma$ if $\Delta$ is fully expanded, $\Gamma \subseteq \Delta$, and there is no $\Delta'$ such that $\Gamma \subseteq \Delta' \subset \Delta$ and $\Delta'$ is fully expanded.

\textbf{Rule (SR)} Given a prestate $\Gamma$ such that (SR) has not been applied to it before, do the following:

1. Add all minimal fully expanded extensions $\Delta$ of $\Gamma$ as states;
2. For each so obtained state $\Delta$, put $\Gamma \Rightarrow \Delta$;
3. If, however, the pretableau already contains a state $\Delta'$ that coincides with $\Delta$, do not create another copy of $\Delta'$, but only put $\Gamma \Rightarrow \Delta'$.

We denote by $\text{states}(\Gamma)$ the (finite) set $\{ \Delta \mid \Gamma \Rightarrow \Delta \}$.

The second type of an edge featuring in our tableaux represents transition relations in the CMAEHS which it attempts to build. Accordingly, this type of edge is represented by single arrows marked with formulae whose presence in the source state requires the presence in the tableau of a target state, reachable by a particular relation. All such formulae have the form $\neg D_A \varphi$ (as can be seen from Definition 3.2). Intuitively, if, say, $\neg D_A \varphi \in \Delta$, then we need some prestate $\Gamma$ containing $\neg \varphi$ to be accessible from $\Delta$ by $R_A \Gamma$; the reason we mark this single arrow not just by coalition $A$, but by formula $\neg D_A \varphi$, is that it helps us remember not just what relation connects states satisfying $\Delta$ and $\Gamma$, but why we had to create this particular $\Gamma$. This information will be needed when we start eliminating prestates and then states.

The second construction rule, (DR), prescribes how to create prestates from states. This rule does not apply to states containing patent inconsistencies (such sets are called \textit{patently inconsistent}), as such states cannot be satisfied in any CMAEHS.

\textbf{Rule (DR)}: Given a state $\Delta$ such that $\neg D_A \varphi \in \Delta$, state $\Delta$ is not patently inconsistent, and (DR) has not been applied to it before, do the following:

1. Create a new prestate $\Gamma = \{ \neg \varphi \} \cup \bigcup A' \subseteq A \{ D_{A'} \psi \mid D_{A'} \psi \in \Delta \} \cup \bigcup A' \subseteq A \{ \neg D_{A'} \psi \mid \neg D_{A'} \psi \in \Delta \}$;
2. Connect $\Delta$ to $\Gamma$ with $\neg D_A \varphi$;
3. If, however, the tableau already contains a prestate $\Gamma' = \Gamma$, do not add to it another copy of $\Gamma'$, but simply connect $\Delta$ to $\Gamma'$ with $\neg D_A \varphi$. 

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When building a tableau for $\theta \in \mathcal{L}$, the construction phase begins with creating a single prestate $\{\theta\}$. Afterwards, we alternate between (SR) and (DR): first, (SR) is applied to the prestates created at the previous stage of the construction, then (DR) is applied to the states created at the previous stage. The construction phase comes to an end when every prestate required to be added to the pretableau has already been added (as prescribed in point 3 of (SR)), or when we end up with states to which (DR) does not apply.

Since we identify states and prestates whenever possible, to prove termination of the construction phase it suffices to show that there are only finitely many possible states and prestates. For that, we use the concept of an extended closure of a formula.

**Definition 5.1** Let $\theta \in \mathcal{L}$. The closure of $\theta$, denoted $\text{cl}(\theta)$, is the least set of formulae such that $\theta \in \text{cl}(\theta)$, $\text{cl}(\theta)$ is closed under subformulae, and the following conditions hold:

- if $D_A \varphi \in \text{cl}(\theta)$ and $A \subseteq A' \subseteq \Sigma$, then $D_{A'} \varphi \in \text{cl}(\theta)$;
- if $C_A \varphi \in \text{cl}(\theta)$, then $D_a (\varphi \land C_A \varphi) \in \text{cl}(\theta)$ for every $a \in A$.

The extended closure of $\theta$, denoted $\text{ecl}(\theta)$, is the least set such that, if $\varphi \in \text{cl}(\theta)$, then $\varphi, \neg \varphi \in \text{ecl}(\theta)$.

It is straightforward to check that $\text{ecl}(\theta)$ is finite for every $\theta$ and that all states and prestates of $P^\theta$ are subsets of $\text{ecl}(\theta)$; hence, their number is indeed finite; hence, the construction phase terminates.

### 5.3 Prestate elimination phase

At this phase, we remove from $P^\theta$ all the prestates and unmarked arrows, by applying the following rule:

(PR) For every prestate $\Gamma$ in $P^\theta$, do the following:

1. Remove $\Gamma$ from $P^\theta$;
2. If there is a state $\Delta$ in $P^\theta$ with $\Delta \xrightarrow{\chi} \Gamma$, then for every state $\Delta' \in \text{states}(\Gamma)$, put $\Delta \xrightarrow{\chi} \Delta'$;

The resultant graph is denoted $T^\theta_0$ and called the initial tableau.

### 5.4 State elimination phase

During this phase, we remove from $T^\theta_0$ states that are not satisfiable in any CMAEHS. Recall, that there are three reasons why a state $\Delta$ of $T^\theta_0$ can turn out to be unsatisfiable: $\Delta$ is patently inconsistent, or satisfiability of $\Delta$ requires satisfiability of some other unsatisfiable “successor” states, or $\Delta$ contains an eventuality that is not realized in the tableau. Accordingly, we have three elimination rules, (E1)–(E3).
Formally, the state elimination phase is divided into stages; we start at stage 0 with $T^θ_0$; at stage $n + 1$, we remove from the tableau $T^θ_n$ obtained at the previous stage exactly one state, by applying one of the elimination rules, thus obtaining the tableau $T^θ_{n+1}$. We state the rules below, where $S^θ_m$ denotes the set of states of $T^θ_m$.

(E1) If $\{ϕ, \neg ϕ\} ⊆ ∆ ∈ S^θ_n$, then obtain $T^θ_{n+1}$ by eliminating $∆$ from $T^θ_n$.

(E2) If $∆$ contains a formula $χ = \neg D_A ϕ$ and all states reachable from $∆$ by single arrows marked with $χ$ have been eliminated at previous stages, obtain $T^θ_{n+1}$ by eliminating $∆$ from $T^θ_n$.

For the third elimination rule, we need the concept of eventuality realization. We say that the eventuality $ξ = \neg C_A ϕ$ is realized at $∆$ in $T^θ_n$ if either $\neg ϕ ∈ ∆$ or there exists in $T^θ_n$ a finite path $∆_0, ∆_1, \ldots, ∆_m$ such that $∆_0 = ∆$, $\neg ϕ ∈ ∆_m$, and for every $0 ≤ i < m$ there exist $χ_i = D_B ψ_i$ such that $B ⊆ A$ and $∆_i χ_i −→ ∆_{i+1}$. We check for realization of $ξ$ by running the following marking procedure that marks all states that realize an eventuality $ξ$ in $T^θ_n$. Initially, we mark all $∆ ∈ S^θ_n$ such that $\neg ϕ ∈ ∆$. Then, we repeatedly do the following: if $∆ ∈ S^θ_n$ is unmarked and there exists at least one $∆'$ such that $∆ D_B ψ_i −→ ∆'$ for some $B ⊆ A$ and $∆'$ is marked, then $∆$ gets marked. The procedure ends when no more states get marked at a current round of marking. Note that marking is carried out with respect to a fixed eventuality $ξ$ and is, therefore, repeated as many times as the number of eventualities in (the states) of a tableau. Now, we can state our last rule.

(E3) If $∆ ∈ S^θ_n$ contains an eventuality $\neg C_A ϕ$ that is not realized at $∆$ in $T^θ_n$, then obtain $T^θ_{n+1}$ by removing $∆$ from $T^θ_n$.

We have so far described individual rules; to describe the state elimination phase as a whole, we must specify the order of their application. First, we apply (E1) to all the states of $T^θ_0$; once this is done, we do not need to apply (E1) again. The cases of (E2) and (E3) are more involved. After having applied (E3) we could have removed all the states accessible from some $∆$ along the arrows marked with some formula $χ$; hence, we need to reapply (E2) to the resultant tableau to remove such $∆$’s. Conversely, after having applied (E2), we could have thrown away some states that were needed for realizing certain eventualities; hence, we need to reapply (E3). Therefore, we need to apply (E3) and (E2) in a dovetailed sequence that cycles through all the eventualities. More precisely, we arrange all eventualities occurring in the tableau obtained from $T^θ_0$ after having applied (E1) in a list: $ξ_1, \ldots, ξ_m$. Then, we proceed in cycles. Each cycle consists of alternatingly applying (E3) to the pending eventuality (starting with $ξ_1$), and then applying (E2) to the resulting tableau, until all the eventualities have been dealt with. These cycles are repeated until no state is removed in a whole cycle. Then, the state elimination phase is over.

The graph produced at the end of the state elimination phase is called the final tableau for $θ$, denoted by $T^θ$ and its set of states is denoted by $S^θ$.

Definition 5.2 The final tableau $T^θ$ is open if $θ ∈ ∆$ for some $∆ ∈ S^θ$; otherwise, $T^θ$ is closed.
The tableau procedure returns “no” if the final tableau is closed; otherwise, it returns “yes” and, moreover, provides sufficient information for producing a finite model satisfying \( \theta \); that construction is sketched in Section 6.

6 Soundness, completeness, and complexity

The soundness of a tableau procedure amounts to claiming that if the input formula \( \theta \) is satisfiable, then the tableau for \( \theta \) is open. To establish soundness of the overall procedure, we use a series of lemmas showing that every rule by itself is sound; the soundness of the overall procedure is then an easy consequence. The proofs of the following two lemmas are straightforward.

**Lemma 6.1** Let \( \Gamma \) be a prestate of \( \mathcal{P}^\theta \) such that \( \mathcal{M}, s \vdash \Gamma \) for some CMAEM \( \mathcal{M} \) and \( s \in \mathcal{M} \). Then, \( \mathcal{M}, s \vdash \Delta \) holds for at least one \( \Delta \in \text{states}(\Gamma) \).

**Lemma 6.2** Let \( \Delta \in S^\theta_0 \) be such that \( \mathcal{M}, s \vdash \Delta \) for some CMAEM \( \mathcal{M} \) and \( s \in \mathcal{M} \), and let \( -\text{D}_A \phi \in \Delta \). Then, there exists \( t \in \mathcal{M} \) such that \( (s, t) \in R_A^\theta \) and \( \mathcal{M}, t \vdash \{ \neg \phi \} \cup \bigcup_{A' \subseteq A} \{ \text{D}_A \psi \mid \text{D}_A \psi \in \Delta \} \cup \bigcup_{A' \subseteq A} \{ -\text{D}_A \psi \mid -\text{D}_A \psi \in \Delta \} \).

**Lemma 6.3** Let \( \Delta \in S^\theta_0 \) be such that \( \mathcal{M}, s \vdash \Delta \) for some CMAEM \( \mathcal{M} \) and \( s \in \mathcal{M} \), and let \( -\text{C}_A \phi \in \Delta \). Then, \( -\text{C}_A \phi \) is realized at \( \Delta \) in \( T^\theta \).

**Proof idea.** As \( -\text{C}_A \phi \) is true at \( s \), there is a path in \( \mathcal{M} \) from \( s \) leading to a state satisfying \( \neg \phi \). As the tableaux organize the exhaustive search, a chain of tableau states corresponding to those states in the model will be produced.

**Theorem 6.4** If \( \theta \in \mathcal{L} \) is satisfiable in a CMAEM, then \( T^\theta \) is open.

**Proof sketch.** Using the preceding lemmas, show by induction on the number of stages in the state elimination process that no satisfiable state can be eliminated due to (E1)–(E3). The claim then follows from Lemma 6.1.

The completeness of a tableau procedure means that if the tableau for a formula \( \theta \) is open, then \( \theta \) is satisfiable in a CMAEM. In view of Theorem 4.5, it suffices to show that an open tableau for \( \theta \) can be turned into a CMAEHS for \( \theta \).

**Lemma 6.5** If \( T^\theta \) is open, then there exists a CMAEHS for \( \theta \).

**Proof sketch.** The CMAEHS \( \mathcal{H} \) for \( \theta \) is built out of the so-called final tree components. Each component is a tree-like CMAES with nodes labeled with states from \( S^\theta \), and is associated with a state \( \Delta \in S^\theta \) and an eventuality \( \xi \in \text{cl}(\theta) \) (such a component is denoted by \( T_{\Delta, \xi} \)). If \( \xi \notin \Delta \), then \( T_{\Delta, \xi} \) is a simple tree, whose root is labeled with \( \Delta \), that has exactly one leaf associated with each formula \( -\text{D}_A \psi \) marking an arrow from \( \Delta \) to some \( \Delta' \in S^\theta \); this leaf is labeled.
by $\Delta'$ and connected to the root by relation $R^D_\Delta$. If $\xi \in \Delta$, take the chain realizing $\chi$ at $\Delta$ and give each node “enough” successors, as prescribed above for simple trees. The crucial fact is that if $\xi'$ is an eventuality in $\Delta$ that is not “realized” inside $T_{\Delta,\xi}$, then $\xi'$ belongs to every leaf of $T_{\Delta,\xi}$. This allows us to stitch up all the $T_{\Delta,\xi}$s into a Hintikka structure. The procedure is recursive. All the eventualities are queued. We start from the component uniquely associated with $\theta$ (say, we take $T_{\Delta,\theta}$ where $\Delta$ is the least numbered state containing $\theta$; such a state exists as the tableau is open) and then replace each leaf of the structure built so far with the component associated with the set marking the leaf and the pending eventuality. The procedure is repeated in cycles until we have attached enough components to realize all eventualities. To obtain a CMAEHS, we put $H(\Delta) = \Delta$ for all $\Delta$'s.

\[ \text{Theorem 6.6 (Completeness)} \] Let $\theta \in \mathcal{L}$ and let $T^0$ be open. Then, $\theta$ is satisfiable in a CMAEM.

\[ \text{Proof.} \] Immediate from Lemma 6.5 and Theorem 4.5.

As for complexity of the procedure, for lack of space, we only state that our procedure runs within $O(k^{2n^2})$ steps, where $n$ is the size of the input formula and $k$ is the number of agents in the language. Therefore, the CMAEL(CD)-satisfiability is in ExpTime, which together with the ExpTime-hardness result from \([7]\) for a fragment of our logic containing, along with individual knowledge modalities, the common knowledge operators for the whole set of agents, implies that CMAEL(CD)-satisfiability is ExpTime-complete.

7 Concluding remarks

We have developed a sound and complete, incremental-tableau-based decision procedure for the full coalitional multiagent epistemic logic CMAEL(CD). We are convinced that this style of tableaux is more intuitive, practically more efficient and more adaptable than the top-down style of tableaux e.g., developed for a fragment of this logic in \([7]\), and therefore is suitable both for manual and automated execution. In particular, it is amenable to extension with operators for strategic abilities of the Alternating-time temporal logic ATL, a tableaux for which were developed in \([6]\). Merging these two systems is a topic of our future work.

Acknowledgments

We gratefully acknowledge the financial support from the National Research Foundation of South Africa through a research grant for the first author, and from the Claude Harris Leon Foundation, funding the second author’s post-doctoral fellowship at the University of the Witwatersrand, during which this research was done.
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A Examples

**Example 1** Let $\theta = \neg D_{\{a,c\}} C_{\{a,b\}} p \land C_{\{a,b\}} (p \land q)$, where $\Sigma = \{a, b, c\}$. To save space, we replace $\theta$ by the set of its conjuncts $\Theta = \{-D_{\{a,c\}} C_{\{a,b\}} p, C_{\{a,b\}} (p \land q)\}$. We also use some heuristics that we did not have space to discuss in the main part of the paper (they will, however, be explained in a follow up work).
The picture on the left below represents the final pretableau for Θ, while the picture on the right represents the initial tableau. Under the pictures we list formulae that occur in the labels of states and prestates.

\[ \chi_0 = \neg D_{\{a,c\}} C_{\{a,b\}} p; \chi_1 = \neg D_a (p \land C_{\{a,b\}} p); \chi_2 = \neg D_b (p \land C_{\{a,b\}} p); \]
\[ \Gamma_0 = \{ \neg D_{\{a,c\}} C_{\{a,b\}} p, C_{\{a,b\}} (p \land q) \}; \]
\[ \Delta_0 = \{ \neg D_{\{a,c\}} C_{\{a,b\}} p, D_a [(p \land q) \land C_{\{a,b\}} (p \land q)], D_b [(p \land q) \land C_{\{a,b\}} (p \land q)] \}; \]
\[ \Gamma_1 = \{ \neg C_{\{a,b\}} p, D_a [(p \land q) \land C_{\{a,b\}} (p \land q)] \}; \]
\[ \Delta_1 = \{ \neg D_a (p \land C_{\{a,b\}} p), D_a [(p \land q) \land C_{\{a,b\}} (p \land q)] \}; \]
\[ \Delta_2 = \{ \neg D_b (p \land C_{\{a,b\}} p), D_b [(p \land q) \land C_{\{a,b\}} (p \land q)] \}; \]
\[ \Delta_3 = \{ \neg (p \land C_{\{a,b\}} p), (p \land q) \land C_{\{a,b\}} (p \land q) \}; \]
\[ \Delta_4 = \{ \neg \neg p, (p \land q) \land C_{\{a,b\}} (p \land q), p, q, p \land q, D_a [(p \land q) \land C_{\{a,b\}} (p \land q)], D_b [(p \land q) \land C_{\{a,b\}} (p \land q)] \}; \]
\[ \Delta_5 = \{ \neg C_{\{a,b\}} p, \neg D_a (p \land C_{\{a,b\}} p), (p \land q) \land C_{\{a,b\}} (p \land q), p, q, p \land q, D_a [(p \land q) \land C_{\{a,b\}} (p \land q)], D_b [(p \land q) \land C_{\{a,b\}} (p \land q)] \}; \]

During the state-elimination phase, the state \( \Delta_3 \) is removed due to (E1), as it contains a patent inconsistency \( (p, \neg p) \). Then, the states \( \Delta_1, \Delta_2, \Delta_4, \) and \( \Delta_5 \) are eliminated due to (E3), as all of them contain the unrealized eventuality \( \neg C_{\{a,b\}} p \). Finally, \( \Delta_0 \) gets eliminated, as it has lost all its successors along the arrow marked with \( \chi_0 \). Thus, the final tableau for \( \Theta \) is an empty graph; therefore, \( \Theta \) is unsatisfiable.

**Example 2** Let \( \theta = C_{\{a,b\}} p \land C_{\{b,c\}} p \land \neg C_{\{a,c\}} p \), where \( \Sigma = \{a,b,c\} \). Once again, to save space, we replace \( \theta \) with \( \Theta = \{ C_{\{a,b\}} p, C_{\{b,c\}} p, \neg C_{\{a,c\}} p \} \). The picture below shows the final pretableau for \( \Theta \); the initial tableau is easily extracted from it, as in the previous example. Formulae that occur in the labels of states and prestates are listed under the picture.
\[ \chi_1 = \lnot D_a(p \land C_{(a,c)}p); \chi_2 = \lnot D_b(p \land C_{(a,c)}p); \]
\[ \Gamma_0 = \{ C_{(a,b)}p, C_{(b,c)}p, \lnot C_{(a,c)}p \}; \]
\[ \Delta_1 = \{ C_{(a,b)}p, C_{(b,c)}p, \lnot C_{(a,c)}p, D_a(p \land C_{(a,b)}p), D_b(p \land C_{(a,b)}p), D_a(p \land C_{(b,c)}p), \lnot D_b(p \land C_{(a,c)}p) \}; \]
\[ \Delta_2 = \{ C_{(a,b)}p, C_{(b,c)}p, \lnot C_{(a,c)}p, D_a(p \land C_{(a,b)}p), D_b(p \land C_{(a,b)}p), D_b(p \land C_{(b,c)}p), \lnot D_a(p \land C_{(a,c)}p) \}; \]
\[ \Gamma_1 = \{ \lnot(p \land C_{(a,c)}p), D_a(p \land C_{(a,b)}p) \}; \]
\[ \Gamma_2 = \{ \lnot(p \land C_{(a,c)}p), D_b(p \land C_{(b,c)}p) \}; \]
\[ \Delta_3 = \{ \lnot C_{(a,c)}p, D_a(p \land C_{(a,b)}p), \lnot D_b(p \land C_{(a,c)}p) \}; \]
\[ \Delta_4 = \{ \lnot C_{(a,c)}p, D_b(p \land C_{(a,b)}p), \lnot D_a(p \land C_{(a,c)}p) \}; \]
\[ \Delta_5 = \{ \lnot p, D_a(p \land C_{(a,b)}p), p \land D_a(p \land C_{(a,b)}p), p, C_{(a,b)}p \}; \]
\[ \Delta_6 = \{ \lnot p, D_b(p \land C_{(b,c)}p), p \land C_{(b,c)}p, p, C_{(b,c)}p \}; \]
\[ \Delta_7 = \{ \lnot C_{(a,c)}p, D_c(p \land C_{(b,c)}p), \lnot D_a(p \land C_{(a,c)}p) \}; \]
\[ \Delta_8 = \{ \lnot C_{(a,c)}p, D_c(p \land C_{(b,c)}p), \lnot D_b(p \land C_{(a,c)}p) \}; \]
\[ \Gamma_3 = \{ \lnot(p \land C_{(a,c)}p) \}; \]
\[ \Delta_9 = \{ \lnot C_{(a,c)}p, \lnot D_a(p \land C_{(a,c)}p) \}; \]
\[ \Delta_{10} = \{ \lnot C_{(a,c)}p, \lnot D_b(p \land C_{(a,c)}p) \}; \]
\[ \Delta_{11} = \{ \lnot p \}. \]

At the state elimination phase, states \( \Delta_5 \) and \( \Delta_6 \) get removed due to \( (E1) \). All other states remain in place; in particular, no states get eliminated due to \( (E3) \), because, from any state one can reach either \( \Delta_9 \) or \( \Delta_{13} \), both of which contain \( \lnot p \), and our only eventuality is \( \lnot C_{(a,c)}p \). Thus, \( \theta \) is satisfiable, and a Hintikka structure for it is readily extracted from the final tableau.