The Cowen-Douglas Theory for Operator Tuples and Similarity

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Abstract
We are concerned with the similarity problem for Cowen-Douglas operator tuples. The unitary equivalence counterpart was already investigated in the 1970’s and geometric concepts including vector bundles and curvature appeared in the description. As the Cowen-Douglas conjecture show, the study of the similarity problem has not been so successful until quite recently. The latest results reveal the close correlation between complex geometry, the corona problem, and the similarity problem for single Cowen-Douglas operators. Without making use of the corona theorems that no longer hold in the multi-variable setting, we prove that the single operator results for similarity remain true for Cowen-Douglas operator tuples as well.

Keywords: The Cowen-Douglas class, Similarity, Complex bundles, Curvature inequality

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1 Introduction
To study equivalence problems for bounded linear operators on Hilbert space to which standard methods do not apply, M. J. Cowen and R. G. Douglas introduced in the late 1970’s, a class of operators with a holomorphic eigenvector bundle structure [20, 21]. Prior to the introduction of the Cowen-Douglas class, the discussion on operator equivalence for even the adjoints of various shift operators (the most-mentioned entities in the study of operators) was non-existent. Their influential work connects concepts and results from complex geometry to the fundamental problem of determining operator equivalence. Note that when one considers operators that are defined on finite-dimensional Hilbert space, the well-known Jordan Canonical Form Theorem and the results by C. Pearcy [60] and W. Specht [63] give a complete answer to this problem.
For a complex separable Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $m \in \mathbb{N}$, let $T = (T_1, \cdots, T_m) \in \mathcal{L}(\mathcal{H})^m$ and $S = (S_1, \cdots, S_m) \in \mathcal{L}(\mathcal{H})^m$ be $m$-tuples of commuting operators on $\mathcal{H}$. If there is a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $UT = SU$, then $T$ and $S$ are said to be unitarily equivalent (denoted by $T \sim_u S$). If there is an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $XT = SX$, then $T$ and $S$ are similar (denoted by $T \sim_s S$).

Given $T = (T_1, \cdots, T_m) \in \mathcal{L}(\mathcal{H})^m$, first define an operator $\mathcal{P}_T : \mathcal{H} \rightarrow \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ by

$$
\mathcal{P}_T h = (T_1 h, \cdots, T_m h),
$$

for $h \in \mathcal{H}$. Let $\Omega$ be a bounded domain of the $m$-dimensional complex plane $\mathbb{C}^m$ and consider $w = (w_1, \cdots, w_m) \in \Omega$. If one sets $T - w := (T_1 - w_1, \cdots, T_m - w_m)$, then it is easily seen that $\ker \mathcal{P}_{T-w} = \bigcap_{i=1}^m \ker(T_i - w_i)$.

**Definition 1.1** For $m, n \in \mathbb{N}$ and $\Omega \subset \mathbb{C}^m$, the Cowen-Douglas class $\mathcal{B}_n^m(\Omega)$ consists of $m$-tuples of commuting operators $T = (T_1, \cdots, T_m) \in \mathcal{L}(\mathcal{H})^m$ satisfying the following conditions:

1. $\text{ran} \mathcal{P}_{T-w}$ is closed for all $w \in \Omega$;
2. $\dim \ker \mathcal{P}_{T-w} = n$ for all $w \in \Omega$; and
3. $\bigvee_{w \in \Omega} \ker \mathcal{P}_{T-w}$ is dense in $\mathcal{H}$.

For an $m$-tuple of commuting operators $T = (T_1, \cdots, T_m) \in \mathcal{B}_n^m(\Omega)$, M. J. Cowen and R. G. Douglas proved in [20, 21] that an associated holomorphic eigenvector bundle $\mathcal{E}_T$ over $\Omega$ of rank $n$ exists, where

$$
\mathcal{E}_T = \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker \mathcal{P}_{T-w}\}, \quad \pi(w, x) = w.
$$

Furthermore, it was shown that two operator tuples $T$ and $\tilde{T}$ in $\mathcal{B}_n^m(\Omega)$ are unitarily equivalent if and only if the vector bundles $\mathcal{E}_T$ and $\mathcal{E}_{\tilde{T}}$ are equivalent as Hermitian holomorphic vector bundles. They also showed that every $m$-tuple $T \in \mathcal{B}_n^m(\Omega)$ can be realized as the adjoint of an $m$-tuple of multiplication operators by the coordinate functions on a Hilbert space of holomorphic functions on $\Omega^* = \{\overline{w} : w \in \Omega\}$.

For $T \in \mathcal{B}_n^m(\Omega)$, if we let $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a holomorphic frame of $\mathcal{E}_T$ and form the Gram matrix $h(w) = (\langle \sigma_j(w), \sigma_i(w) \rangle)_{i,j=1}^n$ for $w \in \Omega$, then the curvature $K_T$ and the corresponding curvature matrix $K_T$ with entries $K_i{}^{ij}, 1 \leq i, j \leq m$, of the bundle $\mathcal{E}_T$ are given by the formulas

$$
K_T(w) = -\sum_{i,j=1}^m \nabla_{\overline{w}_j} \left( h^{-1}(w) \frac{\partial h(w)}{\partial w_i} \right) dw_i \wedge d\overline{w}_j,
$$

$$
K_i{}^{ij}(w) = -\frac{\partial}{\partial w_j} \left( h^{-1}(w) \frac{\partial h(w)}{\partial w_i} \right). \tag{1.1}
$$
Note that we omit the notation $\mathcal{E}$ in $\mathcal{K}_T$ without any ambiguity. Since the above formulas depend on the selection of the holomorphic frame, they are also written as $\mathcal{K}_T(\sigma)$ and $\mathcal{K}_{T,i,j}(\sigma)$, respectively, should the need arise. In the special case of $T \in \mathcal{B}_1^m(\Omega)$, the curvature of the line bundle $\mathcal{E}_T$ can be defined alternately as

$$K_T(w) = -\sum_{i,j=1}^{m} \frac{\partial^2 \log \|\gamma(w)\|^2}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j,$$

where $\gamma$ is a non-vanishing holomorphic section of $E_T$. In [13], S. Biswas, D. K. Keshari, and G. Misra showed that the curvature matrix $K_T$ of any $T \in \mathcal{B}_1^m(\Omega)$ is negative-definite. Moreover, in [20], it was shown that for $T, \tilde{T} \in \mathcal{B}_1^1(\Omega)$, $T \sim_u \tilde{T}$ if and only if $K_T = K_{\tilde{T}}$. The curvature $K_T$, along with certain covariant derivations of the curvature, form a complete set of unitary invariants for an operator $T \in \mathcal{B}_1^n(\Omega)$ and this is another main result of [20]. In [23], R. E. Curto and N. Salinas established a relationship between the class $\mathcal{B}_m^n(\Omega)$ and generalized reproducing kernels to describe when two $m$-tuples are unitarily equivalent. A similarity result for Cowen-Douglas operators in geometric terms such as curvature had been much more difficult to obtain. In fact, the work of D. N. Clark and G. Misra in [18, 19] showed that the Cowen-Douglas conjecture that similarity can be determined from the behavior of the quotient of the entries of curvature matrices was false.

The corona problem of complex analysis is closely related to operator theory and complex geometry [17, 57, 58, 64, 73]. In particular, M. Uchiyama characterized the contractive operators that are similar to the adjoint of some multiplication operator in the work [73] based on the corona theorem due to M. Rosenblum in [62]. The following lemma given by N. K. Nikolski shows how to use the notion of projections $P(w), P(w) = P(w)^2$, to solve the corona problem. The space $H_{E,\pi}^\infty(\Omega)$ denotes the algebra of bounded analytic functions defined on a domain $\Omega \subset \mathbb{C}^m$ whose function values are bounded linear operators from a Hilbert space $E_\pi$ to another one $E$:

**Lemma 1.2** (Nikolski’s Lemma) Let $F \in H_{E,\pi}^\infty(\Omega)$ satisfy $F^*(z)F(z) > \delta^2$ for some $\delta > 0$ and for all $z \in \Omega$. Then $F$ is left invertible in $H_{E,\pi}^\infty(\Omega)$ (i.e., there exists a $G \in H_{E,\pi}^\infty(\Omega)$ such that $GF \equiv I$) if and only if there exists a function $P \in H_{E,\pi}^\infty(\Omega)$ whose values are projections (not necessarily orthogonal) onto $F(z)E$ for all $z \in \Omega$. Moreover, if such an analytic projection $P$ exists, then one can find a left inverse $G \in H_{E,\pi}^\infty(\Omega)$ satisfying $\|G\|_{\infty} \leq \delta^{-1}\|P\|_{\infty}$.

For $T \in \mathcal{B}_1^n(\mathbb{D})$, where $\mathbb{D} \subset \mathbb{C}$ denotes the unit disk, let $\Pi(w)$ denote the orthogonal projection onto $\ker(T-w)$, for each $w \in \mathbb{D}$. In [50], the third author and S. Treil described when a contraction $T \in \mathcal{B}_1^n(\mathbb{D})$ is similar to $n$ copies of $M_z^*$, the adjoint of the shift operator on the Hardy space of the unit disk $\mathbb{D}$, in
terms of the curvature matrices. It was proven that \( T \sim s \bigoplus_{i=1}^{n} M_i^\ast \) if and only if
\[
\left\| \frac{\partial \Pi(w)}{\partial w} \right\|^2_{S_2} - \frac{n}{(1 - |w|^2)^2} \leq \frac{\partial^2 \psi(w)}{\partial w \partial w^*},
\]
for some bounded subharmonic function \( \psi \) defined on \( \mathbb{D} \) and for all \( w \in \mathbb{D} \).
For \( T \in \mathcal{B}_n^1(\mathbb{D}) \), trace \( K_T = -\frac{1}{(1 - |w|^2)^2} \) (see [35, 40]), while trace \( K_{M_i^\ast} = -\frac{1}{(1 - |w|^2)^2} \) for the Hardy shift \( M_i^\ast \). The result was then generalized to other shift operators in [27]. Note that this is consistent with the curvature inequality result of G. Misra given in [54].

The similarity results mentioned above rely, to some extent, on a model theorem and a well-known \( \overline{\partial} \)-method that has been used extensively in recent years to solve numerous versions of the corona problem [33]. For a contraction \( T \in \mathcal{L}(\mathcal{H}) \), the model theory of B. Sz.-Nagy and C. Foias provides the canonical model as a complete set of invariants. Model theorems for operator tuples in \( \mathcal{L}(\mathcal{H})^m \) were also studied under various assumptions [64, 67]. Since one cannot relate the similarity problem to the corona problem easily anymore in the multi-operator setting, we propose an alternative approach in managing the similarity of tuples of operators in \( \mathcal{B}_n^m(\Omega) \) for \( m > 1 \).

Inspired by the previous similarity results, we first give a sufficient condition for the similarity between \( T \in \mathcal{B}_1^m(\mathbb{B}_m) \) and the adjoint of the operator tuple \( M_z = (M_{z_1}, M_{z_2}, \cdots, M_{z_m}) \) on a weighted Bergman space defined on the unit ball \( \mathbb{B}_m = \{ z \in \mathbb{C}^m : |z| < 1 \} \) of \( \mathbb{C}^m \). The characterization is given in terms of the defect operator \( D_T \) corresponding to \( T \). Note that there already exist a number of necessary conditions for similarity – it is a sufficient condition that had been hard to obtain. Throughout the paper, \( M_z = (M_{z_1}, M_{z_2}, \cdots, M_{z_m}) \) defined on a Hilbert space \( \mathcal{H} \) of holomorphic functions on \( \mathbb{B}_m \) will denote the tuple of multiplication operators by the coordinate functions
\[
(M_z f)(z) = z_i f(z),
\]
for \( f \in \mathcal{H} \) and \( z \in \mathbb{B}_m \). It can be checked that \( M_z^\ast = \mathcal{B}_1^m(\mathbb{B}_m) \), where \( M_z^\ast = (M_{z_1}^\ast, M_{z_2}^\ast, \cdots, M_{z_m}^\ast) \).

**Theorem 1.3** Let \( T = (T_1, \cdots, T_m) \in \mathcal{B}_1^m(\mathbb{B}_m) \subset \mathcal{L}(\mathcal{H})^m \) and consider the operator tuple \( M_z^\ast = (M_{z_1}^\ast, \cdots, M_{z_m}^\ast) \) on a weighted Bergman space \( \mathcal{H}_k \), where \( k > m + 1 \). Suppose that \( (I - \sum_{i=1}^{m} T_i^* T_i)^k \geq 0 \) and \( \lim_{j} f_j(T^*, T) h = 0, h \in \mathcal{H} \), where \( f_j(z, w) = \sum_{i=j}^{\infty} e_i(z)(1 - \langle z, w \rangle)^k e_i(w)^* \), for an orthonormal basis \( \{ e_i \}_{i=0}^{\infty} \) for \( \mathcal{H}_k \). If there exist a non-vanishing holomorphic section \( t \) of \( \mathcal{E}_T \) and a unit vector \( \zeta_0 \in \overline{\text{ran} D_T} \) such that
\[
\sup_{w \in \mathbb{B}_m} \left| \frac{\mathcal{D}_T t(w)}{|\mathcal{D}_T t(w), \zeta_0|} \right| < \infty,
\]
then $T \sim_s M_z^\ast$.

Although there is no general corona theorem that works for higher-order domains, a related condition that will be called condition (C) in Section 3 will play a significant role in the formulation of another sufficient condition for similarity in terms of curvature matrices. The space $H^\infty(\Omega)$ denotes the collection of bounded analytic functions defined on the domain $\Omega$.

**Theorem 1.4** Let $T = (T_1, \ldots, T_m) \in B_m^m(\mathbb{B}_m)$ and consider the operator tuple $M_z^\ast = (M_{z_1}^\ast, \ldots, M_{z_m}^\ast) \in B_{n_1}^m(\mathbb{B}_m)$ on a Hilbert space $\mathcal{H}_K$ defined on $\mathbb{B}_m$ with reproducing kernel $K$. Suppose that $n_2 := n/n_1 \in \mathbb{N}$ and that there exist an isometry $V$ and a Hermitian holomorphic vector bundle $E$ over $\mathbb{B}_m$ such that

$$V K_{T, w, \overline{w}_i} V^* - K_{M_z^\ast, w, \overline{w}_i} \otimes I_{n_2} = I_{n_1} \otimes K_{E, w, \overline{w}_j}, \quad I, J \in \mathbb{N}_0^m.$$ 

If the bundle $E$ satisfies condition (C) via $\mathcal{H}_K$, then $T \sim_s \bigoplus_{i=1}^{n_2} M_z^\ast$.

A number of corollaries are given in Section 4. The result of [27] and [50] describing contractive Cowen-Douglas operators that are similar to $M_z^\ast$ using curvature is generalized in the commuting operator tuples setting. Moreover, this description is also used to obtain a sufficient condition for the similarity between arbitrary Cowen-Douglas operator tuples in $B_1^m(\Omega)$. The space $\{S\}'$ denotes the commutant of the operator tuple $S$.

**Theorem 1.5** Let $T = (T_1, \ldots, T_m), S = (S_1, \ldots, S_m) \in B_1^m(\Omega)$ be such that $\{S\}' \cong H^\infty(\Omega)$. Suppose that

$$K_S(w) - K_T(w) = \sum_{i,j=1}^{m} \frac{\partial^2 \psi(w)}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j, \quad w \in \Omega,$$

for some $\psi(w) = \log \sum_{k=1}^{n} |\phi_k(w)|^2$, where $\phi_k$ are holomorphic functions defined on $\Omega$.

If there exists an integer $l \leq n$ satisfying $\frac{\partial \phi_k}{\partial \overline{w}_j} \in H^\infty(\Omega)$ for all $k \leq n$, then $T \sim_s S$, and $K_T \leq K_S$. In particular, when $m = 1$, $T$ and $S$ are unitarily equivalent.

The inequalities and identities given in [13], [50], and [54] involving curvature matrices are extended as well in the final section.

## 2 Preliminaries

### 2.1 Reproducing kernel Hilbert spaces

Let $\mathbb{B}_m = \{z \in \mathbb{C}^m : |z| < 1\}$ be the open unit ball of $\mathbb{C}^m$. The space of all holomorphic functions defined on $\mathbb{B}_m$ will be denoted as $\mathcal{O}(\mathbb{B}_m)$ while $H^\infty(\mathbb{B}_m)$ will stand for the space of all bounded holomorphic functions on
For a function \( f \in \mathcal{O}(\mathbb{B}_m) \), the radial derivative of \( f \) is defined to be 
\[
Rf(z) = \sum_{i=1}^{m} z_i \frac{\partial f}{\partial z_i}. 
\]
Once it is set that \( R^0 f(z) = f(z) \), we have for every \( j \in \mathbb{N} \), \( R^j f(z) = R(R^{j-1} f(z)) \). In particular, for a homogeneous polynomial of degree \( n \), \( R f = nf \). We will also need the familiar multi-index notation \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \). As is well-known, \( |\alpha| = |\alpha_1| + \cdots + |\alpha_m| \) and \( \alpha! = \alpha_1! \cdots \alpha_m! \). For \( \alpha, \beta \in \mathbb{N}_0^m \), \( \alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_m + \beta_m) \) and \( \alpha \leq \beta \) whenever \( \alpha_i \leq \beta_i \) for every \( 1 \leq i \leq m \).

The work \([6, 32, 76]\) offer good references for what follows. For a real number \( k \), one can consider the family of holomorphic function spaces 
\[
\mathcal{H}_k = \left \{ f = \sum_{\alpha \in \mathbb{N}_0^m} a_{\alpha} z^\alpha \in \mathcal{O}(\mathbb{B}_m) : \sum_{\alpha \in \mathbb{N}_0^m} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} (|\alpha| + 1)^{1-k} < \infty \right \}. 
\]
Recall that a reproducing kernel Hilbert space is a Hilbert space \( \mathcal{H} \) of functions on a set \( X \) with the property that the evaluation at each \( x \in X \) is a bounded linear functional on \( \mathcal{H} \). By the Riesz representation theorem, for each \( x \in X \), there exists a function \( k_x \in \mathcal{H} \) such that for all \( f \in \mathcal{H} \), 
\[
\langle f, k_x \rangle = f(x). 
\]
The function \( K : X \times X \rightarrow \mathbb{C} \) defined by \( K(x, y) = k_y(x) \) is called the reproducing kernel of \( \mathcal{H} \). When \( k > 0 \), \( \mathcal{H}_k \) is a reproducing kernel Hilbert space with reproducing kernel 
\[
K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^k}. 
\]
If \( n \in \mathbb{N} \) is such that \( 2n + k - m > 0 \), then the space \( \mathcal{H}_k \) can also be represented as 
\[
\mathcal{H}_k = \left \{ f \in \mathcal{O}(\mathbb{B}_m) : \int_{\mathbb{B}_m} |R^n f|^2 (1 - |z|^2)^{2n+k-m-1} dV(z) < \infty \right \}, \tag{2.1} 
\]
where \( V \) denotes the normalized volume measure on \( \mathbb{B}_m \). The spaces \( \mathcal{H}_k \) are closely related to the analytic Besov-Sobolev spaces \( \mathcal{B}_p^\sigma(\mathbb{B}_m) \). Recall that for \( n \in \mathbb{N}_0 \), \( 0 \leq \sigma < \infty \), \( 1 < p < \infty \), and \( n + \sigma > m/p \), the space \( \mathcal{B}_p^\sigma(\mathbb{B}_m) \) contains \( f \in \mathcal{O}(\mathbb{B}_m) \) with 
\[
\left( \sum_{|\alpha| < n} |\frac{\partial^n f}{\partial z^n}(0)|^p + \int_{\mathbb{B}_m} |R^n f|^p (1 - |z|^2)^{p(n+\sigma)} d\lambda_m(z) \right)^{\frac{1}{p}} < \infty, 
\]
where \( dz \) denotes the Lebesgue measure on \( \mathbb{C}^m \) and \( d\lambda_m(z) = (1 - |z|^2)^{-m-1} dz \) is the invariant measure on \( \mathbb{B}_m \). Well-known examples in this family of spaces include the Dirichlet space \( \mathcal{D}(\mathbb{B}_m) = \mathcal{H}_0 = \mathcal{B}_2^0(\mathbb{B}_m) \), the Drury-Arveson space \( H^2 = \mathcal{H}_1 = \mathcal{B}_2^{1/2}(\mathbb{B}_m) \), the Hardy space \( H^2(\mathbb{B}_m) = \mathcal{H}_m = \mathcal{B}_2^{m/2}(\mathbb{B}_m) \), and the Bergman space \( L^2(\mathbb{B}_m) = \mathcal{H}_{m+1} = \mathcal{B}_2^{(m+1)/2}(\mathbb{B}_m) \). Moreover, \( \mathcal{H}_k \) with
0 < k < 1 give weighted Dirichlet-type spaces while those with k > m + 1 represent weighted Bergman spaces.

2.2 Operator-valued multipliers

As every Hilbert space \( \mathcal{H} \) of functions on a set \( X \) comes with a corresponding multiplier algebra

\[
\text{Mult}(\mathcal{H}) = \{ f : X \to \mathbb{C} : fh \in \mathcal{H} \text{ for all } h \in \mathcal{H} \},
\]

it is natural to consider the multipliers of reproducing kernel Hilbert spaces. For every multiplier \( f \in \text{Mult}(\mathcal{H}) \), there is an associated multiplication operator \( M_f \) defined by \( M_fh = fh \) with \( \| f \|_{\text{Mult}(\mathcal{H})} = \| M_f \| \). In particular, for weighted Bergman spaces \( \mathcal{H}_k \) with \( k > m + 1 \), \( \text{Mult}(\mathcal{H}_k) = H^\infty(\mathbb{B}_m) \). For \( m \geq 2 \), the multiplier norm on the Drury-Arveson space \( \mathcal{H}_1 \) is no longer equal to the supremum norm on the unit ball and therefore, \( \text{Mult}(\mathcal{H}_1) \subsetneq H^\infty(\mathbb{B}_m) \).

Let \( E \) be a Hilbert space. The Hilbert space tensor product \( \mathcal{H}_k \otimes E \) can be regarded as the space of all holomorphic functions \( f : \mathbb{B}_m \to E \) with Taylor series

\[
f(z) = \sum_{\alpha \in \mathbb{N}^m_0} a_\alpha z^\alpha,
\]

where \( a_\alpha \in E \) and

\[
\sum_{\alpha \in \mathbb{N}^m_0} \| a_\alpha \|^2 \frac{\alpha!}{|\alpha|!(|\alpha| + 1)^{1-k}} < \infty.
\]

Now, for Hilbert spaces \( E_1 \) and \( E_2 \), let \( \Phi : \mathbb{B}_m \to \mathcal{L}(E_1, E_2) \) be an operator-valued function. Given \( h \in \mathcal{H}_k \otimes E_1 \), we define a function \( M_{\Phi}h : \mathbb{B}_m \to E_2 \) as

\[
M_{\Phi}h(z) = \Phi(z)h(z), \quad z \in \mathbb{B}_m.
\]

Denote by \( \text{Mult}(\mathcal{H}_k \otimes E_1, \mathcal{H}_k \otimes E_2) \) the space of all \( \Phi \) for which \( M_{\Phi}h \in \mathcal{H}_k \otimes E_2 \) for every \( h \in \mathcal{H}_k \otimes E_1 \). An element \( \Phi \in \text{Mult}(\mathcal{H}_k \otimes E_1, \mathcal{H}_k \otimes E_2) \) is said to be a multiplier and \( M_{\Phi} \) is called an operator of multiplication by \( \Phi \). The space \( \text{Mult}(\mathcal{H}_k \otimes E_1, \mathcal{H}_k \otimes E_2) \) is endowed with the norm \( \| \Phi \| = \| M_{\Phi} \| \). We now list some basic properties of multipliers.

**Lemma 2.1** For a weighted Bergman space \( \mathcal{H}_k \) with \( k > m + 1 \) and a Hilbert space \( E \),

\[
\text{Mult}(\mathcal{H}_k \otimes E, \mathcal{H}_k \otimes \mathbb{C}) = H^\infty_{E \to \mathbb{C}}(\mathbb{B}_m).
\]

**Proof** Taking \( n = 0 \) in (2.1), we have for every \( F \in H^\infty_{E \to \mathbb{C}}(\mathbb{B}_m) \) and \( f(z) \otimes g \in \mathcal{H}_k \otimes E \),

\[
\| M_F(f(z) \otimes g) \|^2_{\mathcal{H}_k \otimes \mathbb{C}} = \int_{\mathbb{B}_m} |f(z) \otimes F(z)g|^2(1 - |z|^2)^{k-m-1} dV(z) \\
\leq \| F \|^2_{\infty} \int_{\mathbb{B}_m} |f(z)|^2 \| g \|^2_E(1 - |z|^2)^{k-m-1} dV(z)
\]
The kernel \( \Phi \in \alpha \)

\[ \text{Let } \Phi = \text{Lemma 2.3} \]

\[ \text{Lemma 2.2 Let } \Phi : \mathbb{B}_m \to \mathcal{L}(E_1, E_2) \text{ be an operator-valued function. If } \Phi \in \text{Mult}(\mathcal{H}_K \otimes E_1, \mathcal{H}_K \otimes E_2), \text{ then} \]

\[ M_\Phi^* (K(\cdot, \overline{w}) \otimes f) = K(\cdot, \overline{w}) \otimes \Phi(\overline{w})^* f, \quad w \in \mathbb{B}_m, f \in E_2. \]

Conversely, if \( \Phi : \mathbb{B}_m \to \mathcal{L}(E_1, E_2) \) and the mapping \( K(\cdot, \overline{w}) \otimes f \mapsto K(\cdot, \overline{w}) \otimes \Phi(\overline{w})^* f \)

extends to a bounded operator \( X \in \mathcal{L}(\mathcal{H}_K \otimes E_2, \mathcal{H}_K \otimes E_1) \), then \( \Phi \in \text{Mult}(\mathcal{H}_K \otimes E_1, \mathcal{H}_K \otimes E_2) \) and \( X = M_\Phi^* \).

\[ \text{Lemma 2.3 Let } M_{z_i, i}^* \text{ be the adjoint of the multiplication tuple } (M_{z_1}, \ldots, M_{z_m}) \text{ on } \mathcal{H}_K \otimes E_i, i = 1, 2 \text{. If } \Phi \in \text{Mult}(\mathcal{H}_K \otimes E_1, \mathcal{H}_K \otimes E_2), \text{ then} \]

\[ M_\Phi^* M_{z_i, i}^* = M_{z_i, i}^* M_\Phi^*. \]

The following lemma, due to J. A. Ball, T. T. Trent, and V. Vinnikov, characterizes \( \text{Mult}(\mathcal{H}_1 \otimes E_1, \mathcal{H}_1 \otimes E_2) \) for the Drury-Arveson space \( \mathcal{H}_1 \). For the proof and additional results, see \[9, 28]:

\[ \text{Lemma 2.4 Let } \Phi : \mathbb{B}_m \to \mathcal{L}(\mathcal{H}_1 \otimes E_1, \mathcal{H}_1 \otimes E_2). \text{ Then the following statements are equivalent:} \]

1. \( \Phi \in \text{Mult}(\mathcal{H}_1 \otimes E_1, \mathcal{H}_1 \otimes E_2) \) with \( \| \Phi \| \leq 1. \)

2. The kernel

\[ R_\Phi(z, w) = \frac{I - \Phi(z)\Phi(w)^*}{1 - \langle z, w \rangle}, \]

is a positive, sesqui-analytic, \( \mathcal{L}(E_2) \)-valued kernel on \( \mathbb{B}_m \times \mathbb{B}_m \), i.e., there is an auxiliary Hilbert space \( \mathcal{H} \) and a holomorphic \( \mathcal{L}(\mathcal{H}, E_2) \)-valued function \( \Psi \) on \( \mathbb{B}_m \) such that for all \( z, w \in \mathbb{B}_m, \)

\[ R_\Phi(z, w) = \Psi(z)\Psi(w)^*. \]

### 2.3 Model theorem

Let \( M_z = (M_{z_1}, \ldots, M_{z_m}) \) be the multiplication tuple on a reproducing kernel Hilbert space \( \mathcal{H}_K \) defined on \( \mathbb{B}_m \) such that for every \( 1 \leq i \leq m, \)

\[ (M_{z_i} f)(z) = z_if(z), \quad f \in \mathcal{H}_K, z \in \mathbb{B}_m. \]

For an \( m \)-tuple of commuting operators \( T = (T_1, \ldots, T_m) \in \mathcal{L}(\mathcal{H})^m \) and a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \), let \( T^\alpha = T_1^{\alpha_1} \cdots T_m^{\alpha_m} \) and \( T^* = (T_1^*, \ldots, T_m^*) \). Suppose that \( 1/K \) is a polynomial and that \( \frac{1}{K}(T^*, T) \geq 0, \)

\[ = \|F\|_\infty^2 \|f(z) \otimes g\|^2_{\mathcal{H}_K \otimes E}. \]

This means that \( \|M_F\| \leq \|F\|_\infty \), and therefore, \( F \in \text{Mult}(\mathcal{H}_k \otimes E, \mathcal{H}_k \otimes \mathbb{C}) \). Conversely, since \( \text{Mult}(\mathcal{H}_k) \subset H^\infty(\mathbb{B}_m), \text{ Mult}(\mathcal{H}_k \otimes E, \mathcal{H}_k \otimes \mathbb{C}) \subset H^\infty_{\mathbb{C} \to \mathbb{C}}(\mathbb{B}_m). \]
where given a polynomial
\[ p(z, \omega) = \sum_{I, J \leq \beta} \alpha_{I, J} z^I \omega^J, \beta \in \mathbb{N}_0^m, \]
we let
\[ p(T^*, T) = \sum_{I, J \leq \beta} \alpha_{I, J} T^* T^J. \]
The defect operator \( D_T \) of \( T \) is then defined to be
\[ D_T = \frac{1}{K}(T^*, T)\hat{\tau}. \]

We next define a mapping \( V : \mathcal{H} \to \mathcal{N} \subset \mathcal{H}_K \otimes \mathcal{H} \) as
\[ Vh = \sum_i e_i(\cdot) \otimes D_T e_i(T^*)^* h, \]
for \( h \in \mathcal{H} \), where \( \mathcal{N} = \text{ran} V \) and \( \{ e_i \}_{i=0}^\infty \) is an orthonormal basis for \( \mathcal{H}_K \).

Then according to the result of C. G. Ambrozie, M. Englis, and V. Müller in [5], \( V \) is a unitary operator satisfying
\[ VT_j = M_{z_j} V \]
for \( 1 \leq j \leq m \). The study of a model theorem for bounded linear operators have been quite extensive and can be found in [1, 2, 5, 7, 8, 24, 25, 56, 61]. The following model theorem for a tuple of commuting operators is stated in [5]:

**Theorem 2.5** Consider the operator tuple \( M_z = (M_{z_1}, \ldots, M_{z_m}) \) on a Hilbert space \( \mathcal{H}_K \) of holomorphic functions with reproducing kernel \( K \) such that \( 1/K \) is a polynomial. For an orthonormal basis \( \{ e_i \}_{i=0}^\infty \) for \( \mathcal{H}_K \), let
\[ f_j(z, w) = \sum_{i=j}^\infty e_i(z) \frac{1}{K}(z, w) e_i(w)^*. \]
Then the following statements are equivalent:

1. \( T = (T_1, \ldots, T_m) \in \mathcal{L}(\mathcal{H})^m \) is unitarily equivalent to the restriction of \( M_z^* \) to an invariant subspace.
2. \( \frac{1}{K}(T^*, T) \geq 0 \) and \( \lim_j f_j(T^*, T)h = 0 \) for \( h \in \mathcal{H} \).

### 3 Similarity in the class \( \mathcal{B}_n^m(\Omega) \)

We first give a sufficient condition for the similarity between operator tuples \( T \in \mathcal{B}_n^m(\mathbb{B}_m) \) and \( M_z^* \) on a weighted Bergman space by using the defect operator \( D_T \) and the model theorem given previously. We then introduce condition (C) for Hermitian holomorphic vector bundles (see Subsection 3.2) and use it together with the curvature and its covariant derivatives to characterize similarity in the class \( \mathcal{B}_n^m(\Omega) \). The similarity of operators inside a specific subclass of \( \mathcal{B}_n^m(\Omega) \) and the uniqueness of decomposition of Cowen-Douglas operators are then discussed.

#### 3.1 Model theorem and similarity

We start by investigating the eigenvector bundle \( E_T \) of \( T \in \mathcal{B}_n^m(\Omega) \).
Lemma 3.1 Let $T = (T_1, \cdots, T_m) \in \mathcal{B}_1^m(\Omega) \subset \mathcal{L}(\mathcal{H})^m$ and consider the operator tuple $M_z = (M_{z_1}, \cdots, M_{z_m})$ on a Hilbert space $\mathcal{H}_K$ of holomorphic functions with reproducing kernel $K$ such that $1/K$ is a polynomial. Suppose that $T$ satisfies either one of the equivalent statements in Theorem 2.5. Then for any $t(w) \in \ker(T - w)$, \[
\|t(w)\|^2 = K(w, \overline{w}) \|D_T t(w)\|^2.
\]

Proof Let $\{e_i\}_{i=0}^\infty$ be an orthonormal basis for $\mathcal{H}_K$. Since $t(w) \in \ker(T - w)$, $f(T)t(w) = f(w)t(w)$ for every $f \in \mathcal{O}(\Omega)$. Defining a mapping $V : \mathcal{H} \to \mathcal{N} \subset \mathcal{H}_K \otimes \mathcal{H}$ as

$$Vh = \sum_i e_i(\cdot) \otimes D_T e_i(T^*)^* h,$$

for $h \in \mathcal{H}$ and $\mathcal{N} = \overline{\text{ran} V}$, we have

$$Vt(w) = \sum_i e_i(\cdot) \otimes D_T e_i(T^*)^* t(w) = \sum_i e_i(\cdot) \otimes D_T e_i(w)^* t(w) = \sum_i e_i(\cdot) e_i(w)^* \otimes D_T t(w) = K(\cdot, \overline{w}) \otimes D_T t(w).$$

From [5], $V$ is unitary and the result follows. \qed

The next theorem is the first of our main results of the paper.

**Theorem 3.2** Let $T = (T_1, \cdots, T_m) \in \mathcal{B}_1^m(\mathbb{B}_m) \subset \mathcal{L}(\mathcal{H})^m$ and consider the operator tuple $M^*_z = (M^*_{z_1}, \cdots, M^*_{z_m})$ on a weighted Bergman space $\mathcal{H}_k$, where $k > m + 1$. Suppose that $(I - \sum_{i=1}^m T_i^* T_i)^k \geq 0$ and $\lim_{j} f_j(T^*, T) h = 0, h \in \mathcal{H}$, where $f_j(z, w) = \sum_{i=j}^\infty e_i(z)(1 - \langle z, w \rangle)^k e_i(w)^*$, for an orthonormal basis $\{e_i\}_{i=0}^\infty$ for $\mathcal{H}_k$. If there exist a non-vanishing holomorphic section $t$ of $\mathcal{E}_T$ and a unit vector $\zeta_0 \in \overline{\text{ran} D_T}$ such that

$$\sup_{w \in \mathbb{B}_m} \|D_T t(w)\|^2 / \|D_T t(w), \zeta_0\|^2 < \infty,$$

(3.1)

then $T \sim_u M^*_z$.

Proof Let $E = \overline{\text{ran} D_T}$ and note by Theorem 2.5 that $T \sim_u M^*_z|_{\mathcal{N}}$, where $\mathcal{N}$ is an invariant subspace of $M^*_z|_{\mathcal{E}_T}$. By Lemma 3.1, we then have

$$\ker(M^*_z|_{\mathcal{N}} - w) = \bigvee \{K(\cdot, \overline{w}) \otimes D_T t(w) : D_T t(w) \in E\}.$$

Given a unit vector $\zeta_0 \in E$, one can select an orthonormal basis $\{\zeta_\alpha\}_{\alpha \geq 0}$ of $E$ to express $D_T t(w)$ as $D_T t(w) = \sum_{\alpha \geq 0} \langle D_T t(w), \zeta_\alpha \rangle \zeta_\alpha$. If we set

$$\eta(w) := D_T t(w) - \langle D_T t(w), \zeta_0 \rangle \zeta_0 \in \mathcal{O}(\mathbb{B}_m) \quad \text{and} \quad \psi(w) := \langle D_T t(w), \zeta_0 \rangle \in \mathcal{O}(\mathbb{B}_m),$$

then $\|D_T t(w)\|^2 = \|\eta(w)\|^2 + |\psi(w)|^2$.

Now define an operator-valued function $F \in H_{\mathcal{E}_T \otimes \mathcal{C}(\mathbb{B}_m)}^\infty$ by

$$F^*(z)(\lambda) := \lambda \frac{\eta(z)}{\psi(z)}, \quad \lambda \in \mathbb{C}, z \in \mathbb{B}_m.$$
Since we know that for a weighted Bergman space $\mathcal{H}_k$ with $k > m + 1$,

$$\text{Mult}(\mathcal{H}_k \otimes E, \mathcal{H}_k \otimes \mathbb{C}) = H^\infty_{E \rightarrow \mathbb{C}}(\mathbb{B}_m),$$

Lemma 2.2 and condition (3.1) yield

$$M_F^*(K(\cdot, \overline{w}) \otimes 1) = K(\cdot, \overline{w}) \otimes F(\overline{w})^*(1) = K(\cdot, \overline{w}) \otimes \frac{\eta(w)}{\psi(w)}, \quad w \in \mathbb{B}_m.$$ 

Hence, $M_F^*$ is bounded and for $w \in \mathbb{B}_m$,

$$\|D_T t(w)\|^2 = \|\eta(w)\|^2 + |\psi(w)|^2 = |\psi(w)|^2 \left(\frac{\|\eta(w)\|^2}{\|\psi(w)\|^2} + 1\right) = |\psi(w)|^2 \left(\frac{\|M_F^*(K(\cdot, \overline{w}) \otimes 1)\|^2}{\|K(\cdot, \overline{w})\|^2} + 1\right).$$

Since $\psi(w) \in \mathcal{O}(\mathbb{B}_m)$, the definition of curvature from (1.1) then gives

$$K_{M^*_z,E\mid N}(w) = -\sum_{i,j=1}^m \frac{\partial^2 \log \|K(\cdot, \overline{w}) \otimes D_T t(w)\|^2}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j$$

$$= -\sum_{i,j=1}^m \frac{\partial^2 \log \|D_T t(w)\|^2}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j - \sum_{i,j=1}^m \frac{\partial^2 \log \|K(\cdot, \overline{w})\|^2}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j$$

$$= -\sum_{i,j=1}^m \frac{\partial^2 \log (\|M_F^*(K(\cdot, \overline{w}) \otimes 1)\|^2 + \|K(\cdot, \overline{w})\|^2)}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j$$

$$= -\sum_{i,j=1}^m \frac{\partial^2 \log (\|I + M_F M_F^* K(\cdot, \overline{w}), K(\cdot, \overline{w})\|)}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j$$

$$= -\sum_{i,j=1}^m \frac{\partial^2 \log (\|I + M_F M_F^* \|^2) K(\cdot, \overline{w})}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j.$$

Finally, let

$$Y := (I + M_F M_F^*)^{\frac{1}{2}}.$$ 

Obviously, $0 \notin \sigma(Y)$ and $YK(\cdot, \overline{w}) \in \ker(YM^*_z Y^{-1} - w)$ so that for any $w \in \mathbb{B}_m$,

$$K_{M^*_z,E\mid N}(w) = -\sum_{i,j=1}^m \frac{\partial^2 \log (\|I + M_F M_F^* \|^2) K(\cdot, \overline{w})}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j = K_{YM^*_z Y^{-1}}(w).$$

This shows that $M^*_z,E\mid N \sim_u YM^*_z Y^{-1}$ and since $T \sim_s M^*_z \mid N$, $T \sim_s M^*_z$ as claimed. 

\[ \square \]

**Remark 3.3** After an obvious modification of the condition $(I - \sum_{i=1}^m T_i^* T_i)^k \geq 0$ and the form of $f_j$ tailored to the reproducing kernel $K$, Theorem 3.2 can be generalized to any operator tuple $M^*_z$ on a reproducing kernel Hilbert space $\mathcal{H}_K$ such that $1/K$ is a polynomial as long as $\text{Mult}(\mathcal{H}_K) = H^\infty(\mathbb{B}_m)$. Moreover, one can use Lemma 2.4 to check the multiplier algebra condition when working on the similarity between a row-contraction $T \in B_m(\mathbb{B}_m)$ and the operator tuple $M^*_z = (M^*_z, \cdots, M^*_z)$ on the Drury-Arveson space $\mathcal{H}_1$. 

$\mathcal{H}_k$ with $k > m + 1,$
3.2 Complex bundles and similarity

Denote by \( \{\sigma_i\}_{i=1}^n \) an orthonormal basis for \( \mathbb{C}^n \) and let a Hilbert space \( \mathcal{H} \) on \( \Omega \) and analytic vector valued functions \( \{f_i\}_{i=1}^n \) over \( \Omega \) be given, where \( \Omega \subset \mathbb{C}^m \). Let \( \mathcal{E} \) be an \( n \)-dimensional Hermitian holomorphic vector bundle over \( \Omega \), \( f_1, \ldots, f_n \) be \( n \) holomorphic cross-sections of \( \mathcal{E} \) which form a frame for \( \mathcal{E} \) on \( \Omega \). For \( w \in \Omega \), set \( \mathcal{E}(w) = \bigvee \{f_1(w), \ldots, f_n(w)\} \) and \( E = \bigvee_{w \in \Omega} \{f_1(w), \ldots, f_n(w)\} \). We will say that condition (C) holds for the Hermitian holomorphic vector bundle \( \mathcal{E} \) via \( \mathcal{H} \) if there exist functions \( F \in H^\infty_{\mathbb{C}^n \rightarrow E}(\Omega) \) and \( G \in H^\infty_{E \rightarrow \mathbb{C}^n}(\Omega) \) such that \( F^\#(\bar{w})(\sigma_i) := F(w)(\sigma_i) = f_i(w) \), \( G^\#(\bar{w})(f_i(w)) := G(w)(f_i(w)) \), \((F^\#)^* \in \text{Mult}(\mathcal{H} \otimes E, \mathcal{H} \otimes \mathbb{C}^n)\), \((G^\#)^* \in \text{Mult}(\mathcal{H} \otimes \mathbb{C}^n, \mathcal{H} \otimes E)\), and \( G^\#(\bar{w})F^\#(\bar{w}) \equiv I \) for all \( w \in \Omega \). Using the curvature and covariant derivatives of complex bundles as well as condition (C), we give a similarity description in the class \( B^m_\mathbb{N}(\mathbb{B}_m) \).

**Lemma 3.4** Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be Hermitian holomorphic bundles over \( \Omega \subset \mathbb{C}^m \) of rank \( n_1 \) and of \( n_2 \), respectively. Then for any \( I, J \in \mathbb{N}_0^m \),

\[
K_{\mathcal{E}_1 \otimes \mathcal{E}_2, w^I m^J} = K_{\mathcal{E}_1, w^I m^J} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathcal{E}_2, w^I m^J}.
\]

**Proof** Let \( \{\phi_1, \phi_2, \ldots, \phi_{n_1}\} \) and \( \{\gamma_1, \gamma_2, \ldots, \gamma_{n_2}\} \) be holomorphic frames of \( \mathcal{E}_1 \) and of \( \mathcal{E}_2 \), respectively. Then \( \{\phi_1 \otimes \gamma_1, \phi_1 \otimes \gamma_2, \ldots, \phi_{n_1} \otimes \gamma_1, \ldots, \phi_{n_1} \otimes \gamma_{n_2}\} \) is a holomorphic frame of \( \mathcal{E}_1 \otimes \mathcal{E}_2 \) and \( h_{\mathcal{E}_1} \otimes h_{\mathcal{E}_2} = h_{\mathcal{E}_1} \otimes h_{\mathcal{E}_2} \) so that

\[
K_{\mathcal{E}_1 \otimes \mathcal{E}_2} = \left( \frac{\partial}{\partial w_j} \left( h_{\mathcal{E}_1}^{-1} \frac{\partial h_{\mathcal{E}_1} \otimes h_{\mathcal{E}_2}}{\partial w_i} \right) \right)_{i,j=1}^m
\]

\[
= \left( \frac{\partial}{\partial w_j} \left( h_{\mathcal{E}_1}^{-1} \otimes h_{\mathcal{E}_2}^{-1} \left( \frac{\partial h_{\mathcal{E}_1}}{\partial w_i} \otimes h_{\mathcal{E}_2} + h_{\mathcal{E}_1} \otimes \frac{\partial h_{\mathcal{E}_2}}{\partial w_i} \right) \right) \right)_{i,j=1}^m
\]

\[
= \left( \frac{\partial}{\partial w_j} \left( h_{\mathcal{E}_1}^{-1} \otimes h_{\mathcal{E}_2}^{-1} \left( \frac{\partial h_{\mathcal{E}_1}}{\partial w_i} \otimes h_{\mathcal{E}_2} \right) \right) \right)_{i,j=1}^m + \left( \frac{\partial}{\partial w_j} \left( h_{\mathcal{E}_1}^{-1} \otimes h_{\mathcal{E}_2}^{-1} \left( h_{\mathcal{E}_1} \otimes \frac{\partial h_{\mathcal{E}_2}}{\partial w_i} \right) \right) \right)_{i,j=1}^m
\]

\[
= \frac{\partial}{\partial w_j} \left( h_{\mathcal{E}_1}^{-1} \frac{\partial h_{\mathcal{E}_1}}{\partial w_i} \otimes h_{\mathcal{E}_2} \right)_{i,j=1}^m + \left( \frac{\partial}{\partial w_j} \left( h_{\mathcal{E}_1}^{-1} h_{\mathcal{E}_1} \otimes h_{\mathcal{E}_2}^{-1} \frac{\partial h_{\mathcal{E}_2}}{\partial w_i} \right) \right)_{i,j=1}^m
\]

\[
= K_{\mathcal{E}_1} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathcal{E}_2}.
\]

It follows that for \( e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}_0^m \) with 1 in the \( j \)-th position,

\[
K_{\mathcal{E}_1 \otimes \mathcal{E}_2, w^e_j} = \frac{\partial}{\partial w_j} K_{\mathcal{E}_1 \otimes \mathcal{E}_2} + \left[ h_{\mathcal{E}_1}^{-1} \frac{\partial h_{\mathcal{E}_1} \otimes h_{\mathcal{E}_2}}{\partial w_j}, K_{\mathcal{E}_1 \otimes \mathcal{E}_2} \right]
\]

\[
= \frac{\partial}{\partial w_j} \left( K_{\mathcal{E}_1} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathcal{E}_2} \right)
\]

\[
+ \left[ h_{\mathcal{E}_1}^{-1} \frac{\partial h_{\mathcal{E}_1}}{\partial w_j} \otimes I_{n_2} + I_{n_1} \otimes h_{\mathcal{E}_2}^{-1} \frac{\partial h_{\mathcal{E}_2}}{\partial w_j}, K_{\mathcal{E}_1} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathcal{E}_2} \right]
\]
The Cowen-Douglas Theory for Operator Tuples and Similarity

\[
\begin{align*}
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, w^j} &= \frac{\partial K_{\mathbf{E}_1 \otimes \mathbf{E}_2}}{\partial w_j} + h_{\mathbf{E}_1}^{-1} \frac{\partial h_{\mathbf{E}_1}}{\partial w_j} K_{\mathbf{E}_1} - K_{\mathbf{E}_1} h_{\mathbf{E}_1}^{-1} \frac{\partial h_{\mathbf{E}_1}}{\partial w_j} \\
&= K_{\mathbf{E}_1, w^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, w^j},
\end{align*}
\]

and

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, \bar{w}^j} = \frac{\partial K_{\mathbf{E}_1 \otimes \mathbf{E}_2}}{\partial \bar{w}_j} = \frac{\partial}{\partial \bar{w}_j} \left( K_{\mathbf{E}_1 \otimes I_{n_2}} + I_{n_1} \otimes K_{\mathbf{E}_2, \bar{w}^j} \right) = K_{\mathbf{E}_1, \bar{w}^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, \bar{w}^j}.
\]

Next, if

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, w^j} = K_{\mathbf{E}_1, w^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, w^j}
\]

and

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, \bar{w}^j} = K_{\mathbf{E}_1, \bar{w}^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, \bar{w}^j}
\]

holds for some \( J \in \mathbb{N}_0^n \), then

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, w^{j+ej}} = \frac{\partial K_{\mathbf{E}_1 \otimes \mathbf{E}_2}}{\partial w_j} + \left[ h_{\mathbf{E}_1}^{-1} \frac{\partial h_{\mathbf{E}_1}}{\partial w_j} + h_{\mathbf{E}_2}^{-1} \frac{\partial h_{\mathbf{E}_2}}{\partial w_j} \right] K_{\mathbf{E}_1, w^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, w^j}.
\]

and

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, \bar{w}^{j+ej}} = \frac{\partial K_{\mathbf{E}_1 \otimes \mathbf{E}_2}}{\partial \bar{w}_j} = \frac{\partial}{\partial \bar{w}_j} \left( K_{\mathbf{E}_1, \bar{w}^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, \bar{w}^j} \right) = K_{\mathbf{E}_1, \bar{w}^{j+ej}} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, \bar{w}^{j+ej}},
\]

for any \( ej = (0, \cdots, 1, \cdots, 0) \in \mathbb{N}_0^n \). Hence,

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, w^j} = K_{\mathbf{E}_1, w^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, w^j}
\]

and

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, \bar{w}^j} = K_{\mathbf{E}_1, \bar{w}^j} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, \bar{w}^j}
\]

for all \( J \in \mathbb{N}_0^n \). Since without loss of generality, we have for \( I, J - e_j \in \mathbb{N}_0^n \),

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, w^{j-e_j}} = K_{\mathbf{E}_1, w^{j-e_j}} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, w^{j-e_j}},
\]

it is easy to see that

\[
K_{\mathbf{E}_1 \otimes \mathbf{E}_2, \bar{w}^{j-e_j}} = K_{\mathbf{E}_1, \bar{w}^{j-e_j}} \otimes I_{n_2} + I_{n_1} \otimes K_{\mathbf{E}_2, \bar{w}^{j-e_j}}.
\]

\( \square \)
The Cowen-Douglas Theory for Operator Tuples and Similarity

We are now ready to prove our second main theorem of the paper.

**Theorem 3.5** Let $T = (T_1, \cdots, T_m) \in B_n(B_m)$ and consider the operator tuple $M_m^* = (M_1^*, \cdots, M_m^*) \in B_n(B_m)$ on a Hilbert space $H_K$ defined on $B_m$ with reproducing kernel $K$. Suppose that $n_2 := n/n_1 \in \mathbb{N}$ and that there exist an isometry $V$ and a Hermitian holomorphic vector bundle $E$ over $B_m$ such that

$$VK_{T,w^j}V^* - K_{M_m^*,w^j} \otimes I_{n_2} = I \otimes K_{E,w^j}, \quad I, J \in \mathbb{N}_0.$$  \hspace{1cm} (3.2)

If the bundle $E$ satisfies condition (C) via $H_K$, then $T \sim \bigoplus_{i=1}^{n_2} M_i^*$.

**Proof** First, from (3.2) and the definition of condition (C), we know that $E$ is $n_2$-dimensional Hermitian holomorphic vector bundle. Without losing generality, assume that $\{f_1, \ldots, f_{n_2}\}$ is a frame of $E$. It is easy to see that the Hermitian holomorphic vector bundle $E_{M_k^*} \otimes E$ is expressed as

$$(E_{M_k^*} \otimes E)(w) = \bigvee_{1 \leq i \leq n_1 \atop 1 \leq j \leq n_2} \{K(\cdot, \overline{w})\overline{\sigma}_i \otimes f_j(w)\}, \quad w \in B_m,$$

where $\{\overline{\sigma}_i\}_{i=1}^{n_1}$ is an orthonormal basis for $\mathbb{C}^{n_1}$. By Lemma 3.4 and (3.2), we then have for some isometry $V$ and for all $I, J \in \mathbb{N}_0$,

$$K_{E_{M_k^*} \otimes E, w^j} = K_{E_{M_k^*}, w^j} \otimes I_{n_2} + I_{n_1} \otimes K_{E, w^j} = V K_{E_{T,w^j}} V^*.$$

It is also proven in [22] that $E_{M_k^*} \otimes E$ is congruent to $E_T$, that is, there is a unitary operator $U$ such that

$$U(E_{M_k^*} \otimes E)(w) = E_T(w), \quad w \in B_m. \hspace{1cm} (3.3)$$

Next, since the Hermitian holomorphic vector bundle $E$ satisfies condition (C), for the holomorphic frame $\{f_1, \ldots, f_{n_2}\}$ of $E$, there exist $F \in H_c(\mathbb{C}^{n_2} \rightarrow E(B_m))$ and $G \in H_c(\mathbb{C}^{n_2} \rightarrow H_K)$ such that $G(\overline{w})F(\overline{w}) = I$ for every $w \in B_m$, where $E = \bigvee_{w \in B_m} \{f_1(w), \ldots, f_{n_2}(w)\}$. A proof similar to the one used in Lemma 2.2 then implies that for $(F(\overline{w}))^* \in \text{Mult}(H_K \otimes E, H_K \otimes \mathbb{C}^{n_2})$,

$$M_{(F(\overline{w}))^*}(K(\cdot, \overline{w})\overline{\sigma}_i \otimes \sigma_j) = K(\cdot, \overline{w})\overline{\sigma}_i \otimes F(\overline{w})(\sigma_j)$$

$$= K(\cdot, \overline{w})\overline{\sigma}_i \otimes F(w)(\sigma_j)$$

$$= K(\cdot, \overline{w})\overline{\sigma}_i \otimes f_j(w),$$

where $\{\sigma_j\}_{j=1}^{n_2}$ is an orthonormal basis for $\mathbb{C}^{n_2}$. In addition, for $f \in H_K$ and $g \in E$,

$$M_{(F(\overline{w}))^*}(f(z) \otimes g) = (F(\overline{w})(z))^* \left( \sum_{i=0}^{\infty} \langle f(z), e_i \rangle e_i \otimes g \right)$$

$$= \sum_{i=1}^{\infty} \langle f(z), e_i \rangle (F(\overline{w})(z))^*(e_i \otimes g)$$

$$= \sum_{i=1}^{\infty} \langle f(z), e_i \rangle e_i \otimes (F(\overline{w})(z))^* g.$$
= f(z) \otimes (F^\#(z))^* g,

where \( \{e_i\}_{i=0}^\infty \) is an orthonormal basis for \( \mathcal{H}_K \). Hence,

\[
\langle f(z) \otimes g, M^*_{(F\#)}(K(\cdot, \overline{w})\tilde{\sigma}_i \otimes \sigma_j) \rangle = \langle M^*_{(F\#)}(f(z) \otimes g), K(\cdot, \overline{w})\tilde{\sigma}_i \otimes \sigma_j \rangle = \langle f(z) \otimes (F^\#(z))^* g, K(\cdot, \overline{w})\tilde{\sigma}_i \otimes \sigma_j \rangle
\]

= \sum_{k=1}^{n_2} \langle f(z)\langle (F^\#(z))^* g, \sigma_k \rangle \otimes \sigma_k, K(\cdot, \overline{w})\tilde{\sigma}_i \otimes \sigma_j \rangle

= \sum_{k=1}^{n_2} \langle f(z)\langle (F^\#(z))^* g, \sigma_k \rangle, K(\cdot, \overline{w})\tilde{\sigma}_i \rangle \langle \sigma_k, \sigma_j \rangle

= \langle f(w)\langle (F^\#(\overline{w}))^* g, \sigma_j \rangle, \tilde{\sigma}_i \rangle

= \langle f(z), K(\cdot, \overline{w})\tilde{\sigma}_i \rangle \langle g, F^\#(\overline{w}) \sigma_j \rangle

= \langle f(z) \otimes g, K(\cdot, \overline{w})\tilde{\sigma}_i \otimes F^\#(\overline{w}) \sigma_j \rangle.

This shows that \( M^*_{(F\#)}(K(\cdot, \overline{w})\tilde{\sigma}_i \otimes \sigma_j) = K(\cdot, \overline{w})\tilde{\sigma}_i \otimes F^\#(\overline{w}) \sigma_j \).

Furthermore, since \( \mathcal{H}_K = \bigvee_{w \in \mathbb{B}^m} \{K(\cdot, \overline{w})\xi : \xi \in \mathbb{C}^{n_1}\} \), we have ran \( M^*_{(F\#)} = \mathcal{H}_K \otimes E \) and \( \|M^*_{(F\#)}\| = \|F^\#\| < \infty \). Combining these results and taking into account the operator \( M^*_{(G\#)} : \mathcal{H}_K \otimes E \rightarrow \mathcal{H}_K \otimes \mathbb{C}^{n_2} \), we obtain

\[
M^*_{(G\#)} M^*_{(F\#)}(K(\cdot, \overline{w})\tilde{\sigma}_i \otimes \sigma_j) = M^*_{(G\#)} (K(\cdot, \overline{w})\tilde{\sigma}_i \otimes F^\#(\overline{w}) \sigma_j)
\]

\[
= K(\cdot, \overline{w})\tilde{\sigma}_i \otimes G^\#(\overline{w})F^\#(\overline{w}) \sigma_j
\]

\[
= K(\cdot, \overline{w})\tilde{\sigma}_i \otimes \sigma_j,
\]

for \( 1 \leq i \leq n_1 \) and \( 1 \leq j \leq n_2 \). Therefore, \( M^*_{(G\#)}, M^*_{(F\#)} \equiv I \) so that \( M^*_{(F\#)} \) is an invertible operator satisfying

\[
M^*_{(F\#)}(E_{M_{Z}^2 \otimes I_{n_2}}(w)) = (E_{M_{Z}^2} \otimes E)(w), \quad w \in \mathbb{B}^m.
\]

From this and (3.3), we conclude that the invertible operator \( UM^*_{(F\#)} \) establishes the similarity between \( T \) and \( \bigoplus_{1}^{n_2} M_{Z}^* \).

The following corollary should be compared to M. Uchiyama’s result in [73]. It is already known that the result holds for weighted Bergman spaces \( \mathcal{H}_k, k > m + 1 \):
Corollary 3.6 Let $T = (T_1, \cdots, T_m) \in \mathcal{B}_1^n(\mathbb{B}_m)$ and consider the operator tuple $M^*_z = (M^*_z, \cdots, M^*_z)$ on a Hilbert space $\mathcal{H}_K$ such that $\text{Mult}(\mathcal{H}_K) = H^\infty(\mathbb{B}_m)$. If $\mathcal{E}_T = \mathcal{E}_{M^*_z} \otimes \mathcal{E}$ for some Hermitian holomorphic vector bundle $\mathcal{E}$ over $\mathbb{B}_m$, where $\mathcal{E}(w) = \sqrt{f(w)}$ and $f$ is bounded by positive constants, then $T \sim_s M^*_z$.

Remark 3.7 Condition (C) is related to the corona theorem. Let $T = (T_1, \cdots, T_m) \in \mathcal{B}_1^n(\mathbb{B}_m)$ and consider the operator tuple $M^*_z = (M^*_z, \cdots, M^*_z)$ on a Hilbert space $\mathcal{H}_K$ defined on $\mathbb{B}_m$. Suppose that $\mathcal{E}_T = \mathcal{E}_{M^*_z} \otimes \mathcal{E}$ for some Hermitian holomorphic vector bundle $\mathcal{E}$ over $\mathbb{B}_m$, where $\mathcal{E}(w) = \sqrt{f(w)}$ is such that $f(w) = (f_1(w), f_2(w), \cdots, f_n(w))^T$ and $f_j \in \text{Mult}(\mathcal{H}_K), 1 \leq j \leq n$. Let there exist a constant $\delta > 0$ satisfying

$$\delta \leq \left( \sum_{j=1}^n |f_j(w)|^2 \right)^{\frac{1}{2}}, \quad w \in \mathbb{B}_m,$$

and a function $F \in H^\infty_{\mathcal{E}_C \rightarrow E}(\mathbb{B}_m)$ such that

$$F^\#(\overline{w})(\lambda) := F(w)(\lambda)(w) = \lambda f(w)$$

and $(F^\#)^* \in \text{Mult}(\mathcal{H}_K \otimes E, \mathcal{H}_K \otimes \mathbb{C})$, for $\lambda \in \mathbb{C}$ and $E = \bigvee_{w \in \mathbb{B}_m} f(w)$. Then condition (3.4) implies the existence of $g_1, \cdots, g_n \in \text{Mult}(\mathcal{H}_K)$ such that

$$\sum_{j=1}^n g_j(w)f_j(w) = 1, \quad w \in \mathbb{B}_m.$$

Setting $G = (g_1, g_2, \cdots, g_n)$, one can proceed as in the proof of Theorem 3.5 to conclude that $T \sim_s M^*_z$.

Corollary 3.8 Let $T \in \mathcal{B}_n^1(\mathbb{D})$ and $S = (M^*_z, \mathcal{H}_K) \in \mathcal{B}_n^1(\mathbb{D})$. If $\text{Mult}(\mathcal{H}_K) = H^\infty(\mathbb{D})$ and $\mathcal{E}_T \sim_u \mathcal{E}_S \otimes \mathcal{E}$, for some Hermitian holomorphic vector bundle $\mathcal{E}$ of rank $n_2 := n/n_1$, then $T \sim_s \bigoplus_{i=1}^{n_2} S_i$ if and only if $\mathcal{E}$ satisfies condition (C) via $\mathcal{H}_K$.

Proof Suppose first that there exists a bounded invertible operator $X$ such that

$$X(\bigoplus_{i=1}^{n_2} S_i) = TX.$$ Then $X : \mathcal{E}_S \otimes \mathbb{C}^{n_2} \rightarrow \mathcal{E}_S \otimes \mathcal{E}$ satisfies

$$X(K(\cdot, \overline{w})\delta_i \otimes \sigma_j) = K(\cdot, \overline{w})\delta_i \otimes f_j(w),$$

for a holomorphic frame $\{f_1, \cdots, f_{n_2}\}$ of $\mathcal{E}$ and orthonormal bases $\{\sigma_j\}_{j=1}^{n_1}$ and $\{\sigma_j\}_{j=1}^{n_2}$ for $\mathbb{C}^{n_1}$ and for $\mathbb{C}^{n_2}$, respectively. Note that since $X$ is a bounded linear operator, the $f_i$ are uniformly bounded on $\mathbb{D}$. Thus, we can define a function $F \in H^\infty_{\mathcal{E}_C \rightarrow E}(\mathbb{D})$ as

$$F(w)\sigma_j = f_j(w), \quad 1 \leq j \leq n_2,$$

where $E = \bigvee_{w \in \mathbb{D}} \{f_j(w) : 1 \leq j \leq n_2\}$. Obviously, the function $F^\#$ defined on $\mathbb{D}$ as $F^\#(w) := F(\overline{w})$ is such that $(F^\#)^* \in H^\infty_{\mathbb{D} \rightarrow \mathbb{C}^{n_2}}(\mathbb{D})$. Moreover, since $\text{Mult}(\mathcal{H}_K \otimes E, \mathcal{H}_K \otimes \mathbb{C}^{n_2}) = H^\infty_{\mathbb{D} \rightarrow \mathbb{C}^{n_2}}(\mathbb{D})$,

$$(F^\#)^* \in \text{Mult}(\mathcal{H}_K \otimes E, \mathcal{H}_K \otimes \mathbb{C}^{n_2}).$$
Next, note from Theorem 3.5 that
\begin{align*}
M_{(F^#)_*}^*(K(\cdot, \bar{w})\sigma_i \otimes \sigma_j) &= K(\cdot, \bar{w})\sigma_i \otimes F^#(w)\sigma_j \\
&= K(\cdot, \bar{w})\sigma_i \otimes (F(w)\sigma_j) \\
&= K(\cdot, \bar{w})\sigma_i \otimes f_j(w),
\end{align*}
and that the operator $M_{(F^#)_*}^*$ has dense range. Since $X$ is invertible, this means that $X = M_{(F^#)_*}^*$. Furthermore, for every $h = K(\cdot, \bar{w})\xi \in \mathcal{H}_K$, $g \in \mathbb{C}^{n_2}$ and $w \in \mathbb{D}$, there exists a $\delta > 0$ such that
\begin{align*}
\langle X^*X(h \otimes g), h \otimes g \rangle &= \langle M_{(F^#)_*}^*(h \otimes g), h \otimes g \rangle \\
&= \|h\|^2 \langle F^*(w)F(w)g, g \rangle \\
&\geq \delta^2 \|g\|^2 \|h\|^2.
\end{align*}
It follows that since $F \in H_{\mathbb{C}^{n_2} \rightarrow E}(\mathbb{D})$ and $F^*(w)F(w) \geq \delta^2 > 0$, there exists a function $G \in H_{\mathbb{C}^{n_2} \rightarrow \mathbb{C}^{n_2}}(\mathbb{D})$ such that $G(w)F(w) \equiv I$, that is, $G^#(w)F^#(w) = I$, for every $w \in \mathbb{D}$.

Conversely, if the complex bundle $E$ satisfies condition (C), then there is a holomorphic frame $\{f_1, \cdots, f_{n_2}\}$ of $\mathbb{C}^{n_2}$ and an orthonormal basis $\{\sigma_1, \cdots, \sigma_{n_2}\}$ of $\mathbb{C}^{n_2}$. Another application of Theorem 3.5 yields a bounded invertible operator $M_{(F^#)_*}^*$:
\begin{align*}
\left(\mathcal{E}_{i=1}^{n_2} E\right)(w) \rightarrow (\mathcal{E}_S \otimes \mathcal{E})(w).
\end{align*}
Now, since $\mathcal{E}_T \sim_u \mathcal{E}_S \otimes \mathcal{E}$, there is a unitary operator $U$ such that $U \left( (\mathcal{E}_S \otimes \mathcal{E})(w) \right) = \mathcal{E}_T(w)$ for $w \in \mathbb{D}$. Then, $UM_{(F^#)_*}^*$ is a bounded invertible operator establishing the similarity between $T$ and $\bigoplus_{i=1}^{n_2} S$. \hfill \Box

The Dirichlet space $\mathcal{D}$ consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined on the unit disk $\mathbb{D} \subset \mathbb{C}$ satisfying $\|f\| = \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty$. It is well-known that the reproducing kernel of $\mathcal{D}$ is given as $K(z, w) = \frac{1}{\overline{w}z} \log \frac{1}{1-\overline{w}z}$, for $w, z \in \mathbb{D}$. Research on similarity on the Dirichlet space $\mathcal{D}$ can be found in [49], for instance. Using the results of [53], we now give a sufficient condition for a Cowen-Douglas operator to be similar to $M_z^*$ on $\mathcal{D}$.

**Corollary 3.9** Let $T \in \mathcal{B}_1^1(\mathbb{D})$ and consider the operator $M_z^* \in \mathcal{B}_1^1(\mathbb{D})$ on the Dirichlet space $\mathcal{D}$. Suppose that $\mathcal{E}_T = \mathcal{E}_{M_z^*} \otimes \mathcal{E}$ for some Hermitian holomorphic vector bundle $\mathcal{E}$ over $\mathbb{D}$, where $\mathcal{E}(w) = \sqrt{f(w)}$ and $f(w) = (f_1(w), \cdots, f_n(w))^T$ for $f_j \in H^\infty(\mathbb{D})$. If
\begin{align*}
\int_{\mathbb{D}} |f(z)|^2 |f_j(z)|^2 U(z) dA(z) \lesssim \|f\|^2_{\mathcal{D}}, \quad f \in \mathcal{D},
\end{align*}
where
\begin{align*}
U(z) &= \int_{\overline{\mathbb{D}}} \log \left| 1 - \frac{w}{z} \right|^2 \frac{|dw|}{2\pi (1 - |w|^2)} + \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \overline{w}z|^2} \frac{|dw|}{2\pi}, \quad z \in \mathbb{D},
\end{align*}
and $dA$ is the normalized area measure, then $T \sim_s M_z^*$.
In order to investigate similarity between tuples of irreducible operators in the Cowen-Douglas class $\mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$, we introduce a subclass denoted $\mathcal{F}\mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$. The subclass $\mathcal{F}\mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$ is the collection of all $T = (T_1, \cdots, T_m) \in \mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$ of the form

$$T = \begin{pmatrix} T_{1,1} & \tilde{T}_{1,2} & \cdots & \tilde{T}_{1,n-1} & \tilde{T}_{1,n} \\ 0 & T_{2,1} & \cdots & \tilde{T}_{2,n-1} & \tilde{T}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{n,1} & \tilde{T}_{n,n} \\ 0 & 0 & \cdots & 0 & T_{m,1} \end{pmatrix}, \cdots, \begin{pmatrix} T_{m,1} & \tilde{T}_{1,2} & \cdots & \tilde{T}_{1,n-1} & \tilde{T}_{1,n} \\ 0 & T_{m,2} & \cdots & \tilde{T}_{2,n-1} & \tilde{T}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{m,n-1} & \tilde{T}_{m,n} \\ 0 & 0 & \cdots & 0 & T_{m,m} \end{pmatrix},$$

where $T_i = (T_{i,1}, \cdots, T_{i,i}) \in \mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$ and the $\tilde{T}_{i,i+1}, 1 \leq i \leq n - 1$, are non-zero operators such that $T_{k,i} T_{i,i+1} = \tilde{T}_{i,i+1} T_{k,i+1}$ for all $1 \leq k \leq m$. The class $\mathcal{F}\mathcal{B}^m_\mathcal{r}(\Omega)$ was defined in [38], and in [39, 44], the corresponding similarity question was considered. We now give a sufficient condition for the similarity between $T$ and $S$, where

$$S = \begin{pmatrix} S_{1,1} & \tilde{S}_{1,2} & \cdots & \tilde{S}_{1,n-1} & \tilde{S}_{1,n} \\ 0 & S_{1,2} & \cdots & \tilde{S}_{2,n-1} & \tilde{S}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S_{n,1} & \tilde{S}_{n,n-1} \\ 0 & 0 & \cdots & 0 & S_{n,n} \end{pmatrix} = (S_{i,j}) \in \mathcal{F}\mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$$

and $S_i = (S_{i,1}, \cdots, S_{i,i})$ is in $\mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$ for $1 \leq i \leq n$.

**Theorem 3.10** Let $T, S \in \mathcal{F}\mathcal{B}^m_\mathcal{r}(\mathbb{B}_m)$, where $S_i \sim_u (M^*_z, \mathcal{H}_{k_i})$ for some $k_i > m+1$ and for all $1 \leq i \leq n$. Suppose that the following conditions hold:

1. $K_{S_i}(w) - K_T(w) = \sum_{i,j=1}^m \frac{\partial^2 \psi(w)}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j$, where $\psi(w) = \log \|f(w)\|^2$ for some analytic vector valued function $f$ over $\mathbb{B}_m$.

2. Condition (C) holds for the Hermitian holomorphic vector bundle $\mathcal{E}$, with $\mathcal{E} = \bigvee f(w)$, via $\mathcal{H}_{k_i}, 1 \leq i \leq n$.

3. There exist functions $\{\phi_i\}_{i=1}^{n-1} \subset GL(H^\infty(\mathbb{B}_m))$ such that for all $1 \leq i < j \leq n$ and $w \in \mathbb{B}_m$,

$$\prod_{k=i}^{j-1} |\phi_k(w)|^2 \frac{\langle \tilde{T}_{i,j} t_i(w), t_i(w) \rangle}{\|t_j(w)\|^2} = \frac{\langle \tilde{S}_{i,j} K_j(\cdot, \overline{w}), K_i(\cdot, \overline{w}) \rangle}{\|K_j(\cdot, \overline{w})\|^2},$$

where $t_n(w) \in \ker(T_n - w)$, $K_n(\cdot, \overline{w}) \in \ker(S_n - w)$, $t_i(w) = \tilde{T}_{i,i+1} t_{i+1}(w)$, and $K_i(\cdot, \overline{w}) = \tilde{S}_{i,i+1} K_{i+1}(\cdot, \overline{w})$ for $1 \leq i \leq n - 1$.

4. $T_{k,i} \tilde{T}_{i,j} = \tilde{T}_{i,j} T_{k,j}$ and $S_{k,i} \tilde{S}_{i,j} = \tilde{S}_{i,j} S_{k,j}$ for all $1 \leq i < j \leq n$ and $1 \leq k \leq m$.

Then $S \sim_u T$.
Proof: We can assume from condition (1) that $\mathcal{E}_{T_1} = \mathcal{E}_S \otimes \mathcal{E}$, where $\mathcal{E}(w) = \sqrt{f(w)}$ for analytic vector valued function $f$ over $\mathbb{B}_m$. Furthermore, $\|t_1(w)\| = \|K_1(\cdot, \overline{w}) \otimes f(w)\|$ for some $t_1(w) \in \ker(T_1 - w)$. Note also that if we let $j = i + 1$ in condition (3), then

$$|\phi_i(w)|^2 \|t_i(w)\|^2 \|u_{i+1}(w)\|^2 = \|K_i(\cdot, \overline{w})\|^2 \|K_{i+1}(\cdot, \overline{w})\|^2. \quad (3.5)$$

Thus, for $2 \leq i \leq n$,

$$\|t_i(w)\|^2 = \frac{\prod_{k=1}^{i-1} |\phi_k(w)|^2 \|K_i(\cdot, \overline{w})\|^2 \|t_i(w)\|^2}{\|K_1(\cdot, \overline{w})\|^2} = \frac{\prod_{k=1}^{i-1} |\phi_k(w)|^2 \|K_i(\cdot, \overline{w})\|^2 \|K_{i+1}(\cdot, \overline{w})\|^2}{\|K_1(\cdot, \overline{w})\|^2} = \frac{\prod_{k=1}^{i-1} |\phi_k(w)|^2 \|K_i(\cdot, \overline{w})\|^2 \|f(w)\|^2}{\|K_1(\cdot, \overline{w})\|^2}.$$

By using the Rigidity Theorem given in [20], we now define the isometries $W_i$, $1 \leq i \leq n$, by

$$W_1t_1(w) := K_1(\cdot, \overline{w}) \otimes f(w) \text{ and } W_i t_i(w) := \prod_{k=1}^{i-1} \phi_k(w)K_i(\cdot, \overline{w}) \otimes f(w). \quad (3.6)$$

Then, for $1 \leq i \leq m$,

$$\begin{pmatrix} T_{i,1} & \tilde{T}_{1,2} & \cdots & \tilde{T}_{1,n} \\ 0 & T_{i,2} & \cdots & \tilde{T}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{i,n} \end{pmatrix} \sim_u \begin{pmatrix} M_1^{2} & W_1 & W_2 \cdot \cdots \cdot W_n \\ 0 & M_2^{2} & W_2 \cdot \cdots \cdot W_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n^{2} \end{pmatrix}, \quad (3.7)$$

where $N_j = \text{ran} W_j$ for $1 \leq j \leq n$. Moreover,

$$\ker(M_i^{2} \mid N_i - w) = \sqrt{\{K_1(\cdot, \overline{w}) \otimes f(w)\}}$$

and

$$\ker(M_i^{2} \mid N_j - w) = \sqrt{\left\{ \prod_{k=1}^{j-1} \phi_k(w)K_j(\cdot, \overline{w}) \otimes f(w) \right\}},$$

for $2 \leq j \leq n$ and $w \in \mathbb{B}_m$. Since $S_i \sim_u (M_i^{2}, \mathcal{H}_{k_i})$ for $1 \leq i \leq n$,

$$\mathcal{K}_S - \mathcal{K}_{T_1} = \mathcal{K}_S - \mathcal{K}_{M_1^{2} \mid N_1} = \mathcal{K}_S - \mathcal{K}_{\mathcal{E}} = \sum_{i,j=1}^{m} \frac{\partial^2 \psi(w)}{\partial w_i \partial w_j} dw_i \wedge dw_j,$$

where $\psi(w) = \log \|f(w)\|^2$. Next, by condition (2) and Lemma 2.2, there is a multiplier $(M^{\#})^* \in \text{Mult}(\mathcal{H}_{k_i} \otimes E, \mathcal{H}_{k_i} \otimes \mathbb{C})$ with $F \in H_{\infty}^{\infty}(\mathbb{B}_m)$ that satisfies

$$M^{\#}_i(K_i(\cdot, \overline{w}) \otimes \lambda) = K_i(\cdot, \overline{w}) \otimes F_{\#}(\overline{w})(\lambda) = K_i(\cdot, \overline{w}) \otimes \lambda f(w), \quad \lambda \in \mathbb{C},$$

and a $G \in H_{\infty}^{\infty}(\mathbb{B}_m)$ so that

$$M^{\#}_{(G^\#)}(K_i(\cdot, \overline{w}) \otimes 1) = M^{\#}_{(G^\#)}(K_i(\cdot, \overline{w}) \otimes F_{\#}(\overline{w})(1)) = K_i(\cdot, \overline{w}) \otimes G_{\#}(\overline{w}) F_{\#}(\overline{w})(1) = K_i(\cdot, \overline{w}) \otimes 1,$$
where \( E = \bigvee_{w \in \mathbb{B}_m} f(w) \). Then there exist invertible operators \( X_i \in \mathcal{L}(\mathcal{H}_{K_i}, \mathcal{N}_i), 1 \leq i \leq n \), such that \( X_i S_i = M_{\mathcal{Z}}^*|_{\mathcal{N}'}, X_i = W_i T_i W_i^* X_i \). It then follows for some \( g \in \mathcal{O}(\mathbb{B}_m) \), that

\[
X_1 K_1(\cdot, \overline{w}) = g(w) K_1(\cdot, \overline{w}) \otimes f(w) \quad \text{and} \quad X_i K_i(\cdot, \overline{w}) = g(w) \prod_{k=1}^{i-1} \phi_k(w) K_i(\cdot, \overline{w}) \otimes f(w),
\]

for all \( 2 \leq i \leq n \) and \( w \in \mathbb{B}_m \). A direct calculation shows that for \( 1 \leq i \leq n-1 \),

\[
X_i \tilde{S}_{i,i+1} = W_i \tilde{T}_{i,i+1} W_{i+1}^* X_{i+1}.
\]

Note that \( T_{k,i} \tilde{T}_{i,j} = \tilde{T}_{i,j} T_{k,j} \) and \( S_{k,i} \tilde{S}_{i,j} = \tilde{S}_{i,j} S_{k,j} \) for all \( 1 \leq i < j \leq n \) and \( 1 \leq k \leq m \). Moreover, there exist functions \( \phi_{i,j}, \varphi_{i,j} \in \mathcal{O}(\mathbb{B}_m) \) such that \( \tilde{T}_{i,j} t_j(w) = \phi_{i,j}(w) t_j(w) \) and \( \tilde{S}_{i,j} K_j(\cdot, \overline{w}) = \varphi_{i,j}(w) K_i(\cdot, \overline{w}) \). Therefore, for \( 1 \leq i < j \leq n \), we get from condition (3) and (3.5) that

\[
\prod_{k=i}^{j-1} |\phi_k(w)|^2 \frac{\phi_{i,j}(w)}{||t_i(w)||^2} = \frac{\varphi_{i,j}(w) ||K_i(\cdot, \overline{w})||^2}{||K_j(\cdot, \overline{w})||^2}
\]

and

\[
\prod_{k=i}^{j-1} |\phi_k(w)|^2 \frac{||t_i(w)||^2}{||t_j(w)||^2} = \frac{||K_i(\cdot, \overline{w})||^2}{||K_j(\cdot, \overline{w})||^2},
\]

that is, \( \phi_{i,j} = \varphi_{i,j} \). Moreover, by (3.6) and (3.8),

\[
W_1 \tilde{T}_{1,i} W_i^* X_1 K_1(\cdot, \overline{w}) = W_1 \tilde{T}_{1,i} W_i^* \left( g(w) \prod_{k=1}^{l-1} \phi_k(w) K_1(\cdot, \overline{w}) \otimes f(w) \right)
\]

and

\[
W_i \tilde{T}_{i,j} W_j^* X_j K_j(\cdot, \overline{w}) = W_i \tilde{T}_{i,j} W_j^* \left( g(w) \prod_{k=1}^{j-1} \phi_k(w) K_j(\cdot, \overline{w}) \otimes f(w) \right)
\]
for $1 < l \leq n$ and $2 \leq i < j \leq n$. Hence, the operator $X := \text{diag}(X_1, \ldots, X_n)$ is invertible and
\[
M^{*}_{\mathcal{N}_i} W_{1} \tilde{T}_{1,2} W_{2}^{*} \cdots W_{1} \tilde{T}_{1,n} W_{n}^{*} = \begin{pmatrix}
(X_1 & X_2 & \cdots & X_n) \\
0 & M^{*}_{\mathcal{N}_i} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M^{*}_{\mathcal{N}_i}
\end{pmatrix}
\]
for all $1 \leq i \leq m$. From this and (3.7), we conclude that $T \sim_{s} S$. \hfill \square

### 3.3 Uniqueness of strongly irreducible decomposition up to similarity

When the Hilbert space $\mathcal{H}$ is finite-dimensional, the Jordan canonical form theorem indicates that every operator on $\mathcal{H}$ can be uniquely written as a direct sum of strongly irreducible operators up to similarity. Is there a corresponding analogue when one considers operators on an infinite-dimensional complex separable Hilbert space $\mathcal{H}$? The notion of a unicellular operator was introduced in [14, 15] and it was shown in [47, 48] that dissipative operators can be written as a direct sum of unicellular operators. In [11, 12, 64–67], every $C_0$-operator on a complex separable Hilbert space was proven to be similar to a Jordan operator. Furthermore, in [26], every bitriangular operator was shown to be quasisimilar to a Jordan operator. The concepts of strong irreducibility and of Banach irreducibility introduced in [30] and [45], respectively, turned out to be equivalent. In [31], the set of irreducible operators was proven to be dense in $\mathcal{L}(\mathcal{H})$ in the sense of Hilbert-Schmidt norm approximations. For the class $\mathcal{B}_n^1(\Omega)$, the work Y. Cao-J. S. Fang-C. L. Jiang [16], C. L. Jiang [41], and C. L. Jiang-X. Z. Guo-K. Ji [42] involve the $K_0$-group of the commutant algebra as an invariant to show that an operator in $\mathcal{B}_n^1(\Omega)$ has a unique strong irreducible decomposition up to similarity.

Let $T \in \mathcal{B}_n^1(\mathbb{D})$ be an $n$-hypercontraction. Denote by $\mathcal{H}_n^1$ the Hilbert space of analytic functions on the unit disk $\mathbb{D}$ with reproducing kernel $K(z, w) = \frac{1}{(1-z \bar{w})^n}$, for $z, w \in \mathbb{D}$. The results in [27] and [35] show that $\bigoplus_{i=1}^{k} T$ is similar to the backward shift operator $\bigoplus_{i=1}^{k} M_{z}^{*}$ on $\bigoplus_{i=1}^{k} \mathcal{H}_n^1$ if and only if there exists a bounded subharmonic function $\varphi$ defined on $\mathbb{D}$ such that
\[
\text{trace} \ k \bigoplus_{i=1}^{k} M_{z}^{*}(w) - \text{trace} \ k \bigoplus_{i=1}^{k} T(w) = kK_{M_{z}^{*}}(w) - kK_T(w) \leq \frac{\partial^2 \varphi(w)}{\partial \bar{w} \partial w}, \quad w \in \mathbb{D}.
\]

Note here that if $\bigoplus_{i=1}^{k} T \sim_{s} \bigoplus_{i=1}^{k} M_{z}^{*}$, then $T \sim_{s} M_{z}^{*}$.
We start with some definitions given in [30, 31].

**Definition 3.11** Let \( \{T\}' = \{X \in \mathcal{L}(\mathcal{H}) : XT = TX\} \) be the commutant of \( T \in \mathcal{L}(\mathcal{H}) \). The operator \( T \) is called **strongly irreducible** if \( \{T\}' \) does not have any nontrivial idempotents. It is called **irreducible** if \( \{T\}' \) does not any contain nontrivial self-adjoint idempotents.

**Definition 3.12** Consider \( T = (M_z^*, \mathcal{H}_K, K) \in \mathcal{B}_n^1(\Omega) \), where \( \mathcal{H}_K \) is an analytic function space with reproducing kernel \( K \).

Proposition 3.13 Let \( T = T_1 \oplus \cdots \oplus T_k, S^* = S_1^* \oplus \cdots \oplus S_k^* \in \mathcal{B}_n^1(\Omega) \), where \( S_i^* \) is the adjoint of the operator of multiplication by \( z \) on a reproducing kernel Hilbert space \( \mathcal{H}_K \) with reproducing kernel \( K_i \) and \( S_i^* \) is strongly irreducible for \( 1 \leq i \leq k \).

Then, there is a permutation \( \pi \) on \( \{1, 2, \cdots, k\} \) such that \( T_i \sim_s \tilde{T}_{\pi(i)} \) for \( 1 \leq i \leq k \).

**Proof** Note first that \( T, S^* \in \mathcal{B}_n^1(\Omega) \). By Lemma 3.4, we have
\[
VK_{T,w^*}V^* = K_{S^*,w^*} + K_{\mathcal{E},w^*} \otimes I_n, \quad 0 \leq i, j \leq n - 1.
\]
Then, there is a permutation \( \pi \) on \( \{1, \cdots, k\} \) such that \( S_i^* \sim_s T_{\pi(i)} \) for \( 1 \leq i \leq k \).
Hence, the operator $U_iM^{*}_{(F^#)}$ is invertible and
\[(U_iM^{*}_{(F^#)})^*(\mathcal{E}_{S^*_i}(w)) = \mathcal{E}_{T_{\pi(i)}}(w), \quad w \in \Omega,\]
so that $S^*_i \sim_s T_{\pi(i)}$ for $1 \leq i \leq k$. \qed

**Remark 3.14** As in the proof of Theorem 3.5, Proposition 3.13 shows that $T$ is similar to $S^*$. The result remains valid for operator tuples in $B^m_n(\Omega)$ possessing a unique irreducible decomposition up to unitary equivalence.

### 4 Applications of Theorem 3.2

Theorem 3.2 yields several sufficient conditions involving curvature for the similarity of certain adjoints of multiplication tuples. For $m > 1$, let $\Omega$ be a bounded domain in $\mathbb{C}^m$. The space $\mathcal{C}^2(\Omega)$ consists of functions defined on $\Omega$ whose second order partial derivatives are continuous. The reader is referred to [4, 51, 59] for the following definitions and results.

**Definition 4.1** A function $u \in \mathcal{C}^2(\Omega)$ is said to be pluriharmonic if it satisfies the $m^2$ differential equations
\[
\frac{\partial^2 u}{\partial w_i \partial w_j} = 0 \quad \text{for} \quad 1 \leq i, j \leq m.
\]

**Definition 4.2** A real-valued function $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\} (u \neq -\infty)$ is said to be plurisubharmonic if
(1) $u(z)$ is upper-semicontinuous on $\Omega$; and
(2) for each $z_0 \in \Omega$ and some $z_1 \in \mathbb{C}^m$ that is dependent on $z_0$, the function $u(z_0 + \lambda z_1)$ is subharmonic with respect to $\lambda \in \mathbb{C}$.

**Definition 4.3** A non-negative function $g : \mathbb{R}^m \rightarrow [0, +\infty)$ is called log-plurisubharmonic if the function $\log g$ is plurisubharmonic.

**Lemma 4.4** If $f$ is a pluriharmonic function on $\Omega$, then both $\log |f|$ and $|f|^p (0 < p < \infty)$ are plurisubharmonic functions on $\Omega$.

**Lemma 4.5** A real-valued function $f \in \mathcal{C}^2(\Omega)$ is plurisubharmonic if and only if
\[
\left( \frac{\partial^2 f(w)}{\partial w_i \partial w_j} \right)_{i,j=1}^m \geq 0 \quad \text{for every} \quad w \in \Omega.
\]

**Lemma 4.6** Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a convex function of two variables, increasing in each variable. If $F$ and $G$ are plurisubharmonic functions, then $\varphi(F, G)$ is also plurisubharmonic.

**Lemma 4.7** Let $f$ and $g$ be log-plurisubharmonic functions. Then $f + g$ is also log-plurisubharmonic.
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Proof Since the mapping \( k \mapsto \log(1 + e^k) \) is convex,
\[
\varphi(x, y) := \log(e^x + e^y) = x + \log(1 + e^{y-x})
\]
is also a convex function of two variables, increasing in each variable. Since \( F := \log f \) and \( G := \log g \) are plurisubharmonic, by Lemma 4.6,
\[
\varphi(F, G) = \log(e^F + e^G) = \log(f + g)
\]
is plurisubharmonic. \( \Box \)

Finally, we need the following definition of an \( n \)-hypercontraction given in [56]:

**Definition 4.8** Let \( T = (T_1, \ldots, T_m) \in \mathcal{L}(\mathcal{H})^m \) be an \( m \)-tuple of commuting operators. Define an operator \( M_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) by
\[
M_T(X) = \sum_{i=1}^{m} T_i^* X T_i, \quad X \in \mathcal{L}(\mathcal{H}).
\]
An \( m \)-tuple \( T \in \mathcal{L}(\mathcal{H})^m \) is called an \( n \)-hypercontraction if
\[
\triangle^{(l)}_T := (I - M_T)^l(I) \geq 0,
\]
for all integers \( l \) with \( 1 \leq l \leq n \). The special case of a 1-hypercontraction corresponds to the usual row contraction.

Note that since
\[
M_T^l(X) = \sum_{\alpha \in \mathbb{N}_0^m |\alpha| = l} \frac{l!}{\alpha! (l - |\alpha|)!} T^{*\alpha} X T^\alpha
\]
for all \( l \geq 0 \), one has
\[
\triangle^{(l)}_T = \sum_{j=0}^{l} (-1)^j \binom{l}{j} M_T^j(I) = \sum_{|\alpha| \leq l} (-1)^{|\alpha|} \frac{l!}{\alpha! (l - |\alpha|)!} T^{*\alpha} X T^\alpha.
\]

We now give a necessary and sufficient condition for the similarity of certain operator tuples in \( \mathcal{B}_1^m(\mathbb{B}_m) \) in terms of curvature and plurisubharmonic functions.

**Theorem 4.9** Let \( T = (T_1, \ldots, T_m) \in \mathcal{B}_1^m(\mathbb{B}_m) \) be an operator tuple on a Hilbert space \( \mathcal{H}_K \) with reproducing kernel \( K(z, w) = \sum_{i=0}^{\infty} a(i)(z_1 \overline{w}_1 + \cdots + z_m \overline{w}_m)^i \), where \( a(i) > 0 \). Consider \( M_T^z = (M_T^z, \ldots, M_T^z) \) on a weighted Bergman space \( \mathcal{H}_k \) with \( k > m + 1 \). Suppose that \( T \) is \( k \)-hypercontractive and that \( \lim_{j} f_j(T^*, T)h = 0, h \in \mathcal{H}_k \), for \( f_j(z, w) = \sum_{i=j}^{\infty} e_i(z)(1 - \langle z, w \rangle)^k e_i(w)^* \) and an orthonormal basis \( \{ e_i \}_{i=0}^{\infty} \) for \( \mathcal{H}_k \). Then \( T \sim_s M_T^z \) if and only if there exists a bounded plurisubharmonic function \( \psi \) such that
\[
\mathcal{K}_{M_T^z}(w) - \mathcal{K}_T(w) = \sum_{i, j=1}^{m} \frac{\partial^2 \psi(w)}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j, \quad w \in \mathbb{B}_m. \quad (4.1)
\]
Proof If $T$ and $M^*_T$ are similar, then there is a bounded invertible operator $X$ such that $X^*T = M^*_X X$. If $\gamma$ is a non-vanishing holomorphic section of $\mathcal{E}_T$, then $\tilde{\gamma} = X \gamma$ is an non-vanishing holomorphic section of $\mathcal{E}_{M^*_T}$, and there are constants $a$ and $b$ such that

$$0 < a \leq \frac{h_T(w)}{h_{M^*_T}(w)} = \left\| \frac{\gamma(w)}{\tilde{\gamma}(w)} \right\|^2 \leq b, \quad w \in \mathbb{B}_m.$$ 

Since $\frac{\gamma(w)}{\tilde{\gamma}(w)}$ is pluriharmonic, Lemma 4.4 shows that $\psi(w) := \log \left\| \frac{\gamma(w)}{\tilde{\gamma}(w)} \right\|^2$ is a bounded pluriharmonic function.

For the converse, let $\gamma$ and $\tilde{\gamma}$ be non-vanishing holomorphic sections of $\mathcal{E}_T$ and $\mathcal{E}_{M^*_T}$, respectively. Since

$$\mathcal{K}_{M^*_T}(w) - \mathcal{K}_T(w) = \sum_{i,j=1}^{m} \frac{\partial^2 \log \left\| \frac{\gamma(w)}{\tilde{\gamma}(w)} \right\|^2}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j,$$

condition (4.1) gives

$$\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \left( \log \left\| \frac{\gamma(w)}{\tilde{\gamma}(w)} \right\|^2 e^{\psi(w)} \right) = 0,$$

for all $1 \leq i, j \leq m$. Therefore, there exists a non-zero function $\phi \in \mathcal{O}(\mathbb{B}_m)$ such that

$$\left\| \frac{\gamma(w)}{\tilde{\gamma}(w)} \right\|^2 = e^{\psi(w)}.$$ 

Since the function $\psi$ is bounded on $\mathbb{B}_m$, there exist constants $\tilde{m}$ and $\tilde{M}$ such that

$$0 < \tilde{m} \leq \frac{||\gamma(w)||^2}{||\phi(w)\tilde{\gamma}(w)||^2} \leq \tilde{M}, \quad w \in \mathbb{B}_m.$$ (4.2)

Note that since $h_T(w) = K(\bar{w}, w) = \sum_{\alpha \in \mathbb{N}_0^m} a(|\alpha|) \frac{|\alpha|!}{\alpha!} w^\alpha \bar{w}^\alpha = \sum_{\alpha \in \mathbb{N}_0^m} \rho(\alpha) w^\alpha \bar{w}^\alpha$,

$$T^* \alpha T^\alpha e_\beta = \begin{cases} \frac{\rho(\alpha - \beta)}{\rho(\beta)} e_\beta, & \text{for } \alpha \leq \beta, \\ 0, & \text{otherwise,} \end{cases}$$

where $\rho(\alpha) = a(|\alpha|) \frac{|\alpha|!}{\alpha!}$ and $\{e_\alpha\}_{\alpha \in \mathbb{N}_0^m}$ denotes an orthonormal basis for $\mathcal{H}_K$. Since the reproducing kernel $K(z, w)$ on a weighted Bergman space $\mathcal{H}_k$ with $k > m + 1$ satisfies

$$\frac{1}{K(\bar{w}, w)} = \sum_{i=0}^{k} b(i) |w|^{2i},$$

where $b(i) = \frac{(-1)^{k+i}}{i!(k-i)!}$, a direct calculation yields

$$\frac{1}{K}(T^*, T)e_\beta = \sum_{\alpha \in \mathbb{N}_0^m} b(|\alpha|) \frac{|\alpha|!}{\alpha!} T^{* \alpha} T^{\alpha} e_\beta = \sum_{\alpha \in \mathbb{N}_0^m} b(|\alpha|) \frac{|\alpha|!}{\alpha!} \frac{\rho(\beta - \alpha)}{\rho(\beta)} e_\beta = \sum_{\alpha \in \mathbb{N}_0^m} b(|\alpha|) \frac{|\alpha|! \beta - |\alpha|! \beta}{\alpha! \beta!} e_\beta.$$ 

Then,

$$\frac{1}{K}(T^*, T)e_\beta = \begin{cases} \sum_{i=0}^{s} b(i) \frac{a(s-i)}{a(s)} e_\beta, & \text{if } \beta = (s, 0, \ldots, 0), 0 \leq s \leq k, \\ \sum_{i=0}^{k} b(i) \frac{a(s-i)}{a(s)} e_\beta, & \text{if } \beta = (s, 0, \ldots, 0), s > k. \end{cases}$$
Note that since \( \frac{1}{k} (\mathbf{T}^*, \mathbf{T}) \geq 0 \) and \( a(s) > 0 \) for every \( s \geq 0 \), \( \sum_{i=0}^{k} b(i)a(s - i) \geq 0 \) when \( 0 \leq s \leq k \) and \( \sum_{i=0}^{k} b(i)a(s - i) \geq 0 \) otherwise. Moreover, \[
\frac{h_T(w)}{h_{M_z}^*(w)} = \left( \sum_{j=0}^{k} b(j)|w|^{2j} \right) \left( \sum_{i=0}^{\infty} a(i)|w|^{2i} \right) = \sum_{l=0}^{k} \sum_{j=0}^{l} b(j)a(l - j)|w|^{2l} + \sum_{l=k+1}^{\infty} \sum_{j=0}^{k} b(j)a(l - j)|w|^{2l}.
\]

We next claim that there exists a constant \( M' \) such that for all \( w \in \mathbb{B}_m \),
\[
0 < \frac{h_T(w)}{h_{M_z}^*(w)} = \left\| \frac{\gamma(w)}{\tilde{\gamma}(w)} \right\|^2 \leq M'.
\]
If not, then \( \left\| \frac{\gamma(w)}{\tilde{\gamma}(w)} \right\|^2 \to \infty \) as \( |w| \to 1 \) so that by (4.2), \( \frac{1}{|\phi(w)|} \to 0 \) as \( |w| \to 1 \). The maximum modulus principle would then imply that \( \frac{1}{|\phi(w)|} = 0 \), a contradiction.

Finally, we have from Lemma 3.1 that \( \frac{h_T(w)}{h_{M_z}^*(w)} = \|D_T \tilde{\gamma}(w)\|^2 \), where \( D_T = \frac{1}{K}(\mathbf{T}^*, \mathbf{T}) \) denotes the defect operator corresponding to \( \mathbf{T} \) and
\[
\sup_{w \in \mathbb{B}_m} \frac{|D_T \tilde{\gamma}(w)|^2}{|D_T \gamma(w), \zeta_0|^2} \leq \frac{M'}{a(0)b(0)} < \infty,
\]
for some unit vector \( \zeta_0 \in \text{ran} D_T \). Using Theorem 3.2, we then conclude that \( \mathbf{T} \sim_s M_z^* \).

The notion of curvature can also be used to describe the similarity of non-contractive \( m \)-tuples. The following results show.

**Proposition 4.10** Let \( T \in \mathcal{B}_i^1(\mathbb{D}) \in \mathcal{L}(\mathcal{H}) \). Suppose that \( \{\phi_j\}_{j=0}^m \subset H^\infty(\mathbb{D}) \) for \( m > 2, 2|\phi_j(w)|^2 > m(m + 1)|\phi_j'(w)|^2 \), and
\[
K_{M_z^*}(w) - K_T(w) = \partial^2 \log \left( 1 - |w|^2 \right) \sum_{j=0}^{m} |\phi_j(w)|^2 + 1, \quad w \in \mathbb{D},
\]
where \( M_z \) is the operator of multiplication by \( z \) on the Hardy space \( H^2(\mathbb{D}) \). Then \( T \sim_s M_z^* \) but \( T \) is not a contraction.

**Proof** Set \( \phi_j(w) := \sum_{i=0}^{\infty} a_{ji} w^i \) and \( M := \max\{\|\phi_0\|_{H^2}, \|\phi_1\|_{H^2}, \ldots, \|\phi_m\|_{H^2}\} \). Denote by \( \{e_i\}_{i=0}^{\infty} \) an orthonormal basis for the space \( \mathcal{H} \) and define an operator \( X \) as \( X e_n := \sum_{j=0}^{m} a_{jn} e_j \) for \( n \geq 0 \). Let \( y = \sum_{n=0}^{\infty} b_n e_n \in \mathcal{H} \). Then,
\[
\|X\| = \sup_{\|y\| = 1} \|Xy\| = \sup_{\|y\| = 1} \left\| X \sum_{n=0}^{\infty} b_n e_n \right\| = \sup_{\|y\| = 1} \left\| \sum_{j=0}^{m} \sum_{n=0}^{\infty} b_n a_{jn} e_j \right\| \leq \left( \sum_{n=0}^{\infty} b_n^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} a_{jn}^2 \right)^{\frac{1}{2}} \leq (m + 1)M,
\]
where \( \sum_{n=0}^{\infty} b_n^2 \leq \sum_{n=0}^{\infty} a_{jn}^2 \leq (m + 1)M \).
and therefore, \( X \) is bounded. For a non-vanishing holomorphic section \( K(z, \overline{w}) = \frac{1}{1 - z^w} \) of \( E_{M^*_w} \), we have
\[
XK(z, \overline{w}) = \sum_{i=0}^{\infty} w^i X(z^i) = \sum_{j=0}^{m} \left( \sum_{i=0}^{\infty} a_{ji} w^i \right) z^j = \sum_{j=0}^{m} \phi_j(w) z^j.
\]

This implies that \( \|XK(\cdot, \overline{w})\|^2 = \sum_{j=0}^{m} |\phi_j(w)|^2 \) and hence,
\[
K_{M^*_w}(w) - K_T(w) = \frac{\partial^2}{\partial w \partial \overline{w}} \log \left( (1 - |w|^2) \sum_{i=0}^{m} |\phi_i(w)|^2 + 1 \right)
= \frac{\partial^2}{\partial w \partial \overline{w}} \log \left( \|XK(\cdot, \overline{w})\|^2 + 1 \right) .
\]

(4.3)

The operator \( Y \) defined by \( Y := (I + X^*X)^{\frac{1}{2}} \) is invertible and \( K_T = K_{YM^*_w}Y^{-1} \).

Therefore, \( T \sim_u Y_{M^*_w}Y^{-1} \), and hence, \( T \sim_u M^*_w \).

Suppose now that \( T \) is a contraction. For \( w \in \mathbb{D} \), set
\[
\mathcal{R}(w) := (1 - |w|^2) \sum_{i=0}^{m} |\phi_i(w)|^2.
\]

Since for every \( 0 \leq j \leq m, \)
\[
\left| \frac{\phi_j(w)}{\phi_j(w)} \right|^2 > \frac{m(m+1)}{2} > 4,
\]
we have \( \left| \frac{\phi_j(w)}{\phi_j(w)} + |w|^2 \right| > 1 \), that is, \( |\phi_j(w)|^2 - |(w\phi_j(w))'|^2 < 0 \). Then,
\[
\frac{\partial^2 \mathcal{R}(w)}{\partial w \partial \overline{w}} = \frac{\partial^2}{\partial w \partial \overline{w}} \left[ (1 - |w|^2) \sum_{j=0}^{m} |\phi_j(w)|^2 \right] = \sum_{j=0}^{m} \left( |\phi_j(w)|^2 - |(w\phi_j(w))'|^2 \right) < 0.
\]

(4.4)

Similarly, since \( \left| \frac{\phi_j'(w)}{\phi_j(w)} \right|^2 < \frac{2}{m(m+1)} \) for \( 0 \leq j \leq m \), it follows that for all \( w \in \mathbb{D} \),\n\[
\sum_{0 \leq i < j \leq m} \left| \frac{\phi_i'(w)}{\phi_j(w)} - \frac{\phi_i'(w)}{\phi_j(w)} \right|^2 \leq 2 \sum_{0 \leq i \leq j \leq m} \left( \left| \frac{\phi_i'(w)}{\phi_j(w)} \right|^2 + \left| \frac{\phi_j'(w)}{\phi_j(w)} \right|^2 \right)
< 2 \sum_{0 \leq i \leq j \leq m} \frac{4}{m(m+1)} = 4 \leq \frac{4}{(1 - |w|^2)^2}.
\]

Therefore, for all \( w \in \mathbb{D} \),
\[
\sum_{0 \leq i < j \leq m} \left( \sum_{k=0}^{m} |\phi_k(w)|^2 \right) \leq \sum_{0 \leq i \leq j \leq m} \left( \left| \frac{\phi_i(w)}{\phi_j(w)} \right|^2 + \left| \frac{\phi_j(w)}{\phi_j(w)} \right|^2 \right)
< \frac{1}{4} \sum_{0 \leq i \leq j \leq m} \left| \frac{\phi_i(w)}{\phi_j(w)} - \frac{\phi_i(w)}{\phi_j(w)} \right|^2 < \frac{1}{(1 - |w|^2)^2}.
\]

Then,
\[
\frac{\partial^2 \log \mathcal{R}(w)}{\partial w \partial \overline{w}} = -\frac{1}{(1 - |w|^2)^2} + \frac{\sum_{i=0}^{m} |\phi_i(w)|^2 - \sum_{i=0}^{m} |\phi_i'(w)|^2}{(\sum_{i=0}^{m} |\phi_i(w)|^2)}\]
\[
+ \frac{\sum_{i=0}^{m} |\phi_i'(w)|^2 - \sum_{i=0}^{m} \phi_i \overline{\phi_i}}{(\sum_{i=0}^{m} |\phi_i(w)|^2)^2}.
\]
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\[
\begin{align*}
&= -\frac{1}{(1 - |w|^2)^2} + \sum_{0 \leq i < j \leq m} |\phi_i \phi_j' - \phi_i' \phi_j|^2 \\
&< 0.
\end{align*}
\]

However, since \(\frac{\partial^2 \log \mathcal{M}(w)}{\partial w \partial \overline{w}} = \frac{\mathcal{M}(w) \frac{\partial^2 \log \mathcal{M}(w)}{\partial w \partial \overline{w}} - \frac{\partial \mathcal{H}(w)}{\partial w} \frac{\partial \mathcal{H}(w)}{\partial \overline{w}}}{\mathcal{M}(w)} < 0,\)

\[
\mathcal{M}(w) \frac{\partial^2 \log \mathcal{M}(w)}{\partial w \partial \overline{w}} - \frac{\partial \mathcal{H}(w)}{\partial w} \frac{\partial \mathcal{H}(w)}{\partial \overline{w}} < 0. 
\]  

Finally, by (4.3)–(4.5), it is easy to see that

\[
K_{M_z^*}(w) - K_T(w) = \frac{\mathcal{M}(w) \frac{\partial^2 \log \mathcal{M}(w)}{\partial w \partial \overline{w}} - \frac{\partial \mathcal{H}(w)}{\partial w} \frac{\partial \mathcal{H}(w)}{\partial \overline{w}} + \frac{\partial^2 \mathcal{M}(w)}{\partial w \partial \overline{w}}}{(\mathcal{M}(w) + 1)^2} < 0.
\]

This contradicts the result given in [54] that for a contraction \(T \in \mathcal{B}_1^1(\mathbb{D}), K_{M_z^*} \geq K_T. \)

\[\square\]

**Corollary 4.11** Let \(T \in \mathcal{B}_1^1(\mathbb{D})\) and denote by \(M_z\) the operator of multiplication by \(z\) on the Hardy space \(H^2(\mathbb{D})\). Suppose that \(\phi \in \mathcal{O}(\mathbb{D})\) is such that for all \(w \in \mathbb{D}, |\phi(w)|^2 > |\phi'(w)|^2\). If

\[
K_{M_z^*}(w) - K_T(w) = -\frac{\left( |\phi(w)|^4 - |\phi'(w)|^2 + |\phi(w) + w\phi'(w)|^2 \right)}{\left( |\phi(w)|^2 (1 - |w|^2) + 1 \right)^2}, \quad w \in \mathbb{D},
\]

then \(T \sim_s M_z^*\), but \(T\) is not a contraction.

**Corollary 4.12** Let \(T \in \mathcal{B}_1^1(\mathbb{D})\) and denote by \(M_z\) be the operator of multiplication by \(z\) on the Hardy space \(H^2(\mathbb{D})\). Suppose that \(\varphi \in \mathcal{O}(\mathbb{D})\) is such that for all \(w \in \mathbb{D}, |\varphi(w)| > 2|\varphi'(w)|\). If

\[
K_{M_z^*}(w) - K_T(w) = \frac{\partial^2}{\partial w \partial \overline{w}} \log \left( (1 - |w|^2) |\varphi(w)|^2 + 1 \right), \quad w \in \mathbb{D},
\]

then \(T \sim_s M_z^*\), but \(T\) is not a contraction.

In the following theorem, we will use log-plurisubharmonic functions to give a sufficient condition for the similarity of tuples in \(\mathcal{B}_1^m(\Omega)\). For an \(m\)-tuple \(T = (T_1, \cdots, T_m) \in \mathcal{L}(\mathcal{H})^m\), let \(\{T_j\}' := \bigcap_{j=1}^m \{T_j\}'.\)

**Theorem 4.13** Let \(T = (T_1, \cdots, T_m), S = (S_1, \cdots, S_m) \in \mathcal{B}_1^m(\Omega)\) be such that \(\{S\}' \cong H^\infty(\Omega)\). Suppose that

\[
K_{S}(w) - K_{T}(w) = \sum_{i,j=1}^m \frac{\partial^2 \psi(w)}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j, \quad w \in \Omega,
\]

for some \(\psi(w) = \log \sum_{k=1}^n |\phi_k(w)|^2\), where \(\phi_k \in \mathcal{O}(\Omega)\). If there exists an integer \(l \leq n\) satisfying \(\frac{\phi_k}{\phi_l} \in H^\infty(\Omega)\) for all \(k \leq n\), then \(T \sim_s S\), and \(K_T \leq K_S\). In particular, when \(m = 1\), \(T\) and \(S\) are unitarily equivalent.
Proof Since $\phi_k \in \mathcal{O}(\Omega)$ for all $1 \leq k \leq n$, $\sum_{i,j=1}^{m} \frac{\partial^2 \log |\phi_k(w)|^2}{\partial w_i \partial w_j} = 0$. Therefore, by Lemmas 4.5 and 4.7, $\sum_{k=1}^{n} |\phi_k(w)|^2$ is log-plurisubharmonic, and

$$K_S(w) - K_T(w) = \left( \frac{\partial^2 \psi(w)}{\partial w_i \partial w_j} \right)_{i,j=1}^{m} \geq 0.$$ 

Now let $t$ be a non-vanishing holomorphic section of $\mathcal{E}_S$. Since $\{S\}' = H^\infty(\Omega)$ and $\{\phi_k\}_1^n \subset H^\infty(\Omega)$ for some $l \leq n$, we assume without loss of generality that $l = n$. Then there exist bounded operators $X_i \in \{S\}'$, $1 \leq i \leq n - 1$, such that

$$X_i t(w) = \frac{\phi_i(w)}{\phi_n(w)} t(w).$$

Define a linear operator $X : \mathcal{H} \to \bigoplus_{i=1}^{n-1} \mathcal{H}$ as

$$Xh = \bigoplus_{i=1}^{n-1} X_i h,$$

for $h \in \mathcal{H}$. Since

$$||X||^2 = \sup_{||h||=1} ||Xh||^2 = \sup_{||h||=1} \left( \sum_{i=1}^{n-1} ||X_i h||^2 \right) \leq \sum_{i=1}^{n-1} ||X_i||^2 < \infty,$$

$X$ is a bounded linear operator. Furthermore, for any $w \in \Omega$,

$$K_S(w) - K_T(w) = \sum_{i,j=1}^{m} \frac{\partial^2 \log |\phi_n(w)|^2}{\partial w_i \partial w_j} \left( \sum_{i=1}^{n-1} \frac{\phi_i(w)}{\phi_n(w)}^2 + 1 \right) dw_i \wedge dw_j$$

$$= \sum_{i,j=1}^{m} \frac{\partial^2 \log \left( ||\phi_n(w)||^2 \cdot \sum_{i=1}^{n-1} \frac{\phi_i(w)}{\phi_n(w)}^2 ||t(w)||^2 + ||t(w)||^2 \right)}{||t(w)||^2} dw_i \wedge dw_j$$

Letting $Y := (I + X^*X)^{\frac{1}{2}}$, we see that $Y$ is invertible and that $K_T = K_{YSY^{-1}}$. Hence, $T \sim \omega YSY^{-1}$ so that indeed, $T \sim S$. \hfill $\square$

### 5 On the Cowen-Douglas conjecture

In [20], M. J. Cowen and R. G. Douglas proved that for $T \in \mathcal{B}_1^1(\Omega)$, where $\Omega \subset \mathbb{C}$, the curvature $K_T$ is a complete unitary invariant. Let $T, S \in \mathcal{B}_1^1(\mathbb{D})$ and suppose that the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ is a $K$-spectral set for $T$ and $S$. The Cowen-Douglas conjecture states that $T$ and $S$ are similar if and only if

$$\lim_{|w| \to 1} K_T^{i,i}(w)/K_S^{i,i}(w) = 1.$$
Consider the operator tuples for every kernel Hilbert space $H$ and constants $C$. One can consider Remark 5.2 which the Cowen-Douglas conjecture holds; nevertheless, they are not similar. We begin with the following lemma given in [53] for which the Cowen-Douglas conjecture holds is some specific cases. We describe a class of operators in $\mathcal{B}_1^m(\mathbb{D})$ for which the Cowen-Douglas conjecture is true.

**Example 5.1** Let $T \in \mathcal{B}_1^m(\mathbb{D})$ and for $\lambda \geq 2$, consider $M^*_z$ on a weighted Bergman space $\mathcal{H}_K$ with reproducing kernel $K(z, w) = \frac{1}{(1 - w\bar{w})^\lambda}$. Suppose that $\mathcal{E}_T = \mathcal{E}_{M^*_z} \otimes \mathcal{E}$, where $\mathcal{E}(w) = \sqrt{f(w)}$, and $\|f(w)\|^2$ is a polynomial in $|w|^2$. If condition (C) holds for the Hermitian holomorphic vector bundle $\mathcal{E}$, then it follows from Lemma 3.4 and Theorem 3.5 that $T \sim_s M^*_z$.

Now, a direct calculation shows that

$$\frac{K_T(w)}{K_{M^*_z}(w)} = 1 + \frac{\partial^2}{\partial w \partial \bar{w}} \log \|f(w)\|^2 = 1 + \frac{(1 - |w|^2)^2}{\lambda} \frac{\partial^2}{\partial w \partial \bar{w}} \log \|f(w)\|^2.$$  

Since $\|f(w)\|^2$ is a polynomial in $|w|^2$, $\frac{\partial^2}{\partial w \partial \bar{w}} \log \|f(w)\|^2$ is bounded above. Hence, $\lim_{|w| \to 1} \frac{K_T(w)}{K_{M^*_z}(w)} = 1$.

**Remark 5.2** One can consider $T = (T_1, \cdots, T_m) \in \mathcal{B}_1^m(\mathbb{B}_m)$ and $M^*_z = (M^*_{z_1}, \cdots, M^*_{z_m})$ on a weighted Bergman space $\mathcal{H}_k$ with $k > m + 1$ to obtain an operator tuple analogue of Example 5.1. Under the same assumptions on $\mathcal{E}$, $T \sim_s M^*_z$.

Moreover, as $|w| \to 1$,

$$\frac{K_{T_i}(w)}{K_{S_i}(w)} = 1 + \frac{(1 - |w|^2)^2}{k(1 - |w|^2 + |w_1|^2)} \frac{\partial^2}{\partial w_i \partial \bar{w}_i} \log \|f(w)\|^2 \to 1.$$  

We next show that the Cowen-Douglas conjecture is false for tuples of commuting operators. As in [19], we construct operator tuples in $\mathcal{B}_n^m(\mathbb{B}_m)$ for which the Cowen-Douglas conjecture holds; nevertheless, they are not similar.

We begin with the following lemma given in [36]:

**Lemma 5.3** Let $T = (T_1, \cdots, T_m), S = (S_1, \cdots, S_m) \in \mathcal{L}(\mathcal{H})^m$ be $m$-tuples of unilateral shift operators with nonzero weight sequences given by $\{\lambda^{(1)}_\alpha, \cdots, \lambda^{(m)}_\alpha\} \in \mathbb{N}_0^m$ and $\{\tilde{\lambda}^{(1)}_\alpha, \cdots, \tilde{\lambda}^{(m)}_\alpha\} \in \mathbb{N}_0^m$, respectively. Then $T \sim_s S$ if and only if there exist constants $C_1$ and $C_2$ such that

$$0 < C_1 \leq \prod_{k=0}^l \lambda^{(i)}_{\alpha + k\epsilon_i}/ \prod_{k=0}^l \tilde{\lambda}^{(i)}_{\alpha + k\epsilon_i} \leq C_2,$$

for every $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^m$, and $1 \leq i \leq m$.

**Example 5.4** Consider the operator tuples $M^*_z = (M^*_{z_1}, \cdots, M^*_{z_m})$ on a reproducing kernel Hilbert space $\mathcal{H}_K$ with reproducing kernel $K(z, w) = \frac{1 - \log(1 - (z, w))}{1 - (z, w)}$ and on
the Drury-Arveson space $\mathcal{H}_1$ with reproducing kernel $\widetilde{K}(z, w) = \frac{1}{1-\langle z, w \rangle}$. To simplify
the notation, we will set $T$ to be $M^\alpha_z$ on $\mathcal{H}_K$ while $\widetilde{T}$ will denote the tuple on $\mathcal{H}_1$.
Since $\tilde{K}(w, w) = \frac{1 - \log(1 - |w|^2)}{1 - |w|^2}
= \sum_{i=0}^{\infty} \frac{|w|^{2i}}{i+1} \left(1 - \sum_{j=0}^{\infty} \frac{(-1)^j (-|w|^2)^j}{j+1} \right)
= 1 + \sum_{n=1}^{\infty} \left(1 + \sum_{i=1}^{n} \frac{1}{i} \right) |w|^{2n}
= \sum_{\alpha \in \mathbb{N}_0^m} \left(1 + \sum_{i=1}^{\alpha} \frac{1}{i} \right) \frac{|\alpha|!}{\alpha!} w^{\alpha}.$

Now set $\tilde{\rho}(\alpha) = \frac{|\alpha|!}{\alpha!}$ and $\rho(\alpha) = \left(1 + \sum_{i=1}^{\alpha} \frac{1}{i} \right) \frac{|\alpha|!}{\alpha!}$. Let $\{e_{\alpha}\}$ and $\{\tilde{e}_{\alpha}\}$ denote orthonormal bases for $\mathcal{H}_K$ and $\mathcal{H}_1$, respectively. From the relation between
reproducing kernels and weight sequences, we have
\[ T_i^* e_{\alpha} = \sqrt{\frac{\rho(\alpha)}{\rho(\alpha + e_i)}} e_{\alpha + e_i}, \quad T_i e_{\alpha} = \sqrt{\frac{\rho(\alpha - e_i)}{\rho(\alpha)}} e_{\alpha - e_i}, \]
and
\[ \tilde{T}_i^* \tilde{e}_{\alpha} = \sqrt{\frac{\tilde{\rho}(\alpha)}{\tilde{\rho}(\alpha + e_i)}} \tilde{e}_{\alpha + e_i}, \quad \tilde{T}_i \tilde{e}_{\alpha} = \sqrt{\frac{\tilde{\rho}(\alpha - e_i)}{\tilde{\rho}(\alpha)}} \tilde{e}_{\alpha - e_i}, \]
for $1 \leq i \leq m$ and $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}_0^m$ with 1 in the $i$-th position. Therefore,
for every $\alpha \in \mathbb{N}_0^m$ and $1 \leq i \leq m$,
\[ \prod_{k=0}^{l-1} \sqrt{\frac{\rho(\alpha + (k+1)e_i)}{\rho(\alpha + (k+1)e_i)}} = \sqrt{\frac{\rho(\alpha)e_{\alpha + (l+1)e_i}}{\rho(\alpha + (l+1)e_i)}} = \frac{1 + \sum_{i=1}^{l} \frac{1}{i}}{1 + \sum_{i=1}^{l+1} \frac{1}{i}}, \]
and by Lemma 5.3, $T$ and $\widetilde{T}$ are not similar.

On the other hand, the definition of curvature yields
\[ \mathcal{K}_{T}(w) = - \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \frac{1 - \log(1 - |w|^2)}{1 - |w|^2} dw_i \wedge d\bar{w}_j, \]
and
\[ \mathcal{K}_{\widetilde{T}}(w) = - \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \frac{1}{1 - |w|^2} dw_i \wedge d\bar{w}_j. \]
Then as $|w| \to 1$,
\[ \frac{K_{T_{z_0}}^{i,i}(w)}{K_{M_z}^{i,i}(w)} = \frac{\partial^2}{\partial w_i \partial \bar{w}_i} \log \frac{1 - \log(1 - |w|^2)}{1 - |w|^2}, \]
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$$= 1 + \frac{\partial^2}{\partial w_i \partial w_i} \log \left( \frac{1 - \log(1 - |w|^2)}{1 - |w|^2} \right) \left( \frac{|w|^2}{1 - |w|^2 + |w_i|^2} \right) \left( \frac{1 - \log(1 - |w|^2)}{1 - |w|^2} \right)^2 \rightarrow 1.$$

6 Further generalizations of single Cowen-Douglas operator results

In this section, we extend additional results given for a single Cowen-Douglas operator to a tuple of commuting operators.

6.1 Inequalities involving the trace of curvature

In \cite{54}, G. Misra gave the curvature inequality for contractions in $B_1^1(\mathbb{D})$. Later in \cite{13}, S. Biswas, D. K. Keshari, and G. Misra generalized the result and presented the following curvature matrix inequality for $m$-tuples in $B_m^m(\mathbb{B}_m)$:

**Lemma 6.1** Let $T \in B_m^m(\mathbb{B}_m)$ be a row-contraction and consider $M^*_z$ on the Dury-Arveson space $\mathcal{H}_1$. Then $K_T(w) \leq K_{M^*_z}(w)$ for $w \in \mathbb{B}_m$.

The above inequality implies that one can use the curvature matrix to determine whether a tuple of operators in $B_m^m(\mathbb{B}_m)$ is a row-contraction. We derive an analogous curvature matrix inequality for an $n$-hypercontraction in $B_n^m(\mathbb{B}_m)$. We first show that the trace of the curvature matrix for $T = (T_1, \cdots, T_m) \in B_n^m(\Omega)$ is independent of the choice of the holomorphic frame of $\mathcal{E}_T$.

**Proposition 6.2** Let $T = (T_1, \cdots, T_m) \in B_n^m(\Omega)$. Suppose that $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ and $\tilde{\sigma} = \{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n\}$ are holomorphic frames of $\mathcal{E}_T$. Then

$$K_{i,j}^T(\tilde{\sigma}) = \phi^{-1} K_{i,j}^T(\sigma) \phi \quad \text{and} \quad \text{trace } K_T(\sigma) = \text{trace } K_T(\tilde{\sigma}),$$

for some invertible holomorphic matrix-valued function $\phi$ on $\Omega$.

**Proof** Since $\sigma$ and $\tilde{\sigma}$ are holomorphic frames of $\mathcal{E}_T$, there is an invertible holomorphic matrix $\phi = (\phi_{ij})_{i,j=1}^n$ such that for all $w \in \Omega$, $(\tilde{\sigma}_1(w), \cdots, \tilde{\sigma}_n(w)) = (\sigma_1(w), \cdots, \sigma_n(w)) \phi(w)$. Therefore,

$$h(w) = (\langle \tilde{\sigma}_j(w), \tilde{\sigma}_i(w) \rangle)_{i,j=1}^n = \phi^*(w) \phi(w).$$
Since \( \phi \) is holomorphic and invertible,
\[
K^{i,j}_T(\tilde{\sigma})(w) = -\frac{\partial}{\partial w_j} \left[ \left( \phi^*(w) h(w) \phi(w) \right)^{-1} \frac{\partial}{\partial w_i} \left( \phi^*(w) h(w) \phi(w) \right) \right] \\
= - \left[ \phi^{-1}(w) \frac{\partial h^{-1}(w)}{\partial w_j} \phi(w) + \phi^{-1}(w) h^{-1}(w) \frac{\partial^2 h(w)}{\partial w_i \partial w_j} \phi(w) \right] \\
= \phi^{-1}(w) K^{i,j}_T(\sigma)(w) \phi(w).
\]
This shows that \( K^{i,j}_T(\tilde{\sigma})(w) \) is similar to \( K^{i,j}_T(\sigma)(w) \) and that
\[
\text{trace } K^{i,j}_T(\tilde{\sigma})(w) = \text{trace } K^{i,j}_T(\sigma)(w).
\]
Thus, for all \( w \in \Omega \),
\[
\text{trace } K_T(\tilde{\sigma})(w) = - \sum_{i,j=1}^{m} \text{trace } \left( \frac{\partial}{\partial w_j} \left( \tilde{h}^{-1}(w) \frac{\partial \tilde{h}(w)}{\partial w_i} \right) \right) \\
= - \sum_{i,j=1}^{m} \text{trace } \left( \frac{\partial}{\partial w_j} \left( h^{-1}(w) \frac{\partial h(w)}{\partial w_i} \right) \right) \\
= \text{trace } K_T(\sigma)(w).
\]
\[\square\]

In [56], V. Müller and F.-H. Vasilescu gave the following description of when an \( n \)-hypercontraction is unitarily equivalent to an adjoint of a multiplication tuple. Let \( M^*_z,n,E \) denote the operator tuple \( M^*_z \) on the space \( H_n \otimes E \):

**Lemma 6.3** Let \( T = (T_1, \cdots, T_m) \in \mathcal{L}(H)^m \) be an \( m \)-tuple of commuting operators and let \( n \) be a positive integer. Then there is a Hilbert space \( E \) and an \( M^*_z,n,E \)-invariant subspace \( N \) of \( H_n \otimes E \) such that \( T \) is unitarily equivalent to \( M^*_z,n,E|_N \) if and only if \( T \) is an \( n \)-hypercontraction with \( \lim_{s \to \infty} M^*_T(I) = 0 \) in the strong operator topology.

Another important tool for our work in the current section comes from the following result by D. K. Keshari in [46]:

**Lemma 6.4** Let \( E \) be a Hermitian holomorphic vector bundle of rank \( n \) over \( \Omega \). Then the curvature matrices of the determinant bundle \( \text{det } E \) and of the vector bundle \( E \) satisfy the equality
\[
K_{\text{det } E} = \text{trace } K_E.
\]

We now give a corresponding curvature inequality that holds for a \( t \)-hypercontractive tuple in the class \( \mathcal{B}_n^m(\mathbb{B}_m) \).

**Theorem 6.5** Let \( t \in \mathbb{N} \). For a \( t \)-hypercontraction \( T = (T_1, \cdots, T_m) \in \mathcal{B}_n^m(\mathbb{B}_m) \),
\[
K_{\text{det } E_T} \leq K_{\text{det } E_n}^*_{M^*_z,t},
\]
where \( M^*_z,t \) is on the space \( H_t \) with reproducing kernel \( K(z,w) = \frac{1}{(1-\langle z,w \rangle)^t} \).
Proof Since $T \in \mathcal{B}^{m}_{n}(\mathbb{B}_{m})$ is a $t$-hypercontraction, by Lemma 6.3, there exist a Hilbert space $E$ and an $\mathbf{M}^{*}_{z,t,E}$-invariant subspace $\mathcal{N} \subset \mathcal{H}_{t} \otimes E$ such that $T \sim u \mathbf{M}^{*}_{z,t,E}|_{\mathcal{N}}$. Let

$$K(\cdot, w) \otimes e(\alpha, w) \in \ker(\mathbf{M}^{*}_{z,t,E}|_{\mathcal{N}} - w) = \bigcap_{i=1}^{n} \ker(M_{zi}^{*}|_{\mathcal{N}} - w).$$

If we denote by $\{a(\alpha)z^\alpha\}_{\alpha \in \mathbb{N}^{m}}$ an orthonormal basis for $\mathcal{H}_{t} \otimes E$, then for every $f \in E(w)$, $\alpha' \in \mathbb{N}^{m}$, and $1 \leq i \leq m$,

$$0 = \langle (M_{zi}^{*} - w) \sum_{\alpha \in \mathbb{N}^{m}} a(\alpha) \otimes e(\alpha, w)z^\alpha w^\alpha, a(\alpha') \otimes f z^{\alpha'} \rangle$$

$$= \sum_{\alpha \in \mathbb{N}^{m}} \langle a(\alpha) \otimes e(\alpha, w)z^\alpha w^\alpha, a(\alpha' + e_{i}) \otimes f z^{\alpha' + e_{i}} \rangle$$

$$- \sum_{\alpha \in \mathbb{N}^{m}} \langle a(\alpha) \otimes e(\alpha, w)z^\alpha w^{\alpha + e_{i}}, a(\alpha') \otimes f z^{\alpha'} \rangle$$

$$= \langle a(\alpha' + e_{i}) \otimes e(\alpha' + e_{i}, w)z^{\alpha' + e_{i}}w^{\alpha' + e_{i}}, a(\alpha' + e_{i}) \otimes f z^{\alpha' + e_{i}} \rangle$$

$$- \langle a(\alpha') \otimes e(\alpha', w)z^{\alpha'}w^{\alpha' + e_{i}}, a(\alpha') \otimes f z^{\alpha'} \rangle$$

$$= \langle e(\alpha' + e_{i}, w) - e(\alpha', w), f \rangle w^{\alpha' + e_{i}}.$$ 

Hence, for every $\alpha, \beta \in \mathbb{N}^{m}$, $e(\alpha, w) = e(\beta, w)$, so that we can set $e(w) := e(\alpha, w)$. Then for all $w \in \mathbb{B}_{m}$,

$$\ker(\mathbf{M}^{*}_{z,t,E}|_{\mathcal{N}} - w) = \bigvee \{ K(\cdot, w) \otimes e(w), e(w) \in E(w) \}.$$ 

Since $\dim(\ker(\mathbf{M}^{*}_{z,t,E}|_{\mathcal{N}} - w)) = \dim(\ker(T - w)) = n$, we can assume that $\{ K(\cdot, w) \otimes e_{i}(w) \}_{i=1}^{n}$ is a basis for $\ker(\mathbf{M}^{*}_{z,t,E}|_{\mathcal{N}} - w)$ and $\mathcal{E}(w) = \bigvee_{1 \leq i \leq n} e_{i}(w)$. Therefore,

$$h(w) = K(\overline{w}, w)\left( e_{j}(w), e_{i}(w) \right)_{i,j=1}^{n} = K(\overline{w}, w)h_{\mathcal{E}}(w),$$

and

$$K_{\text{det } \mathcal{E}_{T}} = \left( -\frac{\partial^{2} \log(K(\overline{w}, w))}{\partial \overline{w}_{j} \partial w_{i}} \right)_{i,j=1}^{n} - \left( \frac{\partial^{2} \log(\det h_{\mathcal{E}}(w))}{\partial \overline{w}_{j} \partial w_{i}} \right)_{i,j=1}^{m} = K_{\text{det } \mathcal{E}_{\mathbf{M}^{*}_{z,t}}} + K_{\text{det } \mathcal{E}}.$$ 

Finally, since it is known from [13] that $K_{\text{det } \mathcal{E}}$ is negative definite, the proof is complete. \hfill $\square$

We next show that the result of [35] holds for the multi-operator case as well. For $T = (T_{1}, \ldots, T_{m}) \in \mathcal{B}^{m}_{n}(\Omega) \subset \mathcal{L}(\mathcal{H})^{m}$, one defines a projection-valued function $\Pi : \Omega \to \mathcal{L}(\mathcal{H})$ that assigns to each $w \in \Omega$, an orthogonal projection $\Pi(w)$ onto $\ker(T - w)$. Define an operator $\Gamma^{*}(w) : \ker(T - w) \to \mathbb{C}^{n}$ by

$$\Gamma^{*}(w)(f) = f(w),$$

where $f \in \ker(T - w)$. Then, $h(w) = \Gamma^{*}(w)\Gamma(w)$ and $\Pi(w) = \Gamma(w)h^{-1}(w)\Gamma^{*}(w)$. 


Theorem 6.6  Let $T \in \mathcal{B}_m^n(\Omega)$ and let $\Pi(w)$ be an orthogonal projection onto $\ker(T - w)$. Then for $w \in \Omega$,

$$\sum_{i=1}^{m} \left\| \frac{\partial \Pi(w)}{\partial w_i} \right\|_{\mathcal{S}_2}^2 = -\text{trace} \ K_T(w),$$

where $\mathcal{S}_2$ denotes the Hilbert-Schmidt class of operators.

Proof  From $\Pi(w) = \Gamma(w)h^{-1}(w)\Gamma^*(w)$, we have for $w \in \Omega$,

$$\frac{\partial \Pi(w)}{\partial w_j} \frac{\partial \Pi(w)}{\partial w_i} = \Gamma(w) \left[ \frac{\partial h^{-1}(w)}{\partial w_j} \frac{\partial h(w)}{\partial w_i} h^{-1}(w) + h^{-1}(w) \frac{\partial^2 h(w)}{\partial w_j \partial w_i} h^{-1}(w) \right] \Gamma^*(w)$$

$$= \Gamma(w) \left[ \frac{\partial}{\partial w_j} \left( h^{-1}(w) \frac{\partial h(w)}{\partial w_i} \right) h^{-1}(w) + h^{-1}(w) \frac{\partial h(w)}{\partial w_j} \frac{\partial h^{-1}(w)}{\partial w_i} \right] \Gamma^*(w)$$

$$= \Gamma(w) \left[ \frac{\partial}{\partial w_j} \left( h^{-1}(w) \frac{\partial h(w)}{\partial w_i} \right) \right] h^{-1}(w) \Gamma^*(w)$$

$$= -\Gamma(w) \left( K^{i,j}_{T}(w) h^{-1}(w) \right) \Gamma^*(w).$$

Hence,

$$\left\| \frac{\partial \Pi(w)}{\partial w_i} \right\|_{\mathcal{S}_2}^2 = \text{trace} \left( \frac{\partial \Pi(w)}{\partial w_i} \frac{\partial \Pi(w)}{\partial w_i} \right)$$

$$= -\text{trace} \left( [K^{i,j}_{T}(w) h^{-1}(w)] \Gamma^*(w) \Gamma(w) \right)$$

$$= -\text{trace} \ K^{i,j}_{T}(w).$$


6.2 Final Remark

For $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, the Rosenblum operator $\sigma_{T_1, T_2} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is defined as

$$\sigma_{T_1, T_2}(X) = T_1 X - X T_2, \quad X \in \mathcal{L}(\mathcal{H}).$$

In [30], F. Gilfeather showed that an operator $T \in \mathcal{L}(\mathcal{H})$ is strongly irreducible if there is no non-trivial idempotent in $\{T\}'$. The following relationship between strong irreducibility and the class $\mathcal{F}\mathcal{B}_2(\Omega)$ is given in [38]:

**Lemma 6.7** An operator $T = \begin{pmatrix} T_1 & T_{1,2} \\ 0 & T_2 \end{pmatrix} \in \mathcal{F}\mathcal{B}_2(\Omega)$ is strongly irreducible if and only if $T_{1,2} \notin \text{ran} \ \sigma_{T_1, T_2}$. 

We conclude with an example illustrating the complexity of the similarity problem. It implies that the trace of the curvature matrix is not always a proper similarity invariant for the class $B^n_n(\Omega)$. Let $T_1$ and $T_2$ be backward shift operators on Hilbert spaces $H_K_1$ and $H_K_2$ defined on the unit disk $\mathbb{D}$ with reproducing kernels $K_1(z,w) = \frac{1}{1-z\overline{w}}$ and $K_2(z,w) = \frac{1}{(1-z\overline{w})^2}$, respectively.

**Example 6.8** Consider $\tilde{T} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \in B_2^2(\mathbb{D})$ and $T = \begin{pmatrix} T_1 & T_1,2 \\ 0 & T_2 \end{pmatrix} \in FB_2^2(\mathbb{D})$. Suppose that $T_{1,2} \notin \text{ran } \sigma_{T_1,T_2}(X) = \{T_1X - XT_2 : X \in \mathcal{L}(H_{K_2},H_{K_1})\}$.

Let us first note that $t_1(w) := K_1(\cdot,\overline{w}) \in \ker(T_1 - w)$ and $t_2(w) := K_2(\cdot,\overline{w}) \in \ker(T_2 - w)$. It can be easily checked that $\{t_1, t_1^t + t_2\}$ is a holomorphic frame of $\mathcal{E}_T$ and that $h_T(w) = \begin{pmatrix} \frac{1}{1-|w|^2} & \frac{\partial}{\partial \overline{w}} \frac{1}{1-|w|^2} + \frac{1}{1-|w|^2} \\ -\frac{\overline{w}}{(1-|w|^2)^2} & -\frac{1}{(1-|w|^2)^2} \end{pmatrix}$. It follows that $\det h_T(w) = \frac{2}{(1-|w|^2)^2}$. Now by Lemma 6.4, we know that $\text{trace } K_T(w) = K_{\det } y_T(w) = -\frac{1}{(1-|w|^2)^2}$.

On the other hand, $\text{trace } K_T(w) = K_{T_{1,2}}(w) + K_{T_{2,2}}(w) = -\frac{4}{(1-|w|^2)^2}$, so that $\text{trace } K_T = \text{trace } K_{\tilde{T}_2}$.

Now, since $T_{1,2} \notin \text{ran } \sigma_{T_1,T_2}$, it follows from Lemma 6.7 that $T$ is strongly irreducible. Thus, $\{T\}'$ has no non-trivial idempotent elements. If $T$ and $\tilde{T}$ were indeed similar, then there has to be an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $T = X^{-1}\tilde{T}X$. If we set $Y := \begin{pmatrix} I_K_1 & 0 \\ 0 & 0 \end{pmatrix}$, then $Y \in \{\tilde{T}\}'$ and $X^{-1}YX$ is a non-trivial idempotent. However,

$$(X^{-1}YX)T = X^{-1}Y\tilde{T}X = X^{-1}\tilde{T}YX = (X^{-1}\tilde{T}X)(X^{-1}YX) = T(X^{-1}YX),$$

so that $X^{-1}YX \in \{T\}'$, and this is a contradiction. Hence, $T$ and $\tilde{T}$ are not similar.

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