DIFFERENTIAL CALCULI ON THE QUANTUM GROUP $SU_q(2)$ AND GLOBAL $U(1)$-COVARIANCE

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Abstract

A variety of three-dimensional left-covariant differential calculi on the quantum group $SU_q(2)$ is considered using an approach based on global $U(1)$-covariance. Explicit representations of possible $q$-Lie algebras are constructed in terms of differential operators. A gauge covariant differential algebra is uniquely determined. The non-standard Leibnitz rule is obtained for a corresponding $q$-Lie algebra.

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1 Introduction

One of the features of non-commutative geometry in the quantum group theory [1-5] is non-uniqueness in defining a differential calculus on the quantum groups and quantum spaces. The bicovariance condition determines a unique differential calculus on the linear quantum groups $GL_q(N)$ (up to symmetry corresponding to the exchange $q \rightarrow \frac{1}{q}$) [6, 7] and provides existence of the corresponding gauge covariant differential algebra [8]. Direct reducing the $GL_q(N)$-bicovariant differential calculus to a case of the special linear quantum group $SL_q(N)$ encounters difficulties connected with a loss of the centrality condition for a quantum determinant. Four-dimensional $4D_{\pm}$ bicovariant and three-dimensional ($3D$) left-covariant differential calculi on the simplest special unitary quantum group $SU_q(2)$ were considered as well using a standard Woronowicz approach [3, 6]. A full consistent construction of the 3D bicovariant differential calculus and a gauge covariant differential algebra on the $SU_q(2)$ are unknown up to now, furthermore, there are strong limitations imposed by no-go theorems [9]. A possible way to solve this problem suggests using a non-standard Leibnitz rule as it was pointed out in ref. [10].

In this paper possible 3D left-covariant differential calculi and gauge covariant differential algebra on the quantum group $SU_q(2)$ are considered in the framework of approach which respects a global $U(1)$-covariance. The group $U(1)$ is a stabilizing subgroup for the quantum group $SU_q(2)$ and the $U(1)$-covariant treatment allows to pass straightforwardly to the description of the quantum sphere $S^2_q \sim SU(2)/U(1)$. In Section 2 we construct explicit representations of $q$-Lie algebras of left-invariant vector fields on $SU_q(2)$ in terms of dif-
ferential operators. The $U(1)$-covariance constraint reduces the variety of possible left-covariant differential calculi on $SU_q(2)$ and leads to a unique gauge covariant differential algebra as it is shown in Section 3. The main commutation relations for the differential algebra agree with ones obtained earlier in the bicovariant formalism [8, 10]. The principal difference of our approach is that we propose a non-standard Leibnitz rule which is consistent with the gauge $SU_q(2)$-covariance. Some discussion of a quantum group gauge Yang-Mills theory is given in Conclusion.

2 Left-invariant vector fields

Following the $R$-matrix formalism [4] the main commutation relation for the generators $T^i_j$ ($i, j = 1, 2$) of the quantum group $SU_q(2)$ are defined by a standard $R$-matrix as follows

$$R_{12}T_1T_2 = T_2T_1R_{12}.$$  

(1)

Let us choose a covariant parametrization for the matrix $T^i_j$

$$T^i_j = \begin{pmatrix} y^1 & x^1 \\ y^2 & x^2 \end{pmatrix} \equiv (y^i x^i),$$  

(2)

where $x^i, y_i$ are generators (coordinates) of the function algebra on the quantum hermitean vector space $U_q^2$ endowed with an involution $\ast : x^i = y_i$ and $SU_q(2)$-comodule structure. The unimodularity condition takes a simple covariant form

$$D \equiv \det_q T^i_j = x_i y^i = 1, \ x_i = \varepsilon_{ij} x^j.$$  

(3)

The $SU_q(2)$ indices are raised and lowered with the invariant metric $\varepsilon_{ij}$ ($\varepsilon_{12} = 1, \varepsilon_{21} = -q^{-1}$).
The parametrization (2) was used in a harmonic formalism [12] of extended superfield supersymmetric theories. The coordinates \((x, y)\) parametrize the quantum sphere \(S^2_q \sim SU(2)/U(1)\) and are just the quantum generalizations of classical harmonic functions \((u^\pm)\) (so called ”harmonics”)

\[
x^i \equiv u^+ i, \quad y^i \equiv u^- i.
\]

The signs \((\pm)\) correspond to charges \((\pm 1)\) of the stabilizing subgroup \(U(1)\) for the quantum group \(SU_q(2)\). To simplify notations we shall not pass to the notations adopted in the harmonic formalism keeping in mind that all geometric objects (like coordinates, derivatives, differential forms etc.) have definite \(U(1)\) charges.

Consider main commutation relations between the coordinates \((x, y)\) and derivatives \(\partial_i \equiv \frac{\partial}{\partial x^i}, \quad \bar{\partial}^i \equiv \frac{\partial}{\partial y_i}\) on the quantum group \(SU_q(2)\):

\[
R_{12}(\partial \xi)_1(\partial \xi)_2 = (\partial \xi)_2(\partial \xi)_1 R_{21},
\]

\[
(\partial \xi)_j^i \equiv \begin{pmatrix} \bar{\partial}_1 & \bar{\partial}_2 \\ \partial_1 & \partial_2 \end{pmatrix},
\]

\[
\partial_i x^k = \gamma_i^k + q Y_{m i}^k x^m \partial_n, \quad \bar{\partial}^i y_j = \delta^i_j + q y_m \bar{\partial}^n \hat{R}_{m i}^{n j},
\]

\[
\partial_i y_j = q (\hat{R}^{-1})_{j i}^k y_k \partial_l, \quad \bar{\partial}^i x^j = \frac{1}{q} \hat{R}_{k l}^{ij} x^k \bar{\partial}^l,
\]

here, we use standard definitions for the matrices \(\hat{R}_{k l}^{ij}, Y_{k l}^{ij}\). The commutation relations (6) do not differ on principle from ones given in ref. [13]. Our choice is motivated by applying manifest covariant tensor notations which are convenient in constructing explicit representations for the \(q\)-Lie algebras. Thus, one implies all geometric objects with upper (lower) indices to be transformed under the quantum group co-action \(\Delta\) like classical co- (contra-) variant tensors. For instance, a
second rank tensor $N^j_i$ will be transformed as follows

$$(N^j_i)' = (T^k_i)^j T^j_l N^l_k \quad (7)$$

(Hereafter the signs $\otimes$ of tensor product are omitted).

Let us define the left-invariant first-order differential operators

$$D^{++} \equiv x_i \bar{\partial}^i, \quad D^{--} \equiv -y_i \partial^i, \quad (8)$$

where $(\pm \pm)$ correspond to $U(1)$ charges $(\pm 2)$. The action of the operators $D^{\pm \pm}$ on the coordinates $(x, y)$ has a simple form

$$D^{++} x^i = 0, \quad D^{--} x^i = y^i,$$

$$D^{++} y_i = x_i, \quad D^{--} y_i = 0. \quad (9)$$

The Leibnitz rule for these differential operators may be written in a convenient form if one considers their action on functions with definite $U(1)$ charges. After some calculations one finds

$$D^{\pm \pm} (f^{(m)} g^{(n)}) = (D^{\pm \pm} f^{(m)}) g^{(n)} + q^{-m} f^{(m)} D^{\pm \pm} g^{(n)}. \quad (10)$$

This is a special feature of quantum group non-commutative geometry that the quantum analogue to classical $U(1)$ generator can be realized as a second-order differential operator. With a little algebra one can write the next expression for the quantum $U(1)$ generator

$$D^0 \equiv -x_i \partial^i - q^2 y_i \bar{\partial}^i + (1 - q^2) x_i y_k \bar{\partial}^k \partial^i. \quad (11)$$

The operator $D^0$ has eigenfunctions which are just the functions with definite $U(1)$ charges

$$D^0 f^{(n)} = \{n\}_q f^{(n)}, \quad (12)$$

$$\{n\}_q \equiv \frac{1 - q^{-2n}}{1 - q^{-2}};$$
where \( \{n\}_q \) is a \( q \)-number. It is not hard to check the following Leibnitz rule for the operator \( D^0 \)

\[
D^0(f^{(m)}g^{(n)}) = (D^0 f^{(m)})g^{(n)} + q^{-2m} f^{(m)} D^0 g^{(n)}. \tag{13}
\]

Reducing the space of functions on \( SU_q(2) \) to the space of functions with a definite \( U(1) \) charge one obtains the covariant description of the coset \( S^2_q \sim SU(2)/U(1) \).

By direct calculating one can verify that the operators \( D^{\pm\pm,0} \) form the generalized \( q \)-Lie algebra of \( SU_q(2) \)

\[
[D^0, D^{++}]_{q^{-4}} = \{2\}_q D^{++},
\]

\[
[D^0, D^{--}]_{q^4} = \{-2\}_q D^{--},
\]

\[
[D^{++}, D^{--}]_{q^2} = D^0,
\]

here, \([A, B]_q \equiv AB - q^s BA\). Note, that the algebra (14) is valid irrespective of whether one imposes the unimodularity constraint \( \mathcal{D} = 1 \). Observe that the action of the operators \( D^{\pm\pm,0} \) is consistent with the constraint \( \mathcal{D} = 1 \), so we have

\[
D^{\pm\pm,0}(\mathcal{D} - 1)f(x, y) \cong 0. \tag{15}
\]

The symbol \( \cong \) means that one has a weak equality which is fulfilled in virtue of commutation relations. We shall treat the algebra (14) as a main \( q \)-Lie algebra of left-invariant vector fields on the quantum group \( SU_q(2) \). A corresponding \( q \)-generalized Jacobi identity is available

\[
[D^0, [D^{++}, D^{--}]_{q^2}] + [D^{++}, [D^{--}, D^0]_{q^{-4}}]_{q^{-2}} + q^2[D^{--}, [D^0, D^{++}]_{q^{-4}}]_{q^{-2}} = 0. \tag{16}
\]

Let us now pass to constructing other possible \( q \)-Lie algebras of left-invariant vector fields on the \( SU_q(2) \). For this purpose we consider differential operators \( \mu, \nu \)

\[
\mu = 1 + (q^2 - 1)y_i \bar{\partial}^i, \quad \nu = 1 + (1 - \frac{1}{q^2})x_i \partial^i. \tag{17}
\]
One can then see that the operators $\mu, \nu$ obey the simple commutation relations
\begin{align*}
\mu D^- &= q^2 D^- \mu, \quad \mu D^+ = \frac{1}{q^2} D^+ \mu, \\
\mu D^0 &= D^0 \mu, \quad \mu \nu = \nu \mu.
\end{align*}
\quad (18)
Similar formulae hold for the operator $\nu$ as well. Using these relations one can find that the operators $D^{++}, D^{--}, D^0$ defined by the next equations
\begin{align*}
D^{++} &= \mu^{-\frac{1}{2}} D^{++}, \quad D^{--} = \nu^{-\frac{1}{2}} D^{--}, \\
D^0 &= \frac{1}{q} \mu \nu D^0 \equiv [\partial^0]_q,
\end{align*}
\quad (19)
generate just the Drinfeld-Jimbo quantum enveloping algebra. To construct other possible $q$-Lie algebras one introduces another differential operators $\Delta^{++}, \Delta^{--}, \Delta^0$ as follows
\begin{align*}
\Delta^{++} &= F(\hat{Z}) D^{++}, \\
\Delta^{--} &= G(\hat{Z}) D^{--}, \\
\Delta^0 &= H(\hat{Z}), \\
\hat{Z} &\equiv (\mu \nu)^{-\frac{1}{2}}, \quad \hat{Z} f^{(n)} = q^n f^{(n)},
\end{align*}
\quad (20)
here, $F, G, H$ – are some operator functions of $\hat{Z}$. The operators $\Delta^{\pm \pm, 0}$ generate a quantum enveloping algebra of left-invariant vector fields with $q$-generalized commutators:
\begin{align*}
\Delta^{++} \Delta^{--} - q^{2p} \Delta^{--} \Delta^{++} &= \frac{q^p}{q - q^{-1}} \hat{Z}^{-p} (\hat{Z} - \hat{Z}^{-1}), \\
\Delta^0 \Delta^{++} - q^{2s} \Delta^{++} \Delta^0 &= \Delta^{++}, \\
\Delta^0 &= \frac{1 - \hat{Z}^s}{1 - q^{2s}},
\end{align*}
\quad (21, 22, 23)
where $s, p$ – are arbitrary integers which determine in part the functions $F, G$. Using the last equation (23) we may express the operator $\hat{Z}$ in terms of $\Delta^0$ and then restrict the arbitrariness of the parameters
s, p by considering only quadratic relations. Having put the quadratic terms from the r.h.s. of the eqn. (21) to the l.h.s. one obtains the possible 3D $q$-Lie algebras of $SU_q(2)$.

A special choice of a $q$-Lie algebra with some assigned Leibnitz rule defines uniquely the differential calculus on the quantum group $SU_q(2)$. Exterior differential 1-forms are treated as dual elements to the left-invariant vector fields.

3 Left-covariant differential algebras

In this section we give description of possible $SU_q(2)$ left-covariant differential algebras with $U(1)$ conserved charge. The gauge covariance condition leads to a unique differential algebra of $SU_q(2)$. At the same time a Leibnitz rule for the exterior differential is not fixed yet. To find the differentiation rules one needs to choose a corresponding $q$-Lie algebra of left-invariant vector fields.

Let us consider the left-invariant Cartan 1-forms $\Omega$ on the quantum group $SU_q(2)$

$$\Omega = dT^{-1}T \equiv \begin{pmatrix} \omega^0 & \omega^{++} \\ \omega^{--} & -q^2\omega^0 \end{pmatrix},$$

(24)

where $\omega^0, \omega^{++}, \omega^{--}$ are the basic left-invariant differential 1-forms with corresponding $U(1)$ charges $(0, +2, -2)$. One defines a gauge transformation as follows

$$T^g = \tilde{T} \tilde{T},$$

$$\Omega^g = \Omega - T^{-1}\tilde{\Omega}T, \quad \tilde{\Omega} \equiv d\tilde{T}^{-1}\tilde{T}.$$ 

(25)

It turns out that the requirement of the global $U(1)$-covariance and the consistence with the quantum group structure determine uniquely
all commutation relations between the differential 1-forms $\omega$ and the coordinates $(x, y)$. As a result we have

\[
\begin{align*}
\omega^{++} x &= q x \omega^{++}, \\
\omega^{--} x &= \frac{1}{q} x \omega^{--} + \frac{1 - q^4}{q^2} y \omega^0, \\
\omega^{++} y &= \frac{1}{q} y \omega^{++}, \\
\omega^{--} y &= q y \omega^{--}, \\
\omega^0 x &= x \omega^0 + (1 - \frac{1}{q^2}) y \omega^{++}, \\
\omega^0 y &= y \omega^0.
\end{align*}
\]

(26)

Similar consideration of commutation relations for the basic differential 1-forms $\omega^{\pm 0}$ leads to the left-covariant algebras parametrized by a real number $\sigma$:

\[
\begin{align*}
\omega^{++} \omega^{++} &= \omega^{--} \omega^{--} = 0, \\
\omega^{++} \omega^0 + q^2 \omega^0 \omega^{++} &= 0, \\
\omega^{--} \omega^0 + \frac{1}{q^2} \omega^0 \omega^{--} &= 0, \\
\omega^{++} \omega^{--} + q^\sigma \omega^{--} \omega^{++} + \frac{q^2(1 - q^\sigma)(1 + q^2)}{q^2 - 1} \omega^0 \omega^0 &= 0, \\
\omega^0 \omega^0 &= \frac{1 - q^2}{q^2(1 + q^2)} \omega^{++} \omega^{--}.
\end{align*}
\]

(27-31)

It should be noted that the algebra defined by eqs. (27-31) is left-covariant irrespective of whether one considers the last relation (31). Requiring the covariance under the gauge transformations and using the additional commutation relation

\[
\tilde{\Omega} \Omega = -q^2 \Omega \tilde{\Omega}
\]

(32)

one finds a unique gauge covariant differential algebra at $\sigma = 4$

\[
\begin{align*}
\omega^{++} \omega^{++} &= \omega^{--} \omega^{--} = 0, \\
\omega^{++} \omega^0 + q^2 \omega^0 \omega^{++} &= 0, \\
\omega^{--} \omega^0 + \frac{1}{q^2} \omega^0 \omega^{--} &= 0, \\
(1 + q^2)^2 \omega^0 \omega^0 &= \frac{1}{q^2} \omega^{++} \omega^{--} + q^2 \omega^{--} \omega^{++}.
\end{align*}
\]

(33)
The equation (31) is not gauge covariant and should be omitted. So defined gauge covariant differential algebra coincides with one obtained in ref. [8], where possible $GL_q(N)$-covariant quantum algebras were studied. It should be noted that our treatment does not contain the condition of vanishing the $GL_q(2)$-invariant

$$C_2 \equiv tr_q(\Omega^2) = 0,$$ (34)

which is not gauge covariant. Here we have used the notion of the $q$-deformed covariant trace [4, 8].

One can rewrite the commutation relations for the gauge covariant differential algebra in terms of the $R$-matrix. Direct checking leads to the next formulae

$$R_{12}dT_1T_2 = T_2dT_1R_{12},$$
$$R_{12}\Omega_2R_{12}^{-1}\Omega_1 + \frac{1}{q^2}\Omega_1R_{12}\Omega_2R_{12}^{-1}$$
$$-\frac{q}{1+q^2+q^4}(E_{12} + (q + \frac{1}{q})\varepsilon_{21})tr_q\Omega^2 = 0,$$ (35)

where

$$E_{ij}^{kl} = \delta_i^k\delta_j^l, \quad \varepsilon_{ij}^{kl} = \varepsilon^{ij}\varepsilon_{kl}.$$ (36)

To construct an exterior differential it is convenient to use the definition based on the dualism between the exterior algebra of differential forms and the $q$-Lie algebra of vector fields. In this way the Leibniz rule is followed straightforwardly and it depends only on a special choice of the $q$-Lie algebra.

Let us start from a general 3D $q$-Lie algebra of left-invariant vector fields $D^a = (D^{++}, D^{--}, D^0)$ on the quantum group $SU_q(2)$ with a Lie bracket

$$[D^a, D^b]_B \equiv D^a D^b - B^{abcd} D^c D^d = C^{abc} D^c.$$ (37)
We consider the braiding matrix $B^{abcd}$ to be unitary, so that it generates a representation of the permutation group. Thus, one can easily define the alteration rules for the tensor algebra of vector fields. Moreover, a generalized Jacobi identity will be available as well.

The basic left-invariant differential 1-forms $\omega^a$ are defined as dual objects by means of the scalar product

$$\omega^a(D^b) = \delta^{ab}.$$  \hspace{1cm} (38)

The action of the exterior differential on arbitrary functions $f$ and differential 1-forms $u$ is defined in analogy with the classical case \cite{13}

$$df(D^a) = D^a f,$$

$$du(D^a, D^b) = -\frac{1}{2} (D^a u(D^b) - B^{abcd} D^c u(D^d) - u([D^a, D^b]_B),$$  \hspace{1cm} (39)

$$du(D^a, D^b) = -B^{abcd} du(D^c, D^d).$$

Rules for the exterior differentiation of the differential $(n > 1)$-forms can be generalized in a similar fashion. The Cartan-Maurer equations have a standard form

$$d\omega^d(D^a, D^b) = \frac{1}{2} C^{abc} \omega^d(D^c).$$  \hspace{1cm} (40)

Note, that although the braiding matrix $B^{abcd}$ determines a corresponding exterior $B$-algebra, nevertheless, commutation relations for the differential 1-forms $\omega^a$ are not specified completely.

As a concrete example we consider the $q$-Lie algebra \cite{14} which is consistent with the gauge covariant algebra of left-invariant differential 1-forms \cite{33}. In this case differentiation rules (39) can be rewritten in a more familiar form after using the explicit tensor representation
for the exterior products of 2-forms $\omega^a \wedge \omega^b$

\[
d = \omega^a D^a, \\
d(\omega^{++} f) = d\omega^{++} f + \beta \omega^0 \omega^{++} D^0 f - (\omega^{(2)0} D^{++} f, \\
d(\omega^{--} f) = d\omega^{--} f + \beta q^2 \omega^0 \omega^{--} D^0 f + q^2 \omega^{(2)0} D^{--} f, \\
d(\omega^0 f) = \omega^{(2)0} f + \beta q^2 \omega^{++} \omega^0 D^{--} f + \beta \omega^{--} \omega^0 D^{++}, \\
\omega^{(2)0} \equiv \frac{1}{1 + q^2} (\omega^{++} \omega^{--} - q^2 \omega^{--} \omega^{++}), \\
\beta \equiv \frac{1 + q^4}{q^2(1 + q^2)}. \\
\] (41)

It should be noted that the formulae (41) involve just three independent basis differential 2-forms $\omega^0 \omega^{++}, \omega^0 \omega^{--}, \omega^{(2)0}$ in the space of exterior 2-forms as in the classical case. The fourth linearly independent basis form $\sigma^0$ is defined as follows

\[
\sigma^0 = \frac{1}{1 + q^2} (\omega^{++} \omega^{--} + q^2 \omega^{--} \omega^{++}). \\
\] (42)

The form $\sigma^0$ takes a non-zero value only for the symmetrical tensor product $D^0 \otimes D^0$:

\[
\sigma^0(D^0, D^0) = \rho, \\
\] (43)

where the number $\rho$ vanishes in the classical limit $q \to 1$. Due to this property the form $\sigma^0$ does not appear in eqs. (41).

Starting from the most general form for the exterior differentiation of 2-forms and taking into account the condition $d^2 = 0$ one can derive the next relations:

\[
d(\omega^{++} \omega^0 f) = -\frac{1}{q^2} \nu D^{++} f, \\
d(\omega^{--} \omega^0 f) = q^2 \nu D^{--} f, \\
d\omega^{(n>2)} = 0, \\
d(\omega^{(2)0} f) = \beta \nu D^0 f, \\
d(\sigma^0 f) = \rho \nu f, \\
\nu \equiv \frac{1}{2} \left( \omega^0 \omega^{++} \omega^{--} - q^2 \omega^0 \omega^{--} \omega^{++} \right), \\
\] (44)
here, \( \nu \) is the volume 3-form.

Having carried out some calculations one can find the explicit expressions for the Cartan-Maurer equations (40)

\[
d\Omega = \Omega^2 - \frac{q^2}{1 + q^2} E \text{tr}_q \Omega^2. \tag{45}
\]

One can see immediately that the r.h.s. of this equation contains only traceless part of the \( \Omega^2 \).

4 Conclusion

One can try to formulate a gauge Yang-Mills theory for the quantum group \( SU_q(2) \) in analogy with the covariant \( GL_q(N) \) version proposed in ref. \[8\]. Differentiation over the matter scalar fields \( \phi^i \) and gauge field \( A^i_j \) in a case of \( SU_q(2) \) will be more complicated due to the non-standard Leibnitz rule. For instance, one has the following differentiation rules for the products of two scalar fields

\[
\begin{align*}
d(\phi^i \phi^j) &= d\phi^i \phi^j + \frac{1}{q^2} \phi^i d\phi^j + \frac{q^2 - 1}{q^2} \varepsilon^{ijk} d\phi^k \phi^k, \\
\bar{d}(\bar{\phi}^i \bar{\phi}^j) &= \bar{d}\bar{\phi}^i \bar{\phi}^j + q^2 \bar{\phi}^i d\bar{\phi}^j, \\
\bar{d}(\bar{\phi}^i \phi^j) &= d\phi^i \phi^j + q^2 \bar{\phi}^i d\phi^j + (q^2 - 1) \bar{\phi}^i \phi^j d\phi^k \phi^k. \tag{46}
\end{align*}
\]

The gauge field \( A^i_j \) satisfies the same commutation relations that right invariant differential 1-forms \( Z = -\frac{1}{q^2} T\Omega T^{-1} \) do. Another possible way toward a consistent gauge Yang-Mills theory corresponds to the differential calculus with a \( q \)-Lie algebra differed from one defined by eqs. (14). Search for a suitable \( q \)-Lie algebra is not carried out directly due to possible non-lexicographic differentiation rules.
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