Dialectical Rough Sets, Parthood and Figures of Opposition

A. Mani

Department of Pure Mathematics
University of Calcutta
9/1B, Jatin Bagchi Road
Kolkata(Calcutta)-700029, India
Email: a.mani.cms@gmail.com
Homepage: http://www.logicamani.in

Abstract. In one perspective, the central problem pursued in this research is that of the inverse problem in the context of general rough sets. The problem is about the existence of rough basis for given approximations in a context. Granular operator spaces were recently introduced by the present author as an optimal framework for anti-chain based algebraic semantics of general rough sets and the inverse problem. In the framework, various subtypes of crisp and non crisp objects are identifiable that may be missed in more restrictive formalism. This is also because in the latter cases the concept of complementation and negation are taken for granted. This opens the door for a general approach to dialectical rough sets building on previous work of the present author and figures of opposition. In this paper dialectical rough logics are developed from a semantic perspective, concept of dialectical predicates is formalized, connection with dialethias and glutty negation established, parthood analyzed and studied from the point of view of classical and dialectical figures of opposition. Potential semantics through dialectical counting based on these figures are proposed building on earlier work by the present author. Her methods become more geometrical and encompass parthood as a primary relation (as opposed to roughly equivalent objects) for algebraic semantics. Dialectical counting strategies over antichains (a specific form of dialectical structure) for semantics are also proposed. [This paper is scheduled to appear as two separate papers (because of length) with some overlap and enhancements.]

Keywords: Rough Objects , Dialectical Rough Semantics , Granular operator Spaces , Rough Mereology , Polytopes of Dialectics , Antichains , Dialectical Rough Counting , Axiomatic Approach to Granules , Constructive Algebraic Semantics , Squares of Opposition

1 Introduction

It is well known that sets of rough objects (in various senses) are quasi or partially orderable. Specifically in classical or Pawlak rough sets [1],
the set of roughly equivalent sets has a Quasi Boolean order on it while the set of rough and crisp objects is Boolean ordered. In the classical semantic domain or classical meta level, associated with general rough sets, the set of crisp and rough objects is quasi or partially orderable. Under minimal assumptions on the nature of these objects, many orders with rough ontology can be associated - these necessarily have to do with concepts of discernibility (This sentence is intended to be read along the lines Associate(Objects, Orders_with_ontology) ). Concepts of rough objects, in these contexts, depend additionally on approximation operators and granulations used. These were part of the motivations of the development of the concept of granular operator spaces by the present author in [2] and developed further in [3,4,5]. The anti-chain based semantics that has been developed in the same paper refers to a specific kind of rough objects. The order in particular may refer to the most relevant parthood relation and dense concept of negation in the context. But rough parthoods need not be partial orders in general [6,7,4]. It suffices to assume, in set-theoretic contexts, that they are binary reflexive and antisymmetric relations in general.

The inverse problem for rough sets is essentially the problem of providing a rough semantics for a set of approximation related information and has been explored by the present author in [6,8] in particular. The solution for classical rough sets may be found in [9]. For more practical contexts, abstract frameworks like rough Y-systems RYS [6] are better suited for problem formulation. However, good semantics may not always be available for RYS. Granular operator spaces and generalizations thereof [3,4], are more suitable for generating other associations by which possible solutions of the inverse problem may be explored because the structures have good semantics associated (In fact, the algebraic semantics of [3] involves distributive lattices without universal negations). Therefore the present paper will focus exclusively on these.

The square of opposition and variants, in modern interpretation, refers to the relation between quantified or modal sentences in different contexts [10,11]. These have been considered in the context of rough sets in [12] from a set theoretical view of approximations of subsets in some rough set theories. The relation of parthood in the context of general rough sets with figures has not been investigated in the literature and so this is taken up in the present research paper (but from a semantic perspective). Connections with dialectical predication and other kinds of opposition are also taken up in the light of recent developments on connections of para-consistency and figures of opposition.
At another level of conception, in the classical semantic domain, various subtypes of objects relate to each other in specific ways through ideas of approximations (however the origin of these approximations may not be known clearly). These ways are shown to interact in dialectical ways to form other semantics under some assumptions. The basic structural schema can be viewed as a generalization of ideas like the square and hexagon of opposition - in fact as a combination of dialectical oppositions involved. This is used for introducing new methods of counting aimed at semantics. In [13], a dialectical rough set theory was developed by the present author using a specific concept of dialectical contradiction that refers to both rough and classical objects. Here related parthood is explored in detail and related to counting for constructive algebraic semantics.

All these are part of a unified whole - the inverse problem and solution strategies in all its generality. Granular operator spaces [2,4] are used as a framework for the problem and as it is a restricted one due to the conditions imposed on the approximations, it makes sense to speak of a part of a unified whole. Importantly no easy negation like operators are definable/defined in the framework and this is also a reason for exploring/identifying dialectical negations. So at one level the entire paper is a contribution towards the possible solutions of the inverse problem and usable frameworks for the same.

The questions and problems that are taken up in this research paper and solved to varying extents also include the following:

1. What may be a proper formalization of a dialectical logic and dialectical opposition?
   - The most interesting problems in relation to these questions are the connection of dialethias, truth gluts and dialectical contradiction and the question of whether the intended interpretation of the objects formalized by the logic should be about state transitions.
   - How do dialethias and dialectical contradiction differ?
2. Are paraconsistent logics proper formalizations of the philosophical intent in Hegelian and Marxist dialectics?
3. What is the connection between parthood in rough contexts and possible dialectical contradictions?
4. What is a rough dialectical semantics and is every parthood based rough semantics a dialectical one?
5. How does parthood relate to figures of opposition?
6. How can counting processes of [6] be generalized to accommodate these in a fruitful way?
This paper is structured as follows. The next section includes background material on rough concepts, posets, granules and parthood. In the third section, the superiority of granular operator spaces over property systems is explained. Dialectical negation and logics are characterized from a critical perspective in the following section. In the fifth section, dialectical rough logics are developed and related parthood are explored. Many examples are provided in the context of the semantic framework used in the sixth section. The following section is about figures of dialectical opposition generated by few specific parthood related statements in rough sets and a proposal for handling pseudo gluts. Dialectical counting in rough contexts are developed in the penultimate section. Some directions are provided in the ninth section.

2 Background

In quasi or partially ordered sets, sets of mutually incomparable elements are called antichains. Some of the basic properties may be found in [14,15]. The possibility of using antichains of rough objects for a possible semantics was mentioned in [16,17,18] and has been developed subsequently in [2]. The semantics developed in the paper is applicable for a large class of operator based rough sets including specific cases of RYS [6] and other general approaches like [19,20,21,22]. In [19,20], negation like operators are assumed in general and these are not definable operations in terms of order related operations in a algebraic sense (Kleene negation is also not a definable operation in the situation).

For basics of rough sets, the reader is referred to [1,23,7]. The present paper is aimed at going beyond the more commonly accepted bounds of general rough sets and related ideas of information, knowledge, beliefs and data. Words like lower or upper approximation operator can mean different concepts in the literature and in general relevant definitions need to be fixed in advance.

If $S$ is any set (in ZFC), then by a lower approximation operator $l$ over $S$ will be a map $l : \wp(S) \rightarrow \wp(S)$ that satisfies:

- $(\forall x \in \wp(S)) \neg (x \subseteq x^l)$ (non-increasing)
- $(\forall x \in \wp(S)) x^l = x^{ll}$ (idempotence)
- $(\forall a, b \in \wp(S)) (a \subseteq b \rightarrow a^l \subseteq b^l)$ (monotonicity)

In the literature on rough sets many variants of the above are also referred to as lower approximations because the concept has to do with what one thinks a lower approximation ought to be.
Over the same set, an upper approximation operator \( u \) shall be a map \( u : \wp(S) \rightarrow \wp(S) \) that satisfies:

\[
(\forall x \in \wp(S)) \quad x \subseteq x^u \quad \text{(increasing)}
\]

\[
(\forall a, b \in \wp(S)) \quad (a \subseteq b \rightarrow a^u \subseteq b^u) \quad \text{(monotonicity)}
\]

In some practical contexts, lower and upper approximation operators may be partial and \( l \) (respectively \( u \)) may be defined on a subset \( S \subset \wp(S) \) instead. In these cases the partial operation need not necessarily be easily completable. Further the properties attributed to approximations may be a matter of discovery. These cases fall under the general class of inverse problems \([8,6]\) where the goal is to see whether the approximations originate or fit a rough evolution (process). More details may be found in the next section.

An element \( x \in \wp S \) will be said to be lower definite (resp. upper definite) if and only if \( x^l = x \) (resp. \( x^u = x \)) and definite, when it is both lower and upper definite. In general rough sets, these ideas of definiteness are insufficient as it can happen that upper approximations of upper approximations are still not upper definite.

The concept of a Rough Y-System \( \text{RYS} \) was introduced by the present author in \([6,24]\) and refined further in \([25]\) and her doctoral thesis as a very general framework for rough sets from an axiomatic granular perspective. The concept is not used in an essential way in the present paper and the reader may skip few remarks concerning the connections. In simplified terms it is a model of any collection of rough/crisp objects with approximation operators and a binary parthood predicate \( P \) as part of its signature.

Possible concepts of rough objects considered in the literature include the following:

- \( x \in S \) is a rough object if and only if \( x^l \neq x^u \) - this definition relates to the idea of regarding all non definite objects as rough objects,
- Any pair of definite subsets of the form \((a, b)\) satisfying \( a \subseteq b \),
- Any pair of subsets of the form \((x^l, x^u)\),
- Sets in an interval of the form \((x^l, x^u)\),
- Sets in an interval of the form \((a, b)\) satisfying \( a \subseteq b \) (when \( a \) and \( b \) are definite subsets),
- Higher order intervals bounded by definite subsets \([6]\),
- \( x \) is a rough object if and only if \( \neg P x^u x^l \) - this definition relates to the idea of regarding all non definite objects in a RYS \([6]\) as rough objects.
The idea of definite and rough objects can be varied substantially even when the approximations have been fixed and the above concepts are based on representation.

Concepts of representation of objects necessarily relate to choice of semantic frameworks. In general, in most contexts, the order theoretic representations are of interest. In operator centric approaches, the problem is also about finding ideal representations.

In simple terms, *granules* are the subsets that generate approximations and *granulations* are the collections of all such granules in the context. For a short overview of divergent views of what they actually are the reader may refer to subsection 2.1.

Granular operator spaces, a set framework with operators introduced by the present author in [2], will be used as considerations relating to antichains will require quasi/partial orders in an essential way. The evolution of the operators need not be induced by a cover or a relation (corresponding to cover or relation based systems respectively), but these would be special cases. The generalization to some rough Y-systems RYS (see [6] for definitions), will of course be possible as a result.

**Definition 1.** A Granular Operator Space \( S \) will be a structure of the form \( S = (S, G, l, u) \) with \( S \) being a set, \( G \) an admissible granulation (defined below) over \( S \) and \( l, u \) being operators : \( \varphi(S) \mapsto \varphi(S) \) (\( \varphi(S) \) denotes the power set of \( S \)) satisfying the following (\( S \) will be replaced with \( S \) if clear from the context. Lower and upper case alphabets will both be used for subsets):

\[
\begin{align*}
 a^l &\subseteq a \& a^u = a^l \& a^u \subseteq a^{uu} \\
 (a \subseteq b \rightarrow a^l \subseteq b^l \& a^u \subseteq b^u) \\
 \emptyset^l = \emptyset \& \emptyset^u = \emptyset \& S^l \subseteq S \& S^u \subseteq S.
\end{align*}
\]

Admissible granulations are granulations \( G \) that satisfy the following three conditions (\( t \) is a term operation formed from the set operations \( \cup, \cap, ^c, 1, \emptyset \)):
\[(\forall a \exists b_1, \ldots, b_r \in G) t(b_1, b_2, \ldots, b_r) = a^l \]
and \[(\forall a)(\exists b_1, \ldots, b_r \in G) t(b_1, b_2, \ldots, b_r) = a^u, \] (Weak RA, WRA)
\[(\forall b \in G)(\forall a \in \wp(S))(b \subseteq a \rightarrow b \subseteq a^l), \] (Lower Stability, LS)
\[(\forall a, b \in G)(\exists z \in \wp(S)) a \subseteq z, b \subseteq z \& z^l = z^u = z, \] (Full Underlap, FU)

**Remarks:**

- The concept of admissible granulation was defined for RYS in \([6]\) using parthoods instead of set inclusion and relative to RYS, \(P = \subseteq, P = \subset\).
- The conditions defining admissible granulations mean that every approximation is somehow representable by granules in a set theoretic way, that granules are lower definite, and that all pairs of distinct granules are contained in definite objects.

On \(\wp(S)\), the relation \(\sqsubseteq\) is defined by
\[A \sqsubseteq B \iff A^l \subseteq B^l \& A^u \subseteq B^u.\] (1)

The rough equality relation on \(\wp(S)\) is defined via \(A \approx B \iff A \sqsubseteq B \& B \sqsubseteq A.\)

Regarding the quotient \(\wp(S) | \approx\) as a subset of \(\wp(S)\), the order \(\sqsubseteq\) will be defined as per
\[\alpha \sqsubseteq \beta \text{ if and only if } \alpha^l \subseteq \beta^l \& \alpha^u \subseteq \beta^u.\] (2)

Here \(\alpha^l\) is being interpreted as the lower approximation of \(\alpha\) and so on. \(\sqsubseteq\) will be referred to as the basic rough order.

**Definition 2.** By a roughly consistent object will be meant a set of subsets of \(S\) with mutually identical lower and upper approximations respectively. In symbols \(H\) is a roughly consistent object if it is of the form \(H = \{A; (\forall B \in H) A^l = B^l, A^u = B^u\}\). The set of all roughly consistent objects is partially ordered by the inclusion relation. Relative this maximal roughly consistent objects will be referred to as rough objects. By definite rough objects, will be meant rough objects of the form \(H\) that satisfy \(\forall A \in H) A^l = A^l \& A^u = A^u\).
However, this definition of rough objects will not necessarily be followed in this paper.

**Proposition 1.** $\in$ is a bounded partial order on $\varphi(S)$. 

**Proof.** Reflexivity is obvious. If $\alpha \in \beta$ and $\beta \in \alpha$, then it follows that $\alpha^l = \beta^l$ and $\alpha^u = \beta^u$ and so antisymmetry holds.

If $\alpha \in \beta$, $\beta \in \gamma$, then the transitivity of set inclusion induces transitivity of $\in$. The poset is bounded by $0 = (\emptyset, \emptyset)$ and $1 = (S^l, S^u)$. Note that $1$ need not coincide with $(S, S)$. □

The concept of generalized granular operator spaces has been introduced in [3,4] as a proper generalization of that of granular operator spaces. The main difference is in the replacement of $\subset$ by arbitrary part of $(P)$ relations in the axioms of admissible granules and inclusion of $P$ in the signature of the structure.

**Definition 3.** A General Granular Operator Space (GSP) $S$ shall be a structure of the form $S = \langle S, G, l, u, P \rangle$ with $S$ being a set, $G$ an admissible granulation (defined below) over $S$, $l, u$ being operators : $\varphi(S) \rightarrow \varphi(S)$ and $P$ being a definable binary generalized transitive predicate (for parthood) on $\varphi(S)$ satisfying the same conditions as in Def.1 except for those on admissible granulations (Generalized transitivity can be any proper non-trivial generalization of parthood (see [18]). $P$ is proper parthood (defined via $Pab$ iff $Pab \& \neg Pba$) and $t$ is a term operation formed from set operations:

\[
(\forall x \exists y_1, \ldots y_r \in G) t(y_1, y_2, \ldots y_r) = x^l, \quad \text{Weak RA, WRA}
\]
\[
(\forall y \in G)(\forall x \in \varphi(S))(P_{yx} \rightarrow P_{yx}^l), \quad \text{(Lower Stability, LS)}
\]
\[
(\forall x, y \in G)(\exists z \in \varphi(S)) Pxz, \quad \& \quad Pyz \& \quad z^l = z^u = z, \quad \text{(Full Underlap, FU)}
\]

2.1 Granules and Granulations

The idea of granular computing is as old as human evolution. Even in the available information on earliest human habitations and dwellings, it is possible to identify a primitive granular computing process (PGCP) at work. This can for example be seen from the stone houses, dating to 3500 BCE, used in what is present-day Scotland. The main features of this and other primitive versions of the paradigm may be seen to be
Problem requirements are not rigid.
• Concept of granules may be vague.
• Little effort on formalization right up to approximately the middle of the previous century.
• Scope of abstraction is very limited.
• Concept of granules may be concrete or abstract (relative all materialist viewpoints).

The precision based granular computing paradigm, traceable to Moore and Shannon’s paper \cite{23}, will be referred to as the \textit{classical granular computing paradigm} CGCP is usually understood as the granular computing paradigm (The reader may note that the idea is vaguely present in \cite{27}). The distinct terminology would be useful to keep track of the differences with other paradigms. CGCP has since been adapted to fuzzy and rough set theories in different ways. An overview is considered in \cite{28}.

Granules may be assumed to subsume the concept of information granules – information at some level of precision. In granular approaches to both rough and fuzzy sets, information granules in this sense are more commonly used in practice. Some of the fragments involved in applying CGCP may be:

• Paradigm Fragment-1: Granules can exist at different levels of precision.
• Paradigm Fragment-2: Among the many precision levels, choose a precision level at which the problem at hand is solved.
• Paradigm Fragment-3: Granulations (granules at specific levels or processes) form a hierarchy (later development).
• Paradigm Fragment-4: It is possible to easily switch between precision levels.
• Paradigm Fragment-5: The problem under investigation may be represented by the hierarchy of multiple levels of granulations.

The not so independent stages of development of the different granular computing paradigms is stated below:

• Classical Primitive Paradigm till middle of previous century.
• CGCP: Since Shannon’s information theory.
• CGCP in fuzzy set theory. It is natural for most real-valued types of fuzzy sets, but even in such domains unsatisfactory results are normal. Type-2 fuzzy sets have an advantage over type-1 fuzzy sets in handling data relating to emotion words, for example, but still far from satisfactory. For one thing linguistic hedges have little to do with numbers. A useful reference would be \cite{29}.
For a long period (up to 2008 or so), the adaptation of CGCP for RST has been based solely on precision and related philosophical aspects. The adaptation is described for example in [30,31]. In the same paper the hierarchical structure of granulations is also stressed. This and many later papers on CGCP (like [28]) in rough sets speak of structure of granulations.

Some Papers with explicit reference to multiple types of granules from a semantic viewpoint include [24,32,33,34].

The axiomatic approach to granularity initiated in [24] has been developed by the present author in the direction of contamination reduction in [6]. The concept of admissible granules, mentioned earlier, was arrived in the latter paper. From the order-theoretic/algebraic point of view, the deviation is in a very new direction relative the precision-based paradigm. The paradigm shift includes a new approach to measures.

Unless the underlying language is restricted, granulations can bear upon theory with unlimited diversity. Thus for example in classical RST, any of the following can be taken as granulations: collection of equivalence classes, complements of equivalence classes, other partitions on the universal set \( S \), set of finite subsets of \( S \) and set of finite subsets of \( S \) of cardinality greater than 2. This is also among the many motivations for the axiomatic approach. A formal simplified version of the axiomatic approach to granules is in [25].

### 2.2 Concepts of Finite Posets

Let \( S \) be a finite poset with \( \#(S) = n < \infty \). The following concepts and notations will be used in this paper:

- If \( \mathcal{X} \) is a collection of subsets \( \{X_i\}_{i \in J} \) of a set \( X \), then a system of distinct representatives SDR for \( \mathcal{X} \) is a set \( \{x_i; i \in J\} \) of distinct elements satisfying \( (\forall i \in J)x_i \in X_i \). Chains are subsets of a poset in which any two elements are comparable. Singletons are both chains and antichains.

- For \( a, b \in S, a \prec b \) shall be an abbreviation for \( b \) covering \( a \) from above (that is \( a < b \) and \( (a \leq c \leq b \rightarrow c = a \text{ or } c = b) \). \( c(S) \) shall be the number of covering pairs in \( S \).

- A chain \( C \) will be said to be saturated if and only if \( a \prec |C| b \) (that is if \( b \) covers \( a \) in the induced order \( \leq_{|C|} \) on \( C \)) implies \( a \prec b \).

- A chain cover of a finite poset \( S \) is a collection \( \mathcal{C} \) of chains in \( S \) satisfying \( \cup \mathcal{C} = S \). It is disjoint if the chains in the cover are pairwise disjoint.
• $S$ has finite width $w$ if and only if it can be partitioned into $w$ number of chains, but not less.

The following theorems are well known:

Theorem 1. • A collection of subsets $\mathcal{F}$ of a finite set $S$ with $\#(\mathcal{F}) = r$ has an SDR if and only if for any $1 \leq k \leq r$, the union of any $k$ members of $\mathcal{F}$ has size at least $k$, that is

$$(\forall X_1, \ldots, X_k \in \mathcal{F}) \ k \leq \#(\bigcup X_i).$$

• Every finite poset $S$ has a disjoint chain cover of width $w = \text{width}(S)$.

Theorem 2. Some known results relating to antichains and lattices are the following:

1. If $X$ is a partially ordered set with longest chains of length $r$ and if it can be partitioned into $k$ number of antichains then $r \leq k$.
2. If $X$ is a finite poset with $k$ elements in its largest antichain, then a chain decomposition of $X$ must contain at least $k$ chains.
3. The poset $AC_m(X)$ of all maximum sized antichains of a poset $X$ is a distributive lattice.
4. For every finite distributive lattice $L$ and every chain decomposition $C$ of $J_L$ (the set of join irreducible elements of $L$), there is a poset $X_C$ such that $L \cong AC_m(X_C)$.

Proof. Proofs of the first three of the assertions can be found in in [35,36] for example. Many proofs of results related to Dilworth’s theorems are known in the literature and some discussion can be found in [35] (pages 126–135).

1. To prove the first, start from a chain decomposition and recursively extract the minimal elements from it to form $r$ number of antichains.
2. This is proved by induction on the size of $X$ across many possibilities.
3. See [35,36] for details.
4. In [15], the last connection between chain decompositions and representation by antichains reveals important gaps - there are other posets $X$ that satisfy $L \cong AC_m(X)$. Further the restriction to posets is too strong and can be relaxed in many ways [37].

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2.3 Rough Sets and Parthood

It is necessary to clarify the nature of parthood even in set-theoretic structures like granular operator spaces. The restriction of the parthood relation to the case when the first argument is a granule is particularly important. The theoretical assumption that objects are determined by their parts, and specifically by granules, may not be reasonable when knowledge of the context is evolving. Indeed, in this situation:

- granulation can be confounded by partial nature of information and noise,
- knowledge of all possible granulations may not be possible and the chosen set of granules may not be optimal for handling partial information, and
- the process has strong connections with apriori understanding of the objects in question.

3 Types of Preprocessing and Ontology

What is perceived is data. Data can occur in real life in various forms and not necessarily in information system form with errors of various kinds. When presented in information system like form, then the appropriateness of rough methods depends on suitability of the abstraction used in defining relations or the process of extraction of covers and related definitions of approximations. But even when data is information system form, there is no guarantee that reasonable abstractions are possible. This is possible, for example, with secondary databases from sociology. Then again there are situations where data is in mixed form with some information being in information system form and some in the form of 'relevant approximation' form.

Databases associated with sportswomen who play badminton, have the form of multi-dimensional information systems about performance in various games, practice sessions, training regimen and more. Video data about all these would also be available. Players tend to perform better in specific conditions and game situations unique to them and may also be able to raise the level of their game under specific kinds of stress. Again the playing of games involves dynamic learning. All of this additional information can be expressed in terms of approximations, especially when the associations are not too perfect. Thus a statement like

Player $A$ is likely to perform at least as well as player $B$ in playing conditions $C$. 
can be translated into $B^l \leq A^l$ where the approximations refer the specific property under consideration. But this information representation has no obvious rough set basis associated and falls under the inverse problem, where the problem is of explaining the approximations from a rough basis.

A substantial part of rough set theory is about data from information systems, but the inverse problem has been known as abstract representation problem at different times. Some examples are [9,33]. The full significance of possible applications had not been looked into in the literature except in some of the present author’s papers [8,43,44,39,40,6]. This is surprising given the fact that abstract formalisms like those of rough orders, abstract approximation spaces, operator based approaches have been considered at various times in the literature as in [22,20]. The goal of those approaches and the property systems considered in [23] and earlier work has been to get to good semantics and possibly logics.

3.1 Granular Operator Spaces and Property Systems

More generally, data can also be expected to be presented in real life predominantly in terms of approximations and partly in the object-attribute-value way of representing things. In this context it is important to note that the idea of property systems or related basic constructors pursued by different authors [23,41,42] was never intended to capture the scenario. The examples in [43], in particular, are abstract ones and the possible problems with basic constructors (when viewed from the perspective of approximation properties satisfied) are issues relating to construction and empirical aspects are missed.

**Definition 4.** A property system $\Pi$ (see [43,23,44,45]) is a triple of the form $(U, P, R)$ with $U$ being a universe of objects, $P$ a set of properties, and $R \subseteq U \times P$ being a manifestation relation subject to the interpretation object $a$ has property $b$ if and only if $(a,b) \in R$. When $P = U$, then $\Pi$ is said to be a square relational system and $\Pi$ then can be read as a Kripke model for a corresponding modal system.

On property systems, basic constructors that may be defined for $A \subseteq U$ and $B \subseteq P$ are

\[
< i >: \varphi(U) \mapsto \varphi(P); \quad < i > (A) = \{ h : (\exists g \in A) (g, h) \in R \} \quad (3)
\]

\[
< e >: \varphi(P) \mapsto \varphi(U); \quad < e > (B) = \{ g : (\exists h \in B) (g, h) \in R \} \quad (4)
\]

\[
[i]: \varphi(U) \mapsto \varphi(P); \quad [i](A) = \{ h : (\forall g \in U)((g, h) \in R \rightarrow g \in A) \} \quad (5)
\]

\[
[e]: \varphi(P) \mapsto \varphi(U); \quad [e](B) = \{ g : (\forall h \in P)((g, h) \in R \rightarrow h \in B) \} \quad (6)
\]
It is known that the basic constructors may correspond to approximations under some conditions and some unclear conditions. Property system are not suitable for handling granularity and many of the inverse problem contexts. The latter part of the statement requires some explanation because suitability depends on the way in which the problem is posed - this has not been looked into comprehensively in the literature.

If all of the data is of the form

Object $X$ is definitely approximated by $\{A_1, \ldots, A_n\}$,

with the symbols $X, A_i$ being potentially substitutable by objects, then the data could in principle be written in property system form with the sets of $A_i$s forming the set of properties $P$ - in the situation the relation $R$ attains a different meaning. This is consistent with the structure being not committed to tractability of properties possessed by objects. Granularity would also be obscure in the situation.

If all of the data is of the form

Object $X$ 's approximations are included in $\{A_1, \ldots, A_n\}$,

then the property system approach comes under even more difficulties. Granular operator spaces and generalized versions thereof [4] in contrast can handle all this.

4 Dialectical Negation

The main questions relating to a definition of dialectical negation or dialectical contradiction at the formal level arise from the following reasons (these are explained below):

- The consensus that dialectical logics must be logics that concern state transitions,
- The view that paraconsistent logics are essentially dialectical logics (see [10]),
- The view that dialectical negation cannot be reduced to classical negation (see [11]). Indeed, in rough sets many kinds of negations and partial negations have been used in the literature (see for example, [17,18,19,20,21,33,50,13] and these lead to many contradictions as in

1. *contradictions* [47] which are not false but that represent topological boundaries;
2. *contradictions* [48] which are not false but lie between an absolute and local false;
3. *contradictions* [43] which lead to at least a paraconsistent and a paracomplete logic.

- The view that dialectical negation is glutty negation (example [51,52])-
an intermediate kind of negation
- The view that only propositional dialectical logics are possible ([53])
- The present author’s position that dialectical contradiction must be expressed by binary predicates or binary logical connectives in general [54,13]. This is arrived at in what follows.

The relationship of an object and its negation may belong to one of three categories (an extension of the classification in [55]):

1. **Cancellation** as in *the attributes do not apply*. The ethical category of the negative as used in natural language is often about this kind of cancellation. It is easy to capture this in logics admitting different types of atomic variables (or formalized for instance by labeled deductive systems [56]). In these the concept of the Negative is usually not an atomic category. Obviously this type of negation carries more information in being a *Not This but Among Those* kind of negation as opposed to the simple *Not This, Not This and Something Else* and weakenings thereof.

2. **Complementation** understood in the sense of classical logic.
3. **Glutty Version** as something intermediate between the above two.

In general rough sets, if $A$ is a subset of attributes and $c$ is set complementation, then the value of more common *negations in a rough sense* include $A^{uc}$, $A^{cl}$, $A^{uc}$, $A^{cla}$. Each of these is a possibly nonempty subset of $A^c$ and the corresponding negation is therefore glutty in a set theoretic sense.

The concept of dialectical negation as a material negation in logics concerning states is a reasonable abstraction of the core concept in Hegelian and Marxist dialectics (though this involves rejection of Hegel’s idealist position). The negation refers to concepts in flux and so a logic concerning the behavior of states rather than static objects would be appropriate. According to Hegel, the world, thought and human reasoning are dynamic and even the idea of true concepts are dynamic in nature. Poorly understood concepts undergo refinement as plural concepts (with the parts being *abstract* in Hegel’s sense) that assume many $H$(Hegelian)- *Contradictory* forms. After successive refinements the resulting forms become reconciled or united as a whole. So for Hegel, $H$-contradictions are
essential for all life and world dynamics. But Hegel’s idealist position permitted only a closed world scheme of things. In Marx’s materialist dialectic, the world is an open ended system and so recursive applications of dialectical contradiction need not terminate or be periodic. All this means that the glutty interpretation of Hegel’s contradiction may well be correct modulo some properties, while Marx’s idea of dialectical contradiction does not reduce to such an interpretation in general. The debate on endurantism and perdurantism is very relevant in the context of Hegelian dialectics because the semantic domain associated is restricted by Hegel’s position on the nature of the world. In rough semantics, especially granular ones, approximations may be seen as transitions and so the pre conditions are met in a sense.

Consider possible concepts of the identity of an apple on a table. For the general class of apples, a set of properties $X$ can be associated. The specific apple in question would also be possessing a set of specific properties $Z$ (say) such as properties like distribution and intensity of colors. Obviously, many of the specific properties will not be true of apples in general. Therefore specific properties would be in dialectical opposition to the general. Note that an agent can have multiple views of what $Z$ and $X$ ought to be and multiple agents would only contribute to the pluralism. In the Hegelian perspective all of this dialectical contradiction must necessarily be resolved in due course (this process may potentially involve non materialist assumptions), while in the Marxist perspective a refined plural that may get resolved would be the result. Thus the fruit on the table may be the Alice variety of *Malus Domestica* (a domestic apple) with many other specific features. The schematics (for an agent) is illustrated in Fig. 1 - $\mathcal{P}$ is a binary predicate with the intended meaning of *dialectically opposed to*.

To see how ideas of unary operations as dialectical negations can fail, consider the color of apples alone. If the collection of all possible colors of apples is known, then the set of all possible colors would be knowable. Negation of a white apple may be definable by complementation in this case. If on the other hand only a few possible sets of colors and some collections of colors are known, then the operational definition can fail (or lack in meaning). This justifies the use of a general binary predicate $\sqsubset$ for expressing dialectical contradiction.

Da Costa et. al consider the heuristics of a possible dialectic logic in [46]. They seem to accept McGill’s interpretation of unity of opposites [53] and restrict themselves to a propositional perspective on the basis
of difficulties with formalizing four of the six principles. This results in a very weak dialectical logic. They are however of the view that:

- Formal logics based on Marxist and Hegelian dialectics intersect the class of paraconsistent logics and there is great scope for deviation and that
- It can be argued that paraconsistent logics represent a desired modification of dialectics because of the latter’s openness and non-rigid formalism

The distinction between static and dynamic dialectical logics within the class of dialectical logics with dialectical contradiction as expressed with the help of an unary operation, may be attributed to Batens [57, 58]. Adaptive logics in that perspective would appear to be more general than dynamic dialectical logics; the main idea being to interpret inconsistencies as consistently as is possible. Key to this class of logics is the concept of tolerance of contradictory statements that are not necessarily reduced in their level of contradictions by way of proof mechanisms. Through this one can capture parts of the thesis-antithesis-synthesis meta principle. All semantic aspects of adaptive logics are intended in a classical perspective as opposed to dialethic logics and these are very closely connected to
paraconsistent logics as well. There are approaches like \[59\] that aim at universal contradiction resolution - these can be said to be Hegelian in nature.

Two of the most common misinterpretations or reductions of the concept of \emph{dialectical opposition} relate to excluding the very possibility of formalizing it and the reduction of dialectical negation to simple negation or opposites. Examples of the latter type include \[60\], \[61\]. Dialectical methods have been very successfully applied in cognitive psychology by Klaus Riegel, Hoffmann and others. Hoffmann \[60\] attempts a formalization of dialectical psychology using a particular form of dialectical reasoning. The method consists in contradiction reduction by synthesis using the Hegelian idea of Sherlock Holmes on the point that \emph{if all that is not possible is removed then what remains must probably be true}. The resulting model is based on difference operations over collections of all subsets of a set and is therefore simplistic.

Some are of the view that Marx worked with normative ideas of concept and so introduction of related ideas in logic are improper. In modern terminology, Marx merely wanted concepts to be grounded in the material and was opposed to idealist positions that were designed for supporting power structures of oppression. This is reflected in Marx’s position on Hegel’s idealism and also, for example, on Wagner’s position \[62\]. Marx and Hegel did not write about formal logic and the normative ideas of \emph{concept} used by both and other authors during the time can be found in great numbers. From the point of view of less normative (or non normative) positions all of these authors implicitly developed concepts at all times. It is also a reasonable position that Marxist methodologies should not be formalized independently of the normative restriction on possible ideas of concept afforded by actualization contexts. This is because it is always a good idea to have good grounding axioms for concepts to the extent that is permitted/possible by the context in question.

In the present author’s opinion a necessary condition on a methodology or theory to qualify as \emph{dialectic in Marx’s sense} is that the idea of concepts used should be well grounded in the actualization contexts of the methodology or theory. In the context of reasoning with vagueness and rough sets, this means that the approach should be granular.

In general, formal versions of dialectical logics can be based on some of the following principles/heuristics.

A Binary Logical predicates (that admit of universal substitution by propositional variables and well formed formulas) that are intended
to signify binary dialectical contradiction are necessary for dialectical logics,

B Unary logical connectives (that admit of universal substitution by propositional variables and well formed formulas) suffice for expressing dialectical contradiction,

C The thesis-antithesis-synthesis principles must necessarily be included in the form of rules in any dialectical logic,

E Higher order quantifiers must be used in the logical formalism in an essential way because dialectical contradictions happen between higher order concepts,

F Dialectical logics should be realizable as fragments of first order predicate logic - this view is typically related to the position that higher order logics are superfluous.

G Dialectical contradiction in whatever representation must be present at each stage of what is defined to constitute dialectical derivation - this abstraction is due to the present author and is not hard to realize.

H All dialectic negations should be dialethic(*) in nature - this is a possibility explored in [55]. Dialethias are statements that can be both true and false simultaneously.

I A logic that permits expression of progression of knowledge states is a dialectical logic.

J a first order logic perspective

K the point of view that dialectical contradiction can be expressed by binary predicates and not by unary operations

L Dialectical logics as paraconsistent logics incorporating contradictions or as inconsistency adaptive logics.

Obviously many of these are mutually incompatible. [K] is particularly incompatible with [B] in the above. [I] can be expected to lead to logics resembling Linear logics. The meaning of dialectical logics that admit representation as a fragment of first order predicate logic will be naturally restricted and some versions are known. Dynamic dialectical logics have been developed as inconsistency adaptive logics by Batens [63] in particular. In the present paper [A] will be preferred as the binary predicate/predication cannot always be reduced to unary negations.

In general it is obvious that given a dynamically changing subject, there will be at least a set of things which are dialectically contradictory to it in many ways. If $a$ is dialectically contradictory to $b$ and $c$ in two different senses, then it is perfectly possible that $b$ is dialectically contradictory to $c$ in some other sense. Further if $X$ is dialectically contradictory to a conjunction of the form $Y \land Z$, then it is possible that $X$ is dialectically contradictory to $Y$ in some other sense and is virtually indifferent to $Z$.
4.1 Dialectical Contradiction and Contradiction

At a more philosophic level the arguments of this section can be expressed in the language of functors, but a set-theoretic semantic approach is better suited for the present purposes. The concept of contradiction and dialethic contradiction make essential use of negation operations (in some general sense), while that of dialectical contradiction when formulated on comparable terms does not necessarily require one. It is necessary to clarify the admissible negations in all this as many variations of the concept of logical negation are known.

Let $S$ be a partially ordered set with at least one extra partial unary operation $f$, a least element $\bot$ and a partial order $\leq$ ($\land, \lor$ being partial lattice infimum and supremum). In a partial algebra, two terms $p, q$, are weakly equal ($p \equiv q$), if both terms are defined then they are equal. Consider the following:

$$x \land f(x) \equiv \bot \quad (N1)$$

$$ (x \leq y \rightarrow f(y) \leq f(x)) \quad (N2)$$

$$x \leq f^2(x) \quad (N3)$$

$$ (x \leq f(y) \rightarrow y \leq f(x)) \quad (N4)$$

$$f^n(x) = f^{n+m}(x) \text{ for some minimal } n, m \in \mathbb{N} \quad (N5)$$

$$f(x \lor y) \equiv f(x) \land f(y) \quad (N6)$$

$$x \land y = \bot \iff y \leq f(x). \quad (N9)$$

These are different conditions that in different combinations can characterize negations. In [64], if an operation satisfies $N1$ and $N2$ over a distributive lattice, then it is taken to be a general negation. This is a reasonable concept for logics dealing with exact information alone as $N1$ does not hold in the algebras of logics of vague, uncertain or approximate reasoning. For example it fails in the algebras of rough logic and generalizations thereof [9,38,6,16].

If $\forall x f^m(x) = f^n(x)$ holds, then the least $n$ such that $m < n$ is called the global period of $f$, $s = n - m$, the Global Pace of $f$ and $(m, n)$, the index of $f$.

It is known that

**Theorem 3.** The following are separately true in the above context when the poset is a distributive lattice:

1. If $N1$, $N2$ are satisfied by $f$, then the index $(0, n)$ for $n > 2$ is not possible.
2. If \( N_1, N_2, N_3 \) are satisfied by \( f \), then \( f(\bot) = T \) is the greatest element of the lattice and \( f(T) = \bot \) is satisfied.
3. \( N_1, N_2, N_3 \) together do not imply \( N_9 \)
4. \( N_9 \) implies \( N_1, N_2, N_3 \).
5. An interior operator \( i \) on a poset is one that satisfies
   - \( i(x) \leq x \),
   - \( (a \leq b \rightarrow i(a) \leq i(b)) \)
   - \( i(i(x)) = i(x) \).

If \( f \) is a regular negation (that is it satisfies \( N_1, N_2, N_3 \)) and \( i \) an interior operator, then \( g = if \) is a negation that satisfies \( g^4 = g^2 \).

Even at these levels of generality, the generalized negations can fail to express the appropriate concept of dialectical contradiction.

If a set-theoretic perspective is opted for then if a set of things \( A \) is dialectically opposed to a set of things \( B \), then it may appear reasonable to expect the set of things dialectically opposed to \( B \) to include \( A \). But if this is the intent then the set of all things dialectically opposed to \( A \) would need to be expressed by \( \sim A \). But in dialectical reasoning it will still be reasonable to say that \( A \) is dialectically opposed to some part of \( \sim A \). For this the use of a unary \( \sim \) can be glaringly insufficient. This is true not only from an algebraic system point of view (when working within a model) but also from perspectives generated by admissible sets of models. Accepting \( N_2 \) is inherently incompatible with accepting \( f \) as a unary dialectical negation operator, especially when a lattice order is expected to be induced by some concept of logical equivalence from the order. \( N_3 \) is perhaps the most necessary property of a dialectical negation operation.

Well-formed formulas of certain derived types alone may admit of a negation (in the sense of being equivalent to one of the negatives). Such a negation is partial. For instance, \( \sim \sim x \) may not be defined in the first place, and some of \( N_1 \sim N_9 \) may hold for such negations. Using such types of negation for expressing dialectical contradictions through compound constructions may be possible in adequately labeled deductive systems.

Dialethic logics are logics which tolerate contradictions and accept true contradictions. To be more specific a dialetheia is a pair of statements of the form \( A \& \neg A \) with each of \( A \) and \( \neg A \) being true. These statements may be interpreted sentences expressed in some language (that may be natural language or a language of thought, or anything). They can be used to formalize only some restricted cases of dialectical reasoning in which a unary dialectical contradiction operation is possible. It is also
possible to reformulate some dialectical contexts as a dialethic process. Priest [65] had indicated the possibility of using dialethic logics as a base for dialectic logics. In [66], Priest develops a dialectical tense logic, where it is possible for a system to exist in both pre and post states during (at the instant of) state transitions. Zeleny [67] in particular has correctly pointed out (from a philosophical perspective) the possible shortcomings of a unary negation based approach. Though the issue of desiring incompatibility between classical logic and a possible dialectical logic is not a justified heuristic. The essential dialetheism principle is however usable in dialectical derivations. Such situations would allow dialectical opposition between proof patterns naturally.

The nature and meaning of negation in a dialethic logic is explained in greater detail in [68,65]. From a philosophical meta perspective the negation of a formula is possibly a collection of formulas that may be representable by a single formula (from a logical perspective). It is with respect to such a negation that dialethic logics must tolerate contradictions and accept true ones. A survey of concepts of contradictions for dialetheism can also be found in [69]. Using any kind of universal paraconsistent system for describing inconsistencies is virtually shown to be an undesirable approach in [58]. In [70] Marxist dialectics is perceived from a dialethic perspective of things. The concept of historical contradictions is claimed to correspond to and be realized as dialethias. The methodological aspect of Marxist dialectics is also ignored by Priest (see [71]) to the point that dialectic is a dialetheia. There are no methodological strictures associated with dialethias except for the requirement that they be real. This approach ignores

- the world view associated with Hegelian-Marxist dialectics,
- the principle of unity of opposites, and
- the basic problem with formalizing dialectical opposition with a unary operator - this is because the negation in dialectics is transient and dependent on the above two points.

In the present author’s view dialethias do exist in the real world - they may be the result of

- missing data/information
- a deliberate disregard for consistency. For example, a large number of people in the news media, religion and politics practice dialethic expression of a crude form and deceit. They may have their motivations for such actions, but those would not be justification for their dialethias. To try and explain away such dialethias through so-called
motivations amounts to interfering in the data (being judgmental in a social sense). So many religious functionaries have been convicted of sexual crimes and most were in harmony with their apparent dialethic behavior (religious texts may be full of contradictions, that only allows for prolonging the derivations on the claim).

The present author agrees with Priest’s claims about dialethias being not resolvable by revision of concepts and that they are better handled as such \[55,71,72\]. However she does not agree at all that the proper way of formalizing Hegelian-Marxist dialectics in all cases is through dialethias. A mathematical formulation of the issue for many sub cases may be possible through rough sets. The *Cold vs Influenza* example considered in the subsection on examples of parthoods throws much light on the matter.

### 4.2 Dialectical Predication Abstraction

At a philosophic level, dialectical predication is a relation between functors in the sense of \[10\]. At a formal model-theoretic semantic level, the best realization is through a binary dialectical predicate \(\square\), that may have limited connections with negation operations (if any). The basic properties that are necessary (not sufficient) of the predicate are the following (with \(\oplus\) standing for aggregation):

\[
\square(a, b) \leftrightarrow \square(b, a) \quad \text{(Commutativity)}
\]
\[
\neg \square(a, a) \quad \text{(Anti-Reflexivity)}
\]
\[
\square(a, b) \rightarrow \square(a \oplus c, b \oplus c) \quad \text{(Aggregation)}
\]

This predicate may be related to unary dialectical negation operations in a simple or complex way. One possibility (that leads to \(N4\)) is the following (for two predicates \(P\) and \(Q\)):

\[
\square(a, b) \iff P(a) \implies \neg Q(b)
\]

The following example illustrates the need for the definitions.

Let \(\{x_n\}\) be a sequence of real numbers. In contexts where reasoning proceeds from the concrete to the general, let

- \(A\) be the statement that *Limit of the sequence is not conceivable*
- \(B\) be the statement that *Statement A is conceivable*
- \(C\) be the statement that *As B is true, the limit of the sequence is conceivable.*
A is dialectically opposed to C, but the scenario does not amount to a dialethia if the entire context has enough information of the process (of B being true) being referred to by C.

4.3 Dialectics from Classification

The intent of this example is to show that

- Strategies for classification of information can be dialectically opposed to each other,
- this information fits into the rough set paradigm, and
- does not involve dialethias.

From an abstract perspective,

- Consider a general process or phenomena C described in abstract terms $A_1, \ldots, A_f$.
- But every extension of the process in reality has additional peculiar properties.
- Suppose further that this leads to not necessarily independent classifications $C_1, C_2, \ldots, C_n$.
- Let interaction between the members of the classes lead to refinement of the classes to the categories $C_1^*, C_2^*, \ldots, C_r^*$.
- This scenario leads to instances of parthood like $PA_1C_1$ and $PC_1C_1^*$.
- With $PC_1C_1^*$ being in dialectical opposition to $PA_1C_1$.

Concrete instances of development over these lines are plentiful. In fact the historical development of any subject in the social sciences that has witnessed significant improvement over the last thirty years or so would fit under this schema. Two diverse contexts where such a dynamics may be envisioned are presented next.

The problem of estimation of income in rural agrarian households depends on how agrarian relations are managed by the model/method used [73]. All of the relations may not be clear without ground level studies over substantial periods of time. Suppose C is about estimate of poverty in a village and $A_1, A_2, A_3$ are abstract categories based on volume of monetary transaction by farmers. Some economists may use this for estimating net income and as an indicator of absence of poverty, while in reality farmers may be having negative income or the sources may not be reliable.

Suppose, based on ground level studies an improved classification $C_1, C_2, C_3, C_4, C_5$ has been arrived at based on estimates of investment,
expenditure, consumption, exchange of labor and other factors. This classification may need to be improved further to take non monetary transactions like barter of goods and resources into account. Thus $C^*_1, C^*_2, C^*_3, C^*_4, C^*_5$ may be arrived at.

$C_1$ definitely takes $A_1$ into account and the latter is a causative factor for the former. This can be expressed by the parthood $PA_1C_1$. $C_1$ is a much stronger causative factor of $C^*_1$. $PC_1C^*_1$ then is dialectically opposed to $PA_1C_1$ - this dialectics represents and also can be used to track the dynamics of the context.

The subject of lesbian sexuality in particular has progressed significantly over the last few decades and can be expressed in such abstract perspectives (see [74] and more recent literature). Women love women in different ways and this variation is significant for sub-classification. The parameters of classification relate to gender expression, sexual state variation, sexual performance, preferences in sexual interaction, routines, mutual communication, lifestyle choices, related social communication and more.

5 Dialectical Rough Sets

Dialectical approach to rough sets was introduced and a more general program was formulated in [13] by the present author. The formulation involves considerable stress on involving multiple kinds of roughly equivalent objects and the dialectical relation between them and this is reflected in the two algebraic logics proposed in the mentioned paper. The dialectical relation is the entire universe in the second semantics and has a central role in possible derivations. The point of the semantics was to include mixed kinds of objects and so ideas of contamination apply differently to the context. The essential content is repeated below (as [13] is a conference paper) and the nature of some possible parthoods involved is defined below.

A pre-rough algebra [9] is an algebra of the form

$$S = \langle S, \cap, \cup, \Rightarrow, L, \neg, 0, 1 \rangle$$

of type $(2, 2, 2, 1, 1, 0, 0)$, which satisfies:

- $\langle S, \cap, \cup, \neg \rangle$ is a De Morgan lattice.
- $\neg \neg a = a$; $L(a) \cap a = L(a)$
- $L(a \cup b) = L(a) \cup L(b)$; $\neg L\neg L(a) = L(a)$
- $LL(a) = L(a)$; $L(1) = 1$; $L(a \cap b) = L(a) \cap L(b)$; $\neg L(a) \cup L(a) = 1$
• If $L(a) \cap L(b) = L(a)$ and $\neg L(-a \cap b) = \neg L(-a)$ then $a \cap b = a$
This is actually a quasi equation.
• $a \Rightarrow b = (\neg L(a) \uplus L(b)) \cap (L(-a) \uplus \neg L(-b))$.

A completely distributive pre-rough algebra is called a *rough algebra*. In all these algebras it is possible to define an operation $\diamond$ by setting $\diamond(x) = \neg L\neg(x)$ for each element $x$.

In this semantics explicit interaction between entities in the rough semantic domain and entities in the classical semantic domain is permitted. The requirement of explicit interaction is naturally tied to objects having a dual nature in the relatively *hybrid* semantic domain. Consequently an object’s existence has dialectical associations. In application contexts, this approach can also be useful in enriching the interaction within the rough semantic domain with additional permissible information from the classical semantic domain. It is also relevant in problems of *relativization of semantics* like the question of equivalence of the rough semantics afforded by an approximation space relative another to the relative semantics afforded by another distinct pair of approximation spaces. But of course, this approach is not intended to be compatible with contamination.

In *rough algebra semantics* it is not possible to keep track of the evolution of rough objects relative to the classical semantics suggested by the Boolean algebra with approximation operators. Conversely in the latter it is not possible to form rough unions and rough intersections relative to the *rough algebra* semantics. These are examples of relative distortions. The CERA semantics (*concrete enriched pre-rough algebra*), which is developed in the next subsection can deal with this, but distortions relative to *super rough semantics* ([8]) are better dealt with CRAD (*concrete rough dialectical algebra* introduced in the last subsection) like methods.

For more on these considerations, the reader is referred to [6] and in the *three-valued perspective* to [48, 47]. In [48], a three-valued sub domain and a classic sub domain formed by the union of the singleton granules of the classification are identified within the the rough domain itself.

Jaskowski’s discussive logic is an example of a subvaluationary approach in that it retains the non truth of contradictions in the face of truth-gluts. Connections with pre-rough, rough algebras and rough logics are well known (see [75]). In particular, Pawlak’s five valued rough logic $R_I$ (see [76]) and $L_R$ ([75]) are not dialethic: though it is possible to know that something is roughly true and roughly false at the same time, it is taken to be *roughly inconsistent* as opposed to being just *true* or *roughly true*. This rejection definitely leads to rejection of other reasoning that leads to it as conclusion. Importantly a large set of logics intended
for capturing rough semantics are paraconsistent and make use of skewed forms of conjunction and disjunction. It can be argued that the latter feature is suggestive of incomplete development of the logics due to inconsistencies in the application of the underlying philosophy (see [77] for example). The 4-valued DDT (see [64]) addresses some of these concerns with a justification for 3-valuedness in some semantics of classical RST. The NMatrix based logic ([78]) provides a different solution by actually avoiding conjunction and disjunction operations (it should, however, be noted that conjunctions and disjunctions are definable operations in the NMatrix based logic). The ability of objects to approximate is called in super rough semantics ([8]) by the present author. These concerns become more acute in the semantics of more general rough sets.

In summary, the main motivations for the approach of the present section are

- to provide a framework for investigating relative distortions introduced by different theories - this is in sharp contrast to the contamination reduction approach [68] of the present author,
- to improve the interface between rough and classical semantic domains in application contexts,
- to investigate relativisation of semantics in the multi source general rough contexts (or equivalently in the general dynamic approximation contexts) - in [79] a distinct semantic approach to the problem is developed by the present author,
- address issues relating to truth and parthood at the semantic level,
- and develop a dialectical logic of rough semantics.

The nature of parthood was not considered in the context by the present author at the time of writing [13]. It is also considered here to specify the nature of oppositions and potential diagrams of opposition.

5.1 Enriched Classical Rough Set Theory

Let $S = \langle S, R \rangle$ be an approximation space with $S$ being a set and $R$ an equivalence. $S$ will be used interchangeably with $S$ and the intended meaning should be clear from the context. If $A \subseteq S$, $A^l = \bigcup\{[x] ; [x] \subseteq A\}$ and $A^u = \bigcup\{[x] ; [x] \cap A \neq \emptyset\}$ are the lower and upper approximation of $A$ respectively. If $A, B \in \wp(S)$, then $A$ is roughly equal to $B$ ($A \approx B$) if and only if $A^l = B^l$ and $A^u = B^u$. $[A]$ shall be the equivalence class (with respect to $\approx$) formed by $A \in \wp(S)$.

The proposed model may be seen as an extension of the pre-rough and rough algebra models in [9]. Here the base set will be taken to be
\( \varphi(S) \cup \varphi(S) \mid \approx \) as opposed to \( \varphi(S) \mid \approx \) (used in the construction of a rough set algebra). The new operations \( \oplus, \odot \) introduced below are intended to correspond to generalized aggregation and commonality operations in the mixed domain. This is not possible in classical rough sets proper.

**Definition 5.** On \( Y = \varphi(S) \cup \varphi(S) \mid \approx \), the operations \( \subseteq, \odot, \odot, \Rightarrow, \rightarrow, \sim \) will be defined as follows: (it is assumed that the operations \( \cup, \cap, \cap, \cap, L, M, \neg, \Rightarrow \) are available on \( \varphi(S) \) and \( \varphi(S) \mid \approx \) respectively. Further

\[
\tau_1 x \Leftrightarrow x \in \varphi(S) \text{ and } \tau_2 x \Leftrightarrow x \in \varphi(S) \mid \approx .
\]

- \( \mathbb{L} x = \begin{cases} x^l & \text{if } \tau_1 x \\ L x & \text{if } \tau_2 x \end{cases} \)

- \( \odot x = \begin{cases} x^u & \text{if } \tau_1 x \\ \neg L \neg x & \text{if } \tau_2 x \end{cases} \)

- \( x \oplus y = \begin{cases} x \cup y & \text{if } \tau_1 x, \tau_1 y \\ [x \cup (\bigcup_{z \in y} z)] & \text{if } \tau_1 x, \tau_2 y \\ [\bigcup_{z \in x} z] \cup y & \text{if } \tau_2 x, \tau_1 y \\ x \cup y & \text{if } \tau_2 x, \tau_2 y \end{cases} \)

- \( x \odot y = \begin{cases} x \cap y & \text{if } \tau_1 x, \tau_1 y \\ [x \cap (\bigcap_{z \in y} z)] & \text{if } \tau_1 x, \tau_2 y \\ [\bigcap_{z \in x} z] \cap y & \text{if } \tau_2 x, \tau_1 y \\ x \cap y & \text{if } \tau_2 x, \tau_2 y \end{cases} \)

- \( \sim x = \begin{cases} x^c & \text{if } \tau_1 x \\ \neg x & \text{if } \tau_2 x \end{cases} \)

- \( x \sim y = \begin{cases} x \cup y^c & \text{if } \tau_1 x, \tau_1 y \\ \bigcup_{z \in y} [x \cup z^c] & \text{if } \tau_1 x, \tau_2 y \\ x \quad \Rightarrow \quad y & \text{if } \tau_2 x, \tau_2 y \\ \bigcup_{z \in x} (z \cup y^c) & \text{if } \tau_2 x, \tau_1 y \end{cases} \)
Definition 6. In the above context a partial algebra of the form

\[ W = \langle \wp(S) \cup \wp(S) | \approx, \sim, \oplus, \odot, \star, 0, 1, \bot, \top \rangle \]

of type \((1, 1, 2, 2, 1, 1, 0, 0, 0, 0)\) will be called a concrete enriched pre-rough algebra (CERA) if a pre-rough algebra structure is induced on \(\wp(S) | \approx\). Concrete enriched rough algebras can be defined in the same manner. If the approximation space is \(X\), then the derived CERA will be denoted by \(\mathfrak{M}(X)\). Note that the two implication-like operations are definable in terms of the others.

Proposition 2. CERAs are well defined because of the representation theory of pre-rough algebras.

Theorem 4. A CERA satisfies all of the following: (The first two conditions essentially state that the \(\tau_i\)s are abbreviations)

\[
\begin{align*}
(x \sim x &= \top \iff \tau_1 x) \quad \text{(type-1)} \\
(\neg x &= \neg x \iff \tau_2 x) \quad \text{(type-2)} \\
\sim\sim x &= x; \ Lx = Lx; \ \star Lx &= Lx \quad \text{(ov-1)} \\
Lx \oplus x &= x; \ Lx \odot x = Lx; \ \star x \odot x &= \star x; \ \star x \odot x &= x \quad \text{(ov-2)} \\
L\star x &= \star x; \ x \odot x = x; \ x \odot x &= x \quad \text{(ov-3)} \\
(\tau_1 x \longrightarrow \sim x \odot x = \top); \ (\tau_2 x \longrightarrow \sim Lx \oplus Lx = 1) \quad \text{(qov-1)} \\
\sim \bot &= \top; \ \sim 0 &= 1 \quad \text{(qov-2)} \\
x \odot (x \odot (x \odot y)) &= x \odot (x \odot y); \ x \odot (x \odot (x \odot y)) = x \odot (x \odot y) \quad \text{(u1)} \\
x \odot y &= y \odot x; \ x \odot y = y \odot x \quad \text{(u2)} \\
(\tau_i x, \tau_i y, \tau_i z \longrightarrow x \odot (y \odot z) = (x \odot y) \odot z); i = 1, 2 \quad \text{(ter(i1))} \\
(\tau_i x, \tau_i y, \tau_i z \longrightarrow x \odot (y \odot z) = (x \odot y) \odot (x \odot z)); i = 1, 2 \quad \text{(ter(i2))} \\
(\tau_i x, \tau_i y, \tau_i z \longrightarrow x \odot (y \odot z) = (x \odot y) \odot z); i = 1, 2 \quad \text{(ter(i3))} \\
(\tau_i x, \tau_i y \longrightarrow x \odot (x \odot y) = x, \sim (x \odot y) = \sim x \odot \sim y); i = 1, 2 \quad \text{(bi(i))} \\
(\tau_1 x, \tau_2 y, x \odot y = y \longrightarrow \star x \odot y = y) \quad \text{(bm)} \\
(\tau_1 x, (1 \odot x = y) \lor (y = x \odot 0) \longrightarrow \tau_2 y) \quad \text{(hra1)}
\end{align*}
\]
**Definition 7.** An abstract enriched pre-rough partial algebra (AERA) will be a partial algebra of the form

\[ S = \langle \mathfrak{X}, \neg, \bowtie, \oslash, \lozenge, \mathfrak{L}, 0, 1, \bot, \top \rangle \]

(of type \((1, 1, 2, 2, 1, 1, 0, 0, 0, 0)\)) that satisfies:

- **RA** \(\text{dom}(\neg)\) along with the operations \((\bowtie, \oslash, \lozenge, \mathfrak{L}, \sim, 0, 1)\) restricted to it and the definable \(\Rightarrow\) forms a pre-rough algebra,
- **BA** \(\mathfrak{X} \setminus \text{dom}(\neg)\) with the operations \((\bowtie, \oslash, \lozenge, \mathfrak{L}, \sim, \top, \bot)\) restricted to it forms a topological boolean algebra (with an interior and closure operator),
- **IN** Given the definitions type-1, type-2, all of \(u_1, u_2, \text{ter}(ij), \text{bi}(i), \text{bm}\) and \(hra\) hold for all \(i, j\).

Note that AERAs are actually defined by a set of quasi equations.

**Theorem 5.** Every AERA \(S\) has an associated approximation space \(X\) (up to isomorphism), such that the derived CERA \(\mathfrak{W}(X)\) is isomorphic to it.

**Proof.** Given \(S\), the topological boolean algebra and the pre-rough algebra part can be isolated as the types can be determined with \(\neg, \bowtie\) and the 0-place operations. The representation theorems for the parts can be found in [80] and [9] respectively.

Suppose \(\mathfrak{W}(Y)\) is a CERA formed from the approximation space \(Y\) (say) determined by the two parts. If \(Y\) is not isomorphic to \(X\) as a relational structure, then it is possible to derive an easy contradiction to the representation theorem of the parts.

Suppose \(\mathfrak{W}(X)\) is not isomorphic to \(S\), then given the isomorphisms between the parts, at least one instance of \(x \bowtie' y \neq x \bowtie y\) or \(x \oslash' y \neq x \oslash y\) (for a type-1 \(x\) and a type-2 \(y\) with \('\) denoting the interpretation in \(\mathfrak{W}(X)\)). But as type-1 elements can be mapped into type-2 elements (using 0 and \(\bowtie\)), this will result in a contradiction to the representation theorem of parts. \(\square\)

### 5.2 Dialectical Rough Logic

A natural dialectical interpretation can be assigned to the proposed semantics. A subset of the original approximation space has a dual interpretation in the classical and rough semantic domain, and an object in the rough semantic domain relates to a set of objects in the classical semantic domain. But in the above semantics it is not possible to
transform objects in the rough domain to objects in the classical domain. So the universe can be taken to be the set of tuples of the form \( \{(x, 0 \oplus x) : \tau_1 x \} \cup \{(y, x) : \tau_2 y, \tau_1 x, x \oplus 0 = y\} = K (x, y \text{ being elements of a CERA}) \). This universe is simply the described dialectical relation between objects in the two domains mentioned above.

**Definition 8.** A concrete rough dialectical algebra (CRAD) will be a partial algebra on \( K \) along with the operations \(+, \cdot, \mathcal{L}^*, \sim, \} \) and 0-place operations \((\top, 1), (1, \top), (0, \perp), (\perp, 0)\) defined by (\# is an abbreviation for Else Undefined)

\[
(a, b) + (c, e) = \begin{cases} 
(a \oplus c, b \oplus e) & \text{if } \tau_1 a, \tau_1 c \\
(a \oplus c, e \oplus a) & \text{if } \tau_1 a, \tau_2 c, (e \oplus a) \oplus 0 = a \oplus c, \# \\
(a \oplus e, c \oplus b) & \text{if } \tau_2 a, \tau_1 c, (c \oplus b) \oplus 0 = a \oplus e, \#
\end{cases}
\]

\[
(a, b) \cdot (c, e) = \begin{cases} 
(a \odot c, b \odot e) & \text{if } \tau_1 a, \tau_1 c \\
(a \odot c, e \odot a) & \text{if } \tau_1 a, \tau_2 c, (e \odot a) \odot 0 = a \odot c, \# \\
(a \odot e, c \odot b) & \text{if } \tau_2 a, \tau_1 c, (c \odot b) \odot 0 = a \odot e, \#
\end{cases}
\]

\[\mathcal{L}^*(a, b) = (\mathcal{L}a, \mathcal{L}b) \; ; \sim (a, b) = (\sim a, \sim b)\]

### 5.3 Parthoods

In CERA related contexts, the universe has the form

\[ W = \wp(S) \cup \wp(S)|_\approx \]

the most natural parthoods are ones defined from the aggregation and commonality operators. Parthoods can also be based on information content and ideas of consistency of comparison.

**Definition 9.** The following parthoods can be defined in the mixed semantic domain corresponding to CERA on \( W \)

\[ P_{\oplus} ab \leftrightarrow [a] \leq [b] \quad \text{(Roughly Consistent)} \]

\[ P_{\odot} ab \leftrightarrow a \oplus b = b \quad \text{(Additive)} \]

\[ P_{\otimes} ab \leftrightarrow a \odot b = a \quad \text{(Common)} \]

\( \leq \) is the lattice order used in the definition of pre-rough algebras. Note that the operations \( \oplus, \odot \) are not really required in the definitions of the last two parthoods which can equivalently be defined using the associated
cases. This is important as one of the goals is to count the objects in specialized ways to arrive at semantics that make sense [6].

In the definition of the base set $K$ of CRAD, $K$ is already a dialectic relation. Still definitions of parthoods over it can make sufficient sense.

**Definition 10.** By the natural parthood relation $P_R$ on $K$ will be meant the following relation:

$$P_R a b \leftrightarrow [e_1 a] \leq [e_1 b] \& [e_2 a] \leq [e_2 b],$$

where the operation $e_i$ gives the $i$th component for $i = 1, 2$.

Admittedly the above definition is not internal to $K$ as it refers to things that do not exist within $K$ at the object level of reasoning.

6 Semantic Frameworks

Since objects are assumed to be definable if at all by sets of attributes, all considerations can be in terms of attributes. For the considerations of the section on counting in antichains on distribution of rough objects and on finite counting to be valid, a minimal set of assumptions are necessary. These are as follows (For other sections, none of FO3, RO2, CO2 will be needed):

- $S$ is a granular operator space. (FO1)
- $S = \wp(S)$. (FO2)
- $(S) = n < \infty$. (FO3)
- $C \subseteq S$ is the set of crisp objects. (CO1)
- $(C) = k$. (CO2)
- $R \subseteq S$ is the set of rough objects defined in a way. (RO1)
- $(R) = n - k$. (RO2)
- $R \cap C = \emptyset$. (RC1)
- there exists a map $\varphi : R \mapsto C^2$. (RO3)
- $(\forall x \in R)(\exists a, b \in C) \varphi(x) = (a, b) \& a \subset b$. (RC2)

Note that

- it makes sense to replace the equality in FO2 with $\subseteq$ and this is done in a separate paper by the present author,
• no further assumptions are made about the nature of \( \varphi(x) \). It is not required that
\[
\varphi(x) = (a, b) \& x^l = a \& x^u = b,
\]
though this happens often and
• RO3 without RC2 can accommodate situations where rough objects are contingently approximated by sets of crisp objects.

The set of crisp objects is necessarily partially ordered. In specific cases, this order may be a lattice, distributive, relatively complemented or Boolean order. Naturally the combinatorial features associated with granular operator space depend on the nature of the partial order and results in situations that are way more involved than the situation encoded by the following simple proposition.

**Proposition 3.** For a fixed value of \( \#(S) = n = \#(\varphi(S)) \) and \( \#(C) = k \), \( R \) must be representable by a finite subset \( K \subseteq C^2 \setminus \Delta_C \), \( \Delta_C \) being the diagonal in \( C^2 \).

The two most extreme cases of the ordering of the set \( C \) of crisp objects correspond to \( C \) forming a chain and \( C \setminus \{0, 1\} \) forming an anti-chain. It must be mentioned that the following valuations and indices are usable for deciding on strategies for counting in the context of decomposition of \( S \) into antichains.

**Definition 11.** When \( S \) is finite, this definition will be admissible.

For \( a, b \in R \), let
\[
\nu(a, b) = \begin{cases} 
0, & \text{if } a^l = b^l \& a^u = b^u \\
1, & \text{if } a^l \neq b^l \& a^u \neq b^u \\
\frac{1}{n}, & \text{if } a^l \neq b^l \& a^u = b^u \\
\frac{1}{n}, & \text{if } a^l = b^l \& a^u \neq b^u.
\end{cases}
\]

By the rough distribution index of \( R \) will be meant the sum
\[
\iota(R, C) = \sum_{a, b \in R} \nu(a, b)
\]
and the relative rough distribution index of \( R \) shall be
\[
\iota^*(R, C) = \frac{\iota(R, C)}{(n - k)^2}
\]
The choice of the transcendental numbers $e$ and $\pi$ in the definition is mainly for keeping track of the number of cases.

**Theorem 6.** When $S$ is finite,

$$0 \leq \iota(R, C) \leq (n - k)^2$$

**Proof.** The lower bounds have been obtained on the assumption that the non crisp elements are mutually roughly equal.

\[ \square \]

The first measure gives an idea of the extent of distribution of non crisp objects over the distribution of the crisp objects and the relative measure is a bad approximation of the idea of seeking comparison across distributions of crisp objects.

### 6.1 Examples of Parthood

Parthood can be defined in various ways in the framework of rough sets in general and granular operator spaces in particular. The rough inclusion defined earlier in the background section is a common example of parthood. Some others have been introduced in Sec 5. The following are more direct possibilities:

- $\mathcal{P}_{ab} \leftrightarrow a^l \subseteq b^l$ (Very Cautious)
- $\mathcal{P}_{ab} \leftrightarrow a^l \subseteq b^u$ (Cautious)
- $\mathcal{P}_{ab} \leftrightarrow a^l \subseteq b^u \setminus b^l$ (Lateral)
- $\mathcal{P}_{ab} \leftrightarrow a^u \subseteq b^u$ (Possibilist)
- $\mathcal{P}_{ab} \leftrightarrow a^u \subseteq b^l$ (Ultra Cautious)
- $\mathcal{P}_{ab} \leftrightarrow a^u \subseteq b^u \setminus b^l$ (Lateral+)
- $\mathcal{P}_{ab} \leftrightarrow a^u \setminus a^l \subseteq b^u \setminus b^l$ (Bilateral)
- $\mathcal{P}_{ab} \leftrightarrow (\forall g \in \mathcal{G})(g \subseteq a \rightarrow g \subseteq b)$ (G-Simple)

All of these are valid concepts of parthoods that make sense in contexts as per availability and nature of information. Very cautious parthood makes sense in contexts in which cost of misclassification is high or the association between properties and objects is confounded by the lack of clarity in the possible set of properties. G-Simple is a version that refers to granules alone and avoids references to approximations.
The above mentioned list of parthoods can be more easily found in decision making contexts in practice. Consider, for example, the nature of diagnosis and treatment of patients in a hospital in war torn Aleppo in the year 2016. The situation was characterized by shortage of medical personnel, damaged infrastructure, large number of patients and possibility of additional damage to infrastructure. Suppose
- patient B has bone fractures and a bullet embedded in their arm and patient C has bone fractures and shoulder dislocation due to a concrete slab in free fall,
- only one doctor is on duty,
- and suppose that only one of the two patients can be treated properly due to resource constraints
- the doctor in question has access to some precise and some unclear diagnostic information on medical conditions,
- all of this data is not in table form.

In the situation, decision making can be based on available information and principles like
- Allocate resources to the patient who is definitely in the worst state - this decision strategy can be corresponded to very cautious parthoods,
- Allocate resources to the patient who seems to be in the worst state - this decision strategy can be corresponded to cautious parthoods,
- Allocate resources to the patient who is possibly in the worst state - this decision strategy can be corresponded to possibilist parthoods.
- Allocate resources to the patient who is likely to show more than the default amount of improvement - this decision strategy can be corresponded to the bilateral parthoods.
- If every symptom or unit complication that is experienced or certainly likely to be experienced by patient 1 is also experienced or is certainly likely to be experienced by patient 2, then prefer treating patient 2 over patient 1 - this decision strategy can be corresponded to the g-simple parthoods with symptoms/unit complications as granules

Parthood can be associated with both dialectic and dialethic statements in a number of ways. Cautious parthood is consistent with instances of the form $P_{ab}$ and $P_{ba}$ and is by itself a dialectic relation within the same domain of discourse. Dialectics between parthoods in different semantic domains are of greater interest and will be considered in subsequent sections.

The apparent parthood relations considered in later sections of this paper typically arise from lack of clarity in specification of properties or due to imprecision (of fuzziness). This is illustrated in the next example.
**Cold vs Influenza** Detection of influenza within 48 hours of *catching it* is important for effective treatment with anti-virals, but often patients fail to understand subtleties in distinguishing between the two. It is also not possible to administer comprehensive medical tests in a timely cost-effective way even in the best of facilities. So ground breaking insights even in restricted contexts can be useful.

The two medical conditions have similar symptoms. These may include

- Fever - as indicated by elevated temperatures,
- Feverishness - as indicated by personal experience (this may not be accompanied by fever),
- Sneezing, running nose, blocked nose,
- Headache of varying intensity,
- cough and
- body pain - this is usually a lot more intense in case of flu (but develops after a couple of days).

Clearly, in this situation, patients in the absence of confirmatory tests can believe both *instances of cold is apparently part of flu* and *instances of flu is apparently part of cold*. These statements are in dialectical contradiction to each other but no dialethias are involved. Cold and flu are also in dialectical contradiction to each other. The usefulness of the formalism of the last sentence is also relevant.

It is another matter that if gluity negations are permitted then *apparently part of* can as well be replaced by *part of*.

7 **Figures of Dialectical Opposition**

The scope of the counting strategies and nature of possible models can be substantially improved when additional dialectical information about the nature of the order-theoretic relation between rough and crisp objects is used. In the literature on generalizations of the square of opposition to rough sets as in [12,81] it is generally assumed that realizations of such relations is the end result of semantic computations. This need not necessarily be so for reasons that will be explained below. In classical rough sets, a subset \( X \) of objects \( O \) results in a tri-partition of \( O \) into the regions \( L(X) \) (corresponding to lower approximation of \( X \)), \( B(X) \) (the boundary region) and \( E(X) \) (the complement of the upper approximation). These form a hexagon of opposition indicated in Fig.2. In more general rough sets, this diagram generalizes to cubes of opposition.

The strategy for the general case used in this paper is illustrated in Fig.3.
To think of the applicability of any dialectical generalized scheme of
the square of opposition and the hexagon of opposition, some idea of
parthood related ordering in the form of the following relations can suffice
(a deeper understanding of ontology is essential for understanding the
vague usage (this is explored in [18]). (Some examples have already been
provided earlier):

**AP** Is Apparently Part of: understood from class, property, expected beh-

avior, or some other perspective.

**APN** Is Apparently not Part of.

**AP0** Is Apparently Neither Part of Nor Not Part of.

**CP** Is Certainly Part of.

**CPN** Is Certainly Not Part of.

**CP0** Is Certainly Neither Part of Nor Not Part of. (This is intended to
convey uncertainty)

**AI** Is Apparently Indistinguishable from.

**CI** Is Certainly Indistinguishable from.

**AW** Is Apparently a Whole of

**AWN** Is Apparently Not a Whole of.

**AW0** Is Apparently Neither a Whole of Nor Not a Whole of.

**CW** Is Certainly a Whole of.

**CWN** Is Certainly not a Whole of.

**CW0** Is Certainly Neither a Whole of Nor Not a Whole of.
By the word apparently, the agent may be referring to the lack of models, properties possessed by the objects, relativised views of the same among other possibilities. For example, the word apparently can refer to the absence of any clear models about connections between diseases in data from a hospital chain in a single city or it can refer to problems caused by lack of data or relativisations about expected state of affairs relative to pre-existing models. Fuzzy and degree valuations of these perceptions are even less justified due to the use of approximate judgments. Predicates like AP0, CP0 are needed for handling indecision (which is likely to be happen often in practice).

As pointed out by a reviewer, AP, APN and AP0 form three-fourths of Belnap’s useful 4-valued logic[82]. It should be noted that while oppositions do not by themselves qualify as a dialectic, dialectical oppositions as a framework that can be sustained by inference procedures for progression of knowledge qualify as a dialectic.

The above set of predicates can be split into the following subsets of interest:

- **Pure Apparence**: AP, APN, AP0, AW,AWN, AW0.
- **Pure Certainty**: CP, CPN, CP0, CW, CWN, CW0.
- **Mixed Apparence**: AP, APN, AP0, AW,AWN, AW0, AI.
- **Mixed Certainty**: CP, CPN, CP0, CW, CWN, CW0, CI.
• Pure: AP, APN, AP0, AW, AWN, AW0, CP, CPN, CP0, CW, CWN, CW0.
• Mixed: Union of all of the above.

Before proceeding further it is necessary to fix the philosophical concepts of *Contradiction, Contrariety, Subcontrariety and Subalternation* because the literature on the concepts is vast and there is much scope for varying the meaning of the concepts (see for example [11,10]). One way of looking at the connection between truth value assignments and sentences, necessary for the diagram to qualify in the square of opposition (generalized) paradigm, is illustrated in Table 1, 2, 3, 4 below (NP means the assignment is not possible):

| α | β | CY(α,β) |
|---|---|---------|
| T | T | NP      |
| T | F | T       |
| F | T | T       |
| F | F | T       |

*Table 1. Contrariety*

| α | β | SCY(α,β) |
|---|---|---------|
| T | T | T       |
| T | F | T       |
| F | T | T       |
| F | F | NP      |

*Table 3. SubContrariety*

| α | β | AN(α,β) |
|---|---|---------|
| T | T | T       |
| T | F | NP      |
| F | T | T       |
| F | F | T       |

*Table 4. SubAlternation*

The $PQ$ semantics tries to take a simplified view of the situation. It may appear that the main problem with the proposal is the lack of suitable logical operators. But this drawback is not likely to be that significant in the counting based approach being adopted in the present paper. The $PQ$ approach in question is to look for the answer to the questions:

• TT: Can the sentences be true together?
• FT: Can the sentences be false together?

After finding those answers, categories are worked out according to Table 5.

But dialectical contradiction requires additional categories that relate to the following questions:
• **Dialethia**: Can any one of the two statements be both true and false together? (Let $\delta(A)$ be the statement that $A$ is both false and true together).

• **Bi-Dialectic**: Is either of the statements in dialectical opposition to the other? (Let $\mathcal{D}(A,B)$ be the statement that $A$ is dialectically opposed to $B$).

• **Dialectic**: Is either of the statements a statement expressing dialectical opposition? (Let $\beta(A)$ be the statement that $A$ expresses dialectical opposition with $\beta$ being a particular associated predicate).

The above realization of the concept of dialethia is pretty clear for implementation, but the latter two forms of dialectic depend on the choice of predicates and so many interpretations would be possible. In a typical concrete case, the parthood(s), the dialectical predicate and figure of opposition should be defined in order to obtain concrete answers.

The Question-Answer Semantic approach (QAS) of [11] is a relatively more complete approach in which the *sense* of a sentence $\alpha$ is an ordered set of questions $Q(\alpha) = \langle q_1(\alpha),\ldots,q_r(\alpha) \rangle$ and its *reference* is an ordered set of answers $A(\alpha) = \langle a_1(\alpha),\ldots,a_r(\alpha) \rangle$. These answers can be coerced to binary form (with answers being *Yes* or *No*).

If the dialectical approach of the present paper is extended to QAS approach, then the number of possible questions (like can question A and B be true together?) becomes very large and suitable subsets that are as efficient as the whole would be of interest. The possibilities are indicated in Table.6.

|   | 0 | 1 |
|---|---|---|
| $\mathcal{D}$ | $\beta$ | $\beta$ |
| $\delta$ | $\delta$ | $\delta$ |

**Table 6. Potential Combinations**

Connectives and operations can be involved in the definition of the predicates, but in the general case the meta concept of *suitable subsets* can
only be roughly estimated and not defined unambiguously. The answer set is not a big issue for handling even the apparent parthood related statements and so after fixing the necessary subsets of the question set, counting procedures can be initiated.

From the point of view of the nature of the truth values, two approaches to the problem can be adopted:

1. keep the concept of truth and falsity fixed and attempt suitable definitions of oppositions and
2. to allow for variation of truth and falsity values. This in general would amount to deviating further from the classical paradigm.

7.1 Classical Case-1: Fixed Truth

When the concept of truth is not allowed to vary beyond the set \{T, F\}, then it is apparently possible to handle the cases involving apparent parthood without special external rules. A natural question that arises in the context of the certain parthoods is about the admissibility of truth values. These aspects are considered in this subsection.

One instructive (but not exhaustive) way is to read CP\(ab\) as all \(a\) with property \(\pi(a)\) and none of properties in \(\neg\pi(a)\) (in the domain of discourse) are part of any \(b\) with property \(\pi(b)\). As a consequence CP\(Nab\) is all \(a\) with property \(\pi(a)\) and none of properties in \(\neg\pi(a)\) (in the domain of discourse) are not part of any \(b\) with property \(\pi(b)\).

In rough sets, the association of objects with properties happen only when mechanisms of associations are explicitly specified or are specifiable. There is much freedom to choose from among different mechanisms of associations in a abstract perspective. In praxis, these choices become limited but rarely do they ever become absent. Implicit in all this is the assumption of stable choice among possible mechanisms of associations. The stability aspect is an important direction of research

If truth tables for determining the nature of opposition is attempted using ideas of state resolution (instead of connectives) then perplexing results may happen. The choice of connectives is in turn hindered by an excess of choice. So the minimalist perspective based on two questions or the QS\(-\)type approach should be preferred.

The following two theorems require interpretation.

**Theorem 7.** The truth tables corresponding to two of the pairs formed from AP, AP\(N\), AP\(0\) have the form indicated in Table.7-8 (\(P \ast Q\) (abbreviated in tables by *) means the resolution of the state relating to \(P\) and \(Q\)).
\begin{center}
\begin{tabular}{ccc}
\hline
AP_{pq} & AP_{pq} & * \\
\hline
T & T & IN \\
T & F & T \\
F & T & T \\
F & F & IN \\
\hline
\end{tabular}
\end{center}

Table 7. Contradiction?

\begin{center}
\begin{tabular}{ccc}
\hline
\begin{tabular}{ccc}
\hline
AP_{pq} & AP_{pq} & * \\
\hline
T & T & IN \\
T & F & T \\
F & T & T \\
F & F & IN \\
\hline
\end{tabular}
\end{tabular}
\end{center}

Table 8. Contradiction?

(IN is an abbreviation for indeterminate.)

\textit{Proof}. The proof is direct, the interpretation is however open and it is possible to read both tables as corresponding to contradiction. \qed

\textbf{Theorem 8.} The truth tables corresponding to the pairs formed from CP, CPN, CP0 and the pair CP, CI have the form indicated in Table.9-12 (NP abbreviates for not possible):

\begin{center}
\begin{tabular}{ccc}
\hline
CP_{pq} & CP_{pq} & * \\
\hline
T & T & NP \\
T & F & T \\
F & T & T \\
F & F & NP \\
\hline
\end{tabular}
\end{center}

Table 9. Contradiction

\begin{center}
\begin{tabular}{ccc}
\hline
CP_{pq} & CP_{pq} & * \\
\hline
T & T & NP \\
T & F & T \\
F & T & T \\
F & F & T \\
\hline
\end{tabular}
\end{center}

Table 10. Contrariety

\begin{center}
\begin{tabular}{ccc}
\hline
CP_{pq} & CP_{pq} & * \\
\hline
T & T & NP \\
T & F & T \\
F & T & T \\
F & F & NP \\
\hline
\end{tabular}
\end{center}

Table 11. Contradiction

\begin{center}
\begin{tabular}{ccc}
\hline
C1_{pq} & CP_{pq} & * \\
\hline
T & T & T \\
T & F & NP \\
F & T & T \\
F & F & T \\
\hline
\end{tabular}
\end{center}

Table 12. Subalternation

\textit{Proof}. The proof consists in checking the possibilities by cases. In the table for CP and CP0, the last line is justified because no possibilities are covered by the last column.

From the safer (and questionable) two question framework, the above two theorems have the following form:

\textbf{Theorem 9.} The answers to the two simultaneity (Sim) questions for the pairs formed from AP, APN, AP0 are in Table.13,14.
\( NP \) is an abbreviation for not possible.

**Proof.** In the table for AP, AP0, TT discloses all possibilities and therefore yields T. Other parts are not hard to prove.

**Theorem 10.** The simultaneity data corresponding to the pairs formed from CP, CPN, CP0 are in Table. 15-18.

**Proof.** In Table 16, for example, the question is can both CPab be false and CP0ab be false? As the situation is impossible, NP is the result.

### 7.2 Case-2: Pseudo Gluts

A minimalist use of assumptions on possible grades of truth in the cases admitting apparent parthood leads to the following diagram of truth values. The figure is biased against falsity because in the face of contradiction agents are expected to be truth seeking - this admittedly is a potentially contestable philosophical statement.

Reading of truth tables in relation to state transition based conjunction is also relevant for dialectical interpretation. But these are not handled by the above tables and will be part of future work.
For the dialectical counting procedures introduced in the next section, the basic contexts are assumed to be very minimalist and possibly naive. In these some meta principles on aggregation of truth can be useful or natural. The states of truth mentioned in Fig. 4 relate to the following meta method of handling apparent truth. These will be referred as Truth State Determining Rules (TSR). In the rules \( \alpha, \beta \) are intended in particular for formulas of the form \( \mathbf{P}ab \) and variants.

Truth State Determining Rules

- If \( \alpha \) is apparently true and \( \beta \) supports it, then \( \alpha \) becomes more true.
- If \( \alpha \) is apparently true and \( \beta \) opposes it, then \( \alpha \) becomes less true.
- If \( \alpha \) is less true (than true is supposed to be) and \( \beta \) opposes it, then \( \alpha \) becomes even less true.
- If \( \alpha \) is apparently false and \( \beta \) opposes it, then \( \alpha \) becomes less false.
In the figure, \( T \) denotes an intermediate truth value that can become stronger \( T^* \), or weaker \( T^\ominus \). This is because operators (apparently) like \textit{less} and \textit{even less} are available.

The above list of rules can be made precise using the distance between vertices in the graph and thus it would be possible to obtain truth values associated with combinations of sentences involving apparent parthood alone.

8 Dialectical Counting

The above dialectical approaches motivate few methods of counting that permit identification of possible isomorphisms in the sense defined below. Those are among the many structures and substructures of interest. \textit{Few} because partials (or fragments) of counting processes can be used in limited ways. Some of these are

- Counting by Dialectical Mereology: This method of counting is intended to be based on the principle that the mereological relation of the object being counted with its predecessors should determine its count and the enumeration should be on convex regular polygons, polyhedrons or generalizations thereof (polytopes) of dialectical and classical opposition.
- Counting by Threes: the relation of the object being counted with its predecessor and the mereological relation of the object with its successor should determine its count - this approach admits of many variations,
- Counting by Reduction to Discernibility: It is possible to count taking increasing scopes of discernibility into account.

In the counting contexts of [6], a pair of integers under contextual rules suffices to indicate the \textit{number} associated with the element in the instance of the counting scheme under consideration. This is because those counting processes basically use a \textit{square of discernibility} with the statements at the vertices being of the following form:

- \texttt{IS.NOT}(a, b) meaning \( a \) is not \( b \).
- \texttt{IS}(a, b) meaning \( a \) is identical with \( b \).
- \texttt{IND}(a, b) meaning \( a \) is indiscernible from \( b \).
- \texttt{DIS}(a, b) meaning \( a \) is discernible from \( b \).
For more general dialectical counting contexts, as in this research, triples of integers suffice because of geometrical properties (see Fig. 5). Typically while counting a finite sequence of elements \( \{x_i\} \) subject to an indiscernibility relation or some other process (as in [6]), three numbers \( o, p, q \) can be associated with each element \( x_i \). The position number \( q = i \), while \( p \) is the position of the last break point from which the count has been in the natural order up to \( o \) - this will be termed the \( p \)-contiguous count. An example from [6], due to the present author, is this counting \( \{1, 1, 2, 1, 3, 2, 4, 1, 4, 3, 1, 5, 2, 5, 1, 6, 2, 6\} \).

The triple corresponding to the second, third and last element are \((2, 1, 2), (1, 2, 3)\) and \((2, 6, 12)\) respectively. In the next definition a name is given to such triples.

**Definition 12.** By a dialectical triple will be meant a triple of positive integers of the form \((o, p, q)\) with \( o \) being the \( p \)-contiguous count at \( q \) (in the counting scheme under consideration). The components will also be referred to as the contiguous, break and position numbers respectively in that order.

**Counting by Dialectical Mereology**

This can be seen as a direct extension of the discernibility based counting of objects considered in [6] in that the methods extend over the types permitted by the mereological assumptions. So this essentially means counting based on an awareness of the position of the successors and predecessors on the polytope associated.
As in case of the discernibility based counting, distinct principles of counting can be integrated and lead to a number of variants. These include the concepts of immediate predecessors, history based perceptions and full history based perceptions. As the intended meaning associated with vertices of the polytopes/polyhedrons can be difficult to remember, the abbreviations used above will be followed. In the immediate predecessor way of counting, the essential steps would be as below.

- Start with two objects \(a, b\).
- Form the parthood related statement \(\Phi(a, b)\) relevant for the figure (here it is implicit that the most relevant polytope has already been figured out by trial). For example \(\Phi(a, b)\) may be the statement that \(a\) is not part of \(b\).
- Assign the tuple \((1, 1, 1)\) to \(a\).
- Based on truth values and dialectical contradiction assign count of \(b\) in the form \((p, q, r)\).
- For the next element \(c\), consider \(\Phi(b, c)\) and proceed as in the previous step.
- Stop when every element occurs in at least one of the \(\Phi(u, v)\).

### Counting by Threes

This method of counting has connections with deep philosophical questions and pragmatic resolutions thereof as reflected in axiomatic systems of mereology. The counting is not invariant across the latter.

- For any two distinct objects \(a, b\), let \(s(a, b)\) be the set of mereological propositional instances satisfied by \(a\) and \(b\) in that order.
- For any three objects \(a, b, c\), form the six sets of the form \(s(\alpha, \beta)\) with \(\alpha, \beta \in \{a, b, c\}\).
- Form a propositional instance \(\chi(s(\alpha, \beta))\) for each pair through the preference order on the propositional types.
- Let \([[v_1, \ldots v_6]](a, b, c)\) be the resulting hexagon.
- The essential strategy at this stage can be
  1. Assign a initial representation \(r_1\) of the form
     \[\{(1, 1, 1), (2, 1, r), (3, 1, r^*)\}\]
     to \([[v_1, \ldots v_6]](a, b, c)\) (it is assumed that the last tuple is determined by the polytope).
  2. If \(e\) is a new object, check the admissibility of the substitutions
     - \([[v_1, \ldots v_6]](a/e, b, c),\)
• $[v_1, \ldots v_6](a, b/e, c)$ and $[v_1, \ldots v_6](a, b/c/e)$.

3. If none of the three are admissible, then increase the break number by 1 and assign $(1, 2, r')$ to $e$. The value $r'$ being determined by the relation of $c$ with $e$.

4. Now consider the new triple $(e, f, g)$ and repeat the previous steps.

• Stop when every object occurs in at least one triple.

Availability of Objects

In practical situations it can happen that objects once counted may be counted again. To include this possibility, both of the above two algorithms can be suitably modified with stopping condition based on availability of resources. The modified processes will be termed low counting by threes and low counting by dialectical mereology.

A not so simple example can be constructed out of the two figures from a image doctoring case [84]. The doctoring can be identified by approximate segmentation and dialectical counting arguments.

Full History Aware Counting

The method of History based primitive counting (HPC) and History Based Perceptive Partial Counting (HPPC) were introduced by the present author in [6]. The HPC method is a way of counting discernible and indiscernible objects in which full memory of objects being counted is retained.
over temporal progression. This does not lead to antichains. For obtaining antichains a modification is proposed in this section.

The HPC method applies to any collection of objects $S$ on which a general indiscernibility relation $R$ is definable, and it need not be an equivalence. The relation Not Part of can also be seen as a general discernibility relation that is not symmetric in general. In the context of counting, antichains and maximal antichains have better dialectical properties.

For simplicity, let the collection be a finite set containing $n$ elements. This means it can be written as a sequence of the form:

$\{x_1, x_2, \ldots, x_k, \ldots, x_n\}$.

Extension to at most countable number of elements would also be possible (as in [6]). It may be noted that the axiom of choice is not really used in the counting process. The adjective primitive is intended to express the minimal use of granularity and related axioms.

**History Based Primitive Counting (HPC)**

In HPC, the relation with all preceding steps of counting is taken into account and proceeds along the following lines. When $R$ is symmetric, the following is the essential algorithm

1. Assign $f(x_1) = 1 = s^0(1_1)$.
2. If $f(x_i) = s^r(1_j)$ and $Rx_i x_{i+1}$, then assign $f(x_{i+1}) = 1_{j+1}$.
3. If $f(x_i) = s^r(1_j)$ and $(\forall k < i + 1) \neg Rx_k x_{i+1}$, then assign $f(x_{i+1}) = s^{r+1}(1_j)$.

**Primitive Counting on Antichains (PCA)**

This proposed method, aimed at antichains, can be implemented at the algorithmic level in much better ways.

- Assign $x_1$ to category $C_1$ and set $f(x_1) = 1_1$.
- If $\neg Rx_1 x_2$ then assign $x_2$ to category $C_1$ and set $f(x_2) = 2_1$ else assign $x_2$ to a new category $C_2$ and set $f(x_2) = 2_2$.
- If $f(x_2) = 2_1$, $\neg Rx_1 x_3$ and $\neg Rx_2 x_3$ then assign $x_3$ to category $C_1$ and set $f(x_3) = 3_1$. If $f(x_2) = 2_1$ and $(Rx_1 x_3$ or $Rx_2 x_3)$ then assign $x_3$ to category $C_3$ and set $f(x_3) = 1_3$. If $f(x_2) = 1_2$ and $\neg Rx_1 x_3$ then assign $x_3$ to category $C_1$ and set $f(x_3) = 2_1$. If $f(x_2) = 1_2$ and $(Rx_1 x_3$ and $\neg Rx_2 x_3)$ then assign $x_3$ to category $C_2$ and set $f(x_3) = 2_2$.
- Proceed recursively under the following conditions:
• No two distinct elements $a, b$ of a category $C_i$ satisfy $Rab$
• For distinct $i, j$, $(\forall a \in C_i)(\exists b \in C_j) Rab$

**Theorem 11.** The following hold as a result of the above procedure.

- **Objects of category $C_1$ form a maximal antichain.**
- **The objects in category $C_i$ would be enumerated by sequences of the form**
  \[1_i, 2_i, \ldots, Q_i. \text{ (with } Q_i = \#(C_i)\text{)}\]
- $\sum Q_i = n$
- **Objects of each category form an antichain.**
- **Each $x_i$ belongs to exactly one of the categories.**

**Proof.** The proof follows from the construction. In the counting of the category $C_1$, all objects of the category are mutually compared and objects outside $C_1$ are $R$-related to at least one object in $C_1$. So $C_1$ is a maximal antichain.

**History Based Primitive Counting on Antichains (HPCA)**

In this variation of PCA, objects will be permitted to belong to multiple categories - with the aspect being reflected in the end result of the counting process. This allows for a enumeration for a maximal antichain decomposition. The steps for this method of counting are as follows:

• Assign $x_1$ to category $C_1$ and set $f(x_1) = 1_1$.
• If $\neg Rx_1x_2$ then assign $x_2$ to category $C_1$ and set $f(x_2) = 2_1$ else assign $x_2$ to a new category $C_2$ and set $f(x_2) = T_2$.
• If $f(x_2) = 2_1$, $\neg Rx_1x_3$ and $\neg Rx_2x_3$ then assign $x_3$ to category $C_1$ and set $f(x_3) = 3_1$. If $f(x_2) = 2_1$ and $(Rx_1x_3$ or $Rx_2x_3)$ then assign $x_3$ to category $C_3$ and set $f(x_3) = T_3$.
• For $j > 2$, if $f(x_j) = k_1$ for $k \leq j$, $\neg Rx_jx_{j+1}$ for all $x \in C_1$ ($C_1$ at this step) then assign $x_{j+1}$ to category $C_1$ and set $f(x_{j+1}) = k_1 + 1$. If $f(x_j) = k_1$ and $(Rx_jx_{j+1}$ for at least an $x \in C_1$ at this stage) then assign $x_{j+1}$ to category $C_{j-k+1}$ and set $f(x_{j+1}) = T_{j+1}$.
• Continue till all objects have been covered for the construction of $C_1$.
The only function of $T_j$ is in locating the start point for the next step.
• Stopping Condition: $\cup C_i = S$ (at this stage) and for $i \neq j$ $C_i \notin C_j$ or all start points are exhausted.
• Find the argument for $f$ for which $j$ in $T_j$ is a minimum
• Assign $x_j$ to the category $C_2$ and set $f(x_j) = 1_2$. 
• Proceed as for $x_1$, but over the set

$$S^{(j)} = \{x_j, x_{j+1}, \ldots, x_n, x_1, x_2, \ldots, x_{j-1}\}$$

to derive all the elements of $C_2$. Continue till stopping condition is satisfied.

**Theorem 12.** HPCA yields a collection of maximal antichains $C_1, \ldots, C_q$ satisfying

$$\bigcup C_i \subseteq S.$$

**Proof.** The order structure initially assumed on $S$, restricts the maximal antichains that may be found by the HPCA algorithm.

It is possible that the order in which the discernible objects are found restricts the discernibility of objects found subsequently. So many maximal antichains are bound to be excluded. Counterexamples are easy to construct. In some cases, the decomposition of $S$ may result in maximal antichains $C_1, C_2, \ldots, C_q$.

The above proof motivates the following definition:

**Definition 13.** A total order $<$ on $S$ will be said to be HPCA coherent if and only if the HPCA procedure generates a set of maximal antichains \{\(C_i : i = 1, 2, \ldots, q\)\} such that $\bigcup C_i = S$.

Such orders provide scope for attaining maximal antichains which in turn correspond to maximal extents of discernible sequences.

**Full History Based Counting on Antichains (FHCA)**

In this method the steps shall be the same as for HPCA if the order is HPCA coherent. If not, then

• Store the maximal antichain $C_1$ as $A_1$.
• Permute the order on $S$ by the permutation $\sigma_1$.
• Compute $C_1$ as per the HPCA algorithm for the new order and store it as $A_2$ - the elements of $A_2$ being numbered as for $C_2$, that is in the form $1_2, 2_2, \ldots, Q_2$.
• Stop if $\bigcup A_i = S$
• Else permute the order on $S$ by a permutation $\sigma_2$ and repeat the above steps to obtain $A_3$ - the elements of $A_3$ being numbered as for $C_3$, that is in the form $1_3, 2_3, \ldots, Q_3$.
• Proceed till stopping criteria is satisfied.
Theorem 13. Let the set of maximal antichains obtained by the FHCA method be \( \mathcal{A} \). \( \mathcal{A} \) need not be associated with a HPCA coherent order \( < \).

Proof. A counterexample can be constructed to prove the theorem.

Aggregation and Counting

Consider the following scenario: Let \( \mathcal{O} \) be a set of objects, well defined on the object-property/attribute perspective of things. This means each object \( x \in \mathcal{O} \) is associable with a unique set attributes \( A_x \) and the existence of \( x \) is defined by \( A_x \) (in symbols \( \Delta(x, A_x) \)). This means the definition of such a structure should be along the following lines:

Definition 14. By a refined object-property system will be meant a system of the form

\[ \mathcal{P} = \langle P, \delta, \nu \rangle, \]

with \( P \subseteq \wp(A) \) being a subset of the powerset of attributes, \( \delta \) a unary relation symbol and \( \nu \) an interpretation of \( \delta \) on \( \wp(A) \). Correctly, the interpretation \( \delta^{\wp(A)} \) must be distinguished from \( \delta \), but for simplicity \( \mathcal{P} \) will be written as \( \langle P, \delta \rangle \).

Now, in this formalism objects have been replaced by subsets of attributes. Counting this kind of system cannot be equivalent to counting sets of rough and crisp objects unless the following properties are satisfied:

\[
(\forall x \in \mathcal{O})(\forall B, C \subseteq A)(\Delta(x, B) \& \Delta(x, C) \rightarrow B = C) \quad \text{(Uniqueness)}
\]
\[
(\forall x \in \mathcal{O})(\exists B \subseteq A) \Delta(x, B) \quad \text{(Existence)}
\]

Further Directions and Remarks

In this research paper all of the following have been accomplished by the present author (apart from the contribution to solving some of the inverse problem contexts)

- Formalization of possible concepts of dialectical contradiction has been done in one possible way using object level predicates.
- The difference between dialethias and dialectical contradiction has been clarified. It has been argued that dialectical contradiction need not be reducible to dialethias or be associated with glutty negation. But the latter correspond to Hegelian dialectical contradiction.
A pair of dialectical rough semantics have been developed over classical rough sets. The nature of parthood is reexamined and used for counting based approaches to semantics.

Opposition in the context of rough set and parthood related sentences is investigated and concepts of dialectical opposition and opposition are generated. Related truth tables show that the classical figures do not work as well for parthood related sentences. This extends previous work on figures of opposition of rough sets in new directions.

Possible diagrams of opposition are used for defining generalized counting process for constructive algebraic semantics. This builds on earlier work of the present author in [6].

The antichain based semantics for general rough sets that has been developed in [2,3] is supplemented with a constructive dialectical counting process and scope for using generalized negations. This has many potential applications in representation and will appear separately.

Subclasses of problems that have been motivated by the present paper relate to

- Formal characterization of conditions that would permit reduction of dialectical contradiction to dialethia.
- Construction of algebraic semantics from dialectical counting using figures of opposition.
- Construction of general rough set algorithms from the antichain based dialectics.
- Algebraic characterization of parthood based semantics as opposed to rough object based approaches. In [3], this problem has been substantially solved by the present author.
- Methods of property extraction by formal interpretation or translation across semantics - this aspect has not been discussed in detail in the present paper and will appear separately.
- Development of logics relating to glutty negation in the context of suitable rough contexts.

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