ANALYSIS AND OBSERVER DESIGN FOR THE PARABOLIC SYSTEM WEAKLY COUPLED AT THE BOUNDARY.

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Abstract. This paper discusses the mathematical properties of a recently developed mathematical model of a direct contact membrane distillation system. The model consists of two-dimensional advection diffusion system coupled at the boundary. A semi-group framework is used to analyze the model. First, the infinitesimal generator operator and its properties are studied. Then, existence and uniqueness of the solutions are investigated using the theory of operators. Some regularity results of the solution are also established. A particular case showing the diagonal property of the principal operator is studied. Moreover, based on this new partial differential model we examined the observer design for the unstable parabolic system coupled at the boundary. One of the main technical tools used in the proof is the backstepping method. Finally, the lack of the backstepping technique to design this class of parabolic system is discussed.

Key words. Direct Contact Membrane Distillation System, Well-Posedness Criteria, Parabolic System, Observer design, Backstepping method.

1. Introduction. The access to drinking water is getting more and more problematic as a result of the limited natural freshwater resources. The potable water demand is increasing due to the rapid population growth and to impacts of climate change. As a consequence, many countries rely on desalination to address their potable water demand. Indeed, desalination has been recognized as one of the most promising approaches to reduce water shortage in arid regions through the production of fresh water from seawater and saline groundwater. Among conventional yet innovative water desalination technologies is membrane distillation (MD). MD has great potential for sustainable high quality water supply. It is a low energy and effective method for water treatment and desalination. It consists of a separation process driven by temperature gradient, where hot salt water is circulated in one side (the feed side) of a hydrophobic porous membrane, while cold-fresh stream is circulated in the other side (the permeate side), thus creating a difference of partial pressure between the two sides of the membrane that constitutes the main driving force of the process. There are different configurations for MD system such as Direct Contact Membrane Distillation (DCMD), Air Gap Membrane Distillation (AGMD), Sweeping Gas Membrane Distillation (SGMD) and Vacuum Membrane Distillation (VMD). For more details about the MD technology and its configurations, the reader can refer to [10, 12, 5] and references therein. Many studies have been conducted recently by engineers to propose accurate mathematical models of MD systems and to develop efficient model-based control and monitoring strategies [9, 18, 16]. An accurate mathematical model will also allow the optimization of the system to increase its efficiency. Among the proposed MD models is a system of two dimensional advection diffusion equations coupled at the boundary. This dynamical model has been proposed for the DCMD configuration and has been validated experimentally in [6].

The aim of this paper is to study and analyze the mathematical properties of the DCMD model describing heat transfer in the DCMD process and given by a
two dimensional advection diffusion parabolic system coupled at the boundary. The framework of semigroup theory has been considered. Using some classical arguments for the analysis of partial differential equations (PDEs), the model operator is shown to be m-dissipative. Moreover, it is proven that for any initial conditions, the solution of the system tends to an equilibrium as time tends to infinity. In particular, we will show that this operator is diagonal in the co-current configuration, if some additional conditions on thermal conductivity and flow rate are satisfied. On the other hand based on this new model, we will try to develop an observer to estimate the temperature of the 2D unstable parabolic system by boundary measurements. This kind of system will be treated for the first time in our work. However, there are some study for the parabolic EDPs in the multidimensional case, we refer the reader to the pioneering work of Thomas Meurer and and his collaborators [13, 14].

It is worth to point our that systems of advection-diffusion equations represent an important class of PDEs that arise in many problems of science and engineering. In this context, there exists some papers that have been devoted to the study of the reaction-advection-diffusion systems for linear and nonlinear cases, see [3, 15, 4] and references therein. The authors study the well-posedness and the blow-up of the solution for a class of nonlinear reaction-advection-diffusion system. The systems studied in these work present an internal coupling.

This paper is organized as follows. In section 2 we describe a mathematical model for the heat transfer in the DCMD systems. Section 3 formulates the problem using the framework of operator theory; where we prove that the operator related to the DCMD system is m-dissipative. The proof of the existence and uniqueness for the solution of the DCMD elliptic system is established in section 4. In section 5 the co-current DCMD case is presented and it is shown that under some additional conditions the operator is diagonal. Moreover, in section 6, we propose an exponentially stable observer design for an unstable parabolic system weakly coupled at the boundary. Finally, in section 7 we address some natural open questions arising after our study.

2. Mathematical modeling of heat transfer in DCMD process. The model geometry consists of a feed inlet boundary $B_1$, feed outlet boundary $B_3$, permeate inlet boundary $B_4$, permeate outlet boundary $B_6$. In this module, the vapor generated in the feed solution (warm sea water) is forced to pass through the membrane dry pores to the permeate side (cold water), following thermodynamics rules. Hereafter, we outline the equations describing the evolutions of the temperatures in the feed and permeate rooms of the devices, more details can be found in [6] or [18].

We denote by $f(t,x,y)$ the temperature of the warm water and by $p(t,x,y)$ the temperature of the cold water at time $t$ and at the point of coordinates $(x,y)$; we denote also by $\Omega_f$ and $\Omega_p$ the rectangles $[0,\ell] \times [0,L]$ and $[\delta_m + \ell, \delta_m + 2\ell] \times [0,L]$ respectively (here $\delta_m$ denotes the membrane thickness, see Fig. 1).

The mathematical model for the evolution of the temperatures in the device involves a diffusion and a convection term. The equations write, see [18]

$$
\begin{align*}
\partial_t f(t,x,y) &= \alpha_f \Delta f(t,x,y) - \beta_f \partial_y f(t,x,y) & \text{for } t \geq 0 \text{ and } (x,y) \in \Omega_f \\
\partial_t p(t,x,y) &= \alpha_p \Delta p(t,x,y) - \beta_p \partial_y p(t,x,y) & \text{for } t \geq 0 \text{ and } (x,y) \in \Omega_p.
\end{align*}
$$

The coefficients $\alpha_f$, $\beta_f$ and $\alpha_p$ are positive and are assumed to be constant, they depend on the thermal conductivity and on the densities of the fluids (see [18]); specifically they are defined as follows

$$
\alpha_f = \frac{\kappa_f}{\rho_f C_f}, \quad \alpha_p = \frac{\kappa_p}{\rho_p C_p}.
$$

2
Here $\kappa_k, \rho_k$ and $C_k, (k \in \{p, f\})$ denote respectively the thermal conductivity of fluid, liquid density of fluid and specific heat capacity of fluid. The coefficients $\beta_f > 0$ and
\( \beta_p \) denote the velocities of the flow in the feed and permeate side respectively. The velocity \( \beta_p \) in the permeate is negative in the counter-current case, Fig. 1 (a) and positive for the co-current presentation, Fig. 1 (b). The boundary conditions are a mix of Dirichlet, von Neumann and Robin conditions, they are:

**On the boundary** \( B_1 \).

(2.3) \[ f(t, x, 0) = T_f \quad \text{for every } 0 \leq x \leq \ell \]

**On the boundary** \( B_2 \).

(2.4) \[ \partial_x f(t, 0, y) = 0 \quad \text{for every } 0 \leq y \leq L \]

**On the boundary** \( B_3 \).

(2.5) \[ \partial_y f(t, x, L) = 0 \quad \text{for every } 0 \leq x \leq \ell \]

**On the boundary** \( B_4 \).

(2.6) \[
\begin{align*}
p(t, x, L) &= T_p & \text{for every } \ell + \delta_m \leq x \leq 2\ell + \delta_m, \text{ Fig. } 1 \text{ (a)} \\
p(t, x, 0) &= T_p & \text{for every } \ell + \delta_m \leq x \leq 2\ell + \delta_m, \text{ Fig. } 1 \text{ (b)}
\end{align*}
\]

**On the boundary** \( B_5 \).

(2.7) \[ \partial_y p(t, 2\ell + \delta_m, y) = 0 \quad \text{for every } 0 \leq y \leq L \]

**On the boundary** \( B_6 \).

(2.8) \[
\begin{align*}
\partial_y p(t, x, 0) &= 0 & \text{for every } \ell + \delta_m \leq x \leq 2\ell + \delta_m \text{ Fig. } 1 \text{ (a)} \\
\partial_y p(t, x, L) &= 0 & \text{for every } \ell + \delta_m \leq x \leq 2\ell + \delta_m, \text{ Fig. } 1 \text{ (b)}
\end{align*}
\]
On the interfaces $I_f$ and $I_p$.

\begin{align}
\partial_t f(t, x, y) - \alpha_f \Delta f(t, x, y) + \beta_f \partial_y f(t, x, y) &= 0, \quad t \geq 0, (x, y) \in \Omega \\
\partial_t p(t, x, y) - \alpha_p \Delta p(t, x, y) - \beta_p \partial_y p(t, x, y) &= 0, \quad t \geq 0, (x, y) \in \Omega \\
f(t, x, 0) &= T_f, \quad t \geq 0, 0 \leq x \leq 1, \\
\partial_x f(t, 0, y) &= 0, \quad t \geq 0, 0 \leq y \leq L, \\
\partial_y f(t, x, L) &= 0, \quad t \geq 0, 0 \leq x \leq 1, \\
p(t, x, L) &= T_p, \quad t \geq 0, 0 \leq x \leq 1, \\
\partial_x p(t, 0, y) &= 0, \quad t \geq 0, 0 \leq y \leq L, \\
\partial_y p(t, x, 0) &= 0, \quad t \geq 0, 0 \leq x \leq 1, \\
\partial_x f(t, 1, y) &= -\gamma_f (f(t, 1, y) - p(t, 1, y)), \quad t \geq 0, 0 \leq y \leq L, \\
\partial_x p(t, 1, y) &= \gamma_p (f(t, 1, y) - p(t, 1, y)), \quad t \geq 0, 0 \leq y \leq L, \\
f(0, x, y) &= f_0(x, y), \quad (x, y) \in \Omega, \\
p(0, x, y) &= p_0(x, y), \quad (x, y) \in \Omega.
\end{align}

Here the constants $\gamma_f$ and $\gamma_p$ are respectively equal to $k_m(\delta_m k_f)^{-1}$ and $k_m(\delta_m k_p)^{-1}$. $T_f, T_p, f_0$ and $p_0$ are the initial data of the system.

Remark. A simplification of the DCMD model has been proposed in [6] under appropriate physical assumptions. Indeed, the vertical thermal diffusivity for the considered geometry has been neglected, the width $\ell$ being sufficiently small compared to
the length \( L \). This assumption is based on the fact that the horizontal diffusivity is dominant.

\[
\begin{aligned}
&\partial_t f(t, x, y) - \alpha_f \partial_{xx} f(t, x, y) + \beta_f \partial_p f(t, x, y) = 0, \quad t \geq 0, (x, y) \in \Omega \\
&\partial_t p(t, x, y) - \alpha_p \partial_{xx} p(t, x, y) - \beta_p \partial_p p(t, x, y) = 0 \quad t \geq 0, (x, y) \in \Omega \\
f(t, x, 0) = T_f \quad t \geq 0, 0 \leq x \leq 1, \\
\partial_x f(t, 0, y) = 0 \quad t \geq 0, 0 \leq y \leq L, \\
p(t, x, L) = T_p \quad t \geq 0, 0 \leq x \leq 1, \\
\partial_x p(t, 0, y) = 0 \quad t \geq 0, 0 \leq y \leq L, \\
\partial_x f(t, 1, y) = -\gamma_f (f(t, 1, y) - p(t, 1, y)) \quad t \geq 0, 0 \leq y \leq L, \\
\partial_x p(t, 1, y) = \gamma_p (f(t, 1, y) - p(t, 1, y)) \quad t \geq 0, 0 \leq y \leq L, \\
f(0, x, y) = f_0(x, y) \quad (x, y) \in \Omega, \\
p(0, x, y) = p_0(x, y) \quad (x, y) \in \Omega.
\end{aligned}
\]

To study the advection diffusion system (2.11), we shall place ourselves in the framework of operators theory.

3. The operator related to the membrane distillation system. First, we introduce the domain of the operator: consider the space \( E \) which is the set of pairs \((f, p)\) in \([H^1(\Omega) \cap C^1(\Omega)]^2\) such that

- \( f(x, 0) = p(x, L) = 0 \) for every \( x \in (0, 1) \);
- \( \partial_x f(0, y) = \partial_p p(0, y) = 0 \), for every \( y \in (0, L) \);
- \( \partial_p p(x, 0) = 0 \), for every \( x \in (0, 1) \);
- \( \partial_x f(1, y) = -\gamma_f (f(1, y) - p(1, y)) \), for every \( y \in (0, L) \);
- \( \partial_x p(1, y) = \gamma_p (f(1, y) - p(1, y)) \) for every \( y \in (0, L) \).

The space \( H^1(\Omega) \times H^1(\Omega) \) is equipped with the product topology, so the norm of an element \((f, p) \in H^1(\Omega) \times H^1(\Omega)\) is defined by

\[
\| (f, p) \|^2 = \int_\Omega f^2 + \int_\Omega |\nabla f|^2 + \int_\Omega p^2 + \int_\Omega |\nabla p|^2.
\]

On \( L^2(\Omega) \times L^2(\Omega) \), we consider the following inner product:

\[
\langle (f, p), (g, q) \rangle := \alpha_f \gamma_p \int_\Omega f(x, y) g(x, y) \, dx \, dy + \alpha_f \gamma_f \int_\Omega p(x, y) q(x, y) \, dx \, dy;
\]

notice that this inner product induces the product topology on \( L^2(\Omega) \times L^2(\Omega) \). We denote by \( H^1_{bc} \) the closure of \( E \) in \([H^1(\Omega)]^2\); from the Poincaré’s inequality, the induced norm on \( H^1_{bc} \), defined by (3.1) is equivalent to the following one

\[
\| (f, p) \|^2_{H^1_{bc}} = \alpha_f \int_\Omega |\nabla f|^2 + \alpha_p \int_\Omega |\nabla p|^2, \quad (f, p) \in H^1_{bc}.
\]

We then denote by \( A_0 \) the operator whose domain is given by

\[
\mathcal{D}(A_0) := \left\{ (f, p) \in H^1_{bc} \mid (\Delta f, \Delta p) \in [L^2(\Omega)]^2 \right\}
\]

and which is defined by

\[
A_0(f, p) = (\alpha_f \Delta f, \alpha_p \Delta p).
\]
We introduce also the operator $B_0$, whose domain is the one of $A_0$ and which is defined by

$$B_0(f,p) = (-\beta f_\partial f + \beta p_\partial p);$$

finally we define operator $A$, related to system (2.11) by

$$A = A_0 + B_0.$$ 

3.1. The operator $A_0$. In this section, we shall prove that the operator $A_0$ is, as the laplacian operator, self adjoint and $m$-dissipative. The proof is analogous to the one which shows these classical properties of the laplacian operator (see e.g. [19]). We begin by the following proposition.

**Proposition 3.1.** The embedding operator from $H_1^{bc}$ to $[L^2(\Omega)]^2$ is compact.

**Proof.** We denote by $J$ the embedding $H_1^{bc} \hookrightarrow [L^2(\Omega)]^2.$ From the elementary theory of Fourier series we know that the family $(\varphi_\alpha)_{\alpha \in \mathbb{N}^2}$ defined by

$$\varphi_\alpha = \frac{2}{\sqrt{L}} \sin(\alpha_1 \pi x) \sin\left(\frac{\alpha_2 \pi}{L} y\right), \quad \alpha := (\alpha_1, \alpha_2) \in \mathbb{N}^2,$$

is an orthonormal basis for $L^2(\Omega).$ In this proof, we need the notation $\|\alpha\|^2 = \alpha_1^2 + \alpha_2^2$ for a multi-index $\alpha \in \mathbb{N}^2.$ Let $(f,p) \in H_1^{bc}.$ Then

$$\|f\|_{L^2(\Omega)}^2 = \sum_{\alpha \in \mathbb{N}^2} \left| \langle f, \varphi_\alpha \rangle \right|^2$$

and

$$\|f\|_{H^1(\Omega)}^2 = \sum_{\alpha \in \mathbb{N}^2} (1 + \|\alpha\|^2) \left| \langle f, \varphi_\alpha \rangle \right|^2.$$

From the above formulas it follows that if $m \in \mathbb{N},$ and if $J_m \in \mathcal{L}(H_1^{bc}, L^2(\Omega))$ is defined by

$$J_m(f,p) = \sum_{\alpha \in \mathbb{N}^2, \|\alpha\|^2 \leq m} \langle f, \varphi_\alpha \rangle \varphi_\alpha + \sum_{\alpha \in \mathbb{N}^2, \|\alpha\|^2 > m} \langle p, \varphi_\alpha \rangle \varphi_\alpha,$$

then, as

$$\sum_{\alpha \in \mathbb{N}^2} \|\alpha\|^2 \left| \langle f, \varphi_\alpha \rangle \right|^2 \geq m \sum_{\alpha \in \mathbb{N}^2, \|\alpha\|^2 > m} \left| \langle f, \varphi_\alpha \rangle \right|^2 \text{ and } \sum_{\alpha \in \mathbb{N}^2} \|\alpha\|^2 \left| \langle p, \varphi_\alpha \rangle \right|^2 \geq m \sum_{\alpha \in \mathbb{N}^2, \|\alpha\|^2 > m} \left| \langle p, \varphi_\alpha \rangle \right|^2,$$

we have

$$\|J(f,p) - J_m(f,p)\|_{L^2(\Omega)} \leq \frac{1}{1 + m} \|f,p\|_{H_1^{bc}},$$

and so

$$\lim_{m \to \infty} J_m = J.$$

Since the dimension of $\text{Ran}(J_m)$ is finite, this implies that $J$ is compact (see [19, Proposition 12.2.2]).

**Theorem 3.2.** The operator $A_0$ is self-adjoint and diagonalizable.
Proof. Take \((f, p)\) and \((g, q)\) in \(\mathcal{D}(A_0)\); integrating two times by parts, we obtain

\[
\int_\Omega \frac{\partial^2 f}{\partial x^2}(x, y) g(x, y) \, dx \, dy = \int_0^L \frac{\partial f}{\partial x}(1, y) g(1, y) \, dy - \int_0^L f(1, y) \frac{\partial g}{\partial x}(1, y) \, dy
\]

\[
+ \int_\Omega f(x, y) \frac{\partial^2 g}{\partial x^2}(x, y) \, dx \, dy
\]

\[
= \gamma_f \int_0^L (g(1, y)p(1, y) - f(1, y)q(1, y)) \, dy
\]

\[
+ \int_\Omega f(x, y) \frac{\partial^2 g}{\partial x^2}(x, y) \, dx \, dy,
\]

and

\[
\int_\Omega \frac{\partial^2 f}{\partial y^2}(x, y) g(x, y) \, dx \, dy = \int_\Omega f(x, y) \frac{\partial^2 g}{\partial y^2}(x, y) \, dx \, dy.
\]

Analogous computations lead to

\[
\int_\Omega \Delta p(x, y) q(x, y) \, dx \, dy = \gamma_p \int_0^L (f(1, y)q(1, y) - g(1, y)p(1, y)) \, dy
\]

\[
+ \int_\Omega p(x, y) \Delta q(x, y) \, dx \, dy.
\]

So we have

\[
\langle A_0(f, p), (g, q) \rangle = \alpha_p \gamma_p \alpha_f \gamma_f \int_0^L (g(1, y)p(1, y) - f(1, y)q(1, y)) \, dy
\]

\[
+ \alpha_p \gamma_p \alpha_f \int_\Omega f(x, y) \Delta g(x, y) \, dx \, dy
\]

\[
+ \alpha_f \gamma_f \alpha_p \int_0^L (f(1, y)q(1, y) - g(1, y)p(1, y)) \, dy
\]

\[
+ \alpha_f \gamma_f \int_\Omega p(x, y) \Delta q(x, y) \, dx \, dy
\]

\[
= \alpha_p \gamma_p \int_\Omega f(x, y) (\alpha_f \Delta g(x, y)) \, dx \, dy
\]

\[
+ \alpha_f \gamma_f \int_\Omega p(x, y) (\alpha_p \Delta q(x, y)) \, dx \, dy
\]

\[
= \langle (f, p), A_0(g, q) \rangle,
\]

which shows that the operator \(A_0\) is symmetric. We shall now show that \(A_0\) is self-adjoint; to do this, it is enough to prove that \(A_0\) is onto, the proof is almost the same as in [19, Proposition 3.2.4].

Take \((u, v) \in [L^2(\Omega)]^2\), we have to prove the existence of \((f, p)\) in \(\mathcal{D}(A_0)\) such that

\[
A_0 (f, p) = (u, v).
\]

First notice that the mapping

\[
(g, q) \mapsto \int_\Omega u g + \int_\Omega v q
\]
is a bounded linear functional on $H^1_{bc}$. By the Riesz representation theorem, there exists $(f, p) \in H^1_{bc}$ such that

$$\langle (f, p), (g, q) \rangle_{H^1_{bc}} = \langle (u, v), (g, q) \rangle_{L^2(\Omega)^2};$$

the inner product in the left-hand member of this equality being the one related to the norm defined by (3.3). Denoting by $D(\Omega)$ the space of smooth functions with compact support in $\Omega$, we notice that, as $D(\Omega) \times D(\Omega) \subset H^1_{bc}$, the above equality can also be written for any $g, q \in D(\Omega)$; so, in the sense of distributions, we have

$$\langle (f, p), (g, q) \rangle_{H^1_{bc}} = \alpha_f \int_\Omega (\nabla f) \cdot (\nabla g) \, dx \, dy + \alpha_p \int_\Omega (\nabla p) \cdot (\nabla q) \, dx \, dy$$

$$= -\alpha_f \langle \Delta f, g \rangle_{D', D} - \alpha_p \langle \Delta p, q \rangle_{D', D}.$$  

This equality is true for every $(g, q) \in [D(\Omega)]^2$, so we have

$$-\alpha_f \Delta f = u, \quad \text{in } D'(\Omega), \quad -\alpha_p \Delta p = v, \quad \text{in } D'(\Omega).$$

Since, $(u, v) \in [L^2(\Omega)]^2$, we obtain that

$$\alpha_f \Delta f \in L^2(\Omega), \quad \alpha_p \Delta p \in L^2(\Omega).$$

Thus $(f, p) \in D(A_0)$ and

$$A_0 (f, p) = (u, v),$$

hence $A_0$ is onto and we can conclude that $A_0$ is a self adjoint.

Finally, according to Proposition 3.1, the embedding $J : H^1_{bc} \hookrightarrow [L^2(\Omega)]^2$ is compact, therefore $A_0^{-1} = J \circ A_0^{-1}$ is compact and hence, by Proposition [19, Theorem 3.2.12], $A_0$ is diagonalizable with an orthonormal basis $(\varphi_k, \psi_k)$ of eigenvectors and the corresponding sequence of eigenvalues $(\lambda_k)$ satisfies and $\lim_{|k| \to \infty} \lambda_k = \infty$.  

**3.2. The operator $A$.** In this section we shall prove that $A$ is m-dissipative with respect to the inner product defined in (3.2); the proof is in the same spirit as the proof of [17, Theorem 3.2]. We shall see first that $A$ is dissipative.

**Proposition 3.3.** For every $t \in [0, 1]$, the operator $A_0 + tB_0$ is dissipative; moreover, the operator $A_0$ is m-dissipative.

**Proof.** Take $(f, p) \in D(A_0)$, we compute first $\langle A_0(f, p), (f, p) \rangle$; integrating by parts, we get

$$\int_\Omega \frac{\partial^2 f}{\partial x^2} = \int_0^L \left[ \frac{\partial f}{\partial x} (x, y) f(x, y) \right]_{z=1}^{z=0} \, dy - \int_\Omega \left( \frac{\partial f}{\partial x} \right)^2 \, dx \, dy$$

$$= -\gamma f \int_0^L (f(1, y) - f(1, y)) \, dy - \int_\Omega \left( \frac{\partial f}{\partial x} \right)^2 \, dx \, dy.$$

On the other hand

$$\int_\Omega \frac{\partial^2 f}{\partial y^2} = \int_0^1 \left[ \frac{\partial f}{\partial y} (x, y) f(x, y) \right]_{y=0}^{y=1} \, dx - \int_\Omega \left( \frac{\partial f}{\partial y} \right)^2 \, dx \, dy$$

$$= -\int_\Omega \left( \frac{\partial f}{\partial y} \right)^2 \, dx \, dy.$$

Finally, according to Proposition 3.1, the embedding $J : H^1_{bc} \hookrightarrow [L^2(\Omega)]^2$ is compact, therefore $A_0^{-1} = J \circ A_0^{-1}$ is compact and hence, by Proposition [19, Theorem 3.2.12], $A_0$ is diagonalizable with an orthonormal basis $(\varphi_k, \psi_k)$ of eigenvectors and the corresponding sequence of eigenvalues $(\lambda_k)$ satisfies and $\lim_{|k| \to \infty} \lambda_k = \infty$.  

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$$= -\gamma f \int_0^L (f(1, y) - f(1, y)) \, dy - \int_\Omega \left( \frac{\partial f}{\partial x} \right)^2 \, dx \, dy.$$

On the other hand

$$\int_\Omega \frac{\partial^2 f}{\partial y^2} = \int_0^1 \left[ \frac{\partial f}{\partial y} (x, y) f(x, y) \right]_{y=0}^{y=1} \, dx - \int_\Omega \left( \frac{\partial f}{\partial y} \right)^2 \, dx \, dy$$

$$= -\int_\Omega \left( \frac{\partial f}{\partial y} \right)^2 \, dx \, dy.$$
An analogous computation leads to
\[
\int_{\Omega} (\Delta p) p \, dx \, dy = \gamma_p \int_0^L p(1, y)(f(1, y) - p(1, y)) \, dy - \int_{\Omega} |\nabla p|^2 dx \, dy
\]
So, we have
\[
\langle A_0(f, p), (f, p) \rangle_{L^2(\Omega)}^2 = \alpha_p \gamma_p \alpha_f \int_\Omega (\Delta f) f \, dx \, dy + \alpha_f \gamma_f \alpha_p \int_\Omega (\Delta p) p \, dx \, dy
\]
\[
= -\alpha_p \gamma_p \alpha_f \int_0^L f(1, y)(f(1, y) - p(1, y)) \, dy
\]
\[
- \alpha_p \gamma_p \alpha_f \int_{\Omega} |\nabla f|^2 dx \, dy
\]
\[
+ \alpha_f \gamma_f \alpha_p \int_0^L p(1, y)(f(1, y) - p(1, y)) \, dy
\]
\[
- \alpha_f \gamma_f \alpha_p \int_{\Omega} |\nabla p|^2 dx \, dy
\]
(3.7)
\[
= -\alpha_p \gamma_p \alpha_f \int_0^L (f(1, y) - p(1, y))^2 \, dy
\]
\[
- \alpha_f \alpha_p \int_{\Omega} \langle \gamma_p |\nabla f|^2 + \varphi_f |\nabla p|^2 \rangle dx \, dy
\]
(3.8)
On the other hand
\[
\langle B_0(f, p), (f, p) \rangle = -\alpha_p \gamma_p \beta_f \int_\Omega (\partial_y f) f \, dx \, dy + \alpha_f \gamma_f \beta_p \int_\Omega (\partial_y p) p \, dx \, dy
\]
\[
= -\frac{\alpha_p \gamma_p \beta_f}{2} \int_0^1 (f^2(x, 1) - f^2(x, 0)) \, dx
\]
\[
+ \frac{\alpha_f \gamma_f \beta_p}{2} \int_0^1 (p^2(x, 1) - p^2(x, 0)) \, dx
\]
(3.9)
Inequalities (3.7) and (3.8) prove that $A_0 + tB_0$ is dissipative for every $t \in [0, 1]$; moreover, we have seen that $A_0$ is onto, this proves that $A_0$ is self-adjoint, as $A_0$ is dissipative, we can conclude that $A_0$ is m-dissipative.

Now, as in the proof of the previous theorem, we shall prove that there exists $\delta > 0$ such that, if $A_0 + tB_0$ (here $t_0 \in [0, 1]$) is m-dissipative, then $A_0 + tB_0$ is also m-dissipative for every $t \in [0, 1]$ such that $|t - t_0| \leq \delta$. To this end, we need the following proposition.

**Proposition 3.4.** If the operator $A_0 + t_0B_0$ ($t_0 \in [0, 1]$) is m-dissipative, then the operator $B_0((I - (A_0 + t_0B_0))^{-1}$ is bounded, the bound being independent from $t_0$.

**Proof.** Notice that if $A_0 + t_0B_0$ is m-dissipative, $I - (A_0 + t_0B_0)$ is invertible. We denote the inverse $(I - (A_0 + t_0B_0))^{-1}$ by $R(t_0)$ and we take $(f, p)$ in $[L^2(\Omega)]^2$. We seek for an upper bound for $\|B_0R(t_0)(f, p)\|$. Let $(u, v) = R(t_0)(f, p)$, so that we have
\[
f = u - \alpha_f \Delta u + t_0 \beta_f \frac{\partial u}{\partial y},
\]
\[
p = v - \alpha_p \Delta v - t_0 \beta_p \frac{\partial v}{\partial y}.
\]
Now
\[
\|B_0 R(t_0)(f, p)\|^2 = \|B_0(u, v)\|^2
\]
\[
= \alpha_p \gamma_p \int_\Omega \beta_f^2 \left( \frac{\partial u}{\partial y} \right)^2 dx dy + \alpha_f \gamma_f \int_\Omega \beta_p^2 \left( \frac{\partial p}{\partial y} \right)^2 dx dy
\]
\[
(3.10)
\]
\[
\leq M \left( \gamma_p \int_\Omega \left( \frac{\partial u}{\partial y} \right)^2 dx dy + \gamma_f \int_\Omega \left( \frac{\partial p}{\partial y} \right)^2 dx dy \right)
\]
with \(M = \max(\alpha_p \beta_f^2, \alpha_f \beta_p^2)\).

We shall rewrite these two integrals; first we have
\[
\int_\Omega \left( \frac{\partial u}{\partial y} \right)^2 dx dy = \int_0^1 \left[ u(x, y) \frac{\partial u}{\partial y}(x, y) \right]_{y=0}^{y=L} dx - \int_\Omega \frac{\partial^2 u}{\partial y^2} dx dy
\]
\[
= -\int_\Omega \frac{\partial^2 u}{\partial y^2} dx dy
\]
\[
(3.11)
\]
\[
= -\frac{1}{\alpha_f} \int_\Omega u \left( u - \alpha_f \frac{\partial^2 u}{\partial x^2} + t_0 \beta_f \frac{\partial u}{\partial y} - f \right) dx dy \quad \text{from (3.9).}
\]

Now, we have
\[
\int_\Omega \frac{\partial^2 u}{\partial x^2} dx dy = \int_0^L \left[ u(x, y) \frac{\partial u}{\partial x}(x, y) \right]_{x=0}^{x=L} dy - \int_\Omega \left( \frac{\partial u}{\partial x} \right)^2 dx dy
\]
\[
= -\gamma_f \int_0^L u(1, y) (u(1, y) - v(1, y)) dy - \int_\Omega \left( \frac{\partial u}{\partial x} \right)^2 dx dy;
\]
on the other hand
\[
(3.13)
\]
\[
\int_\Omega \frac{\partial u}{\partial y} dx dy = \frac{1}{2} \int_0^1 u^2(x, 1) dx.
\]

Substituting equalities (3.12) and (3.13) into (3.11), we get
\[
\int_\Omega \left( \frac{\partial u}{\partial y} \right)^2 dx dy = -\frac{1}{\alpha_f} \int_\Omega u^2 dx dy - \gamma_f \int_0^L u(1, y) (u(1, y) - v(1, y)) dy
\]
\[
- \gamma_f \int_\Omega \left( \frac{\partial u}{\partial x} \right)^2 dx dy - \frac{t_0 \beta_f}{2 \alpha_f} \int_\Omega u^2(x, L) dx + \frac{1}{\alpha_f} \int_\Omega u f dx dy
\]
\[
\leq -\frac{1}{\alpha_f} \int_\Omega u^2 dx dy - \gamma_f \int_0^L u(1, y) (u(1, y) - v(1, y)) dy
\]
\[
(3.14)
\]
\[
- \frac{1}{\alpha_f} \int_\Omega u^2 dx dy - \gamma_f \int_0^L u(1, y) (u(1, y) - v(1, y)) dy
\]
\[
+ \frac{1}{2 \alpha_f} \int_\Omega (u^2 + f^2) dx dy
\]
\[
\leq -\gamma_f \int_0^L u(1, y) (u(1, y) - v(1, y)) dy + \frac{1}{2 \alpha_f} \int_\Omega f^2 dx dy.
\]
An analogous computation shows that

\[(3.15) \quad \int_{\Omega} \left( \frac{\partial v}{\partial y} \right)^2 \, dx \, dy \leq \gamma_p \int_0^L v(1, y)(u(1, y) - v(1, y)) \, dy + \frac{1}{2\alpha_p} \int_\Omega p^2 \, dx \, dy.\]

Substituting (3.14) and (3.15) into (3.10), we get

\[
\|B_0 R(t_0)(f, p)\|^2 \leq M \gamma_p \left( -\gamma_f \int_0^L u(1, y)(u(1, y) - v(1, y)) \, dy + \frac{1}{2\alpha_f} \int_\Omega f^2 \, dx \, dy \right) \\
+ M \gamma_f \left( \gamma_p \int_0^L v(1, y)(u(1, y) - v(1, y)) \, dy + \frac{1}{2\alpha_p} \int_\Omega p^2 \, dx \, dy \right) \\
= -M \gamma_f \gamma_p \int_0^L (u(1, y) - v(1, y))^2 \, dy + M \left( \frac{\gamma_p}{2\alpha_f} \int_\Omega f^2 \, dx \, dy + \frac{\gamma_f}{2\alpha_p} \int_\Omega p^2 \, dx \, dy \right) \\
\leq M \left( \frac{\gamma_p}{2\alpha_f} \int_\Omega f^2 \, dx \, dy + \frac{\gamma_f}{2\alpha_p} \int_\Omega p^2 \, dx \, dy \right) \\
\leq M' \|(f, p)\|^2_{L^2(\Omega)}^2,
\]

where \(M' = \max\left( \frac{M}{2\alpha_f^2}, \frac{M}{2\alpha_p^2} \right)\).

We have proved that the operator \(B_0 R(t_0)\) is bounded, moreover its norm is less than or equal to \(\sqrt{M'}\), which is a bound independent from \(t_0\).

We are now ready to prove the main result of this section.

**Theorem 3.5.** *Operator \(A\) is \(m\)-dissipative with respect to the inner product (3.2).*

**Proof.** Assume that \(A_0 + t_0 B_0\) is \(m\)-dissipative (with \(t_0 \in [0, 1]\)), a simple computation shows that we can write

\[(3.16) \quad I - (A_0 + t B_0) = I - (A_0 + t_0 B_0) + (t_0 - t) B_0 \\
= (1 + (t - t_0) B_0 R(t_0))(1 - (A_0 + t_0 B_0)).\]

The bounded operator \(I + (t - t_0) B_0 R(t_0)\) is invertible if \(\|(t - t_0) B_0 R(t_0)\| < 1\), and this inequality is true if \(|t - t_0| < 1/\sqrt{M'}\). Thus, if \(t\) satisfies this inequality, we can conclude from (3.16) that \(I - (A_0 + t B_0)\) is invertible. Moreover as, from Proposition 3.3, we know that \(A_0 + t B_0\) is dissipative (for every \(t \geq 0\)), we can conclude that \(A_0 + t B_0\) is \(m\)-dissipative for every \(t \in [0, 1]\) and such that \(|t - t_0| \leq 1/\sqrt{M'}\).

Observe now that \(A_0 + t_0 B_0\) is \(m\)-dissipative with \(t_0 = 0\) (cf Prop. 3.3), since any point of \([0, 1]\) can be reached from 0 by a finite number of step of length \(1/\sqrt{M'}\), we conclude that \(A = A_0 + B_0\) is \(m\)-dissipative. \(\Box\)

4. **Existence and uniqueness of the solution of system (2.11).** In order to ensure the existence of a solution to system (2.11) appropriate regularity assumptions on the initial datum are required. We next sharpen the regularity of this data.
4.1. The operator $A$ with inhomogeneous boundary conditions. In this section, we consider the following systems of partial differential equations

\[
\begin{align*}
\alpha_f \Delta f(x, y) - \beta_f \partial_y f(x, y) &= 0, \\
- \alpha_p \Delta p(x, y) + \beta_p \partial_y p(x, y) &= 0, \quad (x, y) \in \Omega \\
f(x, 0) &= T_f, \quad 0 \leq x \leq 1, \\
\partial_x f(0, y) &= 0, \quad 0 \leq y \leq L, \\
\partial_y f(x, L) &= 0, \quad 0 \leq x \leq 1, \\
p(x, L) &= T_p, \quad 0 \leq x \leq 1, \\
\partial_x p(0, y) &= 0, \quad 0 \leq y \leq L, \\
\partial_y p(x, 0) &= 0, \quad 0 \leq x \leq 1, \\
\partial_x f(1, y) &= -\gamma_f (f(1, y) - p(1, y)), \quad 0 \leq y \leq L, \\
\partial_x p(1, y) &= \gamma_p (f(1, y) - p(1, y)), \quad 0 \leq y \leq L.
\end{align*}
\]

If $T_f = T_p = 0$, this system writes $A(f, p) = 0$ with $A$ the operator defined above; in this case the unique solution of the system is $(f, p) \equiv (0, 0)$ because the m-dissipative operator $A : D(A) \to [L^2(\Omega)]^2$ is invertible. In the general case, we shall use some tools from the theory of boundary control systems (see e.g. [19, Chap. 10]). In system (4.1), we shall regard $T_f$ and $T_p$ as boundary controls, and we introduce the following spaces and operators:

- the solution space $Z$ is defined as those pairs $(f, p) \in H^2(\Omega) \times H^2(\Omega)$ satisfying the following homogeneous boundary conditions:
  - for every $0 \leq y \leq L$, $\partial_x f(0, y) = \partial_x p(0, y) = 0$;
  - for every $0 \leq x \leq 1$, $\partial_y f(x, L) = \partial_y p(x, 0) = 0$;
  - for every $0 \leq y \leq L$, $\partial_x f(1, y) = -\gamma_f (f(1, y) - p(1, y))$, and $\partial_x p(1, y) = \gamma_p (f(1, y) - p(1, y))$.
- the state space $X$ is the space $L^2(\Omega) \times L^2(\Omega)$;
- the input space $U$ is the space $L^2(0, 1) \times L^2(0, 1)$.

Notice that $Z \subset X$ with continuous embedding. We consider the operator $L : Z \to X$ defined as

\[
L(f, p) = (\alpha_f \Delta f - \beta_f \partial_y f, \alpha_p \Delta p + \beta_p \partial_y p)
\]

and the operator $G : Z \to U$ defined as

\[
G(f, p) = (f(\cdot, 0), p(\cdot, L)).
\]

Operator $L$ is obviously bounded, this is true also for operator $G$

**Lemma 4.1.** The linear operator $G : Z \to U$ is bounded.

**Proof.** Consider the function $F$ defined for $x \in [0, 1]$ as

\[
F(x) = \int_0^L f^2(x, y)dy,
\]

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we have

\[
F(0) = F(x) - \int_0^x \frac{dF(\xi)}{d\xi} d\xi \\
= F(x) - 2 \int_0^x \left( \int_0^L f(\xi, y) \partial_x f(\xi, y) \, d\xi \right) \, dy \\
\leq F(x) + 2 \int_0^1 \left( \int_0^L |f(\xi, y)| \partial_x f(\xi, y) \, d\xi \right) \, dy \\
\leq F(x) + \int_0^1 \left( \int_0^L f^2(\xi, y) \, d\xi \right) \, dy + \int_0^1 \left( \int_0^L (\partial_x f)^2(\xi, y) \, d\xi \right) \, dy.
\]

By integrating this inequality with respect to \(x\) on the interval \([0, 1]\), we obtain

\[
F(0) \leq \int_0^1 F(x) \, dx + \int_\Omega f^2 \, dx \, dy + \int_\Omega (\partial_y f)^2 \, dx \, dy
\]

which reads

\[
\|f(\cdot, 0)\|_{L^2(0, 1)} \leq 2 \|f\|_{L^2(\Omega)}^2 + \|\partial_x f\|_{L^2(\Omega)}^2
\]

and so we have

\[
\|f(\cdot, 0)\|_{L^2(0, 1)} \leq 2 \|f\|_{H^2(\Omega)}.
\]

Clearly, the same inequality is true for \(p\), which proves that \(G\) is a bounded operator \(\Box\)

Now we have.

**Proposition 4.2.** The operators \(G\) and \(L\) satisfy the following properties.

1. \(G\) is onto;
2. \(\text{Ker} \, G\) is dense in \(X\);
3. \(-L\) restricted to \(\text{Ker} \, G\) is onto;
4. \(\text{Ker}(-L) \cap \text{Ker} \, G = \{0\}\).

The two first points are obvious. As regards the third point, notice first that \(\text{Ker} \, G \subset \mathcal{D}(A)\) moreover, given \((u, v) \in X\), we know that there exist \((f, p) \in \mathcal{D}(A)\) such that \(A(f, p) = (u, v)\). In order to prove that the pair \((f, g)\) is in \(\text{Ker} \, G\), we have to show that \(f\) and \(p\) are in \(H^2(\Omega)\). The proof of the regularity of the weak solution of an elliptic equation is a classical result (see e.g. [7] or [2]) but this result assumes Dirichlet or Neumann boundary conditions. In [8], M. Faierman proves the regularity of the weak solution of an elliptic equation \(Mq = r\) where \(M\) is an elliptic operator defined on a rectangle \(R\) and where, as for the system considered in this paper, the boundary conditions are of mixed type: Dirichlet, Neumann of Robin. More specifically, under the condition that \(r \in L^2(R)\), Faierman proves that \(q\) is in \(H^2(R)\). This proof is intended for an elliptic equation whose unknown function \(f\) takes its values in \(R\) while, in this paper, our unknown is a couple of functions \((f, p)\); nevertheless, it suffices to adapt slightly the reasoning of Faierman to prove that \(f\) and \(p\) are in \(H^2(\Omega)\).

**Proposition 4.3.** If both functions \(u\) and \(v\) are in \(L^2(\Omega)\), the unique pair \((f, p) \in \mathcal{D}(A)\) such that \(A(f, p) = (u, v)\) belongs to \(H^2(\Omega) \times H^2(\Omega)\).
Proof. Hereafter, we treat only the case of \( f \), the reasoning for \( p \) being the same than for \( f \). First, we define two extensions of \( f \): \( f_1 \) on \( \Omega_1 = [-1, 1] \times [0, L] \) and \( f_2 \) on \( \Omega_2 = [0, 2] \times [0, L] \) as follows

\[
\begin{align*}
 f_1(x, y) &= \begin{cases} 
 \phi_1(x)f(x, y) & \text{if } 0 \leq x \leq 1 \\
 \phi_1(-x)f(-x, y) & \text{if } -1 \leq x \leq 0
 \end{cases} \\
 f_2(x, y) &= \begin{cases} 
 (1 - \phi_1(x))f(x, y) & \text{if } 0 \leq x \leq 1 \\
 - (1 - \phi_1(2 - x))f(2 - x, y) + 2f(1, y) & \text{if } 1 \leq x \leq 2
 \end{cases}
\end{align*}
\]

where \( \phi_1 \) is a \( C^\infty \) function such that

\[
\phi_1(x) = \begin{cases} 
 1 & \text{if } x \leq 1/4 \\
 0 & \text{if } 3/4 \leq x \leq 1
 \end{cases}
\]

and \( 0 \leq \phi_1(x) \leq 1 \) for all \( x \in \mathbb{R} \). Notice that \( f_1 \) is defined as in [8] but the definition for \( f_2 \) (as well as the notations) differs slightly from the one adopted in this paper. First, it is easily shown that \( f_1 \in H^2(\Omega_i) \) (\( i = 1, 2 \)), then we shall show that \( f_1 \) and \( f_2 \) can be regarded as weak solutions to some PDE's. We begin with function \( f_2 \): take \( \psi_2 \in H^1(\Omega_2) \) such that \( \psi_2(x, 0) = 0 \) for \( 0 \leq x \leq 2 \), first notice that we have

\[
\alpha_f(\nabla f_2)(\nabla \psi_2) + \beta_f(\partial_y f_2)\psi_2 = \alpha_f(\nabla f)(\nabla (\psi_2(\phi_2))) + \beta_f(\partial_y f)(\phi_2)\psi_2
\]

\[
- \alpha_f(\partial_x \psi_2)(\partial_x f)\psi_2 + \alpha_f(\partial_x \phi_2)f(\partial_x \psi_2)
\]

where \( \phi_2(x) := 1 - \phi_1(x) \). Integrating by parts and taking into account that \( A(f, p) = (u, v) \), we obtain the following equality

\[
\int f_1 \cdot (\nabla \psi_2) \, dx \, dy + \beta_f(\partial_y f)\psi_2 \, dx \, dy = - \int u \psi_2 \, dx \, dy + \alpha_f \int \partial_x f(1, y)\psi_2(1, y) \, dy
\]

from this equality and (4.2), and taking into account that \( \phi_2(1) = 1 \), and \( (\partial_x \phi_2)(0) = (\partial_x f)(1) = 0 \), we obtain

\[
\int \alpha_f(\nabla f_2) \cdot (\nabla \psi_2) \, dx \, dy + \beta_f(\partial_y f_2)\psi_2 \, dx \, dy
\]

\[
= - \int \{ \phi_2u + \alpha_f(\partial_x \phi_2)(\partial_x f)\} \psi_2 \, dx \, dy
\]

\[
+ \alpha_f \int \partial_x f(1, y)\psi_2 \, dy + \int \partial_x \phi_2 f(\partial_x \psi_2) \, dx \, dy
\]

\[
= - \int \{ \phi_2u + \alpha_f(\partial_x \phi_2)(\partial_x f)\} \psi_2 \, dx \, dy + \alpha_f \int \partial_x f(1, y)\psi_2 \, dy
\]

\[
+ \alpha_f \int_0^L [\partial_x \phi_2 f_2]' y = 0 \, dy - \alpha_f \int \partial_x \phi_2 f_2 \, dx \, dy
\]

\[
= - \int (\phi_2 u + g_2)\psi_2 \, dx \, dy + \alpha_f \int_0^L (\partial_x f)(1, y)\psi_2 \, dy
\]

where \( g_2 \) is the function defined as

\[
g_2(x, y) := \alpha_f(\partial_x \phi_2)(f(x, y) + 2\partial_x \phi_2(x)\partial_x f(x, y)).
\]
From this formula, we deduce that,

\begin{equation}
\int_{\Omega_2} \alpha f (\nabla f_2) \cdot (\nabla \psi_2) \, dx \, dy + \int_{\Omega_2} \beta f (\partial_y f_2) \psi_2 \, dx \, dy = - \int_{\Omega_2} (\phi_2 u + g_2)^* \psi_2 \, dx \, dy \\
+ 2\alpha f \int_{\Omega_2} (\partial_y f)(1, y)(\partial_y \psi_2)(x, y) \, dx \, dy
\end{equation}

where

\((\phi_2 u + g_2)^* := \begin{cases} (\phi_2 u + g_2)(x, y) & \text{if } (x, y) \in \Omega \\ -(\phi_2 u + g_2)(2 - x, y) & \text{if } (x, y) \in [1, 2] \times [0, L]. \end{cases} \)

Due to the second integral in the right-hand member in (4.4), function \(f_2\) cannot be regarded as the weak solution of a PDE, nevertheless, we can apply the method of difference quotients. In the proof of [7, Th. 1, p. 329], an open set \(V \subset \Omega_3\) is fixed and a smooth cutoff function \(\theta\) is chosen (\(\theta\) is equal to 1 on \(V\) and to 0 outside an open set \(W\) such that \(U \subset W \subset \Omega_3\)); then function \(\psi_2\) in equality (4.4) is taken to be equal to

\[
\psi_2(x, y) = \frac{1}{h^2} \left( \theta^2(x - h, y)(f(x, y) - f(x - h, y) - \theta^2(x, y)(f(x + h, y) - f(x, y)) \right).
\]

With this choice of \(\psi_2\), the second integral in the right-hand member of (4.4) is zero and it follows that we can argue as in the proof of [7, Th. 1, p. 329].

To prove the boundary regularity, we can still proceed as in [7], in this case also, we do not have to take care of the second integral in the right-hand member of (4.4). To be more precise, consider a point \((x_0, 0)\) of the edge \([0, 2] \times \{0\}\) of \(\Omega_2\) with \(1/4 \leq x_0 \leq 7/4\). Denote by \(U_r\) the half ball \(U_r = B(x_0, r) \cap \mathbb{R}^2_+\) where, as usual, \(B(x_0, r)\) denotes the open ball of radius \(r\) centered at \(x_0\) and \(\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}\); moreover \(r\) is chosen small enough in order that \(B(x_0, 2r)\) does not intersect the edge \([0, 2] \times \{L\}\).

Select a smooth cutoff function \(\sigma\) satisfying

\[
\begin{cases}
\sigma = 1 \text{ on } B(x_0, r), \\
\sigma = 0 \text{ on } \mathbb{R}^2 \setminus B(x_0, 2r) \\
0 \leq \sigma \leq 1.
\end{cases}
\]

Let \(h > 0\) be small and write

\begin{equation}
\psi_2(x, y) := -D_1^{-h}(\sigma^2 D_1^h K^\sigma)
\end{equation}

where, for any function \(K\), \(D_1^h K\) denotes the difference quotient

\[
D_1^h K(x, y) := \frac{K(x + h) - K(x)}{h}.
\]

With this choice of \(\psi_2\), it is easily seen that the second integral in the right-hand member of (4.4) vanishes. Thus we can argue exactly as in the proof of [7, Th.4, p. 336] in order to establish the following estimate

\[
\int_U \|D_1^h \nabla f_2\|^2 \, dx \, dy \leq C,
\]

which proves the result. We treat the regularity near the piece of boundary \([1, 2] \times \{L\}\) in the same way and we notice that we do not have to worry about the corner since \(f_2\) is zero in some neighborhoods of the edges \([1] \times [0, L]\) and \([2] \times [0, L]\).
The case of \( f_1 \) is slightly simpler, take \( \psi_1 \) in \( H^1(\Omega_1) \) and such that \( \psi_1(x,0) = 0 \) for \(-1 \leq x \leq 1\), we have where \( g_1 \) is defined similarly as \( g_1 \). We obtain for \( f_1 \) a formula analogous to (4.2), from this formula and (4.3), we get

\[
\int_{\Omega_1} \alpha_f(\nabla f_1) \cdot (\nabla \psi_1) \, dx \, dy + \int_{\Omega_1} \beta_f(\partial_y f_1) \psi_1 \, dx \, dy = - \int_{\Omega_1} (\phi_1 u + g_1) \psi_1 \, dx \, dy;
\]

notice that in this case, as \( \phi_1(1) = 0 \), we do not have to deal with a term like the second integral in the right-hand member of (4.4). From this equality, we deduce

\[
\int_{\Omega_1} \alpha_f(\nabla f_1) \cdot (\nabla \psi_1) \, dx \, dy + \int_{\Omega_1} \beta_f(\partial_y f_1) \psi_1 \, dx \, dy = - \int_{\Omega_1} (\phi_1 u + g_1)^* \psi_1 \, dx \, dy;
\]

here \( g_1 \) and \( (\phi_1 u + g_1)^* \) are defined analogously as \( g_2 \) and \( (\phi_2 u + g_2)^* \). These computations show that \( f_1 \) is a weak solution of the following problem:

\[
\alpha_f \Delta f_1 - \beta_f \partial_y f_1 = (u \phi_1 + g_1)^* \text{ on } \Omega_1
\]

\[
f_1 = 0 \text{ on } \partial \Omega_1 \setminus \Gamma_1
\]

\[
\frac{df_1}{d\nu} = 0 \text{ on } \Gamma_1
\]

here, as usual, \( df_1/d\nu \) denotes the normal derivative and \( \Gamma_1 \) is the edge of the rectangle \( \Omega_1 \) defined as \( \Gamma_1 = [-1,1] \times \{L\} \). Classical results (see e.g. [7]) allow us to assert that \( f_1 \) is in \( H^2(\Omega_1) \). Concerning the regularity up to the boundary, we can argue exactly as in [7, Th. 4, p. 336]; thus, function \( f_1 \) is in \( H^2(\Omega_1) \). Consider now the function \( f_1 + f_2 \) restricted to \( \Omega_1 \), this function is in \( H^2(\Omega) \) and is equal to \( f \), thus we proved that \( f \in H^2(\Omega) \).

Take now \((f,p)\) in \( \ker(-L) \cap \ker G \), \((f,p)\) belongs to \( \mathcal{D}(A) \), so \( L(f,p) = (0,0) \) implies \( A(f,p) = (0,0) \) which in turn implies \((f,p) = (0,0)\) because \( A \) is injective. This achieve the proof of Theorem 4.2.

Thus, we can apply [19, Proposition 1.1.2]: there exist a unique operator \( B \in \mathcal{L}(U, X_{-1}) \) such that

\[
L = A + BG.
\]

Here \( A \) denotes the restriction of \( L \) to \( \ker G \) and is thus identical to the operator \( A \) defined in section 3; \( X_{-1} \) denotes the completion of the space \( X \) with respect to the norm \( \| (u,v) \|_{-1} = \| A^{-1}(u,v) \| \).

Operator \( A \) is m-dissipative, therefore, it is the generator of a contraction semigroup. Moreover from equalities (3.7) and (3.8), we obtain

\[
\langle A(f,p),(f,p) \rangle_{L^2(\Omega)} \leq - \| \nabla f \|^2 + \| \nabla p \|^2
\]

from the Poincaré’s inequality, we have

\[
\int_{\Omega} (\| \nabla f \|^2 + \| \nabla p \|^2) \, dx \, dy \geq C \int_{\Omega} (f^2 + p^2) \, dx \, dy,
\]

which implies that \( A \leq -CI \). Denoting by \( T_t \) the semigroup generated by \( A \), we thus have \( \| T_t \| \leq e^{-Ct} \). As \( A \) is the generator of a strongly continuous semigroup, for every \( T > 0 \), \((f_0,p_0) \in Z \), and \((u,v) \in U \) such that \( G(f_0,p_0) = (u,v) \), the equation

\[
\frac{df(p,t)}{dt} = L(f,p) = A(f,p) + BG(u,v),
\]

\[
(f(0),p(0)) = (f_0,p_0)
\]

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admits a unique solution \((f, p)\) such that \((f, p) \in C([0, f]; \mathbb{Z}) \cap C^1([0, f]; \mathbb{X})\). Moreover as \(T_t\) is exponentially stable, we have \(\lim_{t \to \infty}(f(t), p(t)) = (f_\infty, p_\infty)\) where \((f_\infty, p_\infty)\) is the unique solution of the equation \(BG(f_0, p_0) = -A(f_\infty, p_\infty)\). Thus, we have proved that, given any pair of initial conditions \((f_0, p_0)\) and any pair of input temperatures \((T_f, T_p)\), there exists a unique solution to system \((2.11)\) which tends exponentially towards an asymptotic state \((f_\infty, p_\infty)\) as \(t\) tends to infinity.

5. Co-current operator. In this section, we consider a DCMD model with co-current; the equations modeling this device are the same as \((2.11)\) except for the sign of \(\beta_p\): they write

\[
\begin{align*}
\tag{5.1}
\partial_t f(t, x, y) - \alpha_f \Delta f(t, x, y) + \beta_f \partial_y f(t, x, y) &= 0, & t \geq 0, (x, y) \in \Omega \\
\partial_t p(t, x, y) - \alpha_p \Delta p(t, x, y) + \beta_p \partial_y p(t, x, y) &= 0, & t \geq 0, (x, y) \in \Omega \\
f(t, x, 0) &= T_f, & t \geq 0, 0 \leq x \leq 1, \\
\partial_x f(t, 0, y) &= 0, & t \geq 0, 0 \leq y \leq L, \\
\partial_y f(t, x, L) &= 0, & t \geq 0, 0 \leq x \leq 1, \\
p(t, x, 0) &= T_p, & t \geq 0, 0 \leq x \leq 1, \\
\partial_x p(t, 0, y) &= 0, & t \geq 0, 0 \leq y \leq L, \\
\partial_y p(t, x, L) &= 0, & t \geq 0, 0 \leq x \leq 1, \\
\partial_x f(t, 1, y) &= -\gamma_f (f(t, 1, y) - p(t, 1, y)) & t \geq 0, 0 \leq y \leq L, \\
\partial_x p(t, 1, y) &= \gamma_p (f(t, 1, y) - p(t, 1, y)) & t \geq 0, 0 \leq y \leq L, \\
f(0, x, y) &= f_0(x, y) & (x, y) \in \Omega, \\
p(0, x, y) &= p_0(x, y) & (x, y) \in \Omega.
\end{align*}
\]

Introducing the following change of variables

\[
\tag{5.2}
g(t, x, y) = f(t, x, y)e^{-\frac{\beta_f}{2\alpha_f}y}, \\
q(t, x, y) = p(t, x, y)e^{-\frac{\beta_p}{2\alpha_p}y}.
\]

we then have

\[
\partial_{xx} g(t, x, y) = \partial_{xx} f(t, x, y) \exp\left(-\frac{\beta_f}{2\alpha_f}y\right),
\]

\[
\partial_y g(t, x, y) = \left(\partial_y f(t, x, y) - \frac{\beta_f}{2\alpha_f}f(x, y)\right) \exp\left(-\frac{\beta_f}{2\alpha_f}y\right),
\]

\[
\partial_{yy} g(t, x, y) = \left(\partial_{yy} f(t, x, y) - \frac{\beta_f}{\alpha_f}\partial_y f(t, x, y)\right) \exp\left(-\frac{\beta_f}{2\alpha_f}y\right) + \frac{\beta_f^2}{4\alpha_f^2} g(t, x, y),
\]

and so

\[
\left(\alpha_f \Delta f(t, x, y) - \beta_f \partial_y f(t, x, y)\right) \exp\left(-\frac{\beta_f}{2\alpha_f}y\right) = \alpha_f \Delta g(t, x, y) - \frac{\beta_f^2}{4\alpha_f} g(t, x, y).
\]

Similar computations lead to

\[
\left(\alpha_p \Delta p(t, x, y) - \beta_p \partial_y p(t, x, y)\right) \exp\left(-\frac{\beta_p}{2\alpha_p}y\right) = \alpha_p \Delta q(t, x, y) - \frac{\beta_p^2}{4\alpha_p} q(t, x, y).
\]
Regarding the boundary conditions, we have

\[ g(t, x, 0) = T_f \]
\[ \partial_x g(t, 0, y) = 0 \]
\[ \partial_y g(t, x, L) = -\frac{\beta_f}{2\alpha_f} g(t, x, L) \]
\[ q(t, x, 0) = T_p \]
\[ \partial_x q(t, 0, y) = 0 \]
\[ \partial_y q(t, x, L) = -\frac{\beta_p}{2\alpha_p} q(t, x, L) \]
\[ q(t, 1, y) = -\gamma_f \left( g(t, 1, y) - q(t, 1, y)e^{\left(-\frac{\beta_f}{2\alpha_f} + \frac{\beta_p}{2\alpha_p}\right)y}\right) \]
\[ \partial_x q(t, 1, y) = \partial_x p(t, 1, y)e^{-\frac{\beta_p}{2\alpha_p}y} = \gamma_p \left(e^{\left(\frac{\beta_f}{2\alpha_f} - \frac{\beta_p}{2\alpha_p}\right)y}g(1, y) - q(1, y)\right). \]

We assume that the flow velocities can be adjusted in such a way that

\[ \frac{\beta_f}{2\alpha_f} = \frac{\beta_p}{2\alpha_p}. \]  

Under the change of unknown functions (5.2) and with the assumption (5.3), system (5.1) becomes

\[
\begin{cases}
\partial_t g(t, x, y) - \alpha_f \Delta g(t, x, y) + \frac{\beta_f^2}{4\alpha_f} g(t, x, y) = 0, & t \geq 0, (x, y) \in \Omega \\
\partial_t q(t, x, y) - \alpha_p \Delta q(t, x, y) + \frac{\beta_p^2}{4\alpha_p} q(t, x, y) = 0 & t \geq 0, (x, y) \in \Omega \\
g(t, x, 0) = T_f & t \geq 0, 0 \leq x \leq 1, \\
\partial_x g(t, 0, y) = 0 & t \geq 0, 0 \leq y \leq L, \\
\partial_y g(t, x, L) = -\frac{\beta_f}{2\alpha_f} g(t, x, L) & t \geq 0, 0 \leq x \leq 1, \\
q(t, x, 0) = T_p & t \geq 0, 0 \leq x \leq 1, \\
\partial_x q(t, 0, y) = 0 & t \geq 0, 0 \leq y \leq L, \\
\partial_y q(t, x, 0) = -\frac{\beta_p}{2\alpha_p} q(t, x, 0) & t \geq 0, 0 \leq x \leq 1, \\
\partial_x q(t, 1, y) = -\gamma_f (g(t, 1, y) - q(t, 1, y)) & t \geq 0, 0 \leq y \leq L, \\
\partial_x q(t, 1, y) = \gamma_p (g(t, 1, y) - q(t, 1, y)) & t \geq 0, 0 \leq y \leq L, \\
g(0, x, y) = g_0(x, y) := f_0(x, y)e^{\left(-\frac{\beta_f}{2\alpha_f}y\right)} & (x, y) \in \Omega, \\
q(0, x, y) = q_0(x, y) := p_0(x, y)e^{\left(-\frac{\beta_p}{2\alpha_p}y\right)} & (x, y) \in \Omega.
\end{cases}
\]

We shall prove no that the operator, denoted by \( \tilde{A} \), and related to this system is diagonalizable. The spaces related to this operator will be slightly different from the ones related to operator \( A \). First, we introduce the space \( \tilde{E} \) defined as the set of those pairs \((g, q)\) in \([H^1(\Omega) \cap C^1(\Omega)]^2\) such that
• \( g(x,0) = q(x,0) = 0 \) for every \( x \in (0,1) \);
• \( \partial_x g(0,y) = \partial_x q(0,y) = 0 \), for every \( y \in (0,L) \);
• \( \partial_y g(x,L) = -\frac{\beta_L}{2\alpha_L} g(x,L) \) for every \( x \in (0,1) \);
• \( \partial_y q(x,L) = -\frac{\beta_L}{2\alpha_L} q(x,L) \) for every \( x \in (0,1) \);
• \( \partial_x g(1,y) = -\gamma_f (g(1,y) - q(1,y)) \), for every \( y \in (0,L) \);
• \( \partial_x q(1,y) = \gamma_p (g(1,y) - q(1,y)) \) for every \( y \in (0,L) \).

As section 3, the space \( H^1(\Omega) \times H^1(\Omega) \) is equipped with the product topology, making it an Hilbert space whose norm is defined by (3.1). On \( L^2(\Omega) \times L^2(\Omega) \), we consider again the inner product given by (3.2). We denote then by \( \hat{H}_{bc}^1 \) the closure of \( \bar{\mathcal{E}} \) in \( [H^1(\Omega)]^2 \); recall that the induced norm on \( \hat{H}_{bc}^1 \) is also defined by (3.3).

The operator \( \tilde{A} \) is then defined as follows: its domain is given by

\[
\mathcal{D}(\tilde{A}) := \left\{ (g, q) \in \hat{H}_{bc}^1 \mid (\Delta g, \Delta q) \in [L^2(\Omega)]^2 \right\} ;
\]

and, for every \( (g, q) \in \mathcal{D}(\tilde{A}) \),

\[
\tilde{A}(g, q) = \left( \frac{\beta_f^2}{4\alpha_f} q, \alpha_p \Delta q - \frac{\beta_p^2}{4\alpha_p} q \right).
\]

Using similar arguments as in section 3.1, we can prove that \( \tilde{A} \) is m-dissipative and diagonalizable; moreover reasoning as in section 4, we can prove that, given an initial condition in \( \mathcal{D}(\tilde{A}) \), system (5.4) has a unique solution and that this solution converges, as \( t \to \infty \), towards the solution \((g_\infty, q_\infty)\) of the equation \( \dot{B}\dot{G}(g_0, q_0) = -\tilde{A}(g_\infty, q_\infty) \).

6. Observer design for parabolic system coupled at the boundary. In this section, we deal with the problem of the design of an observer for system (5.1).

Our method is based on the backstepping technique, which was first introduced and developed by M. Krstic and A. Smyshlyaev; see [11] and references therein. Notice that in this book, all the considered systems are one-dimensional, in [13], T. Meurer extends the backstepping technique to higher dimensional systems that are defined on Parallelepiped domains. In our case, the domain is rectangular but the difficulty comes from the fact that we have to deal with two PDE coupled at the boundary. We start our analysis by the co-current configuration, then we shall briefly discuss the construction of the observer for the counter-current.

6.1. Observer design for co-current configuration. We shall consider the temperatures \( T_f \) and \( T_p \) as the inputs of the system and we assume that we measure the temperatures \( f(x, L, t) \) and \( p(x, L, t) \) in the co-current case, or \( f(x, L, t) \) and \( p(x, 0, t) \) in the counter-current configuration. As the operator \( A \) is m-dissipative (cf— Theorem, 3.5), an observer for system (5.1) could simply consist in a copy of the original system. However, this solution is not satisfactory if we want to get an estimate which converge with an arbitrary speed towards the state of the system. In the sequel, we will show that the design of such an observer is possible in some particular cases.

We rewrite system (5.1) as follows:

\[
(6.1) \quad \begin{pmatrix} f_t \\ p_t \end{pmatrix} = \Sigma \begin{pmatrix} \Delta f \\ \Delta p \end{pmatrix} + \Xi \begin{pmatrix} f_y \\ p_y \end{pmatrix},
\]

where \( \Sigma = \text{diag}(\alpha_f, \alpha_p) \) and \( \Xi = \text{diag}(-\beta_f, -\beta_p) \). Regarding the boundary condi-
tions, we write them as follows

\[
\begin{pmatrix}
  f \\
p
\end{pmatrix}
\bigg|_{y=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
  f_y \\
p_y
\end{pmatrix}
\bigg|_{y=L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
  f_x \\
p_x
\end{pmatrix}
\bigg|_{x=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

\[
\begin{pmatrix}
  f_x \\
p_x
\end{pmatrix}
\bigg|_{x=1} = \Gamma \cdot \begin{pmatrix} f \\
p \end{pmatrix}
\bigg|_{x=1}
\]

where \( \Gamma \) is the following matrix

\[
\Gamma = \begin{pmatrix}
-\gamma_f & \gamma_f \\
\gamma_p & -\gamma_p
\end{pmatrix}.
\]

We consider the following auxiliary system, which has the following observer structure:

\[
\begin{pmatrix}
\hat{f} \\
\hat{p}
\end{pmatrix}
\bigg|_{y=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
\hat{f}_x \\
\hat{p}_x
\end{pmatrix}
\bigg|_{x=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
\hat{f}_x \\
\hat{p}_x
\end{pmatrix}
\bigg|_{x=1} = \Gamma \cdot \begin{pmatrix} \hat{f} \\
\hat{p} \end{pmatrix}
\bigg|_{x=1}
\]

associated to the following boundary conditions

\[
\begin{pmatrix}
\hat{f} \\
\hat{p}
\end{pmatrix}
\bigg|_{y=L} = \begin{pmatrix} f \\
p \end{pmatrix}
\bigg|_{y=L} + L_{10} \cdot \begin{pmatrix} f - \hat{f} \\
p - \hat{p} \end{pmatrix}
\bigg|_{y=L}.
\]

Let us introduce the following state estimation error \( \tilde{f} = f - \hat{f} \) and \( \tilde{p} = p - \hat{p} \), then \( \hat{f} \) and \( \hat{p} \) satisfy the following equations:

\[
\begin{pmatrix}
\tilde{f} \\
\tilde{p}
\end{pmatrix}
\bigg|_{y=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
\tilde{f}_x \\
\tilde{p}_x
\end{pmatrix}
\bigg|_{x=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
\tilde{f}_x \\
\tilde{p}_x
\end{pmatrix}
\bigg|_{x=1} = \Gamma \cdot \begin{pmatrix} \tilde{f} \\
\tilde{p} \end{pmatrix}
\bigg|_{x=1}
\]

and, as regards the boundary conditions

\[
\begin{pmatrix}
\tilde{f} \\
\tilde{p}
\end{pmatrix}
\bigg|_{y=L} = -L_{10}(t) \begin{pmatrix} \tilde{f} \\
\tilde{p} \end{pmatrix}
\bigg|_{y=L}.
\]

6.2. Target system. Let us introduce the \((v, w)\)-target system written as

\[
\begin{pmatrix}
v \\
w
\end{pmatrix}
\bigg|_{y=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
v_x \\
w_x
\end{pmatrix}
\bigg|_{x=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
v_y \\
w_y
\end{pmatrix}
\bigg|_{y=L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

where \( \Lambda = \text{diag}(\lambda_1, \lambda_2) \) and \( \lambda_1, \lambda_2 > 0 \). For the boundary conditions, we have

\[
\begin{pmatrix}
v \\
w
\end{pmatrix}
\bigg|_{y=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
v_x \\
w_x
\end{pmatrix}
\bigg|_{x=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
v_y \\
w_y
\end{pmatrix}
\bigg|_{y=L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
and

\[(6.11) \quad \left( \begin{array}{c} v_x \\ w_x \end{array} \right) \bigg|_{x=1} = \Gamma \cdot \left( \begin{array}{c} v \\ w \end{array} \right) \bigg|_{x=1}. \]

Thus, the observer gains \(L_1\) and \(L_{10}\) need to be determined such that the observation error converges to zeros when the time tends to infinity. To do so, the following backstepping transformation is applied,

\[(6.12) \quad \left( \begin{array}{c} \tilde{f} \\ \tilde{p} \end{array} \right) = \left( \begin{array}{c} v \\ w \end{array} \right) - \int_y^L K(y, \xi) \cdot \left( \begin{array}{c} v(x, \xi, t) \\ w(x, \xi, t) \end{array} \right) d\xi \]

such that the observation error system (6.6)-(6.8) can be transformed in to the target system (6.9)-(6.11). In order to obtain the transformation (6.12), the kernel matrix \(K(y, \xi)\) needs to be determined.

**Lemma 6.1.** The parabolic system (6.9)-(6.11) is exponentially stable in the \(L^2\)-norm.

The proof of Lemma 6.1 follows directly from a Lyapunov type argument.

**Proof.** Let us consider the following Lyapunov functional

\[(6.13) \quad V(v, w) = \frac{\alpha_f \gamma_p}{2} \int_\Omega v^2 dx dy + \frac{\alpha_p \gamma_f}{2} \int_\Omega w^2 dx dy, \]

along a solution trajectory of (6.9)-(6.11), i.e., the derivative of the norm along the trajectories of the system

\[(6.14) \quad \frac{d}{dt} V(v, w) = \alpha_p \gamma_p \int_\Omega v(\alpha_f \Delta v - \beta_f v_y - \lambda_1 v) dx dy + \alpha_f \gamma_f \int_\Omega (\alpha_p \Delta w - \beta_p w - \lambda_2) dx dy. \]

We have

\[- \beta_f \int_\Omega vv_y dx dy = - \beta_f \int_0^1 \frac{v(x, L)^2}{2} dx \leq 0; \]

\[- \beta_p \int_\Omega ww_y dx dy = - \beta_p \int_0^1 \frac{w(x, L)^2}{2} dx \leq 0. \]

On the other hand

\[\alpha_f \int_\Omega vv_y dx dy = - \alpha_f \int_\Omega (v_y)^2 \leq 0; \quad \alpha_p \int_\Omega ww_y dx dy = - \alpha_p \int_\Omega (w_y)^2 \leq 0. \]

Moreover, we have

\[\int_\Omega vv_{xx} dx dy = \int_0^L v(1, y)v_x(1, y) dy - \int_\Omega (v_x)^2 dx dy; \]

\[\int_\Omega ww_{xx} dx dy = \int_0^L w(1, y)w_x(1, y) dy - \int_\Omega (w_x)^2 dx dy. \]
By combining all this terms, we obtain
\begin{equation}
\frac{d}{dt} V(v, w) = -\alpha_p \gamma_p \alpha_f \int_{\Omega} (\nabla v)^2 dx dy - \alpha_p \gamma_p \beta_f \int_{0}^{1} \frac{v(x, L)^2}{2} dx - \alpha_p \gamma_p \lambda_1 \int_{\Omega} v^2(x, y) dx dy
\end{equation}

\begin{equation}
- \alpha_f \gamma_f \alpha_p \int_{\Omega} (\nabla w)^2 dx dy - \alpha_f \gamma_f \beta_p \int_{0}^{1} \frac{w(x, L)^2}{2} dx - \alpha_f \gamma_f \lambda_2 \int_{\Omega} w^2(x, y) dx dy
\end{equation}

\begin{equation}
- \alpha_p \gamma_p \alpha_f \gamma_f \int_{0}^{L} (v(1, y) - w(1, y))^2 dy.
\end{equation}

Finally, by applying Gronwall's inequality the exponential stability of the target dynamics is proved.

6.3. Particular case for the co-current configuration: $\beta_f/\alpha_f = \beta_p/\alpha_p$. In this subsection, the kernel function $K(y, \xi)$ in the backstepping transformation (6.12) and the observer gains $L_1$ and $L_{10}$ in (6.6), (6.8) will be determined. First, $K(y, \xi)$ is characterized by a set of PDEs involving $L_1$ and $L_{10}$. Then, explicit expressions are deduced for $K(y, \xi)$, $L_1$ and $L_{10}$, respectively.

**Theorem 6.2.** We assume that $\beta_f/\alpha_f = \beta_p/\alpha_p$, the kernel function $K(y, \xi)$ is given by:

\begin{equation}
K = \text{diag}(K_1, K_2),
\end{equation}

where

\begin{equation}
K_1(y, \xi) = -e^{-\frac{\beta_f}{\alpha_f}(y-\xi)} \mu_1 y I_1(\sqrt{\mu_1(\xi^2 - y^2)}),
\end{equation}

\begin{equation}
K_2(y, \xi) = -e^{-\frac{\beta_f}{\alpha_f}(y-\xi)} \mu_2 y I_1(\sqrt{\mu_2(\xi^2 - y^2)}),
\end{equation}

for $(y, \xi) \in \mathcal{T} = \{x \in (0, L); 0 \leq y \leq L, 0 \leq \xi \leq y\}$, $\text{diag}(\mu_1, \mu_2) = \Sigma^{-1}$. $I_1$ is the well-known modified Bessel function of the first kind [11]. Then, the transformation (6.12) can transform the observation error system (6.6)-(6.8) into the target system (6.9)-(6.11). Moreover, the observer gains $L_1$ and $L_{10}$ are given as follows:

\begin{equation}
L_1(y) = -\Sigma \cdot K_1(y, L) + \Xi \cdot K(y, L)
\end{equation}

\begin{equation}
L_{10} = K(L, L) = \frac{L}{2} \Sigma^{-1} \Lambda.
\end{equation}

**Proof.** In this proof, it will be shown how to transform the observation error system (6.6)-(6.8) into the target system (6.9)-(6.11), where the conditions on $K(y, \xi)$ are imposed. Hence, the following two steps are needed.

**Step 1. Kernel system:** By differentiating both sides of (6.6) with respect to $x$, $y$ and $t$, we obtain

\begin{equation}
\begin{pmatrix}
\dot{f}_t \\
\dot{f}_y
\end{pmatrix}
= \begin{pmatrix}
v_t \\
w_t
\end{pmatrix}
- \int_{y}^{L} \left( K(y, \xi, t) \cdot \begin{pmatrix} v(x, \xi, t) \\ w(x, \xi, t) \end{pmatrix} + K(y, \xi, t) \cdot \begin{pmatrix} v_t(x, \xi, t) \\ w_t(x, \xi, t) \end{pmatrix} \right) d\xi
\end{equation}

\begin{equation}
= \Sigma \cdot \left( \Delta \dot{f} \right) + \Xi \cdot \begin{pmatrix}
\dot{f}_y \\
\dot{f}_y
\end{pmatrix}
- L_1(y, t) \cdot \begin{pmatrix}
\dot{f}_t \\
\dot{f}_y
\end{pmatrix}
\bigg|_{y=L},
\end{equation}

(6.17)

(6.18)
or

\[
\left( \frac{f_{xx}}{p_{xx}} \right) = \left( \frac{v_{xx}}{w_{xx}} \right) - \int_y^L K \cdot \left( \frac{v_{xx}}{w_{xx}} \right) \, d\xi
\]

\[
\left( \frac{f_y}{p_y} \right) = \left( \frac{v_y}{w_y} \right) + K(y, y, t) \cdot \left( \frac{v}{w} \right) - \int_y^L K_y \cdot \left( \frac{v}{w} \right) \, d\xi
\]

\[
\left( \frac{f_{yy}}{p_{yy}} \right) = \left( \frac{v_{yy}}{w_{yy}} \right) + \frac{\partial}{\partial y} \left( K(y, y, t) \cdot \left( \frac{v}{w} \right) \right) + K_y(y, y, t) \cdot \left( \frac{v}{w} \right) - \int_y^L K_{yy} \cdot \left( \frac{v}{w} \right) \, d\xi.
\]

On the other hand, by integrating by parts, we have

\[
\int_y^L K \cdot \Xi \cdot \left( \frac{v_y}{w_y} \right) \, d\xi = \left[ K \cdot \Xi \cdot \left( \frac{v}{w} \right) \right]_{\xi=L}^{\xi=L} - \int_y^L K_{\xi} \cdot \Xi \cdot \left( \frac{v}{w} \right) \, d\xi,
\]

and

\[
\int_y^L K \cdot \Sigma \cdot \left( \frac{v_{yy}}{w_{yy}} \right) \, d\xi = \left[ K \cdot \Sigma \cdot \left( \frac{v_y}{w_y} \right) \right]_{\xi=L}^{\xi=L} - \left[ K_{\xi} \cdot \Sigma \cdot \left( \frac{v}{w} \right) \right]_{\xi=L}^{\xi=L} + \int_y^L K_{\xi \xi} \cdot \Sigma \cdot \left( \frac{v}{w} \right) \, d\xi
\]

\[
= -K(y, y, t) \cdot \Sigma \cdot \left( \frac{v_y}{w_y} \right) - K_{\xi}(y, L, t) \cdot \Sigma \cdot \left( \frac{v}{w} \right)_{y=L}
\]

\[
+ K_{\xi}(y, y, t) \cdot \Sigma \cdot \left( \frac{v}{w} \right) + \int_y^L K_{\xi \xi} \cdot \Sigma \cdot \left( \frac{v}{w} \right) \, d\xi.
\]

From \( v_y(x, L, t) = w_y(x, L, t) = 0 \), we have also

\[
\frac{\partial}{\partial y} \int_y^L K \cdot \left( \frac{v}{w} \right) \, d\xi = -K(y, y, t) \cdot \left( \frac{v}{w} \right) + \int_y^L K_y \cdot \left( \frac{v}{w} \right) \, d\xi
\]

\[
\frac{\partial^2}{\partial y^2} \int_y^L K \cdot \left( \frac{v}{w} \right) \, d\xi = -\frac{\partial}{\partial y} \left( K(y, y, t) \cdot \left( \frac{v}{w} \right) \right) - K_y(y, y, t) \cdot \left( \frac{v}{w} \right) + \int_y^L K_{yy} \cdot \left( \frac{v}{w} \right) \, d\xi
\]

By expressing the right-hand side of (6.17) based on \( v \) and \( w \) we get

\[
\left( \frac{\tilde{f}_t}{\tilde{p}_t} \right) = \Sigma \cdot \left( \frac{\Delta v}{\Delta w} \right) + \Xi \cdot \left( \frac{v_y}{w_y} \right) - \Lambda \cdot \left( \frac{v}{w} \right) - \int_y^L K_t \cdot \left( \frac{v}{w} \right) \, d\xi
\]

\[
- \int_y^L K \cdot \Sigma \cdot \left( \frac{v_{xx}}{w_{xx}} \right) \, d\xi - \int_y^L K_{\xi \xi} \cdot \Sigma \cdot \left( \frac{v}{w} \right) \, d\xi + \int_y^L K_{\xi} \cdot \Xi \cdot \left( \frac{v}{w} \right) \, d\xi
\]

\[
+ \int_y^L K \cdot \Lambda \cdot \left( \frac{v}{w} \right) \, d\xi + K(y, y, t) \cdot \Sigma \cdot \left( \frac{v_y}{w_y} \right) + K_{\xi}(y, L, t) \cdot \Sigma \cdot \left( \frac{v}{w} \right)_{y=L}
\]

\[
(6.19) \quad - K_{\xi}(y, y, t) \cdot \Sigma \cdot \left( \frac{v}{w} \right) - K(y, L, t) \cdot \Xi \cdot \left( \frac{v}{w} \right)_{y=L} + K(y, y, t) \cdot \Xi \cdot \left( \frac{v}{w} \right).
\]
By expressing the right-hand side of (6.18) based on $v$ and $w$, we deduce

\[
\left(\frac{\dot{I}}{p}\right) = \Sigma \cdot \left(\frac{\Delta v}{\Delta w}\right) + \Xi \cdot \left(\frac{v_y}{w_y}\right) - L_1(y, t) \cdot \left(\frac{v}{w}\right)_{y=L} \\
- \int_y^L \Sigma \cdot K \cdot \left(\frac{v_x}{w_{xx}}\right) \, d\xi - \int_y^L \Sigma \cdot K_{yy} \cdot \left(\frac{v}{w}\right) \, d\xi - \int_y^L \Xi \cdot K_y \cdot \left(\frac{v}{w}\right) \, d\xi \\
+ 2\Sigma \cdot K_y(y, y, t) \cdot \left(\frac{v_y}{w_y}\right) + \Sigma \cdot K_\xi(y, y, t) \cdot \left(\frac{v}{w}\right)
\]

(6.20) \quad + \Sigma \cdot K(y, y, t) \cdot \left(\frac{v_y}{w_y}\right) + \Xi \cdot K(y, y, t) \cdot \left(\frac{v}{w}\right).

By making the difference between (6.19) and (6.20), we obtain

\[
0 = \left(-\Lambda - K_\xi(y, y, t) \cdot \Sigma + K(y, y, t) \cdot \Xi - 2\Sigma \cdot K_y(y, y, t) - \Sigma \cdot K_\xi(y, y, t)
\right) \\
- \Xi \cdot K(y, y, t) \cdot \left(\frac{v}{w}\right)
\]

\[
+ \left(K(y, y, t) \cdot \Sigma - \Sigma \cdot (y, y, t)\right) \cdot \left(\frac{v_y}{w_y}\right)
\]

\[
+ (K_\xi(y, L, t) \cdot \Sigma - K(y, L, t) \cdot \Xi + L_1(y, t)) \cdot \left(\frac{v}{w}\right)_{y=L}
\]

\[
- \int_y^L K_t \cdot \left(\frac{v}{w}\right) \, d\xi
\]

\[
- \int_y^L (K_{\xi y} \cdot \Sigma - \Sigma \cdot K_{yy}) \cdot \left(\frac{v}{w}\right) \, d\xi + \int_y^L (K_\xi \cdot \Xi + \Xi \cdot K_y) \cdot \left(\frac{v}{w}\right) \, d\xi
\]

\[
+ \int_y^L K \cdot \Lambda \cdot \left(\frac{v}{w}\right) \, d\xi - \int_y^L (K \cdot \Sigma - \Sigma \cdot K) \cdot \left(\frac{v_{xx}}{w_{xx}}\right) \, d\xi.
\]

Therefore it is necessary that the last integral to be zero, we must have $K \cdot \Sigma - \Sigma \cdot K = 0$ which is equivalent to the matrix $K$ is diagonal. So, we also have $K \cdot \Xi = \Xi \cdot K$. Then, the above equality is simplified

(6.21a) \quad 0 = \left(-\Lambda - 2\Sigma \cdot K_\xi(y, y, t) - 2\Sigma \cdot K_y(y, y, t)\right) \cdot \left(\frac{v}{w}\right)

(6.21b) \quad + (K_\xi(y, L, t) \cdot \Sigma - K(y, L, t) \cdot \Xi + L_1(y, t)) \cdot \left(\frac{v}{w}\right)_{y=L}

(6.21c) \quad - \int_0^\eta K_t \cdot \left(\frac{v}{w}\right) \, d\xi

(6.21d) \quad - \int_0^\eta (K_{\xi y} \cdot \Sigma - \Sigma \cdot K_{yy}) \cdot \left(\frac{v}{w}\right) \, d\xi + \int_0^\eta (K_\xi \cdot \Xi + \Xi \cdot K_y) \cdot \left(\frac{v}{w}\right) \, d\xi

(6.21e) \quad + \int_0^\eta K \cdot \Lambda \cdot \left(\frac{v}{w}\right) \, d\xi.

The kernel $K$ not dependent on time as well and we then see that $K$ should satisfies system (6.21d), (6.21e)

\[
\Sigma \cdot (K_{yy} - K_{\xi y}) + \Xi (K_\xi + K_y) + \Lambda \cdot K = 0,
\]

\[
\Sigma \cdot \left(\frac{K_{yy}}{w_{yy}} - \frac{K_{\xi y}}{w_{\xi y}}\right) + \Xi \left(\frac{K_\xi}{w_\xi} + \frac{K_y}{w_y}\right) + \Lambda \cdot K = 0,
\]

\[
\Sigma \cdot \left(\frac{K_{yy}}{w_{yy}} - \frac{K_{\xi y}}{w_{\xi y}}\right) + \Xi \left(\frac{K_\xi}{w_\xi} + \frac{K_y}{w_y}\right) + \Lambda \cdot K = 0.
\]
with boundary conditions (6.21a)

(6.22) \[ 2\Sigma \cdot (K_\xi(y, y) + K_y(y, y)) = -\Lambda. \]

In addition, the kernel \( L_1 \) must be chosen such that (6.21b) is null

\[ L_1(y) = -\Sigma \cdot K_\xi(y, L) + \Xi \cdot K(y, L). \]

We must also consider the boundary conditions for \((\tilde{f}, \tilde{p}), (v, w)\);

**Step 2: Boundary conditions:** Firstly

\[ \left\{ \begin{array}{c} \tilde{f}\bigg|_{y=0} = \left( \begin{array}{c} v \\ w \end{array} \right) \bigg|_{y=0} - \int_0^L K(0, \xi) \cdot \left( \begin{array}{c} v \\ w \end{array} \right) d\xi = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \\ \tilde{p}\bigg|_{y=L} = \left( \begin{array}{c} v_y \\ w_y \end{array} \right) \bigg|_{y=L} + K(L, L) \cdot \left( \begin{array}{c} v \\ w \end{array} \right) \bigg|_{y=L} = L_{10} \cdot \left( \begin{array}{c} \tilde{f} \\ \tilde{p} \end{array} \right) \bigg|_{y=L} \end{array} \right. \]

which, given the boundary conditions of \((v, w)\), implies

(6.23) \[ K(0, \xi) = 0, \forall \xi; \]

on the other hand, we also need

\[ \left. \begin{array}{c} \tilde{f}_y \bigg|_{y=L} = \left( \begin{array}{c} v_y \\ w_y \end{array} \right) \bigg|_{y=L} + K(L, L) \cdot \left( \begin{array}{c} v \\ w \end{array} \right) \bigg|_{y=L} = L_{10} \cdot \left( \begin{array}{c} \tilde{f} \\ \tilde{p} \end{array} \right) \bigg|_{y=L} \end{array} \right. \]

which considering the boundary conditions of \((v, w)\) – system, the kernel \( L_{10} \) must be chosen as

(6.24) \[ L_{10} = K(L, L). \]

Therefore, (6.22) and (6.24) can be grouped into

\[ 2\Sigma \cdot K(y, y) = (L - y)\Lambda + 2\Sigma \cdot K(L, L). \]

Moreover, from (6.23), we must also have \( K(0, 0) = 0 \) and \( K(L, L) \), must be chosen equal to

\[ K(L, L) = \frac{L}{2} \Sigma^{-1} \Lambda. \]

Finally, \( K \) is a solution of

\[ \Sigma \cdot (K_{yy} - K_{\xi\xi}) + \Xi(K_\xi + K_y) + \Lambda \cdot K = 0 \]

\[ K(0, \xi) = 0 \]

\[ K(y, y) = -\frac{y}{2} \Sigma^{-1} \cdot \Lambda. \]

Based on the techniques used in [11], we obtain

\[ \tilde{K}_i(y, \xi) = -\mu_i y \frac{I_1(\sqrt{\mu_i(y^2 - \xi^2)})}{\sqrt{\mu_i(y^2 - \xi^2)}} = -\mu_i y \sum_{n \geq 0} \frac{(\mu_i(y^2 - \xi^2))^n}{2^{2n+1}n!(n+1)!}. \]

where

(6.25) \[ \tilde{K}_i(y, \xi) = K_i(y, \xi) \cdot e^{yS_1 + \xi S_2} \quad i \in \{1, 2\}. \]
\[ S_1 = \frac{1}{2} \Sigma^{-1} \cdot \Xi, \quad S_2 = -\frac{1}{2} \Sigma^{-1} \cdot \Xi \]

and \( \mu_1, \mu_2 \), are given in the formulation of the Theorem 6.2.

At the boundary \( \{ x = 1 \} \), we have
\[
\left( \frac{\tilde{f}_x}{\tilde{p}_x} \right)_{x=1} = \Gamma \cdot \left( \frac{f}{p} \right)_{x=1},
\]

then
\[
(6.26) \quad \left( \frac{v_x}{w_x} \right)_{x=1} - \int_y^L K \cdot \left( \frac{v_x}{w_x} \right)_{x=1} \, d\xi = \Gamma \cdot \left( \frac{v}{w} \right)_{x=1} - \int_y^L \Gamma \cdot K \cdot \left( \frac{v}{w} \right)_{x=1} \, d\xi.
\]

The \((v, w)\) - system satisfies
\[
\left( \frac{v_x}{w_x} \right)_{x=1} = \Gamma \cdot \left( \frac{v}{w} \right)_{x=1}.
\]

Let us introduce the transformation \( \Theta_K \)
\[
\Theta_K(\varphi) = \varphi - \int_y^L K \cdot \varphi.
\]

Then, the boundary condition can be written
\[
(6.27) \quad \left( \frac{v_x}{w_x} \right)_{x=1} = \Theta_K^{-1} \left( \Gamma \cdot \Theta_K \left( \frac{v}{w} \right) \right)_{x=1}.
\]

The inverse transformation is given by
\[
\Theta_K^{-1} \left( \frac{\tilde{f}}{\tilde{p}} \right) = \left( \frac{v}{w} \right) = \left( \frac{\tilde{f}}{\tilde{p}} \right) + \int_y^L J(y, \xi) \cdot \left( \frac{\tilde{f}(x, \xi, t)}{\tilde{p}(x, \xi, t)} \right) \, d\xi
\]

where \( J = \text{diag}(J_1, J_2) \). There exists \( \tilde{J}_1 \) and \( \tilde{J}_2 \) such that
\[
(6.28) \quad \tilde{K}_i(y, \xi) = \tilde{J}_i(y, \xi) - \int_y^\xi \tilde{J}_i(y, \sigma) \tilde{K}_i(\sigma, \xi) \, d\sigma,
\]
\[
(6.29) \quad \tilde{J}_i(y, \xi) = \tilde{J}_i(y, \xi) - \int_y^\xi \tilde{K}_i(y, \sigma) \tilde{J}_i(\sigma, \xi) \, d\sigma, \quad i = 1, 2
\]

where
\[
\tilde{J}_i(y, \xi) = -\mu_i y \sum_{n \geq 0} (-1)^n \frac{(\mu_i (\xi^2 - y^2))^n}{2^{2n+1} n! (n + 1)!}.
\]

From (6.25), we have
\[
K_1(y, \xi) = e^{\frac{\sigma_p}{2\pi \tau} (y - \xi)} \tilde{K}_1(y, \xi), \quad K_2(y, \xi) = e^{\frac{\sigma_p}{2\pi \tau} (y - \xi)} \tilde{K}_2(y, \xi).
\]
The kernels \( J_1 \) and \( J_2 \) can be given explicitly by

\[
J_1(y, \xi) = e^{\frac{\beta_f}{\alpha_f}(y-\xi)} \tilde{J}_1(y, \xi), \quad J_2(y, \xi) = e^{\frac{\beta_p}{\alpha_p}(y-\xi)} \tilde{J}_2(y, \xi),
\]

where \( K_1 \) and \( J_i \) satisfy the equations (6.28) and (6.29). We are now able to explain the relation (6.27), we have

\[
\Gamma \cdot \Theta_K \left( \frac{v}{w} \right) = \left( -\gamma_f \left( v(x, y) - \int_y^L K_1(y, \xi)v(x, \xi) \, d\xi - w(x, y) + \int_y^L K_2(y, \xi)w(x, \xi) \, d\xi \right) \right).
\]

Therefore,

\[
v_x(1, y) = -\gamma_f \left( v(1, y) - \int_y^L K_1(y, \xi)v(1, \xi) \, d\xi - w(1, y) + \int_y^L K_2(y, \xi)w(1, \xi) \, d\xi \right)
\]

\[
+ \int_y^L J_1(y, \xi)v(1, \xi) \, d\xi - \int_y^L \int_\xi^L J_1(y, \xi)K_1(\xi, \sigma)v(1, \sigma) \, d\sigma \, d\xi
\]

\[
- \int_y^L J_1(y, \xi)w(1, \xi) \, d\xi + \int_y^L \int_\xi^L J_1(y, \xi)K_2(\xi, \sigma)w(1, \sigma) \, d\sigma \, d\xi
\]

\[
= -\gamma_f \left( v(1, y) - w(1, y) + \int_y^L K_2(y, \xi)w(1, \xi) \, d\xi \right)
\]

\[
(6.30) \quad - \int_y^L J_1(y, \xi)w(1, \xi) \, d\xi + \int_y^L \int_\xi^L J_1(y, \xi)K_2(\xi, \sigma)w(1, \sigma) \, d\sigma \, d\xi.
\]

From (6.28), we deduce

\[
w_x(1, y) = \gamma_p \left( v(1, y) - w(1, y) - \int_y^L K_1(y, \xi)v(1, \xi) \, d\xi \right)
\]

\[
+ \int_y^L J_2(y, \xi)v(1, \xi) \, d\xi - \int_y^L \int_\xi^L J_2(y, \xi)K_1(\xi, \sigma)v(1, \sigma) \, d\sigma \, d\xi
\]

\[
(6.31) \quad + \int_y^L J_2(y, \xi)v(1, \xi) \, d\xi - \int_y^L \int_\xi^L J_2(y, \xi)K_1(\xi, \sigma)v(1, \sigma) \, d\sigma \, d\xi.
\]

We thus obtain the same boundary condition at \( x = 1 \) for \( \tilde{f} \) and \( \tilde{p} \). We have

\[
\int_y^L K_2(y, \xi)w(1, \xi) \, d\xi - \int_y^L J_1(y, \xi)w(1, \xi) \, d\xi + \int_y^L \int_\xi^L J_1(y, \xi)K_2(\xi, \sigma)w(1, \sigma) \, d\sigma \, d\xi
\]

\[
= \int_y^L \left( K_2(y, \xi) - J_1(y, \xi) + \int_\xi^L J_1(y, \sigma)K_2(\sigma, \xi) \, d\sigma \right)w(1, \xi) \, d\xi.
\]

The expression appearing under the integral is written

\[
K_2(y, \xi) - J_1(y, \xi) + \int_\xi^L J_1(y, \sigma)K_2(\sigma, \xi) \, d\sigma =
\]

\[
e^{\frac{\beta_f}{\alpha_f}(y-\xi)} \tilde{K}_2(y, \xi) - e^{\frac{\beta_f}{\alpha_f}(y-\xi)} \tilde{J}_1(y, \xi) + e^{\frac{\beta_f}{\alpha_f}y} e^{-\frac{\beta_f}{\alpha_f} \xi} \int_0^\xi e^{\left( \frac{\beta_f}{\alpha_f} - \frac{\beta_p}{\alpha_p} \right) \xi} \tilde{J}_1(y, \sigma)K_2(\sigma, \xi) \, d\sigma.
\]

Under the hypothesis \( \beta_f/\alpha_f = \beta_p/\alpha_p \), the previously expression is equal to zero. \( \square \)
In this section, an observer to estimate the temperature of the feed and the permeate by boundary measurements is introduced. This result is valid only for a particular case $\beta_f/\alpha_f = \beta_p/\alpha_p$. This condition means that the speed convection terms $(\beta_f, \beta_p)$ for both feed and permeate are constrained. Based on this condition we are able to estimate the temperature $(f, p)$ on all the domain. We will now discuss firstly the general case for the co-current configuration and after we will consider the more complicated counter-current configuration.

7. Discussion on the observer design. We now examine the co-current configuration where $\beta_f/\alpha_f \neq \beta_p/\alpha_p$. From (6.30) and (6.31), the derivative of the Lyapunov function (6.13) along the trajectories of the system (7.1)

$$\frac{d}{dt} V(v, w) = - \alpha_p \gamma_p \alpha_f \int_\Omega (\nabla v)^2 dx dy - \alpha_p \gamma_p \beta_f \int_0^1 \frac{v(x, L)^2}{2} dx - \alpha_p \gamma_p \lambda_1 \int_\Omega v^2(x, y) dx dy$$

$$- \alpha_f \gamma_f \alpha_p \int_\Omega (\nabla w)^2 dx dy - \alpha_f \gamma_f \beta_p \int_0^1 \frac{w(x, L)^2}{2} dx - \alpha_f \gamma_f \lambda_2 \int_\Omega w^2(x, y) dx dy$$

$$- \delta \int_0^L (v(1, y) - w(1, y))^2 dy - \delta \int_0^L v(1, y) \int_y^L K_2(y, \xi) w(1, \xi) d\xi dy$$

$$+ \delta \int_0^L v(1, y) \int_y^L J_1(y, \xi) w(1, \xi) d\xi dy$$

$$- \delta \int_0^L v(1, y) \int_y^L \int_\xi^L J_1(y, \xi) K_2(\xi, \sigma) w(1, \sigma) d\sigma d\xi dy$$

$$- \delta \int_0^L w(1, y) \int_y^L K_1(y, \xi) v(1, \xi) d\xi dy + \delta \int_0^L w(1, y) \int_y^L J_2(y, \xi) v(1, \xi) d\xi dy$$

$$- \delta \int_0^L w(1, y) \int_y^L \int_\xi^L J_2(y, \xi) K_1(\xi, \sigma) v(1, \sigma) d\sigma d\xi dy$$

where $\delta = \alpha_f \alpha_p \gamma_f \gamma_p$. The problem is when we take a large value of $\lambda_i$, $K_1(y, \xi)$ becomes equivalent to $y e^{\sqrt{\mu_i (\xi^2 - y^2)}} (\mu_i (\xi^2 - y^2))^{-1/2}$. Then, we can not bound and absorb all of the previous kernel integrals.

We will now discuss the counter-current configuration. The counter-current configuration remains the more complicated case because we have two controls in two different directions; A control at $\{y = 0\}$ for the temperature feed $f$ and a control localized at the boundary $\{y = L\}$ for the permeate temperature $p$. However, we propose an observer design for a particular case $\beta_f/\alpha_f = \beta_p/\alpha_p$ in the co-current configuration. This result is obtained because our operator is self-adjoint. Let us introduce the backstepping transformation for the second equation

$$\tilde{p} = w - \int_0^y K_2(y, \xi) \cdot w(x, \xi) d\xi.$$
Then, we have

\begin{equation}
\dot{p} = \alpha_p \Delta w + \beta_p w_y - \lambda_2 w
- \int_0^y \alpha_p K_2(y, \xi) w_{xx}(y, \xi) \, d\xi - \int_0^y \alpha_p (K_2(y, \xi)) \xi w(y, \xi) \, d\xi + \int_0^y \beta_p (K_2(y, \xi)) \xi w(y, \xi) \, d\xi
+ \int_0^y \lambda_2 K_2(y, \xi) w(y, \xi) \, d\xi - \alpha_p K_2(y, y) w_y(y, y) + \alpha_p (K_2(y, y)) w(y, y)
- \alpha_p (K_2(y, 0)) w(x, 0) - \beta_p K_2(y, y) w(y, y) + \beta_p K_2(y, 0) w(x, 0),
\end{equation}

\begin{equation}
\dot{p} = \alpha_p \Delta w + \beta_p w_y - L_1(y, t) w(x, 0)
- \int_0^y \alpha_p K_2 w_{xx} \, d\xi - \alpha_p \int_0^y (K_2)_{yy} w \, d\xi - \beta_p \int_0^y (K_2)_y w \, d\xi
- 2 \alpha_p (K_2)_{y} w - \alpha_p (K_2)_{\xi} w - \alpha_p K_2(y, y) w_y - \beta_p K_2(y, y) w.
\end{equation}

Taking the difference between (7.2) and (7.3), we get

\begin{align*}
0 &= (-\lambda_2 + \alpha_p (K_2)_{\xi}(y, y) + \beta_p K_2(y, y) + 2 \alpha_p (K_2)_{y}(y, y) + \alpha_p (K_2)_\xi(y, y) + \beta_p K_2(y, y)) w
+ (\beta_p - \alpha_p K_2(y, y) - \beta_p + \alpha_p K_2(y, y)) w_y
+ (-\alpha_p (K_2)_{\xi}(y, 0) + \beta_p K_2(y, 0) + L_1(y, t)) w(x, 0)
- \int_0^y \alpha_p (K_2)_{\xi} w \, d\xi + \int_0^y \beta_p (K_2)_{\xi} w \, d\xi + \int_0^y \lambda_2 K_2 w \, d\xi + \alpha_p \int_0^y (K_2)_{yy} w \, d\xi + \beta_p \int_0^y (K_2)_y w \, d\xi.
\end{align*}

The kernel $K_2$ satisfies the following PDE

\begin{equation}
\alpha_p ((K_2)_{yy} - (K_2)_{\xi}) + \beta_p ((K_2)_y + (K_2)_{\xi}) + \lambda_2 K_2 = 0,
\end{equation}

with the following boundary condition

\begin{equation}
\alpha_p \frac{\partial K_2(y, y)}{\partial y} = \frac{\lambda_2}{2}.
\end{equation}

The condition $\dot{p}(x, L) = 0$ it leads

\begin{equation}
0 = w(x, L) - \int_0^L K_2(L, \xi) w(x, \xi) \, d\xi,
\end{equation}

which adds a second boundary condition for $K_2$:

\begin{equation}
K_2(L, \xi) = 0.
\end{equation}

Let

\begin{equation}
\tilde{K}_2(y, \xi) = e^{\frac{2p_0}{\alpha_p} (y-\xi)} K_2(y, \xi),
\end{equation}

\section{Conclusions}

In conclusion, the solutions to the wave equation can be expressed in terms of the hyperbolic functions and the given boundary conditions. The problem of finding the solution to the wave equation with specific boundary conditions has been thoroughly discussed.
we see that \( \tilde{K}_2 \) satisfies
\[
\alpha_p \left( (K_2)_{yy} - (K_2)_{\xi \xi} \right) + \lambda_2 \tilde{K}_2 = 0
\]
with the same boundary conditions as \( K_2 \), so we have
\[
K_2(y, \xi) = -\lambda_2 (L - y) \sum_{n=0}^{\infty} \frac{\lambda_2^2 ((L - \xi)^2 - (L - y)^2)^n}{2^{2n+1} n!(n+1)!}
\]
\[
\tilde{K}_2(y, \xi) = -e^{-\frac{\beta_p}{\alpha_p} (y - \xi)} \lambda_2 (L - y) \sum_{n=0}^{\infty} \frac{\lambda_2^2 ((L - \xi)^2 - (L - y)^2)^n}{2^{2n+1} n!(n+1)!}.
\]

However, we need to satisfy the compatibility condition at the boundary \( K_\Gamma = \Gamma K \). This requires that \( K \) commutes with \( \Gamma \) at all points of the domain of \( K \). This condition is not satisfied in the counter-current configuration because the convection speed are in two different directions. In the end of this section we can conclude that the commutativity condition depends on the self-adjoint criterion of our operator. In fact, the operator is self-adjoint only for a particular case \( \beta_f / \alpha_f = \beta_p / \alpha_p \) in the co-current configuration shown in previous section. In that case we can find a Kernel satisfying the commutativity constraint.

8. Conclusion and comments. In this paper, a mathematical analysis of a system of two dimensional advection diffusion equations coupled at the boundary has been provided. This system of equations models the heat transfer in direct contact membrane distillation process. A new formulation of the problem based on semi group framework is introduced. Moreover, the well-posedness criteria for the system is provided using the operator theory. The co-current case has been also analyzed where it has been shown that the operator is diagonalizable. However, based on the backstepping method, we proposed an exponentially stable observer for a very particular case in the co-current configuration to estimate the temperature of an unstable parabolic system weakly coupled at the boundary. Finally, we discussed others configurations by giving some open questions.

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