Microcanonical Approach for the OLA model

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In the present paper it is analyzed a very simple example of pseudoextensive system, the tridimensional system of Linear Coupled Oscillators (OLA Model). The same one constitutes a classical tridimensional system of identical interacting particles by means of harmonic oscillators. This academic problem possesses a complete analytical solution allowing this way that it can find application in modeling some properties of the self-gravitating systems. It is shown that although this is a nonextensive system in the usual sense, it can be dealt in the thermodynamic limit with the usual Boltzmann-Gibbs’ Statistics with an appropriate selection of the representation of the space of the integrals of motion.

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I. INTRODUCTION

In our previous works it was established a general methodology to deal with some nonextensive systems. In the ref. [1] it was addressed the problem of generalizing the extensive postulates of the traditional Thermodynamics in order to extend the application of this theory to the study of some Hamiltonian nonextensive systems. According to our proposition, this can be performed taking into account the self-similarity scaling properties of the systems with the realization of the thermodynamic limit, the limit of many particles, and analyzing the necessary condition for the equivalence of the microcanonical ensemble with the generalized canonical one. In the ref. [2] it was analyzed the most familiar self-similarity scaling laws of the systems, the exponential scaling. Systems with this kind of scaling behavior in the thermodynamic limit can be dealt with the usual Boltzmann-Gibbs’ Statistics if an appropriate selection of the representation of the space of the integrals of movement, \( \mathcal{I}_N \), is taken. That is the reason why we refer those systems as pseudoextensive. The extensive systems are just a particular case of the pseudoextensive systems. It is easy to show that it is sufficient the presence of an additive kinetic part in the system Hamiltonian for the consideration of that system as pseudoextensive.

The present paper will be devoted to the microcanonical analysis of a very simple nonextensive system that allows us a complete analytical study: the tridimensional system of Linear Coupled Oscillators (OLA model). This academic model has been used in the modelation of many systems, and although it is very well-known from the beginning of Mechanics, nowadays it is still considered in the description of some real system, e.i.: quantum dots (see for example in refs. [3–10]). In spite of its simplicity, this model possesses a nonextensive character: it is inhomogeneous, the total energy does not scale with the particle number of the system, etc. We will analyze the scaling properties of its asymptotic accessible volume in order to precise which is its correspondent asymptotic canonical description in the ThL.

II. THE OLA MODEL

The Hamiltonian of this system is given by:

\[ H = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m_i} + \frac{1}{2} k_x x_i^2 + \frac{1}{2} k_y y_i^2 + \frac{1}{2} k_z z_i^2 \right) \]

\[ + \sum_{i<j} \left( -k_x' (x_i - x_j)^2 - k_y' (y_i - y_j)^2 - k_z' (z_i - z_j)^2 \right) \]

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\[ H = \sum_{j=1}^{N} \frac{1}{2m} p_j^2 + \sum_{j,k=1}^{N} \frac{m \omega^2}{4} (r_j - r_k)^2. \] (1)

This is an example very simple of a self-gravitating small system that allows us a complete analytical study. It will develop its microcanonical analysis taking also into account the following integrals of motion:

\[ M = \sum_{j=1}^{N} r_j \times p_j \text{ (angular momentum), and } \mathbf{P}_0 = \sum_{j=1}^{N} p_j \text{ (lineal momentum).} \] (2)

In order to eliminate the divergence in the accessible states density of this model due to the unbound movement of the system mass center, it is also demand the following additional constrains:

\[ \mathbf{P}_0 = \sum_{j=1}^{N} p_j = 0 \text{ and } \mathbf{R}_0 = \sum_{j=1}^{N} r_j = 0, \] (3)

which are consistent with the energy and angular momentum conservation. It is convenient to work with dimensionless variables, so that, it is introduced the following units:

\[ [E] = \hbar \omega, \quad [M] = \hbar, \quad [R] = \left( \frac{\hbar}{m \omega} \right)^{\frac{3}{2}} \text{ and } [P] = (m \hbar \omega)^{\frac{3}{2}}. \] (4)

The calculations will be facilitated using the vectorial convention for the \( \mathbb{R}^{3N} \) vectorial space, which appears in the appendices. It is easy to see that the equations Eq.(1), Eq.(2) and Eq.(3) are rewritten as:

\[ H = \frac{1}{2} P^2 + \frac{1}{2} \left( B^2 R^2 - (B \cdot R)^2 \right), \quad M = R \otimes P, \] (5)

\[ \mathbf{R}_0 = B \cdot R = 0 \text{ and } \mathbf{P}_0 = B \cdot P = 0, \] (6)

where \( R, P \in \mathbb{R}^{3N} \) are the extended system coordinates and linear momentum that together conform the \( N \)-body phase space of the system, and \( B \in \mathbb{R}^N \) with \( B = (1, 1, \ldots, 1) \). The solution of this problem can be easily found through of the generating functional of the distribution:

\[ K(\chi, \upsilon; I, N) = \frac{1}{\hbar \omega} \int \frac{d^{3N} R d^{3N} P}{(2\pi)^{3N}} \exp \left[ i (\chi \cdot R + \upsilon \cdot P) \right] \delta \left[ I - I_N(R, P) \right], \] (7)

where \( I = (E, \mathbf{M}, \mathbf{R}_0, \mathbf{P}_0) \), \( \chi \) and \( \upsilon \) are \( 3N \)-dimensional vectors belonging to \( \mathbb{R}^{3N} \). It is convenient to use the Fourier’s representation of the delta function:

\[ \delta (x_o - x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \exp \left[ \frac{z(x_o - x)}{k} \right], \] (8)

where \( z = \varepsilon + ik \) with \( \varepsilon > 0 \), to rewrite the Eq.(7) as follow:

\[ K(\chi, \upsilon; I, N) = \frac{1}{\hbar \omega} \int \frac{d^{10} k}{(2\pi)^{10}} \exp (k \cdot I) \mathcal{K}(z; \chi, \upsilon), \] (9)

where \( k = (k_1, k_2, i\rho, i\eta) \), and \( z = (z_1, z_2, i\rho, i\eta) \), with \( z_1 = \beta + ik_1, \quad z_2 = \gamma + ik_2 \). The function \( \mathcal{K}(z; \chi, \upsilon) \) is defined by:

\[ \mathcal{K}(z; \chi, \upsilon) = \int \frac{d^{3N} R d^{3N} P}{(2\pi)^{3N}} \exp \left\{ - [z \cdot I_N(R, P) - i\chi \cdot R - i\upsilon \cdot P] \right\}. \] (10)

The above integral is very easy to calculate because it can be reduced to gaussian integrals. The calculations yield:

\[ K(\chi, \upsilon; E, \mathbf{M}, N) = \frac{1}{(2\pi)^3 B^6 \hbar \omega} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left( z_1 E \right) \exp (z_2 \cdot \mathbf{M}) \]
\[
\exp \left( -\frac{B^2 v^2 - (B \cdot v)^2}{2B^2z_1} \right) \exp \left( -\frac{B^2 z_2^2 - (B z_2 \cdot \lambda)^2}{2B^2z_2^2z_1} \right) \exp \left( -\frac{B^2 (z_2 \cdot \lambda)^2 - (B z_2 \otimes \lambda)^2}{2B^2z_2^2z_1^2 - z_2^2} \right) \] (11)

where:
\[
\lambda = \chi + \frac{z_2 \times \nu}{z_1} \] (12)

The above expression must be used in the obtaining of the physical observables. The accessible states density is expressed as:
\[
\Omega (E, M, N) = \frac{1}{(2\pi)^3 B^6 h^7 \omega} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dk_1dk_2k_3}{(2\pi)^4} \exp (z_1 E) \exp (z_2 \cdot M) \] (13)

whose integration yields:
\[
\Omega (E, M, N) = \frac{1}{(2\pi B)^7} M^{3N-7} H_N \left( \frac{E}{BM} \right). \] (14)

The function \( H_N (x) \) is given by:
\[
H_N (x) = (1 - x)^{2N-4} P_{N-2} (1 - x) \sigma (x - 1), \quad \text{where } \sigma (x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \] is the Heaviside function, (15)

and \( P_{N-2} (z) = \sum_{n=0}^{N-2} a_n z^n \), with \( a_n = \frac{(N - 1 + n)!}{(N - 2 - n)! (2N - 4 + n)! (N - 1)!} \frac{(-1)^n}{2^n n!} \). (16)

The accessible volume is obtained multiplying the state density by an appropriate element volume constant \( \delta I \). A reasonable choice for the tridimensional OLA model is:
\[
\delta I = h^7 \omega, \] (17)

and therefore:
\[
W (E, M, N) = \frac{1}{(2\pi B)^7} M^{3N-7} H_N \left( \frac{E}{BM} \right). \] (18)

The integrals of motion space, \( \Im_N \), in the \( \mathcal{R} (E, M, N) \) representation is the \( \mathbb{R}^4 \) cone for a given \( N \):
\[
E^2 - N M^2 \geq 0. \] (19)

In order to access to the scaling laws, it must be obtained the asymptotic dependency of the polynomials coefficients in the Eq.(16). It is easy to show the following asymptotic behavior:
\[
\lim_{N \to \infty} a_n \simeq \frac{1}{2^{2N} N^{3N}} \frac{(-1)^n}{4^n n!} N^n. \] (20)

Therefore, the asymptotic accessible volume is given by the expression:
\[
W_{\text{asymp}} (E, M, N) \propto \left( \frac{E}{2 (N)^{\frac{3}{2}}} \right)^{3N} Q_N \left( \frac{\sqrt{N} M}{E} \right), \] (21)

where the function \( Q_N (x) \) is expressed as:
\[
Q_N (x) = (1 - x)^{2N} \exp \left( N \frac{1-x}{4} \right), \quad \text{with } x \in [0, 1]. \] (22)

It is very easy to see that the tridimensional OLA model possesses an exponential self-similarity scaling laws [1]:
\[ N_o \rightarrow N(\alpha) = \alpha N_o \]
\[ E_o \rightarrow E(\alpha) = \alpha^{\frac{2}{3}}E_o \]
\[ M_o \rightarrow M(\alpha) = \alpha M_o \]
\[ \Rightarrow W_{asym}(\alpha) = \mathcal{F}[W_o, \alpha], \]

where the functional \( \mathcal{F}[W_o, \alpha] \) is given by:
\[ \mathcal{F}[W_o, \alpha] \equiv \exp[\alpha \ln W_o]. \]

where \( \alpha \) is the scaling parameter and \( W_o \) is given by:
\[ W_o = W_{asym}(E_o, M_o, N_o). \]

Thus, the tridimensional OLA model is not extensive, but pseudoextensive \[2\]. In the present case, the generalized Boltzmann’s Principle \[1\] adopts its ordinary form:
\[ S_B = \ln W, \]

his celebrated gravestone epitaph in Vienna. To this entropy form corresponds the usual Shannon-Boltzmann-Gibbs’ extensive entropy:
\[ S_{SBG} = -\sum_k p_k \ln p_k, \]

and therefore, in the thermodynamic limit, the microcanonical description could be substituted equivalently by the usual Boltzmann-Gibbs’ Distribution with a appropriate selection of the representation of the integrals of motion space. This can be performed changing the representation from \( \mathcal{R}(E, M, N) \) to \( \mathcal{R}(E, M, N) \), where:
\[ E = \frac{1}{\sqrt{N}}E. \]

In the thermodynamic limit the microcanonical description of this system should be substituted equivalently by the canonical one. In this case the Boltzmann-Gibbs’ Distribution is:
\[ \omega_c(X; \beta, \gamma, N) = \frac{1}{Z(\beta, \gamma, N)} \exp[-\beta E_N(X) - \gamma \cdot M_N(X)]. \]

Through the Laplace’s Transformation:
\[ Z(\beta, \gamma, N) = \int \exp(-\beta E - \gamma \cdot M) W(E, M, N) \frac{d\mathcal{E} d^3M}{\delta I}, \]

it is easy to show that in the thermodynamic limit both descriptions are equivalent, that is, the Legendre’s Transformation between the thermodynamic potentials of the ensembles is valid:
\[ P(\beta, \gamma, N) \simeq \beta \mathcal{E} + \gamma \cdot M - S_B(\mathcal{E}, M, N), \]

where \( P(\beta, \gamma, N) \) is the Planck’s Potentiak:
\[ P(\beta, \gamma, N) = -\ln Z(\beta, \gamma, N), \]

when exists a unique sharp maximum in the integral argument of the Eq(29). In this maximum, \( (\mathcal{E}_m, M_m) \), it will be satisfied the following conditions:
\[ \beta = \frac{\partial}{\partial \mathcal{E}} S_B(\mathcal{E}_m, M_m, N), \quad \gamma = \frac{\partial}{\partial M} S_B(\mathcal{E}_m, M_m, N), \]

and all the eingenvales of the curvature tensor \[4]\:
\[ K_{\mu \nu} = \partial_{\mu} \partial_{\nu} S_B(\mathcal{E}_m, M_m, N), \]

are negatives [in the Eq(30) \( \partial_{\mu} \equiv \frac{\partial}{\partial I^\mu} \), where \( I^\mu = (\mathcal{E}, M) \)]. Similarly, the generalized Duhem-Gibbs’ relation is also valid in the thermodynamic limit:
where $\mu$ is the chemical potential:

$$
\mu (\beta, \gamma) = \frac{\partial}{\partial N} S_B (\mathcal{E}_m, \mathbf{M}_m, N).
$$

From the Duhem-Gibbs’ relation is easily deduced the following relationship:

$$
P (\beta, \gamma, N) = -\mu (\beta, \gamma) N.
$$

The Boltzmann’s entropy of the tridimensional OLA model in the asymptotic region is given by:

$$
S_B \simeq 3N \ln \left( \frac{\mathcal{E}}{2N} \right) + 2N \ln \left( 1 - \frac{M}{\mathcal{E}} \right) + \frac{1}{4} N \left( 1 - \frac{M}{\mathcal{E}} \right) + O \left( \frac{1}{N} \right),
$$

and therefore, the canonical parameters $\beta$ and $\gamma$, and the chemical potential $\mu$, are given by:

$$
\beta = \frac{1}{4} N \frac{12 \mathcal{E}^2 - 3 \mathcal{E} M - M^2}{\mathcal{E}^2 (\mathcal{E} - M)}, \quad \gamma = -\frac{1}{4} N \frac{9 \mathcal{E} - M}{\mathcal{E} (\mathcal{E} - M)} \mathbf{M}, \quad \mu = \frac{S_B (\mathcal{E}, \mathbf{M}, N)}{N} - 3.
$$

To validate the previous results it must be demanded the concavity of the Boltzmann’s entropy, that is to say, the non-negativity of the $4 \times 4$ curvature tensor:

$$
K_{\mu \nu} = \left( -\frac{1}{4} N \frac{12 \mathcal{E}^2 - 3 \mathcal{E} M + M^2}{\mathcal{E}^2 (\mathcal{E} - M)^2} \mathbf{M}, \mathbf{M} \right) - N \frac{1}{4} \frac{M_{1x1} M_{1x3} M_{1x3} + 2 M_{1x1} M_{1x3} M_{1x3} + M_{1x1} M_{1x3} M_{1x3}}{\mathcal{E}^2 (\mathcal{E} - M)^2} - \frac{1}{4} N \frac{9 \mathcal{E} - M}{\mathcal{E} (\mathcal{E} - M)} \mathbf{M} \mathbf{I}_{3x3}
$$

where $\mathbf{M}_{1x3} = (\mathbf{M}_{3x1})^T = (M_x, M_y, M_z)$, and $\mathbf{I}_{3x3}$ is the $3 \times 3$ transverse unitary matrix:

$$
(\mathbf{I}_{3x3})_{\mu \nu} = \delta_{\mu \nu} - \frac{M_{\mu} M_{\nu}}{\mathbf{M} \mathbf{M}}.
$$

The determinant of the curvature tensor is given by:

$$
\det K = \frac{1}{256} N^4 \frac{15 \mathcal{E}^2 + 18 \mathcal{E} M - M^2}{(\mathcal{E} - M)^4 \mathcal{E}^6 M^2} (9 \mathcal{E} - M)^2 > 0.
$$

From the above result is derived that all the accessible space is appropriately described by the canonical ensemble in the $\mathcal{R}_{\mathcal{E}, \mathbf{M}, N}$ representation: in the Laplace’s Transformation, Eq. (34), there is only one sharp peak. Thus, in the thermodynamic limit both descriptions are identical: all the system accessible space can be deal with the canonical description. This result facilitates so much the analysis of this system. In the canonical description, the Planck’s potential is obtained from the Eq. (35) replacing $Bz_1 \rightarrow \beta$ and $z_2 \rightarrow \gamma$:

$$
P (\beta, \gamma, N) = N \ln \left[ \beta \left( \beta^2 - \gamma^2 \right) \right].
$$

The generating functional in the canonical description:

$$
G_c (\mathcal{E}, \mathbf{v}, \beta, \gamma, N) = \int \frac{d^{3N} Q d^{3N} P}{(2\pi)^{3N}} \omega_c (Q, P; \beta, \gamma, N) \exp (i \mathcal{E} \cdot Q + i v \cdot P),
$$

is also obtained from the Eq. (35):

$$
G_c (\chi, v; \beta, \gamma, N) = \exp \left( \frac{-B^2 v^2 - (B \cdot \mathbf{v})^2}{2B^\beta} \right) \exp \left( \frac{-B^2 \gamma^2 \lambda^2 - (B \gamma \cdot \lambda)^2}{2B^3 \gamma^2 \beta} \right) \exp \left( \frac{-B^2 (\mathbf{\gamma} \times \lambda)^2 - (B \mathbf{\gamma} \otimes \lambda)^2}{2B^3 \gamma^2 (\beta^2 - \gamma^2)} \right).
$$

It is very interesting to known the particle distribution in the space. This observable is easily obtained from the generating functional as follow: setting zero all the components of the vectors $v$ and leaving only one vectorial component in the vector $\chi$, $\chi = (q, 0, ..., 0)$, and performing the following integration:
\[ \rho (\mathbf{r}) = N \int \int \int \frac{dq_{l}dq_{t}}{(2\pi)^{3}} \exp (-ir_{l}q_{l}) \exp (-ir_{t} \cdot q_{t}) \exp \left( -\frac{B_{1}^{2}-1}{2B_{3}} q_{l}^{2} \right) \exp \left( -\frac{B_{1}^{2}-1}{2B_{3}}(\beta^{2} - \gamma^{2}) q_{t}^{2} \right), \quad (45) \]

where \( q_{l} \) and \( q_{t} \) are the longitudinal and the transverse components of the vector \( \mathbf{q} \) with respect to the vector \( \gamma \). The integration yields:

\[ \rho (\mathbf{r}) = \frac{N}{\sqrt{8\pi^{1} \sigma_{l} \sigma_{t}}} \exp \left( -\frac{r_{l}^{2}}{2\sigma_{l}} - \frac{r_{t}^{2}}{2\sigma_{t}} \right), \quad (46) \]

showing that the particle distribution possesses a gaussian shape where the parameters \( \sigma_{l} \) and \( \sigma_{t} \) are given by:

\[ \sigma_{l} = \frac{B_{1}^{2}-1}{B_{3}} \beta, \quad \sigma_{t} = \frac{B_{1}^{2}-1}{B_{3}(\beta^{2} - \gamma^{2})} \beta. \quad (47) \]

Let the parameter \( s \) be introduced as follow:

\[ \frac{r_{l}^{2}}{(\sigma_{l}) s^{2}} + \frac{r_{t}^{2}}{(\sigma_{t}) s^{2}} = 1, \quad (48) \]

parametrizing a family of ellipsoids of revolution which characterizing the surfaces of equal density:

\[ \rho (s) = \frac{N}{\sqrt{8\pi^{3} \sigma^{3}}} \exp \left( -\frac{s^{2}}{2\sigma} \right), \quad (49) \]

where \( \sigma \) is defined by the relation:

\[ \sigma = \sqrt[3]{\sigma_{l} \sigma_{t}^{2}}. \quad (50) \]

The eccentricity of ellipsoids family is constant and given by:

\[ e = \sqrt{1 - \frac{\sigma_{l}}{\sigma_{t}}} = \frac{\gamma}{\beta} = \frac{M}{E}. \quad (51) \]

As it can be seen, the rotation deforms the spherical shape of the distribution. The characteristic size of the distribution is the given by:

\[ l_{c} = \sqrt{\sigma}. \quad (52) \]

From the Eq.(47) it is deduced that the characteristic size decreases with the increasing of the particle number \( N \):

\[ l_{c} \sim 1/\sqrt[4]{N}. \quad (53) \]

This kind of scaling law is not typical in the extensive systems, which are usually scaled proportional to the systems size. In the FIG.1, it is shown different profiles for the particle distributions.

### III. CONCLUSIONS

In the present paper we applied the results of our previous works on study of a very simple model of self-gravitating small system, the tridimensional OLA model. The microcanonical analysis showed that the OLA model is a pseudoextensive system \( \Box \), since it possesses a exponential self-similarity scaling laws in the thermodynamic limit. Systems with this kind of scaling laws could be dealt with the usual Boltzmann-Gibbs’ statistics if an appropriate representation of the system integrals of motion space is selected. Although the model presented before is very simple, the analysis shown that it is necessary the consideration of self-similarity scaling postulates for the analysis of this system, since the same is not extensive.
IV. APPENDIX: THE VECTORIAL CONVENTION.

Let $\mathbb{R}^{3N}$ be the $3N$-dimensional vectorial space, which is the external product of the spaces $\mathbb{R}^3$, $\mathbb{R}^N$. The bases of $\mathbb{R}^{3N}$ are represented by means of the product of the bases of the respective spaces, $\mathbb{R}^3$ and $\mathbb{R}^N$:

$$\vec{E}_{k,s} = \vec{e}_s \vec{E}_k.$$  \hfill (54)

It will be only considered those transformations preserving this property, that is, those transformations which acts in the spaces $\mathbb{R}^3$ and $\mathbb{R}^N$ separately. The vectors of this space can be represented as:

$$V = \sum_{k=1}^{N} \sum_{s=1}^{3} X_{k,s} \vec{e}_s \vec{E}_k = \sum_{k=1}^{N} \vec{x}_k \vec{E}_k = \sum_{s=1}^{3} \vec{X}_s \vec{e}_s \quad \text{where} \quad \vec{x}_k \in \mathbb{R}^3; \vec{X}_s \in \mathbb{R}^N.$$  \hfill (55)

From here and after, it will be adopted the tensorial summation convention. Using the above representation, it can be defined the following external operations:

**Partial and Total Inner Operations:**

- $\mathbb{R}^3 \times \mathbb{R}^{3N} \rightarrow \mathbb{R}^N$: $\vec{a} \cdot \vec{V} = (\vec{a} \cdot \vec{v}_k) \vec{E}_k$,

$$\mathbb{R}^N \times \mathbb{R}^{3N} \rightarrow \mathbb{R}^3$: $\vec{A} \cdot \vec{V} = \left( \vec{A} \cdot \vec{V}_s \right) \vec{e}_s$, \hfill (56)

$$\mathbb{R}^3 \times \mathbb{R}^{3N} \rightarrow \mathbb{R}^N$: $\vec{V} \cdot \vec{W} = (\vec{v}_k \cdot \vec{w}_k) = \left( \vec{V}_s \cdot \vec{W}_s \right)$. \hfill (57)

**Partial Vectorial Product:**

$$\mathbb{R}^3 \times \mathbb{R}^{3N} \rightarrow \mathbb{R}^{3N}: \vec{a} \times \vec{V} = (\vec{a} \times \vec{v}_k) \vec{E}_k.$$ \hfill (58)

**Vectorial-Scalar Product:**

$$\mathbb{R}^{3N} \otimes \mathbb{R}^{3N} \rightarrow \mathbb{R}^3$: $\vec{V} \otimes \vec{W} = (\vec{v}_k \times \vec{w}_k).$ \hfill (59)

From the previous definitions is derived the following identity:

$$\vec{a} \cdot \vec{V} \otimes \vec{W} = \left( \vec{a} \times \vec{V} \right) \cdot \vec{W}. \hfill (60)$$

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