Quasilinear Schrödinger equations with concave and convex nonlinearities

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Abstract
In this paper, we consider the following quasilinear Schrödinger equation

$$-\Delta u - u \Delta (u^2) = k(x) |u|^{q-2} u - h(x) |u|^{s-2} u, \quad u \in D^{1,2}(\mathbb{R}^N),$$

where $1 < q < 2 < s < \infty$. Unlike most results in the literature, the exponent $s$ here is allowed to be supercritical $s > 2 \cdot 2^*$. By taking advantage of geometric properties of a nonlinear transformation $f$ and a variant of Clark’s theorem, we get a sequence of solutions with negative energy in a space smaller than $D^{1,2}(\mathbb{R}^N)$. Nonnegative solution at negative energy level is also obtained.

1 Introduction

In this paper we consider quasilinear stationary Schrödinger equations of the form

$$\begin{cases}
-\Delta u - u \Delta (u^2) = k(x) |u|^{q-2} u - h(x) |u|^{s-2} u, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}$$

(1.1)

where $1 < q < 2 < s < \infty$. This kind of equations arise when we are looking for standing waves $\psi(t, x) = e^{-i\omega t} u(x)$ for the time dependent quasilinear Schrödinger equation

$$i \psi_t = -\Delta \psi - \psi \Delta (|\psi|^2) - \tilde{g}(x, |\psi|^2) \psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Quasilinear Schrödinger equations have captured great interest in the last two decades because they model several important physical phenomena including superfluid film in plasma...
physics, self-trapped electrons in quadratic or hexagonal lattices, see [1, 2] and references therein for more details.

The problem (1.1) possesses a variational structure. Formally, it is the Euler–Lagrange equation of the functional

\[
J(u) = \frac{1}{2} \int (1 + 2u^2) |\nabla u|^2 - \frac{1}{q} \int k |u|^q + \frac{1}{s} \int h |u|^s,
\]

where from now on all integrals are taken over \(\mathbb{R}^N\) unless stated explicitly. However, \(J\) can only be defined on a proper subset of \(D^{1,2} (\mathbb{R}^N)\), hence the standard variational methods could not be applied. To overcome this difficulty, Liu et al. [3] and Colin–Jeanjean [4] introduced a nonlinear transformation \(f\) which converts the quasilinear problem into a semilinear one, and enables us to work with a \(C^1\)-functional

\[
\Phi(v) = J(f(v)) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{q} \int k |f(v)|^q + \frac{1}{s} \int h |f(v)|^s
\]

defined on the whole Sobolev space. Since then, many results about quasilinear Schrödinger equations appear, mainly for 4-superlinear nonlinearities, see [5–8].

In this paper, we study quasilinear Schrödinger equations whose nonlinearity is a combination of concave and convex terms. Elliptic boundary value problems involving concave and convex terms have attracted great attention since the pioneering work of Ambrosetti–Brezis–Cerami [9] and Bartsch–Willem [10] on semilinear problems on bounded domain. However, relatively less have been done for quasilinear Schrödinger equations. It seems that do Ó and Severo [11] is the first work in this direction, see [12] for a more recent result. To apply variational methods, in [11, 12] and most papers on quasilinear Schrödinger equations, the nonlinearity \(g(x, u)\) can at most grow critically, that is

\[
|g(x, u)| \leq C (1 + |u|^{2* - 2})^2,
\]

here \(2^* = 2N/(N - 2)\) is the critical Sobolev exponent. It was pointed out in [3, Remark 3.13] that the exponent \(2 \cdot 2^*\) behaves like a critical exponent for (1.1). The nonlinearities in the above mentioned papers on quasilinear Schrödinger equations are subcritical. For the critical case, one can consult [13–15] and references therein.

On the contrary, in our problem (1.1), no restriction on the power \(s\) is imposed: \(s\) can be greater than \(2 \cdot 2^*\), in this case the nonlinear term \(h(x) |u|^{s - 2} u\) is supercritical. There are also a few papers about supercritical problems, see e.g. [16, 17]. To study supercritical problems, one applies variational methods to get solutions of the subcritical problem obtained by modifying \(g(x, u)\) for \(|u|\) large, then perform \(L^\infty\)-estimate to show that the solutions for the truncated problem have small \(L^\infty\)-norm, therefore they are solutions of the original problem. We can see that \(L^\infty\)-estimate is a crucial step for this approach. Our approach for getting solutions of (1.1) does not require truncation and \(L^\infty\)-estimate.

Removing the quasilinear term \(u \Delta (u^2)\) from (1.1), our equation reduces to a semilinear elliptic equation on \(\mathbb{R}^N\), which was studied by Tonkes [18]. However, in [18] \(s \leq 2^*\) is still required. The result of Tonkes [18] was extended by Liu–Li [19], where a corresponding \(p\)-Laplacian problem is considered, and \(s\) can be greater than the critical Sobolev exponent \(p^*\). Naturally, our strategy to study the supercritical quasilinear problem (1.1) is motivated by Liu and Li [19]. However, due to the above mentioned nonlinear transformation \(f\), more delicate analysis is needed. We will see that the geometric properties of \(f\) play an essential role in our investigation.
To state our result, for $p \in (1, \infty)$ we denote

$$p_0 = \frac{2N}{2N - p(N - 2)}, \quad p' = \frac{p}{p - 1}.$$  

Note that $p'$ is the Hölder conjugate exponent of $p$.

**Theorem 1.1** Assume that

\((k)\) \(k \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), k \geq 0, k \neq 0,\)
\(\)\(\) \(\)\(\)
\((h)\) \(h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), h \geq 0\)
\(\)\(\)\(\)\(\)
are satisfied, then the problem \((1.1)\) has a sequence of solutions \(\{u_n\}\) such that the energy \(J(u_n) < 0\) and \(J(u_n) \to 0\) as \(n \to \infty\).

**Theorem 1.2** Under the assumptions \((k)\) and \((h)\), the problem \((1.1)\) has a nonegative solution \(u\) such that \(J(u) < 0\).

**Remark 1.3** Our Theorem 1.2 is closely related to Miyagaki and Moreira [20], where for \(4 \leq q < s < \infty\), the following problem

$$-\Delta u - \Delta (u^2) = \lambda u + k(x) |u|^{q-2} u - h(x) |u|^{s-2} u, \quad u \in H^1_0(\Omega)$$

on a bounded domain \(\Omega\) is considered; for \(\lambda \in (\lambda^*, \bar{\lambda})\), a nonnegative solution (at negative energy level) is obtained by the Ekeland variational principle and the sub-super solution method.

The paper is organized as follows. In Sect. 2 we review the definition of the transformation \(f\) and present some of its properties which are needed in this paper. Since the exponent \(s\) in \((1.1)\) can be greater than the critical Sobolev exponent \(2 \cdot 2^*\), instead of the usual Sobolev spaces \(H^1(\mathbb{R}^N)\) or \(D^{1,2}(\mathbb{R}^N)\), we introduce a new space \(E\) as the foundation of our functional framework. In Sect. 3 we investigate the geometry and compactness of our energy functional \(\Phi: E \to \mathbb{R}\) and prove our theorems via minimization argument and a variant of Clark’s theorem proved in [21].

## 2 Variational framework

Following Colin and Jeanjean [4] and Liu et al. [3], we make the change of variables by \(u = f(v)\), where \(f\) is an odd function defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}}, \quad f(0) = 0$$

on \([0, +\infty)\). The proof of the following proposition can be found in [4, 22] [some of them are obvious from \((2.1)\)].

**Proposition 2.1** The function \(f\) possesses the following properties:

1. \(f \in C^\infty(\mathbb{R})\) is strictly increasing, therefore is invertible.
2. \(|f(t)| \leq |t|, \quad f'(0) = 1, \quad |f'(t)| \leq 1\) for all \(t \in \mathbb{R}\).
3. \(\left| f(t)f'(t) \right| \leq 1, \quad |f(t)| \leq 2^{1/4} |t|^{1/2}\).
4. There exists a positive constant \(\mu\) such that

$$|f(t)| \geq \mu |t| \text{ for } |t| \leq 1, \quad |f(t)| \geq \mu |t|^{1/2} \text{ for } |t| \geq 1.$$  

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5. For all \( t \in \mathbb{R} \) we have \( f^2(t) \geq f(t) f'(t) t \geq \frac{1}{2} f^2(t) \).

Motivated by Liu and Li [19], let \( E \) be the completion of \( C_0^\infty(\mathbb{R}^N) \) under the norm
\[
\|v\| = \|v\|_D + \|h^{2/s} v\|_{s/2} = \left( \int |\nabla v|^2 \right)^{1/2} + \left( \int h |v|^{s/2} \right)^{2/s},
\]
where \( \|\cdot\|_D \) and \( \|\cdot\|_p \) are the standard \( D^{1,2} \)-norm and \( L^p \)-norm \((p \in [1, \infty])\), respectively. Note that if following Liu and Li [19] directly, one may tend to define the norm as
\[
\|v\| = \|v\|_D + |h^{1/s} v|.
\]
Our definition (2.3) takes the structure of (1.1) and the growth property \( |f(t)| \leq c |t|^{1/2} \) of \( f' \) into account. It turns out that this is the correct choice.

**Remark 2.2** When \( h \equiv 0 \), our space \( E \) reduces to the standard Sobolev space \( D^{1,2}(\mathbb{R}^N) \).

To present the variational framework for our argument, we need the following lemma.

**Lemma 2.3** If \( \phi \in C_0^\infty(\mathbb{R}^N) \), then
\[
\xi = \frac{\phi}{f'(v)} = \sqrt{1 + 2 f^2(v) \phi}
\]
belongs to \( E \).

**Proof** Take \( R > 0 \) such that \( \text{supp} \phi \subset B_R \), where \( B_R \) is the \( R \)-ball in \( \mathbb{R}^N \). Since
\[
(1 + 2 f^2(v))^{s/4} \leq C (1 + |v|^{s/2}),
\]
we have
\[
\left| \int h |\xi|^{s/2} \right| = \left| \int h (1 + 2 f^2(v))^{s/4} |\phi|^{s/2} \right|
\leq C |\phi|^{s/2} \int h (1 + |v|^{s/2})
\leq C |\phi|^{s/2} \left( |h|_1 + \int h |v|^{s/2} \right) < \infty. \quad (2.5)
\]
Now we estimate the \( D^{1,2} \)-norm of \( \xi \). Because \( v \in D^{1,2}(\mathbb{R}^N) \), we have \( v \in L^2_{\text{loc}}(\mathbb{R}^N) \), therefore
\[
\int |\nabla \xi|^2 \leq \int_{B_R} \left| (1 + 2 f^2(v))^{1/2} \nabla \phi + \frac{2 f(v) f'(v) \phi}{\sqrt{1 + 2 f^2(v)}} \nabla v \right|^2
\leq \int_{B_R} \left( (1 + 2 v^2) |\nabla \phi|^2 + 4 |v| |\phi| |\nabla \phi| |\nabla v| + 4 \phi^2 |\nabla v|^2 \right)
\leq m \int_{B_R} \left( (1 + 2 v^2) + 4 |v| |\nabla v| + 4 |\nabla v|^2 \right) < \infty,
\]
where \( m = (|\phi|_\infty + |\nabla \phi|_\infty)^2 \). Combining (2.5) and (2.6) we see that \( \xi \in E \). \( \square \)
By the growth properties of \( f \), it is easy to see that under our assumptions on \( k \) and \( h \), the functional

\[
\Phi(v) = J(f(v)) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{q} \int k |f(v)|^q + \frac{1}{s} \int h |f(v)|^s
\]

is well defined and of class \( C^1 \) on the Banach space \( E \), with derivative given by

\[
\langle \Phi'(v), \xi \rangle = \int \nabla v \cdot \nabla \xi - \int k |f(v)|^{q-2} f(v) f'(v) \xi + \int h |f(v)|^{s-2} f(v) f'(v) \xi
\]

for \( v, \xi \in E \). Moreover, if \( v \) is a critical point of \( \Phi : E \rightarrow \mathbb{R} \), by Lemma 2.3 for \( \phi \in C^\infty_0(\mathbb{R}^N) \) we have \( \xi = \phi/f'(v) \in E \). Hence \( \langle \Phi'(v), \xi \rangle = 0 \) and from which let \( u = f(v) \), by standard computation we get

\[
0 = \frac{d}{dr} \bigg|_{r=0} J(u + t\phi),
\]

which means that \( u \) is a weak solution of the problem (1.1). Therefore, to prove our theorems it suffices to find critical points of \( \Phi : E \rightarrow \mathbb{R} \). This is the task of the next section.

The following proposition justifies our effort to find solutions of (1.1) at negative energy levels.

**Proposition 2.4** Suppose \( s \geq 4 \). If \( v \in E \) is a critical point of \( \Phi \), then \( \Phi(v) \leq 0 \).

**Proof** Let \( c = \Phi(v) \), that is

\[
2c = \int |\nabla v|^2 - \frac{2}{q} \int k |f(v)|^q + \frac{2}{s} \int h |f(v)|^s. \tag{2.7}
\]

Testing \( \Phi'(v) \) by \( v \), we have

\[
0 = \langle \Phi'(v), v \rangle = \int |\nabla v|^2 - \int k |f(v)|^{q-2} f(v) f'(v) v + \int h |f(v)|^{s-2} f(v) f'(v) v. \tag{2.8}
\]

Since \( q < 2, s \geq 4 \), from (2.7), (2.8) and Proposition 2.1(5), we obtain

\[
2c = \int k |f(v)|^{q-2} \left[ f(v) f'(v) v - \frac{2}{q} f^2(v) \right] + \int h |f(v)|^{s-2} \left[ \frac{2}{s} f^2(v) - f(v) f'(v) v \right] \leq 0,
\]

as desired. \( \square \)

**Remark 2.5** The idea of Liu and Li [19], which in turn was inspired by an unpublished preprint [23], has also been employed in [24], where a supercritical Schrödinger–Poisson system

\[
\begin{align*}
-\Delta u + a(x) u + \phi u &= k(x) |u|^{q-2} u - h(x) |u|^{s-2} u & x \in \mathbb{R}^3, \\
-\Delta \phi &= u^2, & \lim_{|x| \to \infty} \phi(x) = 0 & x \in \mathbb{R}^3
\end{align*}
\]

is studied. In [19, 23, 24] no nonlinear transformation like \( f \) is involved, therefore our work is not a trivial application of the idea in these papers.
3 Proof of Theorems 1.1 and 1.2

In this section, we will show that, under our assumptions, \( \Phi : E \to \mathbb{R} \) is coercive and satisfies the Palais–Smale condition, then prove our theorems by minimization method and a variant of the classical Clark’s theorem.

Lemma 3.1 \( \Phi : E \to \mathbb{R} \) is coercive.

Proof Let \( \ell \) be the norm of the embedding \( D^{1,2} (\mathbb{R}^N) \hookrightarrow L^{2^*} (\mathbb{R}^N) \). Since \( q q_0 = 2^* \), for \( v \in E \) we have

\[
\int k |v|^q \leq |k|_{q_0} |v|^q \bigg|_{q_0} = \frac{|k|_{q_0}}{2} |v|_{q_0}^q \leq \ell^q |k|_{q_0} \|v\|^q_D .
\]

(3.1)

If \( \Phi \) is not coercive, there is a sequence \( \{v_n\} \) in \( E \) such that sup \( \Phi(v_n) < +\infty \) and

\[
\|v_n\| = \|v_n\|_D + |h^{2/s} v_n|_{2/s} \to +\infty .
\]

(3.2)

We claim that \( \{v_n\} \) is bounded in \( D^{1,2}(\mathbb{R}^N) \). Otherwise along a subsequence we have \( \|v_n\|_D \to \infty \), using (3.1) and noting \( q < 2 \) we have

\[
\Phi(v_n) \geq \frac{1}{2} \int |\nabla v_n|^2 - \frac{1}{q} \int k |v_n|^q + \frac{1}{s} \int h |f(v_n)|^s
\]

\[
\geq \frac{1}{2} \|v_n\|_D^2 - \frac{1}{q} \ell^q |k|_{q_0} \|v_n\|_D^q + \frac{1}{s} \int h |f(v_n)|^s
\]

\[
\geq \frac{1}{2} \|v_n\|_D^2 - \frac{1}{q} \ell^q |k|_{q_0} \|v_n\|_D^q \to +\infty ,
\]

contradicting sup \( \Phi(v_n) < +\infty \). Therefore sup \( \|v_n\|_D < \infty \) and from (3.2) we have

\[
\int h |v_n|^{2/s} = |h^{2/s} v_n|_{2/s} \to +\infty .
\]

Using (2.2) we get

\[
\int h |f(v_n)|^s = \int_{|v_n| \leq 1} h |f(v_n)|^s + \int_{|v_n| > 1} h |f(v_n)|^s
\]

\[
\geq \mu \int_{|v_n| > 1} h |v_n|^{s/2} = \mu \int h |v_n|^{2/s} - \mu \int_{|v_n| \leq 1} h |v_n|^{2/s}
\]

\[
\geq \mu \int h |v_n|^{2/2} - \mu |h|_1 \to +\infty .
\]

Since \( q < 2 \), we end up at a contradiction:

\[
\Phi(v_n) \geq \frac{1}{2} \|v_n\|_D^2 - \frac{1}{q} \ell^q |k|_{q_0} \|v_n\|_D^q
\]

\[
+ \frac{1}{s} \int h |f(v_n)|^s \to +\infty .
\]

The proof is completed. \( \square \)

Lemma 3.2 Given \( a \in \mathbb{R} \), the function \( \eta : \mathbb{R} \to \mathbb{R} \), \( \eta(t) = |f(t)|^s \), is convex. Hence for \( \alpha, \beta \in \mathbb{R} \) we have

\[
|f(\alpha)|^s \leq |f(\beta)|^s + s |f(\alpha)|^{s-2} f(\alpha) f'(\alpha) (\alpha - \beta) .
\]

(3.3)
Proof Obviously \( \eta \) is smooth and even. For \( t \geq 0 \), because \( s > 2 \), using (2.1) we have

\[
\eta' = s f^{s-1} f' = \frac{s f^{s-1}}{\sqrt{1 + f^2}},
\]

\[
\frac{\eta''}{s} = \left(\frac{f^{s-1}}{\sqrt{1 + f^2}}\right)'
\]

\[
= \frac{(s - 1) f^{s-2} f' \sqrt{1 + f^2} - f^{s-1} f f'}{1 + f^2}
\]

\[
= \frac{(s - 2) f^s f' + (s - 1) f^{s-2} f'}{(1 + f^2)^{3/2}} \geq 0.
\]

Because \( \eta'' \) is also even, we see that \( \eta''(t) \geq 0 \) for all \( t \in \mathbb{R} \), and \( \eta \) is convex. \( \square \)

Lemma 3.3 \( \Phi \) satisfies the Palais–Smale condition.

Proof Let \( \{v_n\} \subset E \) be a \((PS)\) sequence. Lemma 3.2 implies that \( \{v_n\} \) is bounded. Thus \( \{v_n\} \) and \( \{h^{2/s} v_n\} \) are bounded in \( D^{1,2}(\mathbb{R}^N) \) and \( L^{s/2}(\mathbb{R}^N) \), respectively. Up to a subsequence we have

\[
v_n \rightharpoonup v \text{ in } D^{1,2}(\mathbb{R}^N), \quad h^{2/s} v_n \rightharpoonup h^{2/s} v \text{ in } L^{s/2}(\mathbb{R}^N).
\]

(3.4)

From this it is clear that \( v \in E \). Moreover, according to [25, Lemma 1], our condition \((k)\) implies that the functional

\[
\psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}, \quad \psi(v) = \int k |v|^q
\]

(3.5)

is weakly continuous on \( D^{1,2}(\mathbb{R}^N) \), thus

\[
\int k |v_n - v|^q \to 0.
\]

(3.6)

Firstly we want to show

\[
\langle \Phi'(v), v_n - v \rangle \to 0.
\]

(3.7)

Since we don’t know whether our space \( E \) is reflexive, we could not get \( v_n \rightharpoonup v \) in \( E \) and deduce (3.7). Therefore we adapt the following argument.

Because \( |f(t)| \leq |t| \) and \( |f'(t)| \leq 1 \) (see Proposition 2.1), using Hölder inequality and (3.6) we get

\[
\left| \int k |f(v)|^{q-2} f(v) f'(v) (v_n - v) \right| \leq \int k |v|^{q-1} |v_n - v|
\]

\[
\leq \left( \int k |v_n - v|^q \right)^{1/q} \left( \int k |v|^q \right)^{(q-1)/q} \to 0.
\]

(3.8)

Since \( |f(t) f'(t)| \leq 1 \) and \( |f(t)| \leq 2^{1/4} |t|^{1/2} \), noting \( (s/2)' = s/(s - 2) \) we have

\[
\int |h^{1-2/s} |f(v)|^{s-2} f(v) f'(v)|^{(s/2)'} \leq \int h (|f(v)|^{s/2})^{s/(s-2)}
\]

\[
= \int h |f(v)|^s \leq 2^{s/4} \int h |v|^{s/2} < \infty,
\]
that is \( h^{1-2/s} |f(v)|^{s-2} f(v) f'(v) \in L^{(s/2)'}(\mathbb{R}^N) \). Using \( h^{2/s}v_n \to h^{2/s}v \) in \( L^{s/2}(\mathbb{R}^N) \) we get

\[
\int h |f(v)|^{s-2} f(v) f'(v) (v_n - v) = \int h^{2/s} (v_n - v) \cdot h^{1-2/s} |f(v)|^{s-2} f(v) f'(v) \to 0. \tag{3.9}
\]

Combining (3.9) with (3.4) and (3.8) we get

\[
\langle \Phi'(v), v_n - v \rangle = \int \nabla v \cdot \nabla (v_n - v) - \int k |f(v)|^{q-2} f(v) f'(v) (v_n - v) + \int h |f(v)|^{s-2} f(v) f'(v) (v_n - v) \to 0,
\]

our claim (3.7) follows.

Next, using \( |f(t)| \leq |t| \) and \( |f'(t)| \leq 1 \) again, for

\[
\Omega_n := |f(v_n)|^{q-2} f(v_n) f'(v_n) - |f(v)|^{q-2} f(v) f'(v),
\]

we have

\[
|\Omega_n| \leq |f(v_n)|^{q-1} + |f(v)|^{q-1} \leq |v_n|^{q-1} + |v|^{q-1}.
\]

Since \( \{k^{1/q}v_n\} \) is bounded in \( L^q(\mathbb{R}^N) \),

\[
\left| \int k |\Omega_n|^q \right| \leq 2^q \left( \int k (|v_n|^q + |v|^q) \right) \leq 2^{q+1} \sup_n |k^{1/q}v_n|^q =: M < \infty.
\]

Thus using Hölder inequality and (3.6) we deduce

\[
\left| \int k (|f(v_n)|^{q-2} f(v_n) f'(v_n) - |f(v)|^{q-2} f(v) f'(v)) (v_n - v) \right|
\leq \int k^{1/q} |v_n - v| \cdot k^{1/q} |\Omega_n|
\leq \left( \int k |v_n - v|^q \right)^{1/q} \left( \int k |\Omega_n|^q \right)^{1/q'}
\leq M^{1/q'} \left( \int k |v_n - v|^q \right)^{1/q} \to 0. \tag{3.10}
\]

On the other hand, noting that the function

\[
t \mapsto s |f(t)|^{s-2} f(t) f'(t)
\]

is increasing (it is the derivative of the convex function \( \eta \) given in Lemma 3.2), we get

\[
H_n := \int h \left( |f(v_n)|^{s-2} f(v_n) f'(v_n) - |f(v)|^{s-2} f(v) f'(v) \right) (v_n - v) \geq 0.
\]
Now, using \((3.7)\) and \((3.10)\) we get
\[
o(1) = \langle \Phi'(v_n) - \Phi'(v), v_n - v \rangle
\]
\[
= \int |\nabla (v_n - v)|^2
\]
\[
- \int k \left( |f(v_n)|^{q-2} f(v_n) f'(v_n) - |f(v)|^{q-2} f(v) f'(v) \right) (v_n - v)
\]
\[
+ \int h \left( |f(v_n)|^{s-2} f(v_n) f'(v_n) - |f(v)|^{s-2} f(v) f'(v) \right) (v_n - v)
\]
\[
= \int |\nabla (v_n - v)|^2 + H_n + o(1). \tag{3.11}
\]
Consequently, noting \(H_n \geq 0\) we deduce
\[v_n \to v \text{ in } D^{1,2}(\mathbb{R}^N), H_n \to 0.\tag{3.12}\]
Since \(H_n \to 0\), from \((3.9)\) we have
\[
\int h |f(v_n)|^{s-2} f(v_n) f'(v_n) (v_n - v) \to 0.
\]
Replacing \(\alpha\) and \(\beta\) in \((3.3)\) with \(v_n\) and \(v\) respectively, we get
\[
\lim_{n \to \infty} \int h |f(v_n)|^s \leq \int h |f(v)|^s + s \lim_{n \to \infty} \int h |f(v_n)|^{s-2} f(v_n) f'(v_n) (v_n - v)
\]
\[
= \int h |f(v)|^s.
\]
Combining this with the easy consequence
\[
\int h |f(v)|^s \leq \lim_{n \to \infty} \int h |f(v_n)|^s
\]
of \(v_n \to v\) a.e. in \(\mathbb{R}^N\) and Fatou’s lemma, we get
\[
\int h |f(v_n)|^s \to \int h |f(v)|^s. \tag{3.13}
\]
Now, noting the following consequence of \((2.2)\):
\[
h |v_n|^{s/2} \leq h + \frac{1}{\mu^3} h |f(v_n)|^s
\]
and \(h |v_n|^{s/2} \to h |v|^{s/2}\) a.e. in \(\mathbb{R}^N\), by the generalized Lebesgue dominating theorem (see Proposition 3.4 below) and \((3.13)\) we get
\[
\int h |v_n|^{s/2} \to \int h |v|^{s/2}.
\]
That is to say \(|h^{2/s} v_n|^{s/2} \to |h^{2/s} v|^{s/2}\). But \(h^{2/s} v_n \to h^{2/s} v\) in \(L^{s/2}(\mathbb{R}^N)\), we deduce \(h^{2/s} v_n \to h^{2/s} v\) in \(L^{s/2}(\mathbb{R}^N)\). Combining this with \((3.12)\) we get
\[
\|v_n - v\| = \left( \int |\nabla (v_n - v)|^2 \right)^{1/2} + \left( \int h |v_n - v|^{s/2} \right)^{2/s}
\]
\[
= \|v_n - v\|_D + |h^{2/s} v_n - h^{2/s} v|_{s/2} \to 0.
\]
Thus \(v_n \to v\) in \(E\). \[
\square
\]
For the reader’s convenience, we quote the generalized Lebesgue dominating theorem as follow.

**Proposition 3.4** Let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions over the measurable set $\Omega$, $f_n \to f$ a.e. in $\Omega$, $g_n \to g$ a.e. in $\Omega$, $|f_n| \leq g_n$. Then

$$\int_\Omega |f_n - f| \to 0$$

provided $\int_\Omega g_n \to \int_\Omega g$ and $\int_\Omega g < +\infty$.

**Remark 3.5** As is well known, to prove Proposition 3.4 we apply Fatou’s lemma to $F_n := g_n + g - |f_n - f|$.

When $g_n = g$ does not depend on $n$, Proposition 3.4 reduces to the usual Lebesgue dominating theorem.

Having verified the $(PS)$ condition, we need the following variant of Clark’s theorem (see [26] or [27, Theorem 9.1] for the classical Clark’s theorem) to produce the desired solutions of our problem (1.1).

**Proposition 3.6** [21, Lemma 2.4] Let $E$ be a Banach space and $\Phi \in C^1(E, \mathbb{R})$ be an even coercive functional satisfying the $(PS)$ condition and $\Phi(0) = 0$. If for any $n \in \mathbb{N}$, there is an $n$-dimensional subspace $X_n$ and $\rho_n > 0$ such that

$$\sup_{X_n \cap S_{\rho_n}} \Phi < 0,$$

where $S_r = \{ u \in E \mid \|u\| = r \}$, then $\Phi$ has a sequence of critical values $c_n < 0$ satisfying $c_n \to 0$.

**Proof of Theorem 1.1** Given $n \in \mathbb{N}$, let $X_n$ be an $n$-dimensional subspace of $X$, where $X$ is the set of functions in $E$ which vanish in the zero set of $k$. Since the norms $\|\cdot\|$ and $|\cdot|_\infty$ are equivalent on $X_n$, there is $\vartheta > 0$ such that

$$|v|_\infty \leq \vartheta \|v\| \quad \text{for all } v \in X_n.$$

Because $h \in L^1(\mathbb{R}^N)$, we have

$$\left| \int h |v|^{s} \right| \leq |v|_\infty^s \int h \leq \vartheta^s \|v\|^s \|h\|_1 < \infty.$$

Thus we have a well-defined $s$-homogeneous functional $H : X_n \to \mathbb{R}$,

$$H(v) = \int h |v|^s.$$

Using the Lebesgue dominating theorem, it is easy to see that $H$ is continuous.

Since $f'(0) = 1$, there is $\delta \in (0, 1)$ such that

$$\frac{1}{2} |t| \leq |f(t)| \leq |t|, \quad \text{for } t \in [-\delta, \delta]. \quad (3.14)$$

Because $\dim X_n < \infty$, the compactness of

$$X_n \cap S_1 = \{ v \in X_n \mid \|v\| = 1 \}$$
and \( k \not= 0 \) implies that both

\[
A = \inf_{\varphi \in X_n \cap S_1} \int k |\varphi|^q \quad \text{and} \quad B = \sup_{\varphi \in X_n \cap S_1} H(\varphi) = \sup_{\varphi \in X_n \cap S_1} \int h |\varphi|^s
\]  

are finite, \( A > 0, B \geq 0 \). Since the norms \( \| \cdot \| \) and \( | \cdot |_\infty \) on \( X_n \) are equivalent, noting \( q < 2 < s \), we can choose \( \rho_n > 0 \) such that if \( v \in X_n, \| v \| = \rho_n \), then \( |v|_\infty \leq \delta \) and

\[
\theta_n := \frac{1}{2} \rho_n^2 - \frac{A}{2q} \rho_n^q + \frac{B}{s} \rho_n^s < 0.
\]  

(3.16)

Now using (3.14) and (3.15), for \( v \in X_n \cap S_{\rho_n} \) we have \( |v|_\infty \leq \delta \) and

\[
\Phi(v) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{q} \int k |f(v)|^q + \frac{1}{s} \int h |f(v)|^s
\]

\[
\leq \frac{1}{2} \|v\|_D^2 - \frac{1}{2q} \int k |v|^q + \frac{1}{s} \int h |v|^s
\]

\[
\leq \frac{1}{2} \|v\|^2 - \frac{A}{2q} \|v\|^q + \frac{B}{s} \|v\|^s = \theta_n.
\]  

(3.17)

From this, using (3.16) it is clear that

\[
\sup_{X_n \cap S_{\rho_n}} \Phi \leq \theta_n < 0.
\]  

(3.18)

Since our \( \Phi \) is an even coercive functional satisfying the \((PS)\) condition and \( \Phi(0) = 0 \), applying Proposition 3.6 we know that \( \Phi \) has a sequence of critical points \( \{v_n\} \) such that

\[
J(u_n) = \Phi(v_n) < 0, \quad J(u_n) \to 0,
\]

where \( u_n = f(v_n) \) are the desired solutions of (1.1). \( \square \)

**Proof of Theorem 1.2** We know that \( \Phi \) is bounded from below. Since \( \Phi(v_n) = \Phi(|v_n|) \), we may take a minimization sequence \( \{v_n\} \) such that \( v_n \geq 0 \) and

\[
\Phi(v_n) \to c := \inf_E \Phi,
\]

where \( c < 0 \) because from (3.18) we know that \( \Phi \) can take negative values. By Lemma 3.1 we know that \( \{v_n\} \) is bounded in \( D^{1,2}(\mathbb{R}^N) \). Thus we may assume

\[
v_n \rightharpoonup v \text{ in } D^{1,2}(\mathbb{R}^N), \quad v_n \to v \text{ a.e. in } \mathbb{R}^N
\]

for some nonnegative \( v \in E \).

Since \( h |f(v_n)|^s \to h |f(v)|^s \) a.e. in \( \mathbb{R}^N \), using Fatou’s lemma we get

\[
\int |\nabla v|^2 \leq \liminf_{n \to \infty} \int |\nabla v_n|^2, \quad \int h |f(v)|^s \leq \liminf_{n \to \infty} \int h |f(v_n)|^s.
\]  

(3.19)

By the weak continuity of the functional \( \psi \) defined in (3.5), from \( v_n \rightharpoonup v \) in \( D^{1,2}(\mathbb{R}^N) \) we have

\[
\int k |v_n|^q \to \int k |v|^q.
\]

Since \( k |f(v_n)|^q \leq k |v_n|^q \), applying Proposition 3.4 we get
\[
\int |f(v)|^q = \lim_{n \to \infty} \int |f(v_n)|^q.
\]  
(3.20)

From (3.19) and (3.20) we get
\[
\Phi(v) = \frac{1}{2} \int |\nabla v|^2 + \frac{1}{s} \int h |f(v)|^s - \frac{1}{q} \int k |f(v)|^q 
\]
\[
\leq \lim_{n \to \infty} \left( \frac{1}{2} \int |\nabla v_n|^2 + \frac{1}{s} \int h |f(v_n)|^s - \frac{1}{q} \int k |f(v_n)|^q \right)
\]
\[
= \lim_{n \to \infty} \Phi(v_n) = c.
\]

Therefore \( \Phi(v) = c \) and \( v \) is a nonnegative critical point of \( \Phi \).

Since \( f(t) \) has the same sign as \( t, u = f(v) \) is a nonnegative solution of (1.1) at negative energy level \( J(u) = \Phi(v) = c \). \( \Box \)

**Remark 3.7** Under the same assumptions on \( k \) and \( h \), similar results holds for
\[
\begin{cases}
-\Delta u - u \Delta (u^2) = \lambda g(x)u + k(x) |u|^{q-2} u - h(x) |u|^{s-2} u, \\
u \in D^{1,2}(\mathbb{R}^N)
\end{cases}
\]
if \( \lambda \in (\lambda_1^-, \lambda_1^+) \), where \( g \in L^{N/2}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \), \( \lambda_{1}^{\pm} \) are the principle positive/negative eigenvalues of \( -\Delta u = \lambda g(x)u \) on \( D^{1,2}(\mathbb{R}^N) \); see e.g. [28] for discussion about this eigenvalue problem. The reason is that if \( \lambda \in (\lambda_1^-, \lambda_1^+) \) then there is \( \kappa > 0 \) such that
\[
\int |\nabla v|^2 - \lambda \int g f^2(v) \geq \kappa \int |\nabla v|^2,
\]
therefore the additional term \( \int g f^2(v) \) in the functional does not affect the verification of coerciveness. Moreover, similar to the functional \( \psi \) defined in (3.5), the functional \( v \mapsto \int g v^2 \) is also weakly continuous on \( D^{1,2}(\mathbb{R}^N) \); therefore (3.12) remains valid even there is an additional term involving \( g \) in the argument.

Similarly, because of the continuous embedding \( H^1(\mathbb{R}^N) \hookrightarrow D^{1,2}(\mathbb{R}^N) \), replacing the space \( D^{1,2}(\mathbb{R}^N) \) by \( H^1(\mathbb{R}^N) \) in the argument, we can obtain similar results for
\[
\begin{cases}
-\Delta u + V(x)u - u \Delta (u^2) = k(x) |u|^{q-2} u - h(x) |u|^{s-2} u, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]
where \( V \) is a positive potential bounded away from 0.

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