Decompositions of the space of Riemannian metrics on a compact manifold with boundary

Shota Hamanaka*

Abstract

In this paper, for a compact manifold $M$ with non-empty boundary $\partial M$, we give a Koiso-type decomposition theorem, as well as an Ebin-type slice theorem, for the space of all Riemannian metrics on $M$ endowed with a fixed conformal class on $\partial M$. As a corollary, we give a characterization of relative Einstein metrics.

1 Introduction

The study of the differential structure of the space $\mathcal{M}$ of all Riemannian metrics on a closed manifold is one of important studies in geometry. In [13], Ebin particularly has proved a slice theorem for the pullback action of the diffeomorphism group on $\mathcal{M}$. In [20], Koiso has also extended it to an Inverse Limit Hilbert (ILH for brevity)-version. Moreover, he has also studied the conformal action on $\mathcal{M}$, and consequently has proved the following decomposition theorem:

Theorem 1.1 (Koiso’s decomposition theorem [21, Corollary 2.9]). Let $M^n$ be a closed $n$-manifold ($n \geq 3$), $\mathcal{M}$ the space of all Riemannian metrics on $M$ and Diff($M$) the diffeomorphism group of $M$. Set also

$$C^\infty_+(M) := \{ f \in C^\infty(M) \mid f > 0 \text{ on } M \},$$

$$\hat{\mathcal{S}} := \left\{ g \in \mathcal{M} \mid \text{Vol}(M, g) = 1, R_g = \text{const}, \frac{R_g}{n-1} \notin \text{Spec}(-\Delta_g) \right\},$$

where Vol$(M, g)$, $R_g$ and Spec$(-\Delta_g)$ denote respectively the volume of $(M, g)$, the scalar curvature of $g$ and the set of all non-zero eigenvalues of the (non-negative) Laplacian $-\Delta_g$ of $g$. Note that these four spaces become naturally ILH-manifolds. For any $g = f\tilde{g}$ ($f \in C^\infty_+$, $\tilde{g} \in \hat{\mathcal{S}}$) and any smooth deformation $\{g(t)\}_{t \in (-\epsilon, \epsilon)}$ of $g$ for sufficiently small $\epsilon > 0$, then there exist uniquely smooth deformations $\{f(t)\}_{t \in (-\epsilon, \epsilon)}$ of $f$, $\{\phi(t)\}_{t \in (-\epsilon, \epsilon)}$ of $\text{Diff}(M)$ of the identity $id_M$ and $\{g(t)\}_{t \in (-\epsilon, \epsilon)}$ of $\tilde{g}$ with $\delta_g(\tilde{g}'(0)) = 0$ such that

$$g(t) = f(t)\phi(t)^*\tilde{g}(t).$$

Here, $\delta_g(\tilde{g}'(0))$ denotes the divergence $-\nabla^i(\tilde{g}'(0))$, with respect to $g$.

*supported in doctoral program in Chuo University, 2020.
Note that the above decomposition can be replaced by
\[ g(t) = (f(t) \circ \phi(t)) \phi(t)^* \hat{g}(t) \text{ with } \delta_g \left( f'(0) \hat{g}(0) + f(0) \hat{g}'(0) \right) = 0. \]

This theorem has often played an important role in studying geometric structures related to several variational problems on a closed manifold. Hence, extending this to on a manifold with boundary seems to be also important.

From now on, we throughout assume that \( M \) is a compact connected oriented smooth \( n \)-manifold \((n \geq 3)\) with smooth boundary \( \partial M \). Let \( \mathcal{M} \) be the space of all Riemannian metrics on \( M \). In order to obtain a corresponding Koiso-type decomposition theorem on \( M \) with \( \partial M \) to Theorem \([1,1]\) on a closed manifold, we need to fix a suitable boundary condition for each metric \( g \) on \( M \). The variational view point of the Einstein-Hilbert functional, a candidate of such boundary conditions may be the following: For a fixed metric \( g_0 \) on \( M \) with zero mean curvature \( H_{g_0} = 0 \) along \( \partial M \), we fix the boundary condition for each \( g \) on \( M \) as \( [g]_{\partial M} = [g_0]_{\partial M} \) and \( H_g = 0 \) along \( \partial M \) (see Fact \([2.1,2]\) and \([2.4]\)). Here, \( [g]_{\partial M} \) denotes the conformal class of \( g|_{\partial M} \). However, one can notice that this boundary condition is not enough to get a Koiso-type decomposition theorem, even more an Ebin-type slice theorem. Here, we will fix a slightly stronger boundary condition below, which still has a naturality (see Fact \([2.1,1]\)).

We also remark that this boundary condition is very suitable for the Ricci flow on manifolds with boundary (cf. \([3]\)).

Fix a Riemannian metric \( g_0 \) on \( M \) with \( H_{g_0} = 0 \) along \( \partial M \) and set its conformal class \( C := [g_0] \) on \( M \). \( \nu_{g_0} \) denotes the outer unit normal vector field along \( \partial M \) with respect to \( g_0 \). Set also
\[ C^\infty_+(M)_N := \left\{ f \in C^\infty_+(M) \mid \nu_{g_0}(f)|_{\partial M} = 0 \right\}, \]
\[ \mathcal{M}_C := \left\{ g \in \mathcal{M} \mid g = f \cdot g_0 \text{ on } \partial M \text{ for some } f \in C^\infty_+(M)_N \right\}, \]
\[ \mathcal{S}_{C_0} := \left\{ g \in \mathcal{M}_C \mid \text{Vol}(M,g) = 1, R_g = \text{const} \right\}, \]
\[ \mathcal{S}_{C_0}^\prime := \left\{ g \in \mathcal{S}_{C_0} \mid \frac{R_g}{n-1} \notin \text{Spec}(\Delta_g; \text{Neumann}) \right\}, \]
\[ \text{Diff}_{C_0} := \left\{ \phi \in \text{Diff}(M) \mid \phi^* g_0 = f \cdot g_0 \text{ on } \partial M \text{ for some } f \in C^\infty_+(M)_N \right\}, \]
where \( \text{Spec}(\Delta_g; \text{Neumann}) \) denotes the set of all non-zero eigenvalues of \( -\Delta_g \) with the Neumann boundary condition. Note that \( H_g = 0 \) along \( \partial M \) for all \( g \in \mathcal{M}_{C_0} \). Our main result is the following theorem:

**Main Theorem.** For any \( g = f \hat{g} \) \((f \in C^\infty_+(M)_N, \hat{g} \in \mathcal{S}_{C_0}^\prime)\) and any smooth deformation \( \{g(t)\}_{t \in (-\epsilon,\epsilon)} \subset \mathcal{M}_{C_0} \) of \( g \) for sufficiently small \( \epsilon > 0 \), there exist smooth deformations \( \{f(t)\}_{t \in (-\epsilon,\epsilon)} \subset C^\infty_+(M)_N \) of \( f \), \( \{\phi(t)\}_{t \in (-\epsilon,\epsilon)} \subset \text{Diff}_{C_0} \) of \( id_M \) and \( \{\hat{g}(t)\}_{t \in (-\epsilon,\epsilon)} \subset \mathcal{S}_{C_0}^\prime \) of \( \hat{g} \) with \( \delta_g(\hat{g}(0)) = 0 \) such that
\[ g(t) = f(t)\phi(t)^* \hat{g}(t). \]

2
The rest of this paper is organized as follows. In Section 2, we state a Slice theorem for a manifold with boundary with a fixed conformal class on the boundary and prove it. In Section 3, we prepare some necessary lemmas for the proof of Main Theorem. Finally, combining them with Slice theorem, we prove Main Theorem. In Section 4, we give two applications of Main Theorem.

Acknowledgement

I would like to thank my supervisor Kazuo Akutagawa for suggesting the initial direction for my study, his good advice and support.

2 Preliminaries and a slice theorem

Let \( M \) be a compact connected oriented smooth \( n \)-dimensional manifold with non-empty smooth boundary \( \partial M \). Fix a Riemannian metric \( g_0 \) with \( H_{g_0} = 0 \). Here, \( H_{g_0} \) denotes the mean curvature of \( \partial M \) with respect to \( g_0 \). And set \( C := [g_0] \) its conformal class on \( M \). For a given positive definite symmetric \((0,2)\)-type tensor field \( T \) on \( M \), we will write \( T|_{\partial M} \in C|_{\partial M} \) when \( T = f \cdot g_0 \) for some \( f \in C^\infty(M) \) on \( \partial M \). Note that this condition is equivalent to \( \iota^*T = \iota^*(fg_0) \) for some \( f \in C^\infty(M) \), where \( \iota: \partial M \to M \) is the natural inclusion. Moreover, we denote \( T|_{\partial M} \in C_0|_{\partial M} \) when \( T = f \cdot g_0 \) for some \( f \in C^\infty(M) \) on \( \partial M \). This condition is also equivalent to \( \iota^*T = \iota^*(fg_0) \) for some \( f \in C^\infty(M) \). With this understood, we set

\[
M_C := \{ g \in M \mid \[ g|_{\partial M} \] = C|_{\partial M} \},
\]

\[
M_{C_{const}} := \{ g \in M \mid \exists c \in \mathbb{R} \ \text{such that} \ H_g = c \ \text{on} \ \partial M \},
\]

\[
M_0 := \{ g \in M \mid H_g = 0 \ \text{on} \ \partial M \}.
\]

Remark 2.1. In the case that \( \partial M = \emptyset \) (that is, \( M \) is a closed manifold), it is well known that a Riemannian metric on \( M \) is Einstein if and only if it is a critical point of the normalized Einstein-Hilbert functional \( \mathcal{E} \) on the space \( \mathcal{M} : \)

\[
\mathcal{E} : \mathcal{M} \to \mathbb{R}, \ g \mapsto \mathcal{E}(g) := \frac{\int_M R_g dv_g}{\text{Vol}_g(M)^{\frac{n-2}{2}}},
\]

where \( R_g \), \( dv_g \), \( \text{Vol}_g(M) \) denote respectively the scalar curvature, the volume measure of \( g \) and the volume of \( (M,g) \). However, if we consider the analogue of the case of \( \mathcal{E} \) on compact \( n \)-manifold \( M \) with non-empty boundary, then the set of critical points of \( \mathcal{E} \) on the space \( \mathcal{M} \) is empty (see Fact 2.1 below). Hence, in this case, we need to fix a suitable boundary condition for all metrics, and then \( \mathcal{E} \) must be restricted to a subspace of \( \mathcal{M} \).

When \( \partial M \neq \emptyset \), set the several subspaces of \( \mathcal{M} \) below:

\[
\mathcal{M}_{C|_{\partial}} := \{ g \in \mathcal{M} \mid [g|_{\partial M}] = C|_{\partial M} \},
\]

\[
\mathcal{M}_{C_{const}|_{\partial}} := \{ g \in \mathcal{M}_{C|_{\partial}} \mid \exists c \in \mathbb{R} \ \text{s.t.} \ H_g = c \ \text{on} \ \partial M \},
\]

\[
\mathcal{M}_0 := \{ g \in \mathcal{M} \mid H_g = 0 \ \text{on} \ \partial M \}.
\]
\[ \mathcal{M}_{C_0} := \mathcal{M}_{C_0} \cap \mathcal{M}_0 = \{ g \in \mathcal{M}_{C_0} \mid H_g = 0 \text{ on } \partial M \}. \]

A metric \( g \in \mathcal{M} \) is called a **relative** metric if \( g \in \mathcal{M}_0 \). By Fact 2.1 below, it is reasonable to restrict the functional \( E \) to the subspace \( \mathcal{M}_C \) as well as \( \mathcal{M}_{C_0} \) and \( \mathcal{M}_0 \).

**Fact 2.1** ([4] Remark 1, Theorem 1.1, [2] Proposition 2.1). Let \( M, E, \) and \( \mathcal{M} \) be the same as the above. Then the following holds:

1. \( g \in \text{Crit}(E|_{\mathcal{M}_{C_0}}) \) if and only if \( g \) is an Einstein metric with \( H_g = 0 \) (namely, a relative Einstein metric) and \( g|_{\partial M} \in C_s|_{\partial M} \).
2. \( g \in \text{Crit}(E|_{\mathcal{M}_{C_0}}) \) if and only if \( g \) is a relative Einstein metric and \( [g|_{\partial M}] = C_s|_{\partial M} \).
3. \( g \in \text{Crit}(E|_{\mathcal{M}_0}) \) if and only if \( g \) is an Einstein metric with totally geodesic boundary.
4. \( \text{Crit}(E) = \emptyset, \text{Crit}(E|_{\mathcal{M}_C}) = \emptyset, \text{Crit}(E|_{\mathcal{M}_{C_0}}) = \emptyset, \text{Crit}(E|_{\mathcal{M}_{C_0} \cap \mathcal{M}_0}) = \emptyset. \)

Here, for instance, \( \text{Crit}(E) \) and \( \text{Crit}(E|_{\mathcal{M}_C}) \) denote respectively the set of all critical metrics of \( E \) and the set of those of its restriction to \( \mathcal{M}_C \).

From now on, we will concentrate on the spaces \( \mathcal{M}_C \) and \( \mathcal{M}_{C_0} \). For a smooth fibre bundle \( F \), we denote by \( H^k(F) \) the space of all \( W^{k,2} \)-sections. (Note that \( L^2 \)-norm does not depend on the choice of Riemannian metric, hence, we fix a reference metric to define these function spaces.) The Sobolev embedding theorem states that \( H^k(F) \hookrightarrow C^s(F) \) is continuous if \( k > n/2 + s \), see for instance [6]. By the Sobolev embedding, if \( s > n/2, H^s(M \times M) \) (the set of all \( H^s \)-maps from \( M \) to itself) is a Hilbert manifold. Pick \( s > 4 + \frac{n}{2} \) and let \( C_s^* \text{Diff} := \{ \eta \in C^s(M \times M) \mid \eta^{-1} \in C^s(M \times M) \} \) and let \( \text{Diff}^s := H^s(M \times M) \cap C^1 \text{Diff} \). From the Sobolev embedding, \( \text{Diff}^s \) is open in \( H^s(M \times M) \) and hence it is also a Hilbert manifold. Let \( \text{Diff}^s_C := \{ \eta \in \text{Diff}^s \mid (\eta^* g_0)|_{\partial M} \in C_s|_{\partial M} \} = \{ \eta \in \text{Diff}^s \mid \eta^* g_0 = f g_0 \text{ on } \partial M \text{ for some } f \in H^{s-3/2}(C^\infty_+(M)) \}. \)

Then \( \text{Diff}^s_C \) is a Hilbert submanifold of \( \text{Diff}^{s-1} \). And let \( \text{Diff}^s_{C_0} := \{ \eta \in \text{Diff}^s \mid (\eta^* g_0)|_{\partial M} \in C_0|_{\partial M} \} = \{ \eta \in \text{Diff}^s \mid \eta^* g_0 = f g_0 \text{ on } \partial M \text{ for some } f \in H^{s-3/2}(C^\infty_+(M)) \}. \)

Then \( \text{Diff}^s_{C_0} \) is a Hilbert submanifold of \( \text{Diff}^{s-2} \). We denote by \( \text{id}_M \in \text{Diff}^s \) the identity map.

We set \( \mathcal{M}^s := H^s(S^2T^*M) \cap C^0, \mathcal{M}_s \), where \( S^2T^*M \) and \( C^0, \mathcal{M}_s \) the tensor field consisting of all symmetric \((0,2)\)-tensors on \( M \) and the set of all \( C^0 \) metrics respectively. Then \( \mathcal{M}^s \) is a Hilbert manifold modeled on \( H^2(S^2T^*M) \) (by the Sobolev embedding). And we define a closed Hilbert submanifold of \( \mathcal{M}^{s-1} \) as \( \mathcal{M}^s_C := \{ g \in \mathcal{M}^s \mid g|_{\partial M} \in C_{s}|_{\partial M} \} = \{ g \in \mathcal{M}^s \mid g = f g_0 \text{ for some } f \in H^{s-1/2}(C^\infty_+(M)) \text{ on } \partial M \}. \)

Additionally, we set \( \mathcal{M}^s_{C_0} := \{ g \in \mathcal{M}^s \mid g|_{\partial M} \in C_0|_{\partial M} \} = \{ g \in \mathcal{M}^s \mid g = f g_0 \text{ for some } f \in H^{s-1/2}(C^\infty_+(M)) \text{ on } \partial M \}. \)

\( \mathcal{M}^s_{C_0} \) is a Hilbert submanifold of \( \mathcal{M}^{s-2} \). See [13, 28] for more detail about these spaces. Moreover, we denote the pull-back action by \( A : \text{Diff}^{s+1}_C \times \mathcal{M}^s_C \rightarrow \mathcal{M}^s_C \) and for \( g \in \mathcal{M} \), let \( I_g \) be the isotropy subgroup of \( g \) in \( \text{Diff}^s_C \), that is, \( I_g := \{ \eta \in \text{Diff}^s \mid \eta^* g = g \} \).

In this section, we shall prove the following theorem:
Theorem 2.1 (Slice theorem for manifold with boundary). Let \( s > \frac{n}{2} + 4 \) and 
\[
A : \text{Diff}^{s+1}_C \times \mathcal{M}^s_C \longrightarrow \mathcal{M}^s_C
\]
be an usual action by pullback. Then for each \( \gamma \in \mathcal{M}_C \) there exists a submanifold \( S \subset \mathcal{M}^s_C \) containing \( \gamma \), which is diffeomorphic to a ball in a separable real Hilbert space, such that

1. \( \eta \in I_\gamma \Rightarrow A(\eta, S) = S \),
2. \( \eta \in \text{Diff}^{s+1}_C, A(\eta, S) \cap S \neq \emptyset \Rightarrow \eta \in I_\gamma \) and
3. There exists a local section:

\[
\chi : (\text{Diff}^{s+1}_C/I_\gamma \supset U) \longrightarrow \mathcal{D}^{s+1}_C
\]
defined in a neighborhood \( U \) of the identity coset such that if 
\[
F : U \times S \longrightarrow \mathcal{M}^s_C ; (u, t) \mapsto A(\chi(u), t),
\]
then \( F \) is a homeomorphism onto a neighborhood of \( \gamma \). Moreover, the same statement holds when we replace \( \mathcal{M}^s_C \) and \( \text{Diff}^{s+1}_C \) by \( \mathcal{M}^s_{C_0} \) and \( \text{Diff}^{s+1}_{C_0} \) respectively.

First, we get the following lemma:

Lemma 2.1. Under the identification \( T_{id_M} \text{Diff}^{s+1}_C \cong H^{s+1}(TM) \), 
\[
T_{id_M} \text{Diff}^{s+1}_C = \{ X \in H^{s+1}(TM) \mid \exists \rho \in H^s(M), \ g^0 \nabla_i X_j + g^0 \nabla_j X_i = \rho \cdot g_0, \ g_0(X, \nu) = 0 \text{ on } \partial M \},
\]
where \( g^0 \nabla \) and \( X_i \) are respectively the Levi-Civita connection with respect to \( g_0 \) and the \( i \)-th component \( g_{ij} X^j \) of \( X = (X^j) \) in terms of some local coordinates. For \( g \in \mathcal{M}^s_C \),
\[
T_g \mathcal{M}^s_C = \{ h \in H^s(S^2 T^* M) \mid \exists \rho \in H^{s-1/2}(M), h = \rho g \text{ on } \partial M \}.
\]
Here, \( T_{id_M} \text{Diff}^{s+1}_C \) and \( T_g \mathcal{M}^s_C \) represent the tangent spaces respectively. Moreover,
\[
T_{id_M} \text{Diff}^{s+1}_{C_0} = \{ X \in H^{s+1}(TM) \mid g_0(X, \nu) = 0 \text{ on } \partial M, \ \exists \rho \in H^s(C^\infty(M)_N), \ g^0 \nabla_i X_j + g^0 \nabla_j X_i = \rho g_0 \}.
\]
For \( g \in \mathcal{M}^s_{C_0} \), we also have
\[
T_g \mathcal{M}^s_{C_0} = \{ h \in H^s(S^2 T^* M) \mid \exists \rho \in H^{s-1/2}(C^\infty(M)_N), h = \rho g \text{ on } \partial M \}.
\]

Proof. From the definition, the second statement is obvious. For the first statement, we note that the derivative of a diffeomorphism-action via pull-back on metrics \( g \) at \( id_M \) coincides with the Lie derivative of \( g \) under the above identification (see for instance Lemma 6.2 in [13]). Then the first condition comes
from the conformal condition on \( \partial M \) and the second from the fact that any diffeomorphisms map \( \partial M \) to itself.

Here, we also note that \( \partial M \) may have some connected components. (Note that the number of components is finite since \( M \) is compact.) In fact, we assume that \( \partial M = \bigsqcup_{i=1}^{k} \Sigma_i \), where \( \Sigma_i \) is a connected component of \( \partial M \) and \( k \in \mathbb{Z}_{\geq 1} \). Then, under the above identification, \( \eta_i(\Sigma_i) = \Sigma_i \) for all \( t \in (-\epsilon, \epsilon) \) for sufficiently small \( \epsilon > 0 \) as explained below. Here, \( \eta_t \in T_{idM} \text{Diff}^{s+1} \) is the corresponding curve to a tangent vector. Since \( M \) is compact manifold, we can take some open neighborhoods of each \( \Sigma_i \), \( U_i \) such that \( U_i \cap U_j = \emptyset \) for all \( i \neq j \).

Consider \( W(\partial M, U) := \{ f = C^0(M, M) \mid f(\Sigma_i) \subseteq U_i \} \), then this is an open subset of \( C^0(M, M) \) with respect to the compact-open topology. Hence, this is an open neighborhood of \( idM \) in \( C^0(M, M) \). Since \( \eta_0 = idM \) and \( t \mapsto \eta_t \) is continuous, \( \eta_t \in W(\partial M, U) \) for all \( t \) with \( |t| << 1 \). In particular, \( \eta_t(\Sigma_i) = \Sigma_i \) for all \( i \) and \( t \) with \( |t| << 1 \).

The third and fourth are follow in the same way. \( \square \)

From [28] Section 9, \( H^s(TM) \) is linearly isomorphic to a closed subspace of \( H^s(D_n, \mathbb{R}) \), where \( D_n \) is a closed \( n \)-dimensional disc. Therefore we obtain the following lemma in exactly the same way as in [13] Section 3).

**Lemma 2.2** ([13] Section 3)]. \( \text{Diff}_C^s \) and \( \text{Diff}_{C_0} \) are topological groups. Furthermore, for all \( \eta \in \text{Diff}_C^s := \bigcap_{s \geq 0} \text{Diff}_C^s \) and \( \sigma \in \text{Diff}_{C_0} := \bigcap_{s \geq 0} \text{Diff}_{C_0}^s \), the left(right) action \( L_\eta(R_\sigma) : \text{Diff}_C^s \rightarrow \text{Diff}_C^s, \ L_\eta(R_\sigma) : \text{Diff}_{C_0}^s \rightarrow \text{Diff}_{C_0}^s \) are smooth.

**Remark 2.2.** It is well known that any \( C^1 \)-diffeomorphism, which is an isometry of a smooth metric, is smooth (see [13, 10, 23]). Therefore, from the Sobolev embedding, \( L_\gamma \subset \text{Diff}_C \) and \( L_\gamma \) is the same for any such that \( s > \frac{n}{2} + 1 \).

The following lemma follows from Section 5 in [13] and the fact that \( \text{Diff}_C^s \) and \( \text{Diff}_{C_0}^s \) are submanifolds of \( \text{Diff}_C^s \).

**Lemma 2.3** ([13] Section 5]). (1) The inclusion

\[ i : I_\gamma \rightarrow \text{Diff}_C^s \]

is smooth.

(2) \( i \) is embedding, that is for all \( g \in I_\gamma \), its derivative \( D_g i \) is injective and its image is closed in \( T_g \text{Diff}_C^s \).

(3) The composition map:

\[ c : I_\gamma \times \text{Diff}_C^s \rightarrow \text{Diff}_C^s ; \ (g, \eta) \mapsto g \circ \eta \]

is smooth.

(4) Let \( S := \bigcup_{\eta \in \text{Diff}_C^s} T_{idM} R_\eta(\mathcal{F}) \), then \( S \) is a \( C^\infty \) involutive subbundle of \( T\text{Diff}_C^s \), that is,

\[ X, Y \in S \Rightarrow [X, Y] \in S, \]

where \( \mathcal{F} \) is the Lie algebra of \( I_\gamma \). Moreover, the above statements hold that \( \text{Diff}_C^s \) is replaced by \( \text{Diff}_{C_0}^s \).
Using lemma 2.3 and the Frobenius’s theorem (see [22, Chapter 6, Theorem 2]), we obtain a Banach manifold structure on \( \text{Diff}_C^*/I \) and \( \text{Diff}^*_C/I \) (see [13, Proposition 5.8, 5.9]). And we can show the existence of a local section by using this Banach-coordinate-charts:

**Lemma 2.4** ([13, Proposition 5.10]). For any \( I, \eta \in \text{Diff}_C^*/I \), there exists a local section \( \pi : \text{Diff}^*_C \to \text{Diff}_C^*/I \) defined on a neighborhood \( I, \eta \). Moreover, the same statement holds that \( \text{Diff}^*_C \) is replaced by \( \text{Diff}^*_C/I \).

**Proof.** Using the above Banach-coordinate-charts, we can construct a local section exactly same as Proposition 5.10 in [13]. □

For any \( \gamma \in \mathcal{M}_C, \sigma \in \mathcal{M}_{C_0} \), we define

\[
\psi_\gamma : \text{Diff}^{s+1}_C \longrightarrow \mathcal{M}_C^* \ ; \ \psi_\gamma(\eta) := \eta^* \gamma
\]

and

\[
\psi_\sigma : \text{Diff}^{s+1}_C \longrightarrow \mathcal{M}_{C_0}^* \ ; \ \psi_\sigma(\eta) := \eta^* \sigma.
\]

Then, by the definition of \( I, \eta \), it naturally induces \( \phi_\gamma : \text{Diff}^*_C/I \to \mathcal{M}_C^* \) and \( \phi_\sigma^0 : \text{Diff}^*_C/I \to \mathcal{M}_{C_0}^* \). Moreover, from the existence of a local section and the definition, \( \phi^0, \phi_\gamma, \phi_\sigma^0 \) are smooth injective maps.

Here, we will show that \( \phi_\gamma : \text{Diff}^{s+1}_C/I \to \mathcal{M}_C^* \) and \( \phi_\sigma^0 : \text{Diff}^{s+1}_C/I \to \mathcal{M}_{C_0}^* \) are immersions (i.e., its derivation is injective and has closed image).

**Remark 2.3.** We can similarly define \( \phi_\gamma : \text{Diff}(M)^*/I \to \mathcal{M}^* \), but it is not an immersion in general.

Next, we will show that the image of \( D_\eta \psi_\gamma \) is closed in \( T_{\psi_\gamma(\eta)}\mathcal{M}_C^* \) and image of \( D_\eta \psi_\sigma^0 \) is closed in \( T_{\psi_\sigma(\eta)}\mathcal{M}_{C_0}^* \).

As mentioned in the proof of lemma 2.1, under the identification as in lemma 2.1, \( D_{id_M} \psi_\gamma(X) = \mathcal{Z}_X(\gamma) \) (\( X \in T_{id_M} \text{Diff}^{s+1}_C \)), and \( D_{id_M} \psi_\sigma^0(X) = \mathcal{Z}_X(\sigma) \). For simplicity, we put the Lie derivative with respect to \( \gamma \) as \( \alpha \), that is \( \alpha(\bullet) := \mathcal{Z}_X(\gamma) \).

For \( X \in H^{s+\frac{1}{2}}(\partial M), Y_i \in H^{s-\frac{1}{2}}(\partial M), Z_{ij} \in H^{s-\frac{3}{2}}(\partial M) \) \((i, j = 1, \ldots, n - 1)\), we set

\[
A^{s+1}_{(X,Y,Z)} := \{ u \in T_{id_M} \text{Diff}^{s+1}_C \mid u|_{\partial M} = X, \nabla_i u|_{\partial M} = Y_i, \nabla^2_{ij} u|_{\partial M} = Z_{ij} \},
\]

\[
B^{s}_{(X,Y,Z)} := \{ \gamma \in T_{\gamma} \mathcal{M}_C^* \mid \eta(\bullet, \bullet)|_{\partial M} = \gamma(\bullet) \gamma(\bullet, \bullet)|_{\partial M} + \gamma(\bullet, Y(\bullet))|_{\partial M}, \gamma(\bullet, Y(\bullet))|_{\partial M}, \}
\]

where \( \alpha^* \alpha \) is elliptic, we obtain the following boundary estimate (see [10, Theorem 6.6]):

\[7\]
There exists a positive constant $C$ such that for any $u \in A^{s+1}_{(X,Y,Z)}$,
\[
||u||_k \leq C(||(\alpha^* \alpha)u||_{k-2} + ||u||_{k-2} + ||\tilde{X}||_k)
\]
where $\tilde{X}$ is an extension to $M$ of $X$.

For $u \in T_{id}.\text{Diff}^{s+1}_C$ and $h \in T_{\gamma}.\mathcal{M}_C^s$, from the Green’s formula,
\[
(\alpha u, h) = (u, \alpha^* h) + 2(u^\flat, h(\nu, \bullet))_{\partial M},
\]
where $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\partial M}$ are $L^2$-inner product with respect to $\gamma$ and $\gamma|_{\partial M}$. And $\nu$ is the outer unit normal vector along $\partial M$ with respect to $\gamma$.

Since $h \in T_{id}.\text{Diff}^{s+1}_C$, from lemma 2.1, the second term vanishes. Thus we get
\[
(\alpha u, h) = (u, \alpha^* h).
\]
Therefore for all $u, v \in T_{id}.\text{Diff}^{s+1}_C$,
\[
((\alpha^* \alpha)u, v) = (u, (\alpha^* \alpha)v). \tag{1}
\]

From the closed range theorem ([16, Lemma 5.10]), Proposition 6.8 and 6.9 in [13], we get the following:

Lemma 2.5. $\alpha(A^{s+1}_{(X,Y,Z)})$ is a closed subspace of $B^s_{(X,Y,Z)}$ and there is an orthogonal decomposition:
\[
B^s_{(X,Y,Z)} = \text{Im}(\alpha|_{T_{id}.\text{Diff}^{s+1}_C}) \oplus \text{Ker}(\alpha^*|_{B^s_{(X,Y,Z)}}).
\]

From lemma 2.5 we get that
\[
T_{\gamma}.\mathcal{M}_C^s = \text{Im}(\alpha|_{T_{id}.\text{Diff}^{s+1}_C}) + \text{Ker}(\alpha^*|_{T_{\gamma}.\mathcal{M}_C^s}).
\]

On the other hand, from the equation (1),
\[
\text{Im}(\alpha|_{T_{id}.\text{Diff}^{s+1}_C}) \perp \text{Ker}(\alpha^*|_{T_{\gamma}.\mathcal{M}_C^s}).
\]

Hence we get an orthogonal decomposition
\[
T_{\gamma}.\mathcal{M}_C^s = \text{Im}(\alpha|_{T_{id}.\text{Diff}^{s+1}_C}) \oplus \text{Ker}(\alpha^*|_{T_{\gamma}.\mathcal{M}_C^s}).
\]

In particular, $\text{Im} D_{id.}\psi$ is closed subspace of $T_{\gamma}.\mathcal{M}_C^s$. Therefore, for each $\eta \in \text{Diff}^{s+1}_C$, $\text{Im} D_{\eta}\psi^\gamma$ is a closed subspace of $T_{\psi^\gamma(\eta)}.\mathcal{M}_C^s$.

Similarly, we also get the decomposition
\[
T_{\gamma}.\mathcal{M}_{C_0}^s = \text{Im}(\alpha|_{T_{id}.\text{Diff}^{s+1}_{C_0}}) \oplus \text{Ker}(\alpha^*|_{T_{\gamma}.\mathcal{M}_{C_0}^s}).
\]

and that $\text{Im} D_{\eta}\psi^0_\phi$ is a closed subspace of $T_{\psi^0_\phi(\eta)}.\mathcal{M}_{C_0}^s$ for all $\eta \in \text{Diff}^{s+1}_{C_0}$.

Moreover, we can show that $D_{[\psi^\gamma_\phi]}$ and $D_{[\psi^0_\phi]}$ are injective in the same way as Proposition 6.11 in [13]. Consequently, we obtain the following:
Lemma 2.6.

\[ \phi_{\gamma} : \text{Diff}_{C}^{s+1}/I_{\gamma} \rightarrow \mathcal{M}_{C}^{s} \text{ and } \phi_{0}^{0} : \text{Diff}_{C_{0}}^{s+1}/I_{\gamma} \rightarrow \mathcal{M}_{C_{0}}^{s} \]

are smooth injective immersions.

Moreover, we can prove the following in the same way as Proposition 6.13 in [13]:

**Lemma 2.7.** Let \( s > \frac{n}{2} + 4 \), then

\[ \phi_{\gamma} : \text{Diff}_{C}^{s+1}/I_{\gamma} \rightarrow \mathcal{M}_{C}^{s} \]

is a homeomorphism mapping \( \text{Diff}_{C}^{s+1}/I_{\gamma} \) onto a closed subspace of \( \mathcal{M}_{C}^{s} \). Therefore, in particular, \( \phi_{\gamma} \) is an embedding. The same statement also holds when we replace respectively \( \mathcal{M}_{C}^{s}, \text{Diff}_{C}^{s+1} \) and \( \phi \) by \( \mathcal{M}_{C_{0}}^{s}, \text{Diff}_{C_{0}}^{s+1} \) and \( \phi^{0} \).

**Remark 2.4.** The connectedness of \( M \) was used in the proof of lemma 2.7 (see the proof of Proposition 6.13 in [13]).

**Proof of the Theorem 2.1.** The proof is same as in [13]. Because we can show in the same way for \( \mathcal{M}_{C_{0}} \), we only describe about \( \mathcal{M}_{C} \).

For \( \gamma \in \mathcal{M}_{C} \), we set \( O_{\gamma}^{s} := \phi_{\gamma}(\text{Diff}_{C}^{s+1}/I_{\gamma}) \) the orbit of the action \( A \) through \( \gamma \) (where \( \phi_{\gamma} \) is in the lemma 2.7). From Lemma 2.7 this is a closed submanifold of \( \mathcal{M}_{C}^{s} \). Moreover, we define its normal vector bundle:

\[ \nu := \{ V \in T\mathcal{M}_{C}^{s}|O_{\gamma}^{s} \mid (W, V)_{\gamma} = 0 , \forall W \in TO_{\gamma}^{s} \} , \]

where \( (\ast, \ast)_{\gamma} := \int_M \langle \ast, \ast \rangle_{\gamma} dv_{\gamma} \).

Our first step is constructing the normal bundle \( \nu \) of \( O_{\gamma}^{s} \) in \( \mathcal{M}_{C}^{s} \). As stated in [13], this Riemannian metric is strong on \( H^0 \), but is not on \( H^{s} (s \geq 1) \). Thus we do not know automatically that \( \nu \) is a \( C^{\infty} \) subbundle of \( T\mathcal{M}_{C}^{s}|O_{\gamma}^{s} \).

To show this, we shall find a \( C^{\infty} \) surjective vector-bundle-map:

\[ P : T\mathcal{M}_{C}^{s}|O_{\gamma}^{s} \rightarrow TO_{\gamma}^{s} \]

such that \( \text{Ker } P = \nu \) (see [22] Chapter 3, Section 3).

Since, from the proof of the lemma 2.6

\[ T_{\gamma}\mathcal{M}_{C}^{s} = \text{Im } \alpha \oplus \text{Ker } \alpha^{*} . \]

Hence, from the definition of \( O_{\gamma}^{s} \),

\[ \text{Im } \alpha = T_{\gamma}O_{\gamma}^{s} . \]

Thus

\[ \nu_{\gamma}(O_{\gamma}^{s}) = \text{Ker } (\alpha^{*}|_{T_{\gamma}\mathcal{M}_{C}^{s}}) . \]
Moreover, since the weak Riemannian metric \((\ , \ )_{\gamma}\) is invariant under the action of \(\text{Diff}^{C+1}_{\mathbb{C}}\) (Section 4), \(\nu_{\eta, \gamma}(O^s_{\gamma}) = \eta^*(\text{Ker } \alpha^*)\). On the orbit of \(\gamma\), we define
\[
P := \alpha \circ (\alpha^* \circ \alpha)^{-1} \circ \alpha^* : T_{\gamma} \mathcal{M}_{C}^s |_{O^s_{\gamma}} \rightarrow T_{\gamma} O^s_{\gamma}
\]
Thus, as in the same way in [13, Theorem 7.1], we can show that this \(P\) satisfy the above properties.

Next, we shall construct the slice \(S\) of \(\gamma\). To do this, we consider the exponential map of \((\ , \ )_{\gamma}\), \(\exp : T_{\gamma} \mathcal{M}_{C} \rightarrow \mathcal{M}_{C}\). Thus we know the following

**Fact 2.2** (Section 4). This is a smooth map and \(\exp |_{\nu} : \nu \rightarrow \mathcal{M}_{C}^s\) is a diffeomorphism mapping a neighborhood of the zero section of \(\nu\) to a neighborhood of \(O^s_{\gamma}\) in \(\mathcal{M}_{C}^s\). Moreover, since \(A\) is continuous and \(\exp\) and the action of \(\eta\) are commutative, there are a neighborhood of \(O^s_{\gamma}\) and a neighborhood of 0 in \(\nu\) such that
\[
\nu \supset W := \{\eta^*(v) \mid v \in V, \eta \in \chi(U)\}.
\]
Then \(\exp |_{W}\) is a diffeomorphism mapping \(W\) onto a neighborhood of \(\gamma\). Moreover, if necessary, we shall take \(U\) and \(V\) small enough so that \(\exp(W) \cap O^s_{\gamma} = U\).

Consider the strong inner product \((\ , \ )_{\gamma}^s\) on \(H^*(S^2 T^*)\), defined as at the end of Section 4 in [13]. Now let \(\rho_s\) be the metric defined on \(\mathcal{M}_{C}^s\) by \((\ , \ )_{\gamma}^s\). Let \(B^s_{\gamma}\) be the open ball about \(\gamma\) of radius \(r\) with respect to \(\rho_s\). Then, for some positive \(\delta\), \(\exp(W) \supset B^{2\delta}_{\gamma}\). Pick \(U_1 \subset U\), \(\epsilon_1 < \epsilon\) so that if \(W_1 := \{\eta^*(v) \mid v \in V_1, \eta \in \chi(U_1)\}\), then \(\exp(W_1) \subset B^{s}_{\gamma}\).

Then we set
\[
S := \exp(V_1)
\]
and this \(S\) has the three properties of a slice (These are checked in the same way in [13, Section 7 and the proof of Theorem 7.1]).

\[\square\]

3 Main Results

Before starting the proof of Main theorem, we shall line up some basic definitions below:

**Definition 3.1** ([25]). (1) A topological space \(E\) is called ILH-space if \(E\) is an inverse limit of Hilbert spaces \(\{E_i\}_{i \in \mathbb{Z}_{\geq 1}}\), such that \(E_j \subset E_i\) \((i \leq j)\) and each inclusions are bounded linear operators.

(2) A topological space \(X\) is called \(C^k\)-ILH-manifold modeled on \(E\) if \(X\) has the following (a) and (b):

(a) \(X\) is an inverse limit of \(C^k\)-Hilbert manifolds \(\{X_i\}_{i \in \mathbb{Z}_{\geq 1}}\) modeled on \(E_i\) such that \(X_j \subset X_i\) \((i \leq j)\),
(b) For each \( x \in X \) and \( i \), there is an open neighborhood \( X_i \supset U_i(x) \) and homeomorphism \( \psi_i \) from \( U_i(x) \) onto an open subset \( V_i \) in \( E_i \) which gives a \( C^k \)-coordinate around \( x \) in \( X_i \) and satisfies \( U_j(x) \subset U_i(x) \) \( (i \leq j) \), \( \psi_{i+1}(y) = \psi_i(y) \) for all \( y \in U_{i+1}(x) \).

(3) Let \( X \) be a \( C^k \)-ILH-manifold \( (k \geq 1) \) and \( TX_i \) the tangent bundle of \( X_i \). The inverse limit of \( \{TX_i\} \) is called ILH-tangent bundle of \( X \).

(4) Let \( X, Y \) be \( C^k \)-ILH-manifolds. A mapping \( \phi : X \to Y \) is called \( C^l \)-ILH-differentiable \( (l \leq k) \) if \( \phi \) is an inverse limit of \( C^l \)-differentiable mapping \( \{\phi_i\}_{i \in \mathbb{Z}_{\geq 1}} \) (that is, for each \( i \), there exists \( j(i) \) and \( C^l \)-map \( \phi_i : X_{j(i)} \to Y_i \) such that \( \phi_i(x) = \phi_{i+1}(x) \) for all \( x \in X_{j(i+1)} \)).

(5) \( X \) is a ILH-manifold if \( X \) is a \( C^k \)-ILH-manifold for all \( k \geq 0 \).

(6) Let \( X, Y \) be ILH-manifolds. A mapping \( \phi : X \to Y \) is called ILH-differentiable if \( \phi : X \to Y \) is \( C^k \)-ILH-differentiable for all \( k \geq 0 \).

(7) Let \( T_x X_i \) be the tangent space of \( X_i \) at \( x \) and \( T_x X \) the inverse limit of \( \{T_x X_i\} \). Let

\[
D^r \phi_i(x) : \prod_{i=1}^r T_{x} X_{j(i)} \to T_{\phi(x)} Y_i
\]

be the \( r \)-th (Fréchet) derivative of \( \phi_i \) at \( x \). Then, \( \{D^r \phi_i(x)\}_{i \in \mathbb{Z}_{\geq 1}} \) has the inverse limit

\[
\lim_{r \to 0} D^r \phi_i(x) : \prod_{i=1}^r T_x X \to T_{\phi(x)} Y.
\]

It is called \( r \)-th derivative of \( \phi \) and we denote it by \( D^r \phi(x) \).

Let \( M \) and \( g_0 \) be the same as in Section 2 and use the same notations there. As in the closed case, \( \mathcal{M} := \lim_{+} \mathcal{M}^{r}, \text{Diff}(M) := \lim_{+} \text{Diff}^{r} \) naturally become ILH-manifolds and the pullback-action \(\mathcal{A} : \text{Diff}(M) \times \mathcal{M} \to \mathcal{M} \) is ILH-differentiable. Moreover, for a fixed metric \( g_0 \) on \( M \), since each \( \mathcal{M}^{r}_C \) are a submanifold of \( \mathcal{M}^{l-1} \), \( \mathcal{M}_C := \lim_{+} \mathcal{M}^{r}_C \) is an ILH-submanifold of \( \mathcal{M} \) and the inclusion \( \mathcal{M}_C \to \mathcal{M} \) is \( C^\infty \)-differentiable. And, \( \text{Diff}_C := \lim_{+} \text{Diff}^{r}_C \) is an ILH-submanifold of \( \text{Diff}(M) \) and the inclusion \( \text{Diff}_C \to \text{Diff}(M) \) is \( C^\infty \)-differentiable.

Similarly, \( \mathcal{M}_0 := \lim_{+} \mathcal{M}^{r}_0 \) is an ILH-submanifold of \( \mathcal{M} \) and the inclusion \( \mathcal{M}_0 \to \mathcal{M} \) is \( C^\infty \)-differentiable. And, \( \text{Diff}_0 := \lim_{+} \text{Diff}^{r}_0 \) is an ILH-submanifold of \( \text{Diff}(M) \) and the inclusion \( \text{Diff}_0 \to \text{Diff}(M) \) is \( C^\infty \)-differentiable.

Note that the pull-back action \( \mathcal{A} : \text{Diff}_0 \times \mathcal{M}_0 \to \mathcal{M}_0 \) is also \( C^\infty \)-differentiable.

By the Sobolev embedding for fibre bundles over manifold with boundary, we obtain the following (see [6]):

**Lemma 3.1.** Let \( E, F \) be vector bundles over \( M \) and let \( f : E \to F \) be a \( C^\infty \)-differentiable which preserves each fibers.

Let \( s > \frac{2}{q} \). Then the bundle map induced by \( f \)

\[
\phi : H^s(E) \to H^s(F) ; \phi(\alpha) := f \circ \alpha
\]

Page 11
Proof. Same as Lemma 1.1 in [21]. See also [28, Theorem 11.3]. □

Using this lemma and that $\mathcal{M}^r_C$ is a submanifold of $\mathcal{M}$, the following hold in the same as in [21]:

**Lemma 3.2** ([21] Proposition 1.2, Corollary 1.3, Corollary 1.4). Let $s > \frac{a}{2}$.

1. $D : \mathcal{M}^{s+1}_C \times H^{s+1}(T^p\bar{M}) \rightarrow H^{s}(T^q\bar{M})$; $(g, \xi) \mapsto \nabla_g \xi$
is $C^\infty$-differentiable, where $T^p\bar{M}$ is the type $(p, q)$ tensor bundle and $\nabla_g$ is the Levi-Civita connection with respect to $g$.

2. $\mathcal{M}^{s+1}_C \times H^{s+2}(M) \rightarrow H^{s}(M)$; $(g, f) \mapsto \Delta_g f := \nabla_g df$ is $C^\infty$-differentiable.

3. Mappings listed below are $C^\infty$-differentiable:

   - $\mathcal{M}^{s+2}_C \rightarrow H^s(S^2T^* \bar{M})$; $g \mapsto \text{Ric}_g$ (the Ricci curvature of $g$).

   - $\mathcal{M}^{s+2}_C \rightarrow H^{s}(M)$; $g \mapsto R_g$ (the scalar curvature of $g$).

   - $\mathcal{M}^{s+2}_C \rightarrow H^{s+1/2}(\partial M)$; $g \mapsto H_g$ (the mean curvature of $g$ along $\partial M$).

**Remark 3.1.** Since $\mathcal{M}_C^0$ is an ILH submanifold of $\mathcal{M}$, the same statements hold in the above lemma replaced $\mathcal{M}^r_C$ by $\mathcal{M}_C^0$.

Let $r > \frac{a}{2} + 4$. We define

- $\mathcal{M}^{r+1}_C := \{ g \in \mathcal{M}_C^r \mid \text{Vol}_g(M) = 1 \}$,
- $\mathcal{M}^{r+1}_C := \bigcap_r \mathcal{M}^{r+1}_C$,
- $\mathcal{S}^r_C := \{ g \in \mathcal{M}_C^r \mid R_g = \text{const, } H_g = 0 \}$,
- $\mathcal{S}^{r+1}_C := \bigcap_r \mathcal{S}^r_C$,
- $\mathcal{S}^{r+1}_C := \left\{ g \in \mathcal{S}^{r}_C \mid R_g = \text{const, } H_g = 0 \right\}$.

For $\bar{g}, g \in \mathcal{M}^{r+1}_C$, we define

$$\Phi_{\bar{g}}^r(g) : H^r_\bar{g}(M) \longrightarrow \langle (1, -1) \rangle^{1,s} \oplus H^{r-\frac{3}{2}}(\partial M)$$

by

$$f \mapsto \Phi_{\bar{g}}^r(g)(f) := \left( (n-1)(\Delta_g)f - R_g - \Delta_\bar{g}f - \int_M \{ (n-1)(\Delta_\bar{g})^2f - R_\bar{g}\Delta_\bar{g}f \}dV_g \right)$$

$$+ \int_{\partial M} \nu_g \langle -(n-1)\Delta_\bar{g}f - R_\bar{g}f \rangle |_{\partial M} + \nu_g \langle -R_\bar{g}f \rangle |_{\partial M},$$

where $H^r(TM), H^r(T\partial M)$ are defined by fixed $\bar{g}, \nu_\bar{g}$ is the outer unit normal vector along $\partial M$ with respect to $g$, $H^r_\bar{g}(M) := \{ f \in H^r(M) \mid f|_{\partial M} df = 0 \}$, and

$$\langle (1, -1) \rangle^{1,s} := \{ (u, v) \in H^{r-\frac{3}{2}}(\partial M) \oplus H^{r-\frac{5}{2}}(\partial M) \mid \langle (u, v), (1, -1) \rangle_{L^2(\bar{g})} = 0 \}.$$
From Lemma 3.2 and the trace theorem (Appendix B), \((g, f) \mapsto \Phi^r_g(f)\) is a \(C^\infty\) -differentiable map from \(\mathcal{M}^{r,1}_C \times H^r_g(M)\) to \(H^{r-2}_g(M) \oplus H^{r-\frac{3}{2}}(\partial M)\).

And we define

\[
\mathcal{K}^r_C := \{ g \in \mathcal{M}^{r,1}_C \mid \exists \bar{g} \in \mathcal{M}^{r,1}_C \text{ s.t. } \Phi^r_g(\bar{g}) \text{ is an isomorphism} \}.
\]

Then, the following holds:

**Lemma 3.3.** \(\mathcal{K}^r_C\) is an open subset of \(\mathcal{M}^{r,1}_C\).

**Proof.**

\[g \mapsto \Phi^r_g\]

is a diffeomorphism mapping \(\mathcal{M}^{r,1}_C\) to \(L(\mathcal{H}^r_g(M), (1, -1)^{\perp} \oplus H^{r-\frac{3}{2}}(\partial M))\).

On the other hand, the set of all isomorphisms is open in \(L(\mathcal{H}^r_g(M), (1, -1)^{\perp} \oplus H^{r-\frac{3}{2}}(\partial M))\) with respect to the operator-norm. Hence \(\mathcal{K}^r_C\) is an open subset of \(\mathcal{M}^{r,1}_C\).

**Lemma 3.4.** \(\tilde{S}^r_{C_0} = \mathcal{M} \cap \mathcal{K}^r_C \cap \tilde{S}^r_{C_0}\).

**Proof.** (\(\subset\))

Fix \(g \in \tilde{S}^r_{C_0}\).

**surjectivity :**

Firstly, we consider the case that \(R_g \neq 0\).

Given \((F, G, H) \in ((1, -1)^{\perp} \oplus H^{r-\frac{3}{2}}(\partial M))\), we consider two boundary value problem:

\[
-\Delta_g u = F, \quad \nu_g(u) \bigg|_{\partial M} = G, \quad (2)
\]

\[
-\Delta_g v - \frac{R_g}{n-1} v = u, \quad \nu_g(v) \bigg|_{\partial M} = H, \quad (3)
\]

where \(u \in H^{r-2}(M), \ v \in H^r_g(M)\).

We firstly consider \((3)\). For a fixed positive constant \(\alpha \in \mathbb{R}_{>0}\), we consider

\[-\Delta_g v + \alpha v = u, \quad \nu_g(v) \bigg|_{\partial M} = H.\]

Thus we can show that there is a unique solution by using standard variational argument.

Let \(\tilde{L}_g : H^r(M) \rightarrow H^{r-2}(M) \oplus H^{r-\frac{3}{2}}(\partial M)\) be the operator corresponding to the above equation, then this is an elliptic operator in the sense of Definition 20.1.1 in [17]. Therefore, from Theorem 20.1.2 in [17], \(\tilde{L}_g\) is a Fredholm operator. Hence \(\dim \ker \tilde{L}_g < \infty, \dim \coker \tilde{L}_g < \infty\) and \(\text{Im} \tilde{L}_g\) is closed. Moreover, because of
the existence and uniqueness of the above boundary value problem, \( \text{ind}(\tilde{L}_g) = \dim \ker \tilde{L}_g - \dim \text{coker}\tilde{L}_g = 0 \).

We shall back to (3). We consider the corresponding operator:

\[ L_g : H^r(M) \rightarrow H^{r-2}(M) \oplus H^{r-\frac{3}{2}}(\partial M) \ni u \mapsto (-\Delta_g u - \frac{R_g}{n-1} u, \nu_g(u) \big|_{\partial M}), \]

then, this operator is also an elliptic operator. From Theorem 20.1.8 in \([17]\), \(\text{ind}(L_g) = \text{ind}(\tilde{L}_g) \) and since \(g \in \mathcal{C}_0\), \(\ker L_g = \{0\}\). Hence \(\dim \text{coker} L_g = 0\). Therefore \(L_g\) is surjective.

Next, we consider (2). The ellipticity only depends on its principal symbol, thus (2) is also elliptic (Exactly speaking, the operator corresponding to (2) is elliptic). Let

\[ \tilde{L}_g : H^r(M) \rightarrow H^{r-2}(M) \oplus H^{r-\frac{3}{2}}(\partial M) \ni u \mapsto (-\Delta_g u , \nu_g(u) \big|_{\partial M}) \]

be the corresponding operator, then from Theorem 20.1.8 in \([17]\) and the above things,

\(0 = \text{ind}(\tilde{L}_g) = \text{ind}(\tilde{L}_g)\).

Since \(\ker \tilde{L}_g = \mathbb{R} (= \{\text{constant functions}\})\), \(\dim \ker \tilde{L}_g = 1\). Thus \(\dim \text{coker} \tilde{L}_g = 1\). On the other hand, from the Green’s formula,

\[ (F,G) \in \text{Im} \tilde{L}_g \Rightarrow \int_M F \, dv_g - \int_{\partial M} G \, ds_{g|\partial M} = 0. \]

Hence \(\text{coker} \tilde{L}_g \cong \langle (1,-1) \rangle\). Thus \(\text{Im} \tilde{L}_g = \langle (1,-1) \rangle^\perp\). Therefore \(\Phi^r_g(\tilde{g})\) is surjective if \(R_g \neq 0\).

Next, we consider the case that \(R_g = 0\). The above observation implies that (2) has a unique solution up to constants. That is, if we take a solution \(u(F,G)\) of (2), then \(u(F,G) + C\) (\(C\) is arbitrary constant) is also solution of this equation. Hence, for given \(H\), we take a constant \(C\) so that

\[ \int_M u(F,G) + C \, dv_g = - \int_{\partial M} H \, ds_{g|\partial M}. \]

Then, from the above observation, there exists a solution \(v\) of (3). Therefore \(\Phi^r_g(\tilde{g})\) is also surjective if \(R_g = 0\).

in injectivity :

Let \(\Phi^r_g(\tilde{g})(u) = 0\), then

\[ \begin{align*}
(n-1)(\Delta_g)^2 u + R_g \Delta_g u &= 0, \\
\nu_g \left\{ -(n-1)\Delta_g u - R_g u \right\} \big|_{\partial M} &= 0, \\
\nu_g(u) \big|_{\partial M} &= 0.
\end{align*} \]
Multiply the contents in \( \{ \} \) of \((5)\) by the left hand side of \((4)\) and integration it over \(M\). Thus, by integration by parts, \((4)\) and \((5)\),

\[-(n-1)\Delta_g u - R_g u = \text{const}.\]

On the other hand,

\[
\int_M -(n-1)\Delta_g u - R_g u \, dv_g = \int_{\partial M} (n-1)\nu_g(u) \, ds_{g|\partial M} - R_g \int_M u \, dv_g.
\]

Hence, from \((6)\) and \(u \in H^r_g(M)\), the first term of the right hand side is zero. Thus, since \(u \in H^r_g\), the second term also vanish. Therefore, we have

\[-(n-1)\Delta_g u - R_g u = 0.\] (7)

Hence, if \(R_g \neq 0\), then \(u = 0\).

On the other hand, if \(R_g = 0\), we multiply the both sides of \((7)\) by \(u\), integral over \(M\), use the integration by parts and get \(u = \text{const}\). But, since \(u \in H^r_g\), \(u \equiv 0\). Therefore \(\Phi^r_g(g)\) is injective.

\[
(\supset) \text{This inclusion is obvious.} \quad \Box
\]

Lemma 3.5. \(\hat{S}_{C_0} \neq \emptyset\).

Proof. Since \(\dim M \geq 3\), we can construct a negative constant scalar curvature metric \(g\) (see [11]). Thus, since the eigenvalues of \(-\Delta_g\) are positive ([6, Theorem 4.4]), from Lemma 3.4, \(g \in \hat{S}_{C_0}\).

Lemma 3.6. \(\hat{S}^r_{C_0} \cap K^r_{C_0}\) is an ILH-submanifold of \(\mathcal{M}^r_{C,1}\).

Proof. For each \(g \in \hat{S}^r_{C_0} \cap K^r_{C_0}\), we define a map

\[
\Psi : \mathcal{M}^r_{C,1} \rightarrow ((1,-1)^{1/2} \oplus H^{r-2/2}(\partial M))
\]

by

\[
g \mapsto \left( -\Delta_g R_g + \int_M \Delta_g R_g \, dv_g + \int_{\partial M} \nu_g(R_g) \, ds_{g|\partial M}, \nu_g(R_g)|_{\partial M}, \nu_g(tr g_0)|_{\partial M} \right).
\]

From Lemma 3.2, this is a \(C^\infty\)-differentiable map.

First, we note that \(\Psi^{-1}(0) = \hat{S}^r_{C_0}\). In fact, if \(g \in \hat{S}^r_{C_0}\), then \(\nu_g(tr g_0)|_{\partial M} = 0\) since \(g||_{\partial M} \in C||_{\partial M}\), \(H_{g_0} = 0\) and \(H_g = 0\). And it is clear that the first two terms are zero if \(g \in \hat{S}^r_{C_0}\), hence the inclusion of \(\Psi^{-1}(0) \supset \hat{S}^r_{C_0}\) holds.

On the other hand, for \(g \in \Psi^{-1}(0)\), then, since the first and second components are both zero,

\[
\Delta_g R_g = \text{const}
\]

But, since the second component is zero and from the Green’s formula, this constant must be zero. Thus, by multiplying \(\Delta_g R_g\) by \(R_g\) and integrating it
over \( M \), from integration by parts and the fact that the second component is zero,
\[
0 = \int_M |\nabla_g R_g|^2 dg.
\]
Hence \( R_g = \text{const.} \)

Since the third component is zero, \( H_g|_{\partial M} = 0 \) from \( g||_{\partial M} \in C||_{\partial M} \) and the formula under the conformal change.

The derivative of \( \Psi \) at \( g \in \mathcal{S}_{C}^r \cap K_{C}^r \) is calculated as follows:
\[
D_g \Psi (h) = \left( -\Delta_g (\Delta_g tr_g h + \delta_g h - \langle h, \text{Ric}_g \rangle_g) \right)
- \int_M \Delta_g (\Delta_g tr_g h - \delta_g h + \langle h, \text{Ric}_g \rangle_g) dv_{g_0} - \int_{\partial M} \nu_g (\Delta_g tr_g h - \delta_g h + \langle h, \text{Ric}_g \rangle_g) ds_{g_0} |_{\partial M}.
\]
(see \([4, \text{Claim 3.1}], [7, \text{Theorem 1.174}]\)).

Take the variation \( h = fg \ (f \in H^r \Gamma (M)) \). Then, since \( g \in K_{C}^r \), we get \( D_g \Psi \) is surjective.

Therefore, from the Inverse function theorem (\([27\]) \), \( \mathcal{S}_{C_0}^r \cap K_{C_0}^r \) is a submanifold of \( \mathcal{M}_{C,1} \) and the tangent space at \( g \in \mathcal{S}_{C_0}^r \cap K_{C_0}^r \) is \( \text{Ker} D_g \Psi \).

\[ \square \]

**Lemma 3.7.** Let \( C_+^r (M)_N := \{ f \in H^r (M) \mid f > 0 \text{ on } M, \nu_{g_0} (f) = 0 \text{ on } \partial M \} \) and be a map
\[
\chi^r : C_+^r (M)_N \times (\mathcal{S}_{C_0}^r \cap K_{C_0}^r) \longrightarrow \mathcal{M}_{C_0}^r ; (f, g) \mapsto f \cdot g.
\]

Then \( \chi^r \) is \( C^\infty \)-differentiable. Moreover, if \( g \in \mathcal{S}_{C_0}^r \), then \( D_{(f,g)} \chi^r \) is an isomorphism.

**Proof.** It is clear that this is a \( C^\infty \)-differentiable map. In fact,
\[
D_{(f,g)} \chi^r (\phi, h) = fh + \phi g.
\]

Injectivity:

Let \( fh + \phi g = 0 \), then since \( \text{Ker} D_g \Psi \in h = -\phi g =: \tilde{\phi} g \), \( \tilde{\phi} \in \text{Ker} \Phi_{\tilde{g}}^r \). On the other hand, since \( g \in K_{C_0}^r \), \( \tilde{\phi} = 0 \). Hence, since \( f \neq 0 \), \( \phi = 0 \), \( h = 0 \).

Surjectivity:

We shall show it by contradiction.

If \( D_{(f,g)} \chi^r \) is not surjective, then \( \exists \tilde{h} \in (\text{Im} D_{(f,g)} \chi^r)^\perp \setminus \{0\} \)
(since
\[
\text{Im} D_{(f,g)} \chi^r = ftg \left( \mathcal{S}_{C_0}^r \cap K_{C_0}^r \right) + H^r (M)g
\]
is a closed subspace in \( T_{fg} \cdot \mathcal{M}_{C_0}^r \)).

We define an operator on \( (H^r (M)_N g)^\perp \) (which is a closed subspace in \( T_{fg} \cdot \mathcal{M}_{C_0}^r \))
\[
K_g (h) := -\Delta_g tr_g h + \delta_g h - \langle h, \text{Ric}_g \rangle_g.
\]

16
where $H^r(M)_N := \{ f \in H^r(M) \mid \nu_{g_0}(f) = 0 \text{ on } \partial M \}$.

From the Green’s formula,

$$(K_g h, f)_M - (h, K_g^* f)_M = (\nu_g(f), \tr{f} h)_{\partial M} - (f, \nu_g(\tr{f} h))_{\partial M} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
We firstly consider \( h_1 \). Let \( h_1 = \mathcal{L}_X g, \ X \in H^{r+1}(TM) \). Since \( K_g \) is the first derivative of the functional \( g \rightarrow R_g, R_g \) is diffeomorphism invariant and the derivative of the pull-back action of diffeomorphism on \( g \) is \( \mathcal{L}g \) (as mentioning in the proof of lemma 2.1),

\[
K_g h_1 = 0 \text{ on } M.
\]

Hence \( \nu_{g_0}(K_g h) = 0 \) on \( \partial M \).

Finally, we consider \( h_2 \in \text{Ker} \delta_g \). Then \( \delta_g \delta_g h_2 = 0 \) on \( M \).

On the other hand, since \( h_2 \in T_f \mathcal{M}_C^r \), we can write \( h_2 = f \cdot g_0 \) for some \( f \) with \( \nu_{g_0}(f) = 0 \) on \( \partial M \). Then, on \( \partial M \),

\[
\nabla_{\nu_{g_0}} (h, \text{Ric}_g) = \langle \nabla_{\nu_{g_0}} (f g_0), \text{Ric}_g \rangle + \langle f g_0, \nabla_{\nu_{g_0}} \text{Ric}_g \rangle,
\]

where \( \langle , \rangle_g \) denote the natural inner product on the \((0,2)\)-tensor bundle of \( M \) induced by \( g \). Here, as mentioned above, \( f = 0 \) on \( \partial M \). And \( \nabla_{\nu_{g_0}} (f g_0) = (\nu_{g_0}(f)) g_0 + f \nabla_{\nu_{g_0}} g_0 = 0 \) on \( \partial M \) since \( \nu_{g_0}(f) = 0 \) on \( \partial M \). And \( \nabla_{\nu_{g_0}} g_0 = 0 \) on \( M \) (Because \( g_0 \) is parallel with respect to \( g_0 \)). Consequently, we obtain \( \nu_{g_0}(K_g h) = 0 \) on \( \partial M \).

We get the following lemma in the same way as Lemma 2.8 in [20]:

**Lemma 3.8.** Let \( E \) and \( F \) be vector bundles over \( M \) associated with the frame bundle. Any \( \eta \in \text{Diff}_{C_0} \) defines a natural linear map (by pullback)

\[
\eta^*: H^k(E) \longrightarrow H^k(E) \quad (k \geq n/2 + 2).
\]

Let \( r \geq n/2 + 2 \), \( A \subset H^r(E) \) be an open subset and let \( \phi: A \rightarrow H^s(F) \) be a \( C^\infty \)-differentiable map which commutes with the action of \( \text{Diff}_C \). Put \( A^s := A \cap H^s(E) \) \( (s \geq r) \). Then \( \phi(A^s) \subset H^s(F) \) and \( \phi|_{A^s} : A^s \rightarrow H^s(F) \) is \( C^\infty \)-differentiable.

**Theorem 3.1.** \( \tilde{\mathcal{S}}_{C_0} \) is an ILH-submanifold of \( \mathcal{M}_{C_0} \) and the map

\[
\chi: C^\infty_+ (M)_N \times \tilde{\mathcal{S}}_{C_0} \rightarrow \mathcal{M}_{C_0} \ ; \ (f, g) \mapsto f \cdot g
\]

is a local ILH-diffeomorphism into \( \mathcal{M}_{C_0} \).

**Proof.** It can be proved exactly the same as Theorem 2.5 in [21] using lemma 3.7 and 3.8. And note that \( \bigcap_r C_+^r (M)_N = C^\infty_+ (M)_N \).

Since \( \text{Diff}_{C_0} \) and \( \mathcal{M}_{C_0} \) are submanifold of \( \text{Diff}(M) \) and \( \mathcal{M} \) respectively, from Lemma 3.8 and Theorem 2.1 we can obtain the following \( C^\infty \)-version of the Slice theorem exactly same as in [20]:

**Theorem 3.2** \((C^\infty \text{-version of Theorem 2.1}). For all \( g \in \mathcal{M}_{C_0} \) there exists an ILH-submanifold \( S_g \subset \mathcal{M}_{C_0} \) containing \( \gamma \) so that the following holds:

1. \( \eta \in I_g \Rightarrow \eta^* S_g = S_g \),
2. \( \eta \in \text{Diff}_C \), \( \eta^* (S_g) \cap S_g \neq \emptyset \Rightarrow \eta \in I_g \) and

18
(3) There exists a local section defined on an open neighborhood of \([I_g]\):

\[ \exists \chi : (\text{Diff}_{C^0}/I_g \supset U \hookrightarrow \text{Diff}_{C^0}) \]

such that

\[ F : U \times S_g \rightarrow \mathcal{M}_{C^0} : (u, t) \mapsto \chi(u)*t \]

is an ILH-diffeomorphism mapping onto an open neighborhood of \(g\).

Consequently, from this Slice theorem and Theorem 3.1 we can prove Main Theorem in Section 1.

**Proof of Main Theorem.** From Theorem 3.1 we can decompose as

\[ g(t) = f(t)\tilde{g}(t), \]

where \(f(t)\) is a deformation of \(f\) in \(C^\infty_+(M)_N\) and \(\tilde{g}(t)\) is a deformation of \(\tilde{g}\) in \(\tilde{S}_{C^0}\). Moreover, from Theorem 3.2, \(\tilde{g}(t)\) can be decomposed as

\[ \tilde{g}(t) = \phi(t)^*\bar{g}(t) \text{ with } \delta\bar{g}'(0) = 0. \]

Since the scalar curvature is invariant under the action of diffeomorphisms,

\[ R_{\tilde{g}(t)} = R_{\bar{g}(t)} \equiv \text{const}. \]

**Theorem 3.3.** For any \(g = f\tilde{g} \in C^\infty_+(M)_N\), \(\tilde{g} \in \tilde{S}_{C^0}\) and any smooth deformation \(\{g(t)\}_{t \in (-\epsilon, \epsilon)} \subset \mathcal{M}_{C^0}\) of \(g\) for sufficiently small \(\epsilon > 0\), there exists uniquely a smooth deformation \(\{f(t)\}_{t \in (-\epsilon, \epsilon)} \subset C^\infty_+(M)_N\) of \(f\), a smooth one \(\{\phi(t)\}_{t \in (-\epsilon, \epsilon)} \subset \text{Diff}_{C^0}\) of \(id_M\) and a smooth one \(\{\tilde{g}(t)\}_{t \in (-\epsilon, \epsilon)} \subset \tilde{S}_{C^0}\) of \(\tilde{g}\) with \(\delta(f(0)\tilde{g} + f\tilde{g}'(0)) = 0\) such that

\[ g(t) = (f(t) \circ \phi(t))\phi(t)^*\tilde{g}(t). \]

**Proof.** Reverse the order of applying Theorem 3.1 and Theorem 3.2 in the proof of Main Theorem.

### 4 Applications

We use the same notations as those in the above sections. We give two applications of the following.
4.1 Some rigidity theorems for relative constant scalar curvature metrics

In the case of \( \partial M = \emptyset \), a metric \( g \) in a given conformal class \( C \) is called a **Yamabe metric** if \( g \) is a minimizer of the restriction \( \mathcal{E}|_C \) of the normalized Einstein-Hilbert functional \( \mathcal{E} \). The infimum of \( \mathcal{E}|_C \) is called the **Yamabe constant** \( Y(M, C) \) of \( C \). By combining the Koiso’s decomposition theorem with the existence of a Yamabe metric in each conformal class, Böhm-Wang-Ziller proved the following (see the proof of [S Theorem 5.1]):

**Theorem 4.1.** Let \( (M^n, g_\infty) \) be a closed Riemannian manifold of dimension \( n \geq 3 \). Assume that \( g_\infty \) is a unique constant scalar curvature (csc metric for brevity) in its conformal class up to rescaling with \( \lambda_1(-\Delta g_\infty) > \frac{R_{g_\infty}}{n-1} \). Here, \( \lambda_1(-\Delta g_\infty) \) denotes the first non-zero eigenvalue of \( -\Delta g_\infty \). Then each csc metric sufficiently close to \( g_\infty \) with respect to the \( C^\infty \)-topology is a Yamabe metric in its conformal class.

**Remark 4.1.** In the above, the condition \( \lambda_1(-\Delta g_\infty) > \frac{R_{g_\infty}}{n-1} \) implies that \( (M^n, [g_\infty]) \) is not conformally equivalent to the standard \( n \)-sphere \( (S^n, [g_{std}]) \).

On the other hand, using a compactness theorem of the space of all csc metrics in a fixed conformal class (proved by Khuri-Maqures-Schoen [13 Theorem 1.1]), one can also get the following:

**Theorem 4.2.** Let \( (M^n, g_\infty) \) be a closed Riemannian manifold of dimension either \( 3 \leq n \leq 7 \), or both \( 8 \leq n \leq 24 \) and that \( M \) is spin. Assume that \( g_\infty \) is a unique csc metric in its conformal class up to rescaling with \( \lambda_1(-\Delta g_\infty) > \frac{R_{g_\infty}}{n-1} \). Then each csc metric sufficiently close to \( g_\infty \) with respect to the \( C^\infty \)-topology is also a unique csc metric up to rescaling in its conformal class.

We can prove a similar statement below on a manifold with boundary. When \( \partial M \neq \emptyset \), for \( g \in \mathcal{M} \) with \( H_g = 0 \) along \( \partial M \),

\[
Y(M, [g]_0) := \inf_{h \in [g]_0} \mathcal{E}(h)
\]

is called the **relative Yamabe constant** of \([g]_0\). A metric \( h \) with \( H_h = 0 \) along \( \partial M \) is called a **relative Yamabe metric** if \( Y(M, [g]_0) = \mathcal{E}(h) \) (see [4] for more details). Here, \([g]_0\) denotes the relative conformal class of \( g \), that is, \([g]_0 := \{ h \in [g] \mid H_h = 0 \text{ along } \partial M \} = \{ u \cdot g \mid u \in C^\infty_+(M), \nu_g(u)|_{\partial M} = 0 \} \). Then we can prove the following:

**Theorem 4.3.** Let \( (M^n, g_\infty) \) be a compact connected Riemannian manifold of dimension \( n \geq 3 \) with smooth non-empty minimal boundary \( \partial M \) (i.e., \( H_{g_\infty} = 0 \) along \( \partial M \)). Assume that \( g_\infty \) is a unique relative csc metric in \([g_\infty]_0\) up to rescaling with \( \lambda_1(-\Delta g_\infty; \text{Neumann}) > \frac{R_{g_\infty}}{n-1} \), where \( \lambda_1(-\Delta g_\infty; \text{Neumann}) \) denotes the first non-zero eigenvalue of \( -\Delta g_\infty \) with the Neumann boundary condition (see [30] Section 1) and [1] Proposition 2.6)). Moreover, we assume the following: either:

\begin{align*}
\lambda_1(-\Delta g_\infty; \text{Neumann}) &> \frac{R_{g_\infty}}{n-1} \\
\min_{\partial M} R_{g_\infty} &> \frac{2n}{n+2} \lambda_1(-\Delta g_\infty; \text{Neumann}) \\
\end{align*}
(a) \((M^n, g_\infty)\) has a nonumbilic point on \(\partial M\), or

(b) \((M^n, g_\infty)\) is umbilic boundary (therefore, \(\partial M\) is totally geodesic) satisfying that one of the followings (b1)-(b3) holds:

(b1) the Weyl tensor does not vanish identically on \(\partial M\) and \(n \geq 6\),
(b2) \(M\) is locally conformally flat,
(b3) \(n = 3, 4, \text{ or } 5\).

Then each relative csc metric sufficiently close to \(g_\infty\) in \(\mathcal{M}_{C_0}\) with respect to the \(C^\infty\)-topology is a relative Yamabe metric.

**Remark 4.2.** Escobar [15] proved that, for any \((M, g_\infty)\) satisfying either (a) or (b) in the above, then there exists a relative Yamabe metric in \([g_\infty]_0\). Note also that, the condition \(\lambda_1(-\Delta g_\infty; \text{Neumann}) > \frac{R_{g_\infty}}{n-1}\) implies that \((M, [g_\infty]_0)\) is not conformally equivalent to the standard hemisphere \((S^n_+, [g_{\text{std}}]_0)\).

The following follows directly from Main Theorem.

**Corollary 4.1.** Assume that \(g_\infty \in \mathcal{S}_{C_0}\) satisfies \(\lambda_1(-\Delta g_\infty; \text{Neumann}) > \frac{R_{g_\infty}}{n-1}\). Let \(\{g_i\}\) and \(\{\tilde{g}_i := u_i^{-\frac{4}{n-2}} g_i\}\) \(\subset \mathcal{S}_{C_0}\) be sequences each of which converges to \(g_\infty\) with respect to the \(C^\infty\)-topology. Then, except for a finite number of \(i\), \(g_i = \tilde{g}_i\).

**Outline of the proof of Theorem 4.3.** We can assume that \((M, g_\infty)\) has unit volume. Let \(\{g_i\} \subset \mathcal{S}_{C_0}\) be a sequence which converges to \(g_\infty\) with respect to the \(C^\infty\)-topology. For each \(i\), let \(u_i \in C^\infty(M)^N\) be a solution of relative Yamabe problem in \([g_i]_0\), that is, \(\tilde{g}_i := u_i^{-\frac{4}{n-2}} g_i\) is a relative Yamabe metric of \([g_i]_0\) with unit volume. Since \(\tilde{g}_i \in \mathcal{S}_{C_0}\) is a relative Yamabe metric, then the following hold:

\[
||u_i||_{L^\frac{2n}{n-2}(g_i)} = 1, \tag{8}
\]

\[
-\frac{4}{n-2} \Delta_{g_i} u_i + R_{g_i} u_i = Y(M, [g_i]_0) u_i^\frac{n+2}{n-2} \text{ on } M, \tag{9}
\]

\[
\nu_{g_i}(u_i) = 0 \text{ on } \partial M. \tag{10}
\]

By [15], the assumptions in Theorem 4.3 implies that

\[
Y(M, [g_\infty]_0) < Y(S^n_+, [g_{\text{std}}]_0).
\]

Since \(g \mapsto Y(M, [g]_0)\) is continuous with respect to the \(C^2\)-topology,

\[
Y(M, [g_i]_0) < Y(S^n_+, [g_{\text{std}}]_0)
\]

for sufficiently large \(i\). Hence, we can apply the similar argument in the proof of Theorem 5.1 in [8] to that on a manifold with boundary after slight modifications. Then, there exists a subsequence \(\{u_{i_k}\} \subset \{u_i\}\) and \(u_\infty \in C^\infty(M)^N, \ R \in \mathbb{R}\) such that

\[
\tilde{g}_i = u_{i_k}^{-\frac{4}{n-2}} g_{i_k} \rightarrow \tilde{g}_\infty := u_\infty^{-\frac{4}{n-2}} g_\infty \text{ as } k \rightarrow \infty
\]
and

\[
\|u_\infty\|_{L^{\frac{2n}{n-2}}(g_\infty)} = 1, \tag{11}
\]

\[- \frac{n-1}{n-2} \Delta_{g_\infty} u_\infty + R_{g_\infty} u_\infty = \tilde{R} u_\infty^{\frac{n+2}{n-2}} \text{ on } M, \tag{12}
\]

\[\nu_{g_\infty}(u_\infty) = 0 \text{ on } \partial M. \tag{13}\]

Here, the above convergence \(\tilde{g}_i \to \tilde{g}_\infty (k \to \infty)\) is the \(C^\infty\)-convergence with respect to \(g_\infty\). Then, from the regularity theorem\([10]\) and the maximum principle, \(u_\infty \in C^\infty(M)_N\). From \([11]\) and \([12]\), we have \(\tilde{g}_\infty \in \mathcal{G}_C\). Hence, from the uniqueness assumption for \(g_\infty\), \(\tilde{g}_\infty = g_\infty\). Therefore \(\tilde{g}_{ik} = g_{ik}\) from Corollary \([4.1]\) except for finite number of \(k\).

By using a compactness theorem (proved by Discozi-Khuri \([12\, \text{Theorem 1.1}]\)), we can also prove the following:

**Theorem 4.4.** Let \((M^n, g_\infty)\) be a compact connected Riemannian manifold with smooth non-empty totally geodesic boundary \(\partial M\). Assume that either \(3 \leq n \leq 7\), or both \(8 \leq n \leq 24\) and that \(M\) is spin. Let \(g_\infty \in \mathcal{G}_C\) be a unique relative csc metric in \([g_\infty]_0\) up to rescaling with \(\lambda_1(-\Delta_{g_\infty}; \text{Neumann}) > \frac{R_{g_\infty}}{n-1}\). Then each relative csc metric sufficiently close to \(g_\infty\) in \(\mathcal{M}_C\) with respect to the \(C^\infty\)-topology is also a relative unique csc metric up to rescaling in its relative conformal class.

**Proof.** In the proof of Theorem 4.3, we will take \(\tilde{g}_i := u_i^{\frac{4}{n-2}} g_i\) as another relative csc metric in \([g_i]_0\) with unit volume. Then, from the compactness result \([12\, \text{Theorem 1.1}]\), there exist a subsequence \(\{u_{ik}\} \subset \{u_i\}\) and \(u_\infty \in C^\infty(M)_N, \tilde{R} \in \mathbb{R}\) such that

\[\tilde{g}_i = u_{ik}^{\frac{4}{n-2}} g_{ik} \to \tilde{g}_\infty := u_\infty^{\frac{4}{n-2}} g_\infty \text{ as } k \to \infty\]

and

\[\|u_\infty\|_{L^{\frac{2n}{n-2}}(g_\infty)} = 1, \quad - \frac{n-1}{n-2} \Delta_{g_\infty} u_\infty + R_{g_\infty} u_\infty = \tilde{R} u_\infty^{\frac{n+2}{n-2}} \text{ on } M, \quad \nu_{g_\infty}(u_\infty) = 0 \text{ on } \partial M.\]

Then, the same argument as that in the proof of Theorem 4.3 implies that \(\tilde{g}_{ik} = g_{ik}\) except for a finite number of \(k\). This completes the proof. \(\square\)

### 4.2 A characterization of relative Einstein metrics

In the case of \(\partial M = \emptyset\), we recall the *Yamabe invariant* \(Y(M)\) of \(M\) defined by

\[Y(M) := \sup_{C \in \mathcal{C}(M)} Y(M, C) = \sup_{C \in \mathcal{C}(M)} \left( \inf_{g \in C} Y(M, C) \right),\]

where \(\mathcal{C}(M)\) denotes the set of all conformal classes on \(M\). By the Koiso’s decomposition theorem, one can get the following:
Theorem 4.5 (cf. [6] Proposition 5.89). Let \( M^n \) be a closed manifold of dimension \( \geq 3 \) and \( g \) a unique csc metric (up to rescaling) in its conformal class \([g]\). Assume that \( Y(M) \) is attained by \( g \) and that \( \lambda_1(-\Delta_g; \text{Neumann}) > \frac{R_g}{n-1} \). Then, \( g \) is an Einstein metric.

We can prove a similar statement below on a manifold with boundary. Let \( M^n \) be a compact connected Riemannian manifold of dimension \( n \geq 3 \) with non-empty smooth boundary \( \partial M \). For each conformal class \( C \) on \( M \), we define an invariant \( Y(M; C) \):

\[
Y(M; C) := \sup_{C \in \mathcal{C}_C} \inf_{t \in C} \mathcal{E}(\tilde{g}) = \sup_{g \in \mathcal{M}_0} Y(M, [g]_0),
\]

where \( \mathcal{C}_C := \{ \tilde{C} : \text{conformal class on } M \mid \tilde{C}||_{\partial M} = C||_{\partial M} \} \). From the Aubin-type inequality, it holds that \( Y(M; C) \leq Y(S^n, [g_{\text{std}}]) \) (see [1] for instance). Then, we can prove the following:

Theorem 4.6. Fix a conformal class \( C \) on \( M \). Let \( \tilde{g} \) be a unique relative csc metric (up to rescaling) in \([g]_0\) with \( \tilde{g}||_{\partial M} \in C||_{\partial M} \). Assume that \( Y(M; C) \) is attained by \( \tilde{g} \) and that \( \lambda_1(\Delta_{\tilde{g}}; \text{Neumann}) > \frac{R_{\tilde{g}}}{n-1} \). Moreover, assume that \( Y(M; C) < Y(S^n, [g_{\text{std}}]_0) \). Then, \( \tilde{g} \) is a relative Einstein metric.

Proof. From Main Theorem, every critical point of \( \mathcal{E}|_{\mathcal{C}_C} \) is Einstein (cf. [7] Proposition 4.47)). Hence, it is enough to prove that \( \frac{d}{dt}\mathcal{E}(g(t))|_{t=0} = 0 \) for any smooth deformation \( g(t) \) of \( \tilde{g} \) in \( \mathcal{C}_C \).

We now remark that, under the condition \( Y(M, [g]_0) < Y(S^n, [g_{\text{std}}]_0) \), there exists always a relative Yamabe metric in \([g]_0\) (10). Hence, by Theorem 4.3 \( Y(M, [g(t)]_0) = \mathcal{E}(g(t)) \) for sufficiently small \(|t| \ll 1 \). On the other hand, since \( Y(M, [g(t)]) \leq Y(M; C) \) and \( Y(M, [g(0)]_0) = Y(M; C) \), we have

\[
0 = \frac{d}{dt} Y(M, [g(t)])|_{t=0} = \frac{d}{dt} \mathcal{E}(g(t))|_{t=0}.
\]

Therefore, \( \tilde{g} \) is a relative Einstein. \( \square \)

Remark 4.3. From the characterization Theorem 4.6 for relative Einstein metrics, we would like to suggest that the relative Yamabe invariant \( Y(M, \partial M, C|_{\partial M}) \) defined in [4] Section 1:

\[
Y(M, \partial M, C|_{\partial M}) := \sup_{g \in \mathcal{M}_0 \to \partial M} Y(M, [g]_0)
\]

should be replaced by the above \( Y(M; C) = \sup_{g \in \mathcal{M}_0} Y(M, [g]_0) \).

References

[1] K. Akutagawa, Notes on the relative Yamabe invariant, Differential Geometry (ed. by Q. M. Cheng, Josai Univ., Feb. 2001), Josai Mathematical Monographs 3 (2001), 105–113.
[2] K. Akutagawa, An Obata type theorem on compact Einstein manifolds with boundary, preprint, 2020.

[3] K. Akutagawa and B. Botvinnik, Manifolds of positive scalar curvature and conformal cobordism theory, Math. Ann. 324 (2002), 817–840.

[4] K. Akutagawa and B. Botvinnik, The Relative Yamabe Invariant, Comm. Anal. Geom. 10 (2002), 925–954.

[5] K. Akutagawa and S. Hamanaka, The Ricci flow on manifolds with boundary and finite-time extinction, in preparation.

[6] T. Aubin, Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics, 1998.

[7] A. L. Besse, Einstein manifolds, Springer, Berlin, 1987.

[8] C. Böhm, M. Wang and W. Ziller, A variational approach for compact homogeneous Einstein manifolds, Geom. Funct. Anal. 14 (2004), 681–733.

[9] M. Bruveris, Notes on Riemannian geometry on manifolds of maps, Lecture Notes for the IMS Summer School on Mathematics of Shapes, July 4-15, 2016.

[10] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés Riemannienes, J. Funct. Anal. 57 (1984), 154–206.

[11] T. Cruz and F. Vitório, Prescribing the curvature of Riemannian manifolds with boundary, arXiv math/1810.01311, 2018.

[12] M. M. Discozi and M. A. Khuri, Compactness and non-compactness for the Yamabe problem on manifolds with boundary, J. Reine Angew. Math. 724 (2017), 145–201.

[13] D. Ebin, The manifold of Riemannian metrics, Global Analysis, Proc. Sympos. Pure Math. 15 (1968), 11–40.

[14] J. Escobar, Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimates, Comm. Pure Appl. Math. 43 (1990), 857–883.

[15] J. Escobar, The Yamabe problem on manifolds with boundary, J. Diff. Geom. 35 (1992), 21–84.

[16] D. Gilberg and N. S. Trudinger, Elliptic partial differential equations of second order 2nd edition, Springer-Verlag, New York, 1977.

[17] L. Hörmander, The analysis of linear partial differential operators III, Springer-Verlag Berlin Heidelberg New York Tokyo, 1985.

[18] M. A. Khuri, F. C. Marques and R. M. Schoen, A compactness theorem for the Yamabe problem, J. Diff. Geom. 81 (2009), 143–196.
[19] S. Kobayashi and K. Nomizu, Foundations of differential geometry I, Wiley Classics Library, 1996.

[20] N. Koiso, Non-deformability of Einstein metrics, Osaka J. Math 15 (1978), 419–433.

[21] N. Koiso, A decomposition of the space \( M \) of Riemannian metrics on a manifold, Osaka J. Math. 16 (1979), 423–429.

[22] S. Lang, Introduction to differentiable manifolds, Interscience, New York, 1968.

[23] S. B. Myers and N. E. Steenrod, The group of isometries of a Riemannian manifold, Ann. of Math. 40 (1939), 400–416.

[24] L. I. Nicolaescu, Lectures on the geometry of manifolds, World Scientific Pub. Co. Inc., 1996.

[25] H. Omori, On the group of diffeomorphisms on a compact manifold, Global Analysis, Proc. Sympos. Pure Math. 15 (1968), 167–183.

[26] H. Omori, On Banach-Lie groups acting on finite dimensional manifolds, Tohoku Math. J. 30 (1978), 223–250.

[27] H. Omori, Infinite-dimensional Lie groups, Trans. Math. Monographs, 158, Amer. Math. Soc., 2017.

[28] R. S. Palais, Foundations of global non-linear analysis, Benjamin, New York, 1968.

[29] R. S. Palais, On the differentiability of isometries, Proc. Amer. Math. Soc. 8 (1957), 805–807.

[30] R. M. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Topics in calculus of variations (Montecatini Terme, 1987), Lecture Notes in Math. 1365, Springer-Verlag, (1989), 120–154.

E-mail address: a19.fg4w@g.chuo-u.ac.jp
Department Of Mathematics, Chuo University, Tokyo 112-8551, JAPAN