The Power of Cut-Based Parameters for Computing Edge Disjoint Paths

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Abstract

This paper revisits the classical Edge Disjoint Paths (EDP) problem, where one is given an undirected graph \( G \) and a set of terminal pairs \( P \) and asks whether \( G \) contains a set of pairwise edge-disjoint paths connecting every terminal pair in \( P \). Our aim is to identify structural properties (parameters) of graphs which allow the efficient solution of EDP without restricting the placement of terminals in \( P \) in any way. In this setting, EDP is known to remain NP-hard even on extremely restricted graph classes, such as graphs with a vertex cover of size 3.

We present three results which use edge-separator based parameters to chart new islands of tractability in the complexity landscape of EDP. Our first and main result utilizes the fairly recent structural parameter treecut width (a parameter with fundamental ties to graph immersions and graph cuts): we obtain a polynomial-time algorithm for EDP on every graph class of bounded treecutwidth. Our second result shows that EDP parameterized by treecut width is unlikely to be fixed-parameter tractable. Our final, third result is a polynomial kernel for EDP parameterized by the size of a minimum feedback edge set in the graph.

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1 Introduction

EDGE DISJOINT PATHS (EDP) is a fundamental routing graph problem: we are given a graph \( G \) and a set \( P \) containing pairs of vertices (terminals), and are asked to decide whether there is a set of \( |P| \) pairwise edge disjoint paths in \( G \) connecting each pair in \( P \). Similarly to its counterpart, the VERTEX DISJOINT PATHS (VDP) problem, EDP has been at the center of numerous results in structural graph theory, approximation algorithms, and parameterized algorithms [2,8,9,15,17,19,21,22,26].

Both EDP and VDP are \( \text{NP} \)-complete in general [16], and a significant amount of research has focused on identifying structural properties which make these problems tractable. For instance, Robertson and Seymour’s seminal work in the Graph Minors project [22] provides an \( O(n^3) \) time algorithm for both problems for every fixed value of \( |P| \). Such results are often viewed through the more refined lens of the parameterized complexity paradigm [5,7]: there, each problem is associated with a numerical parameter \( k \) (capturing some structural property of the instance), and the goal is to obtain algorithms which are efficient when the parameter is small. Ideally, the aim is then to obtain so-called fixed-parameter algorithms.
for the problem, i.e., algorithms which run in time $f(k) \cdot n^{o(1)}$ where $f$ is a computable function and $n$ the input size; the aforementioned result of Robertson and Seymour is hence an example of a fixed-parameter algorithm where $k = |P|$, and we say that the problem is FPT (w.r.t. this particular parameterization). In cases where fixed-parameter algorithms are unlikely to exist, one can instead aim for so-called XP algorithms, i.e., algorithms which run in polynomial time for every fixed value of $k$.

Naturally, one prominent question that arises is whether we can use the structure of the input graph itself (captured via a structural parameter) to solve EDP and VDP. Here, we find a stark contrast in the difficulty between these two, otherwise closely related, problems. Indeed, while VDP is known to be FPT with respect to the well-established structural parameter treewidth [24], EDP is NP-hard even on graphs of treewidth 3 [9]. What’s worse, the same reduction shows that EDP remains NP-hard even on graphs with a vertex cover of size 3 [9], which rules out fixed-parameter and XP algorithms for the vast majority of studied graph parameters (including, e.g., treedepth and the size of a minimum feedback vertex set).

We note that previous research on the problem has found ways of circumventing these negative results by imposing additional restrictions. Zhou, Tamura and Nishizeki [26] introduced the notion of an augmented graph, which contains information about how terminal pairs need to be connected, and used the treewidth of this graph to solve EDP. Recent work [13] has also observed that EDP admits a fixed-parameter algorithm when parameterized by treewidth and the maximum degree of the graph.

Our Contribution. The aim of this paper is to provide new algorithms and matching lower bounds for solving the Edge Disjoint Paths problem without imposing any restrictions on the number and placement of terminals. In other words, our aim is to be able to identify structural properties of the graph which guarantee tractability of the problem without knowing any information about the placement of terminals. The only positive result known so far in this setting requires us to restrict the degree of the input graph; however, in the bounded-degree setting there is a simple treewidth-preserving reduction from EDP to VDP (see Proposition 1), and so the problem only becomes truly interesting when the input graphs can contain vertices of higher degree.

Our main result is an XP algorithm for EDP when parameterized by the structural parameter treecut width [20,25]. Treecut width is inherently tied to the theory of graph immersions; in particular, it has a similar relationship to graph immersions and cuts as treewidth has to graph minors and separators. Since its introduction, treecut width has been successfully used to obtain fixed-parameter algorithms for problems which are unlikely to be FPT w.r.t. treewidth [11,12]; however, this is the first time that it has been used to obtain an algorithm for a problem that is NP-hard on graphs of bounded treewidth.

One “feature” of algorithmically exploiting treecut width is that it requires the solution of a non-trivial dynamic programming step. In previous works, this was carried out mostly by direct translations into Integer Linear Programming instances with few integer variables [11] or by using network flows [12]. In the case of EDP, the dynamic programming step requires us to solve an instance of EDP with a vertex cover of size $k$ where every vertex outside of the vertex cover has a degree of 2; we call this problem SIMPLE EDP and solve it in the dedicated Section 3. It is worth noting that there is only a very small gap between SIMPLE EDP (for which we provide an XP algorithm) and graphs with a vertex cover of size 3 (where EDP is known to be NP-hard).

In view of our main result, it is natural to ask whether the algorithm can be improved to a fixed-parameter one. After all, given the parallels between EDP parameterized by treecut width (an edge-separator based parameter) and VDP parameterized by treewidth
We use standard terminology for graph theory, see for instance \cite{6}. Given a graph parameterized by the size of a minimum feedback edge set. Finally, in Section 5 we obtain a polynomial kernel for our lower-bound result (Section 3). Section 4 then contains our algorithm for a stronger result: as our final contribution, we obtain a so-called linear kernel \cite{5,7} for EDP parameterized by the size of a minimum feedback edge set.

Having ruled out fixed-parameter algorithms for EDP parameterized by treecut width and in view of previous lower-bound results, one may ask whether it is even possible to obtain such an algorithm for any reasonable parameterization. We answer this question positively by using the size of a minimum feedback edge set as a parameter. In fact, we show an even stronger result: as our final contribution, we obtain a so-called linear kernel \cite{5,7} for EDP parameterized by the size of a minimum feedback edge set.

Organization of the Paper. After introducing the required preliminaries in Section 2, we proceed to introducing SIMPLE EDP, solving it via an XP algorithm and establishing our lower-bound result (Section 3). Section 4 then contains our algorithm for EDP parameterized by treecut width. Finally, in Section 5 we obtain a polynomial kernel for EDP parameterized by the size of a minimum feedback edge set.

2 Preliminaries

We use standard terminology for graph theory, see for instance \cite{6}. Given a graph $G$, we let $V(G)$ denote its vertex set and $E(G)$ its edge set. The (open) neighborhood of a vertex $x \in V(G)$ is the set $\{y \in V(G) : xy \in E(G)\}$ and is denoted by $N_G(x)$. For a vertex subset $X$, the neighborhood of $X$ is defined as $\bigcup_{x \in X} N_G(x) \setminus X$ and denoted by $N_G(X)$; we drop the subscript if the graph is clear from the context. Contracting an edge \((a, b)\) is the operation of replacing vertices $a, b$ by a new vertex whose neighborhood is $(N(a) \cup N(b)) \setminus \{a, b\}$. For a vertex set $A$ (or edge set $B$), we use $G - A$ ($G - B$) to denote the graph obtained from $G$ by deleting all vertices in $A$ (edges in $B$), and we use $G[A]$ to denote the subgraph induced on $A$, i.e., $G - (V(G) \setminus A)$.

A forest is a graph without cycles, and an edge set $X$ is a feedback edge set if $G - X$ is a forest. The feedback edge set number of a graph $G$, denoted by $\text{fes}(G)$, is the smallest integer $k$ such that $G$ has a feedback edge set of size $k$. We use $[i]$ to denote the set $\{0, 1, \ldots, i\}$.

2.1 Parameterized Complexity

A parameterized problem $\mathcal{P}$ is a subset of $\Sigma^* \times \mathbb{N}$ for some finite alphabet $\Sigma$. Let $L \subseteq \Sigma^*$ be a classical decision problem for a finite alphabet, and let $p$ be a non-negative integer-valued function defined on $\Sigma^*$. Then $L$ parameterized by $p$ denotes the parameterized problem \(\{(x, p(x)) \mid x \in L\}\) where $x \in \Sigma^*$. For a problem instance $(x, k) \in \Sigma^* \times \mathbb{N}$ we call $x$ the main part and $k$ the parameter. A parameterized problem $\mathcal{P}$ is fixed-parameter tractable (FPT in short) if a given instance $(x, k)$ can be solved in time $O(f(k) \cdot p(|x|))$ where $f$ is an arbitrary computable function of $k$ and $p$ is a polynomial function; we call algorithms running in this time fixed-parameter algorithms.

Parameterized complexity classes are defined with respect to \textit{fpt-reducibility}. A parameterized problem $\mathcal{P}$ is \textit{fpt-reducible} to $Q$ if in time $f(k) \cdot |x|^{O(1)}$, one can transform an instance $(x, k)$ of $\mathcal{P}$ into an instance $(x', k')$ of $Q$ such that $(x, k) \in \mathcal{P}$ if and only if $(x', k') \in Q$, and $k' \leq g(k)$, where $f$ and $g$ are computable functions depending only on $k$. Owing to
the definition, if \( \mathcal{P} \) fpt-reduces to \( \mathcal{Q} \) and \( \mathcal{Q} \) is fixed-parameter tractable then \( \mathcal{P} \) is fixed-parameter tractable as well. Central to parameterized complexity is the following hierarchy of complexity classes, defined by the closure of canonical problems under fpt-reductions:

\[
\text{FPT} \subseteq \text{W[1]} \subseteq \text{W[2]} \subseteq \cdots \subseteq \text{XP}.
\]

All inclusions are believed to be strict. In particular, \( \text{FPT} \neq \text{W[1]} \) under the Exponential Time Hypothesis.

A major goal in parameterized complexity is to distinguish between parameterized problems which are in \( \text{FPT} \) and those which are \( \text{W[1]} \)-hard, i.e., those to which every problem in \( \text{W[1]} \) is fpt-reducible. There are many problems shown to be complete for \( \text{W[1]} \), or equivalently \( \text{W[1]} \)-complete, including the \text{Multi-Colored Clique (MCC)} problem \([7]\).

We refer the reader to the respective monographs \([4,7,10]\) for an in-depth introduction to parameterized complexity.

### 2.2 Edge Disjoint Path Problem

Throughout the paper we consider the following problem.

**Edge Disjoint Paths (EDP)**

| Input:       | A graph \( G \) and a set \( P \) of terminal pairs, i.e., a set of subsets of \( V(G) \) of size two. |
|--------------|--------------------------------------------------------------------------------------------------|
| Question:    | Is there a set of pairwise edge disjoint paths connecting every set of terminal pairs in \( P \)? |

A vertex which occurs in a terminal pair is called a *terminal*, and a set of pairwise edge disjoint paths connecting every set of terminal pairs in \( P \) is called a *solution*. Without loss of generality, we assume that \( G \) is connected. The \text{Vertex Disjoint Paths (VDP)} problem is defined analogously as EDP, with the sole distinction being that the paths must be vertex-disjoint.

The following proposition establishes a link between EDP and VDP on graphs of bounded degree. Since we will not use the notion of *treewidth* \([23]\) anywhere else in the paper, we refer to the standard textbooks \([4,7]\) for its definition.

▶ **Proposition 1.** There exists a linear-time reduction from EDP to VDP with the following property: if the input graph has treewidth \( k \) and maximum degree \( d \), then the output graph has treewidth at most \( k \cdot d + 1 \).

**Proof.** Let \( (G, P) \) be an instance of EDP where \( G \) has treewidth \( k \) and maximum degree \( d \); let \( V = V(G) \) and \( E = E(G) \). Observe that if any vertex \( v \in V \) occurs in \( P \) more than \( d \) many times, then \( (G, P) \) must be a NO-instance (we assume that \( P \) does not contain tuples in the form \((a, a)\) for any \( a \)).

Consider the graph \( G' \) obtained in the following two-step procedure. First, we subdivide each edge in \( G \) (i.e., we replace that edge with a vertex of degree 2 that is adjacent to both endpoints of the original edge); let \( V' \) be the set of vertices created by such subdivisions. Second, for each vertex \( v = v_1 \in V \) of the original graph \( G \), we create \( d - 1 \) copies \( v_2, \ldots, v_d \) of that vertex and set their neighborhood to match that of \( v_1 \). This construction gives rise to a natural mapping \( \alpha \) from \( G \) to \( G' \) which maps each \( v \in V \) to the set \( v_1, \ldots, v_d \) and each \( e \in E \) to the vertex created by subdividing \( e \). Next, we iteratively process \( P \) as follows: for each \( \{v, w\} \in P \), we add a tuple \( \{v', w'\} \) into the set \( P' \) such that \( v' \in \alpha(v), w' \in \alpha(w) \) and
neither \( v' \) nor \( w' \) occurs in any other pair in \( P' \) (the last condition can be ensured because each vertex in \( v \) has \( d \) copies in \( G' \) but never occurs more than \( d \) times in \( P' \)).

It is now easy to verify that \((G, P)\) is a \textbf{YES}-instance of EDP if and only if \((G', P')\) is a \textbf{YES}-instance of VDP: indeed, such solutions can be converted to each other by applying \( \alpha \) on each path, whereas for the forward direction we simply need to make sure that each path that passes through a vertex \( v \in V \) uses a new vertex from \( \alpha(v) \). Finally, one can convert any tree-decomposition \((T, \mathcal{X})\) of \( G \) of width \( k \) into a tree-decomposition of \( G' \) of width \( k \cdot d + 1 \) by (1) replacing each vertex \( v \) by \( \alpha(v) \) in \( T \), and then (2) by choosing, for each edge \( e = ab \in E \), a bag \( X \supseteq \{a, b\} \), creating a bag \( X' = X \cup \{\alpha(e)\} \), and attaching \( X' \) to \( X \) as a leaf.

We remark that Proposition 3 in combination with the known fixed-parameter algorithm for VDP parameterized by treewidth [24] provides an alternative proof for the fixed-parameter tractability of EDP parameterized by degree and treewidth [13].

### 2.3 Treecut Width

The notion of treecut decompositions was introduced by Wollan [25], see also [20]. A family of subsets \( X_1, \ldots, X_k \) of \( X \) is a near-partition of \( X \) if they are pairwise disjoint and \( \bigcup_{i=1}^{k} X_i = X \), allowing the possibility of \( X_i = \emptyset \).

► **Definition 2.** A treecut decomposition of \( G \) is a pair \((T, \mathcal{X})\) which consists of a rooted tree \( T \) and a near-partition \( \mathcal{X} = \{X_t \subseteq V(G) : t \in V(T)\} \) of \( V(G) \). A set in the family \( \mathcal{X} \) is called a bag of the treecut decomposition.

For any node \( t \) of \( T \) other than the root \( r \), let \( e(t) = ut \) be the unique edge incident to \( t \) on the path to \( r \). Let \( T_u \) and \( T_t \) be the two connected components in \( T - e(t) \) which contain \( u \) and \( t \), respectively. Note that \( (\bigcup_{q \in T_u} X_q, \bigcup_{q \in T_t} X_q) \) is a near-partition of \( V(G) \), and we use \( E_t \) to denote the set of edges with one endpoint in each part. We define the adhesion of \( t \) (\( \text{adh}(t) \)) as \( |E_t| \); if \( t \) is the root, we set \( \text{adh}(t) = 0 \) and \( E(t) = \emptyset \).

The torso of a treecut decomposition \((T, \mathcal{X})\) at a node \( t \), written as \( H_t \), is the graph obtained from \( G \) as follows. If \( T \) consists of a single node \( t \), then the torso of \((T, \mathcal{X}) at \) \( t \) is \( G \). Otherwise let \( T_1, \ldots, T_t \) be the connected components of \( T - t \). For each \( i = 1, \ldots, \ell \), the vertex set \( Z_i \subseteq V(G) \) is defined as the set \( \bigcup_{b \in V(T_i)} X_b \). The torso \( H_t \) at \( t \) is obtained from \( G \) by consolidating each vertex set \( Z_i \) into a single vertex \( z_i \) (this is also called shrinking in the literature). Here, the operation of consolidating a vertex set \( Z \) into \( z \) is to substitute \( Z \) by \( z \) in \( G \), and for each edge \( e \) between \( Z \) and \( v \in V(G) \setminus Z \), adding an edge \( zv \) in the new graph. We note that this may create parallel edges.

The operation of suppressing (also called dissolving in the literature) a vertex \( v \) of degree at most 2 consists of deleting \( v \), and when the degree is two, adding an edge between the neighbors of \( v \). Given a connected graph \( G \) and \( X \subseteq V(G) \), let the 3-center of \((G, X)\) be the unique graph obtained from \( G \) by exhaustively suppressing vertices in \( V(G) \setminus X \) of degree at most two. Finally, for a node \( t \) of \( T \), we denote by \( H_t \) the 3-center of \((H_t, X_t)\), where \( H_t \) is the torso of \((T, \mathcal{X}) \) at \( t \). Let the torso-size \( \text{tor}(t) \) denote \( |H_t| \).

► **Definition 3.** The width of a treecut decomposition \((T, \mathcal{X})\) of \( G \) is \( \max_{e \in V(T)} \{\text{adh}(t), \text{tor}(t)\} \).

The treecut width of \( G \), or \( \text{tcw}(G) \) in short, is the minimum width of \((T, \mathcal{X})\) over all treecut decompositions \((T, \mathcal{X})\) of \( G \).

We conclude this subsection with some notation related to treecut decompositions. Given a tree node \( t \), let \( T_t \) be the subtree of \( T \) rooted at \( t \). Let \( Y_t = \bigcup_{b \in V(T_t)} X_b \), and let \( G_t \)
The Power of Cut-Based Parameters for Computing Edge Disjoint Paths

denote the induced subgraph \( G[Y_t] \). A node \( t \neq r \) in a rooted treecut decomposition is thin if \( \text{adh}(t) \leq 2 \) and bold otherwise.

![Figure 1](image)

Figure 1 A graph \( G \) and a width-3 treecut decomposition of \( G \), including the torso-size (left value) and adhesion (right value) of each node.

While it is not known how to compute optimal treecut decompositions efficiently, there exists a fixed-parameter 2-approximation algorithm which fully suffices for our purposes.

▶ Theorem 4 ([18]). There exists an algorithm that takes as input an \( n \)-vertex graph \( G \) and integer \( k \), runs in time \( 2^{O(k^2 \log k)}n^2 \), and either outputs a treecut decomposition of \( G \) of width at most \( 2k \) or correctly reports that \( \text{tcw}(G) > k \).

A treecut decomposition \((T,X)\) is nice if it satisfies the following condition for every thin node \( t \in V(T) \): \( N(Y_t) \cap (\bigcup_b \text{if } b \text{ is a sibling of } t \text{, } Y_b) = \emptyset \). The intuition behind nice treecut decompositions is that we restrict the neighborhood of thin nodes in a way which facilitates dynamic programming.

▶ Lemma 5 ([11]). There exists a cubic-time algorithm which transforms any rooted treecut decomposition \((T,X)\) of \( G \) into a nice treecut decomposition of the same graph, without increasing its width or number of nodes.

For a node \( t \), we let \( B_t = \{ b \text{ is a child of } t \mid |N(Y_b)| \leq 2 \land N(Y_b) \subseteq X_t \} \) denote the set of thin children of \( t \) whose neighborhood is a subset of \( X_t \), and we let \( A_t = \{ a \text{ is a child of } t \mid a \notin B_t \} \) be the set of all other children of \( t \). The following property of nice treecut decompositions will be crucial for our algorithm.

▶ Lemma 6 ([11]). Let \( t \) be a node in a nice treecut decomposition of width \( k \). Then \( |A_t| \leq 2k + 1 \).

We refer to previous work [11,18,20] for a comparison of treecut width to other parameters. Without loss of generality, we shall assume that \( X_r = \emptyset \).

3 The Simple Edge Disjoint Paths Problem

Before we start working towards our algorithm for solving EDP parameterized by treecut width, we will first deal with a simpler (but crucial) setting for the problem. We call this the Simple Edge Disjoint Paths problem (Simple EDP) and define it below.

| Simple EDP |
|----------------|
| Input: An EDP instance \((G,P)\) such that \( V(G) = A \cup B \) where \( B \) is an independent set containing vertices of degree at most 2. |
| Parameter: \( k = |A| \) |
| Question: Is \((G,P)\) a YES-instance of EDP? |

Notice that every instance of SIMPLE EDP has treecut width at most \( k \), and so it forms a special case of EDP parameterized by treecut width. Indeed, the treecut decomposition
where $T$ is a star, the center bag contains $A$, and each leaf bag contains a vertex from $B$ (except for the root $r$, where $X_r = \emptyset$), has tree cut width at most $k$. This contrasts to the setting where $G$ has a vertex cover of size 3 and all vertices outside the vertex cover have degree 3; the tree cut width of such graphs is not bounded by any constant, and EDP is known to be NP-complete in this setting [3].

The main reason we introduce and focus on Simple EDP is that it captures the combinatorial problem that needs to be solved in the dynamic step of the algorithm for EDP parameterized by tree cut width. Hence, our first task here will be to solve Simple EDP by an algorithm that can later be called as a subroutine.

\textbf{Lemma 7.} Simple EDP can be solved in time $O(|P|^{{k}+1}(k+1)!)$.

\textbf{Proof.} Let $(G, P)$ with partition $A$ and $B$ and $k = |A|$ be an instance of Simple EDP. Let the terminal graph of $G$, denoted by $G^T$, as the graph with vertex set $V$ and edge set $E$.

We will start by simplifying the instance using some simple observations. First we will show that we can remove all vertices in $B$ that are not contained in any terminal pair by adding multi-edges to $G[A]$. Namely, let $v$ be a vertex in $B$ that does not appear in any terminal pair in $P$. If $v$ has no neighbors or at most one neighbor, then $v$ can simply be removed from $G$, and if $v$ has degree two, then we can remove $v$ and add an edge between its two neighbors in $A$. Hence in the following we will assume that all vertices in $B$ occur in at least one terminal pair and that $G[A]$ can contain multi-edges.

The following two observations will be crucial for our algorithm:

1. Consider a path $P$ connecting a terminal pair $v \in P$ in a solution. Because $B$ is an independent set and every vertex in $B$ has degree at most two and is contained in at least one terminal pair in $P$, we obtain that all inner vertices of $P$ are from $A$. Hence, $P$ contains at most $k+2$ vertices and all inner vertices of $P$ are contained in $A$. It follows that $P$ is completely characterized by the sequence of vertices it uses in $A$. Consequently, there are at most $\sum_{r=1}^{k} \binom{k}{r}! \leq (k+1)!$ different types of paths that need to be considered for the connection of any terminal pair.

2. $G^T[B]$ is a disjoint union of paths and cycles. This is because every vertex $v$ of $G$ can be contained in at most $|N_G(v)|$ terminal pairs in $P$ (otherwise we immediately reject) and all vertices in $B$ have degree at most two.

Let $u$ and $v$ be two distinct vertices in $A$. Because $|A| \leq k$, we can enumerate all possible paths between $u$ and $v$ in $G[A]$ in time $O((k+1)!))$. We will represent each such path $H$ as a binary vector $E_H$, whose entries are indexed by all sets of two distinct vertices in $A$, such that $E_H[e] = 1$ if $H$ uses the edge $e$ and $E_H[e] = 0$ otherwise. Moreover, we will denote by $E_{u,v}$ the set $\{ E_H \mid H$ is a path between $u$ and $v$ in $G[A] \}$; intuitively, $E_{u,v}$ captures all possible sets of edges that need to be used in order to connect $u$ to $v$.

Let $S$ be a solution for $(G, P)$. The algorithm represents every solution $S$ for $(G, P)$ as a solution vector $E_S$ of natural numbers whose entries are indexed by all sets $\{ u, v \}$ of two distinct vertices in $A$. More specifically, for two distinct vertices $u$ and $v$ in $A$, $E_S[\{u, v\}]$ is equal to the number of edges between $u$ and $v$ used by the paths in $S$. The algorithm uses dynamic programming to compute the set $L$ of all solution vectors; clearly, $L \neq \emptyset$ if and only if $(G, P)$ is a YES-instance. We compute $L$ in two main steps:

(S1) the algorithm computes the set $L_A$ of all solution vectors for the sub-instance $(G[A], P')$ of $(G, P)$, where $P'$ is the subset of $P$ containing all terminal pairs $\{p, q\}$ with $p, q \in A$.

(S2) the algorithm computes the set of all solution vectors for the sub-instance $(G, P \setminus P')$.

Note that every terminal pair $p$ in $P \setminus P'$ is either completely contained in $B$, in which
case it forms an edge of a path or acycle in $G^T[B]$, or $p$ has one vertex in $A$ and the other vertex in $B$, which is the endpoint of a path in $G^T[B]$. The algorithm now computes the set of all solution vectors for the sub-instance $(G, P \setminus P')$ in two steps:

(S2A) For every cycle $C$ in $G^T[B]$, the algorithm computes the set $L_C$ of all solution vectors for the sub-instance $(G[A \cup V(C)], P_C)$, where $P_C$ is the subset of $P$ containing all terminal pairs $\{p, q\}$ such that $p, q \in C$.

(S2B) For every path $H$ in $G^T[B]$, the algorithm computes the set $L_H$ of all solution vectors for the sub-instance $(G[A \cup V(H)], P_H)$, where $P_H$ is the subset of $P$ containing all terminal pairs $\{p, q\}$ with $\{p, q\} \cap V(H) \neq \emptyset$.

In the end, the set of all hypothetical solution vectors $L'$ for $(G, P)$ is obtained as $L_A \oplus ((\oplus_C$ is a cycle of $G^T[B]L_C) \oplus (\oplus_H$ is a path of $G^T[B]L_H)$), where $\oplus \oplus'$ for two sets $\mathcal{P}$ and $\mathcal{P}'$ of solution vectors is equal to $\{ R + R' \mid R \in \mathcal{P} \land R' \in \mathcal{P}' \}$. Each vector in $L'$ described one possible set of multi-edges in $G[A]$ that can be used to connect all terminal pairs in $P$. In order to compute $L$, one simply needs to remove all vectors from $L'$ which require more multi-edges than are available in $G[A]$; in particular, to obtain $L$ we delete each $S$ from $L'$ such that there exist $u, v \in A$ where $E_S[\{u, v\}]$ exceeds the number of multi-edges between $u$ and $v$ in $G$. The algorithm then returns YES if $L$ is non-empty and otherwise the algorithm returns NO. Note that, as is usually the case with these types of dynamic programming algorithms, the algorithm can also be easily modified to find a solution for $(G, P)$, without increasing its running time.

The set $L_A$ described in step (S1) is computed as follows. Given an arbitrary but fixed ordering $p_1, \ldots, p_{|P'|}$ of the terminal pairs in $P'$, let $P_i$ be the set $\{ p_j \mid 1 \leq j \leq i \}$, for every $i$ with $1 \leq i \leq |P'|$. The algorithm now uses dynamic programming to compute the sets $S_1, \ldots, S_{|P'|}$, where $S_i$ contains the set of all hypothetical solution vectors for the instance $(G[A], P_i)$ as follows. The algorithm starts by setting $T_1$ to be the set $E_{p_1}$. Then for every $i$ with $1 < i \leq |P'|$, the algorithm computes $T_i$ from $T_{i-1}$ as the set $\{ E + E' \mid E \in T_{i-1} \land E' \in E_{p_{i}} \}$. The tables $T_2, \ldots, T_n$ are iteratively computed starting with $T_2$ as follows. For every $n \in N_{G(v_1)}$, and $n_2 \in N_{G(v_2)}$, the table $T_2[v_1,n_2]$ is equal to $E_{n_1,n_2}$. Moreover, for every $i$ with $3 < i < n$, the table $T_i$ is obtained from the table $T_{i-1}$ as follows. For every $n_1 \in N_{G(v_1)}$ and $n_i \in N_{G(v_i)}$, the table $T_i[n_1,n_i]$ is equal to the union of $\{ E + E' \mid E \in T_{i-1}[n_1,n_{i-1}] \land E' \in E_{n_{i-1},n_i} \}$ and $\{ E + E' \mid E \in T_{i-1}[n_1,n_{i-1}] \land E' \in E_{n_i} \}$, where $\{n_1\} = N_{G(v_1)} \setminus \{n_i\}$, $\{n_i\} = N_{G(v_i)} \setminus \{n_1\}$, and $\{n_{i-1},n_{i-1}\} = N_{G(v_{i-1})}$. Finally, the set of all hypothetical solution vectors for the instance $(G[A \cup C], P_C)$ is equal to $\{ E \in T[n_1,n_n] \mid n_1 \in N_{G(v_1)} \land n_n \in N_{G(v_n)} \}$, where $T[n_1,n_n] = \{ E + E' \mid E \in T[n_1,n_n] \land E' \in E_{n_i} \}$ for every $n_1 \in N_{G(v_1)}$, $n_n \in N_{G(v_n)}$, $\{n_1\} = N_{G(v_1)} \setminus \{n_n\}$, and $\{n_n\} = N_{G(v_n)} \setminus \{n_1\}$.

The set $L_H$ described in step (S2B) for a path $H = (v_1, \ldots, v_n)$ of $G^T[B]$ is computed as follows. As for the case of a cycle the algorithm starts by computing the table $T_n$, for which every $n_n \in N_{G(v_1)}$ and $n_n \in N_{G(v_n)}$ contains all solution vectors for the instance $(G[A \cup V(P)] - E^{1,n}, E(P))$, where $E^{1,n}$ contains the edge from $v_1$ to $N_{G(v_1)} \setminus \{n_1\}$ if $N_{G(v_1)} \setminus \{n_1\} \neq \emptyset$ and the edge from $v_n$ to $N_{G(v_n)} \setminus \{n_n\}$ if $N_{G(v_n)} \setminus \{n_n\} \neq \emptyset$. Let $T$ be equal to $T_n$, then the algorithm proceeds as follows. If $P'$ contains a terminal pair $\{v_1,a\}$ with $a \in A$, then for every $n_1 \in N_{G(v_1)}$ and every $n_n \in N_{G(v_n)}$, the algorithm
updates $T[n_1, n_a]$ to be the set $\{E + E' \mid E \in T[n_1, n_a] \land E' \in E_{n_1, a}\}$, where $\{n_1\} = N_G(v_1) \setminus \{n_1\}$. Similarly, if $P'$ contains a terminal pair $\{v_{n_1}, a\}$ with $a \in A$, then for every $n_1 \in N_G(v_1)$ and every $n_a \in N_G(v_a)$, the algorithm updates $T[n_1, n_a]$ to be the set $\{E + E' \mid E \in T[n_1, n_a] \land E' \in E_{n_1, a}\}$, where $\{n_a\} = N_G(v_a) \setminus \{n_a\}$. Finally, the set of all solution vectors $L_H$ for the instance $(G[A \cup V(H)], P')$ is obtained as the set $\{E \in T[n_1, n_a] \mid n_1 \in N_G(v_1) \land n_a \in N_G(v_a)\}$.

This completes the description of the algorithm. To verify correctness, one can observe that each solution vector computed by the algorithm can be traced back to a specific choice of edges (a path) that connects each terminal pair in $P$, and since there are sufficient multi-edges in $G[A]$ to accommodate all the resulting paths, this guarantees the existence of a solution. On the other hand, if a solution exists then it surely has a solution vector, and moreover the algorithm will discover this solution vector by choosing, for each $\{a, b\} \in P$, the entry in $E_{H}$ which corresponds to the $a-b$ path used in the solution.

Finally, we argue the running time bound. Note first that every set of solution vectors computed at any point in the algorithm contains at most $|P|^{\binom{2}{2}}$ elements. Moreover, as argued in (O1) the set $E_{a,v}$ for two distinct vertices $u$ and $v$ in $A$ can be computed in time $O((k + 1)!)$ and contains at most $(k + 1)!$ elements. From this it follows that the time required to compute $L_A$ in (S1) is at most $O(|P|^{\binom{2}{2}}(k + 1)!|P'|)$. Similarly, the time required to compute $L_C$ for a cycle $C$ in $G^T[B]$ in step (S2A) is at most $O(|P|^{\binom{2}{2}}(k + 1)!|P_C|)$ and the time required to compute $L_H$ for a path $H$ in $G^T[B]$ in step (S2B) is at most $O(|P|^{\binom{2}{2}}(k + 1)!|P_H|)$. Hence the time required to compute $L_A$ together with all the sets $L_C$ and $L_H$ for every cycle $C$ and path $H$ of $G^T[B]$ is at most $O(|P|^{\binom{2}{2}}(k + 1)!|P|)$. Finally, combining these sets into $L'$ does not incur an additional run-time overhead since $L'$ can be computed iteratively as part of the computation of the sets $L_A$, $L_C$, and $L_H$.

Notice that Lemma 7 does not provide a fixed-parameter algorithm for SIMPLE EDP. Our second task for this section will be to rule out the existence of such algorithms (hence also ruling out the fixed-parameter tractability of EDP parameterized by treecut width).

Before we proceed, we would like note that this outcome was highly surprising for the authors. Indeed, not only does this “break” the parallel between $\{\text{VDP, treewidth}\}$ and $\{\text{EDP, treecut width}\}$, but inspecting the dynamic programming algorithm for EDP parameterized by treecut width presented in Section 4 reveals that solving SIMPLE EDP is the only step which requires more than “FPT-time”. In particular, if SIMPLE EDP were FPT, then EDP parameterized by treecut width would also be FPT. This situation contrasts the vast majority of dynamic programming algorithms for parameters such as treewidth and clique-width where the complexity bottleneck is usually tied to the size of the records used and not to the computation of the dynamic step.

Our lower-bound result is based on a parameterized reduction from the following problem:

**MULTIDIMENSIONAL SUBSET SUM (MSS)**

**Input:** An integer $k$, a set $S = \{s_1, \ldots, s_n\}$ of item-vectors with $s_i \in \mathbb{N}^k$ for every $i$ with $1 \leq i \leq n$, a target vector $t \in \mathbb{N}^k$, and an integer $\ell$.

**Parameter:** $k$

**Question:** Is there a subset $S' \subseteq S$ with $|S'| \geq \ell$ such that $\sum_{s \in S'} s \leq t$?

The $\text{W}[1]$-hardness of MSS can be obtained by a trivial reduction from the following problem, which was recently shown to be $\text{W}[1]$-hard by Ganian, Ordyniak and Ramanujan [14].
Indeed, given an instance \((k, S, t, \ell)\) of MRSS, it is straightforward to verify that \((k, S, (\sum_{s \in S} s) - t, |S| - \ell)\) is an equivalent instance of MSS; since the reduction preserves the parameter, this shows that MSS is also \(W[1]\)-hard.

**Lemma 8.** \textbf{Simple EDP} is \(W[1]\)-hard.

**Proof.** We provide a parameterized reduction from MSS. Namely, given an instance \((k, S, t, \ell)\) of MSS, we will construct an equivalent instance \((G, P)\) with partition \(A\) and \(B\) and \(|A| = k + 3\) of Simple EDP. For convenience and w.l.o.g. we will assume that all entries of the vectors in \(S\) as well as all entries of the target vector \(t\) are divisible by two; furthermore, we will describe the constructed instance of Simple EDP with multi-edges between vertices in \(A\) (note that these can be replaced by degree-2 vertices in \(B\), similarly as in Lemma 7).

The graph \(G[A]\) has vertices \(a, b, d, d_1, \ldots, d_k\) and the following multi-edges:

\[ |S| - \ell \text{ edges between } a \text{ and } b, \]
\[ \text{for every } i \text{ with } 1 \leq i \leq k, \ell[i] \text{ edges between } d \text{ and } d_i. \]

Moreover, for every \(s \in S\) we construct a gadget \(G(s)\) consisting of:

- the vertices \(v^s, v^s_1, u^s_1, \ldots, v^s_k, u^s_k\) with \(\tilde{s} = \sum_{i=1}^{k} s[i]\),
- two edges \(\{v^s, a\} \) and \(\{v^s, d\}\),
- for every \(i\) with \(1 \leq i \leq \tilde{s}\), two edges \(\{v^s_i, b\}\) and \(\{u^s_i, b\}\),
- for every \(i\) with \(1 \leq i \leq \tilde{s}\) and \(i\) even, two edges \(\{v^s_i, d\}\) and \(\{u^s_i, d\}\),
- for every \(j\) with \(1 \leq j \leq k\) and every \(i\) with \(\sum_{l=1}^{j-1} s[l] < i \leq \sum_{l=1}^{j} s[l]\) and \(i\) odd, two edges \(\{v^s_i, d_j\}\) and \(\{u^s_i, d_j\}\),
- the terminal pair \(\{v^s, v^s_1\}\),
- for every \(i\) with \(1 \leq i \leq \tilde{s}\), a terminal pair \(\{v^s_i, u^s_i\}\),
- for every \(i\) with \(1 \leq i < \tilde{s}\), a terminal pair \(\{u^s_i, v^s_{i+1}\}\).

Then \(G\) consists of the graph \(G[A]\) together with the vertices and edges of the gadget \(G(s)\) for every \(s \in S\); note that \(B\) is the union of the vertices of the gadgets \(G(s)\) for every \(s \in S\). Moreover, \(P\) consists of all terminal pairs of the gadgets \(G(s)\) for every \(s \in S\). This completes the construction of the instance \((G, P)\); an illustration is provided in Figure 2. It remains to show that the instance \((k, S, t, \ell)\) of MSS has a solution if and only if so does the instance \((G, P)\) of EDP.

We start by showing that there are only two ways to connect all terminal pairs of the gadget \(G(s)\) for every \(s \in S\). Figure 2 illustrates the edges used by the two configurations.

**Claim 1.** Let \(S\) be a solution for \((G, P)\), and \(s \in S\). Then either:

\((C1)\) The terminal pair \(\{v^s, v^s_1\}\) is connected by the path \((v^s, a, b, v^s_1)\) and:

- for every \(i\) with \(1 \leq i < \tilde{s}\), the terminal pair \(\{u^s_i, v^s_{i+1}\}\) is connected by the path \((u^s_i, b, v^s_{i+1})\),
- for every \(i\) with \(1 \leq i \leq \tilde{s}\) and \(i\) even, the terminal pair \(\{v^s_i, u^s_i\}\) is connected by the path \((v^s_i, d, u^s_i)\), and
The terminal pair illustrates configuration (C1) and the right side illustrates configuration (C2) as defined in Claim 1; this concludes the argument for the second case.

(C2) The terminal pair \( \{v^a, v^b\} \) is connected by the path \((v^a, d_j, u^b)\), where \( j \) is such that \( \sum_{i=1}^{j-1} s_i < i \leq \sum_{i=1}^{j} s_i \).

Proof. Let \( S \) be a solution for \((G, P)\) and \( s \in G(s) \). Then \( S \) has to connect the terminal pair \( \{v^a, v^b\} \) either by the path \((v^a, a, b, v^b)\) or by the path \((v^a, d_j, v^b)\).

In the former case, the only way to connect the terminal pair \( \{v^a, v^b\} \) is the path \((v^a, d_j, u^b)\), where \( j \) is such that \( \sum_{i=1}^{j-1} s_i < 1 \leq \sum_{i=1}^{j} s_i \). But then the terminal pair \( \{u^1, u^2\} \) can only be connected by the path \((u^1, d_j, v^2)\) and in turn the terminal pair \( \{v^2, v^3\} \) can only be connected by the path \((v^2, d, v^3)\). Since this pattern continues in this manner, this concludes the argument for the first case.

In the latter case, the only way to connect the terminal pair \( \{v^a, v^b\} \) is the path \((v^a, b, u^b)\). But then the terminal pair \( \{u^1, u^2\} \) can only be connected by the path \((u^1, d_j, v^2)\), where \( j \) is such that \( \sum_{i=1}^{j-1} s_i < 1 \leq \sum_{i=1}^{j} s_i \), and in turn the terminal pair \( \{v^2, v^3\} \) can only be connected by the path \((v^2, b, v^3)\). Finally, the terminal pair \( \{v^3, v^4\} \) can then only be connected by the path \((v^3, d, v^4)\), where \( j \) is such that \( \sum_{i=1}^{j-1} s_i < 1 \leq \sum_{i=1}^{j} s_i \). Since this pattern continues in this manner, this concludes the argument for the second case.
Let $S$ be a solution for $(G, P)$ and $s \in S$. It follows from Claim 1 that if $S$ connects the terminal pairs of $G(s)$ according to (C1), then the only edge used from $G[A]$ is the edge $\{a, b\}$. On the other hand, if $S$ connects the terminal pairs in $G(s)$ according to (C2), then $S$ uses $s[i]$ edges between $d$ and $d_j$ for every $i$ with $1 \leq i \leq k$.

Towards showing the forward direction, let $S' \subseteq S$ be a solution for $(k, S, t, \ell)$. W.l.o.g., we can assume that $|S'| = \ell$. We claim that the set of edges disjoint paths $S$, which if $s \in S'$ connects all terminal pairs in $G(s)$ according to (C2) and if $s \in S \setminus S'$ connects all terminal pairs in $G(s)$ according to (C1) is a solution for $(G, P)$. This holds because there are $|S| - \ell$ edges between $a$ and $b$, which are sufficient for the elements in $S \setminus S'$ to be connected according to (C1). Moreover, because $\sum_{s \in S'} s \leq t$, the $t[i]$ edges between $d$ and $d_i$ for every $i$ with $1 \leq i \leq k$, suffices for the elements in $S'$ to be connected according to (C2).

For the reverse direction, let $S$ be a solution for $(G, P)$. We claim that the subset $S'$ of $S$ containing all $s \in S$ such that $S$ connects all terminal pairs in $G(s)$ according to C2 is a solution for $(k, S, t, \ell)$. Because there are at most $|S| - \ell$ edges between $a$ and $b$ in $G[A]$, we obtain that $|S'| \geq \ell$. Moreover, because there are at most $t[i]$ edges between $d$ and $d_i$ in $G[A]$, it follows that $\sum_{s \in S'} s \leq t$. Consequently, $S'$ is a solution for $(k, S, t, \ell)$.

4 An algorithm for EDP for graphs of bounded treecut width

The goal of this section is to provide an XP algorithm for EDP parameterized by treecut-width. The core of the algorithm is a dynamic programming procedure which runs on a nice treecut decomposition $(T, \mathcal{X})$ of the input graph $G$.

4.1 Overview

Our first aim is to define the data table the algorithm is going to dynamically compute for individual nodes of the treecut decomposition; to this end, we introduce two additional notions. For a node $t$, we say that $Y_t$ (or $G_t$) contains an unmatched terminal $s$ if $(s, t) \in P$, $s \in Y_t$ and $t \notin Y_t$; let $U_t$ be the multiset containing all unmatched terminals $Y_t$ (one entry in $U_t$ per tuple in $P$ which contains an unmatched terminal). For a subgraph $H$ of $G$, let $P_H \subseteq P$ denote the subset of terminal pairs whose both endpoints lie in $H$.

Let a record for node $t$ be a tuple $(\delta, I, F, L)$ where:

- $\delta$ is a partitioning of $E_t$ into internal ($I'$), leaving ($L'$), foreign ($F'$) and unused ($U'$);
- $I$ is a set of subsets of size 2, which forms a perfect matching between the edges in $I'$;
- $F$ is a set of subsets of size 2, which forms a perfect matching between the edges in $F'$;
- $L$ is a perfect matching between $U_t$ and the edges in $L'$.

Intuitively, a record captures all the information we need about one possible interaction between a solution to EDP and the edges in $E_t$. In particular, unmatched terminals need to cross between $Y_t$ and $G_t$ using an edge in $E_t$ and $L$ captures the first edge used by a path from an unmatched terminal in the solution, while $I$ and $F$ capture information about paths which intersect with $E_t$ but whose terminals both lie in $Y_t$ and $V(G_t) \setminus Y_t$, respectively. We formalize this intuition below through the notion of a valid record.

Definition 9. A record $\lambda = (\delta, I, F, L)$ is valid for $t$ if $(G^\lambda, P^\lambda)$ is a YES-instance of EDP, where $(G^\lambda, P^\lambda)$ is constructed from $(G_t, P_G_t)$ as follows:
1. For each \( \{\{a, b\}, \{c, d\}\} \in I \) where \( a, c \in Y_t \), add a new vertex into \( G_t \) and connect it to \( a \) and \( c \) by edges (note that if \( a = c \) then this simply creates a new leaf and hence this operation can be ignored).

2. For each \( \{s, \{a, b\}\} \in L \) where \( a \in Y_t \), add a new tuple \( \{s, t'\} \) into \( F_G \) and a new leaf \( t' \) into \( G_t \) adjacent to \( a \).

3. For each \( \{\{a, b\}, \{c, d\}\} \in F \) where \( a, c \in Y_t \), add two new leaves \( b', d' \) into \( G_t \), make them adjacent to \( a \) and \( c \) respectively, and add \( \{b', d'\} \) into \( F_G \).

We are now ready to define our data tables: for a node \( t \in V(T) \), let \( D(t) \) be the set of all valid records for \( t \). We now make two observations. First, for any node \( t \) in a nice treecut decomposition of width \( k \), it holds that there exist at most \( 4^k \cdot k! \) distinct records and hence \( |D(t)| \leq 4^k \cdot k! \); indeed, there are \( 4^k \) possible choices for \( \delta \) and for each such choice and each edge \( e \in E_t \) one has at most \( k \) options of what to match with \( e \). Second, if \( r \) is the root of \( T \), then either \( D(r) = \emptyset \) or \( D(r) = \{(\emptyset, \emptyset, \emptyset, \emptyset)\} \); furthermore, \( (G, P) \) is a **YES**-instance if and only if the latter holds. Hence it suffices to compute \( D(r) \) in order to solve EDP.

The next lemma shows that \( D(t) \) can be computed efficiently for all leaves of \( t \).

**Lemma 10.** There is an algorithm which takes as input \( (G, P) \), a width-\( k \) treecut decomposition \((T, \mathcal{X})\) of \( G \) and a leaf \( t \in V(T) \), runs in time \( k^{O(k^2)} \), and outputs \( D(t) \).

**Proof.** We proceed as follows. For each record \( \lambda \) for \( t \), we construct the instance \( (G^\lambda, P^\lambda) \) as per Definition 9 and check whether \( (G^\lambda, P^\lambda) \) is a **YES**-instance of EDP. Since \( V(G^\lambda) \leq 2k \), a simple brute-force algorithm will suffice here. For instance, one can enumerate all partitions of the at most \( 4k^2 \) edges in \( G^\lambda \), and for each such partition one can check whether this represents a set of edge disjoint paths which forms a solution to \( (G^\lambda, P^\lambda) \). If \( (G^\lambda, P^\lambda) \) is a **YES**-instance of EDP then we add \( \lambda \) into \( D(t) \), and otherwise we do not.

The number of partitions of a set of size \( 4k^2 \) is upper-bounded by \( k^{O(k^2)} \) \([1]\), and \( |D(t)| \leq 4^k \cdot k! \). Hence the runtime of the whole algorithm described above is dominated by \( k^{O(k^2)} \). \( \blacklozenge \)

At this point, all that is left to obtain a dynamic leaves-to-root algorithm which solves EDP is the dynamic step, i.e., computing the data table for a node \( t \in V(t) \) from the data tables of its children. Unfortunately, that is where all the difficulty of the problem lies, and our first step towards handling this task will be the introduction of two additional notions related to records. The first is correspondence, which allows us to associate each solution to \((G, P)\) with a specific record for \( t \); on an intuitive level, a solution corresponds to a particular record if that record precisely captures the “behavior” of that solution on \( E_t \). Correspondence will, among others, be used to later argue the correctness of our algorithm.

**Definition 11.** A solution \( S \) to \((G, P)\) corresponds to a record \( \lambda = (\delta, I, F, L) \) for \( t \) if the conditions 1.-4., stated below hold for every \( a \)-\( b \) path \( S \) in \( \mathcal{S} \) such that \( S \cap E_t \neq \emptyset \). We let \( s = |S \cap E_t| \) and we denote individual edges in \( S \cap E_t \) by \( e_1, e_2, \ldots, e_s \), ordered from the edge nearest to \( a \) along \( S \).

1. If \( a, b \not\in Y_t \), then for each odd \( i \in [s] \), \( F \) contains \((e_i, e_{i+1})\).
2. If \( a, b \in Y_t \), then for each odd \( i \in [s] \), \( I \) contains \((e_i, e_{i+1})\).
3. If \( \{a, b\} \cap Y_t = \{a\} \), then \( L \) contains \((a, e_1) \), and for each even \( i \in [s] \) \( F \) contains \((e_i, e_{i+1})\).
4. There are no elements in \( I, F, L \) other than those specified above.

Note that “restricting” the solution \( \mathcal{S} \) to the instance \((G^\lambda, P^\lambda)\) used in Definition 9 yields also a solution to \((G, P^\lambda)\); in particular, for each path \( S \in \mathcal{S} \) that intersects \( E_t \), one replaces the path segments of \( S \) in \( G \setminus Y_t \) by the newly created vertices to obtain a solution
to \((G^\lambda, P^\lambda)\). Consequently, if \(S\) corresponds to \(\lambda\) then \(\lambda\) must be valid (however, it is clearly not true that every valid record has a solution to the whole instance that corresponds to it). Moreover, since Definition 11 is constructive and deterministic, for each solution \(S\) and node \(t\) there exists precisely one corresponding valid record \(\lambda\).

The second notion that we will need is that of simplification. This is an operation which takes a valid record \(\lambda\) for a node \(t\) and replaces \(G_t\) by a “small representative” so that the resulting graph retains the existence of a solution corresponding to \(S\). Simplification can also be seen as being complementary to the construction of \((G^\lambda, P^\lambda)\) used in Definition 9 (instead of modeling the implications of a record on \(G_t\), we model its implications on \((G - Y_t)\), and will later form an integral part of our procedure for computing valid records for nodes.

**Definition 12.** The simplification of a node \(t\) in accordance with \(\lambda = (\delta, I, F, L)\) is an operation which transforms the instance \((G, P)\) into a new instance \((G', P')\) obtained from \((G - Y_t, P_{G - Y_t})\) as follows:

- For each \(\{s, \{a, b\}\} \in L\) where \((s, t) \in P\) and \(b \notin Y_t\), add \((s, t)\) to \(P'\) and create a vertex \(s\) adjacent to \(b\).
- For each \(\{\{a, b\}, \{c, d\}\} \in I\) where \(a, c \in Y_t\) and \(a \neq c\), add vertices \(a\) and \(c\) into \(G'\) and make them adjacent to \(b\) and \(d\) respectively, and add \((a, c)\) into \(P'\).
- For each \(\{\{a, b\}, \{c, d\}\} \in F\) where \(a, c \in Y_t\) and \(b \neq d\), create a vertex \(e\) and set \(N(e) = \{b, d\}\).

With regards to simplification, observe that every vertex added to \((G - Y_t)\) has degree at most 2 and that simplification can never increase the degree of vertices in \((G - Y_t)\).

**Observation 1.** If there exists a solution to \((G, P)\) which corresponds to a record \(\lambda = (\delta, I, F, L)\) for \(t\), and if \((G', P')\) is the result of simplification of \(t\) in accordance with \(\lambda\), then \((G', P')\) admits a solution. On the other hand, if \((G', P')\) is the result of simplification of \(t\) in accordance with a record \(\lambda\) and if \((G', P')\) admits a solution, then \((G, P)\) also admits a solution.

**Proof.** For the forward direction, consider a solution \(S\) to \((G, P)\) which corresponds to \(\lambda = (\delta, I, F, L)\). By comparing Definition 11 with Definition 12 we observe the following:

1. for each \(s\)-\(t\) path \(P \in S\) such that \(s, t \not\in Y_t\) and \(P \cap E_t \neq \emptyset\), it holds that each path segment of \(P\) in \(Y_t\) begins and ends with a pair of edges in \(F\) and in particular is replaced by a single vertex in \((G', P')\);
2. for each \(s\)-\(t\) path \(P \in S\) such that \(s, t \in Y_t\) and \(P \cap E_t \neq \emptyset\), it holds that each path segment of \(P\) outside of \(Y_t\) begins and ends with a pair of edges in \(I\) and in particular is replaced by a pair of new terminals in \((G', P')\);
3. for each \(s\)-\(t\) path \(P \in S\) such that \(\{s, t\} \cap Y_t = \{s\}\), it holds that the path segment of \(P\) in \(Y_t\) containing \(s\) ends with an edge in \(L\) and is replaced by a new terminal in \((G', P')\), and all other path segments of \(P\) in \(Y_t\) begin and end with a pair of edges in \(F\) and are hence replaced by single vertices in \((G', P')\).

From the above, we observe that \(S\) can be transformed into a solution \(S'\) for \((G', P')\). The backward direction then follows by reversing the above observations; in particular, given a solution \(S'\) for \((G', P')\), we use the fact that \(\lambda\) is valid to expand \(S'\) into a full solution \(S\) to \((G, P)\).
4.2 The Dynamic Step

Let us begin by formalizing our aim for this subsection.

Lemma 13. There is an algorithm which takes as input \((G, P)\) along with a width-
\(k\) treecut decomposition \((T, X)\) of \(G\) and a non-leaf node \(t \in V(T)\) and \(D(t')\) for every child \(t'\) of \(t\), runs in time \(O(k^n)\), and outputs \(D(t)\).

Finally, we introduce two simple reduction rules which will later help us reduce our problem to SIMPLE EDP. The first ensures that two vertices of degree at most 2 are not adjacent to each other.

Reduction Rule 1. Let \((G, P)\) be an instance of EDP containing an edge \(\{a, b\}\) between two vertices of degree at most 2.

1. If \(a\) is not a terminal, then contract \(\{a, b\}\) and replace all occurrences of \(b\) in \(P\) by the new vertex;
2. If \(\{a, b\} \in P\), then contract \(\{a, b\}\) and replace all occurrences of \(a\) and \(b\) in \(P\) by the new vertex;
3. If \(\{a, b\} \notin P\) and each of \(a\) and \(b\) occurs in precisely one element of \(P\), then delete the edge \(\{a, b\}\);
4. Otherwise, reject \((G, P)\).

Proof of Safeness. The safeness of the first three rules is straightforward. As for the fourth rule, let us consider the conditions for when it is applied. In particular, the fourth rule is only called if either \(a\) or \(b\) occurs in three terminal pairs, or if \(a\) occurs in at least one terminal pair and \(b\) in at least two but \(\{a, b\} \notin P\). Clearly, \((G, P)\) is a NO-instance in either of these cases.

The second reduction rule will allow us to replace thin nodes with data tables by small representatives; these representatives will only contain vertices of degree at most 2 adjacent to the original neighborhood of the thin node. The safeness of this rule follows directly from the definition of \(D(t)\) (one simply needs to check each case separately) and hence we do not prove it.

Reduction Rule 2. Let \(t\) be a thin node with non-empty \(D(t)\).

1. If \(E_t = \{\{a, b\}\}\) where \(a \in Y_t\) and if
   \[
   ((\{a, b\} \mapsto L'), \emptyset, \emptyset, \{s, \{a, b\}\}) \notin D(t), \text{ then delete } Y_t \setminus \{s\} \text{ and create the edge } \{s, b\};
   \]
   otherwise, \((\{a, b\} \mapsto U'), \emptyset, \emptyset, \emptyset) \notin D(t) \text{ and we delete } Y_t.
2. If \(E_t = \{\{a, b\}, \{c, d\}\}\) where \(a, c \in Y_t\), \(U_t = \emptyset\) and if
   \[
   ((\{a, b\}, \{c, d\} \mapsto F'), \emptyset, \{\{a, b\}, \{c, d\}\}, \emptyset) \notin D(t), \text{ then delete } Y_t \text{ and create a new vertex } v \text{ adjacent to } b \text{ and } d; \text{ else, if}
   \]
   \[
   ((\{a, b\}, \{c, d\} \mapsto U'), \emptyset, \emptyset, \emptyset) \notin D(t), \text{ then delete } Y_t;
   \]
   otherwise, \((\{a, b\}, \{c, d\} \mapsto I'), \{\{a, b\}, \{c, d\}\}, \emptyset, \emptyset) \notin D(t) \text{ and we delete } Y_t \setminus \{a, c\} \text{ and add } \{a, c\} \text{ into } P.
3. If \(E_t = \{\{a, b\}, \{c, d\}\}\) where \(a, c \in Y_t\), \(U_t = \{s\}\) and if
   \[
   ((\{a, b\} \mapsto L', \{c, d\} \mapsto U'), \emptyset, \emptyset, \{s, \{a, b\}\}) \notin D(t) \text{ and also } ((\{c, d\} \mapsto L', \{a, b\} \mapsto U'), \emptyset, \emptyset, \{s, \{c, d\}\}) \notin D(t), \text{ then delete } Y_t \setminus \{s\} \text{ and make } s \text{ adjacent to } b \text{ and } d;
   \]
   otherwise, \((\{a, b\} \mapsto L', \{c, d\} \mapsto U'), \emptyset, \emptyset, \{s, \{a, b\}\}) \notin D(t) \text{ and we delete } Y_t \setminus \{s\} \text{ and make } s \text{ adjacent to } b.
4. If \( E_t = \{(a, b), (c, d)\} \) where \( a, c \in V_t, U_t = \{s_1, s_2\} \) (not necessarily \( s_1 \neq s_2 \)) and if

- \( (((a, b), (c, d)) \mapsto L'), \emptyset, \emptyset, \{(s_1, \{a, b\}), \{s_2, (c, d)\}\}) \in D(t) \) and also \( (((a, b), (c, d)) \mapsto L'), \emptyset, \emptyset, \{(s_2, \{a, b\}), \{s_1, (c, d)\}\}) \in D(t) \), then add a new vertex \( s' \) adjacent to \( b \) and \( d \), replace all occurrences of \( s_1 \) and \( s_2 \) in \( P \) by \( s' \), and delete \( Y_t \); 
- otherwise, \( (((a, b), (c, d)) \mapsto L'), \emptyset, \emptyset, \{(s_1, \{a, b\}), \{s_2, (c, d)\}\}) \in D(t) \) and we delete \( Y_t \setminus \{s_1, s_2\} \), and make \( s_1 \) adjacent to \( b \) and \( s_2 \) adjacent to \( d \).

5. Otherwise, \((G, P)\) is a NO-instance.

With Lemma 2 and Reduction Rules 1, 2 in hand, we have all we need to handle the dynamic step. It will be useful to recall the definitions of \( A_t \) and \( B_t \), and that \(|A_t| \leq 2k + 1\).

**Proof of Lemma 13** We begin by looping through all of the at most \( 4^k \cdot k! \) distinct records for \( t \); for each such record \( \lambda \), our task is to decide whether it is valid, i.e., whether \((G^\lambda, P^\lambda)\) is a YES-instance. On an intuitive level, our aim will now be to use branching and simplification in order to reduce the question of checking whether \( \lambda \) is valid to an instance of SIMPLE EDP.

In our first layer of branching, we will select a record from the data tables of each node in \( A_t \). Formally, we say that a record-set is a mapping \( \tau : t' \in A_t \mapsto \lambda_{t'} \in D(t') \). Note that the number of record-sets is upper-bounded by \( (4^k \cdot k!)^{O(2k+1)} \), and we will loop over all possible record-sets.

Next, for each record-set \( \tau \), we will apply simplification to each node \( t' \in A_t \) in accordance with \( \tau(t') \), and recall that each vertex \( v \) created by this sequence of simplifications has degree at most 2. Next, we exhaustively apply Reduction Rule 1 to ensure that each such \( v \) is only adjacent to \((V(G) \setminus Y_t) \cup X_t \). At this point, every vertex contained in a bag \( X_{t'} \) for \( t' \in A_t \) has degree at most 2 and is only adjacent to \( X_t \) and \((V(G) \setminus Y_t) \).

Finally, we apply Reduction Rule 2 to replace each thin node by vertices of degree at most 2 adjacent to \( X_t \). At this point, every vertex in \( V(G^\lambda) \setminus X_t \) is of degree at most 2 and only adjacent to \( X_t \), and so \((G^\lambda, P^\lambda)\) is an instance of SIMPLE EDP. All that is left is to invoke Lemma 2 if it is a YES-instance then we add \( \lambda \) to \( D(t) \), and otherwise we do not.

The running time is upper bounded by the branching factor \((4^k \cdot k!)^{O(2k+1)}\) times the time to apply our two reduction rules and the time required to solve the resulting SIMPLE EDP instance. All in all, we obtain a running time of at most \( k^{O(k^2)} \cdot |P|^{O(k^2)} = (k|P|)^{O(k^2)} \).

We conclude the proof by arguing correctness. Assume \( \lambda \) is a valid record. By Definition 2 this implies that \((G^\lambda, P^\lambda)\) admits a solution \( S \). For each child \( t' \in A_t \), \( S \) corresponds to some record \( \lambda_{t'}^S \); consider now the branch in our algorithm which sets \( \tau(t') = \lambda_{t'}^S \).

Then by Observation 1, it follows that each simplification carried out by the algorithm preserves the existence of a solution to \((G^\lambda, P^\lambda)\). Since both our reduction rules are safe, the instance of SIMPLE EDP we obtain at the end of this branch must also be a YES-instance.

On the other hand, assume the algorithm adds a record \( \lambda \) into \( D_t \). This means that the resulting SIMPLE EDP instance was a YES-instance. Then by the safeness of our reduction rules and by the second part of Observation 1, the instance \((G', P')\) obtained by reversing the reduction rules and simplifications was also a YES-instance; in particular \((G^\lambda, P^\lambda)\) is a YES-instance and so \( \lambda \) is a valid record.

We now have all the ingredients we need to prove our main result.

**Theorem 14.** EDP can be solved in time at most \( O(n^3) + k^{O(k^2)} n^2 + (k|P|)^{O(k^2)} n \), where \( k \) is the treecut width of the input graph and \( n \) is the number of its vertices.

**Proof.** We begin by invoking Theorem 3 to compute a treecut decomposition of \( G \) of width at most \( 2k \) and then converting it into a nice treecut decomposition (this takes time \( k^{O(k^2)} n^2 \).
and $O(n^3)$, respectively. Afterwards, we use Lemma 10 to compute $D(t)$ for each leaf of $T$, followed by a recursive leaf-to-root application of Lemma 13. Once we compute $D(r)$ for the root $r$ of $T$, we output YES if and only if $D(r) = \{(\emptyset, \emptyset, \emptyset)\}$.

5 A linear kernel for EDP

The goal of this section is to provide a fixed-parameter algorithm for EDP which exploits the structure of the input graph exclusively. While treecut width cannot be used to obtain such an algorithm, here we show that the feedback edge set number can. More specifically, we obtain a linear kernel for EDP parameterized by the feedback edge set number. Our kernel relies on the following two facts:

▶ Fact 15. A minimum feedback edge set of a graph $G$ can be obtained by deleting the edges of minimum spanning trees of all connected components of $G$, and hence can be computed in time $O(|E(G)| \cdot \log |V(G)|)$.

▶ Fact 16 ([15]). EDP can be solved in polynomial time when $G$ is a forest.

For the purposes of this section, it will be useful to assume that each vertex $v \in V(G)$ occurs in at most one terminal pair, each vertex in a terminal pair has degree 1 in $G$, and each terminal pair is not adjacent to each other. Note that for any instance without these properties, we can add a new leaf vertex for each terminal, attach it to the original terminal, and replace the original terminal in $P$ with the leaf vertex [14,26].

Consider an instance $(G,P)$ of EDP and let $X \subseteq E(G)$ be a minimum feedback edge set $X$. Let $Y$ be the set of all vertices incident to at least one edge from $X$, and let $Q = G - X$. Similarly as before, given a subgraph $H$ of $G$, we say that $H$ contains an unmatched terminal $s$ if $(s,t) \in P$, $s \in V(H)$ and $t \notin V(H)$. We begin with two simple reduction rules which allow us to remove degree 2 vertices and leaves not containing a terminal.

▶ Reduction Rule 3. Let $v \in V(G)$ be such that $|N_G(v)| = 1$. If $v$ is not a terminal, then delete $v$ from $G$.

Proof of Safeness. Since $v$ has degree 1 and is not a terminal, there cannot exist a solution to $(G,P)$ containing a path which uses $va$.

▶ Reduction Rule 4. Let $v,a,b \in V(G)$ be such that $N_G(v) = \{a,b\}$ and $\{a,b\} \notin E$. Then delete $v$ and add the edge $ab$ into $E$. Furthermore, if $\{a,v\}$ or $\{v,b\}$ were in $X$ then add $\{a,b\}$ in $X$.

Proof of Safeness. Observe that any solution to the original instance which uses any edge incident to $v$ must contain a path which traverses through both $av$ and $vb$, and after the reduction rule is applied one can simply replace these two edges in that path by $ab$. Any solution in the reduced instance can be similarly transformed into a solution to the original instance. The same argument also shows that the newly constructed set $X$ is also a feedback edge set in the reduced graph.

Of crucial importance is our third rule, which allows us to prune the instance of subtrees with a single edge to $Y$. For a subgraph $H$ of $G$, recall that $P_H \subseteq P$ denotes the subset of terminal pairs whose both endpoints lie in $H$.

▶ Reduction Rule 5. Let $L$ be a connected component of $G - Y$ such that there exists a single edge $\{\ell \in L, y \in Y\}$ between $L$ and $Y$. 

The Power of Cut-Based Parameters for Computing Edge Disjoint Paths

If $L$ contains no unmatched terminal and $(L, P_L)$ is a YES-instance of EDP, then set $P := P \setminus P_L$ and $G := G \setminus V(L)$.

If $L$ contains precisely one unmatched terminal $s$ where $\{s, t\} \in P$ and the instance $(L, P_L \cup \{s, \ell\})$ is a YES-instance of EDP, then set $P := P \setminus P_L$ and $G := ((V(G) \setminus V(L)) \cup \{s\}, (E(G) \setminus (E(L) \cup \{\ell, y\}) \cup \{y, s\})$.

In all other cases, $(G, P)$ is a NO-instance of EDP.

**Proof of Safeness.** First, we note that a solution can only use the edge $\{\ell, y\}$ (which is the only edge connecting $L$ to $G - L$) for a path connecting an unmatched terminal in $L$. Let us start by arguing the correctness of Point c., which covers about the following three cases. If $L$ contains at least two distinct unmatched terminals, then any solution to $(G, P)$ would require two edge disjoint paths between $L$ and $G - L$ (which, however, do not exist in $G$).

If $L$ contains one unmatched terminal and $(L, P_L \cup \{s, \ell\})$ is a NO-instance of EDP, then it is not possible to find edge disjoint paths which connect terminal pairs in $P_L$ while also connecting $s$ to $t$ (since every path to $t$ must go through $\ell$). Similarly, if $L$ contains zero unmatched terminals and $(L, P_L)$ is a NO-instance of EDP, then $(G, P)$ must also be a NO-instance of EDP.

Next, assume that the conditions of Point a. hold. Since $L$ contains no unmatched terminal, the edge $\{\ell, y\}$ can never be used by any solution and hence it can be removed. Naturally, this results in $L$ being disconnected from $G - L$ and so it suffices to solve $G - L$.

Finally, in the case covered by Point b., every solution to $(G, P)$ must use the edge $\{\ell, y\}$ for an edge disjoint path connecting $s$ to $t$. Hence any solution to the reduced instance in this case implies a solution to $(G, P)$, and similarly any solution to $(G, P)$ can be used to obtain a solution for the reduced instance.

After exhaustive application of Reduction Rules 3 and 4, we observe that each leaf in $Q$ is either in $Y$ or adjacent to a vertex in $Y$. The simple rule below is required to obtain a bound on the number of leaves in $Q$ in the subsequent step.

**Reduction Rule 6.** If $\{a, b\} \in P$ and $a, b$ are leaves in $G$ such that $N(a) = N(b)$, then remove $a$ and $b$ from $G$ and $P$.

**Proof of Safeness.** Every $a-b$ path must use the two unique edges incident to $a$ and $b$, and we may assume without loss of generality that a path never visits a vertex twice.

After exhaustive application of Reduction Rules 3 and 5, we can prove:

**Lemma 17.** If $Q$ contains more than $4|X|$ leaves, then $(G, P)$ is a NO-instance.

**Proof.** Let $Z$ denote the set of leaves in $Q$, and let $Y' = Z \cap Y$. Since $|Y| \leq 2|X|$ it follows that $|Z| > |Y| + 2|X|$ and hence $|Z \setminus Y'| > |Y \setminus Y'| + 2|X|$. For brevity, let $A = Z \setminus Y'$ be the set of leaves excluding the endpoints of our feedback edge set and $B = Y \setminus Y'$ be the set of endpoints of our feedback edge set which are not leaves.

Recall that every vertex in $A$ is adjacent to a vertex in $B$. Let us define a weight function $w(b)$ for each vertex $b \in B$ which is equal to the number of edges in $X$ incident to $b$; note that since each edge in $X$ contributes by adding at most 2 to the weight function, we have $\sum_{b \in B} w(b) \leq 2|X|$. Since $|A| > |B| + 2|X|$ it follows that $|A| \geq |B| + 2|X|$, and $\sum_{b \in B} w(b) = \sum_{b \in B} (w(b) + 1)$, there must exist at least one vertex $c \in B$ such that $w(c) + 1$ is smaller than the number of its neighbors in $A$; in other words, $c$ is adjacent to at least $w(c) + 2$ leaves but is only incident to $w(c) + 1$ edges whose endpoints are not leaves. Since $c$ itself is not a leaf and hence not a terminal, every leaf contains a terminal, and Reduction Rule 6 cannot be applied, we
conclude that it is not possible to route all terminals located in the leaves adjacent to $c$ to their endpoints via edge disjoint paths.

Finally, we put everything together in the proof of the desired theorem.

**Theorem 18.** EDP admits a linear kernel parameterized by the feedback edge set number of the input graph.

**Proof.** Let $(G, P)$ be an instance of EDP; w.l.o.g. we assume that $G$ is and remains connected (note that if $G$ becomes disconnected due to a later application of a reduction rule, one can simply kernelize each connected component separately). We begin by computing a minimum feedback edge set $X$ of $G$ using Fact 15. We then exhaustively apply Reduction Rules 3, 4, 5 and 6; since EDP is polynomial-time tractable by Fact 16, the time required to apply each rule is easily seen to be polynomial.

After no more rules can be applied, we compare the number of leaves in $Q = G - X$ to $|X|$. If $Q$ contains more than $4|X|$ leaves, then we reject in view of Lemma 17. On the other hand, if $Q$ contains at most $4|X|$ leaves, then we claim that $Q$ contains at most $11|X| - 2$ vertices. Indeed, the number of vertices of degree at least 3 in a forest is at most equal to the number of leaves minus two and in particular $Q$ has at most $4|X| - 2$ vertices of degree at least 3. Moreover, due to the exhaustive application of Reduction Rule 4 it follows that the number of degree two vertices is at most $|X|$. And so, by putting together the bounds on $|Y|$ along with the number of vertices of degree 1 and 2 and 3, we obtain $|V(G)| = |V(Q)| \leq 2|X| + 4|X| + |X| + 4|X| - 2$, as claimed.

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