The deformation theory
of local complete intersections

Fourth draft

Angelo Vistoli*

Dipartimento di Matematica
Università di Bologna
Piazza di Porta San Donato 5
40127 Bologna, Italy
E-mail address: vistoli@dm.unibo.it

This is an expository paper on the subject of the title. I tried to make it as self-contained as I could, assuming only basic facts in the theory of schemes and some commutative and homological algebra, and to keep the formalism at a minimum. There is very little pretense of originality on my part; the results are of course very well known, and most of the ideas in the proofs are also known. The possible exceptions are the construction of the obstruction in Section 4, which does not use simplicial techniques, and the proof of the existence of versal deformations in Section 7, in which the relations among the generators are obtained directly from the obstructions.

The treatment is perhaps a little curt, and does not provide much motivation for the results. Many of the details of the proofs are left to the interested reader to fill in, and often I let certain necessary compatibility conditions unstated. I believe that not doing so would substantially increase the length of the exposition without adding much to the reader’s understanding; however, a sufficient number of protests might change my mind (so far, I got none.)

Comments and corrections are very welcome.

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I learned a lot about deformation theory in a course that Hubert Flenner taught in Bologna in July and October 1998, and from discussions with him.

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1 Introduction

A very basic problem in algebraic geometry, which comes in all sorts of guises, is to understand families of objects (varieties, bundles, singularities, maps, ...). This is usually hard. The first step is to study the deformations of a fixed object $X_0$ of the given type, that is, families of objects depending on some parameters $t_1, \ldots, t_r$, such that for $t_1 = \cdots = t_r = 0$ we get exactly $X_0$; we are interested in what happens near $(0, \ldots, 0)$. This is done in three stages.

First of all there is the problem of infinitesimal liftings, which can be illustrated as follows. Suppose that we are given an object $X_0$ over a field $\kappa$, and a deformation $X$ of order $n-1$; we can think of $X$ as obtained by perturbing the definition of $X_0$ by adding a parameter $t$ with $t^n = 0$. This is very vague, of course; in practice the definition usually involves defining $X$ over the ring $\kappa[[t]]/(t^n)$, or some other artinian ring, so that the restriction of $X$ to $\kappa$ is $X_0$, and some other “continuity” condition (often involving flatness) is satisfied (in practice it is very important to look also at deformations on higher dimensional basis, that is, add more than one parameter.) Then the problem is: can we lift $X$ to order $n$? And if we can, how can we describe the liftings? Usually the answer is in two parts: first there is a canonical element $\omega$ of some vector space $V$ such that $\omega = 0$ if and only if a lifting exists. Then there is some other vector space $W$ such that if $\omega = 0$ then $W$ acts on the set of isomorphism classes of liftings making it into a principal homogeneous space (with uncanny regularity $W$ is a cohomology group of a certain algebraic object, and $V$ the cohomology group of the same object in one degree higher.)

The second stage is to look at formal deformations; a formal deformation of $X_0$ can be roughly described as a lifting $X_1$ of $X_0$ to order 1, a lifting $X_2$ of $X_1$ to order 2, and so, to all orders. Here the main result is that, with very weak hypotheses, there is always a formal deformation $V$ (involving several parameters,) called “versal”, such that, very roughly, all other formal deformations are obtained from $V$ (the actual definition is a little technical.) This is defined over a quotient of a power series algebra $R = \kappa[[t_1, \ldots, t_r]]/I$ (or some other complete local ring, when there is no base field); the $t_i$ are the parameters, and $I$ the ideal of relations between the parameters. If the number of parameters is as small as possible then saying that $I = 0$ is equivalent to saying that the infinitesimal deformations are unobstructed, that is, for any $n$, given a deformation to order $n-1$ we can always lift it to order $n$. In practice knowing the ring $R$ gives us a considerable degree of control on the infinitesimal deformations.

The third stage is to pass from formal deformation to actual deformations. In analytic geometry this amounts to passing from formal solution of some equations to analytic solutions, and can usually be done. In an algebraic context it is a much more delicate question, sometimes called the problem of algebraization; it can be solved for curves, but in general not for surfaces.
Despite the fundamental importance of the subject I do not know of any exposition that is both acceptably general and accessible to the average algebraic geometer or number theorist. There is a very thorough discussion of embedded deformations in [Kollár], and deformations of singularities are treated in a very readable way in [Artin 1]; both are highly recommended. Going much beyond these two references is [Illusie]; this is an excellent book, and very carefully written, but uses substantial amounts of simplicial machinery even to define the basic object, the cotangent complex, and the exposition is at the kind of topos-theoretic utmost level of generality that makes most algebraic geometers’ eyes glaze over in a fraction of the time it takes to say “simplicial object associated to a pair of adjoint functors”.

Hopefully, this will change when a book by Buchweitz and Flenner comes out in the next couple of years. It should treat deformations of analytic spaces in full generality, using the definition of the cotangent complex via Tate resolutions. These are much easier to understand and work with than simplicial resolutions; unfortunately at the moment this approach only works in characteristic 0.

In this notes we study the infinitesimal and formal deformation theory of a local complete intersection schemes. We limit ourselves to a few basic results that can be obtained using sheaves of differentials, without resorting to the cotangent complex, or to Tate resolutions, as one has to do to go beyond this simple case. This is not likely to keep most people happy for very long, but the task of giving a reasonably self-contained explanation of the cotangent complex is rather daunting to me.

The question of algebraization of formal deformations is only touched upon very briefly (Proposition 6.3).

Section 2 treats the infinitesimal liftings of local complete intersection subscheme of a given scheme.

Abstract liftings of generically smooth local complete intersection schemes are discussed in Section 4; here we use some standard, and some less standard, facts on extensions of sheaves, proved in Section 3. The construction of the obstruction seems to be new, although I have not searched the literature long enough to be sure.

Section 5 contains some generalizations, most important to the case of local complete intersection maps, and to the case of deformations of pairs.

In Section 6 we define formal deformations of a local complete intersection generically smooth scheme over a field, define the Kodaira-Spencer map and the first obstruction map, and prove some basic properties. Assuming the existence of the base field is absolutely not necessary (a complete local ring would do very well) and is of course a hindrance for applications to arithmetic, but it simplifies to some extent the exposition.

In Section 7 we discuss versal deformations; in particular prove the existence of versal deformations for generically smooth local complete intersection schemes over a field, using the results of Section 4. This proof is different from the one in [Schlessinger], and uses the obstruction theory. It has the merit of illustrating how equations defining versal deformation spaces arise from obstructions.

Section 8 contains the proof of a very important technical lemma.

In Sections 6 and 7 we become a little more formal, and exploit the notion of homomorphism of deformations, which allows to make the treatment somewhat less cluttered and ultimately more transparent. Also, here rings are used, instead of their spectra; I am aware that this might make many algebraic geometers uncomfortable, but I feel that it is more natural in this context.

The theory in these two sections can be extended to any of the other cases considered in Section 5, and beyond. Indeed an axiomatic treatment would be possible (see for example [Artin 2]), and perhaps will be added to a future version of these notes.
2 Liftings of Embedded Local Complete Intersections

(2.1) Notation. The following notation is used here and in Section 4. Let \( A' \) be a noetherian local ring with maximal ideal \( \mathfrak{m}_{A'} \) and residue field \( \kappa = A'/\mathfrak{m}_{A'} \), and \( \mathfrak{a} \subseteq A' \) an ideal such that \( \mathfrak{m}_{A'} \mathfrak{a} = 0 \); then \( \mathfrak{a} \) is a finite-dimensional vector space over \( \kappa \). Set \( A = A'/\mathfrak{a} \). All schemes and morphisms will be defined over \( A' \).

If \( X \) is a scheme over a scheme \( S \), and \( T \) is a subscheme of \( S \), we denote by \( X \mid T \) the inverse image of \( T \) in \( X \). If “\( X \)” denotes a scheme over \( A \), where \( X \) is an arbitrary symbol, we will always set

\[
X_0 = X \mid \text{Spec} \kappa.
\]

Also, to avoid awkward terminology, if \( X \) is a locally closed subscheme of \( M \), we will always talk about the sheaf of ideals of \( X \) in \( M \) meaning the sheaf of ideals of \( X \) in some open subset of \( M \) where \( X \) is closed. More generally we will sometime omit to say that \( M \) has to be restricted to an open subset. This should cause no confusion.

(2.2) Hypotheses. Let \( M' \) be a flat scheme of finite type over \( A' \). Let \( X \) be a local complete intersection subscheme of \( M = M'|_{\text{Spec} A} \) (not necessarily closed); this means that locally the ideal of \( X \) in \( M \) is generated by a regular sequence in \( \mathcal{O}_M \). Assume also that \( X \) is flat over \( A \); then \( X_0 \) is still a local complete intersection in \( M_0 \).

Call \( \mathcal{N}_0 \) the normal bundle to \( X_0 \) in \( M_0 \). If \( \mathcal{I}_0 \) is the ideal of \( X_0 \) in \( M_0 \), then \( \mathcal{C}_0 = \mathcal{I}_0 / \mathcal{I}_0^2 \) is a locally free sheaf on \( X_0 \), called the conormal sheaf to \( X_0 \) in \( M_0 \), and \( \mathcal{N}_0 \) is by definition its dual.

Let us check that indeed \( X_0 \) is still a local complete intersection in \( M_0 \). This is a particular case of the following lemma.

(2.3) Lemma. If \( A \to B \) is a local homomorphism of local noetherian rings, then \( X \times_{\text{Spec} A} \text{Spec} B \) is a local complete intersection in \( M \times_{\text{Spec} A} \text{Spec} B \).

Proof. This is a local problem, so we may assume that \( M \) is affine, \( X \) is closed in \( M \), and the ideal of \( X \) in \( M \) is generated by a regular sequence \( \mathfrak{f} \). The Koszul complex \( \mathcal{K} \) of \( \mathfrak{f} \) is a resolution of \( \mathcal{O}_X \) by flat sheaves over \( A \); but \( \mathcal{O}_X \) is by hypothesis flat over \( A \), so \( \mathcal{K} \otimes_A \mathcal{B} \) is exact in positive degree. This means that the restriction of \( \mathcal{f} \) to \( X \times_{\text{Spec} A} \text{Spec} B \) is still a regular sequence, so \( X \times_{\text{Spec} A} \text{Spec} B \) is a local complete intersection in \( M \times_{\text{Spec} A} \text{Spec} B \).

(2.4) Definition. A lifting of \( X \) to \( M' \) is a subscheme \( X' \) of \( M' \) which is flat over \( A' \) and such that \( X = X' \cap M \).

(2.5) Theorem. (a) Any lifting of \( X \) is a local complete intersection in \( M' \).

(b) There is a canonical element

\[
\omega_{\text{emb}} = \omega_{\text{emb}}(X) \in \mathfrak{a} \otimes_\kappa \mathbb{H}^1(X_0, \mathcal{N}_0),
\]

called the embedded obstruction of \( X \) in \( M \), such that \( \omega_{\text{emb}} = 0 \) if and only if a lifting exists.

(c) If a lifting exists, then there is a canonical action of the group \( \mathfrak{a} \otimes_\kappa \mathbb{H}^0(X_0, \mathcal{N}_0) \) on the set of liftings making it into a principal homogeneous space.

Let us begin the proof with a criterion for a given subscheme \( X' \subseteq M' \) with \( X' \cap M = X \) to be flat over \( A' \). Let \( \mathcal{I}' \) be the ideal of \( X' \) in \( M' \); then \( \mathcal{I}' \mathcal{O}_M = \mathcal{I} \), so that there is a natural surjective map \( \mathcal{I}'/\mathfrak{a} \mathcal{I}' \to \mathcal{I} \).
(2.6) Lemma. The scheme $X'$ is flat over $A'$ if and only if the natural surjective map $\mathcal{I}'/\mathfrak{a}\mathcal{I}' \to \mathcal{I}$ is an isomorphism.

Proof. This statement is local on $X'$, so we assume that all schemes involved are affine and the embedding $X' \hookrightarrow M'$ is closed. There is a short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{O}_{M'} \to \mathcal{O}_{X'} \to 0;$$

so if we tensor over $A'$ with $A$ we get an exact sequence

$$0 = \text{Tor}_1^A(\mathcal{O}_{M'}, A) \to \text{Tor}_1^A(\mathcal{O}_{X'}, A) \to \mathcal{I}'/\mathfrak{a}\mathcal{I}' \to \mathcal{O}_M \to \mathcal{O}_X \to 0$$

which shows that the map $\mathcal{I}'/\mathfrak{a}\mathcal{I}' \to \mathcal{I}$ is an isomorphism if and only if $\text{Tor}_1^A(\mathcal{O}_{X'}, A) = 0$. So if $X'$ is flat then the condition of the theorem is verified; the converse statement is a particular case of Grothendieck’s local criterion of flatness (see for example [Matsumura]), and can be proved very simply as follows. Let $N$ be an arbitrary $A'$ module; we want to show that $\text{Tor}_1^A(\mathcal{O}_{X'}, N) = 0$, assuming this is true for $N = A$. From the exact sequence of Tor’s, it is enough to prove that $\text{Tor}_1^A(\mathcal{O}_{X'}, aN) = \text{Tor}_1^A(\mathcal{O}_{X'}, N/an) = 0$; so we may assume that $an = 0$ (observe that $a(an) = 0$.) In other words, we assume that $N$ is an $A$-module. Then the sequence

$$0 \to \mathcal{I}' \otimes_{A'} N \to \mathcal{O}_{M'} \otimes_{A'} N \to \mathcal{O}_{X'} \otimes_{A'} N \to 0$$

is the same as the sequence

$$0 \to \mathcal{I} \otimes_A N \to \mathcal{O}_M \otimes_A N \to \mathcal{O}_X \otimes_A N \to 0,$$

which is exact because $X$ is flat over $A$.

Now we analyze the local situation; suppose that $M'$ is affine, $X$ is closed in $M$, and the ideal $\mathcal{I}$ of $X$ in $M$ is generated by a regular sequence $f_1, \ldots, f_r$ in $\mathcal{O}_M$.

Let $X'$ be a lifting of $X$, $\mathcal{I}'$ the ideal of $X'$ in $M'$. Choose liftings $f'_1, \ldots, f'_r$ of $f_1, \ldots, f_r$ to $\mathcal{I}$; then from the equality $\mathcal{I}'/\mathfrak{a}\mathcal{I}' = \mathcal{I}$ and the fact that the ideal $a$ is nilpotent we conclude that $f'_1, \ldots, f'_r$ generate $\mathcal{I}'$.

Let us check that $f'_1, \ldots, f'_r$ is a regular sequence in $\mathcal{O}_{M'}$; this will prove part (a). Let $\mathcal{K}'_r$ be the Koszul complex of $f'_1, \ldots, f'_r$; then $\mathcal{K}' = \mathcal{K}'_r \otimes_{A'} A$ is the Koszul complex of $f_1, \ldots, f_r$. We have a homology spectral sequence

$$E_2^{pq} = \text{Tor}_p^A(\mathcal{H}_q(\mathcal{K}'_r), A) \Rightarrow \mathcal{H}_{p+q}(\mathcal{K}_r) = \begin{cases} 0 & \text{if } p + q > 0 \\ \mathcal{O}_X & \text{if } p + q = 0. \end{cases}$$

Notice that $E_2^{p0} = 0$ for $p > 0$, because $\mathcal{H}_0(\mathcal{K}'_r) = \mathcal{O}_{X'}$ is flat over $A'$. From this, and the fact that the abutment is 0 in degree 1, we get that $\mathcal{H}_1(\mathcal{K}'_r) \otimes_{A'} A = E_2^{01} = 0$. This implies that $\mathcal{H}_1(\mathcal{K}'_r) = 0$, and hence $E_2^{p1} = 0$ for all $p$. Analogously one proves that $\mathcal{H}_2(\mathcal{K}'_r) = 0$, and by induction on $q$ that $\mathcal{H}_q(\mathcal{K}'_r) = 0$ for all $q > 0$. This proves that $f'_1, \ldots, f'_r$ is a regular sequence.

Conversely, start from liftings $f'_1, \ldots, f'_r$ of $f_1, \ldots, f_r$ to $\mathcal{O}_{M'}$, and define $X'$ via $\mathcal{O}_{X'} = \mathcal{O}_{M'}/(f'_1, \ldots, f'_r)$. Then we claim that $X'$ is a lifting of $X$ to $M'$; for this we need to show that $\mathcal{I}'/\mathfrak{a}\mathcal{I}' = \mathcal{I}$ (Lemma 2.6). Let $\sum_i a_i f'_i$ be an element of $\mathcal{I}'$ whose image $\sum_i a_i f_i$ in $\mathcal{I}$ is 0. Then because $f_1, \ldots, f_r$ is a regular sequence we can write $(a_1, \ldots, a_r) \in \mathcal{O}_{M}$ as a linear combination of standard relations of the form

$$(0, \ldots, 0, f_{i}, 0, \ldots, 0, -f_i, 0, \ldots, 0).$$
These relations lift to relations

\[(0, \ldots, 0, f_i', 0, \ldots, 0, -f_j', 0, \ldots, 0).\]

among the \(f_i'\). Then \((a_1', \ldots, a_n')\) can be written as a relation among the \(f_i'\), plus an element \((b_1', \ldots, b_{s'}) \in (aO_{M'})^n\), so that \(\sum_{i} a_i'f_i' = \sum_{i} b_i'f_i' \in aI'.\)

So we have proved that the liftings of \(X\) are obtained locally by lifting equations for \(X\). In particular we have proved the following.

**Lemma.** Assume that \(X\) is affine and a complete intersection in \(M\). Then \(X\) has a lifting to \(M'\).

Here by a complete intersection we mean that \(X\) is closed in \(M\), and its ideal is generated by a regular sequence.

Let \(X_1'\) and \(X_2'\) be two liftings of \(X\) to \(M'\); to these we will associate a section \(\nu(X_1', X_2')\) of \(N_0\).

Call \(T_1\) and \(T_2\) the corresponding ideals in \(O_{M'}\). Take a local section \(f\) of \(I\); then \(f\) can be lifted to sections \(f_1'\) and \(f_2'\) of \(I_1\) and \(I_2\) respectively. The difference \(f_1' - f_2'\) is an element of \(aO_{M'} = a \otimes_{A'} O_{M'} = a \otimes_{k} O_{M_0}\). This element does not depend on \(f\) only; if we choose different liftings \(f_1' + g_1'\) and \(f_2' + g_2'\), with \(g_1' \in aI_1'\), then the difference \(f_1' - f_2'\) will change by an element \(g_1' - g_2'\) of \(aI\); so the image of \(f_1' - f_2'\) in \(a \otimes_{k} O_{X_0}\) only depends on \(f\). This construction yields a function \(I \to a \otimes_{k} O_{X_0}\), which we denote by \(\nu(X_1', X_2')\). This function is \(O_{M}\)-linear, and we think of it as a section of

\[\text{Hom}_{O_{M}}(I, a \otimes_{k} O_{X_0}) = \text{Hom}_{O_{M_0}}(I_0, a \otimes_{k} O_{X_0})\]

\[= \text{Hom}_{O_{X_0}}(I_0/I_0', a \otimes_{k} O_{X_0})\]

\[= \text{H}^0(X_0, a \otimes_{k} N_0)\]

\[= a \otimes_{k} \text{H}^0(X_0, N_0).\]

This construction has the following properties.

**Proposition.** To each pair of liftings \(X_1',\ X_2'\) is associated a well defined element

\[\nu_{M'}(X_1', X_2') = \nu(X_1', X_2') \in a \otimes_{k} \text{H}^0(X_0, N_0),\]

with the following properties.

(a) \(\nu(X_1', X_2') = 0\) if and only if \(X_1' = X_2'\).

(b) If \(X_1', X_2'\) and \(X_3'\) are liftings, then

\[\nu(X_1', X_3') = \nu(X_1', X_2') + \nu(X_2', X_3').\]

(c) \(\nu(X_2', X_1') = -\nu(X_1', X_2')\).

(d) Given a lifting \(X'\) and a section \(\nu \in \text{H}^0(X, a \otimes_{A} N_0)\), there is a lifting \(\bar{X}'\) with \(\nu(\bar{X}', X') = \nu\).

(e) If \(Y\) is an open subscheme of \(X\), \(Y_1'\) and \(Y_2'\) are the restrictions of \(X_1'\) and \(X_2'\), then \(\nu(Y_1', Y_2')\) is the restriction of \(\nu(X_1', X_2')\).

(f) Let \(\tilde{M}' \to M'\) be a smooth morphism of flat \(A'\)-schemes of finite type, \(X_1' \hookrightarrow \tilde{M}'\) and \(X_2' \hookrightarrow \tilde{M}'\) embeddings compatible with the embeddings of \(X_1'\) and \(X_2'\) into \(M'\), inducing the same
embedding of \( X \) into \( \tilde{M} = M' \mid \text{Spec} \, \mathbb{A} \). Let \( \tilde{I}_0 \) be the ideal of \( X_0 \) in \( \tilde{M}_0 = M' \mid \text{Spec} \, \mathbb{A} \). Then the homomorphism

\[
\text{Hom}_{\mathcal{O}_X}(\tilde{I}_0/\tilde{I}_0^2, a \otimes \kappa \, \mathcal{O}_{X_0}) \longrightarrow \text{Hom}_{\mathcal{O}_X}(I_0/I_0^2, a \otimes \kappa \, \mathcal{O}_{X_0})
\]

induced by the natural embedding of \( I_0/\tilde{I}_0^2 \) into \( I_0/\tilde{I}_0^2 \) carries \( \nu_{M'}(X'_1, X'_2) \) into \( \nu_M(X'_1, X'_2) \).

**Proof.** We will only give a hint for part (d); the remaining statements are straightforward and left to the reader. The ideal \( \tilde{I}' \) of \( \tilde{M}' \) can be described as follows. A local section \( f' \) of \( \mathcal{O}_{M'} \) is in \( \tilde{I}' \) if and only if its image \( f \) in \( \mathcal{O}_M \) lies in \( I \), and there exists a local section \( f' \) of \( \tilde{I}' \) mapping to \( f \) such that the image of \( f' - f' \) in \( a \otimes \kappa \, \mathcal{O}_{X_0} \) is \( \nu(f) \). One checks that \( \tilde{I}' \) is an ideal of \( \mathcal{O}_{M'} \), and that the subscheme \( X' \subseteq M' \) with ideal \( \tilde{I}' \) is indeed a lifting of \( X \) with \( \nu(\tilde{X}', X') = \nu \).

Theorem 2.5.(c) follows immediately from Proposition 2.8.(a), (b) and (d), and the following elementary fact.

**(2.9) Lemma.** Let \( X \) be a set, \( G \) a group. Let there be given a function \( \phi : X \times X \rightarrow G \) with the following properties.

(a) \( \phi(x_1, x_2) = 1 \) if and only if \( x_1 = x_2 \).

(b) \( \phi(x_1, x_2) \phi(x_2, x_3) = \phi(x_1, x_3) \) for all \( x_1, x_2 \) and \( x_3 \) in \( X \).

(c) For each \( g \in G \) and each \( x \in X \) there exists \( \tilde{x} \in X \) such that \( \phi(\tilde{x}, x) = g \).

Then the element \( \tilde{x} \) in (c) is unique, and \( X \) has the structure of a principal left homogeneous \( G \)-space, by defining \( g \cdot x = \tilde{x} \) for all \( g \in G \) and \( x \in X \).

Let us prove part (b) of the theorem. We may assume that \( X \) is closed in \( M \). Choose a covering \( \mathcal{U} = \{ U_\alpha \} \) of \( M \) by open affine subschemes, such that in each \( U_\alpha \) the subscheme \( X_\alpha = X \cap U_\alpha \) is a complete intersections, and call \( U_\alpha \) the corresponding open subschemes of \( M' \). By Lemma 2.7, each \( X_\alpha \) has a lifting \( X'_\alpha \) in \( M' \); there exists a global lifting if and only if after possibly refining \( \mathcal{U} \) we can choose the \( X'_\alpha \) in such a way that \( X'_\alpha \cap (U'_\alpha \cap U'_\beta) = X'_\beta \cap (U'_\alpha \cap U'_\beta) \) for all \( \alpha \) and \( \beta \). To define the embedded obstruction \( \omega_{\text{emb}} \), choose liftings \( X'_\alpha \) arbitrarily, and set

\[
\nu_{\alpha \beta} = \nu(X'_\alpha, X'_\beta).
\]

Of course this should have been written as

\[
\nu_{\alpha \beta} = \nu_{U'_\alpha \cap U'_\beta}(X'_\alpha \cap (U'_\alpha \cap U'_\beta), X'_\beta \cap (U'_\alpha \cap U'_\beta));
\]

but now as in the future, we will commit a harmless and convenient abuse of language by omitting to indicate the restriction operators. Because of the cocycle relation of Proposition 2.8.(b) we see that \( \{ \nu_{\alpha \beta} \} \) is a Čech 1-cocycle; a lifting exists if and only if it is possible to choose local lifting so that the associated cocycle is 0. If \( X'_\alpha \) are different liftings, we set \( \nu_{\alpha} = \nu(\tilde{X}'_{\alpha}, X'_\alpha) \), \( \nu_{\alpha \beta} = \nu(X'_\alpha, \tilde{X}'_{\beta}) \). Again from Proposition 2.8.(b) and (c) we get that

\[
\nu_{\alpha \beta} = \nu_{\alpha \beta} + \nu_{\alpha} - \nu_{\beta}.
\]

In other words, cocycles associated to different local liftings are cobordant, so the cohomology class \( \omega_{\text{emb}} \in H^1(\mathcal{U}, a \otimes \kappa \, \mathcal{N}) = H^1(X, a \otimes \kappa \, \mathcal{N}) \) of \( \{ \nu_{\alpha \beta} \} \) is independent of the local liftings. Furthermore, if \( \nu_{\alpha \beta} \) is a cocycle in \( \omega_{\text{emb}} \), then there exists a 0-cochain \( \{ \nu_{\alpha} \} \) such that \( \nu_{\alpha \beta} = \nu_{\alpha \beta} + \nu_{\alpha} - \nu_{\beta} \). If we choose liftings \( \tilde{X}_\alpha \) so that \( \nu_{\alpha} = \nu(\tilde{X}_\alpha, X'_\alpha) \) (Proposition 2.8.(d)) we have \( \nu_{\alpha \beta} = \nu(\tilde{X}'_{\beta}, \tilde{X}_\alpha) \). We can state this result as follows.
(2.10) Lemma. The cocycles obtained from different choices of local liftings are exactly the elements of $\omega_{\text{emb}}$.

So a lifting exists if and only if $\omega_{\text{emb}} = 0$, as desired. This concludes the proof of the theorem.

The proof of Theorem 2.5.(b) can be described more conceptually as follows. Consider the sheaf $\mathcal{L}$ of sets on $X$, in which an open subset $U \subseteq X$ is sent to the set of liftings of $U$ to $M'$. Call $i: X_0 \hookrightarrow X$ the embedding. There is an action of the the sheaf $i_*(\mathcal{a} \otimes N_0)$ on $\mathcal{L}$; Theorem 2.5.(c) and Lemma 2.7 imply that $\mathcal{L}$ is a torsor for $i_*(\mathcal{a} \otimes N_0)$. A lifting of $X$ to $M'$ is global section of this torsor, so the obstruction to the existence of a global section is the class $\omega_{\text{emb}}$ of $\mathcal{L}$ in $H^1(X, i_*(\mathcal{a} \otimes N_0)) = \mathcal{a} \otimes N_1(X_0, N_0)$.

The following property of the embedded obstruction $\omega_{\text{emb}}$ follows from its construction and from Proposition 2.8.(f).

(2.11) Lemma. Let $\pi: \widetilde{M}' \rightarrow M'$ be smooth morphism of flat $A'$-schemes of finite type, $j: X \hookrightarrow M'$ and $\overline{j}: X \hookrightarrow \widetilde{M}'$ embeddings such that $\overline{\pi} = j$. Call $N_0$ and $\widetilde{N}_0$ the normal bundles of $X_0$ in $M_0$ and $M_0$ respectively, and $h: H^1(X_0, \widetilde{N}_0) \rightarrow H^1(X_0, N_0)$ the homomorphism obtained from the map $\widetilde{N}_0 \rightarrow N_0$ induced by $\pi$. Then $h$ carries the embedded obstruction of $X$ in $\widetilde{M}'$ to the embedded obstruction of $X$ in $M'$.

This construction has an obvious property of functoriality. Let $B'$ be a local ring, $b \subseteq B'$ be an ideal with $m_{B'}b = 0$, $B = B'/b$. Let $f:A' \rightarrow B'$ a local homomorphism inducing an isomorphism of residue fields, such that $f(a) \subseteq b$. Set $f_*M' = M' \times_{\text{Spec } A'} \text{ Spec } B'$, $f_*M = M \times_{\text{Spec } A} \text{ Spec } B$. $f_*X = X \times_{\text{Spec } A} \text{ Spec } B \subseteq f_*M$. If $X'$ is a lifting of $X$ in $M'$ set $f'_*X' = X' \times_{\text{Spec } A'} \text{ Spec } B' \subseteq f_*M'$; this is a lifting of $f_*X$ in $f_*M'$. Call $g = f \mid _a: a \rightarrow b$ the restriction of $f$.

(2.12) Proposition. (a) Let $X_1'$ and $X_2'$ be liftings of $X$ to $M'$. Then

$$\nu_{f, M'}(f_*X_1', f_*X_2') = (g \otimes \text{id})(\nu_{M'}(X_1', X_2')) \in b \otimes H^0(X_0, N_0).$$

(b) $\omega_{\text{emb}}(f_*X) = (g \otimes \text{id})\omega_{\text{emb}}(X) \in b \otimes H^1(X_0, N_0)$.

The case of local complete intersections in projective spaces is particularly simple.

(2.13) Proposition. Let $X \subseteq P^n_A$ a complete intersection subscheme of codimension $r$, whose ideal is generated by homogeneous polynomials $f_1, \ldots, f_r$. Then any lifting of $X$ to $P^n_{A'}$ is also a complete intersection subscheme, with ideals generated by homogeneous polynomials $f'_1, \ldots, f'_r$ which reduce to $f_1, \ldots, f_r$ modulo $a$.

Proof. Fix liftings $f_1', \ldots, f_r'$ of $f_1, \ldots, f_r$ to homogeneous polynomials with coefficients in $A'$; define these a lifting $X'$ of $X$ to $P^n_{A'}$. If we call $d_1, \ldots, d_r$ the degrees of $f_1, \ldots, f_r$, then the normal bundle $N_0$ decomposes as a direct sum $\sum_{i=1}^r O_{X_0}(d_i)$, so a section of $\mathcal{a} \otimes N_0$ is given by a sequence $(g_1, \ldots, g_r)$ of homogeneous polynomials of degrees $d_1, \ldots, d_r$ with coefficients in $a$. If $X'$ is the subscheme of $P^n_{A'}$ whose ideal is generated by $f'_1 + g_1, \ldots, f'_r + g_r$, an analysis of the construction of $\nu(X', X')$ reveals that $\nu(X', X')$ is precisely $(g_1, \ldots, g_r)$. The result follows from Theorem 2.5.(c).

Here is a typical application of Theorem 2.5.

(2.14) Corollary. Let $\pi: M \rightarrow S$ be a projective morphism, where $S$ is a locally noetherian scheme. Let $s_0 \in S$ be a point, $M_0 = \pi^{-1}(s_0)$. Let $X_0 \subseteq M_0$ be a closed local complete intersection subscheme with normal bundle $N_0$; assume that $H^1(X_0, N_0) = 0$. Then there is an étale neighborhood $s_0 \in U \rightarrow S$ of $s_0$ and a closed subscheme $X \subseteq U \times_S M$ flat over $U$ with $X \cap M_0 = X_0$. 

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Furthermore if $H^0(X_0, N_0) = 0$, then for any other étale neighborhood $s_0 \in U' \to S$, with a closed subscheme $X' \subseteq U' \times_S M$ flat over $U'$ with $X' \cap M_0 = X_0$, there exists a third étale neighborhood $s_0 \in U'' \to S$ with morphisms of neighborhoods $\phi : U'' \to U$ and $\phi' : U'' \to U'$ with $\phi^{-1}(X) = \phi'^{-1}(X')$.

The basic example of this type of situation is a smooth geometrically connected rational curve $X_0$ with self intersection $-1$ on a smooth algebraic surface $M_0$.

**Proof.** Assume that $H^1(X_0, N_0) = 0$. Let $H = \text{Hilb}(M/S) \to S$ be the relative Hilbert scheme, $\xi_0 \in H(\kappa(s_0))$ the point corresponding to $X_0 \subseteq M_0$. Let $\text{Spec} B \hookrightarrow \text{Spec} A$ be a closed embedding of spectra of local artinian rings, with closed point $u_0 \in \text{Spec} B \hookrightarrow \text{Spec} A$, and suppose that it is given a commutative diagram

```
\text{Spec} B \quad \xrightarrow{\beta} \quad H \\
\downarrow \quad \quad \downarrow \\
\text{Spec} A \quad \xrightarrow{\alpha} \quad S
```

such that $\beta(u_0) = \xi_0$, and hence $\alpha(u_0) = s_0$. The morphism $\beta$ corresponds to a subscheme $X_B \subseteq M \times_S \text{Spec} B$ flat over $\text{Spec} B$ such that $X_B \mid u_0 = X_0 \times_{s_0} u_0$. It follows easily from Theorem 2.5.(b), by induction on the length of the kernel of the homomorphism $A \to B$, that there exists a subscheme $X_A \subseteq M \times_S \text{Spec} A$ flat over $\text{Spec} A$ such that $X_A \mid \text{Spec} B = X_B$; furthermore from Theorem 2.5.(c) we get that if $H^0(X_0, N_0) = 0$ then $X_B$ is unique. This means that there exists a morphism $\text{Spec} A \to H$ making the diagram

```
\text{Spec} B \quad \xrightarrow{\beta} \quad H \\
\downarrow \quad \quad \downarrow \\
\text{Spec} A \quad \xrightarrow{\alpha} \quad S
```

commutative; moreover if $H^0(X_0, N_0) = 0$ this morphism is unique. By Grothendieck’s criteria this means that $H$ is smooth at $\xi_0$, and if $H^0(X_0, N_0) = 0$ then it is étale. This implies that there exists an étale neighborhood $s_0 \in U \to S$ and a section $U \to H$ sending $s_0$ to $\xi_0$. By taking for $X \subseteq U \times_S M$ the pullback of the universal subscheme of $H \times_S M$ we have proved the first statement.

For the second statement choose a Zariski neighborhood $\xi_0 \in H' \subseteq H$ which is étale over $S$; by restricting $U$ and $U'$ we may assume that the morphisms $U \to H$ induced by $U' \to H$ induced by the subschemes $X \subseteq U \times_S M$ and $X \subseteq U \times_S M$ have their image in $H'$. Then we can take $U'' = U \times_{H'} U'$.

It is easy to give examples in which the étale neighborhood $U$ can not be taken to be a Zariski neighborhood.
3 Extensions of sheaves

In this section we will discuss briefly the theory of extensions, which we will use to prove Theorem 4.4; I advise the reader to skip it at first and then refer back to it as necessary. Working directly with extensions, instead of elements of groups of extensions, is critical in Section 4, because extensions can be patched together, unlike classes in Ext$^1$.

Let $X$ a topological space with a sheaf of commutative rings $\mathcal{O}$; in this section a sheaf will always be a sheaf of $\mathcal{O}$ modules over $X$, and all homomorphisms will be homomorphisms of sheaves of $\mathcal{O}$-modules. More generally we could work with objects of a fixed abelian category. I hope not to insult the reader by including some very standard definitions.

(3.1) Definition. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves. An extension $(\mathcal{E}, \iota, \kappa)$ of $\mathcal{F}$ by $\mathcal{G}$ is a sheaf $\mathcal{E}$ with two homomorphisms $\iota: \mathcal{G} \to \mathcal{E}$ and $\kappa: \mathcal{E} \to \mathcal{F}$ such that the sequence

$$0 \to \mathcal{G} \xrightarrow{\iota} \mathcal{E} \xrightarrow{\kappa} \mathcal{F} \to 0$$

is exact.

If $(\mathcal{E}_1, \iota_1, \kappa_1)$ and $(\mathcal{E}_2, \iota_2, \kappa_2)$ are extensions $(\mathcal{E}, \iota, \kappa)$ of $\mathcal{F}$ by $\mathcal{G}$, a homomorphism of extension

$$\phi: (\mathcal{E}, \iota, \kappa) \to (\mathcal{E}', \iota', \kappa')$$

is a homomorphism of sheaves $\phi: \mathcal{E} \to \mathcal{E}'$ such that $\phi \iota = \iota'$ and $\kappa' \phi = \kappa$.

We will often talk about an extension $\mathcal{E}$, omitting the homomorphisms $\iota$ and $\kappa$ from the notation; if we need to refer to the them we will call them $\iota_\mathcal{E}$ and $\kappa_\mathcal{E}$.

The content of the five lemma is that any homomorphism of extensions is an isomorphism. With the obvious definition of identities and composition of arrows, extensions form a category, which is a groupoid, i.e., all arrows are invertible. We will denote this category by $\mathcal{E}xt_\mathcal{O}(\mathcal{F}, \mathcal{G})$.

Call $\text{Ext}_\mathcal{O}^1(\mathcal{F}, \mathcal{G})$ the set of isomorphism classes of extensions of $\mathcal{F}$ by $\mathcal{G}$. The link with the usual definition of $\text{Ext}_\mathcal{O}^1(\mathcal{F}, \mathcal{G})$ via injective resolutions is as follows. Take an injective sheaf of $\mathcal{O}$-modules $J$ containing $\mathcal{G}$, and set $Q = J / \mathcal{G}$. Call $\text{Ext}_\mathcal{O}(\mathcal{F}, \mathcal{G})$ the quotient of $\text{Hom}_\mathcal{O}(\mathcal{F}, Q)$ by the subgroup of homomorphism $J \to Q$ which can be lifted to homomorphisms $\mathcal{F} \to J$. Let $\mathcal{E}$ be an extension of $\mathcal{F}$ by $\mathcal{G}$. The embedding $\mathcal{G} \hookrightarrow J$ can be extended to a homomorphism $\mathcal{E} \to J$, because $J$ is injective, which will induce a homomorphism $\mathcal{F} = \mathcal{E} \to Q$. The image of this homomorphism in $\text{Ext}_\mathcal{O}(\mathcal{F}, \mathcal{G})$ only depends on the isomorphism class of $\mathcal{E}$, and the resulting map $\text{Ext}_\mathcal{O}^1(\mathcal{F}, \mathcal{G}) \to \text{Ext}_\mathcal{O}(\mathcal{F}, \mathcal{G})$ is bijective.

This induces a structure of abelian group on $\text{Ext}_\mathcal{O}^1(\mathcal{F}, \mathcal{G})$; this structure can be obtained directly from operations on extensions, as follows.

The identity element corresponds to the split extension $(\mathcal{F} \oplus \mathcal{G}, \iota, \kappa)$, where $\iota(y) = (0, y)$ and $\kappa(x, y) = x$. We will denote the split extension by $0_{\mathcal{F}, \mathcal{G}}$, or simply $0$.

(3.2) Definition. Let $(\mathcal{E}, \iota, \kappa)$ be an extension of $\mathcal{F}$ by $\mathcal{G}$. The opposite $-\mathcal{E}$ is the extension $(\mathcal{E}, -\iota, \kappa)$.

Notice that if $f: \mathcal{E} \to \mathcal{E}'$ is a homomorphism of extensions, then the same sheaf homomorphism $f$ is also a homomorphism from $-\mathcal{E}$ to $-\mathcal{E}'$; we will denote $f$, thought of as a homomorphism from $-\mathcal{E}$ to $-\mathcal{E}'$, by $\Box f$. If we assign to each extension $\mathcal{E}$ the extension $-\mathcal{E}$, and to each homomorphism $f: \mathcal{E} \to \mathcal{E}'$ the homomorphism $\Box f: -\mathcal{E} \to -\mathcal{E}'$, we get a functor from $\mathcal{E}xt_\mathcal{O}(\mathcal{F}, \mathcal{G})$ to itself, whose square is the identity$^\ast$.

$^\ast$ This is a very rare example of an involution in a category, whose square is actually the identity, as opposed to being canonically isomorphic to the identity.
Let \((E_1, \iota_1, \kappa_1), \ldots, (E_r, \iota_r, \kappa_r)\) be extensions of \(F\) by \(G\). Call
\[
\mathcal{A} \subseteq \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r
\]
the subsheaf whose sections over an open subset \(U \subseteq X\) are of the form \((e_1, \ldots, e_r)\), where \(e_i \in E_i(U), \kappa_i(e_1) = \cdots = \kappa_r(e_r)\). Clearly \(\mathcal{A}\) is a subsheaf of \(O\)-modules of \(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r\). Call \(\mathcal{B} \subseteq \mathcal{A}\) the subsheaf of whose sections over \(U \subseteq X\) are of the form \((\iota_1(y_1), \ldots, \iota_r(y_r))\), where \(y_1, \ldots, y_r\) are in \(G(U)\), and \(\sum_{i=1}^r y_i = 0\). Again \(\mathcal{B}\) is a subsheaf of \(O\)-modules of \(\mathcal{A}\). If \((e_1, \ldots, e_r) \in \mathcal{A}(U)\), we will denote the image of \((e_1, \ldots, e_r)\) in \(\mathcal{A}/\mathcal{B}\) by \([e_1, \ldots, e_r]\). Notice that if \(y \in G(U)\), then
\[
[e_1(y), 0, \ldots, 0] = \cdots = [0, \ldots, 0, \iota_r(y)] \in (\mathcal{A}/\mathcal{B})(U).
\]

(3.3) Definition. Let \((E_1, \iota_1, \kappa_1), \ldots, (E_r, \iota_r, \kappa_r)\) be extensions of \(F\) by \(G\). The sum
\[
\sum_{i=1}^r \mathcal{E}_i = \mathcal{E}_1 + \cdots + \mathcal{E}_r
\]
is the extension \((\mathcal{A}/\mathcal{B}, \iota, \kappa)\) of \(F\) by \(G\), where \(\iota: G \to \mathcal{A}/\mathcal{B}\) is defined by \(\iota(y) = [\iota_1(y), 0, \ldots, 0]\), and \(\kappa: \mathcal{A}/\mathcal{B} \to F\) is defined by \(\kappa([e_1, \ldots, e_r]) = \kappa_1(e_1)\).

To avoid confusion it may be appropriate to observe that \(\mathfrak{Ext}_O(F, G)\) is not an additive category, and the sum defined here is neither a categorical sum nor a product.

We leave it to the reader to check that the sum is indeed an extension of \(F\) by \(G\). The sum \(\mathcal{E}_1 + (-\mathcal{E}_2)\) will be denoted with \(\mathcal{E}_1 - \mathcal{E}_2\).

If \(f_1: \mathcal{E}_1 \to \mathcal{E}_1', \ldots, f_r: \mathcal{E}_r \to \mathcal{E}_r'\), are homomorphisms of extensions, there is an induced homomorphism
\[
f_1 + \cdots + f_r = \sum_{i=1}^r f_i: \sum_{i=1}^r \mathcal{E}_i \to \sum_{i=1}^r \mathcal{E}_i'
\]
defined by the obvious rule
\[
\left(\sum_{i=1}^r f_i\right)([e_1, \ldots, e_r]) = [f_1(e_1), \ldots, f_r(e_r)].
\]
This makes the direct sum a functor from \(\mathfrak{Ext}_O(F, G)^r\) to \(\mathfrak{Ext}_O(F, G)\).

The following properties are straightforward to prove and left to the reader.

(3.4) Proposition. (a) If \(E_1, \ldots, E_r, E_{r+1}, \ldots, E_{r+s}\) are extensions of \(F\) by \(G\), there is a functorial isomorphism of extensions
\[
\sum_{i=1}^r \mathcal{E}_i + \sum_{i=r+1}^{r+s} \mathcal{E}_i \simeq \sum_{i=1}^{r+s} \mathcal{E}_i
\]
which sends \([e_1, \ldots, e_r], [e_{r+1}, \ldots, e_{r+s}]\) into \([e_1, \ldots, e_{r+s}]\).

(b) If \(E_1\) and \(E_2\) are extensions, there is functorial isomorphism
\[
\chi_{E_1, E_2}: E_1 \oplus E_2 \simeq E_2 \oplus E_1
\]
which sends \([e_1, e_2]\) into \([e_2, e_1]\).
(c) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are extensions, then
\[-(\mathcal{E}_1 + \mathcal{E}_2) = (-\mathcal{E}_1) + (-\mathcal{E}_2).\]

(d) For each extension $\mathcal{E}$ there are functorial isomorphisms
\[\epsilon_{\mathcal{E}}^0 : \mathcal{E} + \mathcal{O}_{\mathcal{F}, \mathcal{G}} \simeq \mathcal{E}\]
and
\[\epsilon_{\mathcal{E}}^0 : \mathcal{O}_{\mathcal{F}, \mathcal{G}} + \mathcal{E} \simeq \mathcal{E}\]
which send $[e, (x, y)]$ into $e + \iota_{\mathcal{E}}(y)$, and $[(x, y), e]$ into $e + \iota_{\mathcal{E}}(y)$, respectively.

(e) If $\mathcal{E}$ is an extension, then there is a functorial isomorphism of extensions
\[\delta_{\mathcal{E}} : \mathcal{E} - \mathcal{E} \simeq \mathcal{O}_{\mathcal{F}, \mathcal{G}},\]
such that a section $[e_1, e_2]$ of $\mathcal{E} - \mathcal{E}$ is sent to $(\kappa(e_1), y)$, if $y$ is the section of $\mathcal{G}$ such that $\iota_{\mathcal{E}}(y) = e_1 - e_2$.

We will not distinguishing between $\sum_{i=1}^r \mathcal{E}_i + \sum_{i=r+1} \mathcal{E}_i$ and $\mathcal{E}$, but we will use the isomorphism of (a) to identify them.

The functoriality statement in part (d) should be interpreted as saying that if $f : \mathcal{E} \to \mathcal{E}'$ is a homomorphism of extensions, then
\[\epsilon_{\mathcal{E}'}^0 \circ (f + \id_0) = f \circ \epsilon_{\mathcal{E}}^0 : \mathcal{E} + \mathcal{O}_{\mathcal{F}, \mathcal{G}} \to \mathcal{E}',\]
and analogously for $\epsilon^f$. For part (e) it means that $f : \mathcal{E} \to \mathcal{E}'$ is a homomorphism of extensions, then
\[\delta_{\mathcal{E}} \circ (f + \ominus f) = \delta_{\mathcal{E}}'.\]

(3.5) Definition. Let $\mathcal{E}$ be an extension of $\mathcal{G}$. A splitting of $\mathcal{E}$ is a sheaf homomorphism $s : \mathcal{E} \to \mathcal{F}$ such that $s \iota_{\mathcal{E}} = \id_{\mathcal{F}}$.

If we are given a splitting $s : \mathcal{E} \to \mathcal{F}$, then the sheaf homomorphism $f_s : \mathcal{E} \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}$ defined by $f_s(e) = (s(e), \kappa_{\mathcal{E}}(e))$ is an isomorphism of extensions. Conversely, an isomorphism of extensions $\mathcal{E} \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}$ is of the form $f_s$ for a unique splitting $s : \mathcal{E} \to \mathcal{F}$, so we will identify splittings of $\mathcal{E}$ and isomorphisms $\mathcal{E} \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}$.

Given two splittings $s_1 : \mathcal{E}_1 \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}$ and $s_2 : \mathcal{E}_2 \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}$, we can define their sum $\mathcal{E}_1 + \mathcal{E}_2 \to \mathcal{F}$ by the formula $(s_1 + s_2)(e_1, e_2) = s_1(e_1) + s_2(e_2)$; it is readily checked that $s_1 + s_2$ is a well-defined splitting of $\mathcal{E}_1 + \mathcal{E}_2$. In terms of isomorphisms of extensions, we have that
\[f_{s_1 + s_2} = \epsilon_{\mathcal{E}_1}^0 \circ (f_{s_1} + f_{s_2}) : \mathcal{E}_1 + \mathcal{E}_2 \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}.\]

(3.6) Proposition. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be extensions. There is a canonical bijective correspondence between splittings of $\mathcal{E}_1 - \mathcal{E}_2$ and isomorphisms $\mathcal{E}_1 \simeq \mathcal{E}_2$.

Proof. Given an isomorphism $f : \mathcal{E}_1 \to \mathcal{E}_2$ we can associate to it the splitting
\[\delta_{\mathcal{E}_2} \circ (f + \id_{\mathcal{E}_2}) : \mathcal{E}_1 - \mathcal{E}_2 \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}.\]

Conversely, to each splitting $s : \mathcal{E}_1 - \mathcal{E}_2 \to \mathcal{O}_{\mathcal{F}, \mathcal{G}}$ we can associate
\[\epsilon_{\mathcal{E}_2}^0 \circ (s + \id_{\mathcal{E}_2}) \circ (\id_{\mathcal{E}_1} + \delta_{\mathcal{E}_2}) \circ (\epsilon_{\mathcal{E}_1}^0)^{-1} : \mathcal{E}_1 \to \mathcal{E}_2.\]

It is easy to check that these two operations are inverse to each other.
The groups \( \text{Ext}^1_\mathcal{O}(\mathcal{F}, \mathcal{G}) \) are functorial both in \( \mathcal{F} \) and \( \mathcal{G} \), and this functoriality already exists at the level of extensions. Let us begin with the functoriality in \( \mathcal{G} \).

Let \( g: \mathcal{G} \to \mathcal{G}' \) be a homomorphism of sheaves of \( \mathcal{O} \)-modules, and let \( \mathcal{E} \) be an extension of \( \mathcal{F} \) by \( \mathcal{G} \). We define the pushforward \( g_* \mathcal{E} \) of \( \mathcal{F} \) by \( \mathcal{G}' \) as follows. As a sheaf, \( g_* \mathcal{E} \) is the direct sum \( \mathcal{G}' \oplus \mathcal{E} \), divided by the image of \( \mathcal{G} \) under the homomorphism \( \mathcal{G} \to \mathcal{G}' \oplus \mathcal{E} \) which sends a local section \( y \) to \( (g(y), -i_\mathcal{E}(y)) \). The homomorphism \( i_{g_* \mathcal{E}}: \mathcal{G}' \to g_* \mathcal{E} \) sends \( y' \) into the class \([y', 0]\), while \( \kappa_{g_* \mathcal{E}}: g_* \mathcal{E} \to \mathcal{F} \) sends \([y', e]\) into \( \kappa_\mathcal{E}(e) \). It is an easy exercise to show that \( g_* \mathcal{E} \) is an extension of \( \mathcal{F} \) by \( \mathcal{G}' \).

Notice that there a homomorphism \( \phi: \mathcal{E} \to g_* \mathcal{E} \) which sends a local section \( e \) into \([0, e]\). This homomorphism fits into a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{i_\mathcal{E}} & \mathcal{G} & \xrightarrow{i_{g_* \mathcal{E}}} & \mathcal{E} & \xrightarrow{\kappa_\mathcal{E}} & \mathcal{F} & \xrightarrow{\kappa_{g_* \mathcal{E}}} & 0 \\
\downarrow{g} & & \downarrow{\phi} & & \downarrow{\kappa_{g_* \mathcal{E}}} & & \downarrow{\kappa_{g_* \mathcal{E}}} & & \downarrow{0} \\
0 & \xrightarrow{i_{g_* \mathcal{E}}} & \mathcal{G}' & \xrightarrow{i_{g_* \mathcal{E}}} & g_* \mathcal{E} & \xrightarrow{\kappa_{g_* \mathcal{E}}} & \mathcal{F} & & 0,
\end{array}
\]

The extension \( g_* \mathcal{E} \) is the “only” extension with this property, in the following sense.

\[\text{(3.7) Lemma.} \quad \text{Let } \mathcal{E}' \text{ an extension of } \mathcal{F} \text{ by } \mathcal{G}', \text{ and assume that there is a commutative diagram}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{i_\mathcal{E}} & \mathcal{G} & \xrightarrow{i_{g_* \mathcal{E}}} & \mathcal{E} & \xrightarrow{\kappa_\mathcal{E}} & \mathcal{F} & \xrightarrow{\kappa_{g_* \mathcal{E}}} & 0 \\
\downarrow{g} & & \downarrow{\phi} & & \downarrow{\kappa_{g_* \mathcal{E}}} & & \downarrow{\kappa_{g_* \mathcal{E}}} & & \downarrow{0} \\
0 & \xrightarrow{i_{g_* \mathcal{E}}} & \mathcal{G}' & \xrightarrow{i_{g_* \mathcal{E}}} & g_* \mathcal{E} & \xrightarrow{\kappa_{g_* \mathcal{E}}} & \mathcal{F} & & 0,
\end{array}
\]

Then is a unique homomorphism of extensions \( \psi: g_* \mathcal{E} \simeq \mathcal{E}' \) such that \( \psi \phi = \phi' \).

\[\text{Proof.} \quad \text{It is easy to see that if } \psi \text{ exists it must have the form } \psi([y', e]) = \iota_{\mathcal{E}'}(y') + \sigma'(e). \text{ Conversely one checks that this formula yields a well defined homomorphism } \psi \text{ such that } \psi \sigma = \sigma'. \]

This construction has the following properties.

\[\text{(3.8) Proposition.} \quad \text{Let } g, g_1, g_2: \mathcal{G} \to \mathcal{G}' \text{ be homomorphisms of sheaves of } \mathcal{O}-\text{modules, } \mathcal{E}, \mathcal{E}_1 \text{ and } \mathcal{E}_2 \text{ extensions of } \mathcal{F} \text{ by } \mathcal{G}.
\]

(a) If \( 0: \mathcal{G} \to \mathcal{G}' \) is the zero homomorphism, then \( 0_* \mathcal{E} \) is canonically isomorphic to \( 0_{\mathcal{F}, \mathcal{G}'} \).

(b) \( (-g)_* \mathcal{E} = -(g_* \mathcal{E}) \).

(c) There is a canonical isomorphism of extensions of \( \mathcal{F} \) by \( \mathcal{G}' \)

\[
g_* (\mathcal{E}_1 + \mathcal{E}_2) \simeq g_* \mathcal{E}_1 + g_* \mathcal{E}_2.
\]

(d) There is a canonical isomorphism of extensions of \( \mathcal{F} \) by \( \mathcal{G}' \)

\[
(g_1 + g_2)_* \mathcal{E} \simeq g_1_* \mathcal{E} + g_2_* \mathcal{E}.
\]

(e) The boundary homomorphism

\[
\partial: \text{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{G}') \to \text{Ext}^1_\mathcal{O}(\mathcal{F}, \mathcal{G}')
\]

coming from the sequence

\[
0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0
\]

sends \( g \in \text{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{G}') \) into the isomorphism class of \( g_* \mathcal{E} \).

\[\text{Proof.} \quad \text{Part (b) is straightforward.}
\]

For part (c) we apply Lemma 3.7 with \( \mathcal{E}' = g_* \mathcal{E}_1 + g_* \mathcal{E}_2 \), and \( \phi': \mathcal{E}_1 + \mathcal{E}_2 \to \mathcal{E}' \) defined by \( \phi'([e_1, e_2]) = \left[[0, e_1], [0, e_2]\right] \).

For part (d) we take \( \mathcal{E}' = g_1_* \mathcal{E} + g_2_* \mathcal{E}, \phi'([e_1, e_2]) = \left[[0, e_1], [0, e_2]\right] \).

Part (e) is standard.
If \( s: g_* \mathcal{E} \to \mathcal{G}' \) of \( g_* \mathcal{E} \) is a splitting of \( g_* \mathcal{E} \), then the homomorphism \( \sigma = \phi \circ s \) has the property that \( \sigma \circ \iota_{\mathcal{E}} = g \). Conversely, given a homomorphism \( \sigma = \phi \circ s \) such that \( \sigma \circ \iota_{\mathcal{E}} = g \), we get a splitting \( s: g_* \mathcal{E} \to \mathcal{G}' \) by defining \( s([y', e]) = y' + \sigma(e) \).

**Lemma**. There is a natural bijective correspondence of splittings of \( g_* \mathcal{E} \) with homomorphisms \( \sigma = \phi \circ s \) such that \( \sigma \circ \iota_{\mathcal{E}} = g \).

The functoriality in \( \mathcal{F} \) is analogous. Let \( \mathcal{E} \) be an extension of \( \mathcal{F} \) by \( \mathcal{G} \), \( f: \mathcal{F}' \to \mathcal{F} \) be a homomorphism. We define the pullback \( f^* \mathcal{E} \) of \( \mathcal{F}' \) by \( \mathcal{G} \) as the subsheaf of \( \mathcal{F}' \oplus \mathcal{E} \) whose sections are of type \((x', e)\) with \( f(x') = \iota_{\mathcal{E}}(e) \); it is a subsheaf of \( \mathcal{O} \)-modules of \( \mathcal{F}' \oplus \mathcal{E} \). The homomorphism \( \iota_{f^*} \mathcal{E} \) from \( \mathcal{G} \) to \( f^* \mathcal{E} \) sends \( y \) to \((0, \iota_{\mathcal{E}}(g))\), and \( \kappa_{f^*} \mathcal{E} \) from \( f^* \mathcal{E} \) to \( \mathcal{F}' \) sends \((x', e)\) to \( x' \). We leave it to the reader to check that \( f^* \mathcal{E} \) is indeed an extension of \( \mathcal{F}' \) by \( \mathcal{G} \).

The homomorphism \( \phi: f^* \mathcal{E} \to \mathcal{E} \) which sends \((x', e)\) into \( e \) fits into a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{G} & \xrightarrow{\iota_{f^*}} & f^* \mathcal{E} & \xrightarrow{\kappa_{f^*}} & \mathcal{F}' & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{G} & \xrightarrow{\iota_{\mathcal{E}}} & \mathcal{E} & \xrightarrow{\kappa_{\mathcal{E}}} & \mathcal{F} & \rightarrow & 0.
\end{array}
\]

The following results are dual to the results stated for pushforwards. The proofs are left to the reader.

**Lemma**. Let \( \mathcal{E}' \) an extension of \( \mathcal{F}' \) by \( \mathcal{G} \), and assume that there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{G} & \xrightarrow{\iota_{\mathcal{E}'}} & \mathcal{E}' & \xrightarrow{\kappa_{\mathcal{E}'}} & \mathcal{F}' & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{G} & \xrightarrow{\iota_{\mathcal{E}}} & \mathcal{E} & \xrightarrow{\kappa_{\mathcal{E}}} & \mathcal{F} & \rightarrow & 0.
\end{array}
\]

Then is a unique homomorphism of extensions \( \psi: \mathcal{E}' \simeq f^* \mathcal{E}' \) such that \( \phi \psi = \phi' \).

This construction has the following properties.

**Proposition**. Let \( f, f_1, f_2: \mathcal{G} \to \mathcal{G}' \) be homomorphisms of sheaves of \( \mathcal{O} \)-modules, \( \mathcal{E}, \mathcal{E}_1 \) and \( \mathcal{E}_2 \) extensions of \( \mathcal{F} \) by \( \mathcal{G} \).

(a) If \( 0: \mathcal{F}' \to \mathcal{F} \) is the zero homomorphism, then \( 0^* \mathcal{E} \) is canonically isomorphic to \( 0_{\mathcal{F'}, \mathcal{G}} \).

(b) \( (-f)^* \mathcal{E} = -(f^* \mathcal{E}) \).

(c) There is a canonical isomorphism of extensions of \( \mathcal{F}' \) by \( \mathcal{G} \),

\[
f^*(\mathcal{E}_1 + \mathcal{E}_2) \simeq f^* \mathcal{E}_1 + f^* \mathcal{E}_2.
\]

(d) There is a canonical isomorphism of extensions of \( \mathcal{F}' \) by \( \mathcal{G} \),

\[
(f_1 + f_2)^* \mathcal{E} \simeq f_1^* \mathcal{E} + f_2^* \mathcal{E}.
\]

The following concept arises naturally in our construction of the obstruction in Theorem 4.4.(c).

**Definition**. Let \( \mathcal{U} = \{X_\alpha\} \) be an open covering of \( X \). An extension cocycle

\[
(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})
\]

of \( \mathcal{F} \) by \( \mathcal{G} \) on \( \mathcal{U} \) is a collection of extensions \( \{\mathcal{E}_{\alpha\beta}\} \) of \( \mathcal{F} | \ X_\alpha \cap X_\beta \) by \( \mathcal{G} | \ X_\alpha \cap X_\beta \), and isomorphisms \( F_{\alpha\beta\gamma}: \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \simeq \mathcal{E}_{\alpha\gamma} \) on \( X_\alpha \cap X_\beta \cap X_\gamma \), such that for any quadruple \( \alpha, \beta, \gamma, \delta \) we have

\[
F_{\alpha\gamma\delta} \circ (F_{\alpha\beta\gamma} + \text{id}_{\mathcal{E}_{\alpha\gamma}}) = F_{\alpha\beta\delta} \circ (\text{id}_{\mathcal{E}_{\alpha\delta}} + F_{\beta\gamma\delta}).
\]
as a homomorphism \( \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} + \mathcal{E}_{\gamma\delta} \rightarrow \mathcal{E}_{\alpha\delta} \).

There is an obvious notion of isomorphism of extension cocycles. An isomorphism
\[
\phi: (\mathcal{E}_{\alpha\beta}, \{F_{\alpha\beta}\}) \simeq (\mathcal{E}'_{\alpha\beta}, \{F'_{\alpha\beta}\})
\]
consists of a collection of isomorphisms of extensions \( \phi_{\alpha\beta}: \mathcal{E}_{\alpha\beta} \simeq \mathcal{E}'_{\alpha\beta} \) such that
\[
\phi_{\alpha\gamma} \circ F_{\alpha\beta\gamma} = F'_{\alpha\beta\gamma} \circ (\phi_{\alpha\beta} + \phi_{\beta\delta})
\]
for all \( \alpha, \beta \) and \( \gamma \).

Furthermore, extension cocycles can be summed; we define
\[
(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta}\}) + (\{\mathcal{E}'_{\alpha\beta}\}, \{F'_{\alpha\beta}\}) = (\{\mathcal{E}_{\alpha\beta} + \mathcal{E}'_{\alpha\beta}\}, \{\tilde{F}_{\alpha\beta}\})
\]
where
\[
\tilde{F}_{\alpha\beta} = (F_{\alpha\beta\gamma} + F'_{\alpha\beta\gamma}) \circ (\text{id}_{\mathcal{E}_{\alpha\beta}} + \chi_{\mathcal{E}_{\alpha\beta}, \mathcal{E}'_{\alpha\beta}} - \mathcal{E}_{\beta\gamma} + \text{id}_{\mathcal{E}_{\beta\gamma}})
\]
and leave it to the reader to check that this sum is still an extension cocycle.

Consider the set of isomorphism classes of extension cocycles. The operation of sum introduced above makes it into an abelian group.

The zero is represented by the class of the trivial extension cocycle, that is, the cocycle \( (\{0\}, \{\epsilon_0\}) \) in which all the \( \mathcal{E}_{\alpha\beta} \) are trivial, and the \( F_{\alpha\beta\gamma} \) are all \( \epsilon_0: 0 + 0 \rightarrow 0 \). The isomorphism
\[
(\{0\}, \{\epsilon_0\}) + (\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta}\}) \simeq (\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta}\})
\]
is given by
\[
\epsilon_{\mathcal{E}_{\alpha\beta}}: 0 + \mathcal{E}_{\alpha\beta} \simeq \mathcal{E}_{\alpha\beta}.
\]

The inverse of the class of an extension cocycle \( (\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\}) \) is \( (-\mathcal{E}_{\alpha\beta}, \ominus F_{\alpha\beta\gamma}) \), and the isomorphism
\[
(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\}) + (\{-\mathcal{E}_{\alpha\beta}\}, \ominus F_{\alpha\beta\gamma}) \simeq (\{0\}, \{\epsilon_0\})
\]
is given by
\[
\delta_{\mathcal{E}_{\alpha\beta}}: \mathcal{E}_{\alpha\beta} - \mathcal{E}_{\alpha\beta} \simeq 0.
\]

Given a collection of extensions \( \{\mathcal{E}_\alpha\} \) of \( \mathcal{F} | X_\alpha \) by \( \mathcal{G} | X_\alpha \) on \( X_\alpha \), we can define its boundary
\[
\partial \{\mathcal{E}_\alpha\} = (\{\mathcal{E}_\alpha - \mathcal{E}_\beta\}, F_{\alpha\beta\gamma})
\]
where
\[
F_{\alpha\beta\gamma} = (\epsilon_{\mathcal{E}_\alpha} + \text{id}_{-\mathcal{E}_\gamma}) \circ (\text{id}_{\mathcal{E}_\alpha} + \delta_{\mathcal{E}_\beta} + \delta_{-\mathcal{E}_\gamma}) \circ \mathcal{E}_\alpha - \mathcal{E}_\beta + \mathcal{E}_\gamma \rightarrow \mathcal{E}_\alpha - \mathcal{E}_\gamma.
\]

One check that \( \partial \{\mathcal{E}_\alpha\} \) is an obstruction cocycle, and that there is a canonical isomorphism of extension cocycles
\[
\partial \{\mathcal{E}_\alpha + \mathcal{E}'_\alpha\} \simeq \partial \{\mathcal{E}_\alpha\} + \partial \{\mathcal{E}'_\alpha\}
\]
given by
\[
\text{id}_{\mathcal{E}_\alpha} + \chi_{\mathcal{E}_\alpha, -\mathcal{E}_\beta} + \text{id}_{\mathcal{E}'_\alpha} : \mathcal{E}_\alpha + \mathcal{E}'_\alpha - (\mathcal{E}_\beta + \mathcal{E}'_\beta) = \mathcal{E}_\alpha + \mathcal{E}'_\alpha - \mathcal{E}_\beta - \mathcal{E}'_\beta \rightarrow \mathcal{E}_\alpha - \mathcal{E}_\beta + \mathcal{E}'_\alpha - \mathcal{E}'_\beta.
\]
We say that an extension cocycle is a **boundary** if it is isomorphic to the boundary of a collection of extensions \( \{E_\alpha\} \).

The isomorphism classes of boundaries form a subgroup of the group of isomorphism classes of extension cocycles; the quotient group will be called the **group of extension classes**, and will be denoted by \( \mathfrak{E}_O(U; F, G) \).

Let \( U' = \{X'_\alpha\} \) be a refinement of \( U \); there is a function \( \rho: U' \to U \) such that \( X'_\alpha \subseteq X_{\rho(\alpha')} \). An extension cocycle \( \{E_{\alpha, \beta}\} \) on \( U \) induces an extension cocycle \( \{E'_{\alpha', \beta'}\} \), where \( E'_{\alpha', \beta'} \) is the restriction of the extension \( E_{\rho(\alpha') \rho(\beta')} \) to \( X'_{\alpha'} \cap X'_{\beta'} \), and the \( F_{\alpha, \beta} \) are also restricted. Boundaries are obviously brought to boundaries, so we get a restriction map \( \mathfrak{E}_O(U; F, G) \to \mathfrak{E}_O(U'; F, G) \); as we shall see during the proof of Theorem 3.13, this map only depends on \( U \) and \( U' \), and not on the map \( \rho \). Accepting this for the moment, there is a limit abelian group

\[
\mathfrak{E}_O(F, G) = \lim_{U} \mathfrak{E}_O(U; F, G).
\]

**Theorem (3.13)** There is canonical group isomorphism of \( \mathfrak{E}_O(F, G) \) with the kernel of the localization map \( \operatorname{Ext}^2_O(F, G) \to \operatorname{H}^0(X, \operatorname{Ext}^2_O(F, G)) \).

(b) The natural map \( \mathfrak{E}_O(U; F, G) \to \mathfrak{E}_O(F, G) \) is injective.

The group \( \mathfrak{E}_O(F, G) \) can be interpreted as the first cohomology group of the Picard stack \( \mathfrak{E}_{\text{xt}}(F, G) \) of extensions of \( F \) and \( G \) (see [Deligne]); the theorem does not seem to appear in this form in the literature, but, as it was pointed out to me by L. Breen, it was more or less known: see for example [Retakh] and [Ullrich].

**Proof.** We will use the following notation. Let \( E, E_1, E_2 \) be extensions of \( F \) by \( G \), and let \( j: G \to J \) be a homomorphism of sheaves.

If \( \sigma_1: E_1 \to J \) and \( \sigma_2: E_2 \to J \) are homomorphisms with \( \sigma_i \circ \iota_E = j \), we will take their sum \( \sigma_1 + \sigma_2: E_1 + E_2 \to J \) to be the homomorphism defined by the formula \( (\sigma_1 + \sigma_2)([e_1, e_2]) = \sigma_1(e_1) + \sigma_2(e_2) \). If we think of \( \sigma_1 \) as a splitting of \( j_*(\iota_E) \) (Lemma 3.9), then \( \sigma_1 + \sigma_2 \) can be thought of as their sum as a splitting of \( j_*(E_1 + E_2) = j_*(E_1) + j_*(E_2) \) (Proposition 3.8.(c)).

On the other hand, let \( \sigma_1, \sigma_2: E \to J \) be homomorphisms with \( \sigma_i \circ \iota_E = j \). Their difference \( \sigma_1 - \sigma_2: E \to J \) is a homomorphism with \( (\sigma_1 - \sigma_2) \circ \iota_E = 0 \), so there is a unique homomorphism \( \tau: E \to J \) such that \( \tau \circ \iota_E = \sigma_1 - \sigma_2 \). This \( \tau \) we will also denote by \( \sigma_1 - \sigma_2 \).

Finally, if \( \sigma: E \to J \) is such that \( \sigma \circ \iota_E = j \), and \( \tau: F \to J \) is a homomorphism, we will write \( \sigma + \tau \) to mean \( \sigma + \tau \circ \iota_E \).

Let \( J \) be an injective sheaf of \( \mathcal{O} \)-modules containing \( G, Q = J/G \). Call \( j: G \to J \) the inclusion, \( \pi: J \to Q \) the projection. Then the boundary operator

\[
\partial: \operatorname{Ext}^1_{\mathcal{O}}(F, Q) \to \operatorname{Ext}^2_{\mathcal{O}}(F, G)
\]

is an isomorphism, and induces an isomorphism of \( \operatorname{Ext}^1_{\mathcal{O}}(F, Q) \) with \( \operatorname{Ext}^2_{\mathcal{O}}(F, G) \), such the diagram

\[
\begin{array}{ccc}
\operatorname{Ext}^1_{\mathcal{O}}(F, Q) & \longrightarrow & \operatorname{Ext}^2_{\mathcal{O}}(F, G) \\
\downarrow & & \downarrow \\
\operatorname{H}^0(X, \operatorname{Ext}^1_{\mathcal{O}}(F, Q)) & \longrightarrow & \operatorname{H}^0(X, \operatorname{Ext}^2_{\mathcal{O}}(F, G))
\end{array}
\]

where the rows are boundary maps and the columns are localization maps, commutes. Hence the kernel of the left column is isomorphic to the kernel of the right column. But from the spectral sequence

\[
E^{pq}_2 = \operatorname{H}^p(X, \operatorname{Ext}^q_{\mathcal{O}}(F, G)) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}}(F, G)
\]
we get an exact sequence

\[
0 \rightarrow \mathcal{H}^1(X, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})) \rightarrow \text{Ext}_\mathcal{O}^1(\mathcal{F}, \mathcal{Q}) \rightarrow \mathcal{H}^0(X, \text{Ext}_\mathcal{O}^1(\mathcal{F}, \mathcal{Q})),
\]

and an isomorphism of \(\mathcal{H}^1(X, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}))\) with the kernel of the localization map from \(\text{Ext}_\mathcal{O}^2(\mathcal{F}, \mathcal{G})\) to \(\mathcal{H}^0(X, \text{Ext}_\mathcal{O}^2(\mathcal{F}, \mathcal{G}))\). We will prove both statements, and the fact that the restriction map from \(\mathcal{E}_\mathcal{O}(\mathcal{U}; \mathcal{F}, \mathcal{G})\) to \(\mathcal{E}_\mathcal{O}(\mathcal{U}'; \mathcal{F}, \mathcal{G})\) only depends on \(\mathcal{U}\) and \(\mathcal{U}'\), and not on the function \(\mathcal{U}' \rightarrow \mathcal{U}\), by proving the following.

\textbf{(3.14) Lemma.} There is a canonical isomorphism

\[
\mathcal{E}_\mathcal{O}(\mathcal{U}; \mathcal{F}, \mathcal{G}) \simeq \tilde{\mathcal{H}}^1(\mathcal{U}, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{Q}))
\]

which is compatible with restriction maps.

If we remember that the restriction map

\[
\tilde{\mathcal{H}}^1(\mathcal{U}, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{Q})) \rightarrow \tilde{\mathcal{H}}^1(\mathcal{U}', \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{Q}))
\]

independent of the function \(\mathcal{U}' \rightarrow \mathcal{U}\), that the inductive limit of the Čech cohomology groups is the ordinary cohomology, and that the map

\[
\tilde{\mathcal{H}}^1(\mathcal{U}, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{Q})) \rightarrow \mathcal{H}^1(X, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{Q}))
\]

is always injective, we attain the proof of the theorem, together with the statement about the restriction map from \(\mathcal{E}_\mathcal{O}(\mathcal{U}; \mathcal{F}, \mathcal{G})\) to \(\mathcal{E}_\mathcal{O}(\mathcal{U}'; \mathcal{F}, \mathcal{G})\) only depending on \(\mathcal{U}\) and \(\mathcal{U}'\).

\textbf{Proof of 3.14.} Call \(j: \mathcal{G} \rightarrow \mathcal{J}\) the inclusion. Because \(\mathcal{J}\) is injective, for each \(\alpha\) and \(\beta\) we can find a homomorphism \(\sigma_{\alpha\beta}: \mathcal{E}_{\alpha\beta} \rightarrow \mathcal{J}\) such that \(\sigma_{\alpha\beta} \circ \iota_{\mathcal{E}_{\alpha\beta}} = j\). Let us check that we can do it coherently, in the following sense.

\textbf{(3.15) Lemma.} We can find homomorphisms \(\sigma_{\alpha\beta}: \mathcal{E}_{\alpha\beta} \rightarrow \mathcal{J}\) for each \(\alpha, \beta\), such that \(\sigma_{\alpha\beta} \circ \iota_{\mathcal{E}_{\alpha\beta}} = j\), and such that

\[
\sigma_{\alpha\beta} + \sigma_{\beta\gamma} = \sigma_{\alpha\gamma} \circ F_{\alpha\beta\gamma}: \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \rightarrow \mathcal{J}
\]

for each triple \(\alpha, \beta, \gamma\).

We will call such a collection \(\{\sigma_{\alpha\beta}\}\) a function from the extension cocycle \(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\}\) to \(\mathcal{J}\).

\textbf{Proof.} Choose homomorphisms \(\sigma_{\alpha\beta}: \mathcal{E}_{\alpha\beta} \rightarrow \mathcal{J}\) in such a way that \(\sigma_{\alpha\beta} \circ \iota_{\mathcal{E}_{\alpha\beta}} = j\) for all \(\alpha\) and \(\beta\), and consider the homomorphisms

\[
\tau_{\alpha\beta\gamma} = (\sigma_{\alpha\beta} + \sigma_{\beta\gamma}) - \sigma_{\alpha\gamma} \circ F_{\alpha\beta\gamma}: \mathcal{E}_{\alpha\beta} \rightarrow \mathcal{J}.
\]

\textbf{(3.16) Lemma.} The collection \(\{\tau_{\alpha\beta\gamma}\}\) is a Čech 2-cocycle in the sheaf \(\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{J})\), that is,

\[
\tau_{\alpha\beta\gamma} + \tau_{\alpha\gamma\delta} = \tau_{\alpha\beta\delta} + \tau_{\beta\gamma\delta}
\]

for all \(\alpha, \beta, \gamma\) and \(\delta\).

\textbf{Proof.} Let \(x\) be a local section of \(\mathcal{F}\), and choose sections \(e_{\alpha\beta}, e_{\beta\gamma}\) and \(e_{\gamma\delta}\) of \(\mathcal{E}_{\alpha\beta}, \mathcal{E}_{\beta\gamma}\) and \(\mathcal{E}_{\gamma\delta}\) whose image in \(\mathcal{F}\) is \(x\). Set \(e_{\alpha\gamma} = F_{\alpha\beta\gamma}([e_{\alpha\beta}, e_{\beta\gamma}]), e_{\beta\delta} = F_{\beta\gamma\delta}([e_{\beta\gamma}, e_{\gamma\delta}]), e_{\alpha\delta} = F_{\alpha\beta\delta}([e_{\alpha\beta}, e_{\beta\delta}])\). Observe that the cocycle condition on the \(F_{\alpha\beta\gamma}\) (Definition 3.12) is tailor made to give us

\[
e_{\alpha\delta} = F_{\alpha\beta\delta}([e_{\alpha\beta}, e_{\beta\delta}]) = F_{\alpha\gamma\delta}([e_{\alpha\gamma}, e_{\gamma\delta}]).
\]
Then
\[
\tau_{\alpha\beta\gamma} + \tau_{\alpha\gamma\delta} - \tau_{\alpha\beta\delta} - \tau_{\beta\gamma\delta}(x) = \sigma_{\alpha\beta}(e_{\alpha\beta}) + \sigma_{\beta\gamma}(e_{\beta\gamma}) - \sigma_{\alpha\gamma}(F_{\alpha\beta\gamma}(e_{\alpha\beta}, e_{\beta\gamma}))
\]
\[
+ \sigma_{\alpha\gamma}(e_{\alpha\gamma}) + \sigma_{\gamma\delta}(e_{\gamma\delta}) - \sigma_{\alpha\delta}(F_{\alpha\gamma\delta}(e_{\alpha\gamma}, e_{\gamma\delta}))
\]
\[
- \sigma_{\alpha\beta}(e_{\alpha\beta}) - \sigma_{\beta\delta}(e_{\beta\delta}) + \sigma_{\alpha\delta}(F_{\alpha\beta\delta}(e_{\alpha\beta}, e_{\beta\delta}))
\]
\[
- \sigma_{\beta\gamma}(e_{\beta\gamma}) - \sigma_{\gamma\delta}(e_{\gamma\delta}) + \sigma_{\beta\delta}(F_{\beta\gamma\delta}(e_{\beta\gamma}, e_{\gamma\delta}))
\]
\[
= \sigma_{\alpha\beta}(e_{\alpha\beta}) + \sigma_{\beta\gamma}(e_{\beta\gamma}) - \sigma_{\alpha\gamma}(e_{\alpha\gamma})
\]
\[
+ \sigma_{\alpha\gamma}(e_{\alpha\gamma}) + \sigma_{\gamma\delta}(e_{\gamma\delta}) - \sigma_{\alpha\delta}(e_{\alpha\delta})
\]
\[
- \sigma_{\alpha\beta}(e_{\alpha\beta}) - \sigma_{\beta\delta}(e_{\beta\delta}) + \sigma_{\alpha\delta}(e_{\alpha\delta})
\]
\[
- \sigma_{\beta\gamma}(e_{\beta\gamma}) - \sigma_{\gamma\delta}(e_{\gamma\delta}) + \sigma_{\beta\delta}(e_{\beta\delta})
\]
\[
= 0.
\]

This proves the lemma.

Now observe that the sheaf $\text{Hom}_O(F, J)$ is flabby, because $J$ is injective, so its second Čech cohomology group $\tilde{H}^2(U, \text{Hom}_O(F, J))$ is 0. Hence we can find a 1-cocycle $\{\eta_{\alpha\beta}\}$ of $\text{Hom}_O(F, J)$ such that $\tau_{\alpha\beta\gamma} = \tau_{\alpha\beta} + \tau_{\beta\gamma} - \tau_{\alpha\gamma}$. If we set $\tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} + \tau_{\alpha\beta}$ we see easily that the condition
\[
\tilde{\sigma}_{\alpha\beta} + \tilde{\sigma}_{\beta\gamma} = \tilde{\sigma}_{\alpha\gamma} \circ F_{\alpha\beta\gamma} : E_{\alpha\beta} + E_{\beta\gamma} \to J
\]
is satisfied. This proves Lemma 3.16.

Choose a function $\{\sigma_{\alpha\beta}\}$ from $\{E_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\}$ to $J$. The composition of the $\sigma_{\alpha\beta}$ with the projection $\pi, J \to Q$ send $G$ to 0, and therefore induce homomorphisms $\eta_{\alpha\beta} : F \to Q$ satisfying the cocycle condition $\eta_{\alpha\beta} + \eta_{\beta\gamma} = \eta_{\alpha\gamma}$. So we have associated to the extension cocycle $\{\{E_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\}\}$ and the functions $\{\sigma_{\alpha\beta}\}$ an element $\omega$ of the Čech cohomology group $\tilde{H}^1(U, \text{Hom}_O(F, Q))$. Let us check that this element does not depend on the function $\{\sigma_{\alpha\beta}\}$. Let $\{\sigma'_{\alpha\beta}\}$ be another function, and call $\omega'$ the element of $\tilde{H}^1(U, \text{Hom}_O(F, Q))$ associated with $\{\{E_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\}\}$ and $\{\sigma'_{\alpha\beta}\}$. Consider the cocycle $\{\sigma_{\alpha\beta} - \sigma'_{\alpha\beta}\}$ in $\text{Hom}_O(F, J)$. The compositions $\pi \circ (\sigma_{\alpha\beta} - \sigma'_{\alpha\beta})$ define a cocycle in $\text{Hom}_O(F, Q)$ whose cohomology class is clearly $\omega - \omega'$. But $\tilde{H}^1(U, \text{Hom}_O(F, J))$ is 0, because $\text{Hom}_O(F, J)$ is flabby, and so $\omega - \omega' = 0$.

This construction gives a mapping from $\mathcal{E}_O(U; F, G)$ to $\tilde{H}^1(U, \text{Hom}_O(F, Q))$. We leave to the reader to check that it is a homomorphism. Let us prove that it is bijective.

Let $(\{E_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$ be an extension cocycle whose associated cohomology element is 0. Choose a function $\{\sigma_{\alpha\beta}\}$ from $\{\{E_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\}\}$ to $J$, and call $\{\eta_{\alpha\beta}\}$ the associated cocycle in $\text{Hom}_O(F, Q)$. Choose a 0-cochain $\{\eta_a\}$ such that $\eta_{\alpha\beta} = \eta_{\beta\gamma} - \eta_{\alpha\gamma}$ for all $a$ and $\beta$. We think of $J$ as an extension of $G$ by $Q$, and set $\mathcal{E}_a = \eta_a^*J$. Observe that by definition of $\eta_{\alpha\beta}$ the diagram
\[
\begin{array}{cccc}
0 & \to & G & \to & \mathcal{E}_{\alpha\beta} & \to & F & \to & 0 \\
\uparrow & & \uparrow & & \sigma_{\alpha\beta} & & \uparrow & & \eta_{\alpha\beta} & \\
0 & \to & G & \to & J & \to & Q & \to & 0
\end{array}
\]
commutes, so, by Lemma 3.10, there is an induced isomorphism of $\mathcal{E}_{\alpha\beta}$ with $\eta_{\alpha\beta}^*J = (\eta_{\beta\gamma} - \eta_{\alpha\gamma})^*J$. But by Proposition 3.11,(b) and (d) there is a canonical isomorphism of $(\eta_{\beta\gamma} - \eta_{\alpha\gamma})^*J$ with $\eta_{\alpha\beta}^*J - \eta_{\alpha\gamma}^*J = E_{\beta\gamma} - E_{\alpha\gamma}$. We leave it to the reader to show that this collection of isomorphisms is an isomorphism of the extension cocycle $(\{E_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$ with the boundary $\partial\{E_a\}$. 

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To prove surjectivity, let \( \omega \in \check{\text{H}}^1(\mathcal{U}; \mathcal{F}, \mathcal{Q}) \) be a class represented by a cocycle \( \{ \eta_{\alpha \beta} \} \), and set \( \mathcal{E}_{\alpha \beta} = \eta_{\alpha \beta}^* \mathcal{J} \). The isomorphism

\[
F_{\alpha \beta \gamma} : \mathcal{E}_{\alpha \beta} + \mathcal{E}_{\beta \gamma} = \eta_{\alpha \beta}^* \mathcal{J} + \eta_{\beta \gamma}^* \mathcal{J} \simeq (\eta_{\alpha \beta} + \eta_{\beta \gamma})^* \mathcal{J} = \eta_{\alpha \gamma}^* \mathcal{J} = \mathcal{E}_{\alpha \gamma}
\]

is the inverse of the one in Proposition 3.11.(d). One checks that the \( F_{\alpha \beta \gamma} \) satisfy the cocycle condition of Definition 3.12.

It is easy to see that the isomorphism between \( \mathcal{E}_\mathcal{O}(\mathcal{U}; \mathcal{F}, \mathcal{G}) \) and \( \check{\text{H}}^1(\mathcal{U}; \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{Q})) \) that we have just defined is compatible with refinement. This completes the proof of Lemma 3.15.

To finish the proof of Theorem 3.13 we only need to check that the resulting isomorphism between \( \mathcal{E}_\mathcal{O}(\mathcal{F}, \mathcal{G}) \) and the kernel of the localization map \( \text{Ext}^2_\mathcal{O}(\mathcal{F}, \mathcal{G}) \to \text{H}^0(\mathcal{X}, \text{Ext}^2_\mathcal{O}(\mathcal{F}, \mathcal{G})) \) is independent of the choice of \( \mathcal{J} \). This is straightforward and left to the reader.

In order to extend Theorem 4.4 to maps, we need to generalize what has been done above to complexes. There is no problem in defining extensions of complexes, but in general extensions do not represent enough classes in \( \text{Ext}^1 \). This problem, however, will not arise in the case we are interested in, as we will see.

If \( \mathcal{F}' \) and \( \mathcal{G}' \) are complexes of sheaves, we define an extension \((\mathcal{E}', \iota, \kappa)\) of \( \mathcal{F}' \) by \( \mathcal{G}' \) is a complex of sheaves \( \mathcal{E}' \), together with homomorphisms of complexes \( \iota: \mathcal{G}' \to \mathcal{E}' \) and \( \kappa: \mathcal{E}' \to \mathcal{F}' \) such that the sequence

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{G}' & \xrightarrow{\iota} & \mathcal{E}' & \xrightarrow{\kappa} & \mathcal{F}' & \rightarrow & 0
\end{array}
\]

is exact for all integers \( n \).

All of the theory that we have developed above before Theorem 3.13 goes through without changes to this more general case, when we substitute everywhere complexes to sheaves, and homomorphisms of complexes to homomorphism of complexes.

Let \( \mathcal{G} \) be a sheaf, \( \mathcal{F}' \) a complex of sheaves, \( \mathcal{J}' \) an injective resolution of \( \mathcal{G} \). Recall that with \( \mathcal{J}'[n] \) we denote the complex with \( \mathcal{J}'[n]^i = \mathcal{J}^{n+i} \), the differential being \( (\phi^{i+1} - (i+1)\phi^i) \). Then \( \text{Ext}^n_\mathcal{O}(\mathcal{F}', \mathcal{G}) \) is by definition the group of homomorphisms of complexes \( \mathcal{F}' \to \mathcal{J}' \), modulo homotopy. Of course we could take \( \mathcal{G} \) itself to be a complex bounded below, but we will not need this more general case.

Think of \( \mathcal{G} \) itself as a complex, which is zero everywhere except in degree 0, and let \( \mathcal{E}' \) be an extension of \( \mathcal{F}' \) by \( \mathcal{G} \). To this we can associate an element of \( \text{Ext}^1_\mathcal{O}(\mathcal{F}', \mathcal{G}) \) as follows. The embedding of complexes \( \mathcal{G} \to \mathcal{J}' \) can be extended to homomorphism of \( \mathbb{Z} \)-graded sheaves \( \phi: \mathcal{E}' \to \mathcal{J}' \). To \( \phi \) we associate the homomorphism of complexes \( \partial_{\mathcal{J}'}, \phi = \phi \partial_{\mathcal{E}'}: \mathcal{E}' \to \mathcal{J}'[1] \). It sends \( \mathcal{G} \) to 0, and so induces a homomorphism \( \psi: \mathcal{F}' \to \mathcal{J}'[1] \). The homotopy class of this complex does not depend on \( \phi \). Thus we have defined a function from the set of isomorphism classes of extensions of \( \mathcal{F}' \) by \( \mathcal{G} \) to \( \text{Ext}^1_\mathcal{O}(\mathcal{F}', \mathcal{G}) \). By standard arguments one shows that this map is a group homomorphism; but in general it is not injective nor surjective.

**Lemma.** Assume that \( \mathcal{F}'^i = 0 \) for \( i > 0 \). Then every element of \( \text{Ext}^1_\mathcal{O}(\mathcal{F}', \mathcal{G}) \) is represented by a unique isomorphism class of extensions.

**Proof.** Let \( \mathcal{E}' \) be an extension of \( \mathcal{F}' \) by \( \mathcal{G} \) such that the corresponding element in \( \text{Ext}^1_\mathcal{O}(\mathcal{F}', \mathcal{G}) \) is 0. Then I claim that the embedding \( \mathcal{G} \to \mathcal{J}' \) extend to a homomorphism of complexes \( \mathcal{E}' \to \mathcal{J}' \). In fact, extend \( \mathcal{G} \to \mathcal{J}' \) to a homomorphism of graded sheaves \( \phi: \mathcal{E}' \to \mathcal{J}' \), and call \( \psi: \mathcal{F}' \to \mathcal{J}'[1] \) the homomorphism of complexes induced by \( \partial_{\mathcal{J}'}, \phi = \phi \partial_{\mathcal{E}'} \), as above. There is a homomorphism of graded sheaves \( \lambda: \mathcal{F}' \to \mathcal{J}' \) with \( \psi = \partial_{\mathcal{J}'}, \lambda - \lambda \partial_{\mathcal{F}'} \). Then \( \phi' = \phi - \lambda \kappa_{\mathcal{F}'} \) is a homomorphism of complexes extending the embedding \( \mathcal{G} \hookrightarrow \mathcal{J}' \).
The homomorphism $\phi'$ induces a homomorphism of sheaves $\mathcal{H}^0(\mathcal{E}') \to \mathcal{H}^0(\mathcal{J}') = \mathcal{G}$. But $\mathcal{E}^i = 0$ for $i > 0$, so this homomorphism $\mathcal{H}^0(\mathcal{E}') \to \mathcal{G}$ yields a homomorphism $\mathcal{E}' \to \mathcal{G}$ which splits the sequence.

Now start from an element of $\text{Ext}^1_\mathcal{O}(\mathcal{F}', \mathcal{G})$, represented by a homomorphism $f: \mathcal{F}' \to \mathcal{J}'[1]$. Let $\mathcal{I}' \subseteq \mathcal{J}$ be the subcomplex with $\mathcal{I}'^0 = \mathcal{J}'^0$, $\mathcal{I}'^1 = \text{im}(\mathcal{J}'^1 \to \mathcal{J}'^2) = \text{im}(\mathcal{J}'^0 \to \mathcal{J}'^1)$, and $\mathcal{I}'^i = 0$ for $i > 1$. Because of the condition on $\mathcal{F}'$ we see that the homomorphism $f: \mathcal{F}' \to \mathcal{J}'[1]$ factors through $\mathcal{I}'[1]$. There is an extension of $\mathcal{I}'[1]$ by $\mathcal{G}$ given by the diagram

\[
\begin{array}{cccccccc}
\vdots & & & & & & & \\
\downarrow & & & & & & & \\
\deg -2 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & & & & & & \\
\deg -1 & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{J}'^0 & \longrightarrow & \mathcal{I}'^0 & \longrightarrow & 0 \\
\downarrow & & & & & & & & & & \\
\deg 0 & 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{J}'^0 & \longrightarrow & \mathcal{I}'^1 & \longrightarrow & 0 \\
\downarrow & & & & & & & & & & \\
\deg 1 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & & & & & & & & & \\
\vdots & & & & & & & & & & \\
\end{array}
\]

One checks that the pullback of this extension to $\mathcal{F}'$ represents the given class of $\text{Ext}^1_\mathcal{O}(\mathcal{F}', \mathcal{G})$. ♡

With this at our disposal the whole theory in this section can be extended to extensions of complexes with no terms in positive degree by sheaves. The definition of extension cocycle extends to this case also, and Theorem 3.13 remains true, and but the proof has to be changed slightly. Here is the main point. If $\mathcal{A}'$ and $\mathcal{B}'$ are complexes of sheaves, we will use the notation $\mathcal{Hom}_\mathcal{O}(\mathcal{A}', \mathcal{B}')$ to denote the sheaf of honest homomorphisms of complexes of $\mathcal{A}'$ into $\mathcal{B}'$, which is in general very different from $\text{Ext}_\mathcal{O}^0(\mathcal{A}', \mathcal{B}')$, and $\mathcal{Hom}_\mathcal{O}^1(\mathcal{A}', \mathcal{B}')$ to denote the sheaf of homomorphisms of graded sheaves of $\mathcal{A}'$ into $\mathcal{B}'$.

(3.18) Lemma. Let $\mathcal{F}'$ be a complex of sheaves bounded above, $\mathcal{G}$ a sheaf, $\mathcal{J}'$ an injective resolution of $\mathcal{G}$. Then there is a canonical isomorphism of $H^1(X, \mathcal{Hom}_\mathcal{O}(\mathcal{F}', \mathcal{J}'[n]))$ with the kernel of the localization map from $\text{Ext}^{n+1}_\mathcal{O}(\mathcal{F}', \mathcal{G})$ to $H^0(X, \text{Ext}^{n+1}_\mathcal{O}(\mathcal{F}', \mathcal{G}))$.

Proof. Call $A$ the kernel of the localization map $\text{Ext}^{n+1}_\mathcal{O}(\mathcal{F}', \mathcal{G}) \to H^0(X, \text{Ext}^{n+1}_\mathcal{O}(\mathcal{F}', \mathcal{G}))$; let us define a homomorphism from $A$ to $H^1(X, \mathcal{Hom}_\mathcal{O}(\mathcal{F}', \mathcal{J}'[n + 1]))$ as follows. Let $\xi$ be an element of $A$; then $\xi$ is represented by a homomorphism $\phi: \mathcal{F}' \to \mathcal{J}'[n]$. There is an open covering $\{X_\alpha\}$ of $X$ and homomorphisms of graded sheaves $\phi_\alpha: \mathcal{F}'|_{X_\alpha} \to \mathcal{J}'[n]|_{X_\alpha}$ with $\partial_{\mathcal{J}'} \phi_\alpha - \phi_\alpha \partial_{\mathcal{F}'} = \phi|_{X_\alpha}$. Define

$$
\phi_{\alpha \beta} = \phi_\alpha - \phi_\beta \in \text{Hom}_\mathcal{O}(\mathcal{F}'|_{X_\alpha \cap X_\beta}, \mathcal{J}'[n]|_{X_\alpha \cap X_\beta});
$$

then $\{\phi_{\alpha \beta}\}$ is a Čech cocycle in $\mathcal{Hom}_\mathcal{O}(\mathcal{F}', \mathcal{J}')$. It is easy to check that its class

$$
[[\phi_{\alpha \beta}]] \in H^1(X, \mathcal{Hom}_\mathcal{O}(\mathcal{F}', \mathcal{J}'[n]))
$$

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is independent of the choice of the $\phi_\alpha$, and that in this way we obtain an injective group homomorphism from $A$ into $\text{H}^1(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^*, \mathcal{J}^*[n]))$.

Let us check surjectivity. Take a class in $\text{H}^1(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^*, \mathcal{J}^*[n]))$, represented to some cocycle $\phi_{\alpha\beta}$ relative to some open covering $\{X_\alpha\}$ of $X$. The sheaf

$$\mathcal{H}om^0_{\mathcal{O}}(\mathcal{F}^*, \mathcal{J}^*[n]) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^i, \mathcal{J}^{i+n})$$

of homomorphisms of graded groups is a finite direct sum of flabby sheaves, hence all of its higher Čech cohomology groups are 0; so we can find homomorphisms of graded sheaves $\phi_\alpha: \mathcal{F}^* \to \mathcal{J}^*[n]$ on $X_\alpha \cap X_\beta$ with $\phi_{\alpha\beta} = \phi_\beta - \phi_\alpha$. The homomorphisms of complexes $\partial_{\mathcal{J}^*} \phi_\alpha - \phi_\alpha \partial_{\mathcal{F}^*}: \mathcal{F}^* \to \mathcal{J}^*[n+1]$ patch together to yield a homomorphism of complexes $\phi: \mathcal{F}^* \to \mathcal{J}^*[n+1]$. The class in $\text{Ext}^{n+1}(\mathcal{G}^*, \mathcal{G}) \to \text{H}^0(X, \mathcal{E}xt^{n+1}(\mathcal{G}^*, \mathcal{G}))$ maps to our class in $\text{H}^1(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^*, \mathcal{J}^*[n]))$, and this concludes the proof.

Once the have this, to prove the analogue of Theorem 3.13 it is enough to prove the following.

(3.19) Lemma. There is a canonical isomorphism

$$\mathcal{E}_{\mathcal{O}}(\mathcal{U}; \mathcal{F}^*, \mathcal{G}) \simeq \check{\text{H}}^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{J}^*[1]))$$

which is compatible with restriction maps.

Let $\{(\mathcal{E}^*_{\alpha\beta}, F_{\alpha\beta\gamma})\}$ be an extension cocycle of $\mathcal{F}^*$ by $\mathcal{G}$. Extend the homomorphism from $\mathcal{G}$ to $\mathcal{J}^*$ to homomorphisms of graded sheaves $\sigma_{\alpha\beta}: \mathcal{E}^*_{\alpha\beta} \to \mathcal{J}^*$ satisfying the compatibility condition of Lemma 3.15. The homomorphisms of complexes

$$\partial_{\mathcal{J}^*} \sigma_{\alpha\beta} - \sigma_{\alpha\beta} \partial_{\mathcal{F}^*}: \mathcal{E}^*_{\alpha\beta} \to \mathcal{J}^*[1]$$

sends $\mathcal{G}$ to 0, and induces homomorphisms $\tau_{\alpha\beta}: \mathcal{F}^* \to \mathcal{J}^*[1]$ on $X_\alpha \cap X_\beta$ satisfying the cocycle condition. These give the class in $\text{H}^1(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{J}^*[1]))$ associated with $\{(\mathcal{E}^*_{\alpha\beta}, F_{\alpha\beta\gamma})\}$. The rest of the proof of Lemma 3.15 goes through with obvious modifications.

4 Abstract liftings of local complete intersections

In this section we analyze abstract liftings. Notation 2.1 is still in force.

(4.1) Hypotheses. Let $X$ be a flat local complete intersection scheme of finite type over $A$. Assume also that $X_0$ is generically smooth over $\kappa$.

Recall that for $X$ to be a local complete intersection means that if, locally on $X$, we factor the structure morphism $X \to \text{Spec } A$ as an embedding $X \hookrightarrow P$ followed by a smooth morphism $P \to \text{Spec } A$, then $X$ is a local complete intersection in $P$; this condition is independent of the factorization. Also, if $\kappa$ is perfect then for $X_0$ to be generically smooth over $\kappa$ simply means to be reduced.

(4.2) Definition. An abstract lifting of $X$ to $A'$ is a flat scheme $X'$ over $A'$ with an embedding $X \hookrightarrow X'$, which induces an isomorphism of $A$-schemes of $X$ with $X' | _{| \text{Spec } A}$.

(4.3) Definition. Let $X'_1$ and $X'_2$ be two abstract liftings of $X$. An isomorphism of $X'_1$ with $X'_2$ is an isomorphism $X'_1 \simeq X'_2$ of schemes over $A'$ which induces the identity on $X$.

(4.4) Theorem. Call $\Omega_{X_0/\kappa}$ the sheaf of Kähler differentials of $X_0$ relative to $\kappa$. 

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Let \( A \) be an isomorphism of sheaves of \( \mathcal{O} \) on \( X \) isomorphisms between liftings of \( \mathcal{O} \), making it into a principal homogeneous space.

The condition that \( X_0 \) be generically smooth is necessary for the statement to hold. For example, consider that case that \( A = \kappa, A' = \kappa[x]/(x^2), X = X_0 = \text{Spec}(\kappa[t]/(t^2)) \). Then \( X \) has many non-isomorphic abstract lifting to \( A' \), given by \( X' = \text{Spec}(\kappa[x, t]/(t^2 - ax)) \), for \( a \in \kappa \); but \( \text{Ext}^1_{\mathcal{O}_X}(\Omega_{X_0/\kappa}, \mathcal{O}_X) \) is 0.

Let us place ourselves again in the situation of Hypotheses 2.2. We need to understand the isomorphisms between liftings of \( X \) to \( M' \). Let \( \Phi : X'_2 \simeq X'_1 \) be such an isomorphism; \( \Phi \) will yield an isomorphism of sheaves of \( A' \)-algebras \( \phi : \mathcal{O}_{X'_1} \simeq \mathcal{O}_{X'_2} \), inducing the identity on \( \mathcal{O}_X = \mathcal{O}_{X'_1}/a\mathcal{O}_{X'_1} = \mathcal{O}_{X'_2}/a\mathcal{O}_{X'_2} \). Consider the two projections \( \pi_i : \mathcal{O}_{M'} \to \mathcal{O}_{X'_i} \); the difference

\[
D = \pi_2 - \phi \pi_1 : \mathcal{O}_{M'} \to \mathcal{O}_{X'_2}
\]

will have its image inside \( a\mathcal{O}_{X'_2} = a \otimes_{A'} \mathcal{O}_{X'_2} = a \otimes_{\kappa} \mathcal{O}_{X_0} \), so we think of \( D \) as a function \( \mathcal{O}_{M'} \to a \otimes_{\kappa} \mathcal{O}_{X_0} \). If \( f \) and \( g \) are local sections of \( \mathcal{O}_{M'} \), we have

\[
D(fg) = \pi_2(fg) - \phi \pi_1(fg) = \pi_2(f)\pi_2(g) - \phi \pi_1(f)\phi \pi_1(g) + \pi_2(f)\phi \pi_1(g) - \phi \pi_1(f)\phi \pi_1(g) = \pi_2(f)D(g) + D(f)\phi \pi_1(g).
\]

Notice that the two \( \mathcal{O}_{M'} \)-module structures on \( a \otimes_{\kappa} \mathcal{O}_{X_0} = a\mathcal{O}_{X'_2} \) induced by \( \pi_1 \) and by \( \phi \pi_2 \) coincide, so we can think of \( D \) as derivation of \( \mathcal{O}_{M'} \) into the \( \mathcal{O}_{M'} \)-module \( a \otimes_{\kappa} \mathcal{O}_{X_0} \). Because \( \phi \) is \( A' \)-linear we have also that \( D \) kills the elements of \( A' \), so it is an \( A' \)-derivation. Such a derivation will send all the elements of the annihilator of \( \mathcal{O}_{M_0} \) to 0, so we think of \( D \) as a \( \kappa \)-derivation of \( \mathcal{O}_{M_0} \) into \( a \otimes_{\kappa} \mathcal{O}_{X_0} \), that is, as an element of \( \text{Der}_\kappa(\mathcal{O}_{M_0}, a \otimes_{\kappa} \mathcal{O}_{X_0}) = \text{Hom}_{\mathcal{O}_X}(\Omega_{M_0/\kappa} | x_0, a \otimes_{\kappa} \mathcal{O}_{X_0}) \). The derivation \( D \) is not arbitrary.

From the homomorphism \( \mathcal{I}_0/\mathcal{I}_0^2 \to \Omega_{M_0/\kappa} | x_0 \) we get a restriction map

\[
\text{Hom}_{\mathcal{O}_X}(\Omega_{M_0/\kappa} | x_0, a \otimes_{\kappa} \mathcal{O}_{X_0}) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{I}_0/\mathcal{I}_0^2, a \otimes_{\kappa} \mathcal{O}_{X_0}) = H^0(X_0, a \otimes_{\kappa} \mathcal{N}_0).
\]

(4.5) Lemma. The restriction of \( D \) to \( \mathcal{I}_0/\mathcal{I}_0^2 \) is \( \nu(X'_1, X'_2) \).

Proof. Let \( \mathcal{I}_1' \) and \( \mathcal{I}_2' \) be the ideals of \( X'_1 \) and \( X'_2 \) respectively. Take a local section \( f \) of \( \mathcal{I}_1 \), and lift it to two local sections \( f'_1 \) of \( \mathcal{I}_1' \) and \( f'_2 \) of \( \mathcal{I}_2' \). Then

\[
Df = \pi_2(f'_1) - \phi \pi_1(f'_1) = \pi_2(f'_2) = \pi_2(f'_1 - f'_2).
\]

But because \( f'_1 - f'_2 \) is a local section of \( a\mathcal{O}_{M'} = a \otimes_{\kappa} \mathcal{O}_{M_0} \), the homomorphism \( \pi_2 \) will send it into its image in \( a \otimes_{\kappa} \mathcal{O}_{X_0} \), which is by definition \( \nu(X'_1, X'_2)(f) \).
We leave it as an exercise for the interested reader to check that by assigning $D$ to $\Phi$ we get a bijective correspondence between isomorphisms of liftings $X'_2 \simeq X'_1$ with elements of

$$\text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{M_0/\kappa} | x_0, a \otimes_\kappa \mathcal{O}_{X_0})$$

whose image in $\text{H}^0(X_0, a \otimes \mathcal{N}_0)$ is $\nu(X'_1, X'_2)$. This bijective correspondence has the following properties.

(4.6) Proposition. There is a bijective correspondence between isomorphisms of abstract liftings $X'_2 \simeq X'_1$ and elements of $\text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{M_0/\kappa} | x_0, a \otimes_\kappa \mathcal{O}_{X_0})$ whose image in $\text{H}^0(X_0, a \otimes \mathcal{N}_0)$ is $\nu(X'_1, X'_2)$, with the following properties.

(a) The identity $\text{id}_{X'}: X' \simeq X'$ corresponds to $0$.

(b) If $\Phi_1: X'_2 \simeq X'_1$ and $\Phi_2: X'_3 \simeq X'_2$ are isomorphisms of abstract liftings and $D_1$, $D_2$ are the corresponding elements of $\text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{M_0/\kappa} | x_0, a \otimes_\kappa \mathcal{O}_{X_0})$, then the composition $\Phi_1 \circ \Phi_2$ corresponds to $D_1 + D_2$.

(c) Let $Y$ be an open subset of $X$, $Y'_1$ and $Y'_2$ the restrictions of $X'_1$ and $X'_2$ to $Y$, $\Phi: X'_1 \simeq X'_2$ an isomorphism of liftings, corresponding to $D \in \text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{M_0/\kappa} | x_0, a \otimes_\kappa \mathcal{O}_{X_0})$. Then the element of $\text{Hom}_{\mathcal{O}_{Y_0}}(\Omega_{M_0/\kappa} | y_0, a \otimes_\kappa \mathcal{O}_{Y_0})$ corresponding to the restriction $\Phi \mid Y'_1: Y'_1 \rightarrow Y'_2$ is the restriction of $D$.

The rest of the proof is entirely based on the constructions of Section 3, to which the reader should refer for the notation.

Let us put ourselves in the situation of Hypotheses 4.1. Let $X'$ be an abstract lifting of $X$; if we set $M' = X' = X'_1 = X'_2$ in Proposition 4.6 we see that we have proved Theorem 4.4.(b).

Let $X'_1$ and $X'_2$ be two abstract liftings of $X$; we want to know when they are isomorphic. Assume that there exists a smooth morphism $P' \rightarrow \text{Spec} A'$ and an embedding of $X$ into $P = P' \mid_{\text{Spec} A}$ which lifts to embeddings of $X'_1$ and $X'_2$ into $P'$. Choose such liftings; we obtain an $\mathcal{O}_{X_0}$-linear map $\nu(X'_1, X'_2): \mathcal{I}_0/\mathcal{I}_0^2 \rightarrow a \otimes_\kappa \mathcal{O}_{X_0}$.

The following lemma is where we use the hypothesis that $X_0$ be generically smooth.

(4.7) Lemma. The usual sequence

$$0 \rightarrow \mathcal{I}_0/\mathcal{I}_0^2 \rightarrow \Omega_{P_0/\kappa} | x_0 \rightarrow \Omega_{X_0/\kappa} \rightarrow 0$$

is exact.

We will call this sequence the fundamental exact sequence for the embedding of $X$ in $P$.

Proof. This is standard, except for the injectivity of the first arrow. But it is well known that this arrow is injective where $X_0$ is smooth, so its kernel is concentrated on a nowhere dense closed subset of $X_0$, because $X_0$ is generically smooth. Since $\mathcal{I}_0/\mathcal{I}_0^2$ is locally free on $X_0$ and $X_0$ has no embedded point, being a local complete intersection scheme over a field, we see that the kernel must actually be $0$.

If $X_0$ is not generically smooth then the lemma does not hold in general; the remedy is to substitute the complex $\mathcal{I}_0/\mathcal{I}_0^2 \rightarrow \Omega_{P_0/\kappa} | x_0$ for the sheaf $\Omega_{X_0/\kappa}$ in the statement of the theorem, and in the remainder of the proof. This complex is defined up to a canonical isomorphism in the derived category of coherent sheaves on $X_0$, and it is the simplest nontrivial example of a cotangent complex.

We define

$$\mathcal{E}(X'_1, X'_2) \overset{\text{def}}{=} \nu(X'_1, X'_2)_*(\Omega_{P_0/\kappa} | x_0)$$

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as an extension of $\Omega_{X_0/\kappa}$ by $a \otimes_{\kappa} O_{X_0}$. We need to check that this extension is independent of the choices made, so choose two smooth morphisms $P'_1 \to \text{Spec } A'$ and $P'_2 \to \text{Spec } A'$, embeddings $X \to P_i$ and liftings $X'_i \to P'_i$ and $X''_i \to P'_i$. These induce an embedding $X \to X'_1 \times_{\text{Spec } A'} X'_2$ and liftings $X'_1 \to P'_1 \times_{\text{Spec } A'} P'_2$ and $X'_2 \to P'_1 \times_{\text{Spec } A'} P'_2$. Let $C_1$, $C_2$ and $C_{12}$ be the conormal bundles of $X_0$ in $(P_1)_0$, $(P_2)_0$, and $(P'_1 \times_{\text{Spec } A'} P'_2)_0 = (P_1)_0 \times_{\kappa} (P_2)_0$, respectively. Denote by $\nu_i : C_i \to a \otimes_{\kappa} O_{X_0}$ the corresponding sections of the normal bundles, and set $E_i = \nu_{i*} \Omega_{(P_i)_0/\kappa}$, $E_{12} = \nu_{12*} \Omega_{(P_1)_0 \times_{\kappa} (P_2)_0/\kappa}$. There are homomorphisms $\phi_i : C_i \to C_{12}$ fitting into commutative diagrams with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C_1 & \rightarrow & \Omega_{(P_1)_0/\kappa} \mid X_0 & \rightarrow & \Omega_{X_0/\kappa} & \rightarrow & 0 \\
\phi & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C_{12} & \rightarrow & \Omega_{(P_1)_0 \times_{\kappa} (P_2)_0/\kappa} \mid X_0 & \rightarrow & \Omega_{X_0/\kappa} & \rightarrow & 0 \\
\nu_{12} & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & a \otimes_{\kappa} O_{X_0} & \rightarrow & E_{12} & \rightarrow & \Omega_{X_0/\kappa} & \rightarrow & 0
\end{array}
\]

Because of Proposition 2.8.(f) we have that $\nu_{12} \circ \phi_i = \nu_i : C_i \to a \otimes_{\kappa} O_{X_0}$. By Lemma 3.7 the diagram above induces an isomorphism of extensions $\alpha_i : E_i \simeq E_{12}$; we define the canonical isomorphism between $E_2$ and $E_1$ to be $\alpha_{12} = \alpha_1 \circ \alpha_2^{-1}$. One checks that these isomorphisms satisfy the cocycle condition, that is, if $P'_1$, $P'_2$, and $P'_3$ are smooth over $A'$, and are given embeddings $X \to P_i$ and liftings $X'_i \to P'_i$ for $i = 1, 2$ and $j = 1, 2, 3$, then $\alpha_{13} = \alpha_{12} \circ \alpha_{23}$, with the obvious notation.

From the construction above one sees clearly that if $Y$ is an open subset of $X$, $Y'_1$ and $Y'_2$ are the restrictions of $X'_1$ and $X'_2$ to $Y$, then the extension associated to the abstract liftings $Y'_1$ and $Y'_2$ of $Y$ is simply the restriction to $Y$ of the extension associated to $X'_1$ and $X'_2$.

Such a morphism $P' \to \text{Spec } A'$ might not exist globally, but certainly exists locally.

(4.8) **Lemma.** Assume that $X$ is affine, and $X \to A^n_A$ is a closed embedding. If $X'$ is an abstract lifting of $X$, then the embedding $X \to A^n_A$ can be extended to an embedding $X' \to A^n_A$.

**Proof.** Set $X = \text{Spec } B$; the closed embedding $X \to A^n_A$ corresponds to a surjective homomorphism of $A$-algebras $A[x_1, \ldots, x_n] \to B$. Set $X' = \text{Spec } B'$; by choosing liftings of the images of the $x_i$ to $B'$ we obtain a homomorphism of $A'$-algebras $A'[x_1, \ldots, x_n] \to B'$, which is easily shown to be surjective, due once again to the fact that $a$ is nilpotent.

In the general case, let $\{X_\alpha\}$ be a covering of $X$ with open affine subschemes, and call $X'_\alpha$ the restriction of $X'_1$ to $X_\alpha$. For each $X_\alpha$ choose a closed embedding $X_\alpha \to A^n_{A'}$, and extend to embeddings of $X'_\alpha$ into $A^n_{A'}$.

For each $\alpha$ set $E_\alpha = \mathcal{E}(X'_\alpha, X'_\alpha)$. Because of the above there are isomorphisms of extensions of $E_\alpha \mid X_\alpha \cap X_\beta$ with $E_\beta \mid X_\alpha \cap X_\beta$, satisfying the cocycle condition. We use these to glue the $E_\alpha$ together into an extension $E(X'_1, X'_2)$. It is clear that this is defined up to a canonical isomorphism.

(4.9) **Proposition.** Given two abstract liftings $X'_1$ and $X'_2$, there is an extension $E(X'_1, X'_2)$ of $\Omega_{X_0/\kappa}$ by $a \otimes_{\kappa} O_{X_0}$, well defined up to canonical isomorphisms, with the following properties.

(a) If $X'$ is a lifting, $E(X', X')$ is canonically isomorphic to $0_{\Omega_{X_0/\kappa}, a \otimes_{\kappa} O_{X_0}}$.

(b) $E(X'_2, X'_1) = -E(X'_1, X'_2)$.

(c) Given three abstract liftings $X'_1$, $X'_2$ and $X'_3$, there a canonical isomorphism

$$\mathcal{F}(X'_1, X'_2, X'_3) : E(X'_1, X'_2) + E(X'_2, X'_3) \simeq E(X'_1, X'_3).$$

If $X'_1$, $X'_2$, $X'_3$ and $X'_4$ are abstract liftings, then

$$\mathcal{F}(X'_1, X'_2, X'_3) \circ (\mathcal{F}(X'_1, X'_2, X'_3) + \text{id}(X'_2, X'_3)) = \mathcal{F}(X'_1, X'_2, X'_3) \circ \text{id}(X'_1, X'_2) + \mathcal{F}(X'_1, X'_2).$$
as a homomorphism $E(X'_1, X'_2) + E(X'_2, X'_4) + E(X'_4, X'_1) \to E(X'_1, X'_4)$.

(d) If $Y$ is an open subscheme of $X$ and $Y'_1, Y'_2$ are the restrictions of $X'_1$ and $X'_2$ to $Y$, then $E(Y'_1, Y'_2)$ is the restriction of $E(X'_1, X'_2)$ to $Y$.

(e) If there is given a smooth morphism $P' \to \text{Spec} A'$ and an embedding of $X$ into $P = P'|_{\text{Spec} A}$ which lifts to embeddings of $X'_1$ and $X'_2$ into $P'$, then

$$E(X'_1, X'_2) = \nu(X'_1, X'_2)_*(\Omega_{P_0/\kappa} \mid_{X_0})$$

where $\Omega_{P_0/\kappa} \mid_{X_0}$ is considered as an extension of $\Omega_{X_0/\kappa}$ by $a \otimes_{\kappa} O_{X_0}$ via the fundamental exact sequence of the embedding of $X$ in $P$.

(f) Given an extension $E$ of $\Omega_{X_0/\kappa}$ by $a \otimes_{\kappa} O_{X_0}$ and an abstract lifting $X'$ of $X$, there is an abstract lifting $\tilde{X}'_1$ of $X$ such that $E(\tilde{X}', X')$ is isomorphic to $E$.

**Proof.** For part (a), choose embeddings $X_\alpha \subseteq A^n_A$, and liftings $X'_\alpha \subseteq A^n_A$; then $\nu(X'_\alpha, X'_\alpha) = 0$ (Proposition 2.8.(a)) and so the statement follows from Proposition 3.8.(a).

Part (b) follows from Proposition 2.8.(c) and Proposition 3.8.(b).

The isomorphism in part (c) is obtained from Proposition 2.8.(b) and Proposition 3.8.(d). We leave it the reader to check the compatibility condition.

Part (d) is clear.

Part (e) follows from the construction.

The proof of part (f) will be given after Proposition 4.10.

A splitting of $E(X'_1, X'_2)$ corresponds to a collection of splittings of each $E_\alpha$, compatible with the gluing isomorphisms. But by Lemma 3.9 these splittings correspond to homomorphisms of $\Omega_{A^n_A/\kappa}$ into $a \otimes_{\kappa} O_{X_0}$ whose restriction to the conormal sheaf $C_\alpha$ of $X_\alpha$ is $\nu(X'_\alpha, X'_\alpha)$; because of Proposition 4.6 this means exactly a compatible system of isomorphisms of liftings between $X'_\alpha$ and $X'_\alpha$, that is, an isomorphism of abstract liftings of $X'_2$ with $X'_1$.

**Proposition.** There is a natural bijective correspondence of isomorphisms of abstract liftings $X'_2 \simeq X'_1$ with splittings of $E(X'_1, X'_2)$.

If $X'_1, X'_2$ and $X'_3$ are abstract liftings, $\phi_1: X'_2 \simeq X'_1$ and $\phi_2: X'_3 \simeq X'_2$ are isomorphisms corresponding to splittings $s_1$ and $s_2$ of $E(X'_1, X'_2)$ and $E(X'_2, X'_3)$, then the composition $\phi_1 \phi_2$ corresponds to the splitting $(s_1 + s_2) \circ F^{-1}_{X'_1, X'_2, X'_3}$ of $E(X'_1, X'_3)$.

We leave it to the reader to unwind the various definitions and prove the Proposition.

**Proof of 4.9.(f).** Let $\{X_\alpha\}$ be a covering of $X$ by affine open subschemes, and call $X'_\alpha$ the restriction of $X'$ to $X_\alpha$. For each $\alpha$ we choose an embedding $X'_\alpha \hookrightarrow A^n_A$, and call $C_\alpha$ the conormal bundle of $X_\alpha \cap X_0$ in $A^n_A$. The fundamental exact sequence

$$0 \longrightarrow C_\alpha \longrightarrow \Omega_{A^n_A/\kappa} \mid_{X_\alpha \cap X_0} \longrightarrow \Omega_{O_{X_\alpha \cap X_0}} \longrightarrow 0$$

induces an exact sequence of abelian groups

$$\text{Hom}_{O_{X_\alpha \cap X_0}}(C_\alpha, a \otimes_{\kappa} O_{X_\alpha \cap X_0}) \longrightarrow \text{Ext}^1_{O_{X_\alpha \cap X_0}}(\Omega_{X_\alpha/\kappa} \mid_{X_\alpha \cap X_0}, a \otimes_{\kappa} O_{X_\alpha \cap X_0})$$

$$\longrightarrow \text{Ext}^1_{O_{X_\alpha \cap X_0}}(\Omega_{A^n_A/\kappa} \mid_{X_\alpha \cap X_0}, a \otimes_{\kappa} O_{X_\alpha \cap X_0}) = 0$$

where the last group is 0 because $X_\alpha \cap X_0$ is affine and $\Omega_{A^n_A/\kappa} \mid_{X_\alpha \cap X_0}$ is locally free. This means that $\partial$ is surjective, so, by Proposition 3.8.(e), for each $\alpha$ we can find a homomorphism $f_\alpha: C_\alpha \to a \otimes_{\kappa} O_{X_\alpha \cap X_0}$ and an isomorphism $f_\alpha^* \Omega_{A^n_A/\kappa} \mid_{X_\alpha \cap X_0} \cong E \mid_{X_\alpha \cap X_0}$. But Proposition 2.8.(d) implies
that there exists a lifting $\bar{X}_α'$ of $X_α$ in $A^\mathbb{P}_\kappa$ such that $ν(\bar{X}_α', X_α') = f_α$, and from the construction we get an isomorphism of extensions $φ_α: E(\bar{X}_α', X_α') \simeq E |_{X_α \cap X_0}$. Consider the composition

$$φ_α^{-1}φ_β: E(\bar{X}_β', X_β') |_{X_α \cap X_β \cap X_0} \simeq E(\bar{X}_α', X_α') |_{X_α \cap X_β \cap X_0}$$

By Proposition 3.6 the isomorphisms $φ_α^{-1}φ_β$ correspond to splittings of

$$E(\bar{X}_β', X_β') - E(\bar{X}_τ', X_τ') = E(\bar{X}_β', X_β') + E(\bar{X}_τ', \bar{X}_α') \simeq E(\bar{X}_τ', \bar{X}_α')$$

By Proposition 4.10 these splittings yield isomorphisms of liftings $X_β' |_{X_α \cap X_β} \simeq X_α' |_{X_α \cap X_β}$. One proves that they satisfy the cocycle condition: therefore we can glue the various $\bar{X}_α'$ together to find the desired $X_β'$.

If $X_1'$ and $X_2'$ are abstract liftings of $X$, call $e(X_1', X_2')$ the class of $E(X_1', X_2')$ in

$$\text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_0}/\kappa, a \otimes \kappa \mathcal{O}_{X_0}) = a \otimes \kappa \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_0}/\kappa, \mathcal{O}_{X_0})$$

The various properties of the extension $E(X_1', X_2')$ can be translated as follows.

**Proposition (4.11)**. Given two abstract liftings $X_1'$ and $X_2'$, there is well defined element

$$e(X_1', X_2') \in \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_0}/\kappa, a \otimes \kappa \mathcal{O}_{X_0}) = a \otimes \kappa \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_0}/\kappa, \mathcal{O}_{X_0})$$

with the following properties.

(a) $e(X_1', X_2') = 0$ if and only if $X_1'$ and $X_2'$ are isomorphic.

(b) If $X_1'$, $X_2'$ and $X_2'$ are abstract liftings of $X$, then $e(X_1', X_2') + e(X_2', X_3') = e(X_1', X_3')$.

(c) If there exists a smooth morphism $P' \to \text{Spec} A'$ and an embedding of $X$ into $P = P'|_{\text{Spec} A}$ which lifts to embeddings of $X_1'$ and $X_2'$ into $P'$, then

$$e(X_1', X_2') = \partial ν(X_1', X_2'),$$

where

$$\partial: H^0(X_0, \mathcal{N}_0) = \text{Hom}(\mathcal{T}_0/\mathcal{T}_0^2, a \otimes \kappa \mathcal{O}_{X_0}) \to \text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{O}_{X_0}/\kappa, a \otimes \kappa \mathcal{O}_{X_0})$$

is the boundary homomorphism coming from the fundamental exact sequence

$$0 \longrightarrow \mathcal{T}_0/\mathcal{T}_0^2 \longrightarrow \Omega_{P_0/\kappa} |_{X_0} \longrightarrow \Omega_{X_0/\kappa} \longrightarrow 0.$$

**Proof**. Part (a) follows from Proposition 4.10.

Part Proposition 4.11.(b) follows from Proposition 4.9.(c).

Part (c) follows from Proposition 4.9.(e) and Proposition 3.8.(e)

Because of Lemma 2.9 we see that Theorem 4.4.(d) follows.

We still have to prove Theorem 4.4.(c). First we will give an easy proof under a simplifying hypothesis.
(4.12) Hypothesis. Assume that there exists a smooth morphism $P' \to \text{Spec } A'$ and an embedding of $X$ into $P = P' \mid \text{Spec } A$.

The only general case in which I know that Hypothesis 4.12 is true is when $X$ is quasiprojective over $\kappa$. However, quasiprojectivity is not a very natural hypothesis; for example, if $X_0 \subseteq \mathbb{P}_k^2$ is a smooth quartic surface with Picard number 1 and $X$ is a general lifting of $X_0$ to the ring of dual numbers, then the Picard group of $X$ is trivial, so $X$ can not be projective. This problem does not arise for curves, that is, if $A$ is artinian and $X$ is one-dimensional then $X$ is quasiprojective.

Assume that Hypothesis 4.12 holds, and choose such a factorization $X \hookrightarrow P \to \text{Spec } A$. Call $\mathcal{I}_0$ the ideal of $X_0$ in $P_0$, and consider the fundamental exact sequence

$$0 \to \mathcal{I}_0/\mathcal{I}_0^2 \to \Omega_{P_0/\kappa} \mid_{X_0} \to \Omega_{X_0/\kappa} \to 0.$$  

We define the obstruction $\omega_{\text{abs}} \in \text{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, a \otimes_\kappa \mathcal{O}_{X_0}) = a \otimes_\kappa \text{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, \mathcal{O}_{X_0})$ to be the image of the embedded obstruction $\omega_{\text{emb}} \in H^1(X_0, a \otimes_\kappa \mathcal{N}_0) = \text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0/\mathcal{I}_0^2, a \otimes_\kappa \mathcal{O}_{X_0})$ by the boundary map

$$\partial: \text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0/\mathcal{I}_0^2, a \otimes_\kappa \mathcal{O}_{X_0}) \to \text{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, a \otimes_\kappa \mathcal{O}_{X_0})$$

coming from the fundamental exact sequence

$$0 \to \mathcal{I}_0/\mathcal{I}_0^2 \to \Omega_{P_0/\kappa} \mid_{X_0} \to \Omega_{X_0/\kappa} \to 0.$$  

We need to show that $\omega_{\text{abs}}$ does not depend on $P'$, and that it is 0 if and only if an abstract lifting exists. Set $\mathcal{I}_0 = \mathcal{H}om_{\mathcal{O}_{X_0}}(\Omega_{P_0/\kappa} \mid_{X_0}, a \otimes_\kappa \mathcal{O}_{X_0})$, and observe that $\omega_{\text{abs}} = 0$ if and only if $\omega_{\text{emb}}$ is in the image of the map

$$\rho: H^1(X_0, \mathcal{I}_0) = \text{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{P_0/\kappa} \mid_{X_0}, a \otimes_\kappa \mathcal{O}_{X_0}) \to \text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0/\mathcal{I}_0^2, a \otimes_\kappa \mathcal{O}_{X_0}) = a \otimes_\kappa H^1(X_0, \mathcal{N}_0)$$

induced by the fundamental exact sequence.

Notice first of all that if $X$ is affine then $\omega_{\text{emb}} \in H^1(X_0, \mathcal{N}_0) = 0$ vanishes, so a lifting $X'$ exists. Also, the map

$$\partial: \text{Hom}(\mathcal{I}_0/\mathcal{I}_0^2, a \otimes_\kappa \mathcal{O}_{X_0}) \to \text{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, a \otimes_\kappa \mathcal{O}_{X_0})$$

of Proposition 4.11.(c) is surjective, so if $\tilde{X}'$ is another abstract lifting of $X$ there is $\nu \in a \otimes_\kappa H^0(X_0, \mathcal{N}_0)$ with $\partial \nu = e(\tilde{X}', X')$. But such a $\nu$ is of the form $\nu = \nu(\tilde{X}', X')$ for a certain lifting $\tilde{X}' \subseteq P'$ because of Proposition 2.8.(d), and from Proposition 4.11.(c) we get that $e(\tilde{X}', X') = e(\tilde{X}', X')$. Then

$$e(\tilde{X}', \tilde{X}') = e(\tilde{X}', X') - e(\tilde{X}', X') = 0,$$

because of Proposition 4.11. It follows that $\tilde{X}'$ is isomorphic to $\tilde{X}'$. So we have shown that if $X$ is affine then an abstract lifting exists, and any abstract lifting can be embedded in $P'$. The existence of an abstract lifting in the affine case is consistent with the fact that $\text{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, a \otimes_\kappa \mathcal{O}_{X_0}) = 0$, because $\Omega_{X_0/\kappa}$ has projective dimension 1, as a consequence of Lemma 4.7.

Now choose a covering $\{X_\alpha\}$ of $X$ by open affine subsets. Then because of the discussion above an abstract lifting exists if and only if there are liftings $X'_\alpha$ of $X_\alpha$ in $P'$ and isomorphisms $\phi_{\alpha\beta}: X'_\beta \mid_{X_\alpha \cap X_\beta} \simeq X'_\alpha \mid_{X_\alpha \cap X_\beta}$ satisfying the cocycle condition. But invoking Proposition 4.6 and Lemma 2.10 we see that this is true if and only if there exists a cocycle $\{\nu_{\alpha\beta}\}$ in $\omega_{\text{emb}}$, and a collection $\{D_{\alpha\beta}\}$ of elements of $H^0(X_\alpha \cap X_\beta \cap X_0, \mathcal{I}_0)$ such that the restriction of $D_{\alpha\beta}$ to $H^0(X_\alpha \cap X_\beta \cap X_0, \mathcal{N}_0)$ is $\nu_{\alpha\beta}$. But that this is the case if and only if $\omega_{\text{emb}}$ is in the image of $H^1(X_0, \mathcal{I}_0)$, which is what we need.
Let us check that $\omega_{\text{abs}}$ is independent of $P'$ and of the embedding $j: X \hookrightarrow P$. Let $\tilde{j}: X \hookrightarrow \tilde{P} = \tilde{P}' \mid \text{Spec } A$ be another embedding. By the usual method of considering the fiber product $P' \times_{\text{Spec } A} \tilde{P}'$ we may assume that there exists a smooth morphism $\pi': \tilde{P}' \to P'$ such that if we denote by $\pi$ the restriction of $\pi'$ to Spec $A$ we have $\pi \tilde{j} = j$. If $\mathcal{I}_0$ is the ideal of $X_0$ in $P_0$ and $\mathcal{I}_0$ is the ideal of $X_0$ in $\tilde{P}_0$, we get a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & \mathcal{I}_0/R^2 & \to & \Omega_{P_0/k} \mid x_0 & \to & \Omega_{X_0/k} \mid x_0 & \to & 0 \\
 & & \downarrow \pi^* & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_{\tilde{P}_0}/\mathcal{I}_{\tilde{P}_0}^2 & \to & \Omega_{\tilde{P}_0/k} \mid x_0 & \to & \Omega_{X_0/k} \mid x_0 & \to & 0
\end{array}
$$

inducing a commutative diagram

$$
\begin{array}{cccccc}
\text{Ext}^1(\mathcal{I}_0/R^2, a \otimes_k \mathcal{O}_{X_0}) & \to & \text{Ext}^2(\Omega_{X_0/k}, a \otimes_k \mathcal{O}_{X_0}) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Ext}^1(\mathcal{I}_{\tilde{P}_0}/\mathcal{I}_{\tilde{P}_0}^2, a \otimes_k \mathcal{O}_{X_0}) & \to & \text{Ext}^2(\Omega_{X_0/k}, a \otimes_k \mathcal{O}_{X_0}).
\end{array}
$$

According to Lemma 2.11 the first column carries the embedded obstruction of $X$ in $\tilde{P}$ into the embedded obstruction of $X$ in $P$, and so the two images in $\text{Ext}^1(\mathcal{I}_{\tilde{P}_0}/\mathcal{I}_{\tilde{P}_0}^2, a \otimes_k \mathcal{O}_{X_0})$ coincide.

To prove Theorem 4.4.(c) in general, we need the machinery of extension cocycles developed in Section 3. Let $\{X_\alpha\}$ be a covering of $X$ with affine subsets, and for each $\alpha$ let $X'_\alpha$ be an abstract lifting of $X_\alpha$. For each triple $\alpha, \beta$ and $\gamma$ consider the isomorphisms

$$F_{X'_\alpha, X'_\beta, X'_\gamma}: \mathcal{E}(X'_\alpha, X'_\beta) + \mathcal{E}(X'_\beta, X'_\gamma) \simeq \mathcal{E}(X'_\alpha, X'_\gamma)$$

of Proposition 4.9.(c). Then $(\{\mathcal{E}(X'_\alpha, X'_\beta)\}, \{F_{X'_\alpha, X'_\beta, X'_\gamma}\})$ is an extension cocycle of $\Omega_{X_0/k}$ by $a \otimes_k \mathcal{O}_{X_0}$, which we will denote simply by $\{\mathcal{E}(X'_\alpha, X'_\beta)\}$.

Observe that an abstract lifting exists if and only if it is possible to choose the $X'_\alpha$ so that there are isomorphisms of abstract liftings $X'_\beta \mid x_\alpha \cap x_\beta \simeq X'_\alpha \mid x_\alpha \cap x_\beta$ satisfying the cocycle condition. By Proposition 4.10 to give isomorphisms of abstract liftings $X'_\beta \mid x_\alpha \cap x_\beta \simeq X'_\alpha \mid x_\alpha \cap x_\beta$ is equivalent to assigning a collection of splittings $\mathcal{E}(X'_\alpha, X'_\beta) \simeq 0$: it is straightforward to check that the cocycle condition corresponds to asking that the splittings yield an isomorphism of the extension cocycle $\{\mathcal{E}(X'_\alpha, X'_\beta)\}$ with the trivial cocycle $\{0\}$. In other words, an abstract lifting exists if and only if it is possible to choose a collection of abstract liftings $\{X'_\alpha\}$ in such a way that the associated extension cocycle is trivial.

If $\{\tilde{X}'_\alpha\}$ is another collection of abstract liftings, then by Proposition 4.9.(c) and (b), and Proposition 3.4.(b), we get isomorphisms

$$\mathcal{E}(\tilde{X}'_\alpha, X'_\beta) \simeq \mathcal{E}(\tilde{X}'_\alpha, X'_\alpha) + \mathcal{E}(X'_\alpha, X'_\beta) + \mathcal{E}(X'_\beta, \tilde{X}'_\alpha) \simeq \mathcal{E}(X'_\alpha, X'_\beta) + \mathcal{E}(\tilde{X}'_\alpha, X'_\alpha) - \mathcal{E}(\tilde{X}'_\alpha, X'_\beta).$$

One checks that these give an isomorphism of the cocycle $\{\mathcal{E}(\tilde{X}'_\alpha, X'_\beta)\}$ with $\{\mathcal{E}(X'_\alpha, X'_\beta)\} + \partial\{\mathcal{E}(\tilde{X}'_\alpha, X'_\alpha)\}$. This means that the class

$$\omega = \{\mathcal{E}(X'_\alpha, X'_\beta)\} \in \mathcal{E}_{\mathcal{O}_{X_0}}(\Omega_{X_0/k}, a \otimes_k \mathcal{O}_{X_0})$$

is independent of the liftings. What’s more, if $\{\mathcal{E}_{\alpha, \beta}\}$ is a cocycle in $\omega$, then there exist extensions $\mathcal{E}_\alpha$ such that $\{\mathcal{E}_{\alpha, \beta}\}$ is isomorphic to $\{\mathcal{E}(X'_\alpha, X'_\beta)\} + \partial\{\mathcal{E}_\alpha\}$. If we choose liftings $\tilde{X}'_\alpha$ with isomorphisms
\[ \mathcal{E}_\alpha \simeq \mathcal{E}(X'_\alpha, X'_\beta) \] (Proposition 4.9.(f)) then the cocycle \( \{ \mathcal{E}_{\alpha\beta} \} \) is isomorphic to \( \{ \mathcal{E}(X'_{\alpha}, X'_{\beta}) \} \). So it follows that \( \omega = 0 \) if and only if a lifting exists. By Theorem 3.13 the proof is concluded.

This proof could be summarized as follows, for those whose taste runs towards the abstract. From the proof of Theorem 4.4.(d) we get that the category of liftings is a principal bundle stack over the commutative group stack \( \mathcal{E}_P \Omega_{X_0} (\Omega_{X_0/k}, \mathcal{O}_{X_0}) \) of extensions. An abstract lifting is a section of this bundle stack, so the obstruction to its existence is an element of the first cohomology group of \( \mathcal{E}_P \Omega_{X_0} (\Omega_{X_0/k}, \mathcal{O}_{X_0}) \). But Theorem 3.13 says that this group is contained in \( \text{Ext}^2_{\mathcal{O}_{X_0}} (\Omega_{X_0/k}, \mathcal{O}_{X_0}) \).

The class constructed here and the class constructed earlier under Hypothesis 4.12 coincide. The proof of this is omitted.

This construction has an obvious property of functoriality, which will be exploited in Sections 6 and 7. Let \( B' \) be a local ring, \( b \subseteq B' \) be an ideal with \( m_B b = 0 \), \( B = B'/b \). Let \( f: A' \to B' \) a local homomorphism inducing an isomorphism of residue fields, such that \( f(a) \subseteq b \). Set \( f_*X = X \times_{\text{Spec} A} \text{Spec} B \); this scheme \( f_*X \) is a flat local complete intersection on \( A \). If \( X' \) is an abstract lifting of \( X \) we set \( f_*X' = X \times_{\text{Spec} A} \text{Spec} B' \); this is an abstract lifting of \( f_*X \). Any automorphism \( \phi \) of \( X' \) as a lifting induces an automorphism \( f_*\phi \) of \( f_*X' \).

Call \( g = f \mid_a: a \to b \) the restriction of \( f \).

(4.13) Proposition. (a) If \( \phi \) is an automorphism of an abstract lifting \( X' \) corresponding to an element \( \xi \in a \otimes \kappa \text{Hom}_{\mathcal{O}_{X_0}} (\Omega_{X_0/k}, \mathcal{O}_{X_0}) \), the element of \( b \otimes \kappa \text{Hom}_{\mathcal{O}_{X_0}} (\Omega_{X_0/k}, \mathcal{O}_{X_0}) \) corresponding to \( f_*\phi \) is \( (g \otimes \text{id})(\xi) \).

(b) \( \epsilon(f_*X', f_*X') = (g \otimes \text{id})(\epsilon(X'_1, X'_2)) \in b \otimes \text{Ext}^1_{\mathcal{O}_{X_0}} (\Omega_{X_0/k}, \mathcal{O}_{X_0}) \).

(c) \( \omega_{\text{abs}}(f_*X') = (g \otimes \text{id})\omega_{\text{abs}}(X) \in b \otimes \text{Ext}^2_{\mathcal{O}_{X_0}} (\Omega_{X_0/k}, \mathcal{O}_{X_0}) \).

5 Generalizations

Here are two important generalizations.

Fix a scheme \( M' \) of finite type over \( A' \), a scheme \( X \) of finite type over \( A \), such that \( X_0 \) is reduced and generically smooth over \( \kappa \), and a local complete intersection morphism \( f: X \to M = M'|_{\text{Spec} A} \) defined over \( A \). A lifting of the morphism \( f \) consist of an abstract lifting \( X' \) of \( X \), and morphism \( f': X' \to M' \) of \( A' \)-schemes whose restriction to \( X \) is \( f \). There is an obvious notion of isomorphism of liftings; if \( f'_1: X'_1 \to M' \) and \( f'_2: X'_2 \to M' \) are liftings, then an isomorphism of \( f_1 \) with \( f_2 \) is an isomorphism of abstract liftings \( \Phi: X'_1 \simeq X'_2 \) such that \( f'_2 \circ \Phi = f'_1 \).

Let \( f_0: X_0 \to M_0 \) be the restriction of \( f \); \( f_0 \) is again a local complete intersection morphism. We define the complex of differentials \( \Omega^i_{f_0} \) of the morphism \( f_0 \) to be the complex with \( \Omega^0_{f_0} = \Omega_{X_0/k}, \Omega^i_{f_0} = 0 \) for \( i \neq 0, -1 \), the only nontrivial differential \( \partial \Omega^0_{M_0/k} \to \Omega_{X_0/k} \) being the pullback map.

(5.1) Theorem. (a) Any lifting of \( f \) is a local complete intersection morphism.

(b) There is a canonical element \( \omega \in a \otimes \kappa \text{Ext}^2_{\mathcal{O}_{X_0}} (\Omega^i_{f_0}, \mathcal{O}_{X_0}) \), called the obstruction, such that \( \omega = 0 \) if and only if a lifting exists.

(c) If a lifting exists, then there is a canonical action of the group \( a \otimes \kappa \text{Ext}^1_{\mathcal{O}_{X_0}} (\Omega^i_{f_0}, \mathcal{O}_{X_0}) \) on the set of isomorphism classes of liftings making it into a principal homogeneous space.

The proof of this result is essentially the same as the proof of Theorem 4.4 in Section 4.

First one assumes that there is a smooth morphism \( \pi': P' \to M' \) such that \( f: X \to M \) factors through \( P = P'|_{\text{Spec} A} \). Let \( X'_1 \) and \( X'_2 \) be two liftings of \( X \) in \( P' \); the resulting morphisms
$f'_1: X'_1 \to M'$ and $f'_2: X'_1 \to M'$ are both liftings of $f$. Here is a description of the isomorphisms between $f'_1$ and $f'_2$ analogous to Proposition 4.6, with the same proof.

Call $\mathcal{I}_0$ the ideal of the embedding of $X_0$ in $P_0$, and $\pi_0: P_0 \to M_0$ the restriction of $\pi$. The usual homomorphism from $\mathcal{I}_0/\mathcal{I}_0^2$ to $\Omega_{P_0/M_0} |_{x_0}$ yields a restriction map from

$$\text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{P_0/M_0}, a \otimes \kappa \mathcal{O}_{X_0}) \to \Omega^0(X_0, a \otimes N_0).$$

**5.2 Proposition.** There is a bijective correspondence between isomorphisms of liftings $f'_2 \simeq f'_1$ and elements of $\text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{P_0/M_0} |_{x_0}, a \otimes \kappa \mathcal{O}_{X_0})$ whose image in $\Omega^0(X_0, a \otimes N_0)$ is $\nu(X'_1, X'_2)$, with the following properties.

(a) The identity $\text{id}_{X'}: X' \simeq X'$ corresponds to 0.

(b) If $\Phi_1: X'_2 \simeq X'_1$ and $\Phi_2: X'_3 \simeq X'_2$ are isomorphisms of abstract liftings and $D_1$, $D_2$ are the corresponding elements of $\text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{P_0/\kappa} |_{x_0}, a \otimes \kappa \mathcal{O}_{X_0})$, then the composition $\Phi_1 \circ \Phi_2$ corresponds to $D_1 + D_2$.

The role of the fundamental exact sequence of Lemma 4.7 is played by the exact sequence

$$0 \to \mathcal{I}_0/\mathcal{I}_0^2 \to \Omega^0_{\pi_0} |_{x_0} \to \mathcal{O}^*_{f_0} \to 0$$

which looks as follows

```
\begin{array}{cccccccc}
\vdots & & & & & & & \\
deg -2 & 0 & - & 0 & - & 0 & - & 0 \\
& & & & & & & \\
& & & & & & & \\
deg -1 & 0 & - & 0 & - & \pi_0^* \Omega_{M_0/\kappa} |_{x_0} & - & f_0^* \Omega_{M_0/\kappa} \\
& & & & & & & \\
& & & & & & & \\
deg 0 & 0 & - & \mathcal{I}_0/\mathcal{I}_0^2 & - & \Omega_{P_0/\kappa} |_{x_0} & - & \Omega_{X_0/\kappa} \\
& & & & & & & \\
& & & & & & & \\
deg 1 & 0 & - & 0 & - & 0 & - & 0 \\
\end{array}
```

If we observe that for any sheaf of $\mathcal{O}_{X_0}$-modules $\mathcal{G}$ we have

$$\text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{P_0/M_0} |_{x_0}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_{X_0}}(\Omega^*_{\pi_0} |_{x_0}, \mathcal{G}),$$

and we keep into account that, as observed in the last part of Section 3, the theory of extensions of sheaves generalizes to extensions of $\Omega^*_{f_0}$ by sheaves, the proof of Theorem 4.4 goes through almost word for word.

Another situation that arises quite often in practice is when we want to study deformations of a scheme inducing a fixed deformation on a subscheme. For example, in the theory of deformation of pointed curves the scheme is the curve itself, while the subscheme is the union of the distinguished points. Again we may look at embedded deformations or abstract deformations. Here is the embedded setup.

Let $M'$ be a flat scheme of finite type over $A'$, $Z' \subseteq M'$ a closed subscheme, also flat over $A'$. Let $X$ be a local complete intersection subscheme of $M = M' |_{\text{Spec} A}$ containing $Z = Z' |_{\text{Spec} A}$. Assume also that $X$ is flat over $A$. Call $\mathcal{N}_0$ the normal sheaf to $X_0$ in $M_0$, $\mathcal{J}_0$ the ideal of $Z_0$ in $X_0$. A lifting of $X$ to $M'$ relative to $Z'$ is a lifting $X' \subseteq M'$ that contains $Z'$. 

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(5.3) **Theorem.** There is a canonical element

$$\omega_{\text{emb}} \in a \otimes_\kappa \mathbb{H}^1(X_0, N_0 \otimes \mathcal{O}_{X_0} \mathcal{J}_0)$$

called the embedded obstruction of $X$ in $M$, such that $\omega_{\text{emb}} = 0$ if and only if a lifting of $X$ to $M'$ relative to $Z'$ exists.

(a) If a lifting exists, then there is a canonical action of the group

$$a \otimes_\kappa \mathbb{H}^0(X_0, N_0 \otimes \mathcal{O}_{X_0} \mathcal{J}_0)$$
on the set of liftings making it into a principal homogeneous space.

The proof is essentially the same as the the proof of Theorem 2.5. The key point is that if we have two relative liftings $X'_1$ and $X'_2$, then the image of the homomorphism $\nu(X'_1, X'_2) : \mathcal{I}_0/\mathcal{I}_0 \to a \otimes_\kappa \mathcal{O}_{X_0}$ is contained inside $a \otimes_\kappa \mathcal{J}_0$, so $\nu(X'_1, X'_2)$ can be considered as an element of

$$\text{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0/\mathcal{I}_0, a \otimes_\kappa \mathcal{J}_0) = a \otimes_\kappa \mathbb{H}^0(X_0, N_0 \otimes \mathcal{O}_{X_0} \mathcal{J}_0).$$

Here is the setting in the abstract case.

Let $Z'$ be a flat scheme over $A'$, and let set $Z = Z'|\text{Spec } A$. Let $X$ be a flat local complete intersection scheme of finite type over $A$ with a closed embedding $Z \hookrightarrow X$. Assume also that $X_0$ is generically smooth over $\kappa$.

An abstract lifting of $X$ relative to $Z'$ is an abstract lifting $X'$ of $X$ with a closed embedding $Z' \hookrightarrow X'$ extending the given embedding of $Z$ in $X$.

An isomorphism of relative abstract liftings is an isomorphism of abstract lifting inducing the identity on $Z'$.

Again, let $\mathcal{J}_0$ be the ideal of $Z_0$ in $X_0$.

(5.4) **Theorem.** (a) There is a canonical element $\omega_{\text{abs}} \in a \otimes_\kappa \text{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, \mathcal{J}_0)$, called the obstruction, such that $\omega_{\text{abs}} = 0$ if and only if an abstract lifting exists.

(b) If an abstract lifting exists, then there is a canonical action of the group

$$a \otimes_\kappa \text{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, \mathcal{J}_0)$$
on the set of isomorphism classes of abstract liftings making it into a principal homogeneous space.

The proof of Theorem 4.4 goes through with minor variations, substituting $a \otimes_\kappa \mathcal{J}_0$ everywhere for $a \otimes_\kappa \mathcal{O}_{X_0}$. The only point that requires a little care is the analogue of Lemma 4.8. Given two relative abstract liftings $Z' \hookrightarrow X'_1$ and $Z' \hookrightarrow X'_2$, to construct the relative extension we need to compare them locally in a suitable ambient space. For this we need to know that if $X$ is affine, given a closed embedding $X \hookrightarrow A^n_\kappa$, this can be lifted to embeddings $X'_1 \hookrightarrow A^n_\kappa$, and $X'_2 \hookrightarrow A^n_\kappa$, which agree on $Z'$. For this, set $X = \text{Spec } B$, $X'_1 = \text{Spec } B'_1$, $Z = \text{Spec } C$, $Z' = \text{Spec } C'$. The embeddings $Z' \hookrightarrow X'_1$ and $X \hookrightarrow X'_1$, which agree on $Z$, induce a homomorphism of $A'$-algebras $B'_1 \to B \times_C C'$, which is easily shown to be surjective. Then given embedding $X \hookrightarrow A^n_\kappa$ yields a surjective homomorphism of $A'$-algebras $A'[x_1, \ldots, x_n] \to B$. Lift the images of the $x_i$ to $B \times_C C'$, then lift them to $B'_1$ and $B'_2$. The resulting homomorphisms of $A'$-algebras $A'[x_1, \ldots, x_n] \to B'_i$ are surjective, and induce the required liftings $X'_i \hookrightarrow A^n_{A'}$.

Of course we can put together the two generalizations. This is our final and most general setup. It is not in any essential sense harder to treat than the case of Hypotheses 4.1, just a little more confusing.
Let $X$ be a local complete intersection flat scheme of finite type over $A$ such that $X_0$ is generically smooth over $\kappa$, $M'$ and $Z'$ flat schemes over $A'$ with a morphism $j': Z' \to M'$. Let it be given a closed embedding of $Z = Z' |_{\text{Spec } A} \to X = X' |_{\text{Spec } A}$, and morphism $f: X \to M = M' |_{\text{Spec } A}$ such that $f |_Z = j |_Z$. A lifting of $f$ relative to $j'$ is an abstract lifting $Z' \hookrightarrow X'$ relative to $Z'$, together with a morphism $f': X' \to M'$ such that $f' |_X = f$ and $f' |_{Z'} = j'$. Call $\mathcal{J}_0$ the ideal of $\mathcal{O}_0$ in $X_0$.

The notion of isomorphism of relative liftings of $f$ is the obvious one.

(5.5) Theorem. (a) There is a canonical element $\omega \in \mathfrak{a} \otimes_\kappa \text{Ext}^2_{\mathcal{O}_{X_0}}(\Omega^*_{f_0}, \mathcal{J}_0)$, called the obstruction, such that $\omega = 0$ if and only if a lifting exists.

(b) If a lifting exists, then there is a canonical action of the group $\mathfrak{a} \otimes_\kappa \text{Ext}^1_{\mathcal{O}_{X_0}}(\Omega^*_{f_0}, \mathcal{J}_0)$ on the set of isomorphism classes of liftings making it into a principal homogeneous space.

Finally, all the results and the proofs generalize to the case of algebraic spaces, by working with the étale topology instead of the Zariski topology. Also, if we assume that $A'$ is a finite $\mathbf{C}$-algebra they are valid for analytic spaces; the proofs remain the same, if we substituting polydiscs for affine spaces.

6 Formal deformations

We fix a field $\kappa$. In this section and in the next a complete algebra will be a complete noetherian local $\kappa$-algebra with residue field $\kappa$. If $A$ is a complete algebra, we will denote its maximal ideal by $\mathfrak{m}_A$, and set $A_n = A/\mathfrak{m}_A^{n+1}$. An artinian algebra will be a complete algebra which is artinian, or, equivalently, finite over $\kappa$. If $A$ is a complete algebra, the $\kappa$-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$ is called the cotangent space of $A$, the dual space $(\mathfrak{m}_A/\mathfrak{m}_A^2)^\vee$ its tangent space.

A homomorphism of complete algebras is homomorphisms of $\kappa$-algebras; it is automatically local. A homomorphism of complete algebras $f: A \to B$ induces a linear map $f_*: \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$; the dual map $df: (\mathfrak{m}_B/\mathfrak{m}_B^2)^\vee \to (\mathfrak{m}_A/\mathfrak{m}_A^2)^\vee$ will be called the differential of $f$. If $f: A \to B$ is a homomorphism of complete algebras then $f(\mathfrak{m}_A^{n+1}) \subseteq \mathfrak{m}_B^n$ for each $n \geq 0$; we denote by $f_n: A_n \to B_n$ the induced homomorphism.

(6.1) Definition. A deformation $(X, A)$ consists of a complete algebra $A$, a scheme $X_n$ flat and of finite type over $A_n$ for each $n \geq 0$, and a sequence of closed embeddings $X_{n-1} \hookrightarrow X_n$ compatible with with the closed embeddings $\text{Spec } A_{n-1} \to \text{Spec } A_n$, inducing an isomorphism of $X_n |_{\text{Spec } A_{n-1}}$ with $X_{n-1}$.

We will say that $X$ is a deformation of $X_0$ over $A$.

If $X$ and $\tilde{X}$ are deformations of $X_0 = \tilde{X}_0$ over $A$, an isomorphism $\phi: X \simeq \tilde{X}$ of deformations over $A$ is a sequence of isomorphisms $\phi_n: X_n \simeq \tilde{X}_n$ of schemes over $A_n$, such that $\phi_n |_{X_{n-1}} = \phi_{n-1}$, and $\phi_0: X_0 \to \tilde{X}_0$ is the identity.

The objects defined above should be properly called formal deformations, but they are the only types of deformations we will consider.

From now on we fix $X_0$; all deformations will be deformations of the same $X_0$.

Deformations of $X_0$ over $A$ form a category, the arrows being isomorphisms of deformations over $A$.

Let $f: A \to B$ be a homomorphism of complete algebras, $X$ a deformation of $X_0$ over $A$. There is an induced deformation $f_* X$ on $B$, defined by setting $(f_* X)_n = X_n \times_{\text{Spec } A_n} \text{Spec } B_n$; the embeddings $(f_* X)_n \hookrightarrow (f_* X)_{n-1}$ are induced by the embeddings $X_n \hookrightarrow X_{n-1}$. Also, if $\phi: X \simeq \tilde{X}$ is an isomorphism of deformations of $X_0$, then there is an induced isomorphism $f_* \phi: f_* X \simeq f_* \tilde{X}$.
defined in the obvious way. This makes $f_\ast$ into a functor from the category of deformations of $X_0$ over $A$ to deformations of $X_0$ over $B$.

If $f: A \to B$ and $g: B \to C$ are homomorphism of complete algebras and $X$ is a deformation over $A$, then there is a canonical isomorphism $(gf)_\ast X \cong g_\ast f_\ast X$ of deformations over $C$. From now on we identify $(gf)_\ast X$ with $g_\ast f_\ast X$.

The trivial deformation of $X_0$ over a complete algebra $A$ is $X_0^A = f_\ast X_0$, where $f: \kappa \to A$ is the structure homomorphism. Concretely, $(X_0^A)_n = X_0 \times_\kappa \text{Spec} A_n$, and the closed embeddings $(X_0^A)_n \hookrightarrow (X_0^A)_{n-1}$ are induced by the closed embeddings $\text{Spec} A_{n-1} \hookrightarrow \text{Spec} A_n$.

(6.2) Definition. Let $(X, A), (Y, B)$ be deformations of $X_0$. An homomorphism $(\phi, f): (X, A) \to (Y, B)$ of deformations consists of a homomorphism of complete algebras $f: A \to B$ and an isomorphism $\phi: f_\ast X \cong Y$ of deformations over $B$.

A homomorphism $(\phi, f)$ is called surjective if $f$ is surjective.

If $(\phi, f): (X, A) \to (Y, B)$ and $(\psi, g): (Y, B) \to (Z, C)$ are homomorphism of deformations, the composition $(\psi, g) \circ (\phi, f): (X, A) \to (Z, C)$ is $(\psi \circ g_\ast \phi \circ f)$.

Extensions of $X_0$ are the objects of a category in which the arrows are the homomorphisms. The object $(X_0, \kappa)$ is terminal in this category, that is, given a deformation $(X, A)$ there is a unique homomorphism $(X, A) \to (X_0, \kappa)$.

The isomorphisms in the category of deformations are exactly the homomorphisms $(\phi, f)$ where $f$ is an isomorphism.

If $A$ is artinian then $A_n = A$ for $n \gg 0$, so $X_n$ is a flat scheme over $A$ for $n \gg 0$. As one sees immediately, the category of deformations of $X_0$ over $A$ is equivalent to the category of flat schemes $X$ of finite type over $A$, together with closed embeddings of $X_0 \subseteq X$ inducing isomorphisms $X_0 \cong X |_{\text{Spec} \kappa}$; the arrows are isomorphisms of $A$-schemes inducing the identity on $X_0$. From now on we will systematically identify a deformation over an artinian algebra $A$ with the corresponding scheme over $A$.

More generally, if $X$ is a flat scheme of finite type over $A$ with $X |_{\text{Spec} \kappa} = X_0$, then $X$ will induce a deformation of $X_0$ over $A$ by setting $X_n = X |_{\text{Spec} A_n}$, the embeddings $X_{n-1} \hookrightarrow X_n$ being induced by the natural embeddings $\text{Spec} A_{n-1} \hookrightarrow \text{Spec} A_n$. Such a deformation is called algebraic.

Not every deformation is algebraic. For example, one can show that if $X_0 \subseteq \mathbf{P}^3_C$ is a smooth quartic surface, then $X_0$ has non-algebraic deformations over $C[[t]]$.

This problem does not arises for projective curves. More generally we have the following standard fact.

(6.3) Proposition. Assume that $X_0$ is projective, and $H^2(X_0, \mathcal{O}_{X_0}) = 0$. Then every deformation of $X_0$ is algebraic.

Proof. Let $X$ be a deformation of $X_0$. I claim that there for each $n > 0$ the restriction map $\text{Pic} X_n \to \text{Pic} X_{n-1}$ is surjective. In fact, there is an exact sequence

$$0 \longrightarrow (m^0_A/m^{n+1}_A) \otimes_\kappa \mathcal{O}_{X_0} \overset{\alpha}{\longrightarrow} \mathcal{O}_{X_n}^* \overset{\beta}{\longrightarrow} \mathcal{O}_{X_{n-1}}^* \longrightarrow 0$$

in which $\beta$ is the restriction map, and $\alpha$ is defined by identifying

$$(m^0_A/m^{n+1}_A) \otimes_\kappa \mathcal{O}_{X_0} = (m^n_A/m^{n+1}_A) \otimes_A/m^n_A \mathcal{O}_{X_n}$$

with the kernel of the restriction map $\mathcal{O}_{X_n} \to \mathcal{O}_{X_{n-1}}$, which can be done because of the flatness of $X_n$, then setting $\alpha(f) = 1 + f$. The fact that $H^2(X_0, (m^0_A/m^{n+1}_A) \otimes_\kappa \mathcal{O}_{X_0}) = 0$ implies the surjectivity of the map on Picard groups.
Let \( \mathcal{L}_0 \) be a very ample line bundle on \( X_0 \) such that \( H^i(X_0, \mathcal{L}_0) = 0 \) for \( i > 0 \). For each \( n > 0 \) we can choose a line bundle \( \mathcal{L}_n \) on \( X_n \) such that \( \mathcal{L}_n |_{X_{n-1}} \) is isomorphic to \( \mathcal{L}_{n-1} \); by semicontinuity we have that \( H^i(X_n, \mathcal{L}_n) = 0 \) for \( i > 0 \). If \( \pi_n: X_n \to \text{Spec} \, A_n \) is the structure morphism then \( \pi_n^* \mathcal{L}_n \) satisfies base change, and if \( N \) is the dimension of \( H^0(X_0, \mathcal{L}_0) \) over \( \kappa \) the bundles \( \mathcal{L}_n \) induce embeddings \( X_n \to \mathbb{P}^{N-1}_{A_n} \), such that \( X_n \cap \mathbb{P}^{N-1}_{A_{n-1}} = X_{n-1} \). So the system \( \{X_n\} \) can be considered as a formal subscheme of \( \mathbb{P}^{N-1}_A \), and the result follow from Grothendieck’s existence theorem. ♣

I do not know whether this is still true if we do not assume that \( X_0 \) is projective. Elkik proved that deformations of affine schemes with isolated singularities are algebraic ([Elkik]).

From now on we will assume that \( X_0 \) is a generally smooth local complete intersection scheme on \( \kappa \). We set

\[
T^i(X_0) = \text{Ext}^i_{\mathcal{O}_{X_0}}(\Omega_{X_0/\kappa}, \mathcal{O}_{X_0}).
\]

If \( X \) is a deformation, then \( X_n \) is an abstract lifting of \( X_{n-1} \) to \( A_n \), so a deformation of \( X_0 \) can be thought of a sequence of abstract liftings; to these we can apply Theorem 4.4.

(6.4) Proposition. (a) Assume that \( T^0(X_0) = 0 \). Then two deformations of \( X_0 \) over the same algebra admit at most one isomorphism.

(b) Assume that \( T^1(X_0) = 0 \). Then any deformation \( (X, A) \) of \( X_0 \) is isomorphic to the trivial deformation \( X_0^A \).

(c) Assume that \( T^2(X_0) = 0 \). Then if \( (X, A) \) is a deformation of \( X_0 \) and \( f: B \to A \) is a surjective homomorphism of algebras, there exists a deformation \( Y \) of \( X_0 \) over \( B \) and an isomorphism \( f_* Y \cong X \).

Proof. For part (a), notice that because of Theorem 4.4.(b) an isomorphism \( X_{n-1} \cong \tilde{X}_{n-1} \) over \( A_n \) extends in at most one way to an isomorphism \( X_n \cong \tilde{X}_n \).

For part (b), let \( \tilde{X} = X_0^A \) let \( \phi_{n-1}: \tilde{X}_{n-1} \cong X_{n-1} \) be an isomorphism inducing the identity on \( X_0 \). Then \( X_n \) is a lifting of \( X_{n-1} \) to \( A_n \), and we can also think of \( \tilde{X}_n \) as a lifting, via the composition \( \tilde{X}_{n-1} \cong X_{n-1} = X_n \). Then it follows from Theorem 4.4.(d) that the isomorphism \( \phi_{n-1} \) extends to an isomorphism \( \phi_n: \tilde{X}_n \cong X_n \).

Let us prove part (c). Call \( b \) the kernel of \( f \).

Assume first that \( A \) and \( B \) are artinian, so a deformation \( X \) on \( A \) is flat scheme over \( A \). By induction on the least integer \( n \) such that \( m_B^n b = 0 \) we can assume that \( m_B b = 0 \); then the result follows from Theorem 4.4.(c).

In the general case we construct \( Y_n \) by induction on \( n \). Call \( \pi_n: B_n \to A_n = B/(a + m_B^n) \) the projection. For \( n = 0 \) there is no problem, so suppose \( n > 0 \) and that we are given a deformation \( Y_{n-1} \) over \( B_{n-1} = A/(a + m_A^n) \) and a homomorphism \( (\phi_{n-1}, \pi_{n-1}): Y_{n-1} \to X_{n-1} \). We are looking for a deformation \( (Y_n, B_n) \) and a commutative diagram

\[
\begin{array}{ccc}
(Y_n, B_n) & \xrightarrow{(\phi_n, \pi_n)} & (X_n, A_n) \\
\downarrow & & \downarrow \\
(Y_{n-1}, B_{n-1}) & \xrightarrow{(\phi_{n-1}, \pi_{n-1})} & (X_{n-1}, A_{n-1}).
\end{array}
\]

From Corollary 8.2 we get a deformation \( \tilde{Y} \) over \( B/(m_B^n \cap (b + m_B^{n+1})) \) and a commutative diagram

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of deformations
\[
(\widetilde{Y}, B/(m_B^n \cap (b + m_B^{n+1}))) \rightarrow (X_n, A_n)
\]
\[
(Y_{n-1}, B_{n-1}) \rightarrow (X_{n-1}, A_{n-1}).
\]
By the previous case there is a homomorphism of deformations \((Y_n, A/m_A^{n+1}) \rightarrow (\widetilde{Y}, B/(m_B^n \cap (b + m_B^{n+1}))\); this is the deformations we were looking for.

It may happen that \(T^2(X_0) \neq 0\), but still the conclusion of Proposition 6.4.(c) holds. To clarify this we give a definition.

Let \((X, A)\) be a deformation of \(X\) over an artinian algebra \(A\). A small extension of \(A\) is a surjective homomorphism of artinian algebras \(A' \rightarrow A\) whose kernel \(a\) has length 1, and is therefore isomorphic to \(\kappa\). These data determine an element \(\omega_{\text{abs}} \in T^2(X_0) \simeq a \otimes T^2(X_0)\), well defined up to multiplication by a nonzero scalar.

(6.5) Definition. The space of obstructions \(\text{Obs } X_0\) of \(X_0\) is the subspace of \(T^2(X_0)\) generated by the elements \(\omega_{\text{abs}} \in T^2(X_0)\) for all deformations \((X, A)\) and all small extensions \(A' \rightarrow A\) as above.

If \(\text{Obs } X_0 = 0\) we say that \(X_0\) has unobstructed deformations.

(6.6) Proposition. Let \(A'\) be an artinian algebra, \(a \subseteq A'\) an ideal with \(m_{A'} a = 0\), \(A = A'/a\). Let \(X\) be a deformation of \(X_0\) over \(A\); then the obstruction \(\omega_{\text{abs}}(X) \in a \otimes T^2(X_0)\) is contained in \(a \otimes \text{Obs } X_0\).

Proof. Choose a basis \(v_1, \ldots, v_n\) for \(a\) as a vector space over \(\kappa\), and for each \(i\) call \(a_i\) the quotient of \(a\) by the subspace generated by \(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\). Set \(A'_i = A/a_i\); then \(A'_i \rightarrow A\) is a small extension, and by definition the obstruction to lifting \(X\) to \(A'_i\) lies in \((a/a_i) \otimes \text{Obs } X_0 \subseteq (a/a_i) \otimes T^2(X_0)\). According to Proposition 4.13.(c) this obstruction is the image in \(a/a_i \otimes T^2(X_0)\) of \(\omega_{\text{abs}} \in a \otimes T^2(X_0)\), and by elementary linear algebras this implies the thesis.

So we can improve on Proposition 6.4.(c).

(6.7) Proposition. Assume that \(\text{Obs } X_0 = 0\). Then if \((X, A)\) is a deformation of \(X_0\) and \(f: B \rightarrow A\) is a surjective homomorphism of algebras, there exists a deformation \(Y\) of \(X_0\) over \(B\) and an isomorphism \(f^* Y \simeq X\).

The definition of unobstructed variety is relative to then base field; I do not know any specific example, but it seems quite plausible that there may be varieties over a field \(\kappa\) of positive characteristic which are unobstructed, but cannot be lifted to some artinian ring \(A\) with residue field \(\kappa\) (obviously \(A\) can not be a \(\kappa\)-algebra.) On the other hand one can prove that in the situation of Section 4 if the ring \(A'\) is equicharacteristic then the obstruction to lifting \(X\) to \(A'\) lives in \(a \otimes \text{Obs } X_0\).

It happens very frequently that \(T^2(X_0) \neq 0\) but \(\text{Obs } X_0 = 0\).

(6.8) Example. Let \(X_0 \subseteq \mathbb{P}^3_\kappa\) be a smooth surface of degree \(d \geq 6\). Then \(T^2(X_0) \neq 0\) but \(\text{Obs } X_0 = 0\).

Proof. Denote by \(T_{X_0}\) the tangent bundle of \(X_0\); then \(T^i(X_0) = H^i(X_0, T_{X_0})\). We have the sequence
\[
0 \longrightarrow T_{X_0} \longrightarrow T_{\mathbb{P}^3}/x_0 \longrightarrow \mathcal{N}_0 \longrightarrow 0
\]
\[
\| \quad \mathcal{O}_{X_0}(d).
\]

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Since $H^1(X_0, N_0) = 0$ the embedded deformations of $X_0$ in $P^3$ are unobstructed. Assume that we have proved that $H^1(X_0, T_{P^3}|_{X_0}) = 0$. Then the boundary map

$$\partial: H^0(X_0, N_0) \rightarrow H^1(X_0, T_{X_0})$$

is surjective: I claim that this implies that for any deformation $X$ of $X_0$ over an artinian algebra $A$ there is an embedding $X \hookrightarrow P^3_A$ extending the given embedding $X_0 \hookrightarrow P^3_A$. Let $\ell$ be the length of $A$; for $\ell = 1$ we have $A = k$, and the statement is vacuous. Suppose this true when $A$ has length $\ell - 1$, and take an ideal $a \subseteq A$ of length 1. The deformation $Y = X \times_{\text{Spec}A} \text{Spec}A/a$ can be embedded in $P^3_{A/a}$ by inductive hypothesis; since $H^1(X_0, N_0) = 0$ then $Y$ has a lifting $Y'$ in $P^3_A$. Choose another lifting $X'$ of $Y$ in $P^3_A$ such that $\partial(\nu(X', Y')) = e(X, Y')$ (Proposition 2.8(d)). From Proposition 4.11(c) we see that $e(X', Y') = e(X, Y')$ (Proposition 2.8(d)). From Proposition 4.11(c) we see that $e(X', Y') = e(X, Y')$, so that $e(X', X) = e(X', Y') - e(X, Y') = 0$ (Proposition 4.11(b)). Then $X'$ and $X$ are isomorphic (Proposition 4.11(a)), and $X$ can be embedded in $P^3_A$.

Since $H^1(X_0, N_0) = 0$ this proves that the deformations of $X_0$ are unobstructed, hence $\text{Obs} X_0 = 0$.

By Serre duality we have $H^1(X_0, T_{P^1}|_{X_0}) \cong H^1(X_0, \Omega_{P^3}(d - 4)|_{X_0})$. From the twisted restricted Euler sequence

$$0 \longrightarrow \Omega_{P^3}(d - 4)|_{X_0} \longrightarrow \mathcal{O}_{X_0}(d - 5)^4 \longrightarrow \mathcal{O}_{X_0}(d - 4) \longrightarrow 0,$$

the easily proved surjectivity of the induced map $H^0(X_0, \mathcal{O}_{X_0}(d - 5))^4 \rightarrow H^0(X_0, \mathcal{O}_{X_0}(d - 4))$ for $d \geq 5$, and the fact that $H^1(\mathcal{O}_{X_0}(d - 5)) = 0$ we see that $H^1(X_0, T_{P^3}|_{X_0}) = 0$, as claimed.

Finally, by Serre duality $H^2(X_0, T_{P^3}|_{X_0}) \cong H^0(X_0, \Omega_{P^3}(d - 4)|_{X_0})$. Again from the Euler sequence we see that is enough to prove that

$$4 \left( \frac{d - 2}{3} \right) = \dim \kappa H^0(X_0, \mathcal{O}_{X_0}(d - 5))^4 > \dim \kappa H^0(X_0, \mathcal{O}_{X_0}(d - 4)) = \left( \frac{d - 1}{3} \right)$$

for $d \geq 6$, and this is straightforward.

Let us go back to the general situation. Let $(X, A)$ be a deformation of $X_0$. Then $X_1$ is a lifting of $X_0$ to $A_1 = A/m^2_A$, so it can be compared to the trivial lifting $X_0^{A_1} = X_0 \times_{\text{Spec} k} \text{Spec} A_1$.

**Definition.** The Kodaira-Spencer class of $X$ is

$$\kappa_X = e(X_1, X_0^{A_1}) \in (m_A/m^2_A) \otimes \kappa T^1(X_0) \cong \text{Hom}(m_A/m^2_A, T^1(X_0)).$$

The associated linear map $K_{X}: (m_A/m^2_A)^\vee \rightarrow T^1(X_0)$ is called the Kodaira-Spencer map.

Kodaira-Spencer classes and maps have an important functorial property.

**Proposition.** Let $f: A \rightarrow B$ a of complete algebras, $X$ a deformation of $X_0$ on $A$. Then:

(a) $k_{f, X} = (f_\ast \otimes \text{id})(\kappa_X) \in m_B/m^2_B \otimes \kappa T^1(X_0)$;

(b) $K_{f, X} = K_X \circ df: (m_B/m^2_B)^\vee \rightarrow T^1(X_0)$.

**Proof.** The two statements are obviously equivalent; part (a) follows from Proposition 4.13(b). ♦
An alternate and more traditional description of the Kodaira-Spencer map is as follows.

Consider the algebra of dual numbers $\kappa[\epsilon] = \kappa[x]/(x^2)$; call $X_0[\epsilon] = X_0 \times_{\text{Spec } \kappa} \text{Spec } \kappa[\epsilon]$ the trivial deformation. Deformations of $X_0$ on $\kappa[\epsilon]$ are abstract liftings of $X_0$ to $\kappa[\epsilon]$. If $X$ is such a lifting, consider

$$e(X, X_0[\epsilon]) \in (\epsilon) \otimes \kappa T^1(X_0) = T^1(X_0).$$

We get a map from the set of liftings of $X_0$ to $\kappa[\epsilon]$ to $T^1(X_0)$.

If $A$ is a complete algebra and $f: A \to \kappa[\epsilon]$ is a homomorphism of complete algebras, the associated homomorphism of $\kappa$-vector spaces $f_*: m_A/m_A^2 \to (\epsilon) = \kappa$ is an element of $(m_A/m_A^2)^\vee$. In this way we obtain a bijective correspondence of the set of algebra homomorphisms from $A$ to $\kappa[\epsilon]$ with the dual vector space $(m_A/m_A^2)^\vee$.

(6.11) Proposition. Let $X$ be a deformation of $X_0$ over a complete algebra $A$. If $u \in (m_A/m_A^2)^\vee$ and $f: A \to \kappa[\epsilon]$ is the corresponding homomorphism of algebras, then $K_X(u) = e(f_*X, X_0[\epsilon])$.

From this description the linearity of the Kodaira-Spencer map is not obvious.

Proof. Use Proposition 6.10.(b) to reduce to the case $A = \kappa[\epsilon]$, where the statement holds by definition.

The Kodaira-Spencer class of a deformation $X$ is determined by its first-order part $X_1$; conversely the Kodaira-Spencer class determines $X_1$ completely.

(6.12) Proposition. Let $A$ be an artinian algebra with $m_A^2 = 0$. The Kodaira-Spencer class gives a bijective correspondence between isomorphism classes of deformations on $A$ and $m_A \otimes T \cong \text{Hom}(m_A^\vee, T)$.

Proof. This follows immediately from Theorem 4.4.(d).

If $W$ is a vector space, then $\kappa \oplus W$ has a canonical ring structure given by $(a, x)(b, y) = (ab, bx + ay)$. If $w_1, \ldots, w_r$ is a basis of $W$ then

$$\kappa \oplus W = \kappa[[w_1, \ldots, w_r]]/m_\kappa^2[[w_1, \ldots, w_r]].$$

An artinian algebra $A$ with $m_A^2 = 0$ is of the form $\kappa \oplus m_A$.

Assume that $T^1(X_0)$ is finite-dimensional. Set $T = T^1(X_0)$, and consider the artinian algebra $R_1 = \kappa \oplus T^\vee$. Let $V_1$ be the deformation of $X_0$ over $R_1$ corresponding to the identity in $\text{Hom}(T, T) \cong T^\vee \otimes T$ (this is a very important deformation, and we’ll meet it again in the construction of the minimal versal deformation of $X_0$ in Section 7.) Now take the graded algebra $R_2 = \kappa \oplus T^\vee \oplus \text{Sym}^2 T^\vee$, where $\text{Sym}^2 T^\vee$ is the second symmetric power of $T^\vee$. If $t_1, \ldots, t_r$ is a basis for $T^\vee$ then

$$R_2 = \kappa[[t_1, \ldots, t_r]]/m_\kappa^3[[t_1, \ldots, t_r]].$$

Obviously $R_2/\text{Sym}^2 T^\vee = R_1$ and $m_R \text{Sym}^2 T^\vee = 0$.

(6.13) Definition. Assume that $T^1(X_0)$ is finite-dimensional. The first obstruction map of $X_0$ is the linear map

$$Q_{X_0}: \text{Sym}^2 T^1(X_0) \longrightarrow \text{Obs } X_0$$

which corresponds to the obstruction

$$q_{X_0} \in \text{Sym}^2 T^\vee \otimes \text{Obs } X_0 \cong \text{Hom}(\text{Sym}^2 T^1(X_0), \text{Obs } X_0)$$

to lifting $V_1$ from $R_1$ to $R_2$.

This important map induces a vector-valued quadratic form $T \to \text{Obs } X_0$ sending a vector $v \in T$ into $Q_{X_0}(u, v)$. This map has the following interpretation. Consider the algebra $\kappa[\epsilon]$ as before, and choose a vector $u \in T$. Call $X(u)$ the deformation on $\kappa[\epsilon]$ whose Kodaira-Spencer class is $\epsilon \otimes u \in \kappa \epsilon \otimes T$. Set $\kappa[\epsilon'] = \kappa[t]/(t^3)$, with the projection $\kappa[\epsilon'] \to \kappa[\epsilon]$ sending $\epsilon'$ to $\epsilon$. The obstruction to lifting $X(u)$ to $\kappa[\epsilon']$ lives in $\kappa \epsilon'^2 \otimes \text{Obs } X_0 = \text{Obs } X_0$. 

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(6.14) Proposition. The obstruction to lifting $X(u)$ to $\kappa[\epsilon']$ is $Q_{X_0}(u \cdot u)$.

Proof. Think of $u$ as a linear map $u: T^\vee \to \kappa$; there is a unique homomorphism of graded algebras $f: R_2 \to \kappa[\epsilon']$ whose restriction $T \to \kappa\epsilon = \kappa$ is $u$. Call $\phi: \text{Sym}^2 T^\vee \to \kappa\epsilon'^2 = \kappa$ the restriction of $f$; as an element of $\text{Sym}^2 T^\vee = \text{Sym}^2 T$ the map $\phi$ is exactly $u \cdot u$. By Proposition 4.13.(c) the obstruction to lifting $X(u)$ to $\kappa[\epsilon']$ is $\phi(q_{X_0})$; by linear algebra this is equal to $Q_{X_0}(u \cdot u)$. ♣

7 Verbal deformations

(7.1) Definition. A universal deformation of $X_0$ is a deformation $(V, R)$ such that for any deformation $(X, A)$ there is a unique homomorphism of deformations $(\phi, f): (V, R) \to (X, A)$.

So a universal deformation contains all the information about deformations, and is therefore a very good thing to have. Unfortunately universal deformations do not always exist; if $T^0(X_0) \neq 0$, then the trivial deformation $X_0[\epsilon]$ has nontrivial automorphisms (Theorem 4.4.(b)), so there can not exist a unique isomorphism $\phi: f^*V \cong X_0[\epsilon]$, because any such $\phi$ can always be composed with a nontrivial automorphism of $X_0$.

One could feel that imposing the condition of unicity on $\phi$ is being too demanding, and only require unicity for $f$. This is indeed reasonable, from a “functor-theoretic”, as opposed to “stack-theoretic” point of view, and corresponds to the condition of prorepresentability of the functor of isomorphism classes of deformations in [Schlessinger]. Let us call a deformation satisfying this weaker condition a weak universal deformation; these exist for many more schemes $X_0$. For example, assume that $T^0(X_0) \neq 0$ and $T^1(X_0) = 0$, (e.g., when $X_0 = \mathbb{P}^n$ or $X_0 = \mathbb{A}^n, n > 0$). According to Proposition 6.4.(b) all deformations are isomorphic to the trivial deformation, and therefore the trivial deformation $(X_0, \kappa)$ is a weak universal deformation, while a universal deformation does not exists. On the other hand not all projective $X_0$ have weak universal deformation; for example, one can prove that the “banana” curve

\[
\begin{array}{c}
\mathbb{P}^1 \\
\end{array}
\]

consisting of two copies of $\mathbb{P}^1$ glued together at two pairs of rational points does not possess one.

One could think that to obtain an acceptable replacement for the notion of universal deformation it is sufficient to drop the condition of unicity on $f$ and simply require the existence of $f$ and $\phi$; but this turns out to be too weak. For example if $(V, R)$ satisfies this condition then any deformation $(W, S)$ such that there exists a homomorphism $(W, S) \to (V, R)$ also satisfies it.

The correct general notion is the following.

(7.2) Definition. A versal deformation of $X_0$ is a deformation $(V, R)$ such that if $(\eta, p): (X, A) \to (Y, B)$ is a surjective homomorphism of deformations, and $(\psi, g): (V, R) \to (Y, B)$ is a homomorphism, then there exists a homomorphism $(\phi, f): (V, R) \to (Y, B)$ with $(\eta, p) \circ (\phi, f) = (\psi, g)$.

A versal deformation $(V, R)$ is called minimal if the Kodaira-Spencer map $K_V: (m_R/m_R^2)^\vee \to T^1(X_0)$ is an isomorphism.

A minimal versal deformation is often called miniversal.

Actually, the standard definition of a versal deformation only assumes the lifting property above when $A$ and $B$ are artinian. Let us prove that it is equivalent to the definition above.
(7.3) Lemma. Assume that a deformation \((V, R)\) has the property of Definition 7.2 in the case that \(A\) and \(B\) are artinian. Then \((V, R)\) is versal.

Proof. Assume that we are in the situation of Definition 7.2, with \(A\) and \(B\) not necessarily artinian. To give a homomorphism \((\phi, f)\) as above is equivalent to giving a homomorphism

\[(\phi_n, f_n): (V, R) \to (X_n, A_n)\]

for each \(n\) such that the composition of \((\phi_n, f_n)\) with the projection \((X_n, A_n) \to (X_{n-1}, A_{n-1})\) is \((\phi_{n-1}, f_{n-1})\), and

\[(\eta_n, p_n) \circ (\phi_n, f_n) = (\psi_n, g_n)\]

for all \(n\).

Let us assume that we have constructed \((\phi_{n-1}, f_{n-1})\), and consider the commutative diagram

\[
\begin{array}{ccc}
(V, R) & \xrightarrow{(\phi_{n-1}, f_{n-1})} & (X_{n-1}, A_{n-1}) \\
\downarrow{(\psi_n, g_n)} & & \downarrow{(\eta_{n-1}, p_{n-1})} \\
(Y_n, B_n) & \xrightarrow{} & (Y_{n-1}, B_{n-1})
\end{array}
\]

The diagram above induce a homomorphism \((V, R) \to (Z, C)\), where \((Z, C)\) is the fiber product of \((Y_n, B_n)\) and \((X_{n-1}, A_{n-1})\) over \((Y_{n-1}, B_{n-1})\) (Corollary 8.2); this homomorphism can be lifted to a homomorphism \((\phi_n, f_n): (V, R) \to (X_n, A_n)\) with the required properties.

Here are several properties of versal deformations.

(7.4) Proposition. (a) A universal deformation is miniversal.

(b) If \((V, R)\) is a versal deformation and \((X, A)\) is a deformation, there is a homomorphism \((V, R) \to (X, A)\).

(c) If \((V, R)\) is a versal deformation, the Kodaira-Spencer map \(K_V: (m_R/m_R^2)^{\vee} \to T^1(X_0)\) is surjective.

In particular, if \(X_0\) has a versal deformation then \(T^1(X_0)\) is finite dimensional.

(d) Two universal deformations are canonically isomorphic.

(e) Two universal deformations are non-canonically isomorphic.

Proof. Part (b) follows immediately from the definition, if we take \((Y, B) = (X_0, \kappa)\).

For part (c), take an element \(a \in T^1(X_0)\); according to Theorem 4.4.(c), there is a deformation \(X\) on \(\kappa[\varepsilon]\) with \(e(X, X_0[\varepsilon]) = a\). Choose a homomorphism \((\phi, f): (V, R) \to (X, \kappa[\varepsilon])\), and let \(u = f_*: m_R/m_R^2 \to \kappa\) the corresponding element of \((m_R/m_R^2)^{\vee}\). But then \(K_V(u) = a\) because of Proposition 6.11.

For part (a), let \((V, R)\) be universal deformation. To prove that it is versal, let \((\eta, p): (X, A) \to (Y, B)\) be a surjective homomorphism of extensions, \((\psi, g): (V, R) \to (Y, B)\) is a homomorphism. There exists a unique homomorphism \((\phi, f): (V, R) \to (X, A)\). and by unicity \((\eta, p) \circ (\phi, f) = (\psi, g)\).

The Kodaira-Spencer map \(K_V: (m_R/m_R^2)^{\vee} \to T^1(X_0)\) is surjective because of part (c). Let \(u \in (m_R/m_R^2)^{\vee}\) with \(K_V(u) = 0\), and let \(f: R \to \kappa[\varepsilon]\) the corresponding homomorphism. Then \(e(f_*V, X_0[\varepsilon]) = K_V(u) = 0\), so we can choose an isomorphism \(\phi: f_*V \simeq X_0[\varepsilon]\), and we get a homomorphism of extensions \((\phi, f): (V, R) \to (X_0[\varepsilon], \kappa[\varepsilon])\). If we denote by \(k: R \to \kappa[\varepsilon]\) the trivial homomorphism, corresponding to \(0 \in (m_R/m_R^2)^{\vee}\), then \(k_*V = X_0[\varepsilon]\), so \((id, k):(V, R) \to (X_0[\varepsilon], \kappa[\varepsilon])\) is also a homomorphism, hence \(f = k\), and \(u = 0\). It follows that \(K_V\) is injective, and this completes the proof.
Part (d) is standard and easy.

For part (e), take two minimial deformations \((V, R)\) and \((W, S)\). From part (b) we see that there exists homomorphisms \((\phi, f): (V, R) \to (W, S)\) and \((\psi, g): (W, S) \to (V, R)\); but the functoriality of the Kodaira-Spencer map (Proposition 6.10(b)) and the minimality condition insure that \(f_*: m_R/m_R^2 \to m_S/m_S^2\) and \(g_*: m_S/m_S^2 \to m_R/m_R^2\) are isomorphisms. From Lemma 7.5 below we see that \(fg\) and \(gf\) are isomorphisms, so \(f\) and \(g\) are isomorphisms.

\[
(7.5) \text{Lemma.} \quad \text{Let } A \text{ be a complete algebra, } f: A \to A \text{ a homomorphism. If } f_*: m_A/m_A^2 \to m_A/m_A^2 \text{ is surjective then } f \text{ is an automorphism.}
\]

**Proof.** The linear map \(m_A^n/m_A^{n+1} \to m_A^n/m_A^{n+1}\) induced by \(f\) is surjective for all \(n \geq 0\), so \(f_n: A_n \to A_n\) is also surjective, and therefore an isomorphism. Hence

\[
f: A = \lim_n A_n \to \lim_n A_n = A
\]

is also an isomorphism.

The following is a generalization of Proposition 7.4(e), and can be considered as a description of all versal deformations.

\[
(7.6) \text{Proposition.} \quad \text{Let } (V, R) \text{ be a minimal deformation, } t = (t_1, \ldots, t_n) \text{ a sequence of indeterminates, } j: R \to R[[t]] \text{ the inclusion. The deformation } (V, R)[[t]] = (j, V, R[[t]]) \text{ is versal.}
\]

Conversely, if \((W, S)\) is another versal deformation, and \(n\) is the dimension of the kernel of the Kodaira-Spencer map \(K_W: (m_S/m_S^2) \to T^1(X_0)\), then \((W, S)\) is isomorphic to \((V, R)[[t]]\).

**Proof.** Let \((\eta, p): (X, A) \to (Y, B)\) a surjective homomorphism, \((\psi, g): (V, R)[[t]] \to (Y, B)\) a homomorphism. Since \((V, R)\) is versal there will be a homomorphism \((\phi, f'): (V, R) \to (X, A)\) such that \((\eta, p) \circ (\phi, f') = (\psi, g)\).

The homomorphism \(f': R \to A\) can be lifted to a homomorphism \(f: R[[t]] \to A\) such that \(pf = g\); if we choose \(a_i \in m_A\) for each \(i = 1, \ldots, n\) so that \(p(t_i) = a_i\), there is then only a homomorphism \(f: R[[t]] \to A\) such that \(f(x) = f'(x)\) if \(x \in R\), and \(f(t_i) = a_i\) for each \(i\). This homomorphism \(f\) has the desired property.

Since \(f_*j_*V = f'_*V\) the pair \((\phi, f)\) gives a homomorphism \((V, R)[[t]] \to (X, A)\). From the fact that \((\eta, p) \circ (\phi, f') = (\psi, g)\) we get that \(\psi = \eta \circ p_*\phi\); this together with \(pf = g\) implies that \((\eta, p) \circ (\phi, f) = (\psi, g)\), as desired. This proves the first part of the statement.

The diagram below illustrates the proof.

\[
\begin{array}{ccc}
(X, A) & \xrightarrow{(\phi, f')} & (V, R) \xrightarrow{(\text{id}, j)} R[[t]] \\
\downarrow{(\eta, p)} & & \downarrow{(\psi, g)} \\
(Y, B) & \xrightarrow{(\phi, f)} & (X, A)
\end{array}
\]

Now take a versal deformation \((W, S)\). Because of Proposition 7.4(b) there is a homomorphism \((\phi, f'): (V, R) \to (W, S)\); from the functoriality of the Kodaira-Spencer class (Proposition 6.10(b)) and the surjectivity of the Kodaira-Spencer map \(K_W\) (Proposition 7.4(c)) we see that the differential \(df': (m_S/m_S^2) \to (m_R/m_R^2)\) is surjective, so its transpose map \(f'_*: m_R/m_R^2 \to m_S/m_S^2\) is injective. Choose elements \(a_1, \ldots, a_n\) in \(m_S\) whose class in \(m_S/m_S^2\) form a basis for a complement of the image of \(f'_*\), and consider the homomorphism \(f: R[[t]] \to S\) which sends \(x \in R\) into \(f'(x)\), and \(t_i\) into \(a_i\). Let us prove that \((\phi, f): (j_*V, R[[t]]) \to (W, S)\) is an isomorphism. But \(g_1: R[[t]]_1 \to S_1\) is
an isomorphism, so consider the isomorphism of deformations \((\psi_1,g_1)^{-1};(W_1,S_1) \to (j_*V_1,R[|t|]_1)\): this can lifted to a homomorphism \((\psi,g): (W,S) \to (j_*R[|t|])\), because \((W,S)\) is versal. The compositions \(fg\) and \(gf\) induce the identity in \(m_E/m_E^2\) and \(m_R[|t|]/m_R^2[|t|]\) respectively, so from Lemma 7.5 they are isomorphisms. It follows that \(f\) is an isomorphism, and this concludes the proof. ♣

(7.7) Proposition. A miniversal deformation of \(X_0\) is universal if and only if \(T^0(X_0) = 0\).

For this we need a lemma.

Let \(A\) and \(B'\) be artinian algebras, \(b \subseteq B'\) an ideal with \(m_{B'} = 0\). Set \(B = B'/b\), and call \(\pi: B' \to B\) the projection. Let \(f'_1\) and \(f'_2\) be homomorphisms \(A \to B'\) such that \(\pi f'_1 = \pi f'_2 = f\). The difference \(f'_1 - f'_2: A \to b\) is a derivation of the algebra \(A\) into the \(\kappa\)-module \(b\) (this is a standard fact, which we have already used in the proof of Proposition 4.6). So \((f'_1 - f'_2)(m_A^2) = 0\), and \(f'_1 - f'_2\) induces a \(\kappa\)-linear map

\[\Delta(f'_1,f'_2): m_A/m_A^2 \to b.\]

Let \(X\) be a deformation of \(X_0\); then \(f'_1\) and \(f'_2\) are liftings of \(f\) to \(B'\). We want understand when \(f'_1\) and \(f'_2\) are isomorphic; for this we need a formula for

\[e(f'_1X,f'_2X) \in b \otimes T^1(X_0).\]

This turns out to be determined by \(\Delta(f'_1,f'_2)\) and the Kodaira-Spencer class \(k_X \in (m_A/m_A^2) \otimes T^1(X_0)\).

(7.8) Lemma. \(e(f'_1X,f'_2X) = (\Delta(f'_1,f'_2) \otimes id)(k_X) \in b \otimes T^1(X_0)\)

Proof. Set \(V = m_A/m_A^2\). Call \(\rho: A \to A/m_A = \kappa\) the projection, \(D_A: A \to V\) the derivation which sends \(a \in A\) into the class of \(a - \rho(a) \in m_A\). Give to \(A' = A \oplus V\) the obvious ring structure in which \(0 \oplus V\) becomes an ideal with square 0; the multiplication is defined by \((a,x)(b,y) = (ab,bx+ay)\). We call \(\pi: A' \to A\) the projection. The algebra \(A'\) is local, and \(V = 0 \oplus V = \ker \pi \subseteq A'\) is an ideal which is killed by the maximal ideal \(m_{A'} = m_A \oplus V\) of \(A'\). There are two homomorphism of algebras \(i: A \to A'\) and \(u: A \to A'\) defined respectively by \(i(a) = (a,0)\) and \(u(a) = (a,D_A(a))\). Finally, consider the homomorphism of algebras \(F: A' \to B'\) defined by \(F(a,x) = f'_2(a) + \Delta(f'_1,f'_2)(x)\).

Obviously \(F \circ i = f'_2: A \to B'\), while

\[(F \circ u)(a) = F(a,D_A(a))
= f'_2(a) + f'_2(a - \rho(a)) - f'_1(a - \rho(a))
= f'_2(a) + f'_1(a) - \rho(a) - f'_2(a) + \rho(a)
= f'_1(a),\]

so that \(F \circ u = f'_1\).

The deformations \(u_*X\) and \(i_*X\) on \(A'\) are liftings of \(X\); since \(F|_V = \Delta(f'_1,f'_2)\) we get from Proposition 4.13.(b) that \(e(f'_1X,f'_2X) = (\Delta(f'_1,f'_2) \otimes id)e(u_*X,i_*X)\), so it is enough to prove that

\[e(u_*X,i_*X) = k_X.\]

Consider now \(A_1 = A/m_A^2 = \kappa \oplus V\), call \(\sigma: A \to A_1\) the projection. The homomorphism \(h: A' \to A_1\) defined by \(h(a,x) = \rho(a) + x\) has the property that \(h \circ u = \sigma\), while \((h \circ i)(a) = \rho(a) \in A_1\). It follows that \(h_*u_*X = X_1\), while \(h_*i_*X = X_0[\epsilon]\); by Proposition 6.10.(a) applied to the homomorphism \(h\), which sends \(x \in V \subseteq A'\) into \(x \in V \subseteq A_1\), we get that \(e(u_*X,i_*X) = e(X_1,X_0[\epsilon]) \in V \otimes T^1(X_0)\). But \(e(X_1,X_0[\epsilon]) = k_X\) by definition. ♣
Proof of 7.7. Let \((V, R)\) be a deformation and \((\phi, f), (\psi, g): (V, R) \to (X, A)\) two homomorphism; we need to show that \((\phi, f) = (\psi, g)\). If \(f = g\) then \(\phi = \psi\) because of Proposition 6.4.(a), so it is enough to prove the following: if \(f, g: R \to A\) are homomorphisms of complete algebras and \(f_n V\) is isomorphic to \(g_n V\) as a deformation over \(A\), then \(f = g\). Obviously \(f_0 = g_0\); we assume \(n \geq 1\) and prove that if \(f_n^{-1} = g_n^{-1}: R_n^{-1} \to A_n^{-1}\) then \(f_n = g_n: R_n \to A_n\). Then \(f_n = g_n\) for all \(n\), so \(f = g\), as claimed.

Consider \(\Delta(f_n, g_n) : \mathfrak{m}_R / \mathfrak{m}_R^2 \to \mathfrak{m}_A^n / \mathfrak{m}_A^{n+1} \subseteq A_n\). By hypothesis \(f_n X_n\) and \(g_n X_n\) are isomorphic as liftings of \(f_{n-1} X_{n-1} = g_{n-1} X_{n-1}\), so from Lemma 7.8 we get

\[
(\Delta(f_n, g_n) \otimes \text{id})(k_V) = 0 \in \mathfrak{m}_A^n / \mathfrak{m}_A^{n+1} \otimes T^1(X_0).
\]

This is equivalent to saying that the adjoint map

\[
\Delta(f_n, g_n)^\vee : (\mathfrak{m}_A / \mathfrak{m}_A^2)^\vee \to (\mathfrak{m}_R / \mathfrak{m}_R^2)^\vee
\]

composed with the Kodaira-Spencer map \(K_V^\vee: (\mathfrak{m}_R / \mathfrak{m}_R^2)^\vee \to T^1(X_0)\) is 0. But \(K_V\) is an isomorphism, so \(\Delta(f_n, g_n) = 0\), and \(f_n = g_n\), as claimed. ♣

We have seen that if \(X_0\) has a versal deformation then \(T^1(X_0)\) is finite-dimensional (Proposition 7.4.(c)). The main result of this section is that the converse holds.

(7.9) Theorem. If \(T^1(X_0)\) is finite-dimensional then \(X_0\) has a miniversal deformation \((V, R)\).

If \(r = \dim_n T^1(X_0)\) then \(R = \text{of the form } \kappa [[t_1, \ldots, t_r]] / I\) with \(I \subseteq \mathfrak{m}_A^2[[t_1, \ldots, t_r]]\) and the minimal number of generators of \(I\) is the dimension of the space \(\text{Obs } X_0\).

So if \(T^1(X_0)\) is finite-dimensional then the obstruction space \(\text{Obs } X_0\) is finite-dimensional, even thought \(T^2(X_0)\) might not be.

The condition that \(T^1(X_0)\) be finite-dimensional is satisfied for example when \(X\) is proper, or affine with isolated singularities.

Before going to the proof let us draw two consequences.

(7.10) Corollary. Let \((V, R)\) be a miniversal deformation of \(X_0\). Then \(R\) is a power series algebra if and only if \(X_0\) is unobstructed.

The following is a consequence of the theorem and of Proposition 7.7.

(7.11) Corollary. The scheme \(X_0\) has a universal deformation if and only if \(T^0(X_0) = 0\) and \(T^1(X_0)\) is finite-dimensional.

Proof of 7.9. Let us start with a definition.

(7.12) Definition. The order of an artinian algebra \(A\) is the least \(n\) such that \(\mathfrak{m}_A^{n+1} = 0\).

A deformation \((V, R)\) is \(n\)-versal if the condition of Definition 7.2 is satisfied when \(A\) and \(B\) are artinian of order at most \(n\).

Obviously every deformation is 0-versal. A deformation \((V, R)\) is \(n\)-versal of and only if \((V_n, R_n)\) is \(n\)-versal.

The content of Lemma 7.3 is that a deformation is versal if and only if it is \(n\)-versal for all \(n\).

Set \(T = T^1(X_0)\), and let \(t_1, \ldots, t_r\) be a basis for the dual space \(T^\vee\). Assume also that \(\text{Obs } X_0\) is finite-dimensional, with basis \(\omega_1, \ldots, \omega_t\). We do not know a priori that \(\text{Obs } X_0\) is finite-dimensional; if we do not assume this the proof goes through with minor changes in notation. Set

\[
\Lambda = \prod_{i=1}^{\infty} \text{Sym}^i T^\vee = \kappa [[t_1, \ldots, t_r]].
\]

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We will construct $R$ as a quotient of $\Lambda$ by an ideal generated by $\ell$ generators $f_1, \ldots, f_\ell$ in $\Lambda$, with no terms of degree less than 2. If we denote by $f_i^{(n)}$ the part of $f_i$ consisting of terms of degree at most $n$, then $f_i^{(1)} = 0$, and the $f_i^{(n)}$ are constructed by an inductive procedure in such a way that there is an $n$-versal deformation $V_n$ over $R_n = \Lambda/(f_1^{(n)}, \ldots, f_\ell^{(n)}) + m_\Lambda^{n+1}$; then the $f_i$ are defined as the limit of the $f_i^{(n)}$.

We start from the deformation $(V_1, R_1)$ constructed after Proposition 6.12; here $R_1 = \kappa \oplus T' = \Lambda/m_\Lambda^2$, and $V_1$ is a deformation having as the identity $T \to T$ as its Kodaira-Spencer map.

Let us prove that $(V_1, R_1)$ is 1-versal. Let $(X, A)$ and $(Y, B)$ be deformations over artinian algebras of order at most 1 with a surjective homomorphism $(\eta, p): (X, A) \to (Y, B)$, and $(\psi, g): (V_1, R_1) \to (Y, B)$ a homomorphism. The Kodaira-Spencer class $k_X \in m_A \otimes T \simeq \text{Hom}(T', m_A)$ induces a linear map $T' \to m_A$, and an algebra homomorphism $f: R_1 \to A$; by the functoriality of Kodaira-Spencer classes (Proposition 6.10.(a)) and Proposition 6.12 we see that the composition $pf$ is equal to $g$, and $p_*V_1$ is isomorphic to $X$. Fix an isomorphism $\phi': p_*V_1 \simeq X$; if the composition

$$\psi' = \eta \circ p_*\phi' : g_*V_1 = p_*f_*V_1 \to Y$$

is equal to $\psi$ then $(\psi, g) = (\eta, p) \circ (\phi', f)$, and we are done. In general this is not true.

Set $\beta = \eta^{-1} \circ \psi \circ p_*\phi^{-1}: p_*X \simeq p_*X$; then $\beta$ will correspond to an element of $\beta \in m_B \otimes T^0(X_0)$ (Theorem 4.4.(b)). Because of the surjectivity of $p$ the element $\beta$ can be lifted to an element $\overline{\beta} \in m_A \otimes T^0(X_0)$ corresponding to an automorphism $\alpha$ of $X$, and, because of Proposition 4.13.(a), we have $p_*\alpha = \beta$. Set $\phi = \alpha \circ \phi'$; then

$$\eta \circ p_*\phi = \eta \circ p_*\alpha \circ p_*\phi' = \eta \circ \beta \circ p_*\phi' = \eta \circ \eta^{-1} \circ \gamma \circ p_*\phi^{-1} \circ p_*\phi = \psi,$$

so $(\phi, f)$ gives us the lifting.

Next consider the obstruction $q_{X_0} \in \text{Sym}^2 T' \otimes \text{Obs} X_0 = m^{2}_A/m^{3}_A \otimes \text{Obs} X_0$, as in Definition 6.13, which we write as

$$q_{X_0} = \sum_{i=1}^{\ell} f_i^{(2)} \otimes \omega_i,$$

and think of $f_i^{(2)}$ as homogeneous polynomials of order 2. Then we set

$$R_2 = (\kappa \oplus T' \oplus \text{Sym}^2 T')/(f_1^{(2)}, \ldots, f_\ell^{(2)}) = \Lambda/((f_1^{(2)}, \ldots, f_\ell^{(2)}) + m^{3}_\Lambda);$$

because of Proposition 4.13.(c) the obstruction to lifting $V_1$ to $R_2$ vanishes, so we choose an arbitrary lifting $V_2$. This deformation $(V_2, R_2)$ is 2-versal, as we shall see.

In general we proceed by induction. Let us assume that we have lifted $V_1$ to a deformation $(V_{n-1}, R_{n-1})$, with $R_{n-1}$ an algebra of the form

$$R_{n-1} = \left( \bigoplus_{k=0}^{n-1} \text{Sym}^k T' \right)/(f_1^{(n-1)}, \ldots, f_\ell^{(n-1)}) = \Lambda/((f_1^{(n-1)}, \ldots, f_\ell^{(n-1)}) + m^{n}_\Lambda) = \Lambda/I_{n-1}$$

where the $f_i^{(n-1)}$ are polynomials in $t_1, \ldots, t_r$ of degree at most $n - 1$ with no terms of degree less than 2. Suppose also that we know that $R_{n-1}$ is $(n-1)$-versal. Then we look for homogeneous
polynomials $g^{(n)}_i$, $g^{(n)}_\ell$ of degree $n$ so that if we set $f^{(n)}_i = f^{(n-1)}_i + g^{(n)}_i$, the obstruction to lifting $V_{n-1}$ to

$$R_n = \left( \bigoplus_{k=0}^n \text{Sym}^k T^\vee \right) / (f^{(n)}_1, \ldots, f^{(n)}_\ell) = \Lambda / ((f^{(n)}_1, \ldots, f^{(n)}_\ell) + \mathfrak{m}_\Lambda^{n+1}) = \Lambda / I_n$$

vanishes.

To find the $g^{(n)}_i$ set

$$\tilde{R}_n = \Lambda / \mathfrak{m}_\Lambda I_{n-1};$$

the projection $\pi: \tilde{R}_n \to R_{n-1}$ has kernel $I_{n-1}/\mathfrak{m}_\Lambda I_{n-1}$. The obstruction $\omega \in (I_{n-1}/\mathfrak{m}_\Lambda I_{n-1}) \otimes \text{Obs} X_0$ to lifting $V_{n-1}$ to $\tilde{R}_n$ can be written as

$$\omega = \sum_{i=1}^\ell u_i \otimes \omega_i,$$

where the $u_i \in I_{n-1}$ are sums of a homogeneous polynomial of degree $n$ and a linear combination of the $f^{(n-1)}_i$ with coefficients in $\kappa$. Set

$$I_n = \mathfrak{m}_\Lambda I_{n-1} + (u_1, \ldots, u_\ell) = \mathfrak{m}_\Lambda^{n+1} + \mathfrak{m}_\Lambda f^{(n-1)}_1 + \cdots + \mathfrak{m}_\Lambda f^{(n-1)}_\ell + (u_1, \ldots, u_\ell) \subseteq \Lambda$$

and consider the algebra

$$R_n = \Lambda / I_n;$$

with its projection $R_n \to R_{n-1}$. By Proposition 4.13.(c) the obstruction to extending $V_{n-1}$ to $R_n$ vanishes. Let us check that $I_n$ is generated by $\mathfrak{m}_\Lambda^{n+1}$ and by $\ell$ polynomials of the form $f^{(n-1)}_i + g^{(n)}_i$, where $g^{(n)}_i$ is homogeneous of degree $n$.

(7.13) Lemma. We have

$$I_{n-1} = \mathfrak{m}_\Lambda^n + (u_1, \ldots, u_\ell)$$

and

$$I_n = \mathfrak{m}_\Lambda^{n+1} + (u_1, \ldots, u_\ell).$$

Proof. Clearly $\mathfrak{m}_\Lambda^n + (u_1, \ldots, u_\ell) \subseteq I_{n-1}$ and $\mathfrak{m}_\Lambda^{n+1} + (u_1, \ldots, u_\ell) \subseteq I_n$, so we only need to show the reverse inclusions.

Consider the natural algebra homomorphism $\pi: R'_{n-1} \overset{\text{def}}{=} \Lambda / (\mathfrak{m}_\Lambda^n + I_n) \to R_{n-1}$. The algebra $R'_{n-1}$ is a quotient of $R_n$, so there is a lifting of $V_{n-1}$ to $R'_{n-1}$. By hypothesis $(V_{n-1}, R_{n-1})$ is $(n-1)$-versal; hence the projection $\pi$ is split by a algebra homomorphism $\rho: R_{n-1} \to R'_{n-1}$. But $R'_{n-1}$ has order at most $n-1$, and the differential $d\rho$ induces an isomorphism of tangent spaces, whose inverse is $d\rho$; it follows that $\rho$ is surjective, and $\pi$ is an isomorphism. So

$$\mathfrak{m}_\Lambda^n + \mathfrak{m}_\Lambda f^{(n-1)}_1 + \cdots + \mathfrak{m}_\Lambda f^{(n-1)}_\ell + (u_1, \ldots, u_\ell) \overset{\text{def}}{=} \mathfrak{m}_\Lambda^n + I_n = I_{n-1} \overset{\text{def}}{=} \mathfrak{m}_\Lambda^n + (f^{(n-1)}_1, \ldots, f^{(n-1)}_\ell)$$

and we can write

$$f^{(n-1)}_i = h_i + \sum_{j=1}^\ell \alpha_{ij} f^{(n-1)}_j + \sum_{j=1}^\ell \lambda_{ij} u_j$$

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where \( h_i \in m_A^\alpha, \alpha_{ij} \in m_A \) and \( \lambda_{ij} \in \Lambda \). The matrix \( \text{Id}_{\ell \times \ell} - (\alpha_{ij}) \in \text{GL}_n(\Lambda) \) is invertible, and the system of equations above implies that

\[
(\text{Id}_{\ell \times \ell} - (\alpha_{ij})) \begin{pmatrix} f_1^{(n-1)} \\ \vdots \\ f_{\ell}^{(n-1)} \end{pmatrix} \in (m_A^n + (u_1, \ldots, u_{\ell}))^{\otimes \ell}
\]

and therefore \( f_i^{(n-1)} \in m_A^n + (u_1, \ldots, u_{\ell}) \) for all \( i \). This proves the inclusion \( I_{n-1} \subseteq m_A^n + (u_1, \ldots, u_{\ell}) \). From this we also get

\[
m_A^{n+1} + m_A f_1^{(n-1)} + \cdots + m_A f_\ell^{(n-1)} \subseteq m_A^{n+1} + (u_1, \ldots, u_{\ell})
\]

and this proves the inclusion \( I_n \subseteq m_A^{n+1} + (u_1, \ldots, u_{\ell}) \).

So we have made progress; we have managed to express \( R_n \) as the quotient of \( \Lambda/m_A^{n+1} \) by an ideal generated by \( \ell \) polynomial \( u_1, \ldots, u_{\ell} \). This is not quite what we want, as the parts of degree less than \( n \) of these polynomials are not necessarily the \( f_i^{(n-1)} \). To fix this we need an elementary lemma.

**Lemma.** Let \( M \) be a finite module over a local ring \( \Lambda \), \( x_1, \ldots, x_{\ell} \) and \( y_1, \ldots, y_{\ell} \) two sequences of generators. Then there exists an invertible \( \ell \times \ell \) matrix \( (\lambda_{ij}) \in \text{GL}_\ell(\Lambda) \) such that \( y_i = \sum_{j=1}^\ell \lambda_{ij} x_i \) for all \( i \).

**Proof.** If the sequences are minimal then any matrix \( (\lambda_{ij}) \) such that \( y_i = \sum_{j=1}^\ell \lambda_{ij} x_i \) for all \( i \) is invertible, so we are done. In general we permute the \( x_i \) and the \( y_i \) so that \( x_1, \ldots, x_p \) and \( y_1, \ldots, y_p \) are minimal sequences of generators, and write \( y_i = \sum_{j=1}^p \lambda_{ij} x_i \) for all \( 1 \leq i \leq p \), \( y_i = x_i + \sum_{j=1}^p \lambda_{ij} x_j \) for \( p + 1 \leq i \leq \ell \). The resulting matrix

\[
\begin{pmatrix}
\lambda_{11} & \cdots & \lambda_{1p} & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{p1} & \cdots & \lambda_{pp} & 0 & 0 & \cdots & 0 \\
\lambda_{p+11} & \cdots & \lambda_{p+1p} & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{\ell1} & \cdots & \lambda_{\ell p} & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

is invertible.

Now we apply the lemma to the ideal

\[
\frac{m_A^n + (u_1, \ldots, u_{\ell})}{m_A^n} = \frac{m_A^n + (f_1^{(n-1)}, \ldots, f_{\ell}^{(n-1)})}{m_A^n} \subseteq \Lambda_n
\]

and conclude that there is an invertible matrix \( (\lambda_{ij}) \in \text{GL}_{\ell \times \ell}(\Lambda) \) with \( f_i^{(n-1)} \equiv \sum_{j=1}^\ell \lambda_{ij} u_j \) (mod \( m_A^n \)) for all \( i \); therefore there are homogeneous polynomials \( g_i^{(n)} \in \Lambda \) of degree \( n \) such that

\[
f_i^{(n-1)} + g_i^{(n)} \equiv \sum_{j=1}^\ell \lambda_{ij} u_j \pmod{m_A^{n+1}}
\]

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for all \( i \), so that the \( f_i^{(n)} = f_i^{(n-1)} + g_i^{(n)} \) generate \( I_n \), as we wanted.

Lift \( V_{n-1} \) to a deformation \( V_n \) over \( R_n \). We want to prove that \( (V_n,R_n) \) is \( n \)-versal.

We know that \( (V_n,R_n) \) is \( (n-1) \)-versal, because \( R_{n-1} = R_n/m^n_{R_n} \), and \( (V_{n-1},R_{n-1}) \) is \( (n-1) \)-versal. Let \((\eta,p):(X,A) \to (Y,B)\) a surjective homomorphism of deformations with \( A \) and \( B \) algebras of order at most \( n \), \((\varphi,g):(V_n,R_n) \to (Y,B)\) a homomorphism; set \( a = \ker p \).

First of all assume that \( a \subseteq m^n_A \). In this case the induced homomorphism \( A_{n-1} \to B_{n-1} \) is an isomorphism. Lift \( g:R_n = \Lambda/I_n \to B \) to a homomorphism \( F':\Lambda \to A \) by lifting the images of the \( t_i \) in \( B \) to \( A \); clearly \( F'(m^n_{A|I_{n-1}}) \subseteq m_A(m^n_A + a) = 0 \), so \( F' \) induces a homomorphism \( F:R_n = \Lambda/m_AI_{n-1} \to A \). By functoriality the obstruction \( \sum_{i=1}^{\ell} u_i \otimes \omega_i \in I_{n-1}/m_AI_{n-1} \otimes T^1(X_0) \) to lifting \( V_{n-1} \) from \( R_{n-1} \) to \( \tilde{R}_n \) maps to 0 in \( m^n_A \otimes T^2(X_0) \), so \( F'(u_i) = 0 \) for all \( i = 1,\ldots,\ell \), and \( F \) induces a homomorphism \( f':R_n \to A \) which is a lifting of \( g \). The isomorphisms \( \psi: g_*V_n = p_* f'_n V_n \cong Y \) and \( \eta: p_* X \cong Y \) make \( f'_n V_n \) and \( X \) into liftings of \( Y \); if there were an isomorphism of liftings \( \phi: f'_n V_n \cong X \) then \((\eta,p) \circ (\phi,f) = (\psi,g)\). In general there is no such isomorphism, so consider the element \( e(X,f'_n V_n) \in a \otimes T \), and the corresponding linear map \( u:T^\vee \to a \). Now we apply Lemma 7.8: if \( f:R_n \to A \) is a homomorphism such that \( \Delta(f,f') = u \), then \( e(f_*V_n,f'_*V_n) = (u \otimes \text{id})(k_{V_n}) \). But \( k_{V_n} \in T^\vee \otimes T \cong \text{Hom}(T,T) \) is the element corresponding to the identity, so \((u \otimes \text{id})(k_{V_n}) = e(X,f'_n V_n) \), and \( e(f_*V_n,X) = e(f_*V_n,f'_*V_n) - e(X,f'_n V_n) = 0 \). Therefore \( f_*V_n \) and \( X \) are isomorphic as liftings, and the conclusion follows.

In the general case one can factor \( p:A \to B \) as the projection \( \pi:A \to A/(m^n_A \cap a) \) followed by a homomorphism \( A/(m^n_A \cap a) \to B \); if we can lift \((\psi,g):(V,R) \to (Y,B)\) to a homomorphism \( (V,R) \to (\pi_*X,A/(m^n_A \cap a)) \) then it remains to lift along the homomorphism \( \pi \), whose kernel is contained in \( m^n_A \), and this can be done by the previous case.

Consider the cartesian diagram

\[
\begin{array}{ccc}
(\pi_*X,A/(m^n_A \cap a)) & \longrightarrow & (Y,B) \\
\downarrow & & \downarrow \\
(X_{n-1},A_{n-1}) & \longrightarrow & (Y_{n-1},B_{n-1})
\end{array}
\]

(Corollary 8.2). Because \((V_n,R_n)\) is \((n-1)\)-versal we get a lifting of the composition of

\[
(id,\rho) \circ (\psi,g):(V_n,R_n) \to (Y_{n-1},B_{n-1}),
\]

where \( \rho:A \to A_{n-1} \) is the projection, to a homomorphism \( (V_n,R_n) \to (X_{n-1},A_{n-1}) \); from the diagram above we get a lifting \((V_n,R_n) \to (X,A/(m^n_A \cap a)) \), and we conclude that \((V_n,R_n)\) is \( n \)-versal.

So by taking as \( R = \Lambda/I \), where \( I = (f_1,\ldots,f_\ell) \), where each \( f_i \) is the limit of the \( f_i^{(n)} \), we get a deformation \((V,R)\) which is \( n \)-versal for each \( n \). Since every artinian algebra has an order, this means that \((V,R)\) has the property of Definition 7.2 in the case that \( A \) and \( B \) are artinian. Lemma 7.3 implies that \((V,R)\) is versal.

Now we only have to prove that the minimal number of generators of the ideal \( I = (f_1,\ldots,f_\ell) \) is \( \ell = \dim \text{Ob}_{\Lambda} X_0 \). For this we will produce a surjective linear map \((I/m^\Lambda I)^\vee \to \text{Ob}_{\Lambda} X_0 \).

Consider the ideal \( J_n = I/(I \cap m^{n+1}_A) \) in \( \Lambda_n = \Lambda/m^{n+1}_A \); we have \( \Lambda_n/J_n = R_n \). Take the induced surjective morphism

\[
I/m^\Lambda I \to J_n/m^\Lambda J_n.
\]

(7.15) Lemma. The surjective morphism \( I/m^\Lambda I \to J_n/m^\Lambda J_n \) is an isomorphism for \( n \gg 0 \).

Proof. This is equivalent to saying that \( I \cap m^{n+1}_A \subseteq m_A I \) for \( n \gg 0 \), which follows from the Artin-Rees lemma. ♦
Now consider the obvious surjective homomorphism of algebras $\Lambda_n/\mathfrak{m}_\Lambda J_n \to R_n$; the deformation $(V_{n-1}, R_{n-1})$ has an obstruction $\omega \in (J_n/\mathfrak{m}_\Lambda J_n) \otimes \text{Obs} \ X_0$. Let us show that the associated linear map $u: (J_n/\mathfrak{m}_\Lambda J_n)^\vee \to \text{Obs} \ X_0$ is surjective for $n \gg 0$.

In fact we can find a basis $\omega_1, \ldots, \omega_\ell$ of $\text{Obs} \ X_0$ and for each $i$ a small extension of artinian algebras $A_i' \to A_i$ with kernel $\mathfrak{a}_i \simeq \kappa$ and a deformation $(X_i, A_i)$ whose obstruction in $\mathfrak{a}_i \otimes \text{Obs} \ X_0 \simeq \text{Obs} \ X_0$ is exactly $\omega_i$. Now pick a homomorphism $(\phi, f_i): (V_n, R_n) \to (X_i, A_i)$ for some $n \gg 0$, and lift $f_i: R_n \to A_i$ to a homomorphism $f_i': \Lambda_n/\mathfrak{m}_\Lambda J_n \to A_i'$. Call $g_i: J_n/\mathfrak{m}_\Lambda J_n \to a \simeq \kappa$ the restriction of $f_i'$; by the functoriality of the obstruction class (Proposition 4.13.(c)) we have

$$ (g_i \otimes \text{id})(\omega) = \omega_i, $$

which is equivalent to $u(g_i) = \omega_i$. This proves the surjectivity of $u$, and concludes the proof.

In the unobstructed case the miniversal deformations are easy to characterize.

(7.16) Corollary. Assume $\text{Obs} \ X_0 = 0$. A deformation $(X, A)$ is miniversal if and only if $A$ is a power series algebra over $\kappa$ and the Kodaira-Spencer map $K_X: (m_A/m_A^2)^\vee \to T^1(X_0)$ is an isomorphism.

Proof. Let $(V, R)$ be a miniversal deformation constructed above; then $R$ is a power series algebra. Let

$$ (\phi, f): (V', R') \to (V, R) $$

be a homomorphism; then $(X, A)$ is miniversal if and only if $f$ is an isomorphism, and it easy to check that this happens if and only if $A$ is a power series algebra and the differential $df: (m_A/m_A^2)^\vee \to (m_R/m_R^2)^\vee$ is an isomorphism. By Proposition 6.10.(b) $df: (m_A/m_A^2)^\vee \to (m_R/m_R^2)^\vee$ is an isomorphism if and only if $K_X: (m_A/m_A^2)^\vee \to T^1(X_0)$ is an isomorphism, and this concludes the proof.

The following is one of the simplest nontrivial examples of a versal deformation space.

(7.17) Example. Let $X_0 \subseteq \mathbb{A}_\kappa^n$ a hypersurface with isolated singularities, $n \geq 2$. Call

$$ F \in \kappa[x] \overset{\text{def}}{=} \kappa[x_1, \ldots, x_n] $$

generator of the ideal of $X$ in $\mathbb{A}_\kappa^n$, and set $A = \kappa[x]/(F)$. The ideal

$$ J = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right) R \subseteq R $$

generated by the images in $R$ of the partial derivatives of $F$ is called the Jacobian ideal of $X$. Look at the basic exact sequence

$$ 0 \longrightarrow I_0/I_0^2 \longrightarrow \Omega_{\mathbb{A}_\kappa^n/\kappa} \otimes_{\kappa[x]} A \longrightarrow \Omega_{X_0/\kappa} \longrightarrow 0 \quad (1) $$

where $I_0$ is the ideal of $X_0$ in $\kappa[x]$ (Lemma 4.7). The $A$-module $\Omega_{\mathbb{A}_\kappa^n/\kappa} \mid X_0$ is free of rank $n$, generated by $dx_1, \ldots, dx_n$, while $I_0/I_0^2$ is free of rank 1, generated by the class of $F$. The matrix of the differential $I_0/I_0^2 \to \Omega_{\mathbb{A}_\kappa^n/\kappa}$ is the Jacobian matrix of $F$ restricted to $X_0$, so the image of its adjoint $(\Omega_{\mathbb{A}_\kappa^n/\kappa} \mid X_0)^\vee \to (I_0/I_0^2)^\vee$ is $J(I_0/I_0^2)^\vee \simeq J$. We have an exact sequence

$$ 0 \longrightarrow (\Omega_{X_0/\kappa})^\vee \longrightarrow (\Omega_{\mathbb{A}_\kappa^n/\kappa} \otimes_{\kappa[x]} A)^\vee \longrightarrow (I_0/I_0^2)^\vee \longrightarrow \text{Ext}_A^1(\Omega_{X_0/\kappa}, A) \longrightarrow 0 $$

which gives an isomorphism $\text{Ext}_A^1(\Omega_{X_0/\kappa}, A) \simeq A/J$. This shows that the ideal $J$ is the annihilator of $\text{Ext}_A^1(\Omega_{X_0/\kappa}, A)$ in $A$, so it is an invariant of $X_0$ and does not depend on the embedding $X_0 \hookrightarrow \mathbb{A}_\kappa^n$. 47
The dimension \( \mu = \mu(X_0) \) of \( \text{Ext}_A^1(\Omega_{X_0}/\kappa, A) = A/J \) as a \( \kappa \)-vector space is called the Milnor number of \( X_0 \). Choose polynomials \( G_1, \ldots, G_\mu \in \kappa[x] \) whose images \( g_1, \ldots, g_\mu \) in \( A/J \) form a basis, and consider the subscheme \( V \subseteq A^n_{\kappa[[t]]} \), where \( t = (t_1, \ldots, t_\mu) \), whose ideal is generated by

\[
G = F + t_1 G_1 + \cdots + t_\mu G_\mu \in \kappa[[t]][x];
\]

we denote also by \( V \) the formal deformation on \( \Lambda = \kappa[[t]] \) induced by \( V \). I claim that \((V, R)\) is a miniversal deformation of \( X_0 \).

Observe that from the sequence (1) we get that \( \text{Ext}_A^2(\Omega_{X_0}/\kappa, A) = 0 \), so \( X_0 \) is unobstructed. By Corollary 7.16 it is enough to prove that the Kodaira-Spencer map \( K_V : (m_R/m_R^2)^{\vee} \to T^1(X_0) \) is an isomorphism.

Consider the deformation \((V_1, \Lambda_1)\); we want to calculate \( k_V = e(V_1, X_{0A_1}) \). According to Proposition 4.11.(c) this is the image of

\[
\nu(V_1, X_{0A_1}) \in (m_\Lambda/m_\Lambda^2) \otimes \kappa (I_0/I_0^2)^{\vee}
\]

by the linear map

\[
\text{id} \otimes \partial : (m_\Lambda/m_\Lambda^2) \otimes \kappa (I_0/I_0^2)^{\vee} \longrightarrow (m_\Lambda/m_\Lambda^2) \otimes \kappa \text{Ext}_A^1(\Omega_{X_0}/\kappa, A).
\]

Under the isomorphism of \( A \)-modules \( A \cong (I_0/I_0^2)^{\vee} \) the identity corresponds to the element \( (I_0/I_0^2)^{\vee} = \text{Hom}_A(I_0, A) \) which sends \( F \) to 1, so to calculate the element of

\[
(m_\Lambda/m_\Lambda^2) \otimes \kappa (I_0/I_0^2)^{\vee} \cong (m_\Lambda/m_\Lambda^2) \otimes \kappa A
\]

corresponding to \( \nu(V_1, X_{0A_1}) \) it is enough to calculate \( \nu(V_1, X_{0A_1})(F) \). For this we follow the definition of \( \nu(V_1, X_{0A_1}) \) (Section 2). The image of \( G \in \Lambda[x] \) in \( \Lambda_1[x] \) is an element of the ideal of \( V_1 \) in \( A^n_{\Lambda_1} \) mapping to \( F \) in \( \kappa[x] \), while the image of \( F \) in \( \Lambda_1[x] \) is an element of the ideal of \( X_{0A_1} \) mapping to \( F \) in \( \kappa[x] \). The difference

\[
G - F = \sum_{i=1}^\mu t_i G_i \in m_{\Lambda_1}[x] = (m_\Lambda/m_\Lambda^2) \otimes \kappa[x]
\]
/maps to \( \nu(V_1, X_{0A_1})(F) \) in \( (m_\Lambda/m_\Lambda^2) \otimes A \); this means that the image \( k_{V_1} = e(V_1, X_{0A_1}) \) of \( \nu(V_1, X_{0A_1}) \) in \( \text{Ext}_A^1(\Omega_{X_0}/A) = A/J \) is \( \sum_{i=1}^\mu t_i \otimes g_i \), where the \( g_i \) are the classes of the \( G_i \) modulo \( J \). Because by construction these form a basis of \( A/J \), it follows that the Kodaira-Spencer map is an isomorphism, as claimed.

This example can be generalized to complete intersections of positive dimension with isolated singularities in affine spaces.
8 A TECHNICAL LEMMA

Let $A$ be a noetherian commutative ring, $I_1$ and $I_2$ two nilpotent ideals in $A$; set $\tilde{A} = A/(I_1 \cap I_2)$, $A_1 = A/I_1$, $A_2 = A/I_2$, $A_0 = A/(I_1 + I_2)$, and call $\pi_i: A_i \to A_0$ the two projections. Consider the category $\mathcal{F}$ of flat schemes of finite type over $\tilde{A}$. Define the objects of the category $\mathcal{C}$ to be quintuples $(X_1, X_2, X_0, \alpha_1, \alpha_2)$, where $X_i$ is a flat scheme of finite type over $A_i$ and $\alpha_i: X_0 \to X_i$ is a closed embedding of schemes over $A$ inducing an isomorphism $X_i \mid_{\text{Spec} \, A_0} \simeq X_0$. The arrows from $(X_1, X_2, X_0, \alpha_1, \alpha_2)$ to $(Y_1, Y_2, Y_0, \beta_1, \beta_2)$ in $\mathcal{C}$ are triples $(f_1, f_2, f_0)$ of morphisms $f_i: X_i \to Y_i$ of schemes over $A$ such that $\beta_i f_0 = f_i \alpha_i$ for $i = 1, 2$

There is a functor $\Phi: \mathcal{F} \to \mathcal{C}$ which sends a scheme $X \to \text{Spec} \, \tilde{A}$ into

$$(X \mid_{\text{Spec} \, A_1}, X \mid_{\text{Spec} \, A_2}, X \mid_{\text{Spec} \, A_0}, \alpha_1, \alpha_2)$$

where $\alpha_i: X \mid_{\text{Spec} \, A_0} \to X \mid_{\text{Spec} \, A_i}$ is the obvious embedding.

If $X_i = X \mid_{\text{Spec} \, A_i}$ there are embeddings $\iota_1: X_1 \hookrightarrow X$ and $\iota_2: X_2 \hookrightarrow X$ with $\iota_1 \circ \alpha_1 = \iota_2 \circ \alpha_2$, so from a morphism of $A$-schemes $\phi: X \to Z$ we get two morphisms $\psi_1 = \phi \circ \iota_1$ with $\psi_1 \alpha_1 = \psi_2 \alpha_2$.

(8.1) Lemma. The functor $\Phi$ is an equivalence of categories.

Furthermore the construction above yields a bijective correspondence between morphisms of $A$-schemes $\phi: X \to Z$ and pairs of morphisms $\psi_1: X_1 \to Z$ and $\psi_2 = X_2 \to Z$ with $\psi_1 \alpha_1 = \psi_2 \alpha_2$.

Call $\pi_i: A_i \to A_0$ and $\rho_i: \tilde{A} \to A_i$ the projections.

(8.2) Corollary. Suppose that $A$ is a local artinian $\kappa$-algebra with residue field $\kappa$. Let $X^{(i)}$ be a deformation of $X_0$ over $A_i$ for $i = 0, 1, 2$, $(\phi_i, \pi_i): (X^{(i)}, A_i) \to (X^{(0)}, A_0)$ homomorphism of deformation. Then there is a deformation $X$ over $\tilde{A}$ and homomorphisms $(\phi_i, \rho_i): (X, \tilde{A}) \to (X^{(i)}, A_i)$ such that

$$\begin{align*}
(X, \tilde{A}) & \xrightarrow{(\psi_2, \rho_2)} (X^{(2)}, A_2) \\
& \downarrow (\psi_1, \rho_1) \hspace{5cm} \downarrow (\phi_2, \pi_2) \\
(X^{(1)}, A_1) & \xrightarrow{(\phi_1, \pi_1)} (X^{(0)}, A_0)
\end{align*}$$

is a cartesian diagram of deformations.

Proof. We have $\pi_i^* X^{(i)} = X^{(i)} \mid_{\text{Spec} \, A_0}$, so we get an object $(X^{(1)}, X^{(2)}, X^{(0)}, \alpha_1, \alpha_2)$ of $\mathcal{C}$, where $\alpha_i$ is the composition of the isomorphism $\phi_i^{-1}: X^{(0)} \simeq X^{(i)} \mid_{\text{Spec} \, A_0}$ with the embedding $X^{(i)} \mid_{\text{Spec} \, A_0} \hookrightarrow X^{(i)}$. If $X$ is a flat scheme of finite type over $\tilde{A}$ such that $\Phi(X) \simeq (X^{(1)}, X^{(2)}, X^{(0)}, \alpha_1, \alpha_2)$ then we get isomorphisms $\psi_i: \rho_i^* X = X \mid_{\text{Spec} \, A_i} \simeq X^{(i)}$. Then $(X, \tilde{A})$ is the desired fiber product. ♦

Proof of 8.1. We may assume $A = \tilde{A}$. Let us begin with some algebraic preliminaries.

Consider the category $\mathcal{F}_A$ of flat module over $A$, and the category $\mathcal{F}_{\tilde{A}}$, whose objects are of quintuples $M_\ast = (M_1, M_2, M_0, \alpha_1^M, \alpha_2^M)$

where $M_i$ is a flat module over $A_i$, and $\alpha_i^M: M_i \to M_0$ is a homomorphism of $A$-modules inducing an isomorphism $M_i \otimes_A A_0 \simeq M_0$; we will call these objects flat $A$-modules. The homomorphisms of $A$-modules $f_i: M_i \to N_\ast$ are triples $f_\ast = (f_1, f_2, f_0)$ of homomorphisms of $A$-modules $f_i: M_i \to N_i$ such that $f_0 \alpha_i^M = \alpha_i^N f_i$ for $i = 1, 2$.

The homomorphism $f_\ast$ is called surjective if each $f_i$ is surjective.
There is functor $U: \mathcal{F}_A \to \mathcal{F}_A$, which sends a flat module $M$ into

$$M \otimes A_* = (M \otimes_A A_1, M \otimes_A A_2, M \otimes_A A_0, \alpha_0^M, \alpha_1^M)$$

where $\alpha_i: M_1 \otimes_A A_i \to M_0 \otimes_A A_0$ is induced by the projection $\pi_i: A_i \to A_0$. A homomorphism of flat $A$-modules $f: M \to N$ induces a homomorphism

$$U(f) = (f \otimes \text{id}_{A_1}, f \otimes \text{id}_{A_2}, f \otimes \text{id}_{A_0}): M_* \to N_*,$$

and this makes $U$ into a functor.

We want to show that $U$ is an equivalence of categories; let us construct an inverse $V: \mathcal{F}_A \to \mathcal{F}_A$. If $M_*$ is a flat $A_*$-module then we define $V(M_*)$ to be the equalizer of the pair of morphisms $(\alpha_1^M, \alpha_2^M)$, or, in other words, the kernel of the homomorphism of $A$-modules $\delta_M: M_1 \times M_2 \to M_0$ defined by $\delta_M(x_1, x_2) = \alpha_1^M(x_1) - \alpha_2^M(x_2)$. The obvious homomorphism of rings $A \to A_1 \times A_2$ makes the $A_1 \times A_2$-module $M_1 \times M_2$ into an $A$-module, and $V(M_*)$ is an $A$-submodule. If $f_*: M_* \to N_*$ is a homomorphism of $A_*$-modules then we define $V(f_*)$ to be the restriction of $f_1 \times f_2: M_1 \times M_2 \to N_1 \times N_2$, so that we have a commutative diagram with exact rows

$$0 \to V(M_*) \to M_1 \times M_2 \xrightarrow{\delta_M} M_0 \to 0$$

$$0 \to V(N_*) \to N_1 \times N_2 \xrightarrow{\delta_{N*}} N_0 \to 0.$$

This makes $V$ into a functor from $\mathcal{F}_A_*$ into the category $\mathcal{M}_A$ of $A$-modules. We need to show that $V(M_*)$ is flat over $A$ for any flat $A_*$-module $M_*$, and to produce isomorphisms of functors $UV \simeq \text{id}$ and $VU \simeq \text{id}$.

The chinese remainder theorem gives us an exact sequence of $A$-modules

$$0 \to A \to A_1 \times A_2 \xrightarrow{\pi_1-\pi_2} A_0 \to 0$$

which we can tensor with a flat $A$-module $M$ to get an exact sequence

$$0 \to M \to (M \otimes_A A_1) \times (M \otimes_A A_2) \to M \otimes_A A_0 \to 0.$$

This gives a canonical isomorphism of $A$-modules $M \simeq V(M \otimes A_*)$, which yields an isomorphism between the functor $VU$ and the embedding of $\mathcal{F}_A$ into $\mathcal{M}_A$.

Now take a flat $A_*$-module $M_*$, and set $M = V(M_*)$. The homomorphisms $M \to M_1$ and $M \to M_2$ coming from the inclusion $M \subseteq M_1 \times M_2$, and the homomorphism $M \to M_0$ induced by either of the two projections $M_1 \times M_2 \to M_0$ induce homomorphisms of $A_i$-modules $\phi^M_i: M \otimes_A A_i \to M_i$; we want to show that the $\phi^M_i$ are isomorphisms, and that $M$ is flat. Once this is done we can restrict $V$ to a functor $V: \mathcal{F}_A \to \mathcal{F}_A$, and the isomorphisms $\phi_*: UV(M_*) = M \otimes_A A_* \to M_*$ will give an isomorphism of functors $UV \simeq \text{id}$, as claimed.

Choose a free $A$-module $F$ and a surjective homomorphism $f_0: F \otimes A_0 \to M_0$; this lifts to surjective homomorphisms $f_1: F \otimes A_1 \to M_1$ and $f_2: F \otimes A_2 \to M_2$, yielding a surjective homomorphism $f_*: F_* = F \otimes A_* \to M_*$. Call $K_i$ the kernel of $f_i: F_i \to M_i$; because of the flatness of $M_i$ the restriction $\alpha_i^F_*|_{K_i}: K_i \to K_0$ induces an isomorphism $K_i \otimes_A A_0 \simeq K_0$, so that

$$K_* = (K_1, K_2, K_0, \alpha_1^F_*|_{K_1}, \alpha_2^F_*|_{K_2}).$$
is a flat $A_i$-module. Clearly $F = VU(F) = V(F,)$, and $\phi^F_i: F \otimes_A A_i \rightarrow F_i$ is the identity. Set $K = V(K,.)$. We get a commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & K & \rightarrow & K_1 \times K_2 & \rightarrow & K_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & F & \rightarrow & F_1 \times F_2 & \rightarrow & F_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M & \rightarrow & M_1 \times M_2 & \rightarrow & M_0 & \rightarrow & 0 \\
\end{array}
\]

with exact rows, whose last two columns are also exact; this implies that the first column is also exact.

What follows is a familiar argument in commutative algebra. When we tensor it with $A_i$ we get a right exact sequence, which is the top row of a commutative diagram

\[
\begin{array}{cccccccc}
K \otimes A_i & \rightarrow & F \otimes A_i & \rightarrow & M \otimes A_i & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_i & \rightarrow & F_i & \rightarrow & M_i & \rightarrow & 0 \\
\end{array}
\]

with exact rows. This implies that $\phi^M_i$ is surjective; since $M$ is an arbitrary module $\phi^K_i$ will be surjective too, and by diagram chasing we get that $\phi^M_i$ is an isomorphism, as claimed.

Then $\phi^K_i$ will also be an isomorphism; this means that the sequence

\[
0 \rightarrow K \otimes A_i \rightarrow F \otimes A_i \rightarrow M \otimes A_i \rightarrow 0
\]

is exact, so $\text{Tor}^A_1(A_i, M) = 0$. Since $M \otimes_A A_i = M_i$, and $M_i$ is flat over $A_i$, the flatness of $M$ over $A$ follows immediately from the local criterion of flatness (see [Matsumura], or the proof of Lemma 2.6).

Now we go from flat modules to flat algebras. This is straightforward; call $\mathcal{F}^\text{alg}_{A_i}$ the category of flat $A$-algebras, and $\mathcal{F}^\text{alg}_{A_i^*}$ the category of flat $A_i$-algebras; that is, flat $A_i$-modules $M_i$ such that every $M_i$ is endowed with a structure of $A_i$-algebra, so that $\alpha^M_i$ is a homomorphism of $A_i$-algebras. Then if $M$ is an $A$-algebra the $A_i$-module inherits a structure of $A_i$-algebra, so we get a functors $U^\text{alg}: \mathcal{F}^\text{alg}_{A_i} \rightarrow \mathcal{F}^\text{alg}_{A_i^*}$, conversely if $M_i$ is an $A_i$-algebra then $V(M_i)$ is an $A$-subalgebra of $M_i \times M_2$, so $V(M_i)$ has a natural $A$-algebra structure, and we get a functor $V^\text{alg}: \mathcal{F}^\text{alg}_{A_i^*} \rightarrow \mathcal{F}^\text{alg}_{A_i}$; these together give an equivalence of categories.

Also from the construction we get immediately that if $M_i = U(M)$ and $N$ is an $A$-algebra then there is a bijective correspondence between homomorphisms of algebras $\phi: N \rightarrow M$ and pair of homomorphisms of $A$-algebras $\psi_1: N \rightarrow M_1$ and $\psi_2: N \rightarrow M_2$ such that $\alpha^M_i = \psi_1 \circ \psi^M_2$, obtained by composing $\phi$ with the projections $M \rightarrow M_1$ and $M \rightarrow M_2$.

If $M$ is a flat $A$-algebra of finite type then $M \otimes_A A_i$ is of finite type over $A_i$. On the other hand suppose that $M \otimes_A A_i$ is of finite type over $A_i$, and choose a surjective homomorphism $A_1[x_1, \ldots, x_n] \rightarrow M_1 = M/I_1$. By lifting the images of the $x_i$ to $M$ we get a homomorphism $A[x_1, \ldots, x_n] \rightarrow M$, which is easily seen to be surjective, due to the fact that $I_1$ is nilpotent, so $M$ is also of finite type.

This proves Lemma 8.1 for affine schemes. The general case follows from standard patching arguments, which we omit.
9 References

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