Razborov flag algebras as algebras of measurable functions

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Abstract

These are some brief notes on the translation from Razborov’s recently-developed notion of flag algebra ([9]) into the lexicon of functions and measures on certain abstract Cantor spaces (totally disconnected compact metric spaces).

1 The objects of interest

Consider a universal first-order theory $T$ with equality in a language $L$ that contains only predicate symbols; assume $T$ has infinite models. Examples include the theories of undirected and directed graphs and hypergraphs, possibly with loops.

In [9], Razborov develops a formalism for handling the ‘leading order’ statistics of large finite models of such theories. The central objects of his theory are the positive $\mathbb{R}$-homomorphisms of ‘flag algebras’. Here we shall relate these to measures on a subset of the Cantor space $\prod_{i \leq k} K_i^{N_i}$ (where each $K_i$ is itself some Cantor space) which is compact and such that both set and measure are invariant under the canonical coordinate-permuting action of $\text{Sym}_0(\mathbb{N})$. Examples of such Cantor spaces include the spaces of models over $\mathbb{N}$ of certain kinds of theory, and our identification will begin here. In particular we will identify (under some restrictions) a flag algebra with an algebra of measurable functions on the underlying Cantor space of models, and this will lead to an identification of the positive homomorphisms of the flag algebra with certain measures by virtue of classical results of functional analysis.

This will take us through the first three sections of [9]. Section 4, 5 and 6 of [9] relate to a more precise variational analysis of certain examples of these homomorphisms, and we will not discuss this here. For certain special examples of the theories appearing in the study of flag algebras (particularly the theories of hypergraphs and directed hypergraphs),
the associated probability measures on $\prod_{i \leq k} K_i^{N_i}$ fall into the classical study of ‘partial exchangeability’ in probability theory, and quite complete structure theorems describing all such measures are available following work of Hoover [5] [6], Aldous [1, 2, 3] and Kallenberg [7]. However, these general results do not encompass the variational analysis undertaken in Section 5 of [9] and in [10], and it is not clear that adopting the older probabilistic formalism makes much difference to this. We direct the reader to [4] for a survey of this older theory and its relations with combinatorics.

Although it is not assumed in [9], let us here assume for simplicity that the predicates of $T$ have bounded maximum arity, say by $k$. The general case can be recovered as a suitable ‘inverse limit’ in whichever picture is chosen for the description that follows; the necessary modifications are routine. We will also assume that $L$ is countable, although this assumption can also be dropped with only a cosmetic increase in complexity, and will write $L_i$ for the set of predicates in $L$ of arity $i$ for $i \leq k$. Otherwise we adopt the notation and definitions of [9], and have tried to avoid conflict with such further notation as we introduce ourselves.

Given this arity bound, a general countably infinite model of such a theory $T$ can be identified with a non-uniform coloured looped directed hypergraph of maximal rank $k$, in which for each rank $i$ the ‘colour’ of an $i$-tuple $a \in N_i$ is a truth-assignment for $\ell(a)$ for each predicate $\ell \in L$ of arity $i$, and so may be interpreted as a point of $K_i := \{0, 1\}^{L_i}$. Let us work henceforth only with models whose set of variables (or ‘vertices’) is a subset of $N$. Thus we shall view out models of $T$ as points of the Cantor space $\prod_{i \leq k} K_i^{N_i}$ satisfying all additional constraints imposed by the sentences of $T$; we shall denote the subset of these points by $\Omega$. In general, given a model $M$ of $T$ with vertex set $S$ and $S_1 \subseteq S$ we shall write $M|_{S_1}$ for the submodel of $M$ with vertex set $S_1$. If $T$ is free then all points of this space are possible; otherwise the constraints imposed by $T$ carve out some subspace of $\prod_{i \leq k} K_i^{N_i}$, which is an intersection of a family of finite-dimensional subsets corresponding to individual constraints (since any individual interpretation of a sentence in $T$ over some particular finite set of vertices in $N$ simply carves out some clopen subset of $\prod_{i \leq k} K_i^{N_i}$ depending only on those vertices as coordinates). This set of models is therefore a compact subspace of an abstract Cantor space, and so itself an abstract Cantor space. This subspace is also clearly invariant under permutations of the variable set $N$; let us write $\text{Sym}_N$ for the group of finitely-supported such permutations (it will prove convenient to have imposed the finite-support condition when we come to work with this group later). For convenience, if a theory $T$ is clear from the context, we shall refer to a cylinder subset of $\Omega$ to mean the intersection of $\Omega$ with a cylinder subset of the big product space $\prod_{i \leq k} K_i^{N_i}$.

In fact, our analytic formalism will apply to any closed $\text{Sym}_0$-invariant subspace of $\prod_{i \leq k} K_i^{N_i}$ for compact metric spaces $K_1, K_2, \ldots, K_k$, and the reader is free to view $\Omega$ in this gener-
ality.

The relevant data for what follows are the nonempty Sym\textsubscript{0}-invariant compact subset \( \Omega \) together with the canonical action of the finitely-supported permutation group Sym\textsubscript{0} on the vertex set \( \mathbb{N} \), and with its usual topological structure and Borel \( \sigma \)-algebra \( \Sigma \). By mild notational abuse we will identify an element \( g \in \text{Sym}_0 \) directly with the corresponding autohomeomorphism of \( \Omega \).

We need to allow a few more constructions. Henceforth all finite models of \( T \) will be assumed to have vertex sets equal to initial segments of the integers. For a given such finite model, say \( \sigma \) on the vertex set \([k]\), we let \( \Omega_\sigma \) be the subset of those \( \omega \in \Omega \) with \( \omega|_k = \sigma \), and let \( \Sigma_\sigma \) be its Borel \( \sigma \)-algebra (which is just the ideal in \( \Sigma \) of the subsets of \( \Omega_\sigma \subseteq \Omega \) in the usual way). This \( \sigma \) is called a ‘type’ in [9]. We write 0 for the empty type. We now consider isomorphism classes of those finite \( T \)-models that contain a distinguished copy \( \sigma \) as a submodel; that is, of pairs \( F = (M, \theta) \) for \( M \) a finite model and \( \theta \) a particular embedding \( \sigma \hookrightarrow M \). These extensions are called ‘\( \sigma \)-flags’ in [9], and \( \mathcal{F}_\sigma \) is written for the collection of them, \( \mathcal{F}_\ell^\sigma \) for the subcollection of those with vertex set of size \( \ell \) (so \( \mathcal{F}_\ell^\sigma = \emptyset \) unless \( \ell \geq k \)), and we can now specify the obvious notions of embedding and isomorphism for flags.

Restricting our attention to \( \sigma \)-flags \( F = (M, \sigma) \) with \( M \) a model on some \([\ell]\) and \( \sigma \) itself serving as its distinguished copy in \( M \) (so we have implicitly ordered the vertices of \( M \) so that \( \sigma = M|_{[k]} \)), we can identify \( F \) with a finite-dimensional cylinder set \( A_F \) in \( \Omega_\sigma \):

\[
A_F := \{ \omega \in \Omega : \omega|_\ell = M \}
\]

(of course, this is also just \( \Omega_M \) in our earlier notation associating \( \Omega_\sigma \) to \( \sigma \); our choice of different letters reflects the different rôles of \( M \) and \( \sigma \) here). In fact, we could have made any assignment \( F \mapsto A_F \) as above but corresponding to any other choice of locations in \( \mathbb{N} \) for the vertices of \( M \) that are not vertices of \( \sigma \), and it would not matter; the above choice is conveniently concrete.

Occasionally we shall need to relate flags over different types \( \sigma \) and the associated spaces \( \Omega_\sigma \). In general, the appearance of \( \sigma \) as a submodel of \( \sigma' \) does not guarantee that specifically \( \sigma = \sigma'|_{[k]} \) for some \( k \); but we can always identify \( \Omega_{\sigma'} \) as a subspaces of \( \Omega_\sigma \) by suitably reordering the vertices of \( \sigma' \). In [9] this technical matter is more-or-less avoided altogether owing to the early decision to work entirely with isomorphism classes of models; in the picture of the spaces \( \Omega_\sigma \), however, we pay this modest price for the sake of retaining a more concrete picture of the points of this space and (eventually) measures on them, which have more classical and easily-analyzed structures in certain other respects.

Finally, we will want to consider coordinate-permutations that preserve \( \Omega_\sigma \); that is, that
permute only coordinates in $\mathbb{N} \setminus [k]$. Let us call the group of these $\text{Sym}_\sigma \leq \text{Sym}_0$ (even though it actually depends only on $k$).

## 2 Some analysis and measure theory

Having laid out the objects of study in the previous section, we will now recall those additional functional-analytic ideas in terms of which we’ll later give an account of flag algebras. The functional analysis needed does not extend beyond a graduate-level introduction to probability theory and the contents of any good second course on functional analysis; Yosida [12], for example, covers our needs.

Let us write $C(\Omega_\sigma)$ and $\mathcal{M}(\Omega_\sigma)$ for the usual Banach spaces of real-valued continuous functions and signed Radon measures on $\Omega_\sigma$ respectively; the Riesz-Kakutani representation identifies $\mathcal{M}(\Omega_\sigma)$ isometrically with $C(\Omega_\sigma)^*$. Moreover, we write $\mathcal{M}^\sigma$ for the subspace of measures supported on $\Omega_\sigma$ and invariant under finitely-supported permutations of the coordinates in $\mathbb{N} \setminus [k]$; we will call these $\sigma$-exchangeable, and sometimes abbreviate this to just ‘exchangeable’ when $\sigma$ is clear from the context. We write $(\mathcal{M}^\sigma)^\perp$ for the annihilator of this considered as a subspace of $C(\Omega_\sigma)^*$,

$$(\mathcal{M}^\sigma)^\perp = \{ f \in C(\Omega_\sigma) : \langle f, \mu \rangle = 0 \ \forall \mu \in \mathcal{M}^\sigma \};$$

as usual, the dual-of-the-quotient Banach space $(C(\Omega_\sigma)/(\mathcal{M}^\sigma)^\perp)^*$ can be isometrically identified with $\mathcal{M}^\sigma$. Let $q_\sigma$ be the quotient map $C(\Omega_\sigma) \to C(\Omega_\sigma)/(\mathcal{M}^\sigma)^\perp$.

Now, given any $f \in C(\Omega_\sigma)$, it ‘looks the same’ as any $f \circ g$ for $g \in \text{Sym}_\sigma$ to all measures in $\mathcal{M}^\sigma$. Write $T_\sigma$ for the tail $\sigma$-subalgebra $\bigcap_{m \geq k+1} \Sigma_{[k] \cup \{m,m+1,\ldots\}}$ of $\Sigma_\sigma$, and (with a slight abuse of notation) $L^\infty(T_\sigma)$ for the space of bounded $T_\sigma$-measurable functions that are defined $\mu$-a.e. for every $\mu \in \mathcal{M}^\sigma$ and under the equivalence relation of “equality $\mu$-a.e. for every $\mu \in \mathcal{M}^\sigma$”. Clearly these are invariant under the action of $\text{Sym}_\sigma$. By the pointwise ergodic theorem for the amenable group $\text{Sym}_\sigma$ (or any more elementary argument for this very specialized example of a $\text{Sym}_\sigma$-system), we may take the average of the compositions $f \circ g$ over different $g$ to obtain some $T_\sigma$-measurable function $\bar{f}$ on $\Omega_\sigma$ which is defined $\mu$-almost everywhere for every $\mu \in \mathcal{M}^\sigma$ and is invariant under $\text{Sym}_\sigma$, and hence actually specifies a member of $L^\infty(T_\sigma)$. Observe that $\bar{f} = \bar{h}$ for $f, h \in C(\Omega_\sigma)$ if and only if $f - h \in (\mathcal{M}^\sigma)^\perp$, and so our map $f \mapsto \bar{f}$ embeds $C(\Omega_\sigma)/(\mathcal{M}^\sigma)^\perp$ as a subspace $V^\sigma$ of $L^\infty(T_\sigma)$; general nonsense now shows also that this is an isometric embedding, so $V^\sigma$ is closed.

Furthermore, $V^\sigma$ is actually a subalgebra of $L^\infty(T_\sigma)$. To show that it is closed under multiplication, we suppose $f, h \in C(\Omega_\sigma)$, and now consider the products $f \cdot (h \circ g)$ for any
sequence of permutations \( g \) that pushes \( h \) ‘further and further out’, in the following sense: for any \( m \geq 1 \), there are finite subsets \( A, B \subset \mathbb{N} \setminus [k] \) such that \( f \) and \( h \) are uniformly \((1/m)\)-close to functions depending only on vertices in \( A \) and \( B \) (respectively), and now we choose \( g \) that moves all points of \( B \) into \( \mathbb{N} \setminus A \). Letting \( m \to \infty \) this gives a sequence \( g_m \) for which, in terms of their dependence on coordinates, \( f \) and \( h \circ g_m \) are closer and closer to independent.

Now the point is that the quotients \( q_\sigma(f \cdot h \circ g) \) converge in \( C(\Omega_\sigma)/(M^\sigma)_\perp \) to a member that depends only on \( q_\sigma(f) \) and \( q_\sigma(h) \) — this follows from an elementary step-function approximation argument and use of all the permutation invariance. Let us call this the \textbf{asymptotic product} of \( q_\sigma(f) \) and \( q_\sigma(g) \). It is now a routine exercise to check this is actually a \( C^* \)-algebra product on \( C(\Omega_\sigma)/(M^\sigma)_\perp \) and corresponds exactly to the usual product of functions in \( V^\sigma \).

(Alternatively, one can find an actual continuous function on \( \Omega_\sigma \) whose image in \( V^\sigma \) represents this product; let us illustrate one cheap way to do this in case \( \sigma = 0 \). Let \( \psi_1 : 2\mathbb{N} \to \mathbb{N} \) and \( \psi_2 : 2\mathbb{N} + 1 \to \mathbb{N} \) be bijections, and let us use the same letters for the corresponding adjoint maps \( \Omega \to \Omega^{(2\mathbb{N})} \cong \Omega \) and \( \Omega \to \Omega^{(2\mathbb{N}+1)} \cong \Omega \), where we temporarily write \( \Omega^{(S)} \) for the space of models of \( T \) with vertex set \( S \). Then clearly the functions \( f \circ \phi_1 \circ \psi_1 \) and \( g \circ \phi_2 \circ \psi_2 \) depend on disjoint sets of coordinates; their product is the sought representative for \( q_\sigma(f)q_\sigma(g) \).

Thus we have identified \( C(\Omega_\sigma)/(M^\sigma)_\perp \) with a closed subalgebra \( V^\sigma \) of \( L^\infty(T_\sigma) \) (with a newly-defined product). Let us call functions in \( V^\sigma \) \textbf{simple} if they are the images of simple (equivalently, finite-dimensional) functions in \( C(\Omega_\sigma) \). We can describe the simple functions naturally as follows: to any fixed nonempty cylinder set \( A \subseteq \Omega_\sigma \) depending on coordinates in \( J \subset \mathbb{N} \setminus [k] \) corresponds a collection of finite models of the theory \( T \) on the vertex set \( J \) (with some multiplicities), and now the averaged-function \( \overline{T_A}(\omega) \) for \( \omega \in \Omega_\sigma \) is just the sum of the densities with which each of those finite models appears isomorphically as a submodel of \( \omega \) (now summing over the multiplicities). Referring to such a function \( \overline{T_A} \) for \( A \) corresponding to a single model on \( J \) (so that our general \( A \) is a disjoint union of such) as a \textbf{statistics function}, the simple functions in \( V^\sigma \) are now just linear combinations of statistics functions. We write \( V^\sigma_0 \) for the dense subspace of these.

### 3 Description of flag algebras

We will now identify a flag algebra and its homomorphisms with a family of measurable functions and an associated set of probability measures, and show how various results per-
taining to the former translate into facts about the latter. In general we will refer to the two resulting pictures as the ‘flag algebra picture’ and the ‘measure theoretic picture’ respectively. We will follow the sectional structure of Sections 2 and 3 of [9]. We will not broach the more specific optimization problem studied in Sections 4 and 5 (or, for that matter, the more complete result of [10]): while they too can presumably be translated into the measure theoretic picture, aside from rendering the underlying objects in a better-established analytic light this doesn’t seem greatly to change the arguments that are involved.

Informally, the background purpose of [9] is to set up a family of ‘proxies’ for the statistics of large models of \( T \) that enjoy some additional ‘analytic’ structure making them easier to handle for the study of certain kinds of question; two such questions on extremal statistics are then analyzed in these terms in Section 5 of [9] and in the follow-up paper [10]. In [9] these proxies are certain \( \mathbb{R} \)-valued homomorphisms of flag algebras, and the actual elements of the flag algebras are of secondary importance (although they do continue to appear in a supporting role occasionally later in the paper). The manipulation of flag algebras is mostly for the purpose of setting up these homomorphisms. However, in the measure theoretic picture we can say at once what these homomorphisms correspond to — they are the ergodic \( \text{Sym}_\sigma \)-invariant probability measures on \( \Omega_\sigma \) — and so the effort we expend on setting up the correspondence between flag algebras and certain spaces of measurable functions will ultimately be required only to show that the homomorphisms of the former really are identified with a priori-known objects related to the latter.

**Section 2**

In [9], \( \mathbb{R} \mathcal{F}^\sigma \) denotes the free \( \mathbb{R} \)-vectorspace on \( \mathcal{F}^\sigma \). We have already selected for each \( F \in \mathcal{F}^\sigma \) some associated clopen subset \( A_F \subseteq \Omega_\sigma \), and we will now extend this by associating (with some careful normalization) to each member \( \sum_j \lambda_k F_j \in \mathbb{R} \mathcal{F}^\sigma \) a linear combination of the indicator functions \( 1_{A_F} \) of these \( A_F \), each of which is a continuous function since \( A_F \) is clopen. This will define a linear operator \( \Phi : \mathbb{R} \mathcal{F}^\sigma \to C(\Omega_\sigma) \) with image some funny-shaped subspace contained within the space of simple functions in \( C(\Omega_\sigma) \). It turns out that in order to make contact between the calculus of [9] and addition and multiplication of functions on \( \Omega_\sigma \) we need to introduce some nontrivial normalizing constants: our final identification is

\[
F \mapsto \frac{1}{|F|!} 1_{A_F}; \quad \sum_j \lambda_k F_j \mapsto \sum_j \frac{\lambda_k}{|F_j|!} 1_{A_{F_j}}.
\]
The arbitrariness in our choice of subset $A_F$ corresponding to $F$ is reflected in a similar arbitrariness in this linear map $\Phi$ into $C(\Omega_\sigma)$; however, this disappears at the next step, when we define a flag algebra as a quotient of $\mathbb{R}F^\sigma$.

Let $\mathcal{K}^\sigma$ denote the subspace of $\mathbb{R}F^\sigma$ generated by the linear combinations $\tilde{F} - \sum_{F \in F_\ell^\sigma} p(\tilde{F}, F) F$ for different $\ell \geq |V(\tilde{F})|$, and for certain real numbers $p(\tilde{F}, F)$ given in Definition 1 of [9] (we will suppress their exact form here). From that definition one can check at once that, given our chosen normalization in the definition of $\Phi$ above, the values $p(\tilde{F}, F)$ are such that $\Phi(K_\sigma)$ is precisely the following subspace of $\Phi(RF^\sigma)$:

$$\Phi(a) \in \Phi(\mathcal{K}^\sigma) \text{ if and only if, as a simple function on } \Omega_\sigma, \Phi(a) \text{ may be chopped up further into a linear combination of simple functions corresponding to cylinder sets over some common large finite subset of } \mathbb{N}, \text{ say } \Phi(a) = \sum_{t \in J} b_t 1_{B_t}, \text{ such that we can cluster this sum according to some partition } J = \bigcup_{i \in I} J_i,$$

$$\Phi(a) = \sum_{t} b_t 1_{B_t} = \sum_{i \in I} \sum_{t \in J_i} b_t 1_{B_t},$$

so that for each $i \in I$:

- all the $B_t$ for $t \in J_i$ are isomorphic to some fixed $A_{F_i}, F_i \in F^\sigma$;
- and $\sum_{t \in J_i} b_t = 0$.

Alternatively, we can describe this by saying that some re-arrangement of the terms $b_t 1_{B_t}$ by different permutations of $\mathbb{N} \setminus [k]$ sums to zero: that is, there are $g_1, \ldots, g_t \in \text{Sym}_\sigma$ with $\sum_{t} b_t 1_{g_t(B_t)}$ exactly canceling to zero.

It now follows from a little compactness argument that $\mathcal{K}^\sigma$ is also precisely the set of those $a \in \mathbb{R}F^\sigma$ for which $\Phi(a) \in C(\Omega_\sigma)$ is in the annihilator $(M^\sigma)^{\perp}$ (indeed, that $\Phi(\mathcal{K}^\sigma) \subseteq (M^\sigma)^{\perp}$ is immediate; for the opposite inclusion we can argue that if $a \notin \mathcal{K}^\sigma$ then by witnessing our inability to find a decomposition of $\Phi(a)$ as above for cylinder sets over larger and larger finite subset of $\mathbb{N}$, we can extract some sequence of members of $M(\Omega_\sigma)$ that are invariant for the vertex-permutations in some corresponding exhausting sequence of finite subsets of $\text{Sym}_\sigma$ and that converge to some member of $M^\sigma$ that does not annihilate $\Phi(a)$).

We now consider the quotient space $\mathcal{A}^\sigma := \mathbb{R}F^\sigma / \mathcal{K}^\sigma$. By the above, $q_\sigma \circ \Phi$ factors through this quotient to give an injective map $\Psi : \mathcal{A}^\sigma \to V^\sigma$. Moreover, since any finite-dimensional cylinder set contained in $\Omega_\sigma$ is equivalent under $M^\sigma$ to some $A_F$ upon a suitable

\footnote{Notice that here is an appeal to our restriction to finitely-supported permutations.}
permutation of coordinates in $\mathbb{N} \setminus [k]$, the image of $\Psi$ is actually the subspace $V_0^\sigma$ of all simple functions in $V^\sigma$; as such, it is dense.

The next step is to consider products. The definition of a product for $a, b \in \mathbb{R}F^\sigma / A^\sigma$ given in [9] now just translates into the product of $\Psi(a)$ and $\Psi(b)$ as $L^\infty$-functions on $\Omega^\sigma$; and the proof in [9] that this product is well-defined mostly becomes a proof in the measure theoretic picture that this product remains in the image of $\Psi$ (that is, in $V_0^\sigma$). This completes our identification of the flag algebra $A^\sigma$ with the dense subalgebra $V_0^\sigma$ of $V^\sigma$, which is itself a norm-closed Banach subalgebra of $L^\infty(T^\sigma)$. The basic properties of the product contained in Lemma 2.4, for example, are now immediate.

In Subsection 2.2 of [9] is introduced the ‘downward operator’. This applies when we have a submodel $\sigma'$ of $\sigma$, say with vertex set $[k']$ for some $k' \leq k$. By re-labeling the vertices of $\sigma'$ if necessary, we can identify $\Omega^\sigma$ with a clopen subset of $\Omega^{\sigma'}$ (with $A(\sigma^\prime,\sigma)$, in fact); since this subset is clopen, the extension operator $J : C(\Omega^\sigma) \hookrightarrow C(\Omega^{\sigma'})$ obtained by extending a continuous function on $\Omega^\sigma$ to be identically zero elsewhere is well-defined (in particular, its output is still a continuous function). One checks at once that $J$ factorizes through the quotients $C(\Omega^\sigma) \to V^\sigma \subseteq L^\infty(T^\sigma)$ and $C(\Omega^{\sigma'}) \to V^{\sigma'} \subseteq L^\infty(T^{\sigma'})$. This factorization is the measure theoretic picture of the downward operator. Once again, a normalizing factor appears in [9] to make the sums come out right.

Subsection 2.3 turns to the ‘upwards operator’. The definition of this and the properties it enjoys depend more heavily on the precise shape of the theory $T$ than most of the foregoing, and we shall not translate the results of this subsection in detail. Instead, let us examine only a leading special case of this operator.

Given again some type $\sigma$ on $[k]$ extending $\sigma'$ on $[k']$, $k' \leq k$, we now wish to make a passage from $V^{\sigma'}$ to $V^\sigma$ in the following way. Any point of $\Omega^\sigma$ defines a point of $\Omega^{\sigma'}$ simply by ignoring the vertices in $\{k'+1, k'+2, \ldots, k\}$ (and so sending $\mathbb{N} \setminus [k]$ to $\mathbb{N} - (k-k') \setminus [k] = \mathbb{N} \setminus [k']$); this gives a continuous map from $\Omega^\sigma$ to $\Omega^{\sigma'}$, so that composition gives a homomorphism $C(\Omega^{\sigma'}) \to C(\Omega^\sigma)$. This may descend to a map $V^{\sigma'} \to V^\sigma$ which is then necessarily also a homomorphism: this requires that any member of $(M^{\sigma'})^\perp$ be sent to a member of $(M^\sigma)^\perp$ by this composition map, which in some sense tells us that the space of models $\Omega^\sigma$ is still ‘large enough’ to support a sufficiently large collection of $\text{Sym}_\sigma$-invariant measures compared with $\Omega^{\sigma'}$. A formal version of this property appears in the flag algebra picture as a condition that a certain member of $A^\sigma$ is not a zero-divider, and under this assumption the existence of a suitable homomorphism (which translates into the abovementioned factorizability) is proved directly for flag algebras as Theorem 2.6 in [9]; this property is also related to the more immediate property of a theory that it have the ‘amalgamation property’.
This is an example that can be extended to relate the algebras $A^\sigma$ (or $V^\sigma$) arising from two different theories $T_1, T_2$ given an interpretation of one in terms of the other: this interpretation again defines a continuous map from the Cantor space corresponding to one to that corresponding to the other, and again we then face the question of whether the resulting homomorphism of algebras of continuous functions descends under our quotienting operation. The considerations to this purpose in [9] apply in this general setting, but we will not examine the details further here.

When a homomorphism can be obtained, some of the other results that follow on the properties of this map are now translations of certain basic facts for concrete spaces of measurable functions (to which they already appear structurally very similar): Theorem 2.8(a) of [9], for example, asserts in our picture that $E_{\mu}[fh | \Xi] = fE_{\mu}[h | \Xi]$ if $f$ is already $\Xi$-measurable, and 2.8(b) is the rule of iterated conditional expectations.

Section 3

The overall approach of the first three sections of [9] is to define a flag algebra first, and then to obtain a collection of ‘proxies’ or ‘limit objects’ for the statistics of large models of a theory in terms of them. The rôle of these is to be played by the multiplicative functions $\phi : A^\sigma \to \mathbb{R}$ that are non-negative on the image of any single flag $F \in F^\sigma$; the set of these is written $\text{Hom}^+(A^\sigma, \mathbb{R})$.

Having identified the flag algebra $A^\sigma$ as the dense subalgebra $V_0^\sigma$ of the commutative $C^*$-algebra $V^\sigma$ so that the images of single flags correspond to the single statistics functions, we can easily check that non-negativity on these implies non-negativity on any member of $V_0^\sigma$ that is itself a non-negative-valued function; this follows easily since a non-negative simple function can always be written as a linear combination of indicator functions with non-negative coefficients.

We now observe that given the non-negativity of such a $\phi$ it can be extended to a multiplicative linear functional on the whole of $V^\sigma$. We can now, if we wish, apply certain standard representation theorems to this space: either by further exploiting its vector lattice structure following the results of Yosida and Kakutani, as presented in Section XII.5 of [12], or by complexifying the construction so far and using the (arguably more popular) Gelfand-Naimark Theorem. At any rate, this identifies $\phi$ with a point of the spectrum of $V^\sigma$ (for one or other interpretation of ‘spectrum’). In fact, this identification is more-or-less implicit in Remark 4 of Subsection 3.2 of [9], although there we still require some of the basic structure of $\text{Hom}^+(A^\sigma, \mathbb{R})$ to have been identified.

However, given our identification with the measure-theoretic picture, we have an alterna-
tive to the above. Since $V^\sigma \cong C(\Omega_\sigma)/(\mathcal{M}^\sigma)^\perp$, as a linear functional on $V^\sigma$ we can identify $\phi$ with a member of $\mathcal{M}^\sigma$ (uniquely, since $(C(\Omega_\sigma)/(\mathcal{M}^\sigma)^\perp)^* \cong \mathcal{M}^\sigma$): a $\text{Sym}_\sigma$-invariant measure on the space $\Omega_\sigma$ that we started with, rather than a point of some abstractly-produced new space $\text{Spec} V^\sigma$. This has the advantage that we retain the theory itself $T$, coded as it is into the ‘shape’ of the space $\Omega_\sigma$. It is now easy to check that those measures in $\mathcal{M}^\sigma$ that are multiplicative on $V^\sigma$ are precisely the ergodic $\text{Sym}_\sigma$-invariant probability measures on $\Omega_\sigma$ (these are multiplicative on $V^\sigma$ since any member of $V^\sigma$ is $\mu$-a.s. constant if $\mu \in \mathcal{M}^\sigma$ is ergodic). This establishes the identification of the flag algebra homomorphisms with exchangeable measures.

As remarked in Remark 3 of Section 3 of [9], working with arbitrary multiplicative linear functionals on $A^\sigma$ is problematic: the point is that without non-negativity these need not be extendable to the whole of $V^\sigma$ at all (equivalently, they may not be continuous for the norm topology of $V_0^\sigma$). Indeed, the fact, mentioned in Remark 3, that $A^\sigma$ is a non-finitely-generated free commutative algebra over $\mathbb{R}$, is precisely what would allow us to construct a discontinuous such functional using the axiom of choice.

Now the order defined on $A^\sigma$ in Definition 5 of Section 3 of [9] (in terms of the above notion of ‘positivity’ for a homomorphism, which must be introduced first) is precisely the usual pointwise order on $V_0^\sigma$ as a set of real-valued functions; in the setting of abstract flag algebras, where we are unable to define anything ‘pointwise’, the functionals of $\text{Hom}^+(A^\sigma, \mathbb{R})$ are needed as a replacement to formulate this definition. Given this, Theorem 3.1 requires only that composition with a homeomorphism and conditional expectation are non-negative operators between function spaces. Also, the ‘probability-like’ convergence results of Subsection 3.1 (somewhat based on [8]) now really are about the classical vague topology on a set of probability measures.

In Subsection 3.2 of [9] averages of homomorphisms are taken with respect to actual probability measures; in the measure-theoretic picture these become classical ergodic decompositions. Specifically, given some extension $\sigma$ of $\sigma_0$, we know that upon ordering the vertices of $\sigma$ so that $\sigma_0 = \sigma|_{[k_0]}$, then $\Omega_\sigma = A(\sigma_0, \sigma)$ becomes a clopen subset of $\Omega_{\sigma_0}$; and now if $\mu$ is an ergodic $\text{Sym}_{\sigma_0}$-invariant probability measure on $\Omega_{\sigma_0}$ for which $\mu(\Omega_\sigma) > 0$ we may consider the conditioned measure $\mu(\cdot | \Omega_\sigma) := \mu(\cdot \cap \Omega_\sigma)/\mu(\Omega_\sigma)$ (defined free from any measure-theoretic ambiguity, since $\mu(\Omega_\sigma) > 0$). Since $\text{Sym}_\sigma \leq \text{Sym}_{\sigma_0}$ fixes $\Omega_\sigma$, this defines now a $\text{Sym}_\sigma$-invariant probability measure on $\Omega_\sigma$; however, it may not be ergodic under the action of this subgroup $\text{Sym}_\sigma$, and the resulting ‘ensemble’ of homomorphisms obtained in this subsection is simply its ergodic decomposition.

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2Although we note in passing that obtaining them from a prior construction of the function algebras is reminiscent of the alternative route into the theory of integration and measure established by Segal in [11]: an approach that has had lasting consequences for the formulation of ‘non-commutative integration’ in the setting of general von Neumann algebras.
After translation, most of the results of this subsection are identified with the standard results for actual probability measures that they mimic. For example, the argument that a suitable probability measure on $\text{Hom}^+ (\mathcal{A}^\sigma, \mathbb{R})$ exists with barycentre a given member of $\text{Hom}^+ (\mathcal{A}^\sigma', \mathbb{R})$ for Definition 8 now asserts the existence of ergodic decompositions.

Finally, the results of Subsection 3.3 become ordinary inequalities and continuity results for functions and measures. Theorem 3.14 is the conditional Cauchy-Schwartz inequality. Theorems 3.17 and 3.18 relate to iterated conditional expectations and the pointwise order of functions; essentially the same proofs as in [9] are now the proofs of the basic facts about measures and functions.

Theorems 3.15 and 3.16 are more specific to the study of models of a theory $T$, and here the matter of which formalism we choose for their proof seems quite unimportant; we forego giving the measure-theoretic details.

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