Automorphisms of the Weyl manifold

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Abstract

Assume that \( M \) is a smooth manifold with a symplectic structure \( \omega \). Then Weyl manifolds on the symplectic manifold \( M \) are Weyl algebra bundles endowed with suitable transition functions. From the geometrical point of view, Weyl manifolds can be regarded as geometrizations of star products attached to \((M, \omega)\). In the present paper, we are concerned with the automorphisms of the Weyl manifold corresponding to Poincaré-Cartan class \( \frac{1}{ \nu^2 \nu^2 } [ \sum c_\ell \nu^{2 \ell} ] \in \hat{H}^2(M)[[\nu^2]] \). We also construct modified contact Weyl diffeomorphisms corresponding to symplectic diffeomorphisms of the base symplectic manifold.

1 Introduction

It is well known that the concept of Lie group has a long history. It originated from Sophus Lie who initiated the systematic investigation of group germs of continuous transformations. As can be seen in introduction of a monograph by H. Omori [40], S. Lie seemed to be motivated by the followings:

• To construct a theory for differential equation similar to Galois theory.
• To investigate groups such as continuous transformations that leave various geometrical structure invariant.

It is well known that the theory of Lie groups has expanded in two directions:

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\( c_0 \) is a Čech cocycle corresponding to the symplectic structure \( \omega \).

\( ^2 \)In this paper, we often call them lifts of symplectic diffeomorphisms.
(A) theory of finite-dimensional Lie groups and Lie algebras,
(B) theory expanded to include Banach-Lie groups and diffeomorphisms those
elements leave various geometrical structure invariant.

There are a large number of works from standpoint of (A). With respect to (B),
there are also numerous works which are concerned with Banach-Lie groups and
their geometrical and topological properties (cf. [50]). On the other hand, as to
to groups of diffeomorphisms, it was already known in [39] that a Banach-Lie group
acting effectively on a finite-dimensional smooth manifold is necessarily finite-
dimensional. So there is no way to model a group of diffeomorphisms on Banach
spaces as a manifold. Under the situation above, in the end of 1960s, Omori
established theory of infinite-dimensional Lie groups called “ILB-Lie groups” beyond
Banach-Lie groups, taking ILB-chains as model spaces in order to treat
to groups of diffeomorphisms on a manifold (see [10] for the precise definition).
Shortly after his works, Omori et al. [47] introduced the definition of Lie group
modeled on a Fréchet space equipped with a certain property called “regurality”
by relaxing the conditions of ILB-Lie group. Roughly speaking, regularity means
that the smooth curves in the Lie algebra integrate to smooth curves in the Lie
group in a smooth way (see also [29], [40] and [48]). Using this notion, they
studied subgroups of a group of diffeomorphisms, and the group of invertible
Fourier integral operators with suitable amplitude functions on a manifold. For
technical reasons, they assumed that the base manifold is compact (cf. [1], [2],
[3], [28] and [47]). Beyond a compact base manifold, in order to treat groups
of diffeomorphisms on a noncompact manifold, we need more general category
of Lie groups, i.e. infinite-dimensional Lie groups modeled on locally convex
spaces which are Mackey complete (see §2. See also [11] and [22]).

In this paper, we are concerned with the group \( \text{Aut}(M, *) \) of all modified
contact Weyl diffeomorphisms on a contact Weyl manifold over a symplectic
manifold \((M, \omega)\), where a contact Weyl manifold introduced by A. Yoshioka in
[58] is a geometric realization of star product introduced in [6]. In this context,
a modified contact Weyl diffeomorphism is regarded as an automorphism of star
product. As to the group \( \text{Aut}(M, *) \), we have the following.

**Theorem 1.1**

1. Set

\[
\text{Aut}(M, *) = \{ \Phi \in \text{Aut}(M, *) \mid \Phi \text{ induces the base identity map.} \}.
\]

Then \( \text{Aut}(M, *) \) is a Lie group modeled on a Mackey complete locally convex space.

2. Any element \( \Psi \in \text{Aut}(M, *) \) induces a symplectic diffeomorphism on the
base manifold and there exists a group homomorphism \( p \) from \( \text{Aut}(M, *) \)
onto \( \text{Diff}(M, \omega) \), where \( \text{Diff}(M, \omega) \) is the regular Lie group of all symplectic
diffeomorphisms on the symplectic manifold \((M, \omega)\).

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3 “ILB” means inductive limit of Banach spaces.
4 See Definition 5.1 for the precise definition.
3. The group $\text{Aut}(M, \ast)$ is a Lie group modeled on a Mackey complete locally convex space.\footnote{If the base manifold is compact, the model spaces of $\text{Aut}(M, \ast)$ and $\overline{\text{Aut}}(M, \ast)$ are Fréchet spaces.}

4. Under the same assumption above,

$$1 \to \text{Aut}(M, \ast) \to \text{Aut}(M, \ast) \to \text{Diff}(M, \omega) \to 1$$

is a short exact sequence of Lie groups.

5. The groups $\text{Aut}(M, \ast)$ and $\text{Aut}(M, \ast)$ are regular Lie groups.

We note that this result in formal deformation quantization might be regarded as a counterpart of the result of the regular Lie group structure for the group $G_{\mathcal{F}_0}(N)$ of invertible Fourier integral operators with classical symbols of order 0 on a riemannian manifold $N$ in micro-local analysis (see [3], [11] and [47]). Moreover it is also known that the following sequence

$$1 \to G\Psi_0(N) \to G\mathcal{F}_0(N) \to \text{Diff}(S^*N, \theta) \to 1$$

is exact, where $G\Psi_0(N)$ (resp. $\text{Diff}(S^*N, \theta)$) denotes the group of invertible pseudo-differential operators with classical symbols of order 0 (resp. the group of contact diffeomorphisms on the unit cosphere bundle $S^*N$ with the contact structure $\theta$).

Remark that from the point of view of differential geometry, a contact Weyl manifold might be seen as a “prequantum bundle” over a symplectic manifold $(M, \omega)$ where the symplectic structure $\omega$ is not necessarily integral, and a modified contact Weyl diffeomorphism can be regarded as a quantum symplectic diffeomorphism over a “prequantum bundle”.

As is well known, theory of infinite-dimensional Lie algebras including Kac-Moody algebras has made rapid and remarkable progress for the past two decades involving completely integrable system (Sato’s theory), loop groups, conformal field theory and quantum groups. However, it would be rather difficult for me to review this fruitful field. A definite treatment of the infinite-dimensional Lie algebras is found in Kac [18], Tanisaki [53] and Wakimoto [54].

2 Infinite-dimensional Lie groups

In this section we give a survey of regular Lie groups. For the purpose, we first recall Mackey completeness, see the excellent monographs [17], [22] for details.

**Definition 2.1** A locally convex space $E$ is called a Mackey complete (MC for short) if one of the following equivalent conditions is satisfied:

1. For any smooth curve $c$ in $E$ there is a smooth curve $C$ in $E$ with $C' = c$.
2. If $c : \mathbb{R} \to E$ is a curve such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then $c$ is smooth.
3. **Locally completeness**: For every absolutely convex closed bounded subset \( B \), \( E_B \) is complete, where \( E_B \) is a normed space linearly generated by \( B \) with a norm \( p_B(v) = \inf\{\lambda > 0 | v \in \lambda B\} \).

4. **Mackey completeness**: any Mackey-Cauchy net converges in \( E \).

5. **Sequential Mackey completeness**: any Mackey-Cauchy sequence converges in \( E \).

Here a net \( \{x_\gamma\}_{\gamma \in \Gamma} \) is called Mackey-Cauchy if there exists a bounded set \( B \) and a net \( \{\mu_{\gamma,\gamma'}\}_{(\gamma,\gamma') \in \Gamma \times \Gamma} \) in \( \mathbb{R} \) converging to \( 0 \), such that \( x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B = \{\mu_{\gamma,\gamma'} \cdot x | x \in B\} \).

We recall the fundamentals relating to infinite-dimensional differential geometry.

1. Infinite-dimensional manifolds are defined on Mackey complete locally convex spaces in much the same way as ordinary manifolds are defined on finite-dimensional spaces. In this paper, a manifold equipped with a smooth group operation is referred to as a Lie group. Remark that in the category of infinite-dimensional Lie groups, the existence of exponential maps is not ensured in general, and even if an exponential map exists, the local surjectivity of it does not hold (cf. Definition 2.2).

2. A **kinematic tangent vector** (a tangent vector for short) with a foot point \( x \) of an infinite-dimensional manifold \( X \) modeled on a Mackey complete locally convex space \( F \) is a pair \( (x, X) \) with \( X \in F \), and let \( T_x F = F \) be the space of all tangent vectors with foot point \( x \). It consists of all derivatives \( c'(0) \) at \( 0 \) of smooth curve \( c : \mathbb{R} \to F \) with \( c(0) = x \). Remark that operational tangent vectors viewed as derivations and kinematic tangent vectors via curves differ in general. A kinematic vector field is a smooth section of kinematic vector bundle \( TM \to M \).

3. We set \( \Omega^k(M) = C^\infty(L_{\text{skew}}(TM \times \cdots \times TM, M \times \mathbb{R})) \) and call it the space of **kinematic differential forms**, where “skew” denotes “skew-symmetric”. Remark that the space of kinematic differential forms turns out to be the right ones for calculus on manifolds; especially for them the theorem of de Rham is proved.

Next we give the precise definition of regularity (cf. \([29], [40], [47] \) and \([48] \)):

**Definition 2.2** A Lie group \( G \) modeled on a Mackey complete locally convex space \( \mathfrak{g} \) is called a **regular Lie group** if one of the following equivalent conditions is satisfied

1. For each \( X \in C^\infty(\mathbb{R}, \mathfrak{g}) \), there exists \( g \in C^\infty(\mathbb{R}, G) \) satisfying

\[
(2) \quad g(0) = e, \quad \frac{\partial}{\partial t} g(t) = R_{g(t)}(X(t)),
\]

\(6\) A subset \( B \) is called bounded if it is absorbed by every 0-neighborhood in \( E \), i.e. for every 0-neighborhood \( U \), there exists a positive number \( p \) such that \( [0, p] \cdot B \subset U \).
2. For each $X \in C^\infty(\mathbb{R}, \mathfrak{g})$, there exists $g \in C^\infty(\mathbb{R}, G)$ satisfying

\begin{equation}
    g(0) = e, \quad \frac{\partial}{\partial t} g(t) = L_{g(t)}(X(t)),
\end{equation}

where $R(X)$ (resp. $L(X)$) is the right (resp. left) invariant vector field defined by the right(resp. left)-translation of a tangent vector $X$ at $e$.

The following lemma is useful (cf. [22], [29], [47] and [48]):

**Lemma 2.3** Assume that

\begin{equation}
    1 \to N \to G \to H \to 1
\end{equation}

is a short exact sequence of Lie groups with a local smooth section $j$ from a neighborhood $U \subset H$ of $1_H$ into $G$, and $N$ and $H$ are regular. Then $G$ is also regular.

To end this section, we remark that the fundamental properties of principal regular Lie group bundle $(P, G)$ over $M$. Note that these properties are ordinary properties for principal finite-dimensional Lie group bundles.

1. The parallel transformation is well defined.
2. The horizontal distribution $\mathcal{H}$ of a flat connection is integrable, i.e. there exists an integral submanifold for $\mathcal{H}$ at each point.

### 3 Deformation Quantization

Mathematically the concept of quantization originated from H. Weyl [55], who introduced a map from classical observables (functions on the phase space) to quantum observables (operators on Hilbert space). The inverse map was constructed by E. Wigner by interpreting functions (classical observables) as symbols of operators. It is known that the exponent of the bidifferential operator (Poisson bivector) coincides with the product formula of Weyl type symbol calculus developed by L. Hörmander who established the theory of pseudo-differential operators and used them to study partial differential equations (cf. [23] and [38]).

In the 1970s, supported by the mathematical developments above, Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [6] considered quantization as a deformation of the usual commutative product of classical observables into a noncommutative associative product which is parametrized by the Planck constant $\hbar$ and satisfies the correspondence principle. Nowadays deformation quantization, or moreprecisely, star product has gained support from geometers and mathematical physicists. In fact, it plays an important role to give passage from Poisson algebras of classical observables to noncommutative associative algebras of quantum observables. In the approach above, the precise definition of the space of quantum observables and star product is given in the following way(cf. [6]):

\footnote{Remark that this does not give global splitting of the short exact sequence.}
Definition 3.1 A star product of Poisson manifold \((M, \pi)\) is a product \(\ast\) on the space \(C^\infty(M)[[\hbar]]\) of formal power series of parameter \(\hbar\) with coefficients in \(C^\infty(M)\), defined by

\[
f \ast g = fg + \hbar \pi_1(f, g) + \cdots + \hbar^n \pi_n(f, g) + \cdots, \quad \forall f, g \in C^\infty(M)[[\hbar]]
\]

satisfying
(a) \(\ast\) is associative,
(b) \(\pi_1(f, g) = \frac{1}{2\sqrt{-1}} \{f, g\}\),
(c) each \(\pi_n (n \geq 1)\) is a \(C[[\hbar]]\)-bilinear and bidifferential operator,
where \(\{,\}\) is the Poisson bracket defined by the Poisson structure \(\pi\).

A deformed algebra (resp. a deformed algebra structure) is called a star algebra (resp. a star product). Note that on a symplectic vector space \(\mathbb{R}^{2n}\), there exists the “canonical” deformation quantization, the so-called Moyal product:

\[
f \ast g = f \exp \left[\frac{\nu}{2} \partial_x \wedge \partial_y\right] g,
\]

where \(f, g\) are smooth functions of a Darboux coordinate \((x, y)\) on \(\mathbb{R}^{2n}\) and \(\nu = i\hbar\).

The existence and classification problems of star products have been solved by successive steps from special classes of symplectic manifolds to general Poisson manifolds. Because of its physical origin and motivation, the problems of deformation quantization was first considered for symplectic manifolds, however, the problem of deformation quantization is naturally formulated for the Poisson manifolds as well. For example, Etingof and Kazhdan proved every Poisson-Lie group can be quantized in the sense above, and investigated quantum groups as deformation quantization of Poisson-Lie groups. After their works, for a while, there were no specific developments for existence problems of deformation quantization on any Poisson manifold. The situation drastically changed when M. Kontsevich [20] proved his celebrated formality theorem. As a corollary, he showed that deformation quantization exists on any Poisson manifold. (cf. [9], [13], [20], [42], [45], [52] and [58]).

4 Weyl manifold and contact Weyl manifold over a symplectic manifold

4.1 Definition of (contact) Weyl manifold

As mentioned in the introduction, by Omori-Maeda-Yoshioka, for a symplectic manifold, the notion of Weyl manifold was introduced. Later, Yoshioka [58] proposed the notion of contact Weyl manifold as a bridge joining the theory of Weyl manifold and the Fedosov approach to quantization. In order to recall the construction of a contact Weyl manifold, we have to give the precise definitions of fundamental algebras.
Definition 4.1  1. An associative algebra $W$ is called a Weyl algebra if $W$ is formally generated by $\nu, Z^1, \ldots, Z^n, Z^{n+1}, \ldots, Z^{2n}$ satisfying the following commutation relations:

\[ [Z^i, Z^j] = \nu \Lambda_{ij}, \quad [\nu, Z^i] = 0, \]

where $\Lambda = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$, and the product of this algebra is denoted by $\ast$. This algebra has the canonical involution $\bar{\cdot}$ such that

\[ a \ast b = \bar{b} \ast \bar{a}, \quad \bar{\nu} = -\nu, \quad \bar{Z}^i = Z^i. \]

We also define the degree $d$ by $d(\nu^l Z^\alpha) = 2l + |\alpha|$.

2. A Lie algebra $C$ is called a contact Weyl algebra if $C = \tau C \oplus W$ with an additional generator $\tau$ satisfying the following relations:

\[ [\tau, \nu] = 2\nu^2, \quad [\tau, Z^i] = \nu Z^i, \]

and $\bar{\cdot}$ is naturally extended by $\bar{\tau} = \tau$.

Remark that the relation (5) is nothing but the commutation relation of the Moyal product, and called the canonical commutation relation. It is well known that the ordering problem appears when we realize this algebra explicitly. In this paper, we mainly use the Weyl ordering (the Moyal product). See Appendix 6.2, for related topics.

Definition 4.2  1. A $\mathbb{C}[[\nu]]$-linear isomorphism $\Phi$ from $W$ onto $W$ is called a $\nu$-automorphism of Weyl algebra $W$ if

- (a) $\Phi(\nu) = \nu$,
- (b) $\Phi(a \ast b) = \Phi(a) \ast \Phi(b)$,
- (c) $\Phi(\bar{a}) = \bar{\Phi(a)}$.

2. A $\mathbb{C}[[\nu]]$-linear isomorphism $\Psi$ from $C$ onto $C$ is called a $\nu$-automorphism of contact Weyl algebra $C$ if

- (a) $\Psi$ is an algebra isomorphism,
- (b) $\Psi|_W$ is a $\nu$-automorphism of Weyl algebra.

In order to explain the construction of contact Weyl manifolds, it is useful to recall how to construct prequantum line bundles, which play an crucial role in the theory of Souriau-Kostant (geometric) quantization\(^\text{(57)}\). This bundle is constructed in the following way: Let $\omega$ be an integral symplectic structure, then we have $d(\theta_\alpha) = (\delta \omega)_\alpha$, $d(f_{\alpha\beta}) = (\delta \theta)_{\alpha\beta}$, $c_{\alpha\beta\gamma} = (\delta f)_{\alpha\beta\gamma}$, where $U = \{U_\alpha\}$ is a good covering of a symplectic manifold $(M, \omega)$, $f_{\alpha\beta}$ (resp. $\theta_\alpha$) is a local function (resp. a local 1-form) defined on an open set $U_\alpha \cap U_\beta$ (resp. $U_\alpha$), $d$ is the deRham exterior differential operator, and $\delta$ is the Čech coboundary operator. Setting $h_{\alpha\beta} = \exp[2\pi i f_{\alpha\beta}]$, we see that

\[ \theta_\alpha - \theta_\beta = \frac{1}{2\pi i} d \log h_{\alpha\beta}. \]
This equation ensures the existence of a line bundle defined by

\[ L = \prod (U_\alpha \times \mathbb{C}) / h_{\alpha \beta}, \quad \nabla_\xi (\phi_\alpha 1_\alpha) = (\xi \phi_\alpha + 2\pi i \theta_\alpha (\xi) \phi_\alpha) 1_\alpha. \]

This gives the desired bundle with a connection whose curvature equals \( \omega \).

Inspired by the idea above, Yoshioka proposed the notion of contact Weyl manifold and obtained the fundamental results (cf. [58]). To state the precise definition of contact Weyl manifold and theorems related to them, we recall the definitions of Weyl continuation and locally modified contact Weyl diffeomorphism:

**Definition 4.3** Set \((X^1, \ldots, X^n, Y^1, \ldots, Y^n) := (Z^1, \ldots, Z^n, Z^{n+1}, \ldots, Z^{2n})\) (see Definition [7]). Consider the trivial contact Weyl algebra bundle \( C_U := U \times \mathbb{C} \) over a local Darboux chart \((U; (x, y))\). A section

\[ f^\# := f(x + X, y + Y) = \sum_{\alpha \beta} \frac{1}{\alpha! \beta!} \partial^\alpha \partial^\beta f(x, y)X^\alpha Y^\beta \in \Gamma(C_U) \]

determined by a local smooth function \( f \in C^\infty(U) \) is called a Weyl function, and \# : \( f \mapsto f^\# \) is referred to as Weyl continuation. We denote by \( \mathcal{F}_U \) the set of all Weyl functions on \( U \).

A bundle map \( \Phi : C_U \to C_U \) is referred to as a locally modified contact Weyl diffeomorphism if it is a fiberwise \( \nu \)-automorphism of the contact Weyl algebra and its pull-back preserves the set of all Weyl functions \( \mathcal{F}_U \).

**Definition 4.4** Let \( \pi : C_M \to M \) be a locally trivial bundle with a fiber being isomorphic to the contact Weyl algebra over a symplectic manifold \( M \). Take an atlas \( \{(V_\alpha, \varphi_\alpha)\}_{\alpha \in A} \) of \( M \) such that \( \varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbb{R}^{2n} \) gives a local Darboux coordinate for every \( \alpha \in A \). Denote by \( \Psi_\alpha : C_{V_\alpha} \to C_{U_\alpha} \), a local trivialization and by \( \Psi_{\alpha \beta} = \Psi_\beta \Psi_\alpha^{-1} : C_{U_{\alpha \beta}} \to C_{U_{\beta \alpha}} \) the glueing map, where \( C_{V_\alpha} := \pi^{-1}(V_\alpha) \), \( U_{\alpha \beta} := \varphi_\alpha(V_\alpha \cap V_\beta) \), \( U_{\beta \alpha} := \varphi_\beta(V_\alpha \cap V_\beta) \), \( C_{U_{\alpha \beta}} := \Psi_\alpha(C_{V_\alpha} \cap C_{V_\beta}) \), etc. Under the notations above,

\[ (\pi : C_M \to M, \{\Psi_\alpha : C_{V_\alpha} \to C_{U_\alpha}\}_{\alpha \in A}) \]

is called a contact Weyl manifold\(^8\) if the glueing maps \( \Psi_{\alpha \beta} \) are modified contact Weyl diffeomorphisms.

\(^8\) Let \( W_U \) be a trivial Weyl algebra bundle attached to a Darboux coordinate neighborhood. A bundle map \( \Phi : W_U \to W_U \) is referred to as a local Weyl diffeomorphism if it is a fiberwise \( \nu \)-automorphism of the Weyl algebra and its pull-back preserves the set of all Weyl functions \( \mathcal{F}_U \). Originally, using the notion of local Weyl diffeomorphisms, Omori-Maeda-Yoshioka gave the definition of Weyl manifold.

**Definition 4.5** Let \( \pi : W_M \to M \) be a locally trivial bundle with a fiber being isomorphic to the Weyl algebra over a symplectic manifold \( M \). Take an atlas \( \{(V_\alpha, \varphi_\alpha)\}_{\alpha \in A} \) of \( M \) such that \( \varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbb{R}^{2n} \) gives a local Darboux coordinate for every \( \alpha \in A \). Denote by \( \Phi_\alpha : W_{V_\alpha} \to W_{U_\alpha} \) a local trivialization and by \( \Phi_{\alpha \beta} = \Phi_\beta \Phi_\alpha^{-1} : W_{U_{\alpha \beta}} \to W_{U_{\beta \alpha}} \) the glueing map, where \( W_{V_\alpha} := \pi^{-1}(V_\alpha) \), \( U_{\alpha \beta} := \varphi_\alpha(V_\alpha \cap V_\beta) \), \( U_{\beta \alpha} := \varphi_\beta(V_\alpha \cap V_\beta) \), etc.
Theorem 4.6 Let \((M, \omega)\) be an arbitrary (not necessarily integral) symplectic manifold. For any closed form \(\Omega_M(\nu^2) = \omega + \omega_2 \nu^2 + \omega_4 \nu^4 + \cdots\), where \(\nu = \sqrt{-1} \hbar\) is a formal parameter, there exists a contact Weyl manifold \(C_M\) with a connection \(\nabla^Q\) whose curvature equals \(\text{ad}[\frac{1}{\nu} \Omega_M(\nu^2)]\), and the restriction of \(\nabla^Q\) to \(W_M\) is flat, where \(W_M\) is the Weyl algebra bundle associated to \(M\) equipped with the canonical fiber-wise product \(\hat{\ast}\). This bundle \(C_M\) is called a contact Weyl manifold equipped with a quantum connection \(\nabla^Q\). Yoshioka [58] also proved that the connection \(\nabla^Q|_{W_M}\) is essentially the same as a Fedosov connection \(\nabla^W\). It is known (cf. [58] and [42])

Theorem 4.7 There is a bijection between the space of the isomorphism classes of contact Weyl manifolds with quantum connections and \([\omega] + \nu^2 \mathbb{H}^2_{dR}(M)[[\nu^2]]\), which assigns a class \([\Omega_M(\nu^2)] = [\omega + \omega_2 \nu^2 + \cdots]\) to a contact Weyl manifold \((C_M \to M, \{\psi_\alpha\}, \nabla^Q)\).

Proof It is already known that there is a bijection between the space of the isomorphism classes of Weyl manifolds and \([\omega] + \nu^2 \mathbb{H}^2_{dR}(M)[[\nu^2]]\) (cf. subsection 4.2). Generalizing straightly the proof of this result, we can prove Theorem 4.7. □

The flatness of \(\nabla^Q|_{W_M}\) ensures the existence of a linear isomorphism \(\#\) between \(\mathcal{C}^\infty(M)[[\nu]]\) and \(\mathcal{F}_M\) the space of all parallel sections with respect to the quantum connection restricted to \(W_M\). An element of \(\mathcal{F}_M\) is called a Weyl function. Using this map \(\#\), we can recapture a star product in the following way:

\[
(12) \quad f \ast g = \#^{-1}(\#(f) \hat{\ast} \#(g)).
\]

Furthermore, it is known that the following (cf. [42], see also [8] and [15]):

Theorem 4.8 There is a bijection between the space of the equivalence classes of star products and \([\omega] + \nu^2 \mathbb{H}^2_{dR}(M)[[\nu^2]]\).

4.2 Poincaré-Cartan classes (Deligne relative classes)

We begin this subsection with the fundamental facts and definitions. Set \(\tilde{\tau}_U = \tau + \sum z^i \omega_{ij} Z^j\) where \(U \subset \mathbb{R}^{2n}\) is an open subset and \(\omega_{ij} dz^i \wedge dz^j\) stands for the symplectic structure. Then for any modified contact Weyl diffeomorphism, we may set \(\Psi|_{C_U}(\tilde{\tau}_U) = a \tilde{\tau}_U + F\), where \(a \in C^\infty(U)\), \(F \in \Gamma(W_U)\), where \(W_U\) is a trivial bundle \(W_U = U \times W\). Furthermore it is known that the following (cf. Lemma 2.21 in [38]).

\[\Phi_\alpha(W_{V_{\alpha} \cap V_{\beta}}), \text{ etc.} \]

Under the notations above,

\[
(11) \quad (\pi : W_M \to M, \{\Phi_\alpha : W_{V_{\alpha}} \to W_{U_{\alpha}}\}_{\alpha \in A})
\]

is called a Weyl manifold if the glueing maps \(\Phi_{\alpha\beta}\) are local Weyl diffeomorphisms.
Proposition 4.9 Let $U$ be an open set in $\mathbb{R}^{2n}$, $\Psi$ a modified contact Weyl diffeomorphism and $\phi$ the induced map on the base manifold. Then the pullback of $\tilde{\tau}_\phi(U)$ by $\Psi$ can be written as
\begin{equation}
\Psi^*\tilde{\tau}_\phi(U) = \tilde{\tau}_U + f^\# + a(\nu^2),
\end{equation}
for some Weyl functions $f^\# := \#(f) \in \mathcal{F}_U$ with $\bar{f}^\# = f^\#$ and $a(\nu^2) \in C^\infty(U[[\nu^2]])$.

Definition 4.10 A modified contact Weyl diffeomorphism $\Psi$ is called a contact Weyl diffeomorphism (CWD, for short) if
\begin{equation}
\Psi^*\tilde{\tau}_U = \tilde{\tau}_U + f^\#.
\end{equation}

For a contact Weyl diffeomorphism, we obtain the following (see Corollary 2.5 in [42] and Proposition 2.24 in [58]).

Proposition 4.11 Assume that a map $\Psi$ is a contact Weyl diffeomorphism.
1. If the diffeomorphism $\varphi$ on the base map induced by $\Psi$ is the identity, there exists uniquely a Weyl function $g^\#(\nu^2)$ such that $\Psi = \exp\left[\frac{1}{\nu}g^\#(\nu^2)\right]$.
2. $\Psi|_{W_U} = 1$ if and only if there exists an element $c(\nu^2) \in \mathbb{R}[[\nu^2]]$ such that $\Psi = \exp\left[-\frac{1}{\nu}ad(c(\nu^2))\right]$.

From this proposition, we can define the Poincaré-Cartan class in the following way ([42]). Assume that $W_M = \{(W_{U_\alpha}, \Phi_{\alpha\beta})\}$ is a Weyl manifold. Then $\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha}$ is the identity on each $W_{U_{\alpha\beta\gamma}}$. According to 2 of Proposition 4.11 we have
\begin{equation}
\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha} = \exp\left[\frac{1}{\nu}(c_{\alpha\beta\gamma}(\nu^2))\right], \quad (\exists c_{\alpha\beta\gamma}(\nu^2) \in \mathbb{R}[[\nu^2]]).
\end{equation}
We can show that $\{c_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle, and then it defines a Čech 2-class.

Definition 4.12 We refer to this cocycle (resp. class) as the Poincaré-Cartan cocycle (resp. class) and denote it by $\{c_{\alpha\beta\gamma}\}$ (resp. $c(W_M)$).

For the Poincaré-Cartan class we have the following.

Theorem 4.13 For any $c = c^{(0)} + \sum_{i=1}^{\infty} c^{(2i)}\nu^{2i} (c^{(2i)} \in H^2(M; \mathbb{R}))$ such that $[c^{(0)}]$ corresponds to the class of symplectic 2-form, there exists a family of contact Weyl diffeomorphisms $\{\Psi_{\alpha\beta} : C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}\}$, such that $\Psi_{\alpha\beta\gamma}|_{W_{U_{\alpha\beta\gamma}}} = 1$, where $\Psi_{\alpha\beta\gamma} := \Psi_{\alpha\beta}\Psi_{\beta\gamma}\Psi_{\gamma\alpha}$, and $\{c_{\alpha\beta\gamma}(\nu^2)\}$ defines a Čech 2 cohomology class which coincides with $c$. Moreover there is one to one correspondence between the set $\mathfrak{P}(M)$ of Poincaré-Cartan classes and the set $\mathfrak{W}(M)$ of isomorphism classes of Weyl manifolds.
Proof. The proof is already known in [42], but we here give an outline of it for readers. First we show that for any cocycle \( \{ c_{\alpha \beta}(\nu^2) \} \), there exists a Weyl manifold such that \( c(W_M) = \{ c_{\alpha \beta}(\nu^2) \} \). Suppose that \( c = \sum_{k \geq 0} \nu^{2k} c^{(2k)} \in \check{H}^2(M)[[\nu^2]] \) is given. According to the existence theorem of Weyl manifold in [43], we may start with a Weyl manifold \( W^{(0)}_M \) with a Poincaré-Cartan cocycle \( \{ c^{(0)}_{\alpha \beta} \} \), and changing patching Weyl diffeomorphisms we construct a Weyl manifold with a Poincaré-Cartan class \( c \). Let \( \Phi^{*}_{\alpha \beta} : \mathcal{F}(W_{U_{\alpha \beta}}) \to \mathcal{F}(W_{U_{\beta \alpha}}) \) be the glueing Weyl diffeomorphism of \( W^{(0)}_M \) and let \( \Psi^{*}_{\alpha \beta} \) be its extension as a contact Weyl diffeomorphism. Let \( \{ c^{(2k)}_{\alpha \beta} \} \) be a Čech cocycle belonging to \( \langle 2k \rangle \). Since the sheaf cohomology \( H^2(M; \mathcal{E}) \) of the sheaf of germs \( C^\infty \)-functions \( \mathcal{E} \), there is \( h^{(2)}_{\alpha \beta} \in \check{C}^\infty(U_{\alpha \beta}) \) on each \( U_{\alpha \beta} \) such that

\[
-c_{\alpha \beta \gamma}^{(2)} = h^{(2)}_{\alpha \beta} + \phi^* h^{(2)}_{\beta \gamma} + \phi^* h^{(2)}_{\gamma \alpha}.
\]

Replace \( \Psi^{*}_{\alpha \beta} \) by \( \hat{\Psi}^{*}_{\alpha \beta} = \Psi^{*}_{\alpha \beta} e^{ad(\vartheta h_{\alpha \beta})} \) as glueing diffeomorphism for each \( V_{\alpha} \cap V_{\beta} \neq \emptyset \). Then according to the formula \( \hat{\Psi}^{*}_{\beta \alpha} e^{ad(h)} = e^{ad(\hat{\Psi}^{*}_{\alpha \beta} h \vartheta)} \hat{\Psi}^{*}_{\alpha \beta} e^{ad(h)} \), we see

\[
\hat{\Phi}^{*}_{\alpha \beta} \hat{\Phi}^{*}_{\beta \gamma} \hat{\Phi}^{*}_{\gamma \alpha} = \Psi^{*}_{\alpha \beta} \Psi^{*}_{\beta \gamma} \Psi^{*}_{\gamma \alpha} e^{ad(\vartheta h_{\alpha \beta} \vartheta h_{\gamma \alpha})} e^{ad(\vartheta h_{\beta \gamma} \vartheta h_{\gamma \alpha})} e^{ad(\vartheta h_{\alpha \beta} \vartheta h_{\gamma \alpha})},
\]

where we set \( \tilde{h}_{\alpha \beta} = (h^{(2)}_{\alpha \beta})^{\#} + \nu^2 r_{\alpha \beta}^{\#} \) for a function \( r_{\alpha \beta} \in \check{C}^\infty(U_{\alpha \beta})[[\nu^2]] \). By [43], we have

\[
e^{ad(\vartheta h_{\beta \alpha} \vartheta h_{\gamma \alpha})} e^{ad(\vartheta h_{\alpha \beta} \vartheta h_{\gamma \alpha})} e^{ad(\vartheta h_{\gamma \alpha} \vartheta h_{\alpha \beta})} = e^{\nu^2 c^{(2)}_{\alpha \beta} \vartheta ad(\nu^{-1})} \mod \nu^4.
\]

By working on the term \( \nu^4, \nu^6, \ldots \), we can tune up \( r_{\alpha \beta} \) by recursively, so that

\[
e^{ad(\vartheta h_{\beta \alpha} \vartheta h_{\gamma \alpha})} e^{ad(\vartheta h_{\alpha \beta} \vartheta h_{\gamma \alpha})} e^{ad(\vartheta h_{\gamma \alpha} \vartheta h_{\alpha \beta})} = e^{\nu^2 c^{(2)}_{\alpha \beta} \vartheta ad(\nu^{-1})}.
\]

It follows that \( \{ \hat{\Psi}^{*}_{\alpha \beta} \} \) defines a Weyl manifold \( \hat{W}_M \) with the Poincaré-Cartan class \( c^{(0)} + \nu^2 c^{(2)} \). Repeating a similar argument as above, we can replace the condition \( \mod \nu^4 \) in (16) by \( \mod \nu^5 \). Iterating this procedure, we have a Weyl manifold \( W_M \) such that \( c(W_M) = c \in \check{H}^2(M)[[\nu^2]] \).

Next we would like to show that the above construction does not depend on the cocycle chosen. Let \( \{ c_{\alpha \beta \gamma} \}, \{ c'_{\alpha \beta \gamma} \} \) be Poincaré-Cartan cocycles of \( \{ C_U \}, \{ C'_U \} \) respectively, which give same Poincaré-Cartan classes. Then, there exists \( b_{\alpha \beta} \in \mathbb{R}[[\nu^2]] \) on every \( V_{\alpha} \cap V_{\beta} \neq \emptyset \) such that \( b_{\alpha \beta} = b_{\beta \alpha} + c_{\alpha \beta \gamma} - c_{\alpha \beta \gamma} = b_{\alpha \beta} + b_{\beta \gamma} + b_{\gamma \alpha} \). Note that \( b_{\alpha \beta} \) may be replaced by \( b_{\beta \gamma} + b_{\gamma \alpha} \) such that \( c_{\alpha \beta} + c_{\beta \gamma} + c_{\gamma \alpha} = 0 \). Since \( e^{b_{\alpha \beta} \vartheta ad(\nu^{-1})} \) is an automorphism, we can replace \( \Psi_{\alpha \beta} \) by \( \hat{\Psi}_{\alpha \beta} = \Psi_{\alpha \beta} e^{b_{\alpha \beta} \vartheta ad(\nu^{-1})} \). Since \( b_{\alpha \beta} \vartheta ad(\nu^{-1}) \) is the identity on \( \mathcal{F}(W_{U_{\alpha \beta}}) \), this replacement does not change the isomorphism class of \( \mathcal{F}(W_M) \), but it changes the Poincaré-Cartan cocycle from \( \{ c_{\alpha \beta \gamma} \} \) to \( \{ c'_{\alpha \beta \gamma} \} \). This means that the map from the set of Poincaré-Cartan cocycles into the set \( \mathfrak{M}_M \) of isomorphism classes of Weyl
manifolds induces a map $F$ from the set $\mathcal{PC}_M$ of Poincaré-Cartan classes into $\mathcal{M}_M$.

Next we construct the inverse map $\Psi : W'_M \to W_M$ which induces the identity on the base manifold. Equivalently $\Psi^*$ defines an algebra isomorphism of $F(W'_M)$ onto $F(W_M)$. The isomorphism is given by a family $\{\Psi^*_\alpha\}$ of isomorphisms:

$$\Psi^*_\alpha : F(W'_U) \to F(W'_U),$$

each of which induces the identity map on the base space $U_\alpha$ such that

$$\Psi^*_\alpha \Psi^*_\alpha \Psi^*_\beta^{-1} = \tilde{\Psi}^*_{\alpha \beta}.$$ (18)

If we extend $\Psi^*_\alpha$ to a contact Weyl diffeomorphism (cf. subsection 5.3), then the above replacement makes no change of Poincaré-Cartan cocycle. We use the same notation $\Psi^*_\alpha$ for this contact Weyl diffeomorphism. By (18), and Proposition 4.11, we have

$$\Psi^*_\alpha \Psi^*_\alpha \Psi^*_\beta^{-1} e^{\lambda v} = \tilde{\Psi}^*_{\alpha \beta}.$$ (19)

on a contact Weyl algebra bundle. However this type of replacement changes the Poincaré-Cartan cocycle within the same cohomology class. This means that there is a map from $\mathcal{M}_M$ into $\mathcal{PC}_M$, and which is obviously the inverse map of $F$. □

As mentioned in subsection 4.1, there exists a contact Weyl algebra bundle with a connection $\nabla^Q$ such that its first Chern class coincides with Poincaré-Cartan class (cf. [58]). Then $\nabla^Q|W_M$ gives a flat connection on $W_M$ and there is a one to one correspondence $\sigma$ between the space of parallel sections with respect to $\nabla|W_M$ and $C^\infty(M)[[\hbar]]$. Combining this map with fiberwise star product, we can define a star product: $f \ast g = \sigma(\sigma^{-1}(f) \ast \text{fiberwise} \sigma^{-1}(g))$. Hence we obtain Theorem 4.8.

5 A Lie group structure of $\text{Aut}(M, \ast)$

5.1 Fundamental definitions and representatives

With the preliminaries in the previous section, we give the precise definition of $\text{Aut}(M, \ast)$:

**Definition 5.1**

(20) $\text{Aut}(M, \ast) = \{ \Psi : C_M \to C_M | \text{fiber-wise } \nu \text{-automorphism}, \Psi^*(F_M) = F_M \}$,

(21) $\text{Aut}(M, \ast) = \{ \Psi \in \text{Aut}(M, \ast) | \Psi \text{ induces the base identity map. } \}$.

An element of $\text{Aut}(M, \ast)$ is called a modified contact Weyl diffeomorphism (an MCWD for short).
To illustrate automorphisms of a contact Weyl manifold, we consider the automorphisms of a contact Weyl algebra. For any real symplectic matrix $A \in \text{Sp}(n, \mathbb{R})$, set a $\nu$-automorphism of $C$ by $A Z^i = \sum a^i_j Z^j$ and $\hat{A}^\nu = \nu$. Then we easily have $\hat{A}(a, b) = [\hat{A}a, \hat{A}b]$. Conversely, combining the Baker-Campbell-Hausdorff formula with the Poincaré lemma, we have the following.

**Proposition 5.2 ([58])** If $\Psi$ is a $\nu$-automorphism of contact Weyl algebra, there exists uniquely $A \in \text{Sp}(n, \mathbb{R})$, $F \in \{ a = \sum a^i_j Z^j | \alpha = 0, |\alpha| > 0 \}$, $c(\nu^2) = \sum_{i=0}^{\infty} c^{2i} \nu^{2i} \in \mathbb{R}[\nu^2]$, such that $\Phi = \hat{A} \circ e^{ad(1/\nu(\hat{A} c(\nu^2) + F))}$, where $\hat{A} Z^i = \sum a^i_j Z^j$ and $\hat{A}^\nu = \nu$.

**Remark** This $\nu$-automorphism can be seen as a “linear” example appearing in the simplest model of contact Weyl manifolds.

**Proof** The proof of this result was given by Yoshioka [58], however we recall it because of its importance.

Let $m$ be a unique maximal ideal defined by $m = \{ a = \sum_{2\ell + |\alpha| \geq 1} a^{\ell}_\alpha \nu^\ell Z^\alpha \}$. Then we see $\Phi(m) \subset m$, so we have

$$\Phi(Z^i) = \sum a^i_j Z^j + O(2),$$

(22)

where $O(2)$ is the collection of the terms degree $\geq 2$. Applying (22) to the canonical commutation relation $[Z^i, Z^j] = \nu \Lambda^ij$, we see $A \in \text{Sp}(n, \mathbb{R})$ and then we may write

$$\hat{A}^{-1} \circ \Phi(Z^i) = Z^i + g^{(2)}_i + O(3),$$

(23)

where $g^{(2)}_i$ is the term of homogeneous degree 2. Applying (23) to the canonical commutation relation again, we have

$$\nu \partial_{i+n} g^{(2)}_j = \nu \Lambda^{ik} \partial_k g^{(2)}_j = [Z^i, g^{(2)}_j] = [Z^i, g^{(2)}_j] = \nu \Lambda^{ik} \partial_k g^{(2)}_j = \nu \partial_{j+n} g^{(2)}_i,$$

(24)

This is equivalent to

$$d(\sum_j g^{(2)}_j dz^{j+n} + \sum_i g^{(2)}_i dz^{i+n}) = 0.$$

(25)

According to the Poincaré lemma, there exists uniquely $F^{(3)} \in W$ with homogeneous degree 3 such that $1/2[Z^i, F^{(3)}] = g^{(2)}_i$. Therefore we have

$$\hat{A}^{-1} \circ \Phi(Z^i) = e^{ad(1/\nu F^{(3)})}(Z^i) + O(3),$$

(26)
where $O(3)$ is the collection of the terms whose degree $\geq 3$. Repeating the above procedure, we obtain that
\begin{equation}
\hat{A}^{-1} \circ \Phi(Z^i) = e^{ad(\frac{1}{\nu}F(3))} \circ \cdots \circ e^{ad(\frac{1}{\nu}F(k))}(Z^i) + O(k),
\end{equation}
where $O(k)$ is the collection of the terms whose degree $\geq k$. By the Baker-Campbell-Hausdorff formula\footnote{See [43].}, we have
\begin{equation}
\hat{A}^{-1} \circ \Phi(Z^i) = e^{ad(\frac{1}{\nu}F)}(Z^i).
\end{equation}
The uniqueness is inductively verified. Thanks to the argument above, we may assume that $\Psi|_W = \hat{A} \circ e^{ad(\frac{1}{\nu}F)}$. Set
\begin{equation}
\hat{\Phi}(X) = (\hat{A} \circ e^{ad(\frac{1}{\nu}F)})^{-1} \Psi(X).
\end{equation}
Applying (29) to $[\tau, Z^i] = \nu Z^i$, we have
\begin{equation}
\hat{\Psi}(\tau) = \tau + b(\nu^2)
\end{equation}
for some $b(\nu^2) = b_0 + b_2 \nu^2 + \cdots, \ b_2 \in \mathbb{R}$. Put $c(\nu^2) = \sum \nu^{2k} \frac{b_{2k}}{2(1-2k)}$, then we see
\begin{align}
& e^{ad(\frac{1}{\nu}c(\nu^2))}\tau = \tau + b(\nu^2), \\
& e^{ad(\frac{1}{\nu}c(\nu^2))}Z^i = Z^i.
\end{align}
Then we have
\begin{equation}
\Psi = \hat{A} \circ e^{ad(\frac{1}{\nu}(c(\nu^2)+F))}.
\end{equation}
Thus, we see the consequence. \hfill \Box

Next we study the basic properties of a modified contact Weyl diffeomorphism. We recall the fundamental definitions and facts for readers again. Set $\tilde{\tau}_U = \tau + \sum z^i \omega_{ij} Z^j$ where $U \subset \mathbb{R}^{2n}$ is an open subset and $\omega_{ij} dz^i \wedge dz^j$ stands for the symplectic structure. Then for any modified contact Weyl diffeomorphism, we may set $\Psi|_U^C(\tilde{\tau}_U) = a \tilde{\tau}_U + F$, where $a \in C^\infty(U), \ F \in \Gamma(W_U)$, where $W_U$ is a trivial bundle $W_U = U \times W$. Under the notations above, as mentioned before, we have Proposition 4.9 As mentioned in Definition 4.10 a modified contact Weyl diffeomorphism $\Psi$ is called a contact Weyl diffeomorphism (CWD, for short) if
\begin{equation}
\Psi^* \tilde{\tau}_U = \tilde{\tau}_U + f^\#.
\end{equation}
For a contact Weyl diffeomorphism, we obtain the following (see Corollary 2.5 in [12] and Proposition 2.24 in [58]).

**Proposition 5.3** 1. Suppose that $\Psi : C_U \to C_U$ is a contact Weyl diffeomorphism which induces the identity map on the base space. Then, there exists uniquely a Weyl function $f^\#(\nu^2)$ of the form
\begin{equation}
f^\# = f_0 + \nu^2 f_+^\#(\nu^2) \quad (f_0 \in \mathbb{R}, \ f_+(\nu^2) \in C^\infty(U)[[\nu^2]]),
\end{equation}
such that $\Psi = e^{ad(\frac{1}{\nu}(f_0 + \nu^2 f_+^\#(\nu^2)))}$.
2. If $\Psi$ induces the identity map on $W_U$, then there exists a unique element $c(\nu^2) \in \mathbb{R}[[\nu^2]]$ with $c(\nu^2) = c(\nu^2)$, such that $\Psi = e^{ad(\frac{1}{2} c(\nu^2))}$.

Combining Propositions 4.9 and 5.3 for some $g(\nu^2)$, we see that

$$
\Psi \circ e^{ad(\frac{1}{2} g(\nu^2))} = e^{ad(\frac{1}{2} f^#(\nu^2))},
$$

where $f^#(\nu)$ is a Weyl function with the form

$$
f^#(\nu^2) = f_0 + \nu^2 f^#_+(\nu^2) \quad (f_0 \in \mathbb{R}, f_+(\nu^2) \in C^\infty(U[[\nu]])).
$$

Thus we obtain

$$
\Psi = e^{ad(\frac{1}{2} (g(\nu^2) + f^#(\nu^2)))}.
$$

The following is easily verified.

**Proposition 5.4** Suppose that $f(\nu^2)$, $a(\nu^2) \in C^\infty(U[[\nu^2]])$. If

$$
\nu^2 f^#(\nu) = g(\nu^2),
$$

then we have $g_0 = 0$ and $f(\nu^2) = \sum_{i \geq 0} g_i \nu^2i \in \mathbb{R}[[\nu^2]]\nu^2$.

**Proof** The right hand side of (36) has no term containing a factor $X^\alpha Y^\beta$ ($|\alpha| + |\beta| \geq 1$). Hence $\partial^\alpha_X \partial^\beta_Y f = 0$ ($|\alpha| + |\beta| \geq 1$). Then we see $f = \sum c_i \nu^2i$. Thus, $\nu^2 \sum c_i \nu^2i = \sum g_i \nu^2i$ and $g_0 = 0$. □

Using the above proposition, we have

**Proposition 5.5** If

$$
e^{ad(\frac{1}{2} (g + \nu^2 f^#))} = e^{ad(\frac{1}{2} (g' + \nu^2 f'^#))},
$$

then we have

$$
\{g + \nu^2 f^#\} = \{g' + \nu^2 f'^#\}
$$

and

$$
\nu^2 (f - f') = g' - g \in \nu^2 \mathbb{R}[[\nu^2]].
$$

**Proof** Applying $e^{ad(\frac{1}{2} g)}$ to the both hand side, we have

$$
e^{ad(\frac{1}{2} (\nu f^#))} = e^{ad(\frac{1}{2} (-g + g' + \nu f'^#))},
$$

and this implies that the left hand side of the above equality is a contact Weyl diffeomorphism. Then by uniqueness in Proposition 5.3 and Proposition 5.4, we see the consequence. □

Note that this does not induce the identity on the whole of $C_U$. In [58], a notion of modified contact Weyl diffeomorphism is introduced to make a contact Weyl algebra bundle \{\mathbb{C}_U, \Psi_{\alpha\beta}\} by adapting the glueing maps to satisfy the cocycle condition and patching together them.
We have

**Proposition 5.6** For any modified contact Weyl diffeomorphism \( \Psi : C_U \rightarrow C_U \) which induces the identity map on the base space, there exists a Weyl function \( f^\#(\nu^2) \) of the form

\[
f^\#(\nu^2) = f_0 + \nu^2 f_+^\#(\nu^2) \quad (f_0 \in \mathbb{R}, \ f_+^\#(\nu^2) \in C^\infty(U)[[\nu^2]]),
\]

and smooth function \( g(\nu^2) \in C^\infty(U)[[\nu^2]] \) such that \( \Psi = e^{ad(\frac{1}{\nu^2}(g(\nu^2) + f^\#(\nu^2)))) \).

**Remark** Please compare this result with Proposition 5.2.

Furthermore, we have

**Proposition 5.7** Let \( \Psi_{U_{\alpha}} \) (resp. \( \Psi_{U_{\beta}} \)) be a modified contact Weyl diffeomorphism on \( C_{U_{\alpha}} \) (resp. \( C_{U_{\beta}} \)) inducing the identity map on the base manifold. Suppose that

\[
\Psi_{U_{\alpha}}|_{C_{U_{\alpha\beta}}} = \Psi_{U_{\beta}}|_{C_{U_{\beta\alpha}}},
\]

where \( U_{\alpha\beta} := \varphi_\alpha(V_{\alpha} \cap V_{\beta}), \ U_{\beta\alpha} := \varphi_\beta(V_{\alpha} \cap V_{\beta}), \ C_{U_{\alpha}} := \Psi_\alpha(V_{\alpha} \cap V_{\beta}) \) etc.

Then

\[
\Psi_{\alpha\beta}^{-1}\star[(g_\alpha(\nu^2) + \nu^2 f_+^\#(\nu^2))|_{U_{\alpha\beta}}] = (g_\beta(\nu^2) + \nu^2 f_+^\#(\nu^2))|_{U_{\alpha\beta}}.
\]

Thus, patching \( \{g_U + \nu^2 f_+^\#\} \) together we can make a global function \( g + \nu^2 f^\# \in C^\infty(M)[[\nu^2]] + \nu^2 C^\infty(M)^{\#}[[\nu^2]] \). Hence there is a bijection between \( \text{Aut}(M, \star) \) and \( C^\infty(M)[[\nu^2]] + \nu^2 C^\infty(M)^{\#}[[\nu^2]] \).

The propositions mentioned above implicate that the space

\[
\mathcal{C}_c(M) = C^\infty_c(M)[[\nu^2]] + \nu^2 C^\infty_c(M)^{\#}[[\nu^2]]
\]

is a candidate of the model space of \( \text{Aut}(M, \star) \). In fact, the Baker-Campbell-Hausdorff formula shows the smoothness of group operations. Therefore we have the following:

**Theorem 5.8** \( \text{Aut}(M, \star) \) is a Lie group modeled on \( \mathcal{C}_c(M) \).

**Proof** The smoothness of group operations is ensured by the following formula: Put \( H_i(\nu^2) = g_i(\nu^2) + \nu^2 f_+^\#(\nu^2) \ (i = 1, 2) \).

\[
e^{ad(\frac{\nu^2}{\nu^2}H_1(\nu^2))} \circ e^{ad(\frac{1}{\nu^2}H_2(\nu^2))} = e^{ad(\frac{1}{\nu^2}(H_1(\nu^2)+H_2(\nu^2)) + \cdots + \mathcal{B}_m(H_1(\nu^2), H_2(\nu^2)) + \cdots)},
\]

where \( \mathcal{B}_m \) means the general term of the Baker-Campbell-Hausdorff formula \( \square \)

\[\text{See also Definition 1.3.}\]

\[\text{More precisely } \mathcal{B}_m \text{ is given by the following way:}\]

\[
\mathcal{B}_m(\frac{\nu^2}{\nu^2}H_1(\nu^2), \frac{1}{\nu^2}H_2(\nu^2)) \equiv (-1)^{m-1} \sum_{\substack{m-1 \text{th} \text{ term} \\text{of}\mathcal{B}_m}} \nu^{p_1} \nu^{q_1} \cdots \nu^{p_m} \nu^{q_m} (p_1 + q_1 + \cdots + p_m + q_m).
\]

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As to $\mathcal{C}_c(M)$, we have

**Lemma 5.9** The space

$$\mathcal{C}_c(M) = \text{ind lim}_{K: \text{compact}} \left( C_c^\infty(M)[\nu^2] + \nu^2 C_c^\infty(M)[\nu^2] \right)$$

is Mackey complete, where $C_c^\infty(M)$ is the space equipped with the standard locally convex topology.

**Proof** Remark that $C_c^\infty(M,N)$ is a smooth manifold modeled on a Mackey complete locally convex space $C_c^\infty(M \leftarrow f^*TN)$, where $M$ and $N$ are finite-dimensional manifolds. Since limits, direct sums and inductive limits preserve Mackey completeness, $\mathcal{C}_c(M)$ is also a Mackey complete locally convex space. □

In general, we can show the followings.

**Lemma 5.10** If $(E, \star, \| \cdot \|_\rho)$ is a Mackey complete locally convex space with a quasi multiplicative binary operation $\star$, that is,

$$\|f \star g\|_\rho \leq C_\rho \|f\|_\rho \cdot \|g\|_\rho,$$

for some positive number $C_\rho$, then $\sum_{n=0}^{\infty} \frac{f \star \cdots \star f}{n!}$ converges. Set $e_\rho = \sum_{n=0}^{\infty} \frac{f \star \cdots \star f}{n!}$. Then we have

$$\|e_\rho\|_\rho \leq \sum \frac{C_\rho^{n-1} \|f\|\rho^n}{n!}.$$

**Proof** By the assumption, we have

$$\|f \star \cdots \star f\|_\rho \leq C_\rho \|f\|_\rho \cdot \cdots \cdot \|f \star \cdots \star f\|_\rho \leq C_\rho^{n-2} \|f\|_\rho^{n-2} \|f \star f\|_\rho \leq C_\rho^{n-1} \|f\|_\rho^n.$$

Hence we see that $\left\{ \sum_{n=0}^{\ell} \frac{f \star \cdots \star f}{n!} \right\}_{n=1}^{\infty}$ is a Mackey-Cauchy sequence. Set $B = \left\{ \sum_{n=0}^{\ell} \frac{f \star \cdots \star f}{n!} \right\}_{\ell=1}^{\infty}$ is a Mackey-Cauchy sequence. Then by the Mackey completeness, $E_B$ is complete. Hence there exists uniquely an element denoted by $e_\rho = \sum_{n=0}^{\infty} \frac{f \star \cdots \star f}{n!}$ such that

$$\sum_{n=0}^{\ell} \frac{f \star \cdots \star f}{n!} \to e_\rho \in E_B \subset E,$$

and we also have

$$\|\sum_{n=0}^{\infty} \frac{f \star \cdots \star f}{n!}\|_\rho \leq \sum_{n=0}^{\infty} \frac{C_\rho^{n-1} \|f\|_\rho^n}{n!}.$$

$\square$

13 The assumption (45) can be replaced by $\|f \star g\|_\rho \leq C_\rho \|f\|_\rho \cdot \|g\|_\rho$. 

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Lemma 5.11 Let \((E, \star, ||\cdot||_\rho)\) be a Mackey complete locally convex space with a quasi multiplicative binary operation. Then for any smooth curve \(X(t)\) in \(E\), the product integral

\[
\prod e_*^{X(t)} dt = \lim_{n \to \infty} e_*^{X(t_n)\Delta t_n} \cdots e_*^{X(t_1)\Delta t_1} \cdots e_*^{X(t_0)\Delta t_0}
\]

exists.

Proof For \(I = [t_0, t_n]\), set \(\Delta : t_0 < \cdots < t_i < \cdots < t_n\), \(\Delta t_i := t_{i+1} - t_i\), and \(\text{mesh}(\Delta) := \max\{\Delta t_0, \Delta t_1, \ldots, \Delta t_{n-1}\}\). We have to show

\[
e_*^{X(t_n)\Delta t_n} \cdots e_*^{X(t_i)\Delta t_i} \cdots e_*^{X(t_0)\Delta t_0}
\]

is a Mackey-Cauchy net. A direct computation gives the following estimation:

\[
\prod e_*^{X(t_n)} dt \leq \prod e_*^{X(t_{n-1})\Delta t_{n-1}} \cdots e_*^{X(t_1)\Delta t_1} \cdots e_*^{X(t_0)\Delta t_0}
\]

For \(I \subset \mathbb{R}\), set

\[
\prod e_*^{X(t_n)\Delta t_n} \cdots e_*^{X(t_i)\Delta t_i} \cdots e_*^{X(t_0)\Delta t_0} \leq \prod e_*^{X(t_n)\Delta t_n} \cdots e_*^{X(t_i)\Delta t_i} \cdots e_*^{X(t_0)\Delta t_0}
\]

in the last inequality, we used \([44]\). Let \(\Delta : a = s_0 < \cdots < s_{\ell} < \cdots < s_m = b\) be a division of \([a, b]\) and \(\Delta(t) : s_{\ell} = t_0^{(\ell)} < \cdots < t_i^{(\ell)} < \cdots t_{n(t)}^{(\ell)} = s_{\ell+1}\) a subdivision of \(\Delta\). Then

\[
\prod e_*^{X(t_n)\Delta t_n} \cdots e_*^{X(t_i)\Delta t_i} \cdots e_*^{X(t_0)\Delta t_0} \leq \prod e_*^{X(t_n)\Delta t_n} \cdots e_*^{X(t_i)\Delta t_i} \cdots e_*^{X(t_0)\Delta t_0}
\]
In the estimation (\(*\)), we used \([49]\). This implies that

\[
\{ e_\ast X(s_m) \Delta s_m \ast \ldots \ast e_\ast X(s_\ell) \Delta s_\ell \ast \ldots \ast e_\ast X(s_0) \Delta s_0 \}\Delta
\]

is a Mackey-Cauchy net. \(\square\)

Before stating the next lemma, we recall the precise definition of seminorms which we use. The seminorms of \(C^\infty_c(M)\|\nu^2\|\) are defined by:

\[
\| \sum_{\ell=0}^{\infty} \nu^\ell f_\ell \|_{i,K} := \sum_{|\alpha| + 2\ell \leq i} \sum_{p \in K} |\partial_\alpha^\ell f_\ell (p)|, \quad (i \in \mathbb{N})
\]

where \(K\) is a compact subset of \(M\). Then we have the following.

**Lemma 5.12** Set \(f(\nu) = \sum_{k \in \mathbb{N}} f_k \nu^k\) and \(g(\nu) = \sum_{\ell \in \mathbb{N}} f_\ell \nu^\ell\).

\[
\| f(\nu) \ast g(\nu) \|_{i,K} \leq C_{i,K} \| f(\nu) \|_{i,K} \| g(\nu) \|_{i,K}.
\]

**Proof** We may assume that \(K\) is a subset of a Darboux chart \((U; (x, \xi))\).

\[
\| f(\nu) \ast g(\nu) \|_{i,K} = \| f(\nu) e_\ast^k \partial_\xi^a \partial_\xi^b g(\nu) \|_{i,K}
\]

\[
= \sum_{\alpha \beta} \left| \frac{1}{\alpha! \beta!} \partial_\xi^a \partial_\xi^\beta f \cdot \partial_\xi^\alpha g \right|_{i,K}
\]

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Theorem 5.14
curves in the Lie group in a smooth way. Thus we have
and \(34\)). Then we see that smooth curves in the Lie algebra integrate to smooth
quasi multiplicative binary operation.
G
Lemma 5.13
Using this lemma, we easily have
□
Proposition 5.15
5.2 Lifts as modified contact Weyl diffeomorphisms
As will be seen in the next proposition, general modified contact Weyl diffeo-
morphisms are closely related to symplectic diffeomorphisms.

5.2 Lifts as modified contact Weyl diffeomorphisms
For any modified contact Weyl diffeomorphism \(\Psi\), it induces a symplectic diffeomorphism on the base symplectic manifold. Moreover, there exists a group homomorphism \(p\) from \(\text{Aut}(M, \star)\) into \(\text{Diff}(M, \omega)\).
Conversely, we consider the following problem:

**Problem** For any globally defined symplectic diffeomorphism \( \phi : M \to M \), does there exist a globally defined modified contact Weyl diffeomorphism (referred to as a MCW-lift) \( \hat{\phi} \) which induces \( \phi \) ?

To solve the problem above, we need several notations. Let \((M, \omega)\) be a symplectic manifold and \(W_M\) the Weyl algebra bundle over \((M, \omega)\). Set

\[
\nabla_{\text{sym}} := \text{the canonical extension of symplectic connection to } W_M,
\]

\[
\delta := \text{ad}(\frac{1}{\nu} \omega_{ij} dz^i Z^j),
\]

\[
\nabla^W := \nabla|_{W_M} = \nabla_{\text{sym}} - \delta + \text{ad}(\frac{1}{\nu} \gamma) : \text{a Fedosov connection},
\]

\[
\nabla_{W}^W := \nabla_{\text{sym}} + \text{ad}(\frac{1}{\nu} \phi^{-1}\ast G), \quad \text{where } G := \omega_{ij} dz^i Z^j + \gamma,
\]

\[
\{i; D; j\}(\frac{F}{\nu}) := \left( (\text{ad}(\frac{F}{\nu}))^i \circ \text{ad}(\frac{D}{\nu}) \circ (\text{ad}(\frac{F}{\nu}))^j \right).
\]

Here we remark that for any symplectic diffeomorphism \( \phi \) on \( M \) and any section \( \sigma \in W_M \), the pull-back \( \phi^\ast(\sigma) \) is naturally extended in the following way.

\[
\phi^\ast(\sigma(z, \nu, dz)) = \sigma(\phi^\ast(z), \nu, \phi^\ast(dz)).
\]

In order to construct a lift of a symplectic diffeomorphism, we need several formulas.

**Lemma 5.16** Under the notations above,

\[
\nabla^W \circ \phi^\ast(\sigma(z, \nu, dz)) = \phi^\ast \circ (\nabla^W) - \phi^\ast \circ \left( \text{ad}(\frac{1}{\nu} G_{y} - \frac{1}{\nu} \phi^{-1}\ast(G)|_z) \right).
\]

**Proof** By a direct computation, we have

\[
\nabla^W \circ \phi^\ast(\sigma(z, \nu, dz)) = \left( \{ \nabla_{\text{sym}} + \text{ad}(\frac{1}{\nu} G_{y}) \} \circ \phi^\ast \right)\sigma(z, \nu, dz)
\]

\[
\text{symp. conn.}
\]

\[
= \left( \nabla_{\text{sym}} \circ \phi^\ast \right)\sigma(z, \nu, dz) + \text{ad}(\frac{1}{\nu} G_{y}) \phi^\ast(\sigma)(y, \nu, dy)
\]

\[
\text{symp. conn.}
\]

\[
= \left( \phi^\ast \circ \nabla_{\text{sym}} \right)\sigma(z, \nu, dz) + \text{ad}(\frac{1}{\nu} G_{y}) \phi^\ast(\sigma)(y, \nu, dy)
\]

\[
= \left( \phi^\ast \circ \nabla_{\text{sym}} \right)\sigma(z, \nu, dz) + \phi^\ast \left( \text{ad}(\frac{1}{\nu} (\phi^{-1}\ast G_{y})|_z) \sigma(z, \nu, dz) \right).
\]
\[ \phi^* \circ (\nabla^{symp} + \text{ad}(\frac{1}{\nu}G_z)))\sigma(z, \nu, dz) \]
\[ = -\phi^* \circ \text{ad}(\frac{1}{\nu}G_z)\sigma(z, \nu, dz) \]
\[ + \phi^* \left( \text{ad}(\frac{1}{\nu}(\phi^{-1*}G_y)\sigma(z, \nu, dz) \right) \]
\[ = (\phi^* \circ \nabla^W)\sigma(z, \nu, dz) \]
\[ - \phi^* \left( \text{ad}(\frac{1}{\nu}G_z)\sigma(z, \nu, dz) - \text{ad}(\frac{1}{\nu}(\phi^{-1*}G_y)\sigma(z, \nu, dz) \right) \]

Then we obtain the desired formula. \( \square \)

According to the lemma above, we have

**Lemma 5.17** Under the same notations above, for \( n > 0 \),

\[ \nabla^W \circ (\text{ad}_\nu F)^n = \sum_{i+j=n-1} (\text{ad}_\nu F)_i \circ \text{ad}(\nabla^W_F \circ \text{ad}(\frac{1}{\nu}F)^j + (\text{ad}_\nu F)_n \circ \nabla^W. \]

**Proof** First, we can easily verify that

\[ \nabla^W \circ (\text{ad}_\nu F)\sigma = \nabla^W_F \circ [\frac{F}{\nu}, \sigma] \circ \text{alg.conn.} [\nabla^W_F \circ \sigma + [\frac{F}{\nu}, \nabla^W \sigma], \]

for any section \( \sigma \in \Gamma(W_M) \). By using induction on \( n \), we see that

\[ \nabla^W \circ (\text{ad}_\nu F)_n \circ \sigma = \sum_{i+j=k-1} (\text{ad}_\nu F)_i \circ \text{ad}(\nabla^W_F \circ \text{ad}(\frac{1}{\nu}F)^j + (\text{ad}_\nu F)_k \circ \nabla^W \sigma. \]

This completes the proof. \( \square \)

We also have

**Lemma 5.18** Set \( \{i; \nabla^W; j\} = (\text{ad}_\nu F)^i \circ \text{ad}(\nabla^W_F \circ \text{ad}(\frac{1}{\nu}F)^j \right) \)

Then

\[ \nabla^W \circ \exp[\text{ad}(\frac{1}{\nu}F)] = \exp[\text{ad}(\frac{1}{\nu}F)] \circ \nabla^W + \sum_{i+j=k-1} \frac{1}{k!} \left( \sum_{i+j=k-1} \{i; \nabla^W; j\} \right). \]

**Proof** By a direct computation, we see that

\[ \nabla^W \circ \exp[\text{ad}(\frac{1}{\nu}F)] \]
\[ = \nabla^W \circ \sum_{k=0}^{\infty} \frac{1}{k!} (\text{ad}(\frac{1}{\nu}F)_k \]
\[ = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{i+j=k-1} \frac{1}{k!} (\text{ad}(\frac{1}{\nu}F)^i \circ \text{ad}(\nabla^W_F \circ \text{ad}(\frac{1}{\nu}F)^j) + (\text{ad}(\frac{1}{\nu}F)^k \circ \nabla^W. \]

Thus we obtain the desired formula. \( \square \)
Thanks to the lemmas above, we obtain the following.

**Theorem 5.19** Under the same notations,

\[
\nabla^W \circ \phi^* \circ \exp(\text{ad} F) = \phi^* \circ \exp(\text{ad} F) \circ \nabla^W + \phi^* \sum_{k=1}^{\infty} \sum_{i+j=k-1} \{i; \nabla^W; j\}(F) \\
- \phi^* \left( \exp(\text{ad} F) \left( \frac{1}{\nu} (G - \phi^{-1*}(G)) \right) \right).
\]

**Proof** A direct computation with formulas (61) and (63) gives

\[
\nabla^W \circ \phi^* \circ \exp(\text{ad} F) \\
= \{\phi^* \circ \nabla^W - \phi^* \circ \text{ad} \left( \frac{1}{\nu} (G - \phi^{-1*}(G)) \right) \} \circ \exp(\text{ad} F) \\
= \phi^* \circ \nabla^W \circ \exp(\text{ad} F) - \phi^* \circ \text{ad} \left( \frac{1}{\nu} (G - \phi^{-1*}(G)) \right) \circ \exp(\text{ad} F) \\
= \phi^* \circ \exp(\text{ad} F) \circ \nabla^W + \phi^* \sum_{k=1}^{\infty} \sum_{i+j=k-1} \{i; \nu \nabla^W; j\}(F) \\
- \phi^* \circ \exp(\text{ad} F) \left( \frac{1}{\nu} (G - \phi^{-1*}(G)) \right) \\
- \phi^* \left( \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i+j=k-1} \{i; \frac{1}{\nu} (G - \phi^{-1*}(G)); j\}(F) \right) \\
= \phi^* \circ \exp(\text{ad} F) \circ \nabla^W \\
+ \phi^* \sum_{k=1}^{\infty} \sum_{i+j=k-1} \{i; \nabla^W; j\}(F) \\
- \phi^* \left( \exp(\text{ad} F) \left( \frac{1}{\nu} (G - \phi^{-1*}(G)) \right) \right).
\]

\(\square\)

Therefore we have

**Theorem 5.20** Assume that \( F \) satisfies

\[
\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i+j=k-1} \{i; \nabla^W; j\}(F) = \exp(\text{ad} F) \left( \frac{1}{\nu} (G - \phi^{-1*}(G)) \right)
\]

where \( G \) and \( \phi \) is given in (57) and (58). Then we have

\[
\nabla^W \circ \phi^* \circ \exp(\text{ad} F) = \phi^* \circ \exp(\text{ad} F) \circ \nabla^W.
\]

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With a slight modification, we can adapt the iterated argument employed for
the construction of Fedosov connection ([13], see also §6.1) in such a way that
we can apply it to solving the equation (64). Thus, we have

**Theorem 5.21** For any symplectic diffeomorphism \( \phi \) on a symplectic manifold
\((M, \omega)\), there canonically exists an element \( \hat{\phi} \in \text{Aut}(M, \ast) \) which induces
the base map \( \phi \) on \( M \).

**Proposition 5.22** Assume that there exists a map \( j \) from \( \text{Diff}(M, \omega) \) into
\( \text{Aut}(M, \ast) \) satisfying \( p \circ j = \text{identity} \). Then we have a bijection:

\[
\text{Aut}(M, \ast) \cong \text{Diff}(M, \omega) \times \text{Aut}(M, \ast).
\]

**Proof** As mentioned in Proposition 5.14, any element \( \Psi \in \text{Aut}(M, \ast) \) induces
a symplectic diffeomorphism \( \phi = p(\Psi) \) on the base manifold. Set \( \hat{\phi} = j(\phi) \) and
\( \Phi = \hat{\phi}^{-1} \circ \Psi \). By the assumption, \( \Phi \) induces the base identity map. According
to Propositions 5.6 and 5.7, we see \( \Phi = \exp[\text{ad}(\lambda g(\nu^2) + \nu^2 f^\#(\nu^2))] \). \( \square \)

As seen in the proposition above, in order to determine the model space of
\( \text{Aut}(M, \ast) \), we have to determine the model space of \( \text{Diff}(M, \omega) \). Take a diffeomorphism \((\text{pr}_M, \sigma)\) from an open neighborhood \( U_0 \) of the zero section in
\( T^*M \) onto an open neighborhood \( U_2 \) of the diagonal set of \( M \times M \), such that
\( \sigma(0\text{-section}|_x) = x \). Let \( \omega_0 \) be the canonical symplectic structure of \( T^*M \), and
\( \omega_1 := (\text{pr}_M, \sigma)^* (\omega \oplus \omega^-) \), where the reversed symplectic structure of \( \omega \) is denoted
by \( \omega^- \). Since \( \omega_0 \) and \( \omega_1 \) vanish when restricted to the zero section, by virtue of
Moser’s technique (cf. [14]), there exists a diffeomorphism \( \varphi : U_0 \to U_1 \) between
two suitable open neighborhoods \( U_0 \) and \( U_1 \) of the zero section in \( T^*M \) which
is the identity on the zero section and satisfies \( \varphi^* \omega_1 = \omega_0 \). Thus we obtain that

\[
(67) \quad \eta = (\text{pr}_M, \sigma) \circ \varphi : \quad (U_0, \omega_0) \leftrightarrow (U_1, \omega_1) \leftrightarrow (U_2, \omega \oplus \omega^-).
\]

We also see that

\[
\{ \eta^{-1}(x, f(x)) | x \in M \} \text{ is a closed form } \in \Omega^1_1(T^*M) \]

\[14\text{The map } j \text{ is not a Lie group homomorphism in general.} \]
Lemma 5.24  

\[ \{ \eta^{-1}(x, f(x)) \mid x \in M \} \text{ is a Lagrangian submanifold of } (T^*M, \omega_0) \]

5.22, we have  

Combining the Baker-Campbell-Hausdorff formula with Propositions 5.14 and 5.22, we have 

the graph is a Lagrangian submanifold of \((M \times M, \omega \oplus \omega^-)\)

\[ 0 = (Id_M, f)^*(pr_1^*(\omega) - pr_2^*(\omega)) = Id_M^*\omega - f^*\omega \]

\( f \in \text{Diff}_c(M, \omega) \)

Let \( \mathcal{U} \) be an open neighborhood of \( Id_M \) consisting of all \( f \in \text{Diff}(M) \) with compact support satisfying \((Id_M, f)(M) \subset U_2 \) and \( pr_M : \eta^{-1}(\{(x, f(x)) \mid x \in M\}) \to M \) is still a diffeomorphism. For \( f \in \mathcal{U} \), the map \( (Id_M, f) : M \to \text{graph}(f) \subset M \times M \) is the natural diffeomorphism onto the graph of \( f \). According to (67), we can define the smooth chart of \( \text{Diff}(M) \) which is centered at the identity in the following way:

\[ \text{Diff}_c(M) \supset \mathcal{U} \xrightarrow{\Psi} \Omega^1_c(M), \quad \Psi(f) = \eta^{-1}(Id_M, f) : M \to T^*M. \]

Since \( \Omega^1_c(T^*M) \) is Mackey complete (cf. [22]), \( \mathcal{U} \cap \text{Diff}(M, \omega) \) gives a submanifold chart for \( \text{Diff}(M, \omega) \) at \( Id_M \). Moreover, conditions of Definition 5.22 can be shown by the standard argument of ordinary differential equation under a certain identification of \( T^*M \) with \( TM \). Therefore, we have the following.

Theorem 5.23 ([22], [40]) Let \((M, \omega)\) be a finite-dimensional symplectic manifold. Then the group \( \text{Diff}(M, \omega) \) of symplectic diffeomorphisms is a regular Lie group and a closed submanifold of the regular Lie group \( \text{Diff}(M) \) of diffeomorphisms. The Lie algebra of \( \text{Diff}(M, \omega) \) is Mackey complete locally convex space \( \mathfrak{X}_c(M, \omega) \) of symplectic vector fields with compact supports.

Combining the Baker-Campbell-Hausdorff formula with Propositions 5.14 and 5.22, we have

Lemma 5.24 The following maps are smooth:

(i) \( \text{Diff}(M, \omega) \times \text{Aut}(M, \ast) \to \text{Aut}(M, \ast); (\phi, \Psi) \mapsto \hat{\phi}^{-1} \circ \Psi \circ \hat{\phi}, \)

(ii) \( \text{Diff}(M, \omega) \times \text{Diff}(M, \omega) \to \text{Aut}(M, \ast); (\phi, \psi) \mapsto (\hat{\phi} \circ \hat{\psi})^{-1} \circ \hat{\phi} \circ \hat{\psi}, \)

(iii) \( \text{Diff}(M, \omega) \to \text{Aut}(M, \ast); \phi \mapsto \hat{\phi} \circ \hat{\phi}^{-1}. \)

According to Propositions 5.14 and 5.22, \( \mathcal{X}_c(M, \omega) \times \mathcal{X}_c(M) \) is a model space, which is a Mackey complete locally convex space. Let \( \Psi_i = \hat{\psi}_i \circ e^{ad(\frac{1}{2}H_i(\nu^2))} \), where \( H_i(\nu^2) = g_i(\nu^2) + \nu^2 f_i(\nu^2) \) (\( i = 1, 2 \)). Then the multiplication is written in the following way:

\[ \Psi_1 \circ \Psi_2 = \hat{\psi}_1 \circ e^{ad(\frac{1}{2}H_1(\nu^2))} \circ \hat{\psi}_2 \circ e^{ad(\frac{1}{2}H_2(\nu^2))} \]

\[ = \hat{\psi}_1 \circ \hat{\psi}_2 \circ \left( \hat{\psi}_1 \circ \hat{\psi}_2 \right)^{-1} \circ \hat{\psi}_1 \circ \hat{\psi}_2 \]

According to (i) and (ii) of Lemma 5.24, (68) is written as

\[ \left( \hat{\psi}_1 \circ \hat{\psi}_2 \right) \circ e^{ad(\frac{1}{2}H(\psi_1, H_1(\nu^2), H_2(\nu^2)))}, \]

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and we see the smoothness of
\[(\psi_1, \psi_2, H_1(\nu^2), H_2(\nu^2)) \mapsto H(\psi_1, \psi_2, H_1(\nu^2), H_2(\nu^2)).\]

By a similar way, we can verify the smoothness of the inverse operation. Summing up what is mentioned above, we have

**Theorem 5.25** Under the assumption of Proposition [5.22] \(\text{Aut}(M, \ast)\) is a Lie group modeled on a Mackey complete locally convex space \(\mathcal{X}_c(M, \omega) \times \mathcal{C}_c(M)\).

Furthermore, combining the definition of \(\text{Aut}(M, \ast)\) with Proposition [5.15] gives a short exact sequence
\[
1 \to \text{Aut}(M, \ast) \to \text{Aut}(M, \ast) \to \text{Diff}(M, \omega) \to 1.
\]

As mentioned in Theorem [5.23] the group \(\text{Diff}(M, \omega)\) of all symplectic diffeomorphisms is a regular Lie group modeled on a Mackey complete locally convex space \(\mathcal{X}_c(M, \omega)\). Therefore, combining Theorem [5.14] with Lemma [2.3] \(\text{Aut}(M, \ast)\) is a regular Lie group. Thus, we obtain the following.

**Theorem 5.26** Under the same notation above,

1. \(1 \to \text{Aut}(M, \ast) \to \text{Aut}(M, \ast) \to \text{Diff}(M, \omega) \to 1\) is a short exact sequence of Lie groups.

2. \(\text{Aut}(M, \ast)\) and \(\text{Aut}(M, \ast)\) are regular Lie groups.

This completes the proof of Theorem 1.1.

### 5.3 Lifts as local contact Weyl diffeomorphisms

We first remark that we can find a globally defined modified contact Weyl diffeomorphism as a lift of symplectic diffeomorphism. However, in general, we can not find a globally defined contact Weyl diffeomorphism as a lift. In the present subsection, we consider the existence of a locally defined contact Weyl diffeomorphism as a lift of a locally defined symplectic diffeomorphism. Although the following argument seems well known for specialists, we review it for readers’ convenience.

Assume that
\[
(U, z = (z^1, \cdots, z^{2n})), \quad (\phi(U), z' = (z'^1, \cdots, z'^{2n}))
\]
are star-shaped Darboux charts. Then \(\phi|_U\) is expressed as
\[
(z'^1, \cdots, z'^{2n}) = (\phi^1(z), \cdots, \phi^{2n}(z))
\]
and satisfies
\[
\{\phi^i, \phi^j\} = \{\phi^{i+n}, \phi^{j+n}\} = 0, \quad \{\phi^i, \phi^{i+j}\} = \delta^{ij} \quad (1 \leq i, j \leq n),
\]
because φ is a symplectic diffeomorphism defined on U. The Weyl continuations φ⁺ (i = 1, · · · , 2n) only satisfy

\[ [\phi^\#_i, \phi^\#_j] = \nu^3 a^i_{(3)} + \cdots + \nu^{2l+1} a^i_{(2l+1)} + \cdots, \]

(69)

\[ [\phi^\#_i, \phi^{n+j\#}] = -\nu \delta_{ij} a^i_{(3)} + \cdots + \nu^{2l+1} a^i_{(2l+1)} + \cdots, \]

\[ [\phi^{n+i\#}, \phi^{n+j\#}] = \nu^3 a^{n+i,n+j\#} + \cdots + \nu^{2l+1} a^{n+i,n+j\#} + \cdots. \]

However the Jacobi identity holds:

\[ [\phi^s \# [\phi^t \#, \phi^u \#]] + \text{c.p.} = 0, \]

where “c.p.” means “cyclic permutation”. This gives

\[ \{ z^s, a^t_{(3)} \} + \text{c.p.} = \{ \phi^s, a^t_{(3)} \} + \text{c.p.} = 0 \quad (1 \leq s, t, u \leq 2n). \]

Set

\[ \omega'(z') = \frac{1}{2} \sum_{1 \leq i,j \leq n} a^{(3)}_{n+i,n+j}(z')dx'^i \wedge dx'^j \]

(72)

\[ -2a^{(3)}_{n+i,j}(z')dx'^i \wedge dy'^j + a^{(3)}_{i,j}(z')dy'^i \wedge dy'^j \] \quad (z' \in U').

A direct computation shows that (71) is equivalent to dω' = 0. As the proof of Lemma 3.4 in [46], the closedness of ω' above ensures the existence of elements b'_j ∈ C^∞(φ(U))[[p]], (j = 1, · · · , 2n) such that replacing φ^# by

\[ \phi^s_{(1)} = \begin{cases} \phi^i(z) + \nu^2 b'^i_n(\phi(z)), & s = j \\ \phi^{j+n}(z) - \nu^2 b'^j_i(\phi(z)), & s = j + n \end{cases} \quad (1 \leq j \leq n), \]

(73)

shows that \nu^3-components of (69) vanish. Repeating the argument above for the \nu^5-, \nu^7-components gives

\[ \phi_{(\infty)} = (\phi^1_{(\infty)}, \cdots, \phi^{2n}_{(\infty)}), \]

(74)

\[ \phi^i_{(\infty)} = \phi^i(z) + \sum_{p \geq 1} \nu^{2p} g^i_p(z) \]

such that

\[ [\phi^i_{(\infty)}^\#, \phi^j_{(\infty)}^\#] = [\phi^i_{(\infty)}^n, \phi^j_{(\infty)}^n] = 0, \]

\[ [\phi^i_{(\infty)}^\#, \phi^j_{(\infty)}^{n+j}] = -\nu \delta_{ij}, \quad (i, j = 1, \cdots, n). \]

Thus, by Lemma 3.2 in [45], there exists a local Weyl diffeomorphism Φ_U which induces the base map φ_U. We next extend Φ_U to a local contact Weyl diffeomorphism Ψ_U. Set

\[ \Psi^*_U(a) = \begin{cases} \Phi^*_U(a), & (a \in \mathcal{F}_U), \\ \hat{\tau}_U + H, & (a = \hat{\tau}(U)). \end{cases} \]

(75)
where \( H = \sum_m \nu^m h^m \) is an unknown term. \( \Psi_U \) is a contact Weyl diffeomorphism if it satisfies the following equation w.r.t. \( H \):

\[
[\Psi^*_U(\tilde{\tau}_\phi(U)), \Psi_U^*(z'^{i\#})] = \Psi_U^*[\tilde{\tau}_\phi(U), z'^{i\#}].
\]

(76)

As to the equation, we easily have

\[
\text{R.H.S. of (76)} = \Psi_U^*(\nu z'^i) \overset{\text{def}}{=} \nu(\phi^i + B(\nu)),
\]

(77)

where \( B(\nu) = \sum_{l \geq 1} \nu^{2l} g_l \). On the other hand, we also obtain

\[
\text{L.H.S. of (16)}
\]

\[
(2.18) \overset{(2.18) \text{ in [58]}}{=} \nu \left( \sum_l z^l \frac{\partial z'^i}{\partial z^l} \right)^\# + \left[ \sum_m \nu^m h_m(z'^i \circ \phi) + \sum_p \nu^{2p} g_p \right]^\#
\]

\[
+ \left( 2\nu^2 \partial \nu B + \nu (EB) \right)^\#
\]

(78)

where \( E = \nu \sum_{l=1}^{2n} z^l \partial z^l \). As the proof of Theorem 3.6 in [45], comparing the components w.r.t. \( \nu^1, \nu^2, \nu^3, \cdots \) of the both sides splits the equation w.r.t. \( H \) above into infinitely many equations. Since the component of \( \nu \) is

\[
\{h_0, z'^i \circ \phi\} = (z'^i \circ \phi) - \sum z^l \left( \frac{\partial z'^i}{\partial z^l} \right),
\]

(79)

we can find the solution \( h_0 \) for this equation, and then we can solve the infinitely many equations recursively \(^{15}\). Summing up the above, we have

**Proposition 5.27** Take a star-shaped Darboux chart \( U \). For any symplectic diffeomorphism \( \phi : U \to \phi(U) \), there canonically exists a contact Weyl diffeomorphism (CW-lift) \( \hat{\phi} \) which induces \( \phi \).

Then we have

**Corollary 5.28** Assume that a symplectic manifold \( M \) is covered by a star-shaped Darboux chart. Then for any symplectic diffeomorphism \( \phi : M \to M \), there canonically exists a contact Weyl diffeomorphism (CW-lift) \( \hat{\phi} \) which induces \( \phi \).

### 6 Appendices

#### 6.1 Fedosov connection

As seen in the previous section, as to the quantum connection \( \nabla^Q \), it holds that

\[
\nabla^Q |_{W_M} = \nabla^W, \\
\mathcal{F}(W_M) = \{ \text{parallel section w.r.t. } \nabla^Q|_{W_M} \}.
\]

\(^{15}\)Thanks to star-shapedness of \( U \), we can fix \( b_j \) and \( H \) canonically.
Let $\nabla^{sym}$ be a symplectic connection and

$$\delta^{-1}(\mu^m Z^{\alpha} d\bar{z}^\beta) = \left\{ \sum_{i=1}^{2n} dz_i \nu_i Z^m Z^{\alpha} d\bar{z}^\beta \quad (|\alpha| + |\beta| \neq 0), \right.$$  

$$0 \quad (|\alpha| + |\beta| = 0),$$  

where $\nu$ is an inner product. We may write $\nabla Q|_{W_M} = \nabla^{sym} - \delta + r$, where $r$ is a 1-form with $\Gamma(W_M)$ coefficient. Then as in [13], $r$ satisfies the following equation

$$\delta r = R \omega + \nabla^{sym} r + \frac{1}{2\nu} [r, r].$$  

(81)

Or equivalently $r$ satisfies

$$r = \delta^{-1} \{ (\nabla^{sym} + \frac{1}{2\nu} [r, r]) + R \omega \},$$  

(82)

under the assumptions $\deg r \geq 2$, $\delta^{-1} r = 0$, $r_0 = 0$. Set $r_k$ is the term of $r$ degree $k$. Since it is known that this equation can be solved by recursively in the following way

$$r_3 = \delta^{-1} R \omega,$$

$$r_{n+3} = \delta^{-1} \{ \nabla^K r_{n+2} + \frac{1}{\nu} \sum_{l+1}^{k} r_{l+2} * r_{k+2-l} \}. $$

(83)

Until now we did not consider the symplectic action of $G$ on $M$. By the same manner, we have the following.

Proposition 6.1 Suppose that a Lie group $G$ is compact, and for any $g \in G$, $\nabla^{sym} \circ g^* = g^* \circ \nabla^{sym}$, where $\nabla^{sym}$ is a symplectic connection. Then we can construct a quantum connection $\nabla Q$ such that $\nabla Q|_{W_M} \circ g^* = g^* \circ \nabla Q|_{W_M}$.  

6.2 Examples of star exponential

This subsection is devoted to computations of star-exponential functions for quadratic forms (cf. [43], [44]). Let $Z = \langle Z^1, \ldots, Z^{2n} \rangle$, $A[Z] := \langle Z \rangle A \langle Z \rangle$, where $A \in Sym(2n, \mathbb{R})$, i.e. $A$ is a $2n \times 2n$-real symmetric matrix. In order to compute the star exponential function with respect to the Moyal product $e^{\frac{\mu}{i} A[Z]}$, we treat the following evolution equation.

$$\partial_t F = \frac{1}{\mu} A[Z] \ast F, \quad F_0 = e^{\frac{\mu}{i} B[Z]},$$

(84)

where $B \in Sym(2n, \mathbb{R})$, $\mu = -\sqrt{-1} \hbar$. Under the assumption $F(t) = g \cdot e^{\frac{\mu}{i} Q[Z]}$ ($g = g(t)$, $Q = Q(t)$), we would like to find a solution of this equation. Set $\Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $q := \Lambda Q$ and $a := \Lambda A$, then we see that

$$\sum_{l,m,i_1,j_1,i_2,j_2} A_{i_1i_2} A_{j_1j_2}^* Q_{i_1m} Q_{j_2l} Z^m Z^l$$

(85)

Note that this action is not commute with $\nabla Q$ in general. For example, compute and compare $\nabla Q(g^* \tau)$ and $g^* (\nabla Q \tau)$.  

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From now, we use Einstein’s convention. As to the first equation of (84), we see that

\[ L.H.S. \text{ of (84)} = g' e^{\frac{1}{\mu}Q[Z]} + g \frac{1}{\mu} Q'[Z] e^{\frac{1}{\mu}Q[Z]}, \]

\[ R.H.S. \text{ of (84)} = \frac{1}{\mu} A[Z] \cdot F \]

\[ = \frac{1}{\mu} A[Z] \cdot F + \frac{i \hbar}{2} \Lambda_{j_1 i_1} \partial_{i_1} \frac{1}{\mu} A[Z] \cdot \partial_{j_1} F - \frac{\hbar^2}{2 \cdot 4} \Lambda_{j_1 i_1} \Lambda_{i_1 j_2} \partial_{i_1} \partial_{i_2} \frac{1}{\mu} A[Z] \partial_{j_1 j_2} F \]

\[ = \frac{1}{\mu} A[Z] \cdot g e^{\frac{1}{\mu}Q[Z]} - \frac{\mu}{2} A_{i_1 j_1} \left( \frac{2}{\mu} A_{i_1 i_2} Z^l \right) \left( 2g \frac{1}{\mu} Q_{j_1 j_2} Z^m e^{\frac{1}{\mu}Q[Z]} \right) + \]

\[ + \frac{\mu^2}{8} \Lambda_{i_1 j_1} \Lambda_{i_2 j_2} \left( \frac{2}{\mu} A_{i_1 i_2} \right) \times \]

\[ \times \left( 2g \frac{1}{\mu} Q_{j_1 j_2} e^{\frac{1}{\mu}Q[Z]} + 4g \frac{1}{\mu} Q_{j_1 m} \frac{1}{\mu} A[Z] e^{\frac{1}{\mu}Q[Z]} \right). \]

Comparing the coefficient of \( \mu^{-1} \), we obtain

\[ Q'[Z] = A[Z] - 2^4 \Lambda A Q[Z] - Q \Lambda A A Q[Z]. \]

Applying \( \Lambda \) by left, we get

\[ \Lambda Q' = \Lambda A + \Lambda A Q - \Lambda A A Q - \Lambda Q \Lambda A Q \]

\[ = (1 + \Lambda Q) \Lambda A (1 - \Lambda) \]

\[ = (1 + q) \alpha (1 - q). \]

As to the coefficient of \( \mu^0 \), we have

\[ g' = \frac{1}{2} \Lambda_{i_1 j_1} \Lambda_{i_2 j_2} A_{i_1 i_2} \frac{1}{\mu} Q_{j_1 j_2} \]

\[ = -\frac{1}{2} \text{tr}(\Lambda A \cdot \Lambda Q) g \]

\[ = -\frac{1}{2} \text{tr}(\alpha q) \cdot g. \]

Thus the equation (84) is rewritten by

\[ \partial_t g = (1 + q) \alpha (1 - q), \]

\[ \partial_t g = -\frac{1}{2} \text{tr}(\alpha q) \cdot g. \]

We now recall the “Cayley transform.”

**Proposition 6.2** Set \( C(X) := \frac{1}{1 + X} \). Then
1. \( X \in \text{sp}(n, \mathbb{R}) \iff \Lambda X \in \text{Sym}(2n, \mathbb{R}) \),
   and then \( C(X) \in \text{Sp}(n, \mathbb{R}) := \{ g \in M(2n, \mathbb{R}) \mid g^t \Lambda g = \Lambda \} \).

2. \( C^{-1}(g) = \frac{1-g}{1+g} \).

3. \( e^{2\sqrt{-1}a} = c(-\sqrt{-1} \tan(a)) \).

4. \( \log a = 2\sqrt{-1} \arctan(\sqrt{-1}C^{-1}(g)) \).

5. \( \partial_t q = (1 + q)a(1 - q) \iff \partial_t C(q) = -2aC(q) \).

**Proof**  First we remark that

\[
(1 - tX)\Lambda(1 - X) - (1 + tX)\Lambda(1 + X) = \Lambda - tX\Lambda - \Lambda X + tX\Lambda X - (\Lambda - tX\Lambda + \Lambda X + tX\Lambda X) = 0,
\]

if \( tX\Lambda + \Lambda X = 0 \). Hence

\[
(91) \quad (1 - tX)\Lambda(1 - X) - (1 + tX)\Lambda(1 + X) = \Lambda - tX\Lambda - \Lambda X + tX\Lambda X - (\Lambda - tX\Lambda + \Lambda X + tX\Lambda X) = 0,
\]

Take an element \( A \in \text{Sym}(2n, \mathbb{R}) \), and set \( X = \Lambda A \). Then

\[
(92) \quad tC(X)\Lambda C(X) = \Lambda
\]

\[
= t \left( \frac{1 - X}{1 + X} \right) \Lambda \left( \frac{1 - X}{1 + X} \right)
\]

\[
= (1 + tX)^{-1}(1 - tX)\Lambda(1 - X)(1 + X)^{-1}.
\]

Thus, \( X = \Lambda A \in \text{Sp}(n, \mathbb{R}) \). Conversely, assume \( X \in \text{Sp}(n, \mathbb{R}) \). Then

\[
(93) \quad C^{-1}(e^{2\sqrt{-1}a}) = \frac{1 - e^{2\sqrt{-1}a}}{1 + e^{2\sqrt{-1}a}}
\]

\[
= -\sqrt{-1} \frac{e^{2\sqrt{-1}a} - e^{-\sqrt{-1}a}}{2\sqrt{-1}}
\]

\[
= -\sqrt{-1} \tan a.
\]

As to the assertion 4, according to the assertion 3, for \( g = e^{2\sqrt{-1}a} \), we easily have

\[
g = C(-\sqrt{-1} \tan(\frac{1}{2\sqrt{-1}} \log g)).
\]

Then we see that

\[
\log g = 2\sqrt{-1} \arctan(\sqrt{-1}C^{-1}(g)).
\]
Finally we show the assertion 5.

\[(94) \quad C(q)' = \left(\frac{1-q}{1+q}\right)'
\]

\[
= (1 + q)^{-1}(-q)' + (1 + q)^{-1}(-q)'(1 + q)^{-1}(1 - q)
\]

\[
= -(1 + q)^{-1}\{(1 + q)a(1 - q)\} + (1 + q)^{-1}\{(1 + q)a(1 - q)\}(1 + q)^{-1}(1 - q)
\]

\[
= -a(1 - q) - a(1 - q)(1 + q)^{-1}(1 - q)
\]

\[
= -a\left\{1 + \frac{1}{1+q}\right\}(1 - q)
\]

\[
= -2aC(q).
\]

Solving the above equation (94), we have

\[
C(q) = e^{-2at}C(b),
\]

and then

\[
q = C^{-1}(e^{-2at} \cdot C(b)) = C^{-1}(C(-\sqrt{-1}\tan(\sqrt{-1}at) \cdot C(b)).
\]

Thus we obtain

\[(95) \quad Q = -\Lambda \cdot C^{-1}(C(-\sqrt{-1}\tan(\sqrt{-1}\Lambda at)) \cdot C(b)).
\]

We can get \(q\) in the following way.

\[
q = (1 - e^{-2at}C(b))(1 + e^{-2at}C(b))^{-1}
\]

\[
= \left(1 - e^{-2at} \frac{1-b}{1+b}\right) \left(1 - e^{-2at} \frac{1-b}{1+b}\right)
\]

\[
= e^{-at}\{(1+b)^{-1}\{(1+b)^{-1}\}^{-1}\}e^{-at}
\]

\[
= e^{-at}\{e^{at}(1+b) - e^{-at}(1-b)\}\{(1+b)^{-1}\}
\]

\[
\times\{\{(1+b)^{-1}\}^{-1}\{e^{at}(1+b) + e^{-at}(1-b)\}^{-1}(e^{-at})^{-1}
\]

\[
= e^{-at}\{e^{at}(1+b) - e^{-at}(1-b)\}\{e^{at}(1+b) + e^{-at}(1-b)\}\}
\]

\[
= e^{-at}\{e^{at}(1+b) - e^{-at}(1-b)\}\{e^{at}(1+b) + e^{-at}(1-b)\}\}
\]

This determines the phase part \(Q\). Next we compute the amplitude coefficient part \(g\).

First we replace

\[
g' = -\frac{1}{2}Tr(aq) \cdot g
\]

by

\[(96) \quad (\log g)' = -\frac{1}{2}Tr(aq).
\]

Since

\[
Tr\left\{\log\left(\frac{e^{at}(1+b) + e^{-at}(1-b)}{2}\right)\right\}'
\]

\[
= Tr\left\{a\frac{e^{at}(1+b) - e^{-at}(1-b)}{e^{at}(1+b) + e^{-at}(1-b)}\right\}
\]

\[
= Tr(aq),
\]

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we can rewrite (96) as

\[(\log g)' = -\frac{1}{2} Tr \left\{ \log \left( \frac{e^{at}(1+b) + e^{-at}(1-b)}{2} \right) \right\}' \]

Then we have

\[g = \det \left( \frac{e^{at}(1+b) + e^{-at}(1-b)}{2} \right).\]

Setting \(t = 1, a = \Lambda A\) and \(b = \Lambda B\), we get

\[\frac{1}{e^a} A[Z] \ast e^B[Z] = \det \left( \frac{e^{\Lambda A}(1+\Lambda B) + e^{-\Lambda A}(1-\Lambda B)}{2} \right), e^{\Lambda^{-1}|C(\sqrt{1 \tan(\sqrt{-\Lambda A})}) \cdot C(\Lambda B)|} \cdot Z.\]

Setting \(B = 0\), we have

**Theorem 6.3**

\[(98) \frac{1}{e^a} A[Z] = \det \left( \frac{e^{\Lambda A} + e^{-\Lambda A}}{2} \right), e^{\Lambda^{-1}|C(\sqrt{1 \tan(\sqrt{-\Lambda A})})|} \cdot Z.\]

### 6.3 Formality theorem

In this subsection, we recall the basics of \(L_\infty\)-algebras. See [20], [10] and [27] for details.

In the following \(V = \bigoplus_{k \in \mathbb{Z}} V^k\) is a graded vector space, and \([1]\) is the shift-functor, that is, \(V[1]^k = V^{k+1}\). \(V[1] = \bigoplus_k V[1]^k\) is called a shifted graded vector space of \(V\). We set \(C(V) = \bigoplus_{n \geq 1} Sym^n(V)\) where

\[Sym^n(V) = T^n(V)/\{1 \cdot \cdots \cdot (x_1 x_2 - (-1)^{k_1 k_2} x_2 x_1) \otimes \cdots \otimes x_i \in V^{k_i}\}.\]

This space has a coproduct \(\Delta : C(V) \to C(V) \otimes C(V)\) defined in the following way:

\[\Delta(x_1 \cdots x_n) = \sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} \sum_{\sigma \in S_n} \text{sign}(\sigma; x_1 \cdots x_n) (x_{\sigma(1)} \cdots x_{\sigma(k)}) \otimes (x_{\sigma(k+1)} \cdots x_{\sigma(n)}),\]

where \(\text{sign}(\sigma; x_1 \cdots x_n)\) is defined by \(x_{\sigma(1)} \cdots x_{\sigma(n)} = \text{sign}(\sigma; x_1 \cdots x_n)x_1 \cdots x_n\).

This coproduct is coassociative, i.e. \((1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta\). We denote \(k_1 + k_2 + \cdots + k_n\) by \(\deg(x_1 \cdots x_n)\), where \((x_i \in V^{k_i})\).

**Definition 6.4** A map \(f : C(V_1) \to C(V_2)\) is called a coalgebra homomorphism if (1) \(\Delta \circ f = (f \otimes f) \circ \Delta\), (2) \(f\) preserves the grading.

The coderivation is defined in the following way.
Definition 6.5 A map \( \ell : C(V) \to C(V) \) is called a coderivation if the following properties are satisfied

1. \( \ell \) is an odd vector field of degree +1,
2. \( (\ell \otimes id + id \otimes \ell) \circ \Delta = \Delta \circ \ell \), where \((\ell \otimes \ell)(x \otimes y) = (-1)^{\deg x \deg y} \ell(y) \).

We also use the following notation: Set \( f^{(n)} = p \circ f \mid_{Sym^n(V_1)} : Sym^n(V_1) \to V_2 \), and \( \ell^{(n)} = p \circ \ell \mid_{Sym^n(V_1)} : Sym^n(V_1) \to V_2 \), where \( p \) = canonical projection : \( C(V_2) \to V_2 \).

Under the above notation, \( L_\infty \)-algebras and \( L_\infty \)-morphisms are defined in the following way:

**Definition 6.6** An \( L_\infty \)-algebra is a pair \( (V, \ell) \), where \( V \) is a graded vector space and \( \ell \) is a coderivation on the graded coalgebra \( C(V) \), such that \( \ell^2 = 0 \).

**Definition 6.7** An \( L_\infty \)-morphism \( F_* \) between two \( L_\infty \)-algebras \( (V_1, \ell_1) \) and \( (V_2, \ell_2) \) is a coalgebra homomorphism such that \( \ell_2 \circ F_* = F_* \circ \ell_1 \).

**Remark** If \( \ell = \ell^{(1)} + \ell^{(2)} \), and \( d = \ell^{(1)} \), \([x, y] = (-1)^{\deg x - 1} \ell^{(2)}(x, y)\), then \( \ell^2 = 0 \) if and only if

\[
d^2 = 0, \quad d[x, y] = [dx, y] + (-1)^{\deg x - 1}[x, dy],
\]

\[
[[x, y], z] + (-1)^{(x+y)(z+1)}[[z, x], y] + (-1)^{(y+z)(x+1)}[[y, z], x] = 0,
\]

that is, \( (V, \ell) \) is a graded differential Lie algebra.

We next recall the Kontsevich formality theorem [20].

**Differential Graded Lie algebra of \( T_{\text{poly-fields}} \)**

Let \( M \) be a smooth manifold. Set \( T_{\text{poly}}(M) = \bigoplus_{k\geq -1} \Gamma(M, \wedge^{k+1} TM) \), and let \([\cdot, \cdot]_S \) be the Schouten bracket:

\[
[X_0 \wedge \cdots \wedge X_m, Y_0 \wedge \cdots \wedge Y_n]_S = \sum_{i,j} (-1)^{i+j+m}[X_i, Y_j] \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{Y}_j \wedge \cdots,
\]

where \( X_i, Y_i \in \Gamma(M, TM) \). Then, the triple

\[
(T_{\text{poly}}(M)[[\hbar]], d := 0, [\cdot, \cdot] := [\cdot, \cdot]_S)
\]

forms a differential graded Lie algebra. It is well known that for any bivector \( \pi \in \Gamma(M, \wedge^2 TM) \), \( \pi \) is a Poisson structure if and only if

\[
[\pi, \pi]_S = 0.
\]

**Differential Grade Lie algebra of \( D_{\text{poly-fields}} \)**

Let \((A, \bullet)\) be an associative algebra and set \( C(A) = \bigoplus_{k \geq -1} C^k, C^k = Hom(A^{\otimes k+1}, A) \).

For \( \varphi_i \in C^k_i \) \((i = 1, 2)\), we set

\[
\varphi_1 \circ \varphi_2 (a_0 \otimes a_1 \otimes \cdots \otimes a_{k_1+k_2})
\]

\[
= \sum_{i=0}^k (-1)^{ik_2} \varphi_1 (a_0 \otimes \cdots \otimes a_{i-1}
\]

\[
\otimes \varphi_2 (a_i \otimes \cdots \otimes a_{i+k_2}) \otimes a_{i+k_2+1} \otimes \cdots \otimes a_{k_1+k_2}).
\]
Then the Gerstenhaber bracket is defined in the following way:

\[
[\varphi_1, \varphi_2]_G = \varphi_1 \circ \hat{\varphi}_2 - (-1)^{k_1 k_2} \varphi_2 \circ \hat{\varphi}_1
\]  

(101)

and Hochschild coboundary operator \( \delta = \delta_\bullet \) with respect to \( \bullet \) is defined by

\[
\delta_\bullet (\varphi) = (-1)^k [\bullet, \varphi] \quad (\varphi \in C^k).
\]

Then it is known that the triple \((C(A), d := \delta_\bullet, [, , ] := [, , ]_G)\)

is a differential graded Lie algebra.

Let \( M \) be a smooth manifold. Set \( F = C^\infty(M) \), and \( D_{poly}(M)[[\hbar]] \) equals a space of all multidifferential operators from \( F^{\otimes n+1} \) into \( F \). Then \( D_{poly}(M)[[\hbar]] = \oplus_{n \geq -1} D_{poly}(M)[[\hbar]] \) is a subcomplex of \( C(F[[\hbar]]) \). Furthermore, the triple \((D_{poly}(M)[[\hbar]], \delta, [, , ]_G)\) is a differential graded Lie algebra.

**Proposition 6.8** Let \( B \) be a bilinear operator and

\[
f \star g = f \cdot g + B(f, g).
\]

Then the product \( \star \) is associative if and only if \( B \) satisfies

\[
\delta_\bullet B + \frac{1}{2} [B, B]_G = 0.
\]

(102)

Next we recall the moduli space \( \mathcal{M}(C(V[1])) \). For \( b \in V[1] \), set \( e^b = 1 + b + \sum_{r \geq 1} \frac{b^r}{r!} + \cdots \in C(V[1]) \).

**Definition 6.9** \( \ell(e^b) = 0 \) is called a Batalin-Vilkovisky-Maurer-Cartan equation, where \( \ell = d + (-1)^{\deg \circ [\cdot, \cdot]} \).

Using this equation, we define the moduli space as follows:

**Definition 6.10**

\[
\tilde{\mathcal{M}}(C(V[1])) = \{b; \ell(e^b) = 0\},
\]

(103)

\[
\mathcal{M}(C(V[1])) = \tilde{\mathcal{M}}(C(V[1])) / \sim,
\]

(104)

where \( V \) stands for \( T_{poly}(M)[[\hbar]] \) and \( D_{poly}(M)[[\hbar]] \), and \( \sim \) means the gauge equivalence (cf. [20]).

Note that (99) and (102) can be seen as the Batalin-Vilkovisky-Maurer-Cartan equations.

With these preliminaries, we can state precise version of Kontsevich formality theorem:

\[^{17}\text{Strictly speaking, as for formal Poisson bivectors, } \pi_1(h) \sim \pi_2(h) \text{ if there exists a formal vector field } D \in \mathfrak{X}(M)[[\hbar]] \text{ such that } \exp hD \circ \pi_1(h) = \pi_2(h) \circ (\exp hD \circ \exp hD).
\]

On the other hand, as for star-products, \( *_1 \sim *_2 \) if there exists an intertwiner \( T = 1 + \sum_{r \geq 1} \hbar^r T_r \), \( (T_r \text{ : differential operators of order } r) \) such that \( T \circ *_1 = *_2 \circ (T \otimes T) \).
**Theorem 6.11** There exists a map $\mathcal{U}$ such that

$$\mathcal{U} : \mathcal{MC}(C(T_{\text{poly}}(M)[[\hbar]][1])) \cong \mathcal{MC}(C(D_{\text{poly}}(M)[[\hbar]][1])).$$

As a byproduct, we have

**Theorem 6.12** For any Poisson manifold $(M, \omega)$ there exists a formal deformation quantization.

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**References**

[1] Adams, M., Ratiu, T. and Schmid, R. *The Lie group structure of diffeomorphism groups and invertible Fourier integral operators with applications*, Infinite dimensional Groups with applications, (1985), 1-69, Springer.

[2] Adams, M., Ratiu, T. and Schmid, R. *A Lie group structure for Pseudodifferential Operators*, Math. Ann., 273 (1986), 529-551.

[3] Adams, M., Ratiu, T. and Schmid, R. *A Lie group structure for Fourier integral Operators*, Math. Ann., 276 (1986), 19-41.

[4] Arnol’d, V. I. *On a characteristic class entering in quantization conditions*, Func. Anal. Appl. 1 (1967), 1-13.

[5] Banyaga, A. *The structure of classical diffeomorphism groups*, Mathematics and its applications 400 (1997), Kluwer Academic Press.

[6] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D. *Deformation theory and quantization I*, Ann. of Phys. 111 (1978), 61-110.

[7] Belov-Kanel, A. and Kontsevich, M. *Automorphisms of the Weyl algebra*, Lett. Math. Phys., 74 (2005), 181-199.

[8] Deligne, P. *Déformations de l’Algèbre des Fonctions d’une variété Symplectique: Comparison entre Fedosov et De Wilde, Lecomte*, Selecta Math. N.S.1 (1995), 667-697.

[9] De Wilde, M. and Lecomte, P. B. *Existence of star-products and formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. 7 (1983), 487-496.

[10] Dito, G and Sternheimer, D. *Deformation Quantization: Genesis, Developments and Metamorphoses* [math.QA/0201168](https://arxiv.org/abs/math.QA/0201168).

[11] Eighorn, J. and Schmid, R. *Lie groups of Fourier integral operators on open manifolds*, Commun. Analysis. Geom. 9 (2001), no.5, 983-1040.
[12] Etingof, P. and Kazhdan, D. Quantization of Lie bialgebras, I, Selecta Math., New Series, 2 (1996), 1-41.
[13] Fedosov, B. V. A simple geometrical construction of deformation quantization, Jour. Diff. Geom. 40 (1994), 213-238.
[14] Guillemin, V. W. A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues, Adv. Math. 55, (1985), 131-160.
[15] Gutt, S. and Rawnsley, J. Equivalence of star products on a symplectic manifold; an introduction of Deligne’s Cech cohomology classes, Jour. Geom. Phys. 29 (1999), 347-392.
[16] Hamilton, R. The inverse function theorem of Nash and Morser, Bull. Amer. Math. Soc., 7 (1982), 65-225.
[17] Jarchow, H. Locally convex spaces, (1981), Teubner.
[18] Kac, V. Infinite-dimensional Lie algebras, (1990), Cambridge, University Press.
[19] Kontsevich, M. Operads and Motives in Deformation Quantization, Lett. Math. Phys. 48 (1999), 35-72
[20] Kontsevich, M. Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66, (2003), 157-216.
[21] Kontsevich, M. and Soibelman Y. Deformations of algebras over operads and the Deligne conjecture, Math. Phys. Stud. 21 (2000), 255-308.
[22] Kriegl, A. and Michor, P. The convenient setting of Global Analysis, SURV. 53, (1997), Amer. Math. Soc.
[23] Kumano-go, H. Pseudodifferential Operators, MIT, (1982).
[24] Leray, J. Analyse lagrangienne et mechanique quantique, Seminaire du College de France 1976-1977; R.C.P.25, (1978), Strasbourg.
[25] Leslie, J. A. Some Frobenius theorems in global analysis, Jour. Diff. Geom. 2 (1968), 279-297.
[26] Melrose, R. Star products and local line bundles, Annales de l’Institut Fourier, 54 (2004), 1581-1600.
[27] Maeda, Y. and Kajiura, H. Introduction to deformation quantization, Lectures in Math. Sci. The Univ. of Tokyo, 20 (2002), Yurinsya.
[28] Michor, P. Manifolds of smooth maps, II, III, Cahiers topo. et Géom. Diff. XIX-1 (1978), 47-78, XX-3, (1979), 63-86, XXI-3,(1980), 325-337.
[29] Milnor, J. Remarks on infinite dimensional Lie groups, Proc. Summer School on Quantum Gravity, (1983), Les Houches.
[30] Miyazaki, N. On regular Fréchet-Lie group of invertible inhomogeneous Fourier integral operators on $\mathbb{R}^n$, Tokyo Jour. Math. 19, No.1 (1996), 1-38.
[31] Miyazaki, N. A remark on the Maslov form on the group generated by invertible Fourier integral operators, Lett. Math. Phys. 42 (1997), 35-42.
[32] Miyazaki, N. Remarks on the characteristic classes associated with the group of Fourier integral operators, Math. Phys. Stud. 23 (2001), 145-154.

[33] Miyazaki, N. On the integrability of deformation quantized Toda lattice, to appear in Acta Appl. Math.

[34] Miyazaki, N. A Lie Group Structure for Automorphisms of a Contact Weyl Manifold, to appear in Progr. Math. 252, 25-44, Birkhäuser, [http://www.springer.com/west/home/].

[35] Miyazaki, N. Examples of groupoid, submitted.

[36] Miyazaki, N. Lifts of symplectic diffeomorphisms as automorphisms of Weyl algebra bundle compatible with Fedosov connection, in preparation.

[37] Miyazaki, N. Non-trivial cycles in the group of automorphisms of star product, in preparation.

[38] Moyal, J.E. Quantum mechanics as Statistical Theory, Proc. Cambridge Phil. Soc., 45 (1949), 99-124.

[39] Omori, H. and de la Harp, P. About interactions between Banach-Lie groups and finite dimensional manifolds, Jour. Math. Kyoto Univ. 12-3 (1972), 543-570.

[40] Omori, H. Infinite-dimensional Lie groups, Mmono 158 (1995), Amer. Math. Soc.

[41] Omori, H. Physics in Mathematics : Toward Geometrical Quantum Theory, (2004), University of Tokyo Press.

[42] Omori, H., Maeda, Y., Miyazaki, N and Yoshioka, A. Poincaré-Cartan class and deformation quantization of Kähler manifolds, Commun. Math. Phys. 194 (1998), 207-230.

[43] Omori, H., Maeda,Y., Miyazaki,N and Yoshioka, A., Strange phenomena related to ordering problems in quantizations, Jour. Lie Theory vol. 13, no 2 (2003), 481-510.

[44] Omori, H., Maeda, Y., Miyazaki, N. and Yoshioka, A. Star exponential functions as two-valued elements, Progr. Math. 232 (2005), 483-492, Birkhäuser.

[45] Omori, H., Maeda, Y. and Yoshioka, A. Deformation quantization and Weyl manifolds, Adv. Math. 85 (1991), 224-255.

[46] Omori, H., Maeda, Y. and Yoshioka, A. Global calculus on Weyl manifolds, Japan. Jour. Math. 17 (2), (1991), 57-82.

[47] Omori, H., Maeda, Y., Yoshioka, A. and Kobayashi, O. On regular Fréchet-Lie groups IV, Tokyo Jour. Math. 5 (1982), 365-398.

[48] Omori, H., Maeda, Y., Yoshioka, A. and Kobayashi, O. The theory of infinite dimensional Lie groups and its applications, Acta Appl. Math. 3 (1985), 71-105.

[49] Piunikhin, S., Salamon, D. and Schwarz, M. Symplectic Floer-Donaldson theory and quantum cohomology, in “Contact and symplectic geometry”, (1996), 171-200, Cambridge Univ. Press.
[50] Pressley, A. and Segal, G. *Loop groups*, (1988), Clarendon Press, Oxford.
[51] Seidel, P. *$\pi_1$ of symplectic automorphism groups and invertibles in quantum homology rings*, GAFA. 7 (1997), 1046-1095.
[52] Sternheimer, D. *Deformation quantization twenty years after*, AIP Conf. Proc. 453 (1998), 107-145, [q-alg/9809056](http://arxiv.org/abs/q-alg/9809056).
[53] Tanisaki, T. *Lie algebras and Quantum groups*, (2002), Kyoritsu.
[54] Wakimoto, M. *Infinite-dimensional Lie algebras*, (1999), Iwanami.
[55] Weyl, H. *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig, (1928).
[56] Wodzicki, M. *Noncommutative residue, I*. 320-399, Lecture Notes in Math. (1987), Springer.
[57] Woodhouse, N. *Geometric quantization*, Clarendon Press, (1980), Oxford.
[58] Yoshioka, A. *Contact Weyl manifold over a symplectic manifold*, in “Lie groups, Geometric structures and Differential equations”, Adv. Stud. Pure Math. 37(2002), 459-493.