Asymptotic indifference pricing in exponential Lévy models*

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Abstract

Financial markets based on Lévy processes are typically incomplete and option prices depend on risk attitudes of individual agents. In this context, the notion of utility indifference price has gained popularity in the academic circles. Although theoretically very appealing, this pricing method remains difficult to apply in practice, due to the high computational cost of solving the nonlinear partial integro-differential equation associated to the indifference price. In this work, we develop closed form approximations to exponential utility indifference prices in exponential Lévy models. To this end, we first establish a new non-asymptotic approximation of the indifference price which extends earlier results on small risk aversion asymptotics of this quantity. Next, we use this formula to derive a closed-form approximation of the indifference price by treating the Lévy model as a perturbation of the Black-Scholes model. This extends the methodology introduced in a recent paper for smooth linear functionals of Lévy processes to nonlinear and non-smooth functionals. Our closed formula represents the indifference price as the linear combination of the Black-Scholes price and correction terms which depend on the variance, skewness and kurtosis of the underlying Lévy process, and the derivatives of the Black-Scholes price. As a by-product, we obtain a simple explicit formula for the spread between the buyer’s and the seller’s indifference price. This formula allows to quantify, in a model-independent fashion, how sensitive a given product is to jump risk in the limit of small jump size.

Keywords: Lévy process, Utility indifference price, Mean-variance hedging, Asymptotics

*The second author would like to thank Jan Kallsen for insightful discussions and comments.
†The research of the second author is partially supported by the grant of the Government of Russian Federation 14.12.31.0007.
1 Introduction

The celebrated Black-Scholes model, which uses the geometric Brownian motion to describe the dynamics of the assets, is a cornerstone of the modern mathematical finance. However, it fails to reproduce significant features of empirically observed stock returns and option prices, such as fat-tailed distribution and implied volatility smile. For this reason, various extensions of the Black-Scholes framework have been developed in the literature. One popular approach is to replace the geometric Brownian motion with the exponential of a Lévy process.

Lévy processes allow to quantify market risk much more precisely, but the option pricing problem for such processes becomes more involved. Exponential Lévy models typically correspond to incomplete financial markets, meaning that the agents will not necessarily agree on a unique price for a derivative product. Instead, the price at which a market agent will accept to buy or sell a given derivative will depend on his / her risk aversion and preferences. A commonly used pricing paradigm in this context is the indifference pricing approach [17], which states that a fair price $p$ of a contingent claim $H$ for a market agent with utility function $U$ and initial wealth $V_0$ is the one at which the agent is indifferent between entering and not entering the transaction:

$$\max_{\vartheta} \mathbb{E}\left[U\left(V_0 + \int_0^T \vartheta_t dS_t\right)\right] = \max_{\vartheta} \mathbb{E}\left[U\left(V_0 + p + \int_0^T \vartheta_t dS_t - H\right)\right], \quad (1)$$

where $S$ denotes the stock price and the maximum is taken over a suitable set of admissible trading strategies $\vartheta$.

In this paper, we focus more specifically on the exponential (constant absolute risk aversion) utility function $U(x) = -e^{-\alpha x}$, where $\alpha > 0$ is the risk aversion parameter. This leads to a more explicit form for the indifference price:

$$p^\alpha = \frac{1}{\alpha} \log \frac{\min_{\vartheta} \mathbb{E}\left[\exp\left(-\alpha \int_0^T \vartheta_t dS_t + \alpha H\right)\right]}{\min_{\vartheta} \mathbb{E}\left[\exp\left(-\alpha \int_0^T \vartheta_t dS_t\right)\right]}.$$

Additionally, using the so-called minimal entropy martingale measure (MEMM) denoted by $Q^*$ (see equation (6)), the exponential utility indifference price can be expressed through a single optimization problem:

$$p^\alpha = \frac{1}{\alpha} \log \min_{\vartheta} \mathbb{E}^*\left[\exp\left(-\alpha \int_0^T \vartheta_t dS_t + \alpha H\right)\right], \quad (2)$$

where $\mathbb{E}^*$ stands for the expectation under the measure $Q^*$.

Nevertheless, computing the utility indifference price (2) of even a simple European option in an exponential Lévy model boils down to solving a non-linear integro-differential equation (see e.g., [19, 34]), which is a tough numerical problem. This makes this approach unsuitable in a production environment of a bank, where prices must usually be evaluated in real time. For this reason, asymptotic approximations for the indifference price in incomplete markets are of great importance.

One approach is to study the asymptotics when the number of contingent claims or, equivalently, the risk aversion $\alpha$, is small [22, 21, 23, 27, 12, 2]. In particular, for the exponential utility function, it is known [27, 12, 2] that under mild assumptions, as $\alpha$ tends to 0, the indifference price $p^\alpha$...
converges to $\mathbb{E}^*[H]$, the expectation of the option’s pay-off computed under the MEMM, and that
the optimal strategy converges to the quadratic hedging strategy under the MEMM.

However, approximating the indifference price by the expectation under the MEMM fails to take
into account the nonlinear features of the price. For this reason, in [22], the authors compute the
first-order correction to the exponential utility indifference price, and show that it is proportional to
the residual risk of the quadratic hedging strategy under the MEMM:

$$p^\alpha = \mathbb{E}^*[H] + \frac{\alpha}{2} \min_{\vartheta} \mathbb{E}^*\left[\left(\int_0^T \vartheta_t dS_t - H + \mathbb{E}^*[H]\right)^2\right] + o(\alpha), \quad \alpha \to 0. \quad (3)$$

These results are obtained under assumptions which is not straightforward to check for stock price
models with jumps (in particular, Assumption 2 in [22]). Similar results for general path-dependent
claims on a Brownian filtration are obtained in [29] using Malliavin calculus techniques. Our aim
in this paper is therefore to obtain precise approximations for the indifference prices of options in
exponential Lévy models.

First, in Theorem 1, we establish an approximation for the indifference price of the following
form:

$$p = \mathbb{E}^*[H] + \frac{\alpha}{2} \min_{\vartheta} \mathbb{E}^*\left[\left(\int_0^T \vartheta_t dS_t - H + \mathbb{E}^*[H]\right)^2\right] + \text{Error}(\alpha). \quad (4)$$

Unlike previous studies, our formula is non-asymptotic, in the sense that we provide an explicit
bound on the error which is valid for all values of $\alpha$ smaller than a certain positive constant rather
than asymptotically as $\alpha \to 0$. In addition, this formula is proven under assumptions which are
relatively easy to check in exponential Lévy models. The proof of (4) is based on an interplay
between the primal and the dual formulation of the indifference pricing problem to obtain an upper
and a lower bound on the price, and is inspired by similar approaches in [21] and [16].

Next, we use formula (4) to develop a closed form approximation to the exponential utility
indifference price in exponential Lévy models by treating the Lévy model as a perturbation of
the Black-Scholes model (see Theorem 2). In view of our non-asymptotic representation for the
indifference price, this boils down to approximating the expectation of the pay-off under the MEMM,
approximating the residual risk of the quadratic hedging strategy, and controlling the error term in
(4). To this end, we use the approach suggested in a recent paper [7], which consists in introducing a
one-parameter family of Lévy processes $(X^{\lambda}_t)_{t \geq 0}$, where $\lambda = 1$ corresponds to the model of interest,
and $\lambda = 0$ corresponds to the Brownian motion. One can then expand the quantities of interest in $\lambda$
around $\lambda = 0$. In [7] such expansions were developed for the expectation and the residual risk of
the quadratic hedging strategy in the case of European options with smooth $C^\infty$ pay-offs. In our
paper we employ a different technique to prove these expansions for a class of non-smooth pay-offs
including for example the European put option. It is important to note that in problems of this
type, regularity of the pay-off is an essential property which can strongly influence the convergence
rate (compare for example with [15]).

As a result, we obtain an approximate formula for the indifference price of a European option in
an exponential Lévy model as a linear combination of the Black-Scholes price and correction terms
which depend on variance, skewness and kurtosis of the underlying Lévy process as well as high-order
derivatives and a simple integral functional of the Black-Scholes option price. Our method is based
on an interpolation between a Lévy process and a Brownian motion and works well when the Lévy
process in question is “not too far” from the Brownian motion. In a numerical study we compare our approximate formula to the exact value obtained by solving the integro-differential equation in the Merton jump-diffusion model and show that the precision of the approximate formula is good for realistic parameter values.

Our approximate formula can be seen as an extension to the nonlinear utility indifference price of the valuation methodology based on the expansion around a proxy model, which was developed by E. Gobet and collaborators in the diffusion setting (see e.g., [4]) and in [7] in the setting of Lévy processes. It is also related to the “expansion in the model space” technique for utility optimization recently discussed in [25].

An important by-product of our study is a simple explicit approximate formula for the spread between the buyer’s and the seller’s indifference price:

\[ p_s - p_b \approx \alpha \times \frac{1}{4} \left( m_4 - \frac{m_2^2}{\sigma^2} \right) \times \mathbb{E}^{BS} \left[ \int_0^T \left( \frac{S_t^2}{\sigma^2} \frac{\partial^2 P_{BS}(t, S_t)}{\partial S^2} \right)^2 dt \right], \]

where \( \sigma^2 \) is the variance of the Lévy process at time 1, \( m_3 \) and \( m_4 \) are the third and fourth moments of the Lévy measure and \( P_{BS} \) and \( \mathbb{E}^{BS} \) denote the option price and the expectation computed in the Black-Scholes model with volatility \( \sigma \). In our asymptotic regime, therefore, the spread is decomposed into a product of three factors: the risk-aversion which characterizes the economic agent, a factor depending only on the properties of the Lévy model and a factor depending only on the variance \( \sigma^2 \) of the price process and on the properties of the contingent claim whose price is being computed. The last factor therefore provides a model independent measure of the sensitivity of a given product to jump risk in the limit of small jumps occurring at high frequency.

The remainder of the paper is structured as follows. In Section 2 we present the precise mathematical setup for exponential utility maximization and indifference pricing in exponential Lévy models. In Section 3 we derive a non-asymptotic approximation for the utility indifference price, and in Section 4 this formula is used to develop an expansion for the price “in the neighborhood of the Black-Scholes model”. Section 5 presents a numerical study of the performance of our expansion, and Section 6 analyzes specifically the formula for the bid-ask spread. Finally, proofs of technical lemmas are relegated to the appendices.

## 2 Mathematical framework

### Exponential Lévy models

Let \( X \) be a Lévy process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and let \((\mathcal{F}_t)_{t \geq 0}\) be the completed natural filtration of \( X \). We fix a time horizon \( T < \infty \) and consider a price process \( S \) defined for \( t \in [0, T] \) by \( S_t = S_0 \mathcal{E}(X)_t \) where \( S_0 > 0 \) is a constant and \( \mathcal{E} \) denotes the Doob-Dade exponential defined by

\[
\mathcal{E}(X)_t = e^{X_t - \frac{1}{2} [X]_t} \prod_{0 \leq s \leq t; \Delta X_s \neq 0} (1 + \Delta X_s) e^{-\Delta X_s}.
\]

The interest rate is taken to be zero throughout the paper. We make the following standing assumption.
Assumption 1. The process $X$ is not a.s. monotone and there exists $\delta < 1$ such that $X$ satisfies $|\Delta X_t| \leq \delta$ a.s. for all $t \in [0,T]$.

Remark 1. The lower bound on the jumps of $X$ ensures in particular that $S_t > 0$ a.s. for all $t \in [0,T]$. The non-monotonicity ensures that there exists a probability measure $Q$ equivalent to $\mathbb{P}$ under which $S$ is a martingale, which guarantees absence of arbitrage in the model (see e.g., [9, section 9.5]). The upper bound on the jumps is a technical assumption needed in the subsequent developments. It is possible to assume that $\Delta X_t \leq K$ for $K > 1$, however for notational convenience we impose the same bound on the negative and the positive jumps.

Denote by $(\sigma^2, \nu, \gamma)$ the characteristic triplet of $X$ associated with the truncation function $x \mapsto 1_{|x| \leq 1}$. By the Lévy-Khintchine formula, this means that for all $t \in [0,T]$, the characteristic function of $X_t$ is given by

$$E[e^{iuX_t}] = e^{t\psi(u)}, \quad \psi(u) = iu\gamma - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx).$$

Utility indifference pricing

Consider a bounded contingent claim $H = h(S) \in \mathcal{F}_T$, where $h : \mathbb{D}([0,T]) \to \mathbb{R}$ is a mapping defined on the space of càdlàg trajectories. When the financial market is not complete, this claim cannot be perfectly replicated, and therefore the price at which an individual agent will accept to buy / sell the claim will depend on the agent’s attitude towards risk, which may be quantified by a utility function.

In this paper, we focus on the exponential utility function defined by $U(x) = -e^{-\alpha x}$ where $\alpha \in (0,\infty)$ is the risk aversion parameter. Define the set of admissible trading strategies

$$\Theta = \{ \vartheta \in L(S) | \exists L^* \text{ with } E[e^{-\alpha L^*}] < \infty \text{ s.t. } (\vartheta \cdot S)_t \geq L^* \forall t \in [0,T] \text{ a.s.} \}$$

where $L(X)$ is the set of $\mathcal{F}$-predictable $X$-integrable $\mathbb{R}$-valued processes, and $(\vartheta \cdot X)_t := \int_0^t \vartheta_s dX_s$ denotes the stochastic integral with respect to $X$. Other definitions of the set of admissible strategies have been suggested in the literature [12], but the above one appears sufficient in the context of exponential Lévy models and it is somewhat more elementary and easier to check than the ones in [12].

As mentioned in the introduction, the seller’s utility indifference price of the claim $H$ is defined by the implicit relation (1), which, in the case of the exponential utility function, yields the explicit formula

$$p^H_s = \frac{1}{\alpha} \log \frac{\min_{\vartheta \in \Theta} E[\exp(-\alpha(\vartheta \cdot S)_T + \alpha HT)]}{\min_{\vartheta \in \Theta} E[\exp(-\alpha(\vartheta \cdot S)_T)]},$$

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The buyer’s indifference price $p^H_b$ is defined in a similar manner and satisfies $p^H_b = -p^{-H}_s$. In the sequel, we shall focus on the seller’s price and omit the indices $H$ and $s$.

In conclusion, to compute the utility indifference price, we a priori need to solve two optimization problems. As shown below, in the context of exponential Lévy models, the denominator of (5) can be computed explicitly.

Let

$$\ell(u) = \gamma u + \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(dx).$$
Under the assumption that $X$ is not a.s. monotone and has bounded jumps, $\ell(u)$ is well defined for all $u \in \mathbb{R}$, is bounded from below and there exists $\varphi^* \in \mathbb{R}$ such that $\ell(-\alpha \varphi^*) = \inf_u \ell(u)$ (see the proof of Theorem 1 in [33]). Let $\vartheta \in \Theta$ and $\varphi_t = \vartheta_t S_t$. By Itô formula it is easy to show that

$$M_t = e^{-\alpha \int_0^t \varphi_x \, dX_s - \int_0^t \ell(-\alpha \varphi_s) \, ds}$$

is a local martingale. In addition, it is positive and bounded from above by $e^{-\alpha L^*-T \inf_u \ell(u)}$, hence a true martingale. Therefore

$$\mathbb{E}[e^{-\alpha (\vartheta \cdot S)_T}] = \mathbb{E}[e^{-\alpha \int_0^T \varphi_t \, dX_t - \int_0^T \ell(-\alpha \varphi_t) \, dt + \int_0^T \ell(-\alpha \varphi_t) \, dt}]$$

$$\geq \mathbb{E}[e^{-\alpha \int_0^T \varphi_t \, dX_t - \int_0^T \ell(-\alpha \varphi_t) \, dt}] e^{\int_0^T \ell(-\alpha \varphi^*) \, dt} = e^{\int_0^T \ell(-\alpha \varphi^*) \, dt}.$$

On the other hand, the strategy $\vartheta^*_t = \frac{x_t^*}{S_t}$ is admissible. Indeed, as $X$ has bounded jumps, its exponential moments are finite, and by Theorem 25.18 from Sato [31], $\mathbb{E}[e^{-\alpha \varphi^* \inf_{0 \leq t \leq T} X_t}] < \infty$. Thus, taking the strategy $\vartheta^*$ we get equality in the above inequality, which shows that this strategy is optimal. Then we define

$$\frac{d\mathbb{Q}^*}{dp} = \frac{e^{-\alpha \varphi^* X_T}}{\mathbb{E}[e^{-\alpha \varphi^* X_T}]}.$$  \hspace{1cm} (6)

Note that by definition of $\varphi^*$, the measure $\mathbb{Q}^*$ does not depend on $\alpha$. Using Theorem 33.1 from [31], it can be shown that under $\mathbb{Q}^*$, $X$ is a martingale Lévy process with diffusion component volatility $\sigma$ and Lévy measure $\nu^*$, where $\nu^*(dx) = e^{-\alpha \varphi^* x} \nu(dx)$. In particular, it implies that $S$ is a $\mathbb{Q}^*$ martingale. The measure $\mathbb{Q}^*$ is the MEMM for $S$ (see [14]). Using this measure, the utility indifference price writes:

$$p = \frac{1}{\alpha} \log \inf_{\vartheta \in \Theta} \mathbb{E}^*[e^{-\alpha (\vartheta \cdot S)_T - H}].$$  \hspace{1cm} (7)

**Quadratic hedging**

Quadratic (also called mean-variance) hedging consists in finding an initial capital and a hedging strategy which minimize the expected squared P&L (Profit and Loss), that is:

$$\min_{c \in \mathbb{R}, \vartheta \in \Theta'} \mathbb{E}[(c + (\vartheta \cdot S)_T - H)^2],$$

where $\Theta'$ is a suitable class of admissible strategies. We refer to [30], [32], [8] for more details on quadratic hedging and to [20] for the specific setting of exponential Lévy models.

We shall see that the exponential utility indifference price is closely related to quadratic hedging under the measure $\mathbb{Q}^*$. Since $\mathbb{Q}^*$ is a martingale measure, we can define

$$\Theta' = \{ \vartheta \in L(S) : \vartheta \cdot S \text{ is a square integrable martingale}\},$$

and the optimal strategy can be computed as

$$\bar{\vartheta}_t = \frac{d(S,H)_{t \wedge \tau}^\mathbb{Q}^*}{d(S,S)_{t \wedge \tau}^\mathbb{Q}^*}, \quad \text{with} \quad H_t = \mathbb{E}^*[H\mid \mathcal{F}_t].$$

However, it should be noted that the optimal initial capital $c$ may not be interpreted as a price of the claim $H$ since it is equal to the price of the hedging strategy only and does not take into account the unhedged residual risk.
3 An approximation for the indifference price

The goal of this section is to obtain a non-asymptotic approximation for the exponential utility indifference price in terms of the quadratic residual risk (error) under $\mathbb{Q}^*$ in the Lévy model under consideration. The quadratic hedging strategy under $\mathbb{Q}^*$ will be denoted by $\bar{\vartheta}$.

Since $X$ is a Lévy process, the $\mathbb{Q}^*$-martingale $(H_t)_{0 \leq t \leq T}$ has the predictable representation property \cite[paragraph III.4d]{18} and can be written as

$$ H_t = \mathbb{E}^*[H] + \int_0^t \sigma_s dX^c_s + \int_0^t \int_{\mathbb{R}} \gamma_s(z) \tilde{J}_X(ds \times dz), \quad (8) $$

where $X^c$ is the continuous martingale part of the process $X$ under $\mathbb{Q}^*$, $\tilde{J}_X$ is the compensated jump measure of the process $X$ under $\mathbb{Q}^*$, $\sigma_t$ is a predictable process and $\gamma_s(\cdot)$ is a predictable random function.

**Theorem 1.** Assume that there exists a constant $L$ with $2\delta L \alpha < 1$ such that

$$ |H - \mathbb{E}^*[H]| \leq L \quad \text{a.s.,} \quad (9) $$

$$ |\sigma_t| \leq L \quad \text{a.s. for all } t \in [0, T], \quad (10) $$

$$ |\gamma_t(z)| \leq L|z| \quad \text{a.s. for all } t \in [0, T] \text{ and all } z \in \text{supp}\nu. \quad (11) $$

Then there exists a constant $C_{\alpha \delta L} < \infty$ such that for every $\varepsilon \in (0, 1]$ the seller's indifference price of the claim $H$ satisfies

$$ \left| p - \mathbb{E}^*[H] - \frac{\alpha}{2} \mathbb{E}^* \left[ \left( \int_0^T \tilde{\vartheta} dS_s - (H - \mathbb{E}^*[H]) \right)^2 \right] \right| \leq \alpha^{1+\varepsilon} C_{\alpha \delta L} \mathbb{E}^* \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \tilde{\vartheta} dS_s - (H_t - \mathbb{E}^*[H]) \right|^{2+\varepsilon} \right]. $$

The constant $C_{\alpha \delta L}$ can be chosen as

$$ C_{\alpha \delta L} = Ce^{4\alpha L} \lor (1 - 2\alpha \delta L)^{-2}, $$

where $C < \infty$ is a universal constant.

**Remark 2.** The formula of the above theorem is a non-asymptotic approximation formula for the indifference price, which can be used to recover a variety of asymptotic results. For example, observing that $C_{\alpha \delta L}$ is bounded as $\alpha \to 0$, we recover the asymptotics for small risk aversion:

$$ p = \mathbb{E}^*[H] + \frac{\alpha}{2} \mathbb{E}^* \left[ \left( \int_0^T \tilde{\vartheta} dS_s - (H - \mathbb{E}^*[H]) \right)^2 \right] + o(\alpha), \quad \alpha \to 0. $$

**Proof.** Under the assumptions of the Theorem,

$$ \tilde{\vartheta}_t = \frac{1}{S_t} \frac{\sigma_t \gamma_t(z) \nu(dz)}{\sigma^2 + \int_{\mathbb{R}} z^2 \nu(dz)} $$

and therefore $|S_t \tilde{\vartheta}_t| \leq L$ a.s. for all $t \in [0, T]$. We assume without loss of generality that $\mathbb{E}^*[H] = 0$. 


Applying first Lemma 1 below with pay-off $H' = \alpha H$ and bound $L' = \alpha L$, taking the logarithm using the inequality $\log(1 + x) \leq x$ and dividing by $\alpha$, we get

$$p \leq \frac{\alpha}{2} \mathbb{E}^* \left[ \left( \int_0^T \bar{\vartheta}_s dS_s - H \right)^2 \right] + C\alpha^{1+\varepsilon} \mathbb{E}^* \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \bar{\vartheta}_s dS_s - H_t \right|^{2+\varepsilon} \right].$$

Similarly, applying Lemma 2 below yields

$$p \geq \frac{\alpha}{2} \mathbb{E}^* \left[ \left( \int_0^T \bar{\vartheta}_s dS_s - H \right)^2 \right] - C\alpha^{1+\varepsilon} (1 - 2\alpha\delta L)^{-2} \mathbb{E}^* \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \bar{\vartheta}_s dS_s - H_t \right|^{2+\varepsilon} \right].$$

Lemma 1 (Upper Bound). Let $L$ be a constant such that assumptions (9) and (11) are satisfied. Then, there exists a universal constant $C$ such that for every $\varepsilon \in (0,1]$,

$$\inf_{\vartheta \in \Theta} \mathbb{E}^* \left[ e^{-\int_0^T \vartheta_s dS_s + H} \right] \leq 1 + \frac{1}{2} \mathbb{E}^* \left[ \left( \int_0^T \bar{\vartheta}_s dS_s - H \right)^2 \right] + C e^{4L} \mathbb{E}^* \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \bar{\vartheta}_s dS_s - H_t \right|^{2+\varepsilon} \right].$$

Proof. Introduce the stopping time

$$\tau = \inf \{ t \geq 0 : \left| \int_0^t \bar{\vartheta}_s dS_s - H_t \right| \geq 1 \} \wedge T.$$

Since by assumptions, taking into account that $\delta < 1$,

$$\left| \int_0^\tau \bar{\vartheta}_s dS_s - H \right| \leq 3L + 1,$$

the strategy $\vartheta_t = \bar{\vartheta}_1 t$ belongs to $\Theta$ and we get

$$\inf_{\vartheta \in \Theta} \mathbb{E}^* \left[ e^{-\int_0^T \vartheta_s dS_s + H} \right] \leq \mathbb{E}^* \left[ e^{-\int_0^\tau \bar{\vartheta}_s dS_s + H} \right].$$

We shall use a Taylor formula of the following form: for every $\varepsilon \in (0,1)$ and $m < \infty$,

$$e^{\varepsilon x} \leq 1 + x + \frac{x^2}{2} + C|x|^{2+\varepsilon}, \quad \forall x \in [-m, m]$$

with

$$C = \frac{m^{1-\varepsilon} e^m}{6}.$$

Thus, for $C_L = \frac{(3L+1)^{1-\varepsilon} e^{3L+1}}{6}$

$$\inf_{\vartheta \in \Theta} \mathbb{E}^* \left[ e^{-\int_0^T \vartheta_s dS_s + H} \right] \leq 1 + \frac{1}{2} \mathbb{E}^* \left[ \left( \int_0^T \bar{\vartheta}_s dS_s - H \right)^2 \right] + C_L \mathbb{E}^* \left[ \left| \int_0^T \bar{\vartheta}_s dS_s - H \right|^{2+\varepsilon} \right]$$

$$\leq 1 + \frac{1}{2} \mathbb{E}^* \left[ \left( \int_0^T \bar{\vartheta}_s dS_s - H \right)^2 \right] + C_L \mathbb{E}^* \left[ \left| \int_0^T \bar{\vartheta}_s dS_s - H \right|^{2+\varepsilon} \right]$$

$$+ (3L+1)^2 \left( \frac{1}{2} + C_L (3L+1)^\varepsilon \right) \mathbb{Q}^* [\tau < T].$$
Then, by Markov inequality

$$Q^*\left[\tau < T\right] = Q^*\left[\sup_{0 \leq t \leq T} \left| \int_0^t \varrho_s dS_s - H_t \right| > 1\right] \leq \mathbb{E}^*\left[\sup_{0 \leq t \leq T} \left| \int_0^t \varrho_s dS_s - H_t \right|^{2+\varepsilon}\right]$$

so that

$$\inf_{\varrho \in \Theta} \mathbb{E}^*\left[ e^{-\int_0^T \varrho_s dS_s + H} \right] \leq 1 + \frac{1}{2} \mathbb{E}^*\left[ \left(\int_0^T \bar{\varrho}_s dS_s - H\right)^2\right]$$

$$+ \left( C_L + \frac{(3L + 1)^2}{2} + C_L(3L + 1)^2 \varepsilon\right) \mathbb{E}^*\left[ \sup_{0 \leq t \leq T} \left| \int_0^t \bar{\varrho}_s dS_s - H_t \right|^{2+\varepsilon}\right].$$

Now, it is clear that one can choose a universal constant $C$ such that the statement of the Lemma holds true.

**Lemma 2 (Lower Bound).** Let $L$ be a constant with $2L\delta < 1$ such that assumptions (9)–(11) are satisfied. Then, there exists a universal constant $C$ such that for every $\varepsilon \in (0, 1]$, $p \geq \frac{1}{2} \mathbb{E}^*\left[ \left(\int_0^T \bar{\varrho}_s dS_s - H\right)^2\right] - \frac{C}{(1 - 2L\delta)^2} \mathbb{E}^*\left[ \sup_{0 \leq t \leq T} \left| \int_0^t \bar{\varrho}_s dS_s - H_t \right|^{2+\varepsilon}\right].$

**Proof.** From the results of [3], we have$^1$

$$p = \sup_{Q \in \text{EMM}(Q^*)} \left\{ \mathbb{E}^Q[H] - H(Q\|Q^*) \right\},$$

where $\text{EMM}(Q^*)$ denotes the set of martingale measures, equivalent to $Q^*$ and $H(Q\|Q^*)$ is defined by

$$H(Q\|Q^*) = \mathbb{E}^*\left[ \frac{dQ}{dQ^*} \log \frac{dQ}{dQ^*} \right]$$

whenever this quantity is finite and equals $+\infty$ otherwise. Therefore, for any random variable $D > 0$ such that $DQ^*$ is a martingale measure,

$$p \geq \mathbb{E}^*[DH_T] - \mathbb{E}^*[D\log D].$$

(12)

Let $\kappa = \frac{1}{2} - L\delta$, introduce the stopping time

$$\tau_{\kappa} = \inf\{t \geq 0 : \left| \int_0^t \bar{\varrho}_s dS_s - H_t \right| \geq \kappa\} \wedge T$$

and define

$$D = 1 + \int_0^{\tau_{\kappa}} \bar{\varrho}_t dS_t - H_{\tau_{\kappa}}.$$

$^1$This reference provides a duality result for the class of admissible strategies which are bounded from below, but it can easily be extended to our class $\Theta$ using the dominated convergence theorem and the local boundedness of $S$. 

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By construction, $E[D] = 1$ and

$$|D - 1| \leq \kappa + |\bar{\theta}_{\tau_\kappa} - \bar{S}_{\tau_\kappa} - \Delta X_{\tau_\kappa}| + |\Delta H_{\tau_\kappa}| \leq \kappa + 2\delta \leq \frac{1}{2} + L\delta < 1.$$  

Moreover, for a bounded strategy $\bar{\theta}$,

$$E^* \left[ D \int_0^T \bar{\theta}_t dS_t \right] = E^* \left[ D \int_0^{\tau_\kappa} \bar{\theta}_t dS_t \right] = E^* \left[ \left( \int_0^T \bar{\theta}_t dS_t - H_T \right) \int_0^{\tau_\kappa} \bar{\theta}_t dS_t \right] = 0$$

because $\bar{\theta}$ is the optimal quadratic hedging strategy. Therefore, $DQ^*$ is a martingale measure. It remains to compute the right-hand side of (12). For the first term, using the Cauchy-Schwarz inequality and an estimate for $\bar{\theta}_{\tau_\kappa}$, we get

$$E^*[DH_T] = E^*[DH_{\tau_\kappa}] = E^* \left( H_{\tau_\kappa} - \int_0^{\tau_\kappa} \bar{\theta}_t dS_t \right)^2 \geq \mathbb{E}^* \left( H_T - \int_0^T \bar{\theta}_t dS_t \right)^2 - \mathbb{E}^* \left( H_T - \int_0^T \bar{\theta}_t dS_t \right)^2 1_{\tau_\kappa < T} \geq \mathbb{E}^* \left( H_T - \int_0^T \bar{\theta}_t dS_t \right)^2 - \mathbb{E}^* \left[ \left( H_T - \int_0^T \bar{\theta}_t dS_t \right)^{2+\varepsilon} \right] \mathbb{P}[\tau_\kappa < T]^{\frac{2+\varepsilon}{2}} \geq \mathbb{E}^* \left( H_T - \int_0^T \bar{\theta}_t dS_t \right)^2 - \frac{1}{\kappa^\varepsilon} \mathbb{E}^* \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \bar{\theta}_s dS_s - H_t \right|^{2+\varepsilon} \right].$$

The second term in (12) can be estimated using the following Taylor formula: for every $\varepsilon \in (0, 1)$ and $\Delta \in (0, 1)$,

$$x \log x + 1 - x \leq \frac{(x - 1)^2}{2} + C |x - 1|^{2+\varepsilon}, \quad \forall x \in [1 - \Delta, 1 + \Delta] \quad \text{with} \quad C = \frac{\Delta^{1-\varepsilon}}{6(1 - \Delta)^2}.$$  

Then, for $CL_{\delta\varepsilon} = \frac{(1/2 + L\delta)^{1-\varepsilon}}{6(1/2 - L\delta)^2}$,

$$E^*[D \log D] = E^*[D \log D + 1 - D] \leq \frac{1}{2} E^*[(D - 1)^2] + C_{L\delta\varepsilon} \mathbb{E}^*[(D - 1)^{2+\varepsilon}] = \frac{1}{2} \mathbb{E}^* \left( H_{\tau_\kappa} - \int_0^{\tau_\kappa} \bar{\theta}_t dS_t \right)^2 + C_{L\delta\varepsilon} \mathbb{E}^* \left[ \left( H_{\tau_\kappa} - \int_0^{\tau_\kappa} \bar{\theta}_t dS_t \right)^{2+\varepsilon} \right] \leq \frac{1}{2} \mathbb{E}^* \left( H_T - \int_0^T \bar{\theta}_t dS_t \right)^2 + \left( C_{L\delta\varepsilon} + \frac{1}{\kappa^\varepsilon} \right) \mathbb{E}^* \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \bar{\theta}_s dS_s - H_t \right|^{2+\varepsilon} \right].$$

Adding up the estimates for the first and the second term of (12), and choosing the universal constant $C$ appropriately, the proof of the Lemma is complete. \hfill \square
Example 1. Let us check the assumptions of Theorem 1 for the European put option with pay-off $H = (K - S_T)^+$. The process $(H_t)_{0 \leq t \leq T}$ is given by:

$$H_t := \mathbb{E}^*[H | F_t] = P(t, S_t)$$

where $P(t, S) = \mathbb{E}^*[(K - SE(X)_{T-t})^+]$, and under suitable regularity assumptions on the process $X$ (see e.g., [10, Proposition 2]), we have the martingale representation

$$H_t = \mathbb{E}^*[H] + \int_0^t \sigma_s dX_s^c + \int_0^t \int_\mathbb{R} \gamma_s(z) \tilde{J}(ds \times dz)$$

with

$$\sigma_t = \frac{\partial P(t, S_t)}{\partial S} S_t \quad \text{and} \quad \gamma_t(z) = P(t, S_t - (1 + z)) - P(t, S_t).$$

By dominated convergence:

$$\left| S \frac{\partial P(t, S)}{\partial S} \right| = \mathbb{E}^*[SSE(X)_{T-t} 1_{SSE(X)_{T-t} \leq K}] \leq K.$$

On the other hand, for $z \in \text{supp} \nu$,

$$|P(t, S(1 + z)) - P(t, S)| \leq \mathbb{E}^*[|zSSE(X)_{T-t} 1_{SSE(X)_{T-t} \leq (1+z)\wedge 1 \leq K}] \leq \frac{K|z|}{1 - \delta}.$$

4 Indifference price asymptotics in the neighborhood of the Black-Scholes model

Since, as we have seen, the computation of the indifference price can be carried out under the MEMM, in this section, to simplify notation we omit the star in $\mathbb{E}^*$. In other words, we simply assume that all the expectations are taken under the MEMM unless specified otherwise, and that $X$ is a martingale Lévy process with diffusion component volatility $\sigma$ and Lévy measure $\nu$.

In liquid financial markets, jumps are typically small and in most cases the Black-Scholes model provides a correct “order of magnitude” approximation to option prices. Thus it seems reasonable, in these markets, to treat more complex stochastic models as perturbations of the Black-Scholes price and to compute correction terms to this reference value. Our goal in this section is therefore to find an explicit approximation to the indifference price (7) in the situation when the Lévy process $X$ is “close” to the Brownian motion.

To quantify what it means to be close to the Brownian motion, and following a recent paper by Černý, Denkl and Kallsen [7], we artificially introduce a small parameter $\lambda \in (0, 1)$ into the model, by considering the family of stochastic processes

$$X_1^\lambda := \lambda X_1^{1/\lambda^2}, \quad 0 \leq t \leq T.$$

Note that our parameterization is slightly different from the one introduced in [7] because that paper considers Lévy models built using ordinary exponential, whereas we use the Doléans-Dade exponential. As a result, our formulas are somewhat simpler than the ones of [7].

With this parameterization, $X^1 = X$ and, as $\lambda \downarrow 0$, $X_1^\lambda$ converges weakly in Skorokhod topology to the process

$$(\overline{X}_t)_{t \geq 0} = (\overline{W}_t)_{t \geq 0},$$
Theorem 1 (by an argument similar to the one given in Example 1). As for the second assumption, then, as

Assume that

Theorem 2. The following theorem provides an approximation of \( W \) where

\[ \sigma \]

\[ \text{normal inverse Gaussian} \ [1] \] . It is not satisfied by the variance gamma model \([26]\). It is satisfied by most parametric Lévy models used in practice, such as CGMY \([6]\) (with

\[ \sigma^2 = \sigma^2 + \int x^2 \nu(dx) \] , that is,

\[ p_\lambda = \frac{1}{\alpha} \log \inf_{\sigma \in \Theta} \mathbb{E} e^{-\alpha((\sigma, S^\lambda)_T - H^\lambda)} \quad (13) \]

The following theorem provides an approximation of \( p_\lambda \) when \( \lambda \to 0 \) for European pay-offs, that is, we assume that \( H = h(S_T) \) and \( H^\lambda = h(S^\lambda_T) \). In this theorem and below, we let \( P_{BS}(t, S) \) denote the Black-Scholes price of the corresponding option computed with volatility \( \bar{\sigma} \) defined by \( \bar{\sigma}^2 = \sigma^2 + \int x^2 \nu(dx) \), that is,

\[ P_{BS}(t, S) = \mathbb{E} \left[ h \left( S e^{-\frac{\sigma^2}{2}(T-t) + \bar{\sigma} W_{T-t}} \right) \right] , \]

where \( W \) is a standard Brownian motion. When \( t = 0 \) we also write \( P_{BS}(0, S) = P_{BS}(S) \) to shorten notation.

**Theorem 2.** Assume that

- The pay-off function \( h \) is a bounded, almost everywhere differentiable, the derivative \( h' \) has finite variation on \([0, \infty)\) and there exists \( L < \infty \) such that \( |xh'(x)| \leq L \) almost everywhere.
- Either \( \sigma > 0 \) or there exists \( \beta \in (0, 2) \) such that \( \liminf_{r \downarrow 0} \frac{\int_{-r}^r x^2 \nu(dx)}{r^2} > 0 \).

Then, as \( \lambda \to 0 \),

\[ p_\lambda = P_{BS}(S_0) + \frac{\lambda m_3 T}{6} S^3_{BS}(S_0) + \frac{\lambda^2 m_4 T}{24} S^4_{BS}(S_0) \]
\[ + \frac{\lambda^2 m_3^2 T^2}{72} \left\{ 6S^3_{BS}(S_0) + 18S^4_{BS}(S_0) + 9S^5_{BS}(S_0) + S^6_{BS}(S_0) \right\} \]
\[ + \frac{\alpha^2}{8} \left( m_4 - \frac{m_3^2}{\sigma^2} \right) E^{BS} \left[ \int_0^T \left( S \frac{\partial^2 P_{BS}(t, S)}{\partial S^2} \right)^2 dt \right] + o(\lambda^2) \]

where \( m_3 = \int x^3 \nu(dx) \), \( m_4 = \int x^4 \nu(dx) \) and \( E^{BS} \) denotes the expectation computed in the Black-Scholes model with volatility \( \bar{\sigma} \).

**Remark 3.** It is easy to check that the pay-off function of the European put option \( h(x) = (K - x)^+ \) satisfies the first assumption of the Theorem. Moreover, this assumption implies that assumptions of Theorem \([7]\) (by an argument similar to the one given in Example \([7]\)). As for the second assumption, it is satisfied by most parametric Lévy models used in practice, such as CGMY \([6]\) (with \( Y > 0 \)) and normal inverse Gaussian \([7]\). It is not satisfied by the variance gamma model \([20]\).

**Proof.** The proof is based on the non-asymptotic approximation formula of Theorem \([1]\) applied to the process \( S^\lambda \), which takes the form

\[ \left| p_\lambda - \mathbb{E}[H^\lambda] - \frac{\alpha}{2} \mathbb{E} \left[ \left( \int_0^T \tilde{\sigma}_t dS^\lambda_t - (H^\lambda - \mathbb{E}[H^\lambda]) \right)^2 \right] \right| \leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \tilde{\sigma}_s dS_s - (H^\lambda_t - \mathbb{E}[H^\lambda]) \right|^{2+\epsilon} \right] . \]
where \( \bar{\vartheta}^\lambda = \text{argmin}_\vartheta \mathbb{E} \left[ \left( \int_0^T \vartheta_t dS_t^\lambda - (H^\lambda - \mathbb{E}[H^\lambda]) \right)^2 \right] \). The following lemmas provide estimates of the linear part of the price \( \mathbb{E}[H^\lambda] \), the nonlinear part of the price \( \mathbb{E} \left[ \left( \int_0^T \bar{\vartheta}_t^\lambda dS_t^\lambda - (H_t^\lambda - \mathbb{E}[H_t^\lambda]) \right)^2 \right] \) and the residual term in the right-hand side. In these lemmas we suppose that the standing assumptions of the paper hold true. Note that the expansion of the linear part of the price does not require the pay-off function \( h \) to be regular, but in the other two lemmas, regularity is an essential assumption without which the convergence rates as \( \lambda \downarrow 0 \) may be different.

**Lemma 3** (Estimation of the residual term). Let the assumptions of Theorem 2 hold true, let

\[
M^\lambda_t = \int_0^t \bar{\vartheta}_s^\lambda dS_s^\lambda - (H_t^\lambda - \mathbb{E}[H_t^\lambda])
\]

and define

\[
\bar{M}^\lambda_T = \sup_{0 \leq t \leq T} |M^\lambda_t|.
\]

Then \( \forall q > 2 \), as \( \lambda \to 0 \)

\[
\mathbb{E} \left[ (\bar{M}^\lambda_T)^q \right] = O \left( \lambda^q \left( \log \frac{1}{\lambda} \right)^\frac{q}{2} \right).
\]

**Lemma 4** (Estimation of the nonlinear part of the price). Let the assumptions of Theorem 2 hold true. Then, as \( \lambda \to 0 \),

\[
\mathbb{E} \left[ \left( \int_0^T \bar{\vartheta}_t^\lambda dS_t^\lambda - h(S_T^\lambda) \right)^2 \right] = \lambda^2 \left( m_4 - m_2^2 \right) \mathbb{E}^{BS} \left[ \left( S_t^2 \frac{\partial^2 P_{BS}(t,S_t)}{\partial S^2} \right)^2 \right] + o(\lambda^2).
\]

In addition, for the European put option with pay-off function \( h(S_T) = (K - S_T)^+ \),

\[
\mathbb{E}^{BS} \left[ \int_0^T \left( S_t^2 \frac{\partial^2 P_{BS}(t,S_t)}{\partial S^2} \right)^2 dt \right] = \frac{K^2}{2\sigma^2} \int_0^1 \frac{e^{-ud}du}{\sqrt{1-u^2}}
\]

where \( d = \frac{\log \frac{S_0}{K} - \frac{\sigma^2}{2}T}{\sigma \sqrt{T}} \).

**Lemma 5** (Estimation of the linear part of the price). Assume that

- The function \( h \) is measurable with polynomial growth.
- Either \( \sigma > 0 \) or there exists \( \beta \in (0, 2) \) such that \( \liminf_{r \to 1^-} \frac{\int_{-r^2}^{r^2} \frac{\nu(dx)}{r^{2-\beta}}}{r^{2-\beta}} > 0 \).

Then, as \( \lambda \to 0 \),

\[
\mathbb{E}[h(S_T)] = P_{BS}(S_0) + \frac{\lambda m_3 T}{6} S_0^3 P_{BS}^{(3)}(S_0) + \frac{\lambda^2 m_4 T}{24} S_0^4 P_{BS}^{(4)}(S_0) + \frac{\lambda^3 m_5 T^2}{72} \left( 6 S_0^3 P_{BS}^{(3)}(S_0) + 18 S_0^4 P_{BS}^{(4)}(S_0) + 9 S_0^5 P_{BS}^{(5)}(S_0) + S_0^6 P_{BS}^{(6)}(S_0) \right) + o(\lambda^2).
\]
5 Numerical results

In this section, we illustrate numerically the performance of the asymptotic formula of Theorem\(^2\) assuming that the asset price is described by Merton’s jump-diffusion model \(28\) under \(Q^*\). Strictly speaking, this model does not satisfy the standing assumptions of the paper because the (log-normal) jumps are not bounded from above. However, in the numerical implementation discussed below, the Lévy measure is truncated to a bounded domain (which can be chosen sufficiently large so that further increase of the domain does not modify the price).

**Merton’s jump-diffusion model** In this model the stock price is defined by

\[
S_t = S_0 \mathcal{E}(X)_t
\]

where \(W\) denotes standard Brownian motion, jump sizes \((Y_i) \sim N(\gamma, \delta^2)\) are i.i.d. random variables and \((N_t)_{t \geq 0}\) is an independent Poisson process with intensity \(\lambda^M\) accounting for the number of jumps up to time \(t\). The Lévy measure of \(X\) therefore has a density given by

\[
\nu(x) = \frac{\lambda^M}{\delta(x + 1) \sqrt{2\pi}} e^{-\frac{(\log(x + 1) - \gamma)^2}{2\delta^2}}.
\]

**Implementation of the asymptotic formula** In the numerical examples, we let \(\lambda = 1\) and approximate the indifference price by

\[
p = P_{BS}(S_0) + \frac{m_3 T^2}{6} S_0^3 P^{(3)}_{BS}(S_0) + \frac{m_4 T^2}{24} S_0^4 P^{(4)}_{BS}(S_0)
+ \frac{m_3^2 T^2}{72} \left\{ 6 S_0^3 P^{(3)}_{BS}(S_0) + 18 S_0^4 P^{(4)}_{BS}(S_0) + 9 S_0^5 P^{(5)}_{BS}(S_0) + S_0^6 P^{(6)}_{BS}(S_0) \right\}
+ \frac{\alpha \lambda}{8} \left( m_4 - \frac{m_3^2}{\bar{\sigma}^2} \right) \mathbb{E}^{BS} \left[ \int_0^T \left( S_t^2 \frac{\partial^2 P_{BS}(t,S_t)}{\partial S^2} \right)^2 dt \right].
\]

Using the formula which has been justified asymptotically as \(\lambda \to 0\) for a finite nonzero value of \(\lambda\) amounts to use a second-order Taylor expansion of a function at zero to approximate the value of this function at a point \(x \neq 0\). The quality of the approximation does not depend on the specific value of \(x\), but rather on the smoothness of the function between 0 and \(x\). The numerical examples of this section show that the indifference price is indeed smooth as function of \(\lambda\) and that using the formula of Theorem\(^2\) with \(\lambda = 1\) leads to a very precise approximation.

To evaluate the approximate indifference price, one needs to perform three computations.

- Evaluate \(\bar{\sigma}^2\) and the moments of the Lévy measure \(m_3\) and \(m_4\). In Merton’s model these quantities are easily computed from the explicit form of the Lévy measure and are given by

\[
\bar{\sigma}^2 = \sigma^2 + \lambda^M \{ e^{2\gamma + 2\delta^2} - 2 e^{\gamma + \delta^2} + 1 \}
\]

\[
m_3 = \lambda^M \{ e^{3\gamma + \frac{5}{2} \delta^2} - 3 e^{2\gamma + 2\delta^2} + 3 e^{\gamma + \delta^2} - 1 \}
\]

\[
m_4 = \lambda^M \{ e^{4\gamma + 8\delta^2} - 4 e^{3\gamma + \frac{7}{2} \delta^2} + 6 e^{2\gamma + 2\delta^2} - 4 e^{\gamma + \frac{3}{2} \delta^2} + 1 \}
\]
Remark that although the original model has four parameters (since $\mu$ is fixed by the martingale condition), the asymptotic formula only depends on three ‘group’ parameters $\bar{\sigma}^2$, $m_3$ and $m_4$.

- Evaluate the integral in the last line of (14). In our example we consider the put option and evaluate the more explicit form of the integral given in Lemma 4 using a numerical integration algorithm.

- Evaluate the derivatives of the Black-Scholes option price with respect to the underlying up to order 6. The exact explicit formulas for these derivatives are given in Appendix D.

**Partial integro-differential equation and the finite difference scheme**

In this paragraph we briefly describe the HJB equation for the indifference price (see e.g., [19, 34]) as well as the numerical scheme used to solve it. This scheme is inspired by well-studied schemes for linear integro-differential equations [11] and is provided here only for the purpose of illustrating the asymptotic method; its full derivation and the study of its accuracy is out of scope of the present paper.

Let $H_T = (K - S_T)^+$ and assume that $S$ has the dynamics

$$\frac{dS_t}{S_{t-}} = dX_t,$$

where $X$ is a martingale Lévy process with Lévy measure $\nu$ and diffusion coefficient $\sigma$. Then, the indifference price $p(t, S)$ satisfies (omitting the arguments where possible to save space)

$$0 = \frac{\partial p}{\partial t} + \frac{S^2 \sigma^2}{2} \frac{\partial^2 p}{\partial S^2} + \int_{\mathbb{R}} \left( p(t, S(1 + z)) - p - S z \frac{\partial p}{\partial S} \right) \nu(dz)$$

$$+ \min_{\vartheta} \left\{ \frac{\alpha S^2 \sigma^2}{2} \left( \vartheta - \frac{\partial p}{\partial S} \right)^2 + \frac{1}{\alpha} \int_{\mathbb{R}} \left( e^{\alpha(p(t, S(1 + z))) - p - S z \vartheta} - 1 - \alpha(p(t, S(1 + z)) - p - S z \vartheta) \right) \nu(dz) \right\},$$

with terminal condition $p(T, S) = (K - S)^+$. In log-variable $x = \log S$, introducing $P(t, x) = p(t, S)$,

$$0 = \frac{\partial P}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial P}{\partial x} \right) + \int_{\mathbb{R}} \left( P(t, x + z) - P - (e^z - 1) \frac{\partial P}{\partial x} \right) \tilde{\nu}(dz)$$

$$+ \min_{\vartheta} \left\{ \frac{\alpha \sigma^2}{2} \left( \vartheta - \frac{\partial P}{\partial x} \right)^2 + \frac{1}{\alpha} \int_{\mathbb{R}} \left( e^{\alpha(P(t, x + z)) - P - (e^z - 1)\vartheta} - 1 - \alpha(P(t, x + z) - P - (e^z - 1)\vartheta) \right) \tilde{\nu}(dz) \right\},$$

where $\tilde{\nu}$ is the logarithmic transformation of $\nu$.

To discretize this equation we introduce a time grid $t_i = ih$, $i = 0, \ldots, N$ with $h = \frac{T}{N}$, a space grid $x_j = x_0 + jd$, $j = 0, \ldots, 2M$, and represent the Lévy measure $\tilde{\nu}$ as

$$\tilde{\nu}(dx) = \sum_{k=-K}^{K} \tilde{\nu}_k \delta_{kd}(dx),$$

where $K$ is an integer and $\delta$ is the Dirac delta function. Let $P_{t,j}$ denote the approximation of
$P(t, x_j)$. We use the following implicit-explicit scheme:

$$0 = \frac{P_{t+1,j} - P_{t,j}}{h} + \frac{\sigma^2}{2} \left( \frac{P_{t,j+1} + P_{t,j+1} - 2P_{t,j} + P_{t,j+1} - P_{t,j-1}}{d^2} \right) + \sum_{k=-K}^{K} \left( P_{t+1,j+k} - P_{t+1,j} - (e^{kd} - 1) \frac{P_{t+1,j+1} - P_{t+1,j-1}}{2d} \right) \bar{\nu}_k + \min_\vartheta \left\{ \frac{\alpha \sigma^2}{2} \left( \vartheta - \frac{P_{t+1,j+1} - P_{t+1,j-1}}{2d} \right)^2 \right\}$$

In other words, introducing the notation

$$B_j(P_{t+1}) = \sum_{k=-K}^{K} \left( P_{t+1,j+k} - P_{t+1,j} - (e^{kd} - 1) \frac{P_{t+1,j+1} - P_{t+1,j-1}}{2d} \right) \bar{\nu}_k$$

and

$$H_j(P_{t+1}, \vartheta) = \frac{\alpha \sigma^2}{2} \left( \vartheta - \frac{P_{t+1,j+1} - P_{t+1,j-1}}{2d} \right)^2 + \frac{1}{\alpha} \sum_{k=-K}^{K} \left( e^{\alpha(P(t+1,j+k) - P(t+1,j)) - (e^{kd} - 1)\vartheta} \right) - \alpha(P(t+1,j+k) - P(t+1,j)) - (e^{kd} - 1)\vartheta) \bar{\nu}_k,$$

we have for $j = 1, \ldots, 2M - 1$

$$P_{t,j} \left( 1 + \frac{\sigma^2 h}{d^2} \right) - P_{t,j-1} \left( \frac{\sigma^2 h}{2d^2} + \frac{\sigma^2 h}{4d} \right) = P_{t,j+1} \left( \frac{\sigma^2 h}{2d^2} - \frac{\sigma^2 h}{4d} \right) = P_{t+1,j} + hB_j(P_{t+1}) + h \min_\vartheta H_j(P_{t+1}, \vartheta)$$

with boundary conditions

$$P_{t,2M} = (K - e^{x_{2M}})^+ \quad \text{and} \quad P_{t,0} = (K - e^{x_0})^+.$$
6 Bid-ask spread and sensitivity of options to jump risk

As a by-product of the asymptotic formula of Theorem 2 we obtain a simple explicit approximation for the difference between the seller’s and the buyer’s indifference price of a European option, that is, for the bid-ask spread:

\[ p_s - p_b \approx \frac{\alpha}{4} \left( m_4 - \frac{m_2^2}{\sigma^2} \right) \mathbb{E}^{BS} \left[ \int_0^T \left( S_t^2 \frac{\partial^2 P_{BS}}{\partial S^2} (t, S_t) \right)^2 dt \right]. \]

This spread can be seen as a measure of the effect of market incompleteness due to jump risk on the price of a specific option, from the point of view of a specific market agent. It decomposes into a product of three factors, each representing a specific feature of our market model:
• The parameter $\alpha$, which characterizes the risk aversion of the economic agent;
• The factor $m_4 - \frac{m_2^2}{\bar{\sigma}^2}$ which characterizes the specific Lévy model through its variance, skewness and kurtosis;
• The expectation of the integral, which characterizes the specific option, and only depends on the variance of the price process.

The factor

$$E^{BS} \left[ \int_0^T \left( S_t^2 \frac{\partial^2 P_{BS}}{\partial S^2}(t, S_t) \right)^2 dt \right]$$

(15)

can therefore be seen as a model-independent measure of the sensitivity of a specific European option to jump risk, in the limit of small jumps. It is therefore interesting to study the dependence of this measure of jump risk sensitivity on strike and time to maturity.

Figure 2 plots the expectation (15) as function of strike (on the left graph) and as function of time to maturity (on the right graph) for a European put option. We see that the sensitivity to jump risk is maximal for options close to the money, since for far from the money options the exercise probability and therefore also the price and the spread are very small (remember that we are interested in sensitivity to small jumps). Note that actual bid-ask spreads in option markets exhibit similar patterns with a maximum close to the money (see Figure 3), although of course actual bid-ask spreads are influenced by a multitude of factors other than jump risk.

In terms of time to maturity for options which are not at the money, the sensitivity first grows (because the exercise probability increases) and then decays for large maturities due to a ‘central limit theorem’ effect which smoothes out the effect of jumps.
Figure 3: Bid-ask spreads of options on S&P 500 index, observed on Jan 21st, 2006, as function of strike, for time to maturity $T = 42$ days. The underlying index value was $S_0 = 1261.49$.

A Proof of Lemma 3

Let $P_\lambda(t, S) = \mathbb{E}^*[h(S\mathcal{E}(X_\lambda^t)_{T\lambda})]$. By Proposition 2 in [10], under the assumptions of Theorem 2, $P_\lambda(t, S)$ is infinitely differentiable in $t$ and in $S$, and the assumptions of the Lemma imply, in particular, that $|S \frac{\partial P_\lambda}{\partial S}| \leq L$ a.s. for all $t \in [0, T]$. Using the martingale representation of the option price given in [10], we obtain that

$$M_t^\lambda = \int_0^T \left\{ \tilde{\vartheta}_t^\lambda - \frac{\partial P_\lambda}{\partial S} \right\} \sigma S_t^\lambda dW_t + \int_0^T \int_{\mathbb{R}} \left\{ z\tilde{\vartheta}_t^\lambda S_t^\lambda - P_\lambda(t, S_{t^-}^\lambda(1 + z)) + P_\lambda(t, S_{t^-}^\lambda) \right\} \tilde{J}_X^\lambda(dt, dz)$$

and that the quadratic hedging strategy is given by

$$\tilde{\vartheta}_t^\lambda = \frac{\sigma^2 \frac{\partial P_\lambda}{\partial S} + \frac{1}{S_t^\lambda} \int_{\mathbb{R}} z(P_\lambda(t, S_{t^-}^\lambda(1 + z)) - P_\lambda(t, S_{t^-}^\lambda)) \nu_\lambda(dz)}{\sigma^2 + \int_{\mathbb{R}} z^2 \nu_\lambda(dz)} = \frac{\partial P_\lambda}{\partial S} + \frac{1}{\sigma^2 S_t^\lambda} \int_{\mathbb{R}} z\Xi^\lambda(z) \nu_\lambda(dz),$$

where we denote

$$\Xi^\lambda(z) = P_\lambda(t, S_{t^-}^\lambda(1 + z)) - P_\lambda(t, S_t^\lambda) - zS_t^\lambda \frac{\partial P_\lambda}{\partial S}.$$  

By the Burkholder-Davis-Gundy inequality expressed in predictable terms [13], for $q \geq 2$, there exist $c_q, C_q > 0$ such that:

$$\mathbb{E}[\left( \sup_{0 \leq t \leq T} |M_t|^q \right) \leq C_q \mathbb{E}[<M>_T^\frac{q}{2}] + |x|^q * \nu_T^M]$$  

where $\nu_T^M$ is the compensator of the jump measure of the process $M$. The quantities appearing in the right-hand side are explicitly given by

$$<M^\lambda>_T = \int_0^T \left\{ \tilde{\vartheta}_t^\lambda - \frac{\partial P_\lambda}{\partial S} \right\} \sigma^2 (S_t^\lambda)^2 dt + \int_0^T \int_{\mathbb{R}} \left\{ z\tilde{\vartheta}_t^\lambda S_t^\lambda - P_\lambda(t, S_{t^-}^\lambda(1 + z)) + P_\lambda(t, S_{t^-}^\lambda) \right\} \nu_\lambda(dz) dt,$$

$$|x|^q * \nu_T^M = \int_0^T \int_{\mathbb{R}} |z\tilde{\vartheta}_t^\lambda S_t^\lambda - P_\lambda(t, S_{t^-}^\lambda(1 + z)) + P_\lambda(t, S_{t^-}^\lambda)|^q \nu_\lambda(dz) dt.$$
Substituting the expression for $\bar{\partial}^\lambda$ into the first of the above equalities, we further obtain:

$$
\langle M^\lambda \rangle_T = \frac{\sigma^2}{\sigma^4} \int_0^T dt \left( \int_{\mathbb{R}} z \Xi^\lambda_t(z) \nu_\lambda(dz) \right)^2 + \int_0^T \int_{\mathbb{R}} \left( \frac{z}{\sigma^2} \int_{\mathbb{R}} z \Xi_t^\lambda(z) \nu_\lambda(dz) - \Xi_t^\lambda(z) \right)^2 \nu_\lambda(dz) dt
$$

$$
= -\frac{1}{\sigma^2} \int_0^T dt \left( \int_{\mathbb{R}} z \Xi^\lambda_t(z) \nu_\lambda(dz) \right)^2 + \int_0^T \int_{\mathbb{R}} \left( \Xi_t^\lambda(z) \right)^2 \nu_\lambda(dz) dt \leq \int_0^T \int_{\mathbb{R}} \left( \Xi_t^\lambda(z) \right)^2 \nu_\lambda(dz) dt.
$$

(17)

Our first goal is to estimate $\langle M^\lambda \rangle_T$. To this end, we fix $\eta \in (0, T)$ and estimate separately $\langle M^\lambda \rangle_{T-\eta}$ and $\langle M^\lambda \rangle_T - \langle M^\lambda \rangle_{T-\eta}$.

We proceed with the estimation of $\langle M^\lambda \rangle_{T-\eta}$. By Taylor–Lagrange expansion

$$
\Xi^\lambda_t(z) = z^2 \nu_t \int_0^1 \frac{\partial^2 P_\lambda}{\partial S^2}(t, S^\lambda_t(1 + \theta z))(1 - \theta) d\theta.
$$

Note that $\forall z > -1$, $\int_0^1 \frac{1 - \theta}{1 + \theta z} d\theta = \frac{(1 + z) \log(1 + z) - z}{z^2} \leq 1$. Thus denoting $\Gamma_\lambda(t, S) = \frac{\partial^2 P_\lambda}{\partial S^2}(t, S)$

$$
|\Xi^\lambda_t(z)| \leq z^2 S^\lambda_t \max_{S} |\Gamma_\lambda(t, S)|.
$$

We conclude that

$$
\langle M^\lambda \rangle_{T-\eta} \leq \lambda^2 \int_{\mathbb{R}} z^4 \nu_t(dz) \int_0^{T-\eta} \left( \max_S |\Gamma_\lambda(t, S)| \right)^2 dt.
$$

(18)

Furthermore, the term $\langle M^\lambda \rangle_T - \langle M^\lambda \rangle_{T-\eta}$ is controlled by the option’s delta. Indeed, as $|S \frac{\partial P_\lambda}{\partial S}| \leq L$, we deduce that

$$
|P_\lambda(t, S^\lambda_t(1 + z)) - P_\lambda(t, S^\lambda_t)| = \left| \int_0^z S^\lambda_t \frac{\partial P_\lambda}{\partial S}(t, S^\lambda_t(1 + u)) du \right| \leq \frac{L|z|}{1 - \delta},
$$

so that finally

$$
\langle M^\lambda \rangle_T - \langle M^\lambda \rangle_{T-\eta} \leq \int_{T-\eta}^T \int_{\mathbb{R}} (\Xi_t^\lambda(z))^2 \nu_\lambda(dz) dt
$$

$$
\leq 2 \int_{T-\eta}^T \int_{\mathbb{R}} |P_\lambda(t, S^\lambda_t(1 + z)) - P_\lambda(t, S^\lambda_t)|^2 \nu_\lambda(dz) dt + 2 \int_{T-\eta}^T \int_{\mathbb{R}} z^2 (S^\lambda_t)^2 \left( \frac{\partial P_\lambda}{\partial S} \right)^2 \nu_\lambda(dz) dt
$$

$$
\leq 2 \eta L^2 \int_{\mathbb{R}} z^2 \nu_t(dz) \left( \frac{1}{(1 - \delta)^2} + 1 \right) + \eta L^2 \int_{\mathbb{R}} z^2 \nu_t(dz).
$$

(19)

Using (18), (19) and the gamma estimate of Lemma 6, we conclude that there exist two constants $A$ and $B$ such that

$$
\langle M^\lambda \rangle_T \leq \begin{cases} 
\lambda^2 A \sup_{0 \leq t \leq T} (S^\lambda_t)^2 (\log T - \log \eta) + \eta B & \sigma > 0, \\
\lambda^2 A \sup_{0 \leq t \leq T} (S^\lambda_t)^2 \left( (\log T - \log \eta) + \lambda^2 \left( \frac{\sigma^2}{\eta^2} - \frac{1}{\eta^2} \right) \right) + \eta B & \sigma = 0.
\end{cases}
$$
Taking $\eta = \lambda^2$ and assuming $\lambda \leq \sqrt{T} \wedge \frac{1}{2}$, yields, for a constant $C < \infty$,
\[ \langle M^\lambda \rangle_T \leq \lambda^2 C \left\{ 1 - \sup_{0 \leq t \leq T} \langle S^\lambda_t \rangle^2 \log \lambda \right\}. \]

Since $S^\lambda$ is a martingale, by the BDG inequality we have, for all $q \geq 2$,
\[ \mathbb{E}[|S^\lambda_T - S^\lambda_0|^q] \leq \mathbb{E}[\sup_{0 \leq t \leq T} |S^\lambda_t - S^\lambda_0|^q]\]
\[ \leq C_q \mathbb{E} \left[ \left( \sigma^2 + \int_R z^2 \nu(dx) \right)^{q/2} \left( \int_0^T S^2_t\,dt \right)^{q/2} + \lambda^{q-2} \int R z^q \nu(dz) \int_0^T S^2_t\,dt \right], \]
which means that $\mathbb{E}[\sup_{0 \leq t \leq T} |S^\lambda_t|^q]$ is bounded uniformly on $\lambda$, and therefore,
\[ \mathbb{E}[\langle M^\lambda \rangle_T^{q/2}] = O \left( \lambda^q \left( \log \frac{1}{\lambda} \right)^{\frac{q}{2}} \right). \]

Lastly, it remains to estimate the second term in the BGD inequality (16). In a similar manner as for the quadratic variation of $M^\lambda$, we obtain for every $q > 2$ and for some constant $C < \infty$ (which may change from line to line),
\[ |x|^q \ast \nu^{M^\lambda}_T = \int_0^T \int_R \left| \frac{z}{\sigma^2} \int_R z \xi^\lambda_\nu(z) \nu_\lambda(dz) - \xi^\lambda_\nu(z) \right|^q \nu_\lambda(dz)\,dt \]
\[ \leq 2q^{-1} \tilde{\sigma}^{-2q} \lambda^{-q-2} \int_R |z|^q \nu(dz) \int_0^T \left| \int_R z \xi^\lambda_\nu(z) \nu_\lambda(dz) \right|^q \,dt + 2q^{-1} \int_0^T \int_R |\xi^\lambda_\nu(z)|^q \nu_\lambda(dz)\,dt \]
\[ \leq C \lambda^{q-2} \int_0^{T-\eta} |S^\lambda_t| \max_S |\Gamma_\lambda(t, S)||^q\,dt + C \eta \lambda^{q-2} \]
\[ \leq \begin{cases} 
& \lambda^{q-2} C \sup_{0 \leq t \leq T} \langle S^\lambda_t \rangle^q \left( \frac{1}{\eta^{\frac{a-1}{a}}} - \frac{1}{T^{\frac{a-1}{a}}} \right) + \eta \lambda^{q-2} C \quad \sigma > 0, \\
& \lambda^{q-2} C \sup_{0 \leq t \leq T} \langle S^\lambda_t \rangle^q \left( \frac{1}{\eta^{\frac{a-1}{a}}} - \frac{1}{T^{\frac{a-1}{a}}} \right) + \lambda^{2(\frac{q}{a}-1)} \left( \frac{1}{\eta^{\frac{a-1}{a}}} - \frac{1}{T^{\frac{a-1}{a}}} \right) + \eta \lambda^{q-2} C \quad \sigma = 0.
\end{cases} \]

Choosing once again $\eta = \lambda^2$ leads to,
\[ |x|^q \ast \nu^{M^\lambda}_T \leq C \lambda^q (1 + \sup_{0 \leq t \leq T} \langle S^\lambda_t \rangle^q) \]
for some constant $C$, and therefore,
\[ \mathbb{E}[|x|^q \ast \nu^{M^\lambda}_T] = O(\lambda^q) \]
as $\lambda \to 0$. The proof of Lemma 6 is complete.

**Lemma 6** (Estimation of the gamma).

- Let $\sigma > 0$. Then there exists $C < \infty$ such that
\[ \max_S S|\Gamma_\lambda(t, S)| \leq \frac{C}{\sqrt{T-t}}. \]
\( \text{Let } \sigma = 0 \text{ and } \lim_{r \downarrow 0} \frac{f_{r-r_0} \alpha^2 (dx)}{r^{2-\beta}} > 0 \text{ for some } \beta \in (0, 2). \) Then there exists \( C < \infty \) such that

\[
\max_S S|\Gamma_\lambda(t, S)| \leq C \left\{ \frac{1}{\sqrt{T-t}} + \frac{\gamma^2(T-t)^{h-1}}{(T-t)^3} \right\}.
\]

**Proof.** Let

\[
\text{Call}_\lambda^K(t, S) = E[(S \mathcal{E}(X_\lambda)_{T-t} - K)^+].
\]

Under the assumptions of Theorem 2,

\[
h(S) = h(0) + h'(0)S + \int_{(0, \infty]} (S - K)^+ \mu(dK),
\]

where \( \mu \) is a finite measure on \( \mathbb{R}^+ \), defined by \( \mu ((a, b]) = h'(b) - h'(a) \). Since \( \text{Call}_\lambda^K(t, S) \leq S \), by Fubini’s theorem,

\[
\text{P}_\lambda(t, S) = h(0) + h'(0)S + \int_{(0, \infty]} \text{Call}_\lambda^K(t, S) \mu(dK).
\]

Moreover, since \( \frac{\partial \text{Call}_\lambda^K(t, S)}{\partial S} \) is bounded and \( \mu \) is a finite measure, the dominated convergence theorem yields

\[
\frac{\partial \text{P}_\lambda(t, S)}{\partial S} = h'(0) + \int_{(0, \infty]} \frac{\partial \text{Call}_\lambda^K(t, S)}{\partial S} \mu(dK).
\]

Using the Fourier transform representation for call option price in exponential Lévy models, we get the following identity:

\[
\frac{\partial^2 \text{Call}_\lambda^K(t, S)}{\partial S^2} = \frac{1}{2\pi} \int_{\mathbb{R}} K^{iu+1} S^{-iu-2} \Phi_{T-t}^\lambda(-u) du
\]

\[= \frac{1}{2\pi} \int_{\mathbb{R}} K^{iu} S^{-iu-1} \Phi_{T-t}^\lambda(-u - i) du,
\]

where \( \Phi_\lambda^\lambda(u) = E[e^{iu \log(\mathcal{E}(X_\lambda))}] = e^{i\psi_\lambda(u)} \) and \( \psi_\lambda(u) = -\frac{\sigma^2}{2}(u^2 + iu) + \int (e^{iu \log(1+z)} - 1 - iuz) \nu_\lambda(dz) \).

Therefore,

\[
\left| \frac{\partial^2 \text{Call}_\lambda^K(t, S)}{\partial S^2} \right| \leq \frac{1}{2\pi S} \int_{\mathbb{R}} |\Phi_{T-t}^\lambda(-u - i)| du,
\]

and the dominated convergence theorem yields

\[
S|\Gamma_\lambda(t, S)| \leq \frac{C}{2\pi} \int_{\mathbb{R}} |\Phi_{T-t}^\lambda(-u - i)| du = \frac{C}{2\pi} \int_{\mathbb{R}} e^{(T-t)\Re \psi_\lambda(u - i)} du
\]

where \( C = \int_{(0, \infty]} |d\mu| \) and \( \Re \psi_\lambda(u - i) = -\frac{\sigma^2 u^2}{2} + \int (1 + x)(\cos u \log(1 + x)) - 1) \nu_\lambda(dx) \).

Let us study separately the cases \( \sigma > 0 \) and \( \sigma = 0 \). When \( \sigma > 0 \), we directly get

\[
\Re \psi_\lambda(u - i) \leq -\frac{\sigma^2 u^2}{2}
\]
which leads to
\[
S |\Gamma_X(t, S)| \leq \frac{C}{\sigma \sqrt{2\pi(T-t)}}
\]

When \(\sigma = 0\), using \[20\], we get,
\[
S |\Gamma_X(t, S)| \leq \frac{C}{2\pi} \left( \int_{|u|<\ell/\lambda} e^{-c(T-t)u^2} du + \int_{|u|\geq \ell/\lambda} e^{-c|u|\beta/(2\pi)(T-t)} du \right).
\]
\[
\leq \frac{C}{2\pi} \left( \int_{\mathbb{R}} e^{-c(T-t)u^2} du + \int_{\mathbb{R}} e^{-\frac{c|u|\beta}{2\pi}(T-t)} du \right)
\]
\[
= \tilde{C} \left\{ \frac{1}{\sqrt{T-t}} + \frac{\lambda^{\frac{\beta}{2}-1}}{(T-t)^{\frac{\beta}{2}}} \right\},
\]
where \(\tilde{C}\) is a constant.

\[\square\]

**Lemma 7.** Assume that \(\sigma = 0\) and \(\liminf_{r \downarrow 0} \frac{\int_{|x| \leq r} x^2 \nu(dx)}{r^2} > 0\) for some \(\beta \in (0, 2)\). Then, for every \(\ell > 0\) and \(v \in \mathbb{R}\), there exist \(c > 0\) and \(C < \infty\) such that for \(u \in \mathbb{R}\)
\[
\Re \psi_X(u + iv) \leq \begin{cases} C - cu^2 & |u\lambda| \leq \ell \\ C - c|u|^\beta \lambda^{\beta-2} & |u\lambda| > \ell \end{cases}
\]

(20)

**Proof.** Observe first that
\[
\Re \psi_X(u + iv) = \lambda^{-2} \int ((1 + \lambda x)^{-v} \cos(u(1 + \lambda x)) - 1 + v\lambda x) \nu(dx)
\]
\[
= \lambda^{-2} \int ((1 + \lambda x)^{-v}(\cos(u(1 + \lambda x)) - 1) \nu(dx) + v(v + 1) \int x^2 \nu(dx) \int_0^1 (1 + \theta \lambda x)^{-v-2}(1 - \theta) d\theta.
\]
Since \(X\) has jumps bounded by \(\delta\) and \(\lambda \leq 1\), there exists \(C < \infty\) and \(c > 0\) (depending on \(v\)) such that for all \(x\) in the support of \(\nu\),
\[
|v(v + 1) \int x^2 \nu(dx) \int_0^1 (1 + \theta \lambda x)^{-v-2}(1 - \theta) d\theta| \leq C \quad \text{and} \quad |1 + \lambda x|^{-v} \geq c.
\]
Then, using \(1 - \cos(x) = 2(\sin \frac{x}{2})^2 \geq 2(\frac{x}{\pi})^2\) for \(|x| \leq \pi\), we get:
\[
\Re \psi_X(u + iv) \leq C - c \int_{|u\log(1+\lambda x)| \leq \pi} \frac{u^2(\log(1 + \lambda x))^2}{\lambda^2} \nu(dx),
\]
but since \((\log(1 + x))^2 \geq x^2(\log 2)^2\) for \(|x| \leq 1\) we have, for a different \(c > 0\),
\[
\Re \psi_X(u + iv) \leq C - c \int_{|u\log(1+\lambda x)| \leq \pi} (ux)^2 |\lambda x| \leq 1 \nu(dx).
\]
Once again, by the bound on the jumps of \(X\), \(|\log(1 + x)| \leq \frac{|x|}{\log(1 + x)}\) on the support of \(\nu\) and we also have, for a different \(c > 0\),
\[
\Re \psi_X(u + iv) \leq C - c \int_{|x| \leq \frac{\pi}{\log(1 + x)} \wedge 1} u^2 x^2 \nu(dx).
\]
Under the assumption \( \lim \inf \frac{\int_{-r}^{0} x^2 \nu(dx)}{r^{1+\beta}} > 0 \) for some \( \beta \in (0, 2) \), there exist \( r_0 > 0 \) and \( c_0 > 0 \) such that for all \( r < r_0 \), \( \int_{-r}^{0} x^2 \nu(dx) \geq c_0 r^{2-\beta} \). This implies that one can find constants \( \ell > 0 \) and yet another \( c > 0 \) such that

\[
\Re \psi_\lambda (u + iv) \leq \begin{cases} 
C - cu^2 & |u\lambda| \leq \ell \\
C - c|u|^{\beta-2} & |u\lambda| \geq \ell
\end{cases}
\]

Now, by changing \( c \) and \( C \) this inequality can be shown to be true for arbitrary \( \ell \). \( \square \)

**B Proof of Lemma 4**

Using the notation of the proof of Lemma 3, we have

\[
\mathbb{E}\left[ \left( \int_0^T \partial_t^\lambda dS_t^\lambda - h(S_T^\lambda) \right)^2 \right] = \mathbb{E}(M_T^\lambda)^2 = \mathbb{E}(\langle M^\lambda \rangle_T),
\]

where \( \langle M^\lambda \rangle_T \) was computed in (17). From the Fourier transform formula for the call option price

\[
\text{Call}_\lambda^K(t, S) = \frac{1}{2\pi} \int_\mathbb{R} K^{iu+1-R} S_{t}^{-iu+R} \Phi_{T-t}^\lambda(-u-iR) \frac{(1+z)^{-iu+R}-1+(iu-R)z}{(R-iv)(R-1-iv)} du,
\]

we deduce that

\[
\text{Call}_\lambda^K(t, S(1+z)) - \text{Call}_\lambda^K(t, S) - zS \frac{\partial \text{Call}_\lambda^K(t, S)}{\partial S} = \frac{1}{2\pi} \int_\mathbb{R} K^{iu+1-R} S_{t}^{-iu+R} \Phi_{T-t}^\lambda(-u-iR) (1+z)^{-iu+R}-1+(iu-R)z \frac{(1+z)^{-iu+1}-1+(iu-1)z}{iu(iu-1)} du.
\]

Since the fraction under the integral sign is analytic for \( z > -1 \), we can choose \( R = 1 \) in this formula, obtaining

\[
\text{Call}_\lambda^K(t, S(1+z)) - \text{Call}_\lambda^K(t, S) - zS \frac{\partial \text{Call}_\lambda^K(t, S)}{\partial S} = \frac{1}{2\pi} \int_\mathbb{R} K^{iu+1-R} S_{t}^{-iu+1} \Phi_{T-t}^\lambda(-u-i) (1+z)^{-iu+1}-1+(iu-1)z \frac{(1+z)^{-iu+1}-1+(iu-1)z}{iu(iu-1)} du.
\]

and therefore,

\[
\Xi_t^\lambda(z) = \frac{1}{2\pi} \int_{(0,\infty)} \mu(dK) \int_\mathbb{R} K^{iu+1-R} S_{t}^{-iu+1} \Phi_{T-t}^\lambda(-u-i) (1+z)^{-iu+1}-1+(iu-1)z \frac{(1+z)^{-iu+1}-1+(iu-1)z}{iu(iu-1)} du.
\]

By Fubini’s theorem, the expression under the time integral in (17) equals

\[
\frac{1}{4\pi^2} \int_{(0,\infty)} \mu(dK) \int_{(0,\infty)} \mu(d\bar{K}) \int dv \int dv K^{iu} K^{-iu} \Phi_{T-t}^\lambda(-u-i) \Phi_\lambda^K(-u-v-2i) \frac{a_\lambda(u)a_\lambda(v)}{\bar{\sigma}^2} + b_\lambda(u, v),
\]

\( 24 \)
where

\[ a_\lambda(u) = \int_\mathbb{R} \nu_\lambda(dz)z\{1 + z\}^{-iu+1} - 1 + (iu - 1)z = \lambda^{-1} \int_\mathbb{R} \nu(dz)z\{1 + \lambda z\}^{-iu+1} - 1 + (iu - 1)\lambda z \]

and

\[ b_\lambda(u, v) = \int_\mathbb{R} \nu_\lambda(dz)\{(1 + z)^{-1} - 1 + (iu - 1)z\}(1 + \lambda z)^{-iv+1} - 1 + (iv - 1)z \]

\[ = \lambda^{-2} \int_\mathbb{R} \nu(dz)\{(1 + \lambda z)^{-iu+1} - 1 + (iu - 1)\lambda z\}(1 + \lambda z)^{-iv+1} - 1 + (iv - 1)\lambda z \]

From the explicit form of \( \Phi^\lambda \),

\[ \Phi_t^\lambda(u) = e^{t\psi^\lambda(u)}, \quad \psi^\lambda(u) = -\frac{\sigma^2}{2}(u^2 + iu) + \lambda^{-2} \int_\mathbb{R} ((1 + \lambda z)^{iu} - 1 - iu \lambda z) \nu(dz) \]

\[ = -\frac{\sigma^2}{2}(u^2 + iu) - (u^2 + iu) \int \nu(dz) \int_0^1 ((1 + \lambda z \theta)^{iu-2}(1 - \theta)d\theta, \]

we deduce, using the fact that \( \nu \) has bounded support, that for every \( u \in \mathbb{C} \),

\[ \psi^\lambda(u) \to \psi^0(u) = -\frac{u^2 + iu}{2} \left( \sigma^2 + \int_\mathbb{R} z^2 \nu(dz) \right) \quad \text{as} \quad \lambda \to 0. \]

On the other hand, since

\[ (1 + \lambda z)^{-iu+1} - 1 + (iu - 1)\lambda z = iu(iu - 1)\lambda^2 z^2 \int_0^1 (1 + \lambda \theta z)^{-iu-1}(1 - \theta)d\theta, \]

we get that

\[ \lambda^{-1} a_\lambda(u) \to \frac{iu(iu - 1)}{2} \int \nu(dz) \quad \text{and} \quad \lambda^{-2} b_\lambda(u, v) \to -\frac{uv(iu - 1)(iv - 1)}{4} \int \nu(dz) \]

as \( \lambda \to 0 \). Therefore, provided that we can find an integrable bound to apply the dominated convergence theorem, \( \lambda^{-2}(M^\lambda)_T \) converges to

\[ \frac{1}{4} \left( m_4 - \frac{m_3^2}{\sigma^2} \right) \int_{(0, \infty)} \mu(dK) \int_{(0, \infty)} \mu(dK) \int_0^T dt \int_\mathbb{R} dv \int_\mathbb{R} du K^{iu} K^{iv} \]

\[ \times \Phi^0_{T-t}(-u - i) \Phi^0_{T-t}(-v - i) \Phi^0_t(-u - v - 2i) \]

\[ = \frac{1}{4} \left( m_4 - \frac{m_3^2}{\sigma^2} \right) \mathbb{E}^{BS} \left[ \int_0^T \left( S_t \frac{\partial P_{BS}(t, S_t)}{\partial S_t^2} \right)^2 dt \right] \]

as \( \lambda \to 0 \).

We first consider the case \( \sigma > 0 \). Remark first that by the bound on the jumps of \( X \),

\[ \left| \int_0^1 (1 + \lambda \theta z)^{-iu-1}(1 - \theta)d\theta \right| \leq \frac{1}{2(1 - \delta)}, \]

so that it remains to find an integrable (in \( u, v \) and \( t \)) bound for

\[ \Phi^\lambda_{T-t}(-u - i) \Phi^\lambda_{T-t}(-v - i) \Phi^\lambda_t(-u - v - 2i) \]
However, in this case,
\[
\left| \Phi_{T-t}^\lambda(-u-i)\Phi_{T-t}^\lambda(-v-i)\Phi_t^\lambda(-u-v) \right| \leq e^{-\frac{1}{2}(T-t)\sigma^2(u^2+v^2)-\frac{1}{2}t\sigma^2(u+v)^2+t\sigma^2},
\]
which is integrable since
\[
\int_\mathbb{R} du \int_\mathbb{R} dv e^{-\frac{1}{2}(T-t)\sigma^2(u^2+v^2)-\frac{1}{2}t\sigma^2(u+v)^2} = \int_\mathbb{R} du \int_\mathbb{R} dv e^{-\frac{1}{2}T\sigma^2u^2} = \frac{2\pi}{\sigma^2\sqrt{T^2-t^2}}.
\]
Let us now consider the case \( \sigma = 0 \). We shall use the bound (20). In addition,
\[
|b_{\lambda}(u,v)| \leq \lambda^{-2} \left( \int \nu(dz) |(1+\lambda z)^{-iu+1} - 1 + (iu-1)\lambda z|^2 \right)^{\frac{1}{2}}
\times \left( \int \nu(dz) |(1+\lambda z)^{-iv+1} - 1 + (iv-1)\lambda z|^2 \right)^{\frac{1}{2}}
\]
and it is easy to show, using arguments similar to those used to prove the bound (20) that for some constant \( C < \infty \),
\[
\int \nu(dz) |(1+\lambda z)^{-iu+1} - 1 + (iu-1)\lambda z|^2 \leq C\lambda^4 u^2 (u^2 + 1) 1_{|\lambda u| \leq \ell} + C\lambda^2 u^2 1_{|\lambda u| > \ell},
\]
where the constant \( \ell \) may be taken the same as in the bound (20). Similarly, by the Cauchy-Schwarz inequality,
\[
|a_{\lambda}(u)a_{\lambda}(v)| \leq \tilde{\sigma}^2 \lambda^{-2} \left( \int \nu(dz) |(1+\lambda z)^{-iu+1} - 1 + (iu-1)\lambda z|^2 \right)^{\frac{1}{2}}
\times \left( \int \nu(dz) |(1+\lambda z)^{-iv+1} - 1 + (iv-1)\lambda z|^2 \right)^{\frac{1}{2}},
\]
so that to complete the proof it suffices to study the integral
\[
\int_\mathbb{R} dv \int_\mathbb{R} du \left| \Phi_{T-t}^\lambda(-u-i)\Phi_{T-t}^\lambda(-v-i)\Phi_t^\lambda(-u-v-2i) \right| \left( 1_{|\lambda u| \leq \ell} + \frac{1_{|\lambda u| > \ell}}{\lambda \sqrt{1+u^2}} \right) \left( 1_{|\lambda v| \leq \ell} + \frac{1_{|\lambda v| > \ell}}{\lambda \sqrt{1+v^2}} \right).
\]
We shall decompose it into four terms corresponding to integrals over non-disjoint sets (whose union is \( \mathbb{R}^2 \)) \{ |u| \leq 2\ell, |v| \leq 2\ell \}, \{ |u| > 2\ell, |v| \leq \ell \}, \{ |u| \leq \ell, |v| > 2\ell \} and \{ |u| > \ell, |v| > \ell \}, and show that on the first set one can apply the dominated convergence theorem, and the contribution of the three other sets to the limit is zero. On the first set, the integrand is bounded as follows:
\[
\left| \Phi_{T-t}^\lambda(-u-i)\Phi_{T-t}^\lambda(-v-i)\Phi_t^\lambda(-u-v-2i) \right| 1_{|\lambda u| \leq \ell, |\lambda v| \leq \ell} \leq e^{-\left(c(T-t)(u^2+v^2)-ct(u+v)^2\right)},
\]
which is integrable in \( u, v, t \). Hence, by the dominated convergence theorem,
\[
\frac{\lambda^{-2}}{4\pi^2} \int_{(0,\infty)} \mu(dK) \int_{(0,\infty)} \mu(dK) \int_0^T dt \int_{|\lambda u| \leq 2\ell} dv \int_{|u| \leq 2\ell} du K^{iu} K^{iv} \times \Phi_{T-t}^\lambda(-u-i)\Phi_{T-t}^\lambda(-v-i)\Phi_t^\lambda(-u-v-2i) \left( -\frac{a_{\lambda}(u)a_{\lambda}(v)}{\tilde{\sigma}^2} + b_{\lambda}(u,v) \right)
\rightarrow \frac{1}{4} \left( m_4 - \frac{m_2^2}{\tilde{\sigma}^2} \right) \mathbb{E}^BS \int_0^T \left( S_t^2 \frac{\partial P_{BS}(t,S_t)}{\partial S^2} \right)^2 dt
\]
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as $\lambda \to 0$. It remains to show that the other three sets give a zero contribution to the limit.

On the set $\{|u\lambda| > 2\ell, |v\lambda| \leq \ell\}$, the integrand in (21) is bounded by:

$$\frac{\lambda^{-1}}{\sqrt{1 + u^2}} e^{-c(T-t)\lambda^{\beta-2}|u|^\beta - c(T-t)\lambda^{\beta-2}|u+v|^\beta} \leq \frac{\lambda^{-1}}{\sqrt{1 + u^2}} e^{-c(T-t)\lambda^{\beta-2}|u|^\beta - c(T-t)u^2 - ct\lambda^{\beta-2}|v|^\beta} \leq \frac{\lambda^{-1}}{\sqrt{1 + u^2}} e^{-c(T-t)\lambda^{\beta-2}|u|^\beta - cT(\ell^{\beta-2}\lambda_1)|v|^2}.$$

On the other hand,

$$\int_{\{|u\lambda| > 2\ell\}} \frac{du}{\lambda \sqrt{1 + u^2}} e^{-c(T-t)\lambda^{\beta-2}|u|^\beta} \leq 2 \int_{2\ell}^{\infty} \frac{du}{\lambda u} e^{-c(T-t)\lambda^{\beta-2}|u|^\beta} = \frac{1}{\sqrt{T - \ell}} f \left( \frac{T - \ell}{\lambda^2} \right),$$

where

$$f(\theta) = 2\sqrt{\theta} \int_{2\theta^{1/\beta}}^{\infty} \frac{du}{u} e^{-c|u|^\beta}.$$

Note that $f$ is a bounded positive function and $f(\theta) \to 0$ as $\theta \to \infty$. Therefore, by the dominated convergence theorem,

$$\int_0^T \int_{\{|u\lambda| > 2\ell\}} \int_{\{|v\lambda| \leq \ell\}} \left| \Phi_{T-t}^\lambda(-u - i) \Phi_{T-t}^\lambda(-v - i) \Phi_t^\lambda(-u - v - 2i) \right| \frac{1}{\lambda \sqrt{1 + u^2}} dudv = 0$$

as $\lambda \to 0$.

The set $\{|u\lambda| \leq \ell, |v\lambda| > 2\ell\}$ can be dealt with in the same manner. Finally, on the set $\{|u\lambda| > \ell, |v\lambda| > \ell\}$, the integrand in (21) is bounded by

$$\frac{\lambda^{-2}}{(1 + u^2)(1 + v^2)} \left\{ e^{-c(T-t)\lambda^{\beta-2}|u|^\beta - c(T-t)\lambda^{\beta-2}|v|^\beta - ct|u+v|^2} 1_{|u+v| \leq \ell} \right.$$ 

$$\left. + e^{-c(T-t)\lambda^{\beta-2}|u|^\beta - c(T-t)\lambda^{\beta-2}|v|^\beta - ct\lambda^{\beta-2}|u+v|^\beta} \right\} \quad (22)$$

With a change of variable $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$, using the convexity inequality

$$\left| \frac{x+y}{2} \right|^{\beta} + \left| \frac{x-y}{2} \right|^{\beta} \geq c_\beta (|x|^{\beta} + |y|^{\beta}),$$

the integral of the first term above satisfies

$$\int_{|u\lambda| \geq \ell} du \int_{|v\lambda| \geq \ell} dv \frac{\lambda^{-2}}{(1 + u^2)(1 + v^2)} e^{-c(T-t)\lambda^{\beta-2}|u|^\beta - c(T-t)\lambda^{\beta-2}|v|^\beta - ct|u+v|^2} 1_{|u+v| \leq \ell} \leq \int_{|u\lambda| > 2\ell} dx \int_{|v\lambda| > 2\ell} dy \frac{\lambda^{-2}}{(1 + x+y)(1 + |x-y|)} \left| \frac{x+y}{2} \right|^{\beta} + \left| \frac{x-y}{2} \right|^{\beta} \leq \int_{|u\lambda| > \ell} dy \int_{|v\lambda| > \ell} dx \frac{\lambda^{-2}}{(1 + x+y)(1 + |x-y|)} e^{-c_\beta(T-t)\lambda^{\beta-2}|x|^\beta - c_\beta(T-t)\lambda^{\beta-2}|y|^\beta - ct|x|^2} 1_{|x| \leq \ell} \leq \int_{|x| > \ell} dx \frac{C}{\lambda^2} e^{-c_\beta(T-t)\lambda^{\beta-2}|x|^\beta - c_\beta(T-t)\lambda^{\beta-2}|y|^\beta - ct|x|^2},$$

for some constant $C < \infty$, where we used Lemma 2 in [5]. It is clear that this expression converges to 0 as $\lambda \to 0$.  

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Finally, using the same change of variable as above, and once again, Lemma 2 in [5], the integral of the second term in (22) satisfies,

$$\int_{|u\lambda| \geq \ell} du \int_{|v\lambda| \geq \ell} \frac{\lambda^{-2}}{\sqrt{(1 + u^2)(1 + v^2)}} e^{-c(T-t)\lambda^\theta - \frac{|u\lambda|^2}{2} - cT\lambda^\theta - \frac{|v\lambda|^2}{2}} dudv \leq \int_{\lambda|x+y| > 2\ell} dx \int_{\lambda|x-y| > 2\ell} dy \frac{\lambda^{-2}}{\sqrt{(1 + |x+y|)(1 + |x-y|)}} e^{-c(\lambda^\theta - 2)|x|^\beta} \leq \frac{C}{\lambda^3} \int_{|x| > 2\ell} dx e^{-c(\lambda^\theta - 2)|x|^\beta} \leq \frac{C}{\lambda^3} \int_{|x| > 2\ell} dx e^{-c(\lambda^\theta - 2)|x|^\beta}$$

which clearly goes to zero as $\lambda \to 0$.

To perform the computation for the put option pay-off, recall that in the Black-Scholes model, $S_t = S_0 e^{-\frac{T}{2} + \sigma W_t}$ and $\frac{\partial^2 \text{Put}(t,S)}{\partial S^2} = \frac{\phi(d_2(t))}{S_0 \sqrt{T-t}}$ with $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. Therefore,

$$\mathbb{E}^{\text{BS}} \left[ \int_0^T \left( S_t^2 \frac{\partial^2 \text{Put}(t,S)}{\partial S^2} \right)^2 dt \right] = \frac{K^2}{2\pi \sigma^2} \int_0^T \mathbb{E}^{\text{BS}} \left[ e^{-\left( \frac{\log(S_0/K) - \frac{\sigma^2(T-t)}{2}}{\sigma \sqrt{T-t}} \right)^2} \right] dt \leq \frac{K^2}{2\pi \sigma^2} \int_0^T \mathbb{E}^{\text{BS}} \left[ e^{-\left( \frac{\log(S_0/K) - \frac{\sigma^2(T-t)}{2}}{\sigma \sqrt{T-t}} \right)^2} \right] dt \leq \frac{K^2}{2\pi \sigma^2} \int_0^T \mathbb{E}^{\text{BS}} \left[ e^{-\left( \frac{\log(S_0/K) - \frac{\sigma^2(T-t)}{2}}{\sigma \sqrt{T-t}} \right)^2} \right] dt \leq \frac{K^2}{2\pi \sigma^2} \int_0^T \mathbb{E}^{\text{BS}} \left[ e^{-\left( \frac{\log(S_0/K) - \frac{\sigma^2(T-t)}{2}}{\sigma \sqrt{T-t}} \right)^2} \right] dt$$

It remains to perform the explicit integration with the Gaussian density to get the result.

C Proof of Lemma [5]

In this proof, we denote $X^{\text{BS}}_t = \partial W_t$, where $W$ is a standard Brownian motion independent from $X^\lambda$. For $\lambda > 0$, define

$$f_\lambda(t,x) = \mathbb{E}[h(xE(X^{\lambda})_t)\mathcal{E}(X^{\text{BS}})_t],$$

and for $\epsilon > 0$, let

$$h^\epsilon(x) = \mathbb{E}[h(xE(X^{\text{BS}}_\epsilon)_t)], \quad P^{\text{BS}}_t(x) = \mathbb{E}[h^\epsilon(xE(X^{\text{BS}})_t)] = h^\epsilon + T^{-1}(x), \quad f^\epsilon_\lambda(t,x) = \mathbb{E}[h^\epsilon(xE(X^{\lambda})_t)\mathcal{E}(X^{\text{BS}})_t].$$

These functions are well defined because $X^\lambda$ has bounded jumps and therefore all moments of $\mathcal{E}(X^{\lambda})_t$ are finite. Without loss of generality we shall also take $S_0 = 1$ below.

From Itô formula, using item 1 of Lemma [8]

$$P^{\text{BS}}_T(t,S^\lambda_t) = P^{\text{BS}}_T(0,1) + \int_0^T \frac{\partial P^{\text{BS}}_t}{\partial t} dt + \int_0^T \frac{\partial P^{\text{BS}}_t}{\partial S} S^\lambda_t \sigma dW_t + \int_0^T \frac{1}{2} \sigma^2(S^\lambda_t)^2 \frac{\partial^2 P^{\text{BS}}_t}{\partial S^2} dt + \sum \{P^{\text{BS}}_T(t,S^\lambda_{t-}(1+z)) - P^{\text{BS}}_T(t,S^\lambda_{t-}) - zS^\lambda_t \frac{\partial P^{\text{BS}}_T}{\partial S}(t,S^\lambda_{t-})\} J^\lambda(dsdz).$$

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Taking the expectation and using the Black-Scholes equation and Fubini’s theorem justified by item 3 of Lemma 8, we get:

\[
\mathbb{E}[h^\xi(S_T^\lambda)] - P^\xi_{BS}(0,1) = \mathbb{E} \int_0^T \int \left\{ P^\xi_{BS}(t,S_t^\lambda(1+z)) - P^\xi_{BS}(t,S_t^\lambda) - zS_t^\lambda \frac{\partial P^\xi_{BS}}{\partial S} - \frac{1}{2} z^2 \frac{\partial^2 P^\xi_{BS}}{\partial S^2} \right\} \nu_\lambda(dz)dt
\]

\[
= \int_0^T \int \left\{ f^\xi(t,1+z) - f^\xi(t,1) - z(\partial_x f^\xi)(t,1) - \frac{z^2}{2} (\partial^2_x f^\xi)(t,1) \right\} \nu_\lambda(dz)dt
\]

\[
= \int_0^T \int \left\{ \frac{z^3}{6} (\partial^2_x f^\xi)(t,1) + \int_0^1 \frac{z^4}{6} (1-\theta)^3 (\partial^3_x f^\xi)(t,1+z\theta)d\theta \right\} \nu_\lambda(dz)dt
\]

Define \( f^\xi(t,x) = (\partial^3_x f^\xi)(t,x) = \mathbb{E}[h^\xi(x\mathcal{X}(S^\lambda,x),\mathcal{X}(S^\lambda,t_\lambda)) \mid \mathcal{F}_T] \), where \( h^\xi(x) = x^3 (\partial^3_x h^\xi)(x) \). Then, using again a Taylor-Lagrange expansion, we get for all \( t \in [0,T] \)

\[
f^\xi(t,1) = f^\xi(0,1) + \int_0^t \int \left\{ f^\xi(s,1+z) - f^\xi(s,1) - z(\partial_x f^\xi)(s,1) - \frac{z^2}{2} (\partial^2_x f^\xi)(s,1) \right\} \nu_\lambda(dz)ds
\]

\[
= \frac{\partial^3 P^\xi_{BS}}{\partial S^3}(0,1) + \int_0^t \int \frac{\lambda z^3}{2} (1-\theta)^2 (\partial^3_x f^\xi)(s,1+\lambda \theta)d\theta \nu(dz)ds.
\]

Substituting this representation into the above formula, we obtain

\[
\mathbb{E}[h^\xi(S_T^\lambda)] - P^\xi_{BS}(0,1) = \mathbb{E} \int_0^T dt \int \frac{\lambda z^3}{2} \nu(dz) \int_0^1 (1-\theta)^2 (\partial^3_x f^\xi)(t,1+\lambda \theta)d\theta
\]

\[
= \frac{\lambda m_3 T}{6} \frac{\partial^3 P^\xi_{BS}}{\partial S^3}(0,1) + \lambda^2 \int_0^T dt \int \frac{\lambda z^3}{2} \nu(dz) \int_0^1 (1-\theta)^2 (\partial^3_x f^\xi)(t,1+\lambda \theta)d\theta.
\]

Note that \( (\partial^3_x f^\xi)(s,x) = 6(\partial^3_x f^\xi)(s,x) + 18x(\partial^4_x f^\xi)(s,x) + 9x^2(\partial^5_x f^\xi)(s,x) + x^3(\partial^6_x f^\xi)(s,x) \).

Now we use item 5 of Lemma 8 to make \( \varepsilon \) go to zero, obtaining

\[
\mathbb{E}[h^\xi(S_T^\lambda)] = P^\xi_{BS}(0,1) + \frac{\lambda m_3 T}{6} \frac{\partial^3 P^\xi_{BS}}{\partial S^3}(0,1) + \lambda^2 \int_0^T dt \int \frac{\lambda z^3}{2} \nu(dz) \int_0^1 (1-\theta)^2 (\partial^3_x f^\xi)(t,1+\lambda \theta)d\theta.
\]

To finish the proof, we use the dominated convergence theorem (justified by items 3 and 4 of Lemma 8) to show that

\[
\int_0^T dt \int \frac{\lambda z^3}{2} \nu(dz) \int_0^1 (1-\theta)^3 (\partial^4_x f^\xi)(t,1+\lambda \theta)d\theta \to \int_0^T dt \int \frac{\lambda z^4}{6} \nu(dz) \int_0^1 (1-\theta)^3 (\partial^4_x f^\xi)(t,1)d\theta
\]

\[
= \frac{m_4 T}{24} (\partial^4_x f^\xi)(0,1) = \frac{m_4 T}{24} \frac{\partial^4 P^\xi_{BS}}{\partial S^4}(0,1)
\]
Lemma 8. Let the assumptions of Lemma \[\text{hold true. Then, for all 0} \leq k \leq 6\]

1. \(P_{BS}^x \in C^{1,2}([0,T] \times (0,\infty))\).

2. \((\partial_x^k h^\varepsilon)(x)\) exists, is continuous and has polynomial growth in \(x\) for all \(\varepsilon > 0\).

3. \((\partial_x^k f_{\lambda}^x)(t, x)\) exists, is continuous in \(x\) and satisfies

\[|\partial_x^k f_{\lambda}^x|(t, x) \leq C(1 + |x|^n)\]

for some \(n \geq 0\) and a constant \(C\) which does not depend on \(t, \varepsilon\) or \(\lambda\).

4. For all \(t, x\), \((\partial_x^k f_{\lambda}^x)(t, x) \rightarrow (\partial_x^k h^T)(x)\) as \(\lambda \rightarrow 0\).

5. For all \(t, x, \lambda\), \((\partial_x^k f_{\lambda}^x)(t, x) \rightarrow (\partial_x^k f_{\lambda}^x)(t, x)\) as \(\varepsilon \rightarrow 0\).

Proof. Under our assumptions, the random variable

\[
\log(\mathcal{E}(X^\lambda_t)\mathcal{E}(X^{BS})_{T-t+\varepsilon})
\]

admits a density \(p_{\lambda}^x(t, x)\) which can be recovered via Fourier inversion:

\[
p_{\lambda}^x(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \Phi_{T-t+\varepsilon}^BS(u) \Phi_{t}^\lambda(u) du.
\]

By the bound \[\text{[20]}\] and the explicit form of \(\Phi_{BS}\), we conclude that the derivatives of \(p_{\lambda}^x(t, x)\) with respect to \(x\) of any order are continuous and given by

\[
\partial_x^k p_{\lambda}^x(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} (-iu)^k e^{-iux} \Phi_{T-t+\varepsilon}^BS(u) \Phi_{t}^\lambda(u) du.
\]

By Jensen’s inequality and Plancherel’s theorem, for any \(p \geq 0\),

\[
\int_{\mathbb{R}} |\partial_x^k p_{\lambda}^x(t, x)| dx = \left(\int_{\mathbb{R}} |\partial_x^k p_{\lambda}^x(t, x)|^p dx\right)^{\frac{1}{p}} \leq \left(\frac{1}{2\pi} \int |\partial_x^k p_{\lambda}^x(t, x)\|2e^{p|\partial_x^k p_{\lambda}^x(t, x)|} dx\right)^{\frac{1}{2}} \leq \left(\frac{1}{2\pi} \int |\partial_x^k p_{\lambda}^x(t, x)|^2 e^{(2p+2)|x|} dx\right)^{\frac{1}{2}} + \left(\frac{1}{2\pi} \int |\partial_x^k p_{\lambda}^x(t, x)|^2 e^{-(2p+2)|x|} dx\right)^{\frac{1}{2}}
\]

\[
= \left(\frac{1}{2\pi} \int |v-i(p+1)|^2 \Phi_{T-t+\varepsilon}^BS(v-i(p+1)) \Phi_{t}^\lambda(v-i(p+1))^2 dv\right)^{\frac{1}{2}}
\]

\[
+ \left(\frac{1}{2\pi} \int |v+i(p+1)|^2 \Phi_{T-t+\varepsilon}^BS(v+i(p+1)) \Phi_{t}^\lambda(v+i(p+1))^2 dv\right)^{\frac{1}{2}} < \infty.
\]
Consider for example, the second term. Using the bound \([20]\), it satisfies, for some constant \(C < \infty\),
\[
\int_{\mathbb{R}} |v + i(p + 1)|^{2k} |\Phi_{t-T+\varepsilon}^{BS}(v + i(p + 1))|^{2}dv \\
\leq C \int_{\mathbb{R}} (1 + |v|^{2k}) e^{-((T-t+\varepsilon)^2 - ct_0^2)} |\Phi_{t}^{\lambda}(v + i(p + 1))|^{2}dv \\
\leq C \int_{\mathbb{R}} (1 + |v|^{2k}) e^{-((T-t)^2 - ct_0^2)} |\Phi_{t}^{\lambda}(v + i(p + 1))|^{2}dv,
\]
which is easily seen to be bounded uniformly on \(t\). Therefore, \(\int_{\mathbb{R}} |\partial_{x}^{k}p_{\lambda}(t, x)|^{2}dv\) is bounded uniformly on \(t, \varepsilon\) and \(\lambda\).

This means that the function \(f_{\lambda}^{\varepsilon}\) is given by
\[
f_{\lambda}^{\varepsilon}(t, x) = \int_{\mathbb{R}} dz h(x e^{\varepsilon}) p_{\lambda}(t, z).
\]

Instead of the function \(f_{\lambda}^{\varepsilon}\) we shall, for notational convenience, study the function \(\tilde{f}_{\lambda}^{\varepsilon}(t, x) = f_{\lambda}^{\varepsilon}(t, e^{\varepsilon})\), which is therefore given by
\[
\tilde{f}_{\lambda}^{\varepsilon}(t, x) = \int_{\mathbb{R}} dz h(e^{\varepsilon} x) \tilde{p}_{\lambda}(t, z) = \int_{\mathbb{R}} dz h(e^{\varepsilon} z) p_{\lambda}(t, z - x)
\]

By dominated convergence, using the above estimate, we then get that
\[
\partial_{x}^{k}f_{\lambda}^{\varepsilon}(t, x) = (-1)^{k} \int_{\mathbb{R}} dz h(e^{\varepsilon} z) \partial_{x}^{k}p_{\lambda}(t, z - x) = (-1)^{k} \int_{\mathbb{R}} dz h(e^{\varepsilon} z) \partial_{x}^{k}p_{\lambda}(t, z)
\]
exists, is continuous and has exponential growth in \(x\), which means that \(\partial_{x}^{k}f_{\lambda}^{\varepsilon}(t, x)\) has polynomial growth uniformly on \(t, \varepsilon\) and \(\lambda\). This finishes the proof of item \([3]\).

To study the convergence in \(\lambda\), from the polynomial growth of \(h\), \(|h(e^{\varepsilon})| \leq e^{p|x|}\). Then, proceeding similarly to the above, we have
\[
|\partial_{x}^{k}f_{\lambda}^{\varepsilon}(t, x) - \partial_{x}^{k}f_{0}(t, x)| \leq C \int_{\mathbb{R}} e^{p|x|} \left| \partial_{x}^{k}p_{\lambda}(t, z) - \partial_{x}^{k}p_{0}(t, z) \right| \\
\leq C \left( \int_{\mathbb{R}} (1 + |v|^{2k}) |\Phi_{t-T+\varepsilon}^{BS}(v - i(p + 1))|^{2}dv \right)^{1/2} \\
+ C \left( \int_{\mathbb{R}} (1 + |v|^{2k}) |\Phi_{t-T+\varepsilon}^{BS}(v + i(p + 1)) - \Phi_{t}^{BS}(v + i(p + 1))|^{2}dv \right)^{1/2}.
\]

Consider for example the second term. It satisfies
\[
\int_{\mathbb{R}} (1 + |v|^{2k}) e^{-(T-t)^2} |\Phi_{t}^{\lambda}(v + i(p + 1)) - \Phi_{t}^{BS}(v + i(p + 1))|^{2}dv \\
\leq C \int_{\mathbb{R}} (1 + |v|^{2k}) e^{-(T-t)^2} |\Phi_{t}^{\lambda}(v + i(p + 1)) - \Phi_{t}^{BS}(v + i(p + 1))|^{2}dv
\]
Since \(|\Phi_{t}^{\lambda}(v + i(p + 1)) - \Phi_{t}^{BS}(v + i(p + 1))|\) for all \(v\) as \(\lambda \to 0\), and an integrable bound can be found similarly to \([23]\), we conclude using the dominated convergence theorem that the above expression converges to 0 as \(\lambda \to 0\). This finishes the proof of item \([4]\). Other items can be proved in a similar manner.

\(\square\)
D \quad A \text{ general formula for high-order Black-Scholes greeks}

The risk neutral price process dynamics of the stock in the Black-Scholes model with zero interest rate and volatility \( \sigma > 0 \) reads

\[
dS_t = \sigma S_t dW_t,
\]

where \( W \) denotes a standard Brownian motion. In this model, the pricing function of a European call option with strike \( K > 0 \) and maturity \( T > 0 \) given by \( P : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies

\[
P(t,s) = \mathbb{E}[(S_T - K)_+ | S_t = s] = s \Phi(\delta_1(t,s)) - K \Phi(\delta_2(t,s)),
\]

where \( \Phi \) and \( \varphi \) are, respectively, the cumulative distribution function and the density of the standard normal distribution, and the coefficients \( \delta_1 \) and \( \delta_2 \) are defined by

\[
\delta_1(t,s) = \frac{\log \frac{s}{K} + \frac{\sigma^2}{2}(T-t)}{\sigma \sqrt{T-t}}, \quad \delta_2(t,s) = \delta_1(t,s) - \sigma \sqrt{T-t}.
\]

We set for \( n \in \mathbb{N} \),

\[
d_n(t,s) = s^n \frac{\partial^n P}{\partial s^n}(t,s).
\]

The first two cash greeks can be computed by direct differentiation:

\[
d_1(t,s) = s \Phi(\delta_1(t,s)), \quad d_2(t,s) = s^2 \frac{\varphi(\delta_1(t,s))}{s \sigma \sqrt{T-t}}.
\]

For higher order derivatives of European call/put option prices, the following recurrence relation holds for all \( n \geq 0 \):

\[
d_{4+n}(t,s) = \sum_{k=0}^{n} C_n^k D_{n-k}(t,s) d_{2+k}(t,s)
\]

where \( C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!} \) are the binomial coefficients,

\[
D_k(t,s) = (-1)^{k+1} k! \left[ \delta(t,s) - \frac{1}{\sigma^2(T-t)} \sum_{p=1}^{k} \frac{1}{p} \right] \quad \text{and} \quad \delta(t,s) = \frac{\delta_1(t,s)}{\sigma \sqrt{T-t}} + 1.
\]

This recurrence relation leads to the following formulae for the cash greeks up to order 6:

\[
d_3(t,s) = -d_2(t,s) \delta(t,s),
\]

\[
d_4(t,s) = d_2(t,s) \left( \delta(t,s) - \frac{1}{\sigma^2 T} \right) - d_3(t,s) \delta(t,s),
\]

\[
d_5(t,s) = -d_2(t,s) \left( 2 \delta(t,s) - \frac{3}{\sigma^2 T} \right) + 2 d_3(t,s) \left( \delta(t,s) - \frac{1}{\sigma^2 T} \right) - d_4(t,s) \delta(t,s),
\]

\[
d_6(t,s) = d_2(t,s) \left( 6 \delta(t,s) - \frac{11}{\sigma^2 T} \right) - 3 d_3(t,s) \left( 2 \delta(t,s) - \frac{3}{\sigma^2 T} \right)
\]
\[+ 3 d_4(t,s) \left( \delta(t,s) - \frac{1}{\sigma^2 T} \right) - d_5(t,s) \delta(t,s).
\]
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