THE SCHRÖDER-BERNSTEIN PROBLEM FOR MODULES

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Abstract. In this paper we study the Schröder-Bernstein problem for modules. We obtain a positive solution for the Schröder-Bernstein problem for modules invariant under endomorphisms of their general envelopes under some mild conditions that are always satisfied, for example, in the case of injective, pure-injective or cotorsion envelopes. In the particular cases of injective envelopes and pure-injective envelopes, we are able to extend it further and we show that the Schröder-Bernstein problem has a positive solution even for modules that are invariant only under automorphisms of their injective envelopes or pure-injective envelopes.

1. Introduction

The Schröder-Bernstein theorem is a classical result in basic set theory. It states that if $A$ and $B$ are two sets such that there is a one-to-one function from $A$ into $B$ and a one-to-one function from $B$ into $A$, then there exists a bijective map between the two sets $A$ and $B$. This type of problem where one asks if two mathematical objects $A$ and $B$ which are similar in some sense to a part of each other are also similar themselves is usually called the Schröder-Bernstein problem and it has been studied in various branches of Mathematics. The most notable result along this direction is the one due to W. T. Gowers [7] where he constructed an example of two non-isomorphic Banach spaces such that each one is a complemented subspace of the other, thus showing that the Schröder-Bernstein problem has a negative solution for Banach spaces. In the context of modules, this problem was studied by Bumby in [2] where he proved that the Schröder-Bernstein problem has a positive solution for modules which are invariant under endomorphisms of their injective envelope.

The study of modules which are invariant under endomorphisms of their injective envelope goes back to the pioneering work of Johnson and Wong [15]. In order to prove that the Schröder-Bernstein problem has a positive solution for modules which are invariant under endomorphisms of their injective envelope, Bumby first showed that if $M$ and $N$ are two modules such that there is a monomorphism from $M$ to $N$ and a monomorphism from $N$ to $M$, then their injective envelopes are isomorphic, that is, $E(M) \cong E(N)$. As a consequence, he deduced that if $M$ and $N$ are two modules invariant under endomorphisms of their injective envelopes such that there is a monomorphism from $M$ to $N$ and a monomorphism from $N$ to $M$, then $M \cong N$.

Dickson and Fuller in [4] initiated the study of modules which are invariant under all automorphisms of their injective envelope. Inspired by this, modules invariant

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under endomorphisms or, in particular, automorphisms, of their general envelopes were recently introduced in [11]. The objective of this paper is to extend the result of Bumby for general envelopes and obtain Schröder-Bernstein type results for modules invariant under endomorphisms or automorphisms of their envelopes.

Let $X$ be a class of right $R$-modules closed under isomorphisms and direct summands. An $X$-preenvelope of a right module $M$ is a homomorphism $u : M \to X$ with $X \in X$ such that any other homomorphism $g : M \to X'$ with $X' \in X$ factors through $u$. A preenvelope $u : M \to X$ is called an $X$-envelope if it is minimal in the sense that any endomorphism $h : X \to X$ such that $h \circ u = u$ must be an automorphism. An $X$-(pre)envelope $u : M \to X$ is called monomorphic if $u$ is a monomorphism. A class $X$ of right modules over a ring $R$, closed under isomorphisms and direct summands, is called an enveloping class if any right $R$-module $M$ has an $X$-envelope. A module $M$ having a monomorphic $X$-envelope $u : M \to X(M)$ is said to be $X$-automorphism invariant (resp., $X$-endomorphism invariant) if for any automorphism (resp., endomorphism) $\varphi : X(M) \to X(M)$, there exists an endomorphism $f : M \to M$ such that $u \circ f = \varphi \circ u$. It may be observed that when $\varphi$ is an automorphism, then $f$ also turns out to be an automorphism (see [11]).

When $X$ is the class of all injective modules, $X$-automorphism invariant modules are usually just called automorphism-invariant modules and $X$-endomorphism invariant modules are called quasi-injective modules. When $X$ is the class of pure-injective modules, $X$-automorphism invariant modules are usually just called pure-automorphism-invariant modules and $X$-endomorphism invariant modules are called pure-quasi-injective modules.

Let $X$ be an enveloping class and $M, N$, two $X$-endomorphism invariant modules with monomorphic $X$-envelopes. Assume that $N$ is $X$-strongly purely closed and $M$ is an $X$-strongly pure submodule of $N$ (see Section 2 for the definition of these concepts). In this paper we show that if there exists an $X$-strongly pure monomorphism $u : N \to M$, then $M \cong N$. In particular, this shows that the Schröder-Bernstein property holds for modules invariant under endomorphisms of their injective or pure-injective envelopes or for flat modules invariant under endomorphisms of their cotorsion envelopes. In the last section of this paper, we extend this result further for the particular cases of injective envelopes and pure-injective envelopes. We show that a Schröder-Bernstein type result holds for modules that are invariant only under automorphisms of their injective envelopes or pure-injective envelopes.

Throughout this paper, all rings will be associative with a unit element, unless stated otherwise. By an $R$-module, we will always mean a unitary right module over a ring $R$. And we will denote by Mod-$R$, the category of right $R$-modules. We refer to [6 16 19] for any undefined notion used along this paper.

2. The Schröder-Bernstein problem for $X$-endomorphism invariant modules

Let $X$ be an enveloping class of right $R$-modules. We will assume along this paper that every right $R$-module $M$ has a monomorphic $X$-envelope that we are going to denote by $v_M : M \to X(M)$. 

Following the notation in [9, page 14], we are going to say that a homomorphism $u : N \to M$ of right $R$-modules is an $\mathcal{X}$-strongly pure monomorphism if any homomorphism $f : N \to X$, with $X \in \mathcal{X}$, extends to a homomorphism $g : M \to X$ such that $g \circ u = f$. Let us note that $\mathcal{X}$-strongly pure monomorphisms are clearly closed under composition. Moreover, if $X \in \mathcal{X}$, then any $\mathcal{X}$-strongly pure monomorphism $u : X \to M$ splits.

The following characterization of $\mathcal{X}$-strongly pure monomorphisms is straightforward and we state it without any proof.

**Lemma 2.1.** Let $u : N \to M$ be a homomorphism. Then the following are equivalent:

1. $u$ is an $\mathcal{X}$-strongly pure monomorphism.
2. $v_N : N \to X(N)$ factors through $u$.
3. The composition $v_M \circ u : N \to X(M)$ is an $\mathcal{X}$-preenvelope.

Observe that condition (2) implies that any $\mathcal{X}$-strongly pure monomorphism is a monomorphism, as we are assuming that every module has a monomorphic $\mathcal{X}$-envelope.

A submodule $N$ of $M$ will be called an $\mathcal{X}$-strongly pure submodule if the inclusion map $i : N \to M$ is an $\mathcal{X}$-strongly pure monomorphism.

Given a right $R$-module $M$, we will denote by $\text{add}[M]$ the class of all direct summands of finite direct sums of copies of $M$. And we will say that a module $M$ is $\mathcal{X}$-strongly purely closed if any direct limit of splitting monomorphisms among objects in $\text{add}[M]$ is an $\mathcal{X}$-strongly pure monomorphism.

**Example 2.2.** Let us give some examples of $\mathcal{X}$-strongly purely closed modules in which we will be interested in along this paper.

1. Let $\mathcal{X}$ be the class of all injective modules. Then any module is $\mathcal{X}$-strongly purely closed.
2. Let $\mathcal{X}$ be the class of all pure-injective modules. Then any module is $\mathcal{X}$-strongly purely closed.
3. Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair cogenerated by a set (see [14]) and assume that $\mathcal{F}$ is closed under taking direct limits. Then it is known that every module has a monomorphic $\mathcal{C}$-envelope (see e.g. [20]). It is easy to check that any object in $\mathcal{F} \cap \mathcal{C}$ is $\mathcal{C}$-strongly purely closed.

In the proposition below, we describe the endomorphism ring of $\mathcal{X}$-strongly purely closed modules. Recall that a module $M$ is called cotorsion if $\text{Ext}^1(F, M) = 0$ for every flat module $F$. It was shown in [8] that if $M$ is a flat cotorsion right $R$-module and $S = \text{End}(M_R)$, then $S/J(S)$ is a von Neumann regular right self-injective ring and idempotents lift modulo $J(S)$.

**Proposition 2.3.** Let $\mathcal{X}$ be a class of modules closed under isomorphisms and assume any module has a monomorphic $\mathcal{X}$-envelope. Then for any $\mathcal{X}$-strongly purely closed module $X$, $\text{End}(X)$ is a right cotorsion ring.

In particular, $\text{End}(X)/J(\text{End}(X))$ is von Neumann regular right self-injective and idempotents lift modulo $J(\text{End}(X))$.

**Proof.** Let us call $S = \text{End}(X)$. Take any short exact sequence $0 \to S_S \xrightarrow{\iota} L \to F \to 0$ with $F$, a flat right $S$-module. As $F$ is flat, the sequence is pure and thus,
the induced sequence $0 \to S_S \otimes X_R \to L \otimes X_R \to F \otimes X_R \to 0$ is also pure in \text{Mod-}R. On the other hand, we know that $F$ is a direct limit of a family of finitely generated projective modules. Say that $F = \lim_{\to} P_i$. Let us denote by $\delta_i : P_i \to F$ the canonical homomorphisms from $P_i$ to the direct limit. Taking pullbacks, we get the following commutative diagrams

\[
\begin{array}{ccccccccc}
0 & \to & S & \overset{u_i}{\to} & L_i & \overset{\varphi_i}{\to} & P_i & \to & 0 \\
0 & \to & S & \overset{u}{\to} & L & \overset{\delta_i}{\to} & F & \to & 0 \\
\end{array}
\]

in which the upper row splits, since $P_i$ is projective. Moreover, $L = \lim_{\to} L_i$. Applying now the functor $- \otimes S \otimes X$, we get the following commutative diagram in \text{Mod-}R.

\[
\begin{array}{ccccccccc}
0 & \to & S \otimes S X & \overset{u_i \otimes X}{\to} & L_i \otimes X & \overset{\varphi_i \otimes X}{\to} & P_i \otimes X & \to & 0 \\
0 & \to & S \otimes X & \overset{u \otimes X}{\to} & L \otimes X & \overset{\delta \otimes X}{\to} & F \otimes X & \to & 0 \\
\end{array}
\]

We have $L \otimes X = \lim_{\to} L_i \otimes X$ and $F \otimes X = \lim_{\to} P_i \otimes X$, since $- \otimes S X$ commutes with direct limits. Note that $S \otimes S X \cong X$ and $P_i \otimes X$ is isomorphic to a direct summand of a finite direct sum of copies of $X$. This shows that $u \otimes X$ is a direct limit of splitting monomorphisms among modules in \text{add}[X] and, as we are assuming that $X$ is an $\mathcal{X}$-strongly purely closed module, this means that $u \otimes X$ is an $\mathcal{X}$-strongly pure monomorphism. So there exists an $h : L \otimes S X \to S \otimes X$ such that $h \circ (u \otimes X) = 1_{S \otimes X}$. Applying now the functor $\text{Hom}(X, -)$, we get the following diagram in \text{Mod-}S,

\[
\begin{array}{cccccc}
S & \overset{u}{\to} & L \\
\sigma_S & \downarrow & \sigma_L \\
\text{Hom}(X, S \otimes X) & \to & \text{Hom}(X, L \otimes X) \\
\end{array}
\]

in which $\sigma_S$ is an isomorphism, $\sigma_L \circ u = \text{Hom}(X, u \otimes X) \circ \sigma_S$, $\text{Hom}(X, 1_S \otimes X) \circ \sigma_S = \sigma_S$ and $\text{Hom}(X, h) \circ \text{Hom}(X, u \otimes X) = \text{Hom}(X, 1_S \otimes X)$. Therefore, $\sigma_S^{-1} \circ \text{Hom}(X, h) \circ \sigma_L \circ u = 1_S$ and this shows that $u$ splits. Thus, the short exact sequence $0 \to S_S \to^u L \to F \to 0$ splits and hence $\text{End}(X)$ is a right cotorsion ring. Finally, by [8], $\text{End}(X) / J(\text{End}(X))$ is von Neumann regular right self-injective and idempotents lift modulo $J(\text{End}(X)).$ \hfill \qed

We are now ready to prove our first theorem.

**Theorem 2.4.** Let $X \in \mathcal{X}$ be an $\mathcal{X}$-strongly purely closed module and $Y \in \mathcal{X}$, an $\mathcal{X}$-strongly pure submodule of $X$. If there exists an $\mathcal{X}$-strongly pure monomorphism $u : X \to Y$, then $X \cong Y$.\hfill \qed
Proof. As $Y \in \mathcal{X}$, $Y$ must be a direct summand of $X$. Thus, we can find a submodule $H$ of $X$ such that $X = H \oplus Y$. Now

$$X = H \oplus Y \supseteq H \oplus u(X) = H \oplus u(H) \oplus u(Y) \supseteq \ldots$$

and thus, calling $P = \oplus_{i=0}^{\infty} u^i(H)$, we get that $X \supseteq P$. By construction, $P \cap Y = \oplus_{i=1}^{\infty} u^i(H) = u(P)$.

Let $v_{P \cap Y} : P \cap Y \to X(P \cap Y)$ be the $\mathcal{X}$-envelope of $P \cap Y$ and call $w : P \cap Y \to Y$, the inclusion. Note that $Y$ is an $\mathcal{X}$-strongly purely closed module, since it is a direct summand of $X$. And, as $w$ is a directed union of inclusions of direct summands of $Y$, it is an $\mathcal{X}$-strongly pure monomorphism. This means that there exists a $q : Y \to X(P \cap Y)$ such that $q \circ w = v_{P \cap Y}$, since $X(P \cap Y) \in \mathcal{X}$. Similarly, as $Y \in \mathcal{X}$ and $X(P \cap Y)$ is an $\mathcal{X}$-envelope there exists an $h : X(P \cap Y) \to Y$ such that $h \circ v_{P \cap Y} = w$.

In particular, $q \circ h \circ v_{P \cap Y} = v_{P \cap Y}$, as $v_{P \cap Y}$ is an envelope, we deduce that $q \circ h$ is an isomorphism. Therefore, $h$ is a splitting monomorphism and $Q = \text{Im}(h)$ is a direct summand of $Y$. So there exists a submodule $K$ such that $Y = Q \oplus K$.

Now, $X = H \oplus Y = H \oplus (Q \oplus K) = (H \oplus Q) \oplus K$. Thus, $H \oplus Q \in \mathcal{X}$. Moreover, the inclusion $i : P \to H \oplus Q$ may be viewed as $i = (1_H \oplus v_{P \cap Y}) : P = H \oplus (P \cap Y) \to H \oplus X(P \cap Y) \cong H \oplus Q$. So $i$ is an $\mathcal{X}$-strongly pure monomorphism. Now, as $Q \in \mathcal{X}$, we deduce that there exists a $\psi : H \oplus Q \to Q$ such that $\psi \circ i = h \circ v_{P \cap Y} \circ w$. Thus $h^{-1} \circ \psi \circ i = v_{P \cap Y} \circ w$. Note that $u : P \to P \cap Y$ and $h : X(P \cap Y) \to Q$ are isomorphisms. By the same way, as $H \oplus Q \in \mathcal{X}$, there exists a $\varphi : Q \to H \oplus Q$ such that $\varphi \circ h \circ v_{P \cap Y} \circ u = i$. This gives us $\varphi \circ \psi \circ i = i$ and $\psi \circ \varphi \circ h \circ v_{P \cap Y} = v_{P \cap Y}$. On the other hand, as $v_{P \cap Y} : P \cap Y \to X(P \cap Y)$ is an $\mathcal{X}$-envelope and $H \in \mathcal{X}$, we get that $i = (1_H \oplus v_{P \cap Y}) : P = H \oplus (P \cap Y) \to H \oplus Q$ is an $\mathcal{X}$-envelope. As both $i$ and $v_{P \cap Y}$ are envelopes, we deduce that $\varphi \circ \psi$ and $\psi \circ \varphi \circ h$ (and thus, $\psi \circ \varphi$) are automorphisms. Therefore, both $\varphi$ and $\psi$ are isomorphisms. Finally, $\psi \oplus 1_K : X = (H \oplus Q) \oplus K \to Q \oplus K = Y$ is the desired isomorphism. Thus, $X \cong Y$. \hfill \Box

Applying the above theorem to the particular cases of injective envelopes, pure-injective envelopes and cotorsion envelopes, we obtain the following.

Corollary 2.5. Let $E$ be a module.

(1) (Buny, [2]) If $E$ is an injective module and $E'$, an injective submodule of $E$ such that there exists a monomorphism $u : E \to E'$, then $E \cong E'$.

(2) If $E$ is a pure-injective module and $E'$, a pure-injective pure submodule of $E$ such that there exists a pure monomorphism $u : E \to E'$, then $E \cong E'$.

(3) If $E$ is a flat cotorsion module and $E'$, a pure submodule of $E$ such that $E'$ is also flat cotorsion and there exists a pure monomorphism $u : E \to E'$, then $E \cong E'$.

Proof. The above theorem applies to the cases of injective, pure-injective and flat cotorsion modules in view of Example 2.2. \hfill \Box

The following lemma will be used in our next theorem.

Lemma 2.6. A direct summand of an $\mathcal{X}$-endomorphism invariant module is also $\mathcal{X}$-endomorphism invariant.
Proof. Let $M$ be an $\mathcal{X}$-endomorphism invariant module and $N$, a direct summand of $M$. So there exists a module $K$ such that $M = N \oplus K$. Thus, $X(M) = X(N) \oplus X(K)$. Let $f : X(N) \to X(N)$ be an endomorphism of $X(N)$. So $\iota_{X(N)} \circ f \circ \pi_{X(N)}$ is an endomorphism of $X(M)$, where $\iota_{X(N)} : X(N) \to X(M)$ is the inclusion and $\pi_{X(N)} : X(M) \to X(N)$ is the canonical projection. We clearly have $\pi_N \circ \pi_M = \pi_{X(N)} \circ v_M$ and $v_M \circ \iota_N = \iota_{X(N)} \circ v_N$ with $\iota_N : N \to M$, the inclusion and $\pi_N : M \to N$, the canonical projection. As $M$ is $\mathcal{X}$-endomorphism invariant, there exists $h : M \to M$ such that $v_M \circ h = \iota_{X(N)} \circ f \circ \pi_{X(N)} \circ v_M$. We deduce that $g = \pi_N \circ h \circ \iota_N : N \to N$ is an endomorphism of $N$ such that $\pi_N \circ g = f \circ \pi_N$. So $N$ is an $\mathcal{X}$-endomorphism invariant module.

Our next theorem addresses the Schröder-Bernstein problem for modules invariant under endomorphisms of their general envelopes.

**Theorem 2.7.** Let $\mathcal{X}$ be an enveloping class and $M, N$, two $\mathcal{X}$-endomorphism invariant modules with monomorphic $\mathcal{X}$-envelopes $v_M : M \to X(M)$ and $v_N : N \to X(N)$, respectively. Assume that $N$ is $\mathcal{X}$-strongly purely closed and $M$ is an $\mathcal{X}$-strongly pure submodule of $N$. If there exists an $\mathcal{X}$-strongly pure monomorphism $u : N \to M$, then $M \cong N$.

Proof. Let $w'$ be an $\mathcal{X}$-strongly pure monomorphism from $M$ to $N$. As $v_M : M \to X(M)$ is an $\mathcal{X}$-envelope and $v_N \circ w' : M \to X(N)$ is an $\mathcal{X}$-preenvelope, there exists a split monomorphism $f_1 : X(M) \to X(N)$ such that $f_1 \circ v_M = v_N \circ w'$. Similar argument shows that there exists a split monomorphism $f_2 : X(N) \to X(M)$ such that $f_2 \circ v_N = v_M \circ u$. Since the composition $f_2 \circ f_1 : X(M) \to X(M)$ is also a split monomorphism, there exists an endomorphism $g : X(M) \to X(M)$ such that $g \circ (f_2 \circ f_1) = 1_{X(M)}$. Moreover, as $M$ is $\mathcal{X}$-endomorphism invariant, there exists a homomorphism $\delta : M \to M$ such that $v_M \circ \delta = g \circ v_M$. This gives us, $v_M \circ \delta \circ u \circ w' = g \circ v_M \circ u \circ w' = g \circ f_2 \circ v_N \circ w' = g \circ f_2 \circ f_1 \circ v_M = v_M$. Since $v_M$ is a monomorphism, $\delta \circ u \circ w'$ is an automorphism. Therefore, $w'$ is a splitting monomorphism and this yields that $M$ is a direct summand of $N$. Thus, we can find a submodule $H$ of $N$ such that $N = H \oplus M$. Now,

$$N = H \oplus M \supseteq H \oplus u(N) = H \oplus u(H) \oplus u(M) \supseteq \ldots \supseteq \bigoplus_{i=0}^{\infty} u^i(H) \oplus u^i(M) \supseteq \ldots$$

Call $P = \bigoplus_{i=0}^{\infty} u^i(H) = H \oplus (\bigoplus_{i=1}^{\infty} u^i(H)) = H \oplus (P \cap M) \subseteq N$. By construction, $u(P) = P \cap M$. Let $v_{P \cap M} : P \cap M \to X(P \cap M)$ be an $\mathcal{X}$-envelope of $P \cap M$ and $w : P \cap M \to M$ be the inclusion. As $w$ is a directed union of inclusions of direct summands of $M$, it is an $\mathcal{X}$-strongly pure monomorphism. As $v_{P \cap M}$ is an $\mathcal{X}$-envelope, there exists a homomorphism $h : X(P \cap M) \to X(M)$ such that $h \circ v_{P \cap M} = v_M \circ w$. And as $w$ is an $\mathcal{X}$-strongly pure monomorphism, there exists a homomorphism $p : X(M) \to X(P \cap M)$ such that $p \circ v_M \circ w = v_{P \cap M}$. In particular, $p \circ h \circ v_{P \cap M} = v_{P \cap M}$ and since $v_{P \cap M}$ is an $\mathcal{X}$-envelope, $p \circ h = 1_{X(P \cap M)}$. On the other hand, $h \circ p$ is an endomorphism of $X(M)$. As $M$ is $\mathcal{X}$-endomorphism invariant, $(h \circ p)(M) \subseteq M$. This means that, $h \circ p$(M) is a homomorphism from $p(M)$ to $M$.

Now we proceed to show that, $v_{p(M)} : p(M) \to X(P \cap M)$ is an $\mathcal{X}$-envelope and $p(M)$ is $\mathcal{X}$-endomorphism invariant. Let $X' \in \mathcal{X}$ and $f : p(M) \to X'$ be a homomorphism. As $v_M$ is an $\mathcal{X}$-envelope and $X' \in \mathcal{X}$, there exists a homomorphism $\alpha : X(M) \to X'$ such that $\alpha \circ v_M = f \circ p|_M$. Note that, $v_M \circ h|_{p(M)} = h \circ v_{p(M)}$ and $p \circ v_M = v_{p(M)} \circ p|_M$, by the definitions of the homomorphisms. Therefore,
we have $\alpha \circ h : X(P \cap M) \to X'$ with $(\alpha \circ h) \circ v_p(M) = f$. So we deduce that $v_p(M) : p(M) \to X(P \cap M)$ is an $X'$-preenvelope. Moreover, it can be shown that $v_p(M) : p(M) \to X(P \cap M)$ is indeed an $X$-envelope. Now, let $\phi : X(P \cap M) \to X(P \cap M)$ be an endomorphism. As $h \circ \phi \circ p$ is an endomorphism of $X(M)$ and $M$ is $X$-endomorphism invariant, $(h \circ \phi \circ p)(M) \subseteq M$. So we have $\phi(p(M)) \subseteq p(M)$. Thus, $p(M)$ is $X$-endomorphism invariant.

Furthermore, we have $v_p(M) = p \circ h \circ v_p(M) = p \circ v_M \circ h|_{p(M)} = v_p(M) \circ p|_M \circ h|_{p(M)}$ and, as $v_p(M)$ is a monomorphism, we get that $p|_M \circ h|_{p(M)} = 1_{p(M)}$. Therefore, $h|_{p(M)} : p(M) \to M$ is a splitting monomorphism and $Q = Im(h|_{p(M)}) = h \circ p(M)$ is a direct summand of $M$. So there exists a module $K$ such that $M = Q \oplus K$. Again, $N = H \oplus M = H \oplus (Q \oplus K) = (H \oplus Q) \oplus K$ and thus $H \oplus Q$ is an $X$-endomorphism invariant module.

Moreover, the inclusion $i : P \to H \oplus Q$ may be viewed as $i := (1_H \oplus (p|_M \circ w)) : P = H \oplus (P \cap M) \to H \oplus p(M) \cong H \oplus Q$. So $i$ is an $X'$-strongly pure monomorphism. As $H \oplus Q$ is $X$-endomorphism invariant, there exists a $\psi : H \oplus Q \to Q$ such that $\psi \circ i = h|_{p(M)} \circ p|_M \circ w \circ u|_P$, where $h|_{p(M)} : p(M) \to Q$ and $u|_P : P \to P \cap M$ are isomorphisms. On the other hand, as $i$ is an $X'$-strongly pure monomorphism and $H \oplus Q$ is an $X$-endomorphism invariant module, we get that $i : P = H \oplus (P \cap M) \to H \oplus Q$ is an $X$-envelope. Similarly, there exists a homomorphism $\varphi : Q \to H \oplus Q$ such that $\varphi \circ h|_{p(M)} \circ p|_M \circ w \circ u|_P = i$. This means that $\varphi \circ \psi \circ i = i$ and $h^{-1}|_{p(M)} \circ \psi \circ \phi \circ h|_{p(M)} \circ v_{P \cap M} = v_{P \cap M}$. And, as both $i$ and $v_{P \cap M}$ are envelopes, we deduce that $\varphi \circ \psi$ and $h^{-1}|_{p(M)} \circ \psi \circ \phi \circ h|_{p(M)}$ are automorphisms. Thus, it follows that $\psi \circ \phi$ is also an automorphism. Therefore, both $\phi$ and $\psi$ are isomorphisms. Finally, $\psi \oplus 1_K : N = (H \oplus Q) \oplus K \to Q \oplus K = M$ is the desired isomorphism. This completes the proof. \qed

Applying the above theorem to the particular cases of injective envelopes, pure-injective envelopes and cotorsion envelopes, we get the following.

**Corollary 2.8.** Let $M$ and $N$ be two modules.

1. (Bumby, [2]) If $M$ and $N$ are quasi-injective modules such that there is a monomorphism from $M$ to $N$ and a monomorphism from $N$ to $M$, then $M \cong N$.

2. If $M$ and $N$ are pure-quasi-injective modules such that there is a pure monomorphism from $M$ to $N$ and a pure monomorphism from $N$ to $M$, then $M \cong N$.

3. If $M$ and $N$ are flat modules invariant under endomorphisms of their cotorsion envelopes such that there is a pure monomorphism from $M$ to $N$ and a pure monomorphism from $N$ to $M$, then $M \cong N$.

3. **Schröder-Bernstein problem for Automorphism-invariant modules**

Although we do not know if the results from previous section can be extended to modules invariant under automorphisms of their envelopes in the general case, we will study this question for the particular case of injective and pure-injective envelopes in this section. Recently, it has been shown in [1] that if $M$ and $N$ are automorphism-invariant modules of finite Goldie dimension such that there is a monomorphism from $M$ to $N$ and a monomorphism from $N$ to $M$, then $M \cong N$. We
will extend this result and show that the Schröder-Bernstein problem has a positive solution for any automorphism-invariant module.

We will denote the injective envelope of a module \( M \) by \( E(M) \) and \( A \subseteq_e B \) will mean that \( A \) is an essential submodule of \( B \). We can now prove the main result of this section.

**Theorem 3.1.** Let \( M, N \) be automorphism-invariant modules and let \( f : M \to N \) and \( g : N \to M \) be monomorphisms. Then \( M \cong N \).

**Proof.** By Corollary \[\text{[2,5]}\] we know that \( E(M) \cong E(N) \). On the other hand, we have a diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & f(M) & \xrightarrow{u} & N & \xrightarrow{g} & M \\
\downarrow{1_M} & & \downarrow{1_{f(M)}} & & \downarrow{1_N} & & \downarrow{1_M} \\
M & & & & & & \\
\end{array}
\]

in which \( u : f(M) \to N \) is the inclusion. As \( M \) is automorphism-invariant and \( g \circ u \circ f \) is monic, there exists a \( \varphi : M \to M \) such that \( \varphi \circ g \circ u \circ f = 1_M \) (see \[\text{[3]}\] and \[\text{[12]}\]).

And, as \( f : M \to f(M) \) is an isomorphism, this means that \( u : f(M) \to N \) splits and thus, \( f(M) \) is a direct summand of \( N \). Similarly, \( g(N) \) is a direct summand of \( M \).

As \( f : M \to f(M) \) and \( g : N \to g(N) \) are isomorphisms, we know that \( E(f(M)) \cong E(g(N)) \). We proceed to show that \( f(M) \cong g(N) \). Let \( h : E(g(N)) \to E(f(M)) \) be an isomorphism. Call \( M' = h^{-1}(f(M)) \cap g(N) \) and \( N' = h(g(N)) \cap f(M) \). By construction \( h|_{M'} : M' \to N' \) is an isomorphism. Moreover, as \( g(N) \subseteq_e E(g(N)) \), we have that \( h(g(N)) \subseteq_e E(f(M)) \). Similarly, \( h^{-1}(f(M)) \subseteq_e E(g(N)) \).

Therefore, \( M' \subseteq_e E(g(N)) \) and \( N' \subseteq_e E(f(M)) \). In particular, \( M' \subseteq_e g(N) \) and \( N' \subseteq_e f(M) \). We have then

\[
\begin{array}{ccc}
M' & \xrightarrow{h|_{M'}} & N' & \xrightarrow{u_{N'}} & f(M) \\
\downarrow{u_{M'}} & & \downarrow{g(N)} & & \\
M' & \xrightarrow{h|_{M'}} & N' & \xrightarrow{u_{N'}} & f(M) \\
\end{array}
\]

where \( u_{M'} \) and \( u_{N'} \) are inclusions. Moreover, \( g(N) \) is a submodule of \( M \) and \( f(M) \) is isomorphic to \( M \). Therefore, \( f(M) \) is automorphism-invariant and as, \( u_{N'} \circ h|_{M'} \) is monic, there exists a \( \psi : g(N) \to f(M) \) such that \( \psi \circ u_{M'} = u_{N'} \circ h|_{M'} \). Similarly, there exists a \( \varphi : f(M) \to g(N) \) such that \( \varphi \circ u_{N'} = u_{M'} \circ h^{-1}|_{N'} \). Composing, we get the diagram

\[
\begin{array}{ccc}
M' & \xrightarrow{h|_{M'}} & N' & \xrightarrow{h^{-1}|_{N'}} & M' \\
\downarrow{u_{M'}} & & \downarrow{u_{N'}} & & \downarrow{u_{M'}} \\
g(N) & \xrightarrow{\psi} & f(M) & \xrightarrow{\varphi} & g(N) \\
\end{array}
\]
Remark 3.3. The above results suggest that it might be possible to extend Theorem 3.1 in last section to $X$-automorphism invariant modules for which the endomorphism ring of their $X$-envelope is right cotorsion; for instance, to flat modules which are invariant under automorphisms of their cotorsion envelopes. However, our techniques do not seem to work in this more general setting.

Proof. Let us finish this paper by extending the above result to modules which are invariant under automorphisms of their pure-injective envelope. For that, recall that there exists a full embedding $H : \text{Mod}–R \rightarrow \mathcal{D}$ of $\text{Mod}–R$ into a locally finitely presented Grothendieck category $\mathcal{D}$ (normally called the functor category of $\text{Mod}–R$) satisfying the following key properties (see e.g. [18]):

- $H$ has a right adjoint functor $G : \mathcal{D} \rightarrow \text{Mod}–R$.
- An exact sequence
  $$\Sigma \equiv 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$
  in $\text{Mod}–R$ is pure if and only if the induced sequence $H(\Sigma)$ is exact (and pure) in $\mathcal{D}$.
- $H$ identifies $\text{Mod}–R$ with the full subcategory of $\mathcal{D}$ consisting of the all FP-injective objects in $\mathcal{D}$ where an object $D \in \mathcal{D}$ is FP-injective if $\text{Ext}^1(D', D) = 0$ for every finitely presented object $D' \in \mathcal{D}$.
- A module $M \in \text{Mod}–R$ is pure-injective if and only if $H(M)$ is an injective object of $\mathcal{D}$. And $u : M \rightarrow PE(M)$ is the pure-injective envelope of $M$ if and only if $H(u) : H(M) \rightarrow H(PE(M))$ is the injective envelope of $H(M)$ in $\mathcal{D}$.

On the other hand, the locally finitely presented Grothendieck category $\mathcal{D}$ is equivalent to the category of unitary right $S$-modules for a ring with enough idempotents $S$ (see e.g. [19, 52.5(2)]). Recall that a non-unital ring $R$ is said to have enough idempotents if there exists a set of orthogonal idempotents $\{e_i\}_{i \in I}$ in the ring such that $R = \oplus_{i \in I} e_i R = \oplus_{i \in I} e_i$ and a right $R$-module $M$ is called unitary if $MR = M$. We refer to [19, Section 49] for the categorical properties of these unitary modules.

It is easy to check that all the proofs in this paper work for unitary right modules over a ring with enough idempotents. Therefore, identifying $\mathcal{D}$ with $\text{Mod}–S$, we may apply Theorem 3.1 to $\text{Mod}–S$ to get:

**Corollary 3.2.** Let $M, N$ be two modules invariant under automorphisms of their pure-injective envelopes let $f : M \rightarrow N$ and $g : N \rightarrow M$ be pure monomorphisms. Then $M \cong N$.

Proof. In this case, $H(M), H(N)$ are automorphism-invariant objects in $\mathcal{D}$ and $H(f) : H(M) \rightarrow H(N)$ and $H(g) : H(N) \rightarrow H(M)$ are monomorphisms. So $H(M) \cong H(N)$ by Theorem 3.1. And this means that $M \cong G \circ H(M)$ is isomorphic to $N \cong G \circ H(N)$.

Remark 3.3. The above results suggest that it might be possible to extend Theorem 3.1 in last section to $X$-automorphism invariant modules for which the endomorphism ring of their $X$-envelope is right cotorsion; for instance, to flat modules which are invariant under automorphisms of their cotorsion envelopes. However, our techniques do not seem to work in this more general setting.
In regard to this possible extension, our next example shows that we cannot expect to deduce this kind of result from Theorem 3.1, Corollary 3.2 or results in Section 2. Our example shows that there exist flat modules which are invariant under automorphisms of their cotorsion envelopes but they are not invariant under endomorphisms of their cotorsion envelopes, nor under automorphisms of their injective or pure-injective envelopes and therefore, our results cannot be applied to these modules.

Example 3.4. Let $K$ be a field of characteristic zero and $S$, the $K$-algebra constructed in [21, Section 2]. Then $S$ is a right artinian ring which is not right pure-injective. As $S_S$ is artinian, any right $S$-module is cotorsion and thus, is invariant under automorphisms of its cotorsion envelope. Assume that any direct sum of copies of $S_S$ is invariant under automorphisms of its pure-injective envelope. As char($K$) = 0, this means that any direct sum of copies of $S_S$ is also invariant under endomorphisms of its pure-injective envelope (see [11]) and thus, $H(S_S)$ is $\Sigma$-quasi-injective in the functor category $\mathcal{D}$. But then, $E(H(S_S))$ is $\Sigma$-injective (see [6]) and this means that the pure-injective envelope of $S_S$ is $\Sigma$-pure-injective. Therefore, $S_S$ is also $\Sigma$-pure injective as it is a pure submodule of its pure-injective envelope, a contradiction. Thus we conclude that there exists an index set $I$ such that $S_S^{(I)}$ is not invariant under automorphisms of its pure-injective envelope. Call $M_S = S_S^{(I)}$.

Let now $R$ be the ring of all eventually constant sequences over $F_2$, the field of two elements. It is known that $R$ is a von Neumann regular ring, and $R_R$ is an automorphism-invariant module which is not quasi-injective (see [11]). Therefore, it cannot be invariant under endomorphisms of its cotorsion envelope, nor of its pure-injective envelope, either.

Let us consider the ring $R \times S$ and the right $R \times S$-module $R \times M$. Then:

1. $R \times M$ is flat and it is invariant under automorphisms of its cotorsion envelope, since so are $R_R$ and $M_R$.
2. $R \times M$ is not invariant under endomorphisms of its cotorsion envelope, since otherwise so would be $R_R$.
3. $R \times M$ is not invariant under automorphisms of its pure-injective envelope, since otherwise so would be $M_R$.
4. $R \times M$ is not invariant under automorphisms of its injective envelope, since otherwise it would be quasi-injective, as $S$ is an algebra over a field of characteristic zero. And this would mean that $S_S$ would be injective, since it is a direct summand of $M_S$.

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