AN $L^p$-VERSION OF VON NEUMANN DIMENSION FOR BANACH SPACE REPRESENTATIONS OF SOFIC GROUPS

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Abstract. In [7], A. Gournay defined a notion of $L^p$-dimension for $\Gamma$-invariant subspaces of $L^q(\Gamma^{\otimes n})$, with $\Gamma$ amenable. The number $\dim_{L^q} L^p(\Gamma, V)$ is $\dim V$ when $p = q$, and is preserved by a certain class of $\Gamma$-equivariant bounded linear isomorphisms. We develop a notion of $\dim_{L^p, \Sigma}(Y, \Gamma)$ where $Y$ is a Banach space with a uniformly bounded action of a sofic group $\Gamma$ and $\Sigma$ is a sofic approximation. In particular, our definition makes sense for a large class of non-amenable groups. We also develop a notion of $\dim_{S^p, \Sigma}(Y, \Gamma)$ with $\Gamma$ a $\mathcal{R}^\omega$-embeddable group and $S^p$ the space of finite dimensional Schatten $p$-class operators. These numbers are invariant under bounded $\Gamma$-equivariant linear isomorphisms and under the natural translation action of $\Gamma$, $\dim_{L^p}(L^p(\Gamma, V), \Gamma)) = \dim V$ for $1 \leq p \leq 2$. In particular, this shows that $L^p(\Gamma, V)$ is not isomorphic to $L^p(\Gamma, W)$ as a representation of $\Gamma$ if $\dim V \neq \dim W$, and $\Gamma$ is $\mathcal{R}^\omega$-embeddable. In the case of representations which are contained in a multiple of the left regular representation, we show that our dimension agrees with the Murray-von Neumann dimension. Dimensions for certain actions of $\Gamma$ on non-commutative $L^p$-spaces are also computed.

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1. INTRODUCTION

Let $\Gamma$ be a countable discrete group. Suppose that $H$ is a closed $\Gamma$-invariant subspace of $L^2(\Gamma \times N)$, and let $P_H$ be the projection onto $H$, then it is known that the number

$$\dim_{L^1(\Gamma)}(H) = \sum_{n \in N} \langle P_H \delta_{(e,n)}, \delta_{(e,n)} \rangle$$

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obeys the usual properties of dimension,

Property 1: \( \dim_{L(\Gamma)}(H) = \dim_{L(\Gamma)}(K) \), if there is a \( \Gamma \)-equivariant bounded linear bijection from \( H \) to \( K \)

Property 2: \( \dim_{L(\Gamma)}(H \oplus K) = \dim_{L(\Gamma)}(H) + \dim_{L(\Gamma)}(K) \).

Property 3: \( \dim_{L(\Gamma)}(H) = 0 \) if and only if \( H = 0 \),

Property 4: \( \dim_{L(\Gamma)}(\bigcap_{n=1}^{\infty} H_n) = \lim_{n \to \infty} \dim_{L(\Gamma)}(H_n) \), if \( H_{n+1} \subseteq H_n \),

Property 5: \( \dim_{L(\Gamma)}(\bigcup_{n=1}^{\infty} H_n) = \lim_{n \to \infty} \dim_{L(\Gamma)}(H_n) \) if \( H_n \subseteq H_{n+1} \).

We also have

\[
\dim_{L(\Gamma)}(l^2(\Gamma)^{\otimes n}) = n.
\]

Voiculescu in [17] and Gournay in [7] noticed that for amenable groups \( \Gamma \), we can define this dimension as a limit of normalized approximate dimensions of \( F_n \Omega \), with \( F_n \) a Følner sequence, and \( \Omega \subseteq H \). This formula is analogous to the definition of entropy for actions of an amenable group on a topological space or measure space. Gournay noticed that his expression for \( l^2 \)-dimensions makes sense for subspaces of \( l^p(\Gamma, V) \), and defined a similar isomorphism invariant for subspaces of \( l^p(\Gamma, V) \). In particular, Gournay shows that if there is an injective \( \Gamma \)-equivariant linear map of finite type (see [7] for the definition) with closed image from \( l^p(\Gamma, V) \rightarrow l^p(\Gamma, W) \) then \( \dim V \leq \dim W \).

Recently, in [3,9] a theory of entropy for actions of a sofic group on a probability space or topological space has been developed. Using this theory, it was shown for sofic groups \( \Gamma \) that probability measure preserving Bernoulli actions \( \Gamma \acts (X, \mu)^\Gamma \) are not isomorphic if the entropy of \( (X, \mu) \) does not equal the entropy of \( (Y, \nu) \), and that Bernoulli actions are not isomorphic as topological actions if \( |X| \neq |Y| \). We can think of the action of \( \Gamma \) on \( l^p(\Gamma, V) \) as analogous to a Bernoulli action, since both actions are given by translating functions on the group. Combining ideas in Kerr and Li [9] and Voiculescu in [17], we define an isomorphism invariant

\[
\dim_{\Sigma, l^p}(Y, \Gamma)
\]

for a uniformly bounded action of a sofic group on a separable Banach space \( Y \) (all our Banach spaces will be complex, unless explicitly mentioned otherwise).

This definition of dimension has the following properties:

Property 1: \( \dim_{\Sigma, l^p}(Y, \Gamma) \leq \dim_{\Sigma, l^p}(X, \Gamma) \) if there is a equivariant bounded linear map \( X \rightarrow Y \) with dense image.

Property 2: \( \dim_{\Sigma, l^p}(V, \Gamma) \leq \dim_{\Sigma, l^p}(W, \Gamma) + \dim_{\Sigma, l^p}(V/W, \Gamma) \), if \( W \subseteq V \) is a closed \( \Gamma \)-invariant subspace.

Property 3: \( \dim_{\Sigma, l^2}(H, \Gamma) = \dim_{L(\Gamma)}(H) \) if \( H \subseteq l^2(\Gamma \times \mathbb{N}) \) is a closed \( \Gamma \)-invariant subspace.

Property 4: \( \dim_{\Sigma, l^p}(Y \oplus W, \Gamma) \geq \dim_{\Sigma, l^p}(Y, \Gamma) + \dim_{\Sigma, l^p}(W, \Gamma) \) for \( 2 \leq p < \infty \),

where \( \dim \) is a “lower dimension,” and is also an invariant, further

Property 5: \( \dim_{\Sigma, l^p}(l^p(\Gamma, V), \Gamma) = \dim_{\Sigma, l^p}(l^p(\Gamma, V)) = \dim(V) \) for \( 1 \leq p \leq 2 \).

Property 6: \( \dim_{\Sigma, l^p}(X, \Gamma) = 0 \), if \( \Gamma \) either contains an infinite amenable subgroup

or \( \{ |\Lambda| : \Lambda \subseteq \Gamma \text{ is a finite subgroup} \} \) is unbounded.
We also note that for defining $\dim_{p}(Y, \Gamma)$, little about soficity of $\Gamma$ is used, and we can more generally define our invariants associated to a sequence of maps $\sigma_{i}: \Gamma \to \text{Isom}(V_{i})$ where $V_{i}$ are finite-dimensional Banach spaces.

In particular, we can show that $\dim_{\Sigma, p}(Y, \Gamma)$ can be defined for $\mathcal{R}^{\omega}$-embeddable groups $\Gamma$. Because unitaries also act isometrically on the space of Schatten $p$-class operators, we can also define an invariant

$$\dim_{\Sigma, S^{p}}(Y, \Gamma),$$

$S^{p}$ dimension has properties analogous to $l^{p}$ dimension.

Property 1: $\dim_{\Sigma, S^{p}}(Y, \Gamma) \leq \dim_{\Sigma, S^{p}}(X, \Gamma)$ if there is a $\Gamma$-equivariant bounded linear bijection $X \to Y$.

Property 2: $\dim_{\Sigma, S^{p}}(V, \Gamma) \leq \dim_{\Sigma, S^{p}}(W, \Gamma) + \dim_{\Sigma, S^{p}}(V/W, \Gamma)$, if $W \subseteq V$ is a closed $\Gamma$-invariant subspace.

Property 3: $\dim_{\Sigma, p}(H, \Gamma) = \dim_{L(H)}(H)$ if $H \subseteq l^{2}(\Gamma \times \mathbb{N})$ is a closed $\Gamma$-invariant subspace.

Property 4: $\dim_{\Sigma, S^{p}}(Y \oplus W, \Gamma) \geq \dim_{\Sigma, S^{p}}(Y, \Gamma) + \dim_{\Sigma, S^{p}}(W, \Gamma)$ for $2 \leq p < \infty$.

Property 5: $\dim_{\Sigma, S^{p}}(l^{p}(\Gamma, V), \Gamma) = \dim_{V}$ for $1 \leq p \leq 2$.

Property 6: $\dim_{\Sigma, S^{p}}(W, \Gamma) > 0$ if $W \subseteq l^{p}(\Gamma)_{\text{fin}}$ is a nonzero closed invariant and $1 < p \leq 2$, and $\frac{1}{p} + \frac{1}{p'} = 1$. (see Corollary 7.8)

In particular $l^{p}(\Gamma, V)$ is not isomorphic to $l^{p}(\Gamma, W)$ as a representation of $\Gamma$, if $\Gamma$ is $\mathcal{R}^{\omega}$-embeddable and $1 \leq p < \infty$. This extends a result of [7] from amenable groups to $\mathcal{R}^{\omega}$-embeddable groups, and answers a question of Gromov (see [8] page 353) in the case of $\mathcal{R}^{\omega}$-embeddable groups.

We compute other examples as well. In Section 6, we prove that our notion of $l^{2}$-dimension coincides with Murray-von Neumann dimension for representations contained in the infinite direct sum of the left regular representation. We also use our $l^{p}$-dimension to define $l^{p}$-Betti numbers for cocompact actions on CW complexes, and compute $l^{p}$-Betti numbers of free groups.

2. Definition of the Invariants

We recall the definition of sofic and $\mathcal{R}^{\omega}$-embeddable groups (see [13], [3]). For this it is useful to introduce metrics on the symmetric and unitary groups. For $\sigma, \tau \in S_{n}$, we define the Hamming distance by

$$d_{\text{Hamm}}(\sigma, \tau) = \frac{1}{n} |\{ j : \sigma(j) \neq \tau(j)\}|.$$ 

If $A, B \in M_{n}(\mathbb{C})$ we let

$$\langle A, B \rangle = \frac{1}{n} \text{Tr}(B^{*}A),$$

note that $\langle A, B \rangle$ is indeed an inner product. We let $\| . \|_{2}$ denote the Hilbert space norm induced by this inner product. To fix notation we use $\text{Sym}(A)$ for the group of bijections of the set $A$, and we let $S_{n} = \text{Sym}\{1, \cdots, n\}$, finally we let $U(n)$ denote the unitary group of $\mathbb{C}^{n}$, where $\mathbb{C}^{n}$ has the usual inner product.

**Definition 2.1.** Let $\Gamma$ be a countable group. A sofic approximation for $\Gamma$ is a sequence of maps $\sigma_{i}: \Gamma \to S_{d_{i}}$ with $d_{i} \to \infty$, (not assumed to be homomorphisms) which is approximately multiplicatively and approximately free in the sense that

$$d_{\text{Hamm}}(\sigma_{i}(st), \sigma_{i}(s)\sigma_{i}(t)) \to 0, \text{ for all } s, t \in \Gamma$$
We say that $\Gamma$ is sofic if it has a sofic approximation.

One can think of a sofic approximation $\sigma_i$ as above as maps so that if

$$x_1, \ldots, x_n, y_1, \ldots, y_m \in \Gamma,$$

and $a_1, \ldots, a_n, b_1, \ldots, b_m \in \{-1, 1\}$, then with high probability,

$$\sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(j) = \sigma_i(y_1)^{a_1} \cdots \sigma_i(y_n)^{a_n}(j)$$

if $x_1^{a_1} \cdots x_n^{a_n} = y_1^{a_1} \cdots y_n^{a_n}$,

$$\sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(j) \neq \sigma_i(y_1)^{a_1} \cdots \sigma_i(y_n)^{a_n}(j)$$

if $x_1^{a_1} \cdots x_n^{a_n} \neq y_1^{a_1} \cdots y_n^{a_n}$.

The requirement $d_i \to \infty$ is not necessary since one can replace $\sigma_i$ with $\sigma_i^{\otimes k_i}$ where $\sigma_i^{\otimes k_i}: \Gamma \to \text{Sym}(\{1, \ldots, d_i\}^\otimes k_i)$ is given by

$$\sigma_i^{\otimes k_i}(s)(a_1, \ldots, a_k) = (\sigma_i(s)(a_1), \ldots, \sigma_i(s)(a_k)).$$

We require that $d_i \to \infty$ simply for our properties of $l^p$-dimension to behave appropriately. Note that $d_i \to \infty$ is automatic when the group is infinite by our approximate freeness assumption.

A related notion is that of being $\mathcal{R}^\omega$-embeddable.

**Definition 2.2.** Let $\Gamma$ be a countable group. A *embedding sequence* for $\Gamma$ is a sequence of maps $\sigma_i: \Gamma \to U(d_i)$, with $d_i \to \infty$, (not assumed to be homomorphisms) such that

$$\|\sigma_i(st) - \sigma_i(s)\sigma_i(t)\|_2 \to 0 \text{ for all } s, t \in \Gamma$$

$$\frac{1}{d_i} \text{Tr}(\sigma_i(s)^\ast \sigma_i(s)) \to 0 \text{ for all } s \neq s' \text{ in } \Gamma.$$

A group is said to be $\mathcal{R}^\omega$-embeddable if it has a embedding sequence.

The second condition says that if $s \neq s'$, then asymptotically $\sigma_i(s), \sigma_i(s')$ become orthogonal under the inner product which induces $\|\cdot\|_2$. One can formulate a probabilistic interpretation of an embedding sequence analogous to that of a sofic approximation: for any $\varepsilon > 0$, if $x_1, \ldots, x_n, y_1, \ldots, y_m \in \Gamma$, and $a_1, \ldots, a_n, b_1, \ldots, b_m \in \{-1, 1\}$, then if $x_1^{a_1} \cdots x_n^{a_n} = y_1^{b_1} \cdots y_n^{b_n}$,

$$\mathbb{P}\{\xi \in S^{2d_i-1}: \|\sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(\xi) - \sigma_i(y_1)^{b_1} \cdots \sigma_i(y_n)^{b_n}(\xi)\| < \varepsilon\} \to 1,$$

and if $x_1^{a_1} \cdots x_n^{a_n} \neq y_1^{b_1} \cdots y_n^{b_n}$,

$$\mathbb{P}\{\xi \in S^{2d_i-1}: |\langle \sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(\xi), \sigma_i(y_1)^{b_1} \cdots \sigma_i(y_n)^{b_n}(\xi)\| < \varepsilon\} \to 1.$$

This equivalence follows by concentration of measure.

Note that if $\sigma \in S_n$ and $U_\sigma$ is the unitary on $\mathbb{C}^n$ which $\sigma$ induces, we have that

$$d_{\text{Hamm}}(\sigma, \tau) = d_{\text{Hamm}}(\tau^{-1}\sigma, \text{Id}) = 1 - \frac{1}{n} \text{Tr}(U_{\tau^{-1}\sigma}) = 1 - \frac{1}{n} \text{Tr}(U_\tau^* U_\sigma),$$

$$\|U_\sigma - U_\tau\|_2^2 = 2 - 2(1 - d_{\text{Hamm}}(\tau^{-1}\sigma, \text{Id})) = d_{\text{Hamm}}(\sigma, \tau)$$

thus all sofic groups are $\mathcal{R}^\omega$-embeddable.

We will sometimes use an alternate definition of $\mathcal{R}^\omega$-embeddable: a group is $\mathcal{R}^\omega$-embeddable if its group von Neumann algebra embeds into an ultraproduct of matrix algebras. For this alternate condition (and others), and a good introduction to sofic and $\mathcal{R}^\omega$-embeddable groups refer to [13].

We now give examples of sofic groups, and thus $\mathcal{R}^\omega$-embeddable groups, see ([13] for proofs, although most of these can be shown directly).
Example 1. All countable amenable groups are sofic. For instance, if $F_n$ is a Følner sequence for $\Gamma$, we have a map $\sigma_i : \Gamma \to \text{Sym}(F_i)$ given by

$$\sigma_i(s)(x) = \begin{cases} sx & \text{if } sx \in F_n \\ x & \text{otherwise} \end{cases}.$$  

It follows directly from the definition of a Følner sequence that $\sigma_i$ is a sofic approximation.

Example 2. All countable residually sofic groups are sofic. In particular, this includes all free groups and residually amenable groups.

Example 3. Countable locally sofic groups are sofic.

Example 4. By Malcev’s Theorem (see [2] Theorem 6.4.3) all finitely generated linear groups are residually finite, hence sofic. By the preceding example all countable linear groups are sofic.

It is shown in [5] that sofic groups are closed under direct products, taking subgroups, inverse limits, direct limits, free products, and extensions by amenable groups: if $\Lambda \vartriangleleft \Gamma$, if $\Lambda$ is sofic, and $\Gamma/\Lambda$ is amenable, then $\Gamma$ is sofic. It is also known that $R^\omega$-embeddable groups are closed under these operations as well. It is unknown whether all countable groups are sofic. As mentioned earlier, a group is $R^\omega$-embeddable if and only if its group von Neumann algebra embeds into an ultrapower of the hyperfinite $\mathcal{II}_1$ factor. It follows that if the Connes Embedding Conjecture is true, then all countable discrete groups are $R^\omega$-embeddable. Even without the Connes Embedding conjecture we still have many examples of $R^\omega$-embeddable groups.

Definition 2.3. Let $X$ be a Banach space. An action $\Gamma$ on $X$ by is said to be uniformly bounded if there is a constant $C > 0$ such that

$$\|sx\| \leq C\|x\|$$

for all $x \in X$.

We say that a sequence $S = (x_j)_{j=1}^\infty$ in $X$ is dynamically generating, if $S$ is bounded and $\text{Span}\{sx_j : s \in \Gamma, j \in \mathbb{N}\}$ is dense.

If $X$ is a Banach space we shall write $\text{Isom}(X)$ for the group of all linear isometries from $X$ to itself.

Definition 2.4. Let $V$ be a vector space with a pseudonorm $\rho$. If $A \subseteq V$, a linear subspace $W \subseteq V$ is said to $\varepsilon$-contain $A$, denoted $A \subseteq_\varepsilon W$, if for every $v \in A$, there is a $w \in W$ such that $\rho(v - w) < \varepsilon$. We let $d_\varepsilon(A, \rho)$ be the minimal dimension of a subspace which $\varepsilon$-contains $A$.

Definition 2.5. A dimension triple is a triple $(X, \Gamma, \Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i)))$, where $X$ is a separable Banach space, $\Gamma$ is a countable discrete group with a uniformly bounded action on $X$, each $V_i$ is finite-dimensional, and the $\sigma_i$ are functions with no structure assumed on them.

Definition 2.6. Let $(X, \Gamma, \Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i)))$ be a dimension triple. Fix $S = (x_j)_{j=1}^\infty$ a dynamically generating sequence in $X$.

For $e \in E \subseteq \Gamma$ finite, $l \in \mathbb{N}$ let

$$X_{E,l} = \text{Span}\{sx_j : s \in E^l, 1 \leq j \leq l\}.$$
If \( e \in F \subseteq \Gamma \) finite, \( m \in \mathbb{N} \), \( C, \delta, M > 0 \), let \( \text{Hom}_\Gamma(S, F, m, \delta, \sigma_i)_C \) be the set of all linear maps \( T : X_{F,m} \rightarrow V_i \) such that \( \|T\| \leq C \) and
\[
\|T(s_1 \cdots s_k x_j) - \sigma_i(s_1) \cdots \sigma_i(s_k)T(x_j)\| < \delta
\]
if \( 1 \leq j, k \leq m, s_1, \ldots, s_k \in F \). If \( C = 1 \) we shall use \( \text{Hom}_\Gamma(S, F, m, \delta, \sigma_i) \) instead of \( \text{Hom}_\Gamma(S, F, m, \delta, \sigma_i)_1 \).

We shall frequently deal with inducing pseudonorms on \( l^\infty(\mathbb{N}, V) \) from pseudonorms on \( l^\infty(\mathbb{N}) \). For this, we use the following notation: if \( \rho \) is a pseudonorm on \( l^\infty(\mathbb{N}) \) and \( V \) is a Banach space, we let \( \rho_V \) be the pseudonorm on \( l^\infty(\mathbb{N}, V) \) defined by \( \rho_V(f) = \rho(j \mapsto \|f(j)\|) \).

**Definition 2.7.** Let \( \Sigma, S \) be as in the proceeding definition and let \( \rho \) be a pseudonorm on \( l^\infty(\mathbb{N}) \). Let \( \alpha_S : B(X_{F,m}, V_i) \rightarrow l^\infty(\mathbb{N}, V_i) \) be given by \( \alpha_S(T)(j) = \chi_{\{k \leq m\}}(j)T(x_j) \). We let
\[
\hat{d}_\Sigma(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho) = d_{e}(\alpha_S(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i)), \rho_V)
\]
define the *dimension of \( S \) with respect to \( \rho \)* by
\[
f.\dim_{\Sigma}(S, F, m, \delta, \varepsilon, \rho) = \limsup_{i \rightarrow \infty} \frac{1}{\dim V_i} \hat{d}_\Sigma(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho),
\]
where the pairs \( (F, m, \delta) \) are ordered as follows \( (F, m, \delta) \leq (F', m', \delta') \) if \( F \subseteq F' \), \( m \leq m' \), \( \delta \geq \delta' \).

We also use
\[
f.\dim_{\Sigma}(S, F, m, \delta, \varepsilon, \rho) = \liminf_{i \rightarrow \infty} \frac{1}{\dim V_i} \hat{d}_\Sigma(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho),
\]
where the pairs \( (F, m, \delta) \) are ordered as follows \( (F, m, \delta) \leq (F', m', \delta') \) if \( F \subseteq F' \), \( m \leq m' \), \( \delta \geq \delta' \).

In section \( \S \) we will show that
\[
f.\dim_{\Sigma}(S, \rho) = \sup_{\varepsilon > 0} \liminf_{i \rightarrow \infty} \frac{1}{\dim V_i} \hat{d}_\Sigma(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho),
\]
\[
f.\dim_{\Sigma}(S, \rho) = \sup_{\varepsilon > 0} \limsup_{i \rightarrow \infty} \frac{1}{\dim V_i} \hat{d}_\Sigma(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho).
\]

We introduce two other versions of dimension, which will be used to prove that the above notion of dimension does not depend on the generating sequence.

**Definition 2.8.** Let \( X \) be a separable Banach space, we say that \( X \) has the *\( C \)-bounded approximation property* if there is a sequence \( \theta_n : X \rightarrow X \) of finite rank maps such that \( \|\theta_n\| \leq C \) and
\[
\|\theta_n(x) - x\| \rightarrow 0.
\]

We say that \( X \) has the *bounded approximation property* if it has the \( C \)-bounded approximation property for some \( C > 0 \).
Definition 2.9. Let $X$ be a separable Banach space with a uniformly bounded action of a countable discrete group $\Gamma$. Let $q: Y \to X$ be a bounded linear surjective map, where $Y$ is a separable Banach space with the bounded approximation property. A $q$-dynamical filtration is a pair $F = ((a_{s,j})_{(s,j)\in\Gamma\times\mathbb{N}}, (Y_{E,l})_{E\subseteq\Gamma, l\in\mathbb{N}})$ where $a_{s,j} \in Y$, $Y_{E,l} \subseteq Y$ is a finite dimensional linear subspace such that

1. $\sup_{(s,j)} \|a_{s,j}\| < \infty$
2. $q(a_{s,j}) = sq(a_{e_{(s,j)}})$
3. $(q(a_{e_{(s,j)}}))_{j=1}^{\infty}$ is dynamically generating,
4. $Y_{E,l} \subseteq Y_{E',l'}$ if $E \subseteq E', l \leq l'$
5. $\ker(q) = \bigcup_{(E,l)} Y_{E,l} \cap \ker(q)$,

6. $Y_{E,l} = \text{Span}\{a_{s,j} : s \in E^l, 1 \leq j \leq l\} + \ker(q) \cap Y_{E,l}$.

Note that if $X$ has the bounded approximation property and $Y = X$ with $q$ the identity, then a dynamical filtration simply corresponds to a choice of a dynamically generating sequence. In general, if $S = (x_j)_{j=1}^{\infty}$ is a dynamically generating sequence, then there is always a $q$-dynamical filtration $F = ((a_{s,j})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{F,l})$ such that $q(a_{e_{(s,j)}}) = x_j$. Simply choose $a_{s,j}$ such that $\|a_{s,j}\| \leq C\|x_j\|$ and $q(a_{s,j}) = sx_j$ for some $C > 0$. If $(y_j)_{j=1}^{\infty}$ is a dense sequence in $\ker(q)$, we can set

$$Y_{E,l} = \text{Span}\{a_{s,j} : (s,j) \in E^l \times \{1, \cdots, l\}\} + \sum_{j=1}^{l} C \gamma_j.$$

We can always find a Banach space $Y$ with the bounded approximation property and a quotient map $q: Y \to X$, in fact it is a standard exercise that we can choose $Y = l^{1}(\mathbb{N})$.

Definition 2.10. A quotient dimension tuple is a tuple $(Y, q, X, \Gamma, \sigma_i: \Gamma \to \text{Isom}(V_i))$ where $(X, \Gamma, \sigma_i)$ is a dimension triple, $Y$ is a separable Banach space with the bounded approximation property and $q: Y \to X$ is a bounded linear surjection.

Definition 2.11. Let $(Y, q, X, \Gamma, \sigma_i: \Gamma \to \text{Isom}(V_i))$ be a quotient dimension tuple, and let $F = ((a_{s,j})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{F,l})$ be a $q$-dynamical filtration. For $e \in F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $\delta, C > 0$ we let $\text{Hom}_r(F, F, m, \delta, \sigma_i)_C$ be the set of all bounded linear maps $T: Y \to V_i$ such that $\|T\| \leq C$ and

$$\|T(a_{s_1, \cdots, s_k}) - \sigma_i(s_1) \cdots \sigma_i(s_k)T(a_{e_{(s,j)}})\| < \delta$$

and

$$\left\|T_{|_{\ker(q)\cap Y_{F,l}}}\right\| < \delta.$$

As before, if $C = 1$ we will use $\text{Hom}_1(F, F, m, \delta, \sigma_i)$ instead of $\text{Hom}_r(F, F, m, \delta, \sigma_i)_C$.

Again, in the case $X$ has the bounded approximation property, we are simply looking at almost equivariant maps from $\Gamma$ to $V_i$, and this is similar in spirit to the definition of topological entropy in [9]. In the general case, note that genuine equivariant maps from $X$ to $V_i$ would correspond to maps on $Y$ which vanish on the kernel of $q$, and so that

$$T(a_{s_1, \cdots, s_k}) = \sigma_i(s_1) \cdots \sigma_i(s_k)T(a_{e_{(s,j)}}),$$
so we are still looking at almost equivariant maps on \( X \), in a certain sense.

**Definition 2.12.** Fix a pseudonorm \( \rho \) on \( l^\infty(\mathbb{N}) \), let \( (Y, q, X, \Gamma, \Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i))) \) be a quotient dimension tuple, and \( \mathcal{F} \) a \( q \)-dynamical filtration. Let \( \alpha_{\mathcal{F}} : B(Y, V_i) \to l^\infty(\mathbb{N}, V_i) \) be given by \( \alpha_{\mathcal{F}}(\phi) = (\phi(a_{\mathcal{j}}))_{j=1}^{\infty} \) we again use \( \hat{d}_\rho(A, \rho) = d_{\rho}(\alpha_{\mathcal{F}}(A), \rho_{V_i}) \). We define the dimension of \( \mathcal{F} \) with respect to \( \rho, \Sigma \) as follows:

\[
\begin{align*}
\text{f. dim}_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho) &= \limsup_{i \to \infty} \frac{1}{\dim V_i} \hat{d}_\rho(\text{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), \rho), \\
\text{f. dim}_\Sigma(\mathcal{F}, \varepsilon, \rho) &= \inf_{\varepsilon \in F} \text{f. dim}_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho), \\
\text{f. dim}_\Sigma(\mathcal{F}, \rho) &= \sup_{\varepsilon > 0} \text{f. dim}_\Sigma(\mathcal{F}, \varepsilon, \rho).
\end{align*}
\]

Note that unlike \( \text{f. dim}_\Sigma(S, F, m, \delta, \varepsilon, \rho) \) we know that \( \text{f. dim}_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho) \) is smaller when we enlarge \( F \) and \( m \) and shrink \( \delta \), thus the infimum is a limit and there are no issues between equality of limit suprema and limit infima for this definition.

**Definition 2.13.** Let \( Y, X \) be Banach spaces, and let \( \rho \) be a pseudonorm on \( B(X, Y) \). For \( \varepsilon > 0, 0 < M \leq \infty \), and \( A, C \subseteq B(X, Y) \), the set \( C \) is said to \((\varepsilon, M)\)-contain \( A \) if for every \( T \in A \), there is a \( S \in C \) such that \( \|S\| \leq M \) and \( \rho(S - T) < \varepsilon \). In this case we shall write \( A \subseteq_{\varepsilon, M} C \). We let \( d_{\varepsilon, M}(A, \rho) \) be the smallest dimension of a linear subspace which \((\varepsilon, M)\) contains \( A \).

**Definition 2.14.** Let \( (Y, q, X, \Gamma, \sigma_i : \Gamma \to \text{Isom}(V_i)) \) be a quotient dimension tuple. Let \( \mathcal{F} = (a_{\mathcal{j}}, Y_{E,t}) \) be a \( q \)-dynamical filtration. Fix a sequence of pseudonorms of \( \rho_i \) on \( B(X, V_i) \) and \( 0 < M \leq \infty \), set

\[
\begin{align*}
\text{opdim}_{\Sigma, M}(\mathcal{F}, F, m, \delta, \varepsilon, \rho_i) &= \limsup_{i \to \infty} \frac{1}{\dim V_i} d_{\varepsilon, M}(\text{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), \rho_i), \\
\text{opdim}_{\Sigma, M}(\mathcal{F}, \varepsilon, \rho_i) &= \inf_{\varepsilon \in F} \text{opdim}_{\Sigma, M}(\mathcal{F}, F, m, \delta, \varepsilon, \rho_i), \\
\text{opdim}_{\Sigma, M}(\mathcal{F}, \rho_i) &= \sup_{\varepsilon > 0} \text{opdim}_{\Sigma, M}(\mathcal{F}, \varepsilon, \rho_i).
\end{align*}
\]

As before, we shall use

\[
\text{opdim}_{\Sigma, M}(\mathcal{F}, \rho_i), \text{f. dim}_{\Sigma, \infty}(\mathcal{F}, \rho)
\]

for the same definitions as above, but replacing the limit supremum with the limit infimum.

By scaling,

\[
\inf_{0 < M < \infty} \text{opdim}_{\Sigma, M}(\mathcal{F}, \rho_i), \text{opdim}_{\Sigma, \infty}(\mathcal{F}, \rho_i), \text{f. dim}_{\Sigma}(S, \rho), \text{f. dim}_{\Sigma}(\mathcal{F}, \rho)
\]

remain the same when we replace \( \text{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i) \), \( \text{Hom}_\Gamma(S, F, m, \delta, \sigma_i) \), by \( \text{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_C \), \( \text{Hom}_\Gamma(S, F, m, \delta, \sigma_i)_C \), for \( C \) a fixed constant. This will be useful in several proofs.
Note that if \( \rho \) is a pseudonorm on \( l^\infty(\mathbb{N}) \), then we get a pseudonorm \( \rho_{F,i} \) on \( B(Y,V_i) \) by
\[
\rho_{F,i}(T) = \rho(j \mapsto \|T(a_{ej})\}).
\]
Further, for \( 0 < M \leq \infty \)
\[
\text{opdim}_{\Sigma,M}(F,\rho_{F,i}) \geq f. \dim_{\Sigma}(F,\rho).
\]

**Definition 2.15.** A **product norm** \( \rho \) is a norm on \( l^\infty(\mathbb{N}) \) such that
1. \( \rho \) induces a topology stronger than the product topology
2. \( \rho \) induces a topology which agrees with the product topology on \( \{ f \in l^\infty(\mathbb{N}) : \|f\|_\infty \leq 1 \} \).

A typical example are the \( l^p \)-norms:
\[
\rho(f)^p = \sum_{j=1}^{\infty} \frac{1}{2^j}|f(j)|^p.
\]
We shall show that if there is constant \( M > 0 \), depending only on \( Y \), so that if \( F,F' \) are dynamical filtrations of \( q \) and \( S \) is a dynamically generating sequence, then for any two product norms \( \rho,\rho' \),
\[
\text{opdim}_{\Sigma,M}(F,\rho_{F,i}) = \text{opdim}_{\Sigma,M}(F,\rho_{F,i}) = f. \dim_{\Sigma}(F,\rho) = f. \dim_{\Sigma}(F',\rho) = \dim_{\Sigma}(S,\rho),
\]
and the same with \( \dim \) replaced by \( \dim \). In particular all these dimension only depend of the action of \( \Gamma \) on \( X \), and give an isomorphism invariant. When we show all these equalities we let
\[
\dim_{\Sigma}(X,\Gamma)
\]
denote any of these common numbers.

The equality between these dimensions is easier to understand in the case when \( X \) has the bounded approximation property. When \( X \) has the bounded approximation property, we can take \( Y = X, q = \text{Id} \) and then the equality
\[
\text{opdim}_{\Sigma,M}(F,\rho_{F,i}) = f. \dim_{\Sigma}(S,\rho),
\]
says the data of local almost equivariant maps on \( X \) is the same as the data of global almost equivariant maps on \( X \). This is essentially because if we take \( \theta_{E,1} : X \to X_{E,1} \) which tend pointwise to the identity, then any almost equivariant map on \( X_{E,1} \) gives an almost equivariant map on \( X \) by composing with \( \theta_{E,1} \).

Since the maps \( \sigma_i : \Gamma \to \text{Isom}(V_i) \) are not assumed to have any structure, this invariant is uninteresting unless the maps \( \sigma_i \) model the action of \( \Gamma \) on \( X \) in some manner. Thus we note that if \( \Gamma \) is a sofic group, then the maps \( \sigma_i : \Gamma \to S_{d_i} \) model at least the group \( \Gamma \) in a reasonable manner.

Because \( S_n \) acts naturally on \( l^p(n) \) we get an induced sequence of maps \( \sigma_i : \Gamma \to \text{Isom}(l^p(d_i)) \) and the above invariant measures how closely the action of \( \Gamma \) on \( X \) is modeled by these maps. When \( \Gamma \) is sofic, and \( \Sigma = (\sigma_i : \Gamma \to S_{d_i}) \) is a sofic approximation and \( \Sigma^{(p)} = (\sigma_i : \Gamma \to \text{Isom}(l^p(d_i))) \) are the maps induced by the action of \( S_n \) on \( l^p(n) \), we let
\[
\dim_{\Sigma^{(p)}}(X,\Gamma) = \dim_{\Sigma^{(p)}}(X,\Gamma)
\]
\[
\dim_{\Sigma^{(p)}}(X,\Gamma) = \dim_{\Sigma^{(p)}}(X,\Gamma).
\]
Similarly, if $\Gamma$ is $R^\omega$-embeddable, and $\sigma_i: \Gamma \to U(d_i)$ is an embedding sequence, then since $U(d_i)$ is the isometry group of $l^2(d_i)$ we shall let

$$\dim_{\Sigma,l^2}(X,\Gamma) = \dim_{\Sigma}(X,\Gamma)$$

$$\dim_{\Sigma,l^2}(X,\Gamma) = \dim_{\Sigma}(X,\Gamma).$$

Just as $S_n$ acts on commutative $l^p$-Spaces, we have two natural actions of $U(n)$ on non-commutative $L^p$-spaces. Let $S^p(n)$ be $M_n(\mathbb{C})$ with the norm

$$\|A\|_{S^p} = \text{Tr}(|A|^p)$$

where $|A| = (A^* A)^{1/2}$. Then $U(n)$ acts isometrically on $S^p(n)$ by conjugation and by left multiplication. We shall use

$$\dim_{\Sigma,S^p,\text{conj}}(X,\Gamma)$$

for our dimension define above, thinking of $\sigma_i$ as a map into Isom($S^p(n)$) by conjugation and

$$\dim_{\Sigma,S^p,\text{multi}}(X,\Gamma)$$

thinking of $\sigma_i$ as a map into Isom($S^p(n)$) by left multiplication.

One of our main applications will be showing that when $\Gamma$ is $R^\omega$-embeddable

$$\dim_{\Sigma,S^p,\text{conj}}(l^p(\Gamma)^{\oplus n}, \Gamma) = \dim_{\Sigma,S^p,\text{conj}}(l^p(\Gamma)^{\oplus n}, \Gamma) = n,$$

if $1 \leq p \leq 2$, and

$$\dim_{\Sigma,l^p}(l^p(\Gamma)^{\oplus n}, \Gamma) = \dim_{\Sigma,l^p}(l^p(\Gamma)^{\oplus n}, \Gamma) = n,$$

if $1 \leq p \leq 2$. In particular the representations $l^p(\Gamma)^{\oplus n}$ are not isomorphic for different values of $n$, if $\Gamma$ is $R^\omega$-embeddable.

3. Invariance of the Definitions

In this section we show that our various notions of dimension agree. Here is the main strategy of the proof. First we show that there is an $M > 0$, independent of $F$ so that

$$\text{opdim}_{\Sigma,M}(F, \rho_F) = f. \dim_{\Sigma}(F, \rho),$$

the constant $M$ comes from the constant in the definition of bounded approximation property. A compactness argument shows that

$$\text{opdim}_{\Sigma,M}(F, \rho_F)$$

does not depend on the choice of pseudonorm. We then show that

$$\text{opdim}_{\Sigma,\infty}(F, \rho_F)$$

does not depend on the choice of $F$, this is easier than trying to show that

$$f. \dim_{\Sigma}(S, \rho)$$

do not depend on the choice of $S$. This is because the maps used to define

$$\text{opdim}_{\Sigma,\infty}(F, \rho_F)$$

all have the same domain, which makes it easy to switch from one generating set to another, since we can use that generators for $F$ have to be close to linear combinations of generators for $F'$. Then we show that

$$f. \dim_{\Sigma}(F, \rho) = f. \dim_{\Sigma}(S, \rho),$$
this will reduce to showing that if we are given an almost equivariant map \( \phi : Y \to V \) which is small on the kernel of \( q \), then there is a \( T : X' \to V \) with \( X' \subseteq X \) finite dimensional such that \( T \circ q \) is close to \( \phi \) on a prescribed finite set.

First we need a simple fact about spaces with the bounded approximation property.

**Proposition 3.1.** Let \( Y \) be a separable Banach space with the \( C \)-bounded approximation property, and let \( I \) be a countable directed set. Let \( (Y_\alpha)_{\alpha \in I} \) be an increasing net of subspaces of \( Y \) such that

\[
Y = \bigcup_\alpha Y_\alpha.
\]

Then there are finite-rank maps \( \theta_\alpha : Y \to Y_\alpha \) such that \( \|\theta_\alpha\| \leq C \) and

\[
\lim_{\alpha} \|\theta_\alpha(y) - y\| = 0
\]

for all \( y \in Y \).

**Proof.** Fix \( y_1, \cdots, y_k \in Y \) and \( \varepsilon > 0 \). Then there is a finite rank \( \theta : Y \to Y \) such that

\[
\|\theta(y_j) - y_j\| < \varepsilon,
\]

\[
\|\theta\| \leq C.
\]

Write

\[
\theta = \sum_{j=1}^{n} \phi_j \otimes x_j
\]

with \( \phi_j \in Y^* \) and \( x_j \in Y \). If \( \alpha \) is sufficiently large, then we can find \( x'_j \in Y_\alpha \) close enough to \( x_j \) so that if we let

\[
\theta_0 = \sum_{j=1}^{n} \phi_j \otimes x'_j,
\]

\[
\overline{\theta} = \begin{cases} 
\theta_0 & \text{if } \|\theta_0\| \leq C \\
C\frac{\theta_0}{\|\theta_0\|} & \text{otherwise}
\end{cases}
\]

has

\[
\|\overline{\theta}(y_j) - y_j\| < 2\varepsilon.
\]

Now let \( (y_j)_{j=1}^{\infty} \) be a dense sequence in \( Y \), and let

\[
\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots
\]

with \( \alpha_j \in I \) be such that for all \( \beta \in I \), there is a \( j \) such that \( \beta \leq \alpha_j \). By the preceding paragraph, we can inductively construct an increasing sequence \( n_k \) of integers and maps

\[
\theta_k : Y \to Y_{\alpha_{n_k}}
\]

such that

\[
\|\theta_k\| \leq C
\]

\[
\|\theta_k(y_j) - y_j\| \leq 2^{-k} \text{ if } j \leq k.
\]

Set \( \theta_\alpha = \theta_{\alpha_{n_k}} \) if \( k \) is the largest integer such that \( \alpha_{n_k} \) is not bigger than \( \alpha \). Then \( \theta_\alpha \) has the desired properties. \( \square \)
Lemma 3.2. Let \((Y,q,X,\Gamma,\Sigma = (\sigma_i: \Gamma \to \text{Isom}(V_i)))\) be a quotient dimension tuple. Let \(F = ((a_{sj})_{(s,j) \in \Gamma \times \mathbb{N}}, Y_{F,k})\) be a q-dynamical filtration and \(\rho\) a product norm, and let \(C > 0\) be such that \(Y\) has the \(C\)-bounded approximation property.

Fix \(M > C\). Then for any \(V \subseteq Y\) finite-dimensional, there is a \(F \subseteq \Gamma\) finite \(m \in \mathbb{N}\), \(\delta, \varepsilon > 0\) and linear maps

\[L_i: l^\infty(\mathbb{N}, V_i) \to B(Y, V_i)\]

so that if \(\phi \in \text{Hom}\,(F, m, \delta, \sigma_i)\), \(f \in l^\infty(\mathbb{N}, V_i)\) has \(\rho_{V_i}(\alpha_F(\phi) - f) < \varepsilon\), then

\[\|L_i(f)\| \leq M,\]

\[\|L_i(f)\|_{\text{tr}, V} - \phi\|_V < \kappa.\]

Proof. Note that for every \(V\) finite-dimensional there is a \(E \subseteq \Gamma\) finite, \(l \in \mathbb{N}\), such that

\[\sup_{v \in V} \inf_{w \in W_{E,l}} \|v - w\| < \kappa,\]

so we may assume that \(V = W_{E,l}\) for some \(E, l\).

Fix \(\eta > 0\) to be determined later. By the preceding proposition let \(\theta_{F,k}: Y \to Y_{F,k}\) be such that

\[\lim_{(F,k)} \|\theta_{F,k}(y) - y\| = 0\text{ for all }y \in Y.\]

Choose \(F, m\) sufficiently large such that

\[\|\theta_{F,m}|_{Y_{E,l}} - \text{Id}|_{Y_{E,l}}\| \leq \eta.\]

Let \(B_{F,m} \subseteq F_m \times \{1, \cdots, m\}\) be such that \(\{q(a_{sj}) : (s, j) \in B_{F,m}\}\) is a basis for \(X_{F,m} = \text{Span}\{q(a_{sj}) : (s, j) \in F_m \times \{1, \cdots, m\}\}\). Define

\[\widetilde{L}_i: l^\infty(\mathbb{N}, V_i) \to B(X_{F,m}, V_i)\]

by

\[\widetilde{L}_i(f)(q(a_{sj})) = \sigma_i(s)f(j)\text{ for } (s, j) \in B_{F,m}.\]

We claim that if \(\delta > 0, \varepsilon' > 0\) are sufficiently small, \(\phi \in \text{Hom}\,(F, m, \delta, \sigma_i)\) and \(f \in l^\infty(\mathbb{N}, V_i)\) has

\[\rho_{V_i}(f - \alpha_F(\phi)) < \varepsilon',\]

then

\[\|\widetilde{L}_i(f) \circ q|_{Y_{F,m}} - \phi|_{Y_{F,m}}\| \leq \eta.\]

By finite-dimensionality, there is a \(D(F, m) > 0\) such that if \(v \in \ker(q) \cap Y_{F,m}, (d_{tr}) \in \mathbb{C}^{B_{F,m}}\), then

\[\sup\{\|v\|, |d_{tr}|\} \leq D(F, m) \left\|v + \sum_{(t,r) \in B_{F,m}} d_{tr} a_{tr}\right\|.\]

Thus if \(x = v + \sum_{(t,r) \in B_{F,m}} d_{tr} a_{tr}\) with \(v \in \ker(q) \cap Y_{F,m}\) has \(\|x\| = 1\), then

\[\|\widetilde{L}_i(f)(q(x)) - \phi(x)\| \leq D(F, m)\delta + D(F, m) \sum_{(t,r) \in B_{F,m}} \|\phi(a_{tr}) - \sigma_i(t)f(r)\|\]

\[\leq D(F, m)\delta + D(F, m)|F|^m m\delta + \sum_{(t,r) \in B_{F,m}} \|\phi(a_{tr}) - f(r)\|,\]
if \( \delta < \frac{\eta}{2\|F,m\|1 + \|F,m\|} \), and \( \varepsilon' > 0 \) is small enough so that \( \rho(g) < \varepsilon' \) implies
\[
\sum_{(t,r) \in S_{F,m}} |g(r)| < \frac{\eta}{2},
\]
then our claim holds.

So assume that \( \delta, \varepsilon' > 0 \) are small enough so that (1) holds, and set \( L_\varepsilon(f) = \tilde{L}_\varepsilon(f) \circ q_{Y,F,m} \circ \theta_{F,m} \). Then
\[
\|L_\varepsilon(f)\| \leq C(1 + \eta)
\]
and for \( \phi, f \) as above and \( y \in Y_{E,l} \)
\[
\|L_\varepsilon(f)(y) - \phi(y)\| \leq (1 + \eta)\|\theta_{F,m}(y) - y\| + \|\tilde{L}_\varepsilon(f) \circ q(y) - \phi(y)\| \leq (2 + \eta)\|y\|.
\]
So we force \( \eta \) to be small enough so that \( (2 + \eta)\eta < \kappa, C(1 + \eta) < M \).

\[\Box\]

**Lemma 3.3.** Let \( (Y,q,X,\Gamma,\Sigma = (\sigma_i: \Gamma \to \text{Isom}(V_i))) \) be a quotient dimension tuple.

Let \( F = ((a_{sj})_{(s,j) \in \Gamma \times \mathbb{N}}, Y_{E,l}) \) be a q-dynamical filtration, and \( \rho \) a product norm, let \( Y \) have the C-bounded approximation property.

(a) If \( \infty \geq M > C \), then
\[
\dim_{\Sigma}(F,\rho) = \text{opdim}_{\Sigma,M}(F,\rho),
\]
\[
\\dim_{\Sigma}(F,\rho) = \text{opdim}_{\Sigma,M}(F,\rho).
\]

(b) If \( \rho' \) is another product norm then for all \( 0 < M < \infty \),
\[
\text{opdim}_{\Sigma,M}(F,\rho_{F,i}) = \text{opdim}_{\Sigma,M}(F,\rho'_{F,i}),
\]
\[
\text{opdim}_{\Sigma,M}(F,\rho_{F,i}) = \text{opdim}_{\Sigma,M}(F,\rho'_{F,i}).
\]

**Proof.** (a) First note that
\[
\text{opdim}_{\Sigma,M}(F,\rho) \geq \text{opdim}_{\Sigma,\infty}(F,\rho) \geq \dim_{\Sigma}(F,\rho)
\]
so it suffices to handle the case that \( M < \infty \).

Let \( A > 0 \) be such that
\[
\|a_{sj}\| \leq A \text{ for all } (s,j) \in \Gamma \times \mathbb{N}
\]
\[
\|sx\| \leq A\|x\| \text{ for all } s \in \Gamma.
\]

Take \( 1 > \varepsilon > 0 \). Let \( k \) be such that if \( f \in L^\infty(\mathbb{N}), \) and \( \|f\|_{\infty} \leq 1, \) and \( f \) is supported on \( \{n : n \geq k\} \), then \( \rho(f) < \varepsilon. \) Since \( \rho \) induces a topology weaker than the norm topology, we can find a \( \varepsilon > \kappa > 0 \) such that
\[
\rho(f) < \varepsilon
\]
if
\[
\|f\|_{\infty} \leq \kappa.
\]

By Lemma 3.2 let \( e \in F \subseteq \Gamma \) be finite, \( m \in \mathbb{N}, \varepsilon > \varepsilon' > 0, \kappa > \delta > 0 \) and \( L_\varepsilon: \mathbb{L}^\infty(\mathbb{N},V_i) \to B(Y,V_i) \) be such that if \( \phi \in \text{Hom}_F(F,m,\delta,\sigma_i) \) and \( f \in L^\infty(\mathbb{N},V_i) \) has \( \rho_{V_i}(\alpha_F(\phi) - f) < \varepsilon', \) then
\[
\|L_\varepsilon(f)\|_{Y(\varepsilon),k} - \phi\| < \kappa,
\]
\[
\|L_\varepsilon(f)\| \leq M.
\]
Then if \( \phi, f \) are as above we have
\[
\rho_{r,i}(\phi - L_i(f)) \leq (M + 1)\varepsilon + \rho(\chi_{j\leq k}(j))(\|\phi(a_{ej}) - L_i(f)(a_{ej})\|_{j=1}^\infty)
\]
And for \( j \leq k \)
\[
\|\phi(a_{ej}) - L_i(f)(a_{ej})\| \leq A(M + 1)\kappa.
\]
Thus
\[
\rho_{r,i}(\phi - L_i(f)) \leq (M + 1)(A + 1)\varepsilon.
\]
This implies that
\[
d_{(M+1)(A+1)\varepsilon, \mu}(\text{Hom}_{r}(F, F', m', \delta', \sigma_i), \rho_{r,i}) \leq d_{\varepsilon}(\text{Hom}_{r}(F, F', m', \delta', \sigma_i), \rho_{r,i})
\]
for all \( F' \supseteq F, m' \geq m, \) and all \( \delta' < \delta. \) This completes the proof.

(b) This is a simple consequence of the compactness of the \( \| \cdot \|_\infty \) unit ball of \( l^\infty(\mathbb{N}) \) in the product topology.

\( \square \)

**Lemma 3.4.** Let \( (Y, q, X, \Gamma, \sigma_i, \Gamma \to \text{Isom}(V_i)) \) be a quotient dimension tuple. Let \( F, F' \) be two \( q \)-dynamical filtrations. If \( \rho_i \) is any fixed sequence of pseudonorms on \( B(Y, V_i) \), then for all \( 0 < M \leq \infty \),
\[
\text{opdim}_{\Sigma, M}(F, \rho_i) = \text{opdim}_{\Sigma, M}(F', \rho_i),
\]
\[
\text{opdim}_{\Sigma, M}(F, \rho_i) = \text{opdim}_{\Sigma, M}(F', \rho_i),
\]
Proof. Let \( F' = ((a_{s,j}')_{(s,j)\in \Gamma \times \mathbb{N}}, Y_{E,t}'), F = ((a_{s,j})_{(s,j)\in \Gamma \times \mathbb{N}}, Y_{E,t}). \) We do the proof for \( \text{opdim}_{\Sigma, M} \), the other case is proved in the same manner. Let \( C > 0 \) be such that \( \|sx\| \leq C\|x\| \) for all \( s \in \Gamma, \) and such that \( \|a_{s,j}\|, \|a_{s,j}'\| \leq C. \) Fix \( F \subseteq \Gamma \) finite, \( m \in \mathbb{N}, \delta > 0. \) Fix \( \eta > 0 \) which will depend upon \( F, m, \delta \) in a manner to be determined later.

Choose \( E \subseteq \Gamma \) finite \( t \in \mathbb{N}, \) such that for \( 1 \leq j \leq m, s \in F^m \) there are \( c_{j,t,k} \) with \( (t, k) \in E \) and \( v_{s,j} \in Y_{E,t} \cap \ker(q) \) such that
\[
\left\| a_{s,j} - v_{s,j} - \sum_{(t,k)\in E\times \{1, \ldots, l\}} c_{j,t,k}a_{s,tk}' \right\| < \eta,
\]
and so that for every \( w \in Y_{F,m} \cap \ker(q) \) there is a \( v \in Y_{E,t} \cap \ker(q) \) such that \( \|v - w\| < \eta\|w\|. \) Let \( A(\eta) = \sup_{c_{j,t,k}}(\|c_{j,t,k}\|, \sup_{v_{s,j}}(\|v_{s,j}\|)) \)

Set \( m' = 2\max(m, l) + 1, F' = [(F \cup F^{-1} \cup \{e\})(E \cup E^{-1} \cup \{e\})]^{2m'+1}, \) we claim that we can choose \( \delta' > 0, \eta > 0 \) small so that
\[
\text{Hom}_{r}(F', F', m', \delta', \sigma_i) \subseteq \text{Hom}_{r}(F, F, m, \delta, \sigma_i).
\]
If \( T \in \text{Hom}_{r}(F', F', m', \delta', \sigma_i), 1 \leq j, r \leq m, \) and \( s_1, \ldots, s_r \in F \) then
\[
\|T(a_{s_1 \ldots s_r}) - \sigma_i(s_1) \cdots \sigma_i(s_r)T(a_{ej})\| \leq 2\eta + \|T(v_{s,j})\| + \|\sigma_i(s_1) \cdots \sigma_i(s_r)T(v_{s,j})\| + \sum_{(t,k)\in E\times \{1, \ldots, l\}} c_{j,t,k}T(a_{s_1 \ldots s_r,tk}) - \sigma_i(s_1) \cdots \sigma_i(s_r)T(a_{s,tk}') \| \leq 2\eta + \delta' A(\eta) + \delta' A(\eta) + 2|E|A(\eta)\delta'.
\]
By choosing \( \eta < \delta/2 \), and then choosing \( \delta' \) very small we can make the above expression less than \( \delta \). If we also force \( \delta' < \delta/2 \) our choice of \( \eta \) implies that
\[
\|T(w)\| \leq \delta\|w\|
\]
for \( T \) as above and \( w \in Y_{F,t} \cap \ker(q) \). This completes the proof. \( \square \)

Because of the above lemma, the only difficulty in proving that \( \dim_{\Sigma}(F, \rho_{F,i}) \) does not depend on the choice of \( F \) is switching the pseudonorm from \( \rho_{F,i} \) to \( \rho_{F',i} \). Because of this we will investigate how the dimension changes when we switch pseudonorms.

**Definition 3.5.** Let \((Y, q, X, \Gamma, \Sigma) = (\sigma_i : \Gamma \to \text{Isom}(V_i))\), and fix a \( q \)-dynamical filtration \( F \). If \( \rho_i, q_i \) are pseudonorms on \( B(Y, V_i) \) we say that \( \rho_i \) is \((F, \Sigma)\)-weaker than \( q_i \) and write \( \rho_i \preceq_{F, \Sigma} q_i \) if the following holds. For every \( \varepsilon > 0 \), there are \( F \subseteq \Gamma \) finite, \( \delta, \varepsilon' > 0 \), \( m, i_0 \in \mathbb{N} \), and linear maps \( L_i : B(Y, V_i) \to B(Y, V_i) \) for \( i \geq i_0 \) such that if \( \phi \in \text{Hom}_{F, m, \delta, \sigma_i} \) and \( \psi \in B(Y, V_i) \) has \( q_i(\phi - \psi) < \varepsilon' \), then \( \rho_i(\phi - L_i(\psi)) < \varepsilon \). We say that \( \rho_i \) is \((F, \Sigma)\) equivalent to \( q_i \), and write \( \rho_i \sim_{F, \Sigma} q_i \), if \( \rho_i \preceq_{F, \Sigma} q_i \) and \( q_i \preceq_{F, \Sigma} \rho_i \).

**Lemma 3.6.** Let \((Y, X, \Sigma, \Gamma, \Sigma)\) be a quotient dimension tuple and \( F \) a \( q \)-dynamical filtration.

(a) If \( \rho_i, q_i \) are pseudonorms with \( \rho_i \preceq_{F, \Sigma} q_i \), then
\[
\text{opdim}_{\Sigma, \infty}(F, \rho_i) \leq \text{opdim}_{\Sigma, \infty}(F, q_i).
\]

(b) Let \( F' = ((a_{s, j})_{(s, j) \in \Gamma \times \mathbb{N}}) \in F \), \( F = ((a_{s, j})_{(s, j) \in \Gamma \times \mathbb{N}}) \in Y_{E, l} \) be \( q \)-dynamical filtrations. Let \( \rho \) be any product norm. Define a pseudonorm on \( B(Y, V_i) \) by \( \rho_{F, i}(\phi) = \rho((\|\phi(a_{s, j})\|)_{j=1}^{\infty}) \), and similarly define \( \rho_{F', i} \).

**Proof.** Let \( \Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i)) \).

(a) This follows directly follow the definitions.

(b) Let \( C > 0 \) be such that \( Y \) has the \( C \)-bounded approximation property and
\[
\|a_{s, j}\| \leq C
\]
\[
\|a'_{s, j}\| \leq C
\]
\[
\|sx\| \leq C\|x\| \text{ for } s \in \Gamma
\]
for every \( x \in X \), there is a \( y \in Y \) such that \( q(y) = x \) and \( \|y\| \leq C\|x\| \).

Choose \( m \in \mathbb{N} \) such that \( \rho(f) < \varepsilon \) if \( \|f\|_{\infty} \leq 1 \) and \( f \) is supported on \( \{n : n \geq m\} \), and let \( \kappa > 0 \) be such that \( \rho(f) < \varepsilon \) if \( \|f\|_{\infty} \leq \kappa \).

By Lemma 3.2 choose \( F' \supseteq F \) finite \( m \leq m' \in \mathbb{N}, \delta > 0 \) and
\[
\widetilde{L}_i : \ell^\infty(N, V_i) \to B(Y, V_i)
\]
so that if \( f \in \ell^\infty(N, V_i) \) and \( \phi \in \text{Hom}_{F, F', m', \delta, \sigma_i} \) has \( \rho_{V_i}(\alpha_{F}(\phi) - f) < \varepsilon' \) then
\[
\left\| \widetilde{L}_i(f) |_{Y_{i', m}} - \phi |_{Y_{i', m}} \right\| < \kappa,
\]
\[
\| \widetilde{L}_i(f) \| \leq 2C.
\]
Let \( L_i : B(Y, V_i) \to B(Y, V_i) \) be given by \( L_i(\psi) = \widetilde{L}_i(\alpha_{F}(\psi)) \).
Suppose $\phi \in \text{Hom}_T(\mathcal{F}, \mathcal{F}', m', \delta', \sigma_i)$ and $\psi \in B(Y, V_i)$ has $\rho_{\mathcal{F}',i}(\phi - \psi) < \varepsilon'$. Then, for $1 \leq j \leq m$ we have

$$\|\phi(a'_{\varepsilon_j}) - L_\varepsilon(\psi)(a'_{\varepsilon_j})\| \leq C\kappa,$$

Our choice of $m, \kappa$ then imply that $\rho_{\mathcal{F}',i}(\phi - \psi) < 6C\varepsilon$. This completes the proof. \hfill \Box

**Corollary 3.7.** Let $(Y, q, X, \Gamma, \sigma_i : \Gamma \to \text{Isom}(V_i))$ be a quotient dimension tuple. Let $\rho, \rho'$ be two product norms. For any two $q$-dynamical filtrations $\mathcal{F}, \mathcal{F}'$ we have

$$\text{opdim}_{\Sigma,\infty}((\mathcal{F}, \rho_{\mathcal{F},i})) = \text{opdim}_{\Sigma,\infty}((\mathcal{F}', \rho'_{\mathcal{F}',i})) = \text{opdim}_{\Sigma,\infty}((\mathcal{F}', \rho'_{\mathcal{F}',i})),
$$

$$\text{opdim}_{\Sigma,\infty}((\mathcal{F}, \rho_{\mathcal{F},i})) = \text{opdim}_{\Sigma,\infty}((\mathcal{F}', \rho'_{\mathcal{F}',i})).$$

**Proof.** Combining Lemmas 3.3 and 3.4 By Lemma 3.3 we have

$$\text{opdim}_{\Sigma,\infty}((\mathcal{F}', \rho'_{\mathcal{F}',i})) = \text{opdim}_{\Sigma,\infty}((\mathcal{F}', \rho'_{\mathcal{F}',i})) \leq \text{opdim}_{\Sigma,\infty}((\mathcal{F}, \rho_{\mathcal{F},i})).$$

the opposite inequality follows by symmetry. \hfill \Box

Because of the preceding corollary $\dim_{\Sigma}(\mathcal{F}, \rho)$ only depends on the action of $\Gamma$ and the quotient map $q : Y \to X$. Thus we can define

$$\dim_{\Sigma}(q, \Gamma) = \text{opdim}_{\Sigma,\infty}((\mathcal{F}, \rho_{\mathcal{F},i})) = f. \dim_{\Sigma}(\mathcal{F}, \rho)$$

where $\mathcal{F}$ is any $q$-dynamical filtration and $\rho$ is any product norm.

We now proceed to show that $\text{opdim}_{\Sigma,\infty}(q, \Gamma)$ does not depend on $q$, as stated before the idea is to prove that

$$\dim_{\Sigma}(q, \Gamma) = f. \dim_{\Sigma}(S, \rho)$$

where $S$ is any dynamically generating sequence for $X$.

For this, we will prove that we can approximate maps $T$ on $Y$ which almost vanish on the kernel of $q$, by maps on $X$. For the proof, we need the construction of ultraproducts of Banach spaces.

Let $X_n$ be a sequence of Banach spaces and $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ a free ultrafilter. We define the ultraproduct of the $X_n$, written $\prod^\omega X_n$ by

$$\prod^\omega X_n = \{(x_n)_{n=1}^\infty : x_n \in X_n, \sup_n \|x_n\| < \infty\}/\{(x_n)_{n=1}^\infty : x_n \in X_n, \lim_{n \to \omega} \|x_n\| = 0\}.$$

We use $(x_n)_{n \to \omega}$ for the image of $(x_n)_{n=1}^\infty$ under the canonical quotient map to $\prod^\omega X_n$.

If a set $A \subseteq \mathbb{N}$ is in $\omega$, we will say that $A$ is $\omega$-large.

**Lemma 3.8.** Let $X, Y$ be Banach spaces with $X$ finite-dimensional and $q : Y \to X$ a bounded linear surjective map. Let $C > 0$ be such that for all $x \in X$, there is a $y \in Y$ with $\|y\| \leq C\|x\|$ such that $q(y) = x$, and fix $A > C$. Let $I$ be a countable directed set, and $(Y_\alpha)_{\alpha \in I}$ a net of subspaces of $Y$ such that $Y_\alpha \subseteq Y_\beta$ if $\alpha \leq \beta$, and

$$q(Y_\alpha) = X,$$

$$\ker(q) = \bigcup_\alpha Y_\alpha \cap \ker(q).$$
Fix a finite set \( F \subseteq \bigcup_{\alpha} Y_{\alpha} \).

Then for all \( \varepsilon > 0 \), there is a \( \delta > 0 \) and \( \alpha_0 \) with the following property. If \( \alpha \geq \alpha_0 \) and \( W \) is a Banach space with \( T: Y_{\alpha} \to W \) a linear contraction such that
\[
\|T|_{\ker(q) \cap Y_{\alpha}}\| \leq \delta,
\]
then there is a \( S: X \to W \) such that \( \|S\| \leq A \) and
\[
\|T(x) - S \circ q(x)\| \leq \varepsilon,
\]
for all \( x \in F \).

**Proof.** Note that our assumptions imply
\[
Y = \bigcup_{\alpha} Y_{\alpha}.
\]

Fix a countable increasing sequence \( \alpha_n \) in \( I \), such that for every \( \beta \in I \) there is an \( n \) such that \( \beta \leq \alpha_n \). Assume also that \( F \subseteq Y_{\alpha_1} \). Since \( I \) is directed, if the claim is false, then we can find a \( \varepsilon > 0 \) and an increasing sequence \( \beta_n \) with \( \beta_n \geq \alpha_n \) and a \( T_n: Y_{\beta_n} \to W_n \) such that \( \|T_n\| \leq 1 \),
\[
\|T_n|_{\ker(q) \cap Y_{\beta_n}}\| \leq 2^{-n},
\]
and for every \( S: X \to W_n \) with \( \|S\| \leq A \),
\[
\|T_n(x) - S \circ q(x)\| \geq \varepsilon,
\]
for some \( x \in F \).

Fix \( \omega \in \beta\mathbb{N} \setminus \mathbb{N} \) and let
\[
W = \prod_{\alpha} W_{\alpha}.
\]
Define
\[
T: \bigcup_n Y_{\beta_n} \to W
\]
by
\[
T(x) = (T_n(x))_{n=\omega},
\]
note that for any \( k \), the map \( T_n \) is defined on \( Y_{\beta_k} \) for \( n \geq k \), so \( T \) is well-defined. Also
\[
\|T(x)\| \leq \|x\|
\]
\[
T(x) = 0 \text{ on } \bigcup_n Y_{\beta_n} \cap \ker(q).
\]

Our density assumptions imply that \( T \) extends uniquely to a linear map, still denoted \( T \), from \( Y \) to \( W \), which vanishes on the kernel of \( q \). Thus there is \( S: X \to W \) such that \( T = S \circ q \), and our hypothesis on \( C \) implies that \( \|S\| \leq C \).

Since \( X \) is finite dimensional, we can find \( S_n: X \to W_n \) such that \( S(x) = (S_n(x))_{n=\omega} \). Compactness of the unit sphere of \( X \) and a simple diagonal argument show that
\[
C \geq \|S\| = \lim_{n \to \omega} \|S_n\|.
\]
Thus \( B = \{n: \|S_n\| < A\} \) is an \( \omega \)-large set, and by hypothesis
\[
B = \bigcup_{x \in F} \{n \in B: \|T_n(x) - S_n(q(x))\| \geq \varepsilon\}.
\]
Since $B$ is $\omega$-large, there is some $x \in F$ such that
\[
\{ n \in B : \| T_n(x) - S_n(q(x)) \| \geq \varepsilon \}
\]
is $\omega$-large. But then $T(x) \neq S \circ q(x)$, a contradiction.

We now prove a lemma which allows us to treat the limit supremum over $(F, m, \delta)$ in the definition of $f. \text{dim}_\Sigma(S, \rho)$ as a limit.

**Lemma 3.9.** Let $(X, \Gamma, \Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i)))$ be a dimension triple, fix a dynamically generating sequence $S$ in $X$ and $\rho$ a product norm. Then
\[
f. \text{dim}_\Sigma(S, \rho) = \sup_{\varepsilon > 0} \lim \inf \limsup_{\varepsilon > 0} \lim \sup \frac{1}{\dim V_i} \hat{d}_\varepsilon(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho),
\]
\[
f. \text{dim}_\Sigma(S, \rho) = \sup_{\varepsilon > 0} \lim \sup \lim \inf \lim \sup \frac{1}{\dim V_i} \hat{d}_\varepsilon(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho).
\]

**Proof.** Let $S = (x_j)_{j=1}^\infty$. We do the proof for $\text{dim}$ only, the proof for $\text{dim}$ is the same. Fix $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that if $\| f \|_\infty \leq 1$ and $f$ is supported on $\{ n : n \geq k \}$, then $\rho(f) < \varepsilon$. It suffices to show that
\[
f. \text{dim}_\Sigma(S, \rho) \leq \sup_{\varepsilon > 0} \lim \inf \limsup_{\varepsilon > 0} \lim \sup \frac{1}{\dim V_i} \hat{d}_\varepsilon(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho).
\]

Fix $F \subseteq \Gamma$ is finite $m \geq k, \delta > 0$. Then for any $F \subseteq F' \subseteq \Gamma$ finite $m' \geq m, \delta' < \delta$ and $\psi \in \text{Hom}_\Gamma(S, F', m', \delta', \sigma_i)$ we have $\psi \in \text{Hom}_\Gamma(S, F, m, \delta, \sigma_i)$.

Further if $f, g \in l^\infty(\mathbb{N}, V_i)$ are defined by
\[
f(j) = \chi_{\{ n \leq m \}}(j)(\psi(x_j), g(j) = \chi_{\{ n \leq m' \}}(\psi(x_j),
\]
then
\[
\rho(j \mapsto \| f(j) - g(j) \|) < \varepsilon.
\]

Thus
\[
\hat{d}_{2\varepsilon}(\text{Hom}_\Gamma(S, F', m', \delta', \sigma_i), \rho) \leq \hat{d}_{\varepsilon}(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho).
\]

Therefore
\[
f. \text{dim}_\Sigma(S, \varepsilon, \rho) \leq \limsup_{\varepsilon \to 0} \frac{1}{\dim V_i} \hat{d}_{2\varepsilon}(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho).
\]

Since $F, m, \delta$ were arbitrary
\[
f. \text{dim}_\Sigma(S, 2\varepsilon, \rho) \leq \lim \inf \limsup_{\varepsilon > 0} \frac{1}{\dim V_i} \hat{d}_{\varepsilon}(\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho),
\]
and taking the supremum over $\varepsilon > 0$ completes the proof.

**Lemma 3.10.** Let $(Y, q, X, \Gamma, \Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i)))$ be a quotient dimension tuple. Fix a dynamically generating sequence $S$ in $X$, and $\rho$ a product norm.
\[
\text{dim}_\Sigma(q, \Gamma) = f. \text{dim}_\Sigma(S, \rho).
\]
\[
\dim_\Sigma(q, \Gamma) = f. \text{dim}_\Sigma(S, \rho).
\]
Proof. We will only do the proof for \( \dim \).

Let \( S = (x_j)_{j=1}^{\infty} \) and let \( \mathcal{F} = ((a_{E,j})_{(E,j) \in \Gamma \times \mathbb{N}}, Y_{E,l}) \) be a dynamical filtration such that \( q(a_{E,j}) = x_j \). Let \( C > 0 \) be such that

\[
\sup_{(s,j)} \|a_{s,j}\| \leq C
\]

\[
\sup_j \|x_j\| \leq C
\]

\[
\|y\| \leq C,
\]

for every \( x \in X \), there is a \( y \in Y \) such that \( q(y) = x \) and \( \|y\| \leq C\|x\| \), and so that \( Y \) has the \( C \)-bounded approximation property. Let \( \theta_{E,l} : Y \to Y_{E,l} \) be such that \( \|\theta_{E,l}\| \leq C \) and

\[
\lim_{(E,l) \to (F,m)} \|\theta_{E,l}(y) - y\| = 0 \text{ for all } y \in Y.
\]

We first show that

\[
f \cdot \dim_{\Sigma}(q, \Gamma) \geq f \cdot \dim_{\Sigma}(S, \rho).
\]

For this, fix \( \varepsilon > 0 \), and choose \( r \in \mathbb{N} \) such that

\[
\rho(f) < \varepsilon, \text{ if } f \text{ is supported on } \{n : n \geq r\} \text{ and } \|f\|_{\infty} \leq 1,
\]

as before choose \( \kappa > 0 \) such that if \( \|f\|_{\infty} \leq \kappa \), then

\[
\rho(f) < \varepsilon.
\]

Let \( e \in E \subseteq \Gamma \) finite \( l \in \mathbb{N} \) be such that if \( E \subseteq F \subseteq \Gamma \) is finite, and \( k \geq l \) then

\[
\|\theta_{F,k}(a_{E,j}) - a_{E,j}\| < \kappa
\]

for \( 1 \leq j \leq r \).

Now fix \( E \subseteq F \subseteq \Gamma \) finite \( l \leq m \in \mathbb{N}, \delta > 0 \). We claim that we can find \( F' \subseteq F'' \subseteq \Gamma \) finite \( m \leq m' \in \mathbb{N}, \delta > \delta' > 0 \) such that

\[
\text{Hom}_{\Sigma}(S, F', m', \delta', \sigma_1) \circ q |_{Y_{F', m'}} \circ \theta_{F', m'} \subseteq \text{Hom}_{\Sigma}(F, F', m, \delta, \sigma_1)_{C^2}.
\]

For \( T \in \text{Hom}_{\Sigma}(S, F', m', \delta', \sigma_1) \), for \( 1 \leq j, k \leq m \) and \( s_1, \cdots, s_k \in F' \),

\[
\|T \circ q \circ \theta_{F', m'}(a_{s_1 \cdots s_k}) - \sigma_1(s_1) \cdots \sigma_1(s_k)T \circ q \circ \theta_{F', m'}(a_{E,j})\| \\
\leq C\|\theta_{F', m'}(a_{s_1 \cdots s_k}) - a_{s_1 \cdots s_k}\| + C\|\theta_{F', m'}(a_{E,j}) - a_{E,j}\| \\
+ \|T(s_1 \cdots s_k x_j) - \sigma_1(s_1) \cdots \sigma_1(s_k)T(x_j)\| \\
< C\|\theta_{F', m'}(a_{s_1 \cdots s_k}) - a_{s_1 \cdots s_k}\| + C\|\theta_{F', m'}(a_{E,j}) - a_{E,j}\| + \delta'.
\]

Also for \( y \in \ker(q) \cap Y_{F, m} \) we have

\[
\|T \circ q \circ \theta_{F', m'}(y)\| \leq C\|\theta_{F', m'}(y) - y\|.
\]

So it suffices to choose \( \delta' < \min(\delta, \kappa) \) and then \( F' \supseteq F; m' \geq \max(m, l, r) \) such that that

\[
C\|\theta_{F', m'}(a_{s_1 \cdots s_k}) - a_{s_1 \cdots s_k}\| + C\|\theta_{F', m'}(a_{E,j}) - a_{E,j}\| < \delta - \delta',
\]

\[
C\|\theta_{F', m'}|_{Y_{F, m}} - \text{Id} |_{Y_{F, m}}\| < \delta.
\]

for \( 1 \leq j, k \leq m \) and \( s_1, \cdots, s_k \in F \).
Suppose that \( \delta', F', m' \) are so chosen. If \( T \in \text{Hom}_\Gamma(S,F',m',\delta',\sigma_i) \) and \( \phi = T \circ q|_{Y_{F',m'}} \circ \theta_{F',m'} \) then,
\[
\rho_{V_i}(\alpha S(T) - \alpha F(\phi)) \leq (C^2 + 1)\varepsilon + \rho_{V_i}(\chi_{\{j:j \leq r\}}(\alpha S(T) - \alpha F(\phi))
\]
and if \( j \leq r \),
\[
\|\alpha S(T)(j) - \alpha F(\phi)(j)\| = \|T(x_j) - T \circ q \circ \theta_{F',l}(a_{c_j})\| \leq C\kappa + \|T(x_j) - T \circ q(a_{c_j})\| = C\kappa.
\]
Thus
\[
\rho_{V_i}(\alpha S(T) - \alpha F(\phi)) \leq (C^2 + C + 1)\varepsilon.
\]
Therefore
\[
d_i(C^2 + C + 2)\varepsilon(\text{Hom}_\Gamma(S,F',m',\delta',\sigma_i),\rho) \leq d_i(\text{Hom}_\Gamma(F,F,m,\delta,\sigma_i)_{2C^2},\rho).
\]
Since \( F', m' \) can be made arbitrary large and \( \delta' \) arbitrarily small, this implies
\[
f. \dim_{2\varepsilon}(S,\rho,\varepsilon) \leq \limsup_i \frac{1}{\dim V_i} \text{d}_i(C^2 + C + 2)\varepsilon(\text{Hom}_\Gamma(F,F,m,\delta,\sigma_i)_{2C^2},\rho),
\]
taking the limit supremum over \( (F, m, \delta) \) and then the supremum over \( \varepsilon > 0 \),
\[
f. \dim_{2\varepsilon}(S,\rho) \leq f. \dim_{\varepsilon}(q,\Gamma).
\]
For the opposite inequality, fix \( 1 > \varepsilon > 0 \) and let \( r, \kappa, E, l \) be as before. Fix \( E \subseteq F \subseteq \Gamma \) finite, \( m \geq \max(r, l) \) and \( \delta < \min(\kappa, \varepsilon) \).

By Lemma 3 we can find \( \delta' < \delta \), and \( F \subseteq F' \subseteq \Gamma \) finite and \( m \leq m' \in \mathbb{N} \) such that if \( W \) is a Banach space and
\[
T : Y_{F',m'} \to W
\]
has
\[
\|T\| \leq 1,
\]
\[
\|T|_{\ker(q) \cap Y_{F',m'}}\| \leq \delta',
\]
then there is a \( \phi : X_{F,m} \to W \) such that
\[
\|T(a_{s_1 \cdots s_k}) - \phi(s_1 \cdots s_k x_j)\| \leq \delta, \text{ for } 1 \leq j,k \leq m, s_1, \cdots, s_k \in F
\]
and \( \|\phi\| \leq 2C \).

Fix \( T \in \text{Hom}_\Gamma(F,F',m',\delta',\sigma_i) \), and choose \( \phi : X_{F,m} \to V_i \) such that \( \|\phi\| \leq 2C \) and
\[
\|T(a_{s_1 \cdots s_k}) - \phi \circ q(a_{s_1 \cdots s_k})\| \leq \delta, \text{ for } 1 \leq j,k \leq m, s_1, \cdots, s_k \in F.
\]
Thus for \( 1 \leq j,k \leq m \) and \( s_1, \cdots, s_k \in F \) we have
\[
\|\phi(s_1 \cdots s_k x_j) - \sigma_i(s_1) \cdots \sigma_i(s_k) \phi(x_j)\| \leq 2\delta
\]
\[
+ \|T(a_{s_1 \cdots s_k}) - \sigma_i(s_1) \cdots \sigma_i(s_k) T(a_{c_j})\|
\]
\[
< 2\delta + \delta'
\]
\[
< 3\delta.
\]
Thus \( \phi \in \text{Hom}_\Gamma(S,F,m,3\delta,\sigma_i)_{2C} \). Further, for \( 1 \leq j \leq k \)
\[
\|\alpha S(T)(j) - \alpha F(\phi)(j)\| = \|T(a_{c_j}) - \phi \circ q(a_{c_j})\| \leq \kappa,
\]
so
\[
\rho_{V_i}(\alpha S(T) - \alpha F(\phi)) \leq \varepsilon + (2C^2 + C)\varepsilon = (2C^2 + C + 1)\varepsilon.
\]
Thus
\[ f \cdot \dim_\Sigma(\mathcal{F},(2C^2+C+2)\varepsilon,\rho) \leq \lim_{i} \sup \frac{1}{\dim V_i} \hat{d}_{\varepsilon}(\Hom_{\Gamma}(S,F,m,(2C+1)\delta,\sigma_i)_{2C},\rho), \]
and since \( F,m,\delta,\varepsilon \) are arbitrary this completes the proof. \(\square\)

Because of the preceding Lemma and Corollary 3.7, we know that
\[ f \cdot \dim_\Sigma(S,\rho), \dim_\Sigma(S,\rho) \]
only depend upon the action of \( \Gamma \) on \( X \), and are equal. Because of this we will use
\[ \dim_\Sigma(X,\Gamma) = f \cdot \dim_\Sigma(S,\rho) = \dim_\Sigma(q,\Gamma) \]
for any dynamically generating sequence \( S \), and any bounded linear surjective map \( q: Y \to X \), where \( Y \) has the bounded approximation property. We similarly define \( \dim_\Sigma(X,\Gamma) \).

4. Main Properties of \( \dim_\Sigma(X,\Gamma) \)

The first property that we prove is that dimension is decreasing under surjective maps, as in the usual case of finite-dimensional vector spaces.

**Proposition 4.1.** Let \( (Y,\Gamma,\Sigma = (\sigma_i: \Gamma \to \text{Isom}(V_i))) \), \( (X,\Gamma,\Sigma) \) be two dimension tuples. Suppose that there is a \( \Gamma \)-equivariant bounded linear map \( T: Y \to X \), with dense image. Then
\[ \dim_\Sigma(X,\Gamma) \leq \dim_\Sigma(Y,\Gamma). \]
Proof. Let \( S' = (y_j)_{j=1}^{\infty} \) be a dynamically generating sequence for \( Y \). Let \( S = (T(x_j))_{j=1}^{\infty} \), then \( S \) is dynamically generating for \( X \). Then
\[ \Hom_{\Gamma}(S,F,m,\delta,\sigma_i) \circ T \subseteq \Hom_{\Gamma}(S',F,m,\delta,\sigma_i), \]
and
\[ \alpha_{S'}(\phi \circ T) = \alpha_{S}(\phi), \]
so the proposition follows. \(\square\)

We next show that dimension is subadditive under exact sequences. It turns out to be strong of a condition to require that dimension be additive under exact sequences. As noted \cite{1} if \( \dim_{\Sigma,lp} \) is additive under exact sequences and
\[ \dim_{\Sigma,lp}(l^p(\Gamma)^{\oplus n},\Gamma) = n, \]
then we can write the Euler characteristic of a group as an alternating sum of dimensions of \( l^p \) cohomology spaces. But torsion-free cocompact lattices in \( SO(4,1) \) have positive Euler characteristic and their \( l^p \) cohomology vanishes when \( p \) is sufficiently large, so this would give a contradiction.

**Proposition 4.2.** Let \( (V,\Gamma,\Sigma = (\sigma_i: \Gamma \to \text{Isom}(V_i))) \) be a dimension triple. Let \( W \subseteq V \) be a closed \( \Gamma \)-invariant subspace. Then
\[ \dim_\Sigma(V,\Gamma) \leq \dim_\Sigma(V/W,\Gamma) + \dim_\Sigma(W,\Gamma), \]
\[ \dim_{\Sigma,lp}(V,\Gamma) \leq \dim_{\Sigma,lp}(V/W,\Gamma) + \dim_{\Sigma,lp}(W,\Gamma), \]
\[ \dim_{\Sigma,lp}(V^{\oplus n},\Gamma) \leq n \dim_{\Sigma,lp}(V,\Gamma). \]
Proof. Let \( S_2 = (w_j)_{j=1}^\infty \) be a dynamically generating sequence for \( W \), and let 
\( S_1 = (a_j)_{j=1}^\infty \) be a dynamically generating sequence for \( V/W \). Let \( x_j \in V \), be such that \( x_j + W = a_j \), and \( \|x_j\| \leq 2\|a_j\| \). Let \( S \) be the sequence 
\[ x_1, w_1, x_2, w_2, \ldots. \]
we shall use the pseudornoms
\[ \|T\|_{S_1,i} = \sum_{j=1}^\infty \frac{1}{2^j} \|T(w_j)\|, \]
\[ \|T\|_{S_2,i} = \sum_{j=1}^\infty \frac{1}{2^j} \|T(a_j)\|, \]
\[ \|T\|_S = \sum_{j=1}^\infty \frac{1}{2^j} \|T(w_j)\| + \sum_{j=1}^\infty \frac{1}{2^j} \|T(x_j)\|. \]

Let \( \varepsilon > 0 \), and choose \( m \) such that \( 2^{-m} < \varepsilon \). Let \( e \in F_1 \subseteq \Gamma \) be finite, \( m \leq m_1 \in \mathbb{N} \), and \( \delta_1 > 0 \). Let \( \eta > 0 \) to be determined later. By Lemma 3.3, we can find a \( \delta_1 > \delta > 0 \), a \( F_1 \subseteq E \subseteq \Gamma \) finite, and a \( m \leq k \in \mathbb{N} \), so that if \( X \) is a Banach space, and 
\[ T : V_{E,2k} \to X \]
has \( \|T\| \leq 2 \), and 
\[ \|T\|_{W \cap V_{E,k}} \leq \delta, \]
then there is a \( \phi : (V/W)_{F_1,m_1} \to X \) with \( \|\phi\| \leq 3 \), and 
\[ \|\phi(s_1 \cdots s_k a_j) - T(s_1 \cdots s_k x_j)\| < \delta_1, \]
for all \( 1 \leq j, k \leq m_1 \), and \( s_1, \ldots, s_k \in F_1 \).

By finite-dimensionality, we can find a finite set \( F' \supseteq E \subseteq \Gamma \) finite, and a \( m \leq k \in \mathbb{N} \), so that if \( X \) is a Banach space, and 
\[ T : V_{E,2k} \to X \]
has \( \|T\| \leq 2 \), and 
\[ \|T\|_{W \cap V_{E,k}} \leq \delta, \]
then there is a \( \phi : (V/W)_{F_1,m_1} \to X \) with \( \|\phi\| \leq 3 \), and 
\[ \|\phi(s_1 \cdots s_k a_j) - T(s_1 \cdots s_k x_j)\| < \delta_1, \]
for all \( 1 \leq j, k \leq m_1 \), and \( s_1, \ldots, s_k \in F' \), then 
\[ \|T\|_{W \cap V_{E,k}} \leq \delta. \]

Define 
\[ R : \text{Hom}_\Gamma(S, F', 2m', \delta', \sigma_i) \to \text{Hom}_\Gamma(S_2, F', m', \delta', \sigma_i) \]
by 
\[ R(T) = T|_{W_{F',m'}}. \]
Find 
\[ \Theta : \text{im}(R) \to \text{Hom}_\Gamma(S, F', 2m', \delta', \sigma_i) \]
so that \( R \circ \Theta = \text{Id} \).

Then 
\[ (T - \theta(R(T)))(s_1 \cdots s_k x_j) = 0, \]
for all \( 1 \leq j, k \leq m' \), and \( s_1, \ldots, s_k \in F' \). Thus by assumption, we can find a 
\[ \phi : (V/W)_{F_1,m_1} \to V_i, \]
so that \( \|\phi\| \leq 3 \), and 
\[ \|\phi(s_1 \cdots s_k a_j) - T(s_1 \cdots s_k x_j)\| < \delta_1, \]
for all $1 \leq j, k \leq m_1, s_1, \ldots, s_k \in F_1$, in particular,
\[ \|\phi(a_j) - T(w_j)\| < \delta_1, \]
for $1 \leq j \leq m$.

Thus whenever $1 \leq j, k \leq m_1, s_1, \ldots, s_k \in F_1$,
\[ \|\phi(s_1 \cdots s_k a_j) - \sigma_i(s_1) \cdots \sigma_i(s_k)\phi(a_j)\| < 2\delta_1 + \delta' < 3\delta_1. \]

Now suppose that
\[ \alpha_{S_j}(\text{Hom}_R(S_2, F_1, m_1, \delta_1, \sigma_i)) \subseteq \varepsilon G, \]
\[ \alpha_{S_j}(\text{Hom}_R(S_1, F, m, 3\delta_1, \sigma_i)) \subseteq \varepsilon F. \]

Let $E \subseteq l^\infty(\mathbb{N}, V_i)$ be the subspace consisting of all $h$ so that there is an $f \in F, g \in G$ so that
\[ h(2k) = f(k), h(2k - 1) = g(k). \]
Then $\text{dim}(E) \leq \text{dim}(F) + \text{dim}(G)$. It easy to see that
\[ \text{Hom}_R(S, F', m', \delta', \sigma_i) \subseteq 3\varepsilon + \delta_1 m E. \]

So if $\delta_1 m < \varepsilon$, we find that
\[ \text{Hom}_R(S, F_1, m_1, \delta', \sigma_i) \subseteq 4\varepsilon E. \]

From this the first two inequalities follow.

The last inequality is easier and its proof will only be sketched. Let $S = (x_j)_{j=1}^\infty$ be a dynamically generating sequence for $X$, and $y_j = x_q \otimes e_r$ if $j = nq + r$, with $1 \leq r \leq n$, and $x_q \otimes e_r$ is the element of $X^{\oplus n}$ which is zero in all coordinates except for the $r$th, where it is $x_q$. If $F \subseteq \Gamma$ is finite $m \in \mathbb{N}, \delta > 0$, then
\[ \text{Hom}_R(S, F, nm, \delta, \sigma_i) \subseteq \text{Hom}_R(S, F, m, \delta, \sigma_i)^{\oplus n}. \]

The rest of the proof proceeds as above.

We note here that subadditivity is not true for weakly exact sequences, that is sequences
\[ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0, \]
where $X \rightarrow Y$ is injective, $\text{im}(X) = \ker(Y \rightarrow Z)$, and the image of $Y$ is dense in $Z$.

In fact, using $\mathbb{F}_n$ for the free group on $n$ letters $a_1, \ldots, a_n$, we will show in Section \[ \text{that the map} \]
\[ \partial: l^1(\mathbb{F}_n)^{\oplus n} \rightarrow l^1(\mathbb{F}_n), \]
given by
\[ \partial(f_1, \cdots, f_n)(x) = \sum_{j=1}^n f_j(x) - \sum_{j=1}^n f_j(xa_j^{-1}) \]
has dense image and is injective. Since we will also show
\[ \text{dim}_{\Sigma, l^1}(l^1(\mathbb{F}_n)^{\oplus n}, \mathbb{F}_n) = \text{dim}_{\Sigma, l^1}(l^1(\mathbb{F}_n)^{\oplus n}, \mathbb{F}_n) = n, \]
\[ \text{dim}_{\Sigma, l^1}(l^1(\mathbb{F}_n), \mathbb{F}_n) = \text{dim}_{\Sigma, l^1}(l^1(\mathbb{F}_n), \mathbb{F}_n) = 1, \]
this gives a counterexample to subadditivity under weakly exact sequences. This also gives a counterexample to monotonicity under injective maps, though once should note in this case that the map defined above does not have closed image.

For $2 \leq p \leq \infty$, we have a lower bound for direct sums, whose proof requires a few more lemmas.
Lemma 4.3 (\[17\], Lemma 8.5). Let $H_1, H_2$ be Hilbert spaces and let $H = H_1 \oplus H_2$ and let $\Omega_j \subseteq H_j$ and suppose $C_1, C_2 > 0$ are such that $C_1 \leq \|\xi\| \leq C_2$, for all $\xi \in \Omega_j$. If $0 < \delta < C_1$, then
\[
\|Q\| \leq \sqrt{\frac{C_1 - \delta}{C_1}} \|\Omega_1\| + \sqrt{\frac{C_1 - \delta}{C_1}} \|\Omega_2\|.
\]

Proof. By scaling it is easy to see that we may assume $C_1 = C_2 = 1$. Let $P_i$ be the projection onto each $H_i$, and set $\Omega = (\Omega_1 \oplus 0) \cup (0 \oplus \Omega_2)$. Suppose that $V$ is a subspace such that $\Omega \subseteq \delta V$, and let $Q$ be the projection onto $V$ and $T = QP_1Q|_V$.

Define $\Omega'_1 = Q(\Omega_1 \oplus 0), \Omega'_2 = Q(0 \oplus \Omega_2)$.

For $\xi \in (\Omega_1 \oplus 0)$ we have
\[
\|Q\xi\| \leq \delta
\]
so
\[
\|Q\xi\|^2 \geq 1 - \delta^2
\]
thus
\[
\langle TP\xi, Q\xi \rangle = \langle QP_1 Q\xi, Q\xi \rangle \geq \|P_1 Q\xi\|^2 \geq (\|Q\xi\| - \|P_1(1 - Q)\xi\|)^2 \geq \sqrt{1 - \delta^2 - \delta^2}.
\]

So if $T = \int_0^1 t dE(t)$ we have with $\eta = Q\xi$
\[
(\sqrt{1 - \delta^2 - \delta^2})^2 \leq \left\langle \left(1 - \frac{1}{2} E([0, 1/2])\right) \eta, \eta \right\rangle \leq 1 - \frac{1}{2} \|E([0, 1/2])\eta\|^2.
\]

Thus
\[
\|E([0, 1/2])\eta\|^2 \leq 2(1 - (\sqrt{1 - \delta^2 - \delta}^2))
\]
i.e.
\[
\|\eta - E((1/2, 1])\eta\|^2 \leq 2(1 - (\sqrt{1 - \delta^2 - \delta}^2)).
\]

Thus
\[
\Omega'_1 \subseteq \sqrt{2(1 - (\sqrt{1 - \delta^2 - \delta}^2))} E((1/2, 1])V.
\]

Similarly, because $QP_2Q|_V = 1 - T$ we have
\[
\Omega'_2 \subseteq \sqrt{2(1 - (\sqrt{1 - \delta^2 - \delta}^2))} E([0, 1/2])V.
\]

For any projection $P'$ and any $x \in H$ we have $\|x - P'x\|^2 = \|x\|^2 - \|P'x\|^2$. So for all $\xi \in \Omega_1 \oplus 0$ we have since, since $QE((1/2, 1]) = E((1/2, 1])$ (and $E((1/2, 1])Q = E((1/2, 1])$) by taking adjoints that
\[
\|\xi - E((1/2, 1])Q\xi\|^2 = \|\xi - E((1/2, 1])\xi\|^2 = \|\xi\|^2 - \|E((1/2, 1])\xi\|^2 = \|\xi\|^2 - \|Q\xi\|^2 + \|Q\xi\|^2 - \|E((1/2, 1])\xi\|^2 = \|\xi - Q\xi\|^2 + \|Q\xi - E((1/2, 1])Q\xi\|^2 \leq \delta^2 + 2(1 - (\sqrt{1 - \delta^2 - \delta}^2)) < 5\delta.
\]

Thus with a similar proof for $\Omega_2$ we have
\[
\Omega_1 \oplus 0 \subseteq \sqrt{5\delta} E((1/2, 1])V
\]
\[
0 \oplus \Omega_2 \subseteq \sqrt{5\delta} E([0, 1/2])V
\]
since
\[
V = E([0, 1/2])V \oplus E((1/2, 1])V
\]
the desired claim follows.
Lemma 4.4. Let \((X, \Gamma, \Sigma)\) be a dimension triple. Let \(S\) be a dynamically generating sequence in \(X\), and \(\rho\) a product norm such that \(\rho(f) \leq \rho(g)\) if \(|f| \leq |g|\). Set
\[
\rho^{(N)}(f) = \rho(\chi_{j \leq N} f).
\]
Then
\[
f. \dim_{\Sigma}(S, \Gamma, \rho) = \lim_{N \to \infty} f. \dim_{\Sigma}(S, \Gamma, \rho^{(N)}),
\]
\[
f. \dim_{\Sigma}(S, \Gamma, \rho) = \lim_{N \to \infty} f. \dim_{\Sigma}(S, \rho^{(N)}).
\]

Proof. Let \(\Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i))\). Let \((x_j)_{j=1}^\infty, C = \sup_j \|x_j\|\).
Since \(\rho^{(N)} \leq \rho\), for any \(\varepsilon > 0\)
\[
f. \dim_{\Sigma}(S, \varepsilon, \rho^{(N)}) \leq f. \dim_{\Sigma}(S, \varepsilon, \rho) \leq f. \dim_{\Sigma}(S, \rho),
\]
thus
\[
\limsup_{N \to \infty} f. \dim_{\Sigma}(S, \rho^{(N)}) \leq f. \dim_{\Sigma}(S, \rho).
\]
For the opposite inequality, fix \(\varepsilon > 0\) and choose \(N\) such that \(\rho(f) < \varepsilon\) if \(f\) is supported on \(\{k : k \geq N\}\) and \(\|f\|_{\infty} \leq C\). Thus for \(T \in B(X, V_i)\) with \(\|T\| \leq 1\), and \(n \geq N\) we have
\[
|\rho_{V_i} (\alpha_S(T)) - \rho_{V_i}^{(n)} (\alpha_S(T))| = |\rho_{V_i} (\chi_{\{k > n\}} \alpha_S(T))| \leq \varepsilon.
\]
Thus for \(n \geq N\),
\[
f. \dim_{\Sigma}(S, 2\varepsilon, \rho) \leq f. \dim_{\Sigma}(S, \varepsilon, \rho^{(n)}) \leq f. \dim_{\Sigma}(S, \varepsilon, \rho^{(N)}) \leq f. \dim_{\Sigma}(S, \rho^{(N)}),
\]
so
\[
f. \dim_{\Sigma}(S, \varepsilon, \rho) \leq \liminf_{N \to \infty} f. \dim_{\Sigma}(S, \rho^{(N)}).
\]
\(\square\)

For the next lemma, we recall the notion of the volume ratio of a finite-dimensional Banach space. Let \(X\) be an \(n\)-dimensional real Banach space, which we will identify with \(\mathbb{R}^n\) with a certain norm. By an ellipsoid in \(\mathbb{R}^n\) we mean a set which is the unit ball for some Hilbert space norm on \(\mathbb{R}^n\). Let \(B \subseteq \mathbb{R}^n\) be the unit ball of \(X\). We define the volume ratio of \(B\), denoted \(\text{vr}(B)\) by
\[
\text{vr}(B) = \inf \left(\frac{\text{vol}(B)}{\text{vol}(D)}\right)^{1/n},
\]
where the infimum runs over all ellipsoids \(D \subseteq B\). It is known that for any unit ball \(B\) of a Banach space norm on \(\mathbb{R}^n\), there is an ellipsoid \(D^\text{max}\) such that \(D^\text{max} \subseteq B\), and \(D^\text{max}\) has the largest volume of all such ellipsoids. So we have
\[
\text{vr}(B) = \left(\frac{\text{vol}(B)}{\text{vol}(D^\text{max})}\right)^{1/n}.
\]
The main property we will need to know about volume ratio is the following theorem.

Theorem 4.5 (Theorem 6.1,[14]). Let \(B \subseteq \mathbb{R}^n\) be the unit ball for a norm \(\| \cdot \|\) on \(\mathbb{R}^n\). Let \(D \subseteq B\) be an ellipsoid. Set
\[
A = \left(\frac{\text{vol}(B)}{\text{vol}(D)}\right)^{1/n}.
\]
Let \(| \cdot |\) be a norm such that \(D\) is the unit ball of \((\mathbb{R}^n, | \cdot |)\), in particular \(| \cdot | \leq \| \cdot \|\). Then for all \(k = 1, \cdots, n-1\) there is a subspace \(F \subseteq \mathbb{R}^n\) such that \(\dim F = k\) and for every \(x \in F\)
\[(2) \quad |x| \leq (4\pi A)^{\frac{n-1}{n}} \|x\|.
\]
Further if we let \(G_{nk}\) be the Grassmanian manifold of \(k\)-dimensional subspaces of \(\mathbb{R}^n\), then
\[
P(\{F \in G_{nk} : \text{ for all } x \in F, \text{ equation } (2) \text{ holds}\}) > 1 - 2^{-n},
\]
for the unique \(O(n)\)-invariant probability measure on \(G_{nk}\).

What we will actually use is the following corollary.

**Corollary 4.6.** Let \(B \subseteq \mathbb{R}^n\) be the unit ball for a norm \(| \cdot |\) on \(\mathbb{R}^n\), and let \(B^o\) be its polar. Let \(D \subseteq B^o\) be an ellipsoid. Set
\[
A = \left( \frac{\text{vol}(B^o)}{\text{vol}(D)} \right)^{1/n}.
\]
Let \(| \cdot |\) be a norm such that \(D\) is the unit ball of \((\mathbb{R}^n, | \cdot |)\), in particular \(| \cdot | \leq \| \cdot \|\). Then for all \(k = 1, \cdots, n-1\) there is a subspace \(F \subseteq \mathbb{R}^n\) such that \(\dim F = k\) and for every \(x \in \mathbb{R}^n/F^\perp\)
\[(3) \quad \|x\| \|(\mathbb{R}^n/F^\perp, | \cdot |\) \leq (4\pi A)^{\frac{n-k}{n}} |x| ,
\]
where we use \(\| \cdot \|_{(\mathbb{R}^n/F^\perp, | \cdot |)}\) for the quotient norm induced by \(| \cdot |\) and similarly for \(| \cdot |\). Further,
\[
P(\{F \in G_{nk} : \text{ for all } x \in F, \text{ equation } (3) \text{ holds}\}) > 1 - 2^{-n}.
\]
**Proof.** This is precisely the dual of the above theorem. \(\square\)

Here is the main application of the above corollary to dimension theory.

**Theorem 4.7.** Let \(\Gamma\) be a countable group with a uniformly bounded action on separable Banach spaces \(X, Y\). Let \(\Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i))\) with \(\dim V_i < \infty\). Suppose that \(V_i\) is the complexification of a real Banach space \(V_i'\) such that
\[
\sup_{i} \text{vr}(V_i'^*) < \infty,
\]
and there are constants \(C_1, C_2 > 0\) so that
\[
C_1(\|x\|_{V_i'} + \|y\|_{V_i'}) \leq \|x + iy\| \leq C_2(\|x\|_{V_i'} + \|y\|_{V_i'}).
\]
Then the following inequalities hold,
\[
\dim_{\Sigma}(Y_1 \oplus Y_2, \Gamma) \geq \dim_{\Sigma}(Y_1, \Gamma) + \dim_{\Sigma_{V_i}}(Y_2, \Gamma),
\]
\[
\dim_{\Sigma}(Y_1 \oplus Y_2, \Gamma) \geq \dim_{\Sigma}(Y_1, \Gamma) + \dim_{\Sigma(V_i)}(Y_2, \Gamma),
\]
\[
\dim_{\Sigma}(Y_1^\otimes n, \Gamma) \geq n \dim_{\Sigma}(Y_1, \Gamma).
\]
**Proof.** We will do the proof for dim only, the proof of the other claims are the same. Let \(S = (x_n)_{n=1}^\infty, T = (y_n)_{n=1}^\infty\) be dynamically generating sequences, enumerate \(S \cup \{0\} \cup \{0\} \cup T\) by \(x_1, y_1, x_2, y_2, \cdots\), and fix integers \(k, m\). By Lemma 4.4 it suffices to show that for fixed \(m, k \in \mathbb{N}\), and for the pseudonorms \(\rho, \rho_1, \rho_2\) on \(\ell^\infty(\mathbb{N})\) given by
\[
\rho(f) = \left( \sum_{j=1}^{m+k} |f(j)|^2 \right)^{1/2},
\]
\[
\rho_1(f) = \left( \sum_{j=1}^{m} |f(j)|^2 \right)^{1/2},
\]
\[
\rho_2(f) = \left( \sum_{j=1}^{k} |f(j)|^2 \right)^{1/2},
\]
we have
\[
f \cdot \dim_{\mathbb{C}}(S \oplus 0 \oplus 0 \oplus T, \rho) \geq f \cdot \dim_{\mathbb{C}}(S, \rho_1) + f \cdot \dim_{\mathbb{C}}(T, \rho_2).
\]

Fix \( \kappa, \varepsilon > 0 \) and fix \( \eta > 0 \) which will depend upon \( \kappa, \varepsilon \) in a manner to be determined later. By Corollary \[.6 \] there is a constant \( A \), which depends only on \( \kappa, C_1, C_2 \) Hilbert space norms \( |\cdot| \) on \( X_i \), and finite dimensional complex subspaces \( F_i \subseteq V_i^* \) of complex dimension \( (1 - \kappa)(\dim V_i) \) such that
\[
\frac{1}{A} |x| \leq \|x\| \leq A |x|
\]
for all \( x \in V_i/F_i^\perp \). Here, as in the Corollary \[.6 \] we abuse notation by using \( \|x\| \) for the norm on \( X_i/F_i^\perp \) induced by \( \|\cdot\| \), and similarly for \( |\cdot| \).

For \( m' \geq m \in \mathbb{N}, \delta > 0 \) and \( F \subseteq \Gamma \) finite we have
\[
\text{Hom}_{\Gamma}(S, F, 2m', \delta, \sigma_i) \oplus \text{Hom}_{\Gamma}(T, F, 2m', \delta, \sigma_i) \subseteq \text{Hom}_{\Gamma}((S \oplus \{0\}) \cup \{0\} \oplus T, F, m', 2\delta, \sigma_i).
\]
Thus
\[
\tilde{d}_\eta \left( \text{Hom}_{\Gamma}((S \oplus \{0\}) \cup \{0\} \oplus T, F, 2m', 2\delta, \sigma_i) \right) \geq \tilde{d}_\eta \left( \text{Hom}_{\Gamma}(T, F, 2m', \delta, \sigma_i) \right).
\]

Let
\[
K_1 = \{(T(x_1), \cdots, T(x_m)) : T \in \text{Hom}_{\Gamma}(S, F, 2m', \delta, \sigma_i)\}
\]
\[
K_2 = \{(S(y_1), \cdots, S(y_k)) : S \in \text{Hom}_{\Gamma}(S, F, 2m', \delta, \sigma_i)\}.
\]
Then, by definition,
\[
\tilde{d}_\eta \left( \text{Hom}_{\Gamma}(S, F, 2m', \delta, \sigma_i) \oplus \text{Hom}_{\Gamma}(T, F, 2m', \delta, \sigma_i) \right) =
\]
\[
d_\eta \left( K_1 \oplus K_2, \| \cdot \|_{\oplus m} \oplus \| \cdot \|_{\oplus k} \right)
\]
where we use the \( l^2 \)-direct sum.

Let \( \pi_i : V_i \to V_i/F_i^\perp \) be the quotient map and let
\[
G_i = \pi_{i,\mathbb{E}}(K_i),
\]
where \( l = m \) if \( i = 1 \), and \( l = k \) if \( i = 2 \).

Then
\[
d_\eta \left( K_1 \oplus K_2, \| \cdot \|_{\oplus m} \oplus \| \cdot \|_{\oplus k} \right) \geq d_\eta \left( G_1 \oplus G_2, \| \cdot \|_{\oplus m} \oplus \| \cdot \|_{\oplus k} \right) \geq
d_{\Lambda_{\mathbb{E}}}(G_1 \oplus G_2, \| \cdot \|_{\oplus m} \oplus \| \cdot \|_{\oplus k}).
\]
Set
\[
B_i = \left\{ x \in G_i : \lambda A \geq |x| \geq A\frac{\varepsilon}{4} \right\},
\]
where \( l = m \) if \( i = 1 \), and \( l = k \) if \( i = 2 \).

Then
\[
d_{\Lambda_{\mathbb{E}}}(G_1 \oplus G_2, \| \cdot \|_{\oplus m} \oplus \| \cdot \|_{\oplus k}) \geq d_{\max(l,m)(\varepsilon/4)^{-1\sqrt{\rho\Lambda_{\mathbb{E}}}}} \left( B_1, \| \cdot \|_{\oplus m} \right)
\]
\[
+ d_{\max(l,m)(\varepsilon/4)^{-1\sqrt{\rho\Lambda_{\mathbb{E}}}}} \left( B_2, \| \cdot \|_{\oplus k} \right).
\]
Setting \( \eta = \epsilon^{5/3} \) we have
\[
d_\epsilon(K_1 \oplus K_2, \| \cdot \|^{\oplus m} \oplus \| \cdot \|^{\oplus k}) \geq d_\epsilon(B_1, \| \cdot \|^{\oplus m}) + d_\epsilon(B_2, \| \cdot \|^{\oplus k}) \geq d_\epsilon(B_1, \| \cdot \|^{\oplus k}) + d_\epsilon(B_2, \| \cdot \|^{\oplus k}).
\]
Since \( B_i \supseteq \{ x \in C_i : \| x \| \geq \frac{\eta}{2} \} \) we have
\[
d_\epsilon(B_1, \| \cdot \|^{\oplus k}) + d_\epsilon(B_2, \| \cdot \|^{\oplus k}) = d_\epsilon(G_1, \| \cdot \|^{\oplus k}) + d_\epsilon(G_2, \| \cdot \|^{\oplus k}).
\]
Let \( E_i \subseteq (V_i/F_i)_{\oplus l} \) be a linear subspace of minimal dimension which \( \epsilon \)-contains \( C_i \) with respect to \( \| \cdot \|^{\oplus l} \) (\( l = k \), if \( i = 1 \), and \( l = m \) if \( i = 2 \).) Let \( \tilde{E}_i \subseteq V_i \) be a linear subspace such that \( \dim E_i = \dim \tilde{E}_i \) and \( \pi_i^{\oplus l}(\tilde{E}_i) = E_i \). Let \( W_i = \tilde{E}_i + F_i^{\oplus l} \). Then \( W_i \) has dimension at most \( \dim E_i + l \epsilon_i \) with \( \lim_{\epsilon \to \infty} \frac{\epsilon}{\dim W_i} = \kappa \), since \( \dim V_i \to \infty \), and \( K_i \subseteq \epsilon_i \| V_i \). Thus
\[
d_\epsilon(G_1, \| \cdot \|^{\oplus l}) \geq d_\epsilon(K_i, \| \cdot \|^{\oplus l}) - l \epsilon_i.
\]
Since \( \epsilon \to 0 \) as \( \eta \to 0 \) (and vice versa) we conclude that
\[
\dim_\Sigma(S_1 \oplus S_2, \Gamma, \| \cdot \|_{s, \Gamma}) \geq -\kappa(k + m) + \dim_\Sigma(S_1, \Gamma, \| \cdot \|_{s, \Gamma}) + \dim_\Sigma(Y_2, \Gamma, \| \cdot \|_{r, \Gamma}).
\]
Since \( \kappa \) is arbitrary this proves the desired inequality.

**Corollary 4.8.** Let \( 2 \leq p < \infty \).

(a) Let \( \Gamma \) be a sofic group with uniformly bounded actions on separable Banach spaces \( X, Y \) and let \( \Sigma \) be a sofic approximation. Then
\[
\dim_{\Sigma, \Sigma_p}(X \oplus Y, \Gamma) \geq \dim_{\Sigma, \Sigma_p}(X, \Gamma) + \dim_{\Sigma, \Sigma_p}(Y, \Gamma)
\]

(b) Let \( \Gamma \) be an \( \mathcal{R}^\omega \)-embeddable group with uniformly bounded actions on separable Banach spaces \( X, Y \) and let \( \Sigma \) be an embedding sequence. Then
\[
\dim_{\Sigma, \Sigma_p}(X \oplus Y, \Gamma) \geq \dim_{\Sigma, \Sigma_p}(X, \Gamma) + \dim_{\Sigma, \Sigma_p}(Y, \Gamma).
\]

**Proof.** Let \( B_p \) be the unit ball of \( L^p(\{1, \ldots, n\}, \mu_n) \) where \( \mu_n \) is the uniform measure.

It is known that
\[
\sup_n \left( \frac{\text{vol}(B_p)}{\text{vol}(B_2)} \right)^{1/n} < \infty.
\]
Similarly if we let \( C_p \) be the unit ball of \( \{ A \in M_n(C) : A = A^* \} \) in the norm \( \| \cdot \|_{L^p(\frac{1}{n} T_\Gamma)} \), it is known that
\[
\sup_n \left( \frac{\text{vol}(C_p)}{\text{vol}(C_2)} \right)^{1/n^2} < \infty.
\]
Apply the preceding theorem.

We note one last property of \( l^2 \)-dimension for representations, which will only be used in section 6.
Proposition 4.9. Let $H$ be a separable unitary representation of a $\mathcal{R}^{\omega}$-embeddable group $\Gamma$. Let $\Sigma$ be a embedding sequence of $\Gamma$. Suppose that $H = \bigcup_{k=1}^{\infty} H_k$ with $H_k$ increasing, closed invariant subspaces, and that each $H_k$ has a finite dynamically generating sequence. Then
\[
\dim_{\Sigma,l^2}(H, \Gamma) = \sup_k \dim_{\Sigma,l^2}(H_k, \Gamma),
\]
\[
\dim_{\Sigma,l^2}(H, \Gamma) = \sup_k \dim_{\Sigma,l^2}(H_k, \Gamma).
\]

Proof. We will do the proof for $\dim$ only, the other cases are the same. By Proposition 4.2 we know that $\dim_{\Sigma,l^2}$ is monotone for unitary representations, so we only need to show
\[
\dim_{\Sigma,l^2}(H, \Gamma) \geq \sup_k \dim_{\Sigma,l^2}(H_k, \Gamma).
\]

Let $\{\xi_{1}^{(k)}, \ldots, \xi_{r_k}^{(k)}\}$ be unit vectors which dynamically generate $H_k$. Let $S_N$ be the sequence
\[
\xi_1^{(1)}, \ldots, \xi_{r_{1}}^{(1)}, \xi_1^{(2)}, \ldots, \xi_{r_{2}}^{(2)}, \ldots, \xi_1^{(N)}, \ldots, \xi_{r_{N}}^{(N)},
\]
i.e. the $l$th term of $S_N$ is
\[
\xi_{q_l}^{(i)}
\]
if $i$ is the largest integer such that
\[
C_i = \sum_{j \leq i} r_j < l,
\]
and
\[
q_l = l - \sum_{j \leq i} r_j.
\]

Let $S$ be the sequence obtained by the infinite concatenation of the $S_N$'s. We will use $S_N$ to compute $\dim_{\Sigma,l^2}(H_N, \Gamma)$ and $S$ to compute $\dim_{\Sigma,l^2}(H, \Gamma)$, we also use the pseudonorms
\[
\|T\|_{S,i} = \sum_{j=1}^{\infty} \frac{1}{2^j} \|T(\xi_j)\|
\]
and
\[
\|T\|_{S_N,i} = \sum_{j=1}^{\infty} \frac{1}{2^j} \|T(\xi_j)\|.
\]

Fix $\varepsilon > 0$, and let $M$ be such that $2^{-M} < \varepsilon$. Suppose $F \subseteq \Gamma$ is finite, $\delta > 0$ and $m \in \mathbb{N}$ with $m > C_M$. Let $P_M \in B(H)$ be the projection onto $H_M$. Suppose $V$ is a subspace of $B(H_M, \mathbb{C}^{d_i})$ of minimal dimension such that
\[
\text{Hom}_{\Gamma}(S_M, F, m, \delta, \sigma_i) \subseteq \varepsilon,\|\cdot\|_S, V,
\]
let $\bar{V} \subseteq B(H, \mathbb{C}^{d_i})$ be the image of $V$ under the map $T \rightarrow T \circ P_M$. If $T \in \text{Hom}_{\Gamma,l^2}(S, F, m, \delta, \sigma_i)$ then $\bar{T} = T\big|_{H_M}$ is in $\text{Hom}_{\Gamma}(S_M, F, m, \delta, \sigma_i)$, and there exists $\phi \in V$ such that $\|\phi - \bar{T}\|_{S_M,i} < \varepsilon$. Then
\[
\|\phi \circ P - T\|_{S,i} \leq 2 \sum_{n=C_M+1}^{\infty} \frac{1}{2^n} + \|\phi - \bar{T}\|_{S_M,i} \leq 2^{-m+1} + \varepsilon \leq 3\varepsilon.
\]
Thus
\[
\text{Hom}_{\Gamma}(S, F, m, \delta, \sigma_i) \subseteq 3\varepsilon,\|\cdot\|_S, \bar{V},
\]
The corollary is a simple consequence of the above proposition and Theorem 4.7.

**Corollary 4.10.** Let $\Gamma$ be a $\mathcal{R}^\omega$-embeddable group, and let $\Sigma = (\sigma_i : \Gamma \to U(d_i))$ be an embedding sequence. Let $\pi_k : \Gamma \to U(H_k)$ be a representations of $\Gamma$ such that each $\pi_k$ has a finite dynamically generating sequence. Then

$$\dim_{\Sigma,2} \left( \bigoplus_{k=1}^{\infty} \pi_k \right) \leq \sum_{k=1}^{\infty} \dim_{\Sigma,2}(\pi_k)$$

and similarly for $\dim$. Taking the supremum over $\varepsilon > 0$ completes the proof. \(\square\)

**Proof.** The corollary is a simple consequence of the above proposition and Theorem 4.7. \(\square\)

5. **Computation of $\dim_{\Sigma,2}(l^p(\Gamma, V), \Gamma)$, and $\dim_{\Sigma,SP,conj}(l^p(\Gamma, V), \Gamma)$.

In this section we show that if $\Sigma$ is a sofic approximation of $\Gamma$ and $1 \leq p \leq 2$, then

$$\dim_{\Sigma,2}(l^p(\Gamma, V), \Gamma) = \dim V,$$

for $V$ finite dimensional. Similarly if $\Sigma$ is a embedding sequence of $\Gamma$ and $1 \leq p \leq 2$, we show that

$$\dim_{\Sigma,SP,conj}(l^p(\Gamma, V), \Gamma) = \dim V,$$

$$\dim_{\Sigma,2}(L^2(\Gamma, l^2(n)), \Gamma) = n,$$

again for $V$ finite dimensional.

The proof for sofic groups will be relatively simple, but the proof for $\mathcal{R}^\omega$-embeddable groups requires a few more lemmas.

**Lemma 5.1.** Let $A \subseteq \mathbb{N}$, and $\varepsilon > 0$, let $\delta_a : \mathbb{N} \to \mathbb{C}$ be the function which is one on $a$ and zero elsewhere. Also, for a Hilbert space $H, \xi, \eta \in H$, let $\xi \otimes \overline{\eta}(\zeta) = (\xi, \eta)\zeta$. Then

$$d_\epsilon(\{\delta_a\}_{a \in A}, \| \cdot \|_p) \geq |A|(1 - \varepsilon^2)$$

for $1 \leq p \leq 2$, and $\{\delta_a\}_{a \in A}$ regarded as a subset of $l^p(\mathbb{N})$, and

$$d_\epsilon(\{e_a \otimes \overline{e_b}\}_{(a,b) \in A^2}, \| \cdot \|_{L^p(B(l^2(\mathbb{N})), Tr)}) \geq |A|^2(1 - \varepsilon^2)$$

for $1 \leq p \leq 2$, with $\{e_a \otimes \overline{e_b}\}_{(a,b) \in A^2}$ regarded as a subspace of $L^p(B(l^2(\mathbb{N})), Tr)$.

**Proof.** For $1 \leq p \leq 2$,

$$\| \cdot \|_{l^p(\mathbb{N})} \geq \| \cdot \|_{l^2(\mathbb{N})} \geq \| \cdot \|_{L^2(B(l^2(\mathbb{N})), Tr)},$$

thus it suffices to handle the case $p = 2$.

We now proceed as in Lemma 7.8. in [14]. Suppose that $e_1, \cdots, e_n$ is an orthonormal set in a Hilbert space $H$, and suppose that $V \subseteq H$ is a linear subspace such that

$$\{e_j\}_{j=1}^n \subseteq V.$$
Replacing $V$ with its image under the projection onto $\text{Span}\{e_1, \cdots, e_n\}$ we may assume that $V$ is contained in the span of $\{e_1, \cdots, e_n\}$. In this case let

$$Q: \text{Span}\{e_1, \cdots, e_n\} \to \text{Span}\{e_1, \cdots, e_n\}$$

be the projection onto $V$. Then

$$n - \dim V = \text{Tr}(\text{Id} - Q) = \sum_{j=1}^{n} \langle e_j - Qe_j, e_j \rangle = \sum_{j=1}^{n} \|e_j - Qe_j\|^2 \leq n\varepsilon^2.$$ 

Thus

$$\dim V \geq n(1 - \varepsilon^2).$$

Let $\nu$ be the unique $U(n)$ invariant measure on $S^{2n-1}$, for the next lemma we need that if $T: \mathbb{C}^n \to \mathbb{C}^n$ is linear, then

$$\frac{1}{n} \text{Tr}(T) = \int_{S^{2n-1}} \langle T\xi, \xi \rangle d\nu(\xi).$$

This follows from the fact that $\text{Tr}$ is, up to scaling, the unique linear functional on $M_n(\mathbb{C})$ invariant under conjugation by $U(n)$.

**Lemma 5.2.** Let $\Gamma$ be a $\mathcal{R}^\omega$-embeddable group, let $\sigma_i: \Gamma \to U(d_i)$ be an embedding sequence, and fix $E \subseteq \Gamma, m \in \mathbb{N}$ finite. For $j \in \{1, \cdots, m\}, \xi, \eta \in S^{2d_i-1}$ define

$$T_{\xi,j}: l^2(\Gamma \times \{1, \cdots, m\}) \to l^2(d_i),$$

$$T_{\xi,\eta,j}: l^p(\Gamma \times \{1, \cdots, m\}) \to S^p(d_i)$$

by

$$T_{\xi,j}(f) = \sum_{s \in E} f(s,j)\sigma_i(s)\xi,$$

$$T_{\xi,\eta,j}(f) = \sum_{s \in E} f(s,j)\sigma_i(s)\xi \otimes \sigma_i(s)\eta.$$ 

Then for any $\delta > 0$ and $1 \leq p < \infty$,

(a)

$$\lim_{i \to \infty} \mathbb{P}(\{\xi \in S^{2d_i-1} : ||T_{\xi,j} : l^2(\Gamma \times \{1, \cdots, m\}) \to l^2(d_i) || < 1 + \delta\}) = 1,$$

(b)

$$\{(\xi, \eta) \in (S^{2d_i-1})^2 : ||T_{\xi,\eta,j} : l^p(\Gamma \times \{1, \cdots, m\}) || < 1 + \delta\} \supseteq A_i \times A_i,$$

where $A_i \subseteq S^{2d_i-1}$ has $\nu(A_i) \to 1$.

**Proof.** Let $\kappa > 0$ which will depend upon $\delta > 0, p$ in a manner to be determined later. Let

$$A = \bigcap_{s \neq t, s, t \in E} \{\xi \in S^{2d_i-1} : |\sigma_i(s)\xi, \sigma_i(t)\xi| < \kappa\},$$

since

$$\int_{S^{2d_i-1}} \langle \sigma_i(s)\xi, \sigma_i(t)\xi \rangle d\nu(\xi) = \frac{1}{d_i} \text{Tr}(\sigma_i(t)^{-1}\sigma_i(s)) \to 0$$

for $s \neq t$, the concentration of measure phenomenon implies that

$$\nu(A) \to 1.$$
For the proof of (a), (b) we prove that if $\xi, \eta \in A$ then
\[
\|T_{\xi,j}(f)\|_2 \leq 1 + \delta,
\]
\[
\|T_{\xi,\eta,j}(f)\|_{l_2 \rightarrow \ell^p} \leq 1 + \delta,
\]
if $\kappa > 0$ is sufficiently small.

(a) For $f \in l^2(\Gamma \times \{1, \cdots, m\}), \xi \in A$ we have
\[
\|T_{\xi,j}(f)\|_2^2 = \sum_{s,t \in E} T_{\xi}(f(s,j))T_{\xi}(f(t,j))T_{\xi}(\sigma_i(s)\xi, \sigma_i(t)\xi)
\]
\[
\leq \|f\|_2^2 + \sum_{s,t \in E} \|f\|_2^2 \kappa
\]
\[
\leq \|f\|_2^2(1 + \kappa|E|^2)
\]
\[
< (1 + \delta)\|f\|_2^2
\]
if $\kappa < \frac{1}{2|E|^2}$.

(b) Fix $\varepsilon > 0$ to be determined later. If $\kappa$ is sufficiently small, then for any $(\xi, \eta) \in A^2$ we can find $(\xi_s)_{s \in E}(\eta_s)_{s \in E}$ such that $\langle \xi_s, \xi_t \rangle = \delta_{s \neq t}$, $\langle \eta_s, \eta_t \rangle = \delta_{s \neq t}$ and
\[
\|\xi_s - \sigma_i(s)\xi\| < \varepsilon, \|\eta_s - \sigma_i(s)\eta\| < \varepsilon.
\]
Then
\[
\left\|T_{\xi,\eta,j}(f) - \sum_{s \in E} f(s)\xi_s \otimes \eta_s \right\|_p \leq \|f\|_p \sum_{s \in E} \left(\|\xi_s - \sigma_i(s)\xi\| + \|\sigma_i(s)\eta - \eta_s\|\right) < 2|E|\varepsilon.
\]
Note that
\[
\sum_{s \in E} f(s)\xi_s \otimes \eta_s = \sum_{s, t \in E} f(s)f(t)\langle \xi_t, \xi_s \rangle \eta_s \otimes \eta_t
\]
\[
= \sum_{s \in E} |f(s)|^2 \eta_s \otimes \eta_s.
\]
Thus
\[
\left\|\sum_{s \in E} f(s)\xi_s \otimes \eta_s \right\|_p^p = \|f\chi_E\|_p^p \leq \|f\|_p^p.
\]
So if $\varepsilon < \frac{\delta}{2|E|}$ the claim follows. \qed
Proof. We always have
\[ \dim \leq \text{dim} \]
so we will need to get an upper bound for \( \dim \) and a lower bound for \( \text{dim} \).

(a) Let \( \Sigma = \sigma : \Gamma \to S_{d_i} \). We may assume \( V = l^p(n) \). We use the generating sequence \((e_1, \ldots, e_n)\) with \((\delta \otimes e_1, \ldots, \delta \otimes e_n)\) the standard orthonormal basis of \( l^p(n) \). We use the pseudonorm on \( B(l^p(\Gamma, l^p(n)), l^p(d_i)) \) given by
\[ \| \phi \|_{S,i} = \left( \sum_{j=1}^{n} \| \phi(e_j) \|_p^p \right)^{1/p}. \]

We have \( B(l^p(\Gamma, l^p(n)), l^p(d_i)) \) under this pseudonorm is isometric to \( l^p(n d_i) \) so we have
\[ \dim_{\Sigma,l^p}(l^p(\Gamma, l^p(n)), \Gamma) \leq n. \]

Fix \( F \subseteq \Gamma \) finite, \( m \in \mathbb{N} \), \( \delta > 0 \), and let \( E = (F \cup F^{-1} \cup \{ e \})^{2m+1} \). Let \( T_{jk} : l^p(\Gamma, l^p(n)) \to l^p(d_i) \), \( 1 \leq j \leq d_i, 1 \leq k \leq n \) be given by
\[ T_{jk}(f) = \sum_{s \in E} (f(s), e_k^\ast) \sigma_i(s) \delta_j = \sum_{s \in E} (f(s), e_k^\ast) \delta_{\sigma_i(s)(j)}. \]

We use \( e_k^\ast \) for \( e_k \) viewed as an element of \( l^{p'}(n) \) when \( 1/p + 1/p' = 1 \). Let \( A_i \) be the number of \((j, k)\) such that \( T_{jk} \in \text{Hom}_{\Gamma,l^p}(S, F, m, \delta, \sigma_i) \).

We will find a lower bound on the size of \( A_i \). Fix \( \eta > 0 \), now \((j, k)\) is in \( C_i \) if
\[ \| T_{jk} \| p \leq 1 \]
\[ \| T_{jk}(s_1 \cdots s_k e_t) - \sigma_i(s_1) \cdots \sigma_i(s_k) T(e_t) \| p < \delta \]
for all \( s_1, \ldots, s_k \in F, 1 \leq k \leq m, 1 \leq t \leq n \). Let \( B_i \) be the set of \((j, k)\) where the first inequality holds, and \( C_i \) the set of \((j, k)\) where the second inequality holds.

We have
\[ \| T_{jk}(f) \|_p^p = \sum_{r=1}^{d_i} \left| \sum_{s \in E : \sigma_i(s)(j) = r} \langle f(s), e_k^\ast \rangle \right|^p, \]
by soficity, for all large \( i \) and at least \((1 - \eta)d_i \) of the \( j \) we have \( \sigma_i(s)(j) \neq \sigma_i(t)(j) \) if \( s \neq t \) are both in \( E \). For such \( j \) the above sum is at most
\[ \sum_{s \in E} | f(s) | \leq \| f \|_p^p, \]
so \((j, k) \in B_i \) for such \( j \) and so \( |B_i| \geq (1 - \eta)nd_i \).

To estimate the size of \( C_i \), note that
\[ \| T_{jk}(s_1 \cdots s_k e_t) - \sigma_i(s_1) \cdots \sigma_i(s_k) T(e_t) \| p = \]
\[ \| \delta_{\sigma_i(s_1 s_2 \cdots s_k)(j)} \delta_{t=1}^k - \delta_{\sigma_i(s_1 s_2 \cdots s_k)(j)} \delta_{t=k} \|. \]

By soficity
\[ \sigma_i(s_1 s_2 \cdots s_k)(j) = \sigma_i(s_1 \cdots s_k)(j) \]
for all large \( i \), all \( 1 \leq k \leq m, s_1, \ldots, s_k \in F \) and at least \((1 - \eta)d_i \) of the \( j \). Thus \(|C_i| \geq (1 - \eta)d_i n \).

Finally soficity guarantees that for all large \( i \), at least \((1 - \eta)d_i \) of the \( j \) have \( \sigma_i(e)(j) = j \), thus
\[ | \{(j, k) \in A_i : \sigma_i(e)(j) = j \} | \geq (1 - 3\eta)d_i n. \]
Therefore by Lemma 5.1 we have
\[
\dim_{\Sigma,i,p}(l^p(\Gamma,l^p(n)),\Gamma) \geq (1-3\eta)n
\]
for all \(\eta > 0\). Letting \(\eta \to 0\) gives the result.

(b) Let \(\Sigma = (\sigma_i: \Gamma \to U(d_i))\). We will do the case of \(l^2\) dimension first, we use the same pseudonorm as in (a), again the upper bound for dimension is easy.

For the other inequality, fix \(F \subset \Gamma\) finite \(m \in \mathbb{N}, \delta, \eta > 0\) and let
\[
E = [F \cup F^{-1} \cup \{e\}]^{2m+1},
\]
let \(T_{\xi,j}\) be defined as in Lemma 5.2 for this finite set \(E\). Then by Lemma 5.2 and the integration formula
\[
\nu(A) = \int_{U(d_i)} \frac{|\{1 \leq j \leq d_i : Ue_j \in A\}|}{d_i} dU, A \subset S^{2d_i-1},
\]
for all large \(i\) we can find an orthonormal sequence \(\xi_1, \ldots, \xi_t\) in \(l^2(d_i)\) with \(t \geq (1-\eta)d_i\) such that \(T_{\xi,j} = T_{\xi_t,j}\)
\[
\|T_{\xi,j}\|_{l^2 \to l^2} \leq 2,
\]
for \(1 \leq r \leq t\). Extend to an orthonormal basis \(\xi_1, \ldots, \xi_{d_i}\) for \(l^2(d_i)\).

Let \(C_i\) be the set of \((r,j)\) such that \(r \leq t\) and
\[
\|T_{\xi,j}(s_1 \cdots s_k e_l) - \sigma_i(s_1) \cdots \sigma_i(s_k) T_{\xi,j}(e_l)\| < \delta
\]
for all \(s_1, \ldots, s_k \in F, 1 \leq k \leq m, 1 \leq l \leq n\). Since
\[
\|T_{\xi,j}(s_1 \cdots s_k e_l) - \sigma_i(s_1) \cdots \sigma_i(s_k) T_{\xi,j}(e_l)\| = \|\sigma_i(s_1 s_2 \cdots s_k) \delta_i=\xi_j - \sigma_i(s_1) \cdots \sigma_i(s_k) \delta_i=\xi_j\|
\]
and
\[
\|\sigma_i(s_1 s_2 \cdots s_k) - \sigma_i(s_1) \cdots \sigma_i(s_k)\|^2 = \frac{1}{d_i} \sum_{j=1}^{d_i} \|\sigma_i(s_1 s_2 \cdots s_k) \xi_j - \sigma_i(s_1) \cdots \sigma_i(s_k) \xi_j\|^2,
\]
we see that for all large \(i\), at least \((1-\eta)d_in\) of the \((j,r)\) are in \(C_i\).

Now let
\[
A_i = \{(r,j) \in C_i : r \leq t, \|\sigma_i(e)\xi_j - \xi_j\| < \varepsilon\}
\]
since
\[
\|\sigma_i(e) - \Id\|^2 = \frac{1}{d_i} \sum_{j=1}^{d_i} \|\sigma_i(e)\xi_j - \xi_j\|^2
\]
we have \(|A_i| \geq (1-3\eta)n d_i\) for all large \(i\). Since \(T_{\xi,j}(e_l) = (\sigma_i(e))_{\delta_r=\xi_j}\),
\[
d_c(\text{Hom}_{\Gamma,l^2}(S,F,m,\delta,\sigma)_2,\|\cdot\|_{S,i}) \geq d_c(\{\xi_j \otimes e_k : (j,k) \in A_i\}) \geq (1-3\eta)d_in(1-\varepsilon^2).
\]
This implies
\[
\dim_{\Sigma,i,p}(l^2(\Gamma,l^2(n))) \geq n.
\]
Which proves the first half of (b).

We now turn to the second half of (b). Again fix \(\eta > 0\). We will use the same generating sequence as above and the pseudonorm
\[
\|\phi\|_{S,i} = \left(\sum_{j=1}^{n} \|\phi(e_j)\|^p\right)^{1/p},
\]
we may also assume that \( V = l^p(n) \) for some \( n \). Fix \( F \subseteq \Gamma \), finite \( m \in \mathbb{N} \), \( \delta > 0 \) and let \( E = [F \cup F^{-1} \cup \{e\}]^{2m+1} \). For \( 1 \leq k \leq n, \xi, \eta \in S_{2^d,-1} \) let \( T_{\xi,\eta,k} \) be as in Lemma 5.2.

By Lemma 5.2 for all large \( i \) we may find an orthonormal sequence \( \xi_1, \cdots, \xi_t \) with \( t \geq (1 - \eta)d_i \) such that \( \| T_{\xi,\xi_d} \|_{\ell^p \to \ell^p} \leq 2 \), for \( q, r \leq t \), set \( T_{k\ell r} = T_{\xi_k,\xi_r} \).

Let \( C_i \) be the set of all \((k, l, r)\) such that for all \( s_1, \cdots, s_q \in F, 1 \leq q \leq m \)

\[
\| T_{k\ell r}(\sigma_i(s_1) \cdots \sigma_i(s_q)\xi_k \otimes \xi_r - \sigma_i(s_1 \cdots s_q))\xi_k \otimes \xi_r \|_{\ell^p(M_n, C_i, T\epsilon)} < \delta.
\]

From the equality

\[
\| \xi \otimes \xi \|_{\ell^p} = \| \xi \| \| \xi \|,
\]

it follows as in the first half of (b) that \( |C_i| \geq (1 - \eta)nd_i^2 \). Similarly, if

\[
A_i = \{(k, l, r) \in C_i \cap B_i : \| \sigma_i(e) \xi_k \otimes \xi_l - \xi_k \otimes \xi_l \| < \epsilon \}
\]

it follows as in the first half of (b) that \( |A_i| \geq (1 - 3\eta)nd_i^2 \). Now suppose \( V \subseteq B(l^p(\Gamma, l^p(n)), S^p(d_i)) \) is such that

\[
\text{Hom}_\Gamma(S, F, m, \delta, \sigma_i) \subseteq \epsilon, \| s, i \|, V,
\]

and let \( \tilde{V} = \{(T(\delta_i \otimes e_1), \cdots, T(\delta_i \otimes e_n) : T \in V\} \), as before by Lemma 5.1 we have

\[
d_i(\tilde{V}, \| s, i \|) \geq nd_i(1 - 3\eta)(1 - \epsilon^2).
\]

So

\[
\dim_{C_i, \ell^p, \text{multi}}(l^p(\Gamma, l^p(n)), \Gamma) \geq n(1 - 3\eta)
\]

for all \( \eta > 0 \). Letting \( \eta \rightarrow 0 \) completes the proof.

\[\square\]

**Corollary 5.4.** Let \( \Gamma \) be a \( \mathbb{R}^2 \)-embeddable group \( 1 \leq p \leq 2 \). If \( V, W \) are finite dimensional vector spaces with \( \dim V < \dim W \), then there are no \( \Gamma \)-equivariant bounded linear maps from \( l^p(\Gamma, V) \) to \( l^p(\Gamma, W) \) with dense image. Consequently if \( 2 \leq p < \infty \), then there are no \( \Gamma \)-equivariant bounded linear injections from \( l^p(\Gamma, V) \) to \( l^p(\Gamma, W) \).

**Proof.** For \( 1 \leq p \leq 2 \) this is immediate from the above theorem and Proposition 4.1. The other results follow by duality. \[\square\]

**Corollary 5.5.** Let \( \Gamma \) be a countable \( \mathbb{R}^2 \)-embeddable all of whose nontrivial conjugacy classes are infinite. Let \( \pi : \Gamma \rightarrow U(H) \) be representation such that \( \pi \leq \lambda^{\oplus \infty} \). Then for every embedding sequence \( \Sigma \),

\[
\dim_{\Sigma, \ell^p}(\pi) = \dim_{\Sigma, \ell^p}(\pi) = \dim_{L(\Gamma)} \pi.
\]

**Proof.** We have already done the proof in the case \( \pi = \lambda^{\oplus n} \). First suppose that \( \dim_{L(\Gamma)} \pi = \frac{m}{n} \), with \( m, n \in \mathbb{Z} \), \( m \geq 0 \), \( n > 0 \). Then because \( L(\Gamma) \) is a factor, (since \( \Gamma \) is an infinite conjugacy class group) we have

\[
\pi^{\oplus n} \cong \lambda^{\oplus m}.
\]

Thus by Proposition 4.2 and Corollary 4.8

\[
m = \dim_{\Sigma, \ell^p}(\pi^{\oplus n}) \leq nd \dim_{\Sigma, \ell^p}(\pi),
\]

\[
m = \dim_{\Sigma, \ell^p}(\pi^{\oplus n}) \geq n \dim_{\Sigma, \ell^p}(\pi).
\]

This proves the case when \( \dim_{L(\Gamma)} \pi \) is rational. The case \( \dim_{L(\Gamma)} \pi < \infty \) now follows because \( \dim_{\Sigma, \ell^p}, \dim_{\Sigma, \ell^p} \) are monotone by Proposition 4.2. The case \( \dim_{L(\Gamma)} \pi = \infty \) also follows by monotonicity.
We will remove the infinite conjugacy class assumption in the next section, whose proof is a little more technical, and uses heavier operator algebraic machinery.

6. A Proof That $\dim_{\Sigma, 2}(\pi) = \dim_{L(\Gamma)}(\pi)$, for $\mathcal{R}^\omega$-Embeddable $\Gamma$.

The next few lemmas will use purely operator algebraic methods. By a tracial von Neumann algebra we will mean a pair $(M, \tau)$ with $M$ a von Neumann algebra and $\tau$ a normal tracial state on $M$.

**Lemma 6.1.** Let $(M, \tau_M), (N, \tau_N)$ be tracial von Neumann algebra with $\tau_N$ faithful. Let $A \subseteq M$ a weak* dense *-subalgebra containing the identity of $M$. Suppose that $\pi: A \to N$ is a *-homomorphism such that $\tau_N \circ \pi = \tau_M|_A$. Then there is a *-homomorphism $\rho: M \to N$ such that $\rho|_A = \pi$ and $\tau_N \circ \rho = \tau_M$.

**Proof.** Replacing $1_N$ with $\pi(1_M)$ we may assume that $\pi$ is unital. Replacing $N$ with the weak* closure of $\pi(A)$, we may assume that $\pi(A)$ is weak* dense in $N$. For all $x \in A$, $$\langle \pi(x)1, 1\rangle = \tau_N(\pi(x)) = \tau_M(x) = \langle x1, 1\rangle,$$

since $A$ is $\| \cdot \|_2$ dense in $M$, uniqueness of GNS representations imply that there is a unitary $$U: L^2(M, \tau_M) \to L^2(N, \tau_N)$$
such that if $\rho_\tau$ is the GNS rep corresponding to $\tau$, then $$U \rho_{\tau_N} U^* = \rho_{\tau_M}(\pi(x))$$
for all $x \in A$. As $x \to U \rho_{\tau_N}(x) U^*$ may be regarded as a map into $N$ (by faithfulness of $\tau_N$), we have extended our *-homomorphism to a trace-preserving *-homomorphism of $M$ into $N$. $\Box$

For the next corollary we recall the construction of tracial ultraproducts. If $(M_\omega, \tau_\omega)$ are a sequence of tracial von Neumann algebras and $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$ we have a new von Neumann algebra as follows. Set

$$\prod^\omega M_n = \{(x_n)_{n \in \mathbb{N}} : \sup_n \|x_n\| < \infty\}/\{(x_n)_{n \in \mathbb{N}} : \sup_n \|x_n\| < \infty, \lim_{n \to \omega} \tau_n(x_n^*x_n) = 0\}$$

$$\tau_\omega(x) = \lim_{n \to \omega} \tau_n(x_n), \text{ if } x_n \text{ is a representative of } x.$$ 

If $x_n$ is a representative of $x$ we shall write $x = (x_n)_{n \to \omega}$. It is a theorem that $\prod^\omega M_n$ is a von Neumann subalgebra of $B(H)$ where $H = L^2(\prod^\omega M_n, \tau_\omega)$. Whenever we write

$$\prod^\omega M_{k(n)}(\mathbb{C})$$

for a sequence of integers $k(n)$, we always take the unique tracial state on $M_{k(n)}(\mathbb{C})$, i.e. $\frac{1}{k(n)} \text{Tr}$

**Corollary 6.2.** Let $\Gamma$ be a $\mathcal{R}^\omega$-embeddable group with $\Sigma$ a embedding sequence, and fix a free ultrafilter $\omega$ on $\mathbb{N}$. Then for the trace $\{\delta_e, \delta_e\}$ there is a unique trace-preserving embedding $$\pi: L(\Gamma) \to \prod^\omega M_{d_i}(\mathbb{C})$$
such that
\[ \pi(u_s) = (\sigma_i(s))_{i \to \omega} \]
for all \( s \in \Gamma \).

**Proof.** Let \( \Sigma = (\sigma_i : \Gamma \to U(d_i)) \). The hypothesis of \( \mathcal{R}^* \)-embeddablility implies that the sequence of maps
\[ \rho_i : \mathbb{C}[\Gamma] \to M_{d_i}(\mathbb{C}) \]
given by
\[ \rho_i \left( \sum_{s \in \Gamma} c_s u_s \right) = \sum_{s \in \Gamma} c_s \sigma_i(s) \]
is asymptotically trace-preserving and asymptotically a \(*\)-homomorphism. Thus the preceding lemma applies.

\[ \square \]

**Lemma 6.3.** Let \( \Gamma \) be a \( \mathcal{R}^* \)-embeddable group with embedding sequence \( \sigma_i : \Gamma \to U(d_i) \), and let \( R \subseteq L(\Gamma) \) be a \(*\)-subalgebra with a countable basis over \( \mathbb{C} \) and containing \( \mathbb{C}[\Gamma] \). Then there exists \( \rho_i : R \to M_{d_i}(\mathbb{C}) \) linear such that for all \( x, y \in R \),
\[ \|\rho_i(xy) - \rho_i(x)\rho_i(y)\|_2 \to 0, \]
\[ \left| \frac{1}{d_i} \text{Tr}(\rho_i(x)) - \text{Tr}(x) \right| \to 0, \]
\[ \|\rho_i(x^*) - \rho_i(x)^*\|_2 \to 0, \]
and
\[ \rho_i(u_s) = \sigma_i(s) \text{ for } s \in \Gamma, \]
\[ \sup_i \|\rho_i(x)\|_\infty < \infty \text{ for all } x \in R. \]

**Proof.** First some terminology. Let \( (X_\alpha)_{\alpha \in C} \) be variables. By a \(*\)-monomial we shall mean a finite formal product
\[ w = Y_1 \cdots Y_n \]
where each \( Y_i \) is some \( X_{\alpha_i} \) or some \( X_{\alpha_i}^* \). If \( R' \) is a \(*\)-algebra, \( \{y_\alpha\}_{\alpha \in C} \subseteq R' \), by \( w((x_\alpha)_{\alpha \in A}) \) we shall mean the element in \( R' \) obtained by replacing each \( Y_i \) in \( w \) with \( y_\alpha \), if \( Y_i = X_{\alpha_i} \), or replacing \( Y_i \) with \( y_\alpha^* \) if \( Y_i = X_{\alpha_i}^* \).

Let \( \{x_\alpha : a \in A\} \) with \( A \) countable be such that \( \{u_s : s \in \Gamma\} \cup \{x_\alpha : a \in A\} \) is a basis for \( R \) over \( \mathbb{C} \). If the claim is false, then there exists \( \varepsilon > 0 \), a finite set \( F \) of \(*\)-monomials in \( \{X_\alpha : \alpha \in A \cup \Gamma\} \) and a strictly increasing sequence \( k_j \) of integers such that for any function \( f : R \to M_{d_{k_j}}(\mathbb{C}) \), with
\[ f|_{\Gamma} = \sigma_i, \]
\[ \|f(x_\alpha)\|_\infty \leq \|x_\alpha\|_\infty, \]
there is some \( w \in F \) such that one of the following three inequalities hold
\[ \|f(w((u_s)_{s \in \Gamma}, (x_\alpha)_{a \in A})) - w(f((u_s)_{s \in \Gamma}, (x_\alpha)_{a \in A}))\|_2 \geq \varepsilon, \]
\[ \left| \frac{1}{d_{k_j}} \text{Tr}(w(f((u_s)_{s \in \Gamma}, (x_\alpha)_{a \in A}))) - \text{Tr}(w((u_s)_{s \in \Gamma}, (x_\alpha)_{a \in A})) \right| \geq \varepsilon, \]
\[ \|f((w((u_s)_{s \in \Gamma}, (x_\alpha)_{a \in A}))^* - w^*(f((u_s)_{s \in \Gamma}, (x_\alpha)_{a \in A}))\|_2 \geq \varepsilon. \]
Fix $\omega \in \mathbb{N} \setminus \mathbb{N}$, and let $M = \prod_{\omega} M_{d_{\omega}}(\mathbb{C})$. Corollary \ref{cor:trace-preserving-homomorphism} implies that there is a trace-preserving $*$-homomorphism

$$\rho: L(\Gamma) \to M,$$

such that for any finite $F \subseteq \Gamma$,

$$\rho \left( \sum_{a \in F} c_a u_a \right) = \left( \sum_{a \in F} c_a \sigma_i(s) \right)_{i \to \omega}.$$ 

For each $a \in A$, choose $\rho_i(x_a)$ such that $\rho(x_a) = (\rho_i(x_a))_{i \to \omega}$, $\|\rho_i(x_a)\|_{\infty} \leq \|x_a\|_{\infty}$, and define $\rho_i(u_a) = (\sigma_i(s))_{i \to \omega}$. Then, because ultrafilters are closed under intersection, we can find a $\omega$-large set of $i$ such that for all $w \in F$,

$$\|\rho(w((u_s)_{s \in \Gamma}, (x_a)_{a \in A})) - \rho_i(f((u_s)_{s \in \Gamma}, (x_a)_{a \in A}))\|_2 < \varepsilon,$$ 

$$\left| \frac{1}{d_k} w(\rho((u_s)_{s \in \Gamma}, (x_a)_{a \in A})) - \tau_i(w((u_s)_{s \in \Gamma}, (x_a)_{a \in A})) \right| < \varepsilon,$$ 

$$\left\| (\rho_i(w((u_s)_{s \in \Gamma}, (x_a)_{a \in A}))^* - w^*(\rho_i((u_s)_{s \in \Gamma}, (x_a)_{a \in A})) \right\|_2 < \varepsilon,$$

this is a contradiction. \hfill \qed

We will also need a generalization of Lemma \ref{lem:trace-preserving-homomorphism}.

**Lemma 6.4.** Let $H$ be a Hilbert space, and $\eta_1, \cdots, \eta_k$ an orthonormal system in $H$, and $V = \text{Span}\{\eta_j : 1 \leq j \leq k\}$. Let $P \in B(H)$ be a projection, and $P_V$ the projection onto $V$. Then

$$d_{\varepsilon}(\{P(\eta_1), \cdots, P(\eta_k)\}) \geq -k\varepsilon + \text{Tr}(P_V).$$

**Proof.** For a subspace $K \subseteq H$ we let $P_K$ be the projection onto $K$. Let $W$ be a subspace of minimal dimension which $\varepsilon$-contains $\{P(\eta_1), \cdots, P(\eta_k)\}$. Then

$$\text{Tr}(P_W) = \text{Tr}(P_WP) + \text{Tr}(P_W(1-P)) = \text{Tr}(PP_WP) + \text{Tr}((1-P)P_W(1-P)) \geq \text{Tr}(PP_W),$$

similarly

$$\text{Tr}(P_WP) \geq \text{Tr}(PP_WP)P_V = \sum_{j=1}^{k} \langle P_WP(\eta_j), P(\eta_j) \rangle$$

$$\geq -\varepsilon k + \sum_{j=1}^{k} \langle P(\eta_j), P(\eta_j) \rangle$$

$$= -\varepsilon k + \text{Tr}(P_V).$$ \hfill \qed

We are now ready to prove one direction of our desired equality, in the special case of a cyclic representation.

**Proposition 6.5.** Let $\Gamma$ be a $\mathcal{R}^\omega$-embeddable group, with embedding sequence $\Sigma$. Let $\pi: \Gamma \to U(H)$ be a cyclic representation contained in $\lambda^{\oplus \infty}$. Then

$$\dim_{C, \pi}(\pi) \geq \dim_{L(\Gamma)} \pi.$$
Proof. Since \( \pi \) is contained in \( \lambda^{\oplus \infty} \), it extends to a representation, still denoted \( \pi \), of \( L(\Gamma) \). Let \( \eta \in H \) be a cyclic unit vector for \( \Gamma \), then \( \langle \pi(\cdot)\eta, \eta \rangle \) is a normal state on \( L(\Gamma) \) and so equals \( \tau(h) \) for some \( h \in L^1(L(\Gamma), \sigma) \). If we let \( \xi = h^{1/2} \in L^2(L(\Gamma), \tau) = l^2(\Gamma) \) we have that

\[
\langle \pi(x)\eta, \eta \rangle = \langle x\xi, \xi \rangle
\]

for all \( x \in L(\Gamma) \). Thus uniqueness of GNS representations implies that \( \pi \) is isomorphic to the representation contained in \( l^2(\Gamma) \) with cyclic vector \( \xi \). Thus we will assume that \( \pi \) is this representation. Let \( p \in R(\Gamma) \) be the projection onto \( H = L(\Gamma)\xi \). Let \( \Sigma = (\sigma_i : \Gamma \to U(d_i)) \) and \( R \) be the \( * \)-subalgebra of \( R(\Gamma) \) generated by \( p \) and \( \mathbb{C}[\Gamma] \). By Lemma 6.3 we can find a sequence \( \rho_i : R \to M_{d_i}(\mathbb{C}) \) such that

\[
\sup_i \|\rho_i(x)\|_\infty < \infty, \text{ for all } x \in R,
\]

\[
\rho_i(v_s) = \sigma_i(s), \text{ with } v_s \text{ the canonical unitaries generating } R(\Gamma),
\]

\[
\|\rho_i(P(x_1, \cdots, x_n)) - P(\rho_i(x_1), \cdots, \rho_i(x_n))\|_2 \to 0,
\]

(we will not assume they are linear, since we will modify them later) for all \( * \)-polynomials \( P \) and all \( x_1, \cdots, x_n \) in \( R \) and,

\[
\frac{1}{d_i} \text{Tr}(\rho_i(x)) - \tau(x) \to 0, \text{ for all } x \in R.
\]

By functional calculus, we may modify \( \rho_i(p) \) a small amount and assume that \( \rho_i(p) \) is a projection for all \( p, \) set \( p_i = \rho_i(p) \). We shall use the standard abuse of notation and identify an element \( x \in R(\Gamma) \) with its image under the map \( x \to x\delta_e \), under this identification, \( p \) is a cyclic vector for \( L(\Gamma)\xi \). We will use \( S = \{p\} \) to do our computation of \( \dim_{l^2(\pi)} \). Fix \( \delta > \kappa > 0, F \subseteq \Gamma \) finite, \( m \in \mathbb{N}, \) and choose \( F' \subseteq \Gamma \) finite such that

\[
\left\| p - \sum_{s \in F'} p(s)u_s \right\|_2 < \kappa.
\]

Our proof of Theorem 6.3 shows that for any finite \( E, F_1 \subseteq \Gamma, \) \( \delta_1 > 0, m_1 \in \mathbb{N}, \) and for all large \( i, \) there is an orthonormal basis \( \eta_1, \cdots, \eta_d \) of \( l^2(d_i) \) such that \( T_j \in \text{Hom}_{l^2(d_i)}(\{\delta_e\}, S, F_1, m_1, \delta_1, \sigma_j) \) for at least \( (1 - \kappa)d_i \) of the \( j, \) where

\[
T_j f = \sum_{s \in E} f(s)\sigma(s)\eta_j.
\]

For \( s_1, \cdots, s_k \in \Gamma, \) the inequality

\[
\left\| T_j(s_1 \cdots s_k p) - s_1(s_1) \cdots s_1(s_k)T_j(p) \right\|_2 \leq 2\kappa + \sum_{s \in F'} |p(s)||T_j(\delta_{s_1} \cdots s_k) - \sigma_1(s_1) \cdots \sigma_1(s_k)T_j(\delta_{s_k})||_2,
\]

tells us that if \( \kappa \) is small, and we choose \( E, F_1, \Gamma, \delta_1, m_1 \) wisely, then when restricted to \( H, \) we have \( T_j \in \text{Hom}_\pi(\{p\}, F, m, \delta, \sigma_j) \) for all large \( i, \) and at least \( (1 - \kappa)d_i \) of the \( j. \)

Now note that for all \( j, \)

\[
2\kappa > \left\| T_j(p) - \sum_{s \in F'} p(s)T_j(u_s) \right\|_2 = \left\| T_j(p) - \sum_{s \in F'} p(s)\sigma(s)\eta_j \right\|_2.
\]
For all large \( i \) we have
\[
\kappa^2 > \left\| p_i - \sum_{s \in F'} p(s) \sigma_i(s) \right\|_2^2 = \frac{1}{d_i} \sum_{j=1}^{d_i} \left\| p_i \eta_j - \sum_{s \in F'} p(s) \sigma_i(s) \eta_j \right\|_2^2.
\]
Hence, for all large \( i \),
\[
\left\| p_i \eta_j - \sum_{s \in F'} p(s) \sigma_i(s) \eta_j \right\|_2 < \sqrt{\kappa},
\]
for at least \((1 - \kappa)d_i\) of the \( j \). For such \( j \), we have
\[
\left\| p_i \eta_j - T_j(p) \right\|_2 \leq 2\kappa + \sqrt{\kappa}.
\]
Let \( A_i \) be the set of \( j \) where \( T_j \in \text{Hom}_{\pi,\{\{p\}\}}(F, m, \delta, \sigma_i) \) and the above inequality holds. Then our estimates show that for an all large \( i \), we have
\[
|A_i| \geq (1 - 2\kappa)d_i.
\]
Now suppose that \( V \subseteq B(H, l^2(d_i)) \) is a minimal dimensional subspace such that
\[
\{T_j\}_{j \in B_i} \subseteq \epsilon, \| \cdot \|_{S,i,2}, V,
\]
and let \( \hat{V} = \{T(p) : T \in V\} \), then by definition we have
\[
\{T_j(p)\}_{j \in B_i} \subseteq \epsilon \hat{V}.
\]
Thus for all large \( i \) we have
\[
\{p_i \eta_j\}_{j \in B_i} \subseteq \epsilon + 2\kappa + \sqrt{\kappa} \hat{V}.
\]
So Lemma 6.4 tells us that for \( W = \text{Span}\{\eta_j : j \in B_i\} \),
\[
\frac{1}{d_i} \dim V \geq \frac{1}{d_i} \dim \hat{V} \geq -\epsilon - 2\kappa - \sqrt{\kappa} + \frac{1}{d_i} \text{Tr}(P_W p_i),
\]
since \( \dim W^\perp \leq d_i(2\kappa) \) we have
\[
\frac{1}{d_i} \text{Tr}(p_i) = \frac{1}{d_i} \text{Tr}(P_W p_i) + \frac{1}{d_i} \text{Tr}(P_W^\perp p_i) \leq \frac{1}{d_i} \text{Tr}(P_W p_i) + 2\kappa.
\]
Our estimates thus show that
\[
\text{opdim}_{\Sigma,2}(\{p\}, \epsilon, \| \cdot \|_{S,i,2}) \geq -\epsilon - 4\kappa - \sqrt{\kappa} + \lim_{i \to \infty} \frac{1}{d_i} \text{Tr}(p_i),
\]
for all \( \kappa > 0 \). By construction,
\[
\lim_{i \to \infty} \frac{1}{d_i} \text{Tr}(p_i) = \tau_{\Gamma}(p) = \text{dim}_{L(\Gamma)} \pi.
\]
This completes the proof.

We can bootstrap this proposition to a proof of the theorem.

**Theorem 6.6.** Let \( \Gamma \) be a \( \mathcal{R}^\omega \)-embeddable group, and \( \pi : \Gamma \to U(H) \) a representation, such that \( \pi \leq \lambda^{\oplus \infty} \). Then for every embedding sequence \( \Sigma \),
\[
\dim_{\Sigma,2}(\pi) = \text{dim}_{\Sigma,2}(\pi) = \text{dim}_{L(\Gamma)} \pi.
\]
Proof. We already know from Theorem [53] that
\[ \dim_{\Sigma, l^2} \lambda \oplus n = \dim_{\Sigma, l^2} \lambda \oplus n = n. \]

Let us first assume that \( \pi \) is cyclic. As in the proof of the above proposition we have that \( \pi \leq \lambda \) in this case. Let \( \pi' \) be a representation such that \( \lambda = \pi \oplus \pi' \), then by the above proposition we have
\[ 1 = \dim_{\Sigma, l^2} \lambda \geq \dim_{\Sigma, l^2} \pi + \dim_{\Sigma, l^2} \pi' \geq \dim_{L(\Gamma)} \pi + \dim_{L(\Gamma)} \pi' = 1. \]
Thus all the above inequalities must be equalities, in particular
\[ \dim_{\Sigma, l^2} \pi = \dim_{\Sigma, l^2} \pi = \dim_{L(\Gamma)} \pi. \]

In the general case, apply Zorn’s Lemma to write \( \pi = \bigoplus_{n=1}^{\infty} \pi_n \) with \( \pi_n \) cyclic. Then by Corollary 4.10
\[ \dim_{\Sigma, l^2} (\pi) \geq \sum_{n=1}^{\infty} \dim_{\Sigma, l^2} (\pi_n) = \sum_{n=1}^{\infty} \dim_{L(\Gamma)} \pi_n = \dim_{L(\Gamma)} \pi, \]
\[ \dim_{\Sigma, l^2} (\pi) \leq \sum_{n=1}^{\infty} \dim_{\Sigma, l^2} (\pi_n) = \sum_{n=1}^{\infty} \dim_{L(\Gamma)} \pi_n = \dim_{L(\Gamma)} \pi. \]
This completes the proof of the theorem.

\[ \square \]

7. Computation For Representations on Non-Commutative \( L^p \)-Spaces

In this section we compute the dimension of the action of a \( \mathcal{R}^\omega \)-embeddable group \( \Gamma \) on \( L^p(L(\Gamma), \tau) \)q with \( q \) a projection in \( L(\Gamma) \) and \( \tau \) the canonical group trace for \( 2 \leq p < \infty \). By a \( \ast \)-polynomial, we shall mean a finite sum of \( \ast \)-monomials (see Lemma 6.3).

Lemma 7.1. Let \( M \) be a von Neumann algebra and \( \tau: M \to \mathbb{C} \), a faithful normal tracial state. Let \( R \subseteq M \) be a weak\( \ast \)-dense \( \ast \)-subalgebra, and suppose that \( \rho_i: R \to M_d(\mathbb{C}) \) are functions such that
\[ \|\rho_i(P(x_1, \cdots, x_n)) - P(\rho_1(x_1), \cdots, \rho_i(x_n))\|_2 \to 0 \text{ for all } \ast\text{-polynomials } P, \]
\[ \left| \frac{1}{d_i} \text{Tr}(\rho_i(x)) - \tau(x) \right| \to 0, \text{ for all } x \in R, \]
\[ \sup_i \|\rho_i(x)\|_\infty < \infty. \]
Then for \( 1 \leq p < \infty \), and any \( K \subseteq R \), which is compact in the \( \| \cdot \|_p \)-topology we have
\[ \lim_{i \to \infty} \sup_{x \in K} \left| \|\rho_i(x)\|_{L^p(\frac{1}{d_i} \text{Tr})} - \|x\|_{L^p(M, \tau)} \right| = 0. \]

Proof. If the claim was false, we could find a \( \varepsilon > 0 \), an increasing sequence \( i_k \) of integers, \( x_k \in K \) such that
\[ \left| \|\rho_i(x_k)\|_{L^p(\frac{1}{d_i} \text{Tr})} - \|x_k\|_{L^p(M, \tau)} \right| \geq \varepsilon. \]
Let
\[ N = \prod_{\omega} M_{d_{i_\omega}}(\mathbb{C}), \]
and \( \tau_\omega \) the trace on \( N \), set \( x = \lim_{k \to \omega} x_k \). Let
\[ \rho: M \to N \]
be the unique trace-preserving embedding such that \( \rho(a) = (\rho_{i_k}(a))_{k \to \omega} \) for \( a \in R \).
Since the quotient map \( \prod_{M} d_{i_k}(C) \to N \) is a \(*\)-homomorphism, it commutes with continuous functional calculus so
\[ \|x\|_p = \|(\rho_{i_k}(x_k))_{k \to \omega}\|_p = \lim_{k \to \omega} \|\rho_{i_k}(x_k)\|_p, \]
Thus the set of \( k \) where
\[ \|\rho_{i_k}(x_k)\|_{L^p(TM)} - \|x_k\|_{L^p(M, \tau)} < \frac{\varepsilon}{2}, \]
is \( \omega \)-large. But for such \( k \),
\[ \|\rho_{i_k}(x_k)\|_{L^p(TM)} - \|x_k\|_{L^p} < \varepsilon, \]
a contradiction.

For the next application, we need a lower bound on approximation dimension from volume estimates.

**Lemma 7.2.** There is a function \( \kappa: (0, \infty) \times (0, \infty) \to [0, 1] \) such that
\[ \lim_{\varepsilon \to 0} \kappa(\varepsilon, \alpha) = 1, \]
with the following property. Fix a sequence \( V_n \) of \( d(n) \)-complex dimensional normed vector spaces with \( d(n) < \infty \) for all \( n \), and let \( \lambda_n \) be Lebesgue measure on \( V_n \), normalized so that \( \lambda_n(\text{Ball}(V_n)) = 1 \). Suppose that \( A_n \subseteq \text{Ball}(V_n) \) have
\[ \liminf_{n \to \infty} \lambda_n(A_n)^{1/2d(n)} \geq \alpha, \]
then
\[ \liminf_{n \to \infty} \frac{1}{d(n)} \text{d}_\varepsilon(A_n, \| \cdot \|_{V_n}) \geq \kappa(\varepsilon, \alpha). \]

**Proof.** Fix \( 0 < \varepsilon < 1 \). Let \( A_n, V_n, \lambda_n \) be as in the statement of the Lemma. Suppose that
\[ \liminf_{n \to \infty} \frac{1}{d(n)} \text{d}_\varepsilon(A_n, \| \cdot \|_{V_n}) < \kappa. \]
Then for all large \( n \), we can find a subspace \( W_n \subseteq V_n \) of dimension less than \( \kappa d(n) \) so that
\[ A_n \subseteq W_n, \]
that is,
\[ A_n \subseteq W_n + \varepsilon \text{Ball}(V_n). \]
Let \( S \subseteq (1 + \varepsilon) \text{Ball}(W_n) \) be a maximal family of \( \varepsilon \)-separated vectors, i.e. for all \( x, y \in S \) we have \( \|x - y\| \geq \varepsilon \), and \( S \) is not contained in any larger set with
this property. Since the \( \varepsilon/2 \) balls centered at points of \( S \) are disjoint, a volume computation shows that

\[
|S| \leq \left( \frac{2 + 4\varepsilon}{\varepsilon} \right)^{2 \dim W_n}.
\]

Since \( S \) is maximal, it is \( \varepsilon \)-dense in \((1 + \varepsilon) \text{Ball}(W_n)\), hence

\[
A_n \subseteq \bigcup_{x \in S} x + 2\varepsilon \text{Ball}(V_n).
\]

Thus

\[
\lambda_n(A_n) \leq 2^{2d(n)} \varepsilon/2 \left( \frac{2 + 4\varepsilon}{\varepsilon} \right)^{2 \dim(W_n)},
\]

so

\[
\lambda_n(A_n)^{1/2d(n)} \leq 4\varepsilon \left( \frac{2 + 4\varepsilon}{\varepsilon} \right)^{\dim(W_n)/2d(n)} \leq 4\varepsilon^{1-\kappa}(2 + 4\varepsilon)^\kappa.
\]

Thus

\[
\alpha \leq 4\varepsilon^{1-\kappa}(2 + 4\varepsilon)^\kappa,
\]

so

\[
\kappa \geq 1 - \frac{\log \alpha - \log 4 - \log(2 + 4\varepsilon)}{\log \left( \frac{\varepsilon}{2 + 4\varepsilon} \right)}.
\]

Hence we may take

\[
\kappa(\alpha, \varepsilon) = 1 - \frac{\log \alpha - \log 4 - \log(2 + 4\varepsilon)}{\log \left( \frac{\varepsilon}{2 + 4\varepsilon} \right)}.
\]

\[\square\]

**Proposition 7.3.** Let \( \Gamma \) be a \( \mathcal{R}^\omega \) embeddable group, and fix \( 1 \leq p < \infty \). Let \( M = L(\Gamma) \), and the \( \tau \) on \( M \) the canonical group trace, and fix \( q \) a projection in \( M \). Then

\[
\dim_{\Sigma,Sp}(L^p(M, \tau)q, \Gamma) \geq \tau(q).
\]

**Proof.** Let \( R \) be the \( * \)-algebra inside \( L(\Gamma) \) generated by \( \mathbb{C}[\Gamma] \) and \( q \). As in Proposition 6.5 choose \( \rho_i : R \to M_{d_i}(\mathbb{C}) \) such that

\[
\rho_i(u_s) = \sigma_i(s) \text{ for all } s \in \Gamma,
\]

\[
p_i := \rho_i(p) \text{ is a projection for all } i,
\]

\[
\frac{1}{d_i} \text{Tr}(x) - \tau(x) \to 0 \text{ for all } x \in R,
\]

\[
\|\rho_i(P(x_1, \cdots, x_n)) - P(\rho_i(x_1), \cdots, \rho_i(x_n))\|_p \to 0,
\]

for any \( * \)-polynomial \( P \) and \( x_1, \cdots, x_n \in R \).

We will use \( S = \{u_s q \} \) to generate \( L^p(M, \tau)q \). For \( E \subseteq \Gamma \) finite, \( l \in \mathbb{N} \) set

\[
X_{E,l} = \text{Span}\{u_s q : s \in E^l\}.
\]

Fix \( F \subseteq \Gamma \) finite, \( m \in \mathbb{N}, \delta > 0 \). For \( A \in M_{d_i}(\mathbb{C}) \) define \( T_A : X_{F,m} \to L^p(M_{d_i}(\mathbb{C}), \frac{1}{d_i} \text{Tr}) \) by

\[
T_A \left( \sum_{s \in F^m} a_su_s q \right) = \sum_{s \in F^m} a_s \sigma_i(s) \rho_i(q) A.
\]
Note that by Lemma 7.1 if \(i\) is sufficiently large, then for every \((a_s)_{s \in F^m}\),
\[
\left\| \sum_{s \in F^m} a_s \sigma(s) \rho_i(q) A \right\|_p \leq \|A\|_\infty \left\| \sum_{s \in F^m} a_s \sigma(s) \rho_i(q) \right\|_p \leq 2 \|A\|_\infty \left\| \sum_{s \in F^m} a_s u_s q \right\|_p ,
\]
in particular \(T_A\) is well-defined for all large \(i\), and \(\|T_A\| \leq 2 \|A\|_\infty\).

Further if \(s_1, \ldots, s_k \in F\),
\[
\|T_A(u_{s_1} \cdots u_{s_k}) - \sigma_1(s_1) \cdots \sigma_1(s_k) T_A(q)\|_p = \\
\|\sigma_1(s_1) \cdots \sigma_1(s_k) \rho_i(q) A - \sigma_1(s_1) \cdots \sigma_1(s_k) \rho_i(q) A\|_p \leq \\
\|A\|_\infty \|\rho_i(s_1) \cdots \rho_i(s_k) (q) - \sigma_1(s_1) \cdots \sigma_1(s_k) \rho_i(q)\|_p = o(\|A\|_\infty).
\]

Thus \(T_A \in \text{Hom}_\Sigma(S, F, m, \delta, \sigma_{i})_{2}\) for all large \(i\), and all \(A \in \text{Ball}(M_n(\mathbb{C}), \| \cdot \|_\infty)\).

Hence with \(q_i = \rho_i(q)\),
\[
d_c(\text{Hom}_\Sigma(S, F, m, \delta, \sigma_{i})_{2}, \| \cdot \|_\infty) \geq d_c(q_i \text{Ball}(M_n(\mathbb{C}), \| \cdot \|_\infty)) .
\]

It is easy to see that \(d_c(q_i \text{Ball}(M_n(\mathbb{C}))\) computed as a subspace of \(L^p(M_n(\mathbb{C}), \frac{1}{n} \text{Tr})\) is at least \(d_c(q_i \text{Ball}(M_n(\mathbb{C}))\) computed as a subspace of \(q_i L^p(M_n(\mathbb{C}), \frac{1}{n} \text{Tr})\). Since
\[
\left(\frac{\text{vol}(q_i \text{Ball}(M_n(\mathbb{C})), \| \cdot \|_\infty)}{\text{vol}(q_i \text{Ball}(M_n(\mathbb{C})), \| \cdot \|_{L^p(1/n \text{Tr})})}\right)^{\frac{1}{p+1}} = \left(\frac{\text{vol}(\text{Ball}(M_n(\mathbb{C}), \| \cdot \|_\infty))}{\text{vol}(\text{Ball}(M_n(\mathbb{C}), \| \cdot \|_{L^p(1/n \text{Tr})})}\right)^{\frac{1}{p+1}} ,
\]
and the infimum over \(n\) of the right hand side is non-zero, we find by Lemma 7.2 that
\[
\liminf_{i \to \infty} \frac{1}{d_i} \text{Tr}(q_i) d_c(\text{Hom}_\Sigma(S, F, m, \delta, \sigma_{i})_{2}, \| \cdot \|_\infty) \geq \kappa(\epsilon, \alpha) .
\]

Since
\[
\frac{1}{d_i} \text{Tr}(q_i) = \frac{1}{d_i^2} \frac{d_i}{d_i} \text{Tr}(q_i) ,
\]
and
\[
\frac{\text{Tr}(q_i)}{d_i} \to \tau(q) ,
\]
we find that
\[
\dim_{\Sigma}(L^p(M, \tau_{\Gamma}) q, \Gamma) \geq \tau(q) .
\]

\[\square\]

**Corollary 7.4.** Let \(\Gamma\) be a \(\mathcal{R}^\omega\)-embeddable group, and \(1 \leq p < \infty\). Let \(\Sigma\) be an embedding sequence. Let \(\Gamma\) act on \(L^p(L(\Gamma), \tau_{\Gamma})\) by left multiplication. Then for any \(n \in \mathbb{N}\),
\[
\dim_{\Sigma, S^p\text{-mult}}(L^p(L(\Gamma), \tau_{\Gamma})^{\oplus n}, \Gamma) = \dim_{\Sigma, S^p\text{-mult}}(L^p(L(\Gamma), \tau_{\Gamma})^{\oplus n}, \Gamma) = n .
\]

**Proof.** The lower bound is proved exactly as in the preceding theorem. The upper bound follows from the fact that \(L^p(L(\Gamma), \tau_{\Gamma})^{\oplus n}\) can be generated by \(n\) elements.

\[\square\]

Because of superadditivity of dimension (see Corollary 1.8) for \(2 \leq p < \infty\), the same methods of Theorem 6.6 show the following.
Theorem 7.5. Let \( \Gamma \) be a \( \mathcal{R}^\omega \) embeddable group. For \( 2 \leq p < \infty \), and \( q_1, \ldots, q_k \) projections in \( L(\Gamma) \) we have
\[
\dim_{\ast, Sp} \left( \bigoplus_{j=1}^{k} L^p(\Gamma, \tau)q_j, \Gamma \right) = \dim_{\ast, Sp} \left( \bigoplus_{j=1}^{k} L^p(\Gamma, \tau)q_j, \Gamma \right) = \sum_{j=1}^{k} \tau(\langle q_j \rangle).
\]

Let us now show that the above Theorem applies to all closed \( \Gamma \)-invariant subspace of \( L^p(\Gamma, \tau) \).

Proposition 7.6. Let \( \Gamma \) be a countable discrete group, and let \( M = L(\Gamma) \) and \( \tau : M \rightarrow \mathbb{C} \) the canonical group trace. Let \( \{ u_\gamma \}_{\gamma \in \Gamma} \subseteq L(\Gamma) \) be the operator of left-translation by \( \Gamma \), and for \( 1 \leq p < \infty \) let \( \Gamma \) act on \( L^p(M, \tau) \) by left-multiplication by \( u_\gamma \). If \( X \subseteq L^p(M, \tau) \) is a norm-closed \( \Gamma \)-invariant subspace, then there is a projection \( q \) in \( M \) such that \( X = L^p(M, \tau)q \).

Proof. First note that \( M \) acts by left multiplication on \( L^p(M, \tau) \), we first claim that if a subspace \( X \subseteq L^p(M, \tau) \) is \( \Gamma \)-invariant, then it is \( M \)-invariant. For this it suffices, by the Kaplansky Density Theorem, to show that if \( x_n \in C[\Gamma], x \in L(\Gamma) \), have \( \| x_n \|_\infty \leq \| x \|_\infty, \| x_n - x \|_p \rightarrow 0 \), then \( x_n y \rightarrow xy \) for every \( y \in L^p(M, \tau) \).

Because \( \| x_n \|_\infty \leq \| x \|_\infty \), it suffices to note that this is true for \( y \in L(\Gamma) \), where it follows directly from the inequality
\[
\| ab \|_p \leq \| a \|_p \| b \|_\infty.
\]

To prove the proposition, it is enough to show that if \( y \in L^p(M, \tau) \), then \( \overline{M y} = L^p(Mq, \tau) \) for some projection \( q \), and that if \( q_1, q_2 \) are two projections in \( M \), then
\[
L^p(M, \tau)q_1 + L^p(M, \tau)q_2 = L^p(M, \tau)(q_1 \lor q_2).
\]

For the first claim, suppose \( y \in L^p(M, \tau) \), viewing \( y \) as a closed-densely defined unbounded operator on \( L^2(M, \tau) \) affiliated to \( M \), let
\[
y = v |y|,
\]
be the polar decomposition. Since
\[
v^* y = |y|,
\]
we have that
\[
\overline{M y} = \overline{M |y|}.
\]

By functional calculus,
\[
\lim_{\varepsilon \rightarrow 0} \| \chi(0, \varepsilon)(||y||) - \chi(\varepsilon, \infty)(||y||) \|_p = 0,
\]
\[
\chi(\varepsilon, \infty)(||y||) = (||y||^{-1}\chi(\varepsilon, \infty)(||y||))|y| \in \overline{M |y|},
\]

since the operator
\[
||y||^{-1}\chi(\varepsilon, \infty)(||y||) \in M.
\]

Thus
\[
\overline{M |y|} \supset \overline{M \chi(0, \infty)(||y||)},
\]

and because \( \overline{M \chi(0, \infty)(||y||)} = |y| \),
\[
M |y| \subseteq \overline{M \chi(0, \infty)(||y||)}.
\]

Suppose \( q_1, q_2 \) are projections in \( M \). It is clear that
\[
L^p(M, \tau)q_1 + L^p(M, \tau)q_2 \subseteq L^p(M, \tau)(q_1 \lor q_2).
\]
By functional calculus we have
\[ q_1 \lor q_2 = [(1 - q_1) \land (1 - q_1)]^\bot = \]
\[ 1 - \chi_{\{1\}}((1 - q_1)(1 - q_2)(1 - q_1)) = \]
\[ 1 - \lim_{n \to \infty} [(1 - q_1)(1 - q_2)(1 - q_1)]^n, \]
the limit being taken in the \( L^p \) norm. A direct computation shows that
\[ [(1 - q_1)(1 - q_2)(1 - q_1)]^n = 1 + R \]
where \( R \) is a sum of terms which are in \( Mq_1 \) or \( Mq_2 \). This proves the second claim. \( \square \)

**Corollary 7.7.** Let \( \Gamma \) be a \( \mathcal{R}^\omega \)-embeddable group with embedding sequence \( \Sigma = (\sigma_i; \Gamma \to U(d_i)) \), and fix \( 2 \leq p < \infty, n \in \mathbb{N} \). Set \( M = L(\Gamma) \) and \( \tau: M \to \mathbb{C} \) the canonical group trace. If \( X \subseteq L^p(M, \tau)^{\oplus n} \) is \( \Gamma \)-invariant and \( X \neq 0 \), then
\[ \dim_{\Sigma, S^p}(X, \Gamma) > 0. \]

**Proof.** Let \( \pi_j: L^p(M, \tau)^{\oplus n} \to L^p(M, \tau) \) be projection onto the \( j \)th factor. Since \( X \neq 0 \), there is some \( j \) such that \( \pi_j(X) \neq 0 \). By the preceding proposition we can find \( q \in M \) a nonzero projection such that
\[ \pi_j(X) = L^p(M, \tau)q. \]

By Proposition 4.1 and Theorem 7.5 we have
\[ \dim_{\Sigma, S^p}(X, \Gamma) \geq \dim_{\Sigma, S^p}(L^p(M, \tau)q, \Gamma) = \tau(q) > 0. \]

\( \square \)

We will apply this to the usual \( l^p \)-spaces. To do this we will take a “Fourier transform.” Fix a countable discrete group \( \Gamma \), and view the group ring \( \mathbb{C}[\Gamma] \subseteq L(\Gamma) \), by \( \gamma \to u_\gamma \), where \( u_\gamma \) is translation by \( \Gamma \). Define \( \mathcal{F}: l^1(\Gamma) \to C^*_\lambda(\Gamma) \subseteq L(\Gamma) \) by
\[ \mathcal{F}(f) = \sum_{s \in \Gamma} f(s)u_s, \]
we will usually use \( \hat{f} \) for \( \mathcal{F}(f) \). By the triangle inequality,
\[ \| \hat{f} \|_\infty \leq \| f \|_1, \]
and by direct computation
\[ \| \hat{f} \|_2 = \| f \|_2, \]
thus by interpolation
\[ \| \hat{f} \|_p \leq \| f \|_{\frac{p}{p'}} \]
for \( 1 \leq p \leq 2 \), where \( \frac{1}{p} + \frac{1}{p'} = 1. \)

Note that for \( 2 \leq p \leq \infty \),
\[ \mathcal{F}^t: L^p(M, \tau) \to l^p(\Gamma), \]
is given by
\[ \mathcal{F}^t(x)(s) = \tau(xu_s), \]
in particular
\[ \mathcal{F}^t(\mathbb{C}[\Gamma]) = c_0(\Gamma), \]
and thus \( \mathcal{F}^t \) has dense image, so \( \mathcal{F} \) is injective.
Corollary 7.8. Let $\Gamma$ be a $\mathcal{R}^\omega$-embaddable group with embedding sequence $\Sigma = ((\sigma_i): \Gamma \to U(d_i))$, and fix $1 < p \leq 2$, and let $p'$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose that $X \subseteq l^p(\Gamma)^{\oplus n}$ is a closed nonzero $\Gamma$-invariant subspace. Then

$$\dim_{\Sigma, \sigma'}(X, \Gamma) > 0.$$ 

Proof. By injectivity of $\mathcal{F}$ we have

$$\mathcal{F}^{\oplus n}(X) \neq 0,$$

thus by the preceding corollary and Proposition 8.1, we have

$$\dim_{\Sigma, \sigma'}(X, \Gamma) \geq \dim_{\Sigma, \sigma'}(\mathcal{F}^{\oplus n}(X), \Gamma) > 0.$$

\square

8. $l^p$-Betti Numbers of Free Groups

Let $X$ be a CW complex and let $\Delta_n$ be the $n$-simplices of $X$. Suppose that $\Gamma$ acts properly on $X$ with cocompact quotient, preserving the simplicial structure. For $v_0, \cdots, v_n \in X$ let

$$[v_0, v_1, \cdots, v_n]$$

be the simplex spanned by $v_0, \cdots, v_n$. Let

$$V_n(X) = \{(v_0, \cdots, v_n) \in X : [v_0, \cdots, v_n] \in \Delta_n\}.$$

Let $l^p(\Delta_n(X))$ be all functions $f: V_n(X) \to \mathbb{C}$ such that

$$f(v_\sigma(0), \cdots, v_\sigma(n)) = (\text{sgn} \sigma)f(v_0, \cdots, v_n)$$

for $\sigma \in \text{Sym}(\{0, \cdots, n\})$

$$\sum_{[v_0, \cdots, v_n] \in \Delta_n(X)} |f(v_0, \cdots, v_n)|^p < \infty,$$

by our antisymmetry condition the above sum is unchanged if we use a different representative for $[v_0, \cdots, v_n]$. On $l^p(\Delta_n(X))$ we use the norm

$$\|f\|_p = \sum_{v \in \Delta_n(X)} |f(v_0, \cdots, v_n)|^p.$$

Define the discrete differential $\delta: l^p(\Delta_{n-1}(X)) \to l^p(\Delta_n(X))$ by

$$(\delta f)(v_0, \cdots, v_n) = \sum_{j=0}^n (-1)^j f(v_0, \cdots, \hat{v}_j, \cdots, v_n),$$

where the hat indicates a term omitted, note that $\delta f$ satisfies the appropriate antisymmetry condition. Define the $n^{th}$ $l^p$-Cohomology space of $X$ by

$$H^n_{l^p}(X) = \ker(\delta) \cap l^p(\Delta_n(X))/\delta(l^p(\Delta_{n-1}(X)).$$

We define the $l^p$-Betti numbers of $X$ with respect to $\Gamma$ by

$$\beta^{(p)}_{\Sigma, n}(X, \Gamma) = \dim_{\Sigma, l^p}(H^n_{l^p}(X), \Gamma).$$

It is known that if $X$ is contractible and $\pi_1(X/\Gamma) \cong \Gamma$, then the $l^p$-cohomology space only depends upon $\Gamma$, thus we may define

$$H^n_{l^p}(\Gamma) = H^n_{l^p}(X, \Gamma),$$

$$\beta_{\Sigma, n}^{(p)}(\Gamma) = \beta^{(p)}_{\Sigma, n}(X, \Gamma),$$

for such $X$. 

We also consider \( l^p \)-Homology. Define \( \partial: l^p(\Delta_n(X)) \to l^p(\Delta_{n-1}(X)) \) by
\[
\partial f(v_0, \cdots, v_{n-1}) = \sum_{x: [v_0, \cdots, v_{n-1}, x] \in \Delta_n(X)} f(v_0, \cdots, v_{n-1}, x),
\]
by direct computation
\[
(\partial: l^p(\Delta_n(X)) \to l^p(\Delta_{n-1}(X))) = (\delta: l^p(\Delta_{n-1}(X)) \to l^p(\Delta_n(X)))^t,
\]
when \( \frac{1}{p} + \frac{1}{p'} = 1 \). Define the \( l^p \)-Homology of \( X \) by
\[
H^p_n(X) = \frac{\ker(\partial) \cap l^p(\Delta_n(X))}{\partial(l^p(\Delta_{n+1}(X)))}.
\]

We shall be interested in the \( l^p \)-Betti numbers of free groups. Fix \( n \in \mathbb{N} \), and consider the free group \( \mathbb{F}_n \) on \( n \) letters \( a_1, \cdots, a_n \). Let \( G \) be the Cayley graph of \( \mathbb{F}_n \) with respect to \( a_1, \cdots, a_n \), we regard the edges of \( G \) as oriented. Then the topological space \( X \) associated to \( G \) is contractible, since \( G \) is a tree, and has \( \pi_1(X/\mathbb{F}_n) \cong \mathbb{F}_n \), so the \( l^p \)-cohomology of \( G \) is the \( l^p \)-cohomology of \( \mathbb{F}_n \). Let \( E(\mathbb{F}_n) \) denote the edges of \( \mathbb{F}_n \). Then \( l^p(E(\mathbb{F}_n)) \) as defined above is given by all functions \( f: E(\mathbb{F}_n) \to \mathbb{C} \) such that
\[
f(x, s) = -f(s, x) \text{ if } (s, x) \in E(\mathbb{F}_n),
\]
with the norm
\[
\|f\|_p = \sum_{j=1}^n \sum_{x \in \mathbb{F}_n} |f(x, xa_j)|^p < \infty.
\]

Note that this is indeed a norm on \( l^p(E(\mathbb{F}_n)) \), and that \( \mathbb{F}_n \) acts isometrically on \( l^p(E(\mathbb{F}_n)) \) by left translation. Also \( l^p(E(\mathbb{F}_n)) \) is isomorphic to \( l^p(\mathbb{F}_n) \) with respect to this action. If \((x, s) \in E(\mathbb{F}_n)\), we let \( \mathcal{E}_{(x, s)} \) be the function on \( E(\mathbb{F}_n) \) such that
\[
\mathcal{E}_{(x, s)}(y, t) = 0 \text{ if } \{x, s\} \neq \{y, t\},
\]
\[
\mathcal{E}_{(x, s)}(x, s) = 1
\]
\[
\mathcal{E}_{(x, s)}(s, x) = -1.
\]

We think of \( \mathcal{E}_{(x, s)} \) as representing the edge going form \( x \) to \( s \).

Then the discrete differential we defined above
\[
\delta: l^p(\mathbb{F}_n) \to l^p(E(\mathbb{F}_n))
\]
is given by
\[
(\delta f)(x, s) = f(s, x) \text{ if } (x, s) \in E(\mathbb{F}_n).
\]

And the corresponding \( l^p \)-Cohomology space is given by
\[
H^p_1(\mathbb{F}_n) = l^p(E(\mathbb{F}_n))/\delta(l^p(\mathbb{F}_n)).
\]

Also \( \partial: l^p(E(\mathbb{F}_n)) \to l^p(\mathbb{F}_n) \) is given by
\[
(\partial f)(x) = \sum_{j=1}^n f(x, xa_j) - \sum_{j=1}^n f(xa_j^{-1}, x).
\]

Since \( \mathbb{F}_n \) is non-amenable, we know \( l^p(\mathbb{F}_n) \) does not have almost invariant vectors under the translation action of \( \mathbb{F}_n \). Thus, there is a \( C > 0 \) so that
\[
\|\delta f\|_p \geq C\|f\|_p.
\]
So $\delta(l^p(\mathbb{F}_n))$ is closed in $l^p(E(\mathbb{F}_n))$.

In this section, we compute the $l^p$-Betti numbers

$$\beta^{(p)}_{\Sigma,1}(\mathbb{F}_n),$$

for $1 \leq p \leq 2$.

**Lemma 8.1.** Fix $n \in \mathbb{N}$, $1 \leq p < \infty$. Then the image of the elements $\mathcal{E}_{(e,a_1)}, \ldots, \mathcal{E}_{(e,a_{n-1})}$ are dynamically generating for $H^1_{lp}(\mathbb{F}_n)$.

**Proof.** For this, it suffices to show that

$$W = \delta(l^p(\mathbb{F}_n)) + \text{Span}\{\mathcal{E}_{(s,sa_j)} : s \in \mathbb{F}_n, 1 \leq j \leq n-1\}$$

is norm dense in $l^p(E(\mathbb{F}_n))$.

It is enough to show that $E_{(e,a_n)} \in \overline{W}$, and by convexity it is enough to show that $\mathcal{E}_{(e,a_n)}$ is in the weak closure of $W$.

To do this, we shall prove by induction on $k$ that

$$E_{(a_k^n,a_{k+1}^n)} \equiv E_{(a_k^n,a_{k+1}^n)} \mod W,$$

this is enough since

$$E_{(a_k^n,a_{k+1}^n)} \to 0$$

weakly.

The base case $k = 0$ is trivial, so assume the result true for some $k$. Then

$$E_{(a_k^n,a_{k+1}^n)} - \delta(\chi_{(a_{k+1}^n)}) = \sum_{j=1}^{n} E_{(a_k^{n+1},a_{k+1}^n a_j)} + \sum_{j=1}^{n-1} E_{(a_k^{n+1},a_{k+1}^n a_j^{-1})},$$

$$= E_{(a_k^{n+1},a_{k+1}^n)} + \sum_{j=1}^{n-1} a_k^{n+1} E_{(e,a_j)} - \sum_{j=1}^{n-1} a_k^{n+1} a_j E_{(e,a_j)},$$

$$\equiv E_{(a_k^{n+1},a_{k+1}^n)}.$$

Here is a graphical explanation of the above calculation. If we think of the elements of $l^p(E(\mathbb{F}_n))$ as formal sums of oriented edges, then $-\delta(\chi_{(a_{k+1}^n)})$ is a “source” at $a_{k+1}^n$, it is the sum of all edges adjacent to $a_k^{n+1}$, directed away from $a_k^{n+1}$.

Pictured below:

![Graphical Explanation](image)

The above computation can be phrased as follows:

$$-\delta(\chi_{a_{k+1}^n}) + E_{(a_k^n,a_{k+1}^n)} =$$
and the second term on the right-hand side is easily seen to be in the span of translates of $E(e,a_j), j = 1, \ldots, n - 1$.

This completes the induction step. 

We shall prove the analogous claim for $l^p$-Homology of free groups, but we need a few preliminary results. These next few results must be well known, but we include proofs for completeness.

**Lemma 8.2.** Let $\Gamma$ be a non-amenable group with finite generating set $S$. Let $A: l^p(\Gamma) \to l^p(\Gamma)$ be defined by

$$Af = \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s),$$

then for $1 < p < \infty$, there is a constant $C_p < 1$ so that $\|Af\|_p < C_p \|f\|_p$.

**Proof.** For $p = 2$, this is automatic from non-amenability of $\Gamma$. Since $\|Af\|_\infty \leq \|f\|_\infty, \|Af\|_1 \leq \|f\|_1$, the lemma follows by interpolation. \hfill \Box

**Lemma 8.3.** Let $\Gamma$ be a non-amenable group with finite generating set $S$. For $1 < p < \infty$, the operator $\partial \circ \delta: l^p(\Gamma) \to l^p(\Gamma)$, is invertible.

**Proof.** We have that

$$\partial(\delta f)(x) = \sum_{s \in S \cup S^{-1}} f(x) - f(xs) = |S \cup S^{-1}| \left( f(x) - \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s)f(x) \right).$$

By the previous lemma,

$$\left\| \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s) \right\|_{l^p \to l^p} < 1,$$
for $1 < p < \infty$, so this proves that $\partial(\delta)$ is invertible for $1 < p < \infty$.\qed

**Corollary 8.4.** Let $\Gamma$ be a non-amenable group with finite generating set $S$. For $1 < p < \infty$, we have the following Hodge Decomposition:

$$L^p(E(\Gamma)) = \ker(\partial): L^p(E(\Gamma)) \to L^p(\Gamma)) + \delta(L^p(\Gamma)).$$

**Proof.** If $f \in \ker(\partial): L^p(E(\Gamma)) \to L^p(\Gamma)) \cap \delta(L^p(\Gamma))$, write $f = \delta(g)$, then

$$0 = \partial(f) = \partial(\delta(g)),$$

so by the preceding lemma we have that $g = 0$.

If $f \in L^p(E(\Gamma))$, then by the preceding lemma we can find a unique $g$ so that $\partial(f) = \partial(\delta(g))$. Then $f - \delta(g) \in \ker(\partial)$, and

$$f = f - \delta(g) + \delta(g).$$

\qed

**Proposition 8.5.** Let $n \in \mathbb{N}$, and $1 < p < \infty$, then $H^p_\Gamma(F_n)$ can be generated by $n - 1$ elements.

**Proof.** The claim for $n = 1$ is clear since $H^p_\Gamma(\mathbb{Z}) = 0$. We claim that it suffices to do the proof for $n = 2$. For this, let $n > 2$, and let $a_1, \ldots, a_n$ be the generators of $\mathbb{F}_n$. Consider the injective homomorphisms $\phi_j : \mathbb{F}_2 \to \mathbb{F}_n$ for $1 \leq j \leq n - 1$ given by $\phi_j(a_i) = a_{i+j}$. Let $f$ be an element in $L^p(E(\mathbb{F}_2))$ so that $\text{Span}(\mathbb{F}_2) f$ is dense in $\ker(\partial) \cap L^p(E(\mathbb{F}_2))$. Let $f_j \in L^p(E(\mathbb{F}_n))$ be the element defined by

$$f_j(x, y) = \begin{cases} 0, & \text{if one of } x, y \notin \phi_j(\mathbb{F}_2) \\ f(\phi_j^{-1}(x), \phi_j^{-1}(y)), & \text{otherwise.} \end{cases}$$

Then $f_j \in \ker(\partial)$. It is easy to see from the preceding corollary and the fact that $f$ generates $\ker(\partial) \cap L^p(E(\mathbb{F}_2))$, that

$$E_{(e, a_j)} \in \ker(\partial) + \delta(L^p(\mathbb{F}_n)),$$

again by the preceding corollary we find that $f_1, \ldots, f_{n-1}$ generate $\ker(\partial)$. Thus it suffices to handle the case $n = 2$.

We now concentrate on the case $n = 2$, and we use $a, b$ for the generators of $\mathbb{F}_2$. Let $f : E(\mathbb{F}_2) \to \mathbb{R}$ defined by the following inductive procedure. Set

$$f_1 = E_{(e, a)} + E_{(e, b)} + E_{(a^{-1}, e)} + E_{(b^{-1}, e)}.$$

Having constructed $f_1, \ldots, f_n$ so that $f_j$ is supported on the pairs of edges which have word length at most $j$, define $f_{n+1}$ as follows. For each word $w$ of length $n$, let $e_1, e_2, e_3$ be the three oriented edges which have their terminal vertex $w$ and the initial vertex a word of length $n + 1$, and let $e$ be the oriented edge which has its initial vertex $w$ and its terminal vertex a word of length $n - 1$. Define for $j = 1, 2, 3$

$$f_{n+1}(e_j) = \frac{1}{3} f_n(e),$$

and define

$$f_{n+1}(e) = f_n(e)$$

if both vertices of $e$ have length at most $n$. It is easy to see that the $f_n$’s as constructed above converge pointwise to a function $f$ in $L^p(E(\mathbb{F}_2)) \cap \ker(\partial)$ for $1 < p \leq \infty$.

The function $f$ is pictured below:
Set \( V = \text{Span}(F_2 f) + \delta(l^p(F_2)) \).

To show that \( f \) generates \( \ker(\partial) \) it suffices, by the preceding corollary to show that

\[ E(e,a_1), E(e,a_2) \in V. \]

Let \( B_n = \{(x,y) \in G : \|x\|, \|y\| \leq n\} \). For \( n \geq 0 \), let \( g_n : E(F_n) \to \mathbb{C} \), be the function defined by

\[
\chi_{B_n} g_n = \left( \frac{1}{3} \right)^n \left( E(e,a) + E(e,b) + E(a^{-1},e) + E(b^{-1},e) \right),
\]

\[
(1 - \chi_{B_n}) g_n = (1 - \chi_{B_n}) f,
\]

we first show that \( g_n \in \text{Span}(F_2 f) + \delta(l^p(F_2)) \), for all \( n \). We prove this by induction on \( n \), the case \( n = 1 \) being clear since \( g_1 = f \). Suppose the claim true for some \( n \). Then for each word \( w \) of length \( n \), we can add either \( (1/3)^n \delta(\chi(w)) \), or \( -(1/3)^n \delta(\chi(w)) \), to \( f_n \) to make the value on every edge from \( w \) to a word of length \( n + 1 \) zero. This now adds a value of \( \pm (1/3)^n \) to every edge going from a word of length \( n \) to a word of length \( n - 1 \). Now repeat for every word of length \( n - 1 \); add on \( \pm (1/3)^n \delta(\chi(w)) \) for every word \( w \) of length \( n - 1 \) to force a value of 0 on every edge going from a word of length \( n - 1 \) to a word of length \( n \). Repeating this inductively until we get to words of length 1, we find by construction of \( f \) that
$g_n \in \text{Span}(\mathbb{F}_2f) + \delta(l^p(\mathbb{F}_2))$. The first two steps of this process are pictured below:
Since supₙ ∥gₙ∥ₚ < ∞ we find that gₙ converges weakly to
\[ \frac{3}{2} (\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(b^{-1},e)} + \mathcal{E}_{(a^{-1},e)}) \].

Rescaling we find that
\[ \mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(b^{-1},e)} + \mathcal{E}_{(a^{-1},e)} \in V. \]

By adding ±δ(χₑ) and scaling we find that
\[ \mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} \in V, \]
\[ \mathcal{E}_{(e,b^{-1})} + \mathcal{E}_{(e,a^{-1})} \in V. \]

Inductively, we now see that
\[ \mathcal{E}_{(e,a)} + \mathcal{E}_{((ba^{-1})^n - 1, (ba^{-1})^n)} \in V, \]
and taking weak limits proves that
\[ \mathcal{E}_{(e,a)} \in V. \]
Subtracting $\mathcal{E}_{(e,a)}$ from $\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)}$ we find that

$$\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} \in V$$

and thus by $\mathbb{F}_2$-invariance that $V = l^p(E(\mathbb{F}_2))$, this completes the proof.

\[\square\]

**Theorem 8.6.** Fix $n \in \mathbb{N}$, and a sofic approximation $\Sigma$.

(a) The dimension of the $l^p$-cohomology groups of $\mathbb{F}_n$ satisfy

$$\dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = \dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = n - 1, \text{ for } 1 \leq p \leq 2,$$

$$H^m_{l^p}(\mathbb{F}_n) = \{0\} \text{ for } m \geq 2.$$

(b) The dimension of the $l^p$-homology groups of $\mathbb{F}_n$ satisfy:

$$\dim_{\Sigma,l^p}(H^m_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = \dim_{\Sigma,l^p}(H^m_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = n - 1, \text{ for } 1 < p < 2$$

$$H^1_{l^p}(\mathbb{F}_n) = \ker(\partial) \cap l^1(E(\mathbb{F}_n)) = \{0\},$$

$$H^m_{l^p}(\mathbb{F}_n) = 0 \text{ for } m \geq 2.$$

**Proof.** The statements about higher-dimensional homology or cohomology are clear, since we know that the Cayley graph of $\mathbb{F}_n$ is contractible and one-dimensional.

Since the image of $\delta$ is closed, the sequence

$$0 \to l^p(\mathbb{F}_n) \xrightarrow{\delta} l^p(E(\mathbb{F}_n)) \to H^1_{l^p}(\mathbb{F}_n) \to 0$$

is exact, and Proposition 4.2 and Theorem 5.3 imply that

$$n = \dim_{\Sigma,l^p}(l^p(E(\mathbb{F}_n)), \mathbb{F}_n)$$

$$\leq \dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) + \dim_{\Sigma,l^p}(l^p(\mathbb{F}_n))$$

$$= \dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) + 1.$$

Thus

$$\dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) \geq n - 1.$$

On the other hand, by the Lemma 8.1, $H^1_{l^p}(\mathbb{F}_n)$ can be generated by $n - 1$ elements, so

$$\dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) \leq n - 1,$$

which proves the first claim.

For the second claim, by surjectivity of $\partial$ for $1 < p \leq 2$, the sequence

$$0 \to H^1_{l^p}(\mathbb{F}_n) \to l^p(E(\mathbb{F}_n)) \xrightarrow{\partial} l^p(\mathbb{F}_n) \to 0$$

is exact. As in the first half this implies that

$$\dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) \geq n - 1,$$

for $1 < p \leq 2$. The upper bound for $1 < p \leq 2$ also holds by the preceding proposition.

We turn to the last claim. If $x \in \mathbb{F}_n$, because the Cayley graph of $\mathbb{F}_n$ is a tree we can define $\gamma_x$ to be the unique geodesic path from $e$ to $x$. Let $|x| = d(x,e)$, and define

$$A: \mathbb{C}^{E(\mathbb{F}_n)} \to \mathbb{C}^{\mathbb{F}_n}$$

by

$$(Af)(x) = \sum_{|j|} f(\gamma_x(j - 1), \gamma_x(j)),$$

for which we have

$$\dim A = \dim_{\Sigma,l^p}(H^0_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = \dim_{\Sigma,l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = n - 1.$$
note that $\delta(Af) = f$. A direct computation verifies that $A(\mathcal{E}_{(x,xa_j)}) \in l^\infty(F_n)$, thus $\delta(l^\infty(F_n))$ is weak$^*$ dense in $l^\infty(E(F_n))$. By duality $\ker(\partial) \cap l^1(E(F_n)) = \{0\}$, this completes the proof.

9. Triviality In The Case Of Finite-Dimensional Representations

In this section we prove the following.

**Theorem 9.1.** Let $\Gamma$ be a infinite sofic group, and $\Sigma$ a sofic approximation of $\Gamma$. Suppose that $\Gamma$ either contains an infinite amenable subgroups or that $\{|\Lambda| : \Lambda \subseteq \Gamma \text{ is a finite subgroup}\}$ is unbounded. Suppose that $X$ is a finite-dimensional Banach space with a uniformly bounded action of $\Gamma$. Then for every $1 \leq p \leq \infty$,

$$\dim_{\Sigma, l^p}(X, \Gamma) = 0.$$  

At present, it is not clear how to remove the hypothesis on the subgroup structure of $\Gamma$ in the preceding theorem. We mention that one can show the analogous statement if $\Gamma$ is $\mathcal{R}_\omega$-embeddable, provided we make the same assumptions on the subgroups structure of $\Gamma$. Since the proofs are similar to the sofic case, we will focus on the sofic case.

We first outline the proof. We will begin by studying $l^p$-dimension for amenable groups, using the standard technique of averaging over Følner sequences. Using these techniques, we show that for finite $\Gamma$,

$$\dim_{\Sigma, l^p}(X, \Gamma) = \frac{\dim_X}{|\Gamma|}.$$  

This easily implies the case that $\Gamma$ has finite subgroups of unbounded size. We then show that

$$\dim_{\Sigma, l^p}(X, \mathbb{Z}) = 0,$$

if $X$ is finite-dimensional, and that

$$\dim_{\Sigma, l^p}(\mathbb{C}, \Gamma) = 0,$$

if $\Gamma$ is amenable and acts trivially on $\mathbb{C}$. The above three statements will be enough to prove Theorem 9.1 using a compactness argument.

We first show that in the case of an amenable group action, we may assume that the maps we use to compute dimension are only approximately equivariant after cutting down by certain subsets. We formalize this as follows.

**Definition 9.2.** Let $\Gamma$ be a sofic group with a uniformly bounded action on a Banach space $X$. Let $\sigma_i : \Gamma \rightarrow S_{d_i}$ be a sofic approximation, fix $S = \{a_j\}_{j=1}^\infty$ a bounded sequence in $X$. Let $A_i \subseteq \{1, \ldots, d_i\}$, for $F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $\delta > 0$, we set $\text{Hom}_{\Gamma, l^p, (A_i)}(S, F, m, \delta, \sigma_i)$ to be all linear maps $T : X_{F,m} \rightarrow l^p(d_i)$ such that $\|T\| \leq 1$, and for all $1 \leq j, k \leq m$, for all $s_1, \ldots, s_k \in F$ we have

$$\|T(s_1 \ldots s_k a_j) - \sigma_i(s_1) \ldots \sigma_i(s_k) T(a_j)\|_{l^p(A_i)} < \delta.$$  

Set

$$\dim_{\Sigma, l^p}(S, \Gamma, (A_i), \rho) = \sup_{\epsilon > 0} \inf_{F \subseteq \Gamma \text{ finite}} \limsup_{m \rightarrow \infty} \frac{1}{d_i} d\epsilon(\alpha S(\text{Hom}_{\Gamma, l^p, (A_i)}(S, F, m, \delta, \sigma_i)), \rho),$$  

where $\rho$ is any product norm.
Proposition 9.3. Fix a product norm $\rho$ on $l^\infty(\mathbb{N})$. Let $\Gamma$ be a countable amenable group, and $\Sigma = (\sigma_i: \Gamma \to S_{d_i})$ a sofic approximation. Let $A_i \subseteq \{1, \ldots, d_i\}$ be such that

$$\frac{|A_i|}{d_i} \to 1.$$  

Then for any uniformly bounded action of $\Gamma$ on a separable Banach space $X$, for every generating sequence $S$ in $X$, for every product norm $\rho$, and $1 \leq p < \infty$ we have

$$\dim_{\Sigma, l^p}(X, \Gamma) = \dim_{\Sigma, l^p}(S, \Gamma, (A_i), \rho).$$

Proof. Fix $S = (x_j)_{j=1}^\infty$ a dynamically generating sequence for $X$. We first fix some notation, for $E, e \in F \subseteq \Gamma$ finite $m \in \mathbb{N}$ define

$$P_i^{(E)}: B(X_{F,E,m}, l^p(d_i)) \to B(X_{F,m}, l^p(d_i))$$

by

$$P_i^{(E)}(T) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) \circ T \circ s^{-1},$$

then $\|P_i^{(E)}\| \leq 1$. Note that for $s_1, \cdots, s_k \in F$ and $T \in B(X_{F,k}, l^p(d_i))$ that

$$P_i^{(E)}(T)(s_1 \cdots s_k x) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) T(s^{-1}s_1 \cdots s_k) x =$$

$$\frac{1}{|E|} \sum_{s \in s_1^{-1} \cdots s_k^{-1} E} \sigma_i(s_1 \cdots s_k s) T(s^{-1}x).$$

If $B_i \subseteq \{1, \cdots, d_i\}$ is the set of all $j$ such that

$$\sigma_i(s_1 \cdots s_k s)^{-1}(j) = \sigma_i(s)^{-1}\sigma_i(s_1 \cdots s_k)^{-1}(j),$$

for all $s \in E, s_1, \cdots, s_k \in F, 1 \leq k \leq m$, then the above shows that if $T \in B(X_{F,E,m}, l^p(B_i))$ then

$$\|\sigma_i(s_1 \cdot s_k) \circ P_i^{(E)}(T)(x_j) - P_i^{(E)}(T)(s_1 \cdots s_k j)(x_j)\| \leq 2 \frac{|E\Delta s_k^{-1} \cdots s_1^{-1} E|}{|E|} \|T\| \|x_j\|,$$

for $1 \leq j \leq m$.

Suppose $T \in \text{Hom}_{\Gamma,l^p(A_j)}(S, F, m, \delta, \sigma_i)$ $e \in F$ is symmetric $m \geq 2$, and $E \supseteq F$, then

$$P_i^{(E)}(\chi_B T) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) \chi_{B s^{-1}} T \circ s^{-1} =$$

$$\frac{1}{|E|} \sum_{s \in E} \chi_{\sigma_i(s) B} \sigma_i(s) T \circ s^{-1}.$$  

Set $C_i = A_i \cap B_i \cap \bigcap_{s \in E} \sigma_i(s)(A_i \cap B_i)$, then $\frac{|C_i|}{d_i} \to 1$, and for $1 \leq j \leq m$

$$\|P_i^{(E)}(\chi_B T)(x_j) - T(x_j)\|_{ip(C_i)} \leq \frac{1}{|E|} \sum_{s \in E} \|\sigma_i(s) T(s^{-1} x_j) - (x_j)\|_{ip(A_i)} < 2\delta.$$

The claim now easily follows by using a two-sided Følner sequence. □
Corollary 9.4. Let $\Gamma$ be an amenable group with a uniformly bounded action on a separable Banach space $X$. Let $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$, $\Sigma' = (\sigma'_i : \Gamma \to S_{d_i})$ be two sofic approximations, then for all $1 \leq p \leq \infty$,
\[
\dim_{\Sigma,p}(X, \Gamma) = \dim_{\Sigma',p}(X, \Gamma)
\]
for all $1 \leq p < \infty$.

Proof. Because any two sofic embeddings into $\prod S_{d_i}$ are conjugate (see [6]), a simple ultrafilter argument shows that we can find $\tau_i : S_{d_i} \to S_{d_i}$ such that
\[
d_{\text{Hamm}}(\tau_i \sigma_i(s) \tau_i^{-1}, \sigma_i(s)) \to 0.
\]
Replacing $\sigma_i$ by $\tau_i \circ \sigma_i \circ \tau_i^{-1}$, we may assume that
\[
d_{\text{Hamm}}(\sigma_i(s), \sigma'_i(s)) \to 0
\]
for all $s \in \Gamma$. In this case, we can find $A_i, \{1, \cdots, d_i\}$ such that
\[
\left| \frac{|A_i|}{d_i} \right| \to 1
\]
and for all $s_1, \cdots, s_n \in \Gamma$, we have
\[
\sigma_i(s_1 \cdots s_n)(j) = \sigma_i(s_1) \cdots \sigma_i(s_n)(j) = \sigma'_i(s_1) \cdots \sigma'_i(s_n)(j) = \sigma'_i(s_1 \cdots s_n)(j)
\]
for all $j \in A_i$ and all sufficiently large $i$. Thus if $F \subseteq \Gamma$ is finite, $m \in \mathbb{N}$, $\delta > 0$ then for all large $i$,
\[
\text{Hom}_{\Gamma,F,(A_i)}(S, F, m, \delta, \sigma_i) = \text{Hom}_{\Gamma,F,(A_i)}(S, F, m, \delta, \sigma'_i).
\]

Proposition 9.5. Let $\Gamma$ be a finite group acting on a finite-dimensional vector space $X$. For $n \in \mathbb{N}$, let
\[
n = q_n |\Gamma| + r_n
\]
where $0 \leq r_n < |\Gamma|$ and $q_n, r_n \in \mathbb{N}$. Let $A_n$ be a set of size $r_n$ and define a sofic approximation $\Sigma = (\sigma_n : \Gamma \to \text{Sym}(\Gamma \times \{1, \cdots, q_n\}) \prod A_n)$ by
\[
\sigma_n(s)(g, j) = (sg, j) \text{ for } s \in \Gamma, 1 \leq j \leq q_n,
\]
\[
\sigma_n(s)(a) = a \text{ for } a \in A_n.
\]
Then for any $1 \leq p \leq \infty$
\[
\dim_{\Sigma,p}(X, \Gamma) = \dim_{\Sigma,p}(X, \Gamma) = \frac{\dim_{\mathbb{C}} X}{|\Gamma|}.
\]

Proof. Fix a norm on $X$. By finite dimensionality we may use the operator norm on $B(X, l^p(d_i))$ as our pseudonorm, and we replace $\text{Hom}_{\Gamma}(S, F, m, \delta, \sigma_i)$ by the space $\text{Hom}_{\Gamma}(F \times \{a\})$ of all operators $T : X \to l^p(d_i)$ such that
\[
\|T \circ s_1 \cdots s_k - \sigma_i(s_1) \cdots \sigma_i(s_k) \circ T\| < \delta
\]
for all $1 \leq k \leq m, s_1, \cdots, s_k \in \Gamma$.

Let $V_n \subseteq B(X, l^p(n))$ be the linear subspace of all linear operators
\[
T : X \to l^p(\Gamma \times \{1, \cdots, q_n\})
\]
which are equivariant with respect to the $\Gamma$-action. Note that we have norm one projections
\[
B(X, l^p(n)) \to B(X, l^p(\Gamma \times \{1, \cdots, q_n\}))
\]
given by multiplication by $\chi_{\{1, \ldots, q_n\}}$ and by

$$T \mapsto \frac{1}{|\Gamma|} \sum_{s \in \Gamma} \sigma_n(s)^{-1} \circ T \circ s,$$

let $P_n$ denote the composition of these two projections. Since we have a norm one projection form $B(X, l^p(\Gamma)) \to V_n$, a quick application of the Riesz Lemma implies that

$$d_\varepsilon(V_n, \| \cdot \|) \geq \dim V_n = (\dim \mathbb{C} X) q_n,$$

with the norm being the operator norm. Further for $T \in \text{Hom}_r(\Gamma, m, \delta, \sigma_i)$ we have

$$\|P_n(T) - T\|_{B(X, l^p(n))} < \delta.$$

Thus

$$d_\varepsilon(\text{Hom}_r(\Gamma, m, \delta, \sigma_i) \leq (\dim \mathbb{C} X) q_n + r_n,$$

and (4), (5) are enough to imply the proposition.

\[ \square \]

**Corollary 9.6.** Let $\Gamma$ be a finite group acting on a finite-dimensional vector space $X$. For any finite dimensional representation $X$ of $\Gamma$, for any sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$ of $\Gamma$ and $1 \leq p \leq \infty$ we have

$$\dim \Sigma_{l^p}(X, \Gamma) = \frac{\dim \mathbb{C} X}{|\Gamma|}.$$

**Proof.** Take

$$\Sigma' = (\rho_n : \Gamma \to S_{d_i})$$

where $\rho_n$ is defined as in the previous proposition, then use the fact that two sofic approximations into the same size symmetric groups give the same dimension.

\[ \square \]

**Proposition 9.7.** Let $X$ be a finite-dimensional Banach space with a uniformly bounded action of $\mathbb{Z}$. Let $\sigma_n : \mathbb{Z} \to \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ be given by the quotient map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Then for all $1 \leq p \leq \infty$,

$$\dim \Sigma_{l^p}(X, \mathbb{Z}) = 0.$$

**Proof.** By standard averaging tricks, we may assume that $X$ is a Hilbert space and that $\mathbb{Z}$ acts by unitaries. Since $X$ is now a Hilbert space, we will call it $H$ instead. Let $\pi : \mathbb{Z} \to U(H)$ be the representation given by the action of $\Gamma$, and set $U = \pi(1)$. By passing to direct sums, we may assume that $\pi$ is irreducible, so if we fix any $\xi \in H$ with $\|\xi\| = 1$, then $\xi$ is generating. We will take $S = (\xi)$, and as a pseudonorm we take

$$\|T\| = \|T(\xi)\|.$$

Fix $1 > \varepsilon > 0$, and let $\varepsilon > \delta > 0$. Choose $k$ such that $\delta^k < \varepsilon$, (if $p = \infty$ then let $k$ be any integer.) Since $\pi(\mathbb{Z})$ is compact, we can find an integer $m$ such that

$$\|U^{mj} - 1\| < \delta,$$

for $1 \leq j \leq k$. We may assume that $m$ is large enough so that $\{U^j \xi : 1 \leq j \leq m\}$ spans $H$. Let $F = \{-m(2k+1), \ldots, m(2k+1)\}$, finally let $q_n \in \mathbb{N} \cup \{0\}, 0 \leq r_n < k$ be the integers defined by

$$n = q_n mk + r_n.$$
Define $Q_j$, $j = 0, \cdots, k - 1$ by

$$Q_j = \bigcup_{l=1}^{m} \{jm + l, jm + l + mk, \cdots, jm + l + (mk)(q_n - 1)\},$$

pictorially, if we think of $\{1, \cdots, q_n mk\}$ as a rectangle formed out of $mk$ horizontal dots and $q_n$ vertical dots, then $Q_j$ is the rectangle from the $jm + 1^{st}$ horizontal dot to the $(j + 1)m^{th}$ dot. Let $f_j : Q_j \to \mathbb{C}$ be given by

$$f_j(l) = T(\xi)(\sigma_i(m)j)^{-1}(l)).$$

Note that for $1 \leq p < \infty$,

$$\left\| T(\xi) - \sum_{j=0}^{k-1} f_j(l) \right\|_{L^p(\{1, \cdots, q_n mk\})}^{p} = \sum_{j=0}^{k-1} \sum_{x \in Q_j} \left\| T(\xi)(x) - T(\xi)(\sigma_i(m)j)^{-1}(x) \right\|_{L^p(Q_j)}^{p} < 2k\delta^p < 2\varepsilon,$$

similarly for $p = \infty$,

$$\left\| T(\xi) - \sum_{j=0}^{k-1} f_j(l) \right\|_{L^\infty(\{1, \cdots, q_n mk\})} < 2\varepsilon.$$

Finally note that $\sum_{j=0}^{k-1} f_j$ is constant on

$$\{i, i + m, \cdots, i + m(k - 1)\}$$

for each $i \in Q_0$. Thus

$$\tilde{d}_{2\varepsilon}(\text{Hom}_{\Sigma}((\xi), F, m, \delta, \sigma_n)) \leq q_n m + r_n.$$

This implies that

$$\dim_{\Sigma, L^p}(S, F; m, \delta, \sigma_i) \leq \frac{1}{k},$$

and since $k$ becomes arbitrary large when $\delta$ becomes small (or can be made arbitrarily large when $p = \infty$), this completes the proof.

□

**Proposition 9.8.** Let $\Gamma$ be a countably infinite amenable group. Then for the trivial action on $\mathbb{C}$, for any sofic approximation $\Sigma$ of $\Gamma$ and $1 \leq p \leq \infty$, we have

$$\dim_{\Sigma, L^p}(\mathbb{C}, \Gamma) = 0.$$

**Proof.** We will use $S = (1)$ to do our computation, and use the pseudornom on $B(\mathbb{C}, l^p(d_i))$ given by

$$\|T\|_i = \|T(1)\|.$$

Fix $k \in \mathbb{N}$, $\eta > 0$. By Lemma 4.6 in [9], we can find finite subsets $F_1, \cdots, F_l$ of $\Gamma$ with $|F_j| \geq k$, and which has the following property. For all large $i$, there are $C_1, \cdots, C_l \subseteq \{1, \cdots, d_i\}$ such that

$(s, c) \to \sigma_i(s)c$ from $F_j \times C_j \to \sigma_i(F_j)(C_j)$ is bijective

$\{\sigma_i(F_1)C_1, \cdots, \sigma_i(F_l)C_l\}$ are pairwise disjoint.
Let $\varepsilon > 0$, and let $\delta > 0$ which will depend upon $\varepsilon$ in a manner to be determined later. Let

$$F = \bigcup_{j=1}^{l} F_j \cup \bigcup_{j=1}^{l} F_j^{-1} \cup \{e\}^{2012l+1}.$$  

Also, let $m \geq 1$. Fix $T \in \text{Hom}_\Gamma((1), F, m, \delta, \sigma_i)$, and define $f_j : \sigma_i(F_j)C_j \to \mathbb{C}$ by

$$f_j(\sigma_i(s)(c)) = T(1)(c)$$  

for $s \in F_j, c \in C_j$. Then for $B = \bigcup_{j=1}^{l} \sigma_i(F_j)C_j$,

$$\left\| T(1) - \sum_{j=1}^{l} f_j \right\|_{l^p(B)}^p = \sum_{j=1}^{l} \sum_{s \in F_j} \sum_{c \in C_j} \| T(1)(\sigma_i(s)(c)) - T(1)(c) \| < \delta \sum_{j=1}^{l} |F_j| < \varepsilon,$$

if $\delta$ is chosen sufficiently small. Note that $f_j$ is constant on $\sigma_i(F_j)(c)$ for $c \in C_j$ so $f_j$ is inside a subspace $V_j$ of dimension at most $|C_j|$, where $V_j$ is independent of $T$. Thus

$$d_\varepsilon(\text{Hom}_\Gamma((1), F, m, \delta, \sigma_i), \| \cdot \|_i) \leq \left( \sum_{j=1}^{l} |C_j| \right) + \eta d_i.$$  

Since

$$d_i \geq \sum_{j=1}^{l} |F_j||C_j| \geq k \sum_{j=1}^{l} |C_j|,$$

this implies that

$$\dim_{\Sigma,l^p}(\mathbb{C}, \Gamma) \leq \frac{1}{k} + \eta.$$  

Since $k$ and $\eta$ are arbitrary, this implies that

$$\dim_{\Sigma,l^p}(\mathbb{C}, \Gamma) = 0.$$  

\[\square\]

\textbf{Proof of Theorem 9.1.} If $\Lambda$ is a subgroup of $\Gamma$, then

$$\dim_{\Sigma,l^p}(X, \Gamma) \leq \dim_{\Sigma,l^p}(X, \Lambda)$$

and the claim follows from Proposition 9.7 if $\Gamma$ has finite subgroups of unbounded order. So we may thus assume that $\Gamma$ is an infinite amenable group, and that there is an integer $k$ such that $|\Lambda| \leq k$ if $\Lambda$ is a finite subgroup of $\Gamma$.

Without loss of generality, we may assume that $H$ is a finite-dimensional unitary representation of $\Gamma$ and show that

$$\dim_{\Sigma,l^p}(H, \Gamma) = 0.$$  

First note that there is a function $k : (0, \infty) \to \mathbb{N} \cup \{0\}$ with

$$\lim_{\varepsilon \to 0} k(\varepsilon) = \infty$$

such that if $g \in \Gamma$, $\rho : \Gamma \to U(H_0)$ is a unitary representation of $\Gamma$ and $\|\rho(g) - 1\| < \varepsilon$, then either $\rho(g) = \text{Id}$ or the order of $g$ is at least $k(\varepsilon)$.

Let $\pi : \Gamma \to U(H)$ be the homomorphism given by the action of $\Gamma$. We can find $\varepsilon > 0$ sufficiently small so that $k(\varepsilon) > k$, and our assumption thus implies that if
If one of the \( g_n \) has infinite order, then \( \mathbb{Z} \hookrightarrow \Gamma \), and we are done by the case of \( \mathbb{Z} \). Otherwise, our assumption implies that \( \pi(g_n) = 1 \) for all \( n \), and if \( \Lambda = \langle g_n : n \in \mathbb{N} \rangle \) then \( \Lambda \) is an infinite amenable subgroup of \( \Gamma \) which acts trivially. Using subadditivity under direct sums and Proposition 9.8,

\[
\dim_{\Sigma, \ell^p}(X, \Gamma) = 0.
\]

From this, we can recover a counterexample due to A.Gournay. Let \( A \) be a countably infinite abelian group. And let \((x_n)_{n=1}^{\infty}\) be a dense sequence in \( \hat{A} \). Let

\[
V_n = \{ f \in \ell^1(A) : \hat{f}(x_j) = 0, \text{ for } 1 \leq j \leq n \}.
\]

Since the sequence

\[
0 \longrightarrow V_n \longrightarrow \ell^1(A) \longrightarrow \ell^1(A)/V_n \longrightarrow 0,
\]

is exact, we have (for any sofic approximation \( \Sigma \)

\[
1 = \dim_{\Sigma}(\ell^1(A), A) \leq \dim_{\Sigma}(V_n, A) + \dim_{\Sigma, \ell^p}(\ell^1(A)/V_n, A) = \dim_{\Sigma}(V_n, A),
\]

since \( \ell^1(A)/V_n \) is finite-dimensional (in fact, isomorphic to \( \mathbb{C}^n \)). From abstract fourier analysis, it is known that there is a function \( g \in \ell^1(A) \) so that \( \hat{g} \) vanishes only at \( x_1, \ldots, x_n \), further any such \( g \) dynamically generates \( V_n \) by Theorem 7.2.4 of [15]. Thus

\[
\dim_{\Sigma, \ell^p}(V_n, A) = \dim_{\Sigma, \ell^p}(V_n, A) = 1.
\]

Since

\[
\bigcap_{n=1}^{\infty} V_n = \{0\},
\]

we have an example of a decreasing sequence of subspaces of \( \ell^1 \), each with dimension one, whose intersection is trivial. Thus we cannot have continuity of dimension under intersections for \( \ell^1 \)-dimension.

10. Further Questions and Conjectures

As of yet I have been unable to show the following

Conjecture 1. Let \( \dim^G_p \) denote \( p \)-dimension as defined by Gournay. For any \( 1 \leq p < \infty \), for any amenable group \( \Gamma \), and any sofic approximation \( \Sigma \), and \( V \subseteq \ell^p(\Gamma, V) \) a closed \( \Gamma \)-invariant subspace we have

\[
\dim_{\Sigma, \ell^p}(Y, \Gamma) = \dim^G_p(Y, \Gamma).
\]

A good first step to this would be to establish the following.

Conjecture 2. Let \( \Gamma \) be a sofic group and \( \sigma : \Gamma \to S_d \) a sofic approximation, let \( 2 < p < \infty \), and \( V \) a finite dimensional normed vector space. Then

\[
\dim_{\Sigma, \ell^p}(\ell^p(\Gamma, V), \Gamma) = \dim V.
\]

Our method for proving Corollary [7, 4] does not give a computation of \( S^p \) dimension when \( 1 \leq p < 2 \).
**Conjecture 3.** Let $\Gamma$ be a $R^\omega$-embeddable group, and fix $2 < p < \infty$. Then for the action of multiplication of $\Gamma$ on $L^p(L(\Gamma), \tau_\Gamma)^{\otimes n}$, and any embedding sequence $\Sigma$, 
\[ \dim_{\Sigma, S^p} L^p(L(\Gamma), \tau_\Gamma)^{\otimes n} = \dim_{\Sigma, S^p} L^p(L(\Gamma), \tau_\Gamma)^{\otimes n} = n. \]

Again the difficulty in proving this is obtaining a good lower bound for approximate dimensions. Although, it should following from uniqueness of a sofic approximation up to conjugacy (see [6]) I have been unable to show that $l^p$-dimension for amenable groups does not depend upon the choice of sofic approximation.

**Conjecture 4.** Let $\Gamma$ be an amenable group with a uniformly bounded action on a separable Banach space $Y$, and $1 \leq p < \infty$. Then for any two sofic approximations $\Sigma, \Sigma'$ of $\Gamma$ we have 
\[ \dim_{\Sigma, l^p}(X, \Gamma) = \dim_{\Sigma', l^p}(X, \Gamma), \]
\[ \dim_{\Sigma, l^p}(X, \Gamma) = \dim_{\Sigma', l^p}(X, \Gamma). \]

Based on the results of Section 9, if $\Sigma = (\sigma_i : \Gamma \to S_{d_i}), \Sigma' = (\sigma_i : \Gamma \to S'_{d_i})$ we have the desired equality if $d_i \to 1$. Similarly from Section 9 I expect the following to be true:

**Conjecture 5.** Let $\Gamma$ be a infinite sofic group and $\Sigma$ a sofic approximation. Let $V$ be a finite-dimensional representation of $\Gamma$, then for all $1 \leq p < \infty$, 
\[ \dim_{\Sigma, l^p}(V, \Gamma) = 0. \]

The above should also be true if $\Gamma$ is $R^\omega$-embeddable and $l^p$ is replaced by $S^p$. Following the methods in Section 9 it suffices to consider the case $V = \mathbb{C}$ with $\Gamma$ acting trivially.

Following the ideas of Section 7 I conjecture the following:

**Conjecture 6.** Let $\Gamma$ be a sofic group, and fix $1 \leq p \leq 2$. Set $M = L(\Gamma)$, and $\tau$ the canonical group trace. Let $\mathcal{F} : l^p(\Gamma) \to L^{p'}(M, \tau)$, where $1/p + 1/p' = 1$ be the Fourier transform as defined in Section 7. For $X \subseteq l^p(\Gamma)$, let $q \in M$ be a projection such that 
\[ \mathcal{F}(X)^{\| \cdot \|^p} = L^{p'}(M, \tau)q. \]

Then 
\[ \dim_{\Sigma, l^p}(X, \Gamma) \geq \tau(q). \]

Similarly if $\Gamma$ is $R^\omega$-embeddable group and $X, p, p', q, \tau$ are as above, then
\[ \dim_{\Sigma, S^p, \text{conj}}(X, \Gamma) \geq \tau(q). \]

This conjecture is probably too strong to be true in full generality, but it would be interesting to see if even some special cases could be show.

It would also be interesting to see in the above situation when either of 
\[ \dim_{\Sigma, l^p}(X, \Gamma) \leq \tau(q), \]
\[ \dim_{\Sigma, l^p}(X, \Gamma) = \dim_{\Sigma, l^p}(X, \Gamma) = \tau(q) \]
hold.

In particular, if $A$ is a countable discrete abelian group and $\mu$ is the Haar measure on $A$, and $f \in l^p(\Gamma)$, there should be a relation between
\[ \dim_{\Sigma, l^p}(\text{Span}(\Gamma f)^{\| \cdot \|^p}, \Gamma) \]
and 
\[ \mu(\{ x \in \hat{A} : \hat{f}(x) = 0 \}). \]

This would be another instance of “dimension is measure” that is well known in the case \( p = 2 \). We observed the equality between these two quantities in the case \( p = 1 \), and the zero set of \( \hat{f} \) is finite.

Related to these ideas I also conjecture the following

**Conjecture 7.** Let \( \Gamma \) be an \( R^\infty \)-embeddable group and \( 2 \leq p < \infty \), let \( C^*_\lambda(\Gamma) \) be the reduced \( C^* \)-algebra of \( \Gamma \), and \( I \subseteq C^*_\lambda(\Gamma) \) a norm closed left ideal. Regard \( C^*_\lambda(\Gamma) \subseteq L(\Gamma) \) and let \( q \in L(\Gamma) \) be the projection such that

\[ T^{ek^*} = L(\Gamma)q. \]

Then for the action of \( \Gamma \) by left multiplication:

\[ \dim_{\Sigma, SP, \text{multi}} (I, \Gamma) \geq \tau(q). \]

Again either inequality would be nice to know.

In general, it would be nice to see a version of “Fourier duality” between \( l^p \)-dimensions of \( \Gamma \)-invariant subspaces of \( l^p(\Gamma)^{\oplus n} \) and \( S^p \)-dimension of \( \Gamma \)-invariant subspaces of \( L^p(L(\Gamma), \tau_\Gamma)^{\oplus n} \).

Lastly we mention that our definition should generalize to the case of operator spaces although one should probably replace \( \varepsilon \)-dimension with the notion of \( \varepsilon \)-rank defined in [17]. Our method of proof should work to define an invariant for uniformly completely bounded representations of a group on an exact operator space, essentially by replacing spaces with the bounded approximation property by those which are nuclear. It would be nice to see an application of these ideas to representations on operator spaces.

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