Abstract
We test the chronology protection conjecture in classical general relativity by investigating finitely vicious space-times. First we present singularity theorems in finitely vicious space-times by imposing some restrictions on the chronology violating sets. In the theorems we can refer to the location of an occurring singularity and do not assume any asymptotic conditions such as the existence of null infinities. Further introducing the concept of a non-naked singularity, we show that a restricted class of chronology violations cannot arise if all occurring singularities are the non-naked singularities. Our results suggest that the causal feature of the occurring singularities is the key to prevent the appearance of causality violation.
1 Introduction

Whether or not there are causality violating regions in the universe is an interesting problem in physics. Hawking has proposed the chronology protection conjecture which states that the laws of physics prevent closed timelike curves from appearing [1]. He investigated the instability of a Cauchy horizon caused by causality violation and showed that the renormalized stress-energy tensor always diverges at the Cauchy horizon by using quantum field theory. The instabilities of such Cauchy horizons have also been shown in this approach by several authors [2]. On the other hand, Li et al. [3] obtained the opposite result that the Cauchy horizon is stable against the vacuum fluctuations under some conditions. There also remains an unsettled problem that the back reaction of quantum effects on the geometry is difficult to take into account. At present, there seems no consensus on the quantum instability of the Cauchy horizon caused by causality violation.

In the framework of classical general relativity, only a few attempts have so far been made at verifying the chronology protection conjecture. Hawking [1] considered a compactly generated Cauchy horizon that all the past-directed null geodesic generators of the Cauchy horizon are totally imprisoned in a compact set. He showed that such a Cauchy horizon cannot occur under the weak energy condition. Tipler provided a partial answer to the conjecture by showing that the creation of a closed timelike curve leads to singularity formation under the stronger energy condition than the usual energy conditions [4, 5]. Ori [6] discussed the relation between causality violation and the weak energy condition.

In the context of testing the conjecture, one would like to examine the case that causality violation could arise in a finite region of space from a regular initial data by physical process such as gravitational collapse. In such a case space-time may be finitely vicious, whose notion was first introduced by Tipler [5]. Space-times containing time machines are examples of finitely vicious space-times and many models of such space-times have been proposed. Morris et al. [7] suggested a model of a time machine in a wormhole space-time by accelerating one mouth of the wormhole with respect to the other mouth. Deutsch and Politzer [8] constructed a simple model of a time machine (the Deutsch-Politzer time machine) by identifications of spacelike line segments in two-dimensional Minkowski space-time. It is interesting that the null geodesic generators of the boundary of causality violating set in this model are not imprisoned in any compact sets, while those of a compactly generated Cauchy horizon are. The geometrical property of the Doutsch-Politzer time machine in asymptotically flat space-time is investigated by Chamblin et al. [9].

Most recently, Krasnikov [10] proposed a simple model of time machine by modifying the Deutsch-Politzer time machine. Surprisingly, there are no singularities associated with the chronology violation in this model even though the weak energy condition is satisfied. This model demonstrates that the weak energy condition does not necessarily prevent causality violations of the Deutsch-Politzer time machine type from occurring in classical general relativity; the chronology protection conjecture is violated even if this energy condition holds.

In this paper, we test the chronology protection conjecture by investigating finitely vicious space-times under some reasonable conditions, especially, the strong energy condi-
tion. To begin with we define a class of finitely vicious space-times whose causal structures are similar to that of a space-time with the Deutsch-Politzer time machine. Next we show that there exists an inextendible and incomplete causal geodesic in the chronology violating region under physically reasonable conditions. To prove this, we do not assume the existence of any asymptotic regions. Introducing the notion of a non-naked singularity, we finally show that some classes of chronology violations which will be specified by a suitable boundary condition for the chronology violating set cannot arise if all occurring singularities are the non-naked singularities.

In the next section, we consider a geometry of finitely vicious space-time. In Sec. 3, we present our singularity theorems in finitely vicious space-times. On the bases of the results in Sec. 3, we will present our main theorem in Sec. 4. Section 5 is devoted to summary and discussion.

2 A geometry of chronology violation

We shall consider space-times in which causality violating sets (chronology violating sets) are formed in gravitational collapse or “manufactured” as time machines by future technology. In such a space-time causality violation should begin in a finite region of space in the future of a partial Cauchy surface $S$. It is expected that associated with the causality violation a Cauchy horizon $H^+(S)$ with compact spatial section develops in the future of $S$. Such space-times are classified in a finitely vicious space-time by Tipler [4], whose precise definition is given in the following.

It is said that a space-time $(M, g)$ is finitely vicious if it has a hypersurface slicing $S(\tau)$ with the properties

(i) $S$ is one of the slices with $S(0) = S$;

(ii) there is a closed interval $[\tau_1, \tau_2]$ of the slice parameter such that if $\tau \in [\tau_1, \tau_2]$, then $S(\tau) \cap D^+(S)$ is spacelike, and $S(\tau) \cap H^+(S)$ is compact. Also, if $\tau_3, \tau_4$ are any two numbers in $[\tau_1, \tau_2]$ with $\tau_4 \geq \tau_3$, then $S(\tau_4) \cap D^+(S)$ lies to the future of $S(\tau_3) \cap D^+(S)$;

(iii) Let $B$ be the region of space-time in $D^+(S)$ between $S(\tau_1)$ and $S(\tau_2)$ inclusive, and let $\gamma$ be any segment of a generator of $H^+(S)$ with $\gamma \cap S(\tau_2) \neq \emptyset$ and $\gamma \subset B$.

Then $\gamma$ can be extended in $H^+(S) \cap B$ such that the extension intersects each $S(\tau)$ for $\tau \in [\tau_1, \tau_2]$ exactly once.

There are some works on constructions of time machines [4]–[6] in finitely vicious space-times with non-trivial topology. The four-dimensional Deutsch-Politzer time machine [5], constructed below, is one of the simplest models of such time machines. Consider two-dimensional Minkowski space-time $I$ with metric

$$ds^2 = -dt^2 + dx^2,$$

and two horizontal line segments $l_\pm := \{t, x|t = \pm a, |x| < b\}$ on $I$. Remove the points $(|t| = a, |x| = b)$ from $I$ and make cuts along $l_\pm$. Identifying the upper banks of $l_+$ and $l_-$ with the lower banks of $l_-$ and $l_+$, respectively, one can construct a two-dimensional space-time $I'$ containing the two-dimensional Deutsch-Politzer time machine. Then one can obtain the four-dimensional Deutsch-Politzer time machine as a product space-time $M = I' \times Q$, where $Q$ is a closed compact two-dimensional space, like $S^2$ or $T^2$. The causal structure of this space-time is schematically depicted in Fig. 1.
Figure 1: The four-dimensional Deutsch-Politzer time machine $M$ is shown. The hexagon is a chronology violating-region. $S$ is a locally spacelike hypersurface. Each point represents a closed spacelike two-surface $Q$. The points $p$, which correspond to the points $(|t| = a, |x| = b)$ in $I$, are removed.

The chronology violating-region, the interior of the hexagon in Fig. 1, is described as $I^+(q) \cap I^-(q), q \in M$. The boundary of the chronology violating-region consists of portions of $I^+(q)$ and $I^-(q)$. As easily seen in Fig. 1, the space-time is finitely vicious because there is a hypersurface $S$ whose intersection with $I^+(q) \cap I^-(q)$ is compact.

From the above observations, as a simple and reasonable class of finitely vicious space-times, we focus our attention on a finitely vicious space-time $(M, g)$ with a chronology violating set $V$ satisfying the following three conditions:

**Conditions**

1. there is a locally spacelike three-dimensional hypersurface $S$ (with no edge) such that $K := V \cap S$ has a compact closure $\overline{K}$ and the boundary $\partial K$ is equal to $\partial I^+(q) \cap \partial I^-(q)$,
2. every closed causal curve in $\overline{V}$ is a noncontractible curve which intersects $\overline{K}$,
3. $R_{ab}K^aK^b \geq 0$ for every non-spacelike vector $K^a$ and every causal geodesic with a tangent vector $K^a$ contains a point at which $K_{[a}R_{b]cd}K^eK^fK^cK^d \neq 0$.

Condition 1 is a boundary condition for the chronology violating set. Condition 2 is imposed for the simplicity of arguments since we consider closed causal curves associated with non-trivial topology such as the topology in the Deutsch-Politzer time machines. Condition 2 ensures that the causality condition holds on the covering manifold $\tilde{M}$ that will be considered in the following sections. Condition 3 means that the strong energy condition and the generic condition are satisfied.

It should be remarked that in general the chronology violating set is described as the disjoint union of sets of the form $I^+(p) \cap I^-(p), p \in M$ (see Prop. 6.4.1 of Ref. [1]). For simplicity, in the rest of this paper, we shall restrict our discussion to a single component of the chronology violating set without loss of generality and denote the component by $V$, unless there arises confusion.
3 Singularity theorems

In this section, we will investigate finitely vicious space-times satisfying conditions 1, 2, 3 and present theorems which state that there is a singularity in the causality violating-region. Before showing this, we shall explicitly define the statement “there is an incomplete causal geodesic curve (singularity) in a sub-region $N$ in a space-time” as follows.

Definition 1 Consider a submanifold $N$ of a space-time $(M, g)$. If there is an affinely parameterized causal geodesic $\gamma : [0, a) \rightarrow M$ which is inextendible in $M$ and incomplete at the value $a$ such that for each value $t \in [0, a)$ the point $\gamma(t)$ exists in $N$, then we can say that $N$ is causally geodesically incomplete.

Now we shall show that $V$ is causally geodesically incomplete without imposing any singularity conditions (Theorem 2) to begin with. On the basis of Theorem 2, we next present a theorem (Theorem 11) which states that $V$ is causally geodesically incomplete under the strong curvature singularity condition.

To prove the theorems, for the space-time $(M, g)$ under consideration, we often use the covering manifold $\hat{M}$ defined as the set of all pairs $(p, [\lambda])$ where $[\lambda]$ is an equivalence class of curves from a locally spacelike hypersurface (with no edge) $S$ to $p \in M$ homotopic modulo $S$ and $p$ (see Ref. [11], p. 205). Let $\pi : \hat{M} \rightarrow M$ be a projection. Then, for any point $p$ of $M$, there exists an open neighborhood $U$ of $p$ such that $\pi^{-1}(U)$ is a collection of open sets of $\hat{M}$, each mapped diffeomorphically onto $U$ by $\pi$. It is worth noting that each connected component of the image of $S$ is achronal and homeomorphic to $S$ because of the construction of $\hat{M}$. For chronology violating set $V$, a set $\hat{V}$ is defined as the set of all points $\hat{q} \in \hat{M}$ such that $\pi(\hat{q}) \in V$. Similarly, for a subset $N$ in $M$, $\hat{N}$ denotes hereafter the corresponding subsets $\pi^{-1}(N)$ in $\hat{M}$, unless otherwise explicitly stated.

3.1 Singularity Theorem I

We present the following theorem.

Theorem 2 If a finitely vicious space-time $(M, g)$ with a chronology violating set $V$ satisfies conditions 1, 2, 3, then $\hat{V}$ is causally geodesically incomplete.

To prove this theorem, we give the following proposition and lemmas under conditions 1, 2, 3, provided that $\hat{V}$ is causally geodesically complete.

Proposition 3 There is no causal curve which leaves $V$ and then returns to $V$.

Proof. Suppose that there was a causal curve $\gamma$ which left $V$ and reentered $V$, i.e., $\gamma$ had endpoints $p, r \in V$ and contained a point of $M - V$ between $p$ and $r$. Let us consider only the case that $\gamma$ was past-directed from $p$ to $r$ without loss of generality. Then there would be a point $s \in \gamma$ on $\hat{V}$ such that $p > s > r$. Since $V$ can be described as $I^+(p) \cap I^-(p)$, $s$ was in $I^+(p)$ or $I^-(p)$. Consider the case that $s \in I^-(p)$. Because $p \in V$, there would be a closed timelike curve $\gamma_1$ through $p$. Since the causal curve $\gamma_1 + \gamma$ is not a null geodesic, one could obtain a past-directed timelike curve $\gamma_2$ from $p$ to $s$ by deformation of $\gamma_1 + \gamma$.
following Prop. 4.5.10 of Ref. [11]. This contradicts \( s \in \hat{I}^-(p) \). Therefore \( s \in \hat{I}^+(p) \).

Because \( r \in \hat{I}^+(p) \) [recall that \( r \in V = \hat{I}^+(p) \cap \hat{I}^-(p) \)], there would be a past-directed causal curve \( \gamma_3 \) from \( r \) to \( p \). Then there was a past-directed causal curve \( \gamma_4 \) from \( s \) to \( p \) which consists of a segment of \( \gamma \) from \( s \) to \( r \) and \( \gamma_3 \). Then, from Prop. 4.5.10 of Ref. [11], one could obtain a past-directed timelike curve which connects \( s \) with \( p \) by deformation of \( \gamma_4 + \gamma_1 \). This is impossible because \( s \in \hat{I}^+(p) \).

\[\square\]

**Lemma 4** The strong causality condition holds in \( \hat{V} \).

The proof is similar to that of Prop. 6.4.6 of Ref. [11].

**Proof.** Suppose that the strong causality condition was violated at a point \( \hat{p} \in \hat{V} \). Take a convex normal neighborhood \( \hat{U}_p \) of \( \hat{p} \in \hat{V} \) such that \( \hat{U}_p \subset \hat{V} \). Let \( \hat{U}_n \subset \hat{U}_p \) be an infinite sequence of neighborhoods of \( \hat{p} \) such that any neighborhood of \( \hat{p} \) contains all the \( \hat{U}_n \) for \( n \) large enough. For each \( \hat{U}_n \), there would be a future-directed causal curve \( \hat{\lambda}_n \) in \( \hat{V} \) which left \( \hat{U}_n \) and then returned to \( \hat{U}_n \). By Lemma 6.2.1 of Ref. [11], there would be an inextendible causal curve \( \hat{\lambda} \) through \( \hat{p} \) which was a limit curve of \( \hat{\lambda}_n \). It follows from Prop. 4 that \( \hat{\lambda} \subset \hat{V} \) because, for each \( n \), \( \hat{\lambda}_n \subset \hat{V} \). Consider the case that \( \hat{\lambda} \) was a null geodesic. Then \( \hat{\lambda} \) could have a timelike separation, since \( \hat{V} \) was null geodesically complete and the generic condition holds by condition 3. In the case that \( \hat{\lambda} \) was not a null geodesic, \( \hat{\lambda} \) could also have a timelike separation. Thus for any case one could join up some \( \hat{\lambda}_n \) to give a closed causal curve \( \hat{\lambda}_n' \). \( \hat{\lambda}_n' \) was in \( \hat{V} \) by Prop. 4 (recall that each \( \hat{\lambda}_n \) is in \( \hat{V} \) and therefore the closed causal curve \( \hat{\lambda}_n' \) contains a point in \( \hat{V} \)). This is impossible because the covering manifold \( \hat{M} \) contains no closed causal curves in \( \hat{V} \) by the condition 2.

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**Lemma 5** Let \( \hat{K} \) be one of the connected components of \( \pi^{-1}(\hat{K}) \) and consider a future (past) Cauchy horizon \( H^+(\hat{K}) \) [resp. \( H^-(\hat{K}) \)]. If \( H^+(\hat{K}) \cap \hat{V} \) [resp. \( H^-(\hat{K}) \cap \hat{V} \)] is non-empty, then each past-(future-)directed null geodesic generator through a point of \( H^+(\hat{K}) \cap \hat{V} \) [resp. \( H^-(\hat{K}) \cap \hat{V} \)] has no past (future) endpoint and remains in \( \hat{V} \).

**Proof.** Let us consider only \( H^+(\hat{K}) \) without loss of generality. Let \( \hat{\xi} \) be a past-directed null geodesic generator through a point \( \hat{r} \) of \( H^+(\hat{K}) \cap \hat{V} \). Suppose that \( \hat{\xi} \) had a past endpoint \( \hat{s} \), which was in edge(\( \hat{K} \)). By condition 1, \( s = \pi(\hat{s}) \) was in \( \hat{I}^+(r,M) \cap \hat{I}^-(r,M), r = \pi(\hat{r}) \in V \). Then there would be a past-directed causal curve \( \xi = \pi(\hat{\xi}) \subset M \) from \( r \) to \( s \). This is impossible by the same argument in the proof of Prop. 4 since \( s \) was in \( \hat{I}^-(r,M) \). Therefore each null geodesic generator through a point of \( H^+(\hat{K}) \cap \hat{V} \) is past-inextendible.

Suppose that \( \hat{\xi} \) left \( \hat{V} \). Let \( \hat{t} \) be a point of \( (\hat{M} - \hat{V}) \cap \hat{\xi} \) and \( \hat{U}_t \) an arbitrary small neighborhood of \( \hat{t} \). Since \( \hat{t} \) would be in \( H^+(\hat{K}), \hat{I}^-(\hat{t}) \cap \hat{U}_t \cap \text{int}(D^+(\hat{K})) \) is non-empty. Take a point \( \hat{t}' \) in \( \hat{I}^-(\hat{t}) \cap \hat{U}_t \cap \text{int}(D^+(\hat{K})) \). Then one could get a past-directed causal curve \( \hat{\mu} \) from \( \hat{r} \in \hat{V} \) to \( (\hat{K} - \text{edge}(\hat{K})) \subset \hat{V} \) through \( \hat{t} \) and \( \hat{t}' \) because \( \hat{t}' \in \text{int}(D^+(\hat{K})) \). The existence of \( \hat{\mu} \) would imply that a causal curve \( \mu = \pi(\hat{\mu}) \) left \( V \) and then returned to \( V \) in \( M \). This is impossible by Prop. 4. Therefore each past-directed null geodesic generator through a point of \( H^+(\hat{K}) \cap \hat{V} \) remains in \( \hat{V} \).

\[\square\]

**Lemma 6** There exists a future-inextendible timelike curve \( \hat{\gamma} \) in \( \text{int}(D^+(\hat{K})) \) and \( J^-(\hat{\gamma}) \cap J^+(\hat{K}) \) is in \( \hat{V} \).
Proof. There exists a closed timelike curve $\gamma'$ through a point of $K$ because $K \subset V$. Then, in the covering manifold $\hat{M}$, $\hat{\gamma}' = \pi^{-1}(\gamma')$ is a future-inextendible timelike curve from a point of $\text{int}(\hat{K})$.

If $\hat{\gamma}'$ does not intersect $\hat{H}^+(\hat{K})$, $\hat{\gamma}'$ is the desired future-inextendible timelike curve contained in $\text{int}(\hat{D}^+(\hat{K})) \cap \hat{V}$.

Consider the case that $\hat{\gamma}'$ intersects $\hat{H}^+(\hat{K})$. Denote the intersection point by $\hat{p}(\in \hat{V})$ and $J^-(\hat{p}) \cap \hat{K}$ with a compact closure by $\hat{\Sigma}$, respectively. $[I^-(\hat{p}) \cap J^+(\hat{K})] \subset \hat{D}^+(\hat{K})$ because $\hat{p} \in (\hat{H}^+(\hat{K}) - \hat{K})$. This means that $[I^-(\hat{p}) \cap J^+(\hat{K})] \subset \hat{D}^+(\hat{\Sigma})$ by definition of $\hat{\Sigma}$. The past-directed null geodesic generator $\hat{\xi}_p$ of $\hat{H}^+(\hat{K})$ through $\hat{p}$ is a past-inextendible curve in $\hat{V}$ which does not intersect $\hat{\Sigma}$ by Lemma [1]. Then $\hat{\xi}_p$ is also the null geodesic generator of $\hat{H}^+(\hat{\Sigma}) \cap \hat{V}$ because $\hat{\xi}_p \in [I^-(\hat{p}) \cap J^+(\hat{K})] \subset \hat{D}^+(\hat{\Sigma})$.

Suppose that $\hat{H}^+(\hat{\Sigma})$ was compact and hence it could be covered by a finite number of local causality neighborhoods $\hat{U}_i$. Since $\hat{\xi}_p \subset \hat{H}^+(\hat{\Sigma}) \cap \hat{V}$ one could take a subcollection $\{\hat{U}_m\}$ of the neighborhoods $\hat{U}_i$ such that each $\hat{U}_m$ was contained in $\hat{V}$ and $\hat{\xi}_p$ remained in the compact subset $\cup_m \hat{U}_m$ of $\cup_i \hat{U}_i$. This is impossible by Prop. 6.4.7 of Ref. [11] because the strong causality condition holds on each $\hat{U}_m(\subset \hat{V})$ by Lemma [1]. Therefore $\hat{H}^+(\hat{\Sigma})$ is non-compact. Following the proof of Corollary on page 268 of Ref. [11], put a timelike vector field on $\hat{M}$. Because $\hat{\Sigma}(\subset \hat{S})$ is achronal, $\Sigma$ is also achronal. Then, if every integral curve of the vector field which intersected $\hat{\Sigma}$ also intersected $\hat{H}^+(\hat{\Sigma})$, they would define a continuous one-one mapping of $\hat{\Sigma}$ onto $\hat{H}^+(\hat{\Sigma})$ and hence $\hat{H}^+(\hat{\Sigma})$ would be compact. This is a contradiction. Therefore there is a future-directed inextendible timelike curve $\hat{\gamma}$ in $\text{int}(\hat{D}^+(\hat{\Sigma}))$. $\hat{\gamma}$ does not intersect $I^-(\hat{p}) \cap J^+(\hat{K})$, otherwise there would be a point $\hat{s} \in \hat{\gamma}$ such that $(J^-(\hat{s}) \cap \hat{K}) \not\subset \hat{\Sigma}$. Thus $\hat{\gamma}$ is contained in $\text{int}(\hat{D}^+(\hat{\Sigma})) \cap J^-(\hat{p})$ and also in $\text{int}(\hat{D}^+(\hat{K})) \cap J^-(\hat{p})$.

To prove that $J^-(\hat{\gamma}) \cap J^+(\hat{K})$ is also in $\hat{V}$, it is enough to show that $J^-(\hat{p}) \cap J^+(\hat{K})$ does not contain any point of $\hat{M} - \hat{V}$ because $J^-(\hat{\gamma}) \cap J^+(\hat{K}) \subset J^-(\hat{p}) \cap J^+(\hat{K})$. If there was a point of $\hat{M} - \hat{V}$ in $J^-(\hat{p}) \cap J^+(\hat{K})$, there would be a past-directed causal curve from $\hat{p}$ to a point $\hat{r} \in \hat{\Sigma} \cap \hat{\Sigma}$ through a point of $\hat{M} - \hat{V}$ (there is no past-directed causal curve from $\hat{p}$ to a point of edge(\hat{K}) as $\hat{\xi}$ in the proof of Lemma [1]). This is impossible by Prop. [1] because $\hat{p}$ and $\hat{r}$ was in $\hat{V}$.

In the case that $\hat{\gamma}'$ does not intersect $\hat{H}^+(\hat{K})$, it also can be shown that $J^-(\hat{\gamma}') \cap J^+(\hat{K})$ is in $\hat{V}$. \hfill \Box

Lemma 7. Let us suppose that there exists a past-inextendible timelike curve $\hat{\lambda} \in \hat{V}$ through a point of $\text{int}(\hat{T})$, where $\hat{T} := J^-(\hat{\gamma}) \cap \hat{K}$ and $\hat{\gamma}$ is the future-inextendible timelike curve obtained in Lemma [1]. If $\hat{\lambda}$ intersects $H^-(\hat{J}^-\hat{T})$ at a point $\hat{p}(\in \hat{V})$, each null geodesic generator of $[I^+(\hat{p}) \cap J^-(\hat{T})]$ is contained in $E^-\hat{T} \cap \hat{V}$.

Proof. Consider an infinitesimally small neighborhood $\hat{U}$ of $\hat{p}$ and denote a set $[\hat{U} \cap I^+(\hat{p})]$ by $\hat{C}$. Because $\hat{p} \in [H^-\hat{J}^-\hat{T}) - J^-\hat{J}(\hat{T})]$, $\hat{C} \subset \text{int}[D^-\hat{J}^-\hat{T})]$. Thus, every future-inextendible causal curve from any point of $\hat{C}$ intersects $\hat{J}^-\hat{T}$ and also $I^+(\hat{J}^-\hat{T})$ by Lemma 6.6.4 of Ref. [11]. Suppose that there was a future-inextendible null geodesic generator of $J^+(\hat{C}) \cap J^-\hat{T})$. This means that there was a future-inextendible causal
curve from a point of $\hat{C}$ and remained in $J^-(\hat{T})$. This is a contradiction. Therefore every null geodesic generator of $J^+(\hat{C}) \cap J^-(\hat{T})$ has a future endpoint in $\hat{T}$, which means that $[J^+(\hat{C}) \cap J^-(\hat{T})] \subset J^-(\hat{T})$ but $\not\subset I^-(\hat{T})$. By definition of $E^-(\hat{T}) := J^-(\hat{T}) - I^-(\hat{T})$, each null geodesic generator of $[J^+(\hat{C}) \cap J^-(\hat{T})] \subset E^-(\hat{T})$ and hence $[I^+(\hat{p}) \cap J^-(\hat{T})] \subset E^-(\hat{T})$.

Denote $[I^+(\hat{p}) \cap J^-(\hat{T})]$ by $\Sigma$ and the null geodesic generator of $\Sigma$ by $\hat{\eta}$. The future endpoint of $\hat{\eta}$, $\hat{T} \cap \hat{\eta}$, is in $\text{int}(\hat{K})$ (It is impossible that the future endpoint of $\hat{\eta}$ is in $\hat{K}$ by the facts that $\pi(\hat{p})$ is in $V$ and $\hat{K} = \hat{I}^+(\pi(\hat{p})) \cap \hat{I}^-(\pi(\hat{p}))$ as discussed in Lemma 8). Then it follows from Prop. 8 that every null geodesic $\hat{\eta}$ is contained in $\hat{V}$ because $\hat{p}$ and int$(\hat{K})$ are contained in $\hat{V}$.

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**Lemma 8** There is a future and past-inextendible timelike curve in $\text{int}[D(E^-(\hat{T}))] \cap \hat{V}$.

**Proof.** We have only to show that there is a past-inextendible timelike curve $\hat{\lambda}$ in $\text{int}[D^-(E^-(\hat{T}))]$ because of Lemma 8.

There is a closed timelike curve $\lambda$ through a point of $\text{int}(\pi(\hat{T}))$ because $\text{int}(\pi(\hat{T})) \subset V$. This means that there exists a past-inextendible timelike curve $\hat{\lambda} = \pi^{-1}(\lambda)$ in $\hat{V}$ from a point of $\text{int}(\hat{T})$ because $\hat{\lambda}$ is inextendible. Let us consider a Cauchy development $D^-(J^-(\hat{T}))$.

(a) Assume that $\hat{\lambda}$ does not intersect $H^-(\hat{J}^-(\hat{T}))$. Each point of $\hat{\lambda}$ is in $\text{int}[D^-(J^-(\hat{T}))]$ and hence every future-inextendible causal curve from $\hat{\lambda}$ intersects $J^-(\hat{T})$ and also $I^+(\hat{J}^-(\hat{T}))$. Then $[J^+(\hat{\lambda}) \cap J^-(\hat{T})] \subset E^-(\hat{T})$ as shown in the proof of the previous lemma and hence $\hat{\lambda}$ is a desired curve.

(b) Assume that $\hat{\lambda}$ intersects $H^-(\hat{J}^-(\hat{T}))$. Denote the intersection point and the future-directed null geodesic generator of $H^-(\hat{J}^-(\hat{T}))$ through the point by $\hat{p}$ and $\xi_p$, respectively. $\xi_p$ is future-inextendible by Prop.6.5.2 and Prop.6.5.3 of Ref. [11]. Let us consider a set $\Sigma := [I^+(\hat{p}) \cap J^-(\hat{T})]$. By Lemma 8, each null geodesic generator of $\Sigma$ is contained in $E^-(\hat{T}) \cap \hat{V}$. If $\Sigma$ was non-compact, there would be a past-inextendible null geodesic generator $\hat{\eta}$ of $E^-(\hat{T}) \cap \hat{\Sigma}$. This is impossible because the null geodesic generator $\hat{\eta}$ has a past endpoint by condition 3, otherwise one could get a past incomplete null geodesic $\hat{\eta}$ in $\hat{V}$ or a future incomplete null geodesic generator of $J^-(\hat{\eta})$ in $\overline{V}$ since $J^-(\hat{\eta}) \cap J^+(\hat{K}) \subset \hat{V}$ by Lemma 8. Therefore $\Sigma$ is compact.

(b-I) Assume that $\xi_p$ does not intersect $E^-(\hat{T}) \cap \overline{\Sigma}$. It can be shown that $\xi_p$ remains in $\hat{V}$ by the same argument in Lemma 3. Then $H^-(\overline{\Sigma})$ is non-compact by the facts that $\xi_p \subset (\hat{V} \cap H^-(\overline{\Sigma}))$ and the strong causality condition holds in $\hat{V}$ as shown in the proof of Lemma 8. Then one can get a past-inextendible timelike curve $\hat{\lambda}$ in $\text{int}[D^-(E^-(\hat{T}))] \cap \hat{V}$ because of the compactness of $\overline{\Sigma}$ as in Lemma 8.

(b-II) Assume that $\xi_p$ intersects $E^-(\hat{T}) \cap \overline{\Sigma}$. $\xi_p$ is a null geodesic generator of $J^-(\hat{T}) - E^-(\hat{T})$ because $\xi_p$ is future-inextendible (remind that the future-directed null geodesic generators of $J^-(\hat{T}) - E^-(\hat{T})$ are inextendible and also $\xi_p$ and $J^-(\hat{T})$ are achronal). Denote an intersection point of $\xi_p \cap [E^-(\hat{T}) \cap \overline{\Sigma}]$ by $\hat{q} (\in \hat{V})$ and take an arbitrary small
Then one could take a point \( \hat{q} \) such that \( \overline{U_q} \subset \hat{V} \). Since \( \hat{q} \in [J^-(\hat{T}) \cap H^-(J^-(\hat{T}))], \) there is a future-directed causal curve \( \hat{\mu}_0 \) from a point \( \hat{q}_0 (\in \hat{V}) \) of \( J^-(\hat{T}) \cap [\hat{U}_q - D^-(J^-(\hat{T}))] \) to a point of \( \hat{T} \cap \hat{V} \) (\( \hat{\mu}_0 \) cannot have a future endpoint in \( \hat{K} \) as previously discussed \( \hat{q} \) in Lemma [7]). Then from Prop. [3], it follows that \( \hat{\mu}_0 \) is contained in \( \hat{V} \). Consider a collection of local causality neighborhoods \( \hat{U}_n \) covering \( \hat{M} \) as a locally finite atlas and its subcollection \( \{\hat{U}_i\} \) which covers \( H^-(E^-(\hat{T})) \). As discussed in Lemma 8.2.1 of Ref. [11], \( \hat{\mu}_0 \) intersects \( H^-(E^-(\hat{T})) \) at a point \( \hat{r}_1 \in \hat{V} \). Then one can take a neighborhood \( \hat{U}_1 \) of \( \hat{r}_1 \) such that \( \hat{U}_1 \) also is in \( \hat{V} \). Let \( \hat{p}_1 \) be a point of \( J^-(\hat{T}) \cap [\hat{U}_1 - D^-(J^-(\hat{T}))] \). There is a future-inextendible causal curve \( \lambda_1 \) from \( \hat{p}_1 \) which does not intersect either \( J^-(\hat{T}) \) or \( D^-(E^-(\hat{T})) \). Let \( \hat{q}_1 \) be a point on \( \lambda_1 \) not in \( \hat{U}_1 \). Since \( \hat{q}_1 \in J^-(\hat{T}) \) and \( \hat{p}_1 \in \hat{V} \), there is a future-directed causal curve \( \hat{\mu}_1 \) in \( \hat{V} \) which connects \( \hat{q}_1 \) with a point of \( \hat{T} \cap \hat{V} \). By Lemma [4] this curve intersects \( \overline{H^-(E^-(\hat{T}))} \) at a point \( \hat{r}_2 \in \hat{V} \) not in \( \hat{U}_1 \). One can take a neighborhood \( \hat{U}_2 \) of \( \hat{r}_2 \) such that \( \hat{U}_2 \) also is in \( \hat{V} \). Thus one obtain an infinite sequence of points \( \hat{r}_n \in \hat{V} \) such that any subsequence of \( \{\hat{r}_n\} \) does not converge to any point in \( \hat{M} \) by the construction of \( \{\hat{r}_n\} \).

Let us consider a future-directed null geodesic generator \( \hat{\alpha}_n \in H^-(E^-(\hat{T})) \) from \( \hat{r}_n \) \((\hat{\alpha}_m \neq \hat{\alpha}_{m'} \text{ if } m \neq m')\). There are now two different situations to be taken account into; (b-II-i) Every \( \hat{\alpha}_m \) does not have a future endpoint \( \hat{s}_m \). There is a future-inextendible null geodesic generator \( \hat{\alpha}_{m_0} \) of \( H^-(E^-(\hat{T})) \cap \hat{V} \). Then one can obtain a past-inextendible timelike curve \( \hat{\lambda} \) in \( \hat{V} \cap \text{int}[D^-(E^-(\hat{T}))] \) by the same way that we obtained it in the case (b-I).

(b-II-ii) Every \( \hat{\alpha}_m \) has a future endpoint \( \hat{s}_m \). One can get an infinite sequence of points \( \hat{s}_m \in [\text{edge}(E^-(\hat{T})) \cap \hat{V}] \) because \( \hat{\alpha}_m \subset \hat{V} \) as \( \hat{\mu}_n \subset \hat{V} \). Let \( \hat{\beta}_m \) be a null geodesic generator of \( E^-(\hat{T}) \cap \hat{V} \) whose past endpoint is \( \hat{s}_m \). Take a sequence of future endpoints \( \hat{t}_m (\in \hat{T} \cap \hat{V}) \) of \( \hat{\beta}_m \). From the compactness of \( \hat{T} \) one can take a subsequence \( \{\hat{t}_m\} \) of \( \{\hat{t}_m\} \) which converges to a point \( \hat{t} \) of \( \hat{T} \cap \overline{\hat{V}} \). By Lemma. 6.2.1 of Ref. [11] there exists a limit curve \( \hat{\beta} \) through a point \( \hat{t} \) which is also a null geodesic generator of \( E^-(\hat{T}) \). If \( \{\hat{s}_i\} \) did not converge, \( \hat{\beta} \) would be past incomplete or there was a future incomplete null geodesic generator of \( J^-(\hat{T}) \) in \( \overline{\hat{V}} \). Thus, \( \hat{\beta} \) has a point \( \hat{s} \) in \( \text{edge}(E^-(\hat{T})) \) which is a limit point of \( \{\hat{s}_i\} \). Because each null geodesic generator \( \hat{\beta}_l \) is contained in \( \hat{V} \), the segment of the limit curve \( \hat{\beta} \) from \( \hat{s} \) to \( \hat{t} \) is contained in \( \overline{\hat{V}} \). Consider an infinite sequence of null geodesics \( \hat{\alpha}_l \) such that each \( \hat{\alpha}_l (\supset \hat{\alpha}_l) \) is past-inextendible in \( \hat{M} \) and passes through two points \( \hat{s}_l \) and \( \hat{t}_l \). By Lemma 6.2.1 of Ref. [11] through \( \hat{s} \) there exists a limit null geodesic curve \( \hat{\alpha}' \) which is past-inextendible. Since each \( \hat{\alpha}_l \) is a null geodesic generator of \( H^-(E^-(\hat{T})) \), a portion of \( \hat{\alpha}' \) is a null geodesic generator \( \hat{\alpha} \) of \( H^-(E^-(\hat{T})) \) from \( \hat{s} \). Suppose that \( \hat{\alpha} \) had a past endpoint in \( H^-(E^-(\hat{T})) \). Then one could take a point \( \hat{a} \) on \( \hat{\alpha}' \cap [\hat{M} - H^-(E^-(\hat{T}))] \) and the neighborhood \( \hat{U}_a \) of \( \hat{a} \) such that \( \hat{U}_a \subset [\hat{M} - H^-(E^-(\hat{T}))] \). Since \( \hat{a} \) was a limit point of \( \{\hat{\alpha}_l\} \), one could take a point \( \hat{a}_l (< \hat{r}_l) \in \hat{\alpha}_l \cap \hat{U}_a \) for \( l \) large enough. Then a subsequence of \( \{\hat{r}_l\} \) would also converge to a point \( \hat{b} (\hat{s} > \hat{b} > \hat{a}) \) of \( \hat{\alpha} \). This contradicts the fact that any subsequence of \( \{\hat{r}_l\} \) has no limit point as shown above. Therefore \( \hat{\alpha} (= \hat{\alpha}') \) is past-inextendible in \( H^-(E^-(\hat{T})) \).

Because \( \hat{\alpha}_l \subset \hat{V} \), \( \hat{\alpha} \) is contained in \( \overline{\hat{V}} \). Obviously there is a past-inextendible timelike curve \( \hat{\lambda} \) from \( \text{int}(\hat{T}) \cap I^+(\hat{\alpha}) \cap \text{int}[D^-(E^-(\hat{T}))] \). Then it can be also shown that \( \hat{\lambda} \subset \overline{\hat{V}} \) from Prop. [3] as in the case of \( \hat{\mu}_0 \subset \overline{\hat{V}} \). Thus, the proof of the case (b-II) is complete.
Proof of theorem 2. By Lemma 8, there is a future and past-inextendible timelike curve \( \hat{\rho} \) in \( \text{int}[D(E^-(\hat{T}))] \cap \hat{V} \). Then, following the proof of the theorem of Hawking and Penrose (see Ref. [11], p. 269), applied to \( \hat{\rho} \), one can complete the proof of Theorem 2.

Take infinite sequences of points \( \hat{a}_n, \hat{b}_n \) on \( \hat{\rho} \) such that;

(I) \( \hat{a}_{n+1} \in I^-(\hat{a}_n), (\hat{b}_{n+1} \in I^+(\hat{b}_n)) \),

(II) no compact segment of \( \hat{\rho} \) contains more than a finite number of the \( \hat{a}_n (\hat{b}_n) \),

(III) \( \hat{b}_1 \in I^+(\hat{a}_1) \).

Then there would be a timelike geodesic \( \hat{\mu}_n \) of maximum length between \( \hat{a}_n \) and \( \hat{b}_n \) since \( \hat{\rho} \) was contained in the globally hyperbolic set \( \text{int}[D(E^-(\hat{T}))] \cap \hat{V} \). Because \( \hat{T} \) is compact, there would be an inextendible causal geodesic \( \hat{\mu} \) in \( J^- (\hat{\rho}) \cap J^+ (\hat{\rho}) (\subset \hat{V}) \). If \( \hat{\mu} \) was future and past complete in \( \hat{V} \), there would be conjugate points \( \hat{x} \) and \( \hat{y} \) on \( \hat{\mu} \) with \( \hat{y} \in I^+ (\hat{x}) \). This is a contradiction.

We also have the following corollary.

Corollary 9 If the causality violating-region \( \overline{V} \) which satisfies the conditions 1 and 2 is causally geodesically complete, the strong energy condition or the generic condition is violated.

It is remarkable that, to prove Theorem 2, any asymptotic conditions such as the existence of null infinities are not imposed.

3.2 Singularity Theorem II

We have seen that \( \overline{V} \) is causally geodesically incomplete; singularities are formed in \( \overline{V} \). If a space-time singularity is caused by a physically realistic process such as gravitational collapse, the curvatures become unboundedly large near the singularity and consequently all objects falling into it are crushed to be zero volume. This is conjectured by Tipler et al. [13] and independently by Królok [14]. They introduced the notion of a strong curvature singularity [12, 19]. Let us consider an incomplete timelike (null) geodesic terminating at a singularity and take three (two) independent Jacobi fields along the geodesic. Roughly speaking, if the magnitude of a spacelike volume (area) element defined by the exterior product of the independent Jacobi fields becomes zero, then the singularity is called a strong curvature singularity.

As a situation that the strong curvature singularity condition holds, we assume that the space-time is maximal [15], whose definition we shall review below. Consider an affinely parameterized null geodesic \( \lambda(v) \) whose tangent vector is \( K^a \) and their two independent spacelike vorticity free Jacobi fields \( Z_1 \) and \( Z_2 \). Let \( A \) be the area element defined by \( Z_1 \wedge Z_2 \) and introduce the function \( z(v) \) defined by the relation \( z^2 = A \). Then \( z(v) \) satisfies the equation

\[
\frac{d^2 z}{dv^2} + \left( \frac{1}{2} R_{ab} K^a K^b + \sigma_{ab} \sigma^{ab} \right) z = 0,
\]

(2)
where $R_{ab}$ is the Ricci tensor and $\sigma_{ab}$ is the shear of congruence of Jacobi fields along $\lambda$. Consider a null geodesic $\lambda : [0, a) \to M$ which is incomplete at the value $a$ of its affine parameter $v$ and generates an achronal set. It is said that $\lambda(v)$ satisfies the inextendibility condition if for some $v_1 \in (0, a)$ there exists a solution $z(v)$ of Eq. (2) along $\lambda(v)$, with initial conditions: $z(v_1) = 0$ and $\dot{z}(v_1) = 1$, such that $\lim_{v \to a} \inf z(v) = 0$. As shown in Ref. [13], $\lambda(v)$ cannot be extended beyond a point $\lambda(a)$. This condition can be regarded as a strong curvature singularity condition. It is also said that a space-time $(M, g)$ is maximal if the above inextendibility condition is satisfied for any incomplete null geodesic generating an achronal set in $(M, g)$.

In Ref. [13] it is shown that there is no incomplete null geodesic generating an achronal set in a maximal space-time under some physical conditions. From this result we immediately have the following proposition.

**Proposition 10** Let $(M, g)$ be a maximal space-time on which the weak energy condition holds and $N$ a three-dimensional compact acausal set in $(M, g)$. If there is a point $p$ of $H^+(N) - N$ such that $J^-(p) \cap J^+(N)$ contains a future-inextendible timelike curve $\lambda$, then the null geodesic generator of an achronal set $[\hat{J}^-(p) \cap \hat{J}^-(\lambda)] \cap J^+(N)$ is future complete.

In Theorem 2, we showed that $\hat{V}$ is causally geodesically incomplete. Then, it is a possible case that $\text{int}(\hat{V})$ is causally geodesically complete but $\hat{V}$ is not. Indeed, there are three places allowing this case in the proofs of our lemmas. The first place is in Lemma 4, where we assumed that $\hat{V}$ was null geodesically complete and thus the limit curve $\hat{\lambda}$ was complete in $\hat{V}$. Hereafter, for simplicity, we assume that the strong causality condition is satisfied in $\hat{V}$ (condition 2') because this assumption is reasonable for our considering space-time satisfying condition 2. Then one can prove Theorem 2 without assuming the completeness of $\hat{\lambda}$. The second place is where we assumed the future completeness of all the null geodesic generators of $J^-(\hat{\gamma})$ in Lemma 8. The third place is where we assumed the past completeness of $\hat{\beta}$ in Lemma 8.

Now, we can show the following theorem which states that if the considering finitely vicious space-time is maximal, the above case is impossible by combining Corollary 12 and Lemma 13 below with Theorem 2.

**Theorem 11** If a finitely vicious space-time $(M, g)$ with a chronology violating set $V$ is maximal and satisfies conditions 1, 2', 3, then $V$ is causally geodesically incomplete.

Proposition 10 leads to the following Corollary.

**Corollary 12** There is no future incomplete null geodesic generator of $\hat{V} \cap \hat{J}^-(\hat{\gamma}) \cap J^+(\hat{K})$ in a maximal space-time $(M, \hat{g})$, where $\hat{\gamma}$ is the future inextendible timelike curve obtained in Lemma 8.

**Proof.** Suppose that there was a future incomplete null geodesic generator $\hat{\lambda}$ of $\hat{V} \cap \hat{J}^-(\hat{\gamma})$. $\hat{\lambda}$ would also be a null geodesic generator of $\hat{J}^-(\hat{p}) \cap J^+(\hat{K})$ because $J^-(\hat{\gamma}) \cap J^+(\hat{K}) \subset J^-(\hat{p}) \cap J^+(\hat{K}) \subset \hat{V}$ by Lemma 8, where $\hat{p}$ is the point defined in the proof of Lemma 8. Consider a future-directed timelike curve $\hat{\mu}$ in $D^+(\hat{K})$ terminating at a point $\hat{s}$ of $\hat{\lambda}$. Take an infinite sequence of points $\hat{p}_n$ in $\text{int}(D^+(\hat{K}))$ which converge to $\hat{p}$. Then $\hat{J}^-(\hat{p}_n)$
intersects \( \hat{\mu} \) at a point \( \hat{s}_n \) because \( \text{int}(D^+(\hat{K})) \) is causally simple. Denote an achronal null geodesic generator of \( J^-(\hat{\beta}_n) \cap \text{int}(D^+(\hat{K})) \) which connects \( \hat{\rho}_n \) with \( \hat{s}_n \) by \( \xi_n \). Since \( \hat{\rho} \) and \( \hat{s} \) are the limit points of \( \{\hat{\rho}_n\} \) and \( \{\hat{s}_n\} \) respectively, by Lemma 6.2.1 of Ref. [15] \( \{\xi_n\} \) has two limit curves \( \hat{\lambda} \) and \( \hat{\xi}_p \). Therefore one can take a point \( \hat{\xi}_p \) of \( \hat{\xi}_p \) in the past of \( \hat{\rho} \) and a convex normal neighborhood \( \hat{G} \) of \( \hat{\xi}_p \) with compact closure \( \hat{G} \) such that \( \hat{\lambda} \cap \overline{\hat{G}} = \emptyset \). Since the space-time \((\hat{M}, \hat{g})\) is maximal, the future incomplete null geodesic \( \hat{\lambda} : [0, a) \to \hat{M} \) would satisfy the inextendibility condition. Then one can get a contradiction by considering the Jacobi fields along \( \xi_n \) following the proof of the theorem by Rudnicki (see p. 57 of Ref. [13]).

\textbf{Lemma 13} There is a past incomplete null geodesic generator of \( J^+(\hat{\beta}) \cap \hat{V} \) if \( \hat{\beta} \) is past incomplete, where \( \hat{\beta} \) is the null geodesic generator of \( E^-(\hat{T}) \) in Lemma [12].

\textbf{Proof.} Take an arbitrary point \( \hat{w} \) of \( \hat{\beta} \) and consider an infinite sequence of points \( \hat{z}_l(\in \hat{V}) \) on \( \beta_l \) which converge to \( \hat{w} \), where \( \beta_l \) was defined in Lemma [8]. If one takes a future-directed timelike curve \( \hat{\rho}_l \) from \( \hat{z}_l \), \( \hat{\rho}_l \) intersects a point \( \hat{w}_l \) on \( J^+(\hat{\beta}) \). It is obvious that a subsequence of \( \{\hat{w}_l\} \) converges to \( \hat{w} \). Denote a future-directed null geodesic generator of \( J^+(\hat{\beta}) \cap J^-(\hat{T}) \) through a point \( \hat{w}_l \) by \( \hat{k}_l \). Then one can take a subsequence of \( \{\hat{k}_l\} \) that converges to \( \hat{\beta} \).

Suppose that the future endpoint \( \hat{\iota}(\in \hat{T}) \) of \( \hat{\beta} \) was in \( \hat{V} \). Then, \( \hat{\iota} \) was in \( \pi^{-1}(\hat{K}) \) or \( \pi^{-1}(\hat{I}^+(q, M) \cap \hat{I}^-(q, M)) \) by condition 1. Since \( \hat{\beta} \) is the null geodesic generator of \( E^-(\hat{T}) \), \( \hat{\beta} \) (precisely, its extension) is also a null geodesic generator of \( \hat{I}^-(\hat{\gamma}) \cap \hat{V} \), where \( \hat{\gamma} \) is a future-inextendible timelike curve obtained in Lemma [3]. Thus, \( \hat{\beta} \) was also a null geodesic generator of \( \pi^{-1}(\hat{I}^-(q, M)) \). Then \( \hat{\beta} \) cannot remain in \( \hat{V} \) because \( \hat{\iota} \) was in \( \pi^{-1}(\hat{I}^+(q, M) \cap \hat{I}^-(q, M)) \). This means that there was a point \( \hat{b}(\in \hat{V}) \) of \( \hat{\beta} \) in an arbitrary small neighborhood of \( \hat{\iota} \) such that \( \hat{b} \in (\hat{M} - \hat{V}) \). This is impossible because each \( \hat{\beta}_l \) is in \( \hat{V} \). Therefore \( \hat{\iota} \in \hat{V} \) and one can take a neighborhood \( \hat{U}_l \) of \( \hat{\iota} \) such that \( \hat{U}_l \in \hat{V} \). There is a natural number \( l_o \) such that a subsequence of points \( \hat{c}_l \) of \( \hat{k}_l \cap \hat{\omega} \) was in \( \hat{U}_l \) for \( l \geq l_o \). Consider a causal curve which connects \( \hat{z}_l \) with \( \hat{c}_l \) through \( \hat{w}_l \). This causal curve is in \( \hat{V} \) by Prop. [3]. Then each null geodesic generator \( \hat{k}_l \cap \hat{\omega} \) also is in \( \hat{V} \). Since the point \( \hat{w} \) is an arbitrary point of \( \hat{\beta} \), there exists an infinite sequence of past-inextendible null geodesic curves \( \hat{k}_n \) of \( J^+(\hat{\beta}) \cap J^-(\hat{T}) \) in \( \hat{V} \) which converge to \( \hat{\beta} \).

Since \( (M, g) \) is the maximal space-time and \( \hat{\beta} \) is a past incomplete null geodesic generating the achronal surface \( E^-(\hat{T}) \), \( \hat{\beta} \) satisfies the inextendibility condition. By this condition, \( \lim_{v \to a} z(v) = 0 \) is satisfied along \( \hat{\beta} \) in the past direction, where \( v \) is an affine parameter of \( \beta \). Then there is a value \( a'(a > a' > v_1 > 0) \) on \( \hat{\beta} \) such that \( \hat{z}(a') < 0 \) in the past direction. Consider a sequence of the Jacobi fields and the function \( z_n \) along \( \hat{k}_n \) which satisfies Eq. (2). By the continuity there is a natural number \( n_1 \) such that \( \hat{z}_n(n \geq n_1) \) < 0 in the past direction. Thus \( \hat{k}_n(n \geq n_1) \) is also past incomplete in \( \hat{V} \) by the energy condition 3, otherwise \( \hat{k}_n \) had pair conjugate points along \( \hat{k}_n \) which was impossible by the achronality of \( J^+(\hat{\beta}) \cap J^-(\hat{T}) \).

\textbf{Proof of theorem [14].} By Theorem [3] the assumption that \( \hat{V} \) was causally geodesically complete is incorrect. Suppose that \( \hat{V} \) was causally geodesically complete but \( \hat{V} \) was not.
Then, recalling the proof of Theorem 2, one could see that the null geodesic generator of $\dot{J}^{-}(\hat{\gamma})$ in Lemma 8 was future incomplete or $\dot{\beta}$ in Lemma 8 was past incomplete in $\hat{V}$. By Corollary 12 the former case is impossible. By Lemma 13 the latter case contradicts the supposition that $V$ was causally geodesically complete.

4 Chronology protection conjecture and causal feature of singularity

In the previous section we showed that there exists an incomplete causal geodesic in $V$ under the some suitable conditions and the assumption that a finitely vicious space-time is maximal.

In this section we shall test the chronology protection conjecture by examining space-times considered in the previous sections, assuming further that the occurring singularities are physically realistic singularities. Here, we shall say that chronology is protected if $V = \emptyset$.

We begin by inspecting exact solutions of the Einstein equation. One of the most well-known solutions containing both singularities and chronology violating sets is the maximally extended Kerr solution, which represents a rotating black hole. (See Ref. [11], for its metric and the maximal extension of the solution.) A conformal diagram of this solution is illustrated in Fig. 2. Chronology violating set is inside the inner horizons and the space-time is finitely vicious since a hypersurface slicing $\{S(\tau)\}$ can be taken so that the intersection of an $S(\tau)$ and the Cauchy horizon $H^{+}(S)$ for a partial Cauchy surface $S = S(0)$ is compact. As depicted in Fig. 2, this space-time has timelike singularities in the chronology violating set. It can be seen that the singularity is a real, scalar curvature singularity as the curvature scalar polynomial $R_{abcd}R^{abcd}$ diverges there.

From the observation above, one may expect that even in more generic space-time an appearance of chronology violation of the finitely vicious type is accompanied by the formation of timelike (naked) singularities. In other words, from the standpoint of the chronology protection, one may conjecture that if all singularities are spacelike, chronology violation of the type cannot occur. To see that this conjecture is indeed true, we shall introduce the notion of a non-naked singularity, which is a generalization of a spacelike singularity observed, for example, in Schwarzschild space-time.

**Definition 14** We shall say that a future (past) incomplete causal geodesic $\gamma$ terminates at a non-naked singularity if there is a point $p$ on $\gamma$ such that every future (past)-directed causal curve from $p$ terminates at scalar curvature singularities.

A situation of occurrence of a non-naked singularity is illustrated in Fig. 3. The non-naked singularity condition conforms to the belief that the formation of naked (timelike) singularities seems unlikely in a generic space-time in connection with the cosmic censorship conjecture [16]. So this condition may be regarded as a seemingly reasonable condition for physically realistic singularities.
It should be noted that this condition specifies the causal feature of a singularity in question. As depicted in Fig. 3(a), non-naked singularities need not be “everywhere spacelike” and hence a space-time with this singularity needs not be globally hyperbolic even in the case of the absence of causality violation. Thus the condition is a quasi-global condition, so to speak.

Now, under this condition, we shall give a partial answer to the chronology protection conjecture by establishing the following theorem.

**Theorem 15** If every incomplete causal geodesic terminates at non-naked singularities, there is no maximal space-time \((M, g)\) which contains chronology violating set \(V\) satisfying conditions 1, 2', 3.

*Proof.* Suppose that there were \(V\) which satisfied conditions 1, 2', 3. By Theorem 11 there is an incomplete causal geodesic \(\gamma\) in \(V\). Without loss of generality, one can consider \(\gamma\) as a future-directed causal geodesic. Because all the occurring singularities are of the non-naked type, \(\gamma\) has a point \(p\) in the definition 14. On the other hand, there would be a closed timelike curve \(\lambda\) through \(p\) since \(p\) was in the chronology violating set \(V\). This contradicts the fact that all the future-directed causal curves from \(p\) terminate at scalar curvature singularities. \(\square\)

## 5 Concluding remarks

As our main result we presented Theorem 15 which states that a physically reasonable class of chronology violations cannot arise if all occurring singularities are of the non-naked type. This supports the validity of the chronology protection conjecture.

To prove this theorem, we first showed that there exists an incomplete causal geodesic in the chronology violating set \(V\), if \(V\) is non-empty. Although a number of studies have been made on the investigation of the relation between causality violation and singularities \([5, 17, 18]\), little has been known about the locations of the singularities. In Theorems 2 and 11, we succeeded in determining the locations by imposing some restrictions on the chronology violating set.

The non-naked singularity condition in Theorem 15 implies that a version of the cosmic censorship conjecture \([16]\) is correct. Though this conjecture was proved in some seemingly physical cases \([19]\), whether or not this conjecture is correct in a more general context is still an open question. One may say that at least our result suggests that the basic problems behind chronology protection strongly depend on the causal feature of occurring singularities.

As discussed in Ref. [20], in the case of spherically symmetric space-times one can discuss the causal feature of central singularity by using a quasi-local mass \([21]\). We hope that this approach may provide a criterion for determining the causal feature of singularity in more generic cases.
Figure 2: A conformal diagram of a maximally extended Kerr solution along the symmetry axis for non-extreme case. Singularities exist inside the causality violating-regions (the gray regions). Precisely, the zig-zag line in this figure describes the axis passing through, without intersecting, the ring of singularity rather than the singularity itself, since the only causal geodesics reaching the ring singularity are those in the equatorial plane. $S(\tau)$ is a locally spacelike hypersurface and $S(:= S(0))$ is a partial Cauchy surface. $H^+(S)$ is a future Cauchy horizon for $S$. 
Figure 3: (a): A non-naked singularity. $\gamma$ is a future incomplete causal curve which terminates at the non-naked singularity. (b): A naked singularity, which is visible from the future null infinity $I^+$. 

**Acknowledgments**

We are grateful to I. Racz for useful discussion. We would also like to thank A. Hosoya and H. Ishihara for their continuous encouragement. The work is supported in part by the Japan Society for Promotion of Science (K.M.).

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