Energy and Asymptotics of Ricci-Flat 4-Manifolds with a Killing Field

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1 Introduction

We prove three theorems on complete Ricci-flat 4-manifolds with a Killing field. If $N^4$ is such a manifold, we find a bound for $\int_N |Rm|^2$ in terms of an asymptotic property of its Killing field $X$. We also show that if $\int |Rm|^2$ is assumed finite, then the asymptotic condition on $X$ can be eliminated. The first theorem is essentially an application of the Chern-Gauss-Bonnet formula, where the boundary term is estimated using the iteration method of Cheeger-Tian [13]. The second theorem requires an analysis of possible asymptotic degenerations of the Killing field.

If, in addition to the asymptotic condition, the Killing field has no zeros, the manifold is flat. This provides a partial affirmative answer to a conjecture stated in [2]. This is shown by noting that if the orbits of $X$ are unbounded, then since $\int |Rm|^2$ must be either infinite or zero, it must be zero by the universal bound on $\int |Rm|^2$. If the orbits have compact closure, then they lift on the universal cover to unbounded orbits, again forcing $\int |Rm|^2$ to be either zero or infinite, and therefore zero.

In addition to their intrinsic interest, these results have application to the study of Einstein manifolds with negative Einstein constant. This will be explored in subsequent papers.

Before stating our results we must explain the necessary asymptotic condition. The \textit{s-local curvature radius} is

$$r^s_{R}(p) = \sup \{ 0 < r < s \mid |Rm| < r^{-2} \text{ on } B(p, r) \}. \quad (1)$$

We abbreviate $r_{R} = r^\infty_{R}$; compare (1.7) and (2.2) of [13]. Given a vector field $X$, define the \textit{local variation function} of $X$ to be the function $M_X : \mathbb{N}^n \to [1, \infty]$ by

$$M_X(p) = \sup \{ |X(q)| \mid q \in B(p, r_{R}(p)) \} \quad \text{inf} \{ |X(q)| \mid q \in B(p, r_{R}(p)) \}. \quad (2)$$
We say that $X$ is of \textit{bounded local variation} if $\sup M_X < \infty$. Clearly such a vector field has no zeros, and the value of $M_X$ is an asymptotic measure of the behavior of $X$. We modify this definition to account for the presence of a zero locus with finitely many components. If $\Omega$ is a pre-compact domain and $s > 0$, set

$$M_X^{\Omega,s} = \sup_{p \in \Omega} \frac{\sup \{|X(q)| \mid q \in B(p, r_K(p)) \setminus \Omega\}}{\inf \{|X(q)| \mid q \in B(p, r_K(p)) \setminus \Omega\}}.$$  

(3)

If $M_X^{\Omega,s} < \infty$, we say $X$ has bounded $s$-local variation outside $\Omega$. Finally set

$$M_X^\infty(p) = \inf_{s>0} M_X^{B(p,s),s}.$$  

(4)

If $M_X^\infty(p) < \infty$, we say $X$ has \textit{asymptotically bounded local variation}. Of course $M_X^\infty$ is independent of $p$, and the value of $M_X^\infty$ is an asymptotic measure of $X$.

As an example, if $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ on $\mathbb{R}^2$, so $|X| = r = \sqrt{x^2 + y^2}$, then clearly $M_X = \infty$.

If $\Omega$ is any pre-compact domain containing the origin, then $M_X^{\Omega,s} = O(s)$ for large $s$, so $\lim_{s \to \infty} M_X^{\Omega,s} = \infty$. So we see that $\lim_{s \to \infty} M_X^{\Omega,s}$ is a poor asymptotic measure of vector field control. However $M_X^{B(o,s),s} = 1/2$, so $M_X^\infty = 1/2$.

A similar example is the case where $X$ is a Killing field on a Ricci-flat asymptotically locally Euclidean (ALE) manifold. Since curvature falls off like $r^{-2-\delta}$ for some $\delta > 0$ (see, for example, [6]), the vector field $X$ approaches the ideal Euclidean vector field like $r^{-1-\delta/2}$, so we still have $\lim_{s \to \infty} M_X^{\Omega,s} = \infty$ for any domain $\Omega$ that contains the origin. However $M_X^\infty$ again equals $1/2$.

**Theorem 1.1** Assume $(N^4, g)$ is a complete, Ricci-flat 4-manifold with a nowhere-zero Killing field $X$. If $M_X^\infty < \infty$, then $N$ is flat.

**Theorem 1.2** Assume $(N^4, g)$ is a complete, Ricci-flat 4-manifold with a Killing field $X$. Let $\{N_{X,i}\}_{i=1}^m$ be the components of the zero-set of $X$. If $X$ has asymptotically bounded local variation, then $m < \infty$ and

$$\frac{1}{8\pi^2} \int_N |Rm|^2 \leq \sum_{i=1}^m \chi(N_{X,i}) + C \lim_{s \to \infty} s^{-4} \text{Vol}(B(p,s))$$  

(5)

where $p$ is any point in $N$. The constant $C$ depends only on $M_X^\infty$.

In the final theorem, we eliminate the asymptotic condition on the Killing field, but assume that energy is bounded.

**Theorem 1.3** Assume $(N^4, g)$ is a complete, Ricci-flat 4-manifold with a Killing field $X$, and assume $\int |Rm|^2 < \infty$. Then

$$\frac{1}{8\pi^2} \int_N |Rm|^2 = \sum_{i=1}^m \chi(N_{X,i}) - \lim_{s \to \infty} \frac{\text{Vol}(B(p,s))}{\frac{1}{2} \pi^2 s^4}.$$  

(6)
Remark 1.1. As conjectured in [2] and [4], it is likely that Theorems 1.1 and 1.2 hold even if the asymptotic condition on the Killing field is removed. The original rationale may have been via analogy with the Lorenzian case, where geodesically complete space-times with a time-like Killing field are flat (theorem 0.1 of [3]). Based on the work in this paper, it seems likely that, in the Riemannian case, if the Killing field is asymptotically uncontrollable, it should be by having infinitely many “nut” or “bolt” type fixed point sets, in which case the energy will be accounted for in the term, so \( \int |Rm|^2 \) would be infinite. Indeed such metrics were constructed in [5].

Remark 1.2. The estimate (5) is not ideal. For instance if \((N^4, g)\) is the Eguchi-Hanson instanton—in particular \(N^4\) is the tangent bundle over \(S^2\) then \(X\) is Killing field given by angular rotation of each tangent plane. We have \(\chi(N_X) = 2\) and \(\lim s^{-4}Vol B(p, s) = \frac{1}{8} \pi^2\), but \(\frac{1}{8} \pi^2 \int |Rm|^2 = \frac{3}{2}\). On the other hand, the estimate is sharp in the ALF case. Of course Theorems 1.2 and 1.3 should be used in conjunction.

Remark 1.3. Under the strong additional assumption that the manifold is ALE (asymptotically locally Euclidean) or ALF (asymptotically locally flat), the equivalent of Theorem 1.3 has long been known: see, for instance, equation (12) in [16] or the discussion of characteristic classes and instantons in [15]. What we have done is reduce the hypotheses from an asymptotic condition on the topology and the metric—that it approach a model flat metric like \(o(r^{-2})\) in the ALE case or like \(o(r^{-1})\) in the ALF case—to an asymptotic condition on the Killing field alone.

This improvement is not just technical, but is designed for use in analyzing collapsed blow-up limits of metrics with a bound on Ricci curvature. There, very little asymptotic data may be available, but in the low-energy case the collapse proceeds along almost invariant pure N-structures (generalized actions of nilpotent Lie groups, for instance torus actions).

From the theory of collapse with bounded curvature [11] [12] [10], it is known that the locally-defined vector fields \(X\) that constitute these N-structures have \(|\nabla X||X|^{-1}\) controlled, at least away from certain “singular” orbits, which can be dealt with separately.

For instance if the resulting N-structure has rank 1, then (possibly after passing to a finite normal cover) we are precisely in the situation of Theorem 1.1, so any presumptive “blow-up limit” is flat.

Remark 1.4. Here we outline of the proof of our primary result, Theorem 1.2. Comparing the expressions for \(|Rm|^2\) and the Chern-Gauss-Bonnet 4-form \(P_X\),

\[
|Rm|^2 = \frac{1}{6} R^2 + 2|\text{Ric}|^2 + |W|^2
\]

\[
8\pi^2 P_X = \left( \frac{1}{24} R^2 - \frac{1}{2} |\text{Ric}|^2 + \frac{1}{4} |W|^2 \right) dVol,
\]

we have the classic observation that \(\text{Ric} = 0\) implies \(\int |Rm|^2 dVol\) is equivalent to \(\int P_X\).

The Killing field \(X\) can be used to explicitly transgress the curvature 4-form: \(|Rm|^2 dVol = dTP_X\). Unfortunately the 3-form \(TP_X\) is an involved combination of \(X, \nabla X, \) and curvature terms; see Section 2.
Nevertheless integration by parts gives \( \int \varphi |Rm|^2 dVol = \sum \chi(N_{X,i}) - \int d\varphi \wedge TP \) when the support of \( \varphi \) is sufficiently large. The curvature terms in \( \int d\varphi \wedge TP \) can be approximated back in terms of \( (\int |Rm|^2)^{3/4} \); note the improved exponent. The iteration argument of Cheeger-Tian \([13]\) then proceeds; see Section 4. In the end only terms involving \(|\nabla X|/|X|\) remain, and we obtain an estimate that depends only on \( M_X^\infty \) and the asymptotic volume ratio.

**Remark 1.5.** The hypothesis \( Ric = 0 \) can be loosened in three situations.

1. If the trace-free Ricci tensor is sufficiently pinched, say \(|Ric|^2 \leq \frac{1}{24} R^2\), then Theorem 1.2 holds with modified constants. If, in addition, \( Ric \geq 0 \), then Theorem 1.1 also holds.

2. The second case is that of zero scalar curvature half-conformally flat metrics. The Pontryagin 4-form, given in general by \( 12\pi^2 \tau = \frac{1}{4} (|W^+|^2 - |W^-|^2) dVol \), can be transgressed in just the same way as the Euler form. Then \( R^2 = |W^+|^2 = 0 \) implies that

\[
-96\pi^2 \tau - 32\pi^2 \tau_X = |Rm|^2 dVol
\]


(8)

\( P \) can be transgressed, and the process outlined in Remark 1.4 can be repeated precisely. The version of Theorem 1.2 so obtained has a more complicated residue term due to the presence of the Pontryagin form; the expression can be found in [7]. Without \( Ric \geq 0 \), the constant \( C \) depends on both \( M_X^\infty \) and the asymptotic volume doubling constant.

If the metric is scalar flat, half-conformally flat, and has \( Ric \geq 0 \), then we can argue as in Section 5 to obtain Theorem 1.1. Namely, if the Killing field has no zeros and \( M_X^\infty < \infty \), the metric is flat.

3. Finally we consider the Kähler case, where \( |W^+|^2 = \frac{1}{6} R^2 \), so the condition that scalar curvature vanishes puts us precisely in case (ii). Zero scalar curvature Kähler metrics are natural, for instance, in the study of gravitational instantons or blow-up limits of degenerating extremal Kähler metrics.

As in (ii), if \( (M^4, g) \) is a zero scalar curvature Kähler manifold with \( Ric \geq 0 \) and a nowhere-zero Killing field \( X \) with \( M_X^\infty < \infty \), then \( M^4 \) is flat.

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\footnote{The norm here is the metric contraction, which differs from another norm commonly seen in the literature by a factor of \( \frac{1}{3} \).}
2 Transgression Construction

2.1 Transgression construction

Let \( X \) be a nowhere-zero Killing field on an \( n \)-manifold with dual 1-form \( X \). Set \( K = \left| X \right|^{-2} X \otimes \nabla X \); because \( \nabla X \) is antisymmetric, \( K \) is an \( \mathfrak{so}(n) \)-valued 1-form. Letting \( \nabla = \nabla - K \) be a new connection, we see that \( X \) is covariant-constant:

\[
\nabla X = \nabla X - \left| X \right|^{-2} X(X) \nabla X = 0.
\]

(9)

Let \( \{e_1 = X[X]^{-1}, e_2, \ldots, e_n\} \) be a local frame adapted to \( X \) in the sense that \([X, e_i] = 0\). With \( L \) denoting the Lie derivative, the connection 1-forms \( \widetilde{A} = A - K \) obey

\[
L_X \widetilde{A} = 0,
\]

and \( i_X \widetilde{A} = 0 \). Thus the associated curvature 2-forms have a null vector:

\[
i_X \widetilde{F} = i_X d\widetilde{A} + \frac{1}{2} i_X [\widetilde{A}, \widetilde{A}]
= L_X \widetilde{A} = L_X \nabla = 0.
\]

(10)

Letting \( P \) be any symmetric invariant polynomial on \( \mathfrak{so}(n) \) that is homogeneous of order \( \frac{n}{2} \), we know that \( P(F, \ldots, F) \) represents a top characteristic class on \( N^n \), where \( F \) is the curvature of any connection. Because \( \widetilde{F} \) has a null vector, \( P(F, \ldots, F) = 0 \).

Put \( A_t = A - tK \) and \( F_t = dA_t + \frac{1}{2}[A_t, A_t] \). Now \( K = v \otimes h \) for an \( \mathfrak{so}(n) \)-valued function \( h \), so \([K, K] = 0\). Using this we obtain \( D_t K = DK \) and \( F_t = F - t DK \). We compute

\[
\frac{d}{dt} P(F_t, \ldots, F_t) = \frac{n}{2} d \int_0^1 P(K, F_i, \ldots, F_i) dt.
\]

(11)

We set

\[
\mathcal{T}P = \frac{n}{2} \int_0^1 P(K, F_i, \ldots, F_i) dt
\]

(12)

and obtain

\[
P = d \mathcal{T}P.
\]

(13)

In the case \( n = 4 \) we evaluate (12) to get

\[
\mathcal{T}P = 2P(K, F) - P(K, DK).
\]

(14)

2.2 The Chern-Gauss-Bonnet transgression form

We compute \( \nabla K \), an \( \mathfrak{so}(n) \)-valued tensor of type \((0, 2)\), and \( DK \), an \( \mathfrak{so}(n) \)-valued 2-form. For any vector fields \( A, B \), and \( w \) we have

\[
\nabla K(A, B)w = \nabla_A (K(B)w) - K(\nabla_A B)w - K(B)\nabla_A w
= 2\left| X \right|^{-2} \langle \nabla_X X, A \rangle \langle X, B \rangle \nabla_w X + \left| X \right|^{-2} \langle \nabla_A X, B \rangle \nabla_w X + \left| X \right|^{-2} \langle X, B \rangle \nabla^2_{A,w} X.
\]

(15)
With \(dX(A, B) = 2\langle \nabla A, B \rangle\), we therefore have

\[
DK = \left( |X|^{-4} i_X dX \wedge X + \frac{1}{2} |X|^{-2} dX \right) \otimes \nabla X - |X|^{-2} X \wedge \nabla^2 X
\]

and

\[
K \wedge DK = \frac{1}{2} |X|^{-4} X \wedge dX \otimes \nabla X, \quad Tr(K \wedge DK) = -\frac{1}{2} \frac{|\nabla X|^2}{|X|^4} X \wedge dX.
\]

Abusing notation slightly, we have

\[
Tr(K \wedge F) = |X|^{-2} X \wedge \langle F, \nabla X \rangle = |X|^{-2} X \wedge \langle Rm(\cdot, \cdot), \nabla \rangle.
\]

Now assume \(n = 4\) and let \(\mathcal{P} = \mathcal{P}_\chi\) be the Euler 4-form. The symmetric polynomial is

\[
\mathcal{P}_\chi(h_1, h_2) = (8\pi^2)^{-1} Tr(h_1 \ast h_2)\text{ for any } h_1, h_2 \in \mathfrak{so}(4), \text{ where } \ast : \mathfrak{so}(4) \to \mathfrak{so}(4) \text{ is the Hodge star, obtained via the isomorphism } \mathfrak{so}(4) \approx \bigwedge^2 \mathbb{R}^4.
\]

\[
\mathcal{T}\mathcal{P}_\chi = 2\mathcal{P}_\chi(K, F) - \mathcal{P}_\chi(K, DK) = \frac{1}{8\pi^2} \left( 2 |X|^{-2} \langle Rm(\cdot, \cdot), \ast \nabla X \rangle + \frac{1}{2} |X|^{-4} \langle \nabla X, \ast \nabla \rangle X \wedge dX \right).
\]

Obviously this is singular only on the null-set of \(X\).

### 3 Einstein 4-manifolds

We use the hypothesis of asymptotic local boundedness on \(X\) to estimate \(\mathcal{T}\mathcal{P}_\chi\); the primary conclusion is \(26\). The subsection on analytical lemmas collects results specific to manifolds with \(\text{Ric} \geq 0\), in preparation for the Cheeger-Tian process in section 5.

#### 3.1 Implications of the bounded local variation of \(X\)

If \(\Omega \subseteq N\) is any domain, define \(\Omega^{(s)}\) to be the set of points \(x \in N\) with \(\text{dist}(x, \Omega) < s\).

**Lemma 3.1** Given \(\mu > 1\) and \(s_0 > 0\), there exists a constant \(C = C(\mu, s_0, n)\) so that if \((N^n, g)\) is any Riemannian manifold, \(s > s_0\), and

\[
\frac{\sup_{p \in N \setminus \Omega} \{ |X(q)| : q \in B(p, r_K^*(p)) \}}{\inf_{p \in N \setminus \Omega} \{ |X(q)| : q \in B(p, r_K^*(p)) \}} \leq \mu,
\]

1
then
\[
\sup_{p \in \Omega(s)} \frac{\lVert \nabla X(p) \rVert}{\lVert X(p) \rVert} \leq C \left( r^*_R(p) \right)^{-1}.
\] (21)

Proof. First scale the metric so \( r^*_R(p) = 1 \). We can lift to the tangent space, where we obtain a contractible unit ball \( B \) with metric \( g, |\mathbf{Rm}| \leq 1 \), and a Killing field still denoted \( X \). Let \( B = B(o, 1) \) be the ball of radius 1 about the origin in \( T_p M \).

If \( \frac{\lVert \nabla X \rVert}{\lVert X \rVert} \) is not apriori bounded on \( B \), we choose counterexamples \( B_i \) with points \( p_i \in B_i \) so \( \frac{\lVert \nabla X(p_i) \rVert}{\lVert X(p_i) \rVert} \to \infty \). Now scale the metric up until \( \frac{\lVert \nabla X(p_i) \rVert}{\lVert X(p_i) \rVert} = 1 \), and scale \( X \) until \( 1 = \lVert X(p_i) \rVert \). Then the radii of the \( B_i \) are increasingly large, and sectional curvature are uniformly decreasing. Since the \( B_i \) are also contractible, we can pass to a subsequence that converges in the \( C^\infty \) sense to flat \( \mathbb{R}^n \) with a nowhere-vanishing Killing field \( X \) and a point \( p \) with \( \lVert X(p) \rVert = |\nabla X(p)| = 1 \). Note that the quantity on the left of (20) is scale-invariant, and one easily shows that it is lower semi-continuous with respect to this limit. Thus we have flat \( \mathbb{R}^n \) with a non-constant Killing field with \( \lVert X(p) \rVert = |\nabla X(p)| = 1 \), but with \( \sup \{ |X(q)| : q \in \mathbb{R}^n \} \inf \{ |X(q)| : q \in \mathbb{R}^n \} \leq \mu \). (22)

However, the quotient on the left is in fact infinite, as a consequence of the Jacobi equation \( \nabla \dot{\gamma} \nabla \dot{\gamma} X = 0 \) along any geodesic \( \gamma \), which forces \( X \) to grow linearly. Thus we have a contradiction. □

A few immediate consequences are as follows.

Corollary 3.2 We have the following consequences of Lemma 3.1.

- If \( M_X(p) < C \), then there exists some \( C_1 \) so that \( |\nabla X(p)||X(p)|^{-1} < C_1 (r_R(p))^{-1} \).
- If \( M^\Omega_X < C \) and \( \eta > 0 \), there exists some \( C_1 \) so that \( \frac{|\nabla X(p)|}{|X(p)|} < C_1 (r^*_R)^{-1} \) for all \( p \in N \setminus \Omega(s) \).
- If \( M^\infty_X < \infty \), there exists some \( C = C(M^\infty_X) \) so that if \( q \in M \) and \( s \) is sufficiently large (depending on \( q \) and \( M^\infty_X \)), then \( |\nabla X(p)||X(p)|^{-1} < C (r^*_R(p))^{-1} \) for all \( p \in N \setminus B(q, 2s) \).

On Einstein 4-manifolds,
\[
\mathcal{P}_X = \frac{1}{8\pi^2} \left( \frac{1}{24} R^2 + \frac{1}{4} |W|^2 \right) dVol.
\] (23)
Given any $C^\infty$ cutoff function $\varphi$ that is zero on the null-set of $X$, from \([19]\) we obtain

\[\int \varphi P_X = -\int d\varphi \wedge TP_X = -\frac{2}{8\pi^2} \int |X|^{-2} d\varphi \wedge X_\flat \wedge \langle Rm, \ast \nabla X \rangle - \frac{1}{16\pi^2} \int |X|^{-4} \langle \nabla X, \ast \nabla X \rangle 
\] (24)

(this level of detail is actually used in section 5). It is well-known that the singular set of $TP_X$ can be integrated over, provided a residue term of $\sum_i \chi(N_{X,i})$ is added. This can be deduced as a version of the Bott residue formula \([8]\); see also \([7]\). Thus if the set \{ $\varphi = 1$ \} contains all components of the null-set, we obtain

\[\frac{1}{8\pi^2} \int \varphi |Rm|^2 = \sum_{i=1}^m \chi(N_{X,i}) - \frac{1}{4\pi^2} \int |X|^{-2} d\varphi \wedge X_\flat \wedge \langle Rm, \ast \nabla X \rangle - \frac{1}{16\pi^2} \int |X|^{-1} \langle \nabla X, \ast \nabla X \rangle d\varphi \wedge X_\flat \wedge dX_\flat.
\] (25)

From Corollary 3.2 and the fact that $|Rm(p)| < (r^s_X(p))^{-2}$ (any $s > 0$), we obtain

\[\frac{1}{8\pi^2} \int \varphi |Rm|^2 \leq \sum_{i=1}^m \chi(N_{X,i}) + C \int |\nabla \varphi| (r^s_X)^{-3}
\] (26)

where $C$ depends only on $M^{(\varphi=1),s}_X$.

### 3.2 Analytic lemmas

The Cheeger-Tian iteration process may be somewhat unfamiliar, and we run through a version of it, with some simplifications, in \([1]\). First we record a familiar analytic lemma from analysis on $\mathbb{R}^n$ (for instance from \([19]\)). Assuming that metric balls have Euclidean or sub-Euclidean volume growth, the standard proof of Proposition 3.3 goes through with no change. Given a function $f$, define its (Hardy-Littlewood) maximal function on the scale $s$ to be

\[M^s(f)(p) = \sup_{0<r<s} \frac{1}{\text{Vol} \, B_p(r)} \int_{B(p,r)} |f|.
\] (27)

If $\Omega$ is a domain in a Riemannian manifold, set $\Omega^{(s)} = \{ p \in M \mid \text{dist}(p, \Omega) < s \}$.

**Proposition 3.3** Assume $\Omega$ is a pre-compact domain in an $n$-manifold with $\text{Ric} \geq 0$, and assume $0 < \alpha < 1$. There exists a constant $C = C(n, \alpha, \Lambda)$ so that

\[\left( \frac{1}{|\Omega|} \int_{\Omega} (M^s_g)^\alpha \right)^{\frac{1}{\alpha}} \leq C \frac{1}{|\Omega^{(s)}|} \int_{\Omega^{(s)}} |g|.
\]
The next step is to establish a link between the curvature scale \( r_R \) and the maximal function \( M_{|Rm|^2} \). This is obtained via \( \epsilon \)-regularity.

**Proposition 3.4 (Standard \( \epsilon \)-regularity)** There exist constants \( C = C(n) < \infty \), \( \epsilon_0 = \epsilon_0(n) > 0 \) so that if \( M \) has non-negative Ricci curvature and

\[
H \triangleq \frac{1}{r^n} \int_{B(p,r)} |Rm|^\frac{2}{n} \leq \epsilon_0
\]

then \( \sup_{B(p,r/2)} |Rm| < Cr^{-2} H^\frac{1}{2} \).

**Proof.** Among the sources with similar theorems are [22] [18] [1] [20] [21] [14], and [23].

It is convenient to introduce the \( s \)-local energy radius

\[
\rho^s(p) \triangleq \sup \left\{ 0 < r < s \mid \frac{1}{r^{-4}} \int_{B(p,r)} |Rm|^2 \leq \epsilon_0 \right\}.
\]

We use \( \rho(p) \) for \( \rho^\infty(p) \). After possibly choosing \( \epsilon_0 \) to be smaller, Proposition 3.4 directly implies \( r^s_R(p) \geq \frac{1}{2} \rho^s(p) \). A useful way of stating this is the following (see section 4 of [13]).

**Lemma 3.5 (Curvature radius weak estimate)** Assume \((N^n,g)\) has \( \text{Ric} \geq 0 \). Then given any \( s > 0, k \geq 0 \) we have

\[
(r_R(p))^{-k} \leq (r^s_R(p))^{-k} \leq \max \left\{ 2k s^{-k}, \left( 16 \epsilon_0^{-1} M^s_{|Rm|^2}(p) \right)^{\frac{1}{2}} \right\}.
\]

(28)

**Proof.** First if \( \rho^s(p) = s \) then \( r^s_R(p) \geq \frac{1}{2} s \). Assuming \( \rho^s(p) < s \), then directly from the definitions

\[
\epsilon_0 (\rho^s(p))^{-4} = \frac{1}{\text{Vol} B(p,\rho^s(p))} \int_{B(p,\rho^s(p))} |Rm|^2 \leq M^s_{|Rm|^2}(p).
\]

Therefore

\[
(r_R(p))^{-4} \leq (r^s_R(p))^{-4} \leq 16 (\rho^s(p))^{-4} \leq 16 \epsilon_0^{-1} M^s_{|Rm|^2}(p).
\]

**Remark 3.1.** The condition \( \text{Ric} \geq 0 \) can be removed if volume ratios of metric balls are a priori controlled.
4 Proof of Theorem 1.2

We proceed to Cheeger-Tian iteration proper. Abbreviating \( \chi = \sum_i \chi(N_{x,i}) \) and choosing any \( \eta \in (0, 1) \), the estimate (26) gives

\[
\frac{1}{8\pi^2} \int_{\Omega} |\text{Rm}|^2 \, dVol \leq \chi + C\eta^{-1}s^{-1} \int_{\Omega(2\eta s/3) \setminus \Omega(s/3)} (r_K)^{-3} \tag{29}
\]

where we have chosen a cutoff function \( \varphi \) that equals 1 on \( \Omega(s/3) \), zero on \( \Omega(2\eta s/3) \) and has \( |\nabla \varphi| < 6\eta^{-1}s^{-1} \). Then Lemma 3.3 and Proposition 3.3 give

\[
\frac{1}{8\pi^2} \int_{\Omega} |\text{Rm}|^2 \, dVol
\]

\[
\leq \chi + C\eta^{-4}s^{-4} \left| \Omega(2\eta s/3) \setminus \Omega(s/3) \right|
\]

\[
+ C\eta^{-1}s^{-1} \left| \Omega(2\eta s/3) \setminus \Omega(s/3) \right| \frac{1}{4} \left( \int_{\Omega(s/3) \setminus \Omega} |\text{Rm}|^2 \right) \frac{1}{4} \tag{30}
\]

where the notation \( | \cdot | \) indicates Hausdorff 4-volume. It is convenient to choose some \( t > 0 \) and set \( \Omega = B(p, t) \), which we will just abbreviate to \( B(t) \). Since \( \Omega(s\eta) = B(t + s\eta) \), we get

\[
\left| \Omega(2\eta s/3) \setminus \Omega(s/3) \right| < |B(t + sN) \setminus B(t)| \tag{31}
\]

for any \( N \geq \eta \), which will be chosen later. With Young’s inequality,

\[
\frac{1}{8\pi^2} \int_{B(t)} |\text{Rm}|^2 \, dVol \leq \chi + C(s\eta)^{-4}|B(t + sN) \setminus B(t)| + \frac{3}{4} \int_{B(t + s\eta)} |\text{Rm}|^2. \tag{32}
\]

The second integral on the right can be estimated likewise: choosing \( \eta' \in (0, 1) \) we obtain

\[
\frac{1}{8\pi^2} \int_{B(t + s\eta)} |\text{Rm}|^2 \, dVol
\]

\[
\leq \chi + C s^{-4} \left( \eta'^{-4} + \frac{3}{4}(\eta')^{-4} \right)|B(t + sN) \setminus B(t)| + \left( \frac{3}{4} \right)^2 \int_{B(t + s(\eta + \eta'))} |\text{Rm}|^2 \tag{33}
\]

for any \( N \geq \eta + \eta' \). Proceeding inductively, after choosing a sequence \( \{\eta_i\} \) we obtain

\[
\frac{1}{8\pi^2} \int_{B(p, t)} |\text{Rm}|^2 \, dVol \leq \chi + C s^{-4} |B(t + sN) \setminus B(t)| \sum_{i=1}^k \left( \frac{3}{4} \right)^i \eta_i^{-4} \tag{34}
\]

\[
+ \left( \frac{3}{4} \right)^k \int_{B(p, t + s \sum_{i=1}^k \eta_i)} |\text{Rm}|^2
\]

for any \( N \geq \sum_{i=1}^k \eta_i \). Choosing, say, \( \eta_i = \frac{11^{1/4} \cdot 9^{1/4} i^{1/4}}{11i^{1/4} (i + 1)^{1/4}} \), we have that \( \sum \eta_i = 1 \), that \( \sum (\frac{3}{4})^i \eta_i^{-4} \) converges, and that the final term on the right is negligible for large \( k \). Therefore

\[
\frac{1}{8\pi^2} \int_{B(p, t)} |\text{Rm}|^2 \, dVol \leq \chi + C s^{-4} |B(p, t + s) \setminus B(p, t)| \tag{35}
\]
where $C$ depends on $M^{B(p,t),s}_X$. Setting $t = 2s$ and taking the limit $s \to \infty$, we obtain Theorem 1.2.

5 Proof of Theorem 1.1

We have a non-flat, Ricci-flat manifold $n^4$ with a nowhere-zero Killing field $X$ that is asymptotically of bounded local variation; we will prove that $N$ is flat. By passing to the universal cover, we retain the property that $X$ has asymptotically bounded local variation. Let $O_p$ be the generic orbit of a point $p$ under the flow of $X$; its closure is a manifold and may be 1- or 2-dimensional, and may be bounded or unbounded. In the unbounded case, then unless the manifold is flat, it has $\int |Rm|^2 = \infty$, which contradicts Theorem 1.2. So closures of orbits are compact.

By the splitting theorem, $N$ is 1-ended (or else it is a product of a line with a Ricci-flat, therefore flat, 3-manifold). Choose a basepoint $p \in N$ where, possibly after scaling, $r_R(p) = 1$. Let $r_i$ be a sequence of positive numbers with $r_i \not\to \infty$. Let $\Omega_i$ be the saturation of $B(p, r_i)$ by the orbits of $X$; since these orbits are compact, $\Omega_i$ remains compact. Now pass to the universal cover $\tilde{\Omega}_i$, and let $\bar{p}_i$ be a selected point in the pre-image of $p$. We can continue to assume $\tilde{\Omega}_i$ is one-ended, for if an infinite number of the $\tilde{\Omega}_i$ had 2 or more ends, they would converge (in the pointed Gromov-Hausdorff, and $C^\infty$ topologies) to a multiple-ended, Ricci-flat 4-manifold $\tilde{\Omega}_\infty$. But then $\tilde{\Omega}_\infty$ would be flat, contradicting $r_R(\bar{p}_i) = 1$.

Possibly the lift of $X$ to $\tilde{\Omega}_i$ still has bounded orbits. In that case the relative homology sequence gives

$$0 \to H^0(\tilde{\Omega}_i) \xrightarrow{\text{iso}} H^0(\partial \tilde{\Omega}_i) \xrightarrow{\delta^*} H^1(\Omega_i, \partial \tilde{\Omega}_i) \to H^1(\tilde{\Omega}_i) \approx \{0\}$$

Thus $H^1(\tilde{\Omega}_i, \partial \tilde{\Omega}_i) \approx \{0\}$, so by Poincare duality $H^3(\tilde{\Omega}_i) \approx \{0\}$. With $b_k(\Omega_i)$ the $k^{th}$ betti number of $\Omega_i$, we have $\chi(\tilde{\Omega}_i) = 1 + b_2(\tilde{\Omega}_i) \geq 1$. However if $X$ has bounded orbits on $\tilde{\Omega}_i$ then $\chi(\tilde{\Omega}_i) = 0$, a contradiction.

Thus the $X$-orbits on $\tilde{\Omega}_i$ are unbounded. Taking a pointed Gromov-Hausdorff limit of the pairs $(\tilde{\Omega}_i, \bar{p}_i)$, we obtain a limiting pointed 4-manifold $(\tilde{\Omega}_\infty, \bar{p}_\infty)$ with $r_R(\bar{p}) = 1$, with the Killing field $X$ having asymptotically bounded local variation, and with the orbits of $X$ being unbounded. Again this implies $\int |Rm|^2 = \infty$, which again contradicts Theorem 1.2.

6 Proof of Theorem 1.3

We are considering a Ricci-flat manifold $(N^4, g)$ with $\int_N |Rm|^2 < \infty$ and a Killing field $X$. The outline of the proof is as follows: in Step I we deal with the case that the null
set of $X$ has unbounded components; in Step II we deal with the case that $N$ has non-zero asymptotic volume ratio, and in Steps III-VI deal with the case of zero asymptotic volume ratio and compact null-set components.

**Step I: Total geodesy of null set components**

The null set is of course locally totally geodesic. Let $\Sigma$ be a null set component that is unbounded. It has a ray $\gamma : [0, \infty) \to \Sigma$, which we can assume is unit-parametrized, and is locally a geodesic of both $\Sigma$ and $N$, but might not be globally minimizing with respect to $N$. There is at least one “asymptotic” ray, which we denote $\gamma_r$, obtained as a limit of geodesic segments $\gamma_{r_i}$ where $r_i$ is an unbounded sequence of numbers, and $\gamma_{r_i}$ is a geodesic from $\gamma_{\Sigma}(0)$ to $\gamma_{\Sigma}(r_i)$. Let $\phi_t$ be the flow of the vector field $X$. We modify the usual definition of the buseman function to ensure that it is $X$-invariant:

$$b(p) = \lim_{r \to \infty} \left( r - \inf_{t \in \mathbb{R}} \text{dist}(p, \phi_t(\gamma_{\infty}(r))) \right) \quad (36)$$

We have that $b$ is a distance function (meaning it is Lipschitz and $|\nabla b| = 1$ at $C^1$ points) and is $X$-invariant. The $X$-invariance implies $L_X \nabla b = 0$, which means $[X, \nabla b] = 0$, and implies $\nabla b$ is tangent to $\Sigma$ (at $C^1$ points of $\nabla b$).

With Ricci-flatness, we have $0 = |\nabla^2 b|^2 + \langle \nabla \Delta b, \nabla b \rangle$ at smooth points. Let $p \in \Sigma$ be a smooth point of $b$. Put $b^1 = b$ and let $b^2$ be a function on $\Sigma$ so that $b^1, b^2$ are normal coordinates near some chosen point in $\Sigma$. We can spread $b^2$ to a neighborhood of $p$ in $N$ by orthogonal projection (which is locally a smooth submersion). Next pick two other variables $\{w^1, w^2\}$ so that $\{b^1, b^2, w^1, w^2\}$ are a normal coordinate system on a tubular neighborhood of $\Sigma$. By the total geodesy of $\Sigma$ we can assume that

$$\nabla \frac{\partial}{\partial w^i} \frac{\partial}{\partial b^j} = \nabla \frac{\partial}{\partial w^i} \frac{\partial}{\partial w^j} = 0 \quad (37)$$

on $\Sigma$ (not just at $p$). In these coordinates, using the fact that $\nabla b$ is parallel to $\Sigma$ and again that $\Sigma$ is totally geodesic, we see the hessian $\nabla^2 b$ has the form

$$\langle \nabla, \nabla b, \cdot \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \triangle_\Sigma b & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & \beta & \beta & \gamma \end{pmatrix} \quad (38)$$

where $\Delta_\Sigma b$ is the intrinsic Laplacian on $\Sigma$.

Let $r_N(q)$ be the distance from $q$ to $\Sigma$. Now consider the vector field $\nabla_X \nabla b$; clearly $\nabla_X \nabla b = O(r_N^2)$ near $\Sigma$; we shall show that it is $O(r_N^2)$. Letting $x^i$ be any of $b^1, b^2, w^1, w^2$ we have

$$\frac{\partial}{\partial w^i} \left( \nabla_X \nabla b, \frac{\partial}{\partial x^j} \right) = \left\langle \text{Rm} \left( \frac{\partial}{\partial w^i}, X \right) \nabla b, \frac{\partial}{\partial x^j} \right\rangle$$

$$- \left\langle \nabla_{\frac{\partial}{\partial w^i}} \nabla_X \frac{\partial}{\partial x^j} \right\rangle + \left\langle \nabla_X \nabla b, \nabla_{\frac{\partial}{\partial w^i}} \frac{\partial}{\partial x^j} \right\rangle \quad (39)$$

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The right side is a function in $O(r_N)$ for small $r_N$, and recalling that $\frac{\partial}{\partial \psi}$ is orthogonal to $\Sigma$, we can integrate to conclude that $\langle \nabla_X \nabla b, \frac{\partial}{\partial \psi} \rangle = O(r_N^2)$. Now this implies that

$$\left\langle \nabla_X \nabla b, \frac{\partial}{\partial \psi} \right\rangle = O(r_N)$$

(40)

and since (a multiple of) $\nabla_X$ can limit to either of the $\frac{\partial}{\partial \psi}$, we obtain that on $\Sigma$

$$\nabla_{\frac{\partial}{\partial \psi}} \nabla b = 0.$$  

(41)

Using (38) this shows that $|\nabla^2 b|^2 = (\Delta_\Sigma b)^2$ and $\Delta b = \Delta_\Sigma b$ when restricted to $\Sigma$. Thus we have $0 = (\Delta_\Sigma b)^2 + \langle \nabla \Delta_\Sigma b, \nabla b \rangle$. However, the Böcher formula on $\Sigma$ itself gives

$$0 = (\Delta_\Sigma b)^2 + \langle \nabla \Delta_\Sigma b, \nabla b \rangle + K$$

(42)

where $K$ is the intrinsic sectional curvature of $\Sigma$. Thus $K = 0$ at smooth points of $b$ restricted to $\Sigma$.  

This implies $\Sigma$ is intrinsically flat. Since it is 2-dimensional and has a ray, it is a cylinder so has a line. With the buseman construction from above, the usual arguments give the metric splitting of $\mathcal{N}$.

**Step II: Case of non-zero asymptotic volume Ratio**

Second, we verify (6) provided the asymptotic volume ratio is non-zero. In this case, the energy condition implies that curvature falloff is $o(r^{-2})$, and it is well known (see [6] [1] [20]) that the manifold is therefore ALE, and indeed that the curvature falloff is $O(r^{-2-\delta})$.

The vector field $X$ therefore converges to its Euclidean model like $O(r^{-1-\delta/2})$, and so has no zeros outside of some definite radius. Then standard arguments show that

$$\frac{1}{8\pi^2} \int_{\mathcal{N}} |\text{Rm}|^2 = \sum \chi(N_X) - \frac{1}{|\Gamma|}$$

(43)

where $|\Gamma|$ is the order of the quotient group at infinity. Since $|\Gamma|^{-1}$ is in fact the asymptotic volume ratio, equation (6) holds.

**Step III: Case of zero asymptotic volume ratio: use of a blow-up limit**

Next we turn to the case that $\mathcal{N}^4$ has zero asymptotic volume ratio. Let $p \in \mathcal{N}$ be a basepoint; theorem 9.1 of [13] states that $|\text{Rm}| = O(r^{-2})$ where $r$ is the distance to $r$ (notice this is not enough to conclude that the manifold is asymptotically locally flat, as the topology might not be fixed). Consider again (24), written

$$\frac{1}{8\pi^2} \int \varphi |\text{Rm}|^2 = \int d\varphi \wedge \mathcal{T}\mathcal{P}_X,$$

$$\mathcal{T}\mathcal{P}_X = \frac{2}{8\pi^2} |X|^{-2} X \wedge \langle \text{Rm}, *\nabla X \rangle + \frac{1}{16\pi^2} |X|^{-4} \langle \nabla X, *\nabla X \rangle X \wedge dX.$$  

(44)
We shall show that $|TP_X| = O(r^{-3})$ away from a certain “bad” set, which can be dealt with separately.

Pick a sequence $C_i \to \infty$ and a divergent sequence of points $p_i$ with the property that $|TP_X(p_i)| > C_i r^{-3}$. Now possibly $|TP_X(p_i)|$ is completely uncontrollable so we cannot meaningfully rescale to make $|TP_X| = 1$, but we may rescale the metric by, say, $C_i^{-\frac{1}{4}} r^{-2}$ (so $|TP_X(p_i)| > \sqrt{C_i}$). In this sequence of metrics, the basepoint $p$ trails off away from the points $p_i$ to infinity like $C_i^\frac{1}{2}$, and $|Rm| \searrow 0$ in compact domains surrounding $p_i$. Choose a sequence of radii $R_i \nearrow \infty$ so that $|Rm| < 2^{-1}$ on $B(p_i, R_i)$. Because $|TP_X|$ is uncontrolled at $p_i$, then $|\nabla X||X|^{-1}$ must be unbounded. For each $i$, let $X_i$ be the vector field $X$ scaled so that $|\nabla X_i(p_i)| = 1$, so of course $|X_i(p_i)| \searrow 0$ and in the limit $X_i$ has a zero.

\textbf{Step IV: Two blow-up limits}

We consider two pointed sequences, $(B(p_i, R_i), p_i)$ and $(\overline{B}(p_i, R_i), \tilde{p}_i)$ where $\overline{B}(p_i, R_i) \to B(p_i, R_i)$ is the universal cover. By the Klingenberg lemma, there are no short, contractible geodesics loops on $B(p_i, R_i)$, so we have an injectivity radius lower bound on $\overline{B}(p_i, R_i)$. Thus we obtain two limits: a complete length space $(B_\infty, \tilde{p}_\infty) = \lim_i (B(p_i, R_i), \tilde{p}_i)$ and a flat 4-manifold $(\overline{B}_\infty, \bar{p}_\infty) = \lim_i (\overline{B}(p_i, R_i), g, p_i)$ with a Killing field $\overline{X}_\infty$. Further we have $|\nabla \overline{X}_\infty| = 1$ and $|\overline{X}_\infty| = 0$. Clearly the zero locus is has a single component, as, passing to the universal cover (which is flat $\mathbb{R}^n$), the pre-image of the zero locus is either a point (if $\overline{B}_\infty = \mathbb{R}^4$) or a 2-plane.

Now at each stage we have maps $\overline{B}(p_i, R_i) \to B(p_i, R_i)$, and in the limit we retain a map $\overline{M}^k \to M'$, which is via reduction to the orbit space of a pure N-structure [12]. In fact, this N-structure is invariant under $\overline{X}_\infty$, which can be seen from the fact that $X_i$ on $\overline{B}(p_i, R_i)$ is invariant under deck transformations.

\textbf{Step V: The zero-locus of $\overline{X}$ on $\overline{B}_\infty$ is 2-dimensional}

Consider the map $\pi_i : \overline{B}(p_i, R_i) \to B(p_i, R_i)$. Because $\overline{B}(p_i, R_i)$ is Lipschitz close to some domain $\Omega_i \subset \overline{B}_\infty$ and because $\mathbb{R}^4$ covers $\overline{B}_\infty$, the map $\pi_i$ lifts to a map $\pi'_i : \Omega'_i \to B(p_i, R_i)$ where $\Omega'_i$ is some open set in $\mathbb{R}^n$. We may pull back the (almost flat) metric on $\overline{B}(p_i, R_i)$ to $\Omega'_i$ and pull back the Killing field $X_i$ as well, so map $\pi'_i$ is a Riemannian covering map. However the $\Omega'_i \to B(p_i, R_i)$ occurs via identification by deck transformations, each of which has a translational component. Because $\overline{X}_i$ is arbitrarily close to the limiting Killing field $\overline{X}_\infty$ which has a single fixed point, it is impossible that $\overline{X}_i$ is invariant under these transformations.

Thus the vector field $\overline{X}$ has a 2-dimensional null-set. On $\mathbb{R}^4$, any such vector field is a rotation about a 2-plane “axis,” and its perpendicular distribution is integrable. The Frobenius integrability criterion forces $X_i \wedge dX_i = 0$, and because also $Rm = 0$, we have from equation $\textbf{[13]}$ that $TP_X = 0$. In addition, the 2-form $\nabla X$ is decomposable, forcing $\langle \nabla X, *\nabla X \rangle = 0$. All this shows that $|TP_X|$ converges uniformly to zero on domains in $\overline{B}(p_i, R_i)$, and therefore $B(p_i, R_i)$, that miss the null set itself (where $TP_X$ is undefined).
Now consider the original pointed manifold \((N, p)\) again, with its unscaled metric. Let \(\epsilon > 0\), and define \(N(\epsilon)\) to be the domain

\[
N(\epsilon) = \left\{ q \in N \mid \frac{|\nabla X|}{|X|} > \epsilon^{-1}r(q)^{-1} \right\}
\]

(45)

In the convergence scheme above, each \(B(p_i, R_i)\) has non-trivial intersection with \(N(\epsilon)\). However we proved that, in the rescaled metrics, \(|TP| \to 0\) outside \(N(\epsilon)\), which means that \(|TP| = O(r^{-3})\) outside of \(N(\epsilon)\).

**Step VI: Conclusion**

Pick a large radius \(S\) and let \(\varphi^S\) and \(\varphi^\epsilon\) be the following cutoff functions. Let \(\varphi^S\) be the standard \(C^\infty_c\) cutoff function that is zero outside \(B(p, S)\), unity inside \(B(p, S/2)\), and with \(|\nabla \varphi^S| < 10S^{-1}\). Let \(\varphi^\epsilon\) be a cutoff function that is zero on \(N(2\epsilon)\), unity on the compliment of \(N(\epsilon)\). Consider again (24) with the cutoff function \(\varphi^S \varphi^\epsilon\), which is

\[
\frac{1}{8\pi^2} \int \varphi^S \varphi^\epsilon |\text{Rm}|^2 = \int \varphi^S d\varphi^\epsilon \wedge TP_x + \int \varphi^\epsilon d\varphi^S \wedge TP_x.
\]

(46)

Because \(|TP_x| = O(r^{-3})\) outside \(\Omega^\epsilon\), we have

\[
\left| \int \varphi^\epsilon d\varphi^S \wedge TP_x \right| \leq 4S^{-1} \int \varphi^\epsilon |TP_x| \leq CS^{-4} \text{Vol} (B(p, S) \setminus B(p, S/2)).
\]

(47)

Taking the limit as \(S \to \infty\), this vanishes. This leaves

\[
\frac{1}{8\pi^2} \int \varphi^\epsilon |\text{Rm}|^2 = \int d\varphi^\epsilon \wedge TP_x.
\]

(48)

The intersection of the sets \(N(\epsilon)\) as \(\epsilon \to 0\) is just the null-set of \(X\). Recalling that the components of the null set are compact and therefore (eventually) uniformly approximated by components of \(N(\epsilon)\), we have indeed

\[
\frac{1}{8\pi^2} \int |\text{Rm}|^2 = \lim_{\epsilon \to 0} \int d\varphi^\epsilon \wedge TP_x = \sum_i \chi(N_{X,i})
\]

(49)

where the sum is taken over components \(N_X\) of the null set.

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