Numerical experiments with multistep model-predictive control approaches and sensitivity updates for the tracking control of cars

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Abstract: The paper discusses multistep nonlinear model-predictive control (NMPC) schemes for the tracking of a car model along a given reference track. In particular we will compare the numerical performance and robustness of classic single step NMPC, multistep NMPC without re-optimization, multistep NMPC with re-optimization, and multistep NMPC with sensitivity updates instead of a full re-optimization.

Keywords: model predictive control, multistep NMPC, sensitivity updates, car tracking

1. INTRODUCTION

Nonlinear model predictive control (NMPC) is a feedback control paradigm with the capability to take into account control and/or state constraints, compare Rawlings and Mayne (2009); Grüne and Pannek (2011) for a comprehensive overview and analysis. As such it is very powerful, but it relies on the repeated solution of nonlinear optimization problems on a moving time horizon. Especially in online computations the solution of the latter often turns out to be the computational bottleneck of NMPC and efficient numerical techniques are required, compare Diehl et al. (2005). Still, it is often not possible to fully solve these optimization problems within a given time frame. If this is the case, several modifications of the classic NMPC scheme exist. For instance, one could simply stop the iterative optimization procedure as soon as a time budget is consumed and accept the so far obtained result. Another way is to use multistep NMPC schemes, which do not just implement one control step of a computed solution, but more than one steps. This approach gains additional time to solve an optimization problem on a predicted preview horizon. On the downside, since deviations from the reference trajectory are not corrected at these steps, the approach is less robust than the classic NMPC scheme. To overcome this drawback, a re-optimization on the remaining part of the preview horizon can be performed in order to react on intermediate deviations. This leads to a multistep NMPC scheme with re-optimization. Finally, instead of performing a re-optimization, one could instead use parametric sensitivity analysis to update the optimal solution on the preview horizon in the presence of perturbations. This leads to a multistep NMPC scheme with sensitivity updates, compare Zavala et al. (2008). The purpose of the paper is to investigate and compare the classic scheme and the three modifications of the classic NMPC scheme in view of their tracking performance and numerical robustness. A theoretical investigation can be found in Palma (2015).

Throughout, the aim is to construct a feedback control law \( \mu : \mathbb{N} \times X \rightarrow U \) for the constrained control system in discrete time

\[
    x(k+1) = f(x(k), u(k)), \quad k = 0, 1, 2, \ldots, \\
    x(k) \in X, \quad k = 0, 1, 2, \ldots, \\
    u(k) \in U, \quad k = 0, 1, 2, \ldots, \\
    x(0) = x_0,
\]

to track a given reference trajectory \((x_r(k), u_r(k)), k = 0, 1, 2, \ldots\). Herein, \( X \subset \mathbb{R}^n \) and \( U \subset \mathbb{R}^m \) are given sets.

Often, the control system in discrete time can be interpreted as a discretization of a continuous process, where \((x(k), u(k))\) corresponds to the state and the control at time \(t_k = kh\) with sampling time \(h > 0\). Each of the different NMPC schemes yields a feedback control law \( \mu_{N,M} \), where \( N \) denotes the preview horizon and \( M \) the control horizon. Closing the loop by setting \( u(k) = \mu_{N,M}(k, x(K_M(k))) \) with \( K_M(k) \leq k \) yields the closed-loop system

\[
    x(k+1) = f(x(k), \mu_{N,M}(k, x(K_M(k))), \quad k = 0, 1, 2, \ldots, \\
    x(0) = x_0.
\]

The feedback control laws \( \mu_{N,M} \) will be defined in Sections 2 and 3 for the following NMPC versions:

- standard one-step NMPC,
- multistep NMPC,
- multistep NMPC with re-optimization,
- multistep NMPC with sensitivity updates.
Numerical experiments with these schemes are presented in Section 4.

## 2. NMPC SCHEMES

Each of the NMPC schemes require to solve optimal control problems in discrete time on some time horizon \([k_0, k_0 + N]\) of the following type:

\[
\text{OCP}(k_0, x_0, N): \quad \text{Minimize} \quad \sum_{k=k_0}^{k_0+N-1} f_0(k, x(k), u(k))
\]

subject to the constraints

\[
\begin{align*}
  x(k+1) &= f(x(k), u(k)), \quad k = k_0, \ldots, k_0 + N - 1, \\
  x(k) &\in X, \quad k = k_0, \ldots, k_0 + N, \\
  u(k) &\in U, \quad k = k_0, \ldots, k_0 + N - 1, \\
  x(k_0) &= x_0.
\end{align*}
\]

Herein, \(k_0\) denotes the current time of the process, \(x_0\) the current (measured or predicted) state, and the number \(N\) is called preview horizon. Throughout we consider tracking type objective functions. To this end, let a reference trajectory \((x_0(k), u_0(k)), k = 0, 1, 2, \ldots, N\), be given. The function \(f_0\) is then defined by

\[
f_0(k, x, u) = \|x_r(k) - x\|^2_V + \|u_r(k) - u\|^2_W
\]

with weighted norms \(\|y\|^2_V = \sqrt{y^TVy}\), \(\|z\|^2_W = \sqrt{z^TWz}\), where \(V\) and \(W\) are symmetric and positive semi-definite matrices. Throughout it is assumed that OCP\((k_0, x_0, N)\) for any choice of \((k_0, x_0, N)\) is feasible and possesses an optimal solution \((\hat{x}(k), \hat{u}(k))\), \(k = k_0, \ldots, k_0 + N - 1\) (for notational simplicity we omit \(\hat{x}(k_0 + N)\)) throughout, which can be computed by standard techniques, compare Gerdts (2011). The problem of infeasibility could be addressed in practice by relaxation of constraints or by choosing \(N\) sufficiently large.

The classic NMPC algorithm reads as follows and it yields a feedback law \(\mu_N = \mu_{N,1} : X \times X \to U\).

### Algorithm 1. (classic NMPC).

(0) Input: preview horizon \(N\), reference trajectory \((x_r(\cdot), u_r(\cdot))\), weight matrices \(V\) and \(W\). Set \(k = 0\).

(1) Measure state \(x(k) \in X\) at time \(k\).

(2) Solve OCP\((k, x(k), N)\) on time horizon \([k, k + N]\). Let \(\hat{u}(k), \ldots, \hat{u}(k + N - 1)\) be the optimal control.

(3) Define the feedback control \(\mu_N(k, x(k)) := \hat{u}(k)\) and apply it:

\[
x(k + 1) = f(x(k), \mu_N(k, x(k)))
\]

(4) Set \(k \leftarrow k + 1\) and go to (1).

### Remark 1.

Note that the implementation of \(\mu_N(k, x(k))\) typically is delayed by some \(\delta > 0\), where \(\delta\) denotes the time to solve OCP\((k, x(k))\). Alternatively, one could use the predicted state \(x(k + 1)\) in step (3) to solve the next problem OCP\((k + 1, x(k + 1), N)\) already during the step from \(k\) to \(k + 1\). However, the predicted state \(x(k + 1)\) deviates usually from the measured state at \(k + 1\) and hence an update of the computed solution might become necessary. This could be achieved by re-optimization or by sensitivity updates as in Section 3.

The classic NMPC scheme requires to solve the optimal control problem at each time instance. If this is too time consuming, then the following multistep NMPC scheme is useful to reduce the number of optimal control problems to be solved. The idea is to apply not just the control \(\hat{u}(k)\) in step (3) but to apply \(M \leq N\) controls \(\hat{u}(k), \hat{u}(k + 1), \ldots, \hat{u}(k + M - 1)\). The number \(M\) is called control horizon.

### Algorithm 2. (M-multistep NMPC).

(0) Input: preview horizon \(N\), reference trajectory \((x_r(\cdot), u_r(\cdot))\), weight matrices \(V\) and \(W\), control horizon \(M \leq N\). Set \(k = 0\).

(1) Measure state \(x(k) \in X\) at time \(k\).

(2) Solve OCP\((k, x(k), N)\) on time horizon \([k, k + N]\). Let \(\hat{u}(k), \ldots, \hat{u}(k + N - 1)\) be the optimal control.

(3) Define the feedback control

\[
\mu_{N,M}(k + j, x(k)) := \hat{u}(k + j), \quad j = 0, \ldots, M - 1,
\]

and apply it for \(j = 0, \ldots, M - 1:\)

\[
x(k + j + 1) = f(x(k + j), \mu_{N,M}(k + j, x(k)))
\]

(4) Set \(k \leftarrow k + M\) and go to (1).

### Remark 2.

Note that a re-optimization in step (1b) is only necessary, if the measured state at \(k + j\) deviates from the optimal state \(\hat{x}(k + j)\) of the problem OCP\((k + j - 1, x(k + j - 1), N - j - 1)\).
A modification of Algorithm 3, which avoids the solution of the optimal control problems in step (1b), is described in the following Section 3.

3. MULTISTEP NMPC WITH SENSITIVITY UPDATES

The idea of the multistep NMPC scheme with sensitivity updates is to avoid to solve OCP\((k+j+1)\). Instead, the solution of OCP\((k+j+1, x(k+j+1), N-j)\) will be approximated by means of a so-called sensitivity update, which will be the result of a parametric sensitivity analysis of the optimal control problems with respect to the initial states.

3.1 Parametric Sensitivity Analysis

In order to perform the parametric sensitivity analysis, it is convenient to view the optimal control problems as a parametric optimization problem of type

\[ NLP(p): \quad \text{Minimize} \quad J(z,p) \]

with respect to \( z \in \mathbb{R}^{n_z} \) subject to the constraints

\[ H(z,p) = 0, \]
\[ G(z,p) \leq 0. \]

Herein, \( p \in \mathbb{R}^{n_p} \) denotes a parameter, \( J : \mathbb{R}^{n_z} \rightarrow \mathbb{R}, \)
\( H : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_H}, \)
\( G : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_G} \) are at least twice continuously differentiable functions. We are interested in properties of the solution mapping or parameter-to-solution mapping \( p \mapsto z^*(p) \), where \( z^*(p) \) denotes an optimal solution of NLP\((p)\). Conditions under which the solution mapping \( z^* \) depends in a continuously differentiable way on the parameter \( p \) are of particular interest, since in this case a linearization

\[ z^*(p) = z^*(\tilde{p}) + (z^*(\cdot))'(\tilde{p})(p-\tilde{p}) + o(||p-\tilde{p}||) \]

around a nominal parameter \( \tilde{p} \) becomes possible. Neglecting the error term yields the approximate optimal solution \( \bar{z}(p) \) for \( p \) close to \( \tilde{p} \):

\[ z^*(p) \approx \bar{z}(p) := z^*(\tilde{p}) + (z^*(\cdot))'(\tilde{p})(p-\tilde{p}), \]

compare Büskens and Maurer (2001). For \( p \) sufficiently close to \( \tilde{p} \), \( \bar{z}(p) \) may serve as a sufficiently good approximation to the optimal solution \( z^*(p) \) of the perturbed nonlinear optimization problem NLP\((p)\). Note that the evaluation of \( z^*(p) \) requires only a matrix vector multiplication and two vector additions, that is, the computational effort for these operations is negligible. It remains to establish the solution differentiability and the computation of the sensitivity matrix \((z^*(\cdot))'\).

The solution differentiability of the map \( z^* \) was established by Fiacco (1983) with the following sensitivity theorem. The index set of active inequality constraints is given by

\[ A(z,p) := \{ i \mid G_i(z,p) = 0, i \in \{1, \ldots, n_G\} \}. \]

A local minimum \( \hat{z} \) of NLP\((\tilde{p})\) is called strongly regular, if the following properties hold:

(a) \( \hat{z} \) fulfills the linear independence constraint qualification (LICQ), i.e. the gradients \( \nabla_z G_i(\hat{z}, \tilde{p}), i \in A(\hat{z}, \tilde{p}) \), and
\( \nabla_z H_j(\hat{z}, \tilde{p}), j = 1, \ldots, n_H \), are linearly independent.
(b) The KKT conditions hold at \((\hat{z}, \hat{\mu}, \hat{\lambda})\), i.e.

\[ 0 = \nabla_z L(\hat{z}, \hat{\mu}, \hat{\lambda}, \tilde{p}), \quad \hat{\mu} \geq 0, \hat{\mu}^\top G(\hat{z}, \tilde{p}) = 0, \]

where

\[ L(z,\mu,\lambda,p) := J(z,\mu) + \mu^\top G(z,p) + \lambda^\top H(z,p) \]

denotes the Lagrange function of NLP\((p)\) with Lagrange multipliers \( \mu \) and \( \lambda \).
(c) The strict complementarity condition holds:

\[ \hat{\mu}_i - G_i(\hat{z}, \tilde{p}) > 0 \quad \text{for all } i = 1, \ldots, n_G. \]
(d) We have

\[ d^\top \nabla_{zz}^2 L(\hat{z},\hat{\mu},\hat{\lambda},\tilde{p})d > 0 \]

for all \( d \in T_C(\hat{z}, \tilde{p}), d \neq 0 \), where

\[ T_C(z,p) = \left\{ d \mid \nabla_z G_i(z,p)^\top d = 0, i \in A(z,p), \nabla_z H_j(z,p)^\top d = 0, j = 1, \ldots, n_H \right\}. \]

A proof of the following theorem can be found in Fiacco (1983) or (Gerdts, 2011, Theorem 6.1.4).

**Theorem 3.** Let \( J, G, H \) be twice continuously differentiable and \( \hat{p} \) a fixed nominal parameter. Let \( \hat{z} \) be a strongly regular local minimum of NLP\((\hat{p})\) with Lagrange multipliers \( \hat{\lambda} \) and \( \hat{\mu} \).

Then there exist neighborhoods \( B_8(\hat{p}) \) and \( B_8(\hat{z},\hat{\mu},\hat{\lambda}) \), such that NLP\((p)\) has a unique strongly regular local minimum

\[ (z^*(p),\mu^*(p),\lambda^*(p)) \in B_8(\hat{z},\hat{\mu},\hat{\lambda}) \]

for each \( p \in B_8(\hat{p}) \), and \( A(\hat{z},\hat{p}) = A(z^*(p),p) \).

In addition, \((z^*(p),\mu^*(p),\lambda^*(p))\) is continuously differentiable with respect to \( p \) in these neighborhoods with

\[
\begin{pmatrix}
\nabla_{zz}^2 L (G_z')^\top (H_z')^\top \\
\hat{\Xi} \cdot G_z' \hat{\Gamma} 0 \\
H_z' 0 0
\end{pmatrix}
\begin{pmatrix}
(z^*(p))' \\
(\mu^*(\cdot))'(\tilde{p}) \\
(\lambda^*(\cdot))'(\tilde{p})
\end{pmatrix} = - \begin{pmatrix}
\nabla_{zz}^2 L \\
\hat{\Xi} \cdot G_z' \\
H_z'
\end{pmatrix} (1)
\]

where \( \hat{\Xi} = \text{diag}(\hat{\mu}_1, \ldots, \hat{\mu}_{n_G}) \), \( \hat{\Gamma} = \text{diag}(G_1, \ldots, G_{n_G}) \).

Herein, all functions and their derivatives are evaluated at \((\hat{z},\hat{\mu},\hat{\lambda},\tilde{p})\).

**Remark 4.** Please note, that all the assumptions needed to establish solution differentiability can be checked numerically. Results without the strict complementarity condition are derived in Jittorntrum (1984).

3.2 Application to OCP in M-multistep NMPC

For the application of Theorem 3 we assume that the sets \( X \) and \( U \) in the problems OCP\((k,x,N)\) are defined by finitely many inequalities. We exploit the parametric sensitivity analysis to avoid the solution of OCP\((k+j,x(k+j),N-j)\) in step (1b) of Algorithm 3. Instead we approximate its solution by a sensitivity update and arrive at the following algorithm:
Algorithm 4. (M-multistep NMPC with sensitivity upd.).
(0) Input: preview horizon $N$, reference trajectory $(x_{r}(\cdot), u_{r}(\cdot))$, weight matrices $V$ and $W$, control horizon $M \leq N$. Set $k = 0$.
(1) Measure state $x(k) \in X$ at time $k$ and solve $\text{OCP}(k, x(k), N)$ on time horizon $[k, k+N]$. Let $(\hat{x}(k+\ell), \hat{u}(k+\ell), \ell = 0, \ldots, N-1, \text{denote the optimal solution}.
(2) Perform in parallel:
(a) Define the feedback control $\mu_{N,M}(k, x(k)) := \hat{u}(k)$ and apply it
$$x(k+1) = f(x(k), \mu_{N,M}(k, x(k))).$$
(b) For each $j = 1, \ldots, M$ perform a sensitivity analysis of $\text{OCP}(k+j, \hat{x}(k+j), N-j)$ with respect to the parameter $p_j := \hat{x}(k+j)$.
Let $u^j\approx_{\hat{u}}(k+j)$, $\ell = j, \ldots, N-1$, denote the solution mappings according to Theorem 3.
Let $S_j := u^j(k+j)'(\hat{p}_j)$ denote the sensitivity of the nominal control $\hat{u}(k+j)$ of $\text{OCP}(k+j, \hat{x}(k+j), N-j)$ with respect to $p_j$.
(3) For $j = 1, \ldots, M-1$ do
(a) Measure state $x(k+j) \in X$ at time $k+j$.
(b) Define the feedback control
$$\mu_{N,M}(k+j, x(k+j)) := \hat{u}(k+j) + S_j \cdot (x(k+j) - \hat{x}(k+j))$$
and apply it
$$x(k+j+1) = f(x(k+j), \mu_{N,M}(k+j, x(k+j))).$$
(4) Set $k \leftarrow k + M$ and go to (1).

Some remarks are in order. Firstly, the sensitivity analysis is only justified under the assumptions of Theorem 3 for sufficiently small perturbations $x(k+j) \approx \hat{x}(k+j)$. As a consequence, the sensitivity analysis might not provide good approximations for large deviations. In the latter situation, it is recommended to fully re-solve $\text{OCP}(k+j, \hat{x}(k+j), N-j)$ as in Algorithm 3. Still, the updated control in (3b) may serve as an initial guess.

Secondly, it is important to note that the tails $(\hat{x}(k+\ell), \hat{u}(k+\ell), \ell = j, \ldots, N-1, \text{of the optimal solution (}\hat{x}(k+\ell), \hat{u}(k+\ell)), \ell = 0, \ldots, N-1, \text{of OCP}(k, x(k), N)$ are optimal for the problems $\text{OCP}(k+j, \hat{x}(k+j), N-j), j = 1, \ldots, M$, according to Bellman’s optimality principle. Hence, by solving $\text{OCP}(k, x(k), N)$ in step (1), all nominal solutions to the problems $\text{OCP}(k+j, \hat{x}(k+j), N-j), j = 1, \ldots, M$, in step (2b) are known and the initial state of $\text{OCP}(k+j, \hat{x}(k+j), N-j)$, i.e. $\hat{p}_j = \hat{x}(k+j)$, can be viewed as a parameter entering the problem.

Thirdly, please note that the parametric sensitivity analysis in step (2b) yields different solution mappings $u^j(k+j)'(\cdot)$ at the time point $k+j$ is actually exploited in (3b).

It remains to compute the sensitivities $S_j, j = 1, \ldots, M$, in step (2b) in an efficient way. The straightforward way of doing this is to solve equation (1) for each of the problems $\text{OCP}(k+j, \hat{x}(k+j), N-j), j = 1, \ldots, M$, using the nominal solution $(\hat{x}(k+\ell), \hat{u}(k+\ell)), \ell = j, \ldots, N-1$, and the nominal parameter $\hat{p}_j = \hat{x}(k+j)$. Please note that the dimension of the linear equation shrinks with increasing $j$ since the variables $(\hat{x}(k+\ell), \hat{u}(k+\ell)), \ell = 0, \ldots, j-1$, and the constraints at the time points $k+\ell, \ell = 0, \ldots, j-1$, are not present in $\text{OCP}(k+j, \hat{x}(k+j), N-j)$.

Since only the first sensitivity $u^j(k+j)'(\cdot)$ is actually exploited in (3b), solving the full linear systems is not necessary and an alternative and more efficient way is outlined in the sequel. Herein, the sensitivity analysis is merely performed for $\text{OCP}(k, x(k), N)$ with respect to the parameter $p_0 = \hat{x}(k)$. This yields the sensitivity differentials
$$u^j(k+\ell)'(p_0), \ell = 0, \ldots, N-1,$$ at the time points $k+\ell, \ell = 0, \ldots, N-1$, by solving equation (1) once for the nominal solution $(\hat{x}(k+\ell), \hat{u}(k+\ell)), \ell = 0, \ldots, N-1$. If a deviation $p_0 = x(k)$ of $\hat{p}_0$ is detected, the optimal control can be updated by Taylor approximation
$$u^j_0(k+\ell)(p_0) \approx \hat{u}(k+\ell) + u^j_0(k+\ell)'(p_0)(p_0 - \hat{p}_0)$$ for $\ell = 0, \ldots, N-1$ and the state can be predicted by
$$x(k+\ell+1) = f(x(k+\ell), u^j_0(k+\ell)(p_0))$$ for $\ell = 0, \ldots, N-1$.

Unfortunately, the $M$-multistep NMPC algorithm with sensitivity updates requires the sensitivities $S_j = u^j_0(k+j)'(p_0)$ with $\hat{p}_j = \hat{x}(k+j)$ for $j = 1, \ldots, M$, and not the sensitivities $u^j_0(k+j)'(p_0)$ in (2). Hence, a way to compute the $S_j$’s from (2) is sought. To this end, we exploit the dynamics
$$\hat{x}(k+1) = f(\hat{x}(k), \hat{u}(k)).$$

Assumption 5. Let the Jacobian matrix $J_k(\hat{x}(k), \hat{u}(k))$ be non-singular.

Remark 6. Please note that Assumption 5 is satisfied for a sufficiently small step-size $\delta$, if the dynamics are given by a one-step discretization method (e.g. a Runge-Kutta method) for a differential equation, i.e. if $f$ is of type $f(x, u) = x + \delta \Phi(x, u, h)$.

If Assumption 5 holds, Equation (3) can be solved for $\hat{x}(k)$ by the implicit function theorem, which yields the existence of neighborhoods $B_\delta(\hat{x}(k+1))$ and $B_\delta(\hat{x}(k))$ with $\epsilon > 0, \delta > 0$ and a mapping
$$\xi_0 : B_\delta(\hat{x}(k+1)) \rightarrow B_\delta(\hat{x}(k))$$
such that $\hat{x}(k) = \xi_0(\hat{x}(k+1))$ and
$$x(k+1) = f(\xi_0(\hat{x}(k+1)), \hat{u}(k))$$ holds for every $x(k+1) \in B_\delta(\hat{x}(k+1))$. Moreover, by differentiating this identity with respect to $x(k+1)$ we find
$$I = J'_\xi(\xi_0(\hat{x}(k+1)), \hat{u}(k)) : \xi'_0(\hat{x}(k+1))$$ and thus
$$\xi'_0(\hat{x}(k+1)) = J'_\xi(\xi_0(\hat{x}(k+1)), \hat{u}(k))^{-1} = J'_\xi(\hat{x}(k), \hat{u}(k))^{-1}.$$

Note that $\xi'_0(\hat{x}(k+1))$ is the derivative of the initial state $\hat{x}(k)$ with respect to $x(k+1)$.
Note further, that we have the relation
\[ u_1^*(k+1)(x(k+1)) = u_0^*(k+1)(\xi_0(x(k+1))) \]
for every \( x(k+1) \in B_r(\hat{x}(k+1)) \) (eventually after reducing \( \epsilon \) taking into account the neighborhoods of the sensitivity theorem).

Now, by the chain rule we obtain
\[
S_1 = u_1^*(k+1)'(\hat{x}(k+1)) = u_0^*(k+1)'(\hat{x}(k)) \cdot \xi_0(\hat{x}(k+1)) = u_0^*(k+1)'(\hat{x}(k)) \cdot f_\ell'(\hat{x}(k), \hat{u}(k))^{-1}.
\]
This formula allows to compute \( S_1 \) without solving equation (1) for OCP \((k+1, \hat{x}(k+1)), N-1\).

This construction can be repeated for \( j = 2, \ldots, M \) exploiting the relations
\[
u^*_j(k+j)(x(k+j)) = u_0^*(k+j)(\xi_0 \circ \xi_1 \cdots \circ \xi_{j-1}(x(k+j))),
\]
where \( \xi_{j-1} \) satisfies \( \hat{x}(k+j-1) = \xi_{j-1}(\hat{x}(k+j)) \) and \( x(k+j) = f(\xi_{j-1}(x(k+j)), \hat{u}(k+j-1)) \)
holds for every \( x(k+j) \) in some neighborhood of \( \hat{x}(k+j) \). Herein, Assumption 5 has to hold accordingly for \( f_\ell'(\hat{x}(k+\ell), \hat{u}(k+\ell)), \ell = 1, \ldots, M-1 \). Then we obtain
\[
S_j = u_0^*(k+j)'(\hat{x}(k+j)) = u_0^*(k+j)'(\hat{x}(k)) \cdot \prod_{\ell=0}^{j-1} f_\ell'(\hat{x}(k+\ell), \hat{u}(k+\ell))^{-1}.
\]
For a rigorous mathematical stability and performance analysis of the different \( M \)-multistep NMPC schemes we refer the reader to Palma (2015).

4. NUMERICAL EXPERIMENTS

We compare the four NMPC schemes for the problem of tracking the raceline along the testtrack of Oschersleben in Figure 1 with the following kinematic car model:
\[
x'(t) = v(t) \cos \psi(t), \quad x(0) = x_0,
\]
\[
y'(t) = v(t) \sin \psi(t), \quad y(0) = y_0,
\]
\[
\psi'(t) = \frac{v(t)}{\ell} \tan \delta(t), \quad \psi(0) = \psi_0,
\]
\[
v'(t) = u_1(t), \quad v(0) = v_0,
\]
\[
\delta'(t) = u_2(t), \quad \delta(0) = \delta_0.
\]
Herein, \( \ell = 4 \) \( [m] \) denotes the length of the car, \( (x, y) \) the position of the center of the rear axle, \( \psi \) the yaw angle, \( v \) the velocity, and \( \delta \) the steering angle. All numerical experiments have been conducted with \((x_0, y_0, \psi_0, v_0, \delta_0) = (0 \ [m], 0 \ [m], 0 \ [rad], 10 \ [m/s], 0 \ [rad])\), a preview horizon of \( T = 3 \) \( [s] \), \( N = 11 \) grid points (i.e. a step-size of \( h = 0.3 \) \([s]\)), control horizon \( M = 3 \). The initial position \((x, y)\) and the velocity \( v \) in each step of the NMPC schemes are perturbed by adding equally distributed noise in the range \([-0.05, 0.05]\), which is realistic for measurements with a differential GPS system. The total control time horizon was \( T_f = 110 \) \([s]\). The controls are subject to the control bounds \( u_1 \in [-12, 3] \) \([m/s^2]\) and \( u_2 \in [-0.5, 0.5] \) \([rad/s]\). Moreover, the state constraints \( v \in [0, 60] \) \([m/s]\) and \( \delta \in [-0.5, 0.5] \) \([rad]\) have to be obeyed. Throughout, the objective function
\[
\int_0^T \left( \|x(t) - x_r(t)\|^2 + \alpha_2(v(t) - v_r(t))^2 + \alpha_3 \left( \begin{array}{c} u_1(t) - u_{1,r}(t) \\ -u_2(t) - u_{2,r}(t) \end{array} \right) \right)^2 \ dt
\]
was maximized with \( \alpha_1 = 1 \), \( \alpha_2 = 10^{-1} \), and \( \alpha_3 = 10^{-3} \) was used in the NMPC schemes. The optimal control package GCPID-DAE1 Gerds (2013) was used for solving the optimal control problems and performing the sensitivity analysis. The focus of the study is on the robustness and tracking error of the methods, not on the CPU times. For this reason, the sensitivities are computed by solving (1) for simplicity.

Figure 2 shows the tracking error \( \|x - x_r, y - y_r, v - v_r \| \) measured in the \( L_2 \)-norm. The results show that the classic NMPC scheme performs best with regard to the tracking error, followed by the multistep NMPC scheme with re-optimization, the multistep NMPC scheme with sensitivity updates, and the multistep NMPC scheme. This outcome is the expected one since the classic scheme optimizes in each step on the full preview horizon while the multistep scheme optimizes only after \( M \) shifts have been performed. The multistep scheme with re-optimization re-optimizes at least on a shrinking horizon at every shift. This scheme holds true for the multistep scheme with sensitivity updates, but this only provides a Taylor approximation to the optimal solution. The large initial error is due to a large deviation of about 8 \([m]\) in the \( y \)-direction from the initial state of the reference solution.

Figure 3 shows the errors of the \((x,y)\)-position and the velocity for the four NMPC schemes. All schemes are able to track the reference solution at a high precision. Recall that equally distributed noise with an amplitude of 0.1 was added in the NMPC schemes.
Fig. 2. Tracking error for the four different NMPC strategies.

5. CONCLUSION

A numerical study of the classic NMPC scheme, the multistep NMPC scheme, the multistep NMPC scheme with re-optimization, and the multistep NMPC scheme with sensitivity updates was performed and tested for a tracking problem along a racing track with a kinematic car model. The numerical study shows that all approaches are feasible and are able to track the given reference trajectory subject to random noise. Moreover, the results support the expectation that the classic NMPC scheme performs best with regard to the tracking error. It is followed by the multistep scheme with re-optimization and the multistep scheme with sensitivity updates. Finally, the basic multistep scheme yields the largest tracking error of the four approaches.

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