Superconducting vortices in Semilocal Models

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Abstract

It is shown that the SU(2) semilocal model – the Abelian Higgs model with two complex scalars – admits a new class of stationary, straight string solutions carrying a persistent current and having finite energy per unit length. In the plane orthogonal to their direction they correspond to a nontrivial deformation of the embedded Abrikosov-Nielsen-Olesen (ANO) vortices by the current flowing through them. The new solutions bifurcate with the ANO vortices in the limit of vanishing current. They can be either static or stationary. In the stationary case the relative phase of the two scalars rotates at constant velocity, giving rise to an electric field and angular momentum, while the energy remains finite. The new static vortex solutions have lower energy than the ANO vortices and could be of considerable importance in various physical systems (from condensed matter to cosmic strings).

It is difficult to overemphasize the importance of “topological defects” (monopoles, vortex lines, domain walls, etc) resulting from the spontaneous breaking of global or local symmetries in modern physics, we refer to the some of the numerous reviews [1]. There is an interesting class of models in which both global and local symmetries are simultaneously broken, in the most economical way, with the minimal number of scalar fields. These models, dubbed semilocal models, have some remarkable features, see the detailed review [2]. Perhaps the most remarkable point is that semilocal models exhibit vortices, despite the first homotopy group of the vacuum manifold being trivial [3, 4]. The simplest example of such a model is when electromagnetism is coupled to two charged complex scalars transforming as a doublet under a global SU(2) symmetry (extended Abelian Higgs model). This simple case is made quite interesting by noting that it represents the bosonic sector of the Electroweak theory in the limit that the Weinberg angle, \( \theta_W = \pi/2 \) [5]. Furthermore such multicomponent Ginzburg-Landau models are also used in condensed matter theory, for example in the description of unconventional superconductors [6], in modeling neutral two-component plasmas [7]. The Abrikosov-Nielsen-Olesen (ANO) vortex solutions [8], characterized by the mass ratio, \( \beta = m_s/m_v \) (\( m_s \) resp. \( m_v \) denoting the mass of the scalar resp. of the vector fields), are embedded into this theory. The parameter \( \beta \) also distinguishes superconductors of type I (\( \beta < 1 \)) from type II (\( \beta > 1 \)). The case \( \beta = 1 \) is quite special for here, instead of there being a unique vortex solution (the ANO vortex), there is a family of them labelled by a complex number of the same energy, satisfying the Bogomol’nyi equations [9]. The stability of the embedded ANO vortices in Abelian models with an extended Higgs sector was examined in Ref. [4], and it was shown there that they are stable only if \( \beta \leq 1 \). The counterparts of these solutions are the Z-strings in the Standard model which have been shown to be stable for \( \sin^2 \theta_W \gtrsim 0.9 \) [2].

In this Letter we present a new class of straight flux-tube solutions (strings) of the SU(2) symmetric semilocal model which are translationally symmetric along the \( z = x_3 \) direction.
The Noether current associated to the remaining global U(1) symmetry is then simply

$$J_\mu = \frac{1}{g} \epsilon_{\mu\nu\alpha} \partial_\nu |\Phi|^2$$

where $(\omega, x)$ along gauge orbits fields by a nontrivial gauge transformation. In other words a space-time translation moves the stationary vortices are energy per unit length. The energy of the current carrying solutions is

$$E = \frac{1}{2\pi} \int |J_\nu F^{\nu\mu} - \partial_\mu \Phi|^2 d^4x$$

Due to the strong (exponential) localization of the fields the vortices have finite electric field. Due to the strong (exponential) localization of the fields the vortices have finite energy per unit length. The energy of the current carrying solutions is lower than that of the corresponding ANO vortex, therefore they should have better stability properties. When the current vanishes the solution bifurcates with a new solution branch, which carry a nonvanishing current. The Z-string counterparts of the new solutions should also be present in the Electroweak model and they may have better stability properties because of the localizing effect of the current. Our solutions can also be interpreted as superconducting cosmic strings of the bosonic type, but without an additional U(1) gauge field as in the prototype U(1)×U(1) model [10]. There are also some important differences between the superconducting strings in Witten’s model as compared to the semilocal ones. For example the current carrying semilocal strings are strongly localized.

The (suitably rescaled) Lagrangian of the SU(2) semilocal theory can be written as

$$\mathcal{L} = \frac{1}{2g^2} \left\{ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + |D_\mu \Phi|^2 - \frac{\beta^2}{4} (|\Phi|^2 - 1)^2 \right\}$$

where $\Phi = (\phi_1, \phi_2)$, $D_\mu = \partial_\mu - i A_\mu$, $|\Phi|^2 = \phi_1 \bar{\phi}_1 + \phi_2 \bar{\phi}_2$, and $g$ is the coupling constant, which will be set to 1 for simplicity. The breaking of the U(1) gauge symmetry leads to the physical spectrum consisting of a vector particle with mass $m_\nu = g \eta$, two Nambu-Goldstone bosons, and a Higgs scalar with mass $m_h = \beta \eta$, $\eta$ is the vacuum expectation value of the scalar fields. The most general translationally symmetric (in the $x_3 = z$ direction) and stationary Ansatz can be written as

$$A_\mu = (A_0(x_1, x_2), A_i(x_1, x_2)) \quad i = 1, 2, 3$$

$$\phi_1 = f_1(x_1, x_2) \quad \phi_2 = f_2(x_1, x_2) e^{i(\omega_0 t + \omega_3 z)},$$

where $(\omega_0, \omega_3) = \omega_\alpha$, are real parameters. The most important feature of the Ansatz (2) is that the change in the phase of the scalar fields under an infinitesimal translation is compensated by a nontrivial gauge transformation. In other words a space-time translation moves the fields along gauge orbits. Also the Ansatz (2) reduces the global SU(2) symmetry to U(1). The Noether current associated to the remaining global U(1) symmetry is then simply $J_\mu = i(\phi_2 D_\mu \phi_2 - \phi_2 \bar{D}_\mu \bar{\phi}_2)/2$. Therefore the solutions described by the Ansatz (2) have a conserved Noether charge, $I_0$, as well as a current, $I_3$, flowing along the string where

$$I_\alpha = \pm \eta^2 \int d^2 x (\omega_\alpha - A_\alpha) \bar{\phi}_2 \phi_2, \quad \alpha = (0, 3).$$

The equations of motion imply the following “Gauss-law” constraints on $A_\alpha = (A_0, A_3)$ in the $(x_1, x_2)$ plane:

$$\Delta A_\alpha - A_\alpha |\Phi|^2 + \omega_\alpha \bar{\phi}_2 \phi_2 = 0.$$  

Now exploiting the “constraint” equations (4) one can establish the following lower bound on the total energy per unit length:

$$E \geq \pi |n| \eta^2 + |\omega_\alpha I_\alpha|,$$
where \( n \) is an integer determining the magnetic flux. Furthermore from Eqs. (4) it also follows that for any regular solution \( A_0 = \omega_0 A_3/\omega_3 \) (\( \omega_3 \neq 0 \)), i.e. \( A_0 \) is determined by \( A_3 \). This is related to the fact that one can still perform Lorentz transformations (boosts) in the \((t, z)\) plane, which mix the \( A_0 \) and \( A_3 \) components of the vector potential, and also transform the parameters \((\omega_0, \omega_3)\) of the Ansatz (2). In fact it is only their Lorentz invariant combination, \( \omega^2 = \omega_3^2 - \omega_0^2 \), that appears in the equations of motion. Therefore if \( \omega^2 > 0 \), by a Lorentz boost one can always achieve \( \omega_0 = 0 \), \( A_0 = 0 \), i.e. it is sufficient to consider the static case only. It should be pointed out here that the general Ansatz (2) has been also considered by Abraham [11] in the same model. He considered, however, exclusively the Bogomol’nyi equations existing only in the very special case \( \beta = 1 \). As found in Ref. [11] this implies further \( \omega_3 = \omega_0 \) i.e. \( \omega^2 = 0 \), which is referred to as the chiral case, when due to the lack of localization the energy of the solutions exhibited in Ref. [11] (with winding number \( n = 1 \)) diverges.

To simplify further the problem of finding current carrying string solutions of the semilocal theory Eq. (1), we also impose rotational symmetry in the \((x_1, x_2)\) plane, leading finally to the following Ansatz in polar coordinates:

\[
A_0 = a_0(r), A_r = 0, A_\varphi = na(r), A_3 = \omega a_3(r),
\]

\[
\phi_1 = f_1(r)e^{im\varphi}, \quad \phi_2 = f_2(r)e^{in\varphi}e^{i(\omega_0 t + \omega_3 z)},
\]

where \( m = 0, \ldots n - 1 \). In this Letter we shall concentrate on the simplest \( m = 0 \) case, a more exhaustive investigation will be published [12]. The static (i.e. \( a_0 = 0 \)) field equations for \( m = 0 \) become:

\[
\frac{1}{r} (ra'_3)' = a_3|f|^2 - f_2^2, \quad \text{with} \quad ' = d/dr,
\]

\[
\frac{1}{r} (r f'_1)' = f_1 \left[ n^2 \frac{(1-a)^2}{r^2} + \omega^2 a_3^2 - \frac{\beta^2}{2} (1-|f|^2) \right],
\]

\[
\frac{1}{r} (r f'_2)' = f_2 \left[ n^2 a_3^2 + \omega^2 (1-a_3)^2 - \frac{\beta^2}{2} (1-|f|^2) \right].
\]

Eqs. (7) admit a 4-parameter family of local solutions regular at the origin where \( a = a^{(2)} r^2 + O(r^4), a_3 = a^{(0)} + O(r^2), f_1 = f_1^{(1)} r + O(r^3), f_2 = f_2^{(0)} + O(r^2) \) where \( a^{(2)}, a^{(0)}, f_1^{(1)}, f_2^{(0)} \) are free parameters. For \( r \rightarrow \infty \) the regular solutions decay exponentially: \( a = 1 + O(\exp[-br]), a_3 = O(\exp[-c r]), f_1 = 1 + O(\exp[-c r]), f_2 = O(\exp[-br]) \), where the constants, \( b, c \), determining the exponential decay are given as \( b = \min(1, 2\omega), c = \min(2, \beta, 2\omega) \), and they also comprise a 4-parameter family of local solutions. From this asymptotic behaviour it is clear that \( \omega > 0 \) is an essential parameter to ensure exponential localization. The reduced energy functional takes the form:

\[
E = \pi \eta^2 \int_0^\infty rdr \left\{ n^2 \frac{a^2}{r^2} + \omega^2 a_3^2 + |f|^2 + n^2 \left[ \frac{(1-a)^2}{r^2} f_1^2 + \frac{a^2}{r^2} f_2^2 \right] + \omega^2 \left[ f_1^2 a_3^2 + f_2^2 (1-a_3)^2 \right] + \frac{\beta^2}{4} (1-|f|^2)^2 \right\},
\]

from which it is apparent that the exponentially localized solutions have finite energy. In view of the mathematical complexity of Eqs. (7) we have resorted to numerical techniques to obtain regular solutions. Adapting a multishooting procedure [13], we have succeeded in obtaining numerically families of regular solutions of Eqs. (7), representing a new class of vortices; see Fig. 1 for a sample vortex profile. For a given value of \( \beta > 1 \) our solution class is parameterized by \( \omega \), which varies in a finite interval \( 0 < \omega \leq \omega_b(\beta) \). As the parameter \( \omega \) tends to its maximal
value, \(\omega \to \omega_b(\beta)\), the functions \(a_3 \to 0, f_2 \to 0\), while \(a \to a_{\text{ANO}}, f_1 \to f_{\text{ANO}}\), i.e. the new class bifurcates with the corresponding ANO vortex. This bifurcation can be understood by linearizing Eqs. (7) around the ANO solution with \(a_3 \ll 1, f_2 \ll 1\). To first order \(a_3 \equiv 0\) and the linearized \(f_2\) amplitude satisfies

\[
\frac{1}{r} (rf_2')' + \left[ \frac{a_{\text{ANO}}^2}{r^2} - \frac{\beta^2}{2} (1 - f_{\text{ANO}}^2) \right] f_2 = -\omega_b^2(\beta)f_2. \tag{9}
\]

Eq. (9) corresponds precisely to the linearized stability problem of the embedded ANO vortex in the semilocal theory studied by Hindmarsh [4]. In agreement with his results\(^1\), for any \(\beta > 1\) we find a single normalizable solution, \(f_2\), with negative eigenvalue, \(-\omega_b^2(\beta)\). In Ref. [4] this bound state is a sign of instability of the semilocal ANO vortex. In our case the very same bound state indicates the bifurcation of a new branch with that of the ANO solution at \(\omega = \omega_b(\beta)\). Quite importantly the energy of the new solutions (with \(n = 1\)) is always smaller than that of the ANO vortex. Therefore we expect the new solutions of winding number \(n = 1\) with a nonzero current to be stable.

It is instructive to discuss also the \(\beta = \infty\) limiting theory, when the scalar fields are constrained \(|f_1|^2 + |f_2|^2 \equiv 1\), (a gauged \(\mathbf{CP}^1\)-model) whose field equations can be written as:

\[
\frac{1}{r} (ra_3')' = a_3 - \sin^2 \theta, \quad r(\frac{a'}{r})' = a - \cos^2 \theta, \tag{10a}
\]

\[
\frac{1}{r} (r\theta')' = \frac{1}{2} \left[ \frac{\omega^2 (1 - 2a_3) - n^2 \frac{1-2a}{r^2}}{r^2} \right] \sin(2\theta), \tag{10b}
\]

where we have introduced \(f_1 = \cos \theta, f_2 = \sin \theta\). The profile functions of the current carrying vortex solution of Eqs. (10) are qualitatively similar to those of Eqs. (7) (comp. Fig. 1). The energy of the current carrying solutions is finite for any value of \(\omega\), whereas the energy of the \(\beta = \infty\) ANO vortex diverges. This difference is also reflected by the fact that \(\omega_b(\infty) = \infty\), i.e. the parameter \(\omega\) varies in the interval \((0, \infty)\).

The phase space of the new solution class is illustrated on Fig. 2, choosing the total energy, \(E\), the magnitude of the condensate at the origin, \(f_2^{(0)}\) and \(\omega\) as parameters. On Fig. 2 a family of solutions for a fixed value of \(\beta\) corresponds to a curve in the \((f_2^{(0)}, \omega)\) plane, and

\(^1\)Note that our \(\beta^2\) corresponds to \(\beta\) of Ref. [4].
we have indicated the corresponding ANO vortex, located at the endpoint of the curve in the $f_2^{(0)} = 0$ plane where the new family bifurcates with it. The family of solutions for $\beta = \infty$ is represented by the curve in the $f_2^{(0)} = 1$ plane. As the value of $\omega$ decreases from $\omega_b(\beta)$ towards $\omega = 0$, the energy of all superconducting solutions decreases towards the Bogomol’nyi bound, $E = \pi \eta^2$, without ever attaining it, irrespectively of the value of $\beta$. An important point is that solutions with nonzero values of $\omega$ do not have a smooth $\omega \rightarrow 0$ limit, i.e. the point $(f_2^{(0)} = 1, \omega = 0)$ does not belong to the phase space as it is indicated on Fig. 2 by a circle. In the $\beta = \infty$ theory one can obtain the following virial relation

$$\int_0^\infty r dr \left\{ n^2 \frac{a'^2}{r^2} - \omega^2 [a'^2 + (1 - 2a_3) \sin^2 \theta] \right\} = 0,$$

which immediately implies that for $\omega = 0$ the $\beta = \infty$ theory does not admit nontrivial finite energy solutions. For finite values of $\beta$, a suitable change of scale in Eqs. (7) shows that the effective value of the scalar self coupling is given by $\beta_{\text{eff}} = \beta/\omega$. It follows that the limit $\omega \rightarrow 0$ also implies $\beta_{\text{eff}} \rightarrow \infty$, enforcing $|f_1|^2 + |f_2|^2 \rightarrow 1$. This explains the universal behaviour of the solutions in the $\omega \rightarrow 0$ limit. A more detailed analysis shows [12] that all solutions converge pointwise to a limiting configuration corresponding the $\mathbb{CP}^1$ lump analyzed in Ref. [4].

In the limit $\omega \rightarrow 0$ the value of the current, $\mathcal{I}_3$, is getting large and it even seems to diverge while $\omega \mathcal{I}_3 \rightarrow 0$. This behaviour of the current is in sharp contrast with what is found in the U(1)$\times$U(1) model, where the current has a maximal value above which the superconducting string “goes normal” [14]. The $z$-component of the current, $\tilde{I}_3 = I_3/(2\pi \eta^2)$, is depicted on Fig. 3 as a function of the parameter $\omega$.

Since our vortex solutions exist only in type II superconductors ($\beta > 1$) it is natural to ask as to their interaction energy (which is well known to be repulsive for the type II ANO vortices). Since we have superimposed vortex solutions (for $n > 1$) we can ask if it is energetically favourable for them to break up into widely separated constituents. To this
Figure 3: The current, $\tilde{I}_3$ as a function of $\omega$ for $n = 1, \beta = \sqrt{2}, 2, 3, \infty$. (The curves terminate at the smallest value of $\omega$ we could compute)

Figure 4: Energy/winding, $E/(\pi n \eta^2)$, of the $n = 2, 3$ vortices for $\beta = \sqrt{2}$ in function of the current, $\tilde{I}_3$.

end we depict $E(n, \tilde{I}_3)/(\pi n \eta^2)$, the energy per winding number of the superimposed $n = 2$ and $n = 3$ solutions as a function of the current flowing through them on Fig. 4. The graphs on Fig. 4 indicate that the interaction is repulsive up to a certain value of $\tilde{I}_3$, however, at least for the $n = 2$ vortex it becomes attractive when $\tilde{I}_3 > 0.71$. This suggest that the $n = 2$ superimposed vortex for large enough currents becomes stable with respect to breakup into $n = 1$ vortices unlike the type II ANO vortices. If the interaction for separated vortices has also an attractive phase this could have important physical consequences (e.g. the attraction may significatively change the intercommutation of colliding strings). Let us mention here that in a somewhat different two-component Ginzburg-Landau model similar non-monotonic behaviour of $E/n$ has been reported in [15].

Stationary solutions (with $\omega_0 \neq 0$) are obtained by a Lorentz boost of the static ones along the $z$-axis. The stationary vortices obtained this way have momentum, $P = \omega_0 \tilde{I}_3/\omega_3$, angular momentum, $J = -\omega_0 n \tilde{I}_3/\omega_3$, and they are surrounded by a radial electric field $E_r = \omega_0 a'_3$. It is interesting that in our case the electric field is screened, and the total energy of the superconducting vortices stays finite.
In conclusion such current carrying strings provide a new interesting class of defects which may be realized in realistic physical systems described by semilocal models and they may equally find their applications in a cosmological context as cosmic strings and may even be present in the Standard model of Electroweak interactions. Finally we thank G. Volovik for pointing out that twisted vortices have also been investigated and even experimentally observed in superfluid $^3$He [16].

References

[1] A. Vilenkin, E. P. S. Shellard, Cosmic Strings and Other Topological Defects, C.U.P. Cambridge, (1994); M. Hindmarsh, T. W. B. Kibble, Rep. Prog. Phys. 58, 477 (1995).

[2] A. Achúcarro, T. Vachaspati, Phys. Rep. 327, 427 (2000).

[3] T. Vachaspati, A. Achúcarro, Phys. Rev. D 44, 3067 (1991).

[4] M. Hindmarsh, Phys. Rev. Lett. 68, 1263 (1992); Nucl. Phys. B 392, 461 (1993).

[5] T. Vachaspati, Phys. Rev. Lett. 68, 1977 (1992).

[6] M. Sigrist and K. Ueda, Rev. Mod. Phys. 63, 239 (1991).

[7] M. Lübcke, S. M. Nasir, A. Niemi, K. Torokoff, Phys. Lett. B 534, 195 (2002).

[8] A. A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957); H. B. Nielsen, P. Olesen, Nucl. Phys. B 61, 45 (1973).

[9] E. B. Bogomol’nyi, Sov. J. Nucl. Phys. 24, 449 (1976); H. J. de Vega, F. A. Schaposnik, Phys. Rev. D 14, 1100 (1976).

[10] E. Witten, Nucl. Phys. B 249, 557 (1985).

[11] E. Abraham, Nucl. Phys. B 399, 197 (1993).

[12] P. Forgács, S. Reuillon and M. S. Volkov, in preparation.

[13] P. Forgács, N. Obadia and S. Reuillon, Phys. Rev. D 71, 035002 (2005).

[14] A. Babul, T. Piran, D. N. Spergel Phys. Lett. B 202, 307 (1988); R. L. Davis, E. P. S. Shellard Phys. Lett. B 207, 404 (1988); D. Haws, M. Hindmarsh, N. Turok, Phys. Lett. B 209, 255 (1988); C. T. Hill, H. M. Hodges, M. S. Turner, Phys. Rev. D 37, 263 (1988); P. Amsterdamski, P. Laguna-Castillo, ibid 877; P. Peter, Phys. Rev. D 45, 1091 (1992).

[15] E. Babaev and M. Speight, Phys. Rev. B 72, 180502 (2005).

[16] M. M. Salomaa, G. V. Volovik Rev. Mod. Phys. 59, 533 (1987).