Universal trends in interacting two-dimensional Dirac materials

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(Dated: May 21, 2018)

We investigate the extent to which the category of Dirac materials provides general statements about the behavior of both fermionic and bosonic Dirac quasiparticles in the interacting regime. For both quasiparticle types, we find common features in the renormalization of the conical Dirac spectrum by interactions. This feature motivates us to declare that Dirac materials form a separate and well defined category with universal properties. To support this view, we compute the self-energy for both types of quasiparticles with different interactions and collate previous results from the literature whenever necessary. The computations are performed both at zero and finite temperature. Guided by the systematic presentation of our results in Table I, we conclude that long-range interactions generically lead to an increase of the slope of the single particle Dirac cone, whereas short-range interactions leads to a decrease. The quasiparticle statistics does not impact the self-energy correction for long range repulsion, but does affect the behavior of short-range coupled systems, giving rise to different thermal power law contributions.

Keywords: Dirac Materials, Fermions, Bosons, Fermi velocity.

I. INTRODUCTION

The discovery of graphene and other materials with Dirac nodes has led to a strong interest in materials with Dirac dispersion in condensed matter1–3. Conventional metals are described by well-defined quasiparticles which are low-energy excitations from the Fermi surface. In a typical d-dimensional metal, the Fermi surface is a (d−1)-dimensional manifold. In Dirac materials on the other hand, the locus of points with zero quasiparticle energy shrinks to d−2 or d−3 dimensions due to some additional symmetries present in the system4–6. Specifically, for two-dimensional (2D) Dirac materials, Dirac nodes have to be point objects. The low-energy quasiparticles around the Fermi point have linear dispersion and are described by a massless Dirac-like equation. Because of the linear dispersion, the quasiparticles behave like relativistic particles with an effective speed of light dictated by the Fermi velocity. Many realizations of Dirac fermions have been discussed in, for example, the surface states of topological insulators7–9, nodal superconductor10, graphene9 and several artificial electronic systems11–14. The resulting Dirac cone in the spectrum is protected by symmetries and is robust under disorder and interaction. For example, Dirac nodes are protected by parity and time reversal symmetries in graphene, by time reversal and mirror symmetries in topological insulators and by combination of the C4 rotation and gauge transformation in the case of d-wave superconductors in 2D square lattice.15 The Dirac nodes lead to commonly observed universal features of excitations such as the scaling of the Landau levels with the magnetic field as √B, universal metallic and thermal conductivities related to the nodal structure of carriers, suppressed back scattering, and Klein tunneling16. The common properties shared by these systems lead to the concept of Dirac materials as a useful category for explaining their behavior.

Recently, bosonic systems possessing Dirac-like spectra have also been investigated. Plausible physical realizations of bosonic Dirac materials range from artificial honeycomb lattices made out of superconducting grains17 to magnet18, photonic and acoustic crystal19,20 and plasmonic devices21. At the level of the single particle physics, the band structures of bosonic and fermionic Dirac materials are indistinguishable: both are described by point nodes with linearly dispersing quasiparticles parameterized by a velocity. However, several important questions about this new class of materials remain unanswered: Is there a universal behavior of bosonic and fermionic Dirac Materials when interactions are taken into account? What role does the form of the interaction play? And how are these results modified at finite temperature? The first question is particularly important because one might naively assume that the bosonic or fermionic systems will have a strong impact on the interacting behavior. Our result in Sec. III taken together with the existing literature in the field of Dirac materials, allows us to answer these questions. We find that long range interactions lead to an increase in the slope of the noninteracting Dirac cone, whereas short range interactions generically decrease this slope. For Coulomb repulsion, the asymptotic results for Fermions and Bosons are mathematically identical. However, this is not the case for short-range interactions, where different power laws in temperature and momentum are obtained for the self-energy corrections.
The main result of this paper is the calculation of the self-energy for different Dirac quasiparticles in the interacting regime. We compute the self-energy primarily for three different situations: (a) for Dirac Bosons which interact via a long-range dipole-dipole interaction, (b) for Dirac fermions which interact by long-range Coulomb interaction and (c) for Dirac Fermions subject to the half-filled Hubbard model on honeycomb lattice. We emphasize that to the best of our knowledge a complete momentum and frequency dependent self-energy in perturbation theory for the case (c) has not been reported in the literature. Based on our calculation taken together with other results in the literature, we present a summary of our findings in Table I. Previous results on which we rely include the known logarithmic divergence of the Dirac velocity in 2D Fermi-Dirac materials where quasiparticles interact via the long-range Coulomb interaction. Contact interactions in Fermi-Dirac systems have also been investigated. Finally, recent studies by some of the present authors analyzed the role of short-range interactions for Dirac magnons and Cooper pairs. We use the Kramers-Kronig relation to compute the imaginary part of the on-shell self-energy \( \Sigma^+(k) \) in the case of Dirac fermions interacting via short range interaction and Dirac bosons interacting via long range interaction. The notation \( \Sigma^+(k) \) refers to the on-shell renormalization of the upper band of the two-band conical spectrum as explained in Sec. III.

The rest of the paper is structured as follows. In Sec. II we give a brief description of the linear Dirac spectrum for the free bosonic and fermionic Dirac systems. In Sec. III we present the self-energy calculation in various bosonic and fermionic systems with different types of interactions. The behavior of the renormalized Dirac velocity is extracted from the self-energy. We finally discuss our results and conclude in Sec. IV.

### TABLE I: This table gives a summary of the main results discussed in this paper. The self-energy for the positive/upper part of the Dirac cone. The classification is based on short range and long range interactions. The “+” and “−” signs in the cells for \( \Re \Sigma^+(k) \) respectively signify the enhancement and reduction of the renormalized Dirac velocity for the nodal excitations. For short range Dirac fermions \( f_0 \) and \( f_1 \) are temperature independent positive constants as explained in Sec. III (C). The exponent \( \nu \sim 1 \pm 0,2 \) has been evaluated numerically for half-filled Hubbard model on honeycomb lattice. The renormalization of spectrum for Dirac magnons is proportional to \( T^2 \) as explained in Sec. III (E). The renormalization of the bosonic Dirac spectrum with long range interaction is performed considering a nonequilibrium steady state as discussed in Sec. III (D). The \( \Im \Sigma^+(k) \) is computed with the help of the Kramers-Kronig relation. The bold line between the type of the interactions specifies that the distinction is based on the range of interaction. The quasiparticle statistics plays a negligible role for long-range Coulomb interactions, but determines different outcomes (e.g. in thermal power laws and zero temperature results) for short-range interactions. Note: In the long range Coulomb interaction the self-energy proportionality constant is \( \alpha \cdot e^2 \), where \( \alpha \) is the effective fine structure constant and \( e^2 \) refers to the effective charge of the Dirac excitations. For dipolar interaction the proportionality constant is \( \frac{2 \Lambda d^2}{(2 \pi)^{3/2}} \). For short range interactions the exact proportionality constant is not known to our knowledge.

| Quantity | Quasiparticle | Long range interaction | Short range interaction |
|----------|---------------|------------------------|------------------------|
| \( \Re \Sigma^+(k) \) | Dirac fermions | \( T = 0 \) \( \sim + k \log \left( \frac{\Lambda}{\max(k, T)} \right) \) | \( \sim - k f_0 \) |
| | | \( T \neq 0 \) \( \sim + k \log \left( \frac{\Lambda}{\max(k, T)} \right) \), Fig. 5 (a) | \( \sim - (f_0 + f_1 \nu) \), Sec. III (C) |
| | Dirac bosons | \( T = 0 \) \( \sim + k \log \left( \frac{\Lambda}{\max(k, T)} \right) \) (Coulomb) ; \( \sim + k \) (Dipole) | 0 |
| | | \( T \neq 0 \) \( \sim + k \log \left( \frac{\Lambda}{\max(k, T)} \right) \) | \( \sim - T^d \), Sec. III (E) |
| \( \Im \Sigma^+(k) \) | Dirac fermions | \( T = 0 \) \( \sim - k \) | \( \sim + k^2 f_0 \) |
| | | \( T \neq 0 \) \( \sim - \max(k, T) \) | \( \sim + k^2 (f_0 + f_1 \nu) \) |
| | Dirac bosons | \( T = 0 \) \( \sim - k \) (Coulomb) ; \( \sim - k^2 \) (Dipole) | 0 |
| | | \( T \neq 0 \) \( \sim - \max(k, T) \) (Coulomb) | \( \sim T^d \), Sec. III (C) |
II. FERMI AND BOSE DIRAC MATERIALS: SINGLE PARTICLE DESCRIPTION

We begin by discussing the Dirac theory for bosons and fermions on an equal footing and point out their similarities and differences. In the single particle description, statistics does not come into play. Motivated by this fact, we write down a generic tight-binding model on the honeycomb lattice for quasiparticles \( \psi_i \) of either statistics

\[
H = -t \sum_{\langle ij \rangle} \tilde{\psi}_i^\dagger \sigma^k \tilde{\psi}_j + \text{h.c.} + V \sum_{i, \alpha} \hat{n}_i^\alpha \psi_i. \tag{1}
\]

Here \( \tilde{\psi}_i^\dagger \) denotes the quasiparticle creation operator on the sublattice \( \alpha \) at the lattice site \( i \). The notation \( \langle ij \rangle \) denotes the nearest-neighbor lattice indices with \( t \) being the hopping amplitude. We introduce an additional on-site term \( V \). For fermions we identify \( V \) as the chemical potential and for bosons this term is introduced to account for the positive-definite energy. The number operator is \( \hat{n}_i^\alpha = \psi_i^\dagger \psi_i^\alpha \). Define the Fourier transform of the operator \( \psi_i^\dagger \) as \( \tilde{\psi}^\dagger \) and rewrite the Hamiltonian in momentum space

\[
H = \sum_k \left( \psi_k A_k^\dagger \psi_k B_k \right) \left( \begin{array}{c} V \ & -t \gamma_k \alpha \\ -t \gamma_k \alpha & V \end{array} \right) \left( \begin{array}{c} \psi_k A_k \ \psi_k B_k \end{array} \right), \tag{2}
\]

where \( \gamma_k = \sum \epsilon_{i,k}^\alpha = 2 \cos \left( \frac{\sqrt{3} k_y a}{2} \right) e^{i k_x a} + e^{i k_y a/2} \) is the sum over the nearest neighbor vectors. We diagonalize the Hamiltonian in Eq. (2) and obtain the spectrum as \( \epsilon_{k}^\pm = V \pm t |\gamma_k| \). The bipartite structure of the honeycomb lattice gives rise to two bands which cross at the corner of the hexagonal Brillouin zone (i.e. at the K point), as shown in Fig. 1. We approximated the fermionic spectrum by showing only the low energy linear bands around the Dirac crossing point in Fig. 1(b). On the other hand, the complete spectrum for bosons is shown in Fig. 1(a). Both fermionic and bosonic spectrum close to the crossing point (K) are governed by a Dirac-like Hamiltonian \( H_{eff} = v_D \sigma \cdot k \) where \( v_D = \frac{3a}{2} \) is the Dirac velocity and \( \sigma \) are the Pauli matrices.

In this section, we have explicitly shown the derivation of the Dirac Hamiltonian for a tight-binding model on a honeycomb lattice. There are many realizations of non-interacting Dirac materials as discussed in the introduction. The reader can consult the review article by Wehling et al.\(^\text{[24]}\) for more details on the emergence of Dirac structure in different material realizations. Similarities between fermion and boson Dirac Materials are obvious from this discussion. We stress that in addition to the similar linear spectra the whole algebraic structure of the Dirac equation and related symmetries are parallel in the cases of fermions and bosons. The set of physical effects which follows from the single particle Dirac spectrum for fermions will also follow for Dirac bosons. For example the Dirac bosons near the nodal points at K and K' will have opposite chiralities. This fact implies that one would expect a similar Klein paradox in the scattering of Dirac bosons (viz. photonic graphene\(^\text{[23]}\) as one would have for Dirac fermions\(^\text{[22]}\).

III. FERMI AND BOSE DIRAC MATERIALS: MANY BODY EFFECTS

Now we describe the renormalization of the Dirac spectrum due to interactions in bosonic and fermionic systems. The interactions can be broadly divided into two types: (a) long-range and (b) short-range. In most fermionic systems, the quasiparticles are charged and therefore typically have long-range Coulomb interactions. For example, the Dirac electrons in pristine graphene have a long range Coulomb repulsion, which may be screened by disorder and doping. However, short-range interactions are also possible, for example between the surface states of a 3D topological insulator\(^\text{[25]}\). In many bosonic systems the quasiparticles are neutral and therefore, interactions are short ranged, but long-ranged dipole–dipole interactions are also accessible.

The standard technique to calculate the effects of the energy renormalization is to incorporate the particle–particle self-energy into the total Green’s function of the interacting system. In Dyson notation the self-energy is defined as,

\[
\mathcal{G}[\mathbf{k}, \omega]^{-1} = \mathcal{G}_0[\mathbf{k}, \omega]^{-1} - \Sigma[\mathbf{k}, \omega] \tag{3}
\]

where \( \mathcal{G}[\mathbf{k}, \omega], \mathcal{G}_0[\mathbf{k}, \omega] \) are the Green’s functions for the interacting and free system respectively. Specific functional dependence of the self-energy on momentum \( \mathbf{k} \) will then manifest as a modification of the Dirac velocity. Therefore, the central quantity of interest in this paper is the on-shell self-energy \( \Sigma[\mathbf{k}, \omega = \epsilon(\mathbf{k})] = \Sigma(\mathbf{k}) \) for an interacting system. Due to the bipartite structure of the Hamiltonian in Eq. (2) the self-energy \( \Sigma(\mathbf{k}) \) becomes a 2 \times 2 matrix. In the diagonal basis, we can write,

\[
\Sigma(\mathbf{k}) = \begin{pmatrix} \Sigma^+(\mathbf{k}) & 0 \\ 0 & \Sigma^-(\mathbf{k}) \end{pmatrix}
\]

where \( \Sigma^\pm(\mathbf{k}) \) corresponds to renormalization in up and down band of the Dirac spectrum. An exact theoretical evaluation of the self-energy for both fermions and bosons is a difficult task. Therefore, we perform first- and second-order perturbation theory assuming the quasiparticle interaction is weak.

A. Fermi Dirac Material : Long range Coulomb interaction

Starting with fermionic Dirac materials, we discuss the established result of the logarithmic divergence of the Dirac cone in graphene with Coulomb interactions\(^\text{[26]}\). The details of the calculation are given in Appendix A.
Here, we focus on two physical situations: (a) zero temperature diverging Dirac velocity and (b) finite temperature regularized Dirac velocity. In each cases, the self-energy is related to the renormalized Dirac velocity as \( \Sigma(k) \propto \delta v_D(k) \sigma \cdot k \).

a. Zero temperature velocity renormalization: We perform Hartree-Fock approximation for the Coulomb interaction by introducing the Hubbard-Stratonovich field \( \phi_\tau(x) \) (defined in Appendix A). This allows the self-energy associated with the interaction to be computed, and from it we extract the renormalization of the linear Dirac cone. The correction to the Dirac velocity is logarithmically diverging in momentum

\[
\delta v_D(k) \propto \log \frac{\Lambda}{k},
\]

where \( \Lambda \) is the ultraviolet cutoff for the momentum. Hence, the linear Dirac spectrum is reshaped. The logarithmic divergence of the Dirac velocity as \( k \to 0 \) implies that the velocity increases. Relative change of the Dirac cone is shown in Fig. 5(a). This type of divergence obtains even for higher order perturbation theory and is the manifestation of the fact that undoped graphene belongs to the universality class of marginal Fermi liquids.

b. Finite temperature velocity renormalization: A similar finite temperature (\( T \ll T_F \) low temperature, \( T_F \) is the Fermi temperature) (Appendix A) calculation with bare Coulomb interaction softens the velocity divergence as

\[
\delta v_D(k) \propto \log \left( \frac{v_D \Lambda}{2k_B T} \right),
\]

where \( k_B \) is the Boltzmann constant. Temperature behaves as an infrared energy scale to remove the divergence at \( k \to 0 \). Typically, in Fermi systems temperature dependent corrections to physical quantities appear as certain powers of \( \frac{T}{T_F} \). However, for two dimensional Dirac fermion systems we find an apparently unknown temperature correction as \( \delta v_D \propto \log(T) \). We also find that the divergence at \( k \to 0 \) is absent for finite temperature. It is natural to expect that for \( T \to 0 \) the velocity should renormalize according to

\[
\delta v_D(k) \propto \log \left( \frac{v_D \Lambda}{\max(v_D k, 2k_B T)} \right).
\]

B. Fermi Dirac Material: Screened Coulomb interaction

In the previous section, we calculated the renormalization of the Dirac spectrum in undoped graphene due to Coulomb interaction. The range of this interaction is infinite. Real materials always are disordered due to impurities and other defects. It is therefore difficult to observe ideal Coulomb interaction in real materials. For practical purposes, we assume that the effects of these defects will be to truncate the range of Coulomb interaction to a finite amount, implying that quasi-electrons in graphene interact by a Yukawa potential as \( V(r) = \frac{e^2}{\epsilon_0 r} e^{-\kappa r} \), where \( \kappa^{-1} \) is the range of the interaction. We perform a similar Hartree-Fock self-energy calculation for small \( \kappa \ll \Lambda \) (Appendix A). The self-energy can be written as \( \Sigma(k) \propto \delta v_D(k) \sigma \cdot k \). The Dirac velocity is renormalized as

\[
\delta v_D(k) \propto \log \left( \frac{\Lambda + \sqrt{\Lambda^2 + \kappa^2}}{2k} \right).
\]
As we can see from Eq. 7, the renormalized Dirac velocity for Yukawa interaction increases also shown in Table II for relatively long range governed by $\kappa^{-1}$.

C. Fermi Dirac Material : Onsite Hubbard interaction

In this section, we study the renormalization of the Dirac spectrum for the half-filled Hubbard model on honeycomb lattice using second order perturbation theory. The Hubbard Hamiltonian is written as

$$
\mathcal{H} = -t \sum_{\langle i,j \rangle, \sigma} c_{i \sigma}^\dagger c_{j \sigma} + U \sum_i n_{i \uparrow} n_{i \downarrow} - \mu \sum_{i, \sigma} c_{i \sigma}^\dagger c_{i \sigma},
$$

where $\sigma$ is the spin, $t$ and $U$ are the hopping amplitude between the nearest neighbor lattice sites and the strength of the on-site repulsion respectively. The lattice spacing is set to be $a$. Following the discussion in Sec. II, we see the that tight-binding term in Eq. 8 gives rise to a Dirac spectrum as in Fig. 1(b) and that the quasiparticles are governed by a Dirac-like equation parametrized by a Fermi velocity $v_D^0$. Dynamical mean-field theory (DMFT), a non-perturbative treatment of the local interaction $U \sum_i n_{i \uparrow} n_{i \downarrow}$, reveals that the interacting Dirac fermions would have a reduced renormalized Fermi velocity $v_D^R$ as a function of $U$. The numerically evaluated ratio of Fermi velocity $v_D^R/v_D^0$ is shown in Fig. 2. This result has been reproduced from Jafari et al.\(^2\)\(^\text{24}\) Kuroki et. al\(^2\)\(^\text{25}\) has also reported similar reduction of Fermi velocity in the semimetallic phase influenced by the interaction. Several other related works\(^2\)\(^\text{21}\)\(^\text{22}\)\(^\text{27}\) reported a semimetal to insulator transition at stronger interaction. Jafari et. al\(^2\)\(^\text{27}\) noted the critical strength $U_c = 13.29t$ for this transition.

Far from the transition point $U \ll U_C$, the semimetallic phase is a Dirac Fermion liquid parametrized by the renormalized Fermi velocity $v_D^R$. We analyze this phase in the second order perturbation theory assuming $U$ is small compared to $t$. The self-energy corrections to the non-interacting Green’s function can be written up to second order from the relevant Feynmann diagrams shown in Fig. 3. We then extract the renormalization of the Fermi velocity from the self-energy. The contribution of the Hartree diagram in Fig. 3(a) is calculated as $U \langle n_{\downarrow} \rangle$. This constant contribution can be absorbed in the redefinition of chemical potential and does not lead to renormalization of the Dirac velocity.

The bubble diagram in Fig. 3(b), on the other hand, gives a non-monotonic momentum-dependent self-energy correction. We denote the energy of the non-interacting Dirac fermions as $\varepsilon_{sk} = s v_D |k|$, where the label $s = 1$ for the conduction (up) and $s = -1$ for the valence (down) band respectively. A straightforward computation leads to the following expression for the self-energy correction in the band $m = \pm 1$,

$$
\Sigma_{(m)}(k) = \frac{U^2}{4} \sum_{p,q} \sum_{s,s',m'} \frac{f(\varepsilon_{sp}) - f(\varepsilon_{s'p})}{\varepsilon_{mk} - \varepsilon_{m'k'} - \varepsilon_{sp} + \varepsilon_{s'p'} + i\delta} \left[ b(\varepsilon_{sp} - \varepsilon_{s'p'}) + f(\varepsilon_{m'k'}) \right] (1 + s s' \cos \Theta_{pp'}) (1 + m m' \cos \Theta_{kk'}). \quad (9)
$$

where $f(\varepsilon) = \frac{1}{e^{\varepsilon/T} + 1}$ and $b(\varepsilon) = \frac{1}{e^{\varepsilon/T} - 1}$ are the Fermi-Dirac and the Bose-Einstein distributions respectively. $\Theta_{pp'}$ is
FIG. 4: $Re \Sigma^+(k)$ for the conduction band is shown at three different temperatures $0.05t$, $0.1t$ and $0.2t$. The linear momentum dependence of the self-energy is shown by the dashed lines. The onsite potential $U$ is assumed to be $0.1t$ for this numerical calculation.

the angle between $p$ and $p'$ and $\Theta_\mathbf{k}, \mathbf{k'}$ is the angle between $\mathbf{k}$ and $\mathbf{k'}$ respectively. We focus on the renormalization in the conduction band. To be consistent with our notation for the self-energy defined in the beginning of Sec. III we denote the correction $\Sigma_{(m=n=+1)}(k) \equiv \Sigma^+(k)$. The renormalization in the valence band is related to the conduction band by reflection symmetry. We perform the 4-D numerical integration in Eq. (9) and plot $Re \Sigma^+(k)$ for three different temperatures in Fig. 4 and find an approximate linear momentum dependence of $Re \Sigma^+(k)$ for small momentum $ak \ll 1$. We assume that the temperature dependence of the self-energy is governed by some unknown function $f(T/T_F)$ where $T_F$ is the Fermi temperature. Guided by our numerical result and the functional form of the self-energy, we write $Re \Sigma^+(k)$ as

$$Re \Sigma^+(k) \propto -f(T/T_F)U^2k + O(k^2)$$

We assign a negative sign in Eq. (10) guided by our numerical estimate in Fig. 4. We further assume that for small temperature ($T \ll T_F$) the function $f$ can be expanded in $T$ as $f(T/T_F) = f_0 + f_1(T/T_F) + O(T^2)$ where $f_0, f_1,...$ are temperature independent constants and $\nu$ is the first non-vanishing exponent in the Taylor expansion. We numerically analyze $Re \Sigma^+(k)$ for small momentum and find the exponent to be $\nu = 1.0$ with a numerical error estimate of $\pm 0.2$. The ratio of renormalized and non-interacting Dirac velocities $v^0_D/v_D$ is extracted as

$$\frac{v^0_D}{v_D} \propto 1 - (f_0 + f_1 T/T_F)U^2 + O(k^2); \; \nu = 1 \pm 0.2$$

Our numerical analysis predicts a zero temperature renormalization theory is consistent with the DMFT analysis in [22] (shown in Fig. 2). The linear temperature dependence of the velocity as in Eq. (11) is a new finding of our paper.

D. Bose Dirac Material: Long-range interaction

In real materials, long-range interactions among bosons are rare in the broader context of condensed matter physics. However, one can artificially synthesize cold atomic systems by confining the atoms in arbitrary dimensions. Therefore, we examine the role of tunable long-range dipole-dipole interactions between bosonic atoms in quasi-2D. We focus on a typical example: the dipolar Bose gas arranged in honeycomb lattice where all the dipoles point in a preferred direction governed by an applied field. The Hamiltonian for the interacting Bose gas on a lattice is

$$\mathcal{H} = -\frac{1}{2} \sum_{\langle ij \rangle} a_i^\dagger b_j^\dagger \frac{U}{2} \sum_{i,\alpha} n_i^\alpha (n_i^\alpha - 1) + \sum_{ij,\alpha,\beta} V_{ij} n_i^\alpha n_j^\beta,$$

where $U$ denotes the repulsive on-site potential, $V_{ij}$ is the effective dipole-dipole energy in 2D, and $a_i^\dagger (b_i^\dagger)$ is the creation operator on the sublattice A(B). The sublattice indices are denoted by $\alpha = A, B$ and $n_i^\alpha = a_i^\dagger a_i (b_i^\dagger b_i)$ denotes the number of bosons at site $i$. In our example, the bosonic dipolar atoms are strongly confined in $z$ direction. The ansatz for bosonic density for such a configuration of dipolar atoms was chosen as

$$\rho(\mathbf{r}) = \frac{1}{\sqrt{\pi d_z}} \exp\left(-\frac{z^2}{d_z^2}\right) n(x, y),$$

where the density in the plane $n(x, y)$ is normalized to the total number of dipolar atoms and $d_z$ is the confinement. The 3D dipolar interaction $V_{dd}(\mathbf{r})$ between the dipolar atoms in such a configuration is given by

$$V_{dd}(\mathbf{r}) = \frac{3C_{dd}}{4\pi} \frac{1 - 3z^2/\lambda^2}{r^3},$$

where $C_{dd}$ is a constant proportional to the electric/magnetic dipole moment and $r = (x, y, z)$. The factor 3 has been chosen for analytical ease. The effective 2D interaction $V_{ij}$ can be found by integrating $V_{ij}(\mathbf{r})$ along the confined direction $z$. As explained in the previous section, the mean field analysis of on-site potential $U$ will set the value of the ground state for the two band system with positive-definite energy for the bosons (see Fig. I(a)). We focus on analyzing the renormalization of the linear spectrum within the Dirac cone near the crossing point in Fig. I(a). Any isolated bosonic systems in thermal equilibrium always host quasiparticles in the spectrum populated according to the Bose-Einstein function $f_b(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/k_B T} + 1}$ with either zero or negative chemical potential $\mu$. Therefore, thermally excited bosons are always described by a parabolic dispersion near the ground state. However, this restriction can disappear when the bosonic system is coupled to some external pumping source and decay bath. In the steady state of a driven bosonic system a situation might occur where an effective chemical potential $\mu_{\text{Boson}}$ is induced. We assume such an arrangement
where $\mu_{\text{Boson}}$ is near the Dirac crossing point (denoted by the "red" and "blue" dots in Fig. 1 (a)) due to pumping. Possible emergence of such a situation from the interplay of pumping, decay and interactions is an interesting problem in itself but is not the focus of this paper. We treat the somewhat related situation of Dirac bosons with long range Coulomb interaction in the next paragraph.

### a. Dirac bosons with Coulomb interaction

Following the discussion in this section, we consider a nonequilibrium steady state of Dirac bosons where the quasiparticles interact via long range Coulomb repulsion. We point that to compare the renormalization effects on equal footing the effective chemical potential is set at the high energy Dirac crossing point (denoted by the "red" and "blue" dots in Fig. 1 (a)). We assume here that the interaction, occurs between the transient excited states at the Dirac node. Details of such arrangement depend on the microscopic and the precise mechanisms of pumping, relaxations and interaction. Analysis will be given elsewhere. Following analysis identical to that in Sec. III (A), we find similar behavior for the renormalization of the Dirac cone:

$$
\Sigma(k) \propto \begin{cases} 
\sigma \cdot k \log \frac{\Lambda}{k} & ; \quad T = 0 \\
\sigma \cdot k \log \left( \frac{v_D \Lambda}{\max(v_D k, 2k_B T)} \right) & ; \quad T \neq 0
\end{cases}
$$

Continuing our discussion on the renormalization of dipolar bosons, we focus on such nonequilibrium bosonic systems where the effective quasiparticle Hamiltonian near the $K$ point follows from Eq. (12) as

$$
H_K = v_D \int dk \bar{\Psi}_k \sigma \cdot k \Psi_k + \int dq \tilde{V}_q n(q) n(-q),
$$

where $\tilde{V}_q$ is the Fourier transform of $V_{ij}$, $v_D$ is the Dirac velocity for the non-interacting bosons as defined in the previous section, and the spinor $\Psi_k = (a_k^+, b_k^+)$ is composed of bosonic operators and $n(q) = \int dk \bar{\Psi}_k \Psi_k$. The details of the derivation of the effective dipole-dipole interaction $\tilde{V}_q$ from the 3D dipolar potential $V_{dd}(r)$ are
In this equation, \( f \) is Green's function for the free Dirac bosons at the Matsubara frequencies, we find that the one loop diagram in Fig. 7, and after summing over Matsubara frequencies, we have

\[
\tilde{V}_q = \frac{C_{dd}}{2} \left( 2 \frac{q^2}{\sqrt{2\pi}d_z} - \frac{q^2}{2} \exp \left( \frac{q^2d_z^2}{2} \right) \mathrm{erfc} \left( \sqrt{\frac{q^2d_z}{2}} \right) \right). \tag{17}
\]

We introduce a Hubbard-Stratonovich field \( \phi \) to decouple the interacting term in Eq. (16) and rewrite the corresponding Hamiltonian (read out from the action \( S \)) as

\[
\mathcal{H}_K = \mathcal{H}_0 + \int dq \left( \phi_q n(q) + \phi_q \tilde{V}_q^{-1} \phi_{-q} \right). \tag{18}
\]

Now, we set up a one loop perturbation theory with the coupling \( C_{dd} \) in fashion similar to that which we employed for Dirac fermions coupled to gauge fields (analogous to the field \( \phi \) here) in graphene. The retarded Green's function for the free Dirac bosons \( G^0(k, \omega_n) \) and the H-S field \( D^0(q) = \langle \phi_q \phi_{-q} \rangle \) are

\[
G^0(k, \omega_n) = \frac{1}{\omega_n - v_D \sigma \cdot k}, \quad D^0(q) = \tilde{V}_q, \tag{19}
\]

where \( \omega_n \) are the isospin Matsubara frequencies. These definitions allow us to calculate the self-energy for the one loop diagram in Fig. 7 and after summing over the Matsubara frequencies, we find that

\[
\Sigma(k, \omega_n) = \int dq \frac{\sigma \cdot q}{2|q|} \left( 1 + 2f_B(v_D|q|) \right) \tilde{V}_{q-k}. \tag{20}
\]

\[
= -\frac{3C_{dd}}{8} \int dq d\theta (\sigma \cdot q) q' \exp \left( \frac{q'^2d_z^2}{2} \right) \mathrm{erfc} \left( \frac{q'd_z}{\sqrt{2}} \right). \tag{21}
\]

In this equation, \( f_B(\varepsilon) = \frac{1}{e^{\varepsilon/T} - 1} \) is the Bose-Einstein distribution and \( \beta = \frac{1}{k_BT} \). In Eq. (20), we assume the temperature to be small enough \( T \ll \Lambda \) so that the Bose-Einstein distribution for Dirac energy \( f_B(v_D|q|) \sim 0 \) and \( q' = |q - k| \). The angular integration can be evaluated analytically, and using polar coordinates where the angle between the vector \( q \) and \( k \) is \( \gamma - \theta \) we find

\[
q^2 = q^2 + k^2 - 2qk \cos(\gamma - \theta) \approx k^2 t^2 (1 - \frac{1}{t} \cos(\gamma - \theta))^2; \quad t = \frac{q}{k} \gg 1. \tag{21}
\]

For large \( t \) we expand the error function as, \( \mathrm{erfc}(t) = \frac{e^{-t^2}}{\sqrt{\pi}t} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} t^{2n} \) and perform the analytical continuation over the Matsubara frequencies. Hence, we obtain

\[
\Sigma(k) \approx \frac{3C_{dd}}{8(2\pi)^{5/2}\Lambda d_z^2} \sigma \cdot k + O(k^2), \tag{22}
\]

where \( \Lambda \) is the ultraviolet cutoff. The renormalization of the Dirac velocity is then given by the coefficient of the linear \( k \) term, so we have

\[
\delta v_D(k) \approx \frac{3C_{dd}}{8(2\pi)^{5/2}\Lambda d_z^2}. \tag{23}
\]

The renormalized Dirac velocity is larger than the single particle Dirac velocity. The Dirac velocity increases with decreasing \( d_z \) i.e. when we have longer range interaction. This behavior is plotted in Fig. 8. By carefully analyzing Eq. (23), we see that \( d_z \) should be less than or of the order of the lattice constant \( a \) to have a considerable change in the Dirac velocity, which is consistent with our assumption of a quasi 2D system.

We believe that the relative difference of the momentum dependence for the self-energy in the case of Dirac bosons, interacting via Coulomb and dipolar interaction, emerges from the different power law dependencies. In case of Coulomb interaction the interaction behaves as \( 1/r \), where-as dipolar interaction falls off rapidly with \( 1/r^3 \).

### E. Bose Dirac Material : Short range and contact interaction

We now review the effect of contact interactions in bosonic systems. Here we focus on two typical examples of bosonic Dirac materials: Cooper pairs that
are described by an effective Bose-Hubbard model and magnons on a honeycomb lattice. The results are qualitatively similar for both the cases and can be extended to other classes of bosonic Dirac Material.

\textit{a. Mean field description of Dirac bosons:} We start with case of Cooper pairs in superconducting grains which are arranged in a honeycomb lattice. The Bose-Hubbard model for the bosons is described by the Hamiltonian

\[ \mathcal{H} = -t \sum_{\langle ij \rangle} b_i^{\dagger} b_j^B + h.c. + U \sum_{i,\alpha} \left( n_i^\alpha - n_0 \right)^2, \]

where \( b_i^{\dagger} = c_i^{\dagger} c_{i \uparrow} \) is the creation operator for a singlet Cooper pair at site \( i \) and \( c_i^{\dagger} \) is the electron creation operator. \( t \) is the hopping amplitude of the pairs between the sites and \( U \) is an onsite potential. The operator \( n_0 \) denotes the mean Cooper pairs in each grain. As described in our recent work\cite{13} this model provides two branches of collective excitations which are commonly known in the literature as Leggett and Bogoliubov-Anderson-Gorkov (BAG) modes. These two modes together will produce bosonic Dirac dispersions similar to one shown in Fig. \( \text{[a]} \). The Dirac velocity near the crossing point in the Brillouin zone is given by

\[ v_D = \frac{\sqrt{4U \alpha}}{\delta} \]

where \( J \approx n_0 \). We can find the effect of a short range interaction \( U' \sum_{\langle ij \rangle} n_i A_j B_j \) on the Dirac spectrum. Mean field theory predicts that the renormalized Dirac velocity decreases as

\[ \delta v_D/v_D \approx \frac{-3U'}{2U}, \]

for \( U' \sim U \ll J \). Therefore, the linear conical structure remains valid after the short-range interactions are introduced, but the associated Dirac velocity is decreased.

\textit{b. Beyond mean field description of Dirac bosons:} Here we review the spectral renormalization near the Dirac point for magnons in CrBr\(_3\) beyond mean field theory\cite{23}. CrBr\(_3\) is a prototypical ferromagnetic insulator where Cr\(^{3+}\) spins (spin = \( 3/2 \)) are arranged in a single layer honeycomb lattice and interact via Heisenberg exchange. A detailed electronic structure of this material can be found in a recent paper by Wang \emph{et. al.}\cite{23}. The spin exchange Hamiltonian is written as

\[ \mathcal{H} = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \]

Utilizing the Holstein-Primakoff (H-P) transformation, Eq. \( \text{\cite{26}} \) can be written in terms of spin-wave excitations or magnons. The underlying honeycomb structure of CrBr\(_3\) naturally introduces two species of magnons defined on the two sublattices. The non-interacting magnons will produce a Dirac spectrum as described in Sec. \( \text{\cite{13, 23}} \). However, at finite temperature, the magnons will start interacting with each other and a complete non-interacting description will not be possible. Yelon \emph{et. al.}\cite{23} studied the magnon interaction effects in this material using inelastic neutron scattering back in 1970’s. As shown in a recent work\cite{23} by some of the present authors, the magnon–magnon interaction in CrBr\(_3\) will follow keeping the leading higher order terms of the H-P transformation in Eq. \( \text{\cite{26}} \). Assuming that the interactions contained in the Heisenberg Hamiltonian are short ranged\cite{23}, the authors report a low-temperature analysis of the magnon self-energy \( \text{Re} \Sigma(\omega, \mathbf{k}) \), and the renormalized spectrum is shown in Fig. \( \text{[a]} \). From the numerical results, we extracted the Dirac velocity and found that it decreases. The imaginary part of the self-energy for up and down bands are shown in Fig. \( \text{[e]} \).

IV. DISCUSSION AND CONCLUSION

In this paper, we present the first systematic study of the renormalization effects in Dirac materials for various interactions. The results from Sec. III are recorded in Table I and give a unified summary of the effect of interactions on the quasiparticles near the Dirac cone of Dirac materials. It is clear that the form of the quasiparticle interactions qualitatively affects the velocity renormalization. Contact interactions generically lead to a decrease in the Dirac velocity, whereas finite-range interactions (which may be screened or unscreened) generically give an increase. The quasiparticle statistics – i.e. whether we are dealing with bosons or fermions - do affect the power laws fixing the temperature and momentum dependence of the self-energy corrections for short range interactions, but do not influence the logarithmic behavior for long-range Coulomb repulsion.

More specifically, our work represents significant step in developing a detailed understanding of renormalizations in interacting Dirac materials. To our knowledge, a complete analysis of frequency and momentum dependent self-energy \( \Sigma(\mathbf{k}, \omega) \) in a half-filled Hubbard model on honeycomb lattice at finite temperature has not been performed before. In Sec. \( \text{\cite{13, 23}} \), we computed the real part of the finite temperature on-shell self-energy \( \Sigma(\mathbf{k}) \) numerically. We extracted the leading temperature dependence of the velocity as

\[ \delta v_D \sim -(f_0 + f_1 T^\nu), \]

\( \nu = 1 \pm 0.2 \) where \( f_0 \) and \( f_1 \) are temperature independent constants. We have used the Kramers-Kronig relation to compute the imaginary part of \( \Sigma(\mathbf{k}) \) assuming that the later quantity depends on length of the momentum vector \( \mathbf{k} \). Isotropic nature of this renormalization results from the isotropic interaction. This fact has also been used to calculate the imaginary part of the self-energy for other situations in Table II.

To conclude, our results show how the quasiparticles in Dirac materials are modified by various interactions. This analysis opens up the possibilities of both slowing and speeding Dirac quasiparticles. Interaction effects might be engineered and controlled and lead to new technological applications, e.g. spintronics, in heterostructures of van der Waals materials.
The free retarded Green’s function is written as charged Dirac fermions is given by
\[ \text{V} \]
where
\[ \text{V} \]
and introduced the ultraviolet cut-off \( \Lambda \) for \( q \).

\[ \begin{aligned}
\text{V} &= \frac{e^2}{4\pi^2\epsilon_0} \int d\mathbf{q} \frac{\mathbf{\sigma} \cdot \mathbf{q}}{|\mathbf{q} - \mathbf{k}|} \\
&= \frac{e^2}{4\pi^2\epsilon_0} \int d\mathbf{q} \frac{\mathbf{\sigma} \cdot \mathbf{q}}{|\mathbf{q} - \mathbf{k}|} \\
&= \frac{e^2}{4\pi^2\epsilon_0} \int q dq d\theta \frac{\mathbf{\sigma} \cdot \mathbf{\hat{q}}}{\sqrt{\mathbf{q}^2 + \lambda^2 - 2\lambda q \cos(\theta - \theta')}}.
\end{aligned} \]

In the last line we performed transformation of variable \( \lambda = \frac{q}{\mathbf{k}} \) and introduced the angle \( \gamma - \theta \) between the two vectors \( \mathbf{q} \) and \( \mathbf{k} \). We approximate for small \( k \) (large \( \lambda \)) and introduce the ultraviolet cut-off \( \Lambda \) for \( q \) integration.

We arrive at the following expression for the self-energy:
\[ \begin{aligned}
\Sigma_{\text{vd}}(\mathbf{k}, \omega) &= \frac{e^2}{4\pi^2\epsilon_0} k \int d\lambda d\lambda' \text{log} \frac{\lambda}{\Lambda} \left( \cos(\gamma - \theta) \log \Lambda \right) \\
&= \frac{e^2}{4\pi^2\epsilon_0} \int \frac{k}{k'} d\mathbf{q} \rho(\mathbf{q}) \tilde{\rho}(\mathbf{-q}).
\end{aligned} \]

### Appendix B: Fourier transform of Dipole potential

Here, we provide the details for the derivation of the 2D Fourier transform of the potential \( V_{\text{dd}}(\mathbf{r}) \) (see Eq. (14) in main text). The 3D dipole interaction between the atoms is
\[ \begin{aligned}
\mathcal{H}_{dd} &= \int d\mathbf{r} d\mathbf{r}' \rho(\mathbf{r}) V_{\text{dd}}(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \\
&= \int d\mathbf{q} \tilde{V}_{\text{dd}}(\mathbf{q}) \tilde{\rho}(\mathbf{q}) \tilde{\rho}(\mathbf{-q}),
\end{aligned} \]

where \( \tilde{\rho}(\mathbf{q}) \) and \( \tilde{V}_{\text{dd}}(\mathbf{q}) \) are the Fourier transforms of the density of bosons \( \rho(\mathbf{r}) \) and the dipole potential \( V_{\text{dd}}(\mathbf{r}) \) respectively. We use the following identity for the differentials over a sphere
\[ \begin{aligned}
\int_V \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} \frac{1}{4\pi r} d^3r &= - \int_V \frac{\partial}{\partial r_i} A_{rj} d^3r. \\
\int_V \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} \frac{1}{4\pi r} d^3r &= - \int_{\partial V} \frac{r_j}{4\pi r^3} dA.
\end{aligned} \]
Analyzing the symmetry under the exchange of the indices $i, j$ in the above equation leads to

$$ - \int_0^r \frac{r_j}{4\pi r^3} dA_i = \begin{cases} 0, & i \neq j, \\ -\frac{1}{3}, & i = j. \end{cases} \quad (B4) $$

This leads to an operator identity and hence relates the Coulomb potential to the dipole-dipole potential as

$$ -\frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} \frac{1}{4\pi r^3} = \frac{\delta_{ij}}{4\pi r^3} + \frac{1}{3} \delta_{ij} \delta(r). \quad (B5) $$

The above equation can be Fourier transformed as follows ($F$ denotes the action of Fourier transform)

$$ F \left[ \frac{\delta_{ij} - 3\delta_{i} \delta_{j}}{4\pi r^3} \right] = \left( \frac{k_i k_j}{4\pi} F \left[ \frac{1}{r} \right] - \frac{1}{3} \delta_{ij} \right). \quad (B6) $$

We know the Fourier transform of the Coulomb potential in 3D as $F \left[ \frac{1}{r} \right] = \frac{\pi}{2k^3}$. Using all of these results together, we obtain the Fourier transform of dipole-dipole potential $V_{dd}(r)$ in 3D as

$$ \tilde{V}_{dd}(\mathbf{q}) = C_{dd} \left[ \frac{3q^2}{q^2} - 1 \right]. \quad (B7) $$

using the boson density $\rho(\mathbf{r})$ in Eq. [11] we derive the effective 2D dipole potential $\tilde{V}_d(\mathbf{q})$ as,

$$ \mathcal{H}_{dd} = \int \frac{dq_z, dq_{\parallel}}{2\pi} e^{-\frac{q_z^2}{2}} \tilde{V}_{dd}(\mathbf{q})\tilde{n}(\mathbf{q}_{\parallel})\tilde{n}(-\mathbf{q}_{\parallel}) \quad (B8) $$

$$ = C_{dd} \int dq_z, dq_{\parallel} e^{-\frac{q_z^2}{2}} \left( \frac{3q_z^2}{q^2} - 1 \right) \tilde{n}(\mathbf{q}_{\parallel})\tilde{n}(-\mathbf{q}_{\parallel}) $$

$$ = \int dq_{\parallel} \tilde{V}_{aq}\tilde{n}(\mathbf{q}_{\parallel})\tilde{n}(-\mathbf{q}_{\parallel}), $$

where $\mathbf{q} = (q_z, q_{\parallel})$ is the 3D momentum, and $\tilde{V}_{dd}(\mathbf{q}) = C_{dd} \left[ \frac{3q_z^2}{q^2} - 1 \right]$ is the Fourier transform of $V_{dd}(r)$. Substituting the boson density $\rho(\mathbf{r})$ defined in Eq. [13] in Eq. [18] and integrating over $q_z$ we get the effective potential energy as

$$ \tilde{V}_{aq} = C_{dd} \int \left. \frac{2}{\sqrt{2\pi}d_z} - \frac{3}{2} \exp \left( \frac{q_z^2}{2d_z^2} \right) \right|_{q_z=0} \left|^{q_z} \right. $$

$$ \exp \left( -\frac{t^2}{2} \right) dt $$

where $\text{erfc}(x) = 1 - \left( \sqrt{\frac{2}{\pi}} \right) \int_0^x \exp(-t^2) dt$ is the complementary error function and $\mathbf{q}_{\parallel} = (q_x, q_y)$ is the 2D momentum.

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