A Note on Warped Product Almost Quasi-Yamabe Solitons

Adara M. Blaga

Abstract. We consider almost quasi-Yamabe solitons in Riemannian manifolds, derive a Bochner-type formula in the gradient case and prove that under certain assumptions, the manifold is of constant scalar curvature. We also provide necessary and sufficient conditions for a gradient almost quasi-Yamabe soliton on the base manifold to induce a gradient almost quasi-Yamabe soliton on the warped product manifold.

1. Introduction

The notion of Yamabe solitons, which generate self-similar solutions to Yamabe flow [8]:

$$\frac{\partial}{\partial t} g(t) = -\text{scal}(t) \cdot g(t),$$

firstly appeared to L. F. di Cerbo and M. N. Disconzi in [3]. In [4], B.-Y. Chen introduced the notion of quasi-Yamabe soliton which we shall consider in the present paper for a more general case, when the constants are let to be functions.

Let $(M, g)$ be an $n$-dimensional Riemannian manifold ($n > 2$), $\xi$ a vector field and $\eta$ a 1-form on $M$.

Definition 1.1. An almost quasi-Yamabe soliton on $M$ is a data $(g, \xi, \lambda, \mu)$ which satisfy the equation:

$$\frac{1}{2} \mathcal{L}_\xi g + (\lambda - \text{scal}) g + \mu \eta \otimes \eta = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field $\xi$ and $\lambda$ and $\mu$ are smooth functions on $M$.

When the potential vector field of (2) is of gradient type, i.e. $\xi = \text{grad}(f)$, then $(g, \xi, \lambda, \mu)$ is said to be a gradient almost quasi-Yamabe soliton (or a generalized quasi-Yamabe gradient soliton) [9] and the equation satisfied by it becomes:

$$\text{Hess}(f) + (\lambda - \text{scal}) g + \mu df \otimes df = 0.$$  

In the next section, we shall derive a Bochner-type formula for the gradient almost quasi-Yamabe soliton case and prove that under certain assumptions, the manifold is of constant scalar curvature. In the last section we construct an almost quasi-Yamabe soliton on a warped product manifold. Remark that results on warped product gradient Yamabe solitons for certain types of warping functions $f$ have been obtained by W. I. Tokura, L. R. Adriano and R. S. Pina in [10].
2. Gradient almost quasi-Yamabe solitons

Remark that in the gradient case, from (3) we get:
\[ \nabla \xi = -(\lambda - \text{scal}) \xi - \mu \text{df} \otimes \xi. \]

(4)

Therefore, \( \nabla \xi = [\Delta(f) + (n-1)(\lambda - \text{scal})] \xi \), i.e. \( \xi \) is a generalized geodesic vector field with the potential function \( \Delta(f) + (n-1)(\lambda - \text{scal}) \) [6].

Also, if \((\lambda, \mu) = (\text{scal} - 1, 1)\), then \( \xi \) is torse-forming and if \( \mu = 0 \), then \( \xi \) is concircular.

Now we shall get a condition that \( \mu \) should satisfy in a gradient almost quasi-Yamabe soliton \((g, \xi, \lambda, \mu)\).

Taking the scalar product with \( \text{Hess}(\xi) \), from (3) we get:
\[ |\text{Hess}(f)|^2 + (\lambda - \text{scal}) \Delta(f) + \frac{\mu}{2} \xi(\xi^2) = 0 \]

and tracing (3) we obtain:
\[ \Delta(f) + n(\lambda - \text{scal}) + \mu |\xi|^2 = 0. \]

From the above relations we deduce the equation:
\[ n\lambda^2 + (2n \cdot \text{scal} + \mu |\xi|^2)\lambda + n \cdot \text{scal}^2 + \mu |\xi|^2 \cdot \text{scal} - \frac{\mu}{2} \xi(\xi^2) - |\text{Hess}(f)|^2 = 0 \]

which has solution (in \( \lambda \)) if and only if
\[ \mu^2|\xi|^4 + 2n\mu|\xi|^2 + 4n|\text{Hess}(f)|^2 \geq 0 \]

(that is always true for \( \xi \) of constant length).

The next step is to deduce a Bochner-type formula for the gradient almost quasi-Yamabe soliton case.

**Theorem 2.1.** If (3) defines a gradient almost quasi-Yamabe soliton on the \( n \)-dimensional Riemannian manifold \((M, g)\) and \( \eta = df \) is the \( g \)-dual of the gradient vector field \( \xi := \text{grad}(f) \), then:
\[ \frac{1}{2} \Delta(\xi^2) = |\nabla \xi|^2 - \frac{1}{n-1} S(\xi, \xi) - \frac{n-2}{2(n-1)} \mu \nabla \xi(\xi^2) - |\xi|^2 [\xi(\mu) - \frac{n}{n-1} \mu^2 |\xi|^2 - \frac{n^2}{n-1} \lambda \mu + \frac{n^2}{n-1} \mu \cdot \text{scal}]. \]

(5)

**Proof.** First remark that:
\[ \text{trace}(\mu \eta \otimes \eta) = \mu |\xi|^2 \]

and
\[ \text{div}(\mu \eta \otimes \eta) = \frac{\mu}{2} d(|\xi|^2) + \mu \Delta(f) df + d\mu(\xi) df. \]

Taking the trace of the equation (3), we obtain:
\[ \Delta(f) + n(\lambda - \text{scal}) + \mu |\xi|^2 = 0 \]

(6)

and differentiating it:
\[ d(\Delta(f)) + nd\lambda - nd(\text{scal}) + \mu d(|\xi|^2) + |\xi|^2 d\mu = 0. \]

(7)

Now taking the divergence of the same equation, we get:
\[ \text{div}(\text{Hess}(f)) + d\lambda - d(\text{scal}) + \frac{\mu}{2} d(|\xi|^2) + \mu \Delta(f) df + d\mu(\xi) df = 0. \]

(8)
Substracting the relations (8) and (7) computed in $\xi$ and using [2]:

$$\text{div}(\text{Hess}(f)) = d(\Delta(f)) + i_{\xi}\mathfrak{g},$$

$$\text{div}(\text{Hess}(f))(\xi) = \frac{1}{2}\Delta(\langle \xi \rangle^2) - \nabla \langle \xi \rangle^2,$$

we obtain (5). $\square$

**Remark 2.2.** For the case $\mu = 0$, under the assumptions $S(\xi, \xi) \leq (n - 1)|\nabla \xi|^2$ we get $\Delta(\langle \xi \rangle^2) \geq 0$ and from the maximum principle follows that $\langle \xi \rangle^2$ is constant in a neighborhood of any local maximum. If $\langle \xi \rangle$ achieve its maximum, then $S(\xi, \xi) = (n - 1)|\nabla \xi|^2$.

Let us make some remarks on the scalar curvature of $M$.

From (4) we get:

$$R(\cdot, \cdot)\xi = -[d(\lambda - \text{scal}) \otimes I - I \otimes d(\lambda - \text{scal})] - \mu(\lambda - \text{scal})(df \otimes I - I \otimes df) - (d\mu \otimes df - df \otimes d\mu)$$

and

$$R(\cdot, \xi) = d(\lambda - \text{scal}) \otimes I - g \otimes [\text{grad}(\lambda - \text{scal}) - \mu(\lambda - \text{scal})\xi] + \mu(\lambda - \text{scal}) df \otimes I +$$

$$+d\mu \otimes df \otimes \xi - df \otimes df \otimes \text{grad}(\mu)$$

which for $\lambda$ and $\mu$ constant become:

$$R(\cdot, \cdot)\xi = [d(\text{scal}) \otimes I - I \otimes d(\text{scal})] - \mu(\lambda - \text{scal})(df \otimes I - I \otimes df)$$

and

$$R(\cdot, \xi) = -d(\text{scal}) \otimes I + g \otimes [\text{grad}(\text{scal}) + \mu(\lambda - \text{scal})\xi] + \mu(\lambda - \text{scal}) df \otimes I.$$  \hspace{1cm} (9)

Using (9), $R(\xi, \xi)X = 0$ implies:

$$[d(\lambda - \text{scal}) + |\xi|^2 d\mu] \otimes \xi = df \otimes [\text{grad}(\lambda - \text{scal}) + |\xi|^2 \text{grad}(\mu)]$$

which for $\lambda$ and $\mu$ constant becomes:

$$d(\text{scal}) \otimes \xi = df \otimes \text{grad}(\text{scal}).$$

Assume further that $\lambda$ and $\mu$ are constant. Computing the previous relation in $\xi$ and choosing an open subset where $\xi \neq 0$, we deduce:

$$\text{grad}(\text{scal}) = \frac{\xi(\text{scal})}{|\xi|^2} \xi.$$  \hspace{1cm} (11)

Denoting by $h = \frac{\xi(\text{scal})}{|\xi|^2}$, from the symmetry of Hess(\text{scal}) we obtain:

$$dh \otimes df = df \otimes dh$$

which implies:

$$|\xi|^2 dh = \xi(h)df \quad \text{and} \quad |\xi|^2 \text{grad}(h) = \xi(h).$$

A similar result like the one obtained by B.-Y. Chen, S. Deshmukh in [6] for Yamabe solitons can be obtained for quasi-Yamabe solitons, following the same ideas in proving it.

**Theorem 2.3.** Let (3) define a gradient quasi-Yamabe soliton on the connected n-dimensional Riemannian manifold $(M, g)$ ($n > 1$) for $\eta = df$ the g-dual of the unitary vector field $\xi := \text{grad}(f)$. If $\xi(\text{scal})$ is constant along the integral curves of $\xi$ and Hess(\text{scal}) is degenerate in the direction of $\xi$, then $M$ is of constant scalar curvature.
Proof. Under these hypotheses, applying divergence to (11) we obtain:
\[ \Delta(\text{scal}) = \xi(\text{scal}) \Delta(f) = -[n(\lambda - \text{scal}) + \mu] \xi(\text{scal}). \]

(12)

Computing the Ricci operator in \( \xi \), \( Q\xi = -\sum_{i=1}^{n} R(E_i, \xi)E_i \), for \( \{E_i\}_{1 \leq i \leq n} \) a local orthonormal frame field on \( M \), and using (10) we get:
\[ Q\xi = -(n-1)\text{grad}(\text{scal}) + (n-1)\mu(\lambda - \text{scal})\xi \]

(13)

and
\[ S(\xi, \xi) = g(Q\xi, \xi) = -(n-1)\xi(\text{scal}) + (n-1)\mu(\lambda - \text{scal}). \]

(14)

Applying the divergence to (13) we have:
\[ \text{div}(Q\xi) = -(n-1)\Delta(\text{scal}) + (n-1)\mu[(\lambda - \text{scal})\Delta(f) - \xi(\text{scal})]. \]

(15)

Computing the same divergence like:
\[ \text{div}(Q\xi) = \text{div}(S)(\xi) + \langle S, \text{Hess}(f) \rangle, \]

(16)

taking into account the gradient quasi-Yamabe soliton equation, the fact that
\[ \text{div}(S)(\xi) = \frac{\xi(\text{scal})}{2}, \]

the expression of \( S(\xi, \xi) \) from (14) and replacing \( \Delta(\text{scal}) \) from (12), we obtain:
\[ \left[ \frac{1}{2} - n(n-1)(\lambda - \text{scal}) + (n-1)\mu \right] \xi(\text{scal}) = (\lambda - \text{scal})[(1 + n(n-1)\mu)\text{scal} - n(n-1)\lambda \mu]. \]

Differentiating the previous expression along \( \xi \) and taking into account the degeneracy of \( \text{Hess}(\text{scal})(\xi, \xi) = \xi(\xi(\text{scal})) - (V\xi)(\text{scal}) \) in the direction of \( \xi \), after a long computation, we get:
\[ \xi(\text{scal}) \left[ \xi(\text{scal}) + \text{scal}^2 + k_1 \text{scal} + k_2 \right] = 0, \]

where the constants \( k_1 \) and \( k_2 \) are respectively given by:
\[ k_1 := \frac{n+1}{n} \mu + \frac{5}{2n(n-1)}, \quad k_2 := \lambda^2 - \frac{1}{n^2} \mu^2 - \frac{n+1}{n} \lambda \mu - \frac{3\lambda + \mu}{2n(n-1)}. \]

Differentiating again the term in the parantheses along \( \xi \) we get:
\[ \xi(\text{scal}) \left[ 3\text{scal} - \lambda + \frac{1}{n} \mu + \frac{5}{2n(n-1)} \right] = 0 \]

which completes the proof. \( \Box \)

3. Warped product almost quasi-Yamabe solitons

3.1. Warped product manifolds

Consider \((B, g_B)\) and \((F, g_F)\) two Riemannian manifolds of dimensions \( n \) and \( m \), respectively. Denote by \( \pi \) and \( \sigma \) the projection maps from the product manifold \( B \times F \) to \( B \) and \( F \) and by \( \bar{\varphi} := \varphi \circ \pi \) the lift to \( B \times F \) of a smooth function \( \varphi \) on \( B \). In this context, we shall call \( B \) the base and \( F \) the fiber of \( B \times F \), the unique element
of $\chi(B \times F)$ that is $\pi$-related to $X \in \chi(B)$ and to the zero vector field on $F$, the horizontal lift of $X$ and the unique element $V$ of $\chi(B \times F)$ that is $\sigma$-related to $V \in \chi(F)$ and to the zero vector field on $B$, the vertical lift of $V$. Also denote by $\mathcal{L}(B)$ the set of all horizontal lifts of vector fields on $B$, by $\mathcal{L}(F)$ the set of all vertical lifts of vector fields on $F$, by $\mathcal{H}$ the orthogonal projection of $T_{(p,q)}(B \times F)$ onto its horizontal subspace $T_{(p,q)}(B \times \{q\})$ and by $\mathcal{V}$ the orthogonal projection of $T_{(p,q)}(B \times F)$ onto its vertical subspace $T_{(p,q)}(\{p\} \times F)$.

Let $\phi > 0$ be a smooth function on $B$ and

$$ g := \pi^* g_B + (\phi \circ \pi)^2 \sigma^* g_F, $$

(17) be a Riemannian metric on $B \times F$.

**Definition 3.1.** [1] The product manifold of $B$ and $F$ together with the Riemannian metric $g$ defined by (17) is called the warped product of $B$ and $F$ by the warping function $\phi$ (and is denoted by $(M := B \times_{\phi} F, g)$).

In particular, if $\phi = 1$, then the warped product becomes the usual product of the Riemannian manifolds.

For simplification, in the rest of the paper we shall simply denote by $X$ the horizontal lift of $X \in \chi(B)$ and by $V$ the vertical lift of $V \in \chi(F)$.

Notice that the lift on $M$ of the gradient and the Hessian satisfy:

$$ \text{grad}(\tilde{f}) = \text{grad}(f), $$

$$ (\text{Hess}(\tilde{f}))(X, Y) = (\text{Hess}(f))(X, Y), \text{ for any } X, Y \in \mathcal{L}(B), $$

for any smooth function $f$.

Also, the scalar curvatures are connected by the relation [7]:

$$ \text{scal} = \text{scal}_B + \frac{\text{scal}_F}{\phi^2} - \pi \left(2m \frac{\Delta(\phi)}{\phi} + m(m - 1) \frac{\text{grad}(\phi)^2}{\phi^2}\right). $$

(20)

### 3.2. Warped product almost quasi-Yamabe solitons

We shall construct a gradient almost quasi-Yamabe soliton on a warped product manifold.

Let $(B, g_B)$ be an $n$-dimensional Riemannian manifold, $\phi > 0$ a smooth function on $B$ and $f, \mu$ smooth functions on $B$ such that:

$$ \Delta(f) + \mu |\text{grad}(f)|^2 = n \frac{(\text{grad}(f))(\phi)}{\phi}. $$

(21)

In this case, any gradient almost quasi-Yamabe soliton $(g_B, \text{grad}(f), \lambda_B, \mu_B)$ on $(B, g_B)$ is given by $\lambda_B = \text{scal}_B - \frac{(\text{grad}(f))(\phi)}{\phi}$ and $\mu_B = \mu$.

Take $(F, g_F)$ an $m$-dimensional manifold with

$$ \text{scal}_F = \pi^* (\lambda - \lambda_B) \phi^2 + 2m \phi \Delta(\phi) + m(m - 1) |\text{grad}(\phi)|^2 |F|, $$

(22)

where $\pi$ and $\sigma$ are the projection maps from the product manifold $B \times F$ to $B$ and $F$, respectively, $g := \pi^* g_B + (\phi \circ \pi)^2 \sigma^* g_F$ is a Riemannian metric on $B \times F$, $\lambda_B = \text{scal}_B - \frac{(\text{grad}(f))(\phi)}{\phi}$ and $\lambda$ is a smooth function on $B$.

With the above notations, we prove:

**Theorem 3.2.** Let $(B, g_B)$ be an $n$-dimensional Riemannian manifold, $\phi > 0$, $f, \mu$ smooth functions on $B$ satisfying (21) and $(F, g_F)$ an $m$-dimensional Riemannian manifold with the scalar curvature given by (22). Then $(g, \xi, \pi^*(\lambda), \pi^*(\mu))$, where $\xi = \text{grad}(\tilde{f})$, is a gradient almost quasi-Yamabe soliton on the warped product manifold $(B \times_{\phi} F, g)$ if and only if $(g_B, \text{grad}(f), \lambda_B = \text{scal}_B - \frac{(\text{grad}(f))(\phi)}{\phi}, \mu)$ is a gradient almost quasi-Yamabe soliton on $(B, g_B)$.
Proof. The gradient almost quasi-Yamabe soliton \((g, \xi, \pi'(\lambda), \pi'(\mu))\) on \((B \times \varphi, E, g)\) is given by:

\[
Hess(\tilde{f}) + (\pi'(\lambda) - \text{scal})g + \pi'(\mu)d\tilde{f} \otimes d\tilde{f} = 0.
\]  

(23)

Notice that from (20), (21) and (22) we deduce that

\[
\pi'(\lambda) - \text{scal} = \pi'(\lambda_B) - \text{scal}_B,
\]

hence for any \(X, Y \in \mathcal{L}(B)\) we get:

\[
H^f(X, Y) + (\lambda_B - \text{scal}_B)g_B(X, Y) + \mu df(X)df(Y) = 0
\]

i.e. \((g_B, \text{grad}(f), \lambda_B, \mu)\) is a gradient almost quasi-Yamabe soliton on \((B, g_B)\), where \(H^f\) denotes the lift of \(Hess(f)\).

Conversely, notice that the left-hand side term in (23) computed in \((X, V)\), for \(X \in \mathcal{L}(B)\) and \(V \in \mathcal{L}(F)\) vanishes identically and for each situation \((X, Y)\) and \((V, W)\), we can recover the equation (23) from (21) and the fact that \((g_B, \text{grad}(f), \lambda_B, \mu)\) is a gradient almost quasi-Yamabe soliton on \((B, g_B)\). Indeed, taking the trace of (24) we get

\[
\Delta(f) + n(\lambda_B - \text{scal}_B) + \mu|\text{grad}(f)|^2 = 0
\]

and using (21) we obtain

\[
\pi'(\lambda_B) - \text{scal}_B = -\frac{(\text{grad}(f))(\varphi)}{\varphi}.
\]

We know that for any \(V, W \in \mathcal{L}(F)\):

\[
H^f(V, W) = (\text{Hess}(\tilde{f}))(V, W) = g(V_{V}(\text{grad}(\tilde{f})), W) = \pi'\left[\frac{(\text{grad}(f))(\varphi)}{\varphi}\right]|_{\tilde{\varphi}^2}\text{grad}(V, W)
\]

and we deduce that

\[
H^f(V, W) + (\pi'(\lambda_B) - \text{scal}_B)\text{grad}(V, W) = 0.
\]

\(\square\)

Example 3.3. Consider \(M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\),

\[
g_M := \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz)
\]

and \(\xi_M := -\frac{\partial}{\partial z}\).

Let \((g_M, \xi_M, -8, 2)\) be the gradient quasi-Yamabe soliton on the Riemannian manifold \((M, g_M)\) and let \(S^3\) be the 3-sphere with the round metric \(g_S\) (which is Einstein with the Ricci tensor equals to \(2g_S\)). Thus we obtain the gradient quasi-Yamabe soliton \((g, \xi, -2, 2)\) on the "generalized cylinder" \(M \times S^3\), where \(g = g_M + g_S\) and \(\xi\) is the lift on \(M \times S^3\) of the gradient vector field \(\xi_M = \text{grad}(f)\), where \(f(x, y, z) := -\ln z\).

3.3. Some consequences of condition (21)

Let us make some remarks on the class of manifolds that satisfy the condition:

\[
\Delta(f) + \mu|\xi|^2 = n\frac{df(\xi)}{\varphi},
\]

(25)

for \(\varphi > 0\), \(f\) and \(\mu\) smooth functions on the oriented and compact Riemannian manifold \((B, g_B)\) and \(\xi := \text{grad}(f)\).
Remark that if
\[ Hess(f) - \frac{n}{2q} (df \otimes df + d\varphi \otimes df) + \mu df \otimes df = 0, \] (26)
then (25) is satisfied. Computing \( Hess(f)(X, Y) := g_B(\nabla_X \xi, Y) \) we get
\[ \nabla \xi = \frac{n}{2q} (df \otimes \text{grad}(\varphi) + d\varphi \otimes \xi) - \mu df \otimes \xi. \]

Also notice that in this case, if \((g_B, \xi, \lambda_B, \mu)\) is a gradient almost quasi-Yamabe soliton on \((B, g_B)\), then the metric \(g_B\) is precisely
\[ g_B = -\frac{n}{2q(\lambda_B - \text{scal}_B)} (df \otimes df + d\varphi \otimes df) \]
and \(\text{scal}_B = \lambda_B + \frac{d\varphi(\xi)}{\varphi}\).

In what follows, we shall focus on condition (26). We’ve checked that [2]:
\[ |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 = |Hess(f)|^2 - \frac{(\Delta(f))^2}{n}, \]
therefore:
\[ (\text{div}(Hess(f)))(\xi) = \text{div}(Hess(f)(\xi)) - |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 - \frac{(\Delta(f))^2}{n}. \] (27)

Applying the divergence to (26), computing it in \(\xi\) and taking into account that
\[ \text{div}\left(\frac{1}{q} df \otimes d\varphi\right) = \left(\frac{\Delta(f)}{q} - \frac{d\varphi(\xi)}{q^2}\right) d\varphi + \frac{1}{q} \text{div}_\xi \text{grad}(\varphi) g_B \]
and
\[ \text{div}(\mu df \otimes df) = \frac{\mu}{2} d(|\xi|^2) + \mu \Delta(f) df + d\mu(\xi) df, \]
we get:
\[ (\text{div}(Hess(f)))(\xi) = n \left(\frac{\Delta(f)}{q} - \frac{d\varphi(\xi)}{q^2}\right) d\varphi(\xi) + \frac{n}{q} g_B(\nabla \xi \text{grad}(\varphi), \xi) - \frac{\mu}{2} d(|\xi|^2)(\xi) - \mu \Delta(f)|\xi|^2 - d\mu(\xi)|\xi|^2, \] (28)
and we obtain:
\[ \text{div}(Hess(f)(\xi)) = |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 + \frac{(\Delta(f))^2}{n} + n \left(\frac{\Delta(f)}{q} - \frac{d\varphi(\xi)}{q^2}\right) d\varphi(\xi) + \frac{n}{q} g_B(\nabla \xi \text{grad}(\varphi), \xi) - \frac{\mu}{2} d(|\xi|^2)(\xi) - \mu \Delta(f)|\xi|^2 - d\mu(\xi)|\xi|^2. \] (29)

Integrating with respect to the canonical measure on \(B\), we have:
\[ \int_B d(|\xi|^2)(\xi) = \int_B \langle \text{grad}(|\xi|^2), \xi \rangle = - \int_B \langle |\xi|^2, \text{div}(\xi) \rangle = - \int_B |\xi|^2 \cdot \Delta(f). \]
Using:
\[ |\xi|^2 \cdot \Delta(f) = |\xi|^2 \cdot \text{div}(\xi) = \text{div}(|\xi|^2 \xi) - |\xi|^2 \]
From the above observations, we have:

\[ \int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 + (n+1) \int_B \Delta(f) \cdot \frac{d\varphi(\xi)}{\varphi} + \frac{(2-\mu)n+2}{2n} \int_B |\xi|^2 = 0, \tag{30} \]

Assume now that \( \mu \) is constant and consider the product manifold \( B \times F \), in which case (26) and (25) (for \( \varphi = 1 \)) become:

\[ \text{Hess}(f) + \mu df \otimes df = 0 \quad \text{and} \quad \Delta(f) + \mu |\xi|^2 = 0. \tag{31} \]

**Remark 3.4.** i) In the case of product manifold (for \( \varphi = 1 \)), the chosen manifold \((F, g_F)\) is of scalar curvature \( \text{scal}_F = \pi(\lambda - \text{scal}_B)\). In particular, for \( \lambda = \text{scal}_B \), \((F, g_F)\) is locally isometric to an Euclidean space. Moreover, \( \nabla_\xi \xi = -\mu |\xi|^2 \xi \), therefore, \( \xi \) is a generalized geodesic vector field with the potential function \( \Delta(f) \).

ii) For \( \varphi = 1 \) and \( \mu \) constant, we obtain:

\[ \mu^2 \int_B |\xi|^4 = 0 \]

and we can state:

**Corollary 3.5.** Let \((B, g_B)\) be an oriented and compact \( n \)-dimensional Riemannian manifold, \( f \) a smooth function on \( B \) and \( \mu \) a real constant satisfying (31). Then \( \mu = 0 \) hence, \( f \) is harmonic and \( \nabla \xi = 0 \).

**Proposition 3.6.** Let \((B, g_B)\) be an oriented, compact and complete \( n \)-dimensional \((n > 1)\) Riemannian manifold, \( f \) a smooth function on \( B \) and \( \mu \) a real constant satisfying (31). Then \( B \) is conformal to a sphere in the \((n+1)\)-dimensional Euclidean space.

**Proof.** From the above observations, we have:

\[ \int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 = \int_B |\text{Hess}(f)|^2 \int_B (\Delta(f))^2 = \frac{n-1}{n} \mu^2 \int_B |\xi|^4 = 0, \]

so \( \text{Hess}(f) = \frac{\Delta(f)}{n} g_B \) which implies by \([11]\) that \( B \) is conformal to a sphere in the \((n+1)\)-dimensional Euclidean space. \( \square \)

**References**

[1] R. L. Bishop, B. O’Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. 145 (1969), 1–49.
[2] A. M. Blaga, On warped product gradient \( \eta \)-Ricci solitons, Filomat 31, no. 18 (2017), 5791–5801.
[3] L. F. di Cerbo, M. N. Disconzi, Yamabe Solitons, Determinant of the Laplacian and the Uniformization Theorem for Riemann Surfaces, Lett. Math. Phys. 83, no. 1 (2008), 13–18.
[4] B.-Y. Chen, S. Deshmukh, Yamabe and quasi-Yamabe solitons on Euclidean submanifolds, Mediterr. J. Math. (2018) 15: 194.
[5] M. Crasmareanu, A New Approach to Gradient Ricci Solitons and Generalizations, Filomat 32, no. 9 (2018), 3337–3346.
[6] S. Deshmukh, B.-Y. Chen, A note on Yamabe solitons, Balkan J. Geom. Appl. 23, no. 1 (2018), 37–43.
[7] F. Dobarro, B. Unal, Curvature of multiply warped products, J. Geom. Phys. 55, no. 1 (2005), 75–106.
[8] R. S. Hamilton, The Ricci flow on surfaces, Math. and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math. 71 (1988), AMS.
[9] B. L. Neto, H. P. de Oliveira, Generalized quasi Yamabe gradient solitons, Diff. Geom. Appl. 49 (2016), 167–175.
[10] W. I. Tokura, L. R. Adriano, R. S. Pina, On warped product gradient Yamabe soliton, J. Math. Anal. Appl. 473, no. 1 (2019), 201–214.
[11] K. Yano, M. Obata, Conformal changes of Riemannian metrics, J. Differential Geom. 4 (1970), 53–72.