DISCRETE VALUE-DISTRIBUTION OF ARTIN L-FUNCTIONS ASSOCIATED WITH CUBIC FIELDS

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Abstract. Arising from the factorizations of Dedekind zeta-functions of non-Galois cubic fields, we obtain Artin L-functions of two-dimensional irreducible representations. In this paper, we study the distribution of values of such Artin L-functions as the cubic fields are varying. We prove that various mean values of the Artin L-functions are represented by integrals involving a density function which can be explicitly constructed. The result is applied to the study on the distribution of class numbers of cubic fields.

1. Introduction

Let $d$ be a non-square integer satisfying $d \equiv 0, 1 \pmod{4}$, namely, a discriminant of a binary quadratic forms. The quadratic Dirichlet L-function $L(s, \chi_d)$ is the Dirichlet L-function of the character $\chi_d(n) = (\frac{d}{n})$, where $(\frac{d}{n})$ is the Kronecker symbol. The value-distribution of $L(s, \chi_d)$ has been studied by various authors. One of the earliest results was obtained by Chowla and Erdős [11]. They proved the existence of a continuous function $F_\sigma$ for $\sigma > \frac{3}{4}$ such that the limit formula

$$\lim_{X \to \infty} \frac{\# \{0 < d < X \mid L(\sigma, \chi_d) \leq e^a\}}{X/2} = F_\sigma(a)$$

holds for any $a \in \mathbb{R}$. The function $F_\sigma$ is the distribution function of a probability measure on $\mathbb{R}$. Elliott [13, 17] obtained a similar result for negative discriminants:

$$\frac{\# \{-d < X \mid L(1, \chi_d) \leq e^a\}}{X/2} = F_1(a) + O\left(\frac{\sqrt{\log \log X}}{\log X}\right).$$

Furthermore, he showed that the function $F_1$ has a probability density function $Q$ whose Fourier transform is represented as the infinite product

$$\tilde{Q}(\xi) = \prod_p \left[ \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{-i\xi} + \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p}\right)^{-i\xi} + \frac{1}{p} \right],$$

where $p$ runs through all prime numbers. Some related results were also obtained in [14, 15, 16]. These studies led to the idea of comparing the value-distribution of $L(s, \chi_d)$ with a suitable random model, which brought the recent progress in the theory of value-distributions of zeta and L-functions; see [19, 28, 30] for example.

The values of quadratic Dirichlet L-functions at $s = 1$ are connected with class numbers. Let $h_d$ denote the class number of a discriminant $d$ in the narrow sense. We put $\epsilon_d = (u_d + v_d \sqrt{d})/2$ for $d > 0$, where $(u_d, v_d)$ is the fundamental solution of
the Pell equation \(u^2 - d v^2 = 4\). Then we obtain \(L(1, \chi_d)\sqrt{d} = h_d \log \epsilon_d\) for \(d > 0\) and \(L(1, \chi_d)\sqrt{|d|} = \pi h_d\) for \(d < -4\) by Dirichlet’s class number formula. Hence we deduce from (1.1) and (1.2) that

\[
\# \left\{ 0 < d < X \left| h_d \log \epsilon_d \leq e^a \sqrt{d} \right. \right\} \sim \frac{X}{2} F_1(a),
\]

\[
\# \left\{ 0 < -d < X \left| h_d \leq \frac{e^a}{\pi} \sqrt{|d|} \right. \right\} = \frac{X}{2} F_1(a) + O \left( X \frac{\log \log X}{\log X} \right)
\]
as \(X \to \infty\) for \(a \in \mathbb{R}\). Note that there is more classical work on the asymptotic behaviors of \(h_d \log \epsilon_d\) and \(h_d\). Gauss stated without proof that

\[
\sum_{0 < d < X \atop d \equiv 0 \pmod{4}} h_d \log \epsilon_d \sim \frac{\pi^2}{42 \zeta(3)} X^{3/2} \quad \text{and} \quad \sum_{0 < -d < X \atop d \equiv 0 \pmod{4}} h_d \sim \frac{\pi}{42 \zeta(3)} X^{3/2}
\]
as \(X \to \infty\), where \(\zeta(s)\) is the Riemann zeta-function. Furthermore, we have

(1.3) \[
\sum_{0 < d < X} h_d \log \epsilon_d = \frac{\pi^2}{18 \zeta(3)} X^{3/2} + O(X \log X),
\]

(1.4) \[
\sum_{0 < -d < X} h_d = \frac{\pi}{18 \zeta(3)} X^{3/2} + O(X \log X).
\]
The proofs of these formulas were achieved by Siegel [38]. Ayoub [2, 3] studied the case where \(d\) is varying over the set of discriminants of quadratic fields, and he obtained similar asymptotic formulas up to the coefficients of main terms.

1.1. Artin \(L\)-functions associated with cubic fields. Let \(K\) be a non-Galois cubic field with discriminant \(d_K\). We denote by \([K]\) the isomorphism class of cubic fields containing \(K\). Then we define

\[
L_K^+(X) = \{ [K] \mid K \text{ is a non-Galois cubic field with } 0 < \pm d_K < X \}.
\]

Throughout this paper, we write \(K \in L_K^+(X)\) instead of \([K] \in L_K^+(X)\) for simplicity. For every \(K \in L_K^+(X)\), the Dedekind zeta function \(\zeta_K(s)\) is factorized as

(1.5) \[
\zeta_K(s) = L(s, \rho_K) \zeta(s),
\]
where \(\rho_K\) is the standard representation of \(\text{Gal} \left( \overline{K} / \mathbb{Q} \right) \simeq S_3\), and \(L(s, \rho_K)\) is the Artin \(L\)-function of \(\rho_K\). Here, \(\overline{K}\) denotes the Galois closure of \(K\) over \(\mathbb{Q}\), and \(S_3\) is the symmetric group of degree 3. Let \(F = \mathbb{Q}(\sqrt{d_K})\). Then \(\rho_K\) is the representation induced from the non-trivial character of \(\text{Gal} \left( \overline{K} / F \right)\) which is isomorphic to the cyclic group of order 3. The Artin \(L\)-function \(L(s, \rho_K)\) is therefore holomorphic over the whole complex plane. Moreover, the strong Artin conjecture is true for \(L(s, \rho_K)\), i.e. there exists a cuspidal representation \(\pi\) of \(GL_2(\mathbb{A}_\mathbb{Q})\) such that \(L(s, \rho_K) = L(s, \pi)\). Note that \(L(s, \rho_K)\) is self-dual, and thus the value \(L(\sigma, \rho_K)\) is real whenever \(\sigma\) is a real number.

The purpose of this paper is to study the value-distribution of \(L(s, \rho_K)\) as the cubic field \(K\) is varying. The detailed statements of the results are presented in Section 2. In this section, we pick up two of them for comparison with the above results on quadratic Dirichlet \(L\)-functions.
Theorem 1.1. Choose a constant $7/8 < \sigma_1 < 1$ arbitrarily, and let $\sigma > \sigma_1$ be a real number. Then there exists a non-negative $C^\infty$-function $C_\sigma$ such that
\begin{equation}
\# \left\{ K \in L^2_3(X) \mid L(\sigma, \rho_K) \leq e^a \right\} \# L^2_3(X) = \int_{-\infty}^{a} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O(E_\sigma(X))
\end{equation}
for any $a \in \mathbb{R}$, where
\[E_\sigma(X) = \begin{cases} (\log \log X)^2 (\log X)^{-1} & \text{if } \sigma_1 < \sigma \leq 1, \\
(\log \log X)^2 (\log X)^{-1} & \text{if } \sigma > 1. \end{cases}\]
The implied constant in (1.6) depends only on $\sigma$ and the choice of $\sigma_1$. Furthermore, the Fourier transform of $C_\sigma$ is represented as
\[\hat{C}_\sigma(\xi) = \prod_p \frac{1}{1 + p^{-1} + p^{-2}} \left[ \frac{1}{6} \left( 1 - \frac{1}{p^\sigma} \right)^{-2i\xi} + \frac{1}{2} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-i\xi} \right] + \frac{1}{3} \left( 1 + \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} \right)^{-i\xi} + \frac{1}{p} \left( 1 - \frac{1}{p^\sigma} \right)^{-i\xi} + \frac{1}{p^2},\]
where $p$ runs through all prime numbers.

See Theorem 2.3 for more information about the function $C_\sigma$. Theorem 1.1 gives analogues of (1.1) and (1.2) for the Artin $L$-function $L(s, \rho_K)$. The function $C_\sigma$ is similar to Elliott’s density function $Q$ in view of the infinite product representations of their Fourier transforms. Put $D^+ = 4$ and $D^- = 2\pi$. Recall that the relation
\[L(1, \rho_K) = D^+ \frac{h_K R_K}{\sqrt{|d_K|}}\]
holds by the class number formula, where $h_K$ and $R_K$ denote the class number and the regulator of $K$, respectively. As analogues of (1.3) and (1.4), we prove the following asymptotic formulas.

Theorem 1.2. There exists an absolute constant $\delta > 0$ such that
\[\sum_{K \in L^2_3(X)} h_K R_K = cX^{3/2} + O\left( X^{3/2} \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right),\]
\[\sum_{K \in L^2_3(X)} h_K R_K = \frac{6}{\pi} cX^{3/2} + O\left( X^{3/2} \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right),\]
where $c$ is a positive constant represented as
\begin{equation}
c = \frac{\pi^2 \zeta(3)}{432} \prod_p (1 + p^{-2} - 2p^{-3} - 2p^{-4} + 2p^{-6} + p^{-7} - p^{-8}).
\end{equation}

By equality (1.7), Theorem 1.2 is connected with the study on the first moment of values $L(1, \rho_K)$. We further prove that the limit formula
\[\lim_{X \to \infty} \frac{1}{\# L^2_3(X)} \sum_{K \in L^2_3(X) \setminus \mathcal{E}(X)} \Phi(L(\sigma, \rho_K)) = \int_{-\infty}^{\infty} \Phi(x) C_\sigma(x) \frac{dx}{\sqrt{2\pi}}\]
holds for a wide class of test functions, where $\mathcal{E}(X)$ is a subset of $L^2_3(X)$ satisfying at least $\# \mathcal{E}(X) = o(X)$ as $X \to \infty$. See Theorem 2.4 for the strict statement.
1.2. Related topics. The study of this paper is partially motivated by the recent work on “M-functions” by Ihara–Matsumoto. We recall one of the results from [23] in the number field case. Let $F$ be $\mathbb{Q}$ or an imaginary quadratic field. For a prime ideal $f$ of $F$, we define $X(f)$ as the set of all primitive Dirichlet characters $\chi$ on $F$ with conductor $f$. Suppose that the Generalized Riemann Hypothesis (GRH) is true, that is, every Dirichlet $L$-function $L(s, \chi)$ has no zeros in the half plane $\text{Re}(s) > 1/2$. Then $\log L(s, \chi)$ extends to a holomorphic function on $\text{Re}(s) > 1/2$.

**Theorem 1.3** (Ihara–Matsumoto [23]). Assume GRH, and put $|dz| = (2\pi)^{-1}dxdy$ for $z = x + iy$. Then there exists a non-negative $C^\infty$-function $M_\sigma$ such that

$$\lim_{N(f) \to \infty} \frac{1}{\#X(f)} \sum_{\chi \in X(f)} \Phi(\log L(s, \chi)) = \int_{\mathbb{C}} \Phi(z) M_\sigma(z) |dz|$$

for any complex number $s = \sigma + i\tau$ with $\sigma > 1/2$, where $\Phi$ is a function on $\mathbb{C}$ satisfying one of the following:

(i) $\Phi$ is any continuous function on $\mathbb{C}$ with $\Phi(z) \ll e^{a|z|}$ for some $a > 0$,

(ii) $\Phi$ is an indicator function of either a compact subset of $\mathbb{C}$ or the complement of such a subset.

They also showed that a similar result is valid when $\log L(s, \chi)$ is replaced with the logarithmic derivative $(L'/L)(s, \chi)$. In addition, several analogous results were proved in [21, 22, 24], and so on. The results in [22] were proved without GRH but the test functions are restricted to bounded functions. The construction of the function $M_\sigma$ was explained in [22, Section 3], and it matches a density function in the early work of Bohr–Jessen [7, 8] on the value-distribution of $\log \zeta(\sigma + it)$. The density functions such as $M_\sigma$ were named $M$-functions by Ihara [21]. Today there are a lot of variants of Theorem 1.3 and the corresponding $M$-functions; see the survey of Matsumoto [33]. In particular, the $M$-function for the value-distribution of quadratic Dirichlet $L$-functions was studied by Mourtada–Murty [36]. Although their result was proved under GRH, Akbary–Hamieh [1] pointed out that one can remove the assumption by applying an adequate zero density estimate.

Another topic related to the study of this paper is the work on the Artin $L$-function $L(s, \rho_K)$ by Cho and Kim. They studied $L(s, \rho_K)$ not only for cubic fields but also for general $S_n$-fields of degree $n \geq 2$. Here, a number field $K$ of degree $n$ is called an $S_n$-field if the Galois group $\text{Gal}(\overline{K}/\mathbb{Q})$ is isomorphic to the symmetric group $S_n$. Their results were obtained under the strong Artin conjecture for $L(s, \rho_K)$ and the “counting conjecture” [10, Conjecture 3.1] for $S_n$-fields. The truth of the former conjecture is known for $n \leq 4$, and the latter is known for $n \leq 5$. Hence the results for $n = 2, 3, 4$ are unconditional. In [10], they proved an asymptotic formula for integral moments of $\log L(1, \rho_K)$. Here we refer to the result in the cubic case.

**Theorem 1.4** (Cho–Kim [10]). Let $k$ be a non-negative integer. Then we have

$$\frac{1}{\#L_3^2(X)} \sum_{K \in L_3^2(X)} (\log L(1, \rho_K))^k = \tilde{r}(k) + O\left(\frac{1}{\log X}\right),$$

where $\tilde{r}(k)$ is a positive constant which can be explicitly described.
If we assume that \( r(k) \ll c^k \log \log k \) is satisfied for some \( c > 1 \), the method of moments enables us to show the existence of a continuous function \( F \) satisfying

\[
\lim_{X \to \infty} \frac{\# \{ K \in L_3^+(X) \mid L(1, \rho_K) \leq e^a \}}{\# L_3^+(X)} = F(a),
\]

where \( a \) is a point of continuity of \( F \). Theorem 1.1 gives an improvement of this formula, and we remark that it is proved without any assumptions on \( r(k) \). Cho–Kim’s original expression of \( r(k) \) is described in [10, Proposition 5.3], while we obtain another expression \( r(k) = \int_{-\infty}^{\infty} x^k C_1(x) \frac{dx}{\sqrt{2\pi}} \) by formula (1.9); see also Corollary 5.10. We recall another result in [9] on the distribution of values \( L(1, \rho_K) \). Let \( \alpha > 0 \) be a real number. Then it was proved that

\[
\liminf_{X \to \infty} \frac{\# \{ K \in L_3^+(X) \mid |L(1, \rho_K) - \alpha| < \epsilon \}}{\# L_3^+(X)} > 0
\]

for every \( \epsilon > 0 \). This is a cubic analogue of the result of Mishou–Nagoshi [34].

This paper consists of six sections. Section 2 is devoted to giving the statements of the results. First, we present the results for \( \log L(\sigma, \rho_K) \) in Section 2.1. As well as the work of Ihara–Matsumoto, similar results are obtained for \( (L'/L)(\sigma, \rho_K) \), which are presented in Section 2.2. The preliminary lemmas are collected in Section 3. The proofs of the main results begin with the study of the function \( C_\sigma \) of Theorem 1.1. After some preparations in Section 4.1, we construct the density function \( C_\sigma \) in Section 4.2. Several properties of \( C_\sigma \) are proved in Section 4.3. Then, we proceed to the investigation of the mean values of \( \Phi(\log L(\sigma, \rho_K)) \). We study the special case \( \Phi(x) = e^{iz} \) in Section 5, where \( z \) is an arbitrary complex number. Applying the Berry–Esseen inequality, we deduce Theorem 1.1 from the case \( \Phi(x) = e^{iz} \) with \( z = i\xi \in i\mathbb{R} \). Also, Theorem 1.2 is deduced from the case \( \Phi(x) = e^x \) by the class number formula. The proofs of Theorems 1.1 and 1.2 are completed in Section 6. Finally, we prove limit formula (1.9) for general \( \Phi(x) \) at the end of this paper.

2. Statement of results

To begin with, we set up the notation for counting cubic fields with discriminants not exceeding a given quantity. Based on [37, 39], we introduce the notion of the local specifications of cubic fields as follows. Let

\[
\mathcal{A} = \{(111), (21), (3), (1^2 1), (1^3)\}
\]

be the set of symbols. We associate \( a \in \mathcal{A} \) with a diagonal matrix \( A_a \in M_2(\mathbb{C}) \) by putting

\[
A_a = \begin{cases} 
\text{diag}(1, 1) & \text{if } a = (111), \\
\text{diag}(1, -1) & \text{if } a = (21), \\
\text{diag}(\omega, \overline{\omega}) & \text{if } a = (3), \\
\text{diag}(1, 0) & \text{if } a = (1^2 1), \\
\text{diag}(0, 0) & \text{if } a = (1^3),
\end{cases}
\]
where $\omega$ is a primitive cube root of unity. For a prime number $p$, we write the prime ideal decomposition of $p$ in $K$ as $(p) = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_r^{\gamma_r}$. Then, a cubic field $K$ is said to satisfy a local specification $\mathfrak{a} \in \mathfrak{A}$ at $p$ if

(a) for $\mathfrak{a} = (111)$, $p$ is totally splitting in $K$, i.e. $(p) = p_1 p_2 p_3$;
(b) for $\mathfrak{a} = (21)$, $p$ is partially splitting in $K$, i.e. $(p) = p_1 p_2$;
(c) for $\mathfrak{a} = (3)$, $p$ remains inert in $K$, i.e. $(p) = p_1$;
(d) for $\mathfrak{a} = (1^2 1)$, $p$ is partially ramified in $K$, i.e. $(p) = p_2^2$;
(e) for $\mathfrak{a} = (1^3)$, $p$ is totally ramified in $K$, i.e. $(p) = p_1^3$.

The symbol $\mathcal{S}$ is used to denote a collection of local specifications with the following data: (i) a finite set $\text{supp}(\mathcal{S})$ consisting of prime numbers; (ii) an element $\mathcal{S}_p \in \mathfrak{A}$, for each $p \in \text{supp}(\mathcal{S})$. We say that a cubic field $K$ satisfies the local specifications $\mathcal{S} = (\mathcal{S}_p)_p$ if $K$ satisfies $\mathcal{S}_p$ at $p$ for every $p \in \text{supp}(\mathcal{S})$. Then we define

$$L_3^\pm(X, \mathcal{S}) = \{ K \in L_3^\pm(X) \mid K \text{ satisfies the local specifications } \mathcal{S} \}.$$  

Remark that the set $\text{supp}(\mathcal{S})$ may be empty. We define $L_3^\pm(X, \mathcal{S}) = L_3^\pm(X)$ in this case. Next, for a prime number $p$ and $\mathfrak{a} \in \mathfrak{A}$, we put

$$C_p(\mathfrak{a}) = \frac{1}{1 + p^{-1} + p^{-2}} \begin{cases} 
1/6 & \text{if } \mathfrak{a} = (111), \\
1/2 & \text{if } \mathfrak{a} = (21), \\
1/3 & \text{if } \mathfrak{a} = (3), \\
1/p & \text{if } \mathfrak{a} = (1^2 1), \\
1/p^2 & \text{if } \mathfrak{a} = (1^3),
\end{cases}$$

$$K_p(\mathfrak{a}) = \frac{1 - p^{-1/3}}{(1 - p^{-5/3})(1 + p^{-1})} \begin{cases} 
1/6 \cdot (1 + p^{-1/3})^3 & \text{if } \mathfrak{a} = (111), \\
1/2 \cdot (1 + p^{-1/3})(1 + p^{-2/3}) & \text{if } \mathfrak{a} = (21), \\
1/3 \cdot (1 + p^{-1}) & \text{if } \mathfrak{a} = (3), \\
1/p \cdot (1 + p^{-1/3})^2 & \text{if } \mathfrak{a} = (1^2 1), \\
1/p^2 \cdot (1 + p^{-1/3}) & \text{if } \mathfrak{a} = (1^3).
\end{cases}$$

Note that both $\sum_{\mathfrak{a} \in \mathfrak{A}} C_p(\mathfrak{a})$ and $\sum_{\mathfrak{a} \in \mathfrak{A}} K_p(\mathfrak{a})$ are equal to 1. We further define

$$C(\mathcal{S}) = \prod_{p \in \text{supp}(\mathcal{S})} C_p(\mathcal{S}_p) \quad \text{and} \quad K(\mathcal{S}) = \prod_{p \in \text{supp}(\mathcal{S})} K_p(\mathcal{S}_p),$$

where the empty product is interpreted as the value 1. Put $C^+ = 1$, $C^- = 3$ and $K^+ = 1$, $K^- = \sqrt{3}$. Finally, we define $E^\pm(X, \mathcal{S})$ by the equation

$$\# L_3^\pm(X, \mathcal{S}) = C^\pm \frac{1}{12 \zeta(3)} X C(\mathcal{S}) + K^\pm \frac{4 \zeta(1/3)}{5 \Gamma(2/3)\zeta(5/3)} X^{5/6} K(\mathcal{S}) + E^\pm(X, \mathcal{S}).$$

Roberts [37] conjectured $E^\pm(X, \mathcal{S}) = o(X^{5/6})$ as $X \to \infty$. This conjecture was proved to be true by Bhargava–Shankar–Tsimerman [5] and Taniguchi–Thorne [39] independently. In this paper, we assume the following upper bound for $E^\pm(X, \mathcal{S})$.

**Assumption 2.1.** Let $\mathcal{S} = (\mathcal{S}_p)_p$ be a collection of local specifications as above. Then there exist absolute constants $\alpha, \beta$ with $0 < \alpha < 5/6$ such that

$$E^\pm(X, \mathcal{S}) \ll X^{\alpha + \epsilon} \prod_{p \in \text{supp}(\mathcal{S})} p^\beta$$

for each $\epsilon > 0$. The implied constant depends only on $\epsilon$. 

Note that Assumption 2.1 is true at least for $\alpha = 7/9$ and $\beta = 16/9$. Indeed, Taniguchi–Thorne [39] proved

$$E(X, S) \ll_{\epsilon} X^{7/9+\epsilon} \prod_{p \in \text{supp}(S)} p^{e_p(S_p)},$$

where $e_p(a) = 8/9$ for $a = (111), (21), (3)$ and $e_p(a) = 16/9$ for $a = (1^22), (1^3)$. The Grand Riemann Hypothesis (GRH) is used in the sense that every $L(s, \rho_K)$ has no zeros in the half plane $\text{Re}(s) > 1/2$. We may use the following zero density estimate for $L(s, \rho_K)$ as a substitution for GRH.

**Assumption 2.2.** Let $N(\sigma, T; \rho_K)$ count the number of zeros of $L(s, \rho_K)$ satisfying $\text{Re}(\rho) \geq \sigma$ and $|\text{Im}(\rho)| \leq T$ with multiplicity. Then there exists an absolute constant $1/2 < \sigma_1 < 1$ such that the estimate

$$\sum_{K \in L_1^\times(X)} N(\sigma_1, (\log X)^3; \rho_K) \ll X^{1-\delta}$$

holds with an absolute constant $\delta > 0$. We also note that GRH ensures the truth of Assumption 2.2 for any $1/2 < \sigma_1 < 1$ with arbitrary $\delta > 0$. In Section 3.1, we see that one can choose any $\sigma_1 > 7/8$ with a small constant $\delta = \delta(\sigma_1) > 0$ unconditionally, but we remark that the implied constant in (2.5) may depend on the choice of $\sigma_1$ in that case.

### 2.1. Main results.

Let $A_a$ be the matrix attached to a symbol $a \in \mathcal{A}$ as in (2.1). Then we define the function $F(w; a)$ by the power series

$$F(w; a) = \sum_{m=1}^{\infty} \frac{\text{tr}(A_a^m)}{m} w^m$$

which converges for $|w| < 1$. Let $p$ be a prime number. We further define

$$F_{s,p}(z) = \sum_{a \in \mathcal{A}} C_p(a) \exp \left( zF(p^{-s}; a) \right),$$

$$G_{s,p}(z) = \sum_{a \in \mathcal{A}} K_p(a) \exp \left( zF(p^{-s}; a) \right)$$

for $\text{Re}(s) > 0$ and $z \in \mathbb{C}$, where the coefficients $C_p(a)$ and $K_p(a)$ are given by (2.2) and (2.3), respectively. From the above, we define

$$F_s(z) = \prod_p F_{s,p}(z) \quad \text{and} \quad G_s(z) = \prod_p G_{s,p}(z).$$

In Section 4.1, we show that the former product converges for any $\text{Re}(s) > 1/2$ and $z \in \mathbb{C}$, and the latter converges for any $\text{Re}(s) > 2/3$ and $z \in \mathbb{C}$. Let $0 \leq a \leq \infty$. We define the subspace $L^1(\mathbb{R})_a \subset L^1(\mathbb{R})$ as

$$L^1(\mathbb{R})_a = \left\{ F \in L^1(\mathbb{R}) \mid \int_{-\infty}^{\infty} |F(x)|e^{a|x|} \, dx < \infty \right\} \quad \text{for } 0 \leq a < \infty,$$

$$L^1(\mathbb{R})_\infty = \bigcap_{0 \leq a < \infty} L^1(\mathbb{R})_a.$$
Theorem 2.4. Consider the condition without GRH. In fact, we have \( \# \) continuity set of \( K \) if \( S \). Then we define the Fourier transform of \( F \) is the indicator function of a set \( B \), be an absolute constant for which Assumption 2.2 holds. Take \( \tilde{\mathcal{C}}(z) = F(iz) \) holds for any \( z \in \mathbb{C} \). The function \( \mathcal{C}_\sigma \) is everywhere non-negative. Moreover, we have supp\( (\mathcal{C}) = \mathbb{R} \) for \( 2/3 < \sigma \leq 1 \), while \( \mathcal{C}_\sigma \) is compactly supported for \( \sigma > 1 \).

Let \( \sigma > 1/2 \) be a real number. We define two subsets \( \mathcal{A}_\sigma(X), \mathcal{B}_\sigma(X) \subset L^\pm_3(X) \) as

\[
\mathcal{A}_\sigma(X) = \left\{ K \in L^\pm_3(X) \mid \text{there exists a zero } \rho \text{ of } L(s, \rho_K) \text{ such that } \text{Re}(\rho) \geq \sigma \text{ and } \text{Im}(\rho) = 0. \right\},
\]

\[
\mathcal{B}_\sigma(X) = \left\{ K \in L^\pm_3(X) \mid \text{there exists a zero } \rho \text{ of } L(s, \rho_K) \text{ such that } \text{Re}(\rho) \geq \sigma \text{ and } |\text{Im}(\rho)| \leq (\log X)^{\beta}. \right\}.
\]

Then \( \mathcal{A}_\sigma(X) \subset \mathcal{B}_\sigma(X) \) by definition. Note that these subsets are empty under GRH. On the other hand, Assumption 2.2 implies that they are small subsets in \( L^\pm_3(X) \) without GRH. In fact, we have \( \# \mathcal{B}_\sigma(X) \ll X^{1-\delta} \) for \( \sigma \geq \sigma_1 \) by (2.9), while we know that \( \# L^\pm_3(X) \sim X \) holds. We also note that the condition \( L(\sigma, \rho_K) > 0 \) is satisfied if \( K \notin \mathcal{A}_\sigma(X) \). Let \( C(\mathbb{R}) \) denote the class of all continuous functions \( \Phi : \mathbb{R} \to \mathbb{C} \). Then we define two classes of test functions as

\[
C^\exp_\sigma(\mathbb{R}) = \left\{ \Phi \in C(\mathbb{R}) \mid \Phi(x) \ll e^{a|x|} \right\} \quad \text{for } 0 \leq a < \infty,
\]

\[
\mathcal{I}(\mathbb{R}) = \{1_A \mid A \text{ is a continuity set of } \mathbb{R} \} \cup \{1_B \mid B \text{ is a compact subset of } \mathbb{R} \}
\]

\[
\cup \{1_C \mid C \text{ is a compact subset of } \mathbb{R} \},
\]

where \( 1_S \) is the indicator function of a set \( S \subset \mathbb{R} \), and a Borel set \( A \) is called a continuity set of \( \mathbb{R} \) if its boundary \( \partial A \) has Lebesgue measure zero.

Theorem 2.4. Let \( \sigma_1 \) be an absolute constant for which Assumption 2.2 holds. Take a real number \( \sigma > \sigma_1 \) and a subset \( \mathcal{E}(X) \subset L^\pm_3(X) \). For a test function \( \Phi \) on \( \mathbb{R} \), we consider the condition

\[
\lim_{X \to \infty} \frac{1}{\# L^\pm_3(X)} \sum_{K \in \mathbb{L}^\pm_3(X) \setminus \mathcal{E}(X)} \Phi(\log L(\sigma, \rho_K)) = \int_{-\infty}^{\infty} \Phi(x) C_\sigma(x) \frac{dx}{\sqrt{2\pi}}.
\]

Then the following results are valid.

(0) Let \( \sigma > 1 \) and \( \mathcal{E}(X) = \emptyset \). Then (2.11) holds for \( \Phi \in C(\mathbb{R}) \cup \mathcal{I}(\mathbb{R}) \).

(1) Let \( \sigma = 1 \) and \( \mathcal{E}(X) = \emptyset \). Then (2.11) holds for \( \Phi \in C^\exp_\sigma(\mathbb{R}) \cup \mathcal{I}(\mathbb{R}) \) with any \( a \geq 0 \).

(2) Let \( \sigma_1 < \sigma < 1 \) and \( \mathcal{E}(X) = \mathcal{A}_\sigma(X) \). Then (2.11) holds for \( \Phi \in C^\exp_0(\mathbb{R}) \cup \mathcal{I}(\mathbb{R}) \).

(3) Let \( \sigma_1 < \sigma < 1 \) and \( \mathcal{E}(X) = \mathcal{B}_\sigma(X) \). Then (2.11) holds for \( \Phi \in C^\exp_0(\mathbb{R}) \cup \mathcal{I}(\mathbb{R}) \) with any \( a \geq 0 \).
We obtain more precise asymptotic formulas by specializing the test functions to \( \Phi(x) = e^{\pi x} \) with \( z \in \mathbb{C} \). We define
\[
R_\sigma(X) = \begin{cases} 
(\log X)(\log \log X)^{-2} & \text{if } \sigma \geq 1, \\
(\log X)^{(\sigma-\alpha_1)/(1-\sigma_1)}(\log \log X)^{-2} & \text{if } \sigma_1 < \sigma < 1.
\end{cases}
\]

**Theorem 2.5.** Let \( \sigma_1 \) be an absolute constant for which Assumption \( 2.2 \) holds. Take a real number \( \sigma > \sigma_1 \) and a subset \( \mathcal{E}(X) \subset L_3^+(X) \). For a complex number \( z \), we consider the condition that there exists an absolute constant \( \delta > 0 \) such that
\[
\sum_{\mathcal{K} \in L_3^+(X) \setminus \mathcal{E}(X)} L(\sigma, \rho_K)^z = C^+ X^{\frac{z}{12\zeta(3)^2}} F_\sigma(z) + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)
\]
holds, where the implied constant depends only on \( \sigma \). Then the following results hold.

1. Let \( \sigma \geq 1 \) and \( \mathcal{E}(X) = \emptyset \). Then there exists \( b_\sigma > 0 \) depending only on \( \sigma \) such that (2.13) holds for any \( z \in \mathbb{C} \) with \( |z| \leq b_\sigma R_\sigma(X) \).
2. Let \( \sigma_1 < \sigma < 1 \) and \( \mathcal{E}(X) = \mathcal{A}_\sigma(X) \). Then there exists \( b_\sigma > 0 \) depending only on \( \sigma \) such that (2.13) holds for any \( z = i \xi \in i \mathbb{R} \) with \( |\xi| \leq b_\sigma R_\sigma(X) \).
3. Let \( \sigma_1 < \sigma < 1 \) and \( \mathcal{E}(X) = \mathcal{B}_{\sigma_1}(X) \). Then there exists \( b_\sigma > 0 \) depending only on \( \sigma \) such that (2.13) holds for any \( z \in \mathbb{C} \) with \( |z| \leq b_\sigma R_\sigma(X) \).

Such asymptotic formulas for complex moments of \( L \)-functions were proved in several cases. Granville–Soundararajan [19] studied the \( z \)-th moments of \( L(1, \chi_d) \) with \( |z| \ll (\log X)(\log \log X)^{-2} \). Cogdell–Michel [12], Fomenko [18], and Golubeva [20] proved its analogues for automorphic \( L \)-functions. In the case of \( 1/2 < \sigma < 1 \), the results for \( \zeta(\sigma + it) \) in \( t \)-aspect were obtained by Lamzouri [25] and more recently by Lamzouri–Lester–Radziwiłł [30].

**Corollary 2.6.** Let \( \sigma_1 \) be an absolute constant for which Assumption \( 2.2 \) holds, and let \( \sigma > \sigma_1 \). Then we have
\[
\frac{\# \{ K \in L_3^+(X) \setminus \mathcal{A}_\sigma(X) \mid \log L(\sigma, \rho_K) \leq a \}}{\#L_3^+(X)} = \int_{-\infty}^{a} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( \frac{1}{R_\sigma(X)} \right)
\]
for any \( a \in \mathbb{R} \), where the implied constant depends only on \( \sigma \).

**Corollary 2.7.** Let \( r > -2 \) be a fixed real number. There exists an absolute constant \( \delta > 0 \) such that
\[
\sum_{\mathcal{K} \in L_3^+(X)} (h_K R_K)^r = \frac{C^+ \mathcal{F}_1(r)}{12\zeta(3)(D^z)^r} \frac{X^{r/2+1}}{r/2+1} + O \left( X^{r/2+1} \exp \left( -\frac{\delta}{\log \log X} \right) \right),
\]
where the implied constant depends only on \( r \).

Recall that Assumption \( 2.2 \) is true for any \( \sigma_1 > 7/8 \). Hence Corollary 2.6 yields Theorem 1.1. Moreover, Cho–Kim’s result (1.10) can be improved by Corollary 2.6 as follows. Let \( \alpha > 0 \) be a real number. Then it further yields
\[
\lim_{X \to \infty} \frac{\# \{ K \in L_3^+(X) \mid |L(\sigma, \rho_K) - \alpha| < \epsilon \}}{\#L_3^+(X)} = \int_{\log(\alpha+\epsilon)}^{\log(\alpha-\epsilon)} C_\sigma(x) \frac{dx}{\sqrt{2\pi}}
\]
for \( \epsilon > 0 \) small enough. This limit value is positive if \( \sigma \leq 1 \) since \( \text{supp}(C_\sigma) = \mathbb{R} \). Also, Theorem 1.2 is nothing but Corollary 2.7 in the case of \( r = 1 \). See Remark 6.1 for the computation of the coefficient \( c \) in Theorem 1.2.
We obtain the secondary main terms for complex moments of $L(\sigma, \rho_K)$. In place of $R_\sigma(X)$ in Theorem 2.5, we define
\begin{equation}
\tilde{R}_\sigma(X) = (\log X)^{a(\sigma)}(\log \log X)^{-1},
\end{equation}
where $a(\sigma)$ is a small positive constant which we take depending only on $\sigma$ later.

**Theorem 2.8.** Assume GRH, and put
\begin{equation}
\sigma_2 = \max \left\{ \left( \frac{5-6\alpha}{12(1-\alpha)} \right) + \frac{2+2\beta}{3}, \frac{2}{3} \right\}
\end{equation}
with absolute constants $\alpha, \beta$ for which Assumption 2.1 holds. Let $\sigma > \sigma_2$ be a real number. Then there exists a constant $\delta = \delta(\sigma) > 0$ depending only on $\sigma$ such that
\begin{equation}
\sum_{K \in L_\pm^2(X)} L(\sigma, \rho_K)^{+} = C^{\pm} \frac{1}{12\zeta(3)} X F_\sigma(z) + K^{\pm} \frac{4\zeta(1/3)}{5\Gamma(2/3)^3 \zeta(5/3)} X^{5/6} G_\sigma(z) + O \left( X^{5/6-\delta} \right)
\end{equation}
for any $z \in \mathbb{C}$ with $|z| \leq b_\sigma \tilde{R}_\sigma(X)$, where $b_\sigma > 0$ is a constant depending only on $\sigma$. The implied constant also depends only on $\sigma$.

Assumption 2.1 is true for $\alpha = 7/9$ and $\beta = 16/9$. Hence formula (2.16) holds at least for $\sigma > 53/24$ under GRH. Assuming the validity of Assumption 2.1 for smaller $\alpha, \beta$ such that $3\alpha + \beta < 5/2$, we have $\sigma_2 < 1$ in Theorem 2.8. Hence, letting $\sigma = 1$ and $z = 1$ in (2.16), we deduce
\begin{align*}
\sum_{K \in L_+^2(X)} h_K R_K &= c X^{3/2} + k X^{4/3} + O(X^{4/3-\delta}), \\
\sum_{K \in L_-^2(X)} h_K R_K &= \frac{6}{\pi} c X^{3/2} + \frac{6}{\pi} k X^{4/3} + O(X^{4/3-\delta})
\end{align*}
in a similar way of the proof of Corollary 2.7 where $c$ is given by (1.8), and $k$ is a positive constant explicitly computed. We finally remark that the method of this paper do not allow us to take $\sigma \leq 2/3$ in (2.16). In fact, we fail to construct the density function $K_\sigma$ if $\sigma \leq 2/3$; see Remark 4.11.

2.2. **Results for logarithmic derivatives.** In the previous section, we presented the results on the distribution of values $\log L(\sigma, \rho_K)$. We obtain similar results for $(L'/L)(\sigma, \rho_K)$, which are listed in this section. Note that the values $(L'/L)(1, \rho_K)$ are connected with the Euler–Kronecker constants defined by
\begin{equation}
\gamma_K = \lim_{s \to 1} \left( \frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1} \right).
\end{equation}
Indeed, we deduce the relation $(L'/L)(1, \rho_K) = \gamma_K - \gamma$ from (1.5), where $\gamma$ is the Euler–Mascheroni constant. In order to describe the statements of the results, we need modification to the setting in Section 2.1. First, we define
\begin{equation}
F^*(w; a) = \sum_{m=1}^{\infty} \text{tr}(A^m w^m)
\end{equation}
for $|w| < 1$ in place of (2.6). Then, for a prime number $p$, we define

$$\mathcal{F}_{s,p}(z) = \sum_{a \in \mathcal{A}} C_p(a) \exp \left(-z(\log p)F^*(p^{-s}; a)\right),$$

$$\mathcal{G}_{s,p}(z) = \sum_{a \in \mathcal{A}} K_p(a) \exp \left(-z(\log p)F^*(p^{-s}; a)\right)$$

for $\Re(s) > 0$ and $z \in \mathbb{C}$. The infinite product $\mathcal{F}_{s}(z) = \prod_p \mathcal{F}_{s,p}(z)$ converges for any $\Re(s) > 1/2$ and $z \in \mathbb{C}$, while $\mathcal{G}_{s}(z) = \prod_p \mathcal{G}_{s,p}(z)$ converges for any $\Re(s) > 2/3$ and $z \in \mathbb{C}$.

**Theorem 2.9.** (i) For $\sigma > 1/2$, there exists a $C^\infty$-function $C_\sigma$ in $L^1(\mathbb{R})_\infty$ such that $\tilde{C}_\sigma(z) = \mathcal{F}_{\sigma}(iz)$ holds for any $z \in \mathbb{C}$. The function $C_\sigma$ is everywhere non-negative. Moreover, we have $\text{supp}(C_\sigma) = \mathbb{R}$ for $1/2 < \sigma \leq 1$, while $C_\sigma$ is compactly supported for $\sigma > 1$.

(ii) For $\sigma > 2/3$, there exists a $C^\infty$-function $K_\sigma$ in $L^1(\mathbb{R})_\infty$ such that $\tilde{K}_\sigma(z) = \mathcal{G}_{\sigma}(iz)$ holds for any $z \in \mathbb{C}$. The function $K_\sigma$ is everywhere non-negative. Moreover, we have $\text{supp}(K_\sigma) = \mathbb{R}$ for $2/3 < \sigma \leq 1$, while $K_\sigma$ is compactly supported for $\sigma > 1$.

**Theorem 2.10.** Let $\sigma_1$ be an absolute constant for which Assumption 2.2 holds. Take a real number $\sigma > \sigma_1$ and a subset $\mathcal{E}(X) \subset L^2_3(X)$. For a test function $\Phi$ on $\mathbb{R}$, we consider the condition

$$(2.18) \quad \lim_{X \to \infty} \frac{1}{\#L^2_3(X)} \sum_{K \in L^2_3(X) \setminus \mathcal{E}(X)} \Phi \left( \frac{L'}{L}(\sigma, \rho_K) \right) = \int_{-\infty}^{\infty} \Phi(x) C_\sigma(x) \frac{dx}{\sqrt{2\pi}}.$$  

Then the following results are valid.

1. Let $\sigma > 1$ and $\mathcal{E}(X) = \emptyset$. Then (2.18) holds for $\Phi \in C(\mathbb{R}) \cup \mathcal{I}(\mathbb{R})$.

2. Let $\sigma_1 < \sigma \leq 1$ and $\mathcal{E}(X) = \mathcal{A}_\sigma(X)$. Then (2.18) holds for $\Phi \in C^\text{exp}_0(\mathbb{R}) \cup \mathcal{I}(\mathbb{R})$.

3. Let $\sigma_1 < \sigma \leq 1$ and $\mathcal{E}(X) = \mathcal{B}_{\sigma_1}(X)$. Then (2.18) holds for $\Phi \in C^\text{exp}_0(\mathbb{R}) \cup \mathcal{I}(\mathbb{R})$ with any $a \geq 0$.

**Theorem 2.11.** Let $\sigma_1$ be an absolute constant for which Assumption 2.2 holds. Take a real number $\sigma > \sigma_1$ and a subset $\mathcal{E}(X) \subset L^2_3(X)$. For a complex number $z$, we consider the condition that there exists an absolute constant $\delta > 0$ such that

$$(2.19) \quad \sum_{K \in L^2_3(X) \setminus \mathcal{E}(X)} \exp \left( z \frac{L'}{L}(\sigma, \rho_K) \right) = \frac{C^\pm X}{12\zeta(3)} F_{\sigma}^*(z) + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)$$

holds, where the implied constant depends only on $\sigma$. Then the following results hold.

1. Let $\sigma > 1$ and $\mathcal{E}(X) = \emptyset$. Then there exists $b_\sigma > 0$ depending only on $\sigma$ such that (2.19) holds for $z \in \mathbb{C}$ with $|z| \leq b_\sigma R_\sigma(X)$.

2. Let $\sigma_1 < \sigma \leq 1$ and $\mathcal{E}(X) = \mathcal{A}_\sigma(X)$. Then there exists $b_\sigma > 0$ depending only on $\sigma$ such that (2.19) holds for $\mathcal{E}(X) = \mathcal{A}_\sigma(X)$ and $z = i\xi \in i\mathbb{R}$ with $|\xi| \leq b_\sigma R_\sigma(X)$.

3. Let $\sigma_1 < \sigma \leq 1$ and $\mathcal{E}(X) = \mathcal{B}_{\sigma_1}(X)$. Then there exists $b_\sigma > 0$ depending only on $\sigma$ such that (2.19) holds for $z \in \mathbb{C}$ with $|z| \leq b_\sigma R_\sigma(X)$. 
Compare Theorems 2.10 and 2.11 with Theorems 2.4 and 2.5 in Section 2.1. Then we notice that there exist differences between the results for \( \log L(1, \rho_K) \) and \( (L'/L)(1, \rho_K) \). The reason for the differences is explained in Remark 5.7.

**Corollary 2.12.** Let \( \sigma_1 \) be an absolute constant for which Assumption 2.2 holds, and let \( \sigma > \sigma_1 \). Then we have

\[
\# \{ K \in L_3^+(X) \setminus A_\sigma(X) \mid (L'/L)(\sigma, \rho_K) \leq a \} = \int_{-\infty}^{a} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( \frac{1}{R_\sigma(X)} \right)
\]

for any \( a \in \mathbb{R} \), where the implied constant depends only on \( \sigma \).

**Corollary 2.13.** There exists an absolute constant \( \delta > 0 \) such that

\[
\sum_{K \in L_3^+(X) \setminus B_{\sigma_1}(X)} \exp(z \cdot \gamma_K) = \frac{C^+ X}{12 \zeta(3)} F_1^+(z) e^{\gamma z} + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)
\]

for any \( z \in \mathbb{C} \) with \( |z| \leq b(\log X)(\log \log X)^{-2} \), where \( b > 0 \) is an absolute constant.

**Theorem 2.14.** Assume GRH, and take a constant \( \sigma_2 \) as in (2.15). Let \( \sigma > \sigma_2 \) be a real number. Then there exists a constant \( \delta = \delta(\sigma) > 0 \) depending only on \( \sigma \) such that

\[
\sum_{K \in L_3^+(X)} \exp \left( \frac{L'}{L}(\sigma, \rho_K) \right)
\]

\[
= C^+ \frac{1}{12 \zeta(3)} X F_1^+(z) + K^+ \frac{4 \zeta(1/3)}{5 \Gamma(2/3)^{5/3}(5/3)} X^{5/6} G_\sigma^+(z) + O \left( X^{5/6 - \delta} \right)
\]

for any \( z \in \mathbb{C} \) with \( |z| \leq b_\sigma R_\sigma(X) \), where \( b_\sigma > 0 \) is a constant depending only on \( \sigma \). The implied constant also depends only on \( \sigma \).

We often skip the details of the proofs for the case of \( (L'/L)(\sigma, \rho_K) \) since the methods are largely common to \( \log L(\sigma, \rho_K) \) and \( (L'/L)(\sigma, \rho_K) \).

3. Preliminaries

**3.1. Properties of the Artin L-function** \( L(s, \rho_K) \). We first recall that the Artin \( L \)-function \( L(s, \rho_K) \) is represented as the Euler product

\[
L(s, \rho_K) = \prod_p \det (1 - p^{-s} A_p(K))^{-1}
\]

for \( \text{Re}(s) > 1 \), where \( I \in M_2(\mathbb{C}) \) is the identity matrix, and we take \( A_p(K) = A_\alpha \) if \( K \) satisfies a local specification \( \alpha \in \mathcal{A} \) at \( p \). As we mentioned in Section 1.1 there exists a cuspidal representation \( \pi \) such that \( L(s, \rho_K) = L(s, \pi) \). Hence the Artin \( L \)-function \( L(s, \rho_K) \) is continued to an entire function. The complete \( L \)-function

\[
\Lambda(s, \rho_K) = |d_K|^{s/2} \gamma(s, \rho_K) L(s, \rho_K)
\]

satisfies the functional equation \( \Lambda(s, \rho_K) = \Lambda(1 - s, \rho_K) \), where the gamma factor is given by

\[
\gamma(s, \rho_K) = \begin{cases} 
\pi^{-s} \Gamma \left( \frac{s}{2} \right)^2 & \text{if } K \in L_3^+(X), \\
\pi^{-s} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{s+1}{2} \right) & \text{if } K \in L_3^-(X).
\end{cases}
\]

Note that \( \gamma(s, \rho_K) \) are common over the set \( L_3^+(X) \) or \( L_3^-(X) \). Then, we obtain the following zero density estimate for \( L(s, \rho_K) \).
Lemma 3.1. Let $X \geq 1$ and $T \geq 2$. For any $C_0 > 6$, we have
\[
\sum_{K \in L_3^\pm(X)} N(\sigma, T; \rho_K) \ll T^A X^{C_0(1-\sigma)/(2\sigma-1)}
\]
for any $\sigma \geq 3/4$, where $A > 0$ is an absolute constant. The implied constant depends only on the choice of $C_0$.

Proof. Let $S^\pm(X)$ be the set of all cuspidal representations $\pi$ of $GL_2(A_\mathbb{Q})$ such that $L(s, \pi) = L(s, \rho_K)$ holds for some $K \in L_3^\pm(X)$. Note that every $\pi \in S^\pm(X)$ satisfies the Ramanujan–Petersson conjecture. Next, the conductor of $\pi \in S^\pm(X)$ satisfies $\text{Cond}(\pi) \leq X$ since we have $\text{Cond}(\pi) = |d_K|$ if $L(s, \pi) = L(s, \rho_K)$. Furthermore, we obtain $\#S^\pm(X) \ll X$, which is deduced from the fact that there exists a one-to-one correspondence between $S^\pm(X)$ and $L_3^\pm(X)$. Finally, the gamma factors in the functional equations of $L(s, \pi)$ are common in $\pi \in S^\pm(X)$. From the above, the desired estimate follows directly if we apply the zero density estimate of Kowalski–Michel [27, Theorem 2] to the family $S^\pm(X)$. \hfill \Box

We check that Assumption [22] holds for any $7/8 < \sigma_1 < 1$. It is sufficient to consider the case where $\sigma_1 \leq \sigma \leq 1$ since estimate (2.5) is trivial for $\sigma > 1$. Then, taking $C_0 = 6 + \delta$ with $\delta = (8\sigma_1 - 7)/2 > 0$, we obtain
\[
\frac{C_0(1-\sigma)}{2\sigma - 1} = 1 - \frac{(8 + \delta)\sigma - (7 + \delta)}{2\sigma - 1} \leq 1 - \delta.
\]
Hence Lemma 3.1 ensures the validity of Assumption 2.2.

We often consider the logarithms of $L(s, \rho_K)$ for non-real variables $s$. The branch of $\log L(s, \rho_K)$ is determined as follows. First, we define
\[
\log L(s, \rho_K) = \sum_p \sum_{m=1}^{\infty} \frac{\text{tr}(A_p(K)m)}{m p^{-ms}} = \sum_{n=1}^{\infty} \Lambda_K(n) n^{-s}
\]
for $\text{Re}(s) > 1$ from Euler product (3.1). Here, the coefficient $\Lambda_K(n)$ is calculated as $\Lambda_K(p^m) = \text{tr}(A_p(K)^m) \log p$ and $\Lambda_K(n) = 0$ unless $n$ is a prime power. Note that we have $|\Lambda_K(n)| \leq 2\Lambda(n)$ with the usual von Mangoldt function $\Lambda(n)$. Let $D$ denote the right-half plane $\{\sigma + i\tau \mid 1/2 < \sigma \leq \text{Re}(\rho)\}$. We define
\[
G_K = D \setminus \bigcup_{\text{Re}(\rho)>1/2} \{\sigma + i\text{Im}(\rho) \mid 1/2 < \sigma \leq \text{Re}(\rho)\},
\]
where $\rho$ runs through all possible zeros of $L(s, \rho_K)$ with $\text{Re}(\rho) > 1/2$. Then we extend $\log L(s, \rho_K)$ for $s \in G_K$ by the analytic continuation along the horizontal path from right. The region $G_K$ is adequate to define $\log L(s, \rho_K)$, and we note that the subset $A_\sigma(X)$ defined as (2.9) is also expressed as
\[
A_\sigma(X) = \{K \in L_3^\pm(X) \mid \sigma \notin G_K\}.
\]
We can define $(L'/L)(s, \rho_K)$ for $s \in G_K$ since there exist no zeros of $L(s, \rho_K)$ in $G_K$. For $\text{Re}(s) > 1$, we obtain the Dirichlet series expression
\[
\frac{L'}{L}(s, \rho_K) = -\sum_{n=1}^{\infty} \Lambda_K(n) n^{-s}.
\]
From now on, we prove several upper bounds for $\log L(s, \rho_K)$ and $(L'/L)(s, \rho_K)$.
Lemma 3.2. Assume GRH, and let \( \sigma_0 > 1/2 \) be a real number. We take \( K \in L_0^1(X) \) arbitrarily. Let \( s = \sigma + it \) be a complex number with \( \sigma \geq \sigma_0 \) and \( |t| \leq (\log X)^2 \). Then we have

\[
\log L(s, \rho_K) \ll \frac{(\log X)^{2-2\sigma}}{\log \log X} + \log \log X,
\]

\[
\frac{L'}{L}(s, \rho_K) \ll (\log X)^{2-2\sigma} + \log \log X,
\]

where the implied constants depend only on the choice of \( \sigma_0 \).

Proof. Let \( q(s, \rho_K) = |d_K|(|s| + 3)(|s + 1| + 3) \). Assuming GRH, we have

\[
\log L(s, \rho_K) \ll \frac{\{\log q(s, \rho_K)\}^{2-2\sigma}}{(2\sigma - 1) \log \log q(s, \rho_K)} + \log \log q(s, \rho_K),
\]

\[
\frac{L'}{L}(s, \rho_K) \ll \frac{\{\log q(s, \rho_K)\}^{2-2\sigma}}{2\sigma - 1} + \log \log q(s, \rho_K)
\]

for any \( s = \sigma + it \) with \( 1/2 < \sigma \leq 5/4 \) by [25, Theorems 5.17 and 5.19]. Then the results for \( \sigma_0 \leq \sigma \leq 5/4 \) are deduced from these estimates. The cases \( \sigma \geq 5/4 \) are trivial. Indeed, we obtain the inequalities \( |\log L(s, \rho_K)| \leq 2 \log \zeta(5/4) \) and \( |(L'/L)(s, \rho_K)| \leq 2((\zeta'/\zeta)(5/4)| from |\Lambda_K(n)| \leq 2\Lambda(n) \). \( \square \)

Lemma 3.3. Let \( \sigma_1 \) be an absolute constant for which Assumption [22] holds. We take a cubic field \( K \) belonging to \( L_0^1(X)\setminus B_{\sigma_1}(X) \), where the subset \( B_{\sigma_1}(X) \) is obtained from (2.10). Let \( s = \sigma + it \) be a complex number with \( \sigma \geq \sigma_1 + 2(\log \log X)^{-1} \) and \( |t| \leq (\log X)^2 \). Then we have

\[
\log L(s, \rho_K) \ll (\log X)(\log X)^{(1-\sigma)/(1-\sigma_1)} + \log \log X,
\]

\[
\frac{L'}{L}(s, \rho_K) \ll (\log X)(\log X)^{(1-\sigma)/(1-\sigma_1)} + \log \log X,
\]

where the implied constants are absolute.

Proof. The proof is based on the method of Barban [4, Lemma 3]. For simplicity, we write \( \kappa = (\log \log X)^{-1} \). If \( \sigma \geq 1 + \kappa/2 \), then we obtain \( \log L(s, \rho_K) \ll \log \log X \) and \( (L'/L)(s, \rho_K) \ll \log \log X \) by Dirichlet series expressions (3.2) and (3.3). Hence the results follow in this case, and we let \( \sigma_1 + 2\kappa \leq \sigma \leq 1 + \kappa/2 \) bellow. Let \( z_0 = \kappa^{-1} + \kappa + it \) and \( R = \kappa^{-1} + \kappa - \sigma_1 \). Then the function \( (L'/L)(z, \rho_K) \) is holomorphic on \( |z - z_0| < R \) since \( L(z, \rho_K) \) has no zeros in this disk. We define

\[
M(r) = \max_{|z - z_0| = r} \left| \frac{L'}{L}(z, \rho_K) \right|
\]

for \( 0 < r < R \). Let \( r_1 = \kappa^{-1} - 1 \), \( r_2 = \kappa^{-1} + \kappa - \sigma \), and \( r_3 = \kappa^{-1} - \sigma_1 \). Then we have \( 0 < r_1 < r_2 < r_3 < R \). As a consequence of the Hadamard three circles theorem, we obtain the inequality

\[
(3.4) \quad \left| \frac{L'}{L}(s, \rho_K) \right| \leq M(r_1)^{1-a}M(r_3)^a,
\]

where

\[
(3.5) \quad a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} = \frac{1 - \sigma}{1 - \sigma_1} + O\left( \frac{1}{\log \log X} \right).
\]
We then evaluate \( M(r_1) \). Since we have \( \text{Re}(z) \geq 1 + \kappa \) on the circle \(|z - z_0| = r_1\), the estimate \((L'/L)(s, \rho_K) \ll \log \log X\) holds. Therefore we obtain

\[
M(r_1) \leq A \log \log X,
\]

where \( A \geq 1 \) is an absolute constant. Next, we let \( z = x + iy \) be a complex number on the circle \(|z - z_0| = r_3\). By \([25, \text{Proposition}\ 5.7\ (2)]\), we have

\[
\frac{L'}{L}(z, \rho_K) = \sum_{|z - \rho| < 1} \frac{1}{z - \rho} + O\left(\log \{ |d_K| (|y| + 3) \}\right)
\]

with an absolute implied constant, where \( \rho \) runs through zeros of \( L(s, \rho_K) \). For \(|z - z_0| = r_3\), the distance between \( z \) and \( \rho \) is at least \( \kappa \) since \( K \notin \mathcal{B}_\sigma(X) \). We apply the result \([25, \text{Proposition}\ 5.7\ (1)]\) to estimate the number of zeros satisfying \(|z - \rho| < 1\). Then the estimate \((L'/L)(z, \rho_K) \ll \kappa^{-1} \log X\) follows. Therefore we obtain

\[
M(r_3) \leq B \log \log X \log X,
\]

where \( B \geq 1 \) is an absolute constant. Inserting (3.6) and (3.7) to (3.4), we obtain

\[
\left| \frac{L'}{L}(s, \rho_K) \right| \leq A^{1-a} B^a (\log \log X)(\log X)^a.
\]

Note that the inequality \( 0 < a < 1 \) holds since \( r_1 < r_2 < r_3 \). Thus we have absolutely \( A^{1-a} B^a \ll 1 \). Then the desired estimate for \((L'/L)(s, \rho_K)\) follows from (3.5). The result for \( \log L(s, \rho_K) \) is proved as follows. Let \( s_0 = 1 + \kappa/2 + it \). Note that the relation

\[
\log L(s, \rho_K) = \log L(s_0, \rho_K) - \int_\sigma^{1+\kappa/2} \frac{L'}{L}(x + it, \rho_K) \, dx
\]

holds by the definition of \( \log L(s, \rho_K) \). We have \( \log L(s_0, \rho_K) \ll \log \log \log X \), and it is deduced from the estimate on \((L'/L)(s, \rho_K)\) that

\[
\int_\sigma^{1+\kappa/2} \frac{L'}{L}(x + it, \rho_K) \, dx \ll (\log \log X)(\log X)^{(1-\sigma)/(1-\sigma_1)}.
\]

Hence the proof is completed. \( \square \)

**Lemma 3.4.** Take a cubic field \( K \in L_3^+(X) \) arbitrarily. Then we have

\[
\log L(1, \rho_K) \ll \log \log X,
\]

where the implied constant is absolute.

**Proof.** Explicit upper and lower bounds for the value \( L(1, \rho_K) = \text{Res}_{s=1} \zeta_K(s) \) were obtained by Louboutin \([31, 32]\). These results give the inequality

\[
\frac{c_K \alpha_K \exp(-c_K \alpha_K/2)}{2 \log |d_K|} \leq L(1, \rho_K) \leq \left( \frac{e}{4 \log |d_K|} \right)^2,
\]

where \( c_K = 2(\sqrt{2} - 1)^2 \) and \( \alpha_K = \log |d_K|/\log |d_K'| \). We know that \( 1/4 \leq \alpha_K \leq 1/2 \) holds. Hence we conclude \(- \log \log X - 4 \leq \log L(1, \rho_K) \leq 2 \log \log X + 2 \). \( \square \)
3.2. Some power series and their coefficients. Let \( F(w; a) \) and \( F^*(w; a) \) be the functions given by (2.6) and (2.17). For any \( z \in \mathbb{C} \), the functions \( \exp(zF(w; a)) \) and \( \exp(zF^*(w; a)) \) are holomorphic for \( |w| < 1 \). Then, we denote by \( H_r(z; a) \) and \( G_r(z; a) \) the \( r \)-th coefficients of the power series expressions of \( \exp(zF(w; a)) \) and \( \exp(zF^*(w; a)) \) at \( w = 0 \), that is, we let

\[
(3.8) \quad \exp(zF(w; a)) = \sum_{r=0}^{\infty} H_r(z; a)w^r \quad \text{and} \quad \exp(zF^*(w; a)) = \sum_{r=0}^{\infty} G_r(z; a)w^r
\]

for \( z, w \in \mathbb{C} \) with \( |w| < 1 \). The coefficients \( H_r(z; a) \) and \( G_r(z; a) \) are related to \( H_r(z) \) and \( G_r(z) \) introduced by Ihara–Matsumoto [23, Section 1.2], which are determined as follows. Put

\[
F(w) = \log(1 - w)^{-1} = \sum_{m=1}^{\infty} \frac{w^m}{m} \quad \text{and} \quad F^*(w) = \frac{w}{1 - w} = \sum_{m=1}^{\infty} w^m
\]

for \( |w| < 1 \), where \( \log \) denotes the principal branch of logarithm. Then we let

\[
\exp(zF(w)) = \sum_{r=0}^{\infty} H_r(z)w^r \quad \text{and} \quad \exp(zF^*(w)) = \sum_{r=0}^{\infty} G_r(z)w^r.
\]

By definition, \( H_r(z) \) and \( G_r(z) \) are calculated as \( H_0(z) = G_0(z) = 1 \), and for \( r \geq 1 \),

\[
H_r(z) = \frac{1}{r!} (z + 1) \cdots (z + r - 1) \quad \text{and} \quad G_r(z) = \sum_{k=1}^{r} \binom{r-1}{k-1} z^k.
\]

Hence we find \( |H_r(z)| \leq H_r(|z|) \) for any \( z \in \mathbb{C} \), and \( H_r(a) \leq H_r(b) \) for any \( a, b \in \mathbb{R} \) with \( 0 \leq a \leq b \). Similar inequalities are valid for \( G_r(z) \).

**Lemma 3.5.** Let \( a \in \mathbb{A} \). Then we have

\[
|H_r(z; a)| \leq H_r(2|z|) \quad \text{and} \quad |G_r(z; a)| \leq G_r(2|z|)
\]

for any \( r \geq 0 \) and \( z \in \mathbb{C} \).

**Proof.** Let \( \alpha_a \) and \( \beta_a \) be the eigenvalues of the matrix \( A_a \). Then \( F(w; a) \) is equal to \( F(\alpha_a w) + F(\beta_a w) \) by (2.6), and the coefficient \( H_r(z; a) \) is calculated as

\[
H_r(z; a) = \sum_{k=0}^{r} H_k(z)H_{r-k}(z)\alpha_a^k\beta_a^{-k}.
\]

We have \( |\alpha_a| \leq 1 \) and \( |\beta_a| \leq 1 \) by (2.11). Hence we obtain

\[
|H_r(z; a)| \leq \sum_{k=0}^{r} H_k(|z|)H_{r-k}(|z|) = H_r(2|z|)
\]

as desired. The inequality for \( G_r(z; a) \) is proved similarly. \( \square \)

We truncate the power series (3.8) as

\[
(3.9) \quad \exp(zF(w; a)) = \sum_{r=N}^{\infty} H_r(z; a)w^r + E_N,
\]

\[
\exp(zF^*(w; a)) = \sum_{r=N}^{\infty} G_r(z; a)w^r + E_N^*.
\]

Then we deduce \( |E_N|, |E_N^*| \leq 2(4|z| + 2)^N|w|^N \) for any \( |w| \leq (8|z| + 4)^{-1} \) from Lemma 3.5 together with the inequalities \( H_r(|z|), G_r(|z|) \leq 2r(|z| + 1)^r \).
Let $z$ be a complex number. The $z$-th divisor function $d_z(n)$ is defined as the multiplicative function satisfying

$$d_z(p^r) = H_r(z).$$

Then, as usual, $d_k(n)$ is equal to the number of representation of $n$ as the product of $k$ natural numbers if $k > 0$ is an integer. We also define $d^*_z(n)$ by extending

$$d^*_z(p^r) = G_r(z \log p)$$

multiplicatively. For $\Re(s) > 1$, these multiplicative functions satisfy

$$\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta(s)^k$$

and

$$\sum_{n=1}^{\infty} d^*_k(n)n^{-s} = \exp(-\frac{1}{s}).$$

Lemma 3.6. Let $\sigma_0$ be a real number. Let $k \geq 1$ be an integer and $Y \geq 3$. Then there exists absolute constants $C_1, C_2 > 0$ such that

$$\sum_{n=1}^{\infty} d_k(n)n^{-\sigma}e^{-n/Y} \ll (Y^{1-\sigma} + 1)(C_1 \log Y)^{k+1},$$

$$\sum_{n=1}^{\infty} d^*_k(n)n^{-\sigma}e^{-n/Y} \ll (Y^{1-\sigma} + 1)(C_2 \log Y)^{k+1} \exp\left(\frac{\log Y}{\log \log Y}\right)$$

for any $\sigma \geq \sigma_0$, where the implied constants depend only on the choice of $\sigma_0$.

Proof. If $\sigma \geq 1 + (2 \log Y)^{-1}$, we obtain

$$\sum_{n=1}^{\infty} d_k(n)n^{-\sigma}e^{-n/Y} \leq \zeta(1 + (2 \log Y)^{-1})^k \leq (C_1 \log Y)^k$$

by Dirichlet series expression (3.10). Hence the former estimate follows in this case. Then, as usual, estimated as $\Gamma(w)$.

Then, we let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

for any $c > 0$. Taking $c = 1 - \sigma + (\log Y)^{-1}$, we have

$$\zeta(s + w)^kY^w \ll (1 + (\log Y)^{-1})^kY^{1-\sigma} \leq (C_1 \log Y)^kY^{1-\sigma}$$

on $\Re(w) = c$. Furthermore, we recall that the estimate

$$\Gamma(w) \ll |v|^{u-1/2} \exp\left(-\frac{\pi}{2}|v|\right)$$

holds for any $w = u + iv$ with $|v| \geq 1$. Then the integral of $\Gamma(w)$ on $\Re(w) = c$ is estimated as

$$\int_{\Re(w)=c} \Gamma(w) \, dw \ll \int_{1}^{\infty} v^{2-\sigma_0}e^{-v} \, dv + \max_{|v| \leq 1} |\Gamma(c + iv)| \ll_{\sigma_0} 1 + \max_{w \in R} |\Gamma(w)|,$$

where $R$ is the rectangle $\{u + iv \mid (2 \log Y)^{-1} \leq u \leq 2 - \sigma_0, |v| \leq 1\}$. Applying the formula $\Gamma(w) = \Gamma(w + 1)/w$, we obtain $\max_{w \in R} |\Gamma(w)| \ll_{\sigma_0} \log Y$. Hence we have

$$\int_{\Re(w)=c} \Gamma(w) \, dw \ll_{\sigma_0} \log Y.$$
Therefore, the former estimate again follows from (3.11), (3.12), and (3.14). The latter estimate is proved in a similar way. First, we see that the result is trivial if $\sigma \geq 1 + (2 \log \log Y)^{-1}$. For the case of $\sigma_0 \leq \sigma \leq 1 + (2 \log \log Y)^{-1}$, we use the formula

$$\sum_{n=1}^{\infty} d_n^\sigma(n)n^{-\sigma}e^{-n/Y} = \frac{1}{2\pi i} \int_{\text{Re}(w)=c} \exp \left(-k\sum_{i}^\sigma (\sigma + w)\right) \Gamma(w)Y^w \, dw$$

instead of (3.11), where we take $c = 1 - \sigma + (\log \log Y)^{-1}$ in this case. \hfill \Box

3.3. Results from Probability Theory. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $\mathcal{B}(S)$ be the class of the Borel sets of a topological space $S$. Then, for an $S$-valued random variable $X$ on $\Omega$, we denote the distribution of $X$ by

$$\mathbb{P}(X \in A) = \mu(X^{-1}(A)), \quad A \in \mathcal{B}(S),$$

which is a probability measure on $(S, \mathcal{B}(S))$. If $X$ is an $\mathbb{R}$-valued random variable, we further denote by $E[X]$ the expected value, and by $\text{Var}[X]$ the variance of $X$. Let $(X_n)_n$ be a sequence of independent $\mathbb{R}$-valued random variables. It is known that the convergences of two series $\sum_n E[X_n]$ and $\sum_n \text{Var}[X_n]$ imply that $\sum_n X_n$ converges almost surely; see Billingsley [6, Theorem 22.6] for example.

Let $(\phi_n)_n$ be a sequence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The convolution $\phi_i * \phi_j$ is the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined as

$$(\phi_i * \phi_j)(A) = \int_{\mathbb{R}} \phi_i(A-x) \, d\phi_j(x),$$

where we write $A - x = \{ a - x \mid a \in A \}$. Then we obtain $\phi_i * \phi_j = \phi_j * \phi_i$ and $\phi_i * (\phi_j * \phi_k) = (\phi_i * \phi_j) * \phi_k$ by definition. Put $\psi_n = \phi_1 * \cdots * \phi_n$. The convergence of the measures $\psi_n$ as $n \to \infty$ was studied by Jessen–Wintner [26]. Here, we say that $\psi_n$ converges weakly to a probability measure $\psi$ as $n \to \infty$ if we have $\psi_n(A) \to \psi(A)$ for any $A \in \mathcal{B}(\mathbb{R})$ with $\psi(\partial A) = 0$. The convergence is called absolute if it converges in any order of terms of the convolution.

**Lemma 3.7** (Jessen–Wintner [26]). Let $(\phi_n)_n$ be a sequence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Put

$$c(\phi_j) = \int_{\mathbb{R}} x \, d\phi_j(x) \quad \text{and} \quad M(\phi_j) = \int_{\mathbb{R}} x^2 \, d\phi_j(x),$$

and assume $M(\phi_j) < \infty$ for every $j \in \mathbb{N}$. If $\sum_n |c(\phi_n)|$ and $\sum_n M(\phi_n)$ converge, then $\psi_n = \phi_1 * \cdots * \phi_n$ converges weakly to a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $n \to \infty$. Furthermore, the convergence is absolute.

The support of a probability measure $\phi$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is defined as the set

$$\text{supp}(\phi) = \{ x \in \mathbb{R} \mid \phi(A) > 0 \text{ for any } A \in \mathcal{B}(\mathbb{R}) \text{ with } x \in A^i \},$$

where $A^i$ denotes the interior of $A$. We have $\text{supp}(\phi_1 * \phi_2) = \text{supp}(\phi_1) + \text{supp}(\phi_2)$ by definition. Moreover, Jessen and Wintner proved that, if $\phi_1 * \cdots * \phi_n \to \psi$ as $n \to \infty$, then the equality

$$(3.15) \quad \text{supp}(\psi) = \text{supp}(\phi_1) + \text{supp}(\phi_2) + \cdots$$

holds. Here we denote by $A_1 + A_2 + \cdots$ the set of all points $a$ such that $a$ has at least one representation $a = a_1 + a_2 + \cdots$ for some sequence $(a_n)_n$ with $a_n \in A_n$ for each $n$. 

We say that a probability measure $\phi$ on $(\mathbb{R}, B(\mathbb{R}))$ is absolutely continuous if there exists a non-negative integrable function $D$ on $\mathbb{R}$ such that

$$\phi(A) = \int_A D(x) \frac{dx}{\sqrt{2\pi}}$$

for any $A \in B(\mathbb{R})$. The function $D$ is uniquely determined up to the values on a set of Lebesgue measure zero, and is called the density of $\phi$. The absolutely continuity of $\phi$ is associated with the property of the characteristic function $\Lambda(\xi; \phi) = \int_{\mathbb{R}} e^{ix\xi} d\phi(x)$.

Moreover, the density of $\phi$ is determined from $\Lambda(\xi; \phi)$ by Lévy’s inversion formula as follows.

**Lemma 3.8** (Lévy’s inversion formula). Let $\phi$ be a probability measure on $(\mathbb{R}, B(\mathbb{R}))$. Suppose that the characteristic function $\Lambda(\xi; \phi)$ satisfies

$$\int_{\mathbb{R}} |\xi|^k |\Lambda(\xi; \phi)| d\xi < \infty$$

for an integer $k \geq 0$. Then $\phi$ is absolutely continuous with a density given by

$$D(x) = \int_{\mathbb{R}} \Lambda(\xi; \phi) e^{-ix\xi} \frac{d\xi}{\sqrt{2\pi}}$$

(3.16)

Furthermore, the function $D$ belongs to the class $C^k(\mathbb{R})$.

**Proof.** See Billingsley [6, Theorem 26.2] for the case $k = 0$. We obtain the result for $k > 0$ by differentiating under the integral in (3.16). \Box

For a probability measure $\phi$ on $(\mathbb{R}, B(\mathbb{R}))$, the distribution function is defined as

$$F_\phi(t) = \phi((-\infty, t]),$$

which is a non-decreasing continuous function on $\mathbb{R}$ satisfying $F_\phi(t) \to 0$ as $t \to -\infty$ and $F_\phi(t) \to 1$ as $t \to \infty$. The distance between two distribution functions $F_\phi$ and $F_\psi$ is estimated by the following Berry–Esseen inequality.

**Lemma 3.9** (Berry–Esseen inequality). Let $\phi$ and $\psi$ be probability measures on $(\mathbb{R}, B(\mathbb{R}))$. Suppose that $\psi$ is absolutely continuous with a continuous density $D$. Then we have

$$\sup_{t \in \mathbb{R}} |F_\phi(t) - F_\psi(t)| \ll \frac{1}{R} \sup_{x \in \mathbb{R}} D(x) + \int_{-R}^R \left| \frac{\Lambda(\xi; \phi) - \Lambda(\xi; \psi)}{\xi} \right| d\xi$$

for any $R > 0$, where the implied constant is absolute.

**Proof.** Since $D$ is a the density of $\psi$, we have

$$F_\psi(t) = \int_{-\infty}^t D(x) \frac{dx}{\sqrt{2\pi}}$$

for $t \in \mathbb{R}$. Hence $F_\psi'(x) = D(x)$ holds, and thus $F_\psi'(x)$ is bounded on $\mathbb{R}$. Therefore, we obtain the conclusion; see Tenenbaum [40, Theorem 7.16] for example. \Box
3.4. Other preliminary lemmas.

**Lemma 3.10** (Mishou–Nagoshi [34]). Let $H$ be a Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$. Let $(u_n)_n$ be a sequence in $H$ satisfying

(a) $\sum_n \|u_n\|^2 < \infty$
(b) $\sum_n |(u_n, u)| = \infty$ for any $u \in H$ with $\|u\| = 1$.

Then, for any $v \in H$, $k \geq 1$, and $\epsilon > 0$, there exist an integer $N = N(v, k, \epsilon) \geq k$ and numbers $c_k, \ldots, c_N \in \{1, -1\}$ such that

$$\left\| v - \sum_{n=k}^N c_n u_n \right\| < \epsilon.$$ 

Let $\Lambda(\mathbb{R})$ be the subspace of $L^1(\mathbb{R})$ defined as

$$\Lambda(\mathbb{R}) = \{ \Phi \in L^1(\mathbb{R}) \mid \Phi \text{ is continuous and } \tilde{\Phi} \text{ belongs to } L^1(\mathbb{R}) \}.$$

We say that a function $M$ is a good density function on $\mathbb{R}$ if it is a non-negative real valued function in $\Lambda(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} M(x) \frac{dx}{\sqrt{2\pi}} = 1.$$ 

Let $(X_n)_n$ be a sequence of finite sets with probability measures $\omega_n$. We take a function $\ell_n : X_n \to \mathbb{R}$ for each $n \in \mathbb{R}$. Let $M$ be a good density function on $\mathbb{R}$.

Then, for a test function $\Phi$ on $\mathbb{R}$, we consider the condition

$$\lim_{n \to \infty} \sum_{\chi \in X_n} \omega_n \Phi(\ell_n(\chi)) = \int_{\mathbb{R}} \Phi(x) M(x) \frac{dx}{\sqrt{2\pi}}.$$ 

**Lemma 3.11** (Ihara–Matsumoto [23]). With the above notation, the following result holds. Suppose that condition (3.17) holds with $\Phi(x) = e^{ix\xi}$ for any $\xi \in \mathbb{R}$, and that the convergence is uniform in $|\xi| \leq R$ for any fixed real number $R > 0$. Then

(a) (3.17) holds for any bounded continuous function on $\mathbb{R}$.
(b) (3.17) holds for any $\Phi \in C(\mathbb{R})$ with $|\Phi(x)| \leq \phi_0(|x|)$, where $\phi_0(r)$ is a continuous non-decreasing function on $[0, \infty)$ which satisfies $\phi_0(r) > 0$, $\phi_0(r) \to \infty$ as $r \to \infty$, and

$$\sum_{\chi \in X_n} \omega_n \phi_0(|\ell_n(\chi)|)^2 \ll 1,$$

$$\int_{\mathbb{R}} \phi_0(|x|) M(x) \frac{dx}{\sqrt{2\pi}} < \infty.$$ 

(c) (3.17) holds for the indicator function of either a compact subset of $\mathbb{R}$ or the complement of such a subset.

4. The density functions

4.1. Certain Euler products. Let $p$ be a prime number, and let $s, z \in \mathbb{C}$ with $\text{Re}(s) > 0$. We consider the functions $F_{s,p}(z)$ and $G_{s,p}(z)$ defined by (2.7) and (2.8). We begin with studying the convergences of the following infinite products.
Proposition 4.1. (i) Let $\sigma_0 > 1/2$ and $R > 0$. Then the infinite product

\[
F_s(z) = \prod_p F_{s,p}(z)
\]

converges uniformly and absolutely for $\text{Re}(s) \geq \sigma_0$ and $|z| \leq R$.

(ii) Let $\sigma_0 > 2/3$ and $R > 0$. Then the infinite product

\[
G_s(z) = \prod_p G_{s,p}(z)
\]

converges uniformly and absolutely for $\text{Re}(s) \geq \sigma_0$ and $|z| \leq R$.

Proof. Let $\text{Re}(s) \geq \sigma_0$ and $|z| \leq R$, and define $P_1 = P_1(\sigma_0, R) = (8R + 4)^{1/\sigma_0}$. Note that we have $|p^{-s}| \leq (8|z| + 4)^{-1}$ for all $p \geq P_1$. Then we apply formula (2.9) with $N = 2$, which gives

\[
\exp \left( z F(p^{-s}; a) \right) = \sum_{r<2} H_r(z; a)p^{-rs} + O \left( p^{-2\sigma_0} \right)
\]

with the implied constant depending only on $R$. The coefficients $H_r(z; a)$ for $r = 0, 1$ are calculated as $H_0(z; a) = 1$ and $H_1(z; a) = \text{tr}(A_a)$. Hence, inserting them to (2.7) and (2.8), we obtain

\[
F_{s,p}(z) = 1 + O(p^{-\sigma_0-1} + p^{-2\sigma_0}) \quad \text{and} \quad G_{s,p}(z) = 1 + O(p^{-\sigma_0-1/3} + p^{-2\sigma_0})
\]

for $p \geq P_1$. If we have $\sigma_0 > 1/2$, then the series $\sum_p p^{-\sigma_0-1}$ and $\sum_p p^{-2\sigma_0}$ converge. Hence the result of (i) follows. The series $\sum_p p^{-\sigma_0-1/3}$ also converges if we further let $\sigma_0 > 2/3$. Therefore we obtain the result of (ii). \qed

By definition, we find that $F_{s,p}(z)$ and $G_{s,p}(z)$ are holomorphic functions of $s$ for $\text{Re}(s) > 0$ if $z$ is fixed, and they are also holomorphic functions of $z$ if $s$ is fixed with $\text{Re}(s) > 0$. Hence Proposition 4.1 yields the following corollaries.

Corollary 4.2. Let $z$ be a fixed complex number. Then $F_s(z)$ defined by (4.1) is a holomorphic function of $s$ for $\text{Re}(s) > 1/2$. Furthermore, $G_s(z)$ defined by (4.2) is a holomorphic function of $s$ for $\text{Re}(s) > 2/3$.

Corollary 4.3. Let $s$ be a fixed complex number with $\text{Re}(s) > 1/2$. Then $F_s(z)$ defined by (4.1) is a holomorphic function of $z$. Furthermore, $G_s(z)$ defined by (4.2) is a holomorphic function of $z$ if $\text{Re}(s) > 2/3$.

Calculating the local parts $F_{s,p}(z)$ and $G_{s,p}(z)$ more precisely, we obtain several upper bounds for the functions $F_s(z)$ and $G_s(z)$ that we use later.

Proposition 4.4. (i) Let $s = \sigma + it$ and $z$ be complex numbers with $1/2 < \sigma < 1$. Then there exists an absolute constant $c_1 > 0$ such that

\[
|F_s(z)| \leq \exp \left( \frac{c_1}{2(2\sigma - 1)(1-\sigma)} \log(|z|+3)^{1/\sigma} \right).
\]

(ii) Let $s = \sigma + it$ and $z$ be complex numbers with $2/3 < \sigma < 1$. Then there exists an absolute constant $c_2 > 0$ such that

\[
|G_s(z)| \leq \exp \left( \frac{c_2}{3(3\sigma - 2)(1-\sigma)} \log(|z|+3)^{1/\sigma} \right).
\]
Proof. Let $P_1 = P_1(\sigma, |z|) = (8|z| + 4)^{1/\sigma}$. We first consider the contributions of local parts $F_{s,p}(z)$ for $p \geq P_1$. Applying (3.9) again, we deduce

$$\exp \left( zF(p^{-s}; a) \right) = 1 + z \text{tr}(a)p^{-s} + E_a,$$

where $E_a$ is estimated as $|E_a| \leq 8(2|z| + 1)^2p^{-2\sigma}$. Therefore $F_{s,p}(z)$ is calculated as

$$F_a(z) = 1 + \mu + E$$

by (2.7), where we have

$$\mu = \frac{z}{1 + p^{-1} + p^{-2s-1}} \quad \text{and} \quad |E| \leq 8(2|z| + 1)^2p^{-2\sigma}.$$ 

Note that the inequalities $|\mu| \leq 1/8$ and $|E| \leq 1/2$ hold since $p \geq P_1$ implies $(2|z| + 1)p^{-\sigma} \leq 1/4$. Then, we apply the estimate $\log(1 + w) \ll \epsilon |w|$ uniformly for $|w| \leq 1 - \epsilon$ to deduce

$$\log F_{s,p}(z) \ll |z|p^{-\sigma-1} + (|z| + 1)^2p^{-2\sigma}.$$ 

By the prime number theorem, we obtain that the estimate

$$\sum_{p \leq y} p^{-\gamma} \ll \frac{1}{\gamma - 1 \log y}$$

holds for any $1 < \gamma < 2$ with an absolute implied constant. Then we use (4.3) with $\gamma = \sigma + 1$ and $2\sigma$. It is deduced that

$$\prod_{p \geq P_1} F_{s,p}(z) \leq \exp \left( \frac{c_{1,1} (|z| + 3)^{1/\sigma}}{2\sigma - 1 \log(|z| + 3)} \right)$$

with an absolute constant $c_{1,1} > 0$. The contributions of $F_{s,p}(z)$ for $p < P_1$ are estimated as follows. For all prime numbers $p$, we have

$$F(p^{-s}; a) = \log(1 - \alpha_a p^{-s})^{-1} + \log(1 - \beta_a p^{-s})^{-1} \ll p^{-\sigma},$$

where $\alpha_a$ and $\beta_a$ are the eigenvalues of $A_a$. Then, we again apply the prime number theorem to obtain

$$\sum_{p \leq y} p^{-\delta} \ll \frac{1}{1 - \delta \log y}$$

for $0 < \delta < 1$. This yields

$$\prod_{p < P_1} F_{s,p}(z) \leq \exp \left( \frac{c_{1,2} (|z| + 3)^{1/\sigma}}{1 - \sigma \log(|z| + 3)} \right),$$

where $c_{1,2} > 0$ is an absolute constant. By (4.4) and (4.6), we obtain the upper bound for $|F_a(z)|$. In order to prove the result for $G_s(z)$, we see that

$$\log G_{s,p}(z) \ll |z|p^{-\sigma-1/3} + (|z| + 1)^2p^{-2\sigma}$$

holds for $p \geq P_1$. Then, we use estimate (4.3) with $\gamma = \sigma + 1/3$ and $2\sigma$ to obtain

$$\prod_{p \geq P_1} G_{s,p}(z) \leq \exp \left( \frac{c_{2,1} (|z| + 3)^{1/\sigma}}{3\sigma - 2 \log(|z| + 3)} \right)$$
for \( \sigma > 2/3 \) with an absolute constant \( c_{2,1} > 0 \). The contributions of \( G_{s,p}(z) \) for \( p < P_1 \) are estimated by applying \( \text{[4.5]} \) again, and we obtain the conclusion. \( \square \)

**Proposition 4.5.** (i) Let \( \sigma > 1/2 \) be a real number. There exists a positive constant \( c_1(\sigma) \) depending only on \( \sigma \) such that

\[
|F_{\sigma}(i\xi)| \leq \exp \left( -c_1(\sigma) \frac{|\xi|^{1/\sigma}}{\log |\xi|} \right)
\]

for any \( \xi \in \mathbb{R} \) with \( |\xi| \geq 3 \).

(ii) Let \( \sigma > 2/3 \) be a real number. There exists a positive constant \( c_2(\sigma) \) depending only on \( \sigma \) such that

\[
|G_{\sigma}(i\xi)| \leq \exp \left( -c_2(\sigma) \frac{|\xi|^{1/\sigma}}{\log |\xi|} \right)
\]

for any \( \xi \in \mathbb{R} \) with \( |\xi| \geq 3 \).

**Proof.** Let again \( P_1 = P_1(\sigma, |\xi|) = (8|\xi| + 4)^{1/\sigma} \). Then, for any \( p \geq P_1 \), we have

\[
\exp(\xi F(p^{-\sigma}; a)) = 1 + \xi \text{tr}(A) p^{-\sigma} + \xi \text{tr}(A^2) p^{-2\sigma} - \frac{\xi^2}{2} \text{tr}(A^2) p^{-2\sigma} + E_a
\]

by using \( \text{[3.9]} \) with \( N = 3 \) and by calculating the coefficient \( H_2(i\xi; a) \). Here the error term \( E_a \) is estimated as \( |E_a| \leq 16(2|\xi| + 1)^3 p^{-3\sigma} \). Inserting this asymptotic formula to \( \text{[2.7]} \), we obtain

\[
F_{\sigma}(i\xi) = 1 + i\mu_1 + i\mu_{2,1} - \frac{1}{2} \mu_{2,2} + E,
\]

where we put

\[
\mu_1 = \frac{\xi}{1 + p^{-1} + p^{-2} p^{-\sigma - 1}},
\]

\[
\mu_{2,1} = \frac{\xi(1 + p^{-1})}{1 + p^{-1} + p^{-2} p^{-2\sigma}},
\]

\[
\mu_{2,2} = \frac{\xi^2(1 + p^{-1})}{1 + p^{-1} + p^{-2} p^{-2\sigma}},
\]

and we have \( |E| \leq 16(2|\xi| + 1)^3 p^{-3\sigma} \). For \( p \geq P_1 \), we see that the inequalities \( |\mu_1| \leq 1/8, |\mu_{2,1}| \leq 1/16, |\mu_{2,2}| \leq 1/16, \) and \( |E| \leq 1/4 \) are valid. Hence we have

\[
\log F_{\sigma}(i\xi) = i\mu_1 + i\mu_{2,1} - \frac{1}{2} \mu_{2,2} + E + O \left( |\mu_1|^2 + |\mu_{2,1}|^2 + |\mu_{2,2}|^2 + |E|^2 \right),
\]

which gives

\[
\log |F_{\sigma}(i\xi)| = -\frac{\xi^2}{2} \frac{1 + p^{-1}}{1 + p^{-1} + p^{-2} p^{-2\sigma}} + O \left( \xi^2 p^{-2\sigma - 2} + \xi^3 p^{-3\sigma} \right)
\]

since \( \log |F_{\sigma}(i\xi)| = \text{Re} \log F_{\sigma}(i\xi) \). Thus, there exists an absolute constant \( A > 0 \) such that we have

\[
\log |F_{\sigma,p}(i\xi)| \leq -\frac{\xi^2}{4} p^{-2\sigma} + A\xi^2 p^{-2\sigma - 2} + A|\xi|^3 p^{-3\sigma}.
\]
Put $P_2(M) = P_2(M; σ, |ξ|) = (M|ξ|)^{1/σ}$. Then we have $P_2(M) ≥ (3M)^{1/σ}$ for $|ξ| ≥ 3$. Hence we can take a constant $M_σ ≥ 10$ depending only on $σ$ so that

$$P_2(M_σ)^{-2} ≤ \frac{1}{16A}$$

and $A|ξ|P_2(M_σ)^{-σ} ≤ \frac{1}{16A}$ are satisfied. Then we have $P_2(M_σ) ≥ P_1$, and the inequalities

$$Aξ^2p^{-2σ} ≤ \frac{ξ^2}{16p}p^{-2σ}$$

and $A|ξ|^3p^{-3σ} ≤ \frac{ξ^2}{16}p^{-2σ}$ are deduced if $p ≥ P_2(M_σ)$. Hence we have

$$\log |F_{σ,p}(iξ)| ≤ \frac{ξ^2}{8}p^{-2σ}.$$ 

for any $p ≥ P_2(M_σ)$. Applying the prime number theorem, we obtain

$$\sum_{p ≥ P_2(M_σ)} p^{-2σ} ≥ \frac{P_2(M_σ)^{1-2σ}}{\log P_2(M_σ)},$$

where the implied constant depends only on $σ$. Therefore, it follows that

$$|F_{σ,p}(iξ)| ≤ \exp \left( -c_1(σ) \frac{|ξ|^{1/σ} \log |ξ|}{\log |ξ|} \right).$$

On the other hand, we have $|\exp (iξF(p^{-σ}; ξ))| = 1$ since both $ξ$ and $F(p^{-σ}; ξ)$ are real. Hence the inequality $|F_{σ,p}(iξ)| ≤ 1$ holds for all prime numbers $p$. Therefore, the desired estimate for $F_{σ}(iξ)$ follows from (4.8). The result for $G_{σ}(iξ)$ is proved as follows. In this case, we obtain the formula

$$G_{σ}(iξ) = 1 + i\tilde{μ}_1 + i\tilde{μ}_{2,1} - \frac{1}{2}\tilde{μ}_{2,2} + \tilde{E}$$

for $p ≥ P_1$, where

$$\tilde{μ}_1 = ξK_p \left(p^{-1/3} + p^{-2/3} + p^{-1} + 2p^{-4/3} + p^{-5/3}\right)p^{-σ},$$

$$\tilde{μ}_{2,1} = ξK_p \left(1 + 2p^{-1/3} + 2p^{-2/3} + 2p^{-1} + 2p^{-4/3} + p^{-5/3}\right)p^{-2σ},$$

$$\tilde{μ}_{2,2} = ξ^2K_p \left(1 + 2p^{-1/3} + 2p^{-2/3} + 2p^{-1} + 2p^{-4/3} + p^{-5/3}\right)p^{-2σ},$$

with $K_p = (1 - p^{-1/3})(1 - p^{-5/3})^{-1}(1 + p^{-1})^{-1}$ and $|\tilde{E}| ≤ 16(2|ξ| + 1)^3p^{-3σ}$. We let $p ≥ P_2(M_σ)$ with a positive constant $M_σ$ depending only on $σ$. Then, if we let $M_σ$ be large enough, the inequality

$$\log |G_{σ,p}(iξ)| ≤ -\frac{1}{2}\tilde{μ}_{2,2} + Aξ^2p^{-2σ-2/3} + A|ξ|^3p^{-3σ} ≤ -\frac{1}{8}ξ^2p^{-2σ}$$

follows. Hence the upper bound of $|G_{σ}(iξ)|$ is proved by using (4.7) again. 

Note that we can prove similar results for $F_{σ}^*(z)$ and $G_{σ}^*(z)$ of Section 2.2. We omit the proofs since they are obtained by the almost same arguments in the above proofs. Here, we just replace $P_1$ and $P_2(M)$ with

$$P_1^* = P_1^*(a, b) = \{(8a + 4) \log(8a + 4)\}^{1/b},$$

$$P_2^*(M) = P_2^*(M; a, b) = \{(Ma) \log(Ma)\}^{1/b}.$$ 

The statements for $F_{σ}^*(z)$ and $G_{σ}^*(z)$ are as follows.
Proposition 4.6. (i) Let $\sigma_0 > 1/2$ and $R > 0$. Then the infinite product

$$F^*_s(z) = \prod_p F^*_{s,p}(z)$$

converges uniformly and absolutely for $\text{Re}(s) \geq \sigma_0$ and $|z| \leq R$.

(ii) Let $\sigma_0 > 2/3$ and $R > 0$. Then the infinite product

$$G^*_s(z) = \prod_p G^*_{s,p}(z)$$

converges uniformly and absolutely for $\text{Re}(s) \geq \sigma_0$ and $|z| \leq R$.

Proposition 4.7. (i) Let $s = \sigma + it$ and $z$ be complex numbers with $1/2 < \sigma < 1$. Then there exists an absolute constant $c_1 > 0$ such that

$$|F^*_s(z)| \leq \exp \left( \frac{c_1}{2\sigma - 1(1 - \sigma)}(|z| + 3)^{1/\sigma} \log(|z| + 3) \right).$$

(ii) Let $s = \sigma + it$ and $z$ be complex numbers with $2/3 < \sigma < 1$. Then there exists an absolute constant $c_2 > 0$ such that

$$|G^*_s(z)| \leq \exp \left( \frac{c_2}{3\sigma - 2(1 - \sigma)}(|z| + 3)^{1/\sigma} \log(|z| + 3) \right).$$

Proposition 4.8. (i) Let $\sigma > 1/2$ be a real number. There exists a positive constant $c_1(\sigma)$ depending only on $\sigma$ such that

$$|F^*_s(i\xi)| \leq \exp \left( -c_1(\sigma)|\xi|^{1/\sigma}(\log|\xi|)^{(1/\sigma) - 1} \right)$$

for any $\xi \in \mathbb{R}$ with $|\xi| \geq 3$.

(ii) Let $\sigma > 2/3$ be a real number. There exists a positive constant $c_2(\sigma)$ depending only on $\sigma$ such that

$$|G^*_s(i\xi)| \leq \exp \left( -c_2(\sigma)|\xi|^{1/\sigma}(\log|\xi|)^{(1/\sigma) - 1} \right)$$

for any $\xi \in \mathbb{R}$ with $|\xi| \geq 3$.

4.2. Construction of the density functions. Let $\mathcal{A} = \{ A_a \mid a \in \mathcal{A} \}$. Then we take two sequences $X = (X_p)_p$ and $Y = (Y_p)_p$ of independent $\mathcal{A}$-valued random variables, indexed by prime numbers $p$, such that

$$\mathbb{P}(X_p = A_a) = C_p(a) \quad \text{and} \quad \mathbb{P}(Y_p = A_a) = K_p(a).$$

Proposition 4.9. (i) For any $\sigma > 1/2$, the random Euler product

$$L(\sigma, X) = \prod_p \det (I - p^{-\sigma}X_p)^{-1},$$

converges almost surely.

(ii) For any $\sigma > 2/3$, the random Euler product

$$L(\sigma, Y) = \prod_p \det (I - p^{-\sigma}Y_p)^{-1}.$$

converges almost surely.
Proof. The local components of (4.9) and (4.10) are calculated as
\[ \det (I - p^{-\sigma}X_p)^{-1} = 1 + \text{tr}(X_p)p^{-\sigma} + O(p^{-2\sigma}), \]
\[ \det (I - p^{-\sigma}Y_p)^{-1} = 1 + \text{tr}(Y_p)p^{-\sigma} + O(p^{-2\sigma}). \]
Recall that \( \sum_p p^{-2\sigma} \) converges for \( \sigma > 1/2 \). Then it is sufficient to prove that the series \( \sum_p \text{tr}(X_p)p^{-\sigma} \) and \( \sum_p \text{tr}(Y_p)p^{-\sigma} \) converge almost surely for adequate ranges of the real number \( \sigma \). First, we let \( \sigma > 1/2 \). Then we consider the series
\[ (4.11) \quad \sum_p \mathbb{E}[\text{tr}(X_p)p^{-\sigma}] \quad \text{and} \quad \sum_p \text{Var}[\text{tr}(X_p)p^{-\sigma}]. \]
It is deduced that \( \mathbb{E}[\text{tr}(X_p)p^{-\sigma}] \ll p^{-\sigma-1} \) and \( \text{Var}[\text{tr}(X_p)p^{-\sigma}] \ll p^{-2\sigma} \) from the definition of \( X_p \). Furthermore, we recall that the series \( \sum_p p^{-\sigma-1} \) and \( \sum_p p^{-2\sigma} \) converge for \( \sigma > 1/2 \). Hence the convergences of (4.11) follows, and we conclude that the series \( \sum_p \text{tr}(X_p)p^{-\sigma} \) converges almost surely. In order to prove the second statement, we let \( \sigma > 2/3 \). In this case, we see that \( \mathbb{E}[\text{tr}(Y_p)p^{-\sigma}] \ll p^{-\sigma-1/3} \) and \( \text{Var}[\text{tr}(Y_p)p^{-\sigma}] \ll p^{-2\sigma} \) hold. The series \( \sum_p p^{-\sigma-1/3} \) and \( \sum_p p^{-2\sigma} \) converge for \( \sigma > 2/3 \). Hence we obtain the second result. \( \square \)

Proposition 4.9 ensures that \( \log L(\sigma, X) \) (resp. \( \log L(\sigma, Y) \)) gives an \( \mathbb{R} \)-valued random variable for \( \sigma > 1/2 \) (resp. \( \sigma > 2/3 \)). Then we denote by \( \phi_\sigma \) and \( \psi_\sigma \) their distributions, that is,
\[ \phi_\sigma(A) = \mathbb{P}(\log L(\sigma, X) \in A) \quad \text{and} \quad \psi_\sigma(A) = \mathbb{P}(\log L(\sigma, Y) \in A) \]
for \( A \in \mathcal{B}(\mathbb{R}) \). Then \( \phi_\sigma \) and \( \psi_\sigma \) are the probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) with structures of infinite convolutions arising from Euler products (4.9) and (4.10). For each prime number \( p \), we define
\[ \phi_{\sigma,p}(A) = \mathbb{P}(\log L_p(\sigma, X_p) \in A) \quad \text{with} \quad L_p(\sigma, X_p) = \det (I - p^{-\sigma}X_p)^{-1} \]
and
\[ \psi_{\sigma,p}(A) = \mathbb{P}(\log L_p(\sigma, Y_p) \in A) \quad \text{with} \quad L_p(\sigma, Y_p) = \det (I - p^{-\sigma}Y_p)^{-1}. \]

Proposition 4.10. The probability measure \( \phi_\sigma \) is represented as
\[ (4.12) \quad \phi_\sigma = *_p \phi_{\sigma,p} \]
for \( \sigma > 1/2 \), where \( *_p \phi_{\sigma,p} \) denotes the limit measure to which \( \phi_{\sigma,p_1} \cdots * \phi_{\sigma,p_n} \) weakly converges as \( n \to \infty \). Here \( p_n \) describes the \( n \)-th prime number. Also, \( \psi_\sigma \) is represented as \( \psi_\sigma = *_p \psi_{\sigma,p} \) for \( \sigma > 2/3 \). Furthermore, both convergences are absolute.

Proof. By the definition of \( \phi_{\sigma,p} \), we have
\[ c(\phi_{\sigma,p}) = \int_{\mathbb{R}} x \, d\phi_{\sigma,p}(x) = \sum_{a \in \mathcal{A}} C_p(a) F(p^{-\sigma}; a), \]
\[ M(\phi_{\sigma,p}) = \int_{\mathbb{R}} x^2 \, d\phi_{\sigma,p}(x) = \sum_{a \in \mathcal{A}} C_p(a) F(p^{-\sigma}; a)^2 \]
for every prime number \( p \). Hence \( M(\phi_{\sigma,p}) \) is finite for each \( p \). Furthermore, by the asymptotic formula \( F(w; a) = \text{tr}(A_a)w + O(|w|^2) \), we obtain \( c(\phi_{\sigma,p}) \ll p^{-\sigma-1} + p^{-2\sigma} \) and \( M(\phi_{\sigma,p}) \ll p^{-2\sigma} \). Recall that the series \( \sum_p p^{-\sigma-1} \) and \( \sum_p p^{-2\sigma} \) converge for
Therefore, we deduce from Lemma 3.6 that \( \phi_{\sigma,p_1} \ast \cdots \ast \phi_{\sigma,p_n} \) converges weakly to some probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) as \( n \to \infty \). In order to see that the limit measure equals to \( \phi_\sigma \), we examine their characteristic functions. First, the convergence of \( \phi_{\sigma,p_1} \ast \cdots \ast \phi_{\sigma,p_n} \) implies

\[
\Lambda(\xi; *_p \phi_{\sigma,p}) = \lim_{n \to \infty} \Lambda(\xi; \phi_{\sigma,p_1} \ast \cdots \ast \phi_{\sigma,p_n}) = \prod_p \Lambda(\xi; \phi_{\sigma,p}).
\]

Note that the characteristic functions of \( \phi_{\sigma,p} \) are calculated as

\[
\Lambda(\xi, \phi_{\sigma,p}) = \sum_a C_p(a) \exp \left( i\xi f(p^{-\sigma}; a) \right) = \mathcal{F}_{\sigma,p}(i\xi),
\]

where \( \mathcal{F}_{\sigma,p}(i\xi) \) is obtained from (2.7). Therefore we obtain

\[
\Lambda(\xi; *_p \phi_{\sigma,p}) = \mathcal{F}_\sigma(i\xi).
\]

On the other hand, we have

\[
\mathbb{E} \left[ \prod_{j=1}^n L_{p_j}(\sigma, X_{p_j})^{i\xi} \right] = \prod_{j=1}^n \mathbb{E} \left[ L_{p_j}(\sigma, X_{p_j})^{i\xi} \right]
\]

since \( X_p \) are independent. The left-hand side converges to the characteristic function \( \Lambda(\xi; \phi_\sigma) \) as \( n \to \infty \). Also, we recall that the equality \( \mathbb{E} \left[ L_p(\sigma, X_p)^{i\xi} \right] = \mathcal{F}_{\sigma,p}(i\xi) \) holds for every prime number \( p \). Hence, taking \( n \to \infty \), we obtain

\[
\Lambda(\xi; \phi_\sigma) = \mathcal{F}_\sigma(i\xi).
\]

By (4.13) and (4.14), we conclude that \(*_p \phi_{\sigma,p} = \phi_p \) holds as desired. The proof for the case of \( \psi_\sigma \) is similar. Indeed, the difference from \( \phi_\sigma \) is just the coefficients; we use \( K_p(a) \) in place of \( C_p(a) \). Then we obtain the asymptotic formula

\[
(4.15) \quad c(\psi_{\sigma,p}) = \frac{1 - p^{-1/3}}{(1 - p^{-5/3})(1 + p^{-1})} \left( p^{-1/3} + p^{-2/3} + p^{-1} + 2p^{-4/3} + p^{-5/3} \right) p^{-\sigma} + O(p^{-2\sigma}).
\]

Furthermore, we have \( c(\psi_{\sigma,p}) \ll p^{-\sigma-1/3} + p^{-2\sigma} \) and \( M(\psi_{\sigma,p}) \ll p^{-2\sigma} \). Since the series \( \sum_p p^{-\sigma-1/3} \) and \( \sum_p p^{-2\sigma} \) converge for \( \sigma > 2/3 \), we obtain the convergence of \( \psi_{\sigma,p_1} \ast \cdots \ast \psi_{\sigma,p_n} \) as \( n \to \infty \). Moreover, we have

\[
\Lambda(\xi; *_p \psi_{\sigma,p}) = \Lambda(\xi; \psi_\sigma) = \mathcal{G}_\sigma(i\xi)
\]

by the calculations similar to (4.13) and (4.14). Hence we find \( \psi_\sigma = *_p \psi_{\sigma,p} \), which completes the proof.

\[ \square \]

**Remark 4.11.** Jessen–Wintner [26] further proved the converse of Lemma 3.7 is also true if all \( \text{supp}(\phi_n) \) are contained in a fixed interval \( |x| \leq R \). In the case of the probability measure \( \psi_{\sigma,p} \), we see that

\[
\text{supp}(\psi_{\sigma,p}) = \{ F(p^{-\sigma}; a) \mid a \in \mathcal{A} \} \subset [-10, 10].
\]

Therefore, due to (4.15), the condition \( \sigma > 2/3 \) is necessary to the convergence of \( \psi_{\sigma,p_1} \ast \cdots \ast \psi_{\sigma,p_n} \) as \( n \to \infty \).
From the above, we finally arrive at the density functions $C_\sigma$ and $K_\sigma$. For every $k \geq 0$, we obtain

$$\int_{\mathbb{R}} |\xi|^k |\Lambda(\xi; \phi_p)| \, d\xi < \infty$$

as a consequence of Proposition 4.5. Hence, Lemma 3.8 yields that $\phi_\sigma$ is absolutely continuous, and the density is given by

$$C_\sigma(x) = \int_{\mathbb{R}} \Lambda(\xi; \phi_\sigma) e^{-ix\xi} \, \frac{d\xi}{\sqrt{2\pi}}.$$ 

Moreover, the function $C_\sigma$ possesses continuous derivatives of any order. The probability measure $\psi_\sigma$ is also absolutely continuous, whose density is determined from the inversion formula

$$K_\sigma(x) = \int_{\mathbb{R}} \Lambda(\xi; \psi_\sigma) e^{-ix\xi} \, \frac{d\xi}{\sqrt{2\pi}}.$$ 

We obtain density functions $C_\sigma$ and $K_\sigma$ as follows. Note that

$$L'(\sigma, X) = -\sum_{p} \sum_{m=1}^{\infty} (\log p) \text{tr}(X^m_p) p^{-m\sigma}$$

gives an $\mathbb{R}$-valued random variable for $\sigma > 1/2$. We can also define $(L'/L)(\sigma, Y)$ as an $\mathbb{R}$-valued random variable for $\sigma > 2/3$. Let $\phi^*_\sigma$ and $\psi^*_\sigma$ be the distributions of $(L'/L)(\sigma, X)$ and $(L'/L)(\sigma, Y)$, respectively. Then we see that the probability measures $\phi^*_\sigma$ and $\psi^*_\sigma$ are absolutely continuous by Proposition 4.8. We let $C_\sigma$ and $K_\sigma$ be their densities.

**Remark 4.12.** The $M$-function $M_\sigma$ of Ihara–Matsumoto is constructed as an infinite convolution of Schwartz distributions in [22]. As for the density functions in this paper, we obtain similar expressions by Schwartz distributions. For example, $C_\sigma$ is expressed as

$$C_\sigma(x) = *_p C_{\sigma, p}(x)$$

in view of (4.12), where we define a Schwartz distribution $C_{\sigma, p}$ on $\mathbb{R}$ as

$$C_{\sigma, p}(x) = \sum_{a \in \omega} C_p(a) \delta(x - F(p^{-\sigma}; a))$$

for each prime number $p$. Here, $\delta$ denotes the Dirac distribution on $\mathbb{R}$.

4.3. **Proof of Theorems 2.3 and 2.9.** The fact that $C_\sigma$ belongs to the space $L^1(\mathbb{R})_\infty$ is proved as follows. At first, let $a$ be an arbitrary real number. Since $C_\sigma$ is the density of the probability measure $*_p \phi_{\sigma, p}$, we have

$$\int_{\mathbb{R}} e^{iax} C_\sigma(x) \, \frac{dx}{\sqrt{2\pi}} \leq \liminf_{n \to \infty} \int_{\mathbb{R}} e^{iax} d(\phi_{\sigma, p_1} \cdots \phi_{\sigma, p_n})(x)$$

by (2.1) in p.50. Recalling that the equality

$$\int_{\mathbb{R}} e^{iax} d\phi_{\sigma, p}(x) = F_{\sigma, p}(a)$$

Discrete value-distribution of Artin L-functions holds for all prime numbers \( p \), we obtain

\[
\int_{\mathbb{R}} e^{iax} d(\phi_{\sigma,p_1} \ast \cdots \ast \phi_{\sigma,p_n})(x) = \int_{\mathbb{R}} d\phi_{\sigma,p_n}(x_n) \cdots \int_{\mathbb{R}} \exp(a(x_1 + \cdots + x_n)) d\phi_{\sigma,p_1}(x_1) = \prod_{j=1}^{n} F_{\sigma,p_j}(a),
\]

which converges to \( F_\sigma(a) \) as \( n \to \infty \) by Proposition 4.1. Therefore, if \( a \geq 0 \),

\[
\int_{-\infty}^{\infty} e^{ia|x|} C_\sigma(x) \, d\xi \ll \int_{\mathbb{R}} e^{iax} C_\sigma(x) \, dx + \int_{\mathbb{R}} e^{-ax} C_\sigma(x) \, dx \ll F_\sigma(a) + F_\sigma(-a) < \infty
\]
as desired. Moreover, this implies the Fourier transform \( \tilde{C}_\sigma(z) \) is defined for any complex number \( z \), and it is a holomorphic function on \( \mathbb{C} \). On the other hand, \( F_\sigma(z) \) is also a holomorphic function on \( \mathbb{C} \) by Corollary 4.3. Recall that we have \( \tilde{C}_\sigma(i\xi) = \Lambda(\xi; \phi_{\sigma}) = \widetilde{F}_\sigma(\xi) \) for any \( \xi \in \mathbb{R} \) by 4.14. Therefore, by analytic continuation, the equality \( \tilde{C}_\sigma(z) = F_\sigma(iz) \) is valid for any \( z \in \mathbb{C} \).

The remaining work on proving Theorem 2.3 (i) is the calculation of the support of \( C_\sigma \). We have \( \text{supp}(C_\sigma) = \text{supp}(\phi_\sigma) \) by definition. Since \( \phi_\sigma = \phi_{\sigma,p_1} \ast \phi_{\sigma,p_2} \ast \cdots \), we calculate \( \text{supp}(\phi_\sigma) \) as

\[
\text{supp}(\phi_\sigma) = \text{supp}(\phi_{\sigma,p_1}) + \text{supp}(\phi_{\sigma,p_2}) + \cdots
\]

by formula (3.15). For each prime number \( p \), we see that \( \text{supp}(\phi_{\sigma,p}) \) is equal to the set \( \{F(p^{-\sigma}; a) \mid a \in \mathcal{A}\} \). First, we prove that \( \phi_\sigma \) is compactly supported for \( \sigma > 1 \). Since \( |\text{tr}(A_u^n)| \leq 2 \) for any \( a \in \mathcal{A} \), the inequality

\[
\left| \sum_p F(p^{-\sigma}; a_p) \right| \leq \sum_p \sum_{m=1}^{\infty} \frac{|\text{tr}(A_u^n)|}{m} p^{-m\sigma} \leq 2 \log \zeta(\sigma)
\]
is true for any choices of \( a_p \in \mathcal{A} \) if \( \sigma > 1 \). Hence we deduce from 4.16 the conclusion

\[
\text{supp}(\phi_\sigma) \subset [-2 \log \zeta(\sigma), 2 \log \zeta(\sigma)].
\]

Let \( 1/2 < \sigma \leq 1 \). In order to prove \( \text{supp}(\phi_\sigma) = \mathbb{R} \), we use Lemma 3.10 with \( H = \mathbb{R} \) equipped with the usual inner product \( \langle u, v \rangle = uv \). Let \( u_n = p_n^{-\sigma} \). Then conditions (a) and (b) in Lemma 3.10 are easily checked. Take a real number \( \epsilon > 0 \) arbitrarily. Note that we have

\[
\sum_{n=k}^{N} \sum_{m=2}^{\infty} \frac{2}{m} p_n^{-m\sigma} \leq B_\sigma k^{1-2\sigma}
\]
for any \( N \geq k \geq 1 \), where \( B_\sigma > 0 \) is a constant depending only on \( \sigma \). Then we take an integer \( k = k(\epsilon, \sigma) \geq 2 \) so that \( B_\sigma k^{1-2\sigma} < \epsilon \) is satisfied. Let \( x \) be an arbitrary real number. By Lemma 3.10 we obtain an integer \( N = N(\epsilon, \sigma, x) \geq k \) and numbers \( c_1, \ldots, c_N \in \{1, -1\} \) such that

\[
\left| x - \sum_{n=1}^{k-1} \sum_{m=1}^{\infty} \frac{2}{m} p_n^{-m\sigma} \right| \leq \sum_{n=k}^{N} c_n p_n^{-\sigma} \leq \epsilon.
\]
For $1 \leq n \leq N$, we choose a symbol $a_n \in \mathcal{A}$ as follows. For $1 \leq n < k$, let $a_n = (111)$. For $k \leq n \leq N$, we let $a_n = (1^21)$ if $c_n = 1$ and $a_n = (3)$ if $c_n = -1$. Then (4.17) deduces

$$
| x - \sum_{n=1}^{k-1} \sum_{m=1}^{\infty} \frac{\text{tr}(A_{a_n}^m)}{m} p_n^{-m\sigma} - \sum_{n=k}^{N} \text{tr}(A_{a_n}) p_n^{-\sigma} | < \epsilon.
$$

By the choice of $k$, we finally obtain

$$
| x - \sum_{n=1}^{N} F(p_n^{-\sigma}; a_n) | < 2\epsilon,
$$

which means $x$ belongs to the set $\text{supp}(\phi_{\sigma,p}) + \text{supp}(\phi_{\sigma,p_2}) + \cdots$. Therefore, we obtain $\text{supp}(\phi_{\sigma,p}) = \mathbb{R}$, and the proof of Theorem 2.3 (i) is completed. Similar methods are available to prove Theorem 2.3 (ii) and Theorem 2.9 (i), (ii).

5. Complex moments of $L(\sigma, \rho_K)$

We fix the branch of $\log L(s, \rho_K)$ as in Section 3.1. Then we define

$$
g_z(s, \rho_K) = \begin{cases} 
\exp \left( z \log L(s, \rho_K) \right) & \text{(Case I)}, \\
\exp \left( z (L'/L)(s, \rho_K) \right) & \text{(Case II)}
\end{cases}
$$

for $z \in \mathbb{C}$ and $s \in G_K$. For $\text{Re}(s) > 1$, we use (3.2) and (3.3) to deduce the expression

$$
g_z(s, \rho_K) = \prod_{p} \sum_{r=0}^{\infty} \lambda_z(p^r, \rho_K) p^{-rs}.
$$

Here the coefficient $\lambda_z(p^r, \rho_K)$ is determined from $H_r(z; a)$ and $G_r(z; a)$ studied in Section 3.2. Indeed, we have

$$
\lambda_z(p^r, \rho_K) = \begin{cases} 
H_r(z; a) & \text{(Case I)}, \\
G_r(-z \log p; a) & \text{(Case II)}
\end{cases}
$$

by (2.6), (2.17), and (3.8) if $K$ satisfies a local condition $a$ at $p$. Let $k = \lfloor 2|z| \rfloor + 1$, where $|x|$ indicates the largest integer less than or equal to $x$. Then we deduce from Lemma 3.5 that $|\lambda_z(p^r, \rho_K)| \leq d_k(p^r)$ in Case I, and that $|\lambda_z(p^r, \rho_K)| \leq d_k^*(p^r)$ in Case II. We extend $\lambda_z(p^r, \rho_K)$ multiplicatively. Then the inequality

$$
|\lambda_z(n, \rho_K)| \leq \begin{cases} 
d_k(n) & \text{(Case I)}, \\
d_k^*(n) & \text{(Case II)}
\end{cases}
$$

holds for all $n \geq 1$, which gives $\lambda_z(n, \rho_K) \ll n^\epsilon$ for every $\epsilon > 0$ in both cases. Therefore, we obtain the Dirichlet series expression

$$
g_z(s, \rho_K) = \sum_{n=1}^{\infty} \lambda_z(n, \rho_K) n^{-s}
$$

which absolutely converges for any $\text{Re}(s) > 1$. 
5.1. Approximation of the function $g_z(\sigma, \rho_K)$. We approximate $g_z(\sigma, \rho_K)$ by a function which is expressed as a generalized Dirichlet series closely related to (5.2). The goal of this section is the following result.

**Proposition 5.1.** Let $\sigma_1$ be an absolute constant for which Assumption 2.2 holds, and let $\sigma > \sigma_1$. We take a cubic field $K$ belonging to $L^\pm_3(X) \setminus \mathcal{B}_{\sigma_1}(X)$, where the subset $\mathcal{B}_{\sigma_1}(X)$ is defined as (2.4). Let $Y = X^\eta$ with some $\eta > 0$. Then there exists a constant $0 < b(\eta) \leq 1$ depending only on $\eta$ such that

$$g_z(\sigma, \rho_K) = \sum_{n=1}^{\infty} \lambda_z(n, \rho_K)n^{-\sigma}e^{-n/Y} + O\left(\exp\left(-\frac{\eta \log X}{2 \log \log X}\right)\right)$$

for any $z \in \mathbb{C}$ with $|z| \leq b(\eta)R_\sigma(X)$, where $R_\sigma(X)$ is defined as (2.12). The implied constant depends only on $\sigma$ and $\eta$.

In general, we let $\sigma > 1/2$ be a real number and take $c, \kappa > 0$ as

$$c = \max\{1 - \sigma, 0\} + \frac{1}{\log X} \quad \text{and} \quad \kappa = \frac{1}{\log \log X}.$$  

Then we define the functions $g^+_z(\sigma, \rho_K; Y)$ and $g^-_z(\sigma, \rho_K; Y)$ as

$$g^+_z(\sigma, \rho_K; Y) = \frac{1}{2\pi i} \int_{L^+} g_z(\sigma + w, \rho_K) \Gamma(w)Y^w dw.$$ 

Here $L^+$ is the vertical line Re$(w) = c$, and thus $g^+_z(\sigma, \rho_K; Y)$ is defined for any $\sigma > 1/2$ and $K \in L^\pm_3(X)$. On the other hand, the contour $L^- = L_1 + \cdots + L_5$ is given by connecting the points $c - i\infty, c - i(\log X)^2, -\kappa - i(\log X)^2, -\kappa + i(\log X)^2, c + i(\log X)^2, c + i\infty$, in order. Then the function $g^-_z(\sigma, \rho_K; Y)$ is defined for any $\sigma > 1 + (\log \log X)^{-1}$ and $K \in L^\pm_3(X) \setminus \mathcal{B}_{\sigma_1}(X)$. Proposition 5.1 is deduced from the properties of the functions $g^\pm_z(\sigma, \rho_K; Y)$ listed as follows.

**Lemma 5.2.** Let $\sigma_1$ be an absolute constant for which Assumption 2.2 holds, and let $\sigma > \sigma_1$. Suppose that $(\log \log X)^{-1} < \sigma - \sigma_1$ is satisfied, and take a cubic field $K$ belonging to $L^\pm_3(X) \setminus \mathcal{B}_{\sigma_1}(X)$. Then the equality

$$g_z(\sigma, \rho_K) = g^+_z(\sigma, \rho_K; Y) - g^-_z(\sigma, \rho_K; Y)$$

holds for any $z \in \mathbb{C}$ and $Y \geq 1$.

**Proof.** We have Re$(\sigma + w) > 1$ for Re$(w) = c$ by the choice of $c$. Hence formula (5.2) is available to investigate the function $g_z(\sigma + w, \rho_K)$ on the line $L^+$, and we deduce that $|g_z(\sigma + w, \rho_K)|$ is bounded as $|\text{Im}(w)| \to \infty$. Furthermore, we see that $|\Gamma(w)|$ is rapidly decreasing as $|\text{Im}(w)| \to \infty$ by (3.13). Therefore $g_z(\sigma + w, \rho_K)\Gamma(w)Y^w$ is absolutely integrable on the contour $L^+$. Then, we shift the contour to $L^-$. Remark that we have $\sigma - \kappa > \sigma_1$, and that the function $g_z(s, \rho_K)$ is holomorphic in the region Re$(s) > \sigma_1$, $|\text{Im}(s)| < (\log X)^3$ for $K \in L^\pm_3(X) \setminus \mathcal{B}_{\sigma_1}(X)$. Hence we do not encounter any poles of the integrand except for a simple pole at $w = 0$ while sifting the contour. The residue at $w = 0$ is equal to $g_z(\sigma, \rho_K)$. Therefore, we obtain

$$\frac{1}{2\pi i} \int_{L^+} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w dw = \frac{1}{2\pi i} \int_{L^-} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w dw + g_z(\sigma, \rho_K)$$

as desired. \(\Box\)
Lemma 5.3. Let \( \sigma > 1/2 \) be a real number, and take \( K \in L_2^+(X) \) arbitrarily. Then the function \( g_+^z(\sigma, \rho_K; Y) \) is represented as

\[
g_+^z(\sigma, \rho_K; Y) = \sum_{n=1}^{\infty} \lambda_z(n, \rho_K) n^{-\sigma} e^{-n/Y}
\]

for any \( z \in \mathbb{C} \) and \( Y \geq 1 \). If we let \( Y = X^\eta \) with some \( \eta > 0 \), then we have

\[
g_+^z(\sigma, \rho_K; Y) \ll X^\eta
\]

for any \( z \in \mathbb{C} \) with \( |z| \leq R_\sigma(X) \). The implied constant in (5.4) depends only on \( \eta \).

Proof. Recall that we have \( \Re(\sigma + w) > 1 \) on the vertical line \( \Re(w) = c \). Hence Dirichlet series expression (3.2) yields

\[
\int_{\Re(w)=c} g_z(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw = \sum_{n=1}^{\infty} \lambda_z(n, \rho_K) n^{-\sigma} e^{-n/Y}
\]

as desired. In order to prove (5.4) in Case I, we apply inequality (5.1) to deduce

\[
|g_+^z(\sigma, \rho_K; Y)| \leq \sum_{n=1}^{\infty} d_k(n) n^{-\sigma} e^{-n/Y}.
\]

The right-side hand is estimated by the first estimate of Lemma 3.6. Hence we obtain the conclusion

\[
g_+^z(\sigma, \rho_K; Y) \ll (Y^{1-\sigma} + 1) (C_1 \log Y)^{2(|z|+1)} \ll \eta \ X^\eta
\]

if we let \( Y = X^\eta \) and \( |z| \leq R_\sigma(X) \). The result in Case II is obtained from the second estimate of Lemma 3.6.

Lemma 5.4. Let \( \sigma_1 \) be an absolute constant for which Assumption 2.2 holds, and let \( \sigma > \sigma_1 \). Suppose that \( 3(\log \log X)^{-1} \leq \sigma - \sigma_1 \) is satisfied, and take a cubic field \( K \) belonging to \( L_2^+(X) \setminus B_{ \sigma_1}(X) \). Let \( Y = X^\eta \) with some \( \eta > 0 \). Then there exists a constant \( b(\eta) > 0 \) depending only on \( \eta \) such that

\[
g_-^z(\sigma, \rho_K; Y) \ll \exp \left( -\frac{\eta}{2} \frac{\log X}{\log \log X} \right)
\]

for any \( z \in \mathbb{C} \) with \( |z| \leq b(\eta) R_\sigma(X) \), where the implied constant depends only on \( \eta \).

Proof. We first prove the result in Case I. We divide the integral contour \( L^- \) into \( L_1, L_2, \ldots, L_5 \) as above. Then we have \( \Re(\sigma + w) \geq 1 + (\log X)^{-1} \) on \( L_1 \) and \( L_5 \). Hence Dirichlet series expression (3.2) is available to obtain the estimate

\[
\log L(\sigma + w, \rho_K) \ll \log \log X
\]

for \( w \in L_1 \cup L_5 \). Next, if \( w \) lies on \( L_2, L_3, L_4 \), we have \( \Re(\sigma + w) \geq \sigma_1 + 2\kappa \). Then we use Lemma 3.3 in this case. As a result, the estimate

\[
g_z(\sigma + w, \rho_K) \ll \exp \left( A\eta \frac{\log X}{\log \log X} \right)
\]

(5.5)
is valid for any $w \in L^-$, where $A > 0$ is an absolute constant. The integrals of $\Gamma(w)$ are estimated by applying (3.13). Finally, the function $Y^w$ satisfies

$$Y^w \ll \begin{cases} X^\eta \\ \exp\left(-\frac{\eta \log X}{\log \log X}\right) \end{cases}$$

if $w$ lies on $L_1, L_2, L_4, L_5$, and if $w$ lies on $L_3$.

From the above, we obtain

$$\int_{L_1 \cup L_5} g_z(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw \ll \exp\left(-\frac{(\log X)^2}{2 \log \log X}\right),$$

(5.6)

$$\int_{L_2 \cup L_4} g_z(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw \ll \exp\left(-\frac{\eta \log X}{2 \log \log X}\right),$$

$$\int_{L_3} g_z(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw \ll \exp\left(-\frac{\eta \log X}{2 \log \log X}\right)$$

if the constant $b(\eta) > 0$ is small enough, where the implied constants depend only on $\eta$. These estimates yield the desired result in Case I. We obtain the result in Case II similarly, but we should remark that the same estimate as (5.5) does not hold on $L_1$ and $L_5$. Indeed, the upper bound

$$\frac{L'}{L}(\sigma + w) \ll \log X$$

is optimal for $w \in L_1 \cup L_5$, and therefore we know just

$$g_z(\sigma + w, \rho_K) \ll \exp\left(\frac{(\log X)^2}{A \log \log X}\right)$$

on $L_1$ and $L_5$. Nevertheless, estimate (5.6) still holds in this case thanks to the decay of $|\Gamma(w)|$. Hence the difference does not affect the conclusion. □

5.2. Calculation of complex moments. In Case I, we define $\lambda_z(n), \mu_z(n), \nu_z(n)$ for $z \in \mathbb{R}$ as the multiplicative functions satisfying

$$\lambda_z(p^m) = \sum_{a \in \mathcal{O}} C_p(a) H_m(z; a),$$

$$\mu_z(p^m) = \sum_{a \in \mathcal{O}} K_p(a) H_m(z; a),$$

$$\nu_z(p^m) = \sum_{a \in \mathcal{O}} |H_m(z; a)|.$$

We define $\lambda_z(n), \mu_z(n), \nu_z(n)$ in Case II by replacing $H_m(z; a)$ with $G_m(-z \log p; a)$ in the above equalities.

Lemma 5.5. Let $\alpha, \beta$ be absolute constants for which Assumption [2.1] holds. Then, for every $\epsilon > 0$, we have

$$\sum_{K \in L_3^+ (X)} \lambda_z(n, \rho_K) = C^\pm \frac{1}{12\zeta(3)} X \lambda_z(n) + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3) \zeta(5/3)} X^{5/6} \mu_z(n) + O\left(X^{1+\epsilon} \nu_z(n) \eta^\beta\right),$$

(5.7)

where the implied constant depends only on $\epsilon$. 

Proof. Asymptotic formula (5.7) for \( n = 1 \) is directly deduced from Assumption 2.1 with \( \text{supp}(S) = \emptyset \). Then we consider the case where \( n = p_1^{m_1} \cdots p_r^{m_r} > 1 \). For \( A = (a_1, \ldots, a_r) \in A^r \), we denote by \( S(A) \) the collection of local specifications such that \( \text{supp}(S(A)) = \{p_1, \ldots, p_r\} \) and \( S_p(A) = a_j \) for each \( j \). If we suppose that \( K \) satisfies local conditions \( S(A) \), then \( \lambda_z(n, \rho_K) \) is calculated as

\[
\lambda_z(n, \rho_K) = H_{m_1}(z; a_1) \cdots H_{m_r}(z; a_r)
\]

in Case I. Hence we obtain

\[
(5.8) \quad \sum_{K \in L_3^+ (X)} \lambda_z(n, \rho_K) = \sum_{A \in A^r} \sum_{K \in L_3^+ (X, S(A))} \lambda_z(n, \rho_K) = \sum_{A \in A^r} \#L_3^+ (X, S(A)) \prod_{j=1}^r H_{m_j}(z; a_j).
\]

Then we use Assumption 2.2 which gives

\[
\#L_3^+ (X, S(A)) = C^\pm \frac{1}{12 \zeta(3)} X \prod_{j=1}^r C_p(a_j) + K^\pm \frac{4 \zeta(1/3)}{5 \Gamma(2/3)^3 \zeta(5/3)} X^{5/6} \prod_{j=1}^r K_p(a_j)
\]

\[+ O \left( X^{\alpha+\varepsilon} \prod_{j=1}^r \rho_j^\beta \right).
\]

We insert this formula to (5.8). Note that the equality

\[
\sum_{A \in A^r} \prod_{j=1}^r C_p(a_j) H_{m_j}(z; a_j) = \lambda_z(n)
\]

holds, and similar equalities are obtained for \( \mu_z(n) \) and \( \nu_z(n) \). Hence the result follows in Case I. We also obtain the result in Case II by replacing \( H_{m_j}(z; a_j) \) with \( G_{m_j}(-z \log p; a_j) \) in the above argument. \( \square \)

By Proposition 5.1, the left-hand sides of (5.9) and (5.10) are calculated as

\[
(5.9) \quad \sum_{K \in L_3^+ (X) \setminus A_r (X)} g_z(\sigma, \rho_K) = S_1 - S_2 + S_3 + O \left( X \exp \left( -\frac{\eta \log X}{2 \log \log X} \right) \right),
\]

\[
(5.10) \quad \sum_{K \in L_3^+ (X) \setminus B_{r_1} (X)} g_z(\sigma, \rho_K) = S_1 - S_2 + O \left( X \exp \left( -\frac{\eta \log X}{2 \log \log X} \right) \right)
\]

for any \( z \in \mathbb{C} \) with \( |z| \leq b(\eta) R_{\sigma} (X) \), where

\[
S_1 = \sum_{K \in L_3^+ (X)} \sum_{n=1}^\infty \lambda_z(n, \rho_K) n^{-\sigma} e^{-n/Y},
\]

\[
S_2 = \sum_{K \in B_{r_1} (X) \setminus A_r (X)} \sum_{n=1}^\infty \lambda_z(n, \rho_K) n^{-\sigma} e^{-n/Y},
\]

\[
S_3 = \sum_{K \in B_{r_1} (X) \setminus A_r (X)} g_z(\sigma, \rho_K).
\]
Recall that Assumption 2.2 implies \( \#B_{\sigma_1}(X) \ll X^{1-\delta} \) with some \( \delta > 0 \). Hence the terms \( S_2 \) and \( S_3 \) are estimated as

\[
S_2 \ll X^{1-\delta} \max_{K \in L^+_1(X)} |g^+_\infty(\sigma, \rho_K; Y)| \quad \text{and} \quad S_3 \ll X^{1-\delta} \max_{K \in L^+_2(X) \setminus \Lambda_\sigma(X)} |g_\infty(\sigma, \rho_K)|.
\]

The main term is obtained from \( S_1 \), which is calculated by Lemma 5.5 as follows.

**Proposition 5.6.** Let \( \alpha, \beta \) be absolute constants for which Assumption 2.1 holds, and let \( \sigma > 1/2 \) be a real number. If we let \( Y = X^\eta \) with some \( \eta > 0 \), there exists a constant \( 0 < b_\sigma(\eta) \leq 1 \) depending only on \( \sigma \) and \( \eta \) such that

\[
S_1 = C^\pm \frac{X}{12\zeta(3)} F_\sigma(z) + O \left( X \exp \left( -\frac{\eta \log X}{2 \log \log X} + X^{5/6+\eta} + X^{\alpha+(1+\beta)\eta+\epsilon} \right) \right)
\]

for \( z \in \mathbb{C} \) with \( |z| \leq b_\sigma(\eta) R_\sigma(X) \) in Case I. Here, the implied constant depends only on \( \sigma \) and \( \eta \). The result remains valid in Case II if we replace \( F_\sigma(z) \) with \( F_\sigma^*(z) \).

**Proof.** By Lemma 5.5 the term \( S_1 \) is calculated as

\[
S_1 = C^\pm \frac{1}{12\zeta(3)} X \left( \sum_{n=1}^{\infty} \lambda_\infty(n) n^{-\sigma} e^{-n/Y} \right)
\]

\[
+ K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)\zeta(5/3)} X^{5/6} \left( \sum_{n=1}^{\infty} \mu_\infty(n) n^{-\sigma} e^{-n/Y} \right)
\]

\[
+ \left( X^{\alpha+\epsilon} \sum_{n=1}^{\infty} \nu_\infty(n) n^{-\sigma+\beta} e^{-n/Y} \right)
\]

in both cases. First, we investigate \( S_1 \) in Case I. Note that we have

\[
\sum_{r=0}^{\infty} \lambda_\infty(p^r) p^{-rs} = \sum_{\alpha \in \mathbb{C}} C_p(\alpha) \sum_{r=0}^{\infty} H_r(z; \alpha) p^{-rs} = F_{s,p}(z),
\]

where \( F_{s,p}(z) \) is defined as (2.7). Hence the equality

\[
\sum_{n=1}^{\infty} \lambda_\infty(n) n^{-s} = \prod_p F_{s,p}(z) = F_s(z)
\]

holds for any \( s, z \in \mathbb{C} \) with \( \text{Re}(s) > 1/2 \) by Proposition 4.1. This yields

\[
\sum_{n=1}^{\infty} \lambda_\infty(n) n^{-\sigma} e^{-n/Y} = \int_{\text{Re}(w)=c} F_{\sigma+w}(z) \Gamma(w) Y^w \, dw
\]

for any \( \sigma > 1/2 \) and \( c > 0 \). The function \( F_{\sigma+w}(z) \) is a holomorphic function on the half plane \( \text{Re}(w) > 1/2 - \sigma \) by Corollary 4.2. Hence, shifting the integral contour to the left, we obtain

\[
\sum_{n=1}^{\infty} \lambda_\infty(n) n^{-\sigma} e^{-n/Y} = F_\sigma(z) + \int_{\text{Re}(w)=-\kappa_1} F_{\sigma+w}(z) \Gamma(w) Y^w \, dw
\]

for any \( 0 < \kappa_1 < \sigma - 1/2 \). We take

\[
\kappa_1 = \begin{cases} 
2^{-1} \min \{ \sigma - 1/2, 1 - \sigma \} & \text{if } 1/2 < \sigma < 1, \\
\sigma - 1 + (\log \log X)^{-1} & \text{if } \sigma \geq 1.
\end{cases}
\]
so as to keep $1/2 < \Re(\sigma + w) < 1$ on the vertical line $\Re(w) = -\kappa_1$. Then we apply Proposition 4.4 which gives

$$F_{\sigma+w}(z) \ll \exp\left(c_1(\sigma)b_\sigma(\eta)\frac{\log X}{\log \log X}\right)$$

for $\Re(w) = -\kappa_1$ with some $c_1(\sigma) > 0$ depending only on $\sigma$. Furthermore, if the constant $b_\sigma(\eta)$ is small enough, then

$$\int_{\Re(w)=-\kappa_1} F_{\sigma+w}(z)\Gamma(w)Y^w \, dw \ll \exp\left(-\frac{\eta}{2} \frac{\log X}{\log \log X}\right)$$

follows with the implied constant depending only on $\sigma$ and $\eta$. Therefore we obtain

$$\sum_{n=1}^{\infty} \lambda_z(n)n^{-\sigma}e^{-n/Y} = F_{\sigma}(z) + O_{\sigma,\eta}\left(\exp\left(-\frac{\eta}{2} \frac{\log X}{\log \log X}\right)\right). \tag{5.14}$$

Next, we consider the second and the third terms of the right-hand side of (5.12). Recall that the inequality $|H_r(z;a)| \leq d_k(p')$ holds for any $a \in \mathcal{A}$, where we put $k = |2z| + 1$ as before. Hence we find $|\mu_z(p')|, |\nu_z(p')| \leq d_k(p')$ by definition. The inequalities $|\mu_z(n)|, |\nu_z(n)| \leq d_k(n)$ follow since they are multiplicative functions. Therefore, we deduce from Lemma 3.6 that

$$\sum_{n=1}^{\infty} \mu_z(n)n^{-\sigma}e^{-n/Y} \ll (Y^{1-\sigma} + 1)(C_1 \log Y)^{2(|z|+1)} \ll \eta X^\eta, \tag{5.15}$$

$$\sum_{n=1}^{\infty} \nu_z(n)n^{-\sigma+\beta}e^{-n/Y} \ll (Y^{1-\sigma+\beta} + 1)(C_1 \log Y)^{2(|z|+1)} \ll \eta X^{(1+\beta)\eta}. \tag{5.16}$$

Inserting (5.14), (5.15), and (5.16) to (5.12), we finally arrive at the desired conclusion. In Case II, we obtain

$$\sum_{n=1}^{\infty} \lambda_z(n)n^{-\sigma}e^{-n/Y} = F_{\sigma}(z) + O_{\sigma,\eta}\left(\exp\left(-\frac{\eta}{2} \frac{\log X}{\log \log X}\right)\right)$$

instead of (5.14). Furthermore, the second and the third terms are estimated by the latter estimate of Lemma 3.6. Combining them, we deduce the result in Case II.

\textbf{Proof of Theorem 2.7.} First, we prove the statement of (1). Let $\sigma \geq 1$ be a real number. Note that $\mathcal{A}_\sigma(X)$ is empty in this case. We choose a real number $\eta > 0$ small enough to save

$$X^{5/6+\eta} + X^{\alpha+(1+\beta)\eta+\epsilon} \ll X \exp\left(-\frac{\eta}{2} \frac{\log X}{\log \log X}\right) \quad \text{and} \quad X^\eta \ll X^{\delta/2}$$

with the constant $\delta > 0$ satisfying (5.11). In this case, we apply formula (5.9). First, we obtain the upper bound

$$S_2 \ll X^{1-\delta/2} \tag{5.17}$$

from (5.1) and (5.11). Towards the estimate of $S_3$, we evaluate $g_z(\sigma, \rho_K)$ as follows. If $\sigma > 1$, then we have

$$g_z(\sigma, \rho_K) \ll \exp(2|z|\log \zeta(\sigma)) \ll \exp\left(A_\sigma \frac{\log X}{(\log \log X)^2}\right).$$
with a constant $A_\sigma > 0$ depending only on $\sigma$. Furthermore, we obtain
\[ g_z(1, \rho_K) \ll \exp\left( B \log \log X \right) \ll \exp\left( \frac{B \log X}{\log \log X} \right) \]
by Lemma 3.4, where $B > 0$ is an absolute constant. Therefore, the estimate
\[ (5.18) \quad S_3 \ll \epsilon X^{1-\delta+\epsilon} \]
follows for every $\epsilon > 0$ from (5.11). As a result, we obtain
\[ \sum_{K \in L_2^\pm(X)} L(\sigma, \rho_K)^z = \frac{C^\pm X}{12\zeta(3)} F_\sigma(z) + O_{\sigma, \eta} \left( X \exp \left( -\frac{\eta \log X}{2 \log \log X} \right) \right) \]
by Proposition 5.6, (5.17), and (5.18). We proceed to the proof of (2), where we apply formula (5.9) again. Note that upper bound (5.17) remains valid. For the estimate of the term $S_3$, we recall that $\log L(\sigma, \rho_K)$ is real for any $\sigma \in \mathbb{R}$. Hence the equality
\[ |g_i\xi(\sigma, \rho_K)| = |\exp(i\xi \log L(\sigma, \rho_K))| = 1 \]
holds for $\xi \in \mathbb{R}$, which yields
\[ S_3 \ll X^{1-\delta} \]
in the case where $z = i\xi \in i\mathbb{R}$. Using this estimate instead of (5.18), we obtain the conclusion. It is remaining to prove the statement of (3). We apply formula (5.10) in this case, and thus we do not need any estimates for $S_3$. Recall that the term $S_2$ is estimated as (5.17). Therefore the result of (3) follows, and we complete the proof of Theorem 2.5. \qed

**Remark 5.7.** We can prove Theorem 2.11 by a similar argument of the proof of Theorem 2.5. The difference appears only in the case of $\sigma = 1$. Recall that we used Lemma 3.4 to evaluate $g_z(1, \rho_K)$ at $\sigma = 1$ in the proof of Theorem 2.5 (1). However, the method fails in Case II. Indeed, we know just $(L'/L)(1, \rho_K) \ll \log X$ for $K \in L_2^\pm(X)$ without any assumptions. This is why we exclude the case of $\sigma = 1$ in Theorem 2.11 (1). A related difficulty on studying the distribution of values $(L'/L)(1, \chi_d)$ was discussed in the thesis of Mourtada [35]. See also Lamzouri [29] for more information about the Euler–Kronecker constants of quadratic fields.

### 5.3. Secondary terms of the complex moments.

We assume GRH throughout this section. Recall that the subset $B_{\sigma_1}(X)$ obtained from (2.9) is empty in this case. Hence the terms $S_2$ and $S_3$ in formulas (5.9) and (5.10) disappear. Moreover, we see that Propositions 5.1 is improved as a consequence of GRH.

**Proposition 5.8.** Assume GRH, and let $\sigma > 1/2$ be a real number. Take a cubic field $K \in L_3^\pm(X)$ arbitrarily. Let $\kappa_1 = \kappa_1(\sigma)$ be a positive constant with $\sigma - \kappa_1 > 1/2$, which we take depending only on $\sigma$. If we let $Y = X^{\tilde{a}}$ with some $\tilde{a} > 0$, there exists an absolute constant $A > 0$ such that we have
\[ g_z(\sigma, \rho_K) = \sum_{n=1}^{\infty} \lambda_z(n, \rho_K)n^{-\sigma} e^{-n/Y} + O \left( X^{-\eta \kappa_1} \exp \left( A \frac{\log X}{\log \log X} \right) \right) \]
for any $z \in \mathbb{C}$ with $|z| \leq \tilde{R}_\sigma(X)$, where $\tilde{R}_\sigma(X)$ is obtained from (2.14) with a positive constant $a(\sigma) \leq \min \{1, 2\sigma - 1 - 2\kappa_1\}$. Here, the implied constant depends only on $\sigma$ and $\eta$. 
Proof. By Lemmas 5.2 and 5.3 it is sufficient to prove the estimate
\[ g_{\sigma}^{-}(\sigma, \rho_K; Y) \ll X^{-\kappa_1} \exp \left( A \frac{\log X}{\log \log X} \right) \]
for \(|z| \leq \tilde{R}_\sigma(X)\) under GRH. Suppose that \((\log \log X)^{-1} \leq \kappa_1\) is satisfied. Let \(c > 0\) be a real number as in (5.3). Then we change the integral contour \(L^- = L_1 + \cdots + L_5\) to \(\tilde{L}_1 + \cdots + \tilde{L}_5\) which is obtained by connecting the points \(c - i\infty, c - i(\log X)^2, -\kappa_1 - i(\log X)^2, -\kappa_1 + i(\log X)^2, c + i(\log X)^2, c + i\infty\), in order. Note that we do not encounter any poles while shifting the contour as a consequence of GRH. Thus we obtain
\[ (5.19) \quad g_{\sigma}^{-}(\sigma, \rho_K; Y) = \int_{\tilde{L}_1 + \cdots + \tilde{L}_5} g_{\sigma}(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw. \]
The integrals over the contours \(\tilde{L}_1\) and \(\tilde{L}_5\) are estimated as in (5.6). We obtain
\[ \int_{\tilde{L}_1 \cup \tilde{L}_5} g_{\sigma}(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw \ll \eta \exp \left( -(-\log X)^2 \right). \]
Note that \(\tilde{R}_\sigma(X)(\log X)^{2-2(\sigma+w)} \ll \log X \log \log X\) holds for \(w \in \tilde{L}_2 \cup \tilde{L}_3 \cup \tilde{L}_4\) by the choice of the constant \(a(\sigma)\). Then, we apply Lemma 3.2 to bound \(\log L(\sigma + w, \rho_K)\) or \((L'/L)(\sigma + w, \rho_K)\) for \(w \in \tilde{L}_2 \cup \tilde{L}_3 \cup \tilde{L}_4\). As a result, we obtain
\[ \int_{\tilde{L}_2 \cup \tilde{L}_4} g_{\sigma}(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw \ll \exp \left( -(-\log X)^2 \right), \]
\[ \int_{\tilde{L}_3} g_{\sigma}(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw \ll X^{-\kappa_1} \exp \left( A \frac{\log X}{\log \log X} \right) \]
for some \(A > 0\), where the implied constants depend only on \(\sigma\) and \(\eta\). Hence we obtain the desired result from (5.19).

Furthermore, Proposition 5.6 is improved as follows if we let \(\sigma > 2/3\).

Proposition 5.9. Let \(\alpha, \beta > 0\) be absolute constants for which Assumption 2.7 holds, and let \(\sigma > 2/3\) be a real number. Depending only on \(\sigma\), we take two positive constants \(\kappa_j = \kappa_j(\sigma)\) satisfying \(1/2 < \sigma - \kappa_1 < 1\) and \(2/3 < \sigma - \kappa_2 < 1\). If we let \(Y = X^\eta\) with some \(\eta > 0\), there exists a constant \(0 < b_\sigma \leq 1\) depending only on \(\sigma\) such that
\[ S_1 = C^\pm \frac{1}{12\xi(3)} X^{\mathcal{F}_\sigma(z)} + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^2\xi(5/3)} X^{5/6} \mathcal{G}_\sigma(z) \]
\[ + O \left( X^{1-\kappa_1} + X^{5/6-\kappa_2} + X^{\alpha+\eta(1-\sigma+\beta)+\epsilon} + X^{\alpha+\epsilon} \exp \left( \frac{\log X}{\log \log X} \right) \right). \]
for every \(\epsilon > 0\) and for \(z \in \mathbb{C}\) with \(|z| \leq b_\sigma \tilde{R}_\sigma(X)\) in Case I, where \(\tilde{R}_\sigma(X)\) is obtained from (2.14) with \(a(\sigma) \leq 1/2\). Here, the implied constant depends only on \(\sigma, \eta, \) and \(\epsilon\). The result remains valid in Case II if we replace \(\mathcal{F}_\sigma(z)\) and \(\mathcal{G}_\sigma(z)\) with \(\mathcal{F}_{\sigma^*}(z)\) and \(\mathcal{G}_{\sigma^*}(z)\), respectively.

Proof. We begin with formula (5.12) in Case I. Recall that the series in the first term of the left-hand side of (5.12) is calculated as (5.13). Then, we deduce from Proposition 4.4 the estimate
\[ \mathcal{F}_{\sigma+w}(z) \ll \exp \left( c_1(\sigma)b_\sigma \frac{\log X}{\log \log X} \right) \]
for \( \text{Re}(w) = -\kappa_1 \), where \( c_1(\sigma) \) is a positive constant depending only on \( \sigma \). Hence we obtain

\[
\sum_{n=1}^{\infty} \lambda_z(n)n^{-\sigma}e^{-n/Y} = F_\sigma(z) + O \left( X^{-\eta \kappa_1} \exp \left( c_1(\sigma) b_\sigma \frac{\log X}{\log \log X} \right) \right).
\]

Also, the series in the second term of the left-hand side of (5.12) is calculated as

\[
\sum_{n=1}^{\infty} \nu_z(n)n^{-\sigma}e^{-n/Y} = G_\sigma(z) + O \left( X^{-\eta \kappa_2} \exp \left( c_2(\sigma) b_\sigma \frac{(\log X)^{3/4}}{\log \log X} \right) \right),
\]

with some \( c_2(\sigma) > 0 \) depending only on \( \sigma \). Finally, we deduce from Lemma 3.6 the estimate

\[
\sum_{n=1}^{\infty} \nu_z(n)n^{-\sigma+\beta}e^{-n/Y} \ll (X^{\eta(1-\sigma+\beta)} + 1) \exp \left( c_3 b_\sigma (\log X)^{1/2} \right)
\]

similarly to (5.16), where \( c_3 > 0 \) is an absolute constant. Then, if \( b_\sigma > 0 \) is small enough, we obtain the conclusion by combining (5.20), (5.21), and (5.22). The result in Case II is proved in a similar way.

**Proof of Theorems 2.8 and 2.14** By Propositions 5.8 and 5.9, we obtain

\[
\sum_{K \in L^1_1(X)} g_z(\sigma, \rho_K) = C^z + \frac{1}{12\zeta(3)} X F_\sigma(z) + K^z + \frac{4\zeta(1/3)}{5\Gamma(2/3)^3 \zeta(5/3)} X^{5/6} G_\sigma(z) + E
\]

in Case I, where the error term \( E \) is estimated as

\[
E \ll \left( X^{1-\eta \kappa_1} + X^{5/6-\eta \kappa_2} + X^{\alpha + \eta(1-\sigma+\beta)+\epsilon} + X^{\alpha+\epsilon} \right) \exp \left( A \frac{\log X}{\log \log X} \right)
\]

\[
\ll X^{5/6+\epsilon} \left( X^{-\eta \kappa_1+1/6} + X^{-\eta \kappa_2} + X^{-(5/6-\alpha)+\eta(1-\sigma+\beta)} + X^{-(5/6-\alpha)} \right)
\]

for every \( \epsilon > 0 \). Note that the same estimate is valid in Case II if we replace \( F_\sigma(z) \) and \( G_\sigma(z) \) with \( F^*_\sigma(z) \) and \( G^*_\sigma(z) \) in (5.23), respectively. First, we consider the case where \( \sigma \geq 1 + \beta \). In this case, we take two constants \( \kappa_j = \kappa_j(\sigma) > 0 \) with \( 1/2 < \sigma - \kappa_1 < 1 \) and \( 2/3 < \sigma - \kappa_2 < 1 \) arbitrarily. Then, we choose a real number \( \eta \) so that \( \eta \kappa_1 \geq 1/3 \) is satisfied. In this setting, we obtain

\[
E \ll X^{5/6} \left( X^{-\eta \kappa_1+1/6} + X^{-\eta \kappa_2} + X^{-(5/6-\alpha)} \right) \ll X^{5/6+\epsilon-\delta}
\]

with some constant \( \delta = \delta(\sigma) > 0 \) depending only on \( \sigma \). Hence Theorems 2.8 and 2.14 follows for \( \sigma \geq 1 + \beta \) if we let \( \epsilon > 0 \) be small enough. In the case where \( \sigma < 1 + \beta \), we need to choose \( \kappa_1 \) and \( \eta \) more carefully. We suppose \( \sigma_2 < \sigma < 1 + \beta \), where the constant \( \sigma_2 \geq 2/3 \) is given by (2.15). Note that the inequality

\[
\frac{1 - \sigma + \beta}{5 - 6\alpha} \leq \sigma - \frac{1}{2}
\]

holds by the choice of \( \sigma_2 \). Hence we can take a constant \( \kappa_1 = \kappa_1(\sigma) > 0 \) so that

\[
\frac{1}{2} < \sigma - \kappa_1 < 1 \quad \text{and} \quad \frac{1}{6} - \frac{(1-\alpha)\kappa_1}{1-\sigma+\beta+\kappa_1} < 0
\]
are satisfied. Then, we choose a real number \( \eta \) as
\[
\eta = \frac{1 - \alpha}{1 - \sigma + \beta + \kappa_1} > 0
\]
to save \( X^{-\eta \kappa_1 + 1/6} = X^{-(5/6-\alpha)+\eta(1-\sigma+\beta)} \). Finally, we take \( \kappa_2 = \kappa_2(\sigma) > 0 \) with \( 2/3 < \sigma - \kappa_2 < 1 \) arbitrarily. From the above, we conclude
\[
E \ll X^{5/6} \left( X^{-\eta \kappa_1 + 1/6} + X^{-\eta \kappa_2} \right) \ll X^{5/6+\epsilon-\delta},
\]
where \( \delta = \delta(\sigma) > 0 \) is a constant depending only on \( \sigma \). Letting \( \epsilon > 0 \) be small enough, we obtain the desired result. \( \square \)

5.4. Small complex moments. Theorem 2.5 yields the following corollary, which is a generalization of Theorem 1.4 of Cho–Kim [10].

Corollary 5.10. Let \( \sigma_1 \) be an absolute constant for which Assumption 2.2 holds.
For every integer \( k \geq 0 \), there exists an absolute constant \( \delta > 0 \) such that
\[
\sum_{K \in L_\pm^+(X) \setminus E_\sigma(X)} (\log L(\sigma, \rho_K))^k = \frac{C^{\pm} X}{12\zeta(3)} \int_{-\infty}^{\infty} x^k C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)
\]
for any \( \sigma > \sigma_1 \) with the implied constant depending only on \( k \) and \( \sigma \), where the subset \( E_\sigma(X) \) is defined as
\[
E_\sigma(X) = \begin{cases} 
\emptyset & \text{if } \sigma \geq 1, \\
B_{\sigma_1}(X) & \text{if } \sigma_1 < \sigma < 1.
\end{cases}
\]

Proof. We define two complex functions \( F \) and \( G \) as
\[
F(z) = \sum_{K \in L_\pm^+(X) \setminus E_\sigma(X)} \exp \left( z \log L(\sigma, \rho_K) \right),
\]
\[
G(z) = \frac{C^{\pm} X}{12\zeta(3)} \int_{-\infty}^{\infty} e^{zx} C_\sigma(x) \frac{dx}{\sqrt{2\pi}}.
\]

Then the function \( F \) is holomorphic on the whole complex plane. Also, we deduce from Corollary 4.3 that \( G \) is a holomorphic function on \( \mathbb{C} \). Furthermore, applying Theorem 2.5, we obtain
\[
F(z) - G(z) \ll_{\sigma_1} X \exp \left( -\delta \frac{\log X}{\log \log X} \right)
\]
for any \( |z| \leq b_\sigma R_\sigma(X) \), where \( b_\sigma > 0 \) is a constant depending only on \( \sigma \), and \( R_\sigma(X) \) is given by (2.12). Hence, as a consequence of the Cauchy integral formula, we conclude
\[
\sum_{K \in L_\pm^+(X) \setminus E_\sigma(X)} (\log L(\sigma, \rho_K))^k = \frac{C^{\pm} X}{12\zeta(3)} \int_{-\infty}^{\infty} x^k C_\sigma(x) \frac{dx}{\sqrt{2\pi}}
\]
\[
\quad = \frac{d^k}{dz^k} (F(z) - G(z)) \bigg|_{z=0} \ll_{k, \sigma} X \exp \left( -\delta \frac{\log X}{\log \log X} \right)
\]
as desired. \( \square \)
We apply Corollary 5.10 to study the $i\xi$-th moments of $L(\sigma, \rho_K)$ when $\xi$ is a small real number. The result is used in the proof of Corollary 2.6.

**Proposition 5.11.** Let $\sigma_1$ be an absolute constant for which Assumption 2.2 holds. For each $\sigma > \sigma_1$, we take a subset $E_\sigma(X)$ as in (5.24). Then we have

$$
\frac{1}{\#(L_3^0(X) \setminus E_\sigma(X))} \sum_{K \in L_3^0(X) \setminus E_\sigma(X)} L(\sigma, \rho_K)^{i\xi} - \mathcal{F}_\sigma(i\xi) \ll |\xi|
$$

for $\xi \in \mathbb{R}$, where the implied constant depends only on $\sigma$.

**Proof.** Recall that $e^{i\theta} = 1 + O(|\theta|)$ holds uniformly for $\theta \in \mathbb{R}$. Then we have

$$
L(\sigma, \rho_K)^{i\xi} - 1 \ll |\xi| \cdot |\log L(\sigma, \rho_K)|
$$

for any $K \in L_3^0(X) \setminus E_\sigma(X)$. Hence, by Cauchy’s inequality, the estimate

$$
\frac{1}{\#(L_3^0(X) \setminus E_\sigma(X))} \sum_{K \in L_3^0(X) \setminus E_\sigma(X)} L(\sigma, \rho_K)^{i\xi} - 1
$$

$$
\ll |\xi| \left( \frac{1}{\#(L_3^0(X) \setminus E_\sigma(X))} \sum_{K \in L_3^0(X) \setminus E_\sigma(X)} (\log L(\sigma, \rho_K))^2 \right)^{1/2}
$$

holds. Applying Corollary 5.10 with $k = 2$, we further deduce

$$
\frac{1}{\#(L_3^0(X) \setminus E_\sigma(X))} \sum_{K \in L_3^0(X) \setminus E_\sigma(X)} L(\sigma, \rho_K)^{i\xi} - 1 \ll |\xi|,
$$

where the implied constant depends only on $\sigma$. Similarly, we obtain

$$
\mathcal{F}_\sigma(i\xi) - 1 \ll |\xi| \left( \int_{-\infty}^{\infty} x^2 C_\sigma(x) \frac{dx}{\sqrt{2\pi}} \right)^{1/2} \ll_{\sigma} |\xi|.
$$

Therefore the result follows by (5.25) and (5.26). □

We obtain similar results for logarithmic derivatives, but we skip the proof.

**Corollary 5.12.** Let $\sigma_1$ be an absolute constant for which Assumption 2.2 holds. For every integer $k \geq 0$, there exists an absolute constant $\delta > 0$ such that

$$
\sum_{K \in L_3^0(X) \setminus E_\sigma(X)} \left( \frac{L'}{L}(\sigma, \rho_K) \right)^k
$$

$$
= \frac{O \pm X}{12\zeta(3)} \int_{-\infty}^{\infty} x^k C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)
$$

for any $\sigma > \sigma_1$ with the implied constant depending only on $k$ and $\sigma$, where the subset $E_\sigma(X)$ is given by

$$
E_\sigma(X) = \begin{cases} 
\emptyset & \text{if } \sigma > 1, \\
B_{\sigma_1}(X) & \text{if } \sigma_1 < \sigma \leq 1.
\end{cases}
$$

(5.27)
Proposition 5.13. Let $\sigma_1$ be an absolute constant for which Assumption 2.2 holds. For $\sigma > \sigma_1$, we take a subset $\mathcal{E}_\sigma(X)$ as in (5.27). Then we have

$$\frac{1}{\#(L_3^+(X) \setminus \mathcal{E}_\sigma(X))} \sum_{K \in L_3^+(X) \setminus \mathcal{E}_\sigma(X)} \exp \left( i\xi \frac{L'(\sigma, \rho_K)}{L(\sigma, \rho_K)} \right) - F_\sigma'(i\xi) \ll |\xi|$$

for $\xi \in \mathbb{R}$, where the implied constant depends only on $\sigma$.

6. Completion of proofs

Proof of Corollaries 2.6 and 2.12. Let $\phi$ and $\psi$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined as

$$\phi(A) = \frac{\# \left\{ K \in L_3^+(X) \setminus \mathcal{E}_\sigma(X) \mid \log L(\sigma, \rho_K) \in A \right\}}{\#(L_3^+(X) \setminus \mathcal{E}_\sigma(X))},$$

$$\psi(A) = \int_A C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

for $A \in \mathcal{B}(\mathbb{R})$, where $\mathcal{E}_\sigma(X)$ is given by (5.24). Note that $\psi$ is absolutely continuous, and that the density $C_\sigma$ is continuous. Then we apply Lemma 3.4 to obtain

$$F_\phi(a) - F_\psi(a) \ll \frac{1}{R} \sup_{x \in \mathbb{R}} C_\sigma(x) + \int_{-R}^R \left| \Lambda(\xi; \phi) - \Lambda(\xi; \psi) \right| d\xi$$

(6.1)

for any $a \in \mathbb{R}$, where we take $R = b_\sigma R_\sigma(X)$ as in Theorem 2.5. By the definitions of $\phi$ and $\psi$, the left-hand side of (6.1) is calculated as

$$F_\phi(a) - F_\psi(a) = \frac{\# \left\{ K \in L_3^+(X) \setminus \mathcal{A}_\sigma(X) \mid \log L(\sigma, \rho_K) \leq a \right\}}{\#L_3^+(X)}$$

$$- \int_{-\infty}^a C_\sigma(x) \frac{dx}{\sqrt{2\pi}}\right) + O \left( X^{5/6} + X^{1-\delta} \right)$$

(6.2)

since Assumptions 2.1 and 2.2 yield the asymptotic formulas

$$\#(L_3^+(X) \setminus \mathcal{A}_\sigma(X)) = \frac{C^+ X}{12\zeta(3)} + O \left( X^{5/6} + X^{1-\delta} \right),$$

$$\#(L_3^+(X) \setminus \mathcal{E}_\sigma(X)) = \frac{C^- X}{12\zeta(3)} + O \left( X^{5/6} + X^{1-\delta} \right).$$

Also, the integral in the right-hand side of (6.1) is calculated as

$$\int_{-R}^R \left| \frac{\Lambda(\xi; \phi) - \Lambda(\xi; \psi)}{\xi} \right| d\xi = \left( \int_{-R'}^R + \int_{-R'}^{-R'} + \int_{-R'}^{R'} \right) \left| \frac{h(\xi)}{\xi} \right| d\xi$$

for any $0 < R' < R$, where

$$h(\xi) = \Lambda(\xi; \phi) - \Lambda(\xi; \psi)$$

$$= \frac{1}{\#(L_3^+(X) \setminus \mathcal{E}_\sigma(X))} \sum_{K \in L_3^+(X) \setminus \mathcal{E}_\sigma(X)} L(\sigma, \rho_K)^i\xi - F_\sigma(i\xi).$$
We take $R' = (\log X)^{-2}$. Then, by Theorem 2.25 and Proposition 5.11 we estimate $h(\xi)$ as

$$h(\xi) \ll_{\sigma} \begin{cases} |\xi| & \text{for } |\xi| \leq R', \\ \exp \left( -\delta \frac{\log X}{\log \log X} \right) & \text{for } R' \leq |\xi| \leq R. \end{cases}$$

Inserting these estimates to (6.1), we obtain

$$F(\phi(a) - F(\psi(a) \ll 1 + R' + \left( \log \frac{R}{R'} \right) \exp \left( -\delta \frac{\log X}{\log \log X} \right) \ll \frac{1}{R_{\sigma}(X)} \right) \tag{6.3}$$

with the implied constants depending only on $\sigma$. By (6.2) and (6.3), we complete the proof of Corollary 2.6. We can prove Corollary 2.12 by replacing the probability measures $\phi$ and $\psi$ with

$$\phi^*(A) = \# \left\{ K \in L^3_+ \setminus \mathcal{E}_\sigma(X) \mid (L'/L)(\sigma, \rho_K) \in A \right\},$$

$$\psi^*(A) = \int_A C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

in the above arguments, where $\mathcal{E}_\sigma(X)$ is also replaced with (5.27).

Proof of Corollary 2.7. Let $r > -2$ be a fixed real number. By equality (1.7), we deduce that

$$\sum_{K \in L^3_+ \setminus \mathcal{E}_\sigma(X)} \left( \frac{h_K R_K}{\sqrt{|d_K|}} \right)^{r} = \frac{C^\pm F_1(r)}{12 \zeta(3)(D^\pm)^r} X + O_r \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right) \tag{6.4}$$

from Theorem 2.25 with $z = r$ and $\sigma = 1$. For an integer $d > 0$, we put

$$f(d) = \sum_{K \in L^3_+ \setminus \mathcal{E}_\sigma(X) \mid |d_K| = d} \left( \frac{h_K R_K}{\sqrt{|d_K|}} \right)^{r}.$$ 

Then, by the partial summation, we obtain

$$\sum_{K \in L^3_+ \setminus \mathcal{E}_\sigma(X)} (h_K R_K)^r = \sum_{0 \leq d \leq X} f(d)d^{r/2}$$

$$= F(X)X^{r/2} - \frac{r}{2} \int_1^X F(x)x^{r/2-1} \, dx$$

with $F(x) = \sum_{0 \leq d \leq x} f(d)$. Note that the function $F$ is estimated as

$$F(x) = \frac{C^\pm F_1(r)}{12 \zeta(3)(D^\pm)^r} x + O_r \left( x \exp \left( -\delta \frac{\log x}{\log \log x} \right) \right)$$

by (6.4). Inserting it to (6.5), we obtain the desired result.

Remark 6.1. If we let $r = 1$ in Corollary 2.7, we obtain

$$\sum_{K \in L^3_+ \setminus \mathcal{E}_\sigma(X)} h_K R_K = \frac{C^\pm F_1(1)}{D^\pm 18 \zeta(3)} X^{3/2} + O \left( X^{3/2} \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right).$$
Lemma 3.11 are satisfied. Note that the results of Theorems 2.3 and 2.9 imply that

\[ C \]

Proof of Theorems 2.4 and 2.10.

6.2. 

\[ \sigma \]

Corollary 2.13 is a simple application of Theorem 2.11 with

\[ \sigma \]

\[ E \]

with some subset

\[ F \]

expressed as

\[ | \]

\[ convergence \]

\[ C \]

Hence

\[ We \]

\[ have \]

\[ C \]

Therefore, we conclude that

\[ \sum_{K \in L^\pm_3(X)} h_K R_K = \frac{C^\pm}{D^\pm} 4eX^{3/2} + O \left( X^{3/2} \exp \left( -\frac{\delta \log X}{\log \log X} \right) \right) \]

with

\[ c = \frac{\pi^2 \zeta(3) \zeta(2)}{72} \prod_p (1 + p^{-2} - 2p^{-3} - 2p^{-4} + 2p^{-6} + p^{-7} - p^{-8}) \]

as desired in Theorem 1.2.

Proof of Corollary 2.13. Recall that \( \gamma_K \) is obtained as \( \gamma_K = (L'/L)(1, \rho_K) + \gamma \). Then Corollary 2.13 is a simple application of Theorem 2.11 with \( \sigma = 1 \).

6.2. Proof of Theorems 2.4 and 2.10. We first check that the assumptions of Lemma 3.11 are satisfied. Note that the results of Theorems 2.3 and 2.9 imply that \( C_\sigma(x) \) and \( C_\sigma(x) \) belong to the class \( \Lambda(\mathbb{R}) \). Furthermore, we obtain the equalities

\[ \int_{-\infty}^{\infty} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} = \mathcal{F}_\sigma(0) = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} = \mathcal{F}_\sigma^*(0) = 1. \]

Hence \( C_\sigma(x) \) and \( C_\sigma(x) \) are good density functions in the sense of Section 3.4. Let \( \sigma \) be an absolute constant for which Assumption 2.2 holds, and let \( \sigma > \sigma \). For every positive integer \( n \), we take a finite set

\[ X_n = L^\pm_3(n) \setminus E(n) \]

with some subset \( E(X) \subseteq A_\sigma(X) \). Every \( X_n \) is equipped with a probability measure \( \omega_n \) given by \( \omega_n(K) = (\#X_n)^{-1} \) for every \( K \in X_n \). Finally, we define \( \ell_n : X_n \to \mathbb{R} \) as

\[ \ell_n(K) = \begin{cases} \log L(\sigma, \rho_K) & (\text{Case I}), \\ (L'/L)(\sigma, \rho_K) & (\text{Case II}). \end{cases} \]

Note that we have

\[ \#X_n \sim \#L^\pm_3(n) \sim \frac{C^\pm n}{12\zeta(3)} \]

as \( n \to \infty \) since \( \#E(X) \leq \#A_\sigma(X) \ll X^{1-\delta} \) holds by Assumption 2.2. Therefore, with the above setting, we find that (3.17) is equivalent to (2.11) in Case I, and to (2.18) in Case II. Furthermore, Theorems 2.3 (2) and 2.11 (2) imply that condition (3.17) is satisfied with \( E(X) = A_\sigma(X) \) and \( \Phi(x) = e^{ix\xi} \) for any \( \xi \in \mathbb{R} \), and that the convergence is uniform in \( |\xi| \leq R \) for any fixed real number \( R > 0 \).
Proof of Theorems 2.4 (0) and 2.10 (1). Let \( \mathcal{E}(X) \) be empty, and take a continuous function \( \Phi \) on \( \mathbb{R} \) arbitrarily. Then we define a continuous function \( \phi_0 \) on \( [0, \infty) \) as
\[
\phi_0(r) = \max_{|x| \leq r} |\Phi(x)|
\]
for \( r \geq 0 \), which is non-decreasing and satisfies \( \phi_0(r) > 0 \), \( \phi_0(r) \to \infty \) as \( r \to \infty \).

We check that \( \phi_0 \) satisfies conditions (3.18) and (3.19) as follows. Since we have \( \sigma > 1 \), the inequalities
\[
|\log L(\sigma, \rho_K)| \leq 2 \log \zeta(\sigma) \quad \text{and} \quad \left| \frac{L'}{L}(\sigma, \rho_K) \right| \leq 2 \left| \frac{\zeta'}{\zeta}(\sigma) \right|
\]
are valid. Hence we deduce that there exists \( A_{\sigma} > 0 \) depending only on \( \sigma \) such that
\[
\phi_0(\ell_n(K)) \leq \phi_0(A_{\sigma})
\]
for any \( K \in X_n \), which implies that condition (3.18) holds. Then, we recall that \( C_\sigma \) and \( C'_\sigma \) are compactly supported if \( \sigma > 1 \). Therefore (3.19) is also satisfied. By Lemma 3.11 (ii), we conclude that (3.17) holds since \( \Phi \) satisfies \( |\Phi(x)| \leq \phi_0(|x|) \). It is remaining to prove that (3.17) holds for \( \Phi \in I(\mathbb{R}) \). If \( \Phi \) is the indicator function of either a compact subset of \( \mathbb{R} \) or the complement of such a subset, then Lemma 3.11 (iii) yields the result. Next, by Lemma 3.11 (i), we see that the probability measure defined as
\[
\phi_n(A) = \frac{\# \{ K \in X_n \mid \ell_n(K) \in A \}}{\# X_n}
\]
converges weakly to
\[
\phi(A) = \begin{cases} 
\phi_\sigma(A) & \text{(Case I)}, \\
\phi'_\sigma(A) & \text{(Case II)}
\end{cases}
\]
as \( n \to \infty \), where \( \phi_\sigma \) and \( \phi'_\sigma \) are the probability measures whose densities are given by \( C_\sigma \) and \( C'_\sigma \), respectively. Hence we obtain that (3.17) holds if \( \Phi \) is the indicator function of a continuity set of \( \mathbb{R} \). From the above, the proof of Theorems 2.4 (0) and 2.10 (1) are completed.

Proof of Theorem 2.4 (2). Let \( \mathcal{E}(X) \) be empty. In this case, we define the continuous function \( \phi_0 \) as
\[
\phi_0(r) = Ce^{ar},
\]
where \( C \) is some positive constant. Then Theorem 2.5 (1) implies that condition (3.18) is satisfied. Furthermore, (3.19) is valid since \( C_\sigma \) belongs to the space \( L^1(\mathbb{R})_\infty \). Hence Lemma 3.11 (b) yields that (3.17) holds for any continuous function \( \Phi \) satisfying \( |\Phi(x)| \leq Ce^{ax} \). The fact that (3.17) holds for \( \Phi \in I(\mathbb{R}) \) is proved by an similar argument in the case of \( \sigma > 1 \).

Proof of Theorems 2.4 (3) and 2.10 (3). Let \( \mathcal{E}(X) = A_\sigma(X) \). By Lemma 3.11 (a), (3.17) holds hold for any \( \Phi \in C^{\exp}_0(\mathbb{R}) \). Also, we see that (3.17) holds for \( \Phi \in I(\mathbb{R}) \).

Proof of Theorems 2.4 (4) and 2.10 (4). Let \( \mathcal{E}(X) = B_\sigma(X) \). We again take a continuous function \( \phi_0 \) as \( \phi_0(r) = Ce^{ar} \). Then Theorems 2.5 (3) and 2.11 (3) imply that (3.17) holds. As we checked before, condition (3.19) is satisfied. Therefore, Lemma 3.11 (b) yields that (3.17) holds for any \( \Phi \in C^{\exp}_a(\mathbb{R}) \). Finally, we deduce that (3.17) holds for \( \Phi \in I(\mathbb{R}) \) by a similar argument in the case of \( \sigma > 1 \).
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