On weighted measure of inaccuracy for doubly truncated random variables

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ABSTRACT

Recently, authors have studied the weighted version of Kerridge inaccuracy measure for left/right truncated distributions. In the present communication we introduce the notion of weighted interval inaccuracy measure and study it in the context of two-sided truncated random variables. In reliability theory and survival analysis, this measure may help to study the various characteristics of a system/component when it fails between two time points. We study various properties of this measure, including the effect of monotone transformations, and obtain its upper and lower bounds. It is shown that the proposed measure can uniquely determine the distribution function and characterizations of some important life distributions have been provided. This new measure is a generalization of recent weighted residual (past) inaccuracy measure.

1. Introduction

The idea of information theoretic entropy was introduced by Shannon (1948) and Weiner (1949). Shannon was the one who formally introduced entropy, known as Shannon's entropy or Shannon's information measure, into information theory, and characterized the properties of information sources and of communication channels to analyze the outputs of these sources.

Let us consider an absolutely continuous non negative random variable $X$ with probability density function $f$, distribution function $F$, and survival function $F^c = 1 - F$. Then the Shannon's information measure or the differential entropy of $X$ is given by

$$H_X = - \int_0^\infty f(x) \ln f(x) \, dx$$

(1)

which measures the expected uncertainty contained in $f(\cdot)$ about the predictability of an outcome of $X$.

Since the pioneering contributions by Shannon and Weiner, numerous efforts have been made to enrich and extend the underlying information theory. One important development in this direction is inaccuracy measure due to Kerridge (1961) which can be thought of as a generalization of Shannon's entropy. It has been extensively used as a useful tool for measurement of error in experimental results. Nath (1968) extended Kerridge's inaccuracy measure to the case of continuous situation and discussed some properties. If $F(x)$ is the actual distribution
corresponding to the observations and \( G(x) \) is the distribution assigned by the experimenter and \( f, g \) are the corresponding density functions, then the inaccuracy measure is defined as

\[
H_{X,Y} = - \int_0^\infty f(x) \ln g(x) dx
\]  

(2)

It has applications in statistical inference and coding theory. When \( g(x) = f(x) \), then (2) becomes (1), the Shannon’s entropy.

It is well-known that Shannon entropy is a shift independent measure. However, in certain applied contexts it is desirable to deal with shift-dependent information measures. In analogy with Belis and Guiaşu (1968), Di Crescenzo and Longobardi (2006) considered the notion of weighted entropy

\[
H^w_X = - \int_0^\infty x f(x) \ln f(x) dx
\]

(3)

This yields a length-biased shift-dependent information measure assigning greater importance to larger values of \( X \). The use of weighted entropy (3) is also motivated by the need, arising in various communication and transmission problems, of expressing the usefulness of events by means of an information measure. In agreement with Taneja and Tuteja (1986), here we consider the weighted inaccuracy measure

\[
H^w_{X,Y} = - \int_0^\infty x f(x) \ln g(x) dx
\]

(4)

which is a quantitative–qualitative measure of inaccuracy associated with the statement of an experimenter. When \( g(x) = f(x) \), then (4) becomes (3), the weighted entropy. For more properties of quantitative–qualitative measure of inaccuracy one may refer to Prakash and Taneja (1986) and Bhatia and Taneja (1991), among others. The following example illustrates the importance of qualitative characteristic of information as reflected in the definition of weighted inaccuracy measure.

**Example 1.1.** Let \( X_1 \) and \( Y_1 \) denote random lifetimes of two components with probability density functions \( f_1(x) = x/2, \ x \in (0, 2), \) and \( g_1(x) = (2 - x)/2, \ x \in (0, 2), \) respectively. By simple calculations, we have \( H_{X_1,Y_1} = H_{Y_1,X_1} = 3/2. \) But,

\[
H^w_{X_1,Y_1} = \frac{22}{9} \quad \text{and} \quad H^w_{Y_1,X_1} = \frac{5}{9}
\]

Therefore, the inaccuracy measure of the observer for the observations \( X_1 \) (resp. \( Y_1 \)) taking \( Y_1 \) (resp. \( X_1 \)) as corresponding assigned outcomes by the experimenter is identical. Instead, \( H^w_{X_1,Y_1} > H^w_{Y_1,X_1} \), i.e., weighted inaccuracy of the observer for \( (X_1, Y_1) \) is higher than that for \( (Y_1, X_1) \). As a matter of fact, the inaccuracies measured from a quantitative point of view, neglecting the qualitative side, are identical. To distinguish them, we must take into account the qualitative characteristic as given in (4).

Analogous to weighted residual and past entropies, Kumar et al. (2010) and Kumar and Taneja (2012) introduced the notion of weighted residual inaccuracy measure given by

\[
H^w_{X,Y}(t) = - \int_t^\infty x f(x) \ln \left( \frac{g(x)}{G(t)} \right) dx
\]

(5)
and weighted past inaccuracy measure given by
\[
\bar{H}^w(t, t_2) = -\int_{t_1}^{t_2} x \frac{f(x)}{F(t_2) - F(t_1)} \ln \left( \frac{g(x)}{G(t_2) - G(t_1)} \right) dx
\]
and studied their properties in analogy with weighted residual entropy and weighted past entropy, respectively. For \( t = 0 \), (5) reduces to (4) and for \( t = \infty \), (6) reduces to (4). Various aspects of (5) and (6) have been discussed in Kundu (2014).

The rest of the paper is arranged as follows. In Section 2 we introduce the concept of weighted interval inaccuracy measure for doubly truncated random variables. We obtain upper and lower bounds for weighted interval inaccuracy measure. In Section 3 we provide characterizations of quite a few useful continuous distributions based on this newly introduced measure including its uniqueness property. The effect of monotone transformations on the weighted interval inaccuracy measure has been discussed in Section 4.

2. Weighted interval inaccuracy measure

Sometimes, in many practical situations we have information about the lifetime only between two time points, so studying the reliability measures under the condition of doubly truncated random variables is necessary. Recently, Sunoj et al. (2009) and Misagh and Yari (2010, 2012) explored the use of weighted information measures for doubly truncated random variables and obtained some characterization results. Furthermore, Misagh and Yari (2011) explored the use of weighted information measures for doubly truncated random variables. Motivated by this, we introduce the notion of weighted interval inaccuracy measure for doubly truncated random variables.

Let us consider two non negative absolutely continuous doubly truncated random variables \((X|t_1 \leq X \leq t_2)\) and \((Y|t_1 \leq Y \leq t_2)\) where \((t_1, t_2) \in D = \{(u, v) \in \mathbb{R}^2_+ : F(u) < F(v) \text{ and } G(u) < G(v)\}\). Then the interval inaccuracy measure of \(X\) and \(Y\) at interval \((t_1, t_2)\) is given by
\[
H_{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dx
\]
When \(g(x) = f(x)\), we obtain measure of uncertainty for doubly truncated random variable as given in (12) and (13) of Sunoj et al. (2009). Various aspects of interval inaccuracy measure have been discussed in Kundu and Nanda (2015). To construct a shift-dependent dynamic measure of inaccuracy, we use (7) and define weighted interval inaccuracy measure for two-sided truncated random variables.

**Definition 2.1.** The weighted interval inaccuracy measure of \(X\) and \(Y\) at interval \((t_1, t_2)\) is given by
\[
H^w_{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} x \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dx
\]

**Remark 2.1.** Clearly, \(H^w_{X,Y}(0, t) = \bar{H}^w_{X,Y}(t), H^w_{X,Y}(t, \infty) = H^w_{X,Y}(t), \text{ and } H^w_{X,Y}(0, \infty) = H^w_{X,Y}\) as given in (6), (5), and (4), respectively.

The following example clarifies the effectiveness of the weighted interval inaccuracy measure.

**Example 2.1.** Let \(X_1, Y_1\) be the random lifetimes as given in Example 1.1. Also let \(X_2, Y_2\) denote random lifetimes of two components with probability density functions \(f_2(x) = [\text{insert PDF here}], g_2(x) = [\text{insert PDF here}]\).
where the second integral on the right hand side is equal to
\[ t_2 F(t_2) - t_1 F(t_1) - \int_{t_1}^{t_2} F(x)dx, \text{ or } t_1 \bar{F}(t_1) - t_2 \bar{F}(t_2) + \int_{t_1}^{t_2} \bar{F}(x)dx \]
The weighted interval inaccuracy measure can also be written as
\[ H_{w}^{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} \int_{0}^{x} \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dydx \]
\[ = t_1 H_{X,Y}(t_1, t_2) + \int_{t_1}^{t_2} H_{X,Y}(x, t_2)dx \]  \hspace{1cm} (9)
Furthermore,
\[ H_{w}^{X,Y}(t_1, t_2) = t_2 H_{X,Y}(t_1, t_2) - \int_{t_1}^{t_2} H_{X,Y}(t_1, y)dy \]  \hspace{1cm} (10)
where \(H_{X,Y}(t_1, t_2)\) is the interval inaccuracy measure given in (7). Differentiating (9) and (10) with respect to \(t_1\) and \(t_2\), respectively, we obtain
\[
\frac{\partial}{\partial t_1} H_{X,Y}(t_1, t_2) = t_1 \frac{\partial}{\partial t_1} H_{X,Y}(t_1, t_2) \quad \text{and} \quad \frac{\partial}{\partial t_2} H_{X,Y}(t_1, t_2) = t_2 \frac{\partial}{\partial t_2} H_{X,Y}(t_1, t_2)
\]

**Remark 2.2.** Weighted interval inaccuracy measure is increasing (decreasing) in \(t_1\) if and only if the interval inaccuracy measure is increasing (decreasing) in \(t_1\). The result also holds for \(t_2\).  \[
\square
\]

In virtue of Remark 2.2, below we obtain the bounds for the interval inaccuracy measure based on the monotonic behavior of the weighted interval inaccuracy measure. We first give definitions of general failure rate (GFR), general conditional mean (GCM), and geometric vitality function of a random variable \(X\) truncated at two points \(t_1\) and \(t_2\) where \((t_1, t_2) \in D\). For details one may refer to Navarro and Ruiz (1996), Nair and Rajesh (2000), and Sunoj et al. (2009).

**Definition 2.2.** The GFR functions of a doubly truncated random variable \((X|t_1 < X < t_2)\) are defined as a vector \(h^X(t_1, t_2) = (h_1^X(t_1, t_2), h_2^X(t_1, t_2))\) where \(h_i^X(t_1, t_2) = \frac{f_i(t_1)}{F(t_2) - F(t_1)}, i = 1, 2\). Similarly, \(h^Y(t_1, t_2)\) is defined for the random variable \((Y|t_1 < Y < t_2)\).  \[
\square
\]

**Definition 2.3.** The GCM of a doubly truncated random variable \((X|t_1 < X < t_2)\) is defined by
\[
m_X(t_1, t_2) = E(X|t_1 < X < t_2) = \frac{1}{F(t_2) - F(t_1)} \int_{t_1}^{t_2} xf(x)dx
\]
Theorem 2.1. The geometric vitality function for doubly truncated random variable \((X|t_1 < X < t_2)\) is given by
\[
G^*_{X}(t_1, t_2) = E(\ln X|t_1 < X < t_2)
\]
which gives the geometric mean life of \(X\) truncated at two points \(t_1\) and \(t_2\), provided \(E(\ln X)\) is finite. The corresponding weighted version of it is given by \(G^*_{X}(t_1, t_2) = E(X \ln X|t_1 < X < t_2)\).

Proof. Differentiating (8) with respect to \(t_1\) and \(t_2\), we get
\[
\frac{d}{dt_1} G^*_{X}(t_1, t_2) - \ln h_1^Y (t_1, t_2) \leq H_{X,Y} (t_1, t_2) \leq \frac{d}{dt_2} G^*_{X}(t_1, t_2) - \ln h_2^Y (t_1, t_2)
\]
The following proposition gives bounds for the weighted interval inaccuracy measure. The proof follows from (8) and hence omitted.

Proposition 2.1. If \(g(x)\) is decreasing in \(x > 0\), then
\[
-m_X (t_1, t_2) \ln h_1^Y (t_1, t_2) \leq H_{X,Y} (t_1, t_2) \leq -m_X (t_1, t_2) \ln h_2^Y (t_1, t_2)
\]
For increasing \(g(x)\) the above inequalities are reversed.

In the following two theorems we provide upper and lower bounds for the weighted interval inaccuracy measure based on monotonic behavior of the GFR functions of \(Y\). It is to be noted that the similar related results of Kumar et al. (2010) and Kumar and Taneja (2012) can be obtained as a particular case.

Theorem 2.1. For fixed \(t_2\),
(i) if \(h_1^Y (t_1, t_2)\) is decreasing in \(t_1\) then \(H_{X,Y} (t_1, t_2) \geq -m_X (t_1, t_2) \ln h_1^Y (t_1, t_2)\),
and (ii) increasing \(h_1^Y (t_1, t_2)\) in \(t_1\) implies
\[
H_{X,Y} (t_1, t_2) \leq -m_X (t_1, t_2) \ln h_1^Y (t_1, t_2) - \int_{t_1}^{t_2} \frac{xf(x)}{F(t_2) - F(t_1)} \ln \frac{G(t_2) - G(x)}{G(t_2) - G(t_1)} \, dx
\]
Proof. Note that (8) can be written as
\[
H_{X,Y} (t_1, t_2) = -\int_{t_1}^{t_2} \frac{xf(x)}{F(t_2) - F(t_1)} \ln h_1^Y (x, t_2) \, dx + \int_{t_1}^{t_2} \frac{xf(x)}{F(t_2) - F(t_1)} \ln \frac{G(t_2) - G(x)}{G(t_2) - G(t_1)} \, dx
\]
(i) For \(t_1 < x\), \(\ln \frac{G(t_2) - G(x)}{G(t_2) - G(t_1)} \leq 0\) and \(\ln h_1^Y (x, t_2) \leq \ln h_1^Y (t_1, t_2)\) if \(h_1^Y (t_1, t_2)\) is decreasing in \(t_1\). Then, from (11), we obtain
\[
H_{X,Y} (t_1, t_2) \geq -\int_{t_1}^{t_2} \frac{xf(x)}{F(t_2) - F(t_1)} \ln h_1^Y (x, t_2) \, dx
\]
\[
\geq -m_X (t_1, t_2) \ln h_1^Y (t_1, t_2)
\]
(ii) The second part follows easily from (11) on using the fact that \(\ln h_1^Y (x, t_2) \geq \ln h_1^Y (t_1, t_2)\) for \(t_1 < x\).

Example 2.2. Let \(X\) be a non negative random variable with probability density function
\[
f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}
\]
(12)
and $Y$ is uniformly distributed over $(0, a)$. Then $m_X(t_1, t_2) = \frac{1}{(t_2-t_1)}$, and $H^w_{X,Y}(t_1, t_2) = \frac{2(t_2^3+t_1 t_2^2+t_1^2)}{3(t_1+t_2)}$. Note that right hand side of part (ii) is $\geq \frac{2(t_2^3+t_1 t_2^2+t_1^2)}{3(t_1+t_2)}$. It is easily seen that part (ii) of the above theorem is fulfilled. For part (i), let $X$ be uniformly distributed over $[\alpha, \beta]$ and let $Y$ follow Pareto-I distribution given by

$$G(t) = 1 - \frac{\alpha}{t}, \quad t > \alpha (> 0)$$

Then $m_X(t_1, t_2) = \frac{(t_1+t_2)}{2}$, $\alpha < t_1 < t_2 < \beta$ and $h^Y_1(t_1, t_2) = \frac{t_2}{t_1(t_2-t_1)}$, which is decreasing in $t_1$, for fixed $t_2 > 2t_1$. Now

$$H^w_{X,Y}(t_1, t_2) + m_X(t_1, t_2) \ln h^Y_1(t_1, t_2) = 2 \left[ \frac{1}{t_2-t_1} \int_{t_1}^{t_2} x \ln x dx - \frac{(t_1 + t_2)}{2} \ln t_1 \right]$$

and equality holds for $t_1 \rightarrow t_2$. Hence part (i) is also fulfilled.

The proof of the following theorem is analogous to Theorem 2.1 but for completeness we give a brief outline of the proof.

**Theorem 2.2.** For fixed $t_1$, if $h^Y_2(t_1, t_2)$ is decreasing in $t_2$ then

$$H^w_{X,Y}(t_1, t_2) \leq -m_X(t_1, t_2) \ln h^Y_2(t_1, t_2) - \int_{t_1}^{t_2} \frac{x f(x)}{F(t_2)-F(t_1)} \ln \frac{G(x)-G(t_1)}{G(t_2)-G(t_1)} dx.$$  

**Proof.** We write (8) as

$$H^w_{X,Y}(t_1, t_2) = - \int_{t_1}^{t_2} \frac{x f(x) \ln h^Y_2(t_1, x)}{F(t_2)-F(t_1)} dx - \int_{t_1}^{t_2} \frac{x f(x)}{F(t_2)-F(t_1)} \ln \frac{G(x)-G(t_1)}{G(t_2)-G(t_1)} dx.$$  

Hence the result follows from (14) on using the fact that for $x < t_2$, $\ln h^Y_2(t_1, x) \geq \ln h^Y_2(t_1, t_2)$ when $h^Y_2(t_1, t_2)$ is decreasing in $t_2$.

**Example 2.3.** Let $X$ be a non negative random variable with probability density function as given in (12) and let $Y$ be uniformly distributed over $(0, a)$. Since $h^Y_2(t_1, t_2) = h^Y_2(t_1, t_2) = \frac{1}{(t_2-t_1)}$, on using the same argument as in Example 2.2, it can easily be shown that the condition of the above theorem is fulfilled.

**Remark 2.3.** It is not difficult to see from (14) that, for fixed $t_1$, if $h^Y_2(t_1, t_2)$ is increasing in $t_2$ then $H^w_{X,Y}(t_1, t_2) \geq -m_X(t_1, t_2) \ln h^Y_2(t_1, t_2)$. But it also can be shown that for a random variable with support $[0, \infty)$, $h^Y_2(t_1, t_2)$ may not be increasing in $t_2$. This condition can be achieved if either the support of the random variable is $(-\infty, b]$ with $b > 0$ or $[0, b]$ with $b < \infty$.

### 3. Characterizations based on weighted interval inaccuracy measure

In the literature, the problem of characterizing probability distributions has been investigated by many researchers. The standard practice in modeling statistical data is either to derive the appropriate model based on the physical properties of the system or to choose a flexible family of distributions and then find a member of the family that is appropriate to the data. In both the situations it would be helpful if we find characterization theorems that explain the distribution. In fact, characterization approach is very appealing to both theoreticians and applied workers. In this section we show that weighted interval inaccuracy measure can uniquely determine the distribution function. We also provide characterizations of quite a few useful continuous distributions in terms of weighted interval inaccuracy measure.
First we define the proportional hazard rate model (PHRM) and proportional reversed hazard rate model (PRHRM). Let \( X \) and \( Y \) be two random variables with hazard rate functions \( h_F(\cdot), h_G(\cdot) \) and reversed hazard rate functions \( \phi_F(\cdot), \phi_G(\cdot) \), respectively. Then \( X \) and \( Y \) are said to satisfy the PHRM (cf. Cox, 1959), if there exists \( \theta > 0 \) such that \( h_G(t) = \theta h_F(t) \), or equivalently, \( \hat{G}(t) = [\hat{F}(t)]^\theta \), for some \( \theta \). This model has been widely used in analyzing survival data. Similarly, \( X \) and \( Y \) are said to satisfy PRHRM proposed by Gupta et al. (1998) in contrast to the celebrated PHRM with proportionality constant \( \theta > 0 \), if \( \phi_G(t) = \theta \phi_F(t) \). Or, equivalently, \( G(t) = [F(t)]^\theta \), for some \( \theta \). This model is flexible enough to accommodate both monotonic and non monotonic failure rates even though the baseline failure rate is monotonic. In a similar line we extend the concept of proportional (reversed) hazard model for doubly truncated random variables as follows.

**Definition 3.1.** The random variables \( X \) and \( Y \) with hazard rate vectors \( h^X(t_1, t_2) \) and \( h^Y(t_1, t_2) \) are said to satisfy the generalized proportional hazard rate (GPHR) model, if there exists \( \theta > 0 \) such that

\[
h^Y_i(t_1, t_2) = \theta h^X_i(t_1, t_2)
\]

for \( i = 1, 2 \) and \( (t_1, t_2) \in D \).

The general characterization problem is to obtain when the weighted interval inaccuracy measure uniquely determines the distribution function. We consider the following characterization result. For characterization of a distribution by using its GFR functions one may refer to Navarro and Ruiz (1996).

**Theorem 3.1.** For two absolutely continuous non negative random variables \( X \) and \( Y \) satisfying (15), if \( H^w_{X,Y}(t_1, t_2) \) is increasing in \( t_1 \) (for fixed \( t_2 \)) and decreasing in \( t_2 \) (for fixed \( t_1 \)), then \( H^w_{X,Y}(t_1, t_2) \) uniquely determines \( F(x) \).

**Proof.** Differentiating (8) with respect to \( t_i, \ i = 1, 2 \), we have

\[
\frac{\partial}{\partial t_1} H^w_{X,Y}(t_1, t_2) = t_1 h^X_1(t_1, t_2) \left[ H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln h^X_1(t_1, t_2) \right]
\]

and

\[
\frac{\partial}{\partial t_2} H^w_{X,Y}(t_1, t_2) = -t_2 h^X_2(t_1, t_2) \left[ H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln h^X_2(t_1, t_2) \right]
\]

Then for any fixed \( t_1 \) and arbitrary \( t_2 \), \( h^X_1(t_1, t_2) \) is a positive solution of the equation \( \eta(x_{t_2}) = 0 \), where

\[
\eta(x_{t_2}) = t_1 x_{t_2} \left[ H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln x_{t_2} \right] - \frac{\partial}{\partial t_1} H^w_{X,Y}(t_1, t_2)
\]

Similarly, for any fixed \( t_2 \) and arbitrary \( t_1 \), \( h^X_2(t_1, t_2) \) is a positive solution of the equation \( \zeta(y_{t_1}) = 0 \), where

\[
\zeta(y_{t_1}) = t_2 y_{t_1} \left[ H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln y_{t_1} \right] + \frac{\partial}{\partial t_2} H^w_{X,Y}(t_1, t_2)
\]

Differentiating \( \eta(x_{t_2}) \) and \( \zeta(y_{t_1}) \) with respect to \( x_{t_2} \) and \( y_{t_1} \), respectively, we get \( \frac{\partial \eta(x_{t_2})}{\partial x_{t_2}} = t_1 [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + 1 + \ln x_{t_2}] \) and
\[ \frac{\partial \xi(y_1)}{\partial y_1} = t_2 [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + 1 + \ln y_1] \]. Furthermore, second-order derivatives are \[ \frac{\partial^2 \eta(x_2)}{\partial x_2^2} = \frac{t_1}{x_2} > 0 \] and \[ \frac{\partial^2 \zeta(y_1)}{\partial y_1^2} = \frac{t_2}{y_1} > 0 \]. So, both the functions \( \eta(x_2) \) and \( \zeta(y_1) \) are minimized at \( x_{t_2} = \exp[\theta - \ln \theta - 1 - H_{X,Y}(t_1, t_2)] = y_{t_1} \), respectively. Here \( \eta(0) = -\frac{1}{\partial t_1} H_{X,Y}(t_1, t_2) < 0 \), since we assume that \( H_{X,Y}(t_1, t_2) \) is increasing in \( t_1 \), and also, when \( x_{t_2} \to \infty, \eta(x_{t_2}) \to \infty \). Similarly \( \zeta(0) = \frac{1}{\partial t_2} H_{X,Y}(t_1, t_2) < 0 \), and \( \zeta(y_{t_1}) \to \infty \) as \( y_{t_1} \to \infty \). Therefore, both the equations \( \eta(x_{t_2}) = 0 \) and \( \zeta(y_{t_1}) = 0 \) have unique positive solutions \( h_{X,Y}^X(t_1, t_2) \) and \( h_{X,Y}^Y(t_1, t_2) \), respectively. Hence the proof is completed on using the fact that GFR functions uniquely determine the distribution (cf. Navarro and Ruiz, 1996).

Now we provide characterization theorems for some continuous distributions using GFR, GCM, geometric vitality function, and weighted interval inaccuracy measure under PHRM and PRHRM. Below we characterize uniform distribution. Recall that \( \frac{\partial h_{X,Y}^X(t_1, t_2)}{\partial t_2} = -h_{X,Y}^X(t_1, t_2) h_{X,Y}^Y(t_1, t_2) \) and \( \frac{\partial h_{X,Y}^Y(t_1, t_2)}{\partial t_1} = h_{X,Y}^X(t_1, t_2) (\frac{g(t_1)}{g(t_1)} + h_{X,Y}^Y(t_1, t_2)) \).

**Theorem 3.2.** Let \( X \) and \( Y \) be two absolutely continuous random variables satisfying PHRM with proportionality constant \( \theta > 0 \). A relationship of the form

\[ H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_{X,Y}^Y(t_1, t_2) = (1 - \theta) \left[ G_{X,Y}^w(t_1, t_2) - m_X(t_1, t_2) \ln(t_1 - \alpha) \right], \]

(16)

where \( G_{X,Y}^w(t_1, t_2) = E[X \ln(X - \alpha)|t_1 < X < t_2] \) and \( \alpha < t_1 < t_2 < \beta \), holds if and only if \( X \) denotes the random lifetime of a component with uniform distribution over \( (\alpha, \beta) \).

**Proof.** The if part is obtained from (8). To prove the converse, let us assume that (16) holds. Then from definition we can write

\[
- \int_{t_1}^{t_2} x f(x) \ln \frac{g(x)}{G(t_2) - G(t_1)} dx + \ln \frac{g(t_1)}{G(t_2) - G(t_1)} \int_{t_1}^{t_2} x f(x) dx \\
= (1 - \theta) \left[ \int_{t_1}^{t_2} x \ln(x - \alpha) f(x) dx - \ln(t_1 - \alpha) \int_{t_1}^{t_2} x f(x) dx \right].
\]

(17)

Differentiating (17) with respect to \( t_i, i = 1, 2 \) we get, after some algebraic calculations,

\[ g(t_i) = k(t_i - \alpha)^{\theta - 1}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant)} \]

or \( g(t) = k(t - \alpha)^{\theta - 1} \), which gives the required result.

**Corollary 3.1.** Under PRHRM the relation

\[ H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_{X,Y}^Y(t_1, t_2) = (1 - \theta) \left[ G_{X,Y}^w(t_1, t_2) - m_X(t_1, t_2) \ln(t_2 - \alpha) \right] \]

where \( G_{X,Y}^w(t_1, t_2) = E[X \ln(X - \alpha)|t_1 < X < t_2] \) and \( \alpha < t_1 < t_2 < \beta \) characterizes the uniform distribution over \( (\alpha, \beta) \).

Next, we give a theorem which characterizes the power distribution.

**Theorem 3.3.** For two absolutely continuous random variables \( X \) and \( Y \) satisfying PHRM with proportionality constant \( \theta > 0 \), the relation

\[ H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_{X,Y}^Y(t_1, t_2) = (1 - c\theta) \left[ G_{X,Y}^w(t_1, t_2) - m_X(t_1, t_2) \ln t_1 \right] \]

(18)

for all \( 0 < t_1 < t_2 < b \), characterizes the power distribution

\[ F(t) = \begin{cases} t^c, & 0 < t < b, \quad b, c > 0 \\ 0, & \text{otherwise} \end{cases} \]

(19)
Proof. If $X$ follows power distribution as given in (19), then (18) is obtained from (8). To prove the converse, let us assume that (18) holds. Then differentiating with respect to $t_i$, $i = 1, 2$, we get, after some algebraic calculations,

$$g(t_i) = kt_i^{\alpha - 1}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant)}$$

or $g(t) = kt^{\alpha - 1}$, which gives the required result.

**Corollary 3.2.** The relationship

$$H_{X,Y}(t_1, t_2) + m_X(t_1, t_2) \ln h_X^Y(t_1, t_2) = (1 - c\theta) [G_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_2]$$

characterizes the power distribution as given in (19) under PRHRM. □

Below we characterize Weibull distribution under PHRM.

**Theorem 3.4.** Let $X$ and $Y$ be two absolutely continuous random variables satisfying PHRM with proportionality constant $\theta (> 0)$. A relationship of the form

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_X^Y(t_1, t_2) = (1 - p) [G_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_1]$$

$$+ \lambda \theta [m_{X^{p+1}}(t_1, t_2) - t_1^p m_X(t_1, t_2)], \quad (20)$$

where $m_{X^{p+1}}(t_1, t_2) = E(X^{p+1}|t_1 < X < t_2)$, the conditional expectation of $X^{p+1}$, holds for all $(t_1, t_2) \in D$ and $p > 0$ if and only if $X$ follows Weibull distribution

$$F(t) = e^{-\lambda t^p}, \quad t > 0, \quad p > 0$$

**Proof.** The if part is straightforward. To prove the converse, let us assume that (20) holds. Then differentiating with respect to $t_i$, $i = 1, 2$, we get, after some algebraic calculations,

$$g(t_i) = kt_i^{\rho-1} e^{-\lambda t_i^p}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant)}$$

or $g(t) = kt^{\rho-1} e^{-\lambda t^p}$, which gives the required result. □

**Corollary 3.3.** Under PHRM, the relation

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_X^Y(t_1, t_2) = (1 - p) [G_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_2]$$

$$+ \lambda \theta [m_{X^{p+1}}(t_1, t_2) - t_1^p m_X(t_1, t_2)]$$

where $m_{X^{p+1}}(t_1, t_2) = E(X^{p+1}|t_1 < X < t_2)$, the conditional expectation of $X^{p+1}$, characterizes the Weibull distribution as given in the above theorem. □

**Remark 3.1.** Taking $p = 1$ in Theorem 3.4, we obtain the characterization theorem for exponential distribution with mean $1/\lambda$. Similarly, $p = 2$ characterizes the Rayleigh distribution $F(t) = e^{-\lambda t^2}, \quad t > 0$. □

Now we consider Pareto-type distributions which are flexible parametric models and play important role in reliability, actuarial science, economics, finance and telecommunications. Arnold (1983) proposed a general version of this family of distributions called Pareto-IV distribution having the cumulative distribution function

$$F(x) = 1 - \left[ 1 + \left( \frac{x - \mu}{\beta} \right)^\gamma \right]^{-\alpha}, \quad x > \mu \quad (21)$$

where $-\infty < \mu < \infty, \beta > 0, \gamma > 0$, and $\alpha > 0$. This distribution is related to many other families of distributions. For example, setting $\alpha = 1, \gamma = 1$, and $(\gamma = 1, \mu = \beta)$ in (21), one
at a time, we obtain Pareto-III, Pareto-II, and Pareto-I distributions, respectively. Also, taking \( \mu = 0 \) and \( \gamma \to \frac{1}{\gamma} \) in (21), we obtain Burr-XII distribution.

Now we consider Pareto-type distributions for characterization under PHRM. Below we provide characterization of Pareto-I distribution.

**Theorem 3.5.** Let \( X \) and \( Y \) be two absolutely continuous random variables satisfying PHRM with proportionality constant \( \theta (> 0) \). Then the relation

\[
H^w_{X,Y}(t_1, t_2) + m_X(t_1, t_2) \ln h^Y_1(t_1, t_2) = (\alpha \theta + 1) \left[ G^w_X(t_1, t_2) - m_X(t_1, t_2) \ln t_1 \right]
\]

holds for all \( \beta < t_1 < t_2 \) if and only if \( X \) follows Pareto-I distribution given by

\[
F(t) = 1 - \left( \frac{\beta}{t} \right)^\alpha, \quad t > \beta, \alpha, \beta > 0
\]

**Proof.** The if part is straightforward. To prove the converse, let us assume that (22) holds. Then differentiating with respect to \( t_1, i = 1, 2 \), we get, after some algebraic calculations,

\[
g(t_i) = k t^{-\alpha \theta + 1}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant)}
\]

or \( g(t) = k t^{-\alpha \theta + 1} \), which gives the required result. \( \square \)

**Corollary 3.4.** The relation

\[
H^w_{X,Y}(t_1, t_2) + m_X(t_1, t_2) \ln h^Y_2(t_1, t_2) = (\alpha \theta + 1) \left[ G^w_X(t_1, t_2) - m_X(t_1, t_2) \ln(t_1 - \mu + \beta) \right]
\]

characterizes the same distribution under PHRM as mentioned in the above theorem. \( \square \)

We conclude this section by characterizing Pareto-II distribution. The proof is similar to that of Theorem 3.5 and hence omitted.

**Theorem 3.6.** Let \( X \) and \( Y \) be two absolutely continuous random variables satisfying PHRM with proportionality constant \( \theta (> 0) \). Then the relation

\[
H^w_{X,Y}(t_1, t_2) + m_X(t_1, t_2) \ln h^Y_1(t_1, t_2) = (\alpha \theta + 1) \left[ G^w_Z(t_1, t_2) - m_X(t_1, t_2) \ln(t_1 - \mu + \beta) \right]
\]

where \( G^w_Z(t_1, t_2) = E(X \ln(X - \mu + \beta)|t_1 < X < t_2) \) holds for all \( \mu < t_1 < t_2 \) if and only if \( X \) follows Pareto-II distribution given by

\[
F(t) = 1 - \left[ 1 + \left( \frac{t - \mu}{\beta} \right) \right]^{-\alpha}, \quad t > \mu
\]

**Corollary 3.5.** Under PHRM the relation

\[
H^w_{X,Y}(t_1, t_2) + m_X(t_1, t_2) \ln h^Y_2(t_1, t_2) = (\alpha \theta + 1) \left[ G^w_Z(t_1, t_2) - m_X(t_1, t_2) \ln(t_2 - \mu + \beta) \right]
\]

where \( G^w_Z(t_1, t_2) = E(X \ln(X - \mu + \beta)|t_1 < X < t_2) \) and \( \mu < t_1 < t_2 \) characterizes the same distribution as mentioned in the above theorem.

**4. Monotonic transformations**

In this section we study the weighted interval inaccuracy measure under strict monotonic transformations. The following result is a generalization of Theorem 4.1 of Di Cresenzo and Longobardi (2006).
Theorem 4.1. Let \( X \) and \( Y \) be two absolutely continuous non negative random variables. Suppose \( \varphi(x) \) is strictly monotonic, continuous, and differentiable function with derivative \( \varphi'(x) \). Then, for all \( 0 < t_1 < t_2 < \infty \),

\[
H_{w,\varphi(X),\varphi(Y)}^{w}(t_1, t_2) = \begin{cases} 
H_{X,Y}^{w,\varphi}(\varphi^{-1}(t_1), \varphi^{-1}(t_2)) \\
+ E[\varphi(X) \ln \varphi'(X) | \varphi^{-1}(t_1) < X < \varphi^{-1}(t_2)], \ \varphi \text{ strictly increasing} \\
H_{X,Y}^{w,\varphi}(\varphi^{-1}(t_2), \varphi^{-1}(t_1))) \\
+ E[\varphi(X) \ln \{-\varphi'(X)\} | \varphi^{-1}(t_2) < X < \varphi^{-1}(t_1)], \ \varphi \text{ strictly decreasing}
\end{cases}
\]

where

\[
H_{X,Y}^{w,\varphi}(t_1, t_2) = - \int_{t_1}^{t_2} \varphi(x) \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} \, dx
\]

Proof. Let \( \varphi(x) \) be strictly increasing. Then from (4), (5), and (6) we have

\[
H_{w,\varphi(X),\varphi(Y)}^{w} = H_{X,Y}^{w,\varphi} + E[\varphi(X) \ln \varphi'(X)]
\]  

(24)

where \( H_{X,Y}^{w,\varphi} = - \int_{0}^{\infty} \varphi(x) f(x) \ln g(x) \, dx \),

\[
H_{w,\varphi(X),\varphi(Y)}^{w}(t) = H_{X,Y}^{w,\varphi}(\varphi^{-1}(t)) + E[\varphi(X) \ln \varphi'(X) | X > \varphi^{-1}(t)]
\]  

(25)

where \( H_{X,Y}^{w,\varphi}(t) = - \int_{t}^{\infty} \varphi(x) \frac{f(x)}{F(t)} \ln \frac{g(x)}{G(t)} \, dx \), and

\[
H_{w,\varphi(X),\varphi(Y)}^{w}(t) = H_{X,Y}^{w,\varphi}(\varphi^{-1}(t)) + E[\varphi(X) \ln \varphi'(X) | X < \varphi^{-1}(t)]
\]  

(26)

where \( H_{X,Y}^{w,\varphi}(t) = - \int_{0}^{t} \varphi(x) \frac{f(x)}{F(t)} \ln \frac{g(x)}{G(t)} \, dx \). Now, \( H_{w,\varphi(X),\varphi(Y)}^{w} \) can be decomposed as

\[
H_{w,\varphi(X),\varphi(Y)}^{w} = F(\varphi^{-1}(t_1))H_{w,\varphi(X),\varphi(Y)}^{w}(t_1, t_2) + [F(\varphi^{-1}(t_2)) - F(\varphi^{-1}(t_1))]H_{w,\varphi(X),\varphi(Y)}^{w}(t_1, t_2)
\]

\[
+ F(\varphi^{-1}(t_1))E[\varphi(X) \ln \varphi'(X) | X < \varphi^{-1}(t_1)] + F(\varphi^{-1}(t_2))E[\varphi(X) \ln \varphi'(X) | X > \varphi^{-1}(t_2)]
\]

\[
+ F(\varphi^{-1}(t_1))H_{w,\varphi(X),\varphi(Y)}^{w}(\varphi^{-1}(t_1)) + F(\varphi^{-1}(t_2))H_{w,\varphi(X),\varphi(Y)}^{w}(\varphi^{-1}(t_2))
\]

\[
- E[\varphi(X)] \left[ F_{w,\varphi}(\varphi^{-1}(t_1)) \ln G(\varphi^{-1}(t_1)) + F_{w,\varphi}(\varphi^{-1}(t_2)) \ln G(\varphi^{-1}(t_2)) \right]
\]

\[
+ \{ F_{w,\varphi}(\varphi^{-1}(t_1)) - F_{w,\varphi}(\varphi^{-1}(t_2)) \} \ln \{G(\varphi^{-1}(t_2)) - G(\varphi^{-1}(t_1))\}
\]

(27)

where the last three terms on the right hand side of (27) are equal to

\[
H_{w,\varphi(X),\varphi(Y)}^{w} - [F(\varphi^{-1}(t_2)) - F(\varphi^{-1}(t_1))]H_{X,Y}^{w,\varphi}(\varphi^{-1}(t_1), \varphi^{-1}(t_2))
\]

giving the first part of the proof. If \( \varphi(x) \) is strictly decreasing we similarly obtain the second part of the proof. \[\square\]

Remark 4.1. Let \( \varphi_1(x) = F(x) \) and \( \varphi_2(x) = \bar{F}(x) \), with \( \varphi_1 \) and \( \varphi_2 \) satisfying the assumptions of Theorem 4.1. Here \( \varphi_1(X) \) and \( \varphi_2(X) \) are uniformly distributed over \((0, 1)\). Then, for all
\[(t_1, t_2) \in D, \text{ we have} \]
\[H_{F(X), F(Y)}^{w}(t_1, t_2) = H_{X,Y}^{w,F}(F^{-1}(t_1), F^{-1}(t_2)) + E[F(X) \ln f(X)|F^{-1}(t_1) < X < F^{-1}(t_2)] \]
and
\[H_{F(X), F(Y)}^{w}(t_1, t_2) = H_{X,Y}^{w,F}(F^{-1}(t_2), F^{-1}(t_1)) + E[F(X) \ln f(X)|F^{-1}(t_2) < X < F^{-1}(t_1)] \]

**Remark 4.2.** For two absolutely continuous non-negative random variables \(X\) and \(Y\)
\[H_{aX,aY}^{w}(t_1, t_2) = aH_{X,Y}^{w,F}(t_1/a, t_2/a) + m_X(t_1/a, t_2/a) a \ln a \]
for all \(a > 0\) and \(t_1 > 0\). Furthermore, for all \(0 < b < t_1\)
\[H_{X+b,Y+b}^{w}(t_1, t_2) = H_{X,Y}^{w,F}(t_1 - b, t_2 - b) + bH_{X,Y}(t_1 - b, t_2 - b) \]

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**References**

Arnold, B.C. (1983). *Pareto Distributions*. Fairland, MD: International Cooperative Publishing House.

Belis, M., Guiaşu, S. (1968). A quantitative–qualitative measure of information in cybernetic systems. *IEEE Trans. Inf. Theory* 14:593–594.

Bhatia, P.K., Taneja, H.C. (1991). On characterization of quantitative–qualitative measure of inaccuracy. *Inf. Sci.* 56:143–149.

Cox, D.R. (1959). The analysis of exponentially distributed lifetimes with two types of failures. *J. R. Stat. Soc.* 21:411–421.

Di Crescenzo, A., Longobardi, M. (2006). On weighted residual and past entropies. *Scientiae Mathematicae Japonicae* 64:255–266.

Gupta, R.C., Gupta P.L., Gupta, R.D. (1998). Modeling failure time data by Lehman alternatives. *Commun. Stat. Theory Methods* 27(4):887–904.

Kerridge, D.F. (1961). Inaccuracy and inference. *J. R. Stat. Soc.* 23:184–194.

Kumar, V., Taneja, H.C. (2012). On length biased dynamic measure of past inaccuracy. *Metrika* 75:73–84.

Kumar, V., Taneja, H.C., Srivastava, R. (2010). Length biased weighted residual inaccuracy measure. *Metron* LXVIII:153–160.

Kundu, C. (2014). Characterizations based on length–biased weighted measure of inaccuracy for truncated random variables. *Appl. Math.* 59(6):697–714.

Kundu, C., Nanda, A.K. (2015). Characterizations based on measure of inaccuracy for truncated random variables. *Stat. Pap.* 56:619–637.

Misagh, F., Yari, G.H. (2010). A novel entropy-based measure of uncertainty to lifetime distributions characterizations. In: *Proc. ICMS 10*, Ref. No. 100196, Sharjah, UAE.

Misagh, F., Yari, G.H. (2011). On weighted interval entropy. *Stat. Probab. Lett.* 81:188–194.

Misagh, F., Yari, G.H. (2012). Interval entropy and informative distance. *Entropy* 14:480–490.

Nair, K.R.M., Rajesh, G. (2000). Geometric vitality function and its applications to reliability. *IAPQR Trans.* 25(1):1–8.
Nath, P. (1968). Inaccuracy and coding theory. *Metrika* 13:123–135.

Navarro, J., Ruiz, J.M. (1996). Failure rate functions for doubly truncated random variables. *IEEE Trans Reliab.* 45:685–690.

Prakash, O., Taneja, H.C. (1986). Characterization of the quantitative-qualitative measure of inaccuracy for discrete generalized probability distributions. *Commun. Stat. Theory Methods* 15(12):3763–3771.

Shannon, C.E. (1948). A mathematical theory of communications. *Bell Syst. Tech. J.* 27:379–423, 623–656.

Sunoj, S.M., Sankaran, P.G., Maya, S.S. (2009). Characterizations of life distributions using conditional expectations of doubly (interval) truncated random variables. *Commun. Stat. Theory Methods* 38:1441–1452.

Taneja, H.C., Tuteja, R.K. (1986). Characterization of a quantitative-qualitative measure of inaccuracy. *Kybernetika* 22:393–402.

Weiner, N. (1949). *Cybernetics.* New York: The MIT Press and John Wiley.