Quantum generalized cohomology

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Abstract. We construct a ring structure on complex cobordism tensored with \( \mathbb{Q} \), which is related to the usual ring structure as quantum cohomology is related to ordinary cohomology. The resulting object defines a generalized two-dimensional topological field theory taking values in a category of spectra.

Introduction

The conclusion of this paper is that the theory of two-dimensional topological gravity has a remarkably straightforward homotopy-theoretic interpretation in terms of a generalized cohomology theory, completely analogous to the more familiar interpretation of quantum ordinary cohomology as a topological field theory. Two-dimensional gravity originated in attempts to integrate over the space of metrics on a Riemann surface; it was reformulated by Witten in terms of an algebra of generalized Miller-Morita-Mumford characteristic classes for surface bundles. In the interpretation proposed here, this algebra is the coefficient ring of a generalized cohomology theory, and topological gravity becomes a topological-field-theory-like functor, which assigns invariants to families of algebraic curves just as a classical topological field theory assigns invariants to individual curves; in this it resembles algebraic \( K \)-theory, which assigns homotopy-theoretic invariants to families of modules over a ring.

Here is an outline of the argument. After a preliminary section which collects some background information, we define a generalized topological field theory in \S 2 in terms of (homotopy classes of) maps

\[ \tau^n_g : \bar{M}^n_g \to M^{\wedge n} \]

from a compactified moduli space of curves marked with \( n \) points, to \( n \)-fold powers of a module-spectrum \( M \). These maps preserve a monoidal structure (defined geometrically in the domain by glueing curves together at marked points, but defined algebraically in the range); in the language of [14 §3, 17 §1.7], \( \tau^n_* \) is a representation of a certain cyclic operad. The existence of such a representation entails the existence of a (quantum) multiplication on the module-spectrum \( M \) (cf. §2.4); in familiar cases this is the multiplicative structure defined by the WDVV equation, and when \( n = 0 \) we recover Witten’s tau-function for the moduli space of curves.

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That topological gravity and quantum cohomology are closely related is clear from [37], but I suspect that the simplicity of the underlying geometry is not widely understood. The main technical lemma (§2.1) is a kind of splitting theorem (cf. [21, 33]) for generalized Gromov-Witten classes; it is quite natural, in light of Kontsevich’s ideas about stacks of stable maps. These moduli objects have not yet been shown to be smooth for curves of all genus for any variety more complicated than a point [21 §1], so our main example is still conjectural, but in fact smoothness is much more than we need; the constructions of this paper require only that the generalized Gromov-Witten maps of §1.3 be local complete intersection morphisms. This is a convenient working hypothesis, which can be weakened further by elaborating the cohomological formalism; but this paper concerned with the consequences of this assumption, not with proving it. For curves of genus zero the moduli stacks are known to be smooth, if the target manifolds are convex in a suitable sense [4]; this leads to a simple proof (§2.2) of the associativity of the quantum multiplication, and when the defining variety is a point, we can calculate the corresponding coupling constant (§2.3).

Two short appendices discuss some related issues. In particular, there is reason to think that the Virasoro algebra is a ring of ‘quantum generalized cohomology operations’ for the main example. To state this more precisely requires a short digression about representations of the group of antiperiodic loops on the circle, which is included as the first appendix. A second appendix, added after the publication of this paper [in the Proceedings of the Hartford-Luminy Conference on the Operad Renaissance, Contemporary Math. 202 (1997) 407-419], outlines a slight generalization of the main construction of this paper in terms of the physicists’ ‘large phase space’ of deformations of quantum cohomology.

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1. Notation and conventions

1.1. Let \( V \) be a simply-connected projective smooth complex algebraic variety of real dimension \( 2d \), with first Chern class \( c_1(V) \), and let \( H \) denote its second integral homology group \( H_2(V, \mathbb{Z}) \). We will use a rational version

\[
\Lambda = \mathbb{Q}[H \times \mathbb{Z}]
\]

of the Novikov ring [25 §1.8] of \( V \): its elements are Laurent polynomials

\[
\sum_{k \in \mathbb{Z}, \alpha \in H} c_{\alpha, k} \alpha \otimes v^k
\]

with coefficients \( c_{\alpha, k} \in \mathbb{Q} \). This ring has a useful grading, in which \( v \) has (cohomological) degree two, and \( \alpha \) has degree \( 2\langle c_1(V), \alpha \rangle \). We will also use the notation

\[
v_{(k)} = \frac{v^k}{k!}
\]

for the \( k \)-th divided power of \( v \). If \( u : \Sigma \to V \) is a map from a connected oriented surface to \( V \), then the degree of \( u \) is the class \( u_*[\Sigma] \in H \), where \( [\Sigma] \in H_2(\Sigma, \mathbb{Z}) \) is the fundamental class of the surface.
1.2. $MU_\ast(X)$ will denote the complex bordism of a CW-space $X$, and $MU_\Lambda$ is the spectrum representing the homology theory defined on base-pointed finite complexes by

$$X \mapsto MU_\ast(X) \hat{\otimes} \Lambda = [S^\ast, X \wedge MU_\Lambda];$$

the tensor product has been completed, so $MU^\ast(pt) \hat{\otimes} \Lambda$ is the graded ring of formal Laurent series in $v$, with coefficients from the graded group ring $MU^\ast(pt)[H]$. It will be convenient to write $MU_\Lambda(V)$ for the function spectrum $F(V^+, MU_\Lambda)$. [The superscript + indicates the addition of a disjoint basepoint, but this refinement will often be omitted when the space is already encumbered with superscripts.] The fiber product of spaces (or schemes) $X$ and $Y$ over $Z$ will be denoted $X \times_Z Y$, and the product of $MU_\ast^\Lambda(X)$ and $MU_\ast^\Lambda(Y)$ over $MU_\ast^\Lambda(Z)$ will be denoted $MU_\ast^\Lambda(X) \otimes_Z MU_\ast^\Lambda(Y)$. The spectrum $MU_\Lambda(V)$ has an $MU_\Lambda$-algebra structure, and

$$MU_\Lambda(V^n) = MU_\Lambda(V) \wedge_{MU_\Lambda} \cdots \wedge_{MU_\Lambda} MU_\Lambda(V)$$

is its $n$-fold Robinson smash power [31] over $MU_\Lambda$. There is a map

$$\text{Tr}_V : MU_\Lambda(V) \to MU_\Lambda$$

defined by the (complex oriented) projection $V \to pt$, followed by multiplication with $v^d$ (to shift dimensions).

According to Quillen, a proper complex-oriented map $\Phi : P \to M$ between smooth manifolds defines an element $[\Phi]$ of $MU^k(M)$, where $k$ is the codimension of $\Phi$; more generally, a suitably oriented map between geometric cycles, with enough of a normal bundle to possess rational Chern classes, will define an element of $MU^k_\mathbb{Q}(M)$. Contravariant maps in cobordism are defined by fiber products, while covariant map are defined by the obvious compositions. Finally, the bilinear form

$$b_V : MU_\Lambda(V) \wedge MU_\Lambda(V) \to MU_\Lambda(V) \to MU_\Lambda$$

is the composition of the trace with the multiplication map of $MU_\Lambda(V)$.

The graded ring $MU_\Lambda(V)$ is a technical replacement for $MU^\ast_{\mathbb{Q}[v,v^{-1}]}(V \times H)$, which is in some ways more natural; but the latter ring does not help with the usual convergence problems, which (in the present framework) are consequences of the failure of the map $H \times H \to pt$ to be proper.

1.3. A (marked) algebraic curve is stable if its group of automorphisms is finite; $\overline{M}_g^n$ will denote the Deligne-Mumford-Knudsen space of such curves of arithmetic genus $g$, marked with $n$ ordered smooth points. These spaces are compact orbifolds, of complex dimension $3(g-1)+n \geq 0$; cases of low genus are sometimes exceptional. It is useful to understand $n$ to be a finite ordered set (or ordinal number), so that permutations of $n$ can act on $\overline{M}_g^n$. More generally, $\overline{M}_g^n(V, \alpha)$ will denote the stack [4 §3, 12] of stable maps of degree $\alpha$ from a curve of genus $g$ marked with $n$ ordered smooth points, to $V$: there is a morphism from $\overline{M}_g^n(V, \alpha)$ to $\overline{M}_g^n$ which assigns to a map (the stabilization of) its domain, and there is a morphism to $V^n$ which evaluates a map at the marked points. The product of these is a perfect (finite Tor-dimension [11 II §1.2]) proper morphism

$$\Phi_{V, g, \alpha} : \overline{M}_g^n(V, \alpha) \to \overline{M}_g^n \times V^n$$
of stacks. At a point \( u : \Sigma \to V \) of \( \overline{M}_g^u(V, \alpha) \) defined by a smooth \( \Sigma \), the relative tangent space to \( \overline{M}_g^u(V, \alpha) \) over \( \overline{M}_g^u \) defines a \( K \)-theory class
\[
[H^0(\Sigma, u^*T_V)] - [H^1(\Sigma, u^*T_V)]
\]
of complex dimension \( d(1-g) + \langle c_1(V), \alpha \rangle \), where \( d \) is the complex dimension of \( V \). It seems likely that under reasonable hypotheses this map will be a local complete intersection morphism of stacks, and in particular that the \( K \)-theory class of its cotangent complex at a singular point \( (\Sigma, u) \) will equal the holomorphic Euler class of the pullback of \( u^*T_V \) to the normalization of \( \Sigma \). [In fact, Kontsevich [19 §1.4] has already sketched something very close to a local complete intersection structure for this morphism.]. Similarly, let \( \epsilon_{V,g,\alpha}^n(k) \) denote the proper complex-oriented map from \( \overline{M}_g^{n+k}(V, \alpha) \) to \( \overline{M}_g^n \) which forgets the final \( k \) marked points, and define
\[
\Phi_{V,g,\alpha}^n(k) = \Phi_{V,g,\alpha}^n \circ \epsilon_{V,g,\alpha}^n(k) : \overline{M}_g^{n+k}(V, \alpha) \to \overline{M}_g^n \times V^n .
\]
There are also generalizations
\[
\mu_s^V : \overline{M}_g^{r+s}(V, \alpha) \times_V \overline{M}_h^{s+t}(V, \beta) \to \overline{M}_{g+h+s-1}^{r+t}(V, \alpha + \beta) ,
\]
of Knudsen’s gluing morphisms [17]. All these maps represent natural transformations between moduli functors, so their normal bundles are reasonably accessible. In diagrams below, complicated subscripts and superscripts will be suppressed when they are redundant in context.

2. Generalized topological field theories

2.1. We will be interested in generalized Gromov-Witten invariants defined by the cobordism classes of these morphisms. I will assume that these maps are local complete intersection morphisms; such maps between (possibly singular) varieties have most of the topological transversality properties of maps between smooth manifolds. In particular, they have well-behaved Gysin homomorphisms and normal bundles [1 IV §4] and they thus define elements in complex cobordism tensored with the rationals. [A variant approach is discussed below in §2.5.] Our generalized Gromov-Witten invariants are the classes
\[
\phi_{V,g}^n(k) = \sum_{\alpha \in H} [\Phi_{V,g,\alpha}^n(k) \alpha \otimes v(k)] \in MU_A^{2d(n+g-1)}(\overline{M}_g^n \times V^n) ;
\]
permutations of \( k \) define cobordant elements. Summing these classes over \( k \) defines
\[
\tau_{V,g}^n = v^{-d(n+g-1)} \sum_{k \geq 0} \phi_{V,g}^n(k) \in MU_A(\overline{M}_g^n \times V^n) ;
\]
the convergence problems mentioned in the preceding section do not appear when the function \( \alpha \mapsto [\Phi_{V,g,\alpha}^n(k)] \) is supported in a proper cone in \( H \). The tau-function \( \tau_{V,g}^n \) can be interpreted geometrically as the cobordism class of the ‘grand canonical ensemble’ of maps from a curve of genus \( g \) marked with \( n \) ordered smooth points, together with an indeterminate number of further distinct smooth (unordered) points, to \( V \), but we will be more concerned with the homotopy class
\[
\tau_{V,g}^n : \overline{M}_g^n \to MU_A(V^n)
\]
it defines.
**Proposition 2.1:** The diagram
\[
\begin{array}{ccc}
M_g^{r+s} \wedge M_h^{s+t} & \xrightarrow{\mu_s} & M_{g+h+s-1}^{r+t} \\
\tau \wedge \tau & & \tau
\end{array}
\]

\[
\begin{array}{ccc}
\text{MU}_A(V^{r+s}) \wedge \text{MU}_A(V^{s+t}) & \xrightarrow{b_V^*} & \text{MU}_A(V^{r+t})
\end{array}
\]

commutes up to homotopy; alternately,
\[
\mu_s^*(\tau_{V,g+h+s-1}^{r+t}) = v^d \text{Tr}_V(\tau_{V,g}^{r+s} \otimes V^s \tau_{V,h}^{s+t})
\]

**Sketch Proof.** under the standing hypothesis above: The general case reduces by induction to the case \(s = 1\), which can be stated as a coproduct formula
\[
\mu^s \phi^r_{V,g+h} \equiv v^d \sum_{i+j=k} \text{Tr}_V(\phi^r_{V,g}^i \otimes V \phi^s_{V,h}^j).
\]

This can be reformulated as the assertion that the two diagrams
\[
\bigcup_{i+j=k} M_g^{r+i} \times_V M_h^{t+j}(V, \alpha) \xrightarrow{\mu_V} M_{g+h}^{r+i+k}(V, \alpha + \beta)
\]
\[
\bigcup_{\epsilon(i) \times \epsilon(j)} M_g^{r+i} \times V M_h^{t+j}(V, \beta) \xrightarrow{\epsilon(k)} M_{g+h}^{r+i}(V, \alpha + \beta)
\]

and
\[
\bigcup_{\alpha + \beta = \gamma} M_g^{r+i} \times_V M_h^{t+j}(V, \alpha) \xrightarrow{\mu_V} M_{g+h}^{r+i}(V, \gamma)
\]
\[
\bigcup_{\Phi_\alpha \times \Phi_\beta} M_g^{r+i} \times V^{r+i+t} \times M_h^{t+j} \xrightarrow{\mu \times \text{Tr}_V} M_{g+h}^{r+i+t} \times V^{r+t}
\]
are fiber products; the claim follows by stacking the first diagram on top of the second. The bottom diagram describes the stable maps of decomposable curves in terms of the restrictions to their components. In the top diagram, the union is to be taken over partitions of the set with \(k\) elements into subsets of cardinality \(i\) and \(j\); on the level of functors, this diagram asserts that the ways of sprinkling points on a curve decomposed into two components correspond to the ways of sprinkling points on the components separately.

The last assertion is not entirely straightforward, because forgetting marked points may destabilize a genus zero component of a stable marked curve; the morphism \(\Phi_V(k)\) will blow such components down to points. The union in the upper left corner of the diagram is thus not necessarily disjoint: the fiber product is obtained from the disjoint union by identification along certain divisors [28 §3]. The point, however, is that the cobordism classes are defined by maps rather than by subobjects; the fiber product class is equivalent to the sum of the classes defining the disjoint union.

**2.2.** Proposition 2.1 states that the triple \((\tau^*_V, \text{MU}_A(V), b_V)\) defines a topological field theory which takes values in the category of \(\text{MU}_A\)-module spectra, where the usual monoidal structure defined by tensor product of modules over a ring is replaced by the smash product of module-spectra over a ring-spectrum. The domain of this generalized topological field theory is the monoidal category (Stable Curves) with finite ordered sets as objects; morphisms are finite unions of marked curves. [This category, however, does not possess identity maps for its objects.]
Both the domain and range of the generalized topological field theory are topological categories, and \( \tau^*_V \) defines a homotopy class of maps from the space of morphisms of the domain category to the space of morphisms of the range. These homotopy classes preserve the composition of morphisms and thus define a functor. In this generality, we need a version of Proposition 2.1 for Knudsen gluing of two points on a connected curve, but the changes required for this are minor.

The construction involves the bilinear map \( b_V \), but it has not otherwise used the multiplicative structure on \( MU_A(V) \). In fact the morphism
\[
\tau^3_{V,0} : S^0 = \overline{M}^3_0 \to MU_A(V^3)
\]
defines a composition
\[
*_V : MU_A(V) \wedge S^0 \wedge MU_A \xrightarrow{id \wedge \tau^3_{V,0} \wedge id} MU_A(V^3) \xrightarrow{b_V \wedge id} MU_A(V),
\]
and Proposition 2.1 has the following

**Corollary 2.2**: The pair \( (MU_A(V), *_V) \) is a homotopy commutative and homotopy associative ring-spectrum.

**Sketch proof**: Smashing the morphism
\[
A(2, 2) := (id \wedge b_V \wedge id)(\tau^{0+1}_2 \wedge \tau^{1+2}_0) : S^0 = \overline{M}^{2+1}_0 \wedge \overline{M}^{3+2}_0 \to MU_A(V^4)
\]
with the identity map of \( MU_A(V^3) \) defines a map from \( MU_A(V^3) \) to \( MU_A(V^7) \); arranging the seven copies of \( V \) into pairs and applying the trace map \( b_A \) three times defines a collection of maps from \( MU_A(V^3) \) to \( MU_A(V) \) indexed by the possible groupings of the factors. By our conventions \( 2+1 \) and \( 1+2 \) are isomorphic but not equal, so the notation for this associator class emphasizes that it depends on four points partitioned into two subsets, each containing two items. Ignoring obvious involutions, there are three different partitions, corresponding to the maps \( \pi_0, \pi_1, \pi_\infty \) from the 0-manifold \( \overline{M}^{2+1}_0 \times \overline{M}^{3+2}_0 \) to \( \overline{M}^4_0 \) which send it to a degenerate curve of genus zero with two irreducible components, each carrying two marked points (aside from the node); the cross-ratio identifies these configurations with the standard points \( 0, 1, \) and \( \infty \) on the projective line. To verify associativity it suffices to show that the homotopy class \( A(2, 2) \) is independent of the way the four points are partitioned into pairs; but by the proposition, \( A(2, 2) = \tau^4_0 \circ \pi_i \) factors through \( \overline{M}^4_0 \), where the three maps \( \pi_i \) become homotopic.

**2.3.** The morphism
\[
\tau^3_0 = v^{-2d} \sum_{k \geq 0} [\overline{M}^{k+3}_{g-1}(V) \to V^3] v(k)
\]
defining this quantum multiplication is essentially the Gromov-Witten potential [21]. Because the moduli spaces \( \overline{M}^g_0 \) are not defined when \( 3(g-1) + n \) is negative, however, it is not clear that the resulting multiplicative structure on \( MU_A(V) \) possesses a unit. The class
\[
q_V := 1 *_V 1 = v^{-2d}(b_V \otimes id)(\tau^3_{V,0})
\]
is the coupling constant for the topological field theory defined by \( MU_A(V) \); this theory assigns to a connected surface of genus \( g \) with one boundary component, the 2gth power of 1 with respect to the product \( *_V \).
Corollary 2.3: When $V$ is a point, the coupling constant of the resulting topological field theory is

$$q = \sum_{k \geq 0} [\overline{M}_0^{k+3}] v(k) \in MU^0_A(pt),$$

and the quantum product in $MU^*_A(pt)$ is $x \ast y = qxy$.

In this formula $[\overline{M}_0^{k+3}]$ is the cobordism class of the manifold of configurations of $k + 3$ points on a curve of genus zero; the sum is thus a cobordism analogue of Manin’s Hodge-theoretic invariant $\phi$ [26 §0.3.1]. With this structure, $(MU^*_A(pt), \ast_{pt})$ is isomorphic to $MU^*_A(pt)$ with its usual multiplication, by a homomorphism which sends $1 \ast_{2g}$ to $q^{2g}$. More generally, the operation $x \mapsto 1 \ast V x$ is a module isomorphism, and it seems reasonable to hope that $(1 \ast V)^{-1}(1)$ will be a unit for $\ast_V$.

Knudsen glueing defines a pair-of-pants product

$$\mu^+: \overline{M}_g^1 \wedge \overline{M}_h^1 \rightarrow \overline{M}_{g+h}^1$$

and it follows from §2.1 and the arguments above that the diagram

$$\begin{CD}
\overline{M}_g^1 \wedge \overline{M}_h^1 @>\mu^+>> \overline{M}_{g+h}^1 \\
@. \Downarrow \tau \wedge \tau \\
MU_A \wedge MU_A @>\ast_V>> MU_A
\end{CD}$$

is homotopy-commutative; in other words,

$$\tau^1_V : \overline{M}_*^1 \rightarrow MU_A$$

is a kind of homomorphism of monoids. This is probably the most intuitive way to think of the product in quantum cohomology, but from the present point of view it is a conclusion, not a definition.

2.4. A generalized topological field theory has an associated theory of topological gravity, which assigns invariants to proper flat families of stable curves. Such a family $Z$, say of topological type $(g, n)$, is defined by its classifying map to $\overline{M}_g^n$; the pullback of $\tau^0_V$ along this morphism defines a class $\tau_V(Z) \in [Z^+, MU_A(V^n)]$. If (for simplicity) we assume that $V$ is a point, and write $[Z] \in H_*(Z)$ for the fundamental class of $Z$, then the image $\tau_*(Z)$ of $[Z]$ in $H_*(MU_A)$ under the map induced on homology by $\tau(Z)$ is a kind of absolute invariant of the family, obtained by integrating $\tau(Z)$ over $[Z]$. In particular, the vacuum morphism

$$0 \rightarrow 0$$

is defined by the family of arbitrary finite unions of unmarked stable curves; the infinite symmetric product $SP^\infty(\bigsqcup_{g \geq 0} \overline{M}_g^1)$ is a rational model for its parameter space. The resulting absolute invariant

$$\tau = \exp\left(\sum_{g \geq 0} \tau_*(\overline{M}_g^0)\right) \in H_*(MU, Q[v, v^{-1}])$$

is Witten’s tau-function for two-dimensional topological gravity [24]. The point is that the characteristic number homomorphism

$$MU^*(M) \rightarrow H^*(M, H_*(MU))$$
sends the class $[Φ : P \to M]$ to a sum of the form $\sum_I Φ_*(c_I(ν))t^I$, where $ν$ is the stable normal bundle of $Φ$, $c_I(ν)$ is a certain polynomial (indexed by $I = i_1, \ldots$) in its Chern classes, $t^I = t_1^{i_1} \cdots$ is a product of elements in $H_*(MU) = \mathbb{Z}[t_i|i \geq 1]$, and $Φ_*$ is the covariant (Gysin) homomorphism induced by $Φ$. The stable normal bundle of $Φ(k)$ is inverse to the tangent bundle along the fibers, which is the sum of the $k$ line bundles defined by the tangent space to the universal curve at its $k$ marked points, so its Chern polynomials can be expressed as polynomials in the Chern classes of these line bundles. Under the pushdown $Φ(k)_*$, these become polynomials in the Mumford classes.

2.5. Some aspects of Kontsevich-Witten theory suggest that Gromov-Witten invariants can be defined more naturally in $K$-theory than in ordinary cohomology. The algebraic $K$-theory of a reasonable stack, tensored with the rationals, agrees with the algebraic $K$-theory of its quotient space [15 §7], but (perhaps because of this) the $K$-theory of stacks seems to have received little attention otherwise. The following assumes that the standard direct image construction for perfect proper maps of schemes [36 §3.16.4] generalizes to stacks.

Let $π_{g,n}: C^n_g \to \overline{M}_g$ be the universal stable curve; because the range is smooth, this is a perfect proper morphism. Let $C^n_{g,α}(V) := \overline{M}_{g,α}(V) \times_{\overline{M}_g} C^n_g$; from now on I will suppress the subscripts. The projection $π: C(V) \to \overline{M}(V)$ to the first factor, being the pullback of a perfect morphism, is again perfect. Let $U: C(V) \to V$ be the universal evaluation morphism; the vector bundle $U^*T_V$ defines an element of $K(C(V))$, and its hypothetical direct image $\bar{π}_*U^*T_V := ν(Φ_V) ∈ K(\overline{M}(V))$ is a reasonable candidate for the normal bundle of $Φ_V$.

Because $Φ_V$ is itself proper and perfect, we can define generalized Gromov-Witten classes

$$\sum_I Φ_*(m_I(ν(Φ_V)))t^I ∈ K^*(\overline{M}_g × V^n) \otimes_K K_*MU,$$

where $I$ is a multiindex as above, $t^I = \prod_k t_k^{i_k}$ is a basis for $K_*MU$, and $m_I$ denotes the $K$-theory characteristic class associated to the monomial symmetric function by the correspondence which assigns gamma operations [5 V §3] to the elementary symmetric functions. By the Hattori-Stong theorem, such a sum can be identified with a class in the localization $MU^*(\overline{M}_g × V^n)(\mathbb{C}P(1)^{-1})$ of complex cobordism.

If $Φ_V$ is a local complete intersection morphism, this approach to defining Gromov-Witten invariants agrees with the definition in §2.1. In any case some such hypothesis seems to be needed to make the arguments of Prop. 2.1 work.

3. Some questions

3.1. Kontsevich and Witten [20, 24, 37] show that the tau-function for the vacuum state of two-dimensional topological gravity is a lowest weight vector for a certain representation of the Virasoro algebra. This Lie algebra bears a striking resemblance to the Lie algebra defining the Landweber-Novikov algebra of operations in complex cobordism, but the relation between these two structures is not well-understood. I have included as an appendix a construction for the Kontsevich-Witten representation, starting from a representation of a certain loop group of
antiperiodic functions on the circle, following [8]. One point of the appendix is that the representation theory of this loop group is essentially trivial.

On the other hand, the usual complex cobordism functor takes values in the monoidal category of $\mathbb{Z}/2\mathbb{Z}$-graded $G$-equivariant sheaves over the moduli scheme $\text{Spec } MU^*(pt)$ of formal group laws, with the Landweber-Novikov group $G$ of formal coordinate transformations acting by change of coordinate; but this category is equivalent, after tensoring with $\mathbb{Q}[v,v^{-1}]$, to the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. It therefore seems not completely unreasonable to conjecture that the group of antiperiodic loops is a kind of motivic group for the generalized quantum cohomology theory defined by $MU_\Lambda$.

As for products in $V$, the functor $\tau^*_V$ seems to behave very naturally (cf. [19]); in particular, it is reasonable to expect that

$$\tau^0_{V_0 \times V_1, g} = \tau^0_{V_0, g} \otimes_{MU_\Lambda} \tau^0_{V_1, g}.$$  

3.2. The work in this paper was originally motivated by a desire to understand topological gravity and quantum cohomology from the point of view of Floer homotopy theory [6,7], but such questions have been suppressed here. It may be helpful, however, to observe that the circle group $\mathbb{T}$ acts on the universal cover $\tilde{LV}$ of the free loopspace of $V$, with $V \times H$ as fixed point set, so we can think of $MU^*_\Lambda(V)$ as its $t_\mathbb{T}MU^*_\Lambda$-cohomology [16 §15]. The Tate cohomology $t_\mathbb{T}MU^*_\Lambda(\tilde{LV})$ is a rough approximation to the Floer $MU$-homotopy type of $\tilde{LV}$, and we might hope to understand the relation between these invariants as a localization theorem for Tate cohomology.

More specifically, given a compact pointed Riemann surface $(\Sigma, x)$, let

$$(D, 0) \to (\Sigma, x)$$ be a holomorphically embedded closed disk; the boundary $\partial D$ separates the surface into components $\Sigma_0$ and $\Sigma_\infty$, with $x$ the point at infinity, as in [30 §8.11]. Let $\text{Hol}(\Sigma, D; V)$ denote the space of continuous maps from $\Sigma$ to $V$ which are holomorphic on $\Sigma_0$ and $\Sigma_\infty$. This is a manifold, with tangent space

$$H^0(\Sigma_0, u^*TV) \oplus H^0(\Sigma_\infty, u^*TV)$$ at $u \in \text{Hol}(\Sigma, D; V)$; here the sections of the pullback bundles are to be holomorphic on the interior and smooth on the boundary. Restriction to the boundary defines a map $u \mapsto \partial u$ to the free loopspace of $V$, but the homotopy class of $u_\infty$ defines a canonical contraction of $\partial u$, so this restriction map factors naturally through a lift to the universal cover $\tilde{LV}$ of $LV$.

This map is Fredholm, with index equal to the holomorphic Euler characteristic of $u^*TV$. [More precisely: since $u$ will usually not be holomorphic, $u^*TV$ can’t be expected to be holomorphic either; but $u^*TV$ restricts to a holomorphically trivial bundle on an annulus containing $\partial D$, so $u_0^*TV$ extends to a holomorphic bundle $\tilde{u}^*TV$ on $\Sigma$. Then $\chi(\tilde{u}^*TV)$ is the index at $u$.] Moreover, away from maps which collapse $\partial D$ to a point, this map appears to have a good chance to be proper.

We can elaborate this construction, by considering the space $\text{Hol}(\Sigma_x, V)$ of holomorphic disks in $\Sigma$ centered at $x$, together with a map to $V$, continuous and holomorphic away from $\partial D$ as above; since we’re enlarging things, we may as well include trivial disks too. This thickening has the same homotopy type as the preceding space, but now $\mathbb{T}$ acts by rotating loops. More generally, we can allow the moduli of $\Sigma$ to vary as well, thus defining a space of maps over a thickening of
the moduli space $\overline{\mathcal{M}}$. Restriction maps this space to a similar thickening of $\overline{\mathcal{M}} \times \overline{\mathcal{L}}$, defining a candidate for a proper $\mathbb{T}$-equivariant Fredholm map, and thus an element of $\Omega^{-2}_\mathbb{T}(\overline{\mathcal{M}} \times \overline{\mathcal{L}})$, which restricts to the classical Gromov-Witten invariant at the fixed point set of $\mathbb{T}$.

**Appendix I : $\text{MU}^*(pt)$ as a Virasoro-Landweber-Novikov bimodule**

The Virasoro algebra is the Lie algebra of a central extension of the group $\text{D}$ of diffeomorphisms of a circle; it is true generally [30 §13.4] that $\text{D}$ acts projectively on a positive energy representation of a loop group, and in this appendix I will sketch the construction of an action of the double cover $\text{D}(2)$ of $\text{D}$ on the basic representation of the twisted loop group $L_{\text{Twist}} =$ \{ $f \in LT | f = \iota(f)$ \}

of functions from the circle $\mathbb{R}/\mathbb{Z}$ to $\mathbb{T} = \{ z \in \mathbb{C} | |z| = 1 \}$ which are invariant under the involution $\iota(f)(x) = f(x + \frac{1}{2})^{-1}$.

Because the loop functor preserves fibrations, the exact sequence of the exponential function $e^{2\pi i x}$ yields an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow L\mathbb{R} \rightarrow L\mathbb{T}_0 \rightarrow 0$$

of abelian groups with involution, the group on the right being the identity component of the group of untwisted loops. The associated exact sequence

$$0 \rightarrow L\mathbb{R}^{\mathbb{Z}/2\mathbb{Z}} \rightarrow L\text{Twist} \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \rightarrow 0$$

of cohomology groups presents the antiperiodic loops as a canonically split extension of the group $\mathbb{Z}/2\mathbb{Z}$ of constant loops with value plus or minus one, by a vector space of antiperiodic functions.

Now $L\mathbb{T}_0$ contains a subgroup $\mathbb{T}$ of constant loops, and $\text{D}$ contains the subgroup $\text{R}$ of rotations, so the lift of a positive-energy projective unitary representation of $L\mathbb{T}$ to an honest unitary representation of an extension $\widetilde{L}\mathbb{T}$ of $L\mathbb{T}$ by a circle group $C$ restricts to a representation of a semidirect product $\text{R} \times E$, where $E$ is an extension of $\mathbb{Z} \times \mathbb{T}$ by $C$ which splits over the identity component. The irreducible positive-energy projective representations of $L\mathbb{T}$ are classified by their restriction to representations of $\text{R} \times C \times \mathbb{T}$ [30 §9.3]; an irreducible representation of $\mathbb{T}$ is classified by its weight, and the corresponding integer defined by $C$ is the level. However, the identity component of $\widetilde{L}\text{Twist}$ has a trivial subgroup of constant loops: its representation theory is effectively weightless.

There is, however, an interesting basic representation of $\widetilde{L}\text{Twist}$; one construction, modelled on [34 §2], begins with the skew bilinear form defined on $L\mathbb{R}^{\mathbb{Z}/2\mathbb{Z}}$ by

$$B(f_0, f_1) = \frac{2}{\pi} \int_0^1 f_0(x + \frac{1}{2}) f_1(x) dx.$$ 

The group $\text{D}(2)$ of smooth orientation-preserving maps $g$ of $\mathbb{R}$ to itself satisfying $g(x + \frac{1}{2}) = g(x) + \frac{1}{2}$ acts on this symplectic space, by

$$g, f \mapsto g^\frac{1}{2} f \circ g.$$ 

The complexified space of antiperiodic functions admits the decomposition

$$L\mathbb{R}^{\mathbb{Z}/2\mathbb{Z}} \otimes \mathbb{C} = A_+ \oplus A_-,$$
A+ being the subspace of functions on the circle which extend inside the unit disk. There is a standard \[34 \S 9.5\] unitary representation of \(\widetilde{L}^{\text{Twist}}\) on the symmetric algebra \(S(A_+)\) associated to this polarization; the basis

\[ a_n = -\pi i^{-\frac{1}{2}}(-\frac{1}{2})^{n+1} \frac{1}{2}\Gamma(n + \frac{1}{2})^{-1} e((n + \frac{1}{2})x) \]

for the complexification satisfies

\[ B(a_n, a_m) = (2m + 1)\delta_{n+m+1,0} \]

The polarization is defined by a nonstandard complex structure in which conjugation acts by \(\bar{a}_n = -ia_{-n-1}\), making \(iB(a, a)\) a positive-definite Hermitian form on \(A_+\). [This complex structure differs from the standard one by a transformation which is diagonal in the basis \(a_n\); this operator is real but unbounded.] The action of \(D(2)\) on \(L^{Z/2Z}\) makes it reasonable to interpret antiperiodic functions as sections of a bundle of half-densities on the circle; the complexification of this bundle admits the nonvanishing flat section

\[ (2\pi)^{\frac{1}{2}}e(x + \frac{1}{8})(dx)^{\frac{1}{2}} = (dZ)^{\frac{1}{2}} \]

where \(Z = e(x)\). It follows from Euler’s duplication formula that

\[ a_n(dx)^{\frac{1}{2}} = (2n + 1)!! Z^{-n-1}(dZ)^{\frac{1}{2}} \]

when \(n\) is nonnegative.

The Lie algebra of \(D(2)\) now acts on \(S(A_+)\) with generators (cf. \([20 \S 1.2, 37 \S 2]\))

\[ L_k = \frac{1}{4} \sum_{n \in \mathbb{Z}} a_{k-n-1}a_n \quad \text{if} \quad k \neq 0 , \]

\[ = \frac{1}{2} \sum_{n \geq 0} a_{n-1}a_n + \frac{1}{16} \quad \text{if} \quad k = 0 . \]

Convenient polynomial generators \(t_n\) for the algebra \(S(A_+)\), regarded as a ring of holomorphic functions on \(A_-\), can be defined by expanding an element \(f\) of \(A_-\) as

\[ \sum_{n \geq 0} t_n(f) Z^{-n-1}(dZ)^{\frac{1}{2}} ; \]

similar generators \(T_n\), constructed by writing this element as

\[ \sum_{n \geq 0} T_n(f) a_{-n-1} , \]

satisfy the equation

\[ t_n = (2n + 1)!!T_n . \]

The Virasoro generators (which are not derivations) act on these elements so that

\[ L_k T_n = (n - k + \frac{1}{2})T_{n-k} \quad \text{if} \quad n \geq k \]

\[ = 0 \quad \text{otherwise} . \]

On the other hand the group \(G\) of invertible formal power series (under composition) in \(Z^{-1}\) acts on \(A_-\), interpreted as a free module over the ring of power series in \(Z^{-1}\); the Lie algebra of this group is spanned by vector fields

\[ v_k = Z^{-k+1}d/dZ \quad \text{with} \quad k \geq 0 , \]
which act on $S(A_+)$ (as derivations) such that
\[ v_k t_n = (n-k+1)t_{n-k} \quad \text{when} \quad n \geq k \]
\[ = 0 \quad \text{otherwise} . \]

We can thus identify $MU^*(pt)$ with $S(A_+)$ as a comodule over the Landweber-Novikov algebra, in a way which makes it a Virasoro representation as well. [The complex coefficients are only a technical convenience.] The resulting bimodule defines a kind of Morita equivalence of the category of cobordism comodules to the category of representations of $L\widehat{\text{Twist}}$, given the (weak) monoidal structure defined by the fusion product $[35 \S 7]$ of representations.

**Appendix II : the large phase space of deformations**

Some recent advances [9] in quantum cohomology seem to fit very naturally in the homotopy-theoretic framework sketched above. This appendix has been added in July 1998 to the body of the published paper; I have used the opportunity to add references to recent work on the construction of Gromov-Witten invariants discussed in \S 2.1.

\II.1. If $H$ is a commutative Hopf algebra over a ring $k$ and $B$ is a finitely-generated $\mathbb{Z}$-module, then the functor defined on the category of commutative $k$-algebras by

\[ A \mapsto B \otimes \text{Hom}_{\text{alg}}(H, A) \]

is represented by a Hopf $k$-algebra which might be denoted $\otimes^B H$ : if $B$ is free of rank $b$ then a choice of basis defines an isomorphism of $\otimes^B H$ with the $b$-fold tensor product of copies of $H$. A variant of this construction occurs in the theory of vertex operator algebras; in that context $H$ is the algebra of symmetric functions and $B$ is a positive even lattice.

This appendix suggests a conjectural interpretation for a family of deformations of the quantum cohomology of a smooth algebraic variety $V$ in terms of a similar construction, in which the role of $B$ is played by the cohomology of $V$; in the basic example, $H$ is an algebra of Schur $Q$-functions [28]. The parameter space for this family is the ‘large phase space’ of Witten [37 \S 3c], defined when topological gravity is coupled to quantum cohomology; the more usual space of deformations of quantum cohomology proper is then called the ‘small phase space’. In impressionistic terms this large phase space is essentially a tubular neighborhood of the moduli space of holomorphic maps from a Riemann surface to $V$, inside the space of all smooth maps. A two-dimensional quantum field theory is a kind of measure on such a space of smooth (or perhaps continuous) maps, but it seems to be reasonable to think of these measures as supported near the (finite-dimensional) subspace of holomorphic maps. The resulting hybrid structure can thus be interpreted as a homotopy-theoretic family of deformations of a reasonably familiar kind of algebro-geometric object.

This is a summary of work in progress; one way to paraphrase the basic idea is that (even though quantum cohomology is not in any very natural sense a functor), we might interpret it as a cohomology theory taking values in the abelian category of bicommutative Hopf algebras over $\mathbb{Q}$. The interesting examples have further
structure, and the point of this note is that some, at least, of these extra structures seem to have natural interpretations in a Hopf-algebraic context.

I am indebted to Andy Baker and to Ezra Getzler for very helpful discussions of this and related material.

II.2. I will simplify notation here, writing \( \overline{M}^i(V) \) for \( \overline{M}_{g,0}(V) \) when indexing components is unnecessary. Gromov-Witten invariants in the context of algebraic geometry have now been defined rigorously by Behrend and Fantechi [2,3]; related results in the symplectic context have been announced by Li and Tian [23, cf. also 32]. We will be especially interested in the forgetful evaluation maps

\[ \epsilon^n(k) : \overline{M}^{n+k}(V) \to \overline{M}^n(V) \times V^k; \]

the spaces of nonzero tangent vectors at marked points are principal \( C^\infty \)-orbifold bundles over \( \overline{M}'(pt) \) which can be pulled back over the domain of \( \epsilon \), and

\[ u^{-k(d-1)}\epsilon^n_k(k) \in [\overline{M}'(V) \times V^k, BU_+^{p+} \wedge MU, \Lambda] \]

will denote its cobordism class enriched by the memory of the tangent bundles at the forgotten points.

Now let

\[ z = \sum t_{k,i}e^kz_i \in H^*(BT, H_*(V, Q)) , \]

where \( \{z_i\} \) is a basis for \( H_*(V, Q) \), be a homogeneous class of degree zero: we thus interpret the coefficient \( t_{k,i} \) to be an indeterminate of degree \( |z_i| - 2k \). In the language of physics, the elements \( z_i \) are the ‘primary fields’ of a topological field theory defined by the quantum cohomology of \( V \), while \( e^kz_i \) is the \( k \)th ‘topological descendant’ of \( z_i \). Using this terminology we can generalize the constructions of §2 above, replacing the class \( \phi^n_g(k) \) defined there with

\[ \phi^n_g(k; z) = \sum_{\alpha \in H_2(V; \mathbb{Z})} \Phi^n_{g,0}^\alpha(\epsilon^n_{z,0}^\alpha(k) \cap \otimes^k z) \otimes \alpha v(k) \in H^*(\overline{M}^n_g \times V^n, \Lambda) , \]

the cap product being (the \( \mathbb{Q} \)-linear extension of) the Kronecker pairing

\[ H^*(BT, H_*(V, \mathbb{Z})) \otimes H^*(V, H_*(BT, \mathbb{Z})) \to \mathbb{Z} . \]

The product with \( z \) leaves the coproduct formula 2.1 essentially unchanged, and the sum

\[ \tau^n_g(z) = u^{-d(n+g-1)} \sum_{k \geq 0} \phi^n_g(k; z) \]

still defines a generalized topological field theory, and thus a family of multiplications, which specializes when \( z \) is the fundamental class of \( V \) to our previous construction. If, for example, \( V \) is a single point, then \( \tau^0(z) \) is a formal function from \( H^*(BT, \mathbb{Q}) \) to \( H^*(\overline{M}, \mathbb{Q}) \), which can alternately be interpreted as an element of the tensor product of the symmetric algebra on \( H_*(BT, \mathbb{Q}) \) and the cohomology of the moduli space of curves. On the other hand, the composition

\[ BT \to BU \to MU \]

of the Thom map for cobordism with the map induced by the inclusion of the circle in the unitary group defines a canonical isomorphism

\[ S(H_*(BT, \mathbb{Q})) \to H_*(MU, \mathbb{Q}) ; \]

the element \( \tau^0(z) \) thus defines a homomorphism from the homology of the moduli space to \( H_*(MU, \mathbb{Q}) = MU_\bullet \otimes \mathbb{Q} \). In this case there is a unique primary field \( z_0 = 1 \).
and the $t_k$'s defined by its topological descendents are the standard generators of the Landweber-Novikov algebra.

II.3. At the opposite extreme we might suppose that $V$ is nontrivial and set $t_{i,k} = 0$ if $i,k > 0$; in this model there are then nontrivial primary fields, but only the field $z_0 = 1$ has topological descendents. If we identify the symmetric algebra on $H_*(BT, \mathbb{Q})$ with $MU_* \otimes \mathbb{Q}$ as above, then $\tau^n(z)$ becomes the class in $MU_*^n(V^n)$ defined by the space of stable maps from a curve marked with $n$ smooth points, together with an indeterminate number of further distinct but unordered smooth points which map to the subvariety $z$ of $V$. This yields a deformation of the quantum multiplication on $MU_*^n(V)$; its parameter space is the classical ‘small phase space’ of deformations, enlarged slightly (since we permit nontrivial descendents of $z_0$) to yield a theory interpretable in terms of cobordism rather than ordinary cohomology.

When $V$ is a point, for example, the deformation $z \in MU_* \otimes \mathbb{Q}[v, v^{-1}]$ replaces the coupling constant $q = q(v)$ of §2.3 with $q(zv)$.

In the general case (with no hypothesis that the topological descendents of any primary fields vanish) we can interpret the descendants of $z_0 = 1$ as lying in the cobordism ring and rewrite $\tau^n$ as an element of

$$S(H^*(V, H_*(BT, \mathbb{Q}))) \otimes H^*(\overline{M^g} \times V^n, \Lambda),$$

which can be expressed in terms of reduced cohomology as

$$S(H_*(BT, \mathbb{Q})) \otimes S(H^*(V, H_*(BT, \mathbb{Q}))) \otimes H^*(\overline{M^g} \times V^n, \Lambda).$$

This is in turn isomorphic to the symmetric $MU_*$-algebra

$$S_{MU_0}(MU_0^*(V)) \otimes_{MU_0} MU_0^*(\overline{M^g} \times V^n),$$

and we can think of $\tau^n(z)$ as a formal family of deformations, parametrized by (the space underlying) $H^*(V, H_*(BT, \mathbb{Q}))$, of a generalized topological field theory $\tau^n(z) : \overline{M^g} \to F(V^n_+, MU_\Lambda)$.

II.4. Recently Eguchi [9, cf. also 13] and his coworkers have studied an action of the Virasoro algebra on a large phase space model for the quantum cohomology of $\mathbb{C}P(n)$, which generalizes the Virasoro action on what can be interpreted as the (large) quantum cohomology of a point. There is reason to believe [29] that the latter Virasoro action can be understood most naturally not in terms of complex cobordism but instead in terms of cohomology with coefficients in a VOA-like structure defined by the ring $\Delta$ of Schur $Q$-functions; the resulting theory has good integrality properties, and its Virasoro structure is a consequence of Hopf-algebraic properties of $\Delta$. There is a natural (Kontsevich-Witten) genus

$$kw : MU_* \to \Delta[q_1^{-1}]$$

relating these constructions, which is essentially an isomorphism over the rationals, and it is natural to ask if this interpretation of topological gravity can be extended to encompass the quantum cohomology of algebraic varieties.

This is a subject for further research, but it is at least reasonable in terms of the Hopf algebra structures. From that point of view the map defined at the end of §3 identifies the rational homology of $BT$ with the primitives $P_*$ of $\Delta_Q$, which provides an interpretation of the large phase space of deformations for the quantum cohomology of $V$ as the spectrum of $\Delta_Q^{\otimes H(V,\mathbb{Z})}$. Connected, graded Hopf algebras over the rationals are primitively generated, and there is an internal tensor product
which sends the Hopf algebra $H_0$ (resp. $H_1$ with primitives $P_0$ (resp. $P_1$) to the Hopf algebra with primitives $P_0 \otimes P_1$; the Hopf algebra of functions on the large phase space is just this tensor product construction, applied to $\Delta_Q$ and $S(H^*(V))$. An internal tensor product on the category of bicommutative Hopf algebras has become important recently in other parts of algebraic topology, and an integral version of the construction sketched above would seem to be within reach. Questions related to Cartier duality and self-adjointness need to be explored, but it seems likely that these ideas will lead to a VOA-like structure on this large quantum cohomology, at least in relatively simple cases like $\mathbb{C}P(n)$.

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