Viscous damping of $r$-mode oscillations in compact stars with quark matter

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Abstract

We determine characteristic timescales for the viscous damping of $r$-mode oscillations in rapidly rotating compact stars that contain quark matter. We present results for the color-flavor-locked (CFL) phase of dense quark matter, in which the up, down and strange quarks are gapped, as well as the normal (ungapped) quark phase. While the ungapped quark phase supports a temperature window $10^8 \leq T \leq 5 \times 10^9$ K where the $r$-mode is damped even for rapid rotation, the $r$-mode in a rapidly rotating pure CFL star is not damped in the temperature range $10^{10}$ K $\leq T \leq 10^{11}$ K. Rotating hybrid stars with quark matter cores display an instability window whose width is determined by the amount of quark matter present, and they can have large spin frequencies outside this window. Except at high temperatures $T \geq 10^{10}$ K, the presence of a quark phase allows for larger critical frequencies and smaller spin-periods compared to rotating neutron stars. If low-mass X-ray binaries contain a large amount of ungapped or CFL quark matter, then our estimates of the $r$-mode instability suggest that there should be a population of rapidly rotating binaries at $\nu \gg 1000$ Hz which have not yet been observed.

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I. INTRODUCTION

Neutron stars are astrophysical test-beds for the physics of strongly-interacting matter in regimes of density that are not presently accessible either in terrestrial experiments or by numerical studies of lattice-regularized QCD. Therefore, connecting theoretical advances in the knowledge of the equation of state (EoS) of dense matter to astrophysical observations of neutron stars is essential. The EoS, which plays an important role in determining a compact star’s properties and in its evolution with time, is constrained by laboratory data only up to densities of about $2\rho_N$, where $\rho_N \approx 2.6 \times 10^{14} \text{ g cm}^{-3}$ is the saturation density of nuclear matter. At asymptotically high densities where perturbative calculations in QCD are reliable, quark matter is conjectured to be in a color superconducting state commonly called the color-flavor-locked (CFL) phase [1]. Meanwhile, models of low-energy QCD at moderate baryon chemical potentials $\mu_B \gtrsim 1$ GeV in a restricted temperature domain $T \leq T_c \sim 50$ MeV predict a rich patchwork of color superconducting phases in the QCD phase diagram [2, 3, 4, 5, 6], with important consequences for the properties of self-bound quark stars or neutron stars whose core density is large enough for deconfined quark matter to exist there. We refer to the former as strange stars and the latter as hybrid stars throughout this work.

The effects of color superconductivity on the EoS are small, though potentially important for the surface structure of strange stars [7, 8, 9], or the nuclear-quark interface in hybrid stars [10]. More interesting are its effects on transport phenomena, as pairing among the up, down and strange quarks breaks the $SU(3)$ global chiral symmetry spontaneously [1], leading to (pseudo-) Nambu-Goldstone modes that determine the material response to external perturbations. See the recent review Ref. [11] for the effective theory of these Nambu-Goldstone bosons. The pairing pattern that leaves the maximal residual symmetry in the three-flavor case is the CFL phase. In this phase, the up, down and strange quarks participate in Cooper pairing and the order parameter reduces the QCD symmetry $SU(3)_L \times SU(3)_R \times SU(3)_C$ to the diagonal subgroup $SU(3)_{L+R+C}$ locking the color and flavor symmetries. Apart from the 8 pseudo-Nambu-Goldstone bosons associated with the spontaneous breaking of chiral symmetry, other light modes in this phase are the exactly massless Nambu-Goldstone boson (the superfluid phonon), which is a long-wavelength fluctuation of baryon number, and the massless photon, which is effectively a combination of the vacuum photon and a gluon.

Studies of transport coefficients reveal that the thermal conductivity and specific heat of the CFL phase at temperatures much smaller than $T_c$, the critical temperature for pairing, is controlled by the phonon and the photon, while the electrical conductivity is determined by thermally excited electron-positron pairs [12]. In the CFL phase the long-term cooling of the star is controlled by neutrino emission from the scattering and weak decay of phonons [13]. These novel transport phenomena imply that neutrino transport and neutron star cooling curves have the potential to distinguish between the numerous possible patterns of Cooper pairing that might occur at stellar densities [14, 15, 16, 17].

Another set of transport coefficients that are modified by Cooper pairing are the bulk and shear viscosities. Bulk viscosity can lead to energy dissipation when the rate of chemical equilibration in matter is comparable to the driving frequency of the volume perturbation. Shear viscosity determines the relaxation of momentum components perpendicular to the direction of fluid flow. Together, they characterize the material response to compressional and shearing forces that are typically present in rotating and pulsating compact stars, and
serve to damp out large-amplitude oscillatory modes of the fluid. Moreover, since the fluid inside a star is self-gravitating, these oscillations can couple to metric perturbations, causing the star to emit gravitational waves. Indeed, the primary physical interest in exploring viscosities of color superconducting quark matter is to assess their role in damping out the so-called \( r \)-mode, a particular oscillatory mode of the fluid that dissipates the star’s rotational energy by efficiently coupling the angular momentum of the star to gravitational waves \[18, 19, 20\]. The damping timescale, which depends sensitively on the viscosity, and hence on the low-energy modes of the color superconductor determines if the \( r \)-mode will be driven unstable. The unsuppressed growth of the \( r \)-mode eventually saturates due to non-linear effects \[21\] since energy can be transferred from \( r \)-modes to “inertial” oscillation modes which do not couple to gravitational waves. However, the saturation amplitude can still be large enough to emit gravitational wave signals that can be detected directly by Advanced LIGO \[22\].

The observational context of \( r \)-modes is that neutron stars do not typically spin at rates near the maximum allowed frequency, the Kepler frequency \( \Omega_K \). \( r \)-modes offer a possible explanation of this fact: in rotating neutron stars, these modes lose energy through gravitational waves, which carry away angular momentum from the star and act as braking radiation. We mention two distinct settings in which \( r \)-modes can be instrumental in slowing the neutron star’s rotation speed.

(a) In the newly-formed isolated neutron star in the aftermath of a core-collapse supernova: Whether \( r \)-modes are relevant for neutron stars at birth is difficult to determine because the number of rapidly spinning pulsars that can be detected close to their birth is a negligible fraction of the total detectable sample. Limited by this selection effect, most inferences of rotation rates of neutron stars rely on a model of time evolution of radio pulsars from birth (neglecting any contribution from \( r \)-modes), with conflicting results on the birth spin-periods, ranging from milliseconds \[23\] to hundreds of milliseconds \[24\]. An independent method of determining the initial spin period based on correlating a supernova’s observed X-ray luminosity (or upper limits thereof) with the rotational energy loss of the neutron star suggests that a population of millisecond pulsars at birth is ruled out \[25\]. On the other hand, simulations of Type II supernovae including rotational effects predict relatively fast initial spin periods \[26\], which is difficult to reconcile with observed young and slowly rotating pulsars (like the Vela and Crab pulsars). Thus, \( r \)-modes may play an important role in neutron stars at birth, and the impact of \( r \)-modes depends critically on the composition of the neutron star core. The critical temperature for color superconductivity being of the order of a proto-neutron star’s temperature, it is important to understand the \( r \)-mode in the quark superfluid as it can determine the initial spin-down evolution of the strange star or hybrid star.

(b) In old neutron stars spun-up by accretion in binary systems: Millisecond pulsars (typically, pulsars with spin-periods \( \sim 10 \) ms) are old and most of them are observed to be in low-mass X-ray binary systems (LMXBs) \[27\]. The fast rotation is thought to result from spin-up due to steady accretion from a low-mass companion star. This picture has recently received confirmation through the observation of pulsations in LMXBs. Spin frequencies in LMXBs, as determined either from X-ray burst oscillations or from kHz quasi-periodic oscillations, are typically between 300-640 Hz, well below the maximum rotational rate (see Ref. \[28\] for a recent review). The upper limit on the rotation rate of LMXBs has been recently revised by the detections of objects spinning at 716 Hz \[29\] and 1122 Hz \[30\]. However, the statistical significance of the latter observation is not very strong. One possibility
is that accretion generates a mass quadrupole moment which results in gravitational waves and this prevents the star from rotating too quickly \cite{31}. Upper limits on the generation of gravitational waves from this mechanism have already been generated by LIGO \cite{32}. \( r \)-mode oscillations can be an alternative explanation for the limiting frequency. The observation of these fast-rotating objects provides an important constraint on the extent to which \( r \)-mode oscillations can spin down neutron stars. Since inferred temperatures of an LMXB’s interior are much less than \( T_c \) for a quark superfluid, it is pertinent to ask how quark matter inside an accreting neutron star affects the \( r \)-mode and the star’s rotation speed, assuming the quark core is not spun out of existence.

Motivated by these reasons, in this article, we investigate the effect of quark matter on \( r \)-modes in compact stars. In section \textbf{II}, we calculate the \( r \)-mode frequency for perturbed rotating fluids that obey chosen nuclear or quark matter EoS. In section \textbf{III}, we review the viscosities of neutron matter, ungapped and gapped CFL quark matter, which are inputs in a calculation of the damping timescale of the \( r \)-mode. In section \textbf{IV}, we present results for \( r \)-mode damping times in strange and hybrid stars, and discuss the critical spin-frequency in section \textbf{V}. In section \textbf{VI}, we remark on observational consequences for such compact stars and present our conclusions.

\textbf{II. \( r \)-MODES IN HADRONIC AND QUARK MATTER}

Various kinds of pulsation modes exist in neutron stars, classified by the nature of the restoring force (see Ref. \cite{33} for a review). For example, \( p \)-modes are high-frequency (few kHz) pressure waves where fluid oscillations are largely radial, while \( g \)-modes are low-frequency (\( \sim \)few 100 Hz) density waves driven by gravity which tends to smooth out composition and thermal gradients, particularly in proto-neutron stars. The \( r \)-mode is intimately linked to the rotational properties of the star, and the restoring force here (in rotating stars) is the Coriolis force. The modes are termed quasi-normal when they lose energy to gravitational waves and may be classified by the parity of the fluid displacement vector: polar (spheroidal) or axial (toroidal). \( r \)-modes are purely toroidal only for the trivial case of a non-rotating star where they have zero frequency. In a rotating star, the fluid displacement vector corresponding to the \( r \)-mode acquires spheroidal components as well, complicating the mode analysis. However, it is conventional to assume that the modes in rotating stars remain toroidal. It is also conventional to apply the Cowling approximation \footnote{The Cowling approximation is equivalent to neglecting \( \delta \phi_0 \) in Eq. (4). We include \( \delta \phi_0 \), however small, so we do not use this approximation.} in which the back-reaction of the fluid perturbation on the metric, and hence the gravitational potential, is ignored. Previous studies \cite{34,35} have shown that including such effects modifies the \( r \)-mode frequency approximately at the 5% level. Furthermore, if the perturbation is assumed to be isentropic, \( \delta P \) and \( \delta \rho \) obey the same EoS as the unperturbed quantities: \( P \) (the pressure) and \( \rho \) (the density). With these approximations, the \( r \)-mode frequency in the co-rotating frame, to first order in the rotation frequency of the star \( \Omega \), is given by \cite{34}

\begin{equation}
\omega_{\text{rot}} = \frac{2m\Omega}{l(l+1)} + \mathcal{O}(\Omega^3). \tag{1}
\end{equation}

The Cowling approximation is equivalent to neglecting \( \delta \phi_0 \) in Eq. (4). We include \( \delta \phi_0 \), however small, so we do not use this approximation.
Since we are interested in the instability to gravitational wave emission, we restrict ourselves to the “classical” \( r \)-modes of Papaloizou and Pringle \([36]\) for which \( l = m \). An inertial observer measures a \( r \)-mode frequency of

\[
\omega_r^{(0)} = \omega_{\text{rot}} - m\Omega = \left[ \frac{2}{l(l+1)} - 1 \right] m\Omega + \mathcal{O}(\Omega^3),
\]

from which it can be deduced that, for \( l = m \geq 2 \), a counter-rotating mode in the rotating frame appears as co-rotating with the star to a distant inertial observer. Thus, all \( r \)-modes with \( m \geq 2 \) are generically unstable to the emission of gravitational radiation and the \( m = 2 \) \( r \)-mode is the first to go unstable. This is the Chandrasekhar-Friedman-Schutz (CFS) mechanism \([37, 38]\). This instability is active as long as its growth-time is much shorter than the damping time due to the viscosity of stellar matter. Its effect is to slow the rotation rate of a compact star in a short time span of about a year \([20]\). This possibly explains why only slowly rotating pulsars are associated with supernova remnants. The \( r \)-mode instability might not allow millisecond pulsars to be formed after a supernova or an accretion induced collapse of a white dwarf; rather it seems that millisecond pulsars can only be formed by the accretion induced spin-up of old, cold neutron stars. However, in such cases, the neutron matter inside the neutron star could be in a superfluid state, and conclusions for \( r \)-mode damping in a superfluid phase are strongly model-dependent \([39]\).

Studies of \( r \)-mode oscillations provide a unique avenue to sample the star’s density profile in addition to its mass and radius. \( r \)-mode oscillations distinguish between “pure” neutron and hybrid or strange stars with the same mass and radius. To leading order in \( \Omega \), the \( r \)-mode frequency has no dependence on the EoS, Eq. (1). Including second-order rotational effects, one finds the following relation between the mode frequency \( \omega_r \) in the inertial frame and \( \Omega \) \([40]\)

\[
\omega_r = \omega_{\text{rot}} - m\Omega \equiv \kappa\Omega - m\Omega,
\]

\[
\kappa = \frac{2}{m+1} + \frac{\Omega^2}{\pi G \bar{\rho}_0} + \mathcal{O}(\Omega^4),
\]

where \( \kappa_2 \) is obtained explicitly from Eq. (4) below, \( \bar{\rho}_0 \) is the average density of the unperturbed star and \( G \) is Newton’s constant. Ignoring general relativistic effects, \( \bar{\rho}_0 \) is related to the Kepler frequency as \( \Omega_K = \frac{4}{5} \sqrt{\frac{2\pi G \bar{\rho}_0}{m}} \) \([41]\). \( \kappa_2 \) depends on the density profile of the star, as well as density and gravitational perturbations, which are related by the Poisson equation for (Newtonian) gravity. We determine \( \kappa_2 \) from the second-order analysis of \( r \)-modes \([40]\)

\[
\kappa_2 \int_0^R dr \left( \frac{r}{R} \right)^{2m+2} \frac{2m+2}{(m+1)^2} \int_0^r dr \rho_{22} \left( \frac{r}{R} \right)^{2m+2} + 8\pi G \bar{\rho}_0 m \left[ \int_0^r dr \left( \frac{r}{R} \right)^{m+1} \left[ \left( \frac{r}{R} \right)^{m+1} + \delta\Phi_0 \right] \left( \frac{d\rho}{dh} \right)_0 \right],
\]

where \( 0 \leq r \leq R \) is the radial coordinate in the star, \( \rho_{22}(r) \) is the radial dependence of the non-isotropic correction to the density of the rotating configuration to lowest order in \( \Omega \) and \( \delta\Phi_0(r) \) describes the radial dependence of the change in the gravitational potential due to the \( r \)-mode oscillation, again to lowest order in \( \Omega \). The quantity \( h \) is the barotropic enthalpy

\[
h(p) = \int_0^p \frac{dP'}{\rho(P')},
\]
and \((d\rho/dh)_0\) denotes a derivative in the unperturbed star. The function \(\delta\Phi_0(r)\) is determined through the solution of the differential equation

\[
\frac{d^2\delta\Phi_0}{dr^2} + \frac{2}{r} \frac{d\delta\Phi_0}{dr} + \left[ 4\pi G \left( \frac{d\rho}{dh} \right)_0 - \frac{(m+1)(m+2)}{r^2} \right] \delta\Phi_0 = -4\pi G \left( \frac{d\rho}{dh} \right)_0 \left( \frac{r}{R} \right)^{m+1},
\]

whose derivation is detailed in Ref. [40]. The function \(\rho_{22}(r)\) is related to \(\Phi_{22}(r)\), the change in the gravitational potential due to centrifugal deformation to lowest order in \(\Omega\), as

\[
\rho_{22}(r) = \frac{1}{2} \left[ 1 + \frac{1}{4} + (m+1)(m+2) \right] \frac{\delta\Phi_0}{R} \bigg|_{r=R},
\]

and

\[
\frac{d\Phi_{22}}{dr} \bigg|_{r=R} = -\frac{3\Phi_{22}}{R} \bigg|_{r=R}.
\]

Along with the condition that the perturbations vanish at \(r = 0\), Eqs. (6), (7) fix \(\rho_{22}(r)\) and \(\delta\Phi_0(r)\) throughout the star and subsequently \(\kappa_2\) through Eq. (4). We determine \(\kappa_2\) and hence the \(r\)-mode frequency for the \(l = m = 2\) mode since this is the lowest \(l\) that can couple to gravitational waves. We consider four equations of state; the first two are for hadronic matter, the next two are for quark matter.

1. The Akmal-Pandharipande-Ravenhall (APR) hadronic EoS [42] whose microscopic input is based on the Argonne \(v_{18}\) nucleon-nucleon interaction [43] which is calibrated to deuteron properties and vacuum nucleon-nucleon phase shifts for laboratory energies \(E_{\text{lab}}\) up to 350 MeV. The inclusion of the Urbana IX three-body force [44] and a relativistic boost term \(\delta v_b\) [45] successfully reproduces binding energies of several light nuclei.

2. To make contact with earlier works, we also employ a polytropic EoS

\[
P = K\rho^{1+1/n},
\]

where \(K\) is a dimensionful constant for finite values of the polytropic index \(n\). The case \(n = 0\) denotes incompressible matter, which softens with increasing \(n\). In Newtonian gravity, stable configurations exist only for \(\Gamma = 1 + 1/n > 4/3\), i.e., for \(n < 3\). We do not include general relativistic corrections in the structure or in the analysis of \(r\)-modes. Omitting these corrections leads to a smaller mass and larger radius for the compact star (see values of radius \(R\) in Table I). The effect of general relativistic corrections on the \(r\)-mode spectrum remains to be analyzed.

3. An MIT Bag model EoS for charge-neutral self-bound ungapped quark matter, with Bag constant \(B\) and current quark mass \(m_s\). The Bag constant is chosen such that the energy per baryon \(E/A < 930\) MeV at \(P = 0\). The pressure is given by

\[
P = -\frac{3}{2} \sum_{i=u,d,s} \int_0^{k_{Fi}} dk \frac{k^2}{\sqrt{k^2 + m_i^2}} + \frac{\mu_i^4}{12\pi^2} - B.
\]

(10)
For the parameters used in the calculation $m_u \sim m_d \ll m_s \ll \mu_i$, we can write in perturbation

$$P = -B + \frac{\mu_i^4}{12\pi^2} + \sum_{i=u,d,s} \frac{\mu_i^4}{4\pi^2} - \frac{3\mu_i^2 m_s^2}{4\pi^2} + \mathcal{O}(m_u^2, m_d^2, m_s^2), \quad (11)$$

with quark chemical potential

$$\mu_i = \mu_q - Q_i \mu_e, \quad (12)$$

where $Q_i$ is the electric charge and $\mu_q = (\mu_u + \mu_d + \mu_s)/3$ is the average quark chemical potential. Charge neutrality $\partial P/\partial \mu_e = 0$ implies $\mu_e \approx m_s^2/(4\mu_q)$ and we get

$$P \approx \frac{3\mu_q^4}{4\pi^2} - B - \frac{3\mu_q^2 m_s^2}{4\pi^2}, \quad (13)$$

$$n_q = n_u + n_d + n_s = \frac{\partial P}{\partial \mu_q} \approx \frac{3\mu_q^3}{\pi^2} - \frac{3\mu_q m_s^2}{2\pi^2}.$$  

The energy density $\epsilon = -P + n_q \mu_q$ yields

$$\epsilon \approx \frac{9\mu_q^4}{4\pi^2} + B - \frac{3\mu_q^2 m_s^2}{4\pi^2}. \quad (14)$$

Eliminating $\mu_q$ between Eqs. (13) and (14) gives the EoS

$$P \approx \frac{1}{3}(\epsilon - 4B) - \frac{m_s^2}{3\pi} \sqrt{\epsilon - B}, \quad (15)$$

which when compared to the exact EoS, is accurate to about 2% for the entire density range of interest in a strange star made of ungapped quark matter.

4. The CFL phase of quark matter [1] with non-zero quark masses. This phase is characterized by a common quark Fermi momenta $k_F$ and Cooper pairing among all three flavors and colors of quarks. This implies that the number densities of all three quark flavors are equal, which enforces electric charge neutrality even without electrons [2, 46]. The pressure in the CFL phase is given by [47]

$$P_{CFL} = -\frac{3}{\pi^2} \sum_{i=u,d,s} \int_0^{k_F} dk \, k^2 (\sqrt{k^2 + m_i^2} - \mu_q) + \frac{3\Delta^2 \mu_q^2}{\pi^2} - B, \quad (16)$$

where small contributions from the phonon and pseudo-Nambu-Goldstone bosons have been neglected since they have a negligible effect in determining the stellar structure. For realistic quark chemical potentials $\mu_q$, the strange quark's current mass $m_s \gg m_u \sim m_d$ can cause a mismatch in the quark flavor chemical potentials, but as long as the mismatch is much smaller than the superfluid gap $\Delta$, Cooper pairing with a common Fermi momentum $k_F$ is still favored. The condition for a common Fermi momenta $m_s^2 < 4\mu_s \Delta$ [48] is satisfied for the realistic parameters we use in the calculation. The average quark chemical potential $\mu_q = (\mu_u + \mu_d + \mu_s)/3$ determines the total quark number density

$$n_q = n_u + n_s + n_d = \frac{\partial P_{CFL}}{\partial \mu_q} = \frac{3}{\pi^2} (k_F^3 + 2\Delta^2 \mu_q), \quad (17)$$
which is three times the number density for each flavor. In perturbation where \( m_u \sim m_d \ll m_s \ll \mu_q \), we have

\[
k_F = \mu_q - \frac{m_s^2}{6\mu_q} + O(m_u^2, m_d^2, m_s^4),
\]

with pressure and energy density

\[
P_{CFL} \approx \frac{3\mu^4}{4\pi^2} - B + \frac{3\mu^2}{4\pi^2}(4\Delta^2 - m_s^2),
\]

\[
\epsilon = -P_{CFL} + n_q\mu_q \approx \frac{9\mu^4}{4\pi^2} + B - \frac{3\mu^2 m_s^2}{4\pi^2}.
\]

Just as for ungapped quark matter, at leading order in the quark mass expansion, we have

\[
P_{CFL} \approx \frac{1}{3}(\epsilon - 4B) + \frac{4\Delta^2 - m_s^2}{3\pi}\sqrt{\epsilon - B}.
\]

This approximate form based on the expansion in \( m_s^2/\mu_q^2 \) works better than expected due to the accidental combination \( (4\Delta^2 - m_s^2)/\mu_q^2 \) which is numerically small. \( O(m_q^4) \) corrections simply lead to a renormalization of the bag constant \([10]\) and can be ignored. Indeed a comparison to the exact EoS verifies that the approximate form Eq \([20]\) is accurate to within 0.1% for the entire density range of interest in a strange star.

For hybrid stars, we employ the APR EoS for the hadronic phase and either the MIT Bag or CFL EoS. Without modifications, a combination of the equations of state above does not generate a stable stellar configuration which contains a significant amount of quark matter and which also gives a sufficiently large maximum mass when general relativistic effects are included. For this reason, we make two additional modifications. Firstly, we compute the hybrid star profiles using general relativity, even though the expressions for \( r \)-mode properties are derived in the Newtonian limit. While a more consistent approach is desirable, the present approach is not unreasonable because it is the composition of the star, not the gross structural details, which determines the conclusions about the stability of the \( r \)-mode. Secondly, we add a term \( -\frac{3c\mu_q^4}{(4\pi^2)} \) to the pressure, which characterizes the \( O(\alpha_s) \) perturbative QCD correction to the pressure of a non-interacting quark gas \([10, 49]\) with \( 0 < c < 0.3 \).

For hybrid stars, the bag constant and the parameter \( c \) are chosen such that transition density between hadronic and quark matter comes out to be about twice the nuclear saturation density and the maximum mass in general relativity is \( 1.7 \, M_\odot \). For ungapped quark matter, \( B = 75 \, \text{MeV/fm}^3 \) and \( c = 0.083 \), and for gapped quark matter, \( B = 106 \, \text{MeV/fm}^3 \) and \( c = 0.12 \). Unless strange quark matter is absolutely stable at zero pressure, lower transition densities are incommensurate with the nonobservation of deconfined quarks in nuclei. Thus, the procedure outlined above effectively maximizes the amount of quark matter for a given hadronic equation of state in a hybrid star. Configurations with higher transition densities will have smaller quark matter cores, with results that lie between those for pure

\[\text{CFL matter has larger pressure at a fixed density than ungapped matter because of the gap; consequently, the Bag constant value must increase to ensure that the phase transition occurs at the same density in both cases.}\]
hadronic stars and hybrid stars with maximized quark content. Fig. 1 displays the typical density profile for hybrid stars containing ungapped or CFL quark matter for stars with mass $1.4M_\odot$.

We assume that the surface tension between the quark and hadronic phases is large enough so that only one surface is created. An estimate of the critical surface tension and the microscopic nature of the minimal CFL-nuclear interface is given in Ref. [47]. The assumption of a sharp boundary is made in the interest of simplicity, in which case a Maxwell phase construction is sufficient to describe the surface, with pressures and chemical potentials being continuous across the surface but density being discontinuous. A density discontinuity is clearly an idealization of a microscopically thin surface region where, for example, the number density will continuously transition from the larger value in the quark phase to the smaller value in the hadronic phase. This procedure is a good approximation if the surface tension is large enough that the energy gain provided by separating matter into a mixed phase is smaller than the surface energy cost.

![Graph showing density profile of hybrid stars with ungapped and gapped quark matter.](image)

**FIG. 1**: The density profile of hybrid stars containing ungapped (APR+Bag) and gapped (APR+CFL) quark matter that are considered in this work. The transition densities were fixed at $0.32 \text{ fm}^{-3}$. The vertical lines demarcate the homogeneous quark and hadronic phases.

For the equations of state discussed above, results for the $r$-mode frequencies are listed in Table 1. We confirm earlier findings for $r$-mode frequencies in neutron matter described by polytropic EoS, viz., that the $r$-mode frequency for polytropes increases with increasing softness of the EoS (higher polytropic index $n$). For gapped or ungapped quark matter, the
The $r$-mode frequency is similar to matter with a polytropic index $\approx 1.6$. The $r$-mode frequency is robust to variations in the gap $0 < \Delta/(\text{MeV}) < 100$, as long as other parameters are adjusted such that quark matter remains self-bound.

### TABLE I: $\kappa_2$ from Eq. (4) and $m=2$ $r$-mode frequency from Eq. (3) for various equations of state.

We choose the central density such that the star’s mass is $1.4M_\odot$ in each case (unless otherwise specified) while the rotation frequency is chosen to be $\Omega = \Omega_K$, where $\Omega_K$ is EoS dependent. For quark matter in strange stars, we choose $B = 80 \text{ MeV/fm}^3$, $m_s = 100 \text{ MeV}$ and $\Delta = 0 \text{ MeV}$ (ungapped) or $\Delta = 100 \text{ MeV}$ (gapped). For hybrid stars, we choose $B = 75 \text{ MeV/fm}^3$ and $c = 0.083$ (ungapped) and $B = 106 \text{ MeV/fm}^3$ and $c = 0.12$ (gapped). The fractional correction over the 0th-order EoS-independent result $\omega_{r}^{(0)}$ given by Eq. (2) is also displayed.

| EoS                  | $\kappa_2$ | $\omega_{r}^{-1}(\text{ms})$ | $|\omega_{r}/\Omega|$ | $\left|\frac{\omega_{r} - \omega_{r}^{(0)}}{\omega_{r}^{(0)}}\right|$ |
|----------------------|------------|-----------------------------|------------------------|-------------------------------------------------|
| $n = 1$ Polytrope    | 0.298      | 0.155                       | 1.215                  | 0.088                                           |
| $n = 2$ Polytrope    | 0.148      | 0.147                       | 1.275                  | 0.044                                           |
| APR                  | 0.316      | 0.180                       | 1.208                  | 0.094                                           |
| APR ($M=1.8 M_\odot$)| 0.266      | 0.161                       | 1.228                  | 0.079                                           |
| Bag                  | 0.198      | 0.106                       | 1.255                  | 0.059                                           |
| CFL                  | 0.195      | 0.099                       | 1.256                  | 0.058                                           |
| APR + Bag ($M=1.7 M_\odot$) | 0.083 | 0.097                       | 1.301                  | 0.025                                           |
| APR + CFL ($M=1.7 M_\odot$) | 0.099 | 0.100                       | 1.294                  | 0.029                                           |

Since the $r$-mode is really a quasi-normal mode, the mode frequency will generally acquire an imaginary part on account of the energy transmitted in gravitational waves and through dissipative forces in the fluid. To assess the $r$-mode instability and critical rotation frequency of compact stars, a comparison of the viscous damping timescale with the mode-growth timescale due to gravitational wave emission is required. We proceed to a discussion of the role of viscosity in $r$-mode damping.

### III. VISCOSITY OF HADRONIC AND QUARK MATTER

The temperature-dependent bulk viscosity $\zeta$ and shear viscosity $\eta$ of the fluid suppress $r$-mode growth. For hadronic matter, we use simple power-law fits derived in Ref. [50] which were shown to faithfully reproduce results of microscopic calculations of the viscosity [51]. For non-superfluid neutron matter, the dominant contribution to the bulk viscosity comes from the modified Urca process: $n + n \rightarrow n + p + e^- + \bar{\nu}_e$

$$\zeta_{\text{mUrca}} = 6 \times 10^{25} \rho_1^{12} T_9^2 \left(\frac{\kappa\Omega}{\text{Hz}}\right)^{-2} \text{g/(cm s)}, \quad (21)$$

where the dimensionless quantities $\rho_1 = \rho/(10^{15} \text{ g/cm}^3)$ and $T_9 = T/(10^9 \text{ K})$. The dominant contribution to shear viscosity comes from $nn$ scattering and is given by

$$\eta_{nn} = 2 \times 10^{18} \rho_1^{9/4} T_9^{-2} \text{g/(cm s)}. \quad (22)$$
At low temperatures neutron matter is likely to be in a superfluid state. At higher densities in the stellar interior, neutrons pair in the $^3P_2$ state with $T_c \sim 10^8$ K \[52\] while they pair in a $^1S_0$ state with $T_c \sim 5 \times 10^9$ K \[53, 54\] in the outer layers. At temperatures below the critical temperature $T_c$ for superfluidity, the viscosity contributions from modified Urca processes are suppressed \[55\] so that the phonons and Nambu-Goldstone modes corresponding to spontaneous breaking of rotational invariance (“angulons”) in the $^3P_2$ phase \[56\] dominate the viscosity. Ideally, the viscosity estimates [Eqs. (21), (22)] for neutron matter at $T \ll T_c$ should incorporate the effects of superfluidity. We neglect these effects in the interest of simplicity, since the focus is on the quark phases. We note, however, that Lindblom & Mendell \[39\] have shown that $\kappa_2$ (and hence the $r$-mode frequency) for a superfluid neutron star described by the APR EoS differs by less than 0.1% from the non-superfluid case. Furthermore, in computing the viscous damping timescales, these authors have used electron scattering off magnetized vortices, termed mutual friction \[57\], as the dominant source of bulk viscosity, and electron-electron scattering as the dominant source of shear viscosity. The results depend strongly on the model of superfluidity in the core and introduce additional complications in the analysis. We therefore postpone the practical issue of $r$-modes in superfluid neutron matter to future work.

For ungapped quark matter, the bulk viscosity is determined mainly by the weak process $d + s \leftrightarrow u + s$ \[58\], while leptonic contributions can become important at high temperature. \footnote{At high temperatures $T \geq 10^{10}$ K, the semi-leptonic processes $d (or \ s) \rightarrow u + e^- + \bar{\nu}_e$ and $u + e^- \rightarrow d (or \ s) + \nu_e$ can also contribute significantly to the bulk viscosity \[59\]. However, for our parameter choices of $m_s = 100$ MeV and $\omega_r = 1$ kHz, this contribution is negligible.} Since we are only examining stability and not attempting to describe the full evolution of the $r$-mode, we use the approximate expression for the bulk viscosity [Eq. (16) of Ref. \[58\]] which is appropriate for small oscillations of the fluid and when $2\pi T \gg \delta \mu = \mu_s - \mu_d$ (this holds in the range of temperatures of interest in this work). We have

$$\zeta_q = \frac{\alpha T^2}{\omega^2 + \beta T^4} \left[ 1 - \left[ 1 - \exp\left(-\sqrt{\beta T^2/\tau}\right) \right] \frac{2\sqrt{\beta T^2/\tau} \omega^2}{\omega^2 + \beta T^4} \right], \quad (23)$$  

$$\alpha T^2 = \left( \frac{64}{45\pi^3} \right) G^2_F \sin^2 \theta_c \cos^2 \theta_c \mu_d^4 m_s^4 T^2$$

$$= 6.66 \times 10^{20} \left( \frac{\mu_d}{\text{MeV}} \right)^3 \left( \frac{m_s}{\text{MeV}} \right)^4 T^2 \ \text{g/(cm s)}^3,$$

$$\beta T^4 = \frac{36}{\pi^2} \left( \frac{64}{45} \right)^2 G^4_F \sin^4 \theta_c \cos^4 \theta_c \mu_d^6 \left( 1 + \frac{m_s^2}{4\mu_d^2} \right)^2 T^4$$

$$= 3.57 \times 10^{-8} \left( \frac{\mu_d}{\text{MeV}} \right)^6 \left( 1 + \frac{m_s^2}{4\mu_d^2} \right)^2 T^4 \ \text{s}^{-2},$$

where $\omega = \omega_r = 2\pi/\tau$ is the frequency of the perturbation, $G_F$ the Fermi constant and $\theta_c$ the Cabbibo angle. In the bulk viscosity expression, the second term inside the big square bracket is numerically negligible for the parameters of the calculation. Further, this “transient” term drops out when averaged over oscillation cycles leaving only the overall factor in front, similar to the CFL bulk viscosity expression in Eq. \[25\].

The shear viscosity of ungapped quark matter is dominated by quark-quark scattering in
QCD. Neglecting small QED effects, we have

\[ \eta_q \approx 6.99 \times 10^{17} \left( \frac{0.1}{\alpha_s} \right)^{3/2} \left( \frac{\rho}{\rho_0} \right)^{5/3} T_9^{-2} \text{g/(cm s)}. \]  

(24)

In the superfluid CFL phase, the quarks are gapped. The massless phonons and thermally excited light pseudo-Nambu-Goldstone bosons determine the thermodynamic and hydrodynamic properties. Unlike the mesons of the QCD vacuum, the mass ordering of the CFL pseudo-Nambu-Goldstone bosons is “reversed” and the neutral kaon \( K^0 \) is the lightest meson [61]. The contribution to bulk viscosity from phonons alone has been calculated in Ref. [62]. However, the dominant contribution to bulk viscosity comes from flavor changing \( K^0 \) decay, viz., via weak equilibrium processes \( K^0 \leftrightarrow \phi \phi \) and \( \phi K^0 \leftrightarrow \phi \) involving the massless phonon \( \phi \) [63, 64]. It should be noted that in contradistinction to normal matter, superfluid hydrodynamics is described by more than one bulk viscosity. Additional viscosities, commonly labeled \( \zeta_1 \) and \( \zeta_3 \), enter the dissipative terms in the stress-energy tensor, and they multiply factors of \( v_s - v_n \), the difference in the speed of the “superfluid” and “normal” component in the two-fluid model of superfluidity [63, 64]. The superfluid component is non-viscous while the normal component consisting of the phonons and other low energy excitations such as pseudo-Nambu-Goldstone modes is viscous. Due to the large thermal conductivity of neutron and CFL matter [12, 67], we may approximate the star to be isothermal. Consequently, the second sound associated with a non-zero \( v_s - v_n \) and temperature gradient would be damped, and contributions from \( \zeta_1 \) and \( \zeta_3 \) can be ignored in the leading order of this approximation. For the calculations, we employ only the bulk viscosity \( \zeta_2 \) (which becomes the usual bulk viscosity \( \zeta \) in normal matter) from neutral kaon \( K^0 \) decay [64]:

\[ \zeta_{K^0} = C \frac{\gamma_{\text{eff}}}{\omega^2 + \gamma_{\text{eff}}^2}, \]

\[ C = \left( \frac{\partial n_q}{\partial \mu_q} \right)^{-1} \left( \frac{\partial n_K}{\partial \mu_K} \frac{\partial n_q}{\partial \mu_q} - \frac{\partial n_K}{\partial \mu_q} \frac{\partial n_K}{\partial \mu_K} \right)^2, \]

\[ \gamma_{\text{eff}} = \gamma_K \left[ 1 - \left( \frac{\partial n_K}{\partial \mu_q} \right)^2 \left( \frac{\partial n_K}{\partial \mu_q} \frac{\partial n_K}{\partial \delta \mu_K} \right)^{-1} \right], \]

where as before \( \omega = \omega_\tau = 2\pi/\tau \), and \( \gamma_{\text{eff}} \) is the effective kaon width [64]. The number densities for quarks \( n_q \) and neutral kaons \( n_K \) are determined from the thermodynamic pressure \( P = P_{\text{CFL}} + P_K \) where the contribution from kaons \( K^0 \) was added to the pressure \( P_{\text{CFL}} \) from Eq. (16). We choose parameters such that the effective kaon chemical \( \mu_{\text{eff}} = (m_s^2 - m_d^2)/(2\mu_q) \) is smaller than the kaon mass \( m_K \) ensuring that there is no kaon condensation. To compute the bulk viscosity, one considers small departures from equilibrium characterized by a non-zero chemical potential \( \delta \mu_K \) and writes

\[ P_K = -\frac{T}{2\pi^2} \int_0^\infty dk k^2 \ln \left[ 1 - \exp \left( -\frac{E_K - \delta \mu_K}{T} \right) \right], \]

\[ E_K = -\mu_{\text{eff}}^K + \sqrt{\frac{p^2}{3} + m_K^2}. \]
The barred quantities and derivatives in Eq. (25) correspond to equilibrium values $\delta \mu_K = 0$. The width $\gamma_{\text{eff}} \approx \gamma_K$ is determined from the kaon decay rate $[64]$ 

$$\gamma_K = \left( \frac{\partial n_K}{\partial \delta \mu_K} \right)^{-1} \frac{\Gamma_{\text{forward}}(\delta \mu_K = 0)}{T},$$

(27)

$$\Gamma_{\text{forward}} \approx \frac{|V_{ud}V_{us}|^2 G_F^2 f_K^2 f_\phi^2}{9\sqrt{3}\pi} \left( 1 + \frac{m_K^2}{\mu_{\text{eff}}^2 K} \right) \frac{\bar{p}}{\exp\left( \frac{\bar{p}}{\sqrt{3}T} \right)} \exp \left[ \frac{\bar{p}}{\sqrt{3}T} - 1 \right],$$

where $V_{ud}, V_{us}$ are the usual CKM matrix elements. Perturbative estimates for the decay constants yield $[61]$

$$f_K^2 = \frac{21 - 8\ln 2}{18} \frac{\mu_q^2}{2\pi^2},$$

(28)

$$f_\phi^2 = \frac{3}{4\pi^2} \frac{\mu_q^2}{2\pi^2}.$$

In the bulk viscosity calculation we use $\Delta = 100$ MeV, $m_u = 5$ MeV, $m_d = 7$ MeV, $m_s = 100$ MeV and $B = 80$ MeV/fm$^3$ to construct a strange star in the pure CFL phase with mass $1.4 \, M_\odot$. A self-consistent determination from Eq. (20) gives $\mu_q \sim 300$ MeV and $\mu_{\text{eff}} \sim 17$ MeV. Unfortunately, at these moderate densities the kaon mass $m_K$ is not accurately determined but it is expected to be about $15 - 25$ MeV $[68]$. If there is a region in the star where $\delta m = m_K - \mu_{\text{eff}} < 0$, then modifications to the equation of state and viscosities due to kaon condensation would be required $[69]$, but to study a concrete case, we treat $m_K$ as a free parameter and choose $\delta m = m_K - \mu_{\text{eff}} = 1$ MeV, avoiding the issue of kaon condensation. The CFL bulk viscosity drops rapidly with decreasing $T$ at higher values of $\delta m$.

The shear viscosity in the CFL phase has been calculated from phonon scattering $\phi\phi \leftrightarrow \phi\phi$ $[70]$. The contribution from the medium modified photon was shown to be small for temperatures where the phonon viscosity is of interest $[71]$, therefore it is not included in the present calculation. A variational estimate of the linearized Boltzmann equation relevant for shear viscosity gives $[70]$

$$\eta_\phi = 3.745 \times 10^6 \left( \frac{\mu_q}{\text{MeV}} \right)^8 T^{-5} \text{g/(cm s)}.$$

(29)

The shear viscosity falls steeply with increasing temperature and consequently has a small effect on the viscous damping of the $r$-mode at high temperature, see Fig. 2 and Eqs. (36), (39). Although the shear viscosity is undoubtedly more important at low temperatures, the contribution from the phonon becomes irrelevant at very low temperatures. This is because being a Nambu-Goldstone mode, the phonon is derivatively coupled where for typical interaction terms $i\partial_\phi = E_\phi \phi \sim T\phi$ for thermally exited phonons. At very low temperatures $T \ll \mu_q$, then, the phonon scattering cross section is suppressed and the mean free path $\lambda_\phi$ increases. The hydrodynamic expression for the shear viscosity is only relevant when $\lambda_\phi$ is much smaller than the star radius $R$, otherwise the phonon travels through the star without collisions responsible for shearing the fluid flow. An order of magnitude estimate of the relevant phonon mean free path can be obtained from the kinetic theory relation $\eta_\phi \sim n_\phi p\lambda_\phi$, 

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FIG. 2: The temperature dependence of the bulk and shear viscosity of neutron matter from Eqs. (21), (22) (left panel) and that of ungapped and gapped quark matter from Eqs. (23), (24) and Eqs. (25), (29) respectively (right panel). For neutron matter, the energy density is chosen to be $1.5 \times 10^{-4} M_\odot/km^3$ and $\kappa \Omega$ is fixed at 1000/s. For quark matter, the quark chemical potential is chosen to be 310 MeV, $m_s=100$ MeV with $\Delta=100$ MeV for CFL matter, and $\tau = 2\pi/\omega_r$ is fixed at 0.001 s.

where $n_\phi$ is the phonon number density and $p \sim 2.7T/v$ is the thermally averaged phonon momentum with phonon speed $v = 1/\sqrt{3}$. This length scale $\lambda_\phi$ quantify the distance over which the photon must travel to generate shear viscosity due to collision. We estimate

$$\lambda_\phi \sim \frac{n_\phi}{pn_\phi} \approx 2.1 \times 10^9 \left( \frac{\mu_q}{300 \text{ MeV}} \right)^8 T_9^{-9} \text{ km.}$$

(30)

For typical quark chemical potential $\mu_q \sim 300$ MeV, the phonon mean free path $\lambda_\phi$ will exceed the stellar radius at temperatures $T \leq 10^{10}$ K. The calculated damping times for the CFL phase are valid only above this temperature. For $T < 10^{10}$ K, contributions other than from phonons are expected to be relevant for shear viscosity but these have yet to be estimated. For this preliminary work on the CFL $r$-modes, we have included only the phonon contribution.

For hybrid stars, we ignore possible bulk viscosity due to the quark-hadron interface [71], and simply utilize the viscosity expressions above for each phase. This approximation implies that the viscosities for hybrid stars may be underestimated compared to a realistic treatment of the interface/crust.

IV. VISCOSOUS DAMPING OF $r$-MODES

The energy of the $r$-mode is dissipated according to [72]:

$$\frac{dE}{dt} = -(\omega_{rot} - m\Omega)^{2m+1}\omega_{rot} |\delta J_{mm}|^2 - \int d^3r (2\eta \delta \sigma^{ab} \delta \sigma_{ab} + \zeta \delta \sigma \delta \sigma),$$

(31)
where $\delta \sigma = \nabla_a \delta v^a$ is the volume expansion due to the $r$-mode and $\delta \sigma_{ab} = \frac{1}{2} (\nabla_a \delta v_b + \nabla_b \delta v_a - \frac{2}{3} \delta_{ab} \nabla_c \delta v^c)$ is the shear tensor. The first term is the energy radiated in gravitational waves to lowest order in $\Omega$ with $\delta J_{mm}$ being the current multipole

$$\delta J_{mm} \propto R^2 \Omega \int_0^R dr \rho \left( \frac{r}{R} \right)^{m+1}.$$  

(32)

For $m \geq 2$, $\omega_{rot} < m \Omega$, so that the $r$-mode energy grows with gravitational wave emission, triggering the instability. The timescale $\tau$ associated with growth or dissipation is given by

$$\frac{1}{\tau_i} = -\frac{1}{2E} \left( \frac{dE}{dt} \right)_i.$$  

(33)

The energy $E$ of the $r$-mode is given by

$$E = \frac{\pi}{2m} (m+1)^3 (2m+1)! R^4 \Omega^2 \int_0^R dr \rho \left( \frac{r}{R} \right)^{2m+2},$$  

(34)

where we have dropped a proportionality constant that determines the amplitude of the $r$-mode, since it cancels in the evaluation of the damping timescale. Explicitly, the gravitational radiation time scale is

$$\frac{1}{\tau_{GW}} = \frac{32\pi G \Omega^{2m+2}}{c^{2m+3}} \frac{(m-1)^{2m}}{[(2m+1)!]^2} \frac{(m+2)^{2m+2}}{(m+1)} \int_0^R dr \rho r^{2m+2},$$  

(35)

while the shear viscosity time scale is

$$\frac{1}{\tau_{\eta}} = (m-1)(2m+1) \int_0^R dr \rho r^{2m+2} \int_0^R dr \rho r^{2m}.$$  

(36)

We compute the bulk viscosity timescales using a simplified expression for the volume expansion, which is based on approximating the Lagrangian perturbation of the fluid by an Eulerian one

$$\delta \sigma = -i \kappa \Omega \frac{\delta \rho}{\rho}.$$  

(37)

As shown in Ref. [40], while this approximation is only good within an order of magnitude for the bulk viscosity damping timescale, the critical angular velocity, which can be constrained by observations, is hardly modified because the bulk viscosity has a steep temperature dependence. Thus, the critical frequency curves presented in this work are accurate, even though the bulk viscosity timescales are only approximate.

Carrying out angular integrations for the bulk viscosity part of Eq. (31) and using Eq. (37), one obtains

$$\left( \frac{dE}{dt} \right)_\zeta = \frac{1}{2} \int_0^R 2\pi r^2 \zeta \frac{4R^4 \Omega^6}{(1+m)^2 \rho_0^2 \left( \frac{d\rho}{dh} \right)_0} \left( \frac{r}{R} \right)^{m+1} + \delta \Phi_0(r) \right)^2,$$  

(38)

where $\rho_0(r)$ is the density of the unperturbed star. The bulk viscosity damping timescale follows from Eqs. (33), (34) and (38).
For a fixed temperature $T=10^9$ K, and for stars rotating at the Kepler frequency $\Omega = \Omega_K$, all three damping timescales are displayed in Table I. The timescales are also plotted as a function of temperature in Fig. 3. We observe from Fig. 3 that in non-superfluid neutron matter, shear viscosity damps the $r$-mode rapidly only at very low temperatures $T < 10^6$ K, while bulk viscosity damps it only at very high temperatures $T > 10^{10}$ K. In normal quark matter, the damping timescale due to bulk viscosity has a minimum around $T \sim 10^9$ K, corresponding to a maximum in the bulk viscosity. This behavior leads to a high-temperature stability window. In CFL quark matter, the damping timescale due to shear viscosity at low temperature is large. However, the phonon contribution considered here is not reliable below about $10^{10}$ K as mentioned earlier. Other contributions, such as from light-by-light scattering [70], could become important in this regime and damp out the $r$-mode. Keeping this caveat in mind, and noting that the damping timescale due to bulk viscosity is still large except at high temperatures $T > 10^{11}$ K, the $r$-mode in CFL matter is unstable only in a small window around high temperatures $T \sim 5 \times 10^{10}$ K, in marked contrast to either ungapped quark matter or normal neutron matter. It would be interesting to see how the width of this window changes upon inclusion of relevant shear viscosities at temperatures $T < 10^{10}$ K.

![Image of Fig. 3](image_url)

**FIG. 3**: The temperature dependence of the bulk and shear viscosity damping timescales for neutron matter (left panel) and for strange and hybrid star matter (right panel). The rotation rate is the Kepler frequency $\Omega_K$.

**V. CRITICAL ROTATION FREQUENCIES**

The critical rotation frequency $\Omega_c = 1/\tau_c$ of neutron/strange/hybrid stars can be determined by the criterion that at this frequency, the fraction of energy dissipated per unit time exactly cancels the fraction of energy fed into the $r$-mode by gravitational wave emission:

$$\frac{1}{\tau_f} \bigg|_{\Omega_c} = \left[ \frac{1}{\tau_\zeta} + \frac{1}{\tau_\eta} + \frac{1}{\tau_{GW}} \right] \bigg|_{\Omega_c} = 0. \quad (39)$$
TABLE II: Damping times for various equations of state. The central energy density $\bar{\rho}_c$ in each case is chosen to yield a 1.4 $M_\odot$ star (unless otherwise explicitly stated) and the temperature is fixed at $T=10^9$K. $\tau_f$ is the critical or cumulative $r$-mode damping timescale given by $\tau_f = (1/\tau_\zeta + 1/\tau_\eta + 1/\tau_{GW})^{-1}$. The star is taken to be rotating at the Kepler frequency $\Omega_K$. Energy densities are given in units of $10^{-4} M_\odot/\text{km}^3$.

| EoS         | $\bar{\rho}_c$ (km) | $\Omega_K$ (kHz) | $\tau_\zeta$ ($s$) | $\tau_\eta$ ($s$) | $\tau_{GW}$ ($s$) | $\tau_f$ ($s$) |
|-------------|----------------------|-------------------|---------------------|--------------------|--------------------|---------------|
| $n=1$ Polytrope | 5.63                | 12.50             | $2.47 \times 10^1$ | $2.48 \times 10^8$ | $-5.22 \times 10^1$ | $-5.22 \times 10^1$ |
| $n=2$ Polytrope | 19.5                | 12.50             | $6.20 \times 10^4$ | $7.28 \times 10^7$ | $-1.29 \times 10^2$ | $-1.29 \times 10^2$ |
| APR         | 3.60                | 13.74             | $2.48 \times 10^9$ | $3.25 \times 10^8$ | $-9.54 \times 10^1$ | $-9.54 \times 10^1$ |
| APR ($M=1.8 M_\odot$) | 3.97                | 14.06             | $1.79 \times 10^9$ | $3.03 \times 10^8$ | $-3.58 \times 10^1$ | $-3.58 \times 10^1$ |
| Bag         | 4.30                | 9.89              | $7.33 \times 10^7$ | $2.78 \times 10^7$ | $-8.28 \times 10^0$ | $-9.34 \times 10^0$ |
| CFL         | 3.23                | 10.71             | $1.74 \times 10^9$ | $5.38 \times 10^7$ | $-7.48 \times 10^1$ | $1.92 \times 10^0$ |
| APR+Bag ($M=1.7 M_\odot$) | 13.5               | 10.25             | $1.04 \times 10^2$ | $2.38 \times 10^7$ | $-7.96 \times 10^0$ | $-8.62 \times 10^0$ |
| APR+CFL ($M=1.7 M_\odot$) | 10.6               | 10.40             | $7.69 \times 10^9$ | $1.29 \times 10^4$ | $-8.66 \times 10^0$ | $-8.67 \times 10^0$ |

FIG. 4: The critical frequency $\Omega_c$ in units of $\Omega_K$ as a function of temperature for neutron stars, strange stars and hybrid stars. For quark matter, we choose $m_s=100$ MeV and $\Delta=100$ MeV. The bag constant $B=80$ MeV/fm$^3$ for strange stars, while $B=110$ (150) MeV/fm$^3$ for hybrid stars with ungapped (CFL) quark matter. The box represents typical temperatures ($2 \times 10^7$-$3 \times 10^8$ K) and rotation frequencies (300-700 Hz) of the majority of observed LMXBs assuming $\Omega_K = 5500$ Hz.

Stable rotation frequencies at any temperature will be limited by the smaller of the critical frequency or the Kepler limit $\Omega/\Omega_K=1$. As depicted in Fig. 4, the region above the temperature-dependent $\Omega_c$ curve is unstable to $r$-mode oscillations and the star, if it enters this region, will be spun down rapidly to $\Omega < \Omega_c$.

Normal neutron stars are generally unstable to $r$-modes except at very high or very low
temperatures, where the bulk and shear viscosities respectively are effective at damping $r$-mode oscillations. The width of the instability window is not very sensitive to stellar mass, but is noticeably smaller for a softer equation of state (larger polytropic index). The box marked “LMXBs” is representative of typical LMXB core temperatures $2 \times 10^7$ to $3 \times 10^8$ K, with rotation frequencies between 300 and 700 Hz. Strange stars made of ungapped quark matter are more stable than their neutron star counterparts in a window of temperatures $10^8$ K to $5 \times 10^9$ K where the bulk viscosity damping timescale in quark matter is quite small. Strange stars composed of CFL matter are unstable to $r$-mode oscillations in the range $10^{10}$ K $\leq T \leq 10^{11}$ K. Hybrid stars with ungapped quark matter are much like ungapped strange stars, except that the bulk viscosity from the outer shell of hadrons damps the $r$-mode oscillations at higher temperatures. $r$-mode oscillations in hybrid stars with gapped quark matter are unstable in the range $10^9$ K $\leq T \leq 10^{10}$ K. The results for hybrid stars are much like the corresponding strange star model because we have maximized the amount of quark matter by fixing the deconfinement transition density to be small. If this transition density is larger, the critical curves for hybrid stars look more similar to those in the left panel of Fig. 4.

VI. CONCLUSIONS

We have calculated the frequency and viscous damping timescale of the $m = 2$ $r$-mode for compact stars made of hadronic matter, quark matter, and for hybrid stars. To our knowledge, this is the first attempt at studying $r$-mode characteristics for pure or hybrid stars with color superconducting quark matter, particularly the CFL phase. In absolute terms, the $r$-mode frequency is about 50% larger for strange stars in comparison to neutron stars of the same mass with a polytropic EoS, assuming both are rotating at their respective Kepler frequency. The ratio of the $r$-mode frequency to the star’s rotation frequency is larger for strange stars and hybrid stars; this is due to the softer equation of state. It is also evident that EoS effects on the mode frequency are smaller for softer equations of state.

The dominant bulk and shear viscosities of the normal hadronic phase, the ungapped quark phase as well as the gapped CFL quark phase, which was derived recently, are used to estimate viscous damping timescales for homogeneous and hybrid stars. The current results confirm established results for neutron stars; viz. that since neutron stars are stable against the $r$-mode instability only at very high ($T \gtrsim 10^{10}$ K) or very low temperatures ($T \lesssim 10^6$ K), they would be spun down rapidly by the $r$-mode instability shortly after their birth at MeV temperatures, when large neutrino losses cause rapid cooling. The predicted critical frequency for LMXBs with hadronic neutron stars is close to or inside the region given by observations and LMXBs. A more careful analysis taking the effect of superfluidity in neutron matter into account would provide a clearer picture on the presence of $r$-modes assuming no quark matter is present.

Strange stars with non-superfluid quark matter display a stability window between $10^8$ K $< T < 5 \times 10^9$ K where they can spin at a substantial fraction of the Kepler frequency. If LMXBs contain strange stars, then on statistical grounds, we should expect to observe some of them spinning close to their Kepler frequency, unless some mechanism not connected to $r$-modes is responsible for limiting their frequency or the quark phase is more complicated than considered here (ungapped or CFL). If newly-born superconducting strange stars occur in nature, they may still spin down rapidly at birth, provided the cooling timescale is slower than the spin-down timescale. A consistent analysis of spin-down and cooling is currently
under investigation and will be reported in separate work. Hybrid stars tend to be more resilient against the $r$-mode instability, except for a temperature range which is dependent on the quark EoS. This effect is not surprising: in hybrid stars, $r$-modes which would normally be undamped in hadronic matter are damped by the quark core and vice versa. We conclude that hybrid proto-neutron stars are not likely to be damped by $r$-modes at all. This work also has important implications for LMXBs if they are hybrid stars containing a significant amount of deconfined quark matter in the ungapped or CFL phase, or if LMXBs are strange stars with a thin accreted hadronic crust. Based on the estimates of viscous damping timescales, the presence of quark matter in LMXBs implies that either they must contain rapidly rotating stars that have not yet been observed, or have a more complicated phase structure. A similar explanation regarding the LMXB rotating at 1122 Hz was put forth recently [74].

To make the results more concrete, future work will have to address several assumptions more closely. We assumed that the modified Urca process provides the dominant contribution to bulk viscosity while neutron-neutron scattering determines the shear viscosity. In practice, in the core of neutron stars, for $T \lesssim T_c \sim 10^8$ K, neutron matter is likely to be in a triplet-paired superfluid state. This will suppress the modified Urca rate [55], rendering it insignificant at $T \ll T_c$. Singlet pairing among neutrons and/or protons has a larger critical temperature $T_c \sim 5 \times 10^9$ K [54] and can occur in the crust with similar consequences. Under these conditions, it is more realistic to include alternate sources of viscosity along the lines of Ref. [57]. It would also be useful to explicitly check the contributions from the additional viscosities $\zeta_1$ and $\zeta_3$ that are inherent in a 2-fluid model of superfluidity.

We also assume that in the CFL phase, the shear viscosity is from phonons alone, whereas below $T \sim 10^{10}$ K, the damping timescales are not reliable as the phonon mean free path exceeds the stellar size. At these low temperatures, other contributions such as from photons would be more relevant and need to be explicitly computed and included for an accurate treatment of $r$-modes in CFL matter.

In the current work, we considered a simplified model for a neutron/strange star in this work, ignoring realistic features such as a crustal layer, rubbing friction at the quark-hadron interface and possibly the role of other contributions to the bulk and shear viscosity at very high or very low temperatures. Neutron star crusts composed of hadronic matter typically damp $r$-mode oscillations further. Early work in Ref. [75] suggested that damping timescales were decreased by as much as a factor of $10^5 - 10^7$ because of the additional viscosity provided by the neutron star crust. This conclusion has been softened in some later works: Ref. [76] concluded that the associated decrease in the damping timescale was a factor of $10^2 - 10^3$ smaller, and Ref. [77] has pointed out that the damping timescale is sensitive to the crustal composition. The final result may well be that $r$-modes are not as important a mechanism as believed for spinning down fast-rotating compact objects. Ref. [78] has also stressed the importance of updating the multi-component nature of equation of state in order to fully assess the effects of superfluidity. However, this does not change the central conclusion, viz., if $r$-modes do play a role in spinning down neutron stars, the present work demonstrates that this effect is much less efficient in neutron stars containing ungapped or completely gapped quark matter.
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