Research Article

Analytical and Numerical Study of Soret and Dufour Effects on Thermosolutal Convection in a Horizontal Brinkman Porous Layer with a Stress-Free Upper Boundary

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1. Introduction

Natural convection involving binary mixtures in the porous and fluid media is still a relevant field of investigation since it occurs in wide ranges of applications in many engineering problems and natural fields such as geophysics, oil reservoirs, storage of nuclear wastes, operation of solar ponds, chemical reactors, migration of mixtures in fibrous insulation, and metal manufacturing processes. This phenomenon occurs due to temperature differences, concentration differences, or by combination of these two differences [1]. Convection in which the buoyancy forces are due to both temperature and chemical concentration gradients is referred to as thermosolutal or double-diffusive convection.

The fluid flow behavior driven by temperature and concentration gradients has attracted the interest of researchers worldwide through the decades [2]. Different aspects of the problems involving double-diffusive natural convection have been addressed previously [3–10] in the absence of Soret and Dufour effects and neglected also in many other studies since they are of smaller order of magnitude compared to the effects described by Fourier’s and Fick’s laws. Nevertheless, the literature review shows the existence of experiments mainly focused on the determination of the Soret coefficients for mixtures [11–13] or on theoretical aspects to determine the optimal combinations of the governing parameters leading to the separation of species under the effect of thermo-diffusion [14–16]. Other aspects of double-diffusive convection problems in the presence of the Soret effect have been conducted in shallow horizontal cavities by Bourich et al. [17–19]. The results reported in these investigations are concerned with convection.
thresholds [17], reversal of the horizontal concentration gradient [18], and a comparison study between the cases where the enclosure is filled with a clear binary mixture or a saturated porous medium [19]. More recently, Hasnaoui et al. [20] studied numerically double-diffusive natural convection in an inclined enclosure with heat generation and Soret effect using a hybrid lattice Boltzmann-finite difference method. They show that the negative Soret parameter combined with high internal heat generation and a relatively high inclination is important when the objective is to maintain the fluid at a high concentration of species.

Despite the fact that Dufour and Soret effects are of a smaller order of magnitude, they however may play a significant role in double-diffusive flow processes. The list of published works taking into account both these effects is limited. Ojeda et al. [21] studied numerically the influence of an external magnetic field and radiation in free convective Jeffrey fluid flow between parallel porous plates in the presence of Soret and Dufour effects. The governing equations were solved numerically using the shooting method with the 4th Runge–Kutta scheme. The numerical study conducted by Zaho et al. [22] focused on heat and mass transfer generated by natural convection in a porous medium saturated by a viscoelastic fluid and subjected to a magnetic field. Kefayati [23] studied the impact of Soret and Dufour effects on entropy generation due to double diffusion in a square cavity filled with non-Newtonian power-law fluid by using a finite-difference lattice Boltzmann method. He found that the Dufour parameter results in the enhancement of the total irreversibility. A similar study was performed by the same author [24, 25] in the case of inclined cavities. Wang et al. [26] investigated numerically the behavior change of solutions accompanying the increase of the buoyancy ratio. Ren and Chan [27] studied transient double-diffusive convection under the effect of numerous parameters of control including thermoo-diffusion and diffusion-thermo. Balla and Naikoti [28] analyzed numerically double-diffusive free convection heat and solute transfer in an inclined square porous cavity saturated with a fluid in the presence of Soret and Dufour effects. Both these effects were considered by Al-Mudhaf et al. [29] who studied numerically unsteady double-diffusive natural convection inside trapezoidal inclined enclosures filled with an isotropic porous medium and submitted from one side to nonconstant distributions of temperature and concentration. Double-diffusive natural convection of non-Newtonian power-law fluids in an open cavity subjected to a horizontal magnetic field in the presence of Soret and Dufour effects was analyzed by Kefayati [30, 31]. Among others, he showed that the rise of Soret and Dufour parameters enhances the entropy generations due to heat transfer and fluid friction.

The present study is devoted to analytical and numerical investigation of Soret and Dufour effects within a horizontal Brinkman layer with a stress-free upper boundary. The main objective is to bring out the combined effects of Soret and Dufour parameters on thresholds of convection, fluid flow, and heat and mass transfer characteristics. For this shallow enclosure, the parallel flow approximation allows to predict the critical Rayleigh numbers for the onset of supercritical and subcritical convection. In addition, the simplifications allowed by this approximation leads to identify different regions with various parallel flow behaviors. The adopted Brinkman model permits covering the limits of Darcy and the pure fluid media. The study focuses on the discussion of combined effects of Rayleigh number, Dufour parameter, Soret parameter, Dufour parameter, buoyancy ratio, and Darcy number.

2. Mathematical Formulation of the Problem

The domain under study, sketched in Figure 1, is a two-dimensional horizontal porous cavity saturated with a binary mixture. The height of the cavity is $H'$, its length is $L'$ ($A = L'/H' \gg 1$), and its upper horizontal surface is non-deformable and stress-free, while the remaining boundaries are assumed rigid. The short boundaries (vertical walls) of the cavity are maintained adiabatic and impermeable to mass transfer, while its long horizontal walls are subjected to uniform fluxes of heat, $q'$, and mass, $j'$. The porous matrix is assumed isotropic and homogeneous, and the Brinkman–Hazen–Darcy model is adopted. The binary fluid that saturates the porous matrix is assumed to be Newtonian, and it is modeled as a Boussinesq incompressible fluid whose density $\rho$ varies linearly with the temperature $T'$ and the concentration $S'$ as follows:

$$\rho = \rho_0[1 - \beta_T(T' - T_0) - \beta_S(S' - S_0)],$$

where $\rho_0$ is the binary fluid density at temperature and concentration $(T_0, S_0)$ and $\beta_T$ and $\beta_S$ are the thermal and solutal expansion coefficients, respectively.

The dimensional equations governing this problem are written as

$$\nabla \cdot \overrightarrow{\Gamma} = 0,$$  \hspace{1cm} (2)

$$\overrightarrow{\nabla} T = -\frac{K}{\mu} \overrightarrow{V}' + \frac{1}{\rho} (\mu \frac{\partial \overrightarrow{V}'}{\partial t} + \frac{\partial \rho}{\partial t} \overrightarrow{g}),$$  \hspace{1cm} (3)

$$(\rho C)_{f} \frac{\partial T'}{\partial t} + (\rho C)_{f} \overrightarrow{V}' \cdot \overrightarrow{V}' = \nabla \cdot \left( \lambda \nabla T' \right) + (\rho C)_{f} K_{TS} \nabla S',$$  \hspace{1cm} (4)

$$\varepsilon' \frac{\partial S'}{\partial t} + \overrightarrow{V}' \cdot \overrightarrow{V}' S' = \nabla \cdot \left( D_{TS} \nabla S' \right) + K_{ST} \nabla T'.$$  \hspace{1cm} (5)

The reference scales for time, length, velocity, and pressure are $\sigma H'/\alpha$, $H'$, $H'/\alpha$, and $\mu K/\alpha$, respectively. The parameters $\sigma$ and $\alpha$ are defined by $\sigma = (\rho C)_{f}/(\rho C)_{f}$ and $\alpha = \lambda/(\rho C)_{f}$.

The dimensionless temperature and mass fraction are given by $T = T' - T_0/\Delta T'$ and $S = S' - S_0/\Delta S'$ where $\Delta T' = \dot{q}' H'/\lambda$ and $\Delta S' = j' H'/D$.

The governing equations that are solved numerically are based on the vorticity-stream function formulation. In their dimensionless forms [26], they are obtained as follows:

$$\eta \frac{\partial \xi}{\partial t} + \xi = Da \nabla^2 \xi + R_T \left( \frac{\partial T}{\partial x} + N \frac{\partial S}{\partial x} \right),$$  \hspace{1cm} (6)

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \nabla^2 T + Du \nabla^2 S,$$  \hspace{1cm} (7)
The examination of the dimensionless governing equations, equations (6)–(10), and their associated boundary conditions, equations (11a)–(11c), shows that the control parameters are the aspect ratio of the enclosure, $A$, Darcy number, $Da$, the Dufour number, $Du$, the Lewis number, $Le$, the thermal Rayleigh number, $Ra$, and the Soret number, $Sr$. These parameters are defined as follows:

$$A = \frac{L'}{H'},$$

$$Da = \frac{\mu_{ef}K'}{\mu H'^2},$$

$$Du = \frac{K_{T,S} \Delta S'}{a \Delta T'^r},$$

$$Le = \frac{a}{D'},$$

$$Ra = \frac{g \beta_T \Delta T' H'}{\nu v},$$

$$Sr = \frac{K_{ST} \Delta T'}{D \Delta S'}. $$

The Nusselt and Sherwood numbers are, respectively, given by the following expressions:

$$Nu = \frac{1}{\Delta T + Du \Delta S},$$

$$Sh = \frac{1}{\Delta S + Sr \Delta T'},$$

where $\Delta T = T(0,1/2) - T(0,-1/2)$ and $\Delta S = S(0,1/2) - S(0,-1/2)$ are the temperature and concentration differences, evaluated at $x = 0$.

3. Numerical Solution

The numerical solution of the governing equations together with the associated boundary conditions, equations (6) to (11a)–(11c), is obtained using a finite-difference method. The vorticity equation, equation (6), the energy equation, equation (7), and the equation of conservation of species, equation (8), are written in their transient forms since the iterative procedure is performed using the alternate direction implicit (ADI) method for these equations. The stream function field is derived from the resolution of equation (9) using the point successive over-relaxation method (PSOR). The numerical results reported in this paper were obtained with a globally nonuniform grid; the latter is uniform and tight near the confining rigid walls and the free surface to capture the flow details near these boundaries and uniform but coarser elsewhere. The vorticity values on the rigid boundaries are calculated using Wood’s relation [32], while zero is attributed to the vorticity at the free surface. Further details on the numerical method and its validation were given in a previous study by Amahmid et al. [5].

Numerous preliminary numerical tests have been performed to determine the minimum aspect ratio from which the assumption of the parallel flow is corroborated. Within the ranges of variations of the parameters considered in this investigation, it was found that the numerical results are independent of the aspect ratio from the threshold $A = 12$. 

**Figure 1:** Schematic diagram of the studied configuration.
Table 1: Effect of the grid size for \( R_T = 100, N = 1, Sr = 0.5, Du = 0.5, \) and \( Le = 1.1. \)

| Mesh        | \( \psi_0 \) | Nu | Sh |
|-------------|--------------|----|----|
| 161 \times 61 | -4.273       | 2.710 | 2.755 |
| 201 \times 81 | -4.269       | 2.703 | 2.748 |
| 261 \times 121 | -4.269       | 2.701 | 2.745 |
| Analytical results | -4.268       | 2.697 | 2.741 |

Thereby, the numerical results reported here were obtained with \( A = 12 \) and a grid of \( 201 \times 81 \). The effect of the grid size on the results is illustrated in Table 1 (analytical results are also included in this table). It can be seen from this table that the results obtained with the grid \( 201 \times 81 \) differ by less than 0.2% from those corresponding to the finest grid \( 261 \times 121 \). In addition, the grid \( 201 \times 81 \) reproduces the analytical results with relative errors lower than 0.3%.

4. Results and Discussion

4.1. Parallel Flow Analysis. Based on the remarks (verified numerically) allowing simplifications in the case of shallow cavities \( (A > 1) \), the flow is parallel to the long walls of the cavity and the temperature and concentration fields are characterized by a linear and horizontal stratification if we exclude the edges’ effects. Thus, the stream function, the temperature, and the concentration can be approximated as follows:

\[
\psi(x, y) = \psi(y),
\]

\[
T(x, y) = C_T x + \theta_T(y),
\]

\[
S(x, y) = C_S x + \theta_S(y),
\]

the parameters \( C_T \) and \( C_S \) are the unknown constant temperature and concentration gradients in the horizontal direction, respectively.

The investigation by Cormack et al. [33] counts among the early works developing these approximations based on the theory of asymptotic developments for \( A > 1 \). The steady form of the governing equations (6)–(8) while using these approximations can be reduced to the following set of ordinary differential equations:

\[
\frac{d^4 \psi}{dy^4} - \frac{d^2 \psi}{dy^2} + R = 0,
\]

\[
\frac{d^2 \theta_T}{dy^2} = \frac{d\psi}{dy}\left(\frac{C_T - Le Du C_S}{1 - Du Sr}\right),
\]

\[
\frac{d^2 \theta_S}{dy^2} = \frac{d\psi}{dy}\left(\frac{Le C_S - Sr C_T}{1 - Du Sr}\right),
\]

where \( R = -R_T (C_T + NC_S) \).

The corresponding boundary conditions on the horizontal walls become:

\[
\psi = \frac{d^2 \psi}{dy^2} = 0,
\]

\[
\frac{\partial \theta_T}{\partial y} = \frac{Du - 1}{1 - Du Sr}, \quad (18)
\]

\[
\frac{\partial \theta_S}{\partial y} = \frac{Sr - 1}{1 - Du Sr} \quad \text{for} \quad y = \frac{1}{2},
\]

\[
\psi = \frac{d\psi}{dy} = 0,
\]

\[
\frac{\partial \theta_T}{\partial y} = \frac{Du - 1}{1 - Du Sr}, \quad (19)
\]

\[
\frac{\partial \theta_S}{\partial y} = \frac{Sr - 1}{1 - Du Sr} \quad \text{for} \quad y = -\frac{1}{2}.
\]

The analytical integration of equations (15)–(17), together with the associated boundary conditions (18) and (19), leads to the following solutions, respectively, in terms of stream function, horizontal velocity, temperature, and concentration:

\[
\psi(y) = \psi_0 \frac{Z(y)}{Z(0)},
\]

\[
u(y) = \psi_0 \frac{H(y)}{Z(0)},
\]

\[
T(x, y) = C_T x + \frac{Du - 1}{1 - Du Sr} y - \frac{C_T - Le Du C_S}{1 - Du Sr} \psi_0 \frac{S(y)}{Z(0)},
\]

\[
S(x, y) = C_S x + \frac{Sr - 1}{1 - Du Sr} y - \frac{Le C_S - Sr C_T}{1 - Du Sr} \psi_0 \frac{S(y)}{Z(0)}.
\]

The parameter \( \psi_0 \) is the stream function at the center of the cavity, obtained analytically as

\[
\psi_0 = \pm \frac{Z(0)}{2Le \sqrt{2l} (Da)} \left( -b \pm \sqrt{b^2 - 4Le^2 c} \right)^{1/2},
\]

with

\[
b = P - R_T F, \quad (25)
\]

\[
c = Q - R_T F.
\]

The parameters in the previous equation are obtained as
\[ P = 1 + L e^2 + 2 S r D u L e, \]
\[ Q = (1 - D u S r)^2, \]
\[ F_1 = L e^2 + L e N, \]
\[ F_2 = N (D u S r - S r (1 + L e) + L e) - D u (1 + L e - L e S r) + 1. \]

The parameter \( R_0^2 \) is the normalized Rayleigh number, and \( Z (y), H (y), \) and \( S (y) \) are the functions, respectively, given by the following analytical expressions:
\[ Z (y) = \frac{X_1}{\xi} \cosh (\xi y) - \frac{X_2}{\xi} \sinh (\xi y) + y^3 - \chi_3 y - \chi_4, \]
\[ H (y) = \frac{X_1}{\xi} \sinh (\xi y) - \frac{X_2}{\xi} \cosh (\xi y) + 2 y - \chi_3, \]
\[ S (y) = \frac{X_1}{\xi} \sinh (\xi y) + \frac{X_2}{\xi} \cosh (\xi y) - \frac{y^3}{3} + \frac{X_3}{2} y^2 + \chi_4 y. \]

According to equation (21), the velocity cannot be zero in the range \( 0 < y < 1/2 \), which means that only unicellular flows are possible. In addition, the constants \( C_T \) and \( C_S \) are obtained by considering mass and thermal balances across any transversal section of the porous layer [34].
\[
\int_{-1/2}^{1/2} \left[ u T - \frac{\partial T}{\partial x} - D u \frac{\partial S}{\partial x} \right] d y = 0, \tag{31}
\]
\[
\int_{-1/2}^{1/2} \left[ u S - \frac{1}{Le} \left( \frac{\partial S}{\partial x} + S \frac{\partial T}{\partial x} \right) \right] d y = 0. \tag{32}
\]

Substituting equations (22) and (23) into equation (13), the respective analytical expressions of Nusselt and Sherwood numbers are obtained as
\[
Nu = \frac{1}{1 + A (Da) R C_T}, \tag{33}
\]
\[
Sh = \frac{1}{1 + A (Da) R L e C_S}. \tag{34}
\]

By substituting the solutions given by equations (21)–(23) into equations (31) and (32), the final analytical expressions of \( C_T \) and \( C_S \) are obtained as follows:
\[
C_T = -\frac{A (Da) R L e (1 - S r) (L e D u R^2 T (Da) - M D u) - A (Da) R (1 - D u) (M + L e^2 R^2 T (Da))}{(M + R^2 T (Da)) (M + L e^2 R^2 T (Da)) - (L e D u R^2 T (Da) - M D u) (D u R^2 T (Da) - S r M)} \tag{34}
\]
\[
C_S = -\frac{A (Da) R L e (1 - S r) (M + R^2 T (Da)) - A (Da) R (1 - D u) (L e S r R^2 T (Da) - S r M)}{(M + R^2 T (Da)) (M + L e^2 R^2 T (Da)) - (L e D u R^2 T (Da) - M D u) (D u R^2 T (Da) - S r M)} \tag{34}
\]

with \( M = 1 - D u S r \) and \( R = 2 \psi_0 / Z (0) \).
The analysis of equation (24) indicates that up to five different steady-state solutions are possible including the diffusive regime \( (\psi_0 = 0) \). The four other convective solutions depend on the signs \( \pm \) inside and outside the brackets. In fact, the sign \( \pm \) outside the brackets refers to counterclockwise/clockwise circulation, while it refers to convective stable/unstable solutions within the brackets. It is to specify that the “unstable” solutions refer to the analytical solutions that could not be obtained numerically. In addition, from a mathematical point of view, equation (24) shows the existence of supercritical and subcritical bifurcations \([19]\). The supercritical bifurcation is characterized by the transition from the quiescent state to the convective regime through finite amplitude \( \psi_0 = \pm Z(0)\sqrt{-b/2 \text{Le}}\sqrt{2\Gamma(D)} \).

These parallel flow solutions exist only when the two following conditions are satisfied: \((-b \pm \sqrt{b^2 - 4\text{Le}^2c}) > 0\) and \((b^2 - 4\text{Le}^2c) > 0\). The satisfaction of these conditions allows the subdivision of the \((N, Du)\) plane into different regions with specific behaviors, depending on the sign of the parameter \(P\) defined by equation (26).

### 4.1.1. Analytical Delineation of the Different Regions.

On the basis of the analytical solution, up to six regions with specific behaviors are identified in the \((N, Du)\) plane. These regions are illustrated in Figures 2-4 for \( Sr > 0 \), \( Sr < 0 \), and \( Sr = 0 \), respectively.

The region (1) is characterized by the rest state, which means that the parallel flow solution is not possible in this region whatever the value of \(R_f\). Different conditions must be verified to delineate this region in the \((N, Du)\) plane, depending on the sign of \(Sr\).

(i) For \( Sr > 0 \)

\[
\begin{align*}
N &< -\text{Le}, \\
N &< \frac{Du(1 + \text{Le} - \text{Le} Sr) - 1}{Du \text{Sr} - \text{Sr} (1 + \text{Le}) + \text{Le}} \quad \text{for} \quad Du \geq 1 + \text{Le}(Sr - 1), \\
\leq Du &\leq 1 + \text{Le}(Sr - 1).
\end{align*}
\]

(ii) For \( Sr < 0 \)

\[
\begin{align*}
N &< -\text{Le}, \\
N &< \frac{Du(1 + \text{Le} - \text{Le} Sr) - 1}{Du \text{Sr} - \text{Sr} (1 + \text{Le}) + \text{Le}} \quad \text{for} \quad Du \geq 1 + \text{Le}(Sr - 1), \\
\leq Du &\leq 1 + \text{Le}(Sr - 1).
\end{align*}
\]

(iii) For \( Sr = 0 \)

\[
\begin{align*}
N &< -\text{Le}, \\
N &< \frac{Du(1 + \text{Le}) - 1}{\text{Le}} \quad \text{for} \quad Du \geq 1 - \text{Le}, \\
\leq Du &\leq 1 - \text{Le}.
\end{align*}
\]

For the region (2), only the supercritical flow is possible. The corresponding supercritical Rayleigh number, characterizing the transition from the rest state to the convective regime through the zero flow amplitude, is given by

\[
\begin{align*}
N &< -\text{Le}, \\
N &< \frac{Du(1 + \text{Le}) - 1}{\text{Le}} \quad \text{for} \quad Du \geq 1 - \text{Le}.
\end{align*}
\]
\[ R_{T^*}^{\text{sup}} = \frac{R^{\text{sup}}(1 - Du \cdot Sr)^2}{N(Du \cdot Sr - Sr(1 + Le) + Le) - Du(1 + Le - Le \cdot Sr) + 1} \]

The conditions that should be verified to delineate this region in the \((N, Du)\) plane are dependent on the sign of \(Sr\).

(i) For \(Sr > 0\)

\[
\begin{align*}
N < f(Du), & \quad \text{for } Du < \frac{-1 - Le^2}{2Le \cdot Sr}, \\
N < -Le, & \quad \text{for } \frac{-1 - Le^2}{2Le \cdot Sr} \leq Du \leq \frac{Sr(1 + Le) - Le}{Sr}, \\
\frac{Du(1 + Le - Le \cdot Sr) - 1}{Du \cdot Sr - Sr(1 + Le) + Le} \leq N \leq -Le, & \quad \text{for } \frac{Sr(1 + Le) - Le}{Sr} \leq Du \leq 1 + Le(Sr - 1),
\end{align*}
\]

with

\[ f(Du) = \frac{Du^2(2Sr \cdot Le(1 + Le) - Sr \cdot Le^2) + Du(1 + Le + Le^2 + Le^3 - 3Sr \cdot Le - Le^3 \cdot Sr) - 1}{Le \cdot Sr^2 \cdot Du^2 + Du(Sr + 3Sr \cdot Le^2 - 2Sr^2 \cdot Le^2 - 2Sr^3 \cdot Le^2 + 2Le \cdot Sr) + (Le^3 - Sr \cdot Le^3 - Sr - Sr \cdot Le - Sr \cdot Le^3)} \]

(ii) For \(Sr < 0\)

\[
\begin{align*}
\frac{Du(1 + Le - Le \cdot Sr) - 1}{Du \cdot Sr - Sr(1 + Le) + Le} \leq N \leq -Le, & \quad \text{for } Du \leq 1 + Le(Sr - 1). \\
N < -Le, & \quad \text{for } Du \geq 1 - Le,
\end{align*}
\]

(iii) For \(Sr = 0\)

\[
\begin{align*}
N > \frac{Du(1 + Le)}{Le} - \frac{1}{Le}, & \quad \text{for } Du \leq 1 - Le.
\end{align*}
\]

For the region (3), the characteristics are similar to those in region (2). The stable parallel flow solution exists from the
supercritical Rayleigh number given by equation (38). Even if the regions (2) and (3) correspond to the same type of convection (stationary convection), they are however characterized by a difference in terms of asymptotic evolutions at large values of the Rayleigh number.

The conditions that should be verified for this region are dependent on the sign of $Sr$ as follows:

(i) For $Sr > 0$

\[
\begin{align*}
&-Le \leq N < f(Du), \quad \text{for } \frac{-1 - Le^2}{2Le Sr} < Du < Du_1, \\
&N > -Le, \quad \text{for } Du_1 < Du < 1 + Le(Sr - 1), \\
&N > f(Du), \quad \text{for } Du > 1 + Le(Sr - 1),
\end{align*}
\]

with

\[
Du_1 = \frac{g + \sqrt{1 + 10Le^2 - 16Sr L^e^2 + 4Sr^2 L^e^4 - 8Sr L^e^2 + 5L^e^4 - 8Sr L^e^4 + 12L^e^2 + 4Sr^2 L^e^2 + 8Sr L^e^2 + 4Le}}{2Sr L^e},
\]

\[
Du_2 = \frac{g - \sqrt{1 + 10Le^2 - 16Sr L^e^2 + 4Sr^2 L^e^4 - 8Sr L^e^2 + 5L^e^4 - 8Sr L^e^4 + 12L^e^2 + 4Sr^2 L^e^2 + 8Sr L^e^2 + 4Le}}{2Sr L^e},
\]

and $g = 1 + 3Le^2 + 2Le - 2Sr L^e^2 - 2Sr L^e$

(ii) For $Sr < 0$

\[
\begin{align*}
&N > -Le, \quad \text{for } Du \leq 1 + Le(Sr - 1), \\
&N > f(Du), \quad \text{for } 1 + Le(Sr - 1) \leq Du < Du_2.
\end{align*}
\]

(iii) For $Sr = 0$

\[
\begin{align*}
&N > -Le, \quad \text{for } Du < 1 - Le, \\
&N > \max\left(\frac{Du(1 + Le + Le^2 + Le^3)}{Le^3} - \frac{1}{Le}, \frac{Du(1 + Le)}{Le} - \frac{1}{Le}\right), \quad \text{for } Du > 1 - Le.
\end{align*}
\]

For the region (4), both stable and unstable solutions are existing from the subcritical Rayleigh number, but the unstable branch disappears at some threshold of $R_T$. In this region, the resulting solutions correspond to a subcritical bifurcation for which the onset of motion occurs through a finite amplitude.

The critical Rayleigh number, $R_{TC}^{sub}$, above which the parallel flow exists, is obtained from the condition $b^2 - 4Le^2 c = 0$ and its expression is given by

\[
R_{TC}^{sub} = G + 2Le^2 \sqrt{Le^2 F_2^2 + (1 - Du Sr)^2 (Le^2 + Le N) - F_2 (Le^2 + Le N)(1 + Le^2 + 2Sr Du Le)} \quad \text{for } R_T > 0,
\]

where $G = (Le^4 - Le^2) + N (Le - Le^3) + 2N Sr Du (Le^3 - Le^2) + (2N Sr + 2Du)(Le^2 + Le^3)$. The conditions that should be verified for the region (4) in the $(N, Du)$ plane are dependent on the sign of $Sr$ as follows:
(i) For $Sr > 0$

\[\begin{align*}
-\text{Le} < N &< \frac{Du(1 + \text{Le} - \text{Le}Sr) - 1}{Du Sr - \text{Sr}(1 + \text{Le}) + \text{Le}} \\
\text{for } Du &< -\frac{1 - \text{Le}^2}{2\text{Le} Sr} \\
f(Du) < N &< \frac{Du(1 + \text{Le} - \text{Le}Sr) - 1}{Du Sr - \text{Sr}(1 + \text{Le}) + \text{Le}} \\
\text{for } -\frac{1 - \text{Le}^2}{2\text{Le} Sr} &\leq Du \leq \frac{\text{Sr}(1 + \text{Le}) - \text{Le}}{\text{Sr}} \\
N > f(Du), &\text{for } \frac{\text{Sr}(1 + \text{Le}) - \text{Le}}{\text{Sr}} < Du < Du_1,
\end{align*}\]

(ii) For $Sr < 0$

\[\begin{align*}
\frac{Du(1 + \text{Le} - \text{Le}Sr) - 1}{Du Sr - \text{Sr}(1 + \text{Le}) + \text{Le}} < N &< f(Du), \text{ for } 1 + \text{Le}(\text{Sr} - 1) \leq Du < Du_2, \\
N > \frac{Du(1 + \text{Le} - \text{Le}Sr) - 1}{Du Sr - \text{Sr}(1 + \text{Le}) + \text{Le}} &\text{ for } Du_2 \leq Du \leq \frac{\text{Sr}(1 + \text{Le}) - \text{Le}}{\text{Sr}}.
\end{align*}\]

(iii) For $Sr = 0$

\[\begin{align*}
N > -\text{Le}, &\text{ for } Du \leq 1 - \text{Le}, \\
\frac{Du(1 + \text{Le})}{\text{Le}} - \frac{1}{\text{Le}} &< N < \frac{Du(1 + \text{Le} + \text{Le}^2 + \text{Le}^3)}{\text{Le}^3} - \frac{1}{\text{Le}^3}, \text{ for } Du > 1 - \text{Le}.
\end{align*}\]

For the region (5), the behaviors are similar to those of region (4) but the existence range of the unstable branches of the former is extended towards infinite $R_T$.

(i) For $Sr > 0$

\[\begin{align*}
N > \frac{Du(1 + \text{Le} - \text{Le}Sr) - 1}{Du Sr - \text{Sr}(1 + \text{Le}) + \text{Le}} &\text{ for } Du \leq \frac{\text{Sr}(1 + \text{Le}) - \text{Le}}{\text{Sr}}, \\
-\text{Le} < N &< \frac{Du(1 + \text{Le} - \text{Le}Sr) - 1}{Du Sr - \text{Sr}(1 + \text{Le}) + \text{Le}} \text{ for } Du \geq 1 + \text{Le}(\text{Sr} - 1).
\end{align*}\]

(ii) For $Sr < 0$

\[\begin{align*}
-\text{Le} &\leq N \leq \frac{Du(1 + \text{Le} - \text{Le}Sr) - 1}{Du Sr - \text{Sr}(1 + \text{Le}) + \text{Le}} \text{ for } Du \geq 1 + \text{Le}(\text{Sr} - 1), \\
-\text{Le} \leq N &\leq \frac{Du(1 + \text{Le})}{\text{Le}} - \frac{1}{\text{Le}} \text{ for } Du \geq 1 - \text{Le}.
\end{align*}\]

Finally, the region (6) is characterized by the same behavior as that of region (4); the only difference is observed in terms of their asymptotic behaviors. The Soret number appears explicitly in the conditions delineating this region as follows:
\[ f(Du) \leq N \leq -Le, \quad \text{for} \quad Du \leq \frac{Sr(1 + Le) - Le}{Sr} \]  

The expression of \( f(Du) \) was already defined for region (2).

4.2. Effect of \( R_T \). The effect of \( R_T \) on the flow intensity, \( \psi_0 = \psi(0,0) \), the Nusselt number, \( Nu \), and the Sherwood number, \( Sh \), is illustrated in Figures 5(a)–5(c) for \( Sr = 0.6 \), \( Da = 0.01 \), \( Le = 1.1 \), and different combinations of \((N,Du)\).
corresponding to different regions in this plane. The choice of \( \text{Sr} = 0.6 \) allows to cover the six identified regions. Moreover, it can be observed from these figures that the stable analytical solution (solid lines) is in a very good agreement with the numerical results depicted by full circles and obtained by solving the full governing equations.

For the combination \( (N, Du) = (-2, -0.5) \) corresponding to the region (2), only the stable branch exists. The supercritical convection starts from the rest state (characterized by \( \psi = \psi(0, 0) = 0 \), \( \text{Nu} = 1 \), and \( \text{Sh} = 1 \)) at \( R_{Tc}^{\text{Sup}} = 11.12 \). The evolutions of \( \psi \), \( \text{Nu} \), and \( \text{Sh} \) are characterized by asymptotic behaviors, and their respective asymptotic limits are 2.31, 3.45, and 1.64, reached for sufficiently large values of \( R_T \). Note that for this combination of \( (N, Du) \), the asymptotic value of \( \text{Sh} \) indicates that diffusion plays an important role in the mass transfer. At large values of \( R_T \), the analytical expressions of the asymptotic expressions of \( \psi \), \( \text{Nu} \), and \( \text{Sh} \) corresponding to the stable solution are obtained by the following expressions:

\[
\psi_{\text{asy}} = \pm \frac{Z(0)}{2} \frac{N(Du \text{Sr} - \text{Sr}(1 + \text{Le}) + \text{Le}) - Du(1 + \text{Le} - \text{Sr} \text{Le}) + 1}{\Gamma(D)\text{Le}(\text{Le} + N)},
\]

\[
\text{Nu}_{\text{asy}} = \frac{h}{h - A(Da)^2 \chi A(1 - \text{Sr})(\text{Le} \sqrt{\chi} \Gamma(Da) - MDA) - A(Da)^2 (1 - Du)(M + \text{Le}^2 \chi^2 \Gamma(Da))}
\]

\[
\text{Sh}_{\text{asy}} = \frac{h}{h - \chi^2 A(Da)^2 (\text{Le}^2 (1 - \text{Sr})(M + \text{Le}^2 \chi^2 \Gamma(Da)) - \text{Le}(1 - Du)(\text{Le} \sqrt{\chi} \Gamma(Da) - \text{SrM})}
\]

with \( h = (M + \chi A \Gamma(Da))(M + \text{Le}^2 \chi A \Gamma(Da)) - (\text{Le} \sqrt{\chi} A \Gamma(Da) - MDA)(\text{Le} \sqrt{\chi} A \Gamma(Da) - \text{SrM}) \) and \( \chi = \sqrt{(\text{Du} \text{Sr} - \text{Sr}(1 + \text{Le}) + \text{Le} - \text{Du}(1 + \text{Le} - \text{Sr} \text{Le}) + 1)/((\Gamma(Da) \text{Le}(\text{Le} + N)))} \).

For the combination \( (N, Du) = (2, 0.7) \), illustrating the region (3), the behavior is also supercritical and the convection starts from the rest state at \( R_{Tc}^{\text{Sup}} = 11.42 \). The evolution of \( \psi \) vs. \( R_T \) is characterized by a monotonic increase with different rates that depend on the combination \( (N, Du) \). For \( R_T \rightarrow + \infty \), the analytical expression of \( \psi \) is reduced to

\[
\psi_0 = \pm Z(0) \frac{(N + \text{Le})A(Da)}{\Gamma(Da)\text{Le}} R_{Tc}^{1/2}.
\]

Equation (58) shows that \( \psi_0 \) varies as \( R_{Tc}^{1/2} \), while the evolutions of \( \text{Nu} \) and \( \text{Sh} \) are characterized by asymptotic behaviors towards the same limit, which is 4.49 (see Figures 5(b) and 5(c)). This value is deduced from the asymptotic expression given by equation (59) that is common for both \( \text{Nu} \) and \( \text{Sh} \) at large \( R_T \):

\[
\text{Nu} = \text{Sh} = \frac{\Gamma(Da)}{\Gamma(Da) - A(Da)} Z(0)
\]

The region (4) is illustrated by the combination \( (N, Du) = (0.3, 0.7) \) in Figures 5(a)–5(c). Unlike the previous combinations of \( (N, Du) \), for which only the stable branches exist, both stable and unstable solutions exist for this combination in the range \( R_{Tc}^{\text{Sub}} = 45.12 \leq R_T < R_{Tc}^{\text{Sup}} = 83.50 \). This means that the parallel flow solution starts at \( R_{Tc}^{\text{Sub}} = 45.12 \), and the nascent flow corresponds to a state clearly different from the purely diffusive regime characterized by \( \psi_0 = 0 \) and \( \text{Nu} = \text{Sh} = 1 \). For the unstable branch, \( \psi_0/(\text{Nu} \text{ and } \text{Sh}) \) decreases notably/(weakly) by increasing \( R_T \) and disappears for \( R_T \approx R_{Tc}^{\text{Sup}} = 83.50 \).

For the stable branch, the evolution of \( \psi_0 \) versus \( R_T \) is very similar to that described for the combination \( (N, Du) = (2, 0.7) \) but characterized by a delay in terms of \( R_T \). At large values of the last parameter, the expression of \( \psi_0 \) is the same as that established for the region (3) and given by equation (58). For the region (4) also, the evolutions of \( \text{Nu} \) and \( \text{Sh} \) are characterized by asymptotic behaviors at large \( R_T \), leading to the common asymptotic value 4.49, obtained with equation (59) for both \( \text{Nu} \) and \( \text{Sh} \).

The results corresponding to the region (5) are illustrated in Figures 5(a)–5(c) by the combination \( (N, Du) = (0.2, 0.9) \). For this region, both stable and unstable solutions exist for \( R_T \geq R_{Tc}^{\text{Sub}} = 69.16 \) for the considered combination of \( (N, Du) \). In addition, the evolution of the stable branch is similar to that described for region (4). By increasing \( R_T \), the unstable branches are characterized by slow decreases towards asymptotic limits for \( \psi_0 \), \( \text{Nu} \), and \( \text{Sh} \).

The region (6) is illustrated by the combination \( (N, Du) = (1.2, 1.8) \) and shows a behavior different from those already described. More precisely, both stable and unstable solutions for this region exist in the very short range \( R_{Tc}^{\text{Sup}} = 14.793 \leq R_T < R_{Tc}^{\text{Sup}} = 14.797 \). In addition, at large values of \( R_T \), the quantities \( \psi_0 \), \( \text{Nu} \), and \( \text{Sh} \) tend, respectively, to the asymptotic limits 9.70, 5.827, and 4.155. It should be outlined that for the region (6) the behaviors are similar to those already described for the regions (4) and (2), respectively, at low and large values of \( R_T \). Consequently, the analytical expressions of the asymptotic values of \( \psi_0 \), \( \text{Nu} \), and \( \text{Sh} \) are given by equations (55)–(57), respectively. Besides, for the region (6), the Nusselt number becomes negative in the range 27.34 \( \leq R_T \leq 453.99 \). This change in the sign is exclusively attributed to the negative value of the Dufour number and the small temperature differences induced in some ranges of \( R_T \).
4.3 Combined Effects of Dufour and Soret Numbers. This section is devoted to assess the simultaneous effects of thermal-diffusion (Soret effect) and diffusion-thermo (Dufour effect) on fluid flow and heat and mass transfer characteristics. Soret and Dufour effects are defined as the mass flux caused by a temperature difference and the energy flux engendered by concentration difference, respectively [35]. The Soret and Dufour parameters can be widely varied by changing the mean solute concentration or mean temperature of the system.

The effect of the Dufour number on the flow intensity, the Nusselt number, and the Sherwood number is illustrated...
in Figures 6(a)–6(c) for \( R_T = 500, N = -0.2, \) \( \text{Le} = 1.1, \) \( \text{Da} = 0.01 \) and various values of \( \text{Sr}. \) These figures show that, regardless of the value of \( \text{Sr}, \) there exists a critical value of \( \text{Du} \) above which convection disappears. This critical value increases with \( \text{Sr}; \) it rises from 0.124 to 0.897 when \( \text{Sr} \) goes from −0.9 to 0.9.

The effect of \( \text{Sr} \) on \( \psi_0, \) \( \text{Nu}, \) and \( \text{Sh} \) may be drastically affected by the change of \( \text{Du}. \) More specifically, we can observe two different behaviors depending on whether the value of \( \text{Du} \) is above or below critical ranges, which are so narrow that they appear as nodes in Figures 6(a)–6(c). In these critical ranges of \( \text{Du}, \) the effect of thermo-diffusion becomes insignificant. Another mystery of these critical ranges of \( \text{Du} \) lies in the fact that they are not the same for \( \psi_0, \) \( \text{Nu}, \) and \( \text{Sh} \) (compare \([0.6, 0.732]\) for \( \psi_0, \) \([0.016, 0.042]\) for \( \text{Nu}, \) and \([-2.29, -2.25]\) for \( \text{Sh}\)). Due to their very narrowness, the critical ranges will be termed as critical nodes in the following discussion. The critical node corresponding to the curves of \( \psi \) is seen to be largely shifted towards negative values of \( \text{Du}. \) The interaction between convection, thermo-diffusion, and diffusion-thermo is so complex that, without analytical calculations, it was not possible to predict such specific behaviors attributed to the combined effects of the governing parameters. It is worth mentioning that these behaviors were obtained analytically and validated numerically. By focusing on the stable solution (validated numerically), it can be seen that below the critical nodes, the quantities \( \psi_0, \) \( \text{Nu}, \) and \( \text{Sh} \) are increasing functions of \( \text{Sr}. \) Above the critical node corresponding to \( \text{Sh}, \) the latter decreases by increasing \( \text{Sr} \). However, immediately near the critical nodes, \( \psi_0 \) and \( \text{Nu} \) exhibit a behavior similar to that of \( \text{Sh} \) and their evolutions are characterized by complex variations far above these critical nodes. Another important difference observed by comparing heat and mass transfer from Figures 6(b) and 6(c) is the fact that \( \text{Nu} \) decreases by increasing \( \text{Du}, \) while \( \text{Sh} \) increases/decreases by increasing \( \text{Du} \) for \( \text{Sr} < -0.1/(\text{Sr} \geq -0.1). \) Furthermore, the most important variations observed for \( \text{Nu}/\text{Sh} \) when \( \text{Sr} \) is varied are located below/above the critical nodes. Quantitatively, at \( \text{Du} = -3 \) for instance, by increasing \( \text{Sr} \) from −0.9 to 0.9, \( \text{Nu} \) is multiplied by a factor of 4.1 while \( \text{Sh} \) is increased by only 11.44%. For \( \text{Du} = 2, \) by increasing \( \text{Sr} \) from −0.9 to 0.9, \( \text{Sh} \) is divided by 3.2 while \( \text{Nu} \) undergoes a relative decrease of about 14.82%.

The combined effects of \( \text{Du} \) and \( \text{Sr} \) on the temperature and concentration fields are illustrated in Figure 7 for \( R_T = 500, \) \( N = -0.2, \) \( \text{Le} = 1.1, \) \( \text{Da} = 0.01, \) \( \text{Sr} = \pm 0.9, \) and \( \text{Du} = \pm 0.9. \) For the combination \((\text{Du}, \text{Sr}) = (0.9, 0.9),\) it can be seen from Figure 7 that the isotherms are very similar to the isoconcentrations. Even the quantitative comparisons (results not presented) showed that the temperature field is not very different from the concentration one. The temperature and concentration profiles obtained at midwidth of the cavity and plotted in Figure 8 show that the temperature/concentration gradients are important in the vicinity of the horizontal wall, compared to the those (i.e., gradients) around the horizontal median. By changing the combination \((\text{Du}, \text{Sr})\) from \((0.9, 0.9)\) to \((0.9, -0.9),\) we can observe an accentuation/attenuation of the isoconcentration/isotherm distortions. The isotherms are nearly straight lines tilted with respect to the horizontal direction, which means that the temperature field in the cavity (if we except the vicinity of the short walls where the end effect is important) is quasi-linear. The temperature profile at midwidth of the cavity (Figure 8) confirms this behavior. It should be noted that the quasi-linearity exhibited by the temperature field for \((\text{Du}, \text{Sr}) = (0.9, -0.9)\) cannot be explained by the dominance of the diffusive regime. In fact, the diffusive regime leads to linear profiles with the isotherms being parallel to the horizontal walls. However, the isotherms obtained for \((\text{Du}, \text{Sr}) = (0.9, -0.9)\) are strongly tilted with respect to the horizontal direction. This means that this behavior is due to the interaction between convection, conduction, and diffusion-thermo (i.e., Dufour effect). The changes observed in the temperature/concentration field when \((\text{Du}, \text{Sr})\) passes from \((0.9, 0.9)\) to \((-0.9, 0.9)\) are similar to those observed in the concentration/temperature field when \((\text{Du}, \text{Sr})\) passes from \((0.9, 0.9)\) to \((0.9, -0.9). \) Hence, we can observe a quasi-linear behavior for the concentration field at \((\text{Du}, \text{Sr}) = (-0.9, 0.9). \) The combination \((\text{Du}, \text{Sr}) = (-0.9, -0.9)\) leads to the most important variations for the temperature and the concentration among the four combinations considered in this comparison. Quantitatively, by changing \((\text{Du}, \text{Sr})\) from \((0.9, 0.9)\) to \((0.9, -0.9)/(0.9, 0.9)/(0.9, -0.9),\) the temperature difference between the horizontal wall at a fixed cross section passes from 0.1769 to 0.0796/0.2416/2.08. Thus, the temperature difference obtained with the combination \((\text{Du}, \text{Sr}) = (-0.9, -0.9)\) is at least 8.6 times higher than that obtained for the other combinations. A result with the same order is also registered for the concentration difference.

4.4. Effect of Darcy Number. The Darcy number, \( \text{Da}, \) measures the relative importance of the permeability of the porous medium as it is proportional to the latter. It is very small for the well-packed porous media and relatively large for sparsely packed ones.

The effect of \( \text{Da} \) on \( \psi_0, \) \( \text{Nu}, \) and \( \text{Sh} \) is illustrated in Figures 9(a)–9(c) for \( R_T = 100, \) \( \text{Le} = 1.1, \) and different combinations \((\text{Sr}, \text{Du}, \text{N})\) corresponding to different regions. In these figures, it can be seen that the numerical results are in excellent agreement with the analytical ones, corresponding to the stable branches. In addition, regardless of the region considered, the parallel flow solution vanishes when \( \text{Da} \) exceeds a critical value, \( \text{Da}_{cr}, \) that depends on the combination \((\text{Sr}, \text{Du}, \text{N}). \) For the stable branches, the quantities \( \psi_0, \) \( \text{Nu}, \) and \( \text{Sh} \) decrease by increasing the permeability of the porous medium. Similar behaviors were reported in previous studies [5, 36]. Note that the variations versus \( \text{Da} \) of the flow intensity corresponding to the unstable branches are remarkably strong in specific ranges of the latter parameter. However, the effect of \( \text{Da} \) on \( \text{Nu} \) and \( \text{Sh} \) corresponding to the unstable branches is clearly much less important. For the combinations \((0.6, -0.5, -2)\) and \((0.6, 0.7, 2)\) corresponding to the regions 2 and 3 for which only the stable solution exists, the corresponding values of \( \text{Da}_{cr} \) are 0.441 and 0.428, respectively. For these regions (2 and 3) \( \text{Da}_{cr} \) can be computed analytically by solving the following equation:
For the combinations $(0.4, -0.3, 2), (0.9, 0.5, 2),$ and $(0.9, -2, -2),$ corresponding, respectively, to regions 4, 5, and 6, both stable and unstable branches exist for specific ranges of $Da$ and the corresponding critical Darcy numbers, $Dacr$, marking the transition from the parallel flow towards the rest state are $0.4072$, $0.1673$, and $0.2938$, respectively. Note that the bifurcation around $Dacr$ is supercritical for regions 2 and 3 and subcritical for regions 4, 5, and 6 (results not reported here). For regions 4, 5, and 6, $Dacr$ can be computed analytically by solving the following equation:

$$A(Dacr) = \frac{(1 - Du Sr)^2}{R_T (N (Du Sr - Sr (1 + Le) + Le) - Du (1 + Le - Le Sr) + 1)}.$$  \hspace{1cm} (60)$$

For the combinations $(0.4, -0.3, 2), (0.9, 0.5, 2),$ and $(0.9, -2, -2),$ corresponding, respectively, to regions 4, 5, and 6, both stable and unstable branches exist for specific ranges of $Da$ and the corresponding critical Darcy numbers, $Dacr$, marking the transition from the parallel flow towards the rest state are $0.4072$, $0.1673$, and $0.2938$, respectively. Note that the bifurcation around $Dacr$ is supercritical for regions 2 and 3 and subcritical for regions 4, 5, and 6 (results not reported here). For regions 4, 5, and 6, $Dacr$ can be computed analytically by solving the following equation:

$$A(Dacr) = \frac{f + 2Le^2 \sqrt{(1 - Du Sr)^2 (Le^2 + Le N) - F_2^2 (Le^2 + Le N) (1 + Le^2 + 2Sr Du Le) + Le^2 F_2^2}}{R_T (Le^2 + Le N)^2}.$$  \hspace{1cm} (61)$$
with

\[ f = \left( \text{Le}^4 + \text{Le}^3 \right) + N \left( \text{Le} - \text{Le}^3 \right) + 2N \text{Sr} \text{Du} \left( \text{Le}^3 - \text{Le}^2 \right) + \left( 2N \text{Sr} + 2\text{Du} \right) \left( \text{Le}^2 + \text{Le}^3 \right). \]

(62)

For the combination (0.9, −2, −2), a singularity is registered in the curve of Nu at the critical Darcy number \( \text{Da}_{\text{Nu} = \infty} = 0.145 \), as can be seen in Figure 9(b). For the latter combination, the term \( \Delta T + \text{Du} \Delta S \) that represents the inverse of the Nusselt number is zero for \( \text{Da} = \text{Da}_{\text{Nu} = \infty} \), leading to an infinite value for \( \text{Nu} \). This means that, for \( \text{Da} = \text{Da}_{\text{Nu} = \infty} \), the Dufour heat flux induced by the concentration gradient compensates the ordinary heat flux induced by the temperature gradient. Such a compensation is not observed for the mass transfer as \( \text{Sr} \neq \text{Du} \) and \( \text{Le} \neq 1 \).

For the particular combination (0.9, −2, −2), \( \text{Sh} \) induced by the unstable branch (in its existence range) is higher than that induced by the stable branch (see \( \text{Sh} \) behavior in the vicinity of \( \text{Da}_{\text{c},1} \)). Note that \( \psi_0 \) and \( \text{Nu} \) induced by the stable
branches are larger than those corresponding to unstable branches for all the combinations considered in Figures 9(a)–9(b).

5. Conclusion

Combined effects of Soret and Dufour parameters on thermosolutal convection in shallow horizontal Brinkman porous enclosures with stress-free upper surfaces is evaluated analytically and numerically. The analytical results obtained using the parallel flow assumption are validated numerically by solving the full governing equations. The thresholds marking the onset of stationary and finite amplitude convection are determined analytically according to the control parameters. The analytical study shows that the \((N, Du)\) plane can be divided into up to six regions corresponding to different parallel flow regimes. The number, the location, and the extent of these regions are strongly controlled by Dufour and Soret numbers. Depending on the value of the Soret number, the Dufour effect may affect considerably the fluid flow and heat and mass transfer characteristics. By analyzing the effect of Du on \(\psi_0\), \(Nu\), and \(Sh\) for various \(Sr\), we observe the existence of critical nodes (small ranges of \(Du\)) for which these quantities are quasi-independent of \(Sr\). These nodes did not correspond to the same range of \(Du\) for \(\psi_0\), \(Nu\), and \(Sh\).

### Nomenclature

\begin{align*}
A & : \text{Aspect ratio of the enclosure, } L'/H' \\
D & : \text{Diffusion coefficient (m}^2\text{s}^{-1}) \\
D_a & : \text{Darcy number, } \mu_{eff} K/\mu H^2 \\
Du & : \text{Dufour number, } K_{ST} \Delta S'/\alpha \Delta T' \\
g & : \text{Gravitational acceleration (m}^2\text{s}^{-2}) \\
H' & : \text{Height of the enclosure (m)} \\
J & : \text{Constant mass flux per unit area (kg m}^{-2}\text{s}^{-1}) \\
K & : \text{Permeability of the porous medium (m}^2) \\
K_{ST} & : \text{Soret coefficient (kg K}^{-1}\text{m}^{-1}\text{s}^{-1}) \\
K_{TS} & : \text{Dufour coefficient (km}^2\text{s}^{-1}) \\
L' & : \text{Length of the enclosure (m)} \\
Le & : \text{Lewis number, } \alpha/D \\
N & : \text{Buoyancy ratio, } \beta_3 \Delta S'/\beta_T \Delta T' \\
Nu & : \text{Thermal Nusselt number} \\
q' & : \text{Constant heat flux per unit area (W m}^{-2}) \\
R_T & : \text{Thermal Darcy–Rayleigh number, } \rho \beta_T K \Delta T' H'/\alpha v \\
S & : \text{Dimensionless solute concentration, } (S' - S'_0)/\Delta S' \\
S' & : \text{Concentration (kg m}^{-3}) \\
Sh & : \text{Sherwood number} \\
Sr & : \text{Soret number, } K_{ST} \Delta T'/D \Delta S' \\
t & : \text{Dimensionless time, } t' \alpha^{-1} \Delta H^2 \\
T & : \text{Dimensionless temperature, } (T' - T'_0)/\Delta T' \\
T' & : \text{Temperature (K)} \\
u & : \text{Dimensionless horizontal velocity, } u'H'/\alpha \\
w & : \text{Dimensionless vertical velocity, } v'H'/\alpha \\
x & : \text{Dimensionless distance along the } x \text{ axis, } x'/H' \\
y & : \text{Dimensionless distance along the } y \text{ axis, } y'/H' \\
\Delta S' & : \text{Concentration solute difference, } \Delta S'/D \\
\Delta T' & : \text{Temperature difference, } q'H'/k. \\
\end{align*}

### Greek letters

\begin{align*}
\beta_3 & : \text{Solute expansion coefficient} \\
\beta_T & : \text{Thermal expansion coefficient (K}^{-1}) \\
\alpha & : \text{Thermal diffusivity (m}^2\text{s}^{-1}) \\
v & : \text{Kinematic viscosity of the fluid (m}^2\text{s}^{-1}) \\
\psi & : \text{Dimensionless stream function, } \psi'/\alpha \\
\epsilon & : \text{Normalized porosity of the porous medium, } \epsilon'/\sigma \\
\epsilon' & : \text{Porosity of the porous medium} \\
\sigma & : \text{Heat capacity ratio, } (\rho C_p)/(\rho C_f). \\
\end{align*}

### Subscripts

\begin{align*}
c & : \text{Critical value} \\
0 & : \text{Refers to the value taken at the center of the cavity.} \\
\end{align*}

### Superscripts

\begin{align*}
t' & : \text{Dimensional variables} \\
\text{Sub} & : \text{Subcritical} \\
\text{Sup} & : \text{Supercritical.} \\
\end{align*}

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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