Noncommutative fields in curved space

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We consider a noncommutative theory developed in a curved background. We show that the Moyal product has to be conveniently modified and, consequently, some of its old properties are lost compared with the flat case. We also address the question of diffeomorphism symmetry.

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I. INTRODUCTION

Usual noncommutative theories [1] are based on a constant antisymmetric quantity of rank two. Consequently, they do not exhibit the Lorentz symmetry for spacetime dimensions higher than two. In this way, it does not make much sense to look for noncommutative theories invariant by general coordinate transformations or even noncommutative quantum fields in curved background.

The question of Lorentz invariance in noncommutative theories has been recently considered [2], where the antissymmetric tensor is taken as an independent antisymmetric tensor operator. Our purpose in the present paper is to consider the main aspects of noncommutative fields in curved space. Our approach is such that the noncommutativity does not affect the metric but it does on the geometry. For simplicity, we deal with the particular case of $D = 2$ where there is a constant flat limit for the noncommutative antisymmetric tensor. The only difference for dimensions higher than two is that this limit should be spacetime dependent. We deal with real scalar fields, but what will be presented here can be naturally extended for any kind of fields.

Our paper is organized as follows. In Sec. II we present the main aspects of the problem where we see that it is necessity to adopt a more general definition for the Moyal product. In Sec. III we discuss in details the property of commutativity for quadratic terms in the action which is not kept in curved space, and in Sec. IV we consider the question of diffeomorphism symmetry. We left Sec. V for some concluding remarks.

II. PRESENTING THE PROBLEM

Let us first consider the usual action for real scalar fields in curved spacetime at $D = 2$,

$$ S = -\frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. $$

Of course, this theory is invariant under general coordinate transformation. The origin of this symmetry is related to the (first-class) constraints

$$ \pi^{\mu\nu} = \frac{\partial L}{\partial g_{\mu\nu}} = 0, $$

where $L$ is the Lagrangian density corresponding to the action (2.1). The components of $\pi^{\mu\nu}$ given by Eq. (2.2) are three (primary) constraints. Constructing the Dirac Hamiltonian and imposing that these constraints do not evolve in time, one obtain two secondary constraints. Repeating this procedure, no more constraints are obtained and all of them are first-class [3]. According to Castellani [3], primary first-class constraints are related to symmetries of the theory. In the present case, two of them concern to the general coordinate transformation (embodied in the diffeomorphism algebra generated by the two secondary constraints) and the last primary constraints is related to the conformal (Weyl) symmetry.

If one direct follows the general rule of transforming usual theories in noncommutative ones by replacing product of fields by the Moyal product [4], we would have

$$ S = -\frac{1}{2} \int d^2x \sqrt{-g} \ast g^{\mu\nu} \partial_\mu \phi \ast \partial_\nu \phi, $$

where $\ast$ means the Moyal product, which for any two fields $\phi_1$ and $\phi_2$, it is defined as

$$ \phi_1 \ast \phi_2 = \exp \left( \frac{i k}{2} \epsilon^{\mu\nu} \partial_\mu \partial_\nu \right) \phi_1(x) \phi_2(y) \bigg|_{x=y}, $$

and $k$ is a constant with mass dimension minus two and $\epsilon^{\mu\nu}$ is the usual Levi-Civita tensor (in this paper, we adopt the convention $\epsilon^{01} = 1$).

The action (2.3) does not exhibit the diffeomorphism symmetry because the canonical momentum conjugate to $g_{\mu\nu}$ is not a constraint anymore. Further, notice that the Moyal product must also acts on the elements of $g = \det g_{\mu\nu}$, making the relation (2.3) meaningless for any consistent use.
The natural alternative is to consider the derivative operators of the Moyal product in curved space as covariant ones. In this way, they do not affect the spacetime geometry and the inconsistencies mentioned above disappear. So, for the same two fields $\phi_1$ and $\phi_2$, we make

$$\phi_1 \triangleright \phi_2 = \exp\left(\frac{ik}{2\sqrt{-g}} \epsilon^\mu\nu D^\mu D^\nu\right) \phi_1(x) \phi_2(y) \bigg|_{x=y}, \quad (2.5)$$

where the symbol $\triangleright$ is to distinguish from the Moyal product in the flat case. It is important to emphasize that there is no problem related to the commutativity of the covariant derivatives because they act in different spacetime coordinates. Notice that in curved space, even in two dimensions, the characterist antisymmetric tensor $\theta_{\mu\nu}$ is spacetime dependent, $\theta_{\mu\nu}(x) = \sqrt{-g} \epsilon_{\mu\nu}$ (for the contravariant notation, $\theta^{\mu\nu}(x) = \epsilon^{\mu\nu}/\sqrt{-g}$).

Considering the Moyal product given by $[2.3]$, it does not affect products involving the metric tensor. Consequently, instead of the action $[2.5]$ we simply have

$$S = -\frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (2.6)$$

However, the price paid is that some usual properties of the Moyal product are lost. For example, it affects quadratic terms of the action (we are going to analyse this point with details in the next section). It is easy to see that the same occurs for cyclic properties.

III. A FUNDAMENTAL CHARACTERISTIC OF THE NEW MOYAL PRODUCT

The expansion of the exponential operator in $[2.5]$ leads to

$$\int d^2x \epsilon^\mu\nu \epsilon^{\rho\lambda} \epsilon^{\eta\xi} \sqrt{-g} D_\eta D_\rho \partial_\mu \phi_1 D_\xi D_\lambda \partial_\nu \phi_2 = \frac{1}{2} \int d^2x \epsilon^\mu\nu \left( \partial_\mu R D_\nu \partial_\lambda \phi_1 \partial^\lambda \phi_2 + \frac{1}{2} R^2 \partial_\mu \phi_1 \partial_\nu \phi_2 \right), \quad (3.3)$$

$$\int d^2x \epsilon^\mu\nu \epsilon^{\rho\lambda} \epsilon^{\eta\xi} \sqrt{-g} D_\eta D_\rho \partial_\mu \phi_1 D_\xi D_\lambda \partial_\nu \phi_2 = \frac{1}{2} \int d^2x \sqrt{-g} R \left( 3 D_\mu \square \phi_1 D^\mu \phi_2 - 3 D_\alpha D_\beta \partial_\mu \phi_1 D^\alpha D^\beta \partial_\mu \phi_2 \right.$$

$$\left. - R D_\mu \square \partial_\mu \phi_2 - R \partial_\mu \phi_1 D^\mu \phi_2 + R^2 \partial_\mu \phi_1 \partial_\mu \phi_2 \right), \quad (3.4)$$

where $\square$ is the Laplace-Beltrami operator $\square = g^{\mu\nu} D_\mu D_\nu$.

IV. Diffeomorphism Symmetry

An interesting question concerns to symmetries. It is well-known that noncommutative theories exhibit the unitarity problem [4]. This is so due to the presence of higher derivatives. But the problem here is more subtle
because the number of higher derivatives increases indefinitely. This means that the unitarity problem is accompanied by a more fundamental question related to the dynamics of these theories. This is so because dynamics is given by the last order of its time derivative (in lower orders take place momenta and constraints). If there is an infinite number of time derivatives, there is an ambiguity in the limit of these two regions.

A way of understanding this problem in a deeper way consists in analyzing separately the terms of the expansion (isotopically, there is no problem related to the ambiguity of dynamics and constraints) 3. Since there is dynamics for the terms of (3.1), that is simpler than the corresponding ones of (2.1), let us consider the second term of the expansion of expression (3.1). Denoting it by \( S^{(2)} \), and taking \( \phi_1 = \phi_2 = \phi \), we have

\[
S^{(2)} = \frac{1}{2} \int d^2 x \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\rho \partial_\nu \phi D_\lambda \partial_\nu \phi .
\]  

(4.1)

Of course, expression (4.1) can be written in terms of the scalar curvature as given in (3.2). However, for our purposes here, it is better to keep it in the way it is. Let us calculate the momenta. For convenience, we take

\[
\delta S^{(2)} = \int_{t_0}^t dt \int d^2 x \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\rho \partial_\nu \phi \delta (D_\lambda \partial_\nu \phi) ,
\]  

(4.2)

where \( \delta \) means variation of fields under derivatives (because just these terms will contribute to the momenta). In this way, we have for delta term which appears in the equation above

\[
\delta (D_\lambda \partial_\nu \phi) = D_\lambda \partial_\nu \delta \phi - \delta \Gamma^\gamma_{\lambda\nu} \partial_\gamma \phi
\]

\[
= D_\lambda \partial_\nu \delta \phi - g^{\alpha\beta} (\partial_\nu \delta g_{\lambda\beta}) \partial_\alpha \phi - \frac{1}{2} \partial_\beta \delta g_{\lambda\nu} \partial_\alpha \phi ,
\]  

(4.3)

where in the last step it was used the symmetry between \( \lambda \) and \( \nu \) indices, given in (4.3). Developing in a separate way the terms which will appear after replacing (4.3) into (4.2) we have

\[
\int_{t_0}^t dt \int d^2 x \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\rho \partial_\nu \phi D_\lambda \partial_\nu \delta \phi
\]

\[
= \int_{t_0}^t dt \int d^2 x \left\{ \partial_\lambda \left( \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\rho \partial_\mu \phi \delta \phi \right) - \partial_\nu \left( \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\lambda D_\rho \partial_\nu \phi \delta \phi \right) \right\} ,
\]

\[
= \int d^2 x \left\{ \partial_1 \left( \frac{1}{\sqrt{-g}} D_1 \partial_0 \phi \right) + \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\lambda D_\rho \partial_1 \phi \right\} \delta \phi
\]

\[
+ \frac{1}{\sqrt{-g}} D_1 \partial_1 \phi \delta \phi ,
\]  

(4.4)

For the first term involving \( \delta g_{\alpha\beta} \), we obtain

\[
\int_{t_0}^t dt \int d^2 x \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\rho \partial_\mu \phi g^{\alpha\beta} \partial_\nu \delta g_{\lambda\beta} \partial_\alpha \phi
\]

\[
= \int_{t_0}^t dt \int d^2 x \partial_\lambda \left( \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\rho \partial_\mu \phi g^{\alpha\beta} \partial_\nu \delta g_{\lambda\beta} \partial_\alpha \phi \right) ,
\]

\[
= - \int d^2 x \frac{\epsilon^\rho\mu}{\sqrt{-g}} D_\rho \partial_1 \phi g^{\alpha\nu} \partial_\alpha \delta g_{\mu\nu} .
\]  

(4.5)

Similarly, for the last term the result is

\[
\int_{t_0}^t dt \int d^2 x \epsilon^{\mu\nu} \frac{\epsilon^\rho\lambda}{\sqrt{-g}} D_\rho \partial_\mu \phi g^{\alpha\beta} \partial_\nu \partial_\beta \delta g_{\lambda\mu} = \int d^2 x \frac{\epsilon^\lambda^\rho\mu}{\sqrt{-g}} D_\rho \partial_\lambda \phi g^{\alpha\mu} \partial_\alpha \delta g_{\mu\nu} .
\]  

(4.6)

Replacing (4.4), (4.6) into (4.2) we identify the momenta conjugate to \( \phi, \dot{\phi} \) and \( g_{\mu\nu} \) which are the coefficients of the variations of the corresponding fields. Denoting them respectively by \( p, p^{(1)} \) and \( \pi^{\mu\nu} \), we have

\[
p = \partial_1 \left( \frac{1}{\sqrt{-g}} D_1 \partial_0 \phi \right) - \frac{\epsilon^{\mu\nu}}{\sqrt{-g}} D_\mu D_\nu \partial_1 \phi ,
\]

(4.7)

\[
p^{(1)} = \frac{1}{\sqrt{-g}} D_1 \partial_1 \phi ,
\]

(4.8)

\[
\pi^{\mu\nu} = \frac{1}{2} \partial_\alpha \left( \epsilon^{\mu\nu} g^{\rho\sigma} D_\rho \partial_\alpha \phi + \epsilon^{\rho\lambda} \epsilon^{\mu\rho} g^{\alpha\beta} D_\rho \partial_\beta \phi \right) .
\]

(4.9)

Notice that \( \pi^{\mu\nu} \) is symmetric in \( \mu \) and \( \nu \) indices, as it should be. However, the identification of the primary first-class constraints (responsible for the symmetries of the theory) is not so apparent as it was in the case of the action (2.1), where the constraints are simple given by (2.2). Let us explicitly write down the components of \( \pi^{\mu\nu} \).

\[
\pi^{01} = - \frac{1}{2} \sqrt{-g} g^{1\alpha} \partial_\alpha \phi D_1 \partial_1 \phi ,
\]

(4.10)

\[
\pi^{00} = - \frac{1}{2} \sqrt{-g} g^{0\alpha} \partial_\alpha \phi D_1 \partial_1 \phi ,
\]

(4.11)

\[
\pi^{11} = \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \partial_\alpha \phi (2 g^{1\alpha} D_0 \partial_1 \phi + g^{0\alpha} D_0 \partial_0 \phi) .
\]

(4.12)

If one combine (1.8), (4.10) and (4.11), we observe that \( \pi^{01} \) and \( \pi^{00} \) are actually constraints (\( \pi^{11} \) is not). Denoting these constraints by \( \psi \) and \( \chi \) we have

\[
\psi = \pi^{01} + \frac{1}{2} \sqrt{-g} g^{1\alpha} \partial_\alpha \phi p^{(1)} \approx 0 ,
\]

(4.13)

\[
\chi = \pi^{00} + \frac{1}{2} \sqrt{-g} g^{0\alpha} \partial_\alpha \phi p^{(1)} \approx 0 ,
\]

(4.14)
where the symbol $\approx$ means weakly zero \[3\]. Using the fundamental Poisson brackets

$$
\{ \dot{\phi}(x), p^{(1)}(y) \} = \delta(x - y),
$$

$$
\{ g_{\mu\nu}(x), \pi^{\rho\lambda}(y) \} = \frac{1}{2} \left( \delta^\rho_\mu \delta^\lambda_\nu + \delta^\lambda_\mu \delta^\rho_\nu \right) \delta(x - y),
$$

and also

$$
\{ g^{\mu\nu}(x), \pi^{\rho\lambda} \} = \frac{1}{2} \left( g^{\mu\rho} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\rho} \right) \delta(x - y),
$$

obtained from (4.16), we can actually show that $\psi$ and $\chi$ are primary first-class constraints, which are related to the diffeomorphism symmetry. In the case of action \[2\], there was one extra primary first-class constraints responsible for the conformal symmetry. In the present case, this symmetry is missing.

The next step of the present procedure would be the obtainment of secondary constraints (what has to be done in a Hamiltonian formalism keeping velocities \[3\]) and the construction of the diffeomorphism algebra. This is just a question of a hard algebraic work, but we believe that nothing new will appear and the diffeomorphism algebra should be verified, since the theory is invariant by general coordinate transformation.

V. CONCLUSION

In this paper we have studied noncommutative fields in curved space. We consider a formulation where the noncommutativity does not affect the metric but it does on the geometry. This procedure might be taken as an intermediary step to obtain a noncommutative gravitational theory without inconsistencies on the flat space.

We have deal with a spacetime dimensions $D = 2$, where the flat limit of the noncommutative parameter is a constant tensor. Of course, this could have been presented for higher dimensions. We envisage two independent formulations for such theories. One of them is to consider the parameter $\theta_{\mu\nu}$ depending on $x$. In this way, it would be some antisymmetric tensor field. Another possibility, followed by Carlson et al. \[3\], is to consider $\theta_{\mu\nu}$ as an independent spacetime coordinate \[3\].

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APPENDIX A: USEFUL RELATIONS IN CURVED SPACE

For some vector and antisymmetric tensor field ($V^\mu$ and $F^{\mu\nu}$), we have

$$
D_\mu (\sqrt{-g} V^\mu) = \partial_\mu (\sqrt{-g} V^\mu),
$$

$$
D_\nu (\sqrt{-g} F^{\mu\nu}) = \partial_\nu (\sqrt{-g} F^{\mu\nu}).
$$

In the particular case of $D = 2$, the following relations are true

$$
\frac{\epsilon^{\mu\nu}}{\sqrt{-g}} \frac{\epsilon^{\rho\lambda}}{\sqrt{-g}} = g^{\mu\lambda} g^{\nu\rho} - g^{\mu\rho} g^{\nu\lambda},
$$

$$
R_{\mu\nu\rho\lambda} = (g_{\mu\rho} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\rho}) \frac{R}{2},
$$

$$
R_{\mu\nu} = g_{\mu\nu} \frac{R}{2}.
$$

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