POLYNOMIAL ENERGY DECAY RATE OF A 2D PIEZOELECTRIC BEAM WITH MAGNETIC EFFECT ON A RECTANGULAR DOMAIN WITHOUT GEOMETRIC CONDITIONS

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Abstract. In this paper, we investigate the stability of coupled equations modelling a 2D piezoelectric beam with magnetic effect with only one local viscous damping on a rectangular domain without geometric conditions. We prove that the energy of the system decays polynomially with the rate $t^{-1}$.

Keywords. coupled wave equations; viscous damping; $C_0$-semigroup; polynomial stability; rectangular domains.

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1. INTRODUCTION

It is known, since the 19th century, that materials such as quartz, Rochelle salt and barium titanate under pressure produce electric charge/voltage: this phenomenon is called the direct piezoelectric effect and was discovered by brothers Pierre and Jacques Curie in 1880. These same materials, when subjected to an electric field, produce proportional geometric tension. Such a phenomenon is known as the converse piezoelectric effect and was discovered by Gabriel Lippmann in 1881.

Morris and Ozer proposed a piezoelectric beam model with a magnetic effect, based on the Euler-Bernoulli and Rayleigh beam theory for small displacement (the same equations for the model are obtained if Mindlin-Timoshenko small displacement assumptions are used). They considered an elastic beam covered by a piezoelectric material on its upper and lower surfaces, isolated by the edges and connected to an external electrical circuit to feed charge to the electrodes. As the voltage is prescribed at the electrodes, the following Lagrangian is considered

(1.1) $\mathcal{L} = \int_0^T [K - (P + E) + B + W] dt,$

where $K, P + E, B$ and $W$ represent the (mechanical) kinetic energy, total stored energy, magnetic energy (electrical kinetic) of the beam and the work done by external forces, respectively, for a beam with length $L$ and thickness $h$ and considering $v = v(x, t), w = w(x, t)$ and $p = p(x, t)$ as functions that represent the
The authors showed, by using energy method, that the system’s energy decays exponentially. This means that
\(\text{Figure } x\) (1.5)
They assumed that the beam is fixed at \(x = 0\) and free at \(x = L\), and thus they got (from modelling) the following boundary conditions
\[
\begin{align*}
v(0, t) &= \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, \\
p(0, t) &= \beta p_x(L, t) - \gamma \beta v_x(L, t) = -\frac{V(t)}{h}.
\end{align*}
\]
Then, the authors considered \(V(t) = kp_t(L, t)\) (electrical feedback controller) in (1.5) and established strong stabilization for almost all system parameters and exponential stability for system parameters in a null measure set. In [13] Ramos et al. inserted a dissipative term \(\delta v_t\) in the first equation of (1.3), where \(\alpha > 0\) is a constant and considered the following boundary conditions
\[
\begin{align*}
v(0, t) &= \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, \\
p(0, t) &= \beta p_x(L, t) - \gamma \beta v_x(L, t) = 0.
\end{align*}
\]
The authors showed, by using energy method, that the system’s energy decays exponentially. This means that the friction term and the magnetic effect work together in order to uniformly stabilize the system. In [15], the authors considered a one-dimensional piezoelectric beam with magnetic effect damped with a weakly nonlinear feedback in the presence of a nonlinear delay term. They established an energy decay rate under appropriate assumptions on the weight of the delay. In [5], the authors studied the stability of piezoelectric beams with magnetic effects of fractional derivative type and with/without thermal effects of Fourier’s law. They obtained an exponential stability by taking two boundary fractional dampings and additional thermal effects. In [18], the authors studied the longtime behavior of a kind of fully magnetic affected nonlinear multi-dimensional piezoelectric beams with viscoelastic infinite memory. An exponential decay of the solution to the nonlinear coupled PDE’s system is established by the energy estimation method under certain conditions. In [2], the author investigates the stabilization of a system of piezoelectric beams under (Coleman or Pipkin)-Gurtin thermal law with magnetic effect. First, he study the Piezoelectric-Coleman-Gurtin system and he obtain an exponential stability result. Next, he consider the Piezoelectric-Gurtin-Pipkin system and he establish a polynomial energy decay rate of type \(t^{-1}\). In [1], the authors considered a one-dimensional dissipative system of piezoelectric beams with magnetic effect and localized viscous damping. They proved that the system is exponentially stable using a damping mechanism acting only on one component and on a small part of the beam.

What happens if we go to dimension 2? For this aim, let \(\Omega = \square = (0, 1)^2\) be the unit square and \(\partial\Omega = \Gamma_0 \cup \Gamma_1\) (See Figure 1), such that
\[
\begin{align*}
\Gamma_{0,1} &= \{(0) \times (0, 1)\} \cup ((0, 1) \times \{0\}) & \text{and} & \Gamma_{1,1} &= \{(1) \times (0, 1)\} \cup ((0, 1) \times \{1\}).
\end{align*}
\]
In the present work, we consider the fully dynamic magnetic effects in a model for a piezoelectric beam with only one viscous damping on a rectangular domain, whose dynamic behaviour is described by an elasticity
equation and a charge equation coupled via the piezoelectric constants, which is given as follows

\begin{align*}
\frac{\rho v_{tt}(X,t)}{\rho} - \alpha \Delta v(X,t) + \gamma \beta \Delta p(X,t) + d(\cdot)v_t(X,t) &= 0, \quad X = (x,y) \in \Omega, \ t > 0, \\
\frac{\mu p_{tt}(X,t)}{\mu} - \beta \Delta p(X,t) + \gamma \beta \Delta v(X,t) &= 0, \quad X = (x,y) \in \Omega, \ t > 0,
\end{align*}

where \(v(X,t)\) and \(p(X,t)\) respectively denote the transverse displacement of the beam and the total load of the electric displacement along the transverse direction at each point \(x \in \Omega\). The constant coefficients \(\rho, \alpha, \gamma, \mu, \beta > 0\) are the mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, impermeability coefficient of the beam, respectively and satisfy \(\alpha > \gamma^2 \beta\). System (1.7) is subjected to the following boundary conditions

\begin{align*}
\begin{cases}
v(X,t) = p(X,t) = 0, \quad X = (x,y) \in \Gamma_0, \ t > 0, \\
\alpha \frac{\partial v}{\partial \nu}(X,t) - \gamma \beta \frac{\partial p}{\partial \nu}(X,t) = \beta \frac{\partial p}{\partial \nu}(X,t) - \gamma \beta \frac{\partial v}{\partial \nu}(X,t) = 0, \quad X = (x,y) \in \Gamma_1, \ t > 0,
\end{cases}
\end{align*}

Using the fact that \(\alpha = \alpha_1 + \gamma^2 \beta\) (see (1.4)), then the above boundary conditions can be replaced by

\begin{align*}
\begin{cases}
v(X,t) = p(X,t) = 0, \quad X = (x,y) \in \Gamma_0, \ t > 0, \\
\frac{\partial v}{\partial \nu}(X,t) = \frac{\partial p}{\partial \nu}(X,t) = 0, \quad X = (x,y) \in \Gamma_1, \ t > 0.
\end{cases}
\end{align*}

System (1.7) is considered with the following initial data

\begin{align*}
v(X,0) = v_0(X), \quad v_t(X,0) = v_1(X), \quad p(X,0) = p_0(X), \quad p_t(X,0) = p_1(X), \quad X = (x,y) \in \Omega.
\end{align*}

Let \(d \in L^\infty(0,1)\), depending only on \(x\) such that

\begin{align*}
d(x) \geq d_0 > 0 \text{ in } (a,b) \quad \text{and} \quad d(x) = 0 \text{ in } (0,1) \setminus (a,b).
\end{align*}

We set \(\omega_d = (\text{supp } d)^c \times (0,1)\) (See Figure 1).

![Figure 1. Model Describing Ω.](image)

There exist a few results concerning wave equations or coupled wave equations on a rectangular or cylindrical domain with different kinds of damping \([10, 16, 6, 17, 9, 4, 3]\). But to the best of our knowledge, it seems that no result in the literature exists concerning the case of a 2D or multidimensional Piezoelectric beam with local damping.

This paper is organized as follows: the second section is devoted to the case of a rectangular domain. The
well-posedness of the system is proved by using semigroup approach. Next, by combining an orthonormal basis decomposition with frequency-multiplier techniques using appropriate cut-off functions, we prove a polynomial energy decay rate of type $t^{-1}$.

2. Stability of a 2D piezoelectric beam on a rectangular domain

In this section, we study the stability result of a 2D piezoelectric beam with magnetic effect on a rectangular domain.

2.1. Well-Posedness. The energy of System (1.7) – (1.9), is given by

$$E_1(t) = \frac{1}{2} \int_{\Omega_1} (\rho |v|^2 + \alpha |v_1|^2 + \nabla v \cdot \nabla \overline{v}) dX + \frac{1}{2} \int_{\Omega_1} (\mu |p|^2 + \beta |\gamma \nabla p|) dX.$$

**Lemma 2.1.** Let $U = (v, v_1, p, p_1)$ be a regular solution of system (1.7)-(1.9). Then, the energy $E_1(t)$ satisfies the following estimation

$$\frac{d}{dt} E_1(t) = -\int_{\Omega_1} d(x) |v_1(X, t)|^2 dX.$$

**Proof.** Multiplying the first and the second equation of (1.7) by $\overline{v_t}$ et $\overline{y_t}$ respectively, integrating by parts over $\Omega_1$, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_1} |v_1|^2 dX + \int_{\Omega_1} \nabla v \cdot \nabla \overline{v} dX = \gamma \beta \Re \left( \int_{\Omega_1} \nabla p \cdot \nabla \overline{v} dX \right) + \int_{\Omega_1} d(x) |v_1(X, t)|^2 dX = 0$$

and

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_1} |y_t|^2 dX + \int_{\Omega_1} \nabla y \cdot \nabla \overline{y} dX = \gamma \beta \Re \left( \int_{\Omega_1} \nabla u \cdot \nabla \overline{y} dX \right) = 0.$$

Adding (2.2) and (2.3), and using the fact that $\alpha = \alpha_1 + \gamma \beta$, we get the desired equation (2.1). The proof has been completed.

From (2.1), it follows that System (1.7)-(1.9) is dissipative. Now, let us define the energy space $\mathcal{H}$ by

$$\mathcal{H} = (W_1 \times L^2(\Omega_1))^2,$$

where $W_1 = \{ f \in H^1(\Omega_1); f = 0 \text{ on } \Gamma_{0,1} \}$. $\mathcal{H}$ is a Hilbert space, equipped with the inner product defined by

$$\langle U, U_1 \rangle_{\mathcal{H}^2} = \int_{\Omega_1} \alpha_1 \nabla v \cdot \nabla \overline{y_t} + \rho \nabla v \cdot \nabla \overline{y_t} + \beta (\gamma \nabla v - \nabla p) \cdot (\gamma \nabla v - \nabla p) + \mu q \overline{q}, dX,$$

for all $U = (v, z, p, q)^T$ and $U_1 = (v_1, z_1, p_1, q_1)^T \in \mathcal{H}$. The expression $\| \cdot \|_{\mathcal{H}^2}$ will denote the corresponding norm. We define the unbounded linear operator $A_{\square} : D(A_{\square}) \subset \mathcal{H} \rightarrow \mathcal{H}$ by

$$D(A_{\square}) := \left\{ U := (v, z, p, q) \in \mathcal{H}; z, q \in W_1, \Delta v, \Delta z \in L^2(\Omega_1) \text{ and } \frac{\partial v}{\partial v} = \frac{\partial p}{\partial v} = 0 \text{ on } \Gamma_{1,1} \right\}.$$

and

$$A_{\square}(v, z, p, q) = \left(z, \frac{1}{\rho} (\alpha \Delta v - \gamma \beta \Delta p - d p), q, \frac{1}{\mu} (\beta \Delta p - \gamma \beta \Delta v) \right).$$

If $U = (v, v_1, p, p_1)^T$ is the state of System (1.7) – (1.9), then this system is transformed into the first order evolution equation on the Hilbert space $\mathcal{H}$ given by

$$U_t = A_{\square} U, \quad U(0) = U_0,$$

where $U_0 = (v_0, v_1, p_0, p_1)^T$. It is easy to see that for all $U = (v, z, p, q) \in D(A_{\square})$, we have

$$\mathcal{R} \left( \langle A_{\square} U, U \rangle_{\mathcal{H}} \right) = -\int_{\Omega_1} d(x)|z(X)|^2 dX \leq 0,$$

which implies that $A_{\square}$ is dissipative. Now, let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, using the Lax-Milgram Theorem, one proves the existence of $U \in D(A_{\square})$, solution of the equation

$$-A_{\square} U = F.
Then, the unbounded linear operator $A\square$ is $m-$dissipative in the energy space $\mathcal{H}\square$ and consequently $0 \in \rho(A\square)$. Thus, $A\square$ generates a $C_0-$semigroup of contractions $(e^{tA\square})_{t \geq 0}$ following the Lumer-Phillips theorem. The solution of the Cauchy problem (2.7) admits the following representation

$$U(t) = e^{tA\square}U_0, \quad t \geq 0,$$

which leads to the well-posedness of (2.7). Hence, we have the following result.

**Theorem 2.2.** Let $U_0 \in \mathcal{H}\square$, then System (2.7) admits a unique weak solution $U$ satisfying

$$U \in C^0(\mathbb{R}^+, \mathcal{H}\square).$$

Moreover, if $U_0 \in D(A\square)$, then Problem (2.7) admits a unique strong solution $U$ satisfying

$$U \in C^1(\mathbb{R}^+, \mathcal{H}\square) \cap C^0(\mathbb{R}^+, D(A\square)).$$

### 2.2. Polynomial Stability

This subsection is devoted to showing the polynomial stability of System (1.7)-(1.9). Our main result in this subsection is the following theorem.

**Theorem 2.3.** There exists a constant $C > 0$ independent of $U_0$, such that the energy of System (1.7)-(1.9) satisfies the following estimation

$$E(t) \leq \frac{C}{t} \|U_0\|^2_{\mathcal{H}^2(A\square)}, \quad \forall t > 0, \quad \forall U_0 \in D(A\square).$$

To prove this theorem, let us first introduce the following sufficient and necessary condition on the polynomial stability of a semigroup proposed by Borichev-Tomilov in [8] (see also [7], [10], and the recent paper [14]).

**Theorem 2.4.** Assume that $A$ is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t \geq 0}$ on a Hilbert space $H$. If

$$i\mathbb{R} \subset \rho(A),$$

then for a fixed $\ell > 0$ the following conditions are equivalent

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \to \infty} \frac{1}{|\lambda|^\ell} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty.$$ (2.11)

$$\|e^{tA}X_0\|^2_H \leq \frac{C}{t^\ell} \|X_0\|^2_{D(A)}, \quad X_0 \in D(A), \text{ for some } C > 0. \quad \text{(2.12)}$$

According to Theorem 2.4, to prove Theorem 2.3, we need to prove that (2.10) and (2.11) hold, where $\ell = 2$. For the technique, we use the orthonormal basis decomposition. To this aim, let $e_j(y) = \sqrt{2}\sin(\xi_j y)$, where $\xi_j = \frac{2(j+1)\pi}{\ell}, \quad j \in \mathbb{N}^*$ and $y \in (0, 1)$. We may expand $v$ into a series of the form

$$v(X) = \sum_{j=1}^{\infty} v_j(x)e_j(y), \quad (x, y) \in \Omega_1. \quad \text{(2.13)}$$

Similarly, $z$, $p$ and $q$ can be expanded into a series of the same form as that in (2.13) with, respectively, the coefficients $v_j(x)$, $z_j(x)$, $p_j(x)$ and $q_j(x)$. The energy Hilbert space (2.4) is given by

$$\mathcal{H}\square = (\hat{\mathcal{W}}_1 \times L^2(0, 1))^2 \quad \text{(2.14)}$$

where

$$\hat{\mathcal{W}}_1 = \{ f \in H(0, 1); \quad f(0) = 0 \}$$

equipped with the following norm

$$\|U\|^2_{\mathcal{H}\square} = \sum_{j=1}^{\infty} (\alpha_j(\|v_j\|^2 + \xi_j^2\|v_j\|^2) + \rho\|z_j\|^2 + \beta(\|\gamma v_j' - p_j\|^2 + \xi_j^2\|v_j\|^2 + \gamma v_j - p_j\|)^2 + \mu\|q_j\|^2),$$

where $\| \cdot \| := \| \cdot \|_{L^2(0, 1)}$ and (for later) $\| \cdot \|| := \| \cdot \|_{L^\infty(0, 1)}$. This gives rise to the functions

$$v_j, z_j, p_j, q_j \in (H^2(0, 1) \cap \hat{\mathcal{W}}_1)^2 \quad \text{and} \quad v_j'(1) = p_j'(1) = 0, \quad \text{(2.15)}$$
where ”'” represents the derivative with respect to $x$. The operator $A_{\square}$ defined in (2.6) can be written as

\[
(2.16) \quad A_{\square} \left( \begin{array}{c} v_j \\ z_j \\ p_j \\ q_j \end{array} \right) = \left( \begin{array}{c} \frac{1}{\rho} \left( \alpha(v_j'' - \xi_j^2 v_j) - \gamma \beta(p_j'' - \xi_j^2 p_j) - dz_j \right) \\ z_j \\ \frac{1}{\rho} \left( \beta(p_j'' - \xi_j^2 p_j) - \gamma \beta(v_j'' - \xi_j^2 v_j) \right) \end{array} \right).
\]

Proposition 2.5. $i\mathbb{R} \subset \rho(A_{\square})$.

**Proof.** To prove $i\mathbb{R} \subset \rho(A_{\square})$ it is sufficient to prove that $\sigma(A_{\square}) \cap i\mathbb{R} = \emptyset$. Since the resolvent of $A_{\square}$ is compact in $H_{\square}$ then $\sigma(A_{\square}) = \sigma_p(A_{\square})$. In the previous section, we already proved that $0 \in \rho(A_{\square})$. It remains to show that $\sigma(A_{\square}) \cap i\mathbb{R}^+ = \emptyset$. For this aim, suppose by contradiction that there exists a real number $\lambda \neq 0$ and $U = (v, z, p, q)^T \in D(A_{\square}) \setminus \{0\}$ such that

\[
(2.17) \quad A_{\square} U = i\lambda U.
\]

Using the orthonormal basis decomposition, (2.16) and detailing (2.17), we get the following system

\[
(2.18) \quad z_j = i\lambda v_j,
\]

\[
(2.19) \quad i\lambda \rho z_j - \alpha(v_j'' - \xi_j^2 v_j) + \gamma \beta(p_j'' - \xi_j^2 p_j) + dz_j = 0,
\]

\[
(2.20) \quad q_j = i\lambda p_j,
\]

\[
(2.21) \quad i\lambda \mu q_j - \beta(p_j'' - \xi_j^2 p_j) + \gamma \beta(v_j'' - \xi_j^2 v_j) = 0.
\]

From (2.8) and (2.17), we have

\[
(2.22) \quad 0 = \Re \langle i\lambda \|U\|_H_{\square} \rangle = \Re \left( \langle A_{\square} U, U \rangle_{H_{\square}} \right) = -\int_{\Omega_1} d(x)|z(X)|^2 dX = -\int_0^1 d(x)|z_j(x)|^2 dx \leq 0.
\]

Thus, from (2.18), (2.22) and the fact that $\lambda \neq 0$, we have

\[
(2.23) \quad d(x)z_j(x) = 0 \text{ in } (0, 1), \quad \text{and consequently } z_j = v_j = 0 \text{ in } (a, b).
\]

Using the fact that $\alpha = \alpha_1 + \gamma^2 \beta$ and (2.23) in (2.19), we get

\[
(2.24) \quad i\lambda \rho z_j - \alpha_1(v_j'' - \xi_j^2 v_j) - \gamma \left( \beta v_j'' - \xi_j^2 v_j \right) = 0, \quad \text{in } (0, 1).
\]

Combining (2.24) and (2.21), we get

\[
(2.25) \quad i\lambda (\rho z_j + \gamma \mu q_j) - \alpha_1 \left( v_j'' - \xi_j^2 v_j \right) = 0, \quad \text{in } (0, 1).
\]

Using (2.23) in (2.25) and the fact that $\lambda \neq 0$, then using (2.20), we get

\[
(2.26) \quad q_j = p_j = 0 \quad \text{in } (a, b).
\]

Since $v_j, p_j \in H^2(a, b) \subset C^1([a, b])$, we get

\[
(2.27) \quad v_j(\xi) = v_j'(\xi) = p_j(\xi) = p_j'(\xi) = 0 \quad \text{where } \xi \in [a, b].
\]

Inserting (2.18) and (2.20) in (2.25), then combining with (2.21), we get the following system

\[
(2.28) \quad v_j'' - \xi_j^2 v_j = -\frac{\lambda^2}{\alpha_1} (\rho v_j + \gamma \mu p_j), \quad \text{in } (0, 1)
\]

\[
(2.29) \quad p_j'' - \xi_j^2 p_j = -\frac{\lambda^2}{\alpha_1} \left( \gamma \rho v_j + \mu \beta p_j \right), \quad \text{in } (0, 1).
\]

Let $\tilde{U} = (v_j, v_j', p_j, p_j')^T$. From (2.27), we get $\tilde{U}(b) = 0$. Now, system (2.28)-(2.29) can be written in $(b, L)$ as the following

\[
(2.30) \quad \tilde{U}_x = B\tilde{U} \quad \text{in } (b, L),
\]

where $B$ is a $4 \times 4$ constant matrix that depends on $\lambda$. The boundary conditions $\tilde{U}(a) = 0$ and $\tilde{U}(L) = 0$ ensure that $\tilde{U}(b) = 0$. The operator $B$ is continuous in $H_{\square}$ and compact in $C^1([a, L])$. It follows that $\sigma(B) \subset \rho(B)$.
For simplicity, we drop the index $n$ where

$$F(2.41)$$

Using the orthonormal basis decomposition, system (2.39) turns into the system of one-dimensional equations

$$F(2.35)$$

$$F(2.36)$$

$$F(2.37)$$

Inserting (2.35) and (2.37) in (2.36) and (2.38), we get the following system

$$F(2.39)$$

(2.40)

where

$$F(2.41)$$

Using the fact that $\alpha = \alpha_1 + \gamma^2 \beta$ in (2.39), we get

$$F(2.42)$$
Now, combining (2.40) and (2.42), we get

\begin{equation}
\alpha_1 \left( v_j'' - \xi_j^2 v_j \right) = -\lambda^2 \rho v_j - \gamma \lambda^2 \mu p_j + i \lambda d v_j + F_j^1 + \gamma F_j^2.
\end{equation}

Inserting (2.43) in (2.40), we get the following system

\begin{align}
(\lambda^2 \rho - \alpha_1 \xi_j^2) v_j + \alpha_1 v_j'' + \gamma \mu \lambda^2 p_j - i \lambda d v_j &= F_j^3, \\
(\lambda^2 \mu \alpha - \alpha_1 \beta \xi_j^2) p_j + \alpha_1 \beta v_j'' + \rho \gamma \beta \lambda^2 v_j - i \lambda \gamma \beta d v_j &= F_j^4,
\end{align}

where

\begin{equation}
F_j^3 = F_j^1 + \gamma F_j^2 \quad \text{and} \quad F_j^4 = \alpha F_j^2 + \gamma F_j^1.
\end{equation}

Before going on, let us first give the consequence of the dissipativeness property of the solution \((v, z, p, q)\) of the system (2.35)-(2.38).

**Lemma 2.6.** The solution \((v, z, p, q)\) of system (2.34) satisfies the following estimations

\begin{equation}
\sum_{j=1}^{\infty} \left\| \sqrt{d}z_j \right\|^2 = o(\lambda^{-2}) \quad \text{and} \quad \sum_{j=1}^{\infty} \left\| \lambda \sqrt{d}v_j \right\|^2 = o(\lambda^{-2}).
\end{equation}

**Proof.** First, taking the inner product of (2.33) with \(U\) in \(\mathcal{H}\), using the fact that \(\|U\|_{\mathcal{H}} = 1\) and \(\|F\|_{\mathcal{H}} = o(1)\), we get

\begin{equation}
\left\| \sqrt{d}z \right\|^2_{L^2(\Omega)} = -\Re \left( \langle A U, U \rangle_{\mathcal{H}} \right) = \Re \left( \langle (i\lambda I - A)U, U \rangle_{\mathcal{H}} \right) = \lambda^{-2} \Re \left( \langle F, U \rangle_{\mathcal{H}} \right) = o(\lambda^{-2}).
\end{equation}

Thus, by the orthonormal basis decomposition, we get the first estimation in (2.48). Now, multiplying (2.35) by \(\sqrt{d}\) and using the first estimation in (2.48) and that \(\|F\|_{\mathcal{H}} = o(1)\), we get the second estimation in (2.48). The proof has been completed. \(\square\)

For all \(0 < \varepsilon < \frac{b-a}{4}\), we fix the following cut-off functions

- \(\theta_k \in C^2([0,1]), k \in \{1, 2\}\) such that \(0 \leq \theta_k(x) \leq 1\), for all \(x \in [0,1]\) and

\[
\theta_k(x) = \begin{cases} 
1 & \text{if } x \in [a + k\varepsilon, b - k\varepsilon], \\
0 & \text{if } x \in [0, a + (k - 1)\varepsilon] \cup [b + (1 - k)\varepsilon, 1].
\end{cases}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Geometric description of the functions \(\theta_1, \theta_2\) and \(d\).}
\end{figure}

**Lemma 2.7.** The solution \((v, z, p, q)\) of system (2.34) satisfies the following estimations

\begin{equation}
\sum_{j=1}^{\infty} \left\| \lambda p_j \right\|^2_{L^2(D_\varepsilon)} = o(\lambda^{-2}) \quad \text{and} \quad \sum_{j=1}^{\infty} \left\| q_j \right\|^2_{L^2(D_\varepsilon)} = o(\lambda^{-2}),
\end{equation}

where \(D_\varepsilon := (a + \varepsilon, b - \varepsilon)\) with a positive real number \(\varepsilon\) small enough such that \(\varepsilon < \frac{b-a}{4}\).
**Proof.** Multiplying (2.44) by $\beta \theta_1 \overline{\nu_j}$, using integration by parts over $(0, 1)$, and the definition of $\theta_1$, we get

$$\lambda^2 \rho \beta \int_0^1 \theta_1 v_j \overline{\nu_j} dx - \alpha_1 \beta \int_0^1 \theta_1 \xi_j^2 v_j \overline{\nu_j} dx = - \alpha_1 \beta \int_0^1 \theta_1 v_j' \overline{\nu_j} dx$$

(2.51)

Integrating by parts $J_1$, we get

(2.52)

$$J_1 = -\alpha_1 \beta \int_0^1 \theta_1 v_j \overline{\nu_j} dx - \alpha_1 \beta \int_0^1 \theta_1 v_j' \overline{\nu_j} dx.$$

Inserting (2.52) in (2.51), we get

(2.53)

Multiplying (2.45) by $-\theta_1 \overline{\nu_j}$ integrating by parts over $(0, 1)$, we get

(2.54)

Adding the real part of (2.53) and that of (2.54), then taking the sum from 1 to $\infty$, we get

(2.55)

Using Cauchy-Schwarz inequality, (2.48), the fact that $\|U\|_{H^2} = 1$ and $\|F\|_{H^2} = o(1)$, we get

(2.56)

Inserting (2.56) in (2.55) and using (2.48), we get

(2.57)

$$\gamma \mu \beta \sum_{j=1}^\infty \int_0^1 \theta_1 |\lambda p_j|^2 dx \leq \lambda^2 |\rho \beta - \mu \alpha| \sum_{j=1}^\infty \int_0^1 \theta_1 |p_j| |v_j| dx + o(\lambda^{-2}).$$
Using Young inequality, we get
\[
\lambda^2 |p\beta - \mu\alpha| \sum_{j=1}^{\infty} \int_0^1 \theta_1 |p_j||v_j|dx \leq \frac{\gamma \mu \beta}{2} \sum_{j=1}^{\infty} \int_0^1 \theta_1 |\lambda p_j|^2 dx + \frac{(\rho \beta - \mu \alpha)^2}{2 \gamma \mu \beta} \sum_{j=1}^{\infty} \int_0^1 \theta_1 |v_j|^2 dx.
\]

Inserting (2.58) in (2.57) and using (2.48), we get the first estimation in (2.50). Using the first estimation in (2.37) and the fact that \(\|F\|_H = o(1)\), we get the second estimation in (2.50). The proof has been completed. \(\square\)

**Lemma 2.8.** The solution \((v, z, p, q)\) of system (2.34) satisfies the following estimations
\[
\sum_{j=1}^{\infty} \left( \|v_j\|^2_{L^2(D_s)} + \xi_j^2 \|v_j\|^2_{L^2(D_s)} \right) = o(\lambda^{-2}).
\]

**Proof.** Multiplying (2.44) by \(-\theta_1v_j\), integrating by parts over \((0, 1)\), we get
\[
-\rho \int_0^1 \theta_1 |v_j|^2 dx + \alpha \xi_j^2 \int_0^1 \theta_1 |v_j|^2 dx + \alpha \int_0^1 \theta_1 |v_j|^2 dx + \alpha \int_0^1 \theta_1 v_j v_j dx + i \lambda \int_0^1 \theta_1 d|v_j|^2 dx = -\int_0^1 \theta_1 F_j^3 v_j dx.
\]
Taking the sum on \(j\) in (2.60) and using (2.48), we get
\[
\alpha \sum_{j=1}^{\infty} \int_0^1 \theta_1 (|v_j|^2 + \xi_j^2 |v_j|^2) dx = -\alpha \sum_{j=1}^{\infty} \int_0^1 \theta_1 v_j v_j dx + \alpha \sum_{j=1}^{\infty} \int_0^1 \theta_1 p_j v_j dx
\]
(2.61)
\[-\sum_{j=1}^{\infty} \int_0^1 \theta_1 F_j^3 v_j dx + o(\lambda^{-2}).
\]
Using Cauchy-Schwarz inequality, the definition of the function \(\theta_1\), the fact that \(\|U\|_H = 1\), \(\|F\|_H = o(1)\), (2.48) and (2.50), we get
\[
\left| \sum_{j=1}^{\infty} \int_0^1 \theta_1 v_j v_j dx \right| = o(\lambda^{-2}), \quad \left| \sum_{j=1}^{\infty} \int_0^1 \theta_1 p_j v_j dx \right| = o(\lambda^{-2}) \quad \text{and} \quad \left| \sum_{j=1}^{\infty} \int_0^1 \theta_1 F_j^3 v_j dx \right| = o(\lambda^{-3}).
\]
Inserting the above estimations in (2.61), we get (2.59). The proof has been completed. \(\square\)

**Lemma 2.9.** The solution \((v, z, p, q)\) of system (2.34) satisfies the following estimations
\[
\sum_{j=1}^{\infty} \left( \|v_j\|^2_{L^2(D_0)} + \xi_j^2 \|v_j\|^2_{L^2(D_0)} \right) = o(\lambda^{-2}),
\]
where \(D_{2\varepsilon} := (a + 2\varepsilon, b - 2\varepsilon)\) with a positive real number \(\varepsilon\) small enough such that \(\varepsilon < \frac{b-a}{4}\).

**Proof.** Multiplying (2.45) by \(-\theta_2v_j\), integrating by parts over \((0, 1)\), we get
\[
-\mu \int_0^1 \theta_2 |p_j|^2 dx + \alpha \beta \int_0^1 \theta_2 |p_j|^2 dx + \alpha \int_0^1 \theta_2 |p_j|^2 dx + \alpha \int_0^1 \theta_2 p_j v_j dx + i \gamma \beta \int_0^1 \theta_2 d|v_j|^2 dx = -\int_0^1 \theta_2 F_j^3 v_j dx.
\]
Taking the sum on \(j\) in the above equation and using (2.50), we get
\[
\alpha \beta \sum_{j=1}^{\infty} \int_0^1 \theta_2 (|p_j|^2 + \xi_j^2 |p_j|^2) dx = -\alpha \beta \sum_{j=1}^{\infty} \int_0^1 \theta_2 p_j v_j dx + \gamma \beta \lambda^2 \sum_{j=1}^{\infty} \int_0^1 \theta_2 v_j v_j dx
\]
(2.63)
\[-i \gamma \beta \sum_{j=1}^{\infty} \int_0^1 d\theta_2 v_j v_j dx - \sum_{j=1}^{\infty} \int_0^1 \theta_2 F_j^3 v_j dx + o(\lambda^{-2}).
\]
Using Cauchy-Schwarz inequality, the definition of the function \( \theta_2 \), the fact that \( \|U\|_{\mathcal{H}} = 1, \|F\|_{\mathcal{H}} = o(1) \), (2.48) and (2.50), we get

\[
\begin{align*}
\left\| \sum_{j=1}^{\infty} \int_{0}^{1} \theta'_2 p_j v_j dx \right\| &= o(\lambda^{-2}), \\
\lambda^2 \sum_{j=1}^{\infty} \int_{0}^{1} \theta_2 v_j p_j dx &= o(\lambda^{-2}), \\
\lambda \sum_{j=1}^{\infty} \int_{0}^{1} d\theta_2 v_j p_j dx &= o(\lambda^{-3}), \\
\sum_{j=1}^{\infty} \int_{0}^{1} \theta_2 F_j^2 p_j dx &= o(\lambda^{-3}).
\end{align*}
\]

Inserting the above estimations in (2.63) and using the definition of the function \( \theta_2 \), we get (2.62). The proof has been completed. \( \square \)

**Lemma 2.10.** Let \( h \in C^\infty([0,1]) \) such that \( h(0) = h(1) = 0 \). The solution \( (v, z, p, q) \) of system (2.34) satisfies the following estimation

\[
\sum_{j=1}^{\infty} \int_{0}^{1} h'(\rho \lambda^2 - \alpha_1 \xi_j^2) |v_j|^2 dx + \alpha_1 \sum_{j=1}^{\infty} \int_{0}^{1} h' |v_j'|^2 dx + 2 \sum_{j=1}^{\infty} \Re \left( i \lambda \int_{0}^{L} h dv_j v'_j dx \right)
+ \beta \sum_{j=1}^{\infty} \int_{0}^{1} h |\gamma v_j' - p_j|^2 dx = o(\lambda^{-2}).
\]

**Proof.** First, multiplying (2.39) and (2.40) by \(-2h v_j'\) and \(-2h p_j'\), taking the real part, we get

\[
\begin{align*}
\begin{cases}
-2\lambda^2 \rho \Re \left( \int_{0}^{1} h v_j v'_j dx \right) + 2\alpha \xi^2 \Re \left( \int_{0}^{1} h v_j v'_j dx \right) - 2\alpha \Re \left( \int_{0}^{1} h v_j v'_j dx \right) \\
+ 2\gamma \beta \Re \left( \int_{0}^{1} h (v'' - \xi_j^2 v_j) v'_j dx \right) + 2\Re \left( i \lambda \int_{0}^{1} h dv_j v'_j dx \right) = -2\Re \left( \int_{0}^{1} h F_j^2 v'_j dx \right)
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
-2\lambda^2 \mu \Re \left( \int_{0}^{1} h p_j p'_j dx \right) + 2\beta \xi^2 \Re \left( \int_{0}^{1} h p_j p'_j dx \right) - 2\beta \Re \left( \int_{0}^{1} h p_j p'_j dx \right) \\
+ 2\gamma \beta \Re \left( \int_{0}^{1} h (v'' - \xi_j^2 v_j) p'_j dx \right) = -2\Re \left( \int_{0}^{1} h F_j^2 p'_j dx \right).
\end{align*}
\]

Using integration by parts in the above equations and taking the sum on \( j \) from 1 to \( \infty \), we get

\[
\sum_{j=1}^{\infty} \int_{0}^{1} h' (\lambda^2 \rho - \alpha_1 \xi_j^2) |v_j|^2 dx + \alpha_1 \sum_{j=1}^{\infty} \int_{0}^{1} h' |v_j'|^2 dx + 2 \sum_{j=1}^{\infty} \Re \left( i \lambda \int_{0}^{1} h dv_j v'_j dx \right)
+ 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h (v'' - \xi_j^2 v_j) v'_j dx \right) = -2 \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h F_j^2 v'_j dx \right) := J_2
\]

and

\[
\sum_{j=1}^{\infty} \int_{0}^{1} h' (\lambda^2 \mu - \beta \xi_j^2) |p_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h' |p_j|^2 dx + 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h (v'' - \xi_j^2 v_j) p'_j dx \right) = -2 \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h F_j^2 p'_j dx \right).
\]

Adding the above two equations, we get

\[
\begin{align*}
\sum_{j=1}^{\infty} \int_{0}^{1} h' (\lambda^2 \rho - \alpha_1 \xi_j^2) |v_j|^2 dx + \alpha_1 \sum_{j=1}^{\infty} \int_{0}^{1} h' |v_j'|^2 dx + 2 \sum_{j=1}^{\infty} \Re \left( i \lambda \int_{0}^{1} h dv_j v'_j dx \right)
+ \sum_{j=1}^{\infty} \int_{0}^{1} h' (\lambda^2 \mu - \beta \xi_j^2) |p_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h' |p_j|^2 dx + 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h (v'' - \xi_j^2 v_j) p'_j dx \right)
= -2 \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h F_j^2 v'_j dx \right) - 2 \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h F_j^2 p'_j dx \right).
\end{align*}
\]
Using integration by parts in $J_2$, we get

$$J_2 = -2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_0^1 h' p_j' \overline{v_j} \, dx \right) - 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_0^1 h p_j' \overline{v_j} \, dx \right)$$

$$+ 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_0^1 \xi_j^2 h p_j' \overline{v_j} \, dx \right) + 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_0^1 \xi_j^2 h p_j' \overline{v_j} \, dx \right).$$

(2.66)

Inserting (2.66) in (2.65) and using the fact that $\alpha = \alpha_1 + \gamma^2 \beta$, we get

$$\sum_{j=1}^{\infty} \int_0^1 h'(\rho \lambda^2 - \alpha_1 \xi_j^2)v_j|^2 \, dx + \alpha_1 \sum_{j=1}^{\infty} \int_0^1 h'|v_j'|^2 \, dx + 2\sum_{j=1}^{\infty} \Re \left( i\lambda \int_0^L h d v_j \overline{v_j} \, dx \right)$$

$$+ \mu \sum_{j=1}^{\infty} \int_0^1 h'|\lambda p_j|^2 \, dx + \gamma^2 \beta \sum_{j=1}^{\infty} \int_0^1 h'|v_j'|^2 \, dx + \beta \sum_{j=1}^{\infty} \int_0^1 h'|p_j'|^2 \, dx - 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_0^1 h' p_j' \overline{v_j} \, dx \right)$$

$$- \left( \gamma^2 \beta \sum_{j=1}^{\infty} \int_0^1 h' \xi_j^2 |v_j|^2 \, dx + \beta \sum_{j=1}^{\infty} \int_0^1 h' \xi_j^2 |p_j|^2 \, dx - 2\gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_0^1 \xi_j^2 h' p_j' \overline{v_j} \, dx \right) \right)$$

(2.67)

$$= -2 \sum_{j=1}^{\infty} \Re \left( \int_0^1 h F_j^1 \overline{v_j} \, dx \right) - 2 \sum_{j=1}^{\infty} \Re \left( \int_0^1 h F_j^1 \overline{p_j} \, dx \right).$$

$$= J_2$$

$$= J_3$$

It is clear that

$$J_3 = \beta \sum_{j=1}^{\infty} \int_0^1 h' |\gamma v_j' - p_j'|^2 \, dx$$

and

$$J_4 = \beta \sum_{j=1}^{\infty} \int_0^1 h' \xi_j^2 |v_j| \, dx.$$

Inserting (2.68) in (2.67), we get

$$\sum_{j=1}^{\infty} \int_0^1 h'(\rho \lambda^2 - \alpha_1 \xi_j^2)|v_j|^2 \, dx + \alpha_1 \sum_{j=1}^{\infty} \int_0^1 h'|v_j'|^2 \, dx + 2 \sum_{j=1}^{\infty} \Re \left( i\lambda \int_0^L h d v_j \overline{v_j} \, dx \right)$$

$$+ \mu \sum_{j=1}^{\infty} \int_0^1 h'|\lambda p_j|^2 \, dx + \beta \sum_{j=1}^{\infty} \int_0^1 h'|\gamma v_j'|^2 \, dx - \beta \sum_{j=1}^{\infty} \int_0^1 h' \xi_j^2 |v_j| \, dx$$

$$= -2 \sum_{j=1}^{\infty} \Re \left( \int_0^1 h F_j^1 \overline{v_j} \, dx \right) - 2 \sum_{j=1}^{\infty} \Re \left( \int_0^1 h F_j^1 \overline{p_j} \, dx \right).$$

(2.69)

Using the definition of $F^1$ and $F^2$ in (2.41), and the facts that $\sum_{j=1}^{\infty} |v_j|^2 = O(1)$, $\sum_{j=1}^{\infty} |p_j| = O(1)$ and the fact $\|F\|_{\mathcal{H}_\omega} = o(1)$, we get

$$\sum_{j=1}^{\infty} \Re \left( \int_0^1 h F_j^1 \overline{v_j} \, dx \right) = \lambda^{-1} \sum_{j=1}^{\infty} \Re \left( \int h F_j^1 \overline{v_j} \, dx \right) + o(\lambda^{-2})$$

(2.70)

$$\sum_{j=1}^{\infty} \Re \left( \int_0^1 h F_j^2 \overline{p_j} \, dx \right) = \lambda^{-1} \sum_{j=1}^{\infty} \Re \left( \int h F_j^2 \overline{p_j} \, dx \right) + o(\lambda^{-2}).$$
Integrating by parts over \((0, 1)\) in the above estimations and using the fact that \(\sum_{j=1}^{\infty} \|\lambda v_j\|^2 = O(1), \sum_{j=1}^{\infty} \|\lambda p_j\|^2 = O(1)\) and \(\|F\|_{H^0} = o(1)\) in (2.70), we get the following estimations

\[
\begin{align*}
\sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h F_j^1 v_j' dx \right) &= -\lambda^{-1} \sum_{j=1}^{\infty} \Re \left( i \rho \int_{0}^{1} (h' f_j^1 + h(f_j^1)' v_j') dx \right) + o(\lambda^{-2}) = o(\lambda^{-2}) \\
\sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h F_j^2 p_j' dx \right) &= -\lambda^{-1} \sum_{j=1}^{\infty} \Re \left( i \mu \int_{0}^{1} (h' f_j^3 + h(f_j^3)' p_j') dx \right) + o(\lambda^{-2}) = o(\lambda^{-2}).
\end{align*}
\]

(2.71)

Inserting (2.71) in (2.69), we get the desired estimation (2.64). The proof has been completed.

\[\square\]

**Lemma 2.11.** Let \(h \in C^\infty([0, 1])\) such that \(h(0) = h(1) = 0\). The solution \((v, z, p, q)\) of system (2.34) satisfies the following estimation

\[
\begin{align*}
\sum_{j=1}^{\infty} \int_{0}^{1} h'(-\lambda^2 \rho + \alpha_1 \xi_j^2)|v_j|^2 dx + \alpha_1 \sum_{j=1}^{\infty} \int_{0}^{1} h'|v_j'|^2 dx - \mu \sum_{j=1}^{\infty} \int_{0}^{1} h'\lambda p_j|^2 dx \\
+ \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|\gamma v_j - p_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'\xi_j^2|\gamma v_j - p_j|^2 dx + I = o(\lambda^{-2}),
\end{align*}
\]

where

\[
I = \alpha \Re \left( \sum_{j=1}^{\infty} \int_{0}^{1} h'' v_j' v_j' dx \right) + \beta \Re \left( \sum_{j=1}^{\infty} \int_{0}^{1} h'' p_j' p_j' dx \right) \\
- \gamma \beta \Re \left( \sum_{j=1}^{\infty} \int_{0}^{1} h'' p_j' v_j' dx \right) - \gamma \beta \Re \left( \sum_{j=1}^{\infty} \int_{0}^{1} h'' v_j' p_j' dx \right).
\]

**Proof.** Multiplying (2.39) by \(-h' v_j'\) integrating by parts over \((0, 1)\) and taking the real part, we get

\[
\begin{align*}
-\rho \int_{0}^{1} h'|\lambda v_j|^2 dx + \alpha \xi_j^2 \int_{0}^{1} h'|v_j|^2 dx + \alpha \Re \left( \int_{0}^{1} h'' v_j' v_j' dx \right) + \alpha \int_{0}^{1} h'|v_j'|^2 dx \\
+ \gamma \beta \Re \left( \int_{0}^{1} (p_j' - \xi_j^2 p_j) h' v_j' dx \right) = -\Re \left( \int_{0}^{1} h' F_j^1 v_j' dx \right).
\end{align*}
\]

(2.74)

Taking the sum on \(j = 1\) to \(\infty\) in the above equation, we get

\[
\begin{align*}
-\rho \sum_{j=1}^{\infty} \int_{0}^{1} h'|\lambda v_j|^2 dx + \alpha \sum_{j=1}^{\infty} \int_{0}^{1} h'|v_j|^2 dx + \alpha \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h'' v_j' v_j' dx \right) + \alpha \sum_{j=1}^{\infty} \int_{0}^{1} h'|v_j'|^2 dx \\
+ \gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} (p_j' - \xi_j^2 p_j) h' v_j' dx \right) = -\sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^1 v_j' dx \right).
\end{align*}
\]

(2.75)

Multiplying (2.40) by \(-h' p_j'\) integrating by parts over \((0, 1)\) taking the real part and summing the result on \(j = 1\) to \(\infty\), we get

\[
\begin{align*}
-\mu \sum_{j=1}^{\infty} \int_{0}^{1} h'|\lambda p_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|\xi_j^2 p_j|^2 dx + \beta \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h'' p_j' p_j' dx \right) + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|p_j'|^2 dx \\
+ \gamma \beta \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} (v_j' - \xi_j^2 v_j) h' p_j' dx \right) = -\sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^2 p_j' dx \right).
\end{align*}
\]

(2.76)
Summing (2.75) and (2.76), we get

\begin{align}
-\rho \sum_{j=1}^{\infty} & \int_{0}^{1} h'|\lambda v_j|^2 dx + \alpha \sum_{j=1}^{\infty} \int_{0}^{1} h'|\xi_j^2|v_j|^2 dx + \alpha \sum_{j=1}^{\infty} \int_{0}^{1} h'|v_j|^2 dx - \mu \sum_{j=1}^{\infty} \int_{0}^{1} h'|\lambda p_j|^2 dx \\
+\beta \sum_{j=1}^{\infty} & \int_{0}^{1} h'|\xi_j^2|p_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|p_j|^2 dx + J_5 + J_6 + \alpha \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h'' v_j \psi_j^2 dx \right) \\
+\beta \sum_{j=1}^{\infty} & \Re \left( \int_{0}^{1} h'' p_j \psi_j dx \right) = -\sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^1 \psi_j dx \right) - \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^2 \psi_j dx \right).
\end{align}

(2.77)

Integrating by parts \( J_5 \) and \( J_6 \), we get

\begin{align}
J_5 = & -\gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' p_j \psi_j^2 dx \right) - \gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h'' p_j \psi_j dx \right) - \gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} \xi_j^2 h' \psi_j^2 dx \right), \\
J_6 = & -\gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' v_j \psi_j dx \right) - \gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h'' v_j \psi_j dx \right) - \gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} \xi_j^2 h' v_j \psi_j dx \right).
\end{align}

Inserting the above two equations in (2.77), we get

\begin{align}
-\rho \sum_{j=1}^{\infty} & \int_{0}^{1} h'|\lambda v_j|^2 dx + \alpha \sum_{j=1}^{\infty} \int_{0}^{1} h'|\xi_j^2|v_j|^2 dx + \alpha \sum_{j=1}^{\infty} \int_{0}^{1} h'|v_j|^2 dx - \mu \sum_{j=1}^{\infty} \int_{0}^{1} h'|\lambda p_j|^2 dx \\
+\beta \sum_{j=1}^{\infty} & \int_{0}^{1} h'|\xi_j^2|p_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|p_j|^2 dx - 2\gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h'' p_j \psi_j dx \right) + I \\
-2\gamma \sum_{j=1}^{\infty} & \Re \left( \int_{0}^{1} \xi_j^2 h' p_j \psi_j dx \right) = -\sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^1 \psi_j dx \right) - \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^2 \psi_j dx \right),
\end{align}

(2.78)

where \( I \) is defined in (2.73). Using the fact that \( \alpha = \alpha_1 + \gamma^2 \beta \) in (2.78), we get

\begin{align}
\sum_{j=1}^{\infty} & \int_{0}^{1} h'(-\lambda^2 \rho + \alpha_1 \xi_j^2)|v_j|^2 dx + \alpha \sum_{j=1}^{\infty} \int_{0}^{1} h'|v_j|^2 dx - \mu \sum_{j=1}^{\infty} \int_{0}^{1} h'|\lambda p_j|^2 dx \\
+\gamma^2 \sum_{j=1}^{\infty} & \int_{0}^{1} h'|v_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|p_j|^2 dx - 2\gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h'' p_j \psi_j dx \right) \\
+\gamma^2 \sum_{j=1}^{\infty} & \int_{0}^{1} h'|\xi_j^2|v_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|\xi_j^2|p_j|^2 dx - 2\gamma \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} \xi_j^2 h' p_j \psi_j dx \right) \\
+I = & -\sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^1 \psi_j dx \right) - \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^2 \psi_j dx \right).
\end{align}

(2.79)

It is easy to check that

\begin{align}
J_7 = & \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|\gamma v_j - p_j|^2 dx \quad \text{and} \quad J_8 = \beta \sum_{j=1}^{\infty} \int_{0}^{1} h'|\xi_j^2|\gamma v_j - p_j|^2 dx.
\end{align}

Using Cauchy-Schwarz inequality and the facts \( \sum_{j=1}^{\infty} \|\lambda v_j\|^2 = O(1) \), \( \sum_{j=1}^{\infty} \|\lambda p_j\|^2 = O(1) \), the definition of \( F^1 \) and \( F^2 \) given by (2.41), and \( \|F\|_{\mathcal{H}_0} = o(1) \), we get

\begin{align}
\sum_{j=1}^{\infty} & \Re \left( \int_{0}^{1} h' F_j^1 \psi_j dx \right) = o(\lambda^{-2}) \quad \text{and} \quad \sum_{j=1}^{\infty} \Re \left( \int_{0}^{1} h' F_j^2 \psi_j dx \right) = o(\lambda^{-2}).
\end{align}

(2.81)

Inserting (2.80) and (2.81) in (2.79), we get the desired result (2.72). The proof has been completed. \( \square \)
For all $0 < \varepsilon < \frac{b - a}{4}$, we fix the following cut-off functions

- $h_1, h_2 \in C^2([0, 1])$ such that $0 \leq h_1(x) \leq 1$, $0 \leq h_2(x) \leq 1$, for all $x \in [0, 1]$ and

\[
  h_1 = \begin{cases} 
    1 & \text{if } x \in [0, a + 2\varepsilon] \\
    0 & \text{if } x \in [b - 2\varepsilon, 1],
  \end{cases} \quad \text{and} \quad
  h_2 = \begin{cases} 
    0 & \text{if } x \in [0, a + 2\varepsilon] \\
    1 & \text{if } x \in [b - 2\varepsilon, 1].
  \end{cases}
\]

**Lemma 2.12.** The solution $(v, z, p, q)$ of system (2.34) satisfies the following estimation

(2.82) \[ \|U\|_{H^1} = o(1). \]

**Proof.** First, adding (2.64) and (2.72), we get

(2.83) \[ 2\alpha_1 \sum_{j=1}^{\infty} \int_0^1 |h'v_j'|^2 dx + 2\beta \sum_{j=1}^{\infty} \int_0^1 |h'v_j' - p_j'|^2 dx + 2 \sum_{j=1}^{\infty} \Re \left( i\lambda \int_0^1 h v_j \overline{v_j'} dx \right) + I = o(\lambda^{-2}). \]

Now, take $h(x) = xh_1 + (x - 1)h_2$. It is easy to see that

(2.84) \[ h'(x) = xh_1' + h_1 + (x - 1)h_2' + h_2 \quad \text{and} \quad h'' = xh_1'' + 2h_1' + (x - 1)h_2'' + 2h_2'. \]

The aim now is to give an estimation for $I$. We start by using Cauchy-Schwarz inequality and (2.84), (2.48), (2.50), (2.59), (2.62), to get

\[
\left\{ \begin{align*}
\Re \left( \sum_{j=1}^{\infty} \int_0^1 h''v_j' \overline{v_j'} dx \right) & = o(\lambda^{-3}), \\
\Re \left( \sum_{j=1}^{\infty} \int_0^1 h''p_j' \overline{v_j'} dx \right) & = o(\lambda^{-3})
\end{align*} \right. \]

Using the above estimations and (2.73), we get

(2.85) \[ |I| = o(\lambda^{-3}). \]

Setting $\tilde{h} = xh_1' + (x - 1)h_2'$, and using (2.59), (2.62), we get

\[
\sum_{j=1}^{\infty} \int_0^1 \tilde{h}|v_j'|^2 dx = o(\lambda^{-2}) \quad \text{and} \quad \sum_{j=1}^{\infty} \int_0^1 \tilde{h} |v_j' - p_j'|^2 dx = o(\lambda^{-2}).
\]
These estimations, (2.85), (2.84) and (2.83), yield

\[(2.86)\quad a_1 \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |v'_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |\gamma v'_j - p'_j|^2 dx + \sum_{j=1}^{\infty} \Re \left( i \lambda \int_0^1 h d v'_j dx \right) = o(\lambda^{-2}).\]

Using Young inequality and the definitions of the functions \(d\) given by (1.10), \(h\) given at the beginning of the proof of this Lemma and (2.48), we get

\[
\left| \sum_{j=1}^{\infty} \Re \left( i \lambda \int_0^1 h d v'_j dx \right) \right| \leq \lambda \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |v'_j|^2 dx
\]

\[
\leq \lambda \|\sqrt{d}\|_{\infty} \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) \sqrt{d} |v'_j|^2 dx
\]

\[(2.87)\quad \leq \frac{\alpha_1}{2} \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |v'_j|^2 dx + \frac{\|\sqrt{d}\|_{\infty}}{2 \alpha_1} \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |\gamma v'_j|^2 dx
\]

\[
\leq \frac{\alpha_1}{2} \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |v'_j|^2 dx + \frac{\|\sqrt{d}\|_{\infty}}{\alpha_1} \sum_{j=1}^{\infty} d |\gamma v'_j|^2 dx
\]

\[
\leq \frac{\alpha_1}{2} \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |v'_j|^2 dx + o(\lambda^{-2}).
\]

Inserting (2.87) in (2.86), we get

\[(2.88)\quad \frac{\alpha_1}{2} \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |v'_j|^2 dx + \beta \sum_{j=1}^{\infty} \int_0^1 (h_1 + h_2) |\gamma v'_j - p'_j|^2 dx \leq o(\lambda^{-2}).\]

Using the definition of functions \(h_1\) and \(h_2\) and (2.59), (2.62), we get

\[(2.89)\quad \sum_{j=1}^{\infty} \int_0^1 |v'_j|^2 dx = o(\lambda^{-2}) \quad \text{and} \quad \sum_{j=1}^{\infty} \int_0^1 |\gamma v'_j - p'_j|^2 dx = o(\lambda^{-2}).\]

From (2.89), we get

\[(2.90)\quad \sum_{j=1}^{\infty} \int_0^1 |p'_j|^2 dx \leq 2 \sum_{j=1}^{\infty} \int_0^1 |\gamma v'_j - p'_j|^2 dx + 2 \gamma^2 \sum_{j=1}^{\infty} \int_0^1 |v'_j|^2 dx \leq o(\lambda^{-2}).\]

Using Poincaré inequality, (2.89) and (2.90), we get

\[(2.91)\quad \sum_{j=1}^{\infty} \int_0^1 |\lambda v_j|^2 dx = o(1) \quad \text{and} \quad \sum_{j=1}^{\infty} \int_0^1 |\lambda p_j|^2 dx = o(1).\]

Using (2.72), (2.85), (2.91), (2.89), (2.59) and (2.62), we get

\[(2.92)\quad \sum_{j=1}^{\infty} \int_0^1 \xi_j^2 |v'_j|^2 dx = o(1) \quad \text{and} \quad \sum_{j=1}^{\infty} \int_0^1 \xi_j^2 |\gamma v_j - p_j|^2 dx = o(1).\]

Finally, from (2.89)-(2.92), we obtain (2.82). The proof has been completed. \(\square\)

**Proof of Theorem 2.3.** From Lemma 2.12, we have \(\|U\|_{\mathcal{H}} = o(1)\), which contradicts \(\|U\|_{\mathcal{H}} = 1\) in (2.32). This implies that

\[
\limsup_{\lambda \in \mathbb{R}, \ |\lambda| \to \infty} \frac{1}{|\lambda|^2} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.
\]

Finally, according to Theorem 2.4, we obtain the desired result. The proof has been completed.
3. Conclusion and open problems

In this work, the local stabilization of 2D Piezoelectric beam with magnetic effect on a rectangular domain without geometric conditions is considered. The localized damping regions do not satisfy the geometric control condition (GCC). Based on the frequency domain approach with the orthonormal basis decomposition and specific multiplier techniques, we have proved a polynomial energy decay rate of order $t^{-1}$. The open problems that will be posed in this paper are:

(OP1) What happens about the optimality of the polynomial energy decay rate?

(OP2) The case where the damping region does not hit the boundary is still an open problem (See Figure 4).

(OP3) The case where $\Omega$ is an open bounded regular domain (for example of class $C^2$) and the localised damping region does not satisfy the geometric control condition (GCC) condition is still an open problem. For example, (See Figure 5)

\[
\begin{align*}
\Omega &= \left\{ X = (x, y) \in \mathbb{R}^2; \quad r_1 < \| X \|_{Euc} = \sqrt{x^2 + y^2} < r_4 \right\}, \\
\Gamma_0 &= \left\{ X = (x, y) \in \mathbb{R}^2; \quad \| X \|_{Euc} = r_1 \right\}, \\
\Gamma_1 &= \left\{ X = (x, y) \in \mathbb{R}^2; \quad \| X \|_{Euc} = r_4 \right\}.
\end{align*}
\]

and $d$ be a radial function, defined by

$d(r) \geq d_1 > 0$ in $\omega$ and $d(r) = 0$ in $\Omega \setminus \omega$,

where $\omega := \left\{ X = (x, y) \in \mathbb{R}^2; \quad r_2 < \| X \|_{Euc} = \sqrt{x^2 + y^2} < r_3 \right\}$ and $0 < r_1 < r_2 < r_3 < r_4$.

(OP4) The case where $\Omega$ is an open bounded regular domain (for example of class $C^2$) and the localised damping region satisfies the geometric control condition (GCC) condition is still an open problem.

Figure 4. The case where the damping region (red part) is far away from the boundary.
Figure 5. Model Describing $\Omega$, $\Gamma_0$, $\Gamma_1$ and $\omega$ in (OP3).

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