Research Article

An Application of Pascal Distribution Series on Certain Analytic Functions Associated with Stirling Numbers and Sălăgean Operator

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In the present paper, we will observe that the Sălăgean differential operator can be written in terms of Stirling numbers. Furthermore, we find a necessary and sufficient condition and inclusion relation for Pascal distribution series to be in the class $P_k(\lambda, \alpha)$ of analytic functions with negative coefficients defined by the Sălăgean differential operator. Also, we consider an integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

1. Preliminaries

Special functions are used in many applications of physics, engineering, and applied mathematics and statistics. Special polynomials have a close connection with number theory, and one of the most important sets of special numbers is the class of Stirling numbers (of the first and second kind), introduced in 1730 by the Scottish mathematician James Stirling.

In combinatorics, a Stirling number of the second kind (or Stirling partition number) is the number of ways to partition a set of $k$ objects into $j$ nonempty subsets and is denoted by $S(k, j)$ or by $b_{k,j}$ as used in this paper. These numbers occur in the field of mathematics called combinatorics and the study of partitions. In this paper, we will observe that the Sălăgean differential operator $D_k$ can be written in terms of Stirling numbers.

Let $A$ denote the class of analytic functions $f$ in the open unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$ has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Furthermore, let $S$ be a subclass of $A$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad z \in D. \quad (2)$$

For a function $f(z)$ in $A$, we define

$$D^0 f(z) = f(z), \quad (3)$$

$$D^1 f(z) = zf'(z), \quad (4)$$

and in general, we have

$$D^k f(z) = z(D^{k-1} f(z))', \quad k \in \mathbb{N}. \quad (5)$$

The differential operator $D^k$ was introduced by Sălăgean [1].

We note that
$D^k f(z) = z(D^{k-1} f(z))'$

$= b_{k,1} z f'(z) + b_{k,2} z^2 f''(z)$
$+ b_{k,3} z^3 f'''(z) + \cdots + b_{k,k} z^k f^{(k)}(z)$

$= \sum_{j=1}^{k} b_{k,j} z^j f^{(j)}(z), \quad (k \in \mathbb{N}),$  \hfill (6)

where

$b_{k,j} = j b_{k-1,j} + b_{k-1,j-1}$, and $b_{k,1} = b_{k,k} = 1$. \hfill (7)

For example,

(i) If $k = 2$, we have

$D^2 f(z) = z(D f(z))' = z f'(z)$
$+ z^2 f''(z) = b_{2,1} z f'(z) + b_{2,2} z^2 f''(z),$ \hfill (8)

where

$b_{2,1} = b_{2,2} = 1$. \hfill (9)

(ii) If $k = 3$, we have

$D^3 f(z) = z(D^2 f(z))' = z f'(z) + 3z^2 f''(z) + z^3 f'''(z)$
$= b_{3,1} z f'(z) + b_{3,2} z^2 f''(z) + b_{3,3} z^3 f'''(z),$ \hfill (10)

(iii) If $k = 4$, we have

$D^4 f(z) = z(D^3 f(z))' = z f'(z) + 7z^2 f''(z) + 6z^3 f'''(z) + 2z^4 f^{(4)}(z)$
$= b_{4,1} z f'(z) + b_{4,2} z^2 f''(z) + b_{4,3} z^3 f'''(z) + b_{4,4} z^4 f^{(4)}(z),$ \hfill (12)

where

$b_{4,2} = 2b_{3,2} + b_{3,1} = 7, \quad b_{4,3} = 3b_{3,3} + b_{3,2} = 6, \quad b_{4,4} = 1.$ \hfill (13)

(iv) If $k = 5$, we have

$D^5 f(z) = z(D^4 f(z))' = z f'(z) + 15z^2 f''(z) + 25z^3 f'''(z) + 10z^4 f^{(4)}(z) + z^5 f^{(5)}(z)$
$= b_{5,1} z f'(z) + b_{5,2} z^2 f''(z) + b_{5,3} z^3 f'''(z) + b_{5,4} z^4 f^{(4)}(z) + b_{5,5} z^5 f^{(5)}(z),$ \hfill (14)

where

$b_{5,2} = 2b_{4,2} + b_{4,1} = 15,$
$b_{5,3} = 3b_{4,3} + b_{4,2} = 25,$
$b_{5,4} = 4b_{4,4} + b_{4,4} = 40,$
$b_{5,5} = 5.$ \hfill (15)

Table 1 represents the coefficients $b_{k,j}$ of $z^k f^{(k)}(z)$.

Table 1 (see [2]) shows the first few possibilities for Stirling numbers of the second kind. Also, from this table, we note that:

where

$b_{3,2} = 2b_{2,2} + b_{2,1} = 3, \quad b_{3,1} = b_{3,3} = 1.$ \hfill (11)

Furthermore, for $k = 2, 3, 4, 5$, we observe that

(1) $b_{k,j} = j b_{k-1,j} + b_{k-1,j-1}$
(2) $b_{k,k-1} = (k(k-1)/2) = \binom{k}{2}$
(3) $b_{k,2} = 2^{k-1} - 1$
(4) $b_{k,3} = (1/6)(3^k - 3.2^k + 3)$
(5) $b_{k,1} = b_{k,k} = 1$
(6) $b_{k,j} = 0$, when $j > k$.
(7) $b_{p,j} \equiv 0 \pmod{p}$ iff $1 < j < p$, where $p$ is a prime number.
Table 1: Stirling numbers of the second kind.

| $D^0 f(z)$ | $D^1 f(z)$ | $D^2 f(z)$ | $D^3 f(z)$ | $D^4 f(z)$ | $D^5 f(z)$ | $D^6 f(z)$ | $D^7 f(z)$ | $D^8 f(z)$ |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| $D^0 f(z)$ | 1          | 0          | 0          | 0          | 0          | 0          | 0          | 0          |
| $D^1 f(z)$ | 1          | 1          | 0          | 0          | 0          | 0          | 0          | 0          |
| $D^2 f(z)$ | 1          | 3          | 1          | 0          | 0          | 0          | 0          | 0          |
| $D^3 f(z)$ | 1          | 7          | 6          | 1          | 0          | 0          | 0          | 0          |
| $D^4 f(z)$ | 1          | 15         | 25         | 10         | 1          | 0          | 0          | 0          |
| $D^5 f(z)$ | 1          | 63         | 301        | 350        | 140        | 21         | 1          | 0          |
| $D^6 f(z)$ | :          | :          | :          | :          | :          | :          | :          | :          |
| $D^7 f(z)$ | 1          | $b_{k,2}$  | $b_{k,3}$  | $b_{k,4}$  | $b_{k,5}$  | $b_{k,6}$  | $b_{k,7}$  | :          |
| $D^8 f(z)$ | 1          | :          | :          | :          | :          | :          | :          | 1          |

\[
D^3 f(z) = z^2 f''(z) + zf'(z),
\]

\[
z + \sum_{n=2}^{\infty} n^2 a_n z^n = z + \sum_{n=2}^{\infty} [n(n-1) + n]a_n z^n,
\]

(16)

\[
D^3 f(z) = z^3 f'''(z) + 3z^2 f''(z) + zf'(z),
\]

\[
z + \sum_{n=2}^{\infty} n^2 a_n z^n = z + \sum_{n=2}^{\infty} [n(n-1)(n-2) + 3n(n-1) + n]a_n z^n,
\]

(17)

\[
D^4 f(z) = z^4 f^{(4)}(z) + 6z^3 f'''(z) + 7z^2 f''(z) + zf'(z),
\]

\[
z + \sum_{n=2}^{\infty} n^4 a_n z^n = z + \sum_{n=2}^{\infty} [n(n-1)(n-2)(n-3) + 6n(n-1)(n-2) + 7n(n-1) + n]a_n z^n.
\]

(18)

\[
D^5 f(z) = z^5 f^{(5)}(z) + 15z^4 f^{(4)}(z) + 25z^3 f'''(z) + 10z^2 f''(z) + zf'(z),
\]

\[
z + \sum_{n=2}^{\infty} n^5 a_n z^n = z + \sum_{n=2}^{\infty} [n(n-1)(n-2)(n-3)(n-4) + 15n(n-1)(n-2)(n-3) + 10n(n-1)(n-2) + 7n(n-1) + n]a_n z^n,
\]

(19)

From (16)–(19), we conclude that

\[
n = (n-1) + 1 = b_{2,1}(n-1) + b_{2,1},
\]

\[
n^2 = (n-1)(n-2) + 3(n-1) + 1 = b_{3,1}(n-1)(n-2) + b_{3,1}(n-1) + b_{3,1},
\]

\[
n^3 = (n-1)(n-2)(n-3) + 6(n-1)(n-2) + 7(n-1) + 1
\]

\[
= b_{4,1}(n-1)(n-2)(n-3) + b_{4,1}(n-1)(n-2) + b_{4,1}(n-1) + b_{4,1},
\]

(20)

\[
n^4 = (n-1)(n-2)(n-3)(n-4) + 10(n-1)(n-2)(n-3) + 25(n-1)(n-2) + 15(n-1) + 1
\]

\[
= b_{5,1}(n-1)(n-2)(n-3)(n-4) + b_{5,1}(n-1)(n-2)(n-3) + b_{5,1}(n-1)(n-2) + b_{5,1}(n-1) + b_{5,1}.
\]

(21)
In general, we have

\[
n^k = b_{k+1,1} + b_{k+1,2} (n-1) + b_{k+1,3} (n-1) (n-2) + \cdots + b_{k+1,k+1} (n-1) (n-2) (n-3) \cdots (n-k)
\]

\[
= b_{k+1,1} + \sum_{j=1}^{k} b_{k+1,j+1} (n-1) (n-2) (n-3) \cdots (n-j), \quad k = 1, 2, 3, \ldots.
\]  

(22)

For functions \( f \in \mathcal{A} \) given by (1) and \( g \in \mathcal{A} \) given by \( g(z) = z + \sum_{n=2}^{\infty} a_n z^n \), we recall that the well-known Hadamard product of \( f \) and \( g \) is given by

\[(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.
\]  

(23)

For \( \epsilon \in \mathbb{C}/[0] \) and \(-1 \leq \mathfrak{D} \leq \mathfrak{C} \leq 1\), we say that a function \( f \in \mathcal{A} \) is in the class \( \mathcal{A}^\epsilon(\mathfrak{C}, \mathfrak{D}) \) if it satisfies the inequality

\[
\Re \left\{ \frac{(1 - \lambda)z(D^k f(z))' + \lambda z(D^{k+1} f(z))'}{(1 - \lambda)D^k f(z) + \lambda D^{k+1} f(z)} \right\} > \alpha, \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),
\]  

(25)

for some \( 0 < \alpha < 1 \), \( \lambda (0 < \lambda < 1) \), and for all \( z \in \mathbb{D} \). Furthermore, we define the class \( P_k(\lambda, \alpha) \) by

\[
P_k(\lambda, \alpha) = \mathcal{G}(\lambda, \alpha) \cap \mathcal{T}.
\]  

(26)

The class \( P_\mathcal{T}(\lambda, \alpha) \) was introduced and studied by Aouf and Srivastava [4].

We note that, by specializing the parameters \( k \) and \( \lambda \), we obtain the following subclasses:

(i) \( P_0(0, \alpha) = \mathcal{T}^*(\alpha) \) and \( P_0(1, \alpha) = \mathcal{C}(\alpha) \), where \( \mathcal{T}^*(\alpha) \) and \( \mathcal{C}(\alpha) \) represent the classes of starlike functions of order \( \alpha \) and convex functions of order \( \alpha \) with negative coefficients, respectively, introduced and studied by Silverman [5].

(ii) \( P_k(1, \alpha) = \mathcal{C}_k(\alpha) \) (see [4]), where \( \mathcal{C}_k(\alpha) \) represents the class of functions \( f(z) \in \mathcal{T} \) satisfying the inequality

\[
\Re \left\{ \frac{z(D^{k+1} f(z))'}{D^{k+1} f(z)} \right\} > \alpha, \quad (k \in \mathbb{N}_0).
\]  

(27)

(iii) \( P_k(0, \alpha) = \mathcal{T}^*_k(\alpha) \) (see [4]), where \( \mathcal{T}^*_k(\alpha) \) represents the class of functions \( f(z) \in \mathcal{T} \) satisfying the inequality

\[
\Re \left\{ \frac{z(D^k f(z))'}{D^k f(z)} \right\} > \alpha, \quad (k \in \mathbb{N}_0).
\]  

(28)

In statistics and probability, distributions of random variables play a basic role and are used extensively to describe and model a lot of real-life phenomenon; they describe the distribution of the probabilities over the values of the random variable. In recent years, many researchers have examined some important features in the geometric function theory, such as coefficient estimates, inclusion relations, and conditions of being in some known classes, using different probability distributions such as the Poisson, Pascal, Borel, Mittag-Leffler-type Poisson distribution, etc. (see, for example, [6–10]).

The probability density function of a discrete random variable \( X \) which follows the Pascal distribution is given by

\[
\text{Prob}(X = r) = \binom{r + m - 1}{m - 1} \sigma^r (1 - \sigma)^m, \quad r = 0, 1, 2, 3, \ldots
\]  

(29)

Very recently, El-Deeb et al. [11] introduced a power series whose coefficients are probabilities of the Pascal distribution

\[
\Lambda^m_\sigma(z) = z + \sum_{n=2}^{\infty} \frac{n + m - 2}{m - 1} \sigma^{n-1} (1 - \sigma)^m z^n, \quad z \in \mathbb{D}
\]  

(30)

where \( m \geq 1 \) and \( 0 \leq \sigma \leq 1 \) and we note that, by a ratio test, the radius of convergence of above series is infinity. We also define the series

\[
\mathcal{Y}^m_\sigma(z) = 2z - \Lambda^m_\sigma(z)
\]

\[
= z - \sum_{n=2}^{\infty} \frac{n + m - 2}{m - 1} \sigma^{n-1} (1 - \sigma)^m z^n, \quad z \in \mathbb{D}.
\]  

(31)

Now, we considered the linear operator
defined by the Hadamard product
\[ \mathcal{D}_\sigma^m(z) = \mathcal{A}_\sigma^m(z) * f(z) = z + \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} (1 - \sigma)^n a_n z^n, \quad z \in \mathbb{D}. \]

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions, generalized Bessel functions, Struve functions, Poisson distribution series, and Pascal distribution series (see, for example, [12], [13–15], [7–9, 16–23], [24]), we determine a necessary and sufficient condition for \( \gamma_{\sigma}^m(z) \) to be in our class \( \mathcal{P}_k(\lambda, \alpha) \). Furthermore, we give sufficient conditions for \( \gamma_{\sigma}^m(\mathcal{R}(\mathcal{G}, \mathcal{D})) \subset \mathcal{P}_k(\lambda, \alpha) \). Finally, we give conditions for the integral operator \( \mathcal{D}_\sigma^m f(z) = \int_0^z (\gamma_{\sigma}^m(t)/t) \, dt \) belonging to the class \( \mathcal{P}_k(\lambda, \alpha) \).

The following results will be required in our investigation.

Lemma 1 (see [4]). Let the function \( f(z) \) be defined by (2). Then, \( f(z) \in \mathcal{P}_k(\lambda, \alpha) \) if and only if
\[ \sum_{n=2}^{\infty} n^k (n - \alpha)(1 + (n - 1)\lambda) |a_n| \leq 1 - \alpha, \quad z \in \mathbb{D}. \]  
(34)

The result (34) is sharp.

Lemma 2 (see [3]). If \( f \in \mathcal{R}(\mathcal{G}, \mathcal{D}) \) is of the form (1), then
\[ |a_n| \leq (\mathcal{G} - \mathcal{D}) \frac{|e|}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \]  
(35)

The result is sharp for the function
\[ f(z) = \int_0^z \left( 1 + (\mathcal{G} - \mathcal{D}) \frac{e t^{n-1}}{1 + \mathcal{D} t^{n-1}} \right) dt, \quad z \in \mathbb{D}; n \in \mathbb{N} \setminus \{1\}. \]  
(36)

2. Necessary and Sufficient Conditions

By simple calculations, we derive the following relations:

\[ \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} = \frac{1}{(1 - \sigma)^m - 1}, \]
\[ \sum_{n=5}^{\infty} (n - 1)(n - 2)(n - 3)(n - 4) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} = \frac{24\sigma^4 (m + 3)}{(m - 1)(1 - \sigma)^{m+4}}, \]
\[ \sum_{n=4}^{\infty} (n - 1)(n - 2)(n - 3) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} = \frac{6\sigma^3 (m + 2)}{(m - 1)(1 - \sigma)^{m+3}}, \]  
(37)
\[ \sum_{n=3}^{\infty} (n - 1)(n - 2) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} = \frac{2\sigma^2 (m + 1)}{(m - 1)(1 - \sigma)^{m+2}}, \]
\[ \sum_{n=2}^{\infty} (n - 1) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} = \frac{\sigma (m)}{(m - 1)(1 - \sigma)^{m+1}}, \]
and, in general, for \( s = 1, 2, 3, \ldots \), we have

\[
\sum_{n=s+1}^{\infty} (n-1)(n-2)(n-3)\cdots(n-s)\left(\frac{n+m-2}{m-1}\right)^{s+1} = s!\sigma^s \frac{(m+s-1)}{(1-\sigma)^{m+s}}. \tag{38}
\]

Unless otherwise mentioned, we shall assume in this paper that \( 0 \leq \alpha < 1, \ 0 \leq \lambda \leq 1 \) \( m \geq 1 \), and \( 0 \leq \sigma < 1 \).

First of all, with the help of Lemma 1, we obtain the following necessary and sufficient condition for \( \Upsilon^m_\sigma(z) \) to be in \( P_k(\lambda, \alpha) \).

**Theorem 1.** Let \( k \geq 1 \). Then, \( \Upsilon^m_\sigma(z) \in P_k(\lambda, \alpha) \) if and only if

\[
\sum_{j=1}^{k} (\lambda b_{k+3,j+1} + (1-\lambda-\alpha) b_{k+2,j+1} + \alpha(\lambda-1) b_{k+1,j+1}) j! \sigma^j \frac{(m+j-1)}{(1-\sigma)^{m+j}} \\
+ (\lambda b_{k+3,k+2} + (1-\lambda-\alpha)) (k+1)! \sigma^{k+1} \frac{(m+k)}{(1-\sigma)^{m+k+1}} \\
+ \lambda (k+2)! \sigma^{k+2} \frac{(m+k+1)}{(1-\sigma)^{m+k+2}} \\
\leq 1-\alpha. \tag{39}
\]

**Proof.** In view of Lemma 1, we only need to show that \( Q \leq 1-\alpha \), where

\[
Q = \sum_{n=2}^{\infty} n^k (n-\alpha)(1+(n-1)\lambda) \left(\frac{n+m-2}{m-1}\right) \sigma^{n-1} (1-\sigma)^m. \tag{40}
\]

Using (22) and (38), we have

\[
Q = \sum_{n=2}^{\infty} \left[ \lambda n^{k+2} + (1-\lambda-\alpha) n^{k+1} - \alpha (1-\lambda) n^k \right] \left(\frac{n+m-2}{m-1}\right) \sigma^{n-1} (1-\sigma)^m \\
= \sum_{n=2}^{\infty} \left[ \lambda \left( b_{k+3,1} + \sum_{j=1}^{k+2} b_{k+3,j+1} (n-1)(n-2)(n-3)\cdots(n-j) \right) \\
+ (1-\lambda-\alpha) \left( b_{k+2,1} + \sum_{j=1}^{k+1} b_{k+2,j+1} (n-1)(n-2)(n-3)\cdots(n-j) \right) \\
+ \alpha(1-\lambda) \left( b_{k+1,1} + \sum_{j=1}^{k} b_{k+1,j+1} (n-1)(n-2)(n-3)\cdots(n-j) \right) \right] 
\]
\[
\times \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\
= \sum_{n=2}^{\infty} \lambda b_{k+3,1} \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\

+ \sum_{n=2}^{\infty} \lambda b_{k+3,k+3} (n - 1)(n - 2)(n - 3) \cdots (n - (k + 2)) \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\

+ \sum_{n=2}^{\infty} \lambda b_{k+3,k+2} (n - 1)(n - 2)(n - 3) \cdots (n - (k + 1)) \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\

+ \sum_{n=2}^{\infty} \left( \sum_{j=1}^{k} \alpha(\lambda - 1)b_{k+1,j+1} (n - 1)(n - 2)(n - 3) \cdots (n - j) \right) \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\

= \left( \lambda b_{k+3,1} + (1 - \lambda - \alpha \lambda)b_{k+2,1} + \alpha(\lambda - 1)b_{k+1,1} \right) \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\

+ \lambda b_{k+3,k+3} \sum_{n=k+2}^{\infty} (n - 1)(n - 2)(n - 3) \cdots (n - (k + 2)) \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\

+ \left( \lambda b_{k+3,k+2} + (1 - \lambda - \alpha \lambda)b_{k+2,k+2} \right) \sum_{n=k+1}^{\infty} (n - 1)(n - 2)(n - 3) \cdots (n - (k + 1)) \\

\times \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \\

+ \left( \sum_{j=1}^{k} \left( \lambda b_{k+3,j+1} + (1 - \lambda - \alpha \lambda)b_{k+2,j+1} + \alpha(\lambda - 1)b_{k+1,j+1} \right) \right) \\

\times \left( \sum_{n=j+1}^{\infty} (n - 1)(n - 2)(n - 3) \cdots (n - j) \left( \frac{n + m - 2}{m - 1} \right)^{\sigma^{n-1}(1 - \sigma)^m} \right)
\]
\( (1 - \alpha)(1 - (1 - \sigma)^m) \)
\[
+ \lambda (k + 2) \sigma \frac{(m + k + 1)}{(1 - \sigma)^{k+2}}
\]
\[
+ \left( \lambda b_{k+3,k+2} + (1 - \alpha \lambda) \right) \frac{(m + k)}{(1 - \sigma)^{k+1}}
\]
\[
+ \sum_{j=1}^{k} \left( \lambda b_{k+3,j+1} + (1 - \alpha \lambda) b_{k+2,j+1} + \alpha (\lambda - 1) b_{k+1,j+1} \right) \frac{(m + j - 1)}{\sigma^j}
\]
\[\Rightarrow \text{(41)}\]

Therefore, we see that the last expression is bounded above by \( 1 - \alpha \) if (39) is satisfied.

\[\blacksquare\]

### 3. Inclusion Properties

Making use of Lemma 2, we will study the action of the Pascal distribution series on the class \( \mathcal{P}_k(\lambda, \alpha) \).

\[
\begin{align*}
&= (\mathcal{C} - \mathcal{D}) [\mathcal{C}] \kappa \\
&\leq 1 - \alpha.
\end{align*}
\]

\[\text{Proof.}\] In view of Lemma 1, it suffices to show that \( L \leq 1 - \alpha \), where

\[\blacksquare\]
\[
L = \sum_{n=2}^{\infty} n^k (n - \alpha)(1 + (n - 1)\lambda) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} (1 - \sigma)^m |a_n|.
\]

Applying Lemma 2, we find from equations (22) and (38) that

\[
L \leq (C - \Omega) \|e\| \left[ \sum_{n=2}^{\infty} \left( \lambda n^{k+1} + (1 - \lambda - \alpha) n^k - \alpha (1 - \lambda) n^{k-1} \right) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} (1 - \sigma)^m \right]
\]

\[
= (C - \Omega) \|e\| (1 - \sigma)^m \left[ \sum_{n=2}^{\infty} \left[ \lambda b_{k+2,1} + \sum_{j=1}^{k+1} b_{k+2,j+1} (n - 1) (n - 2) (n - 3) \cdots (n - j) \right] \right]
\]

\[
+ (1 - \lambda - \alpha) \left[ b_{k+1,1} + \sum_{j=1}^{k} b_{k+1,j+1} (n - 1) (n - 2) (n - 3) \cdots (n - j) \right] \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} \]

\[
= (C - \Omega) \|e\| (1 - \sigma)^m \left[ \sum_{n=2}^{\infty} \left( \lambda b_{k+2,1} + (1 - \lambda - \alpha \lambda) b_{k+1,1} + \alpha (\lambda - 1) b_{k,1} \right) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1} \right]
\]

\[
+ \lambda b_{k+2,k+2} \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \cdots (n - (k + 1)) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1}
\]

\[
+ \lambda b_{k+2,k+1} \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \cdots (n - k) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1}
\]

\[
+ (1 - \lambda - \alpha) b_{k+1,k+1} \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \cdots (n - k) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1}
\]

\[
+ \left( \lambda \sum_{j=1}^{k} b_{k+2,j+1} + (1 - \lambda - \alpha) \sum_{j=1}^{k} b_{k+1,j+1} + \alpha (\lambda - 1) \sum_{j=1}^{k} b_{k,j+1} \right)
\]

\[
\times \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \cdots (n - j) \left( \frac{n + m - 2}{m - 1} \right) \sigma^{n-1}
\]

\[
= (C - \Omega) \|e\| \left( \left( \lambda b_{k+2,1} + (1 - \lambda - \alpha) b_{k+1,1} + \alpha (\lambda - 1) b_{k,1} \right) (1 - (1 - \sigma)^m) \right)
\]

\[
+ \lambda b_{k+2,k+2} (k + 1)! \sigma^{(k+1)} \left( \frac{m + k}{m - 1} \right) \left( \frac{m + k - 1}{1 - \sigma} \right) + \lambda b_{k+2,k+1} (k)! \sigma^k \left( \frac{m + k - 1}{m - 1} \right) \left( \frac{1 - \sigma}{1 - \sigma} \right)^k
\]

\[
+ \lambda \sum_{j=1}^{k} b_{k+2,j+1} (j)! \sigma^j \left( \frac{m - 1}{m - 1} \right) \left( \frac{1 - \sigma}{1 - \sigma} \right)^j + (1 - \lambda - \alpha) b_{k+1,k+1} (k)! \sigma^k \left( \frac{m + k - 1}{m - 1} \right) \left( \frac{1 - \sigma}{1 - \sigma} \right)^k
\]
\[ +(1 - \lambda - \alpha \lambda) \sum_{j=1}^{k-1} b_{k+j+1} (j)! \sigma^j \left( \frac{m + j - 1}{1 - \sigma} \right) + \alpha (\lambda - 1) \sum_{j=1}^{k-1} b_{k+j+1} (j)! \sigma^j \left( \frac{m - 1}{1 - \sigma} \right) \]

\[ = (\mathcal{G} - \mathcal{D}) \left[ \sum_{j=1}^{k-1} [\lambda b_{k+j+1} + (1 - \lambda - \alpha \lambda)b_{k+j+1} + \alpha (\lambda - 1)b_{k+j+1}] (j)! \sigma^j \left( \frac{m + j - 1}{1 - \sigma} \right) + \lambda (k + 1)! \sigma^{(k+1)} \left( \frac{m - 1}{1 - \sigma} \right) + (1 - (1 - \sigma)^m) \right]. \]

However, this last expression is bounded by \(1 - \alpha\), if (42) holds. This completes the proof of Theorem 2. \(\Box\)

4. An Integral Operator

In this section, we consider the integral operator \(\mathcal{S}_\sigma\) defined by

\[ \mathcal{S}_\sigma f (z) = \int_0^z \frac{\gamma_m^\sigma (t)}{t} \, dt. \]  

**Theorem 3.** Let \(k \geq 2\). Then, the integral operator \(\mathcal{S}_\sigma f (z)\) defined by (45) is in the class \(\mathcal{H}_k (\lambda, \alpha)\) if and only if

\[ \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right)^{m+1} (1 - \sigma)^{m+1} \]

or equivalently,

\[ H = \sum_{n=2}^{\infty} (\lambda n^{k+1} + (1 - \alpha \lambda)n^k + (\lambda - 1)n^{k+1}) \left( \frac{n + m - 2}{m - 1} \right)^{m+1} (1 - \sigma)^{m+1}. \]

The remaining part of the proof of Theorem 3 is similar to that of Theorem 2, and so, we omit the details. \(\Box\)

5. Corollaries and Consequences

By specializing the parameter \(\lambda = 1\) in Theorems 1–3, we obtain the following corollaries.
\[
\sum_{j=1}^{k-1} \left( b_{k+2,j+1} - ab_{k+1,j+1} \right) j! \sigma^j \frac{(m + j - 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^j} \\
+ \left( b_{k+2,k+1} - ab_{k+1,k+1} \right) (k)! \sigma^k \frac{(m + k - 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^k} \\
+ b_{k+2,k+2} (k + 1)! \sigma^{(k+1)} \frac{(m + 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^{(k+1)}} \leq 1 - \alpha.
\]

(52)

Corollary 1. Let \( k \geq 1 \). Then, \( Y^m_\sigma (z) \in C_k (\alpha) \) if and only if

\[
\sum_{j=1}^{k} \left( b_{k+3,j+1} - ab_{k+2,j+1} \right) j! \sigma^j \frac{(m + j - 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^{m+j}} \\
+ \left( b_{k+3,k+2} - a \right) (k + 1)! \sigma^{k+1} \frac{(m + k - 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^{k+1}} \\
+ (1 - \sigma) (1 - \sigma)^m \leq 1 - \alpha.
\]

(50)

Corollary 2. Let \( k \geq 2 \) and \( f \in R^e (C, D) \). Then, \( \mathcal{S}^m_\sigma f (z) \in C_k (\alpha) \) if

\[
(\mathcal{C} - \mathcal{D}) [\mathcal{E}] \mathcal{E} \left[ \sum_{j=1}^{k-1} \left( b_{k+2,j+1} - ab_{k+1,j+1} \right) j! \sigma^j \frac{(m + j - 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^j} \\
+ \left( b_{k+2,k+1} - a \right) (k)! \sigma^k \frac{(m + k - 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^k} \\
+ \left( b_{k+2,k+2} - a \right) (k + 1)! \sigma^{(k+1)} \frac{(m + k - 1)}{m - 1} \frac{(m - 1)}{(1 - \sigma)^{(k+1)}} \right] \\
+ (1 - \sigma) (1 - \sigma)^m \leq 1 - \alpha.
\]

(51)

Corollary 3. Let \( k \geq 2 \). Then, the integral operator \( \mathcal{S}^m_\sigma f (z) \) defined by (45) is in the class \( C_k (\alpha) \) if and only if

By specializing the parameter \( k = 2 \) in Theorems 1–3, we obtain the following corollaries.
Corollary 4. The series $\sum_{m=0}^{\infty} (\frac{m}{m+1})^{m-1} + 2(19\lambda - 5\alpha + \alpha + 6) + 6(9\lambda - \alpha + 1)\sigma^2 + 24\lambda\sigma^4 \leq 1 - \alpha$. 

Corollary 5. Let $f \in \mathcal{R}^c (\mathcal{C}, \mathcal{D})$. Then, $\mathcal{J} f(z) \in \mathbb{P}_2 (\lambda, \alpha)$ if and only if

\[
(4\lambda - 2\alpha + 3 - \alpha)\sigma \left( \frac{m}{m-1} \right) \quad \text{and} \quad 1 - (1 - \sigma)^m \leq 1 - \alpha.
\]

Corollary 6. The integral operator $\mathcal{G} f(z)$ defined by (45) is in the class $\mathbb{P}_2 (\lambda, \alpha)$ if and only if

\[
(4\lambda - 2\alpha + 3 - \alpha)\sigma \left( \frac{m}{m-1} \right) \quad \text{and} \quad 1 - (1 - \sigma)^m \leq 1 - \alpha.
\]

Remark 1. Using relation (22) and Lemma 1, we can obtain new necessary and sufficient conditions and inclusion relations for the Pascal distribution series to be in the class $\mathbb{P}_k (\lambda, \alpha)$ for $k = 3, 4, \ldots$. 

6. Conclusions

The Sălăgean differential operator plays an important role in the geometric function theory. Several authors have used this operator to define and consider the properties of certain known and new classes of analytic univalent functions (see, for example, [25, 26]). In the present paper, and due to the earlier works (see, for example, [11, 16, 18]), we find a necessary and sufficient condition and inclusion relation for the Pascal distribution series to be in the class $\mathbb{P}_k (\lambda, \alpha)$ of analytic functions associated with the Stirling numbers and Sălăgean differential operator. Furthermore, we consider an integral operator related to the Pascal distribution series. Some interesting corollaries and applications of the results are also discussed. Making use of the relation (22) could inspire researchers to find new necessary and sufficient conditions and inclusion relations for the Pascal distribution series to be in different classes of analytic functions with negative coefficients defined by the Sălăgean differential operator.

Data Availability

No data were used to support this study.
Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors’ Contributions

The author read and approved the final manuscript.

References

[1] G. S. Salagean, “Subclasses of univalent functions,” Complex Analysis—Fifth Romanian–Finnish Seminar, vol. 1013, pp. 362–372, 1983.
[2] J. Quaintance and H. W. Gould, Combinatorial Identities for Stirling Numbers (The Unpublished Notes of H.W. Gould), World Scientific Publishing Co., Singapore, 2015.
[3] K. K. Dixit and S. K. Pal, “On a class of univalent functions related to complex order,” Indian Journal of Pure and Applied Mathematics, vol. 26, no. 9, pp. 889–896, 1995.
[4] M. K. Aouf and H. M. Srivastava, “Some families of starlike functions with negative coefficients,” Journal of Mathematical Analysis and Applications, vol. 203, no. 3, pp. 762–790, 1996.
[5] H. Silverman, “Univalent functions with negative coefficients,” Proceedings of the American Mathematical Society, vol. 31, no. 1, pp. 109–116, 1975.
[6] A. Amourah, B. A. Frasin, M. Ahmad, and F. Yousef, “Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions,” Symmetry, vol. 14, no. 1, p. 147, 2022.
[7] B. A. Frasin, G. Murugusundaramoorthy, and M. K. Aouf, “Subclasses of analytic functions associated with Mittag-Leffler-type Poisson distribution,” Palest. J. Math. vol. 11, no. 1, pp. 496–503, 2022.
[8] M. Ghaffar Khan, B. Ahmad, N. Khan et al., “Applications of mittag-leffler type Poisson distribution to a subclass of analytic functions involving conic-type regions,” Journal of Function Spaces, vol. 2021, Article ID 4343163, 9 pages, 2021.
[9] N. Magesh, S. Porwal, and C. Abirami, “Starlike and convex properties for Poisson distribution series,” Stud. Univ. Babeș-Bolyai Math. vol. 63, no. 1, pp. 71–78, 2018.
[10] A. K. Wanas and N. A. J. Al-Ziadi, “Applications of beta negative binomial distribution series on holomorphic functions,” Earthline Journal of Mathematical Sciences, vol. 6, no. 2, pp. 271–292, 2021.
[11] S. M. El-Deeb, T. Bulboaca, and J. Dziok, “Pascal distribution series connected with certain subclasses of univalent functions,” Kyungpook Mathematical Journal, vol. 59, pp. 301–314, 2019.
[12] M. K. Aouf, A. O. Mostafa, and H. M. Zayed, “Some constraints of hypergeometric functions to belong to certain subclasses of analytic functions,” Journal of the Egyptian Mathematical Society, vol. 24, no. 3, pp. 361–366, 2016.
[13] T. Bulboaca and G. Murugusundaramoorthy, “Univalent functions with positive coefficients involving Pascal distribution series, Commun,” Korean Math. Soc. vol. 35, no. 3, pp. 867–877, 2020.
[14] N. E. Cho, S. Y. Woo, and S. Owa, “Uniform convexity properties for hypergeometric functions,” Fractional Calculus and Applied Analysis, vol. 5, no. 3, pp. 303–313, 2002.
[15] H. E. Darwish, A. Y. Lashin, and E. M. Madar, “On subclasses of uniformly starlike and convex functions defined by Struve functions,” International Journal of Open Problems in Complex Analysis, vol. 8, no. 1, pp. 34–43, 2016.
[16] B. A. Frasin, “Subclasses of analytic functions associated with Pascal distribution series,” Advances in the Theory of Nonlinear Analysis and its Application, vol. 4, no. 2, pp. 92–99, 2020.
[17] B. A. Frasin and M. M. Gharbaibeh, “Subclass of analytic functions associated with Poisson distribution series,” Afrika Matematika, vol. 31, no. 7-8, pp. 1167–1173, 2020.
[18] B. A. Frasin, S. R. Swamy, and A. K. Wanas, “Subclasses of starlike and convex functions associated with Pascal distribution series,” Kyungpook Mathematical Journal, vol. 61, pp. 99–110, 2021.
[19] B. A. Frasin, T. Al-Hawary, and F. Yousef, “Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions,” Afrika Matematika, vol. 30, no. 1-2, pp. 223–230, 2019.
[20] T. Janani and G. Murugusundaramoorthy, “Inclusion results on subclasses of starlike and convex functions associated with Struve functions,” Italian Journal of Pure and Applied Mathematics, vol. 32, pp. 467–476, 2014.
[21] E. P. Merkes and W. T. Scott, “Starlike hypergeometric functions,” Proceedings of the American Mathematical Society, vol. 12, no. 6, pp. 885–888, 1961.
[22] A. O. Mostafa, “A study on starlike and convex properties for hypergeometric functions,” Journal of Inequalities in Pure and Applied Mathematics, vol. 10, no. 3, pp. 1–16, 2009.
[23] S. Porwal, “An application of a Poisson distribution series on certain analytic functions,” Journal of Complex Analysis, vol. 2014, Article ID 984135, 3 pages, 2014.
[24] H. Silverman, “Starlike and convexity properties for hypergeometric functions,” Journal of Mathematical Analysis and Applications, vol. 172, no. 2, pp. 574–581, 1993.
[25] M. K. Aouf, R. M. El-Ashwah, A. A. M. Hassan, and A. H. Hassan, “On subordination results for certain new classes of analytic functions defined by using Sălăgean operator,” Bulletin of Mathematical Analysis and Applications, vol. 4, no. 1, pp. 239–246, 2012.
[26] R. M. El-Ashwah, “Subordination results for certain subclass of analytic functions defined by Sălăgean operator,” Acta Universitatis Apulensis, vol. 37, pp. 197–204, 2014.