I. INTRODUCTION

In quantum mechanics the state can represent a quantum system, and the improvement of the state and its determination has vital importance in obtaining any information about that system. Because of the collapse of wave function due to the decoherence, the conventional quantum measurements cannot be directly used in some hot topics of quantum information science such as quantum state based high precision measurements, reconstruction of unknown quantum state, etc. However, the advance of the research in fundamentals of quantum physics provided an effective method to solve the above problems by using the simple and easily manipulable pre- and post-selected quantum weak measurement technique which is characterized by the weak value [1]. In the weak measurement the induced weak value of the observable on the measured system is usually a complex number, and can be beyond the usual range of eigenvalues of that observable. This property of weak value is referred as the amplification effect for weak signal which accompanied by the decrease of the postselection probability. Since the weak signal amplification property experimentally demonstrated in 1991 [2], it have been widely used and solved plenty of fundamental problems in quantum mechanics and related sciences. For details about the weak measurement theory and its applications in weak signal amplification processes, we refer the reader to the recent overview of the field [3, 4].

Another main application of postselected weak measurement technique is quantum state tomography. The significant advantages of postselected weak measurement based state tomography technique than conventional one [5–9] is that in weak measurement technique the tomographic procedures is easy and can get the all global phase information of unknown state than conventional schemes. Since J. Lundeen et al. [10] firstly investigated the reconstruction of transversal spatial wave function of polarized photon beams by using the post-selected weak measurement technique, the direct measurement of unknown quantum states have been studied theoretically and experimentally by using weak and strong measurement techniques [11–29]. In particular, the direct measurement of a photon polarization state in two dimensional system [14] and direct measurement of density matrix of a single photon polarization state in pure and mixed state cases [26] showed the power of weak measurement technique in state determination processes.

Quantum entanglement is a main feature of quantum mechanics, and most of the mysterious phenomena in quantum world caused by entangled systems. Thus, the state determination of entangled systems have significant importance in quantum theory. The direct measurement of general quantum state by using weak measurement has been studied in Refs. [11, 15]. Furthermore, in recent innovative work of Guo-Guang Can et al. [30], they investigated the direct measurement of a two photon entangled state by using postselected weak measurement and used the modular value in reading results in stead of weak value. However, in general, the state of a quantum system is represented by density operator, and the direct measurement of density matrix of an entangled system by using weak measurement technique have not been explicitly studied until now.

In this paper, as an extension of previous works [26, 30], we study the two kinds of reconstruction methods of a two photon entangled state. We take the spatial (paths) and polarization degrees of freedom of unknown entangled state as pointer and measured system, respectively, and the joint (or sequential ) projection operators of two subsystems considered as measured observables of measured system. In first method, we follow the theoretical part of Ref. [30] and use the postselected weak measurement technique to measure the matrix elements of a two photon entangled state. It is noticed that the density matrix elements proportional to the weak values of appropriate joint projection operators of two subsystems, and the pre- and post-selected states are the elements in two mutually unbiased bases. Since the weak measurement of joint projection operators of two subsystems can not be measured directly, it is founded in terms of the modular values of corresponding operators. Based on the theoretical analysis of Ref. [30], the real and imaginary parts of a matrix element can be readout from detection probability after taking appropriate pro-
projection operations before detection on the final state of the pointer.

In the second method, the technique introduced by J. Ludleen et al. \[1\] is used. Three sequential measurements on three projection operators of two subsystems where each complementary to the last are taken to find the matrix elements of a two photon entangled state. It is found that the result of these sequential measurements proportional to the value of matrix elements. In order to read out the value of matrix elements, it is assumed that the spatial degree of freedom of every pointer of two subsystems have \(x\) and \(y\) directional zero mean Gaussian distribution, and initially there have no any correlation between them. After taking the two sequential weak measurements with projection operators where complementary to the last, the weak average equal to the expectation values of products of annihilation operators (can be defined in terms of position and momentum operator) of four Gaussian pointer states. Thus, the real and imaginary parts of corresponding matrix elements can be found by calculating the joint positions and momentum shifts of the final pointer state. Here, we have to mention that previous two sequential weak measurements caused a spatial shifts on different directions of the pointer, respectively.

The rest of the paper is organized as follows: we briefly review the basic concepts of direct measurement of a quantum state by using postselected weak measurement based state tomography technique in Section. \(\text{II}\). In Section. \(\text{III}\), we give the details of two methods to determine the matrix elements of a two photon entangled system, separately, and take comparison between them and discuss their feasibility. We give the conclusion to our study in Section. \(\text{IV}\).

\section{Direct Measurement of a State Via Weak Measurement}

From the quantum mechanics we know that the two dimensional photon polarization state \(|\psi\rangle\) in Hilbert space can be expressed in the \(A = \{|H\}, |V\} \) basis as

\[
|\psi\rangle = \sum_i c_i |i\rangle, \quad i \in \{H, V\}
\]  

(1)

where \(c_i = \langle i |\psi\rangle\) is the probability amplitude. The weak value of projection operator \(\pi_i = |i\rangle \langle i|\) with the preselected and postselected states, \(|\psi\rangle\) and \(|\alpha\rangle\), is defined as

\[
\langle \pi_i \rangle^w_{\alpha} = \frac{\langle \alpha | \pi_i | \psi \rangle}{\langle \alpha | \psi \rangle} = \frac{1}{\nu} c_i.
\]  

(2)

Thus, it is evident that the probability amplitude \(c_i\) of unknown state \(|\psi\rangle\) is directly related with the weak value of projection operator \(\pi_i\), and the state vector \(|\psi\rangle\) can be re-expressed as

\[
|\psi\rangle = \sum_i \nu \langle \pi_i \rangle^w_{\alpha} |i\rangle, \quad \alpha \in (D, A)
\]  

(3)

Here, \(\alpha = D, H\) is the element in \(B = \{|D\} = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), |A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)\) diagonal and anti-diagonal basis, and \(\nu = \frac{\langle \alpha | \psi \rangle}{\langle \alpha | \psi \rangle}\) is independent of \(i\) and can be determined by the normalization condition. Since the real and imaginary parts of weak value \(\langle \pi_i \rangle^w_{\alpha}\) can be found simultaneously\[31\], the unknown state vector \(|\psi\rangle\) can be reconstructed by optical experiments. The most important part of this reconstruction technique is the choice of post-selection, and from the Eq.\(3\) we can know that to determine the unknown pure state vector \(|\psi\rangle\), we can scan only on definite \(|\alpha\rangle\) in \(B\) basis at postselection process. However, if we want to reconstruct the density matrix of a two dimensional unknown state by using weak measurement technique, we have to take scan through all elements in both \(A\) and \(B\) bases since its unknown parameters more than the corresponding pure state. The reconstruction of density matrix of two dimensional system had been studied experimentally in Refs.\[14, 26\]. We have to mention that the two bases \(A\) and \(B\) are mutually unbiased for all basis \(|i\rangle\) in \(A\) and all basis \(|\alpha\rangle\) in \(B\) in two dimensional Hilbert space, i.e. \(|\langle i |\alpha\rangle|^2 = \frac{1}{2}\).

\section{The Methods of Direct Measurement of Density Operator of an Entangled Quantum State}

Let us consider a system consisting of two subsystems, and designate the corresponding state vector as

\[
|\Psi\rangle = \sum_{ij} C_{ij} |i\rangle_1 \otimes |j\rangle_2 = \sum_{i,j} C_{ij} |ij\rangle,
\]  

(4)

where \(i, j \in \{H, V\}\), and \(C_{ij} = \langle ij |\Psi\rangle\) is complex probability amplitude and \(|i\rangle_1\) and \(|j\rangle_2\) represent to subsystem one and subsystem two, respectively. To reconstruct the unknown pure state \(|\Psi\rangle\), we have to find the corresponding amplitudes \(C_{ij}\) and this task is not very easy as single two dimensional pure state case. However, recently the Guo-Guang Can et al.\[30\] successfully accomplished this task by using modular value in stead of weak value of joint projection operators of two subsystems. In general, the state of a quantum system is characterized by density matrix and up to now the determination of density matrix of a two photon entangled state has not been investigated explicitly yet. The matrix elements of an entangled state described by \(\rho\) in \(A' = \{|HH\}, |HV\}, |VH\}, |VV\}\)
basis of two subsystems is given by

\[
\rho = |\Psi\rangle\langle\Psi| = \sum_{j,k} C_{ij} C_{kl}^* |ij\rangle\langle kl| = \sum_{i,j,k,l} \rho_{ij,kl} |ij\rangle\langle kl|
\]

\[
= \begin{pmatrix}
\rho_{HH,HH} & \rho_{HH,HV} & \rho_{HV,HH} & \rho_{HV,HH} \\
\rho_{HH,HV} & \rho_{HH,VV} & \rho_{HV,HV} & \rho_{HV,VV} \\
\rho_{HV,HH} & \rho_{HV,VV} & \rho_{VV,HH} & \rho_{VV,VV} \\
\rho_{HH,VV} & \rho_{HV,VV} & \rho_{VV,HH} & \rho_{VV,VV}
\end{pmatrix},
\]

(5)

where \(\rho_{ij,kl} = \langle ij | \rho | kl\rangle\) is matrix element of \(\rho\) and a complex number, and \(i,j,k,l \in (H,V)\). Thus, to find the complex matrix elements of an entangled state \(\rho\), we have to find the real and imaginary parts of each elements, \(\rho_{ij,kl}\), respectively. Next we will study this problem with two different methods.

A. Method one: Based on modular value scheme

As mentioned in Section II, the weak value of projection operator \(\pi_\alpha = |i\rangle\langle i|\) under the density operator \(\rho\) with postselected state \(|\alpha\rangle\) is defined as

\[
\langle \pi_\alpha \rangle^w = \frac{\langle \alpha | \pi_i | \rho | \alpha \rangle}{\langle \alpha | \rho | \alpha \rangle}.
\]

(6)

Furthermore, if we want to measure the joint projection operators of two subsystems, \(\pi_1^2\pi_2^2 = |i\rangle\langle i|\), \(\pi_1^2 = |j\rangle\langle j|\), and \(\pi_2^2 = |\bar{j}\rangle\langle \bar{j}|\) are the projection operators of subsystem one and two, then the corresponding weak value of \(\pi_1^2\pi_2^2\) under the density operator \(\rho\) with postselected state \(|\alpha\beta\rangle = |\alpha\rangle^1|\beta\rangle^2\) can be written as

\[
\langle \pi_1^2\pi_2^2 \rangle^w = \frac{\langle \alpha \beta | \pi_1^2\pi_2^2 | \rho | \alpha \beta \rangle}{\langle \alpha \beta | \rho | \alpha \beta \rangle}.
\]

(7)

Here, \(i,j,k,l \in (V,H)\) in \(A^\prime\) basis and \(\alpha, \beta \in (D,A)\) in \(B^\prime = \{DD,DA,AD,AA\}\) basis, respectively.

By using the definition of weak value of joint operators, every matrix element of \(\rho\) which is written in Eq.(5) can be expressed in terms of the weak value of joint project operator \(\pi_1^2\pi_2^2\) in \(A^\prime\) basis as

\[
\rho_{ij,kl} = \langle ij | \rho | kl\rangle = \sum_{\alpha\beta} \rho_{\alpha\beta} \langle \alpha \beta | kl \rangle \langle \pi_1^2\pi_2^2 \rangle^w_{\alpha\beta},
\]

(8)

where \(\rho_{\alpha\beta} = \langle \alpha \beta | \rho | \alpha \beta \rangle\) is the probability to find the system in postselected state \(|\alpha\beta\rangle\), \(\langle \pi_1^2\pi_2^2 \rangle^w_{\alpha\beta}\) is independent to the above summation and can be determined by using normalization condition. Thus, if we take weak measurement on joint projection operators \(\pi_1^2\pi_2^2\) in \(A^\prime\) basis following take strong measurement on all elements in \(B^\prime\) basis of both subsystems, respectively, then get the value of every complex elements of density matrix \(\rho\). Furthermore, we can define the density matrix \(\rho\) of an entangled state in \(B^\prime\) as well, and the expressions of its matrix elements can be written as

\[
\rho_{\alpha\beta,\alpha'\beta'} = \langle \beta \alpha | \rho | \alpha'\beta' \rangle = \sum_{ij} \rho_{\alpha\beta} \langle \beta \alpha | \pi_i^{\alpha} \pi_j^{\beta} \rangle \langle \pi_i^{\alpha} \pi_j^{\beta} \rangle^w_{\alpha'\beta'},
\]

(9)

where \(\alpha' \beta'\) also belong to the \(B^\prime\) basis too, i.e., \(\alpha', \beta' \in (D,A)\), and \(\rho_{\alpha'\beta'} = \langle \beta' \alpha' | \rho | \alpha'\beta' \rangle\) is the probability of success for postselection of \(|\alpha'\beta'\rangle\) basis. As shown in Eq. (8) and Eq. (9), to get the matrix elements \(\rho_{ij,kl}\) \((\rho_{\alpha\beta,\alpha'\beta'})\) we should find the weak values \(\langle \pi_1^2\pi_2^2 \rangle^w_{\alpha\beta}\) with consider all elements in bases \(B^\prime(A')\), respectively.

In postselected weak measurement technique, the weak value of nonlocal joint operators can not be obtained exactly and the efficiency is too low for entangled state case[32]. However, in recent study of Guang-Can- Guo [30], they showed that the joint weak values \(\langle \pi_1^2\pi_2^2 \rangle^w_{\alpha\beta}\) can be found in terms of modular values. In remaining part of this subsection, we will calculate the weak values \(\langle \pi_1^2\pi_2^2 \rangle^w_{\alpha\beta}\) to find the matrix elements of \(\rho\).

The modular value of an observable \(\hat{F}\) with pre- and postselected states, \(|\psi_{in}\rangle\) and \(|\psi_{fj}\rangle\) can be written as[33]

\[
\langle \hat{F} \rangle^m_{\psi_{fj}} = \frac{\langle \psi_{fj} | e^{-ig\hat{F}} | \psi_{in}\rangle}{\langle \psi_{fj} | \psi_{in}\rangle},
\]

(10)

where \(g\) is represent the coupling strength between measured system and pointer, and the modular value is valid for any weak and strong coupling cases. If we take \(\hat{F} = \hat{\pi} = |i\rangle\langle i|\) is a projection operator in two dimensional Hilbert space, then

\[
\langle \pi \rangle^m_{\psi_{fj}} = \frac{\langle \psi_{fj} | (\sum_{i} e^{-ig\lambda_i} \pi_i) | \psi_{in}\rangle}{\langle \psi_{fj} | \psi_{in}\rangle} = \frac{\langle \psi_{fj} | (1 - \pi_i + e^{-ig\lambda_i}) | \psi_{in}\rangle}{\langle \psi_{fj} | \psi_{in}\rangle} = 1 + (e^{-ig} - 1) \frac{\langle \psi_{fj} | \pi_i | \psi_{in}\rangle}{\langle \psi_{fj} | \psi_{in}\rangle} = 1 + s \langle \pi_i \rangle^w_{\psi_{fj}},
\]

(11)

where \(\lambda_i = 0, 1\) are eigenvalues of projection operator \(\pi_i\), and \(s = e^{-ig} - 1\).

Furthermore, if we extend our concern to an entangled state composed of two subsystems, i.e., consider \(|\Psi\rangle\langle \Psi|\) as preselection state of the system, then the modular value of projection operators \(\pi_1^2 + \pi_2^2 = |i\rangle_1\langle i| + |j\rangle_2\langle j|\) of total system with postselected state

\[
\rho_{ij,kl} = \langle ij | \rho | kl\rangle = \sum_{\alpha\beta} \rho_{\alpha\beta} \langle \alpha \beta | kl \rangle \langle \pi_1^2\pi_2^2 \rangle^w_{\alpha\beta},
\]

(8)
\[ |\alpha\beta\rangle \text{ can be calculated as} \]
\[
\langle \pi_i^1 + \pi_j^2 \rangle_{\alpha\beta} = \frac{\langle \beta \alpha | e^{-ig(\pi_i^1 + \pi_j^2)} | \Psi \rangle}{\langle \alpha \beta | \Psi \rangle}
= \frac{\langle \beta \alpha | e^{-ig\pi_i^1} e^{-ig\pi_j^2} | \Psi \rangle}{\langle \alpha \beta | \Psi \rangle}
= \frac{\langle \beta \alpha | (1 + s\pi_i^1)(1 + s\pi_j^2) | \Psi \rangle}{\langle \alpha \beta | \Psi \rangle}
= \frac{\langle \beta \alpha | (1 + s\pi_i^1)(1 + s\pi_j^2) | \Psi \rangle}{\langle \alpha \beta | \Psi \rangle}
= 1 + s\langle \sigma_i^w \rangle_{\alpha\beta} + s\langle \sigma_j^w \rangle_{\alpha\beta} + s^2\langle \sigma_i^m \rangle_{\alpha\beta} + s^2\langle \sigma_j^m \rangle_{\alpha\beta} + 1.
\]

From this theoretical result we can deduce that if we can measure the modular values of projection operators \(\pi_i^1, \pi_j^2\) and \(\pi_i^1 + \pi_j^2\), respectively, the weak value of joint operators \(\pi_i^1, \pi_j^2\) can be found easily, after that we can determine the matrix elements \(\rho_{ijkl}\) and \(\rho_{\alpha\beta,\alpha'\beta'}\) of density matrix \(\rho\) by using Eq. (8) and Eq. (9).

As studied in Ref. [30], after a two photon entangled state generated by nonlinear optical devices, during the propagation in space (interferometer for example) the two photons entangled in their polarization degrees of freedom and paths degrees of freedom, respectively. We take the paths degrees of freedom (|↑⟩ and |↓⟩) as pointer, and polarization degrees of freedom (|H⟩ and |V⟩) take as measured system, respectively. Suppose that initially both paths and polarization degrees of two subsystems are entangled but there is no any entanglement between these two degrees of freedoms. Thus, the initial state of the total system can be expressed as
\[
|\Psi_{ms}\rangle = |\varphi\rangle \otimes |\Psi\rangle,
\]
where, |\Psi\rangle is given in Eq. (4) and
\[
|\varphi\rangle = \mu |\uparrow\rangle + \eta |\downarrow\rangle, \quad |\mu|^2 + |\eta|^2 = 1
\]
is correspond to paths degree of freedom of two component systems. Here, |↑⟩ and |↓⟩ represent the first photon in the ↑ path and second photon in ↓ path, respectively. To get modular value of \(\pi_i^1, \pi_j^2\) and \(\pi_i^1 + \pi_j^2\), in Ref. [30] they introduced the three interaction Hamiltonians between two composed pointer state and measured system as
\[
H_1 = g\delta (t - t_0)(\pi_i^1\pi_j^2 = \pi_i^2\pi_j^2),
\]
\[
H_2 = g\delta (t - t_0)\pi_i^1\pi_j^1,
\]
\[
H_3 = g\delta (t - t_0)\pi_i^2\pi_j^2.
\]

Here, \(\pi_1^1 = |\downarrow\rangle \langle \downarrow|\) and \(\pi_1^2 = |\uparrow\rangle \langle \uparrow|\) are represent the projection operators of paths degree of freedom of two subsystems, respectively.

If we consider the intrinsic properties of projection operators of paths degrees of freedom, the evolution operators corresponding to above interaction Hamiltonians becomes as
\[
U_1 = \exp\left[-\frac{i}{\hbar} \int H dt\right] = e^{-ig(\pi_1^1 + \pi_2^2)}
\]
\[
= [1 + (e^{-ig\pi_1^1} - 1]\pi_1^1][1 + (e^{-ig\pi_2^2} - 1]\pi_2^2]
\]
\[
= (e^{-ig\pi_1^1} - 1)\pi_1^1 + (e^{-ig\pi_2^2} - 1)\pi_2^2 + 1
\]
\[
+ [e^{-ig(\pi_1^1 + \pi_2^2)} + 1 - e^{-ig\pi_1^1} - e^{-ig\pi_2^2}]\pi_1^1\pi_2^2,
\]
\[
U_2 = e^{-ig\pi_1^1}\pi_2^2 = 1 + (e^{-ig\pi_1^1} - 1)\pi_2^2,
\]
\[
U_3 = e^{-ig\pi_2^2}\pi_1^1 = 1 + (e^{-ig\pi_2^2} - 1)\pi_1^1,
\]

respectively. In above calculations we use the formula \(e^{\theta\hat{F}} = \sum_n e^{\theta\lambda_n} |\phi_n\rangle \langle \phi_n|\) of operator \(\hat{F}\) with \(\hat{F}|\phi_n\rangle = \lambda_n|\phi_n\rangle\).

Start from the initial state of the total system, Eq. (14), and take the above time evolution operators (see Eqs. (19a-19c)) and post-selection onto \(|\alpha\beta\rangle\) in basis \(\mathcal{B}'\) into account, the final states of the pointers can be obtained as
\[
|\Phi_1\rangle = \mathcal{N}_1 [\eta(\pi_1^1 + \pi_2^2_{\alpha\beta}) | \downarrow \rangle + \mu |\uparrow\rangle],
\]
\[
|\Phi_2\rangle = \mathcal{N}_2 [\eta(\pi_2^2_{\alpha\beta}) | \downarrow \rangle + \mu |\uparrow\rangle],
\]

and
\[
|\Phi_2\rangle = \mathcal{N}_3 [\pi_1^2_{\alpha\beta} | \downarrow \rangle + \mu |\uparrow\rangle],
\]

where \(\mathcal{N}_1 = |\mu|^2 + |\eta(\pi_1^1 + \pi_2^2_{\alpha\beta})|^2 - \frac{1}{2}\mathcal{N}_1 = |\mu|^2 + |\eta(\pi_1^1 + \pi_2^2_{\alpha\beta})|^2 - \frac{1}{2}\mathcal{N}_3 = \mathcal{N}_2 = \mathcal{N}_3 = |\mu|^2 + |\eta(\pi_2^2_{\alpha\beta})|^2 - \frac{1}{2}\) are normalization coefficients, respectively.

If we project these above final states of the pointer onto \(|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)\) and \(|\varphi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)\), respectively, the probabilities to find the final states \(|\Phi_1\rangle, |\Phi_2\rangle\) and \(|\Phi_3\rangle\) on \(|\varphi_1\rangle\) and
in the Lab, then we can find the modular values $P_1 = |\langle \varphi_1 | \Phi_1 \rangle|^2$ (23a) and $P_2 = |\langle \varphi_2 | \Phi_2 \rangle|^2$ (23b) respectively. With these processes finally we can determine the matrix elements of density operator by using Eq.(8) and Eq.(9), respectively.

In Ref.[30], they investigated the direct measurement method of a pure two photon polarization entangled state theoretically and experimentally. In their work to get the complex amplitude $C_{ij}$ in Eq.(2) we only need to scan a definite element of $B'$ basis, i.e.

$$C_{ij} = \chi(\pi_i^j \pi_j^w)_{DD}$$ (27)

where $\chi = \frac{\langle DD | \Psi \rangle}{\langle DD | D \rangle}$ is independent of $|ij\rangle$ and can be obtained by normalization condition. However, to reconstruct the density matrix of two component entangled state in $A'$ basis, we need more strong measurement steps in $B'$ basis to determine every matrix elements.

and $P_3 = |\langle \varphi_3 | \Phi_3 \rangle|^2 = \frac{|\varphi_3|^2}{2} = \frac{|\varphi_2|^2}{2} = \frac{|\varphi_1|^2}{2}$ (23j)

respectively. If we assume that initially the probability of first photon in path $\downarrow$ and second photon in path $\uparrow$ is smaller than the probability of first photon in path $\uparrow$ and second photon in path $\downarrow$, i.e., $|\eta|^2 \ll 1$, then

$$P_1 = |\langle \varphi_1 | \Phi_1 \rangle|^2$$

and $P_2 = |\langle \varphi_2 | \Phi_2 \rangle|^2$ (23c)

$$P_3 = |\langle \varphi_3 | \Phi_3 \rangle|^2 = \frac{|\varphi_3|^2}{2} = \frac{|\varphi_2|^2}{2} = \frac{|\varphi_1|^2}{2} = \frac{|\varphi_1|^2}{2}$$ (23j)

respectively. If we assume that initially the probability of first photon in path $\downarrow$ and second photon in path $\uparrow$ is smaller than the probability of first photon in path $\uparrow$ and second photon in path $\downarrow$, i.e., $|\eta|^2 \ll 1$, then

$$P_1 = \mu |\langle \varphi_1 | \Phi_1 \rangle|^2 + \frac{1}{2} \mu^2$$ (24a)

$$P_2 = \mu |\langle \varphi_2 | \Phi_2 \rangle|^2 + \frac{1}{2} \mu^2$$ (24b)

$$P_3 = \mu |\langle \varphi_3 | \Phi_3 \rangle|^2 + \frac{1}{2} \mu^2$$ (24c)

$$P_4 = \mu |\langle \varphi_4 | \Phi_4 \rangle|^2 + \frac{1}{2} \mu^2$$ (24d)

$$P_5 = \mu |\langle \varphi_5 | \Phi_5 \rangle|^2 + \frac{1}{2} \mu^2$$ (24e)

$$P_6 = \mu |\langle \varphi_6 | \Phi_6 \rangle|^2 + \frac{1}{2} \mu^2$$ (24f)

The probabilities $P_1$ and $P_2$ can be determine by the detectors in the Lab, then we can find the modular values $\pi_i^1, \pi_i^2$ and $\pi_j^1, \pi_j^2$. Finally, the real and imaginary parts of weak value of $\pi_i^1 \pi_j^2$ (Eq.(13)) can be written as

$$\Re[\langle \varphi_i^1 | \varphi_j^2 \rangle] = s^{-2} \eta^{-1} |P_1 - P_3 - P_5 + \eta + \frac{1}{2}|$$ (25)

$$\Im[\langle \varphi_i^1 | \varphi_j^2 \rangle] = \eta^{-1} |P_2 - P_4 - P_6 + \eta + \frac{1}{2}|$$ (26)

respectively. With these processes finally we can determine the matrix elements of density operator by using Eq.(8) and Eq.(9), respectively.

B. Method Two: Based on three sequential measurements [11] scheme

As Lundeen and his co-workers [11] studied, the matrix elements of a quantum system can be obtained by considering the weak measurement of an observable composed of three incompatible projection operator:

$$\Pi_{ij} = \pi_j \pi_D \pi_i.$$ (31)
Here, $\pi_i = |i\rangle\langle i|, \pi_j = |j\rangle\langle j|$ with $i, j \in (H, V)$ in $A$ basis, and $\pi_D = |D\rangle\langle D|$ where $|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ is element in $B$ basis. The basis vectors in $A$ and $B$ are maximally incompatible, and $\langle i|D\rangle = \langle j|D\rangle = \frac{1}{\sqrt{2}}$. The matrix elements $\rho_{ij}$ of unknown density operator $\rho$ can be found as

$$\rho_{ij} = 2\langle \Pi_{ij} \rangle_s = 2\text{Tr}_s[\pi_j \pi_D \pi_i \rho] \quad (32)$$

Since $\Pi_{ij}$ is non-Hermitian, generally the weak average $\langle \Pi_{ij} \rangle_s$ is a complex number. Thus, according to the Eq.(32) we can get the complex density matrix elements of $\rho$ if one can find the $\langle \Pi_{ij} \rangle_s$. In the recent work of Lundeen and his co-workers, they investigated their proposal which introduced in Ref.[11], and experimentally reconstruct the density matrix elements of pure and mixed states of 2-dimensional system[26]. In this study as expansion of their work[26], we will study how to determine the matrix elements of two photon entangled state.

For an entangled state composed of two subsystems, the observable defined in Eq.(31), can be redefined as

$$\Pi_{ij,kl} = \pi_{kl} \pi_{\alpha\beta} \pi_{ijkl} \quad (33)$$

where $\pi_{ijkl} = \pi_k \pi_l \pi_{\alpha\beta} \pi_{ijkl}$, $\pi_k = \pi_k \pi_l \pi_{\alpha\beta} \pi_{ijkl}$, $\pi_l = \pi_k \pi_l \pi_{\alpha\beta} \pi_{ijkl}$, and $\pi_{ijkl} = \pi_k \pi_l \pi_{\alpha\beta} \pi_{ijkl}$ with $i, j, k, l \in (H, V)$ in $A'$ basis, and $\pi_{\alpha\beta} = \pi_{\alpha\beta} \pi_{ijkl}$, $\pi_{ijkl} = \pi_{\alpha\beta} \pi_{ijkl}$ with $\pi_{\alpha\beta} \pi_{ijkl}$. The basis vectors in $A'$ and $B'$ are maximally incompatible, and $\langle ij | \alpha \beta \rangle = \frac{1}{\sqrt{2}}$. The matrix elements $\rho_{ijkl}$ of unknown density operator $\rho$ of an entangled state can be found as

$$\rho_{ijkl} = 4\langle \Pi_{ijkl} \rangle_s = 4\text{Tr}_s[\pi_k \pi_l \pi_{\alpha\beta} \pi_{ijkl} \rho] \quad (34)$$

where $\pi_{\alpha\beta}$ represent the two composed project operator with definite value of $\alpha$ and $\beta$ in $B'$ basis. Here, we will only consider the $\alpha = \beta = D$ case with following the method of Lundeen[11], but other cases such as $\alpha = \beta = A$ also can be used to find the matrix elements with similar procedures described in this study. To find the matrix elements $\rho_{ijkl}$, we have to find the value of $\text{Tr}_s[\pi_k \pi_l \pi_{\alpha\beta} \pi_{ijkl} \rho]$, and remaining part of this subsection we will study this problem.

We assume that the initial state of total system is

$$|\Omega\rangle = |\Phi\rangle \langle \Phi | \otimes \rho,$$

where the initial state of the pointer $|\Phi\rangle = \varphi_1(x_1, y_1) \varphi_2(x_2, y_2)$ is composed by two Gaussian beams of two photons which have $x$ and $y$ transverse spatial distributions separately, i.e.

$$\langle r | \Phi \rangle = \varphi_1(x_1, y_1) \varphi_2(x_2, y_2) \quad (35)$$

where

$$\varphi_1(x_1, y_1) = \left( \frac{1}{2\pi \sigma_{x_1} \sigma_{y_1}} \right)^{\frac{1}{2}} \exp \left( \frac{x_1^2}{4\sigma_{x_1}^2} \right) \exp \left( \frac{y_1^2}{4\sigma_{y_1}^2} \right) \quad (36)$$

and

$$\varphi_2(x_2, y_2) = \left( \frac{1}{2\pi \sigma_{x_2} \sigma_{y_2}} \right)^{\frac{1}{2}} \exp \left( \frac{x_2^2}{4\sigma_{x_2}^2} \right) \exp \left( \frac{y_2^2}{4\sigma_{y_2}^2} \right) \quad (37)$$

are represent the spatial distributions of first and second photons, respectively. $\rho$ is given in Eq.(5), and considered as measured system.

We assume that the interaction Hamiltonian between each pointer and measured systems are

$$H = H_1 + H_2 + H_3 + H_4, \quad (38)$$

with

$$H_1 = g_1 \pi_{1}^2 x_1, H_2 = g_2 \pi_{2}^2 x_2, \quad i, j \in (V, H) \quad (39)$$

and

$$H_3 = g_3 \pi_D^2 y_1, H_4 = g_4 \pi_D^2 y_2, \quad (40)$$

respectively, and $g_n (n = 1, 2, 3, 4)$ represent the coupling strength between the pointer and measuring device, and for simplicity can be taken as equal quantity, i.e., $g_1 = g_2 = g_3 = g_4 = g$. If we assume that the polarizers represented by the projection operator $\pi_{ij}$ and $\pi_{D D}$ causing displacement along $x$ and $y$ directions, respectively, we can get the weak average of $\pi_{D D} \pi_{ij}$ by using the method introduced in Refs.[11, 32] as

$$\langle \pi_{D D} \pi_{ij} \rangle_s = \frac{1}{g^2} \langle a_{2D} a_{1D} a_{2j} a_{1i} \rangle_f, \quad (41)$$

where

$$a_{1i} = x_{1i} + i \frac{2\sigma^2}{\hbar} p_{1x_i}, \quad a_{2j} = x_{2j} + i \frac{2\sigma^2}{\hbar} p_{2x_j}, \quad (42)$$

and

$$a_{1D} = y_{1D} + i \frac{2\sigma^2}{\hbar} p_{1y_D}, \quad a_{2D} = y_{2D} + i \frac{2\sigma^2}{\hbar} p_{2y_D}, \quad (43)$$

are represent the annihilation operators of every spatial transversal components of each photons, respectively, and $\langle f \rangle$ indicate to find the expectation value of variables under the final state of the pointer state. Here, we have to note that $\pi_{ij}$ and $\pi_{D D}$ are non-commute, but as showed in Ref.[34] the Eq.(41) still valid for non-commuting observables if they are measured sequentially as measuring the $\pi_{D D}$ followed by $\pi_{ij}$. Since last measurement is will be taken over the projection operators $\pi_{kl}$ are strong, then

$$\text{Tr}_s[\pi_{kl} \pi_{\alpha\beta} \pi_{ijkl} \rho] = \frac{1}{g^2} \text{Tr}_s[\pi_{kl} a_{2D} a_{1D} a_{2j} a_{1i} \rho]. \quad (44)$$

With these processes we can obtain the matrix elements $\rho_{ijkl}$ of density operator $\rho$ as
\[
\rho_{ij,kl} = 4Tr[\sigma_{kl}D\pi_{Dj}\sigma_{ij}\rho] = 4Tr[\sigma_{kl}a_{1D}a_{2D}a_{1j}a_{2i}\rho] \\
= \frac{4}{g^4} \left( (y_{1D} + i\frac{2\sigma_{x1}^2}{\hbar} p_{1yD}) \left( y_{2D} + i\frac{2\sigma_{x2}^2}{\hbar} p_{2yD} \right) \left( x_{2j} + i\frac{2\sigma_{x2}^2}{\hbar} p_{2xj} \right) \left( x_{1i} + i\frac{2\sigma_{x1}^2}{\hbar} p_{1xi} \right) \right) 
\]

(45)

Then, the real and imaginary parts of the matrix elements of density operator \( \rho \) are

\[
\Re[\rho_{ij,kl}] = \frac{4}{g^4} \left[ (y_{1D}y_{2D}x_{1i}x_{2j})_f - \frac{\sigma^2}{\sigma^2_p} (y_{1D}y_{2D}p_{1xi}p_{2xj})_f - \frac{\sigma^2}{\sigma^2_p} (x_{1i}x_{2j}p_{2yD}p_{1yD})_f + \frac{\sigma^4}{\sigma^2_p} (p_{1xD}p_{2xj}p_{2yD}p_{1yD})_f \\
- \frac{\sigma^2}{\sigma^2_p} (y_{1D}p_{2yD}x_{2j}p_{1xi})_f - \frac{\sigma^2}{\sigma^2_p} (y_{1D}p_{2yD}x_{1i}x_{2j})_f - \frac{\sigma^2}{\sigma^2_p} (y_{2D}p_{1yD}x_{2j}p_{1xi})_f - \frac{\sigma^2}{\sigma^2_p} (y_{2D}p_{1yD}x_{1i}p_{2xj})_f \right] 
\]

(46)

and

\[
\Im[\rho_{ij,kl}] = \frac{4}{g^4} \left[ (y_{1D}y_{2D}x_{1i}x_{2j})_f + (y_{1D}y_{2D}x_{1i}p_{2xj})_f + (x_{1i}x_{2j}y_{1D}p_{2yD})_f + (x_{1i}x_{2j}y_{1D}p_{1yD})_f \\
- \frac{\sigma^2}{\sigma^2_p} (p_{2yD}p_{1yD}x_{2j}p_{1xi})_f - \frac{\sigma^2}{\sigma^2_p} (p_{2yD}p_{1yD}x_{1i}x_{2j})_f - \frac{\sigma^2}{\sigma^2_p} (p_{1xD}p_{2xj}y_{1D}p_{2yD})_f - \frac{\sigma^2}{\sigma^2_p} (p_{1xD}p_{2xj}y_{2D}p_{1yD})_f \right] 
\]

(47)

, respectively. Here, for simplicity we assume that the width of every Gaussian beam is equal to \( \sigma \), i.e., \( \sigma_{x1} = \sigma_{y1} = \sigma_{x2} = \sigma_{y2} = \sigma \), and \( \sigma_p \) is the momentum space width of the pointer state with \( \sigma\sigma_p = \frac{\hbar}{2} \). Based on the experimental results and methods of Ref.[26] for read out the real and imaginary parts of matrix elements of single photon polarization state by measuring the probabilities of transmitted photons through the final polarizers which represented by the projection operators \( \pi_{kl} = |k\rangle \langle k| \) of two subsystems. In general, these probabilities are functions of positions and momenta of two photons, i.e, \( \mathcal{P} = P(x_1, y_1, y_2, x_2, p_{1x}, p_{2x}, p_{1y}, p_{2y}) \). Then, the elements of density operator of two entangled photon state can be reconstructed by determining the expectation values \( \langle \rangle_f \) in Eq.(46) and Eq.(47) via \( \int ABCDPd\tau = \langle ABCD \rangle_f \) respectively.

**IV. CONCLUSION AND REMARKS**

In this study we investigated how to reconstruct the unknown density operator of two component entangled quantum state by using postselected weak measurement method and three sequential measurements where each complementary to the last, respectively, and discussed its feasibility by taking into account the recent related experimental works. The similarity of these methods is that in each scheme we take the weak and strong sequential measurements in two bases during the getting of real and imaginary parts of elements of density operator, respectively, and the postselection is the key to determine which matrix element we want to readout from the final state of the pointer state. However, in second method it is enough to scan over one definite elements in bases \( B' \) but in first method we usually need to scan all elements in \( B' \). Thus, in the same measurement process the method one may need more resources than method two.

Since the Hilbert space of two entangled photon state is larger than single photon case, during the readout of the matrix elements in the Lab of a two entangled photon processes we would need more and some complicated experimental setups and need more resources in method one rather than method two. However, if we consider the wide applications of entangled photon states in every field of quantum theory, it is worthy to study this vital problem. Based on the experimental works of direct measurement of single two dimensional systems and its density operators[26], and direct measurement of pure two entangled photon state[30], we anticipate that in the near future the experts can do experiments by taking those two innovative works into account, and realize the direct measurement of density operator by considering the theoretical results of our current work. In our schemes any matrix elements of an entangled state can be obtained efficiently via proper weak and strong measurements. Since the density operator representation of a quantum state is general than the wave function, if consider the open system cases it suggested that the determination of density operator of an unknown two photon entangled state may have more practical applications rather than the direct measurement of complex probability amplitude of state vector of a pure two photon entangled state.
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