A cut finite element method for a Stokes interface problem

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Abstract

We present a finite element method for the Stokes equations involving two immiscible incompressible fluids with different viscosities and with surface tension. The interface separating the two fluids does not need to align with the mesh. We propose a Nitsche formulation which allows for discontinuities along the interface with optimal a priori error estimates. A stabilization procedure is included which ensures that the method produces a well conditioned stiffness matrix independent of the location of the interface.

Keywords: cut finite element method, Nitsche, Two-phase flow, Stokes, discontinuous viscosity, surface tension, Sharp interface method

1. Introduction

A large number of real world phenomena exhibit strong or weak discontinuities. The application we have in mind is multiphase flow with kinks in the velocity field and jumps in the pressure field as well as in physical parameters, such as viscosity, across interfaces that evolve with time and may undergo topological changes. When simulating such phenomena, discontinuities can occur anywhere relative to a fixed background mesh. Unfortunately, standard finite element methods, as well as finite difference schemes, do not accurately model discontinuities that are not a priori fitted to the mesh. However, letting the mesh conform to the interfaces requires remeshing as the interfaces evolve with time, and leads to significant complications when topological changes such as drop-breakup or coalescence occur. Numerous strategies have been proposed to handle these difficulties.

A common strategy has been to regularize the discontinuities \[1\]. However, this strategy has the drawback that it gives reduced accuracy near the interfaces and consequently requires a very fine mesh in these regions. Methods that allow for discontinuities along interfaces that do not align with the mesh, and hence avoid both regularization and remeshing processes, have become highly attractive and significant efforts have been directed to their development, see e.g. \[2\] \[3\] \[4\] \[5\] \[6\]. In the finite element framework the extended finite element method (XFEM), where the finite element space is enriched so that discontinuities can be captured \[7\], has become a popular alternative. However, in XFEM the conditioning of the problem is sensitive to the position of the interface. Whenever the interface cuts an element in such a way that the ratio between the areas/volumes on one and the other side of the interface becomes very large, the system may become ill-conditioned. In such cases, iterative linear solvers may breakdown. For unsteady problems it is not unusual that such situations occur. In \[8\], this problem is addressed by neglecting basis functions in the XFEM space that have very small support and may cause ill-conditioning. The criterion for the selection of basis functions to neglect then has to be chosen carefully so that accuracy is not lost.

An alternative to the XFEM approach is based on an extension of Nitsche’s method \[9\] for the weak enforcement of essential boundary conditions. This approach was first proposed for an...
elliptic interface problem in [10] and later for a Stokes interface problem in [3]. The idea is to construct the discrete solution from separate solutions defined on each subdomain separated by the interface and at the interface enforce the jump conditions defined on each side of the interface the unfitted finite element methods in [10] [3] can allow for discontinuities internal to the elements with optimal convergence order. However, these methods suffer from ill-conditioning just as XFEM. In [11] and later in [12] a stabilization of the classical Nitsche’s method for the imposition of Dirichlet boundary conditions on a boundary not fitted to the mesh was considered for the Poisson problem and for the Stokes equation, respectively. In these methods the stabilization is applied in the boundary region and optimal convergence order and well-conditioned system matrices are ensured. For the elliptic interface problem other stabilization strategies have been suggested as remedies to the ill-conditioning problem, see e.g. [13, 14]. We also refer to [15] for implementation aspects in three dimensions and [16] for extensions to higher order elements.

In this paper, we propose a robust finite element method which offers a way to accurately and with control of the condition number of the system matrix solve the Stokes interface problem involving two immiscible fluids with different viscosities and surface tension. The model consists of the incompressible Stokes equations in two subdomains each occupied by a fluid. Differences in viscosity between the fluids and the surface tension force poses jump conditions at the interface separating the fluids. The finite element method we propose for this Stokes interface problem enforces the jump conditions at the interface weakly with weighted coefficients in the Nitsche numerical fluxes. We also suggest slight changes to the variational formulation in [3] to reduce spurious velocity oscillations and we include stabilization terms both for the velocity and the pressure that guarantee a well conditioned system matrix. The stabilization terms are consistent least squares terms controlling the jump in the normal gradient across faces between elements in a neighbourhood of the interface. Using the stabilization terms we prove an inf-sup condition and that the resulting stiffness matrix has optimal conditioning. We also prove the inf-sup stability of the method under the condition that the mean value of the pressure in the entire domain is fixed, in contrast to the inf-sup result in [3] which is based on the more restrictive condition that the mean values in each of the two subdomains are fixed. Our inf-sup result is also uniform with respect to the jump in the viscosity. The proposed method is simple to implement as it uses standard continuous linear basis functions with changes only in the variational form.

The outline of this paper is as follows. In Section 2 we formulate the Stokes system and the finite element method. In Section 3 we prove that the method is optimal convergence order. In Section 4 we prove that the condition number is $O(h^{-2})$ independent of the position of the interface relative to the mesh. Finally, in Section 6, we show numerical examples in two space dimensions and compare the method to existing methods. We summarize our results in Section 7.

2. The interface problem and the finite element method

We consider a problem consisting of two immiscible fluids separated by an interface, with the flow described by the incompressible Stokes equations. The Stokes system is a standard model for creeping viscous flow. In this section we present the equations and a finite element method for their approximate solution.

2.1. The two-fluid incompressible Stokes equations

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, with convex polygonal boundary $\partial \Omega$. We assume that two immiscible incompressible fluids occupy subdomains $\Omega_i \subset \Omega$, $i = 1, 2$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$ and that a smooth interface defined by $\Gamma = \partial \Omega_1 \cap \partial \Omega_2$ separates the immiscible fluids.

We consider the following Stokes interface boundary value problem modeling two fluids with different viscosity and with surface tension: find the velocity $u : \Omega \to \mathbb{R}^2$ and the pressure
\( p : \Omega \rightarrow \mathbb{R} \) such that

\[
-\nabla \cdot (\mu \varepsilon(u) - p \mathbf{I}) = f \quad \text{in } \Omega_1 \cup \Omega_2, \\
\nabla \cdot u = 0 \quad \text{on } \Omega_1 \cup \Omega_2, \\
\|u\| = 0 \quad \text{on } \Gamma, \\
\|n \cdot (\mu \varepsilon(u) - p \mathbf{I})\| = \sigma \kappa n \quad \text{on } \Gamma, \\
u = g \quad \text{on } \partial \Omega.
\]

(2.1a) (2.1b) (2.1c) (2.1d) (2.1e)

Here \( \varepsilon(u) = (\nabla u + (\nabla u)^T)/2 \) is the strain tensor, \( \mu = 2\mu_i > 0 \) on \( \Omega_i, i = 1, 2 \) is a piecewise constant viscosity function on the partition \( \Omega_1 \cup \Omega_2, f \in [L^2(\Omega)]^2 \) and \( g \in [H^{1/2}(\partial \Omega)]^2 \) are given functions, \( \sigma \) is the surface tension coefficient, \( \kappa \) is the curvature of the interface, \( n \) is the unit normal to \( \Gamma \), outward-directed with respect to \( \Omega_1 \), and \( \|a\| = (a_1 - a_2)\|_\Gamma \), where \( a_i = a_{\Omega_i}, i = 1, 2 \).

We assume a constant surface tension coefficient \( \sigma \). Hence, from equation (2.1d), it follows that across the interface the shear stress is continuous, i.e.,

\[
\|n \cdot \varepsilon(u)\| \cdot t = 0.
\]

Henceforth, we also denote the outward directed unit normal vector on \( \partial \Omega \) by \( n \). We assume global conservation of mass, that is

\[
\int_{\partial \Omega} g \cdot n ds = 0.
\]

(2.2) (2.3)

We will use the notation \( (\cdot, \cdot)_\omega \) for the \( L^2(\omega) \) inner product on \( \omega \) (and similarly for inner products in \( [L^2(\omega)]^2 \) and \( [L^2(\omega)]^{2 \times 2} \)). We let \( \|v\|_{s, \omega} \) and \( \|v\|_{s, \omega} \) denote the Sobolev norms and seminorms associated with the spaces \( H^s(\omega) \), respectively. For the convergence analysis we will assume that \( u \in [H^2(\Omega_1 \cup \Omega_2)]^2 = \mathcal{W} \) and the pressure space is

\[
\mathcal{V} = \{ p \in H^1(\Omega_1 \cup \Omega_2) : (\mu^{-1} p)_{\Omega_1 \cup \Omega_2} = 0 \}.
\]

(2.4)

2.2. Mesh and assumptions

Let \( \mathcal{K}_h \) be a triangulation of \( \Omega \), generated independently of the location of the interface \( \Gamma \). Introduce the set of all element faces \( \mathcal{F} \) associated with the mesh \( \mathcal{K}_h \), the set of all elements that intersect the interface

\[
\mathcal{K}_\Gamma = \{ K \in \mathcal{K}_h : |\Gamma \cap K| > 0 \},
\]

and the set of all elements on the boundary

\[
\mathcal{K}_{\partial \Omega} = \{ K \in \mathcal{K}_h : |\partial \Omega \cap K| > 0 \}.
\]

(2.5) (2.6)

Define meshes on the subdomains \( \Omega_i, i = 1, 2 \), as follows

\[
\mathcal{K}_{h,i} = \{ K \in \mathcal{K}_h : \exists \ x, y, x \neq y \in K \cap \Omega_i \}
\]

(2.7)

and let

\[
\Omega_{h,i} = \bigcup_{K \in \mathcal{K}_{h,i}} K, \quad \tilde{\omega}_{h,i} = \bigcup_{K \in \mathcal{K}_{h,i}, K \subseteq \Omega_i} K, \quad i = 1, 2.
\]

(2.8)

Note that the interface \( \Gamma \) is allowed to intersect the elements in \( \mathcal{K}_h \) and \( \mathcal{K}_{h,i} \) and that elements intersected by the interface are both in \( \Omega_{h,1} \) and \( \Omega_{h,2} \).

Let \( \mathcal{K}_\Gamma \) be the set of all elements that share two of its faces with elements in \( \mathcal{K}_\Gamma \), let

\[
\mathcal{F}_{\Gamma,i} = \{ F \in \mathcal{F} : F \subseteq \partial K, K \in \mathcal{K}_\Gamma \cup \mathcal{K}_\Gamma, F \cap \Omega_i \neq \emptyset, F \cap \partial \Omega = \emptyset \}
\]

(2.9)

be the set of faces of elements in \( \mathcal{K}_\Gamma \cup \mathcal{K}_\Gamma \), that have a nonempty intersection with \( \Omega_i \), and are not on the boundary \( \partial \Omega \), and let

\[
\omega_{h,i} = \tilde{\omega}_{h,i} \setminus \bigcup_{K \in \mathcal{K}_\Gamma} K, \quad i = 1, 2,
\]

(2.10)
contain the elements in $\tilde{\omega}_{h,i}$ that are not in $\tilde{K}_\Gamma$. We modify the set $\tilde{\omega}_{h,i}$ in order to ensure that there are no elements in $\omega_{h,i}$ that has an element with two edges on the boundary $\partial \omega_{h,i}$ which is a necessary condition for the inf-sup condition condition to hold on $\omega_{h,i}$, see Lemma 3.10. See Fig. 1 for an illustration of the sets $\Omega_{h,i}$, $\omega_{h,i}$, and $F_{\Gamma,i}$, $i=1,2$ in a two-dimensional case.

We make the following assumptions:

- **Assumption 1:** We assume that the triangulation is quasi-uniform, i.e., there exists constants $c_1$ and $c_2$ such that
  \[ c_1 h \leq h_K \leq c_2 h \hspace{1em} \forall K \in \mathcal{K}_h, \]  
  where $h_K$ is the diameter of $K$ and $h = \max K h_K$, and that there are no elements with two edges on the boundary $\partial \Omega$.

- **Assumption 2:** We assume that $\Gamma$ either intersects the boundary $\partial K$ of an element $K \in \mathcal{K}_\Gamma$ exactly twice and each (open) edge at most once, or that $\Gamma \cap K$ coincides with an edge of the element.

- **Assumption 3:** Let $\Gamma_{K,h}$ be the straight line segment connecting the points of intersection between $\Gamma$ and $\partial K$. We assume that $\Gamma_K = \Gamma \cap K$ is a function of length on $\Gamma_{K,h}$; in local coordinates:
  \[ \Gamma_{K,h} = \{(\xi, \eta) : 0 < \xi < |\Gamma_{k,h}|, \eta = 0\} \]  
  and
  \[ \Gamma_K = \{(\xi, \eta) : 0 < \xi < |\Gamma_{k,h}|, \eta = \delta(\xi)\}. \]  

- **Assumption 4:** We assume that for each $K \in \mathcal{K}_\Gamma$ there are elements $K^i \subset \Omega_i$, $i = 1,2$ such that $\overline{K} \cap \overline{K^i} \neq \emptyset$.

- **Assumption 5:** We assume that the mesh coincides with the outer boundary $\partial \Omega$.

Assumptions 2-3 essentially state that the interface is well resolved by the mesh. Except that we in the second assumption also allow the interface to be aligned with a mesh line these two assumptions are as in [10]. Assumption 4 states that each $K \in \mathcal{K}_\Gamma$ shares a face or at least a vertex with an element $K^1 \subset \Omega_1$ and an element $K^2 \subset \Omega_2$. For each $\Omega_i$ this is the same assumption as in [12].
2.3. Penalty parameters and averaging operators

In our finite element method we will make use of the following averaging operators

\[
\{a\} = \kappa_1 a_1 + \kappa_2 a_2, \quad \langle a \rangle = \kappa_2 a_1 + \kappa_1 a_2,
\]

(2.14)

where the weights \( \kappa_1 \) and \( \kappa_2 \) are real numbers satisfying \( \kappa_1 + \kappa_2 = 1 \). We consider the following two cases, in accordance with the intersection options of \( \Gamma \):

- For each element \( K \in \mathcal{K}_\Gamma \) such that \( \Gamma \) intersects the element boundary exactly twice, we have \( |K \cap \Omega_i| = \alpha_i h_K^2 \) and \( |\Gamma \cap K| = \gamma_K h_K \) for some \( \alpha_i, \gamma_K > 0 \), and we define

\[
\kappa_1|_K = \frac{\mu_2 \alpha_1, K}{\mu_1 \alpha_2, K + \mu_2 \alpha_1, K}, \quad \kappa_2|_K = \frac{\mu_1 \alpha_2, K}{\mu_1 \alpha_2, K + \mu_2 \alpha_1, K},
\]

(2.15)

a penalty parameter

\[
\lambda_T|_K = \frac{\eta_K}{h_K},
\]

(2.16)

where

\[
\eta_K = D\{\mu\} + \frac{(1 + B)\gamma_K \mu_1 \mu_2}{A(\mu_1 \alpha_2, K + \mu_2 \alpha_1, K)}, \quad A \in (0, 1), \quad B, D > 0,
\]

(2.17)

and if \( K \) is also an element in \( \mathcal{K}_{\partial \Omega} \) we define a penalty parameter

\[
\lambda_{\partial \Omega}|_{K \cap \Omega_i} = \frac{1}{h_{\Gamma}} \left( G\mu_1 + \frac{2(1 + F)\mu_1 \gamma_{\partial \Omega, K}}{E \gamma_K} \right), \quad E \in \left( 0, \frac{1 - A}{1 + \max(N_{K^i})} \right).
\]

(2.18)

Here, \( F, G > 0 \), \( K^i \) is the closest element to \( K \) that is completely in \( \Omega_i \), and \( N_{K^i} \) is the number of elements in \( \mathcal{K}_{\partial \Omega} \cap \mathcal{K}_\Gamma \) that are not completely in \( \Omega_i \) and have the same element \( K^i \) as the closest element completely in \( \Omega_i \). Furthermore, \( c_q = \max \left( \frac{|K|}{|K^i|}, \frac{|\Gamma^i|}{|\Gamma|} \right) \), where \( F \) are the faces that have to be crossed to pass from \( K \) to \( K^i \) and \( |\partial \Omega \cap K| = |\partial \Omega \cap K^i| = \gamma_{\partial \Omega, K} h_K \).

- If \( \Gamma \cap \overline{K} \) coincides with an element edge, then \( \Gamma \cap \overline{K} \) will also be an edge of another triangle \( T \in \mathcal{K}_\Gamma \) and \( \Gamma \cap \overline{K} = \Gamma \cap \overline{T} \). We may without loss of generality assume that \( T \subset \Omega_i \) and \( K \subset \Omega_2 \). We may write \( |T| = \alpha_T h_T^2 \) and \( |K| = \alpha_K h_K^2 \) for some \( \alpha_T, \alpha_K > 0 \), and \( |\Gamma \cap T| = |\Gamma \cap \overline{T}| = \gamma_T h_T = \gamma_T h_T \) for some \( \gamma_T, \gamma_T > 0 \). We define

\[
\kappa_1|_K = \frac{\mu_2 \alpha_T}{\mu_1 \alpha_K \left( \gamma_T / \gamma_K \right)^2 + \mu_2 \alpha_T}, \quad \kappa_2|_K = \frac{\mu_1 \alpha_K}{\mu_1 \alpha_K + \mu_2 \alpha_T \left( \gamma_K / \gamma_T \right)^2},
\]

(2.19)

a penalty parameter

\[
\lambda_T|_K = \frac{\eta_K}{|\Gamma \cap \overline{K}|},
\]

(2.20)

where

\[
\eta_K = D \gamma_K \{\mu\} + \frac{(1 + B) \mu_1 \mu_2}{A \left( (\mu_1 \alpha_K) / \gamma_K + (\mu_2 \alpha_T) / \gamma_T \right)^2}, \quad A \in (0, 1), \quad B, D > 0,
\]

(2.21)

and if \( K \) is also an element in \( \mathcal{K}_{\partial \Omega} \) we define the penalty parameter

\[
\lambda_{\partial \Omega}|_{K \cap \Omega_i} = \frac{1}{h_{\Gamma}} \left( G\mu_1 + \frac{(1 + F)\mu_1 \gamma_{\partial \Omega, K}}{E \gamma_K} \right), \quad E \in \left( 0, \frac{1 - A}{1 + \max(N_{K^i})} \right).
\]

(2.22)

Here \( F, G > 0 \). For all the elements \( K \in \mathcal{K}_{\partial \Omega} \) that are not intersected by the interface we choose \( \lambda_{\partial \Omega}|_{K \cap \Omega_i} \) as in (2.22). Note that \( 0 \leq \kappa_i \leq 1 \) and \( \kappa_1 + \kappa_2 = 1 \) in both cases.
2.4. The finite element method

Let $\mathcal{K}_h$ be a triangulation of $\Omega$ that satisfies all the assumptions in Section 2.2. We let $V_{h,\mu}$ be the space of continuous piecewise linear polynomials defined on $\mathcal{K}_h$ with $(\mu^{-1}q_h, 1)_{\Omega_1 \cup \Omega_2} = 0$ for all $q_h \in V_{h,\mu}$ and we let

$$V_h = V_{h,1} \oplus V_{h,2}$$

be our pressure space where

$$V_{h,i} = V_{h,\mu}|_{\Omega_{h,i}}, \quad i = 1, 2,$$

i.e. the spaces of restrictions to $\Omega_{h,1}$ and $\Omega_{h,2}$ of functions in $V_{h,\mu}$. A $p_h \in V_h$ consist of a pair $(p_{h,1}, p_{h,2})$, with $p_{h,i} \in V_{h,i}$. For an illustration in a one-dimensional model case, see Fig. 2. Note that $p_h \in V_h$ is double valued on elements in $K_{\Gamma}$ and is allowed to be discontinuous at the interface $\Gamma$.

In the same way as above we construct the velocity space but on a uniform refinement of $\mathcal{K}_h$, denoted by $\mathcal{K}_{h/2}$. The triangulation $\mathcal{K}_{h/2}$ also satisfies all the assumptions in Section 2.2 and we define all the parameters in Section 2.3 on the mesh $\mathcal{K}_{h/2}$. To construct the velocity space we let $W_{h,0}$ be the space of vector valued continuous piecewise linear polynomials on $\mathcal{K}_{h/2}$. We will impose the Dirichlet conditions weakly. Thus, there are no special boundary restrictions at $\partial \Omega$ on the velocity space. We define

$$W_{h,i} = W_{h,0}|_{\Omega_{h,i}}, \quad i = 1, 2$$

and we let

$$W_h = W_{h,1} \oplus W_{h,2}.$$

Each $u_h \in W_h$ consists of a pair $(u_{h,1}, u_{h,2})$, where $u_{h,i} \in W_{h,i}$.

We drop the subscript $h$ and $h/2$ and denote both the velocity and the pressure mesh by $K$. An element $K \in \mathcal{K}$ is an element in $\mathcal{K}_{h/2}$ whenever we have terms involving functions in the velocity space $W_h$ and it is an element in $\mathcal{K}_h$ whenever we have terms involving only functions in the pressure space.

We propose the following Nitsche method: find $(u_h, p_h) \in W_h \times V_h$ such that

$$A_h(u_h, p_h; v_h, q_h) + \varepsilon_u J(u_h, v_h) + \varepsilon_p J(p_h, q_h) = L_h(v_h), \quad \forall (v_h, q_h) \in W_h \times V_h.$$  

Here $A_h(\cdot; \cdot)$ is a bilinear form defined by

$$A_h(w; v, q) = a_h(w, v) + b_h(w, q) - b_h(v, r),$$

where

$$a_h(w, v) = a_h(w, v), \quad b_h(w, q) = b_h(w, q), \quad b_h(v, r) = b_h(v, r).$$

Figure 2: Illustration of the domains, meshes, and spaces in a one-dimensional model case.
where
\[
a_h(w,v) = (\mu\epsilon(w),\epsilon(v))_{\Omega_1 \cup \Omega_2} - [(n \cdot \mu \epsilon(w)), [v]]_\Gamma - ([w], \{n \cdot \mu \epsilon(v)\})_\Gamma \\
- (n \cdot \mu \epsilon(w), v)_{\partial\Omega} - (w \cdot n \cdot \mu \epsilon(v))_{\partial\Omega} \\
+ \lambda_r([w], [v])_\Gamma + \lambda_{\partial\Omega}(w,v)_{\partial\Omega}
\]
(2.29)

\[
b_h(w,q) = (\nabla \cdot w, q)_{\Omega_1 \cup \Omega_2} - ([n \cdot w], \{q\})_\Gamma - (n \cdot w, q)_{\partial\Omega}
\]
(2.30)

\[
b_h(w,q) = -(w, \nabla q)_{\Omega_1 \cup \Omega_2} + ([q], \{w \cdot n\})_\Gamma,
\]
(2.31)

and \(L_h(\cdot)\) is a linear functional defined by
\[
L_h(v) = (f,v)_{\Omega} + (\sigma \kappa, (v \cdot n))_{\Gamma} + (g, \lambda_{\partial\Omega}v - n \cdot \mu \epsilon(v) + nq)_{\partial\Omega}.
\]
(2.32)

In equation (2.27) \(\varepsilon_u\) and \(\varepsilon_p\) are positive constants and the stabilization terms are defined as
\[
J(p_h,q_h) = \sum_{i=1}^{2} \sum_{F \in \mathcal{F}_{T,i}} \mu_i^{-1} h^s \left( \left[ n_F \cdot \nabla p_h, i \right]_F, \left[ n_F \cdot \nabla q_h, i \right]_F \right)_F.
\]
(2.33)

and the component wise extension for vector valued functions \(u_{h,i} = (u_{h,i}^1, u_{h,i}^2)\)
\[
J(u_h,v_h) = \sum_{j=1}^{2} \sum_{F \in \mathcal{F}_{T,i}} \mu_i h^s \left( \left[ n_F \cdot \nabla u_{h,i}^j \right]_F, \left[ n_F \cdot \nabla v_{h,i}^j \right]_F \right)_F.
\]
(2.34)

We choose \(s\) in different ways depending on if the interface cuts the domain boundary or not. We take \(s = 1\) only for those faces that has to be crossed to pass from an element on the boundary \(K \in K_{\partial\Omega}, K \not\subset \Omega_i, K \cap \Omega_i \neq \emptyset\), to the closest element \(K^i \subset \Omega_i\), otherwise \(s = 3\). We have employed the following notation for the jump in a function \(v\) at an interior face \(F\)
\[
[v]_F = v^+ - v^-.
\]
(2.35)

where \(v^\pm(x) = \lim_{t \to 0^\pm} v(x \mp tn_F)\), for \(x \in F\), and \(n_F\) is a fixed unit normal to \(F\).

**Remark.** By integrating the bilinear form \(b_h\) in equation (2.30) by parts we get \(b_h\) in equation (2.31). We recommend to use \(b_h\) in equation (2.31) in simulations since it results in reduced spurious velocities, see the remark in Example 2 of Section 5. The stabilization of the bilinear forms \(a_h\) and \(b_h\) require integration over \(\Omega_i \subset \Omega_{h,i}\). Thus, for elements cut by the interface \(\Gamma\) the integration should be performed only over parts of the elements. The functions \(p_h, i\) and \(u_{h,i}\) are defined on the larger subdomains \(\Omega_{h,i}\) and the stabilization terms (2.33) and (2.34) ensure well defined extensions from \(\Omega_i\) to \(\Omega_{h,i}\).

**Remark.** If the interface does not cut the boundary of the domain \(\Omega\), the stabilization term \(J(p_h,q_h)\) in equation (2.33) is sufficient to prove the inf-sup stability of the method. However, to have control of the condition number of the system matrix independently of the position of the interface relative to the mesh both the stabilization terms \(J(p_h,q_h)\) and \(J(u_h,v_h)\) are needed, see Fig. 9. The sets \(\mathcal{F}_{T,i}\), \(i=1,2\) in equation (2.33) are defined for the pressure mesh \(K_h\) and the two sets \(\mathcal{F}_{T,i}\), \(i=1,2\) in equation (2.34) are defined for the velocity mesh \(K_{h/2}\) and are different. The face \(F\) is always a full face in the underlying mesh. Similar stabilizations were used in the fictitious domain method of [12]. However, the set of faces that are stabilized are slightly different here and we choose \(s\) in equation (2.34) in different ways depending on if the interface cuts the domain boundary or not.

3. Analysis

In this section we will show that the finite element method presented in Section 2.4 has optimal convergence order. We begin by proving the following consistency relation for the finite element formulation (2.27).
Lemma 3.1. Let \((u, p) \in W \times V\) be the solution to the boundary value problem (2.1) and \((u_h, p_h)\) be the solution of the finite element formulation (2.27). Then
\[
A_h(u - u_h, p - p_h; v_h, q_h) = \varepsilon_u J(u_h, v_h) + \varepsilon_p J(p_h, q_h), \quad \forall (v_h, q_h) \in W_h \times V_h.
\] (3.1)

Proof. First, note that since \(\kappa_1 + \kappa_2 = 1\) we have
\[
[a b] = \{a\} [b] + \{a\} \langle b \rangle,
\] (3.2)
hence we can write
\[
\int_{\Gamma} \{n \cdot (\mu \varepsilon(u) - p I)\} \langle v \rangle d\Gamma = \int_{\Gamma} \{n \cdot (\mu \varepsilon(u) - p I)\} \langle v \rangle d\Gamma + \int_{\Gamma} \{n \cdot (\mu \varepsilon(u) - p I)\} \langle v \rangle d\Gamma.
\] (3.3)

Using the interface conditions for the normal stress and the shear stress, equation (2.1d) and (2.2), we have that
\[
\int_{\Gamma} \{n \cdot (\mu \varepsilon(u) - p I)\} \langle v \rangle d\Gamma = \int_{\Gamma} \sigma_k \langle v \cdot n \rangle d\Gamma.
\] (3.4)

Now, multiplying (2.1) by a test function \((v_h, q_h) \in W_h \times V_h\) and integrating by parts, using (3.3) and (3.4), the boundary conditions (2.1e) and (2.3), and that \(u\) is continuous (2.1c) we get
\[
a_h(u, v_h) + b_h(v_h, q_h) - b_h(v_h, p) = L_h(v_h),
\] (3.5)
and the claim follows.

We introduce the following mesh dependent norms
\[
\|v\|^2 = \|\mu^{1/2} \varepsilon(v)\|^2_{0,\Omega_1 \cup \Omega_2} + \|\{n \cdot \mu^{1/2} \varepsilon(v)\}\|^2_{-1/2,h,\Gamma} + \|\{\mu^{1/2} \varepsilon(v)\}\|^2_{1/2,h,\Gamma} \forall v \in W + W_h.
\] (3.6)

\[
\|v_h\|^2_h = \|v_h\|^2 + J(v_h, v_h) \forall v_h \in W_h,
\] (3.7)

\[
\|(v, q)\|^2_h = \|v\|^2 + \|\mu^{-1/2} q\|^2_{0,\Omega_1 \cup \Omega_2} + \|\{\mu^{-1/2} q\}\|^2_{-1/2,h,\Gamma} + \|\mu^{-1/2} q\|^2_{-1/2,h,\partial\Omega} \forall (v, q) \in (W + W_h) \times (V + V_h),
\] (3.8)

\[
\|(v_h, q_h)\|^2_h = \|v_h\|^2_h + \|\mu^{-1/2} q_h\|^2_{0,\Omega_1 \cup \Omega_2} + J(q_h, q_h) \forall (v_h, q_h) \in W_h \times V_h,
\] (3.9)
where \(\mu^{1/2} = \mu \{\mu\}^{-1/2}\), the norms on the trace of a function on \(\Gamma\) are defined by
\[
\|v\|^2_{1/2,h,\Gamma} = \sum_{K \in \mathcal{K}_\Gamma} h_K^{-1} \|v\|^2_{0,\Gamma \cap K},
\] (3.10)
\[
\|v\|^2_{-1/2,h,\Gamma} = \sum_{K \in \mathcal{K}_\Gamma} h_K \|v\|^2_{0,\Gamma \cap K},
\] (3.11)
and similarly for the trace of a function on \(\partial\Omega\). Here \(\mathcal{K}_\Gamma\) is the set of elements that intersect the interface but when a part of \(\Gamma\) coincides with an element edge only one of the two elements sharing that edge belongs to \(\mathcal{K}_\Gamma\). Note that
\[
\langle w, v \rangle_{\Gamma} \leq \|w\|_{1/2,h,\Gamma} \|v\|_{-1/2,h,\Gamma},
\] (3.12)
\[
\langle w, v \rangle_{\partial\Omega} \leq \|w\|_{1/2,h,\partial\Omega} \|v\|_{-1/2,h,\partial\Omega}.
\] (3.13)

We will need the following inverse inequality when proving the inf-sup stability of the finite element method.

Lemma 3.2. Assume that \(K\) is an element in \(\mathcal{K}_\Gamma\) such that, for \(i = 1\) or \(2\), \(|K \cap \Omega_i| = \alpha_i h_K^2\), where \(\alpha_i > 0\) and let \(|\Gamma \cap K| = \gamma_K h_K\). For any function \(v \in W_h\), the following inverse inequality holds
\[
h_K \|\kappa_i \varepsilon(v) \cdot n\|^2_{0,\Gamma \cap K, \Omega_i} \leq \frac{\kappa_i^2 \gamma_K}{\alpha_i} \|\varepsilon(v)\|^2_{0,\Gamma \cap \Omega_i}.
\] (3.14)
Lemma 3.2 follows using that the functions in $W_h$ are linear, cf. [17].

We also state two trace inequalities that we need for proving an approximation result.

**Lemma 3.3.** Let $K \in \mathcal{K}$ and $v \in H^1(K)$. There exists constants $C$ and $\hat{C}$ such that for $s \in \mathbb{R}$

\[
\begin{align*}
    h^{-s}_K \|v\|_{0, \Gamma \cap K}^2 &\leq C(h^{-1-s}_K \|v\|_{0,K}^2 + h^{-s}_K \|v\|_{1,K}^2), \\
    h^{-s}_K \|v\|_{0, \partial K}^2 &\leq \hat{C}(h^{-1-s}_K \|v\|_{0,K}^2 + h^{-s}_K \|v\|_{1,K}^2).
\end{align*}
\]  

(3.15)

Under Assumption 1-3 the first trace inequality follows from Lemma 3 in [10] and a scaling argument. The second trace inequality follows from a standard trace estimate, see [18, Theorem 1.6.6].

We will also need the following estimates:

**Lemma 3.4.** Let $K \in \mathcal{K}$ and $v \in H^1(K)$. Then,

\[
\begin{align*}
    \|\nabla v\|_{0,K}^2 &\leq Ch^{-2}_K \|v\|_{0,K}^2, \\
    \|v\|_{0,\partial K}^2 &\leq Ch^{-1}_K \|v\|_{0,K}^2, \\
    \|v\|_{1,\Gamma \cap K}^2 &\leq Ch^{-1}_K \|v\|_{0,K}^2, \\
    \|v\|_{1,\partial K}^2 &\leq Ch^{-1}_K \|v\|_{0,K}^2.
\end{align*}
\]

(3.16)

(3.17)

(3.18)

The inverse inequality (3.16) follows from [18, Lemma 4.5.3] and the trace inequalities follow from Lemma 3.3 and the inverse inequality (3.18).

3.1. Continuity

We begin by showing the continuity of $a_h(\cdot, \cdot)$ and then we prove the continuity of $A_h(\cdot, \cdot, \cdot, \cdot)$.

**Lemma 3.5.** Let $u \in W + W_h$. There exist a constant $C_{cont}$ such that

\[
    a_h(u, v_h) \leq C_{cont} \|u\| \|v_h\| \quad \forall v_h \in W_h.
\]

(3.19)

**Proof.** For each $K \in \mathcal{K}$ let $|\Gamma \cap K| = \gamma_K h_K$ and $|K| = \alpha_K h_K$ for some $\gamma_K, \alpha_K > 0$. The claim follows by recalling the definition of $a_h(\cdot, \cdot)$, applying the Cauchy-Schwarz inequality, inequality (3.12) and (3.13) to the interface and boundary terms, and noting that $\{n \cdot \mu e(u)\} = \{n \cdot \tilde{\mu}^{1/2} e(u)\}$

\[
    \lambda|_{\Gamma \cap K} \leq \left(C_{\Gamma} \max_{K \in \mathcal{K}} \left(\frac{\gamma_K}{\alpha_K}\right) + D\right) \frac{\mu}{h_K},
\]

\[
    \lambda|_{\partial K \cap \Gamma} \leq \left(C_{\partial K} \max_{K \in \mathcal{K}} \left(\frac{\gamma_{\partial K}}{\alpha_K}\right) + G\right) \frac{\mu}{h_K}.
\]

(3.20)

\[
\square
\]

**Lemma 3.6.** Let $u \in W + W_h$, $v_h \in W_h$, $p \in V + P_h$, $q \in V_h$. There is a constant $C_{contA}$ such that

\[
    A_h(u, p; v_h, q_h) \leq C_{contA} \|(u, p)\| \|(v_h, q_h)\|_h
\]

(3.21)

**Proof.**

\[
A_h(u, p; v_h, q_h) = a_h(u, v_h) + b_h(u, q_h) + b_h(v_h, -p)
\]

(3.22)

(3.23)

**Term I.** Using the continuity property of $a_h(\cdot, \cdot)$ (Lemma 3.5), we have

\[
I \leq C_{cont} \|u\| \|v_h\|.
\]

(3.24)
Term II. Using the Cauchy-Schwarz inequality and the trace inequalities in Lemma 3.4 to bound the contributions from the interface and the boundary we get

\[
II \leq \|\mu^{1/2} \nabla \cdot \mathbf{u}\|_{0,\Omega_1 \cap \Omega_2} \mu^{-1/2} q_h \|_{0,\Omega_1 \cap \Omega_2} + \|\mu^{-1/2} q_h \|_{0,\Omega_1 \cap \Omega_2} + \|\{\mu\}^{-1/2}\{q_h\} \|_{-1/2,h,\Gamma} + \|\mu^{-1/2} q_h \|_{-1/2,h,\partial \Omega}
\]

\[
+ C \|\mathbf{u}\| \left( \|\mu^{-1/2} q_h \|_{0,\Omega_1 \cap \Omega_2} + \|\{\mu\}^{-1/2}\{q_h\} \|_{-1/2,h,\Gamma} + \|\mu^{-1/2} q_h \|_{-1/2,h,\partial \Omega} \right)
\]

\[
\leq C \|\mathbf{u}\| \|\mu^{-1/2} q_h \|_{0,\Omega_1 \cap \Omega_2}.
\]  

(3.25)

Term III.

\[
III \leq C \|v_h\| \left( \|\mu^{-1/2} p \|_{0,\Omega_1 \cap \Omega_2} + \|\{\mu\}^{-1/2}\{p\} \|_{-1/2,h,\Gamma} + \|\mu^{-1/2} p \|_{-1/2,h,\partial \Omega} \right)^{1/2}.
\]  

(3.26)

Summing the estimates of terms I – III and using the definition of the norms \(\|(\cdot, \cdot)\|\) and \(\|(\cdot, \cdot)\|_{h}\) yields the claim.

\[
\square
\]

3.2. Inf-sup stability

We will show that the finite element formulation (3.27) is inf-sup stable. Namely,

**Theorem 3.1.** Let \((u_h, p_h) \in W_h \times V_h\). There is a constant \(C\) such that

\[
\sup_{(v_h, q_h) \in W_h \times V_h} \frac{A_h(u_h, p_h; v_h, q_h) + \varepsilon_u J(u_h, v_h) + \varepsilon_p J(p_h, q_h)}{\|(v_h, q_h)\|_h} \geq C \|(u_h, p_h)\|_h
\]  

(3.27)

First, we need to show the coercivity of \(a_h(\cdot, \cdot, \cdot)\).

**Lemma 3.7.** There exists a positive constant \(C_{\text{coer}}\) such that

\[
C_{\text{coer}} \|\mathbf{v}_h\|_h^2 \leq a_h(\mathbf{v}_h, \mathbf{v}_h) + J(\mathbf{v}_h, \mathbf{v}_h)\quad \forall \mathbf{v}_h \in W_h.
\]  

(3.28)

**Proof.** Let \(K_i = K \cap \Omega_i\). By the definition of \(a_h(\cdot, \cdot, \cdot)\) we have

\[
a_h(\mathbf{v}_h, \mathbf{v}_h) = \sum_{i=1}^{2} \|\mu^{1/2} \mathbf{e}(\mathbf{v}_{h,i})\|_{0,\Omega_i}^2 + \sum_{K \in K_T} \left( \lambda_{\Gamma \cap K} \|\mathbf{v}_h\|_{0,\Gamma \cap K}^2 - 2(\{\mathbf{n} \cdot \mathbf{e}(\mathbf{v}_h)\}, [\mathbf{v}_h])_{\Gamma \cap K} \right)
\]

\[
+ \sum_{i=1}^{2} \sum_{K \in K_T} \left( \lambda_{\Omega \cap K} \|\mathbf{v}_{h,i}\|_{0,\partial \Omega \cap K}^2 - 2(\mathbf{n}_i \cdot \mathbf{e}(\mathbf{v}_{h,i}), [\mathbf{v}_{h,i}])_{\partial \Omega \cap K} \right).
\]  

(3.29)

For each \(K \in K_T\) let \(|\Gamma \cap K| = \gamma_K h_K, |K| = \alpha_K h_k, \text{ and } |K_i| = \alpha_{i,K} h_k\). Using the Cauchy-Schwarz inequality and the geometric-arithmetic inequality we have

\[
\sum_{K \in K_T} \left( \lambda_{\Gamma \cap K} \|\mathbf{v}_h\|_{0,\Gamma \cap K}^2 - 2(\{\mathbf{n} \cdot \mathbf{e}(\mathbf{v}_h)\}, [\mathbf{v}_h])_{\Gamma \cap K} \right) \geq
\]

\[
\sum_{K \in K_T} \left( h_K \lambda_{\Gamma \cap K} - \frac{\delta_{1,K} + \delta_{2,K}}{\gamma_K} \right) h_K^{-1} \|\mathbf{v}_h\|_{0,\Gamma \cap K}^2 - \sum_{i=1}^{2} \sum_{K \in K_T} \frac{|\Gamma \cap K|}{\delta_{i,K}} \|\mathbf{n}_i \cdot \mathbf{e}(\mathbf{v}_{h,i})\|_{0,\Gamma \cap K}^2.
\]  

(3.30)

Here, we take

\[
\delta_{1,K} = \frac{(1 + B)\mu_1 k_{11}^2}{A \alpha_1}, \quad \delta_{2,K} = \frac{(1 + B)\mu_2 k_{22}^2}{A \alpha_2}.
\]  

(3.31)
with \( \alpha_i = \alpha_{i,K} \) and \( \gamma_i = \gamma_K \) when \( \Gamma \cap K \) intersects the element boundary exactly twice, and
\( \alpha_1 = \alpha_T, \alpha_2 = \alpha_K, \gamma_1 = \gamma_T, \gamma_2 = \gamma_K \) when \( \Gamma \cap K \) coincides with an edge shared by element \( T \subset \Omega \) and \( K \subset \Omega \). With our choices of \( \kappa_1, \kappa_2, \) and \( \lambda_T \), using the inverse inequality in Lemma 3.2, \( \mu_i = \mu_i^{1/2}(3/2)^{1/2} \) and that \( \| \mu_i \|_{\text{ess,K}} \geq C \) we obtain

\[
a_h(D(v_h, v_h)) \geq \sum_{i=1}^{2} \left[ (1 - A) \| \mu_i^{1/2} \epsilon(v_h,i) \|_{0, \Omega_i}^2 + BC \| \{ n \cdot \mu_i^{1/2} \epsilon(v_h) \} \|_{1/2, \Omega_i}^2 \right] + D \| \mu_i^{1/2} \|_{1/2, \Omega_i}^2 + \sum_{i=1}^{2} \sum_{K \in \mathcal{K}_{\partial \Omega}} \left( \lambda_i \| \partial \Omega \cap K \|_{2, \partial \Omega \cap K} \| v_{h,i} \|_{0, \partial \Omega \cap K}^2 - 2(\{ n \cdot \mu_i \epsilon(v_h,i) \}, v_{h,i})_{\partial \Omega \cap K} \right).
\]

(3.32)

Applying the Cauchy-Schwarz inequality, the geometric-arithmetic inequality, and the inverse inequality \( \| \mu_i \|_{1/2, \Omega_i} \) on the boundary terms we get

\[
\sum_{i=1}^{2} \sum_{K \in \mathcal{K}_{\partial \Omega}} \left( \lambda_i \| \partial \Omega \cap K \|_{2, \partial \Omega \cap K} \| v_{h,i} \|_{0, \partial \Omega \cap K}^2 - 2(\{ n \cdot \mu_i \epsilon(v_h,i) \}, v_{h,i})_{\partial \Omega \cap K} \right) \geq 0
\]

\[
\sum_{i=1}^{2} \left( h_{\partial \Omega \cap K} \| \partial \Omega \cap K \|_{2, \partial \Omega \cap K} \| v_{h,i} \|_{0, \partial \Omega \cap K}^2 - \frac{\delta_{\partial \Omega \cap K}}{\gamma_{\partial \Omega \cap K}^2} \| \{ n \cdot \mu_i \epsilon(v_h,i) \} \|_{0, \partial \Omega \cap K}^2 \right)
\]

\[
\sum_{i=1}^{2} \sum_{K \in \mathcal{K}_{\partial \Omega}} \gamma_{\partial \Omega \cap K}^2 h_{\partial \Omega \cap K} \| n \cdot \mu_i \epsilon(v_h,i) \|_{0, \partial \Omega \cap K}^2
\]

(3.33)

where \( \gamma_{\partial \Omega \cap K}^2 = | \partial \Omega \cap K \|. \) For each \( K \in \mathcal{K}_{\partial \Omega} \) the following inverse inequality holds

\[
h_{\partial \Omega \cap K} \| n \cdot \epsilon(v_{h,i}) \|_{0, \partial \Omega \cap K}^2 \leq \frac{\gamma_{\partial \Omega \cap K}}{\alpha_K} \| \epsilon(v_{h,i}) \|_{0,K}^2.
\]

(3.34)

For elements \( K \in \mathcal{K}_{\partial \Omega} \) such that \( K \cap \Omega_i \neq \emptyset, K \not\subset \Omega_i, \) let \( \mathcal{F}_{K,K'} \) be the set of all faces that has to be crossed to pass from \( K \) to \( K' \subset \Omega_i \) and \( N_F \) the number of such faces. Our assumptions guarantee that such an element \( K' \) exists and that there are a bounded number of faces in \( \mathcal{F}_{K,K'} \). We use the same idea as in \( \Pi \) and write that

\[
\epsilon(v_{h,i})_{|K} = \epsilon(v_{h,i})_{|K'} + \sum_{F \in \mathcal{F}_{K,K'}} \delta \| n_F \cdot \epsilon(v_{h,i}) \|_F n_F,
\]

(3.35)

where \( \delta = \pm 1 \) with the sign depending on the orientation of \( n_F \) so that the equality holds. We then have

\[
\| \epsilon(v_{h,i}) \|_{0,K}^2 \leq 2 \left( \frac{|K|}{|K'|} \| \epsilon(v_{h,i}) \|_{0,K'}^2 + N_F \frac{|K|}{|F|} \sum_{F \in \mathcal{F}_{K,K'}} \| n_F \cdot \epsilon(v_{h,i}) \|_F^2 \right),
\]

(3.36)

where we have used the Cauchy-Schwarz inequality and the geometric-arithmetic inequality. Due to Assumption 1 (quasi-uniformity) we have that there exists a constant \( c_q = \max \left( \frac{|K|}{|K'|}, \frac{|K|}{|F|} \right) \) and hence

\[
\| \epsilon(v_{h,i}) \|_{0,K}^2 \leq 2 c_q \left( \| \epsilon(v_{h,i}) \|_{0,K'}^2 + N_F h \sum_{F \in \mathcal{F}_{K,K'}} \| n_F \cdot \epsilon(v_{h,i}) \|_F^2 \right).
\]

(3.37)

Let \( N_{K_i} \) be the number of elements \( K \in \mathcal{K}_{\partial \Omega}, K \not\subset \Omega_i \) that have \( K_i \) as the closest element completely in \( \Omega_i \). Taking

\[
\delta_{\partial \Omega \cap K}^2 = \frac{2 c_q (1 + F) \mu_i \gamma_{\partial \Omega \cap K}^2}{E \alpha_K} \quad i = 1, 2,
\]

(3.38)
when $K \cap \Gamma$ intersects the element boundary exactly twice and otherwise

\[ \delta_{\partial \Omega, i, K} = \frac{(1 + F)\mu_i \gamma_{\partial \Omega, K}^2}{\overline{E} \alpha_K} \quad i = 1, 2, \quad (3.39) \]

we have from our choice of $\lambda_{\partial \Omega}$ (equation (3.33)) the inverse inequality (3.34), equation (3.37), and \( \frac{\mu_i \gamma_{\partial \Omega, K}}{\overline{E} \alpha_K} \geq \bar{C} \) that

\[
a_h(v_h, v_h) \geq \sum_{i=1}^{2} (1 - A - E(1 + \max(N_{K_i})) \| \mu_i^{1/2} \epsilon(v_{h,i}) \|_{0, \Omega_i}^2 + BC \| \{ n \cdot \bar{\mu}^{1/2} \epsilon(v_h) \} \|_{-1/2, h, \Gamma}^2
\]
\[
+ D \| \{ \mu \}^{1/2} [v_h] \|_{1/2, h, \Gamma}^2 + F \bar{C} \| n \cdot \mu^{1/2} \epsilon(v_h) \|_{-1/2, h, \partial \Omega}^2
\]
\[
+ G \| \mu^{1/2} v_h \|_{1/2, h, \partial \Omega}^2 - \sum_{i=1}^{2} \| \epsilon(v_{h,i}) \|_{K, F}^2 \| n_F \cdot [\epsilon(v_{h,i})] \|_{F}^2 \quad (3.40)
\]

Note that for each face in $F \in \mathcal{F}_{K, K}$, we can write

\[
[\epsilon(v_{h,i})]_{K} = n_F \cdot [\epsilon(v_{h,i})]_{F} \quad (3.41)
\]

and that each of the terms in equation (3.41) are constant $2 \times 2$ matrices. Hence,

\[
\sum_{F \in \mathcal{F}_{K, K}} \| n_F \cdot [\epsilon(v_{h,i})]_{F} \|_{0, F}^2 \leq \sum_{F \in \mathcal{F}_{K, K}} \| n_F \cdot [\nabla(v_{h,i})]_{F} \|_{0, F}^2.\quad (3.42)
\]

Finally, since $EN_F \leq \frac{(1+A)N_F}{1+\max(N_{K_i})} \leq 1$ we have

\[
\sum_{i=1}^{2} \sum_{F \in \mathcal{F}_{K, K}} \mu_i \| n_F \cdot [\epsilon(v_{h,i})]_{F} \|_{0, F}^2 \leq J(v_h, v_h) \quad (3.43)
\]

and the claim follows.

We will need the following technical lemma.

**Lemma 3.8.** Let $p_h = (p_{h,1}, p_{h,2}) \in \mathcal{V}_h$. There is a constant $C$ such that

\[
\| \mu_i^{-1/2} p_{h,i} \|_{0, \Omega_i}^2 \leq C \left( \| \mu_i^{-1/2} p_{h,i} \|_{0, \Omega_i}^2 + J(p_h, p_h) \right). \quad (3.44)
\]

**Proof.** For elements $K \in \mathcal{K}_i$ that are not entirely in $\Omega_i$, let $\mathcal{F}_{K, K_i}$ be the set of all faces that has to be crossed to pass from $K$ to the closest element $K_i \subset \Omega_i$ and $N_F$ the number of such faces. Assumption 4 guarantees that such an element $K_i$ exists and since the mesh is assumed to be shape regular there are a bounded number of faces in $\mathcal{F}_{K, K_i}$. We can write that

\[
p_{h,i}|_K = p_{h,i}|_{K_i} + \sum_{F \in \mathcal{F}_{K, K_i}} \delta \| n_F \cdot [\nabla p_{h,i}]_{F} \|_{F} \cdot (x - a_F), \quad (3.45)
\]

where $\delta = \pm 1$ with the sign depending on the orientation of $n_F$ so that the equality holds and $a_F$ is the center of gravity of $F$. Taking the square on both sides of identity (3.45), using the
Cauchy-Schwarz inequality and the geometric-arithmetic inequality, we get

\[
\|p_{h,i}\|_{0,K}^2 \leq 2 \left( \frac{|K|}{|K'|} \|p_{h,i}\|_{0,K'}^2 + N_F \sum_{F \in F_{K,K'}} \frac{|K|}{|F|} \|n_F \cdot \nabla p_{h,i}\|_F \|n_F \cdot (x - a_F)\|_{0,F}^2 \right)
\]

\[
\leq 2c_q \left( \|p_{h,i}\|_{0,K}^2 + N_F \sum_{F \in F_{K,K'}} h^3 \|n_F \cdot \nabla p_{h,i}\|_F \|h_{0,F}\|_{0,F}^2 \right),
\]

(3.46)

where the last inequality follows due to Assumption 1 (quasi-uniformity). Let \(N_{K'}\) be the number of elements in \(K_F\) that have \(K'\) as the closest element completely in \(\Omega\). Summing over all elements \(K \in K_F\) and using equation \((3.46)\) for element that are not entirely in \(\Omega\), we obtain

\[
\|\mu_i^{-1/2} p_{h,i}\|_{0,\Omega_{h,i}}^2 \leq \sum_{K \in K_F, K \subset \Omega_i} (1 + 2c_q N_{K'}) \|\mu_i^{-1/2} p_{h,i}\|_{0,K}^2,
\]

\[
+ 2c_q \max(N_{K'}) \sum_{K \in K_F, K \notin \Omega_i, F \in F_{K,K'}} \mu_i^{-1} h^3 \|n_F \cdot \nabla p_{h,i}\|_F \|h\|_{0,F}.
\]

(3.47)

To prove the inf-sup stability of \(b_h(\cdot, \cdot)\) we use some of the ideas in \([19]\). Introduce the piecewise constant function

\[
\bar{p} = \begin{cases} 
\mu_1|\Omega_1|^{-1} & \text{on } \Omega_1 \\
-\mu_2|\Omega_2|^{-1} & \text{on } \Omega_2.
\end{cases}
\]

(3.48)

Let \(M_0 = \text{span}\{\bar{p}\}\). For any \(p_h \in V_h\) we can write

\[
p_h = p_0 + p_{h,0}^\bot, \quad p_0 \in M_0, \quad p_{h,0} ^\bot \in M^\bot_{h,0}.
\]

(3.49)

The functions in \(M^\bot_{h,0}\) satisfy \((p_{h,0}^\bot, 1)_{\Omega_i} = 0, \ i = 1, 2, \text{ see } [19].

**Lemma 3.9.** For any \(p_0 \in M_0\), there exists \(v_{h,0} \in W_h\) such that

\[
b_h(v_{h,0}, p_0) \geq C_{1,p_0} \|\mu_i^{-1/2} p_0\|_{0,\Omega_1 \cup \Omega_2}^2, \quad \|v_{h,0}\|_h \leq C_{2,p_0} \|\mu_i^{-1/2} p_0\|_{0,\Omega_1 \cup \Omega_2}.
\]

(3.50)

**Proof.** Let \(\tilde{p}_0 = \mu^{-1} p_0\), then \((\tilde{p}_0, 1)_{\Omega_1 \cup \Omega_2} = 0\). Let \(I(\tilde{p}_0)\) be the continuous piecewise linear approximation of \(\tilde{p}_0\) which differs from \(\tilde{p}_0\) only in elements \(K \in K_F\). Let \(\alpha\) be the constant for which \((I(\tilde{p}_0) - \alpha, 1)_{\Omega} = 0\) and let \(q_h = I(\tilde{p}_0) - \alpha\). Since the underlying finite element spaces are inf-sup stable there exist \(v_{h,0} \in W_h \cap C(\Omega_1 \cup \Omega_2)\) with \(v_{h,0}|_{\partial\Omega} = 0\) such that

\[
\frac{b_h(v_{h,0}, q_h)}{\|\nabla v_{h,0}\|_{0,\Omega_1 \cup \Omega_2}} \geq C \|q_h\|_{0,\Omega_1 \cup \Omega_2}.
\]

(3.51)

We have

\[
b_h(v_{h,0}, \tilde{p}_0) = \frac{b_h(v_{h,0}, q_h)}{\|\nabla v_{h,0}\|_{0,\Omega_1 \cup \Omega_2}} + \frac{b_h(v_{h,0}, \tilde{p}_0 - q_h)}{\|\nabla v_{h,0}\|_{0,\Omega_1 \cup \Omega_2}}
\]

\[
\geq C \|q_h\|_{0,\Omega_1 \cup \Omega_2} - \sqrt{2} \|\tilde{p}_0 - q_h\|_{0,\Omega_1 \cup \Omega_2}
\]

\[
\geq C \|\tilde{p}_0\|_{0,\Omega_1 \cup \Omega_2} - (\sqrt{2} + C) \|\tilde{p}_0 - q_h\|_{0,\Omega_1 \cup \Omega_2}
\]

\[
\geq \left(C - \sqrt{2}\right) \|\tilde{p}_0 - q_h\|_{0,\Omega_1 \cup \Omega_2} + \left(C - \sqrt{2}\right) \|\tilde{p}_0\|_{0,\Omega_1 \cup \Omega_2}
\]

(3.52)

where we in the last step have used that

\[
|\alpha| = \frac{|(I(\tilde{p}_0) - \tilde{p}_0, 1)_{\Omega_1 \cup \Omega_2}|}{\|1\|_{0,\Omega_1 \cup \Omega_2}^2} \leq \frac{\|I(\tilde{p}_0) - \tilde{p}_0\|_{0,\Omega_1 \cup \Omega_2}}{\|1\|_{0,\Omega_1 \cup \Omega_2}}
\]

(3.53)
and hence
\[ \frac{\|\tilde{p}_0 - q_h\|_{0, \Omega_1 \cup \Omega_2}}{\|\tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2}} \leq 2 \frac{\|I(\tilde{p}_0) - \tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2}}{\|\tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2}} \leq C h^{1/2}. \]  
(3.54)

From the definition of $M_0$ one can see that
\[ \|\mu^{-1/2} p_0\|^2_{0, \Omega_1 \cup \Omega_2} = C(\mu, \Omega) \|\tilde{p}_0\|^2_{0, \Omega_1 \cup \Omega_2}, \quad b_h(v_{h,0}, p_0) = C(\mu, \Omega)b_h(v_{h,0}, \tilde{p}_0), \]  
(3.55)

with $C(\mu, \Omega) = \mu |\Omega_1|^{-1} + \mu_2 |\Omega_2|^{-1}$. We can choose $v_{h,0}$ so that equation (3.52) is satisfied and
\[ \|\nabla v_{h,0}\|_{0, \Omega_1 \cup \Omega_2} = \|\tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2} \] and obtain
\[ b_h(v_{h,0}, p_0) = C(\mu, \Omega)b_h(v_{h,0}, \tilde{p}_0) \geq cC(\mu, \Omega)\|\tilde{p}_0\|^2_{0, \Omega_1 \cup \Omega_2} = c\|\mu^{-1/2} p_0\|^2_{0, \Omega_1 \cup \Omega_2}. \]  
(3.56)

We have
\[ C(\mu, \Omega) \geq \min_{i=1,2} \left( \left|\Omega_i\right| \left(1 + \frac{|\Omega_1|}{|\Omega_2|}\right) \right) \mu_{\max} = \tilde{C} \mu_{\max}, \]  
(3.57)

and
\[ \|v_{h,0}\| \leq \mu_{\max}^{1/2} \|\nabla v_{h,0}\|_{0, \Omega_1 \cup \Omega_2} = \mu_{\max}^{1/2} \|\tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2} \leq \frac{C}{\mu_{\max}} \|\mu^{-1/2} p_0\|_{0, \Omega_1 \cup \Omega_2}. \]  
(3.58)

Finally, note that for $v_{h,0} = (v_{h,1}, v_{h,2}) \in W_h \cap C(\Omega_1 \cup \Omega_2)$ we can choose $v_{h,i} = v_{h,j}$ in $\Omega_{h,i} \setminus \Omega_j$, $j \neq i$ so that $\|v_{h,0}\| \leq C \|v_{h,0}\|$. 

Lemma 3.10. For any $p_{h,0}^i = (p_{h,0}^i, p_{h,0}^i, p_{h,0}^i) \in M_{h,0}$ there exists $v_h \in W_h$ such that
\[ b_h(v_h, p_{h,0}^i) \geq C_1 p_{h,0}^i \|\mu^{-1/2} p_{h,0}^i\|^2_{0, \Omega_{h,1} \cup \Omega_{h,2}} - C_2 p_{h,0}^i J(p_{h,0}^i, p_{h,0}^i) \]  
(3.59)

and $\|v_h\| \leq C \|\mu^{-1/2} p_{h,0}^i\|_{0, \Omega_{h,1} \cup \Omega_{h,2}}$.

Proof. Let $\alpha$ be the constant for which
\[ (p_{h,0}^i - \alpha, 1)_{\Omega_{h,i}} = 0 \]  
(3.60)

and let $q_{h,i} = p_{h,0}^i - \alpha$. Using the fact that the underlying finite element spaces are inf-sup stable with a uniform constant on any polygon of shape regular elements which has no element with two edges on the boundary, see Brezzi-Fortin [20], Proposition 6.1, Page 252, we can for each $q_h = (q_{h,1}, q_{h,2})$ choose $v_{h,\Omega_i} = (v_{h,1}, v_{h,2}) \in W_h$ with $v_{h,i} \in W_{h,i}$ such that $v_{h,i} = 0$ on $\Omega_{h,i} \setminus \omega_{h,i}$ and on $\partial \Omega$ and $v_{h,j} = 0$ for $j \neq i$ so that
\[ b_h(v_{h,\Omega_i}, q_h) \frac{\|v_{h,\Omega_i}\|}{\|v_{h,\Omega_i}\|} \geq C \|\mu^{-1/2} q_{h,i}\|_{0, \omega_{h,i}}. \]  
(3.61)

Using the inverse inequality (3.17) and that $\text{supp}(v_{h,\Omega_i}) = \omega_{h,i}$
\[ J(v_{h,\Omega_i}, v_{h,\Omega_i}) \leq c_q^{-1} \|\mu^{1/2} \nabla v_{h,i}\|^2_{0, \omega_{h,i}}. \]  
(3.62)

which together with Korn’s inequality [21] Eq. (1.19) yields
\[ \|v_{h,\Omega_i}\|^2_h = \|v_{h,\Omega_i}\|^2 + J(v_{h,\Omega_i}, v_{h,\Omega_i}) \leq C \|v_{h,\Omega_i}\|^2. \]  
(3.63)

We can choose $v_{h,\Omega_i}$ so that equation (3.61) and (3.63) are satisfied and $\|v_{h,\Omega_i}\|_h = \|\mu^{-1/2} q_{h,i}\|_{0, \Omega_{h,i}}$. We then have
\[ b_h(v_{h,\Omega_i}, q_h) \geq C_q \|\mu^{-1/2} q_{h,i}\|_{0, \Omega_{h,i}} \|\mu^{-1/2} q_{h,i}\|_{0, \Omega_{h,i}}. \]  
(3.64)
Lemma 3.8 then yields

$$ \| \mu_i^{-1/2} q_{h,i} \|_{0, \Omega_{h,i}}^2 \leq C \left( \| \mu_i^{-1/2} q_{h,i} \|_{0, \omega_{h,i}}^2 + J(q_h, q_h) \right) \leq C \left( C_q^{-1} b_h(v_{h, \Omega_i}, q_h) + J(q_h, q_h) \right). \quad (3.65) $$

Note that $b_h(v_{h, \Omega_i}, q_h) = b_h(v_{h, \Omega_i}, p_{h,0}^+)$, $J(q_h, q_h) = J(p_{h,0}^+, p_{h,0}^+)$ and

$$ \| \mu_i^{-1/2} \alpha \| \leq \left( \| \mu_i^{-1/2} \alpha \|_{0, \Omega_{h,i}} \right) \leq \left( \| \mu_i^{-1/2} \alpha \|_{0, \omega_{h,i}} \right) \leq \| \mu_i^{-1/2} \alpha \|_{0, \omega_{h,i}} \leq \left( \| \mu_i^{-1/2} \alpha \|_{0, \Omega_{h,i}} \right). \quad (3.66) $$

Using equation (3.66) we get

$$ \| \mu_i^{-1/2} q_{h,i} \|_{0, \Omega_{h,i}}^2 \geq \| \mu_i^{-1/2} p_{h,i}^+ \|_{0, \Omega_{h,i}}^2 - \| \mu_i^{-1/2} \alpha \|_{0, \Omega_{h,i}}^2 \ \geq \ \| \mu_i^{-1/2} p_{h,i}^+ \|_{0, \Omega_{h,i}}^2 \left( 1 - \frac{\| \mu_i^{-1/2} \alpha \|_{0, \Omega_{h,i}}^2}{\| \mu_i^{-1/2} \alpha \|_{0, \omega_{h,i}}^2} \right). \quad (3.67) $$

We assume $\| \mu_i^{-1/2} \alpha \|_{0, \Omega_{h,i}} = ch^{1/2}$ and hence equation (3.65) and (3.67) yield

$$ \| \mu_i^{-1/2} p_{h,i}^+ \|_{0, \Omega_{h,i}}^2 \leq C (1 - ch^{1/2})^{-1} \left( C_q^{-1} b_h(v_{h, \Omega_i}, p_{h,0}^+) + J(p_{h,0}^+, p_{h,0}^+) \right). \quad (3.68) $$

From equation (3.66) we also obtain $\| v_{h, \Omega_i} \|_h \leq C \| \mu_i^{-1/2} p_{h,i}^+ \|_{0, \Omega_{h,i}}$. Finally, taking $\bar{v}_h = v_{h, \Omega_i} + v_{h, \Omega_2}$ we have

$$ b_h(\bar{v}_h, p_{h,0}^+) \geq C_1 \sum_{i=1}^2 \| \mu_i^{-1/2} p_{h,i}^+ \|_{0, \Omega_{h,i}}^2 - C_2 J(p_{h,0}^+, p_{h,0}^+) \quad (3.69) $$

and $\| \bar{v}_h \|_h \leq C \sum_{i=1}^2 \| \mu_i^{-1/2} p_{h,i}^+ \|_{0, \Omega_{h,i}}$. \hfill \Box

**Lemma 3.11.** For any $p_h \in V_h$ there exists $v_h \in W_h$ and constants $C_1, C_3 > 0$, and $C_2 \geq 0$ such that

$$ b_h(v_h, p_h) \geq C_1 \| \mu^{-1/2} p_h \|_{0, \Omega_{h,1} \cup \Omega_{h,2}}^2 - C_2 J(p_h, p_h), \quad \| v_h \|_h \leq C_3 \| \mu^{-1/2} p_h \|_{0, \Omega_{h,1} \cup \Omega_{h,2}} \quad (3.70) $$

**Proof.** If $p_h \in V_h$ is piecewise constant, i.e. $p_h = p_0$ where $p_0 \in M_0$, the proof follows from Lemma 3.9 with $C_2 = 0$. Otherwise, we have $p_h = p_0 + p_{h,0}^+$, where $p_0 \in M_0$ and $p_{h,0}^+ \in M_{h,0}^+$. Let $v_{h,0}^+$ be such that $\mu^{-1/2} v_{h,0}^+ \in P_{h,0}^+$ is satisfied and $v_{h,0}^+$ such that Lemma 3.10 is satisfied. For $\alpha > 0$, define $v_h = v_{h,0}^+ + \alpha \bar{v}_h$. Note that $b_h(\bar{v}_h, p_0) = 0$, since $\bar{v}_h$ vanishes on $\Gamma \cup \partial \Omega$ and $p_0$ is constant on each subdomain $\Omega_i$, $i = 1, 2$, and

$$ b_h(v_h, p_h) = b_h(v_h, p_0) + b_h(v_h, p_{h,0}^+) + b_h(\bar{v}_h, p_0) + b_h(\bar{v}_h, p_{h,0}^+) \quad (3.71) $$

since $v_{h,0}^+$ is continuous and vanishes on $\partial \Omega$. Also, $J(p_h, p_h) = J(p_{h,0}^+, p_{h,0}^+)$. Thus,

$$ b_h(v_h, p_h) = b_h(v_h, p_0) + b_h(v_h, p_{h,0}^+) + \alpha b_h(v_h, p_0) + \alpha b_h(\bar{v}_h, p_{h,0}^+) \geq C_1 \mu^{-1/2} p_0 \|_{0, \Omega_{h,1} \cup \Omega_{h,2}}^2 - C_2 \mu^{-1/2} p_0 \|_{0, \Omega_{h,1} \cup \Omega_{h,2}}^2 - C_2 \mu^{-1/2} p_{h,0}^+ \|_{0, \Omega_{h,1} \cup \Omega_{h,2}}^2 \quad (3.72) $$

For sufficiently large $\alpha$. Finally, we also have

$$ \| v_h \|_h \leq \| v_{h,0}^+ \|_h + \alpha \| \bar{v}_h \|_h \leq C \| \mu^{-1/2} p_h \|_{0, \Omega_{h,1} \cup \Omega_{h,2}}. \quad (3.73) $$
We are now ready to prove the inf-sup theorem.

**Proof.** (of Theorem 3.1) Note that if \( p_h \in V_h \) is constant, \( p_h = 0 \) since \( (\mu^{-1} p_h, 1)_{\Omega_1 \cup \Omega_2} = 0 \). Letting \( v_h = u_h \) and using the coercivity of \( a_h(\cdot, \cdot) \) we have
\[
A_h(u_h, p_h; u_h, p_h) + \varepsilon_u J(u_h, u_h) + \varepsilon_p J(p_h, p_h) = a_h(u_h, u_h) + \varepsilon_u J(u_h, u_h)
\geq C_{\text{coer}} \min(1, \varepsilon_u) |||u_h|||^2_h = C_{\text{coer}} \min(1, \varepsilon_u) |||(u_h, p_h)|||^2_h,
\]
and hence the proof follows. Otherwise, let \( v_h \) be such that Lemma 3.11 is satisfied and \(|||v_h|||_h = \mu^{-1/2} p_h|||_{\Omega_{h,1} \cup \Omega_{h,2}}||| \). Then, using the coercivity and continuity of \( a_h(\cdot, \cdot) \) (Lemma 3.5 and 3.7), Cauchy–Schwarz inequality, and the stability of \( b_h(\cdot, \cdot) \) (Lemma 3.11) we have for \( \alpha > 0 \)
\[
A_h(u_h, p_h; u_h - \alpha v_h, p_h) + \varepsilon_u J(u_h, u_h - \alpha v_h) + \varepsilon_p J(p_h, p_h) = a_h(u_h, u_h) + \varepsilon_u J(u_h, u_h) - \alpha (a_h(u_h, v_h) + \varepsilon_u J(u_h, v_h)) + \alpha b_h(v_h, p_h) + \varepsilon_p J(p_h, p_h) \geq C_{\text{coer}} \min(1, \varepsilon_u) |||u_h|||^2_h - \alpha \max(C_{\text{cont}}, \varepsilon_u) (|||u_h|||^2_h + |||v_h|||^2_h + (J(u_h, u_h))^2) / (\max(|||v_h|||_h, (J(v_h, v_h))^2)) + \alpha C_1 \mu^{-1/2} p_h^2 \|_{\partial \Omega_{h,1} \cup \partial \Omega_{h,2}} - C_2 J(p_h, p_h) + \varepsilon_p J(p_h, p_h) \geq \max(D, \max(D_1, D_2, D_3)) (|||u_h|||^2_h + |||v_h|||^2_h)
\]
with \( D = \min(D_1, D_2, D_3) \), where
\[
D_1 = \left(C_{\text{coer}} \min(1, \varepsilon_u) - \alpha \max(C_{\text{cont}}, \varepsilon_u) / \delta\right) > 0,
D_2 = \alpha \left(C_1 - \max(C_{\text{cont}}, \varepsilon_u) / \delta\right) > 0,
D_3 = (\varepsilon_p - \alpha C_2) > 0,
\]
provided \( \delta \) and \( \alpha \) are sufficiently small. Finally, the proof follows using that
\[
|||u_h - \alpha v_h, p_h)|||_h \leq |||u_h, p_h)|||_h + \alpha |||v_h|||_h \leq (1 + \alpha) |||u_h, p_h)|||_h
\]
in equation (3.75). \( \square \)

### 3.3. Approximation properties

In this Section we will show that the spaces \( V_h \) and \( W_h \) have optimal approximation properties on \( H^1(\Omega_1 \cup \Omega_2) \) and \( H^2(\Omega_1 \cup \Omega_2) \), respectively, in the energy norm. In order to construct an interpolation operator we recall that there is an extension operator \( E_i^* : H^s(\Omega_i) \to H^s(\Omega) \), \( i = 1, 2, s \geq 0 \), such that \( E_i^* w|_{\Omega_i} = w_i \) and
\[
|||E_i^* w_i|||_{s, \Omega_i} \leq C |||w_i|||_{s, \Omega}, \quad \forall w_i \in H^s(\Omega_i).
\]
See [22] for further details. Let \( \pi_h : H^s(\Omega) \to V_{h,0} \) be the standard Scott-Zhang interpolation operator [23] and recall the stability property
\[
|||\pi_h w|||_{s, \Omega} \leq C |||w|||_{s, \Omega}, \quad 0 \leq r \leq \min(1, s), \quad \forall w \in H^s(\Omega)
\]
and the approximation property of the interpolation operator
\[
|||w - \pi_h w|||_{r, \mathcal{K}} \leq C h_{\mathcal{K}}^{s-r} |||w|||_{s, N(\mathcal{K})}, \quad 0 \leq r \leq s \leq 2, \quad \forall K \in \mathcal{K}, \forall w \in H^s(\Omega),
\]
where \( \mathcal{N}(\mathcal{K}) \) is the set of elements in \( \mathcal{K} \) sharing at least one vertex with \( K \). We define
\[
\pi^*_{h,i} w_i = \pi_h E_i^* w_i|_{\Omega_{h,i}}, \quad \forall w_i \in H^s(\Omega_i)
\]
and for \( w = (w_1, w_2) \) with \( w_i|_{\Omega_i} \in H^s(\Omega_i) \) we define
\[
\pi^*_{h} w = (\pi^*_{h,1} w_1, \pi^*_{h,2} w_2).
\]
We will use the same interpolant for velocities and pressures. For the velocities \( s = 2, V_{h,0} = W_{h,0} \), and \( \pi^*_{h,i} : H^2(\Omega_i) \to V_{h,i}, i = 1, 2 \) while for the pressures \( s = 1, V_{h,0} = V_{h,\mu} \) and \( \pi^*_{h,i} : H^1(\Omega_i) \to V_{h,i}, i = 1, 2 \). In the norm \(|||\cdot, \cdot|||\) we have the following interpolation error estimate:
Lemma 3.12. It holds that

\[ \| (v - \pi_h^* u, p - \pi_h^* p) \| \leq h^2 (C_u \| u \|_{2, \Omega_1 \cup \Omega_2} + C_p \| p \|_{1, \Omega_1 \cup \Omega_2}). \]  

(3.83)

**Proof.** Recall the definition of the norm \( \| (\cdot, \cdot) \| \) (equation (3.8)). The interface and boundary contributions can be estimated in terms of element contributions by applying the trace inequality in Lemma 3.5. Then, for the element contributions, applying the approximation property of the interpolation operator (3.80), and finally using the stability of the extension operator equation (3.78) yields the desired estimate. \( \square \)

### 3.4. A priori error estimates

**Theorem 3.2.** The following error estimate holds

\[ \| (u - u_h, p - p_h) \| \leq C h \left( \| u \|_{2, \Omega_1 \cup \Omega_2} + \| p \|_{1, \Omega_1 \cup \Omega_2} \right). \]  

(3.84)

**Proof.** We have

\[ \| (u - u_h, p - p_h) \| \leq \| (u - \pi_h^* u, p - \pi_h^* p) \| + \| (\pi_h^* u - u_h, \pi_h^* p - p_h) \|. \]  

(3.85)

Here the first term can be estimated directly using the interpolation error estimate (Lemma 3.12)

\[ \| (u - \pi_h^* u, p - \pi_h^* p) \| \leq C h \left( \| u \|_{2, \Omega_1 \cup \Omega_2} + \| p \|_{1, \Omega_1 \cup \Omega_2} \right). \]  

(3.86)

Turning to the second term we use the inf-sup condition (Theorem 3.1) followed by the consistency relation, Lemma 3.3 to get

\[ \| (\pi_h^* u - u_h, \pi_h^* p - p_h) \| \leq C_s^{-1} \sup_{(v, q) \in W_h \times V_h} \frac{A_h(\pi_h^* u - u_h, \pi_h^* p - p_h; v, q_h) + \epsilon_u J(\pi_h^* u - u_h, v_h) + \epsilon_p J(\pi_h^* p - p_h, q_h)}{\| (v_h, q_h) \|_h} \]  

\[ \leq C_s^{-1} \sup_{(v, q) \in W_h \times V_h} \frac{A_h(\pi_h^* u - u, \pi_h^* p - p; v, q_h) + \epsilon_u J(\pi_h^* u, v_h) + \epsilon_p J(\pi_h^* p, q_h)}{\| (v_h, q_h) \|_h} \]  

(3.87)

Using the continuity of \( A_h(\cdot, \cdot, \cdot, \cdot) \) and that from the Cauchy-Schwarz inequality we have

\[ J(\pi_h^* u, v_h) \leq J(\pi_h^* u, \pi_h^* u)^{1/2} J(v_h, v_h)^{1/2} \leq J(\pi_h^* u, \pi_h^* u)^{1/2} \| (v_h, q_h) \|_h \]  

(3.88)

and similarly for \( J(\pi_h^* p, q_h) \) it follows that

\[ \| (\pi_h^* u - u_h, \pi_h^* p - p_h) \| \leq C_s^{-1} \left( C_{contA} \| (u - \pi_h^* u, p - \pi_h^* p) \| + \epsilon_u J(\pi_h^* u, \pi_h^* u)^{1/2} + \epsilon_p J(\pi_h^* p, \pi_h^* p)^{1/2} \right). \]  

(3.89)

The first term is estimated using the interpolation error estimate. For \( u_i \in H^2(\Omega_i) \) and \( E_i^2 u = (E_i^2 u_1, E_i^2 u_2) \) we have that

\[ J(\pi_h^* u, \pi_h^* u) = J(E_i^2 u - \pi_h^* u, E_i^2 u - \pi_h^* u) \]  

\[ \leq \sum_{i=1}^{2} \| \sum_{F \in J_i} C_{mu} h \| \left[ n_F \cdot \nabla (E_i^2 u_i - \pi_h^* u_i) \right] \| \|_{L_0,F}^2 \]  

\[ \leq \sum_{i=1}^{2} C_{mu} \| \sum_{K \in K_i} (\| E_i^2 u_i - \pi_h^* u_i \|_{L_2,K}^2 + h^2 \| E_i^2 u_i - \pi_h^* u_i \|_{L_0,K}^2) \]  

\[ \leq C h^2 \sum_{i=1}^{2} \| \sum_{K \in K_i} \| E_i^2 u_i \|_{L_2,K}^2 \| \| \leq C h^2 \sum_{i=1}^{2} \mu_i \| u_i \|_{L_2,\Omega_i}^2. \]  

(3.90)
where we have used the Cauchy-Schwarz inequality, the trace inequality in Lemma 3.3, the approximation property of the interpolation operator (equation (3.80)), and finally the stability of the extension operator (equation (3.78)).

Third term in equation (3.89) can be estimated using the inverse estimate

$$h \| n \cdot \nabla \pi^*_h p_i \|^2_{0,F} \leq C \| \nabla \pi^*_h p_i \|^2_{0,K^*_F \cup K^-_F},$$

(3.91)

where $K^*_F$ and $K^-_F$ are the elements sharing face $F$. The inverse estimate together with the stability property of $\pi_h$ equation (3.79) and the stability of the extension operator (equation (3.78)) yield

$$J(\pi_h p, \pi^*_h p) = \sum_{i=1}^2 \sum_{F \in {\mathcal F}_T,i} C \mu_i^{-1} h^3 \| n_F \cdot \nabla \pi^*_h p_i \|^2_{0,F} \leq C h^2 \sum_{i=1}^2 \sum_{F \in {\mathcal F}_T,i} \| \nabla \pi_h \nabla \pi^*_h p_i \|^2_{0,K^*_F \cup K^-_F} \leq C h^2 \sum_{i=1}^2 \sum_{k \in {\mathcal K}} \| \pi_h \nabla \pi^*_h p_i \|^2_{1,k} \leq C h^2 \sum_{i=1}^2 \mu_i^{-1} \| p_i \|^2_{1,\Omega_i}. $$

(3.92)

Collecting the estimates (3.85), (3.86), (3.89), (3.90), and (3.92) the theorem follows. □

An $L^2$-estimate for the velocity can be proven assuming additional regularity and using the Aubin-Nitsche duality argument, following the proof of [3, Proposition 11].

4. Estimate of the condition number

Let $\{ \varphi_i \}_{i=1}^{N_1}$ and $\{ \chi_i \}_{i=1}^{N_2}$ be a standard finite element basis in $W_h$ and $V_h$, respectively. Let $A$ be the stiffness matrix associated with the formulation (2.27). Matrix $A$ has dimension $(N_1 + N_2) \times (N_1 + N_2)$. For the Euclidian norm of a vector $X \in \mathbb{R}^N$ we use the notation $|X|^N = \sum_{i=1}^N X_i^2$.

We recall that the spectral condition number $\kappa(A)$ is defined by

$$\kappa(A) = |A|_{N} |A^{-1}|_{N}. $$

(4.1)

Here $N = (N_1 + N_2)$ and $|A|_{N} = \text{sup}_{|X|_{N}=1} |AX|_{N}$ for $A \in \mathbb{R}^{N \times N}$. The expansion $u_h = \sum_{i=1}^{N_1} U_i \varphi_i$ and $p_h = \sum_{i=1}^{N_2} P_i \chi_i$ define isomorphisms that map $u_h \in W_h$ to $U \in \mathbb{R}^{N_1}$ and $p_h \in V_h$ to $P \in \mathbb{R}^{N_2}$, respectively. We have for $V \in \mathbb{R}^N$ being the concatenation of $U$ and $P$ the following estimate

$$c_1 h^{-1} (|p_h|_{0,\Omega_h,1 \cup \Omega_h,2} + |u_h|_{0,\Omega_h,1 \cup \Omega_h,2}) \leq |V|_{N} \leq c_2 h^{-1} (|p_h|_{0,\Omega_h,1 \cup \Omega_h,2} + |u_h|_{0,\Omega_h,1 \cup \Omega_h,2} + |u_h|_{0,\Omega_h,1 \cup \Omega_h,2}).$$

(4.2)

To derive an estimate of the condition number we first prove a Poincare type inequality in Lemma 4.1 and an inverse estimate in Lemma 4.2. Then the condition number estimates follows from these two lemmas and the approach in [23].

**Lemma 4.1.** The following estimate holds

$$ (|q_h|_{0,\Omega_h,1 \cup \Omega_h,2} + |v_h|_{0,\Omega_h,1 \cup \Omega_h,2}) \leq C \max(\mu_{\text{max}}^{1/2}, \mu_{\text{min}}^{-1/2}) (|v_h|, |q_h|)_h $$

(4.3)

for all $(v_h, q_h) \in W_h \times V_h$.

**Proof.** We have that $|q_h|_{0,\Omega_h,1 \cup \Omega_h,2} \leq \mu_{\text{max}}^{1/2} |q_h|_{0,\Omega_h,1 \cup \Omega_h,2}$ and $|v_h|_{0,\Omega_h,1 \cup \Omega_h,2} \leq C \mu_{\text{min}}^{-1/2} |v_h|_h$. We need to show that $|v_h|_{0,\Omega_h,1 \cup \Omega_h,2} \leq C \mu_{\text{min}}^{-1/2} |v_h|_h$. Let $\phi$ be the solution of the dual problem

$$- \nabla \cdot (\phi) = v_h \text{ in } \Omega, \quad \nabla \cdot \phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial \Omega. $$

(4.4)
We assume that we have elliptic regularity so that
\[ \|\phi\|^2_{2,\Omega} \leq C_\Omega \|v_h\|^2_{0,\Omega}. \] (4.5)

Multiplying the dual problem with \( v_h \), integrating by parts, and using the Cauchy-Schwarz inequality we get
\[
\|v_h\|^2_{0,\Omega \cup \Omega_2} = \langle \epsilon(v_h), \epsilon(\phi) \rangle_{0,\Omega \cup \Omega_2} - \langle \|v_h\| \cdot n \cdot \epsilon(\phi) \rangle_{\Gamma} - \langle v_h, n \cdot \epsilon(\phi) \rangle_{\partial \Omega} \leq 
C \left( \|\epsilon(v_h)\|^2_{0,\Omega \cup \Omega_2} + \|v_h\|^2_{1/2,\Omega,\Gamma} + \|v_h\|^2_{1/2,\partial\Omega} \right)^{1/2} \left( \|\epsilon(\phi)\|^2_{0,\Omega \cup \Omega_2} + \|n \cdot \epsilon(\phi)\|^2_{-1/2,\Omega,\Gamma} \right)^{1/2}
\] (4.6)

Using that \( \|n \cdot \epsilon(\phi)\|^2_{0,\Gamma} \leq C_\Gamma \|\phi\|^2_{2,\Omega} \) and inequality (4.5) we have
\[
\|\epsilon(\phi)\|^2_{0,\Omega \cup \Omega_2} + \|n \cdot \epsilon(\phi)\|^2_{-1/2,\Omega,\Gamma} \leq C \|\phi\|^2_{2,\Omega \cup \Omega_2} \leq C \|v_h\|^2_{0,\Omega \cup \Omega_2}.
\] (4.7)

Thus,
\[
\|v_h\|_{0,\Omega \cup \Omega_2} \leq C \left( \|\epsilon(v_h)\|^2_{0,\Omega \cup \Omega_2} + \|v_h\|^2_{1/2,\Omega,\Gamma} + \|v_h\|^2_{1/2,\partial\Omega} \right)^{1/2} \leq C \mu_{\min}^{-1/2} \|v_h\|. \]
(4.8)

Following the proof of Lemma [3.8] we can show that
\[
\sum_{i=1}^{2} \|v_{h,i}\|_{0,\Omega_{h,i}} \leq C \left( \|v_h\|_{0,\Omega \cup \Omega_2} + \left( \sum_{i=1}^{2} \sum_{F \in \mathcal{F}_h} h^s \|n_F \cdot \nabla v_{h,i}\|_{0,F} \right)^{1/2} \right).
\] (4.9)

Finally, we have that
\[
\sum_{i=1}^{2} \|v_{h,i}\|_{0,\Omega_{h,i}} \leq C \left( \mu_{\min}^{-1/2} \|v_h\| + \mu_{\min}^{-1/2} \left( J(v_h, v_h) \right)^{1/2} \right) \leq C \mu_{\min}^{-1/2} \|v_h\|. \]
(4.10)

Recalling the definition of the norm \( \|(\cdot, \cdot)\|_h \), the claim follows.

\[\Box\]

**Lemma 4.2.** The following estimate holds
\[\|(v_h, q_h)\| \leq C h^{-1} \max(\mu_{\max}^{1/2}, \mu_{\min}^{1/2}) \left( \|q_h\|_{0,\Omega_{h,1} \cup \Omega_{h,2}} + \|v_h\|_{0,\Omega_{h,1} \cup \Omega_{h,2}} \right) \]
for all \((v_h, q_h) \in W_h \times V_h\).

**Proof.** Note that \( \|\mu^{-1/2} q_h\|_{0,\Omega_{h,1} \cup \Omega_{h,2}} \leq \mu_{\min}^{-1/2} \|q_h\|_{0,\Omega_{h,1} \cup \Omega_{h,2}} \). Using the Cauchy-Schwarz inequality and the trace inequality (3.17) we have
\[
J(q_h, q_h) \leq \sum_{i=1}^{2} \mu_{h,i}^{-1} \sum_{F \in \mathcal{F}_{h,i}} C h^3 \|\nabla q_{h,i}\|_{0,F} \|q_{h,i}\|_{0,F} \leq \mu_{\min}^{-1} C h^2 \sum_{i=1}^{2} \sum_{K \in \mathcal{K}_i} C \|\nabla q_{h,i}\|_{0,K} \|q_{h,i}\|_{0,K} \leq \mu_{\min}^{-2} C \sum_{i=1}^{2} \sum_{K \in \mathcal{K}_i} C \|q_{h,i}\|_{0,K} \|q_{h,i}\|_{0,K} \|q_{h,i}\|_{0,K} \|q_{h,i}\|_{0,K} \|q_{h,i}\|_{0,K} \|q_{h,i}\|_{0,K} \|
\] (4.12)

In the same way we obtain
\[
J(v_h, v_h) \leq C \mu_{\max} C h^{s-3} \|v_h\|_{0,\Omega_{h,1} \cup \Omega_{h,2}}^2.
\] (4.13)

The standard inverse inequality (3.16) yields \( \|\mu^{1/2} \epsilon(v_h)\|_{0,\Omega_{1} \cup \Omega_{2}} \leq C h^{-1/2} \mu_{\max}^{-1/2} \|v_h\|_{0,\Omega_{1} \cup \Omega_{2}} \). The Lemma follows using the trace inequalities (3.17) and (3.18) on each of the interface and boundary contributions to \( \|v_h\| \).

\[\Box\]
Theorem 4.1. The following estimate of the spectral condition number of the stiffness matrix holds
\[ \kappa(A) \leq C \max(\mu_{\max}, \mu_{\min}^{-1}) h^{-2}. \] (4.14)

Proof. We need to estimate \(|A|_{N}\) and \(|A^{-1}|_{N}\). Let \(V \in \mathbb{R}^{N}\) and \(W \in \mathbb{R}^{N}\) be the vectors containing the coefficients corresponding to \((v_h, q_h) \in W_h \times V_h\) and \((w_h, r_h) \in W_h \times V_h\), respectively. Starting with \(|A|_{N}\) we have
\[
|A V|_{N} = \sup_{W \in \mathbb{R}^{N}} \frac{(AV, W)|_{N}}{|W|_{N}} = \sup_{(w_h, r_h) \in W_h \times V_h} \frac{A_h(v_h, q_h; w_h, r_h) + \varepsilon u J(v_h, w_h) + \varepsilon p J(q_h, r_h)}{|W|_{N}}. \tag{4.15}
\]

We now use the continuity of \(A_h(\cdot, \cdot; \cdot, \cdot)\) together with that \(||(v_h, q_h)|| \leq C \|\|(v_h, q_h)||\|_{h}\) and the Cauchy-Schwarz inequality to obtain
\[
A_h(v_h, q_h; w_h, r_h) + \varepsilon u J(v_h, w_h) + \varepsilon p J(q_h, r_h) \leq C \max(\mu_{\max}, \mu_{\min}^{-1}) |V|_{N} |W|_{N}. \tag{4.16}
\]

Lemma 4.2 and equation (4.2) then yield
\[
A_h(v_h, q_h; w_h, r_h) + \varepsilon u J(v_h, w_h) + \varepsilon p J(q_h, r_h) \leq C \max(\mu_{\max}, \mu_{\min}^{-1}) |V|_{N} |W|_{N}. \tag{4.17}
\]

Thus, we have the estimate
\[
|A|_{N} = \sup_{V \in \mathbb{R}^{N}} \frac{|AV|_{N}}{|V|_{N}} \leq C \max(\mu_{\max}, \mu_{\min}^{-1}). \tag{4.18}
\]

Next we turn to the estimate of \(|A^{-1}|_{N}\). Using equation (4.2), Lemma 4.1 and the inf-sup stability (Theorem 3.1) we get
\[
|V|_{N} \leq C h^{-1} \left( \|q_h\|_{0, \Omega_h, \cup \Omega_h, 2} + \|v_h\|_{0, \Omega_h, \cup \Omega_h, 2} \right)
\]
\[
\leq C \max(\mu_{\max}^{1/2}, \mu_{\min}^{-1/2}) h^{-1} \|(v_h, q_h)\|_{h}
\]
\[
\leq C \max(\mu_{\max}^{1/2}, \mu_{\min}^{-1/2}) h^{-1} \sup_{(w_h, r_h) \in W_h \times V_h} \frac{A_h(v_h, q_h; w_h, r_h) + \varepsilon u J(v_h, w_h) + \varepsilon p J(q_h, r_h)}{\|(w_h, r_h)\|_{h}}
\]
\[
\leq C \max(\mu_{\max}^{1/2}, \mu_{\min}^{-1/2}) h^{-1} \sup_{W \in \mathbb{R}^{N}} \frac{(AV, W)|_{N}}{|W|_{N}} \frac{|W|_{N}}{\|(w_h, r_h)\|_{h}}
\]
\[
\leq C \max(\mu_{\max}^{1/2}, \mu_{\min}^{-1/2}) h^{-2} |AV|_{N} \left( \|r_h\|_{0, \Omega_h, \cup \Omega_h, 2} + \|w_h\|_{0, \Omega_h, \cup \Omega_h, 2} \right). \tag{4.19}
\]

We conclude that \(|V|_{N} \leq C \max(\mu_{\max}, \mu_{\min}^{-1}) h^{-2} |AV|_{N}\). Setting \(V = A^{-1} W\) we obtain
\[
|A^{-1}|_{N} \leq C \max(\mu_{\max}, \mu_{\min}^{-1}) h^{-2}. \tag{4.20}
\]
Combining estimates (4.18) and (4.20) of \(|A|_{N}\) and \(|A^{-1}|_{N}\) the theorem follows.

5. Numerical examples

We have shown that the proposed finite element method is of optimal convergence order and results in a well-conditioned equation system. In this section we present results using the proposed method (see Section 2.4) for numerical experiments in two space dimensions. We study the convergence rate of the numerical solution and the condition number of the system matrix for three examples. A direct solver is used to solve the linear systems.

Unless stated otherwise, we report the size of the velocity mesh \( h_x \). The pressure mesh is twice as coarse. The parameters \( \kappa_1 \) and \( \kappa_2 \) are chosen according to expression (2.15) and the penalty parameter \( \lambda_\Gamma \) is chosen locally according to expression (2.16). Dirichlet conditions for the velocity are imposed weakly and the penalty parameter \( \lambda_{\partial \Omega} \) enforcing the boundary conditions is chosen according to equation (2.22).

Both the stabilization terms \( J(p_h, q_h) \) and \( J(u_h, v_h) \) are needed in order to have control of the condition number also in the cases when the interface is very close to a mesh line. The stabilization parameter \( \varepsilon_p = 1 \) and \( \varepsilon_u = 10^{-3} \) in all the examples. The errors are not sensitive to these parameters. Also, remember that we have defined \( \mu = 2\mu_i \) in \( \Omega_i, \ i = 1, 2 \).

5.1. Example 1: A continuous problem

We consider a continuous problem presented in [3]. The computational domain is \([0, 1] \times [0, 1]\), the interface is a circle centered in \((0.5, 0.5)\) with radius \(0.3\) and \( \mu = 2 \). The Dirichlet boundary conditions on \( \partial \Omega \) are chosen such that the exact solution is given by \( u = (20xy^3, 5x^4 - 5y^4) \) and \( p = 60x^2y - 20y^3 - 5 \).

In this example we use a regular mesh. We choose \( \lambda_{\partial \Omega} \) according to equation (2.22) with \( G, F, \) and \( E \) such that \( \lambda_{\partial \Omega} = \frac{15}{h} \). Furthermore, we take \( \frac{L + B}{A} = 2 \) and \( D = 0.05 \) in the expression for the penalty parameter \( \lambda_\Gamma \). The condition number of the system matrix and the error depends on these constants. However, we have not optimized these constants.

In Fig. 3 we show the spectral condition number and the error in the pressure as a function of the stabilization parameter \( \varepsilon_p \) for \( h_x = 0.0125 \). As seen in the figure the condition number of the system increases as \( \varepsilon_p \) decreases. Also, a too small \( \varepsilon_p \) results in a condition number that increases rapidly as the mesh size is reduced. However, for \( \varepsilon_p \leq 1 \), the error is not sensitive to the stabilization. Therefore, we have chosen \( \varepsilon_p = 1 \) in our computations. In this example the results for \( \varepsilon_u = 10^{-3} \) and \( \varepsilon_u = 0 \) coincide. Since the interface is not very close to any meshlines we have control of the condition number even when \( \varepsilon_u = 0 \). This is not the case in for example the last example in this section.
The convergence for the velocity and the pressure in the $L^2$ norm is shown in Fig. 4. Since in this example neither the pressure nor the velocity have discontinuities we compare our method with the standard continuous finite element method. Compared to using standard continuous finite element methods we obtain just slightly larger errors for the pressure. However compared to the method in [3] (see Fig. 3 in [3]) we obtain much smaller errors for the pressure. In Fig. 4 we see the optimal second order convergence for the velocity in the $L^2$ norm but for the pressure we observe better convergence than the expected first order measured in the $L^2$ norm. In Fig. 5 we show the spectral condition number. The condition number using the proposed stabilized method grows as $O(h^{-2})$ just as it does for the standard finite element method, which is optimal. For a fixed mesh size the condition number of the system matrix using the proposed method is very close to the condition number of the system matrix using the standard continuous finite element method. We see that the condition number grows erratically with the mesh size when there is no stabilization for the pressure, i.e. $\varepsilon_p = 0$. However, we also see in the figure that the numerically estimated inf-sup constant in case $\varepsilon_p = 0$ is essentially independent of the mesh size. Thus, our numerical results suggest that the inf-sup condition is satisfied in this case even when there is no stabilization.

5.2. Example 2: Static drop

Consider a circular interface $\Gamma$ of radius $R$ in equilibrium in the interior of a domain in two dimensions with $\mu = 2$, $\sigma = 1$ and vanishing $u_b$ on $\partial \Omega$. The exact solution is $u \equiv 0$, $p_1 = 0$, $p_2 = \sigma/R$. This corresponds to a circular fluid drop in equilibrium with the surrounding fluid.

This problem has previously been studied for example in [24] using standard continuous finite element methods. In Figure 6 we compare the pressure approximation using the new method with the results obtained in [24]. In this case $R = 0.5$ and we prescribe the exact curvature $\kappa = 2$.

We use a regular mesh with $h_x = 0.025$ in the velocity mesh. We choose $\lambda_\partial\Omega$ and $\lambda_\Gamma$ as in the previous example. From Table 1 we see that for the new method the magnitude of spurious currents and the error in the pressure are of the order of machine epsilon. However, using a sharp surface tension representation and standard continuous finite element methods the magnitude of spurious currents are large and may lead to unphysical movements of the interface. With standard globally continuous finite element methods the pressure either oscillates or is smeared out depending on if a sharp or regularized surface tension representation is used, see the two leftmost panels of Fig. 6. With the new method the discontinuous pressure is accurately represented even on coarse meshes.
Figure 5: Spectral condition number and the inf-sup constant. Circles (○), Crosses (×), and stars (∗) represent the presented method with $\varepsilon_p = 10^{-2}$, the standard continuous FEM, and the unstabilized method (i.e. $\varepsilon_p = 0$), respectively. Left panel: Condition number versus mesh size. Right Panel: The inf-sup constant versus mesh size.

|              | $\|u_h\|_\infty$ | $\|p - p_h\|_\infty$ | Condition number |
|--------------|-----------------|-----------------|-----------------|
| Regularized force | $\mathcal{O}(10^{-16})$ | 1.0129 | 5.36 \cdot 10^5 |
| Sharp force   | 0.0126          | 1.0164         | 5.36 \cdot 10^5 |
| New method    | $\mathcal{O}(10^{-16})$ | $\mathcal{O}(10^{-16})$ | 6.74 \cdot 10^5 |

Table 1: Spurious velocities, error in the pressure approximation, and the spectral condition number for the static drop. Standard continuous finite elements with a regularized and a sharp approximation of the surface tension force are compared with the new method with $\varepsilon_p = 10^{-1}$.

**Remark.** We would like to emphasize that in order to get the magnitude of spurious currents and the error in the pressure of the order of machine epsilon, even on coarse meshes, it is important to use the bilinear form $b_h$ in equation (2.31). Using $b_h$ in equation (2.30) results in larger spurious currents and errors in the pressure but the errors decrease with mesh refinement. Hence although the two forms $b_h$ in equation (2.30) and (2.31) are mathematically equivalent we obtain a perfect balance between the terms on the left and the right hand side of the variational form when $b_h$ in equation (2.31) is used.

### 5.3. Example 3: Couette flow

We first consider a problem where there is a jump in the pressure due to different fluid viscosities. The computational domain is $[0, L] \times [-0.4, 0.6]$. The interface is the straight line $y = 0$. The viscosity

$$
\mu = \begin{cases} 
200 & y > 0, \\
2 & y < 0,
\end{cases} 
$$

(5.1)

$f = (3, 0)$ and the Dirichlet boundary conditions for the velocity are chosen such that the exact solution is given by

$$
u(x, y) = \left( \frac{L - x^2}{2L}, \frac{xy}{L} \right),$$

(5.2)

$$p(x, y) = \frac{2x}{L} \mu - \frac{\mu_1 + \mu_2}{4}. $$

We take $L = 1$. The interface intersects the boundary of the domain and max($\mathcal{N}_K$) = 1. The penalty parameter $\lambda_{\partial\Gamma}$ is chosen according to equation (2.15) with $E = 0.25$, $F = G = 0.005$ and
Figure 6: Cross section of the pressure approximation for the static drop. The exact curvature $\kappa = 2$ is prescribed. The dotted lines in all figures represent the exact pressure. Left panel: A standard continuous finite element method and a regularized surface tension force is used. Middle panel: A standard continuous finite element method is used with sharp representation of the surface tension force. Right panel: The pressure is approximated using the new finite element method.

the constants in $\lambda_T$ are chosen as $A = 0.3$, $B = D = 0.05$. The condition number depends on these constants but we have not optimized these numbers.

In Fig. 7 we show the approximation of the pressure using the new finite element method and the convergence of the error in the pressure and the velocity. We have as expected first order convergence for the velocity in the $H^1(\Omega_1 \cup \Omega_2)$ norm and a bit better than first order convergence for the pressure in the $L^2(\Omega_1 \cup \Omega_2)$ norm.

Next we consider the problem presented in [2]. The computational domain is $[0, L] \times [0, H]$. The interface is the straight line $x = a$. The jump condition $[\mu D(u) \cdot n - p n] \cdot n = 1$ is imposed at the interface. The Dirichlet boundary conditions on $\partial \Omega$ for the velocity are chosen such that the exact solution is given by

$$u(x, y) = \left( \frac{1}{2\mu L} y(H - y), 0 \right),$$

$$p(x, y) = -\frac{1}{L} x + \mathcal{X}(x - a),$$

where $\mathcal{X}(x - a) = 1$, if $x > a$ and zero otherwise. We let $L = 3$, $H = 1$, $a = 2$, and $\mu = 2$ as in [2].

The approximation of the discontinuous pressure using the proposed method is shown in Fig. 8. The error in the velocity measured in the $H^1$ norm and the error in the pressure measured in the $L^2$ norm are shown for different mesh sizes in Table 2. The results in Table 2 can be compared with the results in [2] Table 2. The errors in the velocity are similar to the result in [2] but the errors in the pressure are much smaller using our method.

In Fig 9 we show the condition number as a function of the relative distance between the interface and the mesh line for different $\varepsilon_u$. We see that the stabilization for the velocity, $J(u_h, v_h)$, is needed in order to obtain a well conditioned system matrix independently of the location of the interface.

6. Conclusions

We have proposed a finite element method which offers a way to accurately solve the Stokes equations involving two immiscible fluids with different viscosities and surface tension. The interface that separates the two fluids can be represented either explicitly, for example as in the
Figure 7: Left Panel: Approximation of the discontinuous pressure in Example 3. The jump in the pressure is due to different fluid viscosities. The mesh does not coincide with the interface. There are 18 grid points along the x-axis in the pressure mesh. Right Panel: The error in the pressure measured in the $L^2$ norm and the error in velocity measured in the $H^1$ norm versus the mesh size $h_x$. The dashed line represents $y = h_x$.

Figure 8: Approximation of the discontinuous pressure in Example 3 using the new method with $\varepsilon_p = 1$. There are 35 grid points along the x-axis in the pressure mesh. The error in the pressure approximation for this simulation is shown in the third row of Table 2.

| $h_x$         | $\|u - uh\|_{H^1(\Omega_1 \cup \Omega_2)}$ | $\|p - ph\|_{L^2(\Omega_1 \cup \Omega_2)}$ | Condition number |
|---------------|------------------------------------------|------------------------------------------|------------------|
| $1.88 \cdot 10^{-1}$ | $2.8742 \cdot 10^{-2}$ | $6.8969 \cdot 10^{-3}$ | $1.55 \cdot 10^4$ |
| $8.82 \cdot 10^{-2}$ | $1.4137 \cdot 10^{-2}$ | $1.6082 \cdot 10^{-3}$ | $6.25 \cdot 10^4$ |
| $4.41 \cdot 10^{-2}$ | $7.6522 \cdot 10^{-3}$ | $4.6902 \cdot 10^{-4}$ | $2.14 \cdot 10^5$ |
| $2.21 \cdot 10^{-2}$ | $3.6411 \cdot 10^{-3}$ | $1.0602 \cdot 10^{-4}$ | $9.25 \cdot 10^6$ |
| $1.10 \cdot 10^{-2}$ | $1.8567 \cdot 10^{-3}$ | $2.7847 \cdot 10^{-5}$ | $3.56 \cdot 10^6$ |

Table 2: Convergence study of the proposed method and the spectral condition number for Example 3.
immersed boundary method, or implicitly as in the level set method. Our method allows for discontinuities across the interface which can be located arbitrarily with respect to a fixed background mesh.

We have used the inf-sup stable P1 iso P2/P1 element and proven that our method is of optimal-order accuracy, and that the stabilization terms $J(u_h, v_h)$ and $J(p_h, q_h)$ guarantee that the condition number of the system matrix is $O(h^{-2})$ independent of the interface location. We expect the method to be applicable also in three space dimensions. For higher-order elements, the stabilization terms $J(u_h, v_h)$ and $J(p_h, q_h)$ will include jumps of derivatives of higher orders, see [12]. One can also include projection operators from [14] into the stabilization to reduce the amount of stabilization and hence the constant in the error. The method we have presented is simple and robust and has properties that are very desirable, in particular for problems with moving interfaces.

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