Fractional-compact numerical algorithms for Riesz spatial fractional reaction-dispersion equations *

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Abstract

It is well known that using high-order numerical algorithms to solve fractional differential equations leads to almost the same computational cost with low-order ones but the accuracy (or convergence order) is greatly improved, due to the nonlocal properties of fractional operators. Therefore, developing some high-order numerical approximation formulas for fractional derivatives play a more important role in numerically solving fractional differential equations. This paper focuses on constructing (generalized) high-order fractional-compact numerical approximation formulas for Riesz derivatives. Then we apply the developed formulas to the one- and two-dimension Riesz spatial fractional reaction-dispersion equations. The stability and convergence of the derived numerical algorithms are strictly studied by using the energy analysis method. Finally, numerical simulations are given to demonstrate the efficiency and convergence orders of the presented numerical algorithms.

Key words: Riesz derivative, Fractional-compact numerical approximation formulas, generating function, Riesz spatial fractional reaction dispersion equations.

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1 Introduction

Riesz derivative \( \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} \) with derivative order \( \alpha \in (1, 2) \) is defined by \[15, 27\]

\[
\frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos \left( \frac{\pi \alpha}{2} \right)} \left( RL D_{a,x}^\alpha + RL D_{x,b}^\alpha \right) u(x), \quad 1 < \alpha < 2, \quad x \in (a, b),
\]

where \( RL D_{a,x}^\alpha \) denotes the left Riemann-Liouville derivative

\[
RL D_{a,x}^\alpha u(x) = \begin{cases} 
\frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dx^2} \int_a^x u(s)ds (x - s)^{\alpha - 1}, & 1 < \alpha < 2, \\
\frac{d^2 u(x)}{dx^2}, & \alpha = 2,
\end{cases}
\]

and \( RL D_{x,b}^\alpha \) the right Riemann-Liouville derivative

\[
RL D_{x,b}^\alpha u(x) = \begin{cases} 
\frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dx^2} \int_x^b u(s)ds (s - x)^{\alpha - 1}, & 1 < \alpha < 2, \\
\frac{d^2 u(x)}{dx^2}, & \alpha = 2.
\end{cases}
\]

The Riesz fractional derivative has been shown to be a suitable tool for modeling Lévy flights whose second moments diverge. But the fractional moments exist. They are often used to analyze the diffusion behaviors of particles \[12, 21, 28, 30\]. So the Riesz spatial fractional reaction-dispersion equations have attracted increasing interest. However, almost all of the fractional differential equations cannot be obtained analytical solutions, hence, more and more studies focus on their numerical solutions in recent decades \[1, 2, 3, 4, 6, 7, 13, 14, 18, 20, 25, 33, 34, 35\]. For the numerical algorithms of such problems, the key step is to construct efficient approximation formula for the Riesz (or Riemann-Liouville) derivatives. Up to now, there have existed some difference formulas to approximate the Riemann-Liouville derivatives and Riesz derivatives. One of the most popular approximation is the first-order (shifted) Grünwald-Letnikov formula \[23, 31\]. Recently, some high-order formulas have been constructed based on (shifted) Grünwald-Letnikov formula, for examples, second-order and third-order weighted and shifted Grünwald difference schemes \[32, 38\]. Some other studies have been also developed to approximate the Riesz derivatives with the help of the fractional centered operator \[24\]. Çelik and Duman \[5\] studied the convergence order of this approximation and applied it to Riesz spatial fractional diffusion equations. Later on, Shen et al. applied it to Riesz spatial fractional advection-dispersion equation and got a weighted difference
scheme \cite{29}. Ding and Li \cite{8, 9, 10} constructed a series of high-order algorithms for Riemann-Liouville (Riesz) derivatives and applied them to the different types of fractional differential equations. It is worth mentioning that we proposed a kind of higher-order numerical approximate formulas for Riemann-Liouville (or Riesz) derivatives based on the novel generating functions in \cite{11}, and obtained an unconditionally stable difference scheme where the 2nd-order approximation formula was applied to Riesz spatial fractional advection-diffusion equations. In addition, other methods, such as L1/L2 approximation methods \cite{36}, (improved) matrix transform methods \cite{16 37}, etc., were also adopted to approximate the Riemann-Liouville (or Riesz) derivatives.

The purpose of this paper is to develop several robust and efficient high-order fractional-compact numerical approximation formulas for Riesz derivatives. The remainder of the paper is organized as follows. In Section 2, we derive two kinds of 3rd-order numerical approximate formulas for the Riesz derivatives. In Section 3, the generalized high-order numerical approximation formulas and their fractional-compact forms for Riemann-Liouville (also Riesz) derivatives are also constructed. In Sections 4 and 5, one of the 3rd-order schemes is applied to solving one- and two-dimension Riesz spatial fractional reaction-dispersion equations, respectively. The stability and convergence analyses of the presented difference schemes are also studied. In Section 6, numerical examples are carried out to confirm the theoretical results and show the efficiency of the proposed schemes. Finally, some conclusions are included in Section 7.

2 Fractional-compact numerical approximation formulas

In this section, we establish fractional-compact numerical schemes for Riesz derivatives, where the ideas and techniques are also suitable for Riemann-Liouville derivatives.

2.1 The third-order fractional-compact formula I

Firstly, we define the following difference operators,

\[ L^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \kappa_{2,\ell}^{(\alpha)} u(x - (\ell - 1)h), \]

and

\[ R^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \kappa_{2,\ell}^{(\alpha)} u(x + (\ell - 1)h). \]
Here, the coefficients read as

$$\kappa_{\alpha, 2, \ell} = \left(\frac{3\alpha - 2}{2\alpha}\right)^\alpha \sum_{m=0}^\ell \left(\alpha - 2\right)^m \frac{\alpha - 2}{3\alpha - 2} \varpi_{\alpha, 1, m} \varpi_{\alpha, 1, \ell - m}, \quad \ell = 0, 1, \ldots$$  \hspace{1cm} (2.1)$$

They can be obtained by the associate generating function

$$W_2(z) = \left(\frac{3\alpha - 2}{2\alpha} - \frac{2(\alpha - 1)}{\alpha} z + \frac{\alpha - 2}{2\alpha} z^2\right)^\alpha,$$

i.e.,

$$\left(\frac{3\alpha - 2}{2\alpha} - \frac{2(\alpha - 1)}{\alpha} z + \frac{\alpha - 2}{2\alpha} z^2\right)^\alpha = \sum_{\ell=0}^\infty \kappa_{\alpha, 2, \ell} z^\ell, \quad |z| < 1.$$  

In particular, the expressions \(\varpi_{\alpha, 1, m}\) in (2.1) are the coefficients of the power series expansion of function \((1 - z)^\alpha\) for \(|z| < 1\). They can be computed recursively

$$\varpi_{\alpha, 1, 0} = 1, \quad \varpi_{\alpha, 1, m} = \left(1 - \frac{1 + \alpha}{m}\right) \varpi_{\alpha, 1, m-1}, \quad m = 1, 2, \ldots$$

Coefficients \(\kappa_{\alpha, 2, \ell} (\ell = 0, 1, \ldots)\) can be calculated by the following recursion formulas

$$\begin{align*}
\kappa_{\alpha, 2, 0} &= \left(\frac{3\alpha - 2}{2\alpha}\right)^\alpha, \\
\kappa_{\alpha, 2, 1} &= \frac{4\alpha(1 - \alpha)}{3\alpha - 2} \kappa_{\alpha, 2, 0}, \\
\kappa_{\alpha, 2, \ell} &= \frac{4\alpha(1 - \alpha)(\alpha - \ell + 1)}{(3\alpha - 2)\ell} \kappa_{\alpha, 2, \ell-1} + \frac{(\alpha - 2)(2\alpha - \ell + 2)}{(3\alpha - 2)\ell} \kappa_{\alpha, 2, \ell-2}, \quad \ell \geq 2.
\end{align*}$$

And these coefficients \(\kappa_{\alpha, 2, \ell} (\ell = 0, 1, \ldots)\) have some important and interesting properties are listed as follows.

**Theorem 2.1** \(\square\) The coefficients \(\kappa_{\alpha, 2, \ell} (\ell = 0, 1, \ldots)\) have the following properties for \(1 < \alpha < 2\),

(i) \(\kappa_{\alpha, 2, 0} = \left(\frac{3\alpha - 2}{2\alpha}\right)^\alpha > 0, \quad \kappa_{\alpha, 2, 1} = \frac{4\alpha(1 - \alpha)}{3\alpha - 2} \kappa_{\alpha, 2, 0} < 0;\)

(ii) \(\kappa_{\alpha, 2, 2} = \frac{\alpha(8\alpha^3 - 21\alpha^2 + 16\alpha - 4)}{(3\alpha - 2)^2} \kappa_{\alpha, 2, 0}\) \(\kappa_{\alpha, 2, 2} < 0 \text{ if } \alpha \in (1, \alpha_1^*), \text{ while } \kappa_{\alpha, 2, 2} \geq 0 \text{ if } \alpha \in [\alpha_1^*, 2), \text{ where } \alpha_1^* = \frac{7}{8} + \frac{\sqrt[3]{621 + 48\sqrt{87}}}{24} + \frac{19}{\sqrt[3]{621 + 48\sqrt{87}}} \approx 1.5333;\)
(iii) \( \kappa_{2,\ell}^{(\alpha)} \geq 0 \) if \( \ell \geq 3 \);
(iv) \( \kappa_{2,\ell}^{(\alpha)} \sim -\frac{\sin(\pi \alpha) \Gamma(\alpha + 1)}{\pi} \ell^{-\alpha - 1} \) as \( \ell \to \infty \);
(v) \( \kappa_{2,\ell}^{(\alpha)} \to 0 \) as \( \ell \to \infty \);
(vi) \( \sum_{\ell=0}^{\infty} \kappa_{2,\ell}^{(\alpha)} = 0 \).

Next, we give the following asymptotic expansion formulas for difference operators \( L_{B_2^\alpha} \) and \( R_{B_2^\alpha} \), which play an important role in establishment of high-order algorithms for Riemann-Liouville derivatives.

**Theorem 2.2** \([11]\) Let \( u(x) \in C^{[\alpha]+n+1}(\mathbb{R}) \) and all the derivatives of \( u(x) \) up to order \( [\alpha] + n + 2 \) belong to \( L_1(\mathbb{R}) \). Then one has

\[
L_{B_2^\alpha} u(x) = RL D_{-\infty,x}^\alpha u(x) + \sum_{\ell=1}^{n-1} \left( \sigma_\ell^{(\alpha)} RL D_{-\infty,x}^{\alpha+\ell} u(x) \right) h^\ell + O(h^n), \quad n \geq 2, \quad (2.2)
\]

and

\[
R_{B_2^\alpha} u(x) = RL D_{x,\infty}^\alpha u(x) + \sum_{\ell=1}^{n-1} \left( \sigma_\ell^{(\alpha)} RL D_{x,\infty}^{\alpha+\ell} u(x) \right) h^\ell + O(h^n), \quad n \geq 2,
\]

hold uniformly on \( \mathbb{R} \). Here coefficients \( \sigma_\ell^{(\alpha)} \) \( (\ell = 1, 2, \ldots) \) satisfy the equation

\[
\frac{e^z}{z^\alpha} W_2(e^{-z}) = 1 + \sum_{\ell=1}^{\infty} \sigma_\ell^{(\alpha)} z^\ell.
\]

Especially, the first three coefficients are

\[
\sigma_1^{(\alpha)} = 0, \quad \sigma_2^{(\alpha)} = -\frac{2\alpha^2 - 6\alpha + 3}{6\alpha}, \quad \sigma_3^{(\alpha)} = \frac{3\alpha^3 - 11\alpha^2 + 12\alpha - 4}{12\alpha^2}.
\]

Define difference operator \( \mathcal{L} \) as

\[
\mathcal{L} u(x) = \left( 1 + \sigma_2^{(\alpha)} h^2 \delta_x^2 \right) u(x),
\]

where \( \delta_x^2 \) is the second-order central difference operator and is defined by \( \delta_x^2 u(x) = \frac{u(x+h)-2u(x)+u(x-h)}{h^2} \). Such \( \mathcal{L} \) can be regarded as a fractional-compact operator by borrowing the appellation of the integer-order case. Accordingly, the main results are enunciated as follows.

**Theorem 2.3** Let \( u(x) \in C^{[\alpha]+4}(\mathbb{R}) \) and all the derivatives of \( u(x) \) up to order \( [\alpha] + 5 \) belong to \( L_1(\mathbb{R}) \). Then there hold

\[
L_{B_2^\alpha} u(x) = \mathcal{L} RL D_{-\infty,x}^\alpha u(x) + O(h^3),
\]
and
\[ R\mathcal{B}_2^\alpha u(x) = \mathcal{L}_{RL} D_{x,+\infty}^\alpha u(x) + O(h^3), \] (2.3)
uniformly for \( x \in \mathbb{R} \).

**Proof.** In equation (2.2), taking \( n = 3 \) and noticing
\[ RL D_{-\infty,x}^{\alpha+2} u(x) = \frac{d^2}{dx^2} \left( RL D_{-\infty,x}^\alpha u(x) \right), \]
then one has
\[ L\mathcal{B}_2^\alpha u(x) = RL D_{-\infty,x}^{\alpha+2} u(x) + \frac{d^2}{dx^2} \left( RL D_{-\infty,x}^\alpha u(x) \right) \sigma_2^{(\alpha)} h^2 + O(h^3) \]
\[ = RL D_{-\infty,x}^{\alpha+2} u(x) + \sigma_2^{(\alpha)} h^2 (\delta_x RL D_{-\infty,x}^\alpha u(x) + O(h^2)) + O(h^3) \]
\[ = L_{RL} D_{-\infty,x}^\alpha u(x) + O(h^3). \]

Using the same method, we can prove that equation (2.3) holds too. All this completes proof.

In particular, if function \( u(x) \) is defined on a bounded interval \((a, b)\) and satisfies \( u(a) = u(b) = 0 \), then we can apply zero-extension to \( u(x) \) such that it is defined on \( \mathbb{R} \). Now we further have the following results.

**Theorem 2.4** Suppose \( u(x) \in C^{[\alpha]+4}((a, b)) \), \( u(a) = u(b) = 0 \) and all the derivatives of \( u(x) \) up to order \([\alpha]+5\) belong to \( L_1((a, b)) \). Then for any \( x \in (a, b) \), one has
\[ L\mathcal{A}_2^\alpha u(x) = \mathcal{L}_{RL} D_{a,x}^\alpha u(x) + O(h^3), \] (2.4)
and
\[ R\mathcal{A}_2^\alpha u(x) = \mathcal{L}_{RL} D_{x,b}^\alpha u(x) + O(h^3). \] (2.5)

Here, operators \( L\mathcal{A}_2^\alpha \) and \( R\mathcal{A}_2^\alpha \) are respectively defined as follows,
\[ L\mathcal{A}_2^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\lfloor \frac{\alpha}{\delta_x} \rfloor +1} \kappa_{2,\ell}^{(\alpha)} u \left( (x - (\ell - 1)h) \right), \]
and
\[ R\mathcal{A}_2^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\lfloor \frac{\alpha}{\delta_x} \rfloor +1} \kappa_{2,\ell}^{(\alpha)} u \left( (x + (\ell - 1)h) \right). \]
Finally, combing equations (1.1), (2.4) with (2.5) gives a 3rd-order fractional-compact numerical approximation formula for Riesz derivatives,

\[ \mathcal{L} \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\frac{1}{2\cos\left(\frac{\pi \alpha}{2}\right)} \left( L \mathcal{A}_2^\alpha u(x) + R \mathcal{A}_2^\alpha u(x) \right) + \mathcal{O}(h^3), \quad 1 < \alpha < 2. \]  

(2.6)

**Remark 1:** When \( \alpha = 2 \), we easily know that \( \sigma_2^{(\alpha)} = \frac{1}{12} \) and \( \sigma_3^{(\alpha)} = 0 \), then (2.6) becomes the following classical fourth-order compact formula for the second order derivative \( \frac{d^2 u(x)}{dx^2} \), that is

\[ \frac{d^2 u(x)}{dx^2} = \left( 1 + \frac{h^2}{12} \right)^{-1} \delta_x^2 u(x) + \mathcal{O}(h^4). \]

### 2.2 The third-order fractional-compact formula II

In this subsection, we continue to develop another numerical approximate formula for Riesz derivatives. If we choose a new generating function

\[ \widetilde{W}_2(z) = \left( \frac{3\alpha + 2}{2\alpha} - \frac{2(\alpha + 1)}{\alpha} z + \frac{\alpha + 2}{2\alpha} z^2 \right)^\alpha, \]

then we define the following difference operators,

\[ L \widetilde{B}_2^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \tilde{\kappa}_{2,\ell}^{(\alpha)} u(x - (\ell + 1)h), \]

and

\[ R \widetilde{B}_2^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \tilde{\kappa}_{2,\ell}^{(\alpha)} u(x + (\ell + 1)h). \]

Here, the coefficients are

\[ \tilde{\kappa}_{2,\ell}^{(\alpha)} = \left( \frac{3\alpha + 2}{2\alpha} \right)^\alpha \sum_{m=0}^{\ell} \left( \frac{\alpha + 2}{3\alpha + 2} \right)^m \varpi_1^{(\alpha)} \varpi_1^{(\alpha)}_{\ell - m}, \quad \ell = 0, 1, \ldots \]

which have following recursion expressions,

\[
\begin{cases}
\tilde{\kappa}_{2,0}^{(\alpha)} = \left( \frac{3\alpha + 2}{2\alpha} \right)^\alpha, \\
\tilde{\kappa}_{2,1}^{(\alpha)} = -\frac{4\alpha(1 + \alpha)}{3\alpha + 2} \tilde{\kappa}_{2,0}^{(\alpha)}, \\
\tilde{\kappa}_{2,\ell}^{(\alpha)} = -\frac{4\alpha(1 + \alpha)(\alpha - \ell + 1)}{(3\alpha + 2)\ell} \tilde{\kappa}_{2,\ell-1}^{(\alpha)} + \frac{(\alpha + 2)(2\alpha - \ell + 2)}{(3\alpha + 2)\ell} \tilde{\kappa}_{2,\ell-2}^{(\alpha)}, \quad \ell \geq 2.
\end{cases}
\]

In particular, these coefficients \( \tilde{\kappa}_{2,\ell}^{(\alpha)} \) also have the following properties.
Theorem 2.5 The coefficients $\tilde{\kappa}_{2,\ell}^{(a)}$ ($\ell = 0, 1, \ldots$) have the following properties for $1 < \alpha < 2$,

(i) $\tilde{\kappa}_{2,1}^{(a)}, \tilde{\kappa}_{2,3}^{(a)} < 0$,

(ii) $\tilde{\kappa}_{2,4}^{(a)} < 0$ if $\alpha \in (1, \alpha_2^*)$, while $\tilde{\kappa}_{2,4}^{(a)} \geq 0$ if $\alpha \in [\alpha_2^*, 2)$, where $\alpha_2^* \approx 1.4917$. $\tilde{\kappa}_{2,5}^{(a)} < 0$ if $\alpha \in (1, \alpha_3^*)$, while $\tilde{\kappa}_{2,5}^{(a)} \geq 0$ if $\alpha \in [\alpha_3^*, 2)$, where $\alpha_3^* \approx 1.4437$;

(iii) $\tilde{\kappa}_{2,\ell}^{(a)} \geq 0$ if $\ell = 0, 2, \text{or } \ell \geq 6$;

(iv) $\tilde{\kappa}_{2,\ell}^{(a)} \sim -\frac{\sin (\pi \alpha)}{\pi} \frac{\Gamma(\alpha + 1)}{\ell^{-\alpha - 1}}$ as $\ell \to \infty$;

(v) $\tilde{\kappa}_{2,\ell}^{(a)} \to 0$ as $\ell \to \infty$;

(vi) $\sum_{\ell=0}^{\infty} \tilde{\kappa}_{2,\ell}^{(a)} = 0$.

Proof. Here, we only consider (ii) and (iii). The others are easily obtained by using almost the same proof as that of Theorem 2.1.

(ii) In view of the definitions of

$$\tilde{\kappa}_{2,4}^{(a)} = \frac{\alpha(\alpha - 1)g_1(\alpha)\tilde{\kappa}_{2,0}^{(a)}}{6(3\alpha + 2)^4 \tilde{\kappa}_{2,0}^{(a)}},$$

and

$$\tilde{\kappa}_{2,5}^{(a)} = \frac{2\alpha(1 - \alpha)(\alpha - 2)g_2(\alpha)\tilde{\kappa}_{2,0}^{(a)}}{15(3\alpha + 2)^5 \tilde{\kappa}_{2,0}^{(a)}},$$

where

$$g_1(\alpha) = 64\alpha^6 + 80\alpha^5 - 101\alpha^4 - 224\alpha^3 - 88\alpha^2 + 64\alpha + 48,$$

and

$$g_2(\alpha) = 64\alpha^6 + 80\alpha^5 - 101\alpha^4 - 224\alpha^3 - 88\alpha^2 + 64\alpha + 48.$$ 

Applying the numerical computations, we easily know that there exist solutions $\alpha_2^* \approx 1.4917$ and $\alpha_3^* \approx 1.4437$ such that $g_1(\alpha^*) = 0$ and $g_2(\alpha^*) = 0$, respectively. Through further analysis, one can obtain the required results.

(iii) For the cases of $\ell = 0, 2$, the direct computation leads to the desired results. For $\ell = 6, 7$, the expressions of them read as

$$\tilde{\kappa}_{2,6}^{(a)} = \frac{\alpha(\alpha - 1)(\alpha - 2)g_3(\alpha)\tilde{\kappa}_{2,0}^{(a)}}{90(3\alpha + 2)^6 \tilde{\kappa}_{2,0}^{(a)}},$$

and

$$\tilde{\kappa}_{2,7}^{(a)} = -\frac{2\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)g_1(\alpha)\tilde{\kappa}_{2,0}^{(a)}}{315(3\alpha + 2)^7 \tilde{\kappa}_{2,0}^{(a)}}.$$
Note that
\[
g_3(\alpha) = 512\alpha^9 - 192\alpha^8 - 2840\alpha^7 - 1811\alpha^6 + 3360\alpha^5 + 4892\alpha^4 + 1088\alpha^3 - 2640\alpha^2 - 2816\alpha - 960 \nonumber\]
\[= -\left(\alpha - 1\right)^{3}\left[\alpha^2(4 - \alpha^2)(512\alpha^2 + 1344\alpha + 1704) + (987\alpha^3 + 7725\alpha^2 + 16422\alpha + 14482)\right] - 6311(\alpha - 1)^2 - 3861\alpha(\alpha - 1) - 1407 < 0, \nonumber\]
and
\[
g_4(\alpha) = 512\alpha^{10} - 64\alpha^9 - 2936\alpha^8 - 2101\alpha^7 + 3619\alpha^6 + 4172\alpha^5 - 2324\alpha^4 - 7216\alpha^3 - 6896\alpha^2 - 3776\alpha - 960 \nonumber\]
\[= -\left(\alpha - 1\right)^{3}\left[\alpha^3(4 - \alpha^2)(512\alpha^2 + 1472\alpha + 1992) + (285\alpha^4 + 5292\alpha^3 + 17145\alpha^2 + 20152\alpha + 29497)\right] - 52076(\alpha - 1)^2 - 39589(\alpha - 1) - 17970 < 0, \nonumber\]
then one easily obtain that \(\tilde{\kappa}_{2,\ell}^{(\alpha)} \geq 0\) for \(\ell = 6, 7\).

As for \(\ell \geq 8\), according to their expressions, one has
\[
\tilde{\kappa}_{2,\ell}^{(\alpha)} = \left(\frac{3\alpha + 2}{2\alpha}\right)^{\alpha} \sum_{m=0}^{\ell} \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{m} \varpi_{1,m}\varpi_{1,\ell-m}^{(\alpha)} \nonumber\]
\[= \left(\frac{3\alpha + 2}{2\alpha}\right)^{\alpha} \left[\varpi_{1,0}\varpi_{1,\ell}^{(\alpha)} + \frac{\alpha + 2}{3\alpha + 2} \varpi_{1,1}\varpi_{1,\ell-1}^{(\alpha)} + \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{2} \varpi_{1,2}\varpi_{1,\ell-2}^{(\alpha)} \right. \nonumber\]
\[\left. + \frac{\alpha + 2}{3\alpha + 2} \varpi_{1,1}\varpi_{1,\ell-1}^{(\alpha)} + \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{\ell} \varpi_{1,0}\varpi_{1,\ell}^{(\alpha)} \right] \nonumber\]
\[+ \left(\frac{3\alpha + 2}{2\alpha}\right)^{\alpha} \sum_{m=3}^{\ell} \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{m} \varpi_{1,m}\varpi_{1,\ell-m}^{(\alpha)} \nonumber\]
\[
\begin{align*}
&= \left(\frac{3\alpha + 2}{2\alpha}\right)^{\alpha} \left[ 1 - \frac{(\alpha + 2)\alpha \ell}{(3\alpha + 2)(\ell - \alpha - 1)} + \frac{(\alpha - 1)(\ell - 1)\alpha \ell}{2(\ell - \alpha - 1)(\ell - \alpha - 2)} \right] \right]^{1,\ell} \\
&\times \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{\ell-2} + \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{8} \left(1 - \frac{(3\alpha + 2)\ell\alpha}{(\alpha + 2)(\ell - \alpha - 1)}\right) \right]^{1,\ell} \\
&\geq \left(\frac{3\alpha + 2}{2\alpha}\right)^{\alpha} \left[ 1 - \frac{(\alpha + 2)\alpha \ell}{(3\alpha + 2)(\ell - \alpha - 1)} + \frac{(\alpha - 1)(\ell - 1)\alpha \ell}{2(\ell - \alpha - 1)(\ell - \alpha - 2)} \right] \right]^{1,\ell} \\
&\times \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{\ell-2} + \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{8} \left(1 - \frac{(3\alpha + 2)\ell\alpha}{(\alpha + 2)(\ell - \alpha - 1)}\right) \right]^{1,\ell} \\
&\geq \left(\frac{3\alpha + 2}{2\alpha}\right)^{\alpha} \left[ 1 - \frac{(\alpha + 2)\alpha \ell}{(3\alpha + 2)(\ell - \alpha - 1)} + \frac{(\alpha - 1)(\ell - 1)\alpha \ell}{2(\ell - \alpha - 1)(\ell - \alpha - 2)} \right] \right]^{1,\ell} \\
&\times \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{\ell-2} + \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{8} \left(1 - \frac{(3\alpha + 2)\ell\alpha}{(\alpha + 2)(\ell - \alpha - 1)}\right) \right]^{1,\ell} \\
&\geq \left(\frac{3\alpha + 2}{2\alpha}\right)^{\alpha} \left[ 1 - \frac{(\alpha + 2)\alpha \ell}{(3\alpha + 2)(\ell - \alpha - 1)} + \frac{(\alpha - 1)(\ell - 1)\alpha \ell}{2(\ell - \alpha - 1)(\ell - \alpha - 2)} \right] \right]^{1,\ell} \\
&\times \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{\ell-2} + \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{8} \left(1 - \frac{(3\alpha + 2)\ell\alpha}{(\alpha + 2)(\ell - \alpha - 1)}\right) \right]^{1,\ell} \\
\end{align*}
\]

Denote
\[
S(\alpha, x) = 1 - \frac{(\alpha + 2)\alpha \ell}{(3\alpha + 2)(\ell - \alpha - 1)} + \frac{(\alpha - 1)(\ell - 1)\alpha \ell}{2(\ell - \alpha - 1)(\ell - \alpha - 2)} \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{2}
\]
\[
+ \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{8} \left(1 - \frac{(3\alpha + 2)\ell\alpha}{(\alpha + 2)(\ell - \alpha - 1)}\right), \quad x \geq 8,
\]

and
\[
P(\alpha, x) = 2(x - \alpha - 1)(x - \alpha - 2)S(\alpha, x)
\]
\[
= 2(x - \alpha - 1)(x - \alpha - 2) - 2\alpha x(x - \alpha - 2) \frac{\alpha + 2}{3\alpha + 2}
\]
\[
+ \alpha x(\alpha - 1)(x - 1) \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{7} - 2\alpha x(x - \alpha - 2)
\]
\[
\times \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{7} + 2(x - \alpha - 1)(x - \alpha - 2) \left(\frac{\alpha + 2}{3\alpha + 2}\right)^{8}.
\]

Then one has
\[
\frac{\partial^2 P(\alpha, x)}{\partial x^2} = \frac{2}{(3\alpha + 2)^8} \left(729\alpha^{10} + 723\alpha^9 - 2516\alpha^8 + 3792\alpha^7
\right.
\]
\[
+ 37952\alpha^6 + 80800\alpha^5 + 89344\alpha^4 + 59648\alpha^3 + 25344\alpha^2
\]
\[
+ 6912\alpha + 1024)
\]
\[
> 0,
\]
for $\alpha \in (1, 2)$, that is to say, $\frac{\partial P(\alpha, x)}{\partial x}$ is an increasing function, i.e., $\left. \frac{\partial P(\alpha, x)}{\partial x} \right|_{x=8} \geq (3\alpha + 2)^8 (15315 \alpha^{10} + 18225 \alpha^9 - 92564 \alpha^8 - 198960 \alpha^7 \\
+ 59456 \alpha^6 + 611680 \alpha^5 + 883200 \alpha^4 + 662784 \alpha^3 \\
+ 301312 \alpha^2 + 86272 \alpha + 13312) > 0.$

It immediately follows that $P(\alpha, x)$ is also an increasing function with respect to $x$ for $1 < \alpha < 2$. So, $P(\alpha, x) > P(\alpha, 8)$. Note that

$$P(\alpha, 8) = \frac{4}{(3\alpha + 2)^8} (22247 \alpha^{10} + 52229 \alpha^9 - 44918 \alpha^8 - 246288 \alpha^7 \\
- 196672 \alpha^6 + 202720 \alpha^5 + 512960 \alpha^4 + 450304 \alpha^3 + 221440 \alpha^2 \\
+ 66816 \alpha + 10752) > 0,$$

for $1 < \alpha < 2$. Then one easily get

$$S(\alpha, x) = \frac{1}{2(x - \alpha - 1)(x - \alpha - 2)} P(\alpha, x) > 0,$$

which implies that $\tilde{\kappa}^{(\alpha)}_{2, \ell} \geq 0$ for $\ell \geq 8$. Combining the former analysis again gives

$$\tilde{\kappa}^{(\alpha)}_{2, \ell} \geq 0,$$

for $\ell \geq 6$. All this ends the proof. ■

Similar to the previous discussion, we can similarly obtain the following results.

**Theorem 2.6** Let $u(x) \in C^{[\alpha]+n+1}(\mathbb{R})$ and all the derivatives of $u(x)$ up to order $[\alpha] + n + 2$ belong to $L_1(\mathbb{R})$. Then

$$L \tilde{B}_2^\alpha u(x) = RLD_{-\infty, x}^{\alpha} u(x) + \sum_{\ell=1}^{n-1} \left( \tilde{\sigma}^{(\alpha)}_{\ell} RLD_{-\infty, x}^{\alpha+\ell} u(x) \right) h^\ell + O(h^n), n \geq 2,$$

and

$$R \tilde{B}_2^\alpha u(x) = RLD_{x, +\infty}^{\alpha} u(x) + \sum_{\ell=1}^{n-1} \left( \tilde{\sigma}^{(\alpha)}_{\ell} RLD_{x, +\infty}^{\alpha+\ell} u(x) \right) h^\ell + O(h^n), n \geq 2,$$
hold uniformly on \( \mathbb{R} \). Here coefficients \( \tilde{\sigma}_\ell^{(\alpha)} (\ell = 1, 2, \ldots) \) satisfy equation 
\[
\frac{e^{-z}}{z^{\alpha}} \tilde{W}_2(e^{-z}) = 1 + \sum_{\ell=1}^{\infty} \tilde{\sigma}_\ell^{(\alpha)} z^\ell, \ |z| < 1. \]
Especially, the first three coefficients are explicitly expressed as
\[
\tilde{\sigma}_1^{(\alpha)} = 0, \quad \tilde{\sigma}_2^{(\alpha)} = -\frac{2\alpha^2 + 6\alpha + 3}{6\alpha}, \quad \tilde{\sigma}_3^{(\alpha)} = \frac{3\alpha^3 + 11\alpha^2 + 12\alpha + 4}{12\alpha^2}.
\]

Define another fractional-compact difference operator \( \tilde{L} \) as
\[
\tilde{L}u(x) = \left(1 + \tilde{\sigma}_2^{(\alpha)} h^2 \delta_x^2\right) u(x),
\]
then the corresponding theorem is stated below.

**Theorem 2.7** Let \( u(x) \in C^{[\alpha]+4}(\mathbb{R}) \) and all the derivatives of \( u(x) \) up to order \( [\alpha] + 5 \) belong to \( L_1(\mathbb{R}) \). Then there hold
\[
{^L L}_2^\alpha u(x) = \tilde{L} {^R L}_2^\alpha u(x) + O(h^3),
\]
and
\[
{^R L}_2^\alpha u(x) = \tilde{L} {^R L}_2^\alpha u(x) + O(h^3),
\]
uniformly for \( x \in \mathbb{R} \).

**Proof.** The proof is almost the same as that of Theorem 2.3, so we omit the proof here or leave to the readers as an exercise.

By the similar technique, another 3rd-order fractional-compact numerical approximation formula for Riesz derivative reads as
\[
\frac{\tilde{L} \partial^\alpha u(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos \left(\frac{\pi \alpha}{2}\right)} \left( {^L L}_2^\alpha u(x) + {^R L}_2^\alpha u(x) \right) + O(h^3),
\]  
(2.7)

where operators \( {^L L}_2^\alpha \) and \( {^R L}_2^\alpha \) are defined by
\[
{^L L}_2^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\lceil \frac{\alpha}{2} \rceil - 1} \tilde{\alpha}_{2,\ell}^{(\alpha)} u \left( x - (\ell + 1) h \right),
\]
and
\[
{^R L}_2^\alpha u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\lceil \frac{\alpha}{2} \rceil - 1} \tilde{\alpha}_{2,\ell}^{(\alpha)} u \left( x + (\ell + 1) h \right).
\]
3 Generalized numerical algorithm formulas and their fractional-compact forms

3.1 Generalized numerical algorithm formulas for Riesz derivatives

At present, for almost all of the numerical algorithms for fractional derivatives, they are all through some of the other grid point values to calculate the value of a particular grid point. However, sometimes we need to calculate the arbitrary point values. At this time, the existing formulas cannot be used, therefore it is necessary to establish some more general numerical algorithm formulas for fractional derivatives. Here, we firstly give the more general numerical algorithm formulas for Riemann-Liouville (and Riesz) derivatives.

Theorem 3.1 (Generalized numerical approximation formula for Riemann-Liouville derivatives) Let \( u(x) \in C^{[\alpha]+p+1}(\mathbb{R}) \) and all the derivatives of \( u(x) \) up to order \([\alpha]+p+2\) belong to \( L_1(\mathbb{R}) \). For any \( s \in \mathbb{R} \) and set

\[
L^\alpha D^-\infty x u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \mu_{p,\ell}^{(\alpha,s)} u(x - (\ell + s)h),
\]

and

\[
R^\alpha D^\infty x u(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \mu_{p,\ell}^{(\alpha,s)} u(x + (\ell + s)h).
\]

Here, the coefficients \( \mu_{p,\ell}^{(\alpha,s)} \) (\( \ell = 0, 1, \ldots \)) can be determined by the following generating functions \( G_{p,s}(z) \)

\[
G_{p,s}(z) = \left( 1 - z \right) + \sum_{k=2}^{p} \frac{\vartheta_{k-1,k-1}^{(\alpha,s)}}{\alpha} (1 - z)^k,\]

that is,

\[
G_{p,s}(z) = \sum_{\ell=0}^{\infty} \mu_{p,\ell}^{(\alpha,s)} z^\ell, \quad |z| < 1,
\]

where the parameters \( \vartheta_{k-1,k-1}^{(\alpha,s)} \) (\( k = 2, 3, \ldots \)) can be obtained by the following equation

\[
G_{k,s} \left( e^{-z} \right) \frac{1}{z^\alpha} \frac{e^{-sz}}{z^\alpha} = 1 - \sum_{\ell=1}^{\infty} \vartheta_{k,\ell}^{(\alpha,s)} z^\ell, \quad k = 1, 2, \ldots
\]

Then the left and right Riemann-Liouville derivative values at any point \( x = x_j + sh \) can be approximated by

\[
RL D^-\infty x u(x)|_{x=x_j+sh} = L^\alpha D^-\infty x u(x_j + sh) + O(h^p), \quad j = 0, 1, \ldots, p \geq 1,
\]
and

$$\text{RL}D_{x, x_j + sh}^\alpha u(x) = \text{RLB}_{p,s}^\alpha u(x_j + sh) + O(h^p), \; j = 0, 1, \ldots, p \geq 1,$$

respectively.

**Proof.** This theorem can be viewed as the extension of Theorem 4 in [11]. The proof can be finished by almost the same method and we omit it here. □

**Remark 2:** The above theorem is called the generalized numerical approximation formula for Riemann-Liouville derivatives, due to any point $x = x_j + sh, (j = 0, 1, \ldots)$ on the real axis can be calculated. Here, the needed value $x$ can be determined by selecting the appropriate parameter $s$, $x_j$ represent the grid point values.

Accordingly, the generalized numerical algorithms for Riesz derivatives can be obtained by,

$$\frac{\partial^\alpha u(x)}{\partial |x|^\alpha} |_{x = x_j + sh} = -\frac{1}{2 \cos \left(\frac{\pi\alpha}{2}\right)} \left[ \text{RLB}_{p,s}^\alpha + \text{LBR}_{p,s}^\alpha \right] u(x_j + sh) + O(h^p), \; p \geq 1.$$

Below, we only study the cases for $p = 2, 3, 4$ in details. Due to the fact that case $p = 1$ is the same as the Grünwald-Letnikov formula, and the fact that the cases for $p \geq 5$ can be similarly obtained in view of the above theorem, we carefully consider cases with $p = 2, 3, 4$ as follows.

(i) $p = 2$

According to Theorem 3.1, we easily know that the generating function for $p = 2$ is

$$G_{2,s}(z) = \left(1 - z + \frac{\alpha + 2s}{2\alpha}(1 - z)^2\right)^\alpha,$$

and the coefficients $\mu_{2,\ell}^{(\alpha,s)} (\ell = 0, 1, \ldots)$ are read as,

$$\mu_{2,\ell}^{(\alpha,s)} = d_{21}^\alpha \sum_{m=0}^\ell d_{22}^m \omega_1^{(\alpha,s)} \omega_1^{(\alpha,s)} (1 - \ell = m), \; \ell = 0, 1 \ldots$$

where,

$$d_{21} = \frac{3\alpha + 2s}{2\alpha}, \; d_{22} = \frac{\alpha + 2s}{3\alpha + 2s}.$$
Furthermore, the coefficients $\mu_{2, \ell}^{(\alpha)} (\ell = 0, 1, \ldots)$ can be obtained by the following recurrence relationships,

$$\begin{cases}
\mu_{2,0}^{(\alpha,s)} = \left( \frac{3\alpha + 2s}{2\alpha} \right)^\alpha, \\
\mu_{2,1}^{(\alpha,s)} = -\frac{4\alpha(\alpha + s)}{3\alpha + 2s} \mu_{2,0}^{(\alpha,s)}, \\
\mu_{2,\ell}^{(\alpha,s)} = \frac{1}{(3\alpha + 2s)^\ell} \left[ -4(\alpha + s)(\alpha - \ell + 1) \mu_{2,\ell-1}^{(\alpha,s)} + (\alpha + 2s)(2\alpha - \ell + 2) \mu_{2,\ell-2}^{(\alpha,s)} \right], \ \ell \geq 2.
\end{cases}$$

(ii) $p = 3$

For this case, the generating function is given below,

$$G_{3,s}(z) = \left( 1 - z + \frac{\alpha + 2s}{2\alpha} (1 - z)^2 + \frac{2\alpha^2 + 6\alpha s + 3s^2}{6\alpha^2} (1 - z)^3 \right)^\alpha,$$

the coefficients $\mu_{3,\ell}^{(\alpha,s)} (\ell = 0, 1, \ldots)$ can be obtained by simple calculations,

$$\mu_{3,\ell}^{(\alpha,s)} = d_{31}^\ell \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\ell_1} \frac{(-1)^{\ell_1+\ell_2} (\ell_1 - \ell_2)!}{\ell_2!(\ell_1 - 2\ell_2)!} d_{32}^{\ell_1-2\ell_2} d_{33}^{\ell_2} \omega_{1,\ell_1-\ell_2}^{(\alpha)} \omega_{1,\ell_1-\ell_2}^{(\alpha)}, \ \ell = 0, 1, \ldots$$

Here,

$$d_{31} = \frac{11\alpha^2 + 12\alpha s + 3s^2}{6\alpha^2}, \quad d_{32} = -\frac{7\alpha^2 + 18\alpha s + 6s^2}{11\alpha^2 + 12\alpha s + 3s^2}, \quad d_{33} = \frac{2\alpha^2 + 6\alpha s + 3s^2}{11\alpha^2 + 12\alpha s + 3s^2}.$$

The recursion relations for coefficients $\mu_{3,\ell}^{(\alpha,s)} (\ell = 0, 1, \ldots)$ read as,

$$\begin{cases}
\mu_{3,0}^{(\alpha,s)} = \left( \frac{11\alpha^2 + 12\alpha s + 3s^2}{6\alpha^2} \right)^\alpha, \\
\mu_{3,1}^{(\alpha,s)} = -\frac{3\alpha(6\alpha^2 + 10\alpha s + 3s^2)}{11\alpha^2 + 12\alpha s + 3s^2} \mu_{3,0}^{(\alpha,s)}, \\
\mu_{3,2}^{(\alpha,s)} = \frac{3\alpha}{2(11\alpha^2 + 12\alpha s + 3s^2)^2} \left( 108\alpha^5 + 360\alpha^4 s - 42\alpha^4 + 408\alpha^3 s^2 
- 112\alpha^3 s + 180\alpha^2 s^3 - 132\alpha^2 s^2 + 27\alpha s^4 - 60\alpha s^3 - 9s^4 \right) \mu_{3,0}^{(\alpha,s)}, \\
\mu_{3,\ell}^{(\alpha,s)} = \frac{1}{(11\alpha^2 + 12\alpha s + 3s^2)^\ell} \left[ -3(6\alpha^2 + 10\alpha s + 3s^2)(\alpha - \ell + 1) \mu_{3,\ell-1}^{(\alpha,s)} + 3(3\alpha^2 + 8\alpha s + 3s^2)(2\alpha - \ell + 2) \mu_{3,\ell-2}^{(\alpha,s)} 
- (2\alpha^2 + 6\alpha s + 3s^2)(3\alpha - \ell + 3) \mu_{3,\ell-3}^{(\alpha,s)} \right], \ \ell \geq 3.
\end{cases}$$
As before, we can also easily get the following generating function for $p = 4$,

$$G_{4,s}(z) = \left( (1 - z) + \frac{\alpha + 2s}{2\alpha} (1 - z)^2 + \frac{2\alpha^2 + 6\alpha s + 3s^2}{6\alpha^2} (1 - z)^3 + \frac{3\alpha^3 + 11\alpha^2 s + 9\alpha s^2 + 2s^3}{12\alpha^3} (1 - z)^4 \right)^\alpha.
$$

By the back-of-the-envelope calculation, we can get the expressions of coefficients $\mu^{(\alpha,s)}_{4,\ell} \ (\ell = 0,1,\ldots)$ as follows,

$$\mu^{(\alpha,s)}_{4,\ell} = d_{41} \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\ell} \sum_{\ell_3=\max\{0,2\ell_2-\ell_1\}} P(\alpha, \ell_1, \ell_2, \ell_3) \varpi^{(\alpha)}_{1,\ell_1-\ell_1} \varpi^{(\alpha)}_{1,\ell_1-\ell_2},
$$

where

$$P(\alpha, \ell_1, \ell_2, \ell_3) = \frac{(-1)^{\ell_1+\ell_2} (\ell_1 - \ell_2)!}{\ell_3! (\ell_2 - 2\ell_3)! (\ell_1 + \ell_3 - 2\ell_2)!} d_{42}^{\ell_1+\ell_3-2\ell_2} d_{43}^{\ell_2-2\ell_3} d_{44}^{\ell_3},$$

and

$$d_{41} = \frac{25\alpha^3 + 35\alpha^2 s + 15\alpha s^2 + 2s^3}{12\alpha^3}, \quad d_{42} = -\frac{23\alpha^3 + 69\alpha^2 s + 39\alpha s^2 + 6s^3}{25\alpha^3 + 35\alpha^2 s + 15\alpha s^2 + 2s^3},$$

$$d_{43} = \frac{13\alpha^3 + 45\alpha^2 s + 33\alpha s^2 + 6s^3}{25\alpha^3 + 35\alpha^2 s + 15\alpha s^2 + 2s^3}, \quad d_{44} = -\frac{3\alpha^3 + 11\alpha^2 s + 9\alpha s^2 + 2s^3}{25\alpha^3 + 35\alpha^2 s + 15\alpha s^2 + 2s^3}.$$
The recursion formulas of coefficient $\mu_{4,\ell}^{(\alpha,s)}$ $(\ell = 0, 1, \ldots)$ are shown below,

$$
\mu_{4,0}^{(\alpha,s)} = \left(\frac{25\alpha^3 + 35\alpha^2s + 15\alpha s^2 + 2s^3}{12\alpha^3}\right)^\alpha,
$$

$$
\mu_{4,1}^{(\alpha,s)} = -\frac{2\alpha}{25\alpha^3 + 35\alpha^2s + 15\alpha s^2 + 2s^3} (24\alpha^3 + 52\alpha^2s + 27\alpha s^2 + 4s^3) \mu_{4,0}^{(\alpha,s)},
$$

$$
\mu_{4,2}^{(\alpha,s)} = \frac{2\alpha}{(25\alpha^3 + 35\alpha^2s + 15\alpha s^2 + 2s^3)^2} (576\alpha^7 + 2496\alpha^6s + 4000\alpha^5s^2
+ 30000\alpha^4s^3 + 1145\alpha^3s^4 + 216\alpha^2s^5 + 16\alpha s^6 - 126\alpha^6 - 441\alpha^5s
- 835\alpha^4s^2 - 699\alpha^3s^3 - 281\alpha^2s^4 - 54\alpha s^5 - 4s^6) \mu_{4,0}^{(\alpha,s)},
$$

$$
\mu_{4,3}^{(\alpha,s)} = \frac{-2\alpha}{(25\alpha^3 + 35\alpha^2s + 15\alpha s^2 + 2s^3)^3} (27648\alpha^{11} + 179712\alpha^{10}s
+ 482688\alpha^9s^2 + 699392\alpha^8s^3 + 602928\alpha^7s^4 + 323448\alpha^6s^5
+ 109062\alpha^5s^6 + 22488\alpha^4s^7 + 2592\alpha^3s^8 + 128\alpha^2s^9 - 18144\alpha^{10}
- 102816\alpha^9s - 278244\alpha^8s^2 - 43564\alpha^7s^3 - 404406\alpha^6s^4
- 228726\alpha^5s^5 - 79722\alpha^4s^6 - 16740\alpha^3s^7 - 1944\alpha^2s^8 - 96\alpha s^9
+ 5496\alpha^9 + 17604\alpha^8s + 29331\alpha^7s^2 + 47500\alpha^6s^3 + 52263\alpha^5s^4
+ 33528\alpha^4s^5 + 12591\alpha^3s^6 + 2748\alpha^2s^7 + 324\alpha s^8 + 16s^9) \mu_{4,0}^{(\alpha,s)},
$$

$$
\mu_{4,\ell}^{(\alpha,s)} = \frac{1}{(25\alpha^3 + 35\alpha^2s + 15\alpha s^2 + 2s^3)\ell} \cdot \left[-2(24\alpha^3 + 52\alpha^2s + 27\alpha s^2 + 4s^3)
\times (\alpha - \ell + 1) \mu_{4,\ell-1}^{(\alpha,s)} + 6(6\alpha^3 + 19\alpha^2s + 12\alpha s^2 + 2s^3)(2\alpha - \ell + 2) \mu_{4,\ell-2}^{(\alpha,s)}
- 2(8\alpha^3 + 28\alpha^2s + 21\alpha s^2 + 4s^3)(3\alpha - \ell + 3) \mu_{4,\ell-3}^{(\alpha,s)}
+ (3\alpha^3 + 11\alpha^2s + 9\alpha s^2 + 2s^3)(4\alpha - \ell + 4) \mu_{4,\ell-4}^{(\alpha,s)}\right], \ \ell \geq 4.
$$

**Remark 3:** It is a remarkable finding that Lubich's method [22] and the modified high-order numerical algorithm formulas in [II] are the special cases of Theorem 3.1 for $s = 0$ and $s = -1$, respectively.

### 3.2 The fractional-compact forms of the generalized numerical algorithms

Firstly, the asymptotic expansion formulas of operators $L^{\alpha}_{p,s}$ and $R^{\alpha}_{p,s}$ are listed as follows, which are the foundations for the establishment of the fractional-compact forms of the generalized numerical algorithms.
Theorem 3.2 Let \( u(x) \in C^{[\alpha]+n+1} (\mathbb{R}) \) and all the derivatives of \( u(x) \) up to order \([\alpha]+n+2\) belong to \( L_1 (\mathbb{R})\). Then for any \( s \in \mathbb{R} \) and \( p \in \mathbb{N} \), one has

\[
L_\mathcal{B}_{p,s}^\alpha u(x) = RL D_{-\infty,x}^{\alpha} u(x) + \sum_{\ell=p}^{n-1} \left( \varrho_{\ell}^{(\alpha,s)} RL D_{-\infty,x}^{\alpha+\ell} u(x) \right) h^\ell + \mathcal{O}(h^n), \quad n \geq p + 1,
\]

and

\[
R_\mathcal{B}_{p,s}^\alpha u(x) = RL D_{x,\infty}^{\alpha} u(x) + \sum_{\ell=p}^{n-1} \left( \varrho_{\ell}^{(\alpha,s)} RL D_{\infty,x}^{\alpha+\ell} u(x) \right) h^\ell + \mathcal{O}(h^n), \quad n \geq p + 1,
\]

hold uniformly on \( \mathbb{R} \). Here the coefficients \( \varrho_{\ell}^{(\alpha,s)} \) \( (\ell = 1, 2, \ldots) \) can be determined by the following equation

\[
\frac{e^{-sz}}{z^\alpha} \mathcal{G}_{p,s}(e^{-z}) = 1 + \sum_{\ell=p}^{\infty} \varrho_{\ell}^{(\alpha,s)} z^\ell, \quad |z| < 1.
\]

Define a generalized fractional difference operator \( \mathcal{J}_{p,s} \) as

\[
\mathcal{J}_{p,s} u(x) = \left( 1 + \varrho_{p}^{(\alpha,s)} h^p \delta_{x}^p \right) u(x),
\]

where the difference operator \( \delta_{x}^p \) is defined by

\[
\delta_{x}^p u(x) = \frac{1}{h^p} \sum_{m=0}^{p} (-1)^m \left( \begin{array}{c} p \\ m \end{array} \right) u \left( x + \left( \frac{p}{2} - m \right) h \right), \quad p \in \mathbb{N}.
\]

Here, note that the facts

\[
RL D_{-\infty,x}^{\alpha+p} u(x) = \frac{d^p}{dx^p} \left( RL D_{-\infty,x}^{\alpha} u(x) \right),
\]

\[
RL D_{x,\infty}^{\alpha+p} u(x) = \frac{d^p}{dx^p} \left( RL D_{x,\infty}^{\alpha} u(x) \right),
\]

and

\[
\frac{d^p u(x)}{dx^p} = \delta_{x}^p u(x) + \mathcal{O}(h^p),
\]

then the generalized fractional-compact numerical algorithm formulas for Riemann-Liouville (Riesz) derivatives are stated below.

Theorem 3.3 Let \( u(x) \in C^{[\alpha]+p+1} (\mathbb{R}) \) and all the derivatives of \( u(x) \) up to order \([\alpha]+p+2\) belong to \( L_1 (\mathbb{R})\). Then there hold

\[
L_\mathcal{B}_{p,s}^\alpha u(x) = \mathcal{J}_{p,s} RL D_{-\infty,x}^{\alpha} u(x) + \mathcal{O}(h^{p+1}),
\]
and
\[ R\mathcal{B}_{p,s}^\alpha u(x) = J_{p,s} RL D_x^{\alpha,+\infty} u(x) + \mathcal{O}(h^{p+1}), \]
uniformly for \( x, s \in \mathbb{R} \) and \( p \in \mathbb{N} \).

Furthermore, one has
\[
J_{p,s} \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} |_{x=x_j+sh} = -\frac{1}{2 \cos \left( \frac{\pi \alpha}{2} \right)} \left[ R\mathcal{B}_{p,s}^\alpha + L\mathcal{B}_{p,s}^\alpha \right] u(x_j + sh) + \mathcal{O}(h^{p+1}).
\]

**Remark 4:** The operators \( \mathcal{L}, \tilde{\mathcal{L}} \) and \( J_{p,s} \) have the relations, \( J_{2,-1} = \mathcal{L} \) and \( J_{2,1} = \tilde{\mathcal{L}} \).

**Remark 5:** It is to be observed that some kinds of \((p+2)\)th-order fractional-compact numerical approximation formulas can be obtained by linear combination of any two different \((p+1)\)th-order fractional-compact schemes, where \( p \geq 2 \). Here, we list them as follows.

Suppose \( u(x) \in C^{[\alpha]+p+1}(\mathbb{R}) \) and all the derivatives of \( u(x) \) up to order \([\alpha]+p+2\) belong to \( L_1(\mathbb{R}) \). Define the fractional-compact operator as
\[
\mathcal{H}_{p,s_1,s_2} u(x) = \left[ \left( l_{p+1} - l_{p+1} \right) + h^p \left( l_{p} l_{p+1} - l_{p} l_{p+1} \right) \right] u(x),
\]
then one has
\[
l_{p+1} L\mathcal{B}_{p,s_1}^\alpha u(x) - l_{p+1} L\mathcal{B}_{p,s_2}^\alpha u(x) = \mathcal{H}_{p,s_1,s_2} RL D_x^{\alpha,+\infty} u(x) + \mathcal{O}(h^{p+2}),
\]
and
\[
l_{p+1} R\mathcal{B}_{p,s_1}^\alpha u(x) - l_{p+1} R\mathcal{B}_{p,s_2}^\alpha u(x) = \mathcal{H}_{p,s_1,s_2} RL D_x^{\alpha,+\infty} u(x) + \mathcal{O}(h^{p+2}),
\]
hold uniformly on \( \mathbb{R} \). Furthermore, a kind of \((p+2)\)th-order fractional-compact numerical approximation formula for Riesz derivative is given by
\[
\mathcal{H}_{p,s_1,s_2} \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos \left( \frac{\pi \alpha}{2} \right)} \left[ l_{p+1} \left( L\mathcal{B}_{p,s_1}^\alpha + R\mathcal{B}_{p,s_1}^\alpha \right) - l_{p+1} \left( L\mathcal{B}_{p,s_2}^\alpha + R\mathcal{B}_{p,s_2}^\alpha \right) \right] u(x) + \mathcal{O}(h^{p+2}).
\]

Particularly, if we choose \( p = 2 \), then the following several commonly four-order fractional-compact schemes can be obtained,
\[
\mathcal{H}_{2,-1,1} \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos \left( \frac{\pi \alpha}{2} \right)} \left[ l_{p+1} \left( L\mathcal{B}_{2,-1}^\alpha + R\mathcal{B}_{2,-1}^\alpha \right) - l_{p+1} \left( L\mathcal{B}_{2,1}^\alpha + R\mathcal{B}_{2,1}^\alpha \right) \right] u(x) + \mathcal{O}(h^4),
\]
(3.1)
\[
\mathcal{H}_{2,0,1} \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos \left(\frac{\pi \alpha}{2}\right)} \left[ \varrho_3^{(\alpha,1)} \left( L B_{2,0}^\alpha + R B_{2,0}^\alpha \right) \\
- \varrho_3^{(\alpha,0)} \left( L B_{2,1}^\alpha + R B_{2,1}^\alpha \right) \right] u(x) + \mathcal{O}(h^4),
\]

and

\[
\mathcal{H}_{2,0,-1} \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos \left(\frac{\pi \alpha}{2}\right)} \left[ \varrho_3^{(\alpha,-1)} \left( L B_{2,0}^\alpha + R B_{2,0}^\alpha \right) \\
- \varrho_3^{(\alpha,0)} \left( L B_{2,-1}^\alpha + R B_{2,-1}^\alpha \right) \right] u(x) + \mathcal{O}(h^4),
\]

where,

\[
\varrho_2^{(\alpha,s)} = -\frac{2 \alpha^2 + 6 \alpha s + 3 s^2}{6 \alpha},
\]

\[
\varrho_3^{(\alpha,s)} = \frac{3 \alpha^3 + 11 \alpha^2 s + 12 s^2 \alpha + 4 s^3}{12 \alpha^2}, \quad s = -1, 0, 1.
\]

4 Application to Riesz spatial fractional reaction-dispersion equation in one space dimension

Here, we consider the one-dimension Riesz spatial fractional reaction-dispersion equation in the following form,

\[
\frac{\partial u(x,t)}{\partial t} = -u(x,t) + K_\alpha \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t), \quad 1 < \alpha < 2,
\]

\[
0 < x < L, \quad 0 < t \leq T,
\]

with initial value condition

\[
u(x,0) = u^0(x), \quad 0 \leq x \leq L,
\]

and boundary value conditions

\[
u(0,t) = \nu(L,t) = 0, \quad 0 < t \leq T,
\]

in which parameter $K_\alpha$ is a positive real constant, $u^0(x)$ and $f(x,t)$ are given suitably smooth functions.
4.1 Derivation of the fractional-compact difference scheme

Let temporal steplength \( \tau = \frac{T}{N} \) and spatial steplength \( h = \frac{L}{M} \), where \( M \) and \( N \) are two positive integers. Define a partition of \([0, T] \times [0, L]\) by \( \Omega = \Omega_\tau \times \Omega_h \) with grids \( \Omega_\tau = \{t_k = k\tau \mid 0 \leq k \leq N\} \) and \( \Omega_h = \{x_j = jh \mid 0 \leq j \leq M\} \). For any grid function \( u_j^k \in \Omega \), denote

\[
\delta_t u_j^{k+\frac{1}{2}} = \frac{u_j^{k+1} - u_j^k}{\tau}, \quad u_j^{k+\frac{1}{2}} = \frac{u_j^{k+1} + u_j^k}{2}.
\]

And set

\[
\delta_x^\alpha = -\frac{1}{2} \cos \left( \frac{\pi \alpha}{2} \right) \left( L A_2^\alpha + R A_2^\alpha \right), \quad 1 < \alpha < 2.
\]

Now we consider equation (4.1) at the point \((x_j, t)\). Then we have

\[
\frac{\partial u(x_j, t)}{\partial t} = -u(x_j, t) + K_\alpha \frac{\partial^\alpha u(x_j, t)}{\partial |x|^\alpha} + f(x_j, t), \quad 0 \leq j \leq M.
\] (4.4)

Operating operator \( \mathcal{L} \) on both sides of (4.3) yields

\[
\mathcal{L} \frac{\partial u(x_j, t)}{\partial t} = -\mathcal{L}u(x_j, t) + K_\alpha \mathcal{L} \frac{\partial^\alpha u(x_j, t)}{\partial |x|^\alpha} + \mathcal{L}f(x_j, t), \quad 0 \leq j \leq M.
\]

Noticing the definition of operator \( \mathcal{L} \) and equation (2.6) gives

\[
\mathcal{L} \frac{\partial u(x_j, t)}{\partial t} = -\mathcal{L}u(x_j, t) + K_\alpha \delta_x^\alpha u(x_j, t) + \mathcal{L}f(x_j, t) + \mathcal{O}(h^3), \quad 0 \leq j \leq M.
\]

Taking \( t = t_{k+\frac{1}{2}} \) and using the Taylor expansion, one has

\[
\mathcal{L} \delta_t u(x_j, t_{k+\frac{1}{2}}) = -\mathcal{L}u(x_j, t_{k+\frac{1}{2}}) + K_\alpha \delta_x^\alpha u(x_j, t_{k+\frac{1}{2}}) + \mathcal{L}f(x_j, t_{k+\frac{1}{2}}) + R_j^k, \quad 0 \leq j \leq M.
\] (4.5)

where there exists a positive constant \( C_1 \) such that

\[
|R_j^k| \leq C_1 (\tau^2 + h^3), \quad 0 \leq k \leq N - 1, \quad 1 \leq j \leq M - 1.
\]

Omitting the high-order terms \( R_j^k \) of (4.5), and letting \( u_j^k \) be the numerical approximation of function \( u(x_j, t_k) \), then we can obtain the following fractional-compact difference scheme for equation (4.1), together with initial and boundary value conditions (4.2) and (4.3) as follows,

\[
\mathcal{L} \delta_t u_j^{k+\frac{1}{2}} = -\mathcal{L}u_j^{k+\frac{1}{2}} + K_\alpha \delta_x^\alpha u_j^{k+\frac{1}{2}} + \mathcal{L}f_j^{k+\frac{1}{2}}, \quad 0 \leq k \leq N - 1, \quad 1 \leq j \leq M - 1,
\] (4.6)

\[
u_j^0 = u^0(x_j), \quad 0 \leq j \leq M,
\] (4.7)

\[
u_j^k = u_j^M = 0, \quad 1 \leq k \leq N.
\] (4.8)
4.2 Analysis of the fractional-compact difference scheme

Let
\[ V_h = \{ v | v = (v_0, v_1, \ldots, v_M), v_0 = v_M = 0 \} \]
be the space of grid functions. Then for any \( u, v \in V_h \), we can define the discrete inner products below,
\[ (u, v) = h \sum_{j=1}^{M-1} u_j v_j, \quad (\delta_x u, \delta_x v) = h \sum_{j=1}^{M} \left( \delta_x u_{j-\frac{1}{2}} \right) \left( \delta_x v_{j-\frac{1}{2}} \right), \]
and associated norms below,
\[ ||u||^2 = (u, u), \quad ||\delta_x u||^2 = (\delta_x u, \delta_x u). \]

Next, we list several lemmas which will be used later on.

**Lemma 4.1** Operator \( L \) is self-adjoint, that is, for any \( u, v \in V_h \), there holds,
\[ (L u, v) = (u, L v). \]

**Proof.** It follows from the definition of operator \( L \) that
\[ (L u, v) = \left( 1 + \sigma_2^{(\alpha)} h^2 \delta_x^2 \right) u, v = (u, v) - \sigma_2^{(\alpha)} h^2 (\delta_x u, \delta_x v) \]
\[ = (u, v) + \sigma_2^{(\alpha)} h^2 (u, \delta_x^2 v) = \left( u, \left( 1 + \sigma_2^{(\alpha)} h^2 \delta_x^2 \right) v \right) \]
\[ = (u, L v). \]

All this ends the proof. \( \blacksquare \)

**Lemma 4.2** For any \( u \in V_h \), there holds that
\[ \left( \frac{4\sqrt{6}}{3} - 3 \right) ||u||^2 \leq (L u, u) \leq ||u||^2, \quad 1 < \alpha < 2. \]

**Proof.** On one hand, note that \( \sigma_2^{(\alpha)} = -\frac{2\alpha^2 - 6\alpha + 3}{6\alpha} \in \left( \frac{1}{12}, 1 - \frac{\sqrt{6}}{3} \right) \] for \( 1 < \alpha < 2 \), then one has
\[ (L u, u) = (u, v) - \sigma_2^{(\alpha)} h^2 (\delta_x u, \delta_x v) = ||u||^2 - \sigma_2^{(\alpha)} h^2 ||\delta_x u||^2 \leq ||u||^2. \]

On the other hand, using the inverse estimate \( ||\delta_x u||^2 \leq \frac{4}{h^2} ||u||^2 \), we reach that
\[ (L u, u) = ||u||^2 - \sigma_2^{(\alpha)} h^2 ||\delta_x u||^2 \geq \left( \frac{4\sqrt{6}}{3} - 3 \right) ||u||^2. \]

This finishes the proof. \( \blacksquare \)
Lemma 4.3 [11] For any \( u \in V_h \), there holds that
\[
(\delta^\alpha_x u, u) \leq 0.
\]

Lemma 4.4 (Grownall’s inequality [26]) Assume that \( \{k_n\} \) and \( \{p_n\} \) are non-negative sequences, and the sequence \( \{\phi_n\} \) satisfies
\[
\phi_0 \leq q_0, \quad \phi_n \leq q_0 + \sum_{\ell=0}^{n-1} p_\ell + \sum_{\ell=0}^{n-1} k_\ell \phi_\ell, \quad n \geq 1,
\]
where \( q_0 \geq 0 \). Then the sequence \( \{\phi_n\} \)
\[
\phi_n \leq \left( q_0 + \sum_{\ell=0}^{n-1} p_\ell \right) \exp \left( \sum_{\ell=0}^{n-1} k_\ell \right)
\]
holds for \( n \geq 1 \).

Now, we turn to study the stability of finite difference scheme (4.6) with (4.7) and (4.8).

Theorem 4.5 The difference scheme (4.6) with (4.7) and (4.8) is unconditionally stable with respect to the initial values.

Proof. Let
\[
\xi^k_j = u^k_j - v^k_j, \quad 0 \leq k \leq N, \quad 0 \leq j \leq M,
\]
where \( u^k_j \) and \( v^k_j \) are the solutions of the following two equations, respectively,
\[
\mathcal{L} \delta_t u^{k+\frac{1}{2}}_{j+\frac{1}{2}} = -\mathcal{L} u^{k+\frac{1}{2}}_j + K_\alpha \delta_x^\alpha u^{k+\frac{1}{2}}_j + \mathcal{L} f^{k+\frac{1}{2}}_j,
\]
\[
0 \leq k \leq N - 1, \quad 1 \leq j \leq M - 1,
\]
\[
u_j^0 = u^0(x_j), \quad 0 \leq j \leq M,
\]
\[
u^k_0 = u^k_M = 0, \quad 1 \leq k \leq N.
\]
and
\[
\mathcal{L} \delta_t v^{k+\frac{1}{2}}_j = -\mathcal{L} v^{k+\frac{1}{2}}_j + K_\alpha \delta_x^\alpha v^{k+\frac{1}{2}}_j + \mathcal{L} f^{k+\frac{1}{2}}_j,
\]
\[
0 \leq k \leq N - 1, \quad 1 \leq j \leq M - 1,
\]
\[
u_j^0 = u^0(x_j) + \varepsilon_j, \quad 0 \leq j \leq M,
\]
\[
u^k_0 = v^k_M = 0, \quad 1 \leq k \leq N.
\]
Subtracting (4.9)-(4.11) from (4.12)-(4.14) gives the perturbation equations as follows,

\[ \mathcal{L} \delta_t \xi_{\frac{n}{2}}^k = -\mathcal{L} \xi_{\frac{n}{2}}^k + K_\alpha \delta_x^0 \xi_{\frac{n}{2}}^k, \]

(4.15)

\[ 0 \leq k \leq N - 1, \quad 1 \leq j \leq M - 1, \]

\[ \xi_j^0 = -\epsilon_j, \quad 0 \leq j \leq M, \]

\[ \xi_0^k = \xi_M^k = 0, \quad 1 \leq k \leq N. \]

Taking the inner product of (4.15) with \( \xi_{\frac{n}{2}}^k \), replacing \( k \) by \( n \), and summing from \( n = 0 \) to \( k - 1 \) yield

\[ \sum_{n=0}^{k-1} \left( \mathcal{L} \delta_t \xi_{\frac{n+1}{2}}, \xi_{\frac{n+1}{2}} \right) + \sum_{n=0}^{k-1} \left( \mathcal{L} \xi_{\frac{n+1}{2}}, \xi_{\frac{n+1}{2}} \right) \]

(4.16)

\[ = K_\alpha \sum_{n=0}^{k-1} \left( \delta_x^0 \xi_{\frac{n+1}{2}}, \xi_{\frac{n+1}{2}} \right), \quad 0 \leq k \leq N - 1, \quad 1 \leq j \leq M - 1. \]

For the first term on the left hand side of (4.16), using Lemma 4.1 leads to

\[ \sum_{n=0}^{k-1} \left( \mathcal{L} \xi_{\frac{n+1}{2}}, \xi_{\frac{n+1}{2}} \right) = \frac{1}{2\tau} \sum_{n=0}^{k-1} \left[ \left( \mathcal{L} \xi_{n+1}, \xi_{n+1} \right) - \left( \mathcal{L} \xi_n, \xi_n \right) \right]. \]

For the second term on the left hand side of (4.16), in view of Lemma 4.2 we have

\[ \sum_{n=0}^{k-1} \left( \mathcal{L} \xi_{\frac{n+1}{2}}, \xi_{\frac{n+1}{2}} \right) \geq 0. \]

For the term on the right hand side of (4.16), it follows from Lemma 4.3 that

\[ K_\alpha \sum_{n=0}^{k-1} \left( \delta_x^0 \xi_{\frac{n+1}{2}}, \xi_{\frac{n+1}{2}} \right) \leq 0. \]

Hence, combining the above discussion, one has

\[ \sum_{n=0}^{k-1} \left[ \left( \mathcal{L} \xi_{n+1}, \xi_{n+1} \right) - \left( \mathcal{L} \xi_n, \xi_n \right) \right] \leq 0, \]

i.e.,

\[ \left( \mathcal{L} \xi_k, \xi_k \right) \leq \left( \mathcal{L} \xi_0, \xi_0 \right). \]
Using Lemma 4.2 again leads to

\[ ||\xi^k|| \leq \frac{\sqrt{5(4\sqrt{6} + 9)}}{5}||\xi^0|| = \frac{\sqrt{5(4\sqrt{6} + 9)}}{5}||\varepsilon||. \]

The proof is thus completed. ■

Finally, we give the convergence result as follows.

**Theorem 4.6** Let \( u_j^k \) and \( u(x,t) \) be the solutions of the finite difference scheme (4.6)-(4.8) and problem (4.1)-(4.3), respectively. Denote \( e_j^k = u_j^k - u(x_j,t_k) \), \( 0 \leq k \leq N, \ 0 \leq j \leq M \). Then there holds

\[ ||e^k|| \leq \frac{\sqrt{6L\tau T}}{6}C_1 \exp\left(\frac{1}{3}T\right) (\tau^2 + h^3), \]

where \( C_1 \) is a constant independent of \( \tau \) and \( h \).

**Proof.** Subtracting (4.1)-(4.3) from (4.6)-(4.8), we get the following error equation

\[ L\delta t e_j^{k+\frac{1}{2}} = -L e_j^{k+\frac{1}{2}} + K_{\alpha} e_j^{k+\frac{1}{2}} + R_j^{k+1}, \quad 0 \leq k \leq N-1, \ 1 \leq j \leq M-1, \]

\[ e_j^0 = 0, \ 0 \leq j \leq M, \]

\[ e_0^k = e_M^k = 0, \ 1 \leq k \leq N. \]

Taking the inner product of (4.17) with \( e^{k+\frac{1}{2}} \), replacing \( k \) by \( n \), and summing up from \( n = 0 \) to \( k-1 \), lead to

\[ \sum_{n=0}^{k-1} \left( L\delta t e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \right) + \sum_{n=0}^{k-1} \left( L e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \right) = K_{\alpha} \sum_{n=0}^{k-1} \left( \delta_x^\alpha e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \right) + \sum_{n=0}^{k-1} \left( R^{n+1}, e^{n+\frac{1}{2}} \right). \]

For the last term on the right hand side of (4.18), we have the following estimate,

\[ \sum_{n=0}^{k-1} \left( R^{n+1}, e^{n+\frac{1}{2}} \right) \leq \frac{1}{4}||e^k||^2 + \frac{1}{2} \sum_{n=0}^{k-1} ||e^n||^2 + \frac{1}{2} \sum_{n=0}^{k-1} ||R^{n+1}||^2. \]

Hence, the following result can be obtained by combining with the above discussion,

\[ (Le^k, e^k) \leq (Le^0, e^0) + \frac{1}{4}||e^k||^2 + \frac{1}{2} \sum_{n=0}^{k-1} ||e^n||^2 + \frac{1}{2} \sum_{n=0}^{k-1} ||R^{n+1}||^2. \]
Noticing
\[ \| R_{n+1}^n \|^2 = h \sum_{j=1}^{M-1} (R_{j+1}^n)^2 \leq C_1^2 L (\tau^2 + h^2)^2 \]
and using Lemma 4.2 give
\[ \| e^k \|^2 \leq \frac{2(16\sqrt{6} + 39)}{1019} \tau \sum_{n=0}^{k-1} \| e^n \|^2 + \frac{2(16\sqrt{6} + 39)}{1019} C_1^2 L (\tau^2 + h^2)^2. \]

Finally, it is easy to obtain the following result by using Lemma 4.3,
\[ \| e^k \|^2 \leq \frac{2(16\sqrt{6} + 39)}{1019} \tau \sum_{n=0}^{k-1} \| e^n \|^2 + \frac{2(16\sqrt{6} + 39)}{1019} C_1^2 L (\tau^2 + h^2)^2. \]
i.e.,
\[ \| e^k \| \leq \sqrt{\frac{6LT}{6} C_1 \exp \left( \frac{1}{12} T \right) (\tau^2 + h^2)}. \]
Therefore, the proof is finished.

\[ \Box \]

5 Application to the two-dimensional equation

In this section, we consider the following two-dimension Riesz spatial fractional reaction-dispersion equation,
\[ \frac{\partial u(x, y, t)}{\partial t} = -u(x, y, t) + K_\alpha \frac{\partial^\alpha u(x, y, t)}{\partial |x|^\alpha} + K_\beta \frac{\partial^\beta u(x, y, t)}{\partial |y|^\beta} + f(x, y, t), (x, y; t) \in \Omega \times (0, T], \]
with the initial value condition
\[ u(x, y, 0) = u^0(x, y), \quad (x, y) \in \Omega, \]
and the boundary value conditions
\[ u(x, y, t) = 0, \quad (x, y; t) \in \partial \Omega \times (0, T], \]
where \( \alpha, \beta \in (1, 2), \Omega = (0, L_a) \times (0, L_b), \) coefficients \( K_\alpha \) and \( K_\beta \) are two positive constants, \( f(x, y, t) \) and \( u^0(x, y) \) are suitably smooth.
5.1 Derivation of the fractional-compact difference scheme

For a given positive integer $N$, denote the timestep size $\tau = \frac{N}{T}$ and grid points $t_k = k\tau$, $0 \leq k \leq N$. Set the space stepsizes $h_a = \frac{L_a}{M_a}$ and $h_b = \frac{L_b}{M_b}$, where $M_a$ and $M_b$ are two positive integers. And the according grid points are $x_i = ih_a$, $0 \leq i \leq M_a$, and $y_j = jh_b$, $0 \leq j \leq M_b$. In addition, let $\Omega_h = \{(x_i, y_j) | 0 \leq i \leq M_a, 0 \leq j \leq M_b\}$, $\Omega_h = \Omega_h \cap \Omega$, and $\partial \Omega_h = \Omega_h \cap \partial \Omega$.

Define the fractional-compact difference operators $L_x$ and $L_y$ as

$$L_x u(x, y, t) = \left(1 + \sigma_2^{(\alpha)} h_a^2 \delta_x^2 \right) u(x, y, t),$$

and

$$L_y u(x, y, t) = \left(1 + \sigma_2^{(\beta)} h_b^2 \delta_y^2 \right) u(x, y, t).$$

For brevity, set

$$L A_{2,x}^\alpha u(x, y, t) = \frac{1}{h_a^\alpha} \sum_{\ell=0}^{[\frac{\alpha}{2}]+1} \kappa_2^{(\alpha)} u \left(x - (\ell - 1)h_a, y, t\right),$$

$$L A_{2,y}^\beta u(x, y, t) = \frac{1}{h_b^\beta} \sum_{\ell=0}^{[\frac{\beta}{2}]+1} \kappa_2^{(\beta)} u \left(x, y - (\ell - 1)h_b, t\right),$$

and

$$R A_{2,x}^\alpha u(x, y, t) = \frac{1}{h_a^\alpha} \sum_{\ell=0}^{[\frac{\alpha}{2}]+1} \kappa_2^{(\alpha)} u \left(x + (\ell - 1)h_a, y, t\right),$$

$$R A_{2,y}^\beta u(x, y, t) = \frac{1}{h_b^\beta} \sum_{\ell=0}^{[\frac{\beta}{2}]+1} \kappa_2^{(\beta)} u \left(x, y + (\ell - 1)h_b, t\right).$$

Then one has

$$L_x \frac{\partial^\alpha u(x, y, t)}{\partial|x|^\alpha} = -\frac{1}{2 \cos \left(\frac{\alpha \pi}{2}\right)} \left( L A_{2,x}^\alpha + R A_{2,x}^\alpha \right) u(x, y, t)$$

$$+ O(h_a^3),$$

and

$$L_y \frac{\partial^\beta u(x, y, t)}{\partial|y|^\beta} = -\frac{1}{2 \cos \left(\frac{\beta \pi}{2}\right)} \left( L A_{2,y}^\beta + R A_{2,y}^\beta \right) u(x, y, t)$$

$$+ O(h_b^3).$$
In order to simplify the expressions, let

$$\delta^\alpha_x = -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left( L^\alpha A_{2,x} + R^\alpha A_{2,x} \right)$$

and

$$\delta^\beta_y = -\frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)}\left( L^\beta A_{2,y} + R^\beta A_{2,y} \right).$$

Accordingly, equations (5.4) and (5.5) can be rewritten as,

$$L^x \frac{\partial x}{\partial x} u(x, y, t) = \delta^\alpha_x u(x, y, t) + O(h_3^3), \quad (5.6)$$

and

$$L^y \frac{\partial y}{\partial y} u(x, y, t) = \delta^\beta_y u(x, y, t) + O(h_3^3). \quad (5.7)$$

Similar to the one-dimension case, utilizing the central difference scheme in time direction and fractional-compact difference formulas (5.6) and (5.7) in space directions, we get

$$L^x L^y \delta t u(x_i, y_j, t^k_{+\frac{1}{2}}) = -L^x L^y u(x_i, y_j, t^k_{+\frac{1}{2}})$$

$$+ K^\alpha L^y \delta^\alpha_x u(x_i, y_j, t^k_{+\frac{1}{2}}) + K^\beta L^x \delta^\beta_y u(x_i, y_j, t^k_{+\frac{1}{2}})$$

$$+ L^x L^y f(x_i, y_j, t^k_{+\frac{1}{2}}) + R^k_{i,j}, \quad 1 \leq i \leq M_x - 1, \quad 1 \leq j \leq M_y - 1, \quad 0 \leq k \leq N - 1,$$  \quad (5.8)

where there exists a positive constant $C_2$ such that

$$|R^k_{i,j}| \leq C_2(\tau^2 + h_x^3 + h_y^3), \quad 1 \leq i \leq M_x - 1, \quad 1 \leq j \leq M_y - 1, \quad 0 \leq k \leq N - 1.$$

Omitting the high-order term $R^k_{i,j}$ and replacing the function $u(x_i, y_j, t^k_{+\frac{1}{2}})$ by its numerical approximation $u^{k+\frac{1}{2}}_{i,j}$ in (5.6), then a finite difference scheme for equations (5.1)-(5.3) is obtained,

$$L^x L^y \delta t u^{k+\frac{1}{2}}_{i,j} = -L^x L^y u^{k+\frac{1}{2}}_{i,j} + K^\alpha L^y \delta^\alpha_x u^{k+\frac{1}{2}}_{i,j} + K^\beta L^x \delta^\beta_y u^{k+\frac{1}{2}}_{i,j}$$

$$+ L^x L^y f^{k+\frac{1}{2}}_{i,j}, \quad 1 \leq i \leq M_x - 1, \quad 1 \leq j \leq M_y - 1, \quad 0 \leq k \leq N - 1,$$  \quad (5.9)

$$u^0_{i,j} = u^0(x_i, y_j), \quad (x_i, y_j) \in \Omega_h,$$  \quad (5.10)

$$u^k_{i,j} = 0, \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq k \leq N.$$  \quad (5.11)
5.2 Analysis of the fractional-compact difference scheme

Almost the same as the one dimensional case, let

\[ V_{h_a,h_b} = \{ v | v = \{ v_{i,j} \} \text{ is a grid function on } \Omega_h \text{ and } v_{i,j} = 0 \text{ if } (x_i, y_j) \in \partial \Omega_h \}, \]

then for any \( u, v \in V_{h_a,h_b} \), we introduce the discrete inner products and corresponding norms below,

\[
(u, v) = h_a h_b \sum_{i=1}^{M_a-1} \sum_{j=1}^{M_b-1} u_{i,j} v_{i,j},
\]

\[
(\delta_x u, \delta_x v) = h_a h_b \sum_{i=1}^{M_a} \sum_{j=1}^{M_b-1} \left( \delta_x u_{i-\frac{1}{2},j} \right) \left( \delta_x v_{i-\frac{1}{2},j} \right),
\]

and

\[
||u||^2 = (u, u), \quad ||\delta_x u||^2 = (\delta_x u, \delta_x u).
\]

The following definition and lemmas are useful for our discussion [19].

**Definition 5.1** If \( A = (a_{ij}) \) is an \( m \times n \) matrix and \( B = (b_{ij}) \) is a \( p \times q \) matrix, then the Kronecker product \( A \otimes B \) is an \( mp \times nq \) block matrix and is denoted by

\[
A \otimes B = \\
\begin{pmatrix}
    a_{11}B & a_{12}B & \cdots & a_{1n}B \\
    a_{21}B & a_{22}B & \cdots & a_{2n}B \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{pmatrix}
\]

**Lemma 5.1** Assume that \( A \in \mathbb{R}^{n \times n} \) has eigenvalues \( \{\lambda_j\}_{j=1}^n \), and that \( B \in \mathbb{R}^{m \times m} \) has eigenvalues \( \{\mu_j\}_{j=1}^m \). Then the mn eigenvalues of \( A \otimes B \) are:

\[
\lambda_1 \mu_1, \ldots, \lambda_1 \mu_m; \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m; \ldots; \lambda_n \mu_1, \ldots, \lambda_n \mu_m.
\]

**Lemma 5.2** Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{r \times s}, C \in \mathbb{R}^{n \times p}, D \in \mathbb{R}^{s \times t} \). Then

\[
(A \otimes B)(C \otimes D) = AC \otimes BD.
\]

Moreover, if \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times m}, I_n \) and \( I_m \) are unit matrices of order \( n, m \), respectively, then matrices \( I_m \otimes A \) and \( B \otimes I_n \) can commute with each other.

**Lemma 5.3** For all \( A \) and \( B \), there holds

\[
(A \otimes B)^T = A^T \otimes B^T.
\]
Lemma 5.4 [11] Denote

$$\mathbf{E}_p^{(\gamma)} = \begin{pmatrix}
\kappa_{2,1}^{(\gamma)} & \kappa_{2,0}^{(\gamma)} & 0 & \ldots & 0 \\
\kappa_{2,2}^{(\gamma)} & \kappa_{2,1}^{(\gamma)} & \kappa_{2,0}^{(\gamma)} & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\kappa_{2,M_p-1}^{(\gamma)} & \kappa_{2,M_p-2}^{(\gamma)} & \ldots & \kappa_{2,2}^{(\gamma)} & \kappa_{2,1}^{(\gamma)}
\end{pmatrix}_{(M_p-1)\times (M_p-1)}.$$

Then matrix \( (\mathbf{E}_p^{(\gamma)} + \mathbf{E}_p^{(\gamma)^T}) \) is semi-negative definite.

Lemma 5.5 For any mesh functions \( \mathbf{u}, \mathbf{v} \in \mathcal{V}_{h_a,h_b} \), there exists a symmetric positive definite operator \( \mathbf{P} \) such that

\[
((K_\alpha \mathcal{L}_y \delta_x^\alpha + K_\beta \mathcal{L}_x \delta_y^\beta) \mathbf{u}, \mathbf{v}) = -(\mathbf{P} \mathbf{u}, \mathbf{P} \mathbf{v}), \quad K_\alpha, K_\beta > 0.
\]

**Proof.** Firstly, we rewrite the inner product \( ((K_\alpha \mathcal{L}_y \delta_x^\alpha + K_\beta \mathcal{L}_x \delta_y^\beta) \mathbf{u}, \mathbf{v}) \) in a matrix form,

\[
((K_\alpha \mathcal{L}_y \delta_x^\alpha + K_\beta \mathcal{L}_x \delta_y^\beta) \mathbf{u}, \mathbf{v}) = h_a h_b v^T S \mathbf{u},
\]

where

\[
S = \frac{K_\alpha}{h_a^2} \left( C_b^{(\beta)} \otimes I_a \right) \left( I_b \otimes D_a^{(\alpha)} \right) + \frac{K_\beta}{h_b^2} \left( I_b \otimes C_a^{(\alpha)} \right) \left( D_b^{(\beta)} \otimes I_a \right),
\]

\[
D_p^{(\gamma)} = -\frac{1}{2 \cos \left( \frac{\pi \gamma}{2} \right)} (\mathbf{E}_p^{(\gamma)} + \mathbf{E}_p^{(\gamma)^T}), \quad p = a, b, \quad \gamma = \alpha, \beta.
\]

Here \( I_p \) is the identity matrix of order \( M_p - 1 \), \( C_p^{(\gamma)} (p = a, b, \gamma = \alpha, \beta) \) has the form

\[
C_p^{(\gamma)} = \begin{pmatrix}
1 - 2\sigma_2^{(\gamma)} & \sigma_2^{(\gamma)} & 0 & \ldots & 0 \\
\sigma_2^{(\gamma)} & 1 - 2\sigma_2^{(\gamma)} & \sigma_2^{(\gamma)} & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \sigma_2^{(\gamma)} & 1 - 2\sigma_2^{(\gamma)} & \sigma_2^{(\gamma)} \\
0 & 0 & \ldots & \sigma_2^{(\gamma)} & 1 - 2\sigma_2^{(\gamma)}
\end{pmatrix}_{(M_p-1)\times (M_p-1)}.
\]
It follows from Lemmas 5.2 and 5.3 that

\[
S^T = \frac{K_\alpha}{h_a^2} (I_b \otimes D_{\alpha}^{(a)})^T \left( C_{\beta}^{(\alpha)} \otimes I_a \right)^T + \frac{K_\beta}{h_b^2} \left( D_{\beta}^{(\beta)} \otimes I_a \right)^T (I_b \otimes C_{\alpha}^{(a)})^T
\]

\[
= \frac{K_\alpha}{h_a^2} (I_b \otimes D_{\alpha}^{(a)}) \left( C_{\beta}^{(\beta)} \otimes I_a \right) + \frac{K_\beta}{h_b^2} \left( D_{\beta}^{(\beta)} \otimes I_a \right) (I_b \otimes C_{\alpha}^{(a)})
\]

\[
= \frac{K_\alpha}{h_a^2} \left( C_{\beta}^{(\beta)} \otimes I_a \right) (I_b \otimes D_{\alpha}^{(a)}) + \frac{K_\beta}{h_b^2} (I_b \otimes C_{\alpha}^{(a)}) \left( D_{\beta}^{(\beta)} \otimes I_a \right)
\]

\[
= S,
\]

namely, \( S \) is a real symmetric matrix.

Besides, one easily knows that all the eigenvalues of matrix \( C_{\gamma}^{(\gamma)} \) are

\[
\lambda_j (C_{\gamma}^{(\gamma)}) = 1 - 4\sigma_{2}^2 \sin^2 \left( \frac{\pi j}{2M_p} \right) \geq \frac{4\sqrt{6}}{3} - 3 > 0, \quad j = 0, 1, \ldots, M_p - 1.
\]

From Lemma 5.4 we see that the eigenvalues of matrix \( D_{\gamma}^{(\gamma)} \) satisfy

\[
\lambda_j (D_{\gamma}^{(\gamma)}) \leq 0, \quad j = 0, 1, \ldots, M_p - 1.
\]

Combining the above analysis and using Lemma 5.1, we can state that matrix \( S \) is semi-negative definite. Accordingly, there exists an orthogonal matrix \( H \) and a diagonal matrix \( \Lambda \) such that

\[
S = -H^T \Lambda H = -\left( \Lambda^{\frac{1}{2}} H \right)^T \left( \Lambda^{\frac{1}{2}} H \right) = -P^T P,
\]

where \( P = \Lambda^{\frac{1}{2}} H \). Therefore, we have

\[
((K_\alpha \mathcal{L}_y \delta_x^{\alpha} + K_\beta \mathcal{L}_x \delta_y^{\beta}) u, v) = h_a h_b v^T S u = -h_a h_b (Pv)^T (Pu) = -(P u, P v),
\]

where \( P \) is the associate operator of matrix \( P \), which is symmetric and semi-positive definite. The proof is thus ended.

**Lemma 5.6** For any mesh functions, \( u, v \in \mathcal{V}_{h_a,h_b} \), there exists a symmetric positive definite operator \( Q \) such that

\[
(\mathcal{L}_x \mathcal{L}_y u, v) = (Q u, Q v).
\]

**Proof.** Similar to Lemma 5.5, the matrix form of inner product \( (\mathcal{L}_x \mathcal{L}_y u, v) \) is

\[
(\mathcal{L}_x \mathcal{L}_y u, v) = h_a h_b v^T T u,
\]

(5.12)
where $T = \left( I_b \otimes C^{(\alpha)}_a \right) \left( C^{(\beta)}_b \otimes I_a \right)$. Here, we easily know that matrix $T$ is symmetric and positive definite by almost the same reasoning of Lemma 5.5. So, there exist an orthogonal matrix $\tilde{H}$ and a diagonal matrix $\tilde{\Lambda}$ such that

$$T = H^T \tilde{\Lambda} H = -\left( \Lambda^\frac{1}{2} \tilde{H} \right)^T \left( \Lambda^\frac{1}{2} \tilde{H} \right) = Q^T Q,$$

in which $Q = \tilde{\Lambda}^\frac{1}{2} \tilde{H}$.

Substitution (5.13) in (5.12) yields

$$(L_x L_y u, v) = h_a h_b v^T T u = h_a h_b (Q v)^T Q u = (Q u, Q v),$$

where $Q$ is a symmetric and positive definite corresponding to matrix $Q$. The proof is completed. ■

Lemma 5.7 For any mesh function $u \in \mathcal{V}_{h_a, h_b}$ and symmetric positive definite operator $Q$, there holds that

$$\left( \frac{4 \sqrt{6}}{3} - 3 \right) ||u|| \leq ||Q u|| \leq ||u||.$$

Proof. From Lemma 5.5, we know that operators $L_x$ and $L_y$ are both symmetric and positive definite. So, there exist two symmetric and positive definite operators $R_x$ and $R_y$ such that

$$(L_x u, v) = (R_x u, R_x v), \quad (L_y u, v) = (R_y u, R_y v)$$

for any mesh functions $u, v \in \mathcal{V}_{h_a, h_b}$.

Therefore, one has

$$||Q u||^2 = (L_x L_y u, u) = (R_x L_y u, R_x u) \geq \left( \frac{4 \sqrt{6}}{3} - 3 \right) (R_x u, R_x u) = \left( \frac{4 \sqrt{6}}{3} - 3 \right) (L_x u, u) \geq \left( \frac{4 \sqrt{6}}{3} - 3 \right)^2 ||u||^2.$$

On the other hand, we also have

$$||Q u||^2 = (L_x L_y u, u) = (R_x L_y u, R_x u) \leq (R_x u, R_x u) = (L_x u, u) \leq ||u||^2.$$
The proof is thus shown. ■

Now, we turn to consider the stability of difference scheme (5.9)–(5.11). Suppose $v_{i,j}^{k}$ is the solution of the following finite difference equation,

$$L_x L_y \frac{\delta t}{2} v_{i,j}^{k+\frac{1}{2}} = -L_x L_y v_{i,j}^{k+\frac{1}{2}} + K_\alpha L_y \frac{\delta x}{2} v_{i,j}^{k+\frac{1}{2}} + K_\beta L_x \frac{\delta y}{2} v_{i,j}^{k+\frac{1}{2}}$$

$$+ L_x L_y f_{i,j}^{k+\frac{1}{2}}, 1 \leq i \leq M_x - 1, 1 \leq j \leq M_y - 1, 0 \leq k \leq N - 1,$n

$$v_{i,j}^{0} = u^0(x_i, y_j) + \varepsilon_{i,j}, \quad (x_i, y_j) \in \overline{\Omega}_h,$n

$$v_{i,j}^{k} = 0, \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq k \leq N. \quad (5.15)$$

Let $\xi_{i,j}^{k} = v_{i,j}^{k} - u_{i,j}^{k}$, then the perturbation equation can be obtained by using equations (5.9)–(5.11) and (5.14)–(5.16),

$$L_x L_y \frac{\delta t}{2} \xi_{i,j}^{k+\frac{1}{2}} = -L_x L_y \xi_{i,j}^{k+\frac{1}{2}} + K_\alpha L_y \frac{\delta x}{2} \xi_{i,j}^{k+\frac{1}{2}} + K_\beta L_x \frac{\delta y}{2} \xi_{i,j}^{k+\frac{1}{2}},$$

$$1 \leq i \leq M_x - 1, 1 \leq j \leq M_y - 1, 0 \leq k \leq N - 1,$n

$$\xi_{i,j}^{0} = \varepsilon_{i,j}, \quad (x_i, y_j) \in \overline{\Omega}_h,$n

$$\xi_{i,j}^{k} = 0, \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq k \leq N. \quad (5.16)$$

Next, the stability is shown below.

**Theorem 5.8** The finite difference scheme (5.9)–(5.11) is unconditionally stable with respect to the initial values.

**Proof.** Taking the inner product of (5.17) with $\xi^{k+\frac{1}{2}}$ gives

$$\left( L_x L_y \frac{\delta t}{2} \xi^{k+\frac{1}{2}}, \xi^{k+\frac{1}{2}} \right) = - \left( L_x L_y \xi^{k+\frac{1}{2}}, \xi^{k+\frac{1}{2}} \right)$$

$$+ \left( (K_\alpha L_y \frac{\delta x}{2} + K_\beta L_x \frac{\delta y}{2}) \xi^{k+\frac{1}{2}}, \xi^{k+\frac{1}{2}} \right).$$

Applying Lemma 5.6 to the first term on the right hand side of equation (5.18), we know that there exists a symmetric and positive definite operator $Q$ such that,

$$- \left( L_x L_y \xi^{k+\frac{1}{2}}, \xi^{k+\frac{1}{2}} \right) = - \left( Q \xi^{k+\frac{1}{2}}, Q \xi^{k+\frac{1}{2}} \right) = - \left\| Q \xi^{k+\frac{1}{2}} \right\|^2 \leq 0. \quad (5.19)$$

For the second term on the right hand side of (5.18), we can also get the following result by using Lemma 5.5,

$$\left( (K_\alpha L_y \frac{\delta x}{2} + K_\beta L_x \frac{\delta y}{2}) \xi^{k+\frac{1}{2}}, \xi^{k+\frac{1}{2}} \right) = - \left\| P \xi^{k+\frac{1}{2}} \right\|^2 \leq 0, \quad (5.20)$$

$$\left( (K_\alpha L_y \frac{\delta x}{2} + K_\beta L_x \frac{\delta y}{2}) \xi^{k+\frac{1}{2}}, \xi^{k+\frac{1}{2}} \right) = - \left\| P \xi^{k+\frac{1}{2}} \right\|^2 \leq 0, \quad (5.20)$$
where \( \mathcal{P} \) is a symmetric and positive definite operator.

Substituting (5.19) and (5.20) into (5.18) leads to

\[
\mathcal{Q} \xi^{k+1} - \mathcal{Q} \xi^k \leq 0.
\]

(5.21)

Replacing \( k \) by \( n \) and summing up \( n \) from 0 to \( k - 1 \) on both sides of (5.21) yield

\[
\mathcal{Q} \xi^k \leq \mathcal{Q} \xi^0.
\]

Utilizing Lemma 5.7 gives

\[
\| \xi \| \leq \frac{(4\sqrt{6} + 9)}{5} \| \xi^0 \| = \frac{(4\sqrt{6} + 9)}{5} \| \epsilon \|.
\]

All this ends the proof.

Finally, we study the convergence for finite difference scheme (5.9)–(5.11).

**Theorem 5.9** Let \( u_{i,j}^k \) and \( u(x, y, t) \) be the solutions of the finite difference scheme (5.9)–(5.11) and problem (5.1)–(5.3), respectively. Denote \( e_{i,j}^k = u_{i,j}^k - u(x_i, y_j, t_k), 0 \leq k \leq N, 0 \leq i \leq M_a, 0 \leq j \leq M_b \). Then there holds

\[
\| e^k \| \leq 32C_2\sqrt{TL_aL_b} \exp(8T) (\tau^2 + h_a^3 + h_b^3),
\]

where \( C_2 \) is a positive constant.

**Proof.** Subtracting (5.9), (5.11) from (5.1)–(5.3) leads to an error system as follows,

\[
\mathcal{L}_x \mathcal{L}_y \delta_t e_{i,j}^{k+\frac{1}{2}} = -\mathcal{L}_x \mathcal{L}_y e_{i,j}^{k+\frac{1}{2}} + K_\alpha \mathcal{L}_y \delta_x e_{i,j}^{k+\frac{1}{2}} + K_\beta \mathcal{L}_x \delta_y e_{i,j}^{k+\frac{1}{2}} + R_{i,j}^k, \quad 1 \leq i \leq M_a - 1, 1 \leq j \leq M_b - 1, 0 \leq k \leq N - 1,
\]

\[
u_{i,j}^0 = 0, \quad (x_i, y_j) \in \overline{\Omega}_h,
\]

\[
u_{i,j}^k = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq k \leq N.
\]

Taking the inner product of (5.22) with \( e^{k+\frac{1}{2}} \) and using the similar method in the proof of Theorem 4.2, we reach that

\[
\| \mathcal{Q} e^{k+1} \|^2 \leq \| \mathcal{Q} e^k \|^2 + 2\tau \left( R^{k+1}, e^{k+\frac{1}{2}} \right)
\]

\[
\leq \| \mathcal{Q} e^k \|^2 + 8\tau \| R^{k+1} \|^2 + \frac{1}{16}\tau \left( \| e^k \|^2 + \| e^{k+1} \|^2 \right)
\]

\[
\leq \| \mathcal{Q} e^k \|^2 + \frac{1}{16}\tau \left( \| e^k \|^2 + \| e^{k+1} \|^2 \right) + 8\tau C_2^2 L_aL_b (\tau^2 + h_a^3 + h_b^3)^2.
\]
Replacing $k$ by $n$ and summing up $n$ from 0 to $k - 1$ on both sides of the above inequality and following Lemma 5.7 we reach that

$$
||e^k||^2 \leq \frac{18}{(16\sqrt{6} - 33)(16\sqrt{6} - 39)} \tau \sum_{n=0}^{k-1} ||e^n||^2
$$

$$
+ \frac{1152}{(16\sqrt{6} - 33)(16\sqrt{6} - 39)} C^2 TL_a L_b (\tau^2 + h^3_a + h^3_b)^2
$$

$$
\leq 16\tau \sum_{n=0}^{k-1} ||e^n||^2 + 1024 C^2 TL_a L_b (\tau^2 + h^3_a + h^3_b)^2.
$$

So we have the following result via Lemma 4.3,

$$
||e^k|| \leq 32 c_2 \sqrt{T L_a L_b} \exp (8T) (\tau^2 + h^3_a + h^3_b).
$$

The proof is finally shown in the end.

6 Numerical examples

In this section, the validity and convergence orders of the numerical algorithms constructed in this paper are demonstrated by several numerical tests.

Example 6.1 Consider function $u(x) = x^2(1 - x)^2$. The exact expression at $x = 0.5$ is given by

$$
\frac{\partial^\alpha u(x)}{\partial |x|^\alpha} \bigg|_{x=0.5} = \frac{1}{2 \cos \left(\frac{\alpha \pi}{2}\right)} \left\{ \frac{\Gamma(3)}{\Gamma(3 - \alpha)} \left(\frac{1}{2}\right)^{1 - \alpha} - \frac{2\Gamma(4)}{\Gamma(4 - \alpha)} \left(\frac{1}{2}\right)^{2 - \alpha} \right. 
$$

$$
\left. + \frac{\Gamma(5)}{\Gamma(5 - \alpha)} \left(\frac{1}{2}\right)^{3 - \alpha} \right\}, \quad x \in [0, 1].
$$

Choosing different spatial stepizes $h$, we compute Riesz derivative of function $u(x)$ using numerical formulas (2.6), (2.7) and (3.1), respectively. Tables 1, 2, and 3 list the absolute errors and numerical convergence orders at $x = 0.5$ for different orders $\alpha$ in (1,2). From these results, one can see that the numerical results are in line with the theoretical order.

Example 6.2 We consider the following one-dimensional Riesz spatial fractional reaction-dispersion equation,

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = -u(x,t) + e^{-12} \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t), & (x,t) \in (0,1) \times (0,1), \\
u(x,0) = 0, & x \in (0,1), \\
u(0,t) = u(1,t) = 0, & t \in (0,1),
\end{cases}
$$
Table 1: The absolute errors and convergence orders of Example 6.1 by numerical formula (2.6).

| α   | h   | the absolute errors   | the convergence orders |
|-----|-----|-----------------------|------------------------|
| 1.1 | 20  | 1.740717e-04          | —                      |
|     | 40  | 2.185595e-05          | 2.9936                 |
|     | 80  | 2.742123e-06          | 2.9947                 |
|     | 160 | 3.434158e-07          | 2.9973                 |
|     | 320 | 4.296784e-08          | 2.9986                 |
| 1.3 | 20  | 1.756079e-04          | —                      |
|     | 40  | 2.198613e-05          | 2.9977                 |
|     | 80  | 2.751417e-06          | 2.9983                 |
|     | 160 | 3.441531e-07          | 2.9991                 |
|     | 320 | 4.303416e-08          | 2.9995                 |
| 1.5 | 20  | 1.377134e-04          | —                      |
|     | 40  | 1.716087e-05          | 3.0045                 |
|     | 80  | 2.143372e-06          | 3.0012                 |
|     | 160 | 2.678606e-07          | 3.0003                 |
|     | 320 | 3.348027e-08          | 3.0001                 |
| 1.7 | 20  | 7.211650e-05          | —                      |
|     | 40  | 8.991024e-06          | 3.0038                 |
|     | 80  | 1.123719e-06          | 3.0002                 |
|     | 160 | 1.404937e-07          | 2.9997                 |
|     | 320 | 1.756457e-08          | 2.9998                 |
| 1.9 | 20  | 1.056422e-05          | —                      |
|     | 40  | 1.364672e-06          | 2.9526                 |
|     | 80  | 1.735867e-07          | 2.9748                 |
|     | 160 | 2.189369e-08          | 2.9871                 |
|     | 320 | 2.749249e-09          | 2.9934                 |
Table 2: The absolute errors and convergence orders of Example 6.1 by numerical formula (2.7).

| α   | h   | the absolute errors | the convergence orders |
|-----|-----|---------------------|------------------------|
| 1.1 | 20  | 5.290778e-02        | —                      |
|     | 40  | 6.548041e-03        | 3.0143                 |
|     | 80  | 8.150577e-04        | 3.0061                 |
|     | 160 | 1.016718e-04        | 3.0030                 |
|     | 320 | 1.269602e-05        | 3.0015                 |
| 1.3 | 20  | 2.033985e-02        | —                      |
|     | 40  | 2.512801e-03        | 3.0169                 |
|     | 80  | 3.117365e-04        | 3.0109                 |
|     | 160 | 3.881109e-04        | 3.0058                 |
|     | 320 | 4.841522e-06        | 3.0029                 |
| 1.5 | 20  | 1.263828e-02        | —                      |
|     | 40  | 1.583605e-03        | 2.9965                 |
|     | 80  | 1.966563e-04        | 3.0095                 |
|     | 160 | 2.448135e-05        | 3.0059                 |
|     | 320 | 3.053477e-06        | 3.0032                 |
| 1.7 | 20  | 7.701877e-03        | —                      |
|     | 40  | 9.893186e-04        | 2.9607                 |
|     | 80  | 1.233352e-04        | 3.0039                 |
|     | 160 | 1.537077e-05        | 3.0043                 |
|     | 320 | 1.917897e-06        | 3.0026                 |
| 1.9 | 20  | 2.787724e-03        | —                      |
|     | 40  | 3.697284e-04        | 2.9145                 |
|     | 80  | 4.637689e-05        | 2.9950                 |
|     | 160 | 5.791288e-06        | 3.0014                 |
|     | 320 | 7.231609e-07        | 3.0015                 |
Table 3: The absolute errors and convergence orders of Example 6.1 by numerical formula (3.1).

| $\alpha$ | $h$ | the absolute errors | the convergence orders |
|----------|-----|---------------------|------------------------|
| 1.1      | 1/20| 8.281680e-07        | —                      |
|          | 1/40| 5.167207e-08        | 4.0025                 |
|          | 1/80| 3.218255e-09        | 4.0050                 |
|          | 1/160| 2.007975e-10       | 4.0025                 |
| 1.3      | 1/20| 8.898742e-07        | —                      |
|          | 1/40| 5.777396e-08        | 3.9451                 |
|          | 1/80| 3.654194e-09        | 3.9828                 |
|          | 1/160| 2.294926e-10       | 3.9930                 |
| 1.5      | 1/20| 5.084772e-07        | —                      |
|          | 1/40| 3.725522e-08        | 3.7707                 |
|          | 1/80| 2.454356e-09        | 3.9240                 |
|          | 1/160| 1.567338e-10       | 3.9690                 |
| 1.7      | 1/20| 1.822972e-07        | —                      |
|          | 1/40| 1.692478e-08        | 3.4291                 |
|          | 1/80| 1.191028e-09        | 3.8289                 |
|          | 1/160| 7.878076e-11       | 3.9182                 |
| 1.9      | 1/20| 9.867011e-08        | —                      |
|          | 1/40| 7.533874e-09        | 3.7111                 |
|          | 1/80| 5.041596e-10        | 3.9014                 |
|          | 1/160| 3.322587e-11       | 3.9235                 |
where the source term \( f(x, t) \) is

\[
2e^t x^6 (1 - x)^6 + \frac{\sin t}{2 \cos \left( \frac{\pi}{2} \alpha \right)} \left\{ \frac{\Gamma(7)}{\Gamma(7 - \alpha)} [x^{6-\alpha} + (1 - x)^{6-\alpha}] - \frac{6\Gamma(8)}{\Gamma(8 - \alpha)} [x^{7-\alpha} + (1 - x)^{7-\alpha}] + \frac{15\Gamma(9)}{\Gamma(9 - \alpha)} [x^{8-\alpha} + (1 - x)^{8-\alpha}] - \frac{20\Gamma(10)}{\Gamma(10 - \alpha)} [x^{9-\alpha} + (1 - x)^{9-\alpha}] + \frac{15\Gamma(11)}{\Gamma(11 - \alpha)} [x^{10-\alpha} + (1 - x)^{10-\alpha}] - \frac{6\Gamma(12)}{\Gamma(12 - \alpha)} [x^{11-\alpha} + (1 - x)^{11-\alpha}] + \frac{\Gamma(13)}{\Gamma(13 - \alpha)} [x^{12-\alpha} + (1 - x)^{12-\alpha}] \right\}.
\]

The exact solution of this equation is \( u(x, t) = e^t x^6 (1 - x)^6 \) and satisfies the according initial and boundary values conditions.

Using numerical scheme (4.6)–(4.8), we present the absolute errors and the corresponding space and time convergence orders with different step sizes in Table 4. It can be found that the convergence orders of scheme (4.6)–(4.8) are almost second- and third-order in time and space directions, respectively, which is in agreement with the theoretical convergence order.

**Example 6.3** We consider the following two-dimensional Riesz spatial fractional reaction-dispersion equation,

\[
\begin{cases}
\frac{\partial u(x, y, t)}{\partial t} = -u(x, y, t) + \pi^{-8} \left( \frac{\partial^\alpha u(x, y, t)}{\partial |x|^\alpha} + \frac{\partial^\beta u(x, y, t)}{\partial |y|^\beta} \right) + f(x, y, t), \\
u(x, y, 0) = 0, \quad (x, y) \in \Omega, \\
u(x, y, t) = 0, \quad (x, y; t) \in \partial \Omega \times (0, 1], \\
u(x, y; t) = 0, \quad (x, y; t) \in \partial \Omega \times (0, 1],
\end{cases}
\]
Table 4: The absolute errors (TAEs), temporal convergence order (TCO) and spatial convergence order (SCO) of Example 6.2 by difference scheme (4.6)–(4.8).

| $\alpha$ | $\tau$, $h$ | TAEs          | TCO   | SCO   |
|---------|-------------|---------------|-------|-------|
| 1.1     | $\tau = \frac{1}{4}, h = \frac{1}{4}$ | 2.984674e-06  | —     | —     |
|         | $\tau = \frac{\sqrt{2}}{16}, h = \frac{1}{8}$ | 3.613655e-07  | 2.0307| 3.0460|
|         | $\tau = \frac{1}{32}, h = \frac{1}{16}$ | 4.685713e-08  | 1.9647| 2.9471|
|         | $\tau = \frac{\sqrt{2}}{128}, h = \frac{1}{32}$ | 5.813993e-09  | 2.0071| 3.0107|
|         | $\tau = \frac{1}{256}, h = \frac{1}{64}$ | 7.321694e-10  | 1.9929| 2.9893|
| 1.3     | $\tau = \frac{1}{4}, h = \frac{1}{4}$ | 2.984597e-06  | —     | —     |
|         | $\tau = \frac{\sqrt{2}}{16}, h = \frac{1}{8}$ | 3.617522e-07  | 2.0296| 3.0445|
|         | $\tau = \frac{1}{32}, h = \frac{1}{16}$ | 4.690406e-08  | 1.9648| 2.9472|
|         | $\tau = \frac{\sqrt{2}}{128}, h = \frac{1}{32}$ | 5.819491e-09  | 2.0072| 3.0107|
|         | $\tau = \frac{1}{256}, h = \frac{1}{64}$ | 7.328387e-10  | 1.9929| 2.9893|
| 1.5     | $\tau = \frac{1}{4}, h = \frac{1}{4}$ | 2.981516e-06  | —     | —     |
|         | $\tau = \frac{\sqrt{2}}{16}, h = \frac{1}{8}$ | 3.616854e-07  | 2.0288| 3.0432|
|         | $\tau = \frac{1}{32}, h = \frac{1}{16}$ | 4.690789e-08  | 1.9646| 2.9468|
|         | $\tau = \frac{\sqrt{2}}{128}, h = \frac{1}{32}$ | 5.819848e-09  | 2.0072| 3.0108|
|         | $\tau = \frac{1}{256}, h = \frac{1}{64}$ | 7.328573e-10  | 1.9929| 2.9894|
| 1.7     | $\tau = \frac{1}{4}, h = \frac{1}{4}$ | 2.974813e-06  | —     | —     |
|         | $\tau = \frac{\sqrt{2}}{16}, h = \frac{1}{8}$ | 3.609314e-07  | 2.0287| 3.0430|
|         | $\tau = \frac{1}{32}, h = \frac{1}{16}$ | 4.683820e-08  | 1.9640| 2.9460|
|         | $\tau = \frac{\sqrt{2}}{128}, h = \frac{1}{32}$ | 5.811874e-09  | 2.0071| 3.0106|
|         | $\tau = \frac{1}{256}, h = \frac{1}{64}$ | 7.318735e-10  | 1.9929| 2.9893|
| 1.9     | $\tau = \frac{1}{4}, h = \frac{1}{4}$ | 2.963689e-06  | —     | —     |
|         | $\tau = \frac{\sqrt{2}}{16}, h = \frac{1}{8}$ | 3.593385e-07  | 2.0293| 3.0440|
|         | $\tau = \frac{1}{32}, h = \frac{1}{16}$ | 4.668265e-08  | 1.9629| 2.9444|
|         | $\tau = \frac{\sqrt{2}}{128}, h = \frac{1}{32}$ | 5.795636e-09  | 2.0066| 3.0098|
|         | $\tau = \frac{1}{256}, h = \frac{1}{64}$ | 7.300171e-10  | 1.9926| 2.9890|
where $\Omega = [0, 1] \times [0, 1]$, and the source term $f(x,y,t)$ is

$$
3e^{2t}x^6(1-x)^6y^6(1-y)^6 + \frac{e^{2t}y^6(1-y)^6}{2\cos\left(\frac{\pi}{2}\alpha\right)} \left\{ \frac{\Gamma(7)}{\Gamma(7-\alpha)} [x^{6-\alpha} + (1-x)^{6-\alpha}] \\
- \frac{6\Gamma(8)}{\Gamma(8-\alpha)} [x^{7-\alpha} + (1-x)^{7-\alpha}] + \frac{15\Gamma(9)}{\Gamma(9-\alpha)} [x^{8-\alpha} + (1-x)^{8-\alpha}] \\
- \frac{20\Gamma(10)}{\Gamma(10-\alpha)} [x^{9-\alpha} + (1-x)^{9-\alpha}] + \frac{15\Gamma(11)}{\Gamma(11-\alpha)} [x^{10-\alpha} + (1-x)^{10-\alpha}] \\
- \frac{6\Gamma(12)}{\Gamma(12-\alpha)} [x^{11-\alpha} + (1-x)^{11-\alpha}] + \frac{\Gamma(13)}{\Gamma(13-\alpha)} [x^{12-\alpha} + (1-x)^{12-\alpha}] \right\} \\
+ \frac{e^{2t}x^6(1-x)^6}{2\cos\left(\frac{\pi}{2}\beta\right)} \left\{ \frac{\Gamma(7)}{\Gamma(7-\beta)} [y^{6-\beta} + (1-y)^{6-\beta}] \\
- \frac{6\Gamma(8)}{\Gamma(8-\beta)} [y^{7-\beta} + (1-y)^{7-\beta}] + \frac{15\Gamma(9)}{\Gamma(9-\beta)} [y^{8-\beta} + (1-y)^{8-\beta}] \\
- \frac{20\Gamma(10)}{\Gamma(10-\beta)} [y^{9-\beta} + (1-y)^{9-\beta}] + \frac{15\Gamma(11)}{\Gamma(11-\beta)} [y^{10-\beta} + (1-y)^{10-\beta}] \\
- \frac{6\Gamma(12)}{\Gamma(12-\beta)} [y^{11-\beta} + (1-y)^{11-\beta}] + \frac{\Gamma(13)}{\Gamma(13-\beta)} [y^{12-\beta} + (1-y)^{12-\beta}] \right\}.
$$

The exact solution of this equation is $u(x,t) = e^{2t}x^6(1-x)^6y^6(1-y)^6$ and satisfies the corresponding initial and boundary values conditions.

We solve this problem through method (5.9)-(5.11) for different values of $\alpha, \beta$. From Table 6 one can see that the convergence orders of scheme (5.9)-(5.11) are $O(\tau^2)$ in temporal direction and $O(h^3_a + h^3_b)$ in spatial directions. It also coincides with the theoretical analysis.

## 7 Conclusion

Based on the novel generating functions, we obtain several kinds of (generalized) high-order fractional-compact numerical approximation formulas for Riemann-Liouville and/or Riesz derivatives with order lying in $(1,2)$. For further checking the efficiency of these high-order formulas, we apply the 3th-order formula to solve one- and two-dimensional Riesz spatial fractional reaction dispersion equations. Both theoretical analysis and numerical tests show that the developed numerical algorithms are efficient and accurate.
Table 5: The absolute errors (TAEs), temporal convergence order (TCO) and spatial convergence order (SCO) of Example 6.3 by difference scheme (5.9)–(5.11).

| $\alpha, \beta$ | $\tau, h_a, h_b$ | TAEs       | TCO  | SCO  |
|----------------|------------------|------------|------|------|
| $\alpha = 1.1, \beta = 1.8$ | $\tau = \frac{1}{4}, h_a = h_b = \frac{1}{4}$ | 7.150284e-09 | —    | —    |
|                 | $\tau = \frac{\sqrt{2}}{16}, h_a = h_b = \frac{1}{8}$ | 8.680618e-10 | 2.0281 | 3.0421 |
|                 | $\tau = \frac{1}{32}, h_a = h_b = \frac{1}{16}$ | 1.155609e-10 | 1.9394 | 2.9091 |
|                 | $\tau = \frac{\sqrt{2}}{128}, h_a = h_b = \frac{1}{32}$ | 1.428060e-11 | 2.0110 | 3.0165 |
| $\alpha = 1.3, \beta = 1.6$ | $\tau = \frac{1}{4}, h_a = h_b = \frac{1}{4}$ | 7.221370e-09 | —    | —    |
|                 | $\tau = \frac{\sqrt{2}}{16}, h_a = h_b = \frac{1}{8}$ | 8.805609e-10 | 2.0239 | 3.0358 |
|                 | $\tau = \frac{1}{32}, h_a = h_b = \frac{1}{16}$ | 1.168858e-10 | 1.9422 | 2.9133 |
|                 | $\tau = \frac{\sqrt{2}}{128}, h_a = h_b = \frac{1}{32}$ | 1.442860e-11 | 2.0121 | 3.0181 |
| $\alpha = 1.5, \beta = 1.5$ | $\tau = \frac{1}{4}, h_a = h_b = \frac{1}{4}$ | 7.219848e-09 | —    | —    |
|                 | $\tau = \frac{\sqrt{2}}{16}, h_a = h_b = \frac{1}{8}$ | 8.823037e-10 | 2.0217 | 3.0326 |
|                 | $\tau = \frac{1}{32}, h_a = h_b = \frac{1}{16}$ | 1.171206e-10 | 1.9422 | 2.9133 |
|                 | $\tau = \frac{\sqrt{2}}{128}, h_a = h_b = \frac{1}{32}$ | 1.445519e-11 | 2.0122 | 3.0183 |
| $\alpha = 1.7, \beta = 1.4$ | $\tau = \frac{1}{4}, h_a = h_b = \frac{1}{4}$ | 7.181389e-09 | —    | —    |
|                 | $\tau = \frac{\sqrt{2}}{16}, h_a = h_b = \frac{1}{8}$ | 8.771397e-10 | 2.0223 | 3.0334 |
|                 | $\tau = \frac{1}{32}, h_a = h_b = \frac{1}{16}$ | 1.165997e-10 | 1.9408 | 2.9112 |
|                 | $\tau = \frac{\sqrt{2}}{128}, h_a = h_b = \frac{1}{32}$ | 1.439652e-11 | 2.0118 | 3.0178 |
| $\alpha = 1.9, \beta = 1.2$ | $\tau = \frac{1}{4}, h_a = h_b = \frac{1}{4}$ | 7.102704e-09 | —    | —    |
|                 | $\tau = \frac{\sqrt{2}}{16}, h_a = h_b = \frac{1}{8}$ | 8.628679e-10 | 2.0274 | 3.0412 |
|                 | $\tau = \frac{1}{32}, h_a = h_b = \frac{1}{16}$ | 1.150916e-10 | 1.9376 | 2.9064 |
|                 | $\tau = \frac{\sqrt{2}}{128}, h_a = h_b = \frac{1}{32}$ | 1.423779e-11 | 2.0100 | 3.0150 |
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