Bihamiltonian elliptic structures

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Introduction

Bihamiltonian structures play an important role in the theory of dynamic systems. In this approach one starts with two Poisson brackets \{,\}_1 and \{,\}_2 on some manifold, such that any linear combination \{,\}_\lambda_1,\lambda_2 = \lambda_1 \{,\}_1 + \lambda_2 \{,\}_2 is also a Poisson bracket. Basing on these brackets one constructs a Hamiltonian system. The construction of the dynamical system basing on these brackets is called Lenard scheme [1,2,3,4]. It provides a family of functions in involution. Namely, let \( C_{\lambda_1,\lambda_2} \) and \( C'_{\mu_1,\mu_2} \) be central elements for the Poisson structure \{,\}_\lambda_1,\lambda_2, then one has

\[ \{ C_{\lambda_1,\lambda_2}, C'_{\mu_1,\mu_2}\}_i = 0 \text{ for } i = 1, 2. \]

In this paper we construct three Poisson structures \{,\}_i (i = 1, 2, 3) on \( C^n \) such that any \{,\}_i is quadratic and any linear combination \{,\}_\lambda_1,\lambda_2,\lambda_3 = \sum \lambda_i \{,\}_i is also a Poisson bracket. We also study symplectic leaves of the Poisson structure \{,\}_\lambda_1,\lambda_2,\lambda_3 and construct central elements.

Let \( E = \mathbb{C}/\Gamma \) be an elliptic curve and \( \eta \in E \). The algebra \( Q_n(E, \eta) \) is generated by \( n \) elements \( \{x_i; i \in \mathbb{Z}/n\mathbb{Z}\} \) with the following defining relations:

\[ \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{1}{\theta_{j-i-r}(-\eta)\theta_r(\eta)} x_{j-r} x_{i+r} = 0, \]

for all \( i, j \in \mathbb{Z}/n\mathbb{Z} \) such that \( i \neq j \). Here \( \{\theta_i(z); i \in \mathbb{Z}/n\mathbb{Z}\} \) are \( \theta \)-functions of order \( n \) with respect to the lattice \( \Gamma \subset \mathbb{C} \). It is known [5,6] that for generic \( \eta \) the algebra \( Q_n(E, \eta) \) has the same size of graded components as the polynomial algebra in \( n \) variables. Hence, for any fixed elliptic curve \( E \) one has the flat deformation of the polynomial algebra. Let \( q_n(E) \) be the corresponding Poisson structure on \( \mathbb{C}^n \). It turns out that the Poisson structure \{,\}_\lambda_1,\lambda_2,\lambda_3 for generic point \( (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 \) is isomorphic to \( q_n(E) \) for certain \( E \). It follows that the algebras \( Q_n(E, \eta) \) are quantizations of the Poisson structures \{,\}_\lambda_1,\lambda_2,\lambda_3 for generic \( \lambda_1, \lambda_2, \lambda_3 \) and one may expect that there is a lot of commuting elements in the algebras \( Q_n(E, \eta) \), which are quantization of commuting elements in \{,\}_\lambda_1,\lambda_2,\lambda_3 provided by Lenard scheme.

Now we describe the contents of the paper. In §1 we construct three compatible quadratic Poisson structures. In §2 we give another description of the Poisson structures in terms of elliptic functions. This construction is called functional realization. In §3 we study symplectic leaves of our Poisson structure and construct Casimir elements. In Appendix we collect some standard facts from the theory of elliptic functions [7].
§1. Three Poisson structures

For any \( n \in \mathbb{C} \) we define three quadratic Poisson structures in the polynomial algebra with infinite number of generators \( \{e_\alpha; \alpha \in \mathbb{Z}\} \). But it will be clear that for \( n \in \mathbb{N} \) the polynomial algebra generated by \( e_0 \) and \( \{e_\alpha; 2 \leq \alpha \leq n\} \) is Poisson subalgebra for all our Poisson structures. We will use the notation: \( S_k(e_\alpha, e_\beta) = \sum_{r=0}^{\infty} e_{\alpha+k \beta} e_{-k \beta} \). It is clear that if \( \alpha + \beta = \alpha' + \beta' \) and \( \alpha \equiv \alpha' \) mod \( k \), then the linear combination \( S_k(e_\alpha, e_\beta) - S_k(e_{\alpha'}, e_{\beta'}) \) contains only finite number of monomials. Define \( \{,\}_{i}, i = 1, 2, 3 \), by the following formulas:

\[
\{e_\alpha, e_\beta\}_1 = \frac{n}{2} (S_1(e_{\alpha+1}, e_\beta) - S_1(e_{\beta+1}, e_\alpha)) + (\alpha - n)e_{\alpha+1}e_\beta - (\beta - n)e_\alpha e_{\beta+1}
\]

\[
\{e_{2\alpha}, e_{2\beta}\}_2 = 0
\]

\[
\{e_{2\alpha+3}, e_{2\beta+3}\}_2 = \frac{n}{8} (S_2(e_{2\beta+2}, e_{2\alpha-2}) - S_2(e_{2\alpha}, e_{2\beta})) + \frac{1}{4} (2\beta + 1) e_{2\alpha} e_{2\beta}
\]

\[
\{e_{2\alpha+3}, e_{2\beta+3}\}_2 = \frac{n}{4} (S_2(e_{2\beta+2}, e_{2\alpha+1}) - S_2(e_{2\alpha+2}, e_{2\beta+1})) - \frac{1}{4} (2\alpha + 1) e_{2\alpha} e_{2\beta+3} + \frac{1}{4} (2\beta + 1) e_{2\alpha+3} e_{2\beta}
\]

\[
\{e_{2\alpha}, e_{2\beta}\}_3 = 0
\]

\[
\{e_{2\alpha}, e_{2\beta+3}\}_3 = \frac{n}{8} (S_2(e_{2\beta}, e_{2\alpha-2}) - S_2(e_{2\alpha}, e_{2\beta-2})) + \frac{1}{2} \beta e_{2\alpha} e_{2\beta-2}
\]

\[
\{e_{2\alpha+3}, e_{2\beta+3}\}_3 = \frac{n}{4} (S_2(e_{2\beta}, e_{2\alpha+1}) - S_2(e_{2\alpha}, e_{2\beta+1})) - \frac{1}{2} \alpha e_{2\alpha-2} e_{2\beta+3} + \frac{1}{2} \beta e_{2\alpha+3} e_{2\beta-2}
\]

**Proposition 1.** For any \( n \in \mathbb{N} \) these formulas (1) define Poisson structures in the polynomial algebra generated by \( e_0 \) and \( \{e_\alpha; 2 \leq \alpha \leq n\} \). Moreover, any linear combination of \( \{,\}_{i}, i = 1, 2, 3 \) is also a Poisson bracket.

**Proof** of this proposition is straightforward. It also follows from the functional construction in the next paragraph, which explains the elliptic nature of this Poisson structure.
§2. Functional realization

Let us fix an integral lattice $\Gamma \subset \mathbb{C}$. We will use the standard notations from the theory of elliptic functions, like $\wp(z), \zeta(z), g_2, g_3$ (see Appendix).

Let $\mathcal{F}$ be the space of elliptic functions in one variable with respect to $\Gamma$ and holomorphic outside $\Gamma$. For $n \in \mathbb{N}$ let $\mathcal{F}_n \subset \mathcal{F}$ be the subspace of functions with poles of order $\leq n$ on $\Gamma$. It is clear that $\dim \mathcal{F}_n = n$. It is known that the functions $\{e_{2\alpha}(z) = \wp(z)^{\alpha}, e_{2\alpha+3}(z) = -\frac{1}{2}\wp(z)^{\alpha}\wp'(z); \alpha \in \mathbb{Z}_{\geq 0}\}$ form a basis of the linear space $\mathcal{F}$. It is clear that for any $\alpha \in \mathbb{N}$ the function $e_{\alpha}(z)$ has a pole of order $\alpha$ in $\Gamma$ with residue 1. The functions $e_0$ and $e_{\alpha}, 2 \leq \alpha \leq n$ form a basis of the space $\mathcal{F}_n$.

It is clear that the symmetric power $S^m \mathcal{F}$ (resp. $S^m \mathcal{F}_n$) is isomorphic to the space of symmetric elliptic functions in $m$ variables $f(z_1, ..., z_m)$ holomorphic outside of the divisors $z_p \in \Gamma, 1 \leq p \leq m$ (resp. with poles of order $\leq n$ on these divisors).

We construct a bilinear operator $\{,\} : \Lambda^2 \mathcal{F} \to S^2 \mathcal{F}$ as follows: for $f,g \in \mathcal{F}$ we set

$$\{f,g\}(x,y) = n(\zeta(x-y) - \zeta(x) + \zeta(y))(f(x)g(y) - f(y)g(x)) - f'(x)g(y) - f'(y)g(x) - f(x)g'(y) - f(y)g'(x)$$

(2)

Proposition 2. The formula (2) defines a Poisson structure on the polynomial algebra $S^* \mathcal{F}$. If $n \in \mathbb{N}$, then $\{\mathcal{F}_n, \mathcal{F}_n\} \subset S^2 \mathcal{F}_n$ and we have the Poisson structure on the polynomial algebra $S^* \mathcal{F}_n$. Moreover, in the basis $\{e_\alpha\}$ we have $\{e_\alpha, e_\beta\} = \{e_\alpha, e_\beta\}_1 + g_2\{e_\alpha, e_\beta\}_2 + g_3\{e_\alpha, e_\beta\}_3$, where $\{,\}_i, i = 1, 2, 3$ are defined by (1).

Proof is a simple calculation with Weierstrass functions using identities (5) from Appendix. It is clear that if $f$ and $g$ are elliptic functions, then l.h.s. of (2) is also an elliptic function in two variables. Moreover, if $f$ and $g$ are holomorphic outside $\Gamma$ with poles of order $\leq n$ on $\Gamma$, then $\{f,g\}$ has the same property. Verification of the Jacobi identity is straightforward. For calculation of this Poisson brackets in the basis $e_\alpha$ one needs only the identities (5) from Appendix.
§3. Symplectic leaves and Casimir elements

For \( p \in \mathbb{N} \) we denote by \( b_{p,n} \) the Poisson algebra which is spanned by the elements \( \{ f(u_1, \ldots, u_p) \psi_1^{\alpha_1} \ldots \psi_p^{\alpha_p}; \alpha_1, \ldots, \alpha_p \in \mathbb{Z}_{\geq 0} \} \) as a linear space, where \( u_1, \ldots u_p, \psi_1, \ldots, \psi_p \) are independent variables and \( f(u_1, \ldots u_p) \) are elliptic functions in variables \( u_1, \ldots, u_p \) with respect to the lattice \( \Gamma \) holomorphic outside the divisors \( u_j \in \Gamma \) and \( u_j - u_k \in \Gamma \). Poisson bracket on \( b_{p,n} \) is defined as follows:

\[
\{ u_\alpha, u_\beta \} = 0, \{ u_\alpha, \psi_\beta \} = \psi_\beta, \{ u_\alpha, \psi_\alpha \} = -\frac{n-2}{2} \psi_\alpha,
\]
\[
\{ \psi_\alpha, \psi_\beta \} = n(\zeta(u_\alpha - u_\beta) - \zeta(u_\alpha) + \zeta(u_\beta))\psi_\alpha\psi_\beta, \tag{3}
\]

where \( \alpha \neq \beta \).

Let us define a linear map \( x_p : \mathcal{F} \to b_{p,n} \) by the formula:

\[
x_p(f) = \sum_{1 \leq \alpha \leq p} f(\psi_\alpha)\psi_\alpha \tag{4}
\]

There is a unique extension of this map to the homomorphism of commutative algebras \( S^*\mathcal{F} \to b_{p,n} \) which we also denote by \( x_p \).

**Proposition 3.** The map \( x_p : S^*\mathcal{F} \to b_{p,n} \) is a homomorphism of the Poisson algebras.

**Proof.** It is easy to check that \( x_p(\{ f, g \}) = \sum_{1 \leq \alpha, \beta \leq p} \{ f, g \}(u_\alpha, u_\beta)\psi_\alpha\psi_\beta \) for any \( f, g \in \mathcal{F} \). This implies the proposition.

If \( 2p < n \), then the formulas (3) define nondegenerate Poisson structure on the open set \( \{ \psi_\alpha \neq 0; u_\alpha, u_\alpha - u_\beta \notin \Gamma \} \). In this case the formula (4) defines symplectic leaves of the Poisson algebra \( S^*\mathcal{F}_n \). Central elements of the Poisson algebra \( S^*\mathcal{F}_n \) belong to \( \ker x_p \) for \( 2p < n \), because the algebra \( b_{p,n} \) is nondegenerated in this case.

One can check that for \( p = \frac{n}{2} - 1 \) for even \( n \) (resp. \( p = \frac{n-1}{2} \) for odd \( n \)) the ideal \( \ker x_p \) is generated by two elements of degree \( \frac{n}{2} \) (resp. by one element of degree \( n \)). Center of the Poisson algebra \( S^*\mathcal{F}_n \) is a polynomial algebra generated by these elements. In fact, our construction of homomorphism \( x_p \) implies that \( \ker x_p \) on the space \( S^{p+1}\mathcal{F}_n \) consists of such elements \( f(z_1, \ldots, z_{p+1}) \in S^{p+1}\mathcal{F}_n \), which are equal to zero on the divisors \( z_j - z_k \in \Gamma, 1 \leq j < k \leq p + 1 \). This allows us to construct central elements from \( \ker x_{p-1} \) of degree \( \frac{n}{2} \) for even \( n \) explicitly.

**Examples:**

1. \( n = 2 \). The Poisson algebra \( S^*\mathcal{F}_2 \) is commutative, one has \( \{ e_0, e_2 \} = 0 \) for \( n = 2 \).

2. \( n = 4 \). Let

\[
C_0^{(2)} = \begin{bmatrix} e_0 & e_2 \\ e_2 & e_4 \end{bmatrix}, C_1^{(2)} = \begin{bmatrix} e_2 & e_3 \\ e_3 & e_4 - \frac{1}{4}g_2e_0 \end{bmatrix} + \frac{1}{4}g_3e_0^2
\]

These elements are central in the algebra \( S^*\mathcal{F}_4 \).
3. \( n = 6 \). Let

\[
C_0^{(3)} = \begin{vmatrix}
e_0 & e_2 & e_3 \\
e_2 & e_4 & e_5 \\
e_3 & e_5 & e_6 - \frac{1}{4} g_2 e_2 - \frac{1}{4} g_3 e_0
\end{vmatrix},
\]

\[
C_1^{(3)} = \begin{vmatrix}
e_2 & e_3 & e_4 & 0 & e_0 & e_2 \\
e_3 & e_4 & e_5 & e_6 & 0 & e_2 \\
e_4 & e_5 & e_6 & e_2 & e_4 & e_6
\end{vmatrix}.
\]

These elements are central in the Poisson algebra \( S^* F_6 \).

Let \((f_{\alpha, \beta}), 1 \leq \alpha, \beta \leq p + 1\) be a \((p + 1) \times (p + 1)\)-matrix of the elements in \( F \). Then \( \det(f_{\alpha, \beta}) \in S^{p+1} F \) defines some element of degree \( p + 1 \). It follows from our definition of \( x_p \), that if rank of \((f_{\alpha, \beta})\) as matrix of functions is equal to 1 (that is \( f_{\alpha, \beta}(z) f_{\alpha', \beta'}(z) = f_{\alpha, \beta'}(z) f_{\alpha', \beta}(z) \) for usual product of functions), then \( \det(f_{\alpha, \beta}) \in \ker x_p \) as element in \( S^{p+1} F \). Moreover, let us extend the definition of the functions \( e_\alpha \) to all integral \( \alpha \). Let \( \tilde{F} \) is the space of functions spanned by \( \{e_\alpha; \alpha \in \mathbb{Z}\} \). Let \( \tilde{x}_p \) be the natural extension of \( x_p \) to \( \tilde{S}^* \tilde{F} \). If \((f_{\alpha, \beta})^{(i)}\) are \((p + 1) \times (p + 1)\)-matrices of the elements from \( \tilde{F} \) such that rank \((f_{\alpha, \beta})^{(i)}\) is equal to 1 for each \( i \), then \( \det(f_{\alpha, \beta})^{(i)} \in \ker \tilde{x}_p \) as element from \( S^{p+1} \tilde{F} \). But if some linear combination \( \Psi = \sum_i \lambda_i \det(f_{\alpha, \beta})^{(i)} \) belongs to \( S^{p+1} \tilde{F} \), then \( \Psi \in \ker x_p \). For example, the elements \( C_1^{(2)}, C_1^{(3)} \) may be written as follow:

\[
C_1^{(2)} = \begin{vmatrix}
e_2 & e_3 & e_0 \\
e_3 & e_4 - \frac{1}{4} g_2 e_0 - \frac{1}{4} g_3 e_2 & + \frac{1}{4} g_3 \\
e_4 & e_5 & e_6
\end{vmatrix},
\]

\[
C_1^{(3)} = \begin{vmatrix}
e_2 & e_3 & e_4 & e_2 & e_4 \\
e_3 & e_4 - \frac{1}{4} g_2 e_0 - \frac{1}{4} g_3 e_2 & e_5 & e_2 & e_4 \\
e_4 & e_5 & e_6 & e_2 & e_6
\end{vmatrix}.
\]

In general case of even \( n \) the construction of central elements \( C_0^{(\frac{n}{2})} \) and \( C_1^{(\frac{n}{2})} \) is similar. Define matrix \((g_{\alpha, \beta}), 1 \leq \alpha, \beta \leq \frac{n}{2}\) as follows: \( g_{1,1} = e_0, g_{1,\alpha} = g_{\alpha,1} = e_\alpha \) for \( \alpha > 1 \); \( g_{\alpha, \beta}(z) = e_\alpha(z) e_\beta(z) \) for \( \alpha, \beta > 1 \). For example, \( g_{2,2} = e_4 \) and \( g_{3,3}(z) = \frac{1}{8} \varphi'(z)^2 - \frac{1}{4} g_2 \varphi(z) - \frac{1}{4} g_3 \), so \( g_{3,3} = e_6 - \frac{1}{4} g_2 e_2 - \frac{1}{4} g_3 e_0 \). We set \( C_0^{(\frac{n}{2})} = \det(g_{\alpha, \beta}) \) as element in \( S^{\frac{n}{2}} F_n \). Define the matrix \((g_{\alpha, \beta})^{(1)}, 1 \leq \alpha, \beta \leq \frac{n}{2}\) as follow: \( g_{1, \alpha}^{(1)} = g_{\alpha,1}^{(1)} = e_{\alpha+1} \) and \( g_{\alpha, \beta}^{(1)}(z) = \frac{e_{\alpha+1}(z) e_{\beta+1}(z)}{e_{2}(z)} \in \tilde{F} \). It is clear that all \( g_{\alpha, \beta}^{(1)} \) belong to \( F \) except \( g_{2,2}^{(1)} \). We have \( g_{2,2}^{(1)} = e_4 - \frac{1}{4} g_2 e_0 - \frac{1}{4} g_3 e_2 \). Let us define the matrix \((g_{\alpha, \beta})^{(2)}, 1 \leq \alpha, \beta \leq \frac{n}{2}\) as follows: \( g_{1,1}^{(2)} = e_{-2}, g_{1,2}^{(2)} = g_{2,1}^{(2)} = e_0, g_{1,\alpha}^{(2)} = g_{\alpha,1}^{(2)} = e_{\alpha-1} \) for \( 3 \leq \alpha \leq \frac{n}{2} \). \( g_{\alpha, \beta}^{(2)}(z) = \frac{g_{\alpha, \beta}^{(1)}(z) g_{\alpha, \beta}^{(1)}(z)}{g_{1,1}^{(2)}(z)} \). We set \( C_1^{(\frac{n}{2})} = \det(g_{\alpha, \beta})^{(1)} + \frac{1}{4} g_3 \det(g_{\alpha, \beta})^{(2)} \). In fact, \( C_1^{(\frac{n}{2})} \in S^{\frac{n}{2}} F_n \) and \( x_{\frac{n}{2}-1}(C_1^{(\frac{n}{2})}) = 0 \).
Let us construct the elements $C^{(n)} \in S^n F_n$ for odd $n$ such that $C^{(n)} \in \ker x^{n-1}_{\frac{n}{2}}$.

It is clear that our elements $C_0^{(\frac{n+1}{2})}$ and $C_1^{(\frac{n+1}{2})}$ from $S^{\frac{n+1}{2}} F_{n+1}$ have a form: $C_0^{(\frac{n+1}{2})} = A_0 + B_0 e_{n+1}$ and $C_1^{(\frac{n+1}{2})} = A_1 + B_1 e_{n+1}$ where $A_0, A_1 \in S^{\frac{n+1}{2}} F_n$ and $B_0, B_1 \in S^{\frac{n-1}{2}} F_n$. We set $C^{(n)} = B_0 C_1^{(\frac{n+1}{2})} - B_1 C_0^{(\frac{n+1}{2})} = B_0 A_1 - B_1 A_0$. It is clear that $C^{(n)} \in \ker x^{n-1}_{\frac{n}{2}}$.

**Proposition 4.** The center of the Poisson algebra $S^* F_n$ is generated by $C_0^{(\frac{n}{2})}$ and $C_1^{(\frac{n}{2})}$ for even $n$. For odd $n$ the center is generated by $C^{(n)}$.

**Proof.** One can check that the elements $C_0^{(\frac{n}{2})}, C_1^{(\frac{n}{2})}$ (resp. $C^{(n)}$) for even (resp. odd) $n$ are central in $S^* F_n$. On the other hand, the quotient algebra $S^* F_n/\langle C_0^{(\frac{n}{2})}, C_1^{(\frac{n}{2})} \rangle$ (resp. $S^* F_n/\langle C^{(n)} \rangle$) is isomorphic to the image of the homomorphism $x^{\frac{n}{2}-1}$ (resp. $x_{\frac{n-1}{2}}$) which is the algebra of functions on the symplectic manifold and has a trivial center. So the center is generated by $C_0^{(\frac{n}{2})}, C_1^{(\frac{n}{2})}$ (resp. $C^{(n)}$).

**Remark.** Considering any two linear combinations of our three Poisson brackets one obtains a bihamiltonian structure. Lenard scheme provides a family of commuting elements from Casimir elements $C_0^{(\frac{n}{2})}, C_1^{(\frac{n}{2})}$ for even $n$ and $C^{(n)}$ for odd $n$. 
Appendix
Elliptic functions

For an integral lattice $\Gamma \subset \mathbb{C}$ the Weierstrass elliptic function is defined as follows:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma'} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right), \text{ where } \Gamma' = \Gamma \setminus \{0\}$$

One has: $\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$, where $g_2$ and $g_3$ depend on the lattice $\Gamma$ only.

The Weierstrass zeta function is defined as follows:

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Gamma'} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

The function $\zeta(z)$ is not elliptic, but one has: $\zeta(z + \omega) = \zeta(z) + \eta(\omega)$, where $\eta: \Gamma \to \mathbb{C}$ is a \mathbb{Z}-linear function. The function $\zeta(z_1 - z_2) - \zeta(z_1) + \zeta(z_2)$ is elliptic in variables $z_1$ and $z_2$. It is clear that $\zeta(-z) = -\zeta(z)$, $\zeta'(z) = -\wp(z)$. One has the following useful decomposition:

$$\wp(z) = \frac{1}{z^2} + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + \ldots$$

We need the following identities:

$$(\zeta(x - y) - \zeta(x) + \zeta(y))(\wp(x) - \wp(y)) = \frac{1}{2}(\wp'(x) + \wp'(y)) \quad (5)$$

$$(\zeta(x - y) - \zeta(x) + \zeta(y))(\wp'(x) - \wp'(y)) = 2\wp(x)^2 + 2\wp(x)\wp(y) + 2\wp(y)^2 - \frac{1}{2}g_2$$

Proof of these identities is standard: to calculate the decomposition in the neighbourhood of the point $x = y = 0$.

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