KOSTANT PARTITION FUNCTION FOR $\text{sl}_4(C)$ AND $\text{sp}_6(C)$

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ABSTRACT
In this paper, we obtain a closed formula for the Kostant’s partition function for the Lie algebras $\text{sl}_4(C)$ and $\text{sp}_6(C)$: Using this function, one can compute the weight multiplicity of irreducible representations of the Lie algebras $\text{sl}_4(C)$ and $\text{sp}_6(C)$.

1. Introduction and preliminaries

One of the important problems in the representation theory of simple Lie algebras is the multiplicity computation of a weight in a finite dimensional complex irreducible representation of a Lie algebra. Let $L$ be a finite dimensional complex semi-simple Lie algebra with a Cartan subalgebra $H$ and root system $\Phi$. Suppose $\lambda$ is an integral dominant weight of $L$ and $V(\lambda)$ is the corresponding irreducible $L$-module. For any other integral dominant weight $\mu$, we denote the multiplicity of $\mu$ in $\lambda$ by $m(\lambda, \mu)$. One way to compute $m(\lambda, \mu)$ is by Kostant’s weight multiplicity formula (Kostant, 1958):

$$m(\lambda, \mu) = \sum_{\sigma \in \mathcal{W}} (-1)^{\ell(\sigma)} \psi(\sigma(\lambda + \rho) - (\mu + \rho)),$$

where $\mathcal{W}$ is the Weyl group, $\ell(\sigma)$ denotes the length of $\sigma$, $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ with $\Phi^+$ being the set of positive roots of $L$. In (1), $\psi$ is the Kostant partition function. For any weight $\gamma$, $\psi(\gamma)$ is the number of ways to write $\gamma$ as a linear combination of positive roots with non-negative integral coefficients. In general, there is no known closed formula for the $\psi$ on arbitrary weights of a Lie algebra. However, there has been some success in the low rank of some classical Lie algebras (Harris & Lauber, 2017; Refaghat & Shahryari, 2012) and for a particular weight (Harris et al., 2018). Moreover, many methods have been used to solve these types of problems (Adiga et al., 2016; Deckhart, 1985; Harris & Lauber, 2017; Kostant, 1958; Sarikaya et al., 2020; Srivastava & Chaudhary, 2015; Srivastava & Saikia, 2020; Zhang et al., 2009). In this paper, we are interested in finding a closed formula for $\psi$ in Lie algebras $\text{sl}_4(C)$ and $\text{sp}_6(C)$. To make different notations of Kostant’s partition function in Lie algebras $\text{sl}_4(C)$ and $\text{sp}_6(C)$, we use a notation $\psi_{sl}$ for Lie algebra $\text{sl}_4(C)$ and $\psi_{sp}$ for Lie algebra $\text{sp}_6(C)$.

Theorem 1.1. Let $\gamma = aR_1 + bR_2 + cR_3$ be a weight of the Lie algebra $\text{sl}_4(C)$ where $a, b$ and $c$ are non-negative integers.

(i) If $b \leq a$ and $b \leq c$, then

$$\psi_{sl}(\gamma) = \frac{1}{6} (b + 1)(b + 2)(b + 3),$$

(ii) If $a \leq c \leq b$, then

$$\psi_{sl}(\gamma) = \left\{ \begin{array}{ll}
\frac{1}{6} (a + 1)(a + 2)(3c - a + 3); & c \leq b - a \\
\frac{1}{6} (a + 1)(a + 2)(3c - a + 3) - \frac{1}{6} A(A + 1)(A + 2); & c > b - a
\end{array} \right.$$

where $A = c + a - b$.

(iii) If $c \leq a \leq b$, then

$$\psi_{sl}(\gamma) = \left\{ \begin{array}{ll}
\frac{1}{6} (c + 1)(c + 2)(3a - c + 3); & a \leq b - c \\
\frac{1}{6} (c + 1)(c + 2)(3a - c + 3) - \frac{1}{6} A(A + 1)(A + 2); & a > b - c
\end{array} \right.$$

where $A = c + a - b$.

(iv) If $a \leq b \leq c$, then

$$\psi_{sl}(\gamma) = \frac{1}{6} (a + 1)(a + 2)(3b - 2a + 3),$$

(v) If $c \leq b \leq a$, then

$$\psi_{sl}(\gamma) = \frac{1}{6} (c + 1)(c + 2)(3b - 2c + 3).$$

Theorem 1.2. Let $\gamma = aR_1 + bR_2 + cR_3$ be a weight of the Lie algebra $\text{sp}_6(C)$, then...
2. Background

In this section, we give a review of a set of notations of Lie algebras \(\mathfrak{sl}_4(\mathbb{C})\) and \(\mathfrak{sp}_6(\mathbb{C})\).

A Cartan subalgebra for \(\mathfrak{sl}_4(\mathbb{C})\) is

\[
H = \{ h = \text{diag}(a_1, a_2, a_3, a_4) | a_i \in \mathbb{C}, \ sum_{i=1}^4 a_i = 0 \}.
\]

For \(i = 1, \ldots, 4\), define a functional \(\mu_i : H \rightarrow \mathbb{C}\) by \(\mu_i(h) = a_i\). So a root system for \(\mathfrak{sl}_4(\mathbb{C})\) is the set

\[
\Phi = \{ \mu_i - \mu_j | 1 \leq i < j \leq 4 \}.
\]

Also, the set

\[
\Pi = \{ R_1 = \mu_1 - \mu_2, \quad R_2 = \mu_2 - \mu_3, \quad R_3 = \mu_3 - \mu_4 \}
\]

is a basis for \(\Phi\). Then, the positive roots are the set

\[
\Phi^+ = \{ \mu_i - \mu_j | 1 \leq i < j \leq 4 \}.
\]

We will denote the elements of \(\Phi^+\) by \(\beta_1, \ldots, \beta_6\). If we write them as a linear combination of simple roots then we have

\[
\begin{align*}
\beta_1 &= R_1, \\
\beta_2 &= R_2, \\
\beta_3 &= R_3, \\
\beta_4 &= R_1 + R_2, \\
\beta_5 &= R_2 + R_3, \\
\beta_6 &= R_1 + R_2 + R_3.
\end{align*}
\]

The notations of \(\mathfrak{sp}_6(\mathbb{C})\) are similar. A Cartan subalgebra for \(\mathfrak{sp}_6(\mathbb{C})\) is

\[
\begin{align*}
H &= \{ h = \text{diag}(a_1, a_2, a_3, -a_1, -a_2, -a_3) | a_i \in \mathbb{C} \},
\end{align*}
\]

and the functional \(\mu_i\) is defined as above. A root system for \(\mathfrak{sp}_6(\mathbb{C})\) is the set

\[
\Phi = \{ \pm \mu_i \pm \mu_j | 1 \leq i, j \leq 3 \} - \{0\},
\]

and the simple roots are the set

\[
\Pi = \{ R_1 = \mu_1 - \mu_2, \quad R_2 = \mu_2 - \mu_3, \quad R_3 = 2\mu_3 \}.
\]

The positive roots of \(\mathfrak{sp}_6(\mathbb{C})\) are

\[
\begin{align*}
\beta_1 &= R_1, \\
\beta_2 &= R_2, \\
\beta_3 &= R_3, \\
\beta_4 &= R_1 + R_2, \\
\beta_5 &= R_2 + R_3, \\
\beta_6 &= R_1 + R_2 + R_3.
\end{align*}
\]

3. Proof of theorem 1.1

Let \(\gamma = aR_1 + bR_2 + cR_3\) be a weight of the Lie algebra \(\mathfrak{sl}_4(\mathbb{C})\). Suppose that we write \(\gamma\) as non-negative integer linear combination of positive roots. Then, we have

\[
\begin{align*}
aR_1 + bR_2 + cR_3 &= r_1\beta_1 + \ldots + r_6\beta_6 \\
&= (r_1 + r_4 + r_6)R_1 + (r_2 + r_4 + r_5 + r_6)R_2 \\
&\quad + (r_3 + r_5 + r_6)R_3,
\end{align*}
\]

so we must have

\[
\begin{align*}
r_1 + r_4 + r_6 &= a, \\
r_2 + r_4 + r_5 + r_6 &= b, \\
r_3 + r_5 + r_6 &= c. \\
\end{align*}
\]

By definition, \(\varphi_{\mathfrak{sl}_4}(\gamma)\) is the number of non-negative integer solutions of system (2). Equivalently, \(\varphi_{\mathfrak{sl}_4}(\gamma)\) is the number of ordered 6-tuple of non-negative integer \((r_1, \ldots, r_6)\) that is the solution of (2). Note that, if we find the three non-negative integers \(r_4, r_5\) and \(r_6\) those are a part of a solution of (2), then we obtain \(r_1, r_2\) and \(r_3\) uniquely (from(2)). Therefore, the number of non-
negative integer solutions of (2) is the number of non-negative integer solutions of the following system.

\[
\begin{aligned}
&\begin{cases}
  r_4 + r_5 + r_6 \leq b, \\
r_4 + r_6 \leq \min\{a, b\}, \\
r_5 + r_6 \leq \min\{b, c\}.
\end{cases}
\end{aligned}
\]

(3)

The following results help us find the number of solutions of (3).

**Lemma 3.1.** (Refaghat & Shahryari, 2012) If \(m\) is a non-negative integer and \(X(m)\) is the number of non-negative integer solutions of the inequality \(x + y \leq m\), then

\[
X(m) = \frac{1}{2} (m + 1)(m + 2).
\]

**Corollary 3.2.** If \(0 \leq n \leq m\), and \(X(m, n)\) is the number of non-negative integer solutions of the system

\[
\begin{aligned}
&\begin{cases}
  x + y \leq m, \\
y \leq n,
\end{cases}
\end{aligned}
\]

then

\[
X(m, n) = \frac{1}{2} (n + 1)(2m - n + 2).
\]

**Proof.** It is easily seen that

\[
X(m, n) = X(m) - X(m - n),
\]

which completes the proof.\(\square\)

**Corollary 3.3.**

\[
\sum_{i=1}^{m} X(m, i) = \frac{1}{6} m(m + 1)(2m + 7).
\]

**Corollary 3.4** If \(0 \leq k \leq m\), then

\[
\sum_{i=1}^{k} X(m, i) = \frac{1}{2} k(k + 3)(m + 1) - \frac{1}{6} k(k + 1)(k + 2).
\]

Also,

\[
\sum_{i=k+1}^{m} X(m, i) = \sum_{i=1}^{m} X(m, i) - \sum_{i=1}^{k} X(m, i).
\]

**Lemma 3.5.** If \(m\) is a non-negative integer and \(Y(m)\) is the number of non-negative integer solutions of the inequality \(x + y + z \leq m\), then

\[
Y(m) = \frac{1}{6} (m + 1)(m + 2)(m + 3).
\]

**Proof.** For all \(i = 0, 1, \ldots, m\), let

\[
S_i = \{(x, y, i)|x, y \in \mathbb{Z}, \ x, y \geq 0, \ x + y + i \leq m\}.
\]

It is clear that \(|S_i| = X(m - i)\). We have

\[
Y(m) = \sum_{i=0}^{m} |S_i|
\]

\[
= \sum_{i=0}^{m} \frac{1}{2} (m - i + 1)(m - i + 2)
\]

\[
= \frac{1}{6} (m + 1)(m + 2)(m + 3).
\]

\(\square\)

**Lemma 3.6.** If \(0 \leq n \leq m\), and \(Y(m, n)\) is the number of non-negative integer solutions of the system

\[
\begin{aligned}
&\begin{cases}
  x + y + z \leq m, \\
x + y \leq n,
\end{cases}
\end{aligned}
\]

then

\[
Y(m, n) = \frac{1}{6} (n + 1)(n + 2)(3m - 2n + 3).
\]

**Proof.** An easy computation shows that

\[
Y(m, n) = (m - n)X(n) + Y(n),
\]

and the lemma follows.\(\square\)

**Lemma 3.7.** If \(0 \leq n \leq k \leq m\), \(Y(m, n, k)\) is the number of non-negative integer solutions of the system

\[
\begin{aligned}
&\begin{cases}
  x + y + z \leq m, \\
x + y \leq n, \\
y + z \leq k,
\end{cases}
\end{aligned}
\]

then

\[
Y(m, n, k) = \frac{1}{6} (n + 1)(n + 2)(3k - n + 3)
\]

for \(k \leq m - n\), and

\[
Y(m, n, k) = \frac{1}{6} (n + 1)(n + 2)(3k - n + 3) - \frac{1}{6} A(A + 1)(A + 2)
\]

for \(k > m - n\), where \(A = k + n - m\).
Proof. Suppose that \( k \leq m - n \). Let
\[
S_i = \{(x, y, i) | x, y \in \mathbb{Z}, \ x, y \geq 0, \ x + y \leq n \},
\]
\[
R_i = \{(x, y, i) | x, y \in \mathbb{Z}, \ x, y \geq 0, \ x + y \leq n, \ y \leq i \}.
\]
It is easy to check that
\[
Y(m, n, k) = \sum_{i=0}^{k-n-1} |S_i| + \sum_{i=0}^n |R_i|
\]
\[
= \sum_{i=0}^{k-n-1} X(n) + \sum_{i=0}^n X(n, i)
\]
\[
= \frac{1}{6} (n+1)(n+2)(3k-n+3).
\]
Now, suppose that \( k > m - n \). Let \( S_i \) as above and
\[
R_i = \{(x, y, i) | x, y \in \mathbb{Z}, \ x, y \geq 0, \ x + y \leq n, \ y \leq k - i \},
\]
\[
T_i = \{(i, y, z) | y, z \in \mathbb{Z}, \ y, z \geq 0, \ y + z \leq k + n - m \},
\]
\[
P = \{(x, y, z) | x, y, z \in \mathbb{Z}, \ x, y, z \geq 0, \ x + y + z \leq k + n - m \}.
\]
Let \( A = k + n - m \). We have
\[
Y(m, n, k) = \sum_{i=0}^{k-n-1} |S_i| + \sum_{i=k-n}^{m-n-1} |R_i| + \sum_{i=0}^{m-k-1} |T_i| + |P|
\]
\[
= \sum_{i=0}^{k-n-1} X(n) + \sum_{i=k-n}^{m-n-1} X(n, k-i) + \sum_{i=0}^{m-k-1} X(A) + Y(A)
\]
\[
= \frac{1}{2} (k-n)(n+1)(n+2) + \frac{1}{3} n(n+1)(2n+7)
\]
\[
- \frac{1}{2} (n+1)A(A+3)
\]
\[
+ \frac{1}{6} A(A+1)(A+2) + \frac{1}{2} (m-k)(A+1)(A+2)
\]
\[
+ \frac{1}{6} (A+1)(A+2)(A+3)
\]
\[
\]
4. Proof of theorem 1.2

Let \( y = aR_1 + bR_2 + cR_3 \) be a weight of the Lie algebra \( \mathfrak{sp}_6(\mathbb{C}) \). As above notations, \( \varphi_{\mathfrak{sp}}(y) \) is the number of ordered 9-tuples of non-negative integer \( (r_1, \ldots, r_9) \) that \( y = \sum_{i=1}^9 r_i \beta_i \), where \( \beta_1, \ldots, \beta_9 \) are positive roots of \( \mathfrak{sp}_6(\mathbb{C}) \). Thus, we must have
\[
\begin{cases}
    r_1 + r_4 + r_6 + r_7 + 2r_8 = a, \\
    r_2 + r_4 + r_5 + r_6 + 2r_7 + 2r_8 + 2r_9 = b, \\
    r_3 + r_5 + r_6 + r_7 + r_8 + r_9 = c.
\end{cases}
\]
By comparing the positive roots of the Lie algebras \( \mathfrak{sl}_4(\mathbb{C}) \) and \( \mathfrak{sp}_6(\mathbb{C}) \), we see that the positive roots \( \beta_1, \ldots, \beta_6 \) of \( \mathfrak{sp}_6(\mathbb{C}) \) are similar to the positive roots of \( \mathfrak{sl}_4(\mathbb{C}) \). Suppose that \( (m, n, k) \) is a triple of non-negative integer that the 9-tuple \( (r_1, \ldots, r_6, m, n, k) \) is a non-negative integer solution of (4). So, we have
\[
y = \sum_{i=1}^6 r_i \beta_i + m \beta_7 + n \beta_8 + k \beta_9.
\]
Since
\[
m \beta_7 + n \beta_8 + k \beta_9 = (m+2n)R_1 + (2m+2n+2k)R_2 + (m+n+k)R_3,
\]
we conclude
\[ \sum_{i=1}^{6} r_i \beta_i = (a - m - 2n)R_1 + (b - 2m - 2n - 2k)R_2 + (c - m - n - k)R_3. \]

Let
\[ \hat{y} = (a - m - 2n)R_1 + (b - 2m - 2n - 2k)R_2 + (c - m - n - k)R_3. \]

An easy verification shows that the number of non-negative integer solutions \((r_1, \ldots, r_6, m, n, k)\) of (4) is equal to \(s_{sl}(\hat{y})\). Therefore, to find the number of non-negative integer solutions of (4), it is sufficient to find the triple \((m, n, k)\) that is a part of solutions of (4). According to the equalities of system (4), if \((r_1, \ldots, r_6, m, n, k)\) is a non-negative integer solution of (4), then we must have
\[ 0 \leq m \leq \min\{a, \frac{1}{2} b\}, \]
\[ 0 \leq n \leq \min\{\frac{1}{2} (a - m), \frac{1}{2} b - m, c - m\}, \]
\[ 0 \leq k \leq \min\{\frac{1}{2} b - m - n, c - m - n\}. \]

Let
\[ \hat{m} = \min\{a, \frac{1}{2} b\}, c, \]
\[ \hat{n} = \min\{\frac{1}{2} (a - m), \frac{1}{2} b - m, c - m\}, \]
\[ \hat{k} = \min\{\frac{1}{2} b - m - n, c - m - n\}. \]

Summarizing, the proof is complete.

5. Conclusion and suggestion

Kostant partition is a very important function in studying representations of Lie algebras. For instance, in Refaghat & Shahryari (2013), the Kostant partition function of Lie algebras of \(\mathfrak{sp}_4(C)\) is used to find representations of symmetry classes of tensors as \(\mathfrak{sp}_4(C)\)-module and thus, the branching rule \(A_5 \rightarrow C_2\) is obtained. Now, this process can be repeated using the Kostant partition function of \(\mathfrak{sp}_6(C)\) to find branching rule \(A_7 \rightarrow C_3\) or similar branching rules. Also, by developing this method, it can be tried to find the exact values of the Kostant partition function for Lie algebras \(\mathfrak{sl}_n(C)\) and \(\mathfrak{sp}_{2n}(C)\).

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Public interest statement

The representation theory of Lie algebras is one of the most important topics in Lie algebras, which is more noticeable in applications of Lie algebra in mathematics and physics. And it can be used to investigate the structure of Lie algebras. In the representation theory of Lie algebras, one of the interesting problems is finding the irreducible submodules of a desired module. One of the most important tools for analyzing a desired module into irreducible submodules is the Kostant’s weight multiplicity formula. In this formula, a function called the Kostant partition function is introduced; in this paper, a closed formula for this function for any desired weight of Lie algebras \(\mathfrak{sl}_4(C)\) and \(\mathfrak{sp}_6(C)\) is obtained.

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