The Energy-Momentum Tensor on Spin\(^c\) Manifolds

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Abstract

On Spin\(^c\) manifolds, we study the Energy-Momentum tensor associated with a spinor field. First, we give a spinorial Gauss type formula for oriented hypersurfaces of a Spin\(^c\) manifold. Using the notion of generalized cylinders, we derive the variationnal formula for the Dirac operator under metric deformation and point out that the Energy-Momentum tensor appears naturally as the second fundamental form of an isometric immersion. Finally, we show that generalized Spin\(^c\) Killing spinors for Codazzi Energy-Momentum tensor are restrictions of parallel spinors.

Key words: Spin\(^c\) structures, Spin\(^c\) Gauss formula, metric variation formula for the Dirac operator, Energy-Momentum tensor, generalized cylinder, generalized Killing spinors.

1 Introduction

In [14], O. Hijazi proved that on a compact Riemannian spin manifold \((M^n, g)\) any eigenvalue \(\lambda\) of the Dirac operator to which is attached an eigenspinor \(\psi\) satisfies

\[
\lambda^2 \geq \inf_M \left( \frac{1}{4} \text{Scal}_M + |\ell\psi|^2 \right),
\]  

(1)
where $\text{Scal}^M$ is the scalar curvature of the manifold $M$ and $\ell^\psi$ is the field of symmetric endomorphisms associated with the field of quadratic forms $T^\psi$ called the Energy-Momentum tensor. It is defined on the complement set of zeroes of the eigenspinor $\psi$, for any vector $X \in \Gamma(TM)$ by

$$T^\psi(X) = \text{Re}<X \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2}>.$$  

Here $\nabla$ denotes the Levi-Civita connection on the spinor bundle of $M$ and “·” the Clifford multiplication. The limiting case of (1) is characterized by the existence of a spinor field $\psi$ satisfying for all $X \in \Gamma(TM)$,

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi.$$  

For $\text{Spin}^c$ structures, the complex line bundle $L^M$ is endowed with an arbitrary connection and hence an arbitrary curvature $i\Omega^M$ which is an imaginary 2-form on the manifold. In terms of the Energy-Momentum tensor the author proved in [25] that on a compact Riemannian $\text{Spin}^c$ manifold any eigenvalue $\lambda$ of the Dirac operator to which is attached an eigenspinor satisfies

$$\lambda^2 \geq \inf_M \left( \frac{1}{4} \text{Scal}^M - \frac{c_n}{4} |\Omega^M| + |\ell^\psi|^2 \right),$$  

(3)

where $c_n = 2[\frac{n}{2}]^\frac{1}{2}$. The limiting case of (3) is characterized by the existence of a spinor field $\psi$ satisfying for every $X \in \Gamma(TM)$,

$$\begin{cases} 
\nabla^\Sigma^M_X \psi = -\ell^\psi(X) \cdot \psi, \\
\Omega^M \cdot \psi = i\frac{c_n}{2} |\Omega^M| \psi.
\end{cases}$$  

(4)

Here $\nabla^\Sigma^M$ denotes the Levi-Civita connection on the $\text{Spin}^c$ spinor bundle and “·” the $\text{Spin}^c$ Clifford multiplication. In [25], the author showed also that the sphere with a special $\text{Spin}^c$ structure is a limiting manifold for (3).

Studying the Energy-Momentum tensor on a Riemannian or semi-Riemannian spin manifolds has been done by many authors, since it is related to several geometric constructions (see [12], [2], [24] and [6] for results in this topic). In this paper we study the Energy-Momentum tensor on Riemannian and semi-Riemannian $\text{Spin}^c$ manifolds. First, we prove that the Energy-Momentum tensor appears in the study of the variations of the spectrum of the Dirac operator:

**Proposition 1.1** Let $(M^n, g)$ be a $\text{Spin}^c$ Riemannian manifold and $g_t = g + tk$ a smooth 1-parameter family of metrics. For any spinor field $\psi \in \Gamma(\Sigma M)$, we have

$$\left. \frac{d}{dt} \right|_{t=0} (D^M r^1_t \psi, r^1_t \psi)_{g_t} = -\frac{1}{2} \int_M <k, T_\psi> dv_g,$$  

(5)
where \((\,,\,) = \int_M \text{Re} (\,,\,) \, dv_g\), the Dirac operator \(D^M_t\) is the Dirac operator associated with \(M_t = (M, g_t)\), \(T^\psi = |\psi|^2 T^\psi = \text{Re} <X \cdot \nabla_X \psi, \psi>\) and \(\tau^1_0 \psi\) is the image of \(\psi\) under the isometry \(\tau^1_0\) between the spinor bundles of \((M, g)\) and \((M, g_t)\).

This was proven in [4] by J. P. Bourguignon and P. Gauduchon for spin manifolds. Using this, we extend to \(\text{Spin}^c\) manifolds a result by Th. Friedrich and E. C. Kim in [8] on spin manifolds:

**Theorem 1.2** Let \(M\) be a \(\text{Spin}^c\) Riemannian manifold. A pair \((g_0, \psi_0)\) is a critical point of the Lagrange functional

\[
\mathcal{W}(g, \psi) = \int_U (\text{Scal}_g^M + \varepsilon \lambda <\psi, \psi>_g - <D_g \psi, \psi>_g) \, dv_g,
\]

\((\lambda, \varepsilon \in \mathbb{R})\) for all open subsets \(U\) of \(M\) if and only if \((g_0, \psi_0)\) is a solution of the following system

\[
\begin{cases}
D_g \psi = \lambda \psi, \\
\text{ric}_g^M - \frac{\text{Scal}_g^M}{2} g = \frac{\varepsilon}{4} T^\psi,
\end{cases}
\]

where \(\text{ric}_g^M\) denotes the Ricci curvature of \(M\) considered as a symmetric bilinear form.

Now, we interprète the Energy-Momentum tensor as the second fundamental form of a hypersurface. In fact, we prove the following:

**Proposition 1.3** Let \(M^n \hookrightarrow (Z, g)\) be any compact oriented hypersurface isometrically immersed in an oriented Riemannian \(\text{Spin}^c\) manifold \((Z, g)\), of constant mean curvature \(H\) and Weingarten map \(W\). Assume that \(Z\) admits a parallel spinor field \(\psi\), then the Energy-Momentum tensor associated with \(\varphi =: \psi|_M\) satisfies

\[2\ell^\varphi = -W.\]

Moreover the hypersurface \(M\) satisfies the equality case in (3) if and only if

\[
\text{Scal}^Z - 2 \text{ric}^Z (\nu, \nu) - c_n |\Omega^M| = 0.
\]

This was proven by Morel in [24] for a compact oriented hypersurface of a spin manifold carrying parallel spinor but in this case the hypersurface \(M\) is directly a limiting manifold for (1) without the condition (6).

Finally, we study generalized Killing spinors on \(\text{Spin}^c\) manifolds. They are characterized by the identity, for any tangent vector field \(X\) on \(M\),

\[
\nabla_X^\Sigma \psi = \frac{1}{2} F(X) \cdot \psi,
\]

\[3\]
where \( F \) is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that
\[
2T^\psi(X,Y) = - \langle F(X), Y \rangle.
\]

These spinors are closely related to the so-called \( T \)-Killing spinors studied by Friedrich and Kim in [9] on spin manifolds. It is natural to ask whether the tensor \( F \) can be realized as the Weingarten tensor of some isometric embedding of \( M \) in a manifold \( Z^{n+1} \) carrying parallel spinors. Morel studied this problem in the case of spin manifolds where the tensor \( F \) is parallel and in [2], the authors studied the problem in the case of semi-Riemannian spin manifolds where the tensor \( F \) is a Codazzi-Mainardi tensor. We establish the corresponding result for semi-Riemannian Spin\(^c\) manifolds:

**Theorem 1.4** Let \((M^n, g)\) be a semi-Riemannian Spin\(^c\) manifold carrying a generalized Spin\(^c\) Killing spinor \( \varphi \) with a Codazzi-Mainardi tensor \( F \). Then the generalized cylinder \( Z := I \times M \) with the metric \( dt^2 + g_t \), where \( g_t(X,Y) = g((Id - tF)^2 X,Y) \), equipped with the Spin\(^c\) structure arising from the given one on \( M \) has a parallel spinor whose restriction to \( M \) is just \( \varphi \).

A characterisation of limiting 3-dimensional manifolds for (3), having generalized Spin\(^c\) Killing spinors with Codazzi tensor is then given.

The paper is organised as follows: In Section 2, we collect basic material on spinors and the Dirac operator on semi-Riemannian Spin\(^c\) manifolds. In Section 3, we study hypersurfaces of Spin\(^c\) manifolds. We derive a spinorial Gauss formula after identifying the restriction of the Spin\(^c\) spinor bundle of the ambient manifold with the Spin\(^c\) spinor bundle of the hypersurface. In Section 4, we define the generalized cylinder of a Spin\(^c\) manifold \( M \) and we collect formulas relating the curvature of a generalized cylinder to geometric data on \( M \). In section 5, we compare the Dirac operators for two different semi-Riemannian metrics, then one first has to identify the spinor bundles using parallel transport. In the last section, we interpret the Energy-Momentum tensor as the second fundamental form of a hypersurface and we study generalized Spin\(^c\) Killing spinors. The author would like to thank Oussama Hijazi for his support and encouragements.

2 The Dirac operator on semi-Riemannian Spin\(^c\) manifolds

In this section, we collect some algebraic and geometric preliminaries concerning the Dirac operator on semi-Riemannian Spin\(^c\) manifolds. Details can be found in [3] and [2]. Let \( r + s = n \) and consider on \( \mathbb{R}^n \) the nondegenerate symmetric bilinear form of
signature \((r, s)\) given by
\[
\langle v, w \rangle := \sum_{j=1}^{r} v_j w_j - \sum_{j=r+1}^{n} v_j w_j,
\]
for any \(v, w \in \mathbb{R}^n\). We denote by \(\text{Cl}_{r,s}\) the real Clifford algebra corresponding to \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), this is the unitary algebra generated by \(\mathbb{R}^n\) subject to the relations
\[
e_j \cdot e_k + e_k \cdot e_j = \begin{cases} -2\delta_{jk} & \text{if } j \leq r, \\ 2\delta_{jk} & \text{if } j > r, \end{cases}
\]
where \((e_j)_{1 \leq j \leq n}\) is an orthonormal basis of \(\mathbb{R}^n\) of signature \((r, s)\), i.e., \(\langle e_j, e_k \rangle = \varepsilon_j \delta_{jk}\) and \(\varepsilon_j = \pm 1\). The complex Clifford algebra \(\text{Cl}^c_{r,s}\) is the complexification of \(\text{Cl}_{r,s}\) and it decomposes into even and odd elements \(\text{Cl}^c_{r,s} = \text{Cl}^e_{r,s} \oplus \text{Cl}^o_{r,s}\). The real spin group is defined by
\[
\text{Spin}(r, s) := \{v_1 \cdot \ldots \cdot v_{2k} \in \text{Cl}_{r,s} \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j \rangle = \pm 1\}.
\]
The spin group \(\text{Spin}(r, s)\) is the double cover of \(\text{SO}(r, s)\), in fact the following sequence is exact
\[
1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Spin}(r, s) \xrightarrow{\xi} \text{SO}(r, s) \longrightarrow 1,
\]
where \(\xi = \text{Ad}_{|\text{spin}(r, s)}\) and \(\text{Ad}\) is defined by
\[
\text{Ad} : \text{Cl}^e_{r,s} \longrightarrow \text{End}(\mathbb{R}^n) \\
w \longrightarrow \text{Ad}_w : v \longrightarrow \text{Ad}_w(v) = w \cdot v \cdot w^{-1}.
\]
Here \(\text{Cl}^e_{r,s}\) denotes the group of units of \(\text{Cl}_{r,s}\). Since \(S^1 \cap \text{Spin}(r, s) = \{\pm 1\}\), we define the complex spin group by
\[
\text{Spin}^c(r, s) = \text{Spin}(r, s) \times_{\mathbb{Z}_2} S^1.
\]
The complex spin group is the double cover of \(\text{SO}(r, s) \times S^1\), this yields to the exact sequence
\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^c(r, s) \xrightarrow{\xi^c} \text{SO}(r, s) \times S^1 \longrightarrow 1,
\]
where \(\xi^c = (\xi, \text{Id}^2)\). When \(n = 2m\) is even, \(\text{Cl}_{r,s}\) has a unique irreducible complex representation \(\chi_{2m}\) of complex dimension \(2^m\), \(\chi_{2m} : \text{Cl}_{r,s} \longrightarrow \text{End}(\Sigma_{r,s})\). If \(n = 2m + 1\) is odd, \(\text{Cl}_{r,s}\) has two inequivalent irreducible representations both of complex dimension \(2^m\), \(\chi_{2m+1}^j : \text{Cl}_{r,s} \longrightarrow \text{End}(\Sigma^j_{r,s})\), for \(j = 0\) or 1, where \(\Sigma^j_{r,s} = \{\sigma \in \Sigma_{r,s} \mid \chi^j_{2m+1}(\omega_{r,s})\sigma = (-1)^j\sigma\}\) and \(\omega_{r,s}\) is the complex volume element
\[
\omega_{r,s} = \begin{cases} i^{m-s} e_1 \cdot \ldots \cdot e_n & \text{if } n = 2m, \\ i^{m-1+s} e_1 \cdot \ldots \cdot e_n & \text{if } n = 2m + 1. \end{cases}
\]
We define the complex spinorial representation \( \rho_n \) by the restriction of an irreducible representation of \( \mathbb{C}l_{r,s} \) to \( \text{Spin}^c(r,s) \):

\[
\rho_n := \begin{cases} 
\chi_{2m|\text{Spin}^c(r,s)} & \text{if } n = 2m, \\
\chi_{2m+1|\text{Spin}^c(r,s)} & \text{if } n = 2m + 1.
\end{cases}
\]

When \( n = 2m \) is even, \( \rho_n \) decomposes into two inequivalent irreducible representations \( \rho_n^+ \) and \( \rho_n^- \), i.e., \( \rho_n = \rho_n^+ + \rho_n^- : \text{Spin}^c(r,s) \to \text{Aut}(\Sigma_{r,s}) \). The space \( \Sigma_{r,s} \) decomposes into \( \Sigma_{r,s} = \Sigma_{r,s}^+ \oplus \Sigma_{r,s}^- \), where \( \omega_{r,s} \) acts on \( \Sigma_{r,s}^+ \) as the identity and minus the identity on \( \Sigma_{r,s}^- \). If \( n = r + s \) is odd and when restricted to \( \text{Spin}^c(r,s) \), the two representations \( \chi_{0}^{2m+1|\text{Spin}^c(r,s)} \) and \( \chi_{+}^{2m+1|\text{Spin}^c(r,s)} \) are equivalent and we simply choose \( \Sigma_{r,s} := \Sigma_{r,s}^0 \). The complex spinor bundle \( \Sigma_{r,s} \) carries a Hemitian symmetric bilinear \( \text{Spin}^c(r,s) \)-invariant form \( (\cdot,\cdot) \), such that

\[
(v \cdot \sigma_1, \sigma_2) = (-1)^{r+1} (\sigma_1, v \cdot \sigma_2) \quad \text{for all } \sigma_1, \sigma_2 \in \Sigma_{r,s} \text{ and } v \in \mathbb{R}^n.
\]

Now, we give the following isomorphism \( \alpha \), which is of particular importance for the identification of the \( \text{Spin}^c \) bundles in the context of immersions of hypersurfaces:

\[
\alpha : \mathbb{C}l_{r,s} \to \mathbb{C}l_{r+1,s}^0 \\
e_j \to \nu \cdot e_j,
\]

where we look at an embedding of \( \mathbb{R}^n \) onto \( \mathbb{R}^{n+1} \) such that \( (\mathbb{R}^n)^\perp \) is spacelike and spanned by a spacelike unit vector \( \nu \).

Let \( N^n \) be an oriented semi-Riemannian manifold of signature \((r,s)\) and let \( P_{SO} \) be the \( SO(r,s) \)-principal bundle of positively space and time oriented orthonormal tangent frames. A complex \( \text{Spin}^c \) structure on \( N \) is a \( \text{Spin}^c(r,s) \)-principal bundle \( P_{\text{Spin}^c} \) over \( N \), an \( S^1 \)-principal bundle \( P_{S^1} \) over \( N \) together with a twofold covering map \( \Theta : P_{\text{Spin}^c} \to P_{SO} \times_N P_{S^1} \) such that

\[
\Theta(ua) = \Theta(u) \xi^c(a),
\]

for every \( u \in P_{\text{Spin}^c} \) and \( a \in \text{Spin}^c(r,s) \), i.e., \( N \) has a \( \text{Spin}^c \) structure if and only if there exists an \( S^1 \)-principal bundle \( P_{S^1} \) over \( N \) such that the transition functions \( g_{\alpha \beta} \times l_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \to SO(r,s) \times S^1 \) of the \( SO(r,s) \times S^1 \)-principal bundle \( P_{SO} \times_N P_{S^1} \) admit lifts to \( \text{Spin}^c(r,s) \) denoted by \( \tilde{g}_{\alpha \beta} \times \tilde{l}_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \to \text{Spin}^c(r,s) \), such that \( \xi^c \circ (\tilde{g}_{\alpha \beta} \times \tilde{l}_{\alpha \beta}) = g_{\alpha \beta} \times l_{\alpha \beta} \). This, anyhow, is equivalent to the second Stiefel-Whitney class \( w_2(N) \) being equal, modulo 2, to the first chern class \( c_1(L^N) \) of the complex line bundle \( L^N \). It is the complex line bundle associated with the \( S^1 \)-principal fibre bundle via the standard representation of the unit circle.
Let $\Sigma N := P_{\text{Spin}^c N} \times_{P_{\text{SO} N}} \Sigma_{r,s}$ be the spinor bundle associated with the spinor representation. A section of $\Sigma N$ will be called a spinor field. Using the cocycle condition of the transition functions of the two principal fibre bundles $P_{\text{Spin}^c N}$ and $P_{\text{SO} N} \times_{N} P_{\mathbb{G}_1} N$, we can prove that

$$\Sigma N = \Sigma' N \otimes (L^N)^{1/2},$$

where $\Sigma' N$ is the locally defined spin bundle and $(L^N)^{1/2}$ is locally defined too but $\Sigma N$ is globally defined. The tangent bundle $TN = P_{\text{SO} N} \times_{P_{\text{SO} N}} \mathbb{R}^n$ where $\rho_0$ stands for the standard matrix representation of $N$, can be seen as the associated vector bundle $TN \simeq P_{\text{Spin}^c N} \times_{P_{\text{SO} N}} \mathbb{R}^n$ where $pr_1$ is the first projection. One defines the Clifford multiplication at every point $p \in N$:

$$T_p N \otimes \Sigma_p N \longrightarrow \Sigma_p N,$$

$$[b, v] \otimes [b, \sigma] \longrightarrow [b, v] \cdot [b, \sigma] := [b, v \cdot \sigma = \chi_n(v)\sigma],$$

where $b \in P_{\text{Spin}^c N}$, $v \in \mathbb{R}^n$, $\sigma \in \Sigma_{r,s}$ and $\chi_n = \chi_{2m}$ if $n$ is even and $\chi_n = \chi_{2m+1}$ if $n$ is odd. The Clifford multiplication can be extended to differential forms. Clifford multiplication inherits the relations of the Clifford algebra, i.e., for $X, Y \in T_p N$ and $\varphi \in \Sigma_p N$ we have $X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi = -2 \langle X, Y \rangle \varphi$. In even dimensions the spinor bundle splits into $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$, where $\Sigma^+ N = P_{\text{Spin}^c N} \times_{\rho_{N}^+} \Sigma_{r,s}^{1,1}$, Clifford multiplication by a non-vanishing tangent vector interchanges $\Sigma^+ N$ and $\Sigma^- N$. The $\text{Spin}^c(r,s)$-invariant nondegenerate symmetric sesquilinear form on $\Sigma_{r,s}$ and $\Sigma_{r,s}^{1,1}$ induces inner products on $\Sigma N$ and $\Sigma^\pm N$ which we again denote by $\langle \cdot, \cdot \rangle$ and it satisfies

$$\langle X \cdot \psi, \varphi \rangle = (-1)^{s+1} \langle \psi, X \cdot \varphi \rangle,$$

for every $X \in \Gamma(TN)$ and $\psi, \varphi \in \Gamma(\Sigma N)$. Additionally, given a connection 1-form $A^N$ on $P_{\mathbb{G}_1} N$, $A^N : T(P_{\mathbb{G}_1} N) \longrightarrow i\mathbb{R}$ and the connection 1-form $\omega^N$ on $P_{\text{SO} N}$ for the Levi-Civita connection $\nabla^N$, we can define the connection $\omega^N \times A^N : T(P_{\text{SO} N} \times_{N} P_{\mathbb{G}_1} N) \longrightarrow so_n \oplus i\mathbb{R} = \text{spin}^c_n$

on the principal fibre bundle $P_{\text{SO} N} \times_{N} P_{\mathbb{G}_1} N$ and hence a covariant derivative $\nabla^{\Sigma N}$ on $\Sigma N$ [7] given locally by

$$\nabla^N_{e_k} \varphi = \left[ b \times s, e_k(\sigma) + \frac{1}{4} \sum_{j=1}^n e_j e_j \cdot \nabla^N_{e_k} e_j \cdot \sigma + \frac{1}{2} A^N(s(e_k)) \sigma \right]$$

$$= e_k(\varphi) + \frac{1}{4} \sum_{j=1}^n e_j e_j \cdot \nabla^N_{e_k} e_j \cdot \varphi + \frac{1}{2} A^N(s(e_k)) \varphi, \quad (9)$$

where $\varphi = [b \times s, \sigma]$ is a locally defined spinor field, $b = (e_1, \ldots, e_n)$ is a local space and time oriented orthonormal tangent frame, $s : U \longrightarrow P_{\mathbb{G}_1} N$ is a local section of
$P_{S^1}N$ and $\tilde{b} \times s$ is the lift of the local section $b \times s : U \rightarrow P_{SO}N \times_N P_{S^1}N$ to the 2-fold covering $\Theta : P_{\text{Spin}^c}N \rightarrow P_{SO}N \times_N P_{S^1}N$. The curvature of $A^N$ is an imaginary valued 2-form denoted by $F_{AN} = dA^N$, i.e., $F_{AN} = i\Omega^N$, where $\Omega^N$ is a real valued 2-form on $P_{S^1}N$. We know that $\Omega^N$ can be viewed as a real valued 2-form on $N$ [7]. In this case $i\Omega^N$ is the curvature form of the associated line bundle $L^N$. The curvature tensor $\mathcal{R}^{\Sigma N}$ of $\nabla^{\Sigma N}$ is given by

$$\mathcal{R}^{\Sigma N}(X,Y)\varphi = \frac{1}{4} \sum_{j,k=1}^{n} \epsilon_j \epsilon_k \left\langle R^N(X,Y)e_j, e_k \right\rangle e_j \cdot e_k \cdot \varphi + \frac{i}{2} \Omega^N(X,Y)\varphi,$$

where $R^N$ is the curvature tensor of the Levi-Civita connection $\nabla^N$. In the Spin$^c$ case, the Ricci identity translates, for every $X \in \Gamma(TN)$, to

$$\sum_{k=1}^{n} \epsilon_k e_k \cdot \mathcal{R}^{\Sigma N}(e_k, X)\varphi = \frac{1}{2} \text{Ric}^N(X) \cdot \varphi - \frac{i}{2} (X \lrcorner \Omega^N) \cdot \varphi.$$  (11)

Here $\text{Ric}^N$ denotes the Ricci curvature considered as a field of endomorphism on $TN$. The Ricci curvature considered as a symmetric bilinear form will be written $\text{ric}^N(Y,Z) = \left\langle \text{Ric}^N(Y), Z \right\rangle$. The Dirac operator maps spinor fields to spinor fields and is locally defined by

$$D^N \varphi = i^n \sum_{j=1}^{n} \epsilon_j e_j \cdot \nabla^{\Sigma N}_{e_j} \varphi,$$

for every spinor field $\varphi$. The Dirac operator is an elliptic operator, formally selfadjoint, i.e., if $\psi$ or $\varphi$ has compact support, then $(D^N \varphi, \psi) = (\varphi, D^N \psi)$, where $(\varphi, \psi) = \int_N \langle \varphi, \psi \rangle \, dv_g$.

### 3 Semi-Riemannian Spin$^c$ hypersurfaces and the Gauss formula

In this section, we study Spin$^c$ structures of hypersurfaces, such as the restriction of a Spin$^c$ bundle of an ambient semi-Riemannian manifold and the complex spinorial Gauss formula.

Let $\mathcal{Z}$ be an oriented $(n + 1)$-dimensional semi-Riemannian Spin$^c$ manifold and $M \subset \mathcal{Z}$ a semi-Riemannian hypersurface with trivial spacelike normal bundle. This means that there is a vector field $\nu$ on $\mathcal{Z}$ along $M$ satisfying $\left\langle \nu, \nu \right\rangle = +1$ and $\left\langle \nu, TM \right\rangle = 0$. Hence if the signature of $M$ is $(r, s)$, then the signature of $\mathcal{Z}$ is $(r + 1, s)$.
Proposition 3.1 The hypersurface $M$ inherits a Spin$^c$ structure from that on $Z$, and we have

\[
\begin{align*}
\Sigma Z|_M & \simeq \Sigma M & \text{if } n \text{ is even}, \\
\Sigma^+ Z|_M & \simeq \Sigma M & \text{if } n \text{ is odd}.
\end{align*}
\]

Moreover Clifford multiplication by a vector field $X$, tangent to $M$, is given by

\[X \cdot \varphi = (\nu \cdot X \cdot \psi)|_M,\]  \hspace{1cm} (12)

where $\psi \in \Gamma(\Sigma Z)$ (or $\psi \in \Gamma(\Sigma^+ Z)$ if $n$ is odd), $\varphi$ is the restriction of $\psi$ to $M$, “$\cdot$” is the Clifford multiplication on $Z$, and “$\cdot$” that on $M$.

Proof: The bundle of space and time oriented orthonormal frames of $M$ can be embedded into the bundle of space and time oriented orthonormal frames of $Z$ restricted to $M$, by

\[
\Phi : P_{SO M} \longrightarrow P_{SO Z|_M} \\
(e_1, \ldots, e_n) \longrightarrow (\nu, e_1, \ldots, e_n).
\]  \hspace{1cm} (13)

The isomorphism $\alpha$, defined in $\S$ yields the following commutative diagram:

\[
\begin{array}{c}
\text{Spin}^c(r, s) \\
\xi^c
\end{array} \quad \Leftarrow \quad \begin{array}{c}
\text{Spin}^c(r + 1, s) \\
\xi^c
\end{array}
\begin{array}{c}
\text{SO}(r, s) \times \mathbb{S}^1 \\
\rightarrow
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\text{SO}(r + 1, s) \times \mathbb{S}^1 \\
\rightarrow
\end{array}
\]

where the inclusion of $\text{SO}(r, s)$ in $\text{SO}(r + 1, s)$ is that which fixes the first basis vector under the action of $\text{SO}(r + 1, s)$ on $\mathbb{R}^{n+1}$. This allows to pull back via $\Phi$ the principal bundle $P_{\text{Spin}^c Z|_M}$ as a Spin$^c$ structure for $M$, denoted by $P_{\text{Spin}^c M}$. Thus, we have the following commutative diagram:

\[
\begin{array}{c}
P_{\text{Spin}^c M} \quad \longrightarrow \quad P_{\text{Spin}^c Z|_M}
\end{array} \\
\begin{array}{c}
P_{SO M} \times_M P_{\mathbb{S}^1 Z|_M} \quad \longrightarrow \quad P_{SO Z|_M} \times_M P_{\mathbb{S}^1 Z|_M}
\end{array}
\]

The Spin$^c(r, s)$-principal bundle $(P_{\text{Spin}^c M}, \pi, M)$ and the $\mathbb{S}^1$-principal bundle $(P_{\mathbb{S}^1 M} =: P_{\mathbb{S}^1 Z|_M}, \pi, M)$ define a Spin$^c$ structure on $M$. Let $\Sigma Z$ be the spinor bundle on $Z$,

\[\Sigma Z = P_{\text{Spin}^c Z} \times_{\rho_{n+1}} \Sigma_{r+1, s},\]

where $\rho_{n+1}$ stands for the spinorial representation of Spin$^c(r + 1, s)$. Moreover, for any spinor $\psi = [b \times s, \sigma] \in \Sigma Z$ we can always assume that $pr_1 \circ \Theta(b \times s) = b$ is a local section of $P_{SO Z}$ with $\nu$ for first basis vector where $pr_1$ is the projection into $P_{SO Z}$. Then we have

\[\psi|_M = [\widetilde{b} \times s|_{U \cap M}, \sigma|_{U \cap M}],\]
where the equivalence class is reduced to elements of Spin$^c(r, s)$. It follows that one can realise the restriction to $M$ of the spinor bundle $\Sigma Z$ as

$$\Sigma Z|_M = P_{\text{Spin}^c} M \times_{\rho_{n+1}\circ\chi} \Sigma_{r+1,s}.$$  

If $n = 2m$ is even, it is easy to check that $\chi^{0}_{2m+1} \circ \alpha = \chi^{0}_{2m+1|_{\text{Cl}_{r+1,s}}}$. Hence $\chi^{0}_{2m+1} \circ \alpha$ is an irreducible representation of $\mathbb{C}l_{r,s}$ of dimension $2^m$, as $\chi^{0}_{2m+1|_{\text{Cl}_{r+1,s}}}$, and finally $\chi^{0}_{2m+1} \circ \alpha \cong \chi_{2m}$. We conclude that

$$\rho_{2m+1} \circ \alpha \cong \rho_{2m}, \quad \text{and} \quad \Sigma Z|_M \cong \Sigma M.$$  

If $n = 2m + 1$ is odd, we know that $\chi^{0}_{2m+1}$ is the unique irreducible representation of $\mathbb{C}l_{r,s}$ of dimension $2^m$ for which the action of the complex volume form is the identity. Since $n+1 = 2m+2$ is even, $\Sigma Z$ decomposes into positive and negative parts, $\Sigma^\pm Z = P_{\text{Spin}^c} Z \times_{\rho_{2m+1} \circ \chi} \Sigma_{r+1,s}^\pm$. It is easy to show that $\chi_{2m+2} \circ \alpha = \chi_{2m+2|_{\text{Cl}_{r+1,s}}}$, but $\chi_{2m+2} \circ \alpha$ can be written as the direct sum of two irreducible inequivalent representations, as $\chi_{2m+2|_{\text{Cl}_{r+1,s}}}$. Hence, we have

$$\chi_{2m+2} \circ \alpha = (\chi_{2m+2} \circ \alpha)^+ \oplus (\chi_{2m+1} \circ \alpha)^-,$$

where $(\chi_{2m+2} \circ \alpha)^\pm (\omega_{r,s}) = \pm \text{Id}_{\mathbb{C}l_{r,s}}$. The representation $\chi^{0}_{2m+1}$ being the unique representation of $\mathbb{C}l_{r,s}$ of dimension $2^m$ for which the action of the volume form is the identity, we get $(\chi_{2m+2} \circ \alpha)^+ \cong \chi^{0}_{2m+1}$. Finally,

$$\rho^+_{2m+2} \circ \alpha \cong \rho_{2m+1} \quad \text{and} \quad \Sigma^+ Z|_M \cong \Sigma M.$$  

Now, Equation (12) follows directly from the above identification.

**Remarks 3.2**

1. The algebraic remarks in the previous section show that if $n$ is odd we can also get $\Sigma^\pm Z|_M \cong \Sigma M$, where the Clifford multiplication by a vector field tangent to $M$ is given by $X \cdot \varphi = -(\nu \cdot X \cdot \psi)|_M$.

2. The connection 1-form defined on the restricted $S^1$-principal bundle $(P_{S^1} M =: P_{S^1} Z|_M, \pi, M)$, is given by

$$A^M = A^2|_M : T(P_{S^1} M) = T(P_{S^1} Z)|_M \to i\mathbb{R}.$$  

Then the curvature 2-form $i\Omega^M$ on the $S^1$-principal bundle $P_{S^1} M$ is given by $i\Omega^M = i\Omega^Z|_M$, which can be viewed as an imaginary 2-form on $M$ and hence as the curvature form of the line bundle $L^M$, the restriction of the line bundle $L^Z$ to $M$.

3. For every $\psi \in \Gamma(\Sigma Z)$, $\psi \in \Gamma(\Sigma^+ Z)$ if $n$ is odd, the real 2-forms $\Omega^M$ and $\Omega^Z$ are related by the following formulas:

$$|\Omega^Z|^2 = |\Omega^M|^2 + |\nu_\psi \Omega^Z|^2,$$  

(14)
\[(\Omega^Z \cdot \psi)_{|M} = \Omega^M \cdot \varphi + (\nu_{,\delta} \Omega^Z) \cdot \varphi. \] (15)

In fact, we can write
\[\Omega^Z = \sum_{i=1}^{n} \Omega^Z(\nu, e_i) \nu \wedge e_i + \sum_{i<j} \Omega^Z(e_i, e_j) e_i \wedge e_j = -(\nu_{,\delta} \Omega^Z) \wedge \nu + \Omega^M,\]
which is (14). When restricting the Clifford multiplication of \(\Omega^Z\) by \(\psi\) to the hypersurface \(M\) we obtain
\[(\Omega^Z \cdot \psi)_{|M} = (\nu \cdot (\nu_{,\delta} \Omega^Z) \cdot \psi)_{|M} + (\Omega^M \cdot \psi)_{|M} = (\nu_{,\delta} \Omega^Z) \cdot \varphi + \Omega^M \cdot \varphi. \] (16)

**Proposition 3.3 (The spinorial Gauss formula)** We denote by \(\nabla^{\Sigma Z}\) the spinorial Levi-Civita connection on \(\Sigma Z\) and by \(\nabla^{\Sigma M}\) that on \(\Sigma M\). For all \(X \in \Gamma(TM)\) and for every spinor field \(\psi \in \Gamma(\Sigma Z)\), then
\[(\nabla^{\Sigma Z} \psi)_{|M} = \nabla^{\Sigma M} \psi - \frac{1}{2} W(X) \cdot \varphi, \] (17)
where \(W\) denotes the Weingarten map with respect to \(\nu\) and \(\varphi = \psi_{|M}\). Moreover, let \(D^Z\) and \(D^M\) be the Dirac operators on \(Z\) and \(M\). Denoting by the same symbol any spinor and it’s restriction to \(M\), we have
\[\nu \cdot D^Z \varphi = \tilde{D} \varphi + \frac{i^n}{2} H \varphi - i^n \nabla^{\Sigma Z} \varphi, \] (18)
where \(H = \frac{1}{n} \text{tr}(W)\) denotes the mean curvature and \(\tilde{D} = D^M \oplus (-D^M)\) if \(n\) is even and \(\tilde{D} = D^M \oplus (\nu_{,\delta} \Omega^Z) \oplus (\nu_{,\delta} \Omega^Z) \cdot \varphi. \]

**Proof:** The Riemannian Gauss formula is given, for every vector fields \(X\) and \(Y\) on \(M\), by
\[\nabla^Z X = \nabla^M X + \langle W(X), Y \rangle \nu. \] (19)
Let \((e_1, e_2, ..., e_n)\) a local space and time oriented orthonormal frame of \(M\), such that \(b = (e_0 = \nu, e_1, e_2, ..., e_n)\) is that of \(Z\). We consider \(\psi\) a local section of \(\Sigma Z\), \(\psi = [b \times s, \sigma] \) where \(s\) is a local section of \(P_{\Sigma Z}\). Using (9), (19) and the fact that \(X(\psi)_{|M} = X(\varphi)\) for \(X \in \Gamma(TM)\), we compute for \(j = 1, ..., n\)
\[\left(\nabla^{\Sigma Z}_{e_j} \psi \right)_{|M} = e_j(\varphi) + \frac{1}{4} \sum_{k=0}^{n} \varepsilon_k (e_k \cdot \nabla^Z_{e_j} e_k \cdot \psi)_{|M} + \frac{1}{2} A^Z(s_*(e_j)) \varphi = e_j(\varphi) + \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k (e_k \cdot \nabla^Z_{e_j} e_k \cdot \psi)_{|M} + \frac{1}{4} (\nu \cdot \nabla^Z_{e_j} \nu \cdot \psi)_{|M} + \frac{1}{2} A^M(s_*(e_j)) \varphi = \nabla^Z_{e_j} \varphi + \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k < W(e_j), e_k > (e_k \cdot \nu \cdot \psi)_{|M} - \frac{1}{4} (\nu \cdot W(e_j) \cdot \psi)_{|M} = \nabla^Z_{e_j} \varphi - \frac{1}{2} (\nu \cdot W(e_j) \cdot \psi)_{|M} = \nabla^Z_{e_j} \varphi - \frac{1}{2} W(e_j) \cdot \varphi. \]
Moreover \((D^2 \psi)|_M = i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j} \Sigma Z \psi)|_M + i^s (\nu \cdot \nabla_{\nu} \Sigma Z \psi)|_M\), and by (17),

\[
i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j} \Sigma Z \psi)|_M = i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j} \Sigma M \varphi) - i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (\nu \cdot e_j \cdot W(e_j) \cdot \psi)|_M
\]

\[
= -i^s \nu \cdot \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \nabla_{\nu} \Sigma M \varphi + i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (\nu \cdot e_j \cdot W(e_j) \cdot \psi)|_M
\]

\[
= -\nu \cdot \tilde{D} \varphi - \frac{i}{2} \text{tr}(W)(\nu \cdot \psi)|_M.
\]

**Proposition 3.4** Let \(Z\) be an \((n + 1)\)-dimensional semi-Riemannian \(\text{Spin}^c\) manifold. Assume that \(Z\) carries a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, i.e., the leaves \(M\) are semi-Riemannian hypersurfaces and there exists a vector field \(\nu\) on \(Z\) perpendicular to the leaves such that \(\langle \nu, \nu \rangle = 1\) and \(\nabla_{\nu}^{Z} \nu = 0\). Then the commutator of the leafwise Dirac operator and the normal derivative is given by

\[
i^{-s}[\nabla_{\nu}^{Z}, \tilde{D}] \varphi = \mathfrak{D}^{W} \varphi - \frac{n}{2} \nu \cdot \text{grad}^M (H) \cdot \varphi + \frac{1}{2} \nu \cdot \text{div}^M (W) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu, \Omega^Z) \cdot \varphi.
\]

Here \(\text{grad}^M\) denotes the leafwise gradient, \(\text{div}^M (W) = \sum_{i=1}^n \varepsilon_i (\nabla_{\nu}^M W)(e_i)\) denotes the leafwise divergence of the endomorphism field \(W\) and \(\mathfrak{D}^{W} \varphi = \sum_{i=1}^n \varepsilon_i \nu \cdot e_i \cdot \nabla_{\nu}^{M} \omega_{(e_i)} \varphi\).

**Proof:** We choose a local oriented orthonormal tangent frame \((e_1, \ldots, e_n)\) for the leaves and we may assume for simplicity that \(\nabla_{\nu}^{Z} e_j = 0\). Now, we compute

\[
i^{-s}[\nabla_{\nu}^{Z}, \tilde{D}] \varphi = \sum_{j=1}^n \varepsilon_j \left( \nabla_{\nu}^{Z} (\nu \cdot e_j \cdot \nabla_{e_j} \Sigma M \varphi) - \nu \cdot e_j \cdot \nabla_{e_j} \nabla_{\nu}^{Z} \varphi \right)
\]

\[
= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left( \nabla_{\nu}^{Z} \nabla_{e_j} \Sigma M \varphi - \nabla_{e_j} \nabla_{\nu}^{Z} \varphi \right)
\]

\[
\overset{[17]}{=} \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left[ \nabla_{\nu}^{Z} (\nabla_{e_j}^{Z} + \frac{1}{2} \nu \cdot W(e_j)) \right] \varphi
\]

\[
- (\nabla_{e_j}^{Z} + \frac{1}{2} \nu \cdot W(e_j)) \nabla_{\nu}^{Z} \varphi
\]

\[
= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left( \mathcal{R}^{Z} (\nu, e_j) + \nabla_{\nu}^{Z} (\nabla_{e_j}^{Z} + \frac{1}{2} \nu \cdot (\nabla_{\nu}^{Z} W)(e_j)) \right) \varphi
\]

\[
\overset{[18]}{=} -\frac{1}{2} \nu \cdot \text{Ric}^{Z} (\nu) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu, \Omega^Z) \cdot \varphi
\]
\[\begin{align*}
&= \frac{1}{2} \nu \cdot \text{Ric}^Z(\nu) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi \\
&+ \sum_{j=1}^{n} \varepsilon_j \nu \cdot e_j \cdot \left( \frac{\nabla^M}{\nabla^W(e_j)} - \frac{1}{2} \nu \cdot W^2(e_j) + \frac{1}{2} \nu \cdot (\nabla^W(e_j)) \right) \varphi
\end{align*}\]

The Riccati equation for the Weingarten map \((\nabla^Z_W)(X) = R^Z(X, \nu)\nu + W^2(X)\) yields
\[
i^{-s} [\nabla^Z_\nu, D] \varphi = -\frac{1}{2} \nu \cdot \text{Ric}^Z(\nu) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi + \partial^W \varphi
+ \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j e_j \cdot (R^Z(e_j, \nu)) \varphi
\]

\[
= -\frac{1}{2} \nu \cdot \text{Ric}^Z(\nu) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi + \partial^W \varphi + \frac{1}{2} \text{Ric}^Z(\nu, \nu) \varphi
\]

\[
= \partial^W \varphi - \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j \text{Ric}^Z(\nu, e_j) \nu \cdot e_j \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi. \tag{20}
\]

The Codazzi-Mainardi equation for \(X, Y, V \in TM\) is given by \(\langle R^Z(X, Y)V, \nu \rangle = \langle (\nabla^M_X V)(Y), V \rangle - \langle (\nabla^M_Y V)(X), V \rangle \). Thus,
\[
\text{ric}^Z(\nu, X) = \sum_{j=1}^{n} \varepsilon_j \langle R^Z(X, e_j) e_j, \nu \rangle
= \sum_{j=1}^{n} \varepsilon_j \left( \langle (\nabla^M_X V)(e_j), e_j \rangle - \langle (\nabla^M_{e_j} V)(X), e_j \rangle \right)
= \text{tr}(\nabla^M_X V) - \langle \text{div}^M(W), X \rangle.
\]

Plugging this into (20) we get
\[
i^{-s} [\nabla^Z_\nu, D] \varphi = \partial^W \varphi - \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j \left( \text{tr}(\nabla^M_{e_j} V) - \langle \text{div}^M(W), e_j \rangle \right) \nu \cdot e_j \cdot \varphi
+ \frac{i}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi.
\]

\[
= \partial^W \varphi - \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j e_j (\text{tr}(W)) \nu \cdot e_j \cdot \varphi + \frac{1}{2} \nu \cdot \text{div}^M(W) \cdot \varphi
\]
\[ + \frac{i}{2} \nu \cdot (\nu \Omega^2) \cdot \varphi. \]

\[ = \mathcal{D}^W \varphi - \frac{n}{2} \nu \cdot \text{grad}^M (H) \cdot \varphi + \frac{1}{2} \nu \cdot \text{div}^M (W) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \Omega^2) \cdot \varphi. \]

4 The generalized cylinder on semi-Riemannian Spin\(^c\) manifolds

Let \( M \) be an \( n \)-dimensional smooth manifold and \( g_t \) a smooth 1-parameter family of semi-Riemannian metrics on \( M \), \( t \in I \) where \( I \subset \mathbb{R} \) is an interval. We define the generalized cylinder by

\[ \mathcal{Z} := I \times M, \]

with semi-Riemannian metric \( g_{\mathcal{Z}} := \langle \cdot, \cdot \rangle = dt^2 + g_t \). The generalized cylinder is an \((n + 1)\)-dimensional semi-Riemannian manifold of signature \((r + 1, s)\) if the signature of \( g_t \) is \((r, s)\).

**Proposition 4.1** There is a 1-1-correspondence between the Spin\(^c\) structures on \( M \) and that on \( \mathcal{Z} \).

**Proof:** As explained in Section 3, Spin\(^c\) structures on \( \mathcal{Z} \) can be restricted to Spin\(^c\) structures on \( M \). Conversely, given a Spin\(^c\) structure on \( M \) it can be pulled back to \( I \times M \) via the projection \( pr_2 : I \times M \rightarrow M \) yields a Spin\(^c\) structure on \( \mathcal{Z} \). In fact, the pull back of the Spin\(^c\)(\( r, s \))-principal bundle \( P_{\text{Spin}^c} M \) on \( M \) gives rise to a Spin\(^c\)(\( r, s \))-principal bundle on \( \mathcal{Z} \) denoted by \( P_{\text{Spin}^c} \mathcal{Z} \).

Enlarging the structure group via the embedding \( \text{Spin}^c(\( r, s \)) \hookrightarrow \text{Spin}^c(\( r + 1, s \)), \) which covers the standard embedding

\[ \text{SO}(r, s) \times S^1 \hookrightarrow \text{SO}(r + 1, s) \times S^1 \]

\[ (a, z) \mapsto \left( \begin{array}{cc} 1 & 0 \\ 0 & a \end{array} \right), z, \]
gives a Spin\(^c\)(\( r + 1, s \))-principal fibre bundle on \( \mathcal{Z} \), denoted also by \( P_{\text{Spin}^c} \mathcal{Z} \). The pull back of the line bundle \( L^M \) on \( M \) defining the Spin\(^c\) structure on \( M \), gives a line bundle \( L^Z \) on \( \mathcal{Z} \) such that the following diagram commutes

\[ \begin{array}{ccc}
L^Z = pr_2^*(L^M) & \longrightarrow & L^M \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{Z} = I \times M & \longrightarrow & M
\end{array} \]
The line bundle $L^Z$ on $Z$ and the $\text{Spin}^c(r+1,s)$-principal fibre bundle $P_{\text{Spin}^c}Z$ on $Z$ yields the $\text{Spin}^c$ structure on $Z$ which restricts to the given $\text{Spin}^c$ structure on $M$.

**Remark 4.2** If $M$ is a $\text{Spin}^c$ Riemannian manifold and if we denote by $i\Omega^M$ the imaginary valued curvature on the line bundle $L^M$, we know that there exists a unique curvature 2-form, denoted by $i\Omega^Z$, on the line bundle $L^Z = pr_2^*(L^M)$, defined by $i\Omega^Z = pr_2^*(i\Omega^M)$. Thus we have

$$\Omega^Z(X,Y) = \Omega^M(X,Y) \quad \text{and} \quad \Omega^Z(\nu,Y) = 0$$

for any $X,Y \in \Gamma(TM)$.

**Proposition 4.3** [2] On a generalized cylinder $Z = I \times M$ with semi-Riemannian metric $g^Z = \langle \cdot, \cdot \rangle = dt^2 + g_t$ we define, in every $p \in M$ and $X,Y \in T_pM$, the first and second derivatives of $g_t$ by

$$\dot{g}_t(X,Y) := \frac{d}{dt}(g_t(X,Y)) \quad \text{and} \quad \ddot{g}_t(X,Y) := \frac{d^2}{dt^2}(g_t(X,Y)).$$

Hence the following formulas hold:

$$\langle W(X),Y \rangle = -\frac{1}{2}\dot{g}_t(X,Y), \quad \langle R^Z(U,V)X,Y \rangle = \langle R^M(U,V)X,Y \rangle$$

$$+ \frac{1}{4} \dot{g}_t(U,X)\dot{g}_t(V,Y) - \dot{g}_t(U,Y)\dot{g}_t(V,X),$$

$$\langle R^Z(X,Y)U,\nu \rangle = \frac{1}{2} (\nabla^M_Y \dot{g}_t)(X,U) - (\nabla^M_X \dot{g}_t)(Y,U),$$

$$\langle R^Z(X,\nu)V \rangle = -\frac{1}{2} (\ddot{g}_t(X,Y) + \dot{g}_t(W(X),Y)), \quad (24)$$

where $X,Y,U,V \in T_pM$, $p \in M$.

## 5 The variation formula for the Dirac operator on Spin$^c$ manifolds

First we give some facts about parallel transport on Spin$^c$ manifolds along a curve $c$. We consider a Riemannian Spin$^c$ manifold $N$, we know that there exists a unique correspondence which associates to a spinor field $\psi(t) = \psi(c(t))$ along a curve $c : I \rightarrow N$ another spinor field $\frac{D}{dt}\psi$ along $c$, called the covariant derivative of $\psi$ along $c$, such that

$$\frac{D}{dt}(\psi + \varphi) = \frac{D}{dt}\psi + \frac{D}{dt}\varphi, \quad \text{for any} \ \psi \ \text{and} \ \varphi \ \text{along the curve} \ c,$$

$$\frac{D}{dt}(f\psi) = f\frac{D}{dt}\psi + (\frac{d}{dt}f)\psi, \quad \text{where} \ f \ \text{is a differentiable function on} \ I,$$
A spinor field \( \psi \) along a curve \( c \) is called parallel when \( \frac{d}{dt} \psi(t) = 0 \) for all \( t \in I \). Now, if \( \psi_0 \) is a spinor at the point \( c(t_0), t_0 \in I \), then there exists a unique parallel spinor \( \varphi \) along \( c \), such that \( \psi_0 = \varphi(t_0) \). The linear isomorphism \( \tau_{t_0}^t \) defined by

\[
\tau_{t_0}^t : \Sigma_{c(t_0)} N \longrightarrow \Sigma_{c(t_1)} N
\]

\[
\psi_0 \longrightarrow \varphi(t_1),
\]

is called the parallel transport along the curve \( c \) from \( c(t_0) \) to \( c(t_1) \). The basic property of the parallel transport on a \( \text{Spin}^c \) manifold is the following: Let \( \psi \) be a spinor field on a Riemannian \( \text{Spin}^c \) manifold \( N \), \( X \in \Gamma(TN) \), \( p \in N \) and \( c: I \longrightarrow N \) an integral curve through \( p \), i.e., \( c(t_0) = p \) and \( \frac{d}{dt} c(t) = X(c(t)) \), we have

\[
(\nabla_X^N \psi)_p = \frac{d}{dt} \left( \tau_{t_0}^t(\psi(t)) \right) \big|_{t=t_0}.
\]

(25)

Now, we consider \( g_t \) a smooth 1-parameter family of semi-Riemannian metrics on a \( \text{Spin}^c \) manifold \( M \) and the generalized cylinder \( Z = I \times M \) with semi-Riemannian metric \( g^Z = \langle \cdot, \cdot \rangle = dt^2 + g_t \). For \( t \in I \) we denote by \( M_t \) the manifold \( (M, g_t) \). Let us write “\( - \)” for the Clifford multiplication on \( Z \) and “\( \cdot \)” for that on \( M_t \). Recall from Section 4 that \( \text{Spin}^c \) structures on \( M \) and \( Z \) are in 1-1-correspondence and \( \Sigma Z |_{M_t} = \Sigma M_t \) as hermitian vector bundles if \( n = r + s \) is even and \( \Sigma^+ Z |_{M_t} = \Sigma M_t \) if \( n \) is odd. For a given \( x \in M \) and \( t_0, t_1 \in I \), parallel transport \( \tau_{t_0}^{t_1} \) on the generalized cylinder \( Z \) along the curve \( c: I \rightarrow I \times M, t \rightarrow (t, x) \) is given by

\[
\tau_{t_0}^{t_1} : \Sigma_{c(t_0)} Z \simeq \Sigma_x M_{t_0} \longrightarrow \Sigma_{c(t_1)} Z \simeq \Sigma_x M_{t_1}.
\]

This isomorphism satisfies

\[
\tau_{t_0}^{t_1}(X\bullet_{t_0} \varphi) = (\xi_{t_0}^t X) \bullet_{t_1} (\xi_{t_0}^{t_1} \varphi),
\]

\[
<\tau_{t_0}^{t_1} \psi, \tau_{t_0}^{t_1} \varphi> = <\psi, \varphi>,
\]

where \( \xi_{t_0}^t : T_{(x, t_0)} Z \simeq T_x M_{t_0} \rightarrow T_{(x, t_1)} Z \simeq T_x M_{t_1} \) is the parallel transport on \( Z \) along the same curve \( c \), \( X \in T_x M_{t_0} \) and \( \psi, \varphi \in \Sigma_x M_{t_0} \).

**Theorem 5.1** On a \( \text{Spin}^c \) manifold \( M \), let \( g_t \) be a smooth 1-parameter family of semi-Riemannian metrics. Denote by \( D^{M_t} \) the Dirac operator of \( M_t \), and \( \mathcal{D}^{g_t} = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j g_t(e_i, e_j) e_i \bullet_e \nabla e_j \). Then for any smooth spinor field \( \psi \) on \( M_{t_0} \) we have

\[
\frac{d}{dt} \bigg|_{t=t_0} \tau_{t_0}^t D^{M_t} \tau_{t_0}^t \psi = \frac{1}{2} \mathcal{D}^{g_{t_0}} \psi + \frac{1}{4} \text{grad}^{M_{t_0}} (\text{tr}_{g_{t_0}} (\hat{g}_{t_0})) \bullet_{t_0} \psi - \frac{1}{4} \text{div}^{M_{t_0}} (g_{t_0}) \bullet_{t_0} \psi.
\]
Proof: The vector field \( \nu := \frac{\partial}{\partial t} \) is spacelike of unit length and orthogonal to the hypersurfaces \( M_t := \{ t \} \times M \). Denote by \( W_t \) the Weingarten map of \( M_t \) with respect to \( \nu \) and by \( H_t \) the mean curvature. If \( X \) is a local coordinate field on \( M \), then \( \langle X, \nu \rangle = 0 \) and \( [X, \nu] = 0 \). Thus

\[
0 = d_\nu \langle X, \nu \rangle = \langle \nabla^Z_\nu X, \nu \rangle + \langle X, \nabla^Z_\nu \nu \rangle = \langle \nabla^Z_\nu X, \nu \rangle + \langle X, \nabla^Z_\nu \nu \rangle = - \langle W_t(X), \nu \rangle + \langle X, \nabla^Z_\nu \nu \rangle = \langle X, \nabla^Z_\nu \nu \rangle
\]

and differentiating \( \langle \nu, \nu \rangle = 1 \) yields \( \langle \nu, \nabla^Z_\nu \nu \rangle = 0 \). Hence \( \nabla^Z_\nu \nu = 0 \), i.e., for \( x \in M \) the curves \( t \mapsto (t, x) \) are geodesics parametrized by arclength. So the assumptions of Proposition 3.4 are satisfied for the foliation \( (M_t)_{t \in I} \). By Remark 4.2, the commutator formula of Proposition 3.4 gives for a section \( \varphi \) of \( \Sigma M_t \), (or \( \Sigma^+ M_t \) if \( n \) is odd)

\[
i^{-s}[\nabla^Z_\nu, D^{M_t}] \varphi = \mathcal{D}^W_t \varphi - \frac{n}{2} \text{grad}^{M_t}(H_t) \bullet_t \varphi + \frac{1}{2} \text{div}^{M_t}(W_t) \bullet_t \varphi. \tag{26}
\]

From Proposition 4.3 we deduce

\[
\text{div}^{M_t}(W_t) = -\frac{1}{2} \text{div}^{M_t}(\dot{g}_t), \quad H_t = -\frac{1}{2n} \text{tr}_{g_t}(\dot{g}_t) \quad \text{and} \quad \mathcal{D}^W_t = -\frac{1}{2} \mathcal{D}^{\dot{g}_t}.
\]

Thus (26) can be rewritten as

\[
i^{-s}[\nabla^Z_\nu, D^{M_t}] \varphi = -\frac{1}{2} \mathcal{D}^{\dot{g}_t} \varphi + \frac{1}{4} \text{grad}^{M_t}(\text{tr}_{g_t}(\dot{g}_t)) \bullet_t \varphi - \frac{1}{4} \text{div}^{M_t}(\dot{g}_t) \bullet_t \varphi. \tag{27}
\]

Now if \( \varphi \) is parallel along the curves \( t \mapsto (t, x) \), i.e., it is of the form \( \varphi(t, x) = \tau^{t_0}_t \psi(t_0, x) \) for some spinor field \( \psi \) on \( M_{t_0} \), then using (25) at \( t = t_0 \), the left hand side of (27) could be written as

\[
i^{-s}[\nabla^Z_\nu, D^{M_t}] \varphi = i^{-s} \nabla^Z_\nu D^{M_t} \varphi = i^{-s} \frac{d}{dt} \bigg |_{t=t_0} \tau^{t_0}_t D^{M_t} \varphi
\]

which gives the variation formula for the Dirac operator.

Corollary 5.2 Let \( (M^n, g) \) be a Spin\(^c\) Riemannian manifold, if we consider the family of metrics defined by \( g_t = g + tk \), where \( k \) is a symmetric \((0,2)\)-tensor, we have

\[
\frac{d}{dt} \bigg |_{t=0} \tau^{t}_t D^{M_t} \tau^{-1}_t = -\frac{1}{2} \mathcal{D}^{k} \psi + \frac{1}{4} \text{grad}^{M}(\text{tr}_{g}(k)) \cdot \psi - \frac{1}{4} \text{div}^{M}(k) \cdot \psi, \tag{29}
\]

where \( \cdot = \bullet_{t=0} \) is the Clifford multiplication on \( M \).

This formula has been proved in [4], Theorem 21 for spin Riemannian manifolds and in [2] for spin semi-Riemannian manifolds.
6 Energy-Momentum tensor on Spin$^c$ manifolds

In this section we study the Energy-Momentum tensor on Spin$^c$ Riemannian manifolds from a geometric point of view. We begin by giving the proofs of Proposition 1.1, Theorem 1.2 and Proposition 1.3.

Proof of Proposition 1.1: Using Equation (29) we calculate
\[
\frac{d}{dt}igg|_{t=0} (\tau^0 D^t \tau^0 \psi, \psi)_{g_t} = \frac{d}{dt}igg|_{t=0} (D^t \tau^0 \psi, \tau^0 \psi)_{g_t}
\]
\[
= -\frac{1}{2} (D^k \psi, \psi)_g
\]
\[
= -\frac{1}{2} \sum_{i,j} k(e_i, e_j)(e_i \cdot \nabla^M e_j \psi, \psi)
\]
\[
= -\frac{1}{2} \int_M <k, T_\psi > dv_g.
\]

Proof of Theorem 1.2: The Proof of this Theorem will be omitted since it is similar to the one given by Friedrich and Kim in [8] for spin manifolds.

Proof of Proposition 1.3: Let $\psi$ be any parallel spinor field on $Z$. Then Equation (17) yields
\[
\nabla_X^M \phi = \frac{1}{2} W(X) \cdot \phi.
\] (30)

Let $(e_1, ..., e_n)$ be a positively oriented local orthonormal basis of $TM$. For $j = 1, ..., n$ we have
\[
\nabla_{e_j}^M \phi = \frac{1}{2} \sum_{k=1}^n W_{jk} e_k \cdot \phi.
\]

Taking Clifford multiplication by $e_i$ and the scalar product with $\phi$, we get
\[
\text{Re}(e_i \cdot \nabla_{e_j}^M \phi, \phi) = \frac{1}{2} \sum_{k=1}^n W_{jk} \text{Re}(e_i \cdot e_k \cdot \phi, \phi).
\]

Since $\text{Re}(e_i \cdot e_k \cdot \phi, \phi) = -\delta_{ik} |\phi|^2$, it follows, by the symmetry of $W$
\[
\text{Re}(e_i \cdot \nabla_{e_j}^M \phi + e_j \cdot \nabla_{e_i}^M \phi, \phi) = -W_{ij} |\phi|^2.
\]

Therefore, $2\ell^\phi = -W$. Using Equation (18) it is easy to see that $\phi$ is an eigenspinor associated with the eigenvalue $-\frac{n}{2} H$ of $D$. Since $\text{Scal}^Z = \text{Scal}^M + 2 \rhoic^Z(\nu, \nu) - n^2 H^2 + |W|^2$ we get
\[
\frac{1}{4} (\text{Scal}^M - c_n |\Omega^M|)^2 + |T^\phi|^2 = \frac{1}{4} (\text{Scal}^Z - 2 \rhoic^Z(\nu, \nu) - c_n |\Omega^M|) + n^2 \frac{H^2}{4}
\]
\[
= n^2 \frac{H^2}{4},
\]
hence $M$ satisfies the equality case in (3) if and only if (6) holds.

**Corollary 6.1** Under the same conditions as Proposition 1.3, if $n = 2$ or 3, the hypersurface $M$ satisfies the equality case in (3) if $\text{Ric}^Z(\nu) = 0$ and $\text{Scal}^Z \geq 0$.

**Proof:** Since $Z$ has a parallel spinor, we have (see [7])

$$|\text{Ric}^Z(\nu)| = |\nu \cdot \Omega^Z|,$$

(31)

$$i(Y \cdot \Omega^Z) \cdot \psi = \text{Ric}^Z(Y) \cdot \psi \quad \text{for every} \quad Y \in \Gamma(\Sigma Z).$$

(32)

For $Y = e_j$ in Equation (32) then taking Clifford multiplication by $e_j$ and summing from $j = 1, ..., n + 1$, we get

$$i \sum_{j=1}^{n+1} e_j \cdot (e_j \cdot \Omega^Z) \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot \text{Ric}^Z(e_j) \cdot \psi = -\text{Scal}^Z \psi.$$ But $2 \Omega^Z \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot (e_j \cdot \Omega^Z) \cdot \psi$, hence we deduce that $\Omega^Z \cdot \psi = i\frac{\text{Scal}^Z}{2} \psi$. By (31) and (15) we obtain $\Omega^M \cdot \varphi = i\frac{\text{Scal}^Z}{2} \varphi$. Since $n = 2$ or 3 we have $|\Omega^M| = \frac{\text{Scal}^Z}{2}$ and Equation (6) is satisfied.

**Corollary 6.2** Under the same conditions as Proposition 1.3 if the restriction of the complex line bundle $L^Z$ is flat, i.e., $L^M$ is a flat complex line bundle ($\Omega^M = 0$), the hypersurface $M$ is a limiting manifold for (3).

**Proof:** Since $\Omega^M = 0$, Equation (15) yields $i \frac{\text{Scal}^Z}{2} \varphi = \Omega^Z \cdot \psi |_M = (\nu \cdot \Omega^Z) \cdot \varphi$. But,

$$i(\nu \cdot \Omega^Z) \cdot \varphi = i(\nu \cdot (\nu \cdot \Omega^Z) \cdot \psi) |_M = (\nu \cdot \text{Ric}^Z(\nu) \cdot \psi) |_M$$

$$= -\text{ric}^Z(\nu, \nu) \varphi + \sum_{j=1}^{n} \text{ric}^Z(\nu, e_j) e_j \cdot \varphi.$$ (33)

Taking the real part of the scalar product of Equation (33) with $\varphi$ yields $\frac{\text{Scal}^Z}{2} = \text{ric}^Z(\nu, \nu)$, hence Equation (6) is satisfied.

Now, let $M$ be a Spin$^c$ Riemannian manifold having a generalized Killing spinor field $\varphi$ with a symmetric endomorphism $F$ on the tangent bundle $TM$. As mentioned in the introduction, it is straightforward to see that $2T^\varphi(X, Y) = -\langle F(X), Y \rangle$. We will study these generalized Killing spinors when the tensor $F$ is a Codazzi-Mainardi tensor, i.e., $F$ satisfies

$$(\nabla^M_X F)(Y) = (\nabla^M_Y F)(X) \quad \text{for} \quad X, Y \in \Gamma(TM).$$

(34)

For this, we give the following lemma whose proof will be omitted since it is similar to Lemma 7.3 in [2].
Lemma 6.3 \[2\] Let \( g_t \) be a smooth 1-parameter family of semi-Riemannian metrics on a Spin\(^c\) manifold \((M^n, g = g_0)\) and let \( F \) be a field of symmetric endomorphisms of \( TM \). We consider the metric \( g_Z = \langle \cdot, \cdot \rangle = dt^2 + g_t \) on \( Z \) such that \( g_t(X, Y) = g((Id - tF)^2 X, Y) \) for all vector fields \( X, Y \) on \( M \). We have \( \langle R^Z(U, \nu) \nu, V \rangle = 0 \) for all vector fields \( U, V \) tangent to \( M \) and if \( F \) satisfies the Codazzi-Mainardi equation then \( \langle R^Z(U, V) W, \nu \rangle = 0 \) for all \( U, V \) and \( W \) on \( Z \).

Proof of Theorem 1.4 We define \( \psi_{(0, x)} := \varphi_x \), via the identification \( \Sigma_x M \cong \Sigma_{(0, x)} Z \) (resp. \( \Sigma^+_{(0, x)} Z \) for \( n \) odd) and \( \psi_{(t, x)} = \tau_t^1 \psi_{(0, x)} \). By Equation (21), the endomorphism \( F \) is the Weingarten tensor of the immersion of \( \{0\} \times M \) in \( Z \) and hence by construction we have for all \( X \in \Gamma(TM) \)

\[
\nabla^Z_X \psi |_{\{0\} \times M} = 0 \quad \text{and} \quad \nabla^Z_\nu \psi = 0. \tag{35}
\]

Since the tensor \( F \) satisfies the Codazzi-Mainardi equation, Lemma 6.3 yields for all \( U, V \) and \( W \in \Gamma(Z) \) and \( g_Z(R^Z(X, \nu) \nu, Y) = 0 \) for all \( X \) and \( Y \) tangent to \( M \). Hence \( R^Z(\nu, X) = 0 \) for all \( X \in \Gamma(TM) \). Let \( X \) be a fixed arbitrary tangent vector field on \( M \). Using (10) and (35) we get

\[
\nabla^Z_\nu \nabla^Z_X \psi = R^Z(\nu, X) \psi = \frac{1}{2} R^Z(X, \nu) \cdot \psi + \frac{i}{2} \Omega^Z(X, \nu) \psi = 0.
\]

Thus showing that the spinor field \( \nabla^Z_X \psi \) is parallel along the geodesics \( \mathbb{R} \times \{x\} \). Now (35) shows that this spinor vanishes for \( t = 0 \), hence it is zero everywhere on \( Z \). Since \( X \) is arbitrary, this shows that \( \psi \) is parallel on \( Z \).

Corollary 6.4 Let \((M^3, g)\) be a compact, oriented Riemannian manifold and \( \varphi \) an eigenspinor associated with the first eigenvalue \( \lambda_1 \) of the Dirac operator such that the Energy-Momentum tensor associated with \( \varphi \) is a Codazzi tensor. \( M \) is a limiting manifold for (3) if and only if the generalized cylinder \( Z^4 \), equipped with the Spin\(^c\) structure arising from the given one on \( M \), is Kähler of positive scalar curvature and the immersion of \( M \) in \( Z \) has constant mean curvature \( H \).

Proof: First, we should point out that every 3-dimensional compact, oriented, smooth manifold has a Spin\(^c\) structure. Now, if \( M^3 \) is a limiting manifold for (3), by Theorem 1.4 the generalized cylinder has a parallel spinor whose restriction to \( M \) is \( \varphi \). Since \( Z \) is a 4-dimensional Spin\(^c\) manifold having parallel spinor, \( Z \) is Kähler (11). Moreover, using (15), we have

\[
\Omega^M \cdot \varphi = \frac{i}{2} \frac{\text{Scal}^Z}{2} \varphi = \frac{i}{2} \frac{\text{Ric}^Z}{2} = \frac{1}{n} \lambda_1 \varphi,
\]

so \( \text{Scal}^Z \geq 0 \). Finally, \( H = \frac{1}{n} \text{tr}(W) = \frac{1}{n} \text{tr}(-2T^c) = -\frac{2}{n} \lambda_1 \), which is a constant. Now if the generalized cylinder is Kähler and \( M \) is a compact hypersurface of constant mean curvature \( H \), thus \( M \) is compact hypersurface immersed in a Spin\(^c\) manifold having parallel spinor with constant mean curvature. Since \( \text{Scal}^Z \geq 0 \) and \( \nu \cdot \Omega^Z = \text{Ric}^Z(\nu) = 0 \), Corollary 6.1 gives the result.
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