Photon inner-product and the Gauss linking number

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Abstract

It is shown that there is an interesting interplay between self-duality, loop representation and knot invariants in the quantum theory of Maxwell fields in Minkowski space-time. Specifically, in the loop representation based on self-dual connections, the measure that dictates the inner-product can be expressed in terms of the Gauss linking number of thickened loops.

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I. INTRODUCTION

Source-free electrodynamics is the simplest physical theory based on connections. From a geometric point of view, a natural set of observables of this theory is given by holonomies of the connection around closed loops. It is then natural to ask if topological invariants associated with loops play a physically significant role in this description. For the observable algebra, the answer is in the affirmative: the fundamental Heisenberg uncertainty relations can be formulated in terms of the Gauss-linking number (see, e.g., [1,2] and references contained therein). One can ask if such a topological invariant of loops also plays a role in the description of the Hilbert space of quantum states. The purpose of this note is to show that the answer is again in the affirmative. Furthermore, the specific calculation we wish to present is based on an interesting interplay between self-duality, loop representation and knot theory and may well be a reflection of a deeper structure that underlies these three notions.

The basic idea is the following. The standard Fock description of photons can be reformulated in terms of loops, so that the states can be regarded as functionals of loops (rather than connections) (see, e.g., [3] and chapter 14 in [4]). There are, however, several such loop representations. In the one most directly related to the Fock-Bargmann representation [3], it is the negative frequency electric field that is diagonal and the conjugate operator represents the holonomy of positive frequency connections. Alternatively, one can work with real electric fields and connections. But then to obtain the Fock representation, the loops have to be thickened. In this paper, we will work with yet another choice: our loop representation will be based on real electric fields but self-dual connections (without any reference to positive and negative frequencies). Again, to get the Fock representation, we will have to thicken our loops. Thus, quantum states are expressed as functionals of thickened loops and the basic operators are the holonomies of self-dual connections and real electric fields. The measure that dictates the inner-product in this representation has a Gaussian form where the exponent is given by the self-linking number of thickened loops.
More precisely, the situation is the following. Given a loop $\alpha$ and a weighting function $f$ on $R^3$, we can define a “canonical thickening” $\alpha_f$ of the loop. The self-linking number of a thickened loop can be computed by adding the Gauss linking numbers of the loops involved in the thickening with weights given by $f$. The measure that dictates the inner product is just the exponential of this self-linking number of $\alpha_f$. Now, as is well-known, the Fock inner product depends on the Minkowski metric. It is quite interesting that one can put all the information about the space-time metric in the construction that associates quantum states with functionals of loops and then express the inner-product itself in terms of the Gauss linking number which is a topological invariant. A further striking fact is that this “coding” of the inner-product information in a topological invariant works only in the loop representation based on self-dual (or anti-self-dual) connections.

The plan of the paper is as follows. In section II, we will collect some mathematical preliminaries. These are used in section III to construct the loop representation based on self-dual connections. The main result then follows in section IV. Since our primary goal is to bring out the interplay between self-duality of connections, loop representations and the linking number, we will keep the functional analytic details to a minimum. However, it should be rather straightforward to see how one can complete our discussion to obtain a rigorous treatment. Throughout the article, we use units where $c = 1$, but write $\hbar$ and $\varepsilon$ explicitly.

II. MATHEMATICAL PRELIMINARIES

This section is divided into two parts. In the first, we recall the phase space formulation of the Maxwell field using self-dual variables and in the second we introduce the notion of “form factors” associated with loops and thickened loops.
A. Self-Dual Variables for the Maxwell Field

Let us begin with a brief summary of the standard phase space formulation of Maxwell fields. Denote by $\Sigma$ a space-like three-plane in Minkowski space, and by $q_{ab}$, the induced positive definite (flat) metric thereon. The configuration variable for the Maxwell field is generally taken to be the connection one-form $A_a(x)$ (the vector potential for the magnetic field) on $\Sigma$. Its canonically conjugate momentum is the electric field $E^a(x)$ on $\Sigma$. ($E^a(x)$ naturally arises as a vector density. However, since we have an underlying metric, $q_{ab}$, which can be used to add or remove density weights, we will ignore density weights in this paper.)

The fundamental Poisson bracket is:

$$\{A_a(x), E^b(y)\} = \delta^b_a \delta^3(x,y) \quad (2.1)$$

The system has one first class constraint, $\partial_a E^a(x) = 0$. One can therefore pass to the reduced phase space by fixing transverse gauge. The true degrees of freedom are then contained in the pair $(A^T_a(x), E^T_a(x))$ of transverse (i.e. divergence-free) vector fields on $\Sigma$. Denote by $\Gamma$ the phase space spanned by these fields. On $\Gamma$, the only non-vanishing fundamental Poisson bracket is:

$$\{A^T_a(x), E^T_b(y)\} = \delta^b_a \delta^3(x,y) - \Delta^{-1} \partial^b \partial_a \delta^3(x,y), \quad (2.2)$$

where $\Delta$ is the Laplacian operator compatible with the flat metric $q_{ab}$. It is convenient to write $A^T_a(x)$ and $E^T_a(x)$ in terms of their Fourier decomposition. Then, the true degrees of freedom are contained in the new dynamical variables $q_j(k), p_j(k)$ with $j = 1, 2$:

$$A^T_a(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \, e^{i k \cdot x} \left( q_1(k)m_a(k) + q_2(k)\bar{m}_a(k) \right) \quad (2.3)$$

$$E^T_a(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \, e^{i k \cdot x} \left( p_1(k)m^a(k) + p_2(k)\bar{m}^a(k) \right) \quad (2.4)$$

where $m_a$ and $\bar{m}_a$ are transverse (complex) vectors satisfying: $m_a k^a = 0$, and $m_a \bar{m}^a = 1$. The Poisson brackets (2.2) for the transverse components are,

$$\{q_i(-k), p_j(k')\} = -\delta_{ij} \delta^3(k,k'), \quad (2.5)$$
while the fact that $A^T_a(x)$ and $E^a_T(x)$ are real translates to the “reality conditions”:

$$\bar{q}_i(k) = q_i(-k) \quad \text{and} \quad \bar{p}_i(k) = p_i(-k). \quad (2.6)$$

In order to construct the self dual connection, we will use $d^T_a(x)$, the transverse vector potential of the electric field ($E^a_T(x) = \epsilon^{abc} \partial_b \bar{e}^c_T(x)$)

$$d^T_a(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{|k|} e^{ik \cdot x} (p_1(k)\bar{m}_a(k) - p_2(k)\bar{m}_a(k)) \quad (2.7)$$

Let us define the self dual connection as

$$\Uparrow A^T_a(x) := -iA^T_a(x) + d^T_a(x) \quad (2.8)$$

We want to use the pair $(\Uparrow A^T_a(x), E^a_T(x))$ as the basic variables. In terms of the $(q_j, p_j)$ coordinates, the self dual connection takes the form,

$$\Uparrow A^T_a(x) = -\frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{|k|} e^{ik \cdot x} [(-p_1(k) + i|k|q_1(k))\bar{m}_a(k) + (p_2(k) + i|k|q_2(k))\bar{m}_a(k)]$$

$$= -\frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{|k|} e^{ik \cdot x} [z_1(k)\bar{m}_a(k) + z_2(k)\bar{m}_a(k)] \quad (2.9)$$

where

$$z_1(k) := -p_1(k) + i|k|q_1(k) \quad \text{and} \quad z_2(k) := p_2(k) + i|k|q_2(k) \quad (2.10)$$

The basic Poisson brackets for the pairs $(z_i(k), p_j(k))$, – the Fourier components of the self-dual connection and the real electric field– are:

$$\{p_i(k), z_j(-k')\} = i|k|\delta_{ij}\delta^3(k, k') \quad (2.11)$$

and the “reality conditions” $(2.6)$ now become:

$$z_1(-k) + \bar{z}_1(k) = -2p_1(-k) \quad \text{and} \quad z_2(-k) + \bar{z}_2(k) = 2p_2(-k) \quad (2.12)$$

Finally, for later convenience, let us examine the self-dual magnetic field $B^a := \epsilon^{abc} \partial_b \Uparrow A_c$.

Its Fourier components of the magnetic field are given by

$$B_1(k) = -z_1(k), \quad \text{and} \quad B_2(k) = z_2(k) \quad (2.13)$$
The reality conditions for the magnetic field read

$$B_j(k) + B_j(-k) = 2p_j(k) \quad (2.14)$$

These conditions will play an important role in the selection of the inner-product in the loop representation.

B. Loops

Let us begin with some definitions. By a loop we shall mean a continuous and piecewise smooth mapping \( \gamma \) from \( S^1 \) to \( \Sigma \), where \( s \in [0, 2\pi] \). Two loops \( \gamma \) and \( \beta \) will be said to be holonomically equivalent if, for every smooth connection \( A_a \), we have \( \oint A_a ds^a = \oint A_a ds^a \). It turns out that two holonomically equivalent loops, \( \gamma \) and \( \beta \), can differ from each other only through: i) reparametrization, \( \gamma(s) = \beta(s') \) for some (orientation-preserving) reparametrization \( s \to s' \) of the curve \( \beta(s) \); ii) retracing identity, \( \gamma = l \cdot \beta \cdot l^{-1} \), where \( l \) is a line segment and \( \cdot \) indicates composition of segments \([5]\). Each equivalence class will be referred to as a holonomic loop. Since loops will primarily enter our discussions through holonomies, it is these equivalence classes –rather than individual loops– that will be directly relevant to our discussion. To keep the notation simple, we will use the same symbols –say \( \gamma \)– to denote both an individual loop and the holonomic loop it defines; the context should suffice to resolve the resulting ambiguity.

An analytic characterization of holonomic loops can be given through certain distributional vector densities, called form factors. Given a loop \( \alpha \), its form factor, \( F^a(\alpha, x) \), is defined via:

$$\int d^3x F^a(\alpha, x) w_a(x) = \oint_{\alpha} w_a ds^a \quad (2.15)$$

Thus, \( F^a(\alpha, x) \) may be more directly expressed as

$$F^a(\alpha, x) = \oint_{\alpha} ds \dot{\alpha}^a(s) \delta^3(x, \alpha(s)) \quad (2.16)$$
where \( \alpha(s) \) is a point on the loop \( \alpha \) at parameter value \( s \) and \( \dot{\alpha}^a(s) \) the tangent vector to \( \alpha \) at \( \alpha(s) \). Note that the form factor \( F^a(\alpha, x) \) is automatically divergence free,

\[
\partial_a F^a(\alpha, x) = 0 ,
\]

because \( \oint_\alpha \partial_a f \, ds^a = 0 \). It is often convenient to perform a Fourier transform to obtain the momentum space representation of \( F^a(\alpha, x) \). We have:

\[
F^a(\alpha, k) := \frac{1}{(2\pi)^{3/2}} \int d^3 x \, e^{-i k \cdot x} F^a(\alpha, x) = \frac{1}{(2\pi)^{3/2}} \oint_\alpha ds \, \dot{\alpha}^a(s) e^{-i k \cdot \alpha(s)}
\]

(2.18)

Let us note a few properties of these form factors. First, two loops \( \alpha \) and \( \beta \) will have the same form factors if and only if they are holonomic. Thus, \( F^a(\alpha, x) \) can be used to characterize holonomic loop \( \alpha \). Next, since \( F^a(\alpha, x) \) is divergence-free its Fourier transform is transverse \( (k_a F^a(\alpha, k) = 0) \). We can write the two independent components as:

\[
F_1(\alpha, k) \equiv F^+(\alpha, k) = \frac{1}{(2\pi)^{3/2}} \oint_\alpha ds \, \dot{\alpha}^a(s) \bar{m}_a(k) e^{-i k \cdot \alpha(s)}
\]

\[
F_2(\alpha, k) \equiv F^-(\alpha, k) = \frac{1}{(2\pi)^{3/2}} \oint_\alpha ds \, \dot{\alpha}^a(s) m_a(k) e^{-i k \cdot \alpha(s)}
\]

(2.19)

so that

\[
F^a(\alpha, k) = F^+(\alpha, k) m^a(k) + F^-(\alpha, k) \bar{m}^a(k)
\]

(2.20)

(We have introduced the \( \pm \) notation because in the quantum theory, \( F_1 \) will capture positive helicity and \( F_2 \) the negative.) This transversality of form factors will play an important role in the loop-representation because it captures in a natural way the gauge invariance of the theory, i.e. the transversality of the photon. The next property follows from the fact that \( F^a(\alpha, x) \) is real. Consequently, its Fourier transform \( F^a(\alpha, k) \) satisfies the “reality condition”

\[
\overline{F_j(\alpha, k)} = -F_j(\alpha, -k)
\]

(2.21)

Finally, given two holonomic loops \( \alpha \) and \( \beta \), we define a new holonomic loop, \( \alpha \# \beta \) as follows: \( \alpha \# \beta = l \cdot \alpha \cdot l^{-1} \cdot \beta \) where \( l \) is any line segment joining a point on \( \alpha \) to a point on \( \beta \). (Because
of its geometric picture $\alpha \# \beta$ is sometimes called the “eye-glass loop.”) Using the definition of the form factors, we now have

$$F_j(\alpha \# \beta, k) = F_j(\alpha, k) + F_j(\beta, k)$$

(2.22)

In order to construct the quantum theory in the loop representation, we will need to thicken the loops appropriately. We will conclude this section by indicating how this can be done. Fix an averaging function $f_r(x)$ such that $\int_{\mathbb{R}^3} d^3 x f_r(x) = 1$, and goes to a delta function when $r \to 0$. A convenient choice is:

$$f_r(y) = \frac{1}{(2\pi r)^{3/2}} \exp \left( -\frac{y^2}{2r} \right)$$

(2.23)

To make the discussion concrete, we will make this choice and thus characterize the thickening completely by a real parameter $r$. (However, the generalization to arbitrary smearing functions is obvious.) Now, given a loop $\alpha$ we take the loop $\alpha + y$ obtained by rigidly shifting the loop by the vector $y^a$,

$$(\alpha + y)^a(s) = \alpha^a(s) + y^a$$

(2.24)

Next, we can average over $y$ using the weight $f_r(y)$ and define a “smeared form factor” via:

$$F^a_r(\alpha, x) := \int d^3 y f_r(y) F^a(\alpha + y, x)$$

$$= \int d^3 y f_r(y) \oint_{\alpha} ds \dot{\alpha}^a(s) \delta^3(x - \alpha(s) - y)$$

$$= \oint_{\alpha} ds \dot{\alpha}^a(s) f_r(x - \alpha(s))$$

(2.25)

Its Fourier transform $F^a_r(\alpha, k)$ satisfies

$$F^a_r(\alpha, k) = \exp \left( -\frac{r^2 k^2}{2} \right) F^a(\alpha, k)$$

(2.26)

We will see that these $F^a_r(\alpha, k)$ can be used as “generalized coordinates” for loops. More precisely, once the weight functions $f_r(y)$ are chosen, we can associate with any loop a transverse, smooth vector field,

$$\alpha \to F^a(k) := F^a_r(\alpha, k).$$

(2.27)
As is well-known, photon states can be expressed as suitable functionals, Φ[F], of smooth vector fields \( F^a(k) \) (in the representation in which the electric field is diagonal). These can be pulled-back to the loop space to yield functionals \( \Psi_r(\alpha) = \Phi[F]|_{F=F_r(\alpha,k)} \). Thus, the entire Fock space of photon states can be expressed in terms of suitable functionals of loops. This fact will be exploited in the next section.

**III. QUANTUM THEORY**

This section is divided into three parts. In the first, we recall a general quantization program (for details, see [6], [7]), in the second we construct a \( \ast \)-algebra of operators based on loop variables and in the third we construct the loop representation.

**A. Quantization Program**

Consider a classical system with phase space \( \Gamma \). To construct the quantum theory, we can proceed in the following steps.

i) Choose a subspace \( \mathcal{S} \) of the space of complex valued functions on \( \Gamma \) which is closed under the Poisson bracket operation and large enough so that any well behaved function on \( \Gamma \) can be expressed as (possible the limit of) a sum of products of elements of \( \mathcal{S} \). Elements of \( \mathcal{S} \) are called *elementary classical variables* and are to have unambiguous quantum analogs.

ii) Associate with each \( f \) in \( \mathcal{S} \) an *elementary quantum operator* \( \hat{f} \) and consider the free associative algebra generated by these abstract operators. Impose on this algebra the (generalized) canonical commutation relations

\[
[\hat{f}, \hat{g}] = i\hbar \{ f, g \}
\]

(3.1)

for all \( f \) and \( g \) in \( \mathcal{S} \). In addition, if the set \( \mathcal{S} \) is over-complete, impose on the algebra also ‘anti-commutation relations’, namely the relations that capture the algebraic relations that exist between elements of \( \mathcal{S} \). For instance if \( f, g \) and \( h = fg \) are all in \( \mathcal{S} \), then \( \hat{f} \cdot \hat{g} + \hat{g} \cdot \hat{f} = 2\hbar \).

Denote the resulting associative algebra by \( \mathcal{A} \).
iii) Introduce an involution, $\ast$, on $\mathcal{A}$ by setting

$$(\hat{f})^\ast = \hat{f}$$

(3.2)

for all elementary variables $f$ (the bar denotes complex conjugation as before) and requiring that $\ast$ satisfies the defining properties of an involution: $(\hat{A} + \lambda \hat{B})^\ast = \hat{A}^\ast + \bar{\lambda} \hat{B}^\ast$; $(\hat{A}\hat{B})^\ast = \hat{B}^\ast \hat{A}^\ast$ and $(\hat{A}^\ast)^\ast = \hat{A}$, for all $\hat{A}, \hat{B}$ in $\mathcal{A}$ and complex numbers $\lambda$. Denote the resulting $\ast$-algebra by $\mathcal{A}^\ast$.

iv) Choose a linear representation of $\mathcal{A}$ on a complex vector space $V$. (The $\ast$-relations are ignored at this step).

v) Introduce on $V$ an inner product $\langle \cdot, \cdot \rangle$ so that the “quantum reality conditions” are satisfied

$$\langle \Psi, \hat{A}\Phi \rangle = \langle \hat{A}^\ast \Psi, \Phi \rangle$$

(3.3)

for all $\Phi, \Psi$ in $V$ and $\hat{A}$ in $\mathcal{A}^\ast$. Thus, it is the $\ast$-relations that are to select the inner product.

The program requires two external inputs: the choice of $\mathcal{S}$ in step (i) and the choice of the carrier space $V$ of the representation in step (iv). If the choices are viable, i.e. if the program can be completed at all, the resulting inner product is unique on each irreducible sector of the representation of $\mathcal{A}$ on $V$. In the framework of this program, the textbook treatments of field theories correspond to choosing for elements of $\mathcal{A}$ the smeared field operators, and, for $V$, the Fock space or, alternatively, suitable functionals of fields. In the loop quantization, on the other hand, one changes this strategy. both $\mathcal{S}$ and $V$ are now constructed from holonomic loops.

B. Algebra based on loop variables

Let us now implement this program for the Maxwell field using loop variables. Let us define the smeared holonomy of self-dual connections as:

$$h_{\nu}[\alpha] := \exp \left( \frac{1}{8} \int d^3x \int_\alpha \, ds \, \hat{\alpha}^a(s) \, \mathcal{A}_a(x) \, f_\nu(x - \alpha) \right)$$

$$= \exp \left( \frac{1}{8} \int d^3x \, F^a_\nu(\alpha, x) \, \mathcal{A}_a(x) \right)$$

(3.4)
or equivalently,
\[
h_r[\alpha] = \exp \left[ -\frac{1}{\epsilon} \int \frac{d^3k}{|k|} (z_1(k) \bar{F}_1(k) + z_2(k) \bar{F}_2(k)) \exp \left( -\frac{\epsilon^2 k^2}{2} \right) \right].
\] (3.5)

Being a function of the self dual connection it can be regarded as a “configuration variable”. As a momentum variable we will take the (real) electric field \( E^a(x) \), or its Fourier transform \( E^a(k) \). (Strictly speaking we should take the smeared observable \( E[f] = \int_\Sigma E^a f_a d^3x \), but this smearing will not be relevant for our results.) Hence, \( h_r[\alpha] \) and \( E^a(k) \) provide us with a (over-) complete coordinatization of the phase space. The space \( S \) of elementary classical variables required in the first step of the quantization program shall be the vector space generated by the \( h_r[\alpha] \) and \( E^a(k) \). It is closed under Poisson-bracket operation because
\[
\{ h_r[\alpha], E^a(k) \} = \frac{i}{\epsilon} F^a_r(\alpha, k) h_r[\alpha]
\] (3.6)

The next step in the quantization program is the construction of the algebra \( \mathcal{A} \) of quantum operators. Let us associate with each \( h_r[\alpha] \) in \( S \) an operator \( \hat{h}_r[\alpha] \) and with each \( E^a(k) \) an operator \( \hat{E}^a(k) \) and consider the associative algebra generated by finite sums of products of these elementary quantum operators. On this algebra impose the commutation relations:
\[
\left[ \hat{h}_r[\alpha], \hat{h}_r[\beta] \right] = 0 \quad ; \quad \left[ \hat{E}^a(k), \hat{E}^a(k') \right] = 0
\]
\[
\left[ \hat{h}_r[\alpha], \hat{E}^a(k) \right] = -\frac{\hbar}{\epsilon} F^a_r(\alpha, k) \hat{h}_r[\alpha]
\] (3.7)

Furthermore, we must incorporate in this quantum algebra the fact that \( h_r[\alpha] \) is overcomplete. i.e. there are algebraic relations among them; \( h_r[\alpha] h_r[\beta] = h_r[\alpha \# \beta] \). This is achieved by imposing on the algebra the relations \( \hat{h}_r[\alpha] \hat{h}_r[\beta] = \hat{h}_r[\alpha \# \beta] \) for all holonomic loops \( \alpha \) and \( \beta \). The result is the algebra \( \mathcal{A} \) of quantum operators.

C. Loop representation

The next step in the program is to choose a vector space \( V \) and a representation of the quantum operators. The procedure involved is generally exploratory. Thus, one does not
specify all the required regularity conditions right in the beginning; the precise definition of spaces considered becomes clear only at the end of the construction. This will also be the case in our construction.

We wish to choose for $V$ a vector space of suitable functionals of loops. As noted at the end of section II B, in the standard electric field representation, one can choose states as suitably regular functionals $\Phi[F]$ of smooth, vector fields $F^a(k)$ which are transverse, i.e., satisfy $F^a(k)k_a = 0$. Now, in section II B (for each choice of a smearing function $f_r$) we set up a mapping $\alpha \mapsto F^a_r(\alpha, k)$ from loops to smooth transverse vector fields in the momentum space. We can just pull back the functionals $\Phi(F)$ via this map to obtain certain functionals $\Psi(\alpha)$ on the loop space:

$$\Psi(\alpha) = \Phi[F]_{F=F_r(\alpha,k)}.$$  \hspace{2cm} (3.8)

(Using the regularity conditions on $\Phi$ that come from the standard electric-field representation, it is not difficult to check that the map has no kernel, i.e., $\Psi(\alpha) = 0$ for all $\alpha$ if and only if $\Phi[F] = 0$.) Since the transverse vector fields $F^a(k)$ have only two components $F^\pm(k)$, from now on we will regard $\Phi$ as functionals of the two fields $F^\pm$.

Thus, for the representation space $V$, we will use the functionals $\Psi$ on the loop space of the form (3.8). Using the procedure that was successful in the loop representation adapted to the positive-frequency connections [3], the action of the basic operators $\hat{h}_r[\alpha]$ and $\hat{E}^a(k)$ will be taken to be:

$$\hat{h}_r[\alpha]\Psi(\gamma) = \Psi(\gamma \cdot \alpha)$$

$$\hat{E}^a(k)\Psi(\gamma) = \frac{ie}{\hbar} F^a(\gamma, k)\Psi(\gamma)$$ \hspace{2cm} (3.9)

As is usual in the loop representation, the electric field is diagonal in the representation. The only non-vanishing commutator between the basic operators is

$$\left[\hat{h}_r[\alpha], \hat{E}^a(k)\right] = -\frac{\hbar e}{c} F^a_r(\alpha, k)\hat{h}_r[\alpha]$$ \hspace{2cm} (3.10)

Finally, for later convenience, we note the action of the magnetic field operators $\hat{B}^\pm$ on these
\[
\hat{B}^\pm(k)\Psi(\alpha) = \pm e|k| \left[ \frac{\delta}{\delta F^\pm(-k)} \Phi[F^\pm(k)] \right] \bigg|_{F^\pm(k) = F^\pm_\alpha(k)},
\] (3.11)

which is nothing but the “loop derivative” evaluated at \( F^\pm \) (see, e.g. [9]).

Our next task is to find an inner-product on \( V \) so that the “quantum reality conditions” (3.3) are satisfied. Let us begin with an inner product of the form

\[
\langle \Psi | \Psi' \rangle := \int \prod_{k,\pm} dF^\pm(k) e^{-T[F^\pm(k)]} \Phi[F^\pm] \Phi'[F^\pm]
\] (3.12)

and determine the measure by imposing the reality conditions. The property (2.21) of form factors implies that \( T[F] \) should be real. It also implies that the reality condition on the electric field is automatically satisfied. The other condition one should impose, namely the quantum version of (2.14) is

\[
\langle \Psi | (\hat{B}^\pm(k))^\dagger \chi \rangle = \langle \chi | \hat{B}^\pm(k) \Psi \rangle
= \langle \Psi | -\hat{B}^\pm(-k) + 2\hat{p}^\pm(-k) | \chi \rangle.
\] (3.13)

Using the form of the operators (3.9) and (3.11) for \( \hat{p}^\pm(k) \) and \( \hat{B}^\pm(k) \), we conclude that the reality condition (2.14) is satisfied if and only if

\[
\frac{\delta T}{\delta F^\pm(k)} = \mp \frac{2\hbar}{e^2|k|} F^\pm(k).
\] (3.14)

The solution to this equation is:

\[
T[F] = -\frac{2\hbar}{e^2} \int \frac{d^3k}{|k|} \left[ |F^+_r(k)|^2 - |F^-_r(k)|^2 \right]
\] (3.15)

Hence, the explicit form of the inner product (3.12) is given by:

\[
\langle \Psi | \Psi' \rangle = \int \prod_{k,\pm} dF^\pm(k) e^{\frac{2\hbar}{e^2} \int \frac{d^3k}{|k|} \left[ (|F^+_r(k)|^2 - |F^-_r(k)|^2) \right]} \Phi[F^\pm] \Phi'[F^\pm]
\] (3.16)

Notice that the basic form of (3.16) is the same as that of the inner-product for a free-field in the configuration (i.e., Schrödinger) representation. There are, however, two important

\footnotetext{Although \( F^\pm(k) \) are complex-valued, they arise as Fourier components of a real field \( F^a(x) \) and}
differences. First, our states are functionals of loops rather than of a configuration field variable (such as the connection or the electric field). Second, for the positive component, the Gaussian is exponentially growing rather than damping. Hence, while we can take the states to be polynomials in $F^-$ as in the Schrödinger representation, we have to assume that they are exponentially damped in their dependence on $F^+$. Thus, for example, we can take elements of $V$ to be the functionals $\psi(\alpha)$ on the loop space of the form:

$$\Psi(\alpha) = P[F_r^{\pm}(\alpha, k)] \exp \left[ -\frac{\hbar}{c^2} \int \frac{d^3k}{|k|} \left( |F_r^+(k)|^2 \right) \right]$$

(3.17)

where $P[F_r^{\pm}(\alpha, k)]$ is a polynomial in $F^{\pm}$. As usual, the Cauchy completion will enlarge this space; the Hilbert space of all states will contain more general functionals. In this description, $F^+$ captures positive helicity while $F^-$ captures the negative helicity of the photon. Thus, as one might have expected from our use of only the self-dual part of the connection, the description is asymmetric in the two helicities.

To summarize, the elementary operators are $\hat{h}_r(\alpha)$ and $\hat{E}^a$. The space of quantum states is given by functionals $\Psi(\alpha)$ of holonomic loops which are normalizable with respect to the inner-product (3.16) and the action of the elementary operators is given by (3.9). For every $r > 0$, this loop representation is naturally isomorphic to the Fock representation\(^2\) (where the isomorphism, however, depends on the value of $r$.) The fact that we are using a loop representation adapted to self-dual connections is reflected in the measure that dictates the inner product (3.16). In the loop representation adapted to positive-frequency fields \(^3\), for example, the measure has the same form but the squares of both $|F^{\pm}|$ appear with negative hence satisfy the reality conditions $F^{\pm}(k) = -F(-k)$. The configuration space underlying our loop representation is thus real and states $\Phi(F^{\pm})$ are arbitrary complex-valued functions of $F^{\pm}$ (i.e., not subject to any “holomorphicity” condition.)

\(^2\)If we let $r$ go to zero, the smearing function $f_r(x)$ tends to the $\delta$-distribution and the thicken loop $\alpha_r$ reduces to the loop $\alpha$. However, now the exponent $T$ in the measure diverges and the loop representation ceases to exist.
signs in the exponent.

IV. MEASURE AND THE GAUSS LINKING NUMBER

Recall that our quantum states are functionals of thickened loops $\alpha_r$, or equivalently, of their form factors $F_r^\pm(\alpha, k)$; it is for technical convenience that in the intermediate stages of calculations that we extended them to functionals on the vector space of all fields $F^\pm(k)$. Therefore, it is instructive to examine the measure that dictates the inner-product also directly in terms of the thickened loops. This is easy to achieve: we can just pull-back the “Gaussian” $\exp -T$ that dictates the inner-product to the space of thickened loops. The result is trivially given by:

$$\exp (-T[F_r(\alpha, k)]) = \exp \left[ \frac{2\hbar}{e^2} \int \frac{d^3k}{|k|} \left( |F^+_r(\alpha, k)|^2 - |F^-_r(\alpha, k)|^2 \right) \right]$$

(4.1)

We will now show that this loop functional can be expressed in terms of the Gauss linking number.

Let us begin by recalling the definition of the linking number. Given non-intersecting loops $\alpha$ and $\beta$ the Gauss linking number $GL(\alpha, \beta)$ between them can be expressed in terms of their form factors as:

$$GL(\alpha, \beta) = \int d^3x F^a(\alpha, x)w_a(\beta, x)$$

(4.2)

where $F^a(\alpha, x)$ is the form factor for $\alpha$ and $w_a(\beta, x)$ is a potential for the form factor of $\beta$: $\epsilon^{abc}\partial_b w_c(\beta, x) = F^a(\beta, x)$. The integral is independent of the specific choice of the potential $w_a(\beta, x)$ because $F^a(\alpha, x)$ is divergence free. Note that neither the definition of the form factor $F^a(\alpha, x)$ nor that of the potential $w_a(\beta, x)$ requires any background fields on the underlying oriented 3-manifold $R^3$; in particular, there is no reference to the 3-metric. (Since $F^a$ is a vector density, the $\epsilon^{abc}$ in the definition of $w_a(\beta, x)$ is the Levi-Civita density which is naturally available on any oriented 3-manifold.) This is to be expected since the Gauss linking number is a topological invariant.
Nonetheless, one can use the flat metric $q_{ab}$ on $\mathbb{R}^3$ to express the linking number in more familiar terms. First, we have the well-known form used by Gauss himself [10]:

$$\mathcal{GL}(\alpha, \beta) := \frac{1}{4\pi} \int ds \int dt \epsilon_{abc} \dot{\alpha}^a(s) \dot{\beta}^b(t) \frac{\alpha^c(s) - \beta^c(t)}{\|\alpha(s) - \beta(t)\|^3}$$  \hspace{1cm} (4.3)

For our purposes, a more convenient form is the one involving the Fourier transforms of the form factors. The Fourier transform of the potential has the form:

$$F^a(\beta, k) = i w_c(\beta, k) k_b \epsilon^{abc}$$

$$= w_c(\beta, k)|k|(m^a \bar{m}^c - \bar{m}^a m^c)$$

$$= |k|(m^a w^+(\beta, k) - \bar{m}^a w^- (\beta, k))$$  \hspace{1cm} (4.4)

whence,

$$F^+(\beta, k) = |k|w^+(\beta, k), \quad F^-(\beta, k) = -|k|w^-(\beta, k).$$  \hspace{1cm} (4.5)

Therefore, the Gauss linking number takes the form

$$\mathcal{GL}(\alpha, \beta) = \int d^3 k F^a(\alpha, k) w_a(\beta, k)$$

$$= \int d^3 k (F^+(\alpha, k) m^a + F^-(\alpha, k) \bar{m}^a) w_a(\beta, k)$$

$$= \int \frac{d^3 k}{|k|}(F^+(\alpha, k) F^+(\beta, k) - F^-(\alpha, k) F^-(\beta, k))$$  \hspace{1cm} (4.6)

Finally, we will need the notion of Gauss number of the thickened loops $\alpha_r$ and $\beta_r$. This is just the total linking number of loops in $\alpha_r$ with those in $\beta_r$:

$$\mathcal{GL}(\alpha_r, \beta_r) := \int d^3 y \int d^3 z f_r(y) f_r(z) \mathcal{GL}(\alpha^a + y^a, \beta^a + z^a)$$

$$= \int \frac{d^3 k}{|k|} (F^+_r(\alpha, k) F^+_r(\beta, k) - F^-_r(\alpha, k) F^-_r(\beta, k))$$  \hspace{1cm} (4.7)

Hence the self-linking number of a thickened loop $\alpha_r$ is given by:

$$\mathcal{GL}(\alpha_r, \alpha_r) = \int \frac{d^3 k}{|k|} [F^+_r(\alpha, k) F^+_r(\alpha, k) - F^-_r(\alpha, k) F^-_r(\alpha, k)]$$  \hspace{1cm} (4.8)

whence the “Gaussian” \hspace{1cm} (4.1) on the space of thickened loops which dictates the inner-product can be expressed as:

$$\exp \left( -T[F_r(\alpha, k)] \right) = \exp \left( \frac{2\hbar}{\epsilon^2} \mathcal{GL}(\alpha_r, \alpha_r) \right)$$  \hspace{1cm} (4.9)
This is the result we were seeking. (Note, incidentally, that the coefficient of the linking number is 2 over the fine structure constant.)

We conclude with a remark. Had we used positive frequency connections \([3]\), for example, the loop functional \([4.1]\) would have been replaced by

\[
\int \frac{d^3k}{|k|} (F^+_r(\alpha, k)F^+_r(\alpha, k) + F^-_r(\alpha, k)F^-_r(\alpha, k))
\]

which has no obvious interpretation in terms of the topology of loops. Similarly, if we had worked in the self-dual connection representation, the measure would have been dictated by a “Gaussian” on the space of connections (see chapter 11.5, especially Eq 42b in \([4]\) and \([11]\)) and would therefore also have had no relation to topological invariants of loops. We need both self-duality of the connection and the loop representation to relate the photon inner product with the Gauss linking number.

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