RAEMACHER SERIES FOR $\eta$-QUOTIENTS

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Abstract. We apply Rademacher’s method in order to compute the Fourier coefficients of a large class of $\eta$-quotients.

1. Background

A partition of an integer $n$ is a multiset of positive integers whose sum is $n$. Let $p(n)$ denote the number of partitions of $n$. The value of $p(n)$ may be computed by brute force for sufficiently small $n$ by simply enumerating all possible partitions and then counting. However, $p(n)$ grows rapidly and brute force computation rapidly becomes intractible. Another technique, pioneered by Euler [1], is to study the properties of the generating function

$Z(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}.$

He showed that

$\frac{1}{Z(q)} = \prod_{n=1}^{\infty} (1-q^n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{(3k-1)/2}.$

This is a result regarding formal series. However, we can interpret these series as complex valued functions in some appropriate domain. Viewed as complex functions, $Z(q)$ and $1/Z(q)$ are both nonvanishing and holomorphic on the open unit disk $D \subset \mathbb{C}$ and cannot be analytically continued beyond $D$. One naturally asks (i) what are the analytic properties of these generating functions, and (ii) what properties of $p(n)$ may be deduced from these analytic properties. With regards to (ii), if we know sufficiently many details regarding the analytic properties of $Z(q)$, $p(n)$ may simply be extracted by performing a Fourier-Laplace transform:

$p(n) = \frac{1}{2\pi i} \int_{\gamma} \frac{Z(q)}{q^{n+1}} dq$

for some suitably chosen contour $\gamma$. The difficulty lies in computing this contour integral.

Before continuing, a word on notation: Given a function $f(q) : D \rightarrow \mathbb{C}$, we may pull back $f(q)$ by the map $q = e^{2\pi i \tau}$ to get a new function $f(e^{2\pi i \tau}) : \mathbb{H} \rightarrow \mathbb{C}$, where $\mathbb{H}$ is the open upper-half of the complex plane. We will often denote $f(e^{2\pi i \tau})$ as simply $f(\tau)$ when no confusion should arise. In order to keep track of which variable we are working with, we will often refer to two copies of $\mathbb{C}$ as the $q$-plane and the $\tau$-plane.

The first progress with regards to (i) was due to Dedekind. Dedekind considered the eponymous function $\eta : D \rightarrow \mathbb{C}$

$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$

where $q = e^{2\pi i \tau}$ [2]. This is simply $1/Z(q)$ with a mysterious additional factor of $q^{1/24}$. Dedekind showed that $\eta(\tau)$ is a modular form of weight $1/2$. That is, for any matrix

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$

$\eta(M \tau) = \text{det}(M)^{-1/2} \eta(\tau)$.
where $\text{SL}_2(\mathbb{Z})$ is the group of integral matrices with determinant +1,

$$\eta \left( \frac{a\tau + b}{c\tau + d} \right) = \epsilon(a,b,c,d) \sqrt{c\tau + d} \cdot \eta(\tau)$$

(6)

where $\epsilon(a,b,c,d) = \epsilon(M)$ is a homomorphism. This phase factor is called a multiplier system. Dedekind computed $\epsilon(M)$. It is given by

$$\epsilon(a,b,c,d) = \begin{cases} 
\exp \left( \frac{\pi i a}{12} \right) & (c = 0, d = 1), \\
\exp \left( \frac{-\pi i a}{12} \right) & (c = 0, d = -1), \\
\exp \left( \pi i \left( \frac{a^2 + d}{12c} - s(d,c) - \frac{1}{4} \right) \right) & (c > 0), \\
\exp \left( \pi i \left( \frac{a^2 + d}{12c} - s(-d,-c) - \frac{1}{4} \right) \right) & (c < 0).
\end{cases}$$

(7)

Here

$$s(h,k) = \sum_{n=1}^{k-1} \frac{1}{k} \left( \frac{hn}{k} - \left\lfloor \frac{hn}{k} \right\rfloor - \frac{1}{2} \right)$$

(8)

is known as a Dedekind sum. In retrospect, Eq. 6 is rather remarkable, and allows us to extract asymptotics of $\eta(\tau)$ for $\tau$ near a given rational $q \in \mathbb{Q} \subset \mathbb{C}$ in terms of the asymptotics of $\eta(\tau)$ near $+i\infty$, which are incredibly simple: As $\tau \rightarrow +i\infty$, that is as $q \rightarrow 0$, $\eta(q) \sim q^{1/24}$. Hardy and Ramanujan [3] followed by Rademacher [4,5] used this in order to carry out the Fourier transform in Eq. 3 and therefore compute $p(n)$. This idea is rather general, and can be used to compute the Fourier coefficients of a wide variety of automorphic forms [6]. Modifications can be used in order to compute the Fourier coefficients of modular forms which are modular under a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. This idea was pioneered by Zuckerman [7]. In this paper we will use such a modification in order to compute the Fourier coefficients of a finite product of modular forms precomposed with multiplication by different scalar factors $M \subset \mathbb{N}$. These forms are modular forms under a congruence subgroup of the modular group, specifically

$$\Gamma_0(\text{lcm}(M)) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \mod \text{lcm}(M) \right\}.$$ 

(9)

So, our result can be obtained using Zuckerman’s method applied to $\Gamma_0(\text{lcm}(M))$. We instead modify Rademacher’s original method in a slightly different, but ultimately equivalent, way so that the calculation may be done for many $\eta$-quotients simultaneously. Our result is also a special case of recent work by Bringmann and Ono [8], but it is easier to simply derive our expressions from scratch.

Specifically, we consider $\eta$-quotients, functions $Z(q)$ of the form

$$Z(\tau) = \prod_{m=1}^{\infty} \eta(m\tau)^{\delta_m}$$

(10)

where $\{\delta_m\}_{m=1}^{\infty}$ is a sequence of integers of which only finitely many are nonzero. Those with $\delta_m \geq 0$ for all $m$ are often called $\eta$-products. These functions have appeared in several different contexts. One context, in the vein of their first, is being related to the generating functions for some partition-like combinatorial quantity. Rademacher’s method has been applied successfully for a wide variety of these cases. See the work of Grosswald [9,10], Haberzete [11], Hagis [12,13,14,15], Hua [16], Iseki [17,18], Livingood [19], Niven [20], as well as more recent work by Sills [21,22] and others [23,24] for examples. Some of the results in these papers are special cases of the main result in this paper, although the proof presented in this paper does not work for many of them. It breaks down when the modular form of interest has zero weight. The proof presented here does work for some of them, for example the result of Sills in [21].

A second context is as the partition functions for 1/2-BPS black holes in CHL models of string theory, defined originally in [25], where the frame shape of the $\eta$-quotient corresponds to the frame shape of the associated $K3$ symplectic automorphism [26,27]. The constants $d(n)$ then have physical interpretation as the exponential of the entropy of a black hole with a given charge. See [28,29,30,31] for computations of black hole entropy in non-reduced rank models using Rademacher series. See [27,32,33] for derivations of CHL 1/2-BPS black hole partition functions. A classification of all CHL models was recently completed [34,35] and the latter work includes a list of all CHL frame shapes. For Type IIB string theory on $K3 \times T^2$ or dual models, the Rademacher series coefficients may be computed by a gravity path integral. See [36,37,38,39,40,41,42] for some recent work in this regard. $\eta$-products have also been of interest in Mathieu moonshine [26,35]. Just
as Rademacher series were used to compute exact black hole entropy from microscopic partition functions for non-CHL models, the original motivation for this work was to compute exact black hole entropy from microscopic partition functions for CHL models.

In order to present our main formula, we need a few preliminary definitions. Let

\[ n_0 = -\frac{1}{24} \sum_{m=1}^{\infty} m \cdot \delta_m. \]

The function \( Z(q) \cdot q^{n_0} \) is holomorphic in the open unit disk \( \mathbb{D} \), and so we may write

\[ Z(q) = q^{-n_0} \sum_{n=0}^{\infty} d(n)q^n. \]

for some coefficients \( d(n) \). From the product formula for \( \eta(q) \) in Eq. 3 each \( d(n) \) is an integer. The main result of this paper is an explicit formula for \( d(n) \) for a large class of sequences \( \{\delta_m\}_{m=1}^{\infty} \). Let

\[\begin{align*}
c_1 &= -\frac{1}{2} \sum_{m=1}^{\infty} \delta_m, \\
c_2(k) &= \prod_{m=1}^{\infty} \left( \frac{\gcd(m,k)}{m} \right)^{\delta_m/2}, \\
c_3(k) &= -\sum_{m=1}^{\infty} \delta_m \frac{\gcd(m,k)^2}{m}, \\
A_k(n) &= \sum_{0 \leq h < k \gcd(h,k)=1} \exp \left[ -2\pi i \left( \frac{h}{k} \cdot n + \frac{1}{2} \sum_{m=1}^{\infty} \delta_m \cdot s \left( \frac{mh}{\gcd(m,k)} : \frac{k}{\gcd(m,k)} \right) \right) \right].
\end{align*}\]

Finally let \( \mathcal{M} \) be the set of \( m \) for which \( \delta_m \) is nonzero. The quantities \( c_1, c_2(k), c_3(k) \) are coefficients which appear in our calculation and formula. The coefficient \( c_1 \) is the negative of the weight of \( Z(\tau) \) as a \( \Gamma_0(\text{lcm}(\mathcal{M})) \)-modular form. The sums \( A_k(n) \) closely resemble – and in some cases are – Kloosterman sums. We will call them Kloosterman-like sums. It can be shown that \( A_k(n) \) is real for all \( k \) and \( n \). With these definitions in hand, we may state

**Theorem 1.1.** If \( c_1 > 0 \) and the periodic function \( g(k) : \mathbb{N} \to \mathbb{R} \) given by

\[ g(k) = \min_{m \in \mathcal{M}} \left( \frac{\gcd(m,k)^2}{m} \right) - \frac{c_3(k)}{24} \]

is non-negative, then for \( n \in \{1, 2, \ldots \} \) such that \( n > n_0 \),

\[ d(n) = 2\pi \left( \frac{1}{24(n - n_0)} \right)^{\frac{c_1+1}{2}} \sum_{k=1}^{\infty} c_2(k)c_3(k) \frac{\pi}{k} A_k(n)k^{-1}I_{1+c_1} \left( \frac{2}{3}c_3(k)(n - n_0) \right) \]

where \( I_{1+c_1} \) is the \((1+c_1)\)th modified Bessel function of the first kind.

It is worth spending a moment to comment on the hypotheses of Theorem 1.1. Rademacher’s proof in [9] works only for modular forms of positive dimension, that is negative weight. Since \( c_1 \) is the dimension of our modular form under the congruence subgroup for which it transforms, our constraint \( c_1 > 0 \) is analogous to the weight constraint of Rademacher. Rademacher and others noted that his formula held for other modular forms, including many of weight zero. The various methods of proof for these extreme cases seem to be substantially more delicate, relying on detailed computations involving Kloosterman sums. Rademacher’s computation of the Fourier coefficients of \( J(\tau) \) [13] [14] is a good example. The story for \( \eta \)-quotients is analogous. Eq. 16 seems to work for many \( \eta \)-quotients with \( c_1 = 0 \) assuming that the second hypothesis regarding \( g(k) \) is satisfied. This is not entirely surprising given recent work by Duncan and others [45] [46] [47].

Similarly, Rademacher’s formula in [9] includes a sum over the polar part of the relevant modular form. For each \( k \), each polar term gives rise to a distinct Bessel function in the Rademacher series. We only get a single Bessel function for each \( k \), as in Eq. 16 when the polar part contains one term. Analogously, the condition that \( g(k) \geq 0 \) implies that the polar part of \( Z(\tau) \) at each cusp of \( \mathbb{H}/\Gamma_0(\text{lcm}(\mathcal{M})) \) contains at most one term. If the polar part of \( Z(\tau) \) at some cusp of \( \mathbb{H}/\Gamma_0(\text{lcm}(\mathcal{M})) \) contains more than one term, the following proof can easily be modified, along with Eq. 16 to yield the correct expression. It will look like Eq. 16 but with additional Bessel functions, one for each term in the polar part.

An outline of this short paper is as follows: The proof of Theorem 1.1 is contained in section 2. In section 3 we check that Eq. 16 has the expected asymptotics. In section 4 we present some numerics.
We will use Rademacher’s modification of the Hardy-Ramanujan-Littlewood circle method to compute \(d(n)\). We present some lemmas which are contained in the original papers [34, 41] without proof. The reader can find proofs of these lemmas in these papers or in many expositions, such as [38], which I personally followed.

As in Rademacher’s computation of the Fourier coefficients of \(1/\eta(\tau)\), we extract \(d(n)\) by performing a Fourier-Laplace transform:

\[
d(n) = \frac{1}{2\pi i} \int_{\gamma} Z(q) \cdot q^{n_0} q^{-\tau} \, dq
\]

where \(\gamma\) is a closed contour winding once around the origin. In essence, that is all there is to it. The rest of the proof is simply computing this integral.

We will define a sequence of suitable contours \(\{\gamma_N\}_{N=1}^{\infty}\), compute the integral in Eq. 17 for \(\gamma = \gamma_N\) up to an error term, take \(N \to \infty\) and show that the error term converges to zero. It is more convenient to define the contours in the \(\tau\)-plane and then map them into the \(q\)-plane. We will denote a pullback of some contour \(\gamma\) in the \(q\)-plane to the \(\tau\)-plane as \(\tau(\gamma)\).

Some preliminary definitions are in order. The \(N\)th Farey sequence \(F_N\) is the finite sequence containing all irreducible fractions in \([0, 1]\) of denominator at most \(N\) in increasing order. The Ford circle \(C(h/k)\) associated with the irreducible fraction \(h/k\) is the circle in the \(\tau\)-plane with center \(h/k + i/2k^2\) and radius \(1/2k^2\). See Fig. 1. We denote by \(q(C(h/k))\) the mapping of \(C(h/k)\) into the \(q\)-plane. Note that \(q(C(0/1)) = q(C(1/1))\). It can be shown that the Ford circles corresponding to consecutive fractions \(h_1/k_2\) and \(h_2/k_2\) in some Farey sequence are tangent at the point

\[
\tau(h_1/k_1, h_2/k_2) = \frac{h_1k_2 + h_2k_1 + i}{k_1^2 + k_2^2}.
\]

For irreducible fractions \(h_0/h_0 < h_1/k_1 < h_2/k_2\) with \(C(h_0/k_0)\) and \(C(h_2/k_2)\) tangent to \(C(h_1/k_1)\) let \(\tau(h_0/k_0, h_1/k_1, h_2/k_2)\) be the arc on \(C(h_1/k_1)\) from the point of tangency with \(C(h_0/k_0)\) to the point of tangency with \(C(h_2/k_2)\) parametrized by arc length. We choose the contour to proceed around the Ford circle clockwise so that the arc does not touch the real line. Likewise for \(h_2/k_2\) such that \(C(h_2/k_2)\) is tangent to \(C(0/1)\) let \(\tau(h_0/k_0, h_2/k_2)\) be the arc on \(C(0/1)\) from the point \(+i\) to the point of tangency with \(C(h_2/k_2)\) parametrized by arc length. Likewise for \(h_0/k_0\) such that \(C(h_0/k_0)\) is tangent to \(C(1/1)\) let \(\tau(h_0/k_0, 0/1)\) be the arc on \(C(0/1)\) from the point of tangency with \(C(h_0/k_0)\) to the point \(1 + i\) parametrized by arc length. If we specify an \(N \in \{1, 2, \ldots\}\) then we may define \(\tau(\gamma_{N,h/k})\) for \(h/k \in F_N\) to be

\[
\tau(\gamma_{N,h/k}) = \begin{cases} 
\tau(h_0/k_0, h_2/k_2) & (h/k = 0/1) \\
\tau(h_0/k_0, h_0/h_2/k_2) & (k \neq 1) \\
\tau(h_0/k_0, 0/1) & (h/k = 1/1)
\end{cases}
\]

where \(h_0/k_0\) is the element in \(F_N\) immediately before \(h/k\) if such an element exists and \(h_2/k_2\) is the element in \(F_N\) immediately after \(h/k\) if such an element exists.

We define \(\tau(\gamma_N)\) to be the concatenation in order of each \(\tau(\gamma_{N,h/k})\) for \(h/k \in F_N\). The contour \(\gamma_N\) is then the mapping of \(\tau(\gamma_N)\) into the \(q\)-plane. This is a concatenation of the contours \(\gamma_{N,h/k}\) for \(h/k \in F_N\). We redefine \(\gamma_{N,0/1}\) to be the concatenation of what we used to call \(\gamma_{N,0/1}\) and \(\gamma_{N,1/1}\). These contours meet in the \(q\)-plane. The full contour \(\gamma_N\) is piecewise smooth and has winding number one about the origin. See Fig. 2 for visualizations of the Rademacher contour \(\gamma_N\) and \(\tau(\gamma_N)\) for various values of \(N\).

We now split up the contour integral in Eq. 17 into a sum of integrals over subcontours:

\[
d(n) = \frac{1}{2\pi i} \sum_{k=1}^{N} \sum_{1 \leq h < k} \int_{\gamma_{N,h/k}} \frac{Z(q)}{q^{n-n_0} + 1} \, dq
\]

where \((h, k)\) is shorthand for \(\gcd(h, k)\). It is convenient to change coordinates within each subcontour integral in order to write them as integrals over similar contours. Note that for irreducible \(h/k\) the coordinate transformation

\[
z = -ik^2 \left( \tau - \frac{h}{k} \right)
\]

or equivalently \(\tau = i \cdot \frac{z}{k^2} + \frac{h}{k}\)
maps the Ford circle $C(h/k)$ in the $τ$-plane to the circle $B_{1/2}(1/2)$ in the $z$-plane with center $1/2$ and radius $1/2$. See Fig. 3. The point $τ(h, k, h_2/k_2)$ is mapped to the point

$$\tilde{z}_{h/k}(h_2/k_2) = \frac{k^2}{k^2 + k_2^2} + i \left( hk - \frac{k^2}{k^2 + k_2^2}(hk + h_2k_2) \right).$$

We moved the $h/k$ into the subscript of $\tilde{z}$ to emphasize that the coordinate transformation depends on $h/k$ and that, for this reason, unlike $τ$, $\tilde{z}$ is not symmetric under interchanging its arguments. The contour $τ(γ_{N,h/k}) = τ(γ_{h_0/k_0, h/k, h_2/k_2})$ is mapped to an arc along $B_{1/2}(1/2)$ from $\tilde{z}_{h/k}(h_0/k_0)$ to $\tilde{z}_{h/k}(h_2/k_2)$, specifically the arc which does not contain the origin. Likewise, the contours $τ(γ_{h_0/k_0, 1/2})$ and $τ(γ_{h_0/k_0, 1/1})$ are mapped together to an arc along $B_{1/2}(1/2)$ from $\tilde{z}_{1/1}(h_0/k_0)$ to $\tilde{z}_{0/1}(h_2/k_2)$, also the arc which does not contain the origin. Let $\tilde{z}_{1,N,h/k}$ be $\tilde{z}_{h/k}(h_0/k_0)$ where $h_0/k_0$ is the element of $F_N$ immediately before $h/k$ if $k \neq 1$ and $\tilde{z}_{1/1}(h_0/k_0)$ where $h_0/k_0$ is the element of $F_N$ immediately before $1/1$ if $k = 1$. For irreducible $h/k$ except $1/1$ let $\tilde{z}_{2,N,h/k}$ be $\tilde{z}_{h/k}(h_2/k_2)$ where $h_2/k_2$ is the element of $F_N$ immediately after $h/k$. It can be checked that

$$\tilde{z}_{1,N,h/k} = \frac{k}{k^2 + k_0^2}(k + ik_0)$$

$$\tilde{z}_{2,N,h/k} = \frac{k}{k^2 + k_2^2}(k - ik_2)$$

where $h_0/k_0 < h/k < h_2/k_2$ are consecutive fractions in $F_N$ or if $h/k = 0/1$ and $h_0/k_0$ is immediately preceding $1/1$ or if $h/k = 1/1$ and $h_2/k_2$ is immediately following $0/1$. See Fig. 3.

Performing the coordinate transformations to Eq. 20

$$d(n) = i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k)=1} \int_{z(γ_{N,h/k})} Z \left( \exp \left( 2\pi \left[ i \cdot \frac{h}{k} - z \cdot \frac{1}{k^2} \right] \right) \right) \exp \left( 2\pi (n - n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right) \right) dz.$$
Here \( z(\gamma_{N,h/k}) \) is the mapping of \( \tau(\gamma_{N,h/k}) \) into the \( z \)-plane. That is, \( z(\gamma_{N,h/k}) \) is the arc of \( B_{1,2} / (1/2) \) which avoids the origin and is from \( \tilde{z}_{1,N,h/k} \) to \( \tilde{z}_{2,N,h/k} \). Tracing through the definitions of \( z \) and \( Z(q) \), the integrands above are holomorphic in the right-half of the \( z \)-plane. We may therefore deform our subcontours from arcs on \( B_{1,2} / (1/2) \) to chords through \( B_{1,2} / (1/2) \). These chords begin at \( \tilde{z}_{1,N,h/k} \) and end at \( \tilde{z}_{2,N,h/k} \). We denote these chords as \( z(N,h/k) \). See Fig. 2. So,

\[
d(n) = \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k) = 1} \int_{\tilde{z}_{1,n}(N,h/k)}^{\tilde{z}_{2}(N,h/k)} Z \left( \exp \left( 2\pi \left[ i \cdot \frac{h}{k} - \frac{z}{k^2} \right] \right) \right) \exp \left( 2\pi(n-n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right) \right) dz.
\]

Before we proceed, we state two geometric lemmas. The first concerns the properties of the chords \( z(N,h/k) \) and the second concerns the properties of arcs on \( B_{1,2} / (1/2) \). Proofs of these are contained in [15].

Lemma 2.1. The chord \( z(N,h/k) \) has length at most \( 2\sqrt{2}k/N \) and on this chord \( |z| \leq \sqrt{2}k/N \). ■

Lemma 2.2. In \( B_{1,2} / (1/2) \) \( \{0\} \), \( \text{Re}(z) \leq 1 \) and \( \text{Re}(1/z) \geq 1 \) with \( \text{Re}(1/z) = 1 \) on the circle itself. On the arcs from 0 to \( \tilde{z}_{1,n},h/k \) and \( \tilde{z}_{2,n},h/k \) to 0, \( |z| \leq \sqrt{2}k/N \) and the length of these arcs is at most \( \pi \sqrt{2}k/N \). ■

By the previous two lemmas, for fixed \( h/k \) as \( N \to \infty \) the chords \( z(N,h/k) \) get shorter and nearer to the origin. As \( z \) approaches the origin, \( \tau = (h/k) + i(z/k^2) \) approaches \( h/k \). We can calculate the asymptotics of \( \eta(\tau) \) as \( \tau \) approaches \( +i\infty \) from the definition of \( \eta(\tau) \). We can then calculate the asymptotics of \( Z(\tau) \) near \( h/k \) in terms of the asymptotics of each \( \eta(m\tau) \) near \( +i\infty \) using the modularity properties of \( \eta(\tau) \). These asymptotics are sufficiently simple to integrate them. This is the key insight in the Hardy-Littlewood-Ramanujan circle method. We now turn to expressing \( \eta(m\tau) \) for \( m \in \{1, 2, \ldots \} \) near \( h/k \) in terms of \( \eta(\tau) \) near \( +i\infty \).

For irreducible fraction \( h/k \in [0,1] \), we may find by the Euclidean algorithm some integer \( H(m,h,k) = H \) such that

\[
mh \equiv -\gcd(m,k) \mod k.
\]

It follows that the matrix \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is in \( SL_2(\mathbb{Z}) \), where

\[
a = H, \quad b = -\frac{1}{k}(mh + \gcd(m,k)), \quad c = \frac{k}{\gcd(m,k)}, \quad d = -\frac{mh}{\gcd(m,k)}.
\]

As a member of the modular group, this matrix induces an action on the upper-half of the complex plane, given by \( M(\tau) = \frac{a\tau + b}{c\tau + d} \) for any \( \tau \in \mathbb{H} \). Under this action,

\[
M(m\tau) = M \left( m \left( \frac{h}{k} + i \cdot \frac{z}{k^2} \right) \right) = \frac{\gcd(m,k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m,k) \right) = \frac{a\tau + b}{c\tau + d}.
\]

Figure 2. The Rademacher contour \( \gamma_N \) for several different values of \( N \), in the \( \tau \) and \( q \) planes.
As \( z \) approaches 0 – and therefore \( \tau \) approaches \( h/k \) – the right hand side goes to positive imaginary infinity, as desired. Using the modular transformation properties of \( \eta(\tau) \),

\[
\eta(m\tau) = \left[ \epsilon(a,b,c,d)(cm\tau + d)^{1/2} \right]^{-1} \eta \left( \frac{am\tau + b}{cm\tau + d} \right).
\]

In this case, \( cm\tau + d = imz/k \gcd(m,k) \) and by Eq. 7

\[
\epsilon(a,b,c,d) = \exp \left( \pi i \left( -\frac{H}{12k} \gcd(m,k) + \frac{mh}{12k} \frac{k}{\gcd(m,k)} - 1 \frac{1}{4} \right) \right),
\]

so that

\[
\eta \left( \frac{h}{k} + \frac{i}{k^2} \right) = \exp \left( \pi i \left( -\frac{H}{12k} \gcd(m,k) + \frac{mh}{12k} \frac{k}{\gcd(m,k)} \right) \right) \times \frac{1}{\sqrt{k \gcd(m,k) \frac{k}{mz}}} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m,k) \right) \right).
\]

The constant \( H = H(m,h,k) \) depends implicitly on \( m \), \( h \), and \( k \). Combining Eq. (32) for all values of \( m \)

\[
Z \left( \frac{h}{k} + \frac{i}{k^2} \right) = \xi(h,k)\omega(h,k) \cdot z^{c_1} \prod_{m=1}^{\infty} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m},
\]

where \( c_1 \) was defined in Eq. 13 and we defined

\[
\xi(h,k) = \prod_{m=1}^{\infty} \sqrt{k \gcd(m,k) \frac{k}{m}} \exp \left( \pi i \left( \frac{mh}{12k} - s \left( \frac{mh}{\gcd(m,k)} \frac{k}{\gcd(m,k)} \right) \right) \right)^{\delta_m}
\]

and

\[
\omega(h,k) = \prod_{m=1}^{\infty} \exp \left( -\frac{\pi i}{12k} H \delta_m \gcd(m,k) \right).
\]
Plugging in the previous formulas into Eq. 26

\[ d(n) = i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k)=1} \xi(h,k) \omega(h,k) \int_{\mathbb{Z}^2(N,h/k)} z^{c_1} \left[ \prod_{m=1}^{\infty} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right. \]

\[ \times \exp \left( 2\pi(n-n_0) \left( \frac{z}{k^2} - i \frac{h}{k} \right) \right) dz. \]

We expect to be able to replace each \( \eta(q) \) in the integrand with \( q^{1/24} \) and accrue a total \( o(1) \) error as \( N \to \infty \). Defining

\[ \text{Er}(N) = \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k)=1} \xi(h,k) \omega(h,k) \int_{\mathbb{Z}^2(N,h/k)} z^{c_1} \exp \left( 2\pi(n-n_0) \left( \frac{z}{k^2} - i \frac{h}{k} \right) \right) \Delta_{h,k}(z) dz, \]

where \( \Delta_{h,k}(z) \) is the difference between the \( \eta \)-quotient and its leading order asymptotics

\[ \Delta_{h,k}(z) = \left[ \prod_{m=1}^{\infty} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right] - \left[ \prod_{m=1}^{\infty} \exp \left( \frac{\pi i \gcd(m,k)}{12} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right], \]

we can write

\[ d(n) = \text{Er}(N) + i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k)=1} \xi(h,k) e^{-2\pi i(n-n_0)\frac{h}{k}} \int_{\mathbb{Z}^2(N,h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi c_3(k)}{12} \right] dz. \]

The second term is the result of replacing each \( \eta \)-function by the appropriate asymptotics, and the first term is the accumulated error from doing so. We wish to show that \( \lim_{N \to \infty} \text{Er}(N) = 0 \).

Our first task is to bound \( \Delta_{h,k}(z) \). We will show that \( \Delta_{h,k}(z) = O(1) \) in \( B_{1/2}(1/2) \) where the bound does not depend on \( h, k, \) or \( z \). The fact that \( \Delta_{h,k}(z) \) does not blow up at the origin expresses the fact that the polar component of \( Z(\tau) \) near \( h/k \) contains at most one term, that which we are approximating it by. Let \( \tilde{\eta}(\tau) = \eta(\tau)/q^{1/24} \), so that

\[ \Delta_{h,k}(z) = \left[ \prod_{m=1}^{\infty} \exp \left( \frac{\pi i \gcd(m,k)}{12} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right] \times \left[ \prod_{m=1}^{\infty} \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right] - 1. \]

Taking the norm,

\[ |\Delta_{h,k}(z)| = \exp \left[ \frac{\pi}{12} c_3(k) \Re \left( \frac{1}{z} \right) \right] \times \left| \prod_{m=1}^{\infty} \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} - 1 \right|. \]

If \( c_3(k) > 0 \), the first term on the right hand side can be large for some \( z \) along \( \mathbb{Z}^2(N,h/k) \). However, we expect the second term on the right hand side to be small. This is the content of the following lemma.

**Lemma 2.3.** For some constant \( D \), which may depend only on \( \{\delta_m\}_{m=1}^{\infty} \), for \( z \) in \( B_{1/2}(1/2)/\{0\} \),

\[ \left| \prod_{m=1}^{\infty} \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} - 1 \right| \leq D \exp \left( -2\pi \Re \left( \frac{1}{z} \right) \min_{m \in M} \left\{ \frac{\gcd(m,k)^2}{m} \right\} \right). \]

**Proof of Lemma 2.3.** First consider

\[ \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{-1}, \]
which by definition is
\[
\sum_{j=0}^\infty p(j) \exp \left( -2\pi j \cdot \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right) \omega(h,m,k,z)^j
\]
where \( p(j) \) is Euler’s partition function and \( \omega(h,m,k,z) = \omega \) is a phase factor. Recall the following upper bound for \( p(j) \), which can be derived by purely classical methods \([2]\)
\[
p(j) = O \left( \exp \left( \pi \sqrt{\frac{2}{3} \cdot j} \right) \right).
\]
The tail of the series in Eq. 44 is therefore bounded above by a convergent geometric series. To be more precise, consider the sum of all but the first two terms in Eq. 44
\[
\exp \left( -2\pi \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right) \omega \sum_{j=2}^\infty p(j) \exp \left( -2\pi (j-1) \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right) \omega^{j-1}.
\]
Using the triangle inequality and Lemma 2.2
\[
\left| \sum_{j=2}^\infty p(j) \exp \left( -2\pi (j-1) \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right) \omega^{j-1} \right| \leq \sum_{j=1}^\infty p(j) \exp \left( -2\pi (j-1) \frac{\gcd(m,k)^2}{m} \right).
\]
Using the classical bound on \( p(j) \) in Eq. 45
\[
\sum_{j=1}^\infty p(j) \exp \left( -2\pi (j-1) \frac{\gcd(m,k)^2}{m} \right) \leq C \sum_{j=2}^\infty \exp \left( \pi \sqrt{\frac{2}{3} \cdot j} - 2\pi (j-1) \cdot \frac{\gcd(m,k)^2}{m} \right)
\]
for some absolute constant \( C \). Therefore, since \( \gcd(m,k)/m \geq 1/m \),
\[
\left| \sum_{j=2}^\infty p(j) \exp \left( -2\pi (j-1) \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right) \omega^{j-1} \right| \leq C \sum_{j=2}^\infty \exp \left( \pi \sqrt{\frac{2}{3} \cdot j} - 2\pi \frac{m}{j} \right) \left( \frac{1}{z} \right) \omega^{j-1}.
\]
This is bounded above by a convergent geometric series, so that for some constant \( C_m > 0 \) which is dependent only on \( m \),
\[
\sum_{j=2}^\infty p(j) \exp \left( -2\pi (j-1) \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right) \omega^{j-1} \leq C_m.
\]
It follows that Eq. 46 is bounded above in magnitude by
\[
C_m \exp \left( -2\pi \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right).
\]
Therefore, replacing the tail in Eq. 44 with this bound,
\[
\tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{-1} = 1 + O(1) \exp \left( -2\pi \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right),
\]
where the \( O(1) \) term satisfies \( |O(1)| \leq C_m + 1 \). Taking the reciprocal of Eq. 52 and using that \( \operatorname{Re}(1/z) \geq 1 \) within \( B_{1/2}(1/2) \), yields
\[
\tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right) = 1 + O(1) \exp \left( -2\pi \frac{\gcd(m,k)^2}{m} \operatorname{Re} \left( \frac{1}{z} \right) \right)
\]
for some other \( O(1) \) term which can be bounded in magnitude depending only on \( m \). Taking the appropriate product of Eq. 52 and Eq. 53 for all \( m \) yields
\[
\prod_{m=1}^\infty \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} = 1 + O(1) \exp \left( -2\pi \min \limits_{m \in \mathcal{M}} \left\{ \frac{\gcd(m,k)^2}{m} \right\} \operatorname{Re} \left( \frac{1}{z} \right) \right),
\]
where $\mathcal{M}$ is the (finite) set of all $m \in \mathbb{N}$ such that $\delta_m$ is nonzero, and the $O(1)$ term is bounded in magnitude depending only on $\{\delta_m\}_{m=1}^{\infty}$. Consequently, for some constant $D > 0$ depending only on $\{\delta_m\}_{m=1}^{\infty}$

\begin{align}
\left| \prod_m \bar{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{m \zeta} \gcd(m,k) \right) \right)^{\delta_m} - 1 \right| &\leq D \exp \left( -2\pi \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m,k)^2}{m} \right\} \Re \left( \frac{1}{z} \right) \right). \tag{55} \\
\end{align}

Using Lemma 2.3

\begin{align}
|\Delta_{h,k}(z)| &\leq D \exp \left( -2\pi \Re \left( \frac{1}{z} \right) \left( \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m,k)^2}{m} \right\} - \frac{c_3(k)}{24} \right) \right) = D \exp \left( -2\pi \Re \left( \frac{1}{z} \right) g(k) \right). \tag{56} \\
\end{align}

One of the hypotheses of Theorem 1.1 is that the function $g(k)$ is non-negative. Then, using Lemma 2.2, we can bound

\begin{align}
|\Delta_{h,k}(z)| &\leq D \exp \left[ -2\pi \min \{ g(k) : k = 1, \ldots, \text{lcm}(\mathcal{M}) \} \right], \tag{57} \\
\end{align}

and the constant on the right hand side depends only on $\{\delta_m\}_{m=1}^{\infty}$. We redefine $D$ to be this constant. Using Eq. 57 and Lemma 2.3, the integral in Eq. 57 is bounded above in magnitude:

\begin{align}
\left| \int_{z^{c_1} \exp \left( 2\pi(n - n_0) \left( \frac{z}{k^2} - i \frac{h}{k} \right) \right) \Delta_{h,k}(z) \, dz \right| &\leq 2D \left( \sqrt{D} \frac{k}{N} \right)^{c_1+1} \exp \left( 2\sqrt{2}\pi(n - n_0) \frac{k}{N} \right). \tag{58} \\
\end{align}

Substituting this into the definition of $\text{Er}(N)$ in Eq. 37 for some constant $C$ depending only on $\{\delta_m\}_{m=1}^{\infty}$

\begin{align}
|\text{Er}(N)| &\leq Ce^{2\sqrt{2}\pi n} N^{-(c_1+1)} \sum_{k=1}^{N} \sum_{0 \leq h < k (h,k)=1} k^{-1}. \tag{59} \\
\end{align}

Since there are at most $k$ terms in the inner sum and $N$ terms in the outer sum, we can bound this

\begin{align}
|\text{Er}(N)| \leq Ce^{2\sqrt{2}\pi n} N^{-c_1}. \tag{60} \\
\end{align}

Since $c_1 > 0$, this shows that $\lim_{N \to \infty} \text{Er}(N) = 0$, as desired. Referring back to Eq. 39 we have shown that

\begin{align}
d(n) &\equiv o(1) + i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k (h,k)=1} \xi(h,k)e^{-2\pi i (n-n_0)h/k} \int_{z^{c_1} \exp \left( 2\pi(n - n_0) \left( k^2 - z \right) \frac{1}{2}\pi \frac{c_3(k)}{12} \right) \, dz}. \tag{61} \\
\end{align}

We now deform our contours back to arcs along $B_{1/2}(1/2)$:

\begin{align}
d(n) &\equiv o(1) + \frac{i}{2} \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k (h,k)=1} \xi(h,k)e^{-2\pi i (n-n_0)h/k} \int_{z^{c_1} \exp \left( 2\pi(n - n_0) \left( k^2 - z \right) \frac{1}{2}\pi \frac{c_3(k)}{12} \right) \, dz}. \tag{62} \\
\end{align}

Our next goal is to show that the main term on the right hand side above,

\begin{align}
i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k (h,k)=1} \xi(h,k)e^{-2\pi i (n-n_0)h/k} \int_{z^{c_1} \exp \left( 2\pi(n - n_0) \left( k^2 - z \right) \frac{1}{2}\pi \frac{c_3(k)}{12} \right) \, dz}, \tag{63} \\
\end{align}

differs from

\begin{align}
i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k (h,k)=1} \xi(h,k)e^{-2\pi i (n-n_0)h/k} \int_{B_{1/2}(1/2)} z^{c_1} \exp \left( 2\pi(n - n_0) \left( k^2 - z \right) \frac{1}{2}\pi \frac{c_3(k)}{12} \right) \, dz, \tag{64} \\
\end{align}

by an $o(1)$ term as $N \to \infty$. The contour in the second expression is traversed clockwise. In other words, we may replace our integrals over incomplete arcs of $B_{1/2}(1/2)$ by integrals over the complete circle $B_{1/2}(1/2)$.
and only accrue a total $o(1)$ error as $N \to \infty$. The former is the latter minus $J_1 + J_2$, where $J_1 = J_1(N)$ and $J_2 = J_2(N)$ are defined by

\begin{equation}
J_1 = i \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k)=1} k^{-2} \xi(h,k) e^{-2\pi i (n-n_0) h/k} \int_{0}^{\xi_1(N,h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi c_3(k)}{12 z} \right] dz,
\end{equation}

\begin{equation}
J_2 = i \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k)=1} k^{-2} \xi(h,k) e^{-2\pi i (n-n_0) h/k} \int_{0}^{\xi_2(N,h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi c_3(k)}{12 z} \right] dz.
\end{equation}

The integrals are on arcs of $B_{1/2}(1/2)$. We bound $J_1$. Using the bound $|\xi(h,k)| \leq k^{-c_1}$,

\begin{equation}
|J_1| \leq \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k)=1} k^{-(2+c_1)} \left( \frac{\sqrt{2} k}{N} \right)^{c_1+1} \left( \frac{1}{N} \right)^{c_1+1} \exp \left[ 2\pi(n-n_0) \right] dz.
\end{equation}

Using Lemma 2.2

\begin{equation}
|J_1| \leq \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k)=1} k^{-1} \left( \frac{1}{N} \right)^{c_1+1} \exp \left[ 2\pi(n-n_0) \right] dz.
\end{equation}

Since the outer sum is over at most $N$ terms and the inner sum is over at most $k$ terms,

\begin{equation}
|J_1| \leq CN^{-c_1} \exp \left[ 2\pi(n-n_0) \right] dz.
\end{equation}

Since $c_1 > 0$, $J_1 = o(1)$, as desired. An identical argument yields $J_2 = o(1)$. Combining all of the previous results,

\begin{equation}
d(n) = o(1) + i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop \gcd(h,k)=1} \xi(h,k) e^{-2\pi i (n-n_0) h/k} \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi c_3(k)}{12 z} \right] dz.
\end{equation}

Taking $N \to \infty$,

\begin{equation}
d(n) = i \sum_{k=1}^{\infty} k^{-2} \sum_{0 \leq h < k \atop \gcd(h,k)=1} \xi(h,k) e^{-2\pi i (n-n_0) h/k} \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi c_3(k)}{12 z} \right] dz.
\end{equation}

Referring to the definition of $\xi(h,k)$ in Eq. 14 and of the Kloosterman-like sum $A_k(n)$ in Eq. 14 this is exactly

\begin{equation}
d(n) = i \sum_{k=1}^{\infty} k^{-(2+c_1)} c_2(k) A_k(n) \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi c_3(k)}{12 z} \right] dz.
\end{equation}

Now we just evaluate this integral

\begin{equation}
I = \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi c_3(k)}{12 z} \right] dz.
\end{equation}

First note that if $c_3(k) = 0$, then the integrand is everywhere holomorphic so that by the Cauchy integral formula $I = 0$. Otherwise, we can rewrite it as

\begin{equation}
I = \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{\pi k}{6} \frac{|c_3(k)|}{|n-n_0|} \left( \frac{z}{k} \sqrt{\frac{24(n-n_0)}{|c_3(k)|}} \right)^{-1} \right] dz
\end{equation}
with the $\pm$ given by $\text{sign}(c_{3}(k))$. We make the substitution

\begin{equation}
    w = \left( \frac{z}{k} \sqrt{\frac{24(n-n_{0})}{|c_{3}(k)|}} \right)^{-1}, \quad z = \left( \frac{w}{k} \sqrt{\frac{24(n-n_{0})}{|c_{3}(k)|}} \right)^{-1}, \quad \mathrm{d}z = -\left( \frac{w^{2}}{k} \sqrt{\frac{24(n-n_{0})}{|c_{3}(k)|}} \right)^{-1} \mathrm{d}w,
\end{equation}

whence

\begin{equation}
    I = -\left( k \sqrt{\frac{|c_{3}(k)|}{24(n-n_{0})}} \right)^{c_{1}+1} \int_{1-i \infty}^{1+i \infty} w^{-(c_{1}+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_{3}(k)|}{6}} (n-n_{0}) (w^{-1} \pm w) \right] \mathrm{d}w.
\end{equation}

We now split into two cases depending on the sign of $c_{3}(k)$. If $c_{3}(k) < 0$, then the integrand decays sufficiently rapidly in the right-half plane such that

\begin{equation}
    I = -\left( k \sqrt{\frac{|c_{3}(k)|}{24(n-n_{0})}} \right)^{c_{1}+1} \lim_{R \to \infty} \int_{S(R)} w^{-(c_{1}+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_{3}(k)|}{6}} (n-n_{0}) (w^{-1} - w) \right] \mathrm{d}w
\end{equation}

where $S(R)$ is the right semicircle of radius $R$, centered at 1, traversed clockwise. The integrand is holomorphic inside this contour, since it does not contain the origin, so that by the Cauchy integral formula $I = 0$. Otherwise, if $c_{3}(k) > 0$, then the integrand decays sufficiently rapidly in the left-half plane such that

\begin{equation}
    I = -\left( k \sqrt{\frac{|c_{3}(k)|}{24(n-n_{0})}} \right)^{c_{1}+1} \int w^{-(c_{1}+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_{3}(k)|}{6}} (n-n_{0}) (w^{-1} + w) \right] \mathrm{d}w
\end{equation}

for any positively oriented closed contour winding once around the origin. We rearrange the terms in the integral slightly:

\begin{equation}
    I = -2\pi i \left( k \sqrt{\frac{|c_{3}(k)|}{24(n-n_{0})}} \right)^{c_{1}+1} \frac{1}{2\pi i} \oint w^{-(c_{1}+1)-1} \exp \left[ \frac{1}{2} \frac{\pi}{k} \sqrt{\frac{2}{3}} |c_{3}(k)|(n-n_{0}) (w^{-1} + w) \right] \mathrm{d}w.
\end{equation}

This integral is a standard form of the modified Bessel function of the first kind \[49, 50]:

\begin{equation}
    I = -2\pi i \left( k \sqrt{\frac{|c_{3}(k)|}{24(n-n_{0})}} \right)^{c_{1}+1} I_{c_{1}+1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} |c_{3}(k)|(n-n_{0}) \right]
\end{equation}

where $I_{c_{1}+1}$ is the modified Bessel function of the first kind of weight $c_{1}$. To summarize,

\begin{equation}
    I = \begin{cases} 
    0 & (c_{3}(k) \leq 0), \\
    -2\pi i \left( k \sqrt{\frac{|c_{3}(k)|}{24(n-n_{0})}} \right)^{c_{1}+1} I_{c_{1}+1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} |c_{3}(k)|(n-n_{0}) \right] & (c_{3}(k) > 0).
    \end{cases}
\end{equation}

Substituting $I$ into Eq. (72) and simplifying, we get our final expression

\begin{equation}
    d(n) = 2\pi \left( \frac{1}{24(n-n_{0})} \right)^{c_{1}+1} \sum_{k=1}^{\infty} c_{2}(k) c_{3}(k)^{c_{1}+1} k^{-1} A_{k}(n) I_{1+c_{1}} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} |c_{3}(k)|(n-n_{0}) \right].
\end{equation}

\[ \square \]

3. Asymptotics

We would like to extract useful asymptotics from Eq. (82). These are contained in the following proposition. For this section we assume that the hypotheses of Theorem 1.1 are satisfied, so that Eq. 10 applies.

**Proposition 3.1.** Let $K \subset \mathbb{N}$ be the set of $k$ that maximize $c_{3}(k)/k^{2}$ and let $c_{3} > 0$ be the maximum value. For any $\epsilon > 0$, there exists some constant $C > 0$ which may depend only on $\{\delta_{m}\}_{m=1}^{\infty}$ such that for all $n \in \mathbb{N}$ with $n > n_{0}$ and

\begin{equation}
    \sum_{k \in K} c_{2}(k) k^{c_{1}} A_{k}(n) \geq \epsilon
\end{equation}
We wish to show that the sum above is dominated by the first term. Consider the rest of the terms, which is bounded above in absolute value by
\[ \sum_{k \in \mathbb{N}} c_2(k) c_3(k) (c_1 + 1)^{-1/2} A_k(n) k^{-1} I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(k) (n - n_0) \right) \]

The dependence on \( \epsilon \) is in the bounding coefficient of the \( O(e^{-C\sqrt{\pi}}) \).

The proof of Prop. 3.1 is straightforward and an exercise in using the asymptotics of the modified Bessel functions.

**Proof.** Note that \( c_3(k) \) is periodic with period \( \text{lcm}(M) \). So, \( K \subseteq \{1, \ldots, \text{lcm}(M)\} \). We first break up the Rademacher series in Eq. (10) into \( \text{lcm}(M) \) sums, one for each possible value of \( k \) modulo \( \text{lcm}(M) \). We then show that each of these smaller sums is exponentially dominated by the leading term. We then absorb the resulting Bessel functions with \( k \notin K \) into those with \( k \in K \). In the following, we will use the result that if \( 0 < a < b \) then \( I(ax) \) is exponentially dominated by \( I(bx) \) for any positive \( x \) and any positive weight modified Bessel function \( I \) of the first kind. So, we first consider for fixed \( b \in \{1, \ldots, \text{lcm}(M)\} \) with \( c_3(b) > 0 \)

\[ \sum_{k \in [b]} c_2(k) c_3(k) (c_1 + 1)^{-1/2} A_k(n) k^{-1} I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(k) (n - n_0) \right) \]

where \([b]\) is the equivalence class of integers modulo \( \text{lcm}(M) \). Because \( c_2(k), c_3(k) \) have period \( \text{lcm}(M) \), this is

\[ c_2(b) c_3(b) (c_1 + 1)^{-1/2} \sum_{k \in [b]} A_k(n) k^{-1} I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) \]

We wish to show that the sum above is dominated by the first term. Consider the rest of the terms,

\[ \sum_{k \in [b]} A_k(n) k^{-1} I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) \]

which is bounded above in absolute value by

\[ \sum_{k \in [b]} \left| I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) \right| \]

Using the expansion of \( I_{1+c_1}(z) \) [50],

\[ I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) = \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2)} \left( \frac{\pi}{2k} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right)^{2j + 1 + c_1} \]

Suppose that \( k_0 \) is a real number satisfying \( 0 < k_0 \leq k \).

\[ \left| I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) \right| \leq \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2)} \left( \frac{\pi}{2k_0} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right)^{2j + 1 + c_1} \left( \frac{k_0}{k} \right)^{2j + 1 + c_1} \]

\[ \leq \left( \frac{k_0}{k} \right)^{1+c_1} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2)} \left( \frac{\pi}{2k_0} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right)^{2j + 1 + c_1} \]

\[ = \left( \frac{k_0}{k} \right)^{1+c_1} I_{1+c_1} \left( \frac{\pi}{k_0} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) \]

Summing over all relevant \( k \),

\[ \sum_{k \in [b]} \left| I_{1+c_1} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) \right| \leq k_0^{1+c_1} \zeta(1 + c_1) I_{1+c_1} \left( \frac{\pi}{k_0} \sqrt{\frac{2}{3}} c_3(b) (n - n_0) \right) \]
Therefore, setting \( k_0 = b + 1/2 \) yields
\[
\sum_{\substack{k \in [b] \\ 0 < k \leq \text{lcm}(M)}} \left| I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right] \right| = O(e^{-C_b \sqrt{n}}) I_{1+c_1} \left[ \frac{\pi}{b} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right]
\]
for some constant \( C_b > 0 \) depending on \( b \). Therefore, the expression in Eq. \( \text{(86)} \) is
\[
c_2(b)c_3(b)^{(c_1+1)/2}(A_b(n)b^{-1} + O(e^{-C_b \sqrt{n}})) I_{1+c_1} \left[ \frac{\pi}{b} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right].
\]
It follows that
\[
\sum_{\substack{k \in [b] \\ 0 < k \leq \text{lcm}(M)}} c_2(k)c_3(k)^{(c_1+1)/2} A_k(n)k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n - n_0)} \right]
\]
is
\[
c_2(b)c_3(b)^{(c_1+1)/2}(A_b(n)b^{-1} + O(e^{-C_b \sqrt{n}})) I_{1+c_1} \left[ \frac{\pi}{b} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right].
\]
We can sum this result for all \( b \in \{1, \ldots, \text{lcm}(M)\} \) with \( c_3(b) > 0 \). We can absorb the terms with \( b \notin K \) into the error term. So, for some constant \( C \) depending only on \( \{\delta_m\}_{m=1}^\infty \),
\[
d(n) = 2\pi \left( \frac{1}{24(n - n_0)} \right)^{\left(\frac{c_1+1}{2}\right) \sum_{k \in K} (A_k(n)k^{-1} + O(e^{-C_b \sqrt{n}}))c_2(k)c_3(k)^{\frac{c_1+1}{2}} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n - n_0)} \right].
\]
Since for \( k \in K \) it is the case that \( c_3(k) = k^2c_3 \),
\[
d(n) = 2\pi \left( \frac{c_3}{24(n - n_0)} \right)^{\left(\frac{c_1+1}{2}\right) I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(n - n_0)} \right] \sum_{k \in K} (A_k(n)k^{-1} + O(e^{-C_b \sqrt{n}}))c_2(k)k^{1+c_1}.
\]
Note that each \( O(e^{-C_b \sqrt{n}}) \) in the sum over \( k \) in Eq. \( \text{(99)} \) is different. Nevertheless, using the assumption in Eq. \( \text{(83)} \)
\[
d(n) = (1 + O(e^{-C_b \sqrt{n}}))2\pi \left( \frac{c_4}{24(n - n_0)} \right)^{\left(\frac{c_1+1}{2}\right) I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(n - n_0)} \right] \sum_{k \in K} A_k(n)c_3^e c_2(k),
\]
as claimed, where now the \( O(e^{-C_b \sqrt{n}}) \) term is bounded in terms of \( \{\delta_m\}_{m=1}^\infty \) and \( \epsilon \).

\[ \square \]

4. Numerics

In this section we numerically test Eq. \( \text{(16)} \) for several \( \eta \)-quotients \( Z(q) \). Here \( d(n, N) \) represents the \( N \)th partial sum of the right hand side of Eq. \( \text{(16)} \) and \( d(n) \) represents the Fourier coefficients of \( Z(q) \cdot q^{\eta n} \). The following \( \eta \)-quotients all satisfy the hypotheses of Theorem \( \text{(1.1)} \). The absolute difference between \( d(n, N) \) and \( d(n) \) for \( n \in \{1, \ldots, 20\} \) with \( n > n_o \) and \( N \in \{1, \ldots, 100\} \) is plotted in Fig. 4 for
\begin{itemize}
  \item (1) \( Z(q) = 1/\eta(4\tau)\eta(\tau)^3 \) (upper left),
  \item (2) \( Z(q) = \eta(4\tau)/\eta(\tau)^3 \) (upper right),
  \item (3) \( Z(q) = 1/\eta(2\tau) \) (middle left),
  \item (4) \( Z(q) = 1/\eta(11\tau)^2\eta(\tau)^2 \) (middle right),
  \item (5) \( Z(q) = 1/\eta(\tau)\eta(22\tau) \) (bottom left),
  \item (6) \( Z(q) = 1/\eta(\tau)\eta(23\tau) \) (bottom right).
\end{itemize}
The convergence of \( d(n, N) \) to \( d(n) \) as \( N \to \infty \) is clear, although a few trends are worth noting. The first is that the convergence of \( d(n, N) \) to \( d(n) \) is rather haphazard. The second is that the rate of convergence of \( d(n, N) \) to \( d(n) \) may depend significantly on \( n \). This appears to be the case for the final few \( \eta \)-quotients above.
Figure 4. Convergence of the $N$th partial sums of Eq. (16) for the listed $\eta$-quotients:

- $Z(q) = 1/\eta(4\tau)\eta(\tau)^3$ (upper left),
- $Z(q) = \eta(4\tau)/\eta(\tau)^3$ (upper right),
- $Z(q) = 1/\eta(2\tau)$ (middle left),
- $Z(q) = 1/\eta(11\tau)\eta(\tau)^2$ (middle right),
- $Z(q) = 1/\eta(\tau)\eta(22\tau)$ (bottom left),
- $Z(q) = 1/\eta(\tau)\eta(23\tau)$ (bottom right).

The vertical axis is $|d(n,N) - d(n)|$ and the horizontal axis is $N$. The vertical axis is scaled logarithmically and the horizontal axis is scaled linearly.

Each line is a plot of $d(n,N)$ for fixed $n$ and variable $N$. The lines are different shades of gray to help visually distinguish them.
Acknowledgements

I would like to thank Prof. Shamit Kachru not only for introducing me to the topic of small black holes, but also for taking the time to patiently and kindly answer my many questions. I am also grateful to the other students in SITP for making it such a welcoming place, and especially Richard Nally and Brandon Rayhaun for their helpful conversations and general advice. Finally, I would like to express my gratitude towards Prof. John Duncan for pointing out an error in an earlier version of this paper. While working on this project I was funded by the Stanford physics department summer research program and a Stanford UAR major grant.

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