NODAL SOLUTIONS FOR A QUASILINEAR SCHRÖDINGER EQUATION WITH CRITICAL NONLINEARITY AND NON-SQUARE DIFFUSION

YINBIN DENG
Department of Mathematics
Huazhong Normal University, Wuhan, 430079, China

YI LI
Department of Mathematics and Statistics
Wright State University, Dayton, OH 45435, USA

XIUJUAN YAN
Department of Mathematics
Huazhong Normal University, Wuhan, 430079, China

Abstract. This paper is concerned with a type of quasilinear Schrödinger equations of the form
\[-\Delta u + V(x)u - p\Delta(|u|^{2p})|u|^{2p-2}u = \lambda|u|^q - 2u + |u|^{2p^*-2}u,\]
where \(\lambda > 0\), \(N \geq 3\), \(4p < q < 2p^*\), \(2^* = \frac{2N}{N-2}\), \(1 < p < +\infty\). For any given \(k \geq 0\), by using a change of variables and Nehari minimization, we obtain a sign-changing minimizer with \(k\) nodes.

1. Introduction and main results. In this paper we consider the existence of \(k\)-node solutions of the following quasilinear Schrödinger equations with pure power nonlinearities:
\[
\left\{
\begin{array}{l}
-\Delta u + V(x)u - p\Delta(|u|^{2p})|u|^{2p-2}u = \lambda|u|^q - 2u + |u|^{2p^*-2}u, \quad x \in \mathbb{R}^N, \\
u \to 0, \text{ as } |x| \to \infty,
\end{array}
\right.
\]
where \(\lambda > 0\), \(N \geq 3\), \(4p < q < 2p^*\), \(2^* = \frac{2N}{N-2}\), \(1 < p < +\infty\). We assume that \(V(x)\) is bounded radially symmetric function and satisfies
\((V_1)\) : \(V \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N), V(0) > 0 \) and \(V'(r) \geq 0\) for all \(r = |x| \in (0, +\infty)\).

Such types of equations have been derived as models of several physical phenomena and have been the subject of extensive study in recent years. For example, solutions to (1.1) are related to the solitary wave solutions for quasilinear Schrödinger equations of the form
\[
i\partial_t z = -\Delta z + W(x)z - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z,
\]
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where \( z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \ W : \mathbb{R}^N \to \mathbb{R} \) is a given potential, \( \kappa \) is a real constant and \( f, h : \mathbb{R}^+ \to \mathbb{R} \) are suitable functions. Quasilinear equation (1.2) appears naturally in mathematical physics and have been derived as modes of several physical phenomena corresponding to various types of \( h \). The case \( h(s) = s \) modes the time evolution of the condensate wave function in super-fluid film ([21, 23]). This equation has been called the superfluid film equation in fluid mechanics by Kurihara [21]. In the case \( h(s) = (1 + s)^{\frac{1}{2}} \), problem (1.2) modes the self-channeling of a high-power ultra short laser in matter (see [7, 10, 14, 34]).

Let \( z(x, t) = \exp(-iEt)u(x) \) in (1.2), we find that \( u(x) \) solves the following elliptic equation
\[
\begin{cases}
-\Delta u + V(x)u - \kappa \Delta h(u^2)u = f(u^2)u, & x \in \mathbb{R}^N, \\
u \to 0, & |x| \to \infty,
\end{cases}
\tag{1.3}
\]
where \( V(x) = W(x) - E \) is a new potential.

Considering the case \( h(s) = s \), problem (1.3) becomes
\[
\begin{cases}
-\Delta u + V(x)u - \kappa \Delta u^2u = f(u^2)u, & x \in \mathbb{R}^N, \\
u \to 0, & |x| \to \infty,
\end{cases}
\tag{1.4}
\]
In this case, \( q \in [4, 22^*] \) if we take \( f(s^2)s \sim |s|^{q-2}s \) as \( s \to \infty \), is called subcritical exponent in spirit of [27]. This case was studied extensively in the past several years. We can refer to the references [12, 16, 24, 25, 26, 27, 28, 33] where positive or sign-changing solutions were obtained by using the variational argument. The case \( q = 22^* \) correspond to a critical growth for (1.4). In fact, it was shown in [27] by using a variational identity that (1.4) has no positive solution in \( H^1(\mathbb{R}^N) \) with \( u^2|\nabla u|^2 \in L^1(\mathbb{R}^N) \) if \( f(u^2)u = |u|^{q-2}u, q \geq 22^* \) and \( V \) satisfies \( \nabla v(x) \cdot x \geq 0 \) in \( \mathbb{R}^N \). As pointed by Liu et al. in [26], the critical case for (1.4) is very interesting. Concerning this case, Moanemi in [30] considered the related singularly perturbed problem and obtained a positive radial solution in the radially symmetric case. Later on, an existence result of positive solutions was given by João Marcos et al. in [6] via Mountain-Pass lemma. Recently, Liu et al. considered the existence of positive solutions for general quasilinear elliptic equations in [22] by perturbation method. Y. Deng, S. Peng and J. Wang in [17] studied (1.4) for the case of critical growth, i.e. \( f(u^2)u = \lambda |u|^{q-2}u + |u|^{2p^*-2}u \), where \( \lambda > 0, \lambda < q < 22^* \). They constructed infinitely many sign-changing solutions for (1.4) by the Nehari method. It is worth pointing out that for the related semilinear equations for \( \kappa = 0 \), the existence of radial sign-changing solutions has been explored thoroughly, we refer the readers to [3, 9, 13, 15, 38] and the references therein.

However, there seems to be little progress on the existence and nonexistence of nontrivial solution for the case when \( h(s) = s^p \ (1 < p < +\infty) \). In this case, problem (1.3) becomes
\[
\begin{cases}
-\Delta u + V(x)u - \kappa p \Delta (|u|^{2p})|u|^{2p-2}u = f(u^2)u & x \in \mathbb{R}^N, \\
u \to 0, & |x| \to \infty,
\end{cases}
\tag{1.5}
\]
and \( q \in [4p, 2p^*) \) if we take \( f(s^2)s \sim |s|^{q-2}s \) as \( s \to \infty \), is called subcritical exponent and \( 2p^* \) is the correspond critical exponent. Comparing with (1.4), the quasilinear term \( \kappa p \Delta (|u|^{2p})|u|^{2p-2}u \) for \( p > 1 \) in problem (1.5) is called the non-square diffusion; meanwhile, the quasilinear term \( \kappa \Delta (u^2)u \) in (1.4) is called the square diffusion.
In this paper, we focus on the case when $h(s) = s^p$ $(1 < p < +\infty)$, $\kappa = 1$ and $f(s^2)s = \lambda|s|^{q-2}s + |s|^{2p^2-2}s$, i.e. problem (1.1). We will construct infinitely many solutions for problem (1.1) with critical growth by the Nehari method. To this end, we introduce some definitions and notations.

For any given set $K \subset \mathbb{R}^N$, we denote $(\int_K |u(x)|^sdx)^{\frac{1}{s}} =: ||u||_{s,K}$ and we write $\int_K$ instead of $\int_{\mathbb{R}^N}$. We use ” $\rightharpoonup$ ” and ” $\rightarrow$ ” to denote the strong and weak convergence in the related function space respectively. $C$ will denote a positive constant unless specified. We denote by $u^+$ and $u^-$ the functions defined by $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$. For a radial domain $\Omega$, we set

$$H^1_0(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) : u(x) = u(|x|) \}$$

and

$$H^1_0(\Omega) = \{ u \in H^1_0(\mathbb{R}^N) : u(x) = u(|x|) \}$$

which is equivalent to the usual norm in $H^1(\mathbb{R}^N)$ by ($V_1$).

The weak form of the equation (1.1) is

$$\int_{\mathbb{R}^N} (1+2p^2)|u|^{4p-2}\nabla u \nabla \phi + p^2(4p^2-2) \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{4p-4}u\phi + \int_{\mathbb{R}^N} V(x)u\phi = \int_{\mathbb{R}^N} k(u)\phi,$nabla u (∇φ + p^2(4p^2-2)…}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$, which is formally the variational formulation of the following functional:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1+2p^2)|u|^{4p-2}||\nabla u||^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 - \int_{\mathbb{R}^N} K(u)$$

where $K(s) = \frac{\lambda}{q} |s|^q + \frac{1}{2p^2} |s|^{2p^2} = K'(s) = \lambda |s|^{q-2}s + |s|^{2p^2-2}s$.

We point out that we cannot apply directly variational method here because $I(u)$ is not well defined in $H^1(\mathbb{R}^N)$. To overcome this difficulty, we generalize the argument developed by Liu-Wang-Wang in [26] and make the change of variables $v = f^{-1}(u)$, where $f$ is defined by the following ODE

$$f'(t) = \frac{1}{(1+2p^2)|f(t)|^{4p-2}}$$

on $[0, +\infty)$,

$$f(-t) = -f(t)$$

on $(-\infty, 0]$.

After the change of variables, $I(u)$ can be reduced to the following functional

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} ||\nabla v||^2 + V(x)f^2(v) - \int_{\mathbb{R}^N} K(f(v)),$nabla u (∇φ + p^2(4p^2-2)…}$$

which is well defined on the usual Sobolev space $H^1(\mathbb{R}^N)$ under suitable assumptions on the potential $V(x)$ and the nonlinearity $K(s)$. Moreover, the nontrivial critical points of the functional $J$ correspond precisely to the nontrivial weak solutions of the following equation

$$-\Delta v = f'(v)[k(f(v)) - V(x)f(v)] \text{ in } \mathbb{R}^N. \tag{1.6}$$

For the convenience, we rewrite equation (1.6) in the following form:

$$\left\{ \begin{array}{l}
-\Delta v + V(x)v = g(x, v) + \frac{1}{p} 4\frac{\lambda}{q} |s|^{q-2}v, \\
v \to 0, \text{ as } |x| \to \infty, 
\end{array} \right. \tag{1.7}$$

\[\]
and the corresponding variational functional is

\[ J(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v|^2 + V(x)|v|^2 \right) \, dx - \frac{4\pi^{1/2}}{p2^{*}} \int_{\mathbb{R}^N} |v|^{2^*} - \int_{\mathbb{R}^N} G(x,v), \quad v \in H^1(\mathbb{R}^N), \]

where

\[ g(x,v) = f'(v)[k(f(v)) - V(x)f(v)] + V(x)v - \frac{1}{p} \frac{4^{1/p} \pi^{1/2}}{2^{*}} |v|^{2^* - 2}, \quad (1.8) \]

and

\[ G(x,v) = \int_0^v g(x,s) \, ds = K(f(v)) + \frac{1}{2} V(x)|v|^2 - \frac{1}{2} V(x)f(v)^2 - \frac{4^{1/p} \pi^{1/2}}{2^{*}} |v|^{2^*}. \quad (1.9) \]

We can easily verify that problem (1.1) is equivalent to problem (1.7) and the nontrivial critical points of \( J(v) \) are the nontrivial solutions of problem (1.7). In the following, we construct infinitely many \( k - \) node solutions for problem (1.7).

For any \( k \in \{0,1,2,\cdots\} \), \( v_\pm \) is said to be a pair of \( k \)-node solution of (1.7) if \( v_\pm \) is a radial solution with the following properties:

(i) \( v_-(0) < 0 < v_+(0) \);
(ii) \( v_\pm \) possess exactly \( k \) nodes \( r_i \) with \( 0 < r_1 < r_2 < \cdots < r_k < +\infty \), and \( v_\pm(r_i) = 0, \, i = 1,2,\cdots, k \).

Our main result can be stated in the following Theorem:

**Theorem 1.1.** Assume that \( V(x) \) satisfies (V1). Then for all \( \lambda > 0 \), problem (1.7) which is equivalent to (1.1), exists at least one pair of \( k \)-node solutions if either \( N \geq 6 \) and \( 4p < q < 2p2^* \) or \( 3 \leq N < 6 \) and \( \frac{2p(N+2)}{N-2} < q < 2p2^* \). Moreover, problem (1.7) still exists at least one pair of \( k \)-node solutions if \( 3 \leq N < 6 \), \( 4p < q < \frac{2p(N+2)}{N-2} \) and \( \lambda \) sufficiently large.

To find nontrivial critical points of \( J \), difficulties lie in two aspects. The first difficulty is caused by the usual lack of compactness since the problem involve critical exponent and are dealt with in the whole \( \mathbb{R}^N \). To overcome this difficulty, we should work out a threshold value of energy under which a (PS) sequence is pre-compact. The second difficulty lies in a new phenomenon in which the nonlinear term \( g(x,s) \) only satisfies \( \lim_{s \to \infty} \frac{g(x,s)}{|s|^{2^* - 1}} = 0 \) instead of the usual subcritical condition \( g(x,s) = o(|s|^t) \), \( 2 < t < 2^* \) at infinity. Further more, as we will see later, the functions \( G(x,v), \, g(x,v)v \) given by (1.8), (1.9) and \( \frac{1}{2} g(x,v)v - G(x,v) \) may change sign. These two new phenomena cause two more difficulties. On one hand, the usual Ambrosetti-Rabinowitz condition is not satisfied. On the other hand, the usual argument to verify that the functional corresponding to (1.7) satisfies the (PS) condition cannot be employed directly. Hence, we need to analyze the exact asymptotic behavior of \( g(x,s) \) and should apply more delicate analysis to the functional corresponding to (1.7).

To construct nodal solutions for (1.7), we will look for a minimizer of a constrained minimization problem in a special space in which each function changes sign \( k \) \( (k \in \{0,1,2,\cdots\}) \) times and then verify that the minimizer is smooth and indeed a solution to (1.7) by analyzing the least energy related to the minimizer. We mention here that the main method to prove our theorem was essentially introduced by Bartsch and Willem in [9], Cao and Zhu in [9] and G. Cerami, S. Solimini and M. Struwe in [11].
It should be mentioned that Shen and Wang [35] studied the problem (1.3) for general \( h(s) \) and an existence result for positive solution was obtained when \( f(s^2)s \) is subcritical. Some interesting results about the existence and nonexistence of solutions for quasilinear elliptic problems can be found in [31, 18, 36, 4, 19].

The paper is organized as follows: in Section 2, we will provide some useful lemmas. The existence of positive solution is proved in Section 3 and Theorem 1.1 is proved in Section 4 respectively.

2. Some preliminary lemmas. In this Section, we give some properties of \( f, g \) and \( G \) which are defined in the introduction.

**Lemma 2.1.** The function \( f(t) \) enjoys the following properties:

1. \( f \) is uniquely defined \( C^\infty \) function and invertible.
2. \( |f'(t)| \leq 1 \) for all \( t \in \mathbb{R} \).
3. \( f(t) \leq |t| \) for all \( t \in \mathbb{R} \).
4. \( \frac{f(t)}{t} \rightarrow 1 \) as \( t \rightarrow 0 \).
5. \( \frac{f'(t)}{\sqrt{t}} \rightarrow 2^\frac{1}{4} \) as \( t \rightarrow +\infty \).
6. \( \frac{f(t)}{t^2} \leq tf'(t) \leq f(t) \) for all \( t \geq 0 \).
7. \( |f^p(t)| \leq 2^\frac{1}{4} |t|^\frac{1}{4} \) for all \( t \in \mathbb{R} \).
8. \( |f^{2p-1}(t)f'(t)| \leq \frac{1}{\sqrt{2^p}} \) for all \( t \in \mathbb{R} \).
9. There exists a positive constant \( C \) such that

\[
|f(t)| \geq \begin{cases} 
C|t|, & |t| \leq 1, \\
C|t|^\frac{1}{4}, & |t| \geq 1.
\end{cases}
\]

Thus, there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
|t| \leq C_1|f(t)| + C_2|f(t)|^2p \quad \text{for all} \quad t \in \mathbb{R}.
\]

**Proof.** Points (1)-(4) are immediate. It’s easy to obtain (5) using L’Hospital rule.

To establish the first inequality of (6), we need to show that, for all \( t \geq 0 \),

\[
f(t)\sqrt{1 + 2p^2|f(t)|^{2p-2}} \leq 2pt.
\]

In this aim we study the function \( g : \mathbb{R}^+ \rightarrow \mathbb{R}, g(t) = 2pt - f(t)\sqrt{1 + 2p^2|f(t)|^{2p-2}}, \)

we have \( g(0) = 0 \), and \( g'(t) = (2p-1)f^2(t) \geq 0 \). Thus the first inequality is proved.

The second one is derived in a similar way.

To prove (7), we use (4), (5), (6). Indeed, according to (4), we have

\[
\lim_{t \to 0^+} \frac{f^p(t)}{\sqrt{t}} \rightarrow 0,
\]

and (6) implies that

\[
\frac{d}{dt} \left( \frac{f^p(t)}{\sqrt{t}} \right) = \frac{f^{p-1}(t)(2ptf'(t) - f(t))}{2t\sqrt{t}} \geq 0 \quad \text{for all} \quad t > 0.
\]

Consequently, the function \( \frac{f^p(t)}{\sqrt{t}} \) is nondecreasing for \( t > 0 \) and from (5) we conclude that

\[
\frac{f^p(t)}{\sqrt{t}} \leq 2^\frac{1}{4} \quad \text{for all} \quad t > 0.
\]
This together with the fact that \( f \) is odd proves (7).

Estimate (8) follows directly from the definition of \( f \) and we can easily get (9) combining (4) and (5). The Lemma is proved.

**Lemma 2.2.** There exist two positive constants \( \alpha \) and \( \beta \) such that
\[
(f(t))^{2p^2} \geq \alpha t^{2^*} - \beta \quad \text{for all} \quad t \geq 0.
\]

**Proof.** From Lemma 2.1(9), there exists \( a, b > 0 \) such that
\[
a f^{2p}(t) + b f(t) - t \geq 0 \quad \text{for all} \quad t \geq 0.
\]

If \( t \geq 1 \), we have that
\[
a f^{2p}(t) + B f^p(t) - t \geq 0 \quad \text{for all} \quad t \geq 1,
\]

since \( f(t) \geq f(1) \), where \( B = b \frac{1}{f^{p-1}(1)} \). Thus
\[
f^p(t) \geq \frac{-B + \sqrt{B^2 + 4at}}{2a} \quad \text{for all} \quad t \geq 1
\]
and
\[
f^{2p}(t) \geq \frac{1}{4a^2} \left[ 2B^2 + 4at - 2b \sqrt{B^2 + 4at} \right] \\
\geq \frac{1}{4a^2} \left[ 2B^2 + 4at - \frac{1}{2} (2b)^2 + B^2 + 4at \right] \\
= \frac{2B^2}{2a} - C.
\]
i.e.
\[
\left( \frac{t}{2a} \right)^{2^*} \leq \left( f^{2p}(t) + C \right)^{2^*} \leq C_1 f^{2p^2}(t) + C_2,
\]
which gives that
\[
f^{2p^2}(t) \geq C_3 t^{2^*} - C_4 \quad \text{for all} \quad t \geq 1. \tag{2.1}
\]

On the other hand, if \( t \leq 1 \), we have that
\[
a f^{2p}(t) + b f(1) - t \geq a f^{2p}(t) + b f(t) - t \geq 0
\]
since \( f(t) \leq f(1) \). Thus
\[
t \leq a f^{2p}(t) + b f(1),
\]
and hence
\[
t^{2^*} \leq C_5 f^{2p^2}(t) + C^{2^*} \leq C_5 f^{2p^2}(t) + C_6,
\]
which implies that
\[
f^{2p^2}(t) \geq C_7 t^{2^*} - C_8 \quad \text{for all} \quad t \leq 1. \tag{2.2}
\]

Our Lemma now follows from (2.1) and (2.2).

**Lemma 2.3.** \( G(x, s) \) and \( g(x, s) \) satisfy the following properties:

\[
(G_1) \quad \lim_{s \to 0} \frac{G(x, s)}{s^2} = 0.
\]
\[
(G_2) \quad \lim_{s \to \infty} \frac{G(x, s)}{s^{2^*}} = 0.
\]
\[
(G_3) \quad \lim_{s \to 0} \frac{g(x, s)}{s} = 0.
\]
Using Lemma 2.1(5) and obstacle when one tries to obtain nontrival solutions. To overcome those obstacle, thus we have

\[ G(x, v) \leq \varepsilon (v^2 + |v|^2) + C_{\varepsilon} |v|^{\gamma}, \]
\[ g(x, v)v \leq \varepsilon (v^2 + |v|^2) + C_{\varepsilon} |v|^{\gamma}. \]

Proof. In fact, we must analyze the terms

\[ \frac{f^q(s)}{s^2} = \left( \frac{f(s)}{s} \right)^2 f^{q-2}(s) \quad \text{and} \quad \frac{f^{2p^2}(s)}{s^2} = \left( \frac{f(s)}{s} \right)^2 f^{2p^2-2}(s) \quad (2.3) \]

Since \( 2 < 4p < q < 2p^2 \), from Lemma 2.1(3)-(4), the two terms in (2.3) converges to zero, as \( s \to 0 \). Thus (G1) holds. Similarly we can prove property (G3). Now by \( q < 2p^2 \) and Lemma 2.1(5), we have

\[ \frac{f^q(s)}{s^2} = \left( \frac{f(s)}{s} \right)^2 s^{q-2} \to 0 \quad \text{as} \quad s \to \infty. \quad (2.4) \]

Also, from Lemma 2.1(3), we have

\[ 0 \leq \frac{1}{2} V(x) \left( \frac{|s|^2}{|s|^2} - \frac{|f(s)|^2}{|s|^2} \right) \leq \frac{1}{2} V(x) \frac{|s|^2}{|s|^2}, \]

thus

\[ \frac{1}{2} V(x) \left( \frac{|s|^2}{|s|^2} - \frac{|f(s)|^2}{|s|^2} \right) \to 0 \quad \text{as} \quad s \to \infty. \quad (2.5) \]

On the other hand, from Lemma 2.1(5), the term

\[ \frac{f^{2p^2}(s)}{s^{2p^2}} = \left( \frac{f(p)}{s} \right)^{2p^2} \to 2^{p^2-2} \quad \text{as} \quad s \to \infty. \quad (2.6) \]

Combining (2.4)-(2.6), we have (G2). Similarly we can prove (G4).

Lemma 2.1 (7) suggest that the functions \( G(x, v), \ g(x, v)v \) and \( \frac{1}{2}g(x, v)v - G(x, v) \) may be sign-changing. As we can see later, this fact causes more new obstacle when one tries to obtain nontrival solutions. To overcome those obstacle, we need to analyze the properties of \( f(t) \) for \( t \) large.

Lemma 2.4. There exists a positive constant \( A \) such that

\[ \left( \frac{f^p(t)}{\sqrt{t}} \right)^4 \geq 2 - At^{-1} \quad \text{for large} \quad t > 0. \quad (2.7) \]

Proof. By the definition of \( f'(t) \), we see \( f'(t)\sqrt{1 + 2p^2|f(t)|^{4p-2}} = 1 \)

\[ t = \int_0^t f'(s)\sqrt{1 + 2p^2|f(s)|^{4p-2}} ds \]
\[ = \int_0^{f(t)} \sqrt{1 + 2p^2|s|^{4p-2}} ds \]
\[ = \frac{f(t)}{2p} \sqrt{1 + 2p^2|f(s)|^{4p-2}} + \frac{2p-1}{2p} \int_0^{f(t)} \frac{ds}{\sqrt{1 + 2p^2|s|^{4p-2}}} \]

Using Lemma 2.1(5) and \( p > 1 \), we deduce that

\[ 1 = \frac{f(t)}{2p\sqrt{t}} \left( \frac{1}{t} + \frac{2p^2 f^{4p-2}(t)}{t} \right)^{\frac{1}{2}} + O(t^{-1}) \]
\[ = \frac{f^p(t)}{2p\sqrt{t}} \left( \frac{1}{t^{2p-2}(t)} + \frac{2p^2 f^{2p}(t)}{t} \right)^{\frac{1}{4}} + O(t^{-1}) \quad \text{as} \quad t \to \infty. \]
Denote \( \omega = \left( \frac{f^p(t)}{\sqrt{t}} \right)^2 \), we obtain that
\[
2p^2\omega^2 + \frac{1}{t^{2p-2}(t)}\omega - 4p^2 + O(t^{-1}) = 0 \quad \text{as} \quad t \to \infty,
\]
which gives that
\[
\omega = -\frac{1}{t^{2p-2}(t)} + \sqrt{\frac{1}{t^{2p-4}(t)} + 8p^2(4p^2 - O(t^{-1}))}
\]
as \( t \to \infty. \)

Thus
\[
\omega^2 = 2 - \frac{1}{8p^4 t^{2p-2}(t)} \sqrt{\frac{1}{t^{2p-4}(t)} + 8p^2(4p^2 - O(t^{-1}))} + \frac{1}{8p^4 t^{4p-4}(t)} - O(t^{-1})
\]
\[
\geq 2 - At^{-1}, \quad \text{for large} \quad t > 0,
\]
i.e.,
\[
\left( \frac{f^p(t)}{\sqrt{t}} \right)^4 \geq 2 - At^{-1} \quad \text{for large} \quad t > 0.
\]

**Remark 2.5.** Using above Lemma we can prove that
\[
\frac{1}{2} \left( \frac{f^p(t)}{\sqrt{t}} \right)^{22^*} \geq 4 \frac{1}{2 - At^{-1}} - Bt^{-1}
\]
(2.8)
f for a positive constant \( B \) if \( t \) is large enough.

In fact, by Lemma 2.4 and noting \((a - b)^\alpha \geq a^\alpha - \alpha a^{\alpha - 1}b\) for all \( 0 \leq b \leq a \) and \( \alpha \geq 1 \), we have
\[
\frac{1}{2} \left( \frac{f^p(t)}{\sqrt{t}} \right)^{22^*} = \frac{1}{2} \left( \left( \frac{f^p(t)}{\sqrt{t}} \right)^4 \right)^{\frac{22^*}{4}} \geq \frac{1}{2} \left( 2 - At^{-1} \right)^{22^*}
\]
\[
\geq \frac{1}{2} \left( 2^{22^*} - \frac{22^*}{2} A2^{22^* - 1}t^{-1} \right)
\]
\[
= 4 \frac{1}{2 - At^{-1}} - Bt^{-1}
\]
for large \( t \).

3. **The existence of positive solutions.** In this section, we prove the existence of positive solution for problem (1.7) which is equivalent to problem (1.1), by using Mountain-Pass Lemma due to Ambrosetti-Rabinowitz [1]. Using Lemmas in Section 2 and proceeding as done in [6] or [17] we can verify that the functional \( J \) exhibits the Mountain-Pass geometry.

**Lemma 3.1.** The functional \( J \) satisfies
(i) there exist \( \alpha, \rho > 0 \) such that \( J(v) \geq \alpha \) for all \( \|v\| = \rho; \)
(ii) there exists \( w \in H^1(\mathbb{R}^N) \), such that \( \|w\| > \rho \) and \( J(w) < 0 \).

As a consequence of Lemma 3.1 and Mountain-Pass lemma, for the constant
\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)) > 0, \quad \text{(3.1)}
\]
where
\[
\Gamma = \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)), \gamma(0) = 0, \gamma(1) \neq 0, J(\gamma(1)) < 0 \},
\]
there exists a \( (PS)_c \) sequence \( \{v_n\} \) in \( H^1(\mathbb{R}^N) \) at the level \( c \), that is,
\[
J(v_n) \to c \quad \text{and} \quad J'(v_n) \to 0, \quad \text{as} \quad n \to \infty.
\]
We will verify that the level value $c$ is in an interval where the $(PS)$ condition hold. To this end, we introduce a well-known fact that the minimization problem

$$S = \inf \{ \| \nabla v \|^2_2 : v \in D^{1,2}(\mathbb{R}^N), |v|^{2^*} = 1 \}$$

has a solution given by $w_c(x) = (N(N - 2)\epsilon)^{\frac{N-2}{2}} \left( \epsilon + |x|^2 \right)^{\frac{N-2}{2}}$, and

$$|\nabla w_c|^2 = |w_c|^{2^*} = S^\frac{2}{N}.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be a radial cut-off function such that $\varphi(|x|) = 1$ for $|x| \leq \rho_\epsilon$, $\varphi(|x|) \in (0, 1)$ for $\rho_\epsilon < |x| < 2\rho_\epsilon$, and $\varphi(|x|) = 0$ for $|x| \geq 2\rho_\epsilon$, where $\rho_\epsilon = \epsilon^\tau$, $\tau \in (\frac{1}{4}, \frac{1}{2})$. Set $\psi_\epsilon(x) = \varphi(x)w_c(x)$, then we get the following estimations (see [2, 11]):

**Lemma 3.2.** $\psi_\epsilon(x)$ satisfies the following estimations: as $\epsilon \to 0$,

$$\int_{\mathbb{R}^N} |\nabla \psi_\epsilon|^2 = S^\frac{2}{N} + O(\epsilon^{\frac{N-2}{2}}), \quad \int_{\mathbb{R}^N} |\psi_\epsilon|^{2^*} = S^\frac{2}{N} + O(\epsilon^{\frac{N}{2}}),$$

$$\int_{\mathbb{R}^N} |\psi_\epsilon| \leq C\epsilon^{\frac{N-2}{2}}, \quad \int_{\mathbb{R}^N} |\psi_\epsilon|^{2^*-1} \leq C\epsilon^{\frac{N-2}{2}}, \quad \int_{\mathbb{R}^N} |\nabla \psi_\epsilon| \leq C\epsilon^{\frac{N-2}{2}}.$$

and

$$\int_{\mathbb{R}^N} |\psi_\epsilon|^2 = \begin{cases} C\epsilon + O(\epsilon^{\frac{N-2}{2}}) & \text{for } N \geq 5, \\ C\epsilon \ln \epsilon + O(\epsilon^{\frac{N-2}{2}}) & \text{for } N = 4, \\ O(\epsilon^{\frac{1}{2}}) & \text{for } N = 3. \end{cases}$$

The following lemma provides the interval where the $(PS)$ condition holds for $J(v)$.

**Lemma 3.3.** $J$ satisfies $(PS)_c$ condition if the level value $c < \frac{1}{2N}p^\frac{N}{2^* - 1}S^\frac{2}{N}.$

The proof of this Lemma is similar to the proof of Lemma 3.3 in [17], so we omit the detail here.

By Lemma 3.1, Lemma 3.3 and Mountain-Pass lemma, we can easily verify the following lemma:

**Lemma 3.4.** Suppose that there exists $v_0 \in H^1_r(\mathbb{R}^N), v_0 \not\equiv 0$ such that

$$\sup_{t \geq 0} J(tv_0) < \frac{1}{2N}p^\frac{N}{2^* - 1}S^\frac{2}{N},$$

then problem (1.7) (and hence (1.1)) possesses at least one positive weak solution.

Now we are ready to prove the following result.

**Theorem 3.5.** Suppose that $V(x)$ satisfies $(V_1)$. Then for all $\lambda > 0$, problem (1.7) has a positive radial solution if either $N \geq 6$ and $4p < q < 2p^{2^*}$ or $3 \leq N < 6$ and $\frac{2p(N+2)}{N-2} < q < 2p^{2^*}$. Moreover, problem (1.7) still exists a positive radial solution if $3 \leq N < 6$, $4p < q \leq \frac{2p(N+2)}{N-2}$ and $\lambda$ sufficiently large.

**Proof.** From Lemma 3.4, we only need to verify that the condition (3.2) holds naturally.

We claim first that for $\epsilon > 0$ small enough, there exists a constant $t_\epsilon > 0$ such that

$$J(t_\epsilon \psi_\epsilon) = \max_{t \geq 0} J(t \psi_\epsilon)$$
and
\[
0 < A_1 < t_\epsilon < A_2 < +\infty \quad \text{for all } \epsilon > 0 \text{ small enough}, \tag{3.3}
\]
where \(A_1\) and \(A_2\) are positive constants independent of \(\epsilon\).

In fact, since \(J(0) = 0, \lim_{t \to \infty} J(t\psi_\epsilon) = -\infty\), there exists \(t_\epsilon > 0\) such that
\[
J(t_\epsilon \psi_\epsilon) = \max_{t \geq 0} J(t\psi_\epsilon) \quad \text{and} \quad \frac{dJ(t\psi_\epsilon)}{dt} \bigg|_{t=t_\epsilon} = 0.
\]

Thus we have
\[
\frac{||\psi_\epsilon||^2}{|\psi_\epsilon|_2^2} - \frac{1}{p} \frac{4^{\frac{4}{p-2}} t_\epsilon^{2^* - 2}}{|t\psi_\epsilon|_2^{2^*}} - \int_{\mathbb{R}^N} \frac{g(t\psi_\epsilon)|\psi_\epsilon|^{2^*}}{t\epsilon|\psi_\epsilon|_2^{2^*}} = 0. \tag{3.4}
\]

By Lemmas 2.3 and Lemma 3.2, for any \(\delta > 0\) there exists a constant \(C > 0\) such that
\[
\int_{\mathbb{R}^N} \frac{g(t\psi_\epsilon)|\psi_\epsilon|^{2^*}}{t\epsilon|\psi_\epsilon|_2^{2^*}} \leq \int_{\mathbb{R}^N} \delta t_\epsilon^{2^*} + \frac{Ct_\epsilon^{2^*} + 2}{t\epsilon|\psi_\epsilon|_2^{2^*}} = \delta t_\epsilon^{2^*} + C \left(S^2 + O(\epsilon^{\frac{N-2}{2}})\right) - \frac{1}{p} \frac{4^{\frac{4}{p-2}} t_\epsilon^{2^* - 2}}{|t\psi_\epsilon|_2^{2^*}}.
\]

Thus,
\[
1 - \frac{1}{p} \frac{4^{\frac{4}{p-2}} t_\epsilon^{2^* - 2}}{|t\psi_\epsilon|_2^{2^*}} - \delta t_\epsilon^{2^* - 2} + o(1) \leq 0 \quad \text{as } \epsilon \to 0.
\]

Thus,
\[
t_\epsilon \geq \left(2\left(\frac{1}{p} \frac{4^{\frac{4}{p-2}}}{t_\epsilon^{2^* - 2}} + \delta\right)\right)^{-\frac{1}{2^* - 2}} \equiv A_1 > 0 \quad \text{if } \epsilon \text{ small enough}.
\]

On the other hand, from (3.4) we have
\[
\frac{1}{p} \frac{4^{\frac{4}{p-2}} t_\epsilon^{2^* - 2}}{|t\psi_\epsilon|_2^{2^*}} \leq \frac{||\psi_\epsilon||^2}{|\psi_\epsilon|_2^{2^*}} + \int_{\mathbb{R}^N} \frac{|g(t\psi_\epsilon)|\psi_\epsilon}{t\epsilon|\psi_\epsilon|_2^{2^*}} \leq 1 + \delta t_\epsilon^{2^* - 2} + o(1) \quad \text{for } \delta > 0 \text{ small enough, as } \epsilon \to 0.
\]

Thus we have
\[
t_\epsilon \leq \left(\frac{1}{p} \frac{4^{\frac{4}{p-2}}}{t_\epsilon^{2^* - 2}} + o(1)\right)^{\frac{1}{2^* - 2}} < A_2 < +\infty \quad \text{if } \epsilon \text{ small enough}.
\]

Next, we are going to estimate \(J(t_\epsilon \psi_\epsilon)\). From Lemma 3.2 and (3.3), we have
\[
J(t_\epsilon \psi_\epsilon) \leq \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^N} (|\nabla \psi_\epsilon|^2 + V(x)\psi_\epsilon^2) - \frac{4^{\frac{4}{p-2}} t_\epsilon^{2^*}}{p2^*} \int_{\mathbb{R}^N} |\psi_\epsilon|^{2^*} - \int_{\mathbb{R}^N} G(x, t_\epsilon \psi_\epsilon)
\leq \left(\frac{t_\epsilon^2}{2} - \frac{4^{\frac{4}{p-2}} t_\epsilon^{2^*}}{p2^*}\right) S^2 + O(\epsilon^{\frac{N-2}{2}}) + \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^N} V(x)\psi_\epsilon^2 - \int_{\mathbb{R}^N} G(x, t_\epsilon \psi_\epsilon).
Since \( Q(t) = \frac{t^2}{2} - \frac{4\pi^{\frac{p}{2}}}{p^{2^*}} t^{2^*} \) has only maximum at \( t = \frac{1}{\sqrt{2}} p^{\frac{N}{2}} \), we have

\[
J(t, \psi_{\epsilon}) \leq \frac{1}{2N} p^{\frac{N}{2} - 1} S_{\frac{N}{2}} + O(\epsilon^{\frac{N}{2} - 2}) + \frac{A_{\epsilon}^2}{2} \int_{\mathbb{R}^N} V(x) \psi_{\epsilon}^2 - \int_{\mathbb{R}^N} G(x, t_x \psi_{\epsilon})
\]
\[
\leq \frac{1}{2N} p^{\frac{N}{2} - 1} S_{\frac{N}{2}} + O(\epsilon^{\frac{N}{2} - 2}) + C \int_{B_{2\rho_{\epsilon}}} \psi_{\epsilon}^2 - \left( \int_{B_{\rho_{\epsilon}}} + \int_{B_{2\rho_{\epsilon}} \setminus B_{\rho_{\epsilon}}} \right) G(x, t_x \psi_{\epsilon}).
\]

In the following, we estimate

\[
\int_{B_{\rho_{\epsilon}}} G(x, t_x \psi_{\epsilon}) \quad \text{and} \quad \int_{B_{2\rho_{\epsilon}} \setminus B_{\rho_{\epsilon}}} G(x, t_x \psi_{\epsilon}).
\]

By Lemma 2.1 (3) and Remark 2.5, we have for \( \epsilon \) small that

\[
\int_{B_{\rho_{\epsilon}}} G(x, t_x \psi_{\epsilon}) = \int_{B_{\rho_{\epsilon}}} \left[ \frac{\lambda}{q} |f(t_x w_{\epsilon})|^q + \frac{1}{2p^{2^*}} |f(t_x w_{\epsilon})|^{2p^{2^*}}
\right.
\]
\[
+ \frac{1}{2} V(x) \left( |t_x w_{\epsilon}|^2 - |f(t_x w_{\epsilon})|^2 \right) - \frac{4^{\frac{1}{p^{2^*}}} |t_x w_{\epsilon}|^{2^*} \right]
\]
\[
\geq \int_{B_{\rho_{\epsilon}}} \left[ \frac{\lambda}{q} |f(t_x w_{\epsilon})|^q + \frac{1}{2} \left( \frac{f(t_x w_{\epsilon})}{|t_x w_{\epsilon}|} \right)^{2^*} - \frac{4^{\frac{1}{p^{2^*}}}}{p^{2^*}} |t_x w_{\epsilon}|^{2^*} \right]
\]
\[
\geq \int_{B_{\rho_{\epsilon}}} \left[ C \lambda |w_{\epsilon}|^{2^*} - C |w_{\epsilon}|^{2^* - 1} \right]
\]
\[
\geq \int_{B_{\rho_{\epsilon}}} \left[ C |w_{\epsilon}| - C |w_{\epsilon}|^{2^* - 1} \right].
\]

Thus

\[
\int_{B_{\rho_{\epsilon}}} G(x, t_x \psi_{\epsilon}) \geq \int_{B_{\rho_{\epsilon}}} (C \omega)^{\frac{2^*}{p^{2^*}} - C \omega_{\epsilon}^{2^* - 1}) \quad \text{for} \quad \epsilon > 0 \quad \text{small.} \quad (3.6)
\]

On the other hand, denote \( \omega = B_{2\rho_{\epsilon}} \setminus B_{\rho_{\epsilon}} \).

\[
\int_{B_{2\rho_{\epsilon}} \setminus B_{\rho_{\epsilon}}} G(x, t_x \psi_{\epsilon}) = \int_{\omega} \left[ \frac{\lambda}{q} |f(t_x \psi_{\epsilon})|^q + \frac{1}{2p^{2^*}} |f(t_x \psi_{\epsilon})|^{2p^{2^*}}
\right.
\]
\[
+ \frac{1}{2} V(x) \left( |t_x \psi_{\epsilon}|^2 - |f(t_x \psi_{\epsilon})|^2 \right) - \frac{4^{\frac{1}{p^{2^*}}} |t_x \psi_{\epsilon}|^{2^*} \right]
\]
\[
\geq \int_{\omega} \left[ \frac{\lambda}{q} |f(t_x \psi_{\epsilon})|^q + \frac{1}{2} \left( \frac{f(t_x \psi_{\epsilon})}{|t_x \psi_{\epsilon}|} \right)^{2^*} - \frac{4^{\frac{1}{p^{2^*}}}}{p^{2^*}} |t_x \psi_{\epsilon}|^{2^*} \right]
\]
\[
= \int_{\omega} \left[ \frac{\lambda}{q} |f(t_x \varphi \omega_{\epsilon})|^q + \frac{1}{2p^{2^*}} |f(t_x \varphi \omega_{\epsilon})|^{2p^{2^*}} - \frac{4^{\frac{1}{p^{2^*}}}}{p^{2^*}} |t_x \varphi \omega_{\epsilon}|^{2^*} \right]
\]
\[
= \int_{\epsilon} \left[ \frac{\lambda}{q} |f(t_x \varphi(r) \omega_{\epsilon}(r))|^q + \frac{1}{2p^{2^*}} |f(t_x \varphi(r) \omega_{\epsilon}(r))|^{2p^{2^*}}
\right.
\]
\[
- \frac{4^{\frac{1}{p^{2^*}}}}{p^{2^*}} |t_x \varphi(r) \omega_{\epsilon}(r)|^{2^*} \right] r^{N-1} dr.
\]
Using the integral mean value theorem, there exists a constant $\theta \in (0, 1)$ such that
\[
\int_{B_{2r_e} \setminus B_{r_e}} G(x, t_e \psi_e) \geq \frac{\lambda}{q} |f(t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau))| q^{\frac{1}{2}}
\]
\[
+ \frac{1}{2p^2} \frac{|f(t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau))|^{2p^2}}{2^{2p^2}}
\]
\[
- \frac{4^{\frac{\lambda}{1 - 2}}} {p^2} |t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau))|^{2} \int_{e^\tau}^{2r^\tau} r^{N-1} dr.
\]
(3.7)

Noting $r_e = e^\tau$, $\tau \in (\frac{1}{4}, \frac{1}{2})$, we deduce
\[
C\epsilon^{-\frac{N-2}{2} - \tau(N-2)} \leq w_e((1 + \theta)e^\tau) \leq C\epsilon^{-\frac{N-2}{2}}.
\]
(3.8)

Hence $(1 + \theta)e^\tau \rightarrow +\infty$ and $\varphi((1 + \theta)e^\tau) \rightarrow c > 0$ as $\epsilon \rightarrow 0$. Now using Remark 2.5 we have that
\[
|f(t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau))| q^{\frac{1}{2}}
\]
\[
= \left| \frac{f_p(t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau))}{\sqrt{t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau)}} \frac{2}{2} |t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau)|^{\frac{1}{p^2}} \right|
\]
\[
\geq C\left[ w_e((1 + \theta)e^\tau) \right]^\frac{2}{p^2},
\]
(3.9)

and
\[
\frac{1}{2p^2} |f(t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau))|^{2p^2} - \frac{4^{\frac{\lambda}{1 - 2}}} {p^2} |t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau))|^{2} \int_{e^\tau}^{2r^\tau} r^{N-1} dr
\]
\[
\geq -C \left[ t_e \varphi((1 + \theta)e^\tau)\omega_e((1 + \theta)e^\tau) \right]^{2\tau - 1}
\]
\[
\geq -C \left[ w_e((1 + \theta)e^\tau) \right]^{2\tau - 1}.
\]
(3.10)

It follows from (3.7)-(3.10) that
\[
\int_{B_{2r_e} \setminus B_{r_e}} G(x, t_e \psi_e) \geq \left\{ C\lambda \left[ \omega_e((1 + \theta)e^\tau) \right]^{\frac{q}{2p}} - C\left[ \omega_e((1 + \theta)e^\tau) \right]^{2\tau - 1} \right\} e^{\tau N}
\]
\[
\geq \left\{ C\lambda \epsilon^{-\frac{N-2}{2} - \tau(N-2)} \right\}^{\frac{q}{2p}} - CC\epsilon^{-\frac{N-2}{2} - \tau(N-2)}
\]
\[
= C\epsilon^{-\frac{N-2}{2} + \frac{N+2}{2} + \frac{N+2}{2} - C}.
\]

If $N \geq 6, q > 4p$. We can take $\tau \in (\frac{1}{4} + \frac{p(N+2)}{2q(N-2)}, \frac{1}{2}) \subset (\frac{1}{4}, \frac{1}{2})$ so that
\[
\left[ \frac{N-2}{4} - \tau(N-2) \right] \cdot \frac{q}{2p} + \frac{N+2}{4} < 0,
\]
(3.11)

which gives
\[
\int_{B_{2r_e} \setminus B_{r_e}} G(x, t_e \psi_e) \geq 0 \text{ for all } \lambda > 0 \text{ and } \epsilon \text{ is small.}
\]
(3.12)

If $N = 3, 4, 5, q > \frac{2p(N+2)}{N-2}$. We can also take $\tau \in (\frac{1}{4} + \frac{p(N+2)}{2q(N-2)}, \frac{1}{2}) \subset (\frac{1}{4}, \frac{1}{2})$ such that (3.11) holds and hence (3.12) holds.
If \( N = 3, 4, 5, 4p < q \leq \frac{2p(N+2)}{N-2} \). We can take \( \lambda = \epsilon^{-\frac{2}{q}} \), \( \tau \in \left( \frac{1}{2}, \frac{1}{4} \right) \) so that
\[
\left[ \frac{N-2}{4} - \tau(N-2) \right] \frac{q}{2p} + \frac{N+2}{4} - \frac{3}{4} < 0,
\]
which again gives (3.12).

From (3.6) and (3.12) we have
\[
J(t, \psi_\epsilon) \leq \frac{1}{2N} p^{\frac{N}{p} - 1} S^{\frac{N}{p}} + O(\epsilon^{\frac{N-2}{2p}}) + C \int_{B_{2\rho \epsilon}} \psi_\epsilon^2 - \int_{B_{\rho \epsilon}} (C\lambda \omega_{\epsilon}^{\frac{N}{p}} - C\omega_{\epsilon}^{2-\frac{1}{2}})
\leq \frac{1}{2N} p^{\frac{N}{p} - 1} S^{\frac{N}{p}} + O(\epsilon^{\frac{N-2}{2p}}) + C \int_{B_{2\rho \epsilon}} \psi_\epsilon^2 - \int_{B_{\rho \epsilon}} C\lambda \omega_{\epsilon}^{\frac{N}{p}} + C\epsilon^{-\frac{N-2}{2p}}
\leq \frac{\epsilon}{2N} p^{\frac{N}{p} - 1} S^{\frac{N}{p}} + I_1.
\]

Noting that
\[
\int_{B_{\rho \epsilon}} \omega_{\epsilon}^{\frac{N}{p}} = \epsilon^{-\left(\frac{N-2}{2p}\right) + \frac{N}{p}} \int_0^{\rho \epsilon} S^{\frac{N-1}{p}} \left(1 + s^2\right)^{\frac{(N-2)s}{2p}} ds,
\]
and
\[
\int_0^{\infty} S^{\frac{N-1}{p}} \left(1 + s^2\right)^{\frac{(N-2)s}{2p}} ds \geq C > 0 \text{ for all } N \geq 3 \text{ and } q > 4p,
\]
we have that
\[
I_1 \leq Ce^{-\frac{N-2}{2p}} - C\lambda \epsilon^{-\left(\frac{N-2}{2p}\right) + \frac{N}{p}} + C \left\{ \begin{array}{ll}
\epsilon + O(\epsilon^{\frac{N-2}{2p}}) & \text{if } N \geq 5, \\
\epsilon |\ln \epsilon| + O(\epsilon) & \text{if } N = 4,
\end{array} \right.
\]
and
\[
O(\epsilon^{\frac{1}{2}}) \text{ if } N = 3.
\]

From (3.13) and (3.14), we only need to prove \( I_1 < 0 \) for small \( \epsilon \).

For the case when \( N \geq 6, q > 4p \), we can verify that \( -\frac{(N-2)q}{8p} + \frac{N}{2} < 1 \leq \frac{N-2}{4} \), which gives that \( I_1 < 0 \) as \( \epsilon > 0 \) sufficiently small;

For the case when \( N = 3, 4, 5, \frac{2p(N+2)}{N-2} < q < \frac{2p}{N-2} \), we can also verify that \( -\frac{(N-2)q}{8p} + \frac{N}{2} < \frac{N-2}{4} \), which gives that \( I_1 < 0 \) as \( \epsilon > 0 \) sufficiently small;

For the case when \( N = 3, 4, 5, \frac{2p(N+2)}{N-2} < q < \frac{2p}{N-2} \), we can take \( \lambda = \epsilon^{-\frac{2}{q}} \), then we have that \( -\frac{(N-2)q}{8p} + \frac{N}{2} - \frac{3}{4} < \frac{N-2}{4} \), which again gives that \( I_1 < 0 \) as \( \epsilon > 0 \) sufficiently small.

Thus (3.2) holds naturally and this complete our proof.

**Remark 3.6.** By the same argument, we can consider the similar problem on a bounded domain \( \Omega \subset \mathbb{R}^N \)
\[
\begin{cases}
-\Delta u + V(x)u - p\Delta(|u|^{2p})|u|^{2p-2}u = \lambda |u|^{q-2}u + |u|^{2p2^*-2}u, & x \in \Omega, \\
u = 0, & \text{on } \partial\Omega.
\end{cases}
\]

The following corollary hold:

**Corollary 3.7.** Suppose that \( 0 \leq V(x) \in C^1(\overline{\Omega}) \). Then for all \( \lambda > 0 \), problem (3.15) has a positive solution if either \( N \geq 6 \), and \( 4p < q < 2p2^* \) or \( 3 \leq N < 6 \) and \( 2p(N+2) < q < 2p2^* \). Moreover, problem (3.15) still exists a positive solution if \( 3 \leq N < 6, 4p < q < \frac{2p(N+2)}{N-2} \) and \( \lambda \) sufficiently large. In particular, if \( \Omega = B_R(0) \) and \( V(x) = V(r), \ V'(r) \geq 0 \) (\( r = |x| \)), the solution is radially symmetric.
4. **The existence of sign-changing solutions.** In this Section, we prove that the existence of node solutions for problem (1.7) by Nehari technique.

Let \( \Omega \) be one of the following three types of domains:

\[
\begin{align*}
\{ x \in \mathbb{R}^N \mid |x| < R_1 \}, \\
\{ x \in \mathbb{R}^N \mid 0 < R_2 \leq |x| < R_3 < +\infty \}, \\
\{ x \in \mathbb{R}^N \mid |x| \geq R_4 > 0 \}.
\end{align*}
\] (4.1)

Set

\[
J_\Omega(v) = \frac{1}{2} \int_\Omega \left( |\nabla v|^2 + V(x)v^2 \right) - \frac{4^{\frac{1}{q-2}}}{p^{\frac{2}{2}}} \int_\Omega |v|^{2^*} - \int_\Omega G(x,v),
\]

\[
\gamma_\Omega(v) = \langle J_\Omega(v), v \rangle = \int_\Omega \left( |\nabla v|^2 + V(x)v^2 - \frac{1}{p} 4^{\frac{1}{q-2}} |v|^{2^*} - g(x,v)v \right)
\]

and

\[
M(\Omega) = \{ v \in H^1_0(\Omega) \mid v \not\equiv 0, v|_{\partial \Omega} = 0, \gamma_\Omega(v) = 0 \}.
\]

Then we have the following lemmas.

**Lemma 4.1.** Let \( \Omega \) be of one of the forms given by (4.1) and \( \tilde{c} = \inf_{v \in M(\Omega)} J_\Omega(v) \). Then

\[
\tilde{c} = \inf_{v \in M(\Omega)} \sup_{t > 0} J_\Omega(tv).
\]

**Proof.** We know that for all \( v \in M(\Omega) \), there exists \( t^* > 0 \), satisfying

\[
J_\Omega(t^* v) = \sup_{t > 0} J_\Omega(tv), \quad \text{and} \quad \left. \frac{dJ_\Omega(tv)}{dt} \right|_{t=t^*} = 0.
\]

We are going to show that \( t^* = 1 \). To this end, we denote

\[
\frac{dJ_\Omega(tv)}{dt} = t \int_\Omega |\nabla v|^2 + \int_\Omega \left( V(x) - \lambda |f(tv)|^{q-2} - |f(tv)|^{2p^*-2} \right) f(tv)f'(tv)v
\]

\[
= t \int_\Omega \left( |\nabla v|^2 + \frac{V(x) - \lambda |f(tv)|^{q-2} - |f(tv)|^{2p^*-2}}{t} f(tv)f'(tv)v \right)
\]

\[
:= t f(t),
\]

and

\[
f'(t)t^2 = \int_\Omega \left[ f'^2(tv) t^2 + f(tv)f''(tv)tv^2 - f(tv)f'(tv)v \right] V(x)
\]

\[
+ \lambda \int_\Omega \left[ -(q-1)f'^2(tv)t^2 - f(tv)f''(tv)tv^2 + f(tv)f'(tv)v \right] |f(tv)|^{q-2}
\]

\[
+ \int_\Omega \left[ -(2p^{2^*-1})f'^2(tv)t^2 - f(tv)f''(tv)tv^2 + f(tv)f'(tv)v \right] |f(tv)|^{2p^*-2}
\]

\[
:= (I) + (II) + (III).
\]

From Lemma 2.1 (6) and the fact that \( f''(tv) = -2p^2(2p-1)f^{4p-3}(t)f'^4(t) \) we have
Let proceeding as done in [6] or [17] we can verify that the functional
\( J \) be of one of the forms given by (4.1). Then
\[
\frac{dJ}{dt} = \inf_{v \in M(\Omega)} J_v(t) = \inf_{v \in M(\Omega)} \sup_{t > 0} J_v(tv).
\]

By a standard argument, we can obtain the following compact lemma on the annular region (see [5]).

**Lemma 4.2.** Let \( \Omega = \{ x \in \mathbb{R}^N \mid 0 < R_1 \leq |x| \leq R_2 \leq +\infty \} \), then \( J_\Omega(v) \) satisfies (PS) condition if \( 4p < q < 2p^* \).

**Lemma 4.3.** Let \( \Omega \) be of one of the forms given by (4.1). Then \( \tilde{c} = \inf_{v \in M(\Omega)} J_\Omega(v) \) can be achieved by a positive function \( w^* \) which is a positive solution of the following problem
\[
\begin{cases}
-\Delta v + V(x)v = g(x,v) + \frac{1}{4} \frac{1}{|x|^s} |v|^{2^*-2}v, & x \in \Omega, \\
v|_{\partial\Omega} = 0,
\end{cases}
\]
under the assumption of Theorem 1.1, where \( g(x,v) \) is defined by (1.8).

**Proof.** Firstly, we prove \( \tilde{c} \) can be attained. Using Lemma 2.1, Lemma 2.3 and proceeding as done in [6] or [17] we can verify that the functional \( J_\Omega \) exhibits the Mountain-Pass geometry:

(i) there exist \( \alpha, \rho > 0 \) such that \( J_\Omega(v) \geq \alpha \) for all \( \|v\| = \rho \);
(ii) there exists \( w \in H^1_0(\Omega) \), such that \( \|w\| > \rho \) and \( J_\Omega(w) < 0 \).

By Lemma 4.1, we deduce that, for any \( v \in M(\Omega) \),

\[
\bar{c} = \inf_{v \in M(\Omega)} J_\Omega(v) = \inf_{v \in M(\Omega)} \sup_{t > 0} J_\Omega(tv) \geq \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\Omega(\gamma(t)) =: c^*.
\]  

(4.3)

where \( \Gamma = \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = w, J_\Omega(\gamma(1)) < 0 \} \).

In order to show that \( \bar{c} \) is attained by \( w^* \) which is the solution of (4.2). We distinguish two cases:

Case 1: \( \Omega = B_R (0 < R \leq +\infty) \).

As done in the proof of Theorem 3.5, we can prove that \( c^* \) is the critical value by Mountain-Pass lemma under the assumptions of Theorem 1.1. Thus there exists \( w^* \in H^1_0(\Omega) \) such that \( J_\Omega(w^*) = c^* \), \( J_\Omega'(w^*) = 0 \). Hence

\[
\bar{c} = \inf_{v \in M(\Omega)} J_\Omega(v) \leq J_\Omega(w^*) = c^*.
\]

(4.4)

Combining (4.3) and (4.4), we obtain \( \bar{c} \) is the critical value and \( J_\Omega(w^*) = \bar{c} \), \( J_\Omega'(w^*) = 0 \). This shows that \( w^* \) is the weak solution of (4.2).

Case 2: \( \Omega = \{ x \in \mathbb{R}^N : 0 < R_1 \leq |x| \leq R_2 \leq +\infty \} \).

By Lemma 4.2 and Mountain-Pass lemma, we can deduce that \( c^* \) is the critical value of \( J_\Omega \). Repeating the the argument of Case 1, we can conclude that \( \bar{c} \) can be attained by \( w^* \) and \( w^* \) is a week solution of (4.2).

Next, we are going to prove that \( w^* > 0 \). Notice that by \( L^p \)-regularity theory [20], we have \( w^* \in H^1_0(\Omega) \cap C^2(\Omega) \). We try to verify that the minimizer of \( \bar{c} \) will not change sign. If the attained function \( w^* \) changes sign in \( \Omega \), then \( (w^*)^+, (w^*)^- \in \mathcal{M}(\Omega) \), where \( (w^*)^+ = \max\{w^*, 0\} \), \( (w^*)^- = -\min\{w^*, 0\} \). Thus

\[
J_\Omega((w^*)^+) < J_\Omega(w^*) = \inf_{M(\Omega)} J_\Omega(v) \leq J_\Omega((w^*)^-),
\]

which is a contradiction. Therefore either \( w^* \geq 0 \) or \( w^* \leq 0 \). Without loss of generality, we can assume \( w^* \geq 0 \). Now we show that \( w^* > 0 \). If there exists \( x_0 \) such that \( w^*(x_0) = 0 \), then \( (w^*)'(x_0) = 0 \) for \( w^* \geq 0 \). By Strong Maximum Principle (e.g., Gilbarg and Trudinger [20]), \( w^* = 0 \) near \( x_0 \) and \( w^* \) will vanish identically, which is impossible since \( w^* \in M(\Omega) \). Hence \( w^* > 0 \) and we complete the proof.

In the following, we will consider the existence of the nodal solutions of (1.7). For any given \( k + 2 \) numbers \( r_j \) \( (j = 0, 1, \cdots, k + 1) \) such that \( 0 = r_0 < r_1 < r_2 < \cdots < r_k < r_{k+1} = +\infty \), denote

\[
\Omega^1 = \{ x \in \mathbb{R}^N : |x| < r_1 \},
\]

\[
\Omega^j = \{ x \in \mathbb{R}^N : r_{j-1} < |x| < r_j \}, \quad j = 2, 3, \cdots, k + 1.
\]

We will always extend \( v_j \in H^1_0(\Omega^j) \) to \( H^1(\mathbb{R}^N) \) by setting \( v \equiv 0 \) for \( x \in \mathbb{R}^N \backslash \Omega^j \) for every \( v_j \), \( j = 1, 2, \cdots, k + 1 \). In this sense, we use \( J(v_j) \) to replace \( J_{\Omega^j}(v_j) \) and \( \gamma(v_j) \) to replace \( \gamma_{\Omega^j}(v_j) \) in the sequel. Define

\[
Y_k^\pm(r_1, r_2, \cdots, r_{k+1}) = \left\{ v \in H^1_0(\mathbb{R}^N) \mid v = \pm \sum_{j=1}^{k+1} (-1)^{j-1} v_j, \quad v_j \geq 0, \quad v_j \neq 0, v_j \in H^1_0(\Omega^j), j = 1, 2, \cdots, k + 1 \right\},
\]

\[
M_k^\pm = \{ v \in H^1_0(\mathbb{R}^N) \mid \exists 0 < r_1 < r_2 < \cdots < r_k < r_{k+1} = +\infty, \quad \text{such that} \quad v \in Y_k^\pm(r_1, r_2, \cdots, r_{k+1}) \text{ and } v_j \in M(\Omega^j), j = 1, 2, \cdots, k + 1 \}.
\]
Note that $M_k^+ \neq \emptyset$, $k = 1, 2, \cdots$. In the following we will always refer $M_k$ and we will drop the “+”. For $M_k^-$, everything could be done exactly in the same way. Set

$$c_k = \inf_{M_k} J(v), \ k = 1, 2, \cdots.$$ 

Let $h(v)$ be the functional defined in $H^1(\mathbb{R}^N)$ by

$$h(v) = \begin{cases} \int_{\mathbb{R}^N} \frac{1}{p} |v|^p + g(x, v) v, & \text{if } v \neq 0, \\ 0, & \text{if } v = 0. \end{cases}$$

**Lemma 4.4.** If $c_k$ is attained, then

$$c_{k+1} < c_k + \frac{1}{2N} p^\frac{N}{N-1} S_N^{\frac{N}{N-1}}$$

under the assumption of Theorem 1.1.

**Proof.** Let $c_k$ be attained by $v$, $r_1$ is the first node of $v$. Set

$$v_0 = \begin{cases} v, & x \in B_{r_1}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$w = \alpha v_0 + \beta \psi_\epsilon.$$ 

We claim that there exist $\alpha^*, \beta^* \in \mathbb{R}$ such that $w^* = \alpha^* v_0 + \beta^* \psi_\epsilon$ has a unique zero point $\bar{r} \in (0, r_1)$ and $(w^*)^+ \in M(\Omega^+), (w^*)^- \in M(\Omega^-)$, where

$$\Omega^+ = \{ \bar{r} \leq |x| \leq r_1 \}, \quad \Omega^- = \{ |x| \leq \bar{r} \}.$$ 

In fact, firstly we denote $P = \{ v \in H^1_\sigma(\mathbb{R}^N) | v \geq 0 \}$, $\Sigma$ be the set of maps such that

i) $\sigma \in C(D, H^1_\sigma(\mathbb{R}^N))$,

ii) $\sigma(s, 0) = 0$, $\sigma(0, t) \in P$, $\sigma(1, t) \in -P$,

iii) $(J \cdot \sigma)(s, 1) \leq 0$, $(h \cdot \sigma)(s, 1) \geq 2$,

where $D = [0, 1] \times [0, 1], \ s, t \in [0, 1]$. Define

$$\sigma = \sigma(s, t) = kt((1-s)v_0 - sv_\epsilon).$$

Then we see that $\sigma \in \Sigma$ for large $k$ and small $\epsilon$ and hence $\Sigma \neq \emptyset$.

Secondly, for any $\sigma \in \Sigma$,

$$h(\sigma^+(x)) - h(\sigma^-(x)) \begin{cases} \geq 0, & \text{for } x \in \{(0, t), t \in [0, 1]\}, \\ \leq 0, & \text{for } x \in \{(1, t), t \in [0, 1]\}, \end{cases}$$

$$h(\sigma^+(x)) + h(\sigma^-(x)) - 2 \begin{cases} \geq 0, & \text{for } x \in \{(s, 1), s \in [0, 1]\}, \\ \leq 0, & \text{for } x \in \{(s, 0), s \in [0, 1]\}. \end{cases}$$

We deduce (by Miranda’s theorem [29]) that there exists $\bar{x} = (\bar{s}, \bar{t}) \in D$ such that

$$h(\sigma^+(\bar{x})) - h(\sigma^-(\bar{x})) = h(\sigma^+(\bar{x})) + h(\sigma^-(\bar{x})) - 2$$

and hence

$$h(\sigma^+(\bar{x})) = h(\sigma^-(\bar{x})) = 1. \quad (4.6)$$

Denote $w^* = \sigma(\bar{x}) = \alpha^* v_0 + \beta^* \psi_\epsilon$, where $\alpha^* = kl(1-\bar{s})$, $\beta^* = -kl\bar{s}$, $\bar{x} \in D$.

Finally, we will prove that $w^*$ has only one zero point $\bar{r} \in (0, r_1)$. Indeed, by standard ODE technique we can deduce that $\frac{d\alpha^*}{dr} < 0$ for $0 < r < r_1$ since $V'(r) \geq 0$. Thus $\alpha^* v_0$ is monotonically decreasing because $\alpha^* > 0$. By the definition of $\psi_\epsilon$ and
the fact that $\beta^*<0$, we can conclude that $\beta^*\psi_\epsilon$ is monotonically increasing. This implies that $w^*$ has only one zero point if $\epsilon$ small enough. By (4.6) we get that $(w^*)^+ \in M(\Omega^+), (w^*)^- \in M(\Omega^-)$.

As a consequence, the claim holds true.

Define $w(x)$ by

$$w(x) = \begin{cases} 
(w^*)^-(x), & x \in B_r, \\
-(w^*)^+(x), & x \in B_{r_1} \setminus B_r, \\
-v(x), & x \in \mathbb{R}^N \setminus B_{r_1}.
\end{cases}$$

Clearly, $w(x) \in M_{k+1}$, thus we have that

$$c_{k+1} \leq J(w(x)) \leq \sup_{\alpha, \beta \in \mathbb{R}} J(\alpha v_0 + \beta \psi_\epsilon) + J(v^*)$$

$$= \sup \{ J(\alpha v_0 + \beta \psi_\epsilon) + J(v^*) \mid \alpha, \beta \in \mathbb{R}, J(\alpha v_0 + \beta \psi_\epsilon) \geq 0 \} ,$$

where

$$v^* = \begin{cases} 
0, & x \in B_{r_1}, \\
v, & x \in \mathbb{R}^N \setminus B_{r_1}.
\end{cases}$$

In order to prove Lemma 4.4, we only need to verify the following inequality

$$\sup_{\alpha, \beta \in \mathbb{R}} J(\alpha v_0 + \beta \psi_\epsilon) + J(v^*) < c_k + \frac{1}{2N} \beta^{N-1} S_N^\frac{N}{N-1}. $$

(4.8)

Now we claim that $\alpha$ and $\beta$ are bounded in the case when $4p < q < 2p2^*$ and $(\alpha, \beta) \in \{(\alpha, \beta) \in \mathbb{R}^2 \mid J(\alpha v_0 + \beta \psi_\epsilon) \geq 0\}$.

Indeed, using Lemma 2.1(3), Lemma 2.2 and the following inequalities (see [15])

$$|c + d|^m \geq |c|^m + |d|^m - C(|c|^{m-1}|d| + |c||d|^{m-1}),$$

for all $c, d \in \mathbb{R}^N, m \geq 1,$ we have

$$0 \leq J(\alpha v_0 + \beta \psi_\epsilon)$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla (\alpha v_0 + \beta \psi_\epsilon)|^2 + V(x)|f(\alpha v_0 + \beta \psi_\epsilon)|^2) - \int_{\mathbb{R}^N} K(f(\alpha v_0 + \beta \psi_\epsilon))$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla (\alpha v_0 + \beta \psi_\epsilon)|^2 + V(x)|\alpha v_0 + \beta \psi_\epsilon|^2) - \frac{1}{2p2^*} \int_{\mathbb{R}^N} |f(\alpha v_0 + \beta \psi_\epsilon)|^{2p2^*}$$

$$\leq C \int_{\mathbb{R}^N} (|\nabla (\alpha v_0 + \beta \psi_\epsilon)|^2 + |\alpha v_0 + \beta \psi_\epsilon|^2) - C \int_{\mathbb{R}^N} |\alpha v_0 + \beta \psi_\epsilon|^2 + C$$

$$\leq C \int_{\mathbb{R}^N} (|\nabla (\alpha v_0 + \beta \psi_\epsilon)|^2 + |\alpha v_0 + \beta \psi_\epsilon|^2) - C \int_{\mathbb{R}^N} (|\alpha v_0|^2 + |\beta \psi_\epsilon|^2)$$

$$+ C \int_{\mathbb{R}^N} (|\alpha|^2 + |\beta|^2) - C(|\alpha|^2 + |\beta|^2)$$

$$+ C \int_{\mathbb{R}^N} (|\alpha|^2 + |\beta|^2) - C(|\alpha|^2 + |\beta|^2)$$

Using the estimations in Lemma 3.2 and Young’s inequality,

$$0 \leq C(|\alpha|^2 + |\beta|^2) - C(|\alpha|^2 + |\beta|^2) + C\epsilon^{\frac{N}{N-2}}(|\alpha|^2 + |\beta|^2) + C.$$

By taking $\epsilon$ small enough such that $\bar{C} = C - C\epsilon^{\frac{N}{N-2}} > 0$, we get that

$$0 \leq C(|\alpha|^2 + |\beta|^2) - C(|\alpha|^2 + |\beta|^2) + C,$$

this immediately implies that $\alpha$ and $\beta$ are bounded.
On the other hand, by Lemma 2.1 (3), (8), (9) and mean value theorem, we have

\[
\begin{align*}
\frac{1}{q} \int_{\mathbb{R}^N} & \left[ |f(\alpha v_0 + \beta \psi_\epsilon)|^q - |f(\alpha v_0)|^q - |f(\beta \psi_\epsilon)|^q \right] \\
= & \int_0^1 \int_{\mathbb{R}^N} \left[ |f(s \alpha v_0 + \beta \psi_\epsilon)|^{q-2} f(s \alpha v_0 + \beta \psi_\epsilon) f'(s \alpha v_0 + \beta \psi_\epsilon) \right. \\
& \left. - |f(s \alpha v_0)|^{q-2} f(s \alpha v_0) f'(s \alpha v_0) \right] \alpha v_0 ds \\
\leq & \int_0^1 \int_{\mathbb{R}^N} \left[ |f(s \alpha v_0 + \beta \psi_\epsilon)|^{q-2} f(s \alpha v_0 + \beta \psi_\epsilon) \right. \\
& \left. - |f(s \alpha v_0)|^{q-2} f(s \alpha v_0) \right] f'(s \alpha v_0 + \beta \psi_\epsilon) \alpha v_0 ds \\
& + \left| \int_0^1 \int_{\mathbb{R}^N} |f(s \alpha v_0)|^{q-2} f(s \alpha v_0) \left[ f'(s \alpha v_0 + \beta \psi_\epsilon) - f'(s \alpha v_0) \right] \alpha v_0 ds \right| \\
\leq & C \int_0^1 \int_{\mathbb{R}^N} \left| f(s \alpha v_0 + \theta_1 \beta \psi_\epsilon) \right|^{q-2} f(s \alpha v_0 + \theta_1 \beta \psi_\epsilon) \left| f'(s \alpha v_0 + \beta \psi_\epsilon) \right| ds \\
& + C \int_0^1 \int_{\mathbb{R}^N} \left| f(s \alpha v_0 + \theta_1 \beta \psi_\epsilon) \right|^{q-2} f(s \alpha v_0 + \theta_2 \beta \psi_\epsilon) \left| f'(s \alpha v_0 + \beta \psi_\epsilon) \right| ds \\
\leq & C \int_0^1 \int_{\mathbb{R}^N} \left| s \alpha v_0 + \theta_1 \beta \psi_\epsilon \right|^{\frac{q-2p-1}{q}} \left| f(s \alpha v_0 + \beta \psi_\epsilon) \right| ds \\
& + C \int_{\mathbb{R}^N} \left| \psi_\epsilon \right| \\
\leq & C \int_{\mathbb{R}^N} \left( \left| \psi_\epsilon \right| + \left| \psi_\epsilon \right|^{\frac{q}{2p}} \right)^{\frac{q}{q-2p-1}} \\
\leq & C \int_{\mathbb{R}^N} \left( \left| \psi_\epsilon \right| + \left| \psi_\epsilon \right|^{2} \right)^{\frac{q}{2p-1}},
\end{align*}
\]

where \( \theta_1, \theta_2 \in (0, 1) \). Similarly, we can verify that

\[
\begin{align*}
\left| \frac{1}{2p^2} \int_{\mathbb{R}^N} \left[ |f(\alpha v_0 + \beta \psi_\epsilon)|^{2p^2} - |f(\alpha v_0)|^{2p^2} - |f(\beta \psi_\epsilon)|^{2p^2} \right] \right| \\
\leq C \int_{\mathbb{R}^N} \left( \left| \psi_\epsilon \right| + \left| \psi_\epsilon \right|^{2q} \right)^{\frac{q}{2p-1}},
\end{align*}
\]

\[
\begin{align*}
\left| \int_{\mathbb{R}^N} \left[ |f(\alpha v_0 + \beta \psi_\epsilon)|^2 - |f(\alpha v_0)|^2 - |f(\beta \psi_\epsilon)|^2 \right] \right| \\
\leq C \int_{\mathbb{R}^N} \left| \psi_\epsilon \right|.
\end{align*}
\]

It follows that

\[
J(\alpha v_0 + \beta \psi_\epsilon) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla(\alpha v_0 + \beta \psi_\epsilon)|^2 + V(x) |f(\alpha v_0 + \beta \psi_\epsilon)|^2 \right) \\
- \int_{\mathbb{R}^N} K(f(\alpha v_0 + \beta \psi_\epsilon)) \\
\leq J(\alpha v_0) + J(\beta \psi_\epsilon) + C \int_{\mathbb{R}^N} \left( |\psi_\epsilon| + |\psi_\epsilon| + \left| \psi_\epsilon \right|^{2^* - 1} \right).
\]
Thus from Lemma 3.2, (3.13) and above inequality we have
\[ J(\alpha v_0 + \beta \psi_r) + J(v^*) \leq c_k + \frac{1}{2N} b^{\frac{N}{2}} - 1 S^\frac{N}{2} + O(\epsilon^\frac{N}{2} + O(\epsilon^\frac{N}{2}) \]
\[ + C \int_{R^N} \psi^2_r - C \int_{R^N} G(x, \psi_r) \]
\[ = c_k + \frac{1}{2N} b^{\frac{N}{2}} - 1 S^\frac{N}{2} + I_1. \]

Next we repeat the last part in the proof of Theorem 3.5 to obtain that \( I_1 < 0 \) and hence
\[ J(\alpha v_0 + \beta \psi_r) + J(v^*) < c_k + \frac{1}{2N} b^{\frac{N}{2}} - 1 S^\frac{N}{2} \] for small \( \epsilon \).

\[ \square \]

**Lemma 4.5.** \( c_k \) is attained under the assumption of Theorem 1.1, where \( k = 0, 1, \cdots \).

The proof of this Lemma is almost a repetition of the proof of Lemma 4.5 in [17], so we omit the detail here.

Now, we are ready to prove the main result.

**Proof of Theorem 1.1.** The main argument is essentially the same as the proof of Theorem 1.1 in [16] and we only give a sketch.

By Lemma 4.5, there exists \( v_k \in M_k \) which attains \( c_k \). We will prove that \( v_k \) is indeed a solution to problem (1.7). To our convenient, we denote \( v := v_k \), thus we get \( k \) nodes: \( r_1, r_2, \cdots, r_k \), \( 0 < r_1 < r_2 < \cdots < r_k < +\infty \). Clearly, \( v \) satisfies (1.7) in \( \{ x \in R^N : |x| \neq r_j, j = 1, 2, \cdots, k \} \). We set \( r = |x| \) and treat (1.7) as an ordinary differential equation. To simplify notation we write \( v(r) \) instead of \( v(|x|) \).

We rewrite (1.7) by
\[ -(r^{N-1}v')' = r^{N-1}(h(r, v) - V v), \] (4.9)
where
\[ h(r, v) = f'(v) \left[ \lambda |f(v)|^{q-2} f(v) + |f(v)|^{2p^2-2} f(v) - V(r) f(v) \right] + V(r) v, \]
and
\[ H(r, v) = \int_0^r h(r, s)ds. \]
We know already that \( v \) is of class \( C^2 \) on
\[ E = \{ r \in (0, +\infty) : r \neq r_j, j = 1, 2, \cdots, k \} \]
and for \( r \in E, v \) satisfies (4.9).

To complete the proof, it suffices to show that \( v \) satisfies (4.9) for all \( r > 0 \), this is the case if and only if
\[ \begin{align*}
v_+'(r) &= \lim_{r \searrow r_j} v'(r) = \lim_{r \nearrow r_j} v'(r) = v'_-, \quad j = 1, 2, \cdots, k. \end{align*} \] (4.10)

To this end, we use an indirect argument. Assume that \( v'_+ \neq v'_- \) and set \( \rho = r_{j-1}, \sigma = r_j, \tau = r_{j+1} \), we may assume that \( v \geq 0 \) on \( [\rho, \sigma] \), \( v \leq 0 \) on \( [\sigma, \tau] \). Now fix \( \delta > 0 \) \( (\delta < \min\{\sigma - \rho, \tau - \sigma\}) \) and define \( \tilde{v} : [\rho, \tau] \to R \) by
\[ \tilde{v}(r) = \begin{cases} v(r), & \text{if } |r - \sigma| \geq \delta, \\ v(\sigma - \delta) + \frac{(r - \sigma + \delta)[v(\sigma + \delta) - v(\sigma - \delta)]}{2\delta}, & \text{if } |r - \sigma| < \delta. \end{cases} \]

Let \( \sigma_0 = \sigma_0(\delta) \in (\sigma - \delta, \sigma + \delta) \) be defined by \( \tilde{v}(\sigma_0) = 0 \). There exist \( \alpha = \alpha(\delta) > 0, \beta = \beta(\delta) > 0 \) such that
This implies that \( \psi \) that \( \psi \)

Next we define \( w : [\rho, \tau] \to \mathbb{R}^N \) by setting

\[
  w(r) = \begin{cases} 
    \alpha \tilde{v}(r), & \rho \leq r \leq \sigma_0, \\
    \beta \tilde{v}(r), & \sigma_0 \leq r \leq \tau.
  \end{cases}
\]

Obviously, \( \psi(v) \leq \psi(w) \), where

\[
  \psi(s) = \int_{\rho}^{s} \left( \frac{1}{2} |s'|^2 + V s^2 - H(r, s) \right) r^{N-1} dr.
\]

Now, estimating \( \psi(w) \) exactly as done in the proof of Theorem 1.1 in [16], we deduce that

\[
  \psi(w) \leq \psi(v) - \frac{\sigma_0 - 1}{4} (v_+ - v_')^2 + o(\delta).
\]

This implies that \( \psi(w) < \psi(v) \) for \( \delta > 0 \) small enough, which contradicts the fact that \( \psi(v) \leq \psi(w) \).

As a consequence, we complete the proof. \( \square \)

REFERENCES

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Func. Anal., 14 (1973), 349–381.

[2] S. Bae, H. O. Choi and D. H. Park, Existence of nodal solutions of nonlinear elliptic equations, Proc. Roy. Soc. Edinburgh Sect., A 137 (2007), 1135–1155.

[3] T. Bartsch and W. D. Wang, Infinitely many radial solutions of a semilinear elliptic problem on \( \mathbb{R}^N \), Arch. Ration. Mech. Anal., 124 (1993), 261–276.

[4] T. Bartsch and Z. Tang, Multibump solutions of nonlinear Schrödinger equations with steep potential well and indefinite potential, Discrete Contin. Dyn. Syst., 33, (2012), 7–26.

[5] G. Bianchi, J. Chabrowski and A. Szulkin, On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent, Nonlinear Anal. TMA, 25 (1995), 41–59.

[6] João M. Bezerra do Ó, Olimpio H. Miyagaki and Sérgio H. M. Soares, Soliton solutions for quasilinear Schrödinger equations with critical growth, J. Differential Equations, 248 (2010), 722–744.

[7] H. Brandi, C. Manus, G. Mainfray, T. Lehner and G. Bonneau, Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma, Phys. Fluids B, 5 (1993), 3539–3550.

[8] H. Brezis and E. Lieb, A relation between pointwise convergence of function and convergence of functional, Proc. Amer. Math. Soc., 88 (1983), 486–490.

[9] D. Cao and X. Zhu, On the existence and nodal character of semilinear elliptic equations, Acta. Math. Sci., 8 (1988), 345–359.

[10] X. L. Chen and R. N. Sudan, Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse in underdense plasma, Phys. Rev. Lett., 70 (1993), 2082–2085.

[11] G. Cerami, S. Solimini and M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, J. Func. Anal., 69 (1986), 289–306.

[12] M. Colin and L. Jeanjean, Solutions for a quasilinear Schrödinger equation: A dual approach, Nonlinear Anal. TMA., 56 (2004), 213–226.

[13] M. Conti, L. Merizzi and S. Terracini, Radial solutions of superlinear equations on \( \mathbb{R}^N \). I. A global variational approach, Arch. Ration. Mech. Anal., 153 (2000), 291–316.

[14] A. De Bouard, N. Hayashi and J. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, Comm. Math. Phys., 189 (1997), 73–105.

[15] Y. Deng, The existence and nodal character of solutions in \( \mathbb{R}^N \) for semilinear elliptic equations involving critical Sobolev exponents, Acta. Math. Sci., 9 (1989), 385–402.
[16] Y. Deng, S. Peng and J. Wang, Infinitely many sign-changing solutions for quasilinear Schrödinger equations in $\mathbb{R}^N$, Commun. Math. Sci., 9 (2011), 859–878.

[17] Y. Deng, S. Peng and J. Wang, Nodal soliton solutions for quasilinear Schrödinger equations with critical exponent, Journal of Mathematical Physics, 54 (2013), 011504.

[18] Y. Deng and W. Shuai, Positive solutions for quasilinear Schrodinger equations with critical growth and potential vanishing at infinity, Commun. Pure Appl. Anal., 13 (2014), 2273–2287.

[19] P. Felmer and C. Torres, Radial symmetry of ground states for a regional fractional nonlinear Schrodinger equation, Commun. Pure Appl. Anal., 13 (2014), 2395–2406.

[20] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin-New York, 1998.

[21] S. Kurihara, Large-amplitude quasi-solitons in superfluid films, J. Phys. Soc. Japan, 50 (1981), 3262–3267.

[22] Xiang-Qing Liu, Jia-Quan Liu and Zhi Qiang Wang, Quasilinear elliptic equations with critical growth via perturbation method, Journal Differential Equations, 254 (2013), 102–124.

[23] E. Laedke, K. Spatschek and L. Stenflo, Evolution theorem for a class of perturbed envelope soliton solutions, J. Math. Phys., 24 (1983), 2764–2769.

[24] J. Liu and Z. Wang, Soliton solutions for quasilinear Schrödinger equations. I., Proc. Amer. Math. Soc., 131 (2003), 441–448.

[25] J. Liu and Z. Wang, Symmetric solutions to a modified nonlinear Schrödinger equation, Nonlinearity, 21 (2008), 121–133.

[26] J. Liu, Y. Wang and Z. Wang, Soliton solutions for quasilinear Schrödinger equations. II, J. Differential Equations, 187 (2003), 473–493.

[27] J. Liu, Y. Wang and Z. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, Comm. Partial Differential Equations, 29 (2004), 879–901.

[28] X. Q. Liu and J. Q. Liu, Quasilinear elliptic equations via perturbation method, Proc. Amer. Math. Soc. 141 (2013), 253–263.

[29] C. Miranda, Un’osservazione su un teorema di Brouwer, Boll. Un. Mat. Ital., 3 (1940), 5–7.

[30] A. Moameni, Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$, J. Differential Equations, 229 (2006), 570–587.

[31] A. Moameni, Soliton solutions for quasilinear Schrodinger equations involving supercritical exponent in $\mathbb{R}^N$, Commun. Pure Appl. Anal., 7 (2008), 89-105.

[32] Z. Nehari, Characteristic values associated with a class of non-linear second-order differential equations, Acta Math., 105 (1961), 141–175.

[33] M. Poppenberg, K. Schmitt and Z. Wang, On the existence of soliton solutions to quasilinear Schrodinger equations, Calc. Var. Partial Differential Equations, 14 (2002), 329–344.

[34] B. Ritchie, Relativistic self-focusing and channel formation in laser-plasma interactions, Phys. Rev. E, 50 (1994), 687–689.

[35] Y. Shen and Y. Wang, Soliton solutions for generalized quasilinear Schrodinger equations, Nonlinear Anal. TMA., 80 (2013), 194–201.

[36] Marco A. S. Souto and Sergio H. M. Soares, Ground state solutions for quasilinear stationary Schrodinger equations with critical growth, Commun. Pure Appl. Anal., 12 (2012), 99–116.

[37] W. A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys., 55 (1977), 149–162.

[38] T. Weth, Energy bounds for entire nodal solutions of autonomous superlinear equations, Calc. Var. Partial Differential Equations, 27 (2006), 421–437.

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E-mail address: ybdeng@mail.ccnu.edu.cn
E-mail address: yi.li@wright.edu
E-mail address: 874010638@qq.com