A new perturbation theory for the superfluid Fermi gas in the molecular Bose-Einstein condensed state

Shina Tan

James Franck Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637

(Dated: December 5, 2021)

We demonstrate how solutions to quantum few-fermion scattering problems can be the point-of-departure of a new treatment of a generalized many-body wave function. Our focus is on a particular ansatz for the ground state wavefunction of a superfluid Fermi gas introduced earlier [cond-mat/0506293]. Our method is perturbative in the sense that the probability amplitudes for few-fermion scattering processes are treated as small quantities; it is also nonperturbative in the sense that whenever such scattering events occur, nonperturbative quantum few-fermion scattering physics dominates. This approach can be viewed as a new diagrammatic methodology, based on a wave function as distinct from a perturbation series in the interparticle interactions. Some generic properties of the wave function are studied, and its parameters in the Bose-Einstein condensed limit are computed beyond mean-field. These results enable us to predict many observable properties of this Fermi gas with well-controlled accuracy, such as the superfluid pairing function, the four-fermion and six-fermion correlation functions, the momentum distribution, and the two-body reduced density matrix, etc.

PACS numbers: 03.75.Ss
Keywords: ground state

I. INTRODUCTION

In this paper we consider a superfluid Fermi gas with strong enough attractive interactions such that molecules of pairs of fermions are formed and Bose-Einstein condensed. A physical example of the system we have in mind is a two-component Fermi gas near a Feshbach resonance, with positive inter-fermionic scattering length. Our goal is to develop a new form of perturbation theory which will enable us to address fundamental and, now, experimentally accessible properties of this gas. One can not determine many observables of this system by simply resorting to the common wisdom of conventional Bose gases. In many instances, the underlying fermionic character must play a profound role. The starting point for the present paper is a wavefunction ansatz introduced earlier [1] for the ground state of a superfluid Fermi gas. In this earlier work we explored the equation of state and its relation to breathing mode experiments. Here we present in considerable detail our theoretical methodology. To demonstrate the predictive power of our theory, we also compute the superfluid pairing function and the four-fermion correlation function. In future work we will address the fermionic momentum distribution, the two-body reduced density matrix, the higher correlation functions as well as other properties.

Because the attractive interactions are strong enough to form bound states, whenever two or more molecules collide, the few-body physics is highly nonperturbative. In this paper we build on the observation that there is, nevertheless, an underlying perturbative component: the interactions are weak in the sense that the probability amplitudes (or component terms of the many-body wave function) associated with collisions are small, provided that the density of the system is relatively low. This is precisely the situation in which standard many-body perturbation techniques, based on weak inter-particle interactions are inappropriate. By contrast, here we develop a new kind of perturbation method, taking full advantage of these small amplitudes. Whenever molecules approach each other, however, the behavior of the constituent particles is treated nonperturbatively. In this way a complete many-body theory fully compatible with the nonperturbative few-body physics will be presented.

Closely related ideas have been used in the companion paper [2] to address the problem of a gas of point-like particles. This is, in some sense, the simpler analog of the bound states problem. In this companion paper [2], the ground state of the dilute Bose gas of structureless particles with quite arbitrary interactions is studied, starting from a general many-body wave function:

\[ |\psi_{\text{boson}}\rangle = \exp \left( \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{k_1 \ldots k_p} \alpha_{k_1}^{(p)} b_{k_1}^\dagger \ldots b_{k_p}^\dagger \right) |0\rangle, \]

where the \( b \)'s are boson annihilation operators, the \( k \)'s are momenta, and \( |0\rangle \) is the particle vacuum. To our surprise, this familiar system has some fundamental properties unknown to people, because the prevailing theories to date (Bogoliubov theory, pseudopotential method, effective field theory, etc) are all low-energy effective theories, capable primarily of describing the physics at the length scale of the superfluid healing length. In the companion paper [2], such a limitation is removed, and the short-distance structure is clarified. As a byproduct, some fundamental theorems derived initially for a Fermi system [3, 4] are successfully extended to a dilute Bose system [2].

In this paper we apply the same approach to a Bose-Einstein condensate of molecules formed out of strongly attractive fermions. Each molecule is a bound state of...
two fermions in two spin states (or any two internal states) $\uparrow\uparrow$ and $\downarrow\downarrow$. In Ref. 1 we presented a generalized many-body wavefunction of the form

$$|\psi_{\text{fermion}}\rangle = \exp\left\{ \sum_{p=2}^{\infty} \frac{1}{p!} \sum_{K_1, K_2, \ldots, K_p} \alpha_{K_1 K_2 \cdots K_p} c_{K_1}^\dagger c_{K_2}^\dagger \cdots c_{K_p}^\dagger |0\rangle \right\},$$

where $p = 2, 4, 6, \cdots$ must be even. The subscript $K_i$ is the shorthand for $k, \sigma$, and $\sigma$ labels the spin state. The coefficient $\alpha_{K_1 \cdots K_p}$ has some elementary properties: 1) it is antisymmetric under the interchange of any two subscripts, 2) it is zero whenever the sum of all its subscripts is nonzero, and 3) it is zero whenever the sum of a nontrivial subset of its subscripts is zero (see Sec. II). (Here the sum of subscripts is defined by summing the momenta and the associated $\sigma$'s separately, and when both sums are zero, the sum of these subscripts is called zero.)

The well-known BCS or Eagles-Leggett (EL) wave function of fermionic superfluids is the lowest order approximation of Eq. (1). This wavefunction and its implications for experiment have been reviewed rather extensively elsewhere. Retaining only the $p = 2$ term, one can verify that $|\psi\rangle$ becomes equivalent to a more familiar form of the EL wave function:

$$|\psi_{\text{EL}}\rangle = \prod_k (u_k + \nu_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger) |0\rangle.$$
Galilean symmetry, \( U(r_1 r_2 r_3 r_4) = 0 \) whenever \( r_1 + r_2 \neq r_3 + r_4 \).

finite-rangedness: there is a finite length scale \( l \) such that \( U(r_1 r_2 r_3 r_4) \) approaches zero sufficiently fast whenever any \( |r_1 - r_j| \gg l \), where \( 1 \leq i, j \leq 4 \).

the interaction supports a nondegenerate \( s \)-wave bound state (the molecule) of two fermions, whose binding energy in the absence of any other particles will be denoted by \( E_m < 0 \) (independent of the center-of-mass velocity of the molecule, due to the Galilean symmetry); the typical distance between the two fermions in the molecule will be denoted by \( r_m \); the zero-velocity scattering length between two molecules, \( a_m \), is positive and finite.

lower bound states do not exist, or they do exist but their formation rates are so low that we can omit them \[2\].

Also, \( H = H^\dagger \Rightarrow U(r_1 r_2 r_3 r_4)^* = U(r_3 r_1 r_2 r_2) \).

For later convenience, we use a symbol \( R \equiv (\sigma r) \), called “spin-spatial vector”; and similarly \( K \equiv (k \sigma) \), called “spin-momentum”. We also define some operations involving them. The “norm-square”:

\[
\| R \|^2 \equiv k^2. \text{ The “addition”: } K_1 + K_2 \equiv (k_1 + k_2, \sigma_1 + \sigma_2), \text{ where each } \uparrow \text{ is counted as } +1 \text{ and each } \downarrow \text{ is counted as } -1; \text{ for any even number of spin-momenta } K_1 \cdots K_p, \text{ if both } \sigma_1 + \cdots + \sigma_p = 0 \text{ and } k_1 + \cdots + k_p = 0, \text{ we say that } K_1 + \cdots + K_p = 0. \text{ The sum of a spin-momentum and a momentum: } K + q \equiv (k + q, \sigma). \text{ The “opposite spin-momentum”: } -K \equiv (-k, -\sigma). \text{ The “integration”: } \int d^3 R \equiv \sum_\sigma \int d^3 r, \text{ and } \int d^3 K/(2\pi)^3 \equiv \sum_\sigma \int d^3 k/(2\pi)^3 \text{; and similarly a summation: } \sum_K \equiv \sum_\sigma \Omega \int d^3 K/(2\pi)^3, \text{ where } \Omega \text{ is the system’s volume. Finally, if } both \sigma_1 + \cdots + \sigma_p = 0 \text{ and } k_1 + \cdots + k_p \sim q, \text{ we say that } K_1 + \cdots + K_p \sim q, \text{ where } q \text{ is some small momentum scale.}

Now we define the function \( U(R_1 R_2 R_3 R_4) \), satisfying:

\[
U(R_1 R_2 R_3 R_4) = -U(R_2 R_3 R_1 R_4) = -U(R_1 R_3 R_2 R_4); \text{ it is zero whenever } \sigma_1 + \sigma_2 \neq 0 \text{ or } \sigma_3 + \sigma_4 \neq 0; U(r_1 \uparrow, r_2 \downarrow, r_3 \uparrow, r_4 \downarrow) \equiv U(r_1 r_2 r_3 r_4).
\]

Equation \[2\] is thus rewritten as

\[
H - \mu \hat{N} = \sum_K \left( k^2/2 - \mu \right) c_K^\dagger c_K + \frac{1}{4} \sum_{K' a} U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} c_{K_k}^\dagger c_{K_{\bar{k}}}^\dagger c_{K_{\bar{k}}} c_{K_k}. \tag{3}
\]

\[
\{c_K^\dagger c_K\} = \delta_{Kk} \text{ and } \{c_K^\dagger c_K\} = 0. \quad U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} = 0 \text{ unless } K_k + K_{\bar{k}} = K_k + K_{\bar{k}}. \quad \text{When this spin-momentum conservation condition is satisfied, } U \text{ is a smooth function of the remaining three independent momenta, as a consequence of the condition that the interaction is finite-ranged. Third,}
\]

\[
U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} = -U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} = -U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} = U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} = U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} + U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} + K_{\bar{k}} + k_{\bar{k}} + k_k + k_k.
\]

where the last equality is due to the Galilean symmetry \((k \text{ is an arbitrary momentum})\). Fourth, in the thermodynamic limit, any nonzero \( U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} \) scales like \[10\]

\[
U_{K_k K_k K_{\bar{k}} K_{\bar{k}}} \sim \Omega^{-1}.
\]

In this paper, an expression like \( x \sim \Omega^2 \) only refers to the behavior of \( x \) for \( \Omega \rightarrow \Omega' \); since the often dimensionful coefficient (independent of \( \Omega \)) is suppressed, one should never use such expression to infer the dimension of \( x \).

B. Wave function

The ground state wave function we have adopted is

\[
|\psi\rangle = \exp\left(\sum_{p=2}^\infty \alpha^{(p)}|0\rangle\right),
\]

where \( p = 2, 4, 6, \ldots \) must be even \[3\], and

\[
\alpha^{(p)} = \frac{1}{p!} \sum_{K_1 \cdots K_p} \alpha^{(p)}_{K_1 \cdots K_p} c_{K_1}^\dagger \cdots c_{K_p}^\dagger
\]

called the \((p/2)\)-th order generator;

\[
\alpha^{(p)\dagger} = \frac{1}{p!} \sum_{K_1 \cdots K_p} \alpha^{(p)}_{K_1 \cdots K_p} c_{K_p} c_{K_{p-1}}^\dagger \cdots c_{K_2}^\dagger
\]

called the \((p/2)\)-th order degenerator. The prefactor \(1/p!\) is for the antisymmetry of the \( c_1^\dagger \)'s (or \( c^\dagger \)'s).

In a very similar manner as in \[3\], we can establish the following properties of \( \alpha^{(p)}_{K_1 \cdots K_p} \).

Firstly, we assume that the ground state is translationally invariant, and that the superfluid pairing occurs in the spin-singlet channel only. So \( \alpha^{(p)}_{K_1 \cdots K_p} = 0 \) whenever \( K_1 + \cdots + K_p \neq 0 \).

Secondly, in the thermodynamic limit, in which the density of fermions

\[
n = N/\Omega
\]
is kept constant, any nonzero coefficient \( \alpha^{(p)}_{K_1 \ldots K_p} \) scales with the system’s volume as
\[
\alpha^{(p)}_{K_1 \ldots K_p} \sim \Omega^{1-p/2}.
\]

Thirdly, \( \alpha^{(p)}_{K_1 \ldots K_p} \) is antisymmetric under the interchange of \( K_i \) and \( K_j \), where \( 1 \leq i < j \leq p \).

### C. Diagrams

This subsection is largely parallel to a similar subsection in \[2\]. The only essential additional feature is due to the antisymmetry of the fermionic wave function.

The evaluation of any physical observables can be reduced to the following general problem.

Consider an arbitrary product \( O = \alpha_1\alpha_2 \ldots \alpha_L \), where each \( \alpha_i \) (\( 1 \leq i \leq L \)) is a basic fermionic operator (creation or annihilation) in the spin-momentum space. The ground state expectation value
\[
\langle O \rangle = \frac{\langle 0 | \exp\left( \sum_{p=2}^{\infty} \alpha^{(p)} \right) | 0 \rangle \langle 0 | \exp\left( \sum_{p'=2}^{\infty} \alpha^{(p')} \right) | 0 \rangle}{\langle 0 | \exp\left( \sum_{p=2}^{\infty} \alpha^{(p)} \right) \exp\left( \sum_{p'=2}^{\infty} \alpha^{(p')} \right) | 0 \rangle}
\]
is calculated using the Wick’s theorem. We assign each fermion operator in the above expression a “time”: the creation operators in the generators are assigned an earliest time \( t_d \), the annihilation operators in the generators a latest time \( t_d \), and \( \alpha_1 \) through \( \alpha_L \) are assigned some intermediate times, \( t_1 \) through \( t_L \), such that \( t_d > t_1 > \cdots > t_L > t_d \) \( \alpha \). The actual magnitudes of these “times” do not matter; only the “causal order” matters.

Equation (5) is thus rewritten as
\[
\langle O \rangle = \frac{\langle 0 | \mathcal{T} \exp\left( \sum_{p=2}^{\infty} \alpha^{(p)} \right) \exp\left( \sum_{p'=2}^{\infty} \alpha^{(p')} \right) | 0 \rangle}{\langle 0 | \mathcal{T} \exp\left( \sum_{p=2}^{\infty} \alpha^{(p)} \right) \exp\left( \sum_{p'=2}^{\infty} \alpha^{(p')} \right) | 0 \rangle},
\]
where \( \mathcal{T} \) is the “time” ordering operator.

The basic formulas to be used in the evaluation of Eq. (6) are
\[
\langle 0 | \mathcal{T} c_{K'}^{d_{K}} c_{K}^{d_{K'}} | 0 \rangle = \langle 0 | \mathcal{T} c_{K}^{d_{K}} c_{K'}^{d_{K'}} | 0 \rangle = 0,
\]
\[
\langle 0 | \mathcal{T} c_{K'}^{d_{K}} c_{K}^{d_{K'}} | 0 \rangle = -\langle 0 | \mathcal{T} c_{K}^{d_{K}} c_{K'}^{d_{K'}} | 0 \rangle = \delta_{KK'} \delta(t - t').
\]

Equation (7) is then expressed as an infinite sum over all the topologically distinct diagrams with the following rules.

- Each \( \alpha_i \) is represented a small solid circle if it is a fermion creation operator, or a small hollow one if it is an annihilation operator, linked by a single solid line, with the designated spin-momentum.

- Each \( p/2 \)-th order generator (degenerate) is represented by a large solid (hollow) circle, linked by \( p \) solid lines; it contributes a factor \( \alpha^{(p)}_{K_1 \ldots K_p} \) (or its complex conjugate).

- Each solid line is associated with a single spin-momentum, and it can only connect an earlier solid object with a later hollow one, in accordance with Eq. (7). The diagram is thus bipartite.

- Impose spin-momentum conservation on all the internal vertices, that is, the sum of all the spin-momenta leaving/entering any generator/degenerate must equal zero. Sum over all the independent loop spin-momenta.

- Each diagram has an overall symmetry factor \( 1/S \), and \( S \) is the number of ways of interchanging components without changing the diagram.

- Vacuum bubbles (disconnected parts which are not connected to any external vertex) are all canceled by the denominator in Eq. (6).

The fermionic operators anticommute under the “time”-ordering operator; the subscripts of \( \alpha^{(p)}_{K_1 \ldots K_p} \) also anticommute. It is easy to show that there is a simple method to determine the sign of a diagram.

Step 1. Divide all the solid lines of the diagram into a collection of continuous trajectories. Each solid line can only be contained by one trajectory, and it can appear in this trajectory only once. Each trajectory is either an external trajectory, or an internal one. Each external trajectory starts from an external vertex \( o^e \), ends at a different external vertex \( o^e \), and is of the form \( v_0 K_1 v_1 K_2 v_2 \ldots K_{M} v_M \), where \( M \geq 1 \), \( v_0 = o^e \), \( v_M = o^e \), and the remaining \( v \)'s are internal vertices. \( K \)'s are spin-momenta, that is, the solid lines. Each internal trajectory starts from an internal vertex \( v_0 \), and must come back to this same vertex; it is of the form \( v_0 K_1 v_1 K_2 v_2 \ldots K_{M} v_M \), where \( v_M = v_0 \), and \( M \) is necessarily an even number (recall that the diagram is bipartite): an internal trajectory can never pass an external vertex. The total number of external trajectories is necessarily \( L/2 \). The \( L \) external vertices is thus divided into \( L/2 \) pairs, and can be written as \( o^e_1 o^e_2 \ldots o^e_{L/2} o^e_{L/2} \); suppose that it takes \( P \) interchanges of pairs of fermionic operators to transform the sequence \( o^e_1 \ldots o^e_L \) to this new one. Suppose that \( X \) of the \( L/2 \) external end points \( (o^e \)'s) are fermion annihilation operators, that is, hollow circles.

Suppose that \( X \) of the \( L/2 \) external end points \( (o^e \)'s) are fermion annihilation operators, that is, hollow circles.

Step 2. For every \( p \)-line internal vertex \( v \), write a separate symbol \( \alpha^{(p)} \) if it is solid, or \( \alpha^{(p)} \) if it is hollow; leave \( p \) blank positions for its subscripts.

Step 3. To each internal vertex \( v \) passed through by each trajectory, assign a pair of ordered subscripts \( K K' \), where the 3-tuple \( K v K' \) (exactly in this order) is a segment of the trajectory. If \( v \) is the starting-ending point of an internal trajectory (containing \( M \) solid lines), assign to it a pair of ordered subscripts \( K_1 K_M \). Multiply the final expression by an overall factor \((-1)^{P+X}\).
FIG. 1: A diagram in the expansion of \((c_{K_1}^1 c_{K_2}^i c_{K_3}^s c_{K_4}^i)\).

There are different ways to divide the same diagram into trajectories, but the results are equal.

We illustrate these rules with a term in the expansion of \((c_{K_1}^i c_{K_2}^o c_{K_3}^s c_{K_4}^i)\), shown in Fig. 1. The diagram can be divided into two trajectories: \(o_1 K_1 A K_2 B K_4 o_4\) and \(o_2 K_2 A K_3 B K_5 C K_5 o_3\), so \((-1)^p = +1, X = 1,\) and \((-1)^p + x = -1.\) The expression with the correct sign is therefore \(-A_{K_1} K_{2} K_{3} K_{4} B_{K_5} K_{5} K_{6} K_{7} C_{K_5} K_{7} = A_{K_1} K_2 K_3 K_6 B_{K_5} K_5 K_6 K_7 C_{K_5} K_7.\) Incorporating the other rules, we get

\[
+ \frac{1}{2} \left( \sum_{K, K_9} \beta_{K_1}^{K_2} K_2 K_3 K_4 \beta_{K_4}^{K_5} K_5 K_6 \alpha_{K_6}^{K_7} \right),
\]

where we have given special names to the coefficients of the first three generators:

\[
\alpha_{K} \equiv \alpha_{K_{2}}^{(2)} - \alpha_{K_{1}}^{(2)}, \quad \beta_{K_{1}}^{K_{2}} K_{3} K_{4} \equiv \alpha_{K_{2}}^{(4)} - \alpha_{K_{1}}^{(4)}, \quad \gamma_{K_{2}}^{K_{1}} K_{3} K_{4} K_{5} \equiv \alpha_{K_{2}}^{(6)} - \alpha_{K_{1}}^{(6)}.
\]

There are quite a few other ways to divide Fig. 1 into trajectories. For instance, \(o_1 K_1 A K_2 o_2, o_3 K_3 C K_5 B K_5 o_4,\) and \(A K_3 B K_4 A,\) where the last trajectory is an internal one; so \((-1)^p = +1, X = 1,\) and \((-1)^p + x = -1.\) The expression with the correct sign is therefore \(-A_{K_1} K_{2} K_{3} K_{4} B_{K_5} K_5 K_6 K_7 C_{K_5} K_7,\) equivalent with the one we wrote above. Even if we reverse the direction of any trajectory, the final result is still the same.

D. Diagrams for \(E - \mu N\)

Like in \([2]\), we use a half-solid-half-hollow diamond to facilitate the diagrammatic calculation of \(E - \mu N = \langle H - \mu N \rangle.\) If each side is attached by a single solid line, the diamond contributes a factor \((k^2/2 - \mu),\) where \(K\) is the spin-momentum flowing through it. If each side is attached by two solid lines, the diamond contributes a factor \(U_{K_1} K_2 K_3 K_4,\) where \(K_1 K_2 \) leave the diamond’s solid side, and \(K_3 K_4\) enter its hollow side.

The above rules are very similar to those in \([2]\). We now describe how to determine the sign of each diagram.

The solid lines in such a diagram can still be divided into a collection of trajectories.

For a trajectory \(v_0 K_1 v_1 \cdots K_M v_M\) \((v_0 = v_M)\) which does not involve the diamond, the rule is the same as in the last subsection: assign an ordered pair of subscripts \(K_i K_{i+1}\) to each \(v_i\) \((1 \leq i \leq M - 1)\), and assign \(K_M K_1\) to \(v_0.\)

There will then be \(L/2\) other, special trajectories \((L = 2\) for the kinetic energy, and \(L = 4\) for the interaction), each of which starts from the diamond and ends at it.

Case 1: \(L = 2,\) and there is one special trajectory \(v_0 K_1 v_1 \cdots K_M v_M\) \((v_0 = v_M)\) are the two sides of the diamond, and \(K_1 = K_M\). Assign an ordered pair of subscripts \(K_1 K_{i+1}\) to each \(v_i\) \((1 \leq i \leq M - 1)\). Multiply the whole expression by \((-1)\).

Case 2: \(L = 4,\) and the two special trajectories are \(s K_1 \cdots K_4 h\) and \(s K_1 \cdots K_4 h\), where \(s\) and \(h\) are the solid and the hollow sides of the diamond, respectively. So each trajectory is the same as the one in Case 1, and the \((-1)\)'s (overall factors contributed by both trajectories) cancel. Assign an ordered pair of subscripts to each vertex (other than the diamond) in the same way as above; the diamond contributes the factor \(U_{K_1} K_2 K_3 K_4\).

Case 3: \(L = 4,\) and each special trajectory \(o_1 \cdots K_M o\) starts from and ends at the same side of the diamond. It can then be treated like an ordinary internal trajectory; the pair \(K_1 K_M\) (exactly in this order) is filled to the first two subscript positions of \(U\) if \(o\) is the solid side, or to the last two subscript positions of \(U\) if \(o\) is the hollow side. The order of the fermion operators in Eqs. \([2]\) and \([3]\) has been chosen to ensure the validity of this simple rule; such order has also affected the rule in Case 2.

E. Thermodynamic orders and a property of the wave function

The theorem derived in \([2]\) concerning the thermodynamic order (see Appendix \([B]\)) of any aforementioned diagram, namely

\[
Q = N_3 - L/2,
\]

is still valid, because the prerequisites of this theorem are unaffected by the change from a bosonic superfluid to a fermionic one; the fermion sign does not affect these thermodynamic orders. \(L\) is the number of external points of the diagram. This theorem describes the scaling behavior of any diagram \(D\) in the thermodynamic limit: \(D \sim Q^L.\)

As a corollary, any diagram in the expansion of \(E - \mu N\) satisfies \(Q = 1\) and is thermodynamically significant.

In almost the same way as in \([2]\), we can also establish a general property of the coefficients \(\alpha^{(p)}_{K_1 \cdots K_4},\) namely it is zero whenever any nontrivial subset of fermions conserve spin-momentum (that is, whenever the sum of a nontrivial subset of the subscripts is zero). The basic idea is that any thermodynamically significant contribution to any physical observable from a coefficient whose nontrivial subset of subscripts conserves spin-momentum (called reducible coefficient) is only present at the “bottleneck” vertex of a dead end (see Appendix \([B]\)), but that dead ends can be absorbed by redefining lower order coefficients. Consequently, there are no dead ends in any diagrams. For more details, see \([2]\).
III. FEW-BODY WAVE FUNCTIONS

In this section we define some few-body functions and discuss some of their properties. They are completely independent of any many-body physics whatsoever.

But the contents listed below are restricted to those that are directly needed in Sec. IV where we do a many-body low-density expansion.

The internal wave function $\phi_K$ of an isolated molecule satisfies $(H - E_m) \sum_K \phi_K^* c^\dagger K \phi_K = 0$.

\[
\sum_{K'} D_{KK'} \phi_K^{*} = 0, \quad \phi_{-K} \equiv -\phi_{K}, \\
\frac{1}{2} \sum_K |\phi_K|^2 = 1, 
\]

where

\[
D_{KK'} \equiv (k^2 - E_m)\delta_{KK'} + \frac{1}{2} U_{K-K', K-K'}
\]

is a hermitian matrix for the relative dynamics of two fermions in opposite spin states isolated from all other fermions. Since $E_m < 0$ and the interaction is finite-ranged, the spatial representation of $\phi$ must be finite ranged, so $\phi_{k\sigma}$ depends smoothly on $k$. Moreover, obviously

\[
\phi_K \sim \Omega^{-1/2}.
\]

Sometimes we elect to abbreviate the spin-momentum $K_i$ as $i$, and $-K_i$ as $i^\prime$, when they do not lead to confusion. $i = 1, 2, 3, \ldots, 9$. Without this notation, some equations would be too lengthy and cumbersome.

The two-molecule zero-speed scattering wave function $\phi_{K_1 K_2 K_3 K_4}^{(4)}$ satisfies

\[
[H - 2E_m - \epsilon^{(4)}] \sum_{1234}^{(4)} \phi_{1234} c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger = 0,
\]

\[
\phi_{1234}^{(4)} = -\phi_{1243}^{(4)} = -\phi_{2134}^{(4)} = -\phi_{2143}^{(4)}, \quad \phi_{1234}^{(4)} = 0 \quad (\text{if } K_1 + \cdots + K_4 \neq 0),
\]

\[
\left[ (k_1^2 + \cdots + k_4^2)/2 - 2E_m - \epsilon^{(4)} \right] \phi_{1234}^{(4)} = \frac{1}{2} \sum_{56} U_{1256} \phi_{5634}^{(4)} + \frac{1}{2} \sum_{56} U_{1256} \phi_{5634}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{1536}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{1536}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{2456}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{2456}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{3456}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{3456}^{(4)} = 0,
\]

where $\epsilon^{(4)} \sim \Omega^{-1}$ is an extremely tiny positive energy, and is always negligible unless two of the four spin-momenta ($K_1 \cdots K_4$) are opposite; Ref. 2 contains a simpler analog. When the distance $r$ between the centers-of-mass of the two molecules is large [$\gg \text{max}(l, r_m)$], the four-fermion wave function (in the coordinate representation) is proportional to $(1 - a_m/r)$ times the internal wave functions of two isolated molecules. Consequently, we have the infrared boundary condition

\[
\phi_{K_1 K_2}^{(4)} = \left( \delta_{q,0} - \frac{4\pi a_m}{\Omega q^2} \right) \phi_{K_1} \phi_{K_2} + O(q^0)
\]

(12d)

when $q$ is small or zero. Here

\[
\phi_{K_1 K_2}^{(4)} \equiv \phi_{K_1 + q/2, -K_1 + q/2}^{(4)}
\]

(13)

The global coefficient of $\phi^{(4)}$ is set by Eq. (12d). Note that $\phi_{K_1 K_2}^{(4)}$ has delta-function singularity at $K_1 = \pm K_2$, but the intensity of the delta function approaches a finite constant when $q \to 0$. The term $O(q^0)$ in Eq. (12d) refers only to the fact that the correction term approaches some function of $K_1 K_2$ which is independent of $q$, when $q \to 0$.

The following decomposition will be useful later:

\[
\phi_{1234}^{(4)} = \phi_{1234}^{(4)} + \delta_{ij}^{(4)} \phi_1 \phi_2 - \delta_{ij}^{(4)} \phi_3 \phi_4 + \delta_{ij}^{(4)} \phi_3 \phi_2 + \delta_{ij}^{(4)} \phi_1 \phi_4,
\]

(14)

where $\phi_{1234}^{(4)} \sim \Omega^{-2}$ contains no delta-function singularity and is also completely antisymmetric. If $K_1 + \cdots + K_4 = 0$ but $K_1 + K_i \neq 0$ (for all $2 \leq i \leq 4$), Eq. (14) becomes

\[
\left[ (k_1^2 + k_2^2 + k_2^2 + k_4^2)/2 - 2E_m \right] \phi_{1234}^{(4)} + \frac{1}{2} \sum_{56} U_{1256} \phi_{5634}^{(4)} + \frac{1}{2} \sum_{56} U_{1256} \phi_{5634}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{1536}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{1536}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{2456}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{2456}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{3456}^{(4)} + \frac{1}{2} \sum_{56} U_{2356} \phi_{3456}^{(4)} = 0.
\]

(15)

A shorthand similar to Eq. (15):

\[
\phi_{K_1 K_2}^{(4)} \equiv \phi_{K_1 + q/2, -K_1 + q/2}^{(4)}
\]

(16)

Obviously $\phi_{12}^{(4)} = -(4\pi a_m/\Omega q^2) \phi_1 \phi_2 + O(q^0)$ as $q \to 0$.

Projector onto the function space orthogonal to $\phi_K$:

\[
P_{K_1 K_2} \equiv \delta_{K_1 K_2} - \phi_{K_1} \phi_{K_2}^{*}/2.
\]

(17)

“Deviation functions”:

\[
d_K \equiv \Omega \lim_{q \to 0} \frac{1}{2} \sum_{K \neq K_4} P_{K K_4} \phi^{(4)}_{K K_4} \phi^{*}_{K K_4}.
\]

(18a)

\[
d_K^* \equiv \Omega \lim_{q \to 0} \frac{1}{2} \sum_{K \neq K_4} P_{K K_4} \phi^{(4)*}_{K K_4} \phi_{K K_4}.
\]

(18b)

\[
g_{K_1 K_2} \equiv \Omega \lim_{q \to 0} \sum_{K \neq K_4} P_{K K_4} \phi^{(4)}_{K K_4} \phi_{K K_4}.
\]

(18c)

\[
g_{K_1 K_2}^* \equiv \Omega \lim_{q \to 0} \sum_{K \neq K_4} P_{K K_4} \phi^{(4)*}_{K K_4} \phi_{K K_4}.
\]

(18d)
These limits exist because the divergent component in $\phi_1^{(4)}(q)$ or $\phi_2^{(4)}(q)$, for $q \neq 0$ but $q \to 0$, is proportional to $\phi_1 \phi_2/q^2$, which is removed by the projector. $d_K$ and $d'_K$ are of the same thermodynamic order as $\phi_K$, but orthogonal to $\phi_K$; $\sum \phi_K d_K = 0$ and $\sum \phi_K d'_K = 0$; likewise, $\sum \phi_1 \phi_2 = 0$. Moreover, $d_{-K} = -d_K$ and $d'_{-K} = -d'_K$; $\bar{q}_1 = g_1 = -g_1$; $g_2 = -g_2 = g_2$; $g_1 \sim \Omega^{-1}$. No evidence for the vanishing of any of these functions is found.

The identity

\[ \sum_{12} U_{1212} \phi_2 = -2(k_1^2 - E_m) \phi_1 \tag{19} \]

and its complex conjugate (note that $U_{1234} = U_{3412}$) are often used (explicitly or implicitly) in the next section.

Two more identities:

\[ \frac{1}{2} \sum_{1234} U_{1234} \phi_1^{(4)} \phi_{1234} + \frac{1}{2} \sum_{1234} U_{1234} \phi_2^{(4)} \phi_{1234} = -2\pi a_m \phi_1 + \frac{1}{\Omega} \sum_{12} D_{12} d_2, \tag{20} \]

\[ \sum_{1234} U_{1234} \phi_1^{(4)} \phi_{1234} = -4\pi a_m / \Omega. \tag{21} \]

Equation (20) is proved in Appendix A, its inner product with $\phi'$ and $d'$, we get (to be used in Secs. IVC and IVB):

\[ \frac{1}{2} \sum_{1234} U_{1234} \phi_1^{(4)} \phi_{1234}^{(4)} + \frac{1}{2} \sum_{1234} U_{1234} \phi_2^{(4)} \phi_{1234}^{(4)} = -2\pi a_m \phi_1 + \frac{1}{\Omega} \sum_{12} D_{12} d_2, \tag{22} \]

\[ \sum_{1234} U_{1234} \phi_1^{(4)} \phi_{1234}^{(4)} = -4\pi a_m / \Omega. \tag{23} \]

The three terms on the left side of Eq. (23) are all real. Using a similar method as in Appendix A, starting from Eq. (15), we can show the following identity:

\[ \frac{1}{2} \sum_{1234} (U_{1234} \phi_1 + U_{1234} \phi_2 (1234) - U_{1234} \phi_1 - U_{1234} \phi_2) + U_{1212} \phi_2 + U_{2112} \phi_2 (1234) = \frac{1}{2} \sum_{12} D_{12} (d' \phi_2 + g' \phi_2) + D_{23} (\phi_1 d' + g' \phi_2) = -2\pi a_m / \Omega \phi_1 \phi_2. \tag{24} \]

when $K_1 \neq \pm K_2$. This will be used in Sec. IVB.

Finally, we have the $(p/2)$-molecule zero-speed scattering wave function ($p = 2, 4, 6, \cdots$), the generalization of Eqs. 10 and 12:

\[ \left[ -\frac{pE_m}{2} - \epsilon(p) \sum_{i=1}^{p} \frac{k^2_i}{2} \right] \phi_{K_1 \cdots K_{p+1}} + \frac{1}{2} \sum_{1 \leq i < j \leq p} U_{K_i K_j K_i K_j} \times \phi_{K_{i+1} \cdots K_{p+1} K_i K_{i+1} \cdots K_{p+1}} = 0 \tag{25} \]

if $K_1 + \cdots + K_p = 0$; $\phi_{K_1 \cdots K_p}$ is antisymmetric under the interchange of any two spin-momenta. If $p = 2$, $\epsilon(p) = 0$. If $p > 4$, $\epsilon(p) \sim \Omega^{-1}$, and this extremely tiny energy is negligible unless the sum of a nontrivial subset of the $p$ spin-momenta vanishes.

IV. DIATOMIC MOLECULES: LOW-DENSITY EXPANSION

At low density, the ground state is a dilute gas of molecules, any two of which are usually - but not always - far apart compared to the size of a molecule. In this section we study the ground state, exploiting such a low density.

A. Low-density orders

If the leading order contribution to a quantity $x$ scales like $n^R \Omega^2$ in the low-density regime, we say that the low-density order of $x$ is $R$, and may schematically write $x \sim n^R$; more details are in Appendix B.

It is reasonable to expect that the ground state energy is, to the lowest order in density, simply $E = E_m N/2$. Then $\mu = E_m/2 + \hbar o c$. [h.o.c. = higher order corrections (Appendix B), and the low-density order of $\mu$ is 0, in contrast with the case of structureless bosons.

Since $\mu < 0$, the low-density order of $k^2 / 2 - \mu$ is 0, for any $K$. The low-density order of $U_{K_1 K_2 K_3 K_4}$ is 0.

The low-density order of $\alpha K$ is 0.5. If we only retain the lowest order term in the exponent in Eq. (14), we get $n_K = |\alpha K|^2 / (1 + |\alpha K|^2)$. At low density, the spin-momentum distribution should approach that of fermions in an isolated molecule times a constant, so the width ($\sim 1/r_m$) is roughly independent of $n$, and the occupation number of each $K$ state should be of the order $n r_m^2 \ll 1$. So $\alpha K \sim n^{0.5}$.

The low-density order of $\beta (K_1 K_2 K_3 K_4)$ (where $K_1 + \cdots + K_2 = 0$) is usually 1 (see below for exceptions), because $|\beta (R_1 R_2 R_3 R_4)|^2$ (norm square of the Fourier transform) - for a given set of locations in a small region independent of $n$ - is approximately associated with the probability of finding two molecules (or rather four fermions) in such a region, and this probability is approximately proportional to $n^2$.

In general, the low-density order of $\alpha_K^{(p)}$ is usually $p/4$. 

Finally, we have the $(p/2)$-molecule zero-speed scattering wave function ($p = 2, 4, 6, \cdots$), the generalization of Eqs. 10 and 12:

\[ \left[ -\frac{pE_m}{2} - \epsilon(p) \sum_{i=1}^{p} \frac{k^2_i}{2} \right] \phi_{K_1 \cdots K_{p+1}} + \frac{1}{2} \sum_{1 \leq i < j \leq p} U_{K_i K_j K_i K_j} \times \phi_{K_{i+1} \cdots K_{p+1} K_i K_{i+1} \cdots K_{p+1}} = 0 \tag{25} \]

if $K_1 + \cdots + K_p = 0$; $\phi_{K_1 \cdots K_p}$ is antisymmetric under the interchange of any two spin-momenta. If $p = 2$, $\epsilon(p) = 0$. If $p > 4$, $\epsilon(p) \sim \Omega^{-1}$, and this extremely tiny energy is negligible unless the sum of a nontrivial subset of the $p$ spin-momenta vanishes.
The first exception to this estimate appears in $\beta_{K_1,K_2,K_3,K_4}$, since it is approximately proportional to the four-fermion wave function describing the zero-velocity scattering of two molecules, in the low density limit. When the two molecules are well separated, their spatial wave function should be of the form $1 - a_m/r$ times the internal wave functions of the molecules, where $r$ is the distance between the molecules. Taking the Fourier transformation, we see that $\beta_{K_1,K_2,K_3,K_4}$ scales like $a_m/q^2$, when $K_1 + K_2 = q \to 0$ (the term $a_4$ is encoded in $\alpha_K$).

If there were no many-body effects, this would be an infrared divergence at small $q$. But again, just like in the case of structureless bosons [2], the divergence is regulated when $q \sim 1/\xi$, where $\xi$ is the superfluid healing length. We will confirm in the concrete calculations that $\xi$ still scales like $n^{-0.5}$. By reducing $q = K_1 + K_2$ from the order $O(1)$ [for any momentum $k$, by $k \sim O(1)$ we mean that $k$ has a finite and positive lower bound when $n \to 0$ to the order $n^{-0.5}$, $\beta_{K_1,K_2,K_3,K_4}$ must increase from the low-density order $n^1$ to the low-density order $n^0$, in accordance with the $1/q^2$ dependence. After this, when $q$ is further reduced, the $1/q^2$ dependence breaks down, and the order of magnitude of $\beta_{K_1,K_2,K_3,K_4}$ will not increase again.

Like in the case of structureless bosons [2], we can extend this result to higher orders, and arrive at a general result

$$\alpha^{(p)}_{K_1,K_2} \sim n^{p/4-4M+1} \quad (1 \leq M \leq p/2),$$

where the $p$ spin-momenta - depending on their values - are divided into $M$ $\ell$-clusters. Each $\ell$-cluster is a collection of (at least two) spin-momenta such that

1. their sum is of the low-density order $\sqrt{n}$ (if $M \geq 2$) or zero (if $M = 1$);
2. the sum of any nontrivial subset of them is of the order $O(1)$ [by this we mean that either the sum of spins does not vanish, or the sum of momenta is of the order $O(1)$].

If $M \geq 2$, the sum of spin-momenta in each $\ell$-cluster must not be zero, or the whole $\alpha^{(p)}_{K_1,K_2}$ vanishes (see Sec. 11). There are still exceptions to the above result, because of the antisymmetry of $\alpha^{(p)}_{K_1,K_2}$. When the difference between two subscripts approaches zero, $\alpha^{(p)}_{K_1,K_2} \to 0$. However, if we regard $O(n^{p/4-4M+1}/\Omega^{1-p/2})$ (see Appendix B) as an upper bound of $\alpha^{(p)}_{K_1,K_2}$, the Fermi statistics does not break it.

Now we decompose the $p/2$-th order (de)generator into the sum of a set of dispersed vertices, like in the case of a dilute gas of structureless bosons [2]. The decomposition of the first three generators are shown in Fig. 2, for the degenerators, we just replace each solid circle with a hollow one. Despite the striking similarity between Fig. 2 and a similar figure in [2], the corresponding coefficients have vastly different orders of magnitude, because of their different thermodynamic orders. Nevertheless, their low-density orders are the same. Since we have proved that the total thermodynamic order of any diagram in the expansion of $E - \mu N$ is 1, we can concentrate on the low-density orders when estimating the orders of magnitude of such diagrams, without worrying about the thermodynamic orders of individual vertices.

In such a decomposition, $\alpha = \alpha^{(2)}$ remains a single term, denoted by $v_1$. $\beta = \alpha^{(4)}$ is decomposed to two terms, $v_2$ and $v_{11}$; the first one is restricted to $k > k_c$, and the second one to $k < k_c$, where $k = \min_{\gamma} \{\gamma \neq 0 \} \equiv (n\eta - 1)/(1/r_m)$, $k_c$ is a momentum scale satisfying

$$\sqrt{n}a_x \ll k_c \ll \min(1/\ell, 1/r_m),$$

where $a_x$ is independent of $n$ and $\Omega$; we will see that $a_x$ is actually $a_m$. Further, $\gamma = \alpha^{(6)}$ is decomposed to three terms (Fig. 2), and we will see that the second term, $v_{21}$, is most important in the low-density regime, similar to the situation for structureless bosons [2].

Such division of the spin-momentum configuration space is crucial for the practical computation. By decomposing diagrams according to such division of the basic vertices, one will be able to analyze the orders of magnitude of diagrams and compute their actual values much more easily. The physical result, however, is independent of such division of the spin-momentum space, and in particular, independent of the choice of $k_c$, as long as the above inequality is satisfied.

The low-density order of the coefficients $\alpha^{(p)}_{K_1,K_2}$ in any dispersed vertex $v$ is

$$R(v) = \frac{p(v)}{4} - N'_1(v) + N'_2(v),$$

where $p(v)$ is the number of solid lines attached to $v$, while $N'_1(v)$ and $N'_2(v)$ are the number of dashed lines connected to $v$.
$N'_i(v)$ is the number of dotted lines contained by $v$, and $N'_j(v)$ is the number of junctions of these dotted lines. If $M \leq 2$ [see Eq. (20)], $N'_i(v) = M - 1$ and $N'_j(v) = 0$; but if $M \geq 3$, $N'_i(v) = M$ and $N'_j(v) = 1$; see Fig. 2. Equation (28) is equivalent to Eq. (26), but will be more convenient in the next subsection.

B. Low-density orders of diagrams

To compute physical observables in the low density regime, it will be much more convenient to decompose the (de)generators in the above way first. The physical observables are then expanded in terms of the dispersed vertices, and we obtain a new set of diagrams containing small internal circles, solid lines, and dotted lines; for $E - \mu N$ we also have a diamond in each diagram, and for $\langle O \rangle$ ($O$ is a product of $L$ basic fermionic operators in spin-momentum space), we have $L$ external points; in this paper we use the term low-density expansion in a narrower sense, to refer to these diagrams.

To obtain the correct value of a physical observable up to a certain order in the low-density expansion, we must take into account all the diagrams for this observable up to this order (called significant diagrams). The omission of any significant diagram may invalidate the result completely. It is therefore crucial to derive a general power-counting formula, using which we can identify all the significant diagrams for an observable up to any certain order.

Such formula is

$$R = N_3 + \sum_{i=1}^{I}(P_i/4 - 1) + N'_j/2, \quad (29)$$

where $R$ is the low-density order of any such diagram (denoted by $D$ below), $N_3$ is its number of disconnected parts, $I$ the number of islands, $P_i$ the $P$-number of the $i$-th island, and $N'_j$ is the number of independent loops in the skeleton diagram. The terms are defined as follows.

If we remove all the dotted lines (if there are any) and their possible junctions (see previous subsection) from $D$, but leave all the other components unchanged, $D$ is changed to $I$ disconnected parts, each of which is called an island. Different islands are either linked by dotted lines only, or completely disconnected. Any small circle (other than the external points) in an island is attached by a certain number of solid lines; the sum of all such numbers over the whole island is called the $P$-number of the island, and the sum over the whole diagram is the $P$-number of the diagram. Finally, if we reduce every entire island in $D$ to a point, but leave the dotted lines (if there are any) and their possible junctions unchanged (the dotted lines are still attached to the islands and junctions), we get the skeleton diagram of $D$. Obviously, if $D$ contains no dotted lines, its skeleton diagram is $N_3$ isolated points and $I = N_3$.

Proof. Only two factors contribute to the low-density order $R$ of $D$:

- each dispersed vertex $v$ contributes $R(v)$ in accordance with Eq. (28);
- each independent small internal momentum $q$ ($\sim \sqrt{\pi}$) which flows through a dotted line contributes 1.5, since the measure $\sum_{|q|\sim \sqrt{\pi}} = \Omega \int q/\langle 2\pi \rangle^3 \sim \sqrt{\pi}$ scales like $n^{1.5}$.

The number of independent $q$’s above is clearly $N'_2$, as momentum is conserved in any part of the diagram. So $R = P/4 - N'_i + N'_j + 1.5N'_2$, where $P$ is the $P$-number of $D$, $N'_i$ the total number of dotted lines, and $N'_j$ the total number of junctions of the dotted lines.

Now turn to the skeleton diagram $D'$. It has $I + N'_i$ vertices (islands and junctions), $N'_i$ lines (dotted lines), $N'_j$ independent loops, and $N'_2 = N_3$ disconnected parts (the number of disconnected parts is not changed when the islands are reduced to points). So using Eq. (28) we get $I + N'_i - N'_j + N'_2 - N_3 = 0$.

Using the above equation to cancel the term $-N'_i + N'_2$ in the expression of $R$, we get Eq. (29) (note that obviously $P = \sum_{i=1}^{I} P_i$).

The central result of the last subsection, Eq. (26), is a special case of Eq. (29).

To better understand $R$, we further study $P_i$. Suppose that the contribution to $P_i$ from all the small solid circles (other than the external points) in the $i$-th island is $P_i^{(+)}$, and that the contribution from all the small hollow circles (other than the external points) in the same island is $P_i^{(-)}$. Suppose also that the number of solid external points in the $i$-th island is $L_i^{(+)}$, and that the number of hollow external points in the same island is $L_i^{(-)}$. Obviously

$$P_i = P_i^{(+)} + P_i^{(-)}, \quad \text{(30a)}$$

$$P_i^{(+)} + L_i^{(+)} = P_i^{(-)} + L_i^{(-)} > 0, \quad \text{(30b)}$$

$$P_i^{(+)}, P_i^{(-)} \in \{0, 2, 4, 6, 8, \ldots \} \quad \text{(30c)}$$

The contribution to the balance of particle number from the diamond (if there is any) is cancelled.

For convenience, we shall call any island which contains two or more external points, an external island; any island which contains no external point, an internal island. Any island which contains a diamond, a Hamiltonian island. Finally, any island which contains neither a diamond nor any external point, a (vacuum) bubble island, since it has the same topological structure as a vacuum bubble.
An corollary of Eq. 29: for any internal island i (a bubble island, or a Hamiltonian island which contains no external points), $P_i$ can only be 4, 8, 12, 16, · · · , and $P_i/4 - 1$ can only be a nonnegative integer. It is this property that makes Eq. 29 particularly useful in the search of all the diagrams in the low-density expansion of an observable up to a certain order.

The bubble island whose $P$-number is 4 has a unique topological structure - a solid circle linked to a hollow one by two solid lines and contributes nothing to $R$; it will be called the simplest bubble.

An alternative (sometimes useful) form of Eq. (29) is

$$R = N_3 + (P_{\text{ext}}/4 - I_{\text{ext}}) + N'_2/2 + (P_{\text{int}}/4 - I_{\text{int}}),$$

(31)

where $I_{\text{ext}}$ and $I_{\text{int}}$ are the numbers of external islands and internal ones, respectively; $P_{\text{ext}}$ and $P_{\text{int}}$ are the total contributions to $P$ from the external islands and internal ones, respectively. $P_{\text{int}}/4 - I_{\text{int}}$ is always a nonnegative integer.

In summary, $D$ is more conveniently viewed as an "archipelago" (islands with weak links) when its low-density order is analyzed. The archipelago structure is a direct consequence of the presence of two sets of well-separated length scales in the problem: the superfluid healing length $\alpha \propto 1/\sqrt{n}$, and the few-body length scales $l$, $r_m$, etc.

When computing its actual value, we of course view $D$ in the "usual" way, namely as a set of dispersed vertices with a diamond on/and external points.

C. $E - \mu N$ to the order $n^3$

In any diagram in the low-density expansion of $E - \mu N$, there can not be any "dangling" dotted line (one whose removal would change such a diagram to two disconnected parts), since dead ends are absent (Sec. III). So if there is any dotted line in such a diagram, $N'_2 \geq 1$ and $R \geq 1.5$ [see Eq. (20)].

So if $R = 1$, there can only be one island, the Hamiltonian island. Moreover, the $P$-number of this island must be 4. So we can only find two diagrams satisfying $R = 1$, in the expansion of $E - \mu N$. They are $T_1$ and $T_2$ shown in Fig. 6.

$$T_1 = \sum_K (k^2/2 - \mu) |\alpha_K|^2,$$

(32a)

$$T_2 = \frac{1}{4} \sum_{KK'} U_{K,-K,K',-K'} \alpha_K^* \alpha_{K'}.$$

(32b)

We then adjust the parameters $\alpha_K$ to minimize $E - \mu N$: $\partial(E - \mu N)/\partial\alpha_K^* = 0$. So

$$k^2 \alpha_K + \frac{1}{2} \sum_{K'} U_{K,-K,K',-K'} \alpha_K = 2\mu \alpha_K + h.o.c.,$$

(33)

where $h.o.c.$ stands for higher order corrections. Since the fermions have formed bound pairs, and the density is low, $\alpha_K$ should be approximately proportional to the internal wave function of an isolated molecule, and $2\mu \approx E_m$. Comparing the above equation with Eq. (11), we find $\alpha_K = \eta \phi_K + h.o.c.$

The number of fermions is $N = \sum_K |\alpha_K|^2 + h.o.c. = 2|\eta|^2 + h.o.c.$, so $\eta = \sqrt{N/2} + h.o.c.$, where we have chosen the phase factor of $\eta$. We thus get

$$\alpha_K = \sqrt{N/2} \phi_K + h.o.c.,$$

(34)

$$\mu = E_m/2 + h.o.c.,$$

(35)

$$E/\Omega = nE_m/2 + h.o.c.,$$

(36)

where $E$ is obtained by solving $dE/dN = \mu$, and

$$N_1 \equiv N_\| \equiv N/2, \quad (37a)$$

$$n_\| \equiv n_\| = n/2 \equiv N/(2\Omega). \quad (37b)$$

D. First cluster-separation theorem

In this subsection and the next, we discuss an elementary property of the system in the low-density regime. It will help to further the low-density expansion.

Consider $p$ spin-momenta $\tilde{K}_1 \cdots \tilde{K}_p$, where $p$ is positive and even, $\tilde{K}_1 + \cdots + \tilde{K}_p = 0$, and the sum of any nontrivial subset of these spin-momenta is nonzero. Suppose that they are divided into $M$ l-clusters ($M \geq 1$), and the $\nu$-th ($1 \leq \nu \leq M$) l-cluster contains $p_\nu$ spin-momenta: $\tilde{K}_{\nu+1} \cdots \tilde{K}_{\nu+p_\nu}$, where $\sum_\nu p_\nu = p$, $e_\nu = p_1 + p_2 + \cdots + p_\nu$, and $b_\nu = e_\nu - p_\nu$. Suppose also that there does not exist any l-cluster other than these $M$. Make the decomposition $\tilde{K}_i = \tilde{K}_{i,\nu} + q_\nu/p_\nu (1 \leq i \leq \tilde{K}_i)$, such that $\sum_\nu q_\nu = \tilde{K}_{i,\nu} = 0$. Consider the ground state expectation value $F_{\tilde{K}_1 \cdots \tilde{K}_p} = \langle e^{\sum_{\nu=1}^M c_{\tilde{K}_{\nu+1}}^* c_{\tilde{K}_{\nu+1}} \cdots c_{\tilde{K}_p}} \rangle$. The first cluster-separation theorem states that there exists some coefficient $\eta$, which depends on $q_1 \cdots q_M$ but not on $\tilde{K}_{i,1} \tilde{K}_{i,2} \cdots \tilde{K}_{i,p_1+1} \cdots \tilde{K}_{i,M} K_{M-1} K_{M-2} \cdots K_{p_M,M}$, such that

$$F_{\tilde{K}_1 \cdots \tilde{K}_p} = \eta \prod_{\nu=1}^M \phi_{\tilde{K}_{\nu+1}}^{(p_\nu)}$$

(38)

with a relative error $\sim n^4$.

There are infinitely many diagrams in the low-density expansion of $F$, but their low-density orders have a lower bound. Consider any one of these diagrams, $D$: $N_3 \geq 1$, $P_{\text{ext}} \geq p$, and $I_{\text{ext}} \leq M$ (it is impossible to separate the $p$ external points into more than $M$ islands, since spin-momentum is conserved at every internal vertex, and the dotted lines can not carry spin or any large momentum), so we deduce from Eq. (31) that $R(D) \geq p/4 - M + 1$. The low-density order of $F$ is therefore $R \equiv p/4 + M - 1$ or higher. One of the leading contributions (with low-density order $R$) to $F$ is $\alpha_{\tilde{K}_1 \cdots \tilde{K}_p}^{(p)}$.

While it is intuitively clear that the right side of Eq. (38) should be the form of the lowest order approximation of $F$, it is not a priori obvious why the next-to-leading order correction (of the order $n^{R+0.5}$) is also...
proportional to such a product of zero-speed scattering wave functions.

The proof is quite involved. In Appendix C we prove this theorem for the cases in which each l-cluster contains only two spin-momenta. To compute $E - \mu N$ up to the order $n^{2.5}$, such an incomplete proof will be sufficient. The complete proof will appear in a future work.

### E. Second cluster-separation theorem

There is a similar formula for the coefficient of the dispersed vertex:

\[
\alpha^{(p)}_{K_1 \cdots K_p} = \eta \prod_{\nu=1}^{M} \phi^{(p_\nu)}_{K_1, \nu} \cdots \phi^{(p_\nu)}_{K_p, \nu}
\]

plus a correction term whose low-density order is $R + 1$ or higher, where $R = p/4 - M + 1$, and the $q$, $K$, $\nu$ and $R$'s are defined in Sec. [IVB](#IVB). The coefficient $\eta$ depends on $q_1, q_2 \cdots q_M$ and $n$, but is independent of the $K$'s.

The nontrivial content of this theorem is that the next-to-leading order correction ($\sim n^{R+0.5}$) to the coefficient of the dispersed vertex is factorizable in the same manner as the leading term.

In Appendix D we give the proof of this theorem for the cases in which each l-cluster contains only two spin-momenta; it builds upon the results already proved in Appendix C and is even more involved. For the determination of $E - \mu N$ up to the order $n^{2.5}$, such special results will suffice. The complete proof will appear later.

This partial proof will simplify the logic in the concrete calculations of $E - \mu N$. More importantly, it enables us to accurately determine some other interesting quantities, such as the many-body effects in the momentum distribution of the fermions.

The second cluster-separation theorem is more fundamental in the present formalism, since the first one and some other similar properties, such as that of $\langle c^\dagger_{K_1+q/2} c_{K_2-q/2} c_{K_3+q/2} c_{K_4+q/2} \rangle$, are its corollaries. The first one is presented for two reasons: 1) it concerns physical observables rather than parameters in a particular calculational framework, and 2) the proof of the second theorem is built upon the proof of the first. Hereafter we use the name “cluster-separation theorem” to refer to the second theorem.

### F. Diagrams for $E - \mu N$ up to the order $n^{2.5}$

In this subsection we identify all the diagrams in the low-density expansion of $E - \mu N$ up to the order $n^{2.5}$.

Any such diagram $D$ has $N_3 = 1$ since it is a connected one. So $R \leq 2.5$ implies $N_2 \leq 3$ [Eq. (29)].

If $N_2 = 0$, there is no dotted line at all in $D$. So there is only one island, the hamiltonian island, whose $P$-number can only be 4 ($R = 1$) or 8 ($R = 2$). This is because any island in the expansion of $E - \mu N$ is an internal one, whose $P$-number must be a multiple of 4 (Sec. [IVB](#IVB)). The two $R = 1$ diagrams are $T_1$ and $T_2$ (Fig. 1), whose sum however has a higher low-density order, $R = 2$. This is the first instance of the “$R+1$”-cancellation mechanism, discussed in the following.

Consider an arbitrary pair of diagrams in the low-density expansion of $E - \mu N$, $D_1$ and $D_2$, whose hamiltonian islands are topologically the same as $T_1$ and $T_2$, respectively: if their other parts are identical, we call them an “$R+1$”-pair. The two diagrams obviously have the same low-density order, $R$. The second cluster-separation theorem implies that the low-density order of their sum is at least $R + 1$.

If $D_1$ and $D_2$ are just $T_1$ and $T_2$, the proof is simple: substituting $\alpha_K = \eta \phi_K + O(n^{1.5})$ (Sec. [IVB](#IVB)) into Eq. (30), and noting that $\mu = E_m/2 + O(n^3)$ (Appendix C), we immediately get this result.

If $D_1$ and $D_2$ contain dotted lines, they can be written in the form [see Fig. 3 and Eq. (27)]

\[
D_1 = \sum_{q<k_1;K} (k^2/2 - E_m/2 - \tilde{\mu} + q^2/8) X^q_{K;K} \quad (40a),
\]

\[
D_2 = \frac{1}{4} \sum_{q<k_1;K_1,K_2} U_{K_1,-K_1,K_2,-K_2} X^q_{K;K_1K_2} \quad (40b),
\]

where $\tilde{\mu} \equiv \mu - E_m/2$, the first subscript of $X^q_{K_1K_2K_3}$ is associated with the creation of a pair of fermions with spin-momenta $K_2 + q/2$ and $-K_2 + q/2$, and the second subscript is associated with the annihilation of a pair with spin-momenta $K_1 + q/2$ and $-K_1 + q/2$. In $D_1$, an average over $K$ and $-K$ has been taken, since $X^q_{K_1K_2}$ is antisymmetric for $K_2 \rightarrow -K_2$ (and similarly for $K_1$), and thus $X^q_{KK}$ is symmetric for $K \rightarrow -K$. In $D_2$, $U$ has Galilean symmetry and is independent of $q$ (Sec. [IIA](#IIA)). Note finally that the signs of $D_1$ and $D_2$ are correct, as is clear if one restrict every trajectory (Secs. [IIA](#IIA), and [IIID](#IIID)) within an island when determining the sign. If the small solid circle associated with the first subscript is contained by a dispersed vertex $v_1 \cdots v_4$, then from our established result we find that $X^q_{K_1K_2K_3} = \phi_{K_1} X^q_{K_1K_2}$, with relative error of the order $n^1$, and the $R + 1$ cancellation between $D_1$ and $D_2$ immediately follows. If the small hollow circle associated with the second subscript
is contained by $v^\dagger_1 \ldots v^\dagger_m$, then $X^a_{K_1 K_2} \phi_{K_1} X^{\alpha a}_{K_2}$ with relative error of the order $n^{-1}$, and the $R + 1$ cancellation follows from the complex conjugate of Eq. 11a.

For $R \leq 2.5$, there is only one “$R + 1$”-pair for which we have not shown the $R + 1$ cancellation, purely due to our incomplete proof of the cluster-separation theorem. This case is shown in Fig. 4. It involves the vertex $v_{11}$ and $v^\dagger_{11}$, and each diagram has $R = 2.5$. However, $\delta K_1 + q/2 - q /2 = \frac{q}{\sqrt{R}} = 0$ is proportional to $\phi K_2$, at the leading order at least, since it is approximately associated with the correlation of four fermions and a molecule which is far away (with distance $\sim \xi$) and the internal structure of that molecule is almost not influenced by the first four fermions. So the sum of these two diagrams is at least of the low-density order $n^3$, and can be omitted in the computation of $E - \mu N$ up to the order $n^{2.5}$.

The counterpart of the “$R+1$” mechanism in the context of structureless bosons [2] is a much more obvious fact that $q^2/2 - \mu_{\text{boson}}$ is of the order $n^{-1}$ when the boson momentum $q = 0$ or $q \sim n^{-1/2}$.

Because of the “$R+1$”-cancellation mechanism, the set of significant diagrams, up to the order $n^{2.5}$, is greatly reduced. If the “$R + 1$”-pair $D_1$ and $D_2$ satisfy $R \geq 2$ each, then their sum is at least of the order $n^3$ and negligible.

If the hamiltonian island is $T_1$ or $T_2$, only the $N^*_2 = 0$ or $N^*_2 = 1$ diagrams are significant, and the bubble islands are all the simplest ones (because of the $R + 1$ mechanism). They are $T_1$, $T_2$, $T_{1a}$ and $T_{2a}$, shown in Fig. 5.

If the hamiltonian island has $P = 8$, then $N^*_2 = 0$ ($R = 2$) or $N^*_2 = 1$ ($R = 2.5$). If $N^*_2 = 0$, the hamiltonian island (the whole diagram) may contain two $v^\dagger_1$'s and two $v^\dagger_1$'s ($T_3$, $T_4$, $T_5$, and $T_6$), or one $v_2$ and two $v^\dagger_1$'s ($T_7$), or two $v^\dagger_1$'s and one $v^\dagger_1$ ($T_8$), or one $v^\dagger_1$ and one $v^\dagger_1$ ($T_9$) and one $v^\dagger_2$ ($T_7$ and $T_{10}$). If $N^*_2 = 1$, we must exhaust all the topologically distinct ways of attaching a dashed line to the hamiltonian island, and obtain the diagrams $T_{3a}$, $T_{3b}$, $T_{3c}$, $T_{3d}$, $T_{4b}$, $T_{4c}$, $T_{4d}$, $T_{5a}$, $T_{5b}$, $T_{5c}$, $T_{5d}$, $T_{6a}$, $T_{6b}$, $T_{6c}$, $T_{6d}$, $T_{7a}$, $T_{7b}$, $T_{8a}$, $T_{8b}$, $T_{9a}$, and $T_{10a}$. The bubble diagrams in these diagrams are all the simplest ones or $R \geq 3.5$ [Eq. 24]. All these diagrams are shown in Fig. 5.

Each dashed line represents a geometric series associated with a variable number of simplest bubbles. Each end of the dashed line is attached by an arbitrary positive even number of solid lines. The concrete meanings of the dashed lines used in Fig. 5 are shown in Fig. 6.

In $T_{3b}$, $T_{3c}$, $T_{4b}$, and $T_{5d}$, a cross is marked on the dashed line, to indicate the absence of the first term in such a series, i.e. $\delta K_1 K_2 K_3 K_4$ (or its complex conjugate), because two of these four spin-momenta happen to be exactly opposite.

G. $E - \mu N$ to the order $n^2$

The diagrams for $E - \mu N$ up to the order $n^2$ are $T_1$ through $T_{10}$ only. Note that $T_{1a}$ and $T_{2a}$ are an “$R+1$”-pair and their sum is of the low-density order $n^{2.5}$.

$$T_1 = \sum_K \left( \frac{k^2}{2} - \mu \right) |\alpha_K|^2,$$  \hspace{1cm} (41a)

![FIG. 4: Two diagrams with $R = 2.5$ each. Their sum should be $\sim n\mu$, but the proof is not yet given. However, trivially this sum is at least of the low-density order $n^3$.](image)

![FIG. 5: Diagrams in the low-density expansion of $E - \mu N$ of molecules (pairs of fermions) up to the order $n^{2.5}$. The composite vertices (each of which contains a dashed line) are defined in Fig. 6, each of them represents a geometric series. In $T_{3b}$, $T_{3c}$, $T_{4b}$, and $T_{5d}$, a cross is marked on the dashed line, to indicate the absence of the first term in such a series, i.e. $\delta K_1 K_2 K_3 K_4$ (or its complex conjugate), because two of these four spin-momenta happen to be exactly opposite.](image)
FIG. 6: Definition of some composite vertices, each of which is a geometric series. The spin-momenta and the momentum $E_{\mu N}$.

$$\beta_{K_1,K_2,K_3} = \delta_{K_1\delta_{K_2\delta_{K_3}}K_4} + \cdots = C_{K,K}^0,$$

where $\sum'$ excludes cases in which the sum of any two subscripts of $\beta$ (or $\beta^*$) is less than $k^*$, and in $T_3$ (and $T_9$) the term containing $\mu$ is omitted since it is $\sim n^3$.

The parameters $\alpha$ and $\beta$ are then adjusted to minimize $E - \mu N$. We first write the terms containing $\beta^*$ (i.e. $T_8 + T_{10}$) in the form $\sum'n'_{\alpha} \beta_{K_1,K_2,K_3} \beta_{K_4,K_5,K_6}$; only the completely antisymmetric component of $\theta$ contributes to the sum, and this component must vanish since it is independent of $\beta^*$ and $\partial(E - \mu N)/\partial \beta^* = 0$. We get

$$[(k_1^2 + k_2^2 + k_3^2 + k_4^2)/2 - 2E_m]\beta_{1234}$$

$$+ \frac{1}{2} \sum_{56} U_{1256}\beta_{5634} + \frac{1}{2} \sum_{56} U_{1356}\beta_{5624} + \frac{1}{2} \sum_{56} U_{1456}\beta_{56236}$$

$$+ \frac{1}{2} \sum_{56} U_{2356}\beta_{56153} + \frac{1}{2} \sum_{56} U_{2456}\beta_{56153} + \frac{1}{2} \sum_{56} U_{3456}\beta_{56152}$$

$$- U_{1234}\alpha_{34} - U_{1324}\alpha_{24} - U_{1423}\alpha_{23} - U_{2314}\alpha_{14} - U_{2413}\alpha_{13} - U_{3412}\alpha_{12} = 0,$$  (42)

The excluded regions in the spin-momentum space for the summation $\sum'_{\alpha}$ contribute terms of the low-density order $n^{1.5}$, which are smaller than the dominant terms by a factor of the order $n^{0.5}$; $\alpha_K$’s lowest order expression is given by Eq. (44). So the comparison between Eq. (42) and Eq. (45) yields

$$\beta_{K_1,K_2,K_3} = N' \phi_{K_1,K_2,K_3} K_4 + O(n^{1.5}\Omega^{-1})$$  (43)

if $\min_{\sigma_i+\sigma_j} = 0, 1 \leq i < j \leq 4 |K_i + K_j| > k_c$.

Now turn to $\partial(E - \mu N)/\partial \alpha_K = 0$. Write $\delta(E - \mu N) = \sum_K \theta^*_{K} \delta_{K} + \cdots$. Only the antisymmetric component of $\theta^*_K$ contributes to the sum, since $\alpha_{-K} = -\alpha_K$. This
component must vanish, and we get (after a little algebra)

\[
\sum_2 \tilde{D}_{12}\alpha_2 = \alpha_1^2 \sum_2 D_{21}\alpha_2^2 + \sum_2 D_{12}|\alpha_2|^2 \alpha_2 \\
+ |\alpha_1|^2 \sum_2 U_{1122}\alpha_2 - \alpha_1 \sum_2 |U_{1212} + U_{1222}|^2 \alpha_2 \\
+ \frac{1}{2} \sum_{234} U_{1234}^{23\alpha_2^2 \beta_{1234}} + \frac{1}{2} \sum_{234} U_{12343\alpha_2^2 \beta_{1234}} (44)
\]

plus correction terms \(\sim n^2\), where \(\tilde{D}_{12} \equiv D_{12} - 2\tilde{\mu}\delta_{12}\) and \(\tilde{\mu} \equiv \mu - E/m/2\). The terms on the right side of Eq. (44) are \(\sim n^{1.5}\), and in this subsection they can be simplified with the lowest order formulas [Eqs. (53) and (55)]; we then solve the equation to obtain more accurate results of \(\alpha_K\) and \(\mu\).

The first term on the right side of Eq. (44) vanishes at the order \(n^{1.5}\), and the remaining terms are further simplified with Eq. (22) (the excluded regions in the spin-momentum space for \(\tilde{\mu}_{234}\) can contribute terms of the low-density order \(n^2\) and are unimportant here):

\[
\sum_2 \tilde{D}_{12}\alpha_2 = \frac{N^{3/2}}{\Omega} \left(-2\pi a_m\phi_1 + \sum_2 D_{12}d_2^2 \right) + O(n^2).
\]

We decompose \(\alpha_K\) into a component parallel to \(\phi_K\) and one orthogonal to \(\phi_K\):

\[
\alpha_K = \eta_1\phi_K + \alpha_1^\perp, \quad \sum_K \phi_K^\perp \alpha_1^\perp = 0, (46)
\]

where \(\eta_1\) is elected as positive, since there is a gauge symmetry: when \(\alpha^{(p)}\) in Eq. (1) is changed by a factor \(e^{ip\theta}\) (\(\theta\) is real and independent of \(p\)), the particle number and the energy expectation values are invariant. Substituting Eq. (46) into Eq. (45), and noting that \(\eta_1 = \sqrt{N_\uparrow + h.o.c.} (\text{Sec. IV C})\), we get

\[
-2\tilde{\mu}n^{1/2}_\uparrow = -2\pi a_m N^{3/2} / \Omega + O(n^2\Omega^{1/2}), \\
\sum_2 D_{12}(\alpha_2^\perp - N^{3/2}2\delta_{2}/\Omega) = 2\pi a_m^\perp + O(n^2).
\]

So

\[
\mu = E_m/2 + \pi a_m n_\uparrow + O(n^{1.5}), (48)
\]

and the first term on the right side of Eq. (47) is \(\sim n^{2.5}\) and negligible; noting also that the spectrum of \(D_{12}\) (when restricted to the subspace orthogonal to \(\phi_K\)) has a lower bound \(|E_m|\), we get

\[
\alpha_1^\perp = n_\uparrow^{3/2} \Omega^{1/2} d_\perp + O(n^2). (49)
\]

Solving \(dE/dn = \mu\), we get

\[
E/\Omega = E_m n_\uparrow + \pi a_m n_\uparrow^2 + O(n^{2.5}). (50)
\]

As a consistency check, we may use the above results for \(\alpha\) and \(\beta\) to calculate \(T_1 + \cdots + T_{10}\), and verify that it is equal to \(E - \mu N = -\pi a_m n_\uparrow^2/\Omega + h.o.c.\).

Firstly, \(T_1 + T_2 = -\tilde{\mu} \sum_K |\alpha_K|^2 + \frac{1}{2} \sum_1^2 \alpha_1^* D_{11}\alpha_2 \sim n^3\), so \(T_1 + T_2 = -2\pi a_m n_\uparrow^2/\Omega + h.o.c.\).

Secondly, \(T_3 + \cdots + T_T = N_\uparrow^2[t(1) + t(2)] + O(n^{2.5}\Omega)\),

where \(t(1) = -(1/2) \sum_1^2 \phi_2 D_{2i}\phi_2^2 \phi_2 = 0\), and \(t(2) = -(1/4) \sum_1^2 U_{1112}\phi_1^2 \phi_2^2 \phi_2 + (1/2) \sum_1^2 U_{1112}\phi_1^2 \phi_2^2 - (1/4) \sum_1^2 U_{1123}\phi_1^2 \phi_2^2 = +\pi a_m/\Omega [\text{see Eq. (23)}]\), so \(T_3 + \cdots + T_T = +\pi a_m n_\uparrow^2/\Omega + h.o.c.\).

Thirdly, \(T_8 + T_9 + T_{10} = (N_\uparrow^2/24) \sum_1^2 \phi'_2 \phi_2 \sum_1^2 \phi_2 = +2(k_2^2 - E_m)\phi_2^2 + 3 \sum_1^2 \sum_1^2 \phi_2 \phi'_2 + O(n^{2.5}\Omega)\), but the antisymmetric component of the expression in the brackets is just the left side of Eq. (10), so \(T_8 + T_9 + T_{10} = O(n^{2.5}\Omega)\).

So \(T_1 + \cdots + T_{10} = -\pi a_m n_\uparrow^2/\Omega + h.o.c.\).

Equation (46) is indeed the well-known equation of state at the mean-field level:

\[
\mu_m = E_m + 4\pi a_m n_m/m_m + h.o.c.,
\]

where \(\mu_m = 2\mu\) is the molecular chemical potential, \(n_m = n_\uparrow\) is the molecular density, and \(m_m = 2m = 2\) is the molecular mass. We have thus successfully constructed a many-body theory which is completely compatible with the exact, nonperturbative solution to the quantum four-fermion problem, with fairly arbitrary finite-range interparticle interaction (\(not\) restricted to a contact interaction for which \(t \to 0\)).

Although common wisdom has had some success on the equation of state, a working many-body theory is needed for the understanding of many other important features of the system. Other predictions of this theory will be presented in Sec. V. Here we just point out that Eq. (19) is directly related to the deviation of \(\langle \epsilon_{c-k_c} \rangle\) from the internal wave function of an isolated molecule. For a two-component Fermi gas near a Feshbach resonance, such deviation is most dramatic in the unitarity limit, in which the width of the internal wave function of an isolated molecule in momentum space \(1/r_m \to 0\), but \(\langle \epsilon_{c-k_c} \rangle\) remains a finite width \(\sim k_F \sim n^{1/3}\). The present theory helps to bridge the two regimes, and describes accurately how the physical picture of the system begins to change, as we start from the BEC limit, going toward the unitary regime.

In the next subsection, we go beyond mean-field.
\[ T_{3b} + T_{3c} = -\frac{1}{2} \left\{ \sum_{2q} (k^2 - E_m) \alpha_2^2 \bar{a}^q_{22} + c.c. \right\}, \]  
\[ T_{4a} + T_{5a} = -\frac{1}{2} \left\{ \sum_{12, q} U_{1122} \alpha_1^* \alpha_2 C_{q22}^q + c.c. \right\}, \]  
\[ T_{4b} + T_{5b} = -\frac{1}{2} \left\{ \sum_{12, q} U_{1122} \alpha_1 \alpha_2^* C_{q21}^q + c.c. \right\}, \]  
\[ T_{4c} + T_{5c} = -\frac{1}{4} \left\{ \sum_{12, q} U_{1122} \alpha_1^2 \alpha_2 A_{q12}^q + c.c. \right\}, \]  
\[ T_{4d} + T_{5d} = -\frac{1}{4} \left\{ \sum_{12, q} U_{1122} \alpha_1 \alpha_2^2 \alpha_2 A_{q22}^q + c.c. \right\}, \]  
\[ T_{6a} = \sum_{12, q} U_{1212} \alpha_1^2 C_{q22}^q, \]  
\[ T_{6b} = \sum_{12, q} U_{1212} \alpha_1^* \alpha_2 C_{q12}^q, \]  
\[ T_{6c} + T_{6d} = \frac{1}{2} \sum_{12, q} U_{1212} \alpha_1 \alpha_2^* A_{q12}^q + c.c., \]  
\[ T_{7a} + T_{8a} = -\frac{1}{4} \left\{ \sum'_{1234, q} U_{1234} A_{q12}^q \beta_{1234} + c.c. \right\}, \]  
\[ T_{7b} = -\frac{1}{4} \left\{ \sum'_{1234, q} U_{1234} \alpha_1^* \alpha_2 \gamma_{1234} \tilde{A}_{q15} + \tilde{A}_{q45}^* \right\}, \]  
\[ T_{8b} = -\frac{1}{4} \left\{ \sum'_{1234, q} U_{1234} \alpha_2^* \gamma_{1234} \tilde{A}_{q15} + \tilde{A}_{q45}^* \right\}, \]  
\[ T_{9a} = \frac{1}{12} \sum'_{1234578, q} \gamma_{1234}\tilde{\gamma}_{1234}^q \times \left[ (1 - \beta_q^2 q/4)^{-1} \right]_{87}, \]  
\[ T_{10a} = \frac{1}{16} \sum'_{12345678, q} \gamma_{12345678, q} \gamma_{1234756348} \times \left[ (1 - \beta_q^2 q/4)^{-1} \right]_{87}, \]  
where the arabic-number subscripts (except those of \( T \)) are spin-momenta, the summation over \( q \neq 0 \) is restricted to \( q < k_c \), corrections of order \( n^3 \) or higher are omitted, c.c. \( \equiv \) complex conjugate,
\[ \gamma_{K_1 K_2 K_3 K_4 K_5} = \gamma_{K_1 + q/4} \cdots K_4 + q/4, K_5 - q/2, K_5 - q/2, \]  
\[ \sum' \] excludes cases in which the sum of any two spin-momenta with opposite spins in the same 4-fermion \( l \)-cluster might be smaller than \( k_c \), and the matrices
\[ C_q = (1 - \beta_q^2 q/4)^{-1} \beta_q \beta_{1q}^2/2, \]  
\[ A_q = (1 - \beta_q^2 q/4)^{-1} \beta_q = \beta_q(1 - \beta_q^2 q/4)^{-1}, \]  
\[ \bar{A}_q = (1 - \beta_q^2 q/4)^{-1} \beta_q q\tilde{a}_q q/4 = A_q - \beta_q. \]  
For any matrix \( M_q \), \( (M_q)_{K_1 K_2} = M_{K_1 K_2}^q \); Equation (D1) defines \( \beta_q \).

Like in Ref. 2, we first determine \( \beta_{1234}^q = \beta_{1234} \), and then \( \beta_{1234} \), and finally determine \( \alpha_1 \).

We first express \( \partial (E - \mu N) / \partial \gamma_{12}^q \) to the leading order, using the lowest order formulas \( \alpha_{K} = \sqrt{N_{T}} \phi_{K} \) and \( h.o.c. \), and

\[ \beta_{12} = x_q \phi_{1} \phi_{2} + h.o.c., \quad x_q = x - q \]
in accordance with the cluster-separation theorem, where \( x_q \) is so-far an unknown number; the second equality follows from the symmetry \( \beta_{12} = \beta_{21}^* \) [Eq. (D1)]. So

\[ \beta_{12}^q = \beta^q_{12}(1 - r_{q})^{-1} \]  

and \( h.o.c. \). \( r_{q} \) is defined by Eq. (D3), and

\[ \sum_8 \left\{ -(x_q \sqrt{N_{T}} / 4) U_{1234} \phi_3 \phi_4 \phi_8 + (1/244) (k^2 - E_m) \gamma_{1234}^q \frac{q}{8} \right\} \]

\[ + (1/16) \sum_5 U_{1256} \gamma_{56348} \frac{q}{8} \]  

where \( \cdots \) means antisymmetrization with respect to the subscripts 1234. Since \( (1 - r_q)^{-1} \) is invertible,

\[ \left\{ -6(2x_q \sqrt{N_{T}} \phi_7) U_{1234} \phi_3 \phi_4 + 2(k^2 - E_m) \gamma_{1234}^q \right\} \]

\[ + 3 \sum_5 U_{1256} \gamma_{56348} \frac{q}{8} \]  

Comparing this equation with Eq. (D5), we find

\[ \gamma_{1234,5} = 2 \sqrt{N_{T}} x_q \phi_5 \phi_{1234}^q + h.o.c. \]  

Next we determine \( \beta_{12}^q (0 < q < k_c) \). We compute \( \partial (E - \mu N) / \partial \beta_{12}^q \) up to the order \( n^3 \), and then determine what value \( \beta_{12}^q \) should take, in order for this partial derivative to vanish. For this purpose \( \alpha_{K} \) (in \( T_{3a}, \cdots, T_{10a} \), \( \gamma_{1234}^q \) (in \( T_{7b}, \cdots, T_{10b} \)), and \( \beta_{1234} \) (in \( T_{7a} + T_{8a} \)) can be retained to the lowest order. After taking the partial derivative, we can retain \( \beta_{12} \) and \( \beta_{12}^* \) in \( \partial (T_{3a} + \cdots + T_{10a}) / \partial \beta_{12}^q \) to the lowest order, since this partial derivative is \( \sim n^3 \). Note, however, that \( \partial (T_{3a} + T_{8a}) / \partial \beta_{12}^q \) is \( \sim n^3 \) and should be treated exactly at this point. Note also that \( T_{8a}, T_{9a}, T_{10a} \) do not contribute to \( \partial (E - \mu N) / \partial \beta_{12}^q \), since in \( T_{8a}, T_{9a}, T_{10a} \), the parameters \( \beta_{1234}^q \) refer to a different region of the spin-momentum configuration space than \( \beta_{12}^q (q < k_c) \); similarly the factor \( \beta_{1234}^q \) in \( T_{8a} \) is independent of the partial derivative.

\[ \partial (T_{3a} / 2 + T_{9a}) / \partial \beta_{12}^q = 0 \]  

up to the order \( n^3 \), since \( \sum(1 - q/2) / \partial (T_{5a} + T_{8a}) / \partial \beta_{12}^q = 0 \), and \( \partial (T_{3a} + T_{9a} + T_{8a}) / \partial \beta_{12}^q = 0 \). Using the above result of \( \gamma \), writing \( A_q = \beta_q (1 - r_q)^{-1} \) and approximating the left factor \( \beta_q \) with its lowest order expression, we easily see that \( \partial (T_{8a} + T_{9a} + T_{10a}) / \partial \beta_{12}^q = 0 \) up to the order \( n^3 \). For the remaining terms, we use the identities

\[ \partial A_{q} / \partial \beta_{12}^q = X_q \leftrightarrow \partial A_{q} / \partial \beta_{12}^q = X_q^q \]  

\[ \partial C_{q} / \partial \beta_{12}^q = (1/2) \beta_q X_q \]  

\[ \partial A_{q} / \partial \beta_{12}^q = (1/4) \beta_q X_q \beta_q. \]
where the dependence of $X_{q'}$ on $K_3 q'$ is suppressed for brevity. We get

$$\partial(E - \mu N)/\partial \beta^{q'}_{09} = \sum_{56,q} S^{q}_{56} X_{56}^{q} + O(n^{1.5} \Omega^{-1}),$$

$$S^{q}_{56} = \frac{1}{4} \sum_{\lambda} \bar{D}^{q}_{51} \beta_{16}^\lambda + N_t \left[ \sum_{5} \phi_{0} S^{(2)}(x_{q}) \phi_{0} + x_{q}^{2} S^{(0)} \phi_{5} \phi_{0} \right],$$

$$\bar{D}^{q}_{12} = (k_{1}^{2} - E_{m} + q^{2}/4 - 2\mu)\Sigma_{12} + (1/2) U_{12},$$

$$S^{(2)} = -\frac{1}{4} \sum_{U} U_{5612}^{1} \phi_{5612}^{2} + \frac{1}{4} U_{5566} \phi_{6} - \frac{1}{4} U_{5566} \phi_{6}^{2},$$

$$S^{(1)} = -\frac{1}{4} \sum_{U} \phi_{5} U_{1515} \phi_{1}^{2} - \frac{1}{2} \sum_{5}^{234} U_{5511} \phi_{1},$$

$$S^{(0)} = -\frac{1}{16} \sum_{U} U_{1122} \phi_{1}^{2} \phi_{2}^{2} + \frac{1}{8} \sum_{U} U_{1212} \phi_{1}^{2} \phi_{2}^{2} - \frac{1}{16} \sum_{U} U_{1234} \phi_{1} \phi_{2} \phi_{3} \phi_{4} = \frac{\pi \epsilon_{m}}{4\Omega},$$

where $S^{(0)}$ is simplified with Eq. \textbf{25} in the end; the sum of any two spin-momenta in the function $\phi'$ is restricted to nonzero. From the definition of $A_{q}$, we can easily show that $X^{q}_{65} = -X^{q}_{65} = -X^{q}_{65}$; in $S^{q}_{56}$ only the component with this same symmetry contributes to the partial derivative, and this component must vanish, since

$$X_{q} = (1 - \beta_{q}^{1}/4)(\partial \beta^{1}_{q}/\partial \beta^{q}_{09})(1 - \beta_{q}^{1}/4)^{-1},$$

where the matrices $(1 - \beta_{q}^{1}/4)^{-1}$ and $(1 - \beta_{q}^{1}/4)^{-1}$ are invertible. The resultant equation is then simplified with the identities in Sec. \textbf{111} in particular Eqs. \textbf{22}, \textbf{110}, and \textbf{24}. The result is

$$\sum_{1} \bar{D}^{q}_{51} \beta^{1}_{16} + \sum_{1} \bar{D}^{q}_{51} \beta^{1}_{51} + 2\pi a_{m} n t \left[ 1 + 4x_{q} + x_{q}^{2} \right] \phi_{5} \phi_{0} - n t \left[ (Dd')_{5} \phi_{5} + \phi_{5} (Dd')_{6} \right] - n t \sum_{1} \beta_{q}^{1} \beta_{q}^{1} = 0,$$

with an absolute error of the low-density order $n^{1.5}$. Here $(Dd')_{5} \equiv 1 \beta_{q}^{1} d_{q}^{1}$. We then do an orthogonal decomposition:

$$\beta_{12}^{q} = x_{q} \phi_{1} \phi_{2} + \beta_{1}^{1} \phi_{2} + \beta_{2}^{1} \phi_{5} + \beta_{2}^{2} \phi_{5},$$

where $\phi_{1} \equiv -\bar{q}$. The third term on the right side of Eq. \textbf{57a} is related to the second by the symmetry $\beta_{12}^{q} = \beta_{12}^{q}$. Substituting Eq. \textbf{57a} into Eq. \textbf{56a}, we get four equations, for the four mutually orthogonal subspaces. The linear kernel in Eq. \textbf{55d} (the sum of the two $D^{q}$’s) has a single eigenvalue of the order $n$ (ie $q^{2}/2 - 4\mu$), associated with the eigenvector $\phi_{1} \phi_{2}$, and all the other eigenvalues are at least about $|E_{m}|$. In the subspaces orthogonal to $\phi_{1} \phi_{2}$, the difference between $D^{q}$ and $D$ is negligible. In the subspace containing $\phi_{1} \phi_{2}$, we may approximate $\mu$ with its lowest order value, $\pi a_{m} n_{t}$ \cite{41}, since at the current order, $x_{q}$ can only be determined to the leading order. We thus get

$$\beta_{12}^{(1)q} = n_{t} (1 + 2x_{q} d_{q} + h.o.c.), \quad (58b)$$

$$\beta_{12}^{(2)q} = n_{t} g_{1} k_{1} + h.o.c.. \quad (58c)$$

The solution to the first equation is

$$x_{q} = -\left( 1 + \xi^{2} q^{2} - \xi q \sqrt{2 + \xi^{2} q^{2}} \right), \quad \xi \equiv (8\pi a_{m} n_{t})^{-1/2}. \quad (59)$$

The solution with norm greater than 1 is unphysical and has been discarded, like in the case of structureless bosons (cf \textbf{21}). The other components of $\beta_{12}^{q}$ remind us that \textit{additional features} are present, in contrast with a gas of structureless bosons.

Now we turn to $\beta_{1234}$ [excluding $\beta_{12}^{q} (q < k_{c})$]. Its leading order expression is determined in Sec. \textbf{114} and we now proceed to the next order ($\sim n^{1.5}$). Only one additional term contributes to $\partial(E - \mu N)/\partial \beta_{1234} = 0$, namely $T_{3a}$.

$$\left\{ 2(k_{1}^{2} - E_{m}) \beta_{1234} + 3 \sum_{56} U_{5656} \beta_{5634} - 6U_{1234} \alpha_{2} \beta^{1} \right\}_{1234} = 0, \quad (60)$$

where $\sum'$ excludes the region occupied by $\beta_{q} (q < k_{c})$, so the last term is in some sense complementary to the second. Since our accuracy goal here is $n^{1.5}$, $A_{34}^{q}$ can be retained to the lowest order, $\phi_{1} \phi_{2} \phi_{3} \phi_{4} (1 - x_{q}^{2})$, and $\alpha_{K}$ should be retained to the next-to-leading order, $\alpha_{K} = n_{q} \phi_{K}$. The component orthogonal to $\phi_{K}$ is $\sim n^{1.5}$ and is negligible here, since it is multiplied by the other $\alpha_{K}$. The lowest order formula $\beta_{1234} = N t \phi_{1234}$ combined with Eq. \textbf{124} (the sum of any two spin-momenta is nonzero here) leads to

$$\beta_{1234} \approx -4a_{m} n_{t} / q^{2} \phi_{3} \phi_{4} \quad (\text{for } 4a_{m} n_{t} / q^{2} \phi_{3} \phi_{4} \quad \text{for } \sqrt{a_{m} n_{t}} \ll q \ll \min(1/1, 1/r_{m}).$$

If the term containing $A_{34}^{q}$ followed this asymptotic formula for all $q < k_{c}$, it would not alter the solution $\beta_{1234} = n_{q}^{2} \phi_{1234}$ at the order $n^{1.5}$. It is the nonzero \textit{difference}, namely

$$A_{34}^{q} = \phi_{3} \phi_{4} (-4a_{m} n_{t} / q^{2}) \phi_{3} \phi_{4} \left( \frac{x_{q}}{1 - x_{q}^{2}} + \frac{1}{2q^{2}} \right),$$

that constitutes an additional driving term (in addition to the term containing $\alpha_{q}$’s). Since this term has the same
structure as the $\alpha$ term, it can be absorbed into the $\alpha$ term, thus yielding the solution

$$\beta_{1234} = \left[ \eta_\alpha^2 + \sum_{0 < q < k_c} \left( \frac{x_q}{1 - x_q^2} + \frac{1}{2\xi^* q^2} \right) \right] \phi_{1234}' .$$

accurate up to the low-density order $n_1^2$. Substituting Eq. (65), and noting that $k_c \gg \sqrt{a_m n_1^2}$, we get

$$\beta_{1234} = \left[ \eta_\alpha^2 + 8\Omega(a_m n_1)^{3/2}/\sqrt{\pi} \right] \phi_{1234}' . \quad (61)$$

Now turn to $\partial(E - \mu N)/\partial \phi_{K}$ = 0. This equation is determined to the order $n_1^2$ in Eq. (44). Here we proceed to the order $n^2$ by including contributions from Eq. (51).

$$\sum_{0 < q < k_c} D_{12} \alpha_2 = f_1^{(1.5)} + f_1^{(2)} + O(n^2),$$

$$f_1^{(2)} = \sum_{0 < q < k_c} \left\{ 4(k_1^2 - E_m) \alpha_1 C_{11}^q + 2(k_1^2 - E_m) \alpha_1^* A_{11}^q + U_{1222} [\alpha_2 (C_{22}^q + C_{11}^q) + \alpha_1 C_{21}^q + \alpha_1^* A_{21}^q + \alpha_2^* A_{22}^q + 2/2] \right. + \left. \sum_{2} U_{2211} [\alpha_1 C_{12}^q + \alpha_1^* A_{12}^q + 1/2] + \sum_{2} U_{1234} (A_{12}^q + A_{21}^q) \right\}_{11},$$

where $\{ \cdots \}_{11}$ retains only the component antisymmetric under $1 \leftrightarrow 1$, and $f_1^{(1.5)}$ are the terms on the right side of Eq. (14). Our accuracy goal for this equation is $n^2$. Each term in $f_1^{(1.5)}$ is $\sim n_1^4$ and is computed to the next-to-leading order, at which $\alpha_K = \eta_\alpha \phi_{K}$ (the components orthogonal to $\phi_{K}$ are negligible here) and $\beta_{1234}$ is given by Eq. (61). Each term in $f_1^{(2)}$ is $\sim n^2$ and only computed to the leading order, at which $\alpha_K = \sqrt{\gamma_1} \phi_{K}$. $C_{12}^q = \phi_1 \phi_2 x_q / (1 - x_q^2)$, $A_{12}^q = \phi_2 \phi_2 x_q / (1 - x_q^2)$, $A_{12}^q = \phi_2 \phi_2 x_q / (1 - x_q^2)$, $A_{12}^q = \sqrt{\gamma_1} \phi_2 \phi_2 (x_q^{(1234)})^2$, $C_{12}^q = \sqrt{\gamma_1} \phi_2 \phi_2 (x_q^{(1234)})^2$. Here $x_q$ is given by Eq. (69).

$$f_1^{(1.5)} = \left\{ \eta_\alpha^2 \left[ -(k_1^2 - E_m) \phi_1^2 + \sum_{2} U_{1122} \phi_1^2 \phi_2 / 2 \right. \right. + \left. \left. 2 \phi_1 \sum_{2} U_{1212} \phi_2^2 \right\}_{11} \right\} \quad (62)$$

accurate up to the low-density order $n^2$, where

$$t \equiv (8/3\sqrt{\pi}) (\alpha_1^3 n_1^3)^{1/2} \sim n^2 \Omega_{1/2},$$

and $\sum'$ excludes three regions in the spin-momentum configuration space: $|K_2 + K_1| < k_c$, $|K_3 + K_1| < k_c$, or $|K_4 + K_1| < k_c$ (the intersection of these regions is smaller by a factor $\propto k_c^3$ and negligible), i.e.,

$$\sum_{2} U_{1234} \phi_1 \phi_1' = \sum_{2} U_{1234} \phi_1 \phi_1'$$

$$- \sum_{2} U_{-K_1,K_1-q,q} \phi_1' \phi_1$$

$$\times \phi_1' K_1-q' \phi_1' K_1+q \phi_1-q/2 - K_1-q$$

$$(the other two small regions).$$

Since $\phi_{1234}' \sim 1/q^2$ at small $q$ (Sec. III), the contributions of these regions to $f_1^{(1.5)}$ scales like $n_1^3 k_c$ and is $\sim n^2$, if we let $k_c$ be $\sqrt{\gamma a_m}$ times a fixed-dimensionless coefficient ($\gg 1$). So we only need to compute them at the leading order; for example, the second term on the right side of the above equation is simplified as $+|\phi_1|^2 \sum_{5} U_{1155} \phi_5 \sum_{0 < q < k_c} 4\alpha_a n_1 / \Omega q^2$.

We can thus change $\sum_{5}$ to $\sum_{5}$ in the last term of Eq. (62), and in the same time add the following correction term ($\sim n^2$) to the right side of Eq. (62):

$$\left\{ |\phi_1|^2 \sum_{2} U_{1122} \phi_2^2 - 2 \phi_1 \sum_{2} U_{1212} \phi_2 \right\}_{11} \sum_{q < k_c} \frac{4\alpha_a n_1^2}{\Omega q^2} \quad (63)$$

which has been retained to the leading order. On the other hand, the terms containing $A_q$ in $f_1^{(2)}$ are also dependent of $k_c$, since $\sqrt{\gamma} \sum_{q < k_c} A_q^2 = |\phi_1 \phi_2 (3t - \sqrt{\gamma} \sum_{q < k_c} 4\alpha_a n_1 / q^2)$). Direct calculation immediately reveals that the sum of the terms $\sim \sum_{q < k_c} 1/q^2$ in $f_1^{(2)}$ is opposite to Eq. (68). $f_1^{(1.5)} + f_1^{(2)}$ is thus independent of $k_c$ at the order $n^2$ (the $\sum$ in $f_1^{(2)}$ can be simplified as $\sum$ since their difference is of the order $n^2$). This independence is not a coincidence, since different terms in this perturbative expansion refer to complementary parts (in spin-momentum configuration space) of an underlying seamless whole. The division between $q < k_c$ and $q > k_c$ is just for the convenience of concrete calculations, and the physical outcome is of course independent of $k_c$.

After straightforward calculation, we get

$$f_1^{(1.5)} + f_1^{(2)} = (\eta_\alpha^3 + 5t) s - \frac{16t^2}{5} \sum_{2} D_{12} |\phi_2|^2 \phi_2 + O(n^2),$$

where $s$ is exactly equal to the left side of Eq. (22). Thus

$$- 2\eta_\alpha \phi_1 + \sum_{2} D_{12} \alpha_2^\dagger = - \phi_1 (\eta_\alpha^3 + 5t) 2\alpha_1 n_1 / \Omega$$

$$+ \sum_{2} D_{12} (|\eta_\alpha^3 + 5t| d_2 / \Omega - (16t/5)) |\phi_2|^2 \phi_2 + O(n^2).$$

The difference between $\sum_{2} D_{12} \alpha_2^\dagger$ and $\sum_{2} D_{12} \alpha_2^\dagger$ is $\sim$
Solving the above equation for the derivatives of \( n \), we get
\[
\bar{\mu} = \eta_a^2 \left[ 1 + (40/3\sqrt{\pi}) (n_1 a_m^3)^{1/2} \right] \pi a_m / \Omega + O(n^2),
\]
and
\[
\alpha_i^\dagger = \eta_a^3 \left[ 1 + (40/3\sqrt{\pi}) (n_1 a_m^3)^{1/2} \right] d_i^1 / \Omega
- (128/15\sqrt{\pi}) (n_1 a_m^3)^{1/2} N_i^3/2 \sum P_{12}^1 \phi_2^1 \phi_2 + O(n^{2.5}).
\]
where \( O(n^2) \) is in the coarse-grained sense; it does not exclude the possibility of a correction term which may behave like \( n^2 \ln(na_m^3) + \text{const.} \).

Like in the case of structureless bosons, \( \eta_a \), we express the particle number \( N \) in terms of the wave function, in order to solve it backwards for \( \eta_a \). Up to the order \( n^{1.5} \), there are only two diagrams for \( N = \sum (c_i^\dagger c_i) \); they are the derivatives of \( T_1 \) and \( T_{1n} \) with respect to \( -\mu \).

\[
N = \sum_{i=1}^{\infty} \left| \alpha_i \right|^2 + 2 \sum_{i=1}^{\infty} C_{i1}^a = 2\eta_a^2 + 2 \sum_{q} \frac{x_q^2}{1 - x_q^2} + O(n^2\Omega)
= 2 \left[ \eta_a^2 + \left( 8/3\sqrt{\pi} \right) (n_1 a_m^3)^{1/2} \right] N_1^1 + O(n^2\Omega).
\]

Solving the above equation for \( \eta_a \), substituting the result back to the formulas we obtained so far, and solving \( dE/dN = \mu \), we get our results, summarized below.

\[
\mu = E_m/2 + \pi a_m n_1 \left[ 1 + (32/3\sqrt{\pi}) (n_1 a_m^3)^{1/2} + \text{h.o.c.} \right],
\]
\[
E/\Omega = E_m n_1 + \pi a_m n_1^2 \times \left[ 1 + (128/15\sqrt{\pi}) (n_1 a_m^3)^{1/2} + \text{h.o.c.} \right],
\]
\[
\alpha_1 = \eta a \phi_1 + \alpha_i^\dagger,
\sum_i \phi_i^1 \alpha_i^\dagger = 0,
\]
\[
\eta_a = N_1^{1/2} \left[ 1 - (4/3\sqrt{\pi}) (n_1 a_m^3)^{1/2} + \text{h.o.c.} \right],
\]
\[
\alpha_i^\dagger = N_1^{3/2} \left[ 1 + (28/3\sqrt{\pi}) (n_1 a_m^3)^{1/2} \right] d_i^1 / \Omega
- (128/15\sqrt{\pi}) (n_1 a_m^3)^{1/2} \sum P_{12}^1 \phi_2^1 \phi_2 + O(n^{2.5}),
\]
\[
\beta_{1234} = N_1 \left[ 1 + (16/3\sqrt{\pi}) (n_1 a_m^3)^{1/2} \right] f_{1234}^4 + O(n^2\Omega^{-1}),
\]
\[
\beta_{12}^q \equiv \eta q \phi_1 \phi_2 + \beta_{12}^q,
\sum_{12} \phi_1^q \phi_2^q \beta_{12}^q = 0,
\]
\[
x_q = -\left( 1 + \xi q^2 - \xi q \sqrt{2 + \xi (q^2)} \right) + \text{h.o.c.},
\]
\[
\beta_{12}^q = n_q \left[ 1 + 2x_q \right] (d_1^2 \phi_2 + \phi_1 d_2) + \phi_1 h_{12} + \text{h.o.c.},
\]
\[
\eta_{12345} = 2\sqrt{N} x_q \phi_{12345} \phi_5 + \text{h.o.c.},
\]
where in Eqs. (65i) and (65j), the first four spin-momenta 1234 must be an l-cluster, thus excluding the cases in which the sum of any two of them is \( \sim \sqrt{na_m^3} \) or zero. In Eqs. (65h),(65i), and (65j), the momentum \( q \ll \min(1/l,1/r_m) \).

As a consistency check, we calculate the sum of all the diagrams in Fig. 5. The calculation is tedious but straightforward. The result is consistent with the value of \( E - \mu N \) from Eqs. (65a) and (65c).

V. PHYSICAL OBSERVABLES OTHER THAN THE EQUATION OF STATE

After a multistep process of logical deductions and analytical calculations, we have now determined the parameters of the wave function [Eq. (11)] to the leading order beyond mean-field, in the low-density regime. We are now in a position to determine various physical observables, by evaluating their expectation values under the wave function.

For every observable computed below, we first use Eq. (29) and the other information in Sec. IV to identify all the diagrams up to a certain order in the low-density expansion; we also note that there are no dead ends.

A. Superfluid pairing function

In this subsection we determine the superfluid pairing function, and discuss the range of densities in which the result is valid.

We decompose it into a component parallel to the internal wave function of an isolated molecule, and one orthogonal to it:

\[
F_K = \langle c_{-K} c_K \rangle \equiv F^\parallel_K + F^\perp_K \equiv \eta_F \phi_K + F^\perp_K,
\]
where \( \sum_K \phi_K^a F^\perp_K = 0 \).

All the diagrams up to the order \( n^2 \) are shown in Fig. 7. Retaining terms to the order \( n^2 \), we get

\[
F_1 = \alpha_1 \left( 1 - |\alpha_1|^2 \right) - \sum_{0 < q < K} \left( \alpha_1^\dagger A_{11} + 2\alpha_1 C_{11}^q \right) + O(n^{2.5}).
\]

Using the established results of the parameters of the many-body wave function, we get

\[
F^\parallel_K = \left[ 1 - \frac{4}{3\sqrt{\pi}} (n_1 a_m^3)^{1/2} + \text{h.o.c.} \right] N_1^{1/2} \phi_K,
\]

\[\text{FIG. 7: Diagrams for the superfluid pairing function } F_K \equiv \langle c_{-K} c_K \rangle \text{ up to the order } n^2.\]
\[
F_1^+ = \left[ 1 + \frac{28}{3\sqrt{\pi}} (n_\uparrow a_\uparrow^3)^{1/2} \right] N_\uparrow^{3/2} \left( d_1'/\Omega - \sum_2 P_{12}^\uparrow |\phi_2|^2 \phi_2 \right) + O(n^{2.5}),
\]
and the latter equation is simplified by Eq. (18a):
\[
F_K^+ = \left[ 1 + \frac{28}{3\sqrt{\pi}} (n_\uparrow a_\uparrow^3)^{1/2} \right] N_\uparrow^{3/2} \left( d_K' \phi_K + O(n^{2.5}) \right). \tag{67b}
\]
The accuracy in the parallel component of \( F_K \) is limited by that of the parallel component of \( \alpha_K \). To increase the accuracy, one must determine \( E - \mu N \) to higher orders.

The amplitude of the component in \( F_K \) parallel to \( \phi_K \) is mapped directly to the well-known condensate-depletion formula
\[
\langle b_0 \rangle = \left[ 1 - \frac{4}{3\sqrt{\pi}} (n_\uparrow a_\uparrow^3)^{1/2} + \text{h.o.c.} \right] \sqrt{N_b}
\]
in the limit of point-like bosons. Here \( N_b \) is the total number of bosons, and \( n_\uparrow \) and \( a_\uparrow \) are the boson density and scattering length, respectively.

The orthogonal deviation of \( F_K \) from \( \eta \phi_K \) is proportional to \( \phi_K \) - a function derived from the two-molecule zero-speed scattering wave function - at the leading and next-to-leading orders. The coefficients are also predicted exactly, in Eq. (18b). The next step is to determine \( d_K \) from Eq. (18a). If the interaction between fermions is zero-range with a large and positive scattering length, we should substitute the four-fermion wave function calculated by Petrov et al. into Eq. (18a), to determine \( d_K \). This step is elementary (involving no many-body physics at all) but tedious; it will be carried out later, by the author himself or by some other people.

The significance of this deviation from the two-body function \( \phi_K \) has been pointed out in the closing paragraphs of Sec. III C. Quantitative comparison between our prediction and experiment is possible in the future.

For ultracold fermionic atoms in two internal states near a wide Feshbach resonance (on the \( a > 0 \) side), since our prediction is based upon the two-molecule scattering physics as studied by Petrov et al. our predictions are relevant to experiment in so far as Petrov et al’s results are relevant to experiment. There is already much experimental evidence supporting the value of \( a_\uparrow \) as predicted by Petrov et al., most notably the cloud size of a trapped molecular Bose-Einstein condensate. On the other hand, the two-molecule physics of Petrov et al’s is never exact except when the molecular density approaches zero; our low-density theory is in a similar situation. We are thus led to believe that the low-density results in this paper are applicable to experimentally accessible densities, just like Petrov et al’s. So the low-density results in this paper are applicable when \( 0 < k_F a \lesssim 1 \). The more detailed upper bound depends on the particular experiment one wants to address, and on one’s accuracy goal. Here \( k_F \equiv (3\pi^2 n)^{1/3} \), and \( a \) is the scattering length between fermions.

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![Diagram](image_url)

**FIG. 8:** Diagrams for the superfluid four-fermion function up to the low-density order \( n^1 \), when the sum of two external spin-momenta is \( \sim \sqrt{n} \). The external points are distinguishable. In the second diagram, the shaded circle stands for various internal structures (either \( N_2^t = 1, 2 \), or there is one bubble island that is not simplest).

Having stated these, we must stress that our results are also valid beyond the range of validity of Petrov et al’s in the sense that ours are valid for other interactions between fermions as well. For other interactions, the two-molecule scattering wave function will be different and \( a_\uparrow \neq 0.6a_\uparrow \). But the results in this paper still apply, provided that the conditions specified in Sec. III A are satisfied.

### B. Superfluid four-fermion function

Let \( K_1 + K_2 + K_3 + K_4 = 0 \), and
\[
F^{(4)}_{K_1 K_2 K_3 K_4} \equiv \langle c_{K_1} c_{K_2} c_{K_3} c_{K_4} \rangle.
\]
Case 1: the sum of two of the four spin-momenta is zero, and the thermodynamically significant contribution to \( F^{(4)} \) equals a product of two \( F^{(2)} \)'s, according to Eq. (9). Here \( F^{(2)} = F_{K} \). For instance, \( F^{(4)}_{K_1 K_2 K_3 K_4} \equiv F_K^+ F_K^+ \) \( K \neq \pm K' \). (If \( K = \pm K' \), \( F^{(4)}_{K K K K} = 0 \).)

Case 2: for all \( i = 2, 3, 4 \), \( K_i \) and \( K_i \) either have parallel spin, or have a nonzero total momenta \( \sim O(n^{1/3}) \). There is only one diagram for \( F^{(4)} \) up to the order \( n^{1/3} \), namely \( \beta_{K_1 K_2 K_3 K_4} \).\n
\[
F^{(4)}_{K_1 K_2 K_3 K_4} = N_\uparrow^\frac{1}{2} \left[ 1 + \frac{16}{3\sqrt{\pi}} (n_\uparrow a_\uparrow^3)^{1/2} \right] \phi_K^+ + O(n^{2/3} \Omega^{-1}) \tag{68}
\]

Case 3: the sum of two of the four spin-momenta is a small momenta \( \sim n^{1/2} \). Consider for instance
\[
F^{(4)q}_{K_1 K_2} = \frac{F^{(4)}_{K_1 + q/2, -K_1 + q/2, K_2 - q/2, -K_2 + q/2}}{q/\sqrt{a_\uparrow}},
\]
where \( q/\sqrt{a_\uparrow} \) is a nonzero constant of order unity (in the low-density limit). If \( K_1 \neq \pm K_2 \), all the diagrams up to the low-density order \( n \) are shown in Fig. 8. Using the part of the cluster-separation theorem that is already proved, together with the established result concerning \( g_{4234} \), we can easily show that the second diagram (containing the shaded circle), whose low-density order is 0.5, is proportional to \( \phi_1 \phi_2 \) up to the order \( n^1 \). So
\[
F^{(4)q}_{12} = (1 - 2|a_1|^2 - 2|a_2|^2) \frac{Aq}{\alpha_2 C_{21}^q} - \frac{Aq}{\alpha_2 C_{12}^q} + \eta|\phi_1|^2 + O(n^{1.5} \Omega^{-1}),
\]
where $\eta^q \sim n^{1/2}$. We first substitute the result of $\beta_{12}^q$ [Eq. (63)] to Eq. (62) to determine $A_{12}^q$ accurately, and then evaluate the other terms to the leading order. At $K_1 = \pm K_2$, there is appropriate delta-function singularity in $F_{k_1 k_2}^{(4)q}$, in accordance with case 1. Combining these subregions of the spin-momentum configuration space, we derive the final result

$$F_{12}^{(4)q} = F_{12}^{(4)q\|} + F_{12}^{(4)q\perp}, \quad \sum_{12} \phi_1 \phi_2 F_{12}^{(4)q\perp} \equiv 0,$$

$$F_{12}^{(4)q\|} = \left( \frac{2 x_q}{1 - x_q^2} + h.a.c. \right) \phi_1 \phi_2, \quad (69a)$$

$$F_{12}^{(4)q\perp} = n_\uparrow \left[ \frac{1 + 2 x_q}{1 - x_q^2} (d_1 \phi_2 + \phi_1 d_2) + g_{12} \right] + O(n^{1.5} \Omega^{-1}). \quad (69b)$$

The physical meaning of these results will be clearer in coordinate space.

C. Superfluid six-fermion function, momentum distribution, two-body reduced density matrix, etc

The six-fermion correlation parameters $\gamma_{12345}^q$ are an indispensable part of the consistent beyond-mean-field theory. This is not surprising, given the fact that three-boson correlation is needed to formulate the correct beyond-mean-field theory of structureless bosons [2]. Leggett (and many others) notices that Bogoliubov wave function does not yield the correct beyond-mean-field equation of state of structureless bosons, since some "bare interactions" are not renormalized in favor of the bosonic scattering length, a problem he considers "a little disturbing" [13], which he then remedies by modifying the Bogoliubov wave function.

Using the pseudopotential as the effective bosonic interaction, one can escape this problem at a price: the correct short-distance behavior of the bosons upon collisions is omitted. For the Fermi superfluid in the molecular Bose-Einstein condensed state, the price of this replacement is even higher: since all the four-fermion scattering physics is replaced by a molecular scattering length, one can no longer describe the many physical observables that involve momentum scales much larger than $\sqrt{\pi a_m}$, such as the many-body effects on the fermionic momentum distribution and the superfluid pairing function.

In the approach of [2] and this paper, such problem is eliminated by strictly applying the power-counting formula to identify all the significant diagrams. When this is done, one finds that six-fermion (or three-boson) correlation terms can not be altogether thrown away as people wish to do. In the same way as the last two subsections, one can derive the associated nonvanishing superfluid six-fermion function:

$$\langle \epsilon_{K_{1}+q/4} \cdots \epsilon_{K_{1}+q/4} \epsilon_{K_{5}-q/2} \epsilon_{K_{5}-q/2} \rangle. \quad (69c)$$

The momentum distribution of fermions, $\langle c_{k}^\dagger c_{K} \rangle$, is another interesting quantity, for two reasons. Firstly, it is directly measurable. Secondly, it is predicted that there are two exact relations between the energy and the momentum distribution of the two-component Fermi gas with large scattering length [3,4], valid for all nonzero values of $k_F a$ (including the unitary regime), and also valid for finite temperatures, few-body systems, etc. Performing the analytic calculation of the momentum distribution in the low-density regime, we can directly test these earlier predictions [3,4]. The results obtained in this paper already enables an unambiguous determination of the fermionic momentum distribution up to the order $n^{2.5}$. It will be presented in a subsequent work.

The two-body reduced density matrix is also observable. It remains to see if there is interesting physics in this quantity. It will be determined analytically in the low-density regime in a subsequent work.

Another quantity to study is the fermionic Green functions ("normal" and "anomalous"). They help us to understand the dynamic behavior of the Fermi superfluid.

The above list is still not exhaustive.

VI. CONCLUDING REMARKS

At low densities, there is a natural coexistence of the characteristics of a conventional Bose gas and that of a Fermi gas in the present theory. If the system is probed at length scales of the order $\xi \sim 1/\sqrt{\pi a_m}$ or longer length scales, one can hardly distinguish it from a usual Bose gas. The equation of state, the condensate depletion, the low-momentum part of the momentum distribution of molecules ($q \sim 1/\xi$), etc, all follow conventional Bogoliubov-Lee-Yang theory. If the observables that involve large momentum scale $q \sim 1/r_m$ are measured, however, one sees many features that are absent in a conventional Bose gas. The momentum distribution of fermions, the deviation of the superfluid pairing function from being proportional to the internal wave function of an isolated molecule, etc, are examples of such observables.

The Bose characteristics is not put in by hand, but forced on us by the fermionic wave function Eq. (1). The ground state energy as derived in this paper is the absolute lower bound, as we have gone through great lengths, making all the possible adjustments of the wave function to minimize it. The wave function itself does not seem to be improvable any more, since all the possible correlations are included. It is the low density that makes higher correlations less important, but their effects are not automatically ignored - instead, using the power-counting formula Eq. (20), we have shown that most of them, except a few, are insignificant in the calculation of the leading order correction to the mean-field equation of state.

Will such Bose characteristics be carried over to the high density regime? It seems that various neutral isotropic superfluids (no matter BEC of structureless bosons, BEC of composite bosons, unitary Fermi superfluid, or BCS superfluids) have very similar struc-
tures at length scales much longer than \( \xi \) (not necessarily \( \propto n^{-1/2} \)). At shorter length scales, their differences show up. However the similarity between the dilute BEC of structureless bosons and dilute BEC of composite bosons is kept at shorter length scales, until \( \max(r_m, l) \), which is much shorter than \( \xi \). For a unitary Fermi superfluid at zero temperature, there is only one length scale \( \xi \sim 1/k_F \sim n^{-1/3} \), so its similarity to other superfluids must be broken at a length scale somewhat longer than \( \xi \).

To formulate the quantitatively well-controlled many-body theory for the Fermi superfluid in the unitary regime, we should probably take advantage of the above partial similarity, and do a momentum scale division at \( k_c \), where \( k_c \) is roughly the highest momentum scale below which the structure of the system is nearly indistinguishable from other superfluids. The structure of the system at \( k > k_c \) needs to be determined nonperturbatively, and smoothly matched to the low-momentum structure.

Besides this project, we have of course the shorter run subject of extending the current low-density theory to higher orders beyond Lee-Yang. We need to first show that Wu’s logarithmic term \( \mathcal{R} \) is present in the equation of state, and then determine the leading differences of the equations of state of molecular condensates and those of other BECs. More concretely, we need to determine the value of the constant \( C \) as defined in Ref. [1]. We derive confidence in these plans from two sources: the diagrammatic theory in this paper (the author has found diagrams in the expansion of \( E - \mu N \) which scale like \( n^3 \ln n \), and Braaten et al’s Effective Field Theory calculations [1], which support the universality of the Wu term, at least for structureless bosons.

In the same time, the fermionic momentum distribution is of high interest. Here we have some exact theorems [3, 4] for the two-component Fermi gas with large scattering length, valid in the low-density as well as unitary regimes. They relate the momentum distribution to the energy, the pressure, and the rate of change of energy during a ramp of the fermionic scattering length [3, 4]. The confirmation of these theorems in a concrete low-density analytic calculation will push our understanding of such a novel Fermi system to a greater depth. This has been done in Ref. [2], for the dilute Bose gas with large scattering length, although the theorems are only approximate in that Bose system [2].

**Acknowledgments**

The author is very grateful to K. Levin for many helpful suggestions, and for communicating the ideas to other researchers. The author thanks T. L. Ho for calling the author’s attention on testing the theorems in Refs. [3, 4] with independent perturbative calculations.

**APPENDIX A: AN IDENTITY SATISFIED BY \( \phi^{(4)} \)**

Proof of Eq. (20). Replacing \( K_1 \cdots K_4 \) in Eq. (123) by \( K_1 + q/2, -K_1 + q/2, K_2 - q/2, -K_2 - q/2 \), respectively, using the Galilean symmetry of \( U \), multiplying the equation by \( \phi^{(4)}_{K_1} \), summing over \( K_2 \), noting the antisymmetry of \( \phi \) and \( \phi^{(4)} \), and noting that dumb momenta can be freely shifted, we get

\[
\sum_{q} \phi^{(4)}_{q/2} + \frac{1}{2} \sum_{q} D_{13} \phi^{(4)}_{2q} + \sum_{q} D_{23} \phi^{(4)}_{2q} + T_{q}^{K_1 + q/2} - T_{q}^{K_1} = 0,
\]

and the infinitesimal \( \epsilon^{(4)} \) is omitted in comparison with \( q^2/2 > 0 \).

Using Eq. (123), we see that the first term of Eq. (A1) approaches \( -2\pi a_m/\Omega \), when \( q \neq 0 \) but \( q \to 0 \).

Obviously \( \sum_{q} D_{14} P_{43}^{q} = D_{14} \), so the second term of Eq. (A1) is rewritten as \( \sum_{q} D_{14} P_{43}^{q} \phi^{(4)}_{q} \phi^{*}_{q} \), and the third term vanishes.

Since \( U \) and \( \phi^{*} \) are smooth functions of momenta (when momentum is conserved by the subscripts), \( T_{K} \equiv \lim_{q \to 0} T_{q}^{K} \) depends continuously on the momentum \( k \), for a given spin \( \sigma \). So in the limit \( q \to 0 \), the last two terms of Eq. (A1) approaches \( T_{K} \), which is zero.

Combining the above results, we see that when \( q \to 0 \), Eq. (20) is obtained.

**APPENDIX B: DEFINITION OF SOME TECHNICAL TERMS**

The terms used in this paper follow convention as much as possible. However some terms in this paper have narrower meanings than convention, some are used in a broader sense, and some are new - due to the needs of the theory. Here we give the precise meanings of some of them (for use in this paper), in alphabetical order.

**Dead end**: an internal part of a diagram that is connected to the remaining part via a single (de)generator. In the present formalism, dead ends are absent because of a basic property of the coefficient \( \phi^{(p)}_{K_1} \): it is zero whenever the sum of a nontrivial subset of the \( p \) spin-momenta is zero.

**Diagram**: an arbitrary collection of vertices and lines; each line has exactly two ends, and they must be attached to two vertices or the same vertex. The number of vertices is denoted by \( N_0 \), the number of lines by \( N_1 \), the number of independent loops by \( N_2 \), and the number of disconnected parts - each part is connected, but any two
different parts are disconnected - by $N_3$. There is a simple universal identity

$$N_0 - N_1 + N_2 - N_3 = 0,$$  \hspace{1cm} \text{(B1)}

This identity is used in the derivation of two basic power-counting formulas, Eqs. (3) and (29).

**External point:** a special vertex representing a fermionic operator (annihilation or creation) in a product whose expectation value we are computing. Each external point is attached by a single solid line associated with a certain spin-momentum. It is a small solid circle if it stands for a creation operator, or a hollow one for an annihilation operator.

**h.o.c.** higher order correction (in the low-density expansion), with strictly greater value of low-density order $R$, in comparison with the term immediately preceding it. 

**Low-density order $R$ of a quantity $x$: if the leading order approximation of $x$ scales like $n^R \Omega^Q$ in the low-density regime, the low-density order of $x$ is said to be $R$. Symbolically we may write $x \sim n^R$. If actually $x \sim n^R \ln(na_n^3)$ or things like that, we may still write $x \sim n^R$ in the slightly broader sense. Here $n$ and $\Omega$ are the particle density and the system’s volume, respectively, and $\Omega$ is assumed to be large enough to avoid finite-size effects. A higher low-density order means a greater value of $R$, and usually indicates a smaller magnitude in the low-density regime.

**Nontrival subset of $M$ items:** a nonempty proper subset, namely with $M_1$ elements such that $1 \leq M_1 \leq M - 1$.

$O(n^R)$: abbreviation of $O(n^R \Omega^Q)$.

$O(n^R \Omega^Q)$: a symbol describing how a quantity (in the low-density regime) scales with $n$ and $\Omega$. The (often dimensionful) coefficient roughly independent of $n$ and $\Omega$ is not shown in this symbol. Things like $n^{-\alpha} \ln(na_n^3) \Omega^Q$ are sometimes possible, and not excluded by this symbol.

**Thermodynamic order $Q$ of a quantity $x$: if $x$ scales like $\Omega^Q$ in the thermodynamic limit (in which $n$ is held constant as usual), the thermodynamic order of $x$ is said to be $Q$. Symbolically we may write $x \sim \Omega^Q$. This notion is obviously orthogonal to the low-density order; it is also valid beyond the low-density regime (in contrast with the low-density order).

No contradiction between $x \sim \Omega^Q$ and $x \sim n^R$ exists - there is no contradiction between “the height of this rectangle is $Q$” and “the width of the rectangle is $R$”. In fact we may use a plane, and map $x$ to a point with coordinates $(R, Q)$ on the plane.

**APPENDIX C: PARTIAL PROOF OF THE FIRST CLUSTER-SEPARATION THEOREM**

First consider $F_K \equiv \langle c_{-K}c_K \rangle$. The theorem states that there exists a coefficient $\eta = \eta_{0.5}^{0.5} + \eta_{1.0}^{1.1}$, such that $F_K = \eta_0^2 |K|$, with an absolute error of the low-density order $n^{1.5}$. The proof follows.

![Diagram](image-url)

**FIG. 9:** Diagrams for $\frac{1}{2} \sum_{K_1, K_2, K_4} U_{K_1, K_2, K_4} \langle c_{K_1}c_{K_2}c_{K_4}c_{-K_1} \rangle$.

(a): the total diagram. The shaded circle stands for the ground state expectation value $\langle c_{K_1}c_{K_2}c_{K_4}c_{-K_1} \rangle$. (b) and (c): two individual diagrams, whose low-density orders are 1.5 and 2, respectively. In (c), if the whole island is reduced to a point, the dotted line connects this same point and forms a loop.

Since the ground state expectation value of $\tilde{H} = H - \mu \tilde{N}$ is stationary under any infinitesimal change of the many-body state (the number of particles may also be changed slightly), we can easily show an equation of motion

$$\langle O \tilde{H} - \tilde{H} O \rangle = 0 \hspace{1cm} \text{(C2)}$$

for any product of fermionic operators, $O$. Substituting Eq. (5) and $O = c_{-K_1}c_{K_1}$ into this formula, we get

$$\sum_{2} D_{12}F_2 = \frac{1}{2} \sum_{234} U_{1234} \langle c_1c_4c_3c_1 \rangle + \frac{1}{2} \sum_{234} U_{1234} \langle c_2c_3c_2 \rangle, \hspace{1cm} \text{(C3)}$$

where $\bar{\mu} = \mu - E_m/2$ is at least of the order $n^{0.5}$ (Sec. [IVC]).

The first term on the right side of Eq. (C3) has a diagrammatic representation in Fig. 9. In Fig. 9(a) the total diagram is shown; the big shaded circle stands for the ground state expectation value $\langle c_{K_1}c_{K_2}c_{K_4}c_{-K_1} \rangle$. In Fig. 9(b)(c) we show, for illustration, two associated individual diagrams, whose low-density orders are 1.5 and 2, respectively [Eq. (29)]; note that $N_2 = 1$ for Fig. 9(c), since its skeleton diagram is a point and a dotted line whose both ends are attached to such a point.

We now show that the low-density orders of the infinitely many individual diagrams associated with Fig. 9(a) have a lower bound, 1.5. In Fig. 9(a), the piece on the left of the shaded circle must be in the same island as the solid line on the right, because the dotted lines (if there are any) can not carry spin or any large momentum, and the separation of the two pieces in two different islands would contradict spin-momentum conservation. Thus we have a single external island $e$. One spin-momentum (i.e., $K_2$) enters the shaded circle, three leave it, and the associated four fermion operators are normally ordered, so $K_2$ must be attached to a degenerator in the big shaded circle; so $P_e^{-1} \geq 2$ and...
The cluster-separation theorem states that there exists $\phi_{K}$ and all the other eigenvalues are at least $|E_{m}| - 2\tilde{\mu} + q^{2}/2 > |E_{m}|/2$. The two terms of $\tilde{H}^{(2)}q$ act (as $\tilde{H}^{(2)}q$) independently on $K_{1}$ and $K_{2}$, respectively. So $\tilde{H}^{(2)}q$ has a single eigenvalue of the order $n^{1}$, associated with the eigenvector $\phi_{K_{1}}\phi_{K_{2}}$; all the other eigenvalues are at least $|E_{m}| - 4\tilde{\mu} + q^{2}/2 > |E_{m}|/2$. So the components of $F_{K_{1}K_{2}}^{a}$ which are orthogonal to $\phi_{K_{1}}\phi_{K_{2}}$ must be of the order $n^{1}$, in order for $\tilde{H}^{(2)}qF_{K}^{a}$ to be of the order $n^{1}$.

On the left side of Eq. (C6), $K_{1}'$ (or $K_{2}'$) may take values close or equal to $\pm K_{3}$ (or $\pm K_{1}$). But the associated contributions are not important. When $K_{1}' + K_{2}' \sim \sqrt{n}$, the associated volume in the $K_{1}'$-space is reduced by a factor of the order $n^{1.5}$. When $K_{1}' + K_{2}'$ is exactly zero, the thermodynamic order of $F_{K_{1}K_{2}}^{a}$ increases by 1, but this is accompanied by the loss of an independent dumb momentum; meanwhile the low-density order of $F_{K_{1}K_{2}}^{a}$ becomes 1, so the net effect on Eq. (C6) is of the same order as those of $e_{i}$ and $f_{i}$.

For more $l$-clusters ($M \geq 3$), each of which contains two spin-momenta, we still find that the equation of motion is dominated by the quasi-independent dynamics of individual clusters, with small “driving terms” [like the ones on the right side of Eq. (C8) or Eq. (C6)], so that $F_{K}$ (at given small momenta $q_{l} \sim \sqrt{n}$) is still proportional to a product of single-molecule internal wave functions, with relative error of the order $n^{1}$.

APPENDIX D: PARTIAL PROOF OF THE SECOND CLUSTER-SEPARATION THEOREM

Consider any diagram $D$ in the low-density expansion of $F_{K} \equiv (c_{K}^{a}K_{1}K_{2})$. Since there are only two external points [$L^{(-)} = 2$, $L^{(+)} = 0$], there is only one external island, and $I_{ext} = 1$, $P_{ext}^{(-)} = P_{ext}^{(+)} + 2$. From Eq. (31) we deduce that $R = 1/2 + P_{ext}^{(-)}/2 + (P_{int}/4 - I_{int}) + N_{2}/2$. If $P_{ext}^{(-)} = 0$, the external island is of a unique structure - a single small solid circle linked to the two external points via two solid lines; moreover, this island must be the whole diagram, since dead ends are absent. If $P_{ext}^{(-)} \geq 2$, $R \geq 1.5$. So $F_{K} = \eta K$ plus correction terms of the order $n^{1.5}$. But we proved in Sec. [1.1.3] that there exists an $\eta$ such that $F_{K} = \eta K\phi_{K}$ plus correction terms of the order $n^{1.5}$. So $\alpha_{K} = \eta K\phi_{K}$ plus correction terms of the order $n^{1.5}$. The simplest case of the second cluster-separation theorem is thus proved.

Now turn to

\[ \beta_{K_{1}K_{2}}^{q} \equiv \beta_{K_{1}+q/2,K_{2}+q/2,K_{2}-q/2,K_{1}-q/2} \]  (D1)
as a function of $K_1K_2$ (where neither $K_1 + K_2$ nor $K_1 - K_2$ is zero or $\sim \sqrt{n}$), for fixed small $q$ ($q \sim \sqrt{n}$). We extract its property from $F_{K_1K_2}$ (see Sec. 11). First identify all the diagrams in the low-density expansion of $F$ satisfying $R = 0$ or $R = 0.5$; using Eq. (31), we find that in such a diagram, there are two external islands (each of which has $P_i = 2$), the internal islands are simplest bubbles, and $N_q' = 0$ for $R = 0$, $N_q' = 1$ for $R = 0.5$. These two sets of diagrams are shown in Fig. 11 (first row). Some diagrams have the same structure after the simplest bubbles are suppressed, and their sum can be represented by a single diagram; each diagram in Fig. 11 is such a sum; a dashed line represents a geometric series associated with a variable number of simplest bubbles. See the first row ($A_{K_1K_2}^q$) of Fig. 2 for the content of the first diagram in Fig. 11.

$$A_{K_1K_2}^q = \beta_{K_1K_2}^q + \sum_{K_3K_4} \beta_{K_1K_2}^q \frac{1}{2} \beta_{K_1K_2}^q \frac{1}{2} \beta_{K_2K_4}^q + \cdots,$$

where $1/2$ is the symmetry factor of a simplest bubble, and the small momenta flowing through two adjacent bubbles have opposite signs (and directions). In the summation over an internal spin-momentum, $egK_3$, $K_3$ may be close or equal to $\pm K_1$ or $\pm K_4$, but the corresponding contributions are negligible for our purpose, since their associated volume in the dumb-momentum space is reduced by a factor of the order $n^{1.5}$. Now introduce matrices $\beta_q$ and $A_q$, whose matrix elements are defined by Eqs. (11) and (D3), respectively: ($\beta_{q})_{K_1K_2} = \beta_{qK_1K_2}$ and similarly for $A_q$. So

$$A_q = \beta_q + \beta_q^4 \beta_q^4 /4 + \beta_q^4 \beta_q^4 \beta_q^4 /16 + \cdots \sim \beta_q (1 - r_q)^{-1},$$

where

$$r_q \equiv \beta_q^4 \beta_q^4 /4$$

is a hermitian matrix, and all its eigenvalues are nonnegative. Also the geometric series in Eq. (D5) should be convergent, so the eigenvalues of $r_q$ should be less than 1. Since $A_q^q = (1 - r_q)^{-1} \beta_q \beta_q /4 \beta_q^4 \beta_q^4 \beta_q^4 /16 + \cdots \sim \beta_q (1 - r_q)^{-1}$, and

$$1 + A_q^q = [(1 + r_q)(1 - r_q)^{-1}]^2,$$

all the eigenvalues of the hermitian matrix $(1 + r_q)(1 - r_q)^{-1}$ are greater than or equal to 1, according to the above properties of $r_q$. So $(1 + r_q)(1 - r_q)^{-1} = \sqrt{1 + A_q^q}$. [The square root of any positive definite hermitian matrix $F$, $\sqrt{F}$, is defined as a hermitian matrix with the same set of eigenvectors as $h$, but its eigenvalues are the (positive) square roots of the corresponding eigenvalues of $h$; this definition is unambiguous even when the spectrum of $h$ has degeneracy; $\sqrt{F}$ depends smoothly on $h$.] So

$$1 + \sqrt{1 + A_q^q} = 2(1 - r_q)^{-1},$$

and

$$\beta_q = 2A_q \left(1 + \sqrt{1 + A_q^q} \right)^{-1}.$$  

(D6)

So $\beta_q$ is a smooth matrix function of the matrix $A_q$.

$B_{K_1K_2}^q$ is the $R = 0.5$ correction to $F$ (the second diagram in the first row of Fig. 11), and $F_{K_1K_2}^q = \eta_q \phi_{K_1} \phi_{K_2} + O(n^{0.5}/\Omega)$. So

$$A_{K_1K_2}^q = \eta_q \phi_{K_1} \phi_{K_2} + O(n^{0.5}/\Omega).$$

(D7)

Substituting this into Eq. (D6), we get [after using Eq. (103)]

$$\beta_{K_1K_2}^q = \frac{2\eta_q}{1 + \sqrt{1 + 4\eta_q^2}} \frac{\eta_q \phi_{K_1} \phi_{K_2} + O(n^{0.5}/\Omega)}{\eta_q \phi_{K_1} \phi_{K_2} + O(n^{0.5}/\Omega)}.$$  

(D8)

This is still one step from the desired result, namely

$$\beta_{K_1K_2}^q = \eta_q \phi_{K_1} \phi_{K_2} + O(n^{1}/\Omega)$$

for some $\eta_q'$. We need to consider the “1-loop” term $B_{K_1K_2}^q$, and show that it is factorizable at the leading order ($\sim n^{0.5}$). But this term contains the dispersed vertex $v_{111}$ (or $v_{111}^\dagger$). So we need to first show that $v_{111}$, and by the way $v_{1111}$, $v_{11111}$, etc., are factorizable at the leading order.

So we turn to $F_{K_1K_2K_3}^{q_1q_2q_3} = (c_{-K_1+q_1/2}c_{-K_2+q_2/2}c_{-K_3+q_3/2}) \sim n^{-0.5}$ (where $q_1 + q_2 + q_3 = 0$, and $q_{1,2,3} \sim \sqrt{n}$). Using Eq. (D1), we see again that diagrams at the lowest two orders have the same external-islands structure. The leading expression is the first diagram in the second row of Fig. 11. It has three dashed lines, linked to the external islands associated with $q_1K_1$, $q_2K_2$, and $q_3K_3$, respectively; suppose that the numbers of simplest bubbles on them are $n_1$, $n_2$, and $n_3$, respectively. Because the small circles contained by a dispersed vertex must be of the same color, $n_{1,2,3}$ must be all even (for $v_{111}$) or all odd (for $v_{111}^\dagger$). After straightforward derivation, we get

$$F_{K_1K_2K_3}^{q_1q_2q_3} = \sum_{K_1K_2K_3} C_{K_1K_1}^{q_1} C_{K_2K_2}^{q_2} C_{K_3K_3}^{q_3} \eta_{q_1} \eta_{q_2} \eta_{q_3}$$

$$+ \frac{1}{8} \sum_{K_1K_2K_3} A_{K_1K_1}^{q_1} A_{K_2K_2}^{q_2} A_{K_3K_3}^{q_3} \eta_{q_1} \eta_{q_2} \eta_{q_3}$$

$$+ B_{K_1K_2K_3}^{q_1q_2q_3} + O(n^{0.5}/\Omega^2).$$

(D9)

FIG. 11: First row: diagrams for $F_{K_1K_2}^q$ up to the order $n^{0.5}$. Second row: diagrams for $F_{K_1K_2K_3}^{q_1q_2}$ up to the order $n$. Each dashed line represents a geometric series associated with a variable number of simplest bubbles. The first diagram corresponds to the first row of Fig. 6.
Here, $\gamma^{q_1 q_2 q_3, G_{K_1 K_2}} = \gamma_{K_1, q_1 / 2} - K_1, q_1 / 2, K_2, q_1 / 2 - K_3, q_1 / 2,$ the matrix $G_{q, q_1} = (1 - \beta_{q_1 q_2} / 4)^{-1}, G_{K_1 K_2}^{q_1 q_2}$ is its matrix element, $\Delta_i = -q_i,$ and $B_{K_1 K_2}^{q_1 q_2} \sim n_i$ is the next-to-leading order contribution (the last three diagrams in the second row of Fig. 11).

The first two terms on the right side of Eq. (D10) are the leading-order expression of $F$, but the second term is proportional to $\phi_{K_1} \phi_{K_2} \phi_{K_3}$ at the leading order, according to Eq. (D7), since $F$ has this same property, we deduce that the first term on the right side of Eq. (D10) is also proportional to $\phi_{K_1} \phi_{K_2} \phi_{K_3}$ at the leading order.

The linear kernel in the first term is the direct product $\phi = \phi_{K_1} \phi_{K_2} \phi_{K_3}$ at the leading order. The skeleton of Fig. 11, which consists of the dispersed vertices we have studied, $B$ is thus factorizable at its own leading order. But $F$ is factorizable at the lowest two orders, and is just the next-to-leading order contribution to $F$. So the sum of the first two terms on the right side of Eq. (D10) is factorizable at the lowest two orders. But the second term itself is factorizable at the lowest two orders, according to the above established property of $A_{K_1 K_2}^{q_1 q_2 q_3}$. So we can deduce that $\phi_{K_1} \phi_{K_2} \phi_{K_3}$ is just the next-to-leading order contribution to $F$.

Now turn to $\gamma^{q_1 q_2 q_3}$, namely the coefficient of the dispersed vertex $v_{111}$. There are two diagrams, $D_1$ and $D_2$, for $F_{K_1 K_2 K_3}^{q_1 q_2 q_3}$, at the leading order. The skeleton of each of them is a tree ($N_2 = 0$). One of them, $D_1$, contains two $v_{111}$'s (or two $v_{111}$'s, or $v_{111}$ and $v_{111}$) so it is factorizable at the leading order according to the established properties of $v_{111}$ and $v_{111}$. The other diagram, $D_2$, contains a single $v_{111}$ (or $v_{111}$) together with variable numbers of $v_{111}$'s and $v_{111}$'s. Because both $D_1$ and $F$ are factorizable at the leading order, $D_2$ is also factorizable. Using the same logic as in the case of $v_{111}$ above, we can then deduce that $v_{111}$ is factorizable at the leading order.

Combining this two findings in the above two paragraphs, we deduce that $\gamma^{q_1 q_2 q_3}$ must be proportional to $\phi_{K_1} \phi_{K_2} \phi_{K_3}$ at the leading order.

Continuing this analysis, we see that for any $M \ll N$, $\alpha_{K_1 \cdots K_M}$ (the coefficient of the dispersed vertex $v_{1 \cdots 1}$) is proportional to $\Pi_{\nu=1}^{M} \phi_{K_{\nu}}$ at the leading order, for any given set of small momenta $q$'s.

So we can now start the second round of logic, to strengthen the above result.

First consider $v_{111}$. In Eq. (D12), the term $B_{K_1 K_2}^{q_1 q_2}$ corresponds to the “1-loop” ($N_2 = 1$) diagram in the first row of Fig. 11 and must be factorizable (ie, proportional to $\phi_{K_1} \phi_{K_2}$ at any given small $q \sim \sqrt{n}$ at its own leading order, according to the established properties of $v_{11}$ and $v_{111}$. But $B$ is just the next-to-leading order contribution to $F$. We have proved in Sec. [4] that $F$ is factorizable at the lowest two orders. So the term $A_{K_1 K_2}^{q_1 q_2}$ must also be factorizable at the lowest two orders, namely there exists an $\eta''_q$ such that $A_{K_1 K_2}^{q_1 q_2} = \eta''_q \phi_{K_1} \phi_{K_2} + O(n_1 \Omega^{-1})$. Substituting this result to Eq. (D10), we see that for $\eta'_q = 2\eta''_q / (1 + \sqrt{1 + 4\eta''_q^2}),$

$$
\beta_{K_1 K_2}^{q_1 q_2} = \eta'_q \phi_{K_1} \phi_{K_2} + O(n_1 \Omega^{-1}).
$$

We then turn to $v_{111}$. In Eq. (D10), the term $B$ is just the second row of Fig. 11 which consist of the dispersed vertices we have studied. $B$ is thus factorizable at its own leading order. But $F$ is factorizable at the lowest two orders, and is just the next-to-leading order contribution to $F$. So the sum of the first two terms on the right side of Eq. (D10) is factorizable at the lowest two orders. But the second term itself is factorizable at the lowest two orders, according to the above established property of $A_{K_1 K_2}^{q_1 q_2}$. So we find that

$$
\gamma^{q_1 q_2 q_3} = \eta''_q \phi_{K_1} \phi_{K_2} \phi_{K_3}
$$

with a relative error of the order $n^1$, for some $\eta''_q q_1 q_2 q_3$.

Extension to more clusters (each of which contains two spin-momenta) is trivial. In general, we have thus proved that

$$
\delta_{K_1 \cdots K_M}^{\alpha_{K_1 \cdots K_M} \cdots q_1 q_2 q_3 = \eta''_q \phi_{K_1} \phi_{K_2} \phi_{K_3} \cdots \phi_{K_M}
$$

with a relative error of the order $n^1$ in the low-density limit, for any $1 \leq M \ll N$, provided that $K_{\nu} \pm K_{\rho}$ are not zero or $\sim \sqrt{n}$ (for all $1 \leq \nu < \rho \leq M$).

[1] Shina Tan and K. Levin, cond-mat/0506923.
[2] Shina Tan, to appear very soon.
[3] Shina Tan, cond-mat/0505200.
[4] Shina Tan, cond-mat/0508320.
[5] Any term $C$ involving a small odd number of fermion creation operators that might appear in such exponent is negligible in the thermodynamic limit, because $C^2 = 0$, and $\exp(C) = 1 + C$, which adds at most a few fermions to the system.
[6] Qijin Chen, J. Stajic, Shina Tan, and K. Levin, Phys. Rep. 412, 1 (2005); Qijin Chen, J. Stajic, and K. Levin, cond-mat/0508630.
[7] T. D. Lee and C. N. Yang, Phys. Rev. 105, 1119 (1957); T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. 106, 1135 (1957).
[8] T. T. Wu, Phys. Rev. 115, 1390 (1959).
[9] In experiment there can be many lower bound states. However for the highest $s$-wave bound state of ultracold fermionic atoms near a wide $s$-wave Feshbach resonance, when the atomic scattering length is large, relaxation processes into lower bound states are suppressed by Fermi statistics [11] and the theory in this paper is applicable.
[10] In this paper, an expression like $x \sim O^2$ only refers to the behavior of $x$ for $\Omega \rightarrow \Omega'$; since the often dimensionful coefficient (independent of $\Omega$) is suppressed, one should never use such expression to infer the dimension of $x$. 
[11] D. S. Petrov, C. Salomon, and G. V. Shlyapnikov, Phys. Rev. Lett. 93, 090404 (2004).
[12] D. S. Petrov, private communication.
[13] A. J. Leggett, New J. Phys. 5, 103 (2003).
[14] E. Braaten, H.-W. Hammer, and T. Mehen, Phys. Rev. Lett. 88, 040401 (2002), and references therein.