Pullback Attractors for the Non-autonomous FitzHugh-Nagumo System on Unbounded Domains

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Abstract

The existence of a pullback attractor is established for the singularly perturbed FitzHugh-Nagumo system defined on the entire space \( \mathbb{R}^n \) when external terms are unbounded in a phase space. The pullback asymptotic compactness of the system is proved by using uniform a priori estimates for far-field values of solutions. Although the limiting system has no global attractor, we show that the pullback attractors for the perturbed system with bounded external terms are uniformly bounded, and hence do not blow up as a small parameter approaches zero.

Key words. pullback attractor, asymptotic compactness, non-autonomous equation.

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1 Introduction

In this paper, we study the dynamical behavior of the non-autonomous FitzHugh-Nagumo equations defined on \( \mathbb{R}^n \):

\[
\frac{\partial u}{\partial t} - \nu \Delta u + \lambda u + h(u) + v = f(t), \tag{1.1}
\]

\[
\frac{\partial v}{\partial t} - \epsilon (u - \gamma v) = g(t), \tag{1.2}
\]

where \( \nu, \lambda, \epsilon \) and \( \gamma \) are positive constants, \( f \) and \( g \) are given functions depending on \( t \), \( h \) is a nonlinear function satisfying a dissipative condition.

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The FitzHugh-Nagumo equations describe the signal transmission across axons in neurobiology, see e.g., [7, 19, 32] and the references therein. The long time behavior of the autonomous FitzHugh-Nagumo equations was studied by several authors in [25, 27, 28, 29, 35] and the references therein. We here intend to investigate the dynamical behavior of the non-autonomous FitzHugh-Nagumo system.

Global attractors for non-autonomous dynamical systems have been extensively studied in the literature, see, e.g., [1, 3, 8, 10, 11, 12, 13, 14, 15, 16, 21, 23, 24, 26, 31, 33, 36, 37]. Particularly, when PDEs are defined in bounded domains, such attractors have been investigated in [8, 12, 13, 14, 16, 21, 26, 36, 37]. In the case of unbounded domains, global attractors for non-autonomous PDEs with almost periodic external terms have been examined in [1, 31, 33]. Notice that almost periodic external terms are bounded in a phase space with respect to time. It seems that attractors for non-autonomous PDEs defined on unbounded domains with unbounded external terms are not well understood. As far as we know, in this case, the existence of attractors was established only for the Navier-Stokes equation by the authors in [10, 11] recently. In this paper, we will prove the existence of attractors for the non-autonomous FitzHugh-Nagumo system defined on the entire space $\mathbb{R}^n$ with unbounded external terms.

Notice that the domain $\mathbb{R}^n$ for system (1.1)-(1.2) is unbounded, and the unboundedness of $\mathbb{R}^n$ introduces a major obstacle for examining the asymptotic compactness of solutions, since Sobolev embeddings are not compact in this case. The difficulty caused by non-compactness of embeddings can be overcome by the energy equation approach, which was introduced by Ball in [4, 5] and then used by several authors for autonomous equations in [20, 22, 30, 34, 39] and for non-autonomous equations in [10, 11, 31]. In this paper, we provide uniform estimates on the far field values of solutions to circumvent the difficulty caused by the unboundedness of the domain. This idea was developed in [38] to prove asymptotic compactness of solutions for autonomous parabolic equations on $\mathbb{R}^n$, and later extended to non-autonomous equations with almost periodic external terms in [1, 33]. The contribution of this paper is to extend the method of using tail estimates to the case of non-autonomous PDEs defined on unbounded domains with unbounded external terms.

We first prove that system (1.1)-(1.2) on $\mathbb{R}^n$ has a pullback attractor when the parameter $\epsilon$ is a small but positive number. Note that the limiting system with $\epsilon = 0$ has no global attractor since $v$ is conserved in this case. Based on this fact, one may guess that the attractors of the perturbed system blow up as $\epsilon \to 0$. In this respect, we will demonstrate that the limiting behavior of the
pullback attractors heavily depends on the behavior of the external terms \( f \) and \( g \). If \( f \) or \( g \) is unbounded in a phase space, then it is very likely that the attractors blow up as \( \epsilon \to 0 \). However, if both \( f \) and \( g \) are bounded, the attractors are uniformly bounded in a phase with respect to all small but positive \( \epsilon \). In other words, in this case, the pullback attractors do not blow up as \( \epsilon \to 0 \).

The paper is organized as follows. In the next section, we recall fundamental concepts and results for pullback attractors for non-autonomous dynamical systems. In Section 3, we derive uniform estimates of solutions for the FitzHugh-Nagumo system for large space and time variables. Section 4 is devoted to the proof of existence of a pullback attractor for the system. In the last section, we discuss the limiting behavior of pullback attractors when \( \epsilon \to 0 \). Particularly, we will show that all attractors for the perturbed system are uniformly bounded in \( H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \) with respect to \( \epsilon \) when external terms are bounded.

The following notations will be used throughout the paper. We denote by \( \| \cdot \| \) and \((\cdot, \cdot)\) the norm and the inner product in \( L^2(\mathbb{R}^n) \) and use \( \| \cdot \|_p \) to denote the norm in \( L^p(\mathbb{R}^n) \). Otherwise, the norm of a general Banach space \( X \) is written as \( \| \cdot \|_X \). The letters \( C \) and \( C_i \) \((i = 1, 2, \ldots)\) are generic positive constants which may change their values from line to line or even in the same line.

## 2 Preliminaries

In this section, we recall some basic concepts related to pullback attractors for non-autonomous dynamical systems. It is worth to notice that these concepts are quite similar to that of random attractors for stochastic systems. We refer the reader to [2, 6, 9, 10, 11, 13, 17, 18, 36] for more details.

Let \( \Omega \) be a nonempty set and \( X \) a metric space with distance \( d(\cdot, \cdot) \).

**Definition 2.1.** A family of mappings \( \{\theta_t\}_{t \in \mathbb{R}} \) from \( \Omega \) to itself is called a family of shift operators on \( \Omega \) if \( \{\theta_t\}_{t \in \mathbb{R}} \) satisfies the group properties:

(i) \( \theta_0 \omega = \omega, \quad \forall \ \omega \in \Omega; \)

(ii) \( \theta_t(\theta_\tau \omega) = \theta_{t+\tau} \omega, \quad \forall \ \omega \in \Omega \quad \text{and} \quad t, \ \tau \in \mathbb{R}. \)

**Definition 2.2.** Let \( \{\theta_t\}_{t \in \mathbb{R}} \) be a family of shift operators on \( \Omega \). Then a continuous \( \theta \)-cocycle \( \phi \) on \( X \) is a mapping

\[ \phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x), \]
which satisfies, for all \( \omega \in \Omega \) and \( t, \tau \in \mathbb{R}^+ \),

(i) \( \phi(0, \omega, \cdot) \) is the identity on \( X \);

(ii) \( \phi(t + \tau, \omega, \cdot) = \phi(t, \theta \tau \omega, \cdot) \circ \phi(\tau, \omega, \cdot) \);

(iii) \( \phi(t, \omega, \cdot) : X \to X \) is continuous.

Hereafter, we always assume that \( \phi \) is a continuous \( \theta \)-cocycle on \( X \), and \( D \) a collection of families of subsets of \( X \):

\[
D = \{ D = \{ D(\omega) \}_{\omega \in \Omega} : D(\omega) \subseteq X \text{ for every } \omega \in \Omega \}.
\]

**Definition 2.3.** Let \( D \) be a collection of families of subsets of \( X \). Then \( D \) is called inclusion-closed if \( D = \{ D(\omega) \}_{\omega \in \Omega} \in D \) and \( \tilde{D} = \{ \tilde{D}(\omega) \subseteq X : \omega \in \Omega \} \) with \( \tilde{D}(\omega) \subseteq D(\omega) \) for all \( \omega \in \Omega \) imply that \( \tilde{D} \in D \).

**Definition 2.4.** Let \( D \) be a collection of families of subsets of \( X \) and \( \{ K(\omega) \}_{\omega \in \Omega} \in D \). Then \( \{ K(\omega) \}_{\omega \in \Omega} \) is called a pullback absorbing set for \( \phi \) in \( D \) if for every \( B \in D \) and \( \omega \in \Omega \), there exists \( t(\omega, B) > 0 \) such that

\[
\phi(t, \theta - t \omega, B(\theta - t \omega)) \subseteq K(\omega) \text{ for all } t \geq t(\omega, B).
\]

**Definition 2.5.** Let \( D \) be a collection of families of subsets of \( X \). Then \( \phi \) is said to be \( D \)-pullback asymptotically compact in \( X \) if for every \( \omega \in \Omega \), \( \{ \phi(t_n, \theta^{-t_n} \omega, x_n) \}_{n=1}^{\infty} \) has a convergent subsequence in \( X \) whenever \( t_n \to \infty \), and \( x_n \in B(\theta^{-t_n} \omega) \) with \( \{ B(\omega) \}_{\omega \in \Omega} \in D \).

**Definition 2.6.** Let \( D \) be a collection of families of subsets of \( X \) and \( \{ A(\omega) \}_{\omega \in \Omega} \in D \). Then \( \{ A(\omega) \}_{\omega \in \Omega} \) is called a \( D \)-pullback global attractor for \( \phi \) if the following conditions are satisfied, for every \( \omega \in \Omega \),

(i) \( A(\omega) \) is compact;

(ii) \( \{ A(\omega) \}_{\omega \in \Omega} \) is invariant, that is,

\[
\phi(t, \omega, A(\omega)) = A(\theta t \omega), \quad \forall t \geq 0;
\]

(iii) \( \{ A(\omega) \}_{\omega \in \Omega} \) attracts every set in \( D \), that is, for every \( B = \{ B(\omega) \}_{\omega \in \Omega} \in D \),

\[
\lim_{t \to \infty} d(\phi(t, \theta^{-t} \omega, B(\theta^{-t} \omega)), A(\omega)) = 0,
\]

where \( d \) is the Hausdorff semi-metric given by \( d(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} \| y - z \|_X \) for any \( Y \subseteq X \) and \( Z \subseteq X \).
The following existence result of a pullback global attractor for a continuous cocycle can be found in \[2, 6, 9, 10, 11, 13, 17, 18\].

**Proposition 2.7.** Let $D$ be an inclusion-closed collection of families of subsets of $X$ and $\phi$ a continuous $\theta$-cocycle on $X$. Suppose that \{\(K(\omega)\), $\omega \in \Omega\} \subseteq D$ is a closed absorbing set for $\phi$ in $D$ and $\phi$ is $D$-pullback asymptotically compact in $X$. Then $\phi$ has a unique $D$-pullback global attractor \{\(A(\omega)\), $\omega \in \Omega\} \subseteq D$ which is given by

$$A(\omega) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)).$$

### 3 Cocycle associated with the FitzHugh-Nagumo system

In this section, we construct a $\theta$-cocycle $\phi$ for the non-autonomous FitzHugh-Nagumo system defined on $\mathbb{R}^n$: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\frac{\partial u}{\partial t} - \nu \Delta u + \lambda u + h(u) + v = f(t), \quad (3.1)$$

$$\frac{\partial v}{\partial t} - \epsilon(u - \gamma v) = \epsilon g(t), \quad (3.2)$$

with the initial data

$$u(x, \tau) = u_\tau(x), \quad v(x, \tau) = v_\tau(x), \quad x \in \mathbb{R}^n, \quad (3.3)$$

where $\nu$, $\lambda$, $\epsilon$ and $\gamma$ are positive constants, $f \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$, $g \in L^2(\mathbb{R}, H^1(\mathbb{R}^n))$, and $h$ is a smooth nonlinear function that satisfies, for some positive constant $C$,

$$h(s)s \geq 0, \quad h(0) = 0, \quad h'(s) \geq -C, \quad s \in \mathbb{R}, \quad (3.4)$$

and

$$|h'(s)| \leq C(1 + |s|^r), \quad s \in \mathbb{R}, \quad (3.5)$$

with $r \geq 0$ for $n \leq 2$ and $r \leq \min(\frac{4}{n}, \frac{2}{n-2})$ for $n \geq 3$.

By a standard method, it can be proved that if $f \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$, $g \in L^2_{\text{loc}}(\mathbb{R}, H^1(\mathbb{R}^n))$ and (3.1)-(3.5) hold true, then problem (3.1)-(3.3) is well-posed in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}$ and $(u_\tau, v_\tau) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, there exists a unique solution $(u, v) \in C([\tau, \infty), L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$. Further, the solution is continuous with respect to initial data $(u_\tau, v_\tau)$ in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. To
construct a cocycle $\phi$ for problem (3.1)-(3.3), we denote by $\Omega = \mathbb{R}$, and define a shift operator $\theta_t$ on $\Omega$ for every $t \in \mathbb{R}$ by

$$\theta_t(\tau) = t + \tau, \quad \text{for all } \tau \in \mathbb{R}.$$ 

Let $\phi$ be a mapping from $\mathbb{R}^+ \times \Omega \times (L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ to $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ given by

$$\phi(t, \tau, (u_\tau, v_\tau)) = (u(t + \tau, \tau, u_\tau), v(t + \tau, \tau, v_\tau)),$$

where $t \geq 0$, $\tau \in \mathbb{R}$, $(u_\tau, v_\tau) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, and $(u, v)$ is the solution of problem (3.1)-(3.3). By the uniqueness of solutions, we find that for every $t, s \geq 0$, $\tau \in \mathbb{R}$ and $(u_\tau, v_\tau) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$,

$$\phi(t + s, \tau, (u_\tau, v_\tau)) = \phi(t, s + \tau, (\phi(s, \tau, (u_\tau, v_\tau)))).$$ 

Then we see that $\phi$ is a continuous $\theta$-cocycle on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In the next two sections, we will investigate the existence of a pullback attractor for $\phi$. To this end, we need to define an appropriate collection of families of subsets of $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

For convenience, if $E \subseteq L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, we denote by

$$\|E\| = \sup_{x \in E} \|x\|_{L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}.$$ 

Let $D = \{D(t)\}_{t \in \mathbb{R}}$ be a family of subsets of $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, i.e., $D(t) \subseteq L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for every $t \in \mathbb{R}$. In this paper, we are interested in a family $D = \{D(t)\}_{t \in \mathbb{R}}$ satisfying

$$\lim_{t \to -\infty} e^{\sigma t} \|D(t)\|^2 = 0,$$ 

where $\sigma$ is a positive number given by

$$\sigma = \frac{1}{2} \epsilon \gamma.$$ 

(3.7)

We write the collection of all families satisfying (3.6) as $\mathcal{D}_\sigma$, that is,

$$\mathcal{D}_\sigma = \{D = \{D(t)\}_{t \in \mathbb{R}} : D \text{ satisfies (3.6)}\}.$$ 

(3.8)

Since $\epsilon$ is small in practice, we assume throughout this paper that

$$\epsilon \leq \epsilon_0 \quad \text{where} \quad \epsilon_0 = \min\{1, \frac{\lambda}{\gamma}\}.$$ 

(3.9)

As we will see later, when we derive uniform estimates of solutions, we need the following conditions for the external terms:

$$\int_{-\infty}^{\tau} e^{\sigma \xi} \|f(\xi)\|^2 d\xi < \infty, \quad \forall \tau \in \mathbb{R},$$ 

(3.10)
and
\[ \int_{-\infty}^{\tau} e^{\sigma \xi} |g(\xi)|^2_{H^1} d\xi < \infty, \quad \forall \tau \in \mathbb{R}. \] (3.11)

In addition, the following asymptotically null conditions are required for proving the asymptotic compactness of solutions:
\[ \lim_{k \to \infty} \int_{-\infty}^{\tau} \int_{|x| \geq k} e^{\sigma \xi} |f(x, \xi)|^2 dxd\xi = 0, \quad \forall \tau \in \mathbb{R}, \] (3.12)
and
\[ \lim_{k \to \infty} \int_{-\infty}^{\tau} \int_{|x| \geq k} e^{\sigma \xi} |g(x, \xi)|^2 dxd\xi = 0, \quad \forall \tau \in \mathbb{R}. \] (3.13)

Notice that conditions (3.10)-(3.13) do not require that \( f \) and \( g \) be bounded in \( L^2(\mathbb{R}^n) \) when \( t \to \pm \infty \). Particularly, these assumptions have no any restriction on \( f \) and \( g \) when \( t \to +\infty \). As a typical example, for any \( f_1 \in L^2(\mathbb{R}^n) \) and \( g_1 \in H^1(\mathbb{R}^n) \), the functions \( f(x, t) = e^{\frac{t}{4} \sigma |t|} f_1(x) \) and \( g(x, t) = e^{\frac{t}{4} \sigma |t|} g_1(x) \) satisfy all conditions (3.10)-(3.13). In this case, \( f \) and \( g \) are indeed unbounded in \( L^2(\mathbb{R}^n) \) as \( t \to \pm \infty \).

It is useful to note that conditions (3.12)-(3.13) imply for every \( \tau \in \mathbb{R} \) and \( \eta > 0 \), there is \( K = K(\tau, \eta) > 0 \) such that
\[ \int_{-\infty}^{\tau} \int_{|x| \geq K} e^{\sigma \xi} |f(x, \xi)|^2 dxd\xi \leq \eta e^{\sigma \tau}, \] (3.14)
and
\[ \int_{-\infty}^{\tau} \int_{|x| \geq K} e^{\sigma \xi} |g(x, \xi)|^2 dxd\xi \leq \eta e^{\sigma \tau}. \] (3.15)

We remark that (3.14) and (3.15) will play a crucial role when we derive uniform estimates on the tails of solutions in the next section.

4 Uniform estimates of solutions

In this section, we derive uniform estimates of solutions of problem (3.1)-(3.3) defined on \( \mathbb{R}^n \) when \( t \to \infty \). These estimates are necessary for proving the existence of a bounded pullback absorbing set and the pullback asymptotic compactness of the \( \theta \)-cocycle \( \phi \) associated with the system. In particular, we will show that the tails of the solutions, i.e., solutions evaluated at large values of \( |x| \), are uniformly small when time is sufficiently large.

We start with the estimates in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \).
Lemma 4.1. Suppose (3.4)-(3.5) and (3.10)-(3.11) hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in D_\sigma$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,

$$
\|u(t, \tau - t, u_0(\tau - t))\|^2 + \|v(t, \tau - t, v_0(\tau - t))\|^2 \leq M e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} (\|f(\xi)\|^2 + \|g(\xi)\|^2) \, d\xi,
$$

and

$$
\int_{\tau-t}^{\tau} e^{\sigma \xi} \|u(\xi, \tau - t, u_0(\tau - t))\|^2 \, d\xi \leq M \int_{-\infty}^{\tau} e^{\sigma \xi} (\|f(\xi)\|^2 + \|g(\xi)\|^2) \, d\xi,
$$

where $M$ is a positive constant depending on the data $(\nu, \lambda, \epsilon, \gamma)$.

Proof. Taking the inner product of (3.1) with $\epsilon u$ in $L^2(\mathbb{R}^n)$, we find that

$$
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \epsilon \nu \|\nabla u\|^2 + \epsilon \lambda \|u\|^2 + \epsilon \int h(u)u + \epsilon \int uv = \epsilon \int f(t)u. \quad (4.1)
$$

Taking the inner product of (3.2) with $v$ in $L^2(\mathbb{R}^n)$, we find

$$
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \epsilon \gamma \|v\|^2 - \epsilon \int uv = \epsilon \int g(t)v. \quad (4.2)
$$

It follows from (4.1)-(4.2) that

$$
\frac{1}{2} \frac{d}{dt} (\epsilon \|u\|^2 + \|v\|^2) + \epsilon \nu \|\nabla u\|^2 + \epsilon \lambda \|u\|^2 + \epsilon \gamma \|v\|^2 + \epsilon \int h(u)u = \epsilon \int f(t)u + \epsilon \int g(t)v. \quad (4.3)
$$

Note that the terms on the right-hand side of (4.3) are bounded by

$$
|\epsilon \int f(t)u| \leq \epsilon \|f(t)\| \|u\| \leq \frac{1}{2} \epsilon \lambda \|u\|^2 + \frac{\epsilon}{2\lambda} \|f\|^2, \quad (4.4)
$$

and

$$
|\epsilon \int g(t)v| \leq \epsilon \|g(t)\| \|v\| \leq \frac{1}{2} \epsilon \gamma \|v\|^2 + \frac{\epsilon}{2\gamma} \|g\|^2. \quad (4.5)
$$

By (4.3)-(4.5) and (3.4), we obtain

$$
\frac{d}{dt} (\epsilon \|u\|^2 + \|v\|^2) + 2\epsilon \nu \|\nabla u\|^2 + \epsilon \lambda \|u\|^2 + \epsilon \gamma \|v\|^2 \leq \frac{\epsilon}{\lambda} \|f\|^2 + \frac{\epsilon}{2\lambda} \|g\|^2;
$$

and hence by (3.7) and (3.8) we have

$$
\frac{d}{dt} (\epsilon \|u\|^2 + \|v\|^2) + 2 \sigma (\epsilon \|u\|^2 + \|v\|^2) + 2 \epsilon \nu \|\nabla u\|^2 \leq \frac{\epsilon}{\lambda} \|f\|^2 + \frac{\epsilon}{2\gamma} \|g\|^2. \quad (4.6)
$$

Multiplying (4.6) by $e^{\alpha t}$ and then integrating between $\tau - t$ and $t$ with $t \geq 0$, we get,

$$
\epsilon \|u(\tau, \tau - t, u_0(\tau - t))\|^2 + \|v(\tau, \tau - t, v_0(\tau - t))\|^2
$$
\[ + \sigma \int_{\tau-t}^\tau e^{\sigma (\xi - \tau)} \left( \|u(\xi, \tau - t, u(\tau - t))\|^2 + \|v(\xi, \tau - t, v_0(\tau - t))\|^2 \right) d\xi \]
\[ + 2e\nu \int_{\tau-t}^\tau e^{\sigma (\xi - \tau)} \|\nabla u(\xi, \tau - t, u(\tau - t))\|^2 d\xi \]
\[ \leq e^{-\sigma \tau} e^{\sigma (\tau - t)} \left( \epsilon \|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2 \right) \]
\[ + \frac{\epsilon}{\lambda} e^{-\sigma \tau} \int_{\tau-t}^\tau e^{\sigma \xi} \|f(\xi)\|^2 d\xi + \frac{\epsilon}{\gamma} e^{-\sigma \tau} \int_{\tau-t}^\tau e^{\sigma \xi} \|g(\xi)\|^2 d\xi \]
\[ \leq e^{-\sigma \tau} e^{\sigma (\tau - t)} \left( \epsilon \|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2 \right) \]
\[ + \frac{\epsilon}{\lambda} e^{-\sigma \tau} \int_{\tau-t}^\tau e^{\sigma \xi} \|f(\xi)\|^2 d\xi + \frac{\epsilon}{\gamma} e^{-\sigma \tau} \int_{\tau-t}^\tau e^{\sigma \xi} \|g(\xi)\|^2 d\xi. \] (4.7)

Notice that \((u_0(\tau - t), v_0(\tau - t)) \in D(\tau - t)\) and \(D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma\). We find that for every \(\tau \in \mathbb{R}\), there exists \(T = T(\tau, D)\) such that for all \(t \geq T\),
\[ e^{\sigma (\tau - t)} \left( \epsilon \|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2 \right) \leq \frac{\epsilon}{\lambda} \int_{-\infty}^\tau e^{\sigma \xi} \|f(\xi)\|^2 d\xi + \frac{\epsilon}{\gamma} \int_{-\infty}^\tau e^{\sigma \xi} \|g(\xi)\|^2 d\xi, \]

which along with (4.7) shows that, for all \(t \geq T\),
\[ \epsilon \|u(t, \tau - t, u_0(\tau - t))\|^2 + \|v(\tau, \tau - t, v_0(\tau - t))\|^2 \]
\[ + \sigma \int_{\tau-t}^\tau e^{\sigma (\xi - \tau)} \left( \|u(\xi, \tau - t, u(\tau - t))\|^2 + \|v(\xi, \tau - t, v_0(\tau - t))\|^2 \right) d\xi \]
\[ + 2e\nu \int_{\tau-t}^\tau e^{\sigma (\xi - \tau)} \|\nabla u(\xi, \tau - t, u(\tau - t))\|^2 d\xi \]
\[ \leq e^{-\sigma \tau} \left( \frac{2\epsilon}{\lambda} \int_{-\infty}^\tau e^{\sigma \xi} \|f(\xi)\|^2 d\xi + \frac{2\epsilon}{\gamma} \int_{-\infty}^\tau e^{\sigma \xi} \|g(\xi)\|^2 d\xi \right), \] (4.8)

which completes the proof. \(\square\)

We will need the following estimates when proving the asymptotic compactness of solutions, which can be derived in a similar manner as Lemma 4.1.

**Lemma 4.2.** Suppose (3.4) - (3.5) and (3.10) - (3.11) hold. Then for every \(\tau \in \mathbb{R}\) and \(D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma\), there exists \(T = T(\tau, D) > 1\) such that for all \(t \geq T\),
\[ \int_{\tau-1}^\tau e^{\sigma \xi} \left( \|u(\xi, \tau - t, u_0(\tau - t))\|^2 + \|v(\xi, \tau - t, v_0(\tau - t))\|^2 \right) d\xi \leq M \int_{-\infty}^\tau e^{\sigma \xi} \left( \|f(\xi)\|^2 + \|g(\xi)\|^2 \right) d\xi, \]
\[ \int_{\tau-1}^\tau e^{\sigma \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi \leq M \int_{-\infty}^\tau e^{\sigma \xi} \left( \|f(\xi)\|^2 + \|g(\xi)\|^2 \right) d\xi, \]
where \(M\) is a positive constant depending on the data \((\nu, \lambda, \epsilon, \gamma)\).
Proof. Note that (4.6) implies that
\[
\frac{d}{dt} (\epsilon ||u||^2 + ||v||^2) + \sigma (\epsilon ||u||^2 + ||v||^2) \leq \frac{\epsilon}{\lambda} ||f||^2 + \frac{\epsilon}{\gamma} ||g||^2.
\] (4.9)
Multiplying (4.9) by $e^{\sigma t}$ and integrating over $(\tau - 1, \tau - t)$ with $t \geq 1$, by repeating the proof of (4.8) we find that there exists $T = T(\tau, D) > 1$ such that for all $t \geq T$,
\[
||u(\tau - 1, \tau - t, u_0(\tau - t))||^2 + ||v(\tau - 1, \tau - t, v_0(\tau - t))||^2 \leq M e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} (||f(\xi)||^2 + ||g(\xi)||^2) d\xi.
\] (4.10)
Multiplying (4.6) by $e^{\sigma t}$ and then integrating over $(\tau - 1, \tau)$, by (4.10) we get that, for all $t \geq T$,
\[
e^{\sigma \tau} (\epsilon ||u(\tau, \tau - t, u_0(\tau - t))||^2 + ||v(\tau, \tau - t, v_0(\tau - t))||^2) + \sigma \int_{\tau - 1}^{\tau} e^{\sigma \xi} (\epsilon ||u(\xi, \tau - t, u_0(\tau - t))||^2 + ||v(\xi, \tau - t, v_0(\tau - t))||^2) d\xi
\]
\[
+ 2 \epsilon \nu \int_{\tau - 1}^{\tau} e^{\sigma \xi} ||\nabla u(\xi, \tau - t, u_0(\tau - t))||^2 d\xi
\]
\[
\leq e^{\sigma (\tau - 1)} (\epsilon ||u(\tau - 1, \tau - t, u_0(\tau - t))||^2 + ||v(\tau - 1, \tau - t, v_0(\tau - t))||^2)
\]
\[
+ \frac{\epsilon}{\lambda} \int_{\tau - 1}^{\tau} e^{\sigma \xi} ||f(\xi)||^2 d\xi + \frac{\epsilon}{\gamma} \int_{\tau - 1}^{\tau} e^{\sigma \xi} ||g(\xi)||^2 d\xi,
\]
which completes the proof. \hfill \square

Lemma 4.3. Suppose (3.4)-(3.5) and (3.10)-(3.11) hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in D_\sigma$, there exists $T = T(\tau, D) > 1$ such that for all $t \geq T$,
\[
||\nabla u(\tau, \tau - t, u_0(\tau - t))||^2 \leq M e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} (||f(\xi)||^2 + ||g(\xi)||^2) d\xi,
\]
where $M$ is a positive constant depending on the data $(\nu, \lambda, \epsilon, \gamma)$.

Proof. Taking the inner product of (3.1) with $-\Delta u$ in $L^2(\mathbb{R}^n)$, we get
\[
\frac{1}{2} \frac{d}{dt} ||\nabla u||^2 + \nu ||\Delta u||^2 + \lambda ||\nabla u||^2 = \int h(u) \Delta u + \int v \Delta u - \int f(t) \Delta u.
\] (4.11)
We now estimate the right-hand side of (4.11). For the last term, we have
\[
| \int f(t) \Delta u | \leq ||f(t)|| ||\Delta u|| \leq \frac{1}{4} \nu ||\Delta u||^2 + \frac{1}{\nu} ||f(t)||^2.
\] (4.12)
For the second term on the right-hand side of (4.11), we have the following bounds
\[ | \int v \nabla u | \leq \| v \| \| \nabla u \| \leq \frac{1}{4} \nu \| \nabla u \| ^2 + \frac{1}{\nu} \| v \| ^2. \]  
(4.13)

Note that by (3.4), the first term on the right-hand side of (4.11) is bounded by
\[ \int h(u) \nabla u = -\int h'(u) |\nabla u| \leq C \| \nabla u \| ^2, \]  
(4.14)

where \( C \) is the constant in (3.4). Then it follows from (4.11)-(4.14) that
\[ \frac{d}{dt} \| \nabla u \| ^2 + \sigma \| \nabla u \| ^2 \leq C \| \nabla u \| ^2 + \frac{2}{\nu} \| v \| ^2 + \frac{2}{\nu} \| f(t) \| ^2. \]  
(4.15)

Multiplying (4.15) by \( e^{\sigma t} \) and then integrating the resulting equality over \((s, \tau)\) with \( \tau - 1 \leq s \leq \tau \), we find that
\[ e^{\sigma \tau} \| \nabla u(\tau, \tau-t, u_0(\tau-t)) \| ^2 \leq e^{\sigma s} \| \nabla u(s, \tau-t, u_0(\tau-t)) \| ^2 \]
\[ + C \int_s^\tau e^{\sigma \xi} \| \nabla u(\xi, \tau-t, u_0(\tau-t)) \| ^2 d\xi \]
\[ + \frac{2}{\nu} \int_s^\tau e^{\sigma \xi} \| v(\xi, \tau-t, v_0(\tau-t)) \| ^2 d\xi + \frac{2}{\nu} \int_s^\tau e^{\sigma \xi} \| f(\xi) \| ^2 d\xi \]
\[ \leq e^{\sigma s} \| \nabla u(s, \tau-t, u_0(\tau-t)) \| ^2 + C \int_{\tau-1}^\tau e^{\sigma \xi} \| \nabla u(\xi, \tau-t, u_0(\tau-t)) \| ^2 d\xi \]
\[ + \frac{2}{\nu} \int_{\tau-1}^\tau e^{\sigma \xi} \| v(\xi, \tau-t, v_0(\tau-t)) \| ^2 d\xi + \frac{2}{\nu} \int_{\tau-1}^\tau e^{\sigma \xi} \| f(\xi) \| ^2 d\xi. \]  
(4.16)

We now integrate (4.16) with respect to \( s \) over \((\tau-1, \tau)\) to get
\[ e^{\sigma \tau} \| \nabla u(\tau, \tau-t, u_0(\tau-t)) \| ^2 \]
\[ \leq \int_{\tau-1}^\tau e^{\sigma s} \| \nabla u(s, \tau-t, u_0(\tau-t)) \| ^2 ds + C \int_{\tau-1}^\tau e^{\sigma \xi} \| \nabla u(\xi, \tau-t, u_0(\tau-t)) \| ^2 d\xi \]
\[ + \frac{2}{\nu} \int_{\tau-1}^\tau e^{\sigma \xi} \| v(\xi, \tau-t, v_0(\tau-t)) \| ^2 d\xi + \frac{2}{\nu} \int_{\tau-1}^\tau e^{\sigma \xi} \| f(\xi) \| ^2 d\xi. \]  
(4.17)

Then it follows from (4.17) and Lemma 4.2 that there is \( T = T(\tau, D) > 1 \) such that for all \( t \geq T \),
\[ e^{\sigma \tau} \| \nabla u(\tau, \tau-t, u_0(\tau-t)) \| ^2 \leq C \int_{-\infty}^\tau e^{\sigma \xi} (\| f(\xi) \| ^2 + \| g(\xi) \| ^2) d\xi, \]
which completes the proof. \( \square \)
Note that Lemma 4.3 shows that system (3.1)-(3.2) has smoothing effect on the $u$ components of solutions. However this is not true for the $v$ components. In order to establish the uniform asymptotic compactness of $v$, we need to decompose $v$ as a sum of two functions: one is regular in the sense it belongs to $H^1(\mathbb{R}^n)$ and the other converges to zero as $t \to \infty$. This splitting technique was used by several authors for the autonomous FitzHugh-Nagumo equations in bounded domains (see, for example [28]).

We split $v$ as $v = v_1 + v_2$ where $v_1$ is the solution of the initial value problem, for $t \geq s$ with $s \in \mathbb{R}$,

$$\frac{\partial v_1}{\partial t} + \epsilon \gamma v_1 = 0, \quad v_1(s) = v_0,$$

and $v_2$ is the solution of

$$\frac{\partial v_2}{\partial t} + \epsilon \gamma v_2 - \epsilon u = \epsilon g(t), \quad v_2(s) = 0.$$  \hspace{1cm} (4.19)

It is evident that $v_1$ satisfies:

$$\|v_1(\tau)\| = e^{-\epsilon \gamma (\tau - s)}\|v_1(s)\|, \quad \text{for all } \tau \geq s.$$  \hspace{1cm} (4.20)

Given $\tau \in \mathbb{R}$ and $t \geq 0$, set $s = \tau - t$. Then we get that

$$\|v_1(\tau)\| = e^{-\epsilon \gamma \tau} e^{\epsilon \gamma (\tau - t)}\|v_0(\tau - t)\|,$$

which implies that $v_1$ converges to zero when $t \to \infty$. Next, we derive uniform estimates for $v_2$ in $H^1(\mathbb{R}^n)$.

**Lemma 4.4.** Suppose (3.4)-(3.5) and (3.10)-(3.11) hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,

$$\|\nabla v_2(\tau, \tau - t, 0)\|^2 \leq M e^{-\sigma \tau} \int_{-\infty}^\tau e^{\sigma \xi} \left(\|f(\xi)\|^2 + \|g(\xi)\|^2_{H^1}\right) d\xi,$$

where $M$ is a positive constant depending on the data $(\nu, \lambda, \epsilon, \gamma)$.

**Proof.** Taking the inner product of (4.19) with $-\Delta v_2$ in $L^2(\mathbb{R}^n)$, we obtain that

$$\frac{d}{dt}\|\nabla v_2\|^2 + 2\epsilon \gamma \|\nabla v_2\|^2 = 2\epsilon (\nabla v_2, \nabla u) - 2\epsilon \int g(t) \Delta v_2 \, dx.$$  \hspace{1cm} (4.21)

The first term on the right-hand side of (4.21) is bounded by

$$|2\epsilon (\nabla v_2, \nabla u)| \leq 2\epsilon \|\nabla v_2\| \|\nabla u\| \leq \frac{1}{4} \epsilon \gamma \|\nabla v_2\|^2 + \frac{4\epsilon}{\gamma} \|\nabla u\|^2.$$  \hspace{1cm} (4.22)
For the second term on the right-hand side of (4.21) we have
\[
|2\epsilon \int g(t) \Delta v_2 \, dx| \leq \frac{1}{4} c_{\gamma} \| \nabla v_2 \|^2 + \frac{4\epsilon}{\gamma} \| \nabla g \|^2.
\] (4.23)

Then it follows from (4.21)-(4.23) that
\[
dt \| \nabla v_2 \|^2 + \sigma \| \nabla v_2 \|^2 \leq \frac{4\epsilon}{\gamma} \| \nabla u \|^2 + \frac{4\epsilon}{\gamma} \| \nabla g \|^2.
\] (4.24)

Multiplying (4.24) by \(e^{\sigma t}\), and then integrating the resulting inequality over \((\tau, \tau - t)\) with \(t \geq 0\), we obtain that
\[
e^{\sigma \tau} \| \nabla v_2(\tau, \tau - t, 0) \|^2 \leq \frac{4\epsilon}{\gamma} \int_{\tau - t}^{\tau} e^{\sigma \xi} \| \nabla u(\xi, \tau - t, u_0(\tau - t)) \|^2 d\xi + \frac{4\epsilon}{\gamma} \int_{\tau - t}^{\tau} e^{\sigma \xi} \| \nabla g(\xi) \|^2 d\xi,
\]
which along with Lemma 4.1 shows that there is \(T = T(\tau, D) > 0\) such that for all \(t \geq T\),
\[
e^{\sigma \tau} \| \nabla v_2(\tau, \tau - t, 0) \|^2 \leq C \int_{-\infty}^{\tau} e^{\sigma \xi}(\| f(\xi) \|^2 + \| \nabla g(\xi) \|^2) d\xi.
\]
The proof is completed. \(\square\)

Next, we establish uniform estimates on the tails of solutions when \(t \to \infty\). We show that the tails of solutions are uniformly small for large space and time variables. These uniform estimates are crucial for proving the pullback asymptotic compactness of the cocycle \(\phi\).

**Lemma 4.5.** Suppose (3.4)-(3.5) and (3.10)-(3.13) hold. Then for every \(\eta > 0\), \(\tau \in \mathbb{R}\) and \(D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_{\sigma}\), there exists \(T = T(\tau, D, \eta) > 0\) and \(K = K(\tau, \eta) > 0\) such that for all \(t \geq T\) and \(k \geq K\),
\[
\int_{|x| \geq k} \left( |u(x, \tau, \tau - t, u_0(\tau - t))|^2 + |v(x, \tau, \tau - t, v_0(\tau - t))|^2 \right) dx \leq \eta,
\]
where \((u_0(\tau - t), v_0(\tau - t)) \in D(\tau - t); K(\tau, \eta)\) depends on \(\tau, \eta\) and the data \((\nu, \lambda, \epsilon, \gamma); T(\tau, D, \eta)\) depends on \(\tau, D, \eta\) and the data \((\nu, \lambda, \epsilon, \gamma)\)

**Proof.** We use a cut-off technique to establish the estimates on the tails of solutions. Let \(\theta\) be a smooth function satisfying \(0 \leq \theta(s) \leq 1\) for \(s \in \mathbb{R}^+\), and
\[
\theta(s) = 0 \text{ for } 0 \leq s \leq 1; \quad \theta(s) = 1 \text{ for } s \geq 2.
\]
Then there exists a constant $C$ such that $|\theta'(s)| \leq C$ for $s \in \mathbb{R}^+$. Taking the inner product of (3.1) with $\varepsilon \theta(\frac{|x|^2}{k^2})u$ in $L^2(\mathbb{R}^n)$, we get

$$
\frac{1}{2} \varepsilon \frac{d}{dt} \int \theta(\frac{|x|^2}{k^2})|u|^2 - \varepsilon \int \theta(\frac{|x|^2}{k^2})u\Delta u + \varepsilon \int \theta(\frac{|x|^2}{k^2})|u|^2 = -\varepsilon \int \theta(\frac{|x|^2}{k^2})h(u)u - \varepsilon \int \theta(\frac{|x|^2}{k^2})uv + \varepsilon \int \theta(\frac{|x|^2}{k^2})uf(t).
$$

(4.25)

Taking the inner product of (3.2) with $\theta(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we find

$$
\frac{1}{2} \frac{d}{dt} \int \theta(\frac{|x|^2}{k^2})|v|^2 + \varepsilon \gamma \int \theta(\frac{|x|^2}{k^2})|v|^2 = \varepsilon \int \theta(\frac{|x|^2}{k^2})uv + \varepsilon \int \theta(\frac{|x|^2}{k^2})vg(t).
$$

(4.26)

Summing up (4.25) and (4.26), by (3.4) we obtain that

$$
\frac{1}{2} \frac{d}{dt} \int \theta(\frac{|x|^2}{k^2}) (|u|^2 + |v|^2) + \varepsilon \int \theta(\frac{|x|^2}{k^2}) (|u|^2 + \gamma|v|^2)
\leq \varepsilon \nu \int \theta(\frac{|x|^2}{k^2})u\Delta u + \varepsilon \int \theta(\frac{|x|^2}{k^2})uf(t) + \varepsilon \int \theta(\frac{|x|^2}{k^2})vg(t).
$$

(4.27)

We now estimate the right-hand side of (4.27). For the second term we have

$$
\varepsilon \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})uf(t) = \varepsilon \int_{|x| \geq k} \theta(\frac{|x|^2}{k^2})uf(t)
\leq \frac{1}{2} \varepsilon \lambda \int_{|x| \geq k} \theta^2(\frac{|x|^2}{k^2}) |u|^2 + \frac{\varepsilon}{2 \lambda} \int_{|x| \geq k} |f(x,t)|^2
\leq \frac{1}{2} \varepsilon \lambda \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^2 + \frac{\varepsilon}{2 \lambda} \int_{|x| \geq k} |f(x,t)|^2.
$$

(4.28)

Similarly, for the last term on the right-hand side of (4.27), we find that

$$
\varepsilon \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})vg(t) \leq \frac{1}{2} \varepsilon \gamma \int \theta(\frac{|x|^2}{k^2}) |v|^2 + \frac{\varepsilon}{2 \gamma} \int_{|x| \geq k} |g(x,t)|^2 dx.
$$

(4.29)

On the other hand, for the first term on the right-hand side of (4.27), by integration by parts, we have

$$
\varepsilon \nu \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})u\Delta u = -\varepsilon \nu \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})|\nabla u|^2 - \varepsilon \nu \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})(\frac{2x}{k^2} \cdot \nabla u)u.
\leq -\varepsilon \nu \int_{k \leq |x| \leq \sqrt{2}k} \theta(\frac{|x|^2}{k^2})(\frac{2x}{k^2} \cdot \nabla u)u \leq \frac{\varepsilon M}{2k} \int_{k \leq |x| \leq \sqrt{2}k} |u||\nabla u| \leq \frac{\varepsilon M}{2k} (\|u\|^2 + ||\nabla u||^2),
$$

(4.30)

where $M$ is independent of $\varepsilon$ and $k$. By (4.27) and (4.28), (4.30), we find that

$$
\frac{d}{dt} \int \theta(\frac{|x|^2}{k^2}) (|u|^2 + |v|^2) + \sigma \int \theta(\frac{|x|^2}{k^2}) (|u|^2 + |v|^2)
\leq \frac{d}{dt} \int \theta(\frac{|x|^2}{k^2}) (|u|^2 + |v|^2) + \sigma \int \theta(\frac{|x|^2}{k^2}) (|u|^2 + |v|^2)
$$
Note that for given \( \eta > 0 \), 

\[
(4.32)
\]

we find that there is 

\[
K_1 = K_1(\tau, \eta) > 0 \quad \text{such that for all} \quad k \geq K_1,
\]

\[
\frac{\epsilon}{\lambda} e^{-\sigma t} \int_{-\infty}^{\tau} \int_{|x| \geq k} e^{\sigma \xi} |f(x, \xi)|^2 \, dx \, d\xi + \frac{\epsilon}{\gamma} e^{-\sigma t} \int_{-\infty}^{\tau} \int_{|x| \geq k} e^{\sigma \xi} |g(x, \xi)|^2 \, dx \, d\xi \leq \left( \frac{1}{\lambda} + \frac{1}{\gamma} \right) \eta. \tag{4.34}
\]

For the last term on the right-hand side of (4.32), it follows from Lemma 1.1 that there is 

\[
T_2 = T_2(\tau, D) > 0 \quad \text{such that for all} \quad t \geq T_2,
\]

\[
\frac{\epsilon M}{k} e^{-\sigma t} \int_{\tau - t}^{\tau} e^{\sigma \xi} \left( \|u(\xi, \tau - t, u_0(\tau - t))\|^2 + \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 \right) \, d\xi
\]

\[
\leq \frac{\epsilon C}{k} e^{-\sigma t} \int_{-\infty}^{\tau} e^{\sigma \xi} \left( \|f(\xi)\|^2 + \|g(\xi)\|^2 \right) \, d\xi.
\]

Therefore, there is 

\[
K_2 = K_2(\tau, \eta) > 0 \quad \text{such that for all} \quad k \geq K_2 \text{ and } t \geq T_2,
\]

\[
\frac{\epsilon M}{k} e^{-\sigma t} \int_{\tau - t}^{\tau} e^{\sigma \xi} \left( \|u(\xi, \tau - t, u_0(\tau - t))\|^2 + \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 \right) \, d\xi \leq \eta. \tag{4.35}
\]
Let $K = \max\{K_1, K_2\}$ and $T = \max\{T_1, T_2\}$. Then by (4.32)-(4.35) we find that there exists a positive constant $C_1$ (independent of $\eta$) such that for all $k \geq K$ and $t \geq T$,

$$
\int \theta \left( \frac{|x|^2}{k^2} \right) \left( |u(x, \tau - t, u_0(\tau - t))|^2 + |v(x, \tau - t, v_0(\tau - t))|^2 \right) \, dx \leq C_1 \eta,
$$

and hence for all $k \geq K$ and $t \geq T$,

$$
\int_{|x| \geq \sqrt{k}} \left( |u(x, \tau - t, u_0(\tau - t))|^2 + |v(x, \tau - t, v_0(\tau - t))|^2 \right) \, dx
\leq \int \theta \left( \frac{|x|^2}{k^2} \right) \left( |u(x, \tau - t, u_0(\tau - t))|^2 + |v(x, \tau - t, v_0(\tau - t))|^2 \right) \, dx \leq C_1 \eta,
$$

which completes the proof.

\section{Existence of pullback attractors}

In this section, we prove, by Proposition 2.7, the existence of a $\mathcal{D}_\sigma$-pullback global attractor for the non-autonomous FitzHugh-Nagumo equations on $\mathbb{R}^n$. To this end, we need to establish the $\mathcal{D}_\sigma$-pullback asymptotic compactness of $\phi$, which is stated as follows.

\begin{lemma}
Suppose (3.1)-(3.5) and (3.10)-(3.13) hold. Then $\phi$ is $\mathcal{D}_\sigma$-pullback asymptotically compact in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}$, $D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma$, and $t_n \to \infty$, $(u_{0,n}, v_{0,n}) \in D(\tau - t_n)$, the sequence $\phi(t_n, \tau - t_n, (u_{0,n}, v_{0,n}))$ has a convergent subsequence in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.
\end{lemma}

\begin{proof}
Given $s \in \mathbb{R}$, $t \geq 0$ and $(u_0, v_0) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, define

$$
\phi_1(t, s, (u_0, v_0)) = (0, v_1(t + s, s, v_0)) \quad \text{and} \quad \phi_2(t, s, (u_0, v_0)) = (u(t + s, s, u_0), v_2(t + s, s, 0)),
$$

where $v_1$ and $v_2$ are solutions to (4.18) and (4.19), respectively, and $(u, v)$ with $v = v_1 + v_2$ is the solution of problem (3.1)-(3.3). It is clear that

$$
\phi(t, s, (u_0, v_0)) = \phi_1(t, s, (u_0, v_0)) + \phi_2(t, s, (u_0, v_0)),
$$

and hence

$$
\phi(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) = \phi_1(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) + \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})). \quad (5.1)
$$

\end{proof}
By (4.20) we get that
\[ \| \phi_1(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \| = e^{-\gamma \tau} e^{\gamma (\tau - t_n)} \| v_0(\tau - t_n) \| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \] (5.2)

Form (5.1)-(5.2) it follows that the sequence \( \phi(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) will have a convergent subsequence in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) as long as \( \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) is precompact. Next we use the uniform estimates on the tails of solutions to establish the precompactness of \( \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), that is, we will prove that for every \( \eta > 0 \), the sequence \( \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) has a finite covering of balls of radii less than \( \eta \). Given \( K > 0 \), denote by
\[ \Omega_K = \{ x : |x| \leq K \} \quad \text{and} \quad \Omega_K^c = \{ x : |x| > K \}. \]

Then by Lemma 4.5 given \( \eta > 0 \), there exist \( K = K(\tau, \eta) > 0 \) and \( T = T(\tau, D, \eta) > 0 \) such that for \( t \geq T \),
\[ \| \phi(t, \tau - t, (u_0(\tau - t), v_0(\tau - t))) \|_{L^2(\Omega_K^c) \times L^2(\Omega_K^c)} \leq \frac{\eta}{8}. \]
Since \( t_n \rightarrow \infty \), there is \( N = N(\tau, D, \eta) > 0 \) such that \( t_n \geq T \) for all \( n \geq N \), and hence we obtain that, for all \( n \geq N \),
\[ \| \phi(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \|_{L^2(\Omega_K^c) \times L^2(\Omega_K^c)} \leq \frac{\eta}{8}. \] (5.3)

It follows from (5.1)-(5.3) that there is \( N_1 = N_1(\tau, D, \eta) \) such that for all \( n \geq N_1 \),
\[ \| \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \|_{L^2(\Omega_K^c) \times L^2(\Omega_K^c)} \leq \frac{\eta}{4}. \] (5.4)

On the other hand, by Lemmas 4.3 and 4.4 there exist \( C = C(\tau, D) > 0 \) and \( N_2(\tau, D) > 0 \) such that for all \( n \geq N_2 \),
\[ \| \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \|_{H^1(\Omega_K) \times H^1(\Omega_K)} \leq C. \] (5.5)

By the compactness of embedding \( H^1(\Omega_K) \hookrightarrow L^2(\Omega_K) \), the sequence \( \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) is precompact in \( L^2(\Omega_K) \times L^2(\Omega_K) \). Therefore, for the given \( \eta > 0 \), \( \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) has a finite covering in \( L^2(\Omega_K) \times L^2(\Omega_K) \) of balls of radii less than \( \eta/4 \), which along with (5.4) shows that \( \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) has a finite covering in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) of balls of radii less than \( \eta \), and thus \( \phi_2(t_n, \tau - t_n, (u_{0,n}, v_{0,n})) \) is precompact in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). The proof is completed. \( \square \)

We are now ready to prove the existence of a pullback attractor for the \( \theta \)-cocycle \( \phi \).

**Theorem 5.2.** Suppose (3.4)-(3.5) and (3.10)-(3.13) hold. Then problem (3.1)-(3.3) has a unique \( D_\sigma \)-pullback global attractor \( \{ A(\tau) \}_{\tau \in \mathbb{R}} \) in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \).
Proposition 2.7 immediately.

\[ \phi(\tau) = 0, \]

Lemma 5.1. Thus the existence of a pullback global attractor for \( \phi \) follows from Proposition 2.7 immediately. \( \square \)

6 Uniform bounds of attractors in \( \epsilon \)

In this section, we investigate the limiting behavior of the random attractor \( \{ \mathcal{A}(\tau) \}_{\tau \in \mathbb{R}} \) for problem \( (3.1)-(3.3) \) when the small parameter \( \epsilon \to 0 \). To indicate the fact that the random attractor depends on \( \epsilon \), hereafter we write the random attractor as \( \{ \mathcal{A}^\epsilon(\tau) \}_{\tau \in \mathbb{R}} \) instead of \( \{ \mathcal{A}(\tau) \}_{\tau \in \mathbb{R}} \). Note that when \( \epsilon = 0 \), \( (3.2) \) reduces to \( \frac{dv}{dt} = 0 \), and hence \( v \) is conserved in this case. This shows that the limiting system with \( \epsilon = 0 \) has no global attractor in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). Based on this fact, one may guess that the random attractor \( \{ \mathcal{A}^\epsilon(\tau) \}_{\tau \in \mathbb{R}} \) blows up as \( \epsilon \to 0 \). In this respect, we will show that the limiting behavior of \( \{ \mathcal{A}^\epsilon(\tau) \}_{\tau \in \mathbb{R}} \) heavily depends on the behavior of the external terms \( f \) and \( g \).

It follows from \( (3.8) \) that for every \( \tau \in \mathbb{R} \), there is \( T = T(\tau) > 0 \) such that for every \( t \geq T \) and \( (u_0, v_0) \in \mathcal{A}^\epsilon(\tau - t) \),

\[
|v(\tau, \tau - t, v_0(\tau - t))|^2 \leq e^{-\sigma \tau} \left( \frac{2\epsilon}{\gamma} \int_{-\infty}^{\tau} e^{\sigma \xi} \|f(\xi)\|^2 d\xi + \frac{2\epsilon}{\lambda} \int_{-\infty}^{\tau} e^{\sigma \xi} \|g(\xi)\|^2 d\xi \right),
\]

which implies that, for \( \tau = 0 \), \( t \geq T \) and \( (u_0, v_0) \in \mathcal{A}^\epsilon(-t) \),

\[
|v(0, -t, v_0(-t))|^2 \leq \frac{2\epsilon}{\lambda} \int_{-\infty}^{0} e^{\sigma \xi} \|f(\xi)\|^2 d\xi + \frac{2\epsilon}{\gamma} \int_{-\infty}^{0} e^{\sigma \xi} \|g(\xi)\|^2 d\xi. \tag{6.1}
\]

Next we illustrate that the right-hand side of \( (6.1) \) is unbounded as \( \epsilon \to 0 \) if \( f \) or \( g \) is unbounded in \( L^2(\mathbb{R}^n) \). To this end, we take

\[
f(x, t) = \sqrt{|t|} f_1(x) \quad \text{and} \quad g(x, t) = \sqrt{|t|} g_1(x), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \tag{6.2}
\]
where \( f_1 \) and \( g_1 \) are given in \( L^2(\mathbb{R}^n) \). It is clear that \( f \) and \( g \) are unbounded in \( L^2(\mathbb{R}^n) \) as \( t \to \pm \infty \).

In this case, the right-hand side of (6.1) is given by

\[
\frac{2\epsilon}{\lambda} \int_{-\infty}^{0} e^{\sigma \xi} \| f(\xi) \|^2 \, d\xi + \frac{2\epsilon}{\gamma} \int_{-\infty}^{0} e^{\sigma \xi} \| g(\xi) \|^2 \, d\xi
\]

\[
= \frac{2\epsilon}{\lambda} \left( \| f_1 \|^2 + \| g_1 \|^2 \right) = 8 \left( \left( \frac{\| f_1 \|^2}{\lambda} + \| g_1 \|^2 \right) \right)
\]

(6.3)

By (6.1)-(6.3) we get that, for \( t \geq T \) and \((u_0, v_0) \in \mathcal{A}(-t)\),

\[
\| v(0, -t, v_0(-t)) \|^2 \leq 8 \frac{\| f_1 \|^2}{\lambda} \left( \frac{\| f_1 \|^2}{\lambda} + \| g_1 \|^2 \right)
\]

(6.4)

Note that the invariance of \( \{ \mathcal{A}^r(\tau) \}_{\tau \in \mathbb{R}} \) implies that

\[
\phi(t, -t, \mathcal{A}^r(-t)) = \mathcal{A}^r(0), \quad \forall \ t \geq 0.
\]

(6.5)

Let \((\tilde{u}, \tilde{v})\) be an arbitrary element in \( \mathcal{A}^r(0) \) and \( t_n \to \infty \). Then it follows from (6.5) that for every \( n \geq 1 \), there exists \((u_{0,n}, v_{0,n}) \in \mathcal{A}^r(-t_n)\) such that

\[
\phi(t_n, -t_n, (u_{0,n}, v_{0,n})) = (\tilde{u}, \tilde{v}),
\]

which implies that

\[
(u(0, -t_n, u_{0,n}), v(0, -t_n, v_{0,n})) = (\tilde{u}, \tilde{v}), \quad \forall \ n \geq 1.
\]

(6.6)

Since \( t_n \to \infty \), there is \( N > 0 \) such that \( t_n \geq T \) for all \( n \geq N \), and hence by (6.4) and (6.6) we have, for all \( n \geq N \),

\[
\| \tilde{v} \|^2 = \| v(0, -t_n, v_{0,n}) \|^2 \leq 8 \frac{\| f_1 \|^2}{\lambda} \left( \frac{\| f_1 \|^2}{\lambda} + \| g_1 \|^2 \right).
\]

(6.7)

Since the right-hand side of (6.7) approaches infinity as \( \epsilon \to 0 \) and \((\tilde{u}, \tilde{v})\) is an arbitrary point in \( \mathcal{A}^r(0) \), we find that the upper bound for \( v \) components of \( \mathcal{A}^r(0) \) becomes unbounded as \( \epsilon \to 0 \).

This shows that it is very likely that \( \mathcal{A}^r(0) \) blows up as \( \epsilon \to 0 \) for such unbounded \( f \) and \( g \) given in (6.2).

Now the question is what happens if \( f \) and \( g \) are bounded in \( L^2(\mathbb{R}^n) \). As we will see later, in this case, we can show that the right-hand side of (6.1) is uniformly bounded in \( \epsilon \), and hence the random attractor does not blows up. To prove this result, we need the uniform estimates of solutions with respect to \( \epsilon \).
In this section, we agree that $K_i$ ($i \in \mathbb{N}$) are any positive constants which depend only on the data $(\nu, \lambda, \gamma)$, but not on $\epsilon$; while $C_i$ ($i \in \mathbb{N}$) are any positive constants which may depend on the parameters $\epsilon$, $\nu$, $\lambda$ and $\gamma$.

**Lemma 6.1.** Suppose $f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$, $g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n))$ and (6.4)-(6.5) hold. Then for every $D = \{D(t)\}_{t \in \mathbb{R}} \in D_\sigma$, $\tau \in \mathbb{R}$ and $t \geq 0$, the following holds for all $\xi \geq \tau - t$,

$$
\|v(\xi, \tau - t, v_0(\tau - t))\|^2 \leq e^{-\sigma \xi} e^{\sigma(\tau - t)} (\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2) + K,
$$

and

$$
\epsilon \int_{\tau - t}^\tau e^{\sigma \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi \leq \frac{1}{2\nu} e^{\sigma(\tau - t)} (\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2) + Ke^{\sigma \tau},
$$

where $K$ is a positive constant depending on the data $(\nu, \lambda, \gamma)$, but not on $\epsilon$ or $\tau$.

**Proof.** Since $f$ and $g$ are bounded in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, respectively, by (4.6) we have

$$
\frac{d}{dt} (\epsilon \|u\|^2 + \|v\|^2) + \sigma (\epsilon \|u\|^2 + \|v\|^2) + 2\epsilon \nu \|\nabla u\|^2 \leq K_1 \epsilon.
$$

Multiplying the above by $e^{\sigma t}$ and then integrating the resulting inequality over $(\tau - t, \xi)$, we get

$$
e^{\sigma \xi} (\epsilon \|u(\xi, \tau - t, u_0(\tau - t))\|^2 + \|v(\xi, \tau - t, v_0(\tau - t))\|^2)
+ 2\epsilon \nu \int_{\tau - t}^\xi e^{\sigma s} \|\nabla u(s, \tau - t, u_0(\tau - t))\|^2 ds
\leq e^{\sigma(\tau - t)} (\epsilon \|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2) + K_1 \epsilon \int_{\tau - t}^\xi e^{\sigma s} ds
\leq e^{\sigma(\tau - t)} (\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2) + K_1 \epsilon \int_{-\infty}^\xi e^{\sigma s} ds
\leq e^{\sigma(\tau - t)} (\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2) + \frac{K_1 \epsilon}{\sigma} e^{\sigma \xi}. \tag{6.8}
$$

Note that $\sigma = \frac{1}{2} \epsilon \gamma$. Then it follows from (6.8) that

$$
\|v(\xi, \tau - t, v_0(\tau - t))\|^2 + 2\epsilon \nu e^{-\sigma \xi} \int_{\tau - t}^\xi e^{\sigma s} \|\nabla u(s, \tau - t, u_0(\tau - t))\|^2 ds
\leq e^{-\sigma \xi} e^{\sigma(\tau - t)} (\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2) + \frac{2K_1}{\gamma}. \tag{6.9}
$$
Particularly, if $\xi = \tau$, by (6.9) we get that
\[
\int_{\tau-t}^{\tau} e^{\sigma \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi \\
\leq \frac{1}{2\nu} e^{-\sigma \tau} e^{\sigma (\tau - t)} \left( \|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2 \right) + \frac{K_1}{\nu \gamma},
\]
which along with (6.9) completes the proof. \qed

**Lemma 6.2.** Suppose $f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$, $g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n))$ and (3.4)-(3.5) hold. Then for every $D = \{D(t)\}_{t \in \mathbb{R}} \in D_\sigma$, $\tau \in \mathbb{R}$ and $t \geq 0$, we have
\[
\|u(\tau, \tau - t, u_0(\tau - t))\|^2 + 2\nu e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi \\
\leq C e^{-\alpha \tau} e^{\alpha (\tau - t)} \left( \|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2 \right) + K,
\]
where $K$ is a positive constant depending on the data $(\nu, \lambda, \gamma)$, but not on $\epsilon$ or $\tau$; while $C$ depends on the data $(\nu, \lambda, \gamma)$ as well as $\epsilon$, but not on $\tau$.

**Proof.** Taking the inner product of (3.4) with $u$ in $L^2(\mathbb{R}^n)$, we get that
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|\nabla u\|^2 + \lambda \|u\|^2 + (h(u), u) = -(u, v) + (f(t), u). \tag{6.10}
\]
Note that the right-hand side of (6.10) is bounded by
\[
\|u\|\|v\| + \|f(t)\|\|u\| \leq \frac{1}{2} \lambda \|u\|^2 + \frac{1}{\lambda} \|v\|^2 + \frac{1}{\lambda} \|f(t)\|^2. \tag{6.11}
\]
By (6.10)-(6.11) and (3.4), we obtain that,
\[
\frac{d}{dt} \|u\|^2 + 2\nu \|\nabla u\|^2 + \lambda \|u\|^2 \leq \frac{2}{\lambda} \|v\|^2 + \frac{2}{\lambda} \|f(t)\|^2. \tag{6.12}
\]
Multiplying (6.12) by $e^{\lambda \xi}$ and then integrating the resulting inequality over $(\tau - t, \tau)$ with $t \geq 0$, we obtain that
\[
\|u(\tau, \tau - t, u_0(\tau - t))\|^2 + 2\nu e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi \\
\leq e^{-\lambda \tau} \|u_0(\tau - t)\|^2 + \frac{2}{\lambda} e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda \xi} \|v(\xi, \tau - t, v_0(\tau - t))\|^2 d\xi + \frac{2}{\lambda} e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda \xi} \|f(\xi)\|^2 d\xi. \tag{6.13}
\]
Note that $f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$. By (6.13) and Lemma 6.1 we find that
\[
\|u(\tau, \tau - t, u_0(\tau - t))\|^2 + 2\nu e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi 
\]
\[
\leq e^{-\lambda t}\|u_0(\tau - t)\|^2 + K_1 e^{-\lambda \tau} \int_{\tau - t}^\tau e^{\lambda \xi} d\xi \\
+ K_2 e^{-\lambda \tau} e^{\sigma(\tau - t)} \left(\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2\right) \int_{\tau - t}^\tau e^{(\lambda - \sigma)\xi} d\xi.
\]
\[
\leq e^{-\lambda t}\|u_0(\tau - t)\|^2 + K_3 + \frac{K_2}{\lambda - \sigma} e^{-\sigma t} \left(\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2\right).
\]
Note that \(\lambda > \sigma\). Then it follows from the above that
\[
\|u(\tau, \tau - t, u_0(\tau - t))\|^2 + 2\nu e^{-\lambda t} \int_{\tau - t}^\tau e^{\lambda \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi
\]
\[
\leq K_3 + \left(1 + \frac{K_2}{\lambda - \sigma}\right) e^{-\sigma t} \left(\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2\right),
\] (6.14)
which completes the proof. \(\square\)

**Lemma 6.3.** Suppose \(f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))\), \(g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n))\) and (6.4)–(6.5) hold. Then for every \(D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma\), \(\tau \in \mathbb{R}\) and \(t \geq 1\), we have
\[
e^{-\lambda \tau} \int_{\tau - 1}^\tau e^{\lambda \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi \leq C e^{-\sigma \tau} e^{\sigma(\tau - t)} \left(\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2\right) + K,
\]
where \(K\) is a positive constant depending on the data \((\nu, \lambda, \gamma)\), but not on \(\epsilon\) or \(\tau\); while \(C\) depends on the data \((\nu, \lambda, \gamma)\) as well as \(\epsilon\), but not on \(\tau\).

**Proof.** By (6.12) we find that
\[
\frac{d}{dt}\|u\|^2 + \lambda \|u\|^2 \leq \frac{2}{\lambda} \|v\|^2 + \frac{2}{\lambda} \|f(t)\|^2.
\] (6.15)
Using \(f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))\) and repeating the proof of (6.14) we can get from (6.15) that
\[
\|u(\tau - 1, \tau - t, u_0(\tau - t))\|^2 \leq C_1 e^{-\sigma \tau} e^{\sigma(\tau - t)} \left(\|u_0(\tau - t)\|^2 + \|v_0(\tau - t)\|^2\right) + K_1.
\] (6.16)
Integrating (6.12) over \((\tau - 1, \tau)\), by Lemma 6.1 we have
\[
e^{\lambda \tau} \|u(\tau, \tau - t, u_0(\tau - t))\|^2 + 2\nu \int_{\tau - 1}^\tau e^{\lambda \xi} \|\nabla u(\xi, \tau - t, u_0(\tau - t))\|^2 d\xi
\]
\[
\leq e^{\lambda(\tau - 1)} \|u(\tau - 1, \tau - t, u_0(\tau - t))\|^2
\]
\[
+ \frac{2}{\lambda} \int_{\tau - 1}^\tau e^{\lambda \xi} \|v(\xi, \tau - t, v_0(\tau - t))\|^2 d\xi + \int_{\tau - 1}^\tau e^{\lambda \xi} \|f(\xi)\|^2 d\xi
\]
\[
\leq e^{\lambda(\tau - 1)} \|u(\tau - 1, \tau - t, u_0(\tau - t))\|^2
\]
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\[
\begin{align*}
+ \frac{2}{\lambda} e^{\sigma(t-t)} \left( \|u_0(t-t)\|^2 + \|v_0(t-t)\|^2 \right) \int_{\tau-1}^{\tau} e^{(\lambda-\sigma)\xi} d\xi + K_2 \int_{\tau-1}^{\tau} e^{\lambda\xi} d\xi \\
\leq e^{\lambda(\tau-1)}\|u(t-1, t-t, u_0(t-t))\|^2 \\
+ \frac{2}{\lambda(\lambda-\sigma)} e^{\lambda(t-t)} \left( \|u_0(t-t)\|^2 + \|v_0(t-t)\|^2 \right) + \frac{K_2}{\lambda} e^{\lambda\tau},
\end{align*}
\]

which along with (6.16) implies that
\[
\begin{align*}
e^{\lambda\tau}\|u(t, t-t, u_0(t-t))\|^2 + 2\nu \int_{\tau-1}^{\tau} e^{\lambda\xi}\|\nabla u(\xi, t-t, u_0(t-t))\|^2 d\xi \\
\leq C_2 e^{\lambda\tau-t} \left( \|u_0(t-t)\|^2 + \|v_0(t-t)\|^2 \right) + K_3 e^{\lambda\tau}.
\end{align*}
\]

Then Lemma 6.3 follows from the above immediately. \(\square\)

Next, we derive uniform estimates in \(\epsilon\) for the \(u\) components of the solutions of problem (3.1)-(3.3) in \(H^1(\mathbb{R}^n)\).

**Lemma 6.4.** Suppose \(f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))\), \(g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n))\) and (3.4) - (3.5) hold. Then for every \(D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma, \tau \in \mathbb{R}\) and \(t \geq 1\), we have
\[
\|\nabla u(\tau, t-t, u_0(t-t))\|^2 \leq C e^{-\sigma t} e^{\sigma(t-t)} \left( \|u_0(t-t)\|^2 + \|v_0(t-t)\|^2 \right) + K,
\]
where \(K\) is a positive constant depending on the data \((\nu, \lambda, \gamma)\), but not on \(\epsilon\) or \(\tau\); while \(C\) depends on the data \((\nu, \lambda, \gamma)\) and \(\epsilon\), but not on \(\tau\).

**Proof.** Note that (4.15) implies that
\[
\frac{d}{dt} \|\nabla u\|^2 \leq K_1 \|\nabla u\|^2 + \frac{2}{\nu} \|v\|^2 + \frac{2}{\nu} \|f(t)\|^2.
\]

Since \(f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))\) we get that
\[
\frac{d}{dt} \|\nabla u\|^2 + \lambda \|\nabla u\|^2 \leq (\lambda + K_1) \|\nabla u\|^2 + \frac{2}{\nu} \|v\|^2 + K_2. \tag{6.17}
\]

Multiplying (6.17) by \(e^{\lambda t}\) and then integrating the resulting inequality over \((s, \tau)\) with \(s \in (\tau-1, \tau)\), we find that for all \(t \geq 1\),
\[
\begin{align*}
e^{\lambda\tau} \|\nabla u(\tau, t-t, u_0(t-t))\|^2 &\leq e^{\lambda s} \|\nabla u(s, t-t, u_0(t-t))\|^2 \\
+ (\lambda + K_1) \int_{s}^{\tau} e^{\lambda\xi} \|\nabla u(\xi, t-t, u_0(t-t))\|^2 d\xi
\end{align*}
\]

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\[ + \frac{2}{\nu} \int_{s}^{\tau} e^{\lambda \xi} \| v(\xi, \tau - t, v_0(t - t)) \|^2 d\xi + K_2 \int_{s}^{\tau} e^{\lambda \xi} d\xi \]
\[\leq e^{\lambda s} \| \nabla u(s, \tau - t, u_0(t - t)) \|^2 + (\lambda + K_1) \int_{\tau - 1}^{\tau} e^{\lambda \xi} \| \nabla u(\xi, \tau - t, u_0(t - t)) \|^2 d\xi \]
\[+ \frac{2}{\nu} \int_{\tau - 1}^{\tau} e^{\lambda \xi} \| v(\xi, \tau - t, v_0(t - t)) \|^2 d\xi + \frac{K_2}{\lambda} e^{\lambda \tau}. \quad (6.18)\]

We now integrate (6.18) with respect to \( s \) on \((\tau - 1, \tau)\) to get
\[ e^{\lambda \tau} \| \nabla u(\tau, \tau - t, u_0(t - t)) \|^2 \leq \int_{\tau - 1}^{\tau} e^{\lambda \xi} \| \nabla u(s, \tau - t, u_0(t - t)) \|^2 ds \]
\[+ (\lambda + K_1) \int_{\tau - 1}^{\tau} e^{\lambda \xi} \| \nabla u(\xi, \tau - t, u_0(t - t)) \|^2 d\xi \]
\[+ \frac{2}{\nu} \int_{\tau - 1}^{\tau} e^{\lambda \xi} \| v(\xi, \tau - t, v_0(t - t)) \|^2 d\xi + \frac{K_2}{\lambda} e^{\lambda \tau}. \quad (6.19)\]

By Lemma 6.3, the first two terms on the right-hand side of (6.19) satisfy
\[ \int_{\tau - 1}^{\tau} e^{\lambda s} \| \nabla u(s, \tau - t, u_0(t - t)) \|^2 ds + (\lambda + K_1) \int_{\tau - 1}^{\tau} e^{\lambda \xi} \| \nabla u(\xi, \tau - t, u_0(t - t)) \|^2 d\xi \]
\[\leq C_1 e^{\lambda \tau - \sigma t} (\| u_0(\tau - t) \|^2 + \| v_0(\tau - t) \|^2) + K_3 e^{\lambda \tau}. \quad (6.20)\]

On the other hand, by Lemma 6.1, for the third term on the right-hand side of (6.19) we have
\[ \frac{2}{\nu} \int_{\tau - 1}^{\tau} e^{\lambda \xi} \| v(\xi, \tau - t, v_0(t - t)) \|^2 d\xi \]
\[\leq \frac{2}{\nu} e^{\sigma(\tau - t)} (\| u_0(\tau - t) \|^2 + \| v_0(\tau - t) \|^2) \int_{\tau - 1}^{\tau} e^{\lambda - \sigma \xi} d\xi + \frac{2}{\nu} K_4 \int_{\tau - 1}^{\tau} e^{\lambda \xi} d\xi \]
\[\leq \frac{2}{\nu(\lambda - \sigma)} e^{\lambda \tau - \sigma t} (\| u_0(\tau - t) \|^2 + \| v_0(\tau - t) \|^2) + \frac{2K_4}{\nu\lambda} e^{\lambda \tau}. \quad (6.21)\]

Then it follows from (6.19)-(6.21) that
\[ e^{\lambda \tau} \| \nabla u(\tau, \tau - t, u_0(t - t)) \|^2 \leq C_2 e^{\lambda \tau - \sigma t} (\| u_0(\tau - t) \|^2 + \| v_0(\tau - t) \|^2) + K_5 e^{\lambda \tau}, \]
which completes the proof. \(\square\)

The following result is concerned with the uniform estimates in \( \epsilon \) for solutions of problem (4.19).
Lemma 6.5. Suppose \( f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)) \), \( g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)) \) and \((3.4)\)–\((3.5)\) hold. Then for every \( D = \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma \), \( \tau \in \mathbb{R} \) and \( t \geq 1 \), we have

\[
\| \nabla v_2(\tau, \tau - t, 0) \|^2 \leq C e^{-\sigma \tau} e^{\sigma (\tau - t)} \left( \| u_0(\tau - t) \|^2 + \| v_0(\tau - t) \|^2 \right) + K,
\]

where \( K \) is a positive constant depending on the data \((\nu, \lambda, \gamma)\), but not on \( \epsilon \) or \( \tau \); while \( C \) depends on the data \((\nu, \lambda, \gamma)\) and \( \epsilon \), but not on \( \tau \).

Proof. By \( g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)) \) and \((4.24)\) we get that

\[
\frac{d}{dt}\| \nabla v_2 \|^2 + \sigma \| \nabla v_2 \|^2 \leq 4 \frac{\epsilon}{\gamma} \| \nabla u \|^2 + \epsilon K_1. \tag{6.22}
\]

Multiplying \((6.22)\) by \( e^{\sigma t} \) and then integrating the resulting inequality over \((\tau - t, \tau)\), we obtain that

\[
e^{\sigma \tau} \| \nabla v_2(\tau, \tau - t, 0) \|^2 \leq \frac{4 \epsilon}{\gamma} \int_{\tau-t}^{\tau} e^{\sigma \xi} \| \nabla u(\xi, \tau - t, u_0(\tau - t)) \|^2 d\xi + \epsilon K_1 \int_{\tau-t}^{\tau} e^{\sigma \xi} d\xi.
\]

Note that \( \sigma = \frac{1}{2} \epsilon \gamma \). Then by Lemma 6.1 we find that

\[
e^{\sigma \tau} \| \nabla v_2(\tau, \tau - t, 0) \|^2 \leq C_1 e^{\sigma (\tau - t)} \left( \| u_0(\tau - t) \|^2 + \| v_0(\tau - t) \|^2 \right) + K_2 e^{\sigma \tau},
\]

which completes the proof. \qed

As an immediate consequence of \((4.20)\) and Lemmas 6.1, 6.2, 6.4 and 6.5 we have the following uniform estimates.

Corollary 6.6. Suppose \( f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)) \), \( g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)) \) and \((3.4)\)–\((3.5)\) hold. Then for every \( D = \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma \), \( \tau \in \mathbb{R} \) and \( t \geq 1 \), we have

\[
\| u(\tau, \tau - t, u_0(\tau - t)) \|^2 _{H^1} + \| v_2(\tau, \tau - t, 0) \|^2 _{H^1} \leq C e^{-\sigma \tau} e^{\sigma (\tau - t)} \left( \| u_0(\tau - t) \|^2 + \| v_0(\tau - t) \|^2 \right) + K,
\]

where \( K \) is a positive constant depending on the data \((\nu, \lambda, \gamma)\), but not on \( \epsilon \) or \( \tau \); while \( C \) depends on the data \((\nu, \lambda, \gamma)\) and \( \epsilon \), but not on \( \tau \).

We are now ready to show that the union of the random attractor \( \{ \mathcal{A}(\tau) \}_{\tau \in \mathbb{R}} \) is bounded in \( H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \).
Theorem 6.7. Suppose \( f \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)) \), \( g \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)) \) and \([3.4], [3.5]\) hold. Let \( \epsilon_0 < \min\{1, \frac{1}{2}\} \) be a fixed positive number. Then the set \( \bigcup_{0 < \epsilon \leq \epsilon_0} \bigcup_{\tau \in \mathbb{R}} A^\epsilon(\tau) \) is bounded in \( H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \). More precisely, there exists a constant \( K \), depending only on the data \((\nu, \lambda, \gamma)\) but not on \( \epsilon \), such that for every \( \tau \in \mathbb{R} \), \( \epsilon \) with \( 0 < \epsilon \leq \epsilon_0 \) and \((u^\epsilon, \nu^\epsilon) \in A^\epsilon(\tau)\),

\[
\|u^\epsilon\|_{H^1(\mathbb{R}^n)} + \|\nu^\epsilon\|_{H^1(\mathbb{R}^n)} \leq K.
\]

Proof. For fixed \( \epsilon > 0 \), denote the solutions of \([4.18] \) and \([4.19] \) by \( v_1^\epsilon \) and \( v_2^\epsilon \), respectively. Then for every \( \tau \in \mathbb{R} \) and \( t \geq 0 \), the solution \((u^\epsilon, \nu^\epsilon)\) of problem \([3.1]-[3.3]\) with initial condition \((u_0, v_0)\) at \( \tau - t \) can be written as, for all \( \xi \geq \tau - t \),

\[
(u^\epsilon(\xi, \tau - t, u_0), \nu^\epsilon(\xi, \tau - t, v_0)) = (u^\epsilon(\xi, \tau - t, u_0), v_2^\epsilon(\xi, \tau - t, 0)) + (0, v_1^\epsilon(\xi, \tau - t, v_0)).
\]  

(6.23)

Take a sequence \( \{t_n\}_{n=1}^\infty \) such that \( t_n \geq 1 \) and \( t_n \rightarrow \infty \). Then given \( \tau \in \mathbb{R} \) and \((u^\epsilon, \nu^\epsilon) \in A^\epsilon(\tau)\), by the invariance of the random attractor, we find that there exists a sequence \( \{(u_0^0(\tau - t_n), v_0^0(\tau - t_n))\} \in A^\epsilon(\tau - t_n) \) such that

\[
(u^\epsilon, \nu^\epsilon) = (u^\epsilon(\tau, \tau - t_n, u_0^0(\tau - t_n)), v^\epsilon(\tau, \tau - t_n, v_0^0(\tau - t_n))).
\]  

(6.24)

It follows from \([6.23] \times [6.24] \) that

\[
(u^\epsilon, \nu^\epsilon) = (u^\epsilon(\tau, \tau - t_n, u_0^0(\tau - t_n)), v_2^\epsilon(\tau, \tau - t_n, 0)) + (0, v_1^\epsilon(\tau, \tau - t_n, v_0^0(\tau - t_n))).
\]  

(6.25)

By Corollary 6.6 we have for all \( n \geq 1 \),

\[
\|u^\epsilon(\tau, \tau - t_n, u_0^0(\tau - t_n))\|_{H^1}^2 + \|v_2^\epsilon(\tau, \tau - t_n, 0)\|_{H^1}^2 \\
\leq Ce^{-\sigma_\tau} e^{\sigma(\tau - t_n)} \left( \|u_0(\tau - t_n)\| \|v_0(\tau - t_n)\| \right) + K,
\]

(6.26)

where \( K \) is a positive constant depending only on the data \((\nu, \lambda, \gamma)\), but not on \( \epsilon \) or \( \tau \). Note that the first term on the right-hand side of \([6.26] \) approaches zero as \( n \rightarrow \infty \), and hence there exists \( N = N(\epsilon, \tau) \) such that for all \( n \geq N \),

\[
\|u^\epsilon(\tau, \tau - t_n, u_0^0(\tau - t_n))\|_{H^1}^2 + \|v_2^\epsilon(\tau, \tau - t_n, 0)\|_{H^1}^2 \leq 2K,
\]

(6.27)

which implies that there is \((\tilde{u}^\epsilon, \tilde{\nu}^\epsilon) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)\) such that, up to a subsequence,

\[
(u^\epsilon(\tau, \tau - t_n, u_0^0(\tau - t_n)), v_2^\epsilon(\tau, \tau - t_n, 0)) \rightarrow (\tilde{u}^\epsilon, \tilde{\nu}^\epsilon) \text{ weakly in } H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n),
\]

(6.28)
as \( n \to \infty \). By (6.27)–(6.28) we have

\[
\|(\tilde{u}^{\epsilon, \tau}, \tilde{v}^{\epsilon, \tau})\|_{H^1 \times H^1} \leq \liminf_{n \to \infty} \|(u^\epsilon(\tau, \tau - t_n), u_0^\epsilon(\tau - t_n), v_2^\epsilon(\tau, \tau - t_n, 0))\|_{H^1 \times H^1} \leq \sqrt{2K}.
\]

(6.29)

Since the weak convergence in \( H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \) implies the weak convergence in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), by (6.28) we have

\[
(u^\epsilon(\tau, \tau - t_n, u_0^\epsilon(\tau - t_n)), v_2^\epsilon(\tau, \tau - t_n, 0)) \to (\tilde{u}^{\epsilon, \tau}, \tilde{v}^{\epsilon, \tau}) \text{ weakly in } L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n).
\]

(6.30)

On the other hand, by (4.20) we find that

\[
\|v_1^\epsilon(\tau, \tau - t_n, v_0^\epsilon(\tau - t_n))\| = e^{-\epsilon\gamma \tau} e^{\epsilon \gamma (\tau - t_n)} \|v_0^\epsilon(\tau - t_n)\| \to 0.
\]

(6.31)

Taking the weak limit of (6.25) in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) as \( n \to \infty \), by (6.30) and (6.31) we obtain

\[
(u^{\epsilon, \tau}, v^{\epsilon, \tau}) = (\tilde{u}^{\epsilon, \tau}, \tilde{v}^{\epsilon, \tau}).
\]

(6.32)

Then it follows from (6.29) and (6.32) that, for every \( \epsilon > 0, \tau \in \mathbb{R} \) and \((u^{\epsilon, \tau}, v^{\epsilon, \tau}) \in \mathcal{A}^\epsilon(\tau),

\[
\|(u^{\epsilon, \tau}, v^{\epsilon, \tau})\|_{H^1 \times H^1} \leq \sqrt{2K}.
\]

Note that \( K \) is independent of \( \epsilon \) and \( \tau \), and thus the proof is completed. \( \square \)

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