Diagonalization of Polynomial-Time Turing Machines
Via Nondeterministic Turing Machine

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Abstract

The diagonalization technique was invented by Cantor to show that there are
more real numbers than algebraic numbers, and is very important in computer
science. In this work, we enumerate all polynomial-time deterministic Turing ma-
chines and diagonalize over all of them by an universal nondeterministic Turing
machine. As a result, we obtain that there is a language \( L_d \) not accepted by any
polynomial-time deterministic Turing machines but accepted by a nondeterministic
Turing machine working within \( O(n^k) \) for any \( k \in \mathbb{N}_1 \), i.e. \( L_d \in NP \). That is, we
present a proof that \( P \) and \( NP \) differs.

Key words: Diagonalization, Polynomial-Time deterministic Turing machine,
Universal nondeterministic Turing machine

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1 Introduction

In 1936, Turing’s seminal work [Tur37] opened the door of computer science, which today develops to many subfields such as computability theory, formal language and automata theory, computational complexity theory, algorithm theory, and so on. On the one hand, Turing’s contributions are so huge that he has been considered as the father of computer. On the other hand, although Turing’s work initiates the study of computer science, he was not concerned with the efficiency of his machines, which is the main topic in computational complexity theory. In fact, Turing’s concern [Tur37] was whether they can simulate arbitrary algorithms given sufficient time.

Mention of computational complexity theory, it is a central subfield of the theoretical foundations of computer science. As mentioned above, the computational complexity theory mainly concerns with the efficiency of Turing machine, or the intrinsic complexity of computational tasks. It specifically deals with fundamental questions such as, what is feasible computation, and what can and can not be computed with a reasonable amount of computational resources in terms of time or space. Additionally, one of the most important classical complexity classes is arguably $NP$, i.e. nondeterministic polynomial-time. This class comprises languages that can be computed in polynomial-time by a nondeterministic Turing machine.

Perhaps, the most basic and fundamental measures of difficulty that appear particularly important are time and space. The fundamental measure of time opened the door to the study of the extremely expressive time complexity class $NP$. The famous Cook-Levin theorem [Coo71, Lev73] shows that this class has complete problems, which states that the Satisfiability is $NP$-complete, that is, Satisfiability is in $NP$ and any other language in $NP$ can be reduced to it in polynomial-time. This result opened the door to the research of the rich theory of $NP$-completeness [Kar72].

In computational complexity theory, the technique of diagonalization led to some of early successes. Historically, it was invented by Cantor to show that there are more real numbers than algebraic numbers, then it was used by Turing to show that the halting problem is undecidable [Tur37], and then refined to prove computational complexity lower bounds. A typical theorem in this area is that more time/space buys more computational power [Coo73, FS07, Pap94], e.g. there are functions computable in time $n^2$, say, which are not computable in time $n$. The heart of such arguments is the existence of a universal Turing machine, which can simulate every other Turing machines with only small loss in efficiency.

The famous $P$ versus $NP$ question is a major question in computational complexity theory, and it is also considered as the most fundamental question in computer science, which asks whether every problem in $NP$ can also be solved in polynomial-time by deterministic Turing machine. It appeared explicitly first in the papers of Cook [Coo71] and Levin [Lev73]. We refer the readers to [Wig07] for more introductions on $P$ versus $NP$ question. As far as we known, many of the methods that have been used to attack the $P$ versus $NP$ question (in direction to $P \neq NP$) have been combinatorial, or algebraic;
this includes, for instance, circuit lower bound, diagonalization, and relativization, but all of these attempts reach a failure, see [Coo00, Wig07]. There is an important doubt: can diagonalization resolve $P$ versus $NP$ question? Researchers realized in the 1970s that diagonalization alone may not resolve the $P$ versus $NP$ question, it needs other new techniques besides diagonalization. To the best of our knowledge, we will say that the above viewpoint is a misunderstanding. In the following, Figure 1 which is from [A1] illustrates the Euler diagram for $P$, $NP$, $NP$-complete and $NP$-Hard set of problems in two possibilities:  

![Euler diagram](image)

**Figure 1:** Euler diagram for $P$, $NP$, $NP$-complete and $NP$-Hard set of problems

Our motivation here is that in [Lad75], Lander constructed a language which is $NP$-intermediate by the method of *lazy diagonalization* under the assumption that $P$ and $NP$ differs. The lazy diagonalization put the $NP$-intermediate language accepted by no polynomial-time deterministic Turing machine, see [Pap94, AB09]. Since it is a widespread belief that $P$ and $NP$ are different, we naturally will consider that can we use an universal nondeterministic Turing machine to diagonalize over all polynomial-time deterministic Turing machines to produce a language accepted by no polynomial-time deterministic Turing machine but accepted by a nondeterministic Turing machine? If so, we can further study whether this language is in $NP$, i.e. whether this nondeterministic Turing machine runs in polynomial-time.

In this work, we put our motivation mentioned above or ideas into practice. We enumerate all polynomial-time deterministic Turing machines and diagonalize over all of them by an universal nondeterministic Turing machine. Generally, people use universal deterministic Turing machine to diagonalize over a list of deterministic Turing machines, for example [Pap94, AB09], or use universal nondeterministic Turing machine to diagonalize over a list of nondeterministic Turing machines, for instance [Coo73]. Using an universal nondeterministic Turing machine to diagonalize a list of deterministic Turing machines appears less often even never except in author’s recent work [Lin21], and it is a new attempt that may lead to successes. As an amazing result, we obtain the following important theorem:
Theorem 1. There exists a language \( L_d \) which is not accepted by all polynomial-time deterministic Turing machines, but accepted by a nondeterministic Turing machine. Furthermore, this nondeterministic Turing machine runs in \( O(n^k) \) for any \( k \in \mathbb{N}_1 \), i.e. \( L_d \in \text{NP} \).

From which it immediately follows that

Corollary 1. \( P \neq \text{NP} \).

and

Corollary 2. \( P \neq \text{PSPACE} \).

1.1 Our approach

The Cook-Levin Theorem is a well known theorem stating that Satisfiability (SAT) is complete for \( \text{NP} \) under polynomial-time many-one reductions. Starting from this, on the one hand, if one wants to prove that \( P \) and \( \text{NP} \) are identical, then one can try to design polynomial-time algorithms for SAT. On the other hand, if one wants to show that \( P \) and \( \text{NP} \) differs, then one may try to prove a super-polynomial lower-bound for SAT.

As a novel thought and attempt, we enumerate all polynomial-time deterministic Turing machines and diagonalize over all of them by an universal nondeterministic Turing machine. Generally, a common practice is to use an universal deterministic Turing machine to diagonalize over a list of deterministic Turing machines [Pap94, AB09], or take advantage of an universal nondeterministic Turing machine to diagonalize over a list of nondeterministic Turing machines [Coo73]. Using an universal nondeterministic Turing machine to diagonalize over all of polynomial-time deterministic Turing machines is a new attempt. We would like to stress that the approach had never been tried by anybody before in the literature.

1.2 Related work

As is well known, a central problem in computational complexity theory is the \( P \) versus \( \text{NP} \) question, which is to determine whether every language accepted by some nondeterministic Turing machine in polynomial-time is also accepted by some deterministic Turing machine in polynomial-time. In this Section, we will review its history and related works. With respect to its importance, we refer the reader to the reference [Coo00, Wig07].

In 1971, Cook [Coo71] introduced a notion of \( \text{NP} \)-completeness as a polynomial-time analog of c.e.-completeness, except that the reduction used was a polynomial-time analog of Turing reducibility rather than of many-one reducibility (see [HR67] Chapter 7). Besides the first well-known \( \text{NP} \)-complete problem of Satisfiability, Cook also showed in [Coo71] that several natural problems, including 3-SAT and subgraph isomorphism are \( \text{NP} \)-complete.

A year later stimulated by the work of Cook [Coo71], Karp [Kar72] used these completeness results to show that 20 other natural problems are \( \text{NP} \)-complete, forcefully demonstrating the importance of the subject. Thus far, there are many problems shown to be \( \text{NP} \)-complete, see excellent reference [GJ79] to this subject. In his paper [Kar72], Karp
also introduced the now standard notation $P$ and $NP$ and redefined $NP$-completeness by using the polynomial-time analog of many-one reducibility, which has become standard. Meanwhile Levin [Lev73], independently of Cook [Coo71] and Karp [Kar72], defined the notion of “universal search problem”, similar to the $NP$-complete problem, and gave six examples, which includes Satisfiability.

Although the $P$ versus $NP$ question was formally defined in the 1970s, there were previous inklings of the problems involved. A mention of the underlying problem occurred in a 1956 letter written by K. Gödel to J. von Neumann. Gödel asked whether theorem-proving could be solved in quadratic or linear time (see [Har89]). We also note that, besides the classical version of the question, there is one expressed in terms of the field of complex numbers, which catches interest in the mathematics community [BCSS98].

1.3 Overview

The remainder of this work is organized as follows: For convenience of the reader, in the next Section we review some notions closely associated with our discussions and fix some notation we will use in the following context; Also, some useful technical lemmas are presented. In Section 3 we provide a method to encode a polynomial-time deterministic Turing machine, so that we can enumerate all the polynomial-time deterministic Turing machines and we will prove that in terms of our enumeration, all polynomial-time deterministic Turing machines are in the enumeration. The proof of our main result is put into Section 4. Some discussions about our result are placed in Section 5 and in Section 6, we recall some background information how our techniques formed and present another proof of that $P$ and $NP$ differs. Finally, we draw some conclusions in the last Section.

2 Preliminaries

In this Section, we describe the notation and notions needed in the following context. We would like to remark that our style of writing from this Section is heavily influenced by that in Aho, Hopcroft and Ullman’s book [AHU74, HU79].

Let $\mathbb{N}$ denote the natural numbers $\{0, 1, 2, 3, \cdots\}$ where $+\infty \notin \mathbb{N}$. Further, $\mathbb{N}_1$ denotes the set of $\mathbb{N} - \{0\}$.

The computation model we use is the Turing machine as it defined in standard textbooks such as [HU69, HU79, HMU06]. Here, we adopt the definition given in [AHU74]:

**Definition 1.** (k-tape deterministic Turing machine, [AHU74]) A k-tape deterministic Turing machine (shortly, DTM) $M$ is a seven-tuple $(Q, T, I, \delta, b, q_0, q_f)$ where:

1. $Q$ is the set of states.

2. $T$ is the set of tape symbols.

3. $I$ is the set of input symbols; $I \subseteq T$.

4. $b \in T - I$, is the blank.
5. $q_0$ is the initial state.

6. $q_f$ is the final (or accepting) state.

7. $\delta$ is the next-move function, maps a subset of $Q \times T^k$ to $Q \times (T \times \{L, R, S\})^k$.
   That is, for some $(k + 1)$-tuples consisting of a state and $k$ tape symbols, it gives a new state and $k$ pairs, each pair consisting of a new tape symbol and a direction for the tape head. Suppose $\delta(q, a_1, a_2, \ldots, a_k) = (q', (a'_1, d_1), (a'_2, d_2), \ldots, (a'_k, d_k))$, and the deterministic Turing machine is in state $q$ with the $i$th tape head scanning tape symbol $a_i$ for $1 \leq i \leq k$. Then in one move the deterministic Turing machine enters state $q'$, changes symbol $a_i$ to $a'_i$, and moves the $i$th tape head in the direction $d_i$ for $1 \leq i \leq k$.

The definition of a nondeterministic Turing machine is similar to that of deterministic Turing machine, except that the next-move function $\delta$ is a mapping from $Q \times T^k$ to subsets of $Q \times (T \times \{L, R, S\})^k$, stated as follows:

**Definition 2.** (k-tape nondeterministic Turing machine, [AHU74]) A $k$-tape nondeterministic Turing machine (shortly, NTM) $M$ is a seven-tuple $(Q, T, I, \delta, b, q_0, q_f)$ where all components have the same meaning as for the ordinary deterministic Turing machine, except that here the next-move function $\delta$ is a mapping from $Q \times T^k$ to subsets of $Q \times (T \times \{L, R, S\})^k$.

In the following, we will refer Turing machine to both the deterministic Turing machine and the nondeterministic Turing machine. And we will often use DTM (respectively, NTM) to denote deterministic (respectively, nondeterministic) Turing machine.

A Turing machine $M$ works in time $T(|x|)$ if for every input $x$ where $|x|$ is the length of $x$, all computations of $M$ on $x$ end in less than $T(|x|)$ steps or moves. Particularly, $\text{DTIME}[T(n)]$ (respectively, $\text{NTIME}[T(n)]$) is the class of languages accepted by the deterministic (respectively, nondeterministic) Turing machine working in time $c \cdot T(n)$ for some constant $c > 0$. The notation $P$ and $NP$ are defined to be the class of languages:

$$P = \bigcup_{k \in \mathbb{N}_1} \text{DTIME}[n^k]$$

and

$$NP = \bigcup_{k \in \mathbb{N}_1} \text{NTIME}[n^k].$$

With respect to the time complexity between $k$-tape nondeterministic (respectively, deterministic) Turing machine and single-tape nondeterministic (respectively, deterministic) Turing machine, we have the following useful Lemmas which play important roles in the following context, extracting from [AHU74], see Lemma 10.1 and Corollary 1 to Lemma 10.1 in [AHU74]:

**Lemma 1.** (Lemma 10.1 in [AHU74]) If $L$ is accepted by a $k$-tape nondeterministic Turing machine of time complexity $T(n)$, then $L$ is accepted by a single-tape nondeterministic Turing machine of time complexity $O(T^2(n))$. $\square$
The deterministic version is as follows

**Corollary 3.** (Corollary 1 in [AHU74] to Lemma 1) If \( L \) is accepted by a \( k \)-tape deterministic Turing machine of time complexity \( T(n) \), then \( L \) is accepted by a single-tape deterministic Turing machine of time complexity \( O(T^2(n)) \). □

The following theorem about efficient simulation is needed a few times, whose proof is present in [HS66], see also [AB09]

**Lemma 2.** There exists a Turing machine \( U \) such that for every \( x, \alpha \in \{0, 1\}^* \), \( U(x, \alpha) = M_\alpha(x) \), where \( M_\alpha \) denotes the Turing machine represented by \( \alpha \). Moreover, if \( M_\alpha \) halts on input \( x \) within \( T(|x|) \) steps then \( U(x, \alpha) \) halts within \( cT(|x|) \log T(|x|) \) moves (steps) \(^1\), where \( c \) is a constant independent of \( |x| \) and depending only on \( M_\alpha \)'s alphabet size, number of tapes, and number of states. □

Finally, more information and premise lemmas will be given along the way to prove our main result.

## 3 Enumeration of Polynomial-Time DTMs

Before coming to the point, we should make a formal definition of a polynomial-time deterministic Turing machine.

**Definition 3.** Formally, a polynomial-time deterministic Turing machine is deterministic Turing machine such that there exists \( k \in \mathbb{N}_1 \), for all input \( x \) of length \( n \) where \( n \in \mathbb{N} \) is arbitrary, \( M(x) \) will halt within \( O(n^k) \) moves. We represent a polynomial-time deterministic Turing machine by a tuple of \((M, k)\) where \( M \) is the deterministic Turing machine itself, and \( k \) is the unique minimal order of some polynomial \( O(n^k) \) such that for any input \( x \) of length \( n \) where \( n \in \mathbb{N} \) is arbitrary, \( M(x) \) will halt within \( O(n^k) \) moves. We call \( k \) the order of \((M, k)\).

**Remark 1.** Obviously, in the above definition, given a polynomial-time deterministic Turing machine \((M, k)\), for any input \( x \) of length \( n \) where \( n \in \mathbb{N} \) is arbitrary, \( M(x) \) will halts within \( O(n^{k+i}) \) moves, where the integer \( i \geq 0 \). But there exists input \( y \) of length \( n \) where \( n \in \mathbb{N} \) is arbitrary, \( M(y) \) does not halt within \( O(n^{k-1}) \) moves.

To obtain our main result we need to enumerate the polynomial-time deterministic Turing machines, that is, assign an ordering to polynomial-time deterministic Turing machines so that for each nonnegative integer \( i \) there is a unique \((M, k)\) associated with \( i \) \(^2\).

To achieve our enumeration, we first use the method presented in [AHU74], p. 407, to encode a deterministic Turing machine into an integer.

Without loss of generality, see [AHU74], we can make the following assumptions about the representation of a single-tape deterministic Turing machine:

\(^1\)In this work, \( \log n \) means \( \log_2 n \).

\(^2\)There are a variety of ways to enumerate all polynomial-time deterministic Turing machines, for instance, see proof of Theorem 14.1 in [Pap94], p. 330; or see [Lad75].
1. The states are named \( q_1, q_2, \ldots, q_s \) for some \( s \), with \( q_1 \) the initial state and \( q_s \) the accepting state.

2. The input alphabet is \{0, 1\}.

3. The tape alphabet is \{\( X_1, X_2, \ldots, X_t \)\} for some \( t \), where \( X_1 = b \), \( X_2 = 0 \), and \( X_3 = 1 \).

4. The next-move function \( \delta \) is a list of quintuples of the form \((q_i, X_j, q_k, X_l, D_m)\), meaning that \( \delta(q_i, X_j) = (q_k, X_l, D_m) \), and \( D_m \) is the direction, \( L \), \( R \), or \( S \), if \( m = 0, 1, \) or \( 2 \), respectively. We assume this quintuple is encoded by the string \( 10^i10^j10^k10^l10^m1 \).

5. The Turing machine itself is encoded by concatenating in any order the codes for each of the quintuples in its next-move function. Additional 1’s may be prefixed to the string if desired. The result will be some string of 0’s and 1’s, beginning with 1, which we can interpret as an integer.

Next, we encode the order of \((M, k)\) to be \( 10^k1 \) so that the tuple \((M, k)\) can be encoded by concatenating the binary string representing \( M \) itself and \( 10^k1 \) together. Now the tuple \((M, k)\) is encoded as a binary string which can be explained as an integer.

By this encoding, any integer which can not be decoded is deemed to represent the trivial Turing machine with an empty next-move function. Every single-tape polynomial-time deterministic Turing machine will appear infinitely often in the enumeration, since given a polynomial-time deterministic Turing machine, we may prefix 1’s at will to find larger and larger integers representing the same set of \((M, k)\). We denote such a polynomial-time DTM by \( \hat{M}_j \) where \( j \) is the integer representing \((M, k)\).

Finally, we remark that the enumeration of all polynomial-time deterministic Turing machines also gives an enumeration of languages in \( P \) (with languages appearing multiple times). In particular, we have the following Theorem 2 is obvious:

**Theorem 2.** Any polynomial-time deterministic Turing machines are in the above enumeration. \( \square \)

### 4 Diagonalization of Polynomial-Time DTMs

We can now design a four-tape NTM \( M_0 \) which treats its input string \( x \) both as an encoding of a tuple \((M, k)\) and also as the input to \( M \). One of the capabilities possessed by \( M_0 \) is the ability to simulate a Turing machine, given its specification. We shall have \( M_0 \) determine whether the polynomial-time deterministic Turing machine \( M \) of time complexity \( O(n^k) \) accepts the input \( x \) without using more than \( O(n^k) \) time. If \( M \) accepts \( x \) within in \( O(n^k) \) time, then \( M_0 \) does not. Otherwise, \( M_0 \) accepts \( x \). Thus, for all \( i \), \( M_0 \) disagrees with the behavior of the polynomial-time DTM in the \( i \)th of enumeration on that input \( x \). We show the following:

**Theorem 3.** There exists a language \( L_d \) accepted by an universal nondeterministic Turing machine \( M_0 \) but by no polynomial-time deterministic Turing machines.
Proof. Let $M_0$ be a four-tape NTM which operates as follows on an input string $x$ of length of $n$.

1. $M_0$ decodes the tuple encoded by $x$. If $x$ is not the encoding of some single-tape polynomial-time DTM $\widehat{M}_j$ for some $j$ then GOTO 5, else determines $t$, the number of tape symbols used by $\widehat{M}_j$; $s$, its number of states; and $k$, its order. The third tape of $M_0$ can be used as “scratch” memory to calculate $t$.

2. Then $M_0$ lays off on its second tape $|x|$ blocks of $\lceil \log t \rceil$ cells each, the blocks being separated by single cell holding a marker #, i.e. there are $(1 + \lceil \log t \rceil)n$ cells in all where $n = |x|$. Each tape symbol occurring in a cell of $\widehat{M}_j$’s tape will be encoded as a binary number in the corresponding block of the second tape of $M_0$. Initially, $M_0$ places $\widehat{M}_j$’s input, in binary coded form, in the blocks of tape 2, filling the unused blocks with the code for the blank.

3. On tape 3, $M_0$ sets up a block of $\lceil (k + 1) \log n \rceil$ cells, initialized to all 0’s. Tape 3 is used as a counter to count up to $n^{k+1}$.

4. By using nondeterminism, $M_0$ simulates $\widehat{M}_j$, using tape 1, its input tape, to determine the moves of $\widehat{M}_j$ and using tape 2 to simulate the tape of $\widehat{M}_j$. The moves of $\widehat{M}_j$ are counted in binary in the block of tape 3, and tape 4 is used to hold the state of $\widehat{M}_j$. If $\widehat{M}_j$ accepts, then $M_0$ halts without accepting. $M_0$ accepts if $\widehat{M}_j$ halts without accepting, or if the counter on tape 3 overflows, $M_0$ halts without accepting.

5. Since $x$ is not encoding of some single-tape DTM. Then $M_0$ sets up a block of $\lfloor 2 \times \log n \rfloor$ cells on tape 3, initialized to all 0’s. Tape 3 is used as a counter to count up to $n^2$. By using its nondeterministic choices, $M_0$ moves as per the path given by $x$. The moves of $M_0$ are counted in binary in the block of tape 3. If the counter on tape 3 overflows, then $M_0$ halts. $M_0$ accepts $x$ if and only if there is a computation path from the start state of $M_0$ leading to the accept state and the total number of moves can not exceed $n^2$, so is within $O(n)$. Note that the number of 2 in $\lfloor 2 \times \log n \rfloor$ is fixed, i.e. it is default.

The NTM $M_0$ described above is of time complexity, say $S$ which is unknown currently. By Lemma 1, $M$ is equivalent to a single-tape NTM of time complexity $O(S^2)$, and it of course accepts some language $L_d$.

Suppose now $L_d$ were accepted by some DTM $\widehat{M}_i$ in the enumeration which is of time complexity $T(n) = O(n^k)$. Then by Corollary 3 we may assume that $\widehat{M}_i$ is a single-tape DTM. Let $\widehat{M}_i$ have $s$ states and $t$ tape symbols. Since $\widehat{M}_i$ appears infinitely often in the

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3. Assume that a deterministic Turing machine of time complexity $T(n)$, then by Lemma 2, the universal simulation can be done within $T(n) \log T(n)$ which is less than $n^{k+1}$ when $T(n) = O(n^k)$.

4. When $M_0$ simulates a DTM, the behavior of $M_0$ is somewhat deterministic, since there is no nondeterministic choices in a DTM.

5. $M_i$ denotes the set of binary strings which encodes $\widehat{M}_i$. We know that we may prefix 1’s at will to find larger and larger integers representing the same set of quintuples of the same DTM $M_i$, thus there are infinitely binary strings of sufficiently long which represents DTM $M_j$. 

9
enumeration, and

\[
\lim_{n \to \infty} \frac{T(n) \log T(n)}{n^{k+1}} = \lim_{n \to \infty} \frac{cn^k \log c + k \log n}{n^{k+1}} \quad (\text{for some constant } c > 0)
\]

\[
= \lim_{n \to \infty} \left( \frac{c \log cn^k}{n^{k+1}} + \frac{ckn \log n}{n^{k+1}} \right)
\]

\[
= 0 < 1,
\]

so, there exists a \(N_0 > 0\) such that for any \(N \geq N_0\),

\[
T(N) \log T(N) < N^{k+1}
\]

which implies that for a sufficiently long \(w\), say \(|w| \geq N_0\), and \(M_w\) denoted by such \(w\) is \(\hat{M}_i\), we have that

\[
T(|w|) \log T(|w|) < |w|^{k+1}.
\]

Thus, on input \(w\), \(M_0\) has sufficient time to simulate \(M_w\) and accepts if and only if \(M_w\) rejects (In simulation a polynomial-time DTM, \(M_0\) only turns off mandatorily when the counter on tape 3 overflows, i.e. the counter \(\geq N^{k+1}\)). But we assumed that \(\hat{M}_i\) accepted \(L_d\), i.e. \(\hat{M}_i\) agreed with \(M_0\) on all inputs. We thus conclude that \(\hat{M}_i\) does not exist, i.e. \(L_d \not\in P\).

In the above, we need say more about (4). Since the universal NTM \(M_0\) diagonalizes over all polynomial-time deterministic Turing machine, so it can flip the answer, i.e. \(M_0(x) = 1 - \hat{M}_j(x)\) directly. If \(M_0\) diagonalizes over all NTMs, then other techniques must be presented.

\[\square\]

**Remark 2.** Generally, to diagonalize over a list nondeterministic Turing machines of time complexity \(T(n)\) within \(L(n)\) time where \(T(n) = o(L(n))\) is more hard, the diagonalization technique from the proof of Theorem 7 does not directly apply. Since a nondeterministic Turing machine that runs in \(O(T(n))\) may have \(2^{O(T(n))}\) branches in its computation. It is unclear how to determine in \(O(L(n))\) time whether or not it accepts and then flip this answer under the assumption that \(O(L(n)) < 2^{O(T(n))}\). That is, we do not know whether \(NP = \text{coNP}\) as observed by Cook [Coo73]. Hence, in his work [Coo73], Cook uses lazy diagonalization technique to prove nondeterministic time hierarchy theorem, see also [AB09]. Fortunately, in Theorem 7, we diagonalize against deterministic Turing machine (of time complexity \(T(n) = O(n^k)\) for any \(k \in \mathbb{N}_1\)) rather than nondeterministic Turing machines.

Next, we are going to show that the universal nondeterministic Turing machine \(M_0\) working in \(O(n^k)\) for any \(k \in \mathbb{N}_1\):
Theorem 4. The universal nondeterministic Turing machine $M_0$ constructed in proof of Theorem 7 runs in $O(n^k)$ for any $k \in \mathbb{N}_1$. That is, $L_d \in NP$.

Proof. On the one hand, when the input $x$ encodes a polynomial-time DTM whose time complexity is $T(n)$, say $T(n) = O(n^k)$, then $M_0$ turns off mandatorily within $O(|x|^{k+1})$ by the construction. This holds for any polynomial-time DTM of time complexity of $O(n^k)$ for any $k \in \mathbb{N}_1$. On the other hand, on input $x$ not encoding for polynomial-time DTM, then the running time of $M_0$ is within $O(|x|)$ by the construction, since $M_0$ turns off mandatorily within $|x|^2$. So $M_0$ is of time complexity of $S(n) = \max\{n^k, n\}$ for any $k \in \mathbb{N}_1$. By Lemma 1, there is a single-tape nondeterministic Turing machine $M'$ which is equivalent to $M_0$ and $M'$ operates in $O(S(n)^2) = O(n^{2k})$ for any $k \in \mathbb{N}_1$. So $M'$ is a nondeterministic Turing machine running in $O(n^k)$ for any $k \in \mathbb{N}_1$, i.e. $L_d \in NP$.

Now we are at the point to present the proof of Theorem 1:

Proof of Theorem 1. It is obvious that Theorem 1 is an immediate consequence of Theorem 7 and Theorem 4. So, Theorem 1 follows.

Remark 3. Originally, we call $M_0$ a polynomial-time NTM. Some experts [For21] argue that $M_0$ is not running in polynomial-time, because it runs in $O(n^k)$ for any $k \in \mathbb{N}_1$. He think that the polynomial-time machine is a fixed mathematical definition, i.e. call a machine runs in polynomial-time iff the machine runs in $O(n^c)$ for some fixed constant $c > 0$. Here, we thanks for their valuable criticisms.

But, the author would like to call such $M_0$ also a “polynomial-time” because that the language $L_d$ accepted by $M_0$ is in $NP$. To see this, we define the family of languages $\{L^i_d\}_{i \in \mathbb{N}_1}$ as follows:

$$L^i_d \triangleq \text{language accepted by } M_0 \text{ runs in } O(n^i) \text{ for fixed } i \text{ where } i \in \mathbb{N}_1.$$

Then it is easy to see that

$$L_d = \bigcup_{i \in \mathbb{N}_1} L^i_d.$$

The following is also obvious:

$$L^i_d \in NTIME[n^i] \text{ for each } i \in \mathbb{N}_1.$$

By the fact that

$$NTIME[n^i] \subseteq NP \text{ for any } i \in \mathbb{N}_1,$$

we can deduce that

$$L_d \in NP = \bigcup_{k \in \mathbb{N}_1} NTIME[n^k].$$

\footnote{See Remark 3.}
In this sense, that we still call $M_0$ a “polynomial-time” machine is rational. In other words, the result $L_d \in NP$ is enough for our discussions.

The above proof logic is inspired indirectly by the following statement in Set Theory:

Let $U$ be the universal, and the family $\{u_i\}_{i \in \mathbb{N}}$ be subsets of $U$, that is, $u_i \subseteq U$ for any $i \in \mathbb{N}$. Then the following holds true:

$$\bigcup_{i \in \mathbb{N}} u_i \subseteq U.$$ 

In a nutshell, what we concern about is that to construct a language $L_d \not\in P$ but $L_d \in NP$. As for the difference between $M_0$ runs within $O(n^c)$ for some fixed $c > 0$ and $M_0$ runs within $O(n^k)$ for any $k \in \mathbb{N}_1$ is not so important to our discussions. In fact, there is no machine can run within $O(n^c)$ for some fixed $c > 0$ accepting the $L_d$ because the mathematicians acknowledge that $\mathbb{N}_1$ is not bounded from above.

5 Don’t suspect that $L_d \in NP$

Some experts do not acknowledge that $L_d \in NP$, for example [For21]. In [For21], expert Lance Fortnow claims that $M_0$ to run in polynomial-time it must run in time $O(n^c)$ for a fixed $c$. But to diagonalize all polynomial-time deterministic Turing machines then $M_0$ needs time $O(n^k)$ for all $k \in \mathbb{N}_1$, including $k > c$. He further assert that the author hence made a common mistake and this mistake is not fixable.

We should stress here that although $M_0$ does not run in $O(n^c)$ for fixed $c > 0$, $L_d$ is indeed in $NP$. To further make some experts believe that $L_d$ is in $NP$, we construct a language $L_s \in P$, but any deterministic Turing machine running within $O(n^k)$ for fixed $k \in \mathbb{N}_1$ can not recognize $L_s$, because it also needs time $O(n^k)$ for all $k \in \mathbb{N}_1$. To begin, we first cite the following:

**Lemma 3.** ([HS66]) If $T_1(n)$ and $T_2(n)$ are time-constructible functions and

$$\lim_{n \to +\infty} \frac{T_1(n) \log T_1(n)}{T_2(n)} = 0,$$

then there is some language accepted in time $T_2(n)$ but not $T_1(n)$ by a deterministic Turing machine. 

**Theorem 5.** There exists a language $L_s$ accepted by an universal deterministic Turing machine $M'_0$ but by no polynomial-time deterministic Turing machine to run within $O(n^k)$ for a fixed $k \in \mathbb{N}_1$.

**Proof.** Let $M'_0$ be a four-tape DTM which operates as follows on an input string $x$ of length $n$.

1. $M'_0$ decodes the tuple encoded by $x$. If $x$ is not the encoding of some single-tape polynomial-time DTM $\widehat{M}_j$ for some $j$ then rejects, else determines $t$, the number of tape symbols used by $\widehat{M}_j$; $s$, its number of states; and $k$, its order. The third tape of $M'_0$ can be used as “scratch” memory to calculate $t$. 

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2. Then $M_0'$ lays off on its second tape $|x|$ blocks of $\lceil \log t \rceil$ cells each, the blocks being separated by single cell holding a marker #, i.e. there are $(1 + \lceil \log t \rceil)n$ cells in all where $n = |x|$. Each tape symbol occurring in a cell of $\hat{M}_j$’s tape will be encoded as a binary number in the corresponding block of the second tape of $M_0'$. Initially, $M_0'$ places $\hat{M}_j$’s input, in binary coded form, in the blocks of tape 2, filling the unused blocks with the code for the blank.

3. On tape 3, $M_0'$ sets up a block of $\lceil (k + 1) \log n \rceil$ cells, initialized to all 0’s. Tape 3 is used as a counter to count up to $n^k + 1$.

4. $M_0'$ simulates $\hat{M}_j$, using tape 1, its input tape, to determine the moves of $\hat{M}_j$ and using tape 2 to simulate the tape of $\hat{M}_j$. The moves of $\hat{M}_j$ are counted in binary in the block of tape 3, and tape 4 is used to hold the state of $\hat{M}_j$. If the counter on tape 3 overflows, $M_0'$ turns off mandatorily and rejects, else $M_0'$ accepts if and only if $\hat{M}_j$ accepts.

The DTM $M_0'$ described above is of time complexity, say $S$ which is unknown currently. By Corollary 3, $M_0'$ is equivalent to a single-tape DTM of time complexity $O(S^2)$, and it of course accepts some language $L_s$.

Suppose now $L_s$ were accepted by some DTM $\hat{M}_i$ in the enumeration which is of time complexity $T(n) = O(n^k)$ for fixed $k \in \mathbb{N}_1$. Then by Corollary 3 we may assume that $\hat{M}_i$ is a single-tape DTM. Let $\hat{M}_i$ have $s$ states and $t$ tape symbols. We consider a single-tape deterministic Turing machine $\hat{M}_j$ which is of time complexity $O(n^k + 1)$. By Lemma 3, there is a language $\mathcal{L}$ accepted $\hat{M}_j$ but not $\hat{M}_i$. Then define the language $\mathcal{L}'$ as follows

$$\mathcal{L}' \triangleq \{ \hat{M}_jw : w \in L \},$$

where $\hat{M}_jw$ denotes the concatenation of $\hat{M}_j$ and $w$. It is obvious that $M_0'$ accepts $\mathcal{L}'$. But $M_0'$ rejects the language $\{ \hat{M}_iw : w \in L \}$. This finishes the proof.

Now we define the family of languages $\{ L_s^i \}_{i \in \mathbb{N}_1}$ as follows:

$$L_s^i \triangleq \text{language accepted by } M_0' \text{ within } O(n^i), \text{ i.e. } M_0' \text{ turns off mandatorily when given an input } w, \text{ if the total number of moves made by } M_0' \text{ exceeds } |w|^{i+1}.$$

Then it is easy to see that

$$L_s^i \in \text{DTIME}[n^i],$$

since $M_0'$ is a single-tape deterministic Turing machine whose total number of moves can not exceed $n^{i+1}$ where $n$ is the length of inputs. By the construction of $M_0'$, we have that

$$L_s = \bigcup_{i \in \mathbb{N}_1} L_s^i,$$

\footnote{In fact, $M_0'$ works in $O(n^k)$ for any $k \in \mathbb{N}_1$}
from which it immediately follows that
\[ L_s \in \bigcup_{i \in \mathbb{N}_1} DTIME[n^i] = P. \]

We assert that no one can deny that \( L_s \not\in P \). But it can not be accepted by any deterministic Turing machine running in time \( O(n^c) \) for a fixed \( c \). Of course, by our constructions of machines \( M_0 \) in Theorem 7 and \( M'_0 \) in Theorem 5, \( L_d \neq L_s \). Such a phenomenon is due to that \( \mathbb{N}_1 \) is not bounded from above, in the author's view.

6 A Note

The first place using this similar techniques is in the author’s work [Lin21], in which it is used to separate two different complexity classes \( DSPACE[S(n)] \) and \( NSPACE[S(n)] \) for some space-constructible function \( S(n) \geq \log n \). The inspirations are drawn from two facts: (1) the author was reading the proof of the space hierarchy for deterministic Turing machine, i.e. Theorem 11.1 in [AHU74] and (2) the author was considering that how to resolve a longstanding open question in automata theory, i.e. the LBA question. Then the idea of using an universal nondeterministic Turing machine to diagonalize a list of deterministic Turing machines of space complexity, say \( S(n) \), naturally appeared in the mind.

After the preliminary of this work posted in an online-archive, we receive some feedbacks from other experts [For21] saying that the Baker-Gill-Solovay oracle [BGS75] basically shows that the techniques of using an universal nondeterministic Turing machine to diagonalize over all polynomial-time deterministic Turing machines cannot work to resolve the \( P \) versus \( NP \) question. The author knows litter about the Baker-Gill-Solovay oracle method, but he recognize that the proof of Theorem 11.1 in Aho, Hopcroft and Ullman’s textbook [AHU74] indeed constructs a language not in \( DSPACE[S_1(n)] \) but in \( DSPACE[S_2(n)] \) assuming that \( \lim_{n \to \infty} \frac{S_1(n)}{S_2(n)} = 0 \). In this Section we pose two questions:

1. The Baker-Gill-Solovay oracle Turing machine is capable to query an extended oracle. Then our first question is that how to encode this function to binary string(s) in similar to encode next-move function of ordinary Turing machine to binary strings, such that when the oracle Turing machine simulated by an universal Turing machine, the binary string(s) encoding the function of to query an extended oracle can be executed by the universal Turing machine step by step?

2. The query operation of Baker-Gill-Solovay oracle Turing machine is in one step. If this oracle Turing machine simulated by an universal Turing machine, is the query operation still in one step?

We by no means answer the above questions, and hence left the (essential) difference between the Baker-Gill-Solovay oracle method and the way using in [AHU74] for future study.

\[ ^8L_d \text{ and } L_s \text{ denote diagonalization language and simulation language, respectively.} \]
Our motivation in this Section is to separate complexity classes \( \text{DTIME} \) and \( \text{NTIME} \). Specifically, we will show the following:

**Theorem 6.** For any \( k \in \mathbb{N}_1 \), there exists a language \( L^k_d \) accepted by an universal nondeterministic Turing machine of time complexity \( O(n^{k+1}) \) but not by any deterministic Turing machines of time complexity \( O(n^k) \). In other words, \( \text{DTIME}[n^k] \subset \text{NTIME}[n^{k+1}] \).

### 6.1 Diagonalization Again

We can now design a four-tape NTM \( M^0 \) which treats its input string \( x \) both as an encoding of a tuple \( (M, k) \) and also as the input to \( M \). We shall have \( M^0 \) determine whether the polynomial-time deterministic Turing machine \( \hat{M}_i \) of time complexity \( O(n^k) \) accepts the input \( x \) without using more than \( O(n^{k+1}) \) time. If \( \hat{M}_i \) accepts \( x \) within in \( O(n^k) \) time, then \( M^0 \) does not. Otherwise, \( M^0 \) accepts \( x \). Thus, for all \( i \), \( M^0 \) disagrees with the behavior of \( \hat{M}_i \) of time complexity \( O(n^k) \) in the \( i \)th of enumeration on that input \( x \), i.e. the following:

**Theorem 7.** Let \( i \in \mathbb{N}_1 \) be arbitrary, then there exists a language \( L^i_d \) accepted by an universal nondeterministic Turing machine \( M^0 \) of time complexity \( O(i^{i+1}) \) but by no deterministic Turing machines of time complexity \( O(n^i) \). In other words, \( \text{DTIME}[n^i] \subset \text{NTIME}[n^{i+1}] \).

**Proof.** Let \( M^0 \) be a four-tape NTM which operates as follows on an input string \( x \) of length of \( n \):

1. \( M^0 \) decodes the tuple encoded by \( x \). If \( x \) is not the encoding of some single-tape polynomial-time DTM \( \hat{M}_j \) for some \( j \) then GOTO 5, else determines \( t \), the number of tape symbols used by \( \hat{M}_j \); \( s \), its number of states; and \( k \), its order. If \( k > i \) then rejects. The third tape of \( M^0 \) can be used as “scratch” memory to calculate \( t \).

2. Then \( M^0 \) lays off on its second tape \( |x| \) blocks of \( \lceil \log t \rceil \) cells each, the blocks being separated by single cell holding a marker \( \# \), i.e. there are \((1 + \lceil \log t \rceil)n \) cells in all where \( n = |x| \). Each tape symbol occurring in a cell of \( \hat{M}_j \)’s tape will be encoded as a binary number in the corresponding block of the second tape of \( M^0 \). Initially, \( M^0 \) places \( \hat{M}_j \)’s input, in binary coded form, in the blocks of tape 2, filling the unused blocks with the code for the blank.

3. On tape 3, \( M^0 \) sets up a block of \( \lceil (i + 1) \log n \rceil \) cells, initialized to all 0’s. Tape 3 is used as a counter to count up to \( n^{i+1} \).

4. By using nondeterminism, \( M^0 \) simulates \( \hat{M}_j \), using tape 1, its input tape, to determine the moves of \( \hat{M}_j \) and using tape 2 to simulate the tape of \( \hat{M}_j \). The moves

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\(^9\)If one can prove the following statement: An universal nondeterministic Turing machine \( U_N \) can simulate a nondeterministic Turing machine \( M_N \) in \( O(T(n)) \) where \( M_N \) runs within \( O(T(n)) \). Then we can show that \( \text{DTIME}[n^k] \subset \text{NTIME}[n^k] \) for any \( k \in \mathbb{N}_1 \).
of $\hat{M}_j$ are counted in binary in the block of tape 3, and tape 4 is used to hold the state of $\hat{M}_j$. If $\hat{M}_j$ accepts, then $M^0$ halts without accepting. $M^0$ accepts if $\hat{M}_j$ halts without accepting. If the counter on tape 3 overflows, $M^0$ halts mandatorily without accepting.

5. Since $x$ is not encoding of some single-tape DTM. Then $M^0$ sets up a block of $[2 \times \log n]$ cells on tape 3, initialized to all 0’s. Tape 3 is used as a counter to count up to $n^2$. By using its nondeterministic choices, $M^0$ moves as per the path given by $x$. The moves of $M^0$ are counted in binary in the block of tape 3. If the counter on tape 3 overflows, then $M^0$ halts. $M^0$ accepts $x$ if and only if there is a computation path from the start state of $M^0$ leading to the accepting state and the total number of moves can not exceed $n^2$, so is within $O(n)$. Note that the number of 2 in $[2 \times \log n]$ is fixed, i.e. it is default.

The NTM $M^0$ described above is of time complexity $S(n) = O(n^{i+1})$ because $M^0$ turns off mandatorily when the total number of moves made by $M^0$ exceeds or equal to $|w|^{i+1}$ for input $w$ if $w$ encodes some single-tape deterministic Turing machines. It of course accepts some language $L^i_d \in NTIME[n^{i+1}]$.

Suppose now $L^i_d$ were accepted by some DTM $\hat{M}_j$ in the enumeration which is of time complexity $T(n) = O(n^i)$. Then by Corollary 3 we may assume that $\hat{M}_j$ is a single-tape DTM. Let $\hat{M}_j$ have $s$ states and $t$ tape symbols. Since $\hat{M}_j$ appears infinitely often in the enumeration, and by Lemma 2, the simulation can be done within $T(n) \log T(n)$ where $T(n) = O(n^i)$:

$$\lim_{n \to \infty} \frac{T(n) \log T(n)}{n^{i+1}} = \lim_{n \to \infty} \frac{cn^i \log c + i \log n}{n^{i+1}} (\text{ for some constant } c > 0)$$

$$= \lim_{n \to \infty} \left( \frac{c \log cn^i}{n^{i+1}} + \frac{cin^i \log n}{n^{i+1}} \right)$$

$$= 0$$

$$< 1,$$

so, there exists a $N_0 > 0$ such that for any $N \geq N_0$,

$$T(N) \log T(N) < N^{i+1}$$

which implies that for a sufficiently long $w$, say $|w| \geq N_0$, and $M_w$ denoted by such $w$ is $\hat{M}_j$, we have that

$$T(|w|) \log T(|w|) < |w|^{i+1}.$$

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$\hat{M}_j$ denotes the set of binary strings which encodes $\hat{M}_j$. We know that we may prefix 1’s at will to find larger and larger integers representing the same set of quintuples of the same DTM $M_j$, thus there are infinitely binary strings of sufficiently long which represents DTM $M_j$. 

Thus, on input $w$, $M^0$ has sufficient time to simulate $M_w$ and accepts if and only if $M_w$ rejects. But we assumed that $\tilde{M}_j$ accepted $L_d^i$, i.e. $\tilde{M}_j$ agreed with $M^0$ on all inputs. We thus conclude that $\tilde{M}_j$ does not exist, i.e. $L_d \not\in \text{DTIME}[n^i]$.

Proof of Theorem 6. Note that Theorem 7 is just another description of Theorem 6, this completes the proof.

Now the Corollary 1 can be simply shown as follows:

Proof of Corollary 1:
Proof. Since the above proof for Theorem 7 is valid for any $i \in \mathbb{N}_1$, we thus have that

$$\text{DTIME}[n^1] \subset \text{NTIME}[n^{1+1}],$$
$$\text{DTIME}[n^2] \subset \text{NTIME}[n^{2+1}],$$
$$\vdots$$
$$\text{DTIME}[n^k] \subset \text{NTIME}[n^{k+1}],$$
$$\vdots$$

from which it immediately follows that

$$\bigcup_{i \in \mathbb{N}_1} \text{DTIME}[n^i] \subset \bigcup_{i \in \mathbb{N}_1} \text{NTIME}[n^{i+1}]. \tag{1}$$

Further note that

$$P = \bigcup_{i \in \mathbb{N}_1} \text{DTIME}[n^i]. \tag{2}$$

By (1) and (2) we have that

$$P \subset \bigcup_{i \geq 2} \bigcap_{i \in \mathbb{N}_1} \text{NTIME}[n^i]. \tag{3}$$

Also note that

$$\text{NTIME}[n^1] \cup \left( \bigcup_{i \geq 2} \bigcap_{i \in \mathbb{N}_1} \text{NTIME}[n^i] \right) = \bigcup_{k \in \mathbb{N}_1} \text{NTIME}[n^k] = NP. \tag{4}$$

Combining (3), and (4) we obtain that

$$P \subset NP$$

thus the Corollary 1 follows.

Return back to Section 6, based on the method of Baker-Gill-Solovay oracle [BGS75], whether there exists an oracle $O$ such that $\text{DTIME}[n^i]^O = \text{NTIME}[n^{i+1}]^O$? We left this question open.
Remark 4. According to [BGS75], we know that $P^{TQBF} = NP^{TQBF}$ where $TQBF$ is polynomial-space complete problem. If such Turing machines (i.e. polynomial-time DTMs with oracle $TQBF$) can be effectively enumerated (that is, we are able to assign an ordering to such a machine so that for each nonnegative integer $i$ there is an unique polynomial-time DTM with oracle $TQBF$ associated with $i$), then we can construct an universal NTM $M_0^{TQBF}$ with oracle $TQBF$, which diagonalizes over all of them. This process will produce a language $L_d^{TQBF}$ not accepted by any polynomial-time DTMs with oracle $TQBF$, but accepted by $M_0^{TQBF}$ to run within $O(n^k)$ for any $k \in \mathbb{N}_1$. This gives $P^{TQBF} \neq NP^{TQBF}$, i.e. $M_0^{TQBF}$ is not in the enumeration. However, by [BGS75]’s result $P^{TQBF} = NP^{TQBF}$, $M_0^{TQBF}$ is also in the enumeration, which is impossible. This raises the doubt that we can not enumerate DTMs with oracle correctly. Of course, if the enumeration is correct, then it possible that the simulation result of $M_0^{TQBF}$ is not correct and flip the incorrect answer. To quote words in [Pap94], such Turing machines with oracle are very unrealistic. To give an another example to illustrate that Turing machine with oracle is unrealistic, we quote the language $L_d$ in [HMU06] (p. 381-382), which is not recursively enumerable, as an oracle. Then $P^{L_d}$ is the language accepted by polynomial-time DTMs with oracle $L_d$, we are unable to enumerate such machines, because the language $L_d$ can not be accepted by any Turing machines (both deterministic and nondeterministic), i.e. $L_d$ is not in the enumeration of all Turing machines, so it is impossible for us to assign an ordering to polynomial-time DTMs with oracle $L_d$ so that for each nonnegative integer $i$ there is an unique polynomial-time DTM with oracle $L_d$ associated with $i$.

7 Conclusions

To summarize, we have shown that there exists a language $L_d$ accepted by some nondeterministic Turing machine but by no polynomial-time deterministic Turing machines. To achieve this, we first encode any single-tape deterministic Turing machine into an integer by using the method presented in [AHU74]. And then we concatenate the binary string representing the single-tape DTM itself and its order together to represent a polynomial-time DTM. Our encoding of polynomial-time DTM is very convenient for us to map a polynomial-time DTM to integer.

Next, we design a four-tape universal nondeterministic Turing machine diagonalizing over all polynomial-time deterministic Turing machines. Theorem 7 illustrates in detail the operation of the universal nondeterministic Turing machine, showing that there is a language $L_d$ accepted by this universal nondeterministic Turing machine, but by no polynomial-time deterministic Turing machines. In Theorem 4, we carefully analyze the running time of the universal nondeterministic Turing machine, showing that this universal nondeterministic Turing machine runs in $O(n^k)$ for any $k \in \mathbb{N}_1$. Combining Theorem 7 and Theorem 4, the Theorem 1 follows.

In Section 6, we use the techniques basically the same as in [Lin21] to give another proof of Corollary 1. We showed that for any $i \in \mathbb{N}_1$, there is a language $L_d^i$ accepted by an universal NTM to run within $O(n^{i+1})$ but not by any single-tape DTMs working within $O(n^i)$. 

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There are many unanswered questions left by this work. For instance, we did not touch the question about the relation between \( NP \) and \( coNP \). Are these two complexity classes the same? Note that although \( P \) and \( NP \) differs and \( P = coP \), it is still possible that \( NP = coNP \), even if most complexity theorists believe that \( NP \neq coNP \). There is a subfield of computational complexity theory, namely the proof complexity, which is devoted to the goal of proving \( NP \neq coNP \). See reference [Coo00, Pap94] for the importance of this topic.

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