PROBABILITY FOR PRIMORDIAL BLACK
HOLES IN HIGHER DIMENSIONAL UNIVERSE

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Abstract
We investigate higher dimensional cosmological models in the semiclassical
approximation with Hartle-Hawking Boundary conditions, assuming a gravi-
tational action which is described by the scalar curvature with a cosmological
constant. In the framework the probability for quantum creation of an infla-
tionary universe with a pair of black holes in a multidimensional universe is
evaluated. The probability for creation of a universe with a spatial section
with $S^1 \times S^{D-2}$ topology is then compared with that of a higher dimensional
de Sitter universe with $S^{D-1}$ spatial topology. It is found that a higher di-
mensional universe with a product space with primordial black holes pair is
less probable to nucleate when the extra dimensions scale factors do not vary
in an inflating universe.

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1 **INTRODUCTION** :

Considerable work has been witnessed in the last two decades on the problem of the creation and subsequent evolution of primordial black holes in different contexts. In quantum cosmology one of the most challenging problems is the existence of primordial black holes (hence forth, PBH). It is now understood that the black holes are formed either, (i) due to gravitational collapse of a massive star when the mass of a given star exceeds about twice that of the solar mass or (ii) due to quantum fluctuation of matter distribution in the early universe. In 1975, the remarkable discovery of Hawking radiation ushered in a new era in black hole physics [1]. The life of the first kind of black holes are long, which are comparable to the age of the universe. Therefore, the only hope of confirming Hawking radiation by observation is through PBH hunting. Recently, Bousso and Hawking [2] (in short, BH) calculated the probability of the quantum creation of a universe with a pair of primordial black holes in (3 + 1) dimensional universe. In the paper they considered two different euclidean space-time: (1) a universe with space-like sections with topology $S^3$ and (2) a universe with space-like section with topology $S^1 \times S^2$, as is obtained in the Schwarzschild–de Sitter solution. With a suitable choice of the time variables, the first one gives an inflationary (de Sitter) universe while the second describes a Nariai universe [3], an inflationary universe with a pair of black holes. BH considered in their paper a massive scalar field which provided an effective cosmological constant for a while through a slow-rolling potential (mass-term). Paul *et al.* [4] following the approach of BH studied the probability of creation of PBH including $R^2$-term in the Einstein action and found that the probability is very much suppressed in
the $R^2$-theory. However, the evolution of a universe at Planck time $t \sim M_P^{-1}$, may be better understood in the framework of quantum gravity. A consistent quantum theory of gravity is yet to emerge. Kaluza and Klein [5] first initiated the study to formulate a unified theory of gravity by introducing an extra dimension. The approach has been revived and considerably generalized after realising that many interesting theories of particle interactions need more than four dimensions for their formulation [6]. It is considered essential to check if consistent cosmological or astrophysical solution, which can accommodate, these theories are also allowed. In this paper we intend to calculate the probability of quantum creation of a higher dimensional universe with $R \times S^{D-1}$ topology and $R \times S^1 \times S^{D-2}$ topology which accommodates a pair of primordial black holes. To calculate the probabilities for the creation of a universe with the two types of topology we use a semiclassical approximation for the evaluation of the euclidean path integrals. The condition that a classical spacetime should emerge, to a good approximation, at a large Lorentzian time was ensured by a choice of the path of the time parameter $\tau$ along the $\tau^{Re}$ axis from 0 to $\frac{\pi}{2H}$ and then continues along the $\tau^{im}$ axis. The semiclassical approximation for the wave-function of the universe is given by

$$\Psi_o[h_{ij}, \Phi_{\partial M}] \approx \sum_n A_n e^{-I_n}$$

(1)

where the sum is over the saddle points of the path integral, and $I_n$ denotes the corresponding Euclidean action. The probability measure of the creation of PBH is

$$P[h_{ab}, \Phi_{\partial M}] \sim e^{-2 I^{Re}}$$

where $h_{ab}$ is the boundary metric and $I^{Re}$ is the real part of the action corresponding to the dominant saddle point, i.e. the classical solution satisfying the Hartle-Hawking
[ henceforth, HH ] boundary conditions [7]. It is also important to explore the possible causes of splitting of a multidimensional universe. It has been shown that the observed 3 physical space may obtained starting from a multidimensional universe by the Freund-Rubin ansatz [8]. Also Candelas and Weinberg [9] observed that a split universe may be obtained by considering fluctuations in massless \((N + 1)\) dimensional matter fields. In this paper we show that the existence of a pair of PBH requires a split universe. However, the probability of creation of such a universe is found to be very low. It is found that a higher dimensional universe nucleates with all of its dimensions expanding equally. However, in course of its evolution, the universe with 3-physical space behave differently from that of the internal space [10]. It was believed that all inflationary models lead to \(\Omega_0 \sim 1\) to a great accuracy. This view was modified after it was discovered that there is a special class of inflaton effective potentials which may lead to a nearly homogeneous open universe with \(\Omega_0 \leq 1\) at the present epoch. Cornish et al. [11, 12] studied the problem of pre-inflationary homogeneity and outlines the possibility of creation of a small, compact, negatively curved universe. We show that a universe with \(S^D\)-topology may give birth to an open inflation.

The paper is organised as follows: in sec. 2 we write the gravitational action for a multidimensional universe and present gravitational instanton solutions and in sec. 3 we use the action to estimate the relative probability of the two types of the universes and in sec. 4 we give a brief discussion.
2 GRAVITATIONAL INSTANTON SOLUTIONS WITH OR WITHOUT A PAIR OF PRIMORDIAL BLACK HOLES:

We consider a higher dimensional Euclidean action given by

\[ I_E = -\frac{1}{16\pi} \int d^Dx \sqrt{g} [R - 2\Lambda] - \frac{1}{8\pi} \int_{\partial M} d^{D-1}x \sqrt{h} K \]

where \( g \) is the D-dimensional Euclidean metric, \( R \) is the Ricci scalar and \( \Lambda \) is the D-dimensional cosmological constant. In the gravitational surface term, \( h_{ab} \) is the boundary metric and \( K = h^{ab} K_{ab} \) is the trace of the second fundamental form of the boundary \( \partial M \) in the metric. The Euclidean action takes the form

\[ I_E = -\frac{1}{16\pi} \int d^{D-1}x \sqrt{h} [K_{ab} K^{ab} + K^2 + D-1 R - 2\Lambda] + \frac{1}{8\pi} \int_{\tau=0} d^{D-1}x \sqrt{h} K \]

(2)

using the no boundary proposal of Hartle-Hawking which is assumed to satisfy at \( \tau = 0 \). Here \( D-1 R \) is the scalar curvature of the (D - 1) dimensional surface.

(A) Topology \( S^D \), the de Sitter spacetime:

In this section we study vacuum solutions of the Euclidean Einstein equation with a cosmological constant in D dimensions. We now look for a solution with spacelike section \( S^n \). Hence we choose the D-dimensional metric ansatz

\[ ds^2 = d\tau^2 + a^2(\tau)d\Omega^2_n \]

(3)

where \( a \) is the scale factor of a D-dimensional universe, \( n = D - 1 \) and \( d\Omega^2_n \) is a line element on the unit (D - 1) sphere. The scalar curvature is given by

\[ R = - \left[ 2n \frac{\ddot{a}}{a} + n(n - 1) \left( \frac{\dot{a}^2}{a^2} - \frac{1}{a^2} \right) \right]. \]
where an overdot denotes differentiation with respect to \( \tau \). We rewrite the action (2) and obtain

\[
I_E = -V_o \int^{\tau_M}_{\tau=0} \left[ 2(D - 1)(D - 2)a^{D-3}(\dot{a}^2 + 1) - 2\Lambda a^{D-1} \right] d\tau + [-2V_o(D-1)\dot{a}a^{D-2}]_{\tau=0}
\]

(4)

where \( V_o = \frac{1}{16\pi^2} \frac{2\pi(D^2+1)}{(D+1)} \), we have eliminated \( \ddot{a} \) by integration by parts. The field equations can now be obtained by varying \( I_E \) with respect to \( a \), giving

\[
2\ddot{a}a + (D - 3) \left[ \frac{\dot{a}^2 - 1}{a^2} \right] + \frac{2\Lambda}{D - 2} = 0.
\]

(5)

The field eq.(5) allows an instanton solution which is given by

\[
a = \frac{1}{H_o} \sin H_o \tau
\]

(6)

where \( H_o^2 = \frac{2\Lambda}{(D-1)(D-2)} \). We note that this solution satisfies the HH no boundary conditions viz., \( a(0) = 0, \dot{a}(0) = 1 \). One can choose a path along the \( \tau^{Re} \) axis to \( \tau = \frac{\pi}{2H} \), the solution describes half of the Euclidean de Sitter instanton \( S^D \). Analytic continuation of the metric (3) to Lorentzian region, \( x_1 \rightarrow \frac{\pi}{2} + i\sigma \), gives

\[
ds^2 = d\tau^2 + a^2(\tau)[-d\sigma^2 + \cosh^2 \sigma d\Omega_{n-1}^2]
\]

(7)

which is a spatially inhomogeneous de Sitter like metric. However, if one sets \( \tau = it \) and \( \sigma = i\frac{\pi}{2} + \chi \), the metric becomes

\[
ds^2 = -dt^2 + b^2(t)[d\chi^2 + \sinh^2 \chi d\Omega_{n-1}^2]
\]

(8)

where \( b(t) = -ia(it) \). It describes the creation of an open inflationary higher dimensional universe. The real part of the Euclidean action corresponding to the solution calculated by following the complex contour of \( \tau \) suggested by BH is given by

\[
I^{Re}_{S^D-1} = -2V_o \sqrt{\frac{(D-1)(D-2)}{(2\Lambda)^{D-2}}} \left[ I_{n-2} - I_n \right]
\]

(9)
where $n = D - 1$ and $I_n = \int_0^\pi \sin^n y \, dy$ with $y = H_0 \tau$. The value of the integral $I_n$ depends on the dimensions of the universe, for odd number of dimensions

$$I_n = \frac{n - 1}{n} \frac{n - 3}{n - 2} \ldots \frac{3}{2} \frac{1}{2} \pi, \text{ for } n \text{ even}$$

and for even number of dimensions

$$I_n = \frac{n - 1}{n} \frac{n - 3}{n - 2} \ldots \frac{4}{3} \frac{2}{3}, \text{ for } n \text{ odd}.$$

With the chosen path for $\tau$, the solution describes half the de Sitter Instanton in a higher dimensional universe with $S^D$ topology, joined to a real Lorentzian hyperboloid of topology $R^1 \times S^{D-1}$. It can be joined to any boundary satisfying the condition $a_{\partial M} > 0$. For $a_{\partial M} > H_0^{-1}$, the wave function oscillates and predicts a classical space-time.

(B) Topology $S^1 \times S^d$:

We consider vacuum solution of the Euclidean Einstein equation with a cosmological constant and look for a universe with $S^1 \times S^d$-spacelike sections as this topology accommodates a pair of black holes. The corresponding ansatz for $(1 + 1 + d)$ dimensions is given by

$$ds^2 = d\tau^2 + a^2(\tau)dx^2 + b^2(\tau)\,d\Omega^2_\text{d}$$

(10)

where $a(\tau)$ is the scale factor of two sphere and $b(\tau)$ is the scale factor of the d-sphere given by the metric

$$d\Omega^2_\text{d} = dx_1^2 + \sin^2 x_1 \, dx_2^2 + \sin^2 x_1 \sin^2 x_2 \, dx_3^2 + \ldots \, d - space.$$ 

The scalar curvature is given by

$$R = - \left[ 2\ddot{a} + 2d\ddot{b} + d(d - 1) \left( \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} \right) + 2d\frac{\dot{b}^2}{ab} \right].$$
The Euclidean action (2) becomes

\[ I_E = -V'_o \int_{\tau=0}^{\tau_{0M}} \left[ d(d-1)ab^{n-2} \left( \dot{b}^2 + 1 \right) + 2d\dot{a}\dot{b}^{d-1} - 2\Lambda ab^d \right] d\tau + V'_o[-2\dot{a}b^d - 2d\dot{a}\dot{b}^{d-1}]_{\tau=0} \]

where \( V'_o = \frac{1}{4\pi} \frac{\pi^{(D-1)/2}}{\Gamma(D/2)} \).

Variation of the action with respect to the scale factors \( a \) and \( b \) gives the following field equations

\[ 2d\ddot{b} + d(d - 1) \left( \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} \right) + 2\Lambda = 0 \quad (11) \]

\[ \frac{\ddot{a}}{a} + (d - 1) \frac{\dot{b}}{b} + \frac{(d - 1)(d - 2)}{2} \left( \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} \right) + (d - 1) \frac{\dot{a}\dot{b}^{d-1}}{ab} + \Lambda = 0. \quad (12) \]

The field eqs. (11) and (12) admit a solution which is given by

\[ a = \frac{1}{H_o} \sin(H_o\tau), \quad b = \sqrt{d-1} H_o^{-1}, \]

\[ H_o^2 = \frac{2}{d} \Lambda \quad (13) \]

for \( n > 1 \) i.e., valid for dimensions \( D > 2 + 1 \). This solution satisfies the HH boundary conditions \( a(0) = 0, \dot{a}(0) = 1, b(0) = b_o, \dot{b}(0) = 0 \). Analytic continuation of the metric (10) to Lorentzian region, i.e., \( \tau \to it \) and \( x \to \frac{\pi}{2} + i\sigma \) yields

\[ ds^2 = -dt^2 + c^2(t)d\sigma^2 + H_o^{-2}d\Omega_{n-1}^2 \quad (14) \]

where \( c(t) = -ia(it) \). In this case the analytic continuation of time and space do not give an open inflationary universe. The corresponding Lorentzian solution is given by

\[ a(\tau^{Im})|_{\tau^{Re}=-\frac{\pi}{2H}} = H^{-1} \cosh H\tau^{Im}, \]

\[ b(\tau^{Im})|_{\tau^{Re}=-\frac{\pi}{2H}} = H^{-1} \]
Its space like sections can be visualised as \((D - 1)\) spheres of radius \(a\) with a ‘hole’ of radius \(b\) punched through the north and south poles. The physical interpretation of the solution is that of \((D - 1)\) - spheres containing two black holes at opposite ends. The black holes have the radius \(H^{-1}\) accelerates away from each other with the expansion of the universe. The real part of the action can now be determined following the contour suggested by BH [2], and it is given by

\[
I_{S^1 \times S^d}^{Re} = - \left[ 2V_0' \left( \frac{d(d - 1)}{2\Lambda} \right)^{d/2} \right]
\]

where \(d = D - 2\). The solution (13) describes a universe with two black holes at the poles of a \((D - 1)\)- sphere. The contribution of the action in this case comes from the surface term only.

3 EVALUATION FOR THE PROBABILITY FOR PRIMORDIAL BLACK HOLES :

In the previous section we have calculated the actions for inflationary universe with or without a pair of black holes. We now compare the probability measure. The probability for creation of a higher dimensional de Sitter universe may be obtained from the action (9). Thus, the probability for nucleation of a higher dimensional universe without PBH is given by

\[
P_{S^{D-1}} \sim \exp \left[ \frac{(D - 1)(D - 2)}{4\pi} \frac{2\pi^{(D+1)/2}}{\Gamma(D+1/2)} \left( \frac{(D - 1)(D - 2)}{2\Lambda} \right)^\frac{D-2}{2} (I_{D-3} - I_{D-1}) \right]
\]

where \(I_n < 1\) and \(I_{D-3} > I_{D-1}\). The action evaluated in this case is always negative. However for an inflationary universe with a pair of black holes the corresponding
probability of nucleation can be obtained from the action (15). The corresponding probability is

\[ P_{S^1 \times S^{D-2}} \sim \exp \left[ \frac{\pi^{(D-1)/2}}{2\Gamma\left(\frac{D-1}{2}\right)} \left( \frac{(D-2)(D-3)}{2\Lambda} \right)^{(D-2)/2} \right] \]  

(17)

for \( D > 3 \). For \( D = 4 \) we recover the result obtained by Bousso and Hawking [2]. However, in a higher dimensional universe the probability of nucleation of a \( S^{D-1} \) - topology is found more than that of a universe with \( S^1 \times S^{D-2} \) - topology in the presence of primordial black holes.

4 DISCUSSIONS :

In this work we have evaluated the probability for primordial black holes pair creation in a higher dimensional universe ( \( D > 4 \) ). In section 2, we have obtained the gravitational instanton solutions in two cases: a universe with (i) \( R \times S^{D-1} \) - topology and a universe with (ii) \( R \times S^1 \times S^{D-2} \) - topology respectively. The Euclidean action is then evaluated. We found that the probability of a universe with \( R \times S^{D-1} \) topology turns out to be much lower than a universe with a split universe with topology \( R \times S^1 \times S^{D-2} \) to begin with. It may be mentioned here that one gets a regular instanton in \( S^D \) - topology if there are no black holes. The existence of black holes restricts such a regular topology. The result obtained here on the probability of creation of a higher dimensional universe with a pair of primordial black holes is found to be strongly suppressed. Such a universe inflates in one particular dimensions whereas the other scale factors do not vary with time. It may be mentioned here that in a de Sitter like multidimensional universe all the scale factors expands equally to begin with. However, analytic continuation of a \( R \times S^{D-1} \) metric considered here
to Lorentzian region leads to an open 3-space and closed extra space. One obtains Hawking-Turok [13,14] type open inflationary universe in this case. In the other type of topology with split universe such a open inflation is not permitted. A detail study of an open inflationary universe will be discussed elsewhere. Finally the result obtained here may be used to answer the question on splitting of a multi dimensional universe.

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References

[1] S. W. Hawking, *Comm. Math. Phys.* **43**, 199 (1975).

[2] R. Bousso and S. W. Hawking, *Phys. Rev.* D **52**, 5659 (1995).

[3] H. Nariai, *Sci. Rep. Tohoku Univ.* **35**, 62 (1951).

[4] B. C. Paul, G. P. Singh, A. Beesham and S. Mukherjee, *Mod. Phys. Letts.* A **13**, 2289 (1998).

[5] T. Kaluza, *Sitz. Preuss. Acad. Wiss.* F **1**, 966 (1921); O. Klein, *Z. Phys.* **37**, 895 (1926).

[6] E. W. Kolb and M. S. Turner, *The Early Universe* (Addission-Wesley Pub. Company, Inc., 1988); J. H. Schwarz, *Superstrings Vol. I and II* (World Scientific, Singapore, 1985).

[7] J. B. Hartle and S. W. Hawking, *Phys. Rev.* D **28**, 2960 (1983).

[8] P. G. O. Freund and M. A. Rubin, *Phys. Letts.* B **97**, 233 (1980).

[9] P. Candelas and S. Weinberg, *Nucl. Phys.* B **237**, 397 (1984).

[10] D. Sahdev, *Phys. Letts.* B **137**, 155 (1984).

[11] N. J. Cornish, D. N. Spergel and G. D. Starkman, *Phys. Rev. Letts.* **77**, 215 (1996); *Class. Quantum Grav.* **15**, 2657 (1998).

[12] N. J. Cornish and D. N. Spergel, ‘*A small Universe after all?’* - astro-ph/9906401.
[13] S. W. Hawking and N. G. Turok, *Open Inflation without false Vacuum*, hep-th/9802030.

[14] N. G. Turok and S. W. Hawking, *Open Inflation, the four form and the cosmological constant*, hep-th/9803156.