Electron and phonon correlations in systems of one-dimensional electrons coupled to phonons

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Electron and phonon correlations in systems of one-dimensional electrons coupled to phonons are studied at low temperatures by emphasizing on the effect of electron-phonon backward scattering. It is found that the $2k_F$-wave components of the electron density and phonon displacement field share the same correlations. Both correlations are quasi-long-ranged for a single conducting chain coupled to one-dimensional or three-dimensional phonons, and they are long-ranged for repulsive electron-electron interactions for a three-dimensional array of parallel one-dimensional conducting chains coupled to three-dimensional phonons.

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I. INTRODUCTION

The physics of one-dimensional (1D) interacting electron systems has attracted lots of interests. The low-energy long-wavelength excitations of such systems can be described by a Tomonaga-Luttinger liquids (TLL) model [1]. Such systems exhibit a critical like behavior (power law) of the charge-density correlations at zero temperature. Thus, 1D electron systems are always at the verge of an instability without being to order [9]. On the other hand, it was noticed by Peierls [2] that due to electron-phonon interactions a charge density wave (CDW) accompanied by a periodic lattice distortion appears in the ground state of a 1D metal, both periods being $\pi/k_F$. For an arbitrary band filling the period is incommensurate with the underlying lattice, where the number of electrons per site is not a simple fractional number (e.g., $1/2$, $1/3$, etc.). Later studies [3] on incommensurate systems of 1D electrons coupled to 1D phonons (1D1D) show that the tendency to form CDW is increased significantly by electron-phonon coupling; however, true long-ranged-order CDW is still missing. In reality 1D electron systems are often embedded in three-dimensional (3D) environment, and phonons in such systems are effectively 3D. Therefore, it is interesting to know if the coupling between 1D electrons and 3D phonons could lead to long-range-order CDW.

For simplicity, we limit this paper to spinless incommensurate systems. Our work focuses on the effect of the electron-phonon coupling at large momentums (i.e., Backward-scattering process). We show that the $2k_F$-wave components of the electron density and phonon displacement field share the same correlations at large distance. This result applies to all the systems we consider in this paper. It is exact and does not depend on the electron-phonon coupling strength, phonon frequency and electron-electron interactions. This result implies that phonon correlations are quasi-long-ranged even for arbitrary weak electron-phonon coupling in 1D1D, in which previous studies [2] showed that electron correlations are quasi-long-ranged.

The first system we consider is a single conducting chain coupled to 3D acoustic phonons (1D3D). We find that electron-phonon coupling in this system does not change the qualitative power-law electron correlations, although it does increase the tendency to form CDW significantly (i.e., make electron correlations decay much slower). Surprisingly, our calculation shows that 1D3D even has less increased tendency to form CDW than 1D1D, given that both the phonon frequency and the electron-phonon coupling strength are the same in these two cases. Furthermore, phonon correlations are also quasi-long-ranged in 1D3D.

The second system we discuss is a 3D array of parallel 1D conducting chains couple to 3D acoustic phonons (3D3D). For arbitrary weak electron-phonon coupling, the system undergoes a quantum phase transition driven by varying electron-electron interactions: for repulsive electron-electron interactions, electrons on different chains are effectively correlated, which leads to long-ranged CDW and hence long-ranged phonon correlations; for attractive electron-electron interactions, electron correlations remain 1D, and the system is effectively a simple sum of 1D1D, which leads to quasi-long-ranged electron and phonon correlations. In the limit of strong coupling and/or low phonon frequency, both electron and phonon correlations are long-ranged for repulsive electron-electron interactions, and interestingly, the phase fluctuations of the $2k_F$-wave components of the electron density and phonon displacement field are not independent but locked together. The 3D3D system has also been considered recently by Artemenko et al [6]. Their work agrees with ours in the limit of strong coupling and/or low phonon frequency.

It was argued [4] that the electron-phonon coupling at small momentums (forward-scattering process) is not important since its effect is of the order $c^2/v_F^2 \ll 1$ ($v_F$ is the Fermi velocity, and $c$ is the sound velocity). However, it was pointed out [8] that in strongly correlated systems coupling to small-momentum acoustic phonons can lead to the Wenzel-Bardeen (WB) singularity [6,7] and hence become important. This singularity is a critical point at which for a critical electron-phonon coupling constant, the attractive electron-phonon interac-
tions cause the system to be unstable. It was shown that the critical point can be reached as one approaches the half-filling for the Hubbard model. In this paper we assume that the system is far away from the WB singularity for a general filling, and the electron-phonon coupling at small momentums is thus not important. In fact we even find that the WB singularity does not exist in 1D3D no matter the filling. This is because the bulk phonon freedom suppresses phonon fluctuations on the chain, thus making the system robust.

This paper is arranged as follows. In Sec. II we introduce the model. In Sec. III we review 1D1D systems. In Sec. IV and Sec. V we discuss 1D3D and 3D3D systems, respectively. Finally some technical details are given in the appendixes.

II. MODEL

In this section we discuss the model for 3D3D, from which the models for 1D1D and 1D3D can be obtained as special cases. Let us start with electrons. Using the TLL description for electrons, the electron density inside the \( j_{th} \) chain can be written in the form

\[
\rho_{e}^{j} = -\frac{1}{\pi} \partial_{x}\phi_{j} + \frac{1}{\pi \alpha} \cos(2k_{F} + 2\phi_{j}),
\]

where \( \phi_{j} \) is a slowly varying phase, \( \hat{x} \) denotes the direction along the chains, \( \alpha \) is the cut-off and of the order of the lattice spacing, \( k_{F} \) is the Fermi momentum. The first term on the right-hand side of (1) is the long-wavelength part of the electron density, the second term is the fast oscillating part. The total electron action is a sum of TLL models

\[
S_{el}/\hbar = \frac{1}{2\pi K} \sum_{j} \int dx d\tau \left[ \frac{1}{v} (\partial_{x}\phi_{j})^{2} + v (\partial_{x}\phi_{j})^{2} \right],
\]

where \( v \) is renormalized Fermi velocity, \( K \) is a dimensionless parameter, which is bigger than 1 for attractive electron-electron interactions and less than 1 for repulsive electron-electron interactions, \( \tau \) is the imaginary time.

Since electrons are restricted on the chains, an effective phonon action in terms of the on-chain freedom is needed. It can be obtained by starting with the standard lattice version of a 3D phonon action and eliminating the off-chain phonon freedom. This effective action is expected to have the following form

\[
S_{ep}^{3D}/\hbar = \frac{1}{2\alpha \hbar} \sum_{j} \int dx d\tau \left[ m (\partial_{x}u_{j})^{2} + K_{x}u_{j}^{2} + K_{\perp} |u_{j} - u_{j+1}|^{2} \right],
\]

where \( \hbar \) is Planck Constant, \( \alpha \) is the lattice spacing, \( b \) is the distance between nearest-neighbor chains, \( m \) is the atom mass, \( \perp \) denotes directions perpendicular to the chains, \( K_{x} \) and \( K_{\perp} \) are the effective spring constants along \( \hat{x} \) and \( \perp \) directions, respectively.

In general the action for electron-phonon coupling can be expressed in the form

\[
S_{ep}/\hbar = \frac{\gamma}{\pi \hbar} \sum_{j} \int dx d\tau \left( (\partial_{x}u_{j}) (\partial_{x}\phi_{j}) + \frac{1}{\alpha} (\partial_{x}u_{j}) \times \cos(2k_{F}x + 2\phi_{j}) \right).
\]

where \( \gamma \) is the coupling constant. Plugging (11) into the above action we obtain

\[
S_{ep}/\hbar = \frac{\gamma}{\pi \hbar} \sum_{j} \int dx d\tau \left[ - (\partial_{x}u_{j}) (\partial_{x}\phi_{j}) + \frac{1}{\alpha} (\partial_{x}u_{j}) \times \cos(2k_{F}x + 2\phi_{j}) \right].
\]

The first piece on the right-hand side of (12) corresponds to the electron-phonon coupling at small momentums, which is responsible for the WB singularity. We will discuss it in section IV. The second piece is the coupling at large momentums, which is the primary focus of this paper. For this latter type of coupling, only the phonon field with wavelength near \( 1/2k_{F} \) is important due to the fast oscillation of the cosine term. We write

\[
u_{j}(x) = \frac{1}{2} e^{2ik_{F}x} \tilde{\psi}_{j}(x) + H.c.,
\]

where \( \tilde{\psi}_{j} \) is a slowly varying complex field. Then the action for the coupling at large momentums can be rewritten as

\[
S_{ep}/\hbar = \frac{i \gamma k_{F}}{2 \pi \hbar \alpha} \sum_{j} \int dx d\tau \left( \tilde{\psi}_{j} e^{-2i\phi_{j}} + H.c. \right).
\]

At this point, it is also convenient to have a phonon action in terms of \( \tilde{\psi}_{j} \). Plugging (13) into (3) we obtain

\[
S_{ph}[\tilde{\psi}_{j}]/\hbar = \frac{1}{2a \hbar} \sum_{j} \int dx d\tau \left( m (\partial_{x}\tilde{\psi}_{j})^{2} + 2K_{x} a^{2} \hbar k_{F} \right)
+ \frac{K_{x}}{2} \tilde{\psi}_{j}^{2} + \frac{K_{\perp}}{2} |\tilde{\psi}_{j} - \tilde{\psi}_{j+1}|^{2}.
\]

For convenience, we rescale the lengths and field such that the actions are fully expressed in terms of dimensionless quantities. An appropriate rescaling is the following

\[
\tau \rightarrow a \tau/(\pi v), \quad x \rightarrow x a \pi,
\]

\[
r_{\perp} \rightarrow b \tau_{\perp}/\pi, \quad \psi_{j} \rightarrow \sqrt{\frac{mv_{j}}{2a \hbar}} \tilde{\psi}_{j}.
\]
After the rescaling, the actions \( S_{el}/\hbar \), \( S_{ph}/\hbar \) and \( S_{ch}/\hbar \) become

\[
S_{el}/\hbar = \frac{1}{2\pi K} \sum_{j} \int dx \, \left[ (\partial_t \phi_j)^2 + (\partial_x \phi_j)^2 \right],
\]

\[
S_{ph}^{3D}(\psi_j)/\hbar = \frac{1}{2} \sum_{j} \int dx \, \left[ |\partial_t \psi_j|^2 + g_1^2 \left( \frac{1}{2n_e} \right)^2 \times |\partial_x \psi_j|^2 + g_1^2 |\psi_j|^2 + \left( \frac{C_\perp}{\pi^2} \right) |\psi_j - \psi_{j+1}|^2 \right],
\]

\[
S_{ch}/\hbar = ig_2 \sum_{j} \int dx \, \psi_j e^{-2i\phi_j} + \text{H.c.},
\]

where \( C_\perp \equiv K_\perp a^2/mv^2 \), the filling factor \( n_e = ak_F/\pi \) (i.e., the averaged number of electrons per lattice site) which is of order 1 for a general filling. Both \( g_1 \) and \( g_2 \) are dimensionless quantities defined as

\[
g_1 = 2n_e \left( \frac{\omega}{v} \right) = (h\omega v_k a) \left( \frac{h\pi v}{a} \right)^{-1},
\]

\[
g_2 = \gamma \sqrt{2\pi} \left( \frac{h^2}{ma^2} \right)^{1/2} \left( \frac{h\pi v}{a} \right)^{-3/2} \left( \frac{an_e}{\alpha} \right),
\]

where \( c \equiv a\sqrt{K_\perp}/m \) is the sound velocity along the chains, \( \omega_{2k_F} \) is the phonon frequency at wave vector \( 2k_F \), \( h^2/ma^2 \) is approximately the ground state energy of an atom with mass \( m \) trapped in an infinite one-dimensional potential well with width \( a \), \( h\pi v/\alpha \) is of the order of the Fermi energy. Typically \( g_1 \) is of order \( 10^{-5} - 10^{-6} \). In the present paper we assume

\[
g_2/g_1 \ll 1,
\]

which can be fulfilled for a general filling. This condition actually implies that the system is far away from the WB instability, so that the electron-phonon coupling at small momentums is not important. For the Hubbard model very close to half-filling, since \( g_1/g_2 \) which is proportional to \( v^1/2 \) drops rapidly to zero \( 15 \), the postulates \( 17 \) breaks down; therefore, the electron-phonon coupling at small momentums could be important. This will be further discussed in section \( 1V \).

### III. A SINGLE CONDUCTING 1D CHAIN COUPLED TO 1D PHONONS

The model for 1D1D can be recovered by restricting the index \( j \) to be 1 in the actions \( 12, 13, 14 \) and setting \( C_\perp \) to be 0. To lighten the notation we drop the index \( j \) in this section. First we want to show a very general result which is exact and does not depend on the values of \( K \), the electron-phonon coupling strength and phonon frequency. We introduce a new complex field \( \psi' \), which, in Fourier space, is related to \( \phi \) and \( \psi \) by

\[
\psi_R(\bar{q}) = \psi_R(\bar{q}) - \frac{2g_2 G_{ph}(\bar{q})}{\sqrt{V}} \int dx \, \sin 2\phi \, e^{i\bar{q} \cdot \vec{x}},
\]

\[
\psi_I(\bar{q}) = \psi_I(\bar{q}) - \frac{2g_2 G_{ph}(\bar{q})}{\sqrt{V}} \int dx \, \cos 2\phi \, e^{i\bar{q} \cdot \vec{x}},
\]

where \( V \equiv Lh\nu/(k_B T a^2) \), \( G_{ph}^{-1}(\bar{q}) = q_\perp^2 + g_1^2 (1/2n_e)^2 q_x^2 + g_2^2 \), the subscripts \( R \) and \( I \) denote the real and imaginary components, respectively. In terms of \( \psi' \) and \( \phi \) the total action can be nicely separated into two parts which are completely decoupled from each other:

\[
S_{\psi'}/\hbar = \sum_{\bar{q}} G_{ph}^{-1}(\bar{q}) \psi'(\bar{q}) \psi'(-\bar{q}),
\]

\[
S_{el}^{1D}/\hbar = \frac{1}{2\pi K} \int dx \, \left[ (\partial_t \phi)^2 + (\partial_x \phi)^2 \right] - \frac{g_2^2}{g_1} \int dx \, \phi_x dx \psi \, G_{ph}(\tau - \tau') \cos [2\phi(x, \tau) - 2\phi(x, \tau')],
\]

where the phonon correlation function is given by

\[
G_{ph}(\tau) \approx e^{-g_2|\tau|}. \tag{22}
\]

To obtain \( 21 \) we have neglected the phonon dispersions \( (\partial_x \phi)^2 \) since the leading order effect on electron correlations comes from the phonon kinetic energy \( (\partial_t \phi)^2 \). Since \( \psi' \) and \( \phi \) are decoupled, from \( 18 \) and \( 19 \) the fluctuations of \( \psi \) can be calculated as

\[
\langle \psi(\bar{q}) \psi(-\bar{q}) \rangle = 16\pi^2 a^2 g_2^2 G_{ph}^2(\bar{q}) \int dx \, \int \frac{d \bar{x}}{(2\pi)^3} \left[ e^{i\bar{q} \cdot \bar{x}} \times \langle \rho_{2k_F}(\bar{r}) \rho_{2k_F}(\bar{0}) \rangle \right] + G_{ph}(\bar{q}),
\]

where \( \rho_{2k_F} \) is the \( 2k_F \)-wave component of the electron density. Since \( G_{ph}(\bar{q}) \) tends toward a constant for \( q \ll g_1 \), this result implies that at large distance (i.e., \( r \gg g_1^{-1} \)) the \( 2k_F \)-wave component of the phonon displacement field has the same correlations as \( \rho_{2k_F} \).

To understand the detailed behavior of the correlations, more calculations are needed. In the limit of weak coupling and/or high phonon frequency (i.e., \( g_2 g_1 K_\perp^{-2} \ll 1 \)), the effort is put on the phonon-mediated effective electron action given by \( 21 \). Since the exponentially decaying phonon correlation \( G_{ph}(\tau) \) imposes a cutoff at \( \tau \approx 1/g_1 \), at the wavelength longer than \( 1/g_1 \) the model \( 21 \) essentially reduces to the one for free electrons but with renormalized \( K \), which we denote as \( K' \).

A Gaussian variational method \( 13 \) and a perturbative renormalization group (RG) \( 2, 13 \) have been applied to this model. Both find that electron-phonon coupling increases the tendency to form CDW. Specifically, in terms
of $K'$ the result is expressed as

$$\left( \frac{K}{K'} \right)^2 - 1 \sim \begin{cases} \left[ \left( \frac{g_2}{g_1} \right) g_1^{K-1} \right]^2 & K < \frac{3}{2} \\ \frac{g_2}{g_1} \ln g_1 & K = \frac{3}{2} \\ \frac{g_2}{g_1} & K > \frac{3}{2} \end{cases}. \quad (24)$$

The details of the calculation are given in appendix \[A\].

To obtain the above result, it has been assumed that the renormalization of $K$ is small, that is, $(K/K')^2 - 1 \ll 1$, which leads to a self-consistent condition. For $K < 1$, this self-consistent condition is just

$$\frac{g_2}{g_1} \ll g_1^{1-K}, \quad (25)$$

which prohibits the application of the Gaussian variational result in the region above the locus $OA$ in Fig. 2.

When the condition $(25)$ is violated, neither the Gaussian variational method nor the perturbative RG can give an analytical result. A new strategy is used. A mean-field variational method nor the perturbative RG can give an analytical result. A new strategy is used. A mean-field variational method nor the perturbative RG can give an analytical result.

The amplitude fluctuations are massive and hence negligible. Based on this belief we now work on the truncated action:

$$S_{\text{total}}^{1D}/\hbar = \frac{1}{2} \int dx d\tau \left\{ \frac{1}{\pi K} \left[ (\partial_x \phi)^2 + (\partial_y \phi)^2 \right] + \psi_0^2 \right. \times \left[ (\partial_x \theta)^2 + g_1^2 \left( \frac{1}{2n_e} \right)^2 (\partial_y \theta)^2 \right] \right\}$$

$$- 2g_2\psi_0 \int dx d\tau \cos (2\phi - \theta), \quad (26)$$

where $\psi_0$ is fixed at the saturated value, and the constant pieces have been thrown away. Hereafter we name this strategy as “fixed-amplitude approximation” just for convenience. In the following we will first estimate the saturated value of $\psi_0$, then calculate the fluctuations of $\phi$ and $\theta$ from the truncated model $(26)$, and finally discuss the limits of the validity of the fixed-amplitude approximation.

We estimate the saturated value of $\psi$ using a mean-field theory. Approximating $\psi$ as uniform both in space and time by doing $\psi \equiv \psi_0$, the total action reduces to

$$S_{\text{total}}^{1D}/\hbar = \frac{1}{2} \int dx d\tau \left\{ \frac{1}{\pi K} \left[ (\partial_x \phi)^2 + (\partial_y \phi)^2 \right] \right.$$

$$\left. - 2g_2\psi_0 \int dx d\tau \cos (2\phi) + \frac{1}{2} \int dx d\tau \ g_1^2\psi_0^2 \right\}, \quad (27)$$

where the electron part of the action is just a Sine-Gordon model. Directly integrating out the field $\phi$ in favor of $\psi_0$ using a Gaussian variational method, we get

$$S_{\text{ph}}^{\text{eff}}[\psi_0]/\hbar = \frac{g_1^2}{2} \int dx d\tau \left[ 2\pi K \left( 2\pi K g_1^{K-2} \psi_0 \right) \frac{\hbar}{\pi} \times (K - 2) + \psi_0^2 \right]. \quad (28)$$

The detailed calculation is shown in appendix \[B\]. The first term in the action $(25)$ is negative, which means that a periodic lattice distortion with period $\pi/k_F$ lowers the electron energy; the second term is the elastic energy for such a lattice distortion and hence positive. To find the ground state, we minimize the action $(28)$ with respect to $\psi_0$. We find

$$\psi_0 = g_1^{-1} \left[ 2(\pi K) \frac{\pi}{\hbar} \left( \frac{g_2}{g_1} \right) \right] \quad (29)$$

for $K < 1$ and $\psi_0 = 0$ for $K > 1$. This indicates that a periodic lattice distortion with period $\pi/k_F$ appears in the ground state for repulsive electron-electron interactions; however, $\psi$ is not long-range-ordered due to its phase fluctuations. Later in this section we will show that $\psi$ is quasi long-range-ordered with power-law correlations. For $K$ not very close to 1, the result given in $(29)$ qualitatively agrees with that in ref. \[3\], in which the electron freedom is eliminated by mapping the model $(27)$ to an exactly solvable field-theoretical model (i.e., Massive Thirring model) and calculating the ground state energy of electrons. In the limit of $K \to 1^-$, $(29)$ shows $\psi_0 \to 0$, and the classic Peierls theory predicts $\psi_0 \sim \exp (-g_1^2/g_2) / g_2$, which is essentially very small.

The above mean-field calculation assumes that the single electron gap opened at the Fermi level is much less than the Fermi energy, that is, $(g_2\psi_0)^{1/(2-K)} \ll 1$. This assumption combined with $(29)$ leads to the postulated condition $(17)$, or equivalently

$$g_1\psi_0 \ll 1. \quad (30)$$

Now we discuss the truncated model $(26)$. This model has been studied previously in the context of liquid crystal \[11\]. It can be shown that the coupling term in this model is either relevant or irrelevant depending on the values of $\psi_0$ and $K$. The critical values of $\psi_0$ and $K$ can be easily calculated in the limit $K \to 0$ (i.e., the electron field $\phi$ is locked) or $\psi_0 \to \infty$ (i.e., the phonon phase field $\theta$ is locked) since in both limits the model reduces to the Sine-Gordon model. We get $K_c = 2$ for $\psi_0 \to \infty$ and $\psi_0^c = \sqrt{n_e/\pi g_1}$ for $K \to 0$. In general, the critical values of $\psi_0$ and $K$ satisfies

$$\frac{n_e}{\pi g_1 \psi_0^c} + 2K = 4. \quad (31)$$

The detailed derivation is given in appendix \[C\]. This function defines a critical line in the $\psi_0^c - K$ plane, which is illustrated in Fig. 1. Thus, for $K < 1$, only for which we can get a non-zero $\psi_0$, we have

$$\psi_0^c \sim g_1^{-\frac{3}{2}}. \quad (32)$$
For $\psi_0 < \psi_0^0$, the cosine term becomes irrelevant, and $\phi$ and $\theta$ are decoupled. For $\psi_0 > \psi_0^0$, the cosine term is relevant, and the fluctuations of $\phi$ and $\theta$ are bound together. This actually implies a self-consistent condition for the fixed-amplitude approximation; that is, the estimated $\psi_0$ must be bigger than $\psi_0^0$ such that phase fluctuations do not drive the cosine term into irrelevance. This cosine term offsets the positive phonon energy and is necessary for obtaining a non-zero term offsets the positive phonon energy and is necessary

\[ \frac{g_2}{g_1} \gg \frac{1 - K}{\sqrt{g_1}}, \]  

(33)

which prohibits the using of the fixed-amplitude approximation in the region below the locus $OA$ in Fig. 2. For the systems satisfying (33) both the effective actions of $\phi$ and $\theta$ can be obtained by making $2\phi = \theta$ in the truncated model (29), and they are given by

\[ S[\Xi] / h = \frac{1}{2\pi K} \int dxd\tau \left[ R(\partial_x \Xi)^2 + B(\partial_x \Xi)^2 \right], \]  

(34)

where $\Xi$ can be either $\phi$ or $\theta/2$, and the parameters $R$ and $B$ are

\[ R = (1 + 4\pi K\psi_0^2)^2, \]  

(35)

\[ B = 1 + \pi K g_1^2 \left( \frac{1}{n_e} \right)^2 \psi_0^2. \]  

(36)

This result is consistent with our earlier conclusion that at large distance the $2k_F$-wave components of the phonon displacement field and electron density share the same correlations. Note that the correction to $B$ comes from the phonon dispersions along the chains and is much less than 1 according to the assumption (30). Also it is of the order of the correction to $R$ multiplied by $g_1^2$ and hence negligible. Plugging (29) into (35) and defining

\[ K' \equiv K / \sqrt{R}, \]  

we get

\[ \left( \frac{K}{K'} \right)^2 - 1 \sim g_1^2 \left( \frac{g_2}{g_1} \right)^2 \left( \frac{1}{2\pi K} \right)^{2}. \]  

(37)

Now let us discuss the validity of the fixed-amplitude approximation. In addition to the self-consistent condition (33) caused by phase fluctuations, there is another one caused by amplitude fluctuations, which is

\[ \langle (\delta \psi_0)^2 \rangle \ll \psi_0^2, \]  

(38)

where $\delta \psi_0$ is the deviation of the amplitude of $\psi$ from the saturated value. For a crude estimation of $\langle (\delta \psi_0)^2 \rangle$, we assume that the effective action for $\delta \psi_0$ (to the quadratic order) has the following form

\[ S[\delta \psi_0] / h = \frac{1}{2\pi^2} \int dxd\tau \left[ \frac{2(1 - K)}{2 - K}(\delta \psi_0)^2 + (\partial_x \delta \psi_0)^2 + g_1^2 \left( \frac{1}{2n_e} \right)^2 (\partial_x \delta \psi_0)^2 \right], \]  

(39)

where the mass term is obtained by expanding the mean-field action (28) around its minimum, and the derivative terms come from the free phonon action. Then after a straightforward calculation, the condition (38) also leads to (33) coincidentally.

In both limits the charge-density correlations can be calculated, given by the non-universal power laws

\[ \langle \rho_{2k_F}(r)\rho_{2k_F}(0) \rangle \sim r^{-2K'}, \]  

(40)

where $r = (x, \tau)$. This implies that in electron-phonon coupling does not change the qualitative TLL behavior,
although the tendency to form CDW is increased since $K'$ is smaller than $K$. Based on the general conclusion given in (23), at large distance phonon correlations are also governed by the same non-universal power laws:

$$\langle \psi(f)\psi(0) \rangle \sim r^{-2K'},$$

(41)

Surprisingly, this implies that in 1D1D while electron-phonon coupling does not have a qualitative effect on electron correlations, it does change phonon correlations qualitatively, that is, the correlations of the $2k_F$-wave component of the phonon displacement field, which are short-ranged in the absence of electron-phonon coupling, become quasi long-ranged. According to the self-consistent conditions given by (25) and (33), there is an intermediate region in which neither of the two approaches apply, and therefore no quantitative result is available. This region is illustrated in figure 2 as the area between the loci $OA$ and $OB$. However, in this region we do expect the same qualitative conclusion.

IV. A SINGLE CONDUCTING 1D CHAIN COUPLED TO 3D PHONONS

In this section we consider a single 1D conducting chain coupled to 3D phonons. In this case the action for electrons and electron-phonon coupling is the same as that in 1D1D, and the phonon action is given in (3). For convenience, we transform the phonon action into its continuum version and perform the dimensional rescaling given in (10) and the field rescaling

$$u' = \sqrt{\frac{mv}{2a\hbar}}u. \quad (42)$$

After this procedure, we obtain

$$S_{\text{ph}}^{3D}(u')/\hbar = \frac{1}{\pi^2} \int d^3r d\tau \left[ (\partial_r u')^2 + g_1^2 \left( \frac{1}{2n_e} \right)^2 \times (\partial_x u')^2 + C_\perp |\nabla_\perp u'|^2 \right]. \quad (43)$$

Since electron-phonon coupling only happens on the chain, an effective phonon action for the on-chain freedom is needed. We integrate out the bulk off-chain phonon freedom in favor of the on-chain one $u'_0(x,\tau) \equiv u'(x,\tau, r'_\perp = 0)$, where we have assumed that the position of the chain is at $r'_\perp = 0$. This leads to an effective 1D phonon action

$$S_{\text{ph}}^{1D}(u'_0)/\hbar = \frac{1}{2} \sum_q G^{-1}(q)|u'_0(q)|^2, \quad (44)$$

where

$$G(q) = \frac{\pi}{8C_\perp} \ln \frac{q_x^2 + \frac{4}{\pi} C_\perp + \left( \frac{q_\perp}{2n_e} \right)^2 q_\perp^2}{\frac{q_x}{2n_e} - \left( \frac{q_\perp}{2n_e} \right)^2 q_\perp^2}. \quad (45)$$

In the limit $C_\perp \to 0$, the renormalized 1D phonon propagator given by (45) reduces to the bare 1D one (i.e., the one which is not renormalized by the bulk phonon freedom) as expected. For $C_\perp$ being non-zero, for small $q_\perp$ the renormalized one has a logarithmic dependence on $q_\perp$ while the bare one has a second-order power-law dependence. This difference has a drastic effect on the electron-phonon coupling at small momentums. We delay this discussion to the later of this section. Now we focus our attention on the electron-phonon coupling at large momentums (i.e., near $2k_F$). Expanding the action (14) around $q_\perp = \pm 2ak_F/\pi, q_r = 0$ (the factor a/π is due to the rescaling), we get

$$S_{\text{ph}}^{1D}(\psi)/\hbar = \frac{1}{4} \sum_{q_\perp q_r} G^{-1} \left( \frac{2ak_F}{\pi} + q_x, q_r \right) |\psi(q_x, q_r)|^2, \quad (46)$$

where the complex phonon field $\psi$ is the slow varying part of the $2k_F$-wave component of the displacement field $u'_0$

$$u'_0(x) = \frac{1}{2} [e^{2ikFx}\psi(x) + H.c.]. \quad (47)$$

The inverse of the propagator in (46), to the lowest order of $q_x, q_r$, is given by

$$G^{-1} \left( \frac{2ak_F}{\pi} + q_x, q_r \right) = 2 \left( g_1^2 + g_3 q_y^2 + g_4 q_x^2 \right) \quad (48)$$

with

$$g_1' = 2 \sqrt{\frac{C_\perp}{\pi} \left( \ln \frac{g_1^2 + 4C_\perp/\pi}{g_1^2} \right)^{-1}}, \quad (49)$$

$$g_3 = \frac{4g_1^2 C_\perp/\pi}{g_3^2 (g_1^2 + 4C_\perp/\pi)} \left( \ln \frac{g_3^2 + 4C_\perp/\pi}{g_3^2} \right)^{-1}, \quad (50)$$

$$g_4 = \frac{\pi^2 g_1^2 g_3^2}{4ak_F^2} \left[ 1 + 4g_1^2 \left( \frac{g_3}{g_1^2} - \frac{1}{g_3^2 + 4C_\perp/\pi} \right) \right]. \quad (51)$$

To get (48), we have thrown away the term linear in $q_x$ (i.e., $q_x |\psi|^2$), which only leads to boundary effects. Plugging (48) into (46) we get

$$S_{\text{ph}}^{1D}(\psi)/\hbar = \frac{1}{2} \int dx d\tau \left[ g_4 |\partial_x \psi|^2 + g_1^2 |\psi|^2 + g_4 |\partial_x \psi|^2 \right]. \quad (52)$$

Therefore, with respect to the electron-phonon coupling at large momentums, the effective model of phonons in 1D3D is qualitatively the same as that in 1D1D but with modified coefficients. Thoughtful reader might have seen this right from the beginning [12]. Note that $g_1', g_3$ and $g_4$ are all increasing functions of $C_\perp$. This is expected since the bulk phonon freedom suppresses phonon fluctuations on the chain.

Since we have shown that the models for 1D3D and 1D1D are essentially the same, the calculations should be
also very similar. Repeating the (virtually same) calculation, we obtain, for 1D3D and repulsive electron-electron interactions,
\[
\left(\frac{K}{K'}\right)^2 - 1 \sim g_2 \frac{\sqrt{g_3}}{g_1} \left(\frac{2K}{g_1^2 g_2^2 (2K-4)}\right)^{3/2} - 2K
\]
in the limit of weak coupling and/or high phonon frequency and
\[
\left(\frac{K}{K'}\right)^2 - 1 \sim g_3 \left(\frac{2K}{g_2^2 g_1^2 (2K-4)}\right)^{3/2} - 2K
\]
in the limit of strong coupling and/or low phonon frequency. It can be verified that both \(K'\) obtained from the two limits are increasing functions of \(C_1\). Therefore, we conclude that a single 1D chain coupled to 3D phonons does not lead to long-range-ordered CDW, instead the increased tendency to form CDW in such systems is even less than that in 1D1D, given that the two dimensionless quantities \(g_1\) and \(g_2\) (or equivalently, the phonon frequency and the coupling strength) are the same in these two cases. This result does not seem obvious just from qualitative arguments. On one hand, the bulk phonon freedom increases the energy cost to form a lattice distortion along the chain, which is manifested in that \(q'_1\) is a increasing function of \(C_1\). This is a negative effect on the increased tendency to form CDW. On the other hand, both \(g_3\) and \(g_4\) are increasing functions of \(C_1\), which enhances the correlations of \(\psi\). This is a positive effect on the increased tendency to form CDW. Only a quantitative calculation can tell that the former effect dominates. In principle we could also think about a single 1D chain coupled to 2D phonons. We expect that in such systems long-range-ordered CDW is still missing, and the increased tendency to form CDW in such systems is even less than that in 1D1D, given that the two dimensionless quantities \(g_1, g_2\) and \(g_3\) are the same in these two cases. In addition, the \(2K_\perp\) - wave components of the phonon displacement field in 1D2D and 1D3D are quasi long-range-ordered due to electron-phonon coupling.

Now we discuss the electron-phonon coupling at small momenta. According to the action given in (45), the action for the electron-phonon coupling at small momenta is given by
\[
S_{el}^s/\hbar = -\frac{\gamma}{\pi\hbar} \int dx \tau (\partial_x u_0) (\partial_x \phi) . \tag{55}
\]
After the dimensional and field rescaling which are given in (10) and (42) respectively, the above action becomes
\[
S_{el}^s/\hbar = -\frac{2\pi \alpha g_2}{a n_e} \int dx \tau (\partial_x u_0) (\partial_x \phi) . \tag{56}
\]
The effective action for the on-chain phonon field is given by (12). Since the total action is quadratic in \(u_0\), we integrate it out. This leaves us an effective electron action
\[
S_{el}^{total} = -\frac{1}{2\pi \hbar} \sum_{\mathbf{q}} \left[ q_x^2 + q_\tau^2 - 4\pi^3 K \left(\frac{\alpha g_2}{a n_e}\right)^2 q_x^4 G(q) \right] \times \phi(-q) \phi(q) . \tag{57}
\]
where \(G(q)\) is the phonon propagator. The first two terms on the right-hand side of (57) are from free electrons, and the third term, which is retarded and negative, is contributed by phonons. First let us treat phonons as pure 1D. Plugging \(G(q)\) as the bare 1D phonon propagator into (57), we get
\[
S_{el}^{total} = \frac{1}{2\pi K} \sum_{\mathbf{q}} \left[ q_x^2 + q_\tau^2 - \frac{2\pi^3 K \alpha^2 g_2^2 q_x^4}{a^2 n_e^2} \left(\frac{\alpha g_2}{a n_e}\right)^2 q_x^4 \right] \times \phi(-q) \phi(q) . \tag{58}
\]
In the limit \(q_\tau \to 0\), the retarded part is proportional to \(q_x^2\) and cancels the first term at the point
\[
\frac{1}{K} = \frac{8\pi^3}{\alpha^2 g_2^2} , \tag{59}
\]
at which the electron density becomes unstable towards long wavelength fluctuations. This singular point is referred to as the Wentzel-Bardeen (WB) singularity. Since \(K\) and \(\alpha/a\) are both of order 1, as long as the postulates (17) is fulfilled, the left-hand side of (59) is much larger than the right-hand side, and the electron-phonon coupling at small momentums can thus be safely neglected. As we mentioned earlier, for the Hubbard model near half-filling, the postulates (17) breaks down, and the WB singularity is reachable; however, in this case the umklapp effect becomes also important, and whether the WB singularity is robust against this additional effect remains an open question. Now we treat phonons as 3D. Plugging (45) into (57), we obtain
\[
S_{el}^{total} = \frac{1}{2\pi K} \sum_{\mathbf{q}} \left[ q_x^2 + q_\tau^2 - \frac{4\pi^4 K}{2C_\perp} \left(\frac{\alpha g_2}{a n_e}\right)^2 q_x^4 \right] \times \ln \left[ q_x^2 + \frac{4\pi C_\perp}{a^2 n_e^2} q_x^4 \right] \phi(-q) \phi(q) . \tag{60}
\]
For small \(q\)’s the negative retarded part in (60) is of the order \(q_\perp^4\) with a logarithmic correction, which is subdominant to \(q_x^4\). This implies that the WB singularity does not exist in 1D3D. This makes sense since the bulk phonon freedom suppresses phonon fluctuations on the chain and thus stabilizes the system against the attractive electron-phonon coupling. Therefore, in a 1D3D system the electron-phonon coupling at small momentums is not important no matter the filling. A similar discussion for 1D2D leads to the same conclusion.

V. 3D ARRAY OF PARALLEL 1D CONDUCTING CHAINS COUPLED TO 3D PHONONS

In the previous two sections we have discussed two special systems: 1D1D and 1D3D. In this section we will...
study a more realistic system: a 3D array of parallel 1D conducting chains coupled to 3D phonons.

In 1D1D we derived an exact relation between phonon and electron correlations, which is given in [23]. Performing an almost identical calculation we also find a similar result in 3D:

$$\langle \psi(\vec{q}) \psi(-\vec{q}) \rangle = 16\pi^4 \alpha^2 g_\perp^2 G_{ph}(\vec{q}) \sum_{\vec{r}_\perp} \int dxd\tau \ e^{i\vec{q}\cdot\vec{r}} \times$$

$$\langle \rho_{2k,\vec{r}}(\vec{r}) \rho_{2k,\vec{r}}(\vec{0}) \rangle + G_{ph}(\vec{q}) \ ,$$

(61)

where $G_{ph}^{-1}(q) \equiv [g_{\perp}^2 + q_x^2 + (g_1/2n_e)^2 q_z^2 + C_{\perp} q_{\perp}^2]/\pi^2$. We have replaced the sum over chain index $j$ by the sum over $\vec{r}_\perp$ with $\vec{r}_\perp = n\pi$ (remember that the distance between nearest-neighbor chains is $\pi$ after the dimensional rescaling given in (11)), $n$ an integer. We will adopt this change hereafter.

For a further understanding of this problem, we follow the logic used in 1D1D and treat this 3D3D problem based on the same two limits. Let us start with the limit $G_{el}$ where both the free energy $F$ and electron correlations, which is given in (23). Per-

$$S_{eff}/\hbar = \frac{S_{el}}{h} - 2g_{\perp}^2 \sum_{\vec{r}_\perp,\vec{r}_\perp'} \int d^2 r_{\perp} d^2 r_{\perp}' d\tau d\tau'$$

$$\{ \cos [2\phi(x, \tau, \vec{r}_\perp) - 2\phi(x, \tau', \vec{r}_\perp')] \times$$

$$G_{ph}(\tau - \tau', \vec{r}_\perp - \vec{r}_\perp') \} ,$$

(62)

where the phonon propagator is given by

$$G_{ph}(\tau, \vec{r}_\perp) = \frac{1}{(2\pi)^4} \int d^2 q_{\perp} dq_{\perp} [$$

$$\frac{\pi^2}{q_{\perp}^2 + C_{\perp} q_{\perp}^2 + g_{\perp}^2} \times$$

$$e^{i(q_{\perp} \cdot \vec{r}_\perp + q_{\perp} \cdot \tau)}] .$$

(63)

We have neglected the phonon dispersions along the chains, which only lead to insignificant correction. The phonon dispersions perpendicular to the chains, although typically very small, are important since they are responsible for the existence of long-range-order CDW. This will be seen later. Now we apply a self-consistent Gaussian variational method [5] to this model. We approximate the action (62) as a quadratic one

$$S_0/\hbar = \frac{1}{2} \sum_{\vec{q}} G_{el}^{-1}(\vec{q}) \phi(-\vec{q}) \phi(\vec{q}) .$$

(64)

Then we try to optimize the propagator $G_{el}^{-1}(\vec{q})$ so as to minimize the variational free energy

$$F_0 = F_0 + \langle S - S_0 \rangle_0 ,$$

(65)

where both the free energy $F_0$ and the average $\langle \rangle_0$ are calculated using the quadratic action $S_0$. Performing the procedure we obtain a self-consistent equation

$$G_{el}^{-1}(\vec{q}) = \frac{1}{(\pi^3 K^2)} (q_x^2 + q_z^2) + 16 < g_{\perp}^2 >^2 \times$$

$$\sum_{\vec{r}_\perp} \int d\tau \ e^{-4(\phi^2)_{0} - (\phi(0,0,0)(0,\tau,\vec{r}_\perp))} \times$$

$$G_{ph}(\tau, \vec{r}_\perp) \left[ 1 - e^{i(q_{\perp} \cdot \vec{r}_\perp + q_{\perp} \cdot \tau)} \right] .$$

(66)

We notice that the right-hand side of the above equation vanishes as $\vec{q} \to 0$. Furthermore, if we naively expand the cosine in the action (62) to the quadratic order of $\phi$, we will get corrections to $|\vec{\nabla}_\perp \phi|^2$ and $|\partial_\perp \phi|^2$ and no corrections to $|\partial_\perp \phi|^2$. Therefore, we assume the following trial solution

$$G_{el}^{-1}(\vec{q}) = \frac{1}{(\pi^3 K^2)} (q_x^2 + q_z^2 + M q_{\perp}^2) .$$

(67)

Hereafter we define $K'$ as the renormalized $K$, given by

$$K' = K/\sqrt{M} .$$

(68)

The self-consistent equation (66) can be solved analytically in the limit $M \ll 1$, $(K/K')^2 - 1 \ll 1$. In this limit the system is highly anisotropic. Electron correlations along the chains decay as a power law at length scales shorter than $1/\sqrt{M}$ and tend towards a non-zero constant at longer length scales, while electron correlations along $\perp$ directions decay very rapidly to the same non-zero constant. Specifically, they are given as

$$e^{-4(\phi^2)_{0} - (\phi(0,0,0)(0,\tau,\vec{r}_\perp))} \times$$

$$= \begin{cases}$$

$$\tau^{-K'/2}, & \vec{r}_\perp = 0, \tau < 1/\sqrt{M} \\

$$M^{K'/4}, & \vec{r}_\perp \neq 0 \ or \ \tau > 1/\sqrt{M} ,$$

(69)

which reduce to the correlations of TLL in the limit $M \to 0$. Plugging the formula (69) into the equation (66) we obtain two coupled self-consistent equations

$$\left( \frac{K}{K'} \right)^2 - 1 = 16\pi K g_{\perp}^2 \int_{-1/\sqrt{M}}^{1/\sqrt{M}} d\tau \tau^{-2K'} \tau^2 G_{ph}(\tau, 0)$$

$$+ 32\pi K g_{\perp}^2 \int_{-1/\sqrt{M}}^{\infty} d\tau \ M^{K'/4} \tau^2 G_{ph}(\tau, 0)$$

$$+ 16\pi K M^{K'/4} g_{\perp}^2 g_{\perp}^{-1}$$

$$M = 16\pi K M^{K'/4} g_{\perp}^2 g_{\perp}^{-1}$$

(70)

where

$$G_{ph}(\tau, 0) = \frac{\pi}{4C_{\perp}} \left[ e^{-\tau^{K'} - \tau^{1-K'}} \sqrt{\frac{\pi}{2g_{\perp}}} \right] .$$

(72)

In the special case $C_{\perp} = 0$, $M$ vanishes, and the second and the third terms on the right-hand side of (70) thus vanish, and then (70) reduces to the result of 1D1D. This is expected since the 3D3D model with $C_{\perp} = 0$ corresponds to a simple sum of 1D1D models. For $C_{\perp}$ being nonzero, $M$ can be easily calculated from (71) as

$$M = \begin{cases}$$

$$0, & K' > 1, \\

$$16\pi K C_{\perp} g_{\perp}^2 g_{\perp}^{-1} \left( \frac{1}{1-K'} - 1 \right), & K' < 1 .$$

(73)
After a more involved, but essentially straightforward calculation, we get, to the lowest order of \((K/K')^2 - 1\),

\[
\left( \frac{K}{K'} \right)^2 - 1 \sim g_2^2 g_1^{2K-4}
\]  

(74)

for \(C_\perp \ll g_1^2\) and

\[
\left( \frac{K}{K'} \right)^2 - 1 \sim \left\{ \frac{1}{M/C_\perp}, \frac{M \ll g_1^2}{M \gg g_1^2} \right\},
\]  

(75)

for \(C_\perp \gg g_1^2\). The solution of \(M\) implies a quantum phase transition at about \(K = 1\) for arbitrary weak electron-phonon coupling. This critical point can also be obtained by calculating the average of the retarded part of the action for a single chain with respect to the free electron action. This average scales as \(L^{2-2K}\) and thus also predicts the critical point \(K = 1\). For \(K < 1\), electrons on different chains are effectively correlated, which leads to long-range-order CDW. The order parameter is the 2\(k_F\)-wave component of the electron density, whose average is non-zero and scales as

\[
\langle \rho_{2k_F} \rangle \sim \frac{M^{K-2}}{2\pi^2}.
\]  

(76)

According to the general result given in (61), the 2\(k_F\)-wave component of the phonon displacement field also becomes long-range-ordered with its expectation value

\[
\langle \psi \rangle \sim \frac{g_2}{g_1^2} M^{K-2}. 
\]  

(77)

For \(K > 1\), electrons on different chains are effectively not correlated, and each behaves as an isolated single 1D conducting chain coupled to 3D phonons, which has been discussed in section IV. For self-consistency, the solutions (73), (74) and (75) have to satisfy \(M \ll 1\), \((K/K')^2 - 1 \ll 1\), which leads to a restriction on the parameters

\[
\begin{align*}
\frac{g_2}{g_1} & \ll g_1^{1-K}, \quad C_\perp \ll g_1^2, \\
\frac{g_2}{g_1} & \ll \frac{g_2}{g_1}, \quad C_\perp \gg g_1^2.
\end{align*}
\]  

(78)

This condition precludes the application of the Gaussian variational result in the region above \(O.A\) in figure 2.

For stronger coupling and/or lower phonon frequency, the result we get from the Gaussian variational method is not valid since the condition (78) is violated. In this case we have to use the fixed-amplitude approximation, which has been discussed in section III. We assume that the important fluctuations of \(\psi\) are its phase fluctuations, and the amplitude of \(\psi\) can be approximated as fixed at its saturated value \(\psi_0\). After this approximation we obtain a simplified model

\[
S_{total}^{3D}/\hbar = -\frac{1}{2\pi^3} \int dx d^2 r_\perp d\tau \left\{ \frac{1}{K} \left[ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right] + \right\}
\]

\[
\left[ \right. (\partial_\tau \theta)^2 + g_1^2 \left( \frac{1}{2n_e} \right)^2 (\partial_x \theta)^2 + C_\perp \left( \nabla_\perp \theta \right)^2 \left. \right\} \times \pi \psi_0^2 - \frac{2g_2 \psi_0}{\pi^2} \int dx d^2 r_\perp d\tau \cos (2\phi - \theta),
\]  

(79)

where we have transformed the action into its continuum version, and the constant pieces have been thrown away. We will calculate the fluctuations of \(\phi\) and \(\theta\) based on this truncated model.

The locally saturated value of \(\psi\) is also estimated by using the mean-field theory, which approximates \(\psi\) as uniform both in space and time. Since there are no bare electron correlations between different chains, the mean-field 3D3D model is a simple sum of the mean-field 1D1D models. Therefore, the mean-field calculation is the same as that in 1D1D, and the result is thus also the same; that is, \(\psi_0\) is nonzero and given by the formula (29) for \(K < 1\), and zero for \(K > 1\). However, unlike in 1D1D, \(\psi\) is now truly long-range-ordered for \(K < 1\). In the following we will include the phase fluctuations and calculate the non-zero expectation value of \(\psi\).

Now we discuss the truncated model (79). It can be shown that the cosine term in the model is always relevant no matter the value of \(\psi_0\), as long as \(K\) is less than 2. The proof is given in appendix C. Therefore, for \(K < 1\) the fluctuations of \(\phi\) and \(\theta\) are always bound together, and both are characterized by the same action

\[
S[\xi]/\hbar = -\frac{1}{2\pi^3} \int dx d^2 r_\perp d\tau \left[ R(\partial_\tau \xi)^2 + B(\partial_x \xi)^2 + \right. \]

\[
\left. M(\nabla_\perp \Xi)^2 \right],
\]  

(80)

where \(\xi\) can be either \(\phi\) or \(\theta/2\), parameters \(R, B\) and \(M\) are given by

\[
R = (1 + 4\pi K \psi_0^2),
\]  

(81)

\[
B = 1 + 4\pi K g_1^2 \left( \frac{1}{2n_e} \right)^2 \psi_0^2,
\]  

(82)

\[
M = C_\perp \pi \psi_0^2.
\]  

(83)

Note that the correction to \(B\) is negligible compared to the correction to \(R\), since \(g_1\) is much less than 1. Plugging the expression of \(\psi_0\) into (81) and (83) and writing \(R\) in terms of \(K'\) using (68), we get

\[
\left( \frac{K}{K'} \right)^2 - 1 \sim \left( g_2^2 g_1^{2K-4} \right) \frac{M}{M},
\]  

(84)

\[
M \sim C_\perp \left( g_2^2 g_1^{2K-4} \right) \frac{(K)}{(K')},
\]  

(85)

which agrees with the result in reference [3]. This result is consistent with the one obtained by the Gaussian variational method in that both predict long-range-ordered
CDW and hence long-range-ordered $\psi$ for $K < 1$. The expectation values of $\rho_{2k}$ and $\psi$ are non-zero and given by the formulae (76) and (77), respectively. Clearly this fixed-amplitude approximation can not give result for attractive electron-electron interactions since the method relies on the condensation of the phonon field $\psi$, which is only possible for repulsive electron-electron interactions according to the mean-field theory. Based on the result of weak coupling and/or high phonon frequency, we suspect that for strong coupling and/or low phonon frequency, there also exists a critical $K$ above which both electron and phonon correlations become quasi long-ranged; however, this critical $K$ could be significantly bigger than 1.

Now let us discuss the validity of the fixed-amplitude approximation. Unlike in 1D1D, based on the above discussions there is no self-consistent condition cause by phase fluctuations in 3D3D. However, the one caused by amplitude fluctuations still exists. After a similar calculation as we did in 1D1D, we find that the fixed-amplitude approximation is self-consistent only if

\[
\begin{cases}
 g_2 \gg g_1^2, \\
g_2 \gg g_1^2 \left( \frac{g_2}{\sqrt{C_\perp}} \right)^{1-K}, \\
 C_\perp \ll g_1^2,
\end{cases}
\]

which prohibits the application of the fixed-amplitude approximation in the region below $OB$ in figure 2.

According to the self-consistent conditions (178) and (179), there is an intermediate region in the parameter space of $g_1$ and $g_2$ in which neither of the approaches applies. This region is illustrated in figure 2 as the area between the loci $OB$ and $OA$. In this region, we can not obtain a quantitatively trustable result; however, we do expect long-ranged electron and phonon correlations for repulsive electron-electron interactions. We also want to point out that for $C_\perp \sim g_1^2$, both approaches give the same result:

\[
M \sim \left( \frac{g_2}{g_1} \right)^{1-K},
\]

\[
\left( \frac{K}{K'} \right)^2 - 1 \sim (g_2^2 g_1^{2-K-4})^{1-K}. \tag{88}
\]

This strongly suggests that in this special case, this quantitative result should also hold in the intermediate region.

So far we have focused on the electron-phonon coupling at large momentums. In section 1V we discussed the effect of the electron-phonon coupling at small momentums in 1D1D and 1D3D. Repeating the (virtually same) calculation, we find that the WB singularity also exists in 3D3D and is given by the formula (179). Again, for a general filling, according to the postulates (177) the system is far away from the WB singularity; therefore, the electron-phonon coupling at small momentums is not important.

VI. CONCLUSION

In this paper we have studied electron and phonon correlations in systems of one-dimensional spinless electrons coupled to phonons at low temperatures. The focus is on the effect of backward electron-phonon scattering. We have been able to obtain quantitative results in the limits of weak coupling and/or high phonon frequency and strong coupling and/or low phonon frequency, which then leads to a qualitative understanding in the intermediate region as well.

It has to be mentioned that our work for a three-dimensional array of one-dimensional conducting chains coupled to three-dimensional phonons does not take into account the effect of the interchain hopping. It is well known that the interchain hopping can break down the TLL behavior which we have assumed for electrons in our calculations. On the other hand, it is not clear if the effect of the interchain hopping will be suppressed in the presence of electron-phonon coupling. To understand these questions, a work which includes both effects of electron-phonon coupling and the interchain hopping will be interesting.

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APPENDIX A: WEAK COUPLING AND/OR HIGH PHONON FREQUENCY IN 1D1D

Since the total action is Gaussian in the phonon field $\psi$, we integrate out the phonon freedom and get an effective action only involving the electron field $\phi$

\[
S_{el}^{1D}/\hbar = \frac{1}{2\pi K} \int dx d\tau \left[ (\partial_x \phi)^2 + (\partial_x \phi)^2 \right]
- \frac{g_1^2}{2} \int dx d\tau d\tau' G_{ph}(\tau - \tau') \cos [2\phi(x, \tau) - 2\phi(x, \tau')]. \tag{A1}
\]

where the phonon correlation function is given by

\[
G_{ph}(\tau) \approx e^{-g_1|\tau|}. \tag{A2}
\]

We have neglect the phonon dispersions $(\partial_x \phi)^2$ since their effect is not important. When we discuss an array of paralleled 1D chains coupled to 3D phonons, we keep the phonon dispersions perpendicular to the wires since they can lead to long-range-order CDW.

First we apply a Gaussian variational calculation to the phonon mediated effective action (A1). We assume a Gaussian variational electron action

\[
S_0 = \sum_{q} G^{-1}_{el}(\bar{q}) \phi^* (\bar{q}) \phi(\bar{q}) \tag{A3}
\]
and try to optimize it to minimize the variational free energy
\[ F_{\text{var}} = F_0 + \langle S_\text{el}^{1D} - S_0 \rangle_0, \]  
where \( F_0 \) and \( \langle \rangle_0 \) are calculated using action \((A3)\). After a straightforward calculation, we obtain
\[ G_{\text{el}}^{-1}(q_x,q_\tau) = \frac{1}{\pi K} \left( \frac{1}{\sqrt{R}} q_\tau^2 + \sqrt{R} q_x^2 \right) \]  
where
\[ G_{\text{el}}(\tau) = \frac{1}{4\pi^2} \int dq_x dq_\tau \ G_{\text{el}}(\vec{q}) e^{i q_\tau \tau}. \]

Since the right-hand side of the above equation vanishes as \( \vec{q} \to 0 \), and the integral is independent of \( q_x \), we assume the following trial solution
\[ G_{\text{el}}^{-1}(q_x,q_\tau) = \frac{1}{\pi K'} \left( \frac{1}{\sqrt{R}} q_x^2 + \sqrt{R} q_\tau^2 \right) \]  
with \( K' = K/\sqrt{R} \). With the electron propagator written in this way, we ensure that the coefficient of \( q_x^2 \) is not renormalized but fixed at \( 1/(\pi K) \). Hereafter we denote \( K' \) and \( K \) as the renormalized and initial value respectively. Then the electron correlation function can be calculated as
\[ G_{\text{el}}(0) - G_{\text{el}}(\tau) = \frac{K'}{2} \ln \left( \frac{K' \alpha \tau + a K'^2}{a K'^2} \right). \]

Plugging this result into Eq. \((A3)\), we obtain
\[ \frac{1}{\pi K} \left[ \left( \frac{K}{K'^2} - 1 \right) q_\tau^2 \right] = \frac{8g_1^2}{g_1} \int_{-\infty}^{+\infty} d\tau \ e^{-\alpha |\tau|} \left( \frac{K' \alpha \tau + a K'^2}{a K'^2} \right)^{-2K'} \times \left[ 1 - \cos(q_\tau \tau) \right]. \]

For \( q_\tau \to 0 \), this leads to
\[ \frac{1}{\pi K} \left( \frac{K}{K'^2} - 1 \right) = 8 \left( \frac{g_1^2}{g_1} \right) \int_{-\infty}^{+\infty} d\tau \ \tau^2 e^{-\alpha |\tau|} \left( \frac{\alpha K'^2 \tau}{a K'^2} \right)^{-2K'}. \]

It is interesting to point out that in the absence of \( e^{-\alpha |\tau|} \) in the integrand on the right-hand side of \((A10)\), the integral would converge for \( K' > 3/2 \) and diverge for \( K' < 3/2 \), and there would be a quantum phase transition at the critical point \( K' = 3/2 \). The two phases are characterized by \( K' \to 0 \) for \( K' < 3/2 \) and finite \( K' \) for \( K' > 3/2 \) respectively. With the presence of \( e^{-\alpha |\tau|} \), the integral in \((A10)\) converges for any \( K' \). Thus, there is no phase transition. Assuming \( K' \) is close to \( K \), calculating \( K' \) to the lowest-order correction \( K' - K \) we get
\[ \frac{K}{K'} - 1 \sim \begin{cases} -\left( \frac{g_1^2}{g_1^2} \right) g_1^{2K-3} & K < \frac{3}{2} \\ \frac{g_1^2}{g_1^2} \ln g_1 & K = \frac{3}{2} \\ -\frac{g_1^2}{g_1^2} & K > \frac{3}{2} \end{cases}. \]

Although there is no phase transition, there exists a sharp crossover at \( K = K_s \) for \( K < 3/2 \) with
\[ K_s = \frac{3}{2 - 2 \ln g_1}. \]

Since both \( g_1^2/g_1 \) and \( g_1^2/g_1 \) are typically much less than 1, and \( g_1^2/g_1 \) gets hardly renormalized. For \( K \to K_s \), \( R \) suddenly becomes larger than 1, (Strictly speaking, formula \((A13)\) doesn’t apply anymore for \( K < K_s \), but it qualitatively tells the trends nonetheless.) and \( v' \) is thus reduced appreciably. In principle this result can be tested by experiments since \( K_s \) can be less than 1 according to \((A12)\).

Now we apply the standard momentum shell renormalization group calculation (RG) to this model. We separate the electron field into fast and slow varying components \( \phi = \phi^> + \phi^< \), where \( \phi^> \) has support in the momentum shell \( a \ell/e^d/a < q_x < a/\alpha, -\infty < q_\tau < \infty \), integrate out the fast varying component \( \phi^> \) and rescale the length with \( x = x' e^d/a, \tau = \tau' e^d/a \) so as to restore the original ultraviolet cutoff. The integration over \( \phi^< \) is achieved perturbatively by expanding the partition function to the first order around the Gaussian fixed point. Performing the above procedure and using the expression \((A7)\) for the propagator, we obtain the following RG flow equations (up to one loop)
\[ \frac{d[K'(\ell)]}{d\ell} = -C_1 K'^3 \frac{g_1^2}{g_1^2} (f(g_1(\ell))) \]  
\[ \frac{d[g_1'(\ell)]}{d\ell} = (3 - 2 K') \frac{g_1^2}{g_1^2}, \]  
\[ \frac{d[g_1(\ell)]}{d\ell} = g_1(\ell), \]  
where \( C_1 \) is a constant of order 1, \( f(x) \) is a step function which is 1 for \( x < 1 \) and 0 for \( x > 1 \). We use \( f(x) \) to cut the renormalization of \( K'(\ell) \) at \( \ell_1 = -\ln (g_1 a/\alpha) \), where the phonon propagator \( G_{\text{ph}}(\tau - \tau') \) becomes effectively \( \delta(\tau - \tau') \). The correspondence between our notation and that of reference \[3\] is: \( Y_{\text{ph}} = g_1/g_2, g_1 \sim a/\xi_{\text{ph}} \).
$\exp(2\xi) \equiv K'$. It seems that the correct eigenvalue in the RG flow equation (3.14b) in reference [3] should be $2 - 2\exp(2\xi)$. Also there are some minor differences between our RG flow equations and those in reference [3]. They are due to different RG procedures and should not affect the physics. The RG flow Eq. (A13) seems to imply a phase transition at $K' = 3/2$; however, since the renormalization of $K'(\ell)$ is utterly cut at $\ell = \ell_1$, $K'$ is always finite even for $K' < 3/2$, so there will be no phase transition at $K' = 3/2$. For $gg_1^{K-2} < 1$, the perturbative RG holds way down to $\ell = \ell_1$ since $g_1^2(\ell_1)/g_1(\ell_1)$ remains less than 1, and we can integrate the flow equations approximately to calculate $K'$ as

$$ (K' - K) \sim -\frac{g_1^2}{g_1} \int_{\ell_0}^{\ell_1} e^{(3-2K)\ell} \, d\ell, \quad \text{(A17)} $$

which essentially gives the Gaussian variational result (A11). For $gg_1^{K-2} > 1$, $g_1^2(\ell)/g_1(\ell)$ becomes larger than 1 at $\ell = \ln \left[g_1(\ell)/g_1^2(\ell)\right]/(3-2K)$, which is less than $\ell_1$. In this case the perturbative RG breaks down before we can calculate $K'$; therefore, a different strategy needs to be used to attack the problem.

**APPENDIX B: CALCULATION OF $\psi_0$**

After the mean-field approximation we get a simplified model

$$ S_{ph}^{1D}/\hbar = \frac{1}{2} \int dx d\tau \frac{1}{\pi K} \left[ (\partial_x \phi)^2 + (\partial_t \phi)^2 \right] - 2g_2\psi_0 \int dx d\tau \cos(2\phi), \quad \text{(B1)} $$

$$ S_{ph}^{1D}/\hbar = \frac{1}{2} \int dx d\tau \ g_1^2(\psi_0^2). \quad \text{(B2)} $$

The partition function can be calculated as

$$ Z = \int D[\phi] D[\phi] e^{-\frac{1}{\hbar} (S_{el}^{1D} + S_{ph}^{1D} + S_{ep}^{1D})} $$

$$ = \int D[\phi] e^{-\frac{S_{ph}^{1D}}{\hbar}} \int D[\phi] e^{-\frac{1}{\hbar} (S_{el}^{1D} + S_{ep}^{1D})} $$

$$ = \int D[\phi] e^{-\frac{1}{\hbar} (S_{el}^{1D} + F_{el}[\psi_0])}, \quad \text{(B3)} $$

where $F_{el}[\psi_0]$ is the free energy of electrons for action $S_{el}^{1D} + S_{ep}^{1D}$ at a fixed $\psi_0$. $S_{el}^{1D} + S_{ep}^{1D}$ is exactly a Sine-Gordon model. Formula (B3) defines an effective action for $\psi_0$

$$ S_{eff}^{1D}[\psi_0] = S_{ph}^{1D} + F_{el}[\psi_0]. \quad \text{(B4)} $$

To calculate $F_{el}[\psi_0]$, we use Gaussian variational method [3]. For any action $S_0$, we have

$$ F_{el}[\psi_0] = \frac{F_0}{\hbar} - \ln \langle e^{-\frac{1}{\hbar} (S_{el}^{1D} + S_{ep}^{1D} - S_0)} \rangle_0. \quad \text{(B5)} $$

where the $\langle \rangle_0$ denotes the average using the action $S_0$. Given the convexity of the exponential [14] one has always

$$ \langle e^{-\frac{1}{\hbar} (S_{el}^{1D} + S_{ep}^{1D} - S_0)} \rangle > e^{-\frac{1}{\hbar} (S_{el}^{1D} + S_{ep}^{1D} - S_0)} \quad \text{(B6)} $$

and thus

$$ F_{el}^{1D} < F_{var} = F_0 + \langle S - S_0 \rangle_0. \quad \text{(B7)} $$

The basic idea is to optimize $S_0$ such that $F_{var}$ gets to $F_{el}^{1D}$ as close as possible. Assuming that $S_0$ is Gaussian

$$ S_0/\hbar = \frac{1}{2} \sum_q \phi(-\bar{q})G^{-1}(\bar{q})\phi(\bar{q}) \quad \text{(B8)} $$

with

$$ G^{-1}(\bar{q}) = \frac{q^2 + D^2}{\pi K}, \quad \text{(B9)} $$

$F_{var}$ can be calculated as

$$ F_{var} = -\sum_q \ln G(\bar{q}) + \frac{1}{2\pi K} \sum_q q^2 G(\bar{q}) - 2g_2\psi_0 \int dx d\tau e^{-2\langle \phi^2 \rangle_0}. \quad \text{(B10)} $$

Minimizing $F_{var}$ with respect to $G(\bar{q})$ by doing $\partial F_{var}/\partial G(\bar{q}) = 0$ leads to a self-consistent equation for $D$

$$ D^2 = 8\pi K g_2\psi_0 \left[ \frac{D}{1 + \sqrt{1 + D^2}} \right] K. \quad \text{(B11)} $$

Assuming that $g_2\psi_0$ is much less than 1 (i.e., the single electron gap is much less than the Fermi energy), we get

$$ D = 2(2\pi K g_2\psi_0)^{1/(2-K)} \quad \text{(B12)} $$

for $K < 2$ and $D = 0$ for $K > 2$. Therefore, $K = 2$ is a naive estimation of the critical value of $K$ at which the KBT transition happens for the Sine-Gordon model (B11). Plugging (B12) into the formula (B10), we get the approximated $F_{el}^{1D}[\psi_0]/\hbar$

$$ F_{el}^{1D}[\psi_0]/\hbar = \frac{1}{2\pi K} \int dx d\tau \left[ -(2-K) \left(2\pi K g_2\psi_0 \right)^{\frac{2-K}{2}} \right], \quad \text{(B13)} $$

where we have thrown out terms which are independent of $\psi_0$. Thus, according to (B4) the effective action for $\psi_0$ is given by

$$ S_{ph}^{1D}[\psi_0]/\hbar = \frac{1}{2\pi K} \int dx d\tau \left[ -(2-K) \left(2\pi K g_2\psi_0 \right)^{\frac{2-K}{2}} + \pi K g_1^2(\psi_0^2) \right]. \quad \text{(B14)} $$

Minimize it with respect to $\psi_0$, we get $\psi_0 = 0$ for $K > 1$ and

$$ \psi_0 = g_1^{-1} \left[ \frac{2(\pi K)^{\frac{1}{2-K}} \left( g_2 / g_1 \right)^{\frac{1}{2}}}{\pi K} \right] \quad \text{(B15)} $$

for $K < 1$. 

APPENDIX C: CALCULATING PHASE FLUCTUATIONS FROM THE FIXED-AMPLITUDE MODELS

We start with the truncated model (20). For convenience, we rewrite it as

\[ S[\Upsilon, \Sigma]/\hbar = \frac{1}{2} \sum_{\vec{q}} \left[ G^{-1}_\phi(\vec{q}) \phi(\vec{q}) \phi(-\vec{q}) + G^{-1}_\theta(\vec{q}) \theta(\vec{q}) \theta(-\vec{q}) \right] - 2g_2 \psi_0 \int dx d\tau \cos(2\phi - \theta) , \]  

(C1)

where

\[ G^{-1}_\phi(\vec{q}) = \frac{1}{2\pi K} (q^2 + q_\tau^2), \]
\[ G^{-1}_\theta(\vec{q}) = \left[ q_1^2 \left( \frac{1}{2\pi g_1} \right)^2 q_\tau^2 + q_\tau^2 \right] \psi_0^2. \]  

(C2)

Then we define two new variables, so that the action in terms of these two variables are decoupled. That is, the action can be expressed as

\[ S[\bar{\Upsilon}, \bar{\Sigma}]/\hbar = \frac{1}{2} \sum_{\vec{q}} G^{-1}_\bar{\Upsilon}(\vec{q}) \bar{\Upsilon}(\vec{q}) \bar{\Upsilon}(-\vec{q}) + \frac{1}{2} \sum_{\vec{q}} G^{-1}_\bar{\Sigma}(\vec{q}) \times \]
\[ \bar{\Upsilon}(\vec{q}) \bar{\Sigma}(-\vec{q}) - 2g_2 \psi_0 \int dx d\tau \cos \bar{\Upsilon}, \]  

(C3)

where

\[ \bar{\Upsilon}(\vec{q}) \equiv 2\phi(\vec{q}) - \theta(\vec{q}), \]
\[ \bar{\Sigma}(\vec{q}) \equiv \phi(\vec{q}) - \frac{2\Upsilon(\vec{q}) G^{-1}_\phi(\vec{q})}{G^{-1}_\phi(\vec{q}) + 4G^{-1}_\theta(\vec{q})}, \]  

(C4)
\[ G^{-1}_\bar{\Upsilon}(\vec{q}) \equiv \frac{G^{-1}_\phi(\vec{q}) G^{-1}_\theta(\vec{q})}{G^{-1}_\phi(\vec{q}) + 4G^{-1}_\theta(\vec{q})}, \]
\[ G^{-1}_\bar{\Sigma}(\vec{q}) \equiv G^{-1}_\phi(\vec{q}) + 4G^{-1}_\theta(\vec{q}). \]  

(C5)

Since \( \bar{\Upsilon} \) and \( \bar{\Sigma} \) are decoupled, using (C1) and (C5) the fluctuations of \( \theta \) and \( \phi \) can be calculated as

\[ \langle \phi(\vec{q}) \phi(-\vec{q}) \rangle = G_\Sigma(\vec{q}) + \frac{4G_\phi^2(\vec{q}) \langle \bar{\Upsilon}(\vec{q}) \bar{\Upsilon}(-\vec{q}) \rangle}{\left[ G^{-1}_\phi(\vec{q}) + 4G^{-1}_\theta(\vec{q}) \right]^2}, \]  

(C8)
\[ \langle \theta(\vec{q}) \theta(-\vec{q}) \rangle = 4G_\Sigma(\vec{q}) + \frac{G^{-2}_\phi(\vec{q}) \langle \bar{\Sigma}(\vec{q}) \bar{\Sigma}(-\vec{q}) \rangle}{\left[ G^{-1}_\phi(\vec{q}) + 4G^{-1}_\theta(\vec{q}) \right]^2}. \]  

(C9)

To calculate \( \langle \bar{\Upsilon}(\vec{q}) \bar{\Upsilon}(-\vec{q}) \rangle \), we need to know whether the cosine term is relevant or irrelevant. Thus, we evaluate

\[ \int dx d\tau \langle \cos \\bar{\Upsilon} \rangle_0 = \int dx d\tau \ e^{-\langle \bar{\Upsilon}^2 \rangle_0^2}, \]  

(C10)

where \( \langle \cdot \rangle_0 \) is the average with respect to the harmonic part of the action (C3). Let us first calculate

\[ \langle \bar{\Upsilon}^2 \rangle_0 = \frac{1}{(2\pi)^2} \int dq_x dq_\tau \ G_\tau(q) \]
\[ = \frac{1}{(2\pi)^2} \int dq_x dq_\tau \ \left\{ \frac{1}{\psi_0^2} \frac{1}{q_x^2 + q_\tau^2} \left[ \frac{1}{2\pi g_1} \right]^2 q_\tau^2 \right\} + \frac{4\pi K}{q_x^2 + q_\tau^2} \]
\[ = \left( \frac{n_c}{\pi g_1 \psi_0^2} + 2K \right) \ln L. \]  

(C11)

Plugging it into (C10) we obtain

\[ \int dx d\tau \langle \cos \bar{\Upsilon} \rangle_0 \sim L^{2-K-\frac{n_c}{2\pi g_1 \psi_0^2}}, \]  

(C12)

which implies a critical line in the \( \psi_0^2 - K \) plane defined by

\[ \frac{n_c}{\pi g_1 \psi_0^2} + 2K = 4. \]  

(C13)

This critical line is illustrated in Fig. 1. The cosine term in the action (C3) is relevant below the critical line and irrelevant above it. In the two limiting cases the function (C13) predicts that \( \psi_0 \rightarrow \infty \) for \( K = 2 \) and \( \psi_0 \sim g_1^{-1/2} \) for \( K = 0 \). For \( K < 1 \), it predicts

\[ \psi_0 \sim g_1^{-1/2}. \]  

(C14)

For \( \psi_0 < \psi_0^c \), the cosine term is irrelevant, and we get

\[ \langle \bar{\Upsilon}(\vec{q}) \bar{\Upsilon}(-\vec{q}) \rangle = G_\tau(\vec{q}), \]  

which combined with (C8) and (C9) leads to

\[ \langle \phi(\vec{q}) \phi(-\vec{q}) \rangle = G_\phi(\vec{q}), \]
\[ \langle \theta(\vec{q}) \theta(-\vec{q}) \rangle = G_\theta(\vec{q}). \]  

(C15)
(C16)

This implies that \( \theta \) and \( \phi \) are effectively decoupled. For \( \psi_0 > \psi_0^c \), \( \langle \bar{\Upsilon}(\vec{q}) \bar{\Upsilon}(-\vec{q}) \rangle \) is massive, and we obtain

\[ \langle \phi(\vec{q}) \phi(-\vec{q}) \rangle = G_\Sigma(\vec{q}), \]
\[ \langle \theta(\vec{q}) \theta(-\vec{q}) \rangle = 4G_\Sigma(\vec{q}). \]  

(C17)
(C18)

which implies that the fluctuations of \( \theta \) and \( \phi \) are bound together.

Now we discuss the model (20). We follow the same routine as the above. Defining \( \bar{\Upsilon} \) and \( \bar{\Sigma} \) using (C4) and (C5), we get a decoupled action:

\[ S[\bar{\Upsilon}, \bar{\Sigma}]/\hbar = \frac{1}{2} \sum_{\vec{q}} G^{-1}_\bar{\Sigma}(\vec{q}) \bar{\Sigma}(\vec{q}) \bar{\Sigma}(-\vec{q}) + \frac{1}{2} \sum_{\vec{q}} G^{-1}_\bar{\Upsilon}(\vec{q}) \times \]
\[ \bar{\Upsilon}(\vec{q}) \bar{\Upsilon}(-\vec{q}) - 2g_2 \psi_0 \int dx d^2r d\tau \cos \bar{\Upsilon}, \]  

(C19)
where $G^{-1}_\gamma (\vec{q})$ and $G^{-1}_\Sigma (\vec{q})$ are defined in (C6) and (C7), respectively. $G^{-1}_\phi (\vec{q})$ and $G^{-1}_\theta (\vec{q})$ are now defined as

$$G^{-1}_\phi (\vec{q}) \equiv \left( \frac{\psi_0}{\pi} \right)^2 \left[ \frac{1}{g_1^2 \left( \frac{1}{2n_e} \right)} q_x^2 + q_\tau^2 + C \right],$$

$$G^{-1}_\theta (\vec{q}) \equiv \left( \frac{\psi_0}{\pi} \right)^2 \left[ \frac{1}{g_1^2 \left( \frac{1}{2n_e} \right)} q_x^2 + q_\tau^2 + C \right].$$

To check whether the cosine term is relevant or not, we evaluate the average of the cosine term for a single chain with respect to the harmonic part of the action. After we calculate

$$\langle \tau^2 \rangle_0 = \frac{1}{(2\pi)^4} \int dq_x dq_\perp dq_\tau \quad G_\gamma (\vec{q})$$

we obtain

$$\int dx d\tau \quad \langle \cos \tau \rangle_0 \sim L^{2-K}.$$