THE TAN $2\Theta$ THEOREM FOR INDEFINITE QUADRATIC FORMS

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Abstract. A version of the Davis-Kahan Tan $2\Theta$ theorem [SIAM J. Numer. Anal. 7 (1970), 1 – 46] for not necessarily semibounded linear operators defined by quadratic forms is proven. This theorem generalizes a recent result by Motovilov and Selin [Integr. Equat. Oper. Theory 56 (2006), 511 – 542].

1. Introduction

In the 1970 paper [3] Davis and Kahan studied the rotation of spectral subspaces for $2 \times 2$ operator matrices under off-diagonal perturbations. In particular, they proved the following result, the celebrated “Tan $2\Theta$ theorem”: Let $A_{\pm}$ be strictly positive bounded operators in Hilbert spaces $\mathcal{H}_{\pm}$, respectively, and $W$ a bounded operator from $\mathcal{H}_-$ to $\mathcal{H}_+$. Denote by

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & -A_- \end{pmatrix} \quad \text{and} \quad B = A + V = \begin{pmatrix} A_+ & W \\ W^* & -A_- \end{pmatrix}$$

the block operator matrices with respect to the orthogonal decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Then

$$\|\tan 2\Theta\| \leq \frac{2\|V\|}{d}, \quad \text{spec}(\Theta) \subset [0, \pi/4),$$

where $\Theta$ is the operator angle between the subspaces $\text{Ran} \ E_A(\mathbb{R}_+)$ and $\text{Ran} \ E_B(\mathbb{R}_+)$ and

$$d = \text{dist}(\text{spec}(A_+), \text{spec}(-A_-))$$

(see, e.g., [8]).

Estimate (1.1) can equivalently be expressed as the following inequality for the norm of the difference of the orthogonal projections $P = E_A(\mathbb{R}_+)$ and $Q = E_B(\mathbb{R}_+)$:

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{d} \right),$$

which, in particular, implies the estimate

$$\|P - Q\| < \frac{\sqrt{2}}{2}.$$  

Independently of the work of Davis and Kahan, inequality (1.3) has been proven by Adamyan and Langer in [1], where the operators $A_{\pm}$ were allowed to be semibounded. The case $d = 0$ has been considered in the work [9] by Kostrykin, Makarov, and Motovilov. In particular, it was proven that there is a unique orthogonal projection $Q$ from the operator interval $[E_B((0, \infty)), E_B([0, \infty))$ such that

$$\|P - Q\| \leq \frac{\sqrt{2}}{2},$$

where $P \in [E_A((0, \infty)), E_A([0, \infty))]$ is the orthogonal projection onto the invariant (not necessary spectral) subspace $\mathcal{H}_+ \subset \mathcal{H}$ of the operator $A$. A particular case of this result has been

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obtained earlier by Adamyan, Langer, and Tretter, in [2]. Recently, a version of the Tan \(2\Theta\) Theorem for off-diagonal perturbations \(V\) that are relatively bounded with respect to the diagonal operator \(A\) has been proven by Motovilov and Selin in [11].

In the present work we obtain several generalizations of the aforementioned results assuming that the perturbation is given by an off-diagonal symmetric form.

Given a sesquilinear symmetric form \(a\) and a self-adjoint involution \(J\) such that the form \(a_J[x, y] := a[x, Jy]\) is a positive definite and

\[ a[x, Jy] = a[ Jx, y], \]

we call a symmetric sesquilinear form \(v\) off-diagonal with respect to the orthogonal decomposition \(H = H_+ \oplus H_-\) with \(H_{\pm} = \text{Ran}(I \pm J)\) if

\[ v[Jx, y] = -v[x, Jy]. \]

Based on a close relationship between the symmetric form \(a[x, y] + v[x, y]\) and the sectorial sesquilinear form \(a[x, Jy] + iv[x, Jy]\) (cf. [11], [13]), under the assumption that the off-diagonal form \(v\) is relatively bounded with respect to the form \(a_J\), we prove

(i) an analog of the First Representation Theorem for block operator matrices defined as not necessarily semibounded quadratic forms,

(ii) a relative version of the Tan \(2\Theta\) Theorem.

We also provide several versions of the relative Tan \(2\Theta\) Theorem in the case where the form \(a\) is semibounded.

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2. **The First Representation Theorem for Off-Diagonal Form Perturbations**

To introduce the notation, it is convenient to assume the following hypothesis.

**Hypothesis 2.1.** Let \(a\) be a symmetric sesquilinear form on \(\text{Dom}[a]\) in a Hilbert space \(\mathcal{H}\). Assume that \(J\) is a self-adjoint involution such that

\[ J \text{ Dom}[a] = \text{Dom}[a]. \]

Suppose that

\[ a[Jx, y] = a[x, Jy] \quad \text{for all} \quad x, y \in \text{Dom}[a_J] = \text{Dom}[a], \]

and that the form \(a_J\) given by

\[ a_J[x, y] = a[x, Jy], \quad x, y \in \text{Dom}[a_J] = \text{Dom}[a]. \]

is a positive definite closed form. Denote by \(m_{\pm}\) the greatest lower bound of the form \(a_J\) restricted to the subspace

\[ \mathcal{H}_{\pm} = \text{Ran}(I \pm J). \]

**Definition 2.2.** Under Hypothesis 2.1, a symmetric sesquilinear form \(v\) on \(\text{Dom}[v] \supset \text{Dom}[a]\) is said to be off-diagonal with respect to the orthogonal decomposition

\[ \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \]
Assume Hypothesis 2.1. Suppose that \( v \) is an \( a \)-bounded off-diagonal with respect to the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) symmetric form. On \( \text{Dom}(b) = \text{Dom}(a) \) introduce the symmetric form

\[
b[x, y] = a[x, y] + v[x, y], \quad x, y \in \text{Dom}(b).
\]

Then

(i) there is a unique self-adjoint operator \( B \) in \( \mathcal{H} \) such that \( \text{Dom}(B) \subset \text{Dom}(b) \) and

\[
b[x, y] = \langle x, By \rangle \quad \text{for all} \quad x \in \text{Dom}(b), \quad y \in \text{Dom}(B).
\]

(ii) the operator \( B \) is boundedly invertible and the open interval \( (-m_-, m_+) \ni 0 \) belongs to its resolvent set.
Proof. (i). Given \( \mu \in (-m_-, m_+) \), on \( \text{Dom}[a_\mu] = \text{Dom}[a] \) introduce the positive closed form \( a_\mu \) by

\[
a_\mu[x, y] = a[x, y] - \mu \langle x, Jy \rangle, \quad x, y \in \text{Dom}[a_\mu],
\]

and denote by \( \mathcal{H}_{a_\mu} \) the Hilbert space \( \text{Dom}[a_\mu] \) equipped with the inner product \( \langle \cdot, \cdot \rangle_\mu = a_\mu[\cdot, \cdot] \). We remark that the norms \( \| \cdot \|_\mu = \sqrt{a_\mu[\cdot, \cdot]} \) on \( \mathcal{H}_{a_\mu} = \text{Dom}[a_\mu] \) are obviously equivalent. Since \( v \) is \( a \)-bounded, one concludes then that

\[
v_\mu := \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v[x]|}{a_\mu[x]} < \infty, \quad \text{for all } \mu \in (-m_-, m_+).
\]

Along with the off-diagonal form \( v \), introduce a dual form \( v' \) by

\[v'[x, y] = \bar{v}[x, Jy], \quad x, y \in \text{Dom}[a].\]

We claim that \( v' \) is an \( a \)-bounded off-diagonal symmetric form. It suffices to show that

\[
v_\mu = v'_\mu < \infty, \quad \mu \in (-m_-, m_+),
\]

where

\[
(2.4) \quad v'_\mu := \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v'[x]|}{a_\mu[x]}.
\]

Indeed, let \( x = x_+ + x_- \) be a unique decomposition of an element \( x \in \text{Dom}[a] \) such that \( x_+ \in \mathcal{H}_{a_\mu} \cap \text{Dom}[a_\mu] \). By Remark 2.3,

\[
v[x] = v[x_+, x_-] + v[x_-, x_+] = 2 \text{Re} \, v[x_+, x_-], \quad x \in \text{Dom}[a].
\]

In a similar way (since the form \( v' \) is obviously off-diagonal) one gets that

\[
v'[x] = \bar{v}[x_+ + x_-, J(x_+ + x_-)] = \bar{v}'[x_+] - \bar{v}'[x_-] - i v'[x_+, x_-] + i v[x_-, x_+]
\]

\[= -i v[x_+, x_-] + i v[x_+, x_-] = 2 \text{Im} \, v[x_+, x_-], \quad x \in \text{Dom}[a].\]

Clearly, from (2.4) it follows that

\[
v'_\mu = 2 \sup_{0 \neq x \in \text{Dom}[a]} \frac{|\text{Im} \, v[x_+, x_-]|}{a_\mu[x]} = 2 \sup_{0 \neq x \in \text{Dom}[a]} \frac{|\text{Re} \, v[x_+, x_-]|}{a_\mu[x]} = v_\mu,
\]

\[\mu \in (-m_-, m_+),\]

which completes the proof of the claim.

Next, on \( \text{Dom}[t_\mu] = \text{Dom}[a] \) introduce the sesquilinear form

\[
t_\mu := a_\mu + i v', \quad \mu \in (-m_-, m_+).
\]

Since the form \( a_\mu \) is positive definite and the form \( v' \) is an \( a_\mu \)-bounded symmetric form, the form \( t \) is a closed sectorial form with the vertex 0 and semi-angle

\[
(2.5) \quad \theta_\mu = \arctan(v'_\mu) = \arctan(v_\mu).
\]

Let \( T_\mu \) be a unique \( m \)-sectorial operator associated with the form \( t_\mu \). Introduce the operator

\[B_\mu = JT_\mu \text{ on } \text{Dom}(B_\mu) = \text{Dom}(T_\mu), \quad \mu \in (-m_-, m_+).\]

One obtains that

\[
\langle x, B_\mu y \rangle = \langle x, JT_\mu \rangle = \langle Jx, T_\mu y \rangle = a_\mu[Jx, y] + i v'[Jx, y]
\]

\[= a[x, y] - \mu \langle Jx, Jy \rangle + i^2 v[Jx, Jy]
\]

\[= a[x, y] - \mu \langle x, y \rangle + v[x, y],
\]

for all \( x \in \text{Dom}[a], \ y \in \text{Dom}(B_\mu) = \text{Dom}(T_\mu) \). In particular, \( B_\mu \) is a symmetric operator on \( \text{Dom}(B_\mu) \), since the forms \( a \) and \( v \) are symmetric, and \( \text{Dom}(B_\mu) = \text{Dom}(T_\mu) \subset \text{Dom}[a] \).
For the real part of the form $t_\mu$ is positive definite with a positive lower bound, the operator $T_\mu$ has a bounded inverse. This implies that the operator $B_\mu = JT_\mu$ has a bounded inverse and, therefore, the symmetric operator $B_\mu$ is self-adjoint on $\text{Dom}(B_\mu)$.

As an immediate consequence, one concludes (put $\mu = 0$) that the self-adjoint operator $B := B_0$ is associated with the symmetric form $b$ and that $\text{Dom}(B) \subset \text{Dom}[a]$.

To prove uniqueness, assume that $B'$ is a self-adjoint operator associated with the form $b$. Then for all $x \in \text{Dom}(B)$ and all $y \in \text{Dom}(B')$ one gets that

$$\langle x, B'y \rangle = b[x, y] = \overline{b[y, x]} = \langle y, Bx \rangle = \langle Bx, y \rangle,$$

which means that $B = (B')^* = B'$.

(ii). From (2.6) one concludes that the self-adjoint operator $B_\mu + \mu I$ is associated with the form $b$ and, hence, by the uniqueness

$$B_\mu = B - \mu I \quad \text{on} \quad \text{Dom}(B_\mu) = \text{Dom}(B) \tag{\text{Remark 2.5}}.$$ 

Since $B_\mu$ has a bounded inverse for all $\mu \in (m_-, m_+)$, so does $B - \mu I$ which means that the interval $(-m_-, m_+)$ belongs to the resolvent set of the operator $B_0$. \hfill $\square$

**Remark 2.5.** In the particular case $v = 0$, from Theorem 2.4 it follows that there exists a unique self-adjoint operator $A$ associated with the form $a$.

For a different, more constructive proof of Theorem 2.4 as well as for the history of the subject we refer to our work [4].

**Remark 2.6.** For the part (i) of Theorem 2.4 to hold it is not necessary to require that the form $\alpha_J$ in Hypothesis 2.1 is positive definite. It is sufficient to assume that $\alpha_J$ is a semi-bounded from below closed form (see, e.g., [12]).

### 3. The Tan $2\Theta$ Theorem

The main result of this work provides a sharp upper bound for the angle between the positive spectral subspaces $\text{Ran} \ E_A(\mathbb{R}_+)$ and $\text{Ran} \ E_B(\mathbb{R}_+)$ of the operators $A$ and $B$ respectively.

**Theorem 3.1.** Assume Hypothesis 2.1 and suppose that $v$ is off-diagonal with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Let $A$ be a unique self-adjoint operator associated with the form $a$ and $B$ the self-adjoint operator associated with the form $b = a + v$ referred to in Theorem 2.4.

Then the norm of the difference of the spectral projections $P = E_A(\mathbb{R}_+)$ and $Q = E_B(\mathbb{R}_+)$ satisfies the estimate

$$\|P - Q\| \leq \sin \left(\frac{1}{2} \arctan v\right) < \frac{\sqrt{2}}{2},$$

where

$$v = \inf_{\mu \in (-m_-, m_+)} v_\mu = \inf_{\mu \in (-m_-, m_+)} \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v[x]|}{a_\mu[x]},$$

with

$$a_\mu[x, y] = a[x, Jy] - \mu \langle x, Jy \rangle, \quad x, y \in \text{Dom}[a_\mu] = \text{Dom}[a].$$

The proof of Theorem 3.1 uses the following result borrowed from [14].

**Proposition 3.2.** Let $T$ be an $m$-sectorial operator of semi-angle $\theta < \pi/2$. Let $T = U|T|$ be its polar decomposition. If $U$ is unitary, then the unitary operator $U$ is sectorial with semi-angle $\theta$.

**Remark 3.3.** We note that for a bounded sectorial operator $T$ with a bounded inverse the statement is quite simple. Due to the equality

$$\langle x, Tx \rangle = \langle |T|^{-1/2}y, U|T|^{1/2}y \rangle = \langle y, |T|^{-1/2}U|T|^{1/2}y \rangle, \quad y = |T|^{1/2}x,$$
the operators $T$ and $|T|^{-1/2}U|T|^{1/2}$ are sectorial with the semi-angle $\theta$. The resolvent sets of the operators $|T|^{-1/2}U|T|^{1/2}$ and $U$ coincide. Therefore, since $U$ is unitary, it follows that $U$ is sectoral with semi-angle $\theta$.

**Proof of Theorem 3.1.** Given $\mu \in (-m_-, m_+)$, let $T_\mu = U_\mu |T_\mu|$ be the polar decomposition of the sectorial operator $T_\mu$ with vertex 0 and semi-angle $\theta_\mu$, with

$$\theta_\mu = \arctan(v_\mu)$$

(as in the proof of Theorem 2.4 (cf. (2.5))). Since $B_\mu = JT_\mu$, one concludes that

$$|T_\mu| = |B_\mu| \quad \text{and} \quad U_\mu = J^{-1} \text{sign}(B_\mu).$$

Since $T_\mu$ is a sectorial operator with semi-angle $\theta_\mu$, by a result in [14] (see Proposition 3.2), the unitary operator $U_\mu$ is sectorial with vertex 0 and semi-angle $\theta_\mu$ as well. Therefore, applying the spectral theorem for the unitary operator $U_\mu$ from (3.1) one obtains the estimate

$$\|J - \text{sign}(B_\mu)\| = \|I - J^{-1} \text{sign}(B_\mu)\| = \|I - U_\mu\| \leq 2 \sin \left( \frac{1}{2} \arctan v_\mu \right).$$

Since the open interval $(-m_-, m_+)$ belongs to the resolvent set of the operator $B = B_0$, the involution $\text{sign}(B_\mu)$ does not depend on $\mu \in (-m_-, m_+)$ and hence one concludes that

$$\text{sign}(B_\mu) = \text{sign}(B_0) = \text{sign}(B), \quad \mu \in (-m_-, m_+).$$

Therefore,

$$\|P - Q\| = \frac{1}{2} \|J - \text{sign}(B)\| = \frac{1}{2} \|J - \text{sign}(B_\mu)\| \leq \sin \left( \frac{1}{2} \arctan v_\mu \right)$$

and, hence, since $\mu \in (-m_-, m_+)$ has been chosen arbitrarily, from (3.2) it follows that

$$\|P - Q\| \leq \inf_{\mu \in (-m_-, m_+)} \sin \left( \frac{1}{2} \arctan v_\mu \right) \leq \sin \left( \frac{1}{2} \arctan v \right).$$

The proof is complete.  

As a consequence, we have the following result that can be considered a geometric variant of the Birman-Schwinger principle for the off-diagonal form-perturbations.

**Corollary 3.4.** Assume Hypothesis 2.1 and suppose that $\nu$ is off-diagonal. Then the form $a_J + \nu$ is positive definite if and only if the $a_J$-relative bound (2.1) of $\nu$ does not exceed one. In this case

$$\|P - Q\| \leq \sin \left( \frac{\pi}{8} \right),$$

where $P$ and $Q$ are the spectral projections referred to in Theorem 3.1.

**Proof.** Since $\nu$ is an $a$-bounded form, one concludes that there exists a self-adjoint bounded operator $V$ in the Hilbert space $\text{Dom}[a]$ such that

$$\nu[x, y] = a_J[x, V^* y], \quad x, y \in \text{Dom}[a].$$

Since $\nu$ is off-diagonal, the numerical range of $V$ coincides with the symmetric about the origin interval $[-\|V\|, \|V\|]$. Therefore, one can find a sequence $\{x_n\}_{n=1}^\infty$ in $\text{Dom}[a]$ such that

$$\lim_{n \to \infty} \frac{\nu[x_n]}{a_J[x_n]} = -\|V\|,$$

which proves that $\|V\| \leq 1$ if and only if the form $a_J + \nu$ is positive definite. If it is the case, applying Theorem 3.1, one obtains the inequality

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \left( \|V\| \right) \right) \leq \sin \left( \frac{\pi}{8} \right)$$

which completes the proof.  

Remark 3.5. We remark that in accordance with the Birman-Schwinger principle, for the form $a_J + \nu$ to have negative spectrum it is necessary that the $a_J$-relative bound $\|\nu\|$ of the perturbation $\nu$ is greater than one. As Corollary 3.4 shows, in the off-diagonal perturbation theory this condition is also sufficient.

4. Two sharp estimates in the semibounded case

In this section we will be dealing with the case of off-diagonal form-perturbations of a semi-bounded operator.

**Hypothesis 4.1.** Assume that $A$ is a self-adjoint semi-bounded from below operator. Suppose that $A$ has a bounded inverse. Assume, in addition, that the following conditions hold:

(i) The spectral condition. An open finite interval $(\alpha, \beta)$ belongs to the resolvent set of the operator $A$. We set

$$
\Sigma_- = \text{spec}(A) \cap (-\infty, \alpha] \quad \text{and} \quad \Sigma_+ = \text{spec}(A) \cap [\beta, \infty].
$$

(ii) Boundedness. The sesquilinear form $\nu$ is symmetric on $\text{Dom}[\nu] \supset \text{Dom}([A]^{1/2})$ and

$$
\nu := \sup_{0 \neq x \in \text{Dom}[a]} \frac{|\nu[x]|}{\| [A]^{1/2} x \|^2} < \infty.
$$

(iii) Off-diagonality. The sesquilinear form $\nu$ is off-diagonal with respect to the orthogonal decomposition $H = H_+ \oplus H_-$, with

$$
H_+ = \text{Ran} E_A((\beta, \infty)) \quad \text{and} \quad H_- = \text{Ran} E_A((\alpha, \infty)).
$$

That is,

$$
\nu[Jx, y] = -\nu[x, Jy], \quad x, y \in \text{Dom}[a],
$$

where the self-adjoint involution $J$ is given by

$$
J = E_A((\beta, \infty)) - E_A((-\infty, \alpha)).
$$

Let $a$ be the closed form represented by the operator $A$. A direct application of Theorem 2.4 shows that under Hypothesis 4.1 there is a unique self-adjoint boundedly invertible operator $B$ associated with the form $b = a + \nu$.

Under Hypothesis 4.1 we distinguish two cases (see Fig. 1 and 2).

**Case I.** Assume that $\alpha < 0$ and $\beta > 0$. Set

$$
d_+ = \text{dist} (\inf(\Sigma_+), 0) \quad \text{and} \quad d_- = \text{dist}(\inf(\Sigma_-), 0),
$$

and suppose that $d_+ > d_-$. 

**Case II.** Assume that $\alpha, \beta > 0$. Set

$$
d_+ = \text{dist}(\inf(\Sigma_+), 0) \quad \text{and} \quad d_- = \text{dist}(\sup(\Sigma_-), 0).
$$

As it follows from the definition of the quantities $d_\pm$, the sum $d_- + d_+$ coincides with the distance between the lower edges of the spectral components $\Sigma_+$ and $\Sigma_-$ in Case I, while in Case II the difference $d_+ - d_-$ is the distance from the lower edge of $\Sigma_+$ to the upper edge of the spectral component $\Sigma_-$. Therefore, $d_+ - d_-$ coincides with the length of the spectral gap $(\alpha, \beta)$ of the operator $A$ in latter case.

We remark that the condition $d_+ > d_-$ required in Case I, holds only if the length of the convex hull of negative spectrum $\Sigma_-$ of $A$ does not exceed the one of the spectral gap $(\alpha, \beta) = (\inf(\Sigma_-), \sup(\Sigma_+))$.

Now we are prepared to state a relative version of the Tan 2$\Theta$ Theorem in the case where the unperturbed operator is semi-bounded or even positive.
Theorem 4.2. In either Cases I or II, introduce the spectral projections
\[ P = E_A((−∞, α]) \quad \text{and} \quad Q = E_B((−∞, α]) \]
of the operators \( A \) and \( B \) respectively.

Then the norm of the difference of \( P \) and \( Q \) satisfies the estimate
\[ ||P − Q|| ≤ \sin \left( \frac{1}{2} \arctan \left[ \frac{2v}{\delta} \right] \right) < \sqrt{2} \]
where
\[ \delta = \frac{1}{\sqrt{d_+d_-}} \begin{cases} d_+ + d_- & \text{in Case I}, \\ d_+ - d_- & \text{in Case II}, \end{cases} \]
and \( v \) stands for the relative bound of the off-diagonal form \( v \) (with respect to \( a \)) given by (4.1).

Proof. We start with the remark that the form \( a − \mu \), where \( a \) is the form of \( A \), satisfies Hypothesis 2.1 with \( J \) given by (4.2). Set
\[ a_\mu = (a − \mu)J, \quad \mu ∈ (α, β), \]
that is,
\[ a_\mu[x, y] = a[x, Jy] − \mu[x, Jy], \quad x, y ∈ \text{Dom}[a]. \]
Notice that \( a_\mu \) is a strictly positive closed form represented by the operators \( JA − J\mu = |A| − \mu J \) and \( JA − \mu J = |A − \mu I| \) in Cases I and II, respectively.

Since \( v \) is off-diagonal, from Theorem 3.1 it follows that
\[ ||E_{A−\mu I}(\mathbb{R}_+) − E_{B−\mu I}(\mathbb{R}_+)|| ≤ \sin \left( \frac{1}{2} \arctan v\mu \right) \quad \text{for all} \quad \mu ∈ (α, β), \]
with
\[ v\mu =: \sup_{0≠x∈\text{Dom}[a]} \frac{|v[x]|}{a_\mu[x]} \]
Since \( v \) is off-diagonal, by Remark 2.3 one gets the estimate
\[ |v[x]| ≤ 2v_0\sqrt{a_0[x_+]a_0[x_-]}, \quad x ∈ \text{Dom}[a], \]
where \( x = x_+ + x_- \) is a unique decomposition of the element \( x \in \text{Dom}[a] \) with \( x_\pm \in \mathcal{H}_\pm \cap \text{Dom}[a] \).

Thus, in these notations, taking into account that

\[
v_0 = v,
\]

where \( v \) is given by (4.1), one gets the bound

\[
v_\mu \leq 2v \sup_{0 \neq x \in \text{Dom}[a]} \frac{\sqrt{a_0[x_+]a_0[x_-]}}{a_\mu[x]}.
\]

Since \( a_\mu \) is represented by \( JA - J\mu = |A| - \mu J \) and \( JA - \mu J = |A - \mu I| \) in Cases I and II, respectively, one observes that

\[
a_\mu[x] = \begin{cases} a_0[x_+] - \mu \|x_+\|^2 + a_0[x_-] + \mu \|x_-\|^2, & \text{in Case I}, \\ a_0[x_+] - \mu \|x_+\|^2 - a_0[x_-] + \mu \|x_-\|^2, & \text{in Case II}.
\end{cases}
\]

Introducing the elements \( y_\pm \in \mathcal{H}_\pm \),

\[
y_\pm := \begin{cases} (|A| + \mu I)^{1/2}x_\pm, & \text{in Case I}, \\ (A - \mu I)^{1/2}x_\pm, & \text{in Case II},
\end{cases}
\]

and taking into account (4.9), one obtains the representation

\[
\frac{\sqrt{a_0[x_+]a_0[x_-]}}{a_\mu[x]} = \frac{\| |A|^{1/2}(|A| - \mu I)^{-1/2}y_+ \|}{\| y_+ \|^2 + \| y_- \|^2} \cdot \frac{\| |A|^{1/2}(-A + \mu I)^{-1/2}y_- \|}{\| y_+ \|^2 + \| y_- \|^2},
\]

valid in both Cases I and II. Using the elementary inequality

\[
\| y_+ \| \| y_- \| \leq \frac{1}{2} \left( \| y_+ \|^2 + \| y_- \|^2 \right),
\]

one arrives at the following bound

\[
\frac{\sqrt{a_0[x_+]a_0[x_-]}}{a_\mu[x]} \leq \frac{1}{2} \| |A|^{1/2}(|A| - \mu I)^{-1/2}|_{\mathcal{H}_+} \| \cdot \| |A|^{1/2}(-A + \mu I)^{-1/2}|_{\mathcal{H}_-} \|.
\]

It is easy to see that

\[
\| |A|^{1/2}(|A| - \mu I)^{-1/2}|_{\mathcal{H}_+} \| \leq \frac{\sqrt{d_+}}{\sqrt{d_+ - \mu}} \quad \mu \in (0, \beta), \quad \text{in Cases I and II},
\]

while

\[
\| |A|^{1/2}(-A + \mu I)^{-1/2}|_{\mathcal{H}_-} \| \leq \begin{cases} \frac{\sqrt{d_-}}{\sqrt{d_- - \mu}}, & \mu \in (0, \beta), \quad \text{in Case I}, \\ \frac{\sqrt{d_-}}{\sqrt{\mu - d_-}}, & \mu \in (\alpha, \beta), \quad \text{in Case II}.
\end{cases}
\]

Choosing \( \mu = \frac{d_+ - d_-}{2} > 0 \) in Case I (recall that \( d_+ > d_- \) by the hypothesis) and \( \mu = \frac{d_+ + d_-}{2} \) in Case II, and combining (4.10), (4.11), (4.12), one gets the estimates

\[
\frac{\sqrt{a_0[x_+]a_0[x_-]}}{a_{d_+ - d_-}[x]} \leq \frac{\sqrt{d_+ d_-}}{d_+ + d_-} \quad \text{in Case I}
\]

and

\[
\frac{\sqrt{a_0[x_+]a_0[x_-]}}{a_{d_+ + d_-}[x]} \leq \frac{\sqrt{d_+ + d_-}}{d_+ - d_-} \quad \text{in Case II}.
\]

Hence, from (4.8) it follows that
\[ \frac{v_{d_+ - d_-}}{2} \leq 2v \sqrt{\frac{d_+ d_-}{d_+ + d_-}} \quad \text{in Case I} \]

and

\[ \frac{v_{d_+ + d_-}}{2} \leq 2v \sqrt{\frac{d_+ d_-}{d_+ - d_-}} \quad \text{in Case II}. \]

Applying (4.6), one gets the norm estimates

\[ \| E_{A - \frac{d_+ - d_-}{2} I} (\mathbb{R}_+) - E_{B - \frac{d_+ - d_-}{2} I} (\mathbb{R}_+) \| \leq \sin \left( \frac{1}{2} \arctan \left[ \frac{2 \sqrt{d_+ d_-}}{v \sqrt{d_+ + d_-}} \right] \right) \]

in Case I and

\[ \| E_{A - \frac{d_+ + d_-}{2} I} (\mathbb{R}_+) - E_{B - \frac{d_+ + d_-}{2} I} (\mathbb{R}_+) \| \leq \sin \left( \frac{1}{2} \arctan \left[ \frac{2 \sqrt{d_+ d_-}}{v \sqrt{d_+ - d_-}} \right] \right) \]

in Case II. In remains to observe that \( \| P - Q \| \), where the spectral projections \( P \) and \( Q \) are given by (4.3), coincides with the left hand side of (4.13) and (4.14) in Case I and Case II, respectively.

The proof is complete. \( \square \)

**Remark 4.3.** We remark that the quantity \( \delta \) given by (4.5) coincides with the relative distance (with respect to the origin) between the lower edges of the spectral components \( \Sigma_+ \) and \( \Sigma_- \) in Case I and it has the meaning of the relative length (with respect to the origin) of the spectral gap \( (d_-, d_+) \) in Case II.

For the further properties of the relative distance and various relative perturbation bounds we refer to the paper [10] and references quoted therein.

We also remark that in Case II, i.e., in the case of a positive operator \( A \), the bound (4.4) directly improves a result obtained in [6], the relative sin \( \Theta \) Theorem, that in the present notations is of the form

\[ \| P - Q \| \leq \frac{v}{\delta}. \]

We conclude our exposition with considering an example of a \( 2 \times 2 \) numerical matrix that shows that the main results obtained above are sharp.

**Example 4.4.** Let \( \mathcal{H} \) be the two-dimensional Hilbert space \( \mathcal{H} = \mathbb{C}^2 \), \( \alpha < \beta \) and \( w \in \mathbb{C} \).

We set

\[ A = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \quad V = \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Let \( v \) be the symmetric form represented by (the operator) \( V \).

Clearly, the form \( v \) satisfy Hypothesis 4.1 with the relative bound \( v \) given by

\[ v = \frac{|w|}{\sqrt{|\alpha \beta|}}, \]

provided that \( \alpha, \beta \neq 0 \). Since \( VJ = -JV \), the form \( v \) is off-diagonal with respect to the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \).

In order to illustrate our results, denote by \( B \) the self-adjoint matrix associated with the form \( \alpha + v \), that is,

\[ B = A + V = \begin{pmatrix} \beta & w \\ w^* & \alpha \end{pmatrix}. \]

Denote by \( P \) the orthogonal projection associated with the eigenvalue \( \alpha \) of the matrix \( A \), and by \( Q \) the one associated with the lower eigenvalue of the matrix \( B \).
It is well known (and easy to see) that the classical Davis-Kahan Tan $2\Theta$ theorem (1.2) is exact in the case of $2 \times 2$ numerical matrices. In particular, the norm of the difference of $P$ and $Q$ can be computed explicitly

\[(4.15) \quad \|P - Q\| = \sin \left(\frac{1}{2} \arctan \left[ \frac{2|w|}{\beta - \alpha} \right] \right).\]

Since, in the case in question,

\[(4.16) \quad v_\mu = \sup_{0 \neq x \in \text{Dom}[a]} \frac{|a[x]|}{a_\mu[x]} = \frac{|w|}{\sqrt{(\beta - \mu)(\mu - \alpha)}}, \quad \mu \in (\alpha, \beta),
\]

from (4.16) it follows that

\[\inf_{\mu \in (\alpha, \beta)} v_\mu = \frac{2|w|}{\beta - \alpha},
\]

(with the infimum attained at the point $\mu = \frac{\alpha + \beta}{2}$).

Therefore, the result of the relative Tan $2\Theta$ Theorem 3.1 is sharp.

It is easy to see that if $\alpha < 0 < \beta$ (Case I), then the equality (4.15) can also be rewritten in the form

\[(4.17) \quad \|P - Q\| = \sin \left(\frac{1}{2} \arctan \left[ \frac{2\sqrt{d_+ d_-}}{d_+ + d_-} v \right] \right),
\]

where $d_+ = \beta$, $d_- = -\alpha$ and $v = \frac{|w|}{\sqrt{\alpha \beta}}$.

If $0 < \alpha < \beta$ (Case II), the equality (4.15) can be rewritten as

\[(4.18) \quad \|P - Q\| = \sin \left(\frac{1}{2} \arctan \left[ \frac{2\sqrt{d_+ d_-}}{d_+ - d_-} v \right] \right),
\]

with $d_+ = \beta$, $d_- = \alpha$, and $v = \frac{|w|}{\sqrt{\alpha \beta}}$.

The representations (4.17) and (4.18) show that the estimate (4.4) becomes equality in the case of $2 \times 2$ numerical matrices and, therefore, the results of Theorem 4.2 are sharp.

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