Classification of sectors of the Cuntz algebras by graph invariants

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Abstract

A unitary equivalence class of endomorphisms of a unital C∗-algebra $A$ is called a sector of $A$. We introduced permutative endomorphisms of the Cuntz algebra $O_N$ in the previous work. Branching laws of permutative representations of $O_N$ by them are computed by directed regular graphs. In this article, we classify sectors associated with permutative endomorphisms of $O_N$ by their graph invariants concretely.

1 Introduction

Super selection sectors play important role in not only local quantum physics ([6, 7]) but also operator algebras ([8]). A sector in the theory of operator algebras is defined as a unitary equivalence class of a unital C∗-algebra. Special sectors are classified by the statistical dimension, which is an additive, positive integer-valued invariant of sectors. We introduced endomorphisms of the Cuntz algebra $O_N$ as follows: Let $S_{N,l}$ be the set of all bijections on the set $\{1, \ldots, N\}^l$. For $\sigma \in S_{N,l}$, let $\psi_\sigma$ be the endomorphism of $O_N$ defined by

$$\psi_\sigma(s_i) \equiv u_\sigma s_i \quad (i = 1, \ldots, N)$$

(1.1)

where $u_\sigma \equiv \sum_{J \in \{1, \ldots, N\}^l} s_{\sigma(J)}(s_J)^*$ and $s_J \equiv s_{j_1} \cdots s_{j_l}$ for $J = (j_1, \ldots, j_l)$. $\psi_\sigma$ is called the $l$-th order permutative endomorphism of $O_N$ by $\sigma$ ([6-1 [2]). Such endomorphism is concrete and naive but, it is unknown whether it satisfies ingredients to define the statistical dimension or not. For example, we show two endomorphisms $\rho_1, \rho_2$ of $O_3$ as follows:

$$\begin{align*}
\rho_1(s_1) &\equiv s_{23,1} + s_{31,2} + s_{12,3}, \\
\rho_1(s_2) &\equiv s_{32,1} + s_{13,2} + s_{21,3}, \\
\rho_1(s_3) &\equiv s_{11,1} + s_{22,2} + s_{33,3}, \\
\rho_2(s_1) &\equiv s_{32,1} + s_{11,2} + s_{12,3}, \\
\rho_2(s_2) &\equiv s_{33,1} + s_{22,2} + s_{13,3}, \\
\rho_2(s_3) &\equiv s_{23,1} + s_{21,2} + s_{31,3}
\end{align*}$$

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where \( s_{ij,k} \equiv s_is_js_k^* \) for \( i,j,k = 1,2,3 \). N.Nakanishi found \( \rho_1 \) by trial and error. By generalizing \( \rho_1 \), we obtain \( \psi_\sigma \) in (1.1).

**Question.** Whether are \( \rho_1 \) and \( \rho_2 \) unitarily equivalent or not?

Branching laws of permutative representations of \( O_N \) by permutative endomorphisms are computed by directed regular graph ([10]). Since such branching laws are invariant up to unitary equivalence of endomorphisms, we can classify sectors by their branching laws. In this article, we introduce invariants of sectors of \( O_N \) in order to classify sectors more easily than computing branching laws directly.

**Theorem 1.1** For \( \sigma \in \mathfrak{S}_{N,l} \), define a non negative integer-valued \( N^l-1 \times N^l-1 \) matrix \( A_\sigma = (a_{JK})_{J,K \in \{1,\ldots,N\}^l-1} \) such that

\[
a_{JK} \equiv \# \{(n,m) \in \{1,\ldots,N\}^2 : \sigma(n,K) = (J,m)\} \tag{1.2}
\]

and non negative real-valued numbers \( c_1[\sigma], c_2[\sigma], c_3[\sigma] \) by

\[
c_1[\sigma] \equiv \text{Tr} A_\sigma, \quad c_2[\sigma] \equiv \frac{1}{2} \text{Tr}(A_\sigma^2 + A_\sigma), \quad c_3[\sigma] \equiv \frac{1}{3} \text{Tr}(A_\sigma^3 + 2A_\sigma). \tag{1.3}
\]

Under these definitions, if \( \psi_\sigma \sim \psi_\eta \), then \( c_i[\sigma] = c_i[\eta] \) for \( i = 1,2,3 \).

In [9], we classify elements in \( \{\psi_\sigma : \sigma \in \mathfrak{S}_{2,2}\} \) completely by computing branching laws for every \( \psi_\sigma \). However, it is not good to compute branching laws for every element in \( \{\psi_\sigma : \sigma \in \mathfrak{S}_{3,2}\} \) because \( \#\mathfrak{S}_{3,2} = 9! \approx 3.6 \times 10^5 \).

By Theorem 1.1, we obtain a classification of this case.

**Theorem 1.2** For \( \sigma \in \mathfrak{S}_{3,2} \) and \( A_\sigma = (a_{ij})_{i,j=1}^3 \) in Theorem 1.1 define \( g_\sigma \) by the directed graph with 3 vertices \( v_1, v_2, v_3 \), 3 outgoing edges and 3 incoming edges, such that the number of edges from \( v_i \) to \( v_j \) is \( a_{ij} \). Then the following holds:

(i) \( g_\sigma \sim g_\eta \) if and only if \( (c_1[\sigma], c_2[\sigma]) = (c_1[\eta], c_2[\eta]) \) where \( g_\sigma \sim g_\eta \) means that there is a permutation matrix \( T \) such that \( TA_\sigma T^{-1} = A_\eta \).

(ii) \( \#(\{g_\sigma : \sigma \in \mathfrak{S}_{3,2}\}/\sim) = 16 \).

(iii) If \( \psi_\sigma \sim \psi_\eta \), then \( g_\sigma = g_\eta \).

By using Theorem 1.2, we answer the previous question. Since \( \rho_1 = \psi_\sigma \) and \( \rho_2 = \psi_\eta \) for permutations \( \sigma, \eta \in \mathfrak{S}_{3,2} \) defined by

\[
\sigma \begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix} = \begin{pmatrix} 23 & 31 & 12 \\ 32 & 13 & 21 \\ 11 & 22 & 33 \end{pmatrix}, \quad \eta \begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix} = \begin{pmatrix} 32 & 11 & 12 \\ 33 & 22 & 13 \\ 23 & 21 & 31 \end{pmatrix},
\]
graphs \( g_\sigma \) and \( g_\eta \) are given as follows:

\[
\begin{array}{c}
g_\sigma \\
g_\eta
\end{array}
\]

Hence \( \rho_1 \not\sim \rho_2 \).

In [2] we show an algorithm to compute branching laws of permutative representations by permutative endomorphisms. In [3] we introduce a new index for more general sectors. We show formulae among graph invariants and such indices. By these formulae, we prove Theorem 1.1. In [4] we show examples. We classify \( \psi_\sigma \) for \( \sigma \in \mathcal{S}_{N,l} \) when \( (N, l) = (2, 2), (2, 3), (3, 2) \). We prove Theorem 1.2 in [4.2] by illustrating all of 16 graphs.

## 2 Automaton computing of branching laws

For \( N \geq 2 \), let \( \mathcal{O}_N \) be the Cuntz algebra ([4]), that is, the C*-algebra which is universally generated by \( s_1, \ldots, s_N \) satisfying \( s_i^* s_j = \delta_{ij} I \) for \( i, j = 1, \ldots, N \) and \( s_1 s_1^* + \cdots + s_N s_N^* = I \). In this article, any representation and endomorphism are assumed unital and *-preserving. Two endomorphisms \( \rho \) and \( \rho' \) are equivalent if there is a unitary \( u \in \mathcal{O}_N \) such that \( \rho' = \text{Ad}_u \circ \rho \). In this case, we denote \( \rho \sim \rho' \).

Let \( \{1, \ldots, N\} \_1^k \equiv \coprod_{k \geq 1} \{1, \ldots, N\}^k \), \( \{1, \ldots, N\}^k \equiv \{ (j_n)_{n=1}^k : j_n = 1, \ldots, N, n = 1, \ldots, k \} \) for \( k \geq 1 \). For \( J = (j_i)_{i=1}^k \in \{1, \ldots, N\}^k_1 \), a representation \( (\mathcal{H}, \pi) \) of \( \mathcal{O}_N \) is \( P(J) \) if there is a unit cyclic vector \( \Omega \in \mathcal{H} \) such that \( \pi(s_j) \Omega = \Omega \) and \( \{ \pi(s_{j_1} \cdots s_{j_k}) \Omega \}_{j_1=1}^k \) is an orthogonal family. Such representation exists uniquely up to unitary equivalence. Hence the symbol \( P(J) \) makes sense as an equivalence class of representations. \( P(J) \) is equivalent to a cyclic permutative representation of \( \mathcal{O}_N \) with a cycle in \([3, 5]\). When \( (\mathcal{H}, \pi) \) is \( P(J) \), we denote \( \pi \circ \psi_\sigma \) by \( P(J) \circ \psi_\sigma \) for \( \psi_\sigma \) in [1.1]. In [9], we show that for each \( J \), there are \( J_1, \ldots, J_m, 1 \leq m \leq N^{l-1} \) such that \( P(J) \circ \psi_\sigma \) is uniquely decomposed into the direct sum of \( P(J_1), \ldots, P(J_m) \) up to unitary equivalence:

\[
P(J) \circ \psi_\sigma \sim P(J_1) \oplus \cdots \oplus P(J_m).
\]

Concrete several branching laws by \( \psi_\sigma \) are given in [9].

According to [10], we show an algorithm to seek \( J_1, \ldots, J_m \) in [2.1] for \( J \) by reducing problem to a semi-Mealy machine as an input (= \( J \)) and outputs
A semi-Mealy machine is a data \((Q, \Sigma, \Delta, \delta, \lambda)\) which consists of nonempty finite sets \(Q, \Sigma, \Delta\) and two maps \(\delta\) from \(Q \times \Sigma^*\) to \(Q\), \(\lambda\) from \(Q \times \Sigma^*\) to \(\Delta^*\) where \(\Sigma^*\) and \(\Delta^*\) are free semigroups generated by \(\Sigma\) and \(\Delta\), respectively and \(\delta(q, wa) = \delta(\delta(q, w), a)\) and \(\lambda(q, wa) = \lambda(q, w)\lambda(\delta(q, w), a)\) for \(q \in Q, \ w \in \Sigma^*\) and \(a \in \Sigma\). For symbols \(a_1, \ldots, a_N, b_1, \ldots, b_N, J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k, r \geq 1\), we denote \(a_J = a_{j_1} \cdots a_{j_k}, b_J = b_{j_1} \cdots b_{j_k}\) and \(a_J^r = a_J \cdots a_J\) \((r\text{-times})\). The Mealy diagram \(D(M)\) of a semi-Mealy machine \(M = (Q, \Sigma, \Delta, \delta, \lambda)\) is a directed graph with labeled edges which has a set \(Q\) of vertices and a set \(E \equiv \{(q, \delta(q, a), a) \in Q \times Q \times \Sigma : q \in Q, a \in \Sigma\}\) of directed edges with labels. The meaning of \((q, \delta(q, a), a)\) is an edge from \(q\) to \(\delta(q, a)\) with a label \(a/\lambda(q, a)\) for \(a \in \Sigma:\)

\[
\begin{array}{c}
q \\xrightarrow{a}(q, a)
\end{array}
\]

A sequence \(C = (q_1, \ldots, q_k)\) in \(Q\) is a \(k\)-cycle in \(M\) by \(a_{j_1} \cdots a_{j_k} \in \Sigma^*\) if \(q_1, \ldots, q_k\) satisfy that \(\delta(q_t, a_{j_t}) = q_{t+1}\) for \(t = 1, \ldots, k - 1\) and \(\delta(q_1, a_{j_k}) = q_1\) when \(k \geq 2\), and \(\delta(q_1, a_{j_1}) = q_1\) when \(k = 1\). We often denote \(C\) by \(q_1 \cdots q_k\) simply. For a cycle \(q_1 \cdots q_k\), we do not assume that \(q_i \neq q_{i'}\) when \(t \neq t'\) in this article. For \(\sigma \in \mathcal{S}_{N,l}\) with \(l \geq 2\) and \(J \in \{1, \ldots, N\}^l\), we define \(\sigma_1(J), \ldots, \sigma_l(J) \in \{1, \ldots, N\}\) by \(\sigma(J) = (\sigma_1(J), \ldots, \sigma_l(J))\) and let \(\sigma_{n,m}(J) \equiv (\sigma_n(J), \ldots, \sigma_m(J))\) for \(1 \leq n < m \leq l\). Define \(\{1, \ldots, N\}^0 = \{0\}\) for convenience.

**Definition 2.1** For \(\sigma \in \mathcal{S}_{N,l}\), define a data \(M_\sigma \equiv (Q, \Sigma, \Delta, \delta, \lambda)\) by three sets \(Q \equiv \{q_K : K \in \{1, \ldots, N\}^{l-1}\}, \Sigma \equiv \{a_j\}_{j=1}^N, \Delta \equiv \{b_j\}_{j=1}^N\) and two maps \(\delta : Q \times \Sigma^* \to Q, \lambda : Q \times \Sigma^* \to \Delta^*\),

\[
\delta(q_K, a_i) \equiv \begin{cases} q_0 & (l = 1), \\ q_{(\sigma^{-1})_{2,i}(K,i)} & (l \geq 2), \end{cases} \quad \lambda(q_K, a_i) \equiv \begin{cases} b_{\sigma^{-1}(i)} & (l = 1), \\ b_{(\sigma^{-1})_{1,i}(K,i)} & (l \geq 2), \end{cases}
\]

for \(i = 1, \ldots, N\) and \(K \in \{1, \ldots, N\}^{l-1}\). \(M_\sigma \equiv (Q, \Sigma, \Delta, \delta, \lambda)\) is called the semi-Mealy machine by \(\sigma \in \mathcal{S}_{N,l}\).

For \(\sigma \in \mathcal{S}_{N,l}\), the Mealy diagram \(D(M_\sigma)\) of \(M_\sigma\) is a directed regular graph with \(N^{l-1}\) vertices, \(N\) outgoing edges and \(N\) incoming edges when we forget labels of \(D(M_\sigma)\). For a given \(J = (j_i)_{i=1}^k \in \{1, \ldots, N\}^k\), define \(R_J \equiv Q_J/\sim\) where \(Q_J \equiv \{q \in Q : \exists n \in N \text{ s.t. } \delta(q, x^n) = q\}, x \equiv a_{j_1} \cdots a_{j_k} \in \Sigma^* \) and \(\sim\) is defined as \(q \sim q'\) if there is \(n \in N \cup \{0\}\) such that \(\delta(q, x^n) = q'\). Then \(R_J\) is the set of all cycles in \(Q\) by the input word \(x\) and \(R_J \neq \emptyset\).

**Theorem 2.2** \((\text{[10]}\text{)}\) Let \(M_\sigma \equiv (Q, \Sigma, \Delta, \delta, \lambda)\) be in Definition 2.1. For \(J \in \{1, \ldots, N\}^k\), let \(p_1, \ldots, p_m \in Q_J\) be all representatives of elements in...
$R_J$ and $r_i \equiv \min \{ n \in \mathbb{N} : \delta(p_i, x^n) = p_i \}$ for $i = 1, \ldots, m$. If $J_1, \ldots, J_m \in \{1, \ldots, N\}_1^*$ are defined by the output word $b_{J_i} = \lambda(p_i, x^{r_i}) \in \Delta^*$ for $i = 1, \ldots, m$, then $P(J) \circ \psi_{\sigma} \sim P(J_1) \oplus \cdots \oplus P(J_m)$.

In Theorem 2.2 if $p_1', \ldots, p_m'$ are another representatives of $R_J$, then the associated $J_1', \ldots, J_m'$ satisfy $P(J_i') \sim P(J_i)$ for each $i = 1, \ldots, m$.

### 3 Cyclic index of endomorphism

For $J = (j_i)_{i=1}^k \in \{1, \ldots, N\}^k$ and $\tau \in \mathbb{Z}_k$, define $\tau(J) \equiv (j_{\tau(i)})_{i=1}^k$. For $J_1, J_2 \in \{1, \ldots, N\}_1^*$, $J_1 \sim J_2$ if there are $k \geq 1$ and $\tau \in \mathbb{Z}_k$ such that $J_{1,} J_2 \in \{1, \ldots, N\}_1^k$ and $\tau(J_1) = J_2$. $P(J_1) \sim P(J_2)$ if and only if $J_1 \sim J_2$.

For $J_1 = (j_i)_{i=1}^k$, $J_2 = (j'_i)_{i=1}^k \in \{1, \ldots, N\}_1^k$, $J_1 \prec J_2$ if $\sum_{l=1}^k (j'_l - j_l)N^k - l \geq 0$. $J \in \{1, \ldots, N\}_1^*$ is minimal if $J \prec J'$ for each $J' \in \{1, \ldots, N\}_1^*$ such that $J \sim J'$. Especially, any element in $\{1, \ldots, N\}$ is minimal.

Let $\text{End}_{\mathcal{O}} N$ be the set of all unital $*$-endomorphisms of $\mathcal{O}_N$. For $\rho \in \text{End}_{\mathcal{O}} N$, assume that

$$\forall J, J' \in \{1, \ldots, N\}_1^*, \exists m(J|\rho|J') \in \{0, 1, 2, \ldots\} \cup \{\infty\} \ s.t. \ P(J) \circ \rho = \bigoplus_{J' \in X_{N,k}} P(J')^{\oplus m(J|\rho|J')}.$$  (3.1)

For such $\rho$, define a non negative integer $c_k(\rho)$ by the sum of multiplicities

$$c_k(\rho) \equiv \sum_{J, J' \in X_{N,k}} m(J|\rho|J') \quad (k \geq 1)$$  (3.2)

where $X_{N,k}$ is the set of all minimal elements in $\{1, \ldots, N\}_1^*$. If $\rho$ satisfies (3.1), then $m(J|\rho|J')$’s are uniquely determined because the l.h.s. of the branching law in (3.1) is also a permutative representation. Hence $c_k(\rho)$ is well-defined. For example, any permutative endomorphism $\rho$ satisfies (3.1) and $c_k(\rho) < \infty$ for each $k \geq 1$. We call $c_k(\rho)$ by the $k$-th cyclic index of $\rho$. Because branching laws are preserved by unitary equivalence, if $\rho$ and $\rho'$ satisfy (3.1) and $\rho \sim \rho'$, then $c_k(\rho) = c_k(\rho')$ for each $k \geq 1$.

For $\rho \in \text{End}_{\mathcal{O}} N$, define $[\rho] \equiv \{ \rho' \in \text{End}_{\mathcal{O}} N : \rho' \sim \rho \}$. $[\rho]$ is the sector of $\mathcal{O}_N$ with the representative $\rho$. We see that $c_k$ is well-defined on $\text{Sect}_{ad}\mathcal{O}_N \equiv \{ [\rho] : \rho \in \text{End}_{\mathcal{O}} N, \rho \text{ is admissible} \}$, that is, $c_k$ is an invariant of sectors of $\mathcal{O}_N$. Define $H_N(\mathcal{O}_N) \equiv \{ (\xi_i)_{i=1}^N \in (\mathcal{O}_N)^N : \xi_1, \ldots, \xi_N \text{ satisfy relations of canonical generators of } \mathcal{O}_N \}$. For $\rho_1, \ldots, \rho_N \in \text{End}_{\mathcal{O}} N$ and $\xi = (\xi_i)_{i=1}^N \in H_N(\mathcal{O}_N)$, define $< \xi|\rho_1, \ldots, \rho_N > \in \text{End}_{\mathcal{O}} N$ by

$$< \xi|\rho_1, \ldots, \rho_N > \equiv \text{Ad} \xi_1 \circ \rho_1 + \cdots + \text{Ad} \xi_N \circ \rho_N.$$
From this, \( \langle \xi | \rho_1, \ldots, \rho_N \rangle = \langle \xi' | \rho_1, \ldots, \rho_N \rangle \) for each \( \xi, \xi' \in H_N(\mathcal{O}_N) \).

[\( \langle \xi | \rho_1, \ldots, \rho_N \rangle = \langle \xi' | \rho_1', \ldots, \rho_N' \rangle \) if \( \rho_i \sim \rho'_i \) for each \( i = 1, \ldots, N \). \( \langle \xi | \rho_1, \ldots, \rho_N \rangle = \langle \xi | \rho_{\sigma(1)}, \ldots, \rho_{\sigma(N)} \rangle \) for each \( \sigma \in \mathfrak{S}_N \). In consequence, the notation

\[
[\rho_1] + \cdots + [\rho_N] \equiv \langle \xi | \rho_1, \ldots, \rho_N \rangle
\]

is well-defined and \( [\rho_1] + \cdots + [\rho_N] = [\rho_{\sigma(1)}] + \cdots + [\rho_{\sigma(N)}] \) for each \( \sigma \in \mathfrak{S}_N \). Further \( [\rho_1] + \cdots + [\rho_N] \) satisfies the associative law \((11)\). We simply denote \( [\rho_1] + \cdots + [\rho_N] \) by \( \rho_1 + \cdots + \rho_N \).

**Proposition 3.1** If \( \rho_1, \ldots, \rho_N \) satisfy \((3.1)\), then \( c_k(\rho_1 + \cdots + \rho_N) = c_k(\rho_1) + \cdots + c_k(\rho_N) \) for each \( k \geq 1 \).

**Proof.** Let \( \rho = \rho_1 + \cdots + \rho_N \) and \( X_{N,*} = \bigcup_{k \geq 1} X_{N,k} \). Because \( P(J) \circ \rho = P(J) \circ \rho_1 + \cdots + P(J) \circ \rho_N \), \( P(J) \circ \rho = \bigoplus_{J' \in X_{N,*}} P(J') \circ \sum_{j=1}^{N} m(J|\rho_j|J') \). From this, \( m(J|\rho|J') = \sum_{j=1}^{N} m(J|\rho_j|J') \). This implies the statement. \( \blacksquare \)

In consequence, \( c_k \) is an additive invariant of sectors for each \( k \geq 1 \).

If \( \rho \) is the canonical endomorphism of \( \mathcal{O}_N \), then \( c_k(\rho) = \#X_{N,k} \cdot N \) for each \( k \geq 1 \). If \( \rho \) is a permutation of canonical generators of \( \mathcal{O}_N \), then \( c_k(\rho) = \#X_{N,k} \) for each \( k \geq 1 \).

For a directed finite graph \( g \) with the set \( V = \{ v_i \}_{i=1}^{m} \) of vertices, \( A = (a_{ij}) \) is the adjacency matrix of \( g \) if \( a_{ij} \) is the number of edges from \( v_i \) to \( v_j \). A sequence \( (v_{j_1}, \ldots, v_{j_k}) \) in \( V \) is a \( k \)-cycle in \( g \) if \( (v_{j_1}, v_{j_2}), \ldots, (v_{j_{k-1}}, v_{j_k}), (v_{j_k}, v_{j_1}) \) are directed edges of \( g \). In this article, we do not assume that \( v_{j_i} \neq v_{j_{i'}} \) when \( i \neq i' \). If \( M = (Q, \Sigma, \Delta, \delta, \lambda) \) is a semi-Mealy machine, then the Mealy diagram \( D(M) \) gives a directed graph by forgetting input/output labels at each edge of \( D(M) \). If \( Q = \{ q_j \}_{j=1}^{m} \), then the Mealy diagram \( D(M) \) of \( M \) gives a graph \( g \) with the adjacency matrix \( A = (a_{ij})_{i,j=1}^{m} \) such that \( a_{ij} = \# \{ l \in \{1, \ldots, N\} : \delta(q_i, a_l) = q_j \} \) for \( i, j = 1, \ldots, m \). For \( \sigma \in \mathfrak{S}_{N,l} \), the adjacency matrix \( A_\sigma = (a_{jK})_{j,K \in \{1, \ldots, N\} \setminus \{ l \}} \) of \( D(M_\sigma) \) is \((12)\).

Define \( c_k(g) \) by the total number of all \( k \)-cycles in \( g \). For example, \( c_1(g) \) is the total number of all 1-cycles, that is, the number of edges which starts a vertex \( v \) and returns \( v \) again. \( c_2(g) \) is the total number of all 2-cycles, that is, the number of paths with length 2 which starts a vertex \( v \) and returns \( v \) again. For the adjacency matrix \( A \) of a directed graph \( g \), the following holds:

\[
\begin{align*}
c_1(g) &= \text{Tr} A, \\
c_2(g) &= \frac{1}{2} \text{Tr}(A + A^2), \\
c_3(g) &= \frac{1}{3} \text{Tr}(A^3 + 2A).
\end{align*}
\] (3.3)
**Proposition 3.2** If $\rho \in E_{N,l}$ and $g$ is the Mealy diagram associated with $\rho$, then $c_k(\rho) = c_k(g)$ for each $k \geq 1$.

**Proof.** By Theorem 2.2 and (3.1), the statement holds.

**Proof of Theorem 1.1.** By (1.3), (3.3) and Proposition 3.2, we see that $c_i[\sigma] = c_i(\psi_\sigma)$ for $i = 1, 2, 3$. Hence the statement holds.

### 4 Example

For $\psi_\sigma$ in (1.1), define

$$E_{N,l} = \{\psi_\sigma \in \text{End}O_N : \sigma \in S_{N,l}\} \quad (l \geq 1). \quad (4.1)$$

We show classifications of elements of $E_{2,2}$, $E_{3,2}$ and $E_{2,3}$. Let $g_{N,l}$ be the set of all directed graphs which is isomorphic to the Mealy diagram associated with $\sigma \in S_{N,l}$. Then $g_{N,l}$ coincides with the set of all directed regular graphs with $N^{l-1}$ vertices, $N$ outgoing edges and $N$ incoming edges.

Define $S_{N,l}(g) = \{\sigma \in S_{N,l} : D(M_\sigma) \sim g\}$, $E_{N,l}(g) = \{\rho \in E_{N,l} : \exists \sigma \in S_{N,l}(g) \text{ s.t. } \rho = \psi_\sigma\}$ and $S\!E_{N,l}(g) = E_{N,l}(g)/\sim$ for $g \in g_{N,l}$. In order to classify elements in $E_{N,l}$, we list up graphs in $g_{N,l}$ at each subsection.

For $\sigma \in S_{N,2}$,

$$\psi_\sigma(s_i) = s_{\sigma(i,1)}s_1^* + \cdots + s_{\sigma(i,N)}s_N^* \quad (i = 1, \ldots, N).$$

The semi-Mealy machine $M_\sigma \equiv (Q, \Sigma, \Delta, \delta, \lambda)$ by $\sigma$ is given as $Q = \{q_j\}_{j=1}^N$, $\Sigma = \{a_j\}_{j=1}^N$, $\Delta = \{b_j\}_{j=1}^N$, $\delta : Q \times \Sigma \to Q$; $\delta(q_i, a_j) = q_{(\sigma^{-1})_2(i,j)}$, $\lambda : Q \times \Delta \to \Sigma$; $\lambda(q_i, a_j) = b_{(\sigma^{-1})_1(i,j)}$ for $i, j = 1, \ldots, N$.

#### 4.1 $E_{2,2}$

$g_{2,2}$ consists of the following 3 graphs:

- **A:**
  ![Diagram A]

- **B:**
  ![Diagram B]

- **C:**
  ![Diagram C]

where these directions are unique up to permutation of vertices. $c_1(g)$ for $g = A, B, C$ are 4, 2, 0, respectively. The graph of the canonical endomorphisms of $O_2$ is A. The graph of the automorphism of permutation of generators is B. In this way, these graphs give sketchy classification of endomorphisms of $O_2$. 

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In [9], we have more explicit results as follows: The number of unitary equivalence classes of elements in $E_{2,2}$ is 16. $G_2 \equiv \text{Aut}O_2 \cap E_{2,2}$ is a subgroup of the automorphism group $\text{Aut}O_2$ of $O_2$ which is isomorphic to the Klein’s four-group. $G_2$ consists of two outer and two inner automorphisms. $E_{2,2} \setminus G_2$ consists of 10 irreducible and 10 reducible endomorphisms. The numbers of equivalence classes are 5 and 9, respectively.

Their application for fermion algebra is given by [1, 2]. These results are given by computing branching laws of every endomorphism concretely. Such method is possible because $\#E_{2,2} = 24$. However this is not effective when one of $N, k$ is greater than equal 3 because $\#E_{N,k}$ is too large to compute every branching laws.

### 4.2 $E_{3,2}$

In order to classify $E_{3,2}$, we list up Mealy diagrams for every elements in $\mathcal{G}_{3,2}$. We see that $M_\sigma = \{q_1, q_2, q_3\}, \{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \delta, \lambda$ for each $\sigma \in \mathcal{G}_{3,2}$. At the beginning, we show the example $\rho_1$ in §2. By the rule of drawing in §2, the Mealy diagram is given as follows (§9):

![Mealy Diagram](image)

By Theorem 2.2 and this diagram, we obtain branching laws by $\rho_1$:

- input $a_1$
- $q_1, q_2, q_3$
- $b_3, b_1, b_2$
- Branching law $P(1) \circ \rho_1 = P(3) \oplus P(12)$

- input $a_1a_2$
- $q_1q_3q_2q_3$
- $b_3b_1b_3b_2b_2$
- Branching law $P(12) \circ \rho_1 = P(113223)$

For general case, the Mealy diagram is a directed regular graphs with 3 vertices, 3 incoming edges and 3 outgoing edges. We classify such graphs. The adjacency matrix $A = (a_{ij})^{3}_{j=1}$ is a $3 \times 3$-matrix with value 0, 1, 2, 3. By regularity, $\sum_{i=1}^{3} a_{ij} = \sum_{i=1}^{3} a_{ji} = 3$. Let $a \equiv a_{11}, b \equiv a_{22}, c \equiv a_{33}$. We assume that $3 \geq a \geq b \geq c \geq 0$ by permutation of $q_1, q_2, q_3$. We see
that the possibilities of \((a, b, c)\) are \((3, 3, 3), (3, 2, 2), (3, 1, 1), (3, 0, 0), (2, 2, 2), (2, 2, 1), (2, 1, 1), (2, 1, 0), (2, 0, 0), (1, 1, 1), (1, 1, 0), (1, 0, 0), (0, 0, 0)\). \(g_{3,2}\) consists of 16 directed graphs as follows:

Table I.

where the direction of each graph without arrows is unique. For each graph in Table I, there are 216 kinds of configuration of output labels on edges. For the Mealy diagrams associated with \(E_{3,2}\), we have the following:
Table II.

| g       | c_1(g) | c_2(g) | #E_{3,2}(g)/216 |
|---------|--------|--------|------------------|
| (3,3,3) | 9      | 18     | 1                |
| (3,2,2) | 7      | 13     | 27               |
| (3,1,1) | 5      | 12     | 27               |
| (3,0,0) | 3      | 15     | 3                |
| (2,2,2) | 6      | 9      | 54               |
| (2,2,1) | 5      | 9      | 162              |
| (2,1,1) | 4      | 7      | 324              |
| (2,1,0) | 3      | 9      | 162              |
| (2,0,0) | 2      | 9      | 54               |
| (1,1,1)a| 3      | 3      | 54               |
| (1,1,1)b| 3      | 6      | 216              |
| (1,1,0) | 2      | 6      | 324              |
| (1,0,0)a| 1      | 4      | 54               |
| (1,0,0)b| 1      | 7      | 162              |
| (0,0,0)a| 0      | 0      | 2                |
| (0,0,0)b| 0      | 6      | 54               |

where the number of diagrams is computed as the case may be.

Proof of Theorem 1.2. By Table II, any element in $E_{3,2}$ belongs to $E_{3,2}(g)$ for some $g$ appearing in Table I because we see that $\sum_g #E_{3,2}(g) = 9! = #E_{3,2}$. By this, Table II and Theorem 1.1, the statement holds.

We show that how the number of $E_{3,2}(2,1,1) \div 216$ is computed. The permutation of vertices gives 6 different diagrams. Assume that $q_1$ has double 1-cycles and there is a directed edge from $q_1$ to $q_2$. The choice of the (input) label of the vertex from $q_1$ to $q_2$ is 3. The choice of label of 1-cycle at $q_2$ is 3. The choice of labels of edges from $q_3$ is 6. Hence the total number of possibilities is given as $6 \times 3 \times 3 \times 6 = 324$.

By Table II, we see that $#E_{3,2}(3,3,3) = 216$. Any element in $E_{3,2}(3,3,3)$ is given as $\rho(x) \equiv s_1\alpha_1(x)s_1^* + s_2\alpha_2(x)s_2^* + s_3\alpha_3(x)s_3^*$ for $x \in O_3$ where $\alpha_1, \alpha_2, \alpha_3 \in \text{Aut}O_3$ are permutations of canonical generators. From this, we can verify that $#E_{3,2}(3,3,3) = \{#\mathcal{G}_3\}^3$ again. The number of equivalence classes in $E_{3,2}(3,3,3)$ is 56.

In consequence, the decomposition $SE_{3,2} = \coprod_{g \in \mathcal{G}_{3,2}} SE_{3,2}(g)$ holds as the disjoint union. Sectors are classified by graphs.
4.3 $E_{2,3}$

For $\sigma \in \mathfrak{S}_{2,3}$, $M_\sigma = (\{q_{11}, q_{12}, q_{21}, q_{22}\}, \{a_1, a_2\}, \{b_1, b_2\}, \delta, \lambda)$, $\delta(q_J, a_i) = q_{(\sigma^{-1})2,3(J,i)}$, $\lambda(q_J, a_i) = b_{(\sigma^{-1})1(J,i)}$ for $J \in \{1, 2\}^2$, $i = 1, 2$. The graph is a regular directed graph with 4 vertices, 2 outgoing edges and 2 incoming edges. $\mathfrak{g}_{2,3}$ consists of 25 directed graphs which are classified by the diagonal part of their adjacency matrices as follows:

Table III.
where the direction of each graph without arrows is unique. For each graph in Table III, there are 16 kinds of configuration of output labels on edges. Along with Table II, we have the following:

Table IV.

| $g$                    | $c_1(g)$ | $c_2(g)$ | $\#E_{2,3}(g)/16$ |
|------------------------|----------|----------|-------------------|
| (2, 2, 2, 2)           | 8        | 12       | 1                 |
| (2, 2, 1, 1)           | 6        | 9        | 24                |
| (2, 2, 0, 0)           | 4        | 10       | 6                 |
| (2, 1, 1, 1)           | 5        | 6        | 64                |
| (2, 1, 1, 0)           | 4        | 7        | 96                |
| (2, 1, 0, 0)           | 3        | 6        | 96                |
| (2, 0, 0, 0)a          | 2        | 6        | 32                |
| (2, 0, 0, 0)b          | 2        | 3        | 8                 |
| (1, 1, 1, 1)a          | 4        | 4        | 96                |
| (1, 1, 1, 1)b          | 4        | 6        | 48                |
| (1, 1, 1, 0)           | 3        | 4        | 384               |
| (1, 1, 0, 0)a          | 2        | 7        | 24                |
| (1, 1, 0, 0)b          | 2        | 4        | 192               |
| (1, 1, 0, 0)c          | 2        | 5        | 192               |
| (1, 1, 0, 0)d          | 2        | 2        | 48                |
| (1, 1, 0, 0)e          | 2        | 3        | 96                |
| (1, 0, 0, 0)a          | 1        | 3        | 384               |
| (1, 0, 0, 0)b          | 1        | 1        | 96                |
| (1, 0, 0, 0)c          | 1        | 4        | 192               |
| (0, 0, 0, 0)a          | 0        | 0        | 6                 |
| (0, 0, 0, 0)b          | 0        | 4        | 96                |
| (0, 0, 0, 0)c          | 0        | 4        | 48                |
| (0, 0, 0, 0)d          | 0        | 1        | 192               |
| (0, 0, 0, 0)e          | 0        | 2        | 96                |
| (0, 0, 0, 0)f          | 0        | 8        | 3                 |

The total number of $\#E_{2,3}(g)/16$ in Table IV is 2520. Hence we can verify that $\#E_{2,3} = 8! = 40320$. We see that $c_3((2, 0, 0, 0)b) = 12$ and $c_3((1, 1, 0, 0)a) = 4$, $c_3((0, 0, 0, 0)b) = c_3((0, 0, 0, 0)c) = 0$, $c_4((0, 0, 0, 0)b) = 14$ and $c_4((0, 0, 0, 0)c) = 12$. In consequence, $SE_{2,3} = \prod_{g \in G_{2,3}} SE_{2,3}(g)$.

Acknowledgement: The authors would like to thank Yuzuru Maeda for his discovery of two graphs, (2, 1, 0) and (2, 2, 1) in Table I of [4,2].

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