QUANTUM HYDRODYNAMICS
WITH NONLINEAR INTERACTIONS

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Dedicated to P. Secchi and A. Valli on the occasion of their 60th birthday

Abstract. In this paper we prove the global existence of large amplitude finite energy solutions for a system describing Quantum Fluids with nonlinear nonlocal interaction terms. The system may also (but not necessarily) include dissipation terms which do not provide any help to get the global existence. The method is based on the polar factorization of the wave function (which somehow generalizes the WKB method), the construction of approximate solutions via a fractional step argument and the deduction of Strichartz type estimates for the approximate solutions. Finally local smoothing and a compactness argument of Lions Aubin type allow to show the convergence.

1. Introduction. In this paper we investigate the existence of global solutions for the following Quantum Hydrodynamics (QHD) system

\[
\begin{cases}
\partial_t \rho + \text{div} J = -\beta \rho^{\sigma+1} \\
\partial_t J + \text{div} \left( \frac{J \otimes J}{\rho} \right) + \nabla P(\rho) + \rho \nabla V = \frac{1}{2} \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \alpha J - \beta \rho^\sigma J - \rho \nabla F \\
- \Delta V = \rho,
\end{cases}
\]

\((t, x) \in \mathbb{R}_+ \times \mathbb{R}^3\), with initial data

\[\rho(0) = \rho_0, \quad J(0) = J_0,\]

which we will assume to be in the energy space (a more precise definition will be given below).

This system describes the dynamics of a quantum fluid, with mass (or charge) density \(\rho\) and momentum (or current) density \(J\), subject to a self-consistent electrostatic potential \(V\), determined by the Poisson equation, and with pressure \(P(\rho)\).

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The term $\frac{1}{2} \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right)$ is called the quantum Bohm potential and $\frac{1}{2} \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right)$ is a nonlinear third-order dispersive term. It can also be interpreted as a quantum correction to the classical pressure (stress tensor). Indeed, with some regularity assumptions, this term can also be written in the following way

$$\frac{1}{2} \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = \frac{1}{4} \nabla \Delta \rho - \text{div}(\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}) = \frac{1}{4} \text{div}(\rho \nabla^2 \log \rho). \quad (2)$$

The terms $-\beta \rho^{\sigma+1}$ and $-\alpha \rho J - \beta \rho^\sigma J$ in the equations for the mass and momentum densities, respectively, are dissipative (relaxation) terms, with $\alpha, \beta$ non-negative constants. Here and throughout the paper we are going to assume $0 \leq \sigma \leq 2$.

QHD models arise in a number of areas in physics, including the description of superfluidity [20], superconductivity [12], the dynamics of Bose-Einstein condensates [9]. In particular the system 1 with $\beta = 0$, $F \equiv 0$ is widely studied in the literature [25, 24, 17, 21, 1, 2, 16] because of its applications in the modeling of semiconductor devices [13]. In this note we also consider the presence of a nonlinear dissipation, with $\beta > 0$, $F \neq 0$. Such terms may be considered as describing some interaction of the quantum fluid with an external gas. In particular in [15] various models regarding Bose Einstein condensation propose a system of Quantum Hydrodynamics supplemented by systems of classical fluids, with exchange terms for the mass and momentum. Our toy model here is aimed at analyzing some of the mathematical difficulties arising in such problems.

For the analysis developed here to be meaningful, we make the following assumptions on $F : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$:

$$\begin{cases}
F \in C^1(\mathbb{R}_+ \times \mathbb{R}^3); \\
|F(\rho, K)| + |\rho \partial_\rho F(\rho, K)| + |\sqrt{\rho} \nabla K F(\rho, K)| \leq C, \quad (\rho, K) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\sqrt{\rho} \nabla_x F \in L^{\infty}_t L^2(\mathbb{R}_+ \times \mathbb{R}^3) \cap L^2_{t,x}(\mathbb{R}_+ \times \mathbb{R}^3).
\end{cases} \tag{3}$$

In this respect, $\rho \nabla F$ can be viewed as a term mimicking the interaction of the quantum fluid with the normal one [15].

We consider the term $F = F(\rho, K)$ as a nonlocal nonlinear function of the unknowns $(\rho, J)$, where along the motion $K$ is given by $K[J](t, x) = \int_{\mathbb{R}^3} \kappa(x - y) J(t, y) \, dy$. We shall assume $\kappa$ is a convolution operator such that

$$K : L^1 \to H^1 \text{ is a bounded operator.} \tag{4}$$

We study the Cauchy problem associated to 1 in the space of energy, namely we require the total mass

$$M[\rho] := \int_{\mathbb{R}^3} \rho(t, x) \, dx,$$

and the total energy

$$E[\rho, J] := \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \sqrt{\rho}|^2 + \frac{1}{2} \frac{|J|^2}{\rho} + f(\rho) + \frac{1}{2} |\nabla V|^2 \, dx$$

to be finite at all times. The internal energy $f(\rho)$ is related to the pressure term $P(\rho)$ through the formula $P(\rho) = \rho f'(\rho) - f(\rho)$. Here for simplicity we assume the internal energy to satisfy a power law, $f(\rho) = \frac{1}{\gamma+1} \rho^{\gamma+1}$, $0 \leq \gamma \leq 2$, so that $P(\rho) = \frac{\gamma}{\gamma+1} \rho^{\gamma+1}$. More general pressure terms could be considered in our analysis, however this goes beyond the scope of our note.

We are interested in finding global in time finite energy weak solutions to the Cauchy problem associated to the system 1.
Definition 1.1 (Finite energy weak solutions). Let \( \rho_0, J_0 \in L^1_{\text{loc}}(\mathbb{R}^3) \), we say the pair \((\rho, J)\) is a finite energy weak solution of the Cauchy problem for \(1\) with initial data
\[
\rho(0) = \rho_0, \quad J(0) = J_0,
\]
in the space-time slab \([0, T] \times \mathbb{R}^3\) if there exist two locally integrable functions
\[
\sqrt{\rho} \in L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\mathbb{R}^3)), \quad \Lambda \in L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\mathbb{R}^3))
\]
such that
\[
\begin{align*}
\rho &:= (\sqrt{\rho})^2, J := \sqrt{\rho} \Lambda; \\
\sqrt{\rho}^{\sigma+1} &\in L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\mathbb{R}^3)) \text{ and } \sqrt{\rho}^{\sigma} \Lambda \in L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\mathbb{R}^3)); \\
\forall \eta \in C^\infty_0([0, T) \times \mathbb{R}^3), \\
\int_0^T \int_{\mathbb{R}^3} \rho \partial_t \eta + J \cdot \nabla \eta - \beta \rho^{\sigma+1} \eta \, dx \, dt + \int_{\mathbb{R}^3} \rho_0(x) \eta(0, x) \, dx &= 0; \\
\forall \zeta \in C^\infty_0([0, T) \times \mathbb{R}^3), \\
\int_0^T \int_{\mathbb{R}^3} J \cdot \partial_t \zeta + \Lambda \otimes \Lambda : \nabla \zeta + P(\rho) \, d\zeta - \rho \nabla V \cdot \zeta - \alpha J \cdot \zeta + \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} : \nabla \zeta \\
+ \frac{1}{4} \rho \Delta \nabla \zeta - \beta \rho^{\sigma} J \cdot \zeta - \rho \nabla F \cdot \zeta \, dx \, dt + \int_{\mathbb{R}^3} J_0(x) \cdot \zeta(0, X) \, dx &= 0; \\
\text{• (finite energy) the total mass and energy defined by} \\
M(t) := \int_{\mathbb{R}^3} \rho(t, x) \, dx, \\
E(t) := \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \sqrt{\rho}|^2 + \frac{1}{2} |\Lambda|^2 + f(\rho) + \frac{1}{2} |\nabla V|^2 \, dx,
\end{align*}
\]
respectively, are finite for every \(t \in [0, T)\);
\[
\text{• (generalized irrotationality condition) for almost every } t \in (0, T) \\
\nabla \wedge J &= 2 \nabla \sqrt{\rho} \wedge \Lambda,
\]
holds in the sense of distributions.

We say \((\rho, J)\) is a global in time finite energy weak solution if we can take \(T = \infty\) in the above definition.

Remark 1. Let us consider for the moment a smooth solution \((\rho, J)\) for which we can write \(J = \rho u\), for some velocity field \(u\), also smooth. Then the last condition in the definition above is equivalent to \(\rho \nabla \wedge u = 0\), i.e., the velocity field \(u\) is irrotational \(\rho \, dx\) almost everywhere. This explains why the last condition is denominated the 
\textit{generalized irrotationality condition}. Let us also remark that the solutions introduced in Definition 1.1 are more general than those obtained by using the WKB ansatz, since in the latter case the velocity field \(u = \nabla S\) is always irrotational. This means the WKB ansatz rules out completely the presence of quantized vortices, which are singularities of the velocity field and appear exactly in the nodal region. Quantized vortices have a very rich structure and they are intensively studied in the physics of superfluids [5], [28]. Hence the finite energy weak solutions we deal with throughout this paper are consistent with quantized vortices and may be a starting point to investigate in a mathematically rigorous way some phenomena in superfluidity.
Remark 2. The term $\rho\nabla F$ in the Definition above, as well as in 3, should be interpreted in the following way:

$$\sqrt{\rho}\partial_j F = \sqrt{\rho}\partial_j F \rho + \sqrt{\rho} \nabla K \cdot \partial_j K \in L^\infty_t L^2_x(R^+ \times R^3) \cap L^2_t L^2_x(R^+ \times R^3).$$

Hence, under the assumptions 3 on $F$ and 4 on $K$, definition 1.1 is well posed.

Our main Theorem in this note is the following one.

**Theorem 1.2.** Let $f(\rho) = \frac{1}{\gamma+1} \rho^{\gamma+1}$, with $0 \leq \gamma \leq 2$, let $0 \leq \sigma \leq 2$ and assume $F = F(\rho, K)$, satisfies 3 and $K$ satisfies 4. For any $\psi_0 \in H^1(R^3)$ let us define

$$\rho_0 := |\psi_0|^2, J_0 := \text{Im}(\bar{\psi_0} \nabla \psi_0),$$

then there exists a global in time finite energy weak solution for 1 with Cauchy data

$\rho(0) = \rho_0, \quad J(0) = J_0.$

Furthermore, for any finite $t \in (0, \infty)$, we have

$$M(t) + \beta \int_0^t \int_{R^3} \rho^{\sigma+1}(t', x) \, dx dt' \leq M(0)$$

$$E(t) + \beta \int_0^t \int_{R^3} \rho^{\sigma} \left( |\nabla \rho|^2 + |\Lambda|^2 + \rho f'(\rho) + \rho V \right) \, dx dt' + \alpha \int_0^t \int_{R^3} |\Lambda|^2 \, dx dt' \lesssim E(0) + \int_0^t \int_{R^3} \rho |\nabla F|^2 \, dx dt'.$$

**Remark 3.** The previous theorem shows the existence of global solutions for 1, however nothing can be said about their uniqueness. This fact is not surprising since in [10] it has been proved for systems of Euler-Korteweg type, including the Quantum Hydrodynamics, the existence of infinitely many weak solutions satisfying the energy inequality.

Let us consider system 1 with $\alpha = 0, F \equiv 0$, then 1 is formally equivalent to the following nonlinear Schrödinger-Poisson system with nonlinear damping

$$\begin{align*}
\begin{cases}
&i \partial_t \psi = -\frac{1}{2} \Delta \psi + V \psi + |\psi|^{2\gamma} \psi - \frac{i}{2} \beta |\psi|^{2\gamma} \psi \\
&- \Delta V = |\psi|^2
\end{cases} \\
\psi(0) = \psi_0.
\end{align*}$$

(7)

The analogy can be easily seen by assuming the so called WKB ansatz, however to make it rigorous we need the polar factorization method developed in [1], [2] (see the discussion in Section 2 below). On the other hand, the terms $-\rho\nabla F - \alpha J$ in the equation for the momentum density destroy somehow this analogy with nonlinear Schrödinger equations. For example, the collision term $-\alpha J$ would introduce, in the wave function description, the following additional self-consistent potential $V = \frac{\alpha}{2} \log \left( \frac{\psi}{\bar{\psi}} \right)$ (see [19]). Unfortunately there is not a well-established theory for NLS equations with such potential, thus we have to deal with those additional terms in a different way. For this purpose we will introduce an operator splitting argument in order to find a sequence of approximate solutions to 1 (see Definition 3.1 below).

**Remark 4.** Actually with these assumptions it is possible to manage the term $\rho\nabla F$ directly inside the Schrödinger by considering $F$ as a perturbation of the external potentials, however we prefer to manage it within the fractional step argument since in the more realistic physical problems, the former method fail.
The paper is organized as follows. In Section 2 we introduce the polar decomposition and discuss the analogy between 7 and 1 with $F \equiv 0, \alpha = 0$. In Section 3 we introduce the operator splitting argument, needed to construct our sequence of approximate solutions, and we prove consistence and compactness of the sequence.

2. Polar decomposition. In this Section we review the polar decomposition method, through which we are going to define the hydrodynamic quantities $(\sqrt{\rho}, \Lambda)$, in terms of the underlying wave function $\psi$, in the framework of finite energy states.

The main advantage of the polar factorization is that vacuum regions are allowed in the theory. More precisely, we write the wave function $\psi$ in terms of its amplitude $\sqrt{\rho} := |\psi|$ and its unitary factor $\phi$, namely a function taking its values in the unitary disk of the complex plane, such that $\psi = \sqrt{\rho} \phi$. In the WKB setting the polar factor would be $\phi = e^{i\phi}$, however this equality holds only in the case of a smooth, nowhere vanishing, wave function. The idea of polar factorization is similar in spirit to the one used by Brenier in [6] to decompose a vector-valued function by means of a gradient of a convex function and a measure preserving map. Our case is much simpler than [6] and it can be studied directly.

Given any function $\psi \in H^1(\mathbb{R}^3)$ we define the set

$$ P(\psi) := \{ \phi \in L^\infty(\mathbb{R}^3) : ||\phi||_{L^\infty} \leq 1, \psi = \sqrt{\rho} \phi \ \text{a.e. in} \ \mathbb{R}^3 \}, $$

where $\sqrt{\rho} := |\psi|$. For any polar factor $\phi \in P(\psi)$, we have $|\phi| = 1 \sqrt{\rho} \text{dx a.e. in} \ \mathbb{R}^3$ and $\phi$ is uniquely defined $\sqrt{\rho} \text{dx a.e. in} \ \mathbb{R}^3$.

The next Lemma uses the polar factor to define the hydrodynamic quantities in terms of the underlying wave function, in the framework of finite energy states. It shows then how this structure is stable in $H^1(\mathbb{R}^3)$ in a sense which will be specified below. Moreover we see that any current density originated from a wave function in $H^1(\mathbb{R}^3)$ satisfies the generalized irrotationality condition.

**Lemma 2.1.** Let $\psi \in H^1(\mathbb{R}^3)$, $\sqrt{\rho} := |\psi|$ its amplitude and let $\phi \in P(\psi)$ be a polar factor associated to $\psi$. Then $\sqrt{\rho} \in H^1(\mathbb{R}^3)$ and we have $\nabla \sqrt{\rho} = \Re(\bar{\phi} \nabla \psi)$. Moreover, if we define $\Lambda := \Im(\bar{\phi} \nabla \psi)$, then $\Lambda \in L^2(\mathbb{R}^3)$ and the following identity holds

$$ \Re(\nabla \bar{\rho} \otimes \nabla \psi) = \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} + \Lambda \otimes \Lambda, \ \text{a.e. in} \ \mathbb{R}^3. \quad (8) $$

Furthermore, if $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ is a strongly converging sequence in $H^1$, say $\psi_n \rightarrow \psi$, then we have

$$ \nabla \sqrt{\rho_n} \rightarrow \nabla \sqrt{\rho}, \ \Lambda_n \rightarrow \Lambda, \ \text{in} \ L^2(\mathbb{R}^3), $$

where $\sqrt{\rho_n} := |\psi_n|$, $\Lambda_n := \Im(\bar{\phi_n} \nabla \psi_n)$, $\phi_n$ being a unitary factor for $\psi_n$. Finally the current density

$$ J := \Im(\bar{\psi} \nabla \psi) = \sqrt{\rho} \Lambda, $$

satisfies

$$ \nabla \wedge J = 2\nabla \sqrt{\rho} \wedge \Lambda, \ \text{a.e. in} \ \mathbb{R}^3. $$

**Proof.** For the proof we address the reader to [1, 2].

Thanks to this Lemma we may show the analogy between equation 7 and 1 with $\alpha = 0, F \equiv 0$. More precisely, we prove the following

**Proposition 1.** Let $\psi_0 \in H^1(\mathbb{R}^3)$ and let $\psi \in C(\mathbb{R}_+; H^1(\mathbb{R}^3))$ be the unique global solution to the Cauchy problem 7. Then $(\sqrt{\rho}, \Lambda)$, with $\sqrt{\rho} := |\psi|, \Lambda := \Im(\bar{\phi} \nabla \psi), \phi$ polar factor for $\psi$, defines a global in time finite energy weak solution to 1 in the case $\alpha = 0, F \equiv 0$, with initial data $p_0 = |\psi_0|^2, J_0 = \Im(\bar{\psi_0} \nabla \psi_0)$. 

\qed
It is known that the Cauchy problem \(7\) is globally well-posed in \(H^1(\mathbb{R}^3)\). In [3] the authors study \(7\) without the nonlocal Poisson potential, however it is straightforward to extend the global well-posedness result to the equation in \(7\) by following the same proof as in [3].

**Proof.** As we said above, the Cauchy problem \(7\) is globally well-posed in \(H^1(\mathbb{R}^3)\), in particular this means the solution is continuous with respect to the initial data. Furthermore, the solution \(\psi\), emanated by a smooth initial datum \(\psi_0\), is smooth. Hence it will suffice to prove the Proposition only for smooth solutions \(\psi\) to \(7\). Let us then consider such a regular solution, and let us define

\[
\sqrt{\rho} := |\psi|, \quad \Lambda := \text{Im}(\overline{\psi} \nabla \psi) = \text{Im}(\overline{\psi_0} \nabla \psi_0),
\]

so that the equation for \(\rho\) in \(1\) is satisfied. Analogously for \(J = \text{Im}(\overline{\psi} \nabla \psi)\), after some calculations, we find

\[
\partial_t J + \text{div}(\Lambda \otimes \Lambda) + \nabla P(\rho) + \rho \nabla V = \frac{1}{4} \nabla \Delta \rho - \beta \rho^\sigma J.
\]

Let us recall formula \(8\), we have

\[
\text{Re}(\overline{\psi} \nabla \psi) = \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} + \Lambda \otimes \Lambda.
\]

If we plug it into the equation for the current density we obtain

\[
\partial_t J + \text{div}((\text{Re}(\nabla \psi \otimes \nabla \psi)) + \frac{\gamma}{\gamma + 1} \nabla \rho^{\gamma + 1} + \rho V = \frac{1}{4} \nabla \Delta \rho - \beta \rho^\sigma J.
\]

which is the second equation in \(1\) thanks to \(2\). Thus the Proposition is proved for \(\psi\) smooth. The case of an arbitrary \(\psi \in C(\mathbb{R}^+; H^1(\mathbb{R}^3))\) is done by considering the equations for \(\rho\) and \(J\) in the weak sense, as in Definition 1.1, and by considering a sequence of smooth solutions \(\psi_n\), for which we already proved the Proposition, converging to \(\psi\). Furthermore, by integrating in space and time the equation for the mass density we find

\[
M(t) + \beta \int_0^t \int_{\mathbb{R}^3} \rho^{\sigma + 1}(t', x) dx dt' = M(0),
\]

so that the total mass is finite at every time. For the energy \(6\) we have

\[
\frac{d}{dt} E(t) = -\alpha \int \rho^\sigma \left[ \frac{2\sigma + 1}{2} |\nabla \sqrt{\rho}|^2 + \frac{1}{2} |\Lambda|^2 + \rho f'(\rho) + \rho V \right] dx,
\]

and consequently,

\[
E(t) + \alpha \int_0^t \int_{\mathbb{R}^3} \rho^\sigma \left[ |\nabla \sqrt{\rho}|^2 + |\Lambda|^2 + \rho f'(\rho) + \rho V \right] dx dt' \lesssim E(0).
\]

\[\square\]

**Corollary 1.** For any \(\psi_0 \in H^1(\mathbb{R}^3)\), let

\[
\rho_0 := |\psi_0|^2, \quad J_0 := \text{Im}(\overline{\psi_0} \nabla \psi_0),
\]

then there exists a global in time finite energy weak solution to the QHD system 1 with $\alpha = 0, F \equiv 0$.

Now we will state a Lemma which will be used in Section 3 for the updating procedure of the approximate solutions. It is a straightforward consequence of the polar factorization, however we will state it here as it will be handy for the application in the fractional step argument.

**Lemma 2.2** (Handy for applications). Let $\psi \in H^1(\mathbb{R}^3)$ and let $\varepsilon, \tau > 0$ be two arbitrary (small) real numbers. Let us furthermore assume Lemma 2.2 application in the fractional step argument.

Then there exists $\tilde{\psi} \in H^1(\mathbb{R}^3)$ such that if $\sqrt{\rho} := |\hat{\psi}|, \hat{\Lambda} := \text{Im}(\bar{\rho} \nabla \hat{\psi})$, with $\hat{\phi}$ polar factor for $\hat{\psi}$, then

\[
\begin{align*}
\sqrt{\tilde{\rho}} &= \sqrt{\rho} + r_{\varepsilon} \\
\tilde{\Lambda} &= (1 - \alpha \tau)\Lambda - \tau \sqrt{\rho} \nabla F(\rho, K) + \Gamma_{\varepsilon} \\
\nabla \tilde{\psi} &= \nabla \psi - i\tau \hat{\phi}(\alpha \Lambda + \sqrt{\rho} \nabla F) + R_{\varepsilon, \tau},
\end{align*}
\]

where $\|\hat{\phi}\|_{L^\infty} \leq 1$ and $\|r_{\varepsilon}\|_{H^1} + \|\Gamma_{\varepsilon}\|_{L^2} \leq \varepsilon$

and $\|R_{\varepsilon, \tau}\|_{L^2} \leq \varepsilon + \tau \|\nabla \psi\|_{L^2}$.

**Proof.** Let $\varepsilon, \tau > 0$ be fixed. For $\psi \in H^1(\mathbb{R}^3)$ we consider a sequence $\{\psi_n\} \subset C^0_0(\mathbb{R}^3)$ of smooth, compactly supported functions, converging strongly to $\psi$ in $H^1$. Because of their regularity we may write

\[
\psi_n(x) = e^{i\theta_n(x)} \sqrt{\rho_n(x)}, \quad \forall x \in \mathbb{R}^3, n \in \mathbb{N},
\]

where $\theta_n$ is a bounded, piecewise smooth function, defined on the compact set $\Omega_n := \{x \in \mathbb{R}^3 : |\psi_n(x)| > 0\}$ and with values in $[-\pi, \pi]$. By Lemma 2.1 we know

\[
\sqrt{\rho_n} \rightarrow \sqrt{\rho} \quad \text{in} \quad H^1, \quad \Lambda_n \rightarrow \Lambda \quad \text{in} \quad L^2.
\]

By using the convergence above and since $F$ satisfies 3, we may choose $n^* = n^*(\varepsilon) \in \mathbb{N}$ such that

\[
\|\psi_n - \psi\|_{H^1} + \|\Lambda_n - \Lambda\|_{L^2} + \|\sqrt{\rho_n} \nabla F_n - \sqrt{\rho} \nabla F\|_{L^2} \leq \varepsilon,
\]

where we denote $F_n := F(\rho_n, K_n), \Lambda_n = \kappa * J_n$. We define

\[
\hat{\psi} := e^{i(1-\alpha \tau)\theta_n} e^{-i\tau F_n} \sqrt{\rho_n}.
\]

We now show that $\hat{\psi}$ satisfies the properties stated in Lemma. A simple calculation yields

\[
\nabla \hat{\psi} = e^{i(1-\alpha \tau)\theta_n} e^{-i\tau F_n} \left( \nabla \sqrt{\rho_n} + i\Lambda_n \right) - i\tau e^{i(1-\alpha \tau)\theta_n} e^{-i\tau F_n} \left( \alpha \Lambda_n + \sqrt{\rho_n} \nabla F_n \right)
\]

\[
= \nabla \psi_n - i\tau e^{i(1-\alpha \tau)\theta_n} e^{-i\tau F_n} \left( \alpha \Lambda_n + \sqrt{\rho_n} \nabla F_n \right) + \left( e^{i(1-\alpha \tau)\theta_n} e^{-i\tau F_n} - e^{i\theta_n} \right) \left( \nabla \sqrt{\rho_n} + i\Lambda_n \right).
\]

For the last term on the right hand side we have

\[
\|e^{i(1-\alpha \tau)\theta_n} e^{-i\tau F_n} - e^{i\theta_n} \|_{L^\infty} \lesssim \tau \left( \alpha \|\theta_n\|_{L^\infty} + \|F_n\|_{L^\infty} \right).
\]

Consequently, if we set $\hat{\phi} = e^{i(1-\alpha \tau)\theta_n} e^{-i\tau F_n}$, we may write

\[
\nabla \hat{\psi} = \nabla \psi - i\tau \hat{\phi}(\alpha \Lambda + \sqrt{\rho} \nabla F) + R_{\varepsilon, \tau}.
\]
where
\[
R_{\varepsilon, \tau} = \nabla \psi_{\varepsilon, \tau} - \nabla \psi - i \varepsilon \phi (\alpha (\Lambda_{\varepsilon, \tau} - \Lambda) + \sqrt{\rho_{\varepsilon, \tau}} \nabla F_{\varepsilon, \tau} - \sqrt{\rho} \nabla F)
\]
\[
+ \left( e^{i(1 - \alpha) \phi_{\varepsilon, \tau}} e^{-i \varepsilon F_{\varepsilon, \tau} - e^{i \phi_{\varepsilon, \tau}}} \right) (\nabla \sqrt{\rho_{\varepsilon, \tau}} + i \Lambda_{\varepsilon, \tau}).
\]
Furthermore,
\[
\tilde{\Lambda} = \text{Im}(\tilde{\phi} \nabla \tilde{\psi}) = (1 - \alpha \tau) \Lambda_{\varepsilon, \tau} - \tau \sqrt{\rho_{\varepsilon, \tau}} \nabla F_{\varepsilon, \tau}.
\]
It is now straightforward to prove that \(\tilde{\psi}\) satisfies (9). Thus the Lemma is proved. \(\square\)

3. Fractional step: Consistency and compactness. In this Section we discuss how to construct the sequence of approximate solutions via the operator splitting argument. We will then show the sequence has the right compactness properties to pass to the limit. As explained in the Introduction the terms \(-\alpha J - \rho \nabla F\) cannot be considered directly in the NLS equation, because they would give ill-posed self-consistent potentials. For this reason, we look for finite energy weak solutions to (1) by means of a fractional step argument: we construct a sequence of approximate solutions for (1), then we show that they have a limit and that this limit is indeed a finite energy weak solution to the Cauchy problem associated to (1).

**Definition 3.1.** Let \(\tau > 0\) be a small parameter, we say \(\{ (\rho^\tau, J^\tau) \}_{\tau > 0}\) is a sequence of approximate solutions for the system (1) with initial data \((\rho_0, J_0) \in L^1_{\text{loc}}(\mathbb{R}^3)\) if there exist locally integrable functions \(\sqrt{\rho^\tau} \in L^2_{\text{loc}}((0, T) \times \mathbb{R}^3)\), \(\Lambda^\tau \in L^2_{\text{loc}}((0, T) \times \mathbb{R}^3)\);

\[
\int_0^T \int_{\mathbb{R}^3} \rho^\tau \partial_t \eta + J^\tau \cdot \nabla \eta - \beta (\rho^\tau)^{\sigma + 1} \eta \, dx \, dt
+ \int_{\mathbb{R}^3} \rho_0(x) \eta(0, x) \, dx = o(\tau), \quad \text{as } \tau \to 0;
\]

\[
\int_0^T \int_{\mathbb{R}^3} J^\tau \cdot \partial_t \zeta + \Lambda^\tau \otimes \Lambda^\tau : \nabla \zeta + P(\rho^\tau) \, \text{div} \zeta - \rho^\tau \nabla V^\tau \cdot \zeta - \alpha J^\tau \cdot \zeta
+ \nabla \sqrt{\rho^\tau} \otimes \nabla \sqrt{\rho^\tau} : \nabla \zeta + \frac{1}{4} \rho^\tau \Delta \text{div} \zeta - \beta (\rho^\tau)^\sigma J^\tau \cdot \zeta - \rho^\tau \nabla F^\tau \cdot \zeta \, dx \, dt
+ \int_{\mathbb{R}^3} J_0(x) \cdot \zeta(0, X) \, dx = o(\tau), \quad \text{as } \tau \to 0;
\]

\[
\text{(finite energy) the total mass and energy defined by}
M^\tau(t) := \int_{\mathbb{R}^3} \rho^\tau(t, x) \, dx,
E^\tau(t) := \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \sqrt{\rho^\tau}|^2 + \frac{1}{2} |\Lambda^\tau|^2 + f(\rho^\tau) + \frac{1}{2} |\nabla V^\tau|^2 \, dx,
\]
respectively, are finite for every \(t \in [0, T] ;\)
are approximate solutions, in the sense of Definition 3.1 to system 1 with \( \alpha \) associated to the wave function \( \psi \), equation (see Remark 5 below), however we will call it in this way because its moments (remaining terms, namely we must solve only need to update the hydrodynamic quantities in order to take into account the remaining terms. Then we start again with the QHD system with \( \alpha = 0, F \equiv 0 \), and so on.

The motivation for this setup of the fractional step is the following: we already know how to solve the QHD system 1 with \( \alpha = 0, F \equiv 0 \), i.e. solve the NLS equation 7 and then we define the hydrodynamic quantities associated to the solution of 7, see Proposition 1. Thus at the end of each time step where we solve 7 we only need to update the hydrodynamic quantities in order to take into account the remaining terms, namely we must solve

\[
\begin{aligned}
\partial_t \rho &= 0 \\
\partial_t J &= -\alpha J - \rho \nabla F.
\end{aligned}
\]

However, let us remark that this simple system must be solved at the level of the wave function description of the dynamics, in order to start again with the NLS equation 7 again at the next step. Consequently we must translate this step into updating the wave function. This is exactly the purpose of Lemma 2.2.

More precisely, let \( \tau > 0 \) be a fixed small parameter, we construct our approximate solution \( \psi^\tau \) in the following way. Let \( \psi_0 \in H^1(\mathbb{R}^3) \), at first step \( k = 0 \) we solve 7 in \([0, \tau) \times \mathbb{R}^3\),

\[
\begin{aligned}
&i\partial_t \psi^\tau = -\frac{1}{2} \Delta \psi^\tau + f'(|\psi^\tau|^2)\psi^\tau + V^\tau \psi^\tau - i\frac{\beta}{2} |\psi^\tau|^{2\sigma} \psi^\tau, \quad (t, x) \in [0, \tau) \times \mathbb{R}^3 \\
&\quad - \Delta V^\tau = |\psi^\tau|^2, \quad (t, x) \in [0, \tau) \times \mathbb{R}^3 \\
&\psi^\tau(0) = \psi_0,
\end{aligned}
\]

Let us define the approximate solution by induction: we assume we already constructed \( \psi^\tau \) in \([k-1] \tau, k \tau) \times \mathbb{R}^3 \), we want to construct \( \psi^{\tau+} \) in the next space-time slab \([k \tau, (k+1) \tau) \times \mathbb{R}^3 \). We invoke Lemma 2.2 with \( \psi = \psi^\tau(k \tau-), \varepsilon = \tau 2^{-k} \| \psi_0 \|_{H^1} \). The \( \psi \) in Lemma will be the updated wave function:

\[
\psi^{\tau}(k \tau+) := \hat{\psi}.
\]

As a consequence we obtain

\[
\begin{aligned}
\sqrt{\rho^\tau(k \tau+)} &= \sqrt{\rho^\tau(k \tau-)} + r_k \\
\Lambda^\tau(k \tau+) &= (1 - \alpha \tau) \Lambda^\tau(k \tau-) - \tau \sqrt{\rho^\tau(k \tau-)} \nabla F^\tau(k \tau-) + \Gamma_k \\
\nabla \psi^\tau(k \tau+) &= \nabla \psi^\tau(k \tau-) - i \tau \dot{\phi}_k (\alpha \Lambda^\tau(k \tau-) + \sqrt{\rho^\tau(k \tau-)} \nabla F^\tau(k \tau-) + R_{\tau,k}.
\end{aligned}
\]

where

\[
\begin{aligned}
&\| r_k \|_{H^1} + \| \Gamma_k \|_{L^2} \leq \tau 2^{-k} \| \psi_0 \|_{H^1} \\
&\| R_{\tau,k} \|_{L^2} \leq \tau 2^{-k} \| \psi_0 \|_{H^1} + \tau \| \nabla \psi^\tau(k \tau-) \|_{L^2}.
\end{aligned}
\]

\[1\]Strictly speaking the approximate solution is given by the hydrodynamic quantities \( (\rho^\tau, J^\tau) \) associated to the wave function \( \psi^\tau \). More precisely, \( \psi^\tau \) is not an approximate solution for any equation (see Remark 5 below), however we will call it in this way because its moments \( (\rho^\tau, J^\tau) \) are approximate solutions, in the sense of Definition 3.1 to system 1 with \( \alpha = 1 \).
Let us remark that, if we had $r_k = 0, r_k = 0$, then
\[
\rho^\tau(k\tau+) = \rho^\tau(k\tau-),
\]
\[
J^\tau(k\tau + 0) = (1 - \tau)J^\tau(k\tau -) - \tau \rho^\tau(k\tau -)\nabla F^\tau(k\tau -),
\]
which is exactly the approximate solution of 12 to first order in $\tau > 0$. Now we
can start again with the Cauchy problem associated to 7 on the space-time slab
$[k\tau, (k + 1)\tau] \times \mathbb{R}^3$, by considering
\[
\psi(k\tau) = \psi^\tau(k\tau+),
\]
as initial condition. Thus we define $\psi^\tau$ on $[k\tau, (k + 1)\tau] \times \mathbb{R}^3$ to be this solution.

With this procedure we construct iteratively $\psi^\tau$ on $[0, \infty) \times \mathbb{R}^3$. By means of
the polar factorization we define $(\sqrt{\rho^\tau}, \Lambda^\tau)$ from $\psi^\tau$.

It is quite straightforward to prove that $(\sqrt{\rho^\tau}, \Lambda^\tau)$ are indeed approximate solutions
for 1. In fact, from the construction of $\psi^\tau$ and 13 it is easily inferred that 10
and 11 are satisfied. On the other hand, for the total mass and energy we have

**Lemma 3.2.** Let $\tau > 0$ be sufficiently close to zero and let $(\sqrt{\rho^\tau}, \Lambda^\tau)$ be the
approximate solution constructed above. Then for any $0 < t < \infty$ we have
\[
M^\tau(t) + \beta \int_0^t \int_{\mathbb{R}^3} (\rho^\tau)^{\sigma+1}(t', x) \, dx \, dt' \leq 2M^\tau(0) + 2\tau \|\psi_0\|_{H^1},
\]
\[
E^\tau(t) + \beta \int_0^t \int_{\mathbb{R}^3} (\rho^\tau)^{\sigma} \left[ |\nabla \sqrt{\rho^\tau}|^2 + |\Lambda^\tau|^2 + \rho^\tau f'(\rho^\tau) + \rho^\tau V^\tau \right] \, dx \, dt' \leq 2E^\tau(0) - \alpha T \left[ \frac{\gamma}{8} \sum_{k=1}^{[\gamma]} \|\Lambda^\tau(k\tau -)\|_{L^2}^2 + 2\tau \sum_{k=1}^{[\gamma]} \int \rho^\tau(k\tau -) |\nabla F^\tau(k\tau -)|^2 \, dx. \right]
\]

Let us remark that the energy bound in Lemma above is nothing but the discrete,
approximate version of the energy inequality for the QHD system 1
\[
E(t) + \beta \int_0^t \int_{\mathbb{R}^3} \rho^\tau \left[ |\nabla \sqrt{\rho^\tau}|^2 + |\Lambda|^2 + \rho f'(\rho) + \rho V \right] \, dx \, dt' \leq E(0)
\]
\[
-\alpha \int_0^t \int_{\mathbb{R}^3} |\Lambda|^2 \, dx \, dt' + \int_0^t \int_{\mathbb{R}^3} \rho |\nabla F|^2 \, dx \, dt'.
\]
Furthermore, let us also remark that the mass and energy dissipation guarantee
that $\sqrt{\rho^{\sigma+1}} \in L^2_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)$, $\sqrt{\rho^\sigma} \Lambda^\tau \in L^2_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)$, as required in Definition
3.1.

Now we need to prove that the sequence of approximate solutions has a limit
and that this limit actually solves (in the weak sense) the QHD system 1. First of
all we show the consistency of approximate solutions, namely that if the sequence
$\{(\sqrt{\rho^\tau}, \Lambda^\tau)\}$ has a strong limit, then this limit is a weak solution to 1.

**Theorem 3.3.** Let us consider a sequence of approximate solutions $\{(\rho^\tau, J^\tau)\}$
constructed via the fractional step method, and let us assume there exist
$\sqrt{\rho} \in L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\mathbb{R}^3)), \Lambda \in L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\mathbb{R}^3))$ such that
\[
\sqrt{\rho^\tau} \to \sqrt{\rho} \quad \text{in } L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\mathbb{R}^3))
\]
\[
\sqrt{\rho^{\sigma+1}} \to \sqrt{\rho^{\sigma+1}} \quad \text{in } L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\mathbb{R}^3))
\]
\[
\Lambda^\tau \to \Lambda \quad \text{in } L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\mathbb{R}^3))
\]
\[
\sqrt{\rho^\sigma} \Lambda^\tau \to \sqrt{\rho^\sigma} \Lambda \quad \text{in } L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\mathbb{R}^3)) /
\]
The \( \rho := (\sqrt{\rho})^2, J := \sqrt{\rho} \Lambda \) is a weak solution to 1 in \( [0, T) \times \mathbb{R}^3 \).

It thus remains to prove that the sequence of approximate solutions has indeed a strong limit, as stated in the hypothesis of the Theorem above. Unfortunately, the a priori bounds in Lemma 3.2 are not sufficient as for those we could only infer that, up to passing to subsequences, \( \{ \psi^\tau \} \) has a weak limit in \( L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^3)) \),

\[ \psi^\tau \rightharpoonup \psi, \quad L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^3)). \]

We need to improve this weak convergence to a strong one: to do that we will show further compactness properties for the family \( \{ \psi^\tau \} \). In what follows we exploit the dispersive properties of the underlying wave function dynamics, i.e. the dispersive estimates associated to the Schrödinger propagator. For this purpose, we first write a Duhamel-like formula, for the gradient of an approximate solution.

**Lemma 3.4.** Let \( \psi^\tau \) be the approximate solution constructed above, then for any \( t \in (0, \infty) \) we have

\[
\nabla \psi^\tau(t) = U(t) \nabla \psi_0 - i \int_0^t U(t-s) \nabla N(\psi^\tau)(s) \, ds \\
- \imath \tau \sum_{k=1}^{[t/\tau]} U(t-k\tau) \left[ \hat{\phi}_k(\alpha \Lambda^\tau(k\tau-) - \sqrt{\rho^\tau(k\tau-)} \nabla F^\tau(k\tau-) \right] \\
+ \sum_{k=1}^{[t/\tau]} U(t-k\tau) R_{k,\tau},
\]

where

\[ N(\psi^\tau) = f'(\abs{\psi^\tau}^2) \psi^\tau + V^\tau \psi^\tau - \frac{\beta}{2} \abs{\psi^\tau}^{2\sigma} \psi^\tau, \]

and \( \hat{\phi}_k, R_{k,\tau} \) are defined in 13.

The Lemma above shows the importance of defining the updating step in the construction of the approximate solutions by means of Lemma 2.2. Indeed, this approximate updating allows us to write formula 15 in a quite neat way. For a more detailed discussion on this point we refer the reader to [1], Remark 21.

At this point we may use the Strichartz estimates for the Schrödinger semigroup. We consider formula 15 and we derive a priori estimates on \( \nabla \psi^\tau \) in the Strichartz spaces \( L^q(0, T; L^r(\mathbb{R}^3)) \), where \((q, r)\) are admissible pairs. We recall that a pair of exponents \((q, r)\) is called admissible if \( \frac{1}{q} = \frac{3}{2} \left( \frac{1}{2} - \frac{1}{r} \right), 2 \leq q \leq \infty, 2 \leq r \leq 6 \). For a more detailed discussion on Strichartz estimates we refer to [14, 18, 7, 27, 22] and references therein. By using a continuity argument and the a priori bounds in Lemma 3.2 we show

**Proposition 2.** Let \( 0 < T < \infty \) be a finite time. Then for any admissible pair \((q, r)\) we have

\[ \| \nabla \psi^\tau \|_{L^q(0, T; L^r(\mathbb{R}^3))} \leq C(E_0, \| \rho_0 \|_{L^1}, T). \]

By the previous Proposition we obtain further integrability properties for the sequence of approximate solutions, namely it is uniformly bounded in \( \tau > 0 \) in the Strichartz spaces \( L^q_t L^r_x \). Now we are going to use such integrability property to show a local smoothing effect [8]. Namely, thanks to the dispersion related to the Schrödinger propagator, the approximate solutions are locally in space more regular than the initial datum.
Proposition 3. Let $0 < T < \infty$ be any finite time. Then
\[
\| \nabla \psi^\tau \|_{L^2([0,T];H^{1/2}_{\text{loc}}(\mathbb{R}^3))} \leq C(E_0, \| \rho_0 \|_{L^1}).
\]

This additional local regularity allows to infer the necessary compactness needed in Theorem 3.3. By using an Aubin-Lions type lemma, due to Rakotoson and Temam [26], we may finally prove:

Theorem 3.5. For any finite time $0 < T < \infty$, the sequence $\nabla \psi^\tau$ is relatively compact in $L^2(0,T;L^2_{\text{loc}}(\mathbb{R}^3))$. More precisely, there exists $\psi \in L^2(0,T;H^1_{\text{loc}}(\mathbb{R}^3))$ such that
\[
\psi = s - \lim_{\tau \to 0} \psi^\tau, \quad \text{in } L^2(0,T;H^1_{\text{loc}}(\mathbb{R}^3)).
\]

As a consequence,
\[
\sqrt{\rho^\tau} \to \sqrt{\rho} \quad \text{in } L^2(0,T;H^1_{\text{loc}}(\mathbb{R}^3))
\]
\[
\Lambda^\tau \to \Lambda \quad \text{in } L^2(0,T;L^2_{\text{loc}}(\mathbb{R}^3)).
\]

By combining the Theorem above and Theorem 3.3, we know that $(\sqrt{\rho}, \Lambda)$ satisfy 1 in the weak sense, in $[0,T] \times \mathbb{R}^3$, for any finite $0 < T < \infty$. Moreover, it is easy to check that the energy for $(\sqrt{\rho}, \Lambda)$ is finite for every time: this follows directly from passing the uniform bounds in Lemma 3.2 to the limit as $\tau \to 0$. Furthermore let us recall $(\sqrt{\rho}, \Lambda)$ are the hydrodynamic quantities associated to $\psi \in L^\infty(\mathbb{R}_+;H^1(\mathbb{R}^3))$, hence by the polar decomposition Lemma they also satisfy the generalized irrotationality condition. We can thus say that $(\sqrt{\rho}, \Lambda)$ define a finite energy weak solution to the QHD system 1. Thus Theorem 1.2 is finally proved.

Remark 5. We should remark here that, despite the fact $\psi$ is the strong limit of the sequence $\{\psi^\tau\}$ and the hydrodynamic quantities $(\sqrt{\rho}, \Lambda)$ associated to $\psi$ solve the QHD system, it is not clear if the wave function $\psi$ solve any nonlinear Schrödinger equation. Indeed, while for $\nabla \psi^\tau$ we can write the Duhamel’s formula 15, we don’t have a similar expression for $\psi^\tau$. In any case, even regarding formula 15 it is not clear if the right hand side has a limit as $\tau \to 0$.

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