The equations of a solid body motion

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Abstract. Traditionally, the equations of motion of a solid body uses the theorem of the center of mass motion and the theorem of change of angular momentum with respect to the center of mass (Koenig's axes). However, in the investigation of a solid body motion in a number of cases, the generalized coordinates are advisable to choose coordinates of another point of a body (pole) does not match the center of mass of the body, and the angles of rotation around this point. We obtain the equations of a solid body motion for these generalized coordinates. This is important for mathematical modeling of the motion of various objects, including vehicles. During the motion of a vehicle, its mass-inertial characteristics may change, including the position of the center of mass and the direction of the main axes of inertia, body mass and moments of inertia.

1. Introduction
Usually, to describe a solid body motion, the theorem of the center of mass motion and the theorem on the change in the angular momentum relative to the center of mass (Koenig's axes) are used [1-7]. In this case, the coordinates of the center of mass and three angles of rotation of the body relative to the Koenig axes (Euler angles, Krylov angles, aircraft angles or rocket angles) are used as generalized coordinates that determine the position of a rigid body.

However, in the investigation of motion of various moving objects (aircraft, ship, walking apparatus [8-9], etc.), it is advisable to choose coordinates of another point of a body (pole) does not match the center of mass of the body, and the angles of rotation around this point. This can be due to various reasons. May be the movement of this very point is of particular interest in the investigation of the dynamics of a solid body motion. A solid body is often viewed as a vehicle motion model. In this case, in the process of motion, the mass-inertial characteristics of the body motion can changes including the center of mass position and the direction of the main axes of inertia, mass and moments of inertia. They can be changes either at the same time during loading (unloading) of the vehicle, or during movement as the result of fuel combustion, icing, and dirt adhesion. In the latter cases, we can assume that the mass-inertial characteristics change slowly, and their derivatives can be considered equal to zero. When simulating the dynamics of a solid body, the origin of the moving coordinate system (pole) associated with the solid body and the direction of the axes remain unchanged. These are the pole and axles used in the vehicle (solid body) motion control system.

In this work, the equations of a solid body motion in these generalized coordinates are obtained in the projection on the axis of a moving coordinate system associated with the body.
2. Motion equations of a free solid body
Consider the motion of a solid body (Figure 1) relative to an inertial (stationary) coordinate system $O\xi\eta\zeta$. A movable coordinate system $Axyz$ is rigidly connected to a solid body. Body mass center point $C$. Let us denote by $m$ the mass of the body, $m_k$ – the mass of the $k$-th point of the body, $J$ – the tensor of inertia in a moving coordinate system, $\vec{v}_A$ – the velocity of the pole $A$, $\vec{v}_C$ – the velocity of the center of mass $C$, $\vec{\omega}$ – the angular velocity of the rigid body, $\vec{F}$ and $\vec{M} = \vec{M}_A$ is the principal vector and the principal moment relative to the pole $A$ of the forces applied to the solid body. The position of each point of the solid body relative to the fixed coordinate system $O\xi\eta\zeta$ is determined by the radius vector $\vec{r}$, and relative to the moving coordinate system $Axyz$ by the radius vector $\vec{\rho}$. Including for the $k$-th point of the rigid body we have $\vec{r}_k$ and $\vec{\rho}_k$, for the center of mass $C$ – $\vec{r}_C$ and $\vec{\rho}_C$, for the pole $A$ – $\vec{r}_A$. Moreover, the vectors $\vec{\rho}_C$ and $\vec{\rho}_k$ are constants.

Figure 1.

$$J = \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{pmatrix}.$$  

For any vector $\vec{b}$, we call the total derivative

$$\frac{d\vec{b}}{dt} = \dot{\vec{b}},$$

the derivative calculated in a fixed coordinate system $O\xi\eta\zeta$.

And the local derivative is called
\[ \frac{\ddot{b}}{dt} = \dot{b}, \]

derivative calculated in a moving coordinate system \( A_{xyz} \).

The total and local derivatives are related by the Boer formula
\[ \frac{d\dot{b}}{dt} = \frac{\ddot{\bar{b}}}{dt} + \bar{\omega} \times \bar{b}, \quad \text{or} \quad \dot{\bar{b}} = \bar{\omega} + \bar{\omega} \times \bar{b}. \]  

(1)

Note that the total and local derivatives of the angular velocity vector coincide
\[ \dot{\bar{\omega}} = \bar{\omega}. \]  

(2)

According to Euler's formula
\[ \bar{v}_C = \bar{v}_A + \bar{\omega} \times \bar{p}_C. \]  

(3)

The theorem on the motion of the center of mass \( m\dot{v}_C = \dot{F} \) taking into account (1)–(3) has the form
\[ m\left( \bar{v}_A + \bar{\omega} \times \bar{v}_A + \bar{\omega} \times \bar{p}_C + \bar{\omega} \times \left[ \bar{\omega} \times \bar{p}_C \right] \right) = \dot{F}. \]  

(4)

The angular momentum vector relative to a fixed point \( O \) is
\[ \bar{K}_O = \sum_k \bar{r}_k \times m_k \bar{v}_k = \sum_k (\bar{r}_A + \bar{p}_k) \times m_k \left( \bar{v}_A + \bar{\omega} \times \bar{p}_k \right). \]

Then by virtue of relation (3), and
\[ \sum_k m_k = m, \quad \sum_k m_k \bar{p}_k = m \bar{p}_C, \quad \sum_k m_k \bar{p}_k \times \left[ \bar{\omega} \times \bar{p}_C \right] = \mathbf{J} \bar{\omega}, \]
we obtain
\[ \bar{K}_O = \bar{r}_A \times m\dot{v}_C + m\bar{p}_C \times \bar{v}_A + \mathbf{J} \bar{\omega}. \]  

(5)

Theorem on the change in angular momentum relative to a fixed point \( O \)
\[ \dot{K}_O = \dot{M}_O = \sum_i \bar{r}_i \times \bar{F}_i = \sum_i \left( \bar{r}_A + \bar{p}_i \right) \times \bar{F}_i = \bar{r}_A \times \dot{F} + \dot{M}_A = \bar{r}_A \times m\dot{v}_C + \dot{M}. \]

Then by virtue of relation (1), (5) and the fact that the tensor of inertia in the moving coordinate system \( A_{xyz} \) is constant, has the form
\[ \bar{r}_A \times m\dot{v}_C + \bar{v}_A \times m\bar{v}_C + m\left[ \bar{\omega} \times \bar{p}_C \right] \times \bar{v}_A + m\bar{p}_C \times \left( \bar{v}_A + \bar{\omega} \times \bar{v}_A \right) + \]
\[ + \mathbf{J} \dot{\bar{\omega}} + \bar{\omega} \times \mathbf{J} \bar{\omega} = \bar{r}_A \times m\dot{v}_C + \dot{M}. \]

Canceling identical terms on the left and right sides and then by virtue of virtue of (3)
\[ \bar{v}_A \times m\bar{v}_C + m\left[ \bar{\omega} \times \bar{p}_C \right] \times \bar{v}_A = \bar{v}_A \times m\left( \bar{v}_A + \bar{\omega} \times \bar{p}_C \right) + m\left[ \bar{\omega} \times \bar{p}_C \right] \times \bar{v}_A = \]
\[ = m\bar{v}_A \times \left[ \bar{\omega} \times \bar{p}_C \right] + m\left[ \bar{\omega} \times \bar{p}_C \right] \times \bar{v}_A = 0. \]
Then by virtue of \(2\), we obtain
\[
\mathbf{J} \ddot{\mathbf{w}} + \mathbf{w} \times \dot{\mathbf{w}} + m \mathbf{p}_C \times (\dot{\mathbf{v}}_A + \mathbf{w} \times \dot{\mathbf{v}}_A) = \mathbf{M}.
\] (6)

Equations (4), (6) must be supplemented with Euler's kinematic equations and equations \(\dot{\mathbf{r}}_A = \mathbf{v}_A\). Note that in these equations \(\mathbf{F}\) and \(\mathbf{M}\) are functions of time, coordinates, and velocities of the pole \(A\), angular coordinates and angular velocity of a solid body.

In the case when the pole \(A\) coincides with the center of mass \(C\), the vector \(\mathbf{p}_C = 0\) and equations (4), (6) have the traditional form of the theorem of the center of mass motion and the theorem on the change in angular momentum with respect to the Koenig axes. In this case, it is advisable to write equations (4) in projection on the axis of a fixed coordinate system. As a result, we have
\[
m\ddot{\mathbf{C}} = \mathbf{F}, \quad \mathbf{J} \ddot{\mathbf{w}} + \mathbf{w} \times \dot{\mathbf{w}} = \mathbf{M}.
\] (7)

Equations (7) are much simpler than equations (4), (6). Nevertheless, when modeling the dynamics of movement and constructing algorithms for controlling the movement of various vehicles, it is advisable to use equations (4), (6) because they allow taking into account changes in the mass-inertial characteristics of an object during its movement.

Equations (4), (6), written in the projection on the axis of the moving coordinate system \(Axyz\), have the form
\[
m[v_x + \omega_z z_C - \omega_x y_C + \omega_y v_z - \omega_z v_y + \omega_y v_z (\omega_z y_C - \omega_x x_C) - \omega_z (\omega_x x_C - \omega_y z_C)] = F_x,
\]
\[
m[v_y + \omega_z x_C - \omega_x z_C + \omega_x v_y - \omega_z v_x - \omega_x (\omega_z y_C - \omega_x x_C) + \omega_z (\omega_y z_C - \omega_x y_C)] = F_y,
\]
\[
m[v_z + \omega_x v_y - \omega_y x_C + \omega_x v_y - \omega_y v_x + \omega_x (\omega_z y_C - \omega_x x_C) - \omega_y (\omega_y z_C - \omega_x y_C)] = F_z,
\]
\[
J_z \omega_z - J_{zy} \omega_y + J_{zy} \omega_x - (J_z - J_y) \omega_x \omega_z + J_{zy} (\omega_z^2 - \omega_y^2) + \omega_y (J_{zy} \omega_x - J_{xy} \omega_z) + \]
\[
+ m[v_x y_C - v_y z_C + y_C (\omega_y v_x - \omega_x v_y) - z_C (\omega_x v_y - \omega_y v_x)] = M_x,
\]
\[
J_y \omega_y - J_{yx} \omega_x - J_{yx} \omega_y + (J_y - J_x) \omega_x \omega_y + J_{yx} (\omega_x^2 - \omega_y^2) + \omega_x (J_{yx} \omega_x - J_{xy} \omega_y) + \]
\[
+ m[v_y z_C - v_z x_C + x_C (\omega_z v_y - \omega_y v_z) + z_C (\omega_x v_y - \omega_y v_x)] = M_y,
\]
\[
J_x \omega_x - J_{zx} \omega_z - J_{zx} \omega_x + (J_x - J_z) \omega_x \omega_z + J_{zx} (\omega_z^2 - \omega_x^2) + \omega_z (J_{zx} \omega_x - J_{zx} \omega_z) + \]
\[
+ m[v_z x_C - v_x y_C + x_C (\omega_z v_x - \omega_x v_z) - y_C (\omega_z v_y - \omega_x v_y)] = M_z.
\] (8)

3. Plane motion of a solid body

In the plane motion of a solid body, when all points of the body move parallel to the plane \(O\xi\eta\) of the fixed coordinate system, which coincides with the plane \(Axy\) of the moving coordinate system. Without loss of generality, we can assume, that the center of mass located in the same plane. Then, \(z_c = 0, \omega_z = 0, \omega_x = \omega, v_z = 0\) and equations of motion (8) have the form
\[
\begin{align*}
    m(\dot{v}_x - \omega y_c - \omega v_y - \omega^2 x_c) &= F_x, \\
    m(\dot{v}_y + \omega x_c + \omega v_x - \omega^2 y_c) &= F_y, \\
    J_z \omega + m(\dot{v}_y x_c - \dot{v}_x y_c + x_c \omega v_x - y_c \omega v_y) &= M_z.
\end{align*}
\]

4. Conclusion

When simulating the dynamics of a solid body motion in several cases (in particular, when simulating the motion of bodies, whose mass-inertial characteristics can change during the motion), it is advisable to use the coordinate of a certain characteristic point of the body (pole) and the angles of rotation of the body around this point as generalized coordinates. In this work, the equations of motion of the body in these generalized coordinates are obtained.

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