Using Strategy Improvement to Stay Alive*

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We design a novel algorithm for solving Mean-Payoff Games (MPGs). Besides solving an MPG in the usual sense, our algorithm computes more information about the game, information that is important with respect to applications. The weights of the edges of an MPG can be thought of as a gained/consumed energy – depending on the sign. For each vertex, our algorithm computes the minimum amount of initial energy that is sufficient for player Max to ensure that in a play starting from the vertex, the energy level never goes below zero. Our algorithm is not the first algorithm that computes the minimum sufficient initial energies, but according to our experimental study it is the fastest algorithm that computes them. The reason is that it utilizes the strategy improvement technique which is very efficient in practice.

1 Introduction

A Mean-Payoff Game (MPG) [12, 15, 19] is a two-player infinite game played on a finite weighted directed graph, the vertices of which are divided between the two players. A play starts by placing a token on some vertex and the players, named Max and Min, move the token along the edges of the graph ad infinitum. If the token is on Max’s vertex, he chooses an outgoing edge and the token goes to the destination vertex of that edge. If the token is on Min’s vertex, it is her turn to choose an outgoing edge. Roughly speaking, Max wants to maximize the average weight of the traversed edges whereas Min wants to minimize it. It was proved in [12] that each vertex \( \nu \) has a value, denoted by \( \nu(v) \), which each player can secure by a positional strategy, i.e., strategy that always chooses the same outgoing edge in the same vertex. To solve an MPG is to find the values of all vertices, and, optionally, also strategies that secure the values.

In this paper we deal with MPGs with other than the standard average-weight goal. Player Max now wants the sum of the weights of the traversed edges, plus some initial value (initial “energy”), to be non-negative at each moment of the play. He also wants to know the minimal sufficient amount of initial energy that enables him to stay non-negative for ever. For different starting vertices, the minimal sufficient initial energy may be different and for starting vertices with \( \nu < 0 \), it is impossible to stay non-negative with arbitrarily large amount of initial energy.

The problem of computation of the minimal sufficient initial energies has been studied under different names by Chakrabarti et al. [5], Lifshits and Pavlov [17], and Bouyer et al. [2]. In [5] it was called the problem of pure energy interfaces, in [17] it was called the problem of potential computation, and in [2] it was called the lower-bound problem. The paper [2] also contains the definition of a similar problem – the lower-weak-upper-bound problem. An instance of this problem contains, besides an MPG, also a bound \( b \). The goal is the same, Max wants to know how much initial energy he needs to stay non-negative forever, but now the energy level is bounded from above by \( b \) and during the play, all increases above this bound are immediately truncated.

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Various resource scheduling problems for which the standard solution of an MPG is not useful can be formulated as the lower-bound or the lower-weak-upper-bound problems, which extends the applicability of MPGs. For example, an MPG can be used to model a robot in a hostile environment. The weights of edges represent changes in the remaining battery capacity of the robot – positive edges represent recharging, negative edges represent energy consuming actions. The bound $b$ is the maximum capacity of the battery. Player Max chooses the actions of the robot and player Min chooses the actions of the hostile environment. By solving the lower-weak-upper-bound problem, we find out if there is some strategy of the robot that allows him to survive in the hostile environment, i.e., its remaining battery capacity never goes below zero, and if there is such a strategy, we also get the minimum initial remaining battery capacity that allows him to survive.

The first algorithm solving the lower-bound problem was proposed by Chakrabarti et al. [5] and it is based on value iteration. The algorithm can also be easily modified to solve the lower-weak-upper-bound problem. The value iteration algorithm was later improved by Chaloupka and Brim in [7], and independently by Doyen, Gentilini, and Raskin [11], extended version of [7] [11] was recently submitted [4]. Henceforward we will use the term “value iteration” (VI) to denote only the improved version from [7] [11]. The algorithms of Bouyer et al. [2] that solve the two problems are essentially the same as the original algorithm from [5]. However, [2] focuses mainly on other problems than the lower-bound and the lower-weak-upper-bound problems for MPGs. Different approach to solving the lower-bound problem was proposed by Lifshits and Pavlov [17], but their algorithm has exponential space complexity, and so it is not appropriate for practical use. VI seems to be the best known approach to solving the two problems.

In this paper, we design a novel algorithm based on the strategy improvement technique, suitable for practical solving of the lower-bound and the lower-weak-upper-bound problems for large MPGs. The use of the strategy improvement technique for solving MPGs goes back to the algorithm of Hoffman and Karp from 1966 [16]. Their algorithm can be used to solve only a restricted class of MPGs, but strategy improvement algorithms for solving MPGs in general exist as well [1] [18] [10]. However, all of them solve neither the lower-bound nor the lower-weak-upper-bound problem (cf. Section 4 first part, last paragraph), our algorithm is the first. Another contribution of this paper is a further improvement of VI.

The shortcoming of VI is that it takes enormous time on MPGs with at least one vertex with $\nu < 0$. Natural way to alleviate this problem is to find the vertices with $\nu < 0$ by some fast algorithm and run VI on the rest. Based on our previous experience with algorithms for solving MPGs [6], we selected two algorithms for computation of the set of vertices with $\nu < 0$. Namely, the algorithm of Björklund and Vorobyov [1] (BV), and the algorithm of Schewe [18] (SW). This gives us two algorithms: VI + BV and VI + SW. However, the preprocessing is not helpful on MPGs with all vertices with $\nu \geq 0$, and it is also not helpful for solving the lower-weak-upper-bound problem for small bound $b$. Therefore, we also study the algorithm VI without the preprocessing.

Our new algorithm based on the strategy improvement technique that we propose in this paper has the complexity $O(|V| \cdot (|V| \cdot \log |V| + |E|) \cdot W)$, where $W$ is the maximal absolute edge-weight. It is slightly worse than the complexity of VI, the same as the complexity of VI + BV, and better than the complexity of VI + SW. We call our algorithm “Keep Alive Strategy Improvement” (KASI). It solves both the lower-bound and the lower-weak-upper-bound problem. Moreover, as each algorithm that solves the lower-bound problem also divides the vertices of an MPG into those with $\nu \geq 0$ and those with $\nu < 0$, which can be used to compute the exact $\nu$ values of all vertices, KASI can be thought of as an algorithm that also solves MPGs in the usual sense. As a by-product of the design of KASI, we improved the complexity of BV and proved that Min may not have positional strategy that is also optimal with respect to the lower-weak-upper-bound problem. Moreover, we describe a way to construct an optimal strategy
for Min with respect to the lower-weak-upper-bound problem.

To evaluate and compare the algorithms VI, VI + BV, VI + SW, and KASI, we implemented them and carried out an experimental study. According to the study, KASI is the best algorithm.

2 Preliminaries

A Mean-Payoff Game (MPG) \cite{12, 15, 19} is given by a triple \( \Gamma = (G, V_{\text{Max}}, V_{\text{Min}}) \), where \( G = (V, E, w) \) is a finite weighted directed graph such that \( V \) is a disjoint union of the sets \( V_{\text{Max}} \) and \( V_{\text{Min}} \), \( w : E \rightarrow \mathbb{Z} \) is the weight function, and each \( v \in V \) has out-degree at least one. The game is played by two opposing players, named Max and Min. A play starts by placing a token on some given vertex and the players then move the token along the edges of \( G \) ad infinitum. If the token is on vertex \( v \in V_{\text{Max}} \), Max moves it. If the token is on vertex \( v \in V_{\text{Min}} \), Min moves it. This way an infinite path \( p = (v_0, v_1, v_2, \ldots) \) is formed. Max’s aim is to maximize his gain: \( \liminf \sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} w(v_i, v_{i+1}) \), and Min’s aim is to minimize her loss: \( \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} w(v_i, v_{i+1}) \). For each vertex \( v \in V \), we define its value, denoted by \( v(v) \), as the maximal gain that Max can ensure if the play starts at vertex \( v \). It was proved that it is equal to the minimal loss that Min can ensure. Moreover, both players can ensure \( v(v) \) by using positional strategies defined below \cite{12}.

A (general) strategy of Max is a function \( \sigma : V^* \cdot V_{\text{Max}} \rightarrow V \) such that for each finite path \( p = (v_0, v_1, \ldots, v_k) \) with \( v_k \in V_{\text{Max}} \), it holds that \( (v_k, \sigma(p)) \in E \). Recall that each vertex has out-degree at least one, and so the definition of a strategy is correct. The set of all strategies of Max in \( \Gamma \) is denoted by \( \Sigma^\Gamma \). We say that an infinite path \( p = (v_0, v_1, v_2, \ldots) \) agrees with the strategy \( \sigma \in \Sigma^\Gamma \) if for each \( v_i \in V_{\text{Max}} \), \( \sigma(v_0, v_1, \ldots, v_k) = v_{k+1} \). A strategy \( \pi \) of Min is defined analogously. The set of all strategies of Min in \( \Gamma \) is denoted by \( \Pi^\Gamma \). Given an initial vertex \( v \in V \), the outcome of two strategies \( \sigma \in \Sigma^\Gamma \) and \( \pi \in \Pi^\Gamma \) is the (unique) infinite path outcome \( \Gamma(v, \sigma, \pi) = (v, v_0, v_1, v_2, \ldots) \) that agrees with both \( \sigma \) and \( \pi \).

The strategy \( \sigma \in \Sigma^\Gamma \) is called a positional strategy if \( \sigma(p) = \sigma(p') \) for all finite paths \( p = (v_0, v_1, \ldots, v_k) \) and \( p' = (v_0', v_1', \ldots, v_k') \) such that \( v_k, v_k' \in V_{\text{Max}} \). For the sake of simplicity, we think of a positional strategy of Max as a function \( \sigma : V_{\text{Max}} \rightarrow V \) such that \( (v, \sigma(v)) \in E \), for each \( v \in V_{\text{Max}} \). The set of all positional strategies of Max in \( \Gamma \) is denoted by \( \Sigma^M_\Gamma \). A positional strategy \( \pi \) of Min is defined analogously. The set of all positional strategies of Min in \( \Gamma \) is denoted by \( \Pi^M_\Gamma \). We define \( G_{\sigma} \), the restriction of \( G \) to \( \sigma \), as the graph \((V, E_{\sigma}, w_{\sigma})\), where \( E_{\sigma} = \{ (u, v) \in E \mid u \in V_{\text{Min}} \lor \sigma(u) = v \} \), and \( w_{\sigma} = w \mid E_{\sigma} \). That is, we get \( G_{\sigma} \) from \( G \) by deleting all the edges emanating from Max’s vertices that do not follow \( \sigma \). \( G_{\pi} \) for a strategy \( \pi \) of Min is defined analogously. For \( \sigma \in \Sigma^M_\Gamma \), we also define \( \Gamma_{\sigma} = (G_{\sigma}, V_{\text{Max}}, V_{\text{Min}}) \), and for \( \pi \in \Pi^M_\Gamma \), \( \Gamma_{\pi} = (G_{\pi}, V_{\text{Max}}, V_{\text{Min}}) \).

The lower-bound problem for an MPG \( \Gamma = (G = (V, E, w), V_{\text{Max}}, V_{\text{Min}}) \) is the problem of finding \( \text{lb}^\Gamma(v) \in \mathbb{N}_0 \cup \{ \infty \} \) for each \( v \in V \), such that:

\[
\text{lb}^\Gamma(v) = \min\{ x \in \mathbb{N}_0 \mid \exists \sigma \in \Sigma^\Gamma \text{ such that } (\forall \pi \in \Pi^\Gamma) \\
( \text{outcome}^\Gamma(v, \sigma, \pi) = (v, v_0, v_1, v_2, \ldots) \land \\
(\forall n \in \mathbb{N}) (x + \sum_{i=0}^{n-1} w(v_i, v_{i+1}) \geq 0) ) \}
\]

where minimum of an empty set is \( \infty \). That is, \( \text{lb}^\Gamma(v) \) is the minimal sufficient amount of initial energy that enables Max to keep the energy level non-negative forever, if the play starts from \( v \). If \( \text{lb}^\Gamma(v) = \infty \), which means that \( v(v) < 0 \), then we say that Max loses from \( v \), because arbitrarily large amount of initial energy is not sufficient. If \( \text{lb}^\Gamma(v) \in \mathbb{N}_0 \), then Max wins from \( v \).

The strategy \( \sigma \in \Sigma^\Gamma \) is an optimal strategy of Max with respect to the lower-bound problem, if it ensures that for each \( v \in V \) such that \( \text{lb}^\Gamma(v) \neq \infty \), \( \text{lb}^\Gamma(v) \) is a sufficient amount of initial energy.
The strategy \( \pi \in \Pi_{\Gamma} \) is an optimal strategy of Min with respect to the lower-bound problem, if it ensures that for each \( v \in V \) such that \( \ell_{\Gamma}(v) \neq \infty \), Max needs at least \( \ell_{\Gamma}(v) \) units of initial energy, and for each \( v \in V \) such that \( \ell_{\Gamma}(v) = \infty \), Max loses.

The lower-weak-upper-bound problem for an MPG \( \Gamma = (G, V, E, W, V_{\text{Min}}, V_{\text{Max}}) \) and a bound \( b \in \mathbb{N}_0 \) is the problem of finding \( \ell_{\text{wub}}^b(v) \in \mathbb{N}_0 \cup \{\infty\} \) for each \( v \in V \), such that:

\[
\ell_{\text{wub}}^b(v) = \min \{x \in \mathbb{N}_0 \mid (\exists \sigma \in \Sigma_{\Gamma})(\forall \pi \in \Pi_{\Gamma}) \:
\begin{align*}
&\text{outcome}^\Gamma(v, \sigma, \pi) = (v = v_0, v_1, v_2, \ldots) \land \\
&(\forall n \in \mathbb{N})(x + \sum_{i=0}^{n-1} w(v_i, v_{i+1}) \geq 0) \land \\
&(\forall n_1, n_2 \in \mathbb{N}_0)(n_1 < n_2 \Rightarrow \sum_{i=n_1}^{n_2-1} w(v_i, v_{i+1}) \geq -b) \} \}
\]

where minimum of an empty set is \( \infty \). That is, \( \ell_{\text{wub}}^b(v) \) is the minimal sufficient amount of initial energy that enables Max to keep the energy level non-negative forever, if the play starts from \( v \), under the additional condition that the energy level is truncated to \( b \) whenever it exceeds the bound. The additional condition is equivalent to the condition that the play does not contain a segment of weight less than \( -b \).

If \( \ell_{\text{wub}}^b(v) = \infty \), then we say that Max loses from \( v \), because arbitrarily large amount of initial energy is not sufficient. If \( \ell_{\text{wub}}^b(v) \in \mathbb{N}_0 \), then Max wins from \( v \). Optimal strategies for Max and Min with respect to the lower-weak-upper-bound problem are defined in the same way as for the lower-bound problem.

It was proved in [2] that both for the lower-bound problem and the lower-weak-upper-bound problem Max can restrict himself only to positional strategies, i.e., he always has a positional strategy that is also optimal. Therefore, we could use the set \( \Sigma_{\Gamma} \) instead of the set \( \Sigma_{\Gamma}^\Gamma \) in the definitions of both the lower-bound problem and the lower-weak-upper-bound problem.

In the rest of the paper, we will focus only on the lower-weak-upper-bound problem, because it includes the lower-bound problem as a special case. The reason is that for each \( v \in V \) such that \( \ell_{\Gamma}(v) < \infty \), it holds that \( \ell_{\Gamma}(v) \leq (|V| - 1) \cdot W \), where \( W \) is the maximal absolute edge-weight in \( G \). It was proved in [2]. Therefore, if we choose \( b = (|V| - 1) \cdot W \), then for each \( v \in V \), \( \ell_{\Gamma}(v) = \ell_{\text{wub}}^b(v) \).

Let \( G = (V, E, W) \) be a weighted directed graph, let \( p = (v_0, \ldots, v_k) \) be a path in \( G \), and let \( c = (u_0, \ldots, u_{r-1}, u_r = u_0) \) be a cycle in \( G \). Then \( w(p) \), the weight of \( p \), and \( w(c) \), the weight of \( c \), are defined in the following way:

\[
w(p) = \sum_{i=0}^{k-1} w(v_i, v_{i+1}) \quad \text{and} \quad w(c) = \sum_{i=0}^{r-1} w(u_i, u_{i+1}).
\]

Let \( \Gamma = (G, V_{\text{Min}}, V_{\text{Max}}) \) be an MPG and let \( D \subseteq V \). Then \( G(D) \) is the subgraph of \( G \) induced by the set \( D \). Formally, \( G(D) = (D, E \cap D \times D, w \upharpoonright D \times D) \). We also define the restriction of \( \Gamma \) induced by \( D \). Since some vertices might have zero out-degree in \( G(D) \), we define \( \Gamma(D) = (G(D), V_{\text{Min}} \cap D, V_{\text{Max}} \cap D) \), where \( G(D) \) is the graph \( G(D) \) with negative self-loops added to all vertices with zero outdegree. That is, we make the vertices with zero out-degree in \( G(D) \) losing for Max in \( \Gamma(D) \) with respect to the the lower-weak-upper-bound problem.

Let \( G = (V, E, W) \) be a graph and let \( B, A \subseteq V \). If we say that “\( p \) is a path from \( v \) to \( B \)” we mean a path with the last vertex and only the last vertex in \( B \), formally: \( p = (v = v_0, \ldots, v_k) \), where \( v_0, \ldots, v_k \in V \setminus B \land v_k \in B \). Furthermore, a path from \( A \) to \( B \) is a path from \( v \) to \( B \) such that \( v \in A \). The term “longest” in connection with paths always refers to the weights of the paths, not the numbers of edges.

Operations on vectors of the same dimension are element-wise. For example, if \( d_0 \) and \( d_1 \) are two vectors of dimension \( |V| \), then \( d_0 < d_1 \) means that for each \( v \in V \), \( d_0(v) \leq d_1(v) \), and for some \( v \in V \), \( d_0(v) < d_1(v) \).

For the whole paper let \( \Gamma = (G, V, E, W, V_{\text{Max}}, V_{\text{Min}}) \) be an MPG and let \( W \) be the maximal absolute edge-weight in \( G \), i.e., \( W = \max_{e \in E} |w(e)| \).
3 The Algorithm

High-level description of our Keep Alive Strategy Improvement algorithm (KASI) for the lower-weak-upper-bound problem is as follows. Let \((\Gamma, b)\) be an instance of the lower-weak-upper-bound problem. KASI maintains a vector \(d \in (\mathbb{Z} \cup \{-\infty\})^V\) such that \(-d \geq 0\) is always a lower estimate of \(lwub_b^{\Gamma}\), i.e., \(-d \leq lwub_b^{\Gamma}\). The vector \(d\) is gradually decreased, and so \(-d\) is increased, until \(-d = lwub_b^{\Gamma}\). The reason why KASI maintains the vector \(d\) rather than \(-d\) is that \(d\) contains weights of certain paths and we find it more natural to keep them as they are, than to keep their opposite values. The algorithm also maintains a set \(D\) of vertices such that about the vertices in \(V \setminus D\) it already knows that they have infinite \(lwub_b^{\Gamma}\) value. Initially, \(d = 0\) and \(D = V\). KASI starts with an arbitrary strategy \(\pi \in \Pi_M^\Gamma\) and then iteratively improves it until no improvement is possible. In each iteration, the current strategy is first evaluated and then improved. The strategy evaluation examines the graph \(G_\pi(D)\) and updates the vector \(d\) so that for each \(v \in D\), it holds

\[-d(v) = lwub_b^\Gamma_{\pi(D)}(v)\]

That is, it solves the lower-weak-upper-bound problem in the restricted game \(\Gamma_\pi(D)\), where Min has no choices. This explains why the restricted game was defined the way it was, because if a vertex from \(D\) has outgoing edges only to \(V \setminus D\), then it is losing for Max in \(\Gamma\). The vertices with the \(d\) value equal to \(-\infty\) are removed from the set \(D\). Since the strategy \(\pi\) is either the first strategy or an improvement of the previous strategy, the vector \(d\) is always decreased by the strategy evaluation and we get a better estimate of \(lwub_b^{\Gamma}\). To improve the current strategy the algorithm checks whether for some \((v, u) \in E\) such that \(v \in V_M\) and \(d(v) > -\infty\) it holds that \(d(v) > d(u) + w(v, u)\). This is called a strategy improvement condition. Such an edge indicates that \(-d(v)\) is not a sufficient initial energy at \(v\), because traversing the edge \(w(v, u)\) and continuing from \(u\) costs at least \(-w(v, u) - d(u)\) units of energy, which is greater than \(-d(v)\) (Recall that \(-d\) is a lower estimate of \(lwub_b^{\Gamma}\)). If there are edges satisfying the condition, the strategy \(\pi\) is improved in the following way. For each vertex \(v \in V_M\) such that there is an edge \((v, u) \in E\) such that \(d(v) > d(u) + w(v, u)\), \(\pi(v)\) is switched to \(u\). If \(v\) has more than one such edge emanating from it, any of them is acceptable. Then, another iteration of KASI is started. If no such edge exists, the algorithm terminates, because it holds that each vertex \(v \in V\) has \(-d(v) = lwub_b^\Gamma(v)\). Detailed description of the algorithm follows.

In Figure 1 is a pseudo-code of the strategy evaluation part of our algorithm. The input to the procedure consists of four parts. The first and the second part form the lower-weak-upper-bound problem instance that the main algorithm KASI is solving, the MPG \(\Gamma\) and the bound \(b \in \mathbb{N}_0\). The third part is the strategy \(\pi \in \Pi_M^\Gamma\) that we want to evaluate and the fourth part of the input is a vector \(d_{-1} \in (\mathbb{Z} \cup \{-\infty\})^V\). The vector \(d_{-1}\) is such that \(-d_{-1}\) is a lower estimate of \(lwub_b^{\Gamma}\), computed for the previous strategy, or, in case of the first iteration of KASI, set by initialization to a vector of zeros. Let \(A = \{v \in V \mid d_{-1}(v) = 0\}\) and \(D = \{v \in V \mid d_{-1}(v) > -\infty\}\), then the following conditions hold.

i. Each cycle in \(G_\pi(D \setminus A)\) is negative

ii. For each \(v \in D \setminus A\), it holds that \(d_{-1}(v) < 0\) and for each edge \((v, u) \in E_\pi\) i.e., for each edge emanating from \(v\) in \(G_\pi\), it holds that \(d_{-1}(v) \geq d_{-1}(u) + w(v, u)\).

From these technical conditions it follows that \(-d_{-1}\) is also a lower estimate of \(lwub_b^\Gamma_{\pi(D)}\) and the purpose of the strategy evaluation procedure is to decrease the vector \(d_{-1}\) so that the resulting vector \(d\) satisfies \(-d = lwub_b^\Gamma_{\pi(D)}\). To see why from (i.) and (ii.) it follows that \(-d_{-1} \leq lwub_b^\Gamma_{\pi(D)}\), consider a path \(p = (v_0, \ldots, v_k)\) from \(D \setminus A\) to \(A\) in \(G_\pi(D)\). From (ii.) it follows that for each \(j \in \{0, \ldots, k - 1\}\), it holds that \(d_{-1}(v_j) \geq d_{-1}(v_{j+1}) + w(v_j, v_{j+1})\). If we sum the inequalities, we get \(d_{-1}(v_0) \geq d_{-1}(v_k) + w(p)\).
Since \( v_k \in A, d_{-1}(v_k) = 0 \) and the inequality becomes \( d_{-1}(v_0) \geq w(p) \). Therefore, each infinite path in \( G_\pi(D) \) starting from \( v \in D \) and containing a vertex from \( A \) has a prefix of weight less or equal to \( d_{-1}(v_0) \). Furthermore, if the infinite path does not contain a vertex from \( A \), weights of its prefixes cannot even be bounded from below, because by (i.), all cycles in \( G_\pi(D \setminus A) \) are negative. All in all, \( -d_{-1} \) is a lower estimate of \( \text{lwub}_b^{\Gamma_x(D)} \).

The conditions (i.) and (ii.) trivially hold in the first iteration of the main algorithm, for \( d_{-1} = 0 \). In each subsequent iteration, \( d_{-1} \) is taken from the output of the previous iteration and an intuition why the conditions hold will be given below.

The output of the strategy evaluation procedure is a vector \( d \in (\mathbb{Z} \cup \{-\infty\})^V \) such that for each \( v \in D \), it holds that \(-d(v) = \text{lwub}_b^{\Gamma_x(D)}(v)\). Recall that \( D = \{v \in V \mid d_{-1}(v) > -\infty\} \).

The strategy evaluation works only with the restricted graph \( G_\pi(D) \) and it is based on the fact that if we have the set \( B_z = \{v \in D \mid \text{lwub}_b^{\Gamma_x(D)}(v) = 0\} \), i.e., the set of vertices where Max does not need any initial energy to win, then we can compute \( \text{lwub}_b^{\Gamma_x(D)} \) of the remaining vertices by computing longest paths to the set \( B_z \). More precisely, for each vertex \( v \in D \setminus B_z \), \( \text{lwub}_b^{\Gamma_x(D)}(v) \) is equal to the absolute value of the weight of a longest path from \( v \) to \( B_z \) in \( G_\pi(D) \) such that the weight of each suffix of the path is greater or equal to \(-b\). If each path from \( v \) to \( B_z \) has a suffix of weight less than \(-b\) or no path from \( v \) to \( B_z \) exists, then \( \text{lwub}_b^{\Gamma_x(D)}(v) = \infty \).

To get some idea about why this holds consider a play winning for Max. The energy level never drops below zero in the play, and so there must be a moment from which onwards the energy level never drops below the energy level of that moment. Therefore, Max does not need any initial energy to win a play starting from the appropriate vertex (Please note that Min has no choices in \( \Gamma_\pi(D) \)), and so \( B_z \) is not empty. For the vertices in \( D \setminus B_z \), in order to win, Max has to get to some vertex in \( B_z \) without exhausting all of his energy. So the minimal sufficient energy to win is the minimal energy that Max needs to get to some vertex in \( B_z \). All paths from \( D \setminus B_z \) to \( B_z \) must be negative (otherwise \( B_z \) would be larger), and so the minimal energy to get to \( B_z \) is the absolute value of the weight of a longest path to \( B_z \) such that the weight of each suffix of the path is greater or equal to \(-b\). If no path to \( B_z \) exists or all such paths have suffixes of weight less than \(-b\), Max cannot win.

Initially, the procedure over-approximates the set \( B_z \) by the set \( B_0 \) of vertices \( v \) with \( d_{-1}(v) = 0 \) that have an edge \((v,u)\) such that \( w(v,u) - d_{-1}(v) + d_{-1}(u) \geq 0 \) emanating from them (line 2), and then iteratively removes vertices from the set until it arrives at the correct set \( B_z \). The vector \(-d_i\) is always a lower estimate of \( \text{lwub}_b^{\Gamma_x(D)} \), i.e., it always holds that \(-d_i \leq \text{lwub}_b^{\Gamma_x(D)} \). Therefore, only vertices \( v \) with \( d_i(v) = 0 \) are candidates for the final set \( B_z \). However, the vertices \( v \) with \( d_i(v) = 0 \) such that for each edge \((v,u)\), it holds that \( w(v,u) - d_i(v) + d_i(u) < 0 \) are removed from the set of candidates. The reason is that since \( d_i(v) = 0 \), the inequality can be developed to \(-w(v,u) - d_i(u) > 0 \), and so if the edge \((v,u)\) is chosen in the first step, then more than zero units of initial energy are needed at \( v \). During the execution of the procedure, \( d_i \) decreases, and so \(-d_i \) increases, until \(-d_i = \text{lwub}_b^{\Gamma_x(D)} \).

In each iteration, the procedure uses a variant of the Dijkstra’s algorithm to compute longest paths from all vertices to \( B_z \) on line 4. Since \( B_z \) is an over-approximation of \( B_z \), the absolute values of the weights of the longest paths are a lower estimate of \( \text{lwub}_b^{\Gamma_x(D)} \). The weights of the longest paths are assigned to \( d_i \). In particular, for each \( v \in B_z, d_i(v) = 0 \). Dijkstra’s algorithm requires all edge-weights be non-positive (Please note that we are computing longest paths). Since edge-weights are arbitrary integers, we apply potential transformation on them to make them non-positive. As vertex potentials we use \( d_{i-1} \), which contains the longest path weights computed in the previous iteration, or, in case \( i = 0 \), is given as input. Transformed weight of an edge \((x,y)\) is \( w(x,y) - d_{i-1}(x) + d_{i-1}(y) \), which is always non-
positive for the relevant edges. In the first iteration of the main algorithm it follows from the condition (ii.), and in the subsequent iterations it follows from properties of longest path weights and the fact that only vertices with all outgoing edges negative with the potential transformation are removed from the candidate set.

The Dijkstra’s algorithm is also modified so that it assigns $-\infty$ to each $v \in D$ such that each path from $v$ to $B_i$ has a suffix of weight less than $-b$. Therefore, the vertices from which $B_i$ is not reachable or is reachable only via paths with suffixes of weight less than $-b$ have $d_i$ equal to $-\infty$. Also, vertices from $V \setminus D$ have $d_i$ equal to $-\infty$. A detailed description of Dijkstra() is in the full version of this paper [3].

On line 5, the variable $i$ is increased (thus the current longest path weights are now in $d_{i-1}$), and on line 6, we remove from $B_{i-1}$ each vertex $v$ that does not have an outgoing edge $(v,u)$ such that $w(v,u) - d_{i-1}(v) + d_{i-1}(u) \geq 0$. Another iteration is started only if $B_i \neq B_{i-1}$. If no vertex is removed on line 6, then $B_i = B_{i-1}$ and the algorithm finishes and returns $d_{i-1}$ as output. The following theorem establishes the correctness of the algorithm. An intuition why the theorem holds was given above. Its formal proof is in the full version of this paper [3].

**Theorem 1** Let $(\Gamma, b)$ be an instance of the lower-weak-upper-bound problem. Let further $\pi \in \Pi^\Gamma_M$ be a positional strategy of Min, and finally let $d_{i-1} \in (\mathbb{Z} \cup \{-\infty\})^V$ be such that for $A = \{v \in V \mid d_{i-1}(v) = 0\}$ and $D = \{v \in V \mid d_{i-1}(v) > -\infty\}$, the conditions (i.) and (ii.) hold. Then for $d := \text{EVALUATESTRATEGY}(\Gamma, b, \pi, d_{i-1})$ it holds that for each $v \in D$, $d(v) = -\text{lub}_{b}^\Gamma(\pi)(v)$.

The complexity of EVALUATESTRATEGY() is $O(|V| \cdot (|V| \cdot \log |V| + |E|))$. Each iteration takes $O(|V| \cdot \log |V| + |E|)$ because of Dijkstra() and the number of iterations of the while loop on lines 3–7 is at most $|V|$, because $B_i \subseteq V$ loses at least one element in each iteration.

In Figure 1 is a pseudo-code of our strategy improvement algorithm for solving the lower-weak-upper-bound problem using EVALUATESTRATEGY(). The input to the algorithm is a lower-weak-upper-bound problem instance $(\Gamma, b)$. The output of the algorithm is the vector $\text{lub}_{b}^\Gamma$. The pseudo-code corresponds to the high-level description of the algorithm given at the beginning of this section.

The algorithm proceeds in iterations. It starts by taking an arbitrary strategy from $\Pi^\Gamma_M$ on line 2 and initializing the lower estimate of $\text{lub}_{b}^\Gamma$ to vector of zeros on line 3. Then it alternates strategy evaluation (line 6) and strategy improvement (lines 7–13) until no improvement is possible at which point the main while-loop on lines 5–13 terminates and the final $d$ vector is returned on line 20. At that point, it holds that for each $v \in V$, $-d_{i-1}(v) = \text{lub}_{b}^\Gamma(v)$. The whole algorithm KASI is illustrated on Example 1.

The following lemmas and theorem establish the correctness of the algorithm.

**Example 1** In Figure 2 is an example of a run of our algorithm KASI on a simple MPG. The MPG is in Figure 3(a). Circles are Max’s vertices and the square is a Min’s vertex. Let’s denote the MPG by $\Gamma$, let
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1proc LOWERWEAKUPPERBOUND(Γ, b)
2i := 0; π₀ := Arbitrary strategy from Π¹
3di−1 := 0
4improvement := true
5while improvement do
6di := EVALUATESTRATEGY(Γ, b, πi, di−1)
7improvement := false
8i := i + 1
9πi := πi−1
10foreach v ∈ VMin do
11if di−1(v) > −∞ then
12foreach (v, u) ∈ E do
13if di−1(v) > di−1(u) + w(v, u) then
14πi(v) := u; improvement := true
15fi
16od
17od
18return −di−1
end

Figure 2: Solving the lower-weak-upper-bound problem

b = 15 and consider a lower-weak-upper-bound problem given by (Γ, b). Min has only two positional strategies, namely, π¹ and π², where π¹(v3) = v₁ and π²(v3) = v₄. Let π = π² be the first selected strategy. For simplicity, we will use the symbols π, d, B, and D without indices, although in pseudocodes these symbols have indices, and the set D of vertices with finite d value is not even explicitly used. Also, if we speak about a weight of an edge, we mean the weight with the potential transformation by d. Figure 3 illustrates the progress of the algorithm. Each figure denoted by (r, s) shows a state of computation right after update of the vector d by Dijkstra(). r is the value of the iteration counter i of LOWERWEAKUPPERBOUND(), and s is the value of the iteration counter i of EVALUATESTRATEGY(). In each figure, the d value of each vertex is shown by that vertex. Edges that do not belong to the current strategy π of Min are dotted. Detailed description of the progress of the algorithm follows. Initially, π = π², d = 0, and D = {v₁, v₂, v₃, v₄}.

There are three vertices in Gπ(D) with non-negative edges emanating from them, namely, v₁, v₂, v₃, and so EVALUATESTRATEGY() takes {v₁, v₂, v₃} as the first set B. After the vector d is updated so that it contains longest path weights to B (Figure 3(0.0)), all vertices in B still have non-negative edges, and so the strategy evaluation finishes and the strategy improvement phase is started. The strategy improvement condition is satisfied for the edge (v₃, v₁) and so π is improved so that π = π¹. This completes the first iteration of KASI and another one is started to evaluate and possibly improve the new strategy π.

Now the vertex v₃ does not have a non-negative edge emanating from it, so it is removed from the set B and the longest path weights are recomputed (Figure 3(1.0)). Please note that the only path from v₄ to B has a suffix of weight less than −b, and so d(v₄) = −∞ and v₄ is removed from the set D. The update to d causes that v₂ does not have a non-negative edge, thus it is also removed from the set B and the vector d is recomputed again (Figure 3(1.1)). This finishes the strategy evaluation and strategy improvement
The first one we have already used: If \( d \) introduced by the strategy improvement are negative with the potential transformation, holds that the strategy improvement results in decrease of the intuition, it remains to show why the conditions still hold after the strategy improvement and why the selected as the current strategy.

Every time line 6 of Lemma 1 follows. The strategy improvement condition is satisfied for the edge \((v_3, v_4)\), and so the strategy \( \pi^2 \) is selected as the current strategy \( \pi \) again. However, this is not the same situation as at the beginning, because the set \( D \) is now smaller. Evaluation of the strategy \( \pi \) results in the \( d \) vector as depicted in Figure 3(2.0). The vertex \( v_3 \) has \( d(v_3) = -\infty \), because \( v_3 \) cannot reach the set \( B \), which also results in removal of \( v_3 \) from \( D \). No further improvement of \( \pi \) is possible, and so \( lwub^1 = d = (0, 12, \infty, \infty) \).

**Lemma 1** Every time line 6 of \textsc{LowerWeakUpperBound}() is reached, \( \Gamma, b, \pi_i \), and \( d_{i-1} \) satisfy the assumptions of Theorem 7. Every time line 2 of \textsc{LowerWeakUpperBound}() is reached and \( i > 0 \), it holds that \( d_i < d_{i-1} \).

A formal proof of Lemma 1 is in the full version of this paper [3]. The proof uses the following facts. The first one we have already used: If \( p \) is a path from \( v \) to \( u \) such that for each edge \((x, y)\) in the path it holds that \( d(x) \geq d(y) + w(x, y) \), then \( d(y) \geq d(u) + w(p) \), and if for some edge the inequality is strict, then \( d(y) > d(u) + w(p) \). The second fact is similar: If \( c \) is a cycle such that for each edge \((x, y)\) in the cycle it holds that \( d(x) \geq d(y) + w(x, y) \), then \( 0 \geq w(c) \), and if for some edge the inequality is strict, then the cycle is strictly negative. Using these facts we can now give an intuition why the lemma holds.

The assumptions of Theorem 7 conditions (i.) and (ii.), are trivially satisfied in the first iteration of \textsc{LowerWeakUpperBound}(), as already mentioned. During execution of \textsc{EvaluateStrategy}(), conditions (i.) and (ii.) remain satisfied, for the following reasons. The \( d \) values of vertices from \( D \) are weights of longest paths to \( B \), and so each edge \((x, y)\) emanating from a vertex from \( D \setminus B \) satisfies \( d(x) \geq d(y) + w(x, y) \). Only vertices with all outgoing edges negative with the potential transformation are removed from the set \( B \), i.e., only the vertices with each outgoing edge \((x, y)\) satisfying \( d(x) > d(y) + w(x, y) \). Using the facts from the previous paragraph, we can conclude that all newly formed cycles in \( G_\pi(D \setminus B) \) are negative and the weights of longest paths to \( B \) cannot increase. So to complete the intuition, it remains to show why the conditions still hold after the strategy improvement and why the strategy improvement results in decrease of the \( d \) vector. This follows from the fact that the new edges introduced by the strategy improvement are negative with the potential transformation.
Lemma 2 The procedure \textsc{LowerWeakUpperBound()} always terminates.

Proof: By Lemma \[1\] \(d_i\) decreases in each iteration. For each \(v \in V\), \(d_i(v)\) is bounded from below by the term \(-(|V| - 1) \cdot W\), because it is the weight of some path in \(G\) with no repeated vertices (Except for the case when \(d_i(v) = -\infty\), but this is obviously not a problem). Since \(d_i\) is a vector of integers, infinite chain of improvements is not possible, and so termination is guaranteed. \[\square\]

Theorem \[2\] is the main theorem of this paper which establishes the correctness of our algorithm. Its proof is in the full version of this paper \[3\]. The key idea of the proof is to define strategies for both players with the following properties. Let \(ds := \text{LowerWeakUpperBound}(\Gamma, b)\). Max’s strategy that we will define ensures that for each vertex \(v \in V\), \(ds(v)\) is a sufficient amount of initial energy no matter what his opponent does, and Min’s strategy that we will define ensures that Max cannot do with smaller amount of initial energy. In particular, for vertices with \(ds(v) = \infty\), the strategy ensures that Max will eventually go negative or traverse a path segment of weight less than \(-b\) with arbitrarily large amount of initial energy. From the existence of such strategies it follows that for each \(v \in V\), \(ds(v) = \text{lwub}_b^\Gamma(v)\), and both strategies are optimal with respect to the lower-weak-upper-bound problem.

The optimal strategy of Max is constructed from the final longest path forest computed by the procedure \textsc{Dijkstra()} and the non-negative (with potential transformation) edges emanating from the final set \(B\). The optimal strategy of Min is more complicated.

There is a theorem in \[2\] which claims that Min can restrict herself to positional strategies. Unfortunately, this is not true. Unlike Max, Min sometimes needs memory. Example \[1\] is a proof of this fact, because none of the two positional strategies of Min guarantees that Max loses from \(v_3\). However, Min can play optimally using the sequence of positional strategies computed by our algorithm. In Example \[1\] to guarantee that Max loses from \(v_3\), Min first sends the play from \(v_3\) to \(v_4\) and when it returns back to \(v_3\), she sends the play to \(v_1\). As a result, a path of weight \(-20\) is traversed and since \(b = 15\), Max loses.

In general, let \(\pi_0, \pi_1, \ldots\) be the sequence of positional strategies computed by the algorithm. Min uses the sequence in the following way: if the play starts from a vertex with finite final \(d\) value and never leaves the set of vertices with finite final \(d\) value, then Min uses the last strategy in the sequence, and it is the best she can do, as stated by Theorem \[1\]. If the play starts or gets to a vertex with infinite final \(d\) value, she uses the strategy that caused that the \(d\) value of that vertex became \(-\infty\), but only until the play gets to a vertex that was made infinite by some strategy with lower index. At that moment Min switches to the appropriate strategy. In particular, Min never switches to a strategy with higher index.

Theorem 2 Let \(ds := \text{LowerWeakUpperBound}(\Gamma, b)\), then for each \(v \in V\), \(ds(v) = \text{lwub}_b^\Gamma(v)\).

The algorithm has a pseudopolynomial time complexity: \(O(|V|^2 \cdot (|V| \cdot \log |V| + |E|) \cdot W)\). It takes \(O(|V|^2 \cdot W)\) iterations until the while-loop on lines \[5\] terminates. The reason is that for each \(v \in V\), if \(d(v) > -\infty\), then \(d(v) \geq -(|V| - 1) \cdot W\), because \(d(v)\) is the weight of some path with no repeated vertices, and so the \(d\) vector can be improved at most \(O(|V|^2 \cdot W)\) times. Each iteration, if considered separately, takes \(O(|V| \cdot (|V| \cdot \log |V| + |E|))\), so one would say that the overall complexity should be \(O(|V|^3 \cdot (|V| \cdot \log |V| + |E|) \cdot W)\). However, the number of elements of the set \(B_i\) in \textsc{EvaluateStrategy()} never increases, even between two distinct calls of the evaluation procedure, hence the amortized complexity of one iteration is only \(O(|V| \cdot \log |V| + |E|)\).

The algorithm can even be improved so that its complexity is \(O(|V| \cdot (|V| \cdot \log |V| + |E|) \cdot W)\). This is accomplished by efficient computation of vertices which will update their \(d\) value in the next iteration so that computational time is not wasted on vertices whose \(d\) value is not going to change. Interestingly enough, the same technique can be used to improve the complexity of the algorithm of Björklund and Vorobyov so that the complexities of the two algorithms are the same. Detailed description of the technique is in the full version of this paper \[3\].
4 Experimental Evaluation

Our experimental study compares four algorithms for solving the lower-bound and the lower-weak-upper-bound problems. The first is value iteration \([7,11]\) (VI). The second and the third are combinations of VI with other algorithm. Finally, the fourth algorithm is our algorithm KASI. We will now briefly describe the algorithms based on VI.

Let \((\Gamma, b)\) be an instance of the lower-weak-upper-bound problem. VI starts with \(d_0(v) = 0\), for each \(v \in V\), and then computes \(d_1,d_2,\ldots\) according to the following rules.

\[
\begin{align*}
    d_{i+1}(v) &= \begin{cases} 
    x = \min_{(v,u) \in E} \max(0,d_i(u) - w(v,u)) & \text{if } v \in V_{\text{Max}} \land x \leq b \\
    x = \max_{(v,u) \in E} \max(0,d_i(u) - w(v,u)) & \text{if } v \in V_{\text{Min}} \land x \leq b \\
    \infty & \text{otherwise}
    \end{cases}
\end{align*}
\]

It is easy to see that for each \(v \in V\) and \(k \in \mathbb{N}_0\), \(d_k(v)\) is the minimum amount of Max’s initial energy that enables him to keep the sum of traversed edges, plus \(d_k(v)\), greater or equal to zero in a \(k\)-step play. The computation continues until two consecutive \(d\) vectors are equal. The last \(d\) vector is then the desired vector \(\text{lwub}_{\Gamma}^b\). If \(b = (|V| - 1) \cdot W\), the algorithm solves the lower-bound problem. The complexity of the straightforward implementation of the algorithm is \(O(|V|^2 \cdot |E| \cdot W)\), which was improved in \([7,11]\) to \(O(|V| \cdot |E| \cdot W)\), which is slightly better than the complexity of KASI.

The shortcoming of VI is that it takes enormous time before the vertices with infinite \(\text{lb}_{\Gamma}^b\) and \(\text{lwub}_{\Gamma}^b\) value are identified. That’s why we first compute the vertices with \(v < 0\) by some fast MPG solving algorithm and then apply VI on the rest of the graph. For the lower-bound problem, the vertices with \(v < 0\) are exactly the vertices with infinite \(\text{lb}_{\Gamma}^b\) value. For the lower-weak-upper-bound problem, the vertices with \(v < 0\) might be a strict subset of the vertices with infinite \(\text{lwub}_{\Gamma}^b\) value, but still the preprocessing sometimes saves a lot of time in practice. It is obvious that on MPGs with all vertices with \(v \geq 0\) the preprocessing does not help at all. It is also not helpful for the lower-weak-upper-bound problem for small bound \(b\).

According to our experiments, partly published in \([6]\), the fastest algorithms in practice for dividing the vertices of an MPG into those with \(v \geq 0\) and \(v < 0\) are the algorithm of Björklund and Vorobyov \([1]\) (BV) and the algorithm of Schewe \([18]\) (SW). The fact that they are the fastest does not directly follow from \([6]\), because that paper focuses on parallel algorithms and computation of the exact \(v\) values.

The original algorithm BV is a sub-exponential randomized algorithm. To prove that the algorithm is sub-exponential, some restrictions had to be imposed. If these restrictions are not obeyed, BV runs faster. Therefore, we decided not to obey the restrictions and use only the “deterministic part” of the algorithm. We used only the modified BV algorithm in our experimental study. We even improved the complexity of the deterministic algorithm from \(O(|V|^2 \cdot |E| \cdot W)\) to \(O(|V| \cdot (|V| \cdot \log |V| + |E|) \cdot W)\) using the same technique as for the improvement of the complexity of KASI which is described in the full version of this paper \([3]\). Since the results of the improved BV were significantly better on all input instances included in our experimental study, all results of BV in this paper are the results of the improved BV.

The complexity of SW is \(O(|V|^2 \cdot (|V| \cdot \log |V| + |E|) \cdot W)\). It might seem that this is in contradiction with the title of Schewe’s paper \([18]\), because if some algorithm is optimal, one would expect that there are no algorithms with better complexity. However, the term “optimal” in the title of the paper refers to the strategy improvement technique. SW is also a strategy improvement algorithm, and the strategy improvement steps in SW are optimal in a certain sense.

We note that any algorithm that divides the vertices of an MPG into those with \(v \geq 0\) and those \(v < 0\) can be used to solve the lower-bound and the lower-weak-upper-bound problem with the help of binary search, but it requires introduction of auxiliary edges and vertices into the input MPG and
repeated application of the algorithm. According to our experiments, BV and SW run no faster than KASI. Therefore, solving the two problems by repeated application of BV and SW would lead to higher runtimes than the runtimes of KASI. If we use the reduction technique from [2], then BV/SW has to be executed \( \Theta(|V| \cdot \log(|V| \cdot \log b)) \) times to solve the lower-bound problem, and \( \Theta(|V| \cdot \log b) \) times to solve the lower-weak-upper-bound. That’s why we compared KASI only with the algorithm VI and the two combined algorithms: VI + BV and VI + SW. The complexities of BV and SW exceed the complexity of VI, and so the complexities of the combined algorithms are the complexities of BV and SW.

4.1 Input MPGs

We experimented with completely random MPGs as well as more structured synthetic MPGs and MPGs modeling simple reactive systems. The synthetic MPGs were generated by two generators, namely SPRAND [9] and TOR [8], downloadable from [14]. The outputs of these generators are only directed weighted graphs, and so we had to divide vertices between Max and Min ourselves. We divided them uniformly at random. The MPGs modeling simple reactive systems we created ourselves.

SPRAND was used to generate the “randx” MPG family. Each of these MPGs contains \( |E| = x \cdot |V| \) edges and consist of a random Hamiltonian cycle and \( |E| − |V| \) additional random edges, with weights chosen uniformly at random from \([1, 10000]\). To make these inputs harder for the algorithms, in each of them, we subtracted a constant from each edge-weight so that the \( \nu \) value of each vertex is close to 0.

TOR was used for generation of the families “sqnc”, “lnc”, and “pnc”. The sqnc and lnc families are 2-dimensional grids with wrap-around, while the pnc family contains layered networks embedded on a torus. We also created subfamilies of each of the three families by adding cycles to the graphs. For more information on these inputs we refer you to [13] or [6]. Like for the SPRAND generated inputs, we adjusted each TOR generated MPG so that the \( \nu \) value of each vertex is close to 0.

As for the MPGs modeling simple reactive systems, we created three parameterized models. The first is called “collect” and models a robot on a ground with obstacles which has to collect items occurring at different locations according to certain rules. Moving and even idling consumes energy, and so the robot has to return to its docking station from time to time to recharge. By solving the lower-bound, or the lower-weak-upper-bound problem for the corresponding MPG, depending on whether there is some upper bound on the robot’s energy, we find out from which initial configurations the robot has a strategy which ensures that it will never consume all of its energy outside the docking station, and we also get some strategy which ensures it. For each “good” initial configuration, we also find out the minimal sufficient amount of initial energy. We note that the energy is not a part of the states of the model. If it was, the problem would be much simpler. We could simply compute the set of states from which Min has a strategy to get the play to a state where the robot has zero energy and it is not in the docking station. However, making the energy part of the states would cause an enormous increase in the number of states and make the model unmanageable.

The second model is called “supply” and models a truck which delivers material to various locations the selection of which is beyond its control. The goal is to never run out of the material so that the truck is always able to satisfy each delivery request. We also want to know the minimal sufficient initial amount of the material.

The third model is called “taxi” and models a taxi which transports people at their request. Its operation costs money and the taxi also earns money. The goal is to never run out of money, and we also want to know the minimal sufficient initial amount of money.

To get MPGs of manageable size, the models are, of course, very simplified, but they are still much closer to real world problems than the synthetic MPGs.
The experiments were carried out on a machine equipped with two dual-core Intel® Xeon® 2.00GHz processors and 16GB of RAM, running GNU/Linux kernel version 2.6.26. All algorithms were implemented in C++ and compiled with GCC 4.3.2 with the “-O2” option.

Table 1 gives the results of our experiments. The first column of the table contains names of the input MPGs. Numbers of vertices and edges, in thousands, are in brackets. The MPGs prefixed by “sqnc”, “lnc”, and “pnc” were generated by the TOR generator. They all contain $2^{18}$ vertices. The MPGs prefixed by “rand” were generated by the SPRAND generator. Both for the rand5 and rand10 family, we experimented with three sizes of graphs, namely, with $2^{18}$ vertices – no suffix, with $2^{19}$ vertices – suffix “b”, and with $2^{20}$ vertices – suffix “h”. Finally, the MPGs prefixed by “collect”, “supply”, and “taxi” are the models of simple reactive systems created by ourselves. For each model, we tried two different values of parameters.

| MPG       | lower-bound | lower-weak-upper-bound |
|-----------|-------------|------------------------|
|           | VI | VI + BV | VI + SW | KASI | VI | VI + BV | VI + SW | KASI  |
| sqnc01    | (262k 524k) | n/a | 31.22 | 55.40 | 17.83 | 13.28 | 19.41 | 43.54 | 9.06  |
| sqnc02    | (262k 524k) | n/a | 13.30 | 20.14 | 11.01 | 2.88  | 10.70 | 17.52 | 3.57  |
| sqnc03    | (262k 525k) | n/a | 3.18  | 3.54  | 1.58  | 0.75  | 3.20  | 3.55  | 1.07  |
| sqnc04    | (262k 532k) | n/a | 9.34  | 11.49 | 8.48  | 1.65  | 8.55  | 10.70 | 2.53  |
| sqnc05    | (262k 786k) | n/a | 10.45 | 14.24 | 4.89  | 1.20  | 10.17 | 13.95 | 1.72  |
| lnc01     | (262k 524k) | n/a | 60.79 | 67.89 | 111.32| 11.31 | 17.49 | 27.85 | 9.06  |
| lnc02     | (262k 524k) | n/a | 13.30 | 20.14 | 11.01 | 2.88  | 10.70 | 17.52 | 3.57  |
| lnc03     | (262k 525k) | n/a | 3.18  | 3.54  | 1.58  | 0.75  | 3.20  | 3.55  | 1.07  |
| lnc04     | (262k 528k) | n/a | 21.05 | 25.28 | 10.63 | 3.53  | 11.65 | 15.64 | 4.24  |
| lnc05     | (262k 786k) | n/a | 10.45 | 14.24 | 4.89  | 1.20  | 10.17 | 13.95 | 1.72  |
| pnc01     | (262k 2097k) | n/a | 24.27 | 16.08 | 3.80  | 1.41  | 24.31 | 16.08 | 1.98  |
| pnc02     | (262k 2097k) | n/a | 25.49 | 15.37 | 3.80  | 1.43  | 25.55 | 15.38 | 1.98  |
| pnc03     | (262k 2098k) | n/a | 23.48 | 17.66 | 3.86  | 1.49  | 23.53 | 17.66 | 2.04  |
| pnc04     | (262k 2101k) | n/a | 27.09 | 29.69 | 4.71  | 1.97  | 27.15 | 29.69 | 2.51  |
| pnc05     | (262k 2359k) | n/a | 19.16 | 20.42 | 4.55  | 1.65  | 19.27 | 20.54 | 2.39  |
| rand5     | (262k 1310k) | n/a | 36.29 | 33.06 | 10.09 | 3.53  | 36.52 | 33.24 | 5.17  |
| rand5b    | (524k 2621k) | n/a | 86.55 | 59.01 | 21.35 | 7.45  | 87.16 | 59.48 | 11.07 |
| rand5h    | (1048k 10482k) | n/a | 23.48 | 17.66 | 3.86  | 1.49  | 23.53 | 17.66 | 2.04  |
| rand10    | (262k 2098k) | n/a | 27.09 | 29.69 | 4.71  | 1.97  | 27.15 | 29.69 | 2.51  |
| rand10b   | (524k 2624k) | n/a | 105.69 | 43.43 | 14.54 | 5.07  | 105.97 | 43.45 | 7.98  |
| rand10h   | (1048k 10485k) | n/a | 140.46 | 110.68 | 29.27 | 11.38 | 140.57 | 110.82 | 17.52 |
| collect1  | (636k 3309k) | 996.08 | 1027.12 | 1032.55 | 5.68 | 531.40 | 544.77 | 563.78 | 4.89  |
| collect2  | (636k 3309k) | 338.56 | 352.45 | 367.12 | 5.70 | 181.35 | 189.17 | 208.52 | 4.89  |
| supply1   | (363k 1014k) | 6956.23 | 16.03 | 109.72 | 1.79 | 7.72  | 8.87  | 102.97 | 1.85  |
| supply2   | (727k 2030k) | 28046.54 | 65.08 | 449.47 | 3.64 | 30.84 | 33.31 | 418.88 | 3.77  |
| taxi1     | (509k 979k) | 11.64 | 12.85 | 13.16 | 1.29 | 0.70  | 1.49  | 2.17  | 1.38  |
| taxi2     | (509k 979k) | 6.00  | 7.03  | 7.51  | 1.29 | 0.70  | 1.49  | 2.17  | 1.38  |

Table 1: Runtimes of the experiments (in seconds)

### 4.2 Results

The experiments were carried out on a machine equipped with two dual-core Intel® Xeon® 2.00GHz processors and 16GB of RAM, running GNU/Linux kernel version 2.6.26. All algorithms were implemented in C++ and compiled with GCC 4.3.2 with the “-O2” option.

Table 1 gives the results of our experiments. The first column of the table contains names of the input MPGs. Numbers of vertices and edges, in thousands, are in brackets. The MPGs prefixed by “sqnc”, “lnc”, and “pnc” were generated by the TOR generator. They all contain $2^{18}$ vertices. The MPGs prefixed by “rand” were generated by the SPRAND generator. Both for the rand5 and rand10 family, we experimented with three sizes of graphs, namely, with $2^{18}$ vertices – no suffix, with $2^{19}$ vertices – suffix “b”, and with $2^{20}$ vertices – suffix “h”. Finally, the MPGs prefixed by “collect”, “supply”, and “taxi” are the models of simple reactive systems created by ourselves. For each model, we tried two different values of parameters.
Each MPG used in the experiments has eight columns in the table. Each column headed by a name of an algorithm contains execution times of that algorithm in seconds, excluding the time for reading input. The term “n/a” means more than 10 hours. The first four columns contain results for the lower-bound problem, the last four columns contain the results for the lower-weak-upper-bound problem, which contains a bound $b$ as a part of the input. If the bound is too high, the algorithms essentially solve the lower-bound problem, and so the runtimes are practically the same as for the lower-bound problem. If the bound is too low, all vertices in our inputs have infinite $\text{lwb}_b^\Gamma$ value, and they become very easy to solve. We tried various values of $b$, and for this paper, we selected as $b$ the average $\text{lb}_b^\Gamma$ value of the vertices with finite $\text{lb}_b^\Gamma$ value divided by 2, which seems to be a reasonable amount so that the results provide insight. We note that smaller $b$ makes the computation of VI and KASI faster. However, the BV and SW parts of VI + BV and VI + SW always perform the same work, and so for $b \ll (|V| − 1) \cdot W$, the combined algorithms are often slower than VI alone.

The table shows that the algorithm KASI was the fastest on all inputs for the lower-bound problem. For the lower-weak-upper-bound problem it was never slower than the fastest algorithm by more than a factor of 2, and for some inputs it was significantly faster. This was true for all values of $b$ that we tried. Therefore, the results clearly suggest that KASI is the best algorithm. In addition, there are several other interesting points.

VI is practically unusable for solving the lower-bound problem for MPGs with some vertices with $\nu < 0$. Except for lnc01–02, collect1–2, and taxi1–2, all input MPGs had vertices with $\nu < 0$. The preprocessing by BV and SW reduces the execution time by orders of magnitude for these MPGs. On the other hand, for the lower-weak-upper-bound problem for the bound we selected, VI is often very fast and the preprocessing slows the computation down in most cases. VI was even faster than KASI on a lot of inputs. However, the difference was never significant, and it was mostly caused by the initialization phase of the algorithms, which takes more time for the more complex algorithm KASI. Moreover, for some inputs, especially from the “collect” family, VI is very slow. VI makes a lot of iterations for the inputs from the collect family, because the robot can survive for quite long by idling, which consumes a very small amount of energy per time unit. However, it cannot survive by idling forever. The $i$-th iteration of VI computes the minimal sufficient initial energy to keep the energy level non-negative for $i$ time units, and so until the idling consumes at least as much energy as the minimal sufficient initial energy to keep the energy level non-negative forever, new iterations have to be started. We believe that this is a typical situation for this kind of application. Other inputs for which VI took a lot of time are: sqnc01, lnc01–02, supply1–2.

Finally, we comment on scalability of the algorithms. As the experiments on the SPRAND generated inputs suggest, the runtimes of the algorithms increase no faster than the term $|V| \cdot |E|$, and so they are able to scale up to very large MPGs.

5 Conclusion

We proposed a novel algorithm for solving the lower-bound and the lower-weak-upper-bound problems for MPGs. Our algorithm, called Keep Alive Strategy Improvement (KASI), is based on the strategy improvement technique which is very efficient in practice. To demonstrate that the algorithm is able to solve the two problems for large MPGs, we carried out an experimental study. In the study we compared KASI with the value iteration algorithm (VI) from [2,11], which we also improved by combining it with the algorithm of Björklund and Vorobyov [1] (BV) and the algorithm of Schewe (SW). KASI is the clear winner of the experimental study.

Two additional results of this paper are the improvement of the complexity of BV, and characteriza-
tion of Min’s optimal strategies w.r.t. the lower-weak-upper-bound problem.

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