Degenerate Sklyanin algebras, Askey–Wilson polynomials and Heun operators

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1. Introduction

Quite some time ago, it was shown [1, 2] that the Askey–Wilson difference operator could be realized as a quadratic expression in the generators of the degenerate Sklyanin algebra (of dimension four). A little earlier Kalnins and Miller [3] used symmetry techniques to derive...
the orthogonality relation of the Askey–Wilson polynomials and identified to that end interesting ladder operators. Over the years the application of the factorization method [4] to these polynomials and the study of their structure relations [5] brought attention to related elements. More recently advances have been made in the elaboration of the theory of $q$-Heun operators and the Heun–Askey–Wilson [6] and rational [7] Heun operators have been identified by focusing on certain raising properties of their actions on appropriate spaces of functions. The purpose of this report is to stress the connections between these topics.

The paper will develop as follows. In section 2 we shall introduce three operators involving $q$-shifts that realize a three-dimensional degenerate Sklyanin algebra $\mathfrak{s}l_3$. These operators will be ubiquitous; it will be observed that their linear combination is diagonal on special Askey–Wilson polynomials in base $q$ and, following [1], that the most general quadratic expression formed with them yields the full Askey–Wilson operator in base $q^2$. The degenerate Sklyanin algebra $\mathfrak{s}l_4$ obtained by Gorsky and Zabrodin will also be introduced. It has $\mathfrak{s}l_3$ as a subalgebra and will be seen to admit a formal embedding of the Askey–Wilson algebra. In section 3, it will be seen that the contiguity operators introduced by Kalnins and Miller in their treatment of the Askey–Wilson polynomials all belong to a model of the degenerate Sklyanin algebra $\mathfrak{s}l_4$. It will also be seen that the $q$-para Racah polynomials [8] support a finite-dimensional representation of the four-dimensional degenerate Sklyanin algebra. Section 4 will indicate how the Askey–Wilson bispectral operators emerge in this context. We shall consider first order $q$-difference operators and identify the conditions for such operators to raise by one the degree of polynomials in the symmetric variable $x = z + z^{-1}$. This will lead to a five dimensional vector space of operators. A basis will consist of one lowering operator, two that stabilize polynomials of a given degree and two that are raising this degree by one. The lowering and stabilizing operators will coincide with the operators realizing $\mathfrak{s}l_3$ introduced in section 2. A combination involving the two raising operators will give the realization of the fourth generator of $\mathfrak{s}l_4$ beyond those of $\mathfrak{s}l_3$. The relations obeyed by these five operators will be found in an appendix A. It will further be seen that the most general quadratic operator in the five basis elements and not raising the degree by more than one is the Heun–Askey–Wilson operator [9]. This parallels the fact that in bispectral situations Heun operators could be defined equivalently as raising operators or as bilinear expressions in the bispectral operators. As will be indicated in the conclusion this approach is paving the way to the definition of Sklyanin-like Heun algebras associated to different degenerations of the Askey–Wilson grid.

2. Realizations of degenerate Sklyanin algebras and Askey–Wilson operators

It is well known that quantum algebras can be realized in terms of $q$-derivatives. In the case of $\mathcal{U}_q(\mathfrak{su}(2))$ for example, the commutation relations

$$\begin{align*}
[\hat{B}, \hat{C}] &= \frac{\hat{A}^2 - \hat{D}^2}{q - q^{-1}}, \quad [\hat{A}, \hat{D}] = 0, \\
\hat{A}\hat{B} &= q\hat{B}\hat{A}, \quad \hat{D}\hat{B} = q\hat{D}\hat{B}, \quad \hat{C}\hat{A} = q\hat{A}\hat{C}, \quad \hat{D}\hat{C} = q\hat{C}\hat{D}
\end{align*}$$

(2.1)

are realized [10, 11] by taking

$$\begin{align*}
\hat{A}^{(\nu)} &= q^{-\nu}T_+, \quad \hat{B}^{(\nu)} = \frac{z}{2(q - q^{-1})(q^{2\nu}T_+ - q^{-2\nu}T_+)}(T_+ - qT_-), \\
\hat{C} &= \frac{2}{(q - q^{-1})z}(T_+ - T_-), \quad \hat{D}^{(\nu)} = q^{\nu}T_-,
\end{align*}$$

(2.2)
where in the case of finite dimensional representations $\nu$ is integer or half-integer (see below) and where $T_+$ and $T_-$ are the $q$-shift operators that act as follows on functions of $z$:

$$T_+ f(z) = f(qz), \quad T_- f(z) = f(q^{-1}z). \quad (2.3)$$

We shall in the following look at models built with operators of the divided difference type.

2.1. The three-dimensional degenerate Sklyanin algebra $\hat{S}^\text{d}$

Let $p = (a, b, c, \alpha, \beta, \gamma)$ be a set of parameters. The generalized three-dimensional Sklyanin algebra $\hat{S}^\text{d}$ as defined in [12] (see also [13]), is given by three generators $u, v, y$ and the relations:

$$uv - auv - cuy = 0, \quad vy - byv - \beta uu, \quad yu - cuy - \gamma vv = 0. \quad (2.4)$$

Consider the operators

$$Y = \frac{1}{z - z^{-1}}(T_+ - T_-),$$
$$U = \frac{1}{z - z^{-1}}(zT_+ - z^{-1}T_-), \quad (2.5)$$
$$V = \frac{1}{z - z^{-1}}(zT_- - z^{-1}T_+).$$

It is readily checked that they satisfy the following relations:

$$VY - qYV = 0, \quad YU - qUY = 0,$$
$$[U, V] = (q - q^{-1})Y^2 \quad (2.6)$$

Under the correspondence \{y, u, v\} $\rightarrow$ \{Y, U, V\}, it is seen that $Y, U, V$ realize a special case of $\hat{S}^\text{d}$ with

$$a = 1, \quad b = c = q, \quad \alpha = (q - q^{-1}), \quad \beta = \gamma = 0. \quad (2.7)$$

We shall henceforth denote this algebra by $\hat{a}^3_\text{d}$. As shown in [12], it corresponds to one of the situations $((a, b, c) \neq (0, 0, 0)$ and $\beta = \gamma = b - c = 0))$ for which the generalized Sklyanin algebra $\hat{S}^\text{d}$ has a polynomial growth Hilbert series and Koszul properties. The algebra $\hat{a}^3_\text{d}$ thus defined possesses a quadratic Casimir elements $\Omega^{(2)}$:

$$\Omega^{(2)} = uv + q^{-1}y^2 \quad (2.8)$$

that takes the value 1 in the realization (2.5) which implies that $UV$ is related to $Y^2$.

2.2. The Askey–Wilson polynomials and algebra

Let us recall that the Askey–Wilson polynomials $p_n(x; a, b, c, d | q)$ defined by

$$\frac{d^n p_n(x; a, b, c, d | q)}{dx^n} = 4\phi_3 \left[ q^{-n}, \frac{abcdq^{-n-1}}{ab}, \frac{az}{ac}, \frac{az^{-1}}{ad} \right| q, q \right]$$

with $x = z + z^{-1}$ are eigenfunctions of the operator $L^{(a,b,c,d)}_q$ [14]

$$L^{(a,b,c,d)}_q p_n(x; a, b, c, d | q) = \lambda_n p_n(x; a, b, c, d | q) \quad (2.9)$$

where $n$ is integer or half-integer (see below) and where $T_+$ and $T_-$ are the $q$-shift operators that act as follows on functions of $z$:

$$T_+ f(z) = f(qz), \quad T_- f(z) = f(q^{-1}z). \quad (2.3)$$

We shall in the following look at models built with operators of the divided difference type.
with eigenvalues
\[ \lambda_n = q^{-n}(1 - q^n)(1 - abcdq^{n-1}). \]  

(2.11)

We use standard notation for the basic hypergeometric functions and \( q \)-shifted factorials [14]. In base \( q \), the Askey–Wilson operator reads
\[ L^{(a,b,c,d)}_q = A^{(a)}(z)T^+ + [A^{(b)}(z) + A^{(c)}(z^{-1})]I + A^{(d)}(z^{-1})T^-. \]  

(2.12)

with
\[ A^{(a)}(z) = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)}. \]  

(2.13)

and where \( I \) is the identity operator.

The Askey–Wilson algebra \( AW(3) \) [15] that encodes the bispectrality of the polynomials \( p_n \) is realized by taking the generators \( K_0 = L^{(a,b,c,d)}_q + (1 + q^{-1}abcd) \) and \( K_1 = x \) to find that the defining relations of \( AW(3) \)
\[ [K_0, K_1]_q = K_2, \quad [K_1, K_2]_q = \mu K_1 + \nu_0 K_0 + \rho_0, \quad [K_2, K_0]_q = \mu K_0 + \nu_1 K_1 + \rho_1, \]  

(2.14)

where \( [A, B]_q = q^{1/2}AB - q^{-1/2}BA \), are verified with the parameters \( \mu, \nu \) and \( \rho \) related to those, \( a, b, c, d \) of the polynomials \( p_n \) (see for instance [16]).

Consider now the following general linear combination of \( Y, U \) and \( V \):
\[ M^{(\alpha,\beta,\gamma)} = \alpha Y + \beta U + \gamma V. \]  

(2.15)

Using (2.5), we see that
\[ M^{(\alpha,\beta,\gamma)} = F(z)T^+ + F(z^{-1})T^- \]  

(2.16)

where
\[ F(z) = \frac{\gamma(1 - az)(1 - bz)}{1 - z^2}. \]  

(2.17)

with
\[ \frac{\alpha}{\gamma} = (a + b), \quad \frac{\beta}{\gamma} = -ab. \]  

(2.18)

Since
\[ F(z) + F(z^{-1}) = \gamma(1 - ab), \]  

(2.19)

we observe that
\[ M^{(\alpha,\beta,\gamma)} = \gamma[L^{(a,b,c,d)}_q + (1 - ab)]. \]  

(2.20)

It follows that the eigenfunctions of a linear combination of the operators \( Y, U, V \) such as \( M^{(\alpha,\beta,\gamma)} \) are special Askey–Wilson polynomials with the property of being ‘symmetric’ when looked at from the dual perspective where variable and degree are exchanged; this is because the diagonal term in \( M^{(\alpha,\beta,\gamma)} \) is constant. Correspondingly, following [16], by taking
\[ K_0 = \frac{1}{\gamma}M^{(\alpha,\beta,\gamma)} \quad \text{and} \quad K_1 = x \]  

(2.21)
we find that the Askey–Wilson algebra relations (2.14) are satisfied with
\[
\mu = 0, \quad \nu_0 = 1, \quad \rho_0 = 0,
\]
\[
\nu_1 = -ab(q - q^{-1})^2, \quad \rho_1 = (1 - q^{-1})(a + b)(ab + q).
\] (2.22)
We shall consider next quadratic expressions in the generators of \( \mathfrak{sl}_3 \).

2.3. The Askey–Wilson operator and \( \mathfrak{sl}_3 \)

An important observation \([1]\) comes from considering the most general quadratic expression in the operators \( \{ Y, U, V \} \) representing \( \mathfrak{sl}_3 \). Let us go over this. Define as before another general linear combination of these operators:
\[
M^{(\delta, \epsilon, \zeta)} = \delta Y + \epsilon U + \zeta V = G(z)T_+ + G(z^{-1})T_-
\] (2.23)
where
\[
G(z) = \frac{\zeta(1 - q^{-1}cz)(1 - q^{-1}dz)}{1 - z^2}
\] (2.24)
with
\[
\frac{\delta}{\zeta} = q^{-1}(c + d), \quad \frac{\epsilon}{\zeta} = -q^{-2}cd.
\] (2.25)
The product \( M^{(\alpha, \beta, \gamma)}M^{(\delta, \epsilon, \zeta)} \) will take the form:
\[
M^{(\alpha, \beta, \gamma)}M^{(\delta, \epsilon, \zeta)} = F(z)G(qz)T_2^2 + \left[ F(z)G(q^{-1}z^{-1}) + F(z^{-1})G(q^{-1}z) \right]I
\]
\[
+ F(z^{-1})G(qz^{-1})T_2^2
\] (2.26)
A straightforward computation shows that for the specific functions \( F(z) \) and \( G(z) \) given in (2.17) and (2.25), the following identity holds:
\[
F(z)G(q^{-1}z^{-1}) + F(z^{-1})G(qz) = -F(z)G(qz) - F(z^{-1})G(qz^{-1}) + \Gamma
\] (2.27)
with \( \Gamma \) a constant given by
\[
\Gamma = \gamma \zeta (abcdq^{-2} - ab - cdq^{-2} + 1).
\] (2.28)
Recalling the expression of \( A^{(2)}(z) \) in (2.13), we see that
\[
F(z)G(qz) = \gamma \zeta A^{(2)}(z)
\] (2.29)
and hence we write
\[
M^{(\alpha, \beta, \gamma)}M^{(\delta, \epsilon, \zeta)} = \gamma \zeta [ L_{q^2}^{(ab, cd)} + (abcdq^{-2} - ab - cdq^{-2} + 1)I].
\] (2.30)
We have thus obtained a factorization of the Askey–Wilson operator \( L_{q^2}^{(ab, cd)} \) as a product of two linear combinations of the generators in the representation (2.5) of the special generalized Sklyanin algebra \( \mathfrak{sl}_3 \).

We also note that
\[
M^{(\alpha, \beta, \gamma)}M^{(\delta, \epsilon, \zeta)} = (\alpha Y + \beta U + \gamma V)(\delta Y + \epsilon U + \zeta V)
\] (2.31)
provides the most general quadratic expression in the three generators \{Y, U, V\}. Taking into account the relations (2.5) between the generators and the expression of \(UV\) (2.47) (and \(VU\)) in terms of \(Y^2\) provided by the value of the Casimir, the product \(\mathcal{M}^{(a,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)}\) can be reduced to:

\[
\mathcal{M}^{(a,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)} = \beta eU^2 + \gamma \zeta V^2 + (\alpha \delta - \beta \zeta q^{-1} - \gamma \epsilon q)Y^2 + (\alpha \epsilon q + \beta \delta)UY \\
+ (\alpha \zeta q^{-1} + \gamma \delta)VY + (\beta \zeta + \gamma \epsilon)\mathcal{I}.
\]  

(2.32)

We thus recover (with a different parameterization) the result of Gorsky and Zabrodin [1] according to which the Askey–Wilson \(q\)-difference operator is a quadratic expression in the generators of \(\mathfrak{sl}_3\). (As a matter of fact this result is presented in [1] in the context of the four-dimensional degenerate Sklyanin algebra to which we shall turn in a moment.)

**Remark 2.1.** The idea of obtaining operators of interest, like the Askey–Wilson one, as quadratic expressions in the generators of fundamental algebras has precedents. Of note is the identification of the Askey–Wilson algebra as a coideal subalgebra of \(U_q(\mathfrak{sl}(2))\) [17] and the use of the realization (2.1) to obtain the difference operator of the big \(q\)-Jacobi polynomials [18] as a generator in this embedding.

Returning to the factorization formula, since \(\gamma\) and \(\zeta\) only occur in the global factor, we may set \(\gamma = \zeta = 1\). Summing up we thus have:

\[
\mathcal{M}^{(a,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)} = \mathcal{L}_{q^2}^{(a,\beta,\gamma)} + (abcdq^{-2} - ab - cdq^{-2} + 1)\mathcal{I}
\]  

(2.33)

with

\[
\alpha = (a + b), \quad \beta = -ab, \\
\delta = q^{-1}(c + d), \quad \epsilon = -q^{-2}cd.
\]  

(2.34)

(2.35)

With the eigenvalues \(\lambda_n\) of \(\mathcal{L}_{q^2}^{(a,\beta,\gamma)}\) given by (2.11) with \(q\) replaced by \(q^2\), it is straightforward to see that the Askey–Wilson polynomials with base \(q^2\) correspondingly verify

\[
[\mathcal{M}^{(a,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)}] p_n(x; a, b, c, d|q^2) = \rho_n \ p_n(x; a, b, c, d|q^2)
\]

(2.36)

with

\[
\rho_n = q^{-2n}(1 - abq^{2n})(1 - cdq^{2n-2}).
\]  

(2.37)

**Remark 2.2.** If we were to consider two linear combinations \(\mathcal{M}^{(a,\beta,\gamma)}\) and \(\mathcal{M}^{(\delta,\epsilon,\zeta)}\) of \(Y, U\) and \(V\) where the roles of the pairs of parameters \((a, b)\) and \((c, d)\) are exchanged with respect to \(\mathcal{M}^{(a,\beta,\gamma)}\) and \(\mathcal{M}^{(\delta,\epsilon,\zeta)}\), namely if we were to take

\[
\tilde{a} = q^{-1}(a + b) = q^{-1}\alpha, \quad \tilde{\beta} = -q^{-2}ab = q^{-2}\beta, \\
\tilde{\delta} = (c + d) = q\delta, \quad \tilde{\epsilon} = -cd = q^2\epsilon,
\]

(2.38)

(2.39)

we would obtain again a factorization of the Askey–Wilson operator \(\mathcal{L}_{q^2}^{(a,\beta,\gamma)}\) of similar form

\[
\mathcal{M}^{(\delta,\epsilon,\zeta)}\mathcal{M}^{(\tilde{a},\tilde{\beta},\tilde{\gamma})} = \mathcal{L}_{q^2}^{(a,\beta,\gamma)} + (abcdq^{-2} - abq^{-2} - cd + 1)\mathcal{I}
\]

(2.40)

This is because \(\mathcal{L}_{q^2}^{(a,\beta,\gamma)}\) is invariant under the permutation of the parameters and the exchanges of the operators \(\mathcal{M}\) with the \(q\)-shifts of the parameters given in (2.38) and in (2.39) simply
amount to permuting the pairs \((a, b)\) and \((c, d)\) in the constant term of the rhs of (2.33). This is in line with the fact that

\[
\mathcal{M}^{(\alpha, \beta, 1)} \mathcal{M}^{(\delta, \epsilon, 1)} - \mathcal{M}^{(\delta, \epsilon, 1)} \mathcal{M}^{(\alpha, \beta, 1)} = (-ab + cd)(1 - q^{-2})\mathcal{I}
\]

as is easily checked using the relations (2.6) as well as (2.47). Conversely,

\[
\mathcal{M}^{(\alpha, \beta, 1)} \mathcal{M}^{(\delta, \epsilon, 1)} - \mathcal{M}^{(\delta, \epsilon, 1)} \mathcal{M}^{(\alpha, \beta, 1)} = (\beta - q^2\epsilon)(1 - q^{-2})\Omega^{(2)}
\]

with the \(\mathcal{M}\)s taken as the linear combinations of the generators, can be seen to package (abstractly) the relations between \(y, a\) and \(v\) (2.4) with parameters (2.7).

2.4. The four-dimensional degenerate Sklyanin algebra \(\mathfrak{ska}_4\)

The four-dimensional degenerate Sklyanin algebra \(\mathfrak{ska}_4\) was obtained in [1] as a limit of the elliptic algebra originally introduced by Sklyanin [10] (see [19] for a mathematically oriented review). It is presented in terms of four generators \(A, B, C, D\) obeying the following homogeneous quadratic relations:

\[
DC = qCD, \quad CA = qAC, \quad [A, D] = \frac{(q - q^{-1})^3}{4} C^2, \quad [B, C] = \frac{A^2 - D^2}{q - q^{-1}},
\]

\[
AB - qBA = qDB - BD = -\frac{q^2 - q^{-2}}{4}(DC - CA).
\]

This algebra possesses two Casimir elements:

\[
\Omega_0 = AD + \frac{(q - q^{-1})^2}{4q} C^2, \quad \Omega_1 = \frac{q^{-1}A^2 + qD^2}{(q - q^{-1})^2} + BC + \frac{q + q^{-1}}{4} C^2.
\]

We note that the subalgebra generated by \(\{A, C, D\}\) is isomorphic to \(\mathfrak{su}_3\).

It was observed [1] that the degenerate Sklyanin algebra contracts to \(\mathcal{U}_0(\mathfrak{su}(2))\); indeed, if one sets \(A = \epsilon\hat{A}, B = \hat{B}, C = \epsilon^2\hat{C}, D = \epsilon\hat{D}\) and let \(\epsilon\) go to zero we see that the relations (2.43) reduce to (2.1). In keeping with the representation theory [10] of the Sklyanin algebra, the finite dimensional representations of its degenerate version are characterized by an integer or half-integer \(\nu\) and are of dimension \((2\nu + 1)\). We know from [1] that these can be realized by associating \(A, B, C, D\) to the following \(q\)-difference operators (we shall not distinguish here the abstract algebra element from its realization):

\[
A = q^{-\nu}U, \quad C = \frac{2}{(q - q^{-1})} Y, \quad D = q^\nu V,
\]

\[
B = \frac{1}{2(q - q^{-1})(1 - z^2)} \times \left[ q^{2\nu}(z^2T_+ - z^{-2}T_-) - q^{-2\nu}(z^2T_+ - z^{-2}T_-) - (q + q^{-1})(T_+ - T_-) \right],
\]

where \(U, V, Y\) are as in (2.5). In this realization the Casimir elements \(\Omega_0\) and \(\Omega_1\) take the following values:

\[
\Omega_0 = 1, \quad \Omega_1 = \frac{q^{2\nu+1} + q^{-2\nu-1}}{(q - q^{-1})^2}.
\]
In light of (2.44) and (2.45), the former relation restates the already observed fact that $UV$ is related to $Y^2$, namely that

$$UV = 1 - q^{-1}Y^2. \quad (2.47)$$

In the realization (2.45), the contraction from the degenerate Sklyanin algebra to $U_q(\mathfrak{su}(2))$ amounts to taking $z$ very large. It is quickly seen that in this limit the divided difference operators $\{A, B, C, D\}$ given above reduce to the $\{A, B, C, D\}$ of (2.2).

Now using the variable $x = z + z^{-1}$, it is readily found that $B$ can be expressed as

$$B = \frac{1}{2(q-q^{-1})} \left[q^{-2\nu}(q^{-1}xU - Ux) + q^{2\nu}(qxV - Vx) - (q + q^{-1})Y\right] \quad (2.48)$$

in terms of the operators (2.5) realizing $\mathfrak{sl}_3$. We shall now indicate how $x$ can be expressed as a formal power series in terms of $A, B, C, D$ by inverting (2.48). In light of the commutation relation $[U, V] = (q - q^{-1})Y^2$ given in (2.5) and the Casimir relation (2.47) we have

$$UV = 1 - q^{-1}Y^2 \quad \text{and} \quad VU = 1 - qY^2. \quad (2.49)$$

It follows that $V$ has an inverse $V^{-1}$ given by the formal power series in $Y$ expressed as follows:

$$V^{-1} = U(1 - qY^2)^{-1} = (1 - q^{-1}Y^2)^{-1}U. \quad (2.50)$$

Using the relations $UX - qxU = -(q - q^{-1})Y$ and $xV - qVx = q(q - q^{-1})Y$, we arrive at

$$x = q^{-2\nu} \left[2B + \left(\frac{q + q^{-1}}{q - q^{-1}} + q^{2\nu} - q^{-2\nu}\right)Y\right] (1 - q^{-4\nu}V^{-1}U)^{-1}V^{-1}. \quad (2.51)$$

As indicated before the Askey–Wilson operator $L^{(a,b,c,d)}_{q^2}$ and $x$ generate the Askey–Wilson algebra. Within the realization in terms of divided difference operators, we saw that $L^{(a,b,c,d)}_{q^2}$ according to (2.33) is obtained as a quadratic expression in the generators of the subalgebra $\mathfrak{sl}_3$ of $\mathfrak{sl}_4$ and just found as per (2.51) that $x$ is in the completion of the latter algebra. We can therefore assert that the Askey–Wilson algebra can be formally embedded in this realization of $\mathfrak{sl}_4$.

3. Contiguity operators of the Askey–Wilson polynomials and the degenerate Sklyanin algebra

In [3], Kalnins and Miller presented an elegant derivation of the weight function of the Askey–Wilson polynomials which is based on symmetry techniques. We here wish to point out that their approach can actually be cast in the framework of degenerate Sklyanin algebras. Central to the treatment in [3] are certain contiguity and ladder operators that will prove familiar. In order to facilitate comparison with the original reference we shall adopt essentially the same notation; we shall however use $q^2$ as the base.

Kalnins and Miller begin their considerations by observing that the Askey–Wilson polynomials satisfy the following contiguity relation

$$\mu^{(a,b,c,d)}_{ab, cd, aq, bq, cq, dq}(x; a, b, c, d|q^2) = q^{-n}(1 - abq^{2n-2})p_n(x; aq^{-1}, bq^{-1}, cq, dq|q^2) \quad (3.1)$$
if \( \mu^{(a,b,c,d)} \) is the following operator:

\[
\mu^{(a,b,c,d)} = \frac{1}{(z - \bar{z}^{-1})} \left( -z^{-1}(1 - a\bar{z}^{-1})z(1 - bq^{-1})T_+ + z(1 - a\bar{z}^{-1})z(1 - bq^{-1})T_- \right). \tag{3.2}
\]

It is further observed that

\[
\mu^{(cq, dq, aq^{-1}, bq^{-1})} p_n(x; aq^{-1}, bq^{-1}, cq, dq|q^2) = q^{-n}(1 - cdq^{2n})p_n(x; a, b, c, d|q^2). \tag{3.3}
\]

We may proceed from here to derive the weight function by requesting that it be such that \( \mu^{(cq, dq, aq^{-1}, bq^{-1}) \mu^{(a,b,c,d)}} \) is the formal adjoint of \( \mu^{(a,b,c,d)} \); this is done in [3]. Let us focus on the fact that in view of (3.1) and (3.3), the Askey–Wilson polynomials are eigenfunctions of \( \mu^{(cq, dq, aq^{-1}, bq^{-1}) \mu^{(a,b,c,d)}} \), namely,

\[
\left[ \mu^{(cq, dq, aq^{-1}, bq^{-1}) \mu^{(a,b,c,d)}} \right] p_n(x; a, b, c, d|q^2) = \tilde{\rho}_n p_n(x; a, b, c, d|q^2), \tag{3.4}
\]

with

\[
\tilde{\rho}_n = q^{-2n}(1 - cdq^{2n})(1 - abq^{2n-2}). \tag{3.5}
\]

Not surprisingly the factorization of the Askey–Wilson operator that this eigenvalue equation entails will coincide with the one described in the preceding section. This is readily established by recognizing that

\[
\mu^{(a,b,c,d)} = (aq^{-1} + bq^{-1})Y = abq^{-2}U + V = \mathcal{M}^{(\alpha, \beta, 1)},
\]

\[
\mu^{(cq, dq, aq^{-1}, bq^{-1})} = (c + d)Y - cdU + V = \mathcal{M}^{(\delta, \epsilon, 1)}. \tag{3.6}
\]

These contiguity operators are thus found to belong to the realization (2.5) of the Sklyanin algebra \( \mathfrak{a} \mathfrak{f} \mathfrak{a} \mathfrak{s} \mathfrak{a} \mathfrak{s} \) and we see that

\[
\mu^{(cq, dq, aq^{-1}, bq^{-1}) \mu^{(a,b,c,d)}} = \mathcal{M}^{(\delta, \epsilon, 1)} \mathcal{M}^{(\alpha, \beta, 1)}, \tag{3.7}
\]

with the connection with the Askey–Wilson operator provided by (2.40); we note moreover that the eigenvalue \( \tilde{\rho}_n \) in (3.5) coincides with the expression obtained from (2.37) under the exchange \((a, b) \leftrightarrow (c, d)\).

Kalnins and Miller consider in addition the lowering operator \( \tau^{(a,b,c,d)} \),

\[
\tau^{(a,b,c,d)} = \frac{1}{z - \bar{z}^{-1}(T_+ - T_-)} = Y, \tag{3.8}
\]

which is nothing else than our operator \( Y \) (or \( C \)). They proceed to find its adjoint \( \tau^{(a,b,c,d)*} \) which reads:

\[
\tau^{(a,b,c,d)*} = \frac{q^{-1}}{z - \bar{z}^{-1}} \left[ (1 - az)(1 - bz)(1 - cz)(1 - dq)T_+ - (1 - az^{-1})(1 - bz^{-1})(1 - cz^{-1})(1 - dq)T_- \right], \tag{3.9}
\]
These operators act as follows on the Askey–Wilson polynomials:

\[
\tau^{(a,b,c,d)}_{-n} p_n(x; a, b, c, d|q^2) = q^n(1 - q^{-2})p_n-1(x; aq, bq, cq, dq|q^2),
\]

\[
\tau^{(a,b,c,d)}_+ p_{n-1}(x; aq, bq, cq, dq|q^2) = -q^{-n}p_n(x; a, b, c, d|q^2).
\]

(3.10)

The key point is that \(\tau^{(a,b,c,d)}\) can be expressed as a linear combination of the generators \(A, B, C\) and \(D\) of the degenerate Sklyanin algebra \(\mathfrak{s}ta\). Let \(e_1 = (a + b + c + d)\), \(e_2 = (ab + ac + ad + bc + bd + cd)\), \(e_3 = abc + abd + acd + bcd\) and \(e_4 = abcd\) be the elementary symmetric functions in the parameters \((a, b, c, d)\), one finds indeed that

\[
\tau^{(a,b,c,d)} = q^{-1}[e_3(e_2)^{-\frac{1}{2}} A - 2(q - q^{-1})(e_4)^\frac{1}{2} B
+ (q - q^{-1})[e_2 - (q + q^{-1})(e_3)^\frac{1}{2}]C + e_1(e_3)^\frac{1}{2} D]
\]

(3.11)

with

\(q^{-\nu} = (abcd)^\frac{1}{2}\).  

(3.12)

We thus observe that the contiguity and raising operators \(\mu^{(a,b,c,d)}\), \(\tau^{(a,b,c,d)}\) and \(\tau^{(a,b,c,d)}\) belong to the realization of the degenerate Sklyanin algebra which is hence represented on the Askey–Wilson polynomials. In general \(\nu\) as given by the relation (3.12) above will not be an integer or half integer and the corresponding representation extends the finite-dimensional one discussed in section 3 to an infinite-dimensional one.

**Proposition 3.1.** The operator \(\mu^{(a,b,c,d)}\), its adjoint \(\mu^{(aq,bq,ac,bc,cd)}\), \(\tau^{(a,b,c,d)}\) and \(\tau^{(a,b,c,d)}\) form a basis equivalent to the set \(\{A, B, C, D\}\) as a representation of the degenerate Sklyanin algebra \(\mathfrak{s}ta\). Their action on the Askey–Wilson polynomials \(p_n(x; a, b, c, d|q^2)\) is provided by (3.1), (3.3) and (3.10) respectively. The connection formula [20, 21] of Askey and Wilson can be used to express these formulae as combinations of polynomials \(p_n(x; a, b, c, d|q^2)\), \(k = 0, 1, \ldots\), with parameters \(a, b, c, d\) fixed, that span the representation space.

Imposing that the representation be finite-dimensional amounts to enforcing the non-conventional truncation condition

\[
(q^2)^{N+1} = abcd, \quad N = 2\nu + 1
\]

(3.13)

for the Askey–Wilson polynomials with base \(q^2\). Quite strikingly this leads to polynomials called \(q\)-para Racah polynomials that have been recently characterized [8] and which are in particular orthogonal on a bilattice composed of two Askey–Wilson grids. We wish to stress this result.

**Proposition 3.2.** The \(q\)-para Racah polynomials with base \(q^2\) realize a basis for a representation of the degenerate Sklyanin algebra of dimension \(N = 2\nu + 1\) with \(\nu\) integer or half-integer.

**Remark 3.1.** Remarkably the operator \(\tau^{(a,b,c,d)}\) also features centrally in Koornwinder’s study [5] of the structure relations of the Askey–Wilson polynomials. These relations amount to raising and lowering relations where in contradistinction with the shift relations that we considered above (following Kalnins and Miller), the parameters are not affected. It is shown in [5] that such a structure relation is obtained when \(\tau^{(a,b,c,d)}\) (denoted by \(L\) in [5] with the factor \(q^{-1}\) omitted) acts upon the Askey–Wilson polynomial \(p_n(x; a, b, c, d|q)\) with base \(q\).
We show below how (3.14) encapsulates the relations (2.43) of J. Phys. A: Math. Theor. It is further recognized in [5] on the basis of results of Rains [22] and Rosen gren [23] that the operator \( \tau^{(a,b,c,d)} \) generates a representation of the degenerate Sklyanin algebra \( \mathfrak{sl}_4 \). This is ascertained from the relation

\[
\tau^{(a,b,c,d)} \tau^{(q,aqb,q^{-1},c,q^{-1}d)} = \tau^{(q,aqb,q^{-1},c,q^{-1}d)} \tau^{(a,b,c,d)}
\]

(3.14)
given in [5] and easily checked from (3.9). As observed by Koornwinder [5], it is the trigonometric specialization of a formula in [22] giving the defining relations of the Sklyanin algebra. We show below how (3.14) encapsulates the relations (2.43) of \( \mathfrak{sl}_4 \).

Consider the expression (3.11) for \( \tau^{(a,b,c,d)} \) as a linear combination of the operators \( A, B, C, D \). Substituting (3.11) in (3.14) and multiplying by \( (e_4)^{\frac{1}{2}}(1 - e^{-1})(ce - d)^{-1} \), one arrives at

\[
0 = (e_4)^{\frac{1}{2}}(q - q^{-1})(a + b) \left[ A^2 - D^2 - (q - q^{-1})(BC - CB) \right]
- 2(e_4)^{\frac{1}{2}}(q - q^{-1})ab(AB - qBA)
- 2(e_4)(q - q^{-1})q^{-1}(BD - qDB)
+ (e_4)^{\frac{1}{2}}(a + b)(qab - q^{-1}cd) \left[ (AD - DA) - \frac{1}{4}(q - q^{-1})^3 C^2 \right]
- (e_4)^{\frac{1}{2}}(a + b)^2 q^2 - ab - cd) q^{-1} + (e_4)^{\frac{1}{2}}(1 + q^{-2}) CD
+ (e_4)^{\frac{1}{2}}(a + b)^2 q^2 - abq - cd) q^{-2} + (e_4)^{\frac{1}{2}}(1 + q^{-2}) DC
- \frac{1}{2}(q - q^{-1}) \left( ab(q + q^{-1})(e_4)^{\frac{1}{2}} + [e_4(2 - q^{-2}) + (b^2 cd + a^2 cd - a^2 b^2 q^2)] \right) AC
- \frac{1}{2}(q - q^{-1}) \left( -abq(q + q^{-1})(e_4)^{\frac{1}{2}} - q^{-1}[e_4(2 - q^2) + (b^2 cd + a^2 cd - a^2 b^2 q^2)] \right) CA.
\]

(3.15)

We shall illustrate how the defining relations of \( \mathfrak{sl}_4 \) can be obtained from (3.15). First choose \( b = -a \) and \( c = 0 \). The equality (3.15) implies

\[
CA = qAC.
\]

(3.16)

Substituting this back in (3.15) and multiplying by \( (e_4)^{-\frac{1}{2}} \) yields

\[
0 = (e_4)^{\frac{1}{2}}(q - q^{-1})(a + b) \left[ A^2 - D^2 - (q - q^{-1})(BC - CB) \right]
- 2(e_4)^{\frac{1}{2}}(q - q^{-1})ab(AB - qBA)
+ (a + b)(qab - q^{-1}cd) \left[ (AD - DA) - \frac{1}{4}(q - q^{-1})^3 C^2 \right]
- 2(e_4)^{\frac{1}{2}}(q - q^{-1})q^{-1}(BD - qDB)
\]
\[
- (e_4)^{(q-q^{-1})} \left( ((a + b)^2 q^2 - ab - cd)q^{-1} + (e_4)^{(1 + q^{-2})} \right) CD \\
+ (e_4)^{(q-q^{-1})} \left( ((a + b)^2 q^2 - abq^{-4} - cd)q^{-2} + (e_4)^{(1 + q^{-2})} q \right) DC \\
+ \frac{(q-q^{-1})(q^2 - q^{-2})}{2} ((e_4)^{(q)} ab - q^{-1}(e_4)^{(q^2)}) CA.
\]

(3.17)

Once again, choose \( c = 0 \) for instance. The equality (3.17) implies

\[
AD - DA = \frac{(q-q^{-1})^2}{4} C^2.
\]

(3.18)

Repeating the same kind of argument, one obtains the other relations (2.43) that define \( \mathfrak{sl}_4 \).

Through the realization that we have considered here, we have observed so far that the degenerate Sklyanin algebra \( \mathfrak{sl}_4 \) is a basic structure underneath the theory of Askey–Wilson polynomials. Much like a supersymmetric Hamiltonian is the ‘square’ of supercharges, the Askey–Wilson operator is quadratic in generators realizing \( \mathfrak{sl}_4 \). We also saw that this is intimately connected to the application of Darboux transformations or of the factorization method [3, 4] to this operator. This approach as we know is based on the identification of raising operators. It has been realized recently that raising properties can provide a unifying principle in the theory of Heun operators [24]. We next take this angle to revisit the Heun–Askey–Wilson operator [6] and sort out the place occupied by the degenerate Sklyanin algebra in this Heun operator picture.

4. S-Heun operators and the Heun–Askey–Wilson operator

The standard Heun operator that defines the ordinary second order differential equation with four regular singularities [25] has the property of raising the degree of polynomials by one. It can also be obtained as a bilinear expression in the bispectral operators of the Jacobi polynomials, namely, multiplication by the variable and the hypergeometric operator [26]. Both viewpoints have been built upon to develop a broad perspective on operators of Heun type and the algebras they realize. The tridiagonalization method based on the hypergeometric operator has been generalized to any bispectral situation and the concept of algebraic Heun operator [27] has emerged in this fashion. In a nutshell this construct amounts to forming the generic bilinear expression in the bispectral operators. The raising property has been used to arrive at Heun operators defined on various lattices. In summary, one looks in this case for the most general second-order operator that raises by one the degree of polynomials on specified grids. Applied to the Askey–Wilson lattice or polynomials, both approaches have led equivalently to the Heun–Askey–Wilson operator [6]. (The Heun–Racah and Heun–Bannai–Ito operators have similarly been obtained [28].)

Let us mention that the Heun–Askey–Wilson operator has been shown [7] to arise as a degeneration of the one-variable Ruijsenaars–van Diejen Hamiltonian [29–31]. It has also been found that this operator can be diagonalized with the help of the algebraic Bethe ansatz [9]. We shall expand this by relating here the Heun–Askey–Wilson operators to our observations on Sklyanin algebras. To that end, we shall first focus on determining the most general first order operators acting on the Askey–Wilson grid that raise the degree of polynomials by one. We shall call them special Heun operators or S-Heun operators for short. These can be viewed as second order operators without diagonal terms. Indeed if the operator (4.5) given below is multiplied by \( T_+ \), we readily see that it takes the form of a first order operator \( A_1(z)T_2^2 + A_2(z) \)
on a grid with base \( q^2 \). Looking for \( S \)-Heun operators is in fact a more basic problem than searching for the generic second order operator with the raising property as a way of arriving directly at the Heun operator of Askey–Wilson type. It is hence not surprising that there will be factorization connotations. This undertaking will reveal that the \( S \)-Heun operators form a five-dimensional space that includes the operators \((A, B, C, D)\) realizing \( sl_2 \). We shall further observe that the Heun–Askey–Wilson operator has a quadratic expression in terms of these \( S \)-Heun operators.

4.1. The \( S \)-Heun operators

Before we apply the raising condition to determine the \( S \)-Heun operators that act through \( q \)-differences on the symmetric variable \( x = z + z^{-1} \), for reference, let us first go over the most simple case of first order differential operators that raise by one the degree of polynomials in the variable \( z \). Consider the operator

\[
S = F(z) \frac{d}{dz} + G(z) \tag{4.1}
\]

and demand that \( Sp_n(z) = \tilde{p}_{n+1}(z) \) with \( p_n \) and \( \tilde{p}_n \) polynomials of degree \( n \). It is readily seen that the most general admissible functions \( F(z) \) and \( G(z) \) are

\[
F(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2, \quad G(z) = \beta_0 + \beta_1 z. \tag{4.2}
\]

\( S \) therefore belongs to a five-dimensional vector space with the following natural basis

\[
L = \frac{d}{dz}, \quad M_1 = 1, \quad M_2 = z \frac{d}{dz}, \quad R_1 = z, \quad R_2 = z^2 \frac{d}{dz}, \tag{4.3}
\]

obtained by setting all coefficients \( \alpha_i \) and \( \beta_i \) equal to zero except for one.

These operators can be combined to form the usual finite-dimensional differential realization of dimension \( 2j + 1 \) on monomials \( z^n, n = 0, 1, \ldots \) of the Lie algebra \( sl_2 \), i.e.:

\[
J_0 = z \frac{d}{dz} = j = M_2 - jM_1, \quad J_+ = z^2 \frac{d}{dz} - 2jz = R_2 - 2jR_1, \quad J_- = \frac{d}{dz} = L. \tag{4.4}
\]

This corresponds to the \( q \to 1 \) limit of the realization of \( U_q(sl_2) \) given in section 2.

Consider now the \( q \)-difference operator

\[
S = A_1(z) T_+ + A_2(z) T_- \tag{4.5}
\]

where \( A_{1,2}(z) \) are functions of \( z \). Note that these \( S \)-Heun operators can be viewed as ‘square roots’ of the general (second order) Heun operators used in [6]. Impose again a raising condition on polynomials \( P_n(x(z)) \) of degree \( n \) in \( x(z) = z + z^{-1} \):

\[
SP_n(x(z)) = \tilde{P}_{n+1}(x(z)) \tag{4.6}
\]

for all \( n = 0, 1, 2, \ldots \).

It is sufficient to check property (4.6) for the elementary Askey–Wilson monomials

\[
\chi_n(z) = z^n + z^{-n}. \tag{4.7}
\]
that is to verify that

\[ S \chi_n(z) = \sum_{k=0}^{n+1} a_{nk} \chi_k(z) \quad (4.8) \]

for some coefficients \(a_{nk}\). Let us look at the action of \(S\) on the two Askey–Wilson monomials \(\chi_n(x)\) of lowest degrees. For \(n = 0\), the raising condition reads

\[ A_1(z) + A_2(z) = a_{00} + a_{10} \chi_1(z) \quad (4.9) \]

and similarly, for \(n = 1\) we have

\[ A_1(z)(zq + z^{-1}q^{-1}) + A_2(z)(zq^{-1} + z^{-1}q) = a_{10} + a_{11} \chi_1(z) + a_{12} \chi_2(z) \quad (4.10) \]

where \(a_{00}, a_{01}, a_{10}, a_{11}, a_{12}\) are arbitrary parameters. Evaluating the action of \(S\) on the higher degree Askey–Wilson monomials does not give rise to new parameters: the higher coefficients \(a_{nk}\) with \(n \geq 2\) are always expressed in terms of the \(a_{0k}\) and \(a_{1k}\). Hence these five parameters account for all the degrees of freedom that the most general \(S\)-Heun operator defined on the Askey–Wilson grid possesses.

Combining (4.9) and (4.10), we find for \(A_1(z)\)

\[ A_1(z) = \frac{\pi_4(z)}{z(1 - z^2)(1 - q^2)} \quad (4.11) \]

where \(\pi_4(z)\) is a polynomial of degree four:

\[ \pi_4(z) = (a_{12}q - a_{01})z^4 + (qa_{11} - a_{00})z^3 - ((1 + q^2)a_{01} - qa_{10})z^2 + q(a_{11} - qa_{00})z + qa_{12} - qa_{01}. \quad (4.12) \]

From the observation that both the lhs and rhs of the system (4.9) and (4.10) are invariant under \(z \to z^{-1}\), it follows that

\[ A_2(z) = A_1(z^{-1}). \quad (4.13) \]

This leads to the following proposition.

**Proposition 4.1.** The most general \(S\)-Heun operators on the Askey–Wilson grid which are required by definition to be of the form (4.5) and to raise by one the degrees of polynomials in \(x = z + z^{-1}\) are specified by the functions \(A_{1,2}(z)\) given in (4.11)–(4.13).

As the operator \(S\) depends on five free parameters, it gives rise as in the differential case to a five-dimensional linear space of \(S\)-Heun operators. A natural basis for this space is formed by three sets which correspond respectively to lowering, stabilizing and raising operators:

(a) Taking \(a_{10} = 1\) as the only non-zero parameter in (4.12) leads to the operator denoted \(L\) which decreases the degree of any polynomial in \(x(z)\) by 1 and changes its parity.

(b) Taking either \(a_{00} = 1\) or \(a_{11} = 1\) as the only non-zero parameter, one obtains stabilizing operators, denoted either \(M_1\) or \(M_2\). Both preserve the degree as well as the parity of any polynomial in \(x(z)\).

(c) The choice \(a_{01} = 1\) and all other parameters equal to 0 leads to the raising operator \(R_1\), while the choice \(a_{01} = q, a_{12} = 1\) and all other parameters 0 yields the operator \(R_2\). Both increase by one the degree of any polynomial in \(x(z)\) and change parity.
For the sake of completeness, we give below the full expressions of these five operators

\[
L = \frac{1}{q - q^{-1}} \frac{1}{z - z^{-1}} (T_+ - T_-),
\]
\[
M_1 = \frac{1}{q - q^{-1}} \frac{1}{z - z^{-1}} ((qz + q^{-1}z^{-1})T_+ - (q^{-1}z + qz^{-1})T_-),
\]
\[
M_2 = \frac{1}{q - q^{-1}} \frac{1}{z - z^{-1}} (z + z^{-1})(T_+ - T_-),
\]
\[
R_1 = \frac{1}{q - q^{-1}} \frac{1}{z - z^{-1}} ((qz + q^{-1}z^{-1})T_- - (q^{-1}z + qz^{-1})T_+),
\]
\[
R_2 = \frac{1}{q - q^{-1}} \left( \frac{q^2}{z - z^{-1}} (z + z^{-1})(zT_- - z^{-1}T_+) - (zT_- + z^{-1}T_+) \right).
\]

(4.14)

**Proposition 4.2.** The operators \(L, M_1, M_2, R_1, R_2\) are linearly independent. They form a basis for the linear space of \(S\)-Heun operators.

Note that the 3 operators \(L, M_1, M_2\) span the three-dimensional subspace of all ‘stabilizing’ \(S\)-Heun operators. This means that any operator \(S = \alpha_0 L + \alpha_1 M_1 + \alpha_2 M_2\) preserves the degree of any polynomial in \(x(z)\), if at least one of \(\alpha_1, \alpha_2\) is nonzero. Comparing (2.5) and (4.14), it is immediate to see that

\[
Y = (q - q^{-1})L, \quad U = M_1 + qM_2, \quad V = M_1 + q^{-1}M_2,
\]

(4.15)

and that the operators \((L, M_1, M_2)\) equivalently realize \(\mathfrak{sl}_3\). We can thus rephrase as follows the observations of subsection 2.3 according to which the Askey–Wilson operator is given as a quadratic expression in the operators \((Y, U, V)\) representing \(\mathfrak{sl}_3\):

**Proposition 4.3.** The Askey–Wilson operator can be given as the most general quadratic combination of the \(S\)-Heun operators \(L, M_1, M_2\) that stabilize the degree of polynomials in \(x(z)\).

We know that the operators \(A, C, D\) in the realization (2.45) of \(\mathfrak{sl}_3\) are proportional to \(U, Y, V\) respectively. It is not difficult to see that \(B\) in that same realization can be given as the following combination of \(L, R_1, R_2\):

\[
B = \frac{(q + q^{-1})((q^{2\nu} - q^{2\nu}) - (q - q^{-1}))}{2(q - q^{-1})}L + \frac{q^{1-2\nu}}{2}R_1 + \frac{(q^{2\nu-1} - q^{1-2\nu})}{2(q - q^{-1})}R_2.
\]

(4.16)

We thus have:

**Proposition 4.4.** The realization (2.45) of \(\mathfrak{sl}_3\) is obtained from linear combinations of \(S\)-Heun operators on the Askey–Wilson grid.

In addition, the operator \(x\) can be constructed as a quadratic polynomial in the elementary \(S\)-Heun operators; we have indeed:

\[
x = \frac{1}{q^2 - q^{-2}} \left( (1 + q^{-4})(qM_2R_2 - R_2M_2) + 2q^{-3}(qM_1R_2 - R_2M_1) \right).
\]

(4.17)
It follows that the Askey–Wilson algebra can be realized by combining quadratically the five basic S-Heun operators.

4.2. Heun–Askey–Wilson and S-Heun operators

We shall now obtain a formula for the Heun–Askey–Wilson operator in terms of S-Heun operators.

Consider the most general quadratic combination of the operators $L$, $M_1$, $M_2$, $R_1$, $R_2$ that raises by one the degree of polynomials on the Askey–Wilson grid and on the other hand, as pointed out at the beginning of this section, this operator was identified in [6] on the one hand as the most general second order $q$-shift operator that raises by one the degree of polynomials on the Askey–Wilson grid and on the other hand, as the tridiagonalization of the Askey–Wilson operator as per the algebraic Heun construct. The presentation obtained here with the S-Heun operators as basic building blocks has the merit of providing, typically, a factorization of $Q_{\text{HAW}}$. Indeed it is seen that the Heun–Askey–Wilson operator can also be written generically in the form:

$$Q_{\text{HAW}} = \alpha_1 L^2 + \alpha_2 LM_2 + \alpha_3 M_1^2 + \alpha_4 M_1 M_2 + \alpha_5 M_2 L + \alpha_6 M_2^2 \quad + \beta_1 M_1 R_1 + \beta_2 R_1 M_1 + \beta_3 R_2 M_2$$

(4.18)

where the $\gamma_i$'s and $\delta_i$'s are arbitrary parameters. On functions $f(z)$ this operator takes the form:

$$Q_{\text{HAW}} f(z) = [A_1(z)T_1^2 + A_1(z^{-1})T_2^2 + A_0(z)I] f(z), \quad (4.19)$$

with

$$A_1(z) = \frac{Q_6(z)}{z(1-z^2)(1-q^{2z^3})}, \quad A_0(z) = -(A_1(z) + A_1(z^{-1})) + p_1(x), \quad (4.20)$$

where $Q_6(z)$ is a generic polynomial of degree 6 in $z$ and $p_1(x)$ is a generic polynomial of degree 1 in the variable $x = z + z^{-1}$. The exact parameters are expressible in terms of those of (4.18):

$$p_1(x) = \beta_2 x + \alpha_3, \quad Q_6(z) = \frac{1}{q^2(q - q^{-1})^2} \sum_{k=0}^{6} r_k z^k, \quad (4.21)$$

$$r_0 = \beta_2 q^4 - \beta_3 q^4 + \beta_1 q^3 + \beta_1 q^2, \quad r_1 = \alpha_3 q^4 - \alpha_4 q^3 + \alpha_6 q^2,$n

$$r_2 = -\beta_3 q^6 + \beta_3 q^5 + 2\beta_2 q^4 + (\alpha_5 + \beta_1)q^3 + (\alpha_2 + \beta_2 - \beta_3)q^2 + \beta_1 q,$n

$$r_3 = -\alpha_4 q^5 + (\alpha_3 + \alpha_6) q^4 + \alpha_1 q^3 + (\alpha_3 + \alpha_6) q^2 - \alpha_4 q,$n

$$r_4 = -\beta_3 q^6 + \beta_3 q^5 + (\alpha_2 + \beta_2 - \beta_3)q^4 + (\alpha_5 + \beta_1)q^3 + 2\beta_2 q^2 + \beta_1 q,$n

$$r_5 = \alpha_6 q^6 - \alpha_4 q^5 + \alpha_3 q^4,$n

$$r_6 = \beta_1 q^4 + \beta_2 q^2.$$n

This formula giving $Q_{\text{HAW}}$ as the most general quadratic combination in the S-Heun operators on the Askey–Wilson grid provides a different characterization of the Heun–Askey–Wilson operator. As pointed out at the beginning of this section, this operator was identified in [6] on the one hand as the most general second order $q$-shift operator that raises by one the degree of polynomials on the Askey–Wilson grid and on the other hand, as the tridiagonalization of the Askey–Wilson operator as per the algebraic Heun construct. The presentation obtained here with the S-Heun operators as basic building blocks has the merit of providing, typically, a factorization of $Q_{\text{HAW}}$. Indeed it is seen that the Heun–Askey–Wilson operator can also be written generically in the form:
\[ Q_{\text{HAW}} = (\xi_1 L + \xi_2 M_1 + \xi_3 M_2)(\eta_1 L + \eta_2 M_1 + \eta_3 M_2 + \eta_4 R_1 + \eta_5 R_2) + \kappa. \] (4.23)

This formula for \( Q_{\text{HAW}} \) should be compared with equation (2.30) that provides the factorization of the Askey–Wilson operator as the product of two \( \mathfrak{sl}_3 \) elements. It is hence manifest from (4.23) that \( Q_{\text{HAW}} \) reduces to the Askey–Wilson operator when \( g = h = 0 \).

5. Conclusion

To conclude, let us first summarize our observations and second offer a brief outlook.

We have considered realizations of the Sklyanin algebras \( \mathfrak{sl}_3 \) and \( \mathfrak{sl}_4 \) in terms of \( q \)-difference operators and we determined the first order operators of that type—the \( S \)-Heun operators—that are the basic constituents of the most general degree raising operator in that class. Within these realizations, our salient observations are:

- The Askey–Wilson operator factorizes as the product of two linear combinations of elements in \( \mathfrak{sl}_3 \);
- In analogy with the dynamical enlargement of a symmetry algebra with the inclusion of ladder operators, the contiguity and shift operators of the Askey–Wilson polynomials have been shown to generate a realization of the degenerate Sklyanin algebra \( \mathfrak{sl}_4 \) which formally includes a realization of the Askey–Wilson algebra.
- The \( q \)-para Racah polynomials (with base \( q^2 \)) have been identified as forming a basis for the finite-dimensional representations of the degenerate Sklyanin algebra \( \mathfrak{sl}_4 \).
- The set of \( S \)-Heun operators is five-dimensional and has a subset that realizes \( \mathfrak{sl}_4 \).
- The operator multiplication by \( x \) has a quadratic expression in terms of the \( S \)-Heun operators.
- The Heun–Askey–Wilson operator can also be written as a quadratic expression in the \( S \)-Heun operators.

With respect to these last two points, let us mention the following. We recall that the algebraic Heun construct gives the Heun–Askey–Wilson operator \( Q_{\text{HAW}} \) as a bilinear operator in \( x \) and the Askey–Wilson operator. In view of the first and next to last points, this implies that \( Q_{\text{HAW}} \) is quartic in the \( S \)-Heun operators, an expression that must be reducible to the quadratic formula (4.18) obtained here.

This study raises a number of questions. Let us mention two: (i) How does the examination of the \( S \)-Heun operators extend when the raising property is applied to rational functions as in [7]? (ii) What other algebraic structures akin to the degenerate Sklyanin algebras will emerge when the \( S \)-Heun operator approach is adapted to other lattices such as for example the quadratic one on which the Wilson polynomials are defined? We plan on addressing these and other related questions in the near future.

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Appendix A. Quadratic algebraic relations for $L, M_1, M_2, R_1, R_2$

The homogeneous quadratic algebraic relations between the five $S$-Heun operators $L, M_1, M_2, R_1, R_2$ are collected below:

\[ [M_1, M_2] = (q + q^{-1})^2 L^2, \tag{A.1} \]
\[ M_1 L - (q + q^{-1})LM_1 = LM_2, \tag{A.2} \]
\[ LM_1 + M_2 L = 0, \tag{A.3} \]
\[ M_1^2 + M_2^2 + (q + q^{-1})M_2 M_1 = 1, \tag{A.4} \]
\[ LR_1 = 1 - M_2^2, \tag{A.5} \]
\[ R_1 L = 1 - M_1^2, \tag{A.6} \]
\[ LR_2 = -2L^2 + q^{-1}M_2^2 + M_1 M_2 + q, \tag{A.7} \]
\[ R_2 L = -2L^2 + qM_2^2 + q^2 M_2 M_1, \tag{A.8} \]
\[ R_1 M_2 + M_1 R_1 = 0, \tag{A.9} \]
\[ M_1 R_2 + R_2 M_2 = 2(q + q^{-1})M_2 L - (q + q^{-1})^2 LM_2, \tag{A.10} \]
\[ qR_2 M_1 - M_1 R_1 = R_2 M_1 + (q^2 + q^{-2})LM_1, \tag{A.11} \]
\[ R_1 M_1 - (q + q^{-1})M_1 R_1 = M_2 R_1 - (q + q^{-1})(q - q^{-1})^2 M_2 L, \tag{A.12} \]
\[ M_2 R_2 - (q + q^{-1})R_2 M_2 = R_2 M_1 + 2(q + q^{-1})M_1 L - (2q^2 + 1 + q^4)LM_1, \tag{A.13} \]
\[ R_2^2 - qR_2 R_1 + q^{-1} R_1 R_2 = -2(q + q^{-1})^2 M_1 M_2 - (q + q^{-1})^3 M_2^2 \]
\[ + 2[(q^2 + q^{-2}) - (q^2 - q^{-2})^2]L^2, \tag{A.14} \]

These relations are checked directly from the expressions of the operators in (4.14). They provide the necessary reorderings to reexpress the most general quadratic combination of the five operators as in (4.18).

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