A REMARK ON THE MULTIPLIERS ON SPACES OF
WEAK PRODUCTS OF FUNCTIONS

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Abstract. If $\mathcal{H}$ denotes a Hilbert space of analytic functions on a region $\Omega \subseteq \mathbb{C}^d$, then the weak product is defined by

$$\mathcal{H} \odot \mathcal{H} = \left\{ h = \sum_{n=1}^{\infty} f_n g_n : \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{H}} \|g_n\|_{\mathcal{H}} < \infty \right\}.$$ We prove that if $\mathcal{H}$ is a first order holomorphic Besov Hilbert space on the unit ball of $\mathbb{C}^d$, then the multiplier algebras of $\mathcal{H}$ and of $\mathcal{H} \odot \mathcal{H}$ coincide.

1. Introduction

Let $d$ be a positive integer and let $R = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i}$ denote the radial derivative operator. For $s \in \mathbb{R}$ the holomorphic Besov space $B_s$ is defined to be the space of holomorphic functions $f$ on the unit ball $B_d$ of $\mathbb{C}^d$ such that for some nonnegative integer $k > s$

$$\|f\|_{k,s}^2 = \int_{B_d} |(I + R)^k f(z)|^2 (1 - |z|^2)^{2(k-s)-1} dV(z) < \infty.$$ Here $dV$ denotes Lebesgue measure on $B_d$. It is well-known that for any $f \in \text{Hol}(\mathbb{B}_d)$ and any $s \in \mathbb{R}$ the quantity $\|f\|_{k,s}$ is finite for some nonnegative integer $k > s$ if and only if it is finite for all nonnegative integers $k > s$, and that for each $k > s$ $\|\cdot\|_{k,s}$ defines a norm on $B_s$, and that all these norms are equivalent to one another, see [2]. For $s < 0$ one can take $k = 0$ and these spaces are weighted Bergman spaces. In particular, $B_{-1/2} = L_a^2(\mathbb{B}_d)$ is the unweighted Bergman space. For $s = 0$ one obtains the Hardy space of $\mathbb{B}_d$ and one has that for each $k \geq 1$ $\|f\|_{k,0}^2$ is equivalent to $\int_{\partial \mathbb{B}_d} |f|^2 d\sigma$, where $\sigma$ is the rotationally invariant probability measure on $\partial \mathbb{B}_d$. We also note that for $s = (d - 1)/2$ we

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have $B_s = H^2_d$, the Drury-Arveson space. If $d = 1$ and $s = 1/2$, then $B_s = D$, the classical Dirichlet space of the unit disc.

Let $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space such that $1 \in \mathcal{H}$. The weak product of $\mathcal{H}$ is denoted by $\mathcal{H} \odot \mathcal{H}$ and it is defined to be the collection of all functions $h \in \text{Hol}(\mathbb{B}_d)$ such that there are sequences $\{f_i\}_{i \geq 1}, \{g_i\}_{i \geq 1} \subseteq \mathcal{H}$ with $\sum_{i=1}^{\infty} \|f_i\|_H \|g_i\|_H < \infty$ and for all $z \in \mathbb{B}_d$, $h(z) = \sum_{i=1}^{\infty} f_i(z)g_i(z)$.

We define a norm on $\mathcal{H} \odot \mathcal{H}$ by

$$\|h\|_* = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_H \|g_i\|_H : h(z) = \sum_{i=1}^{\infty} f_i(z)g_i(z) \text{ for all } z \in \mathbb{B}_d \right\}.$$ 

In what appears below we will frequently take $\mathcal{H} = B_s$, and will use the same notation for this weak product.

Weak products have their origin in the work of Coifman, Rochberg, and Weiss [5]. In the frame work of the Hilbert space $\mathcal{H}$ one may consider the weak product to be an analogue of the Hardy $H^1$-space. For example, one has $H^2(\partial \mathbb{B}_d) \odot H^2(\partial \mathbb{B}_d) = H^1(\partial \mathbb{B}_d)$ and $L^2(\mathbb{B}_d) \odot L^2(\mathbb{B}_d) = L^1(\mathbb{B}_d)$, see [5]. For the Dirichlet space $D$ the weak product $D \odot D$ has recently been considered in [1], [4], [9], [6], and [7]. The space $H^2_d \odot H^2_d$ was used in [10]. For further motivation and general background on weak products we refer the reader to [1] and [9].

Let $\mathcal{B}$ be a Banach space of analytic functions on $\mathbb{B}_d$ such that point evaluations are continuous and such that $1 \in \mathcal{B}$. We use $M(\mathcal{B})$ to denote the multiplier algebra of $\mathcal{B}$,

$$M(\mathcal{B}) = \{ \varphi : \varphi f \in \mathcal{B} \text{ for all } f \in \mathcal{B} \}.$$ 

The multiplier norm $\|\varphi\|_M$ is defined to be the norm of the associated multiplication operator $M_\varphi : \mathcal{B} \rightarrow \mathcal{B}$. It is easy to check and is well-known that $M(\mathcal{B}) \subseteq H^\infty(\mathbb{B}_d)$, and that for $s \leq 0$ we have $M(B_s) = H^\infty(\mathbb{B}_d)$. For $s > d/2$ the space $B_s$ is an algebra [2], hence $B_s = M(B_s)$, but for $0 < s \leq d/2$ one has $M(B_s) \subseteq B_s \cap H^\infty(\partial \mathbb{B}_d)$. For those cases $M(B_s)$ has been described by a certain Carleson measure condition, see [3],[8].

It is easy to see that $M(\mathcal{H}) \subseteq M(\mathcal{H} \odot \mathcal{H}) \subseteq H^\infty$ (see Proposition 3.1). Thus, if $s \leq 0$, then $M(B_s) = M(B_s \odot B_s) = H^\infty$. Furthermore, if $s > d/2$, then $B_s = B_s \odot B_s = M(B_s)$ since $B_s$ is an algebra. This raises the question whether $M(B_s)$ and $M(B_s \odot B_s)$ always agree. We prove the following:

**Theorem 1.1.** Let $s \in \mathbb{R}$ and $d \in \mathbb{N}$. If $s \leq 1$ or $d \leq 2$, then $M(B_s) = M(B_s \odot B_s)$. 


Note that when \( d \leq 2 \), then \( B_s \) is an algebra for all \( s > 1 \). Thus for each \( d \in \mathbb{N} \) the nontrivial range of the Theorem is \( 0 < s \leq 1 \). If \( d = 1 \) then the theorem applies to the classical Dirichlet space of the unit disc and for \( d \leq 3 \) it applies to the Drury-Arveson space.

2. Preliminaries

For \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \) and \( t \in \mathbb{R} \) we write \( e^{it}z = (e^{it}z_1, \ldots, e^{it}z_d) \) and we write \( \langle z, w \rangle \) for the inner product in \( \mathbb{C}^d \). Furthermore, if \( h \) is a function on \( B_d \), then we define \( T_t \) by \((T_t f)(z) = f(e^{it}z)\). We say that a space \( \mathcal{H} \subseteq \text{Hol}(B_d) \) is radially symmetric, if each \( T_t \) acts isometrically on \( \mathcal{H} \) and if for all \( t_0 \in \mathbb{R}, T_t \to T_{t_0} \) in the strong operator topology as \( t \to t_0 \), i.e. if \( \|T_t f\|_\mathcal{H} = \|f\|_\mathcal{H} \) and \( \|T_t f - T_{t_0} f\|_\mathcal{H} \to 0 \) for all \( f \in \mathcal{H} \).

For example, for each \( s \in \mathbb{R} \) the holomorphic Besov space \( B_s \) is radially symmetric when equipped with any of the norms \( \| \cdot \|_{k,s}, k > s \).

It is elementary to verify the following lemma.

**Lemma 2.1.** If \( \mathcal{H} \subseteq \text{Hol}(B_d) \) is radially symmetric, then so is \( \mathcal{H} \odot \mathcal{H} \).

Note that if \( h \) and \( \varphi \) are functions on \( B_d \), then for every \( t \in \mathbb{R} \) we have \((T_t \varphi) h = T_t(\varphi T_{-t} h)\), hence if a space is radially symmetric, then \( T_t \) acts isometrically on the multiplier algebra. For \( 0 < r < 1 \) we write \( f_r(z) = f(rz) \).

**Lemma 2.2.** If \( \mathcal{H} \subseteq \text{Hol}(B_d) \) is radially symmetric, and if \( \varphi \in M(\mathcal{H} \odot \mathcal{H}) \), then for all \( 0 < r < 1 \) we have \( \|\varphi_r\|_{M(\mathcal{H} \odot \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})} \).

**Proof.** Let \( \varphi \in M(\mathcal{H} \odot \mathcal{H}) \) and \( h \in \mathcal{H} \odot \mathcal{H} \), then for \( 0 < r < 1 \) we have

\[
\varphi_r h = \int_{-\pi}^{\pi} \frac{1 - r^2}{|1 - re^{it}|^2} (T_t \varphi) h \frac{dt}{2\pi}.
\]

This implies

\[
\|\varphi_r h\| \leq \int_{-\pi}^{\pi} \frac{1 - r^2}{|1 - re^{it}|^2} \|T_t \varphi\| h \frac{dt}{2\pi} \leq \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})} \|h\|_\ast.
\]

Thus, \( \|\varphi_r\|_{M(\mathcal{H} \odot \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})} \).

3. Multipliers

The following Proposition is elementary.

**Proposition 3.1.** We have \( M(\mathcal{H}) \subseteq M(\mathcal{H} \odot \mathcal{H}) \subseteq \mathcal{H}^\infty \) and if \( \varphi \in M(\mathcal{H}) \), \( \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H})} \).

As explained in the Introduction, the following will establish Theorem 1.1.
Theorem 3.2. Let $0 < s \leq 1$. Then $M(B_s) = M(B_s \odot B_s)$ and there is a $C_s > 0$ such that

$$
\|\varphi\|_{M(B_s \odot B_s)} \leq \|\varphi\|_{M(B_s)} \leq C_s \|\varphi\|_{M(B_s \odot B_s)}
$$

for all $\varphi \in M(B_s)$.

Here for each $s$ we have the norm on $B_s$ to be $\| \cdot \|_{k,s}$, where $k$ is the smallest natural number $> s$.

Proof. We first do the case $0 < s < 1$. Then $k = 1$, and $\|f\|_{B_s}^2 = \int_{B_d} |(I + R)f(z)|^2dV_s(z)$, where $dV_s(z) = (1 - |z|^2)^{1-2s}dV(z)$. For later reference we note that a short calculation shows that $\int_{B_d} |Rf|^2dV_s \leq \|f\|_{B_s}^2$.

We write $\|R\varphi\|_{Ca(B_s)}$ for the Carleson measure norm of $|R\varphi|^2$, i.e.

$$
\|R\varphi\|_{Ca(B_s)}^2 = \inf \left\{ C > 0 : \int_{B_d} |f|^2|R\varphi|^2dV_s \leq C\|f\|_{B_s}^2 \text{ for all } f \in B_s \right\}.
$$

Since $\|\varphi f\|_{B_s}^2 = \int_{B_d} |\varphi(z)(I + R)f(z) + f(z)R\varphi(z)|^2dV_s(z)$ it is clear that $\|\varphi\|_{M(B_s)}$ is equivalent to $\|\varphi\|_\infty + \|R\varphi\|_{Ca(B_s)}$. Thus, it suffices to show that there is a $c > 0$ such that $\|R\varphi\|_{Ca(B_s)} \leq c\|\varphi\|_{M(B_s \odot B_s)}$ for all $\varphi \in M(B_s \odot B_s)$.

First we note that if $b$ is holomorphic in a neighborhood of $\overline{B_d}$ and $h = \sum_{i=1}^\infty f_i g_i \in B_s \odot B_s$, then

$$
\int_{B_d} |(Rh)Rb|dV_s \leq \sum_{i=1}^\infty \int_{B_d} |(Rf_i)g_i Rb|dV_s + \int_{B_d} |(Rg_i)f_i Rb|dV_s
$$

$$
\leq \sum_{i=1}^\infty \|f_i\|_{B_s} \left( \int_{B_d} |g_i Rb|^2dV_s \right)^{1/2} + \|g_i\|_{B_s} \left( \int_{B_d} |f_i Rb|^2dV_s \right)^{1/2}
$$

$$
\leq 2 \sum_{i=1}^\infty \|f_i\|_{B_s} \|g_i\|_{B_s} \|Rb\|_{Ca(B_s)}.
$$

Hence

$$
\int_{B_d} |(Rh)Rb|dV_s \leq 2\|h\|_* \|Rb\|_{Ca(B_s)},
$$

where we have continued to write $\| \cdot \|_*$ for $\| \cdot \|_{B_s \odot B_s}$. 
Let \( \varphi \in M(B_s \odot B_s) \) and let \( 0 < r < 1 \). Then for all \( f \in B_s \) we have \( f^2, \varphi_r f^2 \in B_s \odot B_s \), hence
\[
\int_{B_d} |f|^2 |R \varphi_r|^2 dV_s = \int_{B_d} |R(\varphi_r f^2) - \varphi_r R(f^2)| |R \varphi_r| dV_s
\leq 2(\| \varphi_r f^2 \|_* + \| \varphi \|_{\infty} \| f^2 \|_*) \| R \varphi_r \|_{Ca(B_s)}
\leq 2(\| \varphi \|_{M(B_s \odot B_s)} \| f^2 \|_* + \| \varphi \|_{\infty} \| f^2 \|_*) \| R \varphi_r \|_{Ca(B_s)}
\leq 4 \| \varphi \|_{M(B_s \odot B_s)} \| f \|_{B_s}^2 \| R \varphi_r \|_{Ca(B_s)}.
\]
Next we take the sup of the left hand side of this expression over all \( f \) with \( \| f \|_{B_s} = 1 \) and we obtain \( \| R \varphi_r \|_{Ca(B_s)}^2 \leq 4 \| \varphi \|_{M(B_s \odot B_s)} \| R \varphi_r \|_{Ca(B_s)} \) which implies that \( \| R \varphi_r \|_{Ca(B_s)} \leq 4 \| \varphi \|_{M(B_s \odot B_s)} \) holds for all \( 0 < r < 1 \).
Thus, for \( 0 < s < 1 \) the result follows from Fatou’s lemma as \( r \to 1 \).
If \( s = 1 \), then \( \| f \|_{2,1}^2 \sim \int_{\partial B_d} |(I + R)f(z)|^2 d\sigma(z) \) and the argument proceeds as above. \( \blacksquare \)
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