Climbing Algorithms (Invited Talk)

Leonid A. Levin
https://www.cs.bu.edu/fac/Lnd/
Boston University
Computer Science
Boston, MA, USA

ABSTRACT
NP (search) problems allow easy correctness tests for solutions. Climbing algorithms allow also easy assessment of how close to yielding the correct answer is the configuration at any stage of their run. This offers a great flexibility, as how sensible is any deviation from the standard procedures can be instantly assessed.

An example is the Dual Matrix Algorithm (DMA) for linear programming, variations of which were considered by A.Y. Levin in 1965 and by Yamnitsky and myself in 1982. It has little sensitivity to numerical errors and to the number of inequalities. It offers substantial flexibility and, thus, potential for further developments.

1 INTRODUCTION
A Climbing algorithm A(x) is supplied with an easily computable valuation V(s). A runs by iterating a step transformation T(s) starting from s ← x and ending with output y ← s once the required valuation V(s) = R(x) is reached. The efficiency of the algorithm reflects the effort required to compute T, V and the progress in V toward R(x) each iteration of T assures.

The following account is an example of such algorithm. It adds algebraic details to previously published geometric versions. Some of these details exploit the flexibility provided by its climbing nature to help with worst case-performance by deviating in appropriate cases from the standard procedure. The main point is to emphasize the flexibility assured by this climbing nature as an example that may be useful to follow for some other algorithms and problems.

1.1 The Idea
Notations. 0 indicates definitions. det(C) denotes the determinant of a matrix C; the Euclidean norm of a vector b is |b|. For clarity row vectors may be underlined: 1 = (1, . . . , 1), column overlined: 0 = (0, . . . , 0). I is the identity matrix, e_k, r_k are its k-th row and column. We are to solve a system of inequalities Ax ≥ 0 for a rational vector x, given an m×n integer matrix A. Our inequalities are a_k x ≥ 0 for a_k ≥ e_k A, linearly independent for k ≤ n. Let n and all entries of A be < l bits long, so each a_k has < L = nl bits. By Hahn-Banach Theorem, the system Ax > 0 is inconsistent iff bA = 0 for some vector b ≥ 0 ̸= b. The same holds if |bA| < b/l^4.

The DMA searches for b in the form DB, where matrix B has no negative entries, C ∥ BA = V−1, d ∥ uV = 1D > 0 for a diagonal D, u ∥ ∑k≤n a_k. We must grow bD to > l^4. This growth is hard to keep monotone, so a lower bound log(n! det(DC)) < n log(bD) + 3L is grown instead. It is − log of the volume of the simplex <B with faces Cx = 0, u∥x = 1, vertices V D−1r_k, 0, and center v = V D−1(1/n+1). The original simplex <B starts with B_k,k = 1, B_k,k′ = 0.

It turns out that by incrementing a single entry of B one can always increase In det(DC) by > 1/2n^2, as long as x = v fails the Ax > 0 requirement. This provides an O(n^3L) steps algorithm. Each step takes O(n^3) arithmetic operations, on O(L)-bit numbers and one call of a procedure which points an inequality a_k x ≥ 0 violated by a given solution candidate x. This call is the only operation that may depend on the number m of inequalities, which could even be an infinite family with an oracle providing the violated a_k.

1.2 Some Comparisons
The above bound has n times more steps than Ellipsoid Method (EM). However, the EM is much more demanding with respect to the precision with which the numbers are to be kept. The simplex <B cannot possibly fail to include all solutions of Ax > 0, ux < 1, whatever B with no negative entries is taken. In contrast, the faithful transformation of ellipsoids in the EM is the only guarantee that they include all solutions.

Also, for m = O(n) several Karmarkar-type algorithms have lower polynomial complexity bounds. Yet, they work in the dual space and their bounds are in terms of the number m of inequalities, while the above DMA bound is in terms of n. For DMA, m may even be infinite, e.g., forming a ball instead of a polyhedron. Then dual-space complexity bounds break down, while the DMA complexity is not affected (as long as a simple procedure finds a violated inequality for any candidate x).
To assure fast progress, numbers are kept with $O(L)$ digits. This bound cannot be improved since some consistent systems have no solutions with shorter entries. Yet, this or any other precision is not actually required by DMA. Any rounding (or, indeed, any other deviation from the procedure) can be made any time as long as log det$(DC)$ keeps growing, which is immediately observable. This leaves DMA open to a great variety of further developments. In contrast, an inappropriate rounding in the EM, can yield ellipsoids which, while still shrinking fast, lose any relation to the set of solutions and produce a false negative output.

1.3 A Historical Background

The bound det$(DC)$ is inversely proportional to the volume of $\mathbf{a}^B$, which parallels the EM. Interestingly, in history this parallel went which, while still shrinking fast, lose any relation to the set of solutions. Its center of gravity is checked, and, if it fails some inequality, the corresponding hyperplane cuts out a “half” of the simplex. The process repeats with the resulting polyhedron. Each cut decreases the volume by a constant factor and so, after some number $q(n)$ of steps the remaining body can be re-enclosed in a new smaller simplex. Only a weak upper bound $\eta(n) \approx n \log n$ was proven by A.Y. Levin; it did not preclude the simplex from turning into a too complex to manipulate in polynomial time polyhedron.

Nemirovsky and Yudin replaced simplices with ellipsoids and made $q(n) = 1$. Both they and A.Y. Levin used real numbers and looked for approximate solutions with a given accuracy. Khachian in 1979 [3] modified the EM for rationals and exact solutions. Yamnitsky and myself in 1982 [4] proved $q(n) = 1$ for the original A.Y. Levin’s simplex splitting method. Below, an algebraic version of that geometric algorithm and some implementation improvements are considered and analyzed.

2 THE MAIN ALGORITHM AND ANALYSIS

Let $d = \sum_{k \leq n} d_k$, $C = BA = V$, $d = uV$, $d_k = d_k B$, $c_k = d_k C$, $v_k = \sum_{k \leq n} d_k B$, for some $j, i, s$ let $a_i = u_i$, $v = v_j$, $t = (s^2 - 1)u_i$. Then $C' = (B + \frac{v_i}{d_j})L = (V')^{-1}$, $d_i = u_i V$, $d_k = d_k C$. With $\sigma = 1 + v_i u_j V$, $C' = \sigma C$, det$(\sigma) = (1 + u_i u_j) = 1 + 1/(s^2 - 1)$, $V' = V$, $d_i = d_i V'$. Thus, $d_k = d_k B$, $d_k C$, det$(\sigma) \approx (s^2 - 1)u_i$. Taking $k = j$, $\delta_j = 1 + 1/(s^2 - 1)$, $\delta_j = 1 + 1/(s^2 - 1)$, $\delta_j = 1 + 1/(s^2 - 1)$. Our gain is in $\lambda = \Pi_k \delta_k$ det$(C')/\Pi_k B$. So its volume is $\Pi_k \delta_k V'$. Note that $\eta(n) = ba$ for $b = \Pi_k \delta_k V'$. Its polyhedron $\delta_k V'$. Note that $\eta(n) = ba$ for $b = \Pi_k \delta_k V'$. Its polyhedron $\delta_k V'$.

Now $\delta_k = \sum_{k \leq n} v_k I$ with $a_i v_i \leq 0$, $a_i v_i = \max_k a_i v_k$. Then $d_k \leq \delta_j$, $\sum_{k \leq n} \delta_k = n - \delta_k \leq n$, and $\Pi_k \delta_k \geq \delta_k^{(n-2)}(\delta_j n + (1 - \delta_j)) = (1 - \delta_j)(n - (1 - \delta_j)) = (1 - \delta_j)(n - (1 - \delta_j)) = (1 - \delta_j)(n - (1 - \delta_j))$. So, $\Pi_k \delta_k \geq \delta_k^{(n-2)}(\delta_j n + (1 - \delta_j)) = (1 - \delta_j)(n - (1 - \delta_j)) = (1 - \delta_j)(n - (1 - \delta_j)) = (1 - \delta_j)(n - (1 - \delta_j))$. For $s = n - 1$ and $f(s) s = s f(s - 1)$ this is $f(s) - f(s - 1) > 1/2n^2$.

This > $1/2n^2$ gain holds if $s$ is accurate to $O(L)$ digits, so $t$ can be rounded to $O(l(n))$ significant digits, too.

3 SOME IMPROVEMENTS

Inverting matrices may take cubic time, but when a matrix with a known inverse is moderately modified, Sherman-Morrison formula gives its inverse in $O(n^2)$ steps. In our case the inverse of $C'$ is $V' = \omega - a u V$. Finally, the following occasional deviations from the standard step help the worst-case performance and also illustrate the potential allowed by the flexibility of the algorithm.

Digits. The nodes $v_k$ of the starting simplex $\mathbf{a}^B$ lie in a $4L$ ball. Rounding $t$, DB to $O(L)$ digits preserves the $> 1/2n^2$ gain in steps with max$_k log |v_k| < 4L$. Yet, at some steps a longest edge $(v_j, v_j)$ of $\mathbf{a}^B$ may grow up to $O(NL)$ long. But there would be only $O(n)$ such steps, since they allow large gains in log det$(DC)$ as follows.

Let $w = (v_j - v_j)^T$, $M = \max_k w_k^2$, $m = \max_k k w_k^2$, $t' = \omega - t'/\omega$, $t' = \max(0, t - 1)/\omega$. Let $h_{ij} = (Mu - \omega)$. $\mathbf{a}^B$ has a slice of its projection along $w$ and volume $p(w)/\omega$. Its slice of height $m/\omega = O(4^n)$ cut by $h_{ji} x \geq 0$, $h_{ji} x \leq 0$ encloses $\mathbf{a}^B$. Note that $h_{ji} = b$ for $b = \Pi_k h_{ij} V'$. Then repeat the following. Use $x \in f \in \mathcal{E}$ with maximal $b_{ij}$ to annul $b_{ii}$ via $b_{ii} \leftarrow b_{ii} - b_{ii} f$. This preserves $b_{ii}$, keeps $b_{ii} \geq 0$, and shrinks $S$. As $S$ loses an entry $i$ we use an $f \in \mathcal{E}$ to annul the $i$-th component in all other vectors in $F$ and drop $f$ from $F$.

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