ATOMIC DECOMPOSITIONS FOR HARDY SPACES RELATED TO SCHröDINGER OPERATORS

MARCIN PREISNER

ABSTRACT. Let $U = -\Delta + U$ be a Schrödinger operator on $\mathbb{R}^d$, where $U \in L^1_{loc}(\mathbb{R}^d)$ is a non-negative potential and $d \geq 3$. The Hardy space $H^1(U)$ is defined in terms of the maximal function for the semigroup $K_t^U = \exp(-tL_t^U)$, namely

$$H^1(U) = \left\{ f \in L^1(\mathbb{R}^d) : \| f \|_{H^1(U)} := \sup_{t > 0} \| K_t^U f \|_{L^1(\mathbb{R}^d)} < \infty. \right\}$$

Assume that $U = V + W$, where $V \geq 0$ satisfies the global Kato condition

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(y) |x-y|^{2-d} < \infty.$$ 

We prove that, under certain assumptions on $W \geq 0$, the space $H^1(U)$ admits an atomic decomposition of local type. An atom $a$ for $H^1(U)$ is either of the form $a(x) = |Q^{-1}1_Q(x)|$, where $Q$ are special cubes determined by $W$, or $a$ satisfies the cancellation condition $\int_{\mathbb{R}^d} a(x) w(x) dx = 0$, where $w$ is an $\omega(\Delta + V)$-harmonic function given by $w(x) = \lim_{r \to \infty} K_{r}^U 1(x)$. Furthermore, we show that, in some cases, the cancellation condition $\int_{\mathbb{R}^d} a(x) w(x) dx = 0$ can be replaced by the classical one $\int_{\mathbb{R}^d} a(x) dx = 0$. However, we construct another example, such that the atomic spaces with these two cancellation conditions are not equivalent as Banach spaces.

1. BACKGROUND AND STATEMENT OF RESULTS

1.1. Introduction. Let $U$ be a non-negative, locally integrable function on $\mathbb{R}^d$. In this article we consider the Schrödinger operator given by $-\Delta + U$, where $\Delta$ is the standard Laplacian on $\mathbb{R}^d$ and $U$ is called the potential. Throughout the whole paper we assume that $d \geq 3$.

To be more precise, let us recall what do we mean by the Schrödinger operator. First, define a quadratic form

$$Q^U(f, g) = \int_{\mathbb{R}^d} \nabla f(x) \overline{\nabla g(x)} dx + \int_{\mathbb{R}^d} U(x) f(x) g(x) dx$$

with the domain $\text{Dom}(Q^U) = \{ f \in L^2(\mathbb{R}^d) : \nabla f, \sqrt{U} f \in L^2(\mathbb{R}^d) \}$. This quadratic form is closed, thus it defines the self-adjoint operator $L^U : \text{Dom}(L^U) \to L^2(\mathbb{R}^d)$. In particular,

$$\text{Dom}(L^U) = \left\{ f \in \text{Dom}(Q^U) : \exists h \in L^2(\mathbb{R}^d) \forall g \in \text{Dom}(Q^U) \quad Q^U(f, g) = \int_{\mathbb{R}^d} h(x) g(x) dx \right\}$$

and $L^U f := h$, when $f$ and $h$ are as above. Formally, we write

$$L^U = -\Delta + U.$$ 

Let $(K_t^U)_{t>0}$ be the semigroup generated by $L^U$ on $L^2(\mathbb{R}^d)$. By the Feynman-Kac formula, $K_t^U$ has an integral kernel $K_t^U(x, y)$ satisfying upper-Gaussian bounds, i.e.

$$0 \leq K_t^U(x, y) \leq (4\pi t)^{-d/2} \exp \left( -\frac{|x-y|^2}{4t} \right) = P_t(x-y).$$

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The Hardy space $H^1(L^U)$ associated with $L^U$ is defined as follows. Let
\[
M^U f(x) = \sup_{r > 0} |K_t^U f(x)|
\]
be a maximal operator associated with $(K_t^U)_{t > 0}$. We say that a function $f \in L^1(\mathbb{R}^d)$ belongs to the maximal Hardy space $H^1(L^U)$, when
\[
\|f\|_{H^1(L^U)} := \|M^U f(x)\|_{L^1(\mathbb{R}^d)} < \infty.
\]

In the paper atomic Hardy spaces play a special role. The general definition is as follows. Assume that a family of functions $\mathcal{A} \subseteq L^1(\mathbb{R}^d)$ is given. A function $a \in \mathcal{A}$ will be called an atom and we assume that $\|a\|_{L^1(\mathbb{R}^d)} \leq 1$. We say that a function $f$ belongs to the atomic Hardy space $H^1_{at}(\mathcal{A})$, if
\[
f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),
\]
where $a_j \in \mathcal{A}$, $\lambda_j \in \mathbb{C}$, and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Whenever $f \in H^1_{at}(\mathcal{A})$ we set
\[
\|f\|_{H^1_{at}(\mathcal{A})} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f \text{ as in (1.3)} \right\}.
\]
It is not difficult to check that $H^1_{at}(\mathcal{A})$ is a Banach space and $H^1_{at}(\mathcal{A}) \subseteq L^1(\mathbb{R}^d)$.

In the classical theory of Hardy spaces an important result is the atomic decomposition theorem, see [1], [14]. It asserts that $H^1(-\Delta) = H^1_{at}(\mathcal{A}_{\text{class}})$ and the corresponding norms are equivalent. Here $\mathcal{A}_{\text{class}}$ is the set of classical atoms, that is $a \in \mathcal{A}_{\text{class}}$ if there exist a cube $Q$, such that $\text{supp } a \subseteq Q$ (localization condition), $|a|_{L^\infty} \leq |Q|^{-1}$ (size condition), and $\int_Q a(x) \, dx = 0$ (cancellation condition). By $|S|$ we denote the Lebesgue measure of a set $S$ and $Q = Q(c_Q, r_Q) = \left\{ y = (y_1, \ldots, y_d) \in \mathbb{R}^d : \max_{i=1, \ldots, d} (|c_Q|_i - y_i) < r_Q \right\}$, where $c_Q$ and $r_Q$ are the center and the radius of $Q$, respectively. Denote $d_Q = \text{diam}(Q) = 2\sqrt{d}r_Q$.

The question we shall be concerned with is: whether $H^1(L^U)$ coincides with $H^1_{at}(\mathcal{A})$ for a potential $U$ and a family $\mathcal{A}$? If so, are the norms (1.2) and (1.4) comparable?

There are partial answers to the question above. A general result of Hofmann et. al. [13] gives an atomic and molecular characterizations of $H^1(L^U)$ for any positive potential $U \in L^1_{\text{loc}}(\mathbb{R}^d)$. Also, using [13], Dziubański and Zienkiewicz in [10] proved another general atomic characterization of $H^1(L^U)$. The atoms in [13] are of the form $a = (L^U)^M b$, where $M \geq 1$ is fixed natural number and $b$ satisfies some localization and size conditions, see [13, Theorem 7.1]. Likewise, atoms in [10] are given by $a = K_t^U b - b$ for similar $b$.

Although the approaches just mentioned are useful in many situations, they have also some disadvantages. One of them is that the atoms are images of the operator $L^U$ (or its semigroup) of some function, and they no more satisfy simple geometric conditions (localization, size, cancellation). One would also like to better understand the nature of $H^1(L^U)$ by describing it in terms of simpler, "geometric atoms". In the 90’s Dziubański and Zienkiewicz started studies on atomic decompositions of Hardy spaces for Schrödinger operators. In this paper we continue this approach. For more results of this type see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. Let us finally mention, that this approach was successfully used e.g. for proving Riesz transform characterization of $H^1(L^U)$, while such characterization is not known in general.

Before proceeding to our main results, we present results of [11] and [7], which are the starting point for our considerations.
1.2. The space $H^1(L^V)$. Assume that a potential $V \geq 0$ satisfies

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^{2-d} V(y) \, dy < \infty.$$  

In other words, $(L^V)^{-1} V \in L^\infty(\mathbb{R}^d)$. Let $\omega = \omega(V)$ be a function defined by

$$\omega(x) = \lim_{t \to \infty} \int_{\mathbb{R}^d} K^V_t(x, y) \, dy.$$  

The function $\omega$ is $L^V$-harmonic and satisfies

$$0 < \delta < \omega(x) \leq 1,$$

with some $\delta$ for all $x \in \mathbb{R}^d$, see [11, Lemma 2.1]. It is well-known, see [15], that the integral kernel $K^V_t(x, y)$ has not only upper-Gaussian bounds, but also lower-Gaussian bounds, that is we have $\kappa_1, \kappa_2 > 0$ such that

$$K^V_t(x, y) \geq \kappa_1 t^{-d/2} \exp\left(-\frac{|x-y|^2}{\kappa_2 t}\right).$$  

By definition, a function $a$ is an $\omega$-atom, if there exists a cube $Q$ such that

$$\text{supp } a \subseteq Q, \quad \|a\|_\infty \leq |Q|^{-1}, \quad \text{and } \int_Q a(x) \omega(x) \, dx = 0.$$  

Let $\mathcal{A}_\omega$ be the set of $\omega$-atoms. Corollary 1.2 of [11] states that $H^1(L^V) = H^1_{at}(\mathcal{A}_\omega)$ and

$$\|f\|_{H^1(L^V)} \approx \|f\|_{H^1_{at}(\mathcal{A}_\omega)}.$$  

Let us mention that (S) is satisfied for example when $V$ is compactly supported and $V \in L^p(\mathbb{R}^d)$ for some $p > d/2$. For more general examples, see [6] and [11].

1.3. The space $H^1(L^W)$. For $\theta > 0$ (small) and a cube $Q = Q(c_Q, r_Q)$ denote $Q^* = Q(c_Q, (1+\theta)r_Q)$. Assume a family of cubes $\mathcal{Q}$ is given and there exist $C, \theta > 0$ such that for $Q_1, Q_2 \in \mathcal{Q}$, $Q_1 \neq Q_2$, we have:

$$\cup_{Q \in \mathcal{Q}} \text{cl}(Q) = \mathbb{R}^d,$$

$$|Q_1 \cap Q_2| = 0,$$

$$\text{if } Q_1^{***} \cap Q_2^{***} \neq \emptyset, \text{ then } C^{-1} d_{Q_1} \leq d_{Q_2} \leq C d_{Q_1}.$$  

Observe that, under these assumptions, the family $\{Q^{***} : Q \in \mathcal{Q}\}$ is automatically a finite covering of $\mathbb{R}^d$. In the following, we shortly write that $\mathcal{Q}$ satisfies (G), when it satisfies (G1), (G2), (G3).

Suppose that for a potential $W \geq 0$ and a family $\mathcal{Q}$ as above there exist positive constants $\epsilon, \delta, C$ such that

$$\sup_{y \in Q^{***}} \int_{\mathbb{R}^d} K^W_{2^{n}d_Q^2}(x, y) \, dx \leq Cn^{-1-\epsilon} \quad (Q \in \mathcal{Q}, n \in \mathbb{N}),$$

$$\int_0^{2t} (1_{Q} \ast ... \ast W) * P_t(x) \, ds \leq C \left( \frac{t}{d_Q^2} \right)^{\delta} \quad (x \in \mathbb{R}^d, \ Q \in \mathcal{Q}, t \leq d_Q^2),$$

where $P_t(x - y) = K^0(x, y)$ is the classical heat semigroup, see (1.1). By definition, an $\mathcal{Q}$-atom is a function $a$ such that one of the following holds:

- there exists $Q \in \mathcal{Q}$ and a cube $K \subset Q^{**}$ such that:
  $$\text{supp } a \subseteq K, \quad \|a\|_\infty \leq |K|^{-1}, \quad \int_K a(x) \, dx = 0;$$

- $a(x) = |Q|^{-1} \chi_Q(x)$ for some $Q \in \mathcal{Q}$. 

Let $\mathcal{A}_2$ be a set of $2$-atoms. By Theorem 2.2 of [7] we have that $H^1(L^V) = H^1_{at}(\mathcal{A}_2)$ and
\[ \|f\|_{H^1(L^V)} \approx \|f\|_{H^1_{at}(\mathcal{A}_2)}. \]
A list of examples of potentials $W$ and related families $\mathcal{D}$ can be found in [7]. At this place we shall only mention one simple example, that we shall use later in this paper. Let $t > 0$ and denote by $2^{[t]}$ the family of cubes of radius equal to $t$ that satisfies (G). If $W^{[t]}(x) = t^{-2}$, then the pair $(W^{[t]}, 2^{[t]})$ satisfies (D), (K), (G) with constants independent of $t$.

1.4. Main results. In this paper $V$ always denote a potential satisfying (S) and $\omega$ is related to $V$ by (1.5). Similarly, the pair $W, \mathcal{D}$ always satisfy (D), (K), and (G). Notice, that in $H^1_{at}(\mathcal{A}_2)$ and $H^1_{at}(\mathcal{A}_2)$ two different effects appear. For an atom $a \in \mathcal{A}_2$ (atom for $L^V$) the cancellation condition is w.r.t. the measure $\omega$, not the Lebesgue measure. On the other hand, for $a \in \mathcal{A}_2$, there are “local” atoms, i.e. atoms of the type $|Q|^{-1} \chi_Q(x)$ that do not satisfy any cancellation condition.

The goal of this paper is to study $L^{V+W}$ and its Hardy space $H^1(L^{V+W})$. We shall prove that in atomic decompositions for this space both effects described above appear simultaneously. Define $\mathcal{A}_{\omega,2}$ to be the set of $(\omega, 2)$-atoms, that is functions such that one of the following holds:

- there exists $Q \in \mathcal{D}$ and a cube $K \subseteq Q^\kappa$ such that:
  \[ \text{supp} \ a \subseteq K, \quad \|a\|_\infty \leq |K|^{-1}, \quad \int_K a(x) \omega(x) \, dx = 0, \]
- $a(x) = |Q|^{-1} \chi_Q(x)$ for some $Q \in \mathcal{D}$.

The following theorem gives the atomic characterization of $H^1(L^{V+W})$ in the spirit of [7] and [11].

**Theorem A.** Assume that $d \geq 3$, $V \geq 0$ satisfies (S), and $W \geq 0$ with a family $\mathcal{D}$ satisfy (D), (K), (G). Then
\[ C^{-1} \|f\|_{H^1(L^{V+W})} \leq \|f\|_{H^1_{at}(\mathcal{A}_{\omega,2})} \leq C \|f\|_{H^1(L^{V+W})}. \]
In particular, $H^1(L^{V+W}) = H^1_{at}(\mathcal{A}_{\omega,2})$.

In Theorem A atoms are localized to cubes $Q \in \mathcal{D}$ and the cancellation condition is w.r.t. the measure $\omega(x) \, dx$. However, it is not hard to see that every $(\omega, 2) - \text{atom}$ can be written as a linear combination of just $2$-atoms. Indeed, if $a$ is such that $\text{supp} a \subseteq K \subseteq Q^\kappa$, $\|a\|_\infty \leq |K|^{-1}$, and $\int_K a(x) \omega(x) \, dx = 0$ for $Q \in \mathcal{D}$, then
\[ a(x) = (a(x) - \kappa |Q|^{-1} \chi_Q(x)) + \kappa |Q|^{-1} \chi_Q(x) = b_1(x) + b_2(x), \]
where $\kappa = \int_K a(x) \, dx, \ |x| \leq 1$. Observe that $\text{supp} b_1 \subseteq Q^\kappa$ and $\int_Q b_1(x) \, dx = 0$. Thus both $b_1$ and $b_2$ are multiples of $2$-atoms. What we have just shown is that $a \in H^1_{at}(\mathcal{A}_2)$ and
\[ \|a\|_{H^1_{at}(\mathcal{A}_2)} \leq T, \]
for every $(\omega, 2)$-atom $a$.

The constant $T$ in (1.10) possibly depend on $a$. This lead us to the following question: whether $H^1_{at}(\mathcal{A}_{\omega,2})$ and $H^1_{at}(\mathcal{A}_2)$ are equal as Banach spaces? In Theorem B we prove that, under certain Lipschitz assumption, the answer to this question is positive. However, a more difficult task is to find an example such that $\|f\|_{H^1_{at}(\mathcal{A}_2)} \neq \|f\|_{H^1_{at}(\mathcal{A}_{\omega,2})}$. This is done in Example C.

**Theorem B.** Assume that $0 < \delta \leq \omega \leq 1$, $\mathcal{D}$ satisfies (G), and there exists $\lambda > 0$ such that
\[ |\omega(x) - \omega(y)| \leq C \left( \frac{|x - y|^\delta}{d_Q} \right)^\lambda, \quad (Q \in \mathcal{D}, x, y \in Q^\kappa). \]
Then
\[ \|f\|_{H^1_{at}(\mathcal{A}_2)} \approx \|f\|_{H^1_{at}(\mathcal{A}_{\omega,2})}. \]
As an example that fulfills the assumptions of Theorem B one could take $W^{[1]}$, $\mathcal{Q}^{[1]}$ (see Subsection 1.3) and $\omega = \omega(V)$, with $V$ such that $\text{supp } V \subseteq Q(0,1)$ and $V \in L^p(\mathbb{R}^d)$ for $p > d/2$ (for details see [9]). In this case $\omega$ satisfies global Hölder condition.

**Example C.** Let $\mathcal{Q}^{[1]}$ be as above, and $\omega = \omega(V)$, where $V$ is a potential given in (6.1). There exist a sequence of $(\omega, \mathcal{Q})$-atoms $a_j$, such that

\begin{equation}
\lim_{j \to \infty} \|a_j\|_{H^1_{\omega}(\mathcal{Q})} = \infty.
\end{equation}

In other words, $\|f\|_{H^1_{\omega}(\mathcal{Q})} \neq \|f\|_{H^1_{\omega}(\mathcal{Q})}$.

The paper is organized as follows. Section 2 is devoted to local Hardy spaces. We prove an atomic decomposition for a local version of $H^1(\mathcal{L}^V)$. In Section 3 we prove some auxiliary estimates, most of which are analogues of Lemmas in [7]. In Section 4 and Section 5 we present the proofs of Theorems A and B, respectively. In Section 6 we provide details of Example C and prove (1.13). Finally, in the Appendix we give a proof of $\|f\|_{L^1(\mathbb{R}^d)} \leq \|\sup_{t \leq r} K(t f)\|_{L^1(\mathbb{R}^d)}$.

At the end of this section let us give a short remark. In some papers authors define local atomic spaces in a slightly different manner. The remark below clarify, that different definitions lead to the same atomic Hardy spaces in the sense of equivalent Banach spaces.

**Remark 1.14.** Let us consider $\mathcal{Q}$ and $\omega$ as above and a function $a$ that satisfies:

- there exists $Q \in \mathcal{Q}$ and a cube $K \subseteq Q$ such that:

$$\text{supp } a \subseteq K, \quad 4d_K \geq d_Q, \quad \|a\|_{\infty} \leq |K|^{-1}.$$

For each $a$ as above, we have that $\|a\|_{H^1_{\omega}(\mathcal{Q})} \leq C$, with universal $C$. To see this, one has to write

$$a$$

as a linear combination of $|Q|^{-1} \chi_Q(x)$ and atom with cancellation condition. Therefore, the functions $a$ as above can be substitutes for the atoms of the form $|Q|^{-1} \chi_Q(x)$ in the definition of $\mathcal{A}_{\omega, \mathcal{Q}}$.

2. LOCAL HARDY SPACES

2.1. **Local Hardy spaces.** In this section we put aside $W$ and $\mathcal{Q}$ for a moment and consider only $\mathcal{L}^V$ and related objects. The local version of the maximal operator $M^V_t$ at scale $\tau > 0$ is

$$M^V_t f(x) = \sup_{t \leq r} |K^V_t f(x)|.$$

By definition, a function $f \in L^1(\mathbb{R}^d)$ is in the local Hardy space $h^1_{\omega}(\mathcal{L}^V)$, when $M^V_t f$ is in $L^1(\mathbb{R}^d)$. We set

$$\|f\|_{h^1_{\omega}(\mathcal{L}^V)} := \|M^V_{\tau} f\|_{L^1(\mathbb{R}^d)}.$$

In a special case $V \equiv 0$, the space $h^1_{\omega}(-\Delta)$ is a classical local Hardy space introduced by Goldberg [12]. It follows from [12] that

\begin{equation}
C^{-1} \|f\|_{H^1_{\omega}(\mathcal{A}_{\omega}(\mathcal{Q}))} \leq \|f\|_{h^1_{\omega}(-\Delta)} \leq C \|f\|_{H^1_{\omega}(\mathcal{A}_{\omega}(\mathcal{Q}))},
\end{equation}

where $C$ does not depend on $\tau$. The following proposition is a generalization of (2.1) for $h^1_{\omega}(\mathcal{L}^V)$ localized to a cube of diameter comparable to $\tau$. It will play a crucial role in the proof of Theorem A.

**Proposition 2.2.** Let $Q$ be a cube.

a) Let a be $\omega$--atom, such that $\text{supp } a \subseteq Q^\ast \ast$ or $a(x) = |Q|^{-1} \chi_Q(x)$. Then

\begin{equation}
\|M^V_{\tau} a\|_{L^1(\mathbb{R}^d)} \leq C.
\end{equation}
b) Assume that $\text{supp } f \subseteq Q^*$ and $M_{d_0}^V f(x) \in L^1(\mathbb{R}^d)$. There exist $\lambda_j$ and $a_j$ being either $\omega$-atoms or of the form $|Q|^{-1} \chi_Q(x)$, such that

$$ f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x), \quad \sum_{j=1}^{\infty} |\lambda_j| \leq C \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)}. $$

The constant $C$ above depends only on $d$ and $\theta$ in the definition of $Q^*$.

**Proof.** Assume first that $a$ is $\omega$-atom. Obviously, $M_{d_0}^V a(x) \leq M^V a(x)$, so (2.3) holds by (1.8). In the case when $a = |Q|^{-1} \chi_Q$ we use (1.1) and (2.1), getting

$$ \left\| M_{d_0}^V a \right\|_{L^1(\mathbb{R}^d)} \leq \left\| M_{d_0}^0 a \right\|_{L^1(\mathbb{R}^d)} \leq C. $$

Now, let $f$ be as in the assumptions of b). Set

$$ g(x) = f(x) - K_{d_0}^V f(x), $$

so that

$$ f(x)\omega(x) = g(x)\omega(x) + K_{d_0}^V f(x)\omega(x) = h_1(x) + h_2(x). $$

We claim that $h_1 \in H^1(-\Delta)$ and $h_2 \in h_{d_0}^1(-\Delta)$ with

\begin{align*}
(2.4) & \quad \|h_1\|_{H^1(-\Delta)} \leq C \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)}, \\
(2.5) & \quad \|h_2\|_{h_{d_0}^1(-\Delta)} \leq C \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)}.
\end{align*}

To prove (2.4), observe that

$$ \left\| \sup_{t > d_0^2/2} |K_t^V g| \right\|_{L^1(\mathbb{R}^d)} \leq 2 \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)} < \infty. $$

Likewise,

$$ \left\| \sup_{t > d_0^2/2} |K_t^V g(x)| \right\|_{L^1(\mathbb{R}^d)} \leq C \left\| f \right\|_{L^1(\mathbb{R}^d)} $$

by the argument identical as in the proof of [11, Proposition 6.3]. By Corollary 7.2, $\| f \|_{L^1(\mathbb{R}^d)} \leq \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)}$. Thus $g \in H^1(L^V)$ and, by (1.8), $h_1 = g \cdot \omega \in H^1(-\Delta)$, so (2.4) is proved.

Now, we turn to prove (2.5). It is clear that

$$ h_2(x) = \sum_{K \subseteq \partial d_0^j} K_{d_0}^V f(x)\omega(x) \chi_K(x) = \sum_{K \subseteq \partial d_0^j} h_K(x) $$

and

$$ \|h_K\|_{\infty} \leq C \int_K dQ^d \exp \left( -\frac{|x-y|^2}{2dQ^d} \right) \|f(y)\| \, dy $$

$$ \leq C|Q|^{-1} \exp \left( -\frac{d(Q^*, K)^2}{2dQ^d} \right) \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)}. $$

Clearly, $\text{supp } h_K \subseteq K$, so by using the classical atomic characterization of $h_{d_0}^1(-\Delta)$ we have that

$$ \|h_K\|_{h_{d_0}^1(-\Delta)} \leq C \exp \left( -\frac{d(Q^*, K)^2}{2dQ^d} \right) \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)}. $$

Summing up,

$$ \|h_2\|_{h_{d_0}^1(-\Delta)} \leq C \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)} \sum_{K \subseteq \partial d_0^j} \exp \left( -\frac{d(Q^*, K)^2}{2dQ^d} \right) $$

$$ \leq C \left\| M_{d_0}^V f \right\|_{L^1(\mathbb{R}^d)}, $$
where the last inequality is a simple geometric observation.

Having (2.4) and (2.5) proved, we finish the prove by the following argument. The function $f \cdot \omega$ is supported in $Q^*$ and $f \cdot \omega \in H_{d\phi}^1(-\Delta)$ with $\|f \cdot \omega\|_{H_{d\phi}^1(-\Delta)} \leq C \|M^V f\|_{L^1(\mathbb{R}^d)}$. So, by the classical local characterization of $h_{d\phi}^1(-\Delta)$, $f \cdot \omega = \sum_j \lambda_j a_j$, where $a_j$ are either classical atoms or of the form $|Q|^{-1} \chi_Q(x)$. Moreover, $\sum_j |\lambda_j| \leq C \|M^V f\|_{L^1(\mathbb{R}^d)}$. Then $f = \sum_j \lambda_j b_j$ where $b_j = a_j / \omega$ are either $\omega$-atoms or $b_j = \omega^{-1} |Q|^{-1} \chi_Q$. In the last case, $b_j$ can be decomposed into a linear combination of $|Q|^{-1} \chi_Q$-atom and $\omega$-atom, exactly as in Remark 1.14. $\square$

The following corollary is a “global” version of Proposition 2.2 and can be proved by standard techniques. The details are left to the reader.

**Corollary 2.6.** There exists a constant $C$, independent of $\tau > 0$, such that

$$\|f\|_{H^1_t(\mathbb{R}^d)} \leq \|f\|_{H^1_{at}(\mathcal{A}_0, \partial)}.$$  

In particular, $h^1_{at}(\mathcal{L}^V) = H^1_{at}(\mathcal{A}_0, \partial)$.

### 3. Auxiliary estimates

In this section we present tools and lemmas that will be used in the proof of Theorem A. The proofs of Lemmas 3.4, 3.5, 3.7, 3.8 are very similar to their analogues in [7]. Thus we only provide sketches how to adapt proofs from [7] to our background.

Let $U_1, U_2 \geq 0$ be two potentials. A well-known perturbation formula states that

$$K_{t}^{U_1} - K_{t}^{U_1 + U_2} = \int_0^t K_{t-s}^{U_1} U_2 K_{s}^{U_1 + U_2} \, ds.$$  

For the kernels this reads as

$$K_{t}^{U_1}(x, y) - K_{t}^{U_1 + U_2}(x, y) = \int_0^t \int_{\mathbb{R}^d} K_{t-s}^{U_1}(x, z)V(z)K_{s}^{U_1 + U_2}(z, y) \, dz \, ds.$$  

With a family $\mathcal{Q}$ satisfying (G) we associate a partition of unity $\Phi = \{\phi_Q\}_{Q \in \mathcal{Q}}$ such that

$$0 \leq \phi_Q \in C_c^\infty(Q^*), \quad 1_{\mathbb{R}^d} = \sum_{Q \in \mathcal{Q}} \phi_Q, \quad \|\nabla \phi_Q\|_{\infty} \leq C d_Q^{-1}.$$  

**Lemma 3.4.** Let $U \in L^1_{loc}(\mathbb{R}^d)$ be a positive potential. For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{Q}$,

$$\sup_{t \leq d_Q^2} \left|K^U_t(\phi_Q f)\right|_{L^1((Q^*')^c)} \leq \|\phi_Q f\|_{L^1(\mathbb{R}^d)}.$$  

**Proof.** Let $c_Q$ be the center of $Q$. For $t \leq d_Q^2$, $y \in Q^*$ and $x \notin Q^{**}$ we have

$$\sup_{t \leq d_Q^2} K^U_t(x, y) \leq \sup_{t \leq d_Q^2} C t^{-d/2} \exp\left(-\frac{|x - c_Q|^2}{ct}\right) \leq C d_Q^{-d} \exp\left(-\frac{|x - c_Q|^2}{c d_Q^2}\right).$$  

The lemma follows by integrating the last expression w.r.t. $dx$ on $(Q^{**})^c$. $\square$

**Lemma 3.5.** Assume (K). For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{Q}$,

$$\left\|\sup_{t \leq d_Q^2} \left|\left(K^U_t - K^V_t\right)(\phi_Q f)\right|\right\|_{L^1(\mathbb{R}^d)} \leq C \|\phi_Q f\|_{L^1(\mathbb{R}^d)}.$$  

Sketch of the proof. Using (3.1) we write

\[
(K_t^V - K_t^{V+W})(\phi_Q f) = \int_0^t K_{t-s}^V(W \cdot \mathbb{1}_{(Q^{***})})K_t^{V+W}(\phi_Q f) \, ds \\
+ \int_0^t K_{t-s}^V(W \cdot \mathbb{1}_{(Q^{***})})K_t^{V+W}(\phi_Q f) \, ds.
\]

Both summands can be estimated similarly as in [7, Lemma 3.11]. In order to repeat arguments of [7], one should have in mind that, by (3.2),

\[
K_t^{V+W}(x,y) \leq K_t^U(x,y) \leq P_t(x-y),
\]
where $U$ is either $V$ or $W$. The details are omitted. \(\square\)

For each $Q \in \mathcal{D}$ we set

\[
\mathcal{D}_{loc,Q} = \{Q' \in \mathcal{D} : Q^{***} \cap Q'^{***} \neq \emptyset\},
\]
\[
\mathcal{D}_{glob,Q} = \{Q'' \in \mathcal{D} : Q^{***} \cap Q''^{***} = \emptyset\}.
\]

Roughly speaking, for each $Q$, the set $\mathcal{D}_{loc,Q}$ is the set of cubes $Q' \in \mathcal{D}$ that are "close" to $Q$. For a function $f$ denote

\[
f_{loc,Q} = \sum_{Q' \in \mathcal{D}_{loc,Q}} \phi_{Q'} f, \quad f_{glob,Q} = f - f_{loc,Q}.
\]

The following two lemmas and their proofs are almost identical to [7, Lemma 3.7] and [7, Lemma 3.8]. To see this one only has to use (3.6). The details are left to the reader.

**Lemma 3.7.** For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{D}$,

\[
\sup_{t > 0} \left\| \frac{d}{ds} K_t^{V+W} (\phi_Q f_{loc,Q} - \phi_Q f) \right\|_{L^1(Q^{***})} \leq C \left\| f_{loc,Q} \right\|_{L^1(\mathbb{R}^d)}.
\]

**Lemma 3.8.** Assume (D). For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{D}$,

\[
\sum_{Q \in \mathcal{D}} \left\| \sum_{t = 0}^{\infty} \left( \frac{d}{dt} K_t^{V+W} f_{glob,Q} \right) \right\|_{L^1(Q^{***})} \leq C \left\| f \right\|_{L^1(\mathbb{R}^d)}.
\]

4. PROOF OF THEOREM A

In the proof below, we shall often use the fact that, for $0 \leq U \in L^1_{loc}(\mathbb{R}^d)$ and $\tau > 0$, we have

\[
\left\| U \right\|_{L^1(\mathbb{R}^d)} \leq \left\| M^U_{\tau} f \right\|_{L^1(\mathbb{R}^d)}.
\]

This is a consequence of semigroup property and Gaussian estimates. A detailed proof is given in the Appendix, see Proposition 7.1 and Corollary 7.2.

**First implication.** We start by proving the second inequality of (1.9), that is for a function $f$ such that $\|f\|_{H^1(L^{V+W})} < \infty$ we will find $(\omega, \mathcal{D})$-atoms $a_i$ such that

\[
f(x) = \sum_{i=1}^{\infty} \lambda_i a_i(x) \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i| \leq C \|f\|_{H^1(L^{V+W})}.
\]

Let $\phi_Q$ be as in (3.3), in particular $f = \sum_{Q \in \mathcal{D}} \phi_Q f$. The key estimate is the following.

\[
\sum_{Q \in \mathcal{D}} \left\| \sum_{t = 0}^{\infty} \left( \frac{d}{dt} K_t^{V+W} (\phi_Q f) \right) \right\|_{L^1(\mathbb{R}^d)} \leq C \left\| f \right\|_{H^1(L^{V+W})}.
\]
Now we prove (4.2). By Lemma 3.4 we get that \( \sum_{Q \in 2} \| f \|_{L^1(Q')} \leq C \| f \|_{L^1(\mathbb{R}^d)} \). Now we concentrate our attention on \( Q^{**} \). Notice that

\[
K^V_t (\phi_Q f) = \left[ (K^V_t - K^{V+W}_t) (\phi_Q f) \right] + \left[ K^{V+W}_t (\phi_Q f) - \phi_Q \cdot K^{V+W}_t (f_{loc,Q}) \right] \\
+ \left[ -\phi_Q \cdot K^{V+W}_t (f_{glo,Q}) \right] + \left[ \phi_Q \cdot K^{V+W}_t (f) \right]
\]

where we have used (4.1) and 

\[
A_1 + A_2 + A_3 + A_4.
\]

Notice that \( \phi_Q f_{loc,Q} = \phi_Q f \). Lemmas 3.5, 3.7, 3.8 lead to

\[
\sum_{k=1}^{3} \sum_{Q \in 2} \sup_{t \in d^2_Q} |A_k| \leq C \sum_{Q \in 2} \left( \| \phi_Q \cdot f \|_{L^1(\mathbb{R}^d)} + \| f_{loc,Q} \|_{L^1(\mathbb{R}^d)} + \| f \|_{L^1(\mathbb{R}^d)} \right)
\]

\[
\leq C \| f \|_{L^1(\mathbb{R}^d)} \leq C \| \phi_Q f \|_{L^1(\mathbb{R}^d)}
\]

where we have used (4.1) and

\[
\sum_{Q \in 2} \| f_{loc,Q} \|_{L^1(\mathbb{R}^d)} \leq \sum_{Q \in 2} \left( \sum_{Q' \in 2_{loc,Q}} \| \phi_Q f \|_{L^1(\mathbb{R}^d)} \right) = C \sum_{Q' \in 2} \| \phi_Q f \|_{L^1(\mathbb{R}^d)}
\]

The proof of (4.2) is finished by noticing that

\[
\sum_{Q \in 2} \sup_{t \in d^2_Q} |A_1| \leq C \| f \|_{L^1(\mathbb{R}^d)}.
\]

Having (4.2) proved, we apply Proposition 2.2b to \( \phi_Q f \), obtaining \( \lambda_{j,Q} \) and \( (\omega, \mathcal{D}) \)-atoms \( a_{j,Q} \) such that

\[
\phi_Q(x) f(x) = \sum_{j=1}^{\infty} \lambda_{j,Q} a_{j,Q}(x), \quad \text{with} \quad \sum_{j=1}^{\infty} |\lambda_{j,Q}| \leq C \| f \|_{L^1(\mathbb{R}^d)}.
\]

Therefore,

\[
f(x) = \sum_{j,Q} \lambda_{j,Q} a_{j,Q}(x), \quad \text{with} \quad \sum_{j,Q} |\lambda_{j,Q}| \leq C \| f \|_{L^1(\mathbb{R}^d)}
\]

and the proof of the first part is finished.

**Second implication.** By a standard argument it is enough to prove that

\[
\left\| \sup_{t \in 2^{-m} d^2_Q} |K^{V+W}_t a| \right\|_{L^1(\mathbb{R}^d)} \leq C
\]

for \( a \in \mathcal{A}_{\omega, \mathcal{D}} \). Assume then that \( \supp a \subseteq Q^{**} \), where \( Q \in \mathcal{D} \). By the definition of \( \mathcal{D}_{loc,Q} \) and \( \phi_Q \) it is clear that \( a = a_{loc,Q} \). From (G3) there exists a universal constant \( m \in \mathbb{N} \) such that \( d^2_{Q'} \geq 2^{-m} d^2_Q \) whenever \( Q' \in 2_{loc,Q} \).

\[
\left\| \sup_{t \in 2^{-m} d^2_Q} |K^{V+W}_t a| \right\|_{L^1(\mathbb{R}^d)} \leq \sum_{Q' \in 2_{loc,Q}} \left\| \left( K^{V+W}_t - K^V_t \right) \phi_Q a \right\|_{L^1(\mathbb{R}^d)}
\]

\[
+ \left\| \sup_{t \in d^2_Q} |K^V_t a| \right\|_{L^1(\mathbb{R}^d)}
\]

By Lemma 3.5, the sum is bounded by \( C \| a \|_{L^1(\mathbb{R}^d)} \leq C \). The second summand is bounded by Proposition 2.2a.

What is left is to consider \( t \geq 2^{-m} d^2_{Q'} \). Denote

\[
I_j = [2^j d^2_Q, 2^{j+1} d^2_Q), \quad f_j = [2^{j-1} d^2_Q, 3 \cdot 2^{j-1} d^2_Q].
\]
Note that \( I_j = \{ x + 2^j - 1 d_Q^2 : x \in I_j^j \} \). By (1.1) it is not hard to check that for \( g \in L^1(\mathbb{R}^d) \) we have

\[
\left\| \sup_{t \in I_j^j \cup I_j} |K_t^{J+W} g| \right\|_{L^1(\mathbb{R}^d)} \leq C \left\| g \right\|_{L^1(\mathbb{R}^d)},
\]

where \( C \) does not depend on \( j \) and \( g \). Therefore, for \( j \geq 2 \),

\[
\left\| \sup_{t \in I_j} |K_t^{J+W} a| \right\|_{L^1(\mathbb{R}^d)} \leq \sup_{t \in I_j^j} \left\| K_t^{J+W} \left( K_{2^j-1} g \right) \right\|_{L^1(\mathbb{R}^d)} \leq C \left\| K_{2^j-1} d_Q^2 a \right\|_{L^1(\mathbb{R}^d)} \leq C J_{j-1-\epsilon},
\]

where in the last inequality we have used (D). The proof is finished by noticing that

\[
\left\| \sup_{t \geq 2^{-m} d_Q^2} |K_t^{J+W} a| \right\|_{L^1(\mathbb{R}^d)} \leq \sum_{j=m}^{\infty} \left\| \sup_{t \in I_j} |K_t^{J+W} a| \right\|_{L^1(\mathbb{R}^d)} \leq C \left( m + 2 + \sum_{j=2}^{\infty} J_{j-1-\epsilon} \right) \leq C.
\]

5. PROOF OF THEOREM B

The proof follows by known procedure that uses atomic decompositions. Assume that \( W, V, \mathcal{Q}, \omega \) are given and \( \omega \) satisfies (1.11).

To prove one of the inequalities of (1.12) it is enough to show that

\[
\| a \|_{H^1_{\mathcal{Q}}(\mathcal{A}_\omega, \mathcal{Q})} \leq C
\]

for \( a \in \mathcal{A}_\omega, \mathcal{Q} \). Obviously, if \( a \) is an atom of the form \( a(x) = |Q|^{-1} \chi_Q(x) \), the inequality (5.1) holds with \( C = 1 \). Assume then that \( a \) is such that \( \text{supp} a \subseteq K \subseteq Q^* \), \( Q \in \mathcal{Q} \), \( \| a \|_\infty \leq |K|^{-1} \), \( \int_K a(x) \omega(x) \, dx = 0 \).

Take a sequence of cubes \( G_n \) such that

\[
K = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_N \subseteq Q^*, \quad d_{G_{n+1}} = 2 d_{G_n} \quad (n = 0, \ldots, N - 1),
\]

and \( d_Q \leq 2 d_{G_0} \). Observe that \( N \leq C (\log_2 (d_Q/d_K) + 1) \) and \( a(x) = \sum_{n=0}^{N+2} b_n(x) \), where

\[
\begin{align*}
&b_0(x) = a(x) - t_0 \chi_{G_0}(x), \\
&b_n(x) = t_{n-1} \chi_{G_{n-1}}(x) - t_n \chi_{G_n}(x) \quad (n = 1, \ldots, N), \\
&b_{N+1} = t_N \chi_{G_N}(x) - t_{N+1} |Q|^{-1} \chi_Q(x), \\
&b_{N+2} = t_{N+1} |Q|^{-1} \chi_Q(x).
\end{align*}
\]

The constants \( t_n \) are chosen so that \( \int b_n(x) \, dx = 0 \) for \( n = 0, \ldots, N + 1 \), i.e.

\[
\begin{align*}
t_0 &= |G_0|^{-1} \int_{G_0} a(x) \, dx, \\
t_n &= 2^{-d} t_{n-1} \quad (n = 1, \ldots, N), \\
t_{N+1} &= t_N |G_N|.
\end{align*}
\]

The key estimate, that uses (1.11) and the cancellation property, is the following

\[
|t_0| = |K|^{-1} \omega(c_K)^{-1} \left| \int_K a(x) \left( \omega(c_K) - \omega(x) \right) \, dx \right| 
\leq C |K|^{-2} \int_K \left| \frac{|x - c_K|}{d_Q} \right|^\lambda \, dx 
\leq C |K|^{-1} \left( \frac{d_K}{d_Q} \right)^\lambda 
\leq C 2^{-c_N} |K|^{-1}
\]

Thus \( |t_n| \leq C 2^{-c_N} |G_n|^{-1} \) for \( n = 1, \ldots, N \), and \( |t_{N+1}| \leq C \).
Obviously, $\text{supp} \ b_n \subseteq G_n$ for $n = 0, ..., N$, and $\text{supp} \ b_{N+1} \subseteq Q^{**}$. Moreover,

\[
\|b_0\|_{\infty} \leq |K|^{-1} + |t_0| \leq C |K|^{-1}, \\
\|b_n\|_{\infty} \leq C |t_{n-1}| \leq C 2^{-CN} |G_n|^{-1} \quad (n = 1, ..., N) \\
\|b_{N+1}\|_{\infty} \leq C |Q^{**}|^{-1}.
\]

As a consequence we have that all $b_n$ are multiples of $H^1_{at}(\mathcal{A}_\omega)$-atoms and (5.1) is proved, since

\[
\|a\|_{H^1_{at}(\mathcal{A}_\omega)} \leq N+2 \sum_{n=0}^{N+2} \|b_n\|_{H^1_{at}(\mathcal{A}_\omega)} \leq CN2^{-CN} + 3C \leq C.
\]

For the second inequality one should consider $a \in H^1_{at}(\mathcal{A}_\omega)$ and prove that

\[
\|a\|_{H^1_{at}(\mathcal{A}_\omega)} \leq C.
\]

This can be done in a similar fashion. The details are omitted here.

6. Example C

Denote $c_n = 2^n e_1$ and $C_n = Q(c_n, 1/(2n))$, where $e_1$ denotes the vector $(1,0,...,0)$ in $\mathbb{R}^d$. The potential $V$ that we need for Example C is the following

\[
V(x) = \sum_{k=2}^{\infty} k^2 \chi_{C_k}(x).
\]

**Lemma 6.2.** $V$ satisfies (S).

**Proof.** Let $x \in \mathbb{R}^d$.

\[
\int_{\mathbb{R}^d} V(y) |x - y|^{2-d} \, dy = \sum_{k=2}^{\infty} k^2 \int_{C_k} |x - y|^{2-d} \, dy = \sum_{k=2}^{\infty} I_k.
\]

We have that

\[
I_k \leq k^2 \int_{C_k} |y - c_k|^{2-d} \, dy \leq C \quad (x \in \mathbb{R}^d),
\]

\[
I_k \leq C k^2 \int_{C_k} |x - c_k|^{2-d} \, dy \leq C(k|x - c_k|)^{2-d} \quad (x \notin 2C_k).
\]

Consider $x = (x_1, ..., x_d)$ and let $N \geq 2$ be such that $2^N < x_1 \leq 2^{N+1}$ ($N = 2$ when $x_1 \leq 8$). Then

\[
\sum_{k=2}^{\infty} I_k = \sum_{k=2}^{N-1} I_k + (I_N + I_{N+1}) + \sum_{k=N+2}^{\infty} I_k = A_1 + A_2 + A_3,
\]

with obvious modification when $N = 2$. Obviously, $A_2 \leq C$ by (6.3). Moreover, for $k \neq N$ and $k \neq N+1$, we have that $|x - c_k| \geq C 2^{\max(N,k)}$, so using (6.4) we obtain

\[
A_1 \leq C \sum_{k=2}^{N-2} (k^2 N)^{2-d} \leq C,
\]

\[
A_3 \leq C \sum_{k=N+1}^{\infty} (k^2 k)^{2-d} \leq C.
\]

\[\square\]

For the rest of this section by $\omega$ we mean $\omega(V)$ for $V$ given by (6.1). The following lemma give an essential information about local oscillations of $\omega$. 
Proposition 6.5. Let $c_n$ and $C_n$ be as above,
\[ d_n = c_n + (\tau/n)e_1, \quad D_n = Q(d_n, 1/(2n)). \]
There exists $\tau > 3$, $c_0 > 0$, and $N \in \mathbb{N}$ such that for $n \geq N$ we have
\[ \inf_{x \in D_n, y \in C_n} (\omega(x) - \omega(y)) \geq c_0. \]

Let us remark that $\omega$ satisfying (6.6) cannot fulfill the global Hölder condition. To see this, just observe that \[ |c_n - d_n| \to 0 \text{ and } \omega(d_n) - \omega(c_n) \geq c_0. \]

**Proof.** Recall that $K^\tau(x, y)$ always satisfies upper-Gaussian bounds, see (1.1). By Lemma 6.2, there are also lower-Gaussian bounds. Set $\kappa = \min(\kappa_1, \kappa_2)$, where $\kappa_1, \kappa_2$ are as in (1.7). Put $U_1 = 0, U_2 = V$ in (3.2), integrate w.r.t. $x \in \mathbb{R}^d$, and let $t$ tend to infinity. We obtain that
\[ 1 - \omega(y) = \int_0^\infty \int_{\mathbb{R}^d} V(z)K^\tau_\kappa(z, y) \, dy \, ds. \]

It is enough to show that, for properly chosen $\tau$ and $c_0$, the following estimates hold for $x \in D_n$ and $y \in C_n$.
\[ 1 - \omega(y) = \int_0^\infty \int_{\mathbb{R}^d} V(z)K^\tau_\kappa(z, y) \, dy \, ds \geq 2c_0, \]
\[ 1 - \omega(x) = \int_0^\infty \int_{\mathbb{R}^d} V(z)K^\tau_\kappa(z, x) \, dy \, ds \leq c_0. \]

Fix $n \geq 2$ and $y \in C_n$. By (1.7) and (6.1),
\[
\int_0^\infty \int_{\mathbb{R}^d} V(z)K^\tau_\kappa(z, y) \, dy \, ds \geq c \int_0^\infty \int_{C_n} \kappa n^2 s^{-d/2} \exp\left(-\frac{|z-y|^2}{\kappa s}\right) \, dz \, ds
\]
\[
= c\kappa n^2 \int_{C_n} |z-y|^{2-d} \, dz \int_0^\infty s^{-d/2} \exp\left(-\frac{1}{\kappa s}\right) \, ds
\]
\[
\geq c(d, \kappa) = 2c_0.
\]

Thus (6.7) is proved. For $x \in D_n$,
\[
\int_0^\infty \int_{\mathbb{R}^d} V(z)K^\tau_\kappa(z, x) \, dy \, ds \leq C \sum_{k=2}^\infty k^2 \int_0^\infty \int_{C_k} s^{-d/2} \exp\left(-\frac{|z-x|^2}{4s}\right) \, dz \, ds
\]
\[
\leq Cn^2 \int_{C_n} |z-x|^{2-d} \, dz + Ck^2 \sum_{2 \leq k \neq n} \int_{C_k} |z-x|^{2-d} \, dz
\]
\[
= A_1 + A_2.
\]

Observe that if $x \in D_n$ and $z \in C_n$, then $|x-z| \geq \tau/(2n)$. Therefore,
\[ A_1 \leq C n^2 (\tau/n)^{2-d} n^{-d} = C \tau^{2-d} \leq c_0/2, \]
where the last inequality holds for $\tau$ big enough. Fix such $\tau$. In what follows we consider only $n \geq N_1$, such that $d(c_n, d_n) \leq 1/2$. For such $n$ and $k \neq n$ we have $|z-x| \geq c^{\max(n, k)}$ for $z \in C_k$ and $x \in D_n$. Thus,
\[ A_2 = \sum_{2 \leq k < n} \ldots + \sum_{k > n} \ldots \leq C \sum_{2 \leq k < n} k^2 2^{n(2-d)} k^{-d} + C \sum_{k > n} k^2 2^{k(2-d)} k^{-d}
\]
\[
\leq Cn^{2n(2-d)} + C2^{n(2-d)} \leq c_0/2,
\]
where the last estimate holds for $n \geq N_2$. The proof of (6.8) finished by taking $N = \max(N_1, N_2)$. \qed

Recall that $Q^{[1]}$ consist of cubes of radii equal to 1 that satisfies (G). We are now in position to prove that the spaces $H^1_{at}(\mathcal{A}Q^{[1]})$ and $H^1_{at}(\mathcal{A}Q^{[1]}, \omega)$ are not equivalent as Banach spaces.
Proposition 6.9. There exist a sequence \( a_n \) of \((\mathcal{Q}^1, \omega)\)–atoms such that
\[
\|a_n\|_{H^1_{\omega}(\mathcal{Q}^1)} \geq c \ln n.
\]

Proof. In this proof we use notation already introduced in Section 6. Let us denote
\[
\omega(S) = \int_S \omega(x) \, dx
\]
and
\[
\mu_n = \omega(D_n) - \omega(C_n).
\]
The atoms we are looking for are
\[
a_n(x) = \zeta n^d \left( \mu_n \chi_{C_n}(x) - \chi_{D_n}(x) \right),
\]
where \( \zeta > 0 \) is a constant that will be fixed in a moment.

Let us check that \( a_n \) are \((\mathcal{Q}^1, \omega)\)–atoms. Obviously \( \text{supp} \ a_n \subseteq K_n := Q(c_n, (\tau + 1)/n) \). By the definition of \( \mu_n \), \( \int_{K_n} a_n(x) \omega(x) \, dx = 0 \). Recall that \( |C_n| = |D_n| \), so by (1.6) we get that \( \mu_n \leq \delta^{-1} \). Moreover, by using Proposition 6.5, for \( n \geq N \),
\[
\mu_n \geq \frac{\inf \{ \omega(x) : x \in D_n \}}{\sup \{ \omega(y) : y \in C_n \}} = 1 + \frac{\inf \{ \omega(x) - \omega(y) : x \in D_n, y \in C_n \}}{\sup \{ \omega(y) : y \in C_n \}} \geq 1 + c_0.
\]

What is left is to check the size condition. By choosing proper \( \zeta > 0 \) we can write
\[
\|a_n\|_{\infty} \leq \zeta n^d \delta^{-1} \leq |K_n|^{-1},
\]
so \( a_n \) are indeed \((\mathcal{Q}, \omega)\)–atoms.

Now we prove (6.10). For the collection \( \mathcal{Q}^1 \), the space \( H^1_{\omega}(\mathcal{Q}^1) \) is a classical local Hardy space. Equivalently, the norm can be given by a local maximal operator, see (2.1),
\[
\|f\|_{H^1_{\omega}(\mathcal{Q}^1)} \approx \sup_{r \leq 1} \|K^0_r f\|_{L^1(\mathbb{R}^d)}.
\]

Denote
\[
S_n = \left\{ x \in \mathbb{R}^d : \sqrt{d}/n < |x - c_n| < 1, (x)_1 < (c_n)_1 \right\},
\]

**Figure 1.** The sets \( C_n, D_n, S_n \).
where \((x)_1\) is the first coordinate of \(x \in \mathbb{R}^d\), see Figure 1. Obviously, \(|S_n| = C\). Assume now that \(x \in S_n\) for some \(n\). By (6.11),

\[
K_0^n a_n(x) = \zeta n^d \int_{\mathbb{R}^d} (4\pi t)^{-d/2} \exp \left(-\frac{|x-y|^2}{4t}\right) \left(\mu_n \chi_{C_n}(y) - \chi_{D_n}(y)\right) \, dy
\]

\[
\geq C n^d t^{-d/2} \int_{\mathbb{R}^d} \exp \left(-\frac{|x-y|^2}{4t}\right) \left(\chi_{C_n}(y) - \chi_{D_n}(y)\right) \, dy
\]

\[
+ C n^d t^{-d/2} c_0 \int_{C_n} \exp \left(-\frac{|x-y|^2}{4t}\right) \, dy
\]

\[= A_1 + A_2.\]

We claim that \(A_1 \geq 0\). Indeed, \(D_n = C_n + (\tau/n) e_1\) and for \(x \in S_n\), \(y_1 \in C_n\) and \(y_2 = y_1 + (\tau/n) e_1\) we have \(|y_1 - x| < |y_2 - x|\), c.f. (6.12). We obtain that

\[A_1 = C n^d t^{-d/2} \int_{C_n} \left(\exp \left(-\frac{|x-y|^2}{4t}\right) - \exp \left(-\frac{|x-(y+(\tau/n) e_1)|^2}{4t}\right)\right) \, dy \geq 0.\]

Now we deal with \(A_2\). For \(x \in S_n\) and \(y \in C_n\) we have that \(|x-y| \leq 2|x-c_n|\). Thus,

\[A_2 \geq C t^{-d/2} \exp \left(-\frac{|x-c_n|^2}{t}\right).\]

Taking \(t = |x-c_n|^2 \leq 1\) we obtain that \(\sup_{t \leq 1} A_2 \geq C|x-c_n|^{-d}\). The proof is finished by noticing that

\[\left\|\sup_{t \leq 1} |K_0^n a_n(x)|\right\|_{L^1(S_n)} \geq C \int_{S_n} |x-c_n|^{-d} \, dx \geq C \ln n,
\]

where the last inequality is easily obtained by integrating in spherical coordinates. \(\square\)

7. Appendix

In the Appendix we consider a semigroup \((T_t)_{t \geq 0}\) that has positive integral kernel satisfying (1.1). Obviously, all Schrödinger semigroups \(K_t^U\) with \(0 \leq U \in L^1_{loc}(\mathbb{R}^d)\) satisfy these assumptions.

Our goal is to give a precise proof of the following natural estimate.

**Proposition 7.1.** Assume that \(f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)\). For almost every \(x \in \mathbb{R}^d\),

\[\lim_{t \to 0} T_t f(x) = f(x).\]

**Corollary 7.2.** Let \(0 \leq U \in L^1_{loc}(\mathbb{R}^d)\) and \(\tau > 0\). Then

\[\left\|f\right\|_{L^1(\mathbb{R}^d)} \leq \left\|M_t^U f\right\|_{L^1(\mathbb{R}^d)}.\]

The proof of Proposition 7.1 will be given at the end. We shall start with the following.

**Lemma 7.3.** Assume that \(r > 0\) is given. For a.e. \(x \in \mathbb{R}^d\),

\[
\lim_{t \to 0} \int_{|x-y| > r} T_t f(x, y) \, dy = 0,
\]

\[
(7.4)
\]

\[
\lim_{t \to 0} \int_{|x-y| < r} T_t f(x, y) \, dy = 1.
\]

\[
(7.5)
\]

**Proof.** The equation (7.4) is a simple consequence of (1.1). To prove (7.5) we shall use the fact that \(\lim_{t \to 0} T_t f = f\), where the convergence is in \(L^2(\mathbb{R}^d)\). From \(L^2\) convergence we have a.e. convergence for a subsequence. Applying this to \(f_n(x) = \chi_{Q(0,n)}(x)\), by a diagonal argument, we obtain a sequence \(t_k > 0\) that tends to zero, such that for a.e. \(x \in \mathbb{R}^d\) we have

\[
\lim_{k \to \infty} \int_{\mathbb{R}^d} T_{t_k} f(x, y) \, dy = 1.
\]

\[
(7.6)
\]
Now, we are going to prove (7.6) for arbitrary sequence $s_j$ such that $\lim_{j \to \infty} s_j = 0$. Without loss of generality we can assume that $t_k$ is decreasing. For $j \in \mathbb{N}$, let $k_j$ be such that $t_{k_j-1} < s_j \leq t_{k_j}$ ($k_j = 1$ when $s_j > t_{k_1}$). Then $t_{k_j} = s_j + r_j$, where $\lim_{j \to \infty} t_{k_j} = \lim_{j \to \infty} r_j = 0$. By (1.1) and the semigroup property,

$$
\int_{\mathbb{R}^d} T_{t_{k_j}}(x, y) \, dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T_{s_j}(x,z) T_{r_j}(z,y) \, dz \, dy \leq \int_{\mathbb{R}^d} T_{s_j}(x,z) \, dz \leq 1.
$$

Letting $j \to \infty$, by (7.6), we have that $\lim_{j \to \infty} \int_{\mathbb{R}^d} T_{s_j}(x,z) \, dz = 1$. \hfill \Box

**Proof of Proposition 7.1.** Assume that $f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \subseteq L^1_{\text{loc}}(\mathbb{R}^d)$. By the Lebesgue differentiation theorem

$$
\lim_{s \to 0} |Q(x,s)|^{-1} \int_{Q(x,s)} |f(y) - f(x)| \, dy = 0
$$

for a.e. $x \in \mathbb{R}^d$. Assume that $x \in \mathbb{R}^d$ is such that (7.7), (7.4) and (7.5) are satisfied for all rational $r > 0$. The set of such points has full measure. For $\varepsilon > 0$ fixed, we shall show that $|T_t f(x) - f(x)| < C\varepsilon$ for $t$ small enough. Let $r > 0$ be a fixed rational number such that for $s < r$ we have

$$
\int_{Q(x,s)} |f(y) - f(x)| \, dy \leq \varepsilon |Q(x,s)|.
$$

We can assume that $\sqrt{t} < r$. For such $t$, write

$$
T_t f(x) - f(x) = T_t f(x) \left( \int_{|x-y| < r} T_t(x,y) \, dy - 1 \right) + \int_{|x-y| \geq r} T_t(x,y) f(y) \, dy
$$

$$
+ \int_{|x-y| < \sqrt{t}} T_t(x,y) (f(y) - f(x)) \, dy + \int_{\sqrt{t} < |x-y| < r} T_t(x,y) (f(y) - f(x)) \, dy
$$

$$
= A_1 + A_2 + A_3 + A_4.
$$

By using (7.5), we get that $A_1 < \varepsilon$ for $t$ small enough. For the summand $A_2$ we consider two cases:

- if $f \in L^\infty(\mathbb{R}^d)$, then $|A_2| < \varepsilon$ for $t$ small enough by (7.4),
- if $f \in L^1(\mathbb{R}^d)$, then $|A_2| \leq C t^{-d/2} \exp \left( -r^2/t \right) \|f\|_{L^1(\mathbb{R}^d)} < \varepsilon$ for $t$ small enough.

By (1.1) and (7.8), for $t$ small enough,

$$
A_3 \leq C t^{-d/2} \int_{|x-y| < \sqrt{t}} |f(y) - f(x)| \, dy \leq C\varepsilon.
$$

To estimate $A_4$ denote $N = \left\lceil \log_2 \frac{L}{\sqrt{t}} \right\rceil$, so that $r \leq \sqrt{t}2^N \leq 2r$. Let

$$
R_n = \left\{ x \in \mathbb{R}^d : r2^{-n} < |x-y| < r2^{-n+1} \right\}
$$

for $n = 1, \ldots, N$. By (1.1) and (7.8),

$$
A_4 \leq C t^{-d/2} \sum_{n=1}^{N} \int_{R_n} \exp \left( -\frac{|x-y|^2}{4t} \right) |f(y) - f(x)| \, dy
$$

$$
\leq C t^{-d/2} \sum_{n=1}^{N} \exp \left( -\frac{r2^{-n}}{ct} \right) \int_{R_n} |f(y) - f(x)| \, dy
$$

$$
\leq C\varepsilon \sum_{n=1}^{N} \left( \frac{r2^{-n}}{\sqrt{t}} \right)^d \exp \left( -\frac{r2^{-n}}{c\sqrt{t}} \right) \leq C\varepsilon \frac{\sqrt{t}2^N}{r} \leq C\varepsilon.
$$

\hfill \Box

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MARcin PREISNER

Institut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: preisner@math.uni.wroc.pl