Multipole Moments of Isolated Horizons

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Abstract

To every axi-symmetric isolated horizon we associate two sets of numbers, $M_n$ and $J_n$ with $n = 0, 1, 2, \ldots$, representing its mass and angular momentum multipoles. They provide a diffeomorphism invariant characterization of the horizon geometry. Physically, they can be thought of as the ‘source multipoles’ of black holes in equilibrium. These structures have a variety of potential applications ranging from equations of motion of black holes and numerical relativity to quantum gravity.

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I. INTRODUCTION

Multipole moments play an important role in Newtonian gravity and Maxwellian electrodynamics. Conceptually, there are two distinct notions of multipole moments —source multipoles which encode the distribution of mass (or charge-current), and field multipoles which arise as coefficients in the asymptotic expansions of fields. In Newtonian gravity, the first set is of direct interest to equations of motion of extended bodies while the second determines the gravitational potential outside sources. However, via field equations one can easily relate the two sets of multipoles. In the Maxwell theory, the rate of change of the source dipole moment is directly related to the energy flux measured at infinity. Because of such useful properties, there has been considerable interest in extending these notions to general relativity.

Results of the Maxwell theory were extended to the weak field regime of general relativity —i.e., linearized gravity— quite some time ago. Already in 1916 Einstein obtained the celebrated formula relating the rate of change of the quadrupole moment of the source to the energy flux at infinity [1]. In the fifties, Sachs and Bergmann extended the relation between the source multipoles and asymptotic fields [2]. However, in the framework of exact general relativity, progress has been slow. In the seventies, Geroch, Hansen and others [3] restricted themselves to the stationary context and introduced field multipoles by analyzing the asymptotic structure of suitable geometric fields constructed from the metric and equations they satisfy near spatial infinity. As in electrodynamics —and, in contrast to the situation in the Newtonian theory— they found that there are two sets of multipoles, the mass multipoles $M^{(n)}$ and the angular momentum multipoles $J^{(n)}$. In static situations, all the angular momentum multipoles $J^{(n)}$ vanish and the mass multipoles $M^{(n)}$ are constructed from the norm of the static Killing field which, like the Newtonian gravitational potential, satisfies a Laplace-type equation outside sources. In the (genuinely) stationary context, the $J^{(n)}$ are non-zero and are analogous to the magnetic multipoles in the Maxwell theory. In the Newtonian theory, since the field multipoles are defined as coefficients in the $1/r$ expansion of the gravitational potential, knowing all multipoles one can trivially reconstruct the potential outside sources. In general relativity, there are considerable coordinate ambiguities in performing asymptotic expansions of the metric. Therefore, Geroch and Hansen were led to define their multipoles using other techniques. Nonetheless, Beig and Simon [4] have established that the knowledge of these multipoles does suffice to determine the space-time geometry near infinity. Their construction also shows that, if two stationary space-times have the same multipoles, they are isometric in a neighborhood of infinity.

In the Geroch-Hansen framework, one works on the 3-dimensional manifold of orbits of the stationary Killing field and multipole moments $M^{(n)}$ and the $J^{(n)}$ arise as symmetric, trace-free tensors in the tangent space of the point $\Lambda$ at spatial infinity of this manifold. Now, the vector space of n-th rank, symmetric traceless tensors on $\mathbb{R}^3$ is naturally isomorphic with the vector space spanned by the linear combinations $\sum_{m=-n}^{m=n} a_{nm} Y^m_n(\theta, \phi)$ of spherical harmonics on the unit sphere $S^2$ in $\mathbb{R}^3$. Therefore, each $M^{(n)}$ and $J^{(n)}$ uniquely defines a set of numbers, $M_{nm}$ and $J_{nm}$ with $m \in \{ -n, -n+1, \ldots, n-1, n \}$. Finally, let us consider stationary space-times which are also axi-symmetric (e.g., the Kerr space-time). Then $M_{nm} = M_n \delta_{m,0}$ and $J_{nm} = J_n \delta_{m,0}$, whence multipoles are completely characterized by two sets of numbers, $M_n, J_n$ with $n = 0, 1, \ldots$. This fact will be useful to us because most of our analysis will be restricted to axi-symmetric isolated horizons.

In stationary space-times, then, the situation with field multipoles is quite analogous to
that in the Newtonian theory. The status of source multipoles, on the other hand, has not been so satisfactory. Dixon developed a framework to define source multipoles [5] but the program did not reach the degree of maturity enjoyed by the field multipoles. On the other hand, it is the source multipoles, rather than the field multipoles defined at infinity, that appear to be more directly relevant to the motion of extended bodies in general relativity. (For recent discussions, see [6].) In particular, it would be interesting to obtain useful generalizations of the quadrupole formula to fully relativistic objects such as neutron stars and black holes. Is there, for example, a relation between the flux of energy falling across the horizon and the rate of change of the quadrupole moment of a black hole? To analyze such issues one needs a reliable definition of the source quadrupole moment, and more generally, of source multipoles.

In this paper, we will focus on the problem of defining the analogs of source multipoles for black holes in equilibrium. The isolated horizon framework provides a suitable quasi-local arena for describing such black holes [7–11]. Thus, our task is to introduce an appropriate definition of multipoles which capture distortions of the horizon geometry and of the distribution of angular momentum currents on the horizon itself, and explore properties of these multipoles. Do these multipoles provide a diffeomorphism invariant characterization of the horizon geometry? If so, they may play a significant role in the analysis of equations of motion. Do they suffice to determine space-time geometry in the neighborhood of the horizon in stationary space-times? These are attractive possibilities. But since the horizon lies in a genuinely strong field region, a priori it is not obvious that a useful notion of multipoles can be introduced at all. Indeed, since the problem of defining and analyzing multipoles has turned out to be so difficult already for relativistic fluids, at first glance it may seem hopelessly difficult for black holes. However, a tremendous simplification occurs because black holes are purely geometric objects; one does not have to resolve the messy and difficult issues related to the details of matter sources. We will see that this simplification makes it possible to carry out a detailed analysis and satisfactorily address the issues raised above.

We will restrict our detailed analysis to axi-symmetric (or type II) isolated horizons, where the symmetry restriction applies only to the horizon geometry (and to the pull-back of the Maxwell field to the horizon) and not to the entire space-time. The material is organized as follows. In section II, we recall the relevant facts about isolated horizons and axi-symmetric geometric structures. In section III we define the multipoles $M_n$ and $J_n$ and show that these two sets of numbers provide a complete characterization of the isolated horizon geometry. For the specific case of Maxwell sources, we introduce another pair, $Q_n, P_n$ of electromagnetic multipoles and show that they suffice to determine the pull-back of the Maxwell tensor as well as that of its dual. Finally, the initial value formulation based on two intersecting null surfaces [12] implies that, if there is a stationary Killing field in the neighborhood of the horizon, multipoles also suffice to determine the near horizon geometry [13]. In section IV, we discuss potential extensions and applications of the framework ranging from numerical relativity to quantum gravity.

Unless otherwise stated, in this paper all manifolds and fields will be assumed to be smooth.
II. PRELIMINARIES

A. Isolated horizons

In this sub-section we briefly recall the relevant notions pertaining to isolated horizons [7, 9–11]. This discussion will also serve to fix our notation.

Let us begin with the basic definitions [9]. A non-expanding horizon $\Delta$ is a null, 3-dimensional sub-manifold of the 4-dimensional space-time $(\mathcal{M}, g_{ab})$, with topology $S^2 \times \mathbb{R}$, such that:

i) the expansion $\theta_\ell$ of its null normal $\ell$ vanishes; and,

ii) field equations hold on $\Delta$ with stress energy $T_{ab}$, satisfying the very weak requirement that $-T_{ab}\ell^b$ is a future-directed, causal vector. (Throughout, $\ell^a$ will be assumed to be future pointing.)

Before discussing their consequences, let us note two facts about these assumptions: i) if the expansion vanishes for one null normal, it vanishes for all; and, ii) the condition on the stress energy is satisfied by all the standard matter fields provided they are minimally coupled to gravity. (Non-minimally coupled matter can be incorporated by a small modification of this condition. See, e.g., [14].) Since $\Delta$ is a null 3-surface, its intrinsic ‘metric’ $q_{ab}$ has signature 0,+,+. The definition ensures that $T_{ab}\ell^b \propto \ell^a$ and $L_{\ell} q_{ab} \hat{=} 0$, where $\hat{=}$ denotes equality restricted to points of $\Delta$. In particular, the area of any 2-sphere cross-section is constant on $\Delta$. We will denote it by $a_\Delta$. The definition also implies that the space-time derivative operator $\nabla$ naturally induces a unique derivative operator $D$ on $\Delta$. Furthermore, $D_a^b = \omega_a^b$ for some globally defined 1-form $\omega_a$ on the horizon. This 1-form will play an important role.

The pair $(q_{ab}, D)$ is referred to as the intrinsic geometry of $\Delta$. The notion that the black hole itself is in equilibrium is captured by requiring that this geometry is time independent: An isolated horizon $(\Delta, [\ell])$ is a non-expanding horizon $\Delta$ equipped with an equivalence class of null normals $\ell^a$, where $\ell^a \sim \ell'^a$ if and only if $\ell'^a = c\ell^a$ for a positive constant $c$, such that $[L_\ell, D] \hat{=} 0$ on $\Delta$.

Since $\Delta$ is a null surface, given any one null normal $\ell^a$ in $[\ell]$, we have $\ell^a D_a \ell^b = \kappa_\ell \ell^b$ for some $\kappa_\ell$. (Thus, $\kappa_\ell = \ell^a \omega_a$. ) The requirement $[L_\ell, D] \hat{=} 0$ further implies that $\kappa_\ell$ is constant on $\Delta$. It is referred to as the surface gravity of $\Delta$ with respect to $\ell^a$. Note that if $\ell'^a = c\ell^a$ then $\kappa_{\ell'} = c\kappa_\ell$. Thus, the value of surface gravity refers to a specific null normal; it is not a property of $(\Delta, [\ell])$. However, one can unambiguously say whether the given isolated horizon is extremal (i.e., has $\kappa_\ell = 0$) or non-extremal (i.e., has $\kappa_\ell \neq 0$).

The isolated horizon definition extracts from the notion of the Killing horizon just that ‘tiny’ part which turns out to be essential for black hole mechanics [9, 11] and, more generally, to capture the notion that the horizon is in equilibrium, allowing for dynamical processes and radiation in the exterior region [7]. Indeed, Einstein’s equations admit solutions with isolated horizons in which there is radiation arbitrarily close to the horizons [15]. Finally, note that the definition uses conditions which are local to $\Delta$. Hence, unlike event horizons one does not require the knowledge of full space-time; the notion is not ‘teleological’.

Of particular interest to our analysis is the case when the only matter present at the horizon is a Maxwell field. In this case, only the pull-backs $B_{ab}$ and $E_{ab}$ to $\Delta$, respectively of

\[^1\] Given $(\Delta, g_{ab}, D)$, one can show that, generically, there is an unique equivalence class $[\ell]$ of null normals such that $(\Delta, [\ell])$ is an isolated horizon. However, our analysis will not be restricted to this case.
the Maxwell field $F_{ab}$ and its dual $*F_{ab}$ are needed in the isolated horizon analysis. We will refer to the quadruplet $(q_{ab}, D, B_{ab}, E_{ab})$ as the Einstein-Maxwell geometry of the isolated horizon.

**Remark:** Note that the notion of an isolated horizon $(\Delta, [\ell])$, can be formulated intrinsically, using only those fields which define its geometry, without reference to the full space-time metric or curvature. Specifically: i) the condition $\ell^a$ is a null normal to $\Delta$ is captured in the property that $q_{ab}$ has signature $0,+,+$ with $q_{ab}\ell^b \approx 0$; ii) the condition $\theta^a \approx 0$ can be replaced by $L_\ell q_{ab} \approx 0$; iii) the requirement on the stress-energy refers only to fields $B_{ab}$ and $E_{ab}$; and iv) the condition $[\mathcal{L}_\ell, D] \approx 0$ refers only to $D$ (and not to the full space-time connection). This is why the quadruplet $(q_{ab}, D, B_{ab}, E_{ab})$, defined intrinsically on $\Delta$, was singled out to introduce the notion of Einstein-Maxwell horizon geometry. All equations that are required in the derivation of the laws of mechanics of isolated horizons in Einstein-Maxwell theory [9, 11] as well as the main results of this paper refer just to these fields.

Next, let us examine symmetry groups of isolated horizons. A symmetry of $(\Delta, \ell^a, q_{ab}, D, B_{ab}, E_{ab})$ is a diffeomorphism on $\Delta$ which preserves $q_{ab}, D, B_{ab}$ and $E_{ab}$ and at most rescales $\ell^a$ by a positive constant. It is clear that diffeomorphisms generated by $\ell^a$ are symmetries. So, the symmetry group $G_\Delta$ is at least 1-dimensional. The question is: Are there any other symmetries? At infinity, we generally have a universal symmetry group (such as the Poincaré or the anti-de Sitter) because all metrics under consideration approach a fixed metric (Minkowskian or anti-de Sitter) there. In the case of the isolated horizons, generically we are in the strong field regime and space-time metrics do not approach a universal metric. Therefore, the symmetry group is not universal. However, there are only three universality classes:

i) Type I: the isolated horizon geometry is spherical; in this case, $G_\Delta$ is four dimensional;

ii) Type II: the isolated horizon geometry is axi-symmetric; in this case, $G_\Delta$ is two dimensional;

iii) Type III: the diffeomorphisms generated by $\ell^a$ are the only symmetries; $G_\Delta$ is one dimensional.

Note that these symmetries refer only to the horizon geometry. The full space-time metric need not admit any isometries even in a neighborhood of the horizon. Physically, type II horizons are the most interesting ones. They include the Kerr-Newman horizons as well as their generalizations incorporating distortions (due to exterior matter or other black holes) and hair. Our main results refer to the type II case.

Finally, one can use the field equations to isolate the freely specifiable data which determines the Einstein-Maxwell geometry of an isolated horizon [10, 16]. The analysis is naturally divided in to two cases: $\kappa_\ell \neq 0$ and $\kappa_\ell = 0$. Fix any 2-sphere cross-section $\tilde{\Delta}$ of $\Delta$ and denote by $\tilde{q}^{ab}$ the natural projection operator on $\tilde{\Delta}$. Then, on any type II horizon, we have the following.²

- In the non-extremal case, the free data consists of the projections $\tilde{q}_{ab}, \tilde{\omega}_a, \tilde{B}_{ab}, \tilde{E}_{ab}$ to $\tilde{\Delta}$ (using $\tilde{q}^{ab}$) of $q_{ab}, \omega_a, B_{ab}, E_{ab}$ on $\Delta$. (Recall that $\omega_a$ is defined by $D_a \ell^b = \omega_a \ell^b$.) That is, given the free data on $\tilde{\Delta}$, using the projections of the field equations on $\tilde{\Delta}$, one can

² For convenience of the reader, proofs of these assertions are sketched in Appendix A. Throughout this paper, the term ‘free data’ refers to freely specifiable fields. In order to reconstruct the horizon geometry, in addition to specifying this data one also needs to fix some constants, such as the horizon area $a_\Delta$.
reconstruct the full Einstein-Maxwell geometry \((q_{ab}, D, B_{ab}, E_{ab})\) on \(\Delta\).

- In the extremal case, remarkably, the fields \(q_{ab}, \omega_a, B_{ab}\) and \(E_{ab}\) turn out to be all universal; they are the same as those in the extremal Kerr-Newman geometry \([16]\). The free data consists of \(\tilde{S}_{ab}\), a symmetric, second rank tensor on \(\tilde{\Delta}\), determined by \(D\) via

\[
\tilde{S}_{ab} = q_{ac}q_{bd}D_{cn}n_d
\]

where \(n_a\) is the covariant normal to \(\tilde{\Delta}\) within \(\Delta\), satisfying \(\ell^a n_a = -1\). (Thus, \(\tilde{S}_{ab}\) is the analog of the ‘extrinsic curvature of \(\tilde{\Delta}\) within \(\Delta\).) Given a Kerr-Newman quadruplet \(q_{ab}, \omega_a, B_{ab}\) and \(E_{ab}\) the only freedom in the Einstein-Maxwell horizon geometry lies in the part of \(D\) which determines the ‘extrinsic curvature’ \(\tilde{S}_{ab}\) of \(\Delta\).

These results will play an important role in section 3. Specifically, we will construct multipoles from the free data and show that they provide a diffeomorphism invariant characterization of the free data and hence, via field equations, of the horizon geometry.\(^3\)

**B. Axi-symmetric structures**

In this sub-section we will recall a few facts about axi-symmetric geometries on \(S^2\). This discussion will also make our assumptions explicit.

Let \(S\) be a manifold with the topology of a 2-sphere, equipped with a metric \(\tilde{q}_{ab}\). We will denote by \(\tilde{\epsilon}_{ab}\) the alternating tensor on \(S\) compatible with \(\tilde{q}_{ab}\), by \(a\) the area of \(S\) and by \(R\) its radius, defined by \(a = 4\pi R^2\). We will say that \((S, \tilde{q}_{ab})\) is axi-symmetric if it admits a Killing field \(\phi^a\) with closed orbits which vanishes exactly at two points of \(S\). The two points will be referred to as poles. We will now show that such metric manifolds carry invariantly defined coordinates and discuss properties of the metric coefficients in these coordinates.

Since \(L_{\phi^b}\tilde{\epsilon}_{ab} = 0\), there exists a unique (globally defined) function \(\zeta\) on \(S\) such that

\[
\tilde{D}_a\zeta = \frac{1}{R^2} \tilde{\epsilon}_{ab} \phi^b \quad \text{and} \quad \oint_S \zeta \, \tilde{\epsilon} = 0.
\]

It is clear that \(L_{\phi^b}\zeta \equiv 0\) and \(\tilde{D}_a\zeta\) vanishes only at the poles. Hence, \(\zeta\) is a monotonic function on the 1-dimensional manifold \(\tilde{S}\) of orbits of \(\phi^a\). Now \(\tilde{S}\) is a closed interval and the end points correspond to the two poles. Hence \(\zeta\) monotonically increases from one pole to another. We will say that \(\zeta\) assumes its minimum value at the south pole and the maximum at the north pole.

Next, let us introduce a vector field \(\zeta^a\) on \(S' = (S - \text{poles})\) via:

\[
\tilde{q}_{ab}\zeta^a\phi^b = 0 \quad \text{and} \quad \zeta^a\tilde{D}_a\zeta = 1
\]

Then it follows that

\[
\zeta^a = \frac{R^4}{\mu^2} \tilde{q}^{ab} \tilde{D}_b\zeta,
\]

where \(\mu^2 = \tilde{q}_{ab}\phi^a\phi^b\) is the squared norm of \(\phi^a\). Hence integral curves of \(\zeta^a\) go from the south pole to the north (and \(\zeta^a\) diverges as one approaches the poles). Using \(\zeta^a\), we can

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\(^3\) This general philosophy is rather similar to that used by Janis and Newman to introduce a notion of multipoles using ‘free data’ on null surfaces \([17]\). However, since our analysis is restricted to isolated horizons rather than general null surfaces, our results are free of coordinate ambiguities.
define a preferred affine parameter \( \phi \) of \( \phi^a \) as follows: Fix any one integral curve \( I \) of \( \zeta^a \) in \( S' \) and set \( \phi = 0 \) on \( I \). Thus, on \( I \), we have \( \mathcal{L}_\zeta \phi = 0 \). Now, since \( \phi^a \) is a Killing field and \( \zeta^a \) is constructed uniquely from \( (\tilde{g}_{ab}, \phi^a) \), it follows that \( \mathcal{L}_\phi \zeta^a = 0 \). Hence, we conclude: \( \mathcal{L}_\phi (\mathcal{L}_\zeta \phi) = 0 \). Since \( \mathcal{L}_\zeta \phi = 0 \) on \( I \), it now follows that \( \phi = \text{const} \) on every orbit of \( \zeta^a \). This now implies that the affine parameter \( \phi \) of \( \phi^a \) has the same range on every orbit of \( \phi^a \). Without loss of generality we will assume that \( \phi^a \) is such that \( \phi \in [0, 2\pi) \) on \( S \).

Thus, starting from geometry, we have constructed two coordinates \( \zeta, \phi \) on \( S \) such that \( \phi^a \equiv (\partial/\partial \phi)^a \) and \( \zeta^a \equiv (\partial/\partial \zeta)^a \) are orthogonal. Eqs (2.1) and (2.3) now imply that the metric has the form:

\[
\tilde{q}_{ab} = R^2 (f^{-1} D_a \zeta D_b \zeta + f D_a \phi D_b \phi) \quad \text{and} \quad \tilde{q}^{ab} = \frac{1}{R^2} (f \zeta^a \zeta^b + f^{-1} \phi^a \phi^b) \quad (2.4)
\]

where \( f = \mu^2 / R^2 \). The fact that the area of \( S \) is \( 4\pi R^2 \) now implies that the range of \( \zeta \) is necessarily \([-1, 1] \). Conversely, given any ‘coordinates’ \( \zeta', \phi' \) in \([-1, 1] \times \mathbb{R} / 2\pi \mathbb{Z} \) in which the metric can be expressed in (the primed version of) the form (2.4), it is easy to show that \( \zeta' = \zeta \) and \( \phi' \) is at most a rigid shift of \( \phi \). (For an analogous construction in Newman-Penrose notation, see [18].)

Functions \( \zeta, \phi \) serve as ‘coordinates’ on \( S \) modulo usual caveats: they are ill-defined at the poles and \( \phi \) has a \( 2\pi \) discontinuity on the integral curve \( I \) of \( \zeta^a \). We have to ensure that the metric \( \tilde{q}_{ab} \) is smooth in spite of these coordinate problems. The discontinuity at \( I \) causes no problems. However, poles do require a careful treatment because the norm \( \mu \) of \( \phi^a \) — and hence \( f \) — vanishes there. Smoothness of \( \tilde{q}_{ab} \) at the poles (i.e. absence of conical singularities) imposes a non-trivial condition on \( f \):

\[
\lim_{\zeta \to \pm 1} f'(\zeta) = \mp 2 \quad (2.5)
\]

where ‘prime’ denotes derivative with respect to \( \zeta \). On a metric 2-sphere, we have \( f = 1 - \zeta^2 \) and we can bring the metric to the standard form simply by setting \( \zeta = \cos \theta \). In the general case, \( f \) has the same values and first derivatives at the poles as on a metric 2-sphere. Using l’Hôpital’s rule, one can show that this fact suffices to ensure that the metric (2.4) is smooth at the poles.

Finally, we note a property of these axi-symmetric metrics which will be useful in section III B. A simple calculation shows that the scalar curvature \( \tilde{\mathcal{R}} \) of \( \tilde{q}_{ab} \) is given by:

\[
\tilde{\mathcal{R}}(\zeta, \phi) = -\frac{1}{R^2} f''(\zeta) . \quad (2.6)
\]

By integrating it twice with respect to \( \zeta \) and using the boundary conditions \( (f|_{\zeta=-1}) = 0 \) and \( (f'|_{\zeta=-1}) = 2 \), one can reconstruct the function \( f \) from the scalar curvature:

\[
f = -R^2 \left[ \int_{-1}^{\zeta_1} d\zeta_1 \int_{-1}^{\zeta_2} d\zeta_2 \tilde{\mathcal{R}}(\zeta_2) \right] + 2(\zeta + 1) \quad (2.7)
\]

Thus, thanks to the preferred coordinates admitted by an axi-symmetric geometry on \( S \), given the area \( a \) of \( S \) and the scalar curvature \( \tilde{\mathcal{R}} \), the metric \( \tilde{g}_{ab} \) is completely determined.

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4 To make this argument precise, one should work on the universal covering of the orbits of \( \phi^a \) on \( S' \) but the additional steps are straightforward.
Remark: Coordinates \((ζ, φ)\), determined by the axi-symmetry of \(q_{ab}\), also enable us to define a canonical round, 2-sphere metric \(q_{ab}^0\) on \(S\):
\[
q_{ab}^0 = R^2(f_0^{-1}D_aζ D_bζ + f_o D_aφ D_bφ)
\]  
(2.8)
where \(f_o = 1 − ζ^2\). Note that \(q_{ab}^0\) has the same area element as \(q_{ab}\). This round metric captures the extra structure made available by axi-symmetry in a coordinate invariant way. The availability of \(q_{ab}^0\) enables one to perform a natural spherical harmonic decomposition on \(S\). This fact will play a key role throughout section III.

III. MULTIPOLES OF TYPE II ISOLATED HORIZONS

This section is divided into four parts. In the first three, we restrict ourselves to non-extremal, type II isolated horizons with no matter fields on them. We begin in the first part by defining a set of multipoles, \(I_n, L_n\), starting from the horizon geometry. We then show that one can reconstruct the horizon geometry starting from these two sets of numbers. In the second, we show that if two isolated horizons have the same multipoles \(I_n, L_n\), their geometries are related by a diffeomorphism. Thus, these two sets of numbers provide a convenient diffeomorphism invariant characterization of the horizon geometry. Therefore, we refer to \(I_n, L_n\) as geometric multipoles. In the third part, we rescale these moments by appropriate dimensionful factors to obtain mass and angular momentum multipoles \(M_n, J_n\). Finally, in the fourth part we first discuss electromagnetic multipoles and then summarize the situation on extremal isolated horizons.

A. Geometric Multipoles

Let \((Δ, [ℓ])\) be a non-extremal, type II isolated horizon with an axial Killing field \(φ^a\). In this sub-section, we will ignore matter fields on \(Δ\) and concentrate just on the horizon geometry defined by \(\{q_{ab}, D\}\). Fix a cross-section \(∆\) of \(Δ\). Then, as summarized in section III A, the free data that determine the horizon geometry consists of the pair \(\{q_{ab}, ∆\}\) where \(q_{ab}\) is the intrinsic metric on \(∆\) and \(ω_a\) is the projection on \(∆\) of the 1-form \(ω_a\) on \(Δ\) (defined by \(D_aℓ^b = ω_aℓ^b\)). However, there is some gauge freedom associated with our choice of the cross-section \(∆\) [10]. We will first spell it out and then define multipoles using gauge invariant fields.

For simplicity of presentation, let us fix a null normal \(ℓ^a\) in \([ℓ^a]\). Then, \(∆\) can be regarded as a leaf of a foliation \(u = \text{const}\) such that \(ℓ^a D_a u \equiv 1\). For notational simplicity, we will set \(n_a = −D_a u\) so that \(n_a\) is the covariant normal to the foliation satisfying \(ℓ^n n_a = −1\). The projection operator \(q_{ab}^a b\) on the leaves of this foliation is given by \(q_{a} b = δ^b a + n_a ℓ^b\). Hence, \(q_{ab} = q_{ab}\) and \(ω_a = ω_a + κ ℓ n_a\) as tensor fields on \(Δ\). Since \(L_ℓ q_{ab} \equiv 0\) and \(L_ℓ ω_a \equiv 0\) on any isolated horizon, and since \(L_ℓ n_a \equiv 0\) from the definition of \(n_a\), it follows that \(q_{ab}^a b\) on any one leaf is mapped to that on any other leaf under the natural diffeomorphism (generated by \(ℓ^a\)) relating them. Let us now consider a cross-section \(∆′\) which does not belong to this foliation. Let \(u' = \text{const}\) denote the corresponding foliation. Set \(F = u − u'\). Then, regarded as tensor fields on \(Δ\) the two sets of free data are related by
\[
q_{ab}' = q_{ab} \quad \text{and} \quad ω_a' = ω_a + κ ℓ D_a F
\]  
(3.1)
Thus, under the natural diffeomorphism (defined by the integral curves of $\ell^a$) between $\Delta$ and $\Delta'$, $\tilde{q}_{ab}$ is mapped to $\tilde{q}'_{ab}$ but $\tilde{\omega}_a$ is not mapped to $\tilde{\omega}'_a$: the difference is a gradient of a function. This is the gauge freedom in the free data.

It is therefore natural to consider, in place of $\tilde{\omega}_a$, its curl. From the isolated horizon framework, it is known that for any null tetrad $\ell^a, n^a, m^a, \hat{m}^a$ such that $\ell^a \in [\ell]$, the Weyl components $\Psi_0$ and $\Psi_1$ vanish on $\Delta$ whence $\Psi_2$ is gauge invariant [9, 10], and the curl of $\tilde{\omega}_a$ is given just by $\text{Im}\Psi_2$:

$$D_{[\alpha} \tilde{\omega}_{\beta]} = \text{Im}\Psi_2 \epsilon_{ab} \tag{3.2}$$

where $\epsilon_{ab}$ is the natural area element on $\Delta$ (which satisfies $\epsilon_{ab} \ell^a \cong 0$ and $L_\ell \epsilon_{ab} \cong 0$). Thus, the gauge invariant content of $\tilde{\omega}_a$ is coded in $\text{Im}\Psi_2$.

The second piece of free data is the metric $\tilde{q}_{ab}$ on $\tilde{\Delta}$. In section II B we showed that using an invariant coordinate system $(\zeta, \phi)$, one can completely determine $\tilde{q}_{ab}$ in terms of a number, the area $a$ and a function, its scalar curvature $\tilde{R}$. If, as assumed in this sub-section, the cosmological constant is zero and there are no matter fields on $\Delta$, then $\tilde{R} = -4 \text{Re}\Psi_2$ [8]. Hence the gauge invariant part of the free data that determines the horizon geometry is neatly coded in the Weyl component $\Psi_2$.

It is therefore natural to define multipoles using a complex function $\Phi_\Delta$ on $\Delta$:

$$\Phi_\Delta := \frac{1}{4} \tilde{R} - i \text{Im}\Psi_2. \tag{3.3}$$

(Thus, in absence of matter on $\Delta$, $\Phi_\Delta = -\Psi_2$ while in presence of matter it is given by $\Phi_\Delta \cong -\Psi_2 + (1/4) R_{ab} \tilde{q}_a^b - (1/12) R$, where $R_{ab}$ is the Ricci tensor and $R$ the scalar curvature of the 4-metric at the horizon.) Since all fields are axi-symmetric, using the natural coordinate $\zeta$ on $\Delta$, we are led to define multipoles as:

$$I_n + i L_n := \oint_\Delta \Phi_\Delta Y^0_n(\zeta) \, d^2\tilde{V} \tag{3.4}$$

or,

$$I_n := \frac{1}{4} \oint_\Delta \tilde{R} Y^0_n(\zeta) \, d^2\tilde{V} \quad \text{and} \quad L_n := - \oint_\Delta \text{Im}\Psi_2 Y^0_n(\zeta) \, d^2\tilde{V}.$$  

Here $Y^0_n$ are the $m = 0$ spherical harmonics, subject to the standard normalization:

$$\oint_\Delta Y^0_n Y^0_n \, d^2\tilde{V} = R^2_\Delta \delta_{n,m}, \tag{3.5}$$

where $R_\Delta$ is the horizon radius defined through its area $a_\Delta$ via $a_\Delta = 4\pi R^2_\Delta$. Thus, given any horizon geometry, we can define a set of two numbers, $I_n$ and $L_n$. Recall that $\zeta$ is determined entirely by the metric $\tilde{q}_{ab}$ and the rotational Killing field $\phi^a$. Therefore, it is immediate that if $(\Delta, q_{ab}, D) \text{ and } (\Delta', q'_{ab}, D')$ are related by a diffeomorphism, we have $I_n = I'_n$ and $L_n = L'_n$: the two sets of numbers are diffeomorphism invariant. If the isolated horizon were of type I, $q_{ab}$ would be spherically symmetric and $\text{Im}\Psi_2$ would vanish [9]. Then, the only non-zero multipole would be $I_0$ which, by the Gauss-Bonnet theorem, has a universal value (see below). Given a generic type II horizon, as we saw in section II B, the axi-symmetric structure provides a canonical 2-sphere metric $q_{ab}^\prime$ (which, in the type I case, coincides with the physical metric). The physical geometry has distortion and rotation built in it. The round metric $q_{ab}^\prime$ serves as an invariantly defined ‘background’ against which one can measure distortions and rotations. Multipoles $I_n, L_n$ provide a diffeomorphism invariant
characterization of these. More precisely, they encode the difference between the physical horizon geometry \((q_{ab}, D)\) and the fiducial, type I geometry determined by \((\tilde{q}_{ab}, \tilde{\omega}_a = 0)\).

Finally, note that \(I_n\) and \(L_n\) can not be specified entirely freely but are subject to certain algebraic constraints. The first comes from the Gauss-Bonnet theorem which, in the axisymmetric case, follows from the boundary condition (2.5) on \(f'\) and the expression (2.6) of the scalar curvature in terms of \(f\):

\[
I_0 = \frac{1}{4} \int_\Delta \tilde{R} Y_0^0(\zeta) \, d^2\tilde{V} = \sqrt{\pi}.
\] (3.6)

The second comes directly from the relation (3.2) between \(\text{Im} \Psi_2\) and \(\text{curl} \tilde{\omega}_a\):

\[
L_0 = -\frac{1}{\sqrt{4\pi}} \int_\Delta \text{Im} \Psi_2 \, d^2\tilde{V} = 0
\] (3.7)

The third constraint comes again from (2.5) and (2.6):

\[
I_1 := \frac{\sqrt{3}}{8\sqrt{\pi}} \int_\Delta \tilde{R} \zeta \, d^2\tilde{V} = 0
\] (3.8)

We will show in section III C that (3.7) implies that, as one would physically expect, the ‘angular momentum monopole’ necessarily vanishes and (3.8) implies that the mass dipole vanishes, i.e., that our framework has automatically placed us in the ‘center of mass frame of the horizon’. Next, because \(\Phi_\Delta\) is smooth, these moments have a certain fall-off. Let us assume that \(\Phi_\Delta\) is \(C^k\) (i.e., the space-time metric is \(C^{k+2}\)). Then as \(n\) tends to infinity, \(I_n\) and \(L_n\) must fall off in such a way that

\[
\sum_{n=0}^\infty \sum_{m=0}^{k+1} |n^m(I_n - iL_n)|^2 < \infty
\] (3.9)

Finally, there is a constraint arising from the fact that \(f\) is non-negative and vanishes only at the poles. (This property of \(f\) is essential for regularity of the metric.) Using (2.6) and the definition (3.4) of \(I_n\), one can express \(f\) in terms of \(I_n\). Unfortunately, the resulting restriction on multipoles is quite complicated:

\[
f(\zeta) = 1 - \zeta^2 + \sum_{n=2}^{\infty} \frac{2}{\sqrt{\pi}(2n+1)} \left[ -\frac{1}{2n+3} P_{n+2}(\zeta) + \frac{2(2n+1)}{(2n+3)(2n-1)} P_n(\zeta) - \frac{1}{2n-1} P_{n-2}(\zeta) \right] I_n \geq 0
\] (3.10)

and vanishes only if \(\zeta = \pm 1\), where \(P_n\) are the Legendre polynomials.

Remark: We conclude by noting some simplifications that occur in presence of additional symmetries. Certain space-time metrics, such as the Kerr solutions have a discrete (spatial) reflection symmetry, \((\zeta, \phi) \mapsto (-\zeta, \phi + \pi)\), under which \(\Psi_2 \mapsto \Psi_2^*\). Therefore, in the isolated horizon framework it is interesting to consider the case in which \(\Phi_\Delta \mapsto \Phi_\Delta^*\) and \(\epsilon_{ab} \mapsto \epsilon_{ab}\) under the discrete diffeomorphism \(\zeta \mapsto -\zeta\) on \(\Delta\). Then, since \(Y_n^0\) are even/odd under reflections if \(n\) is even/odd, it follows that \(I_n = 0\) for all odd \(n\) and \(L_n = 0\) for all even \(n\). Next, consider the case in which the isolated horizon is a Killing horizon of a static Killing
field. Then, one can show that $\text{Im}\Psi_2 \equiv 0$ [11]. Hence $L_n = 0$ for all $n$ but in general $I_n$ can be arbitrary, capturing possible distortions in the horizon geometry. Finally, consider type I isolated horizons on which the horizon geometry is spherically symmetric. Then, $\Phi_\Delta$ and $\epsilon_{ab}$ are spherically symmetric. It is then obvious from properties of spherical harmonics $Y_n^0(\zeta)$ that for all $n > 0$, we have $I_n = L_n = 0$. Since $L_0$ always vanishes, in this case the only non-trivial multipole is $I_0 = \sqrt{\pi}$.

B. Reconstruction of the horizon geometry and the uniqueness issue

1. Construction

Let us first show that the knowledge of the area and multipole moments suffices to reconstruct an isolated horizon geometry.

Suppose we are given only the radius $R_\Delta$ and the set $\{I_n, L_n\}$ of geometric multipoles of a $C^k$, non-extremal, type II isolated horizon. Then, one can explicitly construct a $C^k$, non-extremal, type II isolated horizon geometry $(\Delta, q_{ab}, D)$ such that area is given by $\Delta = 4\pi R_\Delta^2$, and geometric multipoles are given by $I_n$ and $L_n$.

Let $S$ be a smooth 2-manifold, topologically $S^2$. Let $(\zeta, \phi)$ be a spherical coordinate system on $S$, with $\zeta \in [-1, 1]$ and $\phi \in [0, 2\pi)$. Define

$$\tilde{R} := \frac{4}{R_\Delta^2} \sum_{n=0}^{\infty} I_n Y_n^0(\zeta)$$

$$\text{Im}\Psi_2 := -\frac{1}{R_\Delta^2} \sum_{n=0}^{\infty} L_n Y_n^0(\zeta).$$

Since $\{I_n, L_n\}$ are multipoles of a $C^k$ isolated horizon, we are guaranteed that the fields $\tilde{R}$ and $\text{Im}\Psi_2$ so defined are $C^k$. Next, define a function $f$ on $S$ via:

$$f(\zeta) := -\left[ R_\Delta^2 \int_{-1}^{\zeta} d\zeta_1 \int_{-1}^{\zeta_1} d\zeta_2 \tilde{R}(\zeta_2) \right] + 2(\zeta + 1)$$

and set

$$\tilde{q}_{ab} := R_\Delta^2 \left( \frac{1}{f(\zeta)} D_a \zeta D_b \zeta + f(\zeta) D_a \phi D_b \phi \right).$$

Since $I_n$ are the multipoles of an isolated horizon, (3.10) is satisfied whence it follows that $f$ is non-negative on $S$ and vanishes only at the ‘poles’, $\zeta = \pm 1$. Hence $\tilde{q}_{ab}$ is a smooth metric except possibly at the poles. Next, $I_0 = \sqrt{\pi}$ and $I_1 = 0$. Interestingly, these conditions imply $f'(\pm 1) = \mp 2$, and therefore ensure that the metric $\tilde{q}_{ab}$ can be smoothly extended to points $\zeta = \pm 1$ of $S$. Next, it is straightforward to verify that the area of $S$ with respect to this metric is given by $4\pi R_\Delta^2$ and its scalar curvature is given by $\tilde{R}$. Let us now turn to the multipoles $L_n$. Since these multipoles also come from a type II isolated horizon geometry, we have $L_0 = 0$. Eq (3.12) now implies $\oint \text{Im}\Psi_2 d^2V = 0$, whence there is a globally defined 1-form $\tilde{\omega}_a$ on $S$ such that

$$D_{[a} \tilde{\omega}_{b]} = \text{Im}\Psi_2 \epsilon_{ab}$$

where $\epsilon_{ab}$ is the alternating tensor on $(S, \tilde{q}_{ab})$. 

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Now consider a 3-manifold \( \Delta = \mathbb{S} \times \mathbb{R} \) and equip it with a vector field \( \ell^a \) along the ‘\( \mathbb{R} \)-direction’. Let \( \pi \) be an embedding of \( \mathbb{S} \) onto a 2-sphere cross-section \( \tilde{\Delta} \) of \( \Delta \). Then, the diffeomorphism \( \pi \) equips \( \tilde{\Delta} \) with fields \( \tilde{q}_{ab}, \tilde{\omega}_a \). This is the free data required to construct a non-extremal isolated horizon geometry \((q_{ab}, D)\) on \( \Delta \) (see \cite{10} or Appendix A). Since the data are axi-symmetric, so is the horizon geometry. Finally, from our definitions of multipole moments, it follows immediately that the multipole moments of this isolated horizon are given by \( \{I_n, L_n\} \).

**Remark:** Our construction in fact suffices to show the following stronger **existence result:**

Given a positive number \( R_\Delta \) and a set of real numbers \( I_n, L_n \) subject to the constraints Eqs (3.6)-(3.10), there is a non-extremal isolated horizon geometry \((\Delta, q_{ab}, D)\) with area \( a_\Delta = 4\pi R_\Delta^2 \) and geometric multipoles \( I_n, L_n \). We chose not to refer to the conditions (3.6)-(3.10) but assumed instead that the \( I_n, L_n \) arise from a type II isolated horizon for simplicity. Although the existence result is stronger, it is difficult to verify if a given set \((I_n, L_n)\) satisfies Eqs (3.9) and especially (3.10). The overall situation is similar at spatial infinity. There, the constraints that must be imposed on field multipoles to ensure that the formal power series converges to a non-degenerate metric are awkward to state and investigations have focussed on reconstruction \cite{3}.

2. **Uniqueness**

The question now is whether the geometry we constructed in section III B 1 is diffeomorphic to the one from which the multipoles were first constructed.

Let \((\Delta, [\ell], q_{ab}, D)\) and \((\Delta', [\ell'], q'_{ab}, D')\) be two type II non-extremal isolated horizons without matter fields, with same area and same geometric multipoles \( I_n, L_n \). Then there is a diffeomorphism from \( \Delta' \) to \( \Delta \) which maps \(([\ell'], q'_{ab}, D') \to ([\ell], q_{ab}, D)\).

Choose null normals \( \ell \in [\ell] \) and \( \ell' \in [\ell'] \) such that \( \kappa_\ell = \kappa_{\ell'} = \kappa \) and consider foliations \( u = \text{const} \) and \( u' = \text{const} \) of \( \Delta \) and \( \Delta' \) which are tangential to \( \phi^a \) and \( \phi'^a \) respectively such that \( \ell^a D_a u = 1 \) and \( \ell'^a D_a u' = 1 \). The intrinsic metrics on these leaves are axi-symmetric.

Hence, as in section II B, we can find invariant coordinates \((\zeta, \phi)\) and \((\zeta', \phi')\) on the two sets of cross-sections. Let \( \pi \) be the diffeomorphism from \( \Delta \) to \( \Delta' \) defined by \((u, \zeta, \phi) \mapsto (u', \zeta', \phi')\). Since the multipoles \( I_n \) are the same, it follows from (3.11) that \( \pi \) maps \( {\mathcal{R}}' \) to \( \mathcal{R} \) and hence \( \tilde{q}'_{ab} \) to \( \tilde{q}_{ab} \). Similarly, since the multipoles \( L_n \) are the same, it follows from (3.12) that \( \pi \) maps \( \text{Im}\Psi_2' \) on \( \Delta' \) to \( \text{Im}\Psi_2 \) on \( \Delta \). But this only implies that \( d\tilde{\omega}' \) is mapped to \( d\tilde{\omega} \), i.e.

\[
\pi^* \tilde{\omega}'_a = \tilde{\omega}_a + \kappa D_a h
\]  

(3.16)

for some function \( h \). Thus, in general, \( \pi \) does not map the free data on \( \Delta' \) to that on \( \Delta \). However, as the discussion in the beginning of sub-section III A suggests, this can be easily remedied by changing the foliation on \( \Delta \). Define \( \bar{u} = u - h \) on \( \Delta \). Then, the projection \( \bar{\omega}_a \) of \( \omega_a \) on the new foliation is given by \( \bar{\omega}_a = \tilde{\omega}_a + \kappa D_a h \). Hence the diffeomorphism \( \bar{\pi} : (\bar{u}, \zeta, \phi) \mapsto (u', \zeta', \phi') \) from \( \Delta \) to \( \Delta' \) does map the primed free data to the barred free data:

\[
\bar{\pi}^* \bar{\omega}'_a = \bar{\omega}_a \quad \text{and} \quad \bar{\pi}^* \bar{q}'_{ab} = \bar{q}_{ab} \equiv q_{ab}
\]  

(3.17)

This isomorphism between the two sets of free data naturally extends to an isomorphism between the two horizon geometries (see \cite{10} or Appendix A).

Thus, together with area, multipoles \( I_n, L_n \) provide a convenient, diffeomorphism invariant characterization of type II horizon geometries.
C. Mass and angular Momentum multipoles

As is obvious from their definition, \( I_n, L_n \) are all dimensionless. Therefore, it is difficult to attribute a direct physical interpretation to them. In this sub-section we will argue that they can be rescaled in a natural fashion to obtain quantities which can be interpreted as mass and angular momentum multipoles \( M_n \) and \( J_n \).

In the isolated horizon framework, the area \( a_\Delta \) is defined geometrically. One then defines the horizon angular momentum \( J_\Delta \) as a surface term in the expression of the generator of rotations, evaluated on a 2-sphere cross-section of the horizon. \( J_\Delta \) is unambiguous because type II horizons come with an axial symmetry [11]. The horizon mass \( M_\Delta \) is also defined using Hamiltonian methods as the generator of a preferred time translation. However, the preferred time translation varies from space-time to space-time. If \( J_\Delta = 0 \), it points along \( \ell^a \); if not, it is a suitable linear combination of \( \ell^a \) and \( \phi^a \), which can be fixed only after one knows the value of \( J_\Delta \) [11]. Thus, \( J_\Delta \) is defined first before one can fix \( M_\Delta \). In the same spirit, we will first define the angular momentum multipoles \( J_n \) and then the mass multipoles \( M_n \).

We begin by recalling a general fact about angular momentum. Fix a space-time \((M, g_{ab})\) and a space-like 2-sphere \( S \) in it. Let \( \varphi \) be any vector field tangential to \( S \). Then, by regarding \( S \) as the inner boundary of a partial Cauchy surface \( M \) [11], one can use the Hamiltonian framework to define a ‘conserved’ quantity \( J_\varphi^S \)

\[
J_\varphi^S = -\frac{1}{8\pi G} \oint_S K_{ab} \varphi^a dS^b
\]

(3.18)

where \( K_{ab} \) is the extrinsic curvature of \( M \). In a general space-time, this quantity is independent of \( M \) if and only if \( \varphi \) is divergence free with respect to the natural area element of \( S \). Thus, for each divergence-free \( \varphi \) on \( S, J_\varphi^S \) depends only on \( S \) and can be interpreted as the \( \varphi^a \)-component of a ‘generalized angular momentum’ associated with \( S \). If \( S \) happens to be a cross-section of \( \Delta \), as one would expect, one can recast this expression in terms of the fields defined by the isolated horizon geometry, making no reference at all to the partial Cauchy surface \( M \) [11]:

\[
J_\varphi^S \equiv -\frac{1}{8\pi G} \oint_S \varphi^a \hat{\omega}_n d^2V \equiv -\frac{1}{4\pi G} \oint_S f [\text{Im} \Psi_2] d^2V.
\]

(3.19)

Here \( f \) is a ‘potential’ for \( \varphi^a \) on \( \Delta \) —given by \( \varphi^a = \epsilon^{ab} D_b f \) — which exists because \( \mathcal{L}_{\varphi} \epsilon_{ab} \hat{=} 0 \). By the isolated horizon boundary conditions it follows that if \( \varphi^a \) is the restriction to \( S \) of a vector field on \( \Delta \) satisfying \( \mathcal{L}_\ell \varphi^a \hat{=} 0 \), then \( J_\varphi^S \) is independent of the 2-sphere cross-section \( S \) used in (3.19).

Thus, on any isolated horizon there is a well-defined notion of a ‘generalized angular momentum’ \( J_\varphi^S \), associated with any divergence free vector field \( \varphi^a \) satisfying \( \mathcal{L}_\ell \varphi^a \hat{=} 0 \). \( \text{Im} \Psi_2 \) plays the role of the ‘angular momentum aspect’. Hence, it is natural to construct the angular momentum multipoles \( J_n \) by rescaling the \( L_n \) with appropriate dimensionful factors. This strategy is supported also by other considerations. First, since \( \text{Im} \Psi_2 \) transforms as a pseudo-scalar under spatial reflections, we will automatically satisfy the criterion that the angular momentum multipoles should transform as pseudo tensors. Second, all angular momentum multipoles would vanish if and only if \( \text{Im} \Psi_2 \hat{=} 0 \) and this is precisely the condition defining non-rotating isolated horizons [9, 11]. Thus, the strategy has an overall coherence.
To obtain the precise expression, let us first recall the situation in magnetostatics in flat space-time. If the current distribution \( j^a \) is axi-symmetric, the \( n \)th magnetic moment \( M_n \) is given by:

\[
M_n = \int r^n P_n(\cos \theta) \vec{\nabla} \cdot (\vec{x} \times \vec{j}) \, d^3 x ,
\]

where \( P_n \) are the Legendre polynomials. If the current distribution is concentrated on the sphere \( S \) defined by \( r = R \), the expression simplifies to:

\[
M_n = -R^{n+1} \int_S (\bar{\epsilon}^{ab} \bar{D}_b P_n(\cos \theta)) \tilde{j}_a \, d^2 V ,
\]

where \( \bar{\epsilon}_{ab} \) is the alternating tensor on the \( r = R \) 2-sphere and \( \tilde{j}_b \) is the projection of \( j_b \) on this 2-sphere. Note that this expression refers just to the axi-symmetric structure on the 2-sphere \( S \) and not to the flat space in which it is embedded. Comparison of (3.19) with (3.21) suggests that we can think of the horizon \( \Delta \) as being endowed with a surface ‘current density’

\[
(\tilde{j})_a = \frac{1}{8\pi G} \bar{\omega}_a
\]

and define the angular momentum (or ‘current’) moments as:

\[
J_n = -\frac{R^{n+1}}{8\pi G} \int_S (\bar{\epsilon}^{ab} \bar{D}_b P_n(\zeta)) \bar{\omega}_a \, d^2 V
\]

\[
= -\sqrt{\frac{4\pi R^{n+1}}{2n+1}} \int_S Y_{n+1}^0(\zeta) \text{Im}\Psi_2 \, d^2 V
\]

\[
= \sqrt{\frac{4\pi R^{n+1}}{2n+1}} L_n
\]

Let us now turn to the mass multipoles, \( M_n \). When all \( J_n \) vanish, we should be left just with \( M_n \). These are then to be obtained by rescaling the multipole moments \( I_n \) by appropriate dimensionful factors. In electrostatics, when the charge density is axi-symmetric, the electric multipoles are defined by

\[
E_n = \int r^n P_n(\cos \theta) \rho \, d^3 x .
\]

When the charge is concentrated on the sphere \( S \) defined by \( r = R \), the expression simplifies to:

\[
E_n = R^n \int_S P_n(\cos \theta) \bar{\rho} \, d^2 V
\]

where \( \bar{\rho} \) is the surface charge density. Again, the final expression refers only to the axi-symmetric structure on the 2-sphere \( S \) and not to the flat space in which it is embedded. Hence we can take it over to type II horizons. What we need is a notion of a ‘surface mass density’. Now, Hamiltonian methods have provided a precise definition of mass \( M_\Delta \) of type II isolated horizons in the Einstein-Maxwell theory [9, 11]. The structure of geometric multipoles \( I_n \) now suggests that we regard \( M_\Delta \) as being ‘spread out’ on the horizon, the ‘surface
density $\tilde{\rho}_\Delta$ being uniformly distributed in the spherical case but unevenly distributed if the horizon is distorted. It is then natural to set\(^5\)

$$\tilde{\rho}_\Delta = \frac{1}{8\pi} M_\Delta \tilde{R} = -\frac{1}{2\pi} M_\Delta \text{Re}\Psi_2. \quad (3.27)$$

This heuristic picture motivates the following definitions:

$$M_n := -\sqrt{\frac{4\pi}{2n+1}} \frac{M_\Delta \tilde{R}_\Delta^n}{2\pi} \oint_S Y_n^0(\zeta) \text{Re}\Psi_2 d^2V$$

$$= \sqrt{\frac{4\pi}{2n+1}} \frac{M_\Delta \tilde{R}_\Delta^n}{2\pi} I_n. \quad (3.28)$$

Here $M_\Delta$ is the isolated horizon mass, which is determined by the horizon radius $R_\Delta$ and angular momentum $J_1$ via:

$$M_\Delta = \frac{1}{2GR_\Delta} \sqrt{R_\Delta^4 + 4G^2 J_1^2}, \quad (3.29)$$

Since our definitions are based on analogies with source multipoles in the Maxwell theory, an important question is whether they have the physical properties we expect in general relativity. We have the following:

- As discussed in section III A, the geometrical multipoles $L_0$ and $I_1$ vanish. Hence it follows that the angular momentum monopole moment $J_0$ vanishes as one would expect on physical grounds, and the mass dipole moment $M_1$ vanishes implying that we are in the center of mass frame.

- By construction, the mass monopole $M_0$ agrees with the horizon mass $M_\Delta$ and, by inspection, the angular momentum dipole moment $J_1$ equals the horizon angular momentum $J_\Delta$, calculated through Hamiltonian analysis [11].

- Restrictions on the multipoles imposed by symmetries of the horizon geometry follow immediately from the remark at the end of section III A:
  i) if the horizon geometry is such that $\Psi_2 \mapsto \Psi_2^*$ under the reflection $\zeta \mapsto -\zeta$, then $M_n = 0$ if $n$ is odd and $J_n = 0$ if $n$ is even. This is in particular the case for the Kerr isolated horizon.
  ii) If $\text{Im}\Psi_2 \equiv 0$, then all angular momentum multipoles vanish. This is in particular the case if $\Delta$ is a Killing horizon in a static space-time.
  iii) If the horizon geometry is spherically symmetric, $M_n = 0$ for all $n > 0$ and $J_n = 0$ for all $n$.

- Our multipoles $M_n, J_n$ are constructed just from the knowledge of the horizon geometry; knowledge of the space-time metric in the exterior region is not required. In particular, there may well be matter sources outside the horizon, responsible for its distortion and the exterior geometry need not even be stationary or asymptotically stable.

\(^5\) Note incidentally that the smaller the principal radii of curvature of the intrinsic geometry, the higher is $\tilde{\rho}_\Delta$. Thus, the situation has a qualitative similarity with the way charge is distributed on the surface of a conductor.
flat. Even when the exterior is stationary and asymptotically flat, there is no a priori reason to expect that these ‘source multipoles’ would agree with the ‘field multipoles’ defined at infinity —unless symmetry principles are involved— because the gravitational field outside the horizon would also act as a source, contributing to the ‘total’ moments at infinity.

It is interesting to analyze the situation in Kerr space-times. Because of symmetries, the horizon mass monopole and angular momentum dipole agree exactly with the corresponding field moments at infinity. What about the mass quadrupole $M_2$ or the angular momentum octupole? As figure 1 shows, while the field quadrupole is not exactly equal to the horizon quadrupole, the difference is insignificant for small values of the Kerr parameter $a = J/M$. Furthermore, even in the extreme limit $a = M$, there is only a factor of 1.4 between the two mass quadrupoles and 1.14 between the angular momentum octupoles. In the large $a$ limit, the effects due to dragging of inertial frames is so large that post-Newtonian and post-Minkowskian approximations fail; in this regime, there are no reliable calculations relating the source and field moments. So a priori it would not have been surprising if there were a factor of a million!

![FIG. 1: Plots of the horizon multipoles (solid lines) and Hansen’s field multipoles (dashed lines) for Kerr space-times as functions of $a/M$. The first figure shows the behavior of $|M_2|/M^3$ where $M_2$ is the mass quadrupole moment and the second shows the behavior of $|J_3|/M^4$ where $J_3$ is the angular momentum octupole. Absolute values are used because of the difference in Hansen’s and our sign conventions. (His mass monopole is $-M$ while ours is $M$.) Quantities plotted are dimensionless in the $G = 1$ units.](image)

- As noted in the Introduction, in vacuum, stationary space-times Hansen’s field moments suffice to determine the geometry in the neighborhood of infinity. Is there an analogous result for the horizon multipoles? The answer turns out to be affirmative. Fix a cross section $S$ of $\Delta$ and consider the future directed, inward pointing null vector field $n^a$ which is orthogonal to $S$, normalized so that $\ell^a n_a = -1$. The null geodesics originating on $S$ with tangent vector $-n^a$ generate a null surface $N$. In the source-free Einstein theory, there is a well-defined initial value problem based on the double null
surfaces, \( \Delta \) and \( \mathcal{N} \) [12]. The freely specifiable data consists of the pair \((\bar{q}_{ab}, \bar{\omega}_a)\) on \( \Delta \) and the Newman-Penrose component \(\Psi_4\) of the Weyl tensor on \( \mathcal{N} \) [7].

Now suppose that the space-time is analytic near \( \Delta \) and admits a stationary Killing field \( t^a \) in the neighborhood of \( \Delta \) which is time-like in the exterior region and becomes null on \( \Delta \). Then, as one might imagine, the initial value problem becomes highly constrained: \(\Psi_4\) on \( \mathcal{N} \) is determined by the horizon geometry [13]. Hence, the horizon multipoles suffice to determine the solution to Einstein’s equation in a past neighborhood of \( \Delta \cup \mathcal{N} \). Thus, qualitatively the result is the same as with the field multipoles.\(^6\)

To summarize, the mass and angular momentum multipoles \( M_n, J_n \) have physically expected properties. This in turn strengthens the heuristic picture we used to fix the dimensionful rescalings of \( I_n, L_n \). We first considered stationary, axi-symmetric charge and current distributions with support on a 2-sphere in the Maxwell theory and expressed the electric and magnetic multipoles using only the axi-symmetric structure on the 2-spheres without reference to Minkowski space-time. We then noted that these structures are available also on type II horizons.\(^7\) Additional structures made available on these horizons by geometric and Hamiltonian methods then led us to our definitions of \( M_n, J_n \). The physical picture is that observers in the exterior region (between the horizon and infinity) can regard the horizon multipoles as arising from an effective (but fictitious) mass density \( \tilde{\rho}_\Delta = -(1/2\pi)M_\Delta \text{Re}\Psi_2 \) and a current density \((\tilde{j}_\Delta)_a = (1/8\pi G)\bar{\omega}_a\) on \( \Delta \). This picture may be useful in physical applications.

**D. Extensions**

So far, we have restricted ourselves to non-extremal, type II isolated horizons which have no matter fields on them. In this sub-section, we will extend our results in various directions.

1. **Maxwell fields**

Let us restrict ourselves to the non-extremal case but allow Maxwell fields on \( \Delta \). Then the Einstein-Maxwell horizon geometry consists of the quadruplet \((\bar{q}_{ab}, D, B_{ab}, E_{ab})\). The presence of matter fields on \( \Delta \) causes a few minor modifications in the discussion of gravitational multipoles. We will first discuss these and then turn to the electromagnetic multipoles.

\(^6\) However, while the domain in which the solution is determined by field multipoles is known to be large [4], the available results on the double-null initial value problem only ensure the existence of the solution in a small, past neighborhood of \( \Delta \cup \mathcal{N} \). But it is quite possible that the double-null initial value results can be significantly strengthened.

\(^7\) Note that the spherical harmonics \( Y_{n}^{\alpha} (\zeta) \) used in the definition of geometric multipoles \( I_n, L_n \) refer to the unique round metric \( \bar{q}_{ab}^\circ \) defined by coordinates \((\zeta, \phi)\) (see (2.8)). They are eigenstates of the Laplacian defined by \( \bar{q}_{ab}^\circ \) and not of the physical metric \( \bar{q}_{ab} \). Therefore, the extension of the Maxwell formulas is natural. As noted in section II B, \( \bar{q}_{ab}^\circ \) is uniquely determined by the axi-symmetry of \( \bar{q}_{ab} \) and the two metrics have the same area element. Therefore, \( \tilde{\rho}_\Delta \) and \((\tilde{j}_\Delta)_a \) can be interpreted as the mass and current densities in terms of either metric.
In the gravitational case, the definition of the basic complex field $\Phi_\Delta$ on $\Delta$ continues to be the same in terms of $\mathcal{R}$ and $\text{Im}\Psi_2$ but changes if we use $\text{Re}\Psi_2$ in place of $\mathcal{R}$:

$$\Phi_\Delta := \frac{1}{4} \mathcal{R} - i \text{Im}\Psi_2 = -\Psi_2 + \frac{1}{4} R_{ab} \tilde{q}^{ab}. \tag{3.30}$$

where the term involving the space-time Ricci tensor can be expressed in terms of $B_{ab}$ and $E_{ab}$ as:

$$R_{ab} \tilde{q}^{ab} = G (B_{ab} \tilde{B}^{ab} + E_{ab} \tilde{E}^{ab}). \tag{3.31}$$

The definition of mass $M_\Delta$ changes from (3.29) to

$$M_\Delta = \sqrt{(R_\Delta^2 + G Q_\Delta^2)^2 + 4 G^2 J_\Delta^2}. \tag{3.32}$$

The subsequent definition of the geometric and physical multipole moments is the same as in section III.A. The reconstruction of the free data from multipoles is also unaffected but there is an additional term involving $B_{ab}$ and $E_{ab}$ in the reconstruction of $D$ from the free data (see Appendix A). Finally, there is a minor modification in the list of properties of $M_n, J_n$ listed in section III.C. This arises because, in presence of a Maxwell field on the horizon, the canonical angular momentum $J_\Delta$ obtained by Hamiltonian methods contains two terms, a gravitational one and an electromagnetic one [11]. The angular momentum dipole moment $J_1$ yields just ‘the gravitational part’ of $J_\Delta$. There does not seem to be a natural generalization of the definition of angular momentum multipoles such that $J_1$ would agree with the full $J_\Delta$.

Let us now turn to the electromagnetic fields. As noted in section II.A, the electromagnetic free data consists of the projections $\tilde{B}_{ab}$ and $\tilde{E}_{ab}$ of $B_{ab}$ and $E_{ab}$ on a cross-section $\Delta$. Therefore, it is natural to define the electromagnetic counterpart of $\Phi_\Delta$:

$$\Phi_{\Delta}^{\text{EM}} := -\frac{1}{2} \epsilon^{ab} [\tilde{E}_{ab} + i \tilde{B}_{ab}] \tag{3.33}$$

and define multipoles via:

$$Q_n = \frac{R^n}{\sqrt{4\pi(2n+1)}} \oint_{\Delta} \text{Re}(\Phi_{\Delta}^{\text{EM}}) Y_n^0(\zeta) d^2\tilde{V} \tag{3.34}$$

$$P_n = \frac{R^n}{\sqrt{4\pi(2n+1)}} \oint_{\Delta} \text{Im}(\Phi_{\Delta}^{\text{EM}}) Y_n^0(\zeta) d^2\tilde{V}. \tag{3.35}$$

Clearly, $Q_0$ and $P_0$ are the electric and magnetic charges of the horizon. Thus, heuristically $\text{Re}(\Phi_{\Delta}^{\text{EM}})/4\pi$ and $\text{Im}(\Phi_{\Delta}^{\text{EM}})/4\pi$ may be thought of as ‘surface charge densities’ on the horizon and charge multipoles capture the non-uniformity in the distributions of electric and magnetic charge densities.

Remark: If non-electromagnetic sources are also present, we can still define the gravitational (and electromagnetic) multipoles as above but the multipoles for other sources have to be defined case by case. Gravitational multipoles again determine the ‘free data’ for the horizon geometry ($\tilde{q}_{ac}, \tilde{q}_{bd}$, $R_{cd}$). However, to reconstruct the horizon geometry from this data, one needs to know those matter fields which determine the components $\tilde{q}_a^c \tilde{q}_b^d R_{cd}$ of the Ricci tensor, evaluated at the horizon.
2. Extremal horizons

Let us now summarize the situation with extremal horizons. In this case, important modifications are needed in the gravitational sector. As noted in section II A, surprisingly, \((q_{ab}, \omega_a, \tilde{B}_{ab}, \tilde{E}_{ab})\) are now universal, determined by the Kerr-Newman family [16]. However, they no longer suffice to determine the derivative operator \(D\) on the horizon because the second rank, symmetric tensor field \(\tilde{S}_{ab} = \tilde{q}_{ac}\tilde{q}_{bd} D^c D^d\) is now free [10] (see Appendix A). Therefore, multipole moments characterizing the free data in the geometry have to be constructed from components of \(\tilde{S}_{ab}\). Again, as with \(\tilde{\omega}_a\), there is a gauge freedom in \(\tilde{S}_{ab}\) associated with the choice of the cross-section \(\tilde{\Delta}\) used to evaluate it. This can be eliminated by fixing the trace of \(\tilde{S}_{ab}\) [13]. Then the gauge invariant information is coded in the two components of the trace-free part of \(\tilde{S}_{ab}\).

Of course, we can still use \(\Phi_{\Delta}\) to introduce the geometric multipoles \(I_n, L_n\) and electromagnetic multipoles \(Q_n, P_n\). But they are universal; the same as in the extremal Kerr-Newman case. However, now there are two additional sets of geometric multipoles obtained by integrating the two independent components of \(\tilde{S}_{ab}\) against \(Y^0_n(\zeta)\). They capture the free data which can vary from one extremal isolated horizon to another. To our knowledge, they do not have a simple, direct physical interpretation.

3. Type III horizons

The notion of the horizon geometry and the calculations leading to the free data do not refer to the axial Killing field at all [10] (see Appendix A). In the above, we restricted ourselves to type II horizons primarily because they are the ones which are physically most interesting. On a non-extremal type III horizon, the complex-valued seed function \(\Phi_{\Delta}\) again captures the gauge invariant part of the free data. However, in absence of axi-symmetry, we no longer have the natural coordinate \(\zeta\). \(\Phi_{\Delta}\) is now a function of two coordinates on \(\tilde{\Delta}\). Nonetheless, we can use the Laplacian operator \(\tilde{q}^{ab} \tilde{D}_a \tilde{D}_b\) on \(\tilde{\Delta}\) to decompose \(\Phi_{\Delta}\) into generalized spherical harmonics. Thus, now the geometric multipoles \(I_{n,m}, L_{n,m}\) are labelled by two integers where \(n\) labels the discrete eigenvalue of the Laplacian and \(m\) is the degeneracy label of eigenvectors with same eigenvalue \(n\) of the Laplacian. Should a physically interesting application of type III horizons arise, it should not be difficult to extend this basic framework further.

There is, however, an important subtlety. In the type II case, the presence of axi-symmetry led us to single out a preferred function \(\zeta\) and our multipoles were defined using the \(Y^0_n(\zeta)\) associated with this \(\zeta\). As noted in section III, these \(Y^0_n\) are spherical harmonics associated with the round 2-sphere metric \(\tilde{q}_{ab}\) of (2.8). In general, they are not eigenfunctions of the Laplacian of the physical metric \(\tilde{q}_{ab}\). Our procedure was essential to ensure that the angular momentum dipole \(J_1\) agrees with the horizon angular momentum \(J_{\Delta}\) defined by Hamiltonian methods [11]. Had we used multipoles through spherical harmonics associated with \(\tilde{q}_{ab}\), this equality would not have held in general. Thus the extra structure available on type II horizons played an important role in our definition of multipoles in sections III A and

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8 In the extremal case, further work is needed to understand the constraint on \((\tilde{q}_{ab}, \tilde{\omega}_a)\) which, in the type II case led to the surprising result that these fields must coincide with those on an extremal Kerr isolated horizon.
III C. Since it is absent on a general type III horizon, multipoles defined using the physical Laplacian will not have all the physically desirable properties discussed in section III C.

IV. DISCUSSION

Source multipoles in Newtonian gravity characterize the way in which mass is distributed, higher multipoles providing a complete description of distortions, i.e., departures from sphericity. Multipole moments defined in this paper do the same for the horizon geometry. In this sense, they can be regarded as ‘source multipoles’ associated with a black hole, distinct from the ‘field multipoles’ defined at infinity. In Newtonian gravity and Maxwellian electrodynamics, there is a simple relation between the two because both these theories are ‘Abelian’: the field does not serve as its own source. In non-Abelian contexts such as Yang-Mills theory and general relativity, the field in the region between the source and infinity itself acts as an effective source. Hence one would expect there to exist two distinct sets of multipoles, one associated with the central gravitating body and the other associated with the entire system. If space-time is stationary, one would expect a simple relation between mass monopoles, if it is axi-symmetric, between angular momentum dipoles and if in addition there is reflection symmetry, one would expect all odd mass multipoles and even angular momentum multipoles to vanish in both regimes. However, in absence of symmetries, one does not expect a simple relation between the two sets in the fully relativistic regimes where post-Newtonian and post-Minkowskian approximations fail. This point seems to have been glossed over in the general relativity literature on applications of multipoles. Since horizon multipoles were unavailable, one typically identified the field multipoles with the multipoles of the black hole and used them in all contexts.

The horizon multipoles defined in this paper are likely to have three sets of applications which we now summarize.

The first is to the problem of equations of motion. In the more recent literature on this subject, multipoles have been used in two different contexts. The first is illustrated by the problem of extreme mass-ratio binaries where a solar mass black hole orbits about a central supermassive one. For this case, there is an interesting framework [19] enabling one to use gravitational wave observations to ‘measure’ the gravitational multipole moments of the central object. In actual calculations, one typically assumes that the small hole is far from the supermassive one. One can then express the space-time metric at the position of the small hole as a series in inverse powers of distance from the central hole using the Hansen field-multipoles as coefficients, and calculate quantities of physical interest. Within the approximations inherent to these calculations, it is entirely appropriate to use the field multipoles and the framework provides a way of measuring these multipoles from the gravitational wave data. However, the approximation will fail if the orbit lies in a genuinely strong field regime. In that case, one may be able to use the near horizon expansion of the metric [7]. Here expansion coefficients refer to the horizon geometry but it should be straightforward to recast them in terms of the horizon multipoles.

The second context is illustrated in the study of how the gravitational field of other objects affects a given, small black hole (see, e.g., [20]). Calculations of the resulting distortion, precession, spin-flips, etc. of the black hole require the knowledge of tidal forces and torques exerted by other bodies on the black hole. Typically, these are calculated from interaction terms involving a coupling of black hole multipoles to perturbations in the Weyl curvature caused by other objects (at the location of the black hole). In these calculations it is
inappropriate to use the *field* multipoles of the small black hole since the coupling occurs at the location of the black hole. What one needs is multipoles associated with the horizon, not coefficients in the asymptotic expansion of the metric produced by the small black hole alone. The horizon multipoles introduced here provide the appropriate notion.

A second set of applications is to binary black hole simulations in numerical relativity. At very early times when the two black holes are widely separated and at late times when the single black hole has settled down, the world tubes of apparent horizons are well approximated, within numerical errors, by isolated horizons. One often wants to compare results of two different numerical simulations, particularly at late times. These comparisons are generally difficult in the strong field regime because the results are tied to the coordinates used. Our mass and angular momentum multipoles provide a diffeomorphism invariant method to compare such results. Furthermore, when there are differences, they provide a tool to interpret them physically.

A more interesting potential application is in the dynamical regimes, where the world tubes of apparent horizons can be modelled by dynamical horizons $H$ [21]. These are naturally foliated by marginally trapped 2-spheres $S_H$. On each $S_H$, there is an obvious, well-defined analog of $\Phi_\Delta$. Therefore, a natural strategy is to use it to define multipoles which would now be *time-dependent*. It would be very interesting to analyze the evolution of these moments both analytically and numerically. For example, there is a well-defined notion of energy flux across any $S_H$ [21]. Is there then an analog of the quadrupole formula *at the horizon*? From the knowledge of the horizon quadrupole and its relation to the Kerr quadrupole, can one gain insight into the maximum amount of energy that can be emitted in gravitational radiation? For non-rotating black holes, early numerical simulations [22] introduced a notion of horizon multipoles, using tools then available. They then calculated a complex frequency describing the time evolution of the quadrupole, where the imaginary part of the frequency captures damping. They found that this frequency coincided with the quasi-normal frequency of the final black hole. Although the calculation of multipoles was somewhat imprecise, the result is simple and intriguing. Interestingly, our horizon multipoles provide a precise formulation of the physical ideas underlying those more intuitive notions of multipoles. One is therefore led to ask: Can one analyze the final stages of mergers using the present, diffeomorphism invariant multipoles and show in detail that the complex frequency is the same as the quasi-normal one? Such a result would provide a deep insight into the relation between the highly non-linear dynamics of the horizon and the quasi-normal ringing associated with the linear regime. Again, the issue can be examined using analytical methods as well as numerical simulations.

The final set of applications is to quantum gravity. To calculate black hole entropy from first principles, one needs to construct an ensemble, where the ‘macroscopic parameters’ describing the system are fixed. To be physically meaningful, these parameters have to be diffeomorphism invariant. For type I horizons, this is straightforward: there is only one gravitational parameter, which can be taken to be the horizon area (or mass). For type II horizons, one can not just work with mass and angular momentum because the horizon may be distorted by various types of hair. Even within the 4-dimensional Einstein-Maxwell theory where no-hair theorems hold, distortions can be caused by matter rings and other black holes. Even if the black hole itself is isolated, one can not automatically rely on uniqueness theorems and say that it must be a Kerr hole because it is physically unreasonable to require that the whole space-time is stationary. Since one wishes to calculate entropy of a black hole in equilibrium, one should only need to require that the horizon geometry is time
independent, not the whole universe. To incorporate physically realistic situations, then, one needs a diffeomorphism invariant characterization of the horizon fields. Multipoles can now serve as the required parameters in the construction of the ensemble. It turns out that they can in fact be used to calculate entropy associated with type II horizons [23].

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APPENDIX A: RECONSTRUCTION OF THE HORIZON GEOMETRY FROM THE FREE DATA

For convenience of the reader, in this Appendix we will collect some results from [10] which play an important role in this paper. Specifically, we will show how the horizon geometry \((q_{ab}, D)\) is determined by the free data via field equations.

Fix a 3-manifold \(\Delta\), topologically \(S^2 \times \mathbb{R}\) and a 2-sphere cross-section \(\tilde{\Delta}\) thereon. We wish to show that, given a pair of fields \((\tilde{q}_{ab}, \tilde{\omega}_a)\) on \(\tilde{\Delta}\), there is a non-extremal isolated horizon structure \(([\ell^a], q_{ab}, D)\), unique up to diffeomorphism, on \(\Delta\) such that \(\tilde{q}_{ab}\) is the projection of \(q_{ab}\) on \(\tilde{\Delta}\); \(D_a \ell^b = \omega_a \ell^b\); \(\tilde{\omega}_a\) is the projection of \(\omega_a\) on \(\tilde{\Delta}\) and \(\omega_a \ell^a \neq 0\). (We will turn to the extremal case at the end.)

Introduce a function \(u\) whose level surfaces provide a foliation of \(\Delta\) by a family of 2-spheres such that \(\tilde{\Delta}\) is a leaf. Let \(n_a = -D_a u\) and define a vector field \(\ell^a\) such that \(\ell^a n_a = -1\). Thus, \(\ell^a\) is transversal to the foliation. Now, given \(\tilde{q}_{ab}\) on \(\tilde{\Delta}\), there is a unique tensor field \(q_{ab}\) on \(\Delta\) such that:

\[
q_{ab} \ell^b = 0, \quad L_{\ell} q_{ab} = 0, \quad \text{and} \quad q_{ab} \tilde{V}^a \tilde{W}^b = \tilde{q}_{ab} \tilde{V}^a \tilde{W}^b
\]

for all vector fields \(\tilde{V}^a, \tilde{W}^a\) tangential to \(\tilde{\Delta}\). This is the required \(q_{ab}\).

The \(D\) we seek is to be the derivative operator on a non-extremal isolated horizon. So, it must satisfy: i) \(D_a q_{bc} \equiv 0\), ii) \([L_{\ell}, D] \equiv 0\), and iii) \(D_a \ell^b = \omega_a \ell^b\) for some 1-form \(\omega_a\) where \(\ell^a \omega_a = \kappa_\ell\) is non-zero. Consider first torsion-free derivative operators \(D\) on \(\Delta\) which are compatible with \(q_{ab}\). They all have the same action on 1-forms \(f_a\) satisfying \(f_a \ell^a = 0\), given by:

\[
D_a f_b = \frac{1}{2} L_{\tilde{f}} q_{ab} + D_{[a f_b]}
\]

where \(\tilde{f}^a\) is any vector field on \(\Delta\) satisfying \(\tilde{f}^a q_{ab} = f_b\). (While \(\tilde{f}^a\) is not unique, one can check that the first term on the right side is insensitive to this ambiguity.) What distinguishes these derivative operators is only their action on \(n_a\). Set

\[
S_{ab} := D_a n_b
\]
Then, $S_{ab}$ is symmetric and $\ell^b S_{ab} = \ell^b D_a n_b = \omega_a$. Thus, the freedom in the definition of $D$ is completely captured by the pair $(\omega_a, \tilde{S}_{ab})$ where $\tilde{S}_{ab}$ is the projection of $S_{ab}$ on the leaves of the foliation (defined by the projection operator $\tilde{q}_a^b = \delta_a^b + n_a \ell^b$).

We now bring in the pull-backs of the field equations to $\Delta$. A simple calculation yields:

$$R_{ab} \ell^b \equiv 2 \ell^a D_{[a} \omega_{b]} ,$$

where the underline denotes the pull-back to $\Delta$. Now, conditions on the stress-energy tensor in the definition of a non-expanding horizon imply that $T_{ab} \ell^a \equiv 0$ whence $R_{ab} \ell^b \equiv 0$. Therefore, via the Cartan identity, this part of the Einstein equation is equivalent to:

$$D_a (\omega_b \ell^b) - L_\ell \omega_a \equiv 0$$

Now, since $[L_\ell, D] \equiv 0$, it follows that $L_\ell \omega_a \equiv 0$, whence $\kappa_\ell$ is constant on $\Delta$. The remaining projection of Einstein’s equations on $\Delta$ determines how $S_{ab}$ ‘evolves’ along $\Delta$:

$$L_\ell \tilde{S}_{ab} \equiv - \kappa_\ell \tilde{S}_{ab} + \tilde{D}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b - \frac{1}{2} \tilde{R}_{ab} + \frac{1}{2} \tilde{q}_a^c \tilde{q}_b^d R_{cd}$$

where $\tilde{D}$ and $\tilde{R}_{ab}$ denote the derivative operator and the Ricci tensor on the leaves of the foliation and $\tilde{\omega}_a$ is the projection of $\omega_a$ on these cross-sections. However, since $[L_\ell, D] \equiv 0$, the left side vanishes. Hence, if $\kappa_\ell \neq 0$, we have:

$$\tilde{S}_{ab} = \frac{1}{\kappa_\ell} \left[ \tilde{D}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b - \frac{1}{2} \tilde{R}_{ab} + \frac{1}{2} \tilde{q}_a^c \tilde{q}_b^d R_{cd} \right] .$$

Thus, given the part of the stress-tensor which determines the projection $\tilde{q}_a^c \tilde{q}_b^d R_{cd}$ of the space-time Ricci tensor on the leaves of the foliation, $\tilde{S}_{ab}$ is completely determined by $\tilde{q}_{ab}$ and $\tilde{\omega}_a$. Since $L_\ell \tilde{q}_{ab} \equiv 0$ and $[L_\ell, D] \equiv 0$, it follows that it suffices to specify $(\tilde{q}_{ab}, \tilde{\omega}_a)$ only on one leaf of the foliation.

To summarize, in the non-extremal case, the field equations pulled-back to $\Delta$ imply that $D$ is completely determined by $\tilde{q}_{ab}, \tilde{\omega}_a, \kappa_\ell$ and $\kappa_\ell$ is a constant on $\Delta$. An examination of the other field equations shows that they do not constrain the horizon geometry; rather, they provide evolution equations off $\Delta$. Thus, given $(\tilde{q}_{ab}, \tilde{\omega}_a)$ on $\Delta$, we can construct a triplet $([\ell], q_{ab}, D)$ on $\Delta = \Delta \times \mathbb{R}$ providing a non-extremal isolated horizon geometry. All such possible triplets are diffeomorphically related because the only freedom in the construction is the choice of $[\ell]$. (As noted in the main text, the value of $\kappa_\ell$ is not fixed on an isolated horizon because of the constant rescaling freedom in $[\ell]$.)

Let us now turn to the extremal case. Analysis is identical until (A6). But, since $\kappa_\ell$ vanishes, $\tilde{S}_{ab}$ decouples from the equation and remains completely unconstrained. However, we now have a constraint on the pair $(\tilde{q}_{ab}, \tilde{\omega}_a)$:

$$\tilde{D}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b - \frac{1}{2} \tilde{R}_{ab} + \frac{1}{2} \tilde{q}_a^c \tilde{q}_b^d R_{cd} = 0$$

This equation is quite complicated and its solutions are not known in the general case. However, on type II horizons in the Einstein-Maxwell theory, a surprising simplification occurs: in this case every solution $(\tilde{q}_{ab}, \tilde{\omega}_a)$ to this equation is diffeomorphically related to
that on the Kerr-Newman extremal horizon [16]. Hence, the freely specifiable data is just \( \tilde{S}_{ab} \).

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