Attractor Universe in the Scalar-Tensor Theory of Gravitation

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In the scalar-tensor theory of gravitation it seems nontrivial to establish if solutions of the cosmological equations in the presence of a cosmological constant behave as attractors independently of the initial values. We develop a general formulation in terms of two-dimensional phase space. We show that there are two kinds of fixed points, one of which is an attractor depending on the coupling constant and equation of state. In the case with a power-law potential in the Jordan frame, we also find new type of inflation caused by the coupling to the matter fluid.

I. INTRODUCTION

Einstein’s General Relativity has proved to be the simplest theory for successful understanding of a number of experiments and observations. Still, on the other hand, there seem to be growing indications that a yet-to-be-discovered scalar field might play fundamental roles in cosmology. A list of the possible sources include the one expected from higher-dimensional theories of gravitation such as superstring/M theory and a scalar field (a volume modulus) which couples to 4-dimensional gravity after compactification, also those in the context of the brane world scenario. We may reasonably expect that realistic consequences of these hypotheses can be implemented in terms of the scalar-tensor theory proposed first by Jordan \(^1\), developed later by Brans and Dicke \(^2\).

One of the recent focuses of the scalar field is aimed particularly at the origin of the dark energy which appears to be required from the observed acceleration of the universe, with the renewed interest of today’s version of the cosmological constant problem, culminating to the twin questions; fine-tuning and the coincidence problem. We argued \(^3,4,5,6\) that the scalar-tensor theory is precisely that causes the behavior, \(\Lambda \sim t^{-2}\) realized numerically by \(10^{-120} \sim (10^{60})^{-2}\) \(^7\), where \(10^{60}\) is today’s age of the universe in units of the Planck time \(\sim 10^{-43}\)s, hence preparing another simple implementation of the scenario of the decaying cosmological constant as discussed in \(^8\) and \(^9\).

Before reaching conclusions to be compared with observations, however, we must go through certain complications including details on the choice of the conformal frames among other things. Also many consequences derive from leaving the Brans-Dicke model \(^2\), as required ultimately by a single technical aspect on the attractor nature of the cosmological solutions. This motivated us to develop a general framework of studying dynamics of the system including the scalar field taking the unique roles of the conformal transformation properly into account.

After a brief introduction of the action of the scalar-tensor theory in both conformal frames, the Jordan and the Einstein frames in Section II, we enter Section III to develop a formulation in the Einstein frame in which we may trace how the cosmological solutions evolve in two-dimensional phase space. We assume the presence of the exponential potential of the scalar field corresponding to the simple cosmological constant in the Jordan frame. It is crucially important to make a right choice of the new extended time-coordinate other than the conventional cosmic time. Also of central importance is to deal with self-autonomous systems.

In Subsection III-A we discuss the fixed points in phase space. We show that there are two different sets of fixed points: one (FP1) is for the well-known universe of scalar-field dominance, and the other (FP2) represents a new type of the universe in which the matter fluid energy is scaled to the potential of the scalar field because of the coupling with the matter fluid. We find that the universe of the latter type always expands in the same way as the radiation-dominant universe in the Einstein frame. In Subsection III-B, we present the stability analysis and the behavior of the attractor solutions. Flows of trajectories in phase space are illustrated for the examples taken from the radiation-dominant universe. Basically the same analysis will be repeated in Section IV now in the Jordan frame.

In Section V we generalize the argument to the power-law potential with a monomial of the scalar field multiplied by the cosmological constant chosen in the preceding section in the Jordan frame. We find a new type of inflation at the fixed point FP2 even for the potential which is too steep to cause inflation without the coupling with the matter fluid. Section VI is devoted to the concluding remarks.

In Appendix A, we discuss the effect of the curvature term not included in the preceding sections. In the subsequent three appendices, we add related discussions on the accelerating universe. Appendix B reveals the presence of a complication in the numerical analysis in the Jordan frame, while in Appendix C we discuss the attractor nature of the scale-invariant model as an alternative to the Brans-Dicke model. The final Appendix D will be devoted to offering another simplified approach to the power-law potential.
II. SCALAR-TENSOR THEORY: JORDAN FRAME VS EINSTEIN FRAME

We discuss cosmology in the scalar-tensor theory of gravitation. We assume the presence of a potential $V(\phi)$ in the Jordan frame. The action is

$$S = S_g + S_m,$$

where

$$S_g = \int d^4x \sqrt{-g} \left[ \frac{\xi}{2} R - \frac{\xi}{2} (\nabla \phi)^2 - V(\phi) \right],$$

$$S_m = \int d^4x \sqrt{-g} L_m(g, \phi),$$

with $\epsilon = \pm 1$ [10]. Note that the matter action $S_m$ is assumed to have no scalar field $\phi$ according to the Brans-Dicke model [2], in which Weak Equivalence Principle (WEP) is intended to be respected. The scalar-tensor theory defined in this way is equivalent to the traditional Jordan-Brans-Dicke theory with an added potential $U(\phi) = V(\phi)$, often expressed as

$$S_g = \int d^4x \sqrt{-g} \left[ \varphi R - \frac{\omega}{\varphi} (\nabla \varphi)^2 - U(\varphi) \right],$$

with the Brans-Dicke constant $\omega = \epsilon/(4\xi)$ and $\varphi = (\xi/2)\phi^2$.

Although we may further extend this type of the scalar-tensor theory with an arbitrary function of the scalar field $\phi$ multiplied with $R$, we confine ourselves to the original simple $\phi$, because it features global scale invariance except generally for the $V(\phi)$ term.

The choice $\epsilon = -1$ in (2.2) is closely related to string theory. The $D$-dimensional action for the zero-modes in the closed string sector is given [11] by

$$S = \frac{1}{2} \int d^Dx \sqrt{-g} e^{-2\Phi} \left[ R(g) + 4(\nabla \Phi)^2 \right],$$

which, re-expressed according to our own sign convention as in (1.30) of [3], corresponds to the first two terms in (2.2) with $\epsilon = -1$ and $\xi = 1/4$, hence $\omega = -1$, by introducing $\phi = 2e^{-\Phi}$.

We can always move to the Einstein frame by a conformal transformation $\Omega^2 g_{\mu\nu} \rightarrow g_{\ast\mu\nu}$ [12, 13, 14]

$$S = S_{\ast} + S_m,$$

where

$$\Omega^2 = \xi \phi^2 = \exp(2\zeta \sigma),$$

with

$$\zeta^2 \equiv (6 + \epsilon \xi^{-1})^{-1} = (6 + 4\omega)^{-1},$$

which defines a canonical scalar field $\sigma$ in the Einstein frame;

$$S = \int d^4x \sqrt{-g_s} \left[ \frac{1}{2} R_s - \frac{1}{2} (\nabla_s \sigma)^2 - V_s(\sigma) \right] + \int d^4x \sqrt{-g_s} L_m(\psi_s, g_s, \sigma),$$

with $V_s(\sigma) = \exp(-4\zeta \sigma) V(\phi)$.

We mark the quantities in the Einstein frame with the subscript $\ast$, while those in the Jordan frame are left unmarked, unless otherwise indicated. This is in accordance with the notation used in [3].

We point out that $\epsilon = -1$, apparently indicating a ghost nature of the non-diagonalized field $\phi$, is a real difficulty only if $\zeta^2$ turns out to be negative implying a negative energy for the diagonalized field $\sigma$. We always assume the condition

$$\zeta^2 > 0.$$

This can be obeyed even if $\epsilon = -1$ if $\xi > 1/6$. Imposing $\xi > 0$, which we assume throughout this paper, due to the required positivity of the energy of tensor gravity, we find that $\epsilon = 1$ allows any $\xi$ but with $\zeta^2 < 1/6$, while $\epsilon = -1$ constrains $\xi > 1/6$ and $\zeta^2 > 1/6$, as displayed graphically in Fig. 1 of [3].

The parameters in (2.5) gives $\zeta^2 = (D - 2)/4$ which is $1/2$ for $D = 4$. If $\epsilon = -1$ and $\xi = 1/6$, we find $\zeta^2 \rightarrow \infty$, implying no kinetic term in the Einstein frame, hence no degree of freedom. We do not consider this choice any further.

Because we assume that no $\phi$ field enters $L_m$ in the Jordan frame, we find that the energy-momentum of the matter fluid is conserved in the Jordan frame;

$$\nabla^\nu T^\mu_{\nu} = 0,$$

for which WEP is respected. The energy-momentum tensor in the Einstein frame is obtained by

$$T^\mu_{\ast\nu} = \exp(-4\zeta \sigma) T^\mu_{\nu},$$

which is no longer conserved;

$$\nabla^\nu T^\mu_{\ast\nu} = -\zeta T, \nabla^\nu \sigma, ,$$

where $T_s = T_{\ast\mu\nu}$. Note that the universal free-fall (UFF) is still maintained, as an expression of WEP.

III. COSMOLOGY WITH A COSMOLOGICAL CONSTANT: ANALYSIS IN THE EINSTEIN FRAME

We discuss cosmology in the scalar-tensor theory with a cosmological constant $\Lambda = V_6$. In this section, we discuss it in the Einstein frame, though the analysis in the Jordan frame is given in the next section.

A. The Basic Equations and the Fixed Points

The metric of isotropic and homogeneous universe is given by the FRW form:

$$ds^2_s = -dt^2_s + a^2_s ds^2_3,$$
where $ds^2$ is the metric of maximally symmetric three-dimensional space with the curvature constant $k = 0$ or $\pm 1$.

The basic equations in the Einstein frame are

$$H^2 + \frac{k}{a^2} = \frac{1}{3} \left( \frac{1}{2} \dot{\sigma}^2 + V_\sigma + \rho_\sigma \right), \quad (3.2)$$

$$\ddot{\sigma} + 3H\dot{\sigma} + \frac{\partial V_\sigma}{\partial \sigma} = \zeta (\rho_\sigma - 3P_\sigma), \quad (3.3)$$

where $H_\sigma = \dot{a}_\sigma/a_\sigma$, $P_\sigma$, and $\rho_\sigma$ are the Hubble expansion parameter, the pressure, and the energy density in the Einstein frame, respectively. The dot implies $d/dt_\sigma$ throughout in the Einstein frame. Eq. (2.14) in the Einstein frame is then re-expressed as

$$\dot{\rho}_\sigma + 3H_\sigma (P_\sigma + \rho_\sigma) = -\zeta \dot{\sigma} (\rho_\sigma - 3P_\sigma). \quad (3.4)$$

Assuming the equation of state $P_\sigma = (\gamma - 1)\rho_\sigma$, we further re-express Eqs. (3.2)-(3.4) into

$$H^2_\sigma + \frac{k}{a^2_\sigma} = \frac{1}{3} \left( \frac{1}{2} \dot{\sigma}^2 + V_\sigma + \rho_\sigma \right), \quad (3.5)$$

$$\ddot{\sigma} + 3H_\sigma \dot{\sigma} - 4\zeta V_\sigma = \zeta (4 - 3\gamma)\rho_\sigma, \quad (3.6)$$

$$\dot{\rho}_\sigma + 3\gamma H_\sigma \rho_\sigma = -\zeta (4 - 3\gamma)\sigma \rho_\sigma. \quad (3.7)$$

We now introduce a new dimension-free time coordinate $\tau_\sigma$ by

$$d\tau_\sigma = 2\sqrt{V_\sigma} dt_\sigma. \quad (3.8)$$

We further introduce $\dot{\mathcal{H}}_\ast = a'_\sigma/a_\sigma$, where the prime is for a differentiation with respect to $\tau_\sigma$. We then put Eqs. (3.5)-(3.7) into the new form

$$3\dot{\mathcal{H}}_\ast^2 + \frac{k}{4V_\sigma a_\sigma^2} = \frac{1}{6} \left[ \dot{\sigma}^2 + \frac{1}{2} \left( 1 + \frac{\rho_\sigma}{V_\sigma} \right) \right], \quad (3.9)$$

$$\sigma'' + 33\zeta \sigma' - \zeta \left[ 2\sigma'^2 + 1 + \frac{(4 - 3\gamma)\sigma}{4V_\sigma} \right] = 0, \quad (3.10)$$

$$\rho'_\sigma + 3\gamma \mathcal{H}_\ast \rho_\sigma = -\zeta (4 - 3\gamma)\sigma \rho_\sigma. \quad (3.11)$$

Focusing on $k = 0$, we differentiate Eq. (3.9) with respect to $\tau_\ast$, to obtain

$$\zeta \mathcal{H}'_\ast = \frac{2 - \gamma}{4} \sigma'^2 + \frac{\gamma}{8} + 2\zeta \sigma' \mathcal{H}_\ast - \frac{3\gamma}{2} \mathcal{H}_\ast^2. \quad (3.12)$$

Here we have used Eqs. (3.9) and (3.10) as well as the equation

$$(\rho_\sigma/V_\sigma)' = -3\gamma (\mathcal{H}_\ast - \zeta \sigma') (\rho_\sigma/V_\sigma), \quad (3.13)$$

which is obtained from Eq. (3.11) and the definition (2.10) of $V_\sigma$.

In the same way we put Eq. (3.10) into

$$\sigma'' = -3\zeta \sigma' + \frac{3\gamma \zeta}{4} \left( 2\sigma'^2 + 1 \right) + 3\zeta (4 - 3\gamma) \mathcal{H}_\ast^2, \quad (3.14)$$

where Eq. (3.9) has been used to obtain the last term on the right-hand side. A set of equations (3.12) and (3.14) gives a self-autonomous system. In fact, by introducing the variables $x$ and $y$ defined by $x = \sigma'$ and $y = \zeta^{-1} \mathcal{H}_\ast$, we derive

$$x' = \frac{3\gamma}{4} \left[ 2\gamma x^2 - 4xy + 4\zeta^2 (4 - 3\gamma) y^2 + \gamma \right] \quad (3.15)$$

$$y' = \frac{1}{8\zeta} \left[ -2(2 - \gamma) x^2 + 16\zeta^2 xy - 12\zeta^2 y^2 + \gamma \right]. \quad (3.16)$$

By choosing $x' = y' = 0$, we find four fixed-points in this system:

$$\begin{align*}
\text{FP1}_\pm: (x_F, y_F) &= (x_1(\pm), y_1(\pm)) = \pm \left( \frac{2\gamma}{\sqrt{3 - 8\zeta^2}}, \frac{1}{2\sqrt{3 - 8\zeta^2}} \right), \\
\text{FP2}_\pm: (x_F, y_F) &= (x_2(\pm), y_2(\pm)) = \pm \left( \frac{\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}}, \frac{\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} \right).
\end{align*}
$$

The fixed points FP1$_\pm$ exist if $\zeta^2 < 3/8$, while the fixed points FP2$_\pm$ exist if $\gamma \geq 4/3$ or if $\gamma < 4/3$ with $\zeta^2 < (2 - \gamma)/(2(4 - 3\gamma))$ (equivalently, $\gamma > 2(4\zeta^2 - 1)/(6\zeta^2 - 1)$). For $\zeta = 1/2$, two types of fixed points coincide to each other. In Fig. 1 we show in which portion of the $\zeta^2$-$\gamma$ plane we find the fixed points.

From (3.9) with $k = 0$, we obtain

$$\rho_\sigma/V_\sigma = 2(6\zeta^2 y^2 - x^2) - 1, \quad (3.19)$$

which is constant at the fixed points.
At the fixed point $FP_{1,\pm}$, we find immediately
\[
(\rho_*/V_*)_{FP1} = 0, \tag{3.20}
\]
while at the fixed point $FP_{2,\pm}$, we obtain
\[
\left(\frac{\rho_*}{V_*}\right)_{FP2} = \frac{2(4\zeta^2 - 1)}{2 - \gamma - 2(4 - 3\gamma)\zeta^2}. \tag{3.21}
\]
This result is consistent with Eq. (3.13), i.e.
\[
\left(\frac{\rho_*}{V_*}\right)' = -3\gamma(\zeta(y - x)\left(\frac{\rho_*}{V_*}\right), \tag{3.22}
\]
the right-hand side of which vanishes at the fixed point $FP_{2,\pm}$, showing that the ratio of the energy density to the potential is constant.

Next we discuss the scale factor and scalar field at the fixed points. We have
\[
\sigma' = x_F, \quad (\ln a_*)' = \zeta y_F, \tag{3.23}
\]
for fixed points.

For $FP_{1,\pm}$, we have $(x_F, y_F) = (x_1^{(\pm)}, y_1^{(\pm)})$, giving
\[
\sigma = \frac{2\zeta}{\sqrt{3 - 8\zeta^2}} \tau_* + \sigma_0, \tag{3.24}
\]
\[
a_* = a_{*0} \exp\left[\frac{\tau_*}{2\sqrt{3 - 8\zeta^2}}\right]. \tag{3.25}
\]
By inverting (3.8) and substituting from (3.23) we obtain
\[
t_* = t_{*0} \exp\left[\frac{4\zeta^2}{\sqrt{3 - 8\zeta^2}} \tau_*\right]. \tag{3.26}
\]
Substituting this back into (3.25) and (3.26) then yields
\[
\sigma = \frac{1}{2\zeta} \ln \left(\frac{t_*}{t_{*0}}\right) + \sigma_0, \tag{3.27}
\]
\[
a_* = a_{*0} \left(\frac{t_*}{t_{*0}}\right)^{1/(8\zeta^2)} \tag{3.28}
\]
In order to fix $\sigma_0$, we go back to the original equations of motion, i.e. setting $\rho_* = 0$ finding
\[
H_*^2 = \frac{1}{3} \left(\frac{1}{2} \dot{\sigma}^2 + \rho_*\right), \tag{3.29}
\]
which gives the value of the scalar field $\sigma_0$ at $t_* = t_{*0}$ as
\[
\exp(-4\zeta\sigma_0) = \frac{3 - 8\zeta^2}{64\zeta^4 t_{*0}^2 V_0}. \tag{3.30}
\]
Eq. (3.28) shows that the solution with $\zeta < 1/(2\sqrt{2})$ gives a power-law inflation.

For the fixed point $FP_{2,\pm}$, we have $(x_F, y_F) = (x_2^{(\pm)}, y_2^{(\pm)})$, replacing (3.27) and (3.28) by
\[
\sigma = \frac{\sqrt{\zeta}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} \tau_* + \sigma_0, \tag{3.31}
\]
\[
a_* = a_{*0} \exp\left[\frac{\zeta \sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} \tau_*\right], \tag{3.32}
\]
respectively. The cosmic time in the Einstein frame is
\[
t_* = t_{*0} \exp\left[\frac{2\zeta \sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} \tau_*\right]. \tag{3.33}
\]
Hence we find
\[
\sigma = \frac{1}{2\zeta} \ln \left(\frac{t_*}{t_{*0}}\right) + \sigma_0, \tag{3.34}
\]
\[
a_* = a_{*0} \left(\frac{t_*}{t_{*0}}\right)^{1/2}. \tag{3.35}
\]
It is important to notice that $a_* \sim t_*^{1/2}$ follows also for dust-dominance. In fact this behavior is true for any equation of state. For the remedy of this unfavorable result, the reader is advised to see Section 4.4.3 of [3] or Section 3.4 of [5].

To fix $\sigma_0$, we use the original Friedmann equation:
\[
H_*^2 = \frac{1}{6} \dot{\sigma}^2 + \frac{V_*}{3} \left(1 + \frac{\rho_*}{V_*}\right), \tag{3.36}
\]
where $\rho_*/V_*$ is a constant given by Eq. (3.21). We then find
\[
\exp(-4\zeta\sigma_0) = \frac{(4\zeta^2 - 1)[2 - \gamma - 2(4 - 3\gamma)\zeta^2]}{16\gamma \zeta^4 t_{*0}^2 V_0}. \tag{3.37}
\]
Note that spacetime in this range is static in the Jordan frame, as will be shown in (4.22).

\textbf{B. Stability Analysis and the Attractors}

Next we analyze stability of the fixed points $FP_{1,\pm}$ and $FP_{2,\pm}$ in the self-autonomous system $3.15$ and $3.16$. 
1. Perturbation analysis

The simplest way is to apply a perturbation analysis. We perturb the variable \((x, y)\) around the fixed point \((x_F, y_F)\) as
\[
\begin{pmatrix}
x
\end{pmatrix} = \begin{pmatrix}
x_F + \delta x \\
y_F + \delta y
\end{pmatrix}.
\]
(3.38)

Inserting Eq. (3.38) into the basic equations (3.15) and (3.16), we find a set of the linear differential equations:
\[
\begin{pmatrix}
\frac{\delta x}{\delta y}
\end{pmatrix}' = \begin{pmatrix}
A_{xx} & A_{xy} \\
A_{yx} & A_{yy}
\end{pmatrix} \begin{pmatrix}
\delta x \\
\delta y
\end{pmatrix},
\]
(3.39)

where the components of the matrix \(A\) is given by
\[
\begin{align*}
A_{xx} &= 3\zeta(\gamma x_F - y_F) \\
A_{xy} &= 3\zeta(-x_F + 2\zeta^2(4 - 3\gamma)y_F) \\
A_{yx} &= -\frac{2\gamma}{\zeta} x_F + 2\zeta y_F \\
A_{yy} &= \zeta(2x_F - 3\gamma y_F).
\end{align*}
\]
(3.40)

Assuming \(\delta x, \delta y \propto e^{\omega \tau}\), we find the equation for the eigenvalue \(\omega\) as
\[
\omega^2 - \text{Tr} A \omega + \det A = 0,
\]
(3.41)

where
\[
\begin{align*}
\text{Tr} A &= \frac{1}{4\zeta^2 - 1} \left[ (x_F - 4\zeta^2 y_F) \\
&\quad + \left\{3\gamma(4\zeta^2 - 1) - (3 - 8\zeta^2)(x_F - y_F) \right\},
\end{align*}
\]
(3.42)

\[
\begin{align*}
\det A &= \frac{3}{2(4\zeta^2 - 1)} \left[ (2 - \gamma - 2(4 - 3\gamma)\zeta^2)(x_F - 4\zeta^2 y_F)^2 \\
&\quad - 2\gamma\zeta^2(3 - 8\zeta^2)(x_F - y_F)^2 \right].
\end{align*}
\]
(3.43)

In this expression, the first terms in (3.42) and (3.43) vanish for \(FP1_\pm\), while the second terms disappear for \(FP2_\pm\). The fixed point is stable in the following two cases:

- Eq. (3.41) has two negative real roots.
- Eq. (3.41) has two complex conjugate roots with a negative real part.

The condition is
\[
\text{Tr} A < 0, \quad \text{and} \quad \det A > 0.
\]
(3.44)

For \(FP1_\pm\), we find
\[
\begin{align*}
(\text{Tr} A)_{FP1_\pm} &= [4(3\gamma + 2)\zeta^2 - 3(\gamma + 1)] y_1^{(\pm)} \\
(\det A)_{FP1_\pm} &= 3[5\zeta^2(3 - 8\zeta^2)(4\zeta^2 - 1)(y_1^{(\pm)})^2]
\end{align*}
\]
(3.45)

For the expanding universe \((FP1_+),\) which we are interested in, the above condition (3.44) gives
\[
\zeta^2 < \frac{1}{4},
\]
(3.47)

For \(FP2_\pm\), we have
\[
\begin{align*}
(\text{Tr} A)_{FP2_\pm} &= -y_2^{(\pm)} \\
(\det A)_{FP2_\pm} &= -3\gamma\zeta^2(3 - 8\zeta^2)(4\zeta^2 - 1)(y_2^{(\pm)})^2
\end{align*}
\]
(3.48)

The stability condition for the expanding universe (the fixed point \(FP2_+\)) gives
\[
\zeta^2 > \frac{1}{4}.
\]
(3.50)

In this way we find that \(FP1_+\) and \(FP2_+\) are the attractors for

\[
\begin{align*}
\gamma^2 < 1/4 \quad \text{and for} \quad \gamma^2 > 1/4 \quad (\text{also with} \quad \gamma^2 = (2 - \gamma)/[2(4 - 3\gamma)]) \quad \text{for the existence of} \quad FP2_+, \quad \text{respectively. For} \quad \gamma^2 = 1/4, \quad \text{two types of fixed points merge with each other, sharing the same behaviors, but like a saddle point rather than an attractor.} \quad \text{Fig. 2} \quad \text{shows in which portion of the} \quad \gamma^2, \gamma \quad \text{plane we have the attractor fixed points.}
\end{align*}
\]

2. Phase space analysis

We also study stability by use of a phase-space analysis of the dynamical system with (3.15) and (3.16). We may discuss global stability rather than local one in the perturbative approach, as we will show shortly. There is a limitation, however, because dependence on the values of \(\gamma, \zeta^2\) is not as simple as shown in (3.45) and in the subsequent equations. We must develop the computation for each of these parameters separately, though without any difficulty in principle. For this reason, we show the following examples of radiation-dominance \((\gamma = 4/3)\) illustrating generic features shared commonly by this type of analyses with any values of the parameters.

For the sake of convenience we start with reproducing (3.15) and (3.16) for \(\gamma = 4/3\):
\[
\begin{align*}
x' &= \zeta(2x^2 - 3xy + 1), \\
y' &= \frac{1}{2\zeta} \left( -\frac{1}{3}x^2 + 4\zeta^2 xy - 4\zeta^2 y^2 + \frac{1}{3} \right),
\end{align*}
\]
(3.51)
which determine how the point \((x, y)\) representing the solution moves with time. The fixed points are \((3.17)\) for FP1\(_\pm\), while \((x_2^{(\pm)}, y_2^{(\pm)}) = (\pm 1, 1)\) from \((3.18)\) for FP2\(_\pm\). We also focus upon FP2\(_\pm\).

A basis for the required analysis is prepared first by drawing the “null curves” for \(x' = 0\) and \(y' = 0\). By choosing the vanishing left-hand sides of \((3.51)\) and \((3.52)\), we find that the former curves, solid (blue), are in fact hyperboloids, whereas the latter ones, dashed (red) curves are either hyperbolic or elliptic depending on \(\zeta^2 > 1/3\) or \(\zeta^2 < 1/3\), respectively, as shown in Figs. 3 and 4 used separately for the analyses of the two choices for \(\zeta^2\). The crossings represent fixed points. Note that there are four fixed points (FP1\(_\pm\), FP2\(_\pm\)) for \(\zeta^2 < 1/3\), while two fixed points (FP2\(_\pm\)) for \(\zeta^2 > 1/3\).

In Fig. 3 (a), we draw two sets of hyperboloids for \(\zeta^2 > 1/3\). It is rather easy to determine which side of each curve implies positive or negative \(x'\) and \(y'\), as shown by the symbols like \(+x\) and \(+y\), for \(x' > 0\) and \(y' > 0\), respectively. We then determine in what direction a point, or better called a trajectory, “flows” with time in a given region in \(x-y\) space bounded by the null curves, as illustrated symbolically by arrows (green). In Fig. 3 (b), we show a typical trajectory represented by a dotted curve which enters the diagram first near the lower-left corner, making a big loop outside the frame of the diagram, then re-entering again, finally spirals into the crossing at FP2\(_+\) \((x = y = 1)\), which corresponds to \((3.34)\) and \((3.35)\). This is the way we now establish our previous numerical results obtained on the basis of the heuristic approach \(3\) and \(4\) to be an authentic attractor in a strict sense.

We notice, on the other hand, that a trajectory, or the solution, may not always converge to a fixed point, straying instead toward infinity, as will be illustrated in Appendix B. Obviously, however, not reaching the point of \(x' = y' = 0\) implies the destination not corresponding to the steady and lasting solution, as given by \((3.34)\) and \((3.35)\), for example. In other words, any solution that survives a long time must come from the attractor.

There is another fixed point FP2\(_-\) \((x = y = -1)\) to which no trajectory flows into as long as we start with a positive \(\rho\), the same sign as \(V_0\). Note that this fixed point corresponds to the contracting universe. The flows shown in Fig. 3 (c) indicates that this is a repeller to be interpreted as the time-reversed point against the attractor at FP2\(_+\).

We encounter another complication, on the other hand, if \(\zeta^2 < 1/3\), for which we have four crossings as illustrated in Fig. 4(a). Focusing upon the behaviors in the upper-right quadrant, we have magnified views at each of the two, one in (b) around \(x = y = 1\), corresponding to the fixed-point solution of FP2 given by \((3.34)\) and \((3.35)\) above, and another shown in (c) now categorized into the type of FP1. The flows in (b) do indicate that the the solution in all directions tends finally to the crossing, a real attractor, while according to those illustrated in (c) there are some narrow strips sandwiched between two null curves in which the trajectory tends away from the...
crossing. This reminds us of a saddle-point potential rather than the purely attractive or repulsive potential in mechanics, hence implying an unstable solution which may not survive a long time, leading ultimately to the same drift toward infinity, as was remarked toward the end of the discussion of the preceding example in Fig. 3. For $1/4 < \zeta^2 < 1/3$, therefore, we may fail to reach the attractor solution with some probability depending on the initial values, or on what portion in the $x$-$y$ plane we started off. We may constrain ourselves to $\zeta^2 > 1/3$, though another *a posteriori* attitude might be suggested: Given what we are at present, right initial values, or right initial locations in phase space, must have been chosen to reach the attractor solution (3.34)-(3.35).

![Graph](image)

**FIG. 4.** An example of $1/4 < \zeta^2 < 1/3$. In the overall diagram (a), we chose an example of $\zeta^2 = 0.2916$, $\xi = 0.3890$. Unlike in Fig. 3 the dashed (red) null curve is an ellipsoid, thus producing four crossings. In addition to the attractor at $\text{FP2}_+(x = y = 1)$, denoted by a blob (green), we have another at $\text{FP1}_+$ ($x = 1.322$, $y = 1.134$) marked by a cross (green), both accompanied with the time-reversal counterparts in the left-lower quadrant ($\text{FP1}_-$ and $\text{FP2}_-$), which we ignore for brevity. In a close-up view (b) around $\text{FP2}_+$, we show an example of a trajectory with the same initial values as considered in Fig. 3(b), certainly spiraling into the fixed point in accordance with the arrows of flows, while we find no trajectories in (c) around another crossing of the type $\text{FP1}_+$. The behaviors of the arrows in the right-upper and left-lower strips remind us of a saddle-point potential resulting in unstable motion.

We apply the same analysis for any values of $\gamma$, finding the similar results. We summarize the power index of scale factor of the attractor solutions ($\text{FP1}_+$ for $\zeta < 1/2$, and $\text{FP2}_+$ for $\zeta > 1/2$) in Fig. 5. We find that the power-law inflation appears when $\zeta^2 < 1/8$. The power exponent of the scale factor in the Einstein frame is always 1/2 for $\text{FP2}_+$, which does not depend on the equation of state of the matter fluid, as derived in (3.35).

**IV. COSMOLOGY IN THE JORDAN FRAME**

We repeat the similar analysis for the same system as in Section III now in the Jordan frame. From (2.1) together with the FRW metric

$$ds^2 = -dt^2 + a^2 ds_j^2,$$  \hspace{1cm} (4.1)

we derive

$$3\xi\dot{\phi}^2 \left( H^2 + \frac{k}{a^2} \right) + 6\xi H \dot{\phi} = \frac{\epsilon}{2} \dot{\phi}^2 + V_0 + \rho$$ \hspace{1cm} (4.2)

$$\xi \left( \phi'' + \dot{\phi}' + 3H \dot{\phi} \right) = \zeta^2 \left( 4V_0 + \rho - 3P \right)$$ \hspace{1cm} (4.3)

$$\dot{\rho} + 3H (P + \rho) = 0$$ \hspace{1cm} (4.4)

where

$$H = \frac{\dot{a}}{a},$$  \hspace{1cm} (4.5)

while $P = P_\ast \exp(4\zeta \sigma)$ and $\rho = \rho_\ast \exp(4\zeta \sigma)$ are the pressure and the energy density, respectively, in the Jordan frame. Note that the quantities in the Jordan frame are all denoted unmarked in contrast to those marked with the subscript * in the Einstein frame. Accordingly, the dots in (4.2)-(4.4) imply differentiation with respect to $t$ instead of $t_\ast$.

Since $ds^2 = \Omega^{-2} ds_\ast^2$ with $\Omega^2 = \xi \dot{\phi}^2 = \exp(2\zeta \sigma)$, we have the relations between the variables in Jordan frame and those in the Einstein frame as

$$dt = \Omega^{-1} dt_\ast,$$  \hspace{1cm} (4.6)

$$a = \Omega^{-1} a_\ast.$$  \hspace{1cm} (4.7)

Note that we do not change the coordinate system when we perform a conformal transformation. However, since we use a cosmic time in each frame, we have to change the time coordinate between $t_\ast$ and $t$ according to (4.6). The Hubble expansion parameters $H$ and $H_\ast$ are defined by each cosmic time as

$$H = \frac{1}{a} \frac{da}{dt}, \quad H_\ast = \frac{1}{a_\ast} \frac{da_\ast}{dt_\ast}.$$  \hspace{1cm} (4.8)
and using Eqs. (4.12), (4.13) and (4.14), we find

\[ H = \Omega^{-1} \left( H + \frac{d \ln \Omega}{dt} \right) \tag{4.9} \]

Introducing new time coordinate \( \tau \) and new scalar field \( \Phi \), which are defined by

\[ d \tau = 2 \xi^{-1/2} V_0^{1/2} \phi^{-1} dt \tag{4.10} \]

\[ \Phi = \ln \phi \tag{4.11} \]

respectively, we rewrite Eqs. (4.2)-(4.4) as

\[ 6 \xi \dot{\gamma}^2 + \frac{3 k \xi e^{2 \Phi}}{2 V_0 a^2} = \frac{1}{\xi^2} (\Phi')^2 + \frac{1}{2} \left( 1 + \frac{\rho}{V_0} \right) \tag{4.12} \]

\[ \Phi'' - 2 (\Phi')^2 + 3 \xi \phi' = - \frac{2}{3} (4 - 3 \gamma) \rho \tag{4.13} \]

\[ \rho' + 3 \gamma (3 \xi - \Phi') \rho = 0 \tag{4.14} \]

where

\[ \xi \equiv \frac{a'}{a} + \Phi' \tag{4.15} \]

and the prime is the derivative with respect to \( \tau \). We also assume the equation of state, \( P = (\gamma - 1) \rho \).

Now, we consider only the spatially flat universe, i.e., \( k = 0 \) (See Appendix A for \( k \neq 0 \)). Taking the derivative of Eq. (4.12) and using Eqs. (4.12), (4.13) and (4.14), we find

\[ \dot{\xi}' = \frac{\gamma - 2}{4 \xi^2} (\Phi')^2 + 2 \xi \phi' - \frac{3 \gamma}{2} \Phi'^2 + \frac{\gamma}{8} \], \tag{4.16} \]

\[ \Phi'' = \frac{3 \gamma}{2} (\Phi')^2 - 3 \xi \phi' + 3 (4 - 3 \gamma) \xi^2 \phi'^2 + \frac{3 \gamma}{4} \xi^2 \tag{4.17} \]

This is again a self-autonomous system with two variables \( x \) and \( y \):

\[ x' = \frac{3 \xi}{4} \left[ 2 \gamma x^2 - 4 y + 4 \xi^2 (4 - 3 \gamma) y^2 + \gamma \right], \tag{4.18} \]

\[ y' = \frac{1}{8 \xi} \left[ -2 (2 - \gamma) x^2 + 16 \xi^2 x y - 12 \gamma \xi^2 y^2 + \gamma \right]. \tag{4.19} \]

where \( x = \zeta^{-1} \Phi' \) and \( y = \zeta^{-1} \phi' \). These equations turn out to be precisely the same as Eqs. (4.15) and (4.16), respectively, implying the same dynamical system, sharing the same fixed points: FP1\( \pm \), FP2\( \pm \).

The energy density is given by

\[ \frac{\rho}{V_0} = 12 \xi^2 - \frac{2}{\xi^2} (\Phi')^2 - 1 \]

\[ = 12 \xi^2 y^2 - 2 x^2 - 1, \tag{4.20} \]

precisely the same as (3.19) for the Einstein frame.

We only show the explicit solutions of the fixed points, because the dynamical properties such as an attractor is the same as that in the Einstein frame. For FP1\( \pm \), we have

\[ a = a_0 (\tau - \tau_0)^{1/\zeta} \tag{4.22} \]

\[ \phi = \pm \frac{2 \xi}{\sqrt{3 - 8 \xi^2}} (\tau - \tau_0) \tag{4.23} \]

\[ \rho = 0 \tag{4.24} \]

This gives a power-law inflation for \( \zeta^2 < 1/8 \).

For FP2\( \pm \), we have

\[ a = a_0, \tag{4.22} \]

\[ \phi = \frac{2 \sqrt{7}}{4 (4 - 3 \gamma) \xi^2} (\tau - \tau_0) \tag{4.23} \]

\[ \rho = \frac{2 (4 \xi^2 - 1)}{2 - 4 \gamma - 2 (4 - 3 \gamma) \xi^2} V_0, \tag{4.24} \]

where \( a_0 \) and \( \tau_0 \) are integration constants. The spacetime is a static Minkowski space, in contrast with the expansion in the Einstein frame as shown by (3.35).

The vacuum solution in radiation-dominance in [8] may be interpreted as the limit \( \zeta^2 \rightarrow 1/4 \) in (4.22)-(4.24). The constant \( a_0 \) was also derived in [18].

V. COSMOLOGY WITH A POWER-LAW POTENTIAL

The analysis in the preceding section can be readily extended to include the power-law potential,

\[ V(\phi) = \lambda_\phi \phi^n. \tag{5.1} \]

in the Jordan frame. An example is provided by extending the action (2.5) in string theory to

\[ S = \frac{1}{2} \int d^D x \sqrt{-g} e^{-2 \Phi} \left( R(g) - 2 \Delta + 4 (\nabla \Phi)^2 \right), \tag{5.2} \]
in which we have added the term of \( \Lambda \), to be corresponded to (5.1) by choosing \( \alpha = 2 \) and \( \lambda_2 = \Lambda / 4 \).

The action equivalent to (5.1) but expressed in the Einstein frame is given by

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R(g) - \frac{1}{2} \left( \nabla \sigma \right)^2 - V_\sigma(\sigma) \right] + \int d^4x \sqrt{-g} L(\sigma, \psi, g),
\]

where

\[
V_\sigma = \exp[-\zeta(4 - \alpha)\sigma] V_0,
\]

with \( V_0 = \lambda_\alpha/|\zeta|^{\alpha/2} \). Since this is of the same form as Eq. (2.9), we repeat the same analysis as before. One of the points to be kept in mind is that the conformal transformation is always the same as (2.6) with (2.7). Hence we have the same energy-momentum conservation law (5.4), as well.

The basic equations for cosmology are then;

\[
H_\ast + \frac{k}{a_\ast^2} = \frac{1}{3} \left( \dot{\sigma}^2 + V_\ast + \rho_\ast \right), \tag{5.5}
\]

\[
\ddot{\sigma} + 3H_\ast \dot{\sigma} + \frac{\partial V_\ast}{\partial \sigma} = \zeta(\rho_\ast - 3P_\ast), \tag{5.6}
\]

\[
\dot{\rho}_\ast + 3H_\ast (P_\ast + \rho_\ast) = -\zeta \dot{\sigma}(\rho_\ast - 3P_\ast). \tag{5.7}
\]

Introducing a new dimension-free time coordinate as

\[
d\tau_\ast = 2\sqrt{V_\ast} dt_\ast, \tag{5.8}
\]

also assuming the equation of state and focusing on \( k = 0 \) as before, we obtain formally the same equations as (3.9)-(3.11), but replace (3.13) by

\[
(\rho_\ast/V_\ast)' = -[3\gamma H_\ast - \zeta(3\gamma - \alpha)\sigma'] (\rho_\ast/V_\ast). \tag{5.9}
\]

Computing in the same way as before, we come to replacing (3.15) and (3.16) by

\[
x' = \frac{\zeta}{4} \left[ 2(3\gamma - \alpha)x^2 + 12\zeta^2(4 - 3\gamma)y^2 - 12xy + 3\gamma - \alpha \right], \tag{5.10}
\]

\[
y' = \frac{1}{8\zeta} \left[ -2(2 - \gamma)x^2 - 12\zeta^2y^2 + 4\zeta^2(4 - \alpha)xy + \gamma \right], \tag{5.11}
\]

respectively, where \( x = \sigma' \) and \( y = \zeta^{-1}H_\ast \) as before.

We find four fixed points as the previous case:

\[
\text{FP1}_\pm : (x_F, y_F) = (x_F^{(\pm)}, y_F^{(\pm)}) = \pm \frac{1}{\sqrt{2(6 - \zeta^2(4 - \alpha)^2)}} \left( \zeta(4 - \alpha), \frac{1}{\zeta} \right),
\]

\[
\text{FP2}_\pm : (x_F, y_F) = (x_F^{(\pm)}, y_F^{(\pm)}) = \pm \frac{1}{\sqrt{6[3\gamma(2 - \gamma) - 2(4 - 3\gamma)(3\gamma - \alpha)\zeta^2]}} (3\gamma, 3\gamma - \alpha). \tag{5.13}
\]

The fixed points \( \text{FP1}_\pm \) exist if

\[
4 \left( 1 - \frac{\sqrt{6}}{4\zeta} \right) < \alpha < 4 \left( 1 + \frac{\sqrt{6}}{4\zeta} \right). \tag{5.14}
\]

For fixed points \( \text{FP2}_\pm \), the constraint for existence is a little more complicated, as we find:

\[
\alpha > 3\gamma \left[ 1 - \frac{2 - \gamma}{2(4 - 3\gamma)} \frac{1}{\zeta^2} \right] \quad \text{for} \quad \gamma < \frac{4}{3}, \tag{5.15}
\]

any values of \( \zeta \) and \( \alpha \) \quad \text{for} \quad \gamma = \frac{4}{3}, \tag{5.16}

\[
\alpha < 3\gamma \left[ 1 + \frac{2 - \gamma}{2(3\gamma - 4)} \frac{1}{\zeta^2} \right] \quad \text{for} \quad \gamma > \frac{4}{3}. \tag{5.17}
\]

Next we analyze the solutions of these fixed points. In what follows, we consider only the expanding universe (\( y_F > 0 \)), i.e. \( \text{FP1}_+ \), while \( \text{FP2}_+ \) for \( \alpha < 3\gamma \) and \( \text{FP2}_- \) for \( \alpha > 3\gamma \).
where
\[ p_* = \frac{2}{\zeta^2(4-\alpha)^2} \]  
(5.23)

In this case, matter does not contribute to the expansion of the universe, i.e. at the fixed point, \( \rho_* = 0 \). The cosmic expansion becomes inflationary if
\[ 4 \left(1 - \frac{\sqrt{2}}{4\zeta}\right) < \alpha < 4 \left(1 + \frac{\sqrt{2}}{4\zeta}\right). \]  
(5.24)

For \( FP2_\pm \), we have
\[ \sigma = \pm \frac{3\gamma \tau_*}{\sqrt{6[3\gamma(2-\gamma) - 2(4-3\gamma)(3\gamma - \alpha)]\zeta^2}} + \sigma_0 \]  
(5.25)
\[ a_* = a_{*0} \exp \left[ \frac{\pm(3\gamma - \alpha) \zeta \tau_*}{2\sqrt{6[3\gamma(2-\gamma) - 2(4-3\gamma)(3\gamma - \alpha)]\zeta^2}} \right]. \]  
(5.26)

The cosmic time \( t_* \) is given by
\[ t_* = t_{*0} \exp \left[ \frac{\pm 3\gamma(4\zeta - \alpha) \tau_*}{2\sqrt{6[3\gamma(2-\gamma) - 2(4-3\gamma)(3\gamma - \alpha)]\zeta^2}} \right]. \]  
(5.27)

Hence the solution of \( FP2_+ \) is described by
\[ \sigma = \frac{2}{\zeta(4-\alpha)} \ln \left( \frac{t_*}{t_{*0}} \right) + \sigma_0 \]  
(5.28)
\[ a_* = a_{*0} \left( \frac{t_*}{t_{*0}} \right)^{p_*}, \]  
(5.29)
where
\[ p_* = \frac{2(3\gamma - \alpha)}{3\gamma(4-\alpha)} = \frac{2}{3\gamma} + \Delta p_* \]  
(5.30)

\[ \Delta p_* = \frac{2(3\gamma - 4)}{3\gamma(4-\alpha)}. \]  
(5.31)

which describes the deviation from conventional power exponent with the adiabatic index \( \gamma \). It is due to the interaction between the matter fluid and the scalar field \( \phi \). It is precisely this interaction that keeps the ratio \( \rho_*/V_* \) constant at \( FP2_\pm \). The value is obtained from Eq. (3.19):\[ \frac{1}{6} \left( \frac{p_*}{V_*} \right)_{FP2} = 2\zeta^2 \left( g_2^{(\pm)} \right)^2 - \frac{(x_2^{(\pm)})^2}{3} - \frac{1}{6} \]
\[ = \frac{2(4-\alpha)^2 + (3\gamma - 4)(4-\alpha) - 3\gamma/\zeta^2}{2(3\gamma - 4)^2 + 2(3\gamma - 4)(4-\alpha) + 3(2-\gamma)/\zeta^2}, \]  
(5.32)

which is consistent with (5.29) for the energy density.

In order for this energy density to be positive, we have to impose the following condition:
\[ \alpha < \frac{1}{2} \left[ (3\gamma + 4) - \sqrt{(3\gamma - 4)^2 + 12\gamma/\zeta^2} \right] \quad \text{for } \quad FP2_+ \]
\[ \alpha > \frac{1}{2} \left[ (3\gamma + 4) + \sqrt{(3\gamma - 4)^2 + 12\gamma/\zeta^2} \right] \quad \text{for } \quad FP2_- . \]

We also find a simple result \( p_* = 1/2 \) follows either by \( \alpha = 0 \) for any \( \gamma \) or by \( \gamma = 4/3 \) for any \( \alpha \). This result may be described by
\[ p_* = \frac{1}{2} + \Delta p', \]  
(5.33)

where
\[ \Delta p' = \frac{(3\gamma - 4)\alpha}{6\gamma(4-\alpha)} - \frac{\alpha}{4} = \frac{\alpha}{4} \]  
(5.34)

We are now looking into more details of the power-law inflation. We find a power-law inflation by \( \sigma \) if the power exponent of the potential \( V_* \) is sufficiently small, i.e. \( \zeta(4-\alpha) < \sqrt{2} \), and if there is no coupling between the matter fluid and the scalar field \( \sigma \) in the Einstein frame. This type of inflation is realized in the fixed points \( FP1_+ \) even when we include the coupling with the matter fluid, which has nothing to do with inflation at \( FP1_+ \). However, the power-law inflation may occur also for \( FP2_\pm \), in which the coupling to the matter fluid is non-negligible. From the condition \( p_* > 1 \) for the power exponent of the scale factor in the Einstein frame, we find the following conditions:
\[ \alpha < \frac{6\gamma}{3\gamma - 2} \quad \text{or} \quad \alpha/2 > 2 \quad \text{for} \quad \gamma < \frac{2}{3} \]  
(5.35)
\[ \alpha > 4 \quad \text{for} \quad \gamma = \frac{2}{3} \]  
(5.36)
\[ 4 < \alpha < \frac{6\gamma}{3\gamma - 2} \quad \text{for} \quad \frac{2}{3} < \gamma < \frac{4}{3} \]  
(5.37)
\[ \text{no case} \quad \text{for} \quad \gamma = \frac{4}{3} \]  
(5.38)
\[ \frac{6\gamma}{3\gamma - 2} < \alpha < 4 \quad \text{for} \quad \gamma > \frac{4}{3} \]  
(5.39)

This is a new type of inflation. The potential itself is too steep to cause inflation, but the matter fluid assist to cause a faster expansion because of its coupling to the scalar field. Note that including the matter fluid with \( \gamma < 2/3 \) means that we can have inflation just by this fluid. But the coupling with the scalar field does not assist such a type of inflation. Rather it will restrict the possibility of inflationary expansion.

Next we have to analyze the stability of the fixed points. We can analyze it by perturbations or by use of the phase diagram, as shown in the previous section. Here we give the perturbation analysis. Inserting the perturbations around the fixed points \( x = x_F + \delta x, y = y_F + \delta y \) in the basic equations (5.10) and (5.11), we find a set of the linear perturbation
equations (3.39) with the components of the matrix being
\[
\begin{align*}
A_{xx} &= (3\gamma - \alpha) x_F - 3y_F \\
A_{xy} &= -3x_F + 6\zeta^2 (4 - 3\gamma) y_F \\
A_{yx} &= \frac{2 - \gamma}{2\zeta^2} x_F + \frac{(4 - \alpha)}{2} y_F \\
A_{yy} &= \frac{(4 - \alpha)}{2} x_F - 3\gamma y_F.
\end{align*}
\] (5.40)

Setting \(\delta x, \delta y \propto e^{\omega \tau}\), we find the eigen equation for \(\omega\) as
\[
\begin{align*}
\text{Tr} A &= (3\gamma + 2 - 3\alpha/2)x_F - 3(\gamma + 1)y_F \quad (5.41) \\
\det A &= \frac{1}{2\zeta^2} \left[ (3\gamma - \alpha)(4 - \alpha) \zeta^2 - 3(2 - \alpha) \right] x_F^2 \\
&+ \ 3(\alpha \gamma - 10\gamma + 8)x_F y_F \\
&+ \ 3 \left[ 3\gamma - \zeta^2 (4 - 3\gamma)(4 - \alpha) \right] y_F^2. \quad (5.42)
\end{align*}
\]

Using these equations, we analyze the stability of the fixed points as before. The result is shown in Fig. 6. The regions denoted by FP1 and FP1-I give the attractor solution of FP1\pm, while those by FP2 and FP2-I correspond to the attractors of FP2\pm. For FP1\pm, the inflationary solutions exist in the case of a flat potential (\(\zeta|4 - \alpha| < 1/2\)), but the new type of inflationary solution also appears for FP2\pm if \(\gamma \neq 4/3\). Each region for inflation is also shown by FP1-1 or FP2-1. This new type of inflation is assisted by a coupling between the scalar field \(\sigma\) and the matter fluid.

In the Jordan frame, we show only the behavior of the fixed points FP1\pm and FP2\pm. After a conformal transformation, we find them:
\[
\begin{align*}
\alpha &\propto t^p \\
\phi &\propto t^q,
\end{align*}
\] (5.43)

where
\[
(p, q) = \begin{cases} 
(2(1 - (4 - \alpha))\zeta^2, 2) & \text{for FP1}_\pm \\
(2 - \alpha)(4 - \alpha)\zeta^2, 2 - \alpha) & \text{for FP2}_\pm \\
-2\alpha, 2 - \alpha) & \text{for FP2}_\pm
\end{cases}
\] (5.45)

The scale factor in the Jordan frame for FP2\pm is not constant except for \(\alpha = 0\). It is expanding for \(\alpha < 0\) or \(\alpha > 2\), while it is contracting for \(0 < \alpha < 2\). For \(\alpha = 2\), we find \(a \propto \exp[-\gamma t]\).

VI. CONCLUDING REMARKS

We have presented a formulation in which we trace the temporal development of cosmological solutions of the scalar-tensor theory in two-dimensional phase space. Thanks to assuming a simple equation of state, in Sections III, IV and V, we have obtained two different sets of fixed points, FP1\pm and FP2\pm. Conditions of obtaining attractors are studied in detail.
We have established the attractor nature of the fixed points. At FP$_2$, the scale factor behaving like a constant and $t^{1/2}$ in the Jordan and the Einstein frames, respectively, for $\zeta^2 > 1/4$ or $\xi < 1/2 (\omega < -1/2)$ with $\epsilon = -1$, when we have a simple cosmological constant in the Jordan frame. This solution is also accompanied with the proportionality between $\rho_*$ and $V_*$, called a scaling behavior, which is going to be replaced by the "interlacing" behavior, as exemplified in Fig. 5.8. of [3], by further extending the model.

An extension to the power-law potential in Section V has shown that the coupling between the scalar field and the matter provides a new type of inflation with $\gamma$ off the conventional choice 4/3, even if the potential of the scalar field is too steep to cause inflationary expansion by itself.

We have also learned that, even confining ourselves to the fixed-point solutions, reaching an attractor can be somewhat complicated if we have another fixed point, as we faced in the example of Fig. 4, though a simple recipe is shown to be effective, i.e., if $\xi = -1$, when we have a mass term.

The most intriguing result in the present scalar-tensor theory is, however, that the static universe in the Jordan frame is an attractor, an unavoidable fate in the presence of the cosmological constant. There is no smooth limit as $V_0 \to 0$. As we re-iterate, the Jordan frame features truly constant masses of microscopic fields, according to the Brans-Dicke model, originally intended to qualify this frame to be physical, allowing a non-static universe in the absence of the cosmological constant. Its presence alters the entire situation, forcing us to accept $ma = \text{const}$. This entails eventually that the universe in the Einstein frame expands in the same rate as the (time-dependent) microscopic length standard. This crisis will be evaded only by leaving the Brans-Dicke model, as was elaborated in [3,4] together with the ensuing consequences. The view that this crisis hinges upon the attractor nature of the solutions is now reinforced even more strongly by our study in this article.

We also emphasize that the extension to the power-law potential leaves the above crisis unsolved. The exponent 1/2 in radiation-dominance in the Einstein frame is a unique consequence of $\gamma = 4/3$ independent of the way $V_0$ is modified by the scalar field. The argument on the power-law inflation also remains unaffected, as we point out, by the structure of the mass term.

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[1] P. Jordan, Schwerkraft und Weltall (Friedrich Vieweg und Sohn, Braunschweig, 1955).
[2] C. Brans and R. H. Dicke, Phys. Rev. 124 (1961), 925.
[3] Y. Fujii and K. Maeda, The Scalar-Tensor Theory of Gravitation (Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, 2003).
[4] Y. Fujii, Phys. Rev. D26 (1982), 2580.
[5] Y. Fujii, Prog. Theor. Phys. 118 (2007), 983.
[6] Y. Fujii, Proc. Workshop on Cold Antimatter Plasmas and Application to Fundamental Physics, Feb. 2008, Naha, Japan. [arXiv:0803.3103 [gr-qc]].
[7] This relation is an outgrowth of the earlier attempt [4]. For more details, see Chapter 4.4.2. of [3], towards the end of Section 4.1 of [3], and Eqs. (16), (17) together with the footnote 12 of [4].
[8] D. Dolgov, In The Very Early Universe, Proc. Nuffield Workshop, ed. G. W. Gibbons and S.T. Siklos (Cambridge University Press, 1982).
[9] K. Freese et al, Nucl. Phys. B287 (1987) 797.
[10] We may also allow $\epsilon = 0$, equivalent to the choice $\xi \to \infty$ and $\zeta^2 = 1/6$, known to be a special situation corresponding to the absence of the kinetic-energy of the scalar field in the Jordan frame but to its presence in the Einstein frame (See J. O’Hanlon, Phys. Rev. Lett. 29 (1972), 137. P. Fiziev, in Gravity, Astrophysics and Strings at the Black Sea (St. Kliment Ohridski University Press, 2005).)
[11] C.G. Callan, D. Friedan, E.J. Martinec, and M.J. Perry, Nucl. Phys. B262 (1985) 593; C.G. Callan, I.R. Klebanov, and M.J. Perry, Nucl. Phys. B278 (1986) 78.
[12] P.W. Higgs, Nuovo Cimento 11 (1959) 816; B. Whitt, Phys. Lett. B145 (1984) 176.
[13] G. Magnano, F. Ferraris, M. Francaviglia, Gen. Rel. Grav. 19 (1987) 465; A. Jakubiec and J. Kijowski, Phys. Rev. D37 (1988) 1406.
[14] K. Maeda, Phys. Rev. D39 (1989) 3159; T. Futamase, K. Maeda, Phys. Rev. D39 (1989) 399.
[15] J.J. Halliwell, Phys. Lett. B185 (1987) 341; J. Yokoyama, K. Maeda, Phys. Lett. B207 (1988) 31.
[16] Y. Kitada, K. Maeda, Phys. Rev. D45 (1992) 1416, Class. Quant. Grav. 10 (1993) 703.
[17] K. Maeda, H. Nishino, Phys. Lett. B154 (1985) 358; B158 (1985) 381; K. Maeda, Class. Quant. Grav. 3 (1986) 233; 651.
[18] C. Wetterich, Nucl. Phys. B302 (1988), 645.
[19] N. Agarwal and R. Bean, Class. Quantum Grav. 25 (2008) 165001.
APPENDIX A: THE EFFECT OF CURVATURE TERM 
(k ≠ 0)

In this Appendix, we study the curvature effect. It may be convenient to analyze the equations rewritten by new variables because the fixed points are constant. We discuss the cases with a cosmological constant and with a power-law potential separately.

1. The case with a cosmological constant

We show the curvature effects both in the Einstein and in the Jordan frames in this order.

a. Curvature term in the Einstein frame

The curvature term is proportional to \( e^{4\zeta_\sigma}/a^2 \). We evaluate the time evolution of this term by

\[
\frac{d \ln (e^{4\zeta_\sigma}/a^2)}{d \tau_*} = 2(2x - y) \tag{A1}
\]

We find the behavior near the fixed point \((x_F, y_F)\) as

\[
\frac{d \ln (e^{4\zeta_\sigma}/a^2)}{d \tau_*} = 2(2x_F - y_F)
\]

\[
= \begin{cases} 
\frac{8^2 - 1}{2\zeta\sqrt{3 - 8^2}} & \text{for } FP_{1+} \\
\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)} & \text{for } FP_2 
\end{cases} \tag{A2}
\]

Hence we find that the curvature term decreases in time for inflationary solution at FP1+ (\(\zeta^2 < 1/8\)). It can be ignored. However, it will grow in time if the universe expands without acceleration. It will becomes important as the same as the usual case.

b. Curvature term in the Jordan frame

The time evolution of the curvature term in the Jordan frame is given by

\[
\frac{d \ln (\phi^2/a^2)}{d \tau} = 2(\Phi' - \phi\zeta) = 2(2\Phi' - \zeta) = 2(2x - y) 
\]

\[
\tag{A3}
\]

We find the behavior near the fixed point \((x_F, y_F)\) as

\[
\frac{d \ln (\phi^2/a^2)}{d \tau} = 2(2x_F - y_F) 
\]

which is exactly the same as Eq. (A2). Hence the curvature term near the fixed point FP1+ is not important for inflationary solution (\(\zeta^2 < 1/8\)).

2. The case with a power-law potential

In this case, we can repeat the same analysis. The curvature term is proportional to \( e^{4(1-\alpha)\zeta_\sigma}/a^2 \). So the time evolution near the fixed point is given by

\[
\frac{d \ln (e^{4(1-\alpha)\zeta_\sigma}/a^2)}{d \tau_*} = \zeta \left[ 2(2x_F - y_F) - \alpha x_F \right] 
\]

\[
= \begin{cases} 
2\zeta^2(2 - \alpha/2)^2 - 1 & \text{for } FP_{1+} \\
\zeta \sqrt{2(6 - \zeta^2(\alpha - 4\zeta^2))} & \text{for } FP_{2+} \\
\sqrt{6(3\zeta^2(2 - \gamma) - 2(4 - 3\gamma)(3\gamma - \alpha)\zeta^2)} & \text{for } FP_{2-} 
\end{cases} \tag{A5}
\]

The curvature term is not important for inflationary solution near the fixed point FP1+ (\(\zeta^2(4 - \alpha)^2 < 2\)) and near the fixed point FP2+ (\(\alpha > 6\gamma/(3\gamma - 2)\) with \(\gamma > 4/3\)) or FP2- (\(\alpha < 6\gamma/(3\gamma - 2)\) with \(2/3 < \gamma < 4/3\)).

APPENDIX B: ADDITIONAL SOLUTIONS FOR RADIATION-DOMINANCE IN THE JORDAN FRAME

We often relied on the numerical approach to solve the cosmological equations in the Jordan frame because the solution is characterized by the simplest aspect of the static universe. We encounter, however, another complicated aspect to be discussed in what follows.

We started conveniently from

\[
6\varphi H^2 = -\frac{1}{2} \dot{\varphi}^2 + V_0 + \rho - 6H\dot{\varphi}, \tag{B1}
\]

\[
\ddot{\varphi} + 3H \dot{\varphi} = 4\zeta^2 V_0, \tag{B2}
\]

\[
\dot{\rho} + 4H\rho = 0, \tag{B3}
\]

where \(\varphi = (\xi/2)\phi^2\) in terms of which (A.3) has been put into a simplified form in (B3), as in [3]. We also write \(V_0 = \Lambda = 1\).

Since \(H\) occurs always without derivative, we may eliminate it by using (B3), for example;

\[
H = -\frac{1}{4\rho} \dot{\rho}, \tag{B4}
\]

Eqs. (B1) and (B2) are then put into

\[
3 \left( \frac{\dot{\rho}}{\rho} \right)^2 \varphi - 12 \frac{\dot{\rho}}{\rho} \dot{\varphi} + 2\zeta^2 \varphi^2 = 8 (V_0 + \rho) \tag{B5}
\]

\[
\ddot{\varphi} - \frac{3}{4\rho} \frac{\dot{\rho}}{\rho} \dot{\varphi} = 4\zeta^2 V_0, \tag{B6}
\]

which are to be solved by giving three initial values of \(\rho, \varphi, \dot{\varphi}\).

We notice, however, that we solve (B5) and (B6) first with respect to \(\dot{\rho}/\rho\). This involves solving an algebraically quadratic equation for \(\dot{\rho}/\rho\), thus producing two differential equations, hence resulting in two separate solutions. An example of numerical solutions is shown in Figs. 7(a) and (b). In
Fig. 7(a), developed basically from Fig. 4.1 of [3], we do find asymptotic behaviors \( H \rightarrow 0, \phi \rightarrow \sqrt{4V_0/(6\xi - 1)}, \rho \rightarrow -3V_0(2\xi - 1)/(6\xi - 1) \) corresponding to an attractor solution. Fig. 7(b) illustrates, on the other hand, the solution of another equation, but sharing the same initial values of \( \rho, \varphi \) and \( \dot{\varphi} \) as discussed in (a). This one represents, however, a shrinking universe taking place in a short time. This type of the second solution occurs nearly always. It even appears as if we are going to lose an opportunity to reach the fixed-point attractor.

![Graphs](image)

**APPENDIX C: DUST-DOMINANCE IN THE SCALE-INVARIANT MODEL**

In sections III and IV we discussed cosmological solutions mainly in the radiation-dominated universe finding a crisis arising from too much time-dependent masses of particles evaded finally by departing from the Brans-Dicke model, even at the risk of WEP violation. The same type of analysis of dust-dominance suffers more seriously because it entails \( a_* \sim t_*^{1/2} \) as shown by (3.35) even for \( \gamma = 1 \). As was discussed in [3], the remedy comes simultaneously from the scale-invariant model intended to overcome the crisis for radiation-dominance. We sketch below how this model provides attractor solutions also for the dust-dominated universe. See Chapter 4.4.3 of [3] and Section 3.4 of [5] for more details.

The field equations for \( \rho \) and \( \varphi \) turn out to be given by (3.6) and (3.7) with the right-hand sides removed to the classical approximation.

As a remarkable difference from (3.35) we find

\[
 a_* = a_{*0} (t_* / t_{*0})^{2/3}, \tag{C1}
\]

in agreement with the conventional law of expansion. Eqs. (3.31) and (3.32) are replaced by

\[
x' = \zeta (2x^2 - 3xy + 1), \tag{C2}
\]

\[
y' = \frac{1}{2\zeta} \left( -\frac{1}{2} x^2 + \frac{1}{4} + \frac{1}{2} \zeta^2 (4x - 3y) y \right), \tag{C3}
\]

respectively. The solutions with (3.34), (C1) and

\[
 \exp (-4\zeta \sigma_0) = \frac{1}{16} \zeta^2, \tag{C4}
\]

in place of (3.37) are obtained for \( x = 1/\sqrt{2}, y = 2\sqrt{2}/3 \), corresponding to an attractor yielding \( x' = y' = 0 \).
A similar distinction between the elliptic and the hyperbolic curves as in radiation-dominance occurs also for \( \zeta^2 < 3/8 \) and \( \zeta^2 > 3/8 \), respectively. The same recipe should apply as mentioned toward the end of section VI.

We add that the scale invariance coming from the absence of dimensional coupling constants has an advantage that \( \sigma \) serves as a massless Nambu-Goldstone boson which will acquire a small mass after the invariance is finally broken explicitly through loops, as discussed in Section 6.3 of [3].

**APPENDIX D: ANOTHER APPROACH TO THE POWER-LAW POTENTIAL**

It seems also useful to apply (B1)-(B3) to the power-law potential to offer a simplified alternative to derive the same common result \( p_\ast = 1/2 \) for radiation-dominance as stated immediately after in (5.30).

We multiply \( V_0 \) in (B1) and (B3) by \( \phi^\alpha \). We search for the solution of the type

\[
a(t) \sim t^p, \quad (D1)
\]

and

\[
\phi(t) \sim t^\beta. \quad (D2)
\]

By substituting them into (B2) modified as above and comparing the exponents of \( t \) we obtain

\[
\beta = \frac{2}{2 - \alpha}, \quad (D3)
\]

implying that \( \alpha = 0 \) corresponds to \( \beta = 1 \). In the modified (B1), on the other hand, we find all the terms other than \( \rho \) to behave like \( t^{2\beta-2} \), while (B3) entails \( \rho \sim t^{-4p} \). For a consistent approach we expect \( 2\beta - 2 = -4p \), or

\[
p = \frac{1 - \beta}{2} = -\frac{\alpha}{2(2 - \alpha)}. \quad (D4)
\]

This point was not properly recognized when it was erroneously stated in Appendix B of [5] that only \( \alpha = 0 \) is consistent with the physically acceptable condition \( p_\ast = 1/2 \) (The coefficients \( \alpha \) and \( \beta \) in Appendix B of [5] are replaced by \( \alpha/2 \) and \( \beta/2 \), respectively, according to the present notation.).

Now from (4.6) and \( \Omega \sim \phi \) combined with (D2), we obtain

\[
dt_\ast = \Omega dt \sim t^{\beta} dt, \quad (D5)
\]

which is integrated to give

\[
t_\ast \sim t^{\beta+1}, \quad (D6)
\]

where we have ignored inessential coefficients for simplicity. In the same context we also use (4.7) to derive

\[
a_\ast = \Omega a \sim t^{\beta+p}. \quad (D7)
\]

Combining this with (D4) and (D6) we identify the right-hand side with \( t_\ast^{2p} \) where

\[
p_\ast = \frac{1}{\beta + 1} (\beta + p) = \frac{1}{2}, \quad (D8)
\]

which turns out to be the same as the result for the purely constant \( V_0 = \Lambda \), in agreement with (5.30) with \( \alpha = 0 \) for any \( \gamma \). This justifies that the present solution to be an attractor.

We also notice that the universe is no longer static in the Jordan frame, as shown in (D4). We may no longer rely on the simplest argument \( am = a_\ast m_\ast = \text{const} \) to leave the BD model. According to Appendix D of [5], particularly its (3.10) and a more general procedure developed there, however, the mass \( m_\ast \) of the matter fields in the Einstein frame is related to \( m \) in the Jordan frame as

\[
m_\ast = \Omega^{-1} m \sim t^{-\beta} m. \quad (D9)
\]

From (D3) and (D6) we find

\[
t^{-\beta} \sim t_\ast^{-\beta/(\beta+1)} = t_\ast^{(1/2)/(1-\alpha/4)}. \quad (D10)
\]

The exponent \( -(1/2)/(1 - \alpha/4) \) is not exactly the same as \( -1/2 \) which would have implied that the universe looks static if measured with respect to the microscopic length standard, but is nevertheless far from zero as expected if the Einstein frame is qualified to be a physical conformal frame for any reasonable choice of \( \alpha \). In this sense the crisis for the purely constant \( V_0 \) as discussed before is not evaded by multiplying it by the scalar field. Departure from the BD model seems still unavoidable. We also add that the argument for the constant \( m_\ast \) with the assumed scale-invariant model remains unaltered by the multiplied scalar field.