Wakimoto realizations of current and exchange algebras

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Abstract

Working at the level of Poisson brackets, we describe the extension of the generalized Wakimoto realization of a simple Lie algebra valued current, $J$, to a corresponding realization of a group valued chiral primary field, $b$, that has diagonal monodromy and satisfies $Kb' = Jb$. The chiral WZNW field $b$ is subject to a monodromy dependent exchange algebra, whose derivation is reviewed, too.

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1 Introduction

Affine Lie algebras are symmetries of interesting integrable systems. In order to perform computations, it is often useful to realize the simple Lie algebra valued current that generates the symmetry and the associated primary fields in terms of free fields. A remarkable family of generalized free field realizations of current algebras are the Wakimoto realizations (see e.g. the reviews in [1, 2]). Recently [3] an explicit formula was found for the Wakimoto realization of the current in the general case. In the context of the WZNW model [4], it is natural to introduce a chiral, group valued primary field, which is related to the current, \( J \), by the differential equation \( Kb' = Jb \) and has diagonal monodromy. The Poisson bracket (PB) (also called “classical exchange algebra”) for such a chiral WZNW field is given in terms of a distinguished solution of the classical dynamical Yang-Baxter equation. The main purpose of this paper is to present an explicit formula that extends the generalized Wakimoto realization of the current algebra to a companion realization of the exchange algebra.

2 Wakimoto realizations of the current algebra

Let \( \mathcal{G} \) denote a simple complex Lie algebra. Consider the loop algebra \( \hat{\mathcal{G}} \) whose elements are \( \hat{\mathcal{G}} \)-valued, smooth, \( 2\pi \)-periodic functions on the real line \( \mathbb{R} \). Let \( \hat{\mathcal{G}} \) be the standard central extension of \( \hat{\mathcal{G}} \). Fixing the value of the dual of the central element to a constant, \( K \), one obtains a hyperplane \( (\hat{\mathcal{G}})^*_K \) in the smooth dual \( (\hat{\mathcal{G}})^* \) of \( \hat{\mathcal{G}} \). The Lie-Poisson bracket on \( (\hat{\mathcal{G}})^* \) restricts to the usual current algebra PB on \( (\hat{\mathcal{G}})^*_K \). Using the model \( (\hat{\mathcal{G}})^*_K = \{ J | J \in \hat{\mathcal{G}} \} \), the current algebra PB has the form

\[
\{ \text{Tr} (T_a J)(x), \text{Tr} (T_b J)(y) \} = \text{Tr} ([T_a, T_b] J)(x) \delta(x - y) + K \text{Tr} (T_a T_b) \delta'(x - y),
\]

where \( \delta(x - y) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{i\pi(x-y)} \), \( \{ T_a \} \) is a basis of \( \mathcal{G} \) and \( \text{Tr} \) stands for an invariant scalar product on \( \mathcal{G} \). To describe the generalized Wakimoto realization of this PB, consider now a triangular decomposition, \( \mathcal{G} = \mathcal{G}_- + \mathcal{G}_0 + \mathcal{G}_+ \), defined by means of some integral gradation of \( \mathcal{G} \). Denote by \( \mathcal{G} \) a connected Lie group whose Lie algebra is \( \mathcal{G} \), and let \( \mathcal{G}_{0,\pm} \subset \mathcal{G} \) be the subgroups corresponding to \( \mathcal{G}_{0,\pm} \subset \mathcal{G} \). Then consider the manifolds \( \mathcal{G}_+ \) and \( \mathcal{G}_- \) whose elements are smooth, \( 2\pi \)-periodic functions on \( \mathbb{R} \) with values in \( \mathcal{G}_+ \) and \( \mathcal{G}_- \), respectively. By means of left translations, identify the cotangent bundle of \( \mathcal{G}_+ \) as \( T^* \mathcal{G}_+ = \mathcal{G}_+ \times \mathcal{G}_- = \{ (\eta_+, i_-) | \eta_+ \in \mathcal{G}_+, \ i_- \in \mathcal{G}_- \} \), and equip it with the canonical symplectic form \(-d \int_0^{2\pi} \text{Tr} (i_- \eta_+^{-1} d\eta_+) \). The corresponding PB on \( T^* \mathcal{G}_+ \) is encoded by

\[
\{ \text{Tr} (V^\alpha i_+)(x), \text{Tr} (V^\beta i_-)(y) \} = \text{Tr} ([V^\alpha, V^\beta] i_-)(x) \delta(x - y),
\]

\[
\{ \text{Tr} (V^\alpha i_+)(x), \eta_+(y) \} = \eta_+(x) V^\alpha \delta(x - y), \quad \{ \eta_+(x), \eta_+(y) \} = 0,
\]

where \( V^\alpha \) is a basis of \( \mathcal{G}_+ \). Now the generalized Wakimoto realization of the current algebra PB based on \( \mathcal{G} \) is given by a Poisson map \( W : (\hat{\mathcal{G}})_K^* \times T^* \mathcal{G}_+ \rightarrow \mathcal{G}_K \), where

\( \dagger \) All subsequent formulas are valid for the normal (split) real form of \( \mathcal{G} \), too.
\((\tilde{G}_0)_K^* = \{i_0\}\) is the space of \(G_0\)-valued currents endowed with the PB
\[
\{\text{Tr} (Y_k i_0)(x), \text{Tr} (Y_l i_0)(y)\} = \text{Tr} ([Y_k, Y_l]i_0)(x)\delta(x - y) + K \text{Tr} (Y_k Y_l)\delta'(x - y) \tag{3}
\]
with a basis \(Y_k\) of \(G_0\). In \([3]\) the following simple formula was obtained:
\[
W : (\tilde{G}_0)_K^* \times T^* \tilde{G}_+ \ni (i_0, \eta_+, i_-) \mapsto J = \eta_+(i_0 - i_-)\eta_+^{-1} + K \eta_+ \eta_-^{-1} \in (\tilde{G})_K^*. \tag{4}
\]
The PBs of \(J\) in \([4]\) follow from the simpler PBs of the constituents \((i_0, \eta_+, i_-)\).

3 The monodromy dependent exchange algebra

In the context of the WZNW model, it is natural to introduce a chiral \(G\)-valued field \(b(x)\) with the following properties:

1. The field \(b(x)\) is related to the current \(J(x)\) by the equation \(J = K b b^{-1}\).

2. The monodromy matrix associated with \(b\) is diagonal: \(b(x + 2\pi) = b(x)e^{\omega}\) with \(\omega \in \mathcal{H} \subset \tilde{G}\), where \(\mathcal{H}\) is a fixed Cartan subalgebra of \(\tilde{G}\).

3. The space of fields \(\{b\}\) has a PB, which is such that it implies the current algebra PB for \(J = K b b^{-1}\) and \(b\) is a primary field:
\[
\{b(x), J^a_n\} = \frac{1}{2\pi} e^{-inx} T_a b(x) \quad \text{for} \quad J^a_n := \frac{1}{2\pi} \int_0^{2\pi} dx e^{-inx} \text{Tr} (T_a J)(x).
\]

A PB satisfying conditions 1–3 is given, for \(0 < |x - y| < 2\pi\), by
\[
\{b_1(x), b_2(y)\} = \frac{1}{2K} (b(x) \otimes b(y)) \left( \text{sign}(y - x)\mathcal{C} + \mathcal{R}(\omega) \right),
\]
\[
\mathcal{R}(\omega) = \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{4} \coth \left( \frac{1}{2} \alpha(\omega) \right) (E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha). \tag{5}
\]

Here \(\Phi\) denotes the set of roots of \((\mathcal{H}, \tilde{G})\), the root vectors \(E_\alpha\) are normalized by \(\text{Tr} (E_\alpha E_{-\alpha}) = \frac{2}{|\alpha|^2}, \mathcal{C} = \sum_a T^a \otimes T_a\) is the tensor Casimir of \(\tilde{G}\) and \(b_1 = b \otimes 1\).

In \([3]\) (see also \([4]\)) the “classical exchange algebra” \([4]\) was obtained by means of a construction of the field \(b(x)\) out of some local coordinates on the phase space of the WZNW model. As a consistency check, it was verified that \(\mathcal{R}(\omega)\) satisfies the following dynamical version of the modified classical Yang-Baxter equation:
\[
[\mathcal{R}_{12}(\omega), \mathcal{R}_{23}(\omega)] + 2 \sum_i H_i^j \frac{\partial}{\partial \omega^j} \mathcal{R}_{23}(\omega) + \text{cycl. perm.} = - \sum_{a,b,c} f^{abc} T_a \otimes T_b \otimes T_c, \tag{6}
\]
where \(\{H_i^j\}\) is basis of \(\mathcal{H}\), \(\omega^i = \text{Tr} (H^i \omega)\) and \([T_a, T_b] = \sum_c f_{abc} T_c\).
Let us comment on the uniqueness of the PB (3) under the conditions 1–3. It is not difficult to see that conditions 1–3 imply that the PB \{b_1(x), b_2(y)\} must be of the form that appears in (3) with some r-matrix \(\mathcal{R}(\omega)\). It also follows that the r-matrix in question must satisfy (3) and it must be neutral with respect to \(\mathcal{H}\). Recently [7] all neutral solutions of (3) have been classified under the additional assumption that \(\mathcal{R} : \mathcal{H} \rightarrow \mathcal{G} \wedge \mathcal{G}\) is a meromorphic function. The parameters that label the general solution contain a regular semisimple subalgebra of \(\mathcal{G}\) and an element of \(\mathcal{H}\). The particular r-matrix in (3) corresponds to the regular subalgebra being equal to \(\mathcal{G}\) and the element of \(\mathcal{H}\) being zero. It can be shown [8] that in the context of the WZNW model no other parameters could be chosen, and conditions 1–3 determine the exchange algebra of the field \(b\) essentially uniquely.

The PB in (3) can actually be derived from a symplectic form [3, 10, 11] on the phase space
\[
\mathcal{M}_G^{\text{Bloch}} = \{ b \in C^\infty(\mathbb{R}, G) \mid b(x + 2\pi) = b(x)e^{i\omega}, \quad \omega \in \mathcal{A} \subset \mathcal{H} \},
\]
where \(\mathcal{A}\) is an open Weyl alcove that contains such \(\omega \in \mathcal{H}\) for which \(\alpha(\omega) \notin i2\pi\mathbb{Z}\) for any \(\alpha \in \Phi\) and the restriction of the map \(\omega \mapsto e^{i\omega}\) to \(\mathcal{A}\) is injective. The symplectic form on \(\mathcal{M}_G^{\text{Bloch}}\) is defined by
\[
\Omega_{G,K}^{\text{Bloch}}(b) = -\frac{K}{2} \int_0^{2\pi} \text{Tr} \left( (b^{-1}db)^{\prime} \wedge \left( b^{-1}db \right)^\prime \right) - \frac{K}{2} \text{Tr} \left( (b^{-1}db)(0) \wedge d\omega \right). \tag{8}
\]
The condition \(\alpha(\omega) \notin i2\pi\mathbb{Z}\) is needed since at the excluded values of \(\omega\) the 2-form in (8) would become singular. The additional restriction of \(\omega\) to \(\mathcal{A}\) ensures that \(\omega\) uniquely parametrizes the monodromy matrix of the “Bloch wave” \(b\), which can be represented in the form \(b(x) = h(x)\exp(x\omega/2\pi)\) with \(h \in \hat{G}\). This parametrization of \(b\) is useful for deriving the PB in (3) by inverting the symplectic form in (8).

Let us recall [9, 10, 11] how the symplectic form (8) arises in the WZNW model. The WZNW model can be described as the Hamiltonian system \((\mathcal{M}, \Omega_K, H_{WZ})\), where
\[
\mathcal{M} = T^*\hat{G} = \{ (g, J) \mid g \in \hat{G}, \; J \in \hat{G} \},
\]
\[
\Omega_K = d \int_0^{2\pi} \text{Tr} \left( Jdg^{-1} \right) + \frac{K}{2} \int_0^{2\pi} \text{Tr} \left( dgg^{-1} \right) \wedge \left( dgg^{-1} \right)^\prime,
\]
and \(H_{WZ} = \frac{1}{2K} \int_0^{2\pi} \text{Tr} \left( J^2 + I^2 \right)\) with \(I = -g^{-1}Jg + Kg^{-1}g^\prime\). The corresponding space of solutions, \(\mathcal{M}^{\text{sol}}\), consists of smooth, \(G\)-valued functions \(g(\sigma, \tau)\) which are 2\(\pi\)-periodic in the space variable \(\sigma\) and satisfy \(\partial_-(\partial_+g \cdot g^{-1}) = 0\), where \(\partial_\pm\) are derivations with respect to the light cone coordinates \(x^\pm = \sigma \pm \tau\). The most general such function has the factorized form \(g(\sigma, \tau) = g_L(x^+)g_R^{-1}(x^-)\) with a pair \((g_L, g_R)\) of \(G\)-valued, smooth, quasiperiodic functions on \(\mathbb{R}\) with equal monodromies: \(g_L(x + 2\pi) = g_L(x)Q\) and \(g_R(x + 2\pi) = g_R(x)Q\) with some \(Q \in \hat{G}\). It is therefore convenient to introduce the set of such pairs, \(\hat{\mathcal{M}} := \{(g_L, g_R)\}\). By associating the elements of the solution space with their initial data at \(\tau = 0\), one can identify \(\mathcal{M}^{\text{sol}}\) with the phase space \(\mathcal{M}\), and thus \(\Omega_K\) yields a symplectic form on \(\mathcal{M}^{\text{sol}}\). Then using the projection \(\theta : \hat{\mathcal{M}} \rightarrow \mathcal{M}^{\text{sol}}\) given by \(\theta : (g_L, g_R) \mapsto g = g_Lg_R^{-1}\), one obtains a
closed 2-form, $\hat{\Omega}_K$, on $\hat{\mathcal{M}}$ by the pull-back. Explicitly, $\hat{\Omega}_K(g_L, g_R) = \Omega_{K}^{\text{chir}}(g_L) - \Omega_{K}^{\text{chir}}(g_R)$ with
\[
\Omega_{K}^{\text{chir}}(g_L) = -\frac{K}{2} \int_0^{2\pi} \text{Tr} \left( g_L^{-1} dg_L \right) \wedge \left( g_L^{-1} dg_L \right)' - \frac{K}{2} \text{Tr} \left( (g_L^{-1} dg_L)(0) \wedge dQ \cdot Q^{-1} \right).
\]

Of course, $\hat{\Omega}_K$ is not a symplectic form on $\hat{\mathcal{M}}$, but its restriction to any local section of the bundle $\theta : \hat{\mathcal{M}} \to \mathcal{M}^{\text{sol}}$ yields a symplectic form. As for the chiral WZNW 2-form $\hat{\Omega}_{K}^{\text{chir}}$, it is not even closed for general monodromy. However, upon the restriction $g_L(x) = b(x)$ with $b \in \mathcal{M}^{\text{Bloch}}_G$, i.e. by imposing the constraint $Q = e^{i\omega}$ with $\omega \in \mathcal{A}$, it becomes the symplectic form in (8). This derivation of $\hat{\Omega}_{K}^{\text{Bloch}}$ can be found in [9, 10, 11] and these papers also contain an outline of the derivation of the exchange algebra (5) from the symplectic form (8). A complete derivation of (5) along these lines is given in [8].

4 Wakimoto realizations of the exchange algebra

We now wish to complete the construction of the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{M}^{\text{Bloch}}_{G_0} \times T^*\tilde{G}_+ & \overset{\hat{W}}{\longrightarrow} & \mathcal{M}^{\text{Bloch}}_G \\
\mathcal{D}_0 \times \text{id} \downarrow & & \downarrow \mathcal{D} \\
(\mathcal{G}_0)^*_K \times T^*\tilde{G}_+ & \overset{\mathcal{W}}{\longrightarrow} & (\mathcal{G})^*_K
\end{array}
\]
(9)

The maps designated by simple arrows have already been described and are Poisson maps. In particular, $\hat{W}$ is the Wakimoto realization of the current algebra defined in (4), and $\mathcal{D}$ operates according to $\mathcal{D} : b \mapsto J = Kb' b^{-1}$. $\mathcal{M}^{\text{Bloch}}_{G_0}$ consists of the $G_0$-valued Bloch waves with regular, diagonal monodromy,
\[
\mathcal{M}^{\text{Bloch}}_{G_0} = \{ \eta_0 \in C^\infty(\mathbb{R}, G_0) \mid \eta_0(x + 2\pi) = \eta_0(x) e^{i\omega}, \ \omega \in \mathcal{A} \subset \mathcal{H} \},
\]
where $\mathcal{A}$ also appears in (7), and is equipped with the symplectic form
\[
\Omega^{\text{Bloch}}_{G_0,K}(\eta_0) = -\frac{K}{2} \int_0^{2\pi} \text{Tr} \left( \eta_0^{-1} d\eta_0 \right) \wedge \left( \eta_0^{-1} d\eta_0 \right)' - \frac{K}{2} \text{Tr} \left( (\eta_0^{-1} d\eta_0)(0) \wedge d\omega \right).
\]

The map $\mathcal{D}_0$ sends $\eta_0$ to $i_0 = K\eta_0 \eta_0^{-1} \in (\mathcal{G}_0)^*_K$. The formula for the mapping
\[
\hat{W} : \mathcal{M}^{\text{Bloch}}_{G_0} \times T^*\tilde{G}_+ \ni (\eta_0, \eta_+, \eta_-) \longmapsto b \in \mathcal{M}^{\text{Bloch}}_G
\]
can be found from the equation $\mathcal{D} \circ \hat{W} = W \circ (\mathcal{D}_0 \times \text{id})$, which requires that
\[
Kb' b^{-1} = \eta_+ (K\eta_0 \eta_0^{-1} - i_- \eta_+^{-1}) + K\eta_+ \eta_+^{-1}.
\]
A solution for $b$ exists that admits a generalized Gauss decomposition. In fact, 

$$b(x) = b_+(x)b_0(x)b_-(x) \quad \text{with} \quad b_{\pm,0}(x) \in G_{\pm,0}$$

is a solution if 

$$b_+ = \eta_+, \quad b_0 = \eta_0 \quad \text{and} \quad K b'_- b^{-1}_- = -\eta_0^{-1} i_- \eta_0.$$

The general solution of the differential equation for $b_-$ can be written in terms of the particular solution $b_P^-$ defined by $b_P^-(0) = 1$ as $b_-(x) = b_P^-(x)S$ with an arbitrary $S \in G_-$. The constant $S = b_-(0)$ has to be determined from the condition that $b$ should have diagonal monodromy. One finds that $b$ has diagonal monodromy, indeed it satisfies $b(x + 2\pi) = b(x)e^{\omega}$, if and only if

$$e^{-\omega} S e^{\omega} = b_P^-(2\pi) S.$$

Inspecting this equation grade by grade using a parametrization $S = e^s$, $s \in G_-$ and the regularity of $\omega$, one sees that it has a unique solution for $S$ as a function of $\omega$ and $b_P^-(2\pi)$. Determining $S$ by this procedure, we now define the mapping

$$\hat{W} : \mathcal{M}_{G_0}^{\text{Bloch}} \times T^* \tilde{G}_+ \ni (\eta_0, \eta_+, i_-) \mapsto b = \eta_+ \eta_0 b_P^- S \in \mathcal{M}_G^{\text{Bloch}}$$

that makes the diagram in (9) commutative.

**Theorem.** The mapping $\hat{W}$ defined in (10) is symplectic, that is,

$$(\hat{W}^* \Omega_{G,K}^{\text{Bloch}})(\eta_0, \eta_+, i_-) = \Omega_{G,K}^{\text{Bloch}}(b = \eta_+ \eta_0 b_P^- S) =\Omega_{K,G_0}(\eta_0) - d \int_0^{2\pi} \text{Tr} (i_- \eta_+^{-1} d\eta_+).$$

The second term on the right hand side is the symplectic form on $T^* \tilde{G}_+$. The proof of the theorem [8] is a straightforward computation. For $G = SL(2)$ the result was already proved in [12, 9, 11] and some other special cases of it can be extracted from [13].

Since a symplectic map is always a Poisson map too, $\hat{W}$ provides us with a realization of the monodromy dependent exchange algebra $\mathcal{G}$ of the $G$-valued Bloch waves in terms of the analogous exchange algebra of the $G_0$-valued Bloch waves and the canonical free fields that may be used to parametrize $T^* \tilde{G}_+$. This generalized free field realization becomes a true free field realization in the principal case, for which $G_0 = H$ is Abelian and $\eta_0$ is the exponential of a $H$-valued free scalar field.

The Wakimoto realizations of the affine Lie algebras in generalized Fock spaces, i.e. at the level of vertex algebras as opposed to the above Poisson algebras, have many applications in conformal field theory (see e.g. [1, 2]). A simple explicit formula for such realizations of the current $J$ is obtained in [3] by quantizing the expression in (4). It would be interesting to also quantize the formula in (10). Another problem for future study is to work out an analogue of the construction presented in this paper for the case of $q$-deformed affine Lie algebras.
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