1. INTRODUCTION

The $H^+_3$ WZNW model plays an important role in the study of string theory on $AdS_3$ (see e.g. MO, KS, GR and references therein). It also has been proposed to be relevant for the study of the integer quantum Hall effect [2], [QH]. One may furthermore view it as a prototypical example for a conformal field theory with continuous spectrum of primary fields (noncompact CFT).

The study of the $H^+_3$ WZNW model was initiated by Gawedzki, who determined the spectrum of the theory from a path-integral calculation of the torus partition function [Ga]. The three point function for the $H^+_3$ WZNW model was calculated in [T1], [T2] by assuming decoupling of singular vectors in degenerate representations of the current algebra. Three point function and current algebra symmetry would fully characterize the whole theory (see [T3] for a discussion of the corresponding issue in the case of Liouville theory). However, so far there was no proof that the three point function from [T1], [T2] actually leads to a consistent theory. Consistency of the theory boils down to proving crossing symmetry of the four-point function that, as shown in [T2], can be uniquely constructed out of the three point functions by means of the current algebra symmetry.

The aim of the present paper is to show that crossing symmetry of the four point function in the $H^+_3$ WZNW model follows from similar properties of a five point function in Liouville theory. A proof of the fact that locality and crossing symmetry hold in Liouville theory was outlined in [T3], building upon [PT]. One may therefore regard the results of [T1], [T2] as providing a solid construction of the $H^+_3$ WZNW model in genus zero, the consistency of which is established in the present note on the basis of results from [T3], [PT].

2. CORRELATION FUNCTIONS IN LIOUVILLE THEORY AND THE $H^+_3$ WZNW MODEL
2.1. **Five point function in Liouville theory**

As indicated above, we will heavily use properties of certain correlation functions in Liouville theory. Let \( V_\alpha(z) \) denote local primary fields in Liouville theory with conformal dimension \( \Delta_\alpha = \alpha(Q - \alpha) \), where \( Q = b + b^{-1} \), where \( b \) is the basic parameter (“coupling constant”) that determines the central charge of the Virasoro algebra according to \( c = 1 + 6Q^2 \).

We shall consider the vacuum expectation value

\[
\Omega_{\text{Liou}}^{(4)}(\alpha_4, \alpha_2, x, z) \equiv \langle V_{\alpha_4}(\infty)V_{\alpha_2}(1)V_{\frac{1}{2}}(x)V_{\alpha_2}(z)V_{\alpha_1}(0) \rangle_{\text{Liou}}.
\]

(1)

It can be represented as follows \([\ZZ][T3]\):

\[
\Omega_{\text{Liou}}^{(4)}(\alpha_4, \alpha_2, x, z) = \sum_{s=\tau, -} \int_0^\infty dP \frac{C(\alpha_4, \alpha_3, \alpha_P - \frac{\alpha_4}{2})C_s(\alpha_P)C(\alpha_P, \alpha_2, \alpha_1)}{\bar{\Upsilon}(\alpha_1 + \alpha_2 + \alpha_3 - Q)\bar{\Upsilon}(\alpha_1 + \alpha_2 - \alpha_3)\bar{\Upsilon}(\alpha_2 + \alpha_3 - \alpha_1)\bar{\Upsilon}(\alpha_3 + \alpha_1 - \alpha_2)} \times \mathcal{F}_{\alpha_P}^s(\alpha_4, \alpha_2 | x, z)\mathcal{F}_{\alpha_P}(\alpha_4, \alpha_2 | \bar{x}, \bar{z}),
\]

(2)

where \( \alpha_P = \frac{Q}{2} + iP \). This expression is composed out of the following ingredients:

- **The three point function** \( C(\alpha_3, \alpha_2, \alpha_1) \) \([\ZZ][\ZZ]\):

\[
C(\alpha_1, \alpha_2, \alpha_3) = \left[ \pi \mu b^2 \right]^{\frac{1}{2}Q - \sum \frac{\alpha_i}{b}} \times \frac{\bar{\Upsilon}(2\alpha_1)\bar{\Upsilon}(2\alpha_2)\bar{\Upsilon}(2\alpha_3)}{\bar{\Upsilon}(\alpha_1 + \alpha_2 + \alpha_3 - Q)\bar{\Upsilon}(\alpha_1 + \alpha_2 - \alpha_3)\bar{\Upsilon}(\alpha_2 + \alpha_3 - \alpha_1)\bar{\Upsilon}(\alpha_3 + \alpha_1 - \alpha_2)}
\]

(3)

The special function \( \bar{\Upsilon}(x) \) that appears in (3) can be defined by means of the integral representation

\[
\log \bar{\Upsilon}(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left( \frac{Q}{2} - x \right)}{\sinh \frac{t}{2} \sinh \frac{t}{2b}} \right].
\]

(4)

- **The structure constants** \( C_s(\alpha) \), \( s = +, - \) are defined from the short-distance asymptotics

\[
V_{\frac{1}{2}}(x)V_{\alpha}(z) \sim \sum_{s=\tau, -} \int_0^\infty dP \frac{C(\alpha_4, \alpha_3, \alpha_P - \frac{\alpha_4}{2})C_s(\alpha_P)C(\alpha_P, \alpha_2, \alpha_1)}{\bar{\Upsilon}(\alpha_1 + \alpha_2 + \alpha_3 - Q)\bar{\Upsilon}(\alpha_1 + \alpha_2 - \alpha_3)\bar{\Upsilon}(\alpha_2 + \alpha_3 - \alpha_1)\bar{\Upsilon}(\alpha_3 + \alpha_1 - \alpha_2)} \times \mathcal{F}_{\alpha_P}^s(\alpha_4, \alpha_2 | x, z)\mathcal{F}_{\alpha_P}(\alpha_4, \alpha_2 | \bar{x}, \bar{z}),
\]

(5)

where \( \alpha_\pm = \alpha - \frac{Q}{2b}, \delta = \frac{\Delta_\pm}{b} \). The \( C_s(\alpha) \), \( s = +, - \) can be recovered as particular residues from the three point function \( C(\alpha_3, \alpha_2, \alpha_1) \) \([\ZZ][\ZZ][\ZZ]\):

\[
C_+(\alpha) = 1, \quad C_-(\alpha) = \left[ \pi \mu b^2 \right]^{\frac{1}{2}Q - \sum \frac{\alpha_i}{b}} \frac{\bar{\Upsilon}(\alpha_2 + \alpha_3 - \alpha_1)\bar{\Upsilon}(\alpha_3 + \alpha_1 - \alpha_2)}{\bar{\Upsilon}(\alpha_1 + \alpha_2 + \alpha_3 - Q)\bar{\Upsilon}(\alpha_1 + \alpha_2 - \alpha_3)\bar{\Upsilon}(\alpha_2 + \alpha_3 - \alpha_1)},
\]

(6)

where \( \gamma(x) = \Gamma(x)/\Gamma(1-x) \).

- **The conformal blocks** \( \mathcal{F}_{\alpha_P}^s(\alpha_4, \alpha_2 | x, z) \), \( s = +, - \): They can be characterized as those solutions to the null vector decoupling equations

\[
\left( b^2 \partial_x^2 + \frac{z(z-1)}{x(x-1)(x-z)} \partial_x + \frac{1 - 2x}{x(x-1)} \partial_x - \frac{\Delta}{x(x-1)} + \frac{\Delta_\alpha_3}{(x-1)^2} + \frac{\Delta_\alpha_2}{(x-z)^2} + \frac{\Delta_\alpha_1}{x^2} \right) \mathcal{F}(x, z) = 0,
\]

(7)
The following identity holds: \( F^{a}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x, z) \sim \), which have the following asymptotic behavior for \( z \to \infty \):

\[
F^{a}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x, z) \sim \frac{\alpha_{i}}{\alpha_{j}} D_{\alpha} - \frac{\alpha_{j}}{\alpha_{i}} F^{a}_{\alpha} (\frac{\alpha_{j}}{\alpha_{i}} | x),
\]

with \( F^{a}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x) \) given by

\[
F^{a}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x) = z^{-\frac{1}{\alpha_{i}}(1 - z)^{\frac{1}{\alpha_{j}}} F(u, v, w, z)}
\]

\[
F^{-}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x) = z^{-\frac{1}{\alpha_{i}}(1 - z)^{\frac{1}{\alpha_{j}}} F(u - w + 1, v - w + 1, 2 - w, z)},
\]

where we have used the notation

\[
u = b^{-1}(\alpha_{i} + \alpha_{j} + \alpha_{k} - 3b^{-1}/2) - 1\]

\[
v = b^{-1}(\alpha_{i} + \alpha_{j} - \alpha_{k} - b^{-1}/2)
\]

For Liouville theory it is possible to prove crossing symmetry and locality \([T3]\). This implies that \( \Omega_{\text{Liou}} (\frac{\alpha_{i}}{\alpha_{j}} | x, z) \) is real analytic on \( \{ (x, z) \in \mathbb{P}_{+}^{1} \times \mathbb{P}_{+}^{1} ; z \neq x \} \), where \( \mathbb{P}_{+}^{1} \equiv \mathbb{P} \setminus \{ 0, 1, \infty \} \), and that the following identity holds:

\[
\Omega^{(4)}_{\text{Liou}} (\frac{\alpha_{i} \alpha_{j} \alpha_{k} \alpha_{l}}{\alpha_{m} \alpha_{n} \alpha_{o} \alpha_{p}} | x, z) = \Omega^{(4)}_{\text{Liou}} (\frac{\alpha_{i} \alpha_{j} \alpha_{k} \alpha_{l}}{\alpha_{m} \alpha_{n} \alpha_{o} \alpha_{p}} | 1 - x, 1 - z).
\]

**Remark.** We do not have an explicit expression for the functions \( F^{a}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x, z) \). However, the free field construction of chiral vertex operators intertwining between three general irreducible highest weight representations of the Virasoro algebra given in \([T3]\) leads to a constructive definition for \( F^{a}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x, z) \) that is manageable enough to allow for the calculation of the monodromies of \( F^{a}_{\alpha} (\frac{\alpha_{i}}{\alpha_{j}} | x, z) \). These turn out to be related to canonical objects from quantum group representation theory, the so-called b-Racah-Wigner coefficients \([T3]\). The identities needed to prove \([10]\) all follow from the definition of the b-Racah-Wigner coefficients in terms of quantum group representation theory \([T3]\), \([T3]\).

2.2. **Four point function in the \( H^{+} \) WZNW model**

The basic parameter in the \( H^{+} \) WZNW model is the level \( k \) of the current algebra

\[
[J_{n}^{0}, J_{m}^{0}] = -k n \delta_{n+m,0}, \]

\[
[J_{n}^{\pm}, J_{m}^{\pm}] = \pm J_{n+m}^{\pm}, \quad [J_{n}^{0}, J_{m}^{\pm}] = 2 J_{n+m}^{0} + k n \delta_{n+m,0}.
\]

It will be useful to write \( k \equiv b^{-2} + 2 \), since \( b \) will turn out to correspond directly to the parameter \( b \) of Liouville theory. The definition of primary fields \( \Phi^{j} (x|z) \) will be the one used in \([T2]\), the conformal dimension being \( \Delta_{j} = -b^{2}(j + 1) \).

We will be interested in the following correlation function in the \( H^{+} \) WZNW model:

\[
\Omega^{(4)}_{\text{WZNW}} (j_{3}, j_{2}, j_{1}, j_{0} | x, z) \equiv \langle \Phi^{j_{3}} (\infty) | \Phi^{j_{2}} (1) | \Phi^{j_{1}} (0) | \Phi^{j_{0}} (0) \rangle_{\text{WZNW}}
\]

The following description was proposed in \([T2]\) for \( \Omega^{(4)}_{\text{WZNW}} (j_{3}, j_{2}, j_{1}, j_{0} | x, z) \):

\[
\Omega^{(4)}_{\text{WZNW}} (j_{3}, j_{2}, j_{1}, j_{0} | x, z) = \int_{0}^{\infty} \frac{dp}{B(p)} D(j_{4}, j_{3}, j_{p}) D(j_{p}, j_{2}, j_{1}) H_{j_{p}} (j_{3}, j_{2}, j_{1} | x, z).
\]

where \( j_{p} = \frac{1}{2} + ip \). The following objects appear in \([T3]\):
The two-point function $B(j)$ is given by the formula
\[
B(j) = - \left( \nu(b) \right)^2 \frac{2j + 1}{\pi} \Gamma(1 + b^2(2j + 1)) \frac{\Gamma(1 - b^2(2j + 1))}{\Gamma(1 + b^2(2j + 1))}.
\] (14)

The expression for $\nu(b)$ can be found in [12], but will not be important in the following.

The three point function $D(j_3, j_2, j_1)$ has the following expression:
\[
D(j_3, j_2, j_1) = \frac{(\nu(b)^2)^j_1 + j_2 + j_3 + 1}{C_w(b) \Gamma(2j_1 + 1) \Gamma(2j_2 + 1) \Gamma(2j_3 + 1)} \frac{\Gamma(2j_1 + 1) \Gamma(2j_2 + 1) \Gamma(2j_3 + 1)}{\Gamma(2j_1 + 1) \Gamma(2j_2 + 1) \Gamma(2j_3 + 1)}.
\] (15)

The special function $\Upsilon_w(j)$ is related to the $\Upsilon$-function via $\Upsilon_w(j) = \Upsilon(-bj)$.

The function $H_j(j_3; j_1 | x | z)$ can be decomposed into conformal blocks $G_j(j_3; j_1 | x | z)$ as follows:
\[
H_j(j_3; j_1 | x | z) = G_j(j_3; j_1 | x | z) G_j(j_3; j_1 | \bar{x} | \bar{z}) - \frac{(2j + 1)^2}{\gamma^2} \gamma(j_3 - j_2 - j_1 j_3 - j_1 + j_2 + 1) \gamma(j_1 + j_2 - j) \times G_{-j-1}(j_3; j_1 | x | z) G_{-j}(j_1; j_2 | \bar{x} | \bar{z})
\] (16)

The conformal blocks $G_j(j_3; j_1 | x | z)$ are uniquely defined as those solutions of the Knizhnik-Zamolodchikov (KZ) equations $(z(1 - \partial_z + b^2 D_x^{(2)}) G = 0$,
\[
D_x^{(2)} = x(x - 1)(x - z) \partial_x^2
- [(\kappa - 1)(x^2 - 2zx + z) + 2j_1 x(z - 1) + 2j_2 x(x - 1) + 2j_3 z(x - 1)] \partial_x
+ 2j_2 \kappa(x - z) + 2j_1 j_2 (z - 1) + 2j_2 j_3 z,
\] (17)

which can be represented by power series of the form
\[
G_j(j_3; j_1 | x | z) = z^{\Delta_{21}(j)} \sum_{n=0}^{\infty} z^n G_j^{(n)}(j_3; j_1 | x),
\] (18)

with initial term $G_j^{(0)}(j_3; j_1 | x)$ given by
\[
G_j^{(0)}(j_3; j_1 | x) = x^{j_3 - j_2} F(j_1 - j_2 - j, j_4 - j_3 - j; -2j; x).
\] (19)

We have used the abbreviations $\kappa = j_1 + j_2 + j_3 - j_4$ and $\Delta_{21}(j) = \Delta_j - \Delta_{j_2} - \Delta_{j_1}$.

Our aim will be to prove that that $\Omega_{\text{WZWN}}(j_3, j_4; x | z)$ is real analytic on $\left\{(x, z) \in \mathbb{P}_+^2 \times \mathbb{P}_+^2; z \neq x\right\}$, and that the following identity (crossing symmetry) holds:
\[
\Omega_{\text{WZWN}}(j_3, j_4; x | z) = \Omega_{\text{WZWN}}(j_3, j_4; 1 - x | 1 - z).
\] (20)
3. PROOF OF CROSSING SYMMETRY

Crossing symmetry for the four point function in the $H_3^+$ WZNW model will follow immediately once the following identity is proven:

$$\Omega_{WZNW}^{(4)}(j_3, j_2 | x, z) = \Omega_{LIOU}^{(4)}(j_3, j_2 | x, z) \Omega_{AUX}^{(4)}(\alpha_1, \alpha_2 | x, z),$$

$$\Omega_{AUX}^{(4)}(j_3, j_2 | x, z) = \pi \frac{C_w(b)}{b^2} \left( \frac{(\mu b)^{b^2}}{\Gamma(b^2-2b^2+1)} \right)^4 \prod_{i=1}^{4} \frac{\Gamma(2j_i+1)}{\Gamma(2\alpha_i)} \times$$

$$\times |x|^{-2b^{-1}+1} - |x|^{-2b^{-1}+1} |z|^{-2b^{-1}+1} |1-z|^{-\gamma_{23}},$$

where we have used the notation

$$s = b^{-1} \left( Q - \sum_{i=1}^{4} \alpha_i + \frac{1}{2b} \right), \quad t = \sum_{i=1}^{4} j_i + 1,$$

and assumed the following identifications between the sets of variables $j_4, \ldots, j_1$ and $\alpha_4, \ldots, \alpha_1$:

$$2\alpha_1 = -b(j_1 + j_2 - j_3 - j_4 - b^{-2} - 1), \quad 2\alpha_3 = -b(j_2 + j_3 - j_4 - b^{-2} - 1),$$

$$2\alpha_2 = -b(j_1 + j_2 + j_3 + j_4 + 1), \quad 2\alpha_4 = -b(j_4 + j_2 - j_1 - j_3 - b^{-2} - 1).$$

Crossing symmetry of $\Omega_{WZNW}^{(4)}(j_3, j_2 | x, z)$ then follows from the corresponding properties of $\Omega_{LIOU}^{(4)}(\alpha_1, \alpha_2 | x, z)$ and $\Omega_{AUX}^{(4)}(j_3, j_2 | x, z)$.

The main ingredient needed to prove identity (21) is the following observation from [FZ]:

**Assume that (i) $j_4, \ldots, j_1$ and $\alpha_4, \ldots, \alpha_1$ satisfy the relations (23), (ii) $\gamma_{23}$ and $\gamma_{12}$ are defined by (22), and (iii) $F(x, z)$ and $G(x | z)$ are related by**

$$F(x, z) = x^{b^{-1}+1} (1-x)^{b^{-1}+1} z^{b^{-1}+1} \gamma_{12} (1-z)^{\frac{1}{2}+\gamma_{23}} G(x | z).$$

$$\mathcal{F}(x, z) will then satisfy the decoupling equations (26) if and only if $G(x | z)$ solves the KZ-equations (27). This observation reduces the proof of (21) to a couple of straightforward verifications. First let us note that

$$G_{j}^{(0)}(j_3, j_2 | x) = x^{b^{-1}+1} z^{b^{-1}+1} \mathcal{F}_{\alpha}^{\gamma_{12}}(\alpha_1, \alpha_2 | x)$$

if the identifications (23) hold, and that

$$2\Delta_n - \Delta_{n+2} = 2\Delta_j - \Delta_{j+2} - 2z^{2\gamma_{12}}$$

if $\gamma_{12}$ is chosen according to (22) and the relation between $j$ and $\alpha$ is assumed to be

$$\alpha = -bj + \frac{1}{2b}.$$
The final point that remains to prove (21) is then to verify the fact that
\[ C(\alpha_4, \alpha_3, \alpha - \frac{1}{2}\alpha_1) \]

therefore suffices to conclude that (28) indeed holds.

In order to prove (21), it therefore suffices to consider the coefficients with which the conformal blocks are multiplied in (2) and (13). Let us first note that
\[ \frac{C(\alpha_4, \alpha_3, \alpha - \frac{1}{2}\alpha_1)}{D(j_4, j_3, j)B^{-1}(j)D(j_2, j_1)} = b \frac{\pi}{\mu} \frac{\gamma(2b^{-2+2b^2})}{2^0 (\nu(b)b^{2b^2})} \prod_{i=1}^{4} \frac{\gamma(2(j_i))}{\gamma((j_i+2))}. \]

The verification of (29) is done by a straightforward calculation using the explicit expressions for $B$, $C$ and $D$ given above, the identifications (23), (27) and the functional equations [29].

The final point that remains to prove (21) is then to verify the fact that
\[ \frac{C(\alpha_4, \alpha_3, \alpha + \frac{1}{2}\alpha_1)C_-(\alpha)}{C(\alpha_4, \alpha_3, \alpha - \frac{1}{2}\alpha_1)C_+(\alpha)} = \frac{(2j_1^2 + 2)\gamma(j_4 - j_3 + j + 1)}{\gamma(2j_1^2 + 2)} \frac{\gamma(j_3 - j_4 + j + 1)}{\gamma(j_2 - j_1 - j)} \frac{\gamma(j_1 - j_2 - j)}{\gamma(j_1 - j_2 - j - 1)}. \]

cf. [10], which is again a matter of straightforward computations using (30).

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\footnote{See \cite{T2} Appendix C for a verification of this statement in the case of the KZ-equation which can easily be adapted to the decoupling equation (\ref{eq:decoupling}).}
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