Zero loci of Bernstein–Sato ideals

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Abstract We prove a conjecture of the first author relating the Bernstein–Sato ideal of a finite collection of multivariate polynomials with cohomology support loci of rank one complex local systems. This generalizes a classical theorem of Malgrange and Kashiwara relating the $b$-function of a multivariate polynomial with the monodromy eigenvalues on the Milnor fibers cohomology.

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1 Introduction

1.1. Let $F = (f_1, \ldots, f_r) : (X, x) \to (\mathbb{C}^r, 0)$ be the germ of a holomorphic map from a complex manifold $X$. The (local) Bernstein–Sato ideal of $F$ is the ideal $B_F$ in $\mathbb{C}[s_1, \ldots, s_r]$ generated by all $b \in \mathbb{C}[s_1, \ldots, s_r]$ such that in a neighborhood of $x$

$$b \prod_{i=1}^{r} f_i^{s_i} = P \cdot \prod_{i=1}^{r} f_i^{s_i+1} \quad (1.1)$$

for some $P \in \mathcal{D}_X[s_1, \ldots, s_r]$, where $\mathcal{D}_X$ is the ring of holomorphic differential operators. Sabbah [22,23] showed that $B_F$ is not zero.

1.2. If $F = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$ is a morphism from a smooth complex affine irreducible algebraic variety, the (global) Bernstein–Sato ideal $B_F$ is defined as the ideal generated by all $b \in \mathbb{C}[s_1, \ldots, s_r]$ such that (1.1) holds globally with $\mathcal{D}_X$ replaced by the ring of algebraic differential operators. The global Bernstein–Sato ideal is the intersection of all the local ones at points $x$ with some $f_i(x) = 0$, and there are only finitely many distinct local Bernstein–Sato ideals, see [1,8].

1.3. It was clear from the beginning that $B_F$ contains some topological information about $F$, e.g. [18,19,22,23]. However, besides the case $r = 1$, it was not clear what precise topological information is provided by $B_F$. Later, a conjec-
ture based on computer experiments was formulated in [10] addressing this problem. In this article we prove this conjecture.

1.4.
Let us recall what happens in the case $r = 1$. If $f : X \to \mathbb{C}$ is a regular function on a smooth complex affine irreducible algebraic variety, or the germ at $x \in X$ of a holomorphic function on a complex manifold, the monic generator of the Bernstein–Sato ideal of $f$ in $\mathbb{C}[s]$ is called the Bernstein–Sato polynomial, or the $b$-function, of $f$ and it is denoted by $b_f(s)$. The non-triviality of $b_f(s)$ is a classical result of Bernstein [5] in the algebraic case, and Björk [6] in the analytic case. One has the following classical theorem, see [16, 17, 21]:

**Theorem 1.4.1** Let $f : X \to \mathbb{C}$ be a regular function on a smooth complex affine irreducible algebraic variety, or the germ at $x \in X$ of a holomorphic function on a complex manifold, such that $f$ is not invertible. Let $b_f(s) \in \mathbb{C}[s]$ be the Bernstein–Sato polynomial of $f$. Then:

(i) (Malgrange, Kashiwara) The set

$$\{\exp(2\pi i \alpha) \mid \alpha \text{ is a root of } b_f(s)\}$$

is the set of monodromy eigenvalues on the nearby cycles complex of $f$.

(ii) (Kashiwara) The roots of $b_f(s)$ are negative rational numbers.

(iii) (Monodromy Theorem) The monodromy eigenvalues on the nearby cycles complex of $f$ are roots of unity.

The definition of the nearby cycles complex is recalled in Sect. 2. In the algebraic case, $b_f(s)$ provides thus an algebraic computation of the monodromy eigenvalues.

1.5.
We complete in this article the extension of this theorem to a finite collection of functions as follows. Let

$$Z(B_F) \subset \mathbb{C}^r$$

be the zero locus of the Bernstein–Sato ideal of $F$. Let $\psi_F \mathbb{C}_X$ be the specialization complex\(^1\) defined by Sabbah [24]; the definition will be recalled in Sect. 2. This complex is a generalization of the nearby cycles complex to a finite collection of functions, the monodromy action being now given by $r$ simultaneous monodromy actions, one for each function $f_i$. Let

$$S(F) \subset (\mathbb{C}^*)^r$$

\(^1\) This is called “le complexe d’Alexander” in [24].
be the support of this monodromy action on $\psi_F \mathbb{C}_X$. In the case $r = 1$, this is the set of eigenvalues of the monodromy on the nearby cycles complex. The support $S(F)$ has a few other topological interpretations, one being in terms of cohomology support loci of rank one local systems, see Sect. 2. Let $\text{Exp} : \mathbb{C}^r \to (\mathbb{C}^*)^r$ be the map $\text{Exp}(\_ \_ \_) = \exp(2\pi i \_ \_)$.

**Theorem 1.5.1** Let $F = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$ be a morphism of smooth complex affine irreducible algebraic varieties, or the germ at $x \in X$ of a holomorphic map on a complex manifold, such that not all $f_i$ are invertible. Then:

(i) $\text{Exp}(Z(B_F)) = S(F)$.

(ii) Every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of type $a_1s_1 + \ldots + a_rs_r + b = 0$ with $a_i \in \mathbb{Q}_{\geq 0}$ and $b \in \mathbb{Q}_{> 0}$. Every irreducible component of $Z(B_F)$ of codimension $> 1$ can be translated by an element of $\mathbb{Z}^r$ inside a component of codimension 1.

(iii) $S(F)$ is a finite union of torsion-translated complex affine subtori of codimension 1 in $(\mathbb{C}^*)^r$.

Thus in the algebraic case, $B_F$ gives an algebraic computation of $S(F)$.

Part (i) was conjectured in [10], where one inclusion was also proved, namely that $\text{Exp}(Z(B_F))$ contains $S(F)$. See also [11, Conjecture 1.4, Remark 2.8].

Regarding part (iii), Sabbah [24] showed that $S(F)$ is included in a finite union of torsion-translated complex affine subtori of codimension 1. Here a complex affine subtorus of $(\mathbb{C}^*)^r$ means an algebraic subgroup $G \subset (\mathbb{C}^*)^r$ such that $G \cong (\mathbb{C}^*)^p$ as algebraic groups for some $0 \leq p \leq r$. In [12], it was proven that every irreducible component of $S(F)$ is a torsion-translated subtorus. Finally, part (iii) was proven as stated in [11].

The first assertion of part (ii), about the components of codimension one of $Z(B_F)$, is due to Sabbah [22,23] and Gyoja [14].

In light of the conjectured equality in part (i), it was therefore expected that part (ii) would hold for $\text{Exp}(Z(B_F))$. This is equivalent to the second assertion in part (ii), about the smaller-dimensional components of $Z(B_F)$, and it was confirmed unconditionally by Maisonobe [20, Résultat 3]. This result of Maisonobe will play a crucial role in this article.

In this article we complete the proof of Theorem 1.5.1 by proving the other inclusion from part (i):

**Theorem 1.5.2** Let $F$ be as in Theorem 1.5.1. Then $\text{Exp}(Z(B_F))$ is contained in $S(F)$. 
The proof uses Maisonobe’s results from [20] and uses an analog of the Cohen-Macaulay property for modules over the noncommutative ring \( \mathcal{D}_X[s_1, \ldots, s_r] \).

1.6.

Algorithms for computing Bernstein–Sato ideals are now implemented in many computer algebra systems. The availability of examples where the zero loci of Bernstein–Sato ideals contain irreducible components of codimension > 1 suggests that this is not a rare phenomenon, see [1]. The stronger conjecture that Bernstein–Sato ideals are generated by products of linear polynomials remains open, [10, Conjecture 1]. This would imply in particular that all irreducible components of \( Z(B_F) \) are linear.

1.7.

In Sect. 2, we recall the definition and some properties of the support of the specialization complex. In Sect. 3 we give the proof of Theorem 1.5.2. Section 4 is an appendix reviewing basic facts from homological algebra for modules over not-necessarily commutative rings.

2 The support of the specialization complex

2.1 Notation

Let \( F = (f_1, \ldots, f_r) : X \to \mathbb{C}^r \) be a holomorphic map on a complex manifold \( X \) of dimension \( n > 0 \). Let \( f = \prod_{i=1}^r f_i, D = f^{-1}(0), U = X \setminus D \).

Let \( i : D \to X \) be the closed embedding and \( j : U \to X \) the open embedding. We are assuming that not all \( f_i \) are invertible, which is equivalent to \( D \neq \emptyset \).

We use the notation \( s = (s_1, \ldots, s_r) \) and \( f^s = \prod_{i=1}^r f_i^{s_i} \), and in general tuples of numbers will be in bold, e.g. \( 1 = (1, \ldots, 1) \), \( \alpha = (\alpha_1, \ldots, \alpha_r) \), etc.

2.2 Specialization complex

Let \( D_b^c(A_D) \) be the derived category of bounded complexes of \( A_D \)-modules with constructible cohomology, where \( A \) is the affine coordinate ring of \( (\mathbb{C}^*)^r \) and \( A_D \) is the constant sheaf of rings on \( D \) with stalks \( A \). Sabbah [24] defined the specialization complex \( \psi_F \mathcal{C}_X \) in \( D_b^c(A_D) \) by

\[
\psi_F \mathcal{C}_X = i^{-1} R j_* R \pi!(j \circ \pi)^{-1} \mathcal{C}_X,
\]

where \( \pi : U \times_{(\mathbb{C}^*)^r} \mathbb{C}^r \to U \) is the first projection from the fibered product obtained from \( F|_U : U \to (\mathbb{C}^*)^r \) and the universal covering map \( \exp : \mathbb{C}^r \to (\mathbb{C}^*)^r \).

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The support of the specialization complex $S(F)$ is defined as the union over all $i \in \mathbb{Z}$ and $x \in D$ of the supports in $(\mathbb{C}^*)^r$ of the cohomology stalks $\mathcal{H}^i(\psi_F \mathbb{C}_X)_x$ viewed as finitely generated $A$-modules.

If $F$ is only given as the germ at a point $x \in X$ of a holomorphic map, by $\psi_F \mathbb{C}_X$ we mean the restriction of the specialization complex to a very small open neighborhood of $x \in X$.

When $r = 1$, that is, in the case of only one holomorphic function $f : X \to \mathbb{C}$, the specialization complex equals the shift by $[-1]$ of Deligne’s nearby cycles complex defined as

$$\psi_f \mathbb{C}_X = i^{-1} R(j \circ \pi)_*(j \circ \pi)^{-1} \mathbb{C}_X.$$

The complex numbers in the support $S(f) \subset \mathbb{C}^*$ are called the monodromy eigenvalues of the nearby cycles complex of $f$.

### 2.3 Cohomology support loci

It was proven in [10,11] that $S(F)$ admits an equivalent definition, without involving derived categories, as the union of cohomology support loci of rank one local systems on small ball complements along the divisor $D$. More precisely,

$$S(F) = \{ \lambda \in (\mathbb{C}^*)^r \mid H^i(U_x, L_\lambda) \neq 0 \text{ for some } x \in D \text{ and } i \in \mathbb{Z} \},$$

where $U_x$ is the intersection of $U$ with a very small open ball in $X$ centered at $x$, and $L_\lambda$ is the rank one $\mathbb{C}$-local system on $U$ obtained as the pullback via $F : U \to (\mathbb{C}^*)^r$ of the rank one local system on $(\mathbb{C}^*)^r$ with monodromy $\lambda_i$ around the $i$-th missing coordinate hyperplane.

If $F$ is only given as the germ at $(X, x)$ of a holomorphic map, $S(F)$ is defined as above by replacing $X$ with a very small open neighborhood of $x$.

For one holomorphic function $f : X \to \mathbb{C}$, the support $S(f)$ is the union of the sets of eigenvalues of the monodromy acting on cohomologies of the Milnor fibers of $f$ along points of the divisor $f = 0$, see [12, Proposition 1.3].

With this description of $S(F)$, the following involutivity property was proven:

**Lemma 2.3.1** ([12, Theorem 1.2]) Let $\lambda \in (\mathbb{C}^*)^r$. Then $\lambda \in S(F)$ if and only if $\lambda^{-1} \in S(F)$.

### 2.4 Non-simple extension loci

An equivalent definition of $S(F)$ was found by [11, §1.4] as a locus of rank one local systems on $U$ with non-simple higher direct image in the category
of perverse sheaves on $X$:

$$S(F) = \left\{ \lambda \in (\mathbb{C}^*)^r \mid \frac{Rj_* L_\lambda[n]}{j_{!*} L_\lambda[n]} \neq 0 \right\},$$

where $L_\lambda$ is the rank one local system on $U$ as in 2.3. This description is equivalent to

$$S(F) = \left\{ \lambda \in (\mathbb{C}^*)^r \mid j! L_\lambda[n] \rightarrow Rj_* L_\lambda[n] \text{ is not an isomorphism} \right\},$$

the map being the natural one.

### 2.5 $\mathcal{D}$-module theoretic interpretation

Recall that for $\alpha \in \mathbb{C}^r$,

$$\mathcal{D}_X[s]f^s$$

is the natural left $\mathcal{D}_X[s]$-submodule of the free rank one $\mathcal{O}_X[s, f^{-1}]$-module $\mathcal{O}_X[s, f^{-1}] \cdot f^s$ generated by the symbol $f^s$. For $r = 1$, see for example Walther [25].

We denote by $D^b_{rh}(\mathcal{D}_X)$ the derived category of bounded complexes of regular holonomic $\mathcal{D}_X$-modules. We denote by $\text{DR}_X : D^b_{rh}(\mathcal{D}_X) \rightarrow D^b_c(\mathbb{C}_X)$ the de Rham functor, an equivalence of categories. The following is a particular case of [27, Theorem 1.3 and Corollary 5.5], see also [4]:

**Theorem 2.5.1** Let $F = (f_1, \ldots, f_r) : X \rightarrow \mathbb{C}^r$ be a morphism from a smooth complex algebraic variety. Let $\alpha \in \mathbb{C}^r$ and $\lambda = \exp(-2\pi i \alpha)$. Let $L_\lambda$ be the rank one local system on $U$ defined as in 2.3, and let $M_\lambda = L_\lambda \otimes_{\mathbb{C}} \mathcal{O}_U$ the corresponding flat line bundle, so that

$$\text{DR}_U(M_\lambda) = L_\lambda[n]$$

as perverse sheaves on $U$. For every integer $k \gg \|\alpha\|$ and $k = (k, \ldots, k) \in \mathbb{Z}^r$, there are natural quasi-isomorphisms in $D^b_{rh}(\mathcal{D}_X)$

$$\mathcal{D}_X[s]f^{s+k} \otimes_{\mathbb{C}[s]} \mathbb{C}_\alpha = j_! M_\lambda,$$

$$\mathcal{D}_X[s]f^{s-k} \otimes_{\mathbb{C}[s]} \mathbb{C}_\alpha = j_* M_\lambda,$$

where $\mathbb{C}_\alpha$ is the residue field of $\alpha$ in $\mathbb{C}^r$. 

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Proposition 2.5.2 With $F$ as in Theorem 2.5.1,

$$S(F) = \text{Exp} \left\{ \alpha \in \mathbb{C}^r \mid \frac{\mathcal{D}_X[s]f^s - k}{\mathcal{D}_X[s]f^{s+k}} \otimes_{\mathbb{C}[s]} \mathbb{C} \neq 0 \text{ for all } k \gg \|\alpha\| \right\}. $$

Proof Applying $\text{DR}_X$ directly to Theorem 2.5.1, one obtains that

$$S(F) = \text{Exp} \left\{ -\alpha \in \mathbb{C}^r \mid \frac{\mathcal{D}_X[s]f^s - k}{\mathcal{D}_X[s]f^{s+k}} \otimes_{\mathbb{C}[s]} \mathbb{C} \neq 0 \text{ in } D^b_{\text{rh}}(\mathcal{D}_X) \text{ for all } k \gg \|\alpha\| \right\}$$

by the interpretation of $S(F)$ from 2.4. Since $j_!M_\lambda \to j_*M_\lambda$ is a morphism of holonomic $\mathcal{D}_X$-modules of same length, the kernel and cokernel must simultaneously vanish or not. Thus, we can replace the derived tensor product with the usual tensor product. We then can replace $-\alpha$ with $\alpha$ by Lemma 2.3.1. $\square$

For related work in a particular case, see [2].

Remark 2.5.3 Note that Theorem 2.5.1 is stated in the algebraic case only. However, the proof from [4,27] extends to the case when $X$ is a complex manifold by replacing $j_!M_\lambda, j_*M_\lambda$ with $\mathcal{M}(D), \mathcal{M}(\ast D)$, respectively, where $\mathcal{M}$ is the analytic $\mathcal{D}_X$-module $\mathcal{D}_X \cdot f^\alpha$ whose restriction to $U$ is $\mathcal{M}_\lambda$. Hence the last proposition also holds in the analytic case.

Since the tensor product is a right exact functor, as a consequence one has the following corollary which also follows from the proof of [10, Proposition 1.7]:

Proposition 2.5.4 If $\alpha$ is in $\mathbb{C}^r$ and

$$\frac{\mathcal{D}_X[s]f^s}{\mathcal{D}_X[s]f^{s+1}} \otimes_{\mathbb{C}[s]} \mathbb{C} \neq 0,$$

then $\text{Exp}(\alpha)$ is in $S(F)$.

This proposition can be interpreted as to say that the difficulty in proving Theorem 1.5.2 is the lack of a Nakayama Lemma for the non-finitely generated $\mathbb{C}[s]$-module $\mathcal{D}_X[s]f^s / \mathcal{D}_X[s]f^{s+1}$.

3 Relative holonomic modules

In this section we will provide necessary conditions for modules over $\mathcal{D}_X[s]$ to obey an analog of Nakayama Lemma, and we will see that $\mathcal{D}_X[s]f^s / \mathcal{D}_X[s]f^{s+1}$
satisfies these conditions at least generically. Using Maisonobe’s results [20], this will prove Theorem 1.5.2.

3.1.

For simplicity, we will assume from now that we are in the algebraic case, namely, $X$ is a smooth complex affine irreducible algebraic variety. We will treat the analytic case at the end.

We define an increasing filtration on the ring $\mathcal{D}_X$ by setting $F_i \mathcal{D}_X$ to consist of all operators of order at most $i$, that is, in local coordinates $(x_1, \ldots, x_n)$ on $X$, the order of $x_i$ is zero and the order of $\partial/\partial x_i$ is one.

We let $R$ be a regular commutative finitely generated $\mathbb{C}$-algebra integral domain. We write

$$\mathcal{A}_R = \mathcal{D}_X \otimes \mathbb{C} R,$$

and if $R = \mathbb{C}[s]$ we write

$$\mathcal{A} = \mathcal{A}_{\mathbb{C}[s]} = \mathcal{D}_X[s].$$

The order filtration on $\mathcal{D}_X$ induces the relative filtration on $\mathcal{A}_R$ by

$$F_i \mathcal{A}_R = F_i \mathcal{D}_X \otimes \mathbb{C} R.$$

The associated graded ring

$$\text{gr} \mathcal{A}_R = \text{gr} \mathcal{D}_X \otimes \mathbb{C} R$$

is a regular commutative finitely generated $\mathbb{C}$-algebra integral domain, and it corresponds to the structure sheaf of $T^* X \times \text{Spec} R$, where $T^* X$ is the cotangent bundle of $X$. Thus $\mathcal{A}_R$ is an Auslander regular ring by Theorem 4.3.2. Moreover, the homological dimension is equal to the Krull dimension of $\text{gr} \mathcal{A}_R$,

$$\text{gl.dim}(\mathcal{A}_R) = 2n + \text{dim}(R),$$

by Propositions 4.3.3, 4.4.2, and 4.5.1.

3.2.

Let $N$ be a left (or right) $\mathcal{A}_R$-module. A good filtration $F$ on $N$ over $R$ is an exhaustive filtration compatible with the relative filtration on $\mathcal{A}_R$ such that the associated graded module $\text{gr} N$ is finitely generated over $\text{gr} \mathcal{A}_R$, cf. 4.2. If $N$ is finitely generated over $\mathcal{A}_R$, then good filtrations over $R$ exist on $N$. We define
the relative characteristic variety of $N$ over $R$ to be the support of $\text{gr} \ N$ inside $T^*X \times \text{Spec} \ R$, denoted by

$$\text{Ch}^{\text{rel}}(N).$$

Equivalently, $\text{Ch}^{\text{rel}}(N)$ is defined by the radical of the annihilator ideal of $\text{gr} \ N$ in $\text{gr} \ A_R$. The relative characteristic variety $\text{Ch}^{\text{rel}}(N)$ and the multiplicities $m_p(N)$ of $\text{gr} \ N$ at generic points $p$ of the irreducible components of $\text{Ch}^{\text{rel}}(N)$ do not depend on the choice of a good filtration for $N$, by 4.2.1.

**Remark 3.2.1** The good filtration $F$ on $N$ localizes, that is, if $S$ is a multiplicatively closed subset of $R$, then

$$F_i(S^{-1}N) = S^{-1}F_i N$$

form a good filtration of $S^{-1}N$ over $S^{-1}R$, and hence

$$\text{gr} \ (S^{-1}N) \simeq S^{-1}\text{gr} \ N.$$

For a finitely generated $A_R$-module $N$, we will denote by $j_{A_R}(N)$, or simply $j(N)$, the grade number of $N$ defined as in 4.3.

**Lemma 3.2.2** Suppose that $N$ is a finitely generated $A_R$-module. Then:

1. $j(N) + \dim(\text{Ch}^{\text{rel}}(N)) = 2n + \dim(R)$;
2. if

$$0 \to N' \to N \to N'' \to 0$$

is a short exact sequence of finitely generated $A_R$-modules, then

$$\text{Ch}^{\text{rel}}(N) = \text{Ch}^{\text{rel}}(N') \cup \text{Ch}^{\text{rel}}(N'')$$

and if $p$ is the generic point of an irreducible component of $\text{Ch}^{\text{rel}}(N)$ then

$$m_p(N) = m_p(N') + m_p(N'').$$

**Proof** Propositions 4.4.2 and 4.5.1 give (1). Proposition 4.2.1 gives (2). $\square$

Note that the lemma does not require, nor does it imply, that $\text{Ch}^{\text{rel}}(N)$ is equidimensional.

**Definition 3.2.3** We say that a finitely generated $A_R$-module $N$ is relative holonomic over $R$ if its relative characteristic variety over $R$ is a finite union

$$\text{Ch}^{\text{rel}}(N) = \bigcup_w \Lambda_w \times S_w$$
where $\Lambda_w$ are irreducible conic Lagrangian subvarieties in $T^*X$ and $S_w$ are algebraic irreducible subvarieties of Spec $R$.

**Lemma 3.2.4** Suppose that $N$ is relative holonomic over $R$. Then:

1. every nonzero subquotient of $N$ is relative holonomic over $R$;
2. if $\text{Ext}_{\mathcal{A}_R}^j (N, \mathcal{A}_R) \neq 0$ for some integer $j$, then $\text{Ext}_{\mathcal{A}_R}^j (N, \mathcal{A}_R)$ is relative holonomic (as a right $\mathcal{A}_R$-module if $N$ is a left $\mathcal{A}_R$-module and vice versa), and

$$\text{Ch}^{rel}(\text{Ext}_{\mathcal{A}_R}^j (N, \mathcal{A}_R)) \subset \text{Ch}^{rel}(N).$$

**Proof** By Proposition 4.2.2, there exist good filtrations on $N$ and $\text{Ext}_{\mathcal{A}_R}^j (N, \mathcal{A}_R)$ such that $\text{gr} (\text{Ext}_{\mathcal{A}_R}^j (N, \mathcal{A}_R))$ is a subquotient of $\text{Ext}_{\mathcal{A}_R}^j (\text{gr} N, \text{gr} \mathcal{A}_R)$. It follows that

$$\text{Ch}^{rel}(\text{Ext}_{\mathcal{A}_R}^j (N, \mathcal{A}_R)) \subset \text{Ch}^{rel}(N).$$

Then part (2) follows from Proposition 3.2.5. Part (1) is proved similarly, using Lemma 3.2.2 (2).

The following is a straight-forward generalization of the algebraic case of [20, Proposition 8] where one replaces $\mathbb{C}[s]$ by $R$:

**Proposition 3.2.5** If $N$ is a finitely generated module over $\mathcal{A}_R$ such that $\text{Ch}^{rel}(N)$ is contained in $\Lambda \times \text{Spec } R$ for some conic Lagrangian, not necessarily irreducible, subvariety $\Lambda$ of $T^*X$, then $N$ is relative holonomic over $R$.

**Proof** The Poisson bracket on $\text{gr} \mathcal{A}_R$ is the $R$-linear extension of the Poisson bracket on $\mathbb{D}_X$. Let $J$ be the radical ideal of the annihilator in $\text{gr} \mathcal{A}_R$ of $\text{gr} N$. By Gabber’s Theorem [7, A.III 3.25], $J$ is involutive with respect to the Poisson bracket on $\text{gr} \mathcal{A}_R$, that is, $\{J, J\} \subset J$. Let $m$ be a maximal ideal in $R$ corresponding to a point $q$ in the image of $\text{Ch}^{rel}(N)$ under the second projection

$$p_2 : T^*X \times \text{Spec } R \to \text{Spec } R.$$

By $R$-linearity of the Poisson bracket, it follows that $J + m \cdot \mathcal{A}_R$ is involutive. Therefore the image $\tilde{J}$ of $J$ in the ring $\text{gr} \mathcal{A}_R \otimes_R R / m \simeq \text{gr} \mathbb{D}_X$ is involutive under the Poisson bracket on $\text{gr} \mathbb{D}_X$. If this ideal would be radical, we could conclude that all the irreducible components of the fiber $\text{Ch}^{rel}(N) \cap p_2^{-1}(q)$ have dimension at least $\dim X$. Note however that the same assertions on involutivity are true for the associated sheaves since the Poisson bracket on
a $\mathbb{C}$-algebra induces a canonical Poisson bracket on the localization of the algebra with respect to any multiplicatively closed subset, cf. [15, Lemma 1.3]. Thus, restricting to an open subset of $\text{Ch}^{\text{rel}}(N)$ where the second projection $p_2$ has smooth reduced fibers, and assuming $q = p_2(y)$ for a point $y$ in this open subset, the involutivity implies that $\dim_y(\text{Ch}^{\text{rel}}(N) \cap p_2^{-1}(q)) \geq \dim X$. By the upper-semicontinuity on $\text{Ch}^{\text{rel}}(N)$ of the function $y \mapsto \dim_y(\text{Ch}^{\text{rel}}(N) \cap p_2^{-1}(p_2(y)))$, every irreducible component of a non-empty fiber $\text{Ch}^{\text{rel}}(N) \cap p_2^{-1}(q)$ has dimension $\geq \dim X$. (So far, this is an elaborate adaptation of proof of the algebraic case of [20, Proposition 5] to the case when $\mathbb{C}[s]$ is replaced by $R$.)

Since $\Lambda$ is equidimensional with $\dim \Lambda = \dim X$, and $\Lambda$ contains every non-empty fiber $\text{Ch}^{\text{rel}}(N) \cap p_2^{-1}(q)$, it follows that $\text{Ch}^{\text{rel}}(N) \cap p_2^{-1}(q)$ is a finite union of some of the irreducible conic Lagrangian subvarieties $\Lambda_w$ of $T^*X$ which are irreducible components of $\Lambda$. Define $S_w$ to be the subset of closed points $q$ in $\text{Spec} R$ such that $\Lambda_w$ is an irreducible component of $\text{Ch}^{\text{rel}}(N) \cap p_2^{-1}(q)$. Then $\text{Ch}^{\text{rel}}(N) = \bigcup_w(\Lambda_w \times S_w)$. Moreover, setting $\lambda_w$ to be a general point of $\Lambda_w$,

$$\{\lambda_w\} \times S_w = \text{Ch}^{\text{rel}}(N) \cap p_2^{-1}(\lambda_w),$$

where $p_1 : T^*X \times \text{Spec} R \to T^*X$ is the first projection. Since the right-hand side is defined in $\text{Spec} R$ by finitely many algebraic regular functions, $S_w$ is Zariski closed in $\text{Spec} R$. It follows that $\text{Ch}^{\text{rel}}(N)$ is relative holonomic over $R$.

\[ \square \]

### 3.3.

Recall from 4.3 the definition of pure modules over $\mathcal{A}_R$. Examples of pure modules are given by the following.

**Definition 3.3.1** We say that a nonzero finitely generated $\mathcal{A}_R$-module $N$ is *Cohen-Macaulay*, or more precisely *$j$-Cohen-Macaulay*, if for some $j \geq 0$

$$\text{Ext}_{\mathcal{A}_R}^k(N, \mathcal{A}_R) = 0 \quad \text{if} \quad k \neq j.$$

**Remark 3.3.2** If $N$ is a Cohen-Macaulay $\mathcal{A}_R$-module, then:

1. $N$ is $j$-pure (see Definition 4.3.4), by Lemma 4.3.5 (2);
2. $\text{Ch}^{\text{rel}}(N)$ is equidimensional of codimension $j$, by Propositions 4.4.1, 4.4.2, and 4.5.1.

**Lemma 3.3.3** If $N$ is relative holonomic over $R$ and $j(N) = n + \dim(R)$, then it is $(n + \dim(R))$-Cohen-Macaulay.
Proof The condition on \( j(N) \) implies that \( N \neq 0 \) by Lemma 3.2.2 (1). If \( \text{Ext}^k_{\mathcal{A}_R}(N, \mathcal{A}_R) \neq 0 \) for some \( k > n + \dim(\text{Spec } R) \), then by Lemma 3.2.4 (2), \( \text{Ext}^k_{\mathcal{A}_R}(N, \mathcal{A}_R) \) is relative holonomic. Hence \( \dim(\text{Chrel}(\text{Ext}^k_{\mathcal{A}_R}(N, \mathcal{A}_R))) \geq n \).

Since \( \mathcal{A}_R \) is an Auslander regular ring, \( j(\text{Ext}^k_{\mathcal{A}_R}(N, \mathcal{A}_R)) \geq k \). This contradicts Lemma 3.2.2 (1). \( \square \)

3.4.

For a finitely generated \( \mathcal{A}_R \)-module \( N \), since \( N \) is also an \( R \)-module, we write

\[ B_N = \text{Ann}_R(N) \]

and denote by \( Z(B_N) \) the reduced subvariety in \( \text{Spec } R \) defined by the radical ideal of \( B_N \). Since in general \( N \) is not finitely generated over \( R \), it is a priori not clear that \( Z(B_N) \) is the \( R \)-module support of \( N \), \( \text{supp}_R(N) \), consisting of closed points with maximal ideal \( m \subset R \) such that the localization \( N_m \neq 0 \).

**Lemma 3.4.1** If \( N \) is relative holonomic over \( R \), then

\[ Z(B_N) = p_2(\text{Chrel}(N)), \]

where \( p_2 : T^*X \times \text{Spec } R \rightarrow \text{Spec } R \) the natural projection. In particular,

\[ Z(B_N) = \text{supp}_R(N). \]

**Proof** For \( R = \mathbb{C}[s] \) and in the analytic setting, this is [20, Proposition 9], whose proof can be easily adapted to our case. Since \( N \) is relative holonomic, \( p_2(\text{Chrel}(N)) \) is closed. Since the contraction of a radical ideal is a radical ideal, the ideal defining \( p_2(\text{Chrel}(N)) \) is \( R \cap \sqrt{\text{Ann}_{\mathcal{A}_R}(\text{gr } N)} \). Hence the first assertion is equivalent to

\[ R \cap \sqrt{\text{Ann}_{\mathcal{A}_R}(\text{gr } N)} = \sqrt{\text{Ann}_R(N)}, \]

where \( R \) is viewed as a \( \mathbb{C} \)-subalgebra of \( \text{gr } \mathcal{A}_R = \text{gr } \mathcal{O}_X \otimes_{\mathbb{C}} R \) via the map \( a \mapsto 1 \otimes a \) for \( a \) in \( R \). Let \( b \) be in \( R \). If \( b^k N = 0 \) for some \( k \geq 1 \), then \( b^k(\text{gr } N) = 0 \) as well. Conversely, if \( b^k(\text{gr } N) = 0 \) for some \( k \geq 1 \), then \( b^k(F_i N) \subset F_{i-1} N \) for all \( i \). Since \( \text{gr } N \) is finitely generated over \( \text{gr } \mathcal{A}_R \), the filtration \( F \) on \( N \) is bounded from below. Then by induction applied to the short exact sequence

\[ 0 \rightarrow F_{i-1} N \rightarrow F_i N \rightarrow \text{gr}_i^F N \rightarrow 0, \]

it follows that for each \( i \) there exist a multiple \( k_i \) of \( k \) such that \( b^{k_i}(F_i N) = 0 \), and \( k_i \) form an increasing sequence. Fix a finite set of generators of \( N \) over
Since $F$ is exhaustive, there exists an index $j$ such that all the generators are contained in $F_j$. Then $b^{kj}N = 0$.

We proved thus the first claim, or equivalently, that $Z(B_N) = \text{supp}_R(\text{gr} N)$.

Hence the second assertion follows from the equality

$$\text{supp}_R(\text{gr} N) = \text{supp}_R(N)$$

which is proved as follows. If $m$ is a maximal ideal in $R$ such that $(\text{gr}_i^F N)_m \neq 0$ for some $i$, then $(F_i N)_m \neq 0$ since localization is an exact functor. Then, again by exactness, $N_m \neq 0$ since $F_i N$ injects into $N$. Thus $\text{supp}_R(\text{gr} N)$ is a subset of $\text{supp}_R(N)$. Conversely, if $N_m \neq 0$, take $i$ to be the minimum integer with the property that $(F_i N)_m \neq 0$ but $(F_{i-1} N)_m = 0$. Then $(\text{gr}_i^F N)_m \neq 0$. □

**Lemma 3.4.2** Suppose that $N$ is relative holonomic over $R$ and $(n + l)$-pure for some $0 \leq l \leq \dim(R)$. If $b$ is an element of $R$ not contained in any minimal prime ideal containing $B_N$, then the morphisms given by multiplication by $b$

$$N \xrightarrow{b} N$$

and

$$\text{Ext}^{n+l}_R(N, \mathcal{A}_R) \xrightarrow{b} \text{Ext}^{n+l}_R(N, \mathcal{A}_R)$$

are injective. Furthermore, there exists a good filtration of $N$ over $R$ so that

$$\text{gr} N \xrightarrow{b} \text{gr} N$$

is also injective.

**Proof** We first prove that $N \xrightarrow{b} N$ is injective. If on the contrary its kernel $K \neq 0$, then by Lemma 3.2.2 (2)

$$\text{Ch}^{\text{rel}}(K) \subset \text{Ch}^{\text{rel}}(N).$$

By purity, we know that $j(K) = j(N) = n + l$. Thanks to Lemma 3.2.2 (1),

$$\dim(\text{Ch}^{\text{rel}}(K)) = \dim(\text{Ch}^{\text{rel}}(N)).$$

By Proposition 4.4.1, we can choose good filtrations on $K$ and $N$ so that both $\text{gr} K$ and $\text{gr} N$ are $(n + l)$-pure over $\mathcal{A}_R$. Hence $\text{Ch}^{\text{rel}}(K)$ and $\text{Ch}^{\text{rel}}(N)$ are equidimensional of dimension $n + \dim(R) - l$, by Propositions 4.4.2 and 4.5.1. In particular, $\text{Ch}^{\text{rel}}(K)$ is a union of some irreducible components of $\text{Ch}^{\text{rel}}(N)$.  

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By the relative holonomicity of $N$, the irreducible components of $\text{Ch}^{\text{rel}}(N)$ are $\Lambda_i \times Z_i$ with $i$ in some finite index set $I$, for some conic irreducible Lagrangian subvarieties $\Lambda_i \subset T^*X$ and some irreducible closed subsets $Z_i \subset \text{Spec } R$. The equidimensionality of $\text{Ch}^{\text{rel}}(N)$ implies that $\dim Z_i = \dim(R) - l$.

By Lemma 3.4.1, $Z(B_N) = \bigcup_{i \in I} Z_i$, and the assumption on $b$ is that $(b = 0)$ does not contain any irreducible component of $Z(B_N)$, where by $(b = 0)$ we mean the reduced closed subset of $\text{Spec } R$ defined by the radical ideal of $b$. We hence have

$$\text{Ch}^{\text{rel}}(K) \not\subset T^*X \times (b = 0).$$

However, since $b$ annihilates $K$, $\text{Ch}^{\text{rel}}(K) \subset T^*X \times (b = 0)$, which is a contradiction.

Similarly, since $\text{gr } N$ is $(n + l)$-pure over $\text{gr } \mathcal{A}_R$, we can run the above argument by replacing $\text{Ch}^{\text{rel}}(K)$ with the support of the kernel of the map

$$\text{gr } N \xrightarrow{b} \text{gr } N$$

to obtain the injectivity of the latter.

By Lemma 3.2.4 (2), $\text{Ext}^{n+l}_{\mathcal{A}_R}(N, \mathcal{A}_R)$ is relative holonomic and

$$\text{Ch}^{\text{rel}}(\text{Ext}^{n+l}_{\mathcal{A}_R}(N, \mathcal{A}_R)) \subset \text{Ch}^{\text{rel}}(N).$$

Since $\text{Ext}^{n+l}_{\mathcal{A}_R}(N, \mathcal{A}_R)$ is always $(n + l)$-pure, cf. Lemma 4.3.5 (1), by a similar argument we conclude that

$$\text{Ext}^{n+l}_{\mathcal{A}_R}(N, \mathcal{A}_R) \xrightarrow{b} \text{Ext}^{n+l}_{\mathcal{A}_R}(N, \mathcal{A}_R)$$

is also injective. \qed

The following is the key technical result of the article. For simplicity, we take $\text{Spec } R$ to be an open set of $\mathbb{C}^r$, the only case we need for the proof of the main result.

**Proposition 3.4.3** Let $\text{Spec } R$ be a nonempty open subset of $\mathbb{C}^r$. Let $N$ be an $\mathcal{A}_R$-module that is relative holonomic over $R$ and $(n + l)$-Cohen-Macaulay over $\mathcal{A}_R$ for some $0 \leq l \leq r$. Then

$$\alpha \in Z(B_N) \text{ if and only if } N \otimes_R \mathbb{C}_\alpha \neq 0,$$

where $\mathbb{C}_\alpha$ is the residue field of the closed point $\alpha \in \text{Spec } R$. 

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Proof We first assume $N \otimes_R C_\alpha \neq 0$. Then $N \otimes_R R_m \neq 0$, where $m \subset R$ is the maximal ideal of $\alpha$ and $R_m$ is the localization of $R$ at $m$. Then $\alpha$ belongs to $\text{supp}_R(N) = Z(B_N)$, by Lemma 3.4.1.

Conversely, we fix a point $\alpha$ in $Z(B_N)$. Since $N$ is $(n+l)$-Cohen-Macaulay, it is in particular $(n + l)$-pure as a module over $\mathcal{A}_R$. By Proposition 4.4.1, we then can choose a good filtration $F$ on $N$ so that $\text{gr} N$ is also pure over $\text{gr} \mathcal{A}_R$. Hence $\text{Ch}^{rel}(N)$ is purely of dimension $n + r - l$. By relative holonomicity and Lemma 3.4.1, $Z(B_N)$ is also purely of dimension $r - l$.

Let us consider the case when $l < r$. We then can choose a linear polynomial $b \in \mathbb{C}[s]$ so that $(b = 0)$ contains $\alpha$, but does not contain any of the irreducible components of $Z(B_N)$. By Lemma 3.4.2, the morphisms given by multiplication by $b$

$$N \xrightarrow{b} N \text{ and } \text{gr} N \xrightarrow{b} \text{gr} N$$

are both injective, the good filtration from Lemma 3.4.2 being constructed in the same way. Thus for every $i$ the vertical maps are injective in the diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & F_{i-1}N & \rightarrow & F_iN & \rightarrow & F_iN/F_{i-1}N & \rightarrow & 0 \\
& & \downarrow{b} & & \downarrow{b} & & \downarrow{b} & & \\
0 & \rightarrow & F_{i-1}N & \rightarrow & F_iN & \rightarrow & F_iN/F_{i-1}N & \rightarrow & 0
\end{array}
$$

and hence by the snake lemma we get an exact sequence

$$0 \rightarrow F_{i-1}N \otimes_R R/(b) \rightarrow F_iN \otimes_R R/(b) \rightarrow \text{gr}_i^F N \otimes_R R/(b) \rightarrow 0.$$  \hfill (3.1)

Note that $b$ is also injective on $N/F_iN$. Indeed, if not, then there exists some $v \in F_jN$ with $j > i$, $v \notin F_{j-1}N$, and $bv \in F_iN$. But then $b$ must annihilate the class of $v$ in $\text{gr}_j^F N$, which contradicts the injectivity of $b$ on $\text{gr} N$. Running a similar snake lemma as above after applying the multiplication by $b$ on the short exact sequence

$$0 \rightarrow F_iN \rightarrow N \rightarrow N/F_iN \rightarrow 0,$$

we obtain a short exact sequence

$$0 \rightarrow F_iN \otimes_R R/(b) \rightarrow N \otimes_R R/(b) \rightarrow (N/F_iN) \otimes_R R/(b) \rightarrow 0.$$  \hfill (3.2)
The injectivity from (3.1) and (3.2) implies that the induced filtration on \( N \otimes_R R/(b) \),
\[
F_i(N \otimes_R R/(b)) = \text{im}(F_iN \to N \otimes_R R/(b)) \simeq F_iN / (F_iN \cap bN),
\]
is the filtration by
\[
F_iN \otimes_R R/(b) \simeq F_iN / bF_iN,
\]
and the surjectivity from (3.1) then implies
\[
\text{gr} (N \otimes_R R/(b)) \simeq \text{gr} N \otimes_R R/(b). \tag{3.3}
\]

By Lemma 3.4.1, \( p_2^{-1}(\alpha) \) intersects non-trivially the support of \( \text{gr} N \), hence the same is true for \( p_2^{-1}(b = 0) \). By Nakayama’s Lemma for the finitely generated module \( \text{gr} N \) over \( \text{gr} \mathcal{A}_R \), we hence have
\[
0 \neq \frac{\text{gr} N}{b \cdot \text{gr} N} \simeq \text{gr} N \otimes_R R/(b).
\]
Together with the isomorphism (3.3), this implies that \( N \otimes_R R/(b) \neq 0 \). Since \( N \otimes_R R/(b) \) is also a finitely generated \( \mathcal{A}_{R/(b)} \)-module and \( \text{gr} (N \otimes_R R/(b)) \) is a finitely generated \( \text{gr} \mathcal{A}_{R/(b)} \)-module, we further conclude from (3.3) that the relative characteristic variety over \( R/(b) \)
\[
\text{Ch}_{\text{rel}}(N \otimes_R R/(b)) = (\text{id}_{T^*X} \times \Delta)^{-1}(\text{Ch}_{\text{rel}}(N)), \tag{3.4}
\]
where \( \Delta : \text{Spec } R/(b) \leftrightarrow \text{Spec } R \) is the closed embedding. Hence \( N \otimes_R R/(b) \) is relative holonomic over \( R/(b) \). By Lemma 3.4.1, we further have
\[
Z(B_{N \otimes_R R/(b)}) = \Delta^{-1}(Z(B_N)).
\]
In particular,
\[
\Delta^{-1}(\alpha) \in Z(B_{N \otimes_R R/(b)}).
\]
Since
\[
N \otimes_R \mathbb{C}_\alpha \simeq N \otimes_R R/(b) \otimes_{R/(b)} \mathbb{C}_{\Delta^{-1}(\alpha)},
\]
where \( \mathbb{C}_{\Delta^{-1}(\alpha)} \) is the residue field of \( \Delta^{-1}(\alpha) \in \text{Spec } R/(b) \), our strategy will be to prove
\[
N \otimes_R \mathbb{C}_\alpha \neq 0
\]
by repeatedly replacing $N$ by $N \otimes_R R/(b)$ and $R$ by $R/(b)$.

To make this work, we need first to prove that $N \otimes_R R/(b)$ remains Cohen-Macaulay over $A_R/(b)$. By taking a free resolution of $N$, one can see that

$$\text{RHom}_{A_R}(N, A_R) \otimes^L_{A_R} A_R/(b) \simeq \text{RHom}_{A_R/(b)}(N \otimes^L_R R/(b), A_R/(b))$$

(3.5)

in the derived category of right $A_R/(b)$-modules. Since the multiplication by $b$ is injective on $N$, we further have

$$\text{RHom}_{A_R/(b)}(N \otimes^L_R R/(b), A_R/(b)) \simeq \text{RHom}_{A_R/(b)}(N \otimes_R R/(b), A_R/(b)).$$

(3.6)

We will use the Grothendieck spectral sequence associated with the left-hand side of (3.5) to compute the Ext modules from the right-hand side of (3.6). Let us assume without harm that $N$ is a left $A_R$-module. Then viewing $\text{Hom}_{A_R}(_, A_R)$ as a covariant right-exact functor on the opposite category of the category of left $A_R$-modules, the composition of the two derived functors $\text{RHom}_{A_R}(_, A_R)$ and $(_, \otimes^L_{A_R})_{A_R/(b)}$ gives us a convergent first quadrant homology spectral sequence

$$E^2_{p,q} = \text{Tor}^p_{A_R}(\text{Ext}^q_{A_R}(N, A_R), A_R/(b)) \Rightarrow \text{Ext}^{-p+q}_{A_R/(b)}(N \otimes_R R/(b), A_R/(b)),$$

by [26, Corollary 5.8.4]. Note that the conditions from loc. cit. are satisfied in our case, since a projective object in the opposite category of the category of left $A_R$-modules is an injective left $A_R$-module $I$, and thus $\text{Hom}_{A_R}(I, A_R)$ is a projective right $A_R$-module, and so acyclic for the left exact functor $(_, \otimes^L_{A_R})_{A_R/(b)}$.

Since $N$ is $(n+l)$-Cohen-Macaulay over $A_R$,

$$\text{Ext}^q_{A_R}(N, A_R) = 0 \quad \text{for } q \neq n+l.$$

Then

$$\text{Tor}^p_{A_R}(\text{Ext}^{n+l}_{A_R}(N, A_R), A_R/(b)) = 0 \quad \text{for } p \neq 0$$

thanks to Lemma 3.4.2, since the complex $A_R \rightarrow A_R$ is a resolution of $A_R/b$. Therefore the above spectral sequence degenerates at $E^2$,

$$\text{Ext}^q_{A_R/(b)}(N \otimes_R R/(b), A_R/(b)) = 0 \quad \text{for } q \neq n+l,$$
and
\[
\text{Ext}^{n+l}_{\mathcal{A}_R/(b)}(N \otimes_R R/(b), \mathcal{A}_R/(b)) \cong \text{Ext}^{n+l}_{\mathcal{A}_R}(N, \mathcal{A}_R) \otimes_{\mathcal{A}_R} \mathcal{A}_R/(b)
\]
\[
\cong \text{Ext}^{n+l}_{\mathcal{A}_R}(N, \mathcal{A}_R) \otimes_R R/(b).
\]

As a consequence, \( N \otimes_R R/(b) \) is \((n + l)\)-Cohen-Macaulay over \( \mathcal{A}_R/(b) \).

Since \( b \) is linear, \( \mathbb{C}^{r-1} \cong \text{Spec } \mathbb{C}[s]/(b) \), and the latter contains \( \text{Spec } R/(b) \) an open subset. We then repeatedly replace \( R \) by \( R/(b) \), \( N \) by \( N \otimes_R R/(b) \), and \( \alpha \) by \( \Delta^{-1}(\alpha) \). Each time \( r \) drops by 1, \( l \) stays unchanged, and \( N \) remains nonzero, relative holonomic, and \((n + l)\)-Cohen-Macaulay. This reduces us to the case \( l = r \).

If \( 0 = l = r \), the claim is trivially true.

We now assume \( 0 < l = r \). Since \( N \) is now relative holonomic and \((n + r)\)-Cohen-Macaulay, hence \((n + r)\)-pure, we have
\[
\text{Ch}^\text{rel}(N) = \sum_w \Lambda_w \times \{p_w\},
\]
where \( p_w \) are points in \( \mathbb{C}^r \). Hence \( Z(B_N) \) is a finite union of points in \( \text{Spec } R \).

Counting multiplicities, by Lemma 3.2.2 (2) we see that \( N \) is of finite length.

We now fix a linear polynomial \( b \in \mathbb{C}[s] \) with \( b(\alpha) = 0 \) but not vanishing at the other points of \( Z(B_N) \). We then have an exact sequence
\[
0 \to K \to N \xrightarrow{b} N \to N \otimes_R R/(b) \to 0,
\]
where \( K \) is the kernel. We claim that \( K \neq 0 \). To see this, chose a polynomial \( c \in \mathbb{C}[s] \) not vanishing at \( \alpha \) but vanishing at all other points of \( Z(B_N) \). Then by Nullstellensatz, there is \( m > 0 \) the smallest power such that \((bc)^m \) is in \( B_N \). On the other hand, \( c^m \) is not in \( B_N \). Taking \( p \geq 1 \) to be the smallest with \( b^p c^m \in B_N \), we see that there exists \( v \) in \( N \), such that \( b^{p-1} c^m v \) is a nonzero element of \( K \).

Since \( K \neq 0 \) and since endomorphisms of modules of finite length are isomorphisms if and only if they are surjective, we have \( N \otimes_R R/(b) \neq 0 \). By Lemma 3.2.4 (1), \( N \otimes_R R/(b) \) is relative holonomic over \( R \), and by Lemma 3.2.2 (2), every irreducible component of its relative characteristic variety over \( R \) is one of the components \( \Lambda_w \times \{p_w\} \) of \( \text{Ch}^\text{rel}(N) \). Since \( b \) annihilates \( N \otimes_R R/(b) \), only the components with \( b(p_w) = 0 \), and hence with \( p_w = \alpha \), appear. We conclude that \( N \otimes_R R/(b) \) is also relative holonomic over \( R/(b) \). By Lemma 3.2.2 (1), we have \( j_{\mathcal{A}_R/(b)}(N \otimes_R R/(b)) = n + r - 1 \). Then by Lemma 3.3.3, \( N \otimes_R R/(b) \) is \((n + r - 1)\)-Cohen-Macaulay over \( \mathcal{A}_R/(b) \).

We therefore can replace \( N \) by \( N \otimes_R R/(b) \), \( R \) by \( R/(b) \), and assume that \( \text{Ch}^\text{rel}(N) = \bigcup_w \Lambda_w \times \{\alpha\} \) for some irreducible conic Lagrangian subvarieties.
\( \Lambda_w \) of \( T^*X \). Repeating this process, each time \( r \) drops by 1, \( N \) remains nonzero, relative holonomic, and \( (n + r) \)-Cohen-Macaulay. The process finishes at the case \( r = 0 \), in which case there is nothing to prove anymore. \( \square \)

**Remark 3.4.4** A result similar to Proposition 3.4.3 is proved by a different method in [3, Appendix B] for \( \mathcal{D}_X[s]/\mathcal{D}_X[f^{s+1}] \) when \( f \) is a reduced free hyperplane arrangement.

### 3.5.

We consider now the left \( \mathscr{A} \)-module

\[
M = \mathcal{D}_X[s]/\mathcal{D}_X[f^{s+1}].
\]

In this case, the annihilator \( B_M \) is the Bernstein–Sato ideal \( B_F \), since \( M \) is a cyclic \( \mathscr{A} \)-module generated by the class of \( f^s \) in \( M \).

It is well-known that the zero locus \( Z(B_F) \) in \( \mathbb{C}^r \) has dimension \( r - 1 \). Indeed, since \( B_F \) is the intersection of the local Bernstein–Sato ideals, by restricting attention to the neighborhood of a smooth point of the zero locus of \( \prod_{i=1}^r f_i \), one reduces the assertion to the case when \( f_i = x_1^{a_i} \) for some \( a_i \in \mathbb{N} \) for all \( i = 1, \ldots r \) with \( a = (a_1, \ldots, a_r) \neq (0, \ldots, 0) \). In this case, the Bernstein–Sato ideal is principal, generated by \( \prod_{j=1}^{a} (a \cdot s + j) \) with \( |a| = a_1 + \ldots + a_r \).

In addition, it is known that every top-dimensional irreducible component of \( Z(B_F) \) is a hyperplane in \( \mathbb{C}^r \) defined over \( \mathbb{Q} \) by \( [22,23] \).

We will use the following result of Maisonobe, which also holds in the local analytic case, cf. 3.6:

**Theorem 3.5.1** (Maisonobe) The \( \mathscr{A} \)-module \( M \) is relative holonomic over \( \mathbb{C}[s] \), has grade number \( j(M) = n + 1 \) over \( \mathscr{A} \), and \( \dim \operatorname{Ch rel}(M) = n + r - 1 \). Every irreducible component of \( Z(B_F) \) of codimension > 1 can be translated by an element of \( \mathbb{Z}^r \) into a component of codimension one.

**Proof** In [20, Résultat 3] it is shown that \( \operatorname{Ch rel}(M) = \bigcup_{i \in I} \Lambda_i \times Z_i \) for some finite set \( I \) with \( \Lambda_i \subset T^*X \) conic Lagrangian, \( Z_i \subset \mathbb{C}^r \) algebraic closed subset of dimension \( \leq r - 1 \). Thus \( M \) is relative holonomic over \( \mathbb{C}[s] \). Lemma 3.4.1 shows that \( Z(B_F) = \bigcup_{i \in I} Z_i \), cf. also the remark after [20, Résultat 2]. Since \( \dim Z(B_F) = r - 1 \), it follows that \( \dim \operatorname{Ch rel}(M) = n + r - 1 \), and hence \( j(M) = n + 1 \) by Lemma 3.2.2 (1). The last claim is contained in the statement of [20, Résultat 3]. \( \square \)

We next observe that over an open subset of \( \mathbb{C}^r \), \( M \) behaves particularly nice:

**Lemma 3.5.2** There exists an open affine subset \( V = \operatorname{Spec} R \subset \mathbb{C}^r \) such that the intersection of \( V \) with each irreducible component of codimension one of
Zero loci of Bernstein–Sato ideals

\( Z(B_F) \) is not empty, and the module \( M \otimes_{\mathbb{C}[s]} R \) is relative holonomic over \( R \) and \((n + 1)\)-Cohen-Macaulay over \( \mathcal{A}_R \).

**Proof** Since \( M \) is relative holonomic over \( \mathbb{C}[s] \), and since good filtrations localize by Remark 3.2.1, it follows that \( M \otimes_{\mathbb{C}[s]} R \) is relative holonomic over \( R \), if \( \text{Spec} \, R \) is a non-empty open subset of \( \mathbb{C}^r \).

Since \( j(M) = n + 1 \),

\[
\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) = 0 \quad \text{for } k < n + 1.
\]

By Auslander regularity of \( \mathcal{A} \), if \( \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) \neq 0 \) for \( k \geq n + 1 \), then

\[
j(\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A})) \geq k.
\]

Note that since \( \text{gl.dim}(\mathcal{A}) \) is finite, there are only finitely many \( k \) with \( \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) \neq 0 \). By Lemma 3.2.4 (2), if \( \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) \neq 0 \), then \( \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) \) is relative holonomic and

\[
\text{Ch}^{\text{rel}}(\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}))) \subset \text{Ch}^{\text{rel}}(M).
\]

By Lemma 3.2.2 (1), when \( k > n + 1 \),

\[
dim(\text{Ch}^{\text{rel}}(\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A})))) < n + r - 1. \tag{3.7}
\]

By relative holonomicity, the irreducible components of \( \text{Ch}^{\text{rel}}(M) \) are \( \Lambda_i \times Z_i \) with \( i \) in some finite index set \( I \), \( \Lambda_i \subset T^*X \) irreducible conic Lagrangian, and \( Z_i \) irreducible closed in \( \mathbb{C}^r \). Then the irreducible components of \( \text{Ch}^{\text{rel}}(\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}))) \) are \( \Lambda_i \times Z'_i \) with \( i \) in some subset \( J \subset I \), and \( Z'_i \) irreducible closed in \( Z_i \). By Lemma 3.4.1 applied to \( M \) and \( \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) \), respectively, we have that \( Z(B_F) = \bigcup_{i \in I} Z_i \), and the support in \( \mathbb{C}^r \) of \( \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) \) is \( \bigcup_{i \in J} Z'_i \). Then \( \dim Z(B_F) = r - 1 \), and \( \dim Z'_i < r - 1 \) for each \( k > n + 1 \) by (3.7). Therefore the support in \( \mathbb{C}^r \) of \( \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) \) is a proper algebraic subset of \( Z(B_F) \) not containing any top-dimensional component of \( Z(B_F) \) if \( k > n + 1 \). Choose \( V = \text{Spec} \, R \) to be an open affine subset of \( \mathbb{C}^r \) away from these proper subsets of \( Z(B_F) \) for all \( k > n + 1 \). Then for any good filtration we have

\[
(\text{gr} \text{Ext}^k_{\mathcal{A}}(M, \mathcal{A})) \otimes_{\mathbb{C}[s]} R = 0
\]

for all \( k > n + 1 \). Since \( R \) is the localization of \( \mathbb{C}[s] \) with respect to some multiplicatively closed subset \( S \), and since good filtrations localize, cf. Remark 3.2.1, we have

\[
\text{gr} \, (S^{-1}\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A})) = 0.
\]
and so
\[ S^{-1}\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) = 0. \]
Since \( S \) is also a multiplicatively closed subset of \( \mathcal{A} \), in the center of \( \mathcal{A} \), and \( M \) is finitely generated over the noetherian ring \( \mathcal{A} \), the Ext module localizes
\[ 0 = S^{-1}\text{Ext}^k_{\mathcal{A}}(M, \mathcal{A}) = \text{Ext}^k_{S^{-1}\mathcal{A}}(S^{-1}M, S^{-1}\mathcal{A}), \]
cf. [26, Lemma 3.3.8] and the proof of [26, Proposition 3.3.10], where one identifies the localization functor \( S^{-1}(\_\_) \) on \( \mathcal{A} \)-modules with the flat extension \( (\_\_) \otimes_{\mathcal{A}} R = (\_\_) \otimes_{\mathbb{C}[s]} R \). Thus \( S^{-1}M = M \otimes_{\mathbb{C}[s]} R \) is \((n + 1)\)-Cohen-Macaulay over \( S^{-1}\mathcal{A} = \mathcal{A}/R \). □

Now Lemma 3.5.2 and Proposition 3.4.3 immediately imply:

**Theorem 3.5.3** For every irreducible component \( H \) of codimension one of \( Z(BF) \) and for every general point \( \alpha \) on \( H \),
\[ M \otimes_{\mathbb{C}[s]} C_{\alpha} \neq 0. \]

### 3.6 Analytic case

Theorem 3.5.3 holds also in the local analytic case. We indicate now the necessary changes in the arguments. The smooth affine algebraic variety \( X \) is replaced by the germ \((X, x)\) of a complex manifold of dimension \( n \). The rings \( R \) stay as before and we let \( Y \) denote the complex manifold underlying the smooth affine complex algebraic variety \( \text{Spec}(R) \). The rings and modules from the algebraic case \( \mathcal{D}_X, \mathcal{A}_R = \mathcal{D}_X \otimes_{\mathbb{C}} R, N, \) etc., have natural analytic versions as sheaves on the complex manifold \( X \), but their role from the previous arguments will be played by the stalks of these sheaves, \( \mathcal{D}_{X,x}, \mathcal{A}_{R,x} = \mathcal{D}_{X,x} \otimes_{\mathbb{C}} R, N_x, \) etc. The role of \( \text{Ch}_{\text{rel}}(N) \) from the algebraic case will be played by \( \text{Ch}_{\text{rel}}(N) \cap \pi^{-1}(\Omega \times Y) \), for a very small open ball \( \Omega \) in \( X \) centered at \( x \). Recall that for a coherent sheaf of \( \mathcal{A}_R \)-modules \( N \) on the complex manifold \( X \), the relative characteristic variety \( \text{Ch}_{\text{rel}}(N) \) is the analytic subspace of \( T^*X \times Y \) defined as the zero locus of the radical of the annihilator of \( N \) in \( \mathcal{A}_R \). With these changes, all the statements in this section hold in the local analytic case as well.

There are however a few special issues arising in this case, since (partial) analytifications of \( \mathcal{A}_R \) and \( N \) are needed in order for the module theory as in the Appendix to capture the analytic structure of \( \text{Ch}_{\text{rel}}(N) \). For a sheaf of \( \mathcal{O}_X \otimes_{\mathbb{C}} R \)-modules \( L \) on the complex manifold \( X \), one defines the (partial) analytification
\[ \tilde{L} = \mathcal{O}_{X \times Y} \otimes_{p^{-1}(\mathcal{O}_X \otimes_{\mathbb{C}} R)} p^{-1}(L), \]
a sheaf of $\mathcal{O}_{X \times Y}$-modules, where $p : X \times Y \to X$ is the first projection. Thus $\widetilde{\mathcal{A}}_R$ is the sheaf of relative differential operators $\mathcal{O}_{X \times Y}/Y$, locally isomorphic to $\mathcal{O}_{X \times Y}[\partial_1, \ldots, \partial_n]$. The analytification of the filtration on $\mathcal{A}_R$ is the natural filtration on $\mathcal{A}_R$, and $\text{gr} \mathcal{A}_R$ is locally isomorphic to $\mathcal{O}_{X \times Y}[\xi_1, \ldots, \xi_n]$, a sheaf of subrings of $\mathcal{O}_{T^*X \times Y}$, where $\xi_i$ are coordinates of the fibers of the natural projection $\pi : T^*X \times Y \to X \times Y$. If $N$ is a coherent sheaf of $\mathcal{A}_R$-modules, then $\widetilde{N}$ is a coherent sheaf of $\widetilde{\mathcal{A}}_R$-modules. Since $(\_)$ is an exact functor, it is is compatible with good filtrations, $\text{gr} \widetilde{N} = \text{gr} \widetilde{N}$, the annihilator in $\text{gr} \mathcal{A}_R$ of $\text{gr} \widetilde{N}$ is the analytification of the annihilator of $\text{gr} N$ in $\mathcal{A}_R$, and the radical $J(\widetilde{N})$ of the former is the analytification $\widetilde{J(N)}$ of the radical of the latter. Then $\text{Ch}^{\text{rel}}(N)$ is the analytic subspace of $T^*X \times Y$ defined by the ideal generated by $J(\widetilde{N})$ in $\mathcal{O}_{T^*X \times Y}$, the full analytification, cf. [7, I.6.21].

Note that there is a natural isomorphism of $\mathbb{C}$-algebras

$$\text{gr} \mathcal{A}_{R,x} \simeq \mathbb{C}[x_1, \ldots, x_n][\xi_1, \ldots, \xi_n] \otimes_{\mathbb{C}} R$$

after choosing local coordinates $x_1, \ldots, x_n$ on $X$ at $x$. This ring is a regular commutative integral domain of dimension $2n + \text{dim}(R)$. Thus all the results in the Appendix apply to this ring, except Proposition 4.5.1 (ii). Indeed, $\text{gr} \mathcal{A}_{R,x}$ has maximal ideals of height less than $\text{dim}(\text{gr} \mathcal{A}_{R,x})$. (For example, the ideal $(1 - x\xi)$ of $\mathbb{C}[x][\xi]$ is maximal of height 1.) On the other hand, our modules are special: $\text{gr} N_x$ is a graded module if $\text{gr} \mathcal{A}_{R,x}$ is given the natural grading in the coordinates $\xi_1, \ldots, \xi_n$. The exact functor $(\_)$ is also faithful on the category of coherent graded $\text{gr} \mathcal{A}_R$-modules:

**Proposition 3.6.1** (Maisonobe [20, Lemme 1]) If $M$ is a coherent $\text{gr} \mathcal{A}_R$-module and $x \in X$, then $M_x = 0$ if and only if there exists an open neighborhood $\Omega$ of $x$ in $X$ such that $\widetilde{M}|_{\Omega \times Y} = 0$.

Thus one obtains, cf. [20, Proposition 2]: for a small enough $\Omega$,

$$j_{\text{gr} \mathcal{A}_{R,x}}(\text{gr} N_x) = \inf_{(x', y) \in \Omega \times Y} j_{(\text{gr} \mathcal{A}_R)(x', y)}((\text{gr} \widetilde{N})(x', y)).$$

The stalks $(\text{gr} \widetilde{N})(x', y)$ determine the local analytic structure at $(x', 0, y)$ of the conical set $\text{Ch}^{\text{rel}}(N)$, since the extension functor from the category of graded coherent sheaves over $\text{gr} \mathcal{A}_R$ into the category of coherent sheaves over $\mathcal{O}_{T^*X \times Y}$ is also faithful besides being exact, by the Nullstellensatz for conical analytic sets, cf. [7, Remark I.1.6.8]. In particular, there is a 1-1 correspondence between conical analytic sets in $T^*X \times Y$ and radical graded coherent ideals in $\text{gr} \mathcal{A}_R$. Therefore the ring $(\text{gr} \mathcal{A}_R)(x', y)$ and the module $(\text{gr} \widetilde{N})(x', y)$ can be replaced by their localization at the unique graded maximal ideal (cf. [9, 1.5]) and in this context Proposition 4.5.1 (ii) does apply. A consequence is that
Lemma 3.2.2 (1) holds indeed with the changes we have mentioned: for a small neighborhood $\Omega$ of $x$,

$$j_{s_R} (N_x) + \dim(\Chrel(N) \cap \pi^{-1}(\Omega \times Y)) = 2n + \dim(R).$$

This is [20, Proposition 2, Théorème 1], where $R = \mathbb{C}[s]$ but the proof applies in general, and we used semicontinuity of the dimension function [13, p.94] to rephrase the statement slightly.

Next, in keeping up with the changes indicated, the condition “regular holonomic” will be replaced by the condition that a coherent module $N$ over $\mathcal{A}_R$ is regular holonomic at $x$, that is, there exists a neighborhood $\Omega$ of $x$ such that $\Chrel(N) \cap \pi^{-1}(\Omega \times Y)$ is as in Definition 3.2.3.

The condition “$j$-Cohen-Macaulay” will be replaced by the condition that $N$ is $j$-Cohen-Macaulay at $x$, that is, $N_x$ is $j$-Cohen-Macaulay. This is equivalent to $N$ being $j$-Cohen-Macaulay on some neighborhood $\Omega$ of $x$, that is, $j$-Cohen-Macaulay at all points $x'$ in $\Omega \cap \text{supp}(N)$. Note that the support of $N$ is an analytic subset of $X$ by Proposition 3.6.1, since the support of $\tilde{N}$ is an analytic subset of $X \times Y$ by the conical property of $\Chrel(N)$. Moreover, $N$ is $j$-Cohen-Macaulay on $\Omega$ if and only if one of the following two equivalent conditions hold for $k \neq j$: $\mathfrak{E}xt^k_{\mathcal{A}_R}(N, \mathcal{A}_R)|_\Omega = 0$; $\mathfrak{E}xt^k_{\mathcal{A}_R}(N, \mathcal{A}_R)|_{x'} = 0$ for all $x' \in \Omega$. Also, $N$ is $j$-Cohen-Macaulay at $x$ if and only if $\tilde{N}$ is $j$-Cohen-Macaulay on $\Omega \times Y$ for some $\Omega \ni x$, by Proposition 3.6.1. This implies, by applying Proposition 4.5.1 in the context mentioned above, that Remark 3.3.2 holds in the local analytic case; in particular, if $N$ is $j$-Cohen-Macaulay at $x$, then $\Chrel(N) \cap \pi^{-1}(\Omega \times Y)$ is equidimensional of codimension $j$.

With the changes we have indicated, the rest of the arguments remain as before, and all statements in this section are true in this case.

### 3.7 Proof of Theorem 1.5.2.

By Theorem 3.5.3 and Proposition 2.5.4, the image under $\text{Exp}$ of a non-empty open subset of each irreducible component of codimension one of $Z(B_F)$ lies in $S(F)$. By the description of $Z(B_F)$ from Theorem 3.5.1 and the paragraphs preceding it, it follows that $\text{Exp}(Z(B_F))$ is included in $S(F)$.

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4 Appendix

We recall some facts for not-necessarily commutative rings from [7, A.III and A.IV] that we use in the proof of the main theorem.

4.1.

Let $A$ be a ring, by which we mean an associative ring with a unit element. Let $\text{Mod}_f(A)$ be the abelian category of finitely generated left $A$-modules.

We say that $A$ is a positively filtered ring if $A$ is endowed with a $\mathbb{Z}$-indexed increasing exhaustive filtration $\{F_i A\}_{i \in \mathbb{Z}}$ of additive subgroups such that $F_i A \cdot F_j A \subset F_{i+j} A$ for all $i, j$ in $\mathbb{Z}$, and $F_{-1} A = 0$. The associated graded object $\text{gr}^F A = \bigoplus_i (F_i A / F_{i-1} A)$ has a natural ring structure. When we do not need to specify the filtration, we write $\text{gr} A$ for $\text{gr}^F A$.

If $A$ is a positively filtered ring such that $\text{gr} A$ is noetherian, then $A$ is noetherian, [7, A.III 1.27]. Here, noetherian means both left and right noetherian.

4.2.

Let $A$ be a noetherian ring, positively filtered. A good filtration on $M \in \text{Mod}_f(A)$ is an increasing exhaustive filtration $F_i M$ of additive subgroups such that $F_i A \cdot F_j M \subset F_{i+j} M$ for all $i, j$ in $\mathbb{Z}$, and such that its associated graded object $\text{gr} M$ is a finitely generated graded module over $\text{gr} A$, cf. [7, A.III 1.29].

Proposition 4.2.1 ([7, A.III 3.20–3.23]) Let $A$ be a noetherian ring, positively filtered.

1. Let $M$ be in $\text{Mod}_f(A)$ with a good filtration. Then the radical of the annihilator ideal in $\text{gr} A$

   \[ J(M) := \sqrt{\text{Ann}_{\text{gr} A}(\text{gr} M)} \]

   and the multiplicities $m_p(M)$ of $\text{gr} M$ at minimal primes $p$ of $J(M)$ do not depend on the choice of a good filtration.

2. If

   \[ 0 \to M' \to M \to M'' \to 0 \]

   is an exact sequence in $\text{Mod}_f(A)$ then

   \[ J(M) = J(M') \cap J(M'') \]

   and if $p$ is a minimal prime of $J(M)$ then

   \[ m_p(M) = m_p(M') + m_p(M''). \]
Note that the last assertion is equivalent to the existence of a $\mathbb{Z}$-valued additive map $m_p$ on the Grothendieck group generated by the finitely generated modules $N$ over $\text{gr} A$ with $J(M) \subset \sqrt{\text{Ann}_{\text{gr} A} N}$, as it is phrased in loc. cit.

**Proposition 4.2.2** ([7, A.IV 4.5]) Let $A$ be a noetherian ring, positively filtered. Let $M$ be in $\text{Mod}_f(A)$ with a good filtration. For every $k \geq 0$, there exists a good filtration on the right $A$-module $\text{Ext}_A^k(M, A)$ such that $\text{gr} (\text{Ext}_A^k(M, A))$ is a subquotient of $\text{Ext}_{\text{gr} A}^k(\text{gr} M, \text{gr} A)$.

### 4.3.

Let $A$ be a noetherian ring. The smallest $k \geq 0$ for which every $M$ in $\text{Mod}_f(A)$ has a projective resolution of length $\leq k$ is called the **homological dimension** of $A$ and it is denoted by $\text{gl.dim}(A)$.

**Definition 4.3.1** For a nonzero $M$ in $\text{Mod}_f(A)$, the smallest integer $k \geq 0$ such that $\text{Ext}_A^k(M, A) \neq 0$ is denoted

$$j_A(M)$$

and it is called the **grade number** of $M$. If $M = 0$ the grade number is taken to be $\infty$.

The ring $A$ is **Auslander regular** if it has finite homological dimension and, for every $M$ in $\text{Mod}_f(A)$, every $k \geq 0$, and every nonzero right submodule $N$ of $\text{Ext}_A^k(M, A)$, one has $j_A(N) \geq k$. This implies the similar condition phrased for right $A$-modules $M$, see [7, A.IV 1.10] and the comment thereafter.

**Theorem 4.3.2** ([7, A.IV 5.1]) If $A$ is a positively filtered ring such that $\text{gr} A$ is a regular commutative ring, then $A$ is an Auslander regular ring.

**Proposition 4.3.3** ([7, A.IV 1.11]) Let $A$ be an Auslander regular ring. Then

$$\text{gl.dim}(A) = \sup\{j_A(M) \mid 0 \neq M \in \text{Mod}_f(A)\}.$$ 

**Definition 4.3.4** A nonzero module $M$ in $\text{Mod}_f(A)$ is **$j$-pure** (or simply, **pure**)$^1$ if $j_A(N) = j_A(M) = j$ for every nonzero submodule $N$.

**Lemma 4.3.5** ([7, A.IV 2.6]) Let $A$ be an Auslander regular ring, $M$ nonzero in $\text{Mod}_f(A)$, and $j = j_A(M)$. Then:

1. $\text{Ext}_A^j(M, A)$ is a $j$-pure right $A$-module;
2. $M$ is pure if and only if $\text{Ext}_A^k(\text{Ext}_A^k(M, A), A) = 0$ for every $k \neq j$. 

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4.4.

We assume now that $A$ is a positively filtered ring such that $\text{gr} \, A$ is a regular commutative ring. Then $A$ is also Auslander regular by Theorem 4.3.2. Moreover, with these assumptions one has the following two results.

**Proposition 4.4.1** ([7, A.IV 4.10 and 4.11]) If $M$ in $\text{Mod}_f(A)$ is $j$-pure, there exists a good filtration on $M$ such that $\text{gr} \, M$ is a $j$-pure $\text{gr} \, A$-module.

**Proposition 4.4.2** ([7, A.IV 4.15]) For any $M$ in $\text{Mod}_f(A)$ and any good filtration on $M$,

$$j_A(M) = j_{\text{gr} \, A}(\text{gr} \, M).$$

4.5.

Lastly, we consider a regular commutative ring $A$. Then $\text{gl} \, \text{dim}(A) = \sup \{ \text{gl} \, \text{dim}(A_m) \mid m \subset A$ maximal ideal $\}$, cf. [6, Ch. 2, 5.20]. We let $\text{dim}(A)$ denote the Krull dimension. For a module $M \in \text{Mod}_f(A)$, $\text{dim}_A(M)$ denotes $\text{dim}(A/\text{Ann}_A(M))$. If $A$ is a regular local commutative ring, then $\text{dim}(A) = \text{gl} \, \text{dim}(A)$, cf. [7, A.IV 3.5].

**Proposition 4.5.1** Let $A$ be a regular commutative ring and $M$ nonzero in $\text{Mod}_f(A)$. Then:

(i) ([7, A.IV 3.4]) $A$ is Auslander regular;

(ii) ([6, Ch. 2, Thm. 7.1]) if $\text{dim}(A_m) = m$ for every maximal ideal $m$ of $A$,

$$j_A(M) + \text{dim}_A(M) = m;$$

(iii) ([7, A.IV 3.7 and 3.8]) $M$ is a pure $A$-module if and only if every associated prime of $M$ is a minimal prime of $M$ and $j_A(M) = \text{dim}(A_p)$ for every minimal prime $p$ of $M$.

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