Cyclic generalizations of two hyperbolic icosahedral manifolds

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Abstract

We discuss two families of closed orientable three-dimensional manifolds which arise as cyclic generalizations of two hyperbolic icosahedral manifolds listed by Everitt. Everitt’s manifolds are cyclic coverings of the lens space $L_{3,1}$ branched over some 2-component links. We present results on covering properties, fundamental groups, and hyperbolic volumes of the manifolds belonging to these families.

keywords: 3-manifold, cyclic branched covering, lens space, links in manifolds

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Introduction

Various examples of three-dimensional spherical, Euclidean, or hyperbolic manifolds arise from pairwise isometrical identifications of faces of convex regular polyhedra in corresponding 3-spaces: $S^3$, $E^3$, or $H^3$. The most famous examples are the spherical and hyperbolic dodecahedral manifolds constructed by Weber and Seifert in 1933 [20]. The whole set of such examples for every spherical, Euclidean, or hyperbolic convex regular polyhedron was listed by Everitt [6]. The list contains eight manifolds $M_{15}, \ldots, M_{22}$, arising from a regular hyperbolic dodecahedron with dihedral angles $2\pi/5$, and six manifolds $M_{23}, \ldots, M_{28}$, arising from a regular hyperbolic icosahedron with dihedral angles $2\pi/3$. It can be checked directly from the gluing schemata that $M_{15}$ is the Weber – Seifert manifold from [20], and $M_{23}$ is the Fibonacci manifold from [8], uniformized by the Fibonacci group $F(2, 10)$. Both manifolds have cyclic symmetries induced by symmetries of the polyhedra, such that $M_{15}$ is an 5-fold cyclic
covering of the 3-sphere $S^3$, branched over the Whitehead link, and $M_{23}$ is the 5-fold cyclic covering of $S^3$, branched over the figure-eight knot. Cyclic generalizations of these manifolds were constructed in [9] and [8]: the $n$-fold strongly-cyclic coverings of $S^3$, branched over the Whitehead link, and the $n$-fold cyclic coverings of $S^3$, branched over the figure-eight knot, respectively. Explicit formulae for hyperbolic volumes of manifolds of these two classes are given in [13] and [12].

It was observed by Cavicchioli, Spaggiari and Telloni [3] that the manifolds $M_{24}$ and $M_{25}$, arising from the $2\pi/3$-icosahedron, are 3-fold cyclic branched coverings of the lens space $L_{3,1}$ branched over some 2-component links. In the present paper we will consider two families of 3-manifolds which are cyclic generalizations of $M_{24}$ and $M_{25}$. One family of manifolds, namely $M_{24}(n)$, $n \geq 1$, is a generalization of the manifold $M_{24}$ from [6]. The manifolds $M_{24}(n)$, where $M_{24}(3) = M_{24}$, were independently constructed by Cavicchioli, Spaggiari and Telloni [4] for $n \geq 3$ and by Kozlovskaya [10, 11] for $n \geq 2$. In both cases $M_{24}(n)$ was defined via pairwise identifications of the faces of a 3-complex. These manifolds are natural generalizations of $M_{24}$ in the following sense: $M_{24}(n)$, $n > 1$, is an $n$-fold strongly-cyclic branched covering of the lens space $L_{3,1}$, branched over the same link as $M_{24}$. The proof of this fact, presented in [3, 4], is based on results by Stevens [18] and by Osborne and Stevens [16]. In Theorem 1 we give a purely topological proof of this fact: we consider a Heegaard diagram for the quotient space of $M_{24}(n)$ by its cyclic symmetry and we reduce it to the standard genus one Heegaard diagram for $L_{3,1}$. The manifold $M_{24}(3)$ is hyperbolic since, by construction, it is obtained from the hyperbolic $2\pi/3$-icosahedron gluing its faces by isometries. It is stated in [4], without proof, that for $n > 3$ the manifolds $M_{24}(n)$ are hyperbolic and a formula for their volumes is given. However the formula turns out to be wrong even for $n = 4$. In fact, in Proposition 4 we will present the correct values of volumes for the initial list of manifolds $M_{24}(n)$, calculated by the computer program Recognizer

Another family of manifolds, namely $M_{25}(n)$, $n \geq 1$, is a generalization of the manifold $M_{25}$ from [6]. As well as $M_{24}$, $M_{25}$ can be constructed from the $2\pi/3$-icosahedron. It is written in [4] p. 931] that “one can construct an analogous $\mathbb{Z}_n$- symmetric description for $M_{25}(n)$ in the same manner” as $M_{24}(n)$, with $M_{25}(3) = M_{25}$. Unfortunately, [4] doesn’t contain an explicit description of $M_{25}(n)$ for arbitrary $n$, but only the presentation for the fundamental group induced by face pairings. In this paper we reconstruct $M_{25}(n)$ by means of the face pairing corresponding to the presentation for $\pi_1(M_{25}(n))$ announced in [4]. We prove that, for

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[1] Three-manifold Recognizer is a computer program developed by the research group of S. Matveev in the Department of Computer Topology and Algebra of Chelyabinsk State University, available on the webpage [http://www.matlas.math.csu.ru](http://www.matlas.math.csu.ru)
1 The family of manifolds $M_{24}(n)$

Denote by $P_3$ an icosahedron with all dihedral angles $2\pi/3$. It is well-known that $P_3$ can be realized in the hyperbolic space $H^3$. Recall that $P_3$ has 12 vertices, 30 edges and 20 faces. We will present $P_3$ as in Fig. 1 where left and right sides, both denoted by $P_1R_1S_1$, are supposed to be identified. The following pairwise identification $\varphi_3$ of the faces of $P_3$, with the given ordering of vertices on the faces, can be found in [6]:

\[
\begin{align*}
    a_i: & A_i \rightarrow \tilde{A}_i & [P_iP_{i+1}Q_i \rightarrow R_{i+2}P_{i+2}Q_{i+1}], & b_i: & B_i \rightarrow \tilde{B}_i & [R_iP_iQ_i \rightarrow S_iS_{i+1}R_{i+1}], \\
    c_i: & C_i \rightarrow \tilde{C}_i & [S_iR_iQ_i \rightarrow R_{i+1}Q_{i+1}], & d: & D \rightarrow \tilde{D} & [P_iP_2P_3 \rightarrow S_3S_1S_2],
\end{align*}
\]

(1)

where $i = 1, 2, 3$ and all indices are taken mod 3. Obviously, the face identification $\varphi_3 = \{a_i, b_i, c_i, d\}$ induces equivalent relations on the sets of vertices, edges, and faces of $P_3$. Since $\varphi_3$ can be realized as isometries of $H^3$, the quotient space $P_3/\varphi_3$ is a compact orientable hyperbolic 3-manifold, that was denoted by $M_{24}$ in [3, 4, 6]. It was shown in [4] that $M_{24}$ has
the following interesting property: it is a 3-fold cyclic branched covering of the lens space $L_{3,1}$ branched over a 2-component link.

To generalize the construction of $M_{24}$, let us consider the complex $P_n$, $n \geq 1$, having $4n$ vertices, $10n$ edges, and $6n + 2$ faces, presented in Fig. 2. In particular, $P_3$ is the icosahedron as above. Define the pairwise identification $\varphi_n$ of faces of $P_n$ by formulae (1) for $i = 1, \ldots, n$, with the following correction:

$$d : D \to \bar{D} \quad [P_1P_2 \ldots P_{n-1}P_n \to S_3S_4 \ldots S_1S_2],$$

and denote the corresponding quotient space by $M_{24}(n)$. The following result was stated in [4] and [10, 11].

**Proposition 1** For each $n \geq 1$ the quotient space $M_{24}(n) = P_n/\varphi_n$ is a manifold.

**Proof.** Denote by $\sigma_k$ for $k = 0, 1, 2, 3$ the number of $k$-dimensional cells in $M_{24}(n)$. Obviously, $\sigma_3 = 1$. Moreover, $\sigma_2 = 3n + 1$, since there are the following classes of equivalent faces: $A_i \equiv A_i$, $B_i \equiv B_i$, $C_i \equiv C_i$, where $i = 1, \ldots, n$, and $D \equiv \bar{D}$. Also, $\sigma_1 = 3n + 1$, since all 1-cells are separated in four types of equivalence classes:

$$\begin{align*}
(I_i) \quad & P_iP_{i+1} \xrightarrow{a_i} R_{i+2}P_{i+2} \xrightarrow{b_{i+2}} S_{i+2}S_{i+3} \xrightarrow{d_{i+1}} P_iP_{i+1}; \\
(II) \quad & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}} S_{i+1}R_{i+1} \xrightarrow{b_{i+1}} P_iQ_i; \\
(III) \quad & Q_iR_i \xrightarrow{c_i} S_iQ_i \xrightarrow{c_i} R_{i+1}S_i \xrightarrow{b_{i+1}} Q_iR_i; \\
(IV) \quad & P_2Q_1 \xrightarrow{a_1} P_3Q_2 \xrightarrow{a_2} \ldots P_nQ_{n-1} \xrightarrow{a_{n-1}} P_1Q_n \xrightarrow{a_n} P_2Q_1,
\end{align*}$$

where $i = 1, \ldots, n$. It is easy to check that, by the action of $\varphi_n$, all vertices of $P_n$ are equivalent, and so, $\sigma_0 = 1$. Thus, the Euler characteristic of the quotient space is $\chi(M_{24}(n)) = 0$, \[\begin{align*}
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and by \[ M_{24}(n) \] \( M_{24}(n) \) is a manifold.

We recall that a presentation for the fundamental group of a closed 3-manifold is geometric if it corresponds to a Heegaard diagram. The following presentation for the fundamental group of \( M_{24}(n) \) was found in [11].

**Proposition 2** The fundamental group of \( M_{24}(n) \), \( n \geq 1 \), has the following geometric presentation:

\[
\pi_1(M_{24}(n)) = \langle a_1, \ldots, a_n; b_1, \ldots, b_n; c_1, \ldots, c_n, d \mid a_1a_2 \ldots a_n = 1, \\
a_ib_{i+2}d^{-1} = 1, \quad a_ic_i^{-1}b_i^{-1} = 1, \quad c_i^2b_i^{-1} = 1, \quad i = 1, \ldots, n \rangle.
\]

(3)

**Proof.** An open Heegaard diagram for \( M_{24}(n) \) arises from Fig. 2 with discs corresponding to faces of \( P_n \) and segments of curves being dual to edges of \( P_n \). Thus, \( \pi_1(M_{24}(n)) \) is generated by \( a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d \). Obviously, passing along curves on the open Heegaard diagram we get defining relations as (I), (II), (III) and (IV) in Proposition 1:

\[ a_ib_i+b_i+2d-1 = 1, \]
\[ a_ic_i^{-1}b_i^{-1} = 1, \]
\[ c_i^2b_i^{-1} = 1, \]
\[ a_1a_2 \ldots a_n = 1, \]

respectively.

\[ \square \]

**Corollary 1** The fundamental group of \( M_{24}(n) \), \( n \geq 1 \), has the following presentation:

\[
\langle c_1, \ldots, c_n \mid \prod_{j=1}^{n} c_j^3 = 1, \quad c_i^2c_{i+1}c_{i+2}^2 = c_{i+1}^2c_{i+2}c_{i+3}^2, \quad i = 1, \ldots, n \rangle.
\]

(4)

**Proof.** Let us start with the group presentation given in Proposition 2. Since \( b_i = c_i^2 \), we get \( a_i = c_i^2c_{i+1} \) for \( i = 1, \ldots, n \), with indices taken mod \( n \). Substituting these expressions into \( a_1 \ldots a_n = 1 \), we get \( \prod_{j=1}^{n}(c_j^2c_{j+1}) = 1 \), and eliminating \( d \) from \( a_ib_{i+2}d^{-1} = 1 \), we get \( d = c_i^2c_{i+1}c_{i+2}^2 \) for \( i = 1, \ldots, n \), with indices taken mod \( n \).

\[ \square \]

A presentation for \( \pi_1(M_{24}(n)) \), which is dual to (3), was given in [4]:

\[
\pi_1(M_{24}(n)) = \langle x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n; u \mid x_1x_2 \ldots x_n = 1, \\
x_iu = y_i, \quad x_iy_{i+2} = z_{i+2}, \quad z_i^2y_{i-1} = 1, \quad i = 1, \ldots, n \rangle.
\]

(5)

Here the relations correspond to the boundaries of the 2-faces, as shown for \( M_{24}(3) \) in Fig. 1.
2 Covering properties and volumes of manifolds \(M_{24}(n)\)

Let \(M, M'\) be compact connected orientable 3-manifolds and \(L'\) a disjoint union of closed curves properly embedded in \(M'\). Let \(p : M \rightarrow M'\) denote a cyclic covering of \(M'\) by \(M\) branched over \(L'\). We say that the branched covering \(p\) is strongly-cyclic if the stabilizer of each point of the singular set \(p^{-1}(L')\) is the whole group of covering transformations.

The following property of \(M_{24}(n)\) was observed in [4]. We will give a new proof, which, unlike the one in [4], does not use results of [16] and [18].

**Theorem 1** For each \(n \geq 2\), the manifold \(M_{24}(n)\) is an \(n\)-fold strongly-cyclic branched covering of the lens space \(L_{3,1}\), branched over a 2-component link. Moreover, \(M_{24}(1)\) is the lens space \(L_{3,1}\).

**Proof.** Denote by \(\rho_n\) the rotational symmetry of \(P_n\) sending \(X_i\) to \(X_{i+1}, i = 1, \ldots, n\), with indices taken mod \(n\), where \(X\) belongs to the set of letters used for the notations of the vertices: \(\{P, Q, R, S\}\). This symmetry induces a cyclic symmetry of the quotient space \(M_{24}(n) = P_n/\varphi_n\), and we denote it by \(\rho_n\), too. The quotient space \(M_{24}(n)/\rho_n\) is an orbifold whose underlying manifold is \(M_{24}(1)\). Its singular set \(L\) consists of two components: \(L = \ell_1 \cup \ell_2\). According to the description of the equivalence classes of the edges given in Proposition 1, the first component \(\ell_1\) corresponds to the class (IV) of edges. The second component \(\ell_2\) corresponds to the axis of rotation \(\rho_n\). Both components have singularity index \(n\). Through the Heegaard diagram of \(M_{24}(n)/\rho_n\), we understand the Heegaard diagram of \(M_{24}(1)\) with information about the singular set \(L\) presented.

The equivalence transformations of Heegaard diagrams from \(M_{24}(n)/\rho_n\) to \(L_{3,1}\) are drawn in Fig. 3. Here the first component \(\ell_1\) of the singular set is represented by a dashed segment connecting the discs \(A\) and \(\bar{A}\); the second component \(\ell_2\) by a dashed segment, connecting the discs \(D\) and \(\bar{D}\). Both components have branching index \(n\). Thus, the branched covering is strongly-cyclic. At the first step we identify the discs \(A\) and \(\bar{A}\), forming an 1-handle. The dashed segment (which also corresponds to \(\ell_1\)), connecting these discs, will give a 2-handle to glue up this 1-handle (i.e. they form a pair of complementary handles). Thus, \(\ell_1\) is a trivial knot; we are not drawing it in the next figures. At the second step we cancel the discs \(D\) and \(\bar{D}\), since they are connected only with the discs \(B\) and \(\bar{B}\), respectively, and the connecting segments are glued together. At the third step we cut along the curve shown by the dotted line to form a new pair of discs \(F\) and \(\bar{F}\) and then identify the discs \(C\) and \(\bar{C}\). After that we easily get a genus one Heegaard diagram, with the discs \(F\) and \(\bar{F}\), which is the standard diagram for the lens space \(L_{3,1}\), where the dotted line represents \(\ell_2\).

\(\square\)
Figure 3: Heegaard diagrams from $M_{24}(n)/\rho_n$ to $L_{3,1}$.

**Proposition 3** $M_{24}(2)$ is the Seifert manifold $(S^2; (3, 1), (3, 2), (3, 2), (1, -1))$.

**Proof.** By Corollary $\mathbf{1}$, $\pi_1(M_{24}(2)) = \langle c_1, c_2 | c_1^3 c_2^3 = 1, c_1^2 c_2 c_1 c_2 = c_1 c_2 c_1^2, c_1 c_2 c_1^2 \rangle$. The second relation is equivalent to $c_2^2 c_1^2 c_2 c_1^2 = c_1 c_2$; after multiplication by $c_1$, we obtain $c_1^2 c_2^2 = c_1 c_2 c_1^2 c_2 c_1^2$ and by using the first relation, i.e. $c_1^3 = c_2^{-3}$, we have $c_1^2 c_2^2 = c_1^3 (c_2 c_1^2)^2$ and thus $c_1^2 c_2^2 = (c_2 c_1^2)^2$. Let us set $c_1 = a$, $b = c_2 c_1^2$ and so $c_2 = ba^{-2}$. Then, we have the following presentation:

$$\pi_1(M_{24}(2)) = \langle a, b | a^3 (ba^{-2})^3 = 1, a^{-2} (ba^{-2})^2 = b^2 \rangle$$

$$= \langle a, b | a^3 a^2 b^2 ba^{-2} = 1, (ba^{-2})^2 = a^2 b^2 \rangle$$

$$= \langle a, b | a^3 b^4 = 1, (ba^{-2})^2 = a^2 b^2 \rangle.$$ 

Consider the Seifert manifold $M = (S^2; (3, 1), (3, 2), (3, 2), (1, -1)) = (S^2; (3, 1), (3, 2), (3, -1))$. The standard presentation of $\pi_1(M)$ is as following (see $[15]$):

$$\langle x, y, z, h | xyz = 1, \ xh = hx, \ yh = hy, \ zh = hz, \ x^3 h = 1, \ y^3 h^2 = 1, \ z^3 h^{-1} = 1 \rangle.$$ 

We obtain $h = z^3$ from the last relation, $y = x^{-1} z^{-1}$ from the first one and we get

$$\pi_1(M) = \langle x, z | xz^3 = z^3 x, \ x^{-1} z^3 = z^3 x^{-1}, \ x^3 z^3 = 1, \ z^6 = (x^{-1} z^{-1})^{-3} \rangle.$$
Since the first two relations come from the third, we have
\[ 
\pi_1(M) = \langle x, z \mid x^3z^3 = 1, \quad z^6 = (zx)^3 \rangle = \langle x, z \mid x^3z^3 = 1, \quad z^5 = xzxz \rangle = \langle x, z \mid x^3z^3 = 1, \quad z^2x^2 = xz^2z^{-1} \rangle = \langle x, z \mid x^3z^3 = 1, \quad z^2x^2 = (xz^{-2})^2 \rangle,
\]
which is the same presentation as above, i.e. the fundamental groups of \( M_{24}(2) \) and \( M \) are isomorphic.

As a consequence, note that, since \( M \) is irreducible, \( M_{24}(2) \) is irreducible, too. Furthermore, since \( M \) is a large Seifert manifold, \( \pi_1(M_{24}(2)) = \pi_1(M) \) contains an infinite cyclic normal subgroup (generated by \( h \), see \cite{13}). Therefore, by a result of \cite{1} and \cite{7} (see also \cite{19}), \( M_{24}(2) \) is Seifert fibered and, consequently, \( M_{24}(2) \) and \( M \) are homeomorphic.

\[ \square \]

Since \( M_{24}(2) \) is \( (S^2; (3, 1), (3, 2), (3, 2), (1, -1)) \), it admits Nil geometry and its first homology group is \( \mathbb{Z}_3 \oplus \mathbb{Z}_4 \). Moreover, \( M_{24}(2) \) can be obtained by Dehn surgeries with parameters \((-1, 2), (-2, 1), (-7, 2)\) on the link \( 6_1^3 \) (chain link), as well as with parameters \((6, 1), (1, 1), (3, 1)\) on \( 6_2^3 \) (Borromean rings).

Theorem 3.1 from \cite{4} states that for \( n \geq 3 \) the manifolds \( M_{24}(n) \) are hyperbolic, although for \( n > 3 \) the authors do not present explicitly any proof of hyperbolicity. Moreover, the authors give the following volume formula: \( \text{vol } M_{24}(n) = (n/3) \cdot (4.686034274\ldots) \). However, even for small \( n > 3 \) the above formula turns out to be wrong. It would be right if the polyhedron \( P_n \) could be obtained by gluing isometrically \( n \) copies of the \( 1/3 \)-piece of the hyperbolic \( 2\pi/3 \)-icosahedron \( P_3 \). This is obviously not true, since the dihedral angle around the image of the axis of rotation \( \rho_n \) in \( P_n/\rho_n \) must be equal to \( 2\pi/n \), that is, it differs from \( 2\pi/3 \) if \( n > 3 \). The correct values of \( \text{vol } M_{24}(n) \) are presented below.

**Proposition 4** The hyperbolic volumes and the first homology groups of \( M_{24}(n) \), for \( 3 \leq n \leq 6 \), are as follows:

| manifold | volume                    | homology group       |
|----------|---------------------------|----------------------|
| \( M_{24}(3) \) | 4.686034273803\ldots | \( \mathbb{Z}_9 \) |
| \( M_{24}(4) \) | 9.702341514665\ldots | \( \mathbb{Z}_3 \oplus \mathbb{Z}_{12} \) |
| \( M_{24}(5) \) | 14.319926985892\ldots | \( \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \) |
| \( M_{24}(6) \) | 18.649157163789\ldots | \( \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{18} \) |

**Proof.** Results are obtained by using the computer program Recognizer. \[ \square \]
3 The family of manifolds $M_{25}(n)$

The following pairwise identification $\psi_3$ of faces of $\mathcal{P}_3$, with notations according to Fig. 4, can be found in [6]:

$$a_i : A_1 \rightarrow \tilde{A}_1 \quad [P_i P_{i+1} Q_i \rightarrow P_{i+2} R_{i+2} Q_{i+2}], \quad b_i : B_1 \rightarrow \tilde{B}_1 \quad [Q_i R_{i+1} P_{i+1} \rightarrow R_{i+2} S_{i+2} S_{i+1}],$$

$$c_i : C_i \rightarrow \tilde{C}_i \quad [Q_{i-1} R_i S_{i-1} \rightarrow S_i Q_i R_i], \quad d : \mathcal{D} \rightarrow \tilde{\mathcal{D}} \quad [P_1 P_2 P_3 \rightarrow S_3 S_1 S_2],$$

(6)

where $i = 1, 2, 3$ and all indices are taken mod 3. The quotient space $\mathcal{P}_3/\psi_3$ is a compact orientable hyperbolic 3-manifold denoted by $M_{25}$ in [3, 4, 6]. It was shown in [4] that $M_{25}$ is a 3-fold cyclic branched covering of the lens space $L_{3,1}$ branched over a 2-component link.

As one can see from Fig. 4, the boundaries of the faces of $\mathcal{P}_3$ are in correspondence with the following relations:

$$x_1 x_2 x_3 = 1, \quad x_i y_i = u, \quad y_i z_i = x_{i-1}, \quad z_{i-1} z_i = y_{i-1}, \quad i = 1, 2, 3. \quad (7)$$

To generalize the construction of $M_{25}$, it is natural to consider the complex $\mathcal{P}_n$, $n \geq 1$, pictured in Fig. 5 and define the pairwise identification $\psi_n$ of the faces of $\mathcal{P}_n$ by formulae (6) for $i = 1, \ldots, n$, with the following correction:

$$d : \mathcal{D} \rightarrow \tilde{\mathcal{D}} \quad [P_1 P_2 \ldots P_{n-1} P_n \rightarrow S_3 S_4 \ldots S_1 S_2]. \quad (8)$$

This is equivalent to the generalization from $M_{25}$ to $M_{25}(n)$ considered in [4], were the face identifications of $\mathcal{P}_n$ are defined by the boundary relations corresponding to the defining relations for the following group presentation:

$$G(n) = \langle x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n; u \mid x_1 x_2 \ldots x_n = 1, \quad x_i y_i = u, \quad y_i z_i = x_{i-1}, \quad z_{i-1} z_i = y_{i-1}, \quad i = 1, \ldots, n \rangle. \quad (9)$$

![Figure 4: Identification $\psi_3$ of faces of $\mathcal{P}_3$.](image)
Let us denote the corresponding quotient space $\mathcal{P}_n/\psi_n$ by $M_{25}(n)$.

**Proposition 5** For each $n \geq 1$, the quotient space $M_{25}(n) = \mathcal{P}_n/\psi_n$ is a manifold.

**Proof.** Let $\sigma_k$ be the number of $k$-cells in $M_{25}(n)$, $k = 0, 1, 2, 3$. Obviously, $\sigma_3 = 1$. Moreover, $\sigma_2 = 3n + 1$, since there are the following classes of equivalent faces: $A_i \equiv \bar{A}_i$, $B_i \equiv \bar{B}_i$, $C_i \equiv \bar{C}_i$, where $i = 1, \ldots, n$, and $D \equiv \bar{D}$. We have the following three types of classes of equivalent edges, with $n$ classes of each type:

(I) $P_i P_{i+1} \xrightarrow{a_i} P_{i+2} R_{i+2} \xrightarrow{b_i+1} S_{i+2} S_{i+3} \xrightarrow{d^{-1}} P_{i+1} P_{i+1};$

(II) $P_{i+1} Q_{i} \xrightarrow{a_i} R_{i+2} Q_{i+2} \xrightarrow{c_{i+2}^{-1}} S_{i+1} R_{i+2} \xrightarrow{b_i^{-1}} P_{i+1} Q_{i};$

(III) $Q_{i-1} R_{i} \xrightarrow{c_i} S_{i} Q_{i} \xrightarrow{c_{i+1}} R_{i+1} S_{i+1} \xrightarrow{b_i^{-1}} Q_{i-1} R_{i};$

where $i = 1, \ldots, n$. Remark that $P_i Q_{i} \xrightarrow{a_i} P_{i+2} Q_{i+2}$. Thus, if $n$ is odd, then the set of edges $\{P_1 Q_1, \ldots, P_n Q_n\}$ will form one class of equivalent edges:

$$P_1 Q_1 \xrightarrow{a_1} P_3 Q_3 \xrightarrow{a_3} \ldots P_n Q_n \xrightarrow{a_n} P_2 Q_2 \xrightarrow{a_2} \ldots P_{n-1} Q_{n-1} \xrightarrow{a_{n-1}} P_1 Q_1.$$ 

Hence $\sigma_1 = 3n + 1$. Moreover, in this case all vertices of $\mathcal{P}_n$ are equivalent, so $\sigma_0 = 1$. If $n$ is even, we get two classes of equivalent edges:

$$P_1 Q_1 \xrightarrow{a_1} P_3 Q_3 \xrightarrow{a_3} \ldots P_{n-1} Q_{n-1} \xrightarrow{a_{n-1}} P_1 Q_1; \quad P_2 Q_2 \xrightarrow{a_2} P_4 Q_4 \xrightarrow{a_4} \ldots P_n Q_n \xrightarrow{a_n} P_2 Q_2.$$ 

Hence $\sigma_1 = 3n + 2$. Moreover, in this case all vertices of $\mathcal{P}_n$ are separated in two classes of equivalence, so $\sigma_0 = 2$. Thus, in both cases, the Euler characteristic of the quotient space is $\chi(M_{25}(n)) = 0$, and so $M_{25}(n)$ is a manifold.

$\square$
**Proposition 6** The fundamental group of $M_{25}(n)$, $n \geq 1$, has the following geometric presentation:

$$
\pi_1(M_{25}(n)) = \langle a_1, \ldots, a_n; b_1, \ldots, b_n; c_1, \ldots, c_n; d \mid a_1a_3 \ldots a_n a_3 \ldots a_n = 1, \\
a_i b_i d^{-1} = 1, a_i c_i^{-1} b_i^{-1} = 1, c_i c_{i+1} b_{i-1}^{-1} = 1, \quad i = 1, \ldots, n \rangle.
$$

(10)

if $n$ is odd, and

$$
\pi_1(M_{25}(n)) = \langle a_1, \ldots, a_n; b_1, \ldots, b_n; c_1, \ldots, c_n; d \mid a_1a_3 \ldots a_n a_3 \ldots a_n = 1, \\
a_i b_i d^{-1} = 1, a_i c_i^{-1} b_i^{-1} = 1, c_i c_{i+1} b_{i-1}^{-1} = 1, \quad i = 1, \ldots, n \rangle.
$$

(11)

otherwise.

**Proof.** If $n$ is odd, an open Heegaard diagram for $M_{25}(n)$ arises from Fig. 5 with the discs corresponding to the faces of $\mathcal{P}_n$ and the segments of curves being dual to the edges of $\mathcal{P}_n$. Thus, $\pi_1(M_{25}(n))$ is generated by $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d$. Obviously, passing along the curves of the Heegaard diagram we get defining relations as (I), (II), and (III) in Proposition 5: $a_i b_i d^{-1} = 1$, $a_i c_i^{-1} b_i^{-1} = 1$, and $c_i c_{i+1} b_{i-1}^{-1} = 1$, with the additional relation $a_1 a_3 \ldots a_n a_3 \ldots a_n = 1$.

If $n$ is even, the discs corresponding to the faces of $\mathcal{P}_n$ still give a complete system of meridian discs for a Heegaard surface $F$ of $M_{25}(n)$. The system of curves, defined on $F$, by the edges dual to the edges of $\mathcal{P}_n$ is proper but not reduced: in fact, by cutting $F$ along these curves, we get two discs. As a consequence, the two systems of curves on $F$ define a generalized Heegaard diagram for $M$ (see [2] for details).

Again this diagram yields a presentation for $\pi_1(M_{25}(n))$, whose generators are $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d$ and whose relations are still (I), (II), and (III) in Proposition 5: $a_i b_i d^{-1} = 1$, $a_i c_i^{-1} b_i^{-1} = 1$, and $c_i c_{i+1} b_{i-1}^{-1} = 1$, with two additional relations: $a_1 a_3 \ldots a_n a_3 \ldots a_n = 1$ and $a_2 a_4 \ldots a_n = 1$.

Therefore, both for $n$ odd and for $n$ even, the presentations are geometric in the sense that they correspond to (generalized) Heegaard diagrams.

\[ \square \]

**Corollary 2** The fundamental group of $M_{25}(n)$, $n \geq 1$, has presentation:

$$
\langle c_1, \ldots, c_n \mid \prod_{j=0}^{k-1} (c_2+2j c_3+2j)^{k-1} \prod_{j=0}^{k-1} (c_3+2j c_4+2j) = 1, \quad c_i c_{i+1} c_{i+2} = c_{i+1} c_{i+2} c_{i+3}, \quad i = 1, \ldots, n \rangle,
$$

(12)
if \( n = 2k + 1 \), and

\[
\langle c_1, \ldots, c_n \mid \prod_{j=0}^{k-1} (c_{2+2j}c_{3+2j}) = 1, \quad \prod_{j=0}^{k-1} (c_{3+2j}c_{4+2j}) = 1, \quad c_ic_{i+1}c_{i+2} = c_{i+1}c_{i+2}c_{i+3}, \quad i = 1, \ldots, n \rangle
\]

if \( n = 2k \), where indices are considered mod \( n \).

**Proof.** Let us start with the group presentation given in Proposition 6. Since \( b_{i-1} = c_ic_{i+1} \), we get \( a_i = c_{i+1}c_{i+2} \) for \( i = 1, \ldots, n \); moreover, \( d = c_{i+1}c_{i+2}c_{i+3} \), for all \( i = 1, \ldots, n \) (all indices are taken mod \( n \)).

\( \square \)

It is stated in [4, p. 391] that for \( n \geq 3 \) the group \( G(n) \) with the presentation (9), where the relations correspond to the boundaries of the 2-faces, as shown for \( M_{25}(3) \) in Fig. 4 is the fundamental group of \( M_{25}(n) \). We observe that \( G(n) \) is isomorphic to \( \pi_1(M_{25}(n)) \) only if \( n \) is odd. This is the case when the quotient space \( P_n/\psi_n \) has exactly one vertex. If there is more than one vertex, more careful considerations are necessary (see, for example, [17, Section 62]).

If \( n = 2k \), then, as was already pointed out above, all vertices of \( P_n \) will form two classes of equivalence (see Fig. 6 for \( M_{25}(4) \) where one class of vertices is marked by ■ and another by ♦) and all its edges will form \( (3n+2) \) classes of equivalence. For example, as we can see from

![Figure 6: Equivalence of faces in the construction of M_{25}(4).](image)

the polyhedral schemata for \( M_{25}(4) \) in Fig. 6, there are two edges in the class \( u \), and two edges in the class \( v \); on the contrary, from the presentation (9), four edges in the class \( u \) would be expected. Actually, for \( n \) even, we must add a relation corresponding to a maximal tree of the 1-skeleton of the complex, i.e. an edge. For instance we suppose \( v = 1 \): indeed \( v \) is an
edge connecting a vertex from class □ with a vertex from class ♦. Therefore, for \( n = 2k \), the fundamental group \( \pi_1(M_{25}(n)) \) is isomorphic to \( H(n) \) with the following presentation:

\[
H(n) = \langle x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n; u \mid x_1x_2\ldots x_n = 1, \\
y_iz_i = x_{i-1}, \quad z_{i-1}z_i = y_{i-1}, \quad i = 1, \ldots, n, \quad x_{2j-1}y_{2j-1} = u, \quad x_{2j}y_{2j} = 1, \quad j = 1, \ldots k \rangle.
\] (14)

4 Covering properties and volumes of manifolds \( M_{25}(n) \)

Proposition 7 The manifold \( M_{25}(2) \) is the lens space \( L_{3,1} \).

Proof. Let us contract the edges of \( P_2 \) labelled by \( z_2 \) to deform it to a fundamental polyhedron whose quotient by \( \psi_2 \) has one vertex (see Fig. 7). We are using the dual on this new polyhedra to get a Heegaard diagram for \( M_{25}(2) \). The transformations of Heegaard diagrams from \( M_{25}(2) \) to \( L_{3,1} \) are drawn in Fig. 8.

Figure 7: Two-vertex and one-vertex fundamental polyhedra for \( M_{25}(2) \).

With regard to the covering properties of \( M_{25}(n) \), we must distinguish again the case \( n \) even from that of \( n \) odd.

Theorem 2 (1) For every \( n \) odd, \( n \geq 3 \), the manifold \( M_{25}(n) \) is an \( n \)-fold strongly-cyclic branched covering of the lens space \( L_{3,1} \), branched over a 2-component link. Moreover, \( M_{25}(1) \) is the lens space \( L_{3,1} \).

(2) For every \( n = 2k \), \( k \geq 2 \), the manifold \( M_{25}(n) \) is a \( n/2 \)-fold strongly-cyclic branched covering of the lens space \( L_{3,1} \), branched over a 3-component link.
Figure 8: Heegaard diagrams from $M_{25}(2)$ to $L_{3,1}$.

Proof. (1) Suppose that $n$ is odd. Denote by $\rho_n$ the rotational symmetry of $\mathcal{P}_n$ sending $X_i$ to $X_{i+1}$ with indices taken mod $n$, $i = 1, \ldots, n$, where $X$ belongs to the set of letters used for the notations of vertices: \{P, Q, R, S\}. This symmetry induces a cyclic symmetry of the quotient space $M_{25}(n) = \mathcal{P}_n/\psi_n$, and we denote it by $\rho_n$, too. The quotient space $M_{25}(n)/\rho_n$ is an orbifold whose underlying manifold is $M_{25}(1)$ with a 2-component singular set $L_{\text{odd}} = \ell_1 \cup \ell_2$. According to the description of the equivalence classes of edges given in Prop. 5, the first component $\ell_1$ of $L_{\text{odd}}$ corresponds to the class of edges $\{P_1Q_1, \ldots, P_nQ_n\}$. Only one element from this class will appear in the quotient space, so $\ell_1$ has singularity index $n$. The second singular curve $\ell_2$, also having singularity index $n$, corresponds to the axis of rotation $\rho$. Through the Heegaard diagram of $M_{25}(n)/\rho_n$ we understand the Heegaard diagram of $M_{25}(1)$ with information about the singular set $L_{\text{odd}}$ presented.
The transformations of Heegaard diagrams from $M_{25}(n)/\rho_n$ to $L_{3,1}$ are drawn in Fig. 9. Here the first component of the singular set is represented by a dashed segment connecting the discs $A$ and $\overline{A}$; the second component by a dashed segment, connecting the discs $D$ and $\overline{D}$. Both components have branching index $n$, thus the cyclic covering is strongly cyclic. At the first step we identify discs $A$ and $\overline{A}$, forming an 1-handle. The dashed segment (which corresponds to the first component of the singular set) connecting these discs will give a 2-handle to glue up this 1-handle (i.e. they form a pair of complementary handles). Thus, the first component is a trivial knot; we are not drawing it in next figures. At the second step we cancel the discs $D$ and $\overline{D}$, since they are connected only with the discs $B$ and $\overline{B}$, respectively, and the connecting segments are glued together. At the third step we cut along the curve represented by the dotted line to form a new pair of discs $F$ and $\overline{F}$ and then identify the discs $C$ and $\overline{C}$. After that, we easily get a genus one Heegaard diagram, with the discs $F$ and $\overline{F}$, which is the standard diagram for the lens space $L_{3,1}$, where the dotted line represents the second component of the singular set $L_{odd}$.

(2) Suppose $n = 2k$, $k \geq 2$. Denote by $\rho_k$ the rotational symmetry of $P_n$ sending $X_i$ to $X_{i+2}$, $i = 1, \ldots, n$, with indices taken mod $n$, where $X$ belongs to the set of letters used for the notations of vertices: $\{P, Q, R, S\}$. This symmetry induces a cyclic symmetry of the
quotient space $M_{25}(n) = \mathcal{P}_n/\psi_n$, and we denote it by $\rho_k$, too. The quotient space $M_{25}(n)/\rho_k$ is an orbifold whose underlying manifold is $M_{25}(2)$. We observe that the singular set $\mathcal{L}_{\text{even}}$ of the orbifold $M_{25}(n)/\rho_k$ has three components: $\mathcal{L}_{\text{even}} = \ell'_1 \cup \ell'_2 \cup \ell'_3$. Indeed, according to the description of the equivalence classes of the edges given in Prop. 5 for the case of $n$ even, the first component $\ell'_1$ corresponds to the class of edges $\{P_1Q_1, P_3Q_3, \ldots, P_{n-1}Q_{n-1}\}$. Only one element from this class will be represented in the quotient space, so it will produce a singular curve with singularity index $k = n/2$. Analogously, the second component $\ell'_2$ corresponds to the class of edges $\{P_2Q_2, P_4Q_4, \ldots, P_nQ_n\}$ and its singularity index equals $k = n/2$, too. The third component $\ell'_3$ corresponds to the axis of rotation $\rho_k$ and its singularity index equals $k = n/2$. Through the Heegaard diagram of $M_{25}(n)/\rho_k$ we understand the Heegaard diagram of $M_{25}(2)$ with information about the singular set $\mathcal{L}_{\text{even}}$ presented. The final step of the proof is based on the fact, proved in Prop. 7, that $M_{25}(2)$ is homeomorphic to $L_{3,1}$.

Theorem 3.1 from [4] states that for $n \geq 3$ the manifolds $M_{25}(n)$ are hyperbolic, although for $n > 3$ the authors do not present explicitly any proof of hyperbolicity. Moreover, they give the following volume formula $\text{vol} M_{25}(n) = (n/3) \cdot (4.686034274\ldots)$. However, the given formula turns out to be wrong even for small $n > 3$: it would be right if the polyhedron $\mathcal{P}_n$ could be obtained by gluing isometrically $n$ copies of the $\frac{1}{3}$-piece of the hyperbolic $2\pi/3$-icosahedron $\mathcal{P}_3$. This is obviously not true, since the dihedral angle around the image of the axis of rotation $\rho_n$ in $\mathcal{P}_n/\rho_n$ must be equal to $2\pi/n$, that is, it differs from $2\pi/3$ if $n > 3$. The correct values of $\text{vol} M_{25}(n)$ for small $n$ are presented below.

**Proposition 8** The hyperbolic volumes and the first homology groups of $M_{25}(n)$, for $n \leq 6$, are as follows:

| manifold | volume      | homology group |
|----------|-------------|----------------|
| $M_{25}(3)$ | 4.686034273803\ldots | $\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$ |
| $M_{25}(4)$ | 3.970289623891\ldots | $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6$ |
| $M_{25}(5)$ | 14.319926985892\ldots | $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$ |
| $M_{25}(6)$ | 14.004768920617\ldots | $\mathbb{Z}_8 \oplus \mathbb{Z}_{72}$ |

**Proof.** The results are obtained by using the computer program Recognizer.
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