Radially symmetric solutions to the Hénon-Lane-Emden system on the critical hyperbola

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Abstract

We use variational methods to study the existence of nontrivial and radially symmetric solutions to the Hénon-Lane-Emden system with weights, when the exponents involved lie on the "critical hyperbola". We also discuss qualitative properties of solutions and nonexistence results.

Keywords: weighted Lane-Emden system, critical hyperbola, Rellich inequality, Sobolev inequality, fourth order ordinary differential equations, Hamiltonian systems.

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1 Introduction

In this paper we discuss existence, non existence and qualitative properties of nontrivial radially symmetric solutions $u, v$ to the following weakly coupled system on the punctured space $\mathbb{R}^n \setminus \{0\}$:

\[
\begin{align*}
-\Delta u &= |x|^a |v|^{p-2} v \\
-\Delta v &= |x|^b |u|^{q-2} u.
\end{align*}
\]
Here $n \geq 2$, $a, b \in \mathbb{R}$, $p, q > 1$ belong to the weighted critical hyperbola

\[
\frac{a + n}{p} + \frac{b + n}{q} = n - 2,
\]  
and satisfy the standard anticoercivity assumption

\[
\frac{1}{p} + \frac{1}{q} < 1.
\]  

The Hénon-Lane-Emden system (1.1) is a largely studied problem. In the autonomous case $a = b = 0$, P.L. Lions proved in [24] the existence of a solution $u \in D^{2,q'}(\mathbb{R}^n), v \in D^{2,p'}(\mathbb{R}^n)$ to

\[
\begin{aligned}
-\Delta u &= |v|^{p-2}v \\
-\Delta v &= |u|^{q-2}u,
\end{aligned}
\]  

under the assumptions $n > n/p + n/q = n - 2 > 0$. We quote also the paper [22] by Hulshof-Van der Vorst, for additional qualitative properties of the pair $u, v$.

The role of the ”critical hyperbola” was first pointed out by Mitidieri [25], [26] for the autonomous case $a = b = 0$ (see also [33]). It turns out that (1.4) has no positive, radial solutions $u, v \in C^2(\mathbb{R}^n)$ if $p, q$ are below the critical hyperbola. On the other hand, Serrin and Zou used shooting methods in [34] to prove that (1.1) admits infinitely many positive radial solutions $u, v$ which tend to 0 as $|x| \to +\infty$, provided that the pair $p, q$ is on or above the critical hyperbola.

The Hénon-Lane-Emden conjecture has been raised in [32] and in [10] for a more general class of higher order system. It says in particular that there is no positive solution for system (1.1) if $p, q$ is under the critical hyperbola. Bidaut-Veron and Giacomini have recently shown in [4] that if $n \geq 3$, $a, b > -2$, then the system (1.1) admits a positive classical radial solution $u, v$ with $u, v$ continuous at the origin if and only if $(p, q)$ is above or on the critical hyperbola. We recall that by Proposition 2.1 in [3], no solution to (1.1) is continuous at the origin if $a \leq -2$ or $b \leq -2$.

Remarkable results about the Hénon-Lane-Emden conjecture have been recently obtained also in [6], [15], [16], [30], [31], [33] and [35].

Finally, we recall that the weighted critical hyperbola enters in a natural way in the context of the solvability of Hardy-Hénon type elliptic systems in bounded domains, see for instance [14] and [7].
In the present paper we first use variational methods to extend the Lions existence theorem to the non autonomous case. Then we discuss nonexistence results. We always assume that (1.2) and (1.3) are satisfied. We limit ourselves to state here some of our results, and we postpone more precise statements to Section 3.

**Existence** If \( a \neq -n \) and \( b \neq -n \), then (1.1) has a nontrivial radial solution \( u, v \) such that
\[
\int_{\mathbb{R}^n} |x|^{\frac{a}{p-1}} |\Delta u|^p \, dx < \infty , \quad \int_{\mathbb{R}^n} |x|^{\frac{b}{q-1}} |\Delta v|^q \, dx < \infty .
\]
Moreover, it holds that
\[
\lim_{|x| \to \infty} |x|^{\frac{b+n}{q}} u(x) = \lim_{x \to 0} |x|^{\frac{b+n}{p}} u(x) = \lim_{|x| \to \infty} |x|^{\frac{a+n}{p}} v(x) = \lim_{x \to 0} |x|^{\frac{a+n}{p}} v(x) = 0, \quad (1.5)
\]
and \( u, v \) are both positive if and only if \( a > -n \) and \( b > -n \).

**Nonexistence** Let \( u, v \) be a solution to (1.1) on \( \mathbb{R}^n \setminus \{0\} \) and assume that either
\[
\begin{align*}
\lim_{|x| \to \infty} |x|^{\frac{b+n}{q}} u(x) & \quad \text{exist and are finite, or} \\
\lim_{x \to 0} |x|^{\frac{b+n}{p}} u(x) & \quad \text{exist and are finite.}
\end{align*}
\]
(1.6)
If \( a \leq -n \) or \( b \leq -n \) and if \( u \geq 0, v \geq 0 \) then \( u \equiv v \equiv 0 \).

Let us briefly describe our approach. It has already been noticed for instance in [3], [4], [6], that radial solutions to (1.1) are in one-to-one correspondence with trajectories \( g, f \) of the Hamiltonian system
\[
\begin{align*}
-g'' + 2Ag' + \Gamma g &= |f|^{p-2}f \quad \text{on } \mathbb{R} \\
-f'' - 2Af' + \Gamma f &= |g|^{q-2}g \quad \text{on } \mathbb{R}
\end{align*}
\]
(1.7)
for suitable constants \( A, \Gamma \in \mathbb{R} \) depending on the data. Notice that (1.7) includes the Schrödinger equation \(-g'' + \Gamma g = |g|^{p-2}g\), whose relevance with the Caffarelli-Kohn-Nirenberg inequality was pointed out by Catrina and Wang in [11]. For \( p = 2 \) the system (1.7) reduces to the fourth order ordinary differential equation
\[
g''' - 2(2A^2 + \Gamma)g'' + \Gamma^2 g = |g|^{q-2}g , \quad (1.8)
\]
3
which is naturally related to second order dilation invariant inequalities of Rellich-Sobolev type, see [2]. Actually the system (1.7) and the equation (1.8) have independent interest because of their applications. We shall not attempt to give a complete list of references. We cite for instance [5] [12], [13] [17], [19]–[21], [23] and references therein. In the monograph [29] by Peletier and Troy one can find several applications and a rich bibliography on these topics.

In Section 2 we use the results in [28] and variational methods to get the existence of solutions $g \in W^{2,p'}(\mathbb{R}), f \in W^{2,q'}(\mathbb{R})$ to (1.7), compare with Theorem 2.1. Then we discuss sign properties of solutions to (1.7) having certain behavior at $-\infty$ and/or at $+\infty$. In Section 3 we obtain our main theorems about (1.1) as corollaries of our results for (1.7).

In the Appendix we indicate a possible non-radial approach to (1.1).

Notation For any integer $n \geq 2$ we denote by $\omega_n$ the $n-1$ dimensional measure of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

Let $q \in [1, +\infty)$ and let $\omega$ be a non-negative measurable function on a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. The weighted Lebesgue space $L^q(\Omega; \omega(x) \, dx)$ is the space of measurable maps $u$ in $\Omega$ with finite norm $(\int_{\Omega} |u|^q \omega(x) \, dx)^{1/q}$. For $\omega \equiv 1$ we simply write $L^q(\Omega)$. As usual, $\| \cdot \|_\infty$ is the $L^\infty$ norm.

For any function $\varphi : \mathbb{R} \to \mathbb{R}$, the notation $\varphi(\pm \infty) = c$ means that there exists $\lim_{s \to \pm \infty} \varphi(s) = c$.

2 A $2 \times 2$ system of ordinary differential equations

In this section we provide conditions for the existence of solutions to (1.7) vanishing at $\pm \infty$ and for the nonexistence of positive solutions having nonnegative limits at $-\infty$ or at $\infty$. We start with an existence result.

Theorem 2.1 Let $p,q \in (1, \infty)$, $A, \Gamma \in \mathbb{R}$ be given, such that $A^2 + \Gamma \geq 0$ and $\Gamma \neq 0$. Assume that (1.3) is satisfied. Then the system (1.7) has a nontrivial solution $g, f$ such that $g \in W^{2,p'}(\mathbb{R})$ and $f \in W^{2,q'}(\mathbb{R})$.

Proof. To simplify notations, we set

$$L_+ \varphi := -\varphi'' + 2A \varphi' + \Gamma \varphi, \quad L_- \varphi = -\varphi'' - 2A \varphi' + \Gamma \varphi.$$
Since \( A^2 + \Gamma \geq 0 \) and \( \Gamma \neq 0 \), from Proposition 5.2 in [28] we have that the infimum

\[
I_{p',q}(A,\Gamma) = \inf_{g \in W^{2,p'}(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} |L+g|^{p'} \, ds}{\left( \int_{\mathbb{R}} |g|^q \, dx \right)^{p'/q}}
\]

is achieved by some \( g \in W^{2,p'}(\mathbb{R}) \) that solves

\[
\int_{\mathbb{R}} |L+g|^{p'-2} L+g L+\psi \, ds = \int_{\mathbb{R}} |g|^{q-2} g\psi \, ds \quad \text{for any } \psi \in W^{2,p'}(\mathbb{R}).
\]

Thus \( g \in W^{2,p'}(\mathbb{R}) \) is a weak solution to the following fourth-order ODE:

\[
L_- \left( (L+g)^{p'-2} L+g \right) = |g|^{q-2} g \quad \text{on } \mathbb{R},
\]

which is equivalent to the system (1.7), by defining \( f = -|L+g|^{p'-2} L+g \). Clearly, \( g,f \in C^2(\mathbb{R}) \). Now we recall that \( \|L-\cdot\|_q' \) is an equivalent norm in \( W^{2,q'}(\mathbb{R}) \) by Proposition 5.2 in [28]. Since \( g \in W^{2,p'}(\mathbb{R}) \mapsto L^q(\mathbb{R}) \), we have \( L_- f = |g|^{q-2} g \in L^q(\mathbb{R}) \), and thus \( f \in W^{2,q'}(\mathbb{R}) \). \( \square \)

**Remark 2.2** One could exchange \( g \) and \( f \) in the proof of Theorem 2.1 to find a solution \( \tilde{g}, \tilde{f} \), such that \( \tilde{f} \in W^{2,q'}(\mathbb{R}) \) achieves \( I_{q',p}(-A,\Gamma) \) and \( \tilde{g} = -|L-\tilde{f}|^{q'-2} L-\tilde{f} \). This argument does not lead to a multiplicity result for (1.7). To simplify notations we set \( m = I_{p',q}(A,\Gamma) \) and \( \tilde{m} = I_{q',p}(-A,\Gamma) \). Since \( |L_- f|^{q'} = |g|^q, \ |f|^p = |L+g|^{p'} \), and since \( g \) achieves \( m \) we find

\[
\tilde{m} \leq \frac{\int_{\mathbb{R}} |L_- f|^{q'} \, ds}{\left( \int_{\mathbb{R}} |f|^p \, ds \right)^{q/p}} = \frac{\int_{\mathbb{R}} |g|^q \, ds}{\left( \int_{\mathbb{R}} |L+g|^p \, ds \right)^{q/p}} = m \frac{p-q'}{q} \frac{q'}{p-q'},
\]

so that \( \tilde{m} \frac{p-q'}{q} \leq m \frac{p-q'}{q} \). In a similar way we get the opposite inequality, and in particular \( \tilde{m} \frac{p-q'}{q} = m \frac{p-q'}{q} \). Moreover, \( \tilde{f} \) achieves \( \tilde{m} \) and \( \tilde{g} \) achieves \( m \).

In order to study the qualitative properties of solutions to (1.7) we take advantage of its Hamiltonian structure. Indeed, the system (1.7) is conservative, and any solution \( g,f \) satisfies

\[
E(g,f) := g' f' - \Gamma g f + \frac{1}{q} |g|^q + \frac{1}{p} |f|^p = \text{constant}. \quad (2.1)
\]
Remark 2.3 Let $g \in W^{2,p'}(\mathbb{R})$, $f \in W^{2,q'}(\mathbb{R})$ be a solution to (1.7). By well known facts about Sobolev spaces, the functions $g, g', f$ and $f'$ are Hölder continuous on $\mathbb{R}$. Thus $g, f \in C^2(\mathbb{R})$. In addition, $g, g', f$ and $f'$ vanish at $\pm \infty$ and hence (2.1) implies
\[
g'f' - \Gamma gf + \frac{1}{q}|g|^q + \frac{1}{p}|f|^p \equiv 0 \quad \text{on } \mathbb{R}.
\] (2.2)

Remark 2.4 Problem (1.7) is equivalent to a $(2 \times 2)$-dimensional first order Hamiltonian system. For $X = (x_1, x_2), Y = (y_1, y_2) \in \mathbb{R}^2$ we set
\[
H(X,Y) = y_1y_2 + A(x_1y_1 - x_2y_2) - (A^2 + \Gamma)x_1x_2 + \frac{1}{q}|x_1|^q + \frac{1}{p}|x_2|^p.
\]
Then $g, f$ solves (1.7) if and only if $X = (g, f), Y = (f' + Af, g' - Ag)$ solves
\[
\begin{cases}
X' = \partial_Y H(X,Y) \\
Y' = -\partial_X H(X,Y).
\end{cases}
\] (2.3)

If $\Gamma \neq 0$ and $\delta := pq - (p + q) > 0$, then $\pm \left(|\Gamma|^{p/\delta}, |\Gamma|^{-q/\delta}\right)$ are equilibrium points for (2.3). Notice that a positive equilibrium exists if and only if $\Gamma > 0$.

From (2.1) we first infer the following a-priori bound on trajectories having null energy.

Proposition 2.5 Let $g, f \in C^2(\mathbb{R})$ be a solution to (1.7) such that $g, g', f$ and $f'$ vanish at infinity. Then
\[
\|g\|_{\infty}^{p-q'} \leq \frac{q}{p'}|\Gamma|^{p'} , \quad \|f\|_{\infty}^{p-q'} \leq \frac{p}{q'}|\Gamma|^{q'}.
\]
In particular, if $\Gamma = 0$ then $g = f \equiv 0$.

Proof. Let $\overline{\sigma} \in \mathbb{R}$ be such that $|g(\overline{\sigma})| = \|g\|_\infty$. Then $g'(\overline{\sigma}) = 0$ and therefore from (2.1) and since $E(g, f) = 0$ we get
\[
\frac{1}{q}\|g\|_\infty^q + \frac{1}{p}|f(\overline{\sigma})|^p = \Gamma f(\overline{\sigma})\|g\|_\infty \leq \frac{|\Gamma|^{p'}}{p'}\|g\|_\infty^{p'} + \frac{1}{p}|f(\overline{\sigma})|^p
\]
by Young’s inequality. The desired a-priori bound on $g$ follows immediately. The estimate on $\|f\|_\infty$ can be obtained in a similar way. \qed
In the remaining part of this section, we study the sign of solutions $g, f$ to (1.7). We distinguish the case $\Gamma > 0$ from the case when $\Gamma$ is nonpositive.

**Theorem 2.6** Let $g, f \in C^2(\mathbb{R})$ be a solution to (1.7), such that $g$ and $f$ vanish at $\pm\infty$ together with their derivatives. If $\Gamma > 0$ then $g \equiv f \equiv 0$ or $gf > 0$ on $\mathbb{R}$.

**Proof.** We start by noticing that $g, f$ satisfies (2.2). In a moment we will prove the following

Claim: $g(s)f(s) \neq 0$ for any $s \in \mathbb{R}$.

Assume that the claim is proved. Then both $g$ and $f$ have constant sign. The function $g$ has at least one critical point $s$. By (2.2), it holds that

$$-\Gamma g(s)f(s) + \frac{1}{q}|g(s)|^q + \frac{1}{p}|f(s)|^p = 0.$$ 

Thus $g(s)f(s) > 0$, and therefore $gf > 0$ everywhere in $\mathbb{R}$, that concludes the proof of the theorem.

It remains to prove the claim. Notice that from (2.2) the following facts follow:

1. If $g'(\xi)f'(\xi) = 0$ then $f(\xi) = g(\xi) = 0$ or $f(\xi)g(\xi) > 0$ (2.4)
2. If $g(\xi)f(\xi) = 0$ then $f(\xi) = g(\xi) = 0 = f'(\xi)g'(\xi)$ or $f'(\xi)g'(\xi) < 0$. (2.5)

By contradiction, assume that $g$ vanishes somewhere. Up to a change of sign and/or inversion $s \mapsto -s$ we can assume that $g$ attains its negative minimum at some $s_1 \in \mathbb{R}$ and that $g$ reaches 0 in $(s_1, \infty)$. Let $s_2$ be the first zero of $g$ in $(s_1, \infty)$. Thus $g < 0$ on $[s_1, s_2)$, $f(s_1) < 0$ by (2.4), and $g'(s_2) \geq 0$. In addition,

$$f'(\bar{s}) = 0 \text{ for some } \bar{s} \in [s_1, s_2), \text{ then } f(\bar{s}) < 0,$$ (2.6)

because of (2.4). Now we prove that

$$g'(s_2)f'(s_2) = 0, \quad f(s_2) = 0, \quad f < 0 \text{ on } [s_1, s_2).$$ (2.7)

If $g'(s_2) = 0$ then (2.7) readily follows from (2.4) and (2.6). If $g'(s_2) > 0$ and $f'(s_2) = f(s_2) = 0$ then (2.6) immediately implies (2.7). In view of (2.5), to conclude the proof of (2.7) we only have to exclude that $g'(s_2) > 0 > f'(s_2)$. We argue by
contradiction. If \( f'(s_2) < 0 \) then \( f(s_2) < 0 \) by (2.6). Since \( g \) is increasing in a neighborhood of \( s_2 \) and since \( g \) decays at infinity, there is a point \( s_3 > s_2 \) such that \( g'(s_3) = 0 \) and \( g > 0 \) on \((s_2, s_3] \). But then \( f(s_3) > 0 \) by (2.4). Since \( f(s_2), f'(s_2) \) are negative, we infer that \( f \) has a minimum \( s_4 \in (s_2, s_3) \), with \( f(s_4) < 0 \). But then \( g(s_4) < 0 \) by (2.4), which is impossible. Thus (2.7) is proved.

In conclusion, we have that \( g, f \) solves the system

\[
\begin{cases}
g'' - 2A g' - \Gamma g = -|f|^{p-2} f \geq 0 & \text{in } (s_1, s_2) \\
f'' + 2A f' - \Gamma f = -|g|^{q-2} g \geq 0 & \text{in } (s_1, s_2) \\
g, f < 0 & \text{in } (s_1, s_2) \\
g(s_2) = f(s_2) = g'(s_2) = f'(s_2) = 0,
\end{cases}
\]

that contradicts the Hopf boundary point lemma. The claim and the theorem are completely proved. \( \square \)

The condition \( \Gamma > 0 \) is also necessary to have the existence of positive solutions vanishing at \( \pm \infty \). In view of Remark 2.3, the next proposition applies in particular to solutions \( g \in W^{2,p'}(\mathbb{R}), f \in W^{2,q'}(\mathbb{R}) \).

**Proposition 2.7** Let \( g, f \in C^2(\mathbb{R}) \) be a solution to (1.7), such that \( g \) and \( f \) vanish at \( \pm \infty \) together with their derivatives. If \( \Gamma \leq 0 \) and \( gf \geq 0 \) on \( \mathbb{R} \) then \( g \equiv f \equiv 0 \).

**Proof.** The trajectory \( g, f \) has null energy, that is, (2.2) holds. In particular, at any critical point \( \bar{s} \) of \( g \) one has that \( |\Gamma| g(\bar{s}) f(\bar{s}) + 1/b |g(\bar{s})|^q + 1/p |f(\bar{s})|^p = 0 \). Thus both \( g \) and \( f \) vanish at \( \bar{s} \). In particular, \( \min g = \max g = 0 \), and the conclusion follows. \( \square \)

We conclude this section with two more nonexistence results in case \( \Gamma \leq 0 \).

**Theorem 2.8** Assume that \( g, f \in C^2(\mathbb{R}) \) solves (1.7) for some \( A \in \mathbb{R}, \Gamma \leq 0 \) and \( p, q \) satisfying (1.3). In addition, assume that

\[
g(-\infty) = c_g \in [0, \infty), \quad f(-\infty) = c_f \in [0, \infty), \quad g \geq 0 \text{ and } f \geq 0 \quad \text{on } \mathbb{R}.
\]

Then \( g \equiv f \equiv 0 \).

**Proof.** First of all we notice that \( g, f \) can not be a nontrivial pair of constant functions by Remark 2.4.
The function \( h := -f' - 2Af \) is increasing in \( \mathbb{R} \), as \( h' = g(s)^{q-1} - \Gamma f \geq 0 \). Thus it has a limit as \( s \to -\infty \). Hence, also \( f' \) has a limit as \( s \to -\infty \). Clearly

\[
f'(\infty) = 0, \tag{2.8}
\]

and therefore from (1.7) we get also

\[
-f''(\infty) = -\Gamma c_f + c_g^{q-1} \geq 0. \tag{2.9}
\]

In a similar way we get that

\[
g'(\infty) = 0, \quad -g''(\infty) = -\Gamma c_g + c_f^{p-1} \geq 0. \tag{2.10}
\]

In particular, from (2.1) and (2.8), (2.10) we infer that

\[
g'f' - \Gamma gf + \frac{1}{q} |g|^q + \frac{1}{p} |f|^p = -\Gamma cgc_f + \frac{1}{q} c_g^q + \frac{1}{p} c_f^p \quad \text{on} \quad \mathbb{R}. \tag{2.11}
\]

**Claim 1:** If \( c_g = c_f = 0 \) then \( g \equiv f \equiv 0 \).

To prove the claim, we notice that the trajectory \( g, f \) satisfies (2.2). If we assume by contradiction that \( g \) or \( f \) do not vanish identically, then there exists \( s_0 \in \mathbb{R} \) such that \( g'(s_0)f'(s_0) < 0 \). To fix ideas, assume that \( f'(s_0) < 0 \). Since \( f \geq 0 \) and \( f(s) \to 0 \) as \( s \to -\infty \), it means that \( f \) must have a positive local maximum \( s_1 < s_0 \). At the point \( s_1 \) the conservation law (2.2) gives

\[
-\Gamma g(s_1)f(s_1) + \frac{1}{q} |g(s_1)|^q + \frac{1}{p} |f(s_1)|^p = 0,
\]

which contradicts \( f(s_1) > 0 \). The claim is proved.

**Claim 2** If \( A \leq 0 \) then \( \Gamma c_f = 0 \) and \( c_g = 0 \).

By contradiction, assume that \( -\Gamma c_f + c_g^{q-1} > 0 \). Then the function \( f \) is strictly concave and decreasing in a neighborhood of \( -\infty \) by (2.9) and (2.8). Thus in particular \( c_f > 0 \), and therefore form the conservation law we get

\[
g'f' - \Gamma gf + \frac{1}{q} |g|^q + \frac{1}{p} |f|^p \geq \frac{1}{p} c_f^p > 0 \quad \text{on} \quad \mathbb{R}. \tag{2.11}
\]

Since \( f \) is bounded from below, it can not be strictly concave on \( \mathbb{R} \). We claim that \( f \) can never be locally convex. Assume that there exists \( s_0 \in \mathbb{R} \) such that \( f''(s_0) > 0 \). Then from (1.7) we have that

\[
-2Af''(s_0) > -\Gamma f(s_0) + g(s_0)^{q-1} \geq 0. \quad \text{Thus,} \quad A < 0
\]
and \( f'(s_0) > 0 \). Since \( f'(s) < 0 \) for \( s << 0 \), then the function \( f \) must have a local minimum \( s_1 \in (-\infty, s_0) \). Thus \( f'(s_1) = 0 \) and \( f''(s_1) \geq 0 \). But then

\[
0 \geq -f''(s_1) = -\Gamma f(s_1) + g(s_1)^{q-1} \geq 0,
\]

which implies \( \Gamma f(s_1) = g(s_1) = 0 \). In particular, \( g'(s_1) = 0 \), and \( g''(s_1) \geq 0 \), since \( s_1 \) is a minimum for \( g \) thanks to the assumption that \( g \geq 0 \). Thus, (1.7) gives

\[
0 \geq -g''(s_1) = f(s_1)^{p-1} \geq 0.
\]

Thus \( f(s_1) = 0 \), contradicting (2.11).

We have proved that \( f'' \leq 0 \) on \( \mathbb{R} \). Thus there exists \( s_0 \in \mathbb{R} \) such that \( f \) is a nonnegative constant on \( [s_0, \infty) \). But then from (1.7) we infer that \( f \equiv g \equiv 0 \) on \( [s_0, \infty) \), as \( \Gamma \leq 0 \). We have reached again a contradiction with (2.11), and the claim is proved.

**Claim 3** If \( A \geq 0 \) then \( \Gamma c_g = 0 \) and \( c_f = 0 \)

It is sufficient to exchange the roles of \( g \) and \( f \), and argue as in Claim 2.

Now we are in position to conclude the proof. By Claim 1, we only have to show that \( c_g = c_f = 0 \). Thus we are done if \( A = 0 \), thanks to Claims 2 and 3. We have to study the case

\[
A < 0, \quad \Gamma = c_g = 0
\]

and the case \( A > 0, \ \Gamma = 0 = c_f \), that can be handled in a similar way. Assume that (2.12) holds. Since \( g \) solves \( -g'' + 2Ag' = f^{p-1} \geq 0 \), then the function \( -g' + 2Ag \) is non decreasing on \( \mathbb{R} \). Hence \( -g' + 2Ag \geq 0 \) by (2.10) and since \( c_g = 0 \). Thus \( g' \leq 2Ag \leq 0 \) on \( \mathbb{R} \) that is, \( g \equiv 0 \) because it is non increasing and nonnegative. The proof is complete.

Since the system (1.7) is invariant with respect to inversion \( s \mapsto -s \), then clearly the next result holds as well.

**Theorem 2.9** Assume that \( g, f \in C^2(\mathbb{R}) \) solves (1.7) for some \( A \in \mathbb{R} \), \( \Gamma \leq 0 \) and \( p, q \) satisfying (1.3). In addition, assume that

\[
g(\infty) = c_g \in [0, \infty), \quad f(\infty) = c_f \in [0, \infty), \quad g \geq 0 \text{ and } f \geq 0 \text{ on } \mathbb{R}.
\]

Then \( g \equiv f \equiv 0 \).
We conclude this section with a result that holds in case $p = 2 < q$.

**Theorem 2.10** Let $q \in (2, \infty)$ and assume that $A^2 + \Gamma \geq 0$, $\Gamma \neq 0$. Up to translations in $\mathbb{R}$, composition with the inversion $s \mapsto -s$ and change of sign, the system

$$
\begin{cases}
-g'' + 2Ag' + \Gamma g = f & \text{on} \ \mathbb{R} \\
-f'' - 2Af' + \Gamma f = |g|^{q-2}g & \text{on} \ \mathbb{R}.
\end{cases}
$$

has a unique nontrivial solution $(g, f)$ such that $g \in H^2(\mathbb{R})$ and $f \in W^{2,q}(\mathbb{R})$. Moreover, $g$ is even, positive and strictly decreasing on $(0, \infty)$, and $f$ is positive if and only if $\Gamma > 0$.

**Proof.** Existence is given by Theorem 2.1. Notice that $g$ is smooth and solves

$$g''' - 2(2A^2 + \Gamma)g'' + \Gamma^2 g = |g|^{q-2}g. \tag{2.13}$$

On the other hand, since $(2A^2 + \Gamma)^2 \geq \Gamma^2$, then Theorem 2.2 in [2] implies that (2.13) has a unique solution $g$ (up to the above transforms), that can be taken to be positive, even and strictly decreasing on $(0, \infty)$. The uniqueness of $f$ is immediate. The last statement concerning the sign of $f$ follows by Theorem 2.6 and Proposiion 2.7. \hfill \Box

**Remark 2.11** Clearly, $f$ is even if and only if $A = 0$.

### 3 The Hénon-Lane-Emden system

In this section we provide conditions for the existence of solutions to (1.1) in suitable energy spaces and for the nonexistence of positive solutions having certain behavior at 0 or at $\infty$.

We start by introducing some weighted Sobolev spaces. Let $\theta \in (1, \infty)$ and $\alpha \in \mathbb{R}$ be given, such that $\alpha \notin \{2\theta - n, np - n\}$. Then we can use the results in [28] to define the Banach space $D^{2,\theta}_r(\mathbb{R}^n; |x|^\alpha dx)$ as the completion of radial functions in $C^2_0(\mathbb{R}^n \setminus \{0\})$ with respect to the norm

$$
\|u\|_\alpha = \left( \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^{\theta} dx \right)^{1/\theta}.
$$
To any pair of radial functions $u, v \in C^2(\mathbb{R}^n \setminus \{0\})$, we associate the pair $g, f \in C^2(\mathbb{R})$ defined by

$$u(x) = |x|^{-\lambda_1} g \left( - \log |x| \right), \quad v(x) = |x|^{-\lambda_2} f \left( - \log |x| \right),$$

where

$$\lambda_1 = \frac{b + n}{q}, \quad \lambda_2 = \frac{a + n}{p}.$$  

We will always assume that $(p, q)$ belongs to the critical hyperbola in $(1.2)$, that is, $\lambda_1 + \lambda_2 = n - 2$.

We introduce also the constants

$$\Gamma = \frac{n + a}{p} \frac{n + b}{q} = \lambda_1 \lambda_2, \quad A = \frac{n - 2}{2} - \lambda_1 = - \frac{n - 2}{2} + \lambda_2.$$  

Notice that

$$A^2 + \Gamma = \left( \frac{n - 2}{2} \right)^2 \geq 0.$$  

A direct computation shows that a radial pair $u, v$ solves $(1.1)$ on $\mathbb{R}^n \setminus \{0\}$ if and only if $g, f$ solves $(1.7)$ with $\Gamma, A$ given by $(3.2)$. Thanks to the results in previous section we first get the next existence theorem.

**Theorem 3.1** Let $n \geq 2$, $a, b \in \mathbb{R} \setminus \{-n\}$ and $p, q > 1$. Assume that $(1.5)$ and $(1.2)$ are satisfied. Then the Hénon-Lane-Emden system $(1.1)$ has a radially symmetric solution

$$u \in \mathcal{D}^{2,p'}_r(\mathbb{R}^n; |x|^\frac{b-p}{p-1} dx), \quad v \in \mathcal{D}^{2,q'}_r(\mathbb{R}^n; |x|^\frac{b-q}{q-1} dx).$$

In addition, $u, v$ satisfies $(1.3)$.

**Proof.** Define $\Gamma, A$ as in $(3.2)$, and notice that $\Gamma \neq 0$, and $A^2 + \Gamma \geq 0$. By Theorem 2.1, we see that there exists $f \in W^{2,p'}(\mathbb{R})$ and $g \in W^{2,q'}(\mathbb{R})$ satisfying $(1.7)$. Now using the Emden-Fowler transformation in $(3.1)$ and the results in [28], we get $u, v$ satisfies $(3.3)$ and solves $(1.1)$. The conclusion readily follows since

$$|x|^{\frac{b-p}{p-1}} u(x) = g \left( - \log |x| \right), \quad |x|^{\frac{b-q}{q-1}} v(x) = f \left( - \log |x| \right)$$

and since $g, f$ vanish at $\pm \infty$.  \[\Box\]
**Theorem 3.2** Let \( n \geq 2 \), \( a, b \in \mathbb{R} \) and \( p, q > 1 \). Assume that (1.3) and (1.2) are satisfied. Let \( u, v \in C^2(\mathbb{R}^n \setminus \{0\}) \) be a radially symmetric solution to (1.1) on \( \mathbb{R}^n \setminus \{0\} \).

i) If \( a > -n \), \( b > -n \) and if \( u, v \) satisfies (1.5), then \( u \equiv v \equiv 0 \) or \( uv > 0 \) on \( \mathbb{R} \).

ii) Assume that (1.6) holds. If \( a \leq -n \) or \( b \leq -n \) and if \( u \geq 0, v \geq 0 \) then \( u \equiv v \equiv 0 \).

**Proof.** Define \( A, \Gamma \) and use the Emden-Fowler transform \((u, v) \mapsto (g, f)\) as before. Notice that \( \Gamma > 0 \) in case i) and \( \Gamma \leq 0 \) in case ii). Then apply Theorem 2.8 and Theorem 2.6.

In the next corollary we emphasize the impact of Theorem 3.2 in case \( n = 2 \), when Theorem 3.1 gives existence on the critical hyperbola whenever \( a, b \neq -2 \).

**Corollary 3.3** Let \( n = 2 \) and \( p, q > 1 \). Assume that (1.3) and (1.2) are satisfied, and in addition assume that \( a, b \neq -2 \). Let \( u, v \in C^2(\mathbb{R}^2 \setminus \{0\}) \) be a radially symmetric and nonnegative solution to (1.1) satisfying (1.6). Then \( u \equiv v \equiv 0 \).

In Theorem 3.2 we saw that the sign of \( \Gamma \) affects the sign of the product \( uv \). However, at least in case \( p = 2 \), the function \( u \) never changes sign, also in case \( \Gamma < 0 \). The next result for problem

\[
\begin{align*}
-\Delta u &= |x|^a v \\
-\Delta v &= |x|^b |u|^{q-2} u.
\end{align*}
\tag{3.4}
\]

is an immediate consequence to Theorem 2.10.

**Theorem 3.4** Let \( n \geq 2 \), \( a, b \in \mathbb{R} \) and \( q > 1 \). Assume that \( a, b \neq -n \) and

\[
\frac{a+n}{2} + \frac{b+n}{q} = n - 2.
\]

is satisfied. Up to dilations, compositions with the Kelvin transform and change of sign, problem (3.4) has a unique nontrivial radial solution \( u \in \mathcal{D}_c^{2,2}(\mathbb{R}^n; |x|^{-a} dx), v \in \mathcal{D}_c^{2,q}(\mathbb{R}^n; |x|^{-q} dx) \). Moreover, \( u \) is positive, and \( v \) is positive if and only if \( a, b > -n \).
Appendix: a non-radial approach

Following Wang [37] and Calanchi-Ruf [7], we notice that (1.1) is formally equivalent to the fourth order equation

$$\Delta(|x|^\alpha|\Delta u|^{\theta-2}\Delta u) = |x|^b|u|^{\theta-2}u \quad \text{(A.1)}$$

where $\theta = p' = \frac{p}{p-1}$ and $\alpha = -\frac{a}{p-1}$. Equation (A.1) is variational. In particular, its nontrivial solutions can be found as critical points for the functional

$$u \mapsto \frac{\int_{\mathbb{R}^n} |x|^\alpha|\Delta u|^\theta dx}{\left(\int_{\mathbb{R}^n} |x|^b|u|^q dx\right)^{\theta/q}}$$

on a suitable function space. Let us introduce the weighted Rellich constant

$$\mu_{\theta}(\alpha) := \inf_{u \in C^2_0(\mathbb{R}^n \setminus \{0\}) \atop u = u(|x|), u \neq 0} \frac{\int_{\mathbb{R}^n} |x|^\alpha|\Delta u|^\theta dx}{\int_{\mathbb{R}^n} |x|^\alpha |u|^{\theta q} dx}.$$  \hspace{1cm} (A.2)

The best constant $\mu_{\theta}(\alpha)$ is explicitly known in few cases. We define $\Gamma$ as in (3.2) and we notice that

$$\Gamma = \left(\frac{n + \alpha}{\theta} - 2\right) \left(\frac{n - \alpha}{\theta}\right)$$ \hspace{1cm} (A.3)

if (1.2) is satisfied. The value of $\mu_2(\alpha)$ (case $\theta = 2$) is known from [13], [8]:

$$\mu_2(\alpha) = \min_{k \in \mathbb{N} \cup \{0\}} |\Gamma + k(n - 2 + k)|^2.$$  

For general $\theta > 1$, Mitidieri proved in [27] that $\mu_{\theta}(\alpha) = |\Gamma|^\theta$, provided that $\Gamma \in \mathbb{R}$.

From now on, we assume that $\mu_\theta(\alpha) > 0$. Then we can define the space $\mathcal{D}^{2,\theta}(\mathbb{R}^n; |x|^\alpha dx)$ as the closure of functions in $C^2_0(\mathbb{R}^n \setminus \{0\})$ with respect to the norm

$$\|u\|^\theta = \int_{\mathbb{R}^n} |x|^\alpha|\Delta u|^\theta dx.$$  

**Lemma A.1** Let $\theta > 1$, $\alpha \in \mathbb{R}$ be given, such that $\mu_{\theta}(\alpha) > 0$. Let $q \geq \theta$ and assume that $q \leq \theta^\alpha := \frac{\theta a}{n-2}$ if $n > 2\theta$. Then there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^n} |x|^\alpha|\Delta u|^\theta dx \leq c \left(\int_{\mathbb{R}^n} |x|^{-n+q - \frac{2q}{\theta} + \alpha} |u|^q dx\right)^{\theta/q}$$

for any $u \in \mathcal{D}^{2,\theta}(\mathbb{R}^n; |x|^\alpha dx)$.  

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Proof. If \( n > 2\theta \) the conclusion readily follows via interpolation with the Sobolev inequality. For a proof in lower dimensions, we use the Emden-Fowler transform \( T : C_c^2(\mathbb{R}^n \setminus \{0\}) \to C_c^2(\mathbb{R} \times S^{n-1}) \), \( T : u \mapsto g \) defined via

\[
u(x) = |x|^{\frac{2\theta-n-\alpha}{\theta}} g \left( -\log |x|, \frac{x}{|x|} \right).
\]

We denote by \( \Delta_\sigma \) the Laplace-Beltrami operator on \( S^{n-1} \) and by and \( g'', g' \) the derivatives of \( g = g(s, \sigma) \) with respect to \( s \in \mathbb{R} \). By direct computation one has that

\[
\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta u|^{\theta} \, dx = \int_{\mathbb{R}} \int_{S^{n-1}} |\Delta_\sigma g + g'' - 2Ag' - \Gamma g|^{\theta} \, d\sigma ds
\]

\[
\int_{\mathbb{R}^n} |x|^{\alpha-2\theta} |u|^{\theta} \, dx = \int_{\mathbb{R}} \int_{S^{n-1}} |g|^{\theta} \, d\sigma ds,
\]

where \( \Gamma \) is given by (A.3) and \( A = \frac{2(\theta-\alpha)+n(\theta-2)}{2g} \). Thus, using the assumption \( \mu_\theta(\alpha) > 0 \), one proves that

\[
\|g\|^\theta := \int_{\mathbb{R}} \int_{S^{n-1}} |\Delta_\sigma g + g'' - 2Ag' - \Gamma g|^{\theta} \, d\sigma ds
\]

is an equivalent norm on \( W^{2,\theta}(\mathbb{R} \times S^{n-1}) \). Therefore, \( T \) can be regarded as an isometry between Banach spaces, and the conclusion readily follows by using the Sobolev embedding \( W^{2,\theta}(\mathbb{R} \times S^{n-1}) \hookrightarrow L^q(\mathbb{R} \times S^{n-1}) \). \( \square \)

Under the assumptions in Lemma [A.1] we have that the infimum

\[
S_{\theta,q}(\alpha) := \inf_{u \in D^{2,\theta}(\mathbb{R}^n), \int_{|x|^\alpha dx} u \neq 0} \frac{\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta u|^{\theta} \, dx}{\left( \int_{\mathbb{R}^n} |x|^{-n+\frac{q-n-2\theta+\alpha}{\theta}} |u|^{q} \, dx \right)^{\theta/q}},
\]

is positive. Notice that for \( n > 2\theta \), \( \alpha = 0 \) and \( q = \theta^{**} \) we have that

\[
S_{\theta,\theta^{**}}(0) = S^{**}(\theta) := \inf_{u \in D^{2,\theta}(\mathbb{R}^n), \int_{|x|^{\theta} dx} u \neq 0} \frac{\int_{\mathbb{R}^n} |\Delta u|^{\theta} \, dx}{\left( \int_{\mathbb{R}^n} |u|^{\theta^{**}} \, dx \right)^{\theta/\theta^{**}}},
\]

which is the best constant in the Sobolev embedding \( D^{2,\theta}(\mathbb{R}^n) \hookrightarrow L^{\theta^{**}}(\mathbb{R}^n) \), see [1], [36]. The next existence results can be proved, for instance, by using the techniques in [8] (proof of Theorem 1.2). We omit details.
Theorem A.2 Let $\theta > 1$, $\alpha \in \mathbb{R}$ be given, in such a way that the infimum in (A.1) is positive. Let $q > \theta$.

i) Assume that $n \geq 3$ and $q < \theta^\ast$ if $n > 2\theta$. Then $S_{\theta,q}(\alpha)$ is achieved.

ii) If $n > 2\theta$ and $S_{\theta,\theta^\ast}(\alpha) < S^\ast(\theta)$ then $S_{\theta,\theta^\ast}(\alpha)$ is achieved.

Remark A.3 Thanks to the results in [27] we know that $\mu_\theta(\alpha) = |\Gamma|^\theta > 0$, whenever $\Gamma > 0$. We suspect that in this case the infimum $S_{\theta,q}(\alpha)$ is always achieved by radial functions. We leave this as an open problem.

From Theorem A.2 one can easily infer sufficient conditions for the existence of (minimal energy) solutions to the Hénon-Lane-Emden system (1.1), whenever $\mu_p'(\alpha) > 0$. More can be said when $p = 2$. From [18], [8] we know that $\mu_2(-a) > 0$ if and only if $-\Gamma$ is not an eigenvalue of the Laplace-Beltrami operator on the sphere, where now

$$
\Gamma = \left(\frac{n + a}{2}\right) \left(\frac{n + b}{q}\right), \quad \frac{a + n}{2} + \frac{b + n}{q} = n - 2.
$$

From now on we assume that

$$
-\left(\frac{n + a}{2}\right) \left(\frac{n + b}{q}\right) \neq k(n - 2 + k) \text{ for any integer } k \geq 0. \quad (A.4)
$$

By ground state solutions to (3.4) we mean solutions $u,v$ such that $u$ achieves the infimum

$$
\inf_{u \in \mathcal{D}^{2,2}(\mathbb{R}^n;|x|^{-a} dx), u \neq 0} \frac{\int_{\mathbb{R}^n} |x|^{-a} |\Delta u|^2 \, dx}{\left(\int_{\mathbb{R}^n} |x|^b |u|^q \, ds\right)^{2/q}}.
$$

For convenience of the reader we summarize here the main results for (3.4) that can be obtained as immediate corollaries of the results in [9].

Theorem A.4 Let $q > 2$ and assume that $q < 2^\ast := \frac{2n}{n-1}$ if $n \geq 5$.

i) If (A.4) holds, then (3.4) has a ground state solution $\overline{u},\overline{v}$.

ii) For every $q > 2$ and for every integer $k \geq 1$ there exists $\delta > 0$ such that if

$$
0 < |\Gamma + k(n - 2 + k)| < \delta
$$

then $\overline{u}$ is not radially symmetric. Thus problem (1.7) has at least two distinct weak solutions.
iii) If $|\Gamma| > \frac{n-1}{q-2} \left(1 + \sqrt{q-1}\right)$ then $\pi$ is not radially symmetric. Thus problem (1.1) has at least two distinct weak solutions.

iv) Assume that $-\Gamma > \frac{n-1}{2}$. Then there exists $q_\alpha > 2$ such that no ground state solution to (3.4) can be positive.

In the limiting case $n \geq 5$ and $q = 2^*$ the problem is more difficult. We limit ourselves to point out some corollaries to the results in [9] in case $n \geq 6$.

**Theorem A.5** Assume $n \geq 6$ and that (A.4) is satisfied. If in addition $|a+2| > 2$, then the problem

\[
\begin{cases}
-\Delta u = |x|^a v & \text{on } \mathbb{R}^n \\
-\Delta v = |x|^b |u|^\frac{n}{n-4} u & \text{on } \mathbb{R}^n.
\end{cases}
\]

(A.5)

has a ground state solution $\bar{u},\bar{v}$. Moreover, the conclusions ii)–iv) in Theorem A.4 still hold. In particular, (A.5) has a radial and a non-radial weak solutions.

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