A Sylvester-Type Matrix Equation over the Hamilton Quaternions with an Application

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Abstract: We derive the solvability conditions and a formula of a general solution to a Sylvester-type matrix equation over Hamilton quaternions. As an application, we investigate the necessary and sufficient conditions for the solvability of the quaternion matrix equation, which involves \( \eta \)-Hermicity. We also provide an algorithm with a numerical example to illustrate the main results of this paper.

Keywords: matrix equation; Hamilton quaternion; \( \eta \)-Hermitian matrix; Moore–Penrose inverse; rank

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1. Introduction

Let \( \mathbb{R} \) stand for the real number field and

\[ \mathbb{H} = \{ u_0 + u_1i + u_2j + u_3k | i^2 = j^2 = k^2 = ijk = -1, \ u_0, u_1, u_2, u_3 \in \mathbb{R} \}. \]

\( \mathbb{H} \) is called the Hamilton quaternion algebra, which is a non-commutative division ring. Hamilton quaternions and Hermitian quaternion matrices have been utilized in statistics of quaternion random signals [1], quaternion matrix optimization problems [2], signal and color image processing, face recognition [3,4], and so on.

Sylvester and Sylvester-type matrix equations have a large number of applications in different disciplines and fields. For example, the Sylvester matrix equation

\[ A_1X + XB_1 = C_1 \]  

and the Sylvester-type matrix equation

\[ A_1X + YB_1 = C_1 \]

have been applied in singular system control [5], system design [6], perturbation theory [7], sensitivity analysis [8], \( H_\alpha \)-optimal control [9], linear descriptor systems [10], and control theory [11]. Roth [12] gave the Sylvester-type matrix Equation (2) for the first time over the polynomial integral domain. Baksalary and Kala [13] established the solvability conditions for Equation (2) and gave an expression of its general solution. In addition, Baksalary and Kala [14] derived the necessary and sufficient conditions for a two-sided Sylvester-type matrix equation

\[ A_{11}X_1B_{11} + C_{11}X_2D_{11} = E_{11} \]

to be consistent. Özgüler [15] studied (3) over a principal ideal domain. Wang [16] investigated (3) over an arbitrary regular ring with an identity element.

Due to the wide applications of quaternions, the investigations on Sylvester-type matrix equations have been extended to \( \mathbb{H} \) in the last decade (see, e.g., [17–24]). They are
applied for signal processing, color-image processing, and maximal invariant semidefinite or neutral subspaces, etc. (see, e.g., [25–28]). For instance, the general solution to Sylvester-type matrix Equation (2) can be used in color-image processing. He [29] derived the matrix Equation (2) as an essential finding. Roman [25] established the necessary and sufficient conditions for Equation (1) to have a solution. Kychei [30] investigated Cramer’s rules to drive the necessary and sufficient conditions for Equation (3) to be solvable. As an extension of Equations (2) and (3), Wang and He [31] gave the solvability conditions and the general solution when it is solvable. It is clear that Equation (5) provides a proper generalization of Equation (4), and we carry out an algorithm with a numerical example to calculate the general solution of Equation (5). As a special case of Equation (5), we also obtain the solvability conditions and the general solution to the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1$$

over the complex number field \( \mathbb{C} \), which can be generalized to \( \mathbb{H} \) and applicable in some Sylvester-type matrix equations over \( \mathbb{H} \) (see, e.g., [29,32]).

We know that in system and control theory, the more unknown matrices that a matrix equation has, the wider its application will be. Consequently, for the sake of developing theoretical studies and the applications mentioned above of Sylvester-type matrix equation and their generalizations, in this paper, we aim to establish some necessary and sufficient conditions for the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4 = B$$

(5)
to have a solution in terms of the rank equalities and Moore–Penrose inverses of some coefficient quaternion matrices in Equation (5) over \( \mathbb{H} \). We derive a formula of its general solution when it is solvable. It is clear that Equation (5) provides a proper generalization of Equation (4), and we carry out an algorithm with a numerical example to calculate the general solution of Equation (5). As a special case of Equation (5), we also obtain the solvability conditions and the general solution for the two-sided Sylvester-type matrix equation

$$A_{11}Y_1B_{11} + A_{22}Y_2B_{22} + A_{33}Y_3B_{33} = T_1.$$  

(6)

To the best of our knowledge, so far, there has been little information on the solvability conditions and an expression of the general solution to Equation (6) by using generalized inverses.

As usual, we use \( A^\dagger \) to denote the conjugate transpose of \( A \). Recall that a quaternion matrix \( A \), for \( \eta \in \{i, j, k\} \), is said to be \( \eta \)-Hermitian if \( A = A^\eta \), where \( A^\eta = -\eta A^\dagger \eta \) [33]. For more properties and information on \( \eta \)-quaternion matrices, we refer to [33]. We know that \( \eta \)-Hermitian matrices have some applications in linear modeling and statistics of quaternion random signals [1,33]. As an application of Equation (5), we establish some necessary and sufficient conditions for the quaternion matrix equation

$$A_1X_1 + (A_1X_1)^\eta + A_2Y_1A_2^\eta + A_3Y_2A_3^\eta + A_4Y_3A_4^\eta = B$$

(7)
to be consistent. Moreover, we derive a formula of the general solution to Equation (7) where \( B = B^\eta \), \( Y_i = Y_i^\eta \) \((i = 1,3)\) over \( \mathbb{H} \).

The rest of this paper is organized as follows. In Section 2, we review some definitions and lemmas. In Section 3, we establish some necessary and sufficient conditions for Equation (5) to have a solution. In addition, we give an expression of its general solution to Equation (5) when it is solvable. In Section 4, as an application of Equation (5), we consider some solvability conditions and the general solution to Equation (7), where \( Y_i = Y_i^\eta \) \((i = 1,3)\). Finally, we give a brief conclusion to the paper in Section 5.

2. Preliminaries

Throughout this paper, \( \mathbb{H}^{m \times n} \) stands for the space of all \( m \times n \) matrices over \( \mathbb{H} \). The symbol \( r(A) \) denotes the rank of \( A \). \( I \) and \( 0 \) represent an identity matrix and a zero matrix of appropriate sizes, respectively. In general, \( A^\dagger \) stands for the Moore–Penrose inverse.
of $A \in \mathbb{H}^{l \times k}$, which is defined as the solution of $AYA = A$, $YAY = Y$, $(AY)^* = AY$ and $(YA)^* = YA$. Moreover, $L_A = I - A^*A$ and $R_A = I - AA^*$ represent two projectors along $A$.

The following lemma is due to Marsaglia and Styan [34], which can be generalized to $\mathbb{H}$.

**Lemma 1** ([34]). Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times k}$, $C \in \mathbb{H}^{l \times n}$, $D \in \mathbb{H}^{l \times k}$ and $E \in \mathbb{H}^{l \times l}$ be given. Then, we have the following rank equality:

$$r \left( \begin{array}{ccc} A & B & BLD \\ RE & C & 0 \\ 0 & 0 & D \end{array} \right) = r \left( \begin{array}{ccc} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{array} \right) - r(D) - r(E).$$

**Lemma 2** ([35]). Let $A \in \mathbb{H}^{m \times n}$ be given. Then,

1. $(A^\eta)^\dagger = (A^\dagger)^\eta$, $(A^\eta)^\dagger = (A^\dagger)^\eta$.
2. $r(A) = r(A^\eta) = r(A^\eta) = r(A^\eta A^\eta) = r(A^\eta A^\eta)$.
3. $(L_A)^\eta = -\eta(L_A) = (L_A)^\eta = L_A^\eta = R_A^\eta$.
4. $(R_A)^\eta = -\eta(R_A) = (R_A)^\eta = R_A = L_A^\eta$.
5. $(AA^\eta)^\eta = (A^\dagger)^\eta A^\eta = (A^\dagger A)^\eta = A^\eta (A^\dagger)^\eta$.
6. $(A^\dagger A)^\eta = A^\eta (A^\dagger)^\eta = (AA^\eta)^\eta = (A^\dagger)^\eta A^\eta$.

**Lemma 3** ([16]). Let $A_{ii}, B_i$ and $C_i$ ($i = 1, 2$) be given matrices with suitable sizes over $\mathbb{H}$, $A_1 = A_{22}L_{A_1}$, $T = R_{B_{11}}B_{22}$, $F = B_{22}L_T$, $G = R_{A_1}A_{22}$. Then, the following statements are equivalent:

1. The system
   $$A_{11}X_{11}B_{11} = C_1, \; A_{22}X_{21}B_{22} = C_2$$
   has a solution.
2. $$A_{ii}A_{ii}^\dagger C_i B_{ii}^\dagger B_{ii} = C_i \; (i = 1, 2)$$

and

$$G(A_{22}C_2B_{22}^\dagger - A_{11}^\dagger C_1B_{11}^\dagger)F = 0.$$

**Lemma 4** ([13]). Let $A_1$, $B_1$ and $C_1$ be given matrices with suitable sizes. Then, the Sylvester-type Equation (2) is solvable if and only if

$$R_{A_1}C_1B_{11} = 0.$$ 

In this case, the general solution to Equation (2) can be expressed as

$$X = A_1^\dagger C_1 - A_1^\dagger U_1B_1 + L_{A_1}U_2, \; Y = R_{A_1}C_1B_{11}^\dagger + A_1A_1^\dagger U_1 + U_3R_{B_1},$$

where $U_1$, $U_2$, and $U_3$ are arbitrary matrices with appropriate sizes.
Lemma 5 ([31]). Let $A_1, B_1, C_3, D_3, C_4, D_4$ and $E_1$ be given matrices over $\mathbb{H}$. Put

$$A = R_{A_1}C_3, \quad B = D_3L_{B_1}, \quad C = R_{A_1}C_4, \quad D = D_4L_{B_1}, \quad E = R_{A_1}E_1L_{B_1}, \quad M = R_A C, \quad N = D L_B, \quad S = C L_M.$$ 

Then, the following statements are equivalent:

(1) Equation (4) has a solution.

(2) \(R_M R_A E = 0, \quad E L_B L_N = 0, \quad R_A E L_D = 0, \quad R_E L_B = 0.\)

(3) \[ r\begin{pmatrix} E_1 & C_4 & C_3 & A_1 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(C_4, C_3, A_1), \]

\[ r\begin{pmatrix} E_1 & A_1 \\ D_3 & 0 \\ D_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} D_3 \\ D_4 \\ B_1 \end{pmatrix} + r(A_1), \]

\[ r\begin{pmatrix} E_1 & C_3 & A_1 \\ D_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_1, C_3) + r\begin{pmatrix} D_4 \\ B_1 \end{pmatrix}, \]

\[ r\begin{pmatrix} E_1 & C_4 & A_1 \\ D_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_1, C_4) + r\begin{pmatrix} D_3 \\ B_1 \end{pmatrix}. \]

In this case, the general solution to Equation (4) can be expressed as

\[ X_1 = A_1^r(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^rT_7B_1 + L_A T_6, \]

\[ X_2 = R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^r + A_1 A_1^rT_7 + T_8 R_{B_1}, \]

\[ X_3 = A^rE^rB^r - A^rC^rM^rE^rB^r - A^rS^rE^rN^rD^rB^r + L_A T_4 + T_5 R_B, \]

\[ X_4 = M^rE^rD^r + S^rE^rN^rD^r + L_M L_S T_1 + L_M L_T R_N + T_3 R_D, \]

where $T_1, \ldots, T_8$ are arbitrary matrices with appropriate sizes over $\mathbb{H}$.

3. Some Solvability Conditions and a Formula of the General Solution

In this section, we establish the solvability conditions and a formula of the general solution to Equation (5). We begin with the following lemma, which is used to reach the main results of this paper.

Lemma 6. Let $A_{11}, B_{11}, C_{11},$ and $D_{11}$ be given matrices with suitable sizes over $\mathbb{H}, \ A_{11} L_{A_{22}} = 0$ and $R_{B_{11}} B_{22} = 0$. Set

$$A_1 = A_{22} L_{A_{11}}, \quad C_{11} = C_2 - A_{22} A_{11}^t C_1 B_{11}^r B_{22}. \quad (9)$$

Then, the following statements are equivalent:

(1) The system (8) is consistent.

(2) \(R_{A_i} C_i = 0, \quad C_i L_{B_{ii}} = 0 \ (i = 1, 2), \quad R_{A_i} C_{11} = 0.\)

(3) \[ A_{ii} A_{ii}^t C_i B_{ii}^r B_{ii} = C_i \ (i = 1, 2), \quad C_1 B_{11}^r B_{22} = A_{11} A_{22}^r C_2. \]
$r(A_{ii}, C_i) = r(A_{ii}), r(B_i) = r(B_{ii})$ (i = 1, 2),
\[
\begin{pmatrix}
C_1 & 0 & A_{11} \\
0 & -C_2 & A_{22} \\
B_{11} & B_{22} & 0
\end{pmatrix}
\]
\[= r(A_{22}) + r(B_{11}).\]

In this case, the general solution to system (8) can be expressed as
\[
X_1 = A_{11}^t C_1 B_{11}^t + L_{A_{11}} A_{22}^t C_2 B_{22}^t + L_{A_{22}} V_1 + V_2 R_{B_{11}} + L_{A_{11}} V_3 R_{B_{22}},
\]
where $V_1, V_2, and V_3$ are arbitrary matrices with appropriate sizes over $\mathbb{F}$.

**Proof.** (1) $\iff$ (2) It follows from Lemma 3 that
\[
G(A_{22}^t C_2 B_{22}^t - A_{11}^t C_1 B_{11}^t)F = 0
\]
$\iff R_{A_i}(A_{11} + A_{22} A_{11}^t A_{11}) A_{22}^t C_2 B_{22}^t - A_{11}^t C_1 B_{11}^t B_{22} = 0$
$\iff R_{A_i} A_{22} A_{11}^t A_{22}^t C_2 B_{22} - A_{22} A_{11}^t C_1 B_{11}^t B_{22} = 0$
$\iff R_{A_i} A_{22} A_{11}^t A_{22}^t C_2 B_{22} - A_{22} A_{11}^t C_1 B_{11}^t B_{22} = 0$
$\iff R_{A_i} A_{22} A_{11}^t A_{22}^t C_2 B_{22} - A_{22} A_{11}^t C_1 B_{11}^t B_{22} = 0$
$\iff R_{A_i} A_{22} A_{11}^t C_2 B_{22} = 0 \iff R_{A_i} C_1 = 0,$

where $G$ and $F$ are given in Lemma 3.

(1) $\Rightarrow$ (3) If the system (8) has a solution, then there exists a solution $X_0$ such that
\[
A_{11} X_0 B_{11} = C_1, A_{22} X_0 B_{22} = C_2.
\]

It is easy to show that
\[
R_{A_{ii}} C_i = 0, C_i L_{B_{ii}} = 0 (i = 1, 2).
\]
Thus, $A_{ii} A_{ii}^t C_i B_{ii}^t = C_i (i = 1, 2)$. It follows from $R_{B_{ii}} B_{22} = 0, A_{11} L_{A_{22}} = 0$ that
\[
C_1 B_{11}^t B_{22} = A_{11} X_0 B_{11} B_{11}^t B_{22} = A_{11} X_0 B_{22} = A_{11} A_{22}^t A_{22} X_0 B_{22} = A_{11} A_{22}^t C_2.
\]

(3) $\Rightarrow$ (2) Since $A_{22} - A_1 = A_{22} A_{11}^t A_{11}$ and $C_1 B_{11}^t B_{22} = A_{11} A_{22}^t C_2$, we have that
\[
R_{A_i} C_1 = R_{A_i} C_2 - R_{A_i} A_{22} A_{11}^t C_1 B_{11}^t B_{22} = R_{A_i} C_2 - R_{A_i} A_{22} A_{11}^t A_{11} A_{22} C_2
\]
\[= R_{A_i} C_2 - R_{A_i} (A_{22} - A_1) A_{22} A_{22}^t C_2 = R_{A_i} C_2 - R_{A_i} A_{22} A_{22}^t C_2 = 0.
\]

(2) $\iff$ (4) It follows from $R_{B_{ii}} B_{22} = 0$ and $A_{11} L_{A_{22}} = 0$ that
\[
r(B_{22}, B_{11}) = r(B_{11}), \begin{pmatrix} A_{11} \\ A_{22} \end{pmatrix} = r(A_{22}).
\]
By Lemma 1,

\[ R_{A_i} C_i = 0 \iff r(R_{A_i} C_i) = 0 \iff r(A_{ii}, C_i) = r(A_{ii}) \ (i = 1, 2), \]
\[ C_i L_{B_{ii}} = 0 \iff r(C_i L_{B_{ii}}) = 0 \iff r(B_{ii}) \ (i = 1, 2), \]
\[ R_{A_i} C_{11} = 0 \iff r(R_{A_i} C_{11}) = 0 \iff r(C_{11}, A_{1}) = r(A_{1}) \]

\[ r \left( \begin{array}{c} C_{11} \\
R_{B_{11}} B_{22} \\
A_{22} L_{A_{11}} \\
0 \end{array} \right) = r(A_{22} L_{A_{11}}) + r(R_{B_{11}} B_{22}) \]
\[ r \left( \begin{array}{ccc}
C_2 - A_{22} A_{11}^t C_1 B_{11}^t B_{22} & A_{22} & 0 \\
B_{22} & 0 & B_{11} \\
0 & A_{11} & 0 \end{array} \right) = r \left( \begin{array}{c} A_{11} \\
A_{22} \end{array} \right) + r(B_{11}, B_{22}) \]
\[ r \left( \begin{array}{ccc}
C_1 & 0 & A_{11} \\
0 & -C_2 & A_{22} \\
B_{11} & B_{22} & 0 \end{array} \right) = r \left( \begin{array}{c} A_{11} \\
A_{22} \end{array} \right) + r(B_{11}, B_{22}) = r(A_{22}) + r(B_{11}). \]

We now prove that \( X_1 \) in (10) is the general solution of the system (8). We prove it in two steps. We show that \( X_1 \) is a solution of system (8) in Step 1. In Step 2, if the system (8) is consistent, then the general solution to system (8) can be expressed as (10).

Step 1. In this step, we show that \( X_1 \) is a solution of system (8). Substituting \( X_1 \) in (10) into the system (8) yields

\[ A_{11} X_1 B_{11} = A_{11} X_0 B_{11}, \ A_{22} X_1 B_{22} = A_{22} X_0 B_{22}, \] (11)

where \( X_0 = A_{11}^t C_1 B_{11}^t + L_{A_{11}} A_{22}^t C_2 B_{22}^t \). Since \( R_{A_{11}} C_1 = 0 \) and \( C_1 L_{B_{11}} = 0 \), we have that

\[ A_{11} X_0 B_{11} = A_{11} A_{11}^t C_1 B_{11}^t + L_{A_{11}} A_{22}^t A_{22}^t C_1 B_{22}^t B_{11} \]
\[ = A_{11} A_{11}^t C_1 B_{11}^t B_{11} + A_{11} L_{A_{11}} A_{11}^t C_1 B_{11} - R_{A_{22}} C_1 B_{22}^t B_{11} = A_{11} A_{11}^t C_1 B_{11}^t B_{11} \]
\[ = -R_{A_{11}} C_1 B_{11}^t B_{11} - C_1 L_{B_{11}} + C_1 = C_1. \]

By

\[ R_{B_{11}} B_{22} = 0, \ R_{A_{22}} C_{22} = 0, \ C_2 L_{B_{22}} = 0 \] and \( C_1 B_{11}^t B_{22} = A_{11} A_{11}^t C_2, \)
we have that

\[ A_{22} X_0 B_{22} = A_{22} (A_{11}^t C_1 B_{11}^t + L_{A_{11}} A_{22}^t C_2 B_{22}^t) B_{22} \]
\[ = A_{22} A_{11}^t C_1 B_{11}^t B_{22} + A_{22} A_{22}^t C_2 B_{22}^t B_{22} - A_{22} A_{11}^t A_{11} A_{22}^t C_2 B_{22}^t B_{22} \]
\[ = C_2 + A_{22} A_{11}^t C_1 B_{11}^t B_{22} - A_{22} A_{11}^t C_1 B_{11}^t B_{22} = C_2. \]

Thus, \( A_{11} X_1 B_{11} = C_1, \ A_{22} X_1 B_{22} = C_2. \) \( X_1 \) is a solution of system (8).

Step 2. In this step, we show that the general solution to the system (8) can be expressed as (10). It is sufficient to show that for an arbitrary solution, say, \( X_{01} \) of (8), \( X_{01} \) can be expressed in form (10). Put

\[ V_1 = X_{01} B_{22} B_{22}^t, \ V_2 = X_{01}, \ V_3 = X_{01} B_{11} B_{11}^t. \]
It follows from $B_{22} = B_{11}^{-1} B_{22}$ and $A_{11} = A_{11} A_{22}^t A_{22}$ that
\[
X_1 = A_{11}^t C_1 B_{11}^t + L A_{11} A_{22}^t C_2 B_{22}^t + L A_{22} V_1 + V_2 R_{B_{11}} + L A_{11} V_3 R_{B_{22}} \\
= A_{11}^t C_1 B_{11}^t + L A_{11} A_{22}^t C_2 B_{22}^t + L A_{22} X_{01} B_{22}^t B_{22}^t + X_{01} R_{B_{11}} + L A_{11} X_{01} B_{11}^t R_{B_{22}} \\
= A_{11}^t C_1 B_{11}^t + L A_{11} A_{22}^t C_2 B_{22}^t + X_{01} B_{22}^t B_{22}^t - A_{11}^t A_{22} X_{01} B_{22}^t B_{22}^t + X_{01} - X_{01} B_{11}^t R_{B_{22}} \\
+ X_{01} R_{B_{11}} B_{11}^t R_{B_{22}} - A_{11}^t A_{11} X_{01} B_{11}^t R_{B_{22}} \\
= A_{11}^t C_1 B_{11}^t + L A_{11} A_{22}^t C_2 B_{22}^t - X_{01} R_{B_{11}} B_{22}^t B_{22}^t + X_{01} - A_{11}^t A_{22} X_{01} B_{22}^t B_{22}^t \\
- A_{11}^t A_{11} X_{01} B_{11}^t R_{B_{22}} + A_{11}^t A_{11} X_{01} B_{11}^t B_{22}^t B_{22}^t \\
= X_{01} + A_{11}^t A_{11} X_{01} B_{11}^t B_{22}^t B_{22}^t - A_{11}^t A_{11} A_{22} X_{01} B_{22}^t B_{22}^t \\
= X_{01} + A_{11}^t A_{11} X_{01} B_{22}^t B_{22}^t - A_{11}^t A_{11} X_{01} B_{22}^t B_{22}^t = X_{01}.
\]

Hence, $X_{01}$ can be expressed as (10). To sum up, (10) is the general solution of the system (8). \[\square\]

Now, we give the fundamental theorem of this paper.

**Theorem 1.** Let $A_i, B_i$ and $B$ ($i = \overline{1, 4}$) be given quaternion matrices with appropriate sizes over $\mathbb{H}$. Set

\[
\begin{align*}
R_{A_i} A_2 &= A_{11}, \quad R_{A_i} A_3 = A_{22}, \quad R_{A_i} A_4 = A_{33}, \quad B_2 L_{B_1} = B_{11}, \quad B_{22} L_{B_{11}} = N_{11}, \\
B_3 L_{B_1} &= B_{22}, \quad B_4 L_{B_1} = B_{33}, \quad R_{A_i} A_{22} = M_1, \quad S_1 = A_{22} L_M, \quad R_{A_i} B L_{B_1} = T_1, \\
C &= R_M R_{A_{11}}, \quad C_1 = CA_{33}, \quad C_2 = R_{A_{11}} A_{33}, \quad C_3 = R_{A_{22}} A_{33}, \quad C_4 = A_{33}, \\
D &= L_{B_{11}} L_{N_{11}}, \quad D_1 = B_{33}, \quad D_2 = B_{33} L_{B_{22}}, \quad D_3 = B_{33} L_{B_{11}}, \quad D_4 = B_{33} D, \\
E_1 &= CT_{11}, \quad E_2 = R_{A_1 T_1} L_{B_{22}}, \quad E_3 = R_{A_2 T_1} L_{B_{11}}, \quad E_4 = T_1 D, \\
C_{11} &= (L_{C_2}, L_{C_4}), \quad D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix}, \quad C_{22} = L_{C_1}, \quad D_{22} = R_{D_2}, \quad C_{33} = L_{C_3}, \\
D_{33} &= R_{D_4}, \quad E_{11} = R_{C_1} C_{22}, \quad E_{22} = R_{C_1} C_{33}, \quad E_{33} = D_{22} L_{D_{11}}, \quad E_{44} = D_{33} L_{D_{11}}, \\
M &= R_{E_{11}} E_{22}, \quad N = E_{44} L_{E_{33}}, \quad F = F_2, \quad F_1 = R_{C_1} F L_{D_{11}}, \quad S = E_{22} L_M, \\
F_{11} &= C_2 L_{C_1}, \quad G_1 = E_2 - C_2 C_4 E_1 D_{11} D_2, \quad F_2 = C_4 L_{C_3}, \quad G_2 = E_4 - C_4 C_4 E_3 D_{11} D_4, \\
F_1 &= C_1^t E_1 D_{11}^t + L_{C_1} C_1^t E_2 D_{11}^t, \quad F_2 = C_3^t E_3 D_{11}^t + L_{C_3} C_3^t E_4 D_{11}^t.
\end{align*}
\]

Then, the following statements are equivalent:

1. Equation (5) is consistent.
2. $R_{C_1} E_i = 0, \quad E_i L_{D_{11}} = 0$ ($i = \overline{1, 4}$), $R_{E_{11}} E_{44} L_{E_{33}} = 0.$ (16)
where $A$ is solvable. According to Equation (28), we have that $Y$ has a solution if and only if there exist $U_i$ such that

$$
\begin{align*}
X_1 &= A_i^† (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) B_i + L_{A_i} U_i + U_i B_i \quad \forall i = 1, 3,
\end{align*}
$$

where $U_i$ are any matrices with appropriate dimensions over $\mathbb{H}$. Hence, Equation (28) has a solution if and only if there exist $Y_i (i = 1, 3)$ in Equation (26) such that Equation (28) is solvable. According to Equation (28), we have that

$$
A_{11} Y_1 B_{11} + A_{22} Y_2 B_{22} + A_{33} Y_3 B_{33} = T_1
$$

Proof. (1) $\iff$ (2) Equation (5) can be written as

$$
A_1 X_1 + X_2 B_1 = B - (A_2 Y_1 B_2 + A_3 Y_2 B_3 + A_4 Y_3 B_4).
$$

Clearly, Equation (5) is solvable if and only if Equation (26) has a solution. By Lemma 4, Equation (26) is consistent if and only if there exist $Y_i (i = 1, 3)$ in Equation (26) such that

$$
R_{A_i} [B - (A_2 Y_1 B_2 + A_3 Y_2 B_3 + A_4 Y_3 B_4)] B_i = 0,
$$

i.e.,

$$
A_{11} Y_1 B_{11} + A_{22} Y_2 B_{22} + A_{33} Y_3 B_{33} = T_1,
$$

where $A_{ii}, B_i (i = 1, 3)$, and $T_1$ are defined by (12). In addition, when Equation (26) has a solution, we get the following:

$$
\begin{align*}
X_1 &= A_i^† (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) B_i + L_{A_i} U_i + U_i B_i, \\
X_2 &= R_{A_i} (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) B_i^† + A_1 A_i^† U_i + U_3 R_{B_i},
\end{align*}
$$

where $U_i (i = 1, 3)$ are any matrices with appropriate dimensions over $\mathbb{H}$. Hence, Equation (28) has a solution if and only if there exist $Y_i (i = 1, 3)$ in Equation (26) such that Equation (28) is solvable. According to Equation (28), we have that

$$
A_{11} Y_1 B_{11} + A_{22} Y_2 B_{22} = T_1 - A_{33} Y_3 B_{33}.
$$
Hence, Equation (28) is consistent if and only if Equation (29) is solvable. It follows from Lemma 5 that Equation (29) has a solution if and only if there exists $Y_3$ in Equation (29) such that

\begin{align*}
R_{M_1}R_{A_{11}}(A_{33}Y_3B_{33} - T_1) &= 0, \quad R_{A_{11}}(T_1 - A_{33}Y_3B_{33})L_{B_{22}} = 0, \\
R_{A_{22}}(T_1 - A_{33}Y_3B_{33})L_{B_{11}} &= 0, \quad (T_1 - A_{33}Y_3B_{33})L_{B_{11}}L_{N_1} = 0,
\end{align*}
\tag{30}

i.e.,

\begin{align*}
C_1 Y_3 D_1 &= E_1, \quad C_2 Y_3 D_2 = E_2, \quad C_3 Y_3 D_3 = E_3, \quad C_4 Y_3 D_4 = E_4,
\end{align*}
\tag{31}

where $C_i$, $D_i$, $E_i$ $(i = 1, 4)$ are defined by (13). When Equation (29) is solvable, we have that

\begin{align*}
Y_1 &= A_{11}^T B_{11}^T - A_{11}^T A_{22}M_1^T T B_{11}^T - A_{11}^T S_1A_{22}^T TN_1^T B_{22}^T B_{11}^T \\
&\quad - A_{11}^T S_1 U_4 R_{N_1} B_{22} B_{11}^T + L_{A_{11}} U_5 + U_6 R_{B_{11}}, \\
Y_2 &= M_1^T T B_{22}^T + S_1^T S_1 A_{22}^T T N_1^T + L_{M_1} L_{S_1} U_7 + U_8 R_{B_{22}} + L_{M_1} U_4 R_{N_1},
\end{align*}

where $A_{ii}$, $B_{ii}$ $(i = 1, 3)$, $M_i$, $N_i$, $S_i$, $T_1$ are defined by (12), $T = T_1 - A_{33}Y_3B_{33}$ and $U_j$ $(j = 4, 8)$ are any matrices with the appropriate dimensions over $\mathbb{H}$.

It is easy to infer that

\begin{align*}
C_1 L_{C_2} &= 0, \quad R_{D_1} D_2 = 0, \quad C_3 L_{C_4} = 0, \quad R_{D_3} D_4 = 0.
\end{align*}
\tag{32}

Thus, according to Lemma 6, we have that the system (31) is consistent if and only if

\begin{align*}
R_{C_i} E_i &= 0, \quad E_i L_{D_i} = 0 \quad (i = 1, 2, 3, 4), \quad R_{F_{11}} G_1 = 0, \quad R_{F_{22}} G_2 = 0.
\end{align*}
\tag{33}

In this case, the general solution to system (31) can be expressed as

\begin{align*}
Y_3 &= F_1 + L_{C_2} V_1 + V_2 R_{D_1} + L_{C_1} V_3 R_{D_2}, \\
Y_3 &= F_2 - L_{C_4} W_1 - W_2 R_{D_3} - L_{C_3} W_3 R_{D_4},
\end{align*}
\tag{34}
\tag{35}

where $F_1$, $F_2$ are defined by (15) and $V_i$, $W_i$ $(i = 1, 3)$ are any matrices with the appropriate dimensions over $\mathbb{H}$. Thus, system (31) has a solution if and only if (33) holds and there exist $V_i$, $W_i$ $(i = 1, 3)$ such that (34) equals to (35), namely

\begin{align*}
(L_{C_2}, L_{C_4}) \begin{pmatrix} V_1 \\ W_1 \end{pmatrix} + (V_2, W_2) \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix} + L_{C_1} V_3 R_{D_2} + L_{C_3} W_3 R_{D_4} &= F, \\
\end{align*}

i.e.,

\begin{align*}
C_{11} \begin{pmatrix} V_1 \\ W_1 \end{pmatrix} + (V_2, W_2) D_{11} + C_{22} V_3 D_{22} + C_{33} W_3 D_{33} &= F, \\
\end{align*}
\tag{36}

where $F$, $C_{11}$ and $D_{ii}$ $(i = 1, 3)$ are defined by (14). It follows from Lemma 5 that Equation (36) has a solution if and only if

\begin{align*}
R_M R_{E_{11}} E &= 0, \quad E L_{E_{33}} L_N = 0, \quad R_{E_{11}} E L_{E_{44}} = 0, \quad R_{E_{22}} E L_{E_{33}} = 0.
\end{align*}
\tag{37}

In this case, the general solution to Equation (36) can be expressed as

\begin{align*}
V_1 &= (I_m, 0) \left[ C_{11}^T (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) - C_{11}^T U_{11} D_{11} + L_{C_{11}} U_{12} \right], \\
W_1 &= (0, I_m) \left[ C_{11}^T (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) - C_{11}^T U_{11} D_{11} + L_{C_{11}} U_{12} \right],
\end{align*}
\[ W_2 = \left[ R_{C_1} (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) D_{11}^t + C_{11} C_{11}^t U_{11} + U_{21} R_{D_{11}} \right] \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \]

\[ V_2 = \left[ R_{C_1} (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) D_{11}^t + C_{11} (C_{11})^t U_{11} + U_{21} R_{D_{11}} \right] \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \]

\[ V_3 = E_{11}^{t} F^t E_{33} - E_{11}^{t} E_{22} M^t F E_{33} - E_{11}^{t} S E_{22}^t F N^t E_{44} E_{33}^t - E_{11}^{t} S U_{31} R N E_{44} E_{33}^t + L_{E_{11}} U_{32} + U_{33} R_{E_{33}}, \]

\[ W_3 = M^t F E_{44} + S^t S E_{22}^t F N^t + L_M L_S U_{41} + L_M U_{31} R_N - U_{42} R_{E_{44}}, \]

where \( U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, \) and \( U_{42} \) are any matrices with the suitable dimensions over \( \mathbb{H} \). \( M, E, N, S, C_{11}, D_{11}, \) and \( E_{ii} (i = 1, 4) \) are defined by (14), \( m \) is the column number of \( A_4 \) and \( n \) is the row number of \( B_4 \). We summarize up that (28) has a solution if and only if (33) and (37) hold. Hence, Equation (5) is solvable if and only if (33) and (37) hold.

In fact, \( R_{C_2} E_2 = 0, E_1 L_{D_3} = 0 \Rightarrow R_{F_{11}} G_1 = 0; R_{C_2} E_4 = 0, E_3 L_{D_3} = 0 \Rightarrow R_{F_{22}} G_2 = 0; R_{C_3} E_3 = 0, E_1 L_{D_1} = 0 \Rightarrow R_M R_{E_{33}} E_0 = 0; R_{C_4} E_4 = 0, E_1 L_{D_1} = 0 \Rightarrow E L_{E_{33}} L_N = 0; R_{C_4} E_4 = 0, E_2 L_{D_3} = 0 \Rightarrow R_{E_{22}} E L_{E_{33}} = 0. \) The specific proof is as follows.

Firstly, we prove that \( R_{C_2} E_2 = 0, E_1 L_{D_3} = 0 \Rightarrow R_{F_{11}} G_1 = 0; R_{C_2} E_4 = 0, E_3 L_{D_3} = 0 \Rightarrow R_{F_{22}} G_2 = 0. \) It follows from Lemma 1 and elementary transformations that

\[ R_{C_2} E_1 = 0 \Leftrightarrow r(E_1, C_1) = r(C_1) = r(C_1 T_1, C A_{33}) = r(C A_{33}) \Leftrightarrow r(T_1, A_{33}, A_{11}, A_{22}) = r(A_{33}, A_{11}, A_{22}), \]  

\[ (38) \]

\[ R_{C_2} E_2 = 0 \Leftrightarrow r(E_2, C_2) = r(C_2) \Leftrightarrow r(T_1 B_{22} \begin{pmatrix} A_{33} \\ 0 \\ A_{11} \end{pmatrix}, 0) = r(A_{33}, A_{11}) + r(B_{22}). \]  

\[ (39) \]

\[ R_{C_2} E_3 = 0 \Leftrightarrow r(E_3, C_3) = r(C_3) \Leftrightarrow r(T_1 B_{11} \begin{pmatrix} A_{33} \\ 0 \\ A_{22} \end{pmatrix}, 0) = r(A_{33}, A_{22}) + r(B_{11}), \]  

\[ (40) \]

\[ R_{C_2} E_4 = 0 \Leftrightarrow r(E_4, C_4) = r(C_4) \Leftrightarrow r(T_1 B_{11} \begin{pmatrix} A_{33} \\ 0 \\ A_{11} \end{pmatrix}, 0) = r(A_{33}) + r(B_{11}), \]  

\[ (41) \]

\[ E_1 L_{D_1} = 0 \Leftrightarrow r(E_1, D_1) = r(T_1 B_{33} \begin{pmatrix} A_{11} \\ 0 \\ A_{22} \end{pmatrix}, 0) = r(A_{11}, A_{22}) + r(B_{33}), \]  

\[ (42) \]

\[ E_2 L_{D_2} = 0 \Leftrightarrow r(E_2, D_2) = r(D_2) \Leftrightarrow r(T_1 B_{33} \begin{pmatrix} A_{11} \\ 0 \\ A_{22} \end{pmatrix}, 0) = r(B_{33}) + r(A_{11}), \]  

\[ (43) \]

\[ E_3 L_{D_3} = 0 \Leftrightarrow r(E_3, D_3) = r(D_3) \Leftrightarrow r(T_1 B_{33} \begin{pmatrix} A_{11} \\ 0 \\ A_{22} \end{pmatrix}, 0) = r(B_{33}) + r(A_{22}), \]  

\[ (44) \]

\[ E_4 L_{D_4} = 0 \Leftrightarrow r(E_4, D_4) = r(D_4) \Leftrightarrow r(T_1 B_{33} \begin{pmatrix} A_{11} \\ 0 \\ A_{22} \end{pmatrix}, 0) = r(B_{33}) + r(A_{22}). \]  

\[ (45) \]

It follows from Lemma 6 and (32) that \( R_{F_{11}} G_1 = 0 \) and \( R_{F_{22}} G_2 = 0 \) are equivalent to

\[ r \begin{pmatrix} E_1 \\ 0 \\ C_1 \\ D_1 \end{pmatrix} = r \begin{pmatrix} C_1 \\ C_2 \\ D_2 \end{pmatrix}, \]  

\[ (46) \]

\[ r \begin{pmatrix} E_3 \\ 0 \\ C_3 \\ D_3 \end{pmatrix} = r \begin{pmatrix} C_3 \\ C_4 \\ D_4 \end{pmatrix}, \]  

\[ (47) \]
According to Lemma 1, we have that

\[
\begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & 0 \\ 0 & -T_1 & 0 & 0 & 0 & A_{11} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & 0 & 0 \end{pmatrix} \Leftrightarrow r \begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & 0 \\ 0 & -T_1 & 0 & 0 & 0 & A_{11} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} 0 & A_{11} & A_{22} & 0 \\ A_{33} & 0 & 0 & A_{11} \\ 0 & 0 & 0 & B_{33} \end{pmatrix} + r \begin{pmatrix} B_{33} & 0 \\ 0 & B_{22} \end{pmatrix}
\]

Thus, it follows from (48) that (46) holds when (39) and (42) hold. Similarly, if (41) and (44) hold, then (47) holds.

Secondly, we prove that \( R_{C_1}E_3 = 0, E_1L_{D_1} = 0 \Rightarrow R_M R_{E_1}E = 0; R_{C_4}E_4 = 0, E_2L_{D_2} = 0 \Rightarrow R_{E_2}E L_{E_3} = 0. \) According to Lemma 5 and (32), we have that (37) are equivalent to

\[
\begin{align*}
\begin{pmatrix} F & L_{C_1} & L_{C_3} \\ R_{D_1} & 0 & 0 \\ R_{D_3} & 0 & 0 \end{pmatrix} &= r(L_{C_1}, L_{C_3}) + r(R_{D_1}, R_{D_3}), \\
\begin{pmatrix} F & L_{C_2} & L_{C_4} \\ R_{D_2} & 0 & 0 \\ R_{D_4} & 0 & 0 \end{pmatrix} &= r(L_{C_2}, L_{C_4}) + r(R_{D_2}, R_{D_4}), \\
\begin{pmatrix} F & L_{C_1} & L_{C_4} \\ R_{D_1} & 0 & 0 \\ R_{D_4} & 0 & 0 \end{pmatrix} &= r(L_{C_1}, L_{C_4}) + r(R_{D_1}, R_{D_4}), \\
\begin{pmatrix} F & L_{C_2} & L_{C_3} \\ R_{D_2} & 0 & 0 \\ R_{D_3} & 0 & 0 \end{pmatrix} &= r(L_{C_2}, L_{C_3}) + r(R_{D_2}, R_{D_3})
\end{align*}
\]

respectively. By Lemma 1, we have that

\[
\begin{pmatrix} F & I & I & 0 & 0 \\ I & 0 & 0 & D_1 & 0 \\ I & 0 & 0 & 0 & D_3 \\ 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 \end{pmatrix} \Leftrightarrow r \begin{pmatrix} F & I & I & 0 & 0 \\ I & 0 & 0 & D_1 & 0 \\ I & 0 & 0 & 0 & D_3 \\ 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} I & D_1 & 0 & 0 & 0 \\ I & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + r \begin{pmatrix} I & I \\ 0 & D_3 \end{pmatrix}
\]

Similarly, we can show that (50)–(52) are equivalent to

\[
\begin{align*}
\begin{pmatrix} E_1 & 0 & C_1 \\ 0 & -E_4 & C_4 \\ D_1 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} C_1 \\ C_4 \\ 0 \end{pmatrix} + r(D_1, D_3), \\
\begin{pmatrix} E_2 & 0 & C_2 \\ 0 & -E_3 & C_3 \\ D_2 & 0 & 0 \end{pmatrix} &= r \begin{pmatrix} C_2 \\ C_3 \\ 0 \end{pmatrix} + r(D_2, D_3),
\end{align*}
\]
we turn to prove that (42) is equivalent to (19). It follows from Lemma 1 and elementary operations to (55) that

\[ r \begin{pmatrix} E_2 & 0 & C_2 \\ 0 & -E_4 & C_4 \\ D_2 & D_4 & 0 \end{pmatrix} = r \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} + r(D_2, D_4). \]  

(56)

Substituting \( C_i, D_i, \) and \( E_i \) (\( i = 1, 3 \)) in (13) into the rank equality (53) and by Lemma 1, we have that

\( \begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & 0 \\ 0 & -T_1 & A_{33} & 0 & 0 & A_{22} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} 0 & A_{11} & A_{22} & 0 \\ A_{33} & 0 & 0 & A_{22} \end{pmatrix} + r(B_{33} 0 B_{11}). \)  

(57)

Hence, it follows from (40) and (42) that (57) holds. Similarly, we can prove that when (41), (42) hold and (43), hold, we can get that (54) and (56) hold, respectively. Thus, Equation (28) has a solution if and only if (16) holds. That is to say, Equation (5) has a solution if and only if (16) holds.

(2) \( \Leftrightarrow \) (3) We prove the equivalence in two parts. In the first part, we want to show that (38) to (45) are equivalent to (17) to (24), respectively. Now, we turn to prove that (42) is equivalent to (19). It follows from the Lemma 1 and elementary transformations that

\[ r \begin{pmatrix} R_{A_1} B L_{B_1} & R_{A_1} A_2 & R_{A_1} A_3 \\ B_4 L_{B_1} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_4 \\ A_2 \end{pmatrix} + r(B_4 B_1). \]  

(38)

Similarly, we can show that (39) to (41) are equivalent to (18) to (20), respectively. Now, we turn to prove that (42) is equivalent to (19). It follows from Lemma 1 and elementary transformations that

\[ r \begin{pmatrix} R_{A_1} T_1 L_{B_2} & 0 & R_{A_1} A_3 \\ B_3 L_{B_2} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_4 \\ A_2 \end{pmatrix} + r(B_4 B_1). \]  

(42)

Similarly, we can show that (43) to (45) are equivalent to (22) to (24). Hence, (38) to (45) are equivalent to (17) to (24), respectively.

Part 2. We want to show that (55) \( \Leftrightarrow \) (25). It follows from Lemma 1 and elementary operations to (55) that

\[ r \begin{pmatrix} R_{A_1} T_1 L_{B_2} & 0 & R_{A_1} A_3 \\ B_3 L_{B_2} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_4 \\ A_2 \end{pmatrix} + r(B_4 B_1). \]  

(55)
Hence, (38) to (45) and (55) are equivalent to (17) to (25), respectively.

Next, we give the formula of general solution to matrix Equation (5) by using Moore–Penrose. According to Theorem 1, we get the following theorem:

**Theorem 2.** Let matrix Equation (5) be solvable. Then, the general solution to matrix Equation (5) can be expressed as

\[
X_1 = A_1^T(B - A_2Y_1B - A_3Y_2B - A_4Y_3B4) - A_1^TU_1B_1 + L_{A_1}U_2,
\]

\[
X_2 = R_{A_1}(B - A_2Y_1B - A_3Y_2B - A_4Y_3B4)B_1^T + A_1A_2^TU_1 + U_3R_{B_1},
\]

\[
Y_1 = A_1^TTB_{11} - A_{11}^TA_{22}^MTB_{11}^T - A_{11}^TS_1R_{N_1}B_{22}B_{11}^T
\]

\[\quad - A_{11}^TS_1U_1R_{N_1}B_{22}B_{11}^T + L_{A_1}U_3 + U_6R_{B_1},\]

\[
Y_2 = M_1^TTB_{22} + S_1^TA_{22}^TN_1T + L_{M_1}S_1U_7 + U_8R_{B_2} + L_{M_1}U_4R_{N_1},
\]

\[
Y_3 = F_1 + L_{C_1}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2},\]

where \( T = T_1 - A_{33}Y_3B_{33}, U_i(i = 1, 8) \) are arbitrary matrices with appropriate sizes over \( \mathbb{H} \),

\[
V_1 = (I_{m}, 0)[C_{11}^T(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^TU_{11}D_{11} + L_{C_1}U_{12}],
\]

\[
W_1 = (0, I_{n})[C_{11}^T(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^TU_{11}D_{11} + L_{C_1}U_{12}],
\]

\[
W_2 = [R_{C_11}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^T + C_{11}C_{11}^TU_{11} + U_{21}R_{D_1}]egin{pmatrix} 0 \\ I_n \end{pmatrix},
\]

\[
V_2 = [R_{C_11}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^T + C_{11}C_{11}^TU_{11} + U_{21}R_{D_1}]egin{pmatrix} I_n \\ 0 \end{pmatrix},
\]

\[
V_3 = E_{11}^TE_{33}^T - E_{11}^TE_{22}M^TE_{33}^T - E_{11}^TS_3E_{44}^TE_{33}^T - E_{11}^TS_4U_{31}R_{N_1}E_{44}^T + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},
\]

\[
W_3 = M^TE_{44}^T + S^TE_{22}^TFN^T + L_{M}S_1U_{41} + L_{M}U_{31}R_{N_1} - U_{42}R_{E_{44}}.
\]

where \( U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, \) and \( U_{42} \) are arbitrary matrices with appropriate sizes over \( \mathbb{H} \), \( m \) is the column number of \( A_4 \) and \( n \) is the row number of \( B_4 \).

**Algorithm with a Numerical Example**

In this section, we give Algorithm 1 with a numerical example to illustrate the main results.

**Algorithm 1** Algorithm for computing the general solution of Equation (5)

1. Input the quaternion matrices \( A_i, B_i \) \( i = 1, 4 \) and \( B \) with conformable shapes.
2. Compute all matrices given by (12)–(15).
3. Check equalities in (16) or (17)–(25). If not, it returns inconsistent.
4. Else, compute \( X_i \), \( Y_j \) \( i = 1, 2, j = 1, 3 \)
Example 1. Consider the matrix Equation (5). Put

\[ A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

Computation directly yields

\[ r \begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(A_2, A_3, A_4, A_1) = 3, \]

\[ r \begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \end{pmatrix} = r(A_2, A_4, A_1) + r(B_3) = 4, \]

\[ r \begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \end{pmatrix} = r(A_3, A_4, A_1) + r(B_2) = 4, \]

\[ r \begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} = r(B_2) + r(A_4, A_1) = 3, \]

\[ r \begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \end{pmatrix} = r(A_2, A_3, A_1) + r(B_4) = 4, \]

\[ r \begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \end{pmatrix} = r(B_3) + r(A_2, A_1) = 3, \]

\[ r \begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} = r(B_2) + r(A_3, A_1) = 3, \]

\[ r \begin{pmatrix} B & A_1 \\ B_2 & 0 \end{pmatrix} = r(B_2) + r(A_1) = 3, \]

\[ r \begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_3 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \\ B_4 & B_4 \end{pmatrix} + r \begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix} = 7. \]

All rank equalities in (17) to (25) hold. Hence, according to Theorem 1, Equation (5) has a solution. Moreover, by Theorem 2, we have that

\[ X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} i & k \\ 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} i & j \\ k & 0 \end{pmatrix}. \]
Remark 1. Chu et al. gave potential applications of the maximal and minimal ranks in the discipline of control theory (e.g., [36–38]). We may consider the rank bounds of the general solution of Equation (5).

4. The General Solution to Equation with $\eta$-Hermicity

In this section, as an application of (5), we establish some necessary and sufficient conditions for quaternion matrix Equation (7) to have a solution and derive a formula of its general solution involving $\eta$-Hermicity.

**Theorem 3.** Let $A_i (i = \frac{1}{4})$ and $B$ be given matrices with suitable sizes over $\mathbb{H}$, $B = B^\eta$. Set

$$
R_A, A_2 = A_{11}, R_A, A_3 = A_{22}, R_A, A_4 = A_{33}, R_A, A_{22} = M_1, S_1 = A_{22}L_{M_1},
$$

$$
R_A, B(R_A^\eta)^T = T_1, C = R_M, R_{A_1}, C_1 = CA_{33}, C_2 = R_{A_1}, A_{33},
$$

$$
C_3 = R_{A_2}, A_{33}, C_4 = A_{33}, E_1 = CT_1, E_2 = R_{A_1}, T_1(R_{A_2})^\eta, E_3 = R_{A_2}, T_1(R_{A_1})^\eta, E_4 = T_1C^\eta,
$$

$$
C_{11} = (L_{C_2}, L_{C_4}), C_{22} = L_{C_1}, C_{33} = L_{C_4}, E_{11} = R_{C_1}, C_{22}, E_{22} = R_{C_1}, C_{33},
$$

$$
M = R_{E_1}, E_{22}, N = (R_{E_2}, E_{11})^\eta, F = F_2 - F_1, E = R_{C_1}, F(R_{C_1})^\eta, S = E_{22}L_{M},
$$

$$
F_{11} = C_2L_{C_1}, G_1 = E_2 - C_2C_1 E_1(C_1^\eta)^tC_3^\eta, F_{22} = C_4L_{C_4}, G_2 = E_4 - C_4C_4 E_3(C_3^\eta)^tC_4^\eta,
$$

$$
F_1 = C_1^\eta E_1(C_1^\eta)^t + L_{C_1}C_1^\eta E_2(C_2^\eta)^t, F_2 = C_3^\eta E_3(C_3^\eta)^t + L_{C_3}C_4^\eta E_4(C_4^\eta)^t.
$$

Then, the following statements are equivalent:

1. Equation (7) is consistent.
2. $R_{C_i} E_i = 0 (i = \frac{1}{4}), R_{E_{22}}, E(R_{E_{22}})^\eta = 0.$
3. $r\begin{pmatrix}
B & A_2 & A_3 & A_4 & A_1 \\
A_2^\eta & 0 & 0 & 0 & 0
\end{pmatrix} = r(A_1) + r(A_2, A_3, A_4, A_1),$

$r\begin{pmatrix}
A_2^\eta & 0 & 0 & 0 \\
A_3^\eta & 0 & 0 & 0
\end{pmatrix} = r(A_2, A_3, A_1) + r(A_4, A_1),$

$r\begin{pmatrix}
A_3^\eta & 0 & 0 & 0 \\
A_4^\eta & 0 & 0 & 0
\end{pmatrix} = r(A_2, A_4, A_1) + r(A_3, A_1),$

$r\begin{pmatrix}
A_4^\eta & 0 & 0 & 0 \\
A_1^\eta & 0 & 0 & 0
\end{pmatrix} = r(A_3, A_4, A_1) + r(A_2, A_1),$

$r\begin{pmatrix}
B & A_2 & A_3 & A_4 & A_1 \\
0 & -B & 0 & A_3 & A_4 & 0 & A_1
\end{pmatrix} = 2r\begin{pmatrix}
A_2 & 0 & A_4 & A_1 & 0 \\
0 & A_3 & A_4 & 0 & A_1
\end{pmatrix}.$
In this case, the general solution to Equation (7) can be expressed as

\[
X_1 = \frac{\bar{X}_1 + (\bar{X}_2)^\dagger}{2}, \quad Y_1 = \frac{\bar{Y}_1 + (\bar{Y}_2)^\dagger}{2}, \quad Y_2 = \frac{\bar{Y}_2 + (\bar{Y}_3)^\dagger}{2}, \quad Y_3 = \frac{\bar{Y}_3 + (\bar{Y}_3)^\dagger}{2},
\]

where \( T \) is a suitable matrix and 

\[
\begin{align*}
V_1 &= (J_{m}, 0) \left[ C_{11}(F - C_{22}V_3C_{33}^{\dagger} - C_{33}W_3C_{22}^{\dagger}) - C_{11}^{\dagger}U_{11}^{\dagger}C_{11}^{\dagger} + L_{C_1}U_{11} \right], \\
W_1 &= (0, I_{m}) \left[ C_{11}(F - C_{22}V_3C_{33}^{\dagger} - C_{33}W_3C_{22}^{\dagger}) - C_{11}^{\dagger}U_{11}^{\dagger}C_{11}^{\dagger} + L_{C_1}U_{11} \right], \\
W_2 &= \begin{bmatrix} R_{C_{11}}(F - C_{22}V_3C_{33}^{\dagger} - C_{33}W_3C_{22}^{\dagger})(C_{11}^{\dagger})^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_1}^{\dagger} \end{bmatrix} \begin{bmatrix} 0 \\ I_{n} \end{bmatrix}, \\
V_2 &= \begin{bmatrix} R_{C_{11}}(F - C_{22}V_3C_{33}^{\dagger} - C_{33}W_3C_{22}^{\dagger})(C_{11}^{\dagger})^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}L_{C_1}^{\dagger} \end{bmatrix} \begin{bmatrix} I_{n} \\ 0 \end{bmatrix}, \\
V_3 &= E_1^{(1)} F(0_{22}^{\dagger})^{\dagger} - E_1^{(1)}E_2^{(1)} M^4 F(0_{22}^{\dagger})^{\dagger} - E_1^{(1)}S E_2^{(1)} FN^4 + E_1^{(1)}(0_{22}^{\dagger})^{\dagger} \\
&\quad - E_1^{(1)}S U_{31} R_N E_1^{(1)}(0_{22}^{\dagger})^{\dagger} + L_{E_1} U_{32} + U_{33} L_{E_2}^{\dagger}, \\
W_3 &= M^4 F(0_{22}^{\dagger})^{\dagger} + S^4 S E_2^{(1)} FN^4 + L_{M} L_{S} U_{41} + L_{M} U_{31} R_N - U_{42} L_{E_1}^{\dagger},
\end{align*}
\]

where \( U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, \) and \( U_{42} \) are any matrices with suitable dimensions over \( H \).

**Proof.** It is easy to show that (7) has a solution if and only if the following matrix equation has a solution:

\[
A_1 \bar{X}_1 + \bar{X}_2 A_1^{\dagger} + A_2 \bar{Y}_1 A_2^{\dagger} + A_3 \bar{Y}_2 A_3^{\dagger} + A_4 \bar{Y}_3 A_4^{\dagger} = B.
\]  
(58)

If (7) has a solution, say, \((X_1, Y_1, Y_2, Y_3)\), then

\[
(\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2, \bar{Y}_3) := (X_1, X_1^{\dagger}, Y_1, Y_2, Y_3)
\]

is a solution of (58). Conversely, if (58) has a solution, say

\[
(\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2, \bar{Y}_3).
\]

It is easy to show that (7) has a solution

\[
(X_1, Y_1, Y_2, Y_3) := \left( \frac{\bar{X}_1 + (\bar{X}_2)^\dagger}{2}, \frac{\bar{Y}_1 + (\bar{Y}_2)^\dagger}{2}, \frac{\bar{Y}_2 + (\bar{Y}_3)^\dagger}{2}, \frac{\bar{Y}_3 + (\bar{Y}_3)^\dagger}{2} \right).
\]

Letting \( A_1 \) and \( B_1 \) vanish in Theorem 1, it yields to the following result.
Corollary 1. Let $A_{ij}, B_{ij}$ ($i = 1, \ldots, 3$), and $T_1$ be given matrices with appropriate sizes over $\mathbb{H}$. Set

$$M_1 = R_{A_{11}}A_{22}, N_1 = B_{22}L_{B_{11}}, S_1 = A_{22}L_{M_1},$$

$$C = R_{M_1}R_{A_{11}}, C_1 = C_{A_{33}}, C_2 = R_{A_{11}}A_{33}, C_3 = R_{A_{22}}A_{33}, C_4 = A_{33},$$

$$D = L_{B_{11}}L_{N_1}, D_1 = B_{33}, D_2 = B_{33}L_{B_{22}}, D_3 = B_{33}L_{B_{11}}, D_4 = B_{33}D,$$

$$E_1 = C_{T_1}, E_2 = R_{A_{11}}T_1L_{B_{22}}, E_3 = R_{A_{22}}T_1L_{B_{11}}, E_4 = T_1D,$$

$$C_{11} = (L_{C_2}, L_{C_4}), D_{11} = \left( \begin{array}{cc} R_{D_1} & 0 \\ R_{D_3} & 0 \end{array} \right), C_{22} = L_{C_1}, D_{22} = R_{D_2}, C_{33} = L_{C_3},$$

$$D_{33} = R_{D_4}, E_{11} = R_{C_{11}}C_{22}, E_{22} = R_{C_{11}}C_{33}, E_{33} = D_{22}L_{D_{11}}, E_{44} = D_{33}L_{D_{11}},$$

$$M = R_{E_{12}}, N = E_{44}L_{E_{33}}, F = F_2 - F_1, E = R_{C_{11}}F, S = E_{22}L_{M_1},$$

$$F_{11} = C_2L_{C_1}, G_1 = E_2 - C_2C_1^\dagger E_1D_1^\dagger D_2, F_{22} = C_4L_{C_3}, G_2 = E_4 - C_4C_3^\dagger E_3D_3^\dagger D_4,$$

$$F_1 = C_1^\dagger E_1D_1^\dagger + L_{C_1}C_1^\dagger E_2D_2^\dagger, F_2 = C_3^\dagger E_3D_3^\dagger + L_{C_3}C_3^\dagger E_4D_4^\dagger.$$

Then, the following statements are equivalent:

1. Equation (6) is consistent.
2. $R_{C_{11}}E_i = 0, E_iL_{D_1} = 0 (i = 1, 4), R_{E_{22}}E_{L_{E_{33}}} = 0.$
3. $r(T_1, A_{11}, A_{22}, A_{33}) = r(A_{11}, A_{22}, A_{33}),$

$$r \left( \begin{array}{c} T_1 \\ B_{11} \\ B_{22} \\ B_{33} \end{array} \right) = r \left( \begin{array}{c} B_{11} \\ B_{22} \\ B_{33} \end{array} \right), r \left( \begin{array}{c} T_1 \\ B_{33} \\ A_{11} \\ A_{22} \end{array} \right) = r(A_{11}, A_{22}) + r(B_{33}),$$

$$r \left( \begin{array}{c} T_1 \\ B_{22} \\ A_{11} \\ A_{33} \end{array} \right) = r(A_{11}, A_{33}) + r(B_{22}),$$

$$r \left( \begin{array}{c} T_1 \\ B_{11} \\ A_{33} \\ A_{22} \end{array} \right) = r(A_{33}, A_{22}) + r(B_{11}), r \left( \begin{array}{c} T_1 \\ B_{11} \\ 0 \\ 0 \end{array} \right) = r \left( \begin{array}{c} B_{11} \\ B_{22} \\ 0 \\ 0 \end{array} \right) = r(A_{11}, A_{33}),$$

$$r \left( \begin{array}{c} T_1 \\ B_{22} \\ 0 \\ 0 \end{array} \right) = r \left( \begin{array}{c} B_{22} \\ B_{33} \\ 0 \\ 0 \end{array} \right) = r(A_{22}, A_{33}),$$

$$r \left( \begin{array}{c} T_1 \\ B_{33} \\ 0 \\ 0 \end{array} \right) = r \left( \begin{array}{c} B_{33} \\ B_{33} \\ 0 \\ 0 \end{array} \right) = r(A_{11}).$$

In this case, the general solution to Equation (6) can be expressed as

$$Y_1 = A_{11}^\dagger T^\dagger B_{11} + A_{11}^\dagger A_{22}M_{11}^\dagger T^\dagger B_{11} - A_{11}^\dagger S_1A_{33}T^\dagger N_{11}B_{22}B_{11}^\dagger + L_{A_{11}}U_5 + U_6R_{B_{11}},$$

$$Y_2 = M_{11}^\dagger T^\dagger B_{22} + S_1^\dagger S_1A_{33}^\dagger T^\dagger N_{11} + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},$$

$$Y_3 = F_1 + L_{C_{11}}V_1 + V_2R_{D_1} + L_{C_{11}}V_3R_{D_2}, or \quad Y_3 = F_2 - L_{C_{11}}W_1 - W_2R_{D_2} - L_{C_{11}}W_3R_{D_4},$$

where $T = T_1 - A_{33}Y_3B_{33}, U_i(i = 1, \ldots, 8)$ are any matrices with suitable dimensions over $\mathbb{H}$,

$$V_1 = (I_m, 0) \left[ C_{11}^\dagger (F - C_{22}V_3D_{22} - C_{33}W_5D_{33}) - C_{11}^\dagger U_1D_{11} + L_{C_{11}}U_{12} \right],$$

$$W_1 = (0, I_m) \left[ C_{11}^\dagger (F - C_{22}V_3D_{22} - C_{33}W_5D_{33}) - C_{11}^\dagger U_1D_{11} + L_{C_{11}}U_{12} \right].$$
\[ W_2 = \left[ R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{11} + C_{11}C_{11}^*U_{11} + U_{21}R_{D_{11}} \right] \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \]

\[ V_2 = \left[ R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{11} + C_{11}C_{11}^*U_{11} + U_{21}R_{D_{11}} \right] \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \]

\[ V_3 = E_{11}^*FE_{33}^* - E_{11}^*E_{22}M^*F_{33}^* - E_{11}^*SE_{22}^*E_{44}^*E_{33} - E_{11}^*S_{U_{31}}R_{N}E_{44}^*E_{33} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}}, \]

\[ W_3 = M^*F_{44}^* + S^*E_{22}^*E_{44}^* + L_{M}L_{5}U_{41} + L_{M}U_{31}R_{N} - U_{42}R_{E_{44}}, \]

\[ U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, \text{ and } U_{42} \text{ are any matrices with suitable dimensions over } \mathbb{H}. \]

5. Conclusions

We have established the solvability conditions and an exact formula of a general solution to quaternion matrix Equation (5). As an application of Equation (5), we also have established some necessary and sufficient conditions for Equation (7) to have a solution and derived a formula of its general solution involving \( \eta \)-Hermicity. The quaternion matrix Equation (5) plays a key role in studying the solvability conditions and general solutions of other types of matrix equations. For example, we can use the results on Equation (5) to investigate the solvability conditions and the general solution of the following system of quaternion matrix equations

\[ A_2Y_1 = C_2, \quad Y_1B_2 = D_2, \]
\[ A_3Y_2 = C_3, \quad Y_2B_3 = D_3, \]
\[ A_4Y_3 = C_4, \quad Y_3B_4 = D_4. \]

\[ G_1Y_1H_1 + G_2Y_2H_2 + G_3Y_3H_3 = G \]

where \( Y_1, Y_2, \) and \( Y_3 \) are unknown quaternion matrices and the others are given.

It is worth mentioning that the main results of (5) are available over not only \( \mathbb{R} \) and \( \mathbb{C} \) but also any division ring. Moreover, inspired by [39], we can investigate Equation (5) in tensor form.

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