**Research Article**

**Super $H$-Antimagic Total Covering for Generalized Antiprism and Toroidal Octagonal Map**

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Let $G$ be a graph and $H\subseteq G$ be subgraph of $G$. The graph $G$ is said to be $(a,d)$-$H$ antimagic total graph if there exists a bijective function $f : V(H) \cup E(H) \rightarrow \{1, 2, 3, \ldots, |V(H)| + |E(H)|\}$ such that, for all subgraphs isomorphic to $H$, the total $H$ weights $W(H) = W(H) = \sum_{x \in V(H)} f(x) + \sum_{y \in E(H)} f(y)$ forms an arithmetic sequence $a, a + d, a + 2d, \ldots, a + (n - 1)d$, where $a$ and $d$ are positive integers and $n$ is the number of subgraphs isomorphic to $H$. An $(a, d)$-$H$ antimagic total labeling $f$ is said to be super if the vertex labels are from the set $\{1, 2, \ldots, |V(G)|\}$. In this paper, we discuss super $(a, d)$-$C_3$-antimagic total labeling for generalized antiprism and a super $(a, d)$-$C_8$-antimagic total labeling for toroidal octagonal map.

1. **Introduction**

All the graphs that we consider in this works are finite, simple, and connected. Let $G$ be a graph with vertex set and edge set denoted by $V(G)$ and $E(G)$, respectively. For the cardinality of vertex set and edge set, we use the notation $|V(G)|$ and $|E(G)|$, respectively. For basic definitions and terminology related to graph theory, the readers can see the book by Gross et al. [1].

A graph labeling is a map $f$ that sends some of the graph elements (vertices or edges or both) to the set of positive integers. If the domain set of $f$ is the set of vertices (edges), then $f$ is called vertex (edge) labeling. If the domain set is $V(G) \cup E(G)$, then $f$ is called total labeling. Let $G$ be a graph and $H_1, H_2, \ldots, H_k$ be subgraphs of $G$. We say that the graph $G$ has an $H_1, H_2, \ldots, H_k$ covering if each edge of $G$ belongs to at least one of the subgraph $H_i$, where $1 \leq i \leq k$. If all $H_i, i = 1, 2, \ldots, k$, are isomorphic to a graph $H$, then such a covering is called $H$ covering of $G$. Suppose that a graph $G$ admits an $H$ covering. The graph $G$ is called $(a, d)H$ antimagic if there exists a bijective function $f : V(H) \cup E(H) \rightarrow \{1, 2, 3, \ldots, |V(H)| + |E(H)|\}$ such that, for all subgraphs isomorphic to $H$, the total $H$ weights,

$$W(H) = W(K) = \sum_{x \in V(K)} f(x) + \sum_{y \in E(K)} f(y),$$

(1)

form an arithmetic sequence $a, a + d, a + 2d, \ldots, a + (n - 1)d$, where $a$ and $d$ are positive integers and $n$ is the number of subgraphs isomorphic to $H$. An $(a, d)$-$H$ antimagic total labeling $f$ is said to be super if the vertex labels are from the set $\{1, 2, \ldots, |V(G)|\}$. If $d = 0$, then $H$ is called $(a, d)$-$H$ antimagic.

Kotzig and Rosa [2] and Enomoto et al. [3] introduced the concept of edge-magic and super edge-magic labeling. Gutierrez and Llado [4] first studied the $H$ (super) magic coverings of a graph $G$. They proved that the cycle $C_m$ and path $P_n$ are $P_m$ super magic for some $m$. The cycle (super) magic behavior of some classes of connected graphs is studied in Llado et al. [5]. They proved that prisms,
windmills, wheels, and books are $C_m$-magic for some $m$. Maryati et al. [6] investigated the G-supermagicness of a disjoint union of $c$ copies of a graph $G$ and showed that the disjoint union of any paths is $cP_n$-supermagic for some $c$ and $m$. Maryati et al. [7] and Salman et al. [8] proved that certain families of trees are path-supermagic. Ngurah et al. [9] proved that triangles, chains, ladders, wheels, and grids are cycle-supermagic.

Inaya et al. [10] firstly introduced the concept of $H$-magic decomposition and $H$-antimagic decomposition. They showed that, for any graceful tree $T$ with $n$ edges, the complete graph $K_{2n+1}$ admits $(a,d)$-$T$ antimagic decomposition for some $a$ and $d$ and all even differences $0 \leq d \leq n + 1$. They also proved that if any tree $T$ with $n$ edges admits a labeling, then the complete bipartite graph $K_n$ admits an $(a,d)$-$T$ antimagic decomposition for some $a$ and $d$ having same parity as $n$. The condition on the existence of $C_{2k}$ super magic decomposition of complete $n$ partite graph and its copies were given by Lian [11]. The $H$-supermagic decomposition of antiprisms is described by Hendy in [12] and the $H$-supermagic decompositions of the lexicographic product of graphs are discussed by Hendy et al. in [13]. In [14], Hendy et al. examined the existence of super $(a,d)$-$H$ magic labeling for toroidal grids and toroidal triangulations. Recently, Fenovcikova et al. [15] proved that wheels are cycle antimagic.

In this paper, we discuss the Super $(a,d)$-$C_3$-antimagic total labeling for generalized antiprism and a Super $(a,d)$-$C_3$-antimagic total labeling for toroidal octagonal map. We proved that the generalized antimagic $A_r^c$ admits $(a,d)$-$C_3$-antimagic total labeling for $d = 0, 1$ and the toroidal octagonal map $O_r^c$ admits a Super $(a,d)$-$C_3$-antimagic total labeling, for $d = 1, 2, \ldots, 7$.

### 2. Results on Super $(a,d)$-$C_3$-Antimagic Total Covering of Generalized Antiprism $A_r^c$

An $r$-sided generalized antiprism $A_r^c$ is defined as a polyhedron which is composed of $s$ parallel copies of some particular $r$-sided polygon and connected by an alternating band of triangles. Figure 1 represents the labeled graph of $A_r^c$. We denote its vertex set and edge set by $V(A_r^c)$ and $E(A_r^c)$, respectively. The vertex set and the edge set of the generalized antimagic $A_r^c$ can be defined as follows:

$$V(A_r^c) = \{x_i^j, 0 \leq i \leq r - 1, 0 \leq j \leq s - 1\},$$

$$E(A_r^c) = \{x_i^j \pm x_i^{j+1}_r, 0 \leq i \leq r - 1, 0 \leq j \leq s - 1\}$$

$$\cup \{x_i^j \pm x_i^{j+1}_r, 0 \leq i \leq r - 1, 0 \leq j \leq s - 2\}$$

The generalized antimagic $A_r^c$ admits a $C_3$ covering. Let $z^j_i$ and $f^j_i$ be the $C_3$ cycles which cover $A_r^c$, where $0 \leq i \leq r - 1$ and $0 \leq j \leq s - 2$. The cycles $z^j_i$ and $f^j_i$ can be defined as

$$z^j_i = x_i^j x_{i+1}^j x_{i+1}^{j+1}_r x_i^j, \quad \text{for } 0 \leq i \leq r - 1, 0 \leq j \leq s - 2,$$

$$f^j_i = x_i^j x_{i+1}^j x_{i+1}^{j+1}_r x_i^j, \quad \text{for } 0 \leq i \leq r - 1, 0 \leq j \leq s - 2.$$

It is easy to observe that $|V(A_r^c)| = rs$ and $|E(A_r^c)| = 3rs - 2r$. We first give an upper bound for $d$ such that $A_r^c$ admits a super $(a,d)$-$C_3$-antimagic covering.

**Theorem 1.** Let $r, s \geq 3$ and $A_r^c$ be a generalized antiprism graph. Then, there is no super $(a,d)$-$C_3$-antimagic covering with $d \geq 6$.

**Proof.** Suppose that $A_r^c$ has a super $(a,d)$-$C_3$-antimagic covering. Let $f : V(A_r^c) \cup E(A_r^c) \rightarrow \{1, 2, 3, \ldots, 4rs - 2r\}$ be a super $(a,d)$-$C_3$-antimagic covering and $\{a_s, a_i + d, a_3 + 2d, \ldots, a_1 + (2rs - 2r - 1)d\}$ be the set of $C_3$ weights. The minimum weight on cycle $C_3$ is at least $12 + 3rs$ which is the sum of the smallest vertex labels $(1, 2, 3)$ and sum of smallest edge labels $(rs + 1, rs + 2, rs + 3)$. Thus,

$$a_3 \geq 12 + 3rs.$$

On the contrary, the maximum possible $C_3$-weight is the sum of three largest possible vertex labels, namely, $rs - 2, rs - 1, rs$, and the third largest possible edge labels from the set, $\{4rs - 2r - 2, 4rs - 2r - 1, 4rs - 2r\}$. Hence, we have

$$a_3 + (2rs - 2r - 1)d \leq 15rs - 6r - 6.$$  

From (4) and (5), an upper bound for the parameter $d$ can be obtained as

$$d \leq \frac{12rs - 16r - 18}{2rs - 2r - 1},$$

$$d \leq 6 - \frac{4r + 6}{2rs - 2r - 1}.$$  

$$d \leq 6.$$

Thus, we have arrived at the desired result.

**Theorem 2.** Let $r, s \geq 3$; then, the generalized antimagic $A_r^c$ admits a super $(9rs - 3r + 4, o)$-$C_3$-antimagic total covering.

**Proof.** Let $\phi : V(A_r^c) \cup E(A_r^c) \rightarrow \{1, 2, 3, \ldots, 4rs - 2r\}$ be a total labeling of generalized antimagic $A_r^c$ defined as follows:

$$\phi(x_i^j) = (jr + 1 + i), \quad \text{for } 0 \leq i \leq r - 1, 0 \leq j \leq s - 1,$$

$$\phi(x_i^j) = (2s - j)r + i, \quad \text{for } 0 \leq i \leq r - 1, 0 \leq j \leq s - 1,$$

$$\phi(x_i^j) = (3s - 2 - j)r + r - i, \quad \text{for } 0 \leq i \leq r - 1, 0 \leq j \leq s - 2,$$

$$\phi(x_i^j) = (4s - 3 - j)r + r - i, \quad \text{for } 0 \leq i \leq r - 1, 0 \leq j \leq s - 2.$$  

Under the labeling $\phi$, the weights of $3$-cycles $z_i^j$ are
And, the weights of 3-cycles $f^i_j$ are

$$W(z^i_j) = \phi(x^i_j) + \phi(x^{i+1}_{i+1}) + \phi(x^i_j x^{i+1}_{i+1}) + \phi(x^i_j x^{i+1}_{i+1}),$$

$$W(z^i_j) = 9rs - 3r + 4, \quad \text{for} \ 0 \leq i \leq r - 1, 0 \leq j \leq s - 2$$

(8)

And, the weights of 3-cycles $f^i_j$ are

$$W(f^i_j) = \phi(x^i_j) + \phi(x^{i+1}_{i+1}) + \phi(x^i_j x^{i+1}_{i+1}) + \phi(x^i_j x^{i+1}_{i+1}) + \phi(x^i_j x^{i+1}_{i+1}),$$

$$W(f^i_j) = 9rs - 3r + 4, \quad \text{for} \ 0 \leq i \leq r - 1, 0 \leq j \leq s - 2$$

(9)
Observe that the weights $W(z_i^j)$ and $W(f_i^j)$ of all cycles $z_i^j$ and $f_i^j$ are equal, and therefore, the resulting labeling is super $(9rs - 3r + 4,0)$-$C_3$ total labeling.

\textbf{Theorem 3.} Let $r, s \geq 3$; then, the generalized antiprism $\mathcal{A}_s^r$ admits a super $(7rs + 4,2)$-antimagic total covering.

\textbf{Proof.} Let $\chi : V(\mathcal{A}_s^r) \cup E(\mathcal{A}_s^r) \rightarrow \{1, 2, 3, \ldots , 4rs - 2r\}$ be a total labeling of generalized antiprism $\mathcal{A}_s^r$ defined as follows.

For $j = \text{even}$, the label on vertices $x_i^j$ is defined as

$$
\chi(x_i^j) = \begin{cases} 1 + i, & \text{for } 0 \leq i \leq r - 1, j = 0, \\
(j + 1)r, & \text{for } 0, 2 \leq j \leq s - 1, \\
jr + i, & \text{for } 1 \leq i \leq r - 1, 2 \leq j \leq s - 1.
\end{cases}
$$

(10)

For $j = \text{odd}$, the label on vertices $x_i^j$ is defined as

$$
\chi(x_i^j) = \begin{cases} jr + 1, & \text{for } 0, 1 \leq j \leq s - 1, \\
(j + 1)r + 1 - i, & \text{for } 1 \leq i \leq r - 1, 1 \leq j \leq s - 1.
\end{cases}
$$

(11)

For $j = \text{even}$, the label on edges $(x_i^j, x_{i+1}^j)$ is defined as

$$
\chi(x_i^j, x_{i+1}^j) = \begin{cases} rs + 1 + i, & \text{for } 0 \leq i \leq r - 1, j = 0, \\
rs + (j + 1)r, & \text{for } 0, 2 \leq j \leq s - 1, \\
rs + jr + i, & \text{for } 1 \leq i \leq r - 1, 2 \leq j \leq s - 1.
\end{cases}
$$

(12)

For $j = \text{odd}$, the label on edges $(x_i^j, x_{i+1}^j)$ is defined as

$$
\chi(x_i^j, x_{i+1}^j) = \begin{cases} rs + jr + 1, & \text{for } 0, 1 \leq j \leq s - 1, \\
rs + (j + 1)r + 1 - i, & \text{for } 1 \leq i \leq r - 1, 1 \leq j \leq s - 1.
\end{cases}
$$

(13)

The label on edges $(x_i^{j+1}, x_{i+1}^{j+1})$ is defined as

$$
\chi(x_i^{j+1}, x_{i+1}^{j+1}) = \begin{cases} (3s - 2)r + 1 + i, & \text{for } 0 \leq i \leq r - 1, j = 0, \\
(3s - 1)r, & \text{for } 0, 1 \leq j \leq s - 1, \\
(3s - 2 - j)r + i, & \text{for } 1 \leq i \leq r - 1, 1 \leq j \leq s - 1.
\end{cases}
$$

(14)

And, the label on edges $(x_i^{j+1}, x_{i+1}^{j+1})$ is defined as

$$
\chi(x_i^{j+1}, x_{i+1}^{j+1}) = 3rs + jr - i, \text{ for } 0 \leq i \leq r - 1, 0 \leq j \leq s - 2.
$$

(15)

Under the labeling $\chi$, the weights of 3-cycle $z_i^j$ are

$$
W(z_i^j) = \chi(x_i^j) + \chi(x_i^{j+1}) + \chi(x_i^{j+1}) + \chi(x_i^{j+1}) + \chi(x_i^{j+1}).
$$

(16)

For $j = \text{even}$, we have

$$
W(z_i^j) = \begin{cases} 7rs + 8 + 2i, & \text{for } 0 \leq i \leq r - 2, j = 0, \\
7rs + 4, & \text{for } i = r - 1, j = 0, \\
7rs + 4jr + 2r + 2, & \text{for } 0, 2 \leq j \leq s - 2, \\
7rs + 4jr + 2 + 2i, & \text{for } 1 \leq i \leq r - 1, 2 \leq j \leq s - 2.
\end{cases}
$$

(17)

For $j = \text{odd}$, we have

$$
W(z_i^j) = \begin{cases} 7rs + 4jr + 4, & \text{for } i = 0, 1 \leq j \leq s - 2, \\
7rs + 4jr + 2r + 4 - 2i, & \text{for } 1 \leq i \leq r - 1, 1 \leq j \leq s - 2.
\end{cases}
$$

(18)

The weight of 3-cycle $f_i^j$ are

$$
W(f_i^j) = \chi(x_i^j) + \chi(x_i^{j+1}) + \chi(x_i^{j+1}) + \chi(x_i^{j+1}) + \chi(x_i^{j+1}).
$$

(19)

For $j = \text{even}$, we have

$$
W(f_i^j) = \begin{cases} 7rs + 2r + 4, & \text{for } i = 0, j = 0, \\
7rs + 4r + 4 - 2i, & \text{for } 0 \leq i \leq r - 1, j = 0, \\
7rs + 4jr + 4 + 2i, & \text{for } 0 \leq i \leq r - 1, 2 \leq j \leq s - 2.
\end{cases}
$$

(20)

For $j = \text{odd}$, we have

$$
W(f_i^j) = \begin{cases} 7rs + 4jr + 4 + 2, & \text{for } i = 0, 1 \leq j \leq s - 2, \\
7rs + 4jr + 2r + 2i, & \text{for } 1 \leq i \leq r - 1, 1 \leq j \leq s - 2.
\end{cases}
$$

(21)

Observe that the weights $W(z_i^j)$ and $W(f_i^j)$ form an arithmetic progression with common difference 2 starting from $7rs + 4, 7rs + 6$ and ending at $11rs - 4r + 2$. This implies that

The defined labeling is a super $(7rs + 4,2)$-$C_3$-antimagic total covering.

\textbf{3. Results on Super $(a,d)$-$C_8$-Antimagic Total Covering of Toroidal Octagonal Planner Map $O_s^r$}

A planar octagonal map is a graph obtained by joining octagons and squares in such a way that they cover the plane. To obtain the toroidal octagonal map, we apply torus identification on octagonal planner map. We denote the toroidal octagonal map with $r$ rows and $s$ column of octagons by $O_s^r$, where $s, r \geq 2$. The planar representation of $O_s^r$ is depicted in Figure 2. The vertex set $V(O_s^r)$ and the edge set $E(O_s^r)$ of octagonal planner map $O_s^r$ can be defined as follows:

$$
V(O_s^r) = \{u_i^j, v_i^j, w_i^j, x_i^j; 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s - 1\},
$$

$$
E(O_s^r) = \{w_i^j, w_i^{j+1}; 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s - 1\}
\cup \{u_i^j, u_i^{j+1}; 0 \leq i \leq r - 1\}
\cup \{v_i^j, w_i^{j+1}; 0 \leq i \leq r - 1\}
\cup \{v_i^j, x_i^{j+1}; 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s - 1\}
\cup \{u_i^j, x_i^{j+1}; 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s - 1\}.
$$

(22)
From the above sets, we have $|V(O')| = 4rs$ and $|E(O')| = 6rs$. We can cover the toroidal octagonal map $O'$ by the 8-sided cycles $C_{8,j}^i$. For $0 \leq j \leq s-1$ and $0 \leq i \leq r-1$, the vertex set and edge set of 8-sided cycles $C_{8,j}^i$ can be defined as

\begin{align}
V(C_{8,j}^i) &= \{w_i, u_i^{r-1}, v_i^{r-1}, w_{i+1}, x_{i+1}, v_i, u_i, x_i; 0 \leq i \leq r - 1, 1 \leq j \leq s - 1\}, \\
E(C_{8,j}^i) &= \{w_i u_{i+1}^{-1}, u_i^{r-1} v_i^{r-1}, w_i^{r-1} w_{i+1}, v_i^{r-1} x_{i+1}, v_i x_i u_i x_i^{-1} w_i; 0 \leq i \leq r - 1, 1 \leq j \leq s - 1\},
\end{align}

(23)

We start by giving an upper bound for $d$ such that $O'$ admits a super $(a, d)$-$C_8$-antimagic covering.

**Theorem 4.** Suppose $O'$ admits a super $(a, d)$-$C_8$-antimagic covering; then, $d \leq 80$.

**Proof.** Suppose $O'$ admits a super $(a, d)$-$C_8$-antimagic covering. Then, the weight on cycle $C_8$ is at least

$$
\sum_{i=1}^{8} + \sum_{i=1}^{8} (4rs + i) = 32rs + 72,
$$

(24)

and the largest weight of $C_8$ is at most

$$
\sum_{i=1}^{8} (4rs + 1 - i) + \sum_{i=1}^{8} (10rs + 1 - i) = 112rs - 56.
$$

(25)

Thus, we have

$$
a + (rs - 1)d \leq 112rs - 56,
$$

(26)

$$
(r - 1)d \leq 112rs - 56 - 32rs - 72,
$$

and

$$
d \leq \frac{80rs - 128}{rs - 1},
$$

(26)

$$
d \leq 80.
$$

\[\Box\]

In the next two theorems, we show that toroidal octagonal map $O'$ admits a super $(a, d)$-$C_8$-antimagic covering for $d = 1, 2, \ldots, 7$.

**Theorem 5.** Let $r, s \geq 2$; then, the toroidal octagonal map $O'$ is super $(a, d)$-$C_8$-antimagic for $d \in \{1, 3, 5, 7\}$.
Proof. Define a total labeling $\varphi_d : V(O_r^d) \cup E(O_r^d) \longrightarrow \{1, 2, 3, \ldots, |V(O_r^d)| + |E(O_r^d)|\}$, where $d \in \{1, 3, 5, 7\}$ as follows:

\[
\begin{align*}
\varphi_d(u_i^j) &= jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_d(v_i^j) &= rs + jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_d(x_i^j) &= 3rs + (s - 1 - j)r + r - i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_d(u_i^j v_i^j) &= 2rs + jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_d(u_i^j v_i^j) &= 4rs + (s - 1 - j)r + r - i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_d(x_i^j, u_i^j, v_i^j) &= 5rm + 2jm + 1 + 2i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_1(v_i^j, x_i^j + 1) &= 8rs + (s - 1 - j)r + r - i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_5(v_i^j, x_i^j + 1) &= 8rs + jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_d(u_i^j, u_i^j + 1) &= 7rs + (s - 1 - j)r + 2r - 2i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_1(v_i^j, u_i^j + 1) &= 7rs + rj + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_1(x_i^j, u_i^j) &= 9rs + (s - 1 - j)r + r - i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\varphi_1(x_i^j, u_i^j) &= 9rs + jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1.
\end{align*}
\]

The total labeling $\varphi_d$ labels the vertices $u_i^j, v_i^j, w_i^j, x_i^j$ from the set $\{1, 2, \ldots, 4rs\}$ and the edges from the set $\{4rs + 1, 4rs + 2, \ldots, 10rs\}$. For $0 \geq i \geq r - 1$ and $0 \geq j \geq s - 1$, the weight of cycles $C_{d, j}$ under $\varphi_d$ is

\[
W_d(C_{d, j}) = \begin{cases} 
68rs + 2r + 10 + jr + i, & \text{for } d = 1, \\
67rs + r + 11 + 3jr + 3i, & \text{for } d = 3, \\
66rs + 12 + 5jr + 5i, & \text{for } d = 5, \\
65rs + 13 + 7jr + 7i, & \text{for } d = 7.
\end{cases}
\]

For the case $d = 1$, we have weights’ set $\{68rs + 2r + 10, 68rs + 2r + 11, \ldots, 69rs + 2r + 9\}$; similarly, for cases $d = 3, 5, 7$, we get the weights from the sets $\{67rs + r + 11, 67rs + r + 12, \ldots, 70rs + r + 8\}$, $\{66rs + r + 12, 66rs + r + 17, \ldots, 71rs + r + 7\}$, and $\{65rs + r + 13, 65rs + r + 20, \ldots, 72rs + r + 5\}$, respectively. Hence, the weights of cycles $C_{d, j}$ form an arithmetic sequence with difference 1, 3, 5, and 7.

\[\square\]
Theorem 6. Let \( r, s \geq 2 \); then, the toroidal map \( O'_r \) is super \((a, d)\)-\(C_{a^r}\)-antimagic for \( d \in \{2, 4, 6\} \).

\[
\begin{align*}
\phi_d(u_i^j) &= jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\phi_d(v_i^j) &= rs + jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\phi_d(x_i^j) &= 3rs + (s - 1 - j)r + r - i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\phi_d(w_i^j) &= 2rs + jr + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\phi_d(u_i^j v_i^j) &= 8rs + (s - 1 - j)r + r - i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\phi_d(x_i^j w_i^j) &= 9rs + (s - 1 - j)r + r - i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1, \\
\phi_d(x_i^j u_i^j) &= 7rs + j + r + 1 + i, \quad 0 \leq i \leq r - 1, 0 \leq j \leq s - 1. \\
\end{align*}
\]

(29)

Proof. Let \( d \in \{2, 4, 6\} \) and \( 0 \leq i \leq r - 1, 0 \leq j \leq s - 1 \). We define a total labeling \( \phi_d \) of \( O'_r \) as follows:

from set \( V(O'_r) \cup E(O'_r) \) to set \( \{1, 2, \ldots, 10rs\} \). For \( 1 \leq i \leq l \) and \( i \geq j \geq k \), the weights of \( C^j_{a^r} \) under the labeling \( \phi_d \) are

\[
W_d(C^j_{a^r}) = \begin{cases} 
75rs - 4r + 8 + 2jr + 2i, & \text{for } d = 2, \\
74rs - 4r + 9 + 4jr + 4i, & \text{for } d = 4, \\
73rs - 4r + 12 + 6jr + 6i, & \text{for } d = 6.
\end{cases}
\]

(30)

For the case \( d = 2 \), we have weights from the set \( \{75rs - 4r + 8, 75rs - 4r + 10, \ldots, 77rs - 4r + 6\} \). Similarly, for cases \( d = 4, 6 \), we get weights from the sets \( \{74rs - 4r + 9, 74rs - 4r + 13, \ldots, 78rs - 4r + 5\} \) and \( \{73rs - 4r + 12, 73rs - 4r + 18, \ldots, 79rs - 4r + 6\} \), respectively. This showed that weights of the cycles \( C^j_{a^r} \) form an arithmetic sequence with difference 2, 4, and 6.

4. Conclusion

In the present paper, we first constructed an upper bound for the parameter \( d \) for super \((a, d)\)-\(C_{a^2}\)-antimagic covering. Secondly, we examined the existence of super \((a, d)\)-\(C_{a^2}\)-antimagic labeling of generalized antiprism \( A^r_k \). We showed that, for \( r, s \geq 3 \) the generalized antiprism \( A^r_k \) had
$(a, d)\text{-}C_3$-antimagic covering for $d \in \{0, 2\}$. Thirdly, we constructed an upper bound for the parameter $d$ for super $(a, d)\text{-}C_8$-antimagic covering. Finally, we examined the existence of super $(a, d)\text{-}C_8$-antimagic labeling of toroidal map $O_s$. We showed that, for $m, n \geq 2$, the toroidal octagonal map $O_s^r$ had $(a, d)\text{-}C_8$-antimagic covering for $d \in \{1, 2, 3, 4, 5, 6, 7\}$. We conclude the paper with the following open problems.

Open problem 1: find other possible bound for parameter $d$ under $(a, d)\text{-}C_3$-antimagic total covering and the corresponding remaining labeling of $d$ for generalized antiprism $A_s^r$

Open problem 2: find other possible bound for parameter $d$ under $(a, d)\text{-}C_8$-antimagic total covering and the corresponding remaining labeling of $d$ for toroidal octagonal map $O_s^r$

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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