AN EXISTENCE AND UNIQUENESS RESULT FOR ORIENTATION-REVERSING HARMONIC DIFFEOMORPHISM FROM $\mathbb{H}^n$ TO $\mathbb{R}^n$

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Abstract. In this paper, we prove an existence and uniqueness theorem for orientation-reversing harmonic diffeomorphisms from $\mathbb{H}^n$ to $\mathbb{R}^n$ with rotational symmetry, which is a generalization of the corresponding result for dimension 2.

1. Introduction

From the results in [16, 14, 4, 2], we know that there is no rotationally symmetric harmonic diffeomorphism between the model spaces $\mathbb{R}^n$ and $\mathbb{H}^n$. Even from $\mathbb{R}^*_n$ to $\mathbb{H}^n$, this is also true [3]. But conversely, from $\mathbb{D}^*_n$ to $\mathbb{C}^*$, it does not hold [3], although Heinz [8] obtained the nonexistence of harmonic diffeomorphism from the unit disc onto the complex plane.

In this paper, we generalize the result [3] to general dimension, to find a rotationally symmetric harmonic diffeomorphism from $\mathbb{H}^n$ to $\mathbb{R}^n$, and to prove that this map is unique up to a combination of dilation and rotation of $\mathbb{R}^n$. All of these is related to the question mentioned by Schoen [15], which is about the existence, or nonexistence, of a harmonic diffeomorphism from the complex plane onto the hyperbolic unit disc. This question has been extensively studied by many people, see for example [17, 7, 11, 12, 5, 18, 11] and the references therein.

Partial results are related to the Nitsche’s type inequalities, see for example [13, 8, 9, 10] and the references therein.

As in [14, 4], let us denote

$$\mathbb{R}^n = (S^{n-1} \times [0, \infty), r^2 d\theta^2 + dr^2)$$
$$\mathbb{H}^n = (S^{n-1} \times [0, \infty), (f(r))^2 d\theta^2 + dr^2),$$

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where \( f(r) = \sinh r \), \((S^{n-1}, d\theta^2)\) is the \((n - 1)\)-dimensional sphere, and denote 
\[
\mathbb{R}^n_+ = \mathbb{R}^n \setminus \{0\} \text{ and } \mathbb{H}^n_+ = \mathbb{H}^n \setminus \{0\}.
\]
These notations are applicable for the whole notes.

We prove first the existence and uniqueness of the following linear ordinary differential equation with the boundary conditions.

**Lemma 1.1.** For \( n \geq 2 \), every solution \( y(r) \) to the following equation (1.1)
\[
y'' + (n - 1) \frac{f'}{f} \cdot y' - (n - 1) \frac{y}{f^2} = 0 \text{ for } r > 0
\]
satisfying the boundary conditions (1.2)
\[
\lim_{r \to 0^+} y(r) = +\infty, \lim_{r \to +\infty} y(r) = 0 \text{ and } y' < 0
\]

is of the form \( y = c \sinh^{1-n} r \) for some positive constant \( c \).

From this lemma, we can get the following result.

**Theorem 1.1.** For \( n \geq 2 \), there is an orientation-reversing harmonic diffeomorphism from \( \mathbb{H}^n_+ \) to \( \mathbb{R}^n_+ \), moreover, it is unique up to a combination of dilation and rotation of \( \mathbb{R}^n \).

This paper is organized as follows. In Section 2, we will prove Lemma 1.1. Theorem 1.1 will be proved in Section 3.

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**2. PROOF OF LEMMA 1.1**

Noting that from \([4\text{, page 12}]\), one can see that \( y = \tanh^{-1} r \) is another solution to equation (1.1) for dimension 2, which is linearly independent to the solution \( \overline{y} = \sinh^{-1} r \). From this fact, one can check that Lemma 1.1 holds easily. But for general dimension \( n \geq 2 \), we did not get a solution which is linearly independent to the solution \( \overline{y} = \sinh^{1-n} r \), so we need to use boundary condition (1.2) to get the uniqueness.

Since \( y(r) > 0 \) for \( r > 0 \), divided by \( y \) in (1.1), we can get
\[
\frac{y''}{y} + (n - 1) \frac{f'}{f} \cdot \frac{y'}{y} - (n - 1) \frac{1}{f^2} = 0.
\]
Setting
\[
x = \frac{y'}{y} \text{ and } z = f \cdot x,
\]
we have \( x' = \frac{y''}{y} - \frac{y'^2}{y^2} \) and \( z' = f \cdot x' + f' \cdot x \). Consequently, equation (1.1) can be rewritten as

\[
x' + x^2 + (n - 1) \frac{f'}{f} \cdot x - (n - 1) \frac{1}{f^2} = 0,
\]

and then

\[
f \cdot z' = -z^2 - (n - 2) f' \cdot z + (n - 1).
\]

Since \( y = \sinh^{1-n} r \) is a solution to (1.1) under condition (1.2), we can see that \( z \) is a solution to (2.2), where \( z = (1 - n) \cosh r \).

Let us study the property of the solution \( z \) to (2.2).

**Lemma 2.1.** If \( y \) is a solution to (1.1) under condition (1.2), then \( z(r) \) is the solution of (2.2) and

\[
\lim_{r \to 0^+} z(r) = 1 - n.
\]

The proof of this result will appear in the later part of this section.

**Corollary 2.1.** Suppose \( z(r) \) is the same as in Proposition 2.1, then we can get

\[
\lim_{r \to 0^+} z^{(2k)}(r) = 1 - n
\]

and

\[
\lim_{r \to 0^+} z^{(2k+1)}(r) = 0
\]

for all \( k = 0, 1, 2, \ldots \).

**Proof.** For simplicity, let us denote \( z^{(j)}(0) \) as \( \lim_{r \to 0^+} z^{(j)}(r) \) for \( j = 0, 1, \ldots \). From Proposition 2.1 we know that the conclusion is true for \( z(0) \). We want to show that \( z'(0) = 0 \). Taking derivative on both sides of (2.2), by elementary computation, we can get

\[
z'(0) = nz'(0),
\]

which implies \( z'(0) = 0 \).

Suppose Corollary 2.1 is true for \( k - 1 \) where \( k \geq 1 \), we need to show that it is true for \( k \). Taking \( 2k \) derivative on both sides of (2.2) and using the facts

\[
f^{(2i)}(0) = 0, f^{(2i+1)}(0) = 1 \text{ and } C^0_{2i+1} + C^2_{2i+1} + \cdots + C^{2i}_{2i+1} = 2^{2i} \text{ for } i \geq 0 \text{ with } C^0_{2s} + C^2_{2s} + \cdots + C^{2s}_{2s} = 2^{2s-1} \text{ for } s \geq 1,
\]

we can get \( z^{(2k)}(0) = 1 - n \). Similarly, we can prove \( z^{(2k+1)}(0) = 0 \).

By induction, the corollary holds.

Now we can prove the following estimation of two solutions to (2.2).
Lemma 2.2. Suppose $z(r)$ is a solution of (2.2) and $w = z - \overline{z}$, where $\overline{z}(r) = -(n - 1) \cosh r$, then there exists a positive constant $\delta$ such that
\[(2.3) \quad \frac{w(r_0)}{r_0^{n-1}} r^{n-1} \leq w(r) \leq \frac{w(r_0)}{r_0^{n+1}} r^{n+1}\]
for $0 < r_0 < r < \delta$.

Proof. Since $z$ and $\overline{z}$ are two solutions of (2.2) and $w = z - \overline{z}$, we have
\[(2.4) \quad f w' = a w,\]
where
\[a(r) = -[z + \overline{z} + (n - 2) f'] = -z(r) + f'(r) > 0.\]
Solving the separable equation (2.4), we can get
\[(2.5) \quad w(r) = w(r_0) e^{\int_{r_0}^{r} a(\tau)/f(\tau) d\tau}.\]
Noting that
\[\lim_{r \to 0^+} a(r) = n\]
and
\[\lim_{r \to 0^+} f(r)/r = 1,\]
we can get
\[\lim_{r \to 0^+} \left( \frac{a(\tau)}{f(\tau)} \right) / \left( \frac{n}{\tau} \right) = 1.\]
So there exists a positive constant $\delta > 0$, such that for $0 < r_0 < \tau < r < \delta$, there holds
\[\frac{n - 1}{\tau} \leq \frac{a(\tau)}{f(\tau)} \leq \frac{n + 1}{\tau}.\]
Substituting into (2.5), we can get (2.3). The conclusion is drawn. \(\square\)

We are ready to prove Lemma 1.1.

Proof of Lemma 1.1. As mentioned above, we know that
\[\overline{y}(r) = \sinh^{1-n} r\]
is a solution to (1.1) satisfying condition (1.2). If $y$ is also a solution to (1.1) and (1.2), then Corollary 2.1 guarantees $\lim_{r \to 0^+} w^{(j)}(r) = 0$ for $j = 0, 1, \cdots$. So one can get for any $\alpha > 0$,
\[\lim_{r \to 0^+} \frac{w(r)}{r^\alpha} = 0.\]
Taking $r_0 \to 0^+$ in Lemma 2.2, we can get
\[w(r) = 0 \text{ for } 0 < r < \delta.\]
Then the uniqueness theorem of O.D.E. implies
\[ w(r) = 0 \text{ for } r > 0. \]
That is to say,
\[ (\ln y)' = (\ln \overline{y})' \text{ for } r > 0. \]
So \( y = c\overline{y} \) for some constant \( c > 0 \). Hence the conclusion is drawn. \( \square \)

In the rest of this section, we want to prove Lemma 2.1. The idea is simple: We find the lower bound of \( \zeta \) first, then get the upper bound, and finally, compute the limit at 0.

Now let us estimate the lower bound of \( \zeta \). For each \( r > 0 \), let us consider a quadratic function
\[
Q(x) = x^2 + (n-1)\frac{f'}{f} \cdot x - (n-1)\frac{1}{f^2}
\]
in \( x \). Clearly, equation (2.1) can be rewritten by
\[
Q(x) = 0
\]
and the roots of \( Q(x) = 0 \) are given by
\[
R_1(r) = \frac{-(n-1)f' - \sqrt{(n-1)^2f'^2 + 4(n-1)}}{2f} < 0
\]
and
\[
R_2(r) = \frac{-(n-1)f' + \sqrt{(n-1)^2f'^2 + 4(n-1)}}{2f} > 0.
\]
We will show that a lower bound for \( x \) is \( R_1 \), that is, \( \zeta \geq fR_1 \). More precisely, we have

**Lemma 2.3.** If \( y(r) \) is a solution to (1.1)(1.2), then we can get
\[
0 > x(r) \geq R_1(r) \text{ for all } r > 0,
\]
or equivalently,
\[
Q(x(r)) \leq 0 \text{ for all } r > 0.
\]
Hence \( x(r) \) is increasing for \( r > 0 \) and
\[
\lim_{r \to 0^+} x(r) = -\infty.
\]

**Proof.** The idea of the proof is similar to that used in Lemma 2.1 [6]. Assume on the contrary, there exists \( \overline{r} > 0 \) such that
\[
x(\overline{r}) < R_1(\overline{r}).
\]
Setting
\[
\Sigma = \{ \omega \in (\overline{r}, +\infty) : x(r) < R_1(r) \text{ holds true for all } \overline{r} < r < \omega \},
\]
it is clear that $\Sigma$ is a closed set in $(\overline{r}, +\infty)$. We shall prove that $\Sigma$ is also a relative open set in $(\overline{r}, +\infty)$ to yield
$$\Sigma = (\overline{r}, +\infty)$$
by connection of $(\overline{r}, +\infty)$. In fact, letting $\omega_0 \in \Sigma$, we have
$$x(r) < R_1(r)$$
holds for all $r \in (\overline{r}, \omega_0)$. So
$$Q(x(r)) > 0 \quad \text{for all } r \in (\overline{r}, \omega_0).$$
Using (2.1), we have $x(r)$ is a strictly monotone decreasing function in $r \in (\overline{r}, \omega_0)$. On the other hand, noting that $R_1(r)$ is monotone non-decreasing in $r \in (\overline{r}, \omega_0)$, we have
$$x(\omega_0) - R_1(\omega_0) < x(\overline{r}) - R_1(\overline{r}) = -\delta < 0$$
for some positive number $\delta$. By continuity, we have $\omega_0$ is an interior point of $\Sigma$. So $\Sigma$ is also relative open in $(\overline{r}, +\infty)$. Hence
$$\Sigma = (\overline{r}, +\infty).$$
Consequently,
$$Q(x(r)) > 0$$
for all $r > \overline{r}$. As a result, $x(r)$ is a strictly monotone decreasing function in $r \in (\overline{r}, +\infty)$. In addition, by the monotonicity of $R_1(r)$, we have
(2.8) $$x(r) - R_1(r) < x(\overline{r}) - R_1(\overline{r}) = -\delta < 0$$
for all $r > \overline{r}$. Using (2.1) and the fact $x - R_2 < 0$, we can get
$$x' = -Q(x) = -[x(r) - R_1(r)][x(r) - R_2(r)] \leq \delta[x(r) - R_2(r)] \leq \delta x(r)$$
(2.9)
for $r > \overline{r}$. So
$$[e^{-\delta r} x(r)]' \leq 0 \quad \text{for } r > \overline{r}.$$ 
Consequently,
(2.10) $$x(r) \leq -C_0 e^{\delta r}$$
for some constant $C_0 > 0$ and $r > \overline{r}$.
Since $f'/f \to 1$ and $f^{-2} \to 0$ as $r \to +\infty$, by (2.1) and (2.6), we can get
$$Q(x(r)) \geq \frac{1}{2} x^2(r)$$
for \( r > M \), where \( M > \pi \) is a large number. As a result,

\[
x' \leq -\frac{1}{2}x^2 \text{ for } r > M.
\]

Consequently,

\[
-(x^{-1})' \leq -\frac{1}{2} \text{ for } r > M.
\]

After integrating over \( r > M \), we get

\[
x(r) \leq \frac{1}{r - M + x^{-1}(M)} \to -\infty
\]
as \( r \to (M - x^{-1}(M)) \). This contradicts the fact that \( x(r) \) is well-defined in \((0, +\infty)\). Hence for \( r > 0 \), we have

\[
0 > x(r) \geq R_1(r).
\]

From these inequalities, one can get \( Q(x) \leq 0 \), so \( x \) is increasing for \( r > 0 \). In addition, condition \((1.2)\) implies \( \ln y(r) \to +\infty \) as \( r \to 0^+ \), so we can get

\[
\liminf_{r \to 0^+} x(r) = -\infty.
\]

Hence \( \lim_{r \to 0^+} x(r) = -\infty \). Therefore the conclusion of the lemma is drawn. \( \square \)

Now we want to get the upper bound for \( z(r) \).

**Lemma 2.4.** If \( y(r) \) is a solution of \((1.1)(1.2)\), then we can obtain

\[
z(r) \leq Z_1
\]

for all \( r > 0 \), where

\[
Z_1 = \frac{-(n-2)f' - \sqrt{(n-2)^2f'^2 + 4(n-1)}}{2} < 0
\]

and

\[
Z_2 = \frac{-(n-2)f' + \sqrt{(n-2)^2f'^2 + 4(n-1)}}{2} > 0
\]

are roots of quadratic form

\[
\tilde{Q}(z) = z^2 + (n-2)f' \cdot z - (n-1).
\]

**Proof.** Similar to the proof of Lemma 2.3. Assume on the contrary, there exists \( \tilde{r} \in (0, +\infty) \) such that

\[
z(\tilde{r}) > Z_1.
\]

Setting

\[
\Sigma = \{ \omega \in (\tilde{r}, +\infty) : z(r) > Z_1 \text{ for all } r \in (\tilde{r}, \omega) \},
\]
we want to show that $\Sigma = (\bar{r}, +\infty)$. In fact, $\Sigma \neq \emptyset$ by continuity. It’s also clearly that $\Sigma$ is a closed subset in $(\bar{r}, +\infty)$. We remains to show that $\Sigma$ is also relative open in $(\bar{r}, +\infty)$. Actually, for $\omega_0 \in \Sigma$, we have $z(r)$ is a strictly monotone increasing function in $r \in (\bar{r}, \omega_0)$ by equation (2.2). On the other hand, since $Z_1(r)$ is a monotone non-increasing function in $r \in (\bar{r}, \omega_0)$, we have

$$0 > z(\omega_0) > Z_1(\omega_0).$$

Consequently, $\omega_0$ is an interior point of $\Sigma$. Hence $\Sigma = (\bar{r}, +\infty)$.

Now we divide this problem into two cases.

Case one: $n = 2$. In this case $Z_1 = -1$, so $z(r) > -1$ for $r > \bar{r}$. Since $z = fx$, one have $x = zf^{-1}$. So

$$(\ln y)' = zf^{-1} > -f^{-1}$$

for $r > \bar{r}$. Hence

$$y(r) \geq y(\bar{r})e^{-\int_{\bar{r}}^r f^{-1}(t)dt}.$$ 

From this, we can get $\lim_{r \to +\infty} y(r) > 0$. This contradicts the boundary condition $\lim_{r \to +\infty} y(r) = 0$.

Case two: $n \geq 3$. Using equation (2.2), we have $z(r)$ is strictly monotone decreasing function in $r \in (0, \bar{r})$. So

$$(2.12) \quad 0 > z(r) \geq -\beta \text{ for } r > \bar{r}$$

for some constant $\beta > 0$. As a result,

$$(2.13) \quad -z^2(r) + n - 1 \geq -\beta^2 + n - 1 \equiv -\overline{\beta}.$$ 

So it follows from equation (2.2) that

$$fz' \geq -(n-2)f' \cdot z - \overline{\beta},$$

or equivalent

$$(f^{n-2}z)' \geq -\overline{\beta}f^{n-3}$$

for all $r > \bar{r}$. Consequently,

$$f^{n-2}(r)z(r) \geq -\overline{\beta} \int_{\bar{r}}^r f^{n-3}(\tau)d\tau + f^{n-2}(\bar{r})z(\bar{r}),$$

or

$$0 > z(r) \geq -\overline{\beta} \frac{\int_{\bar{r}}^r f^{n-3}(\tau)d\tau}{f^{n-2}(r)} + \frac{f^{n-2}(\bar{r})z(\bar{r})}{f^{n-2}(r)} \to 0^-$$

as $r \to +\infty$, where we have used

$$\lim_{r \to +\infty} f'(r) = +\infty.$$
and L’ Hospital’s rule to get
\[
\lim_{{r \to +\infty}} \frac{\int_{{r}}^{r'} f^{n-3}(\tau) d\tau}{f^{n-2}(r)} = \lim_{{r \to +\infty}} \frac{f^{n-3}(r)}{(n-2)f^{n-3}(r)f'(r)} = 0.
\]
Using (2.14) and equation (2.2), we have
\[
f'z' \geq -(n-2)f'z + \left( n - \frac{3}{2} \right),
\]
or
\[
(f^{n-2}z)' \geq \left( n - \frac{3}{2} \right) f^{n-3}
\]
for \( r > K, \) \( K \) large enough. Since \( n \geq 3 \) and \( \lim_{{r \to +\infty}} f(r) = +\infty, \) integrating over \( r > K, \) we can get
\[
f^{n-2}(r)z(r) \geq \left( n - \frac{3}{2} \right) \int_{{K}}^{r} f^{n-3}(\tau) d\tau + f^{n-2}(K)z(K)
\]
(2.15)
\[
\geq \left( n - \frac{5}{3} \right) \int_{{K}}^{r} f^{n-3}(\tau) d\tau
\]
for \( r > M', M' > K \) large enough. Consequently,
\[
z(r) \geq \left( n - \frac{5}{3} \right) \frac{\int_{{r}}^{r'} f^{n-3}(\tau) d\tau}{f^{n-2}(r)} > 0
\]
provided \( r > M'. \) This contradicts the assumption \( z < 0. \)
Combining above results, the lemma is proved. \( \square \)

**Corollary 2.2.** Let \( z \) be the same as in Lemma 2.4, then \( z(r) \) is a monotone non-increasing function for \( r > 0. \).

**Proof.** Noting that \( Z_1(r) \) is the smaller root of quadratic form \( \tilde{Q}(z) \) and (2.2) can be rewritten by
\[
f'z' = -\tilde{Q}(z) = -(z - Z_1)(z - Z_2) \leq 0,
\]
so \( z(r) \) is a monotone non-increasing function in \( r > 0. \) \( \square \)

Now let us prove Lemma 2.1.

**Proof of Lemma 2.1** By Lemma 2.3 and Lemma 2.4, we can get
\[
f \cdot R_1(r) \leq z(r) \leq Z_1(r).
\]
Passing to the limits, we can get
\[
fR_1|_{r \to 0^+} \leq \lim_{r \to 0^+} z(r) \leq Z_1(0) = 1 - n,
\]
(2.17)
where
\[ f \cdot R_1 \big|_{r \rightarrow 0^+} = -\frac{(n-1)f'(0) + \sqrt{(n-1)^2 f''(0) + 4(n-1)}}{2}. \]

We need to show that \( \lim_{r \rightarrow 0^+} z(r) = Z_1(0) \). Assuming on the contrary, by (2.17), we have
\[ \lim_{r \rightarrow 0^+} z(r) < Z_1(0). \]

By continuity, there exist small constants \( r_0 > 0 \) and \( \kappa > 0 \) such that
\[ z(r) < Z_1(r) - \kappa \]
for all \( 0 < r < r_0 \). Substituting into (2.16), we get
\[ z'(r) \leq -\frac{\kappa'}{f(r)} \]
for some positive constant \( \kappa' \). Noting that there exists a positive constant \( C \) such that
\[ 0 < f(r) \leq Cr \]
for all \( 0 < r < r_0 \), after integrating (2.19), we can get
\[ 0 > z(r) \geq z(r_0) + \kappa' \int_r^{r_0} \frac{1}{f(\tau)}d\tau \rightarrow +\infty \]
as \( r \rightarrow 0^+ \). This is impossible. Hence the lemma is proved. 

3. Proof of Theorem 1.1

Proof of Theorem 1.1. If \( u \) is a rotationally symmetric harmonic map from \( \mathbb{H}^n \) onto \( \mathbb{R}^n \), then we can assume \( u(r;\theta) = (y(r),\theta) \) up to a rotation of \( \mathbb{R}^n \). By (1.2) in [4] for example, \( y(r) \) should satisfy the equation (1.1). Furthermore, if \( u \) is an orientation-reversing diffeomorphism, then condition (1.2) is satisfied.

By Lemma 1.1, equation (1.1) with (1.2) has a unique solution up to a dilation. Hence the theorem holds. 

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