Upper bounds for the bondage number of graphs on topological surfaces

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Abstract

The bondage number \( b(G) \) of a graph \( G \) is the smallest number of edges of \( G \) whose removal results in a graph having the domination number larger than that of \( G \). We show that, for a graph \( G \) having the maximum vertex degree \( \Delta(G) \) and embeddable on an orientable surface of genus \( h \) and a non-orientable surface of genus \( k \),

\[
b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}.
\]

This generalizes known upper bounds for planar and toroidal graphs, and can be improved for bigger values of the genera \( h \) and \( k \) by adjusting the proofs.

Key words: Bondage number, Domination number, Topological surface, Embedding on a surface, Euler’s formula

1. Introduction

We consider simple finite non-empty graphs. For a graph \( G \), its vertex and edge sets are denoted, respectively, by \( V(G) \) and \( E(G) \). We also use the following standard notation: \( d(v) \) for the degree of a vertex \( v \) in \( G \), \( \Delta = \Delta(G) \) for the maximum vertex degree of \( G \), \( \delta = \delta(G) \) for the minimum vertex degree of \( G \), and \( N(v) \) for the neighbourhood of a vertex \( v \) in \( G \).
A set $D \subseteq V(G)$ is a dominating set if every vertex not in $D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma(G)$. Clearly, for any spanning subgraph $H$ of $G$, $\gamma(H) \geq \gamma(G)$. The bondage number of $G$, denoted by $b(G)$, is the minimum cardinality of a set of edges $B \subseteq E(G)$ such that $\gamma(G - B) > \gamma(G)$, where $V(G - B) = V(G)$ and $E(G - B) = E(G) \setminus B$. In a sense, the bondage number $b(G)$ measures integrity and reliability of the domination number $\gamma(G)$ with respect to the edge removal from $G$, which may correspond, e.g., to link failures in communication networks.

The bondage number was introduced by Bauer et al. [1] (see also Fink et al. [4]). Two unsolved classical conjectures for the bondage number of arbitrary and planar graphs are as follows.

**Conjecture 1** (Teschner [9]). For any graph $G$, $b(G) \leq \frac{3}{2}\Delta(G)$.

Hartnell and Rall [6] and Teschner [10] showed that for the cartesian product $G_n = K_n \times K_n$, $n \geq 2$, the bound of Conjecture 1 is sharp, i.e. $b(G_n) = \frac{3}{2}\Delta(G_n)$. Teschner [9] also proved that Conjecture 1 holds when $\gamma(G) \leq 3$.

**Conjecture 2** (Dunbar et al. [3]). If $G$ is a planar graph, then $b(G) \leq \Delta(G) + 1$.

The planar graphs are precisely the graphs that can be drawn on the sphere with no crossing edges. A topological surface $S$ can be obtained from the sphere $S_0$ by adding a number of handles or crosscaps. If we add $h$ handles to $S_0$, we obtain an orientable surface $S_h$, which is often referred to as the $h$-holed torus. The number $h$ is called the orientable genus of $S_h$. If we add $k$ crosscaps to the sphere $S_0$, we obtain a non-orientable surface $N_k$. The number $k$ is called the non-orientable genus of $N_k$. Any topological surface is homeomorphically equivalent either to $S_h$ ($h \geq 0$), or to $N_k$ ($k \geq 1$). For example, $S_1$, $N_1$, $N_2$ are the torus, the projective plane, and the Klein bottle, respectively.

A graph $G$ is embeddable on a topological surface $S$ if it admits a drawing on the surface with no crossing edges. Such a drawing of $G$ on the surface $S$ is called an embedding of $G$ on $S$. Notice that there can be many different embeddings of the same graph $G$ on a particular surface $S$. The embeddings can be distinguished and classified by different properties. The set of faces of a particular embedding of $G$ on $S$ is denoted by $F(G)$.
An embedding of \( G \) on the surface \( S \) is a 2-cell embedding if each face of the embedding is homeomorphic to an open disk. In other words, a 2-cell embedding is an embedding on \( S \) that “fits” the surface. This is expressed in Euler’s formulae (1) and (2) of Theorem 3. For example, a cycle \( C_n \) \((n \geq 3)\) does not have a 2-cell embedding on the torus, but it has 2-cell embeddings on the sphere and the projective plane. Similarly, a planar graph may have 2-cell and non-2-cell embeddings on the torus.

The following result is usually known as (generalized) Euler’s formula. We state it here in a form similar to Thomassen [11].

**Theorem 3** (Euler’s Formula, [11]). Suppose a connected graph \( G \) with \(|V(G)| \) vertices and \(|E(G)| \) edges admits a 2-cell embedding having \(|F(G)| \) faces on a topological surface \( S \). Then, either \( S = S_h \) and

\[
|V(G)| - |E(G)| + |F(G)| = 2 - 2h, \tag{1}
\]

or \( S = N_k \) and

\[
|V(G)| - |E(G)| + |F(G)| = 2 - k. \tag{2}
\]

Equation (1) is usually referred to as Euler’s formula for an orientable surface \( S_h \) of genus \( h \), \( h \geq 0 \), and Equation (2) is known as Euler’s formula for a non-orientable surface \( N_k \) of genus \( k \), \( k \geq 1 \).

The orientable genus of a graph \( G \) is the smallest integer \( h = h(G) \) such that \( G \) admits an embedding on an orientable topological surface \( S \) of genus \( h \). The non-orientable genus of \( G \) is the smallest integer \( k = k(G) \) such that \( G \) can be embedded on a non-orientable topological surface \( S \) of genus \( k \). Clearly, in general, \( h(G) \neq k(G) \), and the embeddings on \( S_{h(G)} \) and \( N_{k(G)} \) must be 2-cell embeddings.

Trying to prove Conjecture 2, Kang and Yuan [7] came up with the following upper bound whose simpler topological proof was later discovered by Carlson and Develin [2].

**Theorem 4** ([7, 2]). For any connected planar graph \( G \),

\[
b(G) \leq \min\{8, \Delta(G) + 2\}.
\]

This solves Conjecture 2 in case \( \Delta(G) \geq 7 \). The upper bound of Theorem 4 is for the sphere \( S_0 \) that has orientable genus \( h = 0 \). The proof of Theorem 4 in [2] is topologically intuitive, uses Euler’s formula for the sphere, and allows its authors to establish a partially similar result for the torus.

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Theorem 5 ([2]). For any connected toroidal graph $G$, $b(G) \leq \Delta(G) + 3$.

Notice that the torus $S_1$ has orientable genus $h = 1$. As mentioned in [2], it is sufficient to prove the results of Theorems 4 and 5 for connected graphs because the bondage number of a disconnected graph $G$ is the minimum of the bondage numbers of its components.

In this paper, we prove the following result which generalizes the corresponding upper bounds of Theorems 4 and 5 for any orientable or non-orientable topological surface $S$.

Theorem 6. For a connected graph $G$ of orientable genus $h$ and non-orientable genus $k$,

$$b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}.$$

The upper bound of Theorem 6 follows from Theorems 8 and 9 proved below in Section 2, and can be improved for bigger values of the genera $h$ and $k$ by adjusting the proofs.

2. The bondage number on orientable and non-orientable surfaces

In this section, we prove Theorem 6 by considering orientable and non-orientable surfaces separately. The proofs are done by using Euler’s formulae (1) and (2), counting arguments, and the following result.

Lemma 7 (Hartnell and Rall [6]). For any edge $uv$ in a graph $G$, we have $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$. In particular, this implies that $b(G) \leq \delta(G) + \Delta(G) - 1$ (see also [1, 4]).

Having a graph $G$ embedded on a surface $S$, each edge $e_i = uv \in E(G)$, $i = 1, \ldots, |E(G)|$, can be assigned two weights, $w_i = \frac{1}{d(u)} + \frac{1}{d(v)}$ and $f_i = \frac{1}{m} + \frac{1}{m''}$, where $m'$ is the number of edges on the boundary of a face on one side of $e_i$, and $m''$ is the number of edges on the boundary of the face on the other side of $e_i$. Notice that, in an embedding on a surface, an edge $e_i$ may be not separating two distinct faces, but instead it can appear twice on the boundary of the same face. For example, every edge of a path $P_n$ ($n \geq 2$) embedded on the sphere is on the boundary of a unique face, and it appears exactly twice on the face boundary walk: once for each side of the edge. Clearly, in this case, $m' = m'' = 2(n - 1)$ and $f_i = \frac{2}{m'} = \frac{2}{m''} = \frac{1}{n-1}$. 
Notice that weights \( w_i \) and \( f_i \), \( i = 1, \ldots, |E(G)| \), count the number of vertices of \( G \) and faces of its embedding on \( S \) as follows:

\[
\sum_{i=1}^{|E(G)|} w_i = |V(G)|, \quad \sum_{i=1}^{|E(G)|} f_i = |F(G)|.
\]

Then, by Euler’s formula (1), we have

\[
\sum_{i=1}^{|E(G)|} (w_i + f_i - 1) = |V(G)| + |F(G)| - |E(G)| = 2 - 2h,
\]

or, in other words,

\[
\sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 - \frac{2 - 2h}{|E(G)|} \right) = \sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 + \frac{2h - 2}{|E(G)|} \right) = 0.
\]

Now, each edge \( e_i = uv \in E(G) \), \( i = 1, \ldots, |E(G)| \), can be associated with the quantity \( w_i + f_i - 1 + \frac{2h - 2}{|E(G)|} \) called the oriented curvature of the edge. Also, by Euler’s formula (2), we have

\[
\sum_{i=1}^{|E(G)|} (w_i + f_i - 1) = |V(G)| + |F(G)| - |E(G)| = 2 - k,
\]

or, in other words,

\[
\sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 - \frac{2 - k}{|E(G)|} \right) = \sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 + \frac{k - 2}{|E(G)|} \right) = 0.
\]

Then, each edge \( e_i = uv \in E(G) \), \( i = 1, \ldots, |E(G)| \), can be associated with the quantity \( w_i + f_i - 1 + \frac{k - 2}{|E(G)|} \) called the non-oriented curvature of the edge.

**Theorem 8.** Let \( G \) be a connected graph 2-cell embeddable on an orientable surface of genus \( h \geq 0 \). Then

\[
b(G) \leq \Delta(G) + h + 2.
\]
Proof. Suppose \( G \) is 2-cell embedded on the \( h \)-holed torus \( S_h \). By Lemma 7, if \( G \) has any vertices of degree \( h + 3 \) or less, we have \( \delta(G) \leq h + 3 \), and inequality (3) holds. Therefore, we can assume \( \Delta(G) \geq \delta(G) \geq h + 4 \).

Now, suppose the opposite, \( b(G) \geq \Delta(G) + h + 3 \). Then, by Lemma 7, for any edge \( e_i = uv \), \( i = 1, \ldots, |E(G)| \), we have

\[
d(u) + d(v) - 1 - |N(u) \cap N(v)| \geq b(G) \geq \Delta(G) + h + 3.
\]

This gives

\[
d(u) + d(v) \geq \Delta(G) + h + 4 + |N(u) \cap N(v)|,
\]

and \( d(u) \leq \Delta(G) \), \( d(v) \leq \Delta(G) \). If either \( d(u) \) or \( d(v) \) is equal to \( h + 4 \), then, by (4), the other degree must be equal to \( \Delta(G) \geq h + 4 \), and \( u \) and \( v \) cannot have any common neighbors, so that \( m' \) and \( m'' \) are at least 4 each. Since in this case \( |E(G)| \geq \frac{(h+4)(h+5)}{2} \), such an edge \( e_i = uv \) has a negative oriented curvature:

\[
w_i + f_i - 1 + \frac{2h-2}{|E(G)|} \leq \frac{2}{h+4} + \frac{2}{4} - 1 + \frac{2(2h-2)}{(h+4)(h+5)} = \frac{-8 + h(3-h)}{2(h+4)(h+5)} < 0
\]

for any \( h \geq 1 \), and, in case \( h = 0 \),

\[
w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{2}{|E(G)|} = \frac{-2}{|E(G)|} < 0.
\]

Suppose one of \( d(u) \) and \( d(v) \) is equal to \( h + 5 \), without loss of generality, \( d(u) = h + 5 \). Then, by (4), \( \Delta(G) \geq d(v) \geq \Delta(G) - 1 + |N(u) \cap N(v)| \).

If \( d(v) = h + 4 = \Delta(G) - 1 \), we are in the previous case. Otherwise, we have \( d(v) \geq h + 5 \), and, by (4), at most one of \( m' \) and \( m'' \) can be equal to 3, implying the other is at least 4. Then again, since in this case \( |E(G)| \geq \frac{(h+4)(h+4)+2(h+5)}{2} = \frac{h^2+10h+26}{2} \), the edge \( e_i \) must have a negative oriented curvature:

\[
w_i + f_i - 1 + \frac{2h-2}{|E(G)|} \leq \frac{2}{h+5} + \frac{1}{3} + \frac{1}{4} - 1 + \frac{2(2h-2)}{h^2+10h+26} = \frac{-5h^3-3h^2+52h-266}{12(h+5)(h^2+10h+26)} < 0
\]

for any \( h \geq 1 \), and, in case \( h = 0 \),

\[
w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{2}{|E(G)|} = \frac{-1}{60} - \frac{2}{|E(G)|} < 0.
\]
The only remaining case is when \( d(u) \geq h + 6 \) and \( d(v) \geq h + 6 \). Since \( m' \geq 3 \) and \( m'' \geq 3 \), and, in this case, \( |E(G)| \geq \frac{(h+4)(h+5)+2(h+6)}{2} = \frac{h^2+11h+32}{2} \), the edge \( e_i \) must have a negative oriented curvature:

\[
\frac{w_i+f_i-1+2h-2}{|E(G)|} \leq \frac{2}{h+6} + \frac{2}{3} - 1 + \frac{2(2h-2)}{h^2 + 11h + 32} = \frac{-h^3 + h^2 + 28h - 72}{3(h+6)(h^2 + 11h + 32)} < 0
\]

for any \( h \geq 1 \), and, in case \( h = 0 \),

\[
w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 - \frac{2}{|E(G)|} = \frac{-2}{|E(G)|} < 0.
\]

Summing over all edges \( e_i \in E(G) \) yields

\[
\sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 + \frac{2h-2}{|E(G)|} \right) < 0,
\]

which is a contradiction to Euler’s formula (1) stating

\[
\sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 - \frac{2-2h}{|E(G)|} \right) = |V(G)| + |F(G)| - |E(G)| - (2-2h) = 0.
\]

Thus, \( b(G) \leq \Delta(G) + h + 2 \). \qed

**Theorem 9.** Let \( G \) be a connected graph 2-cell embeddable on a non-orientable surface of genus \( k \geq 1 \). Then

\[
b(G) \leq \Delta(G) + k + 1. \tag{5}
\]

**Proof.** Suppose \( G \) is 2-cell embedded on the sphere with \( k \) crosscaps \( N_k \). By Lemma 7, if \( G \) has any vertices of degree \( k + 2 \) or less, we have \( \delta(G) \leq k + 2 \), and inequality (5) holds. Therefore, we can assume \( \Delta(G) \geq \delta(G) \geq k + 3 \).

Suppose the opposite, \( b(G) \geq \Delta(G) + k + 2 \). Then, by Lemma 7, for any edge \( e_i = uv, i = 1, \ldots, |E(G)| \), we have \( d(u) + d(v) - 1 - |N(u) \cap N(v)| \geq b(G) \geq \Delta(G) + k + 2 \). Then, \( d(u) + d(v) \geq \Delta(G) + k + 3 + |N(u) \cap N(v)| \), and \( d(u) \leq \Delta(G), d(v) \leq \Delta(G) \). If either \( d(u) \) or \( d(v) \) is equal to \( k + 3 \), the other degree must be equal to \( \Delta(G) \geq k + 3 \), and \( u \) and \( v \) cannot have any common neighbors, so that \( m' \) and \( m'' \) are at least 4 each. Since in this case \( |E(G)| \geq \frac{(k+3)(k+4)}{2} \), the non-oriented curvature of the edge \( e_i = uv \) is

\[
w_i + f_i - 1 + \frac{k-2}{|E(G)|} \leq \frac{2}{k+3} + \frac{2}{4} - 1 + \frac{2(k-2)}{(k+3)(k+4)} = \frac{-4 + k(1-k)}{2(k+3)(k+4)} < 0
\]
for any \( k \geq 2 \), and, in case \( k = 1 \),

\[
\sum_{i=1}^{\frac{|E(G)|}{2}} \left( w_i + f_i - 1 - \frac{1}{|E(G)|} \right) \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{1}{|E(G)|} = -1 < 0.
\]

Suppose one of \( d(u) \) and \( d(v) \), let us say \( d(u) \), is equal to \( k + 4 \). Then, \( \Delta(G) \geq d(v) \geq \Delta(G) - 1 + |N(u) \cap N(v)| \). If \( d(v) = k + 3 = \Delta(G) - 1 \), we are in the previous case. Otherwise, we have \( d(v) \geq k + 4 \), and at most one of \( m' \) and \( m'' \) can be equal to 3, implying the other is at least 4. Then again, since in this case \( |E(G)| \geq \frac{(k+3)(k+3)+2(k+4)}{2} = \frac{k^2 + 8k + 17}{2} \), the edge \( e_i \) must have a negative non-oriented curvature:

\[
\sum_{i=1}^{\frac{|E(G)|}{2}} \left( w_i + f_i - 1 - \frac{k - 2}{|E(G)|} \right) \leq \frac{2}{k + 4} + \frac{1}{3} + \frac{1}{4} - 1 + \frac{2(k - 2)}{k^2 + 8k + 17} = -\frac{124 - 5k - 12k^2 - 5k^3}{12(k + 4)(k^2 + 8k + 17)} < 0
\]

for any \( k \geq 2 \), and, in case \( k = 1 \),

\[
\sum_{i=1}^{\frac{|E(G)|}{2}} \left( w_i + f_i - 1 - \frac{1}{|E(G)|} \right) \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{1}{|E(G)|} = -\frac{1}{60} < 0.
\]

The only remaining case is when \( d(u) \geq k + 5 \) and \( d(v) \geq k + 5 \). Since \( m' \geq 3 \) and \( m'' \geq 3 \), and, in this case, \( |E(G)| \geq \frac{(k+3)(k+4)+2(k+5)}{2} = \frac{k^2 + 9k + 22}{2} \), the edge \( e_i \) must have a negative non-oriented curvature:

\[
\sum_{i=1}^{\frac{|E(G)|}{2}} \left( w_i + f_i - 1 - \frac{k - 2}{|E(G)|} \right) \leq \frac{2}{k + 3} + \frac{2}{3} + 1 + \frac{2(k - 2)}{k^2 + 9k + 22} = -\frac{k^3 - 2k^2 + 5k - 38}{3(k + 5)(k^2 + 9k + 22)} < 0
\]

for any \( k \geq 2 \), and, in case \( k = 1 \),

\[
\sum_{i=1}^{\frac{|E(G)|}{2}} \left( w_i + f_i - 1 - \frac{1}{|E(G)|} \right) \leq \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + 1 - \frac{1}{|E(G)|} = \frac{1}{60} < 0.
\]

Summing over all edges \( e_i \in E(G) \) yields

\[
\sum_{i=1}^{\frac{|E(G)|}{2}} \left( w_i + f_i - 1 + \frac{k - 2}{|E(G)|} \right) < 0,
\]

which is a contradiction to Euler’s formula (2) stating

\[
\sum_{i=1}^{\frac{|E(G)|}{2}} \left( w_i + f_i - 1 - \frac{2 - k}{|E(G)|} \right) = |V(G)| + |F(G)| - |E(G)| - (2 - k) = 0.
\]

Thus, \( b(G) \leq \Delta(G) + k + 1 \), and the proof is complete. \( \square \)
3. Conclusions and final remarks

The upper bound of Theorem 6 provides a hierarchy of upper bounds that eventually may help solving Conjecture 1. However, it can be seen that the bounds of Theorems 8 and 9 are not tight for larger values of the genera \( h = h(G) \) and \( k = k(G) \). For example, by adjusting respectively the proofs of Theorems 8 and 9, upper bound (3) can be improved to \( b(G) \leq \Delta(G) + h + 1 \) for \( h \geq 8 \), to \( b(G) \leq \Delta(G) + h \) for \( h \geq 11 \), etc., and upper bound (5) can be improved to \( b(G) \leq \Delta(G) + k \) for \( k \geq 3 \), to \( b(G) \leq \Delta(G) + k - 1 \) for \( k \geq 6 \), etc. It is left to the reader to adjust the proofs and bounds for a particular topological surface of higher genus. The bounds of Theorems 8 and 9 are stated in this form for clarity and simplicity of presentation and proofs for smaller values of \( h \) and \( k \).

In general, one may try to find certain (linear or sublinear) functions of \( h \) and \( k \) to improve the bounds of Theorems 8 and 9 by replacing the terms \( h + 2 \) and \( k + 1 \), respectively, or to provide asymptotically better bounds. For example, simple asymptotic improvements follow from the upper bounds on the minimum vertex degree of graphs embeddable on topological surfaces: it is known that \( \delta(G) \leq \left\lfloor \frac{5 + \sqrt{1 + 48h^2}}{2} \right\rfloor \) for \( h \geq 1 \), \( \delta(G) \leq \left\lfloor \frac{5 + \sqrt{1 + 24k^2}}{2} \right\rfloor \) for \( k \geq 2 \) (e.g., see Sachs [8]), and \( \delta(G) \leq 5 \) for a planar or projective-planar graph, i.e. when \( h = 0 \) or \( k = 1 \). Then, from Lemma 7, we have \( b(G) \leq \Delta(G) + \left\lfloor \frac{3 + \sqrt{1 + 48h^2}}{2} \right\rfloor \) for \( h \geq 1 \) and \( b(G) \leq \Delta(G) + \left\lfloor \frac{3 + \sqrt{1 + 24k^2}}{2} \right\rfloor \) for \( k \geq 1 \), which are better than bounds (3) for \( h \geq 12 \) and (5) for \( k \geq 8 \), respectively. However, for example, an adjusted proof of Theorem 9 gives \( b(G) \leq \Delta + k - 411 = \Delta + 53 \) for \( k = 464 \), which is better than \( b(G) \leq \Delta(G) + \left\lfloor \frac{3 + \sqrt{1 + 24k^2}}{2} \right\rfloor \) in this case. Therefore, adjustments of the proofs of Theorems 8 and 9 can provide better results than some asymptotic improvements by using closed formulae, and it would be interesting to have closed formula or asymptotic improvements providing a certain justification of their quality.

In view of Theorem 4, its proof in [2], and results presented in this paper, it should be reasonable to conjecture that, when \( \Delta(G) \) is sufficiently large, the bondage number \( b(G) \) is bounded by a certain constant depending only on the properties of topological surfaces where \( G \) embeds.

**Conjecture 10.** For a connected graph \( G \) of orientable genus \( h \) and non-orientable genus \( k \), \( b(G) \leq \min\{c_h, c'_k, \Delta(G) + o(h), \Delta(G) + o(k)\} \), where \( c_h \) and \( c'_k \) are constants depending, respectively, on the orientable and non-orientable genera of \( G \).
Since $\delta(G) \leq 5$ for a planar graph $G$, Fischermann et al. [5] ask whether there exist planar graphs of bondage numbers 6, 7, or 8. A class of planar graphs with the bondage number equal to 6 is shown in [2]. Therefore, in the case of planar graphs, we have $6 \leq c_0 \leq 8$. It would be interesting to have an estimation for the constants $c_h$ and $c'_k$ for the torus $S_1$, projective plane $N_1$, and Klein bottle $N_2$.

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References

[1] D. Bauer, F. Harary, J. Nieminen, C.L. Suffel, Domination alteration sets in graphs, Discrete Math. 47 (1983) 153–161.

[2] K. Carlson, M. Develin, On the bondage number of planar and directed graphs, Discrete Math. 306 (2006) 820–826.

[3] J.E. Dunbar, T.W. Haynes, U. Teschner, L. Volkmann, Bondage insensitivity and reinforcement, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998, pp. 471–489.

[4] J.F. Fink, M.S. Jacobson, L.F. Kinch, J. Roberts, The bondage number of a graph, Discrete Math. 86 (1990) 47–57.

[5] M. Fischermann, D. Rautenbach, L. Volkmann, Remarks on the bondage number of planar graphs, Discrete Math. 260 (2003) 57–67.

[6] B.L. Hartnell, D.F. Rall, Bounds on the bondage number of a graph, Discrete Math. 128 (1994) 173–177.

[7] L. Kang, J. Yuan, Bondage number of planar graphs, Discrete Math. 222 (2000) 191–198.

[8] H. Sachs, Einführung in die Theorie der endlichen Graphen, Teil II, Teubner, Leipzig, 1972 (in German).
[9] U. Teschner, A new upper bound for the bondage number of graphs with small domination number, Australas. J. Combin. 12 (1995) 27–35.

[10] U. Teschner, The bondage number of a graph $G$ can be much greater than $\Delta(G)$, Ars Combin. 43 (1996) 81–87.

[11] C. Thomassen, The Jordan-Schönflies theorem and the classification of surfaces, Amer. Math. Monthly 99 (1992) 116-131.