GRAPHICAL TRANSLATORS FOR MEAN CURVATURE FLOW

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Abstract. In this paper we provide a full classification of complete translating graphs in $\mathbb{R}^3$. We also construct $(n - 1)$-parameter families of new examples of translating graphs in $\mathbb{R}^{n+1}$.

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1. Introduction

A translator is a hypersurface $M$ in $\mathbb{R}^{n+1}$ such that

$$t \mapsto M - t e_{n+1}$$

is a mean curvature flow, i.e., such that normal component of the velocity at each point is equal to the mean curvature at that point:

$$\vec{H} = -e_{n+1}^\perp.$$  \hfill (1.1)

If a translator $M$ is the graph of function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, we will say that $M$ is a translating graph; in that case, we also refer to the function $u$ as a translator, and we say that $u$ is complete if its graph is a complete submanifold of $\mathbb{R}^{n+1}$. Thus
$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a translator if and only if it solves the translator equation (the nonparametric form of (1.1)):

$$D_i \left( \frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = -\frac{1}{\sqrt{1 + |Du|^2}}.$$

The equation can also be written as

$$1 + |Du|^2 \Delta u - D_i u D_j u D_{ij} u + |Du|^2 + 1 = 0.$$

In this paper, we classify all complete translating graphs in $\mathbb{R}^3$. In another paper [HIMW18], we construct new families of complete, properly embedded (non-graphical) translators: a two-parameter family of translating annuli, examples that resemble Scherk’s minimal surfaces, and examples that resemble helicoids.

Before stating our classification theorem, we recall the known examples of translating graphs in $\mathbb{R}^3$. First, the Cartesian product of the grim reaper curve with $\mathbb{R}$ is a translator:

$$\mathcal{G} : \mathbb{R} \times (-\pi/2, \pi/2) \to \mathbb{R},$$

$$\mathcal{G}(x, y) = \log(\cos y).$$

We refer to it as the grim reaper surface.

![Figure 1. The grim reaper surface in $\mathbb{R}^3$, and that surface tilted by angle $\theta = -\pi/4$ and dilated by $1/\cos(\pi/4)$.](image-url)
Second, if we rotate the grim reaper surface by an angle \( \theta \in (0, \pi/2) \) about the 
\( y \)-axis and dilate by \( 1/\cos \theta \), the resulting surface is again a translator, given by 
\[
G_\theta : \mathbb{R} \times (-b, b) \to \mathbb{R},
\]

\[
G_\theta(x, y) = \frac{\log(\cos(y \cos \theta))}{\cos^2 \theta} + x \tan \theta,
\]
where \( b = \pi/(2 \cos \theta) \). Note that as \( \theta \) goes from 0 to \( \pi/2 \), the width \( 2b \) of the strip 
goes from \( \pi \) to \( \infty \). We refer to these examples as \textit{tilted grim reaper surfaces}.

Every translator \( \mathbb{R}^3 \) with zero Gauss curvature is (up to translations and up to 
rotations about a vertical axis) a grim reaper surface, a tilted grim reaper surface, 
or a vertical plane. See [MSHS15] or Theorem 2.2 below.

In [CSS07], J. Clutterbuck, O. Schnürer and F. Schulze (see also [AW94]) proved 
for each \( n \geq 2 \) that there is a unique (up to vertical translation) entire, rotationally 
invariant function \( u : \mathbb{R}^n \to \mathbb{R} \) whose graph is a translator. It is called the \textit{bowl soliton}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bowl_soliton.png}
\caption{The bowl soliton. As one moves down, the slope tends 
to infinity, and thus the end is asymptotically cylindrical.}
\end{figure}

In addition to the examples described above, Imanen (in unpublished work) 
proved that for each \( 0 < k < 1/2 \), there is a translator \( u : \Omega \to \mathbb{R} \) with the 
following properties: \( u(x, y) \equiv u(-x, y) \equiv u(x, -y) \), \( u \) attains its maximum at 
(0, 0) \( \in \Omega \), and 
\[
D^2 u(0, 0) = \begin{bmatrix} -k & 0 \\ 0 & -(1 - k) \end{bmatrix}.
\]

The domain \( \Omega \) is either a strip \( \mathbb{R} \times (-b, b) \) or \( \mathbb{R}^2 \). He referred to these examples as \textit{\( \Delta \)-wings}. As \( k \to 0 \), he showed that the examples converge to the grim reaper
surface. Uniqueness (for a given \( k \)) was not known. It was also not known which strips \( \mathbb{R} \times (-b, b) \) occur as domains of such examples. This paper is primarily about translators in \( \mathbb{R}^3 \), but in Section 8 we extend Ilmanen’s original proof to get \( \Delta \)-wings in \( \mathbb{R}^{n+1} \) that have prescribed principal curvatures at the origin. For \( n \geq 3 \), the examples include entire graphs that are not rotationally invariant. In Section 11 we modify the construction to produce a family of \( \Delta \)-wings in \( \mathbb{R}^{n+2} \) over any given slab of width \( > \pi \). See [Wan11] for a different construction of some higher dimensional graphical translators.

Figure 3. The \( \Delta \)-wing of width \( \sqrt{2} \pi \). As \( y \to \pm \infty \), this \( \Delta \)-wing is asymptotic to the tilted grim reapers \( \mathcal{G}_{-\pi} \) and \( \mathcal{G}_{\pi} \), respectively.

The main result of this paper is the following theorem.

**Theorem 1.1.** For every \( b > \pi/2 \), there is (up to translation) a unique complete, strictly convex translator \( u^b : \mathbb{R} \times (-b, b) \to \mathbb{R} \). Up to isometries of \( \mathbb{R}^2 \), the only other complete translating graphs in \( \mathbb{R}^3 \) are the grim reaper surface, the tilted grim reaper surfaces, and the bowl soliton.

We now describe previously known classification results. Spruck and Xiao recently proved the very powerful theorem that every translating graph in \( \mathbb{R}^3 \) is convex [SX17, Theorem 1.1]. Thus it suffices to classify convex examples. L. Shahriyari [Sha15] proved that if \( u : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \) is a complete translator, then \( \Omega \) is (up to rigid motion) one of the following: the plane \( \mathbb{R}^2 \), a halfplane, or a strip \( \mathbb{R} \times (-b, b) \) with \( b \geq \pi/2 \).

In [Wan11], X. J. Wang proved that the only entire convex translating graph is the bowl soliton, and that there are no complete translating graphs defined over halfplanes. Thus by the Spruck-Xiao Convexity Theorem, the bowl soliton is the only complete translating graph defined over a plane or halfplane.
It remained to classify the translators $u : \Omega \to \mathbb{R}$ whose domains are strips. The main new contributions in this paper are:

1. For each $b > \pi/2$, we prove (Theorem 5.7) existence and uniqueness (up to translation) of a complete translator $u^b : \mathbb{R} \times (-b, b) \to \mathbb{R}$ that is not a tilted grim reaper.
2. We give a simpler proof (see Theorem 6.7) that there are no complete graphical translators in $\mathbb{R}^3$ defined over halfplanes in $\mathbb{R}^2$.

We remark that Bourni, Langford, and Tinaglia have recently given a different proof of the existence (but not uniqueness) in (1) \cite{Bourni2018}.

2. Preliminaries

Here we gather the main properties of translators that will be used in this paper.

As observed by Ilmanen \cite{Ilmanen1994}, a hypersurface $M \subset \mathbb{R}^{n+1}$ is a translator if and only if it is minimal with respect to the Riemannian metric

$$g_{ij}(x_1, \ldots, x_{n+1}) = \exp\left(-\frac{2}{n}x_{n+1}\right) \delta_{ij}. $$

Thus we can freely use curvature estimates and compactness theorems from minimal surface theory; cf. \cite[chapter 3]{White2016}. In particular, if $M$ is a graphical translator, then (since vertical translates of it are also $g$-minimal) $(e_3, \nu)$ is a nowhere vanishing Jacobi field, so $M$ is a stable $g$-minimal surface. It follows that any sequence $M_i$ of complete translating graphs in $\mathbb{R}^3$ has a subsequence that converges smoothly to a translator $M$. Also, if a translator $M$ is the graph of a function $u : \Omega \to \mathbb{R}$, then $M$ and its vertical translates form a $g$-minimal foliation of $\Omega \times \mathbb{R}$, from which it follows that $M$ is $g$-area minimizing in $\Omega \times \mathbb{R}$, and thus that if $K \subset \Omega \times \mathbb{R}$ is compact, then the $g$-area of $M \cap K$ is at most $1/2$ of the $g$-area of $\partial K$. In this paper, we will consider various sequences of translators that are manifolds-with-boundary. In the situations we consider, the area bounds described above together with standard compactness theorems for minimal surfaces (such as those in \cite{White1987}) give smooth, subsequential convergence, including at the boundary. (The local area bounds and bounded topology mean that the only boundary singularities that could arise would be boundary branch points. In the situations that occur in this paper, obvious barriers preclude boundary branch points.)

The situation for higher dimensional translating graphs is more subtle; see Section 12.

**Theorem 2.1.** Let $M \subset \mathbb{R}^3$ be a smooth, connected translator with nonnegative mean curvature. If the mean curvature vanishes anywhere, then $M$ is contained in a vertical plane.

**Proof.** As mentioned above, the mean curvature $(\nu, e_3)$ is a Jacobi field. By hypothesis, it is nonnegative. By the strong maximum principle, if it vanishes anywhere, it vanishes everywhere, so that $M$ is contained in $\Gamma \times \mathbb{R}$ for some curve in $\mathbb{R}^2$. The result follows immediately. $\square$

**Theorem 2.2.** Let $M \subset \mathbb{R}^3$ be a complete translator with positive mean curvature. Then it is a graph and the Gauss curvature is everywhere nonnegative. If the Gauss curvature vanishes anywhere, then it vanishes everywhere and $M$ is a grim reaper surface or tilted grim reaper surface.
Proof. Nonnegativity of the Gauss curvature is the main result of [SX17]. If the curvature vanishes anywhere, it vanishes everywhere because $\kappa_1/H$ satisfies a strong maximum principle (where $0 \leq \kappa_1 \leq \kappa_2$ are the principle curvatures). See for example [Whi03, Theorem 3].

The last assertion follows from work of Martin, Savas-Halilaj, and Smoczyk [MSHS15, Theorem B]. We can also prove it directly as follows. Suppose that $M$ is a translator with Gauss curvature 0. By elementary differential geometry, $M$ is a ruled surface and the Gauss map is constant along the straight lines. Consequently if $L$ is a line in $M$, we can find a grim reaper surface (tilted unless $L$ is horizontal) such that $\Sigma$ is tangent to $M$ along $L$. By Cauchy-Kowalevski, $M = \Sigma$. □

Corollary 2.3. Suppose that $M \subset \mathbb{R}^3$ is a complete graphical translator. If $p_i$ is a divergent sequence in $M$, then $M - p_i$ converges smoothly (after passing to a subsequence) to a vertical plane, a grim reaper surface, or a tilted grim reaper surface.

Proof. If $M$ is not strictly convex, then by Theorem 2.2 it is a grim reaper surface or tilted grim reaper surface, and the corollary is trivially true.

Thus suppose that $M$ is strictly convex. Then the Gauss map maps $M$ diffeomorphically to an open subset of the upper hemisphere of $S^2$. Since $M - p_i$ is a stable minimal surface with respect to the Ilmanen metric, a subsequence will converge smoothly to a complete translator $M'$. The Gauss image of $M'$ lies in the boundary of the Gauss image of $M$ and so has no interior. Thus $M'$ has zero Gauss curvature. By Theorems 2.1 and 2.2 $M'$ is a vertical plane or a grim reaper surface or a tilted grim reaper surface. □

We also use the following result of Spruck and Xiao [SX17, Theorem 1.5]:

Theorem 2.4. Suppose that $u : \mathbb{R} \times (-b, b)$ is a complete, strictly convex translator. Then $u(x, y) \equiv u(x, -y)$. Also,

$$u(x + \zeta, y) - u(\zeta, 0)$$

converges smoothly as $\zeta \to -\infty$ to the tilted grim reaper surface

$$\mathcal{G}_\theta : \mathbb{R} \times (-b, b) \to \mathbb{R}$$

where $\theta = \arccos(2b/\pi)$ (see (1.4)), and it converges smoothly as $\zeta \to \infty$ to the tilted grim reaper surface

$$\mathcal{G}_{-\theta} : \mathbb{R} \times (-b, b) \to \mathbb{R}.$$

Spruck and Xiao prove that $u(x, y) \equiv u(x, -y)$ by Alexandrov moving planes. Subsequential convergence of $u(x, y + \zeta) - u(0, \zeta)$ to a tilted grim reaper surface is relatively easy (see Corollary 2.3); the difficult part is showing that such a subsequential limit is a graph over the entire strip $\mathbb{R} \times (-b, b)$ rather than over a smaller strip. Spruck and Xiao overcome that difficulty by an ingenious use of Theorem 2.6 below.

Corollary 2.5. The function $u$ attains its maximum at a point $(x_0, 0)$.

Proof. By Theorem 2.4 $\lim_{|x| \to \infty} u(x, 0) = -\infty$, so the function $x \to u(x, 0)$ attains its maximum at some $x_0$. By symmetry, $Du(x_0, 0) = 0$, so by convexity, $u$ attains its maximum at $(x_0, 0)$. □
The following very useful gradient bound, which plays a crucial role in this paper, is also due to Spruck and Xiao [SX17]:

**Theorem 2.6.** Let $M$ be a translating graph in $\mathbb{R}^{n+1}$, let $\nu$ be the upward pointing unit normal, and suppose that $\eta: \mathbb{R}^{n+1} \to \mathbb{R}$ is an affine function invariant under vertical translations. Then the function $v = \langle e_{n+1}, \nu \rangle^{-1}$ on $M$ cannot have a local maximum at an interior point where $\eta$ is positive.

Note that if $M$ is the graph of $u$, then $v = \sqrt{1 + |Du|^2}$.

**Proof.** From the translator equation (1.1), we see that for $i = 1, \ldots, n$,

$$\Delta x_i = \langle H, e_i \rangle = \langle (e_{n+1})^\perp, e_i \rangle = \langle e_{n+1}, \nabla x_i \rangle$$

where $\Delta$ is the Laplacian on $M$ and $\nabla$ is the gradient on $M$. Thus

$$\tilde{\Delta} x_i = 0 \quad (i = 1, 2, \ldots, n),$$

where $\tilde{\Delta}$ is the operator on $M$ (sometimes called the Drift Laplacian) given by

$$\tilde{\Delta} f = \Delta f - \langle e_{n+1}, \nabla f \rangle.$$

Consequently,

$$\tilde{\Delta} \eta = 0. \quad (2.1)$$

According to Martin, Savas-Halilaj and Smoczyk [MHS15],

$$\tilde{\Delta} v = |A|^2 v + 2 \left| \frac{\nabla v}{v} \right|^2. \quad (2.2)$$

One easily calculates (just as for the Laplacian) that

$$\tilde{\Delta} (\eta v) = (\tilde{\Delta} \eta) v + \eta (\tilde{\Delta} v) + 2 \langle \nabla \eta, \nabla v \rangle. \quad (2.3)$$

(This product rule holds for any two smooth functions on $M$.) Thus by (2.1) and (2.2),

At a critical point (of any function), the Laplacian and the Drift Laplacian are equal. At a critical point of $\eta v$, we have $0 = \nabla \eta v = v \nabla \eta + \eta \nabla v$, or $\nabla \eta = -\frac{2}{v} \nabla v$. Substitution into the last term in (2.3) gives

$$\Delta (\eta v) = \tilde{\Delta} (\eta v) = \eta |A|^2 v > 0,$$

so the critical point is not a local maximum. \qed

3. **Rectangular Boundaries**

Let

$$u = u_{L,b} : [-L, L] \times [-b, b] \to \mathbb{R}$$

be the translator with boundary values 0.

(Existence can be proved in many ways. For example, we can use the continuity method starting with $\lambda = 0$ to find, for each $\lambda \in [0, 1]$, a graph that is minimal with respect to $e^{-\lambda \delta_{ij}}$. The functions are bounded below by 0, and can be bounded above by a vertical translate of the bowl soliton. Since $u_{b,L}$ and its vertical translates form $\gamma$-minimal foliation of $[-L, L] \times [-b, b] \times \mathbb{R}$, we get uniqueness by the maximum principle.)
From the translator equation (1.3), we see that \( u_{L,b} \) has no interior local minima, so \( u_{L,b} \) > 0 on the interior of the rectangle. The function \( u_{L,b} \) is smooth on \([-L, L] \times [-b, b] \) except at the corners. It cannot be \( C^2 \) at the corners because the translator equation is not satisfied there since

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0
\]

at the corners. Nevertheless, \( u_{L,b} \) is \( C^1 \) on \([-L, L] \times [-b, b] \) by the following proposition. (In fact, it is \( C^{1, \alpha} \) for every \( \alpha < 1 \), but we do not need that fact.)

**Proposition 3.1.** Let \( u = u_{L,b} : [-L, L] \times [-b, b] \to \mathbb{R} \) be the translator with boundary values 0.

1. \( u \) is \( C^1 \) everywhere and is smooth except at the four corners.
2. \( \partial u/\partial x \equiv u(-x, y) \equiv u(x, -y) \).
3. \( \partial u/\partial x \) and \( x \) have opposite signs at interior points where \( x \neq 0 \).
4. \( \partial u/\partial y \) and \( y \) have opposite signs at interior points where \( y \neq 0 \).
5. \( |Du(L,y)| \) (which is equal to \( |Du(-L,y)| \)) is a decreasing function of \( |y| \).

**Proof.** Smoothness away from the corners is standard. To prove that \( u \) is \( C^1 \), let \( M \) be the graph of \( u \). Let \( q_i \in M \) converge to the corner \( q = (-L, -a, 0) \). Translate \( M \) by \((L, a, 0)\) and dilate by \(|p_i - q|^{-1}\) to get \( M_i \). After passing to a subsequence, the \( M_i \) converge smoothly (away from the origin) to a surface \( M' \) such that \( M' \) is minimal with respect to the Euclidean metric, the boundary \( \partial M' \) is the union of the \( x \) and \( y \) axes, and \( M' \) lies in the region \( \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0\} \). Thus \( M' \) is a horizontal quarter-plane. This proves (1).

Properties (2), (3), and (4) are proved by Alexandrov moving planes.

Since \( \frac{\partial u}{\partial y} \equiv 0 \) on \( \{L\} \times (0, a] \) and since \( \frac{\partial u}{\partial y} < 0 \) on \( (-L, L) \times (0, a] \), we see that

\[
\frac{\partial^2 u}{\partial x \partial y} \geq 0 \quad \text{on} \quad \{L\} \times [0, a].
\]

Thus

(3.1)

\[
\frac{\partial^2 u}{\partial y \partial x} \geq 0 \quad \text{on} \quad \{L\} \times [0, a].
\]

On that edge, \( |Du| = -\partial u/\partial x \). Thus (by (3.1)) \( |Du(L,y)| \) is a decreasing function of \( y \in [0, a] \). The corresponding statement for \( y \in [-a, 0] \) follows by symmetry. This proves (5). \( \square \)

**Theorem 3.2.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^p \). Suppose there is bounded translator

\[
u : \Omega \times \mathbb{R}^q \to \mathbb{R}
\]

that vanishes on the boundary. Then \( u \) is unique, and therefore \( u \) depends only on the first \( p \) coordinates.

**Proof.** Suppose that \( v \neq u \) is another bounded solution. Choose \((x_i, y_i) \in \Omega \times \mathbb{R}^q\) such that

\[
|u(x_i,y_i) - v(x_i,y_i)| \to \sup |u - v|.
\]

After passing to a subsequence, if we translate the graphs by \((0, -y_i, 0)\), we get convergence (see Theorem 2.2) to translaters \( \hat{u} \) and \( \hat{v} \) that violate the strong maximum principle. \( \square \)
Corollary 3.3. If
\[ \liminf_{L \to \infty} u_{L,b}(0,0) < \infty, \]
then \( b < \pi/2 \).

Proof. Suppose that the liminf is finite. Then there is a sequence \( L(i) \to \infty \) such that \( u_{L(i),b} \) converges smoothly to a translator
\[ u : [-b, b] \times \mathbb{R} \to \mathbb{R} \]
with \( u = 0 \) on the boundary of the strip and \( u(0,0) = \max u < \infty \).

By Theorem 2.6, the maximum of \( w \) occurs at a point \((\tilde{x}, \tilde{y})\) on one of the four edges of the rectangle \([-L, L] \times [0, b] \). It cannot occur on the edge \([-L, L] \times \{b\}\) since the function vanishes there.

By Proposition 3.3, it cannot occur on the edges \( \{L\} \times (0, a) \) or \( \{-L\} \times (0, a) \). Thus it occurs at a point in \([-L, L] \times \{0\}\).

Now suppose that Proposition 3.4 is false. Then there exist sequences \( L(i) \to \infty \) and \( b(i) \leq A \) such that
\[ \max_{[-L(i),L(i)]\times[-b(i),b(i)]} u_{L(i),b(i)} \to \infty. \]

By the foregoing discussion, the maximum is attained at a point \((x_i, 0)\). Let \( M_i \) be the graph of \( u_{L(i),b(i)} \) and let \( p_i = (x_i, 0, u_{L(i),b(i)}(x_i, 0)) \).

Now \( |Du_{L(i),b(i)}(x_i, 0)| \to \infty \), and \( \frac{\partial}{\partial y} u_{L(i),b(i)}(x_i, 0) \equiv 0 \) (by the symmetry), so
\[ \text{Tan}(M_i, p_i) \]
converges to the plane \( x = 0 \).

Translate \( M_i \) by \(-p_i\) to get a translator \( M_i' \). By passing to a subsequence, we can assume that \( M_i' \) converge to a limit translator \( M' \). We can also assume that \( \text{dist}(0, \partial M_i') \) converges to a limit \( \delta \in [0, \infty] \).

By the maximum principle (if \( \delta > 0 \)) or the boundary maximum principle (if \( \delta = 0 \)),
\[ \text{Tan}(M', p) = \text{Tan}(M', 0) = \{x = 0\} \]
for \( p \) in a small neighborhood of 0. Thus there is a neighborhood \( G \) of 0 such that
\[
M' \cap G \subset \{ x = 0 \}.
\]
By unique continuation, \( M' \) contains the entire plane \( \{ x = 0 \} \), which is impossible since \( M' \) is contained in the slab \( \mathbb{R} \times [-b, b] \times \mathbb{R} \).

4. Existence of \( \Delta \)-Wings

**Theorem 4.1.** For every \( b \in \left( \frac{\pi}{2}, +\infty \right) \) there exists a complete translator
\[
u^b : \mathbb{R} \times (-b, b) \to \mathbb{R}
\]
with the following properties:

(a) \( \nu^b(x, y) \equiv \nu^b(-x, y) \equiv \nu^b(x, -y) \).

(b) The Gauss curvature of the graph is everywhere positive.

(c) For \( B \geq b \),
\[
\sup_{(x,y)} (b - |y|) \sqrt{1 + |D\nu^b(x, y)|^2} \leq C(B)
\]
where \( C(B) < \infty \) is as in Proposition 3.4.

**Proof.** As in Section 3, let \( u = u_{L,b} : [-L, L] \times [-b, b] \to \mathbb{R} \) be the solution to the translator equation with boundary values 0.

By Proposition 3.4, \( |Du_{L,b}| \) is bounded (independent of \( L \)) on compact subsets of \( \mathbb{R} \times (-b, b) \). Thus every sequence of \( L \)'s tending to infinity has a subsequence \( L(i) \) such that \( u_{L(i),b} - u_{L(i),b}(0,0) \) converges smoothly to a limit translator
\[
u^b : \mathbb{R} \times (-b, b) \to \mathbb{R}.
\]
(Later we will show that the limit \( \nu^b \) does not depend on the choice of the sequence \( L(i) \); see Proposition 5.4.)

Since \( u_{L(i),b}(0,0) \) tends to infinity (see Corollary 3.3), the graph of \( \nu^b \) is a complete translator. The symmetries
\[(4.1) \quad \nu^b(x, y) \equiv \nu^b(-x, y) \equiv \nu^b(x, -y)\]
of \( \nu^b \) follow from the corresponding symmetries of the \( u_{L,b} \).

Since \( b > \pi/2 \), we see that \( \nu^a \) is not a grim reaper surface. The symmetries (4.1) imply that \( \nu^a \) is not a tilted grim reaper surface. Hence the Gauss curvature of the translator \( \nu^a \) is everywhere positive by Theorem 2.2.

5. Uniqueness of \( \Delta \)-Wings

**Proposition 5.1.** Suppose that
\[
u : \mathbb{R} \times (a, b) \to \mathbb{R},
\]
\[
u^a : \mathbb{R} \times (\hat{a}, \hat{b}) \to \mathbb{R}
\]
are complete, strictly convex translators.

(1) If \( a < \hat{a} < b < \hat{b} \), then \( u - \nu \) has no critical points.

(2) If \( a < \hat{a} < b < \hat{b} \), then \( u - \nu \) has at most one critical point. If there is a critical point, then \( D^2u \neq D^2\nu \) at that point.

**Proof.** This is a direct consequence of the tilted-grim-reaper-like behavior of \( u \) and \( \nu \) as \( |x| \to \infty \) (Theorem 2.4) and a classical theorem of Rado (Theorem 5.3 below).
Corollary 5.2. If \( u, \hat{u} : \mathbb{R} \times (-b, b) \to \mathbb{R} \) are complete translators and if
\[
Du(x_0, y_0) = D\hat{u}(\hat{x}_0, \hat{y}_0),
\]
then \( y_0 = \hat{y}_0 \).

Proof. If \( y_0 \neq \hat{y}_0 \), then the functions \( u \) and \( \hat{u}(x, y) := \hat{u}(x, y + (\hat{y}_0 - y_0)) \) would violate Proposition 5.1. \( \square \)

Theorem 5.3 (Rado’s Theorem). Suppose that \( U \) is a simply connected open subset of \( \mathbb{R}^2 \), and that
\[
f, g : U \to \mathbb{R}
\]
are smooth functions such that the graphs of \( f \) and \( g \) and their vertical translates are minimal with respect to a smooth Riemannian metric.

1. If \( \{f - g = c\} \cap \partial U \) has fewer than 4 ends, then \( U \cap \{f - g = c\} \) has no point at which \( Df = Dg \).

2. If \( \{f - g = c\} \cap \partial U \) has fewer than 6 ends for each \( c \), then \( f - g \) is a Morse function on \( U \) and has at most one critical point.

See [Rad51].

Theorem 5.4 (Uniqueness of Symmetric \( \Delta \)-Wings). For each \( b > \pi/2 \), there exists a unique translator
\[
u : \mathbb{R} \times (-b, b) \to \mathbb{R}
\]
such that \( \nu(x, y) \equiv \nu(-x, y) \).

Proof. We proved existence in Theorem 4.4 so it suffices to prove uniqueness. Suppose that \( u \) and \( \hat{u} \) both satisfy the hypotheses. Let \( 0 < y_0 < a \). Then
\[
\frac{\partial u}{\partial y}(0, y_0) < 0.
\]
Since \( \frac{\partial u}{\partial y}(0, y_0) \) decreases from 0 to \( -\infty \) on the interval \([0, a]\), there is an \( \hat{y}_0 \in [0, a) \) such that
\[
\frac{\partial \hat{u}}{\partial y}(0, \hat{y}_0) = \frac{\partial u}{\partial y}(0, y_0).
\]
By the symmetry assumption,
\[
\frac{\partial u}{\partial x} = \frac{\partial \hat{u}}{\partial x} = 0 \quad \text{on } \{0\} \times (-b, b).
\]
Thus \( Du(0, y_0) = D\hat{u}(0, \hat{y}_0) \). Hence by Corollary 5.2, \( \hat{y}_0 = y_0 \), so
\[
Du(0, y_0) = D\hat{u}(0, \hat{y}_0).
\]
Since \( y_0 \in [0, a) \) was arbitrary, we have proved that \( Du(0, y) \equiv D\hat{u}(0, \hat{y}) \) for all \( y \in [0, a) \). Hence (since \( u(0, 0) = \hat{u}(0, 0) \)) we have \( u \equiv \hat{u} \) by Cauchy-Kowalevski. \( \square \)

Corollary 5.5 (Continuous Dependence). For \( b > \pi/2 \), let \( u^b : \mathbb{R} \times (-b, b) \to \mathbb{R} \) be the unique translator such that \( u^b(0, 0) = 0 \) and such that \( u^b(x, y) \equiv u^b(-x, y) \). Then \( u^b \) depends continuously on \( b \): as \( b \to d \), the function \( u^b \) converges smoothly to \( u^d \).
Proof. Let \( b(i) \to d \), where \( b(i) \) and \( d \) are in the interval \( (\pi/2, \infty) \). By Theorem 4.1(c), \( u^{b(i)} \) converges smoothly (after passing to a subsequence) to a complete translator
\[
u : \mathbb{R} \times (-d, d) \to \mathbb{R}.
\]
By Theorem 5.4, \( u = u^d \). Since the limit is independent of the sequence and subsequence, we are done. \( \Box \)

**Lemma 5.6.** Let \( a < b \) and let
\[
u : \mathbb{R} \times (-a, a) \to \mathbb{R},
\]
\[
u : \mathbb{R} \times (-b, b) \to \mathbb{R}
\]
be complete translators with \( u(0, 0) = v(0, 0) \) and \( Du(0, 0) = Dv(0, 0) = 0 \). Then \( u(x, 0) > v(x, 0) \) for all \( x \neq 0 \).

Proof. Note that \( \frac{\partial u}{\partial y}(x, 0) = 0 = \frac{\partial v}{\partial y}(x, 0) \) by \( (x, y) \to (x, -y) \) symmetry. Thus if \( \frac{\partial u}{\partial x}(x, 0) = \frac{\partial v}{\partial x}(x, 0) \), then \( (x, 0) \) is a critical point of \( u - v \). By Proposition 5.1(2), \( u - v \) has only one critical point (namely \((0, 0)\)). Thus
\[
\frac{\partial u}{\partial x}(x, 0) \neq \frac{\partial v}{\partial x}(x, 0) \text{ for all } x \neq 0.
\]
By Theorem 2.4,
\[
\lim_{x \to \infty} \left( \frac{\partial u}{\partial x}(x, 0) - \frac{\partial v}{\partial x}(x, 0) \right) > 0.
\]
Thus
\[
\frac{\partial u}{\partial x}(x, 0) > \frac{\partial v}{\partial x}(x, 0) \text{ for all } x > 0.
\]
Integrating from 0 gives
\[
u(x, 0) > v(x, 0) \text{ for all } x > 0.
\]
Exactly the same argument shows that \( u(x, 0) > v(x, 0) \) for all \( x < 0 \). \( \Box \)

**Theorem 5.7** (Existence and Uniqueness of \( \Delta \)-Wings). Let \( b > \pi/2 \). Then, modulo translations, there is a unique complete translator \( u : \mathbb{R} \times (-b, b) \to \mathbb{R} \) that is not a tilted grim reaper.

Proof. We already proved existence, so it suffices to prove uniqueness of \( u \). By Corollary 2.5, \( u \) attains its maximum. By translating, we can assume that \( \max u = u(0, 0) = 0 \).

By Lemma 5.6,
\[
u^c(x, 0) \leq u(x, 0) \leq u^a(x, 0) \text{ for all } a < b < c \text{ and all } x \in \mathbb{R}.
\]
Letting \( a \) and \( c \) tend to \( b \) gives (see Corollary 5.5)
\[
u(x, 0) = u^b(x, 0) \text{ for all } x.
\]
Since \( \frac{\partial u}{\partial y}(x, 0) = 0 = u^b_y(x, 0) \) for all \( x \), we see (by Cauchy-Kowalevski, for example) that \( u = u^b \). \( \Box \)
6. Non-existence of translating graphs over half-planes

In this section, we prove that no complete translators in \( \mathbb{R}^3 \) are graphs over a half-plane. The first result that we need in the proof is about the image under the Gauss map of such a translator. Let \( M \) be a complete translator and \( \nu: M \to \mathbb{S}^2 \) its Gauss map. If we assume that \( M \) is a graph, then clearly \( \nu(M) \subseteq H^+ \), where \( H^+ \) represent the upper hemisphere.

The next lemma says that the Gauss image of a complete translating graph is a domain in \( H^+ \) bounded by 0, 1, or 2 great semicircles.

**Lemma 6.1 (Gauss map lemma).** Let \( M \subset \mathbb{R}^3 \) be a complete, strictly convex translator that is the graph of a function \( u: \Omega \subset \mathbb{R}^2 \to \mathbb{R} \). Then the Gauss map is a diffeomorphism from \( M \) onto an open subset \( W \) of the upper hemisphere \( H^+ \).

Furthermore, one of the following holds:

1. \( \nu(M) \) is the entire upper hemisphere.
2. \( \nu(M) \) is one of the two components of \( H^+ \setminus C \), where \( C \) is a great semicircle in \( H^+ \).
3. \( \nu(M) \) is the region between \( C' \) and \( C'' \), where \( C' \) and \( C'' \) are two disjoint great semicircles in \( H^+ \).

**Proof.** That \( \nu \) maps \( M \) diffeomorphically onto its image holds for any complete, strictly convex \( M \). The following two statements are immediate consequences:

(i) If \( p_i \in M \) and if \( \nu(p_i) \) converges to a limit \( v \notin \nu(M) \), then \( |p_i| \to \infty \).

(ii) If \( p_i \in M \) and if \( |p_i| \to \infty \), then all the subsequential limits of \( \nu(p_i) \) lie in \( \partial W \).

Now suppose that \( v \in H^+ \cap \partial W \). Choose \( p_i \in M \) so that \( \nu(p_i) \to v \). By (i), \( |p_i| \to \infty \). By passing to a subsequence, we can assume that \( M - p_i \) converges smoothly to a translator \( M' \). Let \( \nu' \) be the Gauss map of \( M' \). By (ii), \( \nu'(M') \) is contained in \( \partial W \). Thus \( \nu(M') \) has no interior, so \( M' \) has Gauss curvature 0 at all points. Thus \( M' \) is a tilted grim reaper or a vertical plane. Since \( \nu'(0) = v \in H^+ \), \( M' \) must be a tilted grim reaper. Thus \( \nu'(M') \) is a great semicircle \( C \) containing \( v \).

We have shown that every point in \( H^+ \cap \partial W \) lies in a great semicircle contained in \( H^+ \cap \partial W \). Thus \( H^+ \cap \partial W \) is the union of a collection \( \mathcal{C} \) of great semicircles.

If \( \mathcal{C} \) is empty, (1) holds. If \( \mathcal{C} \) has just one semicircle, then (2) holds.

Suppose that \( \mathcal{C} \) contains more than one semicircle. Let \( C' \) and \( C'' \) be two of them. Since \( W \) is connected, it lies in one of the components of \( H^+ \setminus (C' \cup C'') \), and its closure contains \( C' \cup C'' \). It follows immediately that \( C' \) and \( C'' \) do not intersect. We have shown that all the semicircles in \( \mathcal{C} \) are disjoint. Since \( W \) is connected, there cannot be more than two semicircles in \( \mathcal{C} \).

**Corollary 6.2.** Suppose that \( p_i \in M \) and that \( \nu(p_i) \) converges to a point in \( H^+ \cap \partial W \). Let \( C \) be the great semicircle in \( H^+ \cap \partial W \) that contains \( v \). Then \( M - p_i \) converges to the unique tilted grim reaper \( M' \) such that \( \nu'(M') = C \) and \( \nu'(0) = v \) (where \( \nu' \) is the Gauss map of \( M' \).)

In particular, we get convergence without having to pass to a subsequence.

The next lemma is inspired by results of Spruck and Xiao in [SX17].

**Lemma 6.3.** Let \( \Omega \) be a convex open subset of \( \mathbb{R}^2 \) containing \( [0, \infty) \times [-a, a] \).

Suppose that \( u: \Omega \to \mathbb{R} \) is a translator and suppose that \( [0, \infty) \times [-a, a] \) contains a closed, connected, unbounded subset \( C \) such that \( |Du| \) is bounded on \( C \). Then \( |Du| \) is bounded on \( [0, \infty) \times [-a, a] \).
Proof: We can assume that $C$ contains the origin; otherwise replace $C$ by the union of $C$ and a segment joining the origin with a point of $C$. Moreover, we can assume that $C$ has no interior; otherwise replace $C$ by its topological boundary in $\mathbb{R}^2$.

Take $b > a$ so that $J_b := \{0\} \times [-b,b] \subset \Omega$. Let

$$\lambda = \sup_{J_b \cap C} |Du| < \infty.$$ (6.1)

For $\epsilon > 0$, let $L(\epsilon)$ the line

$$y = -b + \epsilon x.$$ (6.2)

Let $W(\epsilon)$ be the union of the bounded components of $\mathbb{R}^2 \setminus (C \cup J_b \cup L(\epsilon))$ that lie above $L(\epsilon)$. Note that $W(\epsilon)$ lies in the triangle $T(\epsilon)$ determined by the lines $x = 0$, $y = a$ and $L(\epsilon)$.

Thus the function $\psi_\epsilon : \overline{W(\epsilon)} \to \mathbb{R}$

$$\psi_\epsilon(p) = \sqrt{1 + |Du(p)|^2} \cdot \text{dist}(p, L(\epsilon))$$

attains its maximum at a point $p_\epsilon$. The point $p_\epsilon$ need not be unique, nor need it depend continuously on $\epsilon$. The point $p_\epsilon$ does not lie on $L(\epsilon)$ because $\psi_\epsilon$ vanishes on $L(\epsilon)$. By Theorem [2.6] it cannot be in the interior of $W(\epsilon)$. Thus $p_\epsilon \in C \cup J_b$ (and in the triangle $T(\epsilon)$.) Then, for every point $p \in \overline{W(\epsilon)}$, we have

$$\psi_\epsilon(p) \leq \psi_\epsilon(p_\epsilon) \leq \lambda \text{dist}(p_\epsilon, L(\epsilon)) \leq \lambda \text{dist}(p_\epsilon, L(0)) \leq 2\lambda b,$$

where $L(0)$ is the horizontal line $y = -b$ and where $\lambda$ is given by (6.1).

Now let $W$ be the union of the connected components of

$$([0, \infty) \times [-b, \infty)) \setminus (C \cup J_b)$$

that lie in below the line $y = b$.

Suppose that $(x, y) \in W$. Then $(x, y) \in \overline{W(\epsilon)}$ for all sufficiently small $\epsilon > 0$, so

$$\psi_\epsilon(x, y) \leq 2\lambda b, \text{ for all such } \epsilon.$$ (6.3)

Letting $\epsilon \to 0$ and using that $\text{dist}((x, y), L(0)) = y + b \geq b - |y|$, then we have

$$\sqrt{1 + |Du(x, y)|^2} \cdot (b - |y|) \leq 2\lambda b, \quad \forall (x, y) \in W.$$ (6.4)

By continuity, the inequality also holds for all $(x, y) \in \overline{W}$. Applying the same argument to the lines $y = b - \epsilon x$ shows that the inequality (6.3) also holds for all $(x, y) \in \overline{W^*}$, where $W^*$ is the union of the components of

$$([0, \infty) \times (-\infty, b]) \setminus (C \cup J_b)$$

that lie above the line $y = -b$. But it is not hard to see

$$\overline{W} \cup \overline{W^*} = [0, \infty) \times [-b, b],$$

so inequality (6.3) holds for every $(x, y) \in [0, \infty) \times [-b, b]$. \qed

Corollary 6.4. Suppose, in addition to the hypotheses of Lemma 6.3, that the graph $M$ of $u$ is complete and that the domain $\Omega$ is a plane or a half-plane. Let $\{x_n\}$ be a sequence of real numbers such that $x_n \to \infty$. Then a subsequence of $M - (x_n, 0, u(x_n, 0))$ converges smoothly to a complete translator $M'$ that is the graph of a function $u' : \Omega' \to \mathbb{R}$, where $\Omega'$ is the limit of the domains $\Omega - (x_n, 0)$.\hfill\Box
Proof. Suppose first that $\Omega = \{(x, y) \in \mathbb{R}^2 : y > c\}$, for some constant $c \in \mathbb{R}$.

For all $\alpha > c$ sufficiently close to $c$ and all $\beta$ sufficiently large, the set $C$ in Lemma 6.3 will be contained in $[0, \infty) \times [\alpha, \beta]$.

Consequently $|D u|$ is bounded above on $[0, \infty) \times [\alpha, \beta]$. So, any subsequential limit $M'$ of the indicated kind is a graph whose domain includes $[0, \infty) \times [\alpha, \beta]$. Since $\alpha$ and $\beta$ are arbitrary, then $M'$ is a graph over $\Omega$.

The other cases are similar, but easier. 

Remark 6.5. The proof of the previous corollary gives a bit more information about the limit function $u'$:

1. If $\Omega$ is a halfplane of the form $\{(x, y) \in \mathbb{R}^2 : y > c\}$, then $|D u'|$ is bounded on strips of the form $\{(x, y) \in \mathbb{R}^2 : \alpha \leq y \leq \beta\}$, provided that $\alpha > c$.
2. If $\Omega$ is the plane, or if $\partial \Omega$ consists of a non-horizontal line, then $\Omega' = \mathbb{R}^2$ and $|D u'|$ is bounded on all horizontal strips $\{(x, y) \in \mathbb{R}^2 : \alpha \leq y \leq \beta\}$.

Proposition 6.6. Suppose that $M$ is a complete, strictly convex translator and that $M$ is a graph of $u : \Omega \rightarrow \mathbb{R}$, where $\Omega$ is either all of $\mathbb{R}^2$ or a halfplane. Then the Gauss image $\nu(M)$ is the entire upper hemisphere $H^+$.

Proof. Suppose not. Then by Lemma 6.1 the boundary of $\nu(M)$ contains a great semicircle $C$ in the upper unit hemisphere. By rotation, we can assume that the endpoints of the semicircle are $(0, 1, 0)$ and $(0, -1, 0)$. Let $\Gamma$ be the set of points in $\Omega$ such that $\frac{\partial u}{\partial y} = 0$, i.e., the inverse image under $\nu$ of $\{v \in H^+ : \langle v, (0, 1, 0) \rangle = 0\}$.

By Lemma 6.1, $\Gamma$ is a smooth, properly embedded curve in $\Omega$. By translation, we may assume that $(0, 0) \in \Gamma$. Note the strict convexity implies that $y \mapsto \frac{\partial u}{\partial y}(x, y)$ is strictly decreasing (for each $x$). Thus if $\Gamma$ intersects the line $\{(x, y) : y \in \mathbb{R}\}$, it intersects it in a single point $(x, y(x))$. That is, we can parametrize $\Gamma$ as

$\{(x, y(x)) : x \in I\}$

where $I$ is an open interval (possibly infinite) containing 0.

By Lemma 6.1 again, as $x$ tends to one of the endpoints of $I$, say the right endpoint, the surfaces $M - (x, y(x), u(x, y(x)))$ converge smoothly to a grim reaper surface $M'$ through $(0, 0, 0)$ such that $\nu'(M') = C$. Thus $M'$ is the graph of a function

$u' : \mathbb{R} \times (-a, a) \rightarrow \mathbb{R}$.

The strip is horizontal because the endpoints of $\nu'(M')$ are $(0, 1, 0)$ and $(0, -1, 0)$. The strip is symmetric about the line $X = \mathbb{R} \times \{0\}$ because

$u'_x(0, 0) = \lim_{y \to 0} \frac{\partial u}{\partial y}(x, y(x)) = 0$.

It follows that the curves $\Gamma - (x, y(x))$ converge smoothly to $X$ as $x$ tends to the right endpoint of $I$. Thus the right endpoint of $I$ is $+\infty$.

For $t > 0$, let $L(t)$ be the line through $(0, 0)$ and $(t, y(t))$, let

$\Gamma[0, t] = \{(x, y) \in \Gamma : 0 \leq x \leq t\} = \{(x, y(x)) : 0 \leq x \leq t\}$

and let $K(t)$ be the closed bounded subset of $[0, t] \times \mathbb{R}$ determined by $\Gamma \cup L(t)$. (Thus for each vertical line $V$ in $[0, t] \times \mathbb{R}$, $V \cap K(t)$ is the closed segment whose endpoints are $V \cap \Gamma$ and $V \cap L(t)$.)
Note that as \( t \to \infty \) the line \( L(t) \) converges to the horizontal line \( X \), and thus \( K(t) \) converges to the set

\[
K := \{(x, y) : x \geq 0, \text{ and } 0 \leq y \leq y(x) \text{ or } y(x) \leq y \leq 0\}.
\]

Let \( p(t) \) be a point where the function

\[
\varphi^t : K(t) \to \mathbb{R},
\]

\[
\varphi^t(p) = \sqrt{1 + |Du(p)|^2} \text{ dist}(p, L(t))
\]

attains its maximum. (Of course \( p(t) \) need not depend continuously on \( t \).)

Now \( p(t) \) cannot be on \( L(t) \) since the function \( \varphi^t \) vanishes on \( L(t) \). By Corollary 2.6, \( p(t) \) cannot be in the interior of \( K(t) \). Thus

\[
(6.4) \quad p(t) \in \Gamma[0,t].
\]

We claim that

\[
(6.5) \quad \alpha := \sup_{t>0} \sqrt{1 + |Du(p(t))|^2} \text{ dist}(p(t), L(t)) < \infty
\]

For suppose to the contrary that we can find a sequence \( t_n \nearrow \infty \) such that

\[
(6.6) \quad \sqrt{1 + |Du(p_n)|^2} \text{ dist}(p_n, L_n) \to \infty
\]

where \( p_n = p(t_n) \) and \( L_n = L(t_n) \). Since \( |Du(p_n)| \to |Du'(0)| < \infty \), we see from (6.6) that

\[
(6.7) \quad \text{dist}(p_n, L_n) \to \infty.
\]

Since \( \Gamma - p_n \) converges smoothly to the line \( X \), (6.7) implies (after passing to a further subsequence) that \( K(t_n) - p_n \) converges to a halfplane \( Q \) bounded by \( X \).

Let \( q \) be a point in the interior of \( Q \). Then, for all sufficiently large \( n \), one has \( p_n + q \in K(t_n) \). Thus

\[
\sqrt{1 + |Du(p_n)|^2} \text{ dist}(p_n, L_n) \geq \sqrt{1 + |Du(p_n + q)|^2} \text{ dist}(p_n + q, L_n)
\]

\[
\geq \sqrt{1 + |Du(p_n + q)|^2} \left( \text{dist}(p_n, L_n) - |q| \right).
\]

Dividing by \( \text{dist}(p_n, L_n) \), which tends to \( 0 \) by (6.6), and letting \( n \to \infty \) gives

\[
\sqrt{1 + |Du'(0)|^2} \geq \sqrt{1 + |Du'(q)|^2}, \quad \forall q \in Q,
\]

which is absurd because the graph of \( u' \) is a tilted grim reaper. This contradiction proves (6.5).

Now consider a point \( p = (x, y(x)) \) in \( \Gamma \cap \{x \geq 0\} \). Then \( p \in K(t) \) for \( t \geq x \), so for all \( t \geq x \),

\[
\sqrt{1 + |Du(x, y(x))|^2} \text{ dist}((x, y(x)), L(t)) \leq \alpha.
\]

Letting \( t \to \infty \) gives

\[
|y(x)| \leq \sqrt{1 + |Du(x, y(x))|^2} |y(x)| \leq \alpha,
\]

and therefore

\[
\sup_{x \geq 0} |y(x)| \leq \alpha.
\]

Now let \( \tilde{M} \) the subsequential limit of \( M - (x_n, 0, u(x_n, 0)) \) as \( x_n \to \infty \). By Corollary 6.4 and Remark 6.5, \( \tilde{M} \) is a complete graph defined over a halfplane or over all of \( \mathbb{R}^2 \). But by Lemma 6.1 since \( M \) is convex, \( \tilde{M} \) is either a vertical plane or a tilted grim reaper, a contradiction. \( \square \)
Theorem 6.7. No complete translator is the graph of a function over a halfplane.

Proof. We prove it by contradiction using Alexandrov moving planes. Suppose there is a complete translator $M$ that is the graph of a function

$$u : \{(x, y) : y > 0\} \to \mathbb{R}.$$ 

By Proposition 6.6, the Gauss map image $\nu(M)$ is the entire upper hemisphere. Thus the only limits of translates of $M$ are vertical planes.

If $p_n = (x_n, y_n, z_n) \in M$ is a divergent sequence with

$$y_n \leq c < \infty,$$

then the sequence $M - p_n$ converges (subsequentially) to a vertical plane passing through the origin. Since the plane is contained in $\{y \geq -c\}$, it must be the plane $\Pi_0 = \{y = 0\}$. Thus

$$\text{If } (x_n, y_n, z_n) \in M \text{ diverges and if } y_n \text{ is bounded, }$$

$$\text{then } \frac{\partial u}{\partial y}(x_n, y_n) \to \infty.$$ 

In particular, there is an $\eta > 0$ such that

$$\frac{\partial u}{\partial y}(x, y) > 0 \text{ for } 0 < \eta < y.$$ 

Let $\mathcal{S}$ be the set of $(x, y, y')$ such that

$$0 < y = y' \text{ and } \frac{\partial u}{\partial y}(x, y) = 0, \text{ or }$$

$$0 < y < y' \text{ and } u(x, y) \geq u(x, y').$$ 

Since $\nu(M)$ is the upper hemisphere, there is a point $(x, y)$ with $Du(x, y) = 0$. Thus $(x, y, y) \in \mathcal{S}$, so $\mathcal{S}$ is nonempty.

Let

$$s = \inf\{(y + y')/2 : (x, y, y') \in \mathcal{S}\}.$$ 

We claim that the infimum is attained. To see this, let $(x_i, y_i, y'_i)$ be a sequence in $\mathcal{S}$ such that

$$(y_i + y'_i)/2 \to s.$$ 

Note by (6.9) that $x_i$ is bounded. It follows (also by (6.9)) that $y_i$ is bounded away from 0. Thus (after passing to a subsequence) $(x_i, y_i, y'_i)$ converges to a limit $(\hat{x}, \hat{y}, \hat{y}')$ in $\mathcal{S}$ with $s = (\hat{y} + \hat{y}')/2$.

Now

$$u(x, y) \leq u(x, 2s - y) \text{ for all } y \in (0, s]$$

with equality at $(\hat{x}, \hat{y})$. By the strong maximum principle (if $\hat{y} < \hat{y}'$) or the strong boundary maximum principle (if $\hat{y} = \hat{y}'$),

$$u(x, y) = u(x, 2s - y) \text{ for all } x \in \mathbb{R} \text{ and } y \in (0, s],$$

which is clearly impossible. \qed
The Classification Theorem

**Theorem 7.1.** For every \( b > \pi/2 \), there is (up to translation) a unique complete, strictly convex translator

\[ u^b : \mathbb{R} \times (-b, b) \to \mathbb{R}. \]

Up to isometries of \( \mathbb{R}^2 \), the only other complete translating graphs are the grim reaper surface, the tilted grim reaper surfaces, and the rotationally symmetric graph (i.e., the bowl soliton).

**Proof.** Let \( u : \Omega \to \mathbb{R} \) be a complete translator that is not a grim reaper surface or tilted grim reaper surface. By Theorem 2.2, its graph is strictly convex.

By [Sha15], \( \Omega \) is a strip, a halfplane, or all of \( \mathbb{R}^2 \). Theorem 5.7 gives existence and uniqueness (up to rigid motion) of complete, strictly convex \( u^b : \mathbb{R} \times (-b, b) \to \mathbb{R} \).

By Theorem 6.7, \( \Omega \) cannot be a halfplane.

It remains only to consider the case when \( \Omega = \mathbb{R}^2 \). X. J. Wang [Wan11] showed that (up to translation) the only entire, convex translator is the bowl soliton. \( \square \)

8. Higher Dimensional \( \Delta \)-Wings with Prescribed Principal Curvatures at the Apex

In this section, we prove the following theorem:

**Theorem 8.1.** Let \( k_1, \ldots, k_n \) be nonnegative numbers whose sum is 1. Then there is an open subset \( \Omega \) of \( \mathbb{R}^n \) and a complete, properly embedded translator \( u : \Omega \to \mathbb{R} \) with the following properties:

1. \( \max u = u(0) = 0 \).
2. \( D^2 u(0) \) is a diagonal matrix whose \( ii \) entry is \( -(k_i) \) for each \( i \).
3. \( u \) is an even function of each of its coordinates.
4. If \( k_i = 0 \), then \( u \) is translation-invariant in the \( e_i \) direction. If \( k_i > 0 \), then \( D_i u(x) \) and \( x_i \) have opposite signs wherever \( x_i \neq 0 \).
5. If \( k_i = k_j \), then \( u \) is rotationally invariant about the plane \( \{ x_i = x_j = 0 \} \).
6. The domain \( \Omega \) of \( u \) is either all of \( \mathbb{R}^n \) or a slab of the form \( \{ x_i < b \} \) for some \( i \). In the latter case, \( k_i > k_j \) for all \( j \neq i \).

**Corollary 8.2.** If \( k_1 = k_2 \geq k_3 \geq \ldots k_n \), then the function \( u \) given by Theorem 8.1 is entire. In particular, there is an \((n-2)\)-parameter family of entire translators \( u : \mathbb{R}^n \to \mathbb{R} \), no two of which are congruent to each other.

This is in sharp contrast to the situation in \( \mathbb{R}^2 \): by the work of X. J. Wang and Spruck-Xiao, the bowl soliton is the only entire translator \( u : \mathbb{R}^2 \to \mathbb{R} \).

**Proof of Theorem 8.1.** Let \( \Delta_n \) be the set of \( n \)-tuples \( a = (a_1, a_2, \ldots, a_n) \) of non-negative numbers such that \( \sum a_i = 1 \). Given \( a = (a_1, \ldots, a_n) \) in \( \Delta_n \) and \( \lambda > 0 \), consider the ellipsoidal region

\[ \mathcal{E}(a, \lambda) = \left\{ x : \sum_{i=1}^{n} a_i x_i^2 \leq R^2 \right\}, \]

where \( R > 0 \) is chosen so that if \( u = u_{a,\lambda} : \mathcal{E}(a, \lambda) \to \mathbb{R} \) is the bounded translator with boundary values 0, then

\[ u(0) = \lambda. \]
Let \( k = (k_1, \ldots, k_n) \) be the principle curvatures of the graph of \( u \) at the maximum. Thus \( D^2u(0) \) is a diagonal matrix whose diagonal entries are \(-k_1, \ldots, -k_n\).

By Theorem 9.2 below, if \( k_i = k_j \), then \( a_i = a_j \), and thus \( u \) is rotationally invariant about the \((n - 2)\)-dimensional axis \( \{x_i = x_j = 0\} \):

\[
(8.2) \quad k_i = k_j \text{ implies } u_{a_{i},\lambda} \text{ is rotationally symmetric about } \{x_i = x_j = 0\}.
\]

Let

\[
F = F_n^\lambda : \Delta_n \to \Delta_n
\]

be the map that maps \( a \) to \( k \).

According to Theorem 3.2, if \( a_i = 0 \), then \( u \) is translation-invariant in the \( e_i \)-direction, and thus \( k_i = 0 \). It follows that \( F_n^\lambda \) maps each face of \( \Delta \) to itself. We can also conclude that \( F_n^\lambda \) restricted to an \((m - 1)\)-dimensional face of \( \Delta_n \) agrees with \( F_m^\lambda \). Thus, for example,

\[
F_n^\lambda(x_1, \ldots, x_{n-1}, 0) = F_{n-1}^\lambda(x_1, \ldots, x_{n-1})
\]

and

\[
F_n^\lambda(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = F_{n-1}^\lambda(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
\]

Consequently, by elementary topology, \( F_n^\lambda : \Delta_n \to \Delta_n \) is surjective.

Now fix a \( k \) in \( \Delta_n \). For each \( \lambda > 0 \), choose an \( a = a(\lambda) \) in \( \Delta_n \) such that

\[
F_n^\lambda(a) = k.
\]

Now take a subsequential limit of

\[
(8.3) \quad u_{a(\lambda),\lambda} - u_{a(\lambda),\lambda}(0)
\]

as \( \lambda \to \infty \).

The result is a complete, properly embedded translator \( u : \Omega \to \mathbb{R} \). Clearly \( \max u = u(0) = 0 \), the Hessian \( D^2u(0) \) has the specified form, and \( u \) is an even function of each coordinate (since the approximating functions \( (8.3) \) have that property.)

As already mentioned, if \( a_i = 0 \), then \( u_{a(\lambda),\lambda} \) is translation-invariant in the \( x_i \)-direction and thus \( D_iu_{a(\lambda),\lambda} \equiv 0 \). On the other hand, if \( a_i > 0 \), then \( D_iu_{a(\lambda)} > 0 \) wherever \( x_i < 0 \) by the Alexandrov moving planes argument. Thus, either way, we have

\[
D_iu_{a(\lambda)} \geq 0 \quad \text{wherever } x_i < 0.
\]

Passing to the limit, we have

\[
(8.4) \quad D_iu \geq 0 \quad \text{wherever } x_i < 0.
\]

By differentiating the translator equation \( (1.3) \) with respect to \( x_i \), we see that \( v = D_iu \) satisfies an linear elliptic PDE of the form

\[
a_{jk}D_{jk}v + b_jD_jv = 0.
\]

Hence by \( (8.4) \) and the strong maximum principle, either \( v \equiv 0 \), in which case \( u \) is translation-invariant in the \( e_i \)-direction and so \( k_i = 0 \), or else

\[
D_iu(= v) > 0 \quad \text{wherever } x_i < 0.
\]

In the latter case, \( D_iu(0) < 0 \) by the Hopf Boundary Point Lemma. That, \( k_i \neq 0 \).

This completes the proof of Assertion (4).

Assertion (5) follows immediately from \( (8.2) \).
To prove Assertion [9.1], we may suppose (by relabelling the variables) that \( k_1 \) is the largest of the principal curvatures:
\[
    k_1 \geq k_i \quad \text{for all } i.
\]

**Case 1:** The entire \( x_1 \)-axis lies in \( \Omega \). By Theorem [9.3] below,
\[
    u(\rho, 0, \ldots, 0) = \min \{ u(x) : |x| = \rho \}
\]
for each \( \rho \geq 0 \). Thus \( u \) is entire.

**Case 2:** \( \Omega \) does not contain the entire \( x_1 \)-axis. Then \( \partial \Omega \) contains a point \( b = (b, 0, \ldots, 0) \). By symmetry, we can assume that \( b > 0 \).

Note by Assertion [4] that if \( x \) is in the domain of \( u \), then so is its projection to each coordinate hyperplane and therefore (iterating) so is its projection to each coordinate axis. Thus \( \Omega \) lies in the region \( \{ x : x_1 \leq b \} \). Now \( (\partial \Omega) \times \mathbb{R} \) is a minimal variety with respect to the Ilmanen metric since it is the limit of translators. Hence \( \partial \Omega \) is a minimal variety with respect to the Euclidean metric. It lies on one side of the plane \( \{ x : x_1 = b \} \) and touches it at \( b \). Hence by the maximum principle, \( \partial \Omega \) contains all of that plane. By symmetry, \( \partial \Omega \) also contains the plane \( \{ x : x_1 = (-b) \} \).

It follows that \( \Omega = \{ x : |x_1| < b \} \).

(If the last sentence is not clear, note that if \( L \) is a line parallel to a coordinate axis, then \( L \cap \Omega \) is connected by Assertion [4].)

Note that \( k_1 > k_i \) for each \( i \in \{ 2, 3, \ldots, n \} \). For if \( k_1 = k_i \), then \( u \) and therefore its domain would be rotationally symmetric about \( \{ x_1 = x_i = 0 \} \), and the slab \( \{ x : |x_1| < b \} \) does not have that symmetry. \( \square \)

### 9. Translating Graphs over Ellipsoidal Domains

In this section, we prove the properties of translating graphs over ellipsoids that were used in the proof of Theorem [8.1]. Throughout this section,
\[
u : \mathcal{E} \to \mathbb{R}
\]
is a bounded translator with boundary values 0, where
\[
    \mathcal{E} = \left\{ x \in \mathbb{R}^n : \sum a_i x_i^2 \leq R^2 \right\}.
\]
The \( a_i \)'s are nonnegative and not all 0, and \( R > 0 \). We let \( k_1, k_2, \ldots, k_n \) be the principal curvatures at \( x = 0 \), so that \( D^2 u(0) \) is the diagonal matrix with diagonal entries \( -k_1, -k_2, \ldots, -k_n \).

We may assume that \( a_i > 0 \) for each \( i \), since if any \( a_i = 0 \), then \( u \) is translation-invariant in that direction, and thus \( u \) is given by a lower-dimensional example.

**Theorem 9.1.** Suppose \( a_1 > a_2 \). Then \( (x_1 D_2 - x_2 D_1)u \) and \( x_1 x_2 \) have the same sign at all points where \( x_1 x_2 \neq 0 \).

Of course if \( a_1 = a_2 \), then (by uniqueness of the solution \( u \)), \( u \) is rotationally symmetric about the subspace \( \{ x_1 = x_2 = 0 \} \), so \( (x_1 D_2 - x_2 D_1)u \equiv 0 \).

**Proof.** Let \( M \) be the graph of \( u \). Recall that \( M \) is stable in the Ilmanen metric \( g \) (since vertical translations give a Jacobi Field that is everywhere positive.) Consequently, if any Jacobi Field on a connected region \( U \) in \( M \) is nonnegative on \( \partial U \), then it is nonnegative on all of \( U \), and if it is positive on some portion of \( \partial U \), then it is positive everywhere in the interior of \( U \) by the strong maximum principle.

Let
\[
f = (x_1 D_2 - x_2 D_1)u.
\]
Note that \( f \) is equivalent to the Jacobi Field coming from rotating \( M \) about the \( \{ x_1 = x_2 = 0 \} \) plane. Consider \( f \) on the region \( Q := \{ x \in \mathcal{E} : x_1 \geq 0, x_2 \geq 0 \} \).

Clearly \( f = 0 \) on the portions of \( \partial Q \) where \( x_1 = 0 \) or \( x_2 = 0 \). On the remaining portion of \( \partial Q \), that is, on \( \partial \mathcal{E} \cap \{ x_1 > 0, x_2 > 0 \} \), we see that \( f > 0 \) since \( a_1 > a_2 \).

Thus \( f > 0 \) everywhere in the interior of \( Q \):

\[(9.1) \quad f > 0 \text{ everywhere in } Q \setminus \{ x_1x_2 = 0 \}.\]

By symmetry, it follows that \( f > 0 \) wherever \( x_1x_2 > 0 \) and that \( f < 0 \) wherever \( x_1x_2 < 0 \). \( \square \)

**Theorem 9.2.** If \( a_1 > a_2 \), then \( k_1 > k_2 \).

**Proof.** Let \( R_\theta : \mathbb{R}^n \to \mathbb{R}^n \) denote rotation by \( \theta \) about the \( x_1x_2 \)-plane. Thus

\[
\frac{d}{d\theta} R_\theta(x) = x_1 e_2 - x_2 e_1.
\]

For each \( \theta \), the function \( u \circ R_\theta \) is also a solution of the translator equation, so

\[
f := \left( \frac{\partial}{\partial \theta} \right)_{\theta=0} u \circ R_\theta = (x_1 D_2 - x_2 D_1) u
\]
solves the linearization of the translator equation (1.3). In particular,

\[
a_{ij} D_{ij} f + b_i D_i f = 0,
\]

where \( a_{ij}(0) = \delta_{ij} \) and \( b_i(0) = 0 \). Thus the lowest nonzero polynomial \( P(x) \) in the Taylor series for \( f \) at 0 is harmonic. Since it has the same sign as the homogeneous harmonic polynomial \( x_1x_2 \), in fact

\[(9.2) \quad P(x) = c x_1x_2 \text{ for some constant } c > 0.\]

Now

\[
u(x) = - \sum_i k_i x_i^2 + O(|x|^3)
\]

so

\[(x_1 D_2 - x_2 D_1) u(x) = (k_1 - k_2) 2x_1x_2 + O(|x|^3).\]

Hence \( k_1 - k_2 > 0 \) by (9.2). \( \square \)

**Theorem 9.3.** Extend \( u \) to all of \( \mathbb{R}^n \) by setting \( u(x) = 0 \) for \( x \notin \mathcal{E} \).

Suppose that

\[k_1 = \max_{i \leq n} k_i,\]

Then for each \( \rho > 0 \),

\[u(\rho e_1) = \min \{ u(x) : |x| = \rho \}.\]

**Proof.** By Theorem 9.2,

\[r_1 \leq r_i \text{ for all } i.\]

We may assume that \( \rho e_1 \) is in \( \mathcal{E} \), as otherwise as otherwise the assertion is trivially true. Hence the entire sphere \( \{ x : |x| = \rho \} \) is contained in \( \mathcal{E} \).

Suppose

\[\min_{\partial B(0, \rho)} u\]

occurs at the point \( x \). By symmetry, we can assume that \( x_i \geq 0 \) for each \( i \).
Let $i > 1$. Define $\tilde{x}$ by
\[
\tilde{x}_1 = \sqrt{x_1^2 + x_i^2},
\tilde{x}_i = 0,
\tilde{x}_j = x_j \quad \text{for} \ j \neq 1, i.
\]
Applying this with $i = 2, 3, \ldots, n$, we see that the minimum is attained at $(\rho, 0, 0, \ldots, 0)$. □

\section{Ellipsoidal Slabs}

As before, $\Delta^n$ is the set of $a = (a_1, \ldots, a_n)$ where each $a_i \geq 0$ and $\sum_i a_i = 1$. Given $a \in \Delta^n$ and $R > 0$, we let
\[
E = \mathcal{E}(a, R) = \left\{ x \in \mathbb{R}^n : \sum_i a_i x_i^2 \leq R^2 \right\}.
\]
For $a \in \Delta^n$ and $b > \pi/2$, we let $u = u_{a,b,R} : \mathcal{E}(a, R) \times [-b, b] \to \mathbb{R}$ be the bounded translator with boundary values 0. Let $k_i = -D_{ij} u_{a,b,R}(0)$. The theorems in this section describe properties of such $u$. In Section 11, we will let $R \to \infty$ to get complete translators defined over the slab $\mathbb{R}^n \times (-b, b)$.

\textbf{Theorem 10.1.} For $u = u_{a,b,R}$,

1. If $a_i = 0$, then $D_i u \equiv 0$.
2. If $a_i > 0$, then $D_i u(x)$ and $x_i$ have opposite signs at all points in the interior of the domain, and $D_i u(0) < 0$.
3. $D_{n+1} u(x)$ and $x_{n+1}$ have opposite signs at all points in the interior of the domain, and $D_{n+1,n+1} u(0) < 0$.
4. If $1 \leq i, j \leq n$ and if $a_i = a_j$, then
   \[(x_i D_j - x_j D_i) u \equiv 0.\]
5. If $1 \leq i, j \leq n$ and $a_j < a_i$, then $k_i > k_j$, and
   \[(x_i D_j - x_j D_i) u \quad \text{and} \quad x_i x_j \quad \text{have the same sign at all points in the interior of } \mathcal{E}.\]

(The theorem does not assert any relationship between $k_i$ and $k_{n+1}$.)

\textbf{Proof.} Assertion (1) follows immediately from Theorem 3.2. Assertions (2) and (3) follow from the Alexandrov moving plane argument. The proofs of Assertions (4) and (5) are almost identical to the proofs of Theorems 9.1 and 9.2. □

\textbf{Theorem 10.2.} If $\eta$ is the inward pointing unit normal on $\partial \mathcal{E}(a, R)$ and if
\[(x_1, x_2, \ldots, x_n) \in \partial \mathcal{E}(a, R),\]
then
\[s \in [-b, b] \mapsto \eta \cdot \nabla u(x_1, x_2, \ldots, x_n, s) = |Du(x_1, x_2, \ldots, x_n, s)|
\]is a decreasing function of $|s|$. 

Proof. Since $D_{n+1}u = 0$ on $(\partial \mathcal{E}) \times [0, b]$ and since $D_{n+1}u < 0$ on interior$(\mathcal{E}) \times (0, b]$, it follows that
$$D_nD_{n+1}u \leq 0 \quad \text{on} \quad (\partial \mathcal{E}) \times (0, b].$$
Hence
$$D_{n+1}D_nu \leq 0 \quad \text{on} \quad (\partial \mathcal{E}) \times [0, b].$$
Likewise,
$$D_{n+1}D_nu \geq 0 \quad \text{on} \quad (\partial \mathcal{E}) \times [-b, 0].$$
\hfill \Box

Theorem 10.3. Let $a \in \Delta^n$, $R \geq 1$, and let
$$u : \mathcal{E} (a, R) \times [-b, b] \to \mathbb{R}$$
be the translator with boundary values 0. Let
$$w(x) = (b - |x_{n+1}|) \sqrt{1 + |Du(x)|^2}.$$ Then
$$\max w \leq C,$$
where $C = C(n, b) < \infty$ depends only on $n$ and $b$.

Proof. Let $\delta(x) = (b - |x_{n+1}|)$.

Suppose the theorem is false. Then there is a sequence of translators
$$u_k : \mathcal{E} (a(k), R(k)) \times [-b, b] \to \mathbb{R}$$
and a sequence points $p(k)$ such that

(10.1) $$\delta(p(k)) \sqrt{1 + |Du_k(p(k))|^2} \to \infty.$$ 

By passing to a subsequence, we can assume that $R(k)$ converges to a limit $R$ in $[1, \infty]$. The case $R < \infty$ is straightforward, so we assume that $R(k) \to \infty$.

We may suppose that $p(k)$ has been chosen to maximize the left side of (10.1). By symmetry, we may suppose that each coordinate of $p(k)$ is $\geq 0$:

(10.2) $$p(k) \in [0, \infty)^{n+1}.$$ 

By Theorem 2.6, $p(k)$ occurs on the boundary of $\mathcal{E} (a(k), R(k)) \times [0, b]$.

Note that $p(k)$ does not lie on $\mathcal{E} (a(k), R(k)) \times \{b\}$ since $\delta \equiv 0$ there. Also, $p(k)$ does not lie on $(\partial \mathcal{E}) \times (0, b]$ by Theorem 10.2. Thus
$$p(k) \in \mathcal{E} (a(k), R(k)) \times \{0\}.$$ 

Let $\tilde{p}(k) = (p(k), u_k(p(k)))$.

Case 1: $\text{dist}(\tilde{p}(k), \partial M(k))$ is bounded above.

Translate $M(k)$ by $\tilde{p}(k)$ to get a surface $M'(k)$. By passing to a subsequence, $M'(k)$ converges to a limit translator $M$. Note that $\partial M$ is nonempty, horizontal, and has corners. It follows (see Theorem 12.2) that the convergence of $M'(k)$ to $M$ is smooth except at the corners of $\partial M'$.

Note that the tangent plane to $M$ at 0 is vertical. By the strong maximum principle (if $0 \notin \partial M$) or the boundary maximum principle (if $0 \in \partial M$), the tangent plane to $M$ is vertical at every point. In particular, if $q \in \partial M$, then $q + te_{n+2} \in M$ for all $t > 0$. But since $\partial M$ has corners, this contradicts the smoothness of $M$. Thus Case 1 cannot occur.

Case 2: $\text{dist}(\tilde{p}(k), \partial M(k))$ is unbounded. By passing to a subsequence, we can assume that the distance tends to infinity.
Translate \( M(k) \) by \(-p(k)\) to get \( M'(k) \).

Note that \( 0 \in M'(k) \). Furthermore, by Theorem 10.1 \( M(k) \) is disjoint from \((0, \infty)^n \times \mathbb{R} \times (0, \infty)\) (10.3)

By passing to a subsequence, we can assume that the \( M'(k) \) converge smoothly to a complete, properly embedded translator \( M \) in \( \mathbb{R}^n \times (-b, b) \times \mathbb{R} \). (In fact, \( M \) is area-minimizing for the Ilmanen metric.)

Note that \( 0 \in M \) and that the tangent plane to \( M \) at \( 0 \) is vertical. Thus by the maximum principle, \( M \) is vertical everywhere. That is, \( M = \Sigma \times \mathbb{R} \)

for a smooth Euclidean minimal hypersurface \( \Sigma \subset \mathbb{R}^n \times (-b, b) \). Also, by (10.3), \( \Sigma \) is disjoint from \((0, \infty)^n \times \mathbb{R} \)

However, using moving catenoids shows that no such surface \( \Sigma \) exists.

(Here is the moving catenoid argument. Let \( \text{Cat} \) be an \( n \)-dimensional catenoid, i.e., a minimal hypersurface of revolution about the \( x_{n+1} \)-axis, bounded by spheres in the plane \( x_{n+1} = b \) and the plane \( x_{n+1} = -b \). Let \( \text{Cat}(s) \) be \( \text{Cat} \) translated by \( s(e_1 + \cdots + e_n) \).

For \( s > 0 \) very large \( \text{Cat}(s) \) is disjoint from \( M \) by (10.4). Thus \( \text{Cat}(s) \) is disjoint from \( M \) for all \( s \), since otherwise the strong maximum principle would be violated at the first point of contact. But this is a contradiction since \( 0 \in M \) and since there is an \( s \) from which \( 0 \in \text{Cat}(s) \).) □

11. Higher Dimensional \( \Delta \)-Wings in Slabs of Prescribed Width

As in the previous section, we let

\[
\begin{align*}
\mathcal{C}(a, R) \times [-b, b] & \to \mathbb{R}
\end{align*}
\]

be the translator with boundary values \( 0 \), and let

\[
\begin{align*}

k_i &= -D_{ii}u(0).
\end{align*}
\]

Now define \( F_{b, R} : \Delta^n \to \Delta^n \)

as follows.

Let

\[
\begin{align*}
F_{b, R}(a) &= \left( k_1, k_2, \ldots, k_n \right) \\
&= \left( \frac{\sum_{i=1}^{n} k_i}{\sum_{i=1}^{n} k_i} \right).
\end{align*}
\]

As before, \( F_{b, R} \) is continuous, and it maps each face of \( \Delta^n \) to itself. Thus \( F_{b, R} \) is surjective.

Fix \( b > \pi/2 \) and \( k \in \Delta^n \).

By surjectivity, for each \( R > 0 \), there is an \( a(R) \in \Delta_n \) such that

\[
F_{b, R}(a(R)) = k.
\]

Now every sequence of \( R \to \infty \) has a subsequence \( R_i \to \infty \) such that

\[
\begin{align*}
u_{a_{R(i)}, b, R(i)}(0) - u_{a(R(i)), b, R(i)}(0)
\end{align*}
\]

converges smoothly to a limit

\[
u : \mathbb{R}^n \times (-b, b) \to \mathbb{R}.
\]
Note the slope bound Theorem 10.3 implies that the domain of \( u \) is the entire slab \( \mathbb{R}^n \times (-b, b) \) rather than some open subset of it.

We have proved

**Theorem 11.1.** Let \( 0 \leq k_1 \leq k_2 \leq \cdots \leq k_n \) with \( \sum_i k_i = 1 \) and let \( b > \pi/2 \). Then there is a complete translator

\[
u : \mathbb{R}^n \times (-b, b) \to \mathbb{R}
\]

such that

\[
u(0) = 0,
\]
\[
u(x) < 0 \text{ for } x \neq 0,
\]

and such that \(-D^2\nu(0)\) is a positive definite diagonal matrix with diagonal entries \( \kappa_1, \ldots, \kappa_{n+1} \) where

\[
(\kappa_1, \ldots, \kappa_n)
\]

is a scalar multiple of \((k_1, \ldots, k_n)\).

If \( k_i = k_j \) (where \( 1 \leq i < j < n \)), then \((x_iD_j - x_jD_i)u \equiv 0\).

**Corollary 11.2.** For each width \( b > \pi/2 \), there is an \((n-1)\)-parameter family of such \( \Delta \)-wings \( u \), no two of which are congruent to each other.

In the special case \( k_1 = k_2 = \cdots = k_n \), then \( u \) is rotationally invariant about the \( x_{n+1} \) axis (since \((x_iD_j - x_jD_i)u \equiv 0\)). In that case, we have the following uniqueness theorem:

**Theorem 11.3.** Let \( b > \pi/2 \). Then there is a unique complete, proper translator

\[
u : \mathbb{R}^n \times (-b, b) \to \mathbb{R}
\]

with the following properties:

1. **(Rotational invariance)** \( u \) is rotationally invariant about the \( x_{n+1} \) axis:

\[
u(x_1, x_2, \ldots, x_n, x_{n+1}) \equiv \nu(\sqrt{x_1^2 + \cdots + x_n^2}, 0, 0, \ldots, x_{n+1}).
\]

2. **(Slope bound)** For every \( \beta < b \),

\[
\sup_{\mathbb{R}^n \times [-\beta, \beta]} |Du| < \infty.
\]

Existence is a special case of Theorem 11.1. The proof of uniqueness is essentially the same as the proof of Theorem 5.4, so we omit it. The arguments in the proof of Theorem 5.4 are very two dimensional, but if we mod out by the symmetry, then we are dealing with a function of two variables. Specifically, we let

\[
U : \mathbb{R} \times (-b, b) \to \mathbb{R},
\]

\[
U(x, y) = u(x, 0, 0, \ldots, 0, y).
\]

Existence (but not uniqueness) of rotationally invariant examples was proved in a different way, using barriers, by Bourni et al. The barriers show that their examples satisfy the slope bound hypothesis in Theorem 11.3.

In general, we do not know whether the surfaces in Theorem 11.1 are convex. However, if a complete translator has no more than two distinct principal curvatures at each point, then it is convex [BLT18, Theorem 3.1]. Hence in the special case \( k_1 = \cdots = k_n \), the surface is convex.
12. Appendix: Compactness Theorems

**Theorem 12.1.** For \( k = 1, 2, \ldots \), let \( \Omega_k \) be a convex open subset of \( \mathbb{R}^n \) and let \( u_k : \Omega_k \to \mathbb{R} \) be a smooth translator. Let \( M_k \) be the graph of \( u_k \). Suppose that \( W \) is a connected open subset of \( \mathbb{R}^n \) such that for each \( k \),
\[
W \times (-k, k)
\]
does not contain any of the boundary of \( M_k \).

Then, after passing to a subsequence, \( M_k \cap (W \times R) \) converges weakly in \( W \times R \) to a translator \( M \) that is \( g \)-area-minimizing. Furthermore, if \( S \) is a connected component of \( M \), then either

1. \( S \) is the graph of a smooth function over an open subset of \( W \) and the convergence to \( S \) is smooth, or
2. \( S = \Sigma \times \mathbb{R} \), where \( \Sigma \) is a variety in \( W \) that is minimal with respect to the Euclidean metric on \( \mathbb{R}^n \). The singular set of \( \Sigma \) has Hausdorff dimension at most \( n - 7 \).

**Proof.** Since \( \Omega_k \) is convex and since \( M_k \) and its vertical translates form a \( g \)-minimal foliation of \( \Omega_k \times \mathbb{R} \), standard arguments (cf. [Mor88, §6.2]) show that \( M_k \) is \( g \)-area-minimizing as an integral current, or even as a mod 2 flat chain.

Thus the standard compactness theorem (cf. [Sim83, §34.5]) gives subsequential convergence (in the local flat topology) to a \( g \)-area-minimizing hypersurface \( M \) (with no boundary in \( W \times I \)). Also, standard arguments show that the support of \( M_k \) converges to the support of \( M \). Hence we will not make a distinction here between the flat chain and its support.

For notational simplicity, let us assume that \( M \) is connected. Clearly, each vertical line intersects \( M \) in a connected set.

**Case 1:** \( M \) contains a vertical segment of some length \( \epsilon > 0 \). Let \( M(s) \) be the result of translating \( M \) vertically by a distance \( s \), where \( 0 < s < \epsilon \). Then by the strong maximum principle of L. Simon [Sim87], \( M = M(s) \). Since this is true for all \( s \) with \( 0 < s < \epsilon \), it follows that \( M = \Sigma \times \mathbb{R} \) for some \( \Sigma \). Since \( \Sigma \times \mathbb{R} \) is \( g \)-area-minimizing, its singular set has Hausdorff dimension at most \( (n + 1) - 7 \), and therefore the singular set of \( \Sigma \) has Hausdorff dimension at most \( n - 7 \). Since \( \Sigma \times \mathbb{R} \) is \( g \)-minimal, \( \Sigma \) must be minimal with respect to the Euclidean metric.

**Case 2:** \( M \) contains no vertical segment. Then \( M \) is the graph of a continuous function \( u \) whose domain is an open subset of \( W \). Let \( \overline{B}(p, r) \) be a closed ball in the domain of \( u \). Then \( \overline{B}(p, r) \) is contained in the domain of \( u_k \) for large \( k \), and \( u_k \) converges uniformly to \( u \) on \( \overline{B}(p, r) \). (The uniform convergence follows from monotonicity.) By Theorem 5.2 of [ES92] (rediscovered in [CM04, Theorem 1]), the convergence is smooth on \( B(p, r) \). \( \square \)

**Theorem 12.2.** For \( k = 1, 2, \ldots \), let \( \Omega_k \) be a convex open subset of \( \mathbb{R}^n \) such that the \( \Omega_k \) converge to an open set \( \Omega \). Let \( u_k : \Omega_k \to \mathbb{R} \) be a translator with boundary values 0, and let \( M_k \) be the graph of \( u_k \). Then, after passing to a subsequence, the \( M_k \) converge smoothly in \( \mathbb{R}^n \times (0, \infty) \) to a smooth translator \( M \). If \( S \) is a connected component of \( M \), then either

1. \( S \) is the graph of a smooth function whose domain is an open subset of \( \Omega \), or
2. \( S = \Sigma \times [0, \infty) \), where \( \Sigma \) is an \( (n - 1) \)-dimensional affine plane in \( \mathbb{R}^n \).
Furthermore, $M$ is a smooth manifold-with-boundary in a neighborhood of every point of $\partial M$ where $\partial M$ is smooth, and the convergence of $M_k$ to $M$ is smooth up the boundary wherever the convergence of $\partial M_k$ to $\partial M$ is smooth.

Proof. Case 1: $S$ contains a point $p$ in $\partial \Omega \times (0, \infty)$. Let $\Sigma$ be the connected component of $\partial \Omega$ such that $p \in \Sigma \times (0, \infty)$. Then by the strong maximum principle \cite{SW89} or \cite{Sim87} or \cite{Whi16b, Theorem 7.3}, $M$ contains all of $\Sigma \times (0, \infty)$, and therefore (since $\Omega$ is convex) $\Sigma$ must be a plane. It follows that the convergence to $S$ is smooth in $\mathbb{R}^n \times (0, \infty)$.

Case 2: $S$ contains no point in $\partial \Omega \times \mathbb{R}$. That is, $S$ is contained in $\Omega \times \mathbb{R}$. Indeed, $S$ is contained in $\Omega \times \mathbb{R}$, so it cannot be translation-invariant in the vertical direction. Thus by Theorem \ref{12.1} $S$ is a smooth graph over an open subset of $\Omega$, and the convergence to $S$ is smooth. The assertions about boundary behavior follow, for example, from the Hardt-Simon Boundary Regularity Theorem \cite{HS79}. (Note that the tangent cone to $M$ at a regular point of $\partial M$ is, after a rotation of $\mathbb{R}^n$, a cone in $\mathbb{R}^{n-1} \times [0, \infty) \times [0, \infty)$.

\[\square\]

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