Primordial perturbations in the Dapor-Liegener model of hybrid loop quantum cosmology

Laura Castelló Gomar, Alejandro García-Quismondo and Guillermo A. Mena Marugán

Instituto de Estructura de la Materia, IEM-CSIC, Serrano 121, 28006 Madrid, Spain

In this work, we extend the formalism of hybrid loop quantum cosmology for primordial perturbations around a flat, homogeneous, and isotropic universe to the new treatment of Friedmann-Lemaître-Robertson-Walker geometries proposed recently by Dapor and Liegener, based on an alternative regularization of the Hamiltonian constraint. In fact, our discussion is applicable also to other possible regularization schemes for loop quantum cosmology, although we specialize our analysis to the Dapor-Liegener proposal and construct explicitly all involved quantum operators for that case.

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I. INTRODUCTION

The theory of primordial perturbations \cite{1,4} in a spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model is the cornerstone of modern cosmology, since it allows us to connect theories of the very early Universe with the most precise data obtained experimentally. Combined with the inflationary paradigm \cite{3,8}, this satisfactorily describes the evolution of the Universe from its primeval stages and explains the formation of structures observed nowadays at large scales \cite{9}. The basic idea is that primordial inhomogeneities around a smooth background emerged from quantum vacuum fluctuations, and then provided the seeds of cosmological structure formation under the action of gravitational instability \cite{10}. Therefore, one expects that both quantum mechanics and general relativity will be ultimately required in order to fully understand the generation and evolution of fluctuations in an accurate cosmological description.

The standard model is based on the conceptual framework of quantum field theory (QFT) in a classical and fixed curved spacetime, where perturbations are viewed as quantum test fields propagating on a given geometry. Present observations of the cosmic microwave background (CMB) broadly lead to a solid confirmation of the predictions of this standard cosmology. Even so, there appear to exist some puzzling discrepancies between the theoretical results and the observations for large angular scales (around low multipoles, close to number 30 and below) \cite{11,12}. At these scales, however, all measurements are affected by the errors caused by cosmic variance. In the wait for new relevant advances in this front, both the polarization signal of the CMB and the search for signals of the gravitational wave background emitted during the inflationary epoch may provide interesting new frontiers in observational cosmology, as they might offer key information about the early Universe and open a window to new physics.

From a fundamental point of view, the hope is that these data may encode information about phenomena caused by the quantum nature of spacetime geometry itself, as quantum gravity effects should be relevant in the extreme conditions experienced in the early stages of the cosmos. Since a definitive quantum theory of gravity has not yet been established satisfactorily, in the last ten years an original approach has been proposed to deal with the quantization of inhomogeneous gravitational models. The so-called hybrid quantum cosmology formalism \cite{14} combines two different types of quantum representations: one of them is based on a genuine quantum theory of geometry, used for the description of (at least) the homogeneous sector of the cosmological system, and the other consists in a more conventional Fock quantum description, employed for the inhomogeneities (typically identified with perturbations) of both the geometry and the matter content. This hybrid approach rests basically on the assumption that there exists a regime of the quantum dynamics, in between a fully quantum gravity regime and the scheme of QFT in fixed curved spacetimes, where the most relevant quantum effects of the geometry are those affecting the zero modes of the homogeneous sector. An important point for the application of this formalism to cosmological perturbations is the mathematically consistent truncation of the total action at quadratic order in the perturbations. Indeed, this truncation of the action allows one to maintain a symplectic structure for the whole system, formed by the homogeneous degrees of freedom and the perturbations, while keeping the restrictions imposed on this system by the gravitational constraints. In general, in the implementation of this hybrid formalism, one assumes that the quantization satisfies the canonical commutation relations inasmuch as the operators for the homogeneous (and isotropic) geometry commute with those representing the homogeneous sector of the matter fields and, in turn, all of these commute with the elementary operators corresponding to the variables that describe the inhomogeneities.

The hybrid approach was first put forward some ten years ago to deal with cosmological universes with compact sections that sustain gravitational waves. More specifically, the approach was introduced to attain a quantum formulation of the so-called Gowdy cosmologies \cite{13,16} for the model with three-toroidal spatial sections and linearly polarized waves \cite{17,20}. The strategy was soon extended to the analysis of the more realistic system of an FLRW universe with cosmological perturbations, discussing both scalar \cite{21,23} and tensor perturbations \cite{24}. In principle, these original works were specialized to the case where the homogeneous sector of the gravitational system is quantized according to the rules of loop quantum cosmology (LQC) \cite{25,26}. LQC is the particularization to cosmological reductions of general relativity of the strategies of loop quantum gravity (LQG) \cite{27,28}, a nonperturbative formalism for the quantization of the Einsteinian theory. However, the hybrid approach can be generalized and adapted to other candidates for the quantum description of the homogeneous sector of cosmological spacetimes, although this fact may not have been sufficiently emphasized in the literature. Actually, the discussion carried out in Ref. \cite{14} for the gauge invariant treatment of the cosmological perturbations in the hybrid approach was already presented in a way that is suitable for this generalization.

This versatility of the hybrid strategy is especially useful in the situation that is found at present in LQC, where the community is discussing possible alternatives for the regularization of the geometric operators that appear in the analysis of homogeneous cosmologies. In practice, these alternatives, originated from regularization ambiguities \cite{29,31} in the symmetry reduction to homogeneity, result in different quantization prescriptions for the homogeneous sector of the geometry. In this sense, they can be regarded as different representations of the operators that describe the effect of the zero modes of the geometry. The discussed property of the hybrid approach (namely that it can
be accommodated to distinct representations of the zero modes) then allows us to study the quantum inclusion of inhomogeneities while permitting the consideration of any of the mentioned alternatives in LQC. Apart from the standard regularization followed so far in LQC \[32\,33\], most of the attention has been focused on a regularization procedure recently put forward by Dapor and Liegener (DL) \[34\], following pioneering works by Thiemann \[29\] and Yang, Ding, and Ma \[25\]. This procedure adopts a different regularization scheme for the Euclidean and Lorentzian parts of the gravitational action, namely the parts without and with quadratic dependence on the extrinsic curvature, respectively \[28\], implementing this regularization before proceeding to the symmetry reduction to homogeneity and isotropy. This alternative for the construction of the Hamiltonian of LQC has been considered in several papers in the last two years \[36\,37\]. The extension to anisotropic universes of the Bianchi type I has also been studied \[44\]. An especially appealing property of the new Hamiltonian constraint obtained in this manner for flat FLRW cosmologies is that the effective solutions typically present two branches with different cosmological behavior: one of them corresponding to an asymptotically de Sitter cosmology, even in the absence of a genuine cosmological constant, and the other one describing an FLRW universe. Therefore, one finds a physical scenario in which a contracting de Sitter regime is followed by a quantum bounce after which there may exist an inflationary era leading to a universe like the one we observe. This suggestive fact, with potential implications for the generation of primordial perturbations, and the other one describing an FLRW universe. Therefore, one finds a physical scenario in which a contracting de Sitter regime is followed by a quantum bounce after which there may exist an inflationary era leading to a universe like the one we observe. This suggestive fact, with potential implications for the generation of primordial perturbations, together with the convenience of studying well-founded regularizations in LQC other than the standard one (which seems desirable given that we do not yet fully understand the relation between LQG and this formalism of quantum cosmology \[15\,49\]), are the main motivations for our interest in investigating the application of the hybrid approach to inhomogeneities in spacetimes that are described quantum mechanically with the DL proposal.

In more detail, in this work we will construct a theoretical framework for the treatment of cosmological perturbations adopting the DL proposal for the quantization of the FLRW sector of the geometry. We will consider inhomogeneous perturbations in a flat FLRW spacetime with a minimally coupled scalar field. We will study both scalar and tensor perturbations. In general, scalar perturbations are physically the most important ones and, conceptually, the most interesting ones, as they couple to energy density fluctuations, and are ultimately responsible for most of the inhomogeneities and anisotropies in the Universe. This also complicates their mathematical description. Additionally, inflation also generates tensor fluctuations in the spatial metric, that can be viewed as gravitational waves. These are not coupled to any other perturbation at the order of our truncations. However, they do induce fluctuations in the CMB which turn out to be a unique signature of the early epochs of the Universe and offer a valuable window on the physics driving in inflation. To deal with the problem of gauge invariance \[50\] at the level of the perturbations while maintaining a canonical set of variables for the entire cosmological model, including the homogeneous FLRW universe, we will follow the formulation elaborated in Ref. \[14\]. In particular, the physical degrees of freedom of the scalar perturbations will be described by Mukhanov-Sasaki (MS) variables \[51\,52\], that are perturbative gauge invariants and are directly related to the comoving curvature perturbations. For the tensor perturbations, which are directly perturbative gauge invariant variables, we will adopt the description of Ref. \[24\]. Finally, for the quantization of the FLRW part of the geometry according to the DL proposal, we will follow the prescription introduced in Ref. \[43\], that extends to this new alternative regularization the quantization strategy that was detailed in Ref. \[33\] for the standard regularization employed in LQC.

The rest of the paper is organized as follows. The basic results of previous works on the gauge invariant description of the perturbed model will be summarized in Sec. \[III\]. At the end of that section, we will specify the resulting zero mode of the Hamiltonian constraint (the only constraint that remains on the system after adopting suitable gauge invariant variables for the perturbations), including terms that are quadratic in the inhomogeneous perturbations. In Sec. \[III\] we will proceed to the quantization of our constrained system (formed by the homogeneous background and the perturbations) following the hybrid approach and adhering to the new formalism of LQC proposed by Dapor and Liegener for the homogeneous geometry. In particular, an alternative representation of the momentum of the scale factor (or, equivalently, of the physical volume) will be constructed. To conclude, we will present a summary of the main results and some further discussions in Sec. \[IV\]. We set the speed of light and the reduced Planck constant equal to one throughout our discussion.

II. GAUGE INVARIANT PERTURBATIONS AROUND A FLAT FLRW COSMOLOGY

The theory of cosmological perturbations is complicated by the issue of gauge invariance (see e.g. Ref. \[50\]). By performing a small amplitude transformation of the spacetime coordinates, one can easily introduce fictitious fluctuations in a homogeneous and isotropic spacetime. These fluctuations are gauge artifacts that carry no physical significance. There are mainly two approaches to deal with these gauge ambiguities. The first one is to perform a gauge fixing by means of the imposition of suitable conditions on the perturbations, such that they completely eliminate the gauge freedom. The second possibility is to work directly with gauge invariant variables to describe the perturbations. The formulation in terms of these gauge invariants is especially convenient for the passage to
the quantum theory, since they allow to reach robust results that are not byproducts of a particular choice of gauge conditions and because they already take into account the effects of the fundamental uncertainties about the gauge sector of the perturbations. In this section, we succinctly review the construction of a gauge invariant description of the perturbations around (flat) FLRW cosmologies and explain how this description can be extended to include the degrees of freedom of the FLRW universes while retaining a canonical formalism for the entire gravitational system. This construction follows the discussion presented in Ref. [14], that was straightforwardly adapted to tensor perturbations in Ref. [24]. We therefore refer the reader to those articles for further details.

A. The model

We will study a gravitational system resulting from the introduction of perturbations around a homogeneous and isotropic background spacetime. The unperturbed cosmology that serves as the starting point of our discussion is a flat FLRW model. We focus here on the case where the spatial sections are compact, with the topology of a three-torus. Given that the dynamics of vacuum flat FLRW cosmologies are trivial, we include matter content that is provided by a minimally coupled scalar field subject to a generic potential.

On top of this homogeneous background, we include inhomogeneities (and anisotropies) that will be treated perturbatively, in a way that we will make clear later in this section. These perturbations, that affect both the spacetime metric and the scalar field, are typically classified according to their transformation rules under the isometries of the spatial sections. In this sense we will distinguish among scalar, vector, and tensor perturbations (if one wants to, extrapolating our discussion to infinitely large compactification scales). It is a well-known fact that, at the lowest nontrivial perturbative order, vector perturbations are purely a gauge artifact in the case where a scalar field is the only matter content. Therefore, from now on, we focus our attention on scalar and tensor perturbations only.

It is possible (and convenient) to recast the perturbed metric functions and scalar field as suitable expansions in a complete set of scalar and tensor harmonics. These harmonics can be constructed out of appropriate combinations of the spatial three-metric, the affine connection compatible with it, and the eigenmodes of the spatial Laplace-Beltrami operator. These modes are characterized by a tuple of integers $\vec{n} \in \mathbb{Z}^3$, which can be understood as a wave vector. In order to circumvent the issue of the repetition of modes if one employs, for instance, real Fourier harmonics, the tuples $\vec{n}$ are typically restricted by the condition that their first nonvanishing component be strictly positive, and the different harmonics with the same value of $\vec{n}$ are distinguished by a parity label $\epsilon = \pm$. Moreover, the introduction of an additional parameter labeling the polarization, $\tilde{\epsilon} = +, \times$, proves useful in the explicit construction of the tensor harmonics [24]. Then, in short, we write down a general perturbation of the metric and the scalar field, and express it in terms of expansions in the scalar and tensor harmonics, resulting in sums over all possible values of $\vec{n}$, $\epsilon$, and $\tilde{\epsilon}$ (the latter affecting only the tensor components). It is important to note that the zero modes, that describe the homogeneous sector of the system, are left out of these sums since they are treated exactly, and not as perturbations (for details and arguments supporting this treatment, consult e.g. Refs. [3, 4, 23, 55]). At this point, the dynamics of the perturbations are encoded in the Fourier coefficients that arise in the harmonic expansions mentioned above. The homogeneous sector, on the other hand, can be parametrized, for instance, using the homogeneous lapse function, $N_0$, the real logarithmic scale factor $\alpha$ of the unperturbed FLRW background, and the zero mode of the scalar field, $\varphi$.

With these expansions at hand, we can compute the corresponding expansion of the gravitational action and truncate the result at the lowest nontrivial order in the Fourier coefficients, i.e., the quadratic one (given that the zero modes of the system have been dealt with in an exact manner). Thus, we henceforth neglect any terms that contribute to perturbative orders higher than this quadratic order in the action. In this way, we obtain the total Hamiltonian of the system $H$ at the mentioned perturbative truncation, which turns out to be a linear combination of constraints (as expected, since this fact reflects the invariance under spatial diffeomorphisms and time reparametrizations that is inherited from the full theory of general relativity). Indeed, we obtain the following Hamiltonian [14, 24]:

$$H = N_0 \left[ H_{|0} + \sum_{\vec{n},\epsilon} H_{|2}^{\vec{n},\epsilon} + \sum_{\vec{n},\epsilon,\tilde{\epsilon}} T^{\vec{n},\epsilon,\tilde{\epsilon}} H_{|2}^{\vec{n},\epsilon,\tilde{\epsilon}} \right] + \sum_{\vec{n},\epsilon} g_{\vec{n},\epsilon} \tilde{H}_{|1}^{\vec{n},\epsilon} + \sum_{\vec{n},\epsilon} k_{\vec{n},\epsilon} \tilde{H}_{|\tilde{\epsilon}}^{\vec{n},\epsilon}. \quad (2.1)$$

The lapse $N_0$ and the functions $g_{\vec{n},\epsilon}$ and $k_{\vec{n},\epsilon}$ (that are Fourier coefficients related to the perturbative expansion of the lapse function and the shift vector in terms of scalar harmonics) are nondynamical, in the sense that the Hamiltonian does not depend on their conjugate momenta, and there is no additional dependence on these functions other than the one explicitly shown above. Therefore, they are simple Lagrange multipliers and their associated classical equations of motion amount to the vanishing of constraints: the zero mode of the Hamiltonian constraint, $H_{|0}$, that corresponds to the unperturbed flat FLRW background, and two terms that collect...
the quadratic contributions from the inhomogeneities \((H_{\alpha}^{\vec{n},\vec{e}}, \text{ sourced by the scalar perturbations}, \text{ and } T H_{[2]}^{\vec{n},\vec{e}}, \text{ coming from the tensor perturbations instead})\). The homogeneous contribution is given by

\[
H_0 = \frac{e^{-3a}}{2} \left[ -\pi_\alpha^2 + \pi_\varphi^2 + 2e^{3a} \bar{W}(\varphi)^2 \right],
\]

where \(\pi_\alpha\) and \(\pi_\varphi\) are the momentum variables canonically conjugate to \(\alpha\) and \(\varphi\), respectively. In addition, \(\bar{W}(\varphi)\) is related with the field potential \(W(\varphi)\) by \(W(\varphi) = \sigma^4 W(\varphi/\sigma)\), where \(\sigma^2 = 4\pi G/(3l_0^2)\) and \(l_0\) is the period of the fundamental cycles of the three-tori, isomorphic to the spatial sections.

On the other hand, the two remaining constraints are similar inasmuch as they are formed by contributions that are linear in the perturbations instead of quadratic. However, their origins are different: \(\tilde{H}_{1}^{\vec{n},\vec{e}}\) results from the perturbation of the Hamiltonian constraint around the unperturbed background in full general relativity, whereas \(\tilde{H}_{[2]}^{\vec{n},\vec{e}}\) arises from the perturbation of the diffeomorphism constraint. We notice that there is no perturbative constraint linear in the tensor perturbations, a fact which can ultimately be traced back to the absence of couplings with tensor matter fields.

To conclude this brief description of our model, it is important to remark that the system is symplectic at the order of the discussed perturbative truncation. The canonical variables that coordinatize the homogeneous phase space are given by \(\{u^a\}_{a=1,2} = \{w^\alpha, u^\beta\}_{a=1,2} = \{\alpha, \varphi; \pi_\alpha, \pi_\varphi\}\). In the inhomogeneous sector, however, two sets can be distinguished: one of them describes the dynamics of the scalar perturbations, \(\{X_{\vec{q},\vec{p}}^{\alpha}, \delta_{\vec{q},\vec{p}}\}_{l=1,2,3} = \{\bar{X}_{\vec{q},\vec{p}}^{\alpha}, X_{\vec{q},\vec{p}}^{\alpha}\}_{l=1,2,3}\), and the other one describes the tensor perturbations, \(\{d_{\vec{q},\vec{p}}^{\alpha,\beta}, \pi_{d_{\vec{q},\vec{p}}^{\alpha,\beta}}\}_{l=1,2,3}\). In both cases, they are composed by the dynamical Fourier coefficients that are involved in the expansions in scalar and tensor harmonics that we have discussed above (and their associated momenta). For the scalar perturbations of the scalar field and the spatial metric there exist three degrees of freedom, corresponding to the different values of the label \(l\), while for the tensor perturbations we have to deal only with one degree of freedom for each polarization, and the only quantity affected is the spatial metric.

### B. Gauge invariant formalism

In this section, we will present a gauge invariant formalism for the description of the perturbations. Therefore, we will focus our attention on the inhomogeneous sector of the system and regard the homogeneous background as a fixed entity (we will reintroduce the dynamics of the homogeneous background into our physical picture at a later point of the discussion).

As we have just commented, the dynamics of the inhomogeneities is encoded in a series of configuration variables \(\{X_{\vec{q},\vec{p}}^{\alpha}, d_{\vec{q},\vec{p}}^{\alpha,\beta}\}\) and their canonically conjugate momenta \(\{X_{\vec{q},\vec{p}}^{\alpha}, \pi_{d_{\vec{q},\vec{p}}^{\alpha,\beta}}\}\). The variables that describe the tensor modes, \(d_{\vec{q},\vec{p}}^{\alpha,\beta}\) and \(\pi_{d_{\vec{q},\vec{p}}^{\alpha,\beta}}\), turn out to be already gauge invariant in the sense of the Bardeen potentials \([50]\). Thus, we only need to focus on the scalar sector. Notice that we can work on the scalar sector separately because the scalar and the tensor perturbations are decoupled in our model at the considered order of truncation. We want to introduce a description in terms of gauge invariant variables, thus avoiding any gauge fixing in the process that could contravene covariance at the studied perturbative order, and with a view to ultimately constructing a formulation that respects the canonical structure of the entire cosmological system.

The concept of gauge invariance employed here is a perturbative one, adapted to our truncation scheme. Perturbative gauge invariants are characterized by being invariant under a perturbative diffeomorphism when the background is regarded as fixed, meaning that, in that situation, they Poisson commute with the generators of perturbative diffeomorphisms: the linear perturbative constraints, \(\tilde{H}_{1}^{\vec{n},\vec{e}}\) and \(\tilde{H}_{[2]}^{\vec{n},\vec{e}}\), that we introduced in the previous section.

The procedure followed in Ref. \([14]\) to achieve a gauge invariant description of the inhomogeneous sector, compatible with the canonical structure and based on previous work by Langlois \([50]\), begins by introducing the so-called Mukhanov-Sasaki (MS) field and the associated MS variables \(v_{\vec{n},\vec{e}}\), identified as the Fourier coefficients of that field \([51–54]\). These are defined as linear combinations of the configuration variables of the scalar sector \(X_{\vec{q}}^{\vec{e}}\) (the coefficients of these linear combinations are functions of the homogeneous background, which are viewed as fixed quantities for the time being). These variables Poisson commute with the two linear perturbative constraints when the homogeneous background is considered nondynamical and, therefore, are perturbative gauge invariants.

Given that \(v_{\vec{n},\vec{e}}\) commutes with both \(\tilde{H}_{1}^{\vec{n},\vec{e}}\) and \(\tilde{H}_{[2]}^{\vec{n},\vec{e}}\), it is actually possible to construct a set of configuration variables for the inhomogeneous phase space out of these three objects. In order to maintain the canonical structure, we only need to Abelianize the algebra of perturbative constraints, so that we get a complete set of commuting variables for the perturbations from the perturbative constraints and the MS field (in this sense, we recall that the algebra of Fourier coefficients that originally served as configuration variables for the scalar perturbations was Abelian, i.e., \(\{X_{\vec{q}}^{\vec{e}}, X_{\vec{q}'}^{\vec{e}}\} = 0\) for all \(l, l' = 1, 2, 3\)). This Abelianization can be performed in a simple manner by replacing \(\tilde{H}_{1}^{\vec{n},\vec{e}}\) with \(\tilde{H}_{1}^{\vec{n},\vec{e}} = \tilde{H}_{1}^{\vec{n},\vec{e}} - 3e^{3a} H_0 X_{\vec{q}}^{\vec{e}}\), where \(X_{\vec{q}}^{\vec{e}}\) are the variables corresponding to the scalar perturbations of the
trace-part of the spatial metric. Making this replacement in the action results in a total Hamiltonian with the same form as before once the lapse function is suitably redefined, \( N_0 \rightarrow N_0 \), to absorb any quadratic contributions from the perturbations \( \mathbb{L} \). It can easily be shown that, indeed, \( \{ \hat{H}^\vec{n}, v_{\vec{n}, \epsilon}, \hat{H}^\vec{n}, v_{\vec{n}, \epsilon} \} \) = 0 = \( \{ H^\vec{n}, v_{\vec{n}, \epsilon} \} \), where \( \{ \cdot, \cdot \}(P) \) denotes the Poisson bracket on the inhomogeneous sector when the dependence on the homogeneous background is ignored.

Thus, the set \( \{ v_{\vec{n}, \epsilon}, \hat{H}^\vec{n}, \hat{H}^\vec{n} \} \) is composed by gauge invariant variables and constraints that commute with one another. Then, in order to achieve the desired description, the only remaining step is to complete the change of perturbative variables into a canonical transformation for the inhomogeneous sector by finding appropriate conjugate variables \( \{ \pi_{v_{\vec{n}, \epsilon}}, C^\vec{n}, C^\vec{n}_{\epsilon} \} \), such that the only nontrivial brackets are \( \{ v_{\vec{n}, \epsilon}, \pi_{v_{\vec{n}, \epsilon}} \}(P) = \{ C^\vec{n}, \hat{H}^{\vec{n}} \}(P) = \{ C^\vec{n}_{\epsilon}, \hat{H}^{\vec{n}} \}(P) = 1 \). These additional variables can be constructed with relative ease (for details regarding their explicit expressions, see Ref. \( \mathbb{L} \)), attaining in this way the wanted canonical change for the scalar perturbations: \( X^\vec{n}_{\epsilon} \rightarrow V^\vec{n}_{\epsilon} \equiv \{ \pi_{v_{\vec{n}, \epsilon}}, \hat{H}^{\vec{n}} \} \), where \( \{ V^\vec{n}_{\epsilon} \}_{l=1,2,3} = \{ v_{\vec{n}, \epsilon}, C^{\vec{n}_{\epsilon}}, C^{\vec{n}_{\epsilon}} \} \) are the new configuration variables and \( \{ \hat{H}^{\vec{n}} \}_{l=1,2,3} = \{ \pi_{v_{\vec{n}, \epsilon}}, \hat{H}^{\vec{n}} \} \) are the new momenta. Notice that we have included the perturbative constraints in the form of momentum variables. This will facilitate their quantum implementation, as we will discuss in Sec. \( \mathbb{III} \).

**C. Redefinition of the gauge invariant variables**

Before reintroducing the dynamics of the homogeneous background, we want to address an extra freedom that exists in the formalism as we have described it so far, namely, the variables that encode the dynamics of the scalar and tensor perturbations are not uniquely fixed and, indeed, we can perform transformations that leave the canonical structure of the perturbations invariant while changing those variables, something which may have important consequences in the quantum theory. In this subsection we discuss a criterion that allows us to eliminate this freedom up to unitary transformations.

In Sec. \( \mathbb{III} \) we will perform a Fock quantization of the inhomogeneities and, in particular, of the MS and tensor fields. If we carry out a partial reduction (and time deparametrization) of the system so that it can be described by a quantum field theory in a curved background, the MS and tensor fields can actually be interpreted as fields propagating in an ultrastatic compact background. There exist a series of works that guarantee the uniqueness, up to unitary equivalence, of the Fock quantization of fields that propagate in such spacetimes and satisfy a dynamical equation of the Klein-Gordon type, with a mass that can be time-dependent \( \mathbb{57–62} \). This result holds when the Fock representation is required to exhibit the two following properties: (i) the vacuum of the quantum theory is invariant under the isometries of the spatial sections and (ii) the quantum evolution associated with the Klein-Gordon equation can be implemented unitarily. Therefore, the symmetries of the background spacetime and the requirement that the dynamical (Heisenberg) evolution be implemented at the quantum level by a unitary operator pick out a single family of Fock representations that are unitarily equivalent. However, for this statement to apply to the case under consideration, the scalar and tensor gauge invariant fields (as well as their associated momenta) need to undergo a suitable transformation, that effectively fixes the freedom mentioned above up to unitary equivalence.

In the case of the scalar perturbations, we find that the MS field, with the mode coefficients \( v_{\vec{n}, \epsilon} \), already satisfies an equation of Klein-Gordon type in its present parametrization. This already eliminates the freedom of performing a rescaling of the field by a function of the homogeneous sector (and the inverse scaling in the canonically conjugate variable that ensures that the canonical structure is respected). There still exists a relevant freedom in our description for the choice of the associated momentum, since in principle we could add to it a term proportional to the MS field (without coupling different modes). The change in question of the corresponding mode coefficients is

\[
\pi_{v_{\vec{n}, \epsilon}} \rightarrow \pi_{v_{\vec{n}, \epsilon}} + F v_{\vec{n}, \epsilon}.
\]

Notice that, provided that the coefficient \( F \) is a function of the homogeneous phase space alone, the Poisson brackets in the inhomogeneous sector would be unaffected. It turns out that there exists a necessary (and sufficient) condition for the unitary implementability of the quantum evolution which allows us to fix this coefficient. Indeed, it is possible to implement the dynamics as a unitary quantum transformation if the MS modes verify the following equation of motion at the considered perturbative order: \( \dot{v}_{\vec{n}, \epsilon} = \pi_{v_{\vec{n}, \epsilon}} \), where the dot denotes the derivative with respect to the time coordinate, \( t \). Then, it is immediate to realize that the zero mode of the Hamiltonian constraint (which generates the time evolution) cannot contain any term linear in \( \pi_{v_{\vec{n}, \epsilon}} \). Hence, we can use the freedom to fix \( F \) to redefine the momentum variables \( \pi_{v_{\vec{n}, \epsilon}} \) in such a way that the zero mode of the Hamiltonian constraint no longer contains any such linear contribution. This procedure removes the ambiguity in the choice of the gauge invariant variables of the scalar perturbations up to unitary transformations.

In the case of the tensor perturbations, the field determined by the mode coefficients \( d_{\vec{n}, \epsilon} \) does not satisfy an equation of Klein-Gordon type. However, an equation of this class governs the dynamics if one performs a scaling:
\[ d\tilde{h}_{\vec{n},\varepsilon} \rightarrow \tilde{d}\tilde{h}_{\vec{n},\varepsilon} = \epsilon^a d\tilde{h}_{\vec{n},\varepsilon}, \text{ and } \pi_{d\tilde{h}_{\vec{n},\varepsilon}} \rightarrow \pi_{d\tilde{h}_{\vec{n},\varepsilon}} = \epsilon^{-a} \pi_{d\tilde{h}_{\vec{n},\varepsilon}}. \]

Then, one is in the same position as in the case above for the scalar perturbations. Let us finally consider the freedom to add an extra term in the momentum, proportional to the configuration of the tensor field:

\[ \pi_{d\tilde{h}_{\vec{n},\varepsilon}} \rightarrow \pi_{d\tilde{h}_{\vec{n},\varepsilon}} + \tilde{F}d\tilde{h}_{\vec{n},\varepsilon}. \] (2.4)

Again, this freedom can be fixed by requiring that the tensor modes satisfy the equation of motion \( \hat{d}\tilde{h}_{\vec{n},\varepsilon} = \pi_{d\tilde{h}_{\vec{n},\varepsilon}}. \)

Moreover, the unitary implementability of the resulting field dynamics, together with the invariance under the spatial isometries, selects a single family of unitarily equivalent Fock representations for the tensor perturbations.

In conclusion, we have designed a canonical description of the inhomogeneous phase space in terms of gauge invariant variables, preserving the covariance at the considered perturbative order. This description is adapted to facilitate the quantum implementation of the linear constraints and is devised to make possible the unitary implementation of the quantum dynamics. Additional restrictions on the properties of the perturbative terms of the quantum Hamiltonian (and hence on the dynamical behavior of the perturbations) can help to further select the Fock representation of the gauge invariants among the family that is allowed by the spatial isometry and the unitary implementability of the field evolution \([63]\). At this stage, we are ready to switch on again the dynamical nature of the background in our formalism and complete the canonical transformation performed in the perturbations to include also the homogeneous sector.

For the sake of simplicity, in the rest of our discussion we will denote the modified momentum variables using the same notation as before: \( \pi_{v_{\vec{n},\varepsilon}} \) and \( \pi_{d\tilde{h}_{\vec{n},\varepsilon}} \) for the scalar and tensor momenta, respectively.

### D. Completion of the canonical transformation in the homogeneous sector

In the previous two sections, we regarded the background homogeneous variables \( \{w^a\}_{a=1,2} \) as fixed, in order to concentrate our attention on the treatment of the inhomogeneous sector. In that sector, we determined a canonical transformation \( \{\tilde{X}_{\vec{n},\varepsilon}^a, \tilde{d}\tilde{h}_{\vec{n},\varepsilon}, \pi_{d\tilde{h}_{\vec{n},\varepsilon}}\} \rightarrow \{\tilde{V}_{\vec{n},\varepsilon}^{\tilde{R},\varepsilon}, \tilde{d}\tilde{h}_{\vec{n},\varepsilon}, \pi_{d\tilde{h}_{\vec{n},\varepsilon}}\}. \) Now, we proceed to extend this transformation, so that it encompasses the background variables as well, which will also undergo a modification \( \{w^a\} \rightarrow \{\tilde{w}^a\} \equiv \{\tilde{a}, \tilde{\varphi}; \pi_{\tilde{a}}, \pi_{\tilde{\varphi}}\}. \)

We seek a transformation that leaves invariant the symplectic structure, the information of which is encoded in the Legendre term of the action, that we call \( K. \) Our aim is to find a set of homogeneous variables \( \{\tilde{w}^a\}_{a=1,2} \) such that \( K \) retains its canonical form when expressed in terms of our new variables for the perturbations:

\[ K = \int dt \left[ \sum_a \bar{w}^a_q \dot{w}^a_p + \sum_{\vec{n},\varepsilon,l} \dot{\tilde{X}}_{\vec{n},\varepsilon}^a \tilde{X}_{\vec{n},\varepsilon}^a + \sum_{\vec{n},\varepsilon,\varepsilon} \dot{\tilde{d}}_{\vec{n},\varepsilon,\varepsilon} \pi_{d\tilde{h}_{\vec{n},\varepsilon,\varepsilon}} \right] = \int dt \left[ \sum_a \bar{w}^a_q \dot{w}^a_p + \sum_{\vec{n},\varepsilon,l} \tilde{V}_{\vec{n},\varepsilon}^{\tilde{R},\varepsilon} \tilde{V}_{\vec{n},\varepsilon}^{\tilde{R},\varepsilon} + \sum_{\vec{n},\varepsilon,\varepsilon} \dot{\tilde{d}}_{\vec{n},\varepsilon,\varepsilon} \pi_{d\tilde{h}_{\vec{n},\varepsilon,\varepsilon}} \right], \] (2.5)

where, according to our truncation, the last identity must hold only up to terms higher than quadratic in the perturbations. As shown in Ref. [14] (see also Refs. [63] [64]), the result is that the homogeneous variables receive backreaction corrections that are quadratic in the perturbations. Indeed, one finds that

\[ w^a - \tilde{w}^a = \sum_{\vec{n},\varepsilon} S \Delta \tilde{w}^a_{\vec{n},\varepsilon} + \sum_{\vec{n},\varepsilon,\varepsilon} T \Delta \tilde{w}^a_{\vec{n},\varepsilon,\varepsilon} = \Delta \tilde{w}^a, \] (2.6)

where \( S \Delta \tilde{w}^a_{\vec{n},\varepsilon} \) and \( T \Delta \tilde{w}^a_{\vec{n},\varepsilon,\varepsilon} \) are quadratic in the scalar and tensor perturbations, respectively.

We must still reexpress the zero mode of the Hamiltonian constraint in terms of the new variables (recall that the perturbative constraints are essentially part of the new perturbative variables). The original expression had the form

\[ H_0(w^a) + \sum_{\vec{n},\varepsilon} \tilde{H}_{\vec{n},\varepsilon}^{\tilde{R},\varepsilon}(w^a, X_{\vec{n},\varepsilon}^a) + \sum_{\vec{n},\varepsilon,\varepsilon} \tilde{T} \tilde{H}_{\vec{n},\varepsilon,\varepsilon}^{\tilde{R},\varepsilon,\varepsilon}(w^a, d\tilde{h}_{\vec{n},\varepsilon,\varepsilon}, \pi_{d\tilde{h}_{\vec{n},\varepsilon,\varepsilon}}), \] (2.7)

where the dependence of the last two terms on the perturbations is quadratic. It is clear that, since \( \tilde{H}_{\vec{n},\varepsilon}^{\tilde{R},\varepsilon} \) and \( \tilde{T} \tilde{H}_{\vec{n},\varepsilon,\varepsilon}^{\tilde{R},\varepsilon,\varepsilon} \) are already quadratic in the perturbations, they adopt the same expression at this perturbative order when rewritten in terms of the new homogeneous variables \( \tilde{w}^a \) (indeed, any correction would contribute to orders higher than quadratic in the perturbations). Nonetheless, this is not the case for \( H_0; \) it will receive second-order corrections. Thus, the zero mode of the Hamiltonian constraint, when recast in terms of the new variables, has a homogeneous term identical to the original one, but replacing \( w^a \) with \( \tilde{w}^a \), and a total, quadratic perturbative contribution given by

\[ \sum_{\vec{n},\varepsilon} \tilde{H}_{\vec{n},\varepsilon}^{\tilde{R},\varepsilon}(\tilde{w}^a, X_{\vec{n},\varepsilon}^a) + \sum_{\vec{n},\varepsilon,\varepsilon} \tilde{T} \tilde{H}_{\vec{n},\varepsilon,\varepsilon}^{\tilde{R},\varepsilon,\varepsilon}(\tilde{w}^a, d\tilde{h}_{\vec{n},\varepsilon,\varepsilon}, \pi_{d\tilde{h}_{\vec{n},\varepsilon,\varepsilon}}) + \sum_b \Delta \tilde{w}^a \frac{\partial H_0}{\partial \tilde{w}^b}(\tilde{w}^a). \] (2.8)
The last term in the previous expression arises from the commented second-order contributions that the background receives from the perturbations as a result of completing the canonical transformation in the homogeneous sector of the system, and effectively corrects \( H_{12}^{R,e} \) and \( T H_{12}^{R,e} \).

Let us rewrite the homogeneous contribution to the constraint as \( H_0 = e^{-3\alpha}(\vec{\pi}_\varphi^2 - \mathcal{H}^{(2)}_0) / 2 \), where \( \mathcal{H}^{(2)}_0 \) can be immediately read from Eq. (2.2),

\[
\mathcal{H}^{(2)}_0 = \pi_\varphi^2 - 2e^{6\alpha}\bar{W}(\dot{\varphi}).
\]  

The resulting zero mode of the Hamiltonian constraint can then be expressed in the form [14, 24]

\[
e^{-3\alpha}\bar{H} \equiv e^{-3\alpha} \left( \pi_\varphi^2 - \mathcal{H}^{(2)}_0 - \Theta_o^S - \Theta_o^T \right).
\]  

The \( \Theta \)-functions introduced in the previous equation are defined as

\[
\Theta_o^S = -\vartheta_o \sum_{\vec{n},\vec{\epsilon}} (V_{q_1}^{\vec{n},\vec{\epsilon}})^2,
\]

\[
\Theta_e^S = -\sum_{\vec{n},\vec{\epsilon}} [(\vartheta_e \omega_n^2 + \vartheta_e^T) (V_{q_1}^{\vec{n},\vec{\epsilon}})^2 + \vartheta_e (V_{p_1}^{\vec{n},\vec{\epsilon}})^2],
\]

\[
\Theta^T = -\sum_{\vec{n},\vec{\epsilon}} [(\vartheta_e \omega_n^2 + \vartheta_e^T) (\vec{d}_{\vec{n},\vec{\epsilon},\vec{\epsilon}})^2 + \vartheta_e (\pi_{d_{\vec{n},\vec{\epsilon},\vec{\epsilon}}})^2],
\]

with \( \omega_n^2 = -4\pi^2|\vec{r}|_n^2/l_0^2 \) and the following \( \vartheta \)-functions, that only depend on the geometric homogeneous variables and the scalar field potential but, importantly, not on the homogeneous momentum of this field:

\[
\vartheta_o = -12e^{4\alpha}\bar{W}''(\dot{\varphi}) \frac{1}{\pi_\varphi^2},
\]

\[
\vartheta_e = e^{2\alpha},
\]

\[
\vartheta_e^T = e^{-2\alpha}\mathcal{H}^{(2)}_0 \left( 19 - 18 \frac{\mathcal{H}^{(2)}_0}{\pi_\varphi^2} \right) + e^{4\alpha}[\bar{W}''(\varphi) - 4\bar{W}(\varphi)],
\]

\[
\vartheta_T^T = e^{-2\alpha}\mathcal{H}^{(2)}_0 - 4e^{4\alpha}\bar{W}(\dot{\varphi}).
\]

Here, the prime denotes the derivative with respect to the zero mode of the scalar field \( \varphi \). We notice that the quadratic contributions to the zero mode of the Hamiltonian constraint only depend on the perturbations through the MS and tensor canonical variables and, therefore, are gauge invariant. For reasons that should be obvious, the sum of these quadratic terms for the scalar and tensor perturbations will be referred to as the MS Hamiltonian and the tensor Hamiltonian, respectively.

It is also worth commenting that additional terms appear in the Hamiltonian when the expression in Eq. (2.8) is explicitly computed. However, these can be reabsorbed through redefinitions of Lagrange multipliers, which receive quadratic contributions from the perturbations \( g_{\vec{n},\vec{\epsilon}} \) and \( k_{\vec{n},\vec{\epsilon}} \) are exclusively affected by the scalar perturbations, while the homogeous lapse function is corrected by terms quadratic in both the scalar and the tensor perturbations; see Refs. [14, 24] for details). This process leads finally to a total Hamiltonian of the form

\[
H = \bar{N}_0 e^{-3\alpha}\bar{H} + \sum_{\vec{n},\vec{\epsilon}} G_{\vec{n},\vec{\epsilon}} V_{p_2}^{\vec{n},\vec{\epsilon}} + \sum_{\vec{n},\vec{\epsilon}} K_{\vec{n},\vec{\epsilon}} V_{p_3}^{\vec{n},\vec{\epsilon}},
\]

where we have adopted an obvious notation for the modified Lagrange multipliers.

We remark as well that there is no linear contribution from \( V_{p_1}^{\vec{n},\vec{\epsilon}} \) or \( \pi_{d_{\vec{n},\vec{\epsilon},\vec{\epsilon}}} \), as we had anticipated in view of the redefinitions of the gauge invariant variables that we performed in Sec. II C. In the only instances in which these momenta appear, they contribute quadratically and are preceded by a mode independent coefficient (namely, the square of the scale factor).

**III. HYBRID LOOP QUANTIZATION**

In this section, we address the quantization of our model following the approach of hybrid loop quantum cosmology or, for short, hLQC [14, 23]. This quantization strategy is based on the assumption that there exists a certain
regime of the quantum dynamics where the most relevant quantum geometric effects are those encoded in the zero modes of the (homogeneous) geometry, while the perturbations may be described using a more standard quantum representation. In view of this hypothesis, it seems reasonable to adopt two different quantum representations: one, of a quantum gravitational nature, for the homogeneous sector and another more conventional for the inhomogeneities, e.g. a QFT-like Fock representation. Thus, our objective is to quantize the symplectic manifold that describes our cosmological system as a whole, employing quantum representations of different nature for the homogeneous and inhomogeneous sectors, and imposing the constraints quantum mechanically (according to Dirac’s program [66]).

We assume a quantization of the homogeneous variables that provides a representation of the canonical commutation relations such that the operators that describe the background FLRW geometry commute with the homogeneous scalar field operators (as it already happens at the level of the Poisson brackets algebra). Let these geometric and scalar field operators be defined on the kinematical Hilbert spaces $\mathcal{H}_{\text{kin}}^{\text{grav}}$ and $\mathcal{H}_{\text{kin}}^{\text{matt}}$, respectively. Then, provided our assumption on the commutation properties of the homogeneous operators, we can write the kinematical Hilbert space associated with the full homogeneous sector as the tensor product of the two mentioned representation spaces. Concerning the perturbations, we assume a Fock quantization (although this formalism is easily extensible to account for different choices [14]) such that the operators representing the basic variables for the inhomogeneities also commute with the homogeneous ones. These, together with a suitable prescription for a symmetric factor ordering upon quantization, are essentially the only building blocks necessary for the general quantum theory that was introduced in Ref. [14], that in fact does not require at all that the methodology be particularized to a concrete quantization of the homogeneous geometry. In this sense, although in the present work we will focus our attention on the case where we select a polymeric representation of the geometry, inspired by LQG, we emphasize that this is only a particular case. Whereas it is especially interesting owing to the physics emerging from LQC and the reasons discussed in the Introduction, it is by no means necessary in order to construct a formalism of hybrid quantum cosmology, that in a more general context might rest on a different quantum representation of the background degrees of freedom.

A. Quantum representation of the homogeneous sector

Let us now detail the quantum representation that we are going to choose for the zero mode of the scalar field and for the homogeneous FLRW geometry. We will also specify the quantum counterpart of the homogeneous contribution to the zero mode of the Hamiltonian constraint, and discuss some ambiguities that appear in this process.

As far as the homogeneous matter scalar field is concerned, we consider a standard Schrödinger representation, in which the operator $\hat{\varphi}$ acts by multiplication and the momentum operator $\hat{\pi}_\varphi$ acts as a generalized derivative. The kinematical Hilbert space corresponding to the matter content is $\mathcal{H}_{\text{kin}}^{\text{matt}} = L^2(\mathbb{R}, d\varphi)$.

Regarding the homogeneous geometry, we henceforth particularize our discussion to the case where it is described quantum mechanically by employing the formalism of LQC [24, 26]. More concretely, we follow the so-called ”improved dynamics prescription” introduced in Ref. [32], which accounts for the existence of a minimum nonvanishing eigenvalue $\Delta$ allowed for the area in LQG [27, 28]. Among the possible factor ordering prescriptions for the quantum representation of the Hamiltonian constraint, we adopt the symmetric prescription put forward in Ref. [33], usually referred to as Martín-Benito–Mena Marugán–Olmedo (or MMO, for short) prescription. Furthermore, we want to study the application of the hybrid formalism to the particular case where the background is described using the DL procedure for the regularization of the homogeneous Hamiltonian constraint, as we will explain below.

In homogeneous and isotropic LQC, instead of describing the geometry using the scale factor and its conjugate momentum, the gravitational degrees of freedom are encoded in the Ashtekar-Barbero $\mathfrak{su}(2)$ gauge connection and the densitized triad, which compose a canonical pair (in the sense that their Poisson bracket is proportional to the identity). However, given the homogeneous and isotropic nature of the spatial sections of the cosmologies under study, all the relevant information is actually contained in two dynamical variables, $c$ and $p$, coming from the connection and the triad, respectively. The consideration of the improved dynamics scheme motivates a change of variables $(c, p) \rightarrow (b, v)$, where $b$ is classically proportional to the expansion rate and $v$ is the physical volume of the Universe, up to a constant multiplicative factor. This new set of variables remains canonical and their Poisson bracket is $\{b, v\} = 2$. Their precise relation to the scale factor and its canonically conjugate momentum is

$$e^\alpha = \left( \frac{3\gamma\sqrt{\Delta}}{2\sigma} |v| \right)^{1/3} = \left( \frac{3l_0}{4\pi G} \right)^{1/2} V^{1/3},$$

$$\pi_\alpha = -\frac{3}{2} bv,$$

where $\gamma$ is the Immirzi parameter and $V = 2\pi G\gamma\sqrt{\Delta}|v|$ is the physical volume of the Universe.
Since the connection does not have a well-defined quantum analog in LQG, one chooses the holonomies of the connection instead as basic variables which, together with the triad, play the role of fundamental operators in the quantum theory. The holonomy elements are given by complex exponentials of $b$. With the typical notation of the improved dynamics formulation, we denote these exponentials by $\hat{N}_{\pm n\bar{\mu}} = \exp(\pm inb/2)$. Although we may allow $n$ to be real, in the rest of our discussion we will mainly restrict our attention to integer values of $n$. The kinematical Hilbert space $\mathcal{H}_{\text{kin}}^{\text{grav}}$ for the geometry of flat FLRW in LQC is formed by the linear span of all the eigenstates of the volume variable $|v\rangle$ with $v \in \mathbb{R}$, Cauchy-completed with respect to the norm defined by the discrete inner product $\langle v|v' \rangle = \delta_{v,v'}^{(2)}$. On the basis states $|v\rangle$, the volume simply acts by multiplication, whereas the holonomy operators produce constant shifts in their label, $\hat{N}_{\pm n\bar{\mu}} |v\rangle = |v \pm n\rangle$.

Let us now discuss the quantization of the Hamiltonian constraint of flat FLRW cosmologies in LQC, which will determine precisely the homogeneous contribution to the zero mode of the quantum Hamiltonian constraint in our perturbed model. The first step in this process is to reexpress the Hamiltonian $H_0$ in terms of the basic variables of the theory, which have a well-defined quantum analog. In the LQC literature, this procedure is usually understood as a regularization process, on account of the fact that it involves the replacement of the connection and its associated curvature tensor by holonomies around closed circuits with a nonvanishing physical area, thereby dealing with ultraviolet divergences of the classical theory. The key point in this regard is that there is not full consensus in the regularization of the Hamiltonian; indeed, different proposals exist in the literature that lead to different loop quantum theories. Actually, there are two prominent regularization proposals in LQC: the most frequent or standard one \cite{32} and the DL proposal \cite{34}.

To comprehend the main difference between these regularization proposals, it is important to understand the basic structure of the Hamiltonian constraint in full general relativity. When written in terms of Ashtekar-Barbero variables, the Hamiltonian constraint is essentially composed by two pieces, namely the Euclidean and the Lorentzian parts, that can respectively be expressed in terms of the curvature of the connection and of the extrinsic curvature (apart from the triad), and that receive their names from the fact that only the first of these pieces appears in Euclidean gravity. Traditionally, the community of LQC has employed a regularization scheme that exploits the high symmetry of the most commonly considered cosmological spacetimes, namely homogeneity and spatial flatness \cite{32,33}. Indeed, in homogeneous and spatially flat scenarios, the Lorentzian part of the Hamiltonian constraint turns out to be classically proportional to the Euclidean part and, thus, the full Hamiltonian can be expressed in terms of the Euclidean part alone. Hence, regularizing the Euclidean part suffices to rewrite the full Hamiltonian as a function of holonomies (and triads) in this kind of systems. However, conceptually, this prescription may not seem totally satisfactory, since it cannot be applied to more general scenarios, where the aforementioned symmetries fail to exist.

As we have commented, recently Dapor and Liegener put forward an alternative regularization scheme which does not rely on these symmetry considerations \cite{34,35,36,37}. Indeed, it is based on the independent regularization of the two terms in the Hamiltonian constraint (through the use of two Thiemann identities). The subsequently modified model of LQC, sometimes referred to as the DL formalism of LQC, appears to lead to physical predictions which differ from the ones attained with the standard formalism of LQC, e.g., concerning the bounce mechanism that is expected to resolve the big bang singularity in the quantum theory. Results of this type raise the question of whether the standard approach to LQC faithfully captures the actual cosmological dynamics and singularity resolution picture within full LQG. In this sense, it seems enlightening to examine alternative loop quantizations of cosmological spacetimes, like the one that results in the DL formalism, in order to analyze whether the standard physical predictions are robust independently of the regularization process adopted to construct the formulation of LQC.

With this motivation in mind, we now study the hybrid quantization of perturbative inhomogeneities that propagate on a homogeneous and isotropic background described by the DL formalism of LQC. The Hamiltonian constraint operator for a flat FLRW cosmology was first constructed and analyzed employing the MMO quantization prescription in Ref. \cite{43}. As shown in that reference, the densitized version of the Hamiltonian operator for the unperturbed flat FLRW background is given by $(\hat{\pi}^2_{\phi} - \hat{H}_0^{(2)})/2$, with \cite{43}

$$
\hat{H}_0^{(2)} = -\left(\frac{3}{4\pi G}\right)^2 \left(\hat{\Omega}_{4\mu}^2 - \frac{1 + \gamma^2}{4\gamma^2} \hat{\Omega}_{4\mu}^2 + \frac{3\bar{\mu}}{2\pi G} \hat{\gamma}^2 \hat{W} \hat{\gamma} \right).
$$

In the previous expression, the field potential operator is to be understood as the multiplicative operator $\hat{W} = \hat{W}(\hat{\phi})$ (like any function of the zero mode of the scalar field, for that matter). The operator $\hat{W}_{n\bar{\mu}}$ (for any integer number $n$)
is defined as
\[ \hat{\Omega}_{n\bar{\mu}} = \frac{1}{4i\sqrt{\Delta}} \hat{V}^{1/2} \left[ \text{sgn}(v), \hat{N}_{n\bar{\mu}} - \hat{N}_{-n\bar{\mu}} \right]_+ \hat{V}^{1/2}, \]  
(3.4)
where \([\cdot, \cdot]_+\) denotes the anticommutator.

This operator, which is densely defined on the tensor product \(H_{\text{grav}}^{\text{grav}} \otimes H_{\text{kin}}^{\text{mat}}\), satisfies a number of properties which are relevant to our present discussion. In the first place, it annihilates the quantum state of vanishing volume (i.e., the quantum analog of the classical singularity) and leaves invariant its orthogonal complement \(H_{\text{kin}}^{\text{grav}} \otimes H_{\text{kin}}^{\text{mat}}\), where \(H_{\text{kin}}^{\text{grav}}\) is the Cauchy completion of the span of the volume eigenstates with a nonzero volume. This decoupling of the singular state, together with the fact that positive and negative volumes are not connected by the repeated action of the constraint, leads to the Hilbert subspaces spanned by the eigenstates with positive or negative volumes being left invariant. Furthermore, the action of the constraint superselects for the FLRW geometry Hilbert subspaces \(H^+\) with support on discrete semilattices of step four \(\{\pm(\varepsilon + 4n), n \in \mathbb{N}\}\) \([43, 44]\), that have a strictly positive minimum \(\varepsilon\) or a strictly negative maximum \(-\varepsilon\) for the volume. For the sake of definiteness, from now on we restrict our discussion to \(H^+\) with a fixed \(\varepsilon \in (0, 4]\).

Since we regard the inhomogeneities of our system as perturbations, it seems reasonable to demand that their introduction does not alter the superselection sectors of the unperturbed model. For this reason, we will take into account the details about the invariant Hilbert subspaces in the quantization of the quadratic contributions to the zero mode of the Hamiltonian constraint, which must preserve these subspaces as well.

B. Fock representation of the inhomogeneities and implementation of the perturbative constraints

For the perturbations, we consider a Fock quantization, which is selected up to unitary equivalence by the criteria of invariance of the vacuum under the spatial isometries and unitary implementability of the quantum dynamics. Additionally, we may require that the chosen Fock quantization satisfy other conditions that would further restrict the representation (for instance, we could demand that the operators constructed out of elements of the Weyl algebra and other relevant operators be well defined at the quantum level).

The representation where the associated creation and annihilation-like variables correspond just to harmonic oscillators of constant frequency \(\omega_0\) belongs to the family of unitarily equivalent Fock quantizations picked out by our criteria of symmetry invariance and dynamical unitarity. Let us consider this representation for simplicity, although one may instead use another representation in its equivalence class with better physical properties, according to our comments above. The corresponding Fock spaces for the MS and tensor modes will be denoted by \(F_S\) and \(F_T\), respectively. An orthonormal basis of these spaces is given by the occupancy-number states, labeled by a positive integer per mode. Creation and annihilation operators act on this basis in the standard way, namely, by increasing or decreasing the occupancy number corresponding to a particular mode by one unit.

In order to complete the quantum description of the system, we still have to represent the constraints and impose them quantum mechanically. Notice that, in the gauge invariant formulation presented in the previous section, the constraints Poisson commute, a fact which allows us to impose them without the introduction of inconsistencies (at least if their quantum analogs commute as well \([60]\)). Let us deal, in the first place, with the linear perturbative constraints. Our classical formalism was already designed to facilitate their imposition at the quantum level. For the part of the inhomogeneous sector parametrized by \(\{V_{l\eta}^{n,\epsilon}\}_{l=2,3}\), we select a quantization such that the momenta (i.e., the linear perturbative constraints under discussion) act as generalized derivatives with respect to the configuration variables \(\{V_{\mu}^{n,\epsilon}\}_{l=2,3}\). In such a quantization, the vanishing of the classical constraints has a straightforward quantum counterpart: the physical states cannot depend on the configuration variables of this part of the inhomogeneous sector, since generalized derivatives with respect to them must be equal to zero. Therefore, imposing these quantum constraints amounts to the restriction to a representation space which is simply \(H_{\text{kin}}^{\text{grav}} \otimes H_{\text{kin}}^{\text{mat}} \otimes F_S \otimes F_T\). Notice, however, that this Hilbert space is not the physical one yet, since the zero mode of the Hamiltonian constraint still remains to be represented and imposed (something considerably more complicated than imposing the linear perturbative constraints).

We focus our attention here on the \emph{densitized} version of the zero mode of the Hamiltonian constraint, \(\hat{H}\) \([33, 43]\). By virtue of Eq. (2.10) and the definition of \(\hat{H}_{\Omega}^{(2)}\) \([33]\), we obtain that a straightforward quantization leads to
\[ \hat{H} = \frac{1}{2} \left( \hat{\pi}_\phi^2 - \hat{\Theta}^{(2)}_\Omega - \hat{\Theta}^S - \hat{\Theta}^T - \Theta^2 \pi_\phi \right), \]  
(3.5)

We notice a slight modification in the quantum representation of the powers of the volume with respect to the one presented in Ref. \([43]\). This is due to the choice of a different prescription for the representation of the inverse of the minimum coordinate length \(\bar{\mu}\). For further details, consult the Appendix of Ref. \([43]\).
where the different operators involved will be constructed in detail in the following subsections.

C. Factor ordering prescriptions

Notice that the presence of classically noncommuting quantities in Eq. (3.5) (in particular, in the last three terms) makes it necessary to specify a proposal for the factor ordering that must be taken upon quantization. For the sake of clarity, we now explain and discuss the details of this proposal. We adopt the following prescriptions:

i. The products of the form \( f(\hat{\varphi})\pi_\varphi \) will be represented quantum mechanically by \( \frac{1}{2}[f(\hat{\varphi}), \hat{\pi}_\varphi]_+ \), where \( f \) is an arbitrary function. In particular, this implies \( \Theta_s^2\pi_\varphi = \frac{1}{2}[\Theta_s^2, \pi_\varphi]_+ \) in Eq. (3.5).

Moreover, in the products of any real power of the volume with any function of \( bv \) (that typically arises in the regularization with holonomy elements), we adopt an algebraic symmetrization for the powers of the volume (or the inverse volume). Explicitly:

ii. The products of the form \( V^r g(bv) \), where \( r \) is a real number and \( g \) is an arbitrary function, will be represented by \( \hat{V}^{r/2}g\hat{V}^{r/2} \).

The only remaining issue to be addressed is the quantization of the functions of \( bv \) themselves, which requires further comments. To begin with, the quantity \( bv \) is classically proportional to the momentum variable associated with the logarithmic scale factor. The formalism presented in the previous sections provides us with a straightforward and natural definition of the quantum analog of the \( S \) function of \( \pi_\varphi \). Indeed, given the definition of \( \hat{H}_0^{(2)} \) in Eq. (2.9) and the DL proposal \( \Theta_s \), we can simply set

\[
\hat{\pi}_\varphi^2 = \hat{H}_0^{(2)} + 2\epsilon^{6\alpha} \hat{W} = \hat{H}_0^{(2)} + 2 \left( \frac{3\ell_0}{4\pi G} \right)^3 \hat{V}^2 \hat{W} = - \left( \frac{3}{4\pi G} \right)^2 \left( \Omega_{2\mu}^2 - \frac{1}{4\gamma^2} \hat{\Omega}_{4\mu}^2 \right),
\]

(3.6)

Since \( \hat{\Omega}_{2\mu} \) and \( \hat{\Omega}_{4\mu} \) produce shifts of two and four units in the label of the volume eigenstates [see Eq. (3.4)], respectively, it is immediate to conclude that this quantum representation of \( \pi_\varphi^2 \) leaves invariant Hilbert subspaces with support on discrete lattices of step four (indeed, it is essentially the Hamiltonian constraint that would correspond to vacuum flat FLRW cosmology).

We note that, while we might adopt for \( \hat{\pi}_\varphi^2 \) the same prescription as in Ref. [14] (that is, we might represent \( \pi_\varphi^2 \) as being proportional to \( \hat{\Omega}_{2\mu}^2 \)), that alternative definition seems less natural and convenient than the one that we have proposed above. On the one hand, even though \( \hat{\Omega}_{2\mu}^2 \) is indeed a representation of the classical quantity \( (bv)^2 \) (up to some constant multiplicative factor), it would not agree with our choice of regularization procedure for the homogeneous Hamiltonian. Furthermore, even if we ignored this issue and admitted the use of different regularization procedures to provide quantum representations of the same object, the definition that we have proposed behaves better upon inversion. Indeed, while zero is known to belong to the spectrum of \( \Omega_{2\mu} \) (which might lead to problems when computing the inverse), our present proposal fares better in this regard. Actually, on physical solutions to the homogeneous Hamiltonian constraint, \( \hat{H}_0^{(2)} \) is nonnegative and, \( \textit{a fortiori} \), \( \hat{H}_0^{(2)} + 2\epsilon^{6\alpha} \hat{W} \) is also nonnegative if so is the scalar field potential, as it is often the situation in the most interesting physical scenarios (e.g., a mass term). Hence, it is not difficult to conclude that the only case in which we might encounter then a problem for physical states is at the zeros of the field potential, and only if the kinetic energy of the scalar field vanishes there quantum mechanically at zeroth-order in the perturbations, a possibility which is in any case much more stringent than the situation that we had found with the alternative representation (another argument supporting our proposed choice of representation can be found at the end of Sec. III D).

Taking into account the above arguments, we choose to represent \( \pi_\varphi^2 \) as in Eq. (3.6) and, thus, any even power of \( \pi_\varphi \) can be quantized in a simple way as follows:

iii. The even powers of the canonical momentum associated with the logarithmic scale factor, \( \pi_{\alpha}^{2k} \) (for any integer \( k \)), will be represented by

\[
(\hat{\pi}_{\alpha}^2)^k = \left( \hat{H}_0^{(2)} + 2\epsilon^{6\alpha} \hat{W} \right)^k = \left( \frac{3}{4\pi G} \right)^{2k} \left( -\hat{\Omega}_{2\mu}^2 + \frac{1}{4\gamma^2} \hat{\Omega}_{4\mu}^2 \right)^k.
\]

(3.7)

The quantum representation of the odd powers of \( \pi_\varphi \), however, is not so immediate. The extra difficulty arises owing to the fact that, unlike the case of \( \hat{\pi}_{\alpha}^2 \), there is no definition of \( \hat{\pi}_\varphi \) that is straightforwardly provided by the formalism.
Thus, we need to introduce one ourselves. Let us denote by $\hat{A}$ the operator that results from the quantization of $\pi_\alpha$. It is obvious that any odd power of the momentum can be rewritten as an even power (for which we already have a representation) times $\pi_\alpha$ itself. Then, we are in a position to adopt an algebraic symmetrization for the even powers and represent the single remaining factor by $\hat{A}$, the form of which is yet to be discussed. As a result,

$$\overline{\pi^{2k+1}_\alpha} = |\hat{\pi}^2_\alpha|^{k/2} |\hat{\pi}^2_\alpha|^{k/2},$$  \hspace{1cm} (3.8)

where $|\hat{A}|$ is the absolute value of the operator $\hat{A}$. We recall that $\hat{\pi}^2_\alpha$ is a nonnegative operator on physical states when the perturbations are absent or ignored, so that in this situation the absolute values in our definition would be spurious.

To conclude, let us discuss a possible way to define $\hat{A}$. A formal restriction on $\hat{A}$ is that it must only produce shifts in the volume which are integer multiples of four. Otherwise, the resulting operator would not leave invariant the superselection sectors $H^\pm$. In Ref. [14], it was suggested to represent $\hat{A}$ by $\hat{\Omega}_{4\mu}/2$, an operator which did not appear in principle in the homogenous Hamiltonian (recall that $H_0$ was constructed in that work using exclusively the standard regularization), although it is often employed in LQC to represent the Hubble parameter. Indeed, this operator has the good property of respecting the superselection sectors of the unperturbed model. Since there is no longer any need to define $\hat{\Omega}_{4\mu}/2$ ad hoc, it seems reasonable to adopt the same representation of $\hat{A}$ within our DL formulation. In total, thus, we adopt the following prescription:

iv. The odd powers of the canonical momentum associated with the logarithmic scale factor $\pi^{2k+1}_\alpha$ (for any integer $k$) will be represented by

$$\overline{\pi^{2k+1}_\alpha} = |\hat{\pi}^2_\alpha|^{k/2} |\hat{\pi}^2_\alpha|^{k/2} = \left(\frac{3}{4\pi G}\right)^{2k} \hat{\Omega}^2_{2\mu} - \frac{1 + \gamma^2}{4\gamma^2} \hat{\Omega}^2_{4\mu} \hat{A} \left(\hat{\Omega}^2_{2\mu} - \frac{1 + \gamma^2}{4\gamma^2} \hat{\Omega}^2_{4\mu}\right)^{k/2},$$  \hspace{1cm} (3.9)

where $\hat{A}$ is an appropriately chosen operator that leaves invariant the Hilbert subspaces with support on discrete semilattices of step four, such as $\hat{\Omega}_{4\mu}/2$.

This completes the characterization of the factor ordering prescriptions required to quantize the contributions arising from the perturbations in the zero mode of the Hamiltonian constraint.

### D. Quantization of the perturbative contributions to the Hamiltonian constraint

Let us finally analyze the result of quantizing the quadratic perturbative contributions to the Hamiltonian constraint of the system employing the proposal that we have detailed in the previous subsection. With this aim, we now represent the functions of the homogeneous phase space given in Eqs. (2.14)-(2.17), as densely defined operators on $H^\pm \otimes L^2(\mathbb{R}, d\varphi)$.

We consider first the function $\vartheta_o$, defined in Eq. (2.14). This is the only instance where an odd power of $\hat{\pi}_\alpha$ appears. Representing the four powers of the scale factor by the appropriate powers of the volume operator and following our prescriptions, we arrive at

$$\hat{\vartheta}_o = -12l_0^2 \hat{W}' \hat{V}^{2/3} \left(\hat{\Omega}^2_{2\mu} - \frac{1 + \gamma^2}{4\gamma^2} \hat{\Omega}^2_{4\mu}\right)^{-1/2} \hat{A} \left(\hat{\Omega}^2_{2\mu} - \frac{1 + \gamma^2}{4\gamma^2} \hat{\Omega}^2_{4\mu}\right)^{-1/2} \hat{V}^{2/3},$$  \hspace{1cm} (3.10)

where $\hat{A}$ is the operator discussed above.

The remaining functions are considerably simpler. The classical function $\hat{\vartheta}_c$ is nothing but the scale of the square factor [see Eq. (2.15)] and, thus, its quantum counterpart is proportional to a power of the volume operator,

$$\hat{\vartheta}_c = \frac{3l_0}{4\pi G} \hat{V}^{2/3}.$$  \hspace{1cm} (3.11)

The two remaining functions are very similar, since one is the analog of the other: $\hat{\vartheta}_t^e$ is found within the context of scalar perturbations and $\hat{\vartheta}_t^t$ is related to tensor perturbations instead. The scalar one, which was introduced in Eq. (2.16), is quantized as

$$\hat{\vartheta}_t^e = \frac{4\pi G}{3l_0} \left[\frac{1}{\hat{V}}\right]^{1/3} \hat{H}_0^{(2)} \left[19 + 32\pi^2 G^2 \left(\hat{\Omega}^2_{2\mu} - \frac{1 + \gamma^2}{4\gamma^2} \hat{\Omega}^2_{4\mu}\right)^{-1} \hat{H}_0^{(2)}\right] \left[\frac{1}{\hat{V}}\right]^{1/3} + \left(\frac{3l_0}{4\pi G}\right)^2 \left(\hat{W}'' - 4\hat{W}\right) \hat{V}^{4/3},$$  \hspace{1cm} (3.12)
where the inverse volume operator is defined in the way which is standard in LQC, namely

$$\left[\frac{1}{V}\right]^{1/3} = \frac{3}{4\pi G\sqrt{\Delta}} \text{sgn}(v) \hat{V}^{1/3} \left(\hat{N}_{-\bar{\mu}} \hat{V}^{1/3} \hat{N}_{\bar{\mu}} - \hat{N}_{\bar{\mu}} \hat{V}^{1/3} \hat{N}_{-\bar{\mu}}\right). \quad (3.13)$$

The quantum counterpart of $\vartheta_T^q$, defined in Eq. (2.17), is less involved and, in fact, does not require the additional factor ordering and polymeric corrections encountered in $\vartheta_T^e$ [see the term between the inverse volume operators in Eq. (3.12)]. Explicitly,

$$\vartheta_T^q = \frac{4\pi G}{3l_0} \left[\frac{1}{V}\right]^{1/3} \hat{H}_0^{(2)} \left[\frac{1}{V}\right]^{1/3} \hat{N}_{-\bar{\mu}} \hat{V}^{1/3} \hat{N}_{\bar{\mu}} - \left(\frac{3l_0}{2\pi G}\right)^2 \hat{W} \hat{V}^{4/3}. \quad (3.14)$$

At this point, we can present a further argument that supports our choice of $\vartheta_T^q$. Classically, the first term of $\vartheta_T^q$ is obtained from the analogous term of $\vartheta_T^e$ in the limit of vanishing potential. Indeed,

$$19 - 18 \frac{\vartheta_T^{(2)}_0}{\vartheta_T^{(2)}_e} = \frac{19\pi^2 - 18\vartheta_T^{(2)}_0}{\vartheta_T^{(2)}_e} = \frac{\pi^2}{\vartheta_T^{(2)}_e} + 36e6\hat{W}, \quad (3.15)$$

that tends to one as $\hat{W} \to 0$, leading to a successful recovery of the first term of Eq. (2.17). This relation trivially holds at the quantum level if we represent $\pi^2$ as proposed in Eq. (3.6). Nonetheless, this classical relation is violated if we keep the proposal of Ref. [14].

In conclusion, we have been able to represent the functions $\vartheta_o$, $\vartheta_e$, $\vartheta_T^q$, and $\vartheta_T^e$ as densely defined operators on $\mathcal{H}_T^+ \otimes L^2(\mathbb{R}, d\hat{\varphi})$, which amounts to providing a quantization of the perturbative contributions to the zero mode of the Hamiltonian constraint (recall that the gauge invariant perturbative variables are represented in terms of creation and annihilation operators on $\mathcal{F}_S \otimes \mathcal{F}_T$). This, together with the quantization of the homogeneous contribution, completes the quantum representation of the whole Hamiltonian at the considered perturbative order. The only remaining step in the program is the imposition of the adjoint of this Hamiltonian constraint on the algebraic dual of a sufficiently small dense domain, that would allow us to determine the physical quantum states that describe our cosmological system.

IV. CONCLUSION

We have discussed the generalization of the hybrid approach for the quantization of cosmological perturbations around a flat FLRW universe, with compact sections, and minimally coupled with a scalar field, to possible alternative regularization schemes in LQC, showing how to combine the Fock quantization of the physical degrees of freedom of the perturbations with the quantum formalism obtained with such regularizations for homogeneous and isotropic spacetimes.

In order to construct our formulation, we have started with a truncation of the action to second order in the perturbations of the metric and the matter fields, treating the zero modes that describe the FLRW cosmologies exactly in this procedure. Following previous work by Langlois [56] and Refs. [14, 24], we have then adopted a set of canonical variables for the perturbations formed by gauge invariants, Abelianized perturbative constraints, and suitable canonical momenta. We have shown that this set can be completed into a canonical one for our whole cosmological system, including the sector of FLRW backgrounds as part of the total phase space. The resulting formulation permits an almost straightforward imposition of the perturbative constraints, leading to the conclusion that physical states may only depend on perturbative gauge invariants and FLRW zero modes, but are still subject to one global constraint: the zero mode of the Hamiltonian constraint of the entire perturbed cosmology. This formulation is robust and valid for any quantum description of the FLRW degrees of freedom that one decides to adopt, as far as one assumes that the canonical Poisson structure that we have obtained is preserved in the passage to quantum commutators.

Supposing that one has at hand a satisfactory Fock quantization of the gauge invariant perturbations and a consistent quantum theory for the FLRW zero modes in which one of the basic operators represent the volume of the compact spatial sections (or, alternatively, their scale factor), the hybrid quantization of the studied system essentially rests on the definition of two geometric operators that are necessary to obtain the quantum representation of the subsisting Hamiltonian constraint. The first one is the operator corresponding to the square of the canonical momentum of the logarithmic scale factor. This operator is already needed to define the Hamiltonian constraint of the unperturbed FLRW cosmology. In other words, it is an operator which is fundamental to attain a quantization
of homogeneous and isotropic universes. Its explicit form depends on the regularization scheme that one adopts for the gravitational Hamiltonian, but once this regularization is chosen or fixed by suitable criteria, the natural choice is to adopt the same operator representation in the perturbative contributions to the global Hamiltonian constraint.

The second geometric operator that appears in these perturbative terms and requires a definition is a representation of the genuine canonical momentum of the logarithmic scale factor, rather than its square. This is important because the already defined square does not contain information about the sign of the momentum. Moreover, while the operator representing the square momentum preserves by construction the superselection sectors that might exist in the homogeneous and isotropic reduction, the conservation of the superselection sectors is a requirement that one must impose on the operator corresponding to the actual momentum if one wants to regard the perturbative contributions to the zero mode of the Hamiltonian constraint as genuine perturbations, not changing the basic structure of the quantum model that describes the background. In spite of the ambiguity introduced by the choice of an operator for this momentum, one can argue that the effect in most of the physical situations of interest is not relevant. Indeed, the operator in question is needed exclusively to quantize the term of the perturbative contributions that contains the momentum of the homogeneous scalar field (actually in a linear way). This term appears only for scalar perturbations, and not for the tensor modes. Moreover, the term contains also a factor that is the derivative of the scalar field potential [see Eq. (2.14)], and that in many of the situations of interest is small, such as if the scalar field is kinetically dominated, as it is usually the most appealing scenario in LQC [67], or in inflationary regimes driven by a cosmological constant.

From our exposition, we see that our formulation can be adapted essentially to any reasonable proposal for the quantization of the flat FLRW cosmologies. Given the remarkable properties of the physical states in LQC, including the avoidance of the big bang singularity (that is replaced with a bounce), our interest has been focused on this quantization procedure. According to our comments, the main ambiguity in the quantum description of the primordial perturbations within this framework is the same that one encounters in homogeneous and isotropic LQC, namely the freedom in the choice of a quantum representation for the geometric part of the Hamiltonian constraint of the FLRW universes, owing to the ambiguity in the choice of a regularization scheme. In this paper we have adopted the DL proposal for this regularization. The remaining freedom that we have encountered in our quantization process can be understood as the selection of certain prescriptions in the factor ordering and the representation of the Hubble parameter (proportional to the momentum of the logarithmic scale factor). This freedom seems much less important in the selection of a quantum theory, since the choice of an operator for the Hubble parameter only affects a term that is not physically relevant in the most interesting physical situations, as we have explained above, and because one would expect that the factor ordering should not affect the fundamental properties of the formalism. To confirm these matters, in a future investigation we plan to study the effective dynamics of the perturbations derived from our hybrid approach and, in that way, shed light on whether there exist further physical or mathematical reasons supporting our choices of quantization prescriptions, as well as to elucidate whether or not the ambiguities that we have discussed are important for the predictions of the formalism, e.g. for the consequences in the cosmic microwave background. In particular, we want to study the properties of the effective mass seen by the gauge invariant perturbations in their propagation, especially around the quantum bounce, comparing these properties with those that are found in the case of LQC with the standard regularization [68].

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[1] D. Langlois, Lectures on inflation and cosmological perturbations, Lect. Notes Phys. 800, 1 (2010).
[2] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Theory of cosmological perturbations, Phys. Rep. 215, 203 (1992).
[3] J. J. Halliwell and S. W. Hawking, Origin of structure in the universe, Phys. Rev. D 31, 1777 (1985).
[4] I. Shirai and S. Wada, Cosmological perturbations and quantum fields in curved space, Nucl. Phys. B 303, 728 (1988).
[5] A. A. Starobinsky, Spectrum of relic gravitational radiation and the early state of the universe, JETP Lett. 30, 682 (1979).
[6] A. H. Guth, Inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D 23, 347 (1981).
[7] A. D. Linde, A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy, and primordial monopole problems, Phys. Rev. Lett. B 108, 389 (1982).
[8] A. D. Linde, Chaotic inflation, Phys. Rev. Lett. B 129, 177 (1983).
A. Ashtekar, T. Pawski, and P. Singh, Quantum nature of the big bang: Improved dynamics, Phys. Rev. D
T. Thiemann, Modern Canonical Quantum General Relativity
T. Thiemann, Quantum spin dynamics (QSD), Classical Quantum Gravity
G. A. Mena Marugán, A brief introduction to loop quantum cosmology, AIP Conf. Proc.
B.-F. Li, P. Singh, and A. Wang, Towards cosmological dynamics from loop quantum gravity, Phys. Rev. D
B.-F. Li, P. Singh, and A. Wang, Qualitative dynamics and inflationary attractors in loop quantum cosmology, Phys. Rev. D
E. Alesci, M. Assanioussi, and J. Lewandowski, A curvature operator for LQG, Phys. Rev. D
R. H. Gowdy, Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: Topologies and boundary conditions, Ann. Phys. 83, 203 (1974).
M. Martín-Benito, L. J. Garay, and G. A. Mena Marugán, Hybrid quantum Gowdy cosmology: Combining loop and Fock quantizations, Phys. Rev. D 78, 083516 (2008).
G. A. Mena Marugán and M. Martín-Benito, Hybrid quantum cosmology: Combining loop and Fock quantizations, Int. J. Mod. Phys. A 24, 2820 (2009).
L. J. Garay, M. Martín-Benito, and G. A. Mena Marugán, Inhomogeneous loop quantum cosmology: Hybrid quantization of the Gowdy model, Phys. Rev. D 82, 044048 (2010).
M. Martín-Benito, G. A. Mena Marugán, and E. Wilson-Ewing, Hybrid quantization: From Bianchi I to the Gowdy model, Phys. Rev. D 82, 084012 (2010).
M. Fernández-Méndez, G. A. Mena Marugán, and J. Olmedo, Hybrid quantization of an inflationary universe, Phys. Rev. D 86, 024003 (2012).
M. Fernández-Méndez, G. A. Mena Marugán, and J. Olmedo, Hybrid quantization of an inflationary model: The flat case, Phys. Rev. D 88, 044013 (2013).
L. Castelló Gomar, M. Fernández-Méndez, G. A. Mena Marugán, and J. Olmedo, Cosmological perturbations in hybrid loop quantum cosmology: Mukhanov-Sasaki variables, Phys. Rev. D 90, 064015 (2014).
F. Benítez Martínez and J. Olmedo, Primordial tensor modes of the early Universe, Phys. Rev. D 93, 124008 (2016).
A. Ashtekar and P. Singh, Loop quantum cosmology: A status report, Classical Quantum Gravity 28, 213001 (2011).
G. A. Mena Marugán, A brief introduction to loop quantum cosmology, AIP Conf. Proc. 1130, 89 (2009).
A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, Classical Quantum Gravity 21, R53 (2004).
T. Thiemann, Modern Canonical Quantum General Relativity (Cambridge University Press, Cambridge, England, 2007).
T. Thiemann, Quantum spin dynamics (QSD), Classical Quantum Gravity 15, 839 (1998).
M. Assanioussi, J. Lewandowski, and I. Mäkinen, New scalar constraint operator for loop quantum gravity, Phys. Rev. D 92, 044042 (2015).
E. Alesci, M. Assanioussi, and J. Lewandowski, A curvature operator for LQG, Phys. Rev. D 89, 124017 (2014).
A. Ashtekar, T. Pawlowski, and P. Singh, Quantum nature of the big bang: Improved dynamics, Phys. Rev. D 74, 084003 (2006).
M. Martín-Benito, G. A. Mena Marugán, and J. Olmedo, Further improvements in the understanding of isotropic loop quantum cosmology, Phys. Rev. D 80, 104015 (2009).
A. Dapor and K. Liegener, Cosmological effective Hamiltonian from full loop quantum gravity dynamics, Phys. Lett. B 785, 506 (2018).
J. Yang, Y. Ding, and Y. Ma, Alternative quantization of the Hamiltonian in loop quantum cosmology, Phys. Lett. B 682, 1 (2009).
B.-F. Li, P. Singh, and A. Wang, Towards cosmological dynamics from loop quantum gravity, Phys. Rev. D 97, 084029 (2018).
B.-F. Li, P. Singh, and A. Wang, Qualitative dynamics and inflationary attractors in loop cosmology, Phys. Rev. D 98, 066016 (2018).
M. Assanioussi, A. Dapor, K. Liegener, and T. Pawlowski, Emergent de Sitter epoch of the quantum cosmos from loop quantum cosmology, Phys. Rev. Lett. 121, 081303 (2018).
M. Assanioussi, A. Dapor, K. Liegener, and T. Pawlowski, Emergent de Sitter epoch of the quantum cosmos: A detailed analysis, Phys. Rev. D 100, 084003 (2019).
B.-F. Li, P. Singh, and A. Wang, Genericness of pre-inflationary dynamics and probability of the desired slow-roll inflation in modified loop quantum cosmologies, Phys. Rev. D 100, 063513 (2019).
I. Agullo, Primordial power spectrum from the Dapor-Liegener model of loop quantum cosmology, Gen. Relativ. Gravit. 50, 91 (2018).
J. de Haro, The Dapor-Liegener model of loop quantum cosmology: A dynamical analysis, Eur. Phys. J. C 78, 926 (2018).
A. García-Quismondo and G. A. Mena Marugán, Martín-Benito—Mena Marugán—Olmedo prescription for the Dapor-Liegener model of loop quantum cosmology, Phys. Rev. D 99, 083505 (2019).
A. García-Quismondo and G. A. Mena Marugán, Dapor-Liegener formalism of loop quantum cosmology for Bianchi I spacetimes, Phys. Rev. D 101, 023520 (2020).
J. Engle, Relating loop quantum cosmology to loop quantum gravity: Symmetric sectors and embeddings, Classical Quantum Gravity 24, 5777 (2007).
[46] J. Brunnemann and T. A. Kosiowski, Symmetry reduction of loop quantum gravity, Classical Quantum Gravity 28, 245014 (2011).
[47] J. Engle, Embedding loop quantum cosmology without piecewise linearity, Classical Quantum Gravity 30, 085001 (2013).
[48] T. Pawlowski, Observations on interfacing loop quantum gravity with cosmology, Phys. Rev. D 92, 124020 (2015).
[49] C. Beetle, J. Engle, M. E. Hogan, and P. Mendoza, Diffeomorphism invariant cosmological sector in loop quantum gravity, Classical Quantum Gravity 34, 225009 (2017).
[50] J. M. Bardeen, Gauge invariant cosmological perturbations, Phys. Rev. D 22, 1882 (1980).
[51] M. Sasaki, Gauge invariant scalar perturbations in the new inflationary universe, Prog. Theor. Phys. 70, 394 (1983).
[52] H. Kodama and M. Sasaki, Cosmological perturbation theory, Prog. Theor. Phys. Suppl. 78, 1 (1984).
[53] V. Mukhanov, Quantum theory of gauge-invariant cosmological perturbations, Zh. Eksp. Teor. Fiz. 94, 1 (1988).
[54] V. Mukhanov, Physical Foundations of Cosmology (Cambridge University Press, Cambridge, England, 2005).
[55] C. Kiefer and M. Kramer, Quantum gravitational contributions to the cosmic microwave background anisotropy spectrum, Phys. Rev. D 73, 084020 (2012).
[56] D. Langlois, Hamiltonian formalism and gauge invariance for linear perturbations in inflation, Classical Quantum Gravity 11, 389 (1994).
[57] J. Cortez, G. A. Mena Marugán, J. Olmedo, and J. M. Velhinho, Criteria for the determination of time dependent scalings in the Fock quantization of scalar fields with a time dependent mass in ultrastatic spacetimes, Phys. Rev. D 86, 104003 (2012).
[58] J. Cortez, G. A. Mena Marugán, J. Olmedo, and J. M. Velhinho, A uniqueness criterion for the Fock quantization of scalar fields with time-dependent mass, Classical Quantum Gravity 28, 172001 (2011).
[59] L. Castelló Gomar, J. Cortez, D. Martín-de Blas, G. A. Mena Marugán, and J. M. Velhinho, Uniqueness of the Fock quantization of scalar fields in spatially flat cosmological spacetimes, JCAP 1211, 001 (2012).
[60] J. Cortez, L. Fonseca, D. Martín-de Blas, and G. A. Mena Marugán, Uniqueness of the Fock quantization of scalar fields under mode preserving canonical transformations varying in time, Phys. Rev. D 87, 044013 (2013).
[61] J. Cortez, D. Martín-de Blas, G. A. Mena Marugán, and J. M. Velhinho, Massless scalar field in de Sitter spacetime: Unitary quantum time evolution, Classical Quantum Gravity 30, 075015 (2013).
[62] L. Castelló Gomar and G. A. Mena Marugán, Uniqueness of the Fock quantization of scalar fields and processes with signature change in cosmology, Phys. Rev. D 89, 084052 (2014).
[63] B. Elizaga Navascués, G. A. Mena Marugán, and T. Thiemann, Hamiltonian diagonalization in hybrid quantum cosmology, Classical Quantum Gravity 36, 18 (2019).
[64] E. J. C. Pinho and N. Pinto-Neto, Scalar and vector perturbations in quantum cosmological backgrounds, Phys. Rev. D 76, 023506 (2007).
[65] F. T. Falciano and N. Pinto-Neto, Scalar perturbations in scalar field quantum cosmology, Phys. Rev. D 79, 023507 (2009).
[66] P. A. M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School Monograph Series Vol. 2 (Dover, New York, 1964).
[67] B. Elizaga Navascués, D. Martín de Blas, and G. A. Mena Marugán, The vacuum state of primordial fluctuations in hybrid loop quantum cosmology, Universe 4, 98 (2018).
[68] B. Elizaga Navascués, D. Martín de Blas, and G. A. Mena Marugán, Time-dependent mass of cosmological perturbations in the hybrid and dressed metric approaches to loop quantum cosmology, Phys. Rev. D 97, 043523 (2018).