ASYMPTOTICS OF PARTIAL DENSITY FUNCTION VANISHING ALONG SMOOTH SUBVARIETY

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Abstract. We study the asymptotic of the partial density function associated to holomorphic section of a positive line bundle vanishing to high orders along a fixed smooth subvariety. Assuming local torus-action-invariance, we describe the forbidden region, generalizing the result of Ross-Singer.

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1. Introduction

Let \((X, L)\) be a polarized compact Kähler manifold of dimension \(m\). We endow \(L\) with a Hermitian metric \(h\) with positive curvature. And we use \(\omega = \frac{i}{2} \Theta_h\) as the Kähler form. By abuse of notation, we still use \(h\) to denote the induced metric on the \(k\)-th power \(L^k\). Then we have a Hermitian inner product on \(H^0(X, L^k)\), defined by

\[
\langle s_1, s_2 \rangle = \int_X h(s_1, s_2) \frac{\omega^m}{m!}
\]

Let \(\{s_i\}\) be an orthonormal basis of \(\mathcal{H}_k = H^0(X, L^k)\). Then the on-diagonal Bergman kernel, also called density of states function, is defined as

\[
\rho_k(z) = \sum |s_i(z)|^2_k.
\]

The on-diagonal Bergman kernel \(\rho_k\) has very nice asymptotic expansion by the results of Tian, Zelditch, Lu, etc. \([29, 30, 14, 6, 15]\). The asymptotic expansion has been found very useful, making \(\rho_k\) an important bridge connecting Kähler geometry to algebraic geometry (see for example \([8, 10]\)).

Let \(V \subset X\) be a closed submanifold of dimension \(n\). Let \(\delta > 0\) be a small number. We denote by \(\mathcal{H}_{k, \delta}^h \subset \mathcal{H}_k\) the subspace consisting of holomorphic sections whose vanishing orders...
along $V$ are no less than $\delta k$. The partial density function $\rho^\delta_k$ is then defined as the density of states function for $H^\delta_k$:

$$\rho^\delta_k(z) = \sum |s_i(z)|^2,$$

where $\{s_i\}$ is an orthonormal basis of $H^\delta_k$. When $V$ is a divisor, the partial density functions have been studied by Zelditch-Zhou [31, 32], Ross-Singer [16] and Coman-Marinescu [7] etc.

It was first shown by Shiffman-Zelditch [22] in the toric case, that there is a ‘forbidden region’ in which the partial density function is exponentially small. Berman [1] showed that there is an open forbidden region $R$ that asymptotically the partial density function is exponentially small on compact subsets and is equal to the usual density function on compact subsets of the complement of $\bar{R}$. In particular, Ross-Singer [16] studied the case when $V$ is smooth divisor, plus the condition that the line bundle and the metric is $S^1$-symmetric in a neighborhood $U$ of $V$. Let $\mu : U \to \mathbb{R}$ be the Hamiltonian of the $S^1$-action, normalized so that $\mu^{-1}(0) = V$. Then Ross-Singer proved a distributional asymptotic expansion for the partial density function and showed that the forbidden region is $\mu^{-1}(0, \delta)$:

**Theorem 1.1** ([16]). For sufficiently small $\delta$, we have

$$\rho^\delta_k \sim \begin{cases} O(k^{-\infty}) & \text{on } \mu^{-1}[0, \delta) \\ \rho_k + O(k^{-\infty}) & \text{on } X \setminus \mu^{-1}[0, \delta) \end{cases},$$

where $\sim$ means the equality holds on any given compact subset of $\mu^{-1}(0, \delta)$ and $X \setminus \mu^{-1}(0, \delta)$ respectively.

The term $k^{-\infty}$ in the theorem means a quantity depending on $k$ that is $O(k^{-m})$ as $k \to \infty$, for all $m \geq 0$. We will adopt the notation used by Donaldson and denote $k^{-\infty}$ by $\varepsilon(k)$. So, in order to avoid confusion, we use $\delta$ for the partial density function instead of $\varepsilon$ used in [16].

In this article, we generalize Ross-Singer’s result to the case when $V$ is a smooth subvariety. Let $T^{m-n} = (S^1)^{m-n}$ denote the $(m-n)$-dimensional torus. We assume that there is a neighborhood $U$ of $V$ that admits a holomorphic $T^{m-n}$-action. We also assume the line bundle $L$ and the metric $h$, hence $\omega$, are invariant under the $T^{m-n}$-action.

Let $\mu : U \to \mathbb{R}^{m-n}$ be the moment map of the torus action, normalized so that $\mu(V) = 0$. We denote by $\mu_i$ the $i$-th component of $\mu$. Then $\nu = \sum_{i=1}^{m-n} \mu_i : U \to \mathbb{R}$ is the moment map for the diagonal $S^1$-action. Our main result is that the forbidden region is $\nu^{-1}(0, \delta)$:

**Theorem 1.2.** For sufficiently small $\delta$, we have

$$\rho^\delta_k \sim \begin{cases} \varepsilon(k) & \text{on } \nu^{-1}[0, \delta) \\ \rho_k + \varepsilon(k) & \text{on } X \setminus \nu^{-1}[0, \delta) \end{cases}.$$
subvariety. When \( \delta \) is small enough, the moment function \( \nu \) is comparable with the distance function squared. One can see the similarity between them.

The asymptotics of off-diagonal Bergman kernel have been extensively used in the value distribution theory of sections of line bundles by Shiffman-Zelditch and others\[2, 3, 4, 5, 18, 19, 20, 21, 22, 23, 24, 11, 12, 13\]. It would be very interesting and important to study the asymptotics of the partial off-diagonal Bergman kernel vanishing along a general subvariety.

The structure of this article is as follows. We will first prove the key estimates on the complex plane. Then we apply the key lemma to the neighborhood \( U \). Then with the help of the Hörmander’s technique, we show that what we get on \( U \) goes directly to \( X \), hence proving the main theorem.

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### 2. One dimensional model

Let \( \omega = f \sqrt{-1} dz \wedge d \bar{z} \) be a smooth Kähler form on \( \mathbb{C} \) invariant under the natural \( S^1 \)-action. Let \( \varphi(|z|) \) be a smooth potential of \( \omega \). By a linear change of coordinates, we can assume that \( \varphi = |z|^2 + O(|z|^4) \). Let \( \mu : \mathbb{C} \to \mathbb{R} \) be the moment map of the \( S^1 \)-action normalized so that \( \mu(0) = 0 \). We will use the notations \( r = |z|, x = |z|^2 \) and \( t = -\log x \). So \( \varphi_{tt} = x \Delta \varphi = xf \).

Then

\[
\mu = \int_0^x f \, dx.
\]

And since \( \lim_{t \to \infty} \varphi_t = 0 \), we have

\[
\varphi_t = -\int_t^\infty \varphi_{tt} \, dt
\]

\[
= \int_0^x f \, dx
\]

\[
= -\mu.
\]

Let \( a \geq 0 \) be an integer, \( f_1(|z|) > 0 \) be a smooth function, we consider the \( k \)-th norm of the holomorphic function \( z^a \):

\[
I_a = \int_{\mathbb{C}} e^{-k \varphi} |z|^{2a} f_1 \sqrt{-1} dz \wedge d \bar{z} = 2\pi \int_{-\infty}^{\infty} e^{-k \varphi - (a+1)t + \log f_1} \, dt.
\]

Denote by \( g_a(t) = -k \varphi - (a+1)t + \log f_1 \). Then \( g_a' = -k \varphi_t - (a+1) + (\log f_1)_t \) and \( g_a'' = -k \varphi_{tt} + (\log f_1)_{tt} \). Since \( f_1 = A + O(|z|^2) \), \( A > 0 \), we have \( (\log f_1)_t = O(\varphi_t) \) and \( (\log f_1)_{tt} = O(\varphi_{tt}) \). So for \( k \) large enough, \( g_a \) is a concave function of \( t \) which attains its only maximum at \( t_a \) satisfying

\[
\varphi_t(t_a) = -\frac{a+1}{k} + \frac{(\log f_1)_t(t_a)}{k}.
\]

Namely,

\[
\mu(t_a) = \frac{a+1}{k} + O\left(\frac{x_a}{k}\right),
\]

when \( x_a < 1 \).
We recall the following basic lemma used in [25]:

**Lemma 2.1.** Let \( f(x) \) be a concave function. Suppose \( f'(x_0) < 0 \), then we have

\[
\int_{x_0}^{\infty} e^{f(x)} \, dx \leq \frac{e^{f(x_0)}}{-f'(x_0)}.
\]

Let \( 0 < R_1 < R < 1 \) be two fixed numbers. When \( \mu(t_a) > R \), we have \( a > Rk - C \) for some \( C \) independent of \( k \), then \( g_a' < 0 \) when \( \mu \leq R \). Let \( \mu(t_R) = R \) and \( \mu(t_{R_1}) = R_1 \), then we let \( t_m = \frac{t_R + t_{R_1}}{2} \). Clearly, \( |g_a'(t_m)| > C_1 k \) for some \( C_1 > 0 \) depending on \( R \) and \( R_1 \) but independent of \( k \) and \( a \). By the concavity of \( g_a \), we have

\[
e^{g_a(t_m) - g_a(t_{R_1})} = \varepsilon(k).
\]

Therefore, by lemma 2.1 we have

\[
\int_{\mu < R_1} e^{g_a(t)} \, dt = \varepsilon(k) \int_{R_1 \leq \mu \leq R} e^{g_a(t)} \, dt.
\]

When \( \mu(t_a) \leq R \), depending on \( a \), there are two cases.

When \( a \geq \sqrt{k} \), \( g_a'(t_a) \) is large, so we can use Laplace’s method to estimate the integral, namely

\[
I_a \approx \frac{(2\pi)^{3/2}}{\sqrt{g_a''(t_a)}}.
\]

More importantly, the mass of the measure \( e^{g_a(t)} \, dt \) is concentrated in a small neighborhood of \( t_a \). More precisely, if \( k \varphi_{t_a}(t) \delta_a^2 > 2 \log^2 k \), then by lemma 2.1 we have

\[
I_a = (1 + \varepsilon(k)) \int_{t_a + \delta_a}^{t_a + \delta_a} e^{g_a(t)} \, dt.
\]

It is easy to see that there is a constant \( C_2 > 0 \) independent of \( k \) such that if \( \delta_a > C_2 \frac{\log k}{\sqrt{a}} \), then the condition \( k \varphi_{t_a}(t) \delta_a^2 > 2 \log^2 k \) is satisfied.

When \( a < \sqrt{k} \), the term \( k \varphi_{t_a}(t) \) may be too small to use Laplace’s method. We have the following

\[
\int_{x < k^{-1/4}} e^{g_a(t)} \, dt \geq e^{-(a+1)} e^{-(a+1)} e^{g_a(t)}
\]

\[
\geq e^{-(a+1)} e^{-(a+1)} e^{-(a+1) + a \log \frac{a+1}{k}}
\]

\[
\geq e^{-5(a+1)} (\frac{a+1}{k})^a.
\]

When \( x = k^{-1/4}, \frac{e^{g_a(t_k)}}{e^{g_a(t_{k-1/4})}} = \varepsilon(k) \). So by lemma 2.1 we have for \( a < \sqrt{k} \),

\[
\int_{\mu \geq 2k^{-1/4}} e^{g_a(t)} \, dt \leq \int_{x \geq k^{-1/4}} e^{g_a(t)} \, dt = \varepsilon(k) I_a.
\]

In summary, we have proved the following
Lemma 2.2. For fixed $0 < R_1 < R < 1$, there exist constants $C, C_1 > 0$ independent of $k$, such that for $k$ large enough we have the following estimations:

1) If $a > Rk - C$,
$$
\int_{\mu < R_1} e^{g_a(t)} dt = \varepsilon(k) \int_{R_1 \leq \mu \leq R} e^{g_a(t)} dt,
$$
where the $\varepsilon(k)$ term is independent of $a$.
2) If $\sqrt{k} \leq a \leq Rk - C$,
$$
\int_{|\mu - \frac{a}{k}| \geq C_1 \sqrt{\frac{\log k}{k}}} e^{g_a(t)} dt = \varepsilon(k) \int_{|\mu - \frac{a}{k}| < C_1 \sqrt{\frac{\log k}{k}}} e^{g_a(t)} dt,
$$
where the $\varepsilon(k)$ term is independent of $a$.
3) If $a < \sqrt{k}$,
$$
\int_{\mu \geq 2k^{-1/4}} e^{g_a(t)} dt = \varepsilon(k) \int_{\mu < 2k^{-1/4}} e^{g_a(t)} dt,
$$
where the $\varepsilon(k)$ term is independent of $a$.

3. PROOF OF THE MAIN THEOREM

By assumption, there is a neighborhood $U$ of $V$ that is covered with charts that admit standard coordinates $(z, w)$ so that $V = \{z = 0\}$, where $z = (z_1, \ldots, z_{m-n}) \in \mathbb{C}^{m-n}$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, and
$$
\lambda \cdot (z, w) = (\lambda z, w),
$$
for $\lambda = (\lambda_1, \ldots, \lambda_{m-n})$, where $\lambda z = (\lambda_1 z_1, \ldots, \lambda_{m-n} z_{m-n})$. We have the following lemma,

Lemma 3.1. There is a $T^{m-n}$-invariant biholomorphism $\Psi$ from the neighborhood $U$ to an open neighborhood of $V$ in the total space of the vector bundle $L_1 \oplus L_2 \cdots \oplus L_{m-n}$, where each $L_i$ is a holomorphic line bundle.

The proof is very similar to that of lemma 5.3 in [17], and for the convenience of the reader, we include a proof here.

Proof. Let $(z_\alpha, w_\alpha)$ and $(z_\beta, w_\beta)$ be two sets of standard coordinates. Then the transition functions between them are necessarily of the form
$$
z_\beta = \lambda_{\alpha \beta}(z_\alpha, w_\alpha) z_\alpha, \quad w_\beta = \tau_{\alpha \beta}(z_\alpha, w_\alpha)
$$
where $\lambda_{\alpha \beta}$ and $\tau_{\alpha \beta}$ are holomorphic and take values in $\mathbb{C}^{(m-n)}$ and $\mathbb{C}^{(m-n)}$ respectively. We understand the torus $T^{m-n}$-action on the $w$-part is trivial. Then both $\lambda_{\alpha \beta}$ and $\tau_{\alpha \beta}$ are $T^{m-n}$-invariant, so their dependence on $z_\alpha$ are trivial. So the transition functions
$$
z_\beta = \lambda_{\alpha \beta}(w_\alpha) z_\alpha
$$
defines a holomorphic vector bundle $E$ of rank $m - n$ over $V$, with structure group $T^{m-n}$. Therefore, $E$ decomposes as $E = L_1 \oplus L_2 \cdots \oplus L_{m-n}$, and $U$ is biholomorphic to an open neighborhood of $V$ in $E$. Clearly, the biholomorphism is $T^{m-n}$-invariant. \qed
We consider $U$ as an open set in $\oplus L_i$ and denote by $\pi : U \to V$ the projection. Then for $A$ small enough, $U$ contains the poly-disc bundle $U_A = \cap_{1 \leq i \leq m-n} \mu_i ^{-1}(x < A)$. For each $p \in V$, the fiber of $U_A$ is an open poly-disc $D_A$ in $\mathbb{C}^{m-n}$. By the torus-invariance, the restriction $\varphi_p$ of $\varphi$ depends only on $|z_1|, \cdots, |z_{m-n}|$, so does the restriction $\omega_p$ of $\omega$. So the monomials $z^a = \prod z_i ^{a_i}$ are all orthogonal to each other on $D_p$, where $a = (a_1, \cdots, a_{m-n})$ is an $(m-n)$-tuple of non-negative integers.

**Lemma 3.2.** Let $W_{\delta} = \{ q \in D_p | \nu(q) < \delta - k^{1/4} \}$. When $\delta$ is small enough, for each holomorphic function $f = \sum \sum a_i \geq \delta k c_2^a$, we have

$$\int_{W_{\delta}} |f|^2 e^{-k\varphi_p} \omega_p^{m-n} = \varepsilon(k) \int_{D_p} |f|^2 e^{-k\varphi_p} \omega_p^{m-n}$$

**Proof.** By orthogonality, $\int |f|^2 e^{-k\varphi_p} \omega_p^{m-n} = \sum \sum a_i \geq \delta k |c_2|^2 \int |z^a|^2 e^{-k\varphi_p} \omega_p^{m-n}$. So it suffices to prove the lemma for the monomials $z^a$ satisfying $\sum a_i \geq \delta k$.

By making $A$ a little smaller, a compactness argument shows that the constants $C$ and $C_1$ in lemma 2.2 can be chosen to be the same for all $z_i$-discs. For each point $p$ in the open set $U$, at least one $1 \leq i \leq m-n$ satisfies the condition that $\mu_i < \frac{4k}{k} - k^{-3/8}$. Therefore, lemma 2.2 shows that the mass of $z^a$ in $W$ is $\varepsilon(k)$ relative to the mass in $D_A \setminus W_{\delta}$. And the theorem is proved.

As a corollary, we can prove the part about the forbidden region of theorem 1.2

**Proof of the first half of theorem 1.2** We can take an arbitrary orthonormal basis $\{s_i\}$ for the space $H_k^{\delta k}$. Then the lemma above shows that each $s_i$ has $L^2$-mass of the size $\varepsilon(k)$ in the region $W_{\delta}$. We recall that for each point $p \in X$, there is an unit section $s_p$, called the peak section, satisfying $\|s_p(p)\|^2 = \rho_k(p)$ and $s_p \perp s$ as long as $s(p) = 0$.

We take the inner product of $s_i$ with the peak sections $s_p$ for each $p \in W$

$$|s_i(p)| = | < s_i, s_p > | \sqrt{\rho_k(p)} \leq \| s_i \| \sqrt{\rho_k(p)}$$

we see that the point-wise norm $|s_i(p)|$ is also $\varepsilon(k)$. Since the dimension of $H_k^{\delta k}$ is $O(k^m)$. We get the conclusion.

We denote by $T_i$ the $i$-th component of $T^{m-n}$. For each $L_i$, we fix a $T_i$-invariant section $s_i$ such that $\mu_i ^{-1}(0) = \{ s_i = 0 \}$. So we can choose local frame $e_i$ of $L_i$, so that $s_i = z_i e_i$ in the standard torus-invariance coordinates.

For each $s \in H^0(V, kL - \otimes L_i^{a_i})$, where $a_i \geq 0$, $\pi^* s \in H^0(U, kL - \otimes L_i^{a_i})$, so $\bar{s} = \pi^* s \otimes \prod s_i ^{a_i} \in H^0(U, L^k)$. We denote by $G_\delta$ the span of all such sections for $\sum a_i \leq \delta k$. Then clearly, for each section $\beta \in H_k^{\delta k}$, we have $\beta |_U \perp G_\delta$. Moreover, since each $\bar{s}$ is represented by a holomorphic function of the form

$$\sum_{\sum a_i \leq \delta k} f_a(w) z^a,$$

by lemma 2.2 each section $\bar{s} \in G_\delta$ has its mass concentrated within the set $Y_\delta = \{ q | \nu(q) \leq \delta + 2(m-n)k^{-1/4} \}$ and decays fast as $\nu$ gets bigger. More precisely, the measure outside $Y_\delta$ is $\varepsilon(k)$ relative to that inside $Y_\delta$. To connect $G_\delta$ to $(H_k^{\delta k})^\perp$, we need to use Hörmander’s $L^2$ technique. The following lemma is well-known, see for example [23].
Lemma 3.3. Suppose \((M, g)\) is a complete Kähler manifold of complex dimension \(n\), \(L\) is a line bundle on \(M\) with hermitian metric \(h\). If
\[
\langle -2\pi i \Theta_h + \text{Ric}(g), v \wedge \bar{v} \rangle_g \geq C|v|^2_g
\]
for any tangent vector \(v\) of type \((1,0)\) at any point of \(M\), where \(C > 0\) is a constant and \(\Theta_h\) is the curvature form of \(h\). Then for any smooth \(L\)-valued \((0,1)\)-form \(\alpha\) on \(M\) with \(\bar{\partial} \alpha = 0\) and \(\int_M |\alpha|^2 dV_g\) finite, there exists a smooth \(L\)-valued function \(\beta\) on \(M\) such that \(\bar{\partial} \beta = \alpha\) and
\[
\int_M |\beta|^2 dV_g \leq \frac{1}{C} |\alpha|^2 dV_g
\]
where \(dV_g\) is the volume form of \(g\) and the norms are induced by \(h\) and \(g\).

Now let \(\chi \in C_0^\infty(U)\) be a non-negative cut-off function, that equals 1 in a neighborhood of the subset \(\{p \in U | \nu(p) \leq \delta + c\}\) for some small \(c > 0\). Then for each \(\bar{s} \in G_\delta\) with \(\| \bar{s} \|^2 = 1\), we take the orthogonal projection of \(\chi \bar{s}\) onto \(H_k\). The way to do so is to solve the \(\bar{\partial}\)-equation
\[
\bar{\partial} v = \bar{\partial} \chi \otimes \bar{s}.
\]
And since \(|\bar{s}|\) is \(\varepsilon(k)\) on the place where \(\bar{\partial} \chi \neq 0\), we can apply the lemma above to see that the global holomorphic section \(\chi \bar{s} - v \in H_k\) has \(\varepsilon(k)\) \(L^2\)-mass outside \(U\) and the \(L^2\)-mass of \(\chi \bar{s} - v - \bar{s}\) within \(U\) is also \(\varepsilon(k)\). So for each \(f \in \mathcal{H}_k^{\delta k}\) with \(\| f \|^2 = 1\), we have
\[
| \langle f, \chi \bar{s} - v \rangle | = \varepsilon(k)
\]
We then take the orthogonal projection of \(\chi \bar{s} - v\) onto \((\mathcal{H}_k^{\delta k})^\perp\) and denote by \(\bar{s}\) the section obtained. So we have
\[
\int_U |\bar{s} - \bar{s}|^2 e^{-k\rho} \omega^m = \varepsilon(k).
\]
We denote by \(J\) the map we just constructed from \(G_\delta\) to \((\mathcal{H}_k^{\delta k})^\perp\). So \(J\) is almost an isometry, namely
\[
\| J(\bar{s}) \|^2 = (1 + \varepsilon(k)) \| \bar{s} \|^2.
\]
Therefore, if we start with an orthonormal basis of \(G_\delta\), we obtain an almost orthonormal basis of \((\mathcal{H}_k^{\delta k})^\perp\). And then after a Gram-Schmidt process, we get an orthonormal basis. We denote by \(\rho_G\) the density function of \(G_\delta\), considered as a function on \(X\) by extension by zero. Since the density function of \((\mathcal{H}_k^{\delta k})^\perp\) is \(\rho_k - \rho_k^{\delta k}\), we have
\[
\int_X (\rho_k - \rho_k^{\delta k} - \rho_G) \omega^m = \varepsilon(k).
\]
Since, \(\rho_G\) has \(\varepsilon(k)\) mass outside \(Y_\delta\), we have
\[
\int_{X \setminus Y_\delta} (\rho_k - \rho_k^{\delta k}) \omega^m = \varepsilon(k).
\]
Recall that by a local construction of the peak section \(s_p\) for any \(p \in X\) (see for example [9]), \(|s_p|\) decays exponentially away from \(p\). More precisely, \(|s_p| = \varepsilon(k)\), when \(d(p, q) > \frac{\log k}{\sqrt{k}}\).
When \(\delta\) is small enough, \(\nu\) is comparable with the distance function \(d^2(p, q)\), so if we denote by \(Y_\delta' = \{q|\nu(q) \leq \delta + 3(m - n)k^{-1/4}\}\), then for any unit \(s \in (\mathcal{H}_k^{\delta k})^\perp\), we have
\[
| \langle s, s_p \rangle | = \varepsilon(k),
\]
for $p \in X \setminus Y'_\delta$. Therefore $|s(p)| = \epsilon(k)$, $p \in X \setminus Y'_\delta$. Since $\lim_{k \to \infty} Y'_\delta = \{ q | \nu(q) \leq \delta \}$, this implies the second part of theorem \ref{thm:1.2}.

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