CHAIN CONDITIONS, ELEMENTARY AMENABLE GROUPS, AND DESCRIPTIVE SET THEORY

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Abstract. We first consider three well-known chain conditions in the space of marked groups: the minimal condition on centralizers, the maximal condition on subgroups, and the maximal condition on normal subgroups. For each condition, we produce a characterization in terms of well-founded descriptive-set-theoretic trees. Using these characterizations, we demonstrate that the sets given by these conditions are co-analytic and not Borel in the space of marked groups. We then adapt our techniques to show elementary amenable marked groups may be characterized by well-founded descriptive-set-theoretic trees, and therefore, elementary amenability is equivalent to a chain condition. Our characterization again implies the set of elementary amenable groups is co-analytic and non-Borel. As corollary, we obtain a new, non-constructive, proof of the existence of finitely generated amenable groups that are not elementary amenable.

1. Introduction

Chain conditions appear frequently in the study of countable groups. These are finiteness conditions that forbid certain infinite sequences of subgroups. One of the most basic such conditions is solubility; however, there are a wide variety of chain conditions. From a geometric group theory perspective, these finiteness conditions ought to restrict the complexity of the groups, e.g. in the case of soluble groups. From a descriptive set theory perspective, however, the chain conditions are non-Borel co-analytic statements and, therefore, either admit “nice” non-chain-condition characterizations - e.g. soluble groups are given by group laws - or describe large and wild classes. In this work, we explore this descriptive-set-theoretic tension in four chain conditions in the space of marked groups.

In the space of marked groups, denoted $G$, we first consider three well known chain conditions: the minimal condition on centralizers, the maximal condition on subgroups, and the maximal condition on normal subgroups. We characterize each of these in terms of well-founded descriptive-set-theoretic trees. This characterization implies the classes in question are large and wild, whereby they do not admit “nice” characterizations.

Theorem 1.1. Each of the subsets of $G$ defined by the minimal condition on centralizers, the maximal condition on subgroups, and the maximal condition on normal subgroups are co-analytic and not Borel. This remains true when restricting to finitely generated groups.

Our techniques additionally give new ordinal-valued isomorphism invariants unbounded below the first uncountable ordinal in the cases of the minimal condition on centralizers and the maximal condition on subgroups. The ordinal-valued isomorphism invariant we obtain in the case of the maximal condition on normal subgroups is not new and has been considered in the literature; cf. [3]. However, our approach is new, and we show this invariant is unbounded below the first uncountable ordinal.
We next consider the set of elementary amenable marked groups. We likewise characterize these in terms of descriptive-set-theoretic trees. It follows elementary amenability is indeed a chain condition.

**Theorem 1.2.** A countable group $G$ is elementary amenable if and only if there is no infinite descending sequence of the form

$$G = G_0 \geq G_1 \geq \ldots \geq G_n \geq \ldots$$

such that for all $n \geq 0$, $G_n \neq \{e\}$ and there is a finitely generated subgroup $K_n \leq G_n$ with $G_{n+1} = [K_n, K_n] \cap H_n$, where $H_n$ is the intersection of the index-$(\leq (n+1))$ normal subgroups of $K_n$.

Our characterization gives two new invariants of elementary amenable groups: the decomposition rank and decomposition degree. We further obtain

**Theorem 1.3.** The sets of elementary amenable groups and finitely generated elementary amenable groups are co-analytic and non-Borel in the space of marked groups.

It is well-known that the set of amenable groups is Borel in the space of marked groups. Thus this gives a non-constructive answer to an old question of M. Day [4], which was open until R. I. Grigorchuk [6] constructed groups of intermediate growth: Are all finitely generated amenable groups elementary amenable?

**Corollary 1.4.** There is a finitely generated amenable group that is not elementary amenable.

The paper is organized as follows. In Section 2 we discuss the basic properties of $\mathcal{G}$ and introduce concepts from descriptive set theory. In Sections 3-5 we analyze the sets of groups satisfying various chain conditions. This introduces our use of descriptive-set-theoretic trees to study the structure of groups as well as the ordinal-valued invariants arising from those trees. In Section 6 we use those same techniques to analyze elementary amenable groups. In Section 7 we prove that the maps used throughout the paper are indeed Borel. Those who are content to believe that our constructions are Borel can safely skip this section without missing any group-theoretic content. Finally, Section 8 discusses some questions arising from this paper not touched upon in earlier sections.

2. Preliminaries

2.1. The space of marked groups. In order to apply the techniques of descriptive set theory to groups, we need to have an appropriate space of groups. Let $\mathbb{F}_\omega$ be the free group on the letters $\{a_i\}_{i \in \mathbb{N}}$; so $\mathbb{F}_\omega$ is a free group on countably many generators with a distinguished set of generators. The power set of $\mathbb{F}_\omega$ may be naturally identified with the Cantor space $\{0,1\}^{\mathbb{F}_\omega} = 2^{\mathbb{F}_\omega}$. It is easy to check the collection of normal subgroups of $\mathbb{F}_\omega$, denoted $\mathcal{G}$, is a closed subset of $2^{\mathbb{F}_\omega}$ and, hence, a compact Polish space. Each $N \in \mathcal{G}$ is identified with a marked group. That is a group $G = \mathbb{F}_\omega / N$ along with a distinguished generating set $\{f_N(a_i)\}_{i \in \mathbb{N}}$ where $f_N : \mathbb{F}_\omega \to G$ is the usual projection; we always denote this projection by $f_N$. For a marked group $G$, we abuse notation and say $G \in \mathcal{G}$; of course, we formally mean $G = \mathbb{F}_\omega / N$ for some $N \in \mathcal{G}$. Since every countable group is a quotient of $\mathbb{F}_\omega$, $\mathcal{G}$ gives a standard Borel space, indeed a compact Polish space, of all countable groups. A sub-basis for this topology is given by sets of the form

$$O_\gamma := \{N \in \mathcal{G} \mid \gamma \in N\},$$
where \( \gamma \in \mathbb{F}_\omega \), along with their complements.

Similar reasoning leads us to define the space of \( m \)-generated marked groups as

\[
\mathcal{G}_m := \bigcap_{i \geq m} \{ N \leq \mathbb{F}_\omega \mid a_i \in N \}.
\]

This is a closed subset of \( \mathcal{G} \) and so is a standard Borel space in its own right. We further let \( \mathcal{G}_fg := \bigcup_{m \geq 1} \mathcal{G}_m \) be the space of finitely generated marked groups. As this is an \( F_\sigma \) subset of the standard Borel space \( \mathcal{G} \), it is itself a standard Borel space.

It is convenient to give the marked groups \( G = \mathbb{F}_\omega/N \) a preferred enumeration. To this end, we fix an enumeration \( \gamma := (\gamma_i)_{i \in \mathbb{N}} \) of \( \mathbb{F}_\omega \). Each \( G \) is thus taken to come with an enumeration \( f_N(\gamma) := (f_N(\gamma_i))_{i \in \mathbb{N}} \); note the enumeration of \( G \) may have many repetitions. When we write \( G \) as \( G = \{g_0, g_1, \ldots\} \), we will always mean this enumeration. Later in the paper we will work with \( \mathbb{N}^{<\mathbb{N}} \), i.e., the set of finite sequences of natural numbers. If \( (s_0, \ldots, s_n) =: s \in \mathbb{N}^{<\mathbb{N}} \), we will write \( \{g_s\} \) for the set \( \{g_{s_0}, \ldots, g_{s_n}\} \). Note that this set may have fewer than \( n + 1 \) elements, e.g. if \( s_0 = s_1 = \ldots = s_n \), or even if the \( s_i \) are distinct but enumerate the same element.

We will often discuss quotients of groups or particular subgroups of groups, and of course we wish to view these as elements of \( \mathcal{G} \). A quotient of a marked group is obviously again a marked group; however, this does not hold for subgroups. The enumeration gives us a preferred way to select markings for subgroups. If \( H \leq \mathbb{F}_\omega/N = G \in \mathcal{G} \), let \( \pi_H : \mathbb{F}_\omega \to \mathbb{F}_\omega \) be induced by mapping the generators \( (a_i) \) as follows:

\[
\pi_H(a_j) := \begin{cases} 
\gamma_j, & \text{if } f_N(\gamma_j) \in H \\
e, & \text{else.}
\end{cases}
\]

We then identify \( H \) with \( \mathbb{F}_\omega/\ker(f_N \circ \pi_H) \). In the case \( H \) has a distinguished finite generating set \( \{g_{i_0}, \ldots, g_{i_k}\} \), we instead define \( \pi_H(a_{i_j}) = \gamma_{i_j} \) and \( \pi_H(a_j) = e \) for \( j \neq i_k \); this streamlines our proofs later. We often appeal to this convention implicitly.

We will consider maps from and on \( \mathcal{G} \). A slogan from descriptive set theory is “Borel = explicit”, meaning if you describe a map “explicitly”, i.e. without an appeal to something like the axiom of choice, it should be Borel. All of the maps we discuss in the next few sections will be “explicit” in this sense, so we will not prove they are Borel when we define them, in order to keep the focus on the group-theoretic aspects of our constructions. We will often use enumerations of groups in our constructions, but this will not require choice since every marked group comes with a preferred enumeration. For those who are interested in the details, we discuss the descriptive-set-theoretic aspects of our constructions in Section 7.

2.2. Descriptive set theory. We are interested in certain types of non-Borel subsets of \( \mathcal{G} \). The following definitions and theorems are all fundamental in descriptive set theory; a standard reference is [11].

Definition 2.1. Let \( X, Y \) be standard Borel spaces. Then \( A \subseteq Y \) is analytic (denoted \( \Sigma_1^1 \)) if there is a Borel map \( f : X \to Y \) and a Borel \( B \subseteq X \) such that \( f(B) = A \). A set \( C \subseteq Y \) is co-analytic (denoted \( \Pi_1^1 \)) if \( Y \setminus C \) is analytic.

Every Borel set is analytic, but any uncountable standard Borel space contains non-Borel analytic sets. It follows there are non-Borel co-analytic sets. The collection of analytic sets is closed under countable unions, countable intersections, Borel preimages, and Borel
images. It follows the collection of co-analytic sets is closed under countable unions, countable intersections, and Borel preimages, although not necessarily Borel images.

**Definition 2.2.** Let $X, Y$ be standard Borel spaces, and $A \subseteq X$, $B \subseteq Y$. We say that $A$ Borel reduces to $B$ if there is a Borel map $f: X \to Y$ such that $f^{-1}(B) = A$.

If $A$ Borel reduces to $B$ and $B$ is Borel, analytic, or co-analytic, then so is $A$. This gives us a method for proving that sets are, for example, co-analytic simply by showing they Borel reduce to a co-analytic set. One important example comes from the space of (descriptive-set-theoretic) trees.

**Definition 2.3.** A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ of finite sequences of natural numbers is a tree if it is closed under initial segments. A sequence $x \in \mathbb{N}^{<\mathbb{N}}$ is a branch of $T$ if for all $n \in \mathbb{N}$, $x|n \in T$. For $s \in T$, $T_s := \{ r \in \mathbb{N}^{<\mathbb{N}} | s \sqsupset r \in T \}$ where “$\sqsupset$” indicates concatenation of finite sequences.

As with groups, we may identify $X \subseteq \mathbb{N}^{<\mathbb{N}}$ with an element $f_X \in 2^{\mathbb{N}^{<\mathbb{N}}}$. We define $Tr := \{ x \in 2^{\mathbb{N}^{<\mathbb{N}}} | x \text{ is a tree} \}$. Then $Tr$ is a closed subset of $2^{\mathbb{N}^{<\mathbb{N}}}$ and so is a standard Borel space. A sub-basis for the topology on $Tr$ is given by sets of the form $O_t := \{ T \in Tr | t \in T \}$, where $t \in \mathbb{N}^{<\mathbb{N}}$, along with their complements.

The following subsets of $Tr$ are of particular interest to us. Let $IF := \{ T \in Tr | T \text{ has a branch} \}$, and $WF := Tr \setminus IF$. We call $WF$ the set of well-founded trees and $IF$ the set of ill-founded trees. One can check that $IF$ is analytic, so $WF$ is co-analytic. The importance of these sets comes from the following fact.

**Theorem 2.4.** [Π] Theorem 27.1] Every analytic set Borel reduces to $IF$. Therefore, every co-analytic set Borel reduces to $WF$.

Thus $A$ is co-analytic if and only if it Borel reduces to $WF$.

We are interested in $WF$ for a second reason. Let $ORD$ denote the class of ordinals. Then for any $T \in WF$ we can define a function $\rho_T: T \to ORD$ inductively as follows. If $t \in T$ has no extensions in $T$, let $\rho_T(t) = 0$. Otherwise let $\rho_T(t) = \sup\{ \rho_T(s) + 1 | t \sqsubset s \}$. We may then define a rank function $\rho: Tr \to ORD$ by

$$\rho(T) = \begin{cases} \rho_T(\emptyset) + 1, & \text{if } T \in WF \\ \omega_1, & \text{else.} \end{cases}$$

For $T = \emptyset$, we define $\rho(T) = 0$. The function $\rho$ is bounded above by $\omega_1$, the first uncountable ordinal. This rank function has a special property.

**Definition 2.5.** Let $X$ be a standard Borel space and $A \subseteq X$. A function $\phi: A \to ORD$ is a $\Pi_1^1$-rank if there are relations $\leq_\phi^\Pi$, $\leq_\phi^\Sigma \subseteq X \times X$ such that $\leq_\phi^\Pi$ is co-analytic, $\leq_\phi^\Sigma$ is analytic, and for all $y \in A$,

$$x \in A \land \phi(x) \leq_\phi^\Pi y \iff x \leq_\phi^\Sigma y \iff x \leq_\phi^\Pi y.$$
Lemma 3.4. Intuitively, Lemma 3.3 holds since our construction is explicit; we delay a rigorous proof until

The map **Lemma 3.3.** Suppose that

- Put $$s$$
- Each

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abelian-by-nilpotent groups; see \[1\] for further discussion. It is not hard to check that a subgroup $$A$$ of $$G$$ satisfies the minimal condition on centralizers by $$\mathcal{M}_G$$.

The class $$\mathcal{M}_G$$ is large, containing abelian groups, linear groups, and finitely generated abelian-by-nilpotent groups; see \[1\] for further discussion. It is not hard to check that a group $$G$$ satisfies the minimal condition on centralizers if and only if it satisfies the maximal condition on centralizers, but our analysis is easier if we think about the minimal version of the chain condition.

Given a group $$G \in \mathcal{G}$$, we construct a tree $$T_G \subseteq \mathbb{N}^{<\mathbb{N}}$$ and associated groups $$G_s \in \mathcal{G}$$ for each $$s \in T_G$$. Each $$G_s$$ will be a centralizer in $$G$$.

- Put $$\emptyset \in T_G$$ and let $$G_\emptyset = G = C_G(\emptyset)$$.
- Suppose that $$s \in T_G$$ and $$G_s = C_G(\{g_s\})$$ has already been defined. If $$C_G(\{g_s\} \cup \{g_i\}) \neq \{e\}$$ and $$C_G(\{g_s\} \cup \{g_i\}) \neq C_G(\{g_i\})$$, then let $$s^{-i} \in T_G$$ and $$G_{s^{-i}} = C_G(\{g_s\} \cup \{g_i\})$$.

**Lemma 3.3.** The map $$\Phi_G: \mathcal{G} \to \mathcal{G}$$ given by $$G \mapsto T_G$$ is Borel.

Intuitively, Lemma 3.3 holds since our construction is explicit; we delay a rigorous proof until Section [7].

**Lemma 3.4.** $$T_G$$ is well-founded if and only if $$G \in \mathcal{M}_G$$. 

**Theorem 2.6.** \[1\] Exercise 34.6] The function $$\rho: WF \to ORD$$ is a $$\Pi^1_1$$-rank.

We may use this fact to create other $$\Pi^1_1$$-ranks in an easy way: Let $$X$$ be a standard Borel space. If $$A \subseteq X$$ Borel reduces to $$WF$$ via $$f$$, then the map $$x \mapsto \rho(f(x))$$ is a $$\Pi^1_1$$-rank.

The most important fact about $$\Pi^1_1$$-ranks for this paper is the following (\[1\] Theorem 35.23)

**Theorem 2.7** (The Boundedness Theorem for $$\Pi^1_1$$-ranks). Let $$X$$ be a standard Borel space, $$A \subseteq X$$ co-analytic, and $$\phi: A \to ORD$$ a $$\Pi^1_1$$-rank. Then

$$A$$ is Borel $$\iff$$ sup$$\{\phi(x) \mid x \in A\} < \omega_1$$.

We will use the Boundedness Theorem to show that certain $$\Pi^1_1$$ sets are not Borel, by showing that the image of their $$\Pi^1_1$$-ranks is unbounded below $$\omega_1$$. To this end, we will often use the following fact about the ranks of trees, which follows immediately from the definition.

**Lemma 2.8.** Suppose $$S, T$$ are trees and $$\phi: S \to T$$ is a map such that $$s \subseteq t \Rightarrow \phi(s) \subseteq \phi(t)$$. (We call such a map **monotone**.) Then $$\rho_S(s) \leq \rho_T(\phi(s))$$ for all $$s \in S$$. In particular $$\rho(S) \leq \rho(T)$$. 

3. **The minimal condition on centralizers**

We wish to show that certain chain conditions give rise to sets of marked groups which are $$\Pi^1_1$$ and not Borel in $$\mathcal{G}$$. We begin by looking at the following chain condition.

**Definition 3.1.** A subgroup $$H$$ of $$G$$ is a **centralizer** in $$G$$ if $$H = C_G(A)$$ for some $$A \subseteq G$$.

**Definition 3.2.** A group $$G$$ satisfies the **minimal condition on centralizers** if there is no strictly decreasing infinite chain $$C_0 > C_1 > \ldots$$ of centralizers in $$G$$. We denote the class of countable groups satisfying the minimal condition on centralizers by $$\mathcal{M}_G$$.
Applying Lemma 3.8, we see the lemma holds for \( \alpha \). Let \( \beta \) be such that \( A = \{g_{a_0}, g_{a_1}, \ldots\} \). By moving to a subsequence if necessary, we may assume that \( C_G(g_{a_0}, \ldots, g_{a_n}) \leq C_G(g_{a_0}, \ldots, g_{a_{n+1}}) \) for all \( n \in \mathbb{N} \). Then \( (a_0, \ldots, a_n) \in T_G \) for all \( n \in \mathbb{N} \), so \( T_G \) has an infinite branch.

**Lemma 3.5.** Let \( H, G \in \mathcal{G} \). If \( H \hookrightarrow G \), then \( \rho(T_H) \leq \rho(T_G) \).

**Proof.** Let \( \alpha : H \hookrightarrow G \), and let \( \psi : \mathbb{N} \to \mathbb{N} \) be such that \( \alpha(h_k) = g_{\psi(k)} \). We define a map \( \phi \) from \( T_H \). Let \( \phi(\emptyset) = \emptyset \). If \( s \in T_H \) and \( s = (s_0, \ldots, s_n) \), let \( \phi(s) = (\psi(s_0), \ldots, \psi(s_n)) \). Clearly \( \phi \) is monotone. Further, if \( s \in T_H \), then \( H_s = C_H(\{h_s\}) \hookrightarrow C_G(\{g_{\phi(s)}\}) \). Since \( H_s \neq \{e\} \), we see that \( C_G(\{g_{\phi(s)}\}) \neq \{e\} \). Also, since \( C_G(\{g_{\phi(s)}\}) \cap \alpha(H) \cong C_H(\{h_s\}) \), we have that \( C_G(\{g_{\phi(s)}\}) \neq C_G(\{g_{\phi(s)}\}) \) for all \( k < |s| \). Thus \( \phi(s) \in T_G \). It follows that \( \phi(T_H) \subseteq T_G \) and by Lemma 2.8, \( \rho(T_H) \leq \rho(T_G) \).

**Corollary 3.6.** If \( G, G' \in \mathcal{G} \) and \( G \cong G' \), then \( \rho(T_G) = \rho(T_{G'}) \).

We thus see that \( \rho(T_G) \) is an isomorphism invariant, so it makes sense to talk about the rank of a group \( G \) with the minimal condition on centralizers, even when not considering a specific marking.

**Definition 3.7.** If \( G \) has the minimal condition on centralizers, then \( \rho(T_G) \) for some (any) marking of \( G \) is called the centralizer rank of \( G \).

We also mention that the above results, except for Lemma 3.3 work with arbitrary enumerations of the group \( G \), not just those that can arise from viewing \( G \) as a marked group. Certain enumerations may be easier to use to calculate \( \rho(T_G) \), and Corollary 3.6 assures us that using these enumerations will not affect the answer. The same will be true of our later constructions. Of course, in this paper Lemma 3.3 and analogous results are of central importance, and so we will continue to work with groups as elements of \( \mathcal{G} \).

We now argue the centralizer rank is unbounded under \( \omega_1 \).

**Lemma 3.8.** For \( A, B \in \mathcal{M}_C \) with \( A \) nonabelian, \( A \times B \in \mathcal{M}_C \) and \( \rho(T_B) < \rho(T_{A \times B}) \).

**Proof.** It is easy to see \( A \times B \in \mathcal{M}_C \). Let \( a \in A \) be noncentral. Then \( C_{A \times B}(\{(a, e)\}) = (A \times B)_i \) for some \( i \in T_{A \times B} \) since the centralizer is not all of \( A \times B \). Further, \( B \cong \{e\} \times B \leq C_{A \times B}(\{(a, e)\}) \), so by Lemma 3.5, \( \rho(T_{A \times B}) \geq \rho(T_B) \). The result now follows.

**Lemma 3.9.** Let \( \{A_i\}_{i \in \mathbb{N}} \) be countable groups. If \( A_i \in \mathcal{M}_C \) for all \( i \in \mathbb{N} \), then there is a group \( A \in \mathcal{M}_C \) such that \( \rho(T_A) \geq \rho(T_{A_i}) \) for all \( i \in \mathbb{N} \).

**Proof.** Let \( A = \ast_{i \in \mathbb{N}} A_i \). By [13, Corollary 4.1.6], which says that centralizers in free products are cyclic or centralizers of a conjugate of a free factor, we have that \( A \in \mathcal{M}_C \). Then Lemma 3.5 implies that \( \rho(T_A) \geq \rho(T_{A_i}) \) for all \( i \in \mathbb{N} \), as desired.

**Lemma 3.10.** For all \( \alpha < \omega_1 \), there is \( G \in \mathcal{M}_C \) such that \( \rho(T_G) \geq \alpha \).

**Proof.** We prove this inductively. Clearly the lemma holds for \( \alpha = 0 \). Suppose \( \alpha = \beta + 1 \) and the lemma holds for \( \beta \). Let \( G \in \mathcal{M}_C \) be such that \( \rho(T_G) \geq \beta \) and \( A \in \mathcal{M}_C \) be nonabelian. Applying Lemma 3.8 we see \( \rho(T_{A \times G}) \geq \beta + 1 \).

Suppose \( \alpha \) is a limit ordinal. Since \( \alpha \) is countable, there is a countable increasing sequence of \( \alpha_i < \alpha \) such that \( \sup_{i \in \mathbb{N}} \alpha_i = \alpha \). Let \( G_i \in \mathcal{M}_C \) be such that \( \rho(T_{G_i}) > \alpha_i \). Then by
Lemma [3.9] there is some $G \in \mathcal{M}_C$ such that $\rho(T_G) > \alpha_i$ for all $i \in \mathbb{N}$. It now follows that $\rho(T_G) \geq \alpha$. 

**Lemma 3.11.** For all $\alpha < \omega_1$, there is a finitely generated $G \in \mathcal{M}_C$ such that $\rho(T_G) \geq \alpha$.

**Proof.** Let $H \in \mathcal{M}_C$ be a group such that $\rho(T_H) \geq \alpha$. Then [10, Corollary on pg. 949] implies that $H$ embeds into a 3-generated group $G \in \mathcal{M}_C$. By Lemma 3.5, $\rho(T_G) \geq \rho(T_H) \geq \alpha$. 

We remark that the result cited in the previous proof uses nothing more complicated than free products with amalgamation, and is similar to the classical Higman–Neumann–Neumann embedding result from [9].

**Theorem 3.12.** $\mathcal{M}_C$ is $\Pi^1_1$ and not Borel in $\mathcal{G}$, and $\mathcal{M}_C \cap \mathcal{G}_{fg}$ is $\Pi^1_1$ and not Borel in $\mathcal{G}_{fg}$.

**Proof.** Let $\Phi_C$ be the Borel map from Lemma 3.3. By Lemma 3.4, $\Phi^{-1}_C(WF) = \mathcal{M}_C$, and since $\Phi_C$ is Borel, $\mathcal{M}_C$ is $\Pi^1_1$. By Lemma 3.10, the ranks of the trees in $\Phi_C(\mathcal{M}_C)$ are unbounded below $\omega_1$, so the $\Pi^1_1$-rank on $\mathcal{M}_C$ given by $G \mapsto \rho(\Phi_C(G))$ is unbounded below $\omega_1$. By Theorem 2.7, we conclude $\mathcal{M}_C$ is not Borel. By Lemma 3.11, the ranks of the trees in $\Phi_C(\mathcal{M}_C \cap \mathcal{G}_{fg})$ are unbounded below $\omega_1$, and by Theorem 2.7, we conclude that $\mathcal{M}_C \cap \mathcal{G}_{fg}$ is also not Borel.

### 4. The maximal condition on subgroups

We next consider a more basic chain condition. Proving the analog of Lemma 3.9 in this context is more complicated, which is why we present it after the previous section.

**Definition 4.1.** A group $G$ satisfies the **maximal condition on subgroups**, abbreviated by saying a group satisfies max, if there is no strictly increasing chain $H_0 < H_1 < H_2 < \ldots$ of subgroups of $G$. Equivalently, a group $G$ satisfies max if all of its subgroups are finitely generated. We denote the class of groups satisfying max as $\mathcal{M}_{\text{max}}$.

Given a group $G \in \mathcal{G}$, we construct a tree $T_G \subseteq \mathbb{N}^{< \mathbb{N}}$ and associated groups $G_s \in \mathcal{G}$ for each $s \in T_G$.

- Put $\emptyset \in T_G$ and let $G_{\emptyset} = \{e\}$.
- Suppose that $s \in T_G$ and $G_s = \{g_s\}$ has already been defined. If $\{g_s \cup \{g_i\}\} \neq \{g_s\}$, then let $s^s \in T_G$ and $G_{s^s} = \{g_s \cup \{g_i\}\}$.

**Lemma 4.2.** The map $\Phi_M: \mathcal{G} \rightarrow Tr$ given by $G \mapsto T_G$ is Borel.

We will prove this in Section 7.

**Lemma 4.3.** $T_G$ is well-founded if and only if $G \in \mathcal{M}_{\text{max}}$.

**Proof.** If $G \in \mathcal{M}_{\text{max}}$, then $T_G$ contains no infinite branches by definition. If $G \notin \mathcal{M}_{\text{max}}$, then there is some infinitely generated subgroup $H \leq G$. There is some increasing sequence $a_0 < a_1 < \ldots$ of natural numbers such that $H = \langle g_{a_0}, g_{a_1}, \ldots \rangle$. We may assume that $\langle g_0, \ldots, g_{a_n} \rangle \leq \langle g_0, \ldots, g_{a_{n+1}} \rangle$ for all $n \in \mathbb{N}$. Then $(a_0, \ldots, a_n) \in T_G$ for all $n \in \mathbb{N}$, so $T_G$ has an infinite branch.

**Lemma 4.4.** Let $H, G \in \mathcal{G}$. If $H \hookrightarrow G$, then $\rho(T_H) \leq \rho(T_G)$.

**Proof.** Let $\psi: \mathbb{N} \rightarrow \mathbb{N}$ be the map such that $h_k = g_{\psi(k)}$. We will define a map $\phi$ from $T_H$. Let $\phi(\emptyset) = \emptyset$. If $s \in \mathbb{N}^{< \mathbb{N}}$ and $s = (s_0, \ldots, s_n)$, let $\phi(s) = (\psi(s_0), \ldots, \psi(s_n))$. Clearly $\phi$ is monotone. Further, if $s \in T_H$, then $H_s \cong G_{\phi(s)}$, so $\phi(T_H) \subseteq T_G$. By Lemma 2.8, $\rho(T_H) \leq \rho(T_G)$.

\[ \square \]
The previous lemma again implies \( \rho(T_G) \) is a group invariant.

**Corollary 4.5.** If \( G, G' \in \mathcal{G} \) and \( G \cong G' \), then \( \rho(T_G) = \rho(T_{G'}) \).

**Definition 4.6.** If \( G \) has the maximal condition on subgroups, then \( \rho(T_G) \) for some (any) marking of \( G \) is called the **subgroup rank** of \( G \).

**Lemma 4.7.** For all groups \( G \in \mathcal{M}_{\text{max}}, G \times \mathbb{Z} \in \mathcal{M}_{\text{max}} \) and \( \rho(T_G) < \rho(T_{G \times \mathbb{Z}}) \).

**Proof.** It is easy to see \( G \times \mathbb{Z} \) satisfies max. For the latter condition, let \( G = \{g_0, g_1, \ldots\} \) and \( G \times \mathbb{Z} = \{a_0, a_1, \ldots\} \). There is some \( k \in \mathbb{N} \) such that \( a_k = (e_G, z) \), where \( \mathbb{Z} = \langle z \rangle \). Let \( \psi: \mathbb{N} \to \mathbb{N} \) be defined such that \( a_{\psi(m)} = (g_m, e_z) \). The map \( \phi \) defined on \( T_G \) given by \( (s_0, \ldots, s_n) \mapsto (\psi(s_0), \ldots, \psi(s_n)) \) is clearly monotone, and further, \( \phi(T_G) \subseteq (T_{G \times \mathbb{Z}})_k \). By Lemma 2.8, \( \rho(T_G) \leq \rho((T_{G \times \mathbb{Z}})_k) < \rho(T_{G \times \mathbb{Z}}) \).

**Lemma 4.8.** Let \( \{A_i\}_{i \in \mathbb{N}} \) be countable groups. If \( A_i \in \mathcal{M}_{\text{max}} \) for all \( i \in \mathbb{N} \), then there is a group \( A \in \mathcal{M}_{\text{max}} \) such that \( \rho(T_A) \geq \rho(T_{A_i}) \) for all \( i \in \mathbb{N} \).

**Proof.** This is a consequence of [15] Theorem 2] of A. Y. Olshanskii, which gives a 2-generated group \( A \) containing each of the \( A_i \) such that every proper subgroup of \( A \) is either contained in a conjugate of some \( A_i \) or is infinite cyclic or infinite dihedral. Thus if every subgroup of each \( A_i \) is finitely generated, then every subgroup of \( A \) is finitely generated, and so \( A \in \mathcal{M}_{\text{max}} \). Since each \( A_i \) is a subgroup of \( A \), Lemma 4.4 implies that \( \rho(T_A) \geq \rho(T_{A_i}) \) for all \( i \in \mathbb{N} \), as desired.

**Lemma 4.9.** For all \( \alpha < \omega_1 \), there is \( G \in \mathcal{M}_{\text{max}} \) such that \( \rho(T_G) \geq \alpha \).

**Proof.** The proof is the same as that of Lemma 3.10 with Lemmas 4.7 and 4.8 referenced at the appropriate places.

**Theorem 4.10.** \( \mathcal{M}_{\text{max}} \) is \( \Pi_1^1 \) and not Borel in \( \mathcal{G} \) and \( \mathcal{G}_{fg} \).

**Proof.** The proof is the same as that of Theorem 3.12 using Lemmas 4.7 and 4.9 where appropriate. The statement is true for \( \mathcal{G}_{fg} \) simply because \( \mathcal{M}_{\text{max}} \subseteq \mathcal{G}_{fg} \).

5. **The maximal condition on normal subgroups**

Given a group \( G \) and a set \( S \subseteq G \), we write \( \langle \langle S \rangle \rangle_G \) to denote the normal closure of \( S \) in \( G \). We suppress the subscript \( G \) when the group is clear from context.

**Definition 5.1.** A group \( G \) satisfies the **maximal condition on normal subgroups**, abbreviated by saying a group satisfies max-n, if there is no strictly increasing chain \( H_0 < H_1 < H_2 < \ldots \) of normal subgroups of \( G \). Equivalently, a group \( G \) satisfies max-n if all of its normal subgroups are the normal closure of finitely many elements of \( G \). We denote the class of groups satisfying max-n as \( \mathcal{M}_n \).

Given \( G \in \mathcal{G} \), we construct a tree \( T_G \subseteq N^{<\mathbb{N}} \) and associated groups \( G_s \in \mathcal{G} \) for each \( s \in T_G \).

- Put \( \emptyset \in T_G \) and let \( G_\emptyset = G \).
- Suppose that \( s \in T_G \) and \( G_s = G/\langle\langle g_s \rangle \rangle \) has already been defined. If \( \langle\langle g_s \cup \{g_i\} \rangle \rangle \neq \langle\langle g_s \rangle \rangle \), then let \( s^{-1}i \in T_G \) and \( G_{s^{-1}i} = G/\langle\langle g_s \cup \{g_i\} \rangle \rangle \).

**Lemma 5.2.** The map \( \Phi_{\mathcal{M}_n}: \mathcal{G} \to \text{Tr} \) given by \( G \mapsto T_G \) is Borel.

We prove this in Section 7.
Lemma 5.3. $T_G$ is well-founded if and only if $G \in \mathcal{M}_n$.

Proof. If $G \in \mathcal{M}_n$, then $T_G$ contains no infinite branches by definition. If $G \notin \mathcal{M}_n$, then there is a normal subgroup $N \trianglelefteq G$ such that $N = \langle \langle g_{a_0}, g_{a_1}, \ldots \rangle \rangle$, with
\[
\langle \langle g_{a_0}, \ldots, g_{a_n} \rangle \rangle \leq \langle \langle g_{a_0}, \ldots, g_{a_{n+1}} \rangle \rangle
\]
for all $n \in \mathbb{N}$. Then $(a_0, \ldots, a_n) \in T_G$ for all $n \in \mathbb{N}$, so $T_G$ has an infinite branch. \qed

Lemma 5.4. If $G \in \mathcal{M}_n$, then $\rho(T_G/N) < \rho(T_G)$ for all nontrivial $N \trianglelefteq G$.

Proof. Since $G$ is max-n, $N = \langle \langle S \rangle \rangle$ for some finite $S = \{s_0, \ldots, s_n\}$. We may assume that $n$ is minimal, so no element of $S$ is in the normal closure of the others. Let $H = G/\langle \langle S \rangle \rangle$ and set $(s_0, \ldots, s_n) =: s \in \mathbb{N}^{<\mathbb{N}}$. Observe that $s \in T_G$ and let $\psi: \mathbb{N} \to \mathbb{N}$ be a map such that $h_k = g_{\psi(k)}\langle \langle S \rangle \rangle$. Then for all $i_0, \ldots, i_k \in \mathbb{N}$,
\[
H/\langle \langle h_{i_1}, \ldots, h_{i_k} \rangle \rangle H \cong G/\langle \langle g_{\psi(i_1)}, \ldots, g_{\psi(i_k)}, S \rangle \rangle G,
\]
and the monotone map $\phi$ on $T_H$ given by $(r_0, \ldots, r_n) \mapsto (\psi(r_0), \ldots, \psi(r_n))$ sends $T_H$ into $(T_G)_s$. By Lemma 2.8, $\rho(T_H) \leq \rho((T_G)_s) < \rho(T_G)$. \qed

We also find that this rank is isomorphism invariant.

Lemma 5.5. If $G, G' \notin \mathcal{G}$ and $G \cong G'$, then $\rho(T_G) = \rho(T_G')$.

Proof. If $G, G' \notin \mathcal{M}_n$, then $\rho(T_G) = \rho(T_G') = \omega_1$. Otherwise suppose that $f: G \to G'$ is an isomorphism. Define $\psi: \mathbb{N} \to \mathbb{N}$ to be a map such that $f(g_n) = g'_{\psi(n)}$. This gives rise to a monotone map from $T_G$ to $T_G'$, so $\rho(T_G) \leq \rho(T_G')$. The same argument with $f^{-1}$ gives that $\rho(T_G') \leq \rho(T_G)$. \qed

Recall that a group is hopfian if it is not isomorphic to any of its quotients. The following corollary is easy enough to prove directly, but it follows immediately from Lemmas 5.4 and 5.5.

Corollary 5.6. If $G \in \mathcal{M}_n$, then $G$ is hopfian.

Unlike the previous invariants, this rank has appeared before in the literature; cf. [3].

Definition 5.7. If $G$ has the maximal condition on normal subgroups, then $\rho(T_G)$ for some (any) marking of $G$ is called the length of $G$.

If we were to follow our template from previous sections, we would move on to analogs of Lemmas 3.8 and 3.9. However, we were unable to prove an analogue of Lemma 3.9 which would take advantage of Lemma 5.4. Such a result would be a sort of dual version of the result of Olshanskii cited in the proof of Lemma 4.8. Specifically, the following question is open to the best of the authors’ knowledge:

Question 5.8. Suppose $\{A_i\}_{i \in \mathbb{N}}$ is a set of normally $k$-generated max-n groups. Is there a max-n group $A$ such that $A \twoheadrightarrow A_i$ for all $i \in \mathbb{N}$?

A positive answer to this question would give us exactly the right analogue of Lemma 3.9. Lacking this, we will use a construction involving (restricted) wreath products. Recall the wreath product of $H$ and $G$ is $H \wr G := H^{<G} \rtimes G$ where $G \simeq H^{<G}$ by shift; in the case $G \simeq X$ for some set $X$, we write $H \wr X : = H^{<X} \rtimes G$. We will see that we can relate $\rho(T_{H\wr G})$ to both $\rho(T_H)$ and $\rho(T_G)$, while Lemma 5.4 alone only gives us information about how $\rho(T_{H\wr G})$ and $\rho(T_G)$ relate.
We will focus on perfect max-n groups with no central factors. A group $G$ is said to have a **central factor** if there are normal subgroups $L \trianglelefteq M$ in $G$ such that $M/L$ is nontrivial and central in $G/L$. Let us call the set of such groups $\mathcal{M}'_n$. Since $\mathcal{M}'_n \subseteq \mathcal{M}_n$, it is enough for our purposes to show that $\rho$ is unbounded below $\omega_1$ on $\mathcal{M}'_n$.

**Lemma 5.9.** Let $S$ be an infinite simple group. For all groups $G \in \mathcal{M}_n$, $G \times S \in \mathcal{M}_n$ and $\rho(T_G) < \rho(T_{G \times S})$. If $G \in \mathcal{M}'_n$, then so is $G \times S$.

**Proof.** It is easy to see that $G \times S \in \mathcal{M}_n$, and since $G$ is a quotient of $G \times S$, Lemma 5.4 implies that $\rho(T_G) < \rho(T_{G \times S})$. If $G$ is perfect, then $G \times S$ is perfect, so for the last statement we need only check that if $G$ has no central factors, then $G \times S$ has no central factors. Suppose that $L, M \trianglelefteq G \times S$ give a central factor. Let $\pi: G \times S \to G$ be the usual projection. Since $G$ has no central factors, $\pi(M) = \pi(L)$. Thus $MS = LS$, so $M = L(S \cap M)$. Since $S$ has no central factors, $S \cap M = S \cap L$. We conclude $M = L$, whereby $G \times S$ has no central factors. \hfill \Box

Lemma 5.9 allows us to find a group in $\mathcal{M}'_n$ with rank greater than a given group in $\mathcal{M}'_n$. However, we also need to be able to find a group in $\mathcal{M}'_n$ with rank greater than a countable family of groups from $\mathcal{M}'_n$. We begin by recalling a lemma from the literature.

**Lemma 5.10 ([2], Lemma 3.6).** Suppose $A, B$ are countable groups and form $G = A \wr B$. If $N \trianglelefteq G$ meets $B$ non-trivially, then $[A, A]^B \leq N$.

Next we look at properties of the ranks of wreath products.

**Lemma 5.11.** Suppose $H$ and $G$ are groups satisfying max-$n$. Then $\rho(T_{H \wr G}) \geq \rho(T_G) + \rho(T_H)$.

**Proof.** For each $h \in H$ define $f_h \in H^<G$ by

$$f_h(g) = \begin{cases} h, & \text{if } g = e \\ e, & \text{else.} \end{cases}$$

Let $H = \{h_0, h_1, \ldots\}$ and let $\psi: \mathbb{N} \to \mathbb{N}$ be a map such that $f_h = g_{\psi(i)}$. We now define a monotone $\phi$ from $T_H$. Put $\phi(\emptyset) = \emptyset$. For non-empty $s \in T_H$, define $\phi$ by

$$(s_0, \ldots, s_k) \mapsto (\psi(s_0), \ldots, \psi(s_k)).$$

We argue $\phi$ maps $T_H$ into $T_{H \wr G}$ by induction on the length of $s \in T_H$. As the base case is immediate, say $s \in T_H$ and $s^\sim k \in T_H$. By construction, we have $\langle \{h_s\} \cup \{h_k\}\rangle_H \neq \langle \{h_s\}\rangle_H$. For all $t \in T_H$, $\langle \{g_{\phi(t)}\}\rangle_{H \wr G} = \langle \{h_t\}\rangle_{H^<G}$ and, therefore, $\langle \{g_{\phi(s)}\} \cup \{g_{\psi(k)}\}\rangle_{H \wr G} \neq \langle \{g_{\phi(s)}\}\rangle_{H \wr G}$. Hence, $\phi(s^\sim k) \in T_{H \wr G}$, and $\phi$ maps $T_H$ into $T_{H \wr G}$.

Now if $s = (s_0, \ldots, s_n) \in T_H$ is a terminal node, then $\langle \{h_s\}\rangle_H = H$. Thus in this case $(H \wr G)/\langle \{g_{\phi(s)}\}\rangle_{H \wr G} \cong G$, so $\rho(T_G) = \rho((T_H \wr G)_{\phi(s)})$ by Lemma 5.5. The result follows. \hfill \Box

**Lemma 5.12.** If $G$ and $H$ have no central factors, then $H \wr G$ has no central factors.

**Proof.** Suppose $L \trianglelefteq M$ gives a central factor of $H \wr G$. Let $\pi: H \wr G \to G$ be the usual projection. Since $G$ has no central factors, we have $\pi(L) = \pi(M)$ and $LH^<G = MH^<G$. Thus, $M = L(H^<G \cap M)$, and it suffices to show $H^<G \cap M \leq L$. Since $H$ has no central factors, it follows similarly to the proof of Lemma 5.9 that $H^F \cap M = H^F \cap L$ for all finite $F \subseteq G$. Hence, $H^<G \cap M \leq L$ verifying the lemma. \hfill \Box

An old theorem of P. Hall is the last thing we require.
**Theorem 5.13** (Hall [7, Theorem 4]). Let $H$ and $G$ be groups satisfying max-$n$. If $H$ has no central factors, then $H \triangleleft G$ satisfies max-$n$.

**Lemma 5.14.** Let $\{A_i\}_{i \in \mathbb{N}}$ be countable groups. If $A_i \in \mathcal{M}_n'$ for all $i \in \mathbb{N}$, then there is a group $A \in \mathcal{M}_n'$ such that $\rho(T_{A_i}) \leq \rho(T_A)$ for all $i \in \mathbb{N}$.

*Proof.* For each $n$, put $G_n := A_n \lhd (\cdots \lhd A_0)$. By making the natural identification, we may assume $G_n \leq G_{n+1}$ for all $n$ and form $A = \bigcup_{n \in \mathbb{N}} G_n$. (Alternatively, one may take the direct limit.) Consider $N \leq A$. Certainly, $N \cap G_n$ is non-trivial for some $n$. Fix such an $n$ and take $k > n$. We now see that $N \cap G_k \leq G_k = A_k \lhd G_{k-1}$ is a normal subgroup that meets $G_{k-1}$ non-trivially. Applying Lemma 5.10, $[A_k, A_k]^{G_{k-1}} \leq N \cap G_k$. Since $A_k$ is perfect, we have $A_k^{G_{k-1}} \leq N$, and it follows $A/N$ is isomorphic to a quotient of $G_n$.

Suppose $(N_i)_{i \in \mathbb{N}}$ is an increasing sequence of normal subgroups of $A$. By the previous paragraph, $A/N_0$ is a quotient of $G_n$ for some $n$. Theorem 5.13 implies that each $G_n$ is a max-$n$ group. Therefore, letting $\pi : A \to A/N_0$ be the usual projection, it must be the case $\pi(N_i) = \pi(N_j)$ for all sufficiently large $i$ and $j$. We conclude that $N_i = N_j$ for all sufficiently large $i$ and $j$ and $G$ satisfies max-$n$.

For each $n$ and $k > n$, define

$$L^i_k = A_k \lhd \cdots \lhd A_{n+2} \lhd A_{n+1} A^{G_n}_{n+1}$$

and put $L^n := \bigcup_{k \geq n} L^n_k$. We see $L^n \leq A$ and $A/L^n \simeq G_n$. By Lemmas 5.11 and 5.4, $\rho(T_{A_n}) \leq \rho(T_{G_n}) \leq \rho(T_A)$ for all $n$.

We finally verify $A$ is perfect and has no central factors. That $A$ is perfect is immediate. It follows from Lemma 5.12 and induction that each $G_n$ has no central factors. Since any factor of $G$ is a factor of $G_n$ for some $n$, it is the case $G$ has no central factors. \hfill $\square$

**Lemma 5.15.** For all $\alpha < \omega_1$, there is $G \in \mathcal{M}_n'$ such that $\rho(T_G) \geq \alpha$.

*Proof.* The proof is the same as that of Lemma 3.10 with Lemmas 5.9 and 5.14 referenced at the appropriate places. \hfill $\square$

The groups given by Lemma 5.15 are not, in general, finitely generated. For finding finitely generated examples, another result of Hall is needed.

**Lemma 5.16** (Hall, [8, cf. Theorem 4]). Let $H$ be a countable group. Then there exists a short exact sequence

$$\{e\} \to M \to G \to \mathbb{Z} \to \{e\}$$

where $G$ is 2-generated, $[M, M] = [H, H]^{\mathbb{Z}}$, and there is $t \in G$ so that the conjugation action of $t$ on $[M, M]$ is by unit shift.

**Corollary 5.17.** For each $\alpha < \omega_1$, there is a finitely generated group $G \in \mathcal{M}_n$ with $\rho(T_G) \geq \alpha$.

*Proof.* Fix $\alpha < \omega_1$ and apply Lemma 5.15 to find a group $H \in \mathcal{M}_n$ with $\rho(T_H) \geq \alpha$. We now apply Lemma 5.16 to find a 2-generated group $G$ with a short exact sequence

$$\{e\} \to M \to G \to \mathbb{Z} \to \{e\}$$

where $[M, M] = [H, H]^{\mathbb{Z}} = H^{\mathbb{Z}}$.

We see that $G/[M, M]$ is a finitely generated metabelian group, hence it satisfies max-$n$ by [7, Theorem 3]. On the other hand, any normal subgroup of $G$ that lies in $[M, M]$ is shift-invariant. Since $H \triangleleft \mathbb{Z}$ is max-$n$, it follows that $H^{\mathbb{Z}}$ is max-$G$; that is to say $H^{\mathbb{Z}}$ is shift-invariant.
has the maximal condition on subgroups invariant under the conjugation action by $G$. We conclude $G$ is max-$n$.

It remains to compute a lower bound for $\rho(T_G)$. Using again the notation from Lemma 5.11 find $\psi : \mathbb{N} \rightarrow \mathbb{N}$ such that for each $k \in \mathbb{N}$ we have $f_{h_k} = g_{\psi(k)}$. Arguing as in Lemma 5.11 we can define a monotone $\phi : T_H \rightarrow T_G$ and by Lemma 2.8 conclude $\alpha \leq \rho(T_H) \leq \rho(T_G)$. □

**Theorem 5.18.** $\mathcal{M}_n$ is $\Pi_1^1$ and not Borel in $\mathcal{G}$, and $\mathcal{M}_n \cap G_{fg}$ is $\Pi_1^1$ and not Borel in $G_{fg}$.

**Proof.** This follows from Theorem 2.7, Lemma 5.2, and Corollary 5.17. □

### 6. Elementary Amenable Groups

Perhaps surprisingly, the property of being elementary amenable may also be characterized by well-founded trees. This in turn gives a chain condition equivalent to elementary amenability.

#### 6.1. Preliminaries

We study the collection of elementary amenable groups. This class is typically defined as follows:

**Definition 6.1.** The collection of **elementary amenable groups**, denoted $EG$, is the smallest collection of countable groups such that

(i) $EG$ contains all finite and abelian groups.
(ii) $EG$ is closed under group extensions.
(iii) $EG$ is closed under countable increasing unions.
(iv) $EG$ is closed under taking subgroups.
(v) $EG$ is closed under taking quotients.

By a result of C. Chou [2, Proposition 2.2], the class of elementary amenable groups is the smallest class of countable discrete groups that satisfies (i),(ii), and (iii) of Definition 6.1. Chou’s theorem suggests a natural ranking of elementary amenable groups. Indeed, define

- $G \in EG_0$ if and only if $G$ is finite or abelian.
- Suppose $EG_\alpha$ is defined. Put $G \in EG_\alpha'$ if and only if there exists $N \subseteq G$ such that $N \in EG_\alpha'$ and $G/N \in EG_\alpha$. Put $G \in EG_\lambda$ if and only if $G = \bigcup_{i \in \mathbb{N}} H_i$ where $(H_i)_{i \in \mathbb{N}}$ is an $\subseteq$-increasing sequence of subgroups of $G$ and $H_i \in EG_\alpha$ for each $i \in \mathbb{N}$. Set $EG_{\alpha+1} := EG_\alpha' \cup EG_\alpha'$.
- For $\lambda$ a limit ordinal, $EG_\lambda := \bigcup_{\beta < \lambda} EG_\beta$.

By a result of D. Osin [17, Lemma 3.2], $\bigcup_{\alpha < \omega_1} EG_\alpha$ is closed under group extension. It now follows from Chou’s theorem $EG = \bigcup_{\alpha < \omega_1} EG_\alpha$, so one may define for $G \in EG$

$$rk(G) := \min\{\alpha \mid G \in EG_\alpha\}.$$ We call $rk(G)$ the **construction rank** of $G$. The construction rank is well behaved with respect to subgroups and quotients.

**Proposition 6.2** (Osin [17, Lemma 3.1]). *For $G \in EG$,*

1. If $H \subseteq G$, then $rk(H) \leq rk(G)$.
2. If $L \trianglelefteq G$, then $rk(G/L) \leq rk(G)$.

Our results here require a fairly well-known embedding result, which generalizes Lemma 5.16.

**Proposition 6.3** (Hall, Neumann, Neumann [14, Theorem 5.1]). *Suppose $K \in EG$. Then there exists $H \in EG$ and a short exact sequence*

$$\{e\} \rightarrow M \rightarrow G \rightarrow \mathbb{Z} \rightarrow \{e\}.$$
where $G$ is 2-generated, $G \in \mathcal{EG}$, $[M, M] = [H, H]_{< \mathbb{Z}}$, and $K$ embeds into $[H, H]$.

We note one additional, easy lemma.

**Lemma 6.4.** If $H$ is a group and $K \leq H_{< \mathbb{Z}}$ is a finite index subgroup, then there is $N > 0$ with $\pi(N, \infty)(K) = H_{< \mathbb{N}}$, where $\pi(N, \infty)$ is the projection onto the coordinates greater than or equal to $N$.

**Proof.** Suppose $K \leq H_{< \mathbb{Z}}$ is a finite index subgroup. Let $f_1, \ldots, f_k$ be left coset representatives for $K$ in $H_{< \mathbb{Z}}$. We may find $N > 0$ so that $f_i(n) = e$ for all $n \geq N$ and $1 \leq i \leq k$. Letting $\pi_{[N, \infty)}$ be the projection onto the coordinates greater than or equal to $N$, it must be the case that $\pi_{[N, \infty)}(K) = H_{< \mathbb{N}}$. \hfill \Box

### 6.2. Decomposition trees

We now define a tree associated to a marked group $G$. Just as in the previous sections, this tree being well-founded or not gives group-theoretic information about $G$, in this case characterizing being an elementary amenable group.

Let $G \in \mathcal{G}$. For $n \geq 0$, put $R_n(G) := \langle g_0, \ldots, g_n \rangle$. For $k \geq 1$, define

$$S_k(G) := [G, G] \cap \bigcap_n N_k(G)$$

where $N_k(G) := \{ N \leq G \mid [G : N] \leq k+1 \}$. For each $l \geq 1$, we now define a tree $T^l(G) \subseteq \mathbb{N}^{< \mathbb{N}}$ and associated groups $G_s \in \mathcal{G}$ as follows:

- Put $\emptyset \in T^l(G)$ and let $G_\emptyset := G$.
- Suppose we have $s \in T^l(G)$ and $G_s$. If $G_s \neq \{ e \}$, put $s^{-1}n \in T^l(G)$ and $G_{s^{-1}n} := S_{|s|+l} (R_n(G_s))$.

We call $T^l(G)$ the **decomposition tree** of $G$ with offset $l$. This tree is always non-empty and a marked group corresponding to a terminal node is the trivial group. Since the composition of the functions $R_n$ and $S_k$ is associative, we obtain a useful observation:

**Observation 6.5.** For $s \in T^l(G)$, $T^l(G)_s = T^{|s|+l}(G_s)$, and for each $r \in T^{|s|+l}(G_s)$, $(G_s)_r = G_{s^{-1}r}$ as marked groups. This implies in particular that if $T^l(G)$ is well-founded, then so is $T^{|s|+l}(G_s)$.

**Lemma 6.6.** The map $\Phi^l : \mathcal{G} \rightarrow \text{Tr}$ given by $G \mapsto T^l(G)$ is Borel for all $l \geq 1$.

As usual, we postpone the proof to Section 7.

**Theorem 6.7.** If $G \in \mathcal{G}$ and $G \in \mathcal{EG}$, then $T^l(G)$ is well-founded for all $l \geq 1$.

**Proof.** We proceed by induction on $\text{rk}(G)$. If $\text{rk}(G) = 0$, then $G$ is finite or abelian. If $G$ is abelian, then $T^l(G) \subseteq \mathbb{N}^2$ for all $l \in \mathbb{N}$ and $T^l(G)$ is well-founded. If $G$ is finite and $|G| = k$, then $T^l(G) \subseteq \mathbb{N}^{k+1}$ for all $l \in \mathbb{N}$. Thus the theorem holds if $\text{rk}(G) = 0$.

Suppose that $G \in \mathcal{EG}$ and $\text{rk}(G) > 0$. Suppose $G = \bigcup_{i \in \mathbb{N}} H_i$ with $\text{rk}(H_i) < \text{rk}(G)$. Since each $R_k(G)$ is finitely generated, for each $k \in \mathbb{N}$ there is some $n_k \in \mathbb{N}$ such that $R_k(G) \leq H_{n_k}$. It follows that $\text{rk}(G_k) \leq \text{rk}(H_{n_k}) < \text{rk}(G)$ for all $k \in \mathbb{N}$. By the inductive hypothesis, $T^{l+1}(G_k)$ is well-founded for all $k$. Observation 6.5 now implies $T^l(G)_k$ is well-founded for all $k \in \mathbb{N}$, and it follows that $T^l(G)$ is well-founded.

Suppose that $G \in \mathcal{EG}$ and $H \leq G$ is such that $\text{rk}(H) < \text{rk}(G)$ and $G/H$ is abelian. Then $[G, G] \leq H$, so $G_k \leq H$ for all $k \in \mathbb{N}$, and so $\text{rk}(G_k) < \text{rk}(G)$ for all $k \in \mathbb{N}$. As before, the inductive hypothesis and Observation 6.5 give that $T^l(G)$ is well-founded.
Suppose now that \( G \in \mathcal{G} \) and \( H \leq G \) is such that \( \text{rk}(H) < \text{rk}(G) \) and \( G/H \) is finite. Let \( m \geq 0 \) be such that \( m + l \geq |G : H| \). Since for all \( s \in \mathbb{N}_m \) and all \( k \in \mathbb{N} \)
\[
|R_k(G_s) : H \cap R_k(G_s)| \leq m + l,
\]
we have
\[
G_{s^\alpha} = S_{m+1}(R_k(G_s)) \leq H \cap G_s.
\]
It follows that \( \text{rk}(G_{s^\alpha}) \leq \text{rk}(H) < \text{rk}(G) \). By our inductive hypothesis, \( T_{s^\alpha}^{k} \) is well-founded for all \( k \). Observation \( \ref{cor:s-flip} \) implies \( T^l(G_{s^\alpha}) \) is well-founded for all \( k \in \mathbb{N} \) and \( s \in \mathbb{N}_m \), hence \( T^l(G) \) is well-founded.

**Theorem 6.8.** If \( G \in \mathcal{G} \) and \( T^l(G) \) is well-founded for some \( l \geq 1 \), then \( G \in \mathcal{G} \).

**Proof.** We induct on the minimal rank as a well-founded tree of \( T^l(G) \) for any \( l \) such that \( T^l(G) \) is well-founded. Call this minimal rank \( \xi(G) \). For the base case, if \( \xi(G) = 1 \), then \( G = \{e\} \) and \( G \in \mathcal{G} \). Suppose the theorem holds for all \( \alpha < \beta \) and \( \xi(G) = \rho(T^l(G)) = \beta \). Consider \( R_i(G) \). Since \( R_i(G) \) is finitely generated, \( N_{i+1}(R_i(G)) \) is finite, and
\[
[|R_i(G), R_i(G)| : G_i] < \infty.
\]
We thus have that \( R_i(G)/G_i \) is finite-by-abelian and, therefore, elementary amenable.

On the other hand, Observation \( \ref{cor:s-flip} \) gives \( \rho(T^{l+1}(G_i)) = \rho(T^l(G_i)) \). Hence, \( \rho(T^{l+1}(G_i)) < \beta \), and we conclude that \( G_i \in \mathcal{G} \) by the inductive hypothesis. As \( \mathcal{G} \) is closed under group extensions and countable increasing unions, \( R_i(G) \in \mathcal{G} \) for all \( i \in \omega \), whereby \( G \in \mathcal{G} \).

Combining these results, we obtain a characterization of elementary amenable groups.

**Theorem 6.9.** Let \( G \) be a marked group. Then the following are equivalent:

(1) \( G \in \mathcal{G} \).

(2) \( T^l(G) \) is well-founded for all \( l \geq 1 \).

(3) \( T^l(G) \) is well-founded for some \( l \geq 1 \).

We can rephrase this to have the form of a chain condition that is independent of the marking. This corollary may thus be taken to be a definition of elementary amenability.

**Corollary 6.10.** A countable group \( G \) is elementary amenable if and only if there is no infinite descending sequence of the form
\[
G = G_0 \geq G_1 \geq \ldots
\]
such that for all \( n \geq 0 \), \( G_n \neq \{e\} \) and there is a finitely generated subgroup \( K_n \leq G_n \) with \( G_{n+1} = [K_n, K_n] \cap H_n \) where \( H_n \) is the intersection of the index-(\( \leq n+1 \)) normal subgroups of \( K_n \).

**Proof.** Suppose \( G \) is elementary amenable, assume \( G \) has some marking, and let
\[
G = G_0 \geq G_1 \geq \ldots
\]
be a sequence as hypothesized. Form \( T^1(G) \), the decomposition tree of \( G \) with offset 1. We now proceed by induction to build \( s_0 \subseteq s_1 \subseteq \ldots \) with \( s_i \in T^1(G) \) and \( |s_i| = i \) such that \( G_i \hookrightarrow G_{s_i} \). The base case is immediate: set \( s_0 = \emptyset \). Suppose we have defined \( s_n \), so \( G_n \hookrightarrow G_{s_n} \). Let \( K_n \leq G_n \) be such that \( G_{n+1} = [K_n, K_n] \cap H_n \) where \( H_n \) is the intersection of the index-(\( \leq n+1 \)) normal subgroups of \( K_n \). Since \( K_n \) is finitely generated, there is \( R_m(G_{s_n}) \) such that \( K_n \hookrightarrow R_m(G_{s_n}) \). We conclude \( G_{n+1} \hookrightarrow G_{s_{n+m}} \). Setting \( s_{n+1} = s_n m \), we have verified the inductive hypothesis.
By Theorem 6.9, the sequence \( s_0 \subset s_1 \subset \ldots \) must eventually terminate, hence there is \( s_n \) such that \( G_{s_n} = \{ e \} \). It follows that the sequence \( G_0 \geq G_1 \geq \ldots \) is finite.

Suppose there are no infinite descending sequences as specified and form \( T^1(G) \). Let \( s_0 \subset s_1 \subset \ldots \) with \( s_i \in T^1(G) \) and \( |s_i| = i \). It suffices to show \( s_0 \subset s_1 \subset \ldots \) terminates. But this is obvious since by construction the sequence of subgroups \( G_{s_0} \geq G_{s_1} \geq \ldots \) is a sequence of subgroups as in the chain condition.

There are two main differences between this and the chain conditions explored in the earlier sections of this paper. First of all, \( G_{n+1} \) is not related to \( G_n \) only by being a subgroup. This is not unheard of; for example when looking at weak chain conditions one requires that \( G_{n+1} \) be an infinite index subgroup of \( G_n \). The second difference is that the definition of \( H_n \) changes with \( n \). As far as we are aware, there are no widely-studied chain conditions defined in this way. That elementary amenability can be recast this way suggests that perhaps there are other interesting chain conditions with this property.

6.3. EG is not Borel. We now study the descriptive-set-theoretic properties of \( EG \). We first isolate two new invariants.

**Lemma 6.11.** Let \( G, H \in \mathcal{G} \) and \( H \hookrightarrow G \). Then for all \( l \geq k \geq 1 \),

\[
\rho\left( T^l(H) \right) \leq \rho\left( T^k(G) \right).
\]

In particular, for \( G, G' \in \mathcal{G} \), if \( G \cong G' \), then

\[
\rho\left( T^l(G) \right) = \rho\left( T^l(G') \right).
\]

**Proof.** We induct on \( \rho(T^k(G)) \) simultaneously for all \( k \). If \( \rho(T^k(G)) = 1 \), then \( G = \{ e \} \), so \( H = \{ e \} \). Suppose the lemma holds for all \( G \) and \( k \) with \( \rho(T^k(G)) < \beta \). Suppose that \( f : H \rightarrow G \) is an embedding. Then for all \( n \geq 0 \), there is some \( k(n) \) so that \( f(R_n(H)) \leq R_{k(n)}(G) \). It follows that \( f(H_n) \leq G_{k(n)} \) for all \( n \geq 0 \) since \( S_{l+1}(G_{k(n)}) \leq S_{k+1}(G_{k(n)}) \). By the inductive hypothesis and Observation 6.5

\[
\rho\left( T^l(H) \right) = \sup_{n \in \mathbb{N}} \left\{ \rho\left( T^{l+1}(H_n) \right) \right\} + 1 \leq \sup_{n \in \mathbb{N}} \left\{ \rho\left( T^{k+1}(G_{k(n)}) \right) \right\} + 1 \leq \rho\left( T^k(G) \right)
\]

The lemma shows the rank of a decomposition tree is independent of the marking. We thus define

**Definition 6.12.** The decomposition rank of \( G \in EG \) is defined to be

\[
\xi(G) := \min_{k \in \omega} \rho\left( T^k(G) \right)
\]

for some (any) marking of \( G \). The decomposition degree is defined to be

\[
\deg(G) = \min \left\{ k \mid \xi(G) = \rho\left( T^k(G) \right) \right\}
\]

for some (any) marking of \( G \).

**Corollary 6.13.** If \( G, H \in \mathcal{G} \) and \( H \hookrightarrow G \), then \( \xi(H) \leq \xi(G) \).

We also see the decomposition rank is well-behaved with respect to quotients.
**Lemma 6.14.** If $G \in \text{EG}$ and $L \trianglelefteq G$, then $\rho\left(T^k(G/L)\right) \leq \rho\left(T^k(G)\right)$ for all $k \geq 1$. In particular, $\xi(G/L) \leq \xi(G)$.

*Proof.* We argue by induction on $\rho\left(T^k(G)\right)$ simultaneously for all $k$. As the base case is immediate, suppose the lemma holds for $\beta$ and $\rho\left(T^k(G)\right) = \beta + 1$. Let $G_i$ be the subgroup corresponding to $i \in T^k(G)$. By the inductive hypothesis,

$$
\rho\left(T^{k+1}(G_i/G_i \cap L)\right) \leq \rho\left(T^{k+1}(G_i)\right).
$$

On the other hand, form $T^k(G/L)$ and let $(G/L)_i$ be the corresponding subgroups. Since the $R_i(G) \trianglelefteq G$ exhaust $G/L$, for each $i$ there is $n(i)$ so that $(G/L)_i \hookrightarrow G_{n(i)}/G_{n(i)} \cap L$. Lemma 6.11 now implies

$$
\rho\left(T^k(G/L)\right) = \sup_{i \in \mathbb{N}} \rho\left(T^{k+1}((G/L)_i)\right) + 1 \leq \sup_{i \in \mathbb{N}} \rho\left(T^{k+1}(G_i)\right) + 1 = \rho\left(T^k(G)\right)
$$

finishing the induction. \qed

The decomposition rank in a fairly straightforward manner tracks the number of extensions and unions applied to produce the group. Indeed, later we shall see the decomposition rank is closely related to the construction rank. The decomposition degree, on the other hand, is currently mysterious. It somehow tracks the size of the finite groups “appearing” in the construction of a elementary amenable groups.

We do not consider the decomposition degree further as it is tangential to our goal. We do study the decomposition rank in detail. Most importantly, we show that on $\text{EG}$ the decomposition rank is unbounded below $\omega_1$.

**Lemma 6.15.** Suppose $G \in \text{EG}$ is non-trivial and $k \geq 1$ is such that $\rho\left(T^k(G)\right) = \xi(G)$. Then

$$
\sup_{i \in \omega} \xi(G_i) + 1 \leq \xi(G)
$$

where $G_i$ is the subgroup of $G$ associated to $i \in T^k(G)$.

*Proof.* By construction, for all $i \in \mathbb{N}$,

$$
\rho\left(T^{k+1}(G_i)\right) + 1 = \rho\left(T^k(G_i)\right) + 1 \leq \rho\left(T^k(G)\right).
$$

Hence,

$$
\sup_{i \in \mathbb{N}} \xi(G_i) + 1 = \sup_{i \in \mathbb{N}} \left\{ \min_{l \in \mathbb{N}} \rho\left(T^l(G_i)\right) \right\} + 1 \leq \sup_{i \in \mathbb{N}} \left\{ \rho\left(T^{k+1}(G_i)\right) \right\} + 1 = \rho\left(T^k(G)\right) = \xi(G)
$$

as desired. \qed

The inequality in Lemma 6.15 may be strict; for example, consider $\text{Sym}_{\text{fin}}(\mathbb{N})$, the group of finitely supported permutations of $\mathbb{N}$. We also point out that Lemma 6.11 does not hold for choices of $k$ such that $\rho(T^k(G)) \neq \xi(G)$. 

Lemma 6.16. For every $K \in \text{EG}$, there is $G \in \text{EG}$ with $\xi(K) < \xi(G)$.

Proof. Let $G \in \text{EG}$ be as given by Proposition 6.3 for $K$, let $k = \deg(G)$, and take $G_i$ to be the subgroup of $G$ corresponding to $i \in T^k(G)$. Since $G$ is finitely generated, we may find $n$ such that $G = R_n(G)$, so $G_n \preceq G$.

Letting $M$ and $H$ be as given by Proposition 6.3, $[H, H] < Z = [M, M] \preceq [G, G]$. Since $G_n \cap [M, M]$ is finite index normal subgroup of $[M, M] = [H, H] < Z$, Lemma 6.4 implies $G_n \cap [M, M]$ has a quotient isomorphic to $[H, H] < N$. We have that $K$ embeds into $[H, H]$, hence Lemma 6.13 and Lemma 6.14 together give $\xi(K) \leq \xi(G_n \cap [M, M]) \leq \xi(G_n)$. In view of Lemma 6.15, we see $\xi(K) < \xi(G)$ proving the lemma.

Our next lemma follows immediately from Corollary 6.13 by taking the direct sum.

Lemma 6.17. Let $\{A_i\}_{i \in \mathbb{N}}$ be countable groups. If $A_i \in \text{EG}$ for all $i \in \mathbb{N}$, then there is $A \in \text{EG}$ such that $\xi(A) \geq \xi(A_i)$ for all $i \in \mathbb{N}$.

Lemma 6.18. For all $\beta < \omega_1$, there is $G \in \text{EG}$ such that $\xi(G) \geq \beta$.

Proof. The proof is the same as that of Lemma 6.10 with Lemmas 6.16 and 6.17 referenced at the appropriate places.

Lemma 6.19. For each $\beta < \omega_1$, there is a finitely generated $G \in \text{EG}$ such that $\xi(G) \geq \beta$.

Proof. Let $H \in \text{EG}$ be a group such that $\xi(H) \geq \beta$. Then Proposition 6.3 implies that $H$ embeds into a 2-generated group $G \in \text{EG}$. By Corollary 6.13, $\xi(G) \geq \xi(H) \geq \beta$.

Theorem 6.20. $\text{EG}$ is a non-Borel $\Pi_1^1$ set in $\mathcal{G}$, and $\text{EG} \cap \mathcal{G}_{fg}$ is a non-Borel $\Pi_1^1$ set in $\mathcal{G}_{fg}$.

Proof. This follows from Theorem 2.7, Lemma 6.6 and Lemma 6.19 along with the facts that $\xi(G) \leq \rho(T^1(G))$ and $\rho \circ \Phi^1$ is a $\Pi_1^1$ rank on $\text{EG}$.

Let $\text{AG} \subseteq \mathcal{G}$ denote the class of countable amenable groups. Via Theorem 6.20, we now may give a non-constructive answer to an old question of Day [4], which was open until Grigorchuk [6] constructed groups of intermediate growth: Is it the case that every amenable group is elementary amenable?

Corollary 6.21. There is a finitely generated amenable group that is not elementary amenable.

Proof. It is well-known that the set $\text{AG}$ is Borel; see Lemma 7.5 for a proof. Hence $\text{AG} \cap \mathcal{G}_{fg}$ is Borel. On the other hand, Theorem 6.20 implies $\text{EG} \cap \mathcal{G}_{fg}$ is not Borel. We conclude that $\text{EG} \cap \mathcal{G}_{fg} \nsubseteq \text{AG} \cap \mathcal{G}_{fg}$.

6.4. Further observations. We first compare $\xi$ and $\text{rk}$ and in the process mostly recover a theorem of Olshanskii and Osin. To compute a lower bound for $\xi$ in terms of $\text{rk}$, a subsidiary lemma is required.

Proposition 6.22. For $G \in \text{EG}$, $\text{rk}(G) \leq 3\xi(G)$. 
Proof. We induct on $\xi(G)$ for the proposition. For the base case, if $\xi(G) = 1$, then $G = \{e\}$, and the inductive hypothesis obviously holds. Suppose the inductive hypothesis holds up to $\beta$. Say $\xi(G) = \beta + 1$ and $\deg(G) = k$. Then $\xi(G_i) \leq \beta$ for each $G_i$ associated to $i \in T^k(G)$, and applying the inductive hypothesis, $\text{rk}(G_i) \leq 3\xi(G_i)$.

On the other hand, $R_i(G)/G_i$ is finite-by-abelian, say an extension of the group $F$ by the group $A$. Let $F_0$ be the inverse image of $F$ in $R_i(G)$ under the canonical homomorphism. Note that $\text{rk}(F_0) \leq \text{rk}(G_i) + 1$, and that $R_i(G)/F_0 \cong A$. Hence,

$$\text{rk}(R_i(G)) \leq (\text{rk}(G_i) + 1) + 1 \leq 3\xi(G_i) + 2.$$ We conclude

$$\text{rk}(G) \leq \sup_{i \in \mathbb{N}} (3\xi(G_i) + 2) + 1 \leq 3(\beta + 1) = 3\xi(G).$$

This finishes the induction, and we have the proposition.

Bounding $\xi$ from above by $\text{rk}$ involves a bit more work. We begin with a general lemma for well-founded trees.

**Lemma 6.23.** Suppose $T$ is a well-founded tree and $\tau \in T$ has length $k$. Then

$$\rho(T) \leq \sup_{|s|=k} \rho(T_s) + k.$$

**Proof.** We argue by induction on $|s|$. For the base case, $|s| = 1$,

$$\rho(T) = \rho_T(\emptyset) + 1 = \sup_{i \in T} (\rho_T(i) + 1) + 1 = \sup_{i \in \mathbb{N}} \rho(T_i) + 1.$$

Supposing the lemma holds up to length $k$,

$$\rho(T) \leq \sup_{|s|=k} \rho(T_s) + k \leq \sup_{|s|=k} \left( \sup_{s^{-1} \in T} \rho(T_{s^{-1}}) + 1 \right) + k \leq \sup_{|s|=k+1} \rho(T_s) + k + 1$$

completing the induction. □

**Proposition 6.24.** For $G \in EG$,

$$\rho(T^1(G)) \leq \omega(\text{rk}(G) + 1).$$

In particular, $\xi(G) \leq \omega(\text{rk}(G) + 1)$.

**Proof.** We argue by induction on $\text{rk}(G)$. For the base case, $\text{rk}(G) = 0$, $G$ is either finite or abelian. So there is $m \geq 1$ such that every element of $T^1(G)$ has length at most $m$. It follows $\rho(T^1(G))$ is finite, which proves the base case.

Suppose the lemma holds up to $\alpha$ and $\text{rk}(G) = \alpha + 1$. Let us consider first the case that the construction rank is given by a countable increasing union; say $G = \bigcup_{n \in \omega} H_n$ with $\text{rk}(H_n) \leq \alpha$ for each $n$. Since $R_i(G)$ is finitely generated, there is $n(i)$ for which $G_i \leq H_{n(i)}$. We apply the inductive hypothesis and Lemma 6.11 to conclude

$$\rho(T^2(G_i)) \leq \rho(T^1(G_i)) \leq \omega(\alpha + 1).$$

Hence,

$$\rho(T^1(G)) = \sup_{i \in \omega} \rho(T^2(G_i)) + 1 \leq \omega \cdot \alpha + \omega + 1 \leq \omega(\alpha + 2),$$

verifying the hypothesis in this case.

We now consider the case $\text{rk}(G)$ is given by a group extension. Suppose $H \leq G$ is such that $\text{rk}(H) = \alpha$ and $\text{rk}(G/H) = 0$. If $G/H$ is abelian, $G_i \leq H$ for each $i$. Hence, $\text{rk}(G_i) \leq \alpha$,
and the desired result follows just as in the increasing union case. Suppose $G/H$ is finite. We may find $k$ such that for all $s \in T^1(G)$ with $|s| = k$, $G_s \leq H$. Applying the inductive hypothesis and Lemma 6.11
$$
\rho \left( T^{k+1}(G_s) \right) \leq \rho \left( T^1(G_s) \right) \leq \omega(\alpha + 1).
$$

Lemma 6.23 now implies
$$
\rho \left( T^1(G) \right) \leq \sup_{|s|=k} \rho \left( T^1(G_s) \right) + k \leq \omega(\alpha + 1) + k \leq \omega(\alpha + 2).
$$

This completes the induction, and we conclude the proposition. \qed

As a corollary to Lemma 6.19 and Proposition 6.24 we obtain a less detailed version of a theorem from the literature.

**Corollary 6.25** (Olshanskii, Osin [16, Corollary 1.6]). For every ordinal $\alpha < \omega_1$, there is $G \in \text{EG} \cap \mathcal{G}_{fg}$ such that $\alpha \leq \text{rk}(G)$. Thus the function $\text{rk}: \text{EG} \cap \mathcal{G}_{fg} \to \text{ORD}$ is unbounded below $\omega_1$.

**Proof.** Suppose for contradiction $\alpha < \omega_1$ is such that $\text{rk}(G) < \alpha$ for all $G \in \text{EG}$. By Proposition 6.24 $\xi(G) \leq \omega(\alpha + 1) < \omega_1$ for all $G \in \text{EG}$ contradicting Lemma 6.19. \qed

In our proof of Theorem 6.20 we use that $\rho \circ \Phi^1$ is a $\Pi^1_1$-rank. It is natural to ask if $\xi$ itself is a $\Pi^1_1$-rank. This is indeed the case.

**Theorem 6.26.** The decomposition rank is a $\Pi^1_1$-rank on $\text{EG}$.

**Proof.** Each of the ranks $\phi_l := \rho \circ \Phi^l$ is a $\Pi_1^1$-rank on $\text{EG}$ where $\Phi^l$ is as defined in Lemma 6.6. Let $\leq_H \subseteq \mathcal{G} \times \mathcal{G}$ and $\leq_\Sigma :\subseteq \mathcal{G} \times \mathcal{G}$ be the relations given by $\phi_l$ a $\Pi^1_1$-rank. We now consider the following relations:

$$
\leq^H_\xi := \bigcup_{N \in \mathbb{N}, \, l \geq N} \leq^H_l \quad \text{and} \quad \leq^\Sigma_\xi := \bigcup_{N \in \mathbb{N}, \, l \geq N} \leq^\Sigma_l
$$

Since co-analytic and analytic sets are closed under countable unions and intersections, $\leq_\xi^H$ is co-analytic and $\leq_\xi^\Sigma$ is analytic. To conclude $\xi$ is a $\Pi^1_1$-rank, it thus remains to show for $H \in \text{EG}$,

$$
G \in \text{EG} \land \xi(G) \leq \xi(H) \iff G \leq^\Sigma_\xi H \iff G \leq^H_\xi H.
$$

Suppose $G \in \text{EG}$ and $\xi(G) \leq \xi(H)$. Letting $M = \max\{\deg(G), \deg(H)\}$, we see that $\rho \left( T^k(G) \right) \leq \rho \left( T^k(H) \right)$ for all $k \geq M$ via Lemma 6.11 hence $\phi_k(G) \leq \phi_k(H)$ for $k \geq M$. It follows $G \leq^H_\xi H$ and $G \leq^\Sigma_\xi H$.

Conversely, suppose $G \leq^H_\xi H$ and $G \leq^\Sigma_\xi H$ and let $M \geq 0$ be such that $G \leq^H_k H$ and $G \leq^\Sigma_k H$ for all $k \geq M$. Immediately, $G \in \text{EG}$. For each $k \geq M$, we further have $\phi_k(G) \leq \phi_k(H)$, and taking $k = \max\{\deg(G), \deg(H), M\}$,

$$
\xi(G) = \phi_k(G) \leq \phi_k(H) = \xi(H).
$$

We conclude that $\xi$ is a $\Pi^1_1$ rank. \qed
Propositions 6.22 and 6.24 combine to give us
\[ \xi(G) \leq \omega(\text{rk}(G) + 1) \leq \omega(3\xi(G) + 1), \]
so \( \text{rk} \) is closely related to a \( \Pi^1_1 \)-rank. Given the close relationship between \( \xi \) and \( \text{rk} \), a second question arises:

**Question 6.27.** Is \( \text{rk} \) a \( \Pi^1_1 \)-rank?

We see that \( \text{rk} \) has the following two properties shared by any \( \Pi^1_1 \)-rank.

1. For all \( \alpha < \omega_1 \), \( \text{rk}^{-1}(\alpha) = \text{EG}_\alpha \) is Borel. (This is proven in Section 7)
2. If \( S \subseteq \text{EG} \) and \( S \) is analytic, then \( \sup_{G \in S} \{ \text{rk}(G) \} < \omega_1 \). (This only requires the upper bound \( \text{rk}(G) \leq 3\xi(G) \).)

By the argument of A. Kechris, R. Solovay, and J. Steel in [12, Section 2.3], one can show that under an extra set-theoretic assumption known as Projective Determinacy, any rank function with these two properties is a \( \Pi^1_1 \)-rank. It is unlikely that whether or not \( \text{rk} \) is a \( \Pi^1_1 \)-rank would require extra set-theoretic assumptions, and so this should be taken as strong evidence for the affirmative.

7. Borel functions on and sets in \( \mathcal{G} \)

In previous sections we made claims that certain maps and sets were Borel, and from this and the Boundedness Theorem 2.7 we concluded that certain subsets of \( \mathcal{G} \) were not Borel. A slogan from descriptive set theory is “Borel = explicit”, meaning if you describe a map or set without an appeal to something like the axiom of choice, it should be Borel. As the maps and sets from previous sections are “explicit” in this sense, we were content to state that they were Borel without further proof. To those not as familiar with descriptive set theory, we offer this section to verify our previous claims.

Recall that \( \mathcal{G} = \{ N \leq \mathbb{F}_\omega \} \), and that we identify \( N \) with the group \( \mathbb{F}_\omega/N \). We make frequent use of the usual projection from \( \mathbb{F}_\omega \) to \( \mathbb{F}_\omega/N \) and always denote this projection by \( f_N \). Every countable group is thus identified with an element of \( \mathcal{G} \). In fact, a given group \( G \) corresponds to many distinct elements of \( \mathcal{G} \), as there are many different surjections of \( \mathbb{F}_\omega \) onto \( G \). Thus we think of the elements of \( \mathcal{G} \) as being marked groups. We fix an enumeration \( (\gamma_i) \) for \( \mathbb{F}_\omega \), and this gives rise to an enumeration of \( \mathcal{G} \) in the obvious way. Let us also enumerate the generators for \( \mathbb{F}_\omega \) as \( (a_i) \). Then \( \mathbb{F}_\omega/N \) is generated by \( (a_iN) \). Recall also that \( \mathcal{G}_{fg} = \bigcup_{n \in \mathbb{N}} \{ N \leq \mathbb{F}_\omega \mid \forall k \geq n \ a_k \in N \} \). This is an \( F_\sigma \) subset of \( \mathcal{G} \). In particular, its Borel sets are precisely those sets of the form \( B \cap \mathcal{G}_{fg} \) where \( B \subseteq \mathcal{G} \) is Borel.

7.1. Borel functions. We start with some easier examples of Borel maps. Recall that the sub-basic open sets of \( \mathcal{G} \) are those of the form \( O_\gamma = \{ N \mid \gamma \in N \} \) and their complements. The Borel \( \sigma \)-algebra on \( \mathcal{G} \) is thus generated by the \( O_\gamma \), so in order to show \( f : \mathcal{G} \to \mathcal{G} \) is Borel, we need only check that \( f^{-1}(O_\gamma) \) is Borel for all \( \gamma \in \mathbb{F}_\omega \).

**Lemma 7.1.** For each \( \delta \in \mathbb{F}_\omega \), there is a Borel map \( Q_\delta : \mathcal{G} \to \mathcal{G} \) such that if \( N \in \mathcal{G} \) with \( \mathbb{F}_\omega/N \cong G \), then \( \mathbb{F}_\omega/Q_\delta(N) \cong G/\langle \langle f_N(\delta) \rangle \rangle \).
Proof. Note that \( G/\langle \langle f_N(\delta) \rangle \rangle \cong F_\omega/\langle \langle N, \delta \rangle \rangle \). Thus the map \( Q_\delta(N) = \langle \langle \delta \rangle \rangle N \) meets our requirements. We need only check that it is Borel. We see that

\[
Q_\delta^{-1}(O_\gamma) = \{ N \in \mathcal{G} \mid \gamma \in \langle \langle \delta \rangle \rangle N \}
= \{ N \in \mathcal{G} \mid \exists g \in \langle \langle \delta \rangle \rangle \ g^{-1}\gamma \in N \}
= \bigcup_{g \in \langle \langle \delta \rangle \rangle} \{ N \in \mathcal{G} \mid g^{-1}\gamma \in N \}
\]

which is open, so we are done. \( \square \)

We can now easily prove Lemma \[Q.2\] .

Proof of Lemma \[Q.2\] . Let \( i \in \mathbb{N} \). Then \( G_i = Q_{G_0}(G) \). By repeated composition, we may define \( Q_s: \mathcal{G} \to \mathcal{G} \) for all \( s \in \mathbb{N}^{<\mathbb{N}} \) all of which are Borel by the previous lemma; we let \( Q_0 \) be the identity.

Now suppose \( t \in \mathbb{N}^{<\mathbb{N}} \) is of the form \( v \upharpoonright i \) with \( v \in \mathbb{N}^{<\mathbb{N}} \) and \( i \in \mathbb{N} \). Consider the basic open set \( O_t := \{ T \in Tr \mid t \in T \} \) of \( Tr \). So \( \Phi_{Mn}^{-1}(O_t) = \{ N \in \mathcal{G} \mid Q_t(N) \neq Q_t(N) \} \), which is clearly Borel. Hence, \( \Phi_{Mn} \) is Borel. \( \square \)

Lemma 7.2. For each \( n \geq 0 \), there is a Borel map \( R_n: \mathcal{G} \to \mathcal{G} \) such that if \( N \in \mathcal{G} \) with \( F_\omega/N \cong G \), then \( F_\omega/R_n(N) \cong \langle g_0, \ldots, g_n \rangle \).

Proof. Let \( \pi_n: F_\omega \to F_\omega \) be induced by mapping the generators \( (a_i)_{i \in \mathbb{N}} \) as follows:

\[
\pi_n(a_i) = \begin{cases} 
\gamma_i & 0 \leq i \leq n \\
1 & \text{otherwise.}
\end{cases}
\]

Suppose that \( N \in \mathcal{G} \) with \( F_\omega/N = G \). Then \( f_N \circ \pi_n: F_\omega \to \langle g_0, \ldots, g_n \rangle \) is also a surjection. We thus define \( R_n \) to be the map sending \( \ker(f_N) \) to \( \ker(f_N \circ \pi_n) \). Since \( F_\omega/\ker(f_N \circ \pi_n) \cong \langle g_0, \ldots, g_n \rangle \), this works as intended. As \( \gamma \in \ker(f_N \circ \pi_n) \) iff \( \pi_n(\gamma) \in \ker(f_N) \), we have

\[
R_n^{-1}(O_\gamma) = \{ M \in \mathcal{G} \mid \pi_n(\gamma) \in M \},
\]

which is the open set \( O_{\pi_n(\gamma)} \). Hence, \( R_n \) is Borel. \( \square \)

Note that the above proof works for subgroups generated by any fixed collection of elements of \( G \), i.e., the same proof shows that the maps for \( G \mapsto G_s \) defined in Section 4 are Borel, so as before, we get Lemma 4.2 as a corollary.

We now move onto proving Lemma 8.3; this follows from the next lemma. Its proof is more involved than the previous two.

Lemma 7.3. For each \( s \in \mathbb{N}^{<\mathbb{N}} \), there is a Borel map \( C_s: \mathcal{G} \to \mathcal{G} \) such that if \( N \in \mathcal{G} \) with \( F_\omega/N \cong G \), then \( F_\omega/C_s(N) \cong C_G(\{ g_s \}) \).

Proof. Suppose that \( N \in \mathcal{G} \) and \( F_\omega/N \cong G \). Define \( \pi_N: F_\omega \to F_\omega \) by

\[
\pi_N(a_j) := \begin{cases} 
\gamma_j & \text{if } f_N(\gamma_j) \in C_G(\{ f_N(\gamma_s) \}) \\
0 & \text{else.}
\end{cases}
\]

Then the map \( f_N \circ \pi_N: F_\omega \to C_G(\{ g_s \}) \) is a surjection, so the map \( N \mapsto \ker(f_N \circ \pi_N) \) works as intended. In order to check it is Borel, we introduce the set

\[
S_j := \{ N \in \mathcal{G} \mid \pi_N(a_j) = \gamma_j \}.
\]
Since $f_N(\gamma_j) \in C_G(\{f_N(\gamma_s)\})$ iff $[\gamma_j, \gamma_s] \in N$ for each $0 \leq i \leq |s|$, 
\[
S_j = \{ N \in \mathcal{G} \mid [\gamma_j, \gamma_s] \in N \text{ for each } 0 \leq i \leq |s| \},
\]
which is an open set.

We now fix a word $\delta = \delta(a_0, \ldots, a_m) \in \mathbb{F}_\omega$ and consider the pre-image of the basic open set $O_\delta$. Our notation $\delta(a_0, \ldots, a_m)$ indicates that the word $\delta$ only uses the letters appearing in the parentheses. We may evaluate $\pi_N(\delta)$ by substituting in the images of $a_0, \ldots, a_m$, so $\pi_N(\delta) = \delta(x_0, \ldots, x_m)$ for some $\overline{x} := (x_0, \ldots, x_m) \in \Omega := \prod_{i=0}^m \{ \gamma_i, e \}$ that depends on $N$. The set of $N \in \mathcal{G}$ such that $\pi_N : (a_0, \ldots, a_n) \mapsto \overline{x}$ for some fixed $\overline{x}$ is the Borel set
\[
S_\overline{x} := \bigcap_{x_j = \gamma_j} S_j \cap \bigcap_{j=0}^m S_{e_j}.
\]
Now since $\delta \in \ker(f_N \circ \pi_N)$ iff $\pi_N(\delta) \in \ker f_N = N$, we see
\[
C_i^{-1}(O_\delta) = \{ N \in \mathcal{G} \mid \pi_N(\delta) \in N \}
= \bigcup_{\overline{x} \in \Omega} \{ N \in \mathcal{G} \mid \delta(\overline{x}) \in N \} \cap S_\overline{x}
\]
which is Borel.

We next show that the maps $S_k$ from Section 4 are Borel. The main idea is the same as in previous lemma.

**Lemma 7.4.** For each $k \geq 1$, there is a Borel map $S_k : \mathcal{G}_fg \to \mathcal{G}$ such that if $\mathbb{F}_\omega/N = G$, then
\[
\mathbb{F}_\omega/S_k(N) \cong [G, G] \cap \bigcap \mathcal{N}_k(G)
\]
where $\mathcal{N}_k(G) := \{ M \unlhd G \mid |G : M| \leq k + 1 \}$.

**Proof.** Suppose $N \in \mathcal{G}_fg$ and $G \cong \mathbb{F}_\omega/N$. Similarly to the previous lemma, we define $\pi_N : \mathbb{F}_\omega \to \mathbb{F}_\omega$ by
\[
\pi_N(a_i) := \begin{cases} 
\gamma_i, & \text{if } f_N(\gamma_i) \in [G, G] \cap \bigcap \mathcal{N}_k(G) \\
e, & \text{else.}
\end{cases}
\]
Define $S_k : \mathcal{G}_fg \to \mathcal{G}$ by $N \mapsto \ker(f_N \circ \pi_N)$; this map behaves as desired. We claim this map is Borel.

Define
\[
\mathcal{N}_k := \{ M \in \mathcal{G}_fg \mid ||\mathbb{F}_\omega : M|| \leq k + 1 \}.
\]
If $N \in \mathcal{G}_fg$, then the collection of index-$\leq k + 1$ subgroups of $\mathbb{F}_\omega/N$ is precisely $\{ MN/N \mid M \in \mathcal{N}_k \}$. Therefore, $f_N(\gamma_i) \in [G, G] \cap \bigcap \mathcal{N}_k(G)$ iff $\gamma_i \in [\mathbb{F}_\omega, \mathbb{F}_\omega] N \cap \bigcap_{M \in \mathcal{N}_k} MN$. As in the previous lemma, we may define
\[
S_i := \{ N \in \mathcal{G}_fg \mid \pi_N(a_i) = \gamma_i \}
= \left\{ N \in \mathcal{G}_fg \mid \gamma_i \in [\mathbb{F}_\omega, \mathbb{F}_\omega] N \cap \bigcap_{M \in \mathcal{N}_k} MN \right\}
= \bigcup_{\delta \in [\mathbb{F}_\omega, \mathbb{F}_\omega]} \left\{ N \in \mathcal{G}_fg \mid \delta^{-1} \gamma_i \in N \right\} \cap \bigcap_{M \in \mathcal{N}_k} \bigcup_{\delta \in M} \left\{ N \in \mathcal{G}_fg \mid \delta^{-1} \gamma_i \in N \right\}.
\]
The last set is Borel since $\mathcal{N}_k$ is countable. Given $\overline{x} := (x_0, \ldots, x_m) \in \Omega := \prod_{i=0}^m \{ \gamma_i, e \}$, we define as before $S_\overline{x}$. 
We now fix a word $\delta = \delta(a_0, \ldots, a_m) \in F_\omega$ and consider the pre-image of the basic open set $O_\delta$. We see
\[ S_k^{-1}(O_\delta) = \{ N \in \mathcal{G} \mid \pi_N(\delta) \in N \} \]
\[ = \bigcup_{\pi \in \Omega} \{ N \in \mathcal{G} \mid \delta(\pi) \in N \} \cap S_{\pi} \]
which is Borel.

Using the lemmas above, we build Borel maps $\Psi^l_s : \mathcal{G} \to \mathcal{G}$ for each $l \in \mathbb{N}$ and $s \in \mathbb{N}^{<\mathbb{N}}$. For $s = \emptyset$, put $\Psi^l_\emptyset = id$. Supposing we have defined $\Psi^l_s$, define $\Psi^l_{s-n}$ by
\[ \Psi^l_{s-n}(G) := S_{|s|+l} \circ R_n(\Psi^l_s(N)). \]
We see that if $s \in T^l(G)$ with $G = F_\omega/N$, then $F_\omega/\Psi^l_s(N) = G_s$. If $s \notin T^l(G)$, then $F_\omega/\Psi^l_s(N) = \{e\}$. 

**Proof of Lemma 7.5.** Fixing $s \in \mathbb{N}^{<\mathbb{N}}$ and $l \in \mathbb{N}$, we have
\[ (\Phi^l)^{-1}(O_s) = \{ N \in \mathcal{G} \mid s \in T^l(F_\omega/N) \}. \]
If $s = \emptyset$, then $(\Phi^l)^{-1}(O_s) = \mathcal{G}$ which is plainly Borel. Else, say $s = r^{-n}$, so
\[ (\Phi^l)^{-1}(O_s) = \{ N \in \mathcal{G} \mid r^{-n} \in T^l(F_\omega/N) \} \]
\[ = \{ N \in \mathcal{G} \mid (F_\omega/N)_r \neq \{e\} \} \]
\[ = (\Phi^l)^{-1}(\mathcal{G} \setminus \{e\}), \]
which is Borel. 

**7.2. Borel sets.** Recall that $AG$ denotes the class of countable amenable groups.

**Lemma 7.5 (Folklore).** The set $AG$ is Borel in $\mathcal{G}$, and therefore, $AG \cap \mathcal{G}_{fg}$ is Borel.

**Proof.** Recall amenable groups are characterized by Følner’s property: A countable group $G$ is amenable if and only if for every finite $F \subseteq G$ and every $n \geq 1$, there is a finite subset $K \subseteq G$ such that
\[ \frac{|xK \Delta K|}{|K|} \leq \frac{1}{n} \]
for all $x \in F$ where $\Delta$ is the symmetric difference.

Letting $P_f(F_\omega)$ be the collection of finite subsets of $F_\omega$, we see
\[ AG = \bigcap_{F \in P_f(F_\omega)} \bigcap_{n \geq 1} \bigcup_{K \in P_f(F_\omega)} \bigcup_{x \in F} \left\{ N \in \mathcal{G} \mid \frac{|f_N(x)f_N(K)\Delta f_N(K)|}{|f_N(K)|} \leq \frac{1}{n} \right\}. \]
It thus suffices to show
\[ \Omega := \left\{ N \in \mathcal{G} \mid \frac{|f_N(x)f_N(K)\Delta f_N(K)|}{|f_N(K)|} \leq \frac{1}{n} \right\} \]
is Borel. It is easy to see requiring $|f_N(K)| = m$ and $|f_N(x)f_N(K)\Delta f_N(K)| = l$ is Borel, hence
\[ \Omega = \bigcup_{\frac{1}{m} \leq \frac{1}{n}} \left\{ N \mid |f_N(x)f_N(K)\Delta f_N(K)| = l \text{ and } |f_N(K)| = m \right\} \]
is Borel. We conclude $AG$ is a Borel set. 

\[ \square \]
We note one last result.

**Theorem 7.6.** $\text{EG}_\alpha$ is Borel for each $\alpha < \omega_1$.

**Proof.** We argue by induction on $\alpha$. The base case is immediate since the collection of abelian and finite groups is Borel.

Suppose the inductive hypothesis holds for $\alpha$. By construction, $\text{EG}_{\alpha+1} = \text{EG}_\alpha^l \cup \text{EG}_\alpha^e$, so it suffices to show $\text{EG}_\alpha^l$ and $\text{EG}_\alpha^e$ are Borel. Letting $R_n$ be the map defined in Lemma 7.2,

$$\text{EG}_\alpha^l = \bigcap_{n \geq 1} R_n^{-1}(\text{EG}_\alpha),$$

and therefore, $\text{EG}_\alpha^l$ is Borel.

For $\text{EG}_\alpha^e$, form

$$\Omega := \{ (N, M) \in \mathcal{G}^2 : N \trianglelefteq M, M/N \in \text{EG}_\alpha, \text{ and } F_{\omega}/M \in \text{EG}_0 \}.$$

Since $\text{EG}_\alpha$ is Borel by the inductive hypothesis, it follows similarly to Lemma 7.4 that $\Omega$ is a Borel set. Furthermore, $\text{EG}_\alpha^e = \pi_1(\Omega)$ where $\pi_1$ is the projection onto the first coordinate, hence $\text{EG}_\alpha^e$ is an analytic set. In view of Proposition 6.24 there $\beta < \omega_1$ such that $\text{EG}_\alpha^e \subseteq (\rho \circ \Phi_1)^{-1}(\beta)$. We thus have that $(\rho \circ \Phi_1)^{-1}(\beta) \setminus \text{EG}_\alpha^e$ is a co-analytic set with $\rho \circ \Phi_1$ a co-analytic rank bounded below $\omega_1$. Applying Theorem 2.7, we conclude that $(\rho \circ \Phi_1)^{-1}(\beta) \setminus \text{EG}_\alpha^e$ is Borel, whereby $\text{EG}_\alpha^e$ is Borel. This completes the induction, and the theorem is proved. \(\square\)

8. **Further remarks**

Our results herein give tools to study groups enjoying any of the other chain conditions in the literature. Perhaps more interestingly, our results suggest new questions concerning elementary amenable groups and groups with the minimal condition on centralizers, maximal condition on subgroups, and maximal condition on normal subgroups.

Most immediately, one desires a better understanding of the various rank functions. In the case of max groups, there are no infinite subgroup rank two groups, the infinite groups with subgroup rank 3 are Tarski monsters, and $\mathbb{Z}$ has rank $\omega + 1$. In the case of max-n, examples of finite rank groups are easy to produce; however, it becomes much less clear for transfinite examples. Following Olshanskii and Osin, cf. [16, Corollary 1.6], we ask

**Question 8.1.** For which ordinals $\alpha$ is there an infinite group in $\mathcal{M}_C (\mathcal{M}_\text{max}, \mathcal{M}_n)$ such that the centralizer rank (subgroup rank, length) is $\alpha$?

In a different direction, showing a set is non-Borel in $\mathcal{G}$ demonstrates there is no “simple” definition of the class. Our techniques give a way to determine if a subset of a set given by a chain condition is Borel and, hence, to determine if it admits a “simple” characterization. In the setting of max-n groups, there is a particularly intriguing question along these lines. By an old result of Hall, a two-step solvable group is max-n if and only if it is finitely generated; this is certainly a Borel condition. On the other hand, no such nice characterization of three-step solvable groups with max-n is known. We thus ask

**Question 8.2.** Is the set of max-n three-step solvable marked groups Borel?

In a similar vein, our results on elementary amenable groups, in a sense, show elementary amenable groups are not “elementary”. One naturally asks
Question 8.3 (Hume). Is there an intermediate “elementary” Borel set between $EG \cap G_{fg}$ and $AG \cap G_{fg}$? More precisely, is there an elementary class $\mathcal{E}(B)$ in the sense of Osin \cite{17} with $B$ ”small” such that $EG \cap G_{fg} \subseteq \mathcal{E}(B) \subseteq AG \cap G_{fg}$ and $\mathcal{E}(B)$ is Borel?

We also arrive at new questions with a descriptive-set-theoretic flavor.

Definition 8.4. Let $Y$ be a standard Borel space. A set $A \subseteq Y$ is $\Pi^1_1$-complete if $A$ is $\Pi^1_1$ and for all $B \subseteq X$ with $X$ standard Borel and $B$ co-analytic, $B$ Borel reduces to $A$.

The idea is that $\Pi^1_1$-complete sets are as complicated as they possibly could be; Theorem \ref{2.4} says that $WF \subseteq Tr$ is $\Pi^1_1$-complete.

Question 8.5. Are any of $\mathcal{M}_C, \mathcal{M}_{\text{max}}, \mathcal{M}_n$, or $EG$ $\Pi^1_1$-complete?

Note that for a positive answer it suffices to show that $WF$ (or some other $\Pi^1_1$-complete set) Borel reduces to these sets. Under an extra set-theoretic assumption known as $\Sigma^1_1$-Determinacy, every $\Pi^1_1$ set which is not Borel is in fact $\Pi^1_1$-complete. We do not expect that extra set-theoretic assumptions should be necessary to prove any of the sets are $\Pi^1_1$-complete; we mention this as evidence that the positive answer is indeed the correct one. It is worth noting the question is a problem in group theory. For example, in the case of $EG$ one must devise a method of building a group from a tree so that well-founded trees give elementary amenable groups and ill-founded trees give non elementary amenable groups.

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