First Exit of Brownian Motion from a One-sided Moving Boundary

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Abstract. We revisit a result of Uchiyama (1980): given that a certain integral test is satisfied, the rate of the probability that Brownian motion remains below the moving boundary $f$ is asymptotically the same as for the constant boundary. The integral test for $f$ is also necessary in some sense.

After Uchiyama’s result, a number of different proofs appeared simplifying the original arguments, which strongly rely on some known identities for Brownian motion. In particular, Novikov (1996) gives an elementary proof in the case of an increasing boundary. Here, we provide an elementary, half-page proof for the case of a decreasing boundary. Further, we identify that the integral test is related to a repulsion effect of the three-dimensional Bessel process. Our proof gives some hope to be generalized to other processes such as FBM.

Mathematics Subject Classification (2010). Primary 60G15; Secondary 60G18.

Keywords. Brownian motion; Bessel process; moving boundary; first exit time; one-sided exit problem.

1. Introduction

This note is concerned with the first exit time distribution of Brownian motion from a so-called moving boundary:

$$\mathbb{P}[B_t \leq f(t), t \leq T], \quad \text{as } T \to \infty,$$

where $B$ is a Brownian motion and $f : [0, \infty) \to \mathbb{R}$ is the “moving boundary”. The question we treat here is follows: for which functions $f$ does the above probability have the same asymptotic rate as in the case $f \equiv 1$? This problem was considered by a number of authors [1–3, 5, 6, 8] and, besides being a classical problem for Brownian motion, has some implications for the so-called KPP equation (see, e.g., [2]), for branching Brownian motion (see, e.g., [1]), and for other questions.

Frank Aurzada and Tanja Kramm were supported by the DFG Emmy Noether programme.
The solution of the problem was given by Uchiyama [8], Gärtner [2], and Novikov [5] independently and can be re-phrased as follows.

**Theorem 1.1.** Let \( f : [0, \infty) \to \mathbb{R} \) be a continuously differentiable function with \( f(0) > 0 \) and

\[
\int_1^\infty |f(t)| t^{-3/2} \, dt < \infty. \tag{1.1}
\]

Then

\[
\mathbb{P} [B_t \leq f(t), t \leq T] \approx T^{-1/2}, \quad \text{as } T \to \infty. \tag{1.2}
\]

If \( f \) is either convex or concave and the integral test (1.1) fails, \( T^{-1/2} \) is not the right order in (1.2).

Here and in the following, we denote \( a(t) \approx b(t) \) if \( c_1 a(t) \leq b(t) \leq c_2 a(t) \) for some constants \( c_1, c_2 \) and all \( t \) sufficiently large.

Even though the above-mentioned problem has been solved by Uchiyama, there have been various attempts to simplify the proof of this result and to give an interpretation for the integral test (1.1). It is the purpose of this note (a) to give a simplified proof of the theorem for the case of a decreasing boundary. Our proof (b) also allows to interpret the integral test as coming from a repulsion effect of the three-dimensional Bessel process and (c) gives hope to be generalized to other processes, contrary to the existing proofs, which all make use of very specific known identities for Brownian motion.

Let us assume for a moment that \( f \) is monotone. Note that the sufficiency part of the theorem can be decomposed into two parts: if \( f' \geq 0 \) one needs an upper bound of the probability in question, while if \( f' \leq 0 \) one needs a lower bound. The first case is much better studied; in particular, Novikov [6] gives a relatively simple proof of the theorem in this case. To the contrary, in case of a decreasing boundary he wonders that “it would be interesting to find an elementary proof of this bound” ([6], p. 723). We shall provide such an elementary proof here.

The remainder of this note is structured as follows. Section 2 contains the proof of the theorem, which now fits on half a page. We also outline the relation to the Bessel process. In Section 3, we list some additional remarks.

### 2. Proof

We give a proof of the following theorem, which concerns the part of Theorem 1.1 related to the decreasing boundary.

**Theorem 2.1.** Let \( f : [0, \infty) \to \mathbb{R} \) be a twice continuously differentiable function with \( f(0) > 0 \).