$S_3$-PERMUTATION ORBIFOLDS OF VIRASORO VERTEX ALGEBRAS

ANTUN MILAS, MICHAEL PENN, CHRISTOPHER SADOWSKI

Abstract. In this paper, a continuation of [27], we investigate the $S_3$-orbifold subalgebra of $(\mathcal{V}_c)^{\otimes 3}$, that is, we consider the $S_3$-fixed point vertex subalgebra of the tensor product of three copies of the universal Virasoro vertex operator algebras $\mathcal{V}_c$. Our main result is construction of a minimal, strong set of generators of this subalgebra for any generic values of $c$. More precisely, we show that this vertex algebra is of type $(2, 4, 6^2, 8^2, 9, 10^2, 11, 12^3)$. We also investigate two prominent examples of simple $S_3$-orbifold algebras corresponding to central charges $c = \frac{1}{3}$ (Ising model) and $c = -\frac{22}{3}$ (i.e. $(2, 5)$-minimal model). We prove that the former is a new unitary $W$-algebra of type $(2, 4, 6, 8)$ and the latter is isomorphic to the affine simple $W$-algebra of type $\mathfrak{g}_2$ at non-admissible level $-\frac{22}{3}$. We also provide another version of this isomorphism using the affine $W$-algebra of type $\mathfrak{g}_2$ coming from a subregular nilpotent element.

1. Introduction

In recent years there has been considerable attention given to the study of various aspects of permutation orbifold algebras of vertex algebras (see [11, 12, 13, 14, 15, 16, 24, 25] and references therein, where the term “orbifold” is used instead of “orbifold (sub)algebra”). Although a lot is known about their representation theory, we know very little about the structure (e.g. their type) of permutation orbifold algebras even for familiar examples of vertex algebra such as Heisenberg, Virasoro, and affine vertex algebras. Apart from a few general results on orbifold algebras being finitely generated with respect to a reductive group (23), there is no known method based on invariant theory for finding a minimal generating set for an orbifold algebra. This is mainly due to non-trivial quantum corrections that depend on the vertex algebra being studied. In our previous works, some with coauthors (27, 28, 26, 23), we were able to construct minimal generating sets for low rank permutation orbifold algebras arising from the Heisenberg, free fermion, Virasoro, and $N = 1, 2$ superconformal algebras. In particular, in [27] we found a minimal generating set for the 3-cycle permutation orbifold algebra associated to the Virasoro vertex algebra for every value of the central charge.

To introduce the problem, we begin by recalling our notation from [27]. For a vertex algebra $V$, the $n$-fold tensor product will be denoted by $V^\otimes n := V \otimes \cdots \otimes V$. The vector space $V^\otimes n$ has a natural vertex operator algebra structure on which the symmetric group $S_n$ acts on $V^\otimes n$ by permuting tensor factors and thus $S_n \subset \text{Aut}(V^\otimes n)$. The $S_n$-invariant subalgebra of $V^\otimes n$, denoted by $(V^\otimes n)^{S_n}$, we call the $S_n$-orbifold algebra of $V$. Throughout this work we only consider the Virasoro vertex algebra. We denote by $\mathcal{V}_c := V_{Vir}(c, 0)$ the universal Virasoro vertex operator algebra of central charge $c$. It is (freely) generated by the weight 2 conformal vector $\omega$ with vertex operator $Y(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ and the operator product expansion (OPE)

$$L(z)L(w) \sim \frac{\partial_w L(w)}{(z-w)} + \frac{2L(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4}.$$ (1.1)

We denote by $\mathcal{L}_c := L_{Vir}(c, 0)$ its unique simple quotient. For $c = c_{p,q} := 1 - \frac{6(p-q)^2}{pq}$, where $p, q \geq 2$ are coprime integers (i.e. minimal models), we have $\mathcal{V}_c \neq \mathcal{L}_c$ and moreover $\mathcal{L}_c$ is a regular vertex algebra. We already know that $\text{Aut}(\mathcal{V}_c^\otimes n) = S_n$ (see for example [27]) and therefore for any subgroup of $G \subset S_n$ we have a fixed point subalgebra $(\mathcal{V}_c^\otimes n)^G$. The key question in this line of research is to describe the structure of $(\mathcal{L}_c^\otimes n)^G$ using a minimal finite set of generators and to find
possible isomorphisms with more familiar \(W\)-algebras. Some results in this direction were already given in [27].

In this paper we consider \(S_3\)-orbifold subalgebra of \((\mathcal{V}_c)^{\otimes 3}\) for a generic central charge \(c\), and two distinguished simple orbifold algebras of central charge \(c = \frac{1}{2}\) and \(c = -\frac{32}{9}\).

This paper is organized as follows: In Section 2 we setup the necessary notation and recall some basic facts about OPEs of fields of vertex algebras. We begin Section 3 by defining a suitable large \(c\) limit of the Virasoro algebra that we denote by \(\mathcal{V}\). This vertex algebra and associated orbifold algebras can be used to obtain information about strong generators of orbifold algebras of \(\mathcal{V}\) at least if \(c\) is generic (see [25]). We first reduce generators to an infinite set of quadratic and cubic generators using standard methods of invariant theory. Then, in the most difficult part of the paper, we remove all but finitely many quadratic and cubic generators. Further reduction removes a few additional generators. The resulting generating set turns out to be minimal. Our main result in this section can be summarized as:

**Theorem 1.1.** For any generic \(c\), including a suitably defined \(c \to \infty\) limit, the \(S_3\)-orbifold subalgebra \((\mathcal{V}_c)^{\otimes 3}\) is strongly generated by vectors of weight \(2, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15\). Moreover, this is also a minimal generating set.

In Section 4 we switch our attention to a specific simple Virasoro vertex algebra, the unitary minimal model \(L_2^\frac{1}{2}\) (Ising VOA), and its \(S_3\)-orbifold subalgebra \((L_2^\frac{1}{2})^{\otimes 3}\). This vertex algebra admits the well-known fermionic realization \(L_2^\frac{1}{2} \cong \mathcal{F}\), so that \(L_2^\frac{1}{2}\) is the even part of the the rank one free fermion vertex algebra \(\mathcal{F}\). Previously the first two authors and Wauchope investigated \((L_2^\frac{1}{2})^{\otimes 3}\) and determined its type [29]. Although it seems now natural to consider a suitable \((\mathbb{Z}_2)^3\)-orbifold of \((L_2^\frac{1}{2})^{\otimes 3}\) to study \((L_2^\frac{1}{2})^{\otimes 3}\) we found this approach to be very cumbersome. For this reason, we do not use fermionic construction here and instead employ the null vectors of weight 6 in \(L_2^\frac{1}{2}\) to obtain additional relations. This allows us to reduce several generators beyond the generic case. Our main result in this direction is

**Theorem 1.2.** The orbifold subalgebra \((L_2^\frac{1}{2})^{\otimes 3}\) is a (unitary) \(W\)-algebra of type \((2, 4, 6, 8)\).

We note that this vertex algebra is not isomorphic to a simple principal \(W\)-algebra of type \(B_4\) (or \(C_4\) by Feigin-Frenkel duality) nor the \(Z_2\)-orbifold of the principal affine algebra of type \(D_4\). Furthermore, this algebra is not isomorphic to the parafermionic orbifold algebra \(\mathcal{N}_{10}(sl_2)^{\mathbb{Z}_2}\), which is also unitary of central charge \(\frac{3}{2}\).

Finally, in Section 5 we consider the \(S_3\)-orbifold algebra of \(L_{-22/5}\), the famous \((2, 5)\)-minimal model, continuing our discussion from [27]. Interestingly this orbifold algebra is related not to one but two affine vertex algebra of type \(G_2\). The next result was previously announced in [27]. We denote by \(W_k(g_2, f_{\text{prin}})\) the simple principal affine \(W\)-algebra associated to \(G_2\) at level \(k\).

**Theorem 1.3.** We have an isomorphism

\[
(L_2^{\otimes 3})^{S_3} \cong W_{-\frac{30}{6}}(g_2, f_{\text{prin}}).
\]

In particular, the \(W\)-algebra on the right-hand side is regular (i.e. rational and lisse)\(^1\).

Observe that the level \(-\frac{30}{6}\) is not \(g_2\) admissible. This gives an example of a regular principal affine \(W\)-algebra outside admissible series.

We also have an isomorphism of simple VOAs \(W_{-\frac{30}{6}}(g_2, f_{\text{prin}}) \cong W_{-\frac{30}{6}}(g_2, f_{\text{prin}})\), which is basically an instance of the Feigin-Frenkel duality of affine \(W\)-algebras. Moreover, using the construction of the subregular affine \(W\)-algebra of type \(G_2\) (this algebra was also studied in J. Fasquel’s PhD thesis [17]) we can easily see that \(W_{-\frac{30}{6}}(g_2, f_{\text{sub}}) \cong L_2^{\otimes 3}\) and therefore we can view \(W_{-\frac{30}{6}}(g_2, f_{\text{prin}})\) as an \(S_3\)-orbifold, i.e we get:

**Corollary 1.1.**

\[
W_{-\frac{30}{6}}(g_2, f_{\text{sub}})^{S_3} \cong W_{-\frac{30}{6}}(g_2, f_{\text{prin}}).
\]

\(^1\)Using this isomorphism we can also show that the affine \(W\)-algebra has precisely 24 irreducible modules.
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2. Setup and Preliminary results

As already mentioned in the introduction, $(\mathcal{V}, Y, \omega, \mathbb{1})$ denotes the universal Virasoro vertex algebra with $\omega = L(-2)\mathbb{1}$ and $\mathbb{1}$ denotes the vacuum vector. Throughout the paper we will be working with $\mathcal{V}_c^{\otimes n}$ with $n = 3$. For convenience, we suppress the tensor product symbol and let

\[ L_i(-m)\mathbb{1} := \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)-\text{factors}} \otimes L(-m)\mathbb{1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(n-i)-\text{factors}} \in \mathcal{V}_c^{\otimes n}, \]

such that $\mathcal{V}_c^{\otimes n} = \langle L(-2)\mathbb{1}, \ldots, L_n(-2)\mathbb{1} \rangle$. Thus $\omega = \omega_1 + \cdots + \omega_n$ is the total conformal vector in $\mathcal{V}_c^{\otimes n}$. Using this notation, the natural action of $S_n$ on $\mathcal{V}_c^{\otimes n}$ is given by permuting tensor factors, that is

\[ \sigma \cdot L_{i_1}(m_1) \cdots L_{i_k}(m_k)\mathbb{1} = L_{\sigma(i_1)}(m_1) \cdots L_{\sigma(i_k)}(m_k)\mathbb{1}, \]

for $1 \leq i_1 \leq n$, $m_j < -1$, and $\sigma \in S_n$.

**Definition 2.1.** We say that $0 \neq v \in V$ is primary of conformal weight $r$ if $L(n)v = 0$, $n \geq 1$ and $L(0)v = rv$. In our work we sometimes consider vertex algebras $V$ strongly generated by the Virasoro vector and several primary fields of conformal weight $r_1, \ldots, r_k$. If so, using physics’ terminology, we say that $V$ is a $W$-algebra of type $(2, r_1, \ldots, r_k)$, where $r_i$ can be repeated several times. If generators of weight $r_i$ are not necessarily primary we still use the same notation but we omit calling $V$ a $W$-algebra.

For a vertex algebra $V$ denote by $\text{gr}(V)$ the associated graded Poisson algebra of $V$ [22]. We have a natural linear isomorphism

\[ \mathcal{V}_c^{\otimes n} \cong \text{gr}(\mathcal{V}_c^{\otimes n}) \cong \mathbb{C}[x_i(m)| 1 \leq i \leq n, m \geq 0] \]

induced by $L_i(-m-2) \mapsto x_i(m)$ for $m \geq 0$. This algebra comes equipped with a derivation $\partial$ such that it is compatible with the translation operator in $\mathcal{V}_c^{\otimes n}$ given by $D(v) = v\mathbb{1}$. Then we have a standard result [22, 24] (cf. [5]).

**Lemma 2.1.** Let $V$ be a vertex algebra with a “good” $\mathbb{Z}_{\geq 0}$ filtration. If $\{a_i|i \in I\}$ generates $\text{gr}(V)$ then $\{a_i|i \in I\}$ strongly generates $V$, where $a_i$ and $\tilde{a}_i$ are related via the natural linear isomorphism described by the $\mathbb{Z}_{\geq 0}$ filtration.

Because our main computation tool is the OPE package [30], throughout we will switch between working directly in the setting of the vertex operator algebra $\mathcal{V}_c^{\otimes n}$ and its copy inside $(\text{End} \mathcal{V}_c^{\otimes n})[[z, z^{-1}]]$ (fields) via the vertex operator map

\[ Y(\cdot, z) : \mathcal{V}_c(n) \rightarrow (\text{End} \mathcal{V}_c^{\otimes n}(n))[[z, z^{-1}]], \]

i.e. we use the field-state correspondence. Under this map we have

\[ u_k(m_1, \ldots, m_k) := \sum_{i=1}^n L_i(-2 - m_1) \cdots L_i(-2 - m_k)\mathbb{1}, \]

\[ U_k(m_1, \ldots, m_k) := Y(u_k(m_1, \ldots, m_k), z) \]

\[ = \frac{1}{(m_1 - 1)!} \cdots \frac{1}{(m_k - 1)!} \sum_{i=1}^n z^m_1 \partial_\omega^m_1 L_i(z) \cdots \partial_\omega^m_i L_i(z), \]

where by $z^m_1$ the normal ordered product and we will often suppress the formal variable $z$ and write $(\partial^m W) := \partial^m_\omega W(z)$, where $W(z)$ is any field. Using this shorthand notation, we recall some

\[ \text{if so we often write } r_i^n, \text{ indicating that there are } n \text{ generators of conformal weight } r_i. \]
basic facts about relations among normal ordered products (here $a, b$ and $c$ are arbitrary vectors in a vertex algebra):

\begin{equation}
\varepsilon_{n}ab_{c}^{n} = \varepsilon_{n}abc + \sum_{k \geq 0} \frac{1}{(k + 1)!} \left( \varepsilon(\partial^{k+1}a)(b_{c}^{*}) + \varepsilon(\partial^{k+1}b)(a_{c}^{*}) \right),
\end{equation}

\begin{equation}
\varepsilon_{n}(a_{\alpha})b_{c}^{n} = \varepsilon(a_{\alpha}b)c_{\alpha}^{*} + \sum_{k = 1}^{n} \frac{n}{k} (a_{n-k}b)(b_{c}^{*}),
\end{equation}

\begin{equation}
(\varepsilon_{n}ab_{c}^{n})_{\alpha}c = \sum_{k \geq 0} \frac{1}{k!} \varepsilon(\partial^{k}a)(b_{\alpha+k}c)_{\alpha}^{*} + \sum_{k \geq 0} b_{\alpha+k}(a_{n-k}c).
\end{equation}

3. The orbifold subalgebra $\mathcal{V}_{c}^{S_{3}}$

3.1. Large $c$ limit of $\mathcal{V}_{c}$. In order to analyze the orbifold subalgebra $\mathcal{V}_{c}^{S_{3}}$ for generic values of $c$ we pass to the generalized free field limiting algebra as in [25]. From the OPE relations for $L(z)$, after rescaling $t := \sqrt{c}$, $\alpha(z) := \frac{L(z)}{t}$, we obtain

\[ \alpha(z)\alpha(w) \sim \frac{1}{(z-w)^4} + \frac{2\alpha(w)}{t(z-w)^2} + \frac{\partial \alpha(w)}{t(z-w)}. \]

Observe that for $t \to +\infty$ the limit is well-defined and we obtain an OPE algebra $\mathcal{V}_{\infty} := \langle \alpha \rangle$, where

\[ \alpha(z)\alpha(w) \sim \frac{1}{(z-w)^4}. \]

Observe that $gr(\mathcal{V}_{\infty}) \cong gr(\mathcal{V}_{c})$. Then using the result of [25], suitably adjusted, applied in our setup, we get:

**Proposition 3.1.** Let $u_{i}, i \in I$ be a strong set of generators of $(\mathcal{V}_{c}^{S_{n}})^{S_{n}}$ then for at most countably many values $c$ of the central charge, there is a strong generating set $t_{i}, i \in I$ of $(\mathcal{V}_{c}^{S_{n}})^{S_{n}}$ with $deg(t_{i}) = deg(u_{i})$.

**Remark 3.1.** Due to order four pole in the OPE $[3.1]$, $\mathcal{V}_{\infty} \neq \mathcal{H}$, where $\mathcal{H}$ is the rank one Heisenberg algebra (notice that the pairing between the modes of $T(z)$ is degenerate!). For this reason we cannot simply use results from [25] on the structure of $\mathcal{H}(3)^{S_{3}}$ to analyze $(\mathcal{V}_{c}^{S_{3}})^{S_{3}}$.

3.2. Computations. Define elements in $\mathcal{V}_{c}^{S_{3}} = \langle \alpha_{1}, \alpha_{2}, \alpha_{3} \rangle$:

\[ T_{0} = \frac{1}{\sqrt{3}}(\alpha_{1} + \alpha_{2} + \alpha_{3}) \]

\[ T_{1} = \frac{1}{\sqrt{3}}(\alpha_{1} + \eta \alpha_{2} + \eta^{2} \alpha_{3}) \]

\[ T_{2} = \frac{1}{\sqrt{3}}(\alpha_{1} + \eta^{2} \alpha_{2} + \eta \alpha_{3}), \]

where $\eta$ is a primitive third root of unity. Under this operation the algebra generated from $T_{0}, T_{1}$ and $T_{2}$ has the following nontrivial OPE

\[ T_{0}(z)T_{0}(w) \sim \frac{1}{(z-w)^4}, \]

\[ T_{1}(z)T_{2}(w) \sim \frac{1}{(z-w)^4}. \]

Further, as this algebra is the limit of a Virasoro vertex operator algebra, we can define the weight of an element from our algebra as the conformal weight of one of its preimages under the limiting procedure. One can easily check that this implies that $wt T_{i} = 2$ and this is well-defined. Now we set $\mathcal{A} = \langle T_{0}, T_{1}, T_{2} \rangle$ and thus we have

\[ \mathcal{V}_{c}^{S_{3}} \cong \mathcal{A} \]
and

\[ (\Omega_{\infty}^{S_3})^{S_3} \cong \mathcal{A}_{S_3}. \]

From (3.3) we have for \( k \geq 0 \)

\[ \left( \partial^m T_1 \right) (\partial^n T_2) = \frac{(-1)^m(m+n+3)!}{3!} \delta_{k,m+n+3} \]

and

\[ \left( \partial^n T_2 \right) (\partial^m T_1) = \frac{(-1)^m(m+n+3)!}{3!} \delta_{k,m+n+3} \]

Now, following [28] we set

\[ W_{m,n} = (\partial^m T_1)(\partial^n T_2), \quad C_{\ell,m,n} = (\partial^m T_1)(\partial^n T_2) \]

and it is clear that the orbifold subalgebra \( \mathcal{A}^{S_3} \) is strongly generated by the fields \( T_0, W_{m,n}, \) and \( C_{\ell,m,n} \) for \( \ell, m, n \geq 0 \). Generators \( W_{m,n} \) are elements of degree two in the associated graded algebra of \( \text{gr}(\mathcal{A}^{S_3}) \) so these generators are called quadratic, likewise \( C_{\ell,m,n} \) are called cubic. Using quantum corrections we will show that \( \mathcal{A}^{S_3} \) has a finite strong set of generators (this is not the case with \( \text{gr}(\mathcal{A}^{S_3}) \)). Let us fix our generating set:

\[ G = \{ T_0, W_{m,n}, C_{\ell,m,n} : \ell, m, n \geq 0 \}. \]

These generators are ordered according to the weight defined earlier. If \( a \in G \) can be written as a linear combination of \( \partial^b \), \( i \geq 1 \) and \( a_1 \cdots a_k \) where \( \text{wt}(a_i) = \text{wt}(b) \), then we say that \( a \) can be expressed in terms of (generators) of lower weight. If so, then \( G \) can be reduced to \( G \setminus a \). Let us explain the strategy of our proof of Theorem 1.1, which is split into two parts. In the first part right below, we show that we can remove all but finitely many cubic generators from \( G \) (and still have a strong generating set). In the second part we consider various quadratic relations together with the remaining cubic generators and show that all but finitely many quadratic generators remain in \( G \).

### 3.3. Cubic Generators

As the first reduction, using techniques involving the translation operator similar to [27] and [21] we see that our orbifold algebra is in fact strongly generated by \( T_0, W_{2m,0}, \) and \( C_{m,n,0} \) for \( m \geq n \geq 0 \) so \( G \) can be replaced by this set. In particular, we have

\[ C_{\ell,m,n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} C_{m+k,n+\ell-k,0}. \]

Further reductions required a bit more work and our major calculation tool will be the identity

\[
\sum_{5} W_{m_1,m_2} C_{n_1,n_2,n_3} = \frac{1}{6} \left( \left( \frac{(-1)^{m_1}}{m_1 + n_1 + 4} + \frac{(-1)^{m_2}}{m_2 + n_1 + 4} \right) C_{m_1+m_2+n_1+4,n_2,n_3} 
+ \left( \frac{(-1)^{m_1}}{m_1 + n_2 + 4} + \frac{(-1)^{m_2}}{m_2 + n_2 + 4} \right) C_{m_1+m_2+n_2+4,n_1,n_3} 
+ \left( \frac{(-1)^{m_1}}{m_1 + n_3 + 4} + \frac{(-1)^{m_2}}{m_2 + n_3 + 4} \right) C_{m_1+m_2+n_3+4,n_1,n_2} \right) + \Psi,
\]

where \( \Psi \) is a linear combination of vectors that are degree 5 in the associated graded algebra. Using this we can construct a parameterized family of expressions

\[
R_{3}(a, m) = (a_1 + a_2 + a_3) \sum_{5} W_{m_1,m_2} C_{n_1,n_2,n_3} + (a_1 + a_3 + a_5) \sum_{5} W_{m_3,m_4} C_{m_1,m_2,m_3} 
- (a_1 + a_2 + a_3 + a_4 + a_5) \sum_{5} W_{m_3,m_4} C_{m_1,m_2,m_5} + a_5 \sum_{5} W_{m_2,m_4} C_{m_1,m_3,m_5} 
- (a_1 + a_4 + a_5) \sum_{5} W_{m_2,m_3} C_{m_1,m_3,m_4} - (a_1 + a_2 + a_3) \sum_{5} W_{m_1,m_3} C_{m_2,m_3,m_4} 
+ a_4 \sum_{5} W_{m_2,m_3} C_{m_1,m_4,m_5} + a_5 \sum_{5} W_{m_1,m_4} C_{m_2,m_3,m_5} 
+ a_1 \sum_{5} W_{m_1,m_2} C_{m_3,m_4,m_5} + a_2 \sum_{5} W_{m_1,m_3} C_{m_2,m_4,m_5},
\]

where \( \sum_{5} \) and \( \sum_{6} \) refer to the degree 5 and 6 in the associated graded algebra, respectively.
where $a = (a_1, \ldots, a_6)$ and $m = (m_1, \ldots, m_5)$. The fact that $R(a, m) = 0$ in the associated graded algebras allows us to create certain quantum corrections in order to write the $C_{a_1, a_2, a_3}$ generators in terms of lower weight generators. Of particular interest will be the four relations $R_1(a, m, n, 4, 1, 0)$, $R_1(a, m, n, 3, 2, 0)$, $R_1(a, m, n, 3, 1, 1)$, and $R_1(a, m, n, 2, 2, 1)$ for $m > n > 4$. The first two of these produce five linearly independent relations (built from choices of the $a_i$), while the last two produce three linearly independent relations. In total, we have 16 total relations at weight $m + n + 12$. This is most fortunate as the expansion of these relations only considers sixteen cubic generating fields $C_{m+n+9-i, 0}$ for $0 \leq i \leq 9$, $C_{m+n+4+i, 5-i, 0}$ for $0 \leq i \leq 3$, $C_{m+n, 9, 0}$, and $C_{m+n, 9, 0, 0}$ and thus these relations may be used to write these generators in terms of lower weight terms. The result is that all cubic generating fields of weight 23 and higher can be written in terms of cubic generators of weight 13 cubic generators we consider the following system of three equations

$$3C_{7,0,0} + 42C_{4,3,0} = 28\partial C_{3,3,0} = 14\partial C_{6,0,0} - 252\partial^2 C_{3,2,0} + 42\partial^3 C_{2,2,0} - 77\partial^4 C_{3,3,0} + 8\partial^3 C_{2,0,0} - 6\partial^2 C_{0,0,0},$$

$$27C_{5,2,0} + 81C_{4,3,0} - 24\partial C_{3,3,0} = 12\partial C_{6,0,0} - 351\partial^2 C_{3,2,0} + 81\partial^3 C_{2,2,0} - 96\partial^4 C_{3,3,0} + 108\partial^3 C_{2,0,0} - 8\partial^2 C_{0,0,0},$$

$$30C_{4,3,0} + 12C_{5,2,0} + 13C_{7,0,0} = R_2(2, 1, 0, 0, 0).$$

Next, one can check that the determinant of the left-hand side is nonzero and thus we can write $C_{7,0,0}$, $C_{5,2,0}$, and $C_{4,3,0}$ in terms of lower weight generators.

The lowest weight “quantum correction” relation occurs at weight 12 where, prior to this final reduction, the necessary generators are $C_{6,0,0}$, $C_{4,2,0}$, and $C_{3,3,0} - C_{5,1,0}$ can immediately be removed by our previous discussion. Next, we have

$$30C_{4,2,0} = -79C_{6,0,0} + R_2(1, 1, 0, 0, 0),$$

leaving only the for $C_{5,0,0}$ and $C_{3,3,0}$, which may not be removed. Interestingly, there is a non-trivial relation involving cubic generators at weight 12 but it only involves terms that have been differentiated. Namely, we have

$$9\partial^3 C_{4,0,0} - 18\partial^2 C_{2,2,0} - 24\partial\partial C_{3,3,0} + 18\partial\partial C_{2,0,0} - \partial^3 C_{0,0,0} = 0.$$
meaning the only weight 11 generator needed is \( C_{5,0,0} \). Similarly, at weight 10, we only need \( C_{4,0,0} \) because

\[
18C_{2,2,0} = 9C_{4,0,0} - 24\partial C_{3,0,0} + 18\partial^2 C_{2,0,0} - \partial^4 C_{0,0,0}.
\]

After all of this, and other similar calculations, the only cubic generators that are required in \( G \) are

\( C_{0,0,0} \), \( C_{m,0,0} \) for \( 2 \leq m \leq 6 \), and \( C_{3,3,0} \).

### 3.4. Quadratic generators.

Now we move to the quadratic terms \( W_{2m,0} \) for \( m \geq 0 \). We first establish some basic facts about \( W_{a,b} \). Using the fact that

\[
\partial W_{a,b} = W_{a+1,b} + W_{a,b+1}
\]

we have using Binomial Theorem

\[
\partial^k W_{a,0} = \sum_{j=0}^{k} \binom{k}{j} W_{a+k-j,j}
\]

and we rewrite \( W_{a,b} \) as

\[
W_{a,b} = \sum_{i=0}^{b} (-1)^{b-i} \binom{b}{i} \partial^i W_{a+b-i,0}
\]

We need the following lemmas:

**Lemma 3.1.** For \( p_1, p_2, m_1, m_2 \geq 0 \) we have

\[
(W_{p_1,p_2})_{(0)} W_{m_1,m_2} = \frac{(-1)^{p_1} + (-1)^{p_2}}{3!} W_{p_1+p_2+m_1+3,m_2} + \frac{(-1)^{p_1} + (-1)^{p_2}}{3!} W_{p_1+p_2+m_2+3,m_1}
\]

and

\[
(W_{p_1,p_2})_{(1)} W_{m_1,m_2} = \frac{(-1)^{p_1}(p_1 + m_1 + 3) + (-1)^{p_2}(p_2 + m_1 + 3)}{3!} W_{p_1+p_2+m_1+2,m_2} + \frac{(-1)^{p_1}(p_1 + m_2 + 3) + (-1)^{p_2}(p_2 + m_2 + 3)}{3!} W_{p_1+p_2+m_2+2,m_1}
\]

**Proof.** This follows immediately by direct computation with (2.5) - (2.7). \( \square \)

Our main tool will be the operator \((W_{0,0})_{(1)}\). In particular, by direct computation using (3.10) we have that

\[
(W_{0,0})_{(1)} W_{a,0} = \frac{a+3}{3} W_{a+2,0} + W_{2,a} = \frac{a+3}{3} W_{a+2,0} + W_{a,2}
\]

Now, using (3.14) we have

\[
W_{a,2} = W_{a+2,0} - 2\partial W_{a+1,0} + \partial^2 W_{a,0}
\]

and so we rewrite (3.18) as:

\[
(W_{0,0})_{(1)} W_{a,0} = \frac{a+3}{3} W_{a+2,0} + W_{a+2,0} - 2\partial W_{a+1,0} + \partial^2 W_{a,0}
\]

\[= \frac{a+6}{3} W_{a+2,0} - 2\partial W_{a+1,0} + \partial^2 W_{a,0} \]

**Lemma 3.2.** For \( a \geq 0 \) and \( k \geq 2 \) we have

\[
(W_{0,0})_{(1)} (\partial^k W_{a,0}) = \sum_{j=0}^{k} \sum_{i=2}^{j} \frac{1}{3} \binom{k}{j} (-1)^{j-i} \left( a + k - j + 3 \right) \binom{j}{i} + (j+3) \binom{j+2}{i} \partial^i W_{a+k+2-i,0}
\]
Proof. We have, by application of (3.13) and then (3.16),

\[(W_{0,0})_{(1)}(\partial^k W_{a,0}) = (W_{0,0})_{(1)} \left( \sum_{j=0}^{k} \binom{k}{j} W_{a+k-j,j} \right)\]

\[= \sum_{j=0}^{k} \binom{k}{j} \frac{1}{3} (a + k - j + 3) W_{a+k-j+2,j} + \sum_{j=0}^{k} \binom{k}{j} \frac{1}{3} (j + 3) W_{a+k-j,j+2}.\]

Now, using (3.14) we have

\[(W_{0,0})_{(1)}(\partial^k W_{a,0}) = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{3} (a + k - j + 3) \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \partial^i W_{a+k+2-i,0} \]

\[+ \sum_{j=0}^{k} \binom{k}{j} \frac{1}{3} (j + 3) \sum_{i=0}^{j+2} \binom{j+2}{i} (-1)^{j+2-i} \partial^i W_{a+k+2-i,0} \]

\[= \sum_{j=0}^{k} \sum_{i=0}^{j} \binom{k}{j} \frac{1}{3} (-1)^{j-i} \left( (a + k - j + 3) \binom{j}{i} + (j + 3) \binom{j+2}{i} \right) \partial^i W_{a+k+2-i,0}.\]

We note that when \(i = 0\) we have

\[(3.22) \quad \sum_{j=0}^{k} \binom{k}{j} \frac{1}{3} (-1)^j (a + k + 6) W_{a+k+2,0} = 0\]

since

\[(3.23) \quad \sum_{j=0}^{k} \binom{k}{j} (-1)^j = 0.\]

Moreover, when \(i = 1\) we have

\[\sum_{j=0}^{k} \frac{1}{3} \binom{k}{j} (-1)^{j-1} \left( (a + k - j + 3) j + (j + 3)(j + 2) \right) \partial W_{a+k+1,0} \]

\[= \sum_{j=0}^{k} \frac{1}{3} \binom{k}{j} (-1)^{j-1} \left( (a + k + 8) j + 6 \right) \partial W_{a+k+1,0}.\]

We note that

\[(3.24) \quad \sum_{j=0}^{k} (-1)^j \binom{k}{j} ((a + k + 8) j + 6) = 0\]

(here we use the fact that \(\sum_{j=0}^{k} (-1)^j \binom{k}{j} P(j) = 0\) when \(P(j)\) is a polynomial of degree \(\leq k\)). Thus our claim is proved.

We have thus shown that when \((W_{0,0})_{(1)}\) acts on a second or higher derivative of \(W_{a,0}\) it introduces no 0-th and 1-st derivatives of terms of the form \(W_{a,n}\).

Lemma 3.3.

\[(3.25) \quad (W_{0,0})_{(1)} \circ W_{a,b} W_{c,d} \circ = \circ ((W_{0,0})_{(1)} W_{a,b}) W_{c,d} \circ + \circ W_{a,b} ((W_{0,0})_{(1)} W_{c,d}) \circ\]

Proof. We use (26) to obtain

\[(W_{0,0})_{(1)} \circ W_{a,b} W_{c,d} \circ = \circ ((W_{0,0})_{(1)} W_{a,b}) W_{c,d} \circ + \circ W_{a,b} ((W_{0,0})_{(1)} W_{c,d}) \circ + ((W_{0,0})_{(0)} W_{a,b})_{(0)} W_{c,d}\]

This completes the proof of Lemma 3.3.
Thus proving our claim. □

Using (3.15) twice we have that

\[
\begin{align*}
((W_{0,0})_{(0)}W_{a,b})_{(0)}W_{c,d} &= \frac{1}{3}(W_{a+3,b} + W_{a,b+3})_{(0)}W_{c,d} \\
&= \frac{1}{3}(W_{a+3,b})_{(0)}W_{c,d} + \frac{1}{3}(W_{a,b+3})_{(0)}W_{c,d} \\
&= \frac{1}{3} \left( (-1)^a + (-1)^b \right) W_{a+3,b+c+3,d} + \frac{(-1)^a + (-1)^b}{3!} W_{a+3+b+c+d,3} \\
&+ \frac{1}{3} \left( (-1)^a + (-1)^b \right) W_{a+b+3+c+3,d} + \frac{(-1)^a + (-1)^b}{3!} W_{a+b+3+d,3} \\
&= 0
\end{align*}
\]

thus proving our claim.

First, we note that the generators \( W_{0,0}, W_{2,0}, \ldots, W_{8,0} \) cannot be removed as there are no relations which allow us to rewrite them as normally ordered polynomials of our remaining cubic generators and normally ordered polynomials of lower weight terms. The following terms, however, can be removed:

\[
\begin{align*}
W_{10,0} &= \frac{17360}{6777} C_{0,0,0,0} \partial C_{1,0,0,0} - \frac{2240}{20331} \partial^2 C_{0,0,0,0} C_{0,0,0,0} - \frac{560}{251} \partial^2 C_{0,0,0,0} C_{0,0,0,0} - \frac{1120}{251} \partial C_{0,0,0,0} C_{0,0,0,0} \\
&+ \frac{10360}{2259} \partial W_{0,0} \partial^2 W_{3,0} - \frac{86744668}{101655} \partial^2 W_{1,0} \partial W_{2,0} - \frac{50002820}{20331} \partial^2 W_{0,0} \partial W_{3,0} \\
&+ \frac{3920}{2259} W_{0,0} \partial W_{1,0} - \frac{350}{251} W_{0,0} \partial W_{5,0} + \frac{458080}{6777} \partial W_{0,0} W_{0,0} W_{1,0} \\
&+ \frac{81200}{6777} \partial W_{0,0} \partial W_{0,0} W_{0,0} - \frac{336}{251} \partial W_{0,0} W_{5,0} + \frac{173270936}{101655} \partial W_{2,0} W_{3,0} - \frac{86874868}{33885} \partial W_{2,0} \partial W_{2,0} \\
&= \frac{10992884}{101655} \partial W_{2,0} \partial W_{3,0} - \frac{560}{2259} \partial W_{2,0} \partial W_{0,0} W_{0,0} + \frac{24975370}{6777} \partial^2 W_{2,0} \partial^2 W_{2,0} \\
&+ \frac{5307442}{33885} \partial^2 W_{2,0} \partial W_{2,0} - \frac{12484430}{20331} \partial^2 W_{0,0} W_{0,0} \partial W_{2,0} - \frac{560}{251} W_{0,0} W_{0,0} W_{2,0} \\
&+ \frac{3290}{2259} W_{0,0} W_{0,0} + \frac{2240}{251} W_{2,0} W_{4,0} + \frac{40600}{6777} W_{3,0} W_{3,0} - \frac{1256080}{6777} W_{0,0} W_{1,0} W_{1,0} \\
&= \frac{5380942}{101655} \partial^3 W_{1,0} \partial W_{2,0} + \frac{34424}{4518} \partial^2 W_{8,0}
\end{align*}
\]
\[ W_{1,2,0} = \frac{149957732453760}{5282271130981} \cdot C_{0,0,0} C_{4,0,0} + \frac{29449282789080}{5282271130981} \cdot C_{0,0,0} \partial C_{3,0,0} + \frac{308674301111491}{5282271130981} \cdot W_{3,0,0,0,0} W_{5,0,0,0} + \frac{90546594348909}{5282271130981} \cdot W_{1,0,0,0,0} \partial W_{4,0,0} + \frac{132056778274525}{5282271130981} \cdot W_{1,0,0,0,0} W_{1,0,0,0} + \frac{404810761267140}{5282271130981} \cdot \partial C_{0,0,0} + \frac{39272157761934891}{5282271130981} \cdot \partial W_{0,0,0,0,0} W_{1,0,0} + \frac{385990604483438891}{5282271130981} \cdot \partial W_{0,0,0,0,0} W_{0,0,0,0,0} W_{0,0,0,0} + \frac{24061687826154562}{5282271130981} \cdot \partial W_{0,0,0,0,0} W_{0,0,0,0,0} + \frac{205038284206527}{5282271130981} \cdot \partial W_{0,0,0,0,0} W_{0,0,0,0,0} W_{0,0,0,0} + \frac{10564542261962}{5282271130981} \cdot \partial W_{0,0,0,0,0} W_{0,0,0,0,0} W_{0,0,0,0} \]

and \( W_{14,0} \), for which we do not display an explicit formula for the sake of brevity. We note that \( W_{14,0} \) can be written purely in terms of generators of the form \( W_{2m,0} \) for \( 0 \leq m \leq 6 \). Importantly,
we note that any \( \partial^k W_{2m,0} \) term which is not part of a product is a second derivative or higher \((k \geq 2)\).

Suppose now that we have rewritten \( W_{a,0} \), a even, as a normally ordered polynomial of lower weight quadratic terms and their derivatives, where any \( \partial^k W_{2m,0} \) term which is not part of a product is a second derivative or higher \((k \geq 2)\). We then have, using (3.21),

\[
W_{a+2,0} = \frac{3}{a + 6} ((W_{0,0})_1 W_{a,0} + 2 \partial W_{a+1,0} - \partial^2 W_{a,0})
\]

First we examine \((W_{0,0})_1 W_{a,0}\). We note that using Lemma 3.3 any normally ordered product of two or more terms of the form \( W_{k,0} \) and their derivatives will itself become a sum of such products. Using Lemma 3.2 we see that with application of \((W_{0,0})_1\) the only \( \partial^k W_{2m,0} \) which \((W_{0,0})_1\) introduces that is not part of a product is a second derivative or higher and no first and 0-th derivatives are introduced. Next, we note that \( \partial W_{a+1,0} \) can be rewritten in terms of \( \partial^2 W_{a,0} \) and a normally ordered polynomial of terms of lower weight and their derivatives. Lastly, we note that \( \partial^2 W_{a,0} \) is of the correct form. Thus, we have eliminated all quadratic generators \( W_{2m,0} \) for \( m \geq 5 \).

To summarize, cubic generators \( C_{0,0,0}, C_{m,0,0} \) for \( 2 \leq m \leq 6 \), \( C_{3,3,0} \), quadratic generators \( W_{0,0}, W_{2,0}, \ldots, W_{8,0} \), and \( T_0 \) form a strong generating set of \( \mathcal{A}^{S_3} \).

Equipped with a strong generating set for \((\mathcal{V}^{S_3})^3\) we also obtain a strong generating set for \((\mathcal{V}^{c,3})^3\) for any generic value using Proposition 3.1. To see that this is in fact a minimal generating set it is sufficient to compare the character \( \text{ch}(\mathcal{A}^{S_3})(q) \) \([27]\) and the "free" character coming from the obtained generators

\[
\frac{1}{(q^2; q)_\infty (q^4; q)_\infty (q^6; q)_\infty (q^8; q)_\infty (q^{10}; q)_\infty (q^{11}; q)_\infty (q^{12}; q)_\infty^3}.
\]

These two \( q \)-series agree \( O(q^{13}) \) so no generator up to weight 12 can be removed from the generating set so constructed a minimal set. This finishes the proof of Theorem 1.1 in the introduction.

**Remark 3.2.** We expect that for all generic values the generators of conformal weight \( > 2 \) can be replaced with primary vectors.

4. The Simple Orbifold \( c = \frac{1}{2} \)

In this section we consider the special case when the initial central charge is \( \frac{1}{2} \), and thus the final central charge is \( \frac{1}{2} \). This has a nice connection to the universal even spin VOAs. We now work inside \( \mathcal{V}^{c,3}_\varnothing \). We begin with the fields

\[
v_i = \varpi L_i L_i^\circ + \frac{93}{64} (\partial L_i) (\partial L_i)^\circ - \frac{33}{16} (\partial^3 L_i) L_i^\circ - \frac{9}{128} \partial^4 L_i
\]

for \( i = 1, 2, 3 \) which are each singular in \( \mathcal{V}^{c,3}_\varnothing \) as they are each singular in their appropriate copies of \( \mathcal{V}_{\varnothing}^c \). From here we define an alternative to the standard generating set of \( \mathcal{V}^{c,3}_\varnothing \) which diagonalizes the action of \( (123) \in S_3 \)

\[
L = \frac{1}{\sqrt{3}} (L_1 + L_2 + L_3)
\]

(4.2)

\[
U_1 = \frac{1}{\sqrt{3}} (L_1 + \eta L_2 + \eta^2 L_3)
\]

\[
U_2 = \frac{1}{\sqrt{3}} (L_1 + \eta^2 L_2 + \eta L_3),
\]

where \( \eta \) is a primitive third root of unity. Next, we set

\[
W_{m+4} = \varpi (\partial^m U_1) U_2^\circ + (-1)^m \varpi (\partial^m U_2) U_1^\circ
\]

(3.3)

and

\[
C_{m+6}^\pm = \varpi (\partial^m U_1) U_1^\circ U_2^\circ \pm (\partial^m U_2) U_2 U_2^\circ
\]

(3.4)
and by [27] we know that \( \left( \varphi_{\mathbb{Z}} \right)^2 \) is strongly generated by \( W_{m_1}, C_{m_2}^+, \) and \( C_{m_3}^- \) for \( m_1 \in \{ 4, 5, 6, 7, 8, 9, 10 \}, m_2 \in \{ 6, 8, 9, 10 \}, \) and \( m_3 \in \{ 6, 8, 9 \} \). Now we transport the singular vectors \( \left( \varphi_{\mathbb{Z}} \right)^2 \) into \( \left( \varphi_{\mathbb{Z}} \right)^2 \) by defining

\[
S = 128 \xi LLL_{12}^\pm + 768 \xi U_1 U_2 U_{12}^\pm + 128 \xi U_1 U_2 U_{12}^\pm + 128 \xi U_2 U_{12}^\pm - 264 \xi L U_1 (\partial^2 U_2)^\pm \\
- 264 \xi U_1 (\partial U_2)^\pm - 264 \xi (\partial^2 L) U_1 U_2 + 186 \xi (\partial L) (\partial L) U_1 U_2 + 372 \xi (\partial U_1) (\partial U_2) U_1 U_2 + 170^4 L
\]

\[
S_1 = 384 \xi L U_{12}^\pm + 384 \xi L U_{12}^\pm + 384 U_1 U_2 U_{12}^\pm - 264 \xi L (\partial^2 U_1) U_2 + 372 \xi (\partial L) (\partial U_1) U_1 U_2^\pm + 170^4 U_1
\]

\[
S_2 = 384 \xi L U_{12}^\pm + 384 \xi L U_{12}^\pm + 384 U_1 U_2 U_{12}^\pm - 264 \xi L (\partial^2 U_2) U_1 + 372 \xi (\partial L) (\partial U_2) U_2 + 170^4 U_2
\]

and observe that \( S, S_1, S_2 \) are all weight 6 and transform in parallel to \( L, U_1, U_2 \) with respect to the \( S_3 \) action. From these parts we define the following \( S_3 \) invariant singular vectors

\[
V_{\pm}^k = \xi U_1 S_{12}^k \pm \xi U_2 S_{12}^k \\
V_{0}^k = \xi (\partial U_1) S_{12}^k \pm \xi (\partial U_2) S_{12}^k \\
V_{10}^k = \xi (\partial^2 U_1) S_{12}^k \pm \xi (\partial^2 U_2) S_{12}^k \\
Q_{10}^k = \xi U_1 U_2 S_{12}^k \pm \xi U_2 U_1 S_{12}^k.
\]

Now, we have everything ready to prove our first result of this section.

**Theorem 4.1.** The simple orbifold \( \left( L^{\mathbb{Z}} \right)_L \) is of type 2,4,5,6,7,8,9 and is strongly generated by \( L \), together with \( W_4, W_5, W_6, W_7, W_8, W_9, \) and \( C_6^+ \).

**Proof.** This result from fairly routine calculations involving the generators described in (4.2) as well as the singular vectors (4.3). For example, the equation

\[
128 C_6^+ = S + 450 \sqrt{3} W_5 - 768 \xi L, W_4^+ - 128 \xi LLL_{12}^\pm - 186 \xi (\partial L) (\partial L) U_1 U_2 + 264 \xi (\partial^2 L) U_1 U_2 - 186 \xi (\partial^2 U_1) U_2 + (-17 + 75 \sqrt{3}) \partial^4 L,
\]

eliminating the need for \( C_6^+ \) from the strong generating set. Similar equations exist to remove the remaining superfluous strong generators. \( \square \)

Now we move our attention to the orbifold \( \left( L^{\mathbb{Z}} \right)_L \) where the additional nontrivial action is given by \( U_1 \leftrightarrow U_2 \). Due to (4.3) and (4.4) it is clear that

\[
W_{2m+4}, C_{m+6}^+ \in \left( L^{\mathbb{Z}} \right)_L
\]

for all \( m \geq 0 \). In fact, of the generators described in Theorem 4.1, we have \( L, W_4, W_6, W_8 \in \left( L^{\mathbb{Z}} \right)_L \) while \( C_6^+ \rightarrow -C_6^-, W_5 \rightarrow -W_5, W_7 \rightarrow -W_7, \) and \( W_9 \rightarrow -W_9 \) under the additional \( Z_2 \) action. In fact, we can check that the fields \( L, W_4, W_6, W_8 \) close under OPE and thus form a subalgebra of \( \left( L^{\mathbb{Z}} \right)_L \). Direct computation shows that OPEs of odd terms are also in the subalgebra generated by \( L, W_4, W_6, W_8 \). From this, it is a routine calculation to construct the appropriate relations that remove the need for generators of the form \( \xi (\partial^2 W_{2m+1}) (\partial^2 W_{2n+1}) \) and \( \xi (\partial^2 W_{2m+1}) (\partial^2 C_{6}^+) \) from the strong generating set.

**Theorem 4.2.** The orbifold algebra \( \left( L^{\mathbb{Z}} \right)_L \) is strongly generated by the fields \( L, W_4, W_6, W_8 \) and is of type 2,4,6,8.

**Remark 4.1.** Using OPE package [30] we have also shown that \( L, W_2, W_4, W_6 \) generators in Theorem 4.2 can be replaced with another strong set of generators \( L, W_4, W_6, W_8 \), where \( W_1 \) is primary vectors under \( L \). Thus this orbifold algebra is a W-algebra.
Now, we move to classify this orbifold using the tools in [21] which requires we correct the weight 4 field to be primary which may be done by
\[ \tilde{W}_4 = \mu(W_4 - \frac{44}{59} LL_0 - \frac{9}{118} \partial^2 L). \]
where \( \mu \) is a parameter to be fixed later. In [21], the universal two parameter algebra, \( \mathcal{W}^e(c, \lambda) \), of type \( \mathcal{W}(2, 4, 6, \ldots) \) was rigorously constructed. This algebra is strongly generated by infinitely many fields in weights 2, 4, 6, \ldots, and weakly generated by a primary weight 4 field which we denote by \( \mathcal{W}_4^\lambda \). We consider this algebra with central charge \( \frac{4}{5} \) to correspond with the central charge of our orbifold. A normalization can be chosen for this field so that

\[ (W_4^4)_{(3)} W_4^4 = 816\lambda W_4^4 - \frac{1088}{21} \left( -\frac{2303\lambda^2}{4} - 1 \right) LL_0^e + \frac{680}{147} \left( -\frac{2303\lambda^2}{4} - 1 \right) 2\partial^2 L. \]

We also calculate

\[ (\tilde{W}_4)_{(3)} \tilde{W}_4 = -\frac{231}{118} \mu \tilde{W}_4 + \frac{18552}{3481} \mu^2 LL_0^e - \frac{6615}{13924} \mu^2 \partial^2 L. \]

Equating (4.7) and (4.8) gives \( \lambda = \frac{27}{2599} \) and \( \mu = -\frac{1088}{3481} \), which does not match any other known algebras of type \( (2, 4, 6, \ldots, N) \). For instance, the \( Z_2 \) orbifold of the \( \mathfrak{sl}_2 \) parafermion algebra \((N_0(\mathfrak{sl}_2))_{Z_2}\) has a \( \lambda \) value of \( \lambda = \frac{11}{27} \). The simple affine \( \mathcal{W} \)-algebras of type \( B \) and \( C \) as well as the \( Z_2 \) orbifold of the affine \( \mathcal{W} \)-algebras of type \( D \) are also of type \( (2, 4, 6, \ldots, N) \) but all have different values for \( \lambda \) when \( k \) is chosen so that their central charge is \( \frac{4}{27} \).

5. The orbifold for \( c = -\frac{22}{27} \) and affine \( \mathcal{W} \)-algebras of type \( G_2 \)

In this part we study the simple \( S_3 \)-orbifold algebra of \( \mathcal{L}_{-\frac{22}{27}} \). We also consider two affine \( \mathcal{W} \)-algebras associated to the exceptional rank two Lie algebra of type \( \mathfrak{g}_2 \). There are four (nontrivial) nilpotent orbits of \( \mathfrak{g}_2 \): short, long, subregular, and regular [14]. In this paper, we consider the regular and subregular and the corresponding affine \( \mathcal{W} \)-algebras. For more about affine \( \mathcal{W} \)-algebras see [20]; see also [17] for more about \( \mathcal{W} \)-algebras of rank 2. As usual \( \mathcal{W}^k(\mathfrak{g}_2, f) \) will denote the universal affine \( \mathcal{W} \)-algebra of type \( G_2 \) and level \( k \) associated to the nilpotent element \( f \). Of course, two affine algebras are isomorphic if nilpotent elements come from the same nilpotent orbit. The unique simple quotient of \( \mathcal{W}^k(\mathfrak{g}_2, f) \) will be denoted by \( \mathcal{W}_k(\mathfrak{g}_2, f) \).

5.1. \( \mathcal{W} \)-algebra \( \mathcal{W}^k(\mathfrak{g}_2, f_{\text{sub}}) \). We choose realization of \( \mathfrak{g}_2 \) using 8 \( \times \) 8 matrices as in [18]. We also choose a nilpotent element \( f_{\text{sub}} \) as in loc.cit. For this element \( f := f_{\text{sub}} \), let \( \{e, f, h\} \) denote the corresponding \( \mathfrak{sl}_2 \) triple. Then with respect to \( \text{ad}(x) \), where \( x = \frac{h}{2} \), we have decompositions

\[ \mathfrak{g} = \sum_{-2 \leq i \leq 2} \mathfrak{g}_i \]

\[ \mathfrak{g}^f = \mathfrak{g}^f_{-2} \oplus \mathfrak{g}^f_{-1} \]

where \( \mathfrak{g}^f_{-2} \) is one-dimensional (this gives a generator of conformal weight 3 inside the \( \mathcal{W} \)-algebra) and \( \mathfrak{g}^f_{-1} \) is 3-dimensional. According to Kac-Wakimoto [20] work \( \mathcal{W}^k(\mathfrak{g}_2, f_{\text{sub}}) \) is of type \( (2^3, 2) \). We denote the conformal generator of weight two by \( L(z) \), with \( E(z) \) and \( F(z) \) two primaries of weight two, and by \( G(z) \) the primary generator of conformal weight 3. Then using the OPE program [30] it is not difficult to obtain relations among generators. As far as we know, these OPE relations first appeared in J. Fasquel’s PhD thesis [17].
Proposition 5.1. The following OPEs hold (we omit OPEs among $L(z)$ and $E(z), F(z), G(z)$ as those are uniquely determined):

\[ E(z)E(w) \sim (10 + 3k)(4 + k)c \frac{z - w}{2(z - w)^3} + \frac{1}{2(z - w)^2} (2(4 + k)(10 + 3k)L(w) - 4(3 + k)F(w)) \]
\[ + \frac{1}{(z - w)^2} ((4 + k)(10 + 3k)(\partial L)(w) - 2(3 + k)(\partial E)(w)) \]
\[ F(z)F(w) \sim -(10 + 3k)(4 + k)c \frac{z - w}{2(z - w)^5} + \frac{1}{(z - w)^4} (-2(4 + k)(10 + 3k)L(w) - 4(3 + k)E(w)) \]
\[ + \frac{1}{(z - w)^3} (-4 + k)(10 + 3k)(\partial L)(w) - 2(3 + k)(\partial E)(w)) \]
\[ G(z)G(w) \sim \frac{2(2 + k)(16 + 5k)}{(z - w)^3} \frac{F(w)}{2(z - w)^2} + \frac{6}{2(z - w)^2} (2 + k)(4 + k)(4 + 3k)(16 + 3k)(\partial^3 L)(w) \]
\[ \frac{1}{z - w} \left( \frac{2}{2(z - w)^2} E(w)F(w)E(w) + 2(4 + k)z L(w)F(w) + 2\partial G(w) + \frac{2}{6} \partial^2 F(w) \right) \]
\[ F(z)G(w) \sim \frac{2(2 + k)(16 + 5k)}{(z - w)^3} \frac{E(w)}{2(z - w)^2} + \frac{6}{2(z - w)^2} (2 + k)(16 + 5k)(10 + 3k)(\partial E)(w) \]
\[ \frac{1}{z - w} \left( \frac{2}{2(z - w)^2} E(w)F(w)E(w) + 2(4 + k)z L(w)E(w) + \frac{2}{6} \partial^2 F(w) \right) \]

where the central charge is $c = -\frac{4k + 21}{2k + 4}$.

Next we specialize $k = -\frac{46}{9}$ in Proposition 5.1. We consider the ideal $I = \langle G \rangle$ generated by the primary element of degree 3. From the OPEs for the quadratic generators, inside $W := W^{-\frac{46}{9}}(g_2, f_{sub})/I$ we can define three conformal vectors:

\[ L_1(z) = -\frac{5}{12}E(z) - \frac{5i}{4\sqrt{3}}F(z) + \frac{1}{3}L(z), \]
\[ L_2(z) = -\frac{5}{12}E(z) + \frac{5i}{4\sqrt{3}}F(z) + \frac{1}{3}L(z), \]
\[ L_3(z) = \frac{5}{6}E(z) + \frac{1}{3}L(z), \]

which mutually commute in $W$ (i.e. $L_i(z)L_j(w) \sim 0, i \neq j$) and each has central charge $c = -\frac{42}{9}$.

Therefore $W$ must be a quotient of $\mathcal{V}^{\text{sing}}_{22/5}$. It is easy to see now that the maximal ideal $I_{\text{max}} \subset W^{-\frac{46}{9}}(g_2, f_{sub})$ must contain singular vectors $v_{\text{sing}}^{(i)}$, $i = 1, 2, 3$ of degree 4 for each of the three copies $\mathcal{V}^{\text{sing}}_{22/5}$. We conclude that $I_{\text{max}} = \langle G, v_{\text{sing}}^{(1)}, v_{\text{sing}}^{(2)}, v_{\text{sing}}^{(3)} \rangle$ and therefore
Proposition 5.2. We have an isomorphism of simple vertex operator algebras

\[ W_{\mathfrak{g}_2}(\mathfrak{g}_2, f_{\text{prin}}) \cong \mathcal{L}^{(3)}_{-22/3}. \]

5.2. \( W \)-algebra \( W^k(\mathfrak{g}_2, f_{\text{prin}}) \). In this part we construct the principal \( W \)-algebra of type \( \mathfrak{g}_2 \). The \( W \)-algebra \( W^k(\mathfrak{g}_2, f_{\text{prin}}) \) is known to be of type \( (2, 6) \), where \( \omega \) is just the conformal vector given in [20]. It is convenient to use the standard parametrization of the level \( k = -h^\vee + \frac{p}{q} = -4 + \frac{p}{q} \), so that the central charge is \( c(k) = -\frac{2(12k-7)p(7p-4q)}{pq} \). We assume here that

\[ (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) \neq 0. \]

If \( k \) one of the four roots of this polynomial (then \( k \) is generic) one has to adjust the weight 6 generator appropriately. We omit this computation here.

Using the standard approach, we have constructed this algebra leading to the following nonzero OPEs for the weight 6 generator, \( W \), with itself. The OPEs between \( L(z) \) and \( W(z) \) are clear, and we have

\[ W(z)W(w) \sim \sum_{n \geq 0} W_{(n)} W_{(n+1)}. \]

Below we give explicit formulas for all the summands. As far as we know these formulas did not appear in the literature.

\[ W_{(11)} = -\frac{2p_0(k)}{81} (7k + 24)(12k + 41) (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) \]

\[ W_{(9)} = \frac{4p_0(k)}{27} (k + 4) (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) L \]

\[ W_{(8)} = \frac{2p_0(k)}{27} (k + 4) (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) \partial L \]

\[ W_{(7)} = \frac{p_1(k)}{108} (-62(k + 4)^2 LL^\vee + 3 (84k^2 + 579k + 1000) \partial^2 L) \]

\[ W_{(6)} = \frac{p_1(k)}{108} \left( -62(k + 4)^2 (\partial L)L^\vee + \frac{2}{3} (84k^2 + 579k + 1000) \partial^3 L \right) \]

\[ W_{(5)} = \frac{280p_3(k)}{9} W + p_2(k) \Lambda^2(L) \]

\[ W_{(4)} = \frac{140p_3(k)}{9} \partial W + p_2(k) \Lambda^4(L) \]

\[ W_{(3)} = p_6(k) \left( \frac{10}{9} (588k^2 + 3929k + 6504) \partial^2 W - \frac{1240}{3} (k + 4)^2 LW \right) + p_4(k) \Lambda^3(L) \]

\[ W_{(2)} = p_5(k) \Omega^2(W, L) + p_4(k) \Lambda^2(L) \]

\[ W_{(1)} = p_7(k) \Omega^1(W, L) + p_6(k) \Lambda^1(L) \]

\[ W_{(0)} = p_7(k) \Omega^0(W, L) + p_6(k) \Lambda^0(L) \]
where
\[ p_0(k) = (k + 4)(2k + 5)(2k + 7)(3k + 10)(7k + 22)(7k + 27)(9k + 27)(9k + 34)(11k + 40) \]
\[ (12k + 37)(15k + 52)(15k + 53)(18k + 65) \]
\[ p_1(k) = (k + 4)^2(2k + 5)(2k + 7)(3k + 10)(7k + 22)(8k + 27)(9k + 34)(11k + 40) \]
\[ (12k + 37)(15k + 52)(18k + 65) (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) \]
\[ p_2(k) = (k + 4)^2(2k + 5)(2k + 7)(3k + 10)(7k + 22)(8k + 27)(9k + 34)(11k + 40) \]
\[ (12k + 37)(15k + 52)(18k + 65) \]
\[ p_3(k) = (k + 4)(2k + 7)(3k + 10)(7k + 20)(12k + 35)(13k + 48)(24k + 89) \]
\[ (3k^2 + 24k + 47) (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) \]
\[ p_4(k) = (k + 4)^2(2k + 7)(3k + 10)(9k + 34)(11k + 40)(12k + 37) \]
\[ p_5(k) = (k + 4)(2k + 7)(3k + 10)(7k + 20)(24k + 89) (3k^2 + 24k + 47) \]
\[ (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) \]
\[ p_6(k) = (k + 4)^2(2k + 5)(2k + 7)(3k + 10)(9k + 34) \]
\[ p_7(k) = (k + 4) (3k^2 + 24k + 47) (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) \]
and the fields \( \Lambda^n(L) \) only depend on \( L \) while every summand in the fields \( \Omega^n(W) \) depends on \( W \).

**Remark 5.1.** In addition to constructing \( W_{\mathfrak{h}}(\mathfrak{g}_2, f_{\text{prin}}) \) directly using quantum Hamiltonian reduction, we in parallel showed that for generic central charge \( c \) there is a unique universal \( W(2, 6) \) algebra. This has been studied previously in the physics literature [9].

With these explicit formulas in hand, we are now ready to prove the main result of this section, also stated in the introduction.

**Theorem 5.1.** We have an isomorphism of rational vertex algebras:
\[ W_{-\frac{3}{4}}(\mathfrak{g}_2, f_{\text{sub}})^{S_3} \cong W_{-\frac{3}{4}}(\mathfrak{g}_2, f_{\text{prin}}). \]

**Proof.** Using Proposition 5.3 and explicit generators of weights 2 and 6 of \( (L_{-22/3})^{S_3} \) obtained in [27], we can explicitly compute OPEs among generators. Computer computation with the OPE package [30] shows that we get identical OPEs also from \( W_{-\frac{3}{4}}(\mathfrak{g}_2, f_{\text{prin}}) \) (we only have to slightly normalize the \( W \) generator given above). Using the universal property for \( W_{\mathfrak{h}}(\mathfrak{g}_2, f_{\text{prin}}) \), we get a vertex algebra map from \( W_{-\frac{3}{4}}(\mathfrak{g}_2, f_{\text{prin}}) \) to \( W_{-\frac{3}{4}}(\mathfrak{g}_2, f_{\text{sub}})^{S_3} \). Since the \( S_3 \)-orbifold of a simple vertex algebra is always simple [10], we see that this map factors through the simple quotient \( W_{-\frac{3}{4}}(\mathfrak{g}_2, f_{\text{prin}}) \). \( \square \)

From the OPEs we can also see that there are also collapsing levels to the Virasoro algebra. Here we slightly abuse the term “collapsing level” originally introduced in an important work of Adamovic et al. [10] to indicate those levels for which the simple minimal affine \( W \)-algebra reduces to an affine vertex algebra.

**Proposition 5.3.** The simple affine \( W \)-algebras \( W_k(\mathfrak{g}_2, f_{\text{prin}}) \) collapses to a (simple) Virasoro vertex algebra if and only if \( k \in \{ -3, -1, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}, -\frac{9}{4}, -\frac{11}{4}, -\frac{13}{4}, -\frac{15}{4}, -\frac{17}{4}, -\frac{19}{4}, -\frac{21}{4} \} \).

**Proof.** We first notice that in order for \( W_k(\mathfrak{g}_2) := W_k(\mathfrak{g}_2, f_{\text{prin}}) \) to collapse to a (simple) Virasoro vertex algebra we must have
\[ p_0(k)(7k + 24)(12k + 41) (336k^2 + 2301k + 3940) (588k^2 + 3991k + 6752) = 0. \]
If \( k \in \{ -3, -1, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}, -\frac{9}{4}, -\frac{11}{4}, -\frac{13}{4}, -\frac{15}{4}, -\frac{17}{4}, -\frac{19}{4}, -\frac{21}{4} \} \) this is quite evident from the fact that every summand in every pole of the OPE of \( W \) with itself will be a normally ordered multiple of \( W \) or one of its derivatives. These are also the only levels for which this occurs for the generic values of the central charge.
The remaining cases are more interesting because the relevant Virasoro algebras are minimal. We will start with the details of \( k = -\frac{46}{18} \), which corresponds to a central charge of \( c = -\frac{22}{11} \). In this case there is a singular vector of conformal weight 8 which is inherited from the subalgebra copy of \( V_{-46/3} \), namely

\[
V_8 = -\frac{757693104671125}{2529990231179046912} \left( 29160\xi^2LLL^2 - 12960\xi(\partial L)(\partial L)L^2 - 42120\xi(\partial^2 L)LL - 5670\xi(\partial^2 L)(\partial^2 L) + 720\xi(\partial^3 L)(\partial L)^2 - 3960\xi(\partial^4 L)L^2 - 139\partial^6 L \right).
\]

Using the above notation we have

\[
\begin{align*}
p_4 \left(-\frac{65}{18}\right) & \Lambda^3(L) = V_8 \\
p_4 \left(-\frac{65}{18}\right) & \Lambda^2(L) = \partial V_8 \\
p_6 \left(-\frac{65}{18}\right) & \Lambda^1(L) = \frac{81}{820} \partial^2 V_8 + \frac{78}{205} L V_8^2 \\
p_6 \left(-\frac{65}{18}\right) & \Lambda^0(L) = -\frac{15}{41} (\partial L)V_8^2 + \frac{27}{82} L (\partial V_8)^2,
\end{align*}
\]

and as such \( L \) is not inside the ideal generated by \( W \), meaning \( W_{-\frac{46}{18}}(g_2) \cong V_{-46/3} \). The cases when \( k = -\frac{44}{18} \) also corresponds to \( c = -\frac{22}{11} \) and gives \( W_{-\frac{44}{18}}(g_2) \cong V_{-46/3} \) similarly. Further the cases when \( k = -\frac{47}{18} \) and \( k = -\frac{45}{18} \) correspond to \( c = -\frac{29}{11} \) for which there is also a Virasoro singular vector of conformal weight 8 leading to a set of equations similar to (5.1), yielding

\[
W_{-\frac{47}{18}}(g_2) \cong V_{-3/5} \quad \text{and} \quad W_{-\frac{45}{18}}(g_2) \cong V_{-3/3}.
\]

The cases with \( k = -\frac{44}{18} \) and \( k = -\frac{47}{18} \) correspond to \( c = -\frac{2}{5} \), where we have a singular vector of weight 10. In each of these cases \( \Lambda^1(L) \) and \( \Lambda^0(L) \) are multiples of the singular vector and its derivative, respectively, and thus

\[
W_{-\frac{44}{18}}(g_2) \cong V_{-23/11} \quad \text{and} \quad W_{-\frac{47}{18}}(g_2) \cong V_{-23/11}.
\]

Finally, the cases with \( k = -\frac{45}{18} \) and \( k = -\frac{29}{18} \) correspond to \( c = -\frac{22}{11} \), for which their is a singular vector of weight 4 leading to equations similar to (5.1) for the appropriate poles, thus

\[
W_{-\frac{45}{18}}(g_2) \cong V_{-22/5} \quad \text{and} \quad W_{-\frac{29}{18}}(g_2) \cong V_{-22/5}.
\]

\[\Box\]

Remark 5.2. The above result also gives a new proof or rationality of four admissible \( g_2 \) minimal models: \((p,q) = (5,8), (7,18), (7,15) \) and \((4,11)\). Using asymptotic properties of characters of \( g_2 \) and Virasoro minimal models, it follows that only two additional \( G_2 \) models are extensions of Virasoro minimal models: \((p,q) = (4,13) \) and \((5,11)\) of level \(-\frac{44}{18}\) and \(-\frac{47}{18}\). For the former we obtain decomposition

\[
W_{-\frac{44}{18}}(g_2) = L_{Vir}(c_{3,26},0) \oplus L_{Vir}(c_{3,26},6),
\]

and for the latter we can write

\[
W_{-\frac{47}{18}}(g_2) = L_{Vir}(c_{11,30},0) \oplus M,
\]

where \( M \) is a \( L(c_{11,30},0) \)-module. Conjecturally, we expect

\[
W_{-\frac{29}{18}}(g_2) = L_{Vir}(c_{11,30},0) \oplus L_{Vir}(c_{11,30},6) \oplus L_{Vir}(c_{11,30},24) \oplus L_{Vir}(c_{11,30},63).
\]

The appearance of the group \( S_3 \) in the setup of \( W^k(g_2,f_{sub}) \) is not a coincidence. It is known that any element of the component group \( A(f) \) of a nilpotent orbit \( O \) induces an automorphism of the affine \( W \)-algebra. Component and fundamental groups are always finite and their complete list can be found in [11, Chapter 8]. In particular for the subregular orbit \( O_{sub} \) of \( g_2 \) we have \( \pi(O_{sub}) = A(f_{sub}) = S_3 \). It is also easy to show using explicit OPEs in Proposition 5.1 that
Aut($W^k(\mathfrak{g}_2, f_{\text{sub}})$) = $S_3$. There are a few more examples of nilpotent orbits of simple Lie algebra that are conjecturally related to permutation orbifolds of $(2, 5)$-minimal models, the largest being $S_5$ for a particular nilpotent orbit $E_8(a_7)$ of $E_8$. Let $f_{s_5} \in E_8(a_7)$. Motivated by numerical evidence we expect that

**Conjecture 1.** We have an isomorphism

$$W_k(\mathfrak{e}_8, f_{s_5}) \cong \mathcal{L}_{-22/5}^{S_5}$$

and moreover

$$W_k(\mathfrak{e}_8, f_{s_5})^{S_5} \cong W_{-144/5}(\mathfrak{e}_8, f_{\text{sub}})$$

where $k$ is a certain level such that $c(k) = -22$.

The vertex algebra $W^k(\mathfrak{e}_8, f_{\text{sub}})$ appeared in the work of Arakawa and van Ekeren, where they determined its type [6]. Another example coming from the principal affine $W$-algebras of type $\mathfrak{f}_4$ was discussed in [27].
\[ \Lambda^0(L) = \left( \frac{5}{108} (k+4)^4 \right) (6400k^2 + 44499k + 77324) \left( 15552k^2 + 104313k + 174820 \right) \delta (\partial L) \mathcal{L} \mathcal{L} \mathcal{L} \mathcal{L} \mathcal{L} \]
\[
\Lambda^1(L) = \left( \frac{1}{54} (k + 4)^4 \right) \left( 6400k^2 + 4460k + 77324 \right) \left( 1552k^2 + 104313k + 174820 \right) \zeta L L L L L L
\]
\[
\quad - \frac{1}{108} (k + 4)^3 \left( 18396661248k^6 + 375252938352k^5 + 3187752964032k^4 + 14435363779845k^3 \\
\quad + 3675141329532k^2 + 49876565020640k + 28189138866400 \right) \zeta (\partial L)(\partial L)L L L L
\]
\[
\quad - \frac{1}{108} (k + 4)^3 \left( 7309080576k^6 + 149628924864k^5 + 127601582004k^4 + 580219172295k^3 \\
\quad + 148369352826k^2 + 20229656112560k + 1148972582400 \right) \zeta (\partial^2 L)L L L L
\]
\[
\quad + \frac{1}{36} (k + 4)^2 \left( 23462695872k^8 + 639247348656k^7 + 7617735370116k^6 + 51859882602369k^5 \\
\quad + 220598608315643k^4 + 600396725500187k^3 + 1021026239675080k^2 + 991925566014784k \\
\quad + 421482454671360 \right) \zeta (\partial^2 L)(\partial L)(\partial L) + \frac{1}{18} (k + 4)^2 \left( 699584576k^8 + 191067584400k^7 \\
\quad + 2283143277456k^8 + 1558734929978k^5 + 66504839400355k^4 + 181569276291374k^3 \\
\quad + 30977674880856k^2 + 301962382197456k + 128756373080640 \right) \zeta (\partial^2 L)(\partial L)L L L L L L
\]
\[
\quad - \frac{1}{324} (k + 4)^2 \left( 186037762176k^8 + 5066868806496k^7 + 60357766817448k^6 + 410736397701210k^5 \\
\quad + 174641693870509k^4 + 475098412326478k^3 + 8075489288448560k^2 + 784121953710694k \\
\quad + 3329918630393760 \right) \zeta (\partial^2 L)(\partial L)(\partial L) + \frac{1}{324} (k + 4)^2 \left( 32936730624k^8 + 889058750400k^7 \\
\quad + 1071012370616k^6 + 72967497795678k^5 + 31065171178949k^4 + 846002556795866k^3 \\
\quad + 143969104976208k^2 + 139956509179644k + 59505015923712k \right)
\]
\[
\quad - \frac{1}{1944} (k + 4) \left( 491100390540k^{10} + 16750248391296k^9 + 25705128041632k^8 + 23372676209080k^7 \\
\quad + 13944382286929770k^6 + 57038997408972443k^5 + 161993956733818346k^4 + 31543150817670400k^3 \\
\quad + 403003320724401472k^2 + 305066845879548032k + 103900852253614080 \right) \zeta (\partial^2 L)(\partial L)L L L L L L
\]
\[
\quad - \frac{9}{2} (k + 4) \left( 175136919552k^{10} + 597988073054k^9 + 91868150287872k^8 + 83625460625968k^7 + 499482980268456k^6 \\
\quad + 2045391804109553k^5 + 5816036283655148k^4 + 113382401413021648k^3 + 145032681121579968k^2 \\
\quad + 10991883952605344k^4 + 3748159354344960 \right) \zeta (\partial^2 L)(\partial^2 L) + \frac{9}{10} (k + 4) \left( 43592137920k^{10} + 1484929453696k^9 \\
\quad + 227573441700192k^8 + 206632085608936k^7 + 1230964163476976k^6 + 50273171379297971k^5 + 14254844428755828k^4 \\
\quad + 277094185622399176k^3 + 353390471229322752k^2 + 267011039945734656k + 90762062273495040 \right) \zeta (\partial^2 L)(\partial L)L L L L L L
\]
\[
\quad - \frac{1}{20} (k + 4) \left( 2019366236160k^{10} + 68818738157568k^9 + 1055174614367328k^8 + 9585390321860400k^7 \\
\quad + 5713132180231718k^6 + 233447722837784347k^5 + 662288994566698810k^4 + 128811182671584368k^3 \\
\quad + 1643727302314709024k^2 + 1242686605881161472k + 42267181080901480 \right) \zeta (\partial^2 L)(\partial^2 L) + \frac{3}{280} (1505562992640k^{12} \\
\quad + 61604411452416k^{11} + 1155160057778496k^{10} + 13125726342075312k^9 + 100656494870524668k^8 \\
\quad + 548820011579297193k^7 + 218160334040666329k^6 + 6370267278173246187k^5 + 1356108982536455432k^4 \\
\quad + 20525603316596375472k^3 + 20966534092725067520k^2 + 12977698548834862080k + 3681061731402547200 \right) \zeta (\partial^2 L)
\]
\[\Lambda^2(L) = \left( -\frac{1}{54} (k+4)^3 (92703744k^4 + 1267774752k^3 + 6500755699k^2 + 14813187576k \\
+ 12656385200) \right)^{\frac{1}{2} (k+4)^2 (253925280k^6 + 519206068k^5 + 44221218606k^4 \\
+ 20081263069k^3 + 51279465293k^2 + 69817481988k^1 + 395952508800) \right)^{\frac{1}{8} (\partial L)(\partial L)_{x}^2} \\
+ \frac{1}{105} (k+4)^2 (120102208k^6 + 43502274144k^5 + 37190340636k^4 + 169559043576k^3 \\
+ 4348182557408k^2 + 5946567147312k + 3388333598400) \right)^{\frac{1}{4} (\partial^3 L)(\partial L)^2_{x}} \\
+ \frac{1}{162} (k+4)^2 (701338176k^6 + 14385000672k^5 + 122924343060k^4 + 560169728659k^3 \\
+ 1435743475008k^2 + 1962381639248k + 1117447262400) \right)^{\frac{1}{16} (\partial^3 L)(\partial L)^2_{x}} \\
- \frac{1}{162} (k+4)^2 (2788079616k^8 + 76356765072k^7 + 914888731176k^6 + 626983859931k^5 \\
+ 2680475964464k^4 + 7340930859221k^3 + 12565146432350k^2 + 12289723016608k \\
+ 5258833811200) \right)^{\frac{1}{4} (\partial^3 L)(\partial^3 L)_{x}} - \frac{1}{64} (k+4)^2 (6614576640k^8 + 180882084096k^7 \\
+ 216383576960k^6 + 1479004447592k^5 + 6317515287320k^4 + 1726838244618k^3 \\
+ 29497304984232k^2 + 28788354819528k + 12290465043200) \right)^{\frac{1}{4} (\partial^3 L)(\partial L)^2_{x}} \\
- \frac{1}{324} (k+4)^2 (985074048k^8 + 269302852224k^7 + 3220621636932k^6 \\
+ 22006242372675k^5 + 93966753547809k^4 + 256756287752401k^3 + 43841231891022k^2 \\
+ 427697924147920k + 182513991043200) \right)^{\frac{1}{8} (\partial^3 L)(\partial L)^2_{x}} \\
\Lambda^3(L) = \left( -\frac{1}{108} (k+4)^3 (92703744k^4 + 1267774752k^3 + 6500755699k^2 + 14813187576k \\
+ 12656385200) \right)^{\frac{1}{4} (\partial L)(\partial L)_{x}^2} \\
+ \frac{1}{108} (k+4)^2 (2112058368k^6 + 4332863808k^5 + 370338214908k^4 + 1688049810869k^3 \\
+ 432770491216k^2 + 591685603252k + 3370337692800) \right)^{\frac{1}{4} (\partial^2 L)(\partial^2 L)_{x}^2} \\
+ \frac{1}{72} (k+4)^2 (18545708160k^8 + 506465186976k^7 + 605207604632k^6 + 412949647054k^5 \\
+ 17613213350245k^4 + 48072489348134k^3 + 81991749335956k^2 + 79898761245686k^1 \\
+ 340581640627200) \right)^{\frac{1}{4} (\partial^3 L)(\partial L)_{x}^2} - \frac{1}{1296} (k+4)^2 (26316057600k^8 + 71948778854k^7 \\
+ 86057093354k^6 + 588029219924k^5 + 2511085699328k^4 + 68619639859086k^3 \\
+ 1171797678662880k^2 + 114327939649436k + 487932447129600) \right)^{\frac{1}{4} (\partial^3 L)(\partial L)^2_{x}} \\
+ \frac{1}{1776} (418211942400k^{10} + 1429587823104k^9 + 21989026204320k^8 + 200412265390992k^7 \\
+ 119860888720990k^6 + 49151648188457845k^5 + 13995840354432511k^4 + 273252685532068176k^3 \\
+ 350073420007844192k^2 + 26574079350327552k + 90771065277696000) \cdot \partial^2 L \right)
\[ \Lambda^4(L) = \left( \frac{5}{27} (k + 4)^2 (23352k^2 + 159815k + 273412) \right) \cdot (\partial L) L L \cdot \]
\[- \frac{1}{12} (k + 4) (204624k^4 + 2806272k^3 + 14435089k^2 + 33007496k + 28308960) \cdot (\partial^2 L)(\partial L) \cdot \]
\[- \frac{1}{324} (k + 4) (2441376k^4 + 33467028k^3 + 172062031k^2 + 393206164k + 33700960) \cdot (\partial^3 L) L \cdot \]
\[+ \frac{1}{432} \left( 1185408k^6 + 24378480k^5 + 208905480k^4 + 954774795k^3 + 2454576540k^2 \right. \]
\[+ 3365465024k + 1922595712) \cdot (\partial^4 L) \]
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Department of Mathematics and Computer Science, Ursinus College

Email address: csadowski@ursinus.edu