ABSTRACT
Nonparametric linear mixed effects models are preferred due to overcome the restrictions of linear models which need to satisfy distributional assumptions. In these models, smoothing approaches are needed to handle nonparametric part and chosen according to the type of data. When there is a measurement error in the nonparametric part, these smoothing techniques become more complicated. In this paper, we propose wavelet approach to smooth nonparametric function under known measurement error in nonparametric linear mixed effects model and then, we predict random effects parameter. Furthermore, a simulation study is done to demonstrate the theoretical findings by comparing with the case ignoring measurement error. The performances are much better for the proposed model than the no measurement error case.

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Longitudinal data which involve repeated observations of the same things at different points in time are used in NLMM. Waves are not commonly handled for longitudinal data. Müller [8] investigated the nonparametric methods. Kernel-type smoothing methods, smoothing spline methods and regression (polynomial) spline methods for longitudinal data have been largely exposed [2, 9–17]. Rice and Wu [17] introduced NLMM for unequally sampled noisy data. For longitudinal data, a NLMM was discussed on wavelet bases via a Bayesian structure by Lu and Huang [18]. Angelini et al. [19] examined wavelet regression analysis in semiparametric mixed models for longitudinal curves. For longitudinal data, a NLMM was introduced NLMM for unequally sampled noisy data. Müller [8] investigated the nonparametric methods.

Although the waves have been considered as a very powerful mathematical tool and become an alternative to smooth nonparametric function under known measurement error. The author created the estimator for parameter of semiparametric regression using the idea described as wavelet approach for nonparametric wavelet estimation Yalaz [22, 25] represented a wavelet approach to estimate partially linear errors in variables model which is a semiparametric model (SPM) and NLMM with variable of nonparametric part has measurement error. Here, $\tau = t + \Delta t$, where $\Delta t$ are iid measurement errors. It is assumed that $\Delta t$ has a known distribution which is proposed by Fan and Truong [26] for NPM.

This study is configured as follows. We define NLMM errors in variables method for one dimensional waves (d = 1) in Section 2, and in Section 3 we propose a predictor of random effects parameter. To analyze finite sample properties some Monte Carlo simulation studies are done in Section 4. In Section 5, conclusions are given.

THEORY

Approximation Kernels and Family of Estimators for Nonparametric Function in NLMM

We get NPM, if random effects part is embedded into the response variable in a NLMM:

$$Y - Zb = g(t) + \epsilon,$$

for $E[\epsilon|t]=0$. Hence, if $b$ is known, nonparametric function can be estimated using nonparametric methods.

Let

$$g(t) = \int \frac{y g_{ij}(t, y) dy}{f(t)} = \frac{(gf)(t)}{f(t)}$$

where $f(t)$ is defined as a classical deconvolution problem. Our main purpose is to estimate $(gf)(t)$. We denote $p(t) = g(t) \times f(t)$ and consider a father wavelet $\varphi$ on the real line satisfying the following conditions described by Chichignoud et al. [21].

- $\varphi$ is compactly supported on $[-A, A]$; $A$ is a positive integer.
Denote \( \varphi_k(t) = \varphi(t-k) \). There exists a positive integer \( N \), such that for any \( t \) and \( k \),

\[
\int_{k=1}^{n} \varphi(t-k) \varphi(y' - k)(y' - t) dy' = \delta_{kl},
\]

where \( \delta_{kl} \) is the Kronecker delta which is defined as Hardle et al. [27]

\[
\delta_{kl} = \begin{cases} 1, & \text{if } l = 0, \\
0, & \text{otherwise.}
\end{cases}
\]

- \( \varphi \) is of class the space of functions having all continuous derivatives \( C^r \), where \( r \geq 2 \).

The associated projection kernel on the space

\[
V_j := \text{span} \{ \varphi_{jk}, k \in \mathbb{Z} \}, \quad j \in \mathbb{N}
\]

is given for any \( t \) and \( y' \) by

\[
K_j(t, y') = \sum_k \varphi_{jk}(t) \varphi_{jk}(y').
\]

where

\[
\varphi_{jk}(t) = 2^j \varphi(2^j t - k), \quad j \in \mathbb{N}, k \in \mathbb{Z}.
\]

Then the projection of \( p(t) \) on \( V_j \) can be written as,

\[
p_j(t) = K_j(p)(t) := \int K_j(t, y') p(y') dy' = \sum_k p_{jk} \varphi_{jk}(t),
\]

where

\[
p_{jk} = \int p(y') \varphi_{jk}(y') dy'.
\]

In Chichignoud et al. [21], the authors adapted the kernel approach proposed by Fan and Truong [26] in their wavelet context and they introduced

\[
\hat{p}_{jk}(t) = \frac{1}{n} \sum_{i=1}^{n} Y_i \times (D_j \varphi)_{jk}(\tau_i) \varphi_{jk}(t),
\]

where \( \varphi(s) \) is the Fourier transform of wavelet \( \varphi_{jk} \) and \( D \) is the deconvolution operator which is demonstrated by \( K \) in Fan and Truong [26] and defined as follows

\[
(D_j \varphi)(\tau) = \int \exp(-i 2^j \tau - k) \varphi(s) \varphi_s(2^j s) ds.
\]

We show the variables as

\[
Y = Y_1, \ldots, Y_n,
\]

where \( Y_i \) is given for any \( i \) and \( Y \) is a \((n \times q) \times 1\) vector of random effects with mean zero and covariance matrix \( D = \text{diag}(D_1, \ldots, D_n) \). Because we translate the
NLMM to LMM, we write the variance-covariance matrix of \( \hat{Y} \) as \( V = ZDZ^T + \Sigma \) and the block diagonal covariance matrix of \( \varepsilon \) is \( \Sigma \). Under (4), we display the joint Gaussian distribution of \( b \) and \( Y \) as

\[
\begin{pmatrix}
  b \\
  \hat{Y}
\end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \\ ZD \\ V \end{pmatrix}, \begin{pmatrix} D & DZ^T \\ ZD & V \end{pmatrix} \right),
\]

and the conditional distribution of \( \hat{Y} \) given \( b \) is \( \hat{Y} \mid b \sim \mathcal{N}(\bar{Z}b, \Sigma) \). Then, we maximize the joint density of \( \hat{Y} \) and \( b \)

\[
f(Y, b) = f(\hat{Y} | b) f(b) = (2\pi)^{-\frac{3}{2}} |\Sigma|^{-\frac{1}{2}} |P|^\frac{1}{2} \exp \left\{ -\frac{1}{2} \right\}
\]

where \( |\cdot| \) describes the determinant of a matrix. We derive \( \log f(Y, b) \) by giving the log function of (5) as follows

\[
\log f(Y, b) = \log f(\hat{Y} | b) + \log f(b) = -\frac{1}{2} \left\{ q \log(2\pi) + \log |\Sigma| + \log |P| + \right\},
\]

By discarding the constant term and the log function, the partial derivative of (6) with respect to the element of \( b \) equal to zero and employing \( \hat{b} \) to denote the solution find

\[
\bar{Z}^T \Sigma^{-1} \bar{Y} - (\bar{Z}^T \Sigma^{-1} \bar{Z} + D^{-1}) \hat{b} = 0.
\]

We compute \( (\bar{Z}^T \Sigma^{-1} \bar{Z} + D^{-1})^{-1} = D - D\bar{Z}^T V^{-1} \bar{Z}D \) via the Sherman-Morrison-Woodbury Theorem ([29]) and after algebraic simplifications, we introduce the wavelet predictor in NLMM as

\[
\hat{b} = D\bar{Z}^T V^{-1} \hat{Y}.
\]

After prediction of random effects parameter, an estimation of nonparametric component \( g(t) \) can be described as

\[
\hat{g}_n(t) = \sum_{k=1}^n \omega_n(t) \left( Y_k - Z \hat{b} \right).
\]

RESULTS AND DISCUSSION

Simulation Study
In this section, we benefit from Monte Carlo simulation approach to demonstrate the finite sample properties of the estimators by using MATLAB.

We consider the model with two random effects \( q = 2 \) as follows;

\[
y_g = b_1 + b_2 time_g + g(t_g) + \varepsilon_{g},
\]

\[
b_1 \sim \mathcal{N}(0, D), \varepsilon_{g} \sim \mathcal{N}(0, \sigma_e^2 I_n),
\]

where \( D = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \) is the AR(1) process with \( \rho = 0.60 \) and \( time_g \) shows time which was taken as the same set of occasions, \( [t_g = j \text{ for } i = 1, \ldots, n, j = 1, \ldots, n] \). We consider \( n = 4, 8, 16, 32 \) subjects and per subject \( n_t = 8 \) observations. Then, we calculate the simulation results with the sample sizes of \( N = \Sigma n_{it} \) where \( N = 32,64,128,256 \). Nonparametric functions are considered as

\[
g(t) = 4.26(\exp(-3.25 t) - 4\exp(-6.5 t) + 3\exp(-9.75 t))
\]

In all examples, the density of both the true regressor \( t \) and the measurement error \( \Delta t \) are chosen as the most common combinations of ordinarily smooth distributions which are summarized in Table 1.

We consider the normal distribution as an example of a supersmooth distribution, and the Laplace (or double exponential) distribution, uniform distribution and beta distribution for the ordinarily smooth case. Because Beta (2,2) and Beta (0.5,2) distributions reflect two different behaviors on [0,1] we use them. Finally, following the asymptotic considerations given in Chichignoud et al. [21], we choose the primary resolution level \( j \) that we have used throughout our simulations as \( j(n) = \log(n) \log(n) + 1 \).

The average values of 100 replicates of the mean squared error of response variable (MSE) and the mean squared

| Table 1: Examples |
|-------------------|
| **Example 1**     | **Example 2**     | **Example 3**     |
| \( t \rightarrow \text{Beta}(2,2) \) | \( t \rightarrow \text{Beta}(0.5,2) \) | \( t \rightarrow \text{Uniform}[0,1] \) |
| \( \Delta t \rightarrow \text{L}(0,0.001) \) | \( \Delta t \rightarrow \text{L}(0,0.001) \) | \( \Delta t \rightarrow \text{L}(0,0.001) \) |
| \( \varepsilon \rightarrow \text{N}(0,0.25) \) | \( \varepsilon \rightarrow \text{N}(0,0.25) \) | \( \varepsilon \rightarrow \text{N}(0,0.25) \) |
| \( \sigma_e^2 = 0.25 \) | \( \sigma_e^2 = 0.25 \) | \( \sigma_e^2 = 0.25 \) |
error of predictor (MSEP) for measurement error case (Our) and ignoring measurement error case (NoME) which are considered in four different sample sizes are given in Table 2 for \( g_1 \) and given in Table 3 for \( g_2 \). When we compare our results with ignoring measurement error case, it can be easily seen that the results are encouraging.

We also compared the finite sample and the asymptotic distributions of our estimator and the estimator ignoring measurement. In Figures (1-3), the abscissa is \( Z = (\text{Var}(g(t)))^{-\frac{1}{2}} (g(t) - E(g(t))) \) and the ordinate is probability. The empirical cumulative distribution function (CDF) curve of the estimator (indicated by dashed line) fits very well with the normal CDF (indicated by solid line) curve and it is better than the NoME case (indicated by dotted line).

**CONCLUSION**

This paper presents wavelet estimation of the NLMM when nonparametric part has measurement error. If the measurement error is known, prediction of random effects parameter using wavelet approach is possible. And so, we introduce the predictor of \( \hat{g} \) based on projection kernels on wavelets and borrowing the ideas of deconvolution technique. We implemented some simulations to illustrate the theoretical results. Because in literature NLMM is not

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**Table 2: Simulation results \( g_1 \)**

| Example | \( n = 32 \) | \( n = 64 \) | \( n = 128 \) | \( n = 256 \) |
|---------|---------------|---------------|---------------|---------------|
|         | MSE          | MSEP          | MSE          | MSEP          |
| Our     | 0.2388       | 0.1910        | 0.6923       | 0.5311        | 2.3513       | 0.2685       |
| NoME    | 0.6536       | 0.5122        | 1.1017       | 0.4781        | 2.2137       | 0.4912        | 4.4953       | 0.4749       |

| Example | \( n = 32 \) | \( n = 64 \) | \( n = 128 \) | \( n = 256 \) |
|---------|---------------|---------------|---------------|---------------|
|         | MSE          | MSEP          | MSE          | MSEP          |
| Our     | 0.3050       | 0.2397        | 0.4782       | 0.3986        | 1.7930       | 0.4912        | 5.0920       | 0.4113       |
| NoME    | 0.6719       | 0.4866        | 1.3092       | 0.6899        | 2.6972       | 0.4104        | 5.1700       | 0.4113       |

**Table 3: Simulation results \( g_2 \)**

| Example | \( n = 32 \) | \( n = 64 \) | \( n = 128 \) | \( n = 256 \) |
|---------|---------------|---------------|---------------|---------------|
|         | MSE          | MSEP          | MSE          | MSEP          |
| Our     | 0.2707       | 0.1387        | 0.4807       | 0.2204        | 0.9833       | 0.2877        | 3.3393       | 0.3815       |
| NoME    | 0.6415       | 0.5112        | 1.1410       | 0.4661        | 2.2870       | 0.4546        | 4.7235       | 0.4069       |
Figure 1. Comparison of the finite sample and asymptotic distributions of the estimator in Example 1.
Figure 2: Comparison of the finite sample and asymptotic distributions of the estimator in Example 2.
Figure 3: Comparison of the finite sample and asymptotic distributions of the estimator in Example 3.
considered using wavelet approach, we compared our work with the circumstances ignoring measurement error. We aimed at estimating the two different regression functions at different three points with three different densities and finally compared these results. It is discussed in the simulation that the resulting rates are comparable to no measurement error case. The performances are much better for the proposed model then the model ignoring measurement errors. Asymptotic normality of proposed predictor is still open one and should be investigated.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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