The $F$-pure threshold of a determinantal ideal

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— Dedicated to Professor Steven Kleiman and Professor Aron Simis on the occasion of their 70th birthdays.

Abstract. The $F$-pure threshold is a numerical invariant of prime characteristic singularities, that constitutes an analogue of the log canonical thresholds in characteristic zero. We compute the $F$-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of a generic matrix.

Keywords: $F$-pure threshold, log canonical threshold, determinantal ideals.

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1 Introduction

Consider the ring of polynomials in a matrix of indeterminates $X$, with coefficients in a field of prime characteristic. We compute the $F$-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of $X$ of a fixed size.

The notion of $F$-pure thresholds is due to Takagi and Watanabe [18], see also Mustaţă, Takagi, and Watanabe [17]. These are positive characteristic invariants of singularities, analogous to log canonical thresholds in characteristic zero. While the definition exists in greater generality – see the above papers – the following is adequate for our purpose:

Definition 1.1. Let $R$ be a polynomial ring over a field of characteristic $p > 0$, with the homogeneous maximal ideal denoted by $m$. For a homogeneous proper ideal $I$, and integer $q = p^e$, set

$$\nu_I(q) = \max \{ r \in \mathbb{N} \mid I^r \not\subset m^{[q]} \}.$$
where $m^{[q]} = \{ a^q \mid a \in m \}$. If $I$ is generated by $N$ elements, it is readily seen that $0 \leq v_I(q) \leq N(q - 1)$. Moreover, if $f \in I^r \setminus m^{[q]}$, then $f^p \in I^{pr} \setminus m^{[pq]}$. Thus,

$$v_I(pq) \geq pv_I(q).$$

It follows that $\left\{ \frac{v_I(p^e)}{p^e} \right\}_{e \geq 1}$ is a bounded monotone sequence; its limit is the $F$-pure threshold of $I$, denoted $\text{fpt}(I)$.

The $F$-pure threshold is known to be rational in a number of cases, see, for example, [2, 3, 4, 9, 16]. The theory of $F$-pure thresholds is motivated by connections to log canonical thresholds; for simplicity, and to conform to the above context, let $I$ be a homogeneous ideal in a polynomial ring over the field of rational numbers. Using “$I$ modulo $p$” to denote the corresponding characteristic $p$ model, one has the inequality

$$\text{fpt}(I \text{ modulo } p) \leq \text{lct}(I) \quad \text{for all } p \gg 0,$$

where $\text{lct}(I)$ denotes the log canonical threshold of $I$. Moreover,

$$\lim_{p \to \infty} \text{fpt}(I \text{ modulo } p) = \text{lct}(I). \quad (1.1.1)$$

These follow from work of Hara and Yoshida [10]; see [17, Theorems 3.3, 3.4].

The $F$-pure thresholds of defining ideals of Calabi-Yau hypersurfaces are computed in [1]. Hernández has computed $F$-pure thresholds for binomial hypersurfaces [11] and for diagonal hypersurfaces [12]. In the present paper, we perform the computation for determinantal ideals:

**Theorem 1.2.** Fix positive integers $t \leq m \leq n$, and let $X$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{F}$ of prime characteristic. Let $R$ be the polynomial ring $\mathbb{F}[X]$, and $I_t$ the ideal generated by the size $t$ minors of $X$.

The $F$-pure threshold of $I_t$ is

$$\text{fpt}(I_t) = \min \left\{ \frac{(m-k)(n-k)}{t-k} \mid k = 0, \ldots, t - 1 \right\}.$$

It follows that the $F$-pure threshold of a determinantal ideal is independent of the characteristic: for each prime characteristic, it agrees with the log canonical threshold of the corresponding characteristic zero determinantal ideal, as computed by Johnson [15, Theorem 6.1] or Docampo [8, Theorem 5.6] using log resolutions as in Vainsencher [19]. In view of (1.1.1), Theorem 1.2 recovers the calculation of the characteristic zero log canonical threshold.
2 The computations

The primary decomposition of powers of determinantal ideals, i.e., of the ideals $I_r^t$, was computed by DeConcini, Eisenbud, and Procesi [7] in the case of characteristic zero, and extended to the case of non-exceptional prime characteristic by Bruns and Vetter [6, Chapter 10]. By Bruns [5, Theorem 1.3], the intersection of the primary ideals arising in a primary decomposition of $I_r^t$ in non-exceptional characteristics, yields, in all characteristics, the integral closure $I_r^t$. We record this below in the form that is used later in the paper:

**Theorem 2.1 (Bruns).** Let $s$ be a positive integer, and let $\delta_1, \ldots, \delta_h$ be minors of the matrix $X$. If

$$h \leq s \text{ and } \sum_i \deg \delta_i = ts,$$

then

$$\delta_1 \cdots \delta_h \in I_r^t.$$

**Proof.** By [5, Theorem 1.3], the ideal $I_r^t$ has a primary decomposition

$$\bigcap_{j=1}^{t} I_j^{(t-j+1)s}. $$

Thus, it suffices to verify that

$$\delta_1 \cdots \delta_h \in I_j^{(t-j+1)s}$$

for each $j$ with $1 \leq j \leq t$. This follows from [6, Theorem 10.4].

We will also need:

**Lemma 2.2.** Let $k$ be the least integer in the interval $[0, t - 1]$ such that

$$\frac{(m - k)(n - k)}{t - k} \leq \frac{(m - k - 1)(n - k - 1)}{t - k - 1};$$

interpreting a positive integer divided by zero as infinity, such a $k$ indeed exists. Set

$$u = t(m + n - 2k) - mn + k^2.$$

Then $t - k - u \geq 0$.

Moreover, if $k$ is nonzero, then $t - k + u > 0$; if $k = 0$, then $t(m + n - 1) \leq mn$. 

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Proof. Rearranging the inequality above, we have
\[ t(m + n - 2k - 1) \leq mn - k^2 - k, \]
which gives \( t - k - u \geq 0 \). If \( k \) is nonzero, then the minimality of \( k \) implies that
\[ t(m + n - 2k + 1) > mn - k^2 + k, \]
equivalently, that \( t - k + u > 0 \). If \( k = 0 \), the assertion is readily verified. \( \square \)

Notation 2.3. Let \( X \) be an \( m \times n \) matrix of indeterminates. Following the notation in [6], for indices
\[ 1 \leq a_1 < \cdots < a_t \leq m \quad \text{and} \quad 1 \leq b_1 < \cdots < b_t \leq n, \]
we set \( [a_1, \ldots, a_t \mid b_1, \ldots, b_t] \) to be the minor
\[
\det \begin{pmatrix}
  x_{a_1b_1} & \cdots & x_{a_tb_t} \\
  \vdots & & \vdots \\
  x_{a_1b_1} & \cdots & x_{a_tb_t}
\end{pmatrix}.
\]
We use the lexicographical term order on \( R = \mathbb{F}[X] \) with
\[ x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{m1} > \cdots > x_{mn}; \]
under this term order, the initial form of the minor displayed above is the product of the entries on the leading diagonal, i.e.,
\[
\text{in } ([a_1, \ldots, a_t \mid b_1, \ldots, b_t]) = x_{a_1b_1}x_{a_2b_2}\cdots x_{a_tb_t}.
\]
For an integer \( k \) with \( 0 \leq k \leq m \), we set \( \Delta_k \) to be the product of minors:
\[
\prod_{i=1}^{n-m+1} [1, \ldots, m \mid i, \ldots, i + m - 1] \\
\times \prod_{j=2}^{m-k} [j, \ldots, m \mid 1, \ldots, m - j + 1] \cdot [1, \ldots, m - j + 1 \mid n - m + j, \ldots, n].
\]
If \( k \geq 1 \), we set \( \Delta'_k \) to be
\[
\Delta_k \cdot [m - k + 1, \ldots, m \mid 1, \ldots, k].
\]
Notice that \( \deg \Delta_k = mn - k^2 - k \) and that \( \deg \Delta'_k = mn - k^2 \). The element \( \Delta_k \) is a product of \( m + n - 2k - 1 \) minors and \( \Delta'_k \) of \( m + n - 2k \) minors.

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Example 2.4. We include an example to assist with the notation. In the case \( m = 4 \) and \( n = 5 \), the elements \( \Delta_2 \) and \( \Delta'_2 \) are, respectively, the products of the minors determined by the leading diagonals displayed below:

\[
\begin{align*}
\Delta_2 &= \prod_{i=1}^{4} \prod_{j=1}^{5} x_{ij}, \\
\Delta'_2 &= \prod_{i=1}^{4} \prod_{j=1}^{5} x_{ij}.
\end{align*}
\]

The initial form of \( \Delta'_2 \) is the square-free monomial

\[
x_{11} x_{12} x_{13} x_{21} x_{22} x_{23} x_{24} x_{31} x_{32} x_{33} x_{34} x_{35} x_{42} x_{43} x_{44} x_{45}.
\]

For arbitrary \( m, n \), the initial form of \( \Delta_0 \) is the product of the \( mn \) indeterminates.

Proof of Theorem 1.2. We first show that for each \( k \) with \( 0 \leq k \leq t - 1 \), one has

\[
fpt(I_t) \leq \frac{(m - k)(n - k)}{t - k}.
\]

Let \( \delta_k \) and \( \delta_t \) be minors of size \( k \) and \( t \) respectively. Theorem 2.1 implies that

\[
\delta_{k-1}^{t-k} \delta_t \in I_{k+1}^t,
\]

and hence that \( \delta_{k-1}^{t-k} I_t \subseteq I_{k+1}^t \). By the Briançon-Skoda theorem, see, for example, [13, Theorem 5.4], there exists an integer \( N \) such that

\[
\left( \delta_{k-1}^{t-k} I_t \right)^N \subseteq I_{k+1}^{(t-k)/l}
\]

for each integer \( l \geq 1 \). Localizing at the prime ideal \( I_{k+1} \) of \( R \), one has

\[
I_{t+1}^{N+l} \subseteq I_{k+1}^{(t-k)/l} R_{k+1} \quad \text{for each} \quad l \geq 1,
\]

as the element \( \delta_k \) is a unit in \( R_{k+1} \). Since \( R_{k+1} \) is a regular local ring of dimension \( (m - k)(n - k) \), with maximal ideal \( I_{k+1} R_{k+1} \), it follows that

\[
I_{t+1}^{N+l} \subseteq I_{k+1}^{(t-k)/l} R_{k+1}
\]

for positive integers \( l \) and \( q = p^e \) satisfying

\[
(t - k)l > (q - 1)(m - k)(n - k).
\]
Returning to the polynomial ring $R$, the ideal $I_{k+1}$ is the unique associated prime of $I_{k+1}^q$; this follows from the flatness of the Frobenius endomorphism, see for example, [14, Corollary 21.11]. Hence, in the ring $R$, we have

$$I_{k+1}^{q+1} \subseteq I_{k+1}^q$$

for all integers $q, l$ satisfying the above inequality. This implies that

$$v_l(q) \leq N + \frac{(q - 1)(m - k)(n - k)}{l - k}.$$

Dividing by $q$ and passing to the limit, one obtains

$$\text{fpt}(I_t) \leq \frac{(m - k)(n - k)}{t - k}.$$

Next, fix $k$ and $u$ be as in Lemma 2.2, and consider $\Delta_k$ and $\Delta'_k$ as in Notation 2.3; the latter is defined only in the case $k \geq 1$. Set

$$\Delta = \begin{cases} 
\Delta_0 & \text{if } k = 0, \\
\Delta_k^u \cdot (\Delta_k')^{t-k-u} & \text{if } k \geq 1 \text{ and } u \geq 0, \\
(\Delta_k')^{t-k+u} \cdot \Delta_{k-1}^{-u} & \text{if } k \geq 1 \text{ and } u < 0,
\end{cases}$$

bearing in mind that $t - k - u \geq 0$ by Lemma 2.2.

We claim that $\Delta$ belongs to the integral closure of the ideal $t^{(m-k)(n-k)}$. This holds by Theorem 2.1, since, in each case,

$$\deg \Delta = t(m - k)(n - k),$$

and $\Delta$ is a product of at most $(m - k)(n - k)$ minors: if $k \geq 1$, then $\Delta$ is a product of exactly $(m - k)(n - k)$ minors, whereas if $k = 0$ then $\Delta$ is a product of $t(m + n - 1)$ minors and, by Lemma 2.2, one has $t(m + n - 1) \leq mn$.

Let $m$ be the homogeneous maximal ideal of $R$. For a positive integer $s$ that is not necessarily a power of $p$, set

$$m^s = (x_{ij}^s \mid i = 1, \ldots, m, \ j = 1, \ldots, n).$$

Using the lexicographical term order from Notation 2.3, the initial forms $\text{in}(\Delta_k)$ and $\text{in}(\Delta'_k)$ are square-free monomials, and

$$\text{in}(\Delta) = \begin{cases} 
\text{in}(\Delta_0)^t & \text{if } k = 0, \\
\text{in}(\Delta_0)^u \cdot \text{in}(\Delta_k')^{t-k-u} & \text{if } k \geq 1 \text{ and } u \geq 0, \\
(\Delta_k')^{t-k+u} \cdot \text{in}(\Delta_{k-1})^{-u} & \text{if } k \geq 1 \text{ and } u < 0.
\end{cases}$$
Thus, each variable $x_{ij}$ occurs in the monomial $in(\Delta)$ with exponent at most $t - k$. It follows that
\[ \Delta \notin \mathfrak{m}^{[t-k+1]}.
\]
As $\Delta$ belongs to the integral closure of $I_t^{(m-k)(n-k)}$, there exists a nonzero homogeneous polynomial $f \in R$ such that
\[ f \Delta^l \in I_t^{(m-k)(n-k)l} \quad \text{for all integers } l \geq 1.
\]
But then
\[ f \Delta^l \in I_t^{(m-k)(n-k)l} \setminus \mathfrak{m}^q\]
for all integers $l$ with $\deg f + l(t - k) \leq q - 1$. Hence,
\[ v_{I_t}(q) \geq (m - k)(n - k)l \quad \text{for all integers } l \text{ with } l \leq \frac{q - 1 - \deg f}{t - k}.
\]
Thus,
\[ v_{I_t}(q) \geq (m - k)(n - k) \left( \frac{q - 1 - \deg f}{t - k} - 1 \right),
\]
and dividing by $q$ and passing to the limit, one obtains
\[ \text{fpt}(I_t) \geq \frac{(m - k)(n - k)}{t - k},
\]
which completes the proof. \(\square\)

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