Effective Field Theories for Superconductors in the Subgap Regime

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We construct effective field theories for superconductors, that are powerful enough to describe low lying sub gap fermion modes localized to vortex cores, and at the same time resemble topological field theories in that there are no bulk degrees of freedom. This is achieved by a kinetic term for fermions that is proportional to the vortex topological charge, and thus vanish in the bulk. We study the case of a spin-less two-dimensional $p_x + i p_y$ superconductor in some detail, and show that the the subgap fermionic spectrum in a single vortex, including the zero mode, has the same features as those obtained from microscopic models. We also show, that in the topological scaling limit our theory becomes a bona fide topological field theory which retains the Majorana modes at the vortex cores, and correctly describes the non-Abelian statistics of such vortices.

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I. INTRODUCTION

In topologically ordered phases of matter, such as quantum Hall (QH) liquids, superconductors, topological insulators and spin liquids, the excitations in the interior of the system are separated from the ground state by an energy gap, thus distinguishing them from ordinary metals or magnets. They however differ in important ways from trivial gapped phases, such as conventional band insulators, in having excitations with exotic quantum numbers, and/or gapless edge modes. An important theoretical approach to these phases is based on topological field theories (TFT), which directly builds in important features of topologically ordered systems, such as the absence of low energy bulk excitations and, in two-dimensional systems, the possibility of fractional braiding statistics.

Prominent examples are the Chern-Simons (CS) theories of hierarchical QH liquids and the BF theories of superconductors (for reviews, see and) and topological insulators.

That the topological CS theories for the abelian QH liquids encode the characteristic gapless bosonic edge excitations has been known for a long time, but, more surprisingly, purely bosonic TFTs can also describe fermionic edge states. A prominent example is the Moore-Read, or pfaffian, QH state, for which a TFT description based on a SU(2) gauge theory, was proposed by Fradkin et al.

An important property of the MR state, which it shares with the spinless 2d $p_x + i p_y$ superconductor is that the fundamental vortices support zero energy Majorana modes. As a consequence, a set of $2n$ vortices at fixed positions define a Hilbert space of dimension $2^n - 1$, and braiding the vortices corresponds to unitary rotations in this space. This is the basis of the non-abelian fractional statistics that has been looked for in experiments and is proposed to be useful in quantum information applications.

The zero modes in the case of $p_x + i p_y$ paired superconductor, is a special example of the vortex subgap modes that occur also for s and d-wave pairing. What makes the p-wave case particularly interesting is that the zero modes are topologically protected. In this context, it is a challenge to formulate effective theories that describes the physics at energies below the superconducting gap. Such theories not only should encode the topological information about quasiparticles and vortices, but also describe the dynamics of the fermionic subgap modes at vortex cores and at edges. The purpose of this letter is to propose such a theory, and to treat the case of p-wave pairing of spinless fermions in two dimensions in sufficient detail to demonstrate the power of our approach.

II. THE BF THEORY

Our starting point is the topological description of superconductors in terms of BF gauge theory which is reviewed in Ref. 3. In this theory, the quasiparticle current $j_a$ couples to a gauge field $a$ and the vortex current $j_v$ to a gauge field $b$. The BF Lagrangian which describes the topological properties of superconductors is, in the language of differential forms, $\mathcal{L}_{BF} = \frac{1}{2} da b - j_a a - j_v b$. In 3d $b_{\mu\nu}$ is an antisymmetric tensor field that couples to the world sheet of the propagating vortex string. In the 2d case, which we will concentrate on in the following, the vortices are point like, and $b_\mu$ is an ordinary gauge field. In standard vector notation we have,

\begin{equation}
\mathcal{L}_{BF} = \frac{1}{\pi} \epsilon^{a\mu\nu} \partial_\mu a_\nu b_\rho - j_a^a a_\mu - j_v^e b_\mu \quad (1)
\end{equation}

In addition to the two local gauge symmetries, this TFT is also invariant under parity ($P$) and time reversal ($T$).

It is known that by supplementing a TFT with non-topological terms, scales are introduced and more of the low energy physics can be described. Adding Maxwell
terms to the topological BF theory, introduces both a London length, and thus a size for the vortices, and a plasma frequency\(^3\). To describe the subgap fermionic states, we need at least the London length, so we shall supplement the lagrangian (1) with the Maxwell terms,

\[
\mathcal{L}_M = \frac{\alpha_1}{2\pi} (\vec{E}^2) - \frac{\alpha_2}{2\pi} (B^2) + \frac{\beta_1}{2\pi} (\vec{E}^2) - \frac{\beta_2}{2\pi} (B^2)
\]

(2)

where \(B^b = e^{ij} \partial_j b^i\) etc., and where the parameters \(\alpha_i\) and \(\beta_i\) are related to the London penetration length \(\lambda_L\), the Debye screening length \(\lambda_D\), the plasma frequency \(\omega_p\) and the vortex energy \(\epsilon_v\) by \(\lambda_L = \sqrt{\alpha_2 / \beta_1}\), \(\lambda_D = \sqrt{\alpha_1 / \beta_2}\), \(\omega_p^2 = \sqrt{\alpha_1 \beta_1}\), and \(\epsilon_v = 1 / \alpha_1\).

Introducing sources, we can solve for the fields in the pure gauge sector, and for a single, static, pointlike vortex source, \(P_s = h\delta^2(\vec{r})\), we find the solution \(B^b = h(2\lambda_2^2)^{-1} K_0(r / \lambda_L)\), \(\vec{E}^b = -h\lambda_2 \vec{\nabla} B^a\) and \(B^b = E_i = 0\) (in polar coordinates \((r, \theta)\)).

We now present our basic idea. Since the subgap fermion modes are all confined either on the edge of the system, or at the core of vortices, we want a theory without any bulk fermionic degrees of freedom. We achieve this, not by introducing confining potentials, but by having the kinetic energy of the fermions vanish in bulk. Inspired by Ref. 12 we make the following ansatz for the fermionic lagrangian,

\[
\mathcal{L}_\psi = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} \partial_\mu a_\nu \psi^\dagger D_\rho \psi - \tilde{\mathcal{H}},
\]

(3)

where \(iD_\mu = i\partial_\mu + a_\mu\). In the case of s-wave pairing the fermion field \(\psi\) must have two spin components, while for a spin polarized \(p\)-wave phase one component suffices. For a static vortex, the kinetic term in (3) is \(\sim B^a \psi^\dagger \partial_0 \psi\), which vanishes exponentially outside the vortices.

The full Hamiltonian is \(\mathcal{H} = -\epsilon_{ij} E_{ij}^a \psi^\dagger D^a \psi + \tilde{\mathcal{H}}\), where the first term vanishes for a static vortex. \(\tilde{\mathcal{H}}\) is to be constructed by a derivative expansion compatible with the symmetries of the underlying microscopic physics. For a static configuration, there are two possible terms with no derivatives on the fermion field, namely \(\Lambda \psi^\dagger \psi\) and \(\mu B^a \psi^\dagger \psi\). The first term is crucial for localizing the subgap states at the vortices, while the second makes no qualitative change and will be neglected. The quasiparticle states in a superconductor are not charge eigenstates, and to incorporate this we need a pairing interaction, which in our case should be of the \(p\)-wave type. Since our fermions are spinless, the lowest derivative pairing interaction possible which involves only the fields \(a\) and \(\psi\), and is invariant under rotations and gauge transformations, is \(\sim \xi E_z^b \psi \partial_z \psi + h.c.,\) where \(z = x + iy\), \(E_z^b = E_z^b - iE_y^b\) and \(\xi\) is the the phase operator, introduced by Dirac\(^{14}\), which can be used to form a gauge invariant, but non-local, order parameter for a superconductor, and which transforms as \(\xi \rightarrow e^{-2\delta\xi}\). For details on how to construct \(\xi\), see appendix C. In summary, we shall use

\[
\tilde{\mathcal{H}} = \Lambda \psi^\dagger \psi - \frac{\delta}{4\pi} \xi(\vec{r}) E_z^b \psi \partial_z \psi + h.c.
\]

(4)

where \(\Lambda\) is an energy density, and \(\delta / 8\pi\) a dimensionless coupling parameter. Without loss of generality we can take \(\Lambda > 0\) and \(\delta > 0\). Note that the presence of \(E_z^b\) in the pairing term is natural since the current is \(\sim \epsilon^{ij} E_j^b\).

### III. THE HAMILTONIAN

We now proceed to construct \(\tilde{\mathcal{H}}\), so to get a realistic spectrum of subgap modes. In the spirit of effective field theory, we make a derivative expansion compatible with the symmetries of the underlying microscopic physics. For a static configuration, there are two possible terms with no derivatives on the fermion field, namely \(\Lambda \psi^\dagger \psi\) and \(\mu B^a \psi^\dagger \psi\). The first term is crucial for localizing the subgap states at the vortices, while the second makes no qualitative change and will be neglected. The quasiparticle states in a superconductor are not charge eigenstates, and to incorporate this we need a pairing interaction, which in our case should be of the \(p\)-wave type. Since our fermions are spinless, the lowest derivative pairing interaction possible which involves only the fields \(a\) and \(\psi\), and is invariant under rotations and gauge transformations, is \(\sim \xi E_z^b \psi \partial_z \psi + h.c.,\) where \(z = x + iy\), \(E_z^b = E_z^b - iE_y^b\) and \(\xi\) is the the phase operator, introduced by Dirac\(^{14}\), which can be used to form a gauge invariant, but non-local, order parameter for a superconductor, and which transforms as \(\xi \rightarrow e^{-2\delta\xi}\). For details on how to construct \(\xi\), see appendix C. In summary, we shall use

\[
\tilde{\mathcal{H}} = \Lambda \psi^\dagger \psi - \frac{\delta}{4\pi} \xi(\vec{r}) E_z^b \psi \partial_z \psi + h.c.
\]

(6)

where \(\Lambda\) is an energy density, and \(\delta / 8\pi\) a dimensionless coupling parameter. Without loss of generality we can take \(\Lambda > 0\) and \(\delta > 0\). Note that the presence of \(E_z^b\) in the pairing term is natural since the current is \(\sim \epsilon^{ij} E_j^b\).

### IV. THE SPECTRUM

It is convenient to write the full hamiltonian in the BdG form, \(\tilde{\mathcal{H}} = \frac{1}{2} \Psi^\dagger h \Psi\) with \(\Psi = (\psi^\dagger, \psi)\), and

\[
h^2 = \frac{1}{2\pi} \left( -\frac{h_0}{3} \{\partial_z, \Delta^*\} - \frac{1}{2} \{\partial_z, \Delta\} \right)
\]

(7)

where \(\Delta = \frac{1}{2} \delta E_z^b\) and,

\[
h_0 = -\epsilon_{ij} E_{ij}^a D^a - \Lambda - \alpha_0 B_a.
\]

(8)

To diagonalize \(\tilde{\mathcal{H}}\), we expand the field operators as, \(\psi(\vec{r}, t) = \sum_n a_n(t) u_n(\vec{r}) + a_n^\dagger(\vec{r}) v_n^*(\vec{r})\), and introduce the eigenspinors \(\phi(\vec{r}) = (u(\vec{r}), v(\vec{r}))^T\). Next we solve the single particle equation for the spinor \(\phi\) in the background of widely separated vortices. We begin by removing the phase of the off-diagonal terms with a gauge transformation. As can be seen in the supplementary material, \(\xi\) is, in Coulomb gauge, up to a constant phase, equal to \(e^{i\rho \theta}\) for a single vortex with strength \(\rho\) situated at the origin. So we make the transformation

\[
\psi \rightarrow e^{i\frac{\rho}{2}(m+1)\theta} \psi.
\]

(9)
values numerically. Since the kinetic term for the mode can be written as \( \phi \) it follows, without loss of generality, that a zero energy

Using the particle hole symmetry of the BdG equations, a confining box, we expect that the spectrum contains an infinite number of bound states localized at the scale \( \lambda_L \), but no continuum states. All our numerical results support this conjecture, and we shall henceforth assume it to be true; we have not tried to find an analytic proof. While we do not yet have an analytic proof of this conjecture, all our numerical results support it and so, henceforth, we shall assume it to be true.

In a type II superconductor the low lying subgap states are however localized on a smaller scale, \( \lambda_S \), and for suitable parameters our model has this feature. Indeed, if we introduce the “confinement” energy scale \( \epsilon_c \) by \( \Delta_b = \epsilon_c / \lambda_L^2 \), an asymptotic analysis gives \( \lambda_S = \sqrt{|m| \delta \Gamma} \lambda_L \) where \( r = (\hbar \omega_p)^2 / (\epsilon_c \epsilon_v) \). Thus we can have \( \lambda_S \ll \lambda_L \), by taking a small coupling parameter \( \delta \), and/or making the confinement scale \( \epsilon_c \) large. In this parameter range we have established numerically that our spectrum shares important qualitative features with the spectra obtained by self-consistent solutions of the full microscopic BdG equations\(^{17}\) (see the figure):

1. The purely angular excitations have a spectrum \( E_{0,l} \approx \Delta_1 + \alpha l \), with \( \alpha \ll \Delta_1 \).
2. The gap \( \Delta_r \) for the radial excitations is larger than \( \Delta_1 \).
3. The energy scale is inversely proportional to the vortex strength.

The presence of infinitely many bound states is an artifact of our model, and only the low lying states should be considered as physical. It is an interesting possibility that adding higher derivative terms could completely remove the high lying states and leaving a finite Hilbert space. We leave this as an open problem.

\section{V. The Topological Scaling Limit}

So far, we have shown that the \( \psi \)BF theory has all the expected subgap features. We now show how the theory \( (4) \) reduces to a truly topological field theory in a proper scaling limit. For this, consider a collection of \( N \) identical vortices of unit strength. The topological scaling limit is defined by taking both the physical length scale, \( \lambda \), and time scale, \( \hbar / E \), to zero at fixed coupling parameters\(^{18}\). We can think of \( \lambda \) as \( \epsilon \), the minimal distance between the vortices, and \( E \) as a cutoff energy below which our theory is to be valid. We define two Majorana fields by,

\( \gamma(r,t) = \frac{1}{2} (\psi + \psi^\dagger) \),  \( \tilde{\gamma}(r,t) = \frac{1}{2i} (\psi - \psi^\dagger) \),  (12)

and substitute in \( (4) \) to get (setting \( j_0 = 0 \)),

\begin{equation}
\mathcal{L}_{\psi BF} = \frac{1}{8 \pi} a d a^\dagger b + \frac{1}{4} \gamma i d \gamma + \frac{1}{4} \tilde{\gamma} i d \tilde{\gamma} - j_e b
\end{equation}

Since \( a_0 = \bar{E}_a = 0 \), the first term in the second line vanishes. Also \( \bar{E}_a = 0 \) means that \( \mathcal{H} \) is the full hamiltonian, so the last term in this line is \( \sim \sum_{E_n < \epsilon} \bar{E}_n a_n^\dagger a_n \) which
vanishes for fixed $E$ since the subgap (just as all energy scales) diverges. Finally, we make the shift $b \rightarrow b + \frac{1}{4} \gamma i d \gamma$ to eliminate the term $a d i \gamma d \gamma$ in favor of $j_i \gamma i d \gamma$. Using the boundary condition (10) it is easy to show that $\gamma_i d \gamma(\bar{0}) = 0$ so this term vanishes for a point vortex. This concludes the demonstration that the topological theory,

$$L_{\gamma BF} = \frac{1}{\pi} \epsilon^{\mu \nu \rho} \partial_\mu a_\nu \left( b_\rho + \frac{1}{4} \gamma i \partial_\rho \gamma \right) - j^\mu_a a_\mu - j^\nu_v b_\nu,$$

proposed in Ref. 12, is retained in the scaling limit.

VI. NONABELIAN STATISTICS

The nonabelian (NA) statistics in the Moore-Read QH state was originally understood in terms of the monodromies in the Ising CFT\(^{19}\), assuming that there are no remaining Berry phases when the wave functions are represented by conformal blocks. Proofs for this assertion were given in later papers\(^{20}\). In the case of the p-wave superconductor, Ivanov\(^{21}\) derived the NA statistics using the BdG formulation of Read and Green\(^{5}\). Also here it is important that, in a suitably chosen gauge, there are no Berry phases, so that the braiding phases of the vortices come entirely from the coupling to the gauge field. Although quite reasonable, this is not easy to show, and it was taken for granted by Ivanov. In a later paper\(^{22}\) Stern et al. addressed this question, and gave plausible arguments for the absence of Berry phases by a more detailed analysis of the vortex cores, using certain mild assumptions about the continuous part of the spectrum. In the ψBF theory there can be no Berry phases, since the fermionic wave functions only have support on the widely separated vortices. Thus, mutatis mutandis, Ivanov’s proof of NA statistics carries over to the ψBF theory, using no extra assumptions.

We now outline a version of the proof that directly yields the Hilbert space for 2N vortices; details will be given separately\(^{23}\). First note that for widely separated vortices, moving along the world lines $\vec{r}_a(t)$, the Majorana field in (12) takes the form, $\gamma(\vec{r}, t) = \sum_{a=1}^{2N} \chi(\vec{r} - \vec{r}_a(t)) \gamma_a(t)$ where, in an obvious notation, $\gamma_a(t) = a_\alpha_a(t) + a^\dagger_{\alpha_a}(t)$. Substituting this in the ψBF Lagrangian, taking the topological scaling limit as above, and using the normalization of $\chi$, we retain the following quantum mechanical Lagrangian

$$L_M = \frac{m}{4} \sum_{a=1}^{2N} \gamma_a(t) i \partial_t \gamma_a(t),$$

where we used the notation, $\gamma_a(t) \equiv \gamma(t, x^\mu_a(t))$. From (15) follows the commutation relations $\{\gamma_a(t), \gamma_b(0)\} = 2\delta_{ab}$. Thus, for an adiabatic motion of vortices in the ψBF theory, by taking the scaling limit, we get a 2N dimensional Clifford algebra at each instant of time. This algebra has a unique irreducible representation up to a similarity transformation $S$, i.e., $\gamma_a(t) = S^{-1} \gamma_a S = y_{ab} \gamma_b(0)$, where $y_{ab} \in SO(2N)$. It follows from the connection to the ψBF model that the operator $S(t)$ is unique and well defined. We can now express the quantum mechanical Lagrangian (15) in terms of $y_{ab}$ as

$$L_M = -\frac{i}{4} (g^{-1} \dot{g})_{ab} \gamma_b(0) \gamma_a(0) = -\frac{i}{4} \text{Tr}(g^{-1} \dot{g} w_i^T q_i)$$

where the final form is obtained by bringing the matrix $y_{ab}(0) \gamma_a(0)$ to canonical form; the weight vector $w_i$, which is formed from the generators of the Cartan subalgebra of $SO(2N)$, depends on the initial state. Quantizing (16), using standard methods based on\(^{24}\), the resulting Hilbert space is a representation space for the spinor representation of $SO(2N)$, which is known to describe Ising type NA anyons\(^{19}\).

VII. FINAL REMARKS

In this letter we proposed the ψBF theory defined by (4) as the proper effective theory for superconductors in the energy range below the superconducting gap. The distinguishing feature of our theory is that the kinetic term for the fermions has support only where the vorticity differs from zero, and as consequence, will be confined to vortex cores. We treated the 2d spinless case with $p_x + ip_y$ pairing in dome detail, but also indicated how to generalize to 3d and to other pairing channels. We are at present working on this. There are also some aspects of the 2d case treated here that remain to be investigated viz., the edge excitations, and the ground state degeneracies on higher genus surfaces. Finally, it is a challenge to derive the ψBF theories from a truly microscopic theory, and also to find a connection to the Chern-Simons formalism in Ref. 8.

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Appendix A: The single particle Hamiltonian

The Lagrangian $L_{\psi BF}$ gives the canonical equal time commutation relations
\[
\{ \psi^\dagger (\vec{r}, t), \psi (\vec{r}', t) \} = \frac{4\pi}{B_a} \delta^2 (\vec{r} - \vec{r}') \tag{A1}
\]
eq 0, etc., and the Hamiltonian
\[
H = \frac{1}{4\pi} \int d^2 x \left( \psi^\dagger \left( \begin{array}{cc} H_0 & -\Delta \partial_z \\ \Delta^* \partial_z & -H_0^* \end{array} \right) \psi \right) .
\]

Expanding the fields:
\[
\psi (\vec{r}, t) = \sum_n a_n(t) u_n (\vec{r}) + a_n^\dagger (t) v_n^* (\vec{r}) ,
\]
and demanding that the mode operators $a_n$ shall satisfy $\{ a_m, a_n^\dagger \} = \delta_{mn}$, the commutation relations (A1) imply the inner product
\[
\langle \psi | \phi \rangle = \frac{1}{4\pi} \int d^2 x B_a (u^* U + v^* V)
\]
between two single particle states $\psi = (u \ v)^T$ and $\phi = (U \ V)^T$. A straightforward calculation shows that in order for $a_n$ to create a state with energy $E_n$, to satisfy $[H, a_n^\dagger] = E_n a_n^\dagger$, there must exist a function $f(\vec{r})$ such that
\[
\left( \begin{array}{cc} H_0 & -\Delta \partial_z + f(\vec{r}) \\ \Delta^* \partial_z - f(\vec{r}) & -H_0^* \end{array} \right) \left( \begin{array}{c} u_n \\ v_n \end{array} \right) = E_n B_a \left( \begin{array}{c} u_n \\ v_n \end{array} \right) .
\]
(A2)

This is very similar to the usual single particle BdG equation, but note the presence of the factor $B_a$ multiplying the energy $E_n$. Since $B_a$ vanish exponentially away from the vortices, this factor will drastically change the spectrum, and eliminate the continuum.

We denote the matrix in (A2) by $h$, and since it has terms that are singular at the origin, some care is needed to define it properly. The problem is reminiscent of giving a proper definition of free anyons, which mathematically amounts to choosing a particular self-adjoint extension of the free particle Hamiltonian. In the present case there turn out to be three possible self-adjoint extensions of the operator $h$, but (for fixed sign of $\Lambda$) only one of them will support a zero mode. For $h$ to be self-adjoint, it must be symmetric, with respect to the inner product, i.e., $\forall \phi, \psi \in D_h$
\[
\langle \psi | H \phi \rangle = \langle H \psi | \phi \rangle ,
\]
which implies that $f(\vec{r}) = -\frac{1}{2} (\partial_z \Delta)$, and substituting this in (A2) gives eq. (7) in the main text.

Appendix B: The boundary condition

Since the operator $h$ has terms that are singular at the origin, there are non-trivial restrictions on its domain $D_h$, i.e., on the allowed boundary conditions at $r = 0$, for it to be self-adjoint.

The condition (A3) must be satisfied as $r \to 0$, but the domain of $h^\dagger$ must also be the same as that of $h$, i.e., $\hat{\psi} \in K/D_h$ such that
\[
\langle \psi | H \phi \rangle = \langle \psi' | \phi \rangle
\]
for all $\phi \in D_h$ and some $\psi' \in K$ (where $K$ is the Hilbert space $L^2[R^2, B_a] \times L^2[R^2, B_a]$).

The domain of the Hamiltonian is spanned by the states,
\[
|\psi_1\rangle = e^{il\theta} \left( e^{-i\frac{1}{2} (m+1) \theta} \alpha_l (r) \\ e^{i\frac{1}{2} (m+1) \theta} \bar{\beta}_l (r) \right)
\]
where for modes with $l \neq 0$ $\alpha_l(r)$ and $\beta_l(r)$ must vanish at $r = 0$ for $H |\psi_1\rangle$ to be normalizable. For the $l = 0$ modes there is no such condition. To derive the proper boundary condition we first exclude a disc or radius $\bar{r}$ around the origin, and then take the limit $\bar{r} \to 0$. Choosing $\psi = (e^{-i\frac{1}{2} (m+1) A_0} \alpha_l (r), e^{i\frac{1}{2} (m+1) B_0} (r))$ and $\phi = (e^{-i\frac{1}{2} (m+1) A_0} (r), e^{i\frac{1}{2} (m+1) B_0} (r))$, we have
\[
\langle H^\dagger \psi | \phi \rangle = \langle \psi | H \phi \rangle = \langle \psi | \phi \rangle - \int d\theta \left[ \frac{r E_b}{2} \delta (A_n^0 \beta_0 - B_n^0 \alpha_0) \right]_{r=\bar{r}} ,
\]
where the last identity follows by partial integration. Since $r E_b$ remains finite as $r \to 0$ we must have $\lim_{\bar{r} \to 0} (A_n^0 \beta_0 - B_n^0 \alpha_0)_{r=\bar{r}} = 0$ for $H$ to be self adjoint. To satisfy this condition, we must restrict $D_h$. The general solution is
\[
\exists s : \lim_{r \to 0} \alpha_0 (r) = s \lim_{r \to 0} \beta_0 (r) .
\]

Since this condition holds also for the $l \neq 0$ modes (they vanish at the origin), the boundary condition for a general state $\psi = (\alpha (\vec{r}), \beta (\vec{r}))$ is
\[
\lim_{r \to 0} \alpha (\vec{r}) = s \lim_{r \to 0} \beta (\vec{r}) .
\]
(Note that the obvious choice i.e., that the wave functions vanish at the origin is a too strong condition in that demanding the integral in (B2) to vanish does not at all restrict the domain of $H^\dagger$, meaning that $H$ is not essentially self-adjoint.) As shown in the main text, only $s = \pm 1$ gives a Hamiltonian which has a zero mode, and the sign is determined by the sign of the coupling $\Lambda$.

Appendix C: The Dirac phase operator

The Dirac phase operator $\xi$ related to $a$ transforms as
\[
\xi \to e^{2i\Lambda} \xi ,
\]
under a gauge transformation $a \to a + d \Lambda$. In the static case treated in this article we may restrict ourselves to
a gauge sector where \( \vec{a} \) is time independent and \( a_0 = 0 \). For the case of zero magnetic field we can define \( \xi \) as the solution to the differential equation

\[
-i\vec{\nabla} \xi = \hat{a} \xi .
\]

(C1)

If \( \hat{a} \) is sufficiently regular at spatial infinity we can put the boundary condition \( \lim_{r \to \infty} \xi = 1 \) and we get

\[
\xi(r) = \exp \left( 2i \int d^2 r' \hat{a}(r') \cdot \vec{\nabla}' G(r', r) \right),
\]

(C2)

where \( G \) satisfies \( -\vec{\nabla}^2 G(r', r) = \delta^2 (r - r') \), and where we also performed an integration by parts. This is the usual expression for the Dirac phase factor.

The formula (C1) is however not applicable when the magnetic field is not identically zero. If the magnetic field only had compact support we could define \( \xi \) by (C1) in the region where the magnetic field is zero, and then analytically continue to the whole plane. In the case relevant for this article the magnetic field does not have compact support, but is exponentially localized with a localization length \( \lambda_L \) around points \( \{ \vec{r}_a \} \). The straightforward generalization to this case would then be

\[
-i\vec{\nabla} \xi = \lim_{\lambda_L \to 0} \hat{a}_{\lambda_L} \xi ,
\]

(C3)

but then we have to specify a family of vector potentials \( \{ \hat{a}_{\lambda_L} \} \). This family should only be defined by the requirement that no gauge transformation is associated with the change of \( \lambda_L \). More precisely, the equation

\[
\int d^2 r' \hat{a}_{\lambda_L}(r') \cdot \vec{\nabla}' G(r', \vec{r}) = \int d^2 r' \hat{a}_{\lambda_L}(r') \cdot \vec{\nabla}' G(r', \vec{r})
\]

should hold for all \( \lambda_L \) and \( \lambda' \).

Returning to the vortex configuration considered in the text, and using Coulomb gauge, we have

\[
\hat{a} = m \hat{\theta} \left( \frac{1}{r} - \frac{1}{\lambda_L} K_1 \left( \frac{r}{\lambda_L} \right) \right) = m \left[ \vec{\nabla} \theta - \frac{1}{\lambda_L} K_1 \left( \frac{r}{\lambda_L} \right) \right]
\]

for a vortex of strength \( m \). Since \( \lim_{\lambda_L \to 0} \frac{1}{\lambda_L} K_1 \left( \frac{r}{\lambda_L} \right) = 0 \) we can read off the solution to (C3), and we get \( \xi = e^{i(m \theta + \alpha)} \), with \( \alpha \) being a real constant which we, without loss of generality, can put to zero.

Appendix D: Numerics

To get the spectrum of the single particle Hamiltonian we project it to a finite Hilberts space in which we diagonalize exactly. Because of the rather unusual inner product (with a measure \( \sim \sqrt{r} \sim K_0 \)), usual basis sets, such as cylindrical waves, will give a generalized eigenvalue problem with a matrix with a very small determinant, implying that the overlap integrals have to be calculated with a very high precision.

To overcome this problem one can either try to find a basis which from the start is close to orthogonal, w.r.t. the inner product, and has sizable matrix elements for the matrix \( h \), or to fine a basis where all integrals can be analytically, and thus be evaluated to a very high precision.

We choose the the second strategy and used the basis functions \( \phi_n = r^a \sqrt{E(r)} \) since all relevant matrix elements can be evaluated using the formula,

\[
\int_{0}^{\infty} r^a K_b(r) K_c(r) dr = \frac{\Gamma \left( \frac{1}{2}(a - b + c + 1) \right) \Gamma \left( \frac{1}{2}(a + b + c + 1) \right) \Gamma \left( \frac{1}{2}(a - b - c + 1) \right) \Gamma \left( \frac{1}{2}(a + b - c + 1) \right)}{2^{2-a} \Gamma(a + 1)}
\]

To estimate the precision of our result, we increased the maximum exponent \( a_{\text{max}} \), from 125 to 250, which gave a relative changes in the eigenvalues of at most \( < 10^{-2} \), as claimed in the main text.