Influence of periodically modulated cavity field on the generation of atomic-squeezed states

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Abstract
We investigate the influence of a periodically time-modulated cavity frequency on the generation of atomic squeezed states for a collection of $N$ two-level atoms confined in a non-stationary cavity with a moving mirror. We show that the two-photon character of the field generated from the vacuum state of the field plays a significant role in producing the atomic or spin squeezed states. We further show that the maximum amount of persistent atomic squeezing is obtained for the initial cavity field prepared in the vacuum state.

Keywords: dynamical Casimir effect, non-stationary cavity, spin-squeezing

1. Introduction

It is well known that a non-classical state of electromagnetic field such as a squeezed state can be generated from vacuum in a non-stationary cavity with an oscillating wall. This phenomenon of photon generation from the vacuum state by a non-adiabatic change in the boundary condition of quantum fields is termed the dynamical Casimir effect (DCE). Several theoretical studies devoted to the generation of photons in a cavity with a vibrating wall resulting in harmonic modulation of the mode frequency have been reported in the literature [1–5]. We also refer to [6] and [7] for review on various aspects of DCE. It has been shown that for an ideal cavity (without cavity dissipation) a maximum number of photons is created when the modulation frequency is twice the unperturbed cavity mode frequency. Moreover, under this resonance condition the non-classical nature such as the squeezing property also increases with time. From an experimental point of view the major hurdle in generating photons by DCE lies in realizing a vibrating cavity with a frequency of the order of a few gigahertz. Furthermore, detection of the generated photon requires coupling the field mode to a detector, which can in turn alter the statistical properties of the radiation field significantly due to back action. In order to investigate the effect of back action on the generated photons, Dodonov and co-workers have carried out a series of studies on the interaction of various model atoms with the field in a cavity with harmonically modulated mode frequency [2, 8–11]. Interaction of a single two-level atom with quantized electromagnetic field inside a cavity with time dependent parameters has also been studied in [12]. Keeping these difficulties in mind, recently two proposals for experimental generation and detection of photons in non-stationary cavities have been reported in [13] and [14]. In [14] the authors proposed that cavities with mechanical vibration in the gigahertz range may be obtained through a film bulk acoustic resonator (FBAR) [15, 16] made from a vibrating aluminum nitride film of thickness equal to half the acoustic wavelength. For detection of photons generated in the cavity these authors proposed to use ultra-cold alkali atoms in their hyperfine states. On the other hand, [13] explored the possibility of using a cavity with a plasma mirror made from a semiconductor slab irradiated by periodic laser pulses for
generation of photons from the vacuum state, and Rydberg atoms with large values of principal quantum number for detection of these photons.

One of the hallmarks of the photons generated in the non-stationary cavity is the presence of two-photon correlation, as they are generated via a two-photon process. It is then natural to explore the possibility of transferring these two-photon correlations to a collection of atoms to generate so-called correlated atomic squeezed (spin squeezed) states [17–21] or atomic entangled states. To this end, in this paper we study the interaction of \( N \) two-level atoms with a quantized single radiation mode of a non-stationary cavity with harmonically oscillating wall. In particular we focus our attention on the effect of modulation of the cavity mode frequency due to harmonic oscillation of the cavity mirror on the squeezing properties of an ensemble of atoms interacting with this cavity mode. Here we wish to note that dynamics of \( N \) two-level atoms in a non-stationary cavity has already been investigated in the context of detection of photons generated in the cavity [11, 13]. On the other hand, in the present work we concentrate on the generation of atomic squeezed state in a non-stationary cavity. We should also remark here that several early proposals for the generation of atomic squeezed states involved transfer of two-photon correlation from the radiation mode to the collection of atoms using photon–atom interaction [21]. In accordance with this idea the generation of an atomic or spin squeezed state by the transfer of two-photon correlation in a multi-mode squeezed vacuum to an ensemble of atoms has already been reported in the literature [22–25].

Before proceeding further we wish to mention that atomic squeezed states are useful quantum resources to improve the precision of measurements in experiments [18, 19, 26, 27] and to study particle correlations and entanglement [28–30]. These states have found application in atomic clocks for reducing quantum noise [18, 19, 31–33] and in quantum information processing [28, 34–36]. The utility of atomic squeezed states in spectroscopy and meteorology has motivated us to explore the possibility of generation of these states in a non-stationary cavity. The rest of the paper is organized in following manner. In section 2 we give a description of the model used in this paper. The results are presented and discussed in section 3 and the paper is concluded in section 4.

2. Model Hamiltonian

In order to study the generation of atomic squeezed or spin squeezed states by DCE we consider a collection of \( N \) two-level atoms interacting with a quantized single mode of the radiation field of a cavity with an oscillating wall. Following [11], the Hamiltonian for the atom plus non-stationary cavity system under the rotating wave approximation is written as

\[
H = \omega_0 J_z + \omega_a(t) a^{\dagger} a + g_0 \left( a J_x + a^{\dagger} J_{-} \right) + i \xi(t) \left( a^{\dagger 2} - a^2 \right)
\]

where \( a \) and \( a^\dagger \) are the cavity annihilation and creation operators respectively, satisfying the commutation relation \([a, a^\dagger] = 1\). The ensemble of \( N \) atoms is described using the picture of collective spin operators \( J_x = \sum_n |e_i\rangle\langle e_i| \) and \( J_z = \sum_n |g_i\rangle\langle g_i| \), where \(|e_i\rangle\) and \(|g_i\rangle\) represent the excited and the ground states of the \( i \)th two-level atom, respectively. The spin operators are dimensionless and satisfy the commutation relations \([J_x, J_-] = 2J_z \) and \([J_x, J_+] = \mp J_z\). The Hilbert space of these atomic operators is spanned by symmetric Dicke states \(|J, M\rangle\) with \( M = -J, -J+1, \ldots, J-1, J \) [37]. The total spin length is given by \( J = N/2 \). The Dicke states are eigenstates of \( J^2 \) and \( J_z \) such that \( J^2 |J, M\rangle = M(J, M) \) and \( J_z |J, M\rangle = M(J, M) \). The lowering and raising operators \( a \) and \( a^{\dagger} \) act on these states as \( J_x |J, M\rangle = J_x |J, M\rangle \) and \( J_z |J, M\rangle = J_z |J, M\rangle \). The parameters \( \omega_a \) and \( g_0 \) denote the atomic transition frequency, cavity mode frequency, and atom–field coupling constant (which is assumed to be real) respectively. The harmonic time dependence of the cavity mode frequency \( \omega_c(t) \) and the last term, which is nonlinear in \( a \) and \( a^\dagger \) arise due to harmonic motion of the cavity boundary [1]. A brief derivation of this nonlinear part of the Hamiltonian is presented in appendix A. For the purpose of calculations, following earlier works, we assume that the cavity mode frequency has a sinusoidal dependence given by \( \omega_c(t) = \omega_0 (1 + e \sin(\Omega t)) \) with the unperturbed frequency \( \omega_0 \), which is set to 1 for simplicity and \( e \) and \( \Omega \) representing the modulation amplitude and the modulation frequency respectively. Furthermore, note that there are basically two kinds of non-adiabatic process occurring in a non-stationary cavity system described by the above Hamiltonian [1]. The first kind is characterized by the \( a^\dagger a \) term, in which the total number of photons inside the cavity does not change. Such a process is known as a zero-photon process. On the other hand, the second kind is represented by the terms \( a^{\dagger 2} \) and \( a^2 \), which are responsible for the generation of squeezed photons from the vacuum state. The last term in the Hamiltonian introduces two-photon correlation in the cavity mode, and in this paper we explore the possibility of transferring this correlation from photons to atoms via atom–photon interaction. Also, \( \xi(t) \) is the effective frequency, which is an arbitrary function of time and is related to \( \omega_a(t) \) as [1]

\[
\xi(t) = \frac{1}{4\omega_a(t)} \frac{d\omega_a(t)}{dt}.
\]

Considering the realistic case of small-amplitude time modulation, i.e. \(|e| \ll 1\), one can obtain the following expression for \( \xi(t) \) from equation (2):

\[
\xi(t) = 2\ddot{\xi}_0 \cos(\Omega t),
\]
where ω = Ω/8 ≪ 1. Note that for ω(t) independent of time the coefficient ξ(t) = 0 and the above Hamiltonian reduces to the Tavis–Cummings Hamiltonian [38], which has been extensively studied in the context of cavity QED.

Having discussed the basic model of a multi-atom system coupled to a field mode of a non-stationary cavity we now briefly describe the squeezing parameter employed in this paper to characterize the atomic squeezing. We note here that several definitions for the spin or atomic squeezing parameters have been proposed in the literature in different contexts [17–21]. For example, according to the definition of Kitagawa and Ueda, a state is spin squeezed only if the variance of one spin component Jι normal to the mean spin vector is less than the variance for a Bloch state (J/2) [17]. In accordance with the definition of Kitagawa and Ueda, the spin squeezing parameter ζS is written as [17]

$$\zeta_S = \sqrt{\frac{\min(\Delta J_\parallel^2)}{J/2}} = \sqrt{\frac{4 \min(\Delta J_\perp^2)}{N}},$$  

where the subscript 1⊥ refers to an axis perpendicular to the mean-spin direction 1∥ = (J∥/J) and the minimization is over all directions 1⊥. This parameter is used to quantify the degree of quantum correlations among the elementary spins. The atomic-squeezing condition in terms of this parameter is given by the condition ζS < 1; i.e., the fluctuation in one direction is reduced. Indeed, it has a very close relation with quantities such as concurrence [34] and negative correlations [39]. On the other hand, the squeezing parameter proposed by Wineland et al [18] was in the context of Ramsey spectroscopy for the determination of transition frequency, and consequently this parameter is also termed spectroscopic squeezing. This squeezing parameter is related to the ratio of fluctuations in the measurement of resonance frequency using an ensemble of atoms in a general atomic state and in a coherent spin state. The spectroscopic squeezing parameter ζR is given by

$$\zeta_R = \sqrt{\frac{\min(\Delta J_{\parallel 1}^2)}{|J|^2}},$$  

These two squeezing parameters are related to each other and it can be seen from their definitions that ζS² ≤ ζR². In this paper we employ both the squeezing parameters mentioned above to characterize the squeezing property of a collection of atoms interacting with the field mode of a non-stationary cavity. To this end we need to calculate averages of angular momentum operators and their second-order moments in the combined state |Ψcom(t)⟩ of atoms plus field satisfying the time-dependent Schrödinger equation

$$\frac{d|\Psi_{\text{com}}(t)\rangle}{dt} = -iH |\Psi_{\text{com}}(t)\rangle.$$  

In order to solve this differential equation we make use of the ansatz

$$|\Psi_{\text{com}}(t)\rangle = \sum_{n,M} C_{n,M}(t) |n\rangle |M\rangle,$$

where C_{n,M} are the time-dependent coefficients and |n⟩ represents the number state, which is an eigenstate of the number operator a†a such that a|n⟩ = √n |n − 1⟩ and a† |n⟩ = √n + 1 |n + 1⟩. For convenience we use the notation |M⟩ = |J, M⟩ since J is constant of motion for H given by equation (1). We further assume that the radiation mode and the collection of atoms are initially uncorrelated and the initial state of the atom plus field system can be written as a direct product

$$|\Psi_{\text{com}}(0)\rangle = \left(\sum_n C_n |n\rangle\right) \otimes |\psi(0)\rangle,$$

where |\psi(0)⟩ = |J = N/2, M = −N/2⟩ represents the initial atomic state in which all the atoms are occupying their ground states and the coefficients C_n are the projections of the initial field state on the number state |n⟩. Using Schrödinger’s equation (equation (6)) with the Hamiltonian H of equation (1) we write the equation of motion for the coefficient C_{n,M} as

$$i\dot{C}_{n,M}(t) = \left[1 + e \sin(Ωt)n + \omega_n M C_{n,M}(t) \right] + gn \left[\sqrt{n} \sqrt{J(J + 1) - M(M + 1)} C_{n-1,M+1}(t) \right] + gn \left[\sqrt{n + 1} \sqrt{J(J + 1) - M(M - 1)} C_{n+1,M-1}(t) \right] + 2iξ_0 \cos(Ωt) \left[\sqrt{n + 1} \sqrt{n + 2} C_{n+2,M}(t) \right] - 2iξ_0 \cos(Ωt) \left[\sqrt{n} \sqrt{n - 1} C_{n-2,M}(t) \right].$$

We wish to point out that in general the above differential equation cannot be solved analytically due to coupling of the coefficient C_{n,M} with infinitely many C_{n-2,M}, C_{n-4,M}, ... and coefficients. Consequently, one needs to solve these coupled differential equations numerically. To implement the numerical method for practical reasons it becomes necessary to truncate the number state basis of the cavity mode. To this end we choose an adequate number of basis states |n⟩ and ensure the convergence of results by increasing the basis size. In the next section we present and discuss the results we have obtained.

3. Results and discussion

We begin this section by presenting the results, which have been obtained by numerically solving equation (9) with N = 20 (J = 10) and considering initially the atomic and the cavity field mode to be in |ψ(0)⟩ = |J, −J⟩ and vacuum state |0⟩ (c_n = 1 for n = 0 and c_n = 0 for n ≠ 0) respectively. The results for the evolution of atomic squeezing parameters ζS(t) and ζR(t) are displayed in figures 1(a) and (b) respectively for various values of modulation amplitude ε with the modulation frequency fixed at Ω = 2.0. This choice of value of modulation frequency is guided by the result that the maximum number of squeezed photons is created in an ideal empty non-stationary cavity when the modulation frequency is twice the unperturbed cavity mode frequency. It can be clearly seen that under this condition both the atomic squeezing parameters
become less than 1 for non-zero values of the modulation amplitude, indicating generation of an atomic or spin squeezed state. On the other hand, for \( \epsilon = 0 \), no atomic squeezing is observed as both \( \xi_\delta(t) \) and \( \xi_R(t) \) remain unity throughout. This constitutes the main result of this paper. Moreover, we observe that the results presented in figures 1(a) and (b) satisfy the inequality \( \xi_\delta^2 < \xi_R^2 \). The last term in the Hamiltonian with \( a_{12}^2 \) and \( a_2^2 \) results in the generation of a squeezed state of the radiation field from the vacuum state via a two-photon process and the two-photon correlation of the squeezed state is transferred to the ensemble of atoms via interaction of the atoms with this cavity field mode. Moreover, from figures 1(a) and (b), we observe that persistent atomic squeezing, which lasts for a longer time, is produced under the resonant condition \( \Omega = 2\omega_0 \), and the magnitude of squeezing also increases with the increase in modulation amplitude. This increase in atomic squeezing is attributed to the enhancement in squeezing characteristic of the cavity field with higher values of modulation amplitude \( \epsilon \). Thus, the modulation amplitude acts as an additional tool for controlling the degree of atomic squeezing generated by photon–atom interaction in a non-stationary cavity.

To get a better insight into the mechanism of the generation of atomic squeezed states via atom–photon interaction in a non-stationary cavity with an oscillating wall, we now focus our attention on a system of two two-level atoms interacting with a single quantized cavity mode. The time dependent Schrodinger equation with the model Hamiltonian \( H \) given by equation (1) for two atoms \( (N = 2) \) can be solved analytically under a particular value of detuning between the cavity mode \( (\omega_0) \) and modulation frequency \( (\Omega) \). Following [10, 11], we find that for \( \Omega = 2 + \delta \) with \( \delta = \pm \sqrt{6} \omega_0 \), and both the atoms initially in their ground states \( (|g_1g_2\rangle) \) and the field mode in the vacuum state \( (|0\rangle) \), only four essential atom plus photon states become coupled and acquire significant probabilities of occupation. Under this condition no more than two photons can be generated from the initial state mentioned above. Therefore, for this non-resonant condition the time dependent atom plus photon state for time \( t > 0 \) can be written as

\[
|\Psi_{\text{com}}(t)\rangle = a_0(t)e^{it}|0, -1\rangle + a_2(t)e^{-it}|2, -1\rangle + b_1(t)e^{it}|1, 0\rangle + d_0(t)e^{-it}|0, 1\rangle \tag{10}
\]

where as mentioned before state \( |n, M\rangle \) denotes the combined state of photons in number eigenstate \( |n\rangle \) and the two-atom system in a collective state \( |M\rangle \) with \( M = -1, 0, 1 \). In terms of the ground \( (|g_1\rangle, |g_2\rangle) \) and excited \( (|e_1\rangle, |e_2\rangle) \) states of the individual atoms these states are represented as \( |0, -1\rangle = |0, g_1g_2\rangle, |2, -1\rangle = |2, g_1g_2\rangle, |1, 0\rangle = 1/\sqrt{2} (|1, g_1e_2\rangle + |1, e_1g_2\rangle), \) and \( |0, 1\rangle = |0, e_1e_2\rangle \). From the numerical solution of the time-dependent Schrodinger equation for \( \delta = \pm \sqrt{6} \omega_0 \) we find that all the coefficients except for the four states mentioned in equation (10) remain zero or negligible, thus validating the use of only four states in equation (10). Therefore, by considering only the four above mentioned states in the dynamics of a two-atom system, we find from our analysis that the four coefficients are given by (see appendix B for derivation)

\[
\begin{align*}
a_0(t) &= \cos \left( \frac{2}{\sqrt{3}} qt \right) \\
a_2(t) &= \frac{1}{\sqrt{3}} \sin \left( \frac{2}{\sqrt{3}} qt \right) \\
b_1(t) &= \frac{1}{\sqrt{2}} \sin \left( \frac{2}{\sqrt{3}} qt \right) \\
d_0(t) &= \frac{1}{\sqrt{6}} \sin \left( \frac{2}{\sqrt{3}} qt \right)
\end{align*}
\tag{11}
\]

with \( q = (2 + \delta)/8 \). It should be noted here that the non-zero values of the coefficients \( a_0(t) \) and \( d_0(t) \) result in generation of a coherent superposition of atomic states \( |g_1g_2\rangle \) and \( |e_1e_2\rangle \) at time \( t > 0 \). To see this more explicitly we write the atomic density matrix from the combined density matrix.
\[ \rho_{\text{com}}(t) = |\Psi_{\text{com}}(t)\rangle \langle \Psi_{\text{com}}| \] by tracing over the field state as
\[ \rho_{\text{atom}}(t) = \left( |a_0(t)|^2 + |a_2(t)|^2 \right) \rho_{-1-1} \]
\[ + |b_1(t)|^2 \rho_{00} + |d_0(t)|^2 \rho_{11} \]
\[ + a_0(t) d_0^*(t) e^{2i \epsilon t} \rho_{-1-1} \]
\[ + a_0^*(t) d_0(t) e^{-2i \epsilon t} \rho_{-1-1} \]  
(12)

with \( \rho_{PQ} = |P\rangle \langle Q| \) and \( P, Q = 1, 0, -1 \) denoting atomic density matrices. The last two non-diagonal terms of the atomic density matrix \( \rho_{\text{atom}}(t) \) arise due to coherent superposition of atomic states \( |g_1 g_2\rangle \) and \( |e_1 e_2\rangle \), and these terms signify the presence of the two-particle correlation in the atomic system. It is well known that the presence of such two-particle correlation is essential for the generation of atomic squeezed states [40–42], and this also makes the atomic squeezing parameter a reliable measure of entanglement [43] for this kind of state. This clearly shows that in the atom–photon interaction the two-photon correlation of the squeezed state is transferred to the collection of atoms, thereby producing correlated atomic states. To verify the generation of the atomic squeezed state we calculate all the required averages of various components of angular momentum in the state given by equation (10). First, we find that the direction of \( \langle \vec{J} \rangle \) does not change with time and remains aligned along the initial \( z \)-direction, and we obtain the following expressions for \( \langle J_z \rangle \) and \( \langle J_z^2 \rangle \).

\[ \langle J_z(t) \rangle = -\frac{5}{6} \sin^2 \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \]  
(13)
\[ \langle J_z^2(t) \rangle = \frac{1}{2} \left[ 1 + \frac{1}{2} \sin^2 \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \right] - \frac{2}{\sqrt{6}} \sin \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \cos \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \cos 2t \]  
(14)

In accordance with the definition of atomic squeezing parameter we calculate the variance of the \( y \)-component of the angular momentum, which is normal to the mean spin vector \( \langle \vec{J} \rangle \), and the corresponding squeezing parameter is found to be

\[ \zeta_y^2(t) = \left[ 1 + \frac{1}{2} \sin^2 \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \right] - \frac{2}{\sqrt{6}} \sin \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \cos \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \cos 2t \right] \]
\[ \left[ 1 + \frac{5}{6} \sin^2 \left( \frac{2}{3} \sqrt{3} \epsilon t \right) \right] \]  
(15)

To explicitly show that the above expression indeed takes a value less than unity we plot \( \zeta_y^2(t) \) given by equation (15) as a function of \( t \) in figure 2, along with the corresponding results obtained by numerically solving equation (24) for comparison. It can be clearly seen from figure 2 that \( \zeta_y^2(t) \) attains a value less than unity in a finite interval of time, clearly demonstrating the generation of atomic squeezed states. Moreover, we also observe that the numerical result for atomic squeezing is quite close to the corresponding result obtained via the analytical expression given by equation (15). Therefore, we conclude that the inclusion of only four states mentioned above for calculation of the dynamics of a two-atom system in a non-stationary cavity under the condition \( \Omega = 2 + \sqrt{6} g_0 \) is quite accurate. It is important to note here that non-zero value of the coefficients \( a_0(t) \) and \( d_0(t) \) is essential for the reduction of fluctuation \( \zeta_y^2 \). On the other hand, in the absence of mirror oscillation \( (\epsilon = 0) \) and for the same initial states of atoms and the field as considered above \( \zeta_y^2 \) (always remains unity, signifying absence of spin squeezing [44]). Therefore, our analysis for the two two-level atoms clearly elucidates how two-photon correlation from the cavity field mode is transferred to the atoms in the non-stationary cavity to generate atomic or spin squeezed states.

Before proceeding further we wish to note here that for a two-atom system with \( \delta = 0 \) (as chosen for the numerical results presented in figure 1) it is not possible to solve the time dependent Schrodinger equation analytically due to coupling of coefficients with an even number of photons leading to an infinite set of differential equations. However due to generation of states with an even number of photons \( (n \geq 2) \) it is expected that higher spin squeezing will be generated.

Having discussed generation of the atomic squeezed state in a non-stationary cavity with the field mode initially in the vacuum state we now consider the case when the cavity mode is initially prepared in a coherent state. In this connection we mention here that atomic squeezing property of a system of \( N \) two-level atoms interacting with a single radiation mode of a stationary cavity (Tavis–Cummings model), which is initially in a coherent state, has already been reported in [44]. Thus it would be interesting to investigate the effect of cavity oscillations, which give rise to extra non-linear terms in the Hamiltonian, on the squeezing property of atoms with the cavity mode in a coherent state. For this purpose once again we solve the coupled differential equation of equation (9) numerically by using Mathematica 9.0. The calculations are carried out for initial atomic state \( |\psi(0)\rangle = |J, -J\rangle \) with \( J = 10 \) and for the field mode in a coherent state \( |\alpha\rangle\)
α = (thick solid line) and ϵ = 2ζ (dashed line). Here α = t = 0 (thin solid line), ϵ = 0.05 (thick solid line) and ϵ = 0.15 (dashed line). We assume ψ(0) = |Ji − J| (J=10) and the cavity field to be initially prepared in a coherent state with an average of one quantum. The other parameters used are g0 = 0.6, Ω = 2 and ω0 = 1.

(cα = e−|α|2/2|√α|) with |α| = 1.0. The results of these calculations for ζR(t) and ζR(t) are shown in figures 3(a) and (b) respectively, where we plot squeezing parameters as functions of dimensionless time (t) for three different values of modulation amplitude ϵ: ϵ = 0 (thin solid line), ϵ = 0.05 (thick solid line) and ϵ = 0.15 (dashed line). Note that the result for ϵ = 0 corresponds to the stationary cavity case described by the Tavis–Cummings model. First, we observe that for all the values of ϵ atomic squeezed states are generated in the time range 0 < t < 1.5 by the interaction of atoms with the cavity radiation mode irrespective of the parameter employed for characterizing atomic squeezing. The comparison of results for different values of ϵ clearly reveals that within this time range atomic squeezing characterized by ζS(t) and ζR(t) shows very little variation with respect to change in ϵ. On the other hand, for longer time scales (t > 1.5) two parameters exhibit distinctly different atomic squeezing behavior. For example, in the longer time scale ζS(t) attains a value less than unity, signifying generation of atomic squeezing, and the degree of squeezing increases with higher values of ϵ. In contrast to this, in the long time scale ζR(t) remains greater than unity, indicating the absence of atomic squeezing.

Furthermore, we note here that comparison of results presented in figure 1 with the corresponding results in figure 3 also shows that for a field mode initially prepared in the vacuum state atomic squeezing is achieved for a significantly longer time scale as compared to the case when the field mode is initially prepared in a coherent state. Consequently, for the vacuum case, unlike a normal cavity-QED experiment with the cavity field in a coherent state, the stringent requirement of precise control of interaction time for generation of the atomic squeezed state is not necessary [44].

The results discussed above on the generation of an atomic squeezed state by DCE can be realized with the help of practical schemes recently proposed for the generation and detection of DCE photons [13, 14]. As mentioned before both the schemes have proposed to use cavities operating in the gigahertz regime. Therefore, the presence of thermal photons will affect the results discussed above. In order to reduce the effect of thermal photons on the generation of squeezed photons the cavity should be cooled below 100 mK. In a realistic cavity-QED system, decay of the cavity mode will also degrade the degree of field squeezing and this in turn will affect the magnitude of atomic squeezing. However, the loss of photons through the cavity mirrors can be minimized by using a high-Q cavity. For example, it has been shown in [13] that using a semiconductor plasma mirror [45] with Q of the order of 107 it is possible to achieve the so-called threshold condition for generation of squeezed photons. Therefore, we expect that a beam of Rydberg atoms interacting with the field mode of this cavity will be able to produce a significant amount of atomic squeezing. Finally, the atomic squeezed states generated in a non-stationary cavity can be experimentally detected by making measurements of the atomic state population using the Ramsey separated field method on the Rydberg atoms [19].

4. Conclusion

In this paper we have studied the effect of periodic time modulation of cavity frequency on the generation of atomic or spin squeezed states. For this purpose we consider a system of N two-level atoms interacting with a single radiation mode of a non-stationary cavity with harmonically vibrating wall. We demonstrate that by modulating the cavity mode it is possible to generate an atomic squeezed state by allowing an ensemble of atoms prepared in their ground states to interact with a cavity mode initially in the vacuum state. In the absence of modulation, no atomic squeezing is observed. Like squeezed photon generation by DCE the efficient generation of atomic squeezed states occurs when the modulation frequency of the cavity mode is twice its own frequency. By studying a simpler system of two two-level atoms confined in a non-stationary cavity we explicitly show that atomic squeezing arises due to generation of coherent superposition of atomic states by transfer of two-photon correlation from the field mode to the atoms. For this two-atom system we also derive an analytical expression for atomic squeezing for a particular value of
detuning between the modulation frequency and the cavity mode by considering the dynamics of a few essential states. We also study the effect of modulation on the generation of the atomic squeezed state for the case in which the cavity mode is initially prepared in a coherent state. Our study clearly reveals that for the cavity mode initially prepared in the vacuum state the degree of atomic squeezing increases with increasing modulation amplitude. Therefore, modulation amplitude acts as an additional factor in controlling the squeezing of spins.

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Appendix A

In general, a single-mode quantized cavity field is equivalent to a harmonic oscillator of unit mass such that its time-dependent Hamiltonian becomes

\[
H_t = \frac{1}{2} (\omega_c(t) \mathcal{Q}^2 + \mathcal{P}^2),
\]

(16)

where \(Q\) and \(P\) are hermitian operators and play the role of canonical position and momentum respectively such that \([Q, P] = i\). Here, \(\omega_c(t)\) represents the time-dependent cavity frequency. The operators \(Q\) and \(P\) can be written in terms of creation (\(a^\dagger\)) and annihilation (\(a\)) operators as

\[
Q = \sqrt{\frac{\omega_c(t)}{2}} (a + a^\dagger),
\]

(17)

\[
P = i \sqrt{\frac{\omega_c(t)}{2}} (a^\dagger - a),
\]

(18)

where the operators \(a\) and \(a^\dagger\) satisfy the commutation relation \([a, a^\dagger] = 1\). As a result, the equations of motion for \(a\) and \(a^\dagger\) take the following form:

\[
\dot{a} = -i\omega_c(t)a + \frac{1}{2\omega_c(t)} \frac{d\omega_c(t)}{dt} a^\dagger, \quad (19)
\]

\[
\dot{a}^\dagger = i\omega_c(t)a^\dagger - \frac{1}{2\omega_c(t)} \frac{d\omega_c(t)}{dt} a. \quad (20)
\]

Thus, the equations for \(a\) and \(a^\dagger\) are coupled to each other. Therefore, the form of the Hamiltonian in terms of \(a\) and \(a^\dagger\) correctly generating the above equations of motion (19) and (20) can be written as

\[
H_t = \omega_c(t) a^\dagger a + i \xi(t) \left( a^{\dagger 2} - a^2 \right), \quad (21)
\]

where \(\xi(t) = \frac{1}{4\omega_c(t)} \frac{d\omega_c(t)}{dt}\). This last term in the above Hamiltonian is responsible for parametric amplification, which helps in generating the squeezed states of the optical cavity field ([46]).

Appendix B

In this appendix we present derivation of equation (11). For this purpose we first go to the interaction picture by transforming \(|\Phi_{\text{com}}(t)\rangle = \exp[- i\Delta \Omega/2(a^\dagger a + J_c)] |\Phi(t)\rangle\). Under this transformation and by using the rotating wave approximation, we get the following expression for the transformed Hamiltonian from equation (1):

\[
H = (\Delta - \delta/2) J_c - \delta/2 a^\dagger a
\]

\[
+ g_0 \left( a J_e + a^\dagger J_- \right) + iq \left( a^{\dagger 2} - a^2 \right) \tag{22}
\]

where \(\Delta = \omega_b - \omega_{01}\), \(\delta = \Omega^2/2 - \omega_{02}\) and \(q = \epsilon(2 + \delta)/8\). For further analysis we choose \(\omega_{01} = 1\). The above Hamiltonian has been obtained for \(\epsilon \ll 1\). We now use the above Hamiltonian for a two-atom system whose state space is spanned by the states \(|J, M\rangle\) with \(J = 1, M = -1, 0, 1\). For this system we expand the wave vector \(|\Phi(t)\rangle\) as (for \(\Delta = 0\))

\[
|\Phi(t)\rangle = \sum_{n=0}^{\infty} e^{i\delta t/2} \left( a_n(t) e^{-i\delta t/2} |n, -1\rangle + b_n(t) |n, 0\rangle + d_n(t) e^{i\delta t/2} |n, 1\rangle \right) \tag{23}
\]

where \(|n, M\rangle\) denotes the combined state of \(n\) photons and the two-atom system in collective state \(|M\rangle\). For notational convenience we denote the collective atomic state \(|J, M\rangle\) by just \(|M\rangle\). By substituting \(|\Phi(t)\rangle\) given by equation (23) in the time-dependent Schrodinger equation we arrive at the following equations for the coefficients:

\[
\dot{a}_n(t) = -ig_0 \sqrt{2n} b_{n-1}(t)
\]

\[
+ q \left( \sqrt{n} (n - 1) e^{-i\delta t} a_{n-2}(t) - \sqrt{(n + 1)} (n + 2) a_{n+2}(t) e^{i\delta t} \right)
\]

\[
b_{n-1}(t) = -ig_0 \sqrt{2(n+1)} a_n(t) - ig_0 \sqrt{2(n-1)} d_{n-2}(t)
\]

\[
+ q \left( \sqrt{n} (n - 1) e^{-i\delta t} b_{n-3}(t) - \sqrt{(n + 1)} b_{n+3}(t) e^{i\delta t} \right)
\]

\[
d_{n-2}(t) = -ig_0 \sqrt{2(n-1)} b_{n-1}(t)
\]

\[
+ q \left( \sqrt{(n - 2)} (n - 3) e^{-i\delta t} d_{n-4}(t) - \sqrt{n} d_{n-4}(t) e^{i\delta t} \right). \tag{24}
\]

In order to solve these coupled differential equations we adopt the method used in [9–11] in the weak modulation limit defined by \(\epsilon \ll g_0\). In accordance with this method we first solve the above equations for \(q = 0\) (stationary cavity case)
and obtain
\[ b_{n-1}(t) = A_n e^{-i\Omega_n t} + B_n e^{i\Omega_n t} \]
\[ a_n(t) = \frac{g_0}{\Omega_n} \left( A_n e^{-i\Omega_n t} - B_n e^{i\Omega_n t} \right) \]
\[ d_{n-2}(t) = \frac{g_0}{\Omega_n} \left( A_n e^{-i\Omega_n t} - B_n e^{i\Omega_n t} \right) + \frac{n}{n-1} C_n \]
\[ \text{(25)} \]
where \( \Omega_n = g_0 \sqrt{2(n-1)} \) and the constants \( A_n, B_n, C_n \) are determined by initial conditions. For \( q \neq 0 \), the solutions given by equation (25) are once again substituted into equation (24) assuming that the coefficients \( A_n, B_n, C_n \) are slowly varying functions of time to obtain differential equations for these coefficients. It is easy to check that for specific values of detuning \( \delta \) some of these coefficients are multiplied by exponential factors with arguments larger than \( q \) and these terms are neglected by invoking the rotating wave approximation. On the other hand, coefficients which are multiplied by time independent factors are retained to obtain simplified effective dynamics. We find that for the initial state \( |\Psi_{\text{com}}(0)\rangle = |0, -1\rangle = |0, g_0 \Omega_2 \rangle \) and \( \delta = g_0 \sqrt{6} \) at most two photons can be created. Under this condition the coefficient \( a_0(t) \) associated with the initial state mentioned above satisfies the coupled differential equations
\[ a_0(t) = -q \sqrt{\frac{3}{4}} A_2(t) \]
\[ A_2(t) = q \sqrt{\frac{3}{4}} a_0(t) \]
\[ \text{(26)} \]
and the non-zero coefficients (terms of the order of \( e/|g_0|^2 \) are neglected) are obtained via the following relations:
\[ a_2(t) = \frac{2}{3} A_2(t) \]
\[ b_1(t) = A_2(t) \]
\[ d_0(t) = \frac{1}{3} A_2(t) \]
\[ \text{(27)} \]
Using equations (26) and (27) we obtain the solution given by equation (11).

References

[1] Law C K 1994 Phys. Rev. A 49 433
[2] Dodonov V V 1995 Phys. Lett. A 207 126
[3] Dodonov V V and Klimov A B 1996 Phys. Rev. A 53 2664
[4] Plunien G, Schutzhold R and Soff G 2000 Phys. Rev. Lett. 84 1882
[5] Crocce M, Dalvit D A R and Mazzitelli F D 2001 Phys. Rev. A 64 013808
[6] Dodonov V V 2010 Phys. Scr. 82 038105
[7] Nation P D, Johansson J R, Blencowe M P and Nori F 2012 Rev. Mod. Phys. 84 1
[8] Dodonov A V, Nardo R L, Migliore R, Messina A and Dodonov V V 2011 J. Phys. B: At. Mol. Opt. Phys. 44 225502
[9] Dodonov A V and Dodonov V V 2012 Phys. Rev. A 85 055805
[10] Dodonov A V and Dodonov V V 2012 Phys. Rev. A 86 063804
[11] Dodonov A V and Dodonov V V 2012 Phys. Rev. A 86 013801
[12] Fedotov A M, Narozhny N B and Lozovik Y E 2000 Phys. Lett. A 274 213
[13] Kawakubo T and Yamamoto K 2011 Phys. Rev. A 83 013819
[14] Kim W, Brownell J H and Onofrio R 2006 Phys. Rev. Lett. 96 200402
[15] Ruby R, Bradley P, Larson J D III and Oshmyansky Y 1999 Electron. Lett. 35 794
[16] Cleland A N and Geller M R 2004 Phys. Rev. Lett. 93 070501
[17] Kitagawa M and Ueda M 1993 Phys. Rev. A 47 5138
[18] Wineland D J et al 1992 Phys. Rev. A 46 R6797
[19] Wineland D J et al 1994 Phys. Rev. A 50 67
[20] Toth G, Knapp C, Guehne O and Briegel H J 2009 Phys. Rev. A 79 042334
[21] Ma J, Wang X, Sun C P and Nori F 2011 Phys. Reports 509 89
[22] Palma G M and Knight P L 1989 Phys. Rev. A 39 1962
[23] Agarwal G S and Purk R R 1994 Phys. Rev. A 50 67
[24] Banerjee A 1996 Phys. Rev. A 54 5327
[25] Vernac L, Pinard M and Giacobino E 2001 Eur. Phys. J. D 17 125
[26] Polzik E S 2008 Nature 453 45
[27] Cronin A D, Schmiedmayer J and Pritchard D E 2009 Rev. Modern Phys. 81 1051
[28] Sorensen A, Duan L M, Cirac J I and Zoller P 2001 Nature 409 63
[29] Bigelow N 2001 Nature 409 27
[30] Guehne O and Toth G 2009 Phys. Rep. 474 1
[31] Turchette Q A et al 1998 Phys. Rev. Lett. 81 3631
[32] Meyer V et al 2001 Phys. Rev. Lett. 86 5870
[33] Leibfried D et al 2004 Science 304 1476
[34] Wang X and Sanders B C 2003 Phys. Rev. A 68 012101
[35] Korbicz J K, Cirac J I and Lewenstein M 2005 Phys. Rev. Lett. 95 120502
[36] Korbicz J K et al 2006 Phys. Rev. A 74 052319
[37] Yi S and Hu H 2006 Phys. Rev. A 73 023602
[38] Arcchi F T, Courtenes E and Gilmore R 1972 Phys. Rev. A 6 2211
[39] Tavis M and Cummings F W 1968 Phys. Rev. 170 379
[40] Ulan-Orgikh D and Kitagawa M 2001 Phys. Rev. A 64 052106
[41] Banerjee A 2000 arXiv:quant-ph/0110032
[42] Zhou L, Song H S and Li C 2004 J. Opt. B: Quantum Semiclass Opt. 6 425
[43] Messick A, Wahidinn M R B, Pah C H and Ficek Z 2004 J. Opt. B: Quantum Semiclass Opt. 6 289
[44] Sorensen A, Duan L M, Cirac J and Zoller P 2001 Nature 409 63
[45] Saito H and Ueda M 1999 Phys. Rev. A 59 3959
[46] Naylor W, Matsuki S, Nishimura T and Kido Y 2009 Phys. Rev. A 80 043835
[47] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)