SINGULAR SPECTRAL SHIFT IS ADDITIVE

N. A. AZAMOV

Abstract. In this note it is proved that the singular part of the spectral shift function is additive. That is, if $H_0, H_1$ and $H_2$ are self-adjoint (not necessarily bounded) operators with trace-class differences, then

$$
\xi^{(s)}_{H_2, H_0} = \xi^{(s)}_{H_2, H_1} + \xi^{(s)}_{H_1, H_0}.
$$

Here, for any $\varphi \in C_c(\mathbb{R})$

$$
\xi^{(s)}_{H_1, H_0}(\varphi) := \int_0^1 \text{Tr}(V \varphi(H_r^{(s)})) \, dr,
$$

where $V = H_1 - H_0$, $H_r = H_0 + rV$ and $H_r^{(s)}$ is the singular part of $H_r$.

1. Introduction

Let $H_0$ be a self-adjoint operator and $V$ be a trace class self-adjoint operator. The Lifshits-Krein spectral shift function ([L], [Kr], see also [Y, Chapter 8] and [S]) is the unique $L_1$-function $\xi_{H_0 + V, H_0}$ such that for any $\varphi \in C_c^{\infty}$ the equality

$$
\text{Tr}(\varphi(H_0 + V) - \varphi(H_0)) = \int \xi_{H_0 + V, H_0}(\lambda) \varphi'(\lambda) \, d\lambda.
$$

holds. Krein also showed in [Kr] that for any self-adjoint operators $H_0, H_1$ and $H_2$ with trace-class differences the equality

$$
\xi_{H_2, H_0} = \xi_{H_2, H_1} + \xi_{H_1, H_0}
$$

holds.

In [BS], Birman and Solomyak proved the following spectral averaging formula for the spectral shift function:

$$
\xi_{H_0 + V, H_0}(\varphi) := \int_0^1 \text{Tr}(V \varphi(H_r)) \, dr, \quad \varphi \in C_c(\mathbb{R})
$$

(note that if $\varphi$ is a function then $\xi$ in $\xi(\varphi)$ denotes a measure, and if $\lambda$ is a number then $\xi$ in $\xi(\lambda)$ denotes a function — density of the absolutely continuous measure $\xi$).

In [Az3] (see also [Az2, Az4]) I introduced the so-called absolutely continuous and singular spectral shift functions $\xi^{(a)}$ and $\xi^{(s)}$ by formulas

$$
\xi^{(a)}_{H_0 + V, H_0}(\varphi) := \int_0^1 \text{Tr}(V \varphi(H_r^{(a)})) \, dr, \quad \varphi \in C_c(\mathbb{R})
$$

2000 Mathematics Subject Classification. Primary 47A55;
and 
\[ \xi_{H_0+V,H_0}(\varphi) := \int_0^1 \text{Tr}(V \varphi(H_r^{(a)})) \, dr, \quad \varphi \in C_c(\mathbb{R}), \]
where \( H_r = H_0 + rV, \) \( H_r^{(a)} \) is the absolutely continuous part of \( H_r \) and \( H_r^{(s)} \) is the singular part of \( H_r \).

The distributions \( \xi^{(s)} \) and \( \xi^{(a)} \) are absolutely continuous finite measures \([Az3]\).

In \([Az3]\) it is proved that for all operators \( V_1 \) from a linear manifold \( \mathcal{A}_0 \subset \mathcal{L}_1 \), which is dense in \( \mathcal{L}_1 \), the equality
\[ \xi^{(a)}_{H_0+V,H_0}(\varphi) = \xi^{(a)}_{H_0+V,H_0}(\varphi) + \xi^{(a)}_{H_0+V_1,H_0}(\varphi). \]
holds for all \( \varphi \in C_c^\infty \). This equality implies similar equality for \( \xi^{(s)} \).

In this note I give a proof of the equality \([1]\) for all trace-class self-adjoint operators \( V \) and \( V_1 \). This implies that for any self-adjoint operator \( H_0 \) and any trace-class self-adjoint operators \( V_1 \) and \( V_2 \) the equality
\[ \xi^{(s)}_{H_0+V_2,H_0}(\varphi) = \xi^{(s)}_{H_0+V_2,H_0}(\varphi) + \xi^{(s)}_{H_0+V_1,H_0}(\varphi) \]
holds.

The additivity property \((2)\) of the singular spectral shift function \( \xi^{(a)} \) combined with the fact that the density \( \xi^{(s)}(\lambda) \) of the measure \( \xi^{(s)} \) is a.e. integer-valued \([Az3]\), suggests that the singular spectral shift function should be interpreted as generalization of spectral flow of eigenvalues (see e.g. \([APS, Ge, Phi, Phi, CP, CP, ACDS, ACS, Az4]\)) to the case of spectral flow inside the essential spectrum.

2. Results

**Theorem 2.1.** Let \( H_0 \) be a self-adjoint operator on \( \mathcal{H} \), let \( V \) be a trace-class self-adjoint operator on \( \mathcal{H} \). If \( V_1, V_2, \ldots \) is a sequence of self-adjoint trace-class operators converging to \( V \) in the trace-class norm, then for any \( \varphi \in C_c \) the equality
\[ \lim_{n \to \infty} \xi^{(a)}_{H_0+V_n,H_0}(\varphi) = \xi^{(a)}_{H_0+V,H_0}(\varphi). \]
holds. Shortly, the absolutely continuous part of the spectral shift function \( \xi^{(a)}_{H_0+V,H_0} \) is weakly-continuous with respect to \( V \in \mathcal{L}_1(\mathcal{H}) \).

**Proof.** We have to prove that for any \( \varphi \in C_c(\mathbb{R}) \) the difference
\[ \int_0^1 \left( \text{Tr} \left( V \varphi(H_0 + rV)^{(a)} \right) - \text{Tr} \left( V_n \varphi(H_0 + rV_n)^{(a)} \right) \right) \, dr \]
goes to 0 as \( n \to \infty \). Since the integrand as a function of \( r \) is bounded by \( 2 \| V \|_1 \| \varphi \|_\infty \) for all large enough \( n \), it follows from the Lebesgue dominated convergence theorem that it is enough to prove that for any fixed \( r \in [0,1] \)
\[ \lim_{n \to \infty} \text{Tr} \left( V_n \varphi(H_0 + rV_n)^{(a)} \right) = \text{Tr} \left( V \varphi(H_0 + rV)^{(a)} \right). \]
Further, since
\[
\text{Tr} \left( V \phi(H_0 + rV)^{(a)} \right) - \text{Tr} \left( V_n \phi(H_0 + rV_n)^{(a)} \right) = \text{Tr} \left( (V - V_n) \phi(H_0 + rV_n)^{(a)} \right) + \text{Tr} \left( V \left( \phi(H_0 + rV)^{(a)} - \phi(H_0 + rV_n)^{(a)} \right) \right)
\]
and since
\[
\left| \text{Tr} \left( (V - V_n) \phi(H_0 + rV_n)^{(a)} \right) \right| \leq \|V - V_n\|_1 \cdot \|\phi\|_\infty \to 0 \text{ as } n \to \infty,
\]
it is enough to prove that
\[
\lim_{n \to \infty} \text{Tr} \left( V \left( \phi(H_0 + rV)^{(a)} - \phi(H_0 + rV_n)^{(a)} \right) \right) = 0.
\]

It follows from [Y, Lemma 6.1.3], that for this it is enough to show that
\[
\text{s- lim}_{n \to \infty} \phi(H_0 + rV_n)^{(a)} = \phi(H_0 + rV)^{(a)},
\]
where the limit is taken in the strong operator topology. We can assume that \( r = 1 \). Let \( H = H_0 + V \), \( H_n = H_0 + V_n \). For self-adjoint operators \( H_0 \) and \( H_1 \), let \( W_\pm(H_1, H_0) \) be wave operators of the pair \( H_0 \) and \( H_1 \) (if they exist) and let \( P^{(a)}(H_0) \) be the orthogonal projection onto the absolutely continuous part of \( H_0 \). Since
\[
W_+(H_n, H) \phi(H^{(a)}) W_+^*(H_n, H) = \phi(H^{(a)}),
\]
it follows that
\[
\phi(H^{(a)}) - \phi(H^{(a)}_n) = \phi(H^{(a)}) - W_+(H_n, H) \phi(H^{(a)}_n) W_+^*(H_n, H) = \left( \phi(H^{(a)}) - W_+(H_n, H) \phi(H^{(a)}) \right) + \left( W_+(H_n, H) \phi(H^{(a)}_n) - W_+(H_n, H) \phi(H^{(a)}) W_+^*(H_n, H) \right) = \left( P^{(a)}(H) - W_+(H_n, H) \right) \phi(H^{(a)}) + W_+(H_n, H) \phi(H^{(a)}) \left( P^{(a)}(H) - W_+^*(H_n, H) \right).
\]

[Y, Theorem 6.3.6] implies that
\[
\text{s- lim}_{n \to \infty} W_+(H_n, H) = P^{(a)}(H)
\]
and
\[
\text{s- lim}_{n \to \infty} W_+^*(H_n, H) = P^{(a)}(H).
\]

It follows from this and (6) that (5) holds.

The proof is complete. \( \square \)

**Theorem 2.2.** The absolutely continuous part of the spectral shift function is additive. That is, if \( H_0 \) is a self-adjoint operator on \( \mathcal{H} \), and if \( V_1, V_2 \) are trace-class self-adjoint operators on \( \mathcal{H} \), then for any \( \phi \in C_c(\mathbb{R}) \) the equality
\[
\xi_{H_0 + V_2, H_0}^{(a)}(\phi) = \xi_{H_0 + V_2, H_0 + V_1}^{(a)}(\phi) + \xi_{H_0 + V_1, H_0}^{(a)}(\phi)
\]
holds.
Proof. Let $H_0$ be a self-adjoint operator on $\mathcal{H}$, and let $V$ and $V_1$ be two trace-class self-adjoint operators on $\mathcal{H}$. We need to show that for any $\varphi \in C^\infty_c$

$$\xi^{(a)}_{H_0+V,H_0}(\varphi) = \xi^{(a)}_{H_0+V,H_0,V_1}(\varphi) + \xi^{(a)}_{H_0+V_1,H_0}(\varphi)$$

By [Az3, Lemma 5.2], for a given trace-class operator $V$ one can choose a frame operator $F$ (see [Az3] for the definition of the frame operator) such that $V \in A(F) \subset L^1(\mathcal{H})$, where $A(F)$ is a dense linear subset of $L^1(\mathcal{H})$ (see [Az3, §5] for the definition of the class $A(F)$).

By [Az3, Theorem 9.12], there exists a dense linear subset $A_0$ (which depends on $H_0$) of $A(F)$, such that for any $\tilde{V} \in A_0$ and any function $\varphi \in C_c(\mathbb{R})$ the equality

$$\xi^{(a)}_{H_0+\tilde{V},H_0}(\varphi) = \xi^{(a)}_{H_0+V,H_0,V_1}(\varphi) + \xi^{(a)}_{H_0+V_1,H_0}(\varphi)$$

holds. By Theorem 2.1

$$\lim_{n \to \infty} \xi^{(a)}_{H_0+V_n,H_0}(\varphi) = \xi^{(a)}_{H_0+V_1,H_0}(\varphi).$$

It directly follows from the definition of $\xi^{(a)}$ that

$$\xi^{(a)}_{H_1,H_0} = - \xi^{(a)}_{H_0,H_1}$$

for any two self-adjoint operators $H_0, H_1$ with trace-class difference. It follows from (11) and Theorem 2.1 that

$$\lim_{n \to \infty} \xi^{(a)}_{H_0+V,H_0+V_n}(\varphi) = \xi^{(a)}_{H_0+V,H_0+V_1}(\varphi).$$

Combining this equality with (9) and (10) completes the proof. 

Corollary 2.3. The singular part of the spectral shift function is additive. That is, if $H_0$ is a self-adjoint operator on $\mathcal{H}$, and if $V_1, V_2$ are trace-class self-adjoint operators on $\mathcal{H}$, then for any $\varphi \in C_c(\mathbb{R})$ the equality

$$\xi^{(s)}_{H_0+V_2,H_0}(\varphi) = \xi^{(s)}_{H_0+V_2,H_0+V_1}(\varphi) + \xi^{(s)}_{H_0+V_1,H_0}(\varphi)$$

holds.

Proof. This follows from Theorem 2.2 and additivity of the Lifshits-Krein spectral shift function. 

SINGULAR SPECTRAL SHIFT IS ADDITIVE

References

[APS] M. Atiyah, V. Patodi, I. M. Singer, *Spectral Asymmetry and Riemannian Geometry. III*, Math. Proc. Camb. Phil. Soc. **79** (1976), 71–99.

[Az] N. A. Azamov, *Infinitesimal spectral flow and scattering matrix*, preprint, arXiv:0705.3282v4.

[Az2] N. A. Azamov, *Pushnitski’s μ-invariant and Schrödinger operators with embedded eigenvalues*, preprint, arXiv:0711.1190v1.

[Az3] N. A. Azamov, *Absolutely continuous and singular spectral shift functions*, preprint, arXiv:submit/0092981.

[Az4] N. A. Azamov, *Spectral shift function in von Neumann algebras*, VDM Verlag, 2010.

[ACDS] N. A. Azamov, A. L. Carey, P. G. Dodds, F. A. Sukochev, *Operator integrals, spectral shift and spectral flow*, Canad. J. Math. **61** (2009), 241–263.

[ACS] N. A. Azamov, A. L. Carey, F. A. Sukochev, *The spectral shift function and spectral flow*, Comm. Math. Phys. **276** (2007), 51–91.

[BS] M. Sh. Birman, M. Z. Solomyak, *Remarks on the spectral shift function*, J. Soviet math. **3** (1975), 408–419.

[CP] A. L. Carey, J. Phillips, *Unbounded Fredholm modules and spectral flow*, Canad. J. Math. **50** (1998), 673–718.

[CP2] A. L. Carey, J. Phillips, *Spectral flow in Fredholm modules, eta invariants and the JLO cocycle*, K-Theory **31** (2004), 135–194.

[Ge] E. Getzler, *The odd Chern character in cyclic homology and spectral flow*, Topology **32** (1993), 489–507.

[Kr] M. G. Krein, *On the trace formula in perturbation theory*, Mat. Sb., **33** 75 (1953), 597–626.

[L] I. M. Lifshits, *On a problem in perturbation theory*, Uspekhi Mat. Nauk **7** (1952), 171–180 (Russian).

[Ph] J. Phillips, *Self-adjoint Fredholm operators and spectral flow*, Canad. Math. Bull. **39** (1996), 460–467.

[Ph2] J. Phillips, *Spectral flow in type I and type II factors — a new approach*, Fields Inst. Comm. **17** (1997), 137–153.

[S] B. Simon, *Trace ideals and their applications: Second Edition*, Providence, AMS, 2005, Mathematical Surveys and Monographs, **120**.

[Y] D. R. Yafaev, *Mathematical scattering theory: general theory*, Providence, R.I., AMS, 1992.

School of Computer Science, Engineering and Mathematics, Flinders University, Bedford Park, 5042, SA Australia.

E-mail address: azam0001@csem.flinders.edu.au