An elementary proof of a power series identity for the weighted sum of all finite abelian $p$-groups

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Abstract

Using combinatorial techniques, we prove that the weighted sum of the inverse number of automorphisms of all finite abelian $p$-groups $\sum_{G} |G|^{-u} |\text{Aut}(G)|^{-1}$ is equal to $\prod_{j=u+1}^{\infty} \left( 1 - \frac{1}{p^j} \right)^{-1}$, where $u$ is a non-negative integer. This result was originally obtained by H. Cohen and H. W. Lenstra, Jr. In this paper we give a new elementary proof of their result.

1. Introduction

Let $G_p$ denote the set of all finite abelian $p$-groups, for a prime number $p$. We will give an elementary combinatorial proof of the following theorem:

Theorem. For a prime $p$ and a non-negative integer $u$, the following holds

$$\sum_{G \in G_p} \frac{1}{|G|^u \cdot |\text{Aut}(G)|} = \prod_{j=u+1}^{\infty} \left( 1 - \frac{1}{p^j} \right)^{-1}.$$ 

The above result was obtained by Cohen and Lenstra in their famous paper [1]. Their approach is more complicated but has the advantage that it generalizes naturally to finite modules over the rings of integers of number fields. In the special case of $u = 0$, a nice combinatorial proof was given by Hall, which can be found in [2]. Our proof is a generalization of the proof of Hall.
2. Proof of the Theorem

For \( m \geq 0 \), let \( a_m \) be the number of partitions of \( m \) with each part at least \( u + 1 \) and for \( i, j \geq 0 \), let \( b_{i,j} \) be the number of partitions of \( i \) with greatest part exactly equal to \( j \).

**Lemma 1.** For all \( m \geq 0 \), the following holds

\[
a_m = \sum_{i+uj=m} b_{i,j}.
\]

**Proof.** We will give a bijection argument. Note that, the number of partitions of \( i \) with greatest part \( j \), where \( i+uj = m \), is equal to the number of partitions of \( i + uj = m \) with greatest part \( j \) occurring at least \( u + 1 \) times. Hence \( \sum_{i+uj=m} b_{i,j} \) is equal to the number of partitions of \( m \) with greatest part occurring at least \( u + 1 \) times. Now, a partition of \( m \) has greatest part occurring at least \( u + 1 \) times if and only if it’s conjugate partition has each part at least \( u + 1 \). This gives a bijection between the partitions of \( m \) with greatest part occurring at least \( u + 1 \) times and the partitions of \( m \) with each part at least \( u + 1 \). Therefore, \( \sum_{i+uj=m} b_{i,j} = a_m \). \( \square \)

**Lemma 2.** For each \( n \geq 0 \), let us define

\[
f_n(q) := \sum_{N=0}^{\infty} b_{N,n} q^N,
\]

which is a formal power series in \( q \). Then,

\[
\prod_{j=u+1}^{\infty} (1 - q^j)^{-1} = \sum_{n=0}^{\infty} f_n(q)q^{nu}.
\]

**Proof.** Note that,

\[
\prod_{j=u+1}^{\infty} (1 - q^j)^{-1} = \sum_{m=0}^{\infty} a_m q^m
\]

since, for each \( m \), the coefficient of \( q^m \) on LHS is equal to the number of partitions of \( m \) with each part at least \( u + 1 \). Then,
\[
\sum_{n=0}^{\infty} f_n(q)^{nu} = \sum_{n=0}^{\infty} \left( \sum_{N=0}^{\infty} b_{N,n}q^N \right) q^{nu} \\
= \sum_{n=0}^{\infty} \sum_{N=0}^{\infty} b_{N,n}q^{N+nu} \\
= \sum_{m=0}^{\infty} \left( \sum_{i+uj=m} b_{i,j} \right) q^m \\
= \sum_{m=0}^{\infty} a_m q^m \\
= \prod_{j=u+1}^{\infty} (1 - q^j)^{-1}.
\]

We also need the following lemma, which computes the cardinality of \(\text{Aut}(G)\), for a finite abelian \(p\)-group \(G\).

**Lemma 3.** Fix a prime \(p\). Suppose \(G\) is a finite abelian \(p\)-group and

\[
G = \prod_{i=1}^{k} (\mathbb{Z}/p^{e_i}\mathbb{Z})^{r_i}
\]

for some \(k \geq 0\), \(e_1 > e_2 > \cdots > e_k > 0\) and \(r_i \geq 0\). Then

\[
|\text{Aut}(G)| = \left( \prod_{i=1}^{k} \left( \prod_{s=1}^{r_i} (1 - p^{-s}) \right) \right) \left( \prod_{1 \leq i, j \leq k} p^{\min(e_i, e_j)r_ir_j} \right).
\]

**Proof.** See [3], Theorem 1.2.10.

Now let us return to the proof of the theorem. We follow a similar argument given in [2] or in the proof of Theorem 2.1.2 of [3]. First, note that, there is an associated partition corresponding to every finite abelian \(p\)-group and corresponding to every partition there is an associated finite abelian \(p\)-group; this comes from writing finite abelian \(p\)-groups uniquely as a product.
of cyclic groups. For example, if we write a finite abelian $p$-group $G$ as,

$$G = \prod_{i=1}^{k} \mathbb{Z}/p^{e_i}\mathbb{Z}$$

where $e_1 \geq e_2 \geq \cdots \geq e_k > 0$, then the associated partition $\lambda$ is given by $\lambda = (e_1, e_2, \ldots, e_k)$. And, corresponding to every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ the associated $p$-group $G_\lambda$ is given by $G_\lambda = \prod_{i=1}^{k} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$. Note that, if $|\lambda|$ denotes the size of the partition $\lambda$, then the order of the $p$-group $G_\lambda$ is given by $|G_\lambda| = p^{k|\lambda|}$.

Let $\lambda := (\lambda_1, \ldots, \lambda_t)$ be a partition of size $n$ and suppose $\lambda' := (\lambda'_1, \ldots, \lambda'_m)$ is it’s conjugate partition. Then, note that, in $G_\lambda$ (as a product of cyclic groups), the factor $\mathbb{Z}/p^i\mathbb{Z}$ occurs exactly $\lambda'_i - \lambda'_{i+1}$ times (where $\lambda'_{m+1} := 0$). Then using Lemma 3 we can write

$$|\text{Aut}(G_\lambda)| = \left( \prod_{i=1}^{m} \left( \prod_{s=1}^{\lambda'_i-\lambda'_{i+1}} (1 - p^{-s}) \right) \right) \left( \prod_{1 \leq i, j \leq m} p^{\min(i,j)(\lambda'_i-\lambda'_{i+1})(\lambda'_j-\lambda'_{j+1})} \right)$$

$$= \left( \prod_{i=1}^{m} \left( \prod_{s=1}^{\lambda'_i-\lambda'_{i+1}} (1 - p^{-s}) \right) \right) p^{\sum_{1 \leq i, j \leq m} \min(i,j)(\lambda'_i-\lambda'_{i+1})(\lambda'_j-\lambda'_{j+1})}$$

$$= \left( \prod_{i=1}^{m} \left( \prod_{s=1}^{\lambda'_i-\lambda'_{i+1}} (1 - p^{-s}) \right) \right) p^{\sum_{i=1}^{m} (\lambda'_i)^2}.$$

Then, setting $q = p^{-1}$, we have

$$\sum_{G \in \mathcal{G}_p} \frac{1}{|G|^n \cdot |\text{Aut}(G)|} = \sum_{n=0}^{\infty} \sum_{G_\lambda \in \mathcal{G}_p \atop |\lambda| = n} \frac{q^{nu}}{|\text{Aut}(G_\lambda)|}$$

$$= \sum_{n=0}^{\infty} q^{nu} \sum_{G_\lambda \in \mathcal{G}_p \atop |\lambda| = n} \left( \prod_{i=1}^{m} \left( \prod_{s=1}^{\lambda'_i-\lambda'_{i+1}} (1 - p^{-s})^{-1} \right) \right) \left( \prod_{i=1}^{m} p^{-\lambda'_i} \right)$$

$$= \sum_{n=0}^{\infty} q^{nu} \sum_{G_\lambda \in \mathcal{G}_p \atop |\lambda| = n} \left( \prod_{i=1}^{m} \left( \prod_{s=1}^{\lambda'_i-\lambda'_{i+1}} (1 - q^{-s})^{-1} \right) \right) \left( \prod_{i=1}^{m} q^{\lambda'_i} \right).$$

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Note that, $\lambda'$ varies over all partitions as $\lambda$ varies over all partitions. Therefore putting $\mu = \lambda'$, we get

$$\sum_{G \in \mathcal{G}_p} \frac{1}{|G| \cdot |\text{Aut}(G)|} = \sum_{n=0}^{\infty} q^n \sum_{G_{\mu} \in \mathcal{G}_p} \left( \prod_{i=1}^{m} \left( \prod_{s=1}^{\mu_i-\mu_{i-1}} (1 - q^s)^{-1} \right) \right) \left( \prod_{i=1}^{m} q^{\mu_i^2} \right)$$

$$= \sum_{n=0}^{\infty} q^n \sum_{G_{\mu} \in \mathcal{G}_p} \left( \prod_{i=1}^{m} \psi_{\mu_i, \mu_{i-1}-\mu_i}(q) \right) \left( \prod_{i=1}^{m} q^{\mu_i^2} \right),$$

where

$$\psi_{a,b}(q) := \frac{\prod_{i=1}^{a+b} (1 - q^i)}{\prod_{i=1}^{a} (1 - q^i) \prod_{i=1}^{b} (1 - q^i)}, \quad \psi_{a,\infty}(q) := \frac{1}{\prod_{i=1}^{a} (1 - q^i)}$$

and $\mu_0 := \infty$; note that, the coefficient of $q^n$ in $\psi_{a,b}(q)$ is the number of partitions of $n$ with height at most $a$ and width at most $b$. Therefore, by Lemma 2, it is enough to show that, for each $n \geq 0$,

$$f_n(q) = \sum_{G_{\mu} \in \mathcal{G}_p} \left( \prod_{i=1}^{m} \psi_{\mu_i, \mu_{i-1}-\mu_i}(q) \right) \left( \prod_{i=1}^{m} q^{\mu_i^2} \right).$$

That is, we need to equate coefficients of $q^N$ on both sides, for each $N \geq 0$. The argument is same as given in [2] or [3].

Note that, coefficient of $q^N$ on LHS is equal to $b_{N,n}$ which is the number of partitions of $N$ with greatest part $n$. Let $\nu$ be a partition of $N$ with greatest part equal to $n$; then, to each such $\nu$ we will associate a partition $\mu$ of size $n$ on the RHS. Consider the conjugate $\nu'$ of $\nu$ and let $D$ be the standard Young diagram of $\nu'$. Note that $\nu'$ has height equal to $n$. Now, define $\mu := (\mu_1, \ldots, \mu_m)$ as follows:

- Define $\mu_1$ to be the largest integer such that $(\mu_1, \mu_1) \in D$.
- For $i \geq 2$, define $\mu_i$ to be the largest integer such that $(\mu_1 + \cdots + \mu_i, \mu_i) \in D$.

(Where $(i, j) \in D$ is defined as the block of $D$ situated at the $i$th row from
top and \( j \)th column from left). Then \(|\mu| = n\). If \( M \) is the number of blocks outside the squares of size \( \mu_i \) then \( M = N - \mu_1^2 - \mu_2^2 - \cdots - \mu_m^2 \). Let \( M_i \) be the number of blocks at the right of the block of size \( \mu_i \), i.e.

\[
M_i := |\{(x, y) \in D : \mu_1 + \cdots + \mu_{i-1} < x < \mu_1 + \cdots + \mu_i, \mu_i < y\}|.
\]

Then the blocks corresponding to \( M_i \) gives a partition of \( M_i \) of height at most \( \mu_i \) and width at most \( \mu_{i-1} - \mu_i \) and hence this contributes to the coefficient of \( q^{M_i} \) in \( \psi_{\mu_i, \mu_{i-1} - \mu_i}(q) \) on RHS. Note that \( M = M_1 + \cdots + M_m \) which implies \( M_1 + \cdots + M_m + \mu_1^2 + \cdots + \mu_m^2 = N \) and hence \( \mu \) contributes to the coefficient of \( q^N \) on RHS.

Note that, the above construction can be reversed. Suppose \( \mu \) is a partition which corresponds to the coefficient of \( q^N \) on RHS such that \( \mu \) is specified by the numbers \( M_i \), where \( M_1 + \cdots + M_m + \mu_1^2 + \cdots + \mu_m^2 = N \), and partitions of \( M_i \) of height at most \( \mu_i \) and width at most \( \mu_{i-1} - \mu_i \). Then we can construct the Young diagram \( D \) and construct the corresponding partition \( \nu \) on LHS. Hence, we conclude that the coefficients of \( q^N \) on both sides are equal and this proves the theorem.

**References**

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