The understanding of developed turbulence, a long standing challenge for mathematical physics, has entered into the third millennium as an unsolved problem. It poses basic questions concerning both the behavior of solutions of hydrodynamical equations and the basic principles of statistical mechanics of systems out of equilibrium. Although we seem far away from the definite answers to these million (or more) dollar questions (Fefferman, 2000), some insight may be gained by studying simpler models showing similar behaviors, often themselves not devoided of direct physical interest. One of such problems concerns the passive transport of scalar quantities such as temperature or tracer $\theta(t,x)$ and pollutant or dye density $\rho(t,x)$ by random flows. The flows are described by a simple random ensemble of velocities $v(t,x)$ that is considered given. Such approach ignores the back-reaction of the scalar on the velocity dynamics as well as many details of that dynamics. It incorporates, however, the basic phenomenological property of velocities exhibiting developed turbulence: their (approximate) statistical scaling

$$v(t,x) - v(t,y) \sim |x-y|^{\alpha}$$

signaled by the scaling behavior of the expectations of the powers of the velocity differences, the so called velocity structure functions,

$$\mathbb{E} |v(t,x) - v(t,y)|^N \approx |x-y|^{\zeta_N}$$

with $\zeta_N$ (for low $N$) not far from the mean field theory (Kolmogorov, 1941) value $\zeta_{Kol} = N/3$ that would correspond to $\alpha = 1/3$. The time-evolution of the scalar is described by the advection-diffusion equation

$$\partial_t \theta + (v \cdot \nabla) \theta - \kappa \nabla^2 \theta = f,$$

where $\kappa$ is the (molecular) diffusivity constant and $f(t,x)$ denotes a scalar source that one may also take random. Given the distributions of the velocities $v$ and of the sources $f$, one inquires about the statistics of solutions $\theta$ of the above equation. As we shall see, very simple distributions of $v$ and $f$ lead to stationary statistical states of scalar exhibiting many features observed not only in realistic turbulent transport but also in statistics of turbulent velocities, in the atmosphere, in aerodynamical tunnels or in sea channels. Those features include persistent dissipation of energy, energy cascades from large scales to small ones or vice versa and intermittency. Simple models allow a better understanding of the origin of such phenomena.

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behaviors and permit to draw some general conclusions, see (Falkovich, Gawedzki & Vergassola, 2001) for an extensive review and bibliography.

LECTURE 1

(transport of scalars by hydrodynamical flow; the role of fluid particle dynamics, single particle diffusion versus two-particle dispersion)

1.1. Solution of the advection-diffusion equation, fluid particles

It is easy to solve the linear equation (1.3) describing the scalar evolution in $d$ space dimensions when the velocity field is sufficiently smooth.

(i). For $\kappa = 0$ and $f = 0$, the scalar $\theta(t, \mathbf{x})$ is simply constant along the characteristics

$$\frac{d\mathbf{x}}{ds} = \mathbf{v}(s, \mathbf{x})$$

which describe the (Lagrangian) trajectories of fluid particles. In other words, the scalar is carried by the flow:

$$\theta(t, \mathbf{x}) = \theta(s, \mathbf{x}_{t,\mathbf{x}}(s)),$$

where $\mathbf{x}_{t,\mathbf{x}}(s)$ is the Lagrangian trajectory that passes at time $t$ through point $\mathbf{x}$. Note that the forward evolution of $\theta$ corresponds to the backward Lagrangian flow.

(ii). In the presence of the source $f$, the scalar is also created or depleted along the trajectory:

$$\theta(t, \mathbf{x}) = \theta(s, \mathbf{x}_{t,\mathbf{x}}(s)) + \int_s^t f(\sigma, \mathbf{x}_{t,\mathbf{x}}(\sigma)) d\sigma.$$

(iii). Finally, when $\kappa \neq 0$, $\mathbf{x}_{t,\mathbf{x}}(s)$ should be taken as the solution of the stochastic ODE

$$d\mathbf{x} = \mathbf{v}(s, \mathbf{x}) ds + \sqrt{2\kappa} d\mathbf{w}$$

for the Lagrangian trajectories perturbed by the $d$-dimensional Brownian motion $\mathbf{w}(s)$ and the right hand side of eq. (1.3) should be averaged over $\mathbf{w}$:

$$\theta(t, \mathbf{x}) = E_{\mathbf{w}} \left( \theta(s, \mathbf{x}_{t,\mathbf{x}}(s)) + \int_s^t f(\sigma, \mathbf{x}_{t,\mathbf{x}}(\sigma)) d\sigma \right).$$

It is then clear that the statistics of $\theta(t, \mathbf{x})$ is determined by the statistics of the (noisy) Lagrangian trajectories. The latter may be captured in two steps. First, we may consider the transition probabilities for the Markov process given by the solution of the stochastic equation (1.4) in a fixed velocity field:

$$P(\mathbf{v}|t, \mathbf{x}; s, dy) = E_{\mathbf{w}} \delta(y - \mathbf{x}_{t,\mathbf{x}}(s)) dy.$$
Note that the solution (1.3) may be rewritten as
\[
\theta(t, x) = \int P(v|t, x; s, dy) \theta(s, y) + \int_s^t d\sigma \int P(v|t, \sigma; \sigma, dy) f(\sigma, y)
\] (1.7)
or, in the more handy operator notation, as
\[
\theta(t) = P(v|t, s) \theta(s) + \int_s^t P(v|t, \sigma) f(\sigma) d\sigma.
\] (1.8)
The latter formul\'s make sense for \( \kappa > 0 \) also in rough (non-Lipschitz) velocities when eq. (1.1) does not have unique solutions. Second, in order to take account of the velocity fluctuations, we may consider the joint probability distributions of \( N \) Lagrangian particles
\[
\mathcal{P}_N(t, x; s, dy) = E \prod_{n=1}^N P(v|t_n, x_n; s_n, dy_n),
\] (1.9)
where \( t = (t_1, \ldots, t_N), \ x = (x_1, \ldots, x_N), \) etc. and \( E \) stands for the average over the velocity ensemble. We shall be interested in the random ensembles of velocities that are stationary, homogeneous and isotropic, i.e. such that the time and space translations and rotations
\[
v(t, x) \mapsto v(t + t_0, R_0 x + x_0)
\]
are implemented by the measure preserving action of the corresponding groups on the probability space of velocities.

1.2. Single-particle diffusion and Richardson dispersion of two particles

What is the statistical behavior of fluid particles in stationary, homogeneous and isotropic turbulent velocities? The rough answers are as follows. For a single particle, one expects a diffusive behavior for sufficiently long times. Note that
\[
x_{0,x}(t) - x = \int_0^t v(\sigma, x_{0,x}(\sigma)) d\sigma \equiv \int_0^t v_L(\sigma, x) d\sigma
\] (1.10)
for solutions of eq. (1.1), where \( v_L(\sigma, x) \) denotes the the velocity along the Lagrangian trajectory passing at time zero through \( x \), the so called Lagrangian velocity. For fixed \( x \), \( v_L(\sigma, x) \) is a stationary process as long as velocities are incompressible. It has zero expectation if the similar property holds for the (Eulerian) velocity \( v(\sigma, x) \) with which it coincides at time zero. If \( v_L(s, x) \) has temporal correlations that decay fast enough, then the integral in (1.10) falls under the Central Limit Theorem behaving effectively as a sum of many independent equally distributed random variables. As a result, \( \mu^{-\frac{1}{2}}x_{0,x}(\mu t) \) tends when \( \mu \to \infty \) to a Brownian motion
\[
\mathcal{P}_1(0, x; \mu t, d(\mu^{\frac{1}{2}}y)) \xrightarrow{\mu \to \infty} e^{\frac{1}{2}t D_0 \nabla^2} (x, y) dy = \frac{1}{(4\pi D_0 t)^{d/2}} e^{-\frac{(x-y)^2}{4D_0 t}} dy,
\] (1.11)
i.e. it becomes the transition probability of diffusion. The diffusion constant is given by the formula (Taylor, 1921)
\[
D_0 = \int_0^\infty \mathbf{E} v_L(0, x) \cdot v_L(\sigma, x) d\sigma.
\] (1.12)
Still, the decay of temporal correlations of the Lagrangian velocity is a nontrivial fact, see (Fannjiang & Papanicolaou, 1996) or (Majda & Kramer, 1999). It does not automatically follow
from similar property of the (Eulerian) velocity $v(\sigma, x)$ with fixed $x$ and may fail altogether in specially prepared velocity ensembles.

For two particles, the important quantity to study is the 2-particle separation $\rho(t)$ defined as the difference $x_{0,x_1}(t) - x_{0,x_2}(t)$ of particle positions. It satisfies the equation

$$\frac{d\rho}{ds} = v(s, x_{0,x_1} + \rho(s)) - v(s, x_{0,x_2}(s))$$

with the initial condition $\rho(0) = x_2 - x_1$. To get a rough idea about the behavior of the particle separation, let us first consider small separations in smoothly varying velocities where the right hand side may be approximated by linear expression $A\rho(s)$. This leads to a solution

$$\rho(t) \simeq e^{At} \rho(0)$$

with the exponential growth if $A$ has eigenvalues with positive real part (positive Lyapunov exponents) that signal the sensitive dependence on initial conditions usually considered as a definition of chaos. Notice that although the nearby trajectories separate exponentially in this case, the very close trajectories take long time to separate and infinitesimally close ones never separate. As the result, the trajectories are still labeled in a continuous way by their initial positions. Such fluid particle trajectory behavior pertains to the so called Batchelor regime of turbulent flows corresponding to short scales dominated by viscous effects.

Suppose now that we solve the equation (1.13) in the regime where the velocity difference behaves like $\rho(s)^\alpha$ with $\alpha < 1$. Such scaling behaviors are observed in the inertial range of scales of turbulent flows where viscous and steering effects are negligible. Passing to the scalar version of eq. (1.13), we obtain

$$\frac{d\rho^2}{ds} = 2\frac{d\rho}{ds} \cdot \rho \propto \rho^{\alpha+1}$$

and, ignoring again the time and point dependence as well as the statistical fluctuations in the proportionality constants, we obtain

$$\rho(t)^{1-\alpha} \simeq \rho(0)^{1-\alpha} + \text{const. } t.$$}

This very rough estimate allows us to expect a power law growth of the 2-particle dispersion (the distance between two particles), wiping out the memory of the initial separation. In particular, this would mean that infinitesimally close trajectories would still separate in a finite time, unlike in smooth velocities, leading to a spontaneous randomness in the Lagrangian flow at $\kappa = 0$. Of course, in non-smooth velocities (e.g. in the Hölder continuous ones) one should not expect existence of the deterministic Lagrangian flow with trajectories labeled by initial conditions since the assumptions of the theorem about the uniqueness of solutions of the ODE (1.1) require Lipschitz continuity of velocities in space. That type of situation pertains to the behavior of turbulent flows at very high (ideally, infinite) Reynolds numbers when the scaling behavior (1.1) extends down to very small (infinitesimal) separations crossing over to the Lipschitz behavior with $\alpha = 1$ only at scales where the viscous effects become important (of order of fractions of millimeter in the turbulent atmosphere). The super-diffusive behavior $\rho^2(t) \propto t^3$ corresponding to (1.16) with the Kolmogorov value $\alpha = 1/3$ has, indeed, been observed phenomenologically for the mean squared 2-particle dispersion. It constitutes the content of the first quantitative law of developed turbulence formulated in (Richardson, 1926)
on the bases of experimental data about the separation of meteorological balloons and smoke particles.

The above arguments ignored the temporal dependence of velocity fields which plays an important role. Nevertheless, the basic conclusion about the possibility of finite time separation of arbitrary close Lagrangian trajectories in spatially rough velocity fields holds even if velocities are completely decorrelated (white) in time, as we shall see below.

The statistics of the separation of two fluid particles may be captured by the relative transition amplitudes

$$P^\text{rel}_2(t, \mathbf{\rho}_0; s, d\mathbf{\rho}) = \int P_2(t, t, \mathbf{x}_1, \mathbf{x}_1 + \mathbf{\rho}_0; s, s, \mathbf{y}_1, d(\mathbf{y}_1 + \mathbf{\rho}))$$

(1.17)

where the integral is over the final position \(\mathbf{y}_1\) of the first particle. A strong version of the Richardson-type super-diffusive behavior for large times may be formulated as the statement about the existence of the limit of the rescaled process \(\mu^{-\alpha/\nu} \mathbf{\rho}(\mu t)\) or of the limit

$$\lim_{\mu \to \infty} P^\text{rel}_2(0, \mathbf{\rho}_0; \mu t, d(\mu^{-\alpha/\nu} \mathbf{\rho}))$$

(1.18)

Note the difference with the expected diffusive scaling (1.11) for a single particle. Again, this type of scaling is not automatically guaranteed by the statistical scaling of the Eulerian velocity differences since there are many points where the naive mean-field type arguments may go wrong. The super-diffusive behavior of the 2-particle dispersion is, in general, even harder to establish than the diffusive behavior of a single particle.

LECTURE 2

(Kraichnan ensemble of velocities; Le Jan-Raimond construction of Lagrangian particle processes; multi-particle statistics)

It is important to have at the disposal a simple model where the ideas about the behavior of fluid particles discussed in the first lecture could be tested rigorously.

2.1. Kraichnan ensemble of velocities

Such a model has arisen from the work initiated in (Kraichnan, 1968). Kraichnan proposed to consider a Gaussian ensemble of velocities decorrelated in time but with the scaling properties in space built in. Gaussian ensembles are completely determined by the 1-point and 2-point functions. One assumes that the 1-point function of \(\mathbf{v}\) vanishes and that

$$\mathbf{E} \mathbf{v}^i(t, \mathbf{x}) \mathbf{v}^j(s, \mathbf{y}) = \delta(t - s) D^{ij}(\mathbf{x} - \mathbf{y}),$$

(2.1)

where

$$D^{ij}(\mathbf{x}) = \int_{|k| < \eta^{-1}} \left( \delta^{ij} - \frac{k^i k^j}{k^2} \right) \frac{e^{i k \cdot \mathbf{x}}}{(k^2 + L^{-2})(\xi + \delta)^2} \, dk.$$  

(2.2)

Decomposing

$$D^{ij} = D_0 \delta^{ij} - d^{ij}(\mathbf{x})$$

(2.3)
with the first term equal to $D^{ij}(0)$, it is not difficult to see that $d^{ij}(x) \xrightarrow{|x| \to \infty} D_0 \delta^{ij}$ with $D_0 = O(L^\xi)$. At short distances
\[ d^{ij}(x) = D_1 [(d + 1) \delta^{ij} |x|^2 - 2 x^i x^j] + O(|x|^4) \quad \text{for} \quad |x| \ll \eta, \tag{2.4} \]
whereas
\[ d^{ij}(x) \xrightarrow{\eta \to 0 \atop L \to \infty} D_2 [(d - 1 + \xi) \delta^{ij} |x|^{\xi} - \xi x^i x^j |x|^{\xi - 2}]. \tag{2.5} \]

The limiting scaling formula (2.5) approximates well $d^{ij}(x)$ in the “inertial interval” $\eta \ll x \ll L$. The scale $\eta$ plays the role of the “viscous scale” within which the fractional scaling of $d^{ij}(x)$ is replaced by a quadratic behavior and scale $L$ of the “integral scale” on which velocities decorrelate. The matrix $d(x)$ describes the correlations of the velocity differences. E.g.
\[ \mathbf{E} \left( (v^j(t, x) - v^j(t, 0)) (v^j(s, x) - v^j(s, 0)) \right) = 2 \delta(t - s) \ d^{ij}(x). \tag{2.6} \]

The Kraichnan ensemble of velocities incorporates on the statistical level the scaling properties (1.1) of velocities in the inertial interval $\eta \ll |x - y| \ll L$. For $\eta > 0$, the Gaussian measure of the ensemble is supported by smooth velocities. This is not the case, however, in the limiting case $\eta = 0$ where the viscous scale is set to zero, mimicking the inviscid or infinite Reynolds number ensemble of turbulent velocities. In this case the ensemble measure is supported on velocities that in their spatial behavior are H"older continuous with an exponent smaller than $\xi/2$ whereas the velocities with the H"older exponent bigger than $\xi/2$ have measure zero. Of course, in their temporal behavior, the Kraichnan velocities behave as white noise or a derivative of the Brownian motion and thus are distributional. We shall have then to modify our mean-field arguments adapting it to such a case.

2.2. Advection-diffusion equation in the Kraichnan velocities

Let us consider first the finite-dimensional analog of the advection-diffusion equation (0.3),
\[ \dot{\theta} = \beta(t) \theta - a \theta, \tag{2.7} \]
where $\theta(t)$ takes values in $\mathbb{R}^D$ and $\beta(t)$ is a skew-symmetric matrix (a counterpart of the operator $-\mathbf{v} \cdot \nabla$) and $a$ a positive symmetric matrix (a counterpart of $-\kappa \nabla^2$). Suppose first that $\beta$ is a smooth function of time. The solution of the above linear equation has the form $\theta(t) = P(\beta|t, s) \theta(s)$ with the propagator $P(\beta|t, s)$ given by the time-ordered exponential
\[ P(\beta|t, s) = \mathcal{T} e^{\int_{s}^{t} (\beta(\sigma) - a) \ d\sigma}, \]
\[ = \sum_{\sigma_n \leq \sigma_1 \leq \ldots \leq \sigma_n \leq \ldots \leq \sigma_n \leq t} \int e^{-(t - \sigma_n) a} \beta(\sigma_n) \ d\sigma_n \ e^{-(\sigma_n - \sigma_{n - 1}) a} \ldots \]
\[ \ldots \beta(\sigma_2) \ d\sigma_2 \ e^{-(\sigma_2 - \sigma_1) a} \beta(\sigma_1) \ d\sigma_1 \ e^{-(\sigma_1 - t) a}. \tag{2.8} \]

for $t > s$. Another way to express the same solution is by a limiting procedure:
\[ P(\beta|t, s) = \lim_{\min(\sigma_{m - 1} - \sigma_{m - 1}) \to 0} \int_{\sigma_n}^{t} (\beta(\sigma) - a) \ d\sigma \int_{\sigma_n}^{t} (\beta(\sigma) - a) \ d\sigma \int_{\sigma_n}^{t} (\beta(\sigma) - a) \ d\sigma \ldots \ e^{\sigma_n} \ e^{\sigma_{n - 1}} \ldots e^{\sigma_1} \tag{2.9} \]
Suppose now that $\beta(t)$ is a white-noise Gaussian process with values in antisymmetric matrices with mean zero and with the 2-point function

$$E \beta^{\alpha \beta}(t) \beta^{\gamma \delta}(s) = \delta(t - s) C^{\alpha \beta, \gamma \delta}. \quad (2.10)$$

Now $\beta d\sigma = dW$ is a differential of a Brownian marix-valued motion $W(t)$ and the integrals in (2.8) become stochastic ones. Consequently, we fall into the standard ambiguity with the choice of the convention for the latter. On the other hand, the formula (2.9) still makes sense with the convergence taking place in any $L^p$ of the Gaussian process with $p < \infty$, as it is not difficult to show. We shall take it as the solution of the equation (2.9) for the white $\beta(t)$. In terms of the standard conventions, this corresponds to the Stratonovich prescription in (2.8) and provides the solution of the Stratonovich stochastic ODE

$$d\theta = dW \circ \theta - a \theta \, dt \quad (2.11)$$

or, equivalently, of the Itô one:

$$d\theta = (dW) \theta - (a + c) \theta \, dt \quad (2.12)$$

with $a^{\alpha \beta} = -\frac{1}{2} C^\alpha \gamma \cdot C^\beta \gamma \beta$ (summation over $\gamma$!). The solution of the latter equation, in turn, is given by the version of eq. (2.8) with $a$ replaced by $(a + c)$ and the integrals interpreted as the Itô stochastic ones and $\beta(\sigma) d\sigma$ as $dW(\sigma)$.

We shall then try to define the the solution of the advection-diffusion equation (1.3) for white in time velocities $v(t, x) \, dt = d\mathbf{V}(t, x)$ as given by (1.8) with $P(v|t, s)$ represented by

$$P(v|t, s) = T e^t \int_{(-d\mathbf{V}(\sigma) \cdot \nabla + \kappa \nabla^2 \, d\sigma}^s e^{-s} \nabla e^{(s - \sigma_n) \kappa \nabla^2} \nabla e^{(\sigma_{n-1} - \sigma_n) \kappa \nabla^2} \nabla e^{(\sigma_{n-2} - \sigma_{n-1}) \kappa \nabla^2} \cdots \nabla e^{(\sigma_2 - \sigma_1) \kappa \nabla^2} \nabla e^{(\sigma_1 - \sigma_2) \kappa \nabla^2} = \sum_{n=0}^{\infty} P_n(v|t, s), \quad (2.13)$$

with the Itô stochastic integrals and $\kappa = \kappa + \frac{1}{2} D_0$. The operator-valued white noise $-d\mathbf{V}(t) \cdot \nabla$ plays the role of $dW(t)$, the operator $\kappa \nabla^2$ the one of $-a$, and $\frac{1}{2} D_0 \nabla^2$ that of $-c$, as is easy to figure out by comparing (2.1) and (2.3) with (2.10).

It is not difficult to check that each term of the series is well defined in the action on a function, say, from $L^\infty(\mathbb{R}^d)$ and, in its dependence on $v$, belongs to $L^p$ of the Gaussian process with $p < \infty$. The only problem is the convergence of the series that was established for $\kappa \geq 0$ ($\kappa = 0$ included!) in (Le Jan & Raimond, 1999). The argument is quite simple. First, for $f \in L^\infty(\mathbb{R}^d)$ and $P_{\leq N}(v|t, s) = \sum_{n=1}^{N} P_n(v|t, s)$ one shows inductively the a priori bound:

$$E |P_{\leq N}(v|t, s) f|^2 \leq P_{0}(t, s) |f|^2, \quad (2.14)$$

where the $0^{th}$-order term $P_{0}(t, s) = e^{(t-s) \kappa \nabla^2}$. But the terms $P_n(v|t, s) f$ are orthogonal with respect to the scalar product in $L^2$ since all the differentials $d\mathbf{V}(\sigma_n)$ are independent in virtue of the Itô convention and their expectations vanish. Hence the bound (2.14) establishes the
convergence of the series $\sum_{n} P_n(v|t,s)f$ in the $L^2$ norm of the velocity process for any $\kappa \geq 0$ together with the limiting bound
\[
E |P(v|t,s)f|^2 \leq P_0(t,s)|f|^2. \tag{2.15}
\]
The continuity of the result in $\kappa \geq 0$ may be also established.

**Proof of estimate (2.14).** For $N = 0$, the bound reduces to the easy estimate $|P_n(t,s)f|^2 \leq P_0(t,s)|f|^2$. Suppose now that (2.14) holds up to $N$. Note that
\[
P_{\leq N+1}(v|t,s)f = P_0(t,s)f - \int_{s}^{t} P_{\leq N}(v|t,\sigma) \, dV(\sigma) \cdot \nabla P_0(\sigma,s)f. \tag{2.16}
\]

Squaring and taking expectations, we obtain
\[
E |P_{\leq N+1}(v|t,s)f|^2 = |P_0(t,s)f|^2 + \int_{s}^{t} d\sigma \, E \left( P_{\leq N}(v|t,\sigma) \otimes P_{\leq N}(v|t,\sigma) \right) \mathcal{K}_2 \left( P_0(\sigma,s) \mathcal{T} \otimes P_0(\sigma,s)f \right), \tag{2.17}
\]
where $\mathcal{K}_2 = D^{ij}(x_1 - x_2) \nabla_{x_1}^i \nabla_{x_2}^j$ is the operator acting on functions on $\mathbb{R}^d \times \mathbb{R}^d$. Rewriting the positive-definite function $D^{ij}(x)$ with the use of the Fourier transform as $\int \hat{D}^{ij}(k) e^{ik \cdot x} \, dk$ with the positive matrix $\hat{D}^{ij}(k) = \sum_{\alpha=1}^{d-1} \hat{\lambda}_i^{\alpha}(k) \hat{\lambda}_j^{\alpha}(k)$, we may present the second term on the right hand side of (2.17) as
\[
\int_{s}^{t} d\sigma \int dk \sum_{\alpha} E |P_{\leq N}(v|t,\sigma)f_{\sigma,k,\alpha}|^2, \tag{2.18}
\]
where $f_{\sigma,k,\alpha}(x) = e^{-ik \cdot x} \hat{\lambda}_i^{\alpha}(k) \nabla_v [P_0(\sigma,s)f](x)$. By the inductive hypothesis, this expression is bounded by
\[
\int_{s}^{t} d\sigma \int dk \sum_{\alpha} P_0(t,\sigma) |f_{\sigma,k,\alpha}|^2 = \int_{s}^{t} d\sigma \int dk \, P_0(t,\sigma) \hat{D}^{ij}(k) \left( \nabla_i P_0(\sigma,s) \mathcal{T} \right) \left( \nabla_j P_0(\sigma,s)f \right)
\]
\[
= D_0 \int_{s}^{t} d\sigma \, P_0(t,\sigma) \, |\nabla P_0(\sigma,s)f|^2 \leq 2\bar{\kappa} \int_{s}^{t} d\sigma \, P_0(t,\sigma) \, |\nabla P_0(\sigma,s)f|^2. \tag{2.19}
\]

Altogether, we obtain
\[
E |P_{\leq N+1}(v|t,s)f|^2 \leq |P_0(t,s)f|^2 + 2\bar{\kappa} \int_{s}^{t} d\sigma \, P_0(t,\sigma) \, |\nabla P_0(\sigma,s)f|^2
\]
\[
= |P_0(t,s)f|^2 - \int_{s}^{t} d\sigma \, \frac{d}{d\sigma} \left( P_0(t,\sigma) \, |P_0(\sigma,s)f|^2 \right) = P_0(t,s) |f|^2 \tag{2.20}
\]
ending the inductive proof of eq. (2.14). \(\square\)

The Chapman-Kolmogorov chain relation $P(v|t,\sigma)P(v|\sigma,s) = P(v|t,s)$ for $s \leq \sigma \leq t$ and the normalization $P(v|t,s)1 = 1$ follow easily. Le Jan and Raimond also proved that the operators $P(v|t,s)$ preserve positivity. This implies that
\[
|P(v|t,s)f| \leq P(v|t,s) |f| \leq P(v|t,s) \|f\|_\infty = \|f\|_\infty. \tag{2.21}
\]
so that $P(v|t,s) f$ are (essentially) bounded. We obtain this way a family of Markov transition probabilities parametrized by the velocities of the Kraichnan ensemble, hence, also a family of Markov processes describing the Lagrangian trajectories (with and without the perturbing noise). Taking the $N \to \infty$ limit in eq. (2.16), we infer that $P(v|t,s)$ satisfies the stochastic (anti-)Itô integral equation

$$P(v|t,s) f = P_0(t,s) f - \int_s^t P(v|\sigma,s) \nabla P_0(\sigma,s) f \, d\sigma.$$  \hspace{1cm} (2.22)

which, upon differentiation over $s$, gives the stochastic versions of the diffusion-advection equation

$$d_s P(v|t,s) f = P(v|t,s) dV(s) \cdot \nabla f - \bar{\kappa} P(v|t,s) \nabla^2 f ds$$ \hspace{1cm} (2.23)

$$= P(v|t,s) \circ dV(s) \cdot \nabla f - \kappa P(v|t,s) \nabla^2 f ds.$$ \hspace{1cm} (2.24)

2.3. $N$-particle processes in Kraichnan velocities

As follows for example from the relation (2.22), the expectation of the transition probability $P(v|t,s)$ coincides with the first term $P_0(t,s) = e^{(t-s)\bar{\kappa}\nabla^2}$ of the series (2.13). Thus, recalling the definition (1.9), we obtain

$$P_1(t,x; s,dy) = e^{(t-s)\bar{\kappa}\nabla^2} f(y).$$ \hspace{1cm} (2.25)

We infer that in the Kraichnan model, a single fluid particle undergoes diffusion for all times with the diffusion constant equal to $\bar{\kappa}$, i.e. to the sum of the molecular diffusivity $\kappa$ and of the “eddy diffusivity” $\frac{1}{2} D_0$, an effective diffusivity due to the random velocities. Recall that $D_0 = O(L^\xi)$ so that the eddy diffusivity is dominated by the integral scale, i.e. by the correlation length of the velocities. The virtue of the use of the stochastic Itô integrals in (2.13) was that it made the regularizing role of the eddy diffusion explicit and permitted uniform treatment of the cases with $\kappa > 0$ and with $\kappa = 0$.

In order to study the statistics of $N$ particles in the Kraichnan model, we have to analyze the joint transition probabilities $P_N(t,x; s, dy)$, see (1.9). If fact, it is enough to look at their equal-time versions $P_N(t,x; s, dy)$. The stochastic (anti-)Itô equation (2.23) implies the relation

$$\frac{d}{ds} \int P_N(t,x; s, dy) f(y) = - \sum_{n<m} \int P_N(t,x; s, dy) D^{ij}(y_n - y_m) \nabla y_n \nabla y_m f(y)$$

$$- \sum_n \int P_N(t,x; s, dy) \bar{\kappa} \nabla^2 y_n f(y)$$ \hspace{1cm} (2.26)

from which one deduces that

$$P_N(t,x; s, dy) = e^{(t-s)\mathcal{M}_N(x,y)} dy$$ \hspace{1cm} (2.27)

for the second order differential operator

$$\mathcal{M}_N = \frac{1}{2} \sum_{n,m} D^{ij}(x_n - x_m) \nabla x_n \nabla x_m + \kappa \sum_n \nabla^2 x_n.$$ \hspace{1cm} (2.28)
It follows that in the Kraichnan model the transition probabilities \( P_N(t, \mathbf{x}; s, d\mathbf{y}) \) still form a Markov family (this is due to the temporal decorrelation of velocities). The probabilities of relative separations of \( N \) particles

\[
P_{\text{rel}}^N(t, \mathbf{x}; s, d\mathbf{y}) = \int P_N(t, \mathbf{x}; d(y_1 + \mathbf{y}), \ldots, d(y_N + \mathbf{y})) ,
\]

(2.29)

with the integral over the translations \( \mathbf{y} \), are given by the exponential function of the operators \( M_N \) that are restrictions of \( M_N \)’s to the translation-invariant sector:

\[
M_N = -\sum_{n<m} (d^{ij}(x_n - x_m) + 2\kappa) \nabla_{x_n^i} \nabla_{x_m^j}
\]

(2.30)

with \( d^{ij}(\mathbf{x}) \) given by (2.3). As we see, in the Kraichnan velocities, \( N \) fluid particles undergo in their relative motion an effective diffusion process with the configuration-dependent diffusivity.

In particular for the probability distribution (1.17) of the relative separation of two fluid particles, we obtain

\[
P_{\text{rel}}^2(t, \rho_0; s, d\mathbf{\rho}) = e^{(t-s)M_2(\rho_0, \mathbf{\rho})} d\mathbf{\rho},
\]

(2.31)

where

\[
M_2 = d^{ij}(\mathbf{\rho}) \nabla_i \nabla_j + 2\kappa \nabla^2.
\]

(2.32)

This operator commutes with the action of the rotation group in \( L^2(\mathbb{R}^d) \) and reduces in the action on functions carrying irreducible representations of \( SO(d) \) (labeled by the angular momentum \( \ell = 0, 1, \ldots \)) to a second order differential operator in the radial variable \( \rho = |\mathbf{\rho}| \).

In particular, the probability distribution \( P_2(t, \rho_0; s, d\mathbf{\rho}) \) of the 2-particle dispersion is given by the exponential function of the restriction \( M_2^{\ell=0} \) of operator \( M_2 \) to the rotation-invariant sector with \( \ell = 0 \).

The above relations will allow us to analyze in the next two lectures the properties of the fluid particles in the Kraichnan ensemble of velocities.

**LECTURE 3**

(fluid particles and advection of scalar in the Batchelor regime of the Kraichnan model: random chaos)

### 3.1. Separation of close particles in smooth Kraichnan velocities

In the region where the distances between the fluid particles are much smaller then the viscous scale \( \eta \), i.e. in the Batchelor regime, we may approximate \( d^{ij} \) as in (2.3). Upon dropping the 4th-order terms (which, strictly speaking pertains to the behavior of infinitesimally close trajectories) and upon setting \( \kappa = 0 \), the generators of the \( N \)-particle processes become

\[
M_N = -D_1 \sum_{n<m} (x_n - x_m)^2 \nabla_{x_n} \cdot \nabla_{x_m} + 2(x_n^i - x_m^i)(x_n^j - x_m^j) \nabla_{x_n^i} \nabla_{x_m^j}.
\]

(3.1)

In particular, for \( N = 2 \),

\[
M_2 = D_1 [\rho^2 \nabla^2 - 2\rho^i \rho^j \nabla_i \nabla_j]
\]

(3.2)
in terms of the separation variable $\rho$. In the rotational invariant sector,

$$M_2^{t=0} = D_1(d-1) \rho^{-d+1} \partial_\rho \rho^{d+1} \partial_\rho.$$  

(3.3)

From the latter expression, one infers easily the probability distribution of the 2-particle dispersion:

$$P_2(0, \rho_0; t, d\rho) = \frac{1}{\sqrt{4\pi D_1(d-1)t}} \exp \left[ -\frac{1}{4 D_1(d-1)t} \left( \ln\left( \frac{\rho}{\rho_0} \right) - D_1(d-1)t \right)^2 \right] d\rho.$$  

(3.4)

As we see, the logarithm of the 2-particle dispersion grows linearly with the rate (the Lyapunov exponent) $\lambda = D_1(d-1)d > 0$. The system exhibits the exponential separation of trajectories, i.e. is chaotic. It is easy to see that

$$\lim_{\rho_0 \to 0} P_2(0, \rho_0; t, d\rho) = \delta(\rho) d\rho,$$  

(3.5)

which is a consequence of the existence of deterministic Lagrangian trajectories in smooth velocities: it signals that the Markov process with the transition probabilities $P(v|t, x; s, dy)$ in a fixed velocity $v$ realization concentrates (for $\kappa = 0$) on deterministic trajectories determined by the initial conditions.

It is not difficult to understand the origin of the positivity of the Lyapunov exponent in the smooth Kraichnan velocities. For such velocities, the equation (1.13) for very close trajectory separation may be approximated by

$$\frac{d\rho}{ds} = \rho \cdot \nabla v(s, x_{0,x_1}(s)).$$  

(3.6)

In the Kraichnan model,

$$\mathbb{E} \nabla_k v^i(t, x) \nabla_\ell v^j(s, y) = \delta(t-s) \nabla_k \nabla_\ell d^{ij}(x-y)$$  

(3.7)

which is independent of $(x - y)$ in the quadratic approximation to $d^{ij}(x)$. In other words, in this approximation, $\nabla_k v^i(t, x) = \gamma^{ik}(t)$ where $\gamma^{ik}(t)$ is a (Gaussian) white noise with the values in the real traceless matrices, with mean zero and covariance

$$\mathbb{E} \gamma^{ik}(t) \gamma^{\ell\ell}(s) = 2 D_1 \delta(t-s) \left( (d+1) \delta^{ij} \delta^{kl} - \delta^{ik} \delta^{j\ell} - \delta^{i\ell} \delta^{jk} \right).$$  

(3.8)

Eq. (3.6) becomes the stochastic differential equation

$$d\rho = d\Gamma \circ \rho$$  

(3.9)

where $\Gamma(s)$ is a Brownian motion on traceless matrices with $d\Gamma(s) = \gamma(s) ds$. The solution is given by

$$\rho(s) = G_\Gamma(s) \rho(t),$$  

(3.10)

where $G_\Gamma(s)$ is a Brownian motion on the group $SL(d)$ of real unimodular matrices satisfying $G_\Gamma(t) = 1$. It follows that in the quadratic approximation to $d^{ij}(x),$

$$\int P(v|t, x_1; s, dy_1) P(v|t, x_2; s, dy_2) f(y_2 - y_1) = f \left( G_\Gamma(s) \rho_0 \right),$$  

(3.11)
where \( \rho_0 = x_2 - x_1 \). Taking the averages of the last identity, we infer that
\[
(\mathcal{P}_2^{rel}(t, s) f)(\rho_0) = E f (G_t(s)\rho_0) .
\] (3.12)

It is not difficult to find the generator of the Brownian motion \( G_t(s) \). Consider the natural actions of \( SL(d) \) and of its subgroup \( SO(d) \) in \( L^2(\mathbb{R}^d) \). Their infinitesimal generators are
\[
H^{ij} = \left( -\rho^i \nabla_j + \frac{1}{d} \delta^{ij} \rho^k \nabla_k \right),
\]
\[
J^{ij} = \left( -\rho^i \nabla_j + \rho^j \nabla_i \right),
\] (3.13)
respectively. A simple algebra shows that
\[
M_2 = D_1 \left[ d H^2 - (d + 1) J^2 \right]
\] (3.14)
where \( H^2 \) and \( J^2 \) are the quadratic Casimirs of \( SL(d) \) and of \( SO(d) \):
\[
H^2 = H^{ij} H^{ji}, \quad J^2 = \frac{1}{2} (J^{ij})^2 .
\] (3.15)

The right hand side of (3.14) (with the Casimirs interpreted as those of the left regular action) gives the generator of the Brownian motion \( G_t(s) \) on \( SL(d) \). It is well known that such Brownian motions, that may be thought of as continuous products of random, independent, identically distributed matrices in \( SL(d) \), lead to positive Lyapunov exponents (a continuous version of the Furstenberg-Kesten or Oseledets Theorems, see e.g. (Arnold, 1998)).

For the \( N \)-point operator \( M_N \) given by (3.11), we similarly obtain:
\[
M_N = D_1 \left[ d G^2_{N-1} - (d + 1) J^2_{N-1} \right]
\] (3.16)
where \( H^2_{N-1} \) and \( J^2_{N-1} \) are the quadratic Casimirs of the diagonal actions of \( SL(d) \) and of \( SO(d) \) on the \((N-1)\) separation variables, e.g. on \( \rho_n = x_n - x_1 \). An alternative expression for \( M_N \) is
\[
M_N = D_1 \left[ d G^2_{N-1} + \frac{d-N+1}{N-1} \Lambda (\Lambda + (N-1)d) - (d + 1) J^2_{N-1} \right]
\] (3.17)
where \( G^2_{N-1} \) is the quadratic Casimir of the action of \( SL(N-1) \) on the index \( n \) of \( \rho^i_n \) with the generators \( G^{nm} = -\rho^i_n \nabla_{\rho^i_m} + \frac{1}{N-1} \delta^{nm} \sum_p x^i_p \nabla x^i_p \) and \( \Lambda = \sum_p x^i_p \nabla x^i_p \) is the generator of the overall dilations. Consider a function \( f \) of \( N-1 \) separations that depends only on the volume \( \rho = \sqrt{\det_{nm}(\rho_n, \rho_m)} \) spanned by \( N-1 \) separation vectors (for \( N - 1 \leq d \)). Since \( \rho \) is \( SL(N-1) \) and \( SO(d) \)-invariant, it follows from (3.17) that \( M_N f \) is still a function of \( \rho \) only and that
\[
(M_N f)(\rho) = D_1 \frac{d-N+1}{N-1} \Lambda (\Lambda + (N-1)d) f(\rho) = D_1 (N-1)(d - N + 1) (\rho^{-d+1} \partial_\rho^{d+1} \partial_\rho) f(\rho).
\] (3.18)

Comparing to (3.13) and (3.14), we infer that the logarithm of \( \rho \) grows linearly with the rate \( D_1 (N-1)(d - N + 1)d \). By definition, the latter gives the sum \( \lambda_1 + \ldots + \lambda_{N-1} \) of the \((N-1)\) biggest Lyapunov exponents so that
\[
\lambda_N = D_1 (d - 2N + 1)d .
\] (3.19)
In the smooth $d$-dimensional Kraichnan flow the $d$ Lyapunov exponents are equidistant, with the sum equal to zero (as required by incompressibility). The relations of the multi-trajectory processes to the harmonic analysis on the groups $SL(N)$ were observed in (Shraiman & Sigia, 1995 and 1996).

3.2 Scalar statistics in the Batchelor regime

One may easily control the stationary state of the scalar developing under the forced transport of Kraichnan velocities in the Batchelor regime. For white in time velocities, in agreement with the preceding discussion, the solution of the forced transport problem is given by eq. (1.8) with the evolution operators $P(v|t,s)$ as constructed above. For simplicity, let us take the zero initial data for the scalar: $\theta(s) = 0$. Then the characteristic function of the time $t$ probability distribution of the scalar,

$$\Phi(h|t,s) = E e^{i\int dx h(x) \theta(t,x)} = E e^{i \int ds \int dx h(x) P(v|t,x;\sigma,dy) f(\sigma,y)}.$$  \hfill (3.20)

Let us take the source $f(s,y)$ to be also a Gaussian process, independent of velocities, with mean zero and covariance

$$E f(t,x) f(s,y) = \delta(t-s) \chi(x-y),$$ \hfill (3.21)

where $\chi$ is a smooth, positive-definite, decaying function on $\mathbb{R}^d$. For such a forcing, the expectation over the source $f$ of the scalar in (3.20) may be easily calculated, leaving us with the expectation over the velocity ensemble:

$$\Phi(h|t,s) = E e^{-\frac{i}{2} \int ds \int dx_1 dx_2 h(x_1) h(x_2) P(v|t,x_1;\sigma,dy_1) P(v|t,x_2;\sigma,dy_2) \chi(y_2-y_1)}.$$ \hfill (3.22)

In the Batchelor regime, we may use the relation (3.11) to rewrite the latter expectation in terms of the one over the Brownian motion $G_t(\sigma)$ on $SL(d)$:

$$\Phi(h|t,s) = E e^{-\frac{i}{2} \int ds \int dx_1 dx_2 h(x_1) h(x_2) \chi(G_t(\sigma)(x_1-x_2))} = E e^{-\int V_h(G_t(\sigma)) d\sigma},$$ \hfill (3.23)

where $V_h(G) = \frac{i}{2} \int dx_1 dx_2 h(x_1) h(x_2) \chi(G(x_1-x_2))$ is a positive potential on group $SL(d)$. With the use of the Feynman-Kac formula, eq. (3.23) may be expressed as

$$\Phi(h|t,s) = \int_{SL(d)} e^{(t-s)[dH^2-(d+1)J^2-V_h]}(1,G) dG$$

$$= 1 - \int_0^{t-s} d\sigma \left( e^{\sigma[dH^2-(d+1)J^2-V_h]} V^\dagger_h(1) \right),$$ \hfill (3.24)

where the last equality follows by integration by parts. It is then easy to show that the limit $\Phi(h) = \lim_{s \to -\infty} \Phi(h|t,s)$ exists and is a characteristic function of a probability measure supported e.g. on the space $S'(\mathbb{R}^d)$ of tempered distributions. The latter describes the stationary state of the forced scalar which, indeed, is not supported by smooth scalar configurations, as reflected by logarithmic singularities at coinciding points of the scalar $N$-point functions. For $\chi$ and $h$ rotation-invariant, potential $V_h$ is left and right $SO(d)$-invariant. On $SO(d) \setminus SL(d) / SO(d)$, the operator $[dH^2-(d+1)J^2]$ reduces the the Calogero-Sutherland integrable Hamiltonian.
and the analysis of the distribution of the random variable \( \int dx \, h(x) \theta(x) \equiv \theta(h) \) reduces to the analysis of the property of its perturbation by the potential proportional to \( V_h \). For example, the exponential rate of decay of the probability density function of \( \theta(h) \) is related to the bound state energy of such a perturbation with a negative coefficient. The details may be found in (Bernard, Gawedzki & Kupiainen, 1998).

**LECTURE 4**

(fluid particles in non-smooth Kraichnan velocity fields; breakdown of deterministic Lagrangian flow; intrinsic stochasticity versus particle aggregation)

4.1. Separation of close particles in non-smooth Kraichnan velocities

In the limiting case \( \eta \to 0 \) of the Kraichnan model, the Lagrangian trajectories exhibit even more dramatic behavior. For \( \eta = 0 \) and \( L = 0 \), when \( d_{ij}(x) \) takes the scaling form of eq. (2.5), and for \( \kappa = 0 \),

\[
M_2^{\ell=0} = D_2(d-1) \rho^{d+1} \partial_\rho \rho^{d-1+\xi} \partial_\rho. \tag{4.1}
\]

The probability distribution of the 2-particle dispersion

\[
P_2(0, \rho_0; t, d\rho) = e^{t M_2^{\ell=0}(\rho_0, \rho)} d\rho \tag{4.2}
\]

may be studied for example using the spectral decomposition of \( M_2^{\ell=0} \). The long-time-large-distance asymptotics of the dispersion follows from the rescaling property

\[
P_2(0, \rho_0; \mu t, d(\mu^{1/(2-\xi)} \rho)) = P_2(0, \mu^{-1/(2-\xi)} \rho_0; t, d\rho) \tag{4.3}
\]

and the easily shown relation

\[
\lim_{\rho_0 \to 0} P_2(0, \rho_0; t, d\rho) \propto \rho^{d-1} t^{-d/(2-\xi)} e^{-\text{const.} \rho^2 t / t} d\rho. \tag{4.4}
\]

In particular, one obtains the Richardson dispersion law in the form

\[
E \rho(t)^2 = \int \rho^2 P_2(0, \rho_0; t, d\rho) = O(t^{\frac{2}{\xi+1}})
\]

for large times, with an exact proportionality for all times in the limit \( \rho_0 \to 0 \). Such a behavior may be predicted by solving the modified version

\[
d\rho^2 \propto dw \rho^{\xi/2+1} \tag{4.5}
\]

of the naive mean field type equation (1.15), where \( w(t) \) is the Brownian motion introduced to account for the temporal decorrelation of Kraichnan velocities. In particular, the growth of \( \rho^2 \) proportional to \( t^3 \) is obtained for \( \xi = 4/3 \).

Note that the limit (4.4) is absolutely continuous with respect to the Lebesque measure \( d\rho \), in contrast to what happens in the Batchelor regime, see eq. (1.5). Such a property excludes the concentration of the transition probabilities \( P(v|t, x; s, dy) \) at a single (\( v \)-dependent) point \( y \) and signifies the breakdown of the deterministic Lagrangian flow with fluid particle trajectories determined by their initial positions in fixed velocity fields. Instead, the Lagrangian trajectories

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form a genuinely stochastic process already in a fixed velocity realization, as predicted above and illustrated in the figure below:

![Diagram](image)

That Markov process, may be obtained by first adding the regularizing noise to the trajectory equation, as in (1.4), and then turning it off, or also directly, by the Le Jan-Raimond construction of the $\kappa = 0$ transition probabilities $P(v|t, x; s, dy)$.

4.2. The role of compressibility

Up to now, we have considered the Kraichnan ensemble with incompressible velocities, the property guaranteed by the presence of the transverse projector $(\delta^{ij} - \frac{k^i k^j}{k^2})$ in the Fourier representation (2.2) of the velocity 2-point function. In order to study the effects of compressibility, one may introduce another parameter besides the roughness exponent $\xi$, the compressibility degree $\wp$. We shall do it by replacing the transverse projector in (2.2) by $[(1 - \wp)\delta^{ij} + (\wp d-1)\frac{k^i k^j}{k^2}]$. The value $\wp = 0$ corresponds to the incompressible case whereas for $\wp = 1$ almost all velocities are gradients, with the intermediate values of $\wp$ interpolating between the two cases. The preceding constructions, in particular the Le Jan-Raimond one, carry over to the case with non-zero $\wp$. The scaling form of the generator of the 2-particle dispersion process becomes now

$$M_2^{t=0} = D_2(d - 1) \rho^{d-\alpha} \partial_\rho \rho^\alpha \partial_\rho$$

with $a = \frac{d+\xi}{1+\omega \xi} - 1$. The definition of the semigroup $e^{tM_2^{t=0}}$ requires a choice of the boundary
condition for the operator $M_2^{t=0}$ at $\rho = 0$. Such a choice is automatically assured by considering first the $\kappa > 0$ case and then sending $\kappa$ to zero. This limiting procedure selects for $\varphi < d/\xi^2$, i.e. for weak compressibility, the eigenfunctions of $M_2^{t=0}$ behaving like $\mathcal{O}(1)$ at $\rho = 0$ whereas for strong compressibility $\varphi > d/\xi^2$ the eigenfunctions that behave like $\mathcal{O}(\rho^{1-a})$ are chosen. The two choices result in very different probability distributions $P_2(0, \rho_0; t, d\rho)$ of the 2-particle dispersion and, in consequence, to dramatically different Lagrangian flows, as first noticed in (Gawedzki & Vergassola, 2000). For $\varphi < d/\xi^2$, in a simple generalization of (4.4),

$$\lim_{\rho_0 \to 0} P_2(0, \rho_0; t, d\rho) \propto \rho^{\alpha-\xi} t^{(\xi-1-\alpha)/(2-\xi)} e^{-\text{const.} \rho^{2-\xi}/t} d\rho$$

from which the conclusions about the intrinsic stochasticity of the Lagrangian flow may be drawn the same way as for the incompressible case. For $\varphi > \xi/d^2$, however,

$$P_2(0, \rho_0; t, d\rho) = P^{\text{reg}}(0, \rho_0; t, d\rho) + p(t, \rho_0) \delta(\rho) d\rho$$

with $P^{\text{reg}}(0, \rho_0; t, d\rho)$ absolutely continuous with respect to the Lebesgue measure $d\rho$ and $p(t, \rho_0) > 0$. When $\rho_0 \to 0$ the regular part tends to zero and $p(t, \rho_0)$ tends to 1 so that one recovers the behavior (4.3). The latter indicates that the Lagrangian flow is deterministic, with trajectories determined by the initial condition in fixed velocity realizations. The presence of the term proportional to the delta-function for $\rho_0 > 0$ signals, however, that, with positive $t$-dependent probability, the trajectories starting at different initial points collapse together by time $t$, as illustrated in the figure on the next page. Such a non-conventional behavior of trajectories is again possible since the theorem about the existence and unicity of trajectories fails for non-Lipschitz velocities. The competition between the tendency of the trajectories in such velocities to separate explosively and their trapping in the regions of strong compression is won by the first trend for weak compressibility and by the second one for the strong one.

### 4.3. Persistence of scalar dissipation

The unusual behaviors of Lagrangian trajectories in rough Kraichnan velocities are the source of phenomena crucial for the scalar transport. One of the implications of the advection-diffusion equation (4.3) in the incompressible velocities is the scalar ”energy” $\int \theta^2$ balance

$$\frac{d}{dt} \int \theta(t, x)^2 dx = -2\kappa \int (\nabla \theta(t, x))^2 dx + 2 \int \theta(t, x) f(t, x) dx$$

with the first term on the right hand describing the dissipation and the second one the injection of scalar energy by the sources. One may naively expect that in the absence of sources the scalar energy is conserved in the limit $\kappa \to 0$. Let us examine this question more closely. Recall that the evolution of scalar is related by eq. (1.7) to the transition probabilities of the (noisy) Lagrangian trajectories. For vanishing sources, one obtains the identity

$$\int dx \int P(v|t, x; s, dy) [\theta(s, y) - \theta(t, x)]^2 = \int \theta(s, y)^2 dy - \int \theta(t, x)^2 dx$$

which follows by developing the square and using eq. (1.7), the normalization of the measure $P(v|t, x; s, dy)$ and the symmetry of the operator $P(v|t, s)$. The left hand side is obviously non-negative which implies that the scalar energy cannot grow. It vanishes if and only if $\theta(s, y) = \theta(t, x)$ on the support of $P(v|t, x; s, y)$ for each $x$. This happens for arbitrary $\theta(s, y)$ if and only if $P(v|t, x; s, dy)$ is supported by exactly one ($x$-dependent) point, i.e. if the Lagrangian trajectories are determined by their single-time positions. The intrinsic stochasticity
of the Lagrangian flow results then in the persistence of scalar energy dissipation in the limit \( \kappa \to 0 \).

The same effect may be seen in averaged quantities, also in compressible Kraichnan velocities. Suppose that at the initial time \( s \) we are given an homogeneous, isotropic scalar distribution with the 2-point function

\[
E \theta(s, y) \theta(s, y + \rho) = F_2(s, \rho).
\]  

(4.11)

Then, if the initial scalar distribution is independent of velocities and in the absence of sources, the scalar 2-point function at the later time \( t \) is given by

\[
F_2(t, \rho_0) \equiv E \theta(t, x) \theta(t, x + \rho_0) = \int F_2(s, \rho) \mathcal{P}_2(t, \rho_0; s, d\rho).
\]  

(4.12)

In particular, the mean scalar energy density

\[
F_2(t, 0) = \int F_2(s, \rho) \lim_{\rho_0 \to 0} \mathcal{P}_2(t, \rho_0; s, d\rho).
\]  

(4.13)
Since $F_2(s, \rho) \leq F_2(s, 0)$ by the Schwartz inequality, it follows that the mean scalar energy density is non-increasing. It is conserved in general only if $\lim_{\rho_0 \to 0} \mathcal{P}_2(t, \rho_0; s, d\rho) = \delta(\rho) d\rho$, i.e. if the trajectories are uniquely determined by their final positions. This is the case for strongly compressible Kraichnan velocities with $\varphi > d/\xi^2$.

**LECTURE 5**

(forced scalar advection; intermittent direct cascade with persistent dissipation and zero mode dominance versus non-intermittent inverse cascade)

When scalar is forced with the random Gaussian source characterized by the isotropic 2-point function (3.21) and if the forcing is independent of velocities and of the initial distribution of the scalar, then the scalar 2-point function evolves according to the relation

$$F_2(t, \rho_0) = \int F_2(s, \rho) \mathcal{P}_2(t, \rho_0; s, d\rho) + \int d\sigma \int \chi(\rho) \mathcal{P}_2(t, \rho_0; \sigma, d\rho),$$  \hspace{1cm} (5.1)

which solves the differential equation

$$\frac{d}{dt} F_2(t, \rho) = M_{2=0}^\ell F_2(t, \rho) + \chi(\rho).$$  \hspace{1cm} (5.2)

Setting $\rho = 0$ in the latter identity, we obtain the averaged scalar energy balance with $M_{2=0}^\ell F_2(t, 0)|_{\rho=0}$ representing the mean dissipation rate $\epsilon$ and $\chi(0)$ the mean injection rate of the scalar energy per unit volume.

The first term in the solution (5.1) decays with time if the initial 2-point function $F_2(s, \rho)$ decays in space. The second term may be interpreted as the lapse of time between moments $s$ and $t$ when the two trajectories starting at distance $\rho_0$ stay in the region of sizable values of the source covariance $\chi$. Its asymptotic behavior depends crucially on what two Lagrangian trajectories do at long times.

### 5.1. Direct versus inverse scalar cascades

In the weakly compressible regime $\varphi < d/\xi^2$, the trajectories separate to a fixed distance in a finite time, Nevertheless, the average time spent by two trajectories within the range of $\chi$ is finite only for $\varphi < \frac{d-2+\xi}{2\xi^2}$ (or $a < 1$). In that case, the scalar 2-point function reaches the stationary form independent of its initial value

$$F_2(\rho) = \int_{-\infty}^t d\sigma \int \chi(\rho) \mathcal{P}_2(t, \rho_0; \sigma, d\rho) = \frac{1}{D_2(d-1)} \int_0^\infty \rho_1^a \int_0^{\rho_1} \rho_2^{a-\xi} \chi(\rho_2) d\rho_2,$$  \hspace{1cm} (5.3)

where the last equality holds for vanishing $\eta$, $L$ and $\kappa$. If $\frac{d-2+\xi}{2\xi^2} \leq \varphi < d/\xi^2$ then, although the trajectories separate to a fixed distance in a finite time, with positive probability they revisit smaller separations. As a result, the average time they spend within the range of $\chi$ and, consequently, the scalar 2-point function diverge when $t \to \infty$. What still reaches the stationary form, however, is the 2-point scalar structure function

$$\mathbf{E} [\theta(t, x) - \theta(t, x + \rho)]^2 \equiv S_2(\rho) \quad \longrightarrow \quad \frac{2}{D_2(d-1)} \int_0^\infty \rho_1^a \int_0^{\rho_1} \rho_2^{a-\xi} \chi(\rho_2) d\rho_2 \quad (5.4)$$
which in the limit exhibits the scaling behavior $\propto \rho^{2-\xi}$ for small $\rho$. In the stationary state, the scalar energy balance reduces to the identity

$$M_2^f(\rho)|_{\rho=0} + \chi(0) = 0$$

(5.5)

expressing the fact that the dissipation and injection balance each other. In particular, the mean dissipation rate $\epsilon$, equal for $\kappa > 0$ to $2\kappa E(\nabla \theta)^2$, is $\kappa$-independent and does not vanish when $\kappa \rightarrow 0$, the phenomenon called the dissipative anomaly. The anomaly, accompanied by the direct scalar energy cascade from the scale on which $\chi(\rho)$ decays (where it is injected) to smaller and smaller scales, is another manifestation of the persistence of the scalar energy dissipation. It is assured by the explosive separation of the Lagrangian trajectories in the whole $\varphi < d/\xi^2$ range.

In the strongly compressible regime $\varphi > d/\xi^2$, the scalar 2-point function does not stabilize but has a constant contribution growing linearly in time with the rate equal to $\chi(0)$ in the $\kappa \rightarrow 0$ limit. No dissipation persists in this limit due to the deterministic character of the Lagrangian trajectories. The injected scalar energy ultimately condenses in the constant mode in the process of the inverse cascade towards longer and longer distances. The 2-point structure function of the scalar, however, reaches the stationary form

$$S_2(\rho) = \frac{2}{D_2(d-1)} \int_0^\rho \rho_1^{-a} d\rho_1 \int_0^\infty \rho_2^{a-\xi} (\chi(0) - \chi(\rho_2)) d\rho_2.$$

(5.6)

with the scaling behaviors $\propto \rho^{1-a}$ for small $\rho$ and $\propto \rho^{2-\xi}$ for large $\rho$.

5.2. Zero mode scenario of intermittency

In the presence of stationary, Gaussian, time decorrelated sources, the higher-point equal-time correlation functions $E \prod_{n=1}^N \theta(t, x_n) \equiv F_N(t; \mathbf{x})$ of scalar satisfy the evolution equations generalizing (5.2):

$$\frac{d}{dt} F_N(t, \mathbf{x}) = M_N F_N(t, \mathbf{x}) + (F_{N-2} \otimes \chi)(t, \mathbf{x}),$$

(5.7)

where

$$(F_{N-2} \otimes \chi)(t, x_1, \ldots, x_N) = \sum_{n<m} F_{N-2}(t, x_1, \ldots, \hat{x}_n, \hat{x}_m) \chi(x_n - x_m).$$

(5.8)

The above relations are solved inductively by the expressions

$$F_N(t, \mathbf{x}) = \int F_N(s, \mathbf{y}) \mathcal{P}_N(t, \mathbf{x}; s, d\mathbf{y}) + \int_s^t d\sigma \int (F_{N-2} \otimes \chi)(\sigma, \mathbf{y}) \mathcal{P}_N(t, \mathbf{x}; \sigma, d\mathbf{y}),$$

(5.9)

compare to (5.1). One expects that for sufficiently weak compressibility those solutions reach stationary form $F_N(\mathbf{x})$ vanishing for odd $N$ and given for even $N$ by the second term on the right hand side with $s = -\infty$. This has been established rigorously for $\varphi = 0$ and all $\kappa \geq 0$ in (Hakulinen, 2002).

An important question concerns the behavior at small $\rho$ of the $\kappa = 0$ stationary state scalar structure functions

$$S_N(t, \rho) = E [\theta(\mathbf{x}) - \theta(\mathbf{x} + \rho)]^N$$

(5.10)
with even $N$. They are given by special combinations of the correlation functions $F_N(x)$. Naive dimensional predictions based on eq. (5.7) would suggested the behavior $S_{2N}(\rho) \propto \rho^{N(2-\xi)}$ since operators $M_N$ have dimension $length^{-\xi}$. This agrees with the scaling of the 2-point function described above and would automatically hold for the higher structure functions if the scalar differences $[\theta(t,x) - \theta(t,x+\rho)]$ were normally distributed. The dimensional predictions, first postulated in (Obukhov, 1949) and (Corrsin, 1951), are in contradiction with experimental turbulent advection measurements which indicate scaling of scalar structure functions with exponents that grow slower than linearly with $N$. Such behavior of the scaling exponents signals more frequent appearance than in normal distributions of large fluctuations of the scalar differences at small separations, the phenomenon called scalar intermittency.

It had been suggested in (Kraichnan, 1994) that the higher scalar structure functions of the Kraichnan model exhibit non-dimensional scaling. It was subsequently realized in (Shraiman & Siggia, 1995), (Gawędzki & Kupiainen, 1995) and (Chertkov, Falkovich, Kolokolov & Lebedev, 1995) that, in the incompressible model, the $\kappa = 0$ higher point functions $F_N(x)$ are dominated at short distances by the contributions from the scaling zero modes $\varphi_N(x)$ of the operators $M_N$ satisfying

$$M_N \varphi_N(x) = 0, \quad \varphi_N(\lambda x) = \lambda^{\xi_N} \varphi_N(x).$$

(5.11)

The scaling dimensions $\zeta_N$ of such modes are not constraint by the dimensional analysis but are accessible perturbatively or numerically (note that the modes annihilated by $M_N$ drop out in the stationary version of eq. (5.11)). The perturbative calculation of such modes gives for even $N$ and general $\varphi$

$$\zeta_N = \frac{N}{2}(2-\xi) - \frac{N(N-2)(1+2\nu)}{2(d+2)} \xi + O(\xi^2).$$

(5.12)

In particular, the zero modes dominate at short distances the structure functions so that

$$S_N(\rho) \propto \rho^{\zeta_N}.$$ 

(5.13)

Such non-dimensional scaling signals the short distance intermittency of scalar advected by the weakly compressible Kraichnan model velocities.

The zero mode dominance of the stationary scalar higher-point functions has been exhibited by the perturbative analysis of the Green functions of operators $M_N$ around $\xi = 0$, $d = \infty$ and $\xi = 2$. The numerical results gave a similar picture for all values of $\xi$, see the figure on the next page representing the values of the 4-point function anomalous exponent $(2\zeta_2 - \zeta_4)$ obtained by Frisch, Mazzino Noullez & Vergassola (1999) in numerical simulations of the three-dimensional (circles) and 2-dimensional (stars) incompressible Kraichnan model.

What is the physical meaning of the zero modes of the operators $M_N$ that dominate the short-distance asymptotics of the scalar $N$-point functions? They are statistically conserved modes of the effective diffusion of Lagrangian trajectories with generators $M_N$. Indeed, the mean value of a translationally-invariant scaling function $\psi_N(x)$ of scaling dimension $\sigma$, viewed as a function of time $t$ positions of $N$ Lagrangian trajectories, is

$$\int \psi_N(y) \, P_N(0,x; t, dy)$$

(5.14)

which, for generic $\psi_N$, grows dimensionally as $O(t^{\sigma/(2-\xi)})$ for large $t$ reflecting the super-diffusive growth $\propto t^{1/(2-\xi)}$ of the distances between the trajectories. But if $\psi_N = \varphi_N$ is a
zero mode of $M_\eta$ then the above expectation is conserved in time (such conserved modes are accompanied by descendent ones whose Lagrangian averages grow slower than dimensionally, see (Bernard, Gawędzki & Kupiainen, 1997).

5.3. Non-intermittency of the inverse cascade

In the strongly compressible phase with the inverse cascade of scalar energy, the behavior of the higher structure functions is different. In fact, only the lower ones stabilize, but the ones that do, scale normally on large distances. In this regime one can find exactly the stationary form of the probability density function of the scalar difference:

$$E \delta \left( \vartheta - \rho^{-\nu} [\vartheta(x) - \vartheta(x + \rho)] \right) \propto [\chi(0) + \text{const.} \vartheta^2]^{-\alpha (\nu)/2 (2 - \xi)}$$

at large distances. Its scaling form indicates that there is no intermittency in the inverse cascade of the scalar (the deviation from the normal distribution is scale-independent). For small $\rho$, however, all the stabilizing structure functions scale as $\rho^{1-a}$ signaling an extreme short distance intermittency.
CONCLUSIONS

As we have seen, the transport of a scalar quantity by velocities distributed according to the Kraichnan ensemble shows two different phases characterized by different direction of the scalar energy cascades and different degrees of intermittency. The phase transition occurs at the value $\varphi = \frac{d}{\xi}$ of the compressibility degree, where the behavior of the Lagrangian trajectories changes drastically from the explosive separation to the implosive aggregation. These two phases are somewhat reminiscent of the behavior of the three-dimensional versus two-dimensional developed turbulence. That suggests that one should put more stress on the Lagrangian methods in studying the latter, not quite a new lesson, see e.g. (Pope, 1994), by with the new twist pointing to the importance of the intrinsically stochastic character of the Lagrangian flow at extreme Reynolds numbers. Of course, the Navier-Stokes and the Euler equations, unlike the scalar advection one, are non-linear, a difference that, certainly, is far from being minor. Also, they describe velocity fields that are temporally correlated and transformed when carried along their own Lagrangian trajectories. Besides, due to pressure, there are non-local interactions present. Some of those effects, however, may be studied already in synthetic velocity ensembles. It seems that the investigation of such ensembles has a potential to teach us important lessons that have to be mastered on the way to an understanding of fully developed turbulence.

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