QUOTIENT CURVES OF THE GK CURVE

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Abstract. For every \( q = \ell^3 \) with \( \ell \) a prime power greater than 2, the GK curve \( \mathcal{X} \) is an \( \mathbb{F}_{q^2} \)-maximal curve that is not \( \mathbb{F}_{q^2} \)-covered by any \( \mathbb{F}_{q^2} \)-maximal Deligne-Lusztig curve. Interestingly, \( \mathcal{X} \) has a very large \( \mathbb{F}_{q^2} \)-automorphism group with respect to its genus. In this paper we compute the genera of a large variety of curves that are Galois-covered by the GK curve, thus providing several new values in the spectrum of genera of \( \mathbb{F}_{q^2} \)-maximal curves.

1. Introduction

Let \( \mathbb{F}_{q^2} \) be a finite field with \( q^2 \) elements where \( q \) is a power of a prime \( p \). An \( \mathbb{F}_{q^2} \)-rational curve, that is a projective, geometrically absolutely irreducible, nonsingular algebraic curve defined over \( \mathbb{F}_{q^2} \), is called \( \mathbb{F}_{q^2} \)-maximal if the number of its \( \mathbb{F}_{q^2} \)-rational points attains the Hasse-Weil upper bound

\[
q^2 + 1 + 2gq,
\]

where \( g \) is the genus of the curve. Maximal curves have interesting properties and have also been investigated for their applications in Coding theory. Surveys on maximal curves are found in \cite{11, 12, 17, 34, 35} and \cite{25, Chapter 10}; see also \cite{9, 10, 16, 29, 31}.

One of the most important problems on maximal curves is the determination of the possible genera of maximal curves over \( \mathbb{F}_{q^2} \), see e.g. \cite{11}. For a given \( q \), the highest value of \( g \) for which an \( \mathbb{F}_{q^2} \)-maximal curve of genus \( g \) exists is \( q(q - 1)/2 \) \cite{26}, and equality holds if and only if the curve is the Hermitian curve with equation

\[
X^{q+1} = Y^q + Y,
\]

up to \( \mathbb{F}_{q^2} \)-birational equivalence \cite{29}.

By a result of Serre, see \cite{27, Prop. 6}, any \( \mathbb{F}_{q^2} \)-rational curve which is \( \mathbb{F}_{q^2} \)-covered by an \( \mathbb{F}_{q^2} \)-maximal curve is also \( \mathbb{F}_{q^2} \)-maximal. This has made it possible to obtain several genera of \( \mathbb{F}_{q^2} \)-maximal curves by applying the Riemann-Hurwitz formula, especially from the Hermitian curve, see \cite{21, 4, 11, 4, 7, 8, 14, 15, 20, 18, 21, 22}. Others have been obtained from the DLS and DLR curves, see \cite{24, 5, 6, 28}.

The problem of the existence of \( \mathbb{F}_{q^2} \)-maximal curves other than \( \mathbb{F}_{q^2} \)-subcovers of the Hermitian curve, the DLS curve, and the DLR curve was solved in \cite{23}, where for every \( q = \ell^3 \) with \( \ell = p^r > 2 \), \( p \) prime, an \( \mathbb{F}_{q^2} \)-maximal curve \( \mathcal{X} \) that is not \( \mathbb{F}_{q^2} \)-covered by any \( \mathbb{F}_{q^2} \)-maximal Deligne-Lusztig curve was described. Throughout

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\end{itemize}
the paper we will refer to $X$ as to the GK curve. It should be noted that the construction in [23] has been recently generalized in [13]; it is still an open problem to determine whether these generalizations of the GK curve are $F_{q^2}$-subcovers of a Deligne-Lusztig curve or not.

One of the most interesting features of the GK curve $X$ is its very large automorphism group with respect to its genus. In this paper we consider quotient curves $X/L$ under the action of a large variety of subgroups $L$ of $\text{Aut}(X)$. By applying the Riemann-Hurwitz formula to the covering $X \to X/L$ a large number of genera of maximal curves is obtained, see Theorems 4.5, 4.6, 5.4, 6.2, 6.3, 6.4, 6.5, 6.6, 7.2, 7.3, 7.7. It should be noted that when $L$ is tame and contains the centrum $\Lambda$ of $\text{Aut}(X)$, then the quotient curve $X/L$ has the same genus of the quotient curve of the Hermitian curve $H$ over $F_{\ell^2}$ under the action of the factor group $L/\Lambda$, see Corollary 3.4. Apart from these cases, formulas for genera of quotient curves $X/L$ appear to provide new values in the spectrum of genera of $F_{q^2}$-maximal curve, cf. Section 8. One of our main tools for the investigation of the tame case is a relationship between the genus of $X/L$ and that of the quotient curve of $H$ with respect to the factor group $L/(L \cap \Lambda)$, see Section 3.

2. THE GK CURVE AND ITS AUTOMORPHISM GROUP

Throughout this paper, $p$ is a prime, $\ell = p^h$ and $q = \ell^3$ with $h \geq 1$, $\ell > 2$. Furthermore, $\mathbb{K}$ denotes the algebraic closure of $\mathbb{F}_{q^2}$.

Let

$$h(X) = \sum_{i=0}^{n} (-1)^{i+1} X^{i(n-1)}.$$  

In the three–dimensional projective space $\text{PG}(3, q^2)$ over $\mathbb{F}_{q^2}$, consider the algebraic curve $X$ defined to be the complete intersection of the surface with affine equation

$$Z^{\ell^2-\ell+1} = Xh(Y),$$

and the Hermitian cone with affine equation

$$Y^{\ell} + Y = X^{\ell+1}.$$ 

Note that $X$ is defined over $\mathbb{F}_{q^2}$ but it is viewed as a curve over $\mathbb{K}$. Moreover, $X$ has degree $\ell^3 + 1$ and possesses a unique infinite point, namely the infinite point $P_\infty$ of the $Y$-axis.

**Theorem 2.1 ([23]).** $X$ is an $\mathbb{F}_{q^2}$-maximal curve with genus $g = \frac{1}{2}((\ell^3+1)(\ell^2-2)+1$.

For notation, terminology and basic results on automorphism groups of curves, we refer to [25] Chapter 11].
For every \( u \in \mathbb{K} \), with \( u \neq 0 \), consider the collineation \( \alpha_u \) of \( \text{PG}(3, \mathbb{F}_{q^2}) \) defined by
\[
\alpha_u : (X, Y, Z, T) \mapsto (uX, uY, Z, uT).
\]
For \( u^{\ell^2+1} = 1 \), \( \alpha_u \) defines an \( \mathbb{F}_{q^2} \)-automorphism of \( \mathcal{X} \). For \( u \neq 1 \), the fixed points of \( \alpha_u \) are exactly the points of the plane \( \pi_0 \) with equation \( Z = 0 \). Since \( \pi_0 \) contains no tangent to \( \mathcal{X} \), the number of fixed points of \( \alpha_u \) with \( u \neq 1 \) is independent from \( u \) and equal to \( \ell^3 + 1 \). Let \( \Lambda = \{ \alpha_u | u^{\ell^2+1} = 1 \} \).

**Theorem 2.2** ([23]). The group \( \Lambda \) is a central subgroup of \( \text{Aut}(\mathcal{X}) \). The quotient curve \( \mathcal{X}/\Lambda \) is the Hermitian curve \( \mathcal{H} \) over \( \mathbb{F}_{2^2} \) with equation \( X^\ell+1 = Y^\ell + Y \). The factor group \( \text{Aut}(\mathcal{X})/\Lambda \) is isomorphic to \( \text{PGU}(3, \ell) \).

If the non-degenerate Hermitian form in the three dimensional vector space \( V(3, \ell^2) \) over \( \mathbb{F}_{2^2} \) is given by \( Y^\ell T + YT^\ell - X^\ell+1 \) then the unitary group \( U(3, \ell) \) is the subgroup of \( \text{GL}(3, \ell^2) \) whose elements \( U = (u_{ij}) \) are determined by the condition \( U^t D \sigma(U) = D \) where
\[
D = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]
and \( \sigma(U) = (u_{ij}^\ell) \). \( U(3, \ell) \) has order \((\ell + 1)(\ell^3 + 1)\ell^2(\ell^2 - 1)\). A diagonal matrix \([u, u, u]\) is in \( U(3, \ell) \) if and only if \( u^{\ell+1} = 1 \), and such matrices form a cyclic subgroup \( C \) of \( U(3, \ell) \).

The (normal) subgroup \( \text{SU}(3, \ell) \) is the subgroup of \( U(3, \ell) \) of index \( \ell + 1 \) consisting of all matrices with determinant 1. A set of generators of \( \text{SU}(3, \ell) \) are given by the following matrices:

For \( b, c \in \mathbb{F}_{2^2} \) such that \( c^\ell + c - b^{\ell+1} = 0 \), and for \( a \in \mathbb{F}_{2^2}, a \neq 0 \),
\[
Q_{(b,c)} = \begin{pmatrix}
1 & 0 & b \\
b^\ell & 1 & c \\
0 & 0 & 1
\end{pmatrix}, \quad R_a = \begin{pmatrix}
a^{-n} & 0 & 0 \\
0 & a^{n-1} & 0 \\
0 & 0 & a
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

\( \text{SU}(3, \ell) \cap C \) is either trivial or is a subgroup of order 3, according as \( \gcd(3, \ell+1) = 1 \) or 3. The center \( Z(U(3, \ell)) \) coincides with \( C \), and \( Z(\text{SU}(3, \ell)) = \text{SU}(3, \ell) \cap C \). In this context, \( \text{PGU}(3, \ell) = U(3, \ell)/C \). A treatise on unitary groups can be found in [33] Section 10].

From each \( U \in U(3, \ell) \) a \((4 \times 4)\)-matrix \( \bar{U} \) arises by adding 0, 0, 1, 0 as a third row and as a third column. Obviously, these matrices \( \bar{U} \) with \( U \in \text{SU}(3, \ell) \) form a subgroup \( \Gamma \) of \( \text{GL}(4, \ell^2) \) isomorphic to \( \text{SU}(3, \ell) \). Since the identity matrix is the only scalar matrix in \( \Gamma \), we can regard \( \Gamma \) as a projective group in \( \text{PGL}(4, \mathbb{F}_{q^2}) \).

It is shown in [23] that the group \( \Gamma \) preserves \( \mathcal{X} \), \( \Lambda \) centralizes \( \Gamma \), and \( \Gamma \cap \Lambda \) is trivial when \( \gcd(3, \ell + 1) = 1 \) while it has order 3 when \( \gcd(3, \ell + 1) = 3 \). Let \( \Lambda_3 \)
be the unique subgroup of $\Lambda$ of order $\frac{\ell^2 - \ell + 1}{3}$. Then by \cite[Lemma 8]{23} $\text{Aut}(\mathcal{X})$ has a subgroup $\Xi$ with

\begin{equation}
\Xi = \begin{cases}
\Gamma \times \Lambda, & \text{when } \gcd(3, \ell + 1) = 1; \\
\Gamma \times \Lambda_3, & \text{when } \gcd(3, \ell + 1) = 3.
\end{cases}
\end{equation}

When $\gcd(3, \ell + 1) = 1$, $\text{Aut}(\mathcal{X}) = \Xi$ holds (see \cite[Thm. 6 (i)]{23}), whereas for $\gcd(3, \ell + 1) = 3$, $\mathcal{X}$ has further $\mathbb{F}_{q^2}$-automorphisms. Let $\rho$ be a primitive $(\ell^3 + 1)$-st root of unity in $\mathbb{F}_{q^2}$, and let $	ilde{E}$ be the diagonal matrix $[\rho^{-1}, \rho^{\ell^2 - \ell}, 1, \rho^{-1}]$. Then $\tilde{E}$ preserves $\mathcal{X}$, normalizes $\Gamma$ and commutes with $\Lambda$. Moreover, $\tilde{E} \notin \Xi$ but $\tilde{E}^3 \in \Xi$.

By \cite[Thm. 6 (ii)]{23}, if $\gcd(3, n + 1) = 3$ then $[\text{Aut}(\mathcal{X}) : \Xi] = 3$ and $\Xi$ is a normal subgroup of $\text{Aut}(\mathcal{X})$. Moreover, $\Gamma$ is a normal subgroup of $\text{Aut}(\mathcal{X})$.

Both $\Gamma$ and $\Lambda$ preserve the set of points lying in the plane of equation $Z = 0$.

**Theorem 2.3** (\cite{23}). The set of $\mathbb{F}_{q^2}$-rational points of $\mathcal{X}$ splits into two orbits under the action of $\text{Aut}(\mathcal{X})$, one is non-tame, has size $\ell^3 + 1$, and consists of the $\mathbb{F}_{q^2}$-rational points on $\mathcal{X}$; the other is tame of size $\ell^3((\ell^3 + 1)(\ell^2 - 1))$. Furthermore, $\text{Aut}(\mathcal{X})$ acts on the non-tame orbit as $\text{PGU}(3, \ell)$ in its doubly transitive permutation representation.

Henceforth, the orbit of size $\ell^3 + 1$ will be denoted as $\mathcal{O}_1$, whereas the orbit of size $\ell^3((\ell^3 + 1)(\ell^2 - 1))$ by $\mathcal{O}_2$. Moreover, the natural projection from $\text{Aut}(\mathcal{X})$ to $\text{PGU}(3, \ell)$ will be denoted by $\pi$. Let $\phi$ be the rational map $\phi : \mathcal{X} \to \mathcal{H}$ defined by $\phi(1 : x : y : z) = (1 : x : y)$.

For a subgroup $L$ of $\text{Aut}(\mathcal{X})$, let $L$ be the subgroup $\pi(L)$ of $\text{PGU}(3, \ell)$.

Throughout the paper we will refer to the following maximal subgroups, defined up to conjugacy, of the group $\text{PGU}(3, \ell)$, viewed as the group of the projectivities of $\mathbb{P}^2(\mathbb{K})$ preserving the Hermitian curve $\mathcal{H}$.

(A) The stabilizer of an $\mathbb{F}_{q^2}$-rational point, of size $\ell^3(\ell^2 - 1)$.
(B) The normalizer of a Singer group, of size $3(\ell^2 - \ell + 1)$. Here a Singer group of $\text{PGU}(3, \ell)$ is a cyclic group of size $\ell^2 - \ell + 1$ stabilizing a point in $\mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{q^2})$.
(C) The self-conjugate triangle stabilizer, of size $6(\ell + 1)^2$.
(D) The non-tangent line stabilizer, of size $\ell(\ell + 1)(\ell^2 - 1)$.

3. Preliminary results

Let $L$ be a subgroup of $\text{Aut}(\mathcal{X})$. Let $\mathcal{X}/L$ be the quotient curve of $\mathcal{X}$ with respect to $L$, and let $g_L$ be its genus. If $L$ is tame, that is $p$ does not divide the order of $L$, then the Hurwitz genus formula for Galois extensions gives

\begin{align}
(6) \quad (\ell^3 + 1)(\ell^2 - 2) = |L|(2g_L - 2) + e_L
\end{align}

with

\begin{align}
(7) \quad e_L = \sum_{P \in \mathcal{X}} (|L_P| - 1),
\end{align}
where $L_P$ is the stabilizer of $P$ in $L$. The aim of this section is to provide a formula which relates $e_L$ to the action of $L$ on the Hermitian curve $\mathcal{H}$, see Proposition 3.1 below. Let $L_\Lambda = L \cap \Lambda$. The factor group $L/L_\Lambda$ is isomorphic to $\bar{L}$, and the action of $L/L_\Lambda$ on the orbits of $\mathcal{X}$ under $\Lambda$ is isomorphic to that of $\bar{L}$ on $\mathcal{H}$.

As to the relation between $e_L$ and the analogous value $\sum_{P \in \mathcal{H}} (|L_P| - 1)$, for $\bar{L}$, by standard arguments from permutation group theory it is not difficult to prove that

$$
(8) \quad \sum_{P \in \mathcal{X}} (|L_P| - 1) \cdot m_P = |L_\Lambda| \left( \sum_{P \in \mathcal{H}} (|\bar{L}_P| - 1) \cdot |\phi^{-1}(\bar{P})| \right) - \sum_{P \in \mathcal{X}} (m_P - |L_\Lambda|),
$$

where $m_P$ denotes the size of the orbit of $P$ under the action of the subgroup of $L$ stabilizing the set $\phi^{-1}(\phi(P))$. However, we will not use (8), as this would require involved computations on $m(P)$. As it has emerged from the literature, a more adequate approach is based on the equality

$$
(9) \quad e_L = \sum_{h \in L, h \neq id} N_h, \quad \text{where } N_h = |\{ P \in \mathcal{X} | h(P) = P \}|
$$

(cf. [20, Eq. 4.7]).

The ramification points of the morphism $\phi : \mathcal{X} \to \mathcal{H}$ are exactly the points in $\mathcal{O}_1$. At these points $\phi$ is fully ramified. The set $\bar{\mathcal{O}}_1$ of the images of the points in $\mathcal{O}_1$ by $\phi$ in $\mathcal{H}$ is precisely the set $\mathcal{H}(\mathbb{F}_\ell)$ of the $\mathbb{F}_\ell$-rational points of $\mathcal{H}$, whereas the image $\bar{\mathcal{O}}_2$ of $\mathcal{O}_2$ by $\phi$ coincides with $\mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_\ell)$. Any point of $\mathcal{H}$ fixed by some non-trivial element in $PGU(3, \ell)$ lies in $\bar{\mathcal{O}}_1 \cup \bar{\mathcal{O}}_2$, see e.g. [20, Prop. 2.2].

In order to compute $e_L$ as in (9), it is convenient to write

$$
(10) \quad N_h = N_h^{(1)} + N_h^{(2)}
$$

with

$$
N_h^{(1)} = |\{ P \in \mathcal{O}_1 | h(P) = P \}|, \quad N_h^{(2)} = |\{ P \in \mathcal{O}_2 | h(P) = P \}|.
$$

**Proposition 3.1.** Let $L$ be a subgroup of $\text{Aut}(\mathcal{X})$, and let $L_\Lambda = L \cap \Lambda$. Let $\bar{\mathcal{O}}_1 = \mathcal{H}(\mathbb{F}_\ell)$ and $\bar{\mathcal{O}}_2 = \mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_\ell)$. Then

$$
e_L = (|L_\Lambda| - 1)(\ell^a + 1) + |L_\Lambda|n_1 + |L_\Lambda|n_2
$$

where

- $n_1$ counts the non-trivial relations $\bar{h}(\bar{P}) = \bar{P}$ with $\bar{h} \in \bar{L}$ when $\bar{P}$ varies in $\bar{\mathcal{O}}_1$, namely

$$
n_1 = \sum_{\bar{h} \in L, \bar{h} \neq id} |\{ \bar{P} \in \bar{\mathcal{O}}_1 | \bar{h}(\bar{P}) = \bar{P} \}|;
$$

- $n_2$ counts the non-trivial relations $\bar{h}(\bar{P}) = \bar{P}$ with $\bar{h} \in \bar{L}$ when $\bar{P}$ varies in $\bar{\mathcal{O}}_2$, each counted with a multiplicity $l_{\bar{h}, \bar{P}}$ defined as the number of orbits of
\[ \phi^{-1}(P) \text{ under the action of } L_\Lambda \text{ that are fixed by an element } h \in \pi^{-1}(\bar{h}). \text{ That is,} \]

\[ n_2 = |L_\Lambda| \sum_{h \in L, h \neq \text{id}} \sum_{P \in \mathcal{O}_2, h(P) = \bar{P}} l_{\bar{h}, P}. \]

**Proof.** Note that

\[ (11) \sum_{h \in L, h \neq \text{id}} N_h = \sum_{k \in L_\Lambda, k \neq \text{id}} N_k + \sum_{h \in L, h \neq \text{id}} N_{hk}, \]

where \( h \in L \) is an element such that \( \pi(h) = \bar{h} \). For each non-trivial element \( k \in L_\Lambda \), \( N_k = |O_1| = (\ell^3 + 1) \) holds. Therefore

\[ (12) \sum_{k \in L_\Lambda, k \neq \text{id}} N_k = (|L_\Lambda| - 1)(\ell^3 + 1). \]

As to the second term in the right hand side of (11), write \( N_h = N_{hk}^{(1)} + N_{hk}^{(2)} \), with \( N_{hk}^{(i)} \) as in (10). As \( k(P) = P \) for each \( P \in \mathcal{O}_1 \), we have

\[ (13) \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} \sum_{k \in L_\Lambda} N_{hk}^{(1)} = |L_\Lambda| \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} |\{ P \in \mathcal{O}_1 \mid \bar{h}(P) = \bar{P} \}|. \]

It remains to compute a sum \( \sum_{h \in L, h \neq \text{id}} \sum_{k \in L_\Lambda} N_{hk}^{(2)}. \) Since \( \phi(k(P)) = \phi(P) \) for each \( P \in \mathcal{X} \), condition \((hk)(P) = P\) yields that \( \bar{h}(\phi(P)) = \phi(P) \). Therefore,

\[ \sum_{h \in L, h \neq \text{id}} \sum_{k \in L_\Lambda} N_{hk}^{(2)} = \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} \sum_{P \in \mathcal{O}_2, h(P) = \bar{P}} \sum_{k \in L_\Lambda} m_{k, \bar{h}, P} \]

where \( m_{k, \bar{h}, P} = |\{ P \in \mathcal{X}, \pi(P) = \bar{P}, (hk)(P) = P \}|. \) By the orbit-stabilizer theorem \( \sum_{k \in L_\Lambda} m_{k, \bar{h}, P} = |L_\Lambda| l_{\bar{h}, P} \), whence

\[ \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} \sum_{k \in L_\Lambda} N_{hk}^{(2)} = |L_\Lambda| \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} \sum_{\bar{P} \in \mathcal{O}_2, \bar{h}(\bar{P}) = \bar{P}} l_{\bar{h}, \bar{P}}. \]

Taking into account (9), (11), (12), (13), this finishes the proof. \( \square \)

The following corollary to Proposition 3.1 will be useful in the sequel.

**Proposition 3.2.** Let \( L \) be a tame subgroup of \( \text{Aut}(\mathcal{X}) \). Assume that no non-trivial element in \( \bar{L} \) fixes a point in \( \mathcal{H} \setminus \mathcal{O}_1 \). Then

\[ g_L = g_{\bar{L}} + \frac{(\ell^3 + 1)(\ell^2 - |L_\Lambda| - 1) - |L_\Lambda|(|L_\Lambda| - \ell - 2)}{2|L|}, \]

where \( g_L \) is the genus of the quotient curve \( \mathcal{H}/\bar{L} \).
Proof. By \([\text{6]}\) and Proposition \([3.1]\),
\[
(\ell^3 + 1)(\ell^2 - 2) = |L|(2g_L - 2) + (|L\Lambda| - 1)(\ell^3 + 1) + |L\Lambda| \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} |\{\bar{P} \in \mathcal{O}_1 \mid \bar{h}(\bar{P}) = \bar{P}\}|.
\]

On the other hand, by the Hurwitz genus formula applied to the covering \(\mathcal{H} \rightarrow \mathcal{H}/\bar{L}\)
\[
\sum_{\bar{h} \in L, \bar{h} \neq \text{id}} |\{\bar{P} \in \mathcal{O}_1 \mid \bar{h}(\bar{P}) = \bar{P}\}| = (\ell^2 - \ell - 2) - |\bar{L}|(2g_L - 2),
\]
whence
\[
(\ell^3 + 1)(\ell^2 - 2) = |L|(2g_L - 2) + (|L\Lambda| - 1)(\ell^3 + 1) + |L\Lambda|(\ell^2 - \ell - 2) - |L|(2g_L - 2).
\]

Then the claim follows by straightforward computation. \(\square\)

When \(\Lambda \subseteq L\), \(l_{\bar{h}, \bar{P}} = 1\) for every \(\bar{h} \in \bar{L}\), and for every \(\bar{P} \in \mathcal{O}_2\) with \(\bar{h}(\bar{P}) = \bar{P}\). Therefore Proposition \([3.1]\) reads as follows.

**Corollary 3.3.** Let \(L\) be a subgroup of \(\text{Aut}(\mathcal{X})\) containing \(\Lambda\). Then
\[
e_L = (\ell^2 - \ell)(\ell^3 + 1) + (\ell^2 - \ell + 1) \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} |\{\bar{P} \in \mathcal{H}(\mathbb{F}_{q^2}) \mid \bar{h}(\bar{P}) = \bar{P}\}|.
\]

We end this section with a result showing that if \(L\) is tame and \(\Lambda \subset L\), then the genus of \(g_L\) is actually the genus of a quotient curve of the Hermitian curve \(\mathcal{H}\).

**Corollary 3.4.** Let \(L\) be a tame subgroup of \(\text{Aut}(\mathcal{X})\) containing \(\Lambda\). Then \(g_L\) coincides with the genus of the quotient curve \(\mathcal{H}/\bar{L}\).

**Proof.** Let \(g_{\mathcal{H}} = \ell(\ell - 1)/2\) be the genus of \(\mathcal{H}\), and let \(g_L\) be the genus of the quotient curve \(\mathcal{H}/\bar{L}\). Then by straightforward computation
\[
(\ell^3 + 1)(\ell^2 - 2) = (\ell^2 - \ell + 1)(2g_{\mathcal{H}} - 2) + (\ell^2 - \ell)(\ell^3 + 1).
\]

Since \(\bar{L}\) is tame,
\[
2g_{\mathcal{H}} - 2 = |\bar{L}|(2g_L - 2) + \sum_{\bar{h} \in L, \bar{h} \neq \text{id}} |\{\bar{P} \in \mathcal{H}(\mathbb{F}_{q^2}) \mid \bar{h}(\bar{P}) = \bar{P}\}|.
\]

Note that \(|L| = (\ell^2 - \ell + 1)|\bar{L}|\). Taking into account \([\text{6}]\) and Corollary \([3.3]\) \(g_L = g_{\bar{L}}\) follows by straightforward computation. \(\square\)

4. **Curves \(\mathcal{X}/L\) with \(\bar{L}\) subgroup of a group of type \((A)\)**

In this section subgroups \(L\) of \(\text{Aut}(\mathcal{X})\) stabilizing a point \(P \in \mathcal{O}_1\) are investigated. Up to conjugacy, we may assume that \(L\) is contained in the stabilizer \(\text{Aut}(\mathcal{X})_{P_\infty}\) of \(P_\infty\) in \(\text{Aut}(\mathcal{X})\). By the orbit-stabilizer theorem the size of \(\text{Aut}(\mathcal{X})_{P_\infty}\) is \(\ell^3(\ell^2 - 1)(\ell^2 - \ell + 1)\). Since \(\text{Aut}(\mathcal{X})_{P_\infty}\) is non-tame, in order to determine the genus of \(\mathcal{X}/L\) we will use Hilbert’s ramification theory, see \([30]\) Ch. III.8.
Let $G$ be a subgroup of $\text{Aut}(\mathcal{X})$ and let $P$ be a point of $\mathcal{X}$. For an integer $i \geq -1$ the $i$-th ramification group of $G$ at $P$ is

$$G_i(P) = \{ h \in G \mid v_{P}(h^*(t) - t) \geq i + 1 \},$$

where $h^* \in \text{Aut}(\mathbb{K}(\mathcal{X}))$ is the pullback of $h$, $v_P$ is the discrete valuation of $\mathbb{K}(\mathcal{X})$ associated to $P$, and $t$ is any $P$-prime element. The group $G_0(P)$ coincides with the stabilizer $\text{Aut}(\mathcal{X})_P$, whereas $G_1(P)$ is the only $p$-Sylow subgroup of $G_0(P)$, see e.g. [30, Prop. III.8.5]. Moreover, there exists a cyclic group $H$ in $G_0(P)$ such that $G_0(P) = G_1(P) \rtimes H$, see [25, Thm. 11.49]. The Hurwitz genus formula together with the Hilbert different formula (see e.g. [30, Thm. III.8.7]) gives

$$2g - 2 = |G|(2g_C - 2) + \sum_{P \in \mathcal{X}} d_P,$$

where $g_C$ denotes the genus of the quotient curve $\mathcal{X}/G$.

Assume that $G = G_0(P_\infty)$, that is every element in $G$ fixes $P_\infty$. Since $\mathcal{X}$ is a maximal curve, no $p$-element in $G$ can fix a point $P$ different from $P_\infty$, see [25, Thm. 9.76] and [25, Thm. 11.133]. Therefore for any $P \neq P_\infty$ the integer $d_P$ in (16) coincides with $G_0(P) - 1$. The following result then holds.

**Lemma 4.1.** For a subgroup $L$ of $\text{Aut}(\mathcal{X})_{P_\infty}$, let $g_L$ be the genus of the quotient curve $\mathcal{X}/L$. Then

$$(\ell^3 + 1)(\ell^2 - 2) = |L|(2g_L - 2) + e_L + \sum_{i=1}^{\infty} (|L_i(P_\infty)| - 1),$$

with $e_L$ as in (7).

We now provide an explicit description of $\text{Aut}(\mathcal{X})_{P_\infty}$. For $a \in \mathbb{F}_{q^2}$ such that $a^{(\ell^2-\ell+1)(\ell^2-1)} = 1$, for $b, c \in \mathbb{F}_{q^2}$ such that $c^\ell + c = b^{\ell+1}$, let $\xi_{a,b,c}$ be the projectivity in $\text{PGL}(4, q^2)$ defined by the matrix

$$\begin{pmatrix}
  a^{\ell^2-\ell+1} & 0 & 0 & b \\
  b^\ell a^{\ell^2-\ell+1} & a^{\ell+1} & 0 & c \\
  0 & 0 & a^{\ell^2} & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}.$$ 

It is easily seen that $\xi_{a,b,c} = \xi_{a,0,0} \xi_{1,b,c}$, with $\xi_{1,b,c} \in \Gamma \cap \text{Aut}(\mathcal{X})_{P_\infty}$. Also, by straightforward computation $\xi_{a,0,0}$ lies in $\text{Aut}(\mathcal{X})_{P_\infty}$. By a trivial counting argument, we then have

$$\text{Aut}(\mathcal{X})_{P_\infty} = \{ \xi_{a,b,c} \mid a^{(\ell^2-\ell+1)(\ell^2-1)} = 1, b, c \in \mathbb{F}_{q^2}, b^{\ell+1} = c^\ell + c \}.$$ 

The elements $\xi_{1,b,c}$ form a subgroup of $\text{Aut}(\mathcal{X})_{P_\infty}$ of size $\ell^3$, therefore the first ramification group of $\text{Aut}(\mathcal{X})_{P_\infty}$ at $P_\infty$ is

$$\{ \xi_{1,b,c} \mid b, c \in \mathbb{F}_{q^2}, b^{\ell+1} = c^\ell + c \}.$$
In order to determine higher ramification groups at $P_\infty$ we need to compute the integer $v_{P_\infty}(h^*(t) - t)$, with $t$ a $P_\infty$-prime element, for automorphisms $h = \xi_{1,b,c}$. By [23, Sect. 4]

$$v_{P_\infty}(x) = -(\ell^3 - \ell^2 + \ell), \quad v_{P_\infty}(y) = -((\ell^3 + 1), \quad v_{P_\infty}(z) = -\ell^3.$$

Therefore, $t$ can be assumed to be the rational function $z/y$. Since

$$\xi_{1,b,c}^*(z) = z, \quad \xi_{1,b,c}^*(y) = y + b^\ell x + c$$

we have that

$$\xi_{1,b,c}^*(z/y) - (z/y) = \frac{c - zb^\ell}{y(y + b^\ell x + c)},$$

whence

$$v_{P_\infty}(\xi_{1,b,c}^*(t) - t) = \begin{cases} \ell^2 - \ell + 2 & \text{if } b \neq 0 \\ \ell^3 + 2 & \text{if } b = 0. \end{cases}$$

The following result is then obtained.

**Proposition 4.2.** Let $L$ be a subgroup of $\text{Aut}(X)_{P_\infty}$. Then

$$L_1(P_\infty) = L_2(P_\infty) = \ldots = L_{\ell^2 - \ell + 1}(P_\infty) = \{\xi_{1,b,c} \mid \xi_{1,b,c} \in L\},$$

and

$$L_{\ell^2 - \ell + 2}(P_\infty) = L_{\ell^2 - \ell + 3}(P_\infty) = \ldots = L_{\ell^3 + 1}(P_\infty) = \{\xi_{1,0,c} \mid \xi_{1,0,c} \in L\}.$$  

For $i > \ell^2 + 1$ the group $L_i(P_\infty)$ is trivial.

As to the computation of $e_L$ in Lemma 4.1, the following fact will be useful.

**Lemma 4.3.** Let $L$ be a subgroup of $\text{Aut}(X)_{P_\infty}$. Then any point of $X$ which is fixed by a non-trivial element in $L$ belongs to $O_1$.

**Proof.** Assume that $\alpha \in L$ fixes a point $P \in X \setminus O_1$. Then $\pi(\alpha)$ is an element of $PGU(3, \ell)$ fixing both the infinite point $P_\infty$ of $H$ and the point $\phi(P)$ in $H \setminus H(\mathbb{F}_\ell)$. Then by [20, Sect. 4] $\pi(\alpha)$ is trivial, that is $\alpha \in \Lambda$. Since any non-trivial element in $\Gamma$ only fixes points in $O_1$, we obtain $\alpha = \text{id}$. \hfill $\square$

In Section 4.1 we will deal with the case $L = \Sigma_1 \times \Sigma_2$, where $\Sigma_1$ is contained in $\Gamma$ and $\Sigma_2$ is a subgroup of $\Lambda$, see Section 4.1. To this end, we determine a subgroup $\Omega$ of $\Gamma \cap \text{Aut}(X)_{P_\infty}$ such that $\Omega \cap \Lambda = \{\text{id}\}$. In Section 4.2, the case $L = \pi^{-1}(\hat{G})$ with $\hat{G}$ a group of type (A) will be dealt with.

Let $\mu_1$ be the highest power of 3 dividing $\ell + 1$. Let

$$\Omega = \{\xi_{a,b,c} \mid a^{\ell^2 - 1} = 1\}.$$

Assume that $\alpha = \xi_{a,b,c} \in \Omega \cap \Lambda$. Then clearly $b = c = 0$ holds, whence $\alpha = \xi_{a,0,0}$ for some $a$ such that $a^{(\ell^2 - 1)/\mu_1} = 1$. Note that $\xi_{a,0,0} \in \Lambda$ implies $a^{\ell^2 - \ell + 1} = 1$. But then $a = 1$ since $\gcd((\ell^2 - 1)/\mu_1, \ell^2 - \ell + 1) = 1$. Then the following holds.
4.1. Let $L = \Sigma_1 \times \Sigma_2$, $\Sigma_1 < \Omega$, $\Sigma_2 < \Lambda$. Let $|\Sigma_1| = mp^{v+w}$, where 

$$p^{v+w} = |\Sigma_1 \cap \{\xi_{1,b,c}\}|, \quad p^v = |\Sigma_1 \cap \{\xi_{1,0,0}\}|,$$

and $m$ is a divisor of $(\ell^2 - 1)/\mu_1$. Let $|\Sigma_2| = d_2$. Then by Lemma 4.1 together with Proposition 3.1 and Lemma 4.3 it follows that 

$$\ell^5 - 2\ell^3 + \ell^2 - 2 = md_2p^{v+w}(2g_L - 2) + (\ell^2 - \ell + 1)(p^{v+w} - 1) + (\ell^3 - \ell^2 + \ell)(p^w - 1)$$

$$+ (d_2 - 1)(\ell^3 + 1) + d_2 \sum_{h \in \pi(\Sigma_1), h \neq id} |\{\bar{P} \in \mathcal{O}_1 | \bar{h}(\bar{P}) = \bar{P}\}|,$$

that is 

$$\ell^5 - 2\ell^3 + \ell^2 - 2 = (2g_L - 1)md_2p^{v+w} = (\ell^3 + 1)(\ell^2 - d_2) - (\ell^2 - \ell + 1)p^w(p^v + \ell) + d_2$$

$$- d_2 \sum_{h \in \pi(\Sigma_1), h \neq id} |\{\bar{P} \in \mathcal{O}_1, \bar{P} \neq \bar{P}_\infty | \bar{h}(\bar{P}) = \bar{P}\}|,$$

(17) 

In order to provide concrete values of genera $g_L$ we are going to consider subgroups of $PGU(3, \ell)\bar{\mathcal{P}}_\infty$ that are known in the literature, see [20] and [2]. To this end it is useful to note that in both papers [20] and [2] the group $PGU(3, \ell)\bar{\mathcal{P}}_\infty$ is denoted as $\mathcal{A}(P_\infty)$, and that the subgroup $\pi(\Omega)$ of $PGU(3, \ell)$ in the notation of both [20] and [2] is the subgroup of index $\mu_1$ in $\mathcal{A}(P_\infty)$ consisting of elements $[a, b, c]$ with $a^{(\ell^2 - 1)/\mu_1} = 1$. It should also be noted that for each $\Sigma_1 < \Omega$ the integer $\sum_{h \in \pi(\Sigma_1), h \neq id} |\{\bar{P} \in \mathcal{O}_1, \bar{P} \neq \bar{P}_\infty | \bar{h}(\bar{P}) = \bar{P}\}|$ is computed in [20]. From (17), together with Theorem 4.4 in [20], it follows that 

$$g_L = \frac{\ell^5 + \ell^2 - (\ell^2 - \ell + 1)p^w(p^v + \ell) - d_2(\ell^3 + (d_1 - 1)p^v - dp^{v+w})}{2md_2p^{v+w}},$$

where $d = \gcd(m, \ell + 1)$. The following result then holds.

**Theorem 4.5.** Let $d_2$ be any divisor of $\ell^2 - \ell + 1$.

(i) Let $p \neq 2$ and $m \mid \ell^2 - 1$ be such that 3 does not divide $m$ and $m > 1$. Let $d = \gcd(m, l + 1)$ and let $s := \min\{r \geq 1 : p^r \equiv 1 \mod \frac{m}{2}\}$. For each $0 \leq w \leq h$, such that $s \mid w$, there exists a subgroup $L$ of $\operatorname{Aut}(X)$ with 

$$g_L = \frac{(\ell^3 + 1)(\ell^2 - p^w) - (\ell^3 - \ell)(d_2 - dd_2)(l - p^w)}{2md_2p^w},$$

(cf. [20] Prop. 4.6).

(ii) Let $p \neq 2$ and $m \mid \ell - 1$. Let $d = \gcd(m, l + 1)$. Let $s$ be the order of $p$ in $(\mathbb{Z}/m\mathbb{Z})^*$, and let 

$$r = \begin{cases} \text{order of } p \text{ in } (\mathbb{Z}/m\mathbb{Z})^*, & \text{m even} \\ s, & \text{m odd} \end{cases}$$
For each $0 \leq v \leq h$ such that $s \mid v$, and for each $0 \leq w \leq h - 1$ such that $r \mid w$, there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with
\[
g_L = \frac{l^5 + l^2 - l^3d_2 - (l^2 - l + 1 - dd_2)p^{v+w} - (l^3 - l^2 + l)p^w - d_2l^p(d - 1)}{2md_2p^{v+w}}
\]
(cf. [2 Thm. 1]).

(iii) Let $p \neq 2$ and $m \mid l^2 - 1$ be such that $m$ does not divide $l - 1$, and $3$ does not divide $m$. Let $d = \gcd(m, l + 1)$. Let $s$ be the order of $p$ in $(\mathbb{Z}/m\mathbb{Z})^*$, and $r$ be the order of $p$ in $(\mathbb{Z}/(m/d)\mathbb{Z})^*$. For each $0 \leq v \leq h$, such that $v \mid 2h$, $v$ does not divide $h$ and $s \mid v$, and for each $\frac{v}{2} \leq w \leq h$, such that $r \mid w$, there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with
\[
g_L = \frac{l^5 + l^2 - l^3d_2 - (l^2 - l + 1 - dd_2)p^{v+w} - (l^3 - l^2 + l)p^w - d_2l^p(d - 1)}{2md_2p^{v+w}}
\]
(cf. [2 Thm. 2]).

(iv) Let $p \neq 2$. For each $0 \leq v \leq h$, and for each $0 \leq w \leq h - 1$, there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with
\[
g_L = \frac{l^5 + l^2 - l^3d_2 - (l^2 - l + 1 - dd_2)p^w(p^v + l) + ld_2(p^v - l^2) - d_2p^v(l - p^w)}{2d_2p^{v+w}}
\]
(cf. [20 Thm. 3.2]).

(v) Let $p = 2$. For all integers $v, w$ with $0 \leq v \leq w < h$ there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with
\[
g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^v + l) + ld_2(2^v - l^2) - d_22^v(l - 2^w)}{d_22^{v+w+1}}
\]
(cf. [20 Cor. 3.4(ii)]).

(vi) Let $p = 2$. For all integers $v, w$ with $w \mid h$, $w \mid v$, $v \mid 2h, 1 \leq v < n$, and $(2^v - 1)/(2^w - 1) \mid (2^h + 1)$, there exist subgroups $L$ of $\text{Aut}(\mathcal{X})$ with
\[
g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^v + l) + ld_2(2^v - l^2) - d_22^v(l - 2^w)}{d_22^{v+w+1}}
\]
for each $v'$ with $0 \leq v' \leq v$. (cf. [20 Cor. 3.4(i), Cor. 3.4(iii)]).

(vii) Let $p = 2$ and $h$ be odd. Let $s \mid h$ and $0 \leq k \leq s$. For each $1 \leq v \leq h - 1$, such that $v = s + k$, and for each $s \leq w \leq h - 1$, there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with
\[
g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^v + l) + ld_2(2^v - l^2) - d_22^v(l - 2^w)}{d_22^{v+w+1}}
\]
(cf. [2 Thm. 4]).
(viii) Let $p = 2$ and $h$ be even and such that $4$ does not divide $h$. Let $s \mid h$ be odd and $0 \leq k \leq s$. For each $1 \leq v \leq h - 1$, such that $v = 2s + k$, and for each $2s \leq w \leq h - 1$, there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^v + l) + ld_2(2^w - l^2) - d_22^w(l - 2^w)}{d_22^{w+1}}$$

(cf. [2] Thm. 5).

(ix) Let $p = 2$ and write $h = 2^e f$, with $e, f \in \mathbb{N}$ and $f \geq 3$ odd. For each divisor $j$ of $f$, let $k_j$ be the order of 2 in $(\mathbb{Z}/j\mathbb{Z})^*$ and $r_j = \frac{\Phi(j)}{k_j}$, where $\Phi$ is the Euler function. For each $1 \leq w \leq h - 2$, such that $w = 2^e \left[1 + \sum_{j \neq 1} l_j k_j\right]$, with $0 \leq l_j \leq r_j$, there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^{w+1} + l) + ld_2(2^{w+1} - l^2) - d_22^{w+1}(l - 2^w)}{d_22^{2w+2}}$$

(cf. [2] Thm. 6).

4.2. $L = \pi^{-1}(\bar{G})$, $\bar{G} < \text{PGU}(3,\ell)_{P_{\infty}}$. Groups $L = \pi^{-1}(\bar{G})$ that have not already been considered in Section 4.1 are groups $\pi^{-1}(\bar{G})$ with $\bar{G}$ containing elements $[a, b, c]$ with $a^{(\ell^2 - 1)/\mu_1} \neq 1$ (again the notation of both [20] and [2] is used to describe elements in $\text{PGU}(3,\ell)_{P_{\infty}}$). Let $|\bar{G}| = \bar{m}p^{v+w}$, with $\gcd(\bar{m}, p) = 1$, and $p^v = |\{[a, b, c] \in G \mid b = 0\}|$. Then it is easily seen that $|L| = mp^{v+w}$ with $m = (\ell^2 - \ell + 1)\bar{m}$, and

$$p^{v+w} = |L \cap \{\xi_{1,b,c}\}|, \quad p^v = |L \cap \{\xi_{1,0,c}\}|.$$

Clearly, $L_{\Lambda} = \Lambda$ holds.

By Lemma 4.4 together with Corollary 3.3 and Lemma 4.3 it follows that

$$\ell^5 - 2\ell^3 + \ell^2 - 2 = mp^{v+w}(2g_L - 2) + (\ell^2 - \ell + 1)(p^{v+w} - 1) + (\ell^5 - \ell^2 + \ell)(p^w - 1) + (\ell^2 - \ell)(\ell^3 + 1) + (\ell^2 - \ell + 1)\sum_{\bar{h} \in G, \bar{h} \neq \text{id}} |\{\bar{P} \in \mathcal{O}_1 \mid \bar{h} (\bar{P}) = \bar{P}\}|.$$

In order to provide concrete values of genera $g_L$ we are going to consider subgroups $\bar{G}$ containing elements $[a, b, c]$ with $a^{(\ell^2 - 1)/\mu_1} \neq 1$ that have been described in either [20] or [2].

**Theorem 4.6.**

(i) Let $p \neq 2$, $\bar{m}$ be an integer such that $\bar{m} \mid l^2 - 1$ and $3 \mid \bar{m}$. Let $d = \gcd(\bar{m}, l + 1)$ and let $s := \min \{r \geq 1 : p^r \equiv 1 \text{ mod } \frac{\bar{m}}{p}\}$. For each $0 \leq w \leq h$, such that $s \mid w$, there exists a subgroup $L$ of $\text{Aut}(\mathcal{X})$ with

$$g_L = \frac{(l - p^w)(l + 1 - d)}{2\bar{m}p^w}$$

(cf. [20] Prop. 4.6).


(ii) Let \( p \neq 2 \) and \( \bar{m} | l^2 - 1 \) be such that \( \bar{m} \) does not divide \( l - 1 \), and \( 3 | \bar{m} \). Let \( d = \gcd(\bar{m}, l + 1) \). Let \( s \) be the order of \( p \) in \( (\mathbb{Z}/\bar{m}\mathbb{Z})^* \), and let

\[
    r = \begin{cases} \text{order of } p \text{ in } (\mathbb{Z}/\bar{m}\mathbb{Z})^*, & \bar{m} \text{ even} \\ s, & \bar{m} \text{ odd} \end{cases}
\]

For each \( 0 \leq v \leq h \), such that \( v \mid 2h \), \( v \) does not divide \( h \) and \( s \mid v \), and for each \( \frac{v}{2} \leq w \leq h \) such that \( r \mid w \), there exists a subgroup \( L \) of \( \text{Aut}(X) \) with

\[
    g_L = \frac{(l^2 - p^{v+w} - lp^w - dp^{v+w} + lp^w)}{2\bar{m}p^{v+w}}
\]

(cf. [2, Thm. 2]).

5. Curves \( X/L \) with \( \tilde{L} \) subgroup of a group of type \( (B) \)

In this section we investigate subgroups \( L \) of \( \text{Aut}(X) \) with \( L = \Sigma_1 \times \Sigma_2 \), where \( \Sigma_1 \) is contained \( \Gamma \), \( \Sigma_2 \) is a subgroup of \( \Lambda \), and \( \tilde{L} \) is a subgroup of a group of type \( (B) \). To this end, we determine a subgroup \( \Omega \) of \( \Gamma \) such that \( \pi(\Omega) \) is contained in a group of type \( (B) \) and \( \Omega \cap \Lambda = \{id\} \).

The construction of \( \Omega \) requires some technical preliminaries, especially on the Singer groups of \( \text{PGU}(3, \ell) \). As in [7], we use a representation of a Singer group up to conjugation in \( \text{GL}(3, q^2) \). This will allow us to deal with diagonal matrices, thus avoiding more involved matrix computation.

By [7, Prop. 4.6] there exists a matrix \( A_1 \) in \( \text{GL}(3, q^2) \setminus \text{GL}(3, \ell^2) \) such that

\[
    A_1 \sigma(A_1) = D_1, \quad \text{with}
\]

\[
    D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Then a matrix \( M \) is in \( \text{SU}(3, \ell) \) if and only if \( M_1 = A_1^{-1}MA_1 \) is such that

\[
    M_1^*D_1\sigma(M_1) = D_1, \quad \det(M_1) = 1.
\]

Also, it is straightforward to check that the points of \( \mathbb{P}^2(\mathbb{K}) \) whose homogeneous coordinates are the columns of \( A_1 \) lie in \( \mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{\ell^2}) \).

Now we construct a group \( \Omega_1 \) of \( \text{GL}(3, q^2) \), contained in the conjugate subgroup of \( \text{SU}(3, \ell) \) by \( A_1 \). Let \( \mu_2 = \gcd(\ell^2 - \ell + 1, 3) \). It is easily seen that either \( \mu_2 = 3 \) or \( \mu_2 = 1 \) according to whether \( \ell \equiv 2 \) (mod 3) or not. Let \( \Pi_{\ell^2-\ell+1}^{\mu_2} \) be the group of \( (\ell^2-\ell+1) \)-th roots of unity. As \( \gcd(\ell^2-\ell+1, \ell + 1) = 1 \), for each \( \lambda \in \Pi_{\ell^2-\ell+1}^{\mu_2} \) there exists a unique \( \tilde{\lambda} \in \Pi_{\ell^2-\ell+1}^{\mu_2} \) with \( \tilde{\lambda}^{\ell+1} = \lambda^{-\ell} \). Let \( \Omega_1 \) be the group generated by \( D_1 \) and

\[
    T = \begin{pmatrix} \bar{w}w & 0 & 0 \\ 0 & \bar{w}w^\ell & 0 \\ 0 & 0 & \bar{w} \end{pmatrix},
\]
where \( w \) is a primitive \((\frac{\ell^2-\ell+1}{\mu_2})\)-th root of unity.

Let \( \Theta = < T > \) and \( \Upsilon_1 = < D_1 > \). It is straightforward to check that \( \Omega_1 = \Theta \times \Upsilon_1 \). Also, every matrix \( M_1 \) in \( \Omega_1 \) satisfies \((18)\), and thus for each \( M_1 \in \Omega_1 \) the matrix \( A_1 M_1 A_1^{-1} \) belongs to \( SU(3, \ell) \).

For a matrix \( M_1 \in \Omega_1 \) let \( \epsilon(M_1) \) be the projectivity in \( PGL(4, \ell^2) \) defined by the \( 4 \times 4 \) matrix obtained from \( A_1 M_1 A_1^{-1} \) by adding \( 0, 0, 1, 0 \) as a third row and as a third column. Then \( \epsilon : \Omega_1 \rightarrow \Gamma \) is an injective group homomorphism. Let \( \Omega = \epsilon(\Omega_1) \).

Let \( \bar{P}_i \) be the point of \( \mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{\ell^2}) \) whose homogeneous coordinates are the elements in the \( i \)-th column of \( A_1 \). Then the subgroup \( \pi(\epsilon(\Theta)) \) is contained in the stabilizer of \( \bar{P}_i \) in \( PGU(3, \ell) \). Also, the group \( \pi(\epsilon(\Upsilon_1)) \) acts regularly on \{ \( \bar{P}_1, \bar{P}_2, \bar{P}_3 \) \}.

We now prove that \( \Omega \cap \Lambda = \{id\} \). Let \( \alpha \in \Omega \cap \Lambda \). Since \( \pi(\alpha) \) fixes every point in \( \mathcal{H} \), \( \alpha \in \epsilon(\Theta) \). Taking into account that every non-trivial element in \( \Gamma \cap \Lambda \) has order \( 3 \), and that \( 3 \) does not divide the order of \( T \), we obtain that \( \alpha = 1 \).

The following result then holds.

**Lemma 5.1.** The subgroup generated by \( \Omega \) and \( \Lambda \) is the direct product \( \Omega \times \Lambda \). The projection \( \pi(\Omega \times \Lambda) \) is a subgroup of a group of type (B) isomorphic to \( \Omega_1 \).

Let \( \bar{P}_i = (x_i, y_i, 1) \). As \( \bar{P}_i \in \mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{\ell^2}) \), up to a rearrangement of the indexes we can assume that \( x_2 = x_1^\ell, y_2 = y_1^\ell \), and \( x_3 = x_1^\ell, y_3 = y_1^\ell \).

For any point \( Q_i = (x_i, y_i, z_0, 1) \) of \( \mathcal{X} \) such that \( \phi(Q_i) = \bar{P}_i \), the image \( \epsilon(T)(Q_i) \) of \( Q_i \) by \( \epsilon(T) \) is the point \( (x_i, y_i, \frac{z_0}{a_{31}x_1 + a_{32}y_1 + a_{33}}, 1) \), where \( (a_{31}, a_{32}, a_{33}) \) is the third row of the matrix \( A_1 T A_1^{-1} \). Let \( s_i = a_{31}x_i + a_{32}y_1 + a_{33} \). Since \( T \) has order \( (\ell^2 - \ell + 1)/\mu_2 \), we have that \( s_i^{(\ell^2 - \ell + 1)/\mu_2} = 1 \). Note that \( s_1^2 = s_2 \) and \( s_3^4 = s_3 \) hold.

**Lemma 5.2.** Any \( s_i \) is a primitive \( \frac{\ell^2 - \ell + 1}{\mu_2} \)-th root of unity.

**Proof.** It is enough to prove the assertion for \( i = 1 \). Assume that \( s_1^m = 1 \). Then also \( s_2^m = 1 \) and \( s_3^m = 1 \) hold. Therefore, \( \epsilon(T)^m \) fixes every point \( Q \) such that \( \phi(Q) = \bar{P}_i \) for some \( i = 1, 2, 3 \). Since \( \ell^2 - \ell + 1 > 3 \), the three lines joining \( P_\infty \) to the points \( (x_i, y_i, 0, 1), i = 1, 2, 3 \), are fixed by \( \epsilon(T)^m \) pointwise. As these three lines are not coplanar, \( \epsilon(T)^m \) is the identical projectivity of \( PG(3, \mathbb{K}) \). This shows that \( \ell^2 - \ell + 1 \) divides \( m \), and the proof is complete. \( \Box \)

Subgroups of the normalizers of a Singer group of \( PGU(3, n) \) have been classified up to conjugacy in \([32, \text{Chapter 4}]\), see also \([8, \text{Lemma 4.1}]\). As a straightforward consequence, the following result holds.

**Lemma 5.3.** The following is a complete list of subgroups of \( \Omega \), up to conjugacy.

(a) For every divisor \( d \) of \( (\ell^2 - \ell + 1)/\mu_2 \), the cyclic subgroup of \( \epsilon(\Theta) \) of order \( d \), i.e. the subgroup generated by \( \epsilon(T)^{(\ell^2 - \ell + 1)/d\mu_2} \).

(b) For every divisor \( d \) of \( (\ell^2 - \ell + 1)/\mu_2 \), the subgroup of order \( 3d \) which is the semidirect product of the cyclic subgroup of \( \epsilon(\Theta) \) of order \( d \) with \( \epsilon(\Upsilon_1) \).

We deal separately with cases (a) and (b) of Lemma \([5.3]\).
5.1. \( L = \Sigma_1 \times \Sigma_2 \), \( \Sigma_1 < \epsilon(\Theta), \Sigma_2 < \Lambda \). Let \( \Sigma_1 = \langle \epsilon(T)^{i_1} \rangle \), with \( i_1 = (\ell^2 - \ell + 1)/d\mu_2 \), and let \( \Sigma_2 \) be the group generated by the projectivity defined by the diagonal matrix \([1, 1, \beta^{i_2}, 1]\), with \( \beta \) a primitive \((\ell^2 - \ell + 1)\)-th root of unity and \( i_2 \) a divisor of \( \ell^2 - \ell + 1 \). Let \( d_2 = (\ell^2 - \ell + 1)/i_2 \). Without loss of generality assume that \( s_1 = \beta^{3k} \), with \( k = 0 \) when \( \mu_2 = 1 \) and \( k > 0 \) for \( \mu_2 = 3 \). In order to compute integers \( l_{\bar{h}, \bar{P}} \) as in Proposition 3.1 we need to investigate the action of \( \Sigma_1 \) on the orbits of \( \phi^{-1}(P_i) \) under \( \Sigma_2 \). Fix \( Q_1 = (x_1, y_1, z_0, 1) \in \phi^{-1}(P_1) \). The orbits of \( \phi^{-1}(P_1) \) under the action of \( \Sigma_2 \) are

\[ \Delta_j = \{(x_1, y_1, \beta^j z_0, 1), (x_1, y_1, \beta^j i_2 z_0, 1), \ldots, (x_1, y_1, \beta^j (d_2 - 1) i_2 z_0, 1)\}, \]

with \( j = 0, \ldots, i_2 - 1 \). For \( 1 \leq t \leq d - 1 \), the orbit \( \Delta_j \) is fixed by \( \epsilon(T)^{t_1} \) if and only if

\[ \beta^j s_1^{t_1} = \beta^{t_1 u_2}, \]

that is,

\[ 3^k t_1 = v(\ell^2 - \ell + 1) - u_2, \]

for some \( u, v \in \mathbb{Z} \).

Equivalently,

\[ \text{(19)} \]

\[ i_2 | 3^k t_1. \]

Similarly it can be proved that \( \epsilon(T)^{t_1} \) fixes any orbit of \( \phi^{-1}(\bar{P}_2) \) (resp. \( \phi^{-1}(\bar{P}_3) \)) under \( \Sigma_2 \) if and only if \( i_2 | 3^k \ell^2 t_1 \) (resp. \( i_2 | 3^k \ell^4 t_1 \)). Since \( \gcd(i_2, \ell) = 1 \), either \( \epsilon(T)^{t_1} \) fixes all the orbits of \( \phi^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}) \) under \( \Sigma_2 \) or none, according to whether \( \text{(19)} \) holds or not.

If \( 3 \) does not divide \( i_2 \), then \( \text{(19)} \) is equivalent to \( i_2 | t_1 \). Therefore, the number of non-trivial elements in \( \Sigma_1 \) fixing one (and hence every) orbit of \( \phi^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}) \) under \( \Sigma_2 \) is equal to the number of common multiples of \( i_1 \) and \( i_2 \) that are strictly less than \((\ell^2 - \ell + 1)/\mu_2\). If \( \text{lcm}(i_1, i_2) \leq (\ell^2 - \ell + 1)/\mu_2 \), then this number is \( \frac{\ell^2 - \ell + 1}{\mu_2 \text{lcm}(i_1, i_2)} - 1 \); if \( \text{lcm}(i_1, i_2) = \ell^2 - \ell + 1 \), then no orbit is fixed by a non-trivial element in \( \Sigma_1 \).

If \( 3 \) divides \( i_2 \), then \( \text{(19)} \) is equivalent to \( \frac{\ell^2}{3} t_1 \). Arguing as in the previous case, it can be deduced that the number of non-trivial elements in \( \Sigma_1 \) fixing one (and hence every) orbit of \( \phi^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}) \) under \( \Sigma_2 \) is \( \frac{\ell^2 - \ell + 1}{\mu_2 \text{lcm}(i_1, i_2/3)} - 1 \).

The last term of \( e_L \) as in Proposition 3.1 can be written as

\[ |L_A| n_2 = |\Sigma_2| \sum_{\bar{h} \in \Sigma_1, \bar{h} \neq \text{id}} \sum_{\bar{P} \in \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}, \bar{h}\bar{P} = \bar{P}} l_{\bar{h}, \bar{P}}; \]

then, it is equal to

\[
\begin{cases}
3(\ell^2 - \ell + 1)(\frac{\ell^2 - \ell + 1}{\mu_2 \text{lcm}(i_1, i_2)} - 1) & \text{if } 3 \nmid i_2, \text{lcm}(i_1, i_2) \leq (\ell^2 - \ell + 1)/\mu_2, \\
0 & \text{if } 3 \nmid i_2, \text{lcm}(i_1, i_2) = \ell^2 - \ell + 1, \\
3(\ell^2 - \ell + 1)(\frac{\ell^2 - \ell + 1}{\mu_2 \text{lcm}(i_1, i_2/3)} - 1) & \text{if } 3 \mid i_2.
\end{cases}
\]

By (9) and Proposition 3.1 the following result holds.
Theorem 5.4. Let $\mu_2 = \gcd(\ell^2 - \ell + 1, 3)$.

- For $i_1$ divisor of $(\ell^2 - \ell + 1)/\mu_2$, $i_2$ divisor of $\ell^2 - \ell + 1$ such that $3 \nmid i_2$ and $\text{lcm}(i_1, i_2) \leq (\ell^2 - \ell + 1)/\mu_2$, there exists a subgroup $L$ of $\text{Aut}(X)$ with
  \[ g_L = \frac{1}{2} \left( (\ell + 2)\mu_2 i_1 i_2 - (\ell + 1)\mu_2 i_1 - \frac{3i_1 i_2}{\text{lcm}(i_1, i_2)} \right) + 1. \]

- For $i_1$ divisor of $(\ell^2 - \ell + 1)/\mu_2$, $i_2$ divisor of $\ell^2 - \ell + 1$ such that $3 \nmid i_2$ and $\text{lcm}(i_1, i_2) = (\ell^2 - \ell + 1)$, there exists a subgroup $L$ of $\text{Aut}(X)$ with
  \[ g_L = \frac{1}{2} \left( (\ell + 2)\mu_2 i_1 i_2 - (\ell + 1)\mu_2 i_1 - \frac{3\mu_2 i_1 i_2}{\ell^2 - \ell + 1} \right) + 1. \]

- For $i_1$ divisor of $(\ell^2 - \ell + 1)/\mu_2$, $i_2$ divisor of $\ell^2 - \ell + 1$ such that $3 \mid i_2$, there exists a subgroup $L$ of $\text{Aut}(X)$ with
  \[ g_L = \frac{1}{2} \left( (\ell + 2)\mu_2 i_1 i_2 - (\ell + 1)\mu_2 i_1 - \frac{3i_1 i_2}{\text{lcm}(i_1, i_2/3)} \right) + 1. \]

5.2. $L = (\Sigma_1 \times \epsilon(\mathcal{T}_1)) \times \Sigma_2$, $\Sigma_1 < \epsilon(\Theta)$, $\Sigma_2 < \Lambda$. Here we assume that $p \neq 3$. Let $\Sigma_1 = \langle \epsilon(T)^{i_1} \rangle$, with $i_1 = (\ell^2 - \ell + 1)/d\mu_2$. Let $\Sigma_2$ be the group generated by the projectivity defined by the diagonal matrix $[1, 1, \beta, 1]$, with $\beta$ a primitive $(\ell^2 - \ell + 1)$-th root of unity and $i_2$ a divisor of $\ell^2 - \ell + 1$; let $d_2 = (\ell^2 - \ell + 1)/i_2$.

The action of $\pi(L)$ on $\mathcal{H}$ is described in [8].

Lemma 5.5 (Proposition 4.2 in [8]). If $\mu_2 = 1$, then the group $\pi(L)$ has 3 short orbits, namely $\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$ and 2 short orbits of size $d$ consisting of $\mathbb{F}_\ell$-rational points of $\mathcal{H}$. If $\mu_2 = 3$, then the only short orbit of $\pi(L)$ is $\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$.

As a consequence, the last term $|L_\Lambda|n_2$ of $e_L$ as in Proposition 3.1 is just

\[ |\Sigma_2| \sum_{\bar{h} \in \Sigma_1, \bar{h} \neq \id} \sum_{\bar{P} \in \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}, \bar{h}(\bar{P}) = \bar{P}} l_{\bar{h}, \bar{P}}, \]

which has already been computed in Section 5.1. We will distinguish the cases $\mu_2 = 1$ and $\mu_2 = 3$.

5.2.1. $\mu_2 = 1$. Apart from $\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$, the group $\pi(L)$ has further 2 short orbits of size $d$ consisting of $\mathbb{F}_\ell$-rational points of $\mathcal{H}$. By (3) and Proposition 3.1 the following result holds.

Theorem 5.6. Let $p \neq 3$, $\gcd(\ell^2 - \ell + 1, 3) = 1$. For $i_1, i_2$ divisors of $(\ell^2 - \ell + 1)$ there exists a subgroup $L$ of $\text{Aut}(X)$ with

\[ g_L = \frac{1}{3} \left( \frac{1}{2} \left( (\ell + 2)i_1 i_2 - (\ell + 1)i_1 - \frac{3i_1 i_2}{\text{lcm}(i_1, i_2)} \right) + 1 \right). \]
5.2.2. \( \mu_2 = 3 \). The only short orbit of \( \pi(L) \) is \( \{ P_1, P_2, P_3 \} \). By \((\S 3.1)\) and Proposition \( 3.1 \) the following result holds.

**Theorem 5.7.** Let \( \gcd(\ell^2 - \ell + 1, 3) = 3 \).

- For \( i_1 \) divisor of \((\ell^2 - \ell + 1)/3, i_2 \) divisor of \( \ell^2 - \ell + 1 \) such that \( 3 \nmid i_2 \) and \( \text{lcm}(i_1, i_2) \leq (\ell^2 - \ell + 1)/3 \), there exists a subgroup \( L \) of \( \text{Aut}(\mathcal{X}) \) with
  \[
  g_L = \frac{1}{2} \left( (\ell + 2)i_1i_2 - (\ell + 1)i_1 - \frac{i_1i_2}{\text{lcm}(i_1, i_2)} \right) + 1.
  \]

- For \( i_1 \) divisor of \((\ell^2 - \ell + 1)/3, i_2 \) divisor of \( \ell^2 - \ell + 1 \) such that \( 3 \nmid i_2 \) and \( \text{lcm}(i_1, i_2) = (\ell^2 - \ell + 1) \), there exists a subgroup \( L \) of \( \text{Aut}(\mathcal{X}) \) with
  \[
  g_L = \frac{1}{2} \left( (\ell + 2)i_1i_2 - (\ell + 1)i_1 - \frac{3i_1i_2}{\ell^2 - \ell + 1} \right) + 1.
  \]

- For \( i_1 \) divisor of \((\ell^2 - \ell + 1)/3, i_2 \) divisor of \( \ell^2 - \ell + 1 \) such that \( 3 \mid i_2 \), there exists a subgroup \( L \) of \( \text{Aut}(\mathcal{X}) \) with
  \[
  g_L = \frac{1}{2} \left( (\ell + 2)i_1i_2 - (\ell + 1)i_1 - \frac{i_1i_2}{\text{lcm}(i_1, i_2/3)} \right) + 1.
  \]

**Remark 5.8.** When \( L = \pi^{-1}(G) \) with \( G \) a group of type (B), then by Corollary \( 3.4 \) the genus \( g_L \) coincides with \( g_{\ell} \). All the possibilities for \( g_L \) are determined in \([8]\). It should be noted that the statement of Proposition 4.2(3) in \([8]\) contains a misprint, as \((q^2 - q + 1 - 3n)/6n\) should read \((q^2 - q + 1 + 3n)/6n\).

6. **Curves \( \mathcal{X}/L \) with \( \bar{L} \) subgroup of a group of type (C)**

In this section we investigate subgroups \( L \) of \( \text{Aut}(\mathcal{X}) \) with \( L = \Sigma_1 \times \Sigma_2 \), where \( \Sigma_1 \) is contained \( \Gamma \), \( \Sigma_2 \) is a subgroup of \( \Lambda \), and \( \bar{L} \) is a subgroup of a group of type (C). To this end, we determine a subgroup \( \Omega \) of \( \Gamma \) such that \( \pi(\Omega) \) is contained in a group of type (C) and \( \Omega \cap \Lambda = \{ \text{id} \} \). The construction of \( \Omega \) requires some technical preliminaries. In particular, a group conjugate to \( SU(3, \ell) \) in \( GL(3, q^2) \) needs to be considered.

Let \( b, c \in \mathbb{F}_{\ell^2} \) be such that \( b^{\ell+1} = c^\ell + c = -1 \), and let
\[
A_2 = \begin{pmatrix}
  0 & 0 & 1 \\
  \frac{c+1}{b} & \frac{-c}{b} & 0 \\
  \frac{1}{b} & \frac{c}{b} & 0
\end{pmatrix}.
\]

Then \( A_2 \) is a matrix in \( GL(3, \ell^2) \) such that \( A_2^\ell D\sigma(A_2) = I_3 \). A matrix \( M \) is in \( SU(3, \ell) \) if and only if \( M_2 = A_2^{-1}MA_2 \) is such that
\[
M_2^i \sigma(M_2) = I_3, \quad \det(M_2) = 1.
\]
Let \( \mu_1 \) be the largest power of 3 dividing \( \ell + 1 \), and let \( \Pi_{\ell+1}^{\mu_1} \) be the group of \((\frac{\ell + 1}{\mu_1})\)-th roots of unity. As \( \gcd(\frac{\ell + 1}{\mu_1}, 3) = 1 \), for each \( \lambda \in \Pi_{\ell+1}^{\mu_1} \) there exists a unique \( \tilde{\lambda} \in \Pi_{\ell+1}^{\mu_1} \) with \( \tilde{\lambda}^3 = \lambda \). Let \( \Omega_2 \) be the the group generated by the matrices

\[
T_1 = \begin{pmatrix} \frac{w}{\overline{w}} & 0 & 0 \\ 0 & \frac{1}{w} & 0 \\ 0 & 0 & \frac{1}{w} \end{pmatrix},
\quad
T_2 = \begin{pmatrix} \frac{1}{w} & 0 & 0 \\ 0 & \frac{w}{\overline{w}} & 0 \\ 0 & 0 & \frac{w}{\overline{w}} \end{pmatrix},
\quad
U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\quad
U_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]

where \( w \) is a primitive \((\frac{\ell + 1}{\mu_1})\)-th root of unity. Let \( \Theta_1 = \langle T_1 \rangle \), \( \Theta_2 = \langle T_2 \rangle \), \( \Upsilon_1 = \langle U_1 \rangle \), \( \Upsilon_2 = \langle U_2 \rangle \), and \( \Upsilon = \langle U_1, U_2 \rangle \).

It is straightforward to check that every matrix in \( \Omega_2 \) satisfies (20), and that \( \Omega_2 = (\Theta_1 \times \Theta_2) \rtimes \Upsilon \). Moreover, \( \Upsilon \) is isomorphic to \( \text{Sym}_3 \). For a matrix \( M_2 \in \Omega_2 \) let \( \epsilon(M_2) \) be the projectivity in \( PGL(4, \ell^2) \) defined by the \( 4 \times 4 \) matrix obtained from \( A_2 M_2 A_2^{-1} \) by adding \( 0, 0, 1, 0 \) as a third row and as a third column. Then \( \epsilon : \Omega_2 \to \Gamma \) is an injective group homomorphism. Let \( \Omega = \epsilon(\Omega_2) \).

Let \( \tilde{P}_i \) be the point in \( PG(2, \ell^2) \) whose homogeneous coordinates are the elements in the \( i \)-th column of \( A_2 \). Then the subgroup \( \pi(\Omega) \) is contained in the stabilizer of the triangle \( \tilde{P}_1 \tilde{P}_2 \tilde{P}_3 \) in \( PGU(3, \ell) \).

We now prove that \( \Omega \cap \Lambda = \{id\} \). Let \( \alpha \in \Omega \cap \Lambda \). Since \( \pi(\alpha) \) fixes pointwise the set of points in \( \mathcal{H} \) belonging to the triangle \( \tilde{P}_1 \tilde{P}_2 \tilde{P}_3 \) in \( PGU(3, \ell) \), \( \alpha \in \epsilon(\Theta_1 \times \Theta_2) \). Taking into account that every non-trivial element in \( \Gamma \cap \Lambda \) has order 3, and that 3 does not divide the order of \( \Theta_1 \times \Theta_2 \) by construction, we obtain that \( \alpha = 1 \).

Therefore, the following lemma holds.

**Lemma 6.1.** The subgroup generated by \( \Omega \) and \( \Lambda \) is the direct product \( \Omega \times \Lambda \). The projection \( \pi(\Omega \times \Lambda) \) is a subgroup of a group of type \( (C) \) isomorphic to \( \Omega_2 \).

Note that the action of \( \pi(\Omega) \) on \( \mathcal{H} \) can be viewed as the action of the group of projectivities defined by the matrices in \( \Omega_2 \) on the set of points of the plane curve with equation \( X^{\ell+1} + Y^{\ell+1} + T^{\ell+1} = 0 \).

For a divisor \( d \) of \( (\ell + 1)/\mu_1 \) and for \( i = 1, 2 \), let \( C^{(i)}_d \) be the subgroup of \( \epsilon(\Theta_i) \) of order \( d \). We consider subgroups \( \Sigma_1 \) of \( \Omega \) of the following types:

- (a) \( C^{(1)}_{d_1} \times C^{(2)}_{d_2} \);
- (b) the cyclic subgroup of order \( (\ell + 1)/(\mu_1 d_1) \) generated by \( \epsilon(T_1)^{d_1} \epsilon(T_2)^{2d_1} \), with \( d_1 \) a divisor of \( (\ell + 1)/\mu_1 \);
- (c) \( (C^{(1)}_{d_1} \times C^{(2)}_{d_2}) \rtimes \Upsilon_2 \);
- (d) \( (C^{(1)}_{d_1} \times C^{(2)}_{d_2}) \rtimes \Upsilon_1 \);
- (e) \( (C^{(1)}_{d_1} \times C^{(2)}_{d_2}) \rtimes \Upsilon \).

Cases (a)-(e) are dealt with separately.

6.1. \( L = \Sigma_1 \times \Sigma_2 \), with \( \Sigma_1 \) as in (a), \( \Sigma_2 \subset \Lambda \), \( |\Sigma_2| = d \). The action of \( \pi(\Sigma_1) \) on \( \mathcal{H} \) was investigated in [20 Example 5.11]. Any non-trivial element in \( \pi(\Sigma_1) \) fixes no
point in \( \mathcal{O}_2 \). Moreover,
\[
\sum_{h \in \pi(\Sigma_1), h \neq \text{id}} |\{ \bar{P} \in \mathcal{O}_1 \mid \bar{h}(\bar{P}) = \bar{P} \}| = (\ell + 1)(d_1 + d_2 + \gcd(d_1, d_2) - 3).
\]
By (6) and Proposition 3.1 the following result holds.

**Theorem 6.2.** Let \( \mu_1 \) be the highest power of 3 which divides \( \ell + 1 \). For any two divisors \( d_1, d_2 \) of \( (\ell + 1)/\mu_1 \), and for any \( d \) divisor of \( \ell^2 - \ell + 1 \), there exists a subgroup \( L \) of \( \text{Aut}(\mathcal{X}) \) with
\[
g_L = 1 + \frac{(l^3 + 1)(l^2 - d - 1) - d(l + 1)(d_1 + d_2 + \gcd(d_1, d_2) - 3)}{2dd_1d_2}.
\]

6.2. \( L = \Sigma_1 \times \Sigma_2 \), with \( \Sigma_1 \) as in (b), \( \Sigma_2 < \Lambda, |\Sigma_2| = d \). By [20, Example 5.10] any non-trivial element in \( \pi(\Sigma_1) \) fixes no point in \( \mathcal{O}_2 \). Moreover,
\[
g_L = 1 + \frac{\mu_1d_1(l - 2)(l + 1)}{2(l + 1)}
\]
when \( (\ell + 1)/(\mu_1d_1) \) is odd, whereas
\[
g_L = 1 + \frac{\mu_1d_1(l - 3)(l + 1)}{2(l + 1)}
\]
when \( (\ell + 1)/(\mu_1d_1) \) is even. By Proposition 3.2 the following result holds.

**Theorem 6.3.** Let \( \mu_1 \) be the highest power of 3 which divides \( \ell + 1 \).

- For any divisor \( d_1 \) of \( (\ell + 1)/\mu_1 \) such that \( (\ell + 1)/(\mu_1d_1) \) is odd, and for every divisor \( d \) of \( \ell^2 - \ell + 1 \), there exists a subgroup \( L \) of \( \text{Aut}(\mathcal{X}) \) with
\[
g_L = \frac{1}{2} \left( \mu_1d_1 \left( \frac{\ell^2 - \ell + 1}{d} \right) \left( \ell^2 - d - 1 \right) + 2 \right).
\]
- For any divisor \( d_1 \) of \( (\ell + 1)/\mu_1 \) such that \( (\ell + 1)/(\mu_1d_1) \) is even, and for every divisor \( d \) of \( \ell^2 - \ell + 1 \), there exists a subgroup \( L \) of \( \text{Aut}(\mathcal{X}) \) with
\[
g_L = \frac{1}{2} \left( \mu_1d_1 \left( \frac{\ell^2 - \ell + 1}{d} \right) \left( \ell^2 - d - 1 \right) - \mu_1d_1 + 2 \right).
\]

6.3. \( p \neq 2, L = \Sigma_1 \times \Sigma_2 \), with \( \Sigma_1 \) as in (c), \( \Sigma_2 < \Lambda, |\Sigma_2| = d \). As \( C^{(1)}_{d_1} \times C^{(2)}_{d_1} \) is a normal subgroup of \( \Sigma_1 \), the subgroup \( \pi(\Sigma_2) \) acts on the quotient curve \( \mathcal{H}/\pi(C^{(1)}_{d_1} \times C^{(1)}_{d_1}) \). Such action is equivalent to the action of the involutory projectivity defined by \( U_2 \) on the plane curve \( \mathcal{E} \) with equation \( X \frac{d_1}{d_1} + Y \frac{d_1}{d_1} + T \frac{d_1}{d_1} = 0 \). The fixed points of \( U_2 \) on \( \mathcal{E} \) are the points on the line \( X = T \), together with \((-1, 0, 1) \) if \( (\ell + 1)/d_1 \) is odd. It is straightforward to check that any point of \( \mathcal{H} \) lying over one of those fixed points of \( \mathcal{X} \) is \( \mathbb{F}_p \)-rational. As \( \mathcal{E} \) is non-singular, the genus \( \bar{g} \) of \( \mathcal{E}/U_2 \) is given by
\[
\left( \frac{\ell + 1}{d_1} - 1 \right) \left( \frac{\ell + 1}{d_1} - 2 \right) - 2 = 2(2\bar{g} - 2) + 2 \left[ (\ell + 1)/(2d_1) \right],
\]
that is
\begin{equation}
\bar{g} = \frac{((\ell + 1)/(d_1) - 2)^2}{4}, \quad \text{if } (\ell + 1)/d_1 \text{ is even},
\end{equation}
and
\begin{equation}
\bar{g} = \frac{((\ell + 1)/(d_1) - 3)((\ell + 1)/(d_1) - 1)}{4}, \quad \text{if } (\ell + 1)/d_1 \text{ is odd}.
\end{equation}

Note that no point in \( \bar{O}_2 \) is fixed by a non trivial element in \( \pi(\Sigma_1) \). Also, since \( \mathcal{E}/U_2 \) is isomorphic to \( \mathcal{H}/\pi(\Sigma_1) \), from Hurwitz’s genus formula it follows
\begin{equation}
\sum_{\bar{h} \in \pi(\Sigma_1), \bar{h} \neq \text{id}} |\{ \bar{P} \in \bar{O}_1 \mid \bar{h}(\bar{P}) = \bar{P} \}| = \ell^2 - \ell - 2 - 2d_1^2(2\bar{g} - 2).
\end{equation}

By (6) and Proposition 3.2 the following result holds.

**Theorem 6.4.** Assume that \( p \neq 2 \). Let \( \mu_1 \) be the highest power of 3 dividing \( \ell + 1 \). For any divisor \( d_1 \) of \( (\ell + 1)/\mu_1 \), and for every divisor \( d \) of \( \ell^2 - \ell + 1 \), there exists a subgroup \( L \) of \( \text{Aut}(\mathcal{X}) \) with
\begin{equation}
g_L = \bar{g} + \frac{(\ell + 1)(\ell - 1)}{4d_1^2} \left( \frac{\ell^2 - \ell + 1}{d} - 1 \right),
\end{equation}
with \( \bar{g} \) as in (21) for \( (\ell + 1)/d_1 \) even, and with \( \bar{g} \) as in (22) for \( (\ell + 1)/d_1 \) odd.

6.4. \( p \neq 3 \), \( L = \Sigma_1 \times \Sigma_2 \), with \( \Sigma_1 \) as in (d), \( \Sigma_2 \leq \Lambda \), \( |\Sigma_2| = d \). As \( C^{(1)}_{d_1} \times C^{(2)}_{d_1} \) is a normal subgroup of \( \Sigma_1 \), the subgroup \( \pi(\Upsilon_1) \) acts on the quotient curve \( \mathcal{H}/\pi(C^{(1)}_{d_1} \times C^{(2)}_{d_1}) \). Such action is equivalent to the action of the projectivity defined by \( U_1 \) on the plane curve \( \mathcal{E} \) with equation \( X^{(\ell+1)} + Y^{(\ell+1)} + T^{(\ell+1)} = 0 \). Let \( f \) be a primitive third root of unity in \( \mathbb{F}_{\ell^2} \). If 3 does not divide \( (\ell + 1)/d_1 \), then the fixed points of \( U_1 \) on \( \mathcal{E} \) are precisely \( (f, f^2, 1) \) and \( (f^2, f, 1) \); otherwise no point on \( \mathcal{E} \) is fixed by \( U_1 \). Arguing as in Section 6.3 we get that
\begin{equation}
\sum_{\bar{h} \in \pi(\Sigma_1), \bar{h} \neq \text{id}} |\{ \bar{P} \in \bar{O}_1 \mid \bar{h}(\bar{P}) = \bar{P} \}| = \ell^2 - \ell - 2 - 2d_1^2(2\bar{g} - 2),
\end{equation}
where
\begin{equation}
\bar{g} = \frac{((\ell + 1)/(d_1) - 1)((\ell + 1)/(d_1) - 2)}{6}, \quad \text{if } 3 \nmid (\ell + 1)/d_1,
\end{equation}
and
\begin{equation}
\bar{g} = \frac{((\ell + 1)/(d_1) - 1)((\ell + 1)/(d_1) - 2) + 4}{6}, \quad \text{if } 3 \mid (\ell + 1)/d_1.
\end{equation}

By (6) and Proposition 3.2 the following result holds.
Theorem 6.5. Assume that $p \neq 3$. Let $\mu_1$ be the highest power of 3 dividing $\ell + 1$. For any divisor $d_1$ of $(\ell + 1)/\mu_1$, and for every divisor $d$ of $\ell^2 - \ell + 1$, there exists a subgroup $L$ of $\text{Aut}(X)$ with

$$g_L = g + \frac{(\ell + 1)(\ell^2 - 1)}{6d_1^2} \left( \frac{\ell^2 - \ell + 1}{d} - 1 \right),$$

with $g$ as in (23) when 3 does not divide $(\ell + 1)/d_1$, and with $g$ as in (24) if $3 \mid (\ell + 1)/d_1$.

6.5. $p \neq 2, 3$, $L = \Sigma_1 \times \Sigma_2$, with $\Sigma_1$ as in (e), $\Sigma_2 < \Lambda$, $|\Sigma_2| = d$. As $C^{(1)}_{d_1} \times C^{(2)}_{d_1}$ is a normal subgroup of $\Sigma_1$, the subgroup $\pi(\Upsilon)$ acts on the quotient curve $\mathcal{H}/\pi(C^{(1)}_{d_1} \times C^{(2)}_{d_1})$. Such action is equivalent to that of all permutations of the coordinates $(X, Y, T)$ on the plane curve $E$ with equation $X^{d_1+1} + Y^{d_1+1} + T^{d_1+1} = 0$. It is straightforward to check that:

- $(X, Y, T) \mapsto (T, Y, X)$ fixes the points on the line $X = T$, together with $(-1, 0, 1)$ if $(\ell + 1)/d_1$ is odd;
- $(X, Y, T) \mapsto (X, T, Y)$ fixes the points on the line $Y = T$, together with $(0, -1, 1)$ if $(\ell + 1)/d_1$ is odd;
- $(X, Y, T) \mapsto (Y, X, T)$ fixes the points on the line $X = Y$, together with $(-1, 1, 0)$ if $(\ell + 1)/d_1$ is odd;
- $(X, Y, T) \mapsto (Y, T, X)$ fixes the points $(f, f^2, 1)$ and $(f^2, f, 1)$ if 3 does not divide $(\ell + 1)/d_1$, and fixes no point if $3 \mid (\ell + 1)/d_1$;
- $(X, Y, T) \mapsto (T, X, Y)$ fixes the points $(f, f^2, 1)$ and $(f^2, f, 1)$ if 3 does not divide $(\ell + 1)/d_1$, and fixes no point if $3 \mid (\ell + 1)/d_1$.

Therefore, for the genus $\bar{g}$ of the quotient curve of $\mathcal{H}/\pi(C^{(1)}_{d_1} \times C^{(1)}_{d_1})$ with respect to $\pi(\Upsilon)$ it turns out that

$$\bar{g} = \frac{1}{12} \left( \left( \frac{\ell + 1}{d_1} \right)^2 - 6 \frac{\ell + 1}{d_1} + o \right),$$

where

$$o = \begin{cases} 
12 \text{ if } 3 \mid (\ell + 1)/d_1, 2 \mid (\ell + 1)/d_1, \\
9 \text{ if } 3 \mid (\ell + 1)/d_1, 2 \nmid (\ell + 1)/d_1, \\
8 \text{ if } 3 \nmid (\ell + 1)/d_1, 2 \mid (\ell + 1)/d_1, \\
5 \text{ if } 3 \nmid (\ell + 1)/d_1, 2 \nmid (\ell + 1)/d_1.
\end{cases}$$

Arguing as in the preceedings Sections, we obtain the following result as a consequence of (6) and Proposition 3.2

Theorem 6.6. Assume that $p \neq 3$. Let $\mu_1$ be the highest power of 3 dividing $\ell + 1$. For any divisor $d_1$ of $(\ell + 1)/\mu_1$, and for every divisor $d$ of $\ell^2 - \ell + 1$, there exists a
subgroup $L$ of $\text{Aut}(X)$ with

$$g_L = \bar{g} + \frac{(\ell + 1)(\ell^2 - 1)}{12d_1^2} \left( \frac{\ell^2 - \ell + 1}{d} - 1 \right),$$

with $\bar{g}$ as in (25).

**Remark 6.7.** When $L = \pi^{-1}(G)$ with $G$ a group of type (C), then by Corollary 3.4 the genus $g_L$ coincides with $g_L$. Some possibilities for $g_L$ are determined in [20] when either $G$ is isomorphic to $\Sigma_1$ as in cases (a)-(b), or $G$ is isomorphic to $\Upsilon$. It should be noted that other possibilities for $g_L$ are computed here, namely the integers $\bar{g}$ as in (21),(22),(23),(24),(25). To our knowledge these integers provide genera of quotient curves of the Hermitian curve that have not appeared in the literature so far.

### 7. Curves $X/L$ with $\bar{L}$ subgroup of a group of type (D)

In this section we will consider the case where $\bar{L}$ is contained in one of the following subgroups $G_1$ and $G_2$ of $\text{PGU}(3, \ell)$ stabilizing the line with equation $X = 0$.

- Let $\Psi_1$ be the subgroup of $\text{GL}(3, \ell^2)$ consisting of matrices

  $$M_{a_1,a_2,a_3,a_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

  with $a_1 = a_1^\ell$, $a_2 = a_4$, $a_3 = -a_3$, $a_2^\ell = -a_2$, $a_1a_4 - a_2a_3 = 1$. Let $\bar{G}_1$ be the subgroup of $\text{PGL}(3, \ell^2)$ of the projectivities defined by the matrices in $\Psi_1$. Clearly $\bar{G}_1$ is isomorphic to $\Psi_1$. It is straightforward to check that $\Psi_1$ is a subgroup of $\text{SU}(3, \ell)$; in particular, $G_1$ is a subgroup of $\text{PGU}(3, \ell)$ preserving the line $X = 0$. Also, it is easily seen that the map

  $$M_{a_1,a_2,a_3,a_4} \mapsto \begin{pmatrix} a_1 \\ a_2 \\ a_3 \lambda^{-1} \\ a_4 \end{pmatrix}$$

  with $\lambda^{\ell-1} = -1$, defines a isomorphism of $\Psi_1$ and $\text{SL}(2, \ell)$, and the action of $\bar{G}_1$ on the points of $\mathcal{H}$ on the line $X = 0$ is isomorphic to the action of $\text{SL}(2, \ell)$ on the projective line $\mathcal{P}G(1, \ell)$.

- Let $\bar{G}_2$ be the subgroup of $\text{GL}(3, \ell^2)$ generated by the projectivities defined by the matrices

  $$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_{a} = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{\ell+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  with $a$ ranging over the set of non-zero elements in $\mathbb{F}_{\ell^2}$. It is easily seen that $\bar{G}_2$ is a subgroup of $\text{PGU}(3, \ell)$ preserving the line with equation $X = 0$, and that $\bar{G}_2$ is isomorphic to the dihedral group of order $2(\ell^2 - 1)$. 

7.1. \textbf{L subgroup of G₁}. Let \( \Omega \) be the subgroup of \( PGL(4, \ell^2) \) consisting of matrices \( M_{a_1,a_2,a_3,a_4}, \) with \( M_{a_1,a_2,a_3,a_4} \in \Psi_1. \) As \( \Psi_1 \) is contained in \( SU(3, \ell), \) we have that \( \Omega \) is actually a subgroup of \( \Gamma. \) It is straightforward to check that \( \Omega \cap \Lambda \) is trivial. Also, \( \bar{G}_1 = \pi(\Omega) \) clearly holds.

**Lemma 7.1.** \textit{Any non-trivial element in \( \bar{G}_1 \) fixes no point of \( \mathcal{H} \) outside the line with equation \( X = 0. \)}

\textbf{Proof.} Assume that \( M_{a_1,a_2,a_3,a_4}(x_1,y_1,t_1)^t = \rho(x_1,y_1,t_1)^t \) for some \( \rho \in \mathbb{K}, \rho \neq 0, \) and for some \( (x_1,y_1,t_1) \in \mathbb{K}^3 \) with \( x_1 \neq 0 \) and \( x_1^{\ell^2+1} = y_1^\ell t_1 + y_1 t_1^\ell. \) Clearly this can only happen for \( \rho = 1 \) and \( y_1 t_1 \neq 0, \) whence we can assume that \( t_1 = 1 \) and that
\[
(a_1 - 1)y_1 + a_2 = 0, \quad a_3 y_1 + (a_4 - 1) = 0.
\]
Taking \( \ell \)-th powers in both equalities we then have
\[
(a_1 - 1)y_1^\ell - a_2 = 0, \quad -a_3 y_1^\ell + (a_4 - 1) = 0.
\]
It is then straightforward to deduce that \( a_1 = 1, \ a_2 = 0, \ a_3 = 0, \ a_4 = 1, \) that is \( M_{a_1,a_2,a_3,a_4} = I_3. \)

Assume that \( L \) is a tame subgroup of \( \text{Aut}(\mathcal{X}) \) with \( L = \Sigma_1 \times \Sigma_2, \) where \( \Sigma_1 < \Omega \) and \( \Sigma_2 < \Lambda. \) Let \( d = |\Sigma_2|. \) By Proposition 3.2 we then have that
\[
g_L = g_\bar{L} + \frac{(\ell^3 + 1)(\ell^2 - d - 1) - d(\ell^2 - d - 2)}{2d|\Sigma_1|}
\]
where \( g_\bar{L} \) is the genus of the quotient curve \( \mathcal{H}/\pi(\Sigma_1). \)

In [8] a number of genera of quotient curves \( \mathcal{H}/\pi(\Sigma_1) \) with \( \Sigma_1 \) a subgroup of \( \Omega \) are computed. The following results are then straightforward consequences of (26) together with Proposition 3.3 in [8].

**Theorem 7.2.** Assume that \( p = 2. \)

Let \( d \) be any divisor of \( \ell^2 - \ell + 1. \) Then there exist subgroups \( L \) of \( \text{Aut}(\mathcal{X}) \) with the following properties.

- For any \( e \mid \ell + 1, \ |L| = de \) and
  \[
g_L = \frac{1}{2e} \left( \frac{\ell^3 + 1}{d}(\ell^2 - d - 1) \right) + 1.
\]

- For any \( e \mid \ell - 1, \ |L| = de \) and
  \[
g_L = \frac{1}{2e} \left( \frac{\ell^3 + 1}{d}(\ell^2 - d - 1) - 2(e - 1) \right) + 1.
\]

**Theorem 7.3.** Assume that \( p \neq 2. \) Let \( d \) be any divisor of \( \ell^2 - \ell + 1. \) Then there exist subgroups \( L \) of \( \text{Aut}(\mathcal{X}) \) with the following properties.
• For any divisor \( e \) of \( (\ell + 1)/2 \), \(|L| = 2de\) and
\[
g_L = \frac{1}{4e} \left( \frac{\ell^3 + 1}{d} (\ell^2 - d - 1) - (\ell + 1) \right) + 1.
\]
• For any divisor \( e \) of \( (\ell + 1)/2 \), \(|L| = 4de\) and
\[
g_L = \frac{1}{8e} \left( \frac{\ell^3 + 1}{d} (\ell^2 - d - 1) - (\ell + 1) - 2o \right) + 1,
\]
with
\[
o = \begin{cases} 
2e & \text{if } \ell \equiv 1 \pmod{4} \\
0 & \text{if } \ell \equiv 3 \pmod{4}.
\end{cases}
\]
• For any divisor \( e \) or \( (\ell - 1)/2 \), \(|L| = 2de\) and
\[
g_L = \frac{1}{4e} \left( \frac{\ell^3 + 1}{d} (\ell^2 - d - 1) - (\ell + 1) - 4e + 4 \right) + 1.
\]
• For any divisor \( e \) or \( (\ell - 1)/2 \), \(|L| = 4de\) and
\[
g_L = \frac{1}{8e} \left( \frac{\ell^3 + 1}{d} (\ell^2 - d - 1) - (\ell + 1) - 2o \right) + 1
\]
with
\[
o = \begin{cases} 
4e - 2 & \text{if } \ell \equiv 1 \pmod{4} \\
2e - 2 & \text{if } \ell \equiv 3 \pmod{4}.
\end{cases}
\]
• When \( p \geq 5 \), \( \ell^2 \equiv 1 \pmod{16} \), \(|L| = 48d\) and
\[
g_L = \frac{1}{96} \left( \frac{\ell^3 + 1}{d} (\ell^2 - d - 1) - (\ell + 1) - 2o \right) + 1,
\]
with
\[
o = \begin{cases} 
46 & \text{if } \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 1 \pmod{3} \\
30 & \text{if } \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 2 \pmod{3} \\
16 & \text{if } \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 1 \pmod{3} \\
0 & \text{if } \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 2 \pmod{3}.
\end{cases}
\]
• When \( p \geq 5 \), \(|L| = 24d\) and
\[
g_L = \frac{1}{48} \left( \frac{\ell^3 + 1}{d} (\ell^2 - d - 1) - (\ell + 1) - 2o \right) + 1,
\]
with
\[
o = \begin{cases} 
22 & \text{if } \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 1 \pmod{3} \\
6 & \text{if } \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 2 \pmod{3} \\
16 & \text{if } \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 1 \pmod{3} \\
0 & \text{if } \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 2 \pmod{3}.
\end{cases}
\]
When $p \geq 7$, $\ell^2 \equiv 1 \pmod{5}$, $|L| = 120d$ and

$$g_L = \frac{1}{240}\left(\frac{\ell^3 + 1}{d}(\ell^2 - d - 1) - (\ell + 1) - 2o\right) + 1$$

with

$$o = \begin{cases} 
118 & \text{if } \ell \equiv 1 \pmod{3}, \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 1 \pmod{5} \\
78 & \text{if } \ell \equiv 2 \pmod{3}, \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 1 \pmod{5} \\
78 & \text{if } \ell \equiv 1 \pmod{3}, \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 1 \pmod{5} \\
48 & \text{if } \ell \equiv 2 \pmod{3}, \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 1 \pmod{5} \\
70 & \text{if } \ell \equiv 1 \pmod{3}, \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 4 \pmod{5} \\
30 & \text{if } \ell \equiv 2 \pmod{3}, \ell \equiv 1 \pmod{4} \text{ and } \ell \equiv 4 \pmod{5} \\
40 & \text{if } \ell \equiv 1 \pmod{3}, \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 4 \pmod{5} \\
0 & \text{if } \ell \equiv 2 \pmod{3}, \ell \equiv 3 \pmod{4} \text{ and } \ell \equiv 4 \pmod{5}
\end{cases}$$

Remark 7.4. It should be noted that by the discussion in [8, Sect. 3], Theorems 7.2 and 7.3 describe genera $g_L$ for all tame subgroups $L = \Sigma_1 \times \Sigma_2$ such that $\Sigma_1$ is a subgroup of $\Omega$ containing the diagonal matrix $J = \left[1, -1, 1, -1\right]$, and $\Sigma_2 < \Lambda$.

7.2. $\bar{L}$ subgroup of $\bar{G}_2$. We determine a subgroup $\Omega$ of $\Gamma$ such that $\pi(\Omega)$ is contained in $\bar{G}_2$ and $\Omega \cap \Lambda = \{id\}$ holds. Let $\mu_1$ be the maximum power of 3 dividing $\ell + 1$, and let $\gamma$ be a primitive $((\ell^2 - 1)/\mu_1)$-th root of unity in $\mathbb{F}_{\ell^2}$. Let $\bar{\Omega}$ the group generated by the projectivities defined by the matrices $V_\gamma = \left[\gamma^{2-\ell}, \gamma^{\ell+1}, \gamma, 1\right]$ and

$$\bar{W} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ 

Lemma 7.5. The group $\pi(\Omega)$ is contained in $\bar{G}_2$.

Proof. The image by $\pi$ of the projectivity defined by $V_\gamma$ is associated to the matrix

$$\begin{pmatrix} \gamma^{2-\ell} & 0 & 0 \\ 0 & \gamma^{\ell+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

This is the matrix $M_a$ for $a = \gamma^{2-\ell}$, as $(\gamma^{2-\ell})^{\ell+1} = \gamma^{-(\ell^2 - 1 - \ell - 1)} = \gamma^{\ell+1}$.

The image by $\pi$ of the projectivity defined by $\bar{W}$ is associated to the matrix

$$W' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Since for $a = -1$ we have $M_aW' = W$, this projectivity belong to $\bar{G}_2$, and the proof is complete. \qed
Lemma 7.6. The intersection $\Omega \cap \Lambda$ is trivial.

Proof. Any element $\alpha$ in $\Omega \cap \Lambda$ is defined by a diagonal matrix of $\Omega$, that is $[\gamma^{i(2-\ell)}, \gamma^{i(\ell+1)}, \gamma^i, 1]$ for some $i$. Moreover, $\gamma^{i(\ell^2-\ell+1)} = 1$. Since $\gamma^{i(\ell^2-1)/\mu_1} = 1$ and $\gcd(\ell^2-\ell+1, (\ell^2-1)/\mu_1) = 1$, this can only happen $\gamma^i = 1$. Then the assertion is proved. \hfill \Box

By the above lemmas, the map $\pi$ defines an isomorphism of $\Omega$ onto the subgroup of $\bar{G}_2$ generated by the projectivities defined by $W$ and $M_a$, with $a$ an $((\ell^2-1)/\mu_1)$-th root of unity. A number of genera of quotient curves $H/\pi(\Sigma_1)$ for $\Sigma_1$ tame subgroup of $\pi(\Omega)$ have been established in [20, Thm. 5.4, Ex. 5.6]. Taking into account Proposition 3.2, we are in a position to compute genera $g_L$ for tame subgroups $L = \Sigma_1 \times \Sigma_2$ with $\Sigma_1 < \Omega$ and $\Sigma_2 < \Lambda$.

Theorem 7.7. Assume that $p \neq 2$. Let $\mu_1$ be the highest power of 3 dividing $\ell + 1$. Let $d$ be any divisor of $\ell^2 - \ell + 1$. Then there exist subgroups $L$ of $\Aut(X)$ with the following properties.

- For any divisor $e$ of $(\ell^2 - 1)/\mu_1$, $|L| = 2ed$ and
  $$g_L = \frac{1}{4e}\left(\frac{(\ell^3 + 1)}{d}(\ell^2 - d - 1) + (\ell + 1)(1 - u - \tilde{u}) + 2(e + u) - \delta\right),$$
  where $u = \gcd(e, \ell + 1)$, $\tilde{u} = \gcd(e, \ell - 1)$,
  $$\delta = \begin{cases} 0 & \text{if } e \text{ divides } (\ell^2 - 1)/2 \\ e & \text{otherwise} \end{cases}.$$
- When $\ell \equiv 1 \pmod{4}$, for any even divisor $e$ of $\ell - 1$, $|L| = 2ed$ and
  $$g_L = \frac{1}{4e}\left(\frac{(\ell^3 + 1)}{d}(\ell^2 - d - 1) - \ell + 3\right).$$
- When $\ell \equiv 3 \pmod{4}$, for any even divisor $e$ of $\ell - 1$, $|L| = 2ed$ and
  $$g_L = \frac{1}{4e}\left(\frac{(\ell^3 + 1)}{d}(\ell^2 - d - 1) - \ell + 2e + 3\right).$$
- When $\ell \equiv 1 \pmod{4}$, for any odd divisor $e$ of $\ell - 1$, $|L| = 2ed$ and
  $$g_L = \frac{1}{4e}\left(\frac{(\ell^3 + 1)}{d}(\ell^2 - d - 1) + 2\right).$$
- When $\ell \equiv 3 \pmod{4}$, for any odd divisor $e$ of $\ell - 1$, $|L| = 2ed$ and
  $$g_L = \frac{1}{4e}\left(\frac{(\ell^3 + 1)}{d}(\ell^2 - d - 1) + 2e + 2\right).$$
Remark 7.8. When $L = \pi^{-1}(\bar{G})$ with $\bar{G}$ a group of type (D), then by Corollary 3.4 the genus $g_L$ coincides with $g_{\bar{L}}$. As already mentioned in the present section, a number of possibilities for $g_{\bar{L}}$ are determined in [8, Prop. 3.3], and in [20, Thm. 5.4].

8. The case $\ell = 5$

To exemplify how the results of this paper provide many new genera for $\mathbb{F}_{q^2}$-maximal curves we consider in this section the case $q = 5^3$. Up to our knowledge, the 32 integers listed in the following table are new values in the spectrum of genera of $\mathbb{F}_{5^6}$-maximal curves.

| $g$  | Ref. | $g$ | Ref. |
|------|------|-----|------|
| 5    | 4.7  | 9   | 4.5(i), 7.7 |
| 14   | 4.5(ii) | 21  | 4.5(ii) |
| 22   | 4.4 | 25  | 6.6 |
| 27   | 7.7  | 37  | 6.2, 6.5, 7.3, 7.7 |
| 38   | 4.5(i), 4.5(ii), 7.3, 7.7 | 70  | 4.5(ii), 5.7 |
| 73   | 6.6  | 74  | 7.3 |
| 76   | 4.5(i), 4.5(ii), 6.2, 6.3, 6.4, 7.3, 7.7 | 77  | 7.7 |
| 86   | 4.5(iv) | 109 | 6.2, 7.7 |
| 121  | 6.3 | 140 | 4.5(ii) |
| 148  | 6.7 | 180 | 6.4, 7.7 |
| 208  | 6.4 | 220 | 4.5(i), 4.5(ii), 6.2, 6.3, 6.4, 7.3, 7.7 |
| 221  | 7.6 | 241 | 6.6 |
| 242  | 7.3 | 282 | 4.5(iv) |
| 361  | 6.2, 7.7 | 362 | 4.5(i), 4.5(ii), 7.3, 7.7 |
| 442  | 4.5(iv), 5.4, 6.2, 6.3 | 484 | 5.7, 6.5 |
| 724  | 4.5(i), 4.5(ii), 6.2, 6.3, 6.4, 7.3, 7.7 | 725 | 7.7 |

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