Existence of a unique positive solution for a singular fractional boundary value problem

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Abstract

In the present work, we discuss the existence of a unique positive solution of a boundary value problem for nonlinear fractional order equation with singularity. Precisely, order of equation $D^\alpha_0 u(t) = f(t, u(t))$ belongs to $(3, 4]$ and $f$ has a singularity at $t = 0$ and as a boundary conditions we use $u(0) = u(1) = u'(0) = u'(1) = 0$. Using fixed point theorem, we prove the existence of unique positive solution of the considered problem.

Keywords: Nonlinear fractional differential equations, singular boundary value problem, positive solution.

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1. Introduction

In this paper, we study the existence and uniqueness of positive solution for the following singular fractional boundary value problem

\[
\begin{cases}
D^\alpha_0 u(t) = f(t, u(t)), \\ u(0) = u(1) = u'(0) = u'(1) = 0,
\end{cases}
\]

where $\alpha \in (3, 4]$, and $D^\alpha_0$ denotes the Riemann-Liouville fractional derivative. Moreover, $f : (0, 1] \times [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0^+} f(t, -) = \infty$ (i.e. $f$ is singular at $t = 0$).

Similar problem was investigated in [1], in case when $\alpha \in (1, 2]$ and with boundary conditions $u(0) = u(1) = 0$. We note as well work [2], where the following problem

\[
\begin{cases}
D^\alpha u + f(t, u, u', D^\mu u) = 0, \\ u(0) = u'(0) = u'(1) = 0,
\end{cases}
\]

was under consideration. Here $\alpha \in (2, 3), \mu \in (0, 1)$ and function $f(t, x, y, z)$ is singular at the value of 0 of its arguments $x, y, z$.

We would like notice some related recent works [3-5], which consider higher order fractional nonlinear equations for the subject of the existence of positive solutions.

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2. Preliminaries

We need the following lemma, which appear in [6].

**Lemma 1.** (Lemma 2.3 of [6]) Given \( h \in C[0, 1] \) and \( 3 < \alpha \leq 4 \), a unique solution of
\[
\begin{cases}
D^\alpha_{0+} u(t) = h(t), & 0 < t < 1 \\
u(0) = u(1) = u'(0) = u'(1) = 0,
\end{cases}
\]
is
\[
\begin{aligned}
\frac{1}{\Gamma(\alpha)} \int_0^1 G(t, s) h(s) ds,
\end{aligned}
\]
where
\[
G(t, s) = \begin{cases}
\frac{(t-s)^{\alpha-1}+(1-s)^{\alpha-2}[(s-t)+(\alpha-2)(1-t)]_{s+t}}{\Gamma(\alpha)}, & 0 \leq s \leq 1 \\
\frac{(1-s)^{\alpha-2}[(s-t)+(\alpha-2)(1-t)]_{s+t}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1
\end{cases}
\]

**Lemma 2.** (Lemma 2.4 of [6]) The function \( G(t, s) \) appearing in Lemma 1 satisfies:
(a) \( G(t, s) > 0 \) for \( t, s \in (0, 1) \);
(b) \( G(t, s) \) is continuous on \([0, 1] \times [0, 1]\).

For our study, we need a fixed point theorem. This theorem uses the following class of functions \( \mathcal{F} \).

By \( \mathcal{F} \) we denote the class of functions \( \varphi : (0, \infty) \to \mathbb{R} \) satisfying the following conditions:
(a) \( \varphi \) is strictly increasing;
(b) For each sequence \( (t_n) \subset (0, \infty) \)
\[
\lim_{n \to \infty} t_n = 0 \Leftrightarrow \lim_{n \to \infty} \varphi(t_n) = -\infty;
\]
(c) There exists \( k \in (0, 1) \) such that \( \lim_{t \to 0^+} t^k \varphi(t) = 0 \).

Examples of functions belonging to \( \mathcal{F} \) are \( \varphi(t) = -\frac{1}{\sqrt{t}}, \varphi(t) = \ln t, \varphi(t) = \ln t + t, \varphi(t) = \ln(t^2 + t) \).

The result about fixed point which we use is the following and it appears in [7]:

**Theorem 3.** Let \((X, d)\) be a complete metric space and \( T : X \to X \) a mapping such that there exist \( \tau > 0 \) and \( \varphi \in \mathcal{F} \) satisfying for any \( x, y \in X \) with \( d(Tx, Ty) > 0 \),
\[
\tau + \varphi(d(Tx, Ty)) \leq \varphi(d(x, y)).
\]

Then \( T \) has a unique fixed point.

3. Main result

Our starting point of this section is the following lemma.

**Lemma 4.** Let \( 0 < \sigma < 1, \ 3 < \alpha < 4 \) and \( F : (0, 1] \to \mathbb{R} \) is continuous function with \( \lim_{t \to 0^+} F(t) = \infty \). Suppose that \( t^\sigma F(t) \) is a continuous function on \([0, 1]\). Then the function defined by
\[
H(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 G(t, s) F(s) ds
\]
is continuous on $[0, 1]$, where $G(t, s)$ is the Green function appearing in Lemma 1.

Proof: We consider three cases:

1. **Case No 1.** $t_0 = 0$.
   It is clear that $H(0) = 0$. Since $t^\sigma F(t)$ is continuous on $[0, 1]$, we can find a constant $M > 0$ such that

   \[
   |t^\sigma F(t)| \leq M \text{ for any } t \in [0, 1].
   \]

   Moreover, we have

   \[
   |H(t) - H(0)| = |H(t)| = \left| \int_0^t G(t, s)F(s)ds \right| = \left| \int_0^t G(t, s)s^{-\sigma}s^\sigma F(s)ds \right| = \\
   \int_0^t \left| \frac{F(s)}{\Gamma(\alpha)} \right|^{\sigma-2} \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha)} ds s^{-\sigma} F(s)ds + \int_0^t \left| t-s \right|^{\alpha-1} s^{-\sigma} F(s)ds \leq \\
   \int_0^t \left( 1 - s \right)^{\alpha-2} \left( s-t \right)^{\alpha-1} s^{-\sigma} ds + \frac{M^{\alpha-2}}{\Gamma(\alpha)} \int_0^t \left( 1 - s \right)^{\alpha-2} s^{-\sigma} ds.
   \]

   Considering definition of Euler’s beta-function, we derive

   \[
   |H(t) - H(0)| \leq \frac{M^{\alpha-2}}{\Gamma(\alpha)} B(1 - \sigma, \alpha - 1) + \frac{M^{\alpha-2}}{\Gamma(\alpha)} B(1 - \sigma, \alpha).
   \]

   From this we deduce that $|H(t) - H(0)| \to 0$ when $t \to 0$.

   This proves that $H$ is continuous at $t_0 = 0$.

2. **Case No 2.** $t_0 \in (0, 1)$.
We take $t_n \to t_0$ and we have to prove that $H(t_n) \to H(t_0)$. Without loss of
generality, we consider $t_n > t_0$. Then, we have

$$
|H(t_n) - H(t_0)| = \left| \int_0^t (t_n - s)^{-\alpha - 1} \left[ (1-s)^{-\alpha - 2} \right] \left[ (s-t_n)^{\alpha - 2} \right] ds \right| 
$$

where

$$
= \left| \int_0^t (t_n - s)^{-\alpha - 2} \left[ (s-t_n)^{\alpha - 2} \right] ds \right| 
$$

By Lebesgue's dominated convergence theorem

$$
\int_0^t (t_n - s)^{-\alpha - 2} \left[ (s-t_n)^{\alpha - 2} \right] ds \to \int_0^t (t - s)^{-\alpha - 2} \left[ (s-t)^{\alpha - 2} \right] ds
$$

as $t_n \to t_0$. Therefore,

$$
\int_0^t (t_n - s)^{-\alpha - 2} \left[ (s-t_n)^{\alpha - 2} \right] ds \to \int_0^t (t - s)^{-\alpha - 2} \left[ (s-t)^{\alpha - 2} \right] ds
$$

as $t_n \to t_0$.

Now, we will prove that $I_n^1 \to 0$ when $n \to \infty$. In fact, as

$$
\left[ (t_n - s)^{\alpha - 1} - (t_0 - s)^{\alpha - 1} \right] s^{-\sigma} \leq \frac{1}{\Gamma(\alpha)} (\alpha - 1) s^{-\sigma} ds
$$

and

$$
\int_0^t 2s^{-\sigma} ds = \frac{2}{\Gamma(1-\sigma)} < \infty.
$$

By Lebesgue’s dominated convergence theorem $I_n^1 \to 0$ when $n \to \infty$.
Lemma 4, we derive

Then Problem (1) has a unique non-negative solution.

Taking \( L \) into account we infer

Denoting \( L(0) = 0 \) and as \( L \) into account we infer

we can prove that continuity of \( H \) at \( t_0 = 1 \).

**Lemma 5.** Suppose that \( 0 < \sigma < 1 \). Then there exists

\[
N = \max_{0 \leq t \leq 1} \int_{0}^{1} G(t, s)s^{-\sigma}ds.
\]

**Proof:** Considering representation of the function \( G(t, s) \) and evaluations of Lemma 4, we derive

\[
\int_{0}^{1} G(t, s)s^{-\sigma}ds = \frac{1}{\Gamma(\alpha)} \left[ t^{\alpha-\sigma} B(1-\sigma, \alpha) - t^{\alpha-1} B(1-\sigma, \alpha -1) + (\alpha-2)B(2-\sigma, \alpha -1) + (\alpha-1)t^{\alpha-2}B(2-\sigma, \alpha -1) \right].
\]

Taking

\[
B(1-\sigma, \alpha) = \frac{\alpha-1}{\alpha-\sigma} B(1-\sigma, \alpha -1); \quad B(2-\sigma, \alpha -1) = \frac{1-\sigma}{\alpha-\sigma} B(1-\sigma, \alpha -1),
\]

into account we infer

\[
\int_{0}^{1} G(t, s)s^{-\sigma}ds = \frac{B(1-\sigma, \alpha -1)}{\Gamma(\alpha)} \left[ \frac{\alpha-1}{\alpha-\sigma} t^{\alpha-\sigma} - \left( 1 + \frac{(\alpha-2)(1-\sigma)}{\alpha-\sigma} \right) t^{\alpha-1} + \frac{(\alpha-1)(1-\sigma)}{\alpha-\sigma} t^{\alpha-2} \right].
\]

Denoting \( L(t) = \int_{0}^{1} G(t, s)s^{-\sigma}ds \), from the last equality one can easily derive that \( L(0) = 0 \), \( L(1) = 0 \). Since \( G(t, s) \geq 0 \), then \( L(t) \geq 0 \) and as \( L(t) \) is continuous on \([0, 1]\), it has a maximum. This proves Lemma 5.

**Theorem 6.** Let \( 0 < \sigma < 1 \), \( 3 < \alpha \leq 4 \), \( f : (0, 1] \times [0, \infty) \) be continuous and \( \lim_{t \to 0^+} f(t, \cdot) = \infty \), \( t^\sigma f(t, y) \) be continuous function on \([0, 1] \times [0, \infty) \). Assume that there exist constants \( 0 < \lambda \leq \frac{1}{N} \), and \( \tau > 0 \) such that for \( x, y \in [0, \infty) \) and \( t \in [0, 1] \)

\[
|t^\sigma f(t, x) - f(t, y)| \leq \frac{\lambda |x - y|}{\left( 1 + \tau \sqrt{|x - y|} \right)^2}.
\]

Then Problem (1) has a unique non-negative solution.

**Proof:** Consider the cone \( P = \{ u \in C[0, 1] : u \geq 0 \} \). Notice that \( P \) is a closed subset of \( C[0, 1] \) and therefore, \((P, d)\) is a complete metric space where

\[
d(x, y) = \sup \{ |x(t) - y(t)| : t \in [0, 1] \} \quad \text{for} \; x, y \in P.
\]

Now, for \( u \in P \) we define the operator \( T \) by

\[
(Tu)(t) = \int_{0}^{1} G(t, s)f(s, u(s))ds = \int_{0}^{1} G(t, s)s^{-\sigma}s^\sigma f(s, u(s))ds.
\]
In virtue of Lemma 4, for $u \in P$, $Tu \in C[0, 1]$ and, since $G(t, s)$ and $t^\sigma f(t, y)$ are non-negative functions, $Tu \geq 0$ for $u \in P$. Therefore, $T$ applies $P$ into itself.

Next, we check that assumptions of Theorem 3 are satisfied. In fact, for $u, v \in P$ with $d(Tu, Tv) > 0$, we have

$$d(Tu, Tv) = \max_{t \in [0, 1]} |(Tu)(t) - (Tv)(t)| =$$

$$= \max_{t \in [0, 1]} \left| \int_0^1 G(t, s)s^{-\sigma} s^{\sigma}(f(s, u(s)) - f(s, v(s))) \, ds \right| \leq$$

$$\leq \max_{t \in [0, 1]} \int_0^1 G(t, s)s^{-\sigma} |f(s, u(s)) - f(s, v(s))| \, ds \leq$$

$$\leq \max_{t \in [0, 1]} \int_0^1 G(t, s)s^{-\sigma} \frac{\lambda[u(s) - v(s)]}{(1 + \tau \sqrt{|u(s) - v(s)|})} \, ds \leq \max_{t \in [0, 1]} \int_0^1 G(t, s)s^{-\sigma} \frac{\lambda d(u, v)}{(1 + \tau \sqrt{d(u, v)})} \, ds =$$

$$= \frac{\lambda d(u, v)}{(1 + \tau \sqrt{d(u, v)})} \max_{t \in [0, 1]} \int_0^1 G(t, s)s^{-\sigma} \, ds = \frac{\lambda d(u, v)}{(1 + \tau \sqrt{d(u, v)})} N \leq \frac{d(u, v)}{(1 + \tau \sqrt{d(u, v)})},$$

where we have used that $\lambda \leq \frac{1}{N}$ and the non-decreasing character of the function $\beta(t) = \frac{t}{(1 + \tau \sqrt{t})}$. Therefore,

$$d(Tu, Tv) \leq \frac{d(u, v)}{(1 + \tau \sqrt{d(u, v)})^2}.$$

This gives us

$$\sqrt{d(Tu, Tv)} \leq \frac{\sqrt{d(u, v)}}{1 + \tau \sqrt{d(u, v)}}$$

or

$$\tau - \frac{1}{\sqrt{d(Tu, Tv)}} \leq -\frac{1}{\sqrt{d(u, v)}}$$

and the contractivity condition of the Theorem 3 is satisfied with the function $\varphi(t) = -\frac{1}{\sqrt{t}}$ which belongs to the class $\mathfrak{F}$.

Consequently, by Theorem 3, the operator $T$ has a unique fixed point in $P$. This means that Problem (1) has a unique non-negative solution in $C[0, 1]$. This finishes the proof.

An interesting question from a practical point of view is that the solution of Problem (1) is positive. A sufficient condition for that solution is positive, appears in the following result:

**Theorem 7.** Let assumptions of Theorem 6 be valid. If the function $t^\sigma f(t, y)$ is non-decreasing respect to the variable $y$, then the solution of Problem (1) given by Theorem 6 is positive.

**Proof:** In contrary case, we find $t^* \in (0, 1)$ such that $u(t^*) = 0$. Since $u(t)$ is a fixed point of the operator $T$ (see Theorem 6) this means that

$$u(t) = \int_0^1 G(t, s)f(s, u(s)) \, ds \text{ for } 0 < t < 1.$$
Particularly,

\[ 0 = u(t^*) = \int_0^1 G(T^*, s)f(s, u(s))ds. \]

Since that \( G \) and \( f \) are non-negative functions, we infer that

\[ G(t^*, s)f(s, u(s)) = 0 \text{ a.e. (s)} \quad (2) \]

On the other hand, as \( \lim_{t \to 0^+} f(t, 0) = \infty \) for given \( M > 0 \) there exists \( \delta > 0 \) such that for \( s \in (0, \delta) \) \( f(s, 0) > M \). Since \( t^\sigma f(t, y) \) is increasing and \( u(t) \geq 0, \)

\[ s^\sigma f(s, u(s)) \geq s^\sigma f(s, 0) \geq s^\sigma M \text{ for } s \in (0, \delta) \]

and, therefore, \( f(s, u(s)) \geq M \) for \( s \in (0, \delta) \) and \( f(s, u(s)) \neq 0 \text{ a.e. (s)} \). But this is a contradiction since \( G(t^*, s) \) is a function of rational type in the variable \( s \) and, consequently, \( G(t^*, s) \neq 0 \text{ a.e. (s)} \). Therefore, \( u(t) > 0 \) for \( t \in (0, 1) \). This finishes the proof.

4. Acknowledgement

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References

[1] J.Caballero, J.Harjani, K.Sadarangani. Positive solutions for a class of singular fractional boundary value problems. Computers and Mathematics with Applications. 62(2011), pp.1325-1332.
[2] S.Stańko. The existence of positive solutions of singular fractional boundary value problems. Computers and Mathematics with Applications. 62(2011), pp.1379-1388.
[3] Z.Bai, W.Sun. Existence and multiplicity of positive solutions for singular fractional boundary problems. Computers and Mathematics with Applications. 63(2012), pp.1369-1381.
[4] J.Xu, Z.Wei, W.Dong. Uniqueness of positive solutions for a class of fractional boundary value problems. Applied Mathematics Letters. 25(2012), pp.590-593.
[5] S.Zhang. Positive solutions to singular boundary value problem for nonlinear fractional differential equation. Computers and Mathematics with Applications. 59(2010), pp.1300-1309.
[6] X.Xu, D.Jiang, C.Yuan. Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation. Nonlinear Analysis 71 (2009), pp. 4676-4688.
[7] D.Wardowski. Fixed points of a new type of contractive mappings in complete metric spaces. Fixed point theory and applications (2012), 2012:14.