SOLVABILITY OF THE $H^\infty$ ALGEBRAIC RICCATI EQUATION IN BANACH ALGEBRAS

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Abstract. Let $R$ be a commutative complex unital semisimple Banach algebra with the involution $\cdot^\star$. Sufficient conditions are given for the existence of a stabilizing solution to the $H^\infty$ Riccati equation when the matricial data has entries from $R$. Applications to spatially distributed systems are discussed.

1. Introduction

In the standard (regular full information) $H^\infty$ control problem, one wants to find an appropriate control input $u$ such that the effect of the disturbance $w$ on the output $z$ is minimized for the control system given by the following dynamics:

\begin{align}
  x'(t) &= Ax(t) + Bu(t) + Ew(t), \\
  z(t) &= Cx(t) + D_1u(t) + D_2w(t),
\end{align}

where $x(t) \in \mathbb{C}^n$, $u(t) \in \mathbb{C}^m$, $w(t) \in \mathbb{C}^\ell$, $z(t) \in \mathbb{C}^p$, and $A, B, E, C, D_1, D_2$ are complex matrices of appropriate sizes. More precisely, one seeks a feedback law

$$u(t) = Fx(t)$$

such that the closed loop system obtained in this manner is internally stable and its transfer function has $H^\infty$ norm strictly less than some a priori given bound $\gamma > 0$. The following result is well-known; see for example [8, Theorem 3.1, p.49]:

**Theorem 1.1.** Consider (1.1), (1.2), and let $\gamma > 0$. Suppose that the tuple $(A, B, C, D_1)$ has no invariant zeros on the imaginary axis, and that $\ker D_1 = \{0\}$. Then the following are equivalent:

1. There exists a matrix $F$ such that the closed loop system obtained by applying the feedback $u(t) = Fx(t)$ is internally stable, and the $H^\infty$ norm of the closed loop transfer function is strictly less than $\gamma$.

2. $D_2^*D_2 < \gamma^2 I$. Moreover, there exists a positive semi-definite solution of the algebraic Riccati equation

$$(1.3) \quad 0 = A^*P + PA + C^*C - \left[ \begin{array}{cc} B^*P + D_1^*C \\ E^*P + D_2^*C \end{array} \right]^* \left[ \begin{array}{cc} D_1^*D_1 & D_1^*D_2 \\ D_2^*D_1 & D_2^*D_2 - \gamma^2 I \end{array} \right]^{-1} \left[ \begin{array}{cc} B^*P + D_1^*C \\ E^*P + D_2^*C \end{array} \right],$$

such that $A_{cl}$ is exponentially stable, that is, $\text{Re}(\lambda) < 0$ for all eigenvalues $\lambda$ of $A_{cl}$, where

$$(1.4) \quad A_{cl} := A - \left[ \begin{array}{cc} B & E \end{array} \right] \left[ \begin{array}{cc} D_1^*D_1 & D_1^*D_2 \\ D_2^*D_1 & D_2^*D_2 - \gamma^2 I \end{array} \right]^{-1} \left[ \begin{array}{cc} B^*P + D_1^*C \\ E^*P + D_2^*C \end{array} \right].$$

1991 Mathematics Subject Classification. Primary 46J05; Secondary 93D15, 58C15, 47N20.

Key words and phrases. Riccati equations, Banach algebras, Systems over rings, $H^\infty$ control, Spatially distributed dynamical systems.
If \( P \) satisfies the conditions in (2) above, then the matrix \( F \) satisfying (1) can be taken as

\[
F := -(D_1^*(I - \gamma^{-2}D_2D_2^*)^{-1}D_1)^{-1}(D_1^*C + B^*P + D_1^*D_2(\gamma^2 I - D_2^*D_2)^{-1}(D_2^*C + E^*P)).
\]

The interest in systems over rings has recently been revived owing to potential applications for control theoretic problems of spatially invariant systems [1]. In particular, a natural question that arises is the following: if the matricial data in the algebraic Riccati equation has entries from a Banach algebra \( R \), then does there exist a solution matrix \( P \) which also has entries from \( R \)? In the context of the algebraic Riccati equation associated with the problem of optimal control for a linear control system with a quadratic cost (LQ problem), this was done recently in [4]. In this article, we continue our study, and show that an analogous result is identical to the one followed in [4], the result in the current article does not follow from [4], and it is not automatic. There are salient differences in the proof:

1. One of the key differences is that the quadratic term of the form \( P^*QP \) in the algebraic Riccati equation has a positive semi-definite matrix \( Q \) in the LQ problem, whereas the matrix \( Q \) is indefinite in the \( \mathcal{H}^\infty \) control problem. In particular, this manifests in that we need to prove the technical result (Proposition 2.1) on the continuous dependence of the stabilizing solution of the \( \mathcal{H}^\infty \) algebraic Riccati equation on the data.

2. The second new feature is the presence of the inverse of a matrix with feedthrough terms \( D_1, D_2 \) in the \( \mathcal{H}^\infty \) algebraic Riccati equation. This results in an extra complication in the application of the inverse function theorem for Banach algebras, namely, the application of Proposition 2.2.

We begin by fixing some notation.

**Notation 1.2.** Throughout the article, \( R \) will denote a commutative, unital, complex, semisimple Banach algebra, which possesses an involution \( \cdot^* \).

On the other hand, the usual adjoint of a matrix \( M = [m_{ij}] \in \mathbb{C}^{p \times m} \) will be denoted by \( M^* \in \mathbb{C}^{m \times p} \), that is, \( M^* = [m_{ji}] \).

\( M(R) \) will denote the maximal ideal space of \( R \), equipped with the weak-* topology. For \( x \in R \), we will denote its Gelfand transform by \( \hat{x} \), that is,

\[
\hat{x}(\varphi) = \varphi(x), \quad \varphi \in M(R), \quad x \in R.
\]

For a matrix \( M \in R^{p \times m} \), whose entry in the \( i \)th row and \( j \)th column is denoted by \( m_{ij} \), we define \( M^* \in R^{m \times p} \) to be the matrix whose entry in the \( i \)th row and \( j \)th column is \( m_{ji}^* \).

Also by \( \hat{M} \) we mean the \( p \times m \) matrix, whose entry in the \( i \)th row and \( j \)th column is the continuous function \( \hat{m}_{ij} \) on \( M(R) \). Summarizing, if \( M = [m_{ij}] \in R^{p \times m} \), then

\[
M^* = [m_{ij}^*] \in R^{m \times p},
\]

\[
\hat{M} = [\hat{m}_{ij}] \in (C(M(R); \mathbb{C}))^{p \times m},
\]

\[
(\hat{M}(\varphi))^* = [\hat{m}_{ji}(\varphi)] \in \mathbb{C}^{p \times m}.
\]

Our main result is the following.

**Theorem 1.3.** Let \( A \in R^{n \times n} \), \( B \in R^{n \times m} \), \( C \in R^{p \times n} \), \( D_1 \in R^{p \times m} \), \( D_2 \in R^{p \times \ell} \), \( E \in R^{n \times \ell} \). Suppose that for all \( \varphi \in M(R) \),
The absolute convergence of this series is established just as in the scalar case.

Then there exists a \( P \) and \( A \)

For \( A \)

Definition 1.4.

In the following we define what is meant by “exponentially stable” in the conclusion (2) in

Then there exists a \( P \in R^{n \times n} \) such that

(1) \( 0 = A^*P + PA + C^*C \)

(2) The matrix

is exponentially stable, and

(3) for all \( \varphi \in M(R) \), \( \hat{P}(\varphi) \) is positive semidefinite.

In the following we define what is meant by “exponentially stable” in the conclusion (2) in

Definition 1.4. Let \( R \) be a commutative, unital, complex, semisimple Banach algebra. If

\( A \in R^{n \times n} \), then let \( M_A : R^n \to R^n \) be the multiplication map by the matrix \( A \), that is, \( v \mapsto Av (v \in R^n) \). Then \( R^{n \times n} \) is a unital complex Banach algebra (for example) with the norm

\[ \|A\| := \|M_A\|_{\mathcal{L}(R^n)} \quad (A \in R^{n \times n}) \]

where \( \mathcal{L}(R^n) \) denotes the set of all continuous linear transformations from \( R^n \) to \( R^n \), and \( R^n \)

is the Banach space equipped (for example) with the norm

\[ \|x\| = \max\{\|x_k\| : 1 \leq k \leq n\} \text{ for } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \]

and \( \mathcal{L}(R^n) \) is equipped with the usual operator norm:

\[ \|M_A\|_{\mathcal{L}(R^n)} = \sup\{\|Av\| : v \in R^n \text{ with } \|v\| \leq 1\}. \]

For \( A \in R^{n \times n} \), we define

\[ e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \]

The absolute convergence of this series is established just as in the scalar case.

\( A \in R^{n \times n} \) is said to be exponentially stable if there exist positive constants \( C \) and \( \epsilon \) such that

\[ \|e^{tA}\| \leq Ce^{-\epsilon t} \text{ for all } t \geq 0. \]
We recall the following two results [4, Lemma 1.6, Prop.1.7].

**Lemma 1.5.** Let $A \in \mathbb{R}^{n \times n}$. Then the following are equivalent:

1. $\lambda$ belongs to the spectrum of $A \in \mathbb{R}^{n \times n}$.
2. $\lambda$ belongs to the spectrum of $M_A \in \mathcal{L}(\mathbb{R}^n)$.
3. $\lambda$ belongs to the spectrum of $\hat{A}(\varphi)$ for some $\varphi \in M(\mathbb{R})$.

**Proposition 1.6.** Let $A \in \mathbb{R}^{n \times n}$. Then $A$ is exponentially stable if and only if

$$\sup\{\Re(\lambda) : \lambda \text{ is an eigenvalue of } \hat{A}(\varphi) \text{ for some } \varphi \in M(\mathbb{R})\} < 0.$$ 

The proof in Section 2 of our main result above is established by taking the Gelfand transform of our equation, and showing that the pointwise solution is continuous. Then we use the Banach algebra operational calculus to ensure that this continuous solution is actually the Gelfand transform of a matrix with entries from the Banach algebra.

In Section 3 we discuss the applications of this result to the control of spatially invariant systems.

2. **Proof of the main result**

We will need the following two results. The first one gives sufficient conditions on the matricial data for the classical $H^\infty$ algebraic Riccati equation to have a positive semi-definite stabilizing solution, and moreover it says the smallest such solution $P$ depends continuously on the matricial data.

**Proposition 2.1.** Let $S$ be the set of matrix tuples $(A, B, C, D_1, D_2, E) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{p \times n} \times \mathbb{C}^{p \times q} \times \mathbb{C}^{n \times \ell} \times \mathbb{C}^{p \times \ell}$ such that

1. $(A, B, D_1)$ is left invertible,
2. there exist matrices $F_1, F_2$ such that $A + BF_1$ is exponentially stable, and
3. $(C + DF_1)(sI - A - BF_1)^{-1}(E + BF_2) + D_2 + D_1F_2\|_\infty < \gamma$,
4. ker $D_1 = \{0\}$.

Then there exists a smallest positive semidefinite, solution $P = P(A, B, C, D_1, D_2, E) \in \mathbb{C}^{n \times n}$ of the Riccati equation (1.3) such that $A_{cl}$ given by (1.4) is exponentially stable. Moreover, the map $(A, B, C, D_1, D_2, E) : S \to \mathbb{C}^{n \times n}$ is continuous.

**Proof.** The proof is the same as that of [8, Lemma 3.1], mutatis mutandis, except we have complex matrices instead of real ones, and transposes there have to be replaced by the Hermitian adjoints. We give repeat the proof here. The existence and uniqueness of the stabilizing solution is well-known; see [8]. The continuity follows in a straightforward manner. The solution $P$ is associated with the stable subspace of a Hamiltonian matrix. Since this Hamiltonian matrix has no eigenvalues on the imaginary axis, the eigenvalues in the open left half plane and the open right half plane are separated, and the existence of a continuous basis for the stable subspace and hence the continuous dependence of the stabilizing solution of the $H^\infty$ algebraic Riccati equation can be found for example in [7].

The next result we will need is the following version of the Implicit Function Theorem in the context of Banach algebras (see [5, p.155]), and this will be used in proving our main results when we need to pass from continuous functions on $M(R)$ to elements of $R$. 

□
Proposition 2.2. Let $h_1, \ldots, h_s$ be continuous functions on $M(R)$. Suppose that $f_1, \ldots, f_r$ in $R$ and $G_1(z_1, \ldots, z_{s+r}), \ldots, G_t(z_1, \ldots, z_{s+r})$ are holomorphic functions with $t \geq s$ defined on a neighbourhood of the joint spectrum

$$
\sigma(h_1, \ldots, h_s, f_1, \ldots, f_r) := \{(h_1(\varphi), \ldots, h_s(\varphi), \hat{f}_1(\varphi), \ldots, \hat{f}_r(\varphi)) : \varphi \in M(R)\},
$$

such that

$$(2.1) \quad G_k(h_1, \ldots, h_s, \hat{f}_1, \ldots, \hat{f}_r) = 0 \text{ on } M(R) \text{ for } 1 \leq k \leq t.$$
(In the above, we have the replacements of
\[ A, A^*, B, B^*, C, C^*, D_1, D_1^*, D_2, D_2^*, E, E^* \]
by the complex variables which are the components of
\[ U_1, U_2, V_1, V_2, W_1, W_2, X_1, X_2, Y_1, Y_2, Z_1, Z_2, \]
respectively. The replacements of the components of \( \Pi \) in the Riccati equation are by the complex variables which are the components of \( \Theta \).) Since the set of invertible matrices is open, it follows that
\[
(X_1, X_2, Y_1, Y_2) \mapsto \begin{bmatrix} X_2X_1 & X_2Y_1 \\ Y_2X_1 & Y_2Y_1 - \gamma^2I \end{bmatrix}^{-1}
\]
is holomorphic on the joint spectrum of \( (D_1, D_1^*, D_2, D_2^*) \). Consequently the components \( G_1, \ldots, G_{n^2} \) of above map \( G \) is holomorphic on the joint spectrum.

There is a continuous solution \( \Pi \) on the maximal ideal space such that for all \( k \),
\[ G_k(\Pi, \hat{A}, \hat{A}^*, \hat{B}, \hat{B}^*, \hat{C}, \hat{C}^*, \hat{D}_1, \hat{D}_1^*, \hat{D}_2, \hat{D}_2^*, \hat{E}, \hat{E}^*) = 0 \]
on \( M(R) \) (that is, condition (2.1) in Proposition 2.2 is satisfied). So we now investigate the Jacobian with respect to the variables in \( \Theta \). The Jacobian with respect to the \( \Theta \) variables at the point
\[
\left( \Pi(\varphi), \hat{A}(\varphi), (\hat{A}(\varphi))^*, \hat{B}(\varphi), (\hat{B}(\varphi))^*, \hat{C}(\varphi), (\hat{C}(\varphi))^*, \hat{D}_1(\varphi), (\hat{D}_1(\varphi))^*, \hat{D}_2(\varphi), (\hat{D}_2(\varphi))^*, \hat{E}(\varphi), (\hat{E}(\varphi))^* \right)
\]
can be verified to be the following linear transformation \( \Lambda \) from \( \mathbb{C}^{n^2} \rightarrow \mathbb{C}^{n^2} \):
\[
\Theta \mapsto \Theta A_{cl}(\varphi) + (A_{cl}(\varphi))^* \Theta.
\]
The set of eigenvalues of \( \Lambda \) consists of the numbers
\[-(\lambda + \mu),\]
where \( \lambda, \mu \) belong to the set of eigenvalues of \( A_{cl}(\varphi) \); see for example [2, Proposition 7.2.3]. But since \( A_{cl}(\varphi) \) is exponentially stable, all its eigenvalues have a negative real part. Hence \(- (\lambda + \mu) \neq 0\) for all \( \lambda, \mu \) belonging to the set of eigenvalues of the matrix \( A_{cl}(\varphi) \). Consequently, the map \( \Lambda \) is invertible from \( \mathbb{C}^{n^2} \) to \( \mathbb{C}^{n^2} \), and its rank is \( n^2 = s \). So by Proposition 2.2 there exists a \( P \in \mathbb{R}^{n \times n} \) such that \( \tilde{P}(\varphi) = \Pi(\varphi) \) for all \( \varphi \in M(R) \). From (2.3), it follows (using the fact that \( R \) is semisimple) that
\[
0 = A^*P + PA + C^*C
\]
\[-[PB + C^*D_1 \quad PE + C^*D_2] \begin{bmatrix} D_1 D_1^* & D_1 D_2^* \\ D_2 D_1^* & D_2 D_2^* - \gamma^2I \end{bmatrix}^{-1} \begin{bmatrix} B^*P + D_1^*C \\ E^*P + D_2^*C \end{bmatrix}.
\]
From the property possessed by the pointwise solutions \( \Pi(\varphi) \) (\( \varphi \in M(R) \)) of the constant complex matricial Riccati equations (2.2), we have that for all \( \varphi \in M(R) \), all eigenvalues of \( A_{cl}(\varphi) \) have a negative real part. But the set-valued map taking a square complex matrix of size \( n \times n \) to its spectrum (a set of \( n \) complex numbers) is a continuous map; see for example [3, II, §5, Theorem 5.14, p.118]. Since \( M(R) \) is compact in the Gelfand topology (the weak-* topology induced on \( M(R) \) considered as a subset of \( L(R; \mathbb{C}) \)), it follows that
\[
\sup \{\text{Re}(\lambda) : \lambda \text{ is an eigenvalue of } A_{cl}(\varphi) \text{ for some } \varphi \in M(R) \} < 0.
\]
From Proposition 1.6, it follows that $A_{cl}$ is exponentially stable. Finally, again by the property possessed by the pointwise solution II, we have that for all $\varphi \in M(R)$, $\hat{P}(\varphi)$ is positive semidefinite. This completes the proof of Theorem 1.3.

We observe that whether or not the assumption (A1) in Theorem 1.3 holds is intimately related to the choice of the involution $\cdot^*$ in the Banach algebra $R$. For some commutative Banach algebras with involutions, this is automatic, namely if it is symmetric; see [4].

3. Application to Spatially Invariant Systems

Finally, we mention the application of our main result to control problems for spatially invariant systems introduced in [1]. The analysis of spatially invariant systems can be greatly simplified by taking Fourier transforms, see [1], [3]. This yields systems described by multiplication operators with symbols belonging to $L^\infty(\mathbb{T})$. The $H^\infty$ control design is to use the feedback $F$, which uses the bounded, self-adjoint, stabilizing solution $P$ to the $H^\infty$ algebraic Riccati equation on the Hilbert space $(L^2(\mathbb{T}))^n$. For the design of implementable controllers it is important that the gain operator have a spatially decaying property (see [1]). This translates into the mathematical question of when the $H^\infty$ algebraic Riccati equation has a stabilizing solution in a suitable subalgebra (for example, $L^1(\mathbb{T})$ is a subalgebra of $L^\infty(\mathbb{T}) \subset L(L^2(\mathbb{T}))$). So the spatially decaying property now translates into finding suitable subalgebras of $L^\infty(\mathbb{T})$, in particular, the weighted Wiener algebras. In [4], p.472-474, an example of a large class of the Wiener subalgebra of $L^\infty(\mathbb{T})$ having the symmetry property (which leads to automatic satisfaction of the condition (A1) in our main result) was given.

Acknowledgements: The author thanks Ruth Curtain for suggesting to the author the question of investigating if the result from [4] is valid also for the $H^\infty$ algebraic Riccati equation, and also for useful discussions on the classical $H^\infty$ problem.

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