Quasiinvariants of Coxeter groups and $m$-harmonic polynomials

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Abstract

The space of $m$-harmonic polynomials related to a Coxeter group $G$ and a multiplicity function $m$ on its root system is defined as the joint kernel of the properly gauged invariant integrals of the corresponding generalised quantum Calogero-Moser problem. The relation between this space and the ring of all quantum integrals of this system (which is isomorphic to the ring of corresponding quasiinvariants) is investigated.

1 Introduction

Let $G$ be any Coxeter group, i.e. a finite group generated by reflections with respect to some hyperplanes in a Euclidean space $V$ of dimension $n$. Let $\Sigma$ be a set of the hyperplanes $\Pi_\alpha : (\alpha, x) = 0$ corresponding to all the reflections $s_\alpha \in G$ and let $A$ be a set of the corresponding (arbitrarily chosen) normals $\alpha$. Let us also consider a $G$-invariant $\mathbb{Z}_{\geq 0}$-valued function $m$ on $\Sigma$ which will be called multiplicity. In other words to any hyperplane $\Pi_\alpha \in \Sigma$ we prescribe a nonnegative integer $m_\alpha$ such that if $\Pi_\alpha = g(\Pi_\beta)$, $g \in G$ then $m_\alpha = m_\beta$. Let $S = S(V)$ be the ring of all polynomials on $V$, $S^G$ be the subring of $G$-invariant polynomials. According to classical Chevalley result $[\mathbb{F}]$, $S^G$ is freely generated by some homogeneous polynomials $\sigma_1, \ldots, \sigma_n$.

The main object of our investigation is the following subring $Q_m = Q_m(\Sigma) \subset S(V)$. It consists of the polynomials $q$ which are invariant up to order $2m_\alpha$ with respect to any reflection $s_\alpha$:

$$q(s_\alpha(x)) = q(x) + o((\alpha, x)^{2m_\alpha})$$  \hfill (1)
near the hyperplane \((\alpha, x) = 0\) for any \(\alpha \in A\). Equivalently, for any \(\alpha \in A\) the normal derivatives \(\partial^s_\alpha q = (\alpha, \frac{\partial}{\partial x})^s q\) must vanish on \(\Pi_\alpha\) for \(s = 1, 3, 5, \ldots, 2m_\alpha - 1\):

\[ \partial^s_\alpha q|_{\Pi_\alpha} = 0. \]

We will call these polynomials \(m\)-quasiinvariants of the Coxeter group \(G\) or simply quasiinvariants.

The rings \(Q_m\) have been introduced in the theory of quantum Calogero–Moser systems by O. Chalykh and one of the authors [2]. It has been shown [2, 3] that for any Coxeter group \(G\) and any integer-valued multiplicity function \(m\) there exists a homomorphism

\[ \varphi_m : Q_m \to D_\Sigma(V), \]

where \(D_\Sigma(V)\) is the ring of all differential operators in \(V\) with rational coefficients from the algebra generated by \((\alpha, x)^{-1}, \alpha \in A\) and constant functions (see the next section for the details). In particular, for \(q = x^2\) (which is obviously invariant and therefore quasiinvariant) the corresponding operator \(\varphi_m(q)\) is the generalised Calogero–Moser operator

\[ L = \Delta - \sum_{\alpha \in A} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \]  

(2)

first introduced by Olshanetsky and Perelomov [4]. It will be more convenient for us to use the gauge transformation \(L = \hat{g}L\hat{g}^{-1}\), where \(\hat{g}\) is the operator of multiplication by \(g = \prod(\alpha, x)^{m_\alpha}\), after which the operator \(L\) takes the form

\[ \mathcal{L} = \Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)^2} \partial_\alpha. \]  

(3)

This gauge is natural from the point of view of the theory of symmetric spaces where such operators appear as the radial parts of the Laplace–Beltrami operators (see e.g. [5]). Let

\[ \chi_m : Q_m \to D_\Sigma(V) \]

be the corresponding gauged version of the homomorphism \(\varphi_m\):

\[ \chi_m(q) = \hat{g}\varphi_m(q)\hat{g}^{-1}. \]

We should mention that for a generic (not necessary integer-valued) multiplicity function there exists an isomorphism (sometimes called as Harish-Chandra isomorphism, see [3, 4]) between \(S^G\) and the ring \(D^G_m\) of \(G\)-invariant quantum integrals of the Calogero-Moser problem (2):

\[ \gamma_m : S^G \cong D^G_m. \]
where
\[ D^G_m = \{ D \in D_\Sigma(V) : [\mathcal{L}, D] = 0, g(D) = D \text{ for all } g \in G \}. \]

As a corollary we have usual integrability for (3) with \( n \) commuting quantum integrals \( \mathcal{L}_1 = \mathcal{L}, \mathcal{L}_2, \ldots, \mathcal{L}_n \) corresponding to some basic invariants \( \sigma_1 = x^2, \sigma_2, \ldots, \sigma_n. \)

An important novelty of [4] was the possibility of the extension of \( \gamma^m \) to a much bigger ring \( Q_\alpha \) in the case when all \( m_\alpha \) are integer, which implies the algebraic integrability of the corresponding quantum Calogero-Moser problem (see [4] for details).

The primary goal of the present paper is to explain that all the information about the additional quantum integrals of Calogero-Moser problem is actually contained in the joint kernel of its standard invariant integrals \( \mathcal{L}_1, \ldots, \mathcal{L}_n. \) More precisely, let us define the space \( H_\alpha \) as the solutions of the following system

\[
\begin{cases}
\mathcal{L}_1 \psi = 0 \\
\mathcal{L}_i \psi = 0 \\
\mathcal{L}_n \psi = 0
\end{cases}
\]

We will show that all the solutions of (4) are polynomial and that all these polynomials are \( m \)-quasiinvariant. We will call these polynomials \( m \)-harmonic. In the case \( m = 0 \) we have the space of the usual harmonic polynomials related to Coxeter group \( G \) (see e.g [5]).

The following linear map \( \pi^m \) from \( Q_\alpha \) to \( H_\alpha \) will play the central role in our considerations. Let us introduce \( m \)-discriminant

\[ w_m = \prod_{\alpha \in A} (\alpha, x)^{2m_\alpha + 1} \]

which obviously is \( m \)-quasiinvariant. We will show that \( w_m \) is also \( m \)-harmonic. Let \( q \) be any quasiinvariant, \( \mathcal{L}_q = \chi_m(q) \) be the corresponding differential operator. The map \( \pi_m \) is defined by the formula

\[ \pi_m(q) = \mathcal{L}_q(w_m). \]

Since \( \mathcal{L}_q \) commutes with \( \mathcal{L}_i \) it preserves the space \( H_\alpha \), so \( \pi_m(q) \in H_\alpha \). The question now is what is the kernel of \( \pi_m \). It is easy to show that the kernel of \( \pi_m \) contains the ideal \( I_m \subset Q_\alpha \) generated by the invariants \( \sigma_1, \ldots, \sigma_n. \)

We conjecture that the following statements are true for any Coxeter group \( G \) and multiplicity function \( m \).

**Conjecture 1** Kernel of \( \pi_m \) coincides with the ideal \( I_m \).

Consider the restriction of the map \( \pi_m \) onto the subspace \( H_\alpha \subset Q_\alpha \).
**Conjecture 2** The linear map

\[ \pi_m|_{H_m} : H_m \to H_m \]

is an isomorphism.

As a corollary we have the following isomorphism

\[ Q_m/I_m \cong H_m, \]

and the fact that \( Q_m \) is generated by \( H_m \) over \( S^G \).

**Conjecture 3** The ring \( Q_m \) is a free module over \( S^G \) generated by any basis in \( H_m \).

In algebraic terminology this implies that \( Q_m \) is a Cohen-Macaulay ring (see e.g. [8]).

We believe that \( Q_m \) is actually a Gorenstein ring. This follows from the following Conjecture 2* which can be considered as a stronger version of Conjecture 2.

Let us introduce the following bilinear form on the space \( H_m \):

\[ \langle p, q \rangle = (L_p L_q w_m)(0), \]

where \( L_p = \chi_m(p), L_q = \chi_m(q) \).

**Conjecture 2* The form \( \langle, \rangle \) on \( H_m \) is non-degenerate.**

This implies Conjecture 2. Indeed, if \( q \in Ker\pi_m|_{H_m} \) then by definition \( L_q(w_m) = 0 \) and therefore \( \langle q, p \rangle = 0 \) for all \( p \in H_m \). Conjecture 2* also implies that the dimensions \( h_k \) of the spaces of \( m \)-harmonics of degree \( k \) satisfy the duality relation

\[ h_k = h_{N-k}, \]

where \( N = \sum(2m_a + 1) \) is the degree of \( w_m \).

In this paper we prove all these conjectures for all the Coxeter groups of rank 2 (i.e. for the dihedral groups \( I_2(N) \)) under additional assumption that all the multiplicities are equal. In this case we describe all the \( m \)-harmonic polynomials explicitly.

As a corollary we show that the Poincare series for the quasiinvariants of dihedral group \( I_2(N) \) is given by

\[ p(Q_m^{I_2(N)}, t) = \frac{1 + 2t^{(mN+1)} + \ldots + 2t^{(mN+N-1)} + t^{(2m+1)N}}{(1 - t^2)(1 - t^N)}. \]

Notice that Conjecture 3 and Chevalley theorem imply that the Poincare series of the quasiinvariants of any Coxeter group has the form

\[ p(Q_m, t) = \frac{P(H_m, t)}{\prod_{i=1}^m(1 - t^d_i)}, \]
where $P(H_m, t)$ is the corresponding Poincare polynomial of the $m$-harmonics:

$$P(H_m, t) = \sum_{k=0}^{N} h_k t^k.$$ 

Some formulas for the polynomials $P(H_m, t)$ have been recently found in [9] (see Concluding remarks).

### 2 Quasiinvariants and quantum integrals of Calogero–Moser systems

Let us first discuss the homomorphism $\chi_m$ in more details. We will need some facts from the theory of multidimensional Baker–Akhiezer functions related to a Coxeter configuration of hyperplanes (see [2], [3], [10]). Let us remind that we are using the gauge which is different from the chosen in these papers by $g(x) = \prod(\alpha, x)^{m_\alpha}$.

For any Coxeter group $G$ and multiplicity function $m$ there exists a Baker–Akhiezer function (BA function) of the form

$$\psi = P(k, x) e^{(k, x)} \quad (5)$$

with the following properties:

- $P(k, x)$ is a polynomial in $k \in V$ and $x \in V$ with the highest term $g(k)g(x) = \prod_{\alpha \in A}(\alpha, k)^{m_\alpha}(\alpha, x)^{m_\alpha}$.

- $\psi$ satisfies the quasiinvariance conditions in $k$-space

$$\psi(s_\alpha(k)) - \psi(k) = o((\alpha, k)^{2m_\alpha}) \text{ near } (\alpha, k) = 0.$$ 

It is known that such function does exist, and is unique and symmetric with respect to $x$ and $k$:

$$\psi(k, x) = \psi(x, k)$$

(see [2], [3]). As it has been explained in [2] for any quasiinvariant $q \in Q_m$ there exists a differential operator $\chi_m(q) = L_q(x, \frac{\partial}{\partial x})$ such that

$$L_q(x, \frac{\partial}{\partial x})\psi(x, k) = q(k)\psi(x, k).$$

The procedure of finding $L_q$ is effective provided the formula for $\psi$ is given. Since for $q = k^2$ we have the (gauged) Calogero-Moser operator [3] we have the following
Theorem 1 For any Coxeter group $G$ and integer-valued multiplicity $m$ there exists a homomorphism $\chi_m : Q_m \to D_{\Sigma}(V)$ mapping the algebra of quasiinvariants $Q_m$ into the commutative algebra of quantum integrals of generalised Calogero–Moser problem.

One can write down the following explicit formula for this homomorphism first suggested by Yu.Berest:

$$\chi_m(q) = c(ad_L)^{d(q)} \hat{q}.$$ (6)

Here $L$ is the gauged Calogero-Moser operator, $ad_L A = LA - AL$, $\hat{q}$ is the operator of multiplication by $q$, $d(q)$ is degree of polynomial $q$ and the constant $c(q) = (2^{d(q)}d(q)!)^{-1}$.

Indeed because of the symmetry of $\psi$ with respect to $x$ and $k$ we have

$$L(k, \frac{\partial}{\partial k})\psi(x,k) = x^2 \psi(x,k).$$

Thus $\psi$ satisfies the so-called bispectral problem in the sense of Duistermaat and Grünbaum and one can use the general identity (1.8) from their paper which states that

$$(ad_L)^r(q)[\psi] = (ad_{x^2})^r(L_q)[\psi].$$

For $r = \deg q$ we arrive at the formula (6).

As we have mentioned in the Introduction the restriction $\chi_m$ onto the subring of invariants $S^G$ gives an isomorphism $\gamma_m$ between $S^G$ and the ring $D_m^G$ of invariant integrals of the Calogero-Moser quantum problem in the gauge $[3]$. The following result shows that the map $\chi_m$ defined in the Theorem 1 is in a certain sense the maximal extension of this map.

Let $D_m$ be the maximal commutative ring of differential operators on $V$ with rational coefficients which contains $D_m^G$ as a subring.

Theorem 2 The map $\chi_m$ is an isomorphism between the ring of $m$-quasiinvariants $Q_m$ and the ring $D_m$.

The proof follows from the following lemma. Let $\sigma_1 = k^2, \sigma_2, ..., \sigma_n$ be some generators of $S^G$, and let $L_1 = \chi_m(\sigma_1) = L, \ldots, L_n = \chi_m(\sigma_n)$ be the corresponding invariant integrals of the Calogero-Moser problem.

Lemma 1 Let $A$ be a differential operator commuting with all $L_i, i = 1, ..., n$. Then $A = L_q = \chi_m(q)$ for some quasiinvariant $q \in Q_m$. 
To prove this let us notice that since $A$ commutes with all $L_i$ it preserves their joint eigenspace $V(k)$ consisting of the solutions of the system

$$\begin{cases}
L_1 \psi = \sigma_1(k)\psi \\
\cdots \\
L_n \psi = \sigma_n(k)\psi
\end{cases} \tag{7}$$

where $k \in V$ is a "spectral" parameter. For generic $k$ this space is spanned by the Baker-Akhiezer functions $\psi(x, g(k)), g \in G$. From the form (5) of this function it follows that $\psi$ itself must be an eigenvector of $A$:

$$A\psi(x, k) = a(k)\psi(x, k)$$

for some polynomial $a(k)$. To show that $a(k)$ is a quasiinvariant let us notice that the left hand side of the last formula satisfies the quasiinvariance conditions in $k$ (see the properties of the BA function) and therefore must the right hand side. A simple analysis shows that $a(k)$ must be a quasiinvariant in that case.

Another relation between quasiinvariants and quantum integrals $L_q$ is given by the following

**Theorem 3** The space $Q_m$ of all $m$-quasiinvariants is invariant under the action of all the operators $L_q, q \in Q_m$.

For the operator $L$ (3) this can be proven by direct local considerations (c.f. [13] where a similar observation has been first made). The fact that the same is true for any $L_q$ now follows from Berest’s formula (3).

### 3 m-harmonic polynomials

Consider again the space $V(k)$ of the solutions of compatible system of equations (3). Let us put now $k$ to be zero, i.e. consider the system

$$\begin{cases}
L_1 \psi = 0 \\
\cdots \\
L_n \psi = 0
\end{cases} \tag{8}$$

We claim that all the solutions of the system (8) are polynomials in $x$. More precisely we have the following

**Theorem 4** For any Coxeter group $G$ and multiplicity function $m$ all the solutions of the system (8) are polynomial. They form the space of dimension $|G|$ where the natural action of $G$ is its regular representation.
When all the multiplicities are zero this is the classical result (see [14, 5]) and the corresponding polynomials are called harmonic. For the general multiplicity $m$ we will call the corresponding solutions of (8) as $m$-harmonic polynomials and denote the space $V(0)$ as $H_m$.

To prove the theorem let us consider first the general system (7). Heckman and Opdam [6] showed that it is equivalent to a holonomic system of the first order of rank $|G|$. Components of this system are $\phi = (\phi_i)$, $\phi_i = q_i(\partial)\psi$, where $q_i$, $i = 1, \ldots, |G|$ is a basis of harmonic polynomials of $G$ (see [6]).

For generic $k$ (more precisely, if $\prod_{\alpha}(k, \alpha) \neq 0$) we can choose the functions $\psi_{\sigma} = \psi(\sigma(k), x), \sigma \in G$, where $\psi$ is the Baker–Akhiezer function (5) from the previous section, as a basis of the correspondent space $V(k)$. Since BA function is regular everywhere as a function of $x$, the same is true for the solutions of (7) if $\prod_{\alpha}(k, \alpha) \neq 0$.

To prove that this is true for any $k$, in particular for $k = 0$, one can argue as follows. Let us consider the natural complex version of the system (7) by assuming simply that $x \in V^C$ and $\psi$ takes values in $C$. Consider a point $x_0$ such that $\prod_{\alpha}(x, \alpha) \neq 0$ and fix the solution of (7) $\psi(k, x; x_0, a)$, $a \in C^{|G|}$ by fixing the initial data in the corresponding holonomic system $\phi_i(x_0) = a_i$. Since the system (7) (and the corresponding holonomic system) is regular in $k$ everywhere $\psi(k, x; x_0, a)$ is analytic in $k$ everywhere for any $x$ such that $\prod_{\alpha}(x, \alpha) \neq 0$. By Hartogs theorem (see e.g. [15]), $\psi(k, x; x_0, a)$ is analytic in $k$ everywhere and for arbitrary $k$ this follows now by standard continuation arguments.

To prove the second statement of the theorem let us notice that a natural action of the group $G$ on the space $V(k)$ for a generic $k$ is regular. This immediately follows from the formula for a basis

$$\psi_{\sigma}(x, k) = \psi(x, \sigma(k)), \quad \sigma \in G$$

in terms of BA function. Indeed, for any $\tau \in G$

$$\psi_{\sigma}(\tau^{-1}(x), k) = \psi(\tau^{-1}(x), \sigma(k)) = \psi(x, (\tau \circ \sigma)(k)) = \psi_{\tau \sigma}(x, k)$$

since $\psi(\tau x, \tau k) = \psi(x, k)$. For arbitrary $k$ this follows now by standard continuation arguments.

\footnote{Strictly speaking the authors of [7] consider the trigonometric case and Weyl groups but all the arguments work for the rational case and any Coxeter group as well. Corresponding holonomic system can be rewritten in explicit way as a version of Knizhnik–Zamolodchikov equation (see [5]).}
4 Quasiinvariants and $m$-harmonic polynomials

The first relation between the space $H_m$ of $m$-harmonic polynomials and the space $Q_m$ of $m$-quasiinvariants is given by the following

**Theorem 5** Any $m$-harmonic polynomial is $m$-quasiinvariant: $H_m \subset Q_m$.

We will prove actually the following more general statement.

**Proposition 1** Any polynomial $p(x)$ belonging to the kernel of the operator

$$L = \Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha$$

is a quasiinvariant.

Let us deduce the quasiinvariance condition for polynomial $p(x)$ at the hyperplane $(\alpha, x) = 0$. Choose an orthogonal coordinate system $(t, y_1, \ldots, y_{n-1})$ such that the first axis is normal to the hyperplane. Then the operator $L$ can be represented as

$$L = \partial_t^2 + \Delta_y - \left( \frac{2m_\alpha}{t} + tf(t^2, y) \right) \partial_t + \sum_{i=1}^{n-1} g_i(t^2, y) \partial_{y_i},$$

where $\Delta_y = \partial_{y_1}^2 + \ldots + \partial_{y_{n-1}}^2$. The functions $f$ and $g_i$ are analytic at $t = 0$ and invariant under reflection $t \rightarrow -t$ with respect to $(\alpha, x) = 0$ due to invariance of the operator $L$ (c.f. [3]). For a polynomial $p(x)$ we also have a similar expansion

$$p = \sum_{i=0}^{\deg p} p_i(y) t^i.$$

Substituting this into the equation $Lp = 0$ we have

$$\left( \partial_t^2 - \frac{2m_\alpha}{t} \partial_t + \Delta_y - tf(t^2, y) \partial_t + \sum_{i=1}^{n-1} g_i(t^2, y) \partial_{y_i} \right) \left( \sum_{i=0}^{\deg p} p_i(y) t^i \right) = 0.$$

Considering all possible terms at $t^{-1}$ in the lefthand side we conclude that $p_1 \equiv 0$.

Considering now the terms at $t$ we come to

$$(6 - 6m_\alpha)p_3 \equiv 0$$

which implies that $p_3 \equiv 0$ if $m_\alpha > 1$. Continuing in this way we obtain

$$p_1 = p_3 = \ldots = p_{2m_\alpha-1} \equiv 0$$
or equivalently
\[ \partial^2_{s^{-1}} p(x)|_{(\alpha, x)=0} = 0 \text{ for } 1 \leq s \leq m_\alpha \]
which are the quasiinvariance conditions. Thus the proposition (and therefore the theorem) is proven.

We know the classical fact (see [14]) that the space \( H_0 \) of usual harmonic polynomials related to a Coxeter group \( G \) is isomorphic to the quotient
\[ H_0 \approx \mathbb{C}[x_1, \ldots, x_n]/I_0 \]
where \( I_0 \) is the ideal generated by the \( G \)-invariants of positive degree. We are going to present some arguments in favour of the following generalisation of (9):
\[ H_m \approx Q_m/I_m, \quad (10) \]
where \( Q_m \) is the ring of all \( m \)-quasiinvariants, \( I_m \subset Q_m \) is its ideal generated by the invariants of positive degree.

Following the classical scheme [14] let us define the map
\[ \pi_m : Q_m \to H_m \]
by the formula
\[ \pi_m(q) = \mathcal{L}_q(w_m) \quad (12) \]
where \( w_m = \prod_\alpha (\alpha, x)^{2m_\alpha + 1} \) and \( \mathcal{L}_q = \chi_m(q) \) is defined by (8). To prove that \( \mathcal{L}_q(w_m) \in H_m \) we will need the following lemma. Let \( A_m \subset Q_m \) be the subspace of antiinvariants, i.e. the quasiinvariants \( q \) satisfying the property
\[ q(s_\alpha(x)) = -q(x) \]
for any reflection \( s_\alpha \in G \).

**Lemma 2** \( A_m \) is a one-dimensional module over \( S^G \) generated by \( w_m \).

It is easy to show that such an antiinvariant is divisible by \( w_m \). Since the quotient is \( G \)-invariant this implies the lemma.

**Lemma 3** The quasiinvariant \( w_m \) is \( m \)-harmonic.

Indeed, since all \( \mathcal{L}_i \) are \( G \)-invariant and preserve the space \( Q_m \) (see theorem 8 above) the polynomials \( \mathcal{L}_i(w_m) \) belong to \( A_m \). Since they have degree less than the degree of \( w_m \) they must be zero.

**Lemma 4** The space \( H_m \) is invariant under the action of \( \mathcal{L}_q \) for any \( q \in Q_m \).

This follows from commutativity of \( \mathcal{L}_q \) and \( \mathcal{L}_i, \ i = 1, \ldots, n \). All this implies
Theorem 6 The formula
\[ \pi_m(q) = L_q(w_m) \]
defines a linear map from \( Q_m \) to \( H_m \).

Let us discuss the properties of the map \( \pi_m \).

Theorem 7 The kernel of \( \pi_m \) contains the ideal \( I_m \).

To prove this let us represent any element \( q \in I_m \) as
\[ q = \sum_s q_s p_s \]
where \( q_s \in Q_m, p_s \in S^G \). We have
\[ L_q(w_m) = \sum_s L_{q_s} L_{p_s}(w_m) = 0 \]
since \( L_{p_s}(w_m) = 0 \) due to lemma 3.

Our first two conjectures (see the Introduction) claim that the kernel of \( \pi_m \) coincides with \( I_m \) and that the restriction of \( \pi_m \) onto \( H_m \):
\[ \pi_m|_{H_m} : H_m \to H_m \]
is an isomorphism. This implies that
\[ Q_m/I_m \cong H_m \]
and that \( Q_m \) is generated by \( H_m \) as a module over \( S^G \). Our third conjecture says that this module is actually free (like in the classical situation \( m = 0 \)).

In the next section we are giving the proofs of all our conjectures for two-dimensional Coxeter groups with constant multiplicity function.

5 Proofs for the dihedral groups

Consider the dihedral group \( G = I_2(N) \) which is a symmetry group of a regular \( N \)-gon. It is generated by the plane reflections with respect to the lines \( \alpha_j^1 x_1 + \alpha_j^2 x_2 = 0 \), \( \alpha_j^1 = -\sin \frac{2\pi j}{N}, \alpha_j^2 = \cos \frac{2\pi j}{N}, j = 0, 1, \ldots N - 1 \). We suppose that the multiplicity function \( m_{\alpha_j^1} \) is equal to \( m \in \mathbb{Z}_{\geq 0} \) for all \( j = 0, \ldots, N - 1 \). It will be convenient for us to use the complex coordinates \( z = x_1 + ix_2, \bar{z} = x_1 - ix_2 \). The generators of the ring of invariants can be chosen as \( \sigma_1 = zz, \sigma_2 = z^N + \bar{z}^N \).

We are going first to investigate the ring \( Q_m \) of \( m \)-quasiinvariants. For \( m = 1 \) this ring has been studied in the paper [16] where in particular the multiplicative
generators of $Q_1$ have been found. We will use some observations from that paper to describe the ring $Q_m$ with general $m$.

Let us consider the following $2(N - 1)$ polynomials:

$$q_j = a_{j0}z^{mN+j} + a_{j1}z^N z^{(m-1)N+j} + \ldots + a_{jm}z^{mN} z^j,$$

$$\bar{q}_j = \bar{a}_{j0}z^{mN+j} + \bar{a}_{j1}z^N z^{(m-1)N+j} + \ldots + \bar{a}_{jm}z^{mN} z^j,$$

$j = 1, \ldots, N - 1$, where the coefficients $a_{js}$ are chosen to satisfy the system of equations

$$\begin{aligned}
(mN + j)a_{j0} + ((m - 2)N + j)a_{j1} + \ldots + (-mN + j)a_{jm} &= 0 \\
(mN + j)^3a_{j0} + ((m - 2)N + j)^3a_{j1} + \ldots + (-mN + j)^3a_{jm} &= 0 \\
\vdots & \vdots \ddots \ddots \vdots \\
(mN + j)^{2m-1}a_{j0} + ((m - 2)N + j)^{2m-1}a_{j1} + \ldots + (-mN + j)^{2m-1}a_{jm} &= 0
\end{aligned}$$

One can easily see that rank of this system is equal to $m$ so for any $j$ the coefficients $a_{js}$ are defined uniquely up to proportionality.

Explicitly polynomial $q_j$ can be given as determinant

$$q_j = \text{det} Z_j$$

of the following matrix

$$Z_j = \begin{pmatrix}
(mN + j) & ((m - 2)N + j) & \ldots & (-mN + j) \\
(mN + j)^3 & ((m - 2)N + j)^3 & \ldots & (-mN + j)^3 \\
\vdots & \vdots & \ddots & \vdots \\
(mN + j)^{2m-1} & ((m - 2)N + j)^{2m-1} & \ldots & (-mN + j)^{2m-1}
\end{pmatrix}.$$

**Proposition 2** The polynomials $q_j, \bar{q}_j$ belong to the space $Q_m$ of quasiinvariants.

**Proof.** Let us introduce the polar coordinates $z = r e^{i\varphi}$, $\bar{z} = r e^{-i\varphi}$. Then from the system of equations defining coefficients $a_{js}$ it obviously follows that $\partial^{2s-1}_\varphi q_j|_{\varphi = \frac{\pi}{k}} = 0$, $\partial^{2s-1}_\varphi \bar{q}_j|_{\varphi = \frac{\pi}{k}} = 0$, $k = 0, \ldots, N - 1$, $s = 1, \ldots, m$. Now the statement follows from the following lemma.

**Lemma 5** For any polynomial $p(x_1, x_2)$, any vector $\alpha = (-\sin \varphi_0, \cos \varphi_0)$ and for arbitrary $m \in \mathbb{Z}_+$ the conditions

$$\partial^{2s-1}_\alpha p|_{(\alpha, x) = 0} = 0, \quad s = 1, \ldots, m$$
are satisfied if and only if the following conditions in polar coordinates hold:

\[ \partial_{\varphi}^{2s-1}p|_{\varphi=\varphi_0} = 0, \quad s = 1, \ldots, m \]

Now let us introduce two more quasiinvariants

\[ q_0 = 1, \quad q_N = (z^N - \bar{z}^N)^{2m+1}. \quad (16) \]

The last polynomial \( q_N \) is actually a basic antiinvariant quasiinvariant (\( w_m \) in our previous notations).

**Theorem 8** The ring \( Q_m \) is a free finitely generated module over its subring \( S^G \subset Q_m \) of invariant polynomials. One can choose the polynomials \( q_0, q_1, \ldots, q_{N-1}, \bar{q}_1, \ldots, \bar{q}_{N-1}, q_N \) as a basis of \( Q_m \) over \( S^G \).

**Proof.** At first let us show that the polynomials \( q_0, q_1, \ldots, q_{N-1}, \bar{q}_1, \ldots, \bar{q}_{N-1}, q_N \) do generate \( Q_m \) over \( S^G \). To prove this we will use induction on a degree of a polynomial \( q \in Q_m \). If \( \deg q = 0 \) then \( q = const = c \), thus \( q = cq_0 \) so we have checked the base of induction. Suppose now that \( \deg q = d \) and \( q = A z^d + B \bar{z}^d + z \bar{z} p_{d-2} \) is an arbitrary quasiinvariant, \( q \notin S^G \). We will use further the following two lemmas.

**Lemma 6** Any \( m \)-quasiinvariant of degree \( d \leq mN \) is actually invariant.

**Proof.** It is enough to prove the lemma for an arbitrary homogeneous polynomial. Let

\[ q = a_0 z^{mN-\sigma} + a_1 z^{mN-\sigma-1} \bar{z} + a_2 z^{mN-\sigma-2} \bar{z}^2 + \ldots + a_{mN-\sigma} z^{mN-\sigma} \in Q_m \quad (17) \]

for some \( \sigma \geq 0 \). According to lemma 5 the conditions of quasiinvariance in polar coordinates are \( \partial_{\varphi}^{2s-1}q = 0 \) for \( \varphi = \frac{2k}{N}, \quad 0 \leq k \leq N - 1 \). We have

\[
(mN - \sigma)^{2s-1} a_0 e^{i \frac{\pi}{N} (mN-\sigma)k} + (mN - \sigma - 2)^{2s-1} a_1 e^{i \frac{\pi}{N} (mN-\sigma-2)k} + \ldots + (mN - \sigma - 4)^{2s-1} a_2 e^{i \frac{\pi}{N} (mN-\sigma-4)k} + \ldots + (mN + \sigma)^{2s-1} a_{mN-\sigma} e^{i \frac{\pi}{N} (-mN+\sigma)k} = 0
\]

Collecting the terms in this sum with equal exponents, we get

\[
\sum_{j=0}^{N-1} \sum_{t \geq 0 \atop j+Nt \leq mN-\sigma} (mN - \sigma - 2(j + Nt))^{2s-1} a_{j+Nt} e^{i \frac{\pi}{N} (mN-\sigma-2(j+Nt))k} = 0
\]

\[
\sum_{j=0}^{N-1} e^{i \frac{\pi}{N} (mN-\sigma-2j)k} \sum_{t \geq 0 \atop j+Nt \leq mN-\sigma} (mN - \sigma - 2(j + Nt))^{2s-1} a_{j+Nt} = 0
\]
Now let us consider these conditions for all possible \( k = 0, 1, \ldots, N - 1 \). We arrive at the Vandermonde-type system with different exponents \( e^{i \frac{\pi}{N}(mN - \sigma - 2j)} \), \( 0 \leq j \leq N - 1 \). Hence, for all \( j = 0, \ldots, N - 1 \) the following property is satisfied

\[
\sum_{t \geq 0, j + Nt \leq mN - \sigma} (mN - \sigma - 2(j + Nt))^{2s-1}a_{j + Nt} = 0 \quad (18)
\]

Let us analyze the conditions \((18)\) for all possible \( s \), \( 1 \leq s \leq m \). We again have a system of Vandermonde type with the exponents \( (mN - \sigma - 2(j + Nt)) \), \( 0 \leq t \leq \frac{mN - \sigma - j}{N} \leq m \). Notice that the exponents corresponding to different \( t \) may coincide only in pairs. The condition for that is

\[
mN - \sigma - (j + Nt_1) = j + Nt_2. \quad (19)
\]

In the case \( j = \sigma = 0 \) the number of equations in \((18)\) is less than number of unknown coefficients \( a_{j + Nt} \). But after collecting the terms in \((18)\) corresponding to equal exponents the number of equations becomes not less than the number of unknowns. We conclude that all nonzero coefficients \( a_{j + Nt} \) can be divided into pairs so that

\[
a_{j + Nt_1} - a_{j + Nt_2} = 0 \quad \text{and also the condition} \quad (19) \quad \text{is satisfied. In terms of polynomial} \quad q \quad \text{this means that it can be represented in the form}
\]

\[
q = \sum_{j=0}^{N-1} \sum_{(t_1, t_2)} a_{j + Nt_1}z^{mN - \sigma - (j + Nt_1)}z^{j + Nt_1} + a_{j + Nt_2}z^{mN - \sigma - (j + Nt_2)}z^{j + Nt_2} = \\
\sum_{j=0}^{N-1} \sum_{(t_1, t_2)} a_{j + Nt_1}(z\bar{z})^{j + Nt_2}(z^{N(t_1 - t_2)} + \bar{z}^{N(t_1 - t_2)}),
\]

where the pairs \((t_1, t_2)\) satisfy \((19)\) and also we suppose that \( t_1 > t_2 \). Hence polynomial \( q \) is an invariant.

**Lemma 7** If \( q = Az^d + B\bar{z}^d + z\bar{z}p_{d-2} \) is \( m \)-quasiinvariant of degree \( d = mN + lN \), \( 1 \leq l \leq m \) then \( A = B \).

**Proof.** We will use the notations and scheme of proof of lemma 6. Let us consider \( q \) of the form \((17)\), now we have \( \sigma = -lN \). As above in lemma 6 the conditions \((18)\) for \( j = 0, \ldots, N - 1 \) should be satisfied. Let us fix \( j = 0 \), we get

\[
\sum_{t=0}^{m + l} ((m + l - 2t)N)^{2s-1}a_{Nt} = 0
\]

or equivalently

\[
\sum_{t=0}^{\left[ \frac{m + l}{2} \right]} ((m + l - 2t)N)^{2s-1}(a_{Nt} - a_{N(m + l - t)}) = 0.
\]
We have got a system of Vandermonde type with different exponents. For \( l \leq m \) from that it follows that \( a_{Nl} = a_{N(m+l-t)} \). If \( t = 0 \) we get \( a_0 = a_{N(m+l)} \) which completes the proof of the lemma.

To continue the proof of the theorem let us represent \( d \) in the form \( d = mN + jN + k \), where \( 0 \leq k < N \). We have to consider few different cases.

a) If \( k \neq 0 \) then \( q - \frac{A}{a_{00}} q_k(z^N + \bar{z}^N)^j - \frac{B}{b_{00}} q_k(z^N + \bar{z}^N)^j = z\bar{z}p_1 \) for some polynomial \( p_1 \in Q_m \), here constants \( a_{k0}, \bar{a}_{k0} \) are defined by the form (13), (14) of the polynomials \( q_k, \bar{q}_k \). Since \( \deg p_1 = d - 2 < d \) we have done the induction step.

b) If \( k = 0, 1 \leq j \leq m \) then lemma 7 states that \( A = B \), hence \( q - A(z^N + \bar{z}^N)^m + j = z\bar{z}p_2 \), where \( p_2 \in Q_m \) and it can be represented as a linear combination of the polynomials \( q_i, \bar{q}_i \) with invariant coefficients. Further, let us represent polynomials \( q \) and \( \bar{q} \) in the form \( (13), (14) \) of the polynomials \( q_i, \bar{q}_i \) with invariant coefficients. To see this let us consider arbitrary nontrivial combination of the polynomials \( q_i, \bar{q}_i \) with invariant coefficients. Since the ring of invariants for the dihedral group is the ring freely generated by two polynomials \( \sigma_1 = z\bar{z} \) and \( \sigma_2 = z^N + \bar{z}^N \), the linear combination takes the form

\[
p_0^1(\sigma_1, \sigma_2)q_0 + p_1^1(\sigma_1, \sigma_2)q_1 + p_1^2(\sigma_1, \sigma_2)q_1 + \ldots +
\]

\[
p_N^1(\sigma_1, \sigma_2)q_{N-1} + p_N^2(\sigma_1, \sigma_2)q_{N-1} + p_N^1(\sigma_1, \sigma_2)q_N = 0,
\]

where for some \( s, \epsilon \) we have \( p_s^\epsilon \neq 0 \). Also we can suppose that \( p_s^\epsilon \) is not divisible by \( \sigma_2 \). Further, let us represent polynomials \( p_j \) as combinations of monomials in \( \sigma_1, \sigma_2 \) and let us move monomials containing \( \sigma_1 \) into righthand side. We have then that

\[
r_0^1(\sigma_2)q_0 + r_1^1(\sigma_2)q_1 + r_1^2(\sigma_2)q_1 + \ldots + r_{N-1}^1(\sigma_2)q_{N-1} + r_N^2(\sigma_2)q_{N-1} + r_N^1(\sigma_2)q_N
\]

is divisible by \( \sigma_1 \) and some polynomial \( r_s^\epsilon \neq 0 \). Let us consider monomials having degree which is equal to \( s \) modulo \( N \). If \( 1 \leq s \leq N - 1 \) then \( r_s^1(\sigma_2)q_s + r_s^2(\sigma_2)q_s \) must be divisible by \( z\bar{z} \), which is impossible as \( r_s^1(\sigma_2)q_s \) contains monomial of the form \( \lambda_1 z^{n_1} \) and does not contain degrees of \( \bar{z} \), and \( r_s^2(\sigma_2)q_s \) contains monomial of the form \( \lambda_2 \bar{z}^{n_2} \) and does not contain degrees of \( z \). If \( s = 0 \) or \( s = N \) then \( r_0^1(\sigma_2) + r_N^1(\sigma_2)(z^{(2m+1)N} - \bar{z}^{(2m+1)N}) \) must be divisible by \( z\bar{z} \), which is possible only if \( r_0^1 = r_N^1 = 0 \) but this is not the case. Thus the theorem is proven.

**Corollary** The Poincaré series for the \( m \)-quasiinvariants of dihedral group \( I_2(N) \) is

\[
p(Q_m, t) = \frac{1 + 2t^{(mN+1)} + \ldots + 2t^{(mN+N-1)} + t^{(2m+1)N}}{(1-t^2)(1-t^N)}.
\]

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Now we are going to show that polynomials $q_i, \bar{q}_i$ are actually $m$-harmonic. This will complete the proof of conjecture 3.

First let us rewrite the operator $\mathcal{L}$ in the complex coordinates. The set of vectors $\alpha$ for the group $I_2(N)$ has the form $\alpha = (-\sin \varphi_k, \cos \varphi_k)$, where $\varphi_k = \frac{2\pi k}{N}$, $k = 0, \ldots, N - 1$. Substituting $\partial_x = \partial_z + \partial_{\bar{z}}$, $\partial_y = i(\partial_z - \partial_{\bar{z}})$ we get

\[
\mathcal{L} = \Delta - 2m \sum_{\alpha} \frac{\partial_x}{(\alpha, x)} = \Delta - 2m \sum_{\alpha} -\sin \varphi_k \partial_x + \cos \varphi_k \partial_y = 4 \partial_z \partial_{\bar{z}} - 2m \sum_{k=0}^{N-1} -\sin \varphi_k \partial_z + (-i \cos \varphi_k - \sin \varphi_k) \partial_{\bar{z}} = 4 \left( \partial_z \partial_{\bar{z}} - m \sum_{k=0}^{N-1} e^{i\varphi_k} \partial_z - e^{-i\varphi_k} \partial_{\bar{z}} \right).
\]

The operator $\mathcal{L} = \mathcal{L}_1$ has a commuting operator $\mathcal{L}_2$ which is also invariant under dihedral group, it is homogeneous of degree $N$ and has the form $\mathcal{L}_2 = \partial_z^N + \partial_{\bar{z}}^N + \text{lower order terms}$.

**Theorem 9** The polynomials $\{[13], [14], [16]\} \in \mathcal{L}_1$ belong to the common kernel of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, i.e. they are $m$-harmonic.

**Proof.** From theorem 3 and degree consideration it follows immediately that $\mathcal{L}_1(q_0) = \mathcal{L}_2(q_0) = 0$. As $q_N$ is an antiinvariant quasivariant of the smallest possible degree and due to invariance of the operators $\mathcal{L}_1, \mathcal{L}_2$, we have $\mathcal{L}_1(q_N) = \mathcal{L}_2(q_N) = 0$.

Let us show now that $\mathcal{L}_2(q_s) = 0$, $1 \leq s \leq N - 1$. Let $\mathcal{L}_2(q_s) = r_s, \mathcal{L}_2(\bar{q}_s) = \bar{r}_s$. Notice that two dimensional space $V_s = \langle q_s, \bar{q}_s \rangle$ is an irreducible representation for the group $G = I_2(N)$. Since operator $\mathcal{L}_2$ being invariant commutes with the action of $G$, then by Schur lemma the kernel of $\mathcal{L}_2|_{V_s}$ is either $V_s$ or 0. In the last case the space $< r_s, \bar{r}_s >$ is an irreducible representation for $G$. But since $\deg r_s = \deg q_s - N < mN$ the polynomials $r_s, \bar{r}_s$ should be invariant according to lemma 6. The contradiction means that $\mathcal{L}_2|_{V_s} = 0$, i.e. $\mathcal{L}_2(q_s) = \mathcal{L}_2(\bar{q}_s) = 0$.

Now let us show that $\mathcal{L}_1(q_s) = \mathcal{L}_1(\bar{q}_s) = 0$. Let $p_s = \mathcal{L}_1(q_s), \bar{p}_s = \mathcal{L}_1(\bar{q}_s)$. As above by Schur lemma either $\mathcal{L}_1|_{V_s} = 0$ or $\mathcal{L}_1|_{V_s}$ is an isomorphism. In the second case representation $< p_s, \bar{p}_s >$ is isomorphic to irreducible representation $V_s$. It is easy to see from the formulas ([13], [14]) that among all the representations $V_t$, $1 \leq t \leq N - 1, t \neq s$ only $V_{N-s}$ is isomorphic to $V_s$, so that $q_{N-s}$ corresponds to $\bar{q}_s$, and $\bar{q}_{N-s}$ corresponds to $q_s$. This implies that $p_s = P(\sigma_1, \sigma_2) q_{N-s}, \bar{p}_s = P(\sigma_1, \sigma_2) q_{N-s}$ for some polynomial $P(x, y)$. But since

\[
p_s = 4 \left( \partial_z \partial_{\bar{z}} - m \sum_{k=0}^{N-1} e^{i\varphi_k} \partial_z - e^{-i\varphi_k} \partial_{\bar{z}} \right) (a_0 z^{mN+s} + a_1 z^{N(m-1)+s} + \ldots + a_m z^{mN}) \right),
\]
the degree of $p_s$ in $\bar{z}$ is less or equal then $mN$ while \( \deg_{\bar{z}} P(\sigma_1, \sigma_2)q_{N-s} > mN \). This means that $L_1|_{V_s} = 0$ so $L_1(q_s) = L_1(\bar{q}_s) = 0$. The theorem is proven.

The theorems 8 and 9 imply that our conjecture 3 is true for any dihedral group and constant multiplicity function. Let us now prove the first two conjectures.

The following lemma will be essential for us. Let $w$ be $m$-discriminant. Let us introduce the corresponding quantum integral $L_w = \chi_m(w)$. We claim that the constant $L_w w$ is non-zero for any $m$.

**Lemma 8** The constant $L_w w$ is determined by the formula

$$L_w(w) = \frac{N^{2m+1}}{2(2m+1)(N-1)} \prod_{j=1}^{2m+1} (2j - 2m - 1) \prod_{\substack{d=1 \in 1 \mod N \not d\equiv 0 \mod N}}^{(2m+1)N} (d - mN),$$

and in particular, is non-zero for any $m \in \mathbb{Z}_{\geq 0}$.

**Remark** When $m = 0$ $L_w = \prod_{\alpha} \partial_\alpha$ and the lemma claims that

$$\left( \prod_{\alpha} \partial_\alpha \right) \left( \prod_{\alpha} (\alpha, x) \right) = \frac{N!}{2N-1}$$

which is a particular case of the following Macdonald identity:

$$\left( \prod_{\alpha} \partial_\alpha \right) \left( \prod_{\alpha} (\alpha, x) \right) = \prod_{i} d_i!$$

where $d_i$ are the degrees of generators $\sigma_i$ of the invariants $S^G$ of a Coxeter group $G$ and all the roots are supposed to be normalised as $(\alpha, \alpha) = 2$.

**Proof of the lemma.** Since by formula (3) $L_w = \frac{1}{2^M M!} \partial \partial L_w$, $M = \deg w = (2m+1)N$ and $L(w) = 0$, we have that $L_w(w) = \frac{1}{2^M M!} L^M(w^2)$. Now let us rewrite operator $L$ in polar coordinates. We have

$$\Delta = \partial_r^2 + \frac{1}{r^2} \partial_\varphi^2 + \frac{1}{r} \partial_r.$$

If $\alpha = (\cos \varphi_0, \sin \varphi_0)$ then

$$\frac{\partial_\alpha}{(\alpha, x)} = \frac{\cos \varphi_0 (\cos \varphi \partial_r - \frac{1}{r} \sin \varphi \partial_\varphi) + \sin \varphi_0 (\sin \varphi \partial_r + \frac{1}{r} \cos \varphi \partial_\varphi)}{r (\cos \varphi_0 \cos \varphi + \sin \varphi_0 \sin \varphi)} = \frac{1}{r} \partial_r - \frac{1}{r^2} \tan(\varphi - \varphi_0) \partial_\varphi.$$
and

\[
\mathcal{L} = \Delta - 2m \sum_{\alpha, x} \frac{\partial_{\alpha}}{r} = \\
\partial_r^2 + \frac{1}{r^2} \partial_\varphi^2 + \frac{1}{r} \partial_r - \frac{2mN}{r} \partial_r - \frac{2mN}{r^2} \cot N\varphi \partial_\varphi.
\]

Let us notice that \( \mathcal{L} \) maps the space \( V^d \) of homogeneous functions of degree \( d \) to \( V^{d-2} \), and the restriction of \( \mathcal{L} \) onto \( V^d \) takes the form

\[
\mathcal{L}|_{V^d} = \frac{1}{r^2}(d^2 - 2mNd + \partial_\varphi^2 - 2mN \cot N\varphi \partial_\varphi).
\]

Now we are going to calculate \( \mathcal{L}^M w^2 \). In the polar coordinates we have

\[
w^2 = \frac{r(2m+1)2N^m (\sin N\varphi)^{4m+2}}{2(N-1)(4m+2)}
\]

and

\[
\mathcal{L}^M (w^2) = \frac{1}{2(N-1)(4m+2)} \left[ \frac{(2m+1)N}{2} \prod_{d=1}^{(2m+1)N} (4d^2 - 4mNd + \partial_\varphi^2 - 2mN \cot(N\varphi) \partial_\varphi) \right] \sin^{4m+2} N\varphi.
\]

Let us introduce new variable \( \phi = N\varphi \). Also due to commutativity we may order the terms in the previous product as follows

\[
\mathcal{L}^M (w^2) = \frac{N^{(4m+2)}N}{2(N-1)(4m+2)} \prod_{d=1}^{(2m+1)N} \prod_{d \not\equiv 0(\mod N)} (4\left(\frac{d}{N}\right)^2 - 4m\frac{d}{N} + \partial_\phi^2 - 2m \cot \phi \partial_\phi) \times \prod_{j=1}^{2m+1} (4j^2 - 4mj + \partial_\phi^2 - 2m \cot \phi \partial_\phi) \sin^{4m+2} \phi. \quad (20)
\]

Now direct calculation shows that

\[
(4j^2 - 4mj + \partial_\phi^2 - 2m \cot \phi \partial_\phi) \sin^{2j} \phi = 2j(2j - 2m - 1) \sin^{2j-1} \phi.
\]

Thus we can simplify (20) as

\[
\mathcal{L}^M (w^2) = \frac{N^{(4m+2)}N}{2(N-1)(4m+2)} \prod_{j=1}^{2m+1} 2j(2j - 2m - 1) \prod_{d=1}^{(2m+1)N} \prod_{d \not\equiv 0(\mod N)} (4\left(\frac{d}{N}\right)^2 - 4m\frac{d}{N} + \partial_\phi^2 - 2m \cot \phi \partial_\phi)c,
\]

where \( c = \prod_{j=1}^{2m+1} 2j(2j - 2m - 1) \). Finally we get

\[
\mathcal{L}^M (w^2) = \frac{N^{(4m+2)}N}{2(N-1)(4m+2)} \prod_{j=1}^{2m+1} 2j(2j - 2m - 1) \prod_{d=1}^{(2m+1)N} (4\left(\frac{d}{N}\right)^2 - 4m\frac{d}{N}) = \]

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\[ N^{4m+2} \prod_{j=1}^{2m+1} 2j(2j-2m-1) \prod_{d \equiv 0 \pmod{N}} d(d-mN) = N^{4m+2} \prod_{j=1}^{2m+1} 2j(2j-2m-1) \times \]
\[
\frac{M!}{N^{2m+1}(2m+1)!} \prod_{d=1 \atop d \not\equiv 0 \pmod{N}}^{(2m+1)N} (d-mN) = M!(2N)^{2m+1} \prod_{j=1}^{2m+1} (2j-2m-1) \prod_{d=1 \atop d \not\equiv 0 \pmod{N}}^{(2m+1)N} (d-mN) \]

Since \( \mathcal{L}_w w = \frac{1}{2^M M!} \mathcal{L}^M (w^2) \) we have

\[ \mathcal{L}_w w = \frac{(2N)^{2m+1}}{2(2m+1)N} \prod_{j=1}^{2m+1} (2j-2m-1) \prod_{d=1 \atop d \not\equiv 0 \pmod{N}}^{(2m+1)N} (d-mN). \]

This proves the lemma.

Now we are ready to prove Conjecture 1.

**Theorem 10** For any dihedral group \( I_2(N) \)

\[ \text{Ker} \pi_m = I_m, \]

where \( \pi_m \) is defined by \([12]\) and \( I_m \) is the ideal in \( \mathbb{Q}_m \) generated by basic invariants \( \sigma_1, \sigma_2. \)

**Proof.** Let us represent an arbitrary quasiinvariant in the form

\[ q = s_0 q_0 + \sum_{j=1}^{N-1} \left( s_j q_j + \bar{s}_j \bar{q}_j \right) + s_N q_N \quad (21) \]

where \( s_j, \bar{s}_j \) are invariants and \( q_j, \bar{q}_j \) are defined by \([13]\), \([14]\). Suppose that \( q \in \text{Ker} \pi_m \), which is \( \mathcal{L}_q w = 0. \) As

\[ \mathcal{L}_q = \mathcal{L}_{s_0} + \sum_{j=1}^{N-1} \left( \mathcal{L}_{s_j} \mathcal{L}_{q_j} + \mathcal{L}_{\bar{s}_j} \mathcal{L}_{\bar{q}_j} \right) + \mathcal{L}_{s_N} \mathcal{L}_{q_N}, \]

the condition \( \mathcal{L}_q w = 0 \) is equivalent to \( \mathcal{L}_{q^H} w = 0, \) where \( q^H \in H \) is a subsum in \( (21) \) corresponding to those \( j \) that \( s_j \) (or \( \bar{s}_j \)) are constants. Since \( q - q^H \in I_m \) it is sufficient to prove that \( \mathcal{L}_h w \neq 0 \) for any \( h \in H. \)

It is sufficient to consider only the homogeneous \( h. \) If \( h = \text{const} \) then the statement obviously holds. When \( h = \text{const} w \) it follows from lemma 8. Suppose now that \( h = \lambda_i q_j + \lambda_2 \bar{q}_j \) and \( \mathcal{L}_h w = 0. \) Let us consider

\[ \mathcal{L}_{q_{N-j}} \mathcal{L}_h w = 0 = \mathcal{L}_{q_{N-j}} h w = \mathcal{L}_{\lambda_1 q_{N-j} q_j + \lambda_2 q_{N-j} \bar{q}_j} w. \]
The formulas (13), (14) show that
\[ \lambda_1 q_{N-j}q_j + \lambda_2 q_{N-j}\bar{q}_j = \lambda_1 a_{N-j}a_{j0}z^{(2m+1)N} + (z\bar{z})p, \]  
(22)
where \( p \) is some polynomial in \( z, \bar{z} \). On the other hand, we should have a general representation (21) for some invariants \( s_i, \bar{s}_i \)
\[ \lambda_1 q_{N-j}q_j + \lambda_2 q_{N-j}\bar{q}_j = \sum (s_i q_i + \bar{s}_i \bar{q}_i) + s_0 q_0 + s_N q_N \]  
(23)
In the last expression the sum \( \sum (s_i q_i + \bar{s}_i \bar{q}_i) \) cannot contain monomials \( z^{(2m+1)N}, \bar{z}^{(2m+1)N} \) as \( s_i, \bar{s}_i \) are nontrivial polynomials of \( z\bar{z} \) which follows from degree considerations.
Suppose that \( \lambda_1 \neq 0 \), then the left hand side of (23) contains \( z^{2m+1} \) and it does not contain \( \bar{z}^{(2m+1)N} \) (see (22)). Hence \( s_N \) must be a nonzero constant \( c \). Now
\[ \mathcal{L}_{q_{N-j}} \mathcal{L}_h w = \mathcal{L}_{\sum (s_i q_i + \bar{s}_i \bar{q}_i) + s_0 w + c \mathcal{L}_{q_N} w} = c \mathcal{L}_{q_N} w \]  
since \( \sum (s_i q_i + \bar{s}_i \bar{q}_i) + s_0 \in I_m \). Due to lemma 8 \( \mathcal{L}_{q_N} w \neq 0 \) so \( \mathcal{L}_{q_{N-j}} \mathcal{L}_h w \neq 0 \) which contradicts the assumption that \( \mathcal{L}_h w = 0 \). This implies that \( \lambda_1 = 0 \). Similarly multiplying \( \mathcal{L}_h w = 0 \) by \( \mathcal{L}_{q_{N-j}} \) we derive that \( \lambda_2 = 0 \) which means that \( h = 0 \). This proves the theorem and conjecture 1 in this case.

Conjecture 2 now simply follows from the previous arguments. Indeed we have shown in the proof of the previous theorem that if \( \mathcal{L}_h w = 0 \) for some \( h \in H_m \) then \( h = 0 \). This implies the following

Theorem 11  For any dihedral group the linear map
\[ \pi_m : H_m \to H_m \]
is an isomorphism.

Let us finally show that conjecture 2* also holds. For that we fix normalisation of basic quasiinvariants as
\[ q_j = z^{mN+j} + z\bar{z}p_j, \]
\[ \bar{q}_j = \bar{z}^{mN+j} + z\bar{z}\bar{p}_j. \]
Since \( q_j, \bar{q}_j \) is divisible by \( z\bar{z} \) it is obvious that \( < q_{j1}, \bar{q}_{j2} >= 0 \). The consideration of the degrees shows that \( < q_{j1}, q_{j2} >= 0 \) if \( j_1 + j_2 \neq N \). Let us calculate \( < q_j, q_{N-j} > \). We have
\[ q_j q_{N-j} = z^{(2m+1)N} + (z\bar{z})\hat{p}_j = \frac{1}{2}(q_N + \sigma_2^{2m+1}) + (z\bar{z})Q_j, \]
where \( \hat{p}_j \) is some polynomial and \( Q_j \) is a quasiinvariant. Hence
\[ < q_j, q_{N-j} >= (\mathcal{L}_{\frac{1}{2}q_N} + \mathcal{L}_{\frac{1}{2}\sigma_2^{2m+1} + z\bar{z}Q_j})w = \frac{1}{2} \mathcal{L}_{q_N} w \]
which is non-zero by lemma 8. This implies the nondegeneracy of the form \( <,> \) and conjecture 2*.

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6 Concluding remarks.

Besides the proof of the conjectures for all Coxeter groups one of the most interesting open problems is the description of the space of all $m$-harmonic polynomials.

In the classical case $m = 0$ this space can be described as the result of the differential operators with constant coefficients applied to the discriminant $w = \prod_{\alpha \in A}(\alpha, x)$ (see [14]). In the general case one can use the operators $L_q$ which correspond to the quasiinvariants but the quasiinvariants themselves need an effective description.

One of the alternative ideas is to use the relation between the system [7] and Knizhnik-Zamolodchikov (KZ) equation discovered by Matsuo and Cherednik (see [17]). The problem with this is that both Matsuo and Cherednik maps are degenerate when the spectral parameter is zero. Fortunately one can modify the KZ equation and define a map which is an isomorphism for all values of the spectral parameters (see [4]). The possibility to use these modified KZ equations for our problems is now under investigation. The latest news in this direction is the calculation of the Poincare polynomials of $m$-harmonics for all Coxeter groups [9]. The answer is written separately for each isotypical component corresponding to a given irreducible representation of the Coxeter group.

Another interesting direction is to develop a similar approach for the rings of integrals of other algebraically integrable quantum problems, in particular for the trigonometric and difference versions of Calogero-Moser problem related to root systems. In trigonometric case the corresponding polynomials satisfy certain difference relations which in the rational limit coincide with the quasiinvariance relations (1) (see [2],[3]). It would be also very interesting to investigate the analogues of harmonic polynomials in relation with the ring of quantum integrals for the generalised Calogero-Moser problems related to the deformed root systems discovered in [18] (see also [10]).

Finally we would like to mention that for the Weyl groups $G$ the space of usual harmonic polynomials can be interpreted as cohomology of the generalised flag varieties [13]. It would be interesting to look at the space of $m$-harmonic polynomials from this point of view. This is also related to an important question about the multiplication structure on $H_m$ induced by the isomorphism with $Q_m/I_m$.

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