The $\{\beta\}$-expansion for Adler function, Bjorken Sum Rule, and the Crewther-Broadhurst-Kataev relation at order $O(\alpha_s^4)$

P. A. Baikov, a S. V. Mikhailov b

aSkobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, 1(2), Leninskie gory, Moscow 119991, Russian Federation
bBogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia

E-mail: baikov@theory.sinp.msu.ru, mikhs@theor.jinr.ru

ABSTRACT: We derive explicit expressions for the elements of the $\{\beta\}$-expansion for the nonsinglet Adler $D_A$-function and Bjorken polarized sum rules $S_{Bjp}$ in the N$^4$LO using recent results by Chetyrkin for these quantities computed within extended QCD including any number of fermion representations. We discuss the properties of the $\{\beta\}$-expansion for $D_A$ and $S_{Bjp}$ at higher orders which follow from the Crewther [1] and the Broadhurst-Kataev [2] relation.

KEYWORDS: Renormalization Group, QCD

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1 Introduction

The knowledge of the detailed structure of the perturbation QCD series (pQCD) is rather important for a variety of problems of which the optimization via the renormalization group (RG) of the truncated series is the most known. The detailed structure means revealing the structure of the pQCD expansion coefficients by means of the $\{\beta\}$-expansion [3–5,7]. We shall explore this structure for the QCD renormalization group invariant (RGI) quantities (having no anomalous dimension) depending on one-scale. To obtain elements of perturbative coefficients within the $\{\beta\}$-expansion, we apply the algebraic approach elaborated in [4]. Their knowledge allows us to pose the problem of optimizing the series [3, 6] and also helps to relate the elements of different RGI quantities, see [4–7]. As examples of these quantities we consider here the physically important Adler function $D(Q^2, \mu^2)$

$$D^{EM}\left(\frac{Q^2}{\mu^2}, a(\mu^2)\right) = \left(\sum_i q_i^2\right) d_F D_{NS}\left(\frac{Q^2}{\mu^2}, a(\mu^2)\right) + \left(\sum_i q_i\right)^2 d_F D_S\left(\frac{Q^2}{\mu^2}, a(\mu^2)\right), \quad (1.1)$$

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and Bjorken polarized sum rules $S^{\text{Bjp}}(Q^2, \mu^2)$,

$$S^{\text{Bjp}}(Q^2) = \frac{1}{6} \left| \frac{g_A}{g_V} \right| \left[ C^{\text{Bjp}}_{\text{NS}} \left( \frac{Q^2}{\mu^2}, a(\mu^2) \right) + \left( \sum_i q_i \right) C^{\text{Bjp}}_S \left( \frac{Q^2}{\mu^2}, a(\mu^2) \right) \right]. \tag{1.2}$$

Here $q_i$ is the electric charge of the quark, $g_A, g_V$ – nucleon axial and vector charges, $d_F$ – the dimension of the standard quark color representation, strong coupling – $a = \alpha_s/(4\pi)$. The perturbation expansions for these quantities are one of the most advanced, see Appendix B. Perturbative expressions for the nonsinglet (NS) coefficient functions of both these quantities at the default choice of scale $\mu^2 = Q^2$ read

$$D_{\text{NS}}(a(\mu^2)) = 1 + \sum_{n \geq 1} a_n^a(\mu^2) d_n, \quad C^{\text{Bjp}}_{\text{NS}}(a(\mu^2)) = 1 + \sum_{n \geq 1} a_n^a(\mu^2) c_n. \tag{1.3}$$

They were obtained in order of $O(a^4)$ in the $\overline{\text{MS}}$-scheme in [8, 9]. We will use here only the NS parts of these quantities, $D = D_{\text{NS}}, C^{\text{Bjp}} = C^{\text{Bjp}}_{\text{NS}}$, omitting the notation NS further in the text. The expansion coefficients $d_n$ and $c_n$ in Eqs.(1.3) are the combinations of the color coefficients only. Let us recall the structure of these perturbation coefficients. The $\{\beta\}$-expansion representation [3] prescribes to decompose $d_n$ or $c_n$, or coefficients of any other RGI quantity as

$$a^0 = 1$$

$$a^1 = d_1[0], \quad (1.4a)$$

$$a^2 = d_2 = \beta_0 d_2[0] + d_2[0], \quad (1.4b)$$

$$a^3 = d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0], \quad (1.4c)$$

$$a^4 = d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1,1] + \beta_2 d_4[0,0,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0], \quad (1.4d)$$

$$\ldots \ldots,$$

where $\beta_i$ are the coefficients of the QCD $\beta$-function

$$\mu^2 \frac{da(\mu^2)}{d\mu^2} = -\beta(a) = -a^2(\mu^2) \sum_{i \geq 0} \beta_i^a a^i(\mu^2). \tag{1.5}$$

Here we shall consider the results up to $O(a^4)$ in Eqs.(1.4), while for the higher orders we shall provide some predictions, e.g. , for the elements of the next fifth order, see the tail of the previous expansion in Eq.(4.1a) and higher in Eq.(4.1b) of Sec.4. The designation $i_0, i_1, \ldots$ of the arguments of $d_n[i_0, i_1, \ldots]$ denotes the exponents of the accompanying factor $\beta_0^{i_0} \beta_1^{i_1} \ldots$. The decompositions in Eqs.(1.4) contain complete knowledge of the strong charge renormalization by using there all possible $\beta_i$-terms in each order of expansion, for details see [3, 5, 6]. This kind of expansion is the essential part of the procedures for the optimization of the perturbation series, e.g. , the decomposition (1.4b) was the starting point of the well-known BLM prescription [10] in NLO. Higher-order calculations are labor intensive and should be used to the maximum effect, i.e. , be optimized.
At NLO of pQCD the decomposition in (1.4b) looks quite evident, the term proportional to $\frac{4}{3}T_R n_f$ is an unambiguous attribute of $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_R n_f$ in $d_2$ or other physical quantity. Whereas in NNLO QCD we can no longer separate the corresponding terms at $\beta_0$ and $\beta_1$ in (1.4c). To sharpen the problem, let us note that in pure gluodynamics, i.e., without quark loops, at $T_R n_f \equiv 0$, it is impossible to carry out this decomposition even for the lowest case of $d_2$. How to obtain elements of the decomposition in higher orders? We solve the problem introducing additional degrees of freedom (d.o.f.), new fields that interact following the universal gauge principle and enter only in intrinsic loops. Using the fermions in the adjoint representation (e.g., MSSM light gluino $\frac{C_A}{2} n_g$) as an additional d.o.f., a simple algebraic scheme to obtain the elements of the $\{\beta\}$-expansion was formulated in [4].

Here we significantly extend this approach based on the results in [11–13]. It should be emphasised that our approach doesn’t use any specifics of $D$ or $C^{Bjp}$ and is universally applied to the expansion of any RGI quantities.

Another subject of our consideration is the Crewther relation [1, 14] that determines the perturbation structure of the product $D \cdot C^{Bjp}$ that is inspired by the conformal symmetry breaking arguments, see Sec.3. This relation was significantly evolved later by D. Broadhurst and A. Kataev in [2]; therefore, we will name it the Crewther-Broadhurst-Kataev (CBK) relation. The general proof of the CBK relation has been discussed in [15].

The usage of the $\{\beta\}$-expansion for CBK relation leads to predictions of mutual relations for the elements of $D$ and $C^{Bjp}$ for any order of pQCD [4, 5]. These confirmations/predictions are further continued and elaborated in Sec.3-4, where we mention also other attempts to obtain the $\{\beta\}$-expansion. Our main results are listed in Conclusion. In Appendices A and B we have collected the important results for $\beta$-function, $D$, and $C^{Bjp}$ obtained in [11, 13].

2 The N$^4$LO $\{\beta\}$-expansion for the Adler D-function and Bjorken polarized SR $C^{Bjp}$

2.1 How to decompose RG invariants

The algebraic scheme, presented in [4], is well algorithmized and appropriate to apply to high loop results. The key role in the scheme plays the set of zeros of $\beta$-function coefficients $\beta_0(\{R\}), \beta_1(\{R\}), \beta_2(\{R\}), \ldots$ and zeros of different set of these $\beta_k$. Here $\{R\}$ means a set of fermion degrees of freedom – d.o.f., appearing in a QCD-like model extended to include any number of different fermion representations of the gauge group, (QCDc), see [13] and Appendix A.1. The more d.o.f. we include in consideration, the more higher order expansion coefficients $d_n$ can be untangled with respect to $d_n[.]$. Let us illustrate this with an example of the $d_4$ decomposition. Suppose we have the expressions for $d_4(x_0, x_1, x_2)$ and for the coefficients $\beta_{0,1,2}(x_0, x_1, x_2)$ that depend on the attributes of d.o.f. $\{x_0, x_1, x_2\}$. Suppose we find a root $(x_{0,0}, x_{1,0}, x_{2,0}):\{\beta_{0,1,2}(x_{0,0}, x_{1,0}, x_{2,0})=0\}$. So, following Eq.(1.4d), we obtain $d_4(x_{0,0}, x_{1,0}, x_{2,0})=d_4[0]$. Taking the reduced condition $\{\beta_{0,1}(x_{0,0}, x_{1,0})=0\}$ for the roots, one keeps two terms in $d_4$: $d_4(x_{0,0}, x_{1,0}, x_2) = d_4[0] + \beta_2(x_{0,0}, x_{1,0}, x_2)d_4[0, 0, 1]$, from which one can extract $d_4[0, 0, 1]$, and so on. Directly following the scheme presented
in [4], Sec.3.3 there, and using the explicit expressions for $D(\{R\})$ and $C^{\text{Bij}}(\{R\})$ that involve new fermions d.o.f. presented in Appendices A, B, one can obtain expressions for all of the elements $d_4[\cdot]$ and $c_4[\cdot]$. Let us emphasize, we need new d.o.f. only to perform the decomposition, after that we return from QCD to the standard QCD, $\{R\} \rightarrow T_{Rn_f}$.

The trace of the general gauge principle, presented first via interactions of different d.o.f. within QCDe, ends up as a structure of the $\{\beta\}$-expansion. The results for the $\{\beta\}$-expansion presented below for $D$ in $O(a^3)$ were first obtained in [3] using calculations with light MSSM gluino (one new d.o.f.) [17, 18]. In this case, the set $\{R\}$ is reduced to the appearance of a pair of attributes $T_{Rn_f}, \frac{C_A}{2}n_\tilde{g}$ in the final expressions. The remaining elements for $C^{\text{Bij}}$ in this order were first restored in [6] using the CBK relation.

### 2.2 Decomposition for the Adler D-function

For the Adler function $D$ the corresponding elements in order $O(a^3)$ read [3, 5, 6]

\[
\begin{align*}
    d_1 &= 3C_F; \\
    d_2[1] &= d_1 \left( \frac{11}{2} - 4\zeta_3 \right) ; \\
    d_2[0] &= d_1 \left( \frac{C_A}{3} - \frac{C_F}{2} \right) ; \\
    d_3[2] &= d_1 \left( \frac{302}{9} - \frac{76}{3}\zeta_3 \right) ; \\
    d_3[0,1] &= d_1 \left( \frac{101}{12} - 8\zeta_3 \right) ; \\
    d_3[1] &= d_1 \left[ C_A \left( -\frac{3}{4} + \frac{80}{3}\zeta_3 - \frac{40}{3}\zeta_5 \right) - C_F \left( 18 + 52\zeta_3 - 80\zeta_5 \right) \right] ; \\
    d_3[0] &= d_1 \left[ C_A^2 \left( \frac{523}{36} - 72\zeta_3 \right) + \frac{71}{3} C_A C_F - \frac{23}{2} C_F^2 \right].
\end{align*}
\]

The results in Eqs.(2.1) were confirmed based on the results obtained within QCDe [13] and briefly outlined in Appendices A and B. Besides these results has been analyzed and discussed in [19] from the point of view of PMC approach. Then we have obtained the $\{\beta\}$-expansion elements in order of $O(a^4)$,

\[
\begin{align*}
    d_4[3] &= C_F \left( \frac{6131}{9} - 406\zeta_3 - 180\zeta_5 \right) ; \\
    d_4[1,1] &= C_F \left( 385 - \frac{1940}{3}\zeta_3 + 144\zeta_3^2 + 220\zeta_5 \right) ; \\
    d_4[2] &= -C_F \left[ C_F \left( \frac{6733}{8} + 1920\zeta_3 - 3000\zeta_5 \right) + C_A \left( \frac{20929}{144} - \frac{12151}{6}\zeta_3 + 792\zeta_3^2 + 1050\zeta_5 \right) \right] ; \\
    d_4[0,0,1] &= C_F \left( \frac{355}{6} + 136\zeta_3 - 240\zeta_5 \right) ;
\end{align*}
\]
\[ d_4[1] = C_F \left[ - C_F^2 \left( \frac{447}{2} - 42\zeta_3 - 4920\zeta_5 + 5040\zeta_7 \right) + \frac{3301}{4} - 678\zeta_3 - 2280\zeta_5 + 2520\zeta_7 \right] + C_A C_F \left( \frac{16373}{36} - \frac{17513}{3}\zeta_3 + 2592\zeta_3^2 + 3030\zeta_5 - 420\zeta_7 \right) \]; (2.2e)

\[ d_4[0, 1] = -C_F \left[ C_A \left( \frac{139}{12} + \frac{1054}{3}\zeta_3 - 460\zeta_5 \right) + C_F \left( \frac{251}{4} + 144\zeta_3 - 240\zeta_5 \right) \right] \]; (2.2f)

\[ d_4[0] = \tilde{d}_4[0] + \delta d_4 \]
\[ = C_F^4 \left( \frac{4157}{8} + 96\zeta_3 \right) - C_A C_F^3 \left( \frac{2409}{2} + 432\zeta_3 \right) + C_A^2 C_F^2 \left( \frac{3105}{4} + 648\zeta_3 \right) + C_A^3 C_F \left( \frac{68047}{48} + \frac{8113}{2}\zeta_3 - 7110\zeta_5 \right) + \delta d_4 ; \] (2.2g)

\[ \delta d_4 = -16 \left[ n_f \frac{d^{abcd}_{E}[d^{abcd}_{E}}{a_F} (13 + 16\zeta_3 - 40\zeta_5) + \frac{d^{abcd}_{A}[d^{abcd}_{A}}{a_F} (-3 + 4\zeta_3 + 20\zeta_5) \right]. \] (2.2h)

Equation (2.2a) for \( d_4[3] \) should be compared with the term at \( C_F T_3^3 \) in Eq.(10) in [8] and even with the early general formulae for renormalon chain contributions obtained in [2, 16].

### 2.3 Decomposition for the Bjorken SR \( C^{Bjp} \)

Here we confirm the results for \( c_{2-3}[] \) up to the order \( O(a^3) \) [6] through direct calculations not using the CBK relation (following CBK this was done in [6]),

\[ c_1 = -3 C_F; \] (2.3a)

\[ c_2[1] = 2c_1; c_2[0] = \left( \frac{1}{3} C_A - \frac{7}{2} C_F \right) c_1; \] (2.3b)

\[ c_3[2] = \frac{115}{18} c_1; c_3[0, 1] = c_1 \left( \frac{59}{12} - 4\zeta_3 \right); \] (2.3c)

\[ c_3[1] = -c_1 \left[ C_F \left( \frac{166}{9} - \frac{16}{3}\zeta_3 \right) + C_A \left( \frac{215}{36} - 32\zeta_3 + \frac{40}{3}\zeta_5 \right) \right]; \] (2.3d)

\[ c_3[0] = c_1 \left[ C_A^2 \left( \frac{523}{36} - 72\zeta_3 \right) + \frac{65}{3} C_F C_A + \frac{C_F^2}{2} \right]. \] (2.3e)

The elements \( c_4[] \) in order \( O(a^4) \) are obtained based on the results for \( C^{Bjp}(R) \) [13] presented in Appendices A and B following the same universal procedure as it was applied to the \( D \) case in Sec.2.2.
\[ c_4[3] = -C_F \frac{605}{9}; \quad (2.4a) \]
\[ c_4[1,1] = C_F \left( \frac{715}{8} + 677 \zeta_3 - 220 \zeta_5 \right); \quad (2.4b) \]
\[ c_4[2] = C_F \left[ C_F \left( \frac{8057}{24} - 96 \zeta_3 \right) + C_A \left( \frac{28615}{144} - \frac{4105}{6} \zeta_3 - 24 \zeta_5^2 + 370 \zeta_5 \right) \right]; \quad (2.4c) \]
\[ c_4[0,0,1] = C_F \left( -\frac{146}{3} - 148 \zeta_3 + 240 \zeta_5 \right); \quad (2.4d) \]
\[ c_4[1] = C_F \left[ C_F^2 \left( -\frac{1478}{3} - 824 \zeta_3 + 1520 \zeta_5 \right) - \right. \]
\[ \left. C_A C_F \left( \frac{1177}{12} - \frac{5888}{3} \zeta_3 + \frac{2000}{3} \zeta_5 + 1680 \zeta_7 \right) - \right. \]
\[ \left. C_A^2 \left( \frac{3829}{72} - 2286 \zeta_3 + \frac{6250}{3} \zeta_5 - 420 \zeta_7 \right) \right]; \quad (2.4e) \]
\[ c_4[0,1] = C_F \left[ C_A \left( \frac{109}{4} + \frac{1006}{3} \zeta_3 - 460 \zeta_5 \right) + C_F \left( \frac{1399}{12} - 100 \zeta_3 \right) \right]; \quad (2.4f) \]
\[ c_4[0] = \tilde{c}_4[0] + \delta c_4 \]
\[ = -C_F^2 \left( \frac{4823}{8} + 96 \zeta_3 \right) + C_A C_F^3 \left( \frac{3201}{2} + 432 \zeta_3 \right) - C_A^2 C_F^2 \left( \frac{2055}{4} + 1944 \zeta_3 \right) - \]
\[ C_A^3 C_F \left( \frac{68047}{48} + \frac{8113}{2} \zeta_3 - 7110 \zeta_5 \right) + \delta c_4; \quad (2.4g) \]
\[ \delta c_4 = -\delta d_4 = 16 \left[ n_f \frac{d_{abcd} d_{abcd}}{d_F} (13 + 16 \zeta_3 - 40 \zeta_5) + \frac{d_A d_{abcd} d_{abcd}}{d_F} (-3 + 4 \zeta_3 + 20 \zeta_5) \right]. \quad (2.4h) \]

Let us remark that we first face here the d.o.f. dependence of the elements \( d_n[.] \), see the terms \( n_f \) in \( c_4[0] \), (2.4h), and \( d_4[0] \), (2.2h). This kind of \( n_f \) dependence is related to the special contributions from the “box” subgraph rather than the \( a \)-renormalization. These terms \( \delta c_4 = -\delta d_4 \) were obtained within the total expressions for \( c_4[0], d_4[0] \) in [20], they were independently extracted in the form of Eq. (2.4h) in [4], see Appendix A therein. The other part of this “box” subgraph contributes to the charge renormalization (see, e.g., [21]). The \( \delta c_4, \delta d_4 \) are the single elements at \( O(a^4) \) order depended on the d.o.f. from the extended QCDe along with the \( \beta \)-function coefficients \( \beta_{0-2} \). All other elements \( c_i[.], d_i[.] \) at \( i \leq 4 \) are universal, while the effects of the inclusion of these d.o.f. are accumulated only in the \( \beta \)-function coefficients appearing in front of them.
3 The constraints on $C^{\text{Bj}}$ and $D_A$ elements from the CBK relation

3.1 The structure of the CBK relation

The $\{\beta\}$-expansion for the quantities $D, C^{\text{Bj}}$ can serve for effective verification of the CBK relation (CBKR) [2, 5],

$$D(a) \cdot C^{\text{Bj}}(a) = 1 + \beta(a) \cdot K(a),$$  \hspace{1cm} (3.1a)
$$K(a) = \sum_{n=1} a^{n-1} K_n,$$  \hspace{1cm} (3.1b)

where $K_n = K_n(\beta_0, \beta_1, \ldots)$ in Eq.(3.1b) are polynomials in $\beta_i$. The structure of the RHS of Eq.(3.1a) can be traced from the evident representation for the LHS,

$$D(a) \cdot C^{\text{Bj}}(a) = 1 + \sum_{n>2} a^n \left[d_n + c_n + \sum_{i>1} d_i c_{n-i}\right],$$  \hspace{1cm} (3.2)

when the condition for the coefficients $c_1 = -d_1$ is explicitly satisfied, see Appendix B. Further, one should compare the second terms in the RHS of Eq.(3.1a) and Eq.(3.2). It is clear that the coefficients $K_n$ should have the same $\{\beta\}$-structure, Eq.(1.4), as for $d_k(c_k)$ but after factorization of the common $\beta(a)$. We show these expansions for $K_{1,2,3}$ that are sufficient up to $O(a^4),

$$K_1 = K_1[1],$$  \hspace{1cm} (3.3a)
$$K_2 = K_2[1] + \beta_0 K_2[2],$$  \hspace{1cm} (3.3b)
$$K_3 = K_3[1] + \beta_0 K_3[2] + \beta_0^2 K_3[3] + \beta_1 K_3[1,1].$$  \hspace{1cm} (3.3c)

Keep in mind that the meaning of the arguments of $K_n[i_0, i_1, \ldots]$ is a bit different from that for the original elements $d_n[\cdot]$; it denotes the exponents of all the accompanying $\beta^i_0, \beta^i_1, \ldots$, including also those that appear in the common factor $\beta(a)$ in the RHS of (3.1a). For future considerations let us present the decomposition also for the following $K_4$ term,

$$K_4 = K_4[1] + \beta_0 K_4[2] + \beta_0^2 K_4[3] + \beta_0^3 K_4[4] + \beta_2 K_4[1,0,1] + \beta_1 \beta_0 K_4[3,1] + \beta_1 K_4[0,2] + \beta_1 K_4[1,1] + \beta_1 \beta_0 K_4[2,1].$$  \hspace{1cm} (3.3d)

The elements $K_n[\cdot]$ of the $\{\beta\}$-expansion help to get rid of the traces of fermion d.o.f. that will be absorbed into the coefficients $\beta_i$. The factorization of the common $\beta(a)$, being put in base, allows one to relate the different elements of the $\{\beta\}$-expansion of $d_4(d_n)$ to the corresponding elements $c_4(c_n)$ and vice versa.

3.2 The “conformal” $\mathbb{1}$ of the CBK relation

In the conformal limit the coefficients $\beta_i = 0$ and the CBKR (3.1a) returns to its initial form $[1]$ with only $\mathbb{1}$ in the RHS that shows the restoration of conformal symmetry. In this case, one gets the reduced factors $D \rightarrow D_0$, $C^{\text{Bj}} \rightarrow C^{\text{Bj}}_0$ of the product in Eq.(3.1a) (or in Eq.(3.2)); so the corresponding series $D_0(a)$ and $C^{\text{Bj}}_0(a)$ are inverse. The condition
$C_0^{\text{Bip}} \cdot D_0 = I$ is related to the $d_n[0], c_n[0]$ elements in every order (see Eq.(2.8) in [4] and [6]),

$$c_n[0] + d_n[0] = - \sum_{l=1}^{n-1} d_l[0] c_{n-l}[0], \quad (3.4)$$

and leads to an explicit closed solution with respect to $c_k[0]$ and vice versa ($c \leftrightarrow d$)

$$c_k[0] = (-)^k \det[D_0^{(k)}] \equiv (-)^k \begin{vmatrix} d_1 & 1 & 0 & \ldots & 0 \\ d_2 & d_1 & 1 & \ldots & 0 \\ d_3 & d_2 & d_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ d_{k-1} & \ldots & \ldots & d_1 & 1 \\ d_k & d_{k-1} & d_{k-2} & \ldots & d_2 & d_1 \end{vmatrix}. \quad (3.5)$$

Here $D_0^{(k)}$ is the matrix that consists of only $d_i \equiv d_i[0]$ elements. The general relation (3.5) can be treated as a prediction for $C^{\text{Bip}}$ by means of $D$ (and vice versa) that is based on the $\{\beta\}$-expansion and the conformal part of CBKR. In the fourth order of $a$, relation (3.5) reduces to

$$d_4[0] + c_4[0] = \tilde{d}_4[0] + \tilde{c}_4[0] = 2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^3 \quad (3.6a)$$

$$= 3C_F^2 \left[ 132C_F C_A - \frac{111}{4} C_F^2 + \left( \frac{175}{2} - 432 \zeta_3 \right) C_A^2 \right], \quad (3.6b)$$

that is fulfilled automatically (underlined terms), as well as all the previous “zero” element’s sums $d_i[0] + c_i[0]$ at $i \leq 4$. In the LHS of Eq.(3.6a) we use the components from Eqs.(2.2g, 2.4g), while in the RHS – the components $d_i[0]$ from Eq.(2.1). The result for the sum in the LHS of Eq.(3.6a) was already predicted in [5] (Eq.(21) there) and in [4], see Appendix A therein. In the following fifth order the relation (3.5) leads to the prediction for the sum in the LHS (doubly underlined terms) obtained in the fourth order in the RHS,

$$d_5[0] + c_5[0] = 2d_1 d_4[0] + 2d_3[0] d_2[0] - 3d_3[0] d_1^2 + 4d_2[0] d_1^3 - 3d_2[0] d_1 - d_1^4 \quad (3.7a)$$

$$= d_1 \left[ C_F^2 C_F^2 27 \left( 43 + 128 \zeta_3 \right) + C_F^4 \left( \frac{2485}{2} + 192 \zeta_3 \right) - C_A C_F^2 \left( 3097 + 864 \zeta_3 \right) + C_A^2 C_F \left( \frac{206233}{72} + 7969 \zeta_3 - 14220 \zeta_5 \right) + 2 \delta d_4 \right]. \quad (3.7b)$$

In general, see [4], if the elements $d_k[0]$ are known up to an order $(n - 1)$, then the sum $d_n[0] + c_n[0]$ in order $n$ is

$$c_n[0] + d_n[0] = (-)^n \begin{vmatrix} d_1 & 1 & 0 & \ldots & 0 \\ d_2 & d_1 & 1 & \ldots & 0 \\ d_3 & d_2 & d_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ d_{n-1} & \ldots & \ldots & d_1 & 1 \\ 0 & d_{n-1} & d_{n-2} & \ldots & d_2 & d_1 \end{vmatrix}. \quad (3.8)$$
or at $c \equiv d$, as it follows from (3.5).

### 3.3 The conformal symmetry-breaking term of the CBK relation

On the other hand, one can apply in the RHS of Eq. (3.1a) the second term proportional to $\beta(a)$ that expresses the break of conformal symmetry. The factorization of the whole $\beta(a)$ leads to the chains of equalities [5, 6] for the first terms $K_n[1]$ in the representation (3.3). These equalities can be easily traced from the terms under the sum in the RHS of Eq. (3.2). Really, for such factorization the factors appearing in front of the successive coefficients $\beta_i$ should be equal to one another for any position of $i$ – the argument of the factor. For the combination $d_n + c_n$ in the sum in (3.2) it leads

$$
\beta_0 \beta_1 \beta_2 \ldots \\
K_1[1] = K_2[0, 1] = K_3[0, 0, 1] = \ldots = \\
c_2[1] + d_2[1] = c_3[0, 1] + d_3[0, 1] = c_4[0, 0, 1] + d_4[0, 0, 1] = 3C_F \left( \frac{7}{2} - 4\zeta_3 \right)
$$

(3.9a)

$$
c_3[0, 0, 1] + d_3[0, 0, 1] = c_4[0, 0, 1] + d_4[0, 0, 1] = c_n[0, 0, \ldots, 1] + d_n[0, 0, \ldots, 1].
$$

(3.9b)

Then, the factorization of the linear $\beta_i$ terms leads to equalities for the subleading corrections $K_{2,3,4,...}[1]$ (see the first terms in the RHS of Eq. (3.3)).

$$
K_2[1] = K_3[0, 1] = K_4[0, 0, 1] = \ldots = \\
= c_3[1] + d_3[1] + d_4[c_2[1] - d_2[1]] = \\
= c_4[0, 1] + d_4[0, 1] + d_4(c_3[0, 1] - d_3[0, 1]) = \\
= C_F^2 \left( -\frac{397}{6} - 136\zeta_3 + 240\zeta_5 \right) + C_F C_A \left( \frac{47}{3} - 16\zeta_3 \right)
$$

(3.10a)

$$
= c_n[0, 0, \ldots, 1] + d_n[0, 0, \ldots, 1] + d_4(c_{n-1}[0, 0, \ldots, 1] - d_{n-1}[0, 0, \ldots, 1]).
$$

(3.10b)

Equations (3.9a, 3.10a) are fulfilled automatically; the current $O(a^4)$ results are checked and underlined; see the corresponding elements in Eq. (2.2) and Eq. (2.4). Equations (3.9b, 3.10b) can serve as natural predictions for any highest orders, the next order $O(a^5)$ results are doubly underlined. Following this line, Eq. (3.11a) provides predictions for the $c_5[0, 1] + d_5[0, 1]$ for 6-loop results in the RHS of Eq. (3.11b) and further,

$$
K_3[1] = K_4[0, 1] = \ldots = \\
= c_4[1] + d_4[1] + d_4(c_3[1] - d_3[1]) + d_2[0]c_2[1] + d_2[1]c_2[0] = \\
= C_A^2 C_F \left( \frac{3213}{8} - \frac{10655}{3}\zeta_3 + 2592\zeta_3^2 + \frac{2840}{3}\zeta_5 \right) - \\
C_F^3 \left( \frac{2471}{12} - 488\zeta_3 + 5720\zeta_5 - 5040\zeta_7 \right) + \\
C_A C_F^2 \left( \frac{4591}{6} + \frac{2306}{3}\zeta_3 - \frac{8120}{3}\zeta_5 + 840\zeta_7 \right)
$$

(3.11a)
\[ = c_5[0, 1] + d_5[0, 1] + d_1 \left( c_4[0, 1] - d_4[0, 1] \right) + d_2[0] c_3[0, 1] + c_2[0] d_3[0, 1] = \ldots \tag{3.11b} \]
\[ = c_{n+1}[0, \ldots, 1] + d_{n+1}[0, \ldots, 1] + d_1 \left( c_n[0, \ldots, 1] - d_n[0, \ldots, 1] \right) + \\
\quad \quad \quad \quad d_2[0] c_{n-1}[0, \ldots, 1] + c_2[0] d_{n-1}[0, \ldots, 1]. \tag{3.11c} \]

Similarly, the next subleading \( K_4[1] \) and its chain of equations can be constructed. We present its expression here for future results and also because it contains the last term linear in \( \beta_i \) from Eq.\((2.2,2.4)\),

\[ K_4[1] = c_5[1] + d_5[1] + d_1 \left( c_4[1] - d_4[1] \right) + (d_2[0] c_3[1] + d_3[1] c_2[0]) \left( [1] \leftrightarrow [0] \right), \tag{3.12} \]
while in the general case, we obtain

\[ K_n[1] = c_{n+1}[1] + d_{n+1}[1] + d_1 \left( c_n[1] - d_n[1] \right) + \sum_{k=2}^{n-1} (d_k[0] c_{n+1-k}[1] + c_k[0] d_{n+1-k}[1]). \tag{3.13} \]

These examples demonstrate that the elements of the \( \{ \beta \} \)-expansion provide appropriate building blocks to analyse and construct the CBK relation. Following this way we exhausted all the cases linear in \( \beta_i \).

Let us consider now the content of the contributions to \( K_{2,3} \) that are higher in exponents of \( \beta_i \). Taking in the RHS of Eq.\((3.2)\) the \( \{ \beta \} \)-expansion for the terms \( c_n \) and \( d_i \), we come to obvious expressions for the elements \( K_{2,3}[] \),

\[ K_2[2] = c_3[2] + d_3[2] = C_F \left( \frac{163}{2} - 76 \zeta_3 \right), \tag{3.14a} \]
\[ K_3[1,1] = c_4[1,1] + d_4[1,1] = C_F \left( \frac{2365}{8} - 421 \zeta_3 + 144 \zeta_3^2 \right), \tag{3.14b} \]
\[ K_3[2] = c_4[2] + d_4[2] + d_1 \left( c_3[2] - d_3[2] \right) + d_2[1] c_2[1] \\
\quad \quad \quad \quad = C_F \left[ \zeta_A \left( \frac{427}{8} + 1341 \zeta_3 - 816 \zeta_3^2 - 680 \zeta_3 \right) \right. \\
\quad \quad \quad \quad \quad \quad \quad - C_F \left( \frac{11573}{12} + 1716 \zeta_3 - 3000 \zeta_5 \right) \right], \tag{3.14c} \]
\[ K_3[3] = c_4[3] + d_4[3] = C_F \left( 614 - 406 \zeta_3 - 180 \zeta_5 \right). \tag{3.14d} \]

For every second equality in Eqs.\((3.14)\) we have used explicit results from Eqs.\((2.1,2.2)\) for \( d_i[.] \) and from Eqs.\((2.3,2.4)\) for \( c_i[.] \).

It is easy to represent those parts of \( K_n[i] \) that are generating by the renormalon chain contributions possessed the maximum powers of \( (a\beta_0)^n \) in each order. The elements \( d_n[n-1] \) and \( c_n[n-1] \) are known explicitly from the results in \([2,16]\) and one does not need new d.o.f. for it. The corresponding parts, e.g., marked with dashed lines in \((3.14)\), can be
obtained from the expression for $K(a)$, (see Eq. (3.1), at $n \geq 2$), taken in order of $O(a^n)$

$$\beta(a) \cdot K(a) \Rightarrow a^2 \beta_0 \cdot \left\{ (a \beta_0)^{n-2} (c_n[n-1] + d_n[n-1]) + (a \beta_0)^{n-2} \beta_0^{-1} \right\} \cdot \left( d_1 (c_{n-1}[n-2] - d_{n-1}[n-2]) + \sum_{m=2}^{n-2} c_m[m-1]d_{n-m}[n-m-1] \right) \right\}. \quad (3.15)$$

The first term in (3.15) were presented explicitly in [2] (Table 2) in notation $\frac{4}{3}T_R n_f \rightarrow -\beta_0$.

4 What can we expect for the $\{\beta\}$-expansion in N$^{5}$LO and beyond

Let us consider the general structure of the $\{\beta\}$-expansion starting with the 6 loop result in Eq.(4.1) in order $a^5$, $n=5$,

$$a^5 \quad d_5 = \beta_0^{a_4} d_5[4] + \beta_2 \beta_0 d_5[1, 0, 1] + \beta_1^2 d_5[0, 2] + \beta_1 \beta_0^2 d_5[2, 1] + \beta_0 d_5[0, 0, 0, 1] + \beta_0^3 d_5[3] + \beta_1 \beta_0 d_5[1, 1] + \beta_2 d_5[0, 0, 1] + \beta_0^2 d_5[2] + \beta_1 d_5[0, 1] + \beta_0 d_5[1] + d_5[0],$$

$$a^n \quad d_n = \left\{ \beta_0^{n-1} d_n[n-1] + \ldots + \beta_{(n-2)} d_n[0, \ldots, 1] + \ldots + d_n[0] \right\}_{N(n)} \quad (4.1a)$$

$$a^n \quad d_n = \left\{ \beta_0^{n-1} d_n[n-1] + \ldots + \beta_{(n-2)} d_n[0, \ldots, 1] + \ldots + d_n[0] \right\}_{N(n)} \quad (4.1b)$$

To obtain the number of elements in this order, we count them with $\beta_0^i$ up to $\beta_3$ in the first line of Eq.(4.1a), this coincides with the number of partitions $p(5 - 1) = 5$. The other terms in (4.1a) repeat the structure of the result in the previous order at $n = 4$; therefore, the complete number of terms at $n = 5$ is ($p(5 - 1) + 7 = 12$, see the discussion in [4]. Generally speaking, for the term $d_n$ of order of $n$ in Eq.(4.1b), one should count new terms from $\beta_0^{(n-1)}$ up to $\beta_{(n-1)-1}$ that gives their number $p(n-1)$, whereas the complete number $N(n)$ of all the terms is evidently the sum $N(n) = \sum_{l=0}^{(n-1)} p(l)$ that leads to the series

$$N(n) = \sum_{l=0}^{(n-1)} p(l) = \{1, 2, 4, 7, 12, 19, \ldots, 97, \ldots\} \sim \frac{\sqrt{6n}}{\pi} \cdot p(n) + \ldots. \quad (4.2)$$

at $n = \{1, 2, 3, 4, 5, 6, \ldots, 10, \ldots\}$

Appropriate smooth approximations for $N(n)$ under true asymptotics are presented, e.g., in [22] and references therein.

The equations for $c_5[\cdot] + d_5[\cdot]$ following from the CBKR are collected in the table below, the expressions for $c_5[4]$ and $d_5[4]$ separately are known from [2].
The elements of $c_5(d_5)$ are formed by a variety of 6-loop diagrams that get contributions from the intrinsic box- and pentagon-subgraphs with gluon legs. These subgraphs introduce to $c_5[0](d_5[0])$ a specific $n_{j,r}$-dependence similar to the one that appeared for $c_4[0](d_4[0])$ and that does not relate to the charge renormalization. Indeed, the new color coefficients (see the definition in Eq. (A.6)) $d_{F,r}^{abcde} d_{F,r}^{abcde} / d_F$ (gluon pentagon inside), $n_{j,r}^{abcde} d_{F,r}^{abcde} / d_F$ (fermion pentagon inside) enter into $c_5[0]$ together with the contributions from the box-like graphs. The $c_5[0]$ can be obtained from $c_5$ following the scheme discussed for $d_4[0]$ in Sec. 2.1. To extract the $c_5[0]$ one can use the roots of the set of equation $\{ \beta_{0,1,2,3}(\{x_i\}) = 0 \}$, where variables $\{x_0, x_1, x_2, x_3\}$ are the attributes of d.o.f., see the bold terms in Eq. (A.8) in Appendices A.2 and B. Then substituting $\{c_{1,2,3,4,5}[0]\}$ in Eq. (3.5) or in Eq. (3.7b), one can restore $d_5[0]$. The element $c_5[0,0,0,1]$ at $\beta_3$ can then be extracted from the equation $c_5(x_{0,0}, x_{1,0}, x_{2,0}, x_{3}) = \beta_3(x_{0,0}, x_{1,0}, x_{2,0}, x_{3}) c_5[0,0,0,1] + c_5[0]$, while its counterpart $d_5[0,0,0,1]$ – from the prediction in Eq. (3.9b). The other elements can be found following the procedure discussed in [4] and briefly sketched in Sec. 2.1.

A notable attempt to obtain the elements $d_n,c_n$ of the $\{\beta\}$-expansion just for $D$ and $C$ based on the interpretation of the RHS of CBKR was done in [20, 23]. There, firstly, a special “two-fold” form of the RHS of CBKR was proposed that includes one of the sums over the powers of $(\beta(a)/a)^j$ and “works up to $N^{3,4}\text{LO}$”. Secondly, if we assume now such a “two-fold” form for each of the factors $D$ and $C$ separately, which is the strong sufficient condition for CBKR to be satisfied, one can get all the elements in $N^{3,4}\text{LO}$. Their results for elements of the $\{\beta\}$-expansion for $D, C$ do not coincide with ours in these orders, while the results for the sums of elements like $c_{n}[0] + d_{n}[0]$ in Eq. (3.8) coincides with ours because their method is already based on CBKR. The mentioned chain of suggestions allows the authors to obtain 8 elements out of 12 [23] in the $N^{3}\text{LO}$ of $\{\beta\}$-expansion owing to the strength of these suggestions. We believe that the method needs further verification.

5 Conclusion

We have considered here the problem of obtaining the structure of QCD corrections and its elements by means of the $\{\beta\}$-expansion for the renormalization group invariant quantities: the nonsinglet parts of the Adler $D$-function and the Bjorken polarized sum rule $C_{\text{Bip}}^\alpha$ – both in order of $O(a^4)$. The explicit results for the elements $d_n[,]$, $c_n[,]$ of this expansion are obtained in Sec. 2 based on the extended QCD model with any number of fermion representations (with single coupling constant), QCD, which work as new degrees of freedom. This our approach to constructing the $\{\beta\}$-expansion is universally applied to any renormalization group invariant quantity [4]. Note here that the generalization of the $\{\beta\}$-expansion to the renormalization group covariant quantities was made and discussed.
in [7]. The explicit knowledge of the elements of the \{\beta\}-expansion:

(i) gives a possibility to perform various kinds of optimization of the perturbation series for a variety of important physical quantities, e.g., related with the Adler \(D\)-function [6];

(ii) taken together with the Crewther-Broadhurst-Kataev relation [1, 2], they allow one to establish nontrivial relations between the afore-mentioned quantities for high orders (higher than \(a^4\)) and verify the mutual reliability of the results in less orders.

We show and discuss these relations in Sec. 3 and specially for \(N^5\)LO in Sec. 4. Our results satisfy all of the above suggested tests.

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A The \(\beta\)-function in \(O(\alpha_s^4)\) at the extended fermion sector

A.1 QCD extended with several fermion representations – QCDe

The Lagrangian of a QCD-like model [11, 13], extended to include several fermion representations of the gauge group is given by

\[
\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\lambda} (\partial_{\mu} A_{a\mu})^2 + \partial_{\mu} \bar{c}_a \partial_{\mu} c^a + g_s f^{abc} \partial_{\mu} A_b^a \partial_{\mu} c^c \\
+ \sum_{r=1}^{N_{\text{rep}}} \sum_{q=1}^{n_{f,r}} \left\{ \frac{i}{2} \bar{\psi}_{q,r} \partial \psi_{q,r} - m_{q,r} \bar{\psi}_{q,r} \psi_{q,r} + g_s \bar{\psi}_{q,r} \hat{A}_a^a T^{a,r} \psi_{q,r} \right\},
\] (A.1)

with the gluon field strength tensor

\[
G_{\mu\nu}^a = \partial_{\mu} A_{a\nu}^a - \partial_{\nu} A_{a\mu}^a + g_s f^{abc} A_{\mu}^b A_{\nu}^c.
\] (A.2)

The index \(r\) specifies the fermion representation; and the index \(q\), the fermion flavour, \(\psi_{q,r}\) is the corresponding fermion field and \(m_{q,r}\) is the corresponding fermion mass. The number of fermion flavours in the representation \(r\) is \(n_{f,r}\) for any of the \(N_{\text{rep}}\) fermion representations.

The generators \(T^{a,r}\) of each fermion representation \(r\) fulfill the defining anticomuting relation of the Lie Algebra corresponding to the gauge group:

\[
\left[ T^{a,r}, T^{b,r} \right] = i f^{abc} T^{c,r}
\] (A.3)

with the structure constants \(f^{abc}\). We have one quadratic Casimir operator \(C_{F,r}\) for each fermion representation \(r\), defined through

\[
T^{a,r} T_{kj}^{a,r} = \delta_{ij} C_{F,r},
\] (A.4)

and \(C_A\) for the adjoint representation. The dimensions of the fermion representations are given by \(d_{F,r}\) and the dimension of the adjoint representation by \(N_A\). The traces of different representations are defined as

\[
T_{F,r} \delta^{ab} = \text{Tr} \left( T^{a,r} T^{b,r} \right) = T_{ij}^{a,r} T^{b,r}_{ji}.
\] (A.5)
At the four-loop level we also encounter higher order invariants in the gauge group factors which are expressed in terms of symmetric tensors

\[ d_{R}^{a_{1}a_{2}...a_{n}} = \frac{1}{n!} \sum_{\text{perm } \pi} \text{Tr} \{ T_{a_{1}(1),R} T_{a_{2}(2),R} \ldots T_{a_{n}(n),R} \} , \]  

(A.6)

where R can be any fermion representation r, denoted as \( R = \{ F, r \} \), or the adjoint representation, \( R = A \), where \( T_{bc}^{a,A} = -if^{abc} \). The standard QCD corresponds to \( N_{\text{rep}} = 1 \), while at \( N_{\text{rep}} > 1 \) the first fermion representation will be considered as a special one – the standard QCD, in what follows with

\[ C_{F,1} \equiv C_{F}, \: d_{F,1} \equiv d_{F}, \: n_{F,1} \equiv n_{F}, \: T^{a_{1}} \equiv T^{a}, \: d_{\pi}^{abcd} \equiv d_{\pi}^{abcd} . \]  

(A.7)

**A.2 The \( \beta \)-function in \( O(\alpha_{s}^{4}) \)**

Here we provide the results for the four-loop \( \beta \)-function of the QCDe coupling \( \alpha \) with an arbitrary number of fermion representations. The number of active fermion flavours of the representation \( i \) are denoted by \( n_{f,i} \), the Casimir operators – \( C_{A} \) and \( C_{F,i} \), and the traces by \( T_{F,i} \), \( d_{R}^{a_{1}a_{2}...a_{n}} \). These components form special contributions for d.o.f., those of them that are revealed in \( O(\alpha^{4}) \) calculations are highlighted in bold below

\[ nT = \sum_{i} n_{f,i} T_{F,i}, \: nTCk = \sum_{i} n_{f,i} T_{F,i} C_{F,i}^{k}, \: nd^{abcd} = \sum_{i} n_{f,i} d_{F,i}^{abcd} . \]  

(A.8)

where \( k \) in \( nTCk \) is a number. They enter in

\[ \beta_{0} = \frac{11}{3} C_{A} - \frac{4}{3} nT; \]  

(A.9a)

\[ \beta_{1} = \frac{34}{3} C_{A}^{2} - 4 \left[ nTC1 + \frac{5}{3} C_{A}(nT) \right] ; \]  

(A.9b)

\[ \beta_{2} = \frac{2857}{54} C_{A}^{3} + 2(nTC2) - 205 \frac{n}{9} C_{A}(nT)(nTC1) - \frac{1415}{27} C_{A}^{2}(nT) + \]

\[ nT \left[ \frac{44}{9} C_{A}(nTC1) + \frac{158}{27} C_{A}(nT) \right] ; \]  

(A.9c)

\[ \beta_{3} = \left( \frac{150653}{486} - \frac{44}{9} \zeta_{3} \right) C_{A}^{4} - \left( \frac{80}{9} - \frac{704}{3} \zeta_{3} \right) C_{A}^{2} d_{AA} - \sum_{i} n_{f,i} d_{FA,i} + \]

\[ \left( \frac{46(nTC3)}{81} - \frac{4204}{27} - \frac{352}{9} \zeta_{3} \right) C_{A}^{2}(nT) + \left( \frac{7073}{243} - \frac{656}{9} \zeta_{3} \right) C_{A}^{3}(nTC1) + \frac{512}{9} - \frac{1664}{3} \zeta_{3} \sum_{i} n_{f,i} d_{FA,i} + \]

\[ \left( \frac{184}{3} - \frac{64}{9} \zeta_{3} \right) (nTC1)^{2} - \left( \frac{304}{27} + \frac{128}{9} \zeta_{3} \right) (nT)(nTC2) + \left( \frac{17152}{243} + \frac{448}{9} \zeta_{3} \right) C_{A}(nT)(nTC1) + \left( \frac{7930}{81} + \frac{224}{9} \zeta_{3} \right) C_{A}^{2}(nT)^{2} - \]

\[ \left( \frac{704}{9} - \frac{512}{3} \zeta_{3} \right) \sum_{i,j} n_{f,i} n_{f,j} d_{FF,i,j} + (nT)^{2} \left[ \frac{1232}{243} C_{A}(nTC1) + \frac{424}{243} C_{A}(nT) \right] . \]  

(A.9d)
where $T_R = \frac{1}{2}$, $C_F = \frac{N_c^2 - 1}{2N_c}$, $C_A = N_c$, $N_A = 2C_FC_A = N_c^2 - 1$. Apart from the mentioned Casimir operators the following invariants appear in our results:

\[
\begin{align*}
    d_{AA} &= \frac{g_{\mu\nu\rho\sigma}}{N_A}, \quad d_{FA,i} = \frac{g_{\mu\nu\rho\sigma}}{N_A}, \quad d_{FF,i} = \frac{g_{\mu\nu\rho\sigma}}{N_A}, \quad (A.10) \\
    \tilde{d}_{FA} &= \frac{g_{\mu\nu\rho\sigma}}{d_F}, \quad \tilde{d}_{FF,i} = \frac{g_{\mu\nu\rho\sigma}}{d_F}, \quad (A.11)
\end{align*}
\]

where $r$ is fixed and $i, j$ will be summed over all fermion representations.

**B The Adler D-function and Bjorken SR results in $O(\alpha_s^4)$ at the extended fermion sector**

Here we present the results for the coefficients $d_i$ of the Adler $D$-function obtained in the framework of QCDe in [13], see Appendix A.1.

\[
\begin{align*}
    d_1 &= 3C_F; \quad (B.1) \\
    d_2 &= -\frac{3}{2}C_F^2 + C_FC_A \left( \frac{123}{2} - 44\zeta_3 \right) - 2C_F(nT)(11 - 8\zeta_3); \quad (B.2) \\
    d_3 &= -\frac{69}{2}C_F^2 + C_F^2 \left[ C_A \left( -127 - 572\zeta_3 + 880\zeta_5 \right) + (nT)(72 + 208\zeta_3 - 320\zeta_5) \right] + \\
    &\quad \quad C_FC_A \left( \frac{9045}{54} - \frac{10948}{9} \zeta_3 - \frac{440}{3} \zeta_5 \right) + C_FC_A(nT) \left( -\frac{31040}{27} + \frac{1768}{9} \zeta_3 + \frac{160}{3} \zeta_5 \right) + \\
    &\quad \quad C_F(nT)^2 \left( \frac{4832}{27} - \frac{1216}{9} \zeta_3 \right); \quad (B.3) \\
    d_4 &= C_F^4 \left( \frac{4157}{8} + 96\zeta_3 \right) + \\
    &\quad \quad C_F^3 \left[ C_A \left( -2024 - 278\zeta_3 + 18040\zeta_5 - 18480\zeta_7 \right) - nT(-298 + 56\zeta_3 + 6560\zeta_5 - 6720\zeta_7) \right] + \\
    &\quad \quad C_F^2 \left[ C_A \left( -\frac{592141}{72} - \frac{87850}{3} \zeta_3 + \frac{104080}{3} \zeta_5 + 9240\zeta_7 \right) \right] + \\
    &\quad \quad C_A(nT) \left( \frac{67925}{9} + \frac{61912}{3} \zeta_3 - \frac{83680}{3} \zeta_5 - 3360\zeta_7 \right) + \\
    &\quad \quad (nT)^2 \left( -\frac{13466}{9} - \frac{10240}{3} \zeta_3 + \frac{16000}{3} \zeta_5 \right) + nTC1(251 + 576\zeta_3 - 960\zeta_5) +
\end{align*}
\]
\[ C_F \left[ C_A^3 \left( \frac{52207039}{972} - \frac{912446}{27} \zeta_3 - \frac{155990}{9} \zeta_5 + 4840\zeta_3^2 - 1540\zeta_7 \right) \right. \]
\[ \left. + C_A^2(nT) \left( \frac{4379861}{81} + \frac{275488}{9} \zeta_3 + \frac{150440}{9} \zeta_5 - 1408\zeta_3^2 + 560\zeta_7 \right) \right. \]
\[ \left. + C_A(nT)^2 \left( \frac{1363372}{81} - \frac{83624}{9} \zeta_3 - \frac{43520}{9} \zeta_5 - 128\zeta_3^2 \right) \right. \]
\[ \left. + C_A(nTC1) \left( 2112\zeta_3^2 - 7792\zeta_3 - 400\zeta_5 + \frac{375193}{54} \right) \right. \]
\[ \left. + \left( nT \right)^3 \left( \frac{392384}{243} + \frac{25984}{27} \zeta_3 + \frac{1280}{3} \zeta_5 \right) \right. \]
\[ \left. + (nT)(nTC1) \left( \frac{63250}{27} - 2784\zeta_3 + 768\zeta_5^2 \right) + nTC2 \left( \frac{355}{3} + 272\zeta_3 - 480\zeta_5 \right) \right] \]
\[ - 16 \left[ \sum_r n_{f,r} \bar{d}_{FF,r} \cdot (13 + 16\zeta_3 - 40\zeta_5) + \bar{d}_{FA} \cdot (-3 + 4\zeta_3 + 20\zeta_5) \right]. \quad (B.4) \]

The results for \( c_k \) of the Bjorken SR in QCDe,
\[ c_1 = -3C_F; \quad (B.5) \]
\[ c_2 = \frac{21}{2}C_F^2 - 23C_AC_F + 8C_F(nT); \quad (B.6) \]
\[ c_3 = -3C_F^3 + C_F^2 \left[ C_A \left( \frac{1241}{9} - \frac{176}{3} \zeta_3 \right) - nT \left( \frac{664}{9} - \frac{64}{3} \zeta_3 \right) \right] + C_FC_A^2 \left( -\frac{10874}{27} + \frac{440}{3} \zeta_5 \right) + \]
\[ C_FC_A(nT) \left( \frac{7070}{27} + \frac{48\zeta_3}{3} - \frac{160}{3} \zeta_5 \right) - C_F(nT)^2 \frac{920}{27} + C_F(nTC1)(59 - 48\zeta_3); \quad (B.7) \]
\[ c_4 = -C_F^4 \left( \frac{4823}{8} + 96\zeta_3 \right) + \]
\[ C_F^3 \left[ -C_A \left( \frac{3707}{18} + \frac{7768}{3} \zeta_3 - \frac{16720}{3} \zeta_5 \right) + nT \left( \frac{5912}{9} + \frac{3296}{3} \zeta_3 - \frac{6080}{3} \zeta_5 \right) \right] + \]
\[ C_F^2 \left[ C_A^2 \left( \frac{1071641}{216} + \frac{25456}{9} \zeta_3 - \frac{22000}{9} \zeta_5 - 6160\zeta_7 \right) \right. \]
\[ \left. - C_A(nT) \left( \frac{106081}{27} + \frac{9104}{9} \zeta_3 - \frac{8000}{9} \zeta_5 - 2240\zeta_7 \right) + \right. \]
\[ \left. (nT)^2 \left( \frac{16114}{27} - \frac{512}{3} \zeta_3 \right) - nTC1 \left( \frac{1399}{3} - 400\zeta_3 \right) \right] + \]
\[
C_F \left[ C_A^3 \left( -\frac{8004277}{972} + \frac{4276}{9} \zeta_3 + \frac{25090}{9} \zeta_5 - \frac{968}{3} \zeta_3^2 + 1540 \zeta_7 \right) + \right.
\]
\[
C_A^2(n_T) \left( \frac{1238827}{162} + 236 \zeta_3 - \frac{14840}{9} \zeta_5 + \frac{704}{3} \zeta_3^2 - 560 \zeta_7 \right) - \]
\[
C_A(n_T)^2 \left( \frac{165283}{81} + \frac{688}{9} \zeta_3 - \frac{320}{3} \zeta_5 + \frac{128}{3} \zeta_3^2 \right) + \]
\[
C_A(n_TC1) \left( \frac{124759}{54} - 1280 \zeta_3 - 400 \zeta_5 \right) + \]
\[
\frac{38720}{243} (n_T)^3 - (n_T)(n_TC1) \left( \frac{19294}{27} - 480 \zeta_3 \right) - n_TC2 \left( \frac{292}{3} + 296 \zeta_3 - 480 \zeta_5 \right) \] +
\[
16 \sum_r n_{f,r} \bar{d}_{FF,r} \cdot (13 + 16 \zeta_3 - 40 \zeta_5) + \bar{d}_{FA} \cdot (-3 + 4 \zeta_3 + 20 \zeta_5) \right] . \quad (B.8)
\]

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