A note on the fractional Hardy inequality

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Abstract
We give a direct proof of fractional Hardy inequality by means of Littlewood–Paley decomposition and properties of singular homogeneous kernels of degree \(-d\). A refinement when \(q > 2\) is proved.

Keywords Hardy inequality · Littlewood–Paley decomposition · Fractional Sobolev spaces

Mathematics Subject Classification 46E35 · 39B62

The classical Hardy inequality states that when \(d \geq 3\)

\[
\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \tag{0.1}
\]

and it is clearly of fundamental importance in analysis. There are of course many different proofs of (0.1), the simplest one consists in restrict by density to \(D(\mathbb{R}^d \setminus \{0\})\), to observe that

\[
\frac{1}{|x|^2} = -\frac{1}{2} x \cdot \nabla \left( \frac{1}{|x|^2} \right),
\]

then to integrate by parts and eventually to apply Cauchy–Schwarz inequality.

A natural extension of (0.1) is in the framework of fractional Sobolev spaces \(\dot{H}^s(\mathbb{R}^d)\). In this setting the following Hardy-type inequality holds

\[
\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2s}} \, dx \leq C ||f||_{\dot{H}^s(\mathbb{R}^d)}^2 \tag{0.2}
\]

provided that \(0 \leq s < \frac{d}{2}\). For a compact and nice proof of (0.2) we quote Theorem 2.57 in [1] and the proof given by Tao in the Appendix of [16] while for an improvement involving Besov spaces we quote [2].

If one is interested in proving an \(L^q\) estimate for \(||f||\) we need to recall the definition of the homogeneous Sobolev norm \(||f||_{\dot{W}^{s,q}(\mathbb{R}^d)}\) which is defined as \(|||D|^s f||_{L^q(\mathbb{R}^d)}\) where

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In this note we give a direct proof and a refinement when \( q > 2 \) for the following class of Hardy-type inequalities that generalize the fractional Hardy inequality (0.2).

**Theorem 0.1** (Fractional Hardy inequality) Let \( 0 < s < \frac{d}{q} \), \( 1 < q < \infty \) and \( f \in \dot{W}^{s,q}(\mathbb{R}^d) \), then

\[
\left\| \frac{f}{|x|^s} \right\|_{L^q(\mathbb{R}^d)} \leq \mathcal{C}(d, s, q) \| f \|_{\dot{W}^{s,q}(\mathbb{R}^d)}.
\]

(0.3)

The explicit value of the constant \( \mathcal{C}(d, s, q) \) in (0.3) is due to Herbst [11]. The proof of (0.3) goes back to the end of the fifties of the last century thanks to the work of Stein and Weiss [15] who proved an even more general version of (0.3) called Stein–Weiss inequality given by

\[
\left( \int_{\mathbb{R}^d} \left( |T_\lambda f(x)| |x|^{-\beta} \right)^q \, dx \right)^{\frac{1}{q}} \leq \mathcal{C}(d, q, p, \lambda) \left( \int_{\mathbb{R}^d} \left( |f(x)| |x|^\alpha \right)^p \, dx \right)^{\frac{1}{p}}
\]

(0.4)

where

\[ T_\lambda f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\lambda} \, dy \quad 0 < \lambda < d, \]

and

\[ 0 < \lambda < d, \ 1 < p < \infty, \ \alpha < \frac{d}{p'}, \ p \leq q < \infty, \ \beta < -\frac{d}{q}, \ \alpha + \beta \geq 0, \]

\[ \frac{1}{q} = \frac{1}{p} + \left( \frac{\lambda + \alpha + \beta}{d} \right) - 1. \]

The fact that (0.4) implies (0.3) follows by the fact that \( T_\lambda f = c |D|^{-s} f \), with \( \lambda = d - s \), \( c = \pi^{d/2} \Gamma((d-\lambda)/2) \) and choosing \( p = q \) and \( \alpha = 0, \beta = s \).

In order to state our result we recall the standard definition for Homogeneous Besov norm \( \| \cdot \|_{\dot{B}^{s}_{p,q}} \) and Tribel–Lizorkin norm \( \| \cdot \|_{\dot{F}^{s}_{p,q}} \) (see e.g. [8] for general references).

Let \( f \) be a tempered distribution such that \( \hat{\mathcal{f}} \in L^1_{loc} \) and \( P_N(f) \) the Littlewood–Paley projector on the dyadic frequency \( N \), i.e.

\[ P_N(f)(\xi) = \psi_N(\xi) \hat{\mathcal{f}}(\xi) \quad \text{where} \quad \psi_N(\xi) = \psi\left( \frac{\xi}{N} \right) \]

and \( \sum_{N \in 2\mathbb{Z}} \psi_N = 1 \), then we define

\[
\| f \|_{\dot{B}^{s}_{p,q}} = \left( \sum_{N \in 2\mathbb{Z}} \| N^s P_N(f) \|_{L^p}^q \right)^{\frac{1}{q}},
\]

\[ \| f \|_{\dot{F}^{s}_{p,q}} = \left\| \left( \sum_{N \in 2\mathbb{Z}} |N^s P_N(f)(x)|^q \right)^{\frac{1}{q}} \right\|_{L^p}. \]

Our result is a direct proof of the following

**Theorem 0.2** Let \( 0 < s < \frac{d}{q} \), \( 1 < q < \infty \) then

\[
\left\| \frac{f}{|x|^s} \right\|_{L^q(\mathbb{R}^d)} \leq \mathcal{C}(d, s, q) \| f \|_{\dot{B}^{s}_{p,q}(\mathbb{R}^d)},
\]

(0.5)

with the following corollary
Corollary 0.1  Let $0 < s < \frac{d}{q}$, if $1 < q \leq 2$ then
\[
\| f \|_{L^q(|x|^2)} \leq C(d, s, q) \| f \|_{W^{s,q}(\mathbb{R}^d)},
\]  
and if $q > 2$
\[
\| f \|_{L^q(|x|^2)} \leq C(d, s, q) \| f \|_{\dot{W}^{s,q}(\mathbb{R}^d)} \| f \|_{\dot{F}_{q,2(q-1)}^s(\mathbb{R}^d)}.
\]

The fact that $\| \frac{f}{|x|^q} \|_{L^q(\mathbb{R}^d)}$ can be controlled by homogeneous Besov norms is not a novelty, a proof of Theorem 0.2 can be found in [18], see also [19]. Here we present a direct proof using the Schur test. We shall remark that our corollary when $q > 2$ is a refinement of Hardy inequality (0.3). Indeed we have when $2(q-1) > 2$
\[
\| f \|_{\dot{F}_{q,2(q-1)}^s(\mathbb{R}^d)} \leq \| f \|_{\dot{F}_{q,2}^s(\mathbb{R}^d)} \sim \| f \|_{\dot{W}^{s,q}(\mathbb{R}^d)}
\]
\[
\| f \|_{\dot{F}_{q,2(q-1)}^s(\mathbb{R}^d)} \leq \| f \|_{\dot{F}_{q,2}^s(\mathbb{R}^d)} \sim \| f \|_{\dot{W}^{s,q}(\mathbb{R}^d)}
\]
thanks to square function estimate
\[
\| |f| \|_{\dot{F}_{q,2}^s} = \left\| \left( \sum_{N \in \mathbb{Z}} |N^s P_N(f)(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \sim \| |||D^s f|||_{L^q(\mathbb{R}^d)}.
\]
The case $1 < q < 2$ is proved by duality and it requires proving the $L^q$ continuity for singular homogeneous kernels of degree-$d$. This fact is well known and is Lemma 2.1 in [15]. We underline however that our strategy in proving Theorem 0.2 permits to skip the more delicate lemmas in the Stein and Weiss paper [15] that are needed to prove (0.3).

As a final comment, recalling that $||D|f| = \sum_{j=1}^d R_j(\partial_{x_j} f)$ with $R_j$ the Riesz transform defined as $(R_j f)(\xi) = -i \frac{\xi_j}{|\xi|} \hat{u}(\xi)$ and that hence $|||D|f|||_{L^q(\mathbb{R}^d)} \lesssim |||f|||_{L^q(\mathbb{R}^d)}$ when $1 < q < \infty$, we get

Corollary 0.2  Let $2 < q < d$ then
\[
\| f \|_{L^q(|x|^2)} \leq C(d, s, q) \| \nabla f \|_{L^q(\mathbb{R}^d)} \| f \|_{\dot{F}_{q,2(q-1)}^s(\mathbb{R}^d)},
\]

We underline that Corollary 0.2 is a refinement of the classical Hardy inequality involving $\nabla f$
\[
\| f \|_{L^q(\mathbb{R}^d)} \leq \left( \frac{q}{d-q} \right) \| \nabla f \|_{L^q(\mathbb{R}^d)},
\]
by the fact that $|||f|||_{\dot{F}_{q,2(q-1)}^s(\mathbb{R}^d)} \leq \| f \|_{\dot{F}_{q,2}^s(\mathbb{R}^d)} \lesssim \| \nabla f \|_{L^q(\mathbb{R}^d)}$. In the literature there is a lot of interest in proving improvements for (0.9), typically such improvement (in bounded or unbounded domains) are in the direction to add a negative term in r.h.s of (0.9), see e.g. [3–7, 9, 10, 13]. Our refinement, although obtained with different techniques, is more in the spirit of [2, 17], i.e. to control r.h.s. of (0.9) with terms that are smaller (up to a multiplicative constant) than the Sobolev norms.
1 Proof of Theorem 0.2

A key argument in our proof is given by the following well known version of Schur test

**Proposition 1.1** Let \( \alpha_{N,R} \geq 0 \), with \( N, R \in 2^\mathbb{Z} \), 1 < \( q < \infty \), then

\[
\sum_{R} \left( \sum_{N} \alpha_{N,R} C_N \right)^q \lesssim \sum_{N} (C_N)^q
\]

provided there exists a sequence of positive numbers \( p_N \) such that

\[
\left( \sum_{N} \alpha_{N,R} p_N^{\frac{q}{q'}} \right)^{\frac{q}{q'}} \lesssim p_R
\]

(1.1)

\[
\sum_{R} \alpha_{N,R} p_R \lesssim p_N.
\]

(1.2)

**Proof** By Holder's inequality with conjugated exponent \((q, q')\)

\[
\sum_{N} \alpha_{N,R} C_N = \sum_{N} \alpha_{N,R}^{\frac{1}{q}} p_N^{\frac{1}{q'}} C_N \leq \left( \sum_{N} \alpha_{N,R} p_N^{\frac{q}{q'}} \right)^{\frac{1}{q'}} \left( \sum_{N} \alpha_{N,R} C_N^{\frac{q}{q}} \right)^{\frac{1}{q}}
\]

we get

\[
\sum_{R} \left( \sum_{N} \alpha_{N,R} C_N \right)^q \lesssim \sum_{R} \left( \sum_{N} \alpha_{N,R} p_N^{\frac{q}{q'}} \right)^{\frac{q}{q'}} \left( \sum_{N} \alpha_{N,R} C_N^{\frac{q}{q}} \right)^{\frac{1}{q}}
\]

that, thanks to (1.1) and Fubini, implies

\[
\sum_{R} \left( \sum_{N} \alpha_{N,R} C_N \right)^q \lesssim \sum_{R} p_R \left( \sum_{N} \alpha_{N,R} C_N^{\frac{q}{q}} \right) = \sum_{N} C_N^{\frac{q}{q}} \left( \sum_{R} \alpha_{N,R} p_R \right).
\]

Now by (1.2) we conclude

\[
\sum_{R} \left( \sum_{N} \alpha_{N,R} C_N \right)^q \lesssim \sum_{N} C_N^{\frac{q}{q}} p_N = \sum_{N} C_N^{\frac{q}{q}}.
\]

\( \square \)

The strategy of the proof for is an adaptation of proof of Hardy inequality in the case \( q = 2 \) given by Tao [16], i.e. to prove the following estimate

\[
\int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^q} dx \lesssim \sum_{N} N^{qs} ||P_N f||^q_{L^q(\mathbb{R}^d)}
\]

(1.3)

where \( P_N f \) are the classical Littlewood–Paley projectors with \( N \) a dyadic number.

We divide \( \mathbb{R}^d \) in dyadic shells obtaining

\[
\int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^q} dx = \sum_{R \in 2^\mathbb{Z}} \int_{\mathbb{R}^d \setminus [x] \leq R} \frac{|f(x)|^q}{|x|^q} dx \lesssim \sum_{R \in 2^\mathbb{Z}} \frac{1}{R^{sq}} \int_{\mathbb{R}^d \setminus [x] \leq R} |f|^q dx.
\]

(1.4)

\( \square \) Springer
such that using the Littlewood-Paley decomposition we get

$$\sum_{R \in 2Z} \frac{1}{R^{sq}} \int_{\{\frac{R}{2} \leq |x| \leq R\}} |f|^q \, dx \leq \sum_{R \in 2Z} R^{-sq} \left( \sum_{N \in 2Z} \left( \int_{\{\frac{R}{2} \leq |x| \leq R\}} |P_N(f)|^q \right)^\frac{1}{q} \right)^q.$$  \tag{1.5}

By the Bernstein inequality $$||PN(f)||_{L^\infty(\mathbb{R}^d)} \leq N^d ||PN(f)||_{L^q(\mathbb{R}^d)}$$ it follows that

$$\left( \int_{\frac{R}{2} < |x| < R} |P_N(f)|^q \right)^\frac{1}{q} \leq R^d ||P_N(f)||_{L^\infty} \leq (NR)^{\frac{d}{q}} ||P_N(f)||_{L^q}, \tag{1.6}$$

and clearly

$$\left( \int_{\frac{R}{2} < |x| < R} |P_N(f)|^q \right)^\frac{1}{q} \leq ||PNf||_{L^q},$$

such that we get

$$\int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^{qs}} \, dx \lesssim \sum_{R} R^{-qs} \left( \sum_{N} \min\{1, (NR)^\frac{d}{q} \} \|PNf\|_{L^q} \right)^q = \sum_{R} \left( \sum_{N} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s} \} \|NsPNf\|_{L^q} \right)^q.$$

The last step is to apply the Schur test given by Proposition 1.1 in order to conclude that

$$\sum_{R} \left( \sum_{N} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s} \} \|NsPNf\|_{L^q} \right)^q \leq \sum_{N \in 2Z} N^{sq} ||P_N(f)||_{L^q}^q$$

$$= \sum_{N \in 2Z} N^{sq} \int_{\mathbb{R}^d} |P_N(f)|^q = \int_{\mathbb{R}^d} \sum_{N \in 2Z} N^{sq} |P_N(f)|^q.$$

Notice that

$$\sum_{N > \frac{1}{R}} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} + \sum_{N \leq \frac{1}{R}} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\}$$

$$= R^{-s} \sum_{N > \frac{1}{R}} N^{-s} + R^{\frac{d}{q}-s} \sum_{N \leq \frac{1}{R}} N^{\frac{d}{q}-s} \lesssim 1$$

such that (arguing in the same way when summing over $R$)

$$\sum_{N} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} \lesssim 1 \tag{1.7}$$

$$\sum_{R} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} \lesssim 1. \tag{1.8}$$

The hypothesis for Schur test given by Proposition 1.1 are hence fulfilled by choosing $\alpha_{N,R} = \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\}$ and $p_N = 1$ in Proposition 1.1. This proves (0.3).
2 Proof of Corollary 0.1

In Theorem 0.2 we proved the following estimate

\[ \int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^s} \, dx \lesssim \sum_N N^{qs} \|P_N f\|_{L^q}^q \]  

(2.1)

where \( P_N f \) are the classical Littlewood–Paley projectors with \( N \) a dyadic number. First we prove that (2.1) implies the Fractional Hardy inequality. We have two cases: \( q \geq 2 \), \( q < 2 \).

**Case** \( q \geq 2 \):

Thanks to (2.1) we derive

\[ \sum_N N^{qs} \|P_N f\|_{L^q}^q = \int_{\mathbb{R}^d} \sum_N N^{sq} |P_N f(x)|^q \, dx \leq \int_{\mathbb{R}^d} \left( \sum_N |N^s P_N f(x)|^2 \right)^{q/2} \, dx \]

from the elementary inequality \((\sum_i a_i^{p_1})^{1/p_1} \leq (\sum_i a_i^{p_2})^{1/p_2}\) with \( p_1 \geq p_2 \), obtaining

\[ \int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^s} \, dx \lesssim \sum_N N^{qs} \|P_N f\|_{L^q}^q \]

\[ \leq \int_{\mathbb{R}^d} \left( \sum_N |N^s P_N f(x)|^2 \right)^{q/2} \, dx \sim \|D^s f\|_{L^q}^q \]

where the last equivalence is nothing but the classical square function estimate, see for instance [14].

To prove (0.7) we notice that

\[ \int_{\mathbb{R}^d} \sum_N N^{2s} |P_N f(x)|^q \, dx \]

\[ \leq \int_{\mathbb{R}^d} \left( \sum_N N^{2s} |P_N f(x)|^2 \right)^{q/2} \left( \sum_N N^{2s(q-1)} |P_N f(x)|^{2(q-1)} \right)^{1/2} \, dx \]

\[ \leq \left( \int_{\mathbb{R}^d} \left( \sum_N N^{2s} |P_N f(x)|^2 \right)^{q/2} \, dx \right) \left( \int_{\mathbb{R}^d} \left( \sum_N N^{2s(q-1)} |P_N f(x)|^{2(q-1)} \right)^{q/(2q-1)} \, dx \right)^{q-1/q} \]

by applying twice the Holder’s inequality, first in the serie with conjugated exponent \((2, 2)\) and then in the integral with conjugated exponent \((q, \frac{q}{q-1})\). By definition

\[ \left( \int_{\mathbb{R}^d} \left( \sum_N N^{2s(q-1)} |P_N f(x)|^{2(q-1)} \right)^{q/(2q-1)} \, dx \right)^{q-1/q} = \|f\|_{F_{q,2(q-1)}}^{q-1}. \]

**Case** \( q < 2 \):
For the case $q < 2$ we use the dual characterization of $L^q$ norms, i.e.

$$
\left\| \frac{f}{|x|^s} \right\|_{L^q} = \sup_{\|g\|_{L^{q'}} = 1} \left\langle \frac{f(x)}{|x|^s}, g \right\rangle = \sup_{\|g\|_{L^{q'}} = 1} \left\langle f(x), \frac{g(x)}{|x|^s} \right\rangle
$$

$$
= \sup_{\|g\|_{L^{q'}} = 1} \left\langle |D|^{-s} |D|^s f(x), \frac{g(x)}{|x|^s} \right\rangle = \sup_{\|g\|_{L^{q'}} = 1} \left\langle |D|^s f, |D|^{-s} \left( \frac{g(x)}{|x|^s} \right) \right\rangle
$$

$$
\leq \left\| |D|^s f \right\|_{L^q} \left\| |D|^{-s} \left( \frac{g(x)}{|x|^s} \right) \right\|_{L^{q'}}.
$$

Now we aim to prove that

$$
\left\| |D|^{-s} \left( \frac{g(x)}{|x|^s} \right) \right\|_{L^{q'}(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}(\mathbb{R}^d)}, \tag{2.2}
$$

for all $g \in L^{q'}$ with $q' > 2$ such that we could conclude that

$$
\left\| \frac{f}{|x|^s} \right\|_{L^q(\mathbb{R}^d)} = \sup_{\|g\|_{L^{q'}} = 1} \left\langle \frac{f(x)}{|x|^s}, g \right\rangle \lesssim \|D|^s f\|_{L^q(\mathbb{R}^d)}.
$$

Now we prove (2.2). We recall that $|D|^{-s} f \sim \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} dy$, see Theorem 5.9 in [12], such that we have (renaming $q'$ by $q$ to simplify the notation)

$$
|D|^{-s} \left( \frac{g(x)}{|x|^s} \right) \sim \int_{\mathbb{R}^d} \frac{g(y)}{|x-y|^{d-s} |y|^s} dy \lesssim \left| \int_{\mathbb{R}^d} \frac{|g(y)|}{|y|^s |x-y|^{d-s}} dy \right|^q
$$

$$
\lesssim \int_{\mathbb{R}^d} \left| \frac{|g(y)|}{|y|^s |x-y|^{d-s}} \right|^q dy + \int_{\mathbb{R}^d} \left| \frac{|g(y)|}{|y|^s |x-y|^{d-s}} \right|^q dy
$$

$$
\lesssim \int_{\mathbb{R}^d} \left| \frac{|g(y)|}{|y|^s |x-y|^{d-s}} \right|^q dy + \int_{\mathbb{R}^d} \left| \frac{|g(y)|}{|y|^s |x-y|^{d-s}} \right|^q dy
$$

$$
\lesssim \int_{\mathbb{R}^d} \left| \frac{|g(y)|}{|y|^s |x-y|^{d-s}} \right|^q dy + \int_{\mathbb{R}^d} \left| \frac{|g(y)|}{|y|^s |x-y|^{d-s}} \right|^q dy
$$

$$
:= |S_1(g)|^q + |S_2(g)|^q.
$$

By previous estimates using Paley–Littlewood decomposition and the square function equivalence we get when $q > 2$

$$
\int_{\mathbb{R}^d} |S_1(g)|^q dx \sim \int_{\mathbb{R}^d} \left| \frac{|D|^{-s} |g(x)|}{|x|^s} \right|^q dx \lesssim \left\| |D|^s (|D|^{-s} |g|) \right\|_{L^q(\mathbb{R}^d)}^q = \|g\|_{L^q(\mathbb{R}^d)}^q.
$$

Concerning $\|S_2(g)\|_{L^q}$ we follow the strategy of Stein and Weiss [15] proving the $L^q$ continuity for singular homogeneous kernels of degree-$d$. The proof of this fact is Lemma 2.1 in [15] that we show for reader convenience. First notice that $|y| \leq \frac{|x|}{2}$ implies

$$
|x - y| \geq |x| - |y| \geq \frac{|x|}{2},
$$

such that

$$
\int_{|y| \leq \frac{|x|}{2}} \frac{|g(y)|}{|x-y|^{d-s} |y|^s} dy \lesssim \int_{|y| \leq \frac{|x|}{2}} \frac{|g(y)|}{|y|^s |x|^{d-s}} dy. \tag{2.3}
$$
Now we introduce following [15] the function,

$$K(x, y) = \begin{cases} |y|^s |x|^{d-s} |y| \leq \frac{|x|}{2} \\ 0 \text{ otherwise} \end{cases}$$

and

$$Ug(x) := \int_{|y| \leq \frac{|x|}{2}} \frac{|g(y)|}{|y|^s |x|^{d-s}} dy = \int_{\mathbb{R}^d} K(|x|, |y|)|g(y)| dy.$$ 

To conclude the proof it suffices hence to show that

$$\int_{\mathbb{R}^d} |Ug|^q dx \lesssim \int |g|^q dx.$$ 

Fixing $\eta \in S^{d-1}$ and calling $|x| = R$ we define

$$U_\eta g(R) := \int_0^{+\infty} r^{d-1} K(R, r) \cdot |g(r \eta)| dr,$$

such that

$$Ug(x) = \int_{\mathbb{R}} K(|x|, |y|)|g(y)| dy = \int_0^{+\infty} \left( \int_{S^{d-1}} K(R, r)|g(r \eta)| d\sigma_\eta \right) r^{d-1} dr$$

$$= \int_{S^{d-1}} \int_0^{+\infty} K(R, r)|g(r \eta)| r^{d-1} dr d\sigma_\eta = \int_{S^{d-1}} U_\eta g(R) d\sigma_\eta.$$ 

By the substitution $r = tR$ we obtain

$$U_\eta g(R) = \int_0^{+\infty} K(R, tR) |g(tR \eta)| R^{d-1} t^{d-1} R dt$$

$$= \int_0^{+\infty} K(1, t) |g(tR \eta)| t^{d-1} dt,$$

thanks to the fact that $K$ is homogeneous of degree $-d$, i.e. that

$$K(\lambda x, \lambda y) = |\lambda|^{-d} K(|x|, |y|).$$

Let $h$ be the function in $L^q((0, +\infty); R^{d-1} dR)$ of unitary norm such that

$$\left( \int_0^{+\infty} |U_\eta g(R)|^q R^{d-1} dR \right)^{\frac{1}{q}} = \int_0^{+\infty} U_\eta g(R) h(R) R^{d-1} dR$$

$$= \int_0^{+\infty} \left\{ \int_0^{+\infty} K(1, t) |g(tR \eta)| t^{d-1} dt \right\} R^{d-1} h(R) dR$$

$$= \int_0^{+\infty} K(1, t) t^{d-1} \left\{ \int_0^{+\infty} |g(tR \eta)| h(R) R^{d-1} dR \right\} dt$$

$$\leq \int_0^{+\infty} K(1, t) t^{d-1} \left\{ \int_0^{+\infty} |g(tR \eta)|^q R^{d-1} dR \right\}^{\frac{1}{q}} dt$$

$$= \left( \int_0^{+\infty} K(1, t) t^{d-1-\frac{d-1}{q}} dt \right) \cdot \left\{ \int_0^{+\infty} |g(R \eta)|^q R^{d-1} dR \right\}^{\frac{1}{q}}$$

$$= \left( \int_0^{1} t^{d-\frac{d}{q}-1-s} dt \right) \cdot \left\{ \int_0^{+\infty} |g(R \eta)|^q R^{d-1} dR \right\}^{\frac{1}{q}} =: J \cdot \left\{ \int_0^{+\infty} |g(R \eta)|^q R^{d-1} dR \right\}^{\frac{1}{q}}.$$
where the last integral $J$ converges due to the fact that by our assumptions $s < \frac{d}{q}$ (remember that we skipped $q'$ with $q$).

Now we estimate $L^q(\mathbb{R}^d)$ norm of $Ug$. By Jensen inequality

$$|U g(R)|^q = \left| \int_{S^{d-1}} |U_{\eta} g(R)| d\sigma_{\eta} \right|^q \leq \{ |S^{d-1}| \}^{q-1} \int_{S^{d-1}} |U_{\eta} g|^q d\sigma_{\eta},$$

such that integrating with respect to the measure $R^{d-1} dR$ we get

$$\int_0^{+\infty} |U g(R)|^q R^{d-1} dR$$

$$\leq J^q |S^{d-1}|^{q-1} \left( \int_0^{+\infty} \left\{ \int_{S^{d-1}} |U_{\eta} g(R)|^q d\sigma_{\eta} \right\} R^{d-1} dR \right)$$

$$= J^q |S^{d-1}|^{q-1} \int_{S^{d-1}} \int_0^{+\infty} |U_{\eta} g(R)|^q R^{d-1} dR d\sigma_{\eta}$$

$$\leq J^q |S^{d-1}|^{q-1} \int_{S^{d-1}} \int_0^{+\infty} |g(R \eta)|^q R^{d-1} dR d\sigma = J^q |S^{d-1}|^{q-1} \int_{\mathbb{R}^d} |g(x)|^q dx.$$

By the fact that $U f(x)$ is radial we can conclude that

$$\int_{\mathbb{R}^d} |U g(x)|^q dx = |S^{d-1}| \cdot \int_0^{+\infty} |U g(R)|^q R^{d-1} dR \leq J^q |S^{d-1}|^{q-1} \int_{\mathbb{R}^d} |g(x)|^q dx.$$

This concludes the proof in the case $q < 2$.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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