Exact-\(m\)-majority terms

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Abstract. We say that an idempotent term \(t\) is an **exact-\(m\)**-majority term if \(t\) evaluates to \(a\), whenever the element \(a\) occurs exactly \(m\) times in the arguments of \(t\), and all the other arguments are equal.

If \(m < n\) and some variety \(V\) has an \(n\)-ary exact-\(m\)-majority term, then \(V\) is congruence modular. For certain values of \(n\) and \(m\), for example, \(n = 5\) and \(m = 3\), the existence of an \(n\)-ary exact-\(m\)-majority term neither implies congruence distributivity, nor congruence permutability.

Near-unanimity terms have been around in universal algebra starting from the 70’s in the past century \cite{1,10} and recently have played an important role in tractability problems \cite{8,9}. Recent results about near-unanimity terms include \cite{2,8,9}. Further references can be found in the quoted works.

A ternary near-unanimity term is a majority term. Curiously enough, the opposite notion of a minority term has proved quite interesting \cite{7}. In the final section of \cite{8} we generalized the notion of a minority term to a “lone-dissent” term. A lone-dissent term \(u\) returns an element appearing just once among its arguments, provided all the other arguments are equal. In contrast, in the same situation, a near-unanimity term returns the element appearing all but one time. In both cases, we are in the situation in which the arguments of \(u\) are always chosen from a pair of elements. If \(u\) returns the element appearing \(n - 1\) times, where \(u\) is \(n\)-ary, then \(u\) is near-unanimity term. If \(u\) returns the element appearing once, then \(u\) is a lone-dissent term. Thus the two notions have the following common generalization.

**Definition 1.** Suppose that \(0 < m \leq n\). An \(n\)-ary term \(u\) is an **exact-\(m\)**-majority term (in some algebra or some variety) if the equations

\[
u(x_1, x_2, \ldots, x_n) = x
\]

hold, whenever only the variables \(x\) and \(y\) occur in \(\{x_1, x_2, \ldots, x_n\}\) and the set \(\{i \leq n \mid x_i = x\}\) has exactly \(m\) elements.

We shall see that the two situations \(m < \frac{n}{2}\) and \(m > \frac{n}{2}\) are rather different. In both cases congruence modularity follows (if \(m < n\)). On the other hand,
the existence of an exact-$m$-majority term implies congruence permutability for $m < \frac{n}{2}$, while it never implies congruence permutability, for $m > \frac{n}{2}$.

Remark 2. (a) Notice that if (Eq. 1) holds in some variety, we can take $x = y$ in (Eq. 1), hence every exact-$m$-majority term is idempotent. An $n$-ary term $u$ is idempotent if and only if it is an exact-$n$-majority term, in the present terminology.

(b) It follows immediately from the above remark that if $m < n$, then having an exact-$m$-majority term is a nontrivial idempotent Maltsev condition (compare [11] and [4, Lemma 9.4(3)]). As mentioned, we shall prove the stronger result that the existence of an exact-$m$-majority term implies congruence modularity.

(c) If $u$ is an $n$-ary term, then $u$ is a near-unanimity term if and only if $u$ is an exact-$(n-1)$-majority term. A term $u$ is a lone-dissent term [8] if and only if $u$ is an exact-1-majority term. In this terminology, a minority term [7] is a 3-ary lone-dissent term.

(d) If $n$ is even, then a variety $V$ has an exact-$\frac{n}{2}$-majority term if and only if $V$ is trivial.

(e) Intuitively, the notion of a (non exact) $m$-majority term could appear more natural; however we shall soon see that this “non exact” notion provides nothing essentially new, since it turns out to be equivalent to the existence of a majority term—possibly, of distinct arity. If $\frac{n}{2} < m \leq n$ we say that an $n$-ary term $u$ is an $m$-majority term if $u(x_1, x_2, \ldots, x_n) = x$ holds, whenever $|\{ i \leq n \mid x_i = x \}| \geq m$.

If $m < n$ and $u$ is an $m$-majority term, then $v(x_1, x_2, \ldots, x_{m+1}) = u(x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+1}, \ldots, x_{m+1})$ is an $m+1$-ary near-unanimity term, hence the notion of a (non exact) $m$-majority term seems to have little interests. Let us point out, however, that the notion has been used, letting $m$ vary, in order to construct the main counterexample in [8]. See [8, Section 2].

Let $(m, n)$ denote the greatest common divisor of $m$ and $n$.

**Proposition 3.** Suppose that $n \geq 3$ and $0 < m < n$.

1. If $m < \frac{n}{2}$ and some variety $V$ has an $n$-ary exact-$m$-majority term, then $V$ is congruence permutable.
2. If $m > \frac{n}{2}$, then there is some variety $V$ with an $n$-ary exact-$m$-majority term and which is not congruence permutable.
3. If $k|(m, n)$ and some variety $V$ has an $n$-ary exact-$m$-majority term, then $V$ has an $\frac{n}{k}$-ary exact-$\frac{m}{k}$-majority term.
4. If $n - m$ divides $n$ and some variety $V$ has an $n$-ary exact-$m$-majority term, then $V$ has an $\frac{n}{n-m}$-ary near-unanimity term, in particular, $V$ is congruence distributive.
5. If $hm \equiv 1 \pmod{q}$ and $n = m+kq$, for some $h, q, k \in \mathbb{N}$, then the term $u(x_1, x_2, \ldots, x_n) = hx_1 + hx_2 + \cdots + hx_n$ is an $n$-ary exact-$m$-majority term in an abelian group of exponent dividing $q$ (in additive notation).
In particular, for such values of $m$ and $n$, if $q > 1$ the existence of an $n$-ary exact-$m$-majority term does not imply congruence distributivity.

Proof. (1) If $m < \frac{n}{2}$ and $u$ is an $n$-ary exact-$m$-majority term, consider the term $t(x, y, z) = u(x, \ldots, x, y, \ldots, y, z, \ldots, z)$, where both $x$ and $z$ appear $m$ times and $y$ appears $n - 2m$ times. Notice that $n - 2m > 0$, since $m < \frac{n}{2}$. Then $t$ is a Maltsev term witnessing congruence permutability.

(2) If $m > \frac{n}{2}$, then in lattices the term $u_{m,n}(x_1, \ldots, x_n) = \prod_{|J| = m} \sum_{i \in J} x_i$ ($J$ varying on subsets of $\{1, \ldots, n\}$) is an exact-$m$-majority term, actually, an $m$-majority term. However, lattices are not congruence permutative.

(3) If $u$ witnesses the assumptions and $\ell = \frac{n}{k}$, let $t(x_1, x_2, \ldots, x_\ell) = u(x_1, x_1, \ldots, x_2, x_2, \ldots, x_\ell, x_\ell, \ldots)$, where each variable appears $k$ times on the right-hand side.

(4) Take $k = n - m$ in (3), then notice that $\frac{m}{k} + 1 = \frac{n}{k}$.

(5) If $x$ appears $m$ times in the arguments of the term, the sum of the corresponding summands gives $hm$, that is, $x$, since $hm \equiv 1 \pmod{q}$ and the group is abelian of exponent dividing $q$. If the only other variable is $y$, it occurs $kq$ times, hence the outcome is 0, again since the group is abelian and its exponent divides $q$. □

Examples 4. (a) The existence of a 5-ary exact-3-majority term does not imply congruence permutability, by Proposition 3(2). It does not imply congruence distributivity, either, by Proposition 3(5), taking $h = k = 1$ and $q = 2$.

Henceforth, if $V_{3,5}$ is the variety with a 5-ary operation satisfying the equations for a 5-ary exact-3-majority term, then $V_{3,5}$ is neither congruence permutable nor congruence distributive. On the other hand, we shall show that every variety with an exact-$m$-majority term is congruence modular. Hence $V_{3,5}$ seems to be an interesting example of a congruence modular variety which is neither congruence permutable nor congruence distributive.

(b) If $m \neq \frac{n}{2}$, then there is a nontrivial algebra (hence a nontrivial variety) with an $n$-ary exact-$m$-majority term. Actually, every set admits an $n$-ary exact-$m$-majority operation.

Let $A$ be any nonempty set and fix some $a \in A$. Define

$$u(a_1, \ldots, a_n) = \begin{cases} b & \text{if } |\{ i \leq n \mid a_i = b \}| = m \text{ and } |\{ a_i \mid i \leq n \}| = 2, \\ b & \text{if } a_1 = a_2 = \cdots = a_n = b, \\ a & \text{otherwise.} \end{cases}$$

Since $m \neq \frac{n}{2}$, the first clause provides a good definition.

(c) There is no nontrivial group $G$ with a 6-ary exact-2-majority term. To prove this, let us use multiplicative notation. A group term is a product of variables raised to some power; if some term is evaluated for just one element $g$ and for the identity, the outcome of the term is $g$ raised to the sum of the
powers of the occurrences of the corresponding variables. If $t$ is a 6-ary exact-2-majority term, then $t(g, g, e, e, e, e) = g$ and $t(e, e, g, g, e, e) = g$, for every element $g \in G$. By the preceding comment, if $h$ is the sum of the exponents of all the occurrences of the first two variables, then $g^h = g$ in $G$. Similarly $g^k = g$, where $k$ is the sum of the exponents of all the occurrences of the third and fourth variables. Hence $t(g, g, g, g, e, e)$ evaluates to $g^{h+k} = g^h g^k = g^2$ in $G$. But also $t(g, g, g, g, e, e) = e$, by the majority assumption, thus $G$ has exponent 2, since the above argument applies to every $g \in G$. Since every group of exponent 2 is abelian, every term of $G$ can be represented as a product of variables. Since $G$ is nontrivial of exponent 2 and $t(g, g, g, g, e, e) = g$, for every $g \in G$, then either $t$ does not depend on the first variable, or $t$ does not depend on the second variable. Similarly, since $t(e, e, g, g, e, e) = g$, then either $t$ does not depend on the third variable, or $t$ does not depend on the fourth variable. Say, $t$ does not depend on the first and on the third variables. Since $G$ is non-trivial, then there is $g \in G$ with $g \neq e$, but then we have $t(g, e, g, e, e, e) = e \neq g$, a contradiction.

Notice that we have used only 4 instances of the majority rule (among 15 total instances).

(d) By Proposition 3(3), every variety with a 6-ary exact-2-majority term has a 3-ary exact-1-majority term, i.e., a minority term. In a group of exponent 2 the term $xyz$ is a minority term. On the other hand, by the previous item, no nontrivial group has a 6-ary exact-2-majority term. Thus there is a variety with a 3-ary exact-1-majority term but without a 6-ary exact-2-majority term.

**Theorem 5.** Suppose that $n \geq 3$ and $0 < m < n$. Then every variety with an $n$-ary exact-$m$-majority term is congruence modular.

**Proof.** If $m < \frac{n}{2}$, then $V$ is congruence permutative, by Proposition 3(1), hence $V$ is congruence modular. If $m = \frac{n}{2}$, then $V$ is a trivial variety, by Remark 2(d).

It remains to deal with the case $m > \frac{n}{2}$, so let $u$ be an $n$-ary exact-$m$-majority term in this case. Let $k = n - m$ and $h$ be the remainder of the division of $n$ by $k$ (the proof below works also in case $h = 0$; anyway, the case $h = 0$ is already covered by Proposition 3(4)). Let $x^k$ be an abbreviation for the expression “$x, x, \ldots, x$”, with $k$ occurrences of $x$. Consider the terms

\[
d_1(x, y, z) = u(x^h, x^k, x^l, \ldots, x^k, x^k, x^k, y^k, z^k)
\]
\[
d_2(x, y, z) = u(x^h, x^k, x^k, x^l, \ldots, x^k, x^k, y^k, z^k, z^k)
\]
\[
d_3(x, y, z) = u(x^h, x^k, x^k, x^k, \ldots, x^k, y^k, z^k, z^k, z^k)
\]
\[
\ldots
\]
\[
d_{\ell-2}(x, y, z) = u(x^h, x^k, y^k, z^k, \ldots, z^k, z^k, z^k, z^k, z^k)
\]
\[
d_{\ell-1}(x, y, z) = u(x^h, y^k, z^k, \ldots, z^k, z^k, z^k, z^k, z^k)
\]
\[
q(x, y, z) = u(x^h, y^k-z^{k-h}, z^k, z^k, \ldots, z^k, z^k, z^k, z^k, z^k)
\]
were $\ell$ is the integer quotient of the division of $n$ by $k$. Notice that $\ell > 1$, since $m > \frac{n}{2}$. The above terms satisfy $d_i(x, z, z) = d_{i+1}(x, x, z)$, for $1 \leq i < \ell - 1$. If $\ell = 2$, the terms to be considered are

\[
d_1(x, y, z) = u(x^h, y^k, z^k) \\
q(x, y, z) = u(x^h, y^{k-h}, z^h, z^k)
\]

All the above terms satisfy $d_{\ell-1}(x, z, z) = q(x, z, z)$. Due to the exact $m$ ($= n - k$) majority rule, the above terms also satisfy $d_i(x, y, x) = x$, for $1 \leq i \leq \ell - 1$, $x = d_1(x, x, z)$ and $q(x, x, z) = z$. In the terminology from [6] p. 205, the terms $d_1, \ldots, d_{\ell-1}, q$ are directed Gumm terms, and the existence of such a sequence of terms implies congruence modularity, by the easy part of [6] Theorem 1.1, Clause 3).

Remark 6. If $m < n$, an $n$-ary exact-$m$-majority term is a $\Delta$-special cube term, in the terminology of [3] Definition 2.7, for some appropriate $\Delta$. Indeed, if we write down all the equations defining an $n$-ary exact-$m$-majority term, we get a matrix with $k = \binom{n}{m}$ rows, and only the last column is constantly $x$. Hence a variety with an $n$-ary exact-$m$-majority term has a $k$-edge term, by [3] Theorem 2.12. In particular, [3] Theorem 4.2 furnishes another proof of congruence modularity.

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