A LEVEL RAISING RESULT FOR MODULAR GALOIS REPRESENTATIONS MODULO PRIME POWERS.

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Abstract. In this work we provide a level raising theorem for mod $\lambda^n$ modular Galois representations. It allows one to see such a Galois representation that is modular of level $N$, weight 2 and trivial Nebentypus as one that is modular of level $Np$, for a prime $p$ coprime to $N$, when a certain local condition at $p$ is satisfied. It is a generalization of a result of Ribet concerning mod $\ell$ Galois representations.

1. Introduction

Let $N$ and $k$ be positive integers, $S_k(\Gamma_0(N))$ be the space of modular forms of level $N$ and weight $k$, and $T_k(N)$ be the $\mathbb{Z}$-algebra of Hecke operators acting faithfully on this space. Let also $R$ be a complete Noetherian local ring with maximal ideal $m_R$ and residue field of characteristic $\ell > 0$. A (weak) eigenform of level $N$ and $k$ with with coefficients in $R$ is then defined to be a ring homomorphism $\theta : T_k(N) \rightarrow R$ (One can find a discussion on the various notions of modularity modulo prime powers as well as a comparison between them in [CKW11]). We will denote by $\bar{\theta}$ its composition with $R \rightarrow R/m_R$, i.e. the residual reduction of $\theta$. Then one has the following theorem of Carayol (Theorem 3 in [Car94]):

**Theorem 1.1** (Carayol). Let $k \geq 2$ and $N > 4$ or assume that 6 is invertible in $R$ (i.e. that $\ell \geq 5$). If the representation attached to $\bar{\theta}$ is absolutely irreducible, then one can attach a Galois representation $\rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(R)$ to $\theta$ in the following sense: For every prime $q \nmid N\ell$, $\rho$ is unramified at $q$ and

$$\text{tr}(\rho(\text{Frob}_q)) = \theta(T_q).$$

A representation that arises in the way described by the previous theorem is called modular. If one wants to explicitly mention a specific eigenform $\theta$ due to which the representation $\rho$ is modular one can say that $\rho$ is attached to or associated with $\theta$.

One can then ask if the converse is true: Given a Galois representation $\rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(R)$, when is it modular? Furthermore can one have a hold on what the level and weight of this eigenform will be?

Let $p$ be a rational prime. Then one has a natural inclusion map

$$S_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)) \rightarrow S_k(\Gamma_0(Np))$$

whose image is called the $p$-old subspace. This subspace is stable under the action of $T_k(Np)$ and so is its orthogonal complement through the so-called Peterson product. We call this complementary subspace the $p$-new subspace and we denote by $T_k^{p-new}(Np)$ the quotient of $T_k(Np)$ that acts faithfully on it. We will call this quotient the $p$-new quotient of $T_k(Np)$. There is also the $p$-old
quotient that is defined in the obvious way. Fix another rational prime \( \ell \). Assume \( \mathcal{O} \) is the ring of integers of a number field and \( \lambda \) a prime above \( \ell \). Here we prove the following level raising result:

**Theorem 1.2.** Let \( n \geq 2 \) be an integer and \( \rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}/\mathcal{O}^n) \) be a continuous Galois representation that is modular, associated with a Hecke map \( \theta : \mathbb{T}_2(N) \rightarrow \mathcal{O}/\mathcal{O}^n \), and residually absolutely irreducible. Let also \( p \) be a prime such that \((\ell N, p) = 1\) and assume that \( \text{tr}(\rho(\text{Frob}_p)) \equiv \pm(p + 1) \mod \mathcal{O}^n \). Then \( \rho \) is also associated with a Hecke map \( \theta' : \mathbb{T}_2(Np) \rightarrow \mathcal{O}/\mathcal{O}^n \) which is new at \( p \), i.e. \( \theta' \) factors through the \( \mathbb{T}_2^{n\text{-new}}(Np) \).

**Remark:** For \( n = 1 \) this is Theorem 1 of [Rib90b].

**Remark:** The theorem does not exclude the case \( \ell | N \).

**Remark:** As with the case \( n = 1 \), one can also prove the theorem in the case \( p = \ell \) by assuming the condition \( \theta(T_p) \equiv \pm(p + 1) \mod \mathcal{O}^n \) instead of the one involving the trace of the representation.

**Remark:** Notice that even if the Hecke map that makes \( \rho \) modular in the first place lifts to characteristic 0, i.e. comes from a classical eigenform, there is no guarantee that the Hecke map of level new at \( p \) that one obtains in the end lifts too.

**Corollary 1.3.** Let \( p \) be as in Theorem 1.2. Then there exist infinitely many primes \( p \) (coprime to \( N \)) such that \( \rho \) is modular of level \( Np \), new at \( p \).

**Proof.** Immediate consequence of Lemma 7.1 in [Rib90a]. \( \square \)

In what follows we set \( \mathbb{T}_N := \mathbb{T}_2(N) \) and \( \mathbb{T}_{NP} := \mathbb{T}_2(Np) \). We will also denote the \( p \)-th Hecke operator in \( \mathbb{T}_{NP} \) by \( U_p \) in order to emphasize the different way of acting compared to the one in \( \mathbb{T}_N \).

## 2. Jacobians of Modular Curves

In this section we gather the necessary results from [Rib90b] that we will need in the proof of the main result.

Let \( N \) be a positive integer. Let \( X_0(N) \subset \text{Pic}^0(X_0(N)) \) be the modular curve of level \( N \) and \( J_0(N) := \text{Pic}^0(X_0(N)) \) its Jacobian. There is a well defined action of the Hecke operators \( T_n \) on \( X_0(N) \) and hence, by functoriality, on \( J_0(N) \) too. The dual of \( J_0(N) \) carries an action of the Hecke algebra as well and can be identified with \( S_2(\Gamma_0(N)) \). This implies that one has a faithful action of \( \mathbb{T}_N \) on \( J_0(N) \).

Let now \( p \) be a prime not dividing \( N \). In the same way one has an action of Hecke operators on \( X_0(Np) \) and its Jacobian \( J_0(Np) \) and the latter admits a faithful action of \( \mathbb{T}_{NP} \). The interpretation of \( X_0(N) \) and \( X_0(Np) \) allows us to define the two natural degeneracy maps \( \delta_1, \delta_p : X_0(Np) \rightarrow X_0(N) \) and their pullbacks \( \delta_1^*, \delta_p^*: J_0(N) \rightarrow J_0(Np) \).

There is a map

\[
\alpha : J_0(N) \times J_0(N) \rightarrow J_0(Np), \quad (x, y) \mapsto \delta_1^*(x) + \delta_p^*(y).
\]

whose image is by definition the \( p \)-old subvariety of \( J_0(Np) \). We will denote this by \( A \). This map \( \alpha \) is almost Hecke-equivariant:

\[
\alpha \circ T_q = T_q \circ \alpha \text{ for every prime } q \neq p,
\]

\[
\alpha \circ \begin{pmatrix} T_p & p \\ -1 & 0 \end{pmatrix} = U_p \circ \alpha
\]

Of course, the first one makes sense only if one interprets the operator \( T_q \) as acting diagonally on \( J_0(N) \times J_0(N) \). Consider also the kernel \( \text{Sh} \) of the map \( J_0(N) \rightarrow J_1(N) \) induced by \( X_1(N) \rightarrow \)}
X_0(N). If we inject it into J_0(N) \times J_0(N) via x \mapsto (x,-x) then its image, which we will denote by \Sigma, is the kernel of the previous map \alpha (see Proposition 1 in [Rib90b]). Furthermore Sh, and therefore \Sigma too, are annihilated by the operators \eta_r = T_r - (r + 1) \in \mathbb{T}_N for all primes r \nmid N p. (see Proposition 2 in [Rib90b]). We make a small parenthesis here to introduce a useful notion.

**Definition 2.1.** A maximal ideal \textbf{m} of the Hecke algebra \mathbb{T}_N is called Eisenstein if it contains the operator \textbf{T}_r - (r + 1) for almost all primes r.

We need a few more definitions and facts (see Corollary in [Rib90b] and the discussion after that):

Let \Delta be the kernel of \left(\frac{1 + p}{T_p}, \frac{T_p}{1 + p}\right) \in M^{2 \times 2}(\mathbb{T}_N) acting on \text{J}_0(N) \times \text{J}_0(N). \Delta is finite and comes equipped with a perfect \mathbb{G}_m-valued skew-symmetric pairing \omega. Furthermore \Sigma is a subgroup of \Delta, self orthogonal, and \Sigma \subset \Sigma^\perp \subset \Delta. One can also see \Delta/\Sigma and therefore its subgroup \Sigma^\perp/\Sigma, as a subgroup of \text{A}.

Let B be the p-new subvariety of \text{J}_0(N p). It is a complement of A, i.e. \text{A} + \text{B} = \text{J}_0(N p) and \text{A} \cap \text{B} is finite. The Hecke algebra acts on it faithfully through its p-new quotient and it turns out (see Theorem 2 in [Rib90b]) that

\begin{equation}
\text{A} \cap \text{B} \cong \Sigma^\perp/\Sigma.
\end{equation}

as groups, with the isomorphism given by the map \alpha.

3. Proof of Theorem 1.2

Let \theta : \mathbb{T}_N \longrightarrow \mathcal{O}/\lambda^n be the eigenform associated with \rho, \hat{\theta} : \mathbb{T}_N \longrightarrow \mathcal{O}/\lambda its reduction mod \lambda (which is associated with \overline{\rho}, the mod \lambda reduction of \rho) and let I and \textbf{m} be the kernels of \theta and \hat{\theta} respectively. It will be enough to find a weak modular form \theta' : \mathbb{T}_2(N p) \longrightarrow \mathcal{O}/\lambda^n that agrees with \theta on \text{T}_q for all primes q \neq p (i.e. they define the same Galois representation) and factors through \mathbb{T}_2^{\text{p-new}}(N p)(i.e. new at p). In what follows we will be writing Ann(M) instead of Ann_{\Sigma/\mathbb{T}_N}(M) to denote the annihilator of a \mathbb{T}_N-module M.

Let us begin with the following auxiliary result:

**Lemma 3.1.** \textbf{m} is the only maximal ideal of \mathbb{T}_N containing I.

**Proof.** We will equivalently show that \mathbb{T}_N/I is local. The proof actually works for any Artinian ring injecting into a local ring.

By the definition of I, \mathbb{T}_N/I injects in \mathcal{O}/\lambda^n. Since \mathbb{T}_N/I is Artinian it decomposes into the product of its localizations at its prime (actually maximal) ideals, which are finitely many, say s = 1. The set containing the identity \varepsilon_1 of each component then forms a complete set (i.e. \sum_{i=1}^s e_i = 1) of pairwise orthogonal (i.e. e_i e_j = 0 for \ 1 \neq j \leq s) non-trivial (i.e. \varepsilon_i \neq 0, 1) idempotents for \mathbb{T}_N/I. The set \{\bar{e}_1, \ldots, \bar{e}_s\} of their image through the injection of \mathbb{T}_N/I into \mathcal{O}/\lambda^n is clearly a complete set of pairwise orthogonal non-trivial idempotents too. This implies that \mathcal{O}/\lambda^n is isomorphic to \prod_{i=1}^s \bar{e}_i(\mathcal{O}/\lambda^n). But this cannot happen unless s = 1 since \mathcal{O}/\lambda^n is local. Since s = 1 we get that \mathbb{T}_N/I is local. \hfill \Box
We define:

\[ V_\ell = J_0(N)[I], \]
\[ V_m = J_0(N)[m]. \]

We have that \( m \subseteq \text{Ann}(V_m) \) by the definition of \( V_m \). But \( m \) is maximal so \( m = \text{Ann}(V_m) \). We also have that \( \text{Ann}(V_\ell) \subseteq \text{Ann}(V_m) = m \), so \( m \) is in the support of \( \text{Ann}(V_\ell) \). Since the representation \( \rho \), which is the reduction of \( \bar{\rho} \), is the singleton \( \{ m \} \).

Now consider the composite map

\[ \theta(T_p) \equiv -(p + 1) \mod \lambda^n. \]

(5)

Now consider the composite map

\[ J_0(N) \to J_0(N) \times J_0(N) \xrightarrow{\Delta} A \subseteq J_0(Np), \]

where the first map is the diagonal embedding (in the case of \( \text{tr}(\rho(\text{Frob}_p)) \equiv p + 1 \mod \lambda^n \) we pick the anti-diagonal map) and the second is the map \( \alpha \) defined in the previous section. By abuse of notation, we will also denote by \( V_\ell \) the image of \( V_T \) in \( J_0(N) \times J_0(N) \) via the diagonal embedding.

We then claim that its intersection with \( \Sigma \) is zero: Assume that it is not, and denote it by \( V_T' \). It is easy to see that \( V_T' \) is preserved by the action of \( \mathbb{T}_N \) so it can be seen as a \( \mathbb{T}_N \)-module: For an \((x, y) \in V_T' \) we have (using relation (2))

\[ \alpha(T_q(x, x)) = T_q(\alpha(x, y)) = T_q(0) = 0 \quad \text{for primes } q \neq p \]

and (using relation (5))

\[ \alpha(T_p(x, x)) = \alpha(T_p(x, x)) = \alpha(-(p + 1)x, -(p + 1)x) = -(p + 1)\alpha(x, x) = 0. \]

In the case where \( \theta(T_p) \equiv p + 1 \mod \lambda^n \), the elements of \( V_T' \) are of the form \((x, -x)\) but the reasoning is the same. Since \( \Sigma \) is annihilated by almost all operators \( T_r - (r + 1) \), \( V_T' \) is annihilated by almost all of them too. This implies that every maximal ideal containing \( \text{Ann}(V_T') \) is Eisenstein. But \( \text{Ann}(V_T) \subseteq \text{Ann}(V_T') \) so \( V_T \) has an Eisenstein ideal in its support. On the other hand the only maximal ideal in the support of \( V_T \) is \( m \) which is non-Eisenstein, so we get a contradiction. One can therefore see \( V_T \) as a subgroup of \( A \) and we will abuse notation to denote its image through the above map by \( V_T' \).

We have the following Lemma:

**Lemma 3.2.** \( V_\ell \) is stable under the action of \( \mathbb{T}_{Np} \) and the action is given by a ring homomorphism \( \theta^\ell : \mathbb{T}_{Np} \to \mathcal{O}/\lambda^n. \)

**Proof.** This is nothing but a straightforward calculation:

First note that the action of \( \mathbb{T}_N \) on \( V_\ell \) factors through \( \mathbb{T}_N/I \) so we obtain a map \( \theta(\mathbb{T}_N/I) \to \text{End}(V_\ell) \). Let \( y \) be a non-trivial element of the image of \( V_\ell \) in \( A \). Then there exists \( x \in V_\ell \) such that \( \alpha(x, x) = y \). Let now \( q \) be a prime other than \( p \). In view of relation (2) and we have that:

\[ T_q(y) = T_q(\alpha(x, x)) = \alpha(T_q(x), T_q(x)) = \alpha(\theta(T_q)x, \theta(T_q)y) = \theta(T_q)x = \theta(T_q)y. \]

For \( q = p \) we have (using relation (3) and (5)):

\[ U_p(y) = U_p(\alpha(x, x)) = \alpha(\begin{bmatrix} T_p & 0 \\ -I & 0 \end{bmatrix} (x, x)^T) = \alpha(T_p(x) + px, -x) = \alpha(\theta(T_p)x + px, -x) = \alpha(-x, -x) = -\alpha(x, x) = -y. \]
It turns out that $y$ is an eigenvector and that the action of $\mathbb{T}_{Np}$ on it defines a ring homomorphism $\theta' : \mathbb{T}_{Np} \to \mathcal{O}/\lambda^n$ via:

$$\theta'(T_q) = \theta(T_q) \quad \text{for all primes } q \neq p \text{ and } \theta'(U_p) = -1$$

To treat the other case one has to keep in mind for the formulas above that $y = \alpha(x, -x)$ and proceed in the same way to get the same result except that $U_p(y) = y$ this time and therefore $\theta'(U_p) = 1$. \hfill $\square$

**Remark:** Since the $\theta$ and $\theta'$ actually agree on almost all primes, it is clear that they are associated with the same Galois representation, so $\theta'$ is the candidate map we were looking for.

To finish of the proof of the main result it remains to show that the map factors through the $p$-new quotient of the Hecke algebra. To this end, it is enough to show that $V_I$, when viewed as a subgroup of $J_0(Np)$ is a subgroup of $(A \cap B)$. We again proceed according to Ribet. It is easy to see that $V_I$, when considered as a subgroup of $J_0(N) \times J_0(N)$, is a subgroup of $\Delta$. Let $\tilde{V}_I$ be the image of $V_I$ in $\Delta/\Sigma^\perp$. Then, in view of \[1\], we just need to show that $\tilde{V}_I$ is trivial.

First notice that $\tilde{V}_I$ is preserved by the action of $\mathbb{T}_N$. For this it is enough to check that if $z \in V_I \cap \Sigma^\perp$ then $T_q(z) \in \Sigma^\perp$ and $T_p(z) \in \Sigma^\perp$ (clearly they will also be in $V_I$). Let $x \in \Sigma$. We then have the following: $\omega(x, T_q(z)) = \omega(T_q(x, z))$. Now the subalgebras of generated by $T_q$ and $T_q^\vee$ are isomorphic (see p444 in \[Rib90a\]). Since the subalgebra generated by $T_q$ preserves $\Sigma$ as shown in \[6\], we get that $T_q^\vee(x) \in \Sigma$ and therefore that $\omega(T_q^\vee(x, z)) = 0$. Finally, using \[5\] again, $\omega(x, T_q(z)) = \omega(x, -(p+1)z) = -(p+1)\omega(x, z) = 0$.

Now according to Ribet in the proof of Lemma 2 in \[Rib90a\], $\Delta/\Sigma^\perp$ is dual to $\Sigma$ which is annihilated by almost all operators $T_i - (r + 1)$, so $\Delta/\Sigma^\perp$, and therefore $\tilde{V}_I$, is annihilated by them too. This implies that any maximal ideal containing $\text{Ann}(\tilde{V}_I)$ is Eisenstein. Recall that $V_I$ is not Eisenstein. Now assume for contradiction that $\tilde{V}_I$ is non-zero. Since $\text{Ann}(V_I)$ contains $\text{Ann}(V_I)$, we get that the support of $V_I$ also contains Eisenstein ideals. This is the desired contradiction that completes the proof of Theorem 1.2.

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