THE CHERN-SIMONS ACTION IN NON-COMMUTATIVE GEOMETRY

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Abstract. A general definition of Chern-Simons actions in non-commutative geometry is proposed and illustrated in several examples. These examples are based on “space-times” which are products of even-dimensional, Riemannian spin manifolds by a discrete (two-point) set. If the *algebras of operators describing the non-commutative spaces are generated by functions over such “space-times” with values in certain Clifford algebras the Chern-Simons actions turn out to be the actions of topological gravity on the even-dimensional spin manifolds. By constraining the space of field configurations in these examples in an appropriate manner one is able to extract dynamical actions from Chern-Simons actions.

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1. **Introduction**

During the past several years, topological field theories have been the subject of a lot of interesting work. For example, deep connections between three-dimensional, topological Chern-Simons theories [1], or, equivalently, two-dimensional, chiral conformal field theories [2], on one hand, and a large family of invariants of links, including the famous Jones polynomial, and of three-manifolds [1], on the other hand, have been discovered. Other topological field theories have been invented to analyze e.g. the moduli space of flat connections on vector bundles over Riemann surfaces or to elucidate the Donaldson invariants of four-manifolds. These topological field theories are formulated as theories over some classical (topological or differentiable) manifolds.

Connes has proposed notions of non-commutative spaces generalizing, for example, the notion of a classical differentiable manifold [4]. His theory is known under the name of “non-commutative geometry”. Dubois-Violette [3] and Connes have proposed to study field theories over non-commutative spaces. In joint work with J. Lott [4], Connes has found a construction of the classical action of the standard model, using tools of non-commutative geometry, which yields a geometrical interpretation of the scalar Higgs field responsible for the “spontaneous breaking of the electroweak gauge symmetry”. In fact, the Higgs field appears as a component of a generalized gauge field (connection 1-form) associated with the gauge group, $SU(2)_w \times U(1)_{em}$, of electroweak interactions. This is accomplished by formulating gauge theory on a generalized space consisting of two copies of standard Euclidean space-time the “distance” between which is determined by the weak scale. Although the space-time model underlying the Connes-Lott construction is a commutative space, it is not a classical manifold, and analysis on space-times of the Connes-Lott type requires some of the tools of non-commutative geometry.

The results of Connes and Lott have been reformulated and refined in [6,7] and extended to grand-unified theories in [8]. In [9], G. Felder and the authors have proposed some form of non-commutative Riemannian geometry and applied it to derive an analogue of the Einstein-Hilbert action in non-commutative geometry.

Our aim in this article is to attempt to do some steps towards a synthesis between the different developments just described. Some of our results have been described in our review paper [10]. We start by presenting a general definition of the Chern-Simons action in non-commutative geometry, (Section 2). Our definition is motivated by some results of Quillen [11] and is based on joint work with O. Grandjean [12]. In Section 3, we discuss a first family of examples. In these examples, the non-commutative space is described in terms of a $^*$-algebra of matrix-valued functions over a Connes-Lott type “space-time”, i.e., over a commutative space consisting of two copies of an even-dimensional, differentiable spin manifold. The Chern-Simons actions on such non-commutative spaces turn out to be actions of topological gauge- and gravity theories, as studied in [13,14]. In Section 3, the dimension of the continuous, differentiable spin manifold is two, i.e., we consider
products of Riemann surfaces by discrete sets, and our Chern-Simons action is based on
the Chern-Simons 3-form.

In Section 4, we consider two- and four-dimensional topological theories derived from
a Chern-Simons action based on the Chern-Simons 5-form.

In Section 5, we describe connections of the theories found in Section 4 with four-
dimensional gravity and supergravity theories.

In Section 6, we suggest applications of our ideas to string field theory [15], and we
draw some conclusions.

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matters, [12].

2. Elements of non-commutative geometry.

This section is based on Connes’ theory of non-commutative geometry, as described
in [4], and on results in [9,10,11,12].

We start by recalling the definition of a special case of Connes’ general definition of
non-commutative spaces. A real, compact non-commutative space is defined by the data
\((\mathcal{A}, \pi, H, D)\), where \(\mathcal{A}\) is a *algebra of bounded operators containing an identity element, \(\pi\)
is a *representation of \(\mathcal{A}\) on \(H\), where \(H\) is a separable Hilbert space, and \(D\) is a selfadjoint
operator on \(H\), with the following properties:

(i) \([D, \pi(a)]\) is a bounded operator on \(H\), for all \(a \in \mathcal{A}\). [This condition determines the
analogue of a differentiable structure on the non-commutative space described by \(\mathcal{A}\).]

In the following, we shall usually identify the algebra \(\mathcal{A}\) with the *subalgebra \(\pi(\mathcal{A})\)
of the algebra \(B(H)\) of all bounded operators on \(H\); (we shall thus assume that the
kernel of the representation \(\pi\) in \(\mathcal{A}\) is trivial). We shall often write “\(a\)” for both, the
element \(a\) of \(\mathcal{A}\) and the operator \(\pi(a)\) on \(H\).

(ii) \((D^2 + \mathbb{I})^{-1}\) is a compact operator on \(H\). More precisely, \(\exp(-\varepsilon D^2)\) is trace-class, for
any \(\varepsilon > 0\).

Given a real, compact non-commutative space \((\mathcal{A}, \pi, H, D)\), one defines a differential
algebra, \(\Omega_D(\mathcal{A})\), of forms as follows: 0-forms (“scalars”) form a *algebra with identity,
\(\Omega^0_D(\mathcal{A})\), given by \(\pi(\mathcal{A})\); \(n\)-forms form a linear space, \(\Omega^n_D(\mathcal{A})\), spanned by equivalence classes
of operators on \(H\),

\[
\Omega^n_D(\mathcal{A}) := \Omega^n(\mathcal{A})/Aux^n, \quad (2.1)
\]
where the linear space $\Omega^n(A)$ is spanned by the operators

$$\{ \sum_i a_i^0[D, a_i^1] \cdots [D, a_i^n] : a_j^i \in A \equiv \pi(A), \forall i, j \}, \quad (2.2)$$

and $Aux^n$, the space of “auxiliary fields” [5], is spanned by operators of the form

$$Aux^n := \left\{ \sum_i [D, a_i^0][D, a_i^1] \cdots [D, a_i^n] : \sum_i a_i^0[D, a_i^1] \cdots [D, a_i^n] = 0, a_j^i \in A \right\}.$$ \quad (2.3)

Using the Leibniz rule

$$[D, ab] = [D, a]b + a[D, b], \quad a, b \in A,$$ \quad (2.4)

and

$$[D, a]^* = -[D, a^*], \quad a \in A,$$ \quad (2.5)

we see that the spaces $\Omega^n(A)$ are $A$-bimodules closed under the involution $*$ and that

$$Aux := \oplus Aux^n$$ \quad (2.6)

is a two-sided ideal in

$$\Omega(A) := \oplus \Omega^n(A),$$ \quad (2.7)

closed under the operation $*$. Thus, for each $n$, $\Omega^n_D(A)$ is an $A$-bimodule closed under $*$. It follows that

$$\Omega_D(A) := \oplus \Omega^n_D(A)$$ \quad (2.8)

is a *algebra of equivalence classes of bounded operators on $H$, with multiplication defined as the multiplication of operators on $H$. Since $A = \Omega^0(A) = \Omega^0_D(A)$ is a *subalgebra of $\Omega_D(A)$ containing an identity element, $\Omega_D(A)$ is a unital *algebra of equivalence classes (mod $Aux$) of bounded operators on $H$ which is an $A$-bimodule.

The degree of a form $\alpha \in \Omega^n_D(A)$ is defined by

$$deg(\alpha) = n, \quad n = 0, 1, 2, \cdots.$$ \quad (2.9)

Clearly, $deg(\alpha^*) = deg(\alpha)$, by (2.4), (2.5). With this definition of $deg$, $\Omega_D(A)$ is $\mathbb{Z}$-graded. If $\alpha$ is given by

$$\alpha = \sum_i a_i^0[D, a_i^1] \cdots [D, a_i^n](mod Aux^n) \in \Omega^n_D(A)$$

we set

$$d\alpha := \sum_i [D, a_i^0][D, a_i^1] \cdots [D, a_i^n] \in \Omega^{n+1}_D(A).$$ \quad (2.10)
The map
\[ d : \Omega^n_D(A) \to \Omega^{n+1}_D(A), \quad \alpha \mapsto d\alpha \] (2.11)
is a \( \mathbb{C} \)-linear map from \( \Omega_D(A) \) to itself which increases the degree of a form by one and satisfies
\[ d(\alpha \cdot \beta) = (d\alpha) \cdot \beta + (-1)^{\deg \alpha} \alpha \cdot (d\beta), \] (2.12)
for any homogeneous element \( \alpha \) of \( \Omega_D(A) \) (Leibniz rule) and
\[ d^2 = 0. \] (2.13)
Hence \( \Omega_D(A) \) is a differential algebra which is a \( \mathbb{Z} \)-graded complex.

These notions are described in detail (and in a more general setting) in [4].

In non-commutative geometry, vector bundles over a non-commutative space described by a \( ^* \) algebra \( \mathcal{A} \) are defined as finitely generated, projective left \( \mathcal{A} \)-modules. Let \( E \) denote (the “space of sections” of) a vector bundle over \( \mathcal{A} \). A connection \( \nabla \) on \( E \) is a \( \mathbb{C} \)-linear map
\[ \nabla : E \to \Omega^1_D(A) \otimes \mathcal{A} E \] (2.14)
with the property that (with \( da = [D,a] \), for all \( a \in \mathcal{A} \))
\[ \nabla(as) = da \otimes \mathcal{A} s + a \nabla s, \] (2.15)
for arbitrary \( a \in \mathcal{A}, s \in E \). The definition of \( \nabla \) can be extended to the space
\[ \Omega_D(E) = \Omega_D(A) \otimes \mathcal{A} E \] (2.16)
in a canonical way, and, for \( s \in \Omega_D(E) \) and a homogeneous form \( \alpha \in \Omega_D(A) \),
\[ \nabla(\alpha s) = (d\alpha)s + (-1)^{\deg \alpha} \alpha \nabla s. \] (2.17)
Thanks to (2.14) - (2.17), it makes sense to define the curvature, \( R(\nabla) \), of the connection \( \nabla \) as the \( \mathbb{C} \)-linear map
\[ R(\nabla) := -\nabla^2 \] (2.18)
from \( \Omega_D(E) \) to \( \Omega_D(E) \). Actually, it is easy to check that \( R(\nabla) \) is \( \mathcal{A} \)-linear, i.e. \( R(\nabla) \) is a tensor.

A trivial vector bundle, \( E^{(N)} \), corresponds to a finitely generated, free left \( \mathcal{A} \)-module, i.e., one that has a basis \( \{s_1, \ldots, s_N\} \), for some finite \( N \) called its dimension. Then
\[ E^{(N)} \simeq \mathcal{A} \oplus \cdots \oplus \mathcal{A} \equiv \mathcal{A}^n, \]
(with \( N \) summands). The affine space of connections on \( E^{(N)} \) can be characterized as follows: Given a basis \( \{s_1, \ldots, s_N\} \) of \( E^{(N)} \), there are \( N^2 \) 1-forms \( \rho_\beta^\alpha \in \Omega^1_D(\mathcal{A}) \), the components of the connection \( \nabla \), such that
\[ \nabla s_\alpha = -\rho_\beta^\alpha \otimes \mathcal{A} s_\beta, \] (2.19)
Then
\[ \nabla(a^\alpha s_\alpha) = da^\alpha \otimes_A s_\alpha - a^\alpha \rho^\beta_\alpha \otimes_A s_\beta, \tag{2.20} \]
by (2.15). Furthermore, by (2.18) and (2.20),
\[ R(\nabla)(a^\alpha s_\alpha) = - \nabla(da^\alpha \otimes_A s_\alpha - a^\alpha \rho^\beta_\alpha \otimes_A s_\beta) \]
\[ = - (d^2 a^\alpha \otimes_A s_\alpha + da^\alpha \rho^\beta_\alpha \otimes_A s_\beta - da^\alpha \rho^\beta_\alpha \otimes_A s_\beta - a^\alpha d\rho^\beta_\alpha \otimes_A s_\beta - a^\alpha \rho^\gamma_\alpha \rho^\beta_\gamma \otimes_A s_\beta) \]
\[ = a^\alpha (d\rho^\beta_\alpha + \rho^\gamma_\alpha \rho^\beta_\gamma) \otimes_A s_\beta. \tag{2.21} \]
Thus, the curvature tensor \( R(\nabla) \) is completely determined by the \( N \times N \) matrix \( \theta \equiv (\theta^\beta_\alpha) \) of 2-forms given by
\[ \theta^\beta_\alpha = d\rho^\beta_\alpha + \rho^\gamma_\alpha \rho^\beta_\gamma. \tag{2.22} \]
The curvature matrix \( \theta \) satisfies the **Bianchi identity**
\[ d\theta + \rho \theta - \theta \rho \equiv (d\theta^\beta_\alpha + \rho^\gamma_\alpha \theta^\beta_\gamma - \theta^\gamma_\alpha \rho^\beta_\gamma) = 0. \tag{2.23} \]
If one introduces a new basis
\[ \tilde{s}_\alpha = M^\beta_\alpha s_\beta, \ M^\beta_\alpha \in \mathcal{A}, \ \alpha, \beta = 1, \cdots, N, \tag{2.24} \]
where the matrix \( M \equiv (M^\beta_\alpha) \) is invertible, then the components, \( \tilde{\rho} \), of \( \nabla \) in the new basis \( \{\tilde{s}_1, \cdots, \tilde{s}_N\} \) of \( E^{(N)} \) are given by
\[ \tilde{\rho} = M \rho M^{-1} - dM \cdot M^{-1}, \tag{2.25} \]
and the components of the curvature \( R(\nabla) \) transform according to
\[ \tilde{\theta} = M \theta M^{-1}, \tag{2.26} \]
as one easily checks.

Given a basis \( \{s_1, \cdots, s_N\} \) of \( E^{(N)} \), one may define a **Hermitian structure** \( \langle \cdot, \cdot \rangle \) on \( E^{(N)} \) by setting
\[ \langle s_\alpha, s_\beta \rangle = \delta_{\alpha\beta} \mathbb{I}, \tag{2.27} \]
with
\[ \langle a^\alpha s_\alpha, b^\beta s_\beta \rangle = a^\alpha \langle s_\alpha, s_\beta \rangle (b^\beta)^* = \sum_\alpha a^\alpha (b^\alpha)^* \in \mathcal{A}. \tag{2.28} \]
The definition of $\langle \cdot , \cdot \rangle$ can be extended canonically to $\Omega_D(E^{(N)})$, and there is then an obvious notion of “unitary connection” on $E^{(N)}$: $\nabla$ is unitary iff

$$d \langle s, s' \rangle = \langle \nabla s, s' \rangle - \langle s, \nabla s' \rangle .$$  \hfill (2.29)

This is equivalent to the condition that

$$\rho_\alpha^\beta = (\rho_\beta^\alpha)^* ,$$  \hfill (2.30)

where the $\rho_\alpha^\beta$ are the components of $\nabla$ in the orthonormal basis $\{s_1, \cdots, s_N\}$ of $E^{(N)}$.

In the examples studied in Sections 3 through 5, we shall consider unitary connections on trivial vector bundles, in particular on “line bundles” for which $N = 1$. A (unitary) connection $\nabla$ on a line bundle $E^{(1)} \simeq A$ is completely determined by a (selfadjoint) 1-form $\rho \in \Omega^1_D(A)$.

The data $(A, \pi, H, D)$ defining a non-commutative space with differentiable structure is also called a Fredholm module. Following [4], we shall say that the Fredholm module $(A, \pi, H, D)$ is $(d, \infty)$-summable if

$$tr \left( (D^2 + 1)\| \right)^{-p/2} < \infty, \text{ for all } p > d .$$  \hfill (2.31)

Let $Tr_\omega(\cdot)$ denote the so-called Dixmier trace on $B(H)$ which is a positive, cyclic trace vanishing on trace-class operators; see [4]. We define a notion of integration of forms, $\int$, by setting

$$\int \alpha := Tr_\omega (\alpha | D |^{-d}) ,$$  \hfill (2.32)

for $\alpha \in \Omega(A) = \oplus \Omega^n(A)$; (see (2.2), (2.7)). If $d = \infty$ but $exp (-\varepsilon D^2)$ is trace class, for any $\varepsilon > 0$ (as assumed), we set

$$\int \alpha := \lim_{\varepsilon \downarrow 0} \frac{tr(\alpha exp(-\varepsilon D^2))}{tr(exp(-\varepsilon D^2))} ,$$  \hfill (2.33)

(on forms $\alpha$ which are “ analytic elements” for the automorphism group determined by the dynamics $exp(itD^2), t \in \mathbb{R}$; see [12]) and $\lim_{\varepsilon \downarrow 0}$ denotes a limit defined in terms of a kind of “Cesaro mean” described in [4]. Then

$$\int \alpha \beta = \int \beta \alpha ,$$  \hfill (2.34)

i.e., $\int$ is cyclic; it is also a non-negative linear functional on $\Omega(A)$. It can thus be used to define a positive semi-definite inner product on $\Omega(A)$: For $\alpha$ and $\beta$ in $\Omega(A)$, we set

$$\langle \alpha, \beta \rangle = \int \alpha \beta^* .$$  \hfill (2.35)
Then the closure of $\Omega(A)$ (mod kernel of $(\cdot, \cdot)$) in the norm determined by $(\cdot, \cdot)$ is a Hilbert space, denoted by $L^2(\Omega(A))$. Given an element $\alpha \in \Omega^n(A)$, we can now define a canonical representative, $\alpha^\perp$, in the equivalence class $\alpha \mod \text{Aux}^n$ in the norm determined by $(\cdot, \cdot)$ as the unique (modulo the kernel of $(\cdot, \cdot)$) operator in $\alpha \mod \text{Aux}^n$ which is orthogonal to Aux$^n$ in the scalar product $(\cdot, \cdot)$ given by (2.35); (Aux$^n$ has been defined in eq. (2.3)). Then, for $\alpha$ and $\beta$ in $\Omega^D(A)$, we set

$$(\alpha, \beta) := (\alpha^\perp, \beta^\perp) \equiv \int \alpha^\perp(\beta^\perp)^*, \quad (2.36)$$

and this defines a positive semi-definite inner product on $\Omega^D(A)$. The closure of $\Omega^D(A)$ (mod kernel of $(\cdot, \cdot)$) in the norm determined by $(\cdot, \cdot)$ is the Hilbert space of “square-integrable differential forms”, denoted by $\Lambda_D(A)$.

In order to define the Chern-Simons forms and Chern-Simons actions in non-commutative geometry, it is useful to consider a trivial example of the notions introduced, so far. Let $I$ denote the interval $[0,1] \subset \mathbb{R}$. Let $A_1 = C^\infty(I)$ be the algebra of smooth functions, $f(t)$, on the open interval $(0,1)$ which, together with all their derivatives in $t$, have (finite) limits as $t$ tends to 0 or 1. Let $H_1 = L^2(I) \otimes \mathbb{C}^2$ denote the Hilbert space of square-integrable (with respect to Lebesgue measure, $dt$, on $I$) two-component spinors, and $D_1 = i \frac{\partial}{\partial t} \otimes \sigma_1$ the one-dimensional Dirac operator (with appropriate selfadjoint boundary conditions), where $\sigma_1, \sigma_2$ and $\sigma_3$ are the usual Pauli matrices. A representation $\pi_1$ of $A_1$ on $H_1$ is defined by setting

$$\pi_1(a) = a \otimes \mathbb{I}_2, \quad a \in A_1. \quad (2.37)$$

The geometry of $I$ is then coded into the space $(A_1, \pi_1, H_1, D_1)$. The space of 1-forms is given by

$$\Omega^1_{D_1}(A_1) = \{ \omega \otimes \sigma_1 : \omega = \sum_i a^i \partial_i b^i; a^i, b^i \in A_1 \}. \quad (2.38)$$

The space, $\Omega^2_{D_1}(A_1)$, of 2-forms is easily seen to be trivial, and the cohomology groups vanish. The Fredholm module $(A_1, \pi_1, H_1, D_1)$ is $\mathbb{Z}_2$-graded. The $\mathbb{Z}_2$-grading, $\gamma$, is given by

$$\gamma = \mathbb{I} \otimes \sigma_3, \quad (2.39)$$

and $[\gamma, \pi_1(a)] = 0$, for all $a \in A_1$, while

$$\{ \gamma, D_1 \} \equiv \gamma D_1 + D_1 \gamma = 0. \quad (2.40)$$

Using this trivial example, we may introduce the notion of a “cylinder over a non-commutative space”: Let $(A, \pi, H, D)$ be an arbitrary non-commutative space, and let $(A_1, \pi_1, H_1, D_1)$ be as specified in the above example. Then we define the cylinder over $(A, \pi, H, D)$ to be given by the non-commutative space $(\tilde{A}, \tilde{\pi}, \tilde{H}, \tilde{D})$, where

$$\tilde{H} = H \otimes H_1, \quad \tilde{\pi} = \pi \otimes \pi_1, \quad \tilde{A} = A \otimes A_1, \quad (2.41)$$
and
\[ \tilde{D} = \mathbb{1} \otimes D_1 + D \otimes \gamma, \]
with \( \gamma \) as in (2.39). The space \((\tilde{A}, \tilde{\pi}, \tilde{H}, \tilde{D})\) is \( \mathbb{Z}_2 \)-graded: We define
\[ \Gamma = \mathbb{1} \otimes (\mathbb{1} \otimes \sigma_2), \]
(2.43)
\[ \tilde{D}_1 := \mathbb{1} \otimes D_1, \tilde{D}_2 = D \otimes \gamma. \]
Then
\[ \{ \Gamma, \tilde{D}_1 \} = \{ \Gamma, \tilde{D}_2 \} = \{ \Gamma, \tilde{D} \} = \{ \tilde{D}_1, \tilde{D}_2 \} = 0, \]
(2.44)
and
\[ [\Gamma, \tilde{\pi}(\tilde{a})] = 0, \text{ for all } \tilde{a} \in \tilde{A}. \]
(2.45)
It is easy to show (see [12]) that arbitrary sums of operators of the form
\[ \tilde{a}_0[\tilde{D}_{\varepsilon_1}, \tilde{a}_1] \cdots [\tilde{D}_{\varepsilon_n}, \tilde{a}_n], \quad \varepsilon_1, \ldots, \varepsilon_n = 1, 2, \]
(2.46)
belong to \( \Omega^k(\tilde{A}) \). Furthermore, if two or more of the \( \varepsilon_i \)'s take the value 1 then the operator defined in (2.46) belongs to \( \text{Aux}^n \).

We define integration, \( \int_{\tilde{\mathcal{A}}} (\cdot) \), on \((\tilde{A}, \tilde{\pi}, \tilde{H}, \tilde{D})\) by setting, for any \( \alpha \in \Omega(\tilde{A}) \),
\[ \int_{\tilde{\mathcal{A}}} \alpha := \int_0^1 dt \int \text{Tr}_{C2}(\alpha(t)), \]
(2.47)
where \( \alpha(t) \) is a 2×2 matrix of elements of \( \Omega(\mathcal{A}) \). The integral \( \int_{\tilde{\mathcal{A}}} (\cdot) \) is positive semi-definite and cyclic on the algebra \( \Omega(\tilde{A}) \). We are now prepared to define the Chern-Simons forms and Chern-Simons actions in non-commutative geometry, (for connections on trivial vector bundles). Let \((\mathcal{A}, \pi, H, D)\) be a real, compact non-commutative space with a differentiable structure determined by \( D \). Let \( E = E^{(N)} \simeq \mathcal{A}^N \) be a trivial vector bundle over \( \mathcal{A} \), and let \( \nabla \) be a connection on \( E \). By (2.19), \( \nabla \) is completely determined by an \( N \times N \) matrix \( \rho = (\rho_{\alpha}^\beta) \) of 1-forms. By (2.21), the curvature of \( \nabla \) is given by the \( N \times N \) matrix of 2-forms
\[ \theta = d\rho + \rho^2, \]
where \( d \) is the differential on \( \Omega_D(\mathcal{A}) \) defined in (2.10). Following Quillen [11], we define the Chern-Simons \((2n-1)\)-form associated with \( \nabla \) as follows: Let \( \nabla_0 \) denote the flat connection on \( E \) corresponding to an \( N \times N \) matrix \( \rho_0 \) of 1-forms which, in an appropriate gauge, vanishes. We set
\[ \rho_t = t\rho + (1-t)\rho_0 = t\rho, \]
(2.48)
for \( \rho_0 = 0 \), corresponding to the connection \( \nabla_t = t\nabla + (1-t)\nabla_0 \). The curvature of \( \nabla_t \) is given by the matrix \( \theta_t \) of 2-forms given by
\[ \theta_t = d\rho_t + \rho_t^2 = t d\rho + t^2 \rho^2. \]
The Chern-Simons \((2n-1)\)-form associated with \(\nabla\) is then given by
\[
\vartheta^{2n-1}(\rho) := \frac{1}{(n-1)!} \int_0^1 dt \, \rho \, \theta_t^{n-1}.
\] (2.49)

For \(n = 2\), we find
\[
\vartheta^3(\rho) = \frac{1}{2} \{ \rho d\rho + \frac{2}{3} \rho^3 \},
\] (2.50)
and, for \(n = 3\),
\[
\vartheta^5(\rho) = \frac{1}{6} \{ \rho d\rho d\rho + \frac{3}{4} \rho^3 d\rho + \frac{3}{4} \rho (d\rho)^2 + \frac{3}{5} \rho^5 \}.
\] (2.51)

In order to understand where these definitions come from and how to define Chern-Simons actions, we extend \(E\) to a trivial vector bundle over the cylinder \((\tilde{\mathcal{A}}, \tilde{\pi}, \tilde{H}, \tilde{D})\) over \((\mathcal{A}, \pi, H, D)\): We set
\[
\tilde{E} = E \otimes C^\infty(I) \otimes \mathbb{I}_2 \simeq \tilde{\mathcal{A}}^N.
\] (2.52)

We also extend the connection \(\nabla\) on \(E\) to a connection \(\tilde{\nabla}\) on \(\tilde{E}\) by interpolating between \(\nabla\) and the flat connection \(\nabla_0\): By (2.39), (2.41) and (2.42), a 1-form in \(\Omega^1_{\tilde{D}}(\tilde{\mathcal{A}})\) is given by
\[
\rho(t) = \begin{pmatrix} \rho(t) & \phi(t) \\ \phi(t) & -\rho(t) \end{pmatrix},
\]
where \(\rho(t) \in \Omega^1_D(\mathcal{A})\) for all \(t \in I\). Thus
\[
\tilde{\rho}_\alpha^\beta(t) := t \begin{pmatrix} \rho_\alpha^\beta & 0 \\ 0 & -\rho_\alpha^\beta \end{pmatrix}, \quad \alpha, \beta = 1, \ldots, N,
\] (2.53)
is an element of \(\Omega^1_D(\tilde{\mathcal{A}})\). We define \(\tilde{\nabla}\) to be the connection on \(\tilde{E}\) determined by the matrix \(\tilde{\rho}(t) = (\tilde{\rho}_\alpha^\beta(t))\) of 1-forms defined in (2.53). Let \(\tilde{d}\) be the differential on \(\Omega^1_D(\tilde{\mathcal{A}})\) defined as in (2.10), (with \(\mathcal{A}\) replaced by \(\tilde{\mathcal{A}}\) and \(D\) replaced by \(\tilde{D}\)). By (2.42)
\[
\tilde{d}\rho = \begin{pmatrix} 0 & -i\rho \\ i\rho & 0 \end{pmatrix} + t \begin{pmatrix} d\rho & 0 \\ 0 & d\rho \end{pmatrix},
\] (2.54)
with \(\rho = (\rho_\alpha^\beta)\). Hence the curvature of \(\tilde{\nabla}\) is given by the matrix of 2-forms \(\tilde{\theta}\), with
\[
\tilde{\theta}(t) = \theta_t \otimes \mathbb{I}_2 + \rho \otimes \sigma_2,
\] (2.55)
where
\[
\theta_t = td\rho + t^2 \rho^2.
\]
commutes with \( \hat{D} \) and with \( \tilde{\pi}(\tilde{A}) \) and hence with \( \Omega(\tilde{A}) \). It also commutes with the \( \mathbb{Z}_2 \)-grading \( \Gamma = \mathbb{I} \otimes (\mathbb{I} \otimes \sigma_2) \), (as defined in (2.43)). We now define a graded trace \( \tau_\varepsilon(\cdot) \) on \( \Omega(\tilde{A}) \) by setting

\[
\tau_\varepsilon(\alpha) := \oint (\varepsilon \Gamma \alpha), \quad \alpha \in \Omega(\tilde{A}),
\]

(2.56)

where \( \oint (\cdot) \) is given by (2.47). It is easy to show that

\[
\tau_\varepsilon(\alpha) = 0 \quad \text{if } \deg \alpha \text{ is odd},
\]

(2.57)

and

\[
\tau_\varepsilon([\alpha, \beta]) = 0, \quad \text{for all } \alpha, \beta \in \Omega(\tilde{A}),
\]

(2.58)

where

\[
[\alpha, \beta] = \alpha \cdot \beta - (-1)^{\deg \alpha \deg \beta} \beta \cdot \alpha
\]

is the graded commutator.

Using the Bianchi identity,

\[
d\theta^n + [\rho, \theta^n] = 0,
\]

which follows from eq. (2.23) by induction, and the graded cyclicity of \( \tau_\varepsilon \) (see (2.57), (2.58)) one shows that

\[
\tau_\varepsilon((\tilde{\theta}^n)\perp) = n! \tau_\varepsilon((d\phi^{2n-1}(\bar{\rho}))\perp),
\]

(2.59)

where \( \alpha\perp \) is the canonical representative in the equivalence class \( \alpha \) (mod \( \text{Aux}^m \)) in \( \Omega^m_D(\tilde{A}) \) orthogonal to \( \text{Aux}^m \), for any \( m = 1, 2, \cdots \), (as explained after eq. (2.35)). The calculation proving (2.59) is indicated in [11]; (see also [12] for details). In fact, eq. (2.59) is a general identity valid for arbitrary connections on trivial vector bundles over a non-commutative space and arbitrary graded traces [11].

In the case considered here, the l.h.s. of eq. (2.59) can be rewritten in the following interesting way:

\[
\tau_\varepsilon((\tilde{\theta}^n)\perp) = \int_0^1 dt \int Tr_{C^2}(\varepsilon \Gamma (\tilde{\theta}^n(t))\perp)
\]

\[
= n \int_0^1 dt \int Tr_{C^2}((\varepsilon \otimes \mathbb{I}_2) \Gamma (\rho \otimes \sigma_2) (\theta_t^{n-1} \otimes \mathbb{I}_2)).
\]

(2.60)

This is shown by plugging eq. (2.55) for \( \tilde{\theta}(t) \) into the expression in the middle of (2.60) and noticing that (1) all terms contributing to \( \tilde{\theta}^n(t) \) with more than one factor proportional to \([\tilde{D}_1, \tilde{a}]\), i.e., with more than one factor of the form \( \rho \otimes \sigma_2 \), are projected out when passing from \( \tilde{\theta}^n(t) \) to \( (\tilde{\theta}^n(t))\perp \), (see the remark following eq. (2.46)), and (2) \( Tr_{C^2}((\varepsilon \otimes \mathbb{I}_2) \Gamma (\theta_t^{n} \otimes \mathbb{I}_2)) = 0 \).
Evaluating the trace, $\text{Tr}_{\mathbb{C}^2}$, on the r.h.s. of (2.60) and recalling the definition (2.49) of the Chern-Simons form, we finally conclude that

$$
\tau_\varepsilon((\tilde{\theta}^n)^\perp) = 2n \int_0^1 dt \int (\varepsilon \rho \tilde{\theta}^{n-1})^\perp
= 2n! \int (\varepsilon (\varphi^{2n-1}(\rho))^\perp).
$$

(2.61)

**Remark.** The r.h.s. of (2.59) can actually be rewritten as

$$
n! \int \text{Tr}_{\mathbb{C}^2}(1 \otimes \sigma_3 (\varphi^{2n-1}(\bar{\rho}(1)))^\perp);
$$

see [12].

**Chern-Simons actions.** $I_\varepsilon$, in non-commutative geometry are defined by setting

$$
I_\varepsilon^{2n-1}(\rho) := \kappa \int (\varepsilon (\varphi^{2n-1}(\rho))^\perp),
$$

(2.62)

where $\kappa$ is a constant. Using (2.50) and (2.51) and using the properties of $\varepsilon$ and the cyclicity of $f(\cdot)$, we find

$$
I_\varepsilon^3(\rho) = \frac{\kappa}{2} \int (\varepsilon (\rho^2 + \frac{2}{3} \rho^3)^\perp),
$$

and

$$
I_\varepsilon^5(\rho) = \frac{\kappa}{6} \int (\varepsilon (\rho^2 + \frac{3}{2} \rho^3 d\rho + \frac{3}{5} \rho^5)^\perp).
$$

(2.63)

A particularly important special case is obtained by choosing the operator $\varepsilon$ to belong to $\Omega(A)$. Since $\varepsilon$ commutes with $D$ and with $\pi(A)$, this implies that $\varepsilon$ belongs to the centre of the algebra $\Omega(A)$. In the examples discussed in the remainder of this paper, this property is always assumed.

One point of formula (2.61) and generalizations thereof, discussed in [12] (and involving “higher-dimensional cylinders”), is that it enables us to define differences of Chern-Simons actions even when the underlying vector bundle is non-trivial. If $\nabla_0$ denotes a fixed reference connection on a vector bundle $E$ over a non-commutative space $(\mathcal{A}, \pi, H, D)$ and $\nabla$ is an arbitrary connection on $E$ we set

$$
\int (\varepsilon (\varphi^{2n-1}(\nabla))^\perp) := \tau_\varepsilon((\tilde{\theta}^n)^\perp) + \text{const.},
$$

(2.64)

where $\tilde{\theta}$ is the curvature of a connection $\tilde{\nabla}$ on a vector bundle $\tilde{E}$ over the cylinder $(\tilde{\mathcal{A}}, \tilde{\pi}, \tilde{H}, \tilde{D})$ interpolating between $\nabla$ and $\nabla_0$, and the constant on the r.h.s. of (2.64) is related to the choice of $\tilde{\nabla}$ and of the Chern-Simons action associated with $\nabla_0$.

Formulas (2.62) and (2.64) are helpful in understanding the topological nature of Chern-Simons actions.
Next, we propose to discuss various concrete examples and indicate some applications to theories of gravity.

3. Some “three-dimensional” Chern-Simons actions.

We consider a “Euclidean space-time manifold” $X$ which is the Cartesian product of a Riemann surface $M_2$ and a two-point set, i.e. $X$ consists of two copies of $M_2$. The algebra $\mathcal{A}$ used in the definition of the non-commutative space considered in this section is given by

$$\mathcal{A} = C^\infty(M_2) \otimes A_0,$$

where $A_0$ is a finite-dimensional, unital $\ast$-algebra of $M \times M$ matrices. The Hilbert space $H$ is chosen to be

$$H = H_0 \oplus H_0,$$

where

$$H_0 = \mathbb{C}^N \otimes L^2(S) \otimes \mathbb{C}^M,$$

and $L^2(S)$ is the Hilbert space of square-integrable spinors on $M_2$ for some choices of a spin structure and of a (Riemannian) volume form on $M_2$.

The representation $\pi$ of $\mathcal{A}$ on $H$ is given by

$$\pi(a) = \begin{pmatrix} \mathbb{1}_N \otimes a & 0 \\ 0 & \mathbb{1}_N \otimes a \end{pmatrix},$$

for $a \in \mathcal{A}$.

We shall work locally over some coordinate chart of $M_2$, but we do not describe how to glue together different charts (this is standard), and we shall write “$M_2$” even when we mean a coordinate chart of $M_2$. Let $g = (g_{\mu\nu})$ be some fixed, Riemannian reference metric on $M_2$, and $(e^a_{\mu})$ a section of orthonormal 2-frames, $\mu, a = 1, 2$. Let $\gamma^1, \gamma^2$ denote the two-dimensional Dirac matrices satisfying

$$\{\gamma^a, \gamma^b\} = \gamma^a \gamma^b + \gamma^b \gamma^a = 2 \delta^{ab},$$

and

$$\gamma^5 = \gamma^1 \gamma^2.$$

The matrices $\mathbb{1}_N \otimes \gamma^a$ will henceforth also be denoted by $\gamma^a$. Let $D$ denote the covariant Dirac operator on $\mathbb{C}^N \otimes L^2(S)$ corresponding to the Levi-Civita spin connection determined by $(e^a_{\mu})$ and acting trivially on $\mathbb{C}^N$. Let $K$ denote an operator of the form

$$K = k \otimes \mathbb{1} \otimes \mathbb{1},$$
where \( k \) is some real, symmetric \( N \times N \) matrix. The vector space \( \mathbb{C}^N \) and the matrix \( k \) do not play any interesting role in the present section but are introduced for later convenience. Let \( \phi_0 \) be a hermitian \( M \times M \) matrix \((\neq \mathbb{1}_M)\). The operator \( D \) on \( H \) required in the definition of a non-commutative space is chosen as

\[
D = \begin{pmatrix}
\partial \otimes \mathbb{1}_M & i\gamma^5 K \otimes \phi_0 \\
-i\gamma^5 K \otimes \phi_0 & \partial \otimes \mathbb{1}_M
\end{pmatrix},
\]

(3.8)

Then (locally on \( M_2 \)) the space of 1-forms, \( \Omega^1_D(\mathcal{A}) \), (the “cotangent bundle”) is a free, hermitian \( \mathcal{A} \)-bimodule of dimension 3, with an orthonormal basis given by

\[
\varepsilon^a = \begin{pmatrix}
\gamma^a \otimes \mathbb{1}_M \\
0 \\
\gamma^a \otimes \mathbb{1}_M
\end{pmatrix}, \quad a = 1, 2,
\]

(3.9)

and

\[
\varepsilon^3 = \begin{pmatrix}
0 \\
\gamma^5 \otimes \mathbb{1}_M \\
0
\end{pmatrix}
\]

(3.10)

and the hermitian structure is given by the normalized trace, \( tr \), on \( \mathbb{M}_N(\mathbb{C}) \otimes \text{Cliff} \). Then, for \( a, b = 1, 2, 3 \),

\[
\langle \varepsilon^a, \varepsilon^b \rangle = tr(\varepsilon^a (\varepsilon^b)^*) = \delta^{ab} \mathbb{1}_M.
\]

We define a central element \( \varepsilon \in \Omega^3(\mathcal{A}) \) by setting

\[
\varepsilon = \varepsilon^1 \varepsilon^2 \varepsilon^3 = \begin{pmatrix}
0 \\
-\mathbb{1} \\
0
\end{pmatrix}.
\]

(3.11)

It is trivial to verify that \( \varepsilon \) commutes with the operator \( D \) and with \( \pi(\mathcal{A}) \), and, since \( \varepsilon^1, \varepsilon^2 \) and \( \varepsilon^3 \) belong to \( \Omega^1(\mathcal{A}) \), \( \varepsilon \) belongs to \( \Omega^3(\mathcal{A}) \).

A 1-form \( \rho \) has the form

\[
\rho = \sum_j \pi(a^j) [D, \pi(b^j)], \quad a^j, b^j \in \mathcal{A},
\]

(3.12)

and, without loss of generality, we may impose the constraint

\[
\sum_j a^j b^j = \mathbb{1}.
\]

(3.13)

Then

\[
\rho = \begin{pmatrix}
A \\
-i\gamma^5 K \phi \\
A
\end{pmatrix},
\]

(3.14)

where \( A = \sum_j a^j (\partial b^j) \), and \( \phi + \phi_0 = \mathbb{1}_N \otimes (\sum_j a^j \phi_0 b^j) \). The 1-form \( \rho \) given in \( 3.14 \) determines a connection, \( \nabla \), on the “line bundle” \( E \simeq \mathcal{A} \). The three-dimensional Chern-Simons action of \( \nabla \) is then given by

\[
I_\varepsilon^3(\rho) = \frac{k}{2} \int (\varepsilon (\rho d\rho + \frac{2}{3} \rho^3)^+) = \frac{k}{2} Tr_\omega (\varepsilon (\rho d\rho + \frac{2}{3} \rho^3)^+ D^{-2}),
\]

(3.15)
as follows from eqs. (2.63) and (2.32); (the Fredholm module \((A, \pi, H, D)\) is \((2, \infty)\)-summable!).

In order to proceed in our calculation, we must determine the spaces of “auxiliary fields” \(Aux^n\), for \(n = 1, 2, 3\). Clearly \(Aux^1 = 0\). To identify \(Aux^2\), we consider a 1-form

\[
\rho = \sum_j \pi(a^j)[D, \pi(b^j)] = \begin{pmatrix} A & i\gamma^5 K \phi \\ -i\gamma^5 K \phi & A \end{pmatrix}.
\]

Then

\[
d\rho = \sum_j [D, \pi(a^j)][D, \pi(b^j)] = \begin{pmatrix} \frac{1}{2} \gamma^{\mu \nu} \partial_\mu A_\nu + X, & -i\gamma^5 \gamma^{\mu \nu} K(\partial_\mu \phi + A_\mu \phi_0 - \phi_0 A_\mu) \\ i\gamma^5 \gamma^{\mu \nu} K(\partial_\mu \phi + A_\mu \phi_0 - \phi_0 A_\mu), & \frac{1}{2} \gamma^{\mu \nu} \partial_\mu A_\nu + X \end{pmatrix},
\]

where \(X = \mathbb{I}_N \otimes (\sum_j a^j \partial_\mu \partial_\nu b^j) + \partial_\mu A_\mu\) is an arbitrary element of \(\mathbb{I}_N \otimes A\), and \(\gamma^{\mu \nu} := [\gamma^\mu, \gamma^\nu]\). Hence

\[
Aux^2 \simeq \pi(A).
\] (3.16)

Next, let \(\eta \in \Omega^2(A)\). Then one finds that

\[
d\eta \bigg|_{\eta=0} = \varepsilon \begin{pmatrix} \gamma^{\mu} X_\mu & i\gamma^5 K X \\ \gamma^{\mu} X_\mu & \gamma^{\mu} X_\mu \end{pmatrix},
\] (3.17)

where \(X_\mu\) and \(X\) are arbitrary elements of \(\mathbb{I}_N \otimes A\). Thus, in a 3-form \(\vartheta\), terms proportional to \(\gamma^{\mu} \otimes \mathbb{I}_M\) in the off-diagonal elements and terms proportional to \(\gamma^5 K \otimes \mathbb{I}_M\) in the diagonal elements must be discarded when evaluating \(\vartheta^\perp\).

Next, we propose to check under what conditions the Chern-Simons action \(I^3_\varepsilon(\rho)\) is gauge-invariant. In eq. (2.25) we have seen that, under a gauge transformation \(M \in \pi(A)\), \(\rho\) transforms according to

\[
\rho \mapsto \tilde{\rho} = M \rho M^{-1} - (dM) M^{-1} = g^{-1} \rho g + g^{-1} d\rho,
\] (3.18)

with \(g = M^{-1}\). From this equation and the cyclicity of “integration”, \(\int (\cdot)\), we deduce that

\[
I^3_\varepsilon(\tilde{\rho}) = I^3_\varepsilon(\rho) + \frac{\kappa}{2} \int (\varepsilon \{ dg^{-1} \rho d\rho + g^{-1} d\rho dg - \frac{1}{3} (g^{-1} dg)^3 \})^\perp.
\] (3.19)

The second term on the r.h.s. of (3.19) is equal to

\[
\int_{M^2} d\text{vol} tr \left( \varepsilon \{ [D, g^{-1}] \rho [D, g] + g^{-1} d\rho [D, g] - \frac{1}{3} (g^{-1} [D, g])^3 \}^\perp \right).
\] (3.20)
Here, and in the following, \( tr(\cdot) \) denotes a normalized trace, \( (tr(\Pi) = 1) \). A straightforward calculation shows that

\[
[D, g] = \begin{pmatrix}
\bar{\partial} g & i\gamma^5 K(\phi_0 g - g\phi_0) \\
-i\gamma^5 K(\phi_0 g - g\phi_0) & \bar{\partial} g
\end{pmatrix},
\tag{3.21}
\]

and expression (3.20) is found to be given by

\[
-i tr K \int_{M_2} \partial_{\mu} tr \left[ g^{-1} \phi_0 \partial_{\nu} g + A_{\nu}(\phi_0 - g\phi_0 g^{-1}) - (g\partial_{\nu} g^{-1} \phi_0 + g^{-1} \partial_{\nu} g\phi_0) \right] dx^{\mu} \wedge dx^{\nu},
\tag{3.22}
\]

which vanishes if \( \partial M_2 = \phi \) (i.e., \( M_2 \) has no boundary).

**Remark.** Had we considered a more general setting with \( \mathcal{A} = C^\infty(M_2) \otimes \mathcal{A}_1 \oplus C^\infty(M_2) \otimes \mathcal{A}_2 \), where \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are two independent matrix algebras, and \( \pi(a) = \Pi_N \otimes a \), for \( a \in \mathcal{A} \), then, with \( \varepsilon \) chosen as above, \( I^3_\varepsilon(\rho) \) would fail to be gauge-invariant.

Thus the condition for \( I^3_\varepsilon(\rho) \) to be gauge-invariant is that \( \partial M_2 = \phi \) and that the non-commutative space is invariant under permuting the two copies of \( M_2 \) (of the space \( X_c \)), i.e., the elements of \( \pi(\mathcal{A}) \) commute with the operator \( \varepsilon \) defined in (3.11).

Under this condition one finds, after a certain amount of algebra, that

\[
I^3_\varepsilon(\rho) = i \kappa \int_{M_2} tr (\Phi F),
\tag{3.23}
\]

where

\[
\Phi = K(\phi + \phi_0),
\]

and

\[
F = (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}]) dx^{\mu} \wedge dx^{\nu}.
\tag{3.24}
\]

We note that \( I^3_\varepsilon(\rho) \) is obviously gauge-invariant and topological, i.e., metric-independent. Since the “Dirac operator” \( D \) depends on the reference metric \( g \) on \( M_2 \), the metric-independence of \( I^3_\varepsilon \) is not, a priori, obvious from its definition (3.15).

Let us consider the special case where

\[
\mathcal{A}_o = \mathbb{M}_3(\mathbb{R}),
\tag{3.25}
\]

the algebra of real \( 3 \times 3 \) matrices. Then

\[
I^3_\varepsilon(\rho) = i \kappa \int_{M_2} \left( \sum_{A=1}^3 \Phi^A F^A_{\mu\nu} \right) dx^{\mu} \wedge dx^{\nu},
\tag{3.26}
\]

where \( F^A_{\mu\nu} = \partial_{\mu} A^A_{\nu} - \partial_{\nu} A^A_{\mu} + \varepsilon^{ABC} A^A_{[\mu} B A_{\nu]}^C \), with \( A = a, 3, a = 1,2 \). Setting

\[
A^a_{\mu} = e^a_{\mu}, \quad A^3_{\mu} = \frac{1}{2} \omega^a_{\mu} \varepsilon_{ab} \equiv \omega_{\mu}^{ab}
\tag{3.27}
\]
one observes that the action $I^3_\varepsilon$ is the one of two-dimensional topological gravity introduced in [13]. Varying $I^3_\varepsilon$ w.r. to $\Phi^A$ one obtains the zero-torsion and constant-curvature conditions:

$$\varepsilon^{\mu\nu} F^a_{\mu\nu} \equiv \varepsilon^{\mu\nu} T^a_{\mu\nu} = \varepsilon^{\mu\nu} (\partial_\mu e^a_\nu + \frac{1}{2} \sum_b \omega_\mu e^{ab}_b e^b_\nu) = 0$$

$$\varepsilon^{\mu\nu} F^3_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu} R_{\mu\nu} e_{ab} = \frac{1}{2} \varepsilon^{\mu\nu} (\partial_\mu \omega_\nu + 2 \varepsilon a e^a_\mu e^b_\nu) = 0. \quad (3.28)$$

Variation of $I^3_\varepsilon$ with respect to $A^A_\mu$ implies that $\Phi^A$ is covariantly constant, i.e.,

$$D_\mu \Phi^A = \partial_\mu \Phi^A + \varepsilon^{ABC} A^B_\mu \Phi^C = 0. \quad (3.29)$$

The space of solutions of (3.28) and (3.29) is characterized in [13].

These results suggest that the study of Chern-Simons actions in non-commutative geometry is worthwhile.

### 4. Some “five-dimensional” Chern-Simons actions

In this section, we consider non-commutative space $(A, \pi, H, D)$ with “cotangent bundles” $\Omega^1_D(A)$ that are free, hermitian $A$-bimodules of dimension 5, and we evaluate the “five-dimensional” Chern-Simons action, $I^5_\varepsilon(\rho)$, defined in eq. (2.63), for connections on the “line bundle” $E = E^{(1)} \simeq A$. We shall consider algebras $A$ generated by matrix-valued functions on Riemann surfaces or on four-dimensional spin manifolds. We start with the analysis of the latter example.

(I) We choose $X = M_4 \times \{-1, 1\}$, where $M_4$ is a four-dimensional, smooth Riemannian spin manifold. The non-commutative space $(A, \pi, H, D)$ is chosen as in Sect. 3, except that $M_2$ is replaced by $M_4$, the $2 \times 2$ Dirac matrices $\gamma^1, \gamma^2$ are replaced by the $4 \times 4$ Dirac matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$, and $\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$. The definition of the “Dirac operator” $D$ is analogous to that in eq. (3.8).

An orthonormal basis for $\Omega^1_D(A)$ is then given (locally on $M_4$) by

$$\varepsilon^a = \begin{pmatrix} \gamma^a \otimes \mathbb{I}_M & 0 \\ 0 & \gamma^a \otimes \mathbb{I}_M \end{pmatrix}, a = 1, \ldots, 4, \quad \varepsilon^5 = \begin{pmatrix} 0 & \gamma^5 \otimes \mathbb{I}_M \\ -\gamma^5 \otimes \mathbb{I}_M & 0 \end{pmatrix}, \quad (4.1)$$

and

$$\varepsilon = \varepsilon^1 \varepsilon^2 \varepsilon^3 \varepsilon^4 \varepsilon^5 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \quad (4.2)$$

similarly as in (3.11).
Again, we must determine the spaces $Aux^n$, $n = 1, 2, 3, 4, 5$, of “auxiliary fields”. The most important one is $Aux^5$. To determine it, let us consider a vanishing element $\eta$ of $\Omega^4(A)$ and compute $d\eta$, as given by eq. (2.10). After a certain amount of labouring one finds that
\[
d\eta \big|_{\eta=0} = \varepsilon \left( \gamma^{\mu\nu\rho} X_{\mu\nu\rho} + \gamma^\mu (K^2 X_\mu + Y_\mu), \quad i\gamma^5 (\gamma^{\mu\nu} K X_{\mu\nu} + K^3 X + KY),\right.
\[
\left. -i\gamma^5 (\gamma^{\mu\nu} K X_{\mu\nu} + K^3 X + KY), \quad \gamma^{\mu\nu\rho} X_{\mu\nu\rho} + \gamma^\mu (K^2 X_\mu + Y_\mu) \right),
\]
where $X_{\mu\nu\rho}, X_\mu, X_\mu, X_\mu, X_\mu, Y_\mu$ and $Y$ are arbitrary elements of $\mathbb{I}_N \otimes A$, and $\gamma^{\mu\nu\rho} = \sum_{a,b,c} \delta^{\mu\nu\rho}_{a,b,c} \gamma^a \gamma^b \gamma^c$. By (4.3), the passage from an element $\vartheta \in \Omega^5(A)$ to $\vartheta^{\perp}$ amounts to discarding all terms proportional to $\gamma^{\mu\nu\rho} \otimes \mathbb{I}_M, K^2 \gamma^\mu \otimes \mathbb{I}_M \gamma^\mu \otimes \mathbb{I}_M$ from off-diagonal elements of $\vartheta$ and all terms proportional to $K \gamma^5 \gamma^{\mu\nu} \otimes \mathbb{I}_M, K^3 \gamma^5 \otimes \mathbb{I}_M$ and $K \gamma^5 \otimes \mathbb{I}_M$ from the diagonal elements of $\vartheta$. Now we start understanding the useful role played by the matrix $K$.

It is then easy to evaluate $I^5_\varepsilon(\rho)$, with $\rho$ given by
\[
\rho = \begin{pmatrix} A & i\gamma^5 K \phi \\ -i\gamma^5 K \phi & A \end{pmatrix}, \quad A = \gamma^\mu A_\mu.
\] Using eq. (2.63), the result is
\[
I^5_\varepsilon(\rho) = i \frac{3\kappa}{4} \int_{M_4} Tr (\Phi F \wedge F),
\] where $\Phi = K(\phi + \phi_0)$, and $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$, with $F_{\mu\nu}$ the curvature, or field strength, of $A_\mu$. Provided that $\partial M_4 = \emptyset$, $I^5_\varepsilon$ is gauge-invariant and topological (metric-independent), as expected. The field equation obtained by varying $I^5_\varepsilon$ w.r. to $\Phi$ is
\[
\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 0.
\] Setting $\Phi$ to a constant, $I^5_\varepsilon$ turns out to be the action of four-dimensional, topological Yang-Mills theory [16] before gauge-fixing.

(II) We choose $X = M_2 \times \{-1,1\}$ and $A, \pi$ and $H$ as above, but the operator $D$ is given by
\[
D = \begin{pmatrix} \partial & K \gamma^\alpha \phi_0^\alpha \\ -K \gamma^\alpha \phi_0^\alpha & \partial \end{pmatrix},
\] where $\partial = \gamma^1 \partial_1 + \gamma^2 \partial_2$, and $\alpha = 3, 4, 5$. The matrices $\gamma^1, \ldots, \gamma^4$ are antihermitian $4 \times 4$ Dirac matrices, and, only in this paragraph, $\gamma^5 = i \gamma^1 \gamma^2 \gamma^3 \gamma^4$, so that $\gamma^5$ is now antihermitian, too, rather than hermitian (as in the rest of this paper). Locally on $M_2$, the cotangent bundle $\Omega^1_D(A)$ is a free, hermitian $A$-bimodule of dimension 5, with an orthonormal basis given by
\[
\varepsilon^a = \begin{pmatrix} \gamma^a \otimes \mathbb{I}_M & 0 \\ 0 & \gamma^a \otimes \mathbb{I}_M \end{pmatrix}, a = 1, 2, \quad \varepsilon^a = \begin{pmatrix} 0 & i\gamma^\alpha \otimes \mathbb{I}_M \\ -i\gamma^\alpha \otimes \mathbb{I}_M & 0 \end{pmatrix}, \alpha = 3, 4, 5,
\]
and \( \varepsilon \) is taken to be
\[
\varepsilon = \varepsilon^1 \varepsilon^2 \varepsilon^3 \varepsilon^4 \varepsilon^5 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}.
\] (4.9)

A 1-form \( \rho = \sum_j \pi(a^j)(D, \pi(b^j)) \) has the form
\[
\rho = \begin{pmatrix} A & K\gamma^\alpha \phi_\alpha \\ -K\gamma^\alpha \phi_\alpha & A \end{pmatrix},
\] (4.10)

with \( A = \sum_j a^j \partial b^j \) and \( \phi_\alpha + \phi_\alpha_0 = \sum_j a^j \phi_\alpha_0 b^j \). Evaluating \( d\rho \) as in eq. (2.10), one finds that
\[
d\rho = \begin{pmatrix} \gamma^{\mu\nu} \partial_\mu A_\nu - K^2 \gamma^{\alpha\beta} L_{\alpha\beta} + X - K^2 L_\alpha^\alpha, & K^2 \gamma^\alpha \gamma^\beta D_\mu^0 \phi_\alpha \\ -K\gamma^\mu \gamma^\alpha D^0_\mu \phi_\alpha, & \gamma^{\mu\nu} \partial_\mu A_\nu - K^2 \gamma^{\alpha\beta} L_{\alpha\beta} + X - K^2 L_\alpha^\alpha \end{pmatrix},
\] (4.11)

where
\[
L_{\alpha\beta} = \phi_\alpha \phi_\beta + \phi_\alpha \phi_\beta + \sum_j a^j [b^j, \phi_\alpha \phi_\beta],
\]
\[
X = -\sum_j a^j \partial^2 b^j + \partial^\mu A_\mu,
\]
and
\[
D^0_\mu \phi_\alpha = \partial_\mu \phi_\alpha + A_\mu \phi_\alpha_0 - \phi_\alpha_0 A_\mu.
\] (4.12)

For simplicity we assume that
\[
[\phi_\alpha, \phi_\beta] = 0, \quad \text{and} \quad \phi_\alpha \phi_\alpha_0^\alpha = 1.
\] (4.13)

Since we may assume that \( \sum a^j b^j = 1 \), we then have that \( L_{\alpha\beta} = \phi_\alpha \phi_\beta + \phi_\alpha \phi_\beta \), for \( L_{[\alpha\beta]} \) and \( L_{\alpha}^\alpha \) appearing in (4.11), which is not an auxiliary field. A tedious calculation then yields the formula
\[
I^5_\varepsilon(\rho) = 2\kappa \int_{M_3} \varepsilon^{\mu\nu\varepsilon^{\alpha\beta\gamma}} tr K^3 \left[ (L_{\alpha\beta} \phi_\gamma + \phi_\alpha L_{\beta\gamma}) \partial_\mu A_\nu \right.
\]
\[
- \phi_\alpha D^0_\mu \phi_\beta D^0_\nu \phi_\gamma + A_\mu L_{\beta\gamma} D^0_\nu \phi_\gamma + A_\mu D^0_\nu \phi_\alpha L_{\beta\gamma} \\
+ \frac{3}{2} \phi_\alpha A_\mu A_\nu L_{\beta\gamma} + \frac{3}{2} \phi_\alpha \phi_\beta \phi_\gamma \partial_\mu A_\nu \\
- \frac{3}{2} A_\mu \phi_\alpha A_\nu L_{\beta\gamma} + \frac{3}{2} A_\mu A_\nu \phi_\alpha L_{\beta\gamma} \\
- \frac{3}{2} \phi_\alpha A_\mu \phi_\beta D^0_\nu \phi_\gamma + \frac{3}{2} \phi_\alpha \phi_\beta A_\mu D^0_\nu \phi_\gamma \\
+ \frac{3}{2} A_\mu \phi_\alpha \phi_\beta D^0_\nu + 3 A_\mu A_\nu \phi_\alpha \phi_\beta \phi_\gamma \\
- \frac{3}{2} \phi_\alpha A_\mu \phi_\beta A_\nu \phi_\gamma \right] d^2 x.
\] (4.14)
If \( \partial M_2 = \emptyset \), and after further algebraic manipulations, the action (4.14) can be shown to have the manifestly gauge-invariant form

\[
I_5^\varepsilon (\rho) = 2 \kappa \int_{M_2} \varepsilon^{\alpha \beta \gamma} tr \left[ -\Phi_\alpha (D_\mu \Phi_\beta) (D_\nu \Phi_\gamma) \right. \\
\left. + 2 \Phi_\alpha \Phi_\beta \Phi_\gamma (\partial_\mu A_\nu + A_\mu A_\nu) \right] dx^\mu \wedge dx^\nu,
\]

where \( \Phi_\alpha = K_\alpha (\phi_\alpha + \phi_0 \alpha) \).

If the constraints (4.13) are not imposed then one must explicitly determine \( A_\mu x^5 \), in order to derive an explicit expression for \( I_5^\varepsilon \). The result is that (4.15) still holds.

It is remarkable that all the Chern-Simons actions derived in eqs. (3.23), (4.5) and (4.15) can be obtained from Chern-Simons actions for connections on vector bundles over classical, commutative manifolds by dimensional reduction. For example, setting \( M_3 = M_2 \times S^1 \) and \( \phi := A_3 \), and assuming that \( A_1, A_2 \) and \( A_3 \) are independent of the coordinate (angle) parametrizing \( S^1 \), we find that

\[
I_3 (A) = i \kappa' \int_{M_3} tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \\
= i \kappa' \int_{M_2} tr (\phi F),
\]

where \( F = (\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]) \ dx^1 \wedge dx^2 \). Setting \( \kappa' = \kappa \ tr K \), (4.16) reduces to (3.23). Similarly, reducing a classical, five-dimensional Chern-Simons action to four dimensions, with \( M_5 = M_4 \times S^1 \), results in

\[
I_5 (A) = i \kappa' \int_{M_5} tr \left( A \wedge dA \wedge dA + \frac{3}{2} A \wedge A \wedge A \wedge dA \\
+ \frac{3}{5} A \wedge A \wedge A \wedge A \wedge A \right) \\
= i \frac{3 \kappa'}{4} \int_{M_4} tr (\phi F \wedge F),
\]

with \( \phi := A_5 \), and \( A_1, \cdots, A_5 \) independent of the angle parametrizing \( S^1 \). Thus we recover (4.5). Finally, dimensionally reducing \( I_5 (A) \) to a two-dimensional surface (setting \( M_5 = M_2 \times S^1 \times S^1 \times S^1 \)) reproduces the action (4.15).

The advantage of the non-commutative formulation is that it automatically eliminates all excited modes corresponding to a non-trivial dependence of the gauge potential \( A \) on angular variables.
5. Relation to four-dimensional gravity and supergravity

Chern-Simons actions are topological actions. In order to obtain dynamical actions from Chern-Simons actions, one would have to impose constraints on the field configuration space. In this section, we explore this possibility. As a result, we are able to derive some action functionals of four-dimensional gravity and supergravity theory.

We propose to impose a constraint on the scalar multiplet \( \Phi \) appearing in the Chern-Simons action (4.5). The non-commutative space \((A, \pi, H, D)\) is chosen as in example (I) of Sect. 4; (see also Sect. 3). Let us compute the curvature 2-form, \( \theta = (d\rho + \rho^2) \), of a connection \( \nabla \) on the line bundle \( E \simeq A \) given by a 1-form \( \rho \) as displayed in eq. (4.4). Then

\[
\theta = \left( \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} + (K^2)^\perp \left( (\phi + \phi_0)^2 - \phi_0^2 \right), -K i \gamma^5 \gamma^\mu D_\mu (\phi + \phi_0), \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} + (K^2)^\perp \left( (\phi + \phi_0)^2 - \phi_0^2 \right) \right),
\]

where \((K^2)^\perp = K^2 - (tr K^2) \mathbb{I} \); (recall that \( tr(\cdot) \) is normalized: \( tr(\mathbb{I}) = 1 \)). The appearance of \((K^2)^\perp \) is due to the circumstance that when passing from \( d\rho + \rho^2 \) to \((d\rho + \rho^2)^\perp \) terms proportional to \( \mathbb{I}_N \) must be removed. Let \( ptr(\cdot) \) denote the partial trace over the Dirac-Clifford algebra. Then

\[
ptr(\theta) = (K^2)^\perp \left( (\phi + \phi_0)^2 - \phi_0^2 \right),
\]

and we shall impose the constraint

\[
ptr(\theta) = 0. \tag{5.3}
\]

Choosing \( \phi_0 \) to satisfy \( \phi_0^2 = \mathbb{I} \), and renaming \( \phi + \phi_0 \) to read \( \phi \), the constraint (5.3) becomes

\[
\phi^2 = \mathbb{I}, \tag{5.4}
\]

provided \((K^2)^\perp \neq 0 \).

As our matrix algebra \( A_0 \) (see eq. (3.1)) we choose

\[
A_0 = \text{real part of Cliff } (SO(4)). \tag{5.5}
\]

We propose to show that, for this choice of \( A_0 \) and assuming that the constraint (5.4) is satisfied, the Chern-Simons action (4.5) is the action of the metric-independent (first-order) formulation of four-dimensional gravity theory.

Let \( \Gamma_1, \cdots, \Gamma_4 \) denote the usual generators of \( A_0 \), (i.e., \( 4 \times 4 \) Dirac matrices in a real representation), and \( \Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \). Then

\[
\{ \Gamma_a, \Gamma_b \} = -2\delta_{ab}, \quad \Gamma_a^* = -\Gamma_a, \quad a, b = 1, \cdots, 4,
\]

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and $\Gamma_5^* = \Gamma_5$. A basis for $\mathcal{A}_0$ is then given by $\mathbb{I}_4, \Gamma_a, a = 1, \cdots, 4, \Gamma_5, \Gamma_{ab}, a, b = 1, \cdots, 4,$ and $\Gamma_a \Gamma_5$. For a 1-form $\rho$ as in eq. (4.4), we may expand the gauge potential $A$ and the scalar field $\phi$ in the basis of $\mathcal{A}_0$ just described:

$$A = \gamma^\mu \left( A_\mu^0 \mathbb{I} + A_\mu^a \Gamma_a + A_\mu^{ab} \Gamma_{ab} + A_\mu^5 \Gamma_5 + A_\mu^{a5} \Gamma_a \Gamma_5 \right), \quad (5.6)$$

and

$$\phi = \left( \phi^0 \mathbb{I} + \phi^a \Gamma_a + \phi^{ab} \Gamma_{ab} + \phi^5 \Gamma_5 + \phi^{a5} \Gamma_a \Gamma_5 \right). \quad (5.7)$$

In this section, we only consider \underline{unitary} connections on $E \equiv E^{(1)} \simeq A$; see eq. (2.29). By (2.30), this is equivalent to hermiticity of $\rho$. This implies that

$$A_\mu^0 = -A_\mu^0, \quad A_\mu^a = A_\mu^a, \quad A_\mu^{ab} = -A_\mu^{ab}, \quad A_\mu^5 = -A_\mu^5, \quad (5.8)$$

and

$$\phi^0 = \overline{\phi^0}, \quad \phi^a = -\overline{\phi^a}, \quad \phi^{ab} = -\overline{\phi^{ab}}, \quad \phi^5 = \overline{\phi^5}, \quad \phi^{a5} = \overline{\phi^{a5}}, \quad (5.9)$$

where $\bar{z}$ denotes the complex conjugate of $z$. Since $\mathcal{A}_0$ is chosen to be real, the coefficients of $A$ and $\phi$ should be chosen to be real. It then follows from (5.8) and (5.9) that

$$A = \gamma^\mu \left( \frac{1}{2\kappa} e_\mu^a \Gamma_a + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \right) \quad (5.10)$$

and

$$\phi = \phi^0 + \phi^5 \Gamma_5 + \phi^{a5} \Gamma_a \Gamma_5, \quad (5.11)$$

where we have set $A_\mu^a =: \frac{1}{2\kappa} e_\mu^a$, and $A_\mu^{ab} =: \frac{1}{4} \omega_\mu^{ab}$, and $\kappa^{-1}$ is the Planck scale.

Imposing the constraint that $tr_{\mathcal{A}_0}(\varepsilon \rho) = 0$ implies that

$$\phi^0 = 0. \quad (5.12)$$

Constraints (5.12) and (5.4) then yield the condition

$$(\phi^5)^2 + (\phi^{a5})^2 = 1. \quad (5.13)$$

Under a gauge transformation $M \equiv g^1$, $\rho$ transforms according to

$$\rho \mapsto M \rho M^{-1} - (dM) M^{-1}, \quad M \in \pi(A),$$

see (2.25), which implies the transformation law

$$\phi \mapsto g^{-1} \phi g, \quad g = \exp \frac{1}{2} \left( \Lambda^a \Gamma_a + \Lambda^{ab} \Gamma_{ab} \right), \quad (5.14)$$
where $\Lambda^a$ and $\Lambda^{ab}$ are smooth functions on $M_4$. The infinitesimal form of (5.14) reads

$$\delta \phi^5 = - \sum_a \Lambda^a \phi^a,$$

$$\delta \phi^a = - \Lambda^a \phi^5 - \sum_b \Lambda^{ab} \phi^b. \quad (5.15)$$

From this it follows that, locally, we can choose a gauge such that

$$\phi^a = 0. \quad (5.16)$$

In this gauge, the constraint (5.13) has the solutions

$$\phi^5 = \pm 1. \quad (5.17)$$

The action (4.5) then becomes

$$I_5^\varepsilon (\rho) = \pm k \int_{M_4} \text{tr} (\Gamma_5 F \wedge F), \quad (5.18)$$

(with $k = i^{3n} 4$, in the notation of Sect. 4). Next, we expand the field strength $F_{\mu\nu}$ in our Clifford algebra basis which yields

$$F_{\mu\nu} = - \frac{1}{2\kappa} F^a_{\mu\nu} \Gamma_a + \frac{1}{4} F^{ab}_{\mu\nu} \Gamma_{ab}, \quad (5.19)$$

where

$$F^a_{\mu\nu} = \partial_\mu e^a_\nu + \omega^a_{\mu b} e^b_\nu - (\mu \leftrightarrow \nu), \quad (5.20)$$

$$F^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu + \omega^a_{\mu c} \omega^{cb}_\nu + \frac{1}{\kappa^2} e^a_\mu e^b_\nu - (\mu \leftrightarrow \nu), \quad (5.21)$$

and the indices $a, b, \cdots$ are raised and lowered with the flat metric $\eta_{ab} = -\delta_{ab}$.

The only non-vanishing contribution to (5.18) comes from the trace $tr(\Gamma_5 \Gamma_{ab} \Gamma_{cd}) = \varepsilon_{abcd}$, and $I_5^\varepsilon$ is found to be given by

$$I_5^\varepsilon = \pm k \int_{M_4} \varepsilon_{abcd} (R^{ab}_{\mu\nu} + \frac{2}{\kappa^2} e^a_\mu e^b_\nu) (R^{cd}_{\rho\sigma} + \frac{2}{\kappa^2} e^c_\rho e^d_\sigma) \times dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \quad (5.22)$$

where

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu + \omega^a_{\mu c} \omega^{cb}_\nu - (\mu \leftrightarrow \nu). \quad (5.23)$$

Interpreting $\omega^{ab}_\mu$ as the components of a connection on the spinor bundle over $M_4$, $R^{ab}_{\mu\nu}$ are the components of its curvature, and $F^a_{\mu\nu}$ are the components of its torsion, as is well known from the Cartan structure equations.
Setting the variation of $I_5^\varepsilon$ with respect to $\omega^{ab}_\mu$ to zero, we find that the torsion of $\omega$ vanishes:

$$F^\alpha_{\mu\nu} = 0, \quad \text{for all } \mu, \nu \text{ and } a. \quad (5.24)$$

If the frame $(e^a_\mu)$ is invertible, (5.24) can be solved for $\omega^{ab}_\mu$:

$$\omega^{ab}_\mu = \frac{1}{2} \left( \Omega_{\mu ab} - \Omega_{ab \mu} + \Omega_{b \mu a} \right), \quad (5.25)$$

where

$$\Omega_{ab}^\phantom{ab}c = e^\mu_a e^\nu_b \left( \partial_\mu e^c_\nu - \partial_\nu e^c_\mu \right).$$

Substituting (5.25) back into (5.22) yields a functional that depends only on the metric

$$g_{\mu\nu} = e^a_\mu e^a_\nu, \quad (5.26)$$

and is given by

$$I_5^\varepsilon = \pm k \int_{M_4} d^4x \sqrt{g} \left[ \left( 4 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) + \frac{16}{\kappa^2} R + \frac{96}{\kappa^4} \right] \quad (5.27)$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor, $R_{\mu\nu}$ is the Ricci tensor, and $R$ is the scalar curvature determined by the metric $g_{\mu\nu}$ given in (5.26). The term in round brackets on the r.h.s. of (5.27) yields the topological Gauss-Bonnet term for $M_4$, the second term yields the Einstein-Hilbert action, and the last term is a cosmological constant.

Next, we show how to derive a metric-independent formulation of four-dimensional supergravity from the action $I_5^\varepsilon$ given in eq. (4.5). For this purpose we choose the algebra $A_0$ in (3.1) to be a graded algebra [18]:

$$A_0 = \text{real part of SU}(4 \mid 1) \quad (5.28)$$

This algebra is generated by graded $5 \times 5$ matrices preserving the quadratic form

$$(\vartheta^\alpha)^* C_{\alpha\beta} \vartheta^\beta - z^* z, \quad (5.29)$$

where $C_{\alpha\beta}$ is an antisymmetric matrix and $\vartheta^\alpha$ is a Dirac spinor. At this point, one must note that we are leaving the conventional framework of non-commutative geometry, since, for $A_0$ as in (5.28), the algebra $A$ is not a *algebra of operators. But let us try to proceed and find out what the result is.

Let $\rho$ be a 1-form as in eq. (4.4). Then the matrix elements $A_\mu$ and $\phi$ of $\rho$ have the graded matrix representation

$$\phi = \begin{pmatrix} \Pi^\alpha_\beta & \lambda_\alpha \\ \bar{\lambda}^\alpha & \Pi_1 \end{pmatrix}, \quad (5.30)$$
\[ A_\mu = \left( \begin{array}{c} M_{\mu\alpha}^\beta \sqrt{\kappa} \psi_{\mu\alpha} \\ -\sqrt{\kappa} \psi_{\mu}^{\alpha} B_\mu \end{array} \right). \] (5.31)

The reality conditions for \( \phi \) and \( A_\mu \) imply that \( \lambda_\alpha \) and \( \psi_{\mu\alpha} \) are Majorana spinors:

\[ \lambda_\alpha = C_{\alpha\beta} \bar{\lambda}^\beta, \quad \psi_{\mu\alpha} = C_{\alpha\beta} \bar{\psi}_{\mu}^{\beta}. \]

Furthermore, one finds that

\[ \Pi^\beta \alpha = \left( \frac{1}{4} \Pi^0 I + \Pi^5 \Gamma_5 + \Pi^{a5} \Gamma_a \Gamma_5 \right) \beta \alpha, \]

\[ M_{\mu\alpha}^\beta = \left( \frac{1}{2\kappa} \epsilon^e_\mu \Gamma_a + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \right) \beta \alpha, \] (5.32)

and

\[ B_\mu = 0. \]

We shall now impose the constraints

\[ Str (\varepsilon \rho) = 0, \] (5.33)

\[ Str (\theta) = 0, \] (5.34)

and

\[ Str \left( \varepsilon \left( \rho d\rho + \frac{2}{3} \rho^3 \right) \right) = 0, \] (5.35)

along with \( Str \phi_0^2 = 1 \), and \( Str \phi_0 = 0 \). Here \( Str(\cdot) \) denotes the graded trace on \( A_0 \). Renaming \( \phi + \phi_0 \) to read \( \phi \), these constraints imply that

\[ \Pi^0 = \Pi_1, \] (5.36)

\[ -\frac{3}{4} \Pi_1^2 + 4((\Pi^5)^2 - \sum_a (\Pi^{a5})^2) + \bar{\lambda} \lambda = 1, \] (5.37)

and

\[ (K^3) \perp Str(\phi^3) = 0, \] (5.38)

where \( (K^3) \perp \) is defined so as to satisfy \( tr(K(K^3) \perp) = 0 \).

In order to determine the dynamical contents of a theory with an action \( I_\varepsilon^5 \) given by (4.5), \( A_0 \) as in (5.28) and constraints (5.36) through (5.38), it is convenient to work in a special gauge, the unitary gauge. Consider a gauge transformation

\[ g = \exp \left( \begin{array}{cc} \Lambda^\alpha_\mu & \sqrt{\kappa} \varepsilon_\alpha \\ -\sqrt{\kappa} \varepsilon^\alpha & 0 \end{array} \right), \] (5.39)
where $\Lambda^\beta_\alpha = \frac{1}{2} (\Lambda^a \Gamma_a + \Lambda^{ab} \Gamma_{ab})^\beta_\alpha$. The transformation law of $\phi$ is then given by $\phi \mapsto g^{-1} \phi g$. From this we find the infinitesimal gauge transformations of the fields $\Pi$ and $\lambda$:

\[
\delta \Pi_1 = 2 \sqrt{\kappa} \varepsilon \lambda,
\]
\[
\delta \Pi^5 = + \Lambda^a \Pi^5 + \frac{\sqrt{\kappa}}{2} \varepsilon \Gamma^5 \lambda,
\]
\[
\delta \Pi^a_5 = - \Lambda^{ab} \Pi^b_5 - \Lambda^a \Pi^5_5 - \frac{1}{2} \varepsilon \Gamma^a \Gamma^5 \lambda,
\]
\[
\delta \lambda_\alpha = \sqrt{\kappa} \left( - \frac{3}{4} \Pi_1 + \Pi^5 \Gamma_5 + \Pi^{a5} \Gamma_{a} \Gamma_5 \right) \varepsilon_\alpha - \frac{1}{2} \left( \Lambda^a \Gamma_a + \Lambda^{ab} \Gamma_{ab} \right)^\beta_\alpha \lambda_\beta. \tag{5.40}
\]

Thus, locally, we can choose the gauge

\[
Pi^a_5 = 0, \quad \text{and} \quad \lambda_a = 0. \tag{5.41}
\]

The constraints (5.37) and (5.38) then reduce to

\[
- \frac{3}{4} \Pi^2_1 + 4 (\Pi^5)^2 = 1,
\]
\[
\Pi_1 \left( - \frac{5}{16} \Pi^2_1 + (\Pi^5)^2 \right) = 0. \tag{5.42}
\]

These equations have the solutions

\[
\Pi_1 = 0, \quad \Pi^5 = \pm \frac{1}{2}, \tag{5.43}
\]

and

\[
\Pi_1 = \pm \sqrt{2}, \quad \Pi^5 = \pm \sqrt{\frac{5}{8}}. \tag{5.44}
\]

We further study the first solution. Inserting it into the action (4.5), we arrive at the expression

\[
I^5_\varepsilon = \pm \frac{k}{2} \int_{M_4} \text{Str} \left( \left( \Gamma^5 0 0 \right) F_{\mu \nu} F_{\rho \sigma} \right) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \tag{5.45}
\]

where

\[
F_{\mu \nu} = \begin{pmatrix}
\frac{1}{4} F_{\mu \nu}^a \Gamma_{ab} + \frac{1}{2 \kappa} F_{\mu \nu}^a \Gamma_a, & \sqrt{\kappa} \psi_{\mu \nu}^a \\
- \sqrt{\kappa} \psi_{\mu \nu}^a & 0
\end{pmatrix}, \tag{5.46}
\]

with

\[
F_{\mu \nu}^a = \partial_\mu e_\nu^a + \omega_\mu^a e_\nu^b - \frac{\kappa^2}{2} \bar{\psi}_{\mu} \Gamma^a \psi_{\nu} - (\mu \leftrightarrow \nu),
\]
\[
F_{\mu \nu}^{ab} = R_{\mu \nu}^{ab} + \frac{1}{\kappa^2} \left( e_\mu^a e_\nu^b - e_\mu^b e_\nu^a \right) + \kappa \bar{\psi}_{\mu} \Gamma^{ab} \psi_{\nu},
\]
\[
\psi_{\mu \nu \alpha} = \partial_\mu \psi_{\nu \alpha} + \frac{1}{4} \omega_{\mu \nu}^{ab} (\Gamma_{ab} \psi_\nu)_\alpha + \frac{1}{2 \kappa} e_{\mu}^a (\Gamma_a \psi_\nu)_\alpha - (\mu \leftrightarrow \nu). \tag{5.47}
\]
After some further manipulations and evaluating all the traces, one obtains the elegant result that the action reduces to that proposed in [17], namely

$$I_5^\varepsilon = \pm k \int_{M_4} \left[ \frac{1}{4} \varepsilon_{abcd} F_{\mu\nu}^{ab} F_{\rho\sigma}^{cd} + \alpha \kappa \tilde{\psi}_\mu \Gamma_5 \psi_\rho \right]$$

$$\times \ dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma,$$  

(5.48)

where \(\alpha\) is some constant introduced for later convenience, but here \(\alpha = 1\). Substituting eqs. (5.47) into (5.48), one obtains that

$$I_5^\varepsilon = \pm k \int_{M_4} \left[ \frac{1}{4} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} + 2 \kappa R_{\mu\nu}^{ab} (\bar{\psi}_\mu \Gamma^{cd} \psi_\sigma) + \kappa^2 (\bar{\psi}_\mu \Gamma^{ab} \psi_\nu) (\bar{\psi}_\rho \Gamma^{cd} \psi_\sigma) \right] dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

$$+ \frac{4}{\kappa^2} \int_{M_4} d^4x \sqrt{g} e^{a}_\mu e^{b}_\nu (R_{\mu\nu}^{ab} + \kappa (\bar{\psi}_\mu \Gamma^{ab} \psi_\nu))$$

$$+ 4 \alpha \kappa \int_{M_4} (D_\mu \bar{\psi}_\nu) \Gamma^5 (D_\rho \psi_\sigma) \ dx^\mu \wedge \cdots \wedge dx^\sigma$$

$$+ 4 \alpha \int_{M_4} (\bar{\psi}_\mu \Gamma_\nu \Gamma^5 D_\rho \psi_\sigma) \ dx^\mu \wedge \cdots \wedge dx^\sigma$$

$$+ \frac{2 \alpha}{\kappa} \int_{M_4} d^4x \sqrt{g} \bar{\psi}_\mu \Gamma^{\mu\nu} \psi_\nu + \frac{24}{\kappa^4} \int_{M_4} d^4x \sqrt{g}\right\},$$  

(5.49)

where

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \psi_\nu.$$

After Fierz reshuffling, the term quartic in the gravitino field \(\psi_\mu\) disappears. The remaining terms describe massive supergravity with a Gauss-Bonnet term. It is an interesting fact that the action (5.48), with \(\alpha = 2\) (!), is invariant under the same supersymmetry transformation obtained form the variation of \(\Pi(\rho)\), except for \(\delta \omega_\mu^{ab}\) which is chosen to preserve the constraint [18]:

$$F_{\mu\nu}^a = 0.$$  

(5.50)

The supersymmetry transformations can be read by substituting (5.10) into eq. (3.18):

$$\delta e^a_\mu = \kappa \bar{\varepsilon} \Gamma^{a} \psi_\mu,$$

$$\delta \psi_\mu = (\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} + \frac{1}{2\kappa} e^{a}_\mu \Gamma^{a}) \varepsilon,$$  

(5.51)

and, for \(F_{\mu\nu}^{ab}\) and \(\psi_{\mu\nu}\), they are

$$\delta F_{\mu\nu}^{ab} = \kappa \bar{\varepsilon} \Gamma^{ab} \psi_{\mu\nu}^a,$$

$$\delta \psi_{\mu\nu} = - \frac{1}{4} F_{\mu\nu}^{ab} \left( \Gamma_{ab} \varepsilon \right).$$  

(5.52)
When $\alpha = 2$ the action (5.48) becomes invariant under the transformations (5.51) with the constraint (5.50), and the action corresponds to de Sitter supergravity where the cosmological constant and the gravitino mass-like term are fixed with respect to each other. In this case the action (5.48) simplifies to

$$I_{sg} = -\left[ \int_{M_4} d^4 x \varepsilon^{\mu\nu\rho\sigma} \left( \frac{1}{4} \varepsilon_{abcd} R^{ab}_{\mu\nu} R_{cd}^{\rho\sigma} + 8 \bar{\psi}_\mu \Gamma_5 \Gamma_\nu D_\rho \psi_\sigma \right) + 4 \int d^4 x e \left( e^a_\mu e^b_\nu R^{ab}_{\mu\nu} + \frac{2}{\kappa} \bar{\psi}_\mu \Gamma_\nu \psi_\nu + \frac{6}{\kappa^4} \right) \right].$$

(5.52)

The first term in (5.52) is a topological invariant and can be removed from the action without affecting its invariance. After rescaling

$$e_\mu^a \rightarrow r e_\mu^a,$$
$$\psi_{\mu\alpha} \rightarrow \sqrt{r} \psi_{\mu\alpha},$$
$$I_{sg} \rightarrow 8r^2 I_{sg},$$

(5.53)

and taking the limit $r \rightarrow 0$ the action (5.52) reduces to that of $N=1$ supergravity [19]:

$$I_{sg} = -\frac{1}{2\kappa^2} \int_{M_4} d^4 x e e^a_\mu e^b_\nu R^{ab}_{\mu\nu} - \int_{M_4} d^4 x \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \Gamma_5 \Gamma_\nu D_\rho \psi_\sigma.$$ 

(5.54)

The significance of the constraint (5.50) and the choice $\alpha = 2$ in the non-commutative construction is not clear to us. It would be helpful to better understand this point.

If we had worked instead with the solution (5.44), then additional terms which are dynamically trivial will be present. We shall not present the details for this case.

6. Conclusions and outlook.

In this paper, we have shown how to construct Chern-Simons forms and Chern-Simons actions in real, non-commutative geometry; (more detailed results will appear in [12]). We have illustrated the general, mathematical results of Sect. 2 by discussing a number of examples. These examples involve non-commutative spaces described by *algebras of matrix-valued functions over even-dimensional spin manifolds. As expected, the Chern-Simons actions associated with these spaces are manifestly topological (metric-independent). By imposing constraints on the field configurations on which these action functionals depend (and choosing convenient gauges) we have been able to derive the metric-independent, first-order formulation of four-dimensional gravity theory from a Chern-Simons action over a “five-dimensional” non-commutative space. By extending the mathematical framework, formally, to allow for graded algebras, we have also recovered an action functional for supergravity.
It would appear to be of interest to study Chern-Simons actions for more general non-commutative spaces, e.g. those considered in [8], and to derive from them theories of interest to physics. In this regard, one should recall that a rather profound theory has the form of a Chern-Simons theory: Witten’s open string field theory [15]. We are presently attempting to formulate that theory within Connes’ mathematical framework of non-commutative geometry, using a variant of the formalism developed in Sect. 2.

On the mathematical side, it appears to be of interest to better understand the topological nature of Chern-Simons actions over general non-commutative spaces, to understand the connection between the material presented in Sect. 2 and the theory of characteristic classes in non-commutative geometry and cyclic cohomology, see [3,10], and, most importantly, to learn how to quantize Chern-Simons theories in non-commutative geometry, in order to construct new topological field theories.

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