Hawking radiation as quantum tunneling from a noncommutative Schwarzschild black hole

Kourosh Nozari\textsuperscript{1,2} and S Hamid Mehdipour\textsuperscript{1,3}

\textsuperscript{1} Department of Physics, Faculty of Basic Sciences, University of Mazandaran, PO Box 47416-1467, Babolsar, Iran
\textsuperscript{2} Research Institute for Astronomy and Astrophysics of Maragha, PO Box 55134-441, Maragha, Iran
\textsuperscript{3} Islamic Azad University, Lahijan Branch, PO Box 1616 Lahijan, Iran

E-mail: knozari@umz.ac.ir and h.mehdipour@umz.ac.ir

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Abstract
We study the tunneling process through the quantum horizon of a Schwarzschild black hole in noncommutative spacetime. This is done by considering the effect of smearing of the particle mass as a Gaussian profile in flat spacetime. We show that even in this noncommutative setup there will be no correlation between the different modes of radiation, which reflects the fact that information does not come out continuously during the evaporation process at least at late time. However, due to spacetime noncommutativity, information might be preserved by a stable black hole remnant.

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1. Introduction

In 1975, Hawking proposed a scenario in which a black hole can radiate from its event horizon as a black body with a purely thermal spectrum at the temperature $T_H = \frac{\hbar c^3}{2\pi^2 \kappa G}$, utilizing the procedure of quantum field theory in curved spacetime ($\kappa$ is the surface gravity that demonstrates the strength of the gravitational field near the black hole surface). This leads us to a non-unitary quantum evolution which maps a pure state to a mixed state. In 2000, Parikh and Wilczek [2] proposed the method of null-geodesic to derive the Hawking temperature as a quantum tunneling process. In this quantum tunneling framework, the form of the corrected radiation is not exactly thermal which yields a unitary quantum evolution. However, their form of the correction for emission is not adequate by itself to retrieve information since it fails to find the correlations between the emission rates of different modes in the black hole radiation spectrum. Possibly, spacetime noncommutativity [3–5], that is, an inherent trait of the manifold by itself and the fact that spacetime points might be noncommutative, opens the way for finding a solution to the black hole information paradox that can be solved by the black
hole ceasing to decay beyond a minimal mass $M_0$. In 2003 Smailagic and Spallucci [6–8] postulated a new attractive model of noncommutativity in terms of coherent states, which satisfies Lorentz invariance, unitarity and UV finiteness of quantum field theory. In 2005, Nicolini, Smailagic and Spallucci (NSS) [9] by using this method found the generalized line element of Schwarzschild spacetime based on the coordinate coherent state noncommutativity. It has been shown that the generalized line element does not permit the black hole to decay lower than $M_0$. Thus, the evaporation process finishes when the black hole approaches a Planck size remnant with zero temperature, which does not diverge at all, rather reaches a maximum value before shrinking to absolute zero. Since spacetime noncommutativity can eliminate some kind of divergences (which appear in general relativity), and is also an intrinsic property of the manifold itself (even in the absence of gravity), we hope to cure a step further and modify the tunneling paradigm utilizing the noncommutative field theory. In this manner, we would like to proceed with the Parikh–Wilczek tunneling process by using a fascinating formulation of noncommutativity of coordinates which is carried out by the Gaussian distribution of coherent states.

2. The noncommutative Schwarzschild black hole

A valuable test of spacetime noncommutativity is its possibly observable effects on the properties of black holes. To inquire into this issue, one requires prosperously building the noncommutativity corresponding to the general relativity. Although this issue has been considered in the literature [10], no perfect and wholly convincing theory of this model yet exists. There are many formulations of noncommutative field theory established upon the Weyl–Wigner–Moyal $\ast$-product [11] which lead to a downfall in finding a solution to some prominent difficulties, such as Lorentz invariance breaking, defeat of unitarity and UV divergences of quantum field theory. The incidence of noncommutativity at an observable scale has the ability to lead to important effects in the expected properties of the black holes. Although a perfect noncommutative theory of gravity does not yet exist, it becomes essential to model the noncommutativity effects in the frame of the commutative general relativity. Recently, the authors in [6–8] have regarded a physically inspired and obedient type of the noncommutativity amendments to Schwarzschild black hole solutions (coordinate coherent state formalism), which can be released from the difficulties mentioned above. In this formalism, general relativity in its common commutative case as characterized by the Einstein–Hilbert action remains appropriate. If noncommutativity effects can be made to behave in a perturbative manner, then this becomes defensible, at least to a good approximation. The authors in [10] have really demonstrated that the leading noncommutativity amendments to the form of the Einstein–Hilbert action are at least of second order in the noncommutativity parameter, $\theta$. Interestingly, the generalization of quantum field theory by noncommutativity based on the coordinate coherent state formalism also cures the short distance behavior of point-like structures [6–9] (see also [12]). In this formalism, the particle mass $M$, instead of being completely localized at a point, is distributed throughout a region of linear size $\sqrt{\theta}$. The implementation of this argument leads to the substitution of a position Dirac-delta function (which describes point-like structures) with a Gaussian profile describing smeared structures. On the other hand, the mass density of a static, spherically symmetric, particle-like gravitational source cannot be a delta function distribution, but will be given by a Gaussian distribution of minimal width $\sqrt{\theta}$ as follows,

$$\rho_\theta(r) = \frac{M}{(4\pi \theta)^{\frac{3}{2}}} \exp \left(-\frac{r^2}{4\theta}\right).$$  (1)
The Schwarzschild solution of the Einstein equations associated with these smeared mass Gaussian function sources leads to the line element

\[ ds^2 = -\left(1 - \frac{2M_\theta}{r}\right)dt^2 + \left(1 - \frac{2M_\theta}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \]  

(2)

where the smeared mass distribution is implicitly given in terms of the lower incomplete Gamma function as

\[ M_\theta = \int_0^r \rho_\theta(r)4\pi r^2 \, dr = \frac{2M}{\sqrt{\pi}} \Gamma\left(3 + \frac{r^2}{4\theta}\right) = \frac{2M}{\sqrt{\pi}} \int_0^{\frac{r}{2\sqrt{\pi}}} t^2 e^{-t} \, dt. \]  

(3)

(Hereafter we set the fundamental constants equal to unity; \(\hbar = c = k_B = 1\).) In the limit of \(\theta \to 0\), one recovers the complete Gamma function \(\Gamma\left(\frac{3}{2}\right)\),

\[ \lim_{\theta \to 0} M_\theta = M, \]  

(4)

and the modified Schwarzschild solution reduces to the ordinary Schwarzschild solution. The line element (2) characterizes the geometry of a noncommutative inspired Schwarzschild black hole. The radiating behavior of such a modified Schwarzschild black hole can now be investigated and can easily be shown by plotting \(g_{00}\) as a function of \(r\), for different values of \(M\) (hereafter, for plotting the figures we set the value of the noncommutativity parameter equal to unity; \(\theta = 1\)). Figure 1 shows that the coordinate noncommutativity leads to the existence of a minimal non-zero mass that a black hole (due to Hawking radiation and evaporation) can...
shrink to. The event horizon of this line element can be found where \( g_{00}(r_H) = 0 \), which is implicitly written in terms of the upper incomplete Gamma function as

\[
r_H = 2M_0(r_H) = 2M \left(1 - \frac{2}{\sqrt{\pi}} \Gamma \left(\frac{3}{2}, \frac{r_H^2}{4\theta}\right)\right) .
\]  
(5)

The noncommutative Schwarzschild radius versus the mass can approximately be calculated by setting \( r_H = 2M \) into the upper incomplete Gamma function as

\[
r_H = 2M \left(\mathcal{E} \left(\frac{M}{\sqrt{\theta}}\right) - \frac{2M}{\sqrt{\pi} \theta} \exp \left(-\frac{M^2}{\theta}\right)\right),
\]  
(6)

where \( \mathcal{E}(x) \) shows the Gauss error function defined as

\[
\mathcal{E}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt .
\]

For very large masses, \( \mathcal{E} \left(\frac{M}{\sqrt{\theta}}\right) \) tends to unity and the second term on the right will exponentially be reduced to zero and one retrieves the classical Schwarzschild radius, \( r_H \approx 2M \).

When such a noncommutative black hole radiates, its temperature can be calculated to find

\[
T_H = \frac{1}{4\pi} \frac{d g_{00}}{d M} \bigg|_{r_H=r_H} = M \left[\frac{\mathcal{E} \left(\frac{r_H^2}{2\theta}\right)}{2\pi r_H^2} - \exp \left(-\frac{r_H^2}{\theta}\right) \left(\frac{r_H + 2\theta}{r_H}\right)\right].
\]  
(7)

For the commutative case, \( \frac{M}{\sqrt{\theta}} \to \infty \), one recovers the classical Hawking temperature, \( T_H = \frac{1}{4\pi M} \). The numerical calculation of the modified Hawking temperature as a function of the mass is presented in figure 2. In this modified (noncommutative) version, not only does \( T_H \) not diverge at all but it also reaches a maximum value before dropping to absolute zero at the minimal non-zero mass, \( M = M_0 \approx 1.9 \), that the black hole shrinks to.

To find the analytical form of the modified (noncommutative) entropy, \( S_{NC} \), we should note that our calculation for obtaining the modified Hawking temperature, equation (7), is exact and no approximation has been made. But there is no analytical solution for entropy from the first law of classical black hole thermodynamics \( dM = T_H \, dS \) with \( T_H \) given as (7), even if we set \( r_H = 2M \) in this relation. Nevertheless, to obtain an approximate analytical form of entropy we can use the following expression as an approximation to the noncommutative Hawking temperature

\[
T_H = \frac{1}{4\pi r_H} ,
\]  
(8)

where \( r_H \) is given by equation (6). Eventually, the entropy of the black hole can be obtained in analytical form using the first law,

\[
S_{NC} = \int \frac{dM}{T_H} = 2\pi \int \frac{dM}{\kappa(M)} = 4\pi M^2 \mathcal{E} \left(\frac{M}{\sqrt{\theta}}\right) - 6\pi \theta \mathcal{E} \left(\frac{M}{\sqrt{\theta}}\right) + 12\sqrt{\pi} \theta M \exp \left(-\frac{M^2}{\theta}\right),
\]  
(9)

where \( \kappa(M) \) is the noncommutative surface gravity at the horizon and is given by

\[
\kappa(M) = \left[4M \left(\mathcal{E} \left(\frac{M}{\sqrt{\theta}}\right) - \frac{2M}{\sqrt{\pi} \theta} \exp \left(-\frac{M^2}{\theta}\right)\right)\right]^{-1}.
\]  
(10)

Behavior of the entropy \( S_{NC} \) as a function of the mass is depicted in figure 3. As this figure shows, at the final stage of the black hole evaporation, the black hole ceases to radiate and its entropy reaches zero and the existence of a minimal non-zero mass is clear again. In the large mass limit, i.e., \( \frac{M}{\sqrt{\theta}} \gg 1 \), one recovers the standard Bekenstein–Hawking entropy plus...
Figure 2. Black hole temperature, $T_H$, as a function of $M$. The existence of a minimal non-zero mass and disappearance of divergence are clear.

Figure 3. Black hole entropy, $S_{NC}$, as a function of $M$. Note that the figure is plotted approximately by equation (24).

$\theta$-corrections, which leads to

$$S_{NC} = 4\pi M^2 + 12\sqrt{\pi\theta}M \exp\left(-\frac{M^2}{\theta}\right).$$

(11)
We set \( \theta' = 1 \) (which is not exactly the same as \( \theta = 1 \)). This figure is the same as figure 1 with the feasibility of having extremal configuration with one degenerate event horizon when \( M = M_0 \approx 2.2 \) (i.e., the existence of a minimal non-zero mass), and no event horizon when the mass of the black hole is smaller than \( M_0 \). Also as the figure again displays, the distance between the horizons will grow with increasing black hole mass (two event horizons) in the same way as the Gaussian profile. These features show the lack of responsiveness of these consequences to the Gaussian formalism of the smearing.

It should be noted that if we had picked out a different form for the probability of matter distribution, instead of distribution (1), solely the smeared mass distribution \( M_\theta \) would be altered, however the general properties would be directed to entirely comparable consequences to those above. For instance, we consider a Lorentzian distribution of the smeared particle

\[
\rho_{\theta'}(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta'^2)}. \tag{12}
\]

Here the noncommutativity parameter, \( \theta' \), is actually not identical to \( \theta \). The smeared mass distribution is now given by

\[
M_{\theta'} = \int_0^r \rho_{\theta'}(r) 4\pi r^2 \, dr = \frac{2M}{\pi} \left( \tan^{-1} \left( \frac{r}{\sqrt{\theta'}} \right) - \frac{r \sqrt{\theta'}}{(r^2 + \theta'^2)} \right). \tag{13}
\]

In the limit of \( \theta' \) going to zero, we get \( M_{\theta'} \rightarrow M \). As expected, the smeared mass Lorentzian distribution, \( M_{\theta'} \), has the same limiting properties and is completely comparable to the smeared mass Gaussian distribution, \( M_\theta \), qualitatively. Then, many of the outcomes that we achieved remain applicable if we take the other kind of probability distribution (see [13]). The lack of responsiveness of these consequences to our Gaussian formalism of the smearing can easily be exhibited by plotting \( g_{00} \) as a function of \( r \) for different values of \( M \) (see figure 4). We set \( \theta' = 1 \) (which is not equivalent to \( \theta = 1 \)). Comparing these results with the results of figure 1 demonstrates the close similarity of consequences in these two setups at least to...
asymptotic values \( r \). In both situations a minimum \( g_{00} \) occurs at a comparatively small \( r \) with slightly comparable values of \( M \). The \( M_0 \) value is seen to be entirely similar in the two situations. The pre-eminent distinction in the two approaches is comprehended to take place around less than \( M_0 \) where there is mainly the responsiveness to noncommutativity influences and the detailed form of the matter distribution. However, we actually should not have confidence in the details of our modeling when \( r \) is excessively small. In fact, in the area where noncommutativity effects precisely begin to be sensed, the detailed nature of the sharpened mass distribution is not actually explored. In a recent paper [14], we have reported some outcomes about the extraordinary thermodynamical treatment for the Planck-sized black hole evaporation, i.e., when \( M \) is less than \( M_0 \). In this way, one encounters some uncommon thermodynamical features going to negative entropy, negative temperature and abnormal heat capacity where the mass of the black hole becomes of the order of Planck mass or less. It is also in this extreme situation that the majority of the distinctions between, e.g., the Gaussian, Lorentzian or some other forms of the smeared mass distribution would be anticipated to begin to appear. Therefore, we will henceforth use the Gaussian-smeared mass distribution in our calculations just under the circumstance that \( M \geq M_0 \).

3. Quantum tunneling near the horizon

We are now ready to discuss the quantum tunneling process in the noncommutative framework. In accordance with [17], one can express the general spherically-symmetric line element in the form

\[
d s^2 = -[N_t(t, r) \, dt]^2 + L(t, r)^2 \left[ dr + N_r(t, r) \, dt \right]^2 + R(t, r)^2 \, d\Omega^2.
\]  

(14)

When we insert this expression into the Einstein–Hilbert action, due to some restrictions and advantages, e.g., no time derivative in the action, invariance of the action under reparametrization and no singular behavior at the horizon, one finds

\[
\begin{align*}
N_t(t, r) &= N_t(r) \\
N_r(t, r) &= N_r(r) \\
L(t, r) &= r \\
R(t, r) &= 1.
\end{align*}
\]

To describe the noncommutative quantum tunneling process where a particle moves in dynamical geometry and passes through the horizon without singularity on the path, we should use a coordinate system that, unlike Schwarzschild coordinates, is not singular at the horizon. These simple choices for \( L \) and \( R \) mentioned above (first indicated by Painlevé\(^4\)) can prepare this purpose. Thus for a noncommutative Schwarzschild solution one can easily acquire

\[
\begin{align*}
N_t(r) &= 1 \\
N_r(r) &= \sqrt{\frac{2M}{r}}.
\end{align*}
\]

\(^3\) Note that the foundation of these propositions possibly occurs as a result of the fractal nature of spacetime at very short distances. Theories such as \( E \)-infinity [15] and scale relativity [16] which are at the basis of fractal structure of spacetime at very short distances may prepare an appropriate framework to treat thermodynamics of these very short distance systems.

\(^4\) Painlevé coordinates [18] are especially convenient choices, which are obtained by definition of a new noncommutative time coordinate, \( dt = dt_s + \frac{i}{\sqrt{2M}} \, dr \), where \( t_s \) is the Schwarzschild time coordinate. Note that only the Schwarzschild time coordinate is transformed. Both the radial coordinate and angular coordinates remain the same.
The noncommutative line element now immediately reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2M_{\theta}}{r}\right)\mathrm{d}r^2 + 2\sqrt{\frac{2M_{\theta}}{r}}\mathrm{d}r\mathrm{d}r + r^2\mathrm{d}\Omega^2. \quad (15)$$

The metric in these new coordinates is now stationary, non-static, and there are neither coordinate nor intrinsic singularities (due to noncommutativity). The equation of motion for a massless particle (the radial null geodesic) is $\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}t} = \pm N_t - N_r$, where the upper sign (lower sign) corresponds to an outgoing (ingoing) geodesic respectively. Since the horizon, $r = r_H$, is concluded from the condition $N_t(r_H) - N_r(r_H) = 0$, in the vicinity of the horizon $N_t - N_r$ is considered as

$$N_t - N_r \simeq (r - r_H)\kappa(M) + O((r - r_H)^2). \quad (16)$$

If we suppose that $r$ increases toward the future, then the above equations should be modified by the particle’s self-gravitation effect. Kraus and Wilczek [19] studied the motion of particles in the s-wave as spherical massless shells in dynamical geometry and developed self-gravitating shells in Hamiltonian gravity. Further elaborations were performed by Parikh and Wilczek [2]. On the other hand, Shankaranarayanan et al have applied the tunneling approach to obtain the Hawking temperature in different coordinates within a complex paths scenario [20]. This technique has been successfully applied to obtain a global temperature for multi-horizon spacetimes [21]. In this paper, we are going to develop the Parikh–Wilczek method for the noncommutative coordinate coherent states.

We keep the total ADM mass $M$ of the spacetime fixed, and allow the hole mass to fluctuate, due to the fact that we take into account the response of the background geometry to an emitted quantum of energy $E$ which moves in the geodesics of a spacetime with $M$ replaced by $M - E$. Thus we should replace $M$ by $M - E$ both in equations (15) and (16).

Since the characteristic wavelength of the radiation is always haphazardly small near the horizon due to the infinite blueshift there, the wavenumber reaches infinity and therefore the WKB approximation is reliable close to the horizon. In the WKB approximation, the probability of tunneling or emission rate for the classically forbidden region as a function of the imaginary part of the particle’s action at a stationary phase would take the following form:

$$\Gamma \sim \exp(-2\mathrm{Im}I). \quad (17)$$

To calculate the imaginary part of the action we consider a spherical shell consisting of component massless particles each of which travels on a radial null geodesic. We use these radial null geodesics like an s-wave outgoing positive energy particle which passes through the horizon outward from $r_{in}$ to $r_{out}$ to compute $\mathrm{Im}I$, as follows (under the condition that $r_{in} > r_{out}$, where we should have: $r_{in} = \frac{4M}{\sqrt{M - E}}\gamma(\frac{3}{2}, \frac{r_H^2}{4M})$ and $r_{out} = \frac{4(M - E)}{\sqrt{M - E}}\gamma(\frac{3}{2}, \frac{r_H^2}{4M})$):

$$\mathrm{Im}I = \mathrm{Im}\int_{r_{in}}^{r_{out}} p_r \mathrm{d}r = \mathrm{Im}\int_{r_{in}}^{r_{out}} \int_0^{p_r} \mathrm{d}p_r \mathrm{d}r. \quad (18)$$

One can alter the integral variable from momentum in favor of energy by using Hamilton’s equation $\dot{r} = \frac{\partial H}{\partial p_r}$, where the Hamiltonian is $H = M - E'$. We now evaluate the integral without writing the explicit form for the radial null geodesic. The $r$ integral can be done first by deforming the contour,

$$\mathrm{Im}I = \mathrm{Im}\int_M^{M - E} \int_{r_{in}}^{r_{out}} \frac{\mathrm{d}r}{\dot{r}} \mathrm{d}H = -\mathrm{Im}\int_0^{E} \int_{r_{in}}^{r_{out}} \frac{\mathrm{d}E'}{(r - r_H)\kappa(M - E')}. \quad (19)$$

The $r$ integral has a pole at the horizon which lies along the line of integration and this yields $(-\pi i)\times$ the residue. Therefore,

$$\mathrm{Im}I = \pi \int_0^{E} \frac{\mathrm{d}E'}{\kappa(M - E')}. \quad (20)$$
Here, reutilizing the first law of black hole thermodynamics, \( dM = \frac{\kappa}{2\pi} dS \), one can find the imaginary part of the action as \(^{22}\)

\[
\text{Im} I = -\frac{1}{2} \int_{S_{NC}(M)}^{S_{NC}(M-E)} dS = -\frac{1}{2} \Delta S_{NC}.
\] (21)

Hawking radiation as tunneling from the black hole event horizon was also investigated in the context of string theory \(^{22}\), and it was exhibited that the emission rates in the high energy correspond to a difference between counting of states in the microcanonical and canonical ensembles. In fact, the emission rates in the tunneling approach just to first order in \( E \) replace the Boltzmann factor in the canonical ensemble \( \Gamma \sim \exp(-\beta E) \), which is described by the inverse temperature as the coefficient \( \beta \). So, the emission rates in the high energy are proportional to \( \exp(\Delta S) \) (see also \(^{23}\)),

\[
\Gamma \sim \exp(-2 \text{Im} I) \sim \frac{e^{\Delta S_{NC}}}{e^{\Delta S}} = \exp(\Delta S) = \exp[S(M-E) - S(M)].
\] (22)

where \( \Delta S \) is the difference in black hole entropies before and after emission. In other words, at higher energies the emission probability depends on the final and initial numbers of microstates available to the system. Thus, at higher energies the emission spectrum cannot be precisely thermal due to the fact that the high energy corrections arise from the physics of energy conservation with noncommutativity corrections. In this model, one takes into account the back-reaction results in a finite separation between the initial and final radius as a result of self-gravitation effects of outgoing shells, which is the classically forbidden trajectory, i.e., the barrier. On the other hand, according to energy conservation the tunneling barrier is produced by a change in the radius (the decrease in the black hole horizon) just by the emitted particle itself.

Let us now insert our result for noncommutative black hole entropy, equation (9), into the above equation and write the new noncommutative-corrected tunneling probability as follows:

\[
\Gamma \sim \exp(\Delta S_{NC}) = \exp[S_{NC}(M-E) - S_{NC}(M)]
\]

\[
= \exp\left(\frac{4\pi}{(M-E)^2} - \frac{3}{2} \frac{\theta}{\sqrt{\theta}} \right) e^{\left(\frac{M-E}{\sqrt{\theta}}\right)} + 12\sqrt{\pi} \theta (M-E) \exp\left(-\frac{(M-E)^2}{\theta}\right)
\]

\[
- 4\pi \left[ M^2 - \frac{3}{2} \frac{\theta}{\sqrt{\theta}} \right] e^{\left(\frac{M}{\sqrt{\theta}}\right)} - 12\sqrt{\pi} \theta M \exp\left(-\frac{M^2}{\theta}\right).
\] (23)

It is simply observed that to linear order in \( E \), two expressions for \( \Gamma \) in the microcanonical and canonical ensembles coincide. So, manifestly the emission rate (23) deviates from the pure thermal emission but is consistent with an underlying unitary quantum theory \(^{24}\). We must note that the tunneling probability can also be derived by writing out the explicit metric in the tunneling computation, which would lead to the same result in spite of more complicated calculations.

At this stage, we want to demonstrate that there are no correlations between emitted particles even with the inclusion of the noncommutativity corrections at least at late-times. (However, there might be short-time correlations between the quanta emitted earlier and the quanta emitted later on which decay to zero after the black hole is equilibrated at late times.) This means it can be exhibited that the probability of tunneling of two particles of energy \( E_1 \) and \( E_2 \) is precisely similar to the probability of tunneling of one particle with their compound energies, \( E = E_1 + E_2 \), i.e.

\[
\Delta S_{E_1} + \Delta S_{E_2} = \Delta S_{E_1 + E_2} \Rightarrow \chi(E_1 + E_2; E_1, E_2) = 0,
\] (24)

\[ \]
where the emission rate for the first quantum emitted, $E_1$, yields
\[
\Delta S_{E_1} = \ln \Gamma_{E_1} = 4\pi \left[ \left( M - E_1 \right)^2 - \frac{3}{2} \theta \right] \mathcal{E} \left( \frac{M - E_1}{\sqrt{\theta}} \right) + 12\sqrt{\pi \theta} (M - E_1) \exp \left( -\frac{(M - E_1)^2}{\theta} \right)
- 4\pi \left[ M^2 - \frac{3}{2} \theta \right] \mathcal{E} \left( \frac{M}{\sqrt{\theta}} \right) - 12\sqrt{\pi \theta} M \exp \left( -\frac{M^2}{\theta} \right).
\]

(25)

and similarly the emission rate for the second quantum emitted, $E_2$, is given by
\[
\Delta S_{E_2} = \ln \Gamma_{E_2} = 4\pi \left[ \left( (M - E_1) - E_2 \right)^2 - \frac{3}{2} \theta \right] \mathcal{E} \left( \frac{(M - E_1) - E_2}{\sqrt{\theta}} \right)
+ 12\sqrt{\pi \theta} ((M - E_1) - E_2) \exp \left( -\frac{(M - E_1)^2 - (E_1 - E_2)^2}{\theta} \right)
- 4\pi \left[ (M - E_1)^2 - \frac{3}{2} \theta \right] \mathcal{E} \left( \frac{M - E_1}{\sqrt{\theta}} \right)
- 12\sqrt{\pi \theta} (M - E_1) \exp \left( -\frac{(M - E_1)^2}{\theta} \right).
\]

(26)

Finally, the emission rate for a single quantum emitted with the same total energy, $E$, is given by
\[
\Delta S_{(E_1+E_2)} = \ln \Gamma_{(E_1+E_2)} = 4\pi \left[ \left( M - (E_1 + E_2) \right)^2 - \frac{3}{2} \theta \right] \mathcal{E} \left( \frac{M - (E_1 + E_2)}{\sqrt{\theta}} \right)
+ 12\sqrt{\pi \theta} (M - (E_1 + E_2)) \exp \left( -\frac{(M - (E_1 + E_2))^2}{\theta} \right)
- 4\pi \left[ M^2 - \frac{3}{2} \theta \right] \mathcal{E} \left( \frac{M}{\sqrt{\theta}} \right) - 12\sqrt{\pi \theta} M \exp \left( -\frac{M^2}{\theta} \right).
\]

(27)

It can be easily confirmed that these probabilities of emission are uncorrelated. On the other hand, the statistical correlation function, $\chi(E; E_1, E_2)$, is zero which leads to independence of different modes of radiation during evaporation. Hence, in this method the form of the corrections to back-reaction effects even with inclusion of noncommutativity effects is not adequate in itself to retrieve information because there are no correlations between different modes at least at late times and information does not come out with the Hawking radiation (for reviews on resolving the so-called information loss paradox, see [25–27]). Nevertheless, the noncommutativity effect is adequate by itself to preserve information due to the fact that in the noncommutative framework the black hole does not evaporate completely and this leads to the existence of a minimal non-zero mass (e.g., a Planck-sized remnant containing information) that the black hole can reduce to. So information might be preserved in this remnant.

In string theory and loop quantum gravity, the entropy of black hole has been achieved by direct microstate counting as follows (in units of the Planck scale):
\[
S_{QG} = 4\pi M^2 + \alpha \ln(16\pi M^2) + O \left( \frac{1}{M^2} \right).
\]

(28)
It was recently suggested by the authors of [28] that the Planck scale corrections to the black hole radiation spectrum via tunneling can be written as

$$
\Gamma \sim \exp(\Delta S_{QG}) = \exp(S_{QG}(M - E) - S_{QG}(M)) = \left(1 - \frac{E}{M}\right)^{2\alpha} \exp\left(-8\pi ME \left(1 - \frac{E}{2M}\right)\right).
$$

(29)

Since loop quantum gravity anticipates a negative value for $\alpha$ (see, e.g., [29]) which yields a diverging emission rate if $E \to M$, this leads to non-suppression of the black hole emission (although the suppression can only occur when $\alpha > 0$, which is not recommended at least by loop quantum gravity). But our outcome is actually sensible, comparing the noncommutative result for the emission rate (equation (23)) with the quantum gravity result (equation (29)) shows that the noncommutative result is reasonably successful in ceasing the black hole emission when $(M - E) \to M_0$. In fact, the cases $(M - E) < M_0$ are the noncommutativity-forbidden regions that the tunneling particle cannot traverse through. Therefore, the limit $(M - E) \to 0$ cannot be applied by our process because of the existence of non-vanishing mass at the final phase of black hole evaporation.

4. Summary and remarks

We summarize this paper with some remarks. In this paper, generalization of the standard Hawking radiation via tunneling through the event horizon based on the solution of equation (17) within the context of coordinate coherent state noncommutativity has been studied and then the new corrections of the emission rate based on spacetime noncommutativity have been achieved. In this study, we see that there are no correlations between the tunneling rates of different modes in the black hole radiation spectrum even in the noncommutative framework at least at late times. In our opinion, if we really believe the idea of stable black hole remnants due to the fact that there are some exact continuous global symmetries in nature [30], then we should accept that the information remains inside the black hole and can be retained by a stable Planck-size remnant. In principle, there are four main outcomes of the black hole evaporation:

- the black hole can evaporate completely, and information would disappear from our world.
- The black hole can completely disappear, but information emerges in the final burst of radiation when the black hole shrinks to the Planck size.
- There are correlations between different modes of radiation during the evaporation such that information appears continuously through them.
- The black hole never disappears completely, and information would be preserved in a stable black hole remnant.

Indeed, it is not conceivable to date to give a clear answer to the question of the black hole information paradox and this is reasonable because there is no complete self-consistent quantum theory of evaporating black holes (and gravity). In this paper, we have studied the reliability of the fourth conjecture within a noncommutative framework. We have shown that although there is no correlation between the tunneling rates of different modes in the black hole radiation spectrum in noncommutative spacetime at least at late times, noncommutativity has the capability of overcoming the information loss paradox via existence of a stable black hole remnant. At this stage we should stress that there is another point of view on using relation (17). There is a problem here known as the ‘factor 2 problem’. Some authors such as Chowdhury [31]...
and Pilling [32] have argued that relation (17) is not invariant under canonical transformations but the same formula with a factor of 1/2 in the exponent is canonically invariant. As a final remark we emphasize that some authors have treated black hole thermodynamics in the noncommutative framework adopting a coordinate noncommutativity against the coherent state approach (see [33] and references therein). A question then arises: what is the difference between these two procedures? The standard way to handle noncommutative problems is through the use of the Weyl–Wigner–Moyal ⋆-product. This means using complex number commuting coordinates and shifting non-commutativity in the product between functions. This is mathematically correct, but it is physically useless since any model written in terms of star product, even the simplest field theory, becomes non-local and it is not obvious how to handle the non-local quantum field theory. One proposed approach is perturbation in the $\theta$ parameter [34]. This is physically sensible since once expanded up to a given order in $\theta$, the resulting field theory becomes local. The smeared picture of particles based on coordinate coherent states defines complex number coordinates as quantum mean values of the original non-commuting ones between coordinate coherent states. In other words, in this setup one can see commuting coordinates as a classical limit (in the quantum mechanical sense) of the non-commuting ones. In this framework, the emergent semi-classical geometry keeps memory of its origin. For example, free propagation of a point-like object is described by a minimal width Gaussian wavepacket as has been considered in our setup. So, the difference between two approaches lies in the definition of quantum field theoretical propagators.

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Note added. After we completed this work, Banerjee et al reported a similar treatment of the problem [35].

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