1. Introduction and Results

Let $G$ be any group and $n$ a non-negative integer. For any two elements $a$ and $b$ of $G$, we define inductively $[a, b^n]$ the $n$-Engel commutator of the pair $(a, b)$, as follows:

$$[a, 0] := a, \quad [a, b] := [a, 1] := a^{-1}b^{-1}ab \text{ and } [a, b^n] := [[a, b^{n-1}], b^n]$$

for all $n > 0$.

An element $x$ of $G$ is called right (left, resp.) $n$-Engel if $[x, g^n] = 1$ ($[g, x^n] = 1$, resp.) for all $g \in G$. We denote by $R_n(G)$ ($L_n(G)$, resp.) the set of all right (left, resp.) $n$-Engel elements of $G$. A group $G$ is called $n$-Engel if $G = R_n(G)$ or equivalently $G = R_n(G)$. It is clear that $R_0(G) = 1$, $R_1(G) = Z(G)$ the center of $G$, and W.P. Kappe [2] (implicitly) proved $R_2(G)$ is a characteristic subgroup of $G$. L.C. Kappe and Ratchford [8] have shown that $R_3(G)$ is a subgroup of $G$ whenever $G$ is a metabelian group, or $G$ is a center-by-metabelian group such that $[\gamma_k(G), \gamma_j(G)] = 1$ for some $k, j \geq 2$ with $k + j - 2 \leq n$ and $n \geq 3$. Macdonald [9] has shown that the inverse or square of a right 3-Engel element need not be right 3-Engel. Nickel [15] generalized Macdonald’s result to all $n \geq 3$. In fact he constructed a group with a right $n$-Engel element $a$ neither $a^{-1}$ nor $a^2$ is a right $n$-Engel element. The construction of Nickel’s example was guided by computer experiments and arguments based on commutator calculus. Although Macdonald’s example shows that $R_3(G)$ is not in general a subgroup of $G$, Heineken [12] has already shown that if $A$ is the subset of a group $G$ consisting of all elements $a$ such that $a^{\pm 1} \in R_3(G)$, then $A$ is a subgroup if either $G$ has no element of order 2 or $A$ consists only of elements having finite odd order. Newell [13] proved that the
normal closure of every right 3-Engel element is nilpotent of class at most 3. In Section 2 we prove that if $G$ is a 2'-group, then $R_3(G)$ is a subgroup of $G$. Nickel’s example shows that the set of right 4-Engel elements is not a subgroup in general (see also first Example in Section 4 of [1]). In Section 3, we prove that if $G$ is a locally nilpotent $\{2,3,5\}'$-group, then $R_4(G)$ is a subgroup of $G$.

Traustason [17] proved that any locally nilpotent 4-Engel group $H$ is Fitting of degree at most 4. This means that the normal closure of every element of $H$ is nilpotent of class at most 4. More precisely he proved that if $H$ has no element of order 2 or 5, then $H$ has Fitting degree at most 3. Now by a result of Havas and Vaughan-Lee [4], one knows any 4-Engel group is locally nilpotent and so Traustason’s result is true for all 4-Engel groups. In Section 3, by another result of Traustason [18] we show that the normal closure of every right 4-Engel element in a locally nilpotent $\{2,3,5\}'$-group, is nilpotent of class at most 7.

Newman and Nickel [12] have shown that for every $n \geq 5$ there exists a nilpotent group $G$ of class $n + 2$ containing a right $n$-Engel element $a$ and an element $b$ such that $[b, n, a]$ has infinite order. As we mentioned above, Nickel [15] has shown that for every $n \geq 3$ there exists a nilpotent group of class $n + 2$ having a right $n$-Engel element $a$ and an element $b$ such that $[a^{-1}, n, b] = [a^2, n, b] \neq 1$. We have checked that the latter element in Nickel’s example is of finite order whenever $n \in \{5, 6, 7, 8\}$. In Section 4 using the group constructed by Newman and Nickel we show that there exists a nilpotent group $G$ of class $n + 2$ such that $x \in R_n(G)$ and both $[x^{-1}, n, a]$ and $[x^k, n, a]$ have infinite order for every integer $k \geq 2$.

In [1] the following question has been proposed:

**Question 1.1.** Let $n$ be a positive integer. Is there a set of prime numbers $\pi_n$ depending only on $n$ such that the set of right $n$-Engel elements in any nilpotent or finite $\pi_n'$-group forms a subgroup?

In Section 4 we negatively answers Question 1.1.

As far as we know there is no published example of a group whose set of (bounded) right Engel elements do not form a subgroup. But for the set of bounded left Engel elements there are some evidences supporting this idea that the subgroup-ness of bounded left Engel elements of an arbitrary group should be false. We finish the paper by proving that at least one of the following happens:

1. There is an infinite finitely generated $k$-Engel group of exponent $n$ for some positive integer $k$ and some 2-power number $n$.
2. There is a group generated by finitely many bounded left Engel elements which is not an Engel group, where by an Engel group we mean a group in which for every two elements $x$ and $y$, there exists an integer $k = k(x, y) \geq 0$ such that $[x, k, y] = 1$.

Throughout the paper we have frequently use GAP nq package of Werner Nickel. All given timings were obtained on an Intel Pentium 4-1.70GHz processor with 512 MB running Red Hat Enterprise Linux 5.
2. Right 3-Engel elements

Throughout for any positive integer $k$ and any group $H$, $\gamma_k(H)$ denotes the $k$th term of the lower central series of $H$. The main result of this section implies that $R_3(G)$ is a subgroup of $G$ whenever $G$ is a 2′-group. Newell [13] proved that

**Theorem 2.1.** Let $G = \langle a, b, c \rangle$ be a group such that $a, b \in R_3(G)$. Then

1. $\langle a, c \rangle$ is nilpotent of class at most 5 and $\gamma_5(\langle a, c \rangle)$ has exponent 2.
2. $G$ is nilpotent of class at most 6.
3. $\gamma_5(G)$ has exponent 10. Furthermore $[a, c, b, c, c]^2 \in \gamma_6(G)$.
4. $\gamma_6(G)$ has exponent 2.

**Theorem 2.2.** Let $G$ be a group such that $\gamma_5(G)$ has no element of order 2. Then $R_3(G)$ is a subgroup of $G$.

**Proof.** Let $a, b \in R_3(G)$ and $c \in G$. We first show that $a^{-1} \in R_3(G)$. We have

\[
[a^{-1}, c, c, c] = [[a, c, a^{-1}][a, c, c, c]] = [a, c, a^{-1}, c][a, c, c, c][a, c, c, c]
\]

Therefore by Theorem 2.1 (2), $a^{-1} \in R_3(G)$. On the other hand

\[
[ab, c, c, c] = [[a, c][a, b, c]][c, c, c]
\]

Now by Theorem 2.1 $[a, c, b, c, c]^2 \in \gamma_6(G)$ and thus $ab \in R_3(G)$. □

By Theorem 2.2, we know that $(x, y, z)$ is nilpotent if $x, y \in R_3(G)$ and $z \in G$. We now construct the largest nilpotent group $H = \langle a, b, c \rangle$ such that $a, b \in R_3(H)$ and $c \in H$, by nq package.

**Second Proof of Theorem 2.2** By Theorem 2.1 we know that $(x, y, z)$ is nilpotent if $x, y \in R_3(G)$ and $z \in G$. We now construct the largest nilpotent group $H = \langle a, b, c \rangle$ such that $a, b \in R_3(H)$ and $c \in H$, by nq package.

LoadPackage("nq"); # nq package of Werner Nickel
F:=FreeGroup(4); a1:=F.1; b1:=F.2; c1:=F.3; x:=F.4;
L:=F/[LeftNormedComm([a1,x,x,x]),LeftNormedComm([b1,x,x,x])];
H:=NilpotentQuotient(L,[x]);
a:=H.1; b:=H.2; c:=H.3; d:=LeftNormedComm([a^{-1},c,c,c]);
e:=LeftNormedComm([a*b,c,c,c]); Order(d); Order(e);
C:=LowerCentralSeries(H); d in C[5]; e in C[5];

Then if we consider the elements $d = [a^{-1}, c, c, c]$ and $e = [ab, c, c, c]$ of $H$, we can see by above command in GAP that $d$ and $e$ are elements of $\gamma_5(H)$ and have orders 2 and 4, respectively. So, in the group $G$, we have $d = e = 1$. This completes the proof. □

Note that, the second proof of Theorem 2.2 also shows the necessity of assuming that $\gamma_5(G)$ has no element of order 2.

3. Right 4-Engel elements

Our main result in this section is to prove the following.
Theorem 3.1. Let $G$ be a $\{2,3,5,°\}$-group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and any $x \in G$. Then $R_4(G)$ is a subgroup of $G$.

Proof. Consider the ‘freest’ group, denoted by $U$, generated by two elements $u,v$ with $u$ a right 4-Engel element. We mean this by the group $U$ given by the presentation

$$\langle u, v | [u, x]^4 = 1 \text{ for all words } x \in F_2 \rangle,$$

where $F_2$ is the free group generated by $u$ and $v$. We do not know whether $U$ is nilpotent or not. Using the $\text{nq}$ package shows that the group $U$ has a largest nilpotent quotient $M$ with class 8. By the following code, the group $M$ generated by a right 4-Engel element $a$ and an arbitrary element $c$ is constructed. We then see that the element $[a^{-1}, c, c, c, c]$ of $M$ is of order 375 = $3^2 \times 5^3$. Therefore the inverse of a right 4-Engel element of $G$ is again a right 4-Engel element. The following code in GAP gives a proof of the latter claim. The computation was completed in about 24 seconds.

```gap
F:=FreeGroup(3); a1:=F.1; b1:=F.2; x:=F.3;
U:=F/[LeftNormedComm([a1,x,x,x,x])];
M:=NilpotentQuotient(U,[x]);
a:=M.1; c:=M.2;
h:=LeftNormedComm([a^{-1},c,c,c,c]);
Order(h);
```

We now show that the product of every two right 4-Engel elements in $G$ is a right 4-Engel element. Let $a, b \in R_4(G)$ and $c \in G$. Then we claim that $H = \langle a, b, c \rangle$ is nilpotent of class at most $7$. ($*$)

By induction on the nilpotency class of $H$, we may assume that $H$ is nilpotent of class at most 8. Now we construct the largest nilpotent group $K = \langle a_1, b_1, c_1 \rangle$ of class 8 such that $a_1, b_1 \in R_4(K)$.

```gap
F:=FreeGroup(4);A:=F.1; B:=F.2; C:=F.3; x:=F.4;
W:=F/[LeftNormedComm([A,x,x,x,x]),LeftNormedComm([B,x,x,x,x])];
K:=NilpotentQuotient(W,[x],8);
LowerCentralSeries(K);
```

The computation took about 22.7 hours. We see that $\gamma_8(K)$ has exponent 60. Therefore, as $H$ is a $\{2,3,5,°\}$-group, we have $\gamma_8(H) = 1$ and this completes the proof of our claim ($*$).

Therefore we have proved that any nilpotent group without elements of orders 2, 3 or 5 which is generated by three elements two of which are right 4-Engel, is nilpotent of class at most 7.

Now we construct, by the $\text{nq}$ package, the largest nilpotent group $S$ of class 7 generated by two right 4-Engel elements $s,t$ and an arbitrary element $g$. Then one can find by GAP that the order of $[st, g, g, g, g]$ in $S$ is 300. Since $H$ is a quotient of $S$, we have that $[ab, c, c, c, c]$ is of order dividing 300 and so it is trivial, since $H$ is a $\{2,3,5,°\}$-group. This completes the proof. $\square$

Corollary 3.2. Let $G$ be a $\{2,3,5,°\}$-group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and for any $x \in G$. Then $R_4(G)$ is a nilpotent group of class at most 7. In particular, the normal closure of every right 4-Engel element of group $G$ is nilpotent of class at most 7.
Proof. By Theorem 3.1, $R_4(G)$ is a subgroup of $G$ and so it is a 4-Engel group. In [18] it is shown that every locally nilpotent 4-Engel $\{2, 3, 5\}'$-group is nilpotent of class at most 7. Therefore $R_4(G)$ is nilpotent of class at most 7. Since $R_4(G)$ is a normal set, the second part follows easily. □

Therefore, to prove that the normal closure of any right 4-Engel element of a $\{2, 3, 5\}'$-group $G$ is nilpotent, it is enough to show that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and for any $x \in G$. It may be surprising that Newell [13] has had a similar obstacle to prove that the normal closure of a right 3-Engel element is nilpotent in any group.

**Corollary 3.3.** In any $\{2, 3, 5\}'$-group, the normal closure of any right 4-Engel element is nilpotent if and only if every 3-generator subgroup in which two of the generators can be chosen to be right 4-Engel, is nilpotent.

Proof. By Corollary 3.2, it is enough to show that a $\{2, 3, 5\}'$-group $H$ is nilpotent whenever $a, b \in R_4(H)$, $x \in H$ and both $\langle a \rangle^H$ and $\langle b \rangle^H$ are nilpotent. Consider the subgroup $K = \langle a \rangle^H \langle b \rangle^H$ which is nilpotent by Fitting’s theorem. Now we prove that $K$ is finitely generated. We have $K = \langle a, b \rangle$ and since $a$ and $b$ are both right 4-Engel, it is well-known that

$$\langle a \rangle^x = \langle a, a^x, a^{x^2}, a^{x^3} \rangle \quad \text{and} \quad \langle b \rangle^x = \langle b, b^x, b^{x^2}, b^{x^3} \rangle,$$

and so

$$K = \langle a, a^x, a^{x^2}, a^{x^3}, b, b^x, b^{x^2}, b^{x^3} \rangle.$$

It follows that $H$ satisfies maximal condition on its subgroups as it is (finitely generated nilpotent)-by-cyclic. Now by a famous result of Baer [2] we have that $a$ and $b$ lie in the $(m + 1)$th term $\zeta_m(H)$ of the upper central series of $H$ for some positive integer $m$. Hence $H/\zeta_m(H)$ is cyclic and so $H$ is nilpotent. This completes the proof. □

We conclude this section with the following interesting information on the group $M$ in the proof of Theorem 3.1. In fact, for the largest nilpotent group $M = \langle a, b \rangle$ relative to $a \in R_4(M)$, we have that $M/T$ is isomorphic to the largest (nilpotent) 2-generated 4-Engel group $E(2, 4)$, where $T$ is the torsion subgroup of $M$ which is a $\{2, 3, 5\}$-group. Therefore, in a nilpotent $\{2, 3, 5\}'$-group, a right 4-Engel element with an arbitrary element generate a 4-Engel group. This can be seen by comparing the presentations of $M/T$ and $E(2, 4)$ as follows. One can obtain two finitely presented groups $G_1$ and $G_2$ isomorphic to $M/T$ and $E(2, 4)$, respectively by GAP:

```gap
MoverT := FactorGroup(M, TorsionSubgroup(M));
E24 := NilpotentEngelQuotient(FreeGroup(2), 4);
iso1 := IsomorphismFpGroup(MoverT); iso2 := IsomorphismFpGroup(E24);
G1 := Image(iso1); G2 := Image(iso2);
```

Next, we find the relators of the groups $G_1$ and $G_2$ which are two sets of relators on 13 generators by the following command in GAP:

```gap
r1 := RelatorsOfFpGroup(G1); r2 := RelatorsOfFpGroup(G2);
```

Now, save these two sets of relators by the LogTo command of GAP in a file and go to the file to delete the terms as

<identity ...>
in the sets \( R_1 \) and \( R_2 \). Now call these two modified sets \( R_1 \) and \( R_2 \). We show that \( R_1 = R_2 \) as two sets of elements of the free group \( f \) on 13 generators \( f_1, f_2, \ldots, f_{13} \).

\[
f := \text{FreeGroup}(13);
\]

\[
f_1 := f.1; f_2 := f.2; f_3 := f.3; f_4 := f.4; f_5 := f.5; f_6 := f.6;
\]

\[
f_7 := f.7; f_8 := f.8; f_9 := f.9; f_{10} := f.11; f_{12} := f.12; f_{13} := f.13;
\]

Now by \texttt{Read} function, load the file in \texttt{GAP} and type the simple command \( R_1 = R_2 \). This gives us \texttt{true} which shows \( G_1 \) and \( G_2 \) are two finitely presented groups with the same relators and generators and so they are isomorphic. We do not know if there is a guarantee that if someone else does as we did, then he/she finds the same relators for \( F_p \) groups \( G_1 \) and \( G_2 \), as we have found. Also we remark that using function \texttt{IsomorphismGroups} to test if \( G_1 \cong G_2 \), did not give us a result in less than 10 hours and we do not know whether this function can give us a result or not.

We summarize the above discussion as following.

**Theorem 3.4.** Let \( G \) be a nilpotent group generated by two elements, one of which is a right 4-Engel element. If \( G \) has no element of order 2, 3 or 5, then \( G \) is a 4-Engel group of class at most 6.

### 4. Right \( n \)-Engel elements for \( n \geq 5 \)

In this section we show that for every \( n \geq 5 \) there is a nilpotent group \( G \) of class \( n + 2 \) containing elements \( a \) and \( x \in R_n(G) \) such that both \( [x^k, a] \) and \( [x^{-1}, a] \) have infinite order for all integers \( k \geq 2 \).

Note that by Nickel’s example \[15\], for every \( n \geq 3 \) we have already had a nilpotent group \( K \) of class \( n + 2 \) containing a right \( n \)-Engel element \( x \) such that \( [x^{-1}, y] = [x^2, y] \neq 1 \) for some \( y \in K \) i.e., neither \( x^2 \) nor \( x^{-1} \) are right \( n \)-Engel.

We have checked by \texttt{nq} package of Nickel in \texttt{GAP} that \( [x^{-1}, y] = [x^2, y] \) is of finite order whenever \( n \in \{5, 6, 7, 8\} \). In fact,

\[
\begin{align*}
(1) \quad & a([x^{-1}, y]) = 3, \text{ NqRuntime=}1.7 \text{ Sec} \\
(2) \quad & a([x^{-1}, y]) = 7, \text{ NqRuntime=}54.8 \text{ Sec} \\
(3) \quad & a([x^{-1}, y]) = 4, \text{ NqRuntime=}1702 \text{ Sec} \\
(4) \quad & a([x^{-1}, y]) = 9, \text{ NqRuntime=}56406 \text{ Sec}
\end{align*}
\]

Newman and Nickel \[12\] constructed a group \( H \) as follows. Let \( F \) be the relatively free group, generated by \( \{a, b\} \) with nilpotency class \( n + 2 \) and \( \gamma_4(F) \) abelian. Let \( M \) be the (normal) subgroup of \( F \) generated by all commutators in \( a, b \) with at least 3 entries \( b \) and the commutators \( [b, n+1 \ a] \) and \( [b, n \ a, b] \). Then \( H = \frac{F}{M} \). Note that the normal closure of \( b \) in \( H \) is nilpotent of class 2.

We denote the generators of \( H \) by \( a, b \) again. Put

\[
t = [b, n \ a], \quad u_j = [b, n-1-j \ a, b_j a], \quad 0 \leq j \leq n-2,
\]

\[
u = \prod_{j=0}^{n-2} u_j, \quad v = [u_{n-2}, a], \quad w = \prod_{j=0}^{n-3} [u_j, a]
\]

and let \( N \) be the subgroup \( \langle tuw, t^2 w, uw \rangle \). Then \( aN \) is a right \( n \)-Engel element in \( \frac{H}{N} \) and \( [b, n \ a]N \) has infinite order in \( \frac{H}{N} \).

Now let \( H \) be the above group and \( N_0 := \langle u, uv, vt^{-1} \rangle \). First, note that \( N_0 \) is a normal subgroup of \( H \). For, clearly \( t, v, w \in Z(H) \) and \( u^b = u \). Also it is not hard
we have $u^b_i = u_j[u_j, a]$ and thus $u^a = uvw$. This means that $N_0^a = N_0$ and so $N_0$ is a normal subgroup of $H$. Now we can state our main result of this section:

**Theorem 4.1.** $[b, n, a] N_0 = [b^{-2}, m, a] N_0$ and it has infinite order in $\frac{H}{N_0}$ and $[b^{-1}, m, h] \in N_0$ for all $h \in H$. Furthermore $[b^{-k}, m, a] = v^{(k)} N_0$ for all $k \geq 2$.

**Remark 4.2.** As in [12], the proof of Theorem 4.1 involves a series of commutator calculations based, as usual, on the basic identities as following, which are mentioned in [12]. We bring them here for reader’s convenience.

1. $[g, cd] = [g, d][g, c][g, c, d]$.
2. $[cd, g] = [c, g][c, d][d, g]$.
3. $[c^{-1}, d] = [c, d, c^{-1}][c, d]^{-1}$.
4. $[c, d^{-1}] = [c, d, d^{-1}][c, d]^{-1}$.
5. $[hk, h_1, \ldots, h_s] = [h, h_1, \ldots, h_s]$ for every $k$ in $\gamma_{n+3-s}(H)$ and arbitrary $h_1, \ldots, h_s \in H$.
6. $[g, d, c] = [g, d, c][g, c, d][g, d, c]$, where $k$ is a product of commutators of weight at least 4 with entries $g, c$ and $d$.
7. $[a, n, hk] = [a, n, h]$ for all $h \in H$ and $k \in \gamma_3(H)$.
8. $[g, d^k] = [g, d^k][g, d][g, d^k]$.

**Proof of Theorem 4.1.** By Remark 4.2(7), we may assume that $h$ is of the form $a^\alpha b^\beta [b, a]^\gamma$. The following calculations may depend to the signs of $\alpha$ and $\beta$; we here outline only the case in which $\alpha$ and $\beta$ are positive.

$$[b^{-1}, m, h] = [b^{-1}, n, a^\alpha b^\beta [b, a]^\gamma]$$
$$= [b^{-1}, n, a^\alpha b^\beta] \prod_{j=0}^{n-1} [b^{-1}, n, a^\alpha b^\beta, [b, a]^\gamma, j, a^\alpha b^\beta]$$
$$= [b^{-1}, n, a^\alpha b^\beta]\left(b[ [b, a], n, a^{-1} a]\prod_{j=0}^{n-2} [b_{n-1-j}, a, [b, a], j, a]\right)^{-a^n-1}.$$

Since
$$[b, [b, a], n, a^{-1} a] = [[b, a], b^{-1}, a^{-1} a]$$
$$= [b, a, b, n, a^{-1} a]^{-1}$$
$$= v^{-1}$$
and by Remark 4.2(5) and (6)

$$[b, n, a, j, a] = [b, n, a, b, j, a]^{-1} [b, n, a, b, j, a]$$
we have

$$\prod_{j=0}^{n-2} [b_{n-1-j}, a, [b, a], j, a] = \prod_{j=0}^{n-2} [b_{n-1-j}, a, b, j, a]^{-1} [b_{n-1-j}, a, b, j, a]$$
$$= \prod_{j=0}^{n-3} [b_{n-1-j}, a, b, j, a]^{-1} \prod_{j=0}^{n-2} [b_{n-1-j}, a, b, j, a]$$
$$= v.$$
Therefore

\[ [b^{-1}_m a^n b^\beta [b, a]^{-\gamma}] = [b^{-1}_m a^n b^\beta] (v^{-1}v)^{-\alpha^{-1} \gamma} = [b^{-1}_m a^n b^\beta]. \]

On the other hand by Remark 3.2 (8) we have

\[ [b^{-1}_m a^n b^\beta] = [b^{-1}_m a^n] \prod_{j=0}^{n-2} [b^{-1}_m a_{n-1-j}^n b^\beta a^\alpha] \]

\[ = [b^{-1}_m a^n] \left[ b^{-1}_m a_{n+1} \right] \left( \prod_{j=0}^{n-2} (b_{n-1-j} a, b_{j+1} a) \right)^{-\alpha^{-1} \beta} \]

\[ \times \left[ b, a, b_{n-2} a \right]^{-\alpha^{-2} \beta} \left[ b_{n-1} a, b \right]^{-\alpha^{-2} \beta} \]

\[ = (v^{-1})^\alpha v^{-\alpha^{-1} \beta} (v) \left( v^{-1} \right)^\alpha \left( v^{-1} \right)^\beta. \]

Therefore \( b^{-1}_n N_0 \) is a right \( n \)-Engel element in \( \frac{H}{N_0} \). This completes the second part of the theorem.

Since \( (t, u, v, w) \) is a free abelian group of rank 4, it is clear that \([b, n a] N_0 \) has infinite order. On the other hand

\[ [b^{-2}_m a] = [[b^{-1}_m a] [b^{-1}_m a, b^{-1}_m] [b^{-1}_m a, b^{-1}_m, a, b^{-1}_m, [b, a]_n, a] [b^{-1}_m, a]] \mod N_0 \]

\[ \equiv [b, a, b_{n-1} a] \mod N_0 \]

\[ \equiv v \mod N_0. \]

Since \( vt^{-1} \in N_0 \) we have \([b, n a] N_0 = t N_0 = v N_0 = [b^{-2}_m a] N_0 \). Now let \( k \geq 2 \), \( f(1) = 0 \) and \( f(k) = (k - 1) + f(k - 1) = \left( \frac{k}{2} \right) \). Then

\[ [b^{-k}_m a] = [[b^{-1}_m a] [b^{-1}_m, a, b^{-1} - k - 1] [b^{-1}_m, a, b^{-(k-1)}_m, a]] \mod N_0 \]

\[ \equiv [b, a, b^{(k-1)}_m, a, a] \mod N_0 \]

This completes the proof. \( \square \)

Now we answer negatively Question 1.4 which has been proposed in 11.

Let \( T \) be the torsion subgroup of \( H/N_0 \) and \( x = b N_0 T \) and \( y = a N_0 T \). Then the group \( M = H/N_0 T = (x, y) \) is a torsion free, nilpotent of class \( n + 2 \), \( x \in R_n(M) \) and both \([x^{-1}, y] \) and \([x^{k}, a] \) are of infinite order for all integers \( k \geq 2 \). Since, for any given prime number \( p \), a finitely generated torsion-free nilpotent group is residually finite \( p \)-group, it follows that for any prime number \( p \) and integer \( k \geq 2 \), there is a finite \( p \)-group \( G(p, k) \) of class \( n + 2 \) containing a right \( n \)-Engel element \( t \) such that both \( t^k \) and \( t^{-1} \) are not right \( n \)-Engel. This answers negatively Question 1.4.
5. Subgroupness of the set of (bounded) Left Engel elements of a group

Let \( n = 2^k \geq 2^{48} \) and \( B(X, n) \) be the free Burnside group on the set \( X = \{ x_i \mid i \in \mathbb{N} \} \) of the Burnside variety of exponent \( n \) defined by the law \( x^n = 1 \). Lemma 6 of [11] states that the subgroup \( \langle x_{2k-1}^{n/2}, x_{2k}^{n/2} \mid k = 1, 2, \ldots \rangle \) of \( B(X, n) \) is isomorphic to \( B(X, n) \) under the map \( x_{2k-1}^{n/2} x_{2k}^{n/2} \rightarrow x_k, \ k = 1, 2, \ldots \) Therefore the subgroup \( G := \langle x_1^{n/2}, x_2^{n/2}, x_3^{n/2}, x_4^{n/2} \rangle \) is generated by four elements of order 2, contains the subgroup \( H = \langle x_1^{n/2}, x_2^{n/2}, x_3^{n/2}, x_4^{n/2} \rangle \) isomorphic to the free 2-generator Burnside group \( B(2, n) \) of exponent \( n \). One knows the tricky formulae

\[
[x, k y] = [x, y]^{(-1)^{k-1}2^{k-1}}
\]

holding for all elements \( x \) and all elements \( y \) of order 2 in any group and all integers \( k \geq 1 \). It follows that the group \( G \) can be generated by four left 49-Engel elements of \( G \). Thus

\[
G = (L_{49}(G)) = (L(G)) = (\overline{L}(G)),
\]

where \( L(H) \) (resp.) denotes the set of (bounded, resp.) left Engel elements of a group \( H \).

Suppose, if possible, \( G \) is an Engel group. Then \( H \) is also an Engel group. Let \( Z \) and \( Y \) be two free generators of \( H \). Thus \( [Z, k Y] = 1 \) for some integer \( k \geq 1 \). Since \( H \) is the free 2-generator Burnside group of exponent \( n \), we have that every group of exponent \( n \) is a \( k \)-Engel group. Therefore, \( G \) is an infinite finitely generated \( k \)-Engel group of exponent \( n \), as \( H \) is infinite by a celebrated result of Ivanov [10]. Hence, we have proved that

**Proposition 5.1.** At least one of the following happens.

1. There is an infinite finitely generated \( k \)-Engel group of exponent \( n \) for some positive integer \( k \) and 2-power number \( n \).
2. There is a group \( G \) such that \( L(G) = \overline{L}(G) \) and \( L(G) \) is not a subgroup of \( G \).

We believe that the subgroup \( H \) cannot be an Engel group, but we are unable to prove it.

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