The discovery of integral quantum Hall plateaus in graphene \( [1] \) poses several questions: Does the fractional quantum Hall effect (FQHE) also occur in graphene, and, if so, what is its character? Does it resemble the integral quantum Hall effect (IQHE) in graphene? How does it compare to the FQHE in GaAs? What other physics does interaction produce? Even if the Zeeman interaction produces a Fermi-point pseudospin = \( \bar{\eta} = \bar{\eta} + (e/c)\bar{A} \). Letting \( z = x - iy \), choosing the symmetric gauge \( \bar{A} = (-By/2,Bx/2) \), and introducing the Landau level raising and lowering operators, \( a^\dagger = (\bar{z}/2 - 2\bar{\eta}_z)/\sqrt{2} \) and \( a = (\bar{z}/2 + 2\bar{\eta}_z)/\sqrt{2} \), the Hamiltonian becomes

\[
H = \frac{\sqrt{2}\hbar v_F}{iB}
\begin{pmatrix}
0 & a^\dagger \\
-a^\dagger & 0
\end{pmatrix}.
\]

The eigenvalue problem is conveniently formulated in terms of \( H^2 \), which can be represented as LL number operators:

\[
H^2 = \frac{2\hbar^2 v_F^2}{l_B^2}
\begin{pmatrix}
a^\dagger a + 1 & 0 \\
0 & a^\dagger a'
\end{pmatrix}.
\]

Any \( \Psi = (\alpha\eta_{n-1,m_1}^\dagger \beta\eta_{n,m_2}) \) is an eigenvector of \( H^2 \), where \( \eta_{n,m} \) are the standard LL eigenfunctions in GaAs (\( n = 0, 1, \cdots \) is the LL index, and \( m \) is the angular momentum index). \( H\Psi = E_n\Psi \) requires \( m_1 = m_2 \) and fixes \( \beta/\alpha \), with the result that normalized eigenvectors take the form (\( \mathrm{sgn}(0) = 0 \) by convention)

\[
\Psi(n\neq0,m) = \frac{1}{\sqrt{2}} (-\mathrm{sgn}(n)i\eta_{|n|,1,m},
\eta_{|n|,1,m}),
\]

\[
\Psi(0,m) = \begin{pmatrix}
0 \\
\eta_{0,m}
\end{pmatrix},
\]

\[
E_n = \mathrm{sgn}(n)\sqrt{2\hbar v_F^2 eB|n|/c}.
\]

Before discussing the FQHE in graphene, we describe how interactions are expected to affect the integral QHE in graphene. For noninteracting electrons, when both the spin and the pseudospin degeneracies are present, the Hall plateaus have been predicted and seen at

\[
R_H = h/(je^2), \quad j = \ldots, -6, -2, 2, 6, 10, 14, \ldots
\]
For magnetic fields large enough to resolve the spin bands the Hall plateaus occur at

$$R_H = \frac{\hbar}{(je^2)}, \quad j = \ldots, -2, -1, 0, 1, 2, 4, 6, 8, \ldots$$ (8)

where \( j = 0 \) implies a band insulator. However, all integral plateaus should, in principle, become possible for interacting electrons. Consider a magnetic field large enough to lift the spin degeneracy at a filling \( \nu = 2n + 1 \) with \( n = 1, 2, 3, \ldots \) where no QHE occurs for noninteracting electrons. For interacting electrons, the rotational symmetry in pseudospin space is spontaneously broken due to pseudospin exchange, with the pseudospin magnetization picking an arbitrary direction. While this system supports gapless pseudospin-wave excitation, the charged excitations have a gap, thus producing a Hall plateau.

**FQHE in the \( n = 0 \) Landau level of graphene:** The unusual electronic dispersion of graphene around \( E = 0 \) reflects in both the structure of Landau levels and in the offset of the Hall conductance staircase. Nonetheless, the actual wave functions of electrons in the \( n = 0 \) manifold are identical to those in the conventional lowest Landau level of GaAs. Consequently, the FQHE in the \( n = 0 \) Landau level of graphene at a high magnetic field maps onto FQHE of electrons in GaAs with zero Zeeman energy, as corroborated and extended by the numerical results described below.

The basic physics in the \( n = 0 \) graphene LL, therefore, is the same as in GaAs. Each electron captures an even number (2\( p \)) of quantized vortices to become a composite fermion (CF). The Berry phases generated by the vortices effectively cancel part of the external magnetic field, so that the dynamics of composite fermions are governed by a reduced magnetic field \( B^* = B - 2\rho \phi_0 / \hbar c \), where \( \rho \) is the particle density in the \( n = 0 \) Landau level and \( \phi_0 = \hbar c / e \) is the flux quantum. The \( n = 0 \) Landau level of electrons splits into Landau-like levels of composite fermions, whose filling factor \( \nu^* \), in terms of the electron filling \( \nu \), is given by \( \nu = \nu^*/(2p \pm 1) \).

The IQHE of composite fermions for \( \nu^* = n \) produces sequences of fractions:

$$\nu = n/(2p \pm 1).$$ (9)

The origin of gap, i.e. the energy required to promote a composite fermion into a higher CF-Landau level, is different for even and odd values of \( n \). For even \( n \), the ground state is a pseudospin singlet, with \( n/2 \) CF-Landau levels for each component of the pseudospin occupied. For odd \( n \), the ground state is partially pseudospin polarized; no QHE would occur here if the composite fermions did not interact, but the residual interaction between composite fermions opens a gap. To the extent the residual interaction is weak, one expects fractions with even numerators to be more robust than those with odd numerators. The excitation energies for the GaAs FQHE in the zero Zeeman energy limit apply to graphene FQHE within the \( n = 0 \) level. Calculated gaps to creation of a far-separated charged quasiparticle/quasihole pair at \( \nu = 1/3 \) and (unpolarized) \( \nu = 2/5 \) are 0.07 and 0.04 \( e^2/\ell_B \), respectively; the larger gap at 1/3 indicates the significance of inter-CF interactions.

So long as electrons are confined to the \( n = 0 \) Landau level, they have no memory of the Dirac nature of the zero-field dispersion, with some surprising consequences for the FQHE. The CF-cyclotron energy opens up approximately linearly with \( B^* \), as expected for composite fermions with a parabolic dispersion, even though the cyclotron energy of electrons in graphene scales anomalously with \( B \).

In GaAs, the sequence of FQHE states at \( \nu = n/(2p \pm 1) \) terminates as \( n \to \infty \) in a composite-fermion Fermi sea at \( \nu = 1/2p \), where the effective magnetic field vanishes. For zero Zeeman energy, variational calculations favor the spin singlet Fermi sea, so graphene should have a pseudospin-singlet CF Fermi sea at \( \nu = 1/2p \). The CF Fermi sea in GaAs has been successfully modeled as an ordinary Fermi sea with parabolic dispersion, which allows one to deduce an effective mass for composite fermions. The same should be true of the CF Fermi sea in graphene, in spite of the fact that electrons in graphene make a Dirac sea at zero magnetic field and have no effective mass. The singlet nature of the CF Fermi sea can be ascertained through a measurement of the Fermi wave vector, as was accomplished in GaAs sys-
tems by various geometric means. \(^\text{10}\)  

**FQHE in the \(n = 1\) Landau level of graphene:**  
The nature of FQHE depends on the Haldane pseudopotentials. In GaAs, the FQHE is essentially restricted to the lowest LL: very few fractions are seen in \(n = 1\), and almost none in higher LLs. The mapping between GaAs and graphene does not hold in higher Landau levels, so different behaviors are expected. We first evaluate the Coulomb matrix elements within the \(n\)th graphene Landau level. Write \(|n_1, m_1; n_2, m_2; \ldots; n_N, m_N\rangle\) for the product state \(\eta_{n_1, m_1} \otimes \eta_{n_2, m_2} \otimes \cdots \otimes \eta_{n_N, m_N}\) and \(|n_1, m_1; n_2, m_2; \ldots; n_N, m_N\rangle\rangle = \Psi^{(n_1, m_1)} \otimes \cdots \otimes \Psi^{(n_N, m_N)}\). Then,  
\[
4\langle n, m_1; n, m_2 \parallel V \parallel n, m_3; n, m_4 \rangle = \\
\langle n - 1, m_1; n, m_2 \parallel V \parallel n - 1, m_3; n, m_4 \rangle + \\
\langle n, m_4; n - 1, m_2 \parallel V \parallel n, m_3; n - 1, m_4 \rangle + \\
\langle n - 1, m_1; n - 1, m_2 \parallel V \parallel n - 1, m_3; m_4 \rangle.
\]

By conservation of angular momentum, these matrix elements are all proportional to \(\delta_{m_1 + m_2, m_3 + m_4}\). The problem of electrons in the \(n\)th graphene LL thus formally maps into that of the lowest GaAs LL (with two pseudospins) for the product state \(\eta_{n_1, m_1} \otimes \eta_{n_2, m_2} \otimes \cdots \otimes \eta_{n_N, m_N}\) and \(|n_1, m_1; n_2, m_2; \ldots; n_N, m_N\rangle\rangle\) for \(\Psi^{(n_1, m_1)} \otimes \cdots \otimes \Psi^{(n_N, m_N)}\). Then,  
\[
4\langle n, m_1; n, m_2 \parallel V \parallel n, m_3; n, m_4 \rangle = \\
= \langle n - 1, m_1; n, m_2 \parallel V \parallel n - 1, m_3; n, m_4 \rangle + \\
\langle n, m_4; n - 1, m_2 \parallel V \parallel n, m_3; n - 1, m_4 \rangle + \\
\langle n - 1, m_1; n - 1, m_2 \parallel V \parallel n - 1, m_3; m_4 \rangle.
\]

Pseudoskyrmions: In GaAs quantum wells, the excitations of the \(\nu = 1\) state for exactly zero Zeeman splitting are not simple particle-hole excitations but spin textures called skyrmions \(^{13, 14, 15}\), in which half of the spins are reversed. However, the skyrmion size rapidly decreases with increasing Zeeman energy; experimentally, skyrmions typically have 3 to 5 flipped spins. \(^{16, 17}\) No skyrmions occur at \(\nu = 3, 5, \ldots\). CF skyrmions are believed to be relevant near \(\nu = 1/3\) at very small Zeeman energies. \(^{18, 19, 20}\)
fermions. For 2/5 the gaps, $\Delta_{2/5}^{(0)} = 0.051(1)$ and $\Delta_{2/5}^{(1)} = 0.062(1)$, were obtained from the trial wave functions of the CF theory, evaluated by a procedure described in Ref. [22].

We summarize our principal conclusions. The lowest LL FQHE of graphene in the large Zeeman energy limit is equivalent to the lowest LL FQHE in GaAs in the zero Zeeman energy limit, terminating into a pseudospin-singlet Fermi sea at half filling. The effective interaction in Eq. (11) is shown to be more favorable to CF formation in the $|n| = 1$ LL of graphene than in the $n = 1$ LL of GaAs. The gaps at $\nu = 1/3, 2/5$ are calculated and FQHE is predicted at $\nu = 2/3$ due to reverse flux attachment. In contrast to GaAs, skyrmions are predicted to occur at $\nu = 1.3$ in the $n = 0, 1, 2$ LLs.

While completing the manuscript, we learned from Philip Kim of the observation of new integral plateaus by Zhang et al. [23]. Several recent theory preprints have examined similar topics, including a field theory approach that obtains results similar to ours [24] and derivations of the Coulomb pseudopotentials for graphene [25]. Support by the National Science Foundation under grants no. DMR-0240458, DMR-0305035 and ECS-0609243 is gratefully acknowledged.

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FIG. 1: (a) Pseudopotentials for $n = 0$ and $n = 1$ Landau levels in graphene and GaAs. (b)-(d) Activation gaps at $\nu = 1/3$ and $2/5$ in several graphene LLs. All energies are given in units of $e^2/\epsilon l_B$, where $\epsilon$ is the dielectric constant of the host semiconductor and $l_B$ is the magnetic length. The gaps in (b) and (c) refer to the energy required to create a pseudoskyrmion-antiskyrmion pair. Gaps to pseudospin conserving (cons.) and pseudospin reversed (rev.) excitations are given in (d).
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