Superfluid vortex multipoles and soliton stripes on a torus

J. D’Ambroise,1 R. Carretero-González,2 P. Schmelcher,3,4 and P.G. Kevrekidis5

1 Department of Mathematics, Computer & Information Science, State University of New York (SUNY) College at Old Westbury, Westbury, NY, 11568, USA
2Nonlinear Dynamical Systems Group, Computational Sciences Research Center, and Department of Mathematics and Statistics, San Diego State University, San Diego, California 92182-7720, USA
3Center for Optical Quantum Technologies, Department of Physics, University of Hamburg, Luruper Chaussee 149, 22761 Hamburg Germany
4The Hamburg Centre for Ultrafast Imaging, University of Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
5Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA, 01003, USA

We study the existence, stability, and dynamics of vortex dipole and quadrupole configurations in the nonlinear Schrödinger (NLS) equation on the surface of a torus. For this purpose we use, in addition to the full two-dimensional NLS on the torus, a recently derived [Phys. Rev. A 101, 053606 (2021)] reduced point-vortex particle model which is shown to be in excellent agreement with the full NLS evolution. Horizontal, vertical, and diagonal stationary vortex dipoles are identified and continued along the torus aspect ratio and the chemical potential of the solution. Windows of stability for these solutions are identified. We also investigate stationary vortex quadrupole configurations. After eliminating similar solutions induced by invariances and symmetries, we find a total of 16 distinct configurations ranging from horizontal and vertical aligned quadrupoles, to rectangular and rhomboidal quadrupoles, to trapezoidal and irregular quadrupoles. The stability for the least unstable and, potentially, stable quadrupole solutions is monitored both at the NLS and the reduced model levels. Two quadrupole configurations are found to be stable on small windows of the torus aspect ratio and a handful of quadrupoles are found to be very weakly unstable for relatively large parameter windows. Finally, we briefly study the dark soliton stripes and their connection, through a series of bifurcation cascades, with steady state vortex configurations.

PACS numbers:

I. INTRODUCTION

The study of atomic Bose-Einstein condensates (BECs) offers a pristine setting to explore the interplay of nonlinear dynamical phenomena and quantum mechanical effects [13]. A major thrust of associated experimental and theoretical efforts has consisted of the exploration of coherent structures supported by the interplay of effective nonlinearity and dispersion in such systems, both at the mean-field level but also beyond [4]. More specifically, relevant studies and a wide range of experiments have focused on bright solitons [5–8] in attractively-interacting condensates, dark solitons in self-repulsive species [9–16], gap solitons [17], as well as multi-component structures [18]. While the above have been prototypically one-dimensional states, higher dimensional structures such as vortices [19, 20] and vortex rings [21] have also attracted significant attention in their own right.

Naturally, this activity has been mostly focused in the prototypical settings of parabolic (but also often periodic) traps in one- and higher dimensions, that have been the typical settings of experiments so far [13]. However, recent years, have seen a surge of activity as concerns the exploration of BECs in 2D surfaces. In the last year alone, multiple papers explored the dynamics of vortices and vortex-antivortex papers in spherical surface, shell-shaped systems [23, 24], following up on earlier related work not only on spherical surfaces [25], but also on cylindrical surfaces, planar annuli and sectors, as well as cones [26, 27]. The topic of the dynamics of vortices on curved surfaces is one that bears considerable history motivated by a variety of settings in fluid [28] and super-fluid [29] physics. In terrestrial BEC settings, the realization of such experiments suffers from aspects such as the gravitational sag. However, the recent activity in the newly launched Cold Atomic Laboratory (CAL) aboard the International Space Station seems to hold considerable promise in this direction and indeed is specifically aiming to implement a hollow bubble geometry [30, 31].

This, in turn, paves the way for the broader study of pattern dynamics (including of topologically charged states such as vortices) in nontrivial geometry and topology-featuring setups. It should be also noted that this is in addition to the remarkable recent developments towards confining and manipulating atoms via adiabatic potentials, which, in turn, can also lead to a diverse variety of traps for ultracold atoms; see, e.g., Ref. [32] for a relevant review.

In the present work, our platform of choice will be the surface a torus, i.e., the simplest compact and multiply connected surface. This is motivated by the above developments, the torus’ nontrivial topological structure, by the proposal by experimental groups of the realization of

*URL: http://nlds.sdsu.edu
an optical lattice on its surface \[33\] and by the recent formulation of the effective vortex particle dynamics on its surface \[34\]. More concretely, we extend our earlier considerations in the realm of bright solitary waves \[34\] to the self-repulsive condensate setting and the primarily vortical (but also dark soliton stripe) set of structures that can arise therein, while also respecting the periodicity of the torus in both of its angular directions. The fundamental work of Ref. \[34\] has set the stage by providing a description at the level of ordinary differential equations (ODEs) for the vortex particles. Yet, the latter work still left many questions unanswered. For instance, while this description is applicable at large chemical potentials, it is useful to explore the nature of the existence, stability and dynamics of multi-vortex structures as a function of the chemical potential (which is also a proxy for the atom number), but also as a function of the torus geometric parameters such as the ratio of the minor to the major axis. Furthermore, while the ODEs were derived, the potential equilibria of those and the associated stability and phase portraits were not explored even for the most prototypical case of a vortex pair. Indeed, there are further significant multi-vortex configurations that are relevant to consider such as the vortex quadrupoles. Additionally, as we will see below, the vortex patterns also bear connections (through their bifurcations) to states involving dark solitonic stripes that are of interest in their own right. Last but by no means least, it is also particularly meaningful to compare the ODE results with direct partial differential equation (PDE) simulations, to explore the validity and also potential limitations of the approach.

More concretely, our work is organized as follows. In Sec. \[II\] we introduce the original, full, spatio-temporal NLS model on the torus and briefly review (for completeness) the main aspects of its reduction to an effective point-vortex model, as obtained in Ref. \[34\]. Section \[III\] presents the bulk of our results by studying the existence of vortex- and stripe-bearing solutions, their stability and dynamics. In particular, in Sec. \[III A\] we exhaustively analyze the existence, stability, and dynamics for vortex dipole configurations. We find a total of 4 different stationary dipole solutions, two of which are fully stable. We also extend these solutions by adding extra phase windings along the toroidal and poloidal directions. Section \[III B\] is devoted to the study of quadrupole configurations where we identify a total of 16 distinct ones. A couple of these quadrupoles are found to be stable within small parameter windows while a handful of quadrupoles are found to be very weakly unstable for relatively large parameter windows. In Sec. \[III C\] we briefly study the existence and stability of dark soliton configurations and connect some of them, via bifurcation cascades, with steady state vortex patterns. Finally, a summary and conclusions of our work, together with some possible avenues for future research, are given in Sec. \[IV\].

In what follows, we define the torus aspect ratio \(\alpha \equiv r/R\).
B. Steady states and Stability

In Sec. III we will construct and study several steady state solutions for the above NLS on the torus. These steady states (standing waves) are found by separating space and time variables according to \(\psi(\phi, \theta, t) = \varphi_0(\phi, \theta) e^{-i\mu t}\), where \(\mu\) is often referred to as the chemical potential and corresponds to the temporal frequency of the solution (as well as to the density of its background). Thus, the steady state NLS for \(\varphi_0(\phi, \theta)\), parametrized by \(\mu\), reads

\[
-\frac{1}{2} \Delta \varphi_0 - \sigma |\varphi_0|^2 \varphi_0 - \mu \varphi_0 = 0. \tag{4}
\]

After suitably identifying steady states (for numerical details, see below), it is relevant to study their dynamical stability properties by extracting the corresponding stability spectra. Therefore, we follow perturbed solutions starting by a steady state solution \(\varphi_0\) as per Eq. (4) and perturb it with an infinitesimal perturbation according to

\[
\psi = \left\{ \varphi_0 + \varepsilon \left[ a e^{\lambda^* t} + b e^{\lambda t} \right] \right\} e^{-i\mu t}, \tag{5}
\]

where \(a = a(\phi, \theta)\) and \(b = b(\phi, \theta)\) correspond to the spatial eigenmodes with eigenvalue \(\lambda\). Then, the so-called Bogolyubov-de Gennes (BdG) stability spectra is obtained by solving the linearization equation (i.e., to order \(\varepsilon^1\)):

\[
\begin{bmatrix}
M_1 & M_2 \\
-M_2^* & M_1^*
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
= -i\lambda
\begin{bmatrix}
a \\
b
\end{bmatrix}, \tag{6}
\]

where \(M_1 = \frac{1}{2} \Delta + \mu + 2\sigma |\varphi_0|^2\) and \(M_2 = \sigma \varphi_0^2\). By construction, the spectrum obtained from Eq. (6) will respect the Hamiltonian symmetry such that if \(\lambda\) is an eigenvalue, so is \(-\lambda\), \(\lambda^*\), and \(-\lambda^*\) where the asterisk stands for complex conjugation. Therefore, any eigenvalue such that \(\text{Re}(\lambda) > 0\) will correspond to an instability—an exponential instability if \(\text{Im}(\lambda) = 0\) and an oscillatory instability if \(\text{Im}(\lambda) \neq 0\). In the latter case, the exponential growth is also accompanied by an oscillatory dynamics of the solution.

In the present work we focus primarily on stationary solutions composed of multi-vortex configurations. Similar to the focusing case of Ref. [35], there will be special vortex locations on the torus corresponding to stationary configurations. It is important to mention that, due to the periodic nature of the domain, only configurations with zero total charge are allowed. Therefore, only configurations with the same number of positively and negatively charge vortices are possible within the toroidal geometry. Therefore, we will focus on the lowest order ones that correspond to vortex *dipoles* and *quadrupoles*. As indicated also in the introduction, even for the former there are a lot of important features to explore including their stationary configurations and associated stability, while the latter have not been considered at all previously, to the best of our knowledge. The periodicity of the domain also allows for the existence of dark soliton stripe configurations provided they appear in pairs to allow for the individual \(\pi\) phase jumps of each dark soliton to accumulate to a whole \(2\pi\) phase jump. Nonetheless, as we will see in Sec. III C configurations with odd number of dark soliton stripes are also possible by adding extra phase windings (perpendicular to the stripes) to respect the periodicity. Finally, it is also relevant to note that the Laplacian operator in Eq. (3) is translationally invariant along the toroidal \(\phi\)-direction. Thus, steady state solutions will generate an entire family of possible \(\phi\)-translates—unless the steady states is already homogeneous in the \(\phi\)-direction such as all horizontal (toroidal) dark soliton stripe configurations.

C. Reduced point-vortex particle model

In tandem with the procurement of steady states and their characterization (stability), we will study the corresponding elements in the dynamically reduced model where the vortices are considered as point-particles, as per the fundamental work of Ref. [34]. Note that, in comparison to the latter, we are using here adimensional variables which correspond to the models in these works with \(\hbar = M = 1\) where \(M\) is the mass of the particles forming the atomic BEC. The model is cast as a set of ODEs on the vortex positions. This point-vortex model assumes no internal (density) structure for the vortices and that the only effects come from vortex-vortex phase interactions and, importantly, the curvature effects from the toroidal substrate where they are embedded. Such point-vortex models have been shown to be accurate in the (sufficiently) large \(\mu\)-limit [34, 19]. Indeed, what we will conclude here, as well, is that when \(R\) and \(r\) are of order unity, a value of \(\mu = 5\) (and sometimes even as low as \(\mu = 1\)) is sufficiently large such that the reduced model gives an accurate static (steady states), stability, and dynamical representation of the full NLS model on a torus.

In the large \(\mu\)-limit, one only considers the phase of the vortices and how the superposition of the phases from all other vortices advects the position of each vortex through the identification of the local fluid velocity as the gradient of the wavefunction’s phase at that point. The key towards setting up the point-vortex model on the torus lies in taking into consideration two crucial effects that are absent in the standard NLS model on a flat and infinite domain. Namely, (i) the effects of the periodic boundary conditions and (ii) the effects of the torus’ curvature. The periodic boundary conditions are accounted for by placing ‘ghost’ (or mirror) vortices outside the domain accounting for the effects that a particular vortex has on itself through the boundaries as well as the effects of the other vortices through the boundaries. This cumulative process results in an infinite sum for these contributions that can be represented in the form of Weierstrass functions or in terms of Jacobi-\(\theta\) functions; see Ref. [37] and
where cation, namely Λ = Λ\((\phi, \theta)\) one through \[\text{(8)}\]

\[
\begin{align*}
\phi &= \frac{u}{c}, \\
\theta &= 2\tan^{-1}\left[\frac{R + r}{R - r} \tan\left(\frac{v}{2r}\right)\right],
\end{align*}
\]

where \(c \equiv \sqrt{R^2 - r^2}\). Then, it is possible to show that these new \((u, v)\) coordinates, with squared line element \(ds^2 = \Lambda^2(du^2 + dv^2)\), are indeed isothermal (i.e., local ones in which the metric is conformal to the Euclidean) if the local scale factor satisfies

\[
\Lambda = \frac{c}{R - r \cos(v/r)}.
\]

Note that the scale factor only depends on the poloidal location, namely \(\Lambda = \Lambda(v)\), since the system is translationally invariant along the toroidal direction. The isothermal coordinates \((u, v)\) are then defined on the periodic rectangle \([-\pi c, \pi c] \times [-\pi r, \pi r]\).

Taking the periodic and curvature effects and defining the complex coordinate \(w\), the work of Ref. \[34\] gives the explicit form for the wavefunction’s phase \(\Phi\) associated to a set of vortex dipoles composed of a total zero charge configuration of vortices with charges \(q_n\) at (isothermal) positions \(w_n\). Namely, \(\Phi(w) = \text{Im}(\mathcal{F}(w))\) where \(\mathcal{F}(w) = \sum q_n F(w, w_n)\) and

\[
F(w, w_n) \equiv \ln\left[\frac{w - w_n}{2c}\right] - \frac{\text{Re}(w_n)}{2\pi r^c} w.
\]

And \(\partial_1(w, p)\) is the first Jacobi-\(\theta\) function evaluated at \(w\) with some \(p \equiv e^{-\pi r/c}(0 < p < 1)\). The first Jacobi-\(\theta\) function may be written as the following infinite sum

\[
\partial_1(w, p) = 2p^{1/4} \sum_{n=0}^{\infty} (-1)^n p^{n(n+1)} \sin[(2n + 1)w].
\]

Note that this implementation of \(\partial_1\) requires, for typical numerical values used in this manuscript, less than a dozen terms for this infinite sum to converge to machine (double) precision. From this overall phase imparted by all the vortex dipoles, one can explicitly write equations of motion for the individual vortices through the fluid velocity that, in turn, is equivalent to the gradient of the phase. Thus, the \(n\)-th vortex will experience a velocity \(V_n\) given by the gradient of the phase imprinted by the other vortices. Expressing the velocity in complex coordinates \(V_n = \dot{u}_n + i\dot{v}_n\) yields \(\dot{u}_n = \text{Im}(\Omega_n)/\Lambda(v_n)\) and \(\dot{v}_n = \text{Re}(\Omega_n)/\Lambda(v_n)\) where

\[
\Omega_n = \frac{1}{\Lambda(v_n)} \left[\sum_{m \neq n} q_m f(w_n, w_m) + i q_n \Lambda'(v_n) - \frac{q_n u_n}{2\pi r^c}\right],
\]

where

\[
f(w, w_n) \equiv \partial_{w} F(w, w_n).
\]

Note that the sum excludes self-interacting terms (i.e., \(m \neq n\)). By construction, the function \(f(w, w_n)\) is periodic in both the imaginary (vertical) and real (horizontal) directions. The vertical periodicity is captured from the explicit periodicity of the Jacobi-\(\theta\) function itself, while the horizontal periodicity is achieved by judiciously adding a linear term in the horizontal direction, see last term in Eq. \[\text{(9)}\], that ensures continuity of the velocity in the periodic domain \[34\]. In the next section, the point-vortex model, cast through the explicit velocity formulation of Eq. \[\text{(11)}\], is validated against numerical results from the full NLS \[2\]. This point-vortex model will also be instrumental in finding stationary dipole and quadrupole solutions.

### III. NUMERICAL RESULTS

In order to find branches of solutions as the system parameters are varied it is usually sufficient to find a single element of the branch and then apply numerical continuation to extend each branch over these parameters and study their existence and stability as the system parameters (mostly the chemical potential \(\mu\) and the torus aspect ratio \(\alpha\)) are varied. Thus, let us now leverage the results from the previous section to find particular stationary vortex configurations (dipoles and quadrupoles) for the reduced ODE model \[\text{(11)}\] and from there construct approximate steady states for the full NLS \[2\] that can be used with a fixed-point iteration scheme (cf. Newton’s method) to find numerically exact steady states. Once a particular steady state of (even) \(N\) vortices with charges \(q_n\) and locations \((u_n, v_n)\) is identified in the reduced ODE model, we construct an approximate initial wavefunction seed \(\varphi_0\) by ‘superimposing’ individual vortex-like guesses as follows

\[
\varphi_0(u, v) = \sqrt{\mu} \cdot e^{\Phi(u, v)} \cdot \Pi_{n=1}^{N} A(u - u_n, v - v_n),
\]

containing the following ingredients. (i) The background level is fixed so that, away from vortices, the density \(|\varphi_0|^2\) tends to \(\mu\). (ii) The global phase \(\Phi(u, v)\) is prescribed by the ODE model as per \(\Phi(w) = \text{Im}(\sum q_n F(w, w_n))\) with \(F\) defined in Eq. \[\text{(9)}\]. (iii) Each vortex (absolute value) profile is approximated by

\[
A(u, v) = \text{tanh}(\sqrt{\mu} \sqrt{u^2 + v^2}),
\]

centered at each of the vortex locations \((u_n, v_n)\). In a similar vein, one can also construct approximate dark soliton stripe solutions as follows

\[
\varphi_0(u, v) = \sqrt{\mu} \cdot \Pi_{n=1}^{N} B(u_n, v_n)
\]

where \(B(u_n, v_n) = A(u - u_n, 0)\) for dark stripes aligned along the poloidal direction and \(B(u_n, v_n) = A(0, v - v_n)\)
FIG. 2: (Color online) Steady state solutions continued from the vertical dipole-in configuration. The modulus squared (top) and the phase (bottom) of the solutions are plotted on the surface of the torus. The different solutions correspond to the values of $\mu$ indicated in the panels and $R = 12$ while $\alpha = 0.4$ for the top group of panels and $\alpha = 0.7$ for the bottom group of panels.

for dark stripes aligned along the toroidal direction. After a particular initial seed is constructed in the isothermal coordinates, it is converted to toroidal coordinates $(\phi, \theta)$ as per the transformation (7) or to Cartesian coordinates $(x, y) = (R\phi, r\theta)$ on the surface of the torus. Note that, as per Eq. (1), $(X, Y, Z)$ describes the Cartesian coordinates for a point on the torus in 3D space while $(x, y)$ describes the Cartesian 2D coordinates on the surface of the torus. Steady states are then found using Newton’s method by discretizing space using second order, central, finite differences (FDs) and separating real and imaginary parts. We use a 2D grid of $N_\phi \times N_\theta$ mesh points to discretize the wavefunction giving rise to a Newton matrix of size $2N_\phi N_\theta \times 2N_\phi N_\theta$. In our numerics below we typically use $N_\phi = N_\theta = 490$. Similarly, for the numerical stability results, we use the same FD discretization in space to cast the eigenvalue-eigenfunction problem (6) as a standard eigenvalue-eigenvector problem for the resulting stability matrix of size $2N_\phi^2 N_\theta^2 \times 2N_\phi^2 N_\theta^2$. Finally, for the numerical integration of the full NLS (2) we use again the same FD discretization in space and a standard fourth order Runge-Kutta (RK4) in time.

A. Vortex Dipoles

1. Vortex Dipoles: steady states

Through the reduced ODE model, one can browse the entire phase space of solutions for vortex dipoles given by a +1 vortex at location $(\phi_1, \theta_1)$ and a −1 vortex at $(\phi_2, \theta_2)$. Since the system is translationally invariant in the toroidal direction, the original phase space $(\phi_1, \theta_1, \phi_2, \theta_2)$ can be reduced, without loss of generality, to $(\phi, \theta_1, -\phi, \theta_2)$ by centering the solution about the toroidal axis with $\phi = (\phi_1 + \phi_2)/2$. A numerically exhaustive search for steady states in the three-dimensional reduced space $(\phi, \theta_1, \theta_2)$, using a standard fixed point iteration method (nonlinear least squares with a Levenberg-Marquardt algorithm), is then straightforward and yields four different types of stationary dipoles. These correspond to:

- **Vertical dipole-in.** This solution corresponds to a vertically (poloidally) aligned dipole with $\phi_1 = \phi_2$ and $\theta_1 = -\theta_2$. We dub this solution to be ‘in’ as the value of $|\theta_1| = |\theta_2|$ is closer to $\theta = \pi$, the inner part of the torus, than to the outer part with $\theta = 0$. Figure 2 depicts several steady state solutions...
continued from the vertical dipole for $R = 12$, a couple of values of $\alpha$, and for different values of $\mu$.

- **Horizontal dipole-in.** This solution corresponds to a horizontally (toroidally) aligned dipole with $\theta_1 = \theta_2 = \pi$, i.e., on the inside of the torus; see Fig. 4.

- **Horizontal dipole-out.** This solution corresponds to a horizontally (toroidally) aligned dipole

with $\theta_1 = \theta_2 = 0$, i.e., on the outside of the torus; see Fig. 4.

- **Diagonal dipole.** This solution corresponds to a diagonal dipole with $(\phi_2, \theta_2) = -(\phi_1, \theta_1)$ owing to a non-trivial balance of all the vortex velocity components; see Fig. 5. This is, arguably, the least intuitively expected among the different solutions.

At an intuitive level, one can argue that the main phenomenology involves a combination of different factors. On the one hand, a well-known fact stemming from their nonlinear, phase-induced interaction is that two vortices in Euclidean space will travel parallel to each other (in a direction perpendicular to their line of sight). The curvature arising from the toroidal geometry leads to that...
FIG. 8: (Color online) Stationary vertical dipole solutions with increasing horizontal windings $W_x$ (and $W_y = 0$) for $\alpha = 0.7$, $\mu = 1$, and $R = 12$. The left and right columns depict, respectively, the density and phase of the solutions in Cartesian coordinates. The location of the vortices corresponding to the ODE model are overlayed to the density plot using blue squares. The top row corresponds to the standard vertical dipole-in without extra winding ($W_x = 0$) and each successive row corresponds to a stationary vertical dipole solutions with increasing horizontal winding number $W_x$, with the vortices getting closer to compensate for the additional speed in the opposite direction provided by the winding.

feature being relevant now in the isothermal coordinates (where locally the metric is conformal to the Euclidean one). On the other hand, the topology of the torus and its periodic boundary conditions come into play and effectively create an equal and opposite velocity at these suitably selected distances, creating the potential for the steady states that we consider herein.

It is possible to analyze further the statics and dynamics of the vertical and horizontal dipoles as, per the reduced ODE model, any initial condition corresponding to (i) a symmetric ($v_1 = -v_2$) vertical dipole or to (ii) a horizontal dipole will always remain (i) a symmetric vertical dipole or (ii) a horizontal dipole. Figures 6 and 7 show, respectively, the instantaneous velocities for vertical dipoles-in and the planar phase space $(u, v)$ [or $(\phi, \theta)$] for horizontal dipoles. A full explanation of the results for vertical dipoles presented in Fig. 7 is given below in Sec. III A 2.

With regards to the vertical steady state dipole, this solution exists as there is no poloidal contribution to the vortex velocities (velocity contribution from curvature effects is purely toroidal and vortex-vortex velocity contribution is purely toroidal) and the toroidal contribution from curvature balances the vortex-vortex contribution (also toroidal). In fact, as Fig. 6 suggests, for each value of $\alpha$ there seems to be a single vertical dipole distance that balances all velocity contributions leading to a steady state. The figure also suggests that the steady state vertical dipole is always unstable as, e.g., perturbations along the poloidal direction will naturally result in (i) a constant toroidal velocity of the dipole if the vortex perturbations are equal and opposite in the poloidal direction, and, for generic perturbations, (ii) a toroidal velocity imbalance will start deviating exponentially from the stationary state. This instability of the vertical dipole-in will be revisited, for both PDE and ODE models, in Sec. III A 2.

On the other hand, as depicted in Fig. 7, the planar phase space for horizontal dipoles is much richer than the one corresponding to vertical dipoles. The figure suggests that there exist two horizontal dipoles as we described above: one at $\theta = 0$ (horizontal dipole-out, denoted by black dots in the figure) and one at $\theta = \pi$ (horizontal dipole-in, denoted by black plus symbols). Furthermore, the figure suggests that the horizontal dipole-out is (neutral) stable as a center within the relevant phase portrait, while the horizontal dipole-in is unstable (i.e., a saddle point). As for the vertical dipoles, the stability for horizontal dipoles will be covered, for both PDE and ODE models, in Sec. III A 2.

By following the construction of steady state vertical
dipoles as per Fig. 8 it is apparent that there is no vertical dipole that could be dubbed ‘out’. This fact can be rationalized by noting that the velocity contribution from curvature close to \( \theta = 0 \) (i.e., on the outer part of the torus) and the velocity contribution from the vortex-vortex interaction are in the same direction. Nonetheless, let us note that it is possible, in the NLS model, to add extra phase windings along the toroidal and poloidal directions provided that one does respect the periodicity of the domain. In fact, any NLS configuration \( \psi(\phi, \theta) \) can always be multiplied by a phase term with \( W_x \) bearing an extra \( 2\pi \) winding in the toroidal direction and \( W_y \) an extra \( 2\pi \) winding in the poloidal direction without violating the periodicity of the domain. Thus, inspired by the vertical and horizontal dipoles described above, one can take each of these solutions and multiply it by a phase terms as follows:

\[
\psi(\phi, \theta) \rightarrow \psi(\phi, \theta) \cdot e^{iW_x \phi} \cdot e^{iW_y \theta},
\]

where the windings \( W_x \) and \( W_y \) are integers. In a sense these structures bear two sets of topological charges, with one stemming from the charge of the vortex constituents, while the second arises through the potential windings along the toroidal or poloidal (or both) directions around the torus. Using this idea, we took the vertical-in and horizontal-out steady state dipoles and progressively applied, respectively, horizontal and vertical windings in tandem with the fixed point iteration scheme (Newton) to obtain families of dipoles with higher windings. We showcase examples of higher-winding vertical and horizontal dipoles-in Figs. 8 and 9, respectively. It is interesting to note that the vertical dipole steady state with \( W_x \geq 1 \) gives rise to vertical dipoles that could be dubbed ‘out’ as, in order to balance the extra vertical winding the vortices have to get close to each other around \( \theta = 0 \).

It is possible to leverage the reduced equations of motion to include the effects of vertical and horizontal windings. This relies on assuming that both toroidal and poloidal contributions to the phase windings in isothermal coordinates are accounted for by linear phase gradients. Under this assumption, we incorporate linear phase windings that gain \( 2\pi W_x \) and \( 2\pi W_y \) along, respectively, the horizontal and vertical directions. These windings are captured by adding corresponding linear terms in \( \Omega, \) of Eq. (11) as follows:

\[
\Omega_n \rightarrow \Omega_n + \frac{1}{\Lambda(v_n)} \left[ i \frac{W_x}{c} + \frac{W_y}{r} \right].
\]

The corresponding fixed points obtained from this extended reduced model with vertical and horizontal windings are depicted by the blue squares in the different panels of, respectively, Figs. 8 and 9. As evidenced from these figures, this extended reduced ODE accurately predicts the stationary locations of vortex dipole configurations including vertical and horizontal windings. Furthermore, as we will see in the next section, this extended reduced ODE will also accurately describe the stability properties of these stationary dipole configurations.

2. Vortex dipoles: stability

Equipped with the four steady state dipole solutions described in the previous section, let us now study their stability properties at the full NLS and reduced ODE levels. Figure 10 depicts the stability spectra for the four dipole solutions in both the full NLS (large colored dots) and the reduced ODE model (black curves) as \( \mu \) is varied. We note that the stability eigenvalues \( \lambda = \lambda_r + i\lambda_i \) determine the stability of the corresponding solutions as follows: (i) \( \lambda_r = 0 \) corresponds to a (neutrally) stable solution, (ii) \( \lambda_r \neq 0 \) and \( \lambda_i = 0 \) corresponds to an exponential instability, and (iii) \( \lambda_r \neq 0 \) and \( \lambda_i \neq 0 \) corresponds to an oscillatory instability. As Fig. 10 indicates, for the parameters used (namely \( \alpha = 0.4 \) and \( R = 12 \)), the vertical-out and horizontal-out dipoles are stable while the horizontal-in and diagonal dipoles are unstable. Importantly, the figure also evidences the striking match of the reduced ODE model and the PDE findings, with the former very accurately capturing the eigenvalues as-
FIG. 11: (Color online) Stability for dipole configurations as $\alpha$ is varied for $\mu = 1$ and $R = 12$. The columns correspond, from left to right, to the constant, vertical dipole-in, horizontal dipole-in, horizontal dipole-out, and diagonal dipole configurations. The top two rows depict, respectively, the density and phase of the steady state PDE solution for $\alpha = 0.5$ in Cartesian coordinates. The location of the vortices corresponding to the ODE model are overlayed to the density plot using blue squares. The third row of panels depicts the real part of the linearization eigenvalues $\lambda_r$ where the large (light blue) dots correspond to the full BdG spectra computation of the PDE while the black solid line depicts the corresponding results from the ODE model. The bottom two rows depict the imaginary part of the stability eigenvalues $\lambda_i$ where, again, the large (orange) dots correspond to the PDE while the black solid line corresponds to the ODE. The fourth row of panels is a zoomed in version of the bottom row.

FIG. 12: (Color online) Same as Fig. 11 but for $\mu = 5$. Notice the better match between the PDE and ODE results as the latter is obtained for the point-vortex model in the large $\mu$ limit.

associated with the motion of the vortices. Recall that the reduced ODE model is predicated on the condition of the vortices representing a point particle. This is certainly true in the large $\mu$ limit where the size of the vortex cores (tantamount to the healing length of $\propto 1/\sqrt{2\mu}$) tends to zero. However, even for relatively small values of $\mu$, the particle model prediction for the eigenvalues remains remarkably accurate. Indeed, even moderate values of $\mu \gtrsim 3$ converge such that the relative error for $\lambda_r$ and $\lambda_i$ are always less than 1%. However, it is important to mention that configurations bearing vortices that are closer than a few times their width will not be accurately cap-
For the stationary vertical dipoles without any winding ($W_x = 0$; left panels) and with one horizontal winding ($W_x = 1$; right panels) for $R = 12$ and $\mu = 5$; see top two rows of Fig. 8. Same layout and parameters as in Fig. 12. Note that the addition of a horizontal winding destabilizes the vertical dipole.

To complement the stability results we include in Fig. 13 the stability spectra of the vertical dipole solution alongside its corresponding dipole with a winding $W_x = 1$. As the figure suggests, adding a winding completely changes the stability picture by destabilizing the vertical dipole solution. This is in line with the general expectation that higher winding wave patterns are less likely to be dynamically robust than lower winding ones. Furthermore, we again obtain a remarkable agreement of the stability spectra results between the full NLS and, in this case, the extended reduced ODE model (17) including vertical and horizontal windings. Further studies on solutions bearing windings, including mixed combinations of vertical and horizontal windings, at the NLS and ODE levels are outside the scope of the present work and are thus left for future explorations.

### 3. Vortex dipoles: dynamics

In this section we present some results pertaining the dynamics of vortex dipole configurations. For instance, Fig. 7 depicts the dynamics for horizontal dipoles from the reduced ODE model. Interestingly, the stable and unstable manifolds of the unstable (saddle) horizontal dipole-in coincide in a homoclinic orbit and define a separatrix between oscillating (librating) solutions around the horizontal dipole-out and rotating solutions that wrap along the poloidal direction. In the case of $\alpha = 0.7$ we also include two NLS orbits obtained from initial positions as indicated in the figure and a phase profile given, as before, by \( \Phi(w) = \text{Im}(\sum_n q_n F(w, w_n)) \). The NLS orbits were extracted by doing a local fit of the extrema of the vorticity, defined as the curl of the velocity, with the latter defined as is standard in NLS settings.

The purple and green NLS orbits correspond, respectively, to typical oscillating and rotating orbits. Aligned with the stability results, the reduced ODE model accurately captures the nonlinear evolution for these orbits. This again supports the conclusion that the reduced ODE is an accurate, qualitative and quantitative, depiction.
not only for the statics and stability (as seen earlier), but also for the dynamics of vortex orbits in the torus.

In Fig. 14 we depict the dynamics ensuing from the destabilization of unstable dipole configurations. Specifically, the top left and right panels depict one period for, respectively, an oscillating and a rotating orbit. As discussed before, the horizontal dipole-in steady state corresponds to a saddle (cf. phase spaces of Fig. 7) whose separatrices separate regions with oscillating and rotating orbits. The oscillating orbit was obtained by slightly and symmetrically perturbing the vortices in the poloidal direction. Similarly, the rotating orbit was obtained by slightly and symmetrically perturbing in the toroidal direction. The bottom panel in Fig. 14 depicts a typical destabilization of the diagonal dipole. In this case, as the symmetry is already broken from the steady state, the destabilization dynamics follows windings along both toroidal and poloidal directions. Since the diagonal dipole has an angle that is close to horizontal (toroidal), the dipole has a relatively fast poloidal speed and a relatively slow toroidal drift. The ensuing orbit will be generically a quasi-periodic orbit (unless the windings along toroidal and poloidal are commensurate to each other).

B. Vortex Quadrupoles

1. Vortex Quadrupoles: steady states

While there exist only four steady state vortex dipoles, as the number of vortices is increased, a larger zoo of possibilities arises. Motivated by the remarkable agreement of the reduced ODE model with the original NLS, we have proceeded to leverage its use to identify possible quadrupole solutions in the full NLS model. Even when using the reduced ODE model, an exhaustive (ordered) search for quadrupoles (and higher order tuples), as it was performed for the vortex dipoles-in Sec. III A, is a challenging task. This is because of the commonly referred to “curse of dimensionality”. While for the vortex dipole, after eliminating the toroidal translational invariance, the reduced ODE model is left with three degrees of freedom, for the vortex quadrupole one has (after eliminating the translational invariance) seven degrees of freedom. Therefore, an exhaustive search over the whole phase space is computationally prohibitive. Thus, we revert to randomly sampling initial conditions over this seven-dimensional space (for all other parameters fixed; namely $R$ and $\alpha$) and using a standard fixed point iteration (nonlinear least squares with a Levenberg-Marquardt algorithm) to converge to the ‘closest’ steady state solution. Using several million initial conditions we were able to detect 16 distinct quadrupole configurations.
as depicted in Fig. 15 for \( R = 12 \) and \( \alpha = 0.5 \). By distinct we mean here that we have eliminated all the equivalent solutions (not only through toroidal translations but also) through symmetries associated with reflections across \( \theta = 0 \), symmetries associated with reversing the vortex charges, and permutations of the vortex labels. We have therefore obtained a rich palette of quadrupole solutions as depicted in Fig. 15. It is worth mentioning that these solutions have been ordered \( Q_1, \ldots, Q_{16} \) from the least unstable to the most unstable one (for the parameters at hand; namely \( R = 12 \) and \( \alpha = 0.5 \)). They include horizontal (\( Q_1 \)) and vertical aligned quadrupoles (\( Q_9 \)), rectangular quadrupoles (\( Q_2, Q_3, Q_6, Q_{12}, \) and \( Q_{16} \)), rhomboidal quadrupoles (\( Q_8, Q_{10}, Q_{11}, \) and \( Q_{15} \)), trapezoidal quadrupoles (\( Q_5, Q_{13}, \) and \( Q_{14} \)), and, somewhat surprisingly, irregular quadrupoles (\( Q_4 \) and \( Q_7 \)).

2. Vortex Quadrupoles: stability

Let us now comment on the stability for the obtained quadrupole solutions. We start by analyzing the stability obtained from the reduced ODE model.

**FIG. 16**: (Color online) Stability spectra \( (\lambda_r, \lambda_i) \) corresponding to the quadrupole solutions of Fig. 15 obtained from the reduced ODE model. Note that each row of panels has a different scaling for the real part of the eigenvalue as indicated. The configurations are ordered from least unstable to most unstable. In particular, the first three quadrupoles present a very weak instability of \( O(10^{-3}) \) for this value of the parameters \( (R = 12 \) and \( \alpha = 0.5) \).

**FIG. 17**: (Color online) Stability spectra as \( \alpha \) is varied in the reduced ODE model for the first (\( Q_1 \); top), second (\( Q_2 \); middle) and third (\( Q_3 \); bottom) quadrupole solutions (see Fig. 15) for \( R = 12 \). Blue and red curves correspond to the real \( (\lambda_r) \) and imaginary \( (\lambda_i) \) parts of the eigenvalue. The top-left insets depict the configurations for \( \alpha = 0.5 \). The \( Q_1 \) and \( Q_3 \) quadrupole solutions were the only ones found to have stability intervals as \( \alpha \) was varied. These stability intervals are depicted by the thick black lines. The bottom inset for \( Q_3 \) corresponds to a zoom around the stability interval.

...and thus the ordering of the quadrupole solutions as we have posited it, changes as the parameters of the system are varied. Therefore, it is relevant to look at the associated spectra as the parameters are varied. Particularly revealing for some of these solutions are the spectra continuation as the torus aspect ratio \( \alpha \) is varied. For compactness, we only show these full ODE spectra for the first three quadrupole configurations in Fig. 17. These configurations, as shown in Fig. 16, have a very weak instability \( (\max(\lambda_r) < 0.01) \) for \( \alpha = 0.5 \) (and \( R = 12 \)) and thus are potential candidates to be completely stable as the parameters are varied. As Fig. 17 shows, for \( R = 12 \), \( Q_2 \) is always unstable, however this instability is rather weak as it merely reaches \( \max(\lambda_r) \approx 0.0017 \) around \( \alpha \approx 0.73 \). On the other hand, \( Q_1 \) and \( Q_3 \) are not only weakly unstable for most values of \( \alpha \), but, importantly, they can be rendered stable on respective windows of the
We now turn to the study of the stability of the quadrupole configurations from the full NLS model and compare the latter with the results from the reduced ODE model. In particular, Fig. 18 shows the stability results from both the reduced ODE and the full NLS models for the Q1, Q2, and Q3 configurations as $\alpha$ is varied for $R = 12$ and $\mu = 1$. It is remarkable that, even with this moderately small value of $\mu$ ($\mu = 1$), the reduced ODE model is capable to recover the main stability eigenvalues of the full NLS for most parameter values. For instance, the complicated bifurcation scenario displayed by Q3 is perfectly captured by the ODE model (including the bifurcation leading to the narrow stability window around $\alpha \approx 0.66$). In general, we expect that the ODE model is able to properly capture the instabilities corresponding to the destabilization of the vortex positions. However, it is conceivable that there exists instabilities for full NLS solutions that are not captured by the reduced ODE model as the corresponding eigendirections might not be part of the space spanned by the latter. Surprisingly however, as it can be observed from the spectrum of Q2 in Fig. 18, the NLS configurations seem to stabilize for $\alpha \lesssim 0.48$ while the reduced ODE model predicts that the solution is (weakly) unstable for all values of $\alpha$. This tends to suggest that the actual spatial extent of the vortices, and their mutual interaction through the curvature of the background, may be responsible for this stabilization since the point-vortex model, relevant for the $\mu \to \infty$ limit, does not display this stabilization effect.

Finally in Fig. 19 we depict the convergence of the stability spectra for the Q1 and Q3 quadrupole configurations as the chemical potential $\mu$ is increased for $R = 12$ and $\alpha = 0.4$ and $\alpha = 0.7$ as indicated. Same layout as in Fig. 10.

parameter $\alpha$. Specifically, as the figure shows, for $R = 12$, the Q1 solution is stable for $0.84 \lesssim \alpha < 1$ and the Q2 solution is stable for the narrow interval $0.656 \lesssim \alpha \lesssim 0.664$ (see bottom inset for Q3). In addition to the stability windows for Q1 and Q3, it is also worth mentioning that the reduced model predicts that a few quadrupole solutions have relatively weak instabilities. For instance, as it can be seen in Fig. 16 the quadrupoles Q1 and Q2 have a max($\lambda_r$) $\simeq 10^{-3}$, while Q3 has a max($\lambda_r$) $\simeq 6 \times 10^{-3}$, and while Q4 has a max($\lambda_r$) $\simeq 9 \times 10^{-3}$, all for $\alpha = 0.5$ and $R = 12$. Therefore, for the parameter combinations that we explored, although we only found Q1 and Q3 to possess stability windows, the rather weak instabilities presented by about half of the quadrupole configurations (max($\lambda_r$) $< 0.02$), suggest that they have the potential to be long-lived solutions in the original NLS equation (cf. dynamics for Q1 in Sec. III B 3) and the corresponding physical experiments.
Let us now follow the destabilization dynamics for quadrupole configurations. Figure 20 depicts the dynamics ensuing for the first three quadrupole steady state configurations (Q1, Q2, and Q3). In particular, the top-left panel depicts the destabilization of the Q2 quadrupole for \( \alpha = 0.7 \) giving rise to approximately (about half of) a periodic orbit. Top-right: destabilization of the Q3 quadrupole for \( \alpha = 0.4 \). Bottom-left: apparently stable periodic orbit for a perturbed Q1 quadrupole for \( \alpha = 0.5 \) for \( 0 \leq t \leq 4000 \). Bottom-right: the apparently stable periodic orbit for a perturbed Q1 quadrupole for \( \alpha = 0.7 \). Eventually destabilizes for longer times \( 0 \leq t \leq 28000 \). The colored surface depicts the initial density and the overlaid curves correspond to the trajectory traces from the negative (red) and positive (blue) vortices. Please see Q2-R12-alpha07-mu1.gif, Q3-R12-alpha04-mu1.gif, Q1-R12-alpha05-mu1.gif, and Q1-R12-alpha07-mu1-t28K.gif in the Supplemental Material for respective movies depicting the evolution of the density and phase.

3. Vortex Quadrupoles: dynamics

Although in the case of the dipole the motions we have encountered seem to be prototypically quasi-periodic, in the case of quadrupole configurations, there are six degrees of freedom and despite the presence of a conserved energy, there is potential for complex (chaotic) behavior. Studies along these lines are outside the scope of the present paper and will be presented in future work.

C. Dark Soliton Stripes

Finally, let us briefly examine some of the properties of dark soliton stripe configurations and their connection with some of the vortex configurations of the previous sections. For instance, as Fig. 3 suggest, as \( \mu \) is decreased, the horizontal dipole-in seems to degenerate, as the vortices eventually merge, into a single horizontal stripe. It is important to first understand the nature of this single stripe. As is the case for vortices, in order to satisfy the periodicity of the domain, dark soliton stripes should appear in pairs as each one will contribute to a \( \pi \) phase jump for a total of a \( 2\pi \) phase jump conducive to a periodic domain. However, it is possible
to seed a single dark soliton if one adds a winding to add (or counter) the $\pi$ phase jump of a single stripe. Thus, it is in principle possible to construct single horizontal stripe configurations that are consonant the periodicity by adding a vertical winding $W_y = n + 1/2$ with $n$ integer. Likewise single vertical stripes are possible when adding a half-integer horizontal winding $W_x = n + 1/2$. In fact, it is possible to construct any number of odd stripes by adding half-integer windings. On the other hand, similarly to our discussion for vortices, from the perspective of the torus periodicity, it is possible for our system to feature even numbers of parallel dark stripes (without external winding) whose total phase change adds up to a multiple of $2\pi$.

In Fig. 21 we depict more details of the bifurcation between the horizontal-in single stripe and the corresponding dipole solutions. In particular, we follow the effective power of the solutions by computing:

$$P = \int_{0}^{2\pi} (\mu - |\psi|^2) \, dS,$$

where the surface element on the torus is $dS = |\hat{\phi} \times \hat{\theta}| \, d\theta d\phi$. Here hats denote the unit vectors in the different (toroidal and poloidal) directions. Thus, $|\hat{\phi} \times \hat{\theta}| = R + r \cos(\theta)$. As the top two panels of Fig. 21 evidenced, the horizontal-in dipole bifurcates from the horizontal-in single stripe at $\mu \approx 0.17$. The bottom two rows of panels in Fig. 21 depict sample solutions before and after the bifurcation. The phase panels indicate that the stripe has indeed an extra vertical winding of $W_y = -1/2$ and that the vortices of the dipole merge as $\mu$ decreases towards the bifurcation at $\mu \approx 0.17$.

In a similar fashion as the horizontal dipole-in bifurcates from the single horizontal-in stripe, multiple other bifurcations are present involving single and double (and triple, etc.), in and out, vertical and horizontal dark stripes and vortex configurations. In fact, we have observed (not shown here) mixed bifurcations where, as $\mu$ increases, a double horizontal stripe, containing an in and an out stripe, first features a bifurcation towards a mixed state containing a vortex out configuration and a stripe in which, subsequently, after further increase in $\mu$, displays a bifurcation for the stripe in leading to a multi-vortex state. An in-depth analysis of the possible bifurcations involving the above configurations, albeit interesting, falls outside of scope of the present work and will be reported elsewhere. Nonetheless, we present here a collection of examples that showcase some of the most basic bifurcations and ensuing dynamics involving dark soliton stripes.

Figure 22 depicts the bifurcation spectra for the horizontal-in single stripe (left) and the vertical double stripe (right). The spectra for both stripe configurations display a series (cascade) of bifurcations as $\mu$ departs from zero. Each bifurcation is associated with the creation of an offspring configuration where each stripe is replaced by vortices. The higher the bifurcation in the cascade the more vortices are produced. This cascading bifurcation is akin to the bifurcation of dark soliton stripes and rings in parabolically trapped BECs as reported, e.g., in Ref. [39]. In Fig. 23 we portray elements of these bifurcations by perturbing the stripe steady states by the eigenfunction corresponding to the most unstable eigenvalue as shown in Fig. 24. The figure depicts the stripe
steady states (left), their corresponding most unstable eigenfunction (middle), and the steady state perturbed by the eigenfunction (right). In these cases we normalized the eigenfunction such that its maximum density (norm) coincided with the chemical potential $\mu$ of the steady state and we added it using a large perturbation prefactor equal to two. This was done for presentation purposes to exaggerate the visual effects of the perturbation (for the actual dynamical runs presented in Fig. 24 we used a small prefactor for the relevant perturbation of $10^{-3}$). As Fig. 23 shows, the different eigenfunctions bend the stripes into snaking modes with a higher number of relevant undulations as $\mu$ increases and the higher bifurcations in the cascade are reached. Specifically, the first two rows of panels in the figure present the first two unstable modes of the horizontal-in single stripe each giving rise to an aligned quadrupole and hexapole, respectively, after the system is left to evolve as depicted in the first two rows of Fig. 24. On the other hand, the third and fourth rows of panels in Fig. 23 depict the first two excited modes of the double vertical stripe which in this case give rise, respectively, to a Q2-like quadrupole (see $t = 250$) and an octupole (see $t = 120$). The corresponding destabilization dynamics are shown in the third and fourth rows of Fig. 24. Note that the destabilization along the second mode of the vertical double stripe (fourth row of panels) initially generates a vortex octupole ($t = 120$). However, as time progresses, the outer two vortex pairs on each side of the torus merge and produce two “lumps” that travel, in opposite directions, along the toroidal direction until they are eventually destroyed and contribute to background radiation. These lumps correspond to solitonic structures dubbed Jones-Roberts solitons [40], which have been observed in recent BEC experiments [41]. These are quite interesting to explore in their own right in the realm of traveling solutions in the torus setting. While our exploration herein has been restricted (due to their extensive wealth, as we have tried to argue) to stationary states, it does not escape us that such traveling waveforms, including ones involving vorticity are of particular interest in their own right for future studies.

IV. CONCLUSIONS AND OUTLOOK

In this work we have attempted to give a systematic and extensive (although by no means exhaustive) study of the existence, stability and dynamics of dark and vortical structures in the nonlinear Schrödinger (NLS) equation on the surface of a torus as the torus aspect ratio $\alpha$ and the chemical potential $\mu$ of the solutions are varied. We chiefly study vortex dipoles and quadrupoles but also touch upon dark soliton stripes and their connections to the former (through bifurcation cascades). To gain insight into the statics, stability, and dynamics of vortex configurations on the full NLS model, we have importantly leveraged the key insights offered by a remarkably successful (as we illustrate) reduced particle model, introduced in Ref. [34], based upon assuming vortices without internal structure (i.e., point-vortices) that incorporates both vortex-vortex interactions and the effects of space curvature on the surface of the torus. We also extend this reduced particle model to incorporate vertical and horizontal phase windings that induce monotonic flows along the, respectively, poloidal and toroidal directions of the torus. We show that this (fundamental within this setting) reduced particle model is extremely accurate at predicting the statics, stability, and dynamics of dipole configurations even for moderate values of the chemical potential. We have also discussed the potential limitations of the model in regimes of low chemical potential,
or, in some cases, small enough aspect ratios. The balance between vortex-vortex interactions and the curvature effects gives rise to four different types of dipole solutions: (i) the vertical dipole-in, (ii) the horizontal dipole-in (vortices close to the inside part of the torus), (iii) the horizontal dipole-out (vortices close to the outside part of the torus), and (iv) the diagonal dipole. The vertical-in and horizontal-out dipoles are (neutrally) stable for a wide range of parameters while the horizontal-out and diagonal dipoles are chiefly unstable. The source of the relevant (stability or) instability via an eigendirection associated with the relative motion of the vortices has been identified and the related unstable dynamics also elucidated. Nonetheless, for thick tori (large torus aspect ratio) it is possible to stabilize the diagonal vortex dipole for sufficiently large chemical potentials.

We also explore vortex quadrupole configurations. We found 16 different quadrupoles ranging from horizontal and vertical aligned quadrupoles, to rectangular and rhomboidal quadrupoles, to trapezoidal quadrupoles and, even some irregular quadrupoles. All these solutions are continued and monitored for stability as the torus aspect ratio is varied within the reduced ODE model. Out of these 16 quadrupoles we found two that exhibited windows of stability, upon variations of the aspect ratio. We also found a handful of stationary quadrupole solutions with very weak instabilities indicating that it may be possible to observe them as long-lived solutions at the full NLS level. Relevant stability considerations were also presented at the PDE level and indeed, surprisingly it was found that some configurations, such as the rectangular Q2 could be more robust at the latter level and indeed, surprisingly it was found that some configurations, such as the rectangular Q2 could be more robust at the latter level and indeed, genuinely stable for sizeable parametric intervals therein. Finally, we briefly touch upon dark soliton stripe configurations. These stripes can be single or double (or triple, etc.), horizontal or vertical, and be centered about the torus—inner or outer side of the torus. Particularly interesting are the bifurcations of vortex configurations from these stripes as µ increases from the low density limit in a series of bifurcating cascades corresponding to increasing number of vortices, reminiscent of ones emerging for stripe, as well as ring configurations in regular two-dimensional parabolically confined BECs.

A natural extension of this work would be to study vortex configurations with higher number of vortices. It would be indeed interesting to see if configurations for higher number of vortices can be rendered stable for the right parameter windows in a manner akin to what we detected for a couple of the quadrupole configurations (Q1 and Q3). It is likely that configurations with higher number of vortices will be difficult to be stabilized. Indeed, we encountered some such configurations transiently in our dynamics (e.g., stemming from unstable stripes). It would also be interesting to understand in more detail how some full NLS solutions seem to be more stable than their effective ODE model counterparts (cf. Q2 quadrupole). On the other hand, it would also be relevant to study in more detail the bifurcation cascades between the different stripes—single or double (or triple, etc.), in or out, vertical or horizontal—and stationary vortex configurations. This is an intricate endeavor as these cascades are very much dependent on the aspect ratio of the torus (α). For instance, a very thin torus (α → 0) will preferentially promote the merger of vortices along the poloidal direction. However, when the two torus radii are similar (α → 1) both mergers along toroidal and poloidal directions will nontrivially compete.

Finally, further leveraging of the reduced ODE model could be employed to study the existence and stability of periodic vortex orbits in a manner akin to vortex choreographies in the plane [42]. Indeed, an example of such periodic solutions, returning to themselves upon running around the torus are traveling solutions, such as the lump ones spontaneously encountered herein. Given their potential connection to so-called KP-lumps [43], this is an interesting direction in its own right. Indeed, given the success of the particle model herein, exploring additional directions such as the potential ordered and chaotic orbits [44] at the low-dimensional dynamical systems level could also hold some appeal. Lastly, we hope that this fruitful comparison of ODE and PDE dynamics at the level of the torus will springboard further related comparative studies in the context of other nontrivial geometric settings, including spherical, cylindrical and conical shells, among others.

Acknowledgments

J.D.A. gratefully acknowledges computing support based on the Army Research Office ARO-DURIP Grant W911NF-15-1-0403. R.C.G. gratefully acknowledges support from the US National Science Foundation under Grants PHY-1603058 and PHY-2110038. This material is based upon work supported by the US National Science Foundation under Grants DMS-1809074, PHY-2110030 (P.G.K.).

[1] L.P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity*, Oxford University Press (Oxford, 2016).
[2] C.J. Pethick and H. Smith, *Bose-Einstein condensation in dilute gases*, Cambridge University Press (Cambridge, 2002).
[3] P.G. Kevrekidis, D.J. Frantzeskakis and R. Carretero-González, *The Defocusing Nonlinear Schrödinger Equation*, SIAM (Philadelphia, 2015).
[4] See, e.g., N.P. Proukakis, *Beyond Gross-Pitaevskii Mean-Field Theory* in P.G. Kevrekidis, D.J. Frantzeskakis and R. Carretero-González, *Emergent Nonlinear Phenomena*
