LOCAL TERMS OF THE MOTIVIC VERDIER PAIRING

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ABSTRACT. We prove an analogue of a theorem of Varshavsky in the motivic stable homotopy category, which states that for a contracting correspondence, the local terms of the Verdier pairing agree with the naive local terms. We also show that some $A^1$-enumerative invariants, such as the local $A^1$-Brouwer degree and the Euler class with support, can be interpreted as local terms.

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1. INTRODUCTION

In Grothendieck’s approach to the Weil conjectures via the theory of étale cohomology, a trace formula, analogous to the Lefschetz fixed-point theorem in algebraic topology, plays a crucial role: it states that the number of rational points of an algebraic variety over a finite field can be computed as the alternating sum of the traces of the Frobenius morphism acting on (compactly supported) étale cohomology groups ([Gro66]). This formula is later generalized to what is now called the Lefschetz-Verdier formula ([SGA5, III Corollaire 4.7]), expressing the proper covariance of a cohomological pairing called the Verdier pairing. In both cases, the computation of the local contributions to the global trace is an interesting but difficult problem. The case of curves has been discussed in [SGA5, IIIb]; in the topological case, the work of [GM93] provides a practical formula for a class of maps called weakly hyperbolic; over finite fields, the problem is studied at length in [Pin92] and [Fuj97], where a conjecture of Deligne is eventually proved; in [Var07] the computation of local terms is generalized to contracting correspondences.

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Recent works on enumerative invariants given by intersection numbers in the framework of motivic homotopy theory (for example [Hoy15], [Lev17], [LR20], [KW19]) have lead to a theory of \(\mathbb{A}^1\)-enumerative geometry, where one define and study arithmetic refinements of classical invariants in terms of symmetric bilinear forms. The Verdier pairing in the motivic context, which can be reduced to a (generalized) trace map (see Definition 3.2.2 and Remark 3.2.7), is considered in [Ols16] and [Cis19] for \(\acute{e}tale\) motives, and generalized in [JY20] to the motivic stable homotopy category, assuming resolution of singularities.

The main goal of this paper is to prove the following analogue of Varshavsky’s theorem ([Var07, Theorem 2.1.3]):

**Theorem 1.0.1** (see Theorem 4.1.9). For a contracting correspondence (see Definition 4.1.1 and Definition 4.1.7), the local terms agree with the naive local terms.

Here “local term” is the local contribution to the trace (Definition 4.1.7); the “naive local term” can be understood as follows, in the case where the correspondence is given by an endomorphism: if \(X\) is a scheme, \(f : X \to X\) is an endomorphism of \(X\), \(K \in \mathcal{SH}_c(X)\) is a constructible motivic spectrum over \(X\), and \(u : f^*K \to K\) is a map, then for every fixed point \(x\) of \(X\), the local term \(LT_x(u)\), if it is well defined, is the trace of the induced map \(u_x : K|_x \to K|_x\). The basic ideas of the proof are very close to [Var07], with the following ingredients:

1. The use of additivity of traces ([Var07, Proposition 1.5.10]), which states that the trace map is additive along distinguished triangles. Note that the general statement fails for symmetric monoidal triangulated categories ([Per05]), and the proof of [Var07, Proposition 1.5.10] uses a variant for the filtered derived category. Our treatment here (Lemma 3.2.6) uses the language of higher categories, for which we refer to [JY20, §4] for a more detailed discussion.
2. The geometric fact that the under the deformation to the normal cone, a contracting correspondence becomes supported at the zero section of the normal cone ([Var07, Remark 2.1.2 (b))].

A major difficulty in adapting Varshavsky’s proof into the motivic setting is the use of the nearby cycle functor, which in the motivic context is not known to be as powerful as in the \(\acute{e}tale\) setting ([Ayo07a, §3], [Ayo07b]). In this paper we circumvent this difficulty by using two previously unexploited constructions in the literature:

1. The specialization map constructed in [DJK18, 4.5.6], which is modeled on Fulton’s specialization map on Chow groups ([Ful98, §20.3]). The construction uses the intersection theory developed in [DJK18], and allows us to perform the specialization of correspondences (3.3.12) using the deformation to the normal cone à la Fulton (4.2).
2. The trace map over a base scheme, which was first introduced in [YZ19] and [JY20] under some smoothness and transversality assumptions, and generalized in [LZ20] to the even singular base schemes. We show that after minor changes, the arguments of [LZ20] also work for motives (3.1), which enables us to define the trace map over a base scheme instead of a field (Definition 3.2.2). This not only makes it easier to work with the deformation of correspondences, but also gives us the flexibility to work over a base, opening paths for possible further generalizations.

In Section 4.3 we show that some \(\mathbb{A}^1\)-enumerative invariants in the literature, such as the local \(\mathbb{A}^1\)-Brouwer degree and the Euler class with support, can be interpreted as local terms. So one may expect to apply our main result to give some explicit computations of these invariants.
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2. Preliminaries

2.1. Notations and conventions.

2.1.1. All schemes are assumed quasi-compact and quasi-separated, and all morphisms of schemes are separated of finite type. Regular schemes are assumed noetherian.

2.1.2. Throughout the paper, we fix $\mathcal{T}$ a motivic $\infty$-category ([Kha16]). The prototype is the motivic stable homotopy category $\text{SH}$, and by [DG20, Theorem 7.14] it turns out that $\text{SH}$ is the universal $\infty$-category which satisfies the six functor formalism.

2.1.3. If $f : X \to Y$ is a morphism and $K \in \mathcal{T}(Y)$, we denote $K|_X = f^*K \in \mathcal{T}(X)$.

2.1.4. If $f : X \to Y$ is a local complete intersection morphism ([Ful98, §6.6]), we denote by $\tau_f \in K_0(X)$ the virtual tangent bundle of $f$ ([DJK18, Example 2.3.10]).

2.1.5. For $f : X \to S$, let $K_{X/S} = f^!1_S$, and

\[
\mathbb{D}_{X/S} : \mathcal{T}(X) \to \mathcal{T}(X)
M \mapsto \text{Hom}(M, K_{X/S}).
\]

2.1.6. We say that a perfect field $k$ satisfies strong resolution of singularities if the following conditions hold:

1. For every separated integral scheme $X$ of finite type over $k$, there exists a proper birational surjective morphism $X' \to X$ with $X'$ regular;
2. For every separated integral regular scheme $X$ of finite type over $k$ and every nowhere dense closed subscheme $Z$ of $X$, there exists a proper birational surjective morphism $b : X' \to X$ such that $X'$ is regular, $b$ induces an isomorphism $b^{-1}(X - Z) \simeq X - Z$, and $b^{-1}(Z)$ is a strict normal crossing divisor in $X'$.

2.2. Bivariant theory in motivic categories.

2.2.1. If $X \to S$ is a separated morphism of finite type and $v \in K(X)$ is a virtual vector bundle, define the $v$-twisted bivariant spectrum as the mapping spectrum

\[
H(X/S, v) = \text{Map}(Th(v), K_{X/S}).
\]

In particular we denote $H(X/S) = H(X/S, 0) = \text{Map}(1_X, K_{X/S})$. In what follows we recall its functorialities established in [DJK18].
2.2.2. Base change. ([DJK18, 2.2.7 (1)]) For any Cartesian square

\[
\begin{array}{cccc}
Y & \to & X \\
\downarrow q & & \downarrow f \\
Z & \to & S
\end{array}
\]

there are canonical maps

\[
\Delta^* : H(Z/S) \to H(Y/X)
\]

(2.2.2.2)

\[
\Delta'^* : H(X/S) \to H(Y/Z)
\]

(2.2.2.3)

2.2.3. Proper push-forward. ([DJK18, 2.2.7 (2)]) If \( p : X \to Y \) is a proper morphism, then there is a push-forward map \( p_* : H(X/S) \to H(Y/S) \).

2.2.4. (Refined) pullback. ([DJK18, Definition 4.2.5]) Consider a Cartesian square as in (2.2.2.1). Assume that \( p \) is local complete intersection, that is, can be factored as a regular closed immersion followed by a smooth morphism. Then there is a map

\[
\Delta^! : H(X/S) \to H(Y/S, g^* \tau_p)
\]

(2.2.4.1)

induced by the refined fundamental class of \( p \). In particular, if \( q : Y \to X \) is étale, then we have \( q^* : H(X/S) \to H(Y/S) \) ([DJK18, 2.2.7 (3)]).

**Lemma 2.2.5.** Consider a Cartesian square as in (2.2.2.1) Assume that \( p \) is a regular closed immersion and \( 1_S \) is \( p \)-pure. Then the following diagram is commutative:

\[
\begin{array}{ccc}
H(X/S) & \xrightarrow{\Delta'^*} & H(Y/Z) \\
\downarrow \Delta & & \downarrow \sim \\
H(Y/S, -g^* NZS) & & 
\end{array}
\]

(2.2.5.1)

**Proof.** This follows from the definition of the map \( \Delta^! \) in (2.2.4.1) and [DJK18, 2.2.13]. \( \square \)

2.2.6. Specialization. ([DJK18, 4.5.6]) Let \( i : Z \to S \) be a regular closed immersion and let \( j : U \to S \) be the open complement. Assume that there is a null-homotopy \( e(N_ZS) \simeq 0 \). Let \( f : X \to S \) be a separated morphism of finite type, and consider the Cartesian square

\[
\begin{array}{cccc}
X_Z & \xrightarrow{i_X} & X & \xrightarrow{j_X} X_U \\
j_Z \downarrow & & \downarrow f & \downarrow j_U \\
Z & \xleftarrow{i} & S & \xleftarrow{j} U.
\end{array}
\]

(2.2.6.1)

For any object \( A \in \mathcal{T}(X) \), we have the composition

\[
i_{X*}(i_X^* A \otimes Th(-N_ZS)|_{X_Z}) \to i_{X*}i_X^! A \to A \to i_{X*}i_X^* A
\]

(2.2.6.2)

where the first map is induced by the refined fundamental class of \( i \) ([DJK18, Definition 4.2.5]). By the self-intersection formula ([DJK18, Corollary 4.2.3]), the map (2.2.6.2) agrees with the multiplication by the class \( f_Z^* e(N_ZS) \), which is null-homotopic by assumption. By the localization sequence, we obtain a natural transformation of functors

\[
i_{X*}(i_X^* A \otimes Th(-N_ZS)|_{X_Z}) \to j_X^! j_X^* A
\]

(2.2.6.3)
Now assume that $\mathbb{1}_S$ is $i$-pure. Then the map (2.2.6.3) induces the following specialization map:

\begin{equation}
H(X_U/U) = H(X_U/S) \to H(X_Z/S, -(N_{ZS})_{X_Z}) \simeq H(X_Z/Z).
\end{equation}

**Lemma 2.2.7.** Assume that $i : Z \to S$ has a smooth retraction $S \to Z$. Assume that $X \to S$ is a topologically trivial family, i.e. there exists a $Z$-scheme $Y$ and an isomorphism $X_{\text{red}} \simeq (Y \times_Z S)_{\text{red}}$. Then the following composition is an isomorphism:

\begin{equation}
H(Y/Z) \xrightarrow{(2.2.2.3)} H(Y \times_Z U/U) \simeq H(X_U/U) \xrightarrow{(2.2.6.4)} H(X_Z/Z).
\end{equation}

**Proof.** It follows from the localization sequence that for any $T$-scheme $W$, the canonical morphism $W_{\text{red}} \to W$ induces an isomorphism $H(W_{\text{red}}/T) \simeq H(W/T)$. Therefore to prove the claim we may assume that $X = Y \times_k S$ is a trivial family. By construction of the map (2.2.6.4), it suffices to show that the composition

\begin{equation}
H(Y/Z) \xrightarrow{\Delta_2^*} H(Y \times_Z S/S) \xrightarrow{\Delta_1^*} H(Y/S, -(N_{ZS})_Y) \simeq H(Y/Z)
\end{equation}

is identity, where $\Delta_1$ and $\Delta_2$ are Cartesian squares in the diagram

\begin{equation}
\begin{array}{ccc}
Y & \to & Y \\
\downarrow \Delta_1 & & \downarrow \Delta_2 \\
Z & \to & S \\
\end{array}
\end{equation}

By Lemma 2.2.5, the map (2.2.7.2) agrees with the composition

\begin{equation}
H(Y/Z) \xrightarrow{\Delta_2^*} H(Y \times_Z S/S) \xrightarrow{\Delta_1^*} H(Y/Z)
\end{equation}

which is identity since the formation of the map (2.2.2.3) is compatible with composition of squares. \hfill $\square$

3. **Relative Künneth formulas, correspondences and the trace**

3.1. **Relative Künneth formulas.** In this section we prove Künneth formulas over a base scheme. Our arguments are similar to [LZ20, §2.1].

**Definition 3.1.1 (JY20, Definition 2.1.7).** Let $f : X \to S$ be a morphism of schemes and $K \in \mathcal{T}(X)$. We say that $K$ is strongly locally acyclic over $S$ if for any Cartesian square

\begin{equation}
\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
\downarrow g & & \downarrow f \\
T & \xrightarrow{p} & S \\
\end{array}
\end{equation}

and any object $L \in \mathcal{T}(T)$, the canonical map $K \otimes f^*p_*L \to q_*(q^*K \otimes g^*L)$ is an isomorphism. We say that $K$ is universally strongly locally acyclic (abbreviated as USLA) over $S$ if for any morphism $T \to S$, the base change $K|_{X \times_S T}$ is strongly locally acyclic over $T$.

The category of USLA objects form a localizing subcategory. Typical examples are given by dualizable objects for smooth morphisms:

**Lemma 3.1.2.** If $f : X \to S$ is smooth, any dualizable object in $\mathcal{T}(X)$ is USLA over $S$.

**Proof.** This follows from smooth base change and [FHM03, Proposition 3.2]. \hfill $\square$

Further examples are given by any constructible object when $S$ is the spectrum of a field, assuming resolution of singularities:
Lemma 3.1.3. Let $k$ be a field such that one of the following conditions is satisfied:

1. $k$ is a perfect field which satisfies strong resolution of singularities (2.1.6);
2. The motivic $\infty$-category $\mathcal{T}$ is $\mathbb{Z}[1/p]$-linear, where $p$ is the exponential characteristic of $k$.

Then for every separated $k$-scheme of finite type $X$, every object of $\mathcal{T}(X)$ is USLA over $k$.

Proof. This is basically [JY20, Corollary 2.1.14]. Note that the proof in the second case in loc. cit. requires the existence of a premotivic adjunction $\mathcal{SH} \rightleftarrows \mathcal{T}$, which is automatic by [DG20, Theorem 7.14].

Lemma 3.1.4. Let $Y \to S$ be a morphism and let $M \in \mathcal{T}(Y)$ be USLA over $S$. Let $f : X \to X'$ be a separated $S$-morphism of finite type, and let $f_Y = f \times_S id_Y : X \times_S Y \to X' \times_S Y$.

1. For any $L \in \mathcal{T}(X)$, there is a canonical isomorphism

\[
(3.1.4.1) \quad f_* L \boxtimes_S M \simeq f_{Y*}(L \boxtimes_S M).
\]

2. For any $L' \in \mathcal{T}(X')$, there is a canonical isomorphism

\[
(3.1.4.2) \quad f_! L' \boxtimes_S M \simeq f_{Y!}(L' \boxtimes_S M).
\]

Proof. (1) This is a reformulation of the definition of being USLA.

2. We may assume that $f$ is smooth or a closed immersion. The smooth case follows from relative purity. If $f$ is a closed immersion, let $j$ be the complementary open immersion and let $j_Y = j \times_S id_Y$. Then we have a commutative diagram

\[
\begin{array}{ccc}
\overset{f_!}{f_! L'} \boxtimes_S M & \longrightarrow & f_* L' \boxtimes_S M \\
\downarrow & & \downarrow \\
\overset{f_Y}{f_Y(L' \boxtimes_S M)} & \longrightarrow & f_{Y*}(L' \boxtimes_S M)
\end{array}
\]

\[
\begin{array}{ccc}
& & \overset{1}{-1} \\
\overset{f_Y}{f_Y(L' \boxtimes_S M)} & \longrightarrow & f_{Y*}(L' \boxtimes_S M) \\
\downarrow & & \downarrow \\
\overset{f_Y}{f_{Y*}j_Y(L' \boxtimes_S M)} & \longrightarrow & f_{Y*}j_Y(L' \boxtimes_S M)
\end{array}
\]

where both rows are distinguished triangles. The middle vertical map is an isomorphism, and the right vertical map is an isomorphism by (1). Therefore by five lemma the left vertical map is also an isomorphism.

The following corollary is the special case of (3.1.4.2) for $L' = 1$:

Corollary 3.1.5. Let $Y \to S$ be a morphism and let $M \in \mathcal{T}(Y)$ be USLA over $S$. Then for any separated $S$-scheme $X$ of finite type, there is a canonical isomorphism

\[
(3.1.5.1) \quad \mathcal{K}_{X/S} \boxtimes_S M \simeq p_Y^! M
\]

where $p_Y : X \times_S Y \to Y$ is the projection.

Proposition 3.1.6. Let $Y \to S$ be a morphism and let $M \in \mathcal{T}(Y)$ be USLA over $S$. Let $X$ be a separated $S$-scheme of finite type, and denote by $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ the projections. Then for any constructible object $L \in \mathcal{T}_c(X)$, there is a canonical isomorphism

\[
(3.1.6.1) \quad \mathbb{D}_{X/S}(L) \boxtimes_S M \simeq Hom(p_X^* L, p_Y^! M).
\]

Proof. We may assume that $L = \phi_* \mathbb{1}_W$ for some proper morphism $\phi : W \to X$. Denote by $\phi_Y : W \times_S Y \to X \times_S Y$ the base change. Then we have

\[
(3.1.6.2) \quad \phi_* \mathcal{K}_{W/S} \boxtimes_S M \simeq \phi_{Y*}(\mathcal{K}_{W/S} \boxtimes_S M) \simeq \phi_{Y*} Hom(\mathbb{1}_{W \times_S Y}, \phi_Y^! p_Y^! M) \simeq Hom(p_X^* \phi_* \mathbb{1}_W, p_Y^! M)
\]
where the first isomorphism follows from Lemma 3.1.4 and the second isomorphism from Corollary 3.1.5.

\[ \square \]

3.2. **Correspondences and trace.** Using Proposition 3.1.6, we can define the trace map over a base as in [JY20, Proposition 3.2.8].

3.2.1. For \( X \to S \) a morphism, denote by \( p_1, p_2 : X \times_S X \to X \) the projections. A **correspondence** is a morphism of the form \( c : C \to X \times_S X \). We denote by \( c_1, c_2 : C \to X \) the compositions of \( c \) with \( p_1 \) and \( p_2 \). Consider the following Cartesian diagram
\[
\begin{array}{ccc}
\text{Fix}(c) & \xrightarrow{c'} & X \\
\downarrow & & \downarrow \delta_{X/S} \\
C & \xrightarrow{\alpha} & X \times_S X.
\end{array}
\]

Let \( K \in \mathcal{T}_c(X) \) be USLA over \( S \). A (cohomological) **correspondence over** \( c \) is a map of the form \( u : c_1^!K \to c_2^!K \). Given such a map, we have the composition
\[
\mathbb{1}_C \xrightarrow{u} \mathbb{X} \text{Hom}(c_1^!K, c_2^!K) \simeq c_1^!\text{Hom}(p_1^*K, p_2^!K) \overset{\text{(3.2.1.1)}}{\simeq} c_1^!(\mathbb{D}_{X/S}(K) \otimes_S K).
\]

which gives rise to the following map
\[
\begin{align*}
\delta_{X/S}^* & \equiv \mathbb{1}_{\text{Fix}(c)} \xrightarrow{\mathbb{1}_C \otimes \mathbb{1}_C} \delta_{X/S}^* \mathbb{C}_C \otimes \mathbb{C}_C^!(\mathbb{D}_{X/S}(K) \otimes_S K) \\
& \simeq \mathbb{D}_{X/S}(K) \otimes K \simeq K \otimes \mathbb{D}_{X/S}(K) \to \mathcal{K}_{X/S}.
\end{align*}
\]

Note that every \( S \)-endomorphism \( f : X \to X \) can be viewed as a correspondence via the transpose of the graph morphism \( X \xrightarrow{(f, \text{id})} X \times_S X \).

**Definition 3.2.2.** For a correspondence \( u : c_1^!K \to c_2^!K \), we denote by \( \text{Tr}(u/S) : \mathbb{1}_{\text{Fix}(c)} \to \mathcal{K}_{\text{Fix}(c)/S} \) the map obtained by adjunction from the map (3.2.1.3). This construction gives rise to a canonical map called the **trace map**
\[
\begin{align*}
\text{Map}(c_1^!K, c_2^!K) & \xrightarrow{\text{Tr}(-/S)} \text{H}(\text{Fix}(c)/S).
\end{align*}
\]

More generally, we have the following twisted variant, constructed in the same fashion: if \( v \) is a virtual vector bundle on \( C \), then the same construction gives rise to a canonical map
\[
\begin{align*}
\text{Map}(c_1^*K, c_2^!K \otimes \text{Th}(v)) & \xrightarrow{\text{Tr}(-/S)} \text{H}(\text{Fix}(c)/S, -v|_{\text{Fix}(c)}).
\end{align*}
\]

3.2.3. For a regular scheme \( S \), we say that \( \mathcal{T} \) satisfies constructibility for \( S \) if the six functors between separated \( S \)-schemes of finite type preserve constructible objects; we say that \( \mathcal{T} \) satisfies local duality for \( S \) if for any separated \( S \)-scheme of finite type \( X \), the object \( \mathcal{K}_{X/S} \in \mathcal{T}(X) \) is a dualizing object ([CD19, Definition 4.4.19]).

3.2.4. For \( S \) a regular excellent scheme and \( \mathcal{T} \) a continuous motivic \( \infty \)-category ([DFKJ20, Definition A.2]), the constructibility and local duality condition hold in the following cases:

1. The scheme \( S \) is defined over a perfect field which satisfies strong resolution of singularities (2.1.6);
2. The scheme \( S \) is defined over a field of exponential characteristic \( p \), and the motivic \( \infty \)-category \( \mathcal{T} \) is \( \mathbb{Z}[1/p] \)-linear;
3. The motivic \( \infty \)-category \( \mathcal{T} \) is \( \mathbb{Q} \)-linear.
By virtue of [DG20, Theorem 7.14], the first case is [Ayo07a, Théorèmes 2.2.37 and 2.3.73]; the second case is [EK20, Corollary 2.1.7] and [BD17, Theorem 2.4.9]; the last case is [DFKJ20, Propositions 3.3 and 3.4].

3.2.5. The following lemma states the additivity of traces along distinguished triangles, where the structure of higher categories plays a key role. We refer to [JY20, §4] for a detailed discussion.

**Lemma 3.2.6 (Additivity of traces).** Assume that $S$ is a regular excellent scheme such that $\mathcal{T}$ satisfies constructibility and local duality for $S$. Let $L \to M \to N$ be a cofiber sequence in $\mathcal{T}(X)$ of USLA objects over $S$, and let

\begin{equation}
\begin{array}{c}
c_1^1L \to c_1^1M \to c_1^1N \\
\downarrow u_L \downarrow u_M \downarrow u_N \\
c_2^1L \to c_2^1M \to c_2^1N
\end{array}
\end{equation}

be a morphism of cofiber sequences (in the $\infty$-categorical sense). Then there is a canonical homotopy between $Tr(u_M/S)$ and $Tr(u_L/S) + Tr(u_N/S)$ as maps $\mathbb{1}_{Fix(c)} \to K_{Fix(c)/S}$.

**Proof.** The additivity is proved in [JY20, Proposition 4.2.6] when $S$ is the spectrum of a field, using the language of motivic derivators ([Ayo07a, Définition 2.4.48]); by a similar argument the proof also works for a general base scheme $S$. On the other hand, every motivic $\infty$-category gives rise to a homotopy derivator ([Bal18]), where all the relevant structures are transported, so the additivity also holds in this framework.

**Remark 3.2.7.** We may readily extend the trace map to the Verdier pairing over a base $\langle u, v \rangle_S$, in the same fashion as [SGA5, III 4.1], [JY20, Definition 3.1.8] or [LZ20, §2.4]. It turns out that the computation of the Verdier pairing reduces to that of the trace map via the identity $\langle u, v \rangle_S = \langle vu, 1 \rangle_S$, see [JY20, Proposition 3.2.5], and the additivity in Lemma 3.2.6 can also be extended as in [JY20, Theorem 4.2.8]. In this paper we focus on trace maps.

3.3. **Operations on correspondences.** We discuss four types of operations on correspondences: base change, pullback, push-forward and specialization, and show that they are all compatible with the trace map.

3.3.1. **Base change.** Let $c : C \to X \times_S X$ and $T \to S$ be two morphisms. Let $Y = X \times_X T$. Then there is a canonical morphism $c_T : C_T = C \times_S T \to Y \times_Y Y$. Let $K \in \mathcal{T}(X)$. Given a correspondence $u : c_1^1K \to c_2^1K$, we have the following composition

\begin{equation}
\begin{array}{c}
c_{T1}^*K|_Y = (c_1^*K)|_{C_T} \xrightarrow{u} (c_2^*K)|_{C_T} \xrightarrow{e_{T2}^1K|_Y} c_{T2}^1K|_Y.
\end{array}
\end{equation}

This construction gives rise to a canonical map

\begin{equation}
\begin{array}{c}
Map(c_1^1K, c_2^1K) \to Map(c_{T1}^*K|_Y, c_{T2}^1K|_Y).
\end{array}
\end{equation}

We have a canonical Cartesian square

\begin{equation}
\begin{array}{c}
\begin{array}{ccc}
Fix(c_T) & \to & T \\
\downarrow & & \downarrow \\
Fix(c) & \to & S
\end{array}
\end{array}
\end{equation}

so by 2.2.2 there is a canonical map

\begin{equation}
\begin{array}{c}
\Delta^* : H(Fix(c)/S) \to H(Fix(c_T)/T).
\end{array}
\end{equation}
The following lemma states that base change maps are compatible with the trace map, which generalizes [JY20, Proposition 6.2.16]:

**Lemma 3.3.2.** Let $K \in T_c(X)$ be USLA over $S$. Then the following diagram is commutative:

$$
\begin{array}{c}
\text{Map}(c_1^*K, c_2^*K) \\
\downarrow^{(3.3.1.2)} \\
\text{Map}(c_T^*K|_Y, c_T^*K|_Y)
\end{array} \xrightarrow{\text{Tr}_{(-)/S}} \begin{array}{c}
H(Fix(c)/S) \\
\downarrow^{(3.3.1.4)} \\
H(Fix(c_T)/T).
\end{array}
$$

(3.3.2.1)

3.3.3. **Push-forward.** Let $f: X \to Y$ be a morphism of $S$-schemes. Let $c: C \to X \times_S X$ and $d: D \to Y \times_S Y$ be two correspondences. Let $p: C \to D$ be a morphism such that the following diagram is commutative:

$$
\begin{array}{ccc}
C & \xrightarrow{c} & X \times_S X \\
\downarrow{p} & & \downarrow{\Delta} \\
D & \xrightarrow{d} & Y \times_S Y.
\end{array}
$$

(3.3.3.1)

Assume that one of the following conditions hold:

1. The following commutative square is Cartesian:

$$
\begin{array}{ccc}
C & \xrightarrow{c_1} & X \\
\downarrow{p} & & \downarrow{f} \\
D & \xrightarrow{d_1} & Y.
\end{array}
$$

(3.3.3.2)

2. Both morphisms $p$ and $f$ are proper.

Then there is a natural transformation of functors $d_1^*f_1 \to p_1c_1^*$. Then for any $K \in T(X)$ and any correspondence $u: c_1^*K \to c_2^*K$, we have the following composition

$$
d_1^*f_1K \to p_1c_1^*K \xrightarrow{u} p_1c_2^*K \to d_2^*f_1K.
$$

(3.3.3.3)

This construction gives rise to a canonical map

$$
\Delta_1: \text{Map}(c_1^*K, c_2^*K) \to \text{Map}(d_1^*f_1K, d_2^*f_1K).
$$

(3.3.4)

3.3.4. Assume that both $p$ and $f$ are proper. We have a canonical commutative square

$$
\begin{array}{ccc}
\text{Fix}(c) & \xrightarrow{c'} & X \\
\downarrow{q} & & \downarrow{f} \\
\text{Fix}(d) & \xrightarrow{d'} & Y.
\end{array}
$$

(3.3.4.1)

with $q$ proper, which by 2.2.3 induces a canonical map

$$
q_*: H(\text{Fix}(c)/S) \to H(\text{Fix}(d)/S).
$$

(3.3.4.2)

**Lemma 3.3.5.** Let $f: X \to Y$ be a proper morphism of $S$-schemes. For any object $K \in T(X)$ USLA over $S$, the object $f_*K$ is also USLA over $S$.

**Proof.** For any Cartesian square

$$
\begin{array}{ccc}
X_T & \xrightarrow{g} & Y_T \\
\downarrow{t} & & \downarrow{q} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

(3.3.5.1)
and any object \( L \in \mathcal{T}(T) \), we have canonical isomorphisms

\[
(3.3.5.2) \quad f^*K \otimes p^*r_*L \simeq f_* (K \otimes f^*p^*r_*L) \simeq f_* t_* (t^*K \otimes g^*q^*L) = s_* g_* (t^*K \otimes g^*q^*L)
\]

where we use the properness of \( f \) and \( g \) and the fact that \( K \) is USLA over \( S \). The same property also holds after any base change, which implies that \( f_*K \) is USLA over \( S \). □

The following lemma states that proper push-forwards are compatible with the trace map:

**Lemma 3.3.6.** Let \( K \in \mathcal{T}_c(X) \) be USLA over \( S \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Map}(c_1^*K, c_2^*K) & \xrightarrow{\text{Tr}(-/S)} & H(Fix(c)/S) \\
(3.3.3.4) & & (3.3.4.2) \\
\text{Map}(d_1^*f_*K, d_2^*f_*K) & \xrightarrow{\text{Tr}(-/S)} & H(Fix(d)/S).
\end{array}
\]

where the lower horizontal map is well-defined by Lemma 3.3.5.

Lemma 3.3.6 is a particular case of the Lefschetz-V erdier formula. For details of the proof, see [SGA5, III 4.4], [Var07, Proposition 1.2.5], [YZ19, Theorem 3.3.2], [Cis19, Theorem 3.2.18] or [LZ20, Theorem 2.20].

**Remark 3.3.7.** Let \( p : X \to S \) be a smooth proper morphism. By Lemma 3.1.2, the object \( 1_X \) is USLA over \( S \). In diagram (3.3.3.1), consider the case \( C = X, D = Y = S, f = p, c = \delta_{X/S} \) and \( d = id_S \). Then Lemma 3.3.6 applied to the identity map recovers the motivic Gauss-Bonnet formula ([DJK18, Theorem 4.6.1], [LR20, Theorem 5.3]), see also [JY20, Remark 5.1.11 (2)].

3.3.8. **Pullback.** Consider the situation of Diagram (3.3.3.1). Assume that one of the following conditions hold:

1. The following commutative square is topologically Cartesian:

\[
(3.3.8.1) \quad \begin{array}{ccc}
C & \xrightarrow{c_1} & X \\
p \downarrow & & \downarrow f \\
D & \xrightarrow{d_2} & Y
\end{array}
\]

that is, the canonical morphism \( C_{\text{red}} \to (D \times_Y X)_{\text{red}} \) is an isomorphism;

2. Both morphisms \( p \) and \( f \) are étale.

Then there is a natural transformation of functors \( p^*d_2^* \to c_1^*f^* \). Then for any \( K \in \mathcal{T}(Y) \) and any correspondence \( u : d_1^*K \to d_2^*K \), we have the following composition

\[
(3.3.8.2) \quad c_1^*f^*K = p^*d_1^*K \xrightarrow{u} p^*d_2^*K \to c_2^*f^*K.
\]

This construction gives rise to a canonical map

\[
(3.3.8.3) \quad (-)_{|\Delta} : \text{Map}(d_1^*K, d_2^*K) \to \text{Map}(c_1^*f^*K, c_2^*f^*K).
\]
3.3.9. Assume that both \( p \) and \( f \) are \( \acute{e}tale \). We have a canonical commutative square

\[
\begin{array}{ccc}
\text{Fix}(c) & \overset{q}{\rightarrow} & X \\
\downarrow & & \downarrow \\
\text{Fix}(d) & \overset{f}{\rightarrow} & Y
\end{array}
\]

with \( q \) \( \acute{e}tale \), which by 2.2.4 induces a canonical map

\[
q^* : H(\text{Fix}(d)/S) \rightarrow H(\text{Fix}(c)/S).
\]

**Lemma 3.3.10.** Let \( f : X \rightarrow Y \) be a smooth morphism of \( S \)-schemes. For any object \( K \in T(Y) \) USLA over \( S \), the object \( f^*K \) is also USLA over \( S \).

The proof of Lemma 3.3.10 is very similar to Lemma 3.3.5 and is left as an exercise. The following lemma states that \( \acute{e}tale \) pullbacks are compatible with the trace map:

**Lemma 3.3.11.** Let \( K \in T_c(Y) \) be USLA over \( S \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Map}(d_1^*K, d_2^*K) & \overset{\text{Tr}(\cdot)/S}{\rightarrow} & H(\text{Fix}(d)/S) \\
\downarrow & & \downarrow \\
\text{Map}(c_1^*f^*K, c_2^*f^*K) & \overset{\text{Tr}(\cdot)/S}{\rightarrow} & H(\text{Fix}(c)/S)
\end{array}
\]

where the lower horizontal map is well-defined by Lemma 3.3.10.

The proof of Lemma 3.3.11 is quite straightforward, see [SGA5, 4.2.6].

3.3.12. **Specialization.** Let \( S \rightarrow Z \) be a smooth morphism together with a section \( i : Z \rightarrow S \). Then \( i \) is a regular closed immersion, and by [EGA4, Proposition 17.2.5] is a canonical isomorphism

\[
i^*T_{S/Z} \cong N_ZS.
\]

Assume that there is a null-homotopy \( e(N_ZS) \cong 0 \). Let \( j : U \rightarrow S \) be the open complement, so \( U \) is also smooth over \( Z \). Let \( f : X \rightarrow S \) and \( c : C \rightarrow X \times_S X \) be two morphisms, and denote by \( j_X : X_U \rightarrow S \) and \( i_X : X_Z \rightarrow X \) the base change of \( i \) and \( j \). We have the following Cartesian diagrams:

\[
\begin{array}{ccc}
C_Z & \overset{i_C}{\rightarrow} & C & \overset{j_C}{\leftarrow} & C_U \\
\downarrow c_Z & & \downarrow c & & \downarrow c_U \\
X_Z \times_S X_Z & \rightarrow & X \times_S X & \leftarrow & X_U \times_U X_U \\
\downarrow i & & \downarrow j & & \downarrow j \\
X_Z & \overset{i_X}{\rightarrow} & X & \overset{j_X}{\rightarrow} & X_U \\
\downarrow f_Z & & \downarrow f & & \downarrow f_U \\
Z & \overset{i}{\rightarrow} & S & \overset{j}{\rightarrow} & U
\end{array}
\]

As in (2.2.6.3), there is a natural transformation of functors

\[
i_C!(i_C^*(-) \otimes Th(-N_ZS)_{C_Z}) \rightarrow j_Cj_C^*(-).
\]
Now let $K \in \mathcal{T}(X)$ and let $u : c^*_U K|_{X_U} \to c^*_U K|_{X_U}$ be a correspondence. By adjunction $u$ corresponds to a map

$$(3.3.12.5) \quad c_2! j_! j^*_2 c^*_1 K = j_X|c_{U2}|c^*_1 j^*_X K \to K$$

which, by composition with the map $(3.3.12.4)$, gives rise to a map

$$(3.3.12.6) \quad i_X|c_{Z2}|c^*_2 Z_1 (i^*_X K \otimes Th(-N_Z S)|_{X_Z}) = c_2! i_! (i^*_X c^*_1 K \otimes Th(-N_Z S)|_{X_Z})$$

$$(3.3.12.6) \quad \xrightarrow{(3.3.12.4)} c_2! j_! j^*_2 c^*_1 K \xrightarrow{(3.3.12.5)} K.$$

Let $c_Z : C \to X \times_Z X$ be the composition

$$(3.3.12.7) \quad C \xrightarrow{c_Z} X \times_S X \xrightarrow{\delta'} \xrightarrow{X \times_Z X}.$$ We would like to deduce from $(3.3.12.6)$ a correspondence over this morphism $c_Z$, not the morphism $c_Z$ itself. Now we have a canonical Cartesian square

$$(3.3.12.8) \quad X_Z \times_S X_Z \xrightarrow{\delta'} \xrightarrow{X \times_Z X_Z} \xrightarrow{S \times_S S}$$

where $\delta_{S/Z}$ is a regular closed immersion, whose normal bundle is $T_{S/Z}$ by definition. So by [DJK18, 4.2.5 and 4.3.1] there is a natural transformation

$$(3.3.12.9) \quad \delta' (\delta^* A \otimes Th(-T_{S/Z})|_{X_Z \times_S X_Z}) \to A$$

induced by the refined fundamental class of $\delta_{S/Z}$. Composing $(3.3.12.9)$ with $(3.3.12.6)$ and using the isomorphism $(3.3.12.1)$ we obtain a map

$$(3.3.12.10) \quad i_X|c_{Z2}|c^*_2 Z_1 (i^*_X K \otimes Th(-N_Z S)|_{X_Z}) \xrightarrow{(3.3.12.6)} K$$

from which we deduce a correspondence over $c_Z$

$$(3.3.12.11) \quad c^*_2 Z_1 K|_{X_Z} \xrightarrow{i_1} c^*_2 i^*_X K \xrightarrow{i_1} c^*_2 K|_{X_Z}$$

where the first map is obtained from $(3.3.12.10)$ by adjunction, and the second map is the natural transformation $i_1 = i^*_X i_X i^*_X \to i^*_X$. This construction gives rise to a canonical map

$$(3.3.12.12) \quad Map(c^*_U K|_{X_U}, c^*_U K|_{X_U}) \to Map(c^*_2 Z_1 K|_{X_Z}, c^*_2 Z_2 K|_{X_Z}).$$

Note that we have $Fix(c_Z) = Fix(c)$ and there is a Cartesian diagram

$$(3.3.12.13) \quad Fix(c_Z) \xrightarrow{\delta} Fix(c) \xrightarrow{\delta} Fix(c_U)$$

$$(3.3.12.14) \quad H(Fix(c_U)/U) \to H(Fix(c)/Z) = H(Fix(c_Z)/Z).$$

The following lemma states that specializations are compatible with the trace map:
Lemma 3.3.13. Let $K \in \mathcal{T}_c(X)$ be USLA over $S$. Then the following diagram is commutative:

\[
\begin{array}{c}
\text{Map}(c_U^1 \cdot K_{|X_U}, c_U^1 K_{|X_U}) \xrightarrow{\text{Tr}(-/U)} H(Fix(c_U)/U) \\
\downarrow \quad \quad \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{Map}(c_Z^1 \cdot K_{|X_Z}, c_Z^1 K_{|X_Z}) \xrightarrow{\text{Tr}(-/Z)} H(Fix(c_Z)/Z)
\end{array}
\]

(3.3.13.1)

Proof. Use the fact that both vertical maps are deduced from the natural transformation (2.2.6.3), and some diagram chase. □

4. LOCAL TERMS

In this section we fix a morphism of schemes $X \to S$.

4.1. Contracting correspondences and local terms.

Definition 4.1.1. Let $c : C \to X \times_S X$ be a correspondence. A closed subset $Z \subset X$ is $c$-invariant if $c_1(c_2^{-1}(Z)) \subset Z$.

If $Z$ is a closed subscheme of $X$ defined by an ideal sheaf $(\mathcal{I})$, we say that $c$ stabilizes $Z$ if $c_1(c_2^{-1}(Z)) \subset X$ is scheme-theoretically contained in $Z$, i.e. if $c_1^\#(\mathcal{I}) \subset c_2^\#(\mathcal{I}) \cdot \mathcal{O}_C$. We say that $c$ is contracting near $Z$ if $c$ stabilizes $Z$ and there exists $n$ such that $c_1^\#(\mathcal{I})^n \subset c_2^\#(\mathcal{I})^{n+1} \cdot \mathcal{O}_C$.

In particular, if $c$ stabilizes $Z$ then $Z$ is $c$-invariant.

Proposition 4.1.2. Let $Z$ be a closed subscheme of $X$ such that $c$ is contracting near $Z$. Let $K \in \mathcal{T}_c(X)$ be USLA over $S$ such that $K_{|Z} \simeq 0$. The for correspondence $u : c_1^c K \to c_2^c K$ over $c$, there is a null homotopy $\text{Tr}(u/S) \simeq 0$.

The proof of Proposition 4.1.2 will be given in 4.2.5.

4.1.3. If $i : Z \to X$ is the inclusion of a $c$-invariant closed subscheme, let $c_{|Z} : c_2^{-1}(Z)_{\text{red}} \to Z \times_S Z$ be the restriction of $c$. Then the commutative square

\[
\begin{array}{c}
c_2^{-1}(Z)_{\text{red}} \xrightarrow{c_{|Z}} Z \times_S Z \\
\downarrow \Delta_Z \quad \quad \quad \downarrow \quad i \times_S i
\end{array}
\]

(4.1.3.1)

satisfies the first condition of pullback in 3.3.8. It follows that, for every $K \in \mathcal{T}(X)$ and every correspondence $u : c_1^c K \to c_2^c K$ over $c$, the pullback by $\Delta_Z$ defines a map

\[
u_{|\Delta_Z} : c_1^c K_{|Z} \to c_2^c K_{|Z}.
\]

In addition, the square (4.1.3.1) satisfies the second condition of push-forward in 3.3.3, so the push-forward of $\nu_{|\Delta_Z}$ defines a map

\[
\Delta_Z ! \nu_{|\Delta_Z} : c_1^c i^* K_{|Z} \to c_2^c i^* K_{|Z}.
\]

(4.1.3.3)
On the other hand, let \( j : U \to X \) be the open complement of \( i \), then we have \( c_i^{-1}(U) \subset c_2^{-1}(U) \). Let \( c|_U : c_1^{-1}(U) \to U \times_S U \) be the restriction of \( c \). Then the commutative square

\[
\begin{array}{ccc}
c_1^{-1}(U) & \xrightarrow{c|_U} & U \times_S U \\
\downarrow & & \downarrow j \times_S j \\
C & \xrightarrow{c} & X \times_S X \\
\end{array}
\]

satisfies the second condition of pullback in \( 3.3.8 \). It follows that, for every \( K \in T(X) \) and every correspondence \( u : c_1^K \to c_2^K \) over \( c \), the pullback by \( \Delta_U \) defines a map \( u_{|\Delta_U} : c_1^u|_{U1}|U \to c_1^u|_{U2}|U \). In addition, the square \( 4.1.3.1 \) satisfies the first condition of push-forward in \( 3.3.3 \), so the push-forward of \( u_{|\Delta_U} \) defines a map

\[
\Delta_U|u_{|\Delta_U} : c_1^u|j_!|K|U \to c_2^j|_!|K|U.
\]

The following additivity result is the analogue of \([\text{Var}07, \text{Proposition 1.5.10}]\), which we can prove using higher category theory thanks to Lemma 3.2.6:

**Lemma 4.1.4.** Assume that \( S \) is a regular excellent scheme such that \( T \) satisfies constructibility and local duality for \( S \) \((3.2.3)\). Let \( K \in T_c(X) \) be such that both \( j_!|K|U \) and \( i_!|K|Z \) are USLA over \( S \), and let \( u : c_1^K \to c_2^K \) be a correspondence over \( c \). Then the traces of the maps \( 4.1.3.3 \) and \( 4.1.3.5 \) satisfy

\[
\text{Tr}(u/S) = \text{Tr}(\Delta^Z|u_{|\Delta^Z}/S) + \text{Tr}(\Delta^U|u_{|\Delta^U}/S).
\]

**Proof.** The localization triangle gives rise to a canonical cofiber sequence in \( T_c(X) \) of USLA objects over \( S \)

\[
j_!|K|U \to K \to i_!|K|Z
\]

and the construction in \( 4.1.3 \) gives a morphism of cofiber sequences

\[
\begin{array}{ccc}
c_1^u|j_!|K|U & \to & c_1^u|K| \to c_1^u|i_!|K|Z \\
\downarrow \Delta^U|u_{|\Delta^U} & & \downarrow u \\
c_2^j|_!|K|U & \to & c_2^j|K| \to c_2^j|i_!|K|Z
\end{array}
\]

since all functors in the construction are functors in a highly structured sense. We conclude by applying Lemma 3.2.6. \( \square \)

Note that by Lemma 3.1.3, if \( S \) is the spectrum of a field the USLA condition is automatic if we assume resolution of singularities. The following lemma is proved in \([\text{Var}07, \text{Theorem 2.1.3 (a)}]\):

**Lemma 4.1.5.** If \( c \) is contracting near \( Z \) and \( \text{Fix}(c) \) is connected, then the canonical closed immersion \( i_c : \text{Fix}(c)|Z| \to \text{Fix}(c) \) induces an isomorphism \((i_c)_{\text{red}} : (\text{Fix}(c)|Z|)_{\text{red}} \simeq (\text{Fix}(c))_{\text{red}} \). In particular, the push-forward map \( i_{\text{red}} : H(\text{Fix}(c)|Z)/S \to H(\text{Fix}(c)/S) \) is an isomorphism.

4.1.6. We now deal with the general case of not necessarily \( c \)-invariant subschemes.

**Definition 4.1.7.** (1) Let \( Z \) be a closed subscheme of \( X \). Let \( W \) be the complement of the closure of \( c_1^{-1}(Z) \) and \( c_1^{-1}(Z) \) in \( C \). Then \( W \) is the largest subset of \( C \) such that \( Z \) is \( c|W \)-invariant \((\text{[Var}07, \text{Lemma 1.5.3}]\)). For \( K \in T(X) \) and \( u : c_1^K \to c_2^K \) a correspondence over \( c \), let \( c|W : W \to X \times_S X \) and \( u_{|\Delta_W} : c|W^*|K \to c|W^2|K \) be the restrictions, and let

\[
c|Z = (c|W)|Z : c|W^2(Z)_{\text{red}} \to Z \times_S Z,
\]
Let \( W \subseteq C \) be such that \( c_W \) is contracting near \( Z \) over \( S \). Then for every correspondence \( u : c^\ast_X K c_2 K \), we have \( \text{Tr}_\beta(u/S) = i_c \text{Tr}_{\beta'}(u_Z/S) \in H(Fix(c)/S) \). In particular, if \( \beta \) is in addition proper over \( S \), then \( LT_{\beta}(u/S) = \text{LT}_\beta(u_Z/S) \in \text{End}(1_S) \).

**Proof.** Let \( W \subseteq C \) be an open neighborhood of \( Fix(c) \) such that \( c_W \) is contracting near \( Z \), then \( Fix(c_W) = Fix(c) \). Therefore by replacing \( c \) by \( c_W \) and \( u \) by \( u_{|\Delta_W} \), we may assume that \( c \) is contracting near \( Z \). Also by replacing \( C \) by the open subscheme \( C \setminus (Fix(c) \setminus \beta) \), we may assume that \( \beta = Fix(c) \). By Lemma 4.1.4 and Lemma 3.3.6 we have

\[
\text{Tr}_\beta(u) = \text{Tr}_\beta(\Delta_Z u_{|\Delta_Z}) + \text{Tr}_\beta(\Delta_{U} u_{|\Delta_U}) = i_c \text{Tr}_{\beta'}(u_Z) + \text{Tr}_\beta(\Delta_{U} u_{|\Delta_U}).
\]

By Proposition 4.1.2 we have \( \text{Tr}_\beta(\Delta_{U} u_{|\Delta_U}) \simeq 0 \), which finishes the proof. \( \square \)

**Remark 4.1.10.** (1) The element \( LT_{\beta}(u/S) \) is usually called the (true) local term, while \( LT_{\beta'}(u_Z/S) \) is called the naive local term. For example (see [Var07, Example 1.5.7]), if \( S \) is the spectrum of a field and \( c_2 \) is quasi-finite, then for each closed point \( x \) of \( X \), the set \( Fix(c_{1x}) = c_{1x}^{-1}(x) \cap c_{2x}^{-1}(x) \) is finite. Each point \( y \in Fix(c_{1x}) \) determines an endomorphism \( u_y : K_{1x} \to K_{1x} \), and we have \( LT_y(u_x/S) = \text{Tr}(u_{y}/S) \) is the usual (categorical) trace.

(2) Theorem 4.1.9 can be generalized to the twisted trace map (3.2.2.2), by slightly modifying the proof.

(3) Recently Varshavsky further generalized his results to the case of transversal intersections in [Var20]. However, in the motivic setting, we do not know if in general a family of correspondences has constant trace in each fiber, as in Proposition 2.5 of loc. cit.: this problem may be related to specializations of quadratic forms, see [DJK18, Remark 4.5.5]. Therefore it is not clear if a similar approach is possible.

### 4.2. Deformation of correspondences.

#### 4.2.1. If \( Z \to X \) is a closed immersion, let \( D_Z X = Bl_{Z \times 0}(X \times \mathbb{A}^1_{\mathbb{R}}) - Bl_{Z \times 0}(X \times 0) \) be the (affine) deformation to the normal cone ([Ful98, §5.1], [Ros96, §10], [DJK18, 3.2.3]). Explicitly, \( D_Z X \) can be defined as the spectrum over \( \mathcal{O}_X \) of the Rees algebra

\[
\sum_n T^n \cdot t^{-n} \subset \mathcal{O}_X[t, t^{-1}]
\]
where $\mathcal{I}$ is the ideal sheaf defining $Z$ in $X$. The following lemma is an analogue of [Var07, Lemma 1.4.3]:

**Lemma 4.2.2.** Let $f : Y \to X$ be a morphism of schemes, let $Z$ be a closed subscheme of $X$, and let $W$ be a closed subscheme of $f^{-1}(Z)$. Then the morphism $f$ lifts to a unique morphism $D_Z(f) : D_W Y \to D_Z X$.

**Proof.** Let $\mathcal{I}'$ be the ideal sheaf defining $W$ in $Y$, then we have $f^\#(\mathcal{I}) \subset \mathcal{I}'$ by assumption, which gives an inclusion $f^\#(\sum_n \mathcal{I}^n \cdot t^{-n}) \subset \sum_n \mathcal{I}'^n \cdot t^{-n}$, and the result follows. \qed

4.2.3. Let $c : C \to X \times_S X$ be a correspondence and let $Z$ be a closed subscheme of $X$. By Lemma 4.2.2, $c$ lifts to a correspondence $D_Z(c) : D_{c^{-1}(Z \times_S Z)} C \to D_Z X \times_{\mathbb{A}^1_S} D_Z X$.

Over $\mathbb{G}_m$, this is $c_{\mathbb{G}_m} = c \times id_{\mathbb{G}_m} : C \times \mathbb{G}_m \to (X \times_S X) \times \mathbb{G}_m$.

Over $0$, this is $N_Z(c) : N_{c^{-1}(Z \times_S Z)} C \to N_Z X \times_S N_Z X$ constructed in (3.3.12.7).

**Lemma 4.2.4.**

1. The correspondence $c$ is contracting near $Z$ if and only if $c$ stabilizes $Z$ and the image of $N_Z(c)_1$ is set-theoretically supported at the zero section $Z \subset N_Z X$.

2. There is a canonical closed immersion $D_{c^{-1}(Z \times_S Z)} Fix(c) \to Fix(D_Z(c))$. If $c$ is contracting near $Z$, then the canonical isomorphism $(Fix(D_Z(c)))_{\text{red}} \simeq (Fix(c) \times \mathbb{A}^1)_{\text{red}}$ that is, $Fix(D_Z(c))$ is a topologically trivial family over $\mathbb{A}^1_S$.

**Proof.** Let $\mathcal{I}'$ be the ideal sheaf defining $c^{-1}(Z \times_S Z)$ in $C$. Then the map $N_Z(c)_1 : N_{c^{-1}(Z \times_S Z)} C \to N_Z X$ is given by the map

$$(4.2.4.1) \quad \oplus_{n \geq 0}((\mathcal{I})^n/(\mathcal{I}^n+1) \to \oplus_{n \geq 0}((\mathcal{I}')^n/(\mathcal{I}')^{n+1})$$

induced by $c_1 : C \to X$, and the first claim follows. For the second claim, see [Var07, Corollary 1.4.5] and Lemma 4.1.5. \qed

4.2.5. **Proof of Proposition 4.1.2.** Consider the deformation construction in 4.2.3. By Lemma 3.3.2 and Lemma 3.3.13, we have the following commutative diagram

$$\begin{array}{ccc}
\text{Map}(c_1^* K, c_2^* K) & \xrightarrow{\text{Tr}(-/S)} & H(Fix(c)/S) \\
(3.3.12) & & (3.3.14) \\
\text{Map}(c_{\mathbb{G}_m}^1 K_{X \times_S m}, (c_{\mathbb{G}_m})_2^1 K_{X \times S \mathbb{G}_m}) & \xrightarrow{\text{Tr}(-/S \times \mathbb{G}_m)} & H(Fix(c) \times \mathbb{G}_m/S \times \mathbb{G}_m) \\
(3.3.12.12) & & (3.3.12.14) \\
\text{Map}(N_Z(c)_1^* K_{|N_Z X}, N_Z(c)_2^* K_{|N_Z X}) & \xrightarrow{\text{Tr}(-/S)} & H(Fix(N_Z(c))/S).
\end{array}$$

By Lemma 4.2.4 and Lemma 2.2.7, the composition of the two right vertical maps $H(Fix(c)/S) \to H(Fix(N_Z(c))/S)$ is an isomorphism. By Lemma 4.2.4, the image of $N_Z(c)_1$ is set-theoretically supported at the zero section $Z \subset N_Z X$, so $N_Z(c)_1^* K_{|N_Z X} \simeq 0$ since $K_{|Z} \simeq 0$ by assumption. We conclude that the map

$$(4.2.5.2) \quad \text{Map}(c_1^* K, c_2^* K) \xrightarrow{\text{Tr}(-/S)} H(Fix(c)/S)$$

is null-homotopic, which finishes the proof.

4.3. **$\mathbb{A}^1$-enumerative invariants as local terms.** In this section we interpret some invariants defined in $\mathbb{A}^1$-enumerative geometry as local terms.
4.3.1. Recall that if \( f : X \to S \) is a smooth morphism, then we define \( M_S(X) = f_! \mathbb{1}_S \). If \( U \) is an open subscheme of \( X \), we define \( M_S(X/U) \) to be the cofiber of \( M_S(U) \to M_S(X) \) ([CD19, 2.3.14]). Explicitly, if \( i : Z \to X \) is the immersion of the reduced closed complement of \( U \), then \( M_S(X/U) = f_i^! f_j^! \mathbb{1}_S \).

4.3.2. Let \( S \) be a scheme and let \( c_1 : C \to X \) be a morphism of smooth \( S \)-schemes. Let \( s : S \to X \) be a section of \( X \). Consider the commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{c} & S \\
\downarrow{s} & \searrow{\Delta} & \swarrow{\delta} \\
C \times X & \xrightarrow{\delta_c} & S
\end{array}
\]

where the square \( \Delta \) is Cartesian. Then there is a canonical map

\[
\xi(\Delta) : M_S(C/C - C_s) \to M_S(X/X - S)
\]

given by the composition

\[
p_1s_1s_1^!s_1^! \mathbb{1}_S = c_1s_1^!s_1^!q_1^! \mathbb{1}_S \to c_1s_1^!s_1^!q_1^! \mathbb{1}_S \to s_1^!q_1^! \mathbb{1}_S = q_1^! \mathbb{1}_S
\]

which by adjunction can be rewritten as a map

\[
\xi(\Delta) : Th(\tau_{c_1}|C_s) \to c_1^! \mathbb{1}_S
\]

so that \( \xi(\Delta) \in H(C_s/S, \tau_{c_1}|C_s) \).

**Lemma 4.3.3.** Under the assumptions above, the class \( \xi(\Delta) \) agrees with the refined fundamental class \( \Delta^* \eta_{c_1} \).

*Proof.* This follows from the construction of the map \( \Delta^* \) in (2.2.2.2), and the description of the fundamental class of the morphism \( c_1 \) between smooth \( S \)-schemes. \( \square \)

4.3.4. Now consider the correspondence where \( c_2 = s \circ p : C \to X \)

\[
c : C \xrightarrow{(c_1, s \circ p)} X \times_S X
\]

and we have canonically \( Fix(c) = C_s \). Consider the twisted cohomological correspondence over \( c \)

\[
u : c_1^! \mathbb{1}_X = 1_C \xrightarrow{\eta_{c_2}} c_2^! \mathbb{1}_X \otimes Th(-\tau_{c_2})
\]

induced by the fundamental class of \( c_2 \). It has a trace \( Tr(u/S) \in H(C_s/S, \tau_{c_2}|C_s) \) defined in (3.3.12). Note that we have \( \tau_{c_2} \simeq \tau_{c_1} + c_1^! T \) and therefore by virtue of (3.3.12) we have a canonical identification

\[
\tau_{c_2}|C_s \simeq \tau_{c_1}|C_s + s_1^*c_1^! \tau_{c_2} \simeq \tau_{c_1}|C_s + c_1^! \tau_{c_2} \simeq \tau_{c_1}|C_s.
\]

**Proposition 4.3.5.** Under the identification (4.3.4.3), the trace \( Tr(u/S) \) agrees with the map \( \xi(\Delta) \) in (4.3.2.4).

*Proof.* Denote by \( \delta = \delta_{X/S} : X \to X \times_S X \). The trace of the map (4.3.4.2) is given by the composition

\[
Th(\tau_{c_2}|C_s) \to c_1^! s_1! c_1! Th(\tau_{c_2}|C_s) \simeq c_1^! s_1! \delta^* c_1! Th(\tau_{c_2}) \to c_1^! s_1! \delta^* c_1! c_1^! \mathbb{1}_X
\]

\[
\to c_1^! s_1^! \delta^* s_1^! \mathbb{1}_X \simeq c_1^! s_1^! q_1^! \mathbb{1}_S = c_1^! \mathbb{1}_S.
\]
The result then follows from the associativity of fundamental classes ([DJK18, Theorem 3.3.2]), and the fact that the restriction of the fundamental class of \( s \circ q \) to \( S \) is trivial.

\[ \square \]

**Example 4.3.6.** The construction above recovers the following \( \mathbb{A}^1 \)-enumerative invariants:

1. **\( \mathbb{A}^1 \)-Brouwer degree:** if \( X \) is a vector bundle over \( S \), \( C \) is an open subscheme of \( X \), and if \( \beta \) is an open subset of \( C \), which is proper over \( S \), then \( LT_\beta(u/S) \) recovers the local \( \mathbb{A}^1 \)-Brouwer degree ([KW19, Definition 11], [BW20, Definition 7.1]), which we can see from the description in (4.3.2.2).

2. **Euler class with support:** assume that \( C = S \), \( X \) is a vector bundle over \( S \), \( s : S \to X \) is the zero section and \( c_1 : S \to X \) is another section. Then \( C_s \) is the zero locus of \( c_1 \), and \( \xi(\Delta) \) is the Euler class with support defined in [DJK18, 3.2.10] or [BW20, Definition 5.12]. If \( \beta \) is an open subset of \( C \), which is proper over \( S \), then \( LT_\beta(u/S) \) recovers the local contribution to the Euler class with support ([Lev17, §2]).

**References**

[Ayo07a] J. Ayoub, Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, Astérisque No. 314-315 (2007). ↑2, ↑18

[Ayo07b] J. Ayoub, The motivic vanishing cycles and the conservation conjecture, in Algebraic cycles and motives, Volume 1. Selected papers of the EAGER conference, Leiden, Netherlands, August 30–September 3, 2004 on the occasion of the 75th birthday of Professor J. P. Murre. London Mathematical Society Lecture Note Series 343, 3-54 (2007). ↑2

[BW20] T. Bachmann, K. Wickelgren, \( \mathbb{A}^1 \)-Euler classes: six functors formalisms, dualities, integrality and linear subspaces of complete intersections, arXiv:2002.01848. ↑18

[Bal18] E. Balzin, Reedy Model Structures in Families, arXiv:1803.00681. ↑8

[BD17] M. Bondarko, F. Déglise, Dimensional homotopy i-structures in motivic homotopy theory, Adv. Math. 311 (2017), 91-189. ↑8

[Cis19] D.-C. Cisinski, Cohomological methods in intersection theory, to appear in the Clay lecture notes from the LMS-CMI Research School “Homotopy Theory and Arithmetic Geometry: Motivic and Diophantine Aspects”. ↑2, ↑10

[CD19] D.-C. Cisinski, F. Déglise, Triangulated categories of motives, Springer Monographs in Mathematics. Springer, Cham, 2019. ↑7, ↑17

[DFJK20] F. Déglise, J. Fasel, A. Khan, F. Jin, On the rational motivic homotopy category, arXiv:2005.10147. ↑7, ↑8

[DJK18] F. Déglise, F. Jin, A. Khan, Fundamental classes in motivic homotopy theory, to appear in J. Eur. Math. Soc. ↑2, ↑3, ↑4, ↑10, ↑12, ↑15, ↑18

[DG20] B. Drew, M. Gallauer, The Universal Six-Functor Formalism, arXiv:2009.13610. ↑3, ↑6, ↑8

[EGA4] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie, rédigés avec la collaboration de Jean Dieudonné, Inst. Hautes Études Sci. Publ. Math. No. 32 (1966). ↑11

[EK20] E. Elmanto, A. Khan, Perfection in motivic homotopy theory, Proc. Lond. Math. Soc. 120 (2020), no. 1, 28-38. ↑8

[FHM03] H. Fausk, P. Hu, J. P. May, Isomorphisms between left and right adjoints, Theory Appl. Categ. 11 (2003), No. 4, 107-131. ↑5

[Fer05] D. Ferrand, On the non additivity of the trace in derived categories, arXiv:math/0506589, 2005. ↑2

[Fuj97] K. Fujiwara, Rigid geometry, Lefschetz-Verdier trace formula and Deligne’s conjecture, Invent. Math. 127, No. 3, 489-533 (1997). ↑1

[Ful98] W. Fulton, Intersection theory, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin, 1998. ↑2, ↑3, ↑15

[GM93] M. Goresky, R. MacPherson, Local contribution to the Lefschetz fixed point formula, Invent. Math. 111, No. 1, 1-33 (1993). ↑1

[Gro66] A. Grothendieck, Formule de Lefschetz et rationalité des fonctions L, Sémin. Bourbaki Vol. 9, 17e année (1964/1965), Exp. No. 279, 15 p. (1966). ↑1

[Hoy15] M. Hoyois, A quadratic refinement of the Grothendieck-Lefschetz-Verdier trace formula, Algebraic & Geometric Topology 14 (2015), no. 6, 3603-3658. ↑2

[JY20] F. Jin, E. Yang, Küneth formulas for motives and additivity of traces, to appear in Adv. Math.. ↑2, ↑5, ↑6, ↑7, ↑8, ↑9, ↑10
J. Kass, K. Wickelgren, *The class of Eisenbud-Khimshiashvili-Levine is the local $A^1$-Brouwer degree*, Duke Math. J. **168** (2019), no. 3, 429-469. ↑, ↑

A. Khan, *Motivic homotopy theory in derived algebraic geometry*, Ph.D. thesis, Universität Duisburg-Essen, 2016, available at https://www.preschema.com/thesis/. ↑

M. Levine, *Aspects of enumerative geometry with quadratic forms*, arXiv:1703.03049. ↑, ↑

M. Levine, A. Raksit, *Motivic Gauss-Bonnet formulas*, Algebra Number Theory, **14** (7):1801-1851, 2020. ↑, ↑

Q. Lu, W. Zheng, *Categorical traces and a relative Lefschetz-Verdier formula*, arXiv:2005.08522. ↑, ↑, ↑

M. Olsson, *Motivic cohomology, localized Chern classes, and local terms*, Manuscripta Math. **149** (2016), no. 1-2, 1-43. ↑

R. Pink, *On the calculation of local terms in the Lefschetz-Verdier trace formula and its application to a conjecture of Deligne*, Ann. Math. (2) **135**, No. 3, 483-525 (1992). ↑

M. Rost, *Chow groups with coefficients*, Doc. Math. **1**, 319-393 (1996). ↑

A. Grothendieck, *Cohomologie l-adique et fonctions L*, Séminaire de géométrie algébrique du Bois-Marie 1965-66 (SGA 5). Avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou, et J.-P. Serre. Springer Lecture Notes, Vol. 589. Springer-Verlag, Berlin-New York, 1977. ↑, ↑, ↑, ↑, ↑

Y. Varshavsky, *Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara*, Geom. Funct. Anal. **17**, No. 1, 271-319 (2007). ↑, ↑, ↑, ↑, ↑, ↑

Y. Varshavsky, *Local terms for transversal intersections*, arXiv:2003.06815. ↑

E. Yang, Y. Zhao, *On the Relative Twist Formula of $\ell$-adic Sheaves*, Acta. Math. Sin.-English Ser. (2019). ↑, ↑

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