DECAY ESTIMATES FOR HIGHER ORDER ELLIPTIC OPERATORS

HONGLIANG FENG, AVY SOFFER, ZHAO WU AND XIAOHUA YAO

Abstract. This paper is mainly devoted to study time decay estimates of the higher-order Schrödinger type operator $H = (-\Delta)^m + V(x)$ in $\mathbb{R}^n$ for $n > 2m$ and $m \in \mathbb{N}$. For certain decay potentials $V(x)$, we first derive the asymptotic expansions of resolvent $R_V(z)$ near zero threshold with the presence of zero resonance or zero eigenvalue, as well identify the resonance space for each kind of zero resonance which displays different effects on time decay rate. Then we establish Kato-Jensen type estimates and local decay estimates for higher order Schrödinger propagator $e^{-itH}$ in the presence of zero resonance or zero eigenvalue. As a consequence, the endpoint Strichartz estimate and $L^p$-decay estimates can also be obtained. Finally, by a virial argument, a criterion on the absence of positive embedded eigenvalues is given for $(-\Delta)^m + V(x)$ with a repulsive potential.

1. Introduction

1.1. Backgrounds and problems. Consider the higher-order Schrödinger type operators in $\mathbb{R}^n$:

$$H = (-\Delta)^m + V(x), \quad H_0 = (-\Delta)^m,$$

where $n > 2m$, $m \in \mathbb{N}$ and $V(x)$ is a real-valued function satisfying $|V(x)| \lesssim (1 + |x|)^{-\beta}$ for some $\beta > 0$.

It was well-known that the higher-order elliptic operator $P(D) + V$ has been extensively studied as general Hamiltonian operator by many people in different contexts. For instance, one can see Schechter [Sch71] for spectral theory, Kuroda [Kur78], Agmon [Agm75], Hörmander [Ho05] for scattering theory, Davies [Dav97], Davies and Hinz [DH98b], Deng et al [DDY14] for semigroup theory, and as well [HS15, HSI7, BS91, Mou81, SYY18] for many other interesting contents. In this paper, we are interested in establishing some dispersive estimates for the higher-order Schrödinger type operator $H = (-\Delta)^m + V$ with $m \geq 2$, among which, including Kato-Jensen type estimates, local decay estimates, Strichartz estimate and $L^p$-decay estimates. We also show that the presence of zero resonance or eigenvalue of $H$ will affect the time decay rate of $e^{-itH}$. For classical Schrödinger operator $-\Delta + V$ (i.e. $m = 1$), recall that in the last thirty years, dispersive estimates of Schrödinger operator have been one of the key topics, which were applied broadly to nonlinear Schrödinger equations, see e.g. [Caz03, Tao06, JSS91, KT98, Sch07, Sim18a, Sim18b] and references therein.

In the sequel, let us first review the famous Kato-Jensen estimates in [JK79] for Schrödinger operator $-\Delta + V$ in $\mathbb{R}^3$, where Jensen and Kato first established the time asymptotic expansion:

$$e^{-it(-\Delta + V)} = \sum_{j=0}^{N} e^{it\lambda_j} P_j + t^{-1/2}C_{-1} + t^{-3/2}C_0 + \cdots$$

1

Date: April 30, 2019.

Key words and phrases. Higher-order Schrödinger type operator, lower energy asymptotic expansion, zero-resonance, Kato-Jensen decay estimates, Strichartz estimates, absence of positive eigenvalue.
as \( t \to \infty \) in the weighted Lebesgue spaces \( B(L^2_s, L^2_{s'}) \) with suitable \( s, s' > 0 \). Here the \( \lambda_j \) are negative eigenvalues of \(-\Delta + V\) with the associated projection \( P_j \). The spectral property at zero threshold of \(-\Delta + V\) affects the leading term of the asymptotic expansion. In [JK79], they pointed out that \( C_{-1} = 0 \) if zero is a regular point, and the operator \( C_{-1} \) does not vanish if zero is purely a resonance of \(-\Delta + V\). In particular, the following four cases were discussed: zero is a regular point; zero is purely a resonance; zero is purely an eigenvalue; zero is both resonance and eigenvalue. Zero is a regular point of \(-\Delta + V\) means zero is neither an eigenvalue nor a resonance.

To obtain (1.1), their fundamental idea in [JK79] is to deduce the lower energy expansion and higher energy decay estimate of resolvent \( R(-\Delta + V; z) \). The lower energy expansion means the asymptotic expansion of perturbed resolvent \( R(-\Delta + V; z) \) as \( z \to 0 \) in \( B(L^2_s, L^2_{s'}) \). Higher energy decay estimate means the decay estimate of \( R(-\Delta + V; z) \) as \( z \to \infty \) in \( B(L^2_s, L^2_{s'}) \). As a consequence, the following pointwise decay estimates can be obtained:

\[
\left\| (1 + |x|)^{-\sigma} e^{-it(-\Delta+V)} P_{ac}(-\Delta + V)(1 + |x|)^{-\sigma} \right\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \lesssim (1 + |t|)^{-3/2} \tag{1.2}
\]

provided that zero is a regular point of \(-\Delta + V\). The formula (1.1) also shows that, in the presence of zero resonance or eigenvalue, the time decay rate of Kato-Jensen decay estimates is slower than (1.2) in the regular case. Indeed, if zero is purely a resonance of \(-\Delta + V\), then the time decay rate is \((1 + |t|)^{-1/2}\) by (1.1). For the similar results of \( n = 4 \) and \( n \geq 5 \), see Jensen’s works [Jen84] and [Jen80] respectively. Also see [JN01] for \( n = 1 \) and \( n = 2 \). Soon afterward, Murata in [Mur82] generalized Kato and Jensen’s work [JK79] to a certain class of \( P(D) + V \). In [Mur82], Murata required that \( P(D) \) satisfies

\[
(\nabla P)(\xi_0) = 0, \quad \det \left[ \partial_i \partial_j P(\xi) \right]_{\xi_0} \neq 0. \tag{1.3}
\]

However, the polyharmonic operators \( H_0 = (-\Delta)^m \) do not satisfy the nondegenerate condition (1.3) at zero but the case \( m = 1 \). For the fourth order Schrödinger operator \((-\Delta)^2 + V\), i.e. the case \( m = 2 \), the first two authors and the last author in [FSY18] established the Kato-Jensen decay estimates with the time decay rate is \((1 + |t|)^{-n/4}\) for \( n \geq 5 \) and \((1 + |t|)^{-5/4}\) for \( n = 3 \) in the regular case. Recently, the authors in [FWY18] further studied Kato-Jensen decay estimates in non-regular case for \( d \geq 5 \). Thus, in the following, we mainly deal with the higher order Schrödinger operator \((-\Delta)^m + V\) for all \( m \geq 3 \). For instance, when \( 3m - 1 \leq n \leq 4m \) and \( n \) is odd, we have established the complete asymptotic formulas even in the presence of zero resonance or eigenvalue as \( t \to \infty \) (see Subsection 1.2 below):

\[
e^{-itH} P_{ac}(H) = \begin{cases} 
|t|^{-n/2m} B + o(|t|^{-n/2m}), & 0 \text{ is a regular point;} \\
|t|^{-(2 - n/2m)} \frac{1}{m} B_j + o(|t|^{-(2 - n/2m)}), & 0 \text{ is the j-th kind of resonance;} \\
|t|^{-1/2m} B_{k+1} + o(|t|^{-1/2m}), & 0 \text{ is an eigenvalue.}
\end{cases} \tag{1.4}
\]

In order to prove the asymptotic formula (1.4), we will study the resolvent asymptotic expansion of \( R_V(z) \) of \( H = (-\Delta)^m + V \) by the following symmetric identity:

\[
R_V(z) = R_0(z) - R_0(z) v(U + vR_0(z)v^{-1} vR_0(z).
\]

Here \( v(x) = |V(x)|^{1/2} \) and \( U = \text{sign}(V(x)) \). The key point is to deduce the asymptotic expansion of \((U + vR_0(z)v^{-1}\) with the presence of zero resonance or eigenvalue. Due to the degenerate of \( H_0 = (-\Delta)^m \) at zero, the classifications of zero resonance are more complex
than \(-\Delta + V\), we need to iterate a few more steps than Schrödinger operator. The additional iterative steps lead us to split the resonance space into several subspaces. Heuristically, we need to split the whole resonance space into several subspaces depending on the process of the inverse expansion of \(U + vR_0(z)v\) as \(z\) approaches zero. Recall that, Jensen and Kato in [JK79] for \(-\Delta + V\) did not need to split the resonance space.

Besides zero resonance or eigenvalue, in the decay estimates (1.4), we also assume that \(H = (-\Delta)^m + V\) does not exist any positive eigenvalue embedded into the absolutely continuous spectrum. For Schrödinger operator \(-\Delta + V\), Kato in the famous work [Kat59] first showed that \(-\Delta + V\) has no positive eigenvalues if the potential \(V(x)\) decays fast enough at infinity (e.g. \(o(|x|^{-1})\)). The result was further generalized by Agmon [Agm70], Simon [Sim69], Froese, Herbst, M. Hoffmann and T. Hoffmann [FHHH82] et al. In particular, Ionescu and Jerison in [IJ03] showed such criterion on absence of positive eigenvalues of Schrödinger operator with integrable potentials \(V \in L^{n/2}(\mathbb{R}^n)\). Koch and Tataru [KT06] proved the same result for \(V \in L^{(n+1)/2}(\mathbb{R}^n)\), where \((n + 1)/2\) is the highest possible integrable exponent due to the counterexample in [I03].

For higher order operator \(P(D) + V\), the situations are much more complicated than the second order operator. Since there exit some examples even with compactly supported smooth potentials such that the positive eigenvalues appear, so it would be interesting and useful to establish some effective criterion for \(P(D) + V\) on absence of positive embedded eigenvalue. In the sequel, a simple criterion can be given on absence of positive eigenvalues for the higher order operators by a virial argument, which works for repulsive potentials. Besides, we also notice that for a general selfadjoint operator \(H\), in the famous work [JK79] for \(-\Delta\) it did not need to split the resonance space. The additional \(L^2\) spaces: 

\[
L^2_s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : (1 + |\cdot|^s)f \in L^2(\mathbb{R}^n) \right\}.
\]

Let us first give Kato-Jensen estimates for \(H = (-\Delta)^m + V(x)\) assuming that zero is neither a resonance nor an eigenvalue.

**Theorem 1.1.** Let \(n > 2m\) and \(H = (-\Delta)^m + V(x)\) with \(|V(x)| \lesssim (1 + |x|)^{-\beta}\) for some \(\beta > n\). Assume that \(H\) has no positive embedded eigenvalue and \(0\) is a regular point. \(P_{ac}(H)\) denotes the projection onto the absolutely continuous spectrum space of \(H\). Then for any \(s, s' > n/2\), we have

\[
\left\| e^{-itP_{ac}(H)} \right\|_{B(s,-s')} \lesssim (1 + |t|)^{-\frac{n}{2m}}, \quad t \in \mathbb{R}.
\]

It was quite known that the presence of zero resonance or zero eigenvalue affect the time decay rate of Kato-Jensen decay estimates. Recall that Schrödinger operator \(-\Delta + V\) does not exist resonance at zero for \(n > 4\). Similarly, if \(n > 4m\) and \(m \geq 2\), then \(H = (-\Delta)^m + V\) do not exist zero resonance (see Remark 1.4 below), and if \(n \leq 4m\), then \(H = (-\Delta)^m + V\) may have zero resonances. However, since the polyhamonic operators \((-\Delta)^m\) are degenerate
at zero for $m \geq 2$, the resonance space for $H = (-\Delta)^m + V$ is more complex than Schrödinger operator $-\Delta + V$.

In the sequel, we will define resonance space of $(-\Delta)^m + V$ for $n \leq 4m$ and then give different resonance types, which lead to different decay rates of time. For $\sigma \in \mathbb{R}$, let $W_{\sigma}(\mathbb{R}^n)$ denotes the intersection space

$$W_{\sigma}(\mathbb{R}^n) = \bigcap_{s > \sigma} L^2_{-s}(\mathbb{R}^n).$$

Note that, $W_{\sigma_1}(\mathbb{R}^n) \supset W_{\sigma_2}(\mathbb{R}^n)$ if $\sigma_1 > \sigma_2$. Especially, $W_0(\mathbb{R}^n) \supset L^2(\mathbb{R}^n)$.

**Definition 1.2.** For $n \leq 4m$ and $H = (-\Delta)^m + V$ with $|V(x)| \lesssim (1 + |x|)^{-\beta}$ for some $\beta > 0$. If there exists some $\psi(x) \in W_{2m - \frac{n}{2}}(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$ such that $H\psi(x) = 0$ in the distributional sense, then we say $0$ is a resonance of $H$. If $H\psi(x) = 0$ for some nonzero $\psi \in L^2(\mathbb{R}^n)$, then we say $0$ is an eigenvalue of $H$.

We can further define the $k$ different types of resonance, where $k$ is a positive integer by

$$k = k(n, m) = \begin{cases} \frac{4m - n + 1}{2}, & n \leq 4m, \ n \text{ odd}; \\ \frac{4m - n + 2}{2}, & n \leq 4m, \ n \text{ even}. \end{cases}$$ (1.5)

For $l \in \mathbb{N}$, if

$$\psi(x) \in W_{2m - \frac{n}{2} - (l-1)}(\mathbb{R}^n) \setminus W_{2m - \frac{n}{2} - l}(\mathbb{R}^n), \ 1 \leq l \leq k - 1$$
satisfies $H\psi(x) = 0$ in the distributional sense, then we say zero is the $l$-th kind of resonance of $H$. If $\psi(x) \in W_{1/2}(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$ ($n$ odd) or $\psi(x) \in W_0(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$ ($n$ even), then we say zero is the $k$-th kind of resonance of $H$.

**Remark 1.3.** Note that $H\psi = 0$ in the distributional sense if and only if $(1 + R_0(0)V)\psi = 0$ with $R_0(0) = (-\Delta)^{-m}$. Denote

$$\mathfrak{M}_s = \{\psi \in L^2_{-s} : (1 + R_0(0)V)\psi = 0\}; \quad \mathfrak{M}_s = \{\psi \in L^2_s : (1 + VR_0(0))\psi = 0\}.$$

Since $\mathfrak{M}_s$ and $\mathfrak{M}_s$ are independent of $s \in (2m - \frac{n}{2}, \beta + \frac{n}{2} - 2m)$ due to the compactness of $R_0(0)V$, the zero resonance function must belong to the intersection space $W_{\sigma}(\mathbb{R}^n) = \bigcap_{s > \sigma} L^2_{-s}(\mathbb{R}^n)$ for $\sigma = 2m - n/2$. Recall that, for Schrödinger operator $-\Delta + V$, $\sigma = 1/2$ when $n = 3$ and $\sigma = 0$ when $n = 4$. Also, $-\Delta + V$ has only one resonance type (i.e. $k = 1$) for $n = 3, 4$. When $n \geq 5$, $-\Delta + V$ does not exist resonance at zero. See [JK79, Jen80, Jen84].

**Remark 1.4.** For $n > 4m$, $H = (-\Delta)^m + V$ does not exist resonance at zero. Indeed, let $\psi \in L^2_{-s}(\mathbb{R}^n)$ with some $s > 0$ satisfying $H\psi = 0$. Since the Riesz potential $R_0(0) = (-\Delta)^{-m}$ is bounded from $L^2_{2}(\mathbb{R}^n)$ to $L^2_{-s}(\mathbb{R}^n)$ with $s, s' \geq 2m$ and $s, s' \geq 0$, then $R_0(0)$ will pull $V\psi \in L^2_{-s}(\mathbb{R}^n)$ into $L^2_{-s}(\mathbb{R}^n)$ if $\beta - s \geq 2m$ by the identity $\psi = -R_0(0)V\psi$. Hence $\psi \in L^2_{s}(\mathbb{R}^n)$ and $H = (-\Delta)^m + V$ does not exist zero resonance if $n > 4m$.

Now, we state the Kato-Jensen decay estimates of $H = (-\Delta)^m + V$ with the presence of zero resonance or eigenvalue.

**Theorem 1.5.** Let $n > 2m$ and $H = (-\Delta)^m + V(x)$ with $|V(x)| \lesssim (1 + |x|)^{-\beta}$ for some $\beta > 0$. Assume that $H$ has no positive embedded eigenvalue. $P_{ac}(H)$ denotes the projection onto the absolutely continuous spectrum space of $H$.

(I) For $n > 4m$. Let $\beta > n + 4$ and $s, s' > \frac{n}{2} + 2$. If $0$ is an eigenvalue of $H$, then

$$\|e^{-itH}P_{ac}(H)\|_{B(s, -s')} \lesssim (1 + |t|)^{2 - \frac{n}{2m}}, \quad t \in \mathbb{R}.$$
(II) For $n \leq 4m$ and $n$ is odd. Let $\beta > n + 4k$, $s, s' > \frac{n}{2} + 2k$, where $2k = 4m - n + 1$ and $1 \leq k \leq \lfloor \frac{m}{2} \rfloor + 1$. Then

- If $0$ is the $j$-th kind of resonance of $H$ with $1 \leq j \leq k$, then
  \[
  \|e^{-itH}P_{ac}(H)\|_{B(s,-s')} \lesssim (1 + |t|)^{-\frac{4m+2-n-2j}{2m}}, \quad t \in \mathbb{R}.
  \]
- If $0$ is an eigenvalue of $H$, then
  \[
  \|e^{-itH}P_{ac}(H)\|_{B(s,-s')} \lesssim (1 + |t|)^{-\frac{4m}{m}}, \quad t \in \mathbb{R}.
  \]

(III) For $n \leq 4m$ and $n$ is even. Let $\beta > n + 4k + 2$, $s, s' > \frac{n}{2} + 2k + 1$, where $2k = 4m - n + 2$ and $1 \leq k \leq \lfloor \frac{m}{2} \rfloor + 1$. Then

- If $0$ is the $j$-th kind of resonance of $H$ with $1 \leq j \leq k - 1$, then
  \[
  \|e^{-itH}P_{ac}(H)\|_{B(s,-s')} \lesssim (1 + |t|)^{-\frac{4m+2-n-2j}{m}}, \quad t \in \mathbb{R}.
  \]
- If $0$ is the $k$-th kind of resonance or an eigenvalue of $H$, then
  \[
  \|e^{-itH}P_{ac}(H)\|_{B(s,-s')} \lesssim (\ln |t|)^{-1}, \quad |t| > 1, \quad t \in \mathbb{R}.
  \]

Remark 1.6. In the following Table 1, we list the known results of Kato-Jensen decay estimates for $H = (-\Delta)^m + V$. Moreover, if zero is a regular point of $H$, for any order $2 \leq m \in \mathbb{N}$ and all dimension $n \geq 1$, one can obtain the Kato-Jensen decay estimate for $e^{-itH}P_{ac}(H)$ by the similar process as in this paper. In the non-regular cases, i.e. zero is a resonance or an eigenvalue $H$, for $m \geq 2$ and $n$ in the remainder range, the Kato-Jensen decay estimates are still open at present.

| Order | Dimension | Results |
|-------|-----------|---------|
| $m = 1$ | $n \geq 1$ | $n = 1, 2$: [JN01]; $n = 3$: [JK79]; $n = 4$: [Jen84]; $n \geq 5$: [Jen80] |
| $m = 2$ | $n \geq 4$ | $n = 4$: [GT18]; $n \geq 5$: [FSY18; FWY18] |
| $m$ odd | $n \geq 3m - 1$ | Regular case: Theorem 1.1; Non-regular cases: Theorem 1.5 |
| $m$ even | $n \geq 3m$ | Regular case: Theorem 1.1; Non-regular cases: Theorem 1.5 |

Table 1. Known results of Kato-Jensen decay estimates

Next, we state the local decay estimates which is related to Kato’s $\mathcal{H}$-smooth perturbation theory. Recall that for a general selfadjoint operator $\mathcal{H}$, a closed operator $A$ is $\mathcal{H}$-smooth if and only if

\[
\sup_{z \in \mathbb{R}, \phi \in D(A^*) \& \|\phi\| = 1} \left| \langle A^*\phi, \text{Im}[R_{\mathcal{H}}(z)]A^*\phi \rangle \right| < \infty,
\]

where $R_{\mathcal{H}}(z) = (\mathcal{H} - z)^{-1}$. There are many equivalent characterizations of $A$ is $\mathcal{H}$-smooth in [RS78]. Moreover, if the above uniformly bound holds without taking the imaginary part, then $A$ is called to be $\mathcal{H}$-supersmooth. Kato’s $\mathcal{H}$-smooth perturbation theory is not only used in the spectral analysis, but also widely applied in the nonlinear dispersive problems. See
e.g. [RS78, DF08, D’A15, Sim18a, Sim18b] and references therein. As a direct consequence of the resolvent estimates, we will prove that \((1 + |x|)^{-\sigma}P_{ae}(H)\) is \(H\)-supersmooth. Then it follows immediately that the following local decay estimate of \(H = (-\Delta)^m + V\) holds, which can be applied to establish the endpoint Strichartz estimates (see Section 6 below).

**Theorem 1.7.** For \(n > 2m\) and \(H = (-\Delta)^m + V\) with \(|V(x)| \lesssim (1 + |x|)^{-\beta}\) for some \(\beta > n\). Assume 0 is a regular point of \(H\) and \(H\) has no positive embedded eigenvalue. Then for \(\phi \in L^2(\mathbb{R}^n)\) and \(\sigma > n/2\), we have

\[
\left\| \langle x \rangle^{-\sigma}e^{-itH}P_{ac}(H)\phi \right\|_{L^2_tL^2_{x}(\mathbb{R}^{n+1})} \lesssim \|\phi\|_{L^2(\mathbb{R}^n)}. \tag{1.6}
\]

where \(\langle x \rangle = (1 + |x|^2)^{1/2}\).

**Remark 1.8.** We remark that the above estimate (1.6) also holds even if 0 is an eigenvalue of \(H\). Indeed, \(R_V(z)\) is the Laplace transform of \(e^{-itH}\), i.e.

\[
\left| \langle x \rangle^{-\sigma}P_{ac}(H)R_V(z)\langle x \rangle^{-\sigma} \right| = \left| \int_0^\infty \langle x \rangle^{-\sigma}P_{ac}(H)e^{-itH}\langle x \rangle^{-\sigma}e^{itz}dt \right|, \quad \text{Im}z > 0.
\]

The integral is uniformly bounded provided \(n > 6m\) by (I) of Theorem 1.5, which implies estimate (1.6).

**Remark 1.9.** Note that, we obtain the Kato-Jensen decay estimates and local decay estimates under the assumption that \(H = (-\Delta)^m + V\) does not exists positive embedded eigenvalue. In fact, if \(H\) exists one positive eigenvalue \(\lambda_0 > 0\), one can also obtain the Kato-Jensen decay estimates and local decay estimates for \(\bar{H}\) by the positive commutator methods and Mourre’s theory. Where \(\bar{H} = PH\bar{P}\) and \(\bar{P} = 1 - P_{\lambda_0}\). \(P_{\lambda_0}\) is the orthogonal projection onto eigen-subspace related to \(\lambda_0\). In [LS15], Larenas and Soffer established the Kato-Jensen type decay estimates for general Hamiltonian \(\mathcal{H}\) using the commutator methods. In [GLS16], they applied the approach in [LS15] to Schrödinger operator. They start from the local decay estimate while the Mourre estimates implies this estimate, please see [Mon81, ABG96]. In order to ensure Mourre’s theory can be applied to \(\bar{H} = PH\bar{P}\), one need \(\bar{H} \in C^2(\mathcal{A})\) where conjugator \(\mathcal{A} = -\frac{i}{2}(x \cdot \nabla + \nabla \cdot x)\), see [ABG96, LS15]. Note that with the projection \(\bar{P}\), operator \(\bar{H} = PH\bar{P}\) only has continuous spectrum in the interval \((\lambda_0 - \epsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \epsilon)\).

Hence, the Mourre estimate can hold for \(\bar{H}\) in the neighborhood of \(\lambda_0\). Moreover, the Mourre estimate implies the local decay estimate and the limiting absorption principle hold for \(\bar{H}\) in the neighborhood of \(\lambda_0\). See e.g. [Mon81, MW11, LS15] and Amrein, Boutet de Monvel and Georgescu’s book [ABG96]. Furthermore, we will discuss that \(H = (-\Delta)^m + V\) does not exist positive embedded eigenvalue for certain class of potential in Section 7 below.

Note that the following \(L^1 - L^\infty\) estimates of free propagator \(e^{-it(-\Delta)^m}\) always hold:

\[
\left\| e^{-it(-\Delta)^m}u \right\|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{m}{2n}}\|u\|_{L^1(\mathbb{R}^n)}, \quad t \neq 0.
\]

Hence it would be a natural problem to establish the \(L^1 - L^\infty\) estimates for higher order Schrödinger operators with some potentials. For \(H = (-\Delta)^2 + V\), in [ESY18], the first two authors and the last author applied the Kato-Jensen decay estimates and local decay estimate to obtain the \(L^1(\mathbb{R}^3) - L^\infty(\mathbb{R}^3)\) estimate, which is not optimal. Green and Toprak in [GT18] further studied the \(L^1 - L^\infty\) estimate for \(e^{-itH}\) in 4-dimension in the regular case and the cases of zero resonance. For Schrödinger operators \(-\Delta + V\), Journé, Soffer and Sogge in [JSS91] first established the \(L^1 - L^\infty\) estimate in the regular case when \(n \geq 3\). For \(n \leq 3\), one
can see Schlag and Goldberg [GS04], Schlag [Sch05], Rodnianski and Schlag [RS04] and so on. Yajima [Yaj95] also proved the references therein.

In the sequel, as a consequence of the Kato-Jensen decay estimates, we can establish the following $L^1 \cap L^2 - L^{\infty} + L^2$ decay estimate (Ginibre argument) in the presence of zero resonance or eigenvalue.

**Theorem 1.10.** For $H = (-\Delta)^m + V$ with $|V(x)| \precsim (1 + |x|)^{-\beta}$ for some $\beta > 0$. Assume that $H$ has no positive embedded eigenvalue.

(I) For $n > 2m$, let $\beta > n$. If $0$ is a regular point of $H$, then
\[
\|e^{-itH} P_{ac}(H)u\|_{L^2 + L^\infty} \precsim (1 + |t|)^{-\frac{n}{2m}}\|u\|_{L^2 \cap L^1}, \quad t \in \mathbb{R}.
\]

(II) For $n > 4m$, let $\beta > n + 4$. If $0$ is an eigenvalue of $H$, then
\[
\|e^{-itH} P_{ac}(H)u\|_{L^2 + L^\infty} \precsim (1 + |t|)^{\frac{2m - n}{2m}}\|u\|_{L^2 \cap L^1}, \quad t \in \mathbb{R}.
\]

(III) For $2m < n \leq 4m$ and $n$ is odd. Let $\beta > n + 4k$, where $2k = 4m - n + 1$ and $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil + 1$.

- If $0$ is the $j$-th kind of resonance of $H$ with $1 \leq j \leq k - 1$, then
  \[
  \|e^{-itH} P_{ac}(H)u\|_{L^2 + L^\infty} \precsim (1 + |t|)^{-\frac{n}{2m} + \frac{k-1}{m}}\|u\|_{L^2 \cap L^1}, \quad t \in \mathbb{R}.
  \]

- If $0$ is the $k$-th kind of resonance or an eigenvalue of $H$, then
  \[
  \|e^{-itH} P_{ac}(H)u\|_{L^2 + L^\infty} \precsim (1 + |t|)^{-\frac{1}{2m}}\|u\|_{L^2 \cap L^1}, \quad t \in \mathbb{R}.
  \]

(IV) For $2m < n \leq 4m$ and $n$ is even. Let $\beta > n + 4k + 2$, where $2k = 4m - n + 2$ and $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil + 1$.

- If $0$ is the $j$-th kind of resonance of $H$ with $1 \leq j \leq k - 1$, then
  \[
  \|e^{-itH} P_{ac}(H)u\|_{L^2 + L^\infty} \precsim (1 + |t|)^{-\frac{n}{2m} + \frac{k-1}{m}}\|u\|_{L^2 \cap L^1}, \quad t \in \mathbb{R}.
  \]

- If $0$ is the $k$-th kind of resonance or an eigenvalue of $H$, then
  \[
  \|e^{-itH} P_{ac}(H)u\|_{L^2 + L^\infty} \precsim (1 + \ln |t|)^{-1}\|u\|_{L^2 \cap L^1}, \quad |t| > 1, t \in \mathbb{R}.
  \]

Finally, we will give some results on the absence of positive embedded eigenvalue for $h(D) + V$, where $h \geq 0$ is a homogeneous real-valued function of order $\varrho$.

**Theorem 1.11.** Let $H_h = h(D) + V$, where $h$ is a nonnegative real-valued function satisfying $h(\gamma \xi) = \gamma^\varrho h(\xi)$ with $\varrho > 0$ and $\gamma > 0$. $V(x)$ is a real-valued function on $\mathbb{R}^n$. Suppose that $V$ is $h(D)$-bounded with relative bound less than one. Then we have the following conclusions:

(i) If $V(x)$ is repulsive, that is $V(\gamma x) \leq V(x)$ for all $\gamma > 1$ and $x \in \mathbb{R}^n$, then $H_h$ has no bound states, i.e. the point spectrum $\sigma_p(H_h) \cap \mathbb{R} = \emptyset$.

(ii) If $V(x)$ is homogeneous of degree $-\nu$ with $0 < \nu < \varrho$, that is $V(\gamma x) = \gamma^{-\nu} V(x)$, then $H_h$ has no nonnegative eigenvalues, i.e. the point spectrum $\sigma_p(H_h) \cap [0, +\infty) = \emptyset$. 
(iii) If there exists a multiplication operator $V$ on $L^2(\mathbb{R}^n)$ with $\mathcal{D}(V) \supset \mathcal{D}(h(D))$, such that for all $\phi \in \mathcal{D}(h(D))$,

$$s - \lim_{\theta \to 1} (\theta - 1)^{-1} (V(\theta \cdot) - V) \phi(x) = V \phi(x),$$

where $\mathcal{D}(h(D))$ and $\mathcal{D}(V)$ are the self-adjoint domain of $h(D)$ and $V$ respectively. Moreover, if for some $a > 0$, $V$ satisfies,

$$h(D) - \frac{1}{\varrho}(1 + a)\mathcal{V} - aV \geq 0,$$

then $H_h$ has no strictly positive eigenvalues, i.e. $\sigma_p(H_h) \cap (0, +\infty) = \emptyset$.

**Remark 1.12.** The conclusion (i) can be applied to the potential $V(x)$ satisfying $x \cdot \nabla V(x) \leq 0$. The conclusion (iii) can be applied to certain homogeneous potentials $V(x)$ of degree $-\varrho$. Indeed, if $V(x)$ does, then $V = -\varrho V$. Thus,

$$h(D) - \frac{1}{\varrho}(1 + a)\mathcal{V} - aV = h(D) + V \geq 0.$$ 

For instance, if consider the higher order polyhamonic operators $(-\Delta)^m - c|x|^{-2m}$ on $L^2(\mathbb{R}^n)$, by Rellich’s type inequality (see e.g. [DH98a]): if $1 < p < \infty$ and $n > 2mp$, then

$$\| |x|^{-2m}u(x) \|_{L^p(\mathbb{R}^n)} \leq c(m, p, n) \| \Delta^m \|_{L^p(\mathbb{R}^n)},$$

where

$$c(m, p, n) = p^{2m} \left( \prod_{l=1}^{m} (n - 2lp)(2(l - 1)p + (p - 1)n) \right)^{-1}.$$ 

Thus $-c|x|^{-2m}$ is the $(-\Delta)^m$-bounded with relative bound less than one if $n > 4m$ and $0 < c < C(m, 2, n)$. Hence Theorem 1.11 (iii) can apply to the higher order operator $(-\Delta)^m - c|x|^{-2m}$.

The paper is organized as follows. In Section 2 we state the asymptotic expansions of $R_V(z)$ as $z \to 0$ with presence of zero resonance or eigenvalue and the classification of resonance spaces. In Section 3 we show the higher energy decay estimates. In Section 4 we give the proof of local decay estimate and derive the asymptotic expansions in time for $e^{-itH} P_{ac}(H)$ with presence of zero resonance. As a consequence, we get the Kato-Jensen type decay estimate. In Section 5 and Section 6 as applications of Kato-Jensen type decay estimate and local decay estimate, we prove the $L^p$-decay estimates and the endpoint Strichartz estimates for $H = (-\Delta)^m + V$. In Section 7 we construct examples to show that $H_h = h(D) + V$ exists positive eigenvalue and give the proof of Theorem 1.11. In the last section, we show the iterated processes of how to derive the expansion of $R_V(z)$ and the proof of classification of resonance spaces.

2. Resolvent asymptotic expansions at zero threshold

In this section, we derive the asymptotic expansions of $R_V(z)$ as $z \to 0$. 
2.1. Free resolvent asymptotic expansions. For \( z \in \mathbb{C} \setminus \mathbb{R}^+ \), denote
\[
R_0(z) = (H_0 - z)^{-1} = ((-\Delta)^m - z)^{-1}, \quad R_V(z) = (H - z)^{-1} = ((-\Delta)^m + V - z)^{-1}.
\]
For free resolvent \( R_0(z) \) with \( z \in \mathbb{C} \setminus \mathbb{R}^+ \) and \( 0 < \arg(z) < 2\pi \), we have the following decomposition identity
\[
R_0(z) = \left( (-\Delta)^m - z \right)^{-1} = \frac{1}{mz} \sum_{\ell=0}^{m-1} z_\ell \left( -\Delta - z_\ell \right)^{-1}, \quad z_\ell = \frac{\kappa}{\pi} \frac{e^{i\ell \pi}}{m}.
\]
Identity (2.1) shows that the free resolvent \( R_0(z) \) of \((-\Delta)^m\) is a linear combination of the free resolvent of Laplacian. Thus we can obtain the asymptotic expansions of \( \text{free resolvent of Laplacian} \)
\[
\sum_{\ell=0}^{m-1} z_\ell \left( -\Delta - z_\ell \right)^{-1}.
\]
with chosen \( 0 < \arg(z) < 2\pi \). Recall the asymptotic expansions of the free resolvent of Laplacian \( R(-\Delta; \zeta) := (-\Delta - \zeta)^{-1} \) as follows, see [JK79, Jen80, Jen84].

**Lemma 2.1.** For \( \zeta \in \mathbb{C} \setminus [0, \infty) \) and \( \Im \zeta^{1/2} > 0 \).

(i) If \( n \geq 3 \) and \( n \) is odd, then
\[
R(-\Delta; \zeta) = \sum_{j=0}^{\infty} (i\zeta^{1/2})^j G_{j}^{odd}
\]
where the operators \( G_{j}^{odd} \) are given by the following integral kernels:
\[
G_{j}^{odd}(x, y) = \frac{(-1)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}n_j} |x - y|^{j+2-n}
\]
with \( n_j = \sum_{\ell=0, \ell \geq (n-3)/2 - j} (n-3)/2 + \ell \mid \ell \mid (n-3)/2 - \ell \mid \ell + j - (n-3)/2 \mid \).

(ii) If \( n \geq 4 \) and \( n \) is even, then
\[
R(-\Delta; \zeta) = \sum_{j=0}^{\infty} \sum_{\ell=0}^{1} \zeta^j (\ln \zeta)^\ell G_{j}^{\ell,e}
\]
For \( 0 \leq j \leq \frac{n}{2} - 2 \), the operators \( G_{j}^{\ell,e} \) are given by the following integral kernels:
\[
G_{j}^{0,e}(x, y) = \pi^{-\frac{n}{2}} \left( \frac{\frac{n}{2} - j - 2)!}{j!} 4^{-j-1} |x - y|^{2j+2-n}; \quad G_{j}^{1,e}(x, y) = 0.
\]
For \( j \geq \frac{n}{2} - 1 \), the operators \( G_{j}^{k,e} \) are given by the following integral kernels:
\[
G_{j}^{0,e}(x, y) = (4\pi)^{-n/2} \left[ \varphi(j + 1) + \varphi(j + 2 - n/2) \right] \frac{(-1)^j - n/2}{j!(n/2 - 1 + j)!} |x - y|^{2j+2-n}
\]
\[
- 2(4\pi)^{-n/2} (-1)^j + 1 - n/2 \ln(|x - y|/2) j!(n/2 - 1 + j)! |x - y|^{2j+2-n}
\]
\[
+ \frac{i}{4} (4\pi)^{-n/2+1} \frac{1}{j!(n/2 - 1 + j)!} (-1)^j + 1 - n/2 |x - y|^{2j+2-n}.
\]
where \( \varphi(1) = -1, \quad \varphi(\ell) = \sum_{j=1}^{\ell-1} \frac{1}{j} - \kappa \) and \( \kappa \) is the Euler’s constant.
Remark 2.2. If $j$ is odd and $0 < j < n - 2$, then $n_j = 0$, see [Jen80, Lemma 3.3]. For any $j = 0, 1, 2, \ldots$, the operators $G_j^{\operatorname{odd}}, G_j^{\ell,e} \in B(s, -s')$ with $s, s'$ depend on $j$ and $n$.

Based on the expansions of $R(-\Delta; \zeta)$ and the resolvent identity (2.1), we obtain the formally expansions of $R_0(z)$ directly.

Lemma 2.3. For $z \in \mathbb{C} \setminus [0, +\infty)$ and $0 < \arg(z) < 2\pi$, we obtain the following formally expansions of $R_0(z)$:

(i) If $n \geq 3$ and $n$ is odd, then

$$R_0(z) = \sum_{n=0}^{\infty} (\ln z)^m \frac{j+1-m}{m} G_j^{\ell,e}, \quad (2.2)$$

with $g_j = i^j$ for $j \in 2m\mathbb{N} - 2$ and $g_j = \frac{1}{m} \frac{j+1-m}{m} e^{i(\frac{1}{2} + \frac{1}{m})\pi} \pi$ for $j \notin 2m\mathbb{N} - 2$.

(ii) If $n \geq 4$ and $n$ is even, then

$$R_0(z) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{1} (\ln z)^m \frac{j+1-m}{m} G_j^{\ell,e}, \quad (2.3)$$

where $G_j^{0,e} = g_j^0 G_j^{0,e} + g_j^{1,0} G_j^{1,e}$ and $G_j^{1,e} = g_j^{1,1} G_j^{1,e}$. Here,

$$g_j^0 = g_j^{1,1} = 1, \quad g_j^{1,0} = \frac{i(m-1)\pi}{m}, \quad j \in m\mathbb{N} - 1;$$

$$g_j^0 = g_j^{1,1} = 0, \quad g_j^{1,0} = \frac{-2i\pi}{1 - e^{i(\frac{1}{2} + \frac{1}{m})\pi}}, \quad j \notin m\mathbb{N} - 1.$$

Notice that there are factors $i^j - (-i)^j$ and 0 in the expansions of $R_0(z)$, then so many terms can be cancelled. According to Remark 2.2, we obtain the above formal expansions in the following sense. For simplifying the notation, we let $z = \mu^{2m}$ with $0 < \arg(\mu) < \pi/m$. Denote $R_0(\mu^{2m}; x, y)$ be the convolution kernel of $R_0(\mu^{2m})$.

Proposition 2.4. For $\mu \in \mathbb{C} \setminus [0, +\infty)$ and $0 < \arg(\mu) < \pi/m$, we obtain the following expansions for $0 < |\mu| \ll 1$ in $B(s, -s')$ with $s, s' > \frac{n}{2} + 2$:

(i) If $n > 4m$ and $n$ is odd, denotes $\mathbb{R} = [\frac{n}{2m}]$. Then we have

$$R_0(\mu^{2m}; x, y) = a_0|x-y|^{2m-n} + \sum_{j=1}^{R-1} a_j \mu^{2mj}|x-y|^{2m(j+1)-n} + a_R \mu^{n-2m}|x-y|^0, \quad (2.4)$$

$$+ a_{R+1} \mu^{n-2m+2}|x-y|^2 + E_1(\mu; x, y)$$

where $a_n, a_{R+1} \in \mathbb{C} \setminus \mathbb{R}$ and $a_j \in \mathbb{R} \setminus \{0\}$ for $0 \leq j \leq \mathbb{R} - 1$. Furthermore, for $s, s' > \frac{n}{2} + 2$,

$$E_1(\mu; x, y) \in B(s, -s') \quad \text{and} \quad \|E_1(\mu; x, y)\|_{B(s, -s')} = O(\mu^{n-2m+2}).$$

(ii) If $n > 4m$ and $n$ is even, denotes $\mathbb{R} = [\frac{n}{2m}]$. Then we have

$$R_0(\mu^{2m}; x, y) = b_0|x-y|^{2m-n} + \sum_{j=1}^{R-1} b_j \mu^{2mj}|x-y|^{2m(j+1)-n} + b_R \mu^{n-2m}|x-y|^0, \quad (2.5)$$

$$+ b_{R+1} \mu^{n-2m+2}|x-y|^2 + E_2(\mu; x, y)$$
where \( b_{\mathbb{R}}, b_{\mathbb{R}+1} \in \mathbb{C} \setminus \mathbb{R} \) and \( b_j \in \mathbb{R} \setminus \{0\} \) for \( 0 \leq j \leq \mathbb{R} - 1 \). Furthermore, for \( s, s' > \frac{n}{2} + 2 \),

\[
E_2(\mu; x, y) \in B(s, -s') \quad \text{and} \quad \|E_2(\mu; x, y)\|_{B(s, -s')} = O(\mu^{n-2m+2}).
\]

(iii) If \( 2m < n \leq 4m \) and \( n \) is odd, let \( n = 4m + 1 - 2k \) with \( 1 \leq k \leq m \). Then we have

\[
R_0(\mu^{2m}; x, y) = c_0|y - y|^{2m-n} + \sum_{j=1}^{k} c_j \mu^{2(m+j-k)-1}|x - y|^{2j-2} + c_{k+1} \mu^{2m}|x - y|^{2k-1} + E_3(\mu; x, y)
\]

where \( c_j \in \mathbb{C} \setminus \mathbb{R} \) for \( 1 \leq j \leq k \) and \( c_0, c_{k+1} \in \mathbb{R} \setminus \{0\} \). Furthermore, for \( s, s' > \frac{n}{2} + 2k \),

\[
E_3(\mu; x, y) \in B(s, -s') \quad \text{and} \quad \|E_3(\mu; x, y)\|_{B(s, -s')} = O(\mu^{2m+1}).
\]

(iv) If \( 2m < n \leq 4m \) and \( n \) is even, let \( n = 4m + 2 - 2k \) with \( 1 \leq k \leq m \). Then we have

\[
R_0(\mu^{2m}; x, y) = d_0|y - y|^{2m-n} + \sum_{j=1}^{k-1} d_j \mu^{2(m+j-k)}|x - y|^{2j-2} + \mu^{2m} g(\mu)|x - y|^{2k-2}
\]

\[
+ d_{k+1} \mu^{2m}|x - y|^{2k-2} \ln(|x - y|) + E_4(\mu; x, y)
\]

where \( g(\mu) = d_k \ln(\mu) + c_k, d_j \in \mathbb{C} \setminus \mathbb{R} \) for \( 1 \leq j \leq k-1 \) and \( d_0, d_k, d_{k+1} \in \mathbb{R} \setminus \{0\}, c_k \in \mathbb{C} \setminus \mathbb{R} \) that can be calculated by Lemma 2.3 and identity (2.1). Furthermore, for \( s, s' > \frac{n}{2} + 2k + 1 \),

\[
E_4(\mu; x, y) \in B(s, -s') \quad \text{and} \quad \|E_4(\mu; x, y)\|_{B(s, -s')} = O(\mu^{2m+2}).
\]

Proof. The proof follows from [Jen80, Lemma 2.3] and [Jen84, Lemma 3.5, Lemma 3.9], since we derive the expansions of \( R_0(\mu^{2m}) \) by the expansions of \( R(-\Delta; \zeta) \).

\[\square\]

2.2. Asymptotic expansions of \( R_V(z) \) around zero. For \( R_V(\mu^{2m}) \), we apply the following symmetric resolvent identity to derive the asymptotic expansions as \( \mu \to 0 \).

\[
R_V(\mu^{2m}) = R_0(\mu^{2m}) - R_0(\mu^{2m})vM(\mu)^{-1}vR_0(\mu^{2m}),
\]

where \( v(x) = |V(x)|^{1/2}, \)

\[
M(\mu) = U + vR_0(\mu^{2m})v
\]

and \( U(x) = \text{sign}(V(x)) \). Furthermore, let \( w(x) = U(x)v(x) \), then

\[
wR_V(\mu^{2m})w = U - M(\mu)^{-1}.
\]

Note that, for \( 0 < |\mu| \ll 1 \), the identity (2.9) shows \( R_V(\mu^{2m}) \) in \( B(s, -s) \) with suitable \( s > 0 \) has the same asymptotic behaviors of \( M(\mu)^{-1} \) in \( B(0, 0) \) if \( w(x)(1 + |x|)^s \in L^\infty(\mathbb{R}^n) \).

Since the number of expansion terms depends on dimension \( n \) for a fixed order \( m \), we divide the dimension into three intervals to derive the asymptotic expansion of \( R_V(\mu^{2m}) \).

Case 1: \( n > 4m \). By Proposition 2.4 (i) and (ii), we have the following expansions of \( R_0(\mu^{2m}; x, y) \) in \( B(s, -s') \) with \( s, s' > \frac{n}{2} + 2 \):

\[
R_0(\mu^{2m}; x, y) = G_0(x, y) + \sum_{j=1}^{N-1} \mu^{2mj}G_j(x, y) + \mu^{n-2m}G_N(x, y)
\]

\[
+ \mu^{n-2m+2}G_{N+1}(x, y) + E_0(\mu; x, y)
\]
where

\[
G_0(x, y) = \begin{cases} 
  a_0 |x - y|^{2m-n}, & n > 4m \text{ odd;} \\
  b_0 |x - y|^{2m-n}, & n > 4m \text{ even;}
\end{cases}
\]

\[
G_j(x, y) = \begin{cases} 
  a_j |x - y|^{2m(j+1)-n}, & n > 4m \text{ odd;} \\
  b_j |x - y|^{2m(j+1)-n}, & n > 4m \text{ even;}
\end{cases}
\]

for \( 1 \leq j \leq N - 1 \),

\[
G_N(x, y) = \begin{cases} 
  a_N |x - y|^0, & n > 4m \text{ odd;} \\
  b_N |x - y|^0, & n > 4m \text{ even;}
\end{cases}
\]

\[
G_{N+1}(x, y) = \begin{cases} 
  a_{N+1} |x - y|^2, & n > 4m \text{ odd;} \\
  b_{N+1} |x - y|^2, & n > 4m \text{ even;}
\end{cases}
\]

\[
E_0(\mu; x, y) = \begin{cases} 
  E_1(\mu; x, y), & n > 4m \text{ odd;} \\
  E_2(\mu; x, y), & n > 4m \text{ even.}
\end{cases}
\]

Substituting (2.10) into the symmetric resolvent identity (2.8), we have

\[
M(\mu) = U + vG_0v + \sum_{j=1}^{N-1} \mu^{2mj}vG_jv + \mu^{n-2m}vG_Rv + \mu^{n-2m+2}vG_{N+1}v + vE_0(\mu)v. \tag{2.11}
\]

Recall that \( a_N, a_{N+1} \in \mathbb{C} \setminus \mathbb{R} \). Note that we need to expand to the order \( \mu^{n-2m} \) here. The reason is that \( G_j(0 \leq j \leq N - 1) \) are selfadjoint operators which has no contribution to the spectral density.

Denotes \( T_0 = U + vG_0v \). If \( T_0 \) is invertible on \( L^2(\mathbb{R}^n) \), then we say zero is a regular point of \( H = (-\Delta)^m + V \). Otherwise, if \( T_0 \) is not invertible, let \( S_1 \) to be the Riesz projection onto \( \ker(T_0) \) as an operator on \( L^2(\mathbb{R}^n) \). Then \( T_0 + S_1 \) is invertible on \( L^2(\mathbb{R}^n) \). Accordingly, let \( D_0 = (T_0 + S_1)^{-1} \). If \( T_0 \) is not invertible and \( T_1 = S_1vG_1vS_1 \) is invertible, then we say zero is a “resonance” of \( H \). In the case \( n > 4m \), the subspace \( S_1L^2(\mathbb{R}^n) \) is actually the zero eigenspace of \( H \). We will show \( T_1 \) is invertible on \( L^2(\mathbb{R}^n) \), see Theorem 2.5 and Proposition 2.10 below. Hence, \( H = (-\Delta)^m + V \) does not exist zero resonance if \( n > 4m \).

Next, we give the expansions of \( R_V(\mu^{2m}) \) in \( B(s, -s') \) with \( s, s' > \frac{n}{2} + 2 \).

**Theorem 2.5.** For \( n > 4m \), let \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) with some \( \beta > n + 4 \). Then for \( 0 < |\mu| \ll 1 \), we have the following expansions of \( R_V(\mu^{2m}) \) in \( B(s, -s') \):

(i) If 0 is a regular point of \( H \), then

\[
R_V(\mu^{2m}) = A_0 + \sum_{l=1}^{N-1} \mu^{2ml}A_l + \alpha_N\mu^{n-2m}A_R + \alpha_{N+1}\mu^{n-2m+2}A_{R+1} + O(\mu^{n-2m+2})
\]

where \( A_l \in B(s, -s') \) are selfadjoint operators and \( \alpha_N, \alpha_{N+1} \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( A_0 = G_0 - G_0v(T_0)^{-1}vG_0 \).

(ii) If 0 is an eigenvalue of \( H \), then

\[
R_V(\mu^{2m}) = \frac{A_1^e}{\mu^{2m}} + \sum_{j=2}^{N-1} \mu^{2m(j-1)}A_j^e + \alpha_N^e\mu^{n-6m}A_N^e + \alpha_{N+1}^e\mu^{n-6m+2}A_{N+1}^e + O(\mu^{n-6m+2})
\]

where \( A_j^e \in B(s, -s') \) are selfadjoint operators and \( \alpha_N^e, \alpha_{N+1}^e \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( A_1^e = -G_0v(S_1vG_1vS_1)^{-1}vG_0 \).
Case 2: $n = 4m + 1 - 2k$ with $k = 1, 2, \cdots, m$. (i.e. $2m < n \leq 4m$ and $n$ odd). In this case, by Proposition 2.4 (iii), we have the following expansions of $R_0(\mu^{2m}; x, y)$ in $B(s, -s')$ with $s, s' > n/2 + 2k$:

\[
R_0(\mu^{2m}; x, y) = G_0(x, y) + c_1 \mu^{2(m-k)+1} I + \sum_{j=2}^{k} c_j \mu^{2(m+j-k)-1} G_j(x, y) + \mu^{2m} G_{k+1}(x, y) + E_3(\mu; x, y)
\]

(2.12)

where

\[
\begin{align*}
G_0(x, y) &= c_0 |x - y|^{2m-n}, \\
G_j(x, y) &= |x - y|^{2j-2}, \quad 2 \leq j \leq k; \\
G_{k+1}(x, y) &= c_{k+1} |x - y|^{2k-1}.
\end{align*}
\]

Let $P$ be the projection onto the span of $v$, i.e. $P = \|v\|^{-1}_{L^2(\mathbb{R}^n)} \langle v, \cdot \rangle$. Thus $P$ is a self-adjoint operator with $\dim(P) = 1$. Substituting (2.12) into (2.11), we have

\[
M(\mu) = U + vG_0v + c_1 \mu^{2(m-k)+1} P + \sum_{j=2}^{k} c_j \mu^{2(m+j-k)-1} vG_jv + \mu^{2m} vG_{k+1}v + vE_3(\mu)v.
\]

(2.13)

Depending on the inverse processes, now we give the equivalent definition of each kinds of zero resonance of $H$.

**Definition 2.6.** Let $T_0 = U + vG_0v$.

1. If $T_0$ is invertible on $L^2(\mathbb{R}^n)$, then we call 0 is a regular point of $H$.
2. Assume that $T_0$ is not invertible on $L^2(\mathbb{R}^n)$. Let $S_1$ be the Riesz projection onto $\ker(T_0)$, then $T_0 + S_1$ is invertible on $L^2(\mathbb{R}^n)$. If $T_0$ is not invertible and $T_1 := S_1 P S_1$ is invertible on $S_1 L^2(\mathbb{R}^n)$, then we call 0 is the first kind of resonance of $H$.
3. Assume that $T_1$ is not invertible on $S_1 L^2(\mathbb{R}^n)$. Let $S_2$ be the Riesz projection onto $\ker(T_1)$, then $T_1 + S_2$ is invertible on $S_1 L^2(\mathbb{R}^n)$. If $T_1$ is not invertible and $T_2 := S_2 vG_2 v S_2$ is invertible on $S_2 L^2(\mathbb{R}^n)$, then we call 0 is the second kind of resonance of $H$.
4. For $3 \leq j \leq k$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Assume that $T_{j-1} := S_{j-1} vG_{j-1} v S_{j-1}$ is not invertible on $S_{j-1} L^2(\mathbb{R}^n)$. Let $S_j$ be the Riesz projection onto $\ker(T_{j-1})$, then $T_{j-1} + S_j$ is invertible on $S_{j-1} L^2(\mathbb{R}^n)$. If $T_{j-1}$ is not invertible and $T_j := S_j v G_j v S_j$ is invertible on $S_j L^2(\mathbb{R}^n)$, then we call 0 is the $j$-th kind of resonance of $H$.
5. Assume that $T_k$ is not invertible on $S_k L^2(\mathbb{R}^n)$. Let $S_{k+1}$ be the Riesz projection onto $\ker(T_k)$, then $T_k + S_{k+1}$ is invertible on $S_k L^2(\mathbb{R}^n)$. In this case, the operator $T_{k+1} := S_{k+1} v G_{k+1} v S_{k+1}$ is always invertible, then we say there is a $(k+1)$-th kind of “resonance” at zero.

**Remark 2.7.** (1) Definition 2.6 is equivalent to the Definition 2.4 actually. Indeed, we will identify all the subspaces $S_j L^2(\mathbb{R}^n)$, see Proposition 2.11 and Proposition 2.12. From the proof of the identification processes, these two definitions are equivalent, see subsection 8.2 below. Furthermore, the last kind of zero “resonance” is actually zero eigenvalue.

(2) The projections $S_j$ ($1 \leq j \leq k + 1$) are finite rank operators. Indeed, $T_0 = U + vG_0v$ which is a compact perturbation of the invertible operator $U$, thus the Fredholm alternative theorem guarantees that $S_j$ is of finite rank. By Definition 2.6, we have $S_{k+1} \leq S_k \leq \cdots \leq S_2 \leq S_1$, hence $S_j$ ($1 \leq j \leq k + 1$) are both of finite rank.
(3) By Definition 2.6, let \( D_j = (T_j + S_{j+1})^{-1} \) for \( 0 \leq j \leq k \), then we have
\[
S_{j+1}D_j = D_jS_{j+1} = S_{j+1}, \quad 0 \leq j \leq k;
\]
\[
S_jD_j = D_jS_j = D_j, \quad 1 \leq j \leq k.
\]

(4) For Schrödinger operator \(-\Delta + V\) in 3 and 4 dimensional cases, i.e. \( m = 1 \) with \( n = 3, 4 \), we have \( k = 1 \). Thus \(-\Delta + V\) has only one kind resonance for \( n = 3, 4 \). Recall that \(-\Delta + V\) does not exist zero resonance for \( n \geq 5 \), see [Jen80]. Thus Definition 2.6 matches the resonance definition of Kato and Jensen in [JK79, Jen80].

Now, we give the asymptotic expansions of \( R_V(\mu^{2m}) \) in \( B(s, -s') \) with suitable chosen \( s, s' \in \mathbb{R} \) with the presence of each kind of zero resonance. Note that, the following expansions imply that how the zero resonance affect the behavior of spectral density \( dE(\lambda) \) of \( H = (-\Delta)^m + V \) as \( \lambda \) close to 0.

**Theorem 2.8.** For \( n = 4m + 1 - 2k \) with \( k \in \mathbb{N} \) chosen as follows, let \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) with some \( \beta > n + 4k \). Then for \( 0 < |\mu| \ll 1 \), we have the following expansions of \( R_V(\mu^{2m}) \) in \( B(s, -s') \) with \( s, s' > \frac{n}{2} + 2k \):

(i) For \( 1 \leq k \leq m \), if \( 0 \) is a regular point of \( H \), then we have
\[
R_V(\mu^{2m}) = B_0 + \sum_{\ell=1}^{k} \beta_\ell \mu^{2(m+\ell-k)-1} B_\ell + \mu^{2m} B_{k+1} + O(\mu^{2m+})
\]
where \( B_\ell \in B(s, -s') \) are selfadjoint operators with \( s, s' > \frac{n}{2} + 2k \) and \( \beta_\ell \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( B_0 = G_0 - G_0v(T_0)^{-1}vG_0 \).

(ii) For \( 1 \leq k \leq \lfloor \frac{m}{2} \rfloor + 1 \), if \( 0 \) is the \( j \)-th kind of resonance of \( H \) with \( 1 \leq j \leq k \), then we have
\[
R_V(\mu^{2m}) = \frac{\beta_0 B_0^j}{\mu^{2(m-k-j+1)-1}} + \sum_{\ell=1}^{k-j} \frac{\beta_\ell^j B_\ell^j}{\mu^{2(m-k-j-\ell+1)-1}} + \sum_{\ell=k-j+1}^{2m-3k+3j-2} \frac{\beta_\ell^j B_\ell^j}{\mu^{2m-3k+3j-\ell+1}} + O(\mu),
\]
where \( B_\ell^j \in B(s, -s') \) are selfadjoint operators and \( \beta_\ell^j \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( B_0^1 = -G_0v(T_j)^{-1}vG_0 \).

(iii) For \( 1 \leq k \leq \lfloor \frac{m}{2} \rfloor + 1 \), if \( 0 \) is an eigenvalue of \( H \), then we have
\[
R_V(\mu^{2m}) = \frac{B_0^{k+1}}{\mu^{2m}} + \sum_{\ell=1}^{2m-1} \frac{\beta_\ell^{k+1} B_\ell^{k+1}}{\mu^{2m-\ell}} + \frac{\beta_{k+1} B_{2m}}{\mu^{2m}} + O(\mu),
\]
where \( B_\ell^{k+1} \in B(s, -s') \) are selfadjoint operators and \( \beta_{k+1} \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( B_0^{k+1} = -G_0v(T_{k+1})^{-1}vG_0 \).

**Case 3:** \( n = 4m + 2 - 2k \) with \( k = 1, 2, \cdots, m \). (i.e. \( 2m < n \leq 4m \) and \( n \) even). In this case, by Proposition 2.3 (iv), we have the following expansions of \( R_0(\mu^{2m}; x, y) \) in \( B(s, -s') \) with \( s, s' > \frac{n}{2} + 2k + 1 \):
\[
R_0(\mu^{2m}; x, y) = G_0(x, y) + d_1 \mu^{2(m-k+1)} I + \sum_{j=2}^{k-1} d_j \mu^{2(m+j-k)} G_j(x, y)
\]
\[
+ \mu^{2m} g(\mu) G_k(x, y) + \mu^{2m} G_{k+1}(x, y) + E_4(\mu; x, y)
\]
(2.14)
where
\[ G_0(x, y) = d_0|x - y|^{2m-n}; \]
\[ G_j(x, y) = |x - y|^{2j-2}, \quad 2 \leq j \leq k; \]
\[ G_{k+1}(x, y) = d_{k+1}|x - y|^{2k-2}\ln(|x - y|). \]

Substituting (2.14) into identity (2.8), we have
\[ M(\mu) = U + vG_0v + \tilde{d}_1\mu^{2(m-k+1)}P + \sum_{j=2}^{k-1} d_j\mu^{2(m+j-k)}vG_jv \]
\[ + \mu^{2m}g(\mu)vG_kv + \mu^{2m}vG_{k+1}v + vE_4(\mu)v. \]

The definition of resonance for this case is the same as Definition 2.6 by replacing the representations of \( G_j \) in the corresponding case for \( n \) is even. Next, we give the expansions of \( R_V(\mu) \) in \( B(s, -s') \) with \( s, s' > \frac{m}{2} + 2k \).

**Theorem 2.9.** For \( n = 4m + 2 - 2k \) with \( k \in \mathbb{N} \) chosen as follows, let \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) with some \( \beta > n + 4k + 2 \). Then for \( 0 < |\mu| \ll 1 \), we have the following expansions of \( R_V(\mu^{2m}) \) in \( B(s, -s') \) with \( s, s' > \frac{m}{2} + 2k \):

(i) For \( 1 \leq k \leq m \), if 0 is a regular point of \( H \), then we have
\[ R_V(\mu^{2m}) = C_0 + \sum_{l=1}^{k-1} \mu^{2(m+l-k)}\tau_lC_l + \mu^{2m}g(\mu)C_k + \mu^{2m}C_{k+1} + O(\mu^{2m+}) \]
where \( C_l \in B(s, -s') \) are selfadjoint operators and \( \tau_l \in \mathbb{C} \setminus \mathbb{R} \) for all \( l \). Furthermore, \( C_0 = G_0 - G_0v(T_0)^{-1}vG_0 \).

(ii) For \( 1 \leq k \leq \left[ \frac{m}{2} \right] + 1 \), if 0 is the \( j \)-th kind of resonance of \( H \) with \( 1 \leq j \leq k-1 \), then we have
\[ R_V(\mu^{2m}) = \frac{\tau_j^i}{\mu^{2(m-k+j)}} + \sum_{\ell=1}^{k-j} \frac{\tau_j^{i,0}C_{j,0}^{i}}{\mu^{2(m-k-j+\ell)}} + \sum_{\ell=k-j+1}^{m-k+j-1} \left( \frac{\tau_j^{i,0}C_{j,0}^{i}}{\mu^{2(m-k-j-\ell)}} + \frac{\ln(\mu)\tau_j^{i,1}C_{j,1}^{i}}{\mu^{2(m-k-j-\ell)}} \right) \]
\[ + \ln(\mu)\tau_j^{i}C_{m-k+j,1}^{i}C_{m-k+j,1}^{i} + \tau_j^{i}C_{m-k+j,0}^{i}C_{m-k+j,0}^{i} + O(\mu^{0+}), \]
where \( C_{j,0}^{i}, C_{j,1}^{i} \in B(s, -s') \) are selfadjoint operators and \( \tau_j^{i,0}, \tau_j^{i,1} \in \mathbb{C} \setminus \mathbb{R} \) for all \( \ell \). Furthermore, \( C_{j,0}^{i} = -G_0v(T_0)^{-1}vG_0 \).

(iii) For \( 1 \leq k \leq \left[ \frac{m}{2} \right] + 1 \), if 0 is the \( k \)-th kind of resonance of \( H \), then we have
\[ R_V(\mu^{2m}) = \frac{\tau_k^{1,0}C_{k,0}^{1}}{\mu^{2m}(g(\mu))^2} + \frac{\tau_k^{1,2}C_{k,2}^{1}}{\mu^{2m}(g(\mu))^2} + \sum_{\ell=1}^{m-1} \left( \frac{\tau_k^{1,0}C_{k,0}^{1}}{\mu^{2(m-\ell)}} + \frac{\tau_k^{1,1}C_{k,1}^{1}}{\mu^{2(m-\ell)}(g(\mu))^2} + \frac{\tau_k^{1,2}C_{k,2}^{1}}{\mu^{2(m-\ell)}(g(\mu))^2} \right) \]
\[ + \tau_k^{1,0}C_{m,0}^{k} + (g(\mu))^{-1}\tau_k^{1,1}C_{m,1}^{k} + (g(\mu))^{-2}\tau_k^{1,2}C_{m,2}^{k} + O(\mu^{0+}), \]
where \( C_{j,0}^{i}, C_{j,1}^{i}, C_{j,2}^{i} \in B(s, -s') \) are selfadjoint operators and \( \tau_k^{1,0}, \tau_k^{1,1}, \tau_k^{1,2} \in \mathbb{C} \setminus \mathbb{R} \) for all \( \ell \). Furthermore, \( C_{k,0}^{k} = -G_0v(T_k)^{-1}vG_0 \).
(iv) For $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$, if 0 is an eigenvalue of $H$, then we have

\[ R_{V}(\mu^{2m}) = \frac{C_{0,0}^{k+1}}{\mu^{2m}} + \frac{\tau_{0,1}^{k+1}C_{0,1}^{k+1}}{\mu^{2m}g(\mu)} + \sum_{\ell=1}^{m-1} \left( \frac{\tau_{\ell,0}^{k+1}C_{\ell,0}^{k+1}}{\mu^{2m-\ell}} + \frac{\tau_{\ell,1}^{k+1}C_{\ell,1}^{k+1}}{\mu^{2m}g(\mu)} \right) \]

\[ + \frac{k+1}{\mu_{m,0}^{k+1}}C_{m,1}^{k+1} + (g(\mu))^{-1}\tau_{\ell,1}^{k+1}C_{\ell,1}^{k+1} + O(\mu^{0+}), \]

where $C_{\ell,0}^{k+1}, C_{\ell,1}^{k+1} \in B(s, -s')$ are selfadjoint operators and $\tau_{\ell,0}^{k+1}, \tau_{\ell,1}^{k+1} \in \mathbb{C} \setminus \mathbb{R}$ for all $\ell$. Furthermore, $C_{0,0}^{k+1} = -G_{0}(T_{k+1})^{-1}vG_{0}$.

2.3. Identification of zero resonance spaces. In the above subsection, we obtain the asymptotic expansion of $R_{V}(\mu^{2m})$ as $\mu \to 0$ with the presence of zero resonance or eigenvalue. For different dimensional cases, the number of the kind of zero resonance is different by Definition 2.6. In this subsection, we identify all the kinds of zero resonance spaces.

**Proposition 2.10.** For $n > 4m$, let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4$. Then $\phi \in S_{1}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\phi = Uv$ with $\psi \in L^{2}(\mathbb{R}^{n})$ such that $H\psi = 0$ holds in the distributional sense and

\[ \psi(x) = -c \int_{\mathbb{R}^{n}} \frac{v(y)\phi(y)}{|x - y|^{n-2m}} dy. \]

**Proposition 2.11.** For $n = 4m + 1 - 2k$ with $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$ and $n$ is odd, assume that $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4k$. Then the following statements hold:

(i) $\phi(x) \in S_{1}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\phi(x) = Uv(x)\psi(x)$ where $\psi(x) \in W_{2m-\frac{n}{2}}(\mathbb{R}^{n})$ satisfies $H\psi(x) = 0$ in the distributional sense.

(ii) For $2 \leq j \leq k$, $\phi(x) = Uv(x)\psi(x) \in S_{j}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\psi(x) \in W_{2m-\frac{n}{2}-j}(\mathbb{R}^{n})$ satisfies $H\psi(x) = 0$ in the distributional sense.

(iii) $\phi(x) = Uv(x)\psi(x) \in S_{k+1}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\psi(x) \in L^{2}(\mathbb{R}^{n})$ satisfies $H\psi(x) = 0$ in the distributional sense.

**Proposition 2.12.** For $n = 4m + 2 - 2k$ with $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$ and $n$ is even, assume that $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4k + 2$. Then the following statements hold:

(i) $\phi(x) \in S_{1}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\phi(x) = Uv(x)\psi(x)$ where $\psi(x) \in W_{2m-\frac{n}{2}}(\mathbb{R}^{n})$ satisfies $H\psi(x) = 0$ in the distributional sense.

(ii) For $2 \leq j \leq k - 1$, $\phi(x) = Uv(x)\psi(x) \in S_{j}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\psi(x) \in W_{2m-\frac{n}{2}-j}(\mathbb{R}^{n})$ satisfies $H\psi(x) = 0$ in the distributional sense.

(iii) $\phi(x) = Uv(x)\psi(x) \in S_{k}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\psi(x) \in W_{0}(\mathbb{R}^{n})$ satisfies $H\psi(x) = 0$ in the distributional sense.

(iv) $\phi(x) = Uv(x)\psi(x) \in S_{k+1}L^{2}(\mathbb{R}^{n}) \setminus \{0\}$ if and only if $\psi(x) \in L^{2}(\mathbb{R}^{n})$ satisfies $H\psi(x) = 0$ in the distributional sense.

The proofs of Proposition 2.10–2.12 are placed in the last section.

**Remark 2.13.** (1) For $H = (-\Delta)^{m} + V$ and $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4$, Proposition 2.10 shows that $H$ does not exist zero resonance if $n > 4m$.

(2) If $S_{1} = 0$, then 0 is a regular point of $H$. In the case of $3m \leq n \leq 4m$, if $S_{1} = S_{k+1}$, then 0 is purely an eigenvalue of $H$; if $S_{k+1} = 0$, then 0 is purely a resonance of $H$; if $S_{k+1} \neq 0$, then $S_{j}L^{2}(\mathbb{R}^{n})$ for $1 \leq j < k + 1$ contain the mixed state i.e. 0 is both eigenvalue and resonance of $H$. 

3. Higher energy decay estimates of $R_V(z)$

This section is devoted to the two aims. The first one is to obtain the higher energy decay estimates of $R_V(z)$. The other one is to study the boundary behavior of $R_V(z)$, which is to show that the following limit

$$R_V(\lambda \pm i0) = s - \lim_{\epsilon \to 0} R_V(\lambda \pm i\epsilon)$$

exists in $B(s, -s')$ with suitable $s, s' > 0$.

3.1. Higher energy decay estimates. In this section, we aim to study the decay rate of $R_V(z)$ in $B(s, -s')$ as $z$ goes to infinity with some suitable $s, s' > 0$. Recall the symmetric resolvent identity (2.8):

$$R_V(z) = R_0(z) - R_0(z)vM(z)^{-1}vR_0(z).$$

If $M(z) = U + vR_0(z)v$ has uniform bounded inverse in $L^2(\mathbb{R}^n)$, then $R_V(z)$ has the same decay rate as the $R_0(z)$ in $B(s, -s')$ as $z \to \infty$.

The free resolvent decomposition identity (2.1) shows that one can obtain higher energy decay estimates of $R_0(z)$ from the respect estimate of $R(-\Delta; \zeta)$. The following result is the fundamental Agmon-Kato estimate on decay rate for the free resolvent of Schrödinger operator $R(-\Delta; \zeta)$ as $\zeta$ goes to infinity in the weighted Lebesgue norms. It plays a crucial role in time-decay estimates of the solution to Schrödinger equation.

**Lemma 3.1.** ([KK12 Theorem 16.1]) For $\zeta \in \mathbb{C} \setminus [0, +\infty)$, $\ell = 0, 1, 2, \ldots$, any $s, s' > \ell + \frac{1}{2}$ and any $a > 0$, then

$$\|R^{(\ell)}(-\Delta; \zeta)\|_{B(s, -s')} \leq C(s, a)|\zeta|^{-\ell/2}, \ |\zeta| \geq a.$$

Note that the proof of Theorem 16.1 in [KK12] does not depend on the dimension $n$. The following is the similar conclusion for $H_0 = (-\Delta)^m$.

**Proposition 3.2.** For $z \in \mathbb{C} \setminus [0, +\infty)$, $k = 0, 1, 2, \ldots$, any $s, s' > \ell + \frac{1}{2}$ and $a > 0$, then

$$\|R_0^{(\ell)}(z)\|_{B(s, -s')} \leq C(s, a)|z|^{-\frac{(2m-1)(\ell+1)}{2m}}, \ |z| \geq a. \quad (3.1)$$

**Proof.** Firstly, we prove decay estimate (3.1) for $\ell = 0$. By identity (2.1) and Lemma 3.1 we have

$$\|R_0(z)\|_{B(s, -s')} \leq \frac{1}{m|z|} \sum_{\ell=0}^{m-1} |z\ell| \cdot \|R(-\Delta; z\ell)\|_{B(s, -s')} \leq C(s, a)|z|^{-\frac{(2m-1)}{2m}}.$$

Now we check decay estimate (3.1) for $k \geq 1$. For $R_0(z)$ we have the recurrent relations

$$zR_0^{(\ell)}(z) = -\ell R_0^{(\ell-1)}(z) + \frac{1}{2m} [x \cdot \nabla, R_0^{(\ell-1)}(z)]. \quad (3.2)$$

By an similar induction process as in [PSY13], we get the desired estimate (3.1). \qed

Next, we prove that $M(z) = 1 + vR_0(z)v$ has uniform bounded inverse in $L^2(\mathbb{R}^n)$. Note that, $V$ is $H_0$–relative bounded under the assumption of $V(x)$ in Theorem 1.3. Thus there exists a finite constant $V_0 \in \mathbb{R}$, such that for any $\lambda \in \mathbb{R} \setminus [V_0, +\infty)$, $H - \lambda = (-\Delta)^m + V - \lambda > 0$, then we have $\mathbb{C} \setminus [V_0, +\infty) \subset \rho(H)$ where $\rho(H)$ is the resolvent set of $H$. 
Lemma 3.3. Let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 2$. Then for $z \in \mathbb{C} \setminus [0, +\infty)$, $vR_0(z)v \in B(0, 0)$ are compact operators. Moreover, $M(z) = U + vR_0(z)v$ is invertible in $L^2(\mathbb{R}^n)$ for $z \in \mathbb{C} \setminus [V_0, +\infty)$.

Proof. By the free resolvent decomposition identity (2.1), we have

$$vR_0(z)v = \frac{1}{mz} \sum_{\ell=0}^{m-1} z_{\ell} v (\Delta - z_{\ell})^{-1} v.$$ 

Thus we only need to show $v(-\Delta - z_{\ell})^{-1} v$ is compact operator in $L^2(\mathbb{R}^n)$. Since $z \in \mathbb{C} \setminus [0, +\infty)$ and recall that we choose $0 < \text{arg}(z) < 2\pi$, thus $z_{\ell} \neq 0$ and $\text{Im}(z_{\ell}^{1/2}) \geq 0$. Furthermore, from [Jen80, Lemma 3.1], we know the integral kernel $k(z_{\ell}; x, y)$ of $(-\Delta - z_{\ell})^{-1}$ satisfies the following bound

$$|k(z_{\ell}; x, y)| \lesssim (|x - y|^{2-n} + |x - y|^{(1-n)/2}) (1 + |z_{\ell}|)^{(n-3)/4}, \quad n \geq 3. \quad (3.3)$$

Thus one can check that the Hilbert-Schmidt norm of $v(-\Delta - z_{\ell})^{-1} v$ is finite under the assumption on $V(x)$.

Next, we show that $M(z) = U + vR_0(z)v$ is invertible by Fredholm’s alternative theorem. We claim that $(H - z) \psi = 0$ only has trivial solution in $L^2(\mathbb{R}^n)$. In fact, for $z \in \mathbb{C} \setminus \mathbb{R}$, if $\psi \neq 0$, then

$$\text{Im}((H - z)\psi, \psi) = -\text{Im} z(\psi, \psi) \neq 0.$$ 

Since $(H - z)\psi, \psi = (H\psi, \psi) - z(\psi, \psi)$ and $(H\psi, \psi) \in \mathbb{R}$, hence $(H - z)\psi \neq 0$ if $\psi \neq 0$. For $z \in \mathbb{C} \setminus \mathbb{R}$ and $\text{Re}(z) < V_0$, then

$$\text{Re}((H - z)\psi, \psi) \geq (V_0 - z)(\psi, \psi) \neq 0$$

provided $\psi \neq 0$. Thus $(H - z)\psi \neq 0$ if $\psi \neq 0$.

Note that operator $M(z) = U + vR_0(z)v$ is the symmetric form of $1 + VR_0(z)$. Since $M(z)\psi = 0$ implies that $(H - z)\psi = 0$, thus $M(z)\psi = 0$ only has zero solution. Hence, Fredholm’s alternative theorem tells that $M(z) = U + vR_0(z)v$ is invertible in $L^2(\mathbb{R}^n)$.

Using Proposition 3.2 and Lemma 3.3, we obtain the following higher energy decay estimates for $R_V(z)$.

Proposition 3.4. For $\ell = 0, 1, 2, \ldots$, let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 2 + 2\ell$. For $z \in \mathbb{C} \setminus [V_0, +\infty)$, any $s, s' \geq \ell + \frac{1}{2}$ and $a > 0$, then

$$\|R_V^{(\ell)}(z)\|_{B(s, -s')} \leq C(s, a)|z|^{-\frac{(2m-1)(1+\ell)}{2m}}. \quad (3.4)$$

Proof. For $\ell = 0$, by identity (2.3) and Proposition 3.2, the above bound (3.4) holds by the uniformly boundedness of $M(z)^{-1}$ for large $z \in \mathbb{C} \setminus [V_0, +\infty)$ in $L^2$. It is equivalent to prove that for large $z \in \mathbb{C} \setminus [0, +\infty)$,

$$\|f\|_{L^2(\mathbb{R}^n)} \leq C\|(U + vR_0(z)v)f\|_{L^2(\mathbb{R}^n)}.$$ 

In fact, by the triangle inequality we have

$$\|f\|_{L^2} - \|vR_0(z)vf\|_{L^2} \leq \|(U + vR_0(z)v)f\|_{L^2} \leq \|f\|_{L^2} + \|vR_0(z)vf\|_{L^2}.$$ 

By the decay estimate (3.1), thus for $|z|$ large enough, we have

$$\|vR_0(z)vf\|_{L^2} \leq C(s, a)|z|^{-\frac{2m-1}{2m}}\|f\|_{L^2} \leq \frac{1}{4}\|f\|_{L^2}.$$
For $\ell \geq 1$, differentiating (2.8) $\ell$ times in $z$, we have

$$R^{(\ell)}_V(z) = R^{(\ell)}_0(z) - \sum_{\ell_1 + \ell_2 + \ell_3 = \ell} R^{(\ell_1)}_0(z) v \frac{d^{\ell_2} z}{d\zeta^{\ell_2}} \left( M(z)^{-1} \right) v R^{(\ell_3)}_0(z).$$

Note that the derivative term $\frac{d^{\ell_2} z}{d\zeta^{\ell_2}} \left( M(z)^{-1} \right)$ is the linear combination of terms such as $(M(z)^{-1})^j M^{(l)}(z)$ with $0 \leq j, l < \ell_2$. By the representation of $M(z)$, we know $M^{(l)}(z) = v R^{(l)}_0(z) v$ for $l \geq 1$. Since $v(x) (1 + |x|)^{\beta/2} \in L^\infty$ under the assumption of $V(x)$, thus (3.4) holds by the mathematical induction. □

3.2. Limiting absorption principle of $R_V(z)$. The limiting absorption principle means the existence and continuity of the resolvent in the continuous spectrum. The continuity of the resolvent for Schrödinger operator in the weighted Sobolev norms was established by Agmon [Agm75]. Hörmander [Hö05] also considered such problem for general selfadjoint operator $P(D)$ with real coefficient.

Denote by $C^+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \}$ and $C^- = \{ z \in \mathbb{C} : \text{Im} z < 0 \}$. Define $\Xi$ be the disjoint union of $C^+$ and $C^-$ with the identified points $z \leq 0$. Recall the limiting absorption principle results of free Schrödinger operator $R(-\Delta; \zeta)$ (see [JK79, Jen80, Jen84]), we have:

**Lemma 3.5.** ([JK79 Theorem 8.1]) For $k \geq 0$ and $s, s' > k + \frac{1}{2}$, we have $R^{(k)}(-\Delta; \zeta) \in B(s, -s')$ is analytic for $\zeta \in \Xi \setminus \{0\}$. Furthermore, the boundary values

$$R^{(k)}(-\Delta; \lambda \pm i \epsilon) = \lim_{\epsilon \downarrow 0} R^{(k)}(-\Delta; \lambda \pm i \epsilon) \in B(s, -s')$$

exist for any $\lambda \in (0, \infty)$. The decay estimate (3.1) can be extended from $\zeta \in C \setminus [0, +\infty)$ to $\zeta \in \Xi \setminus \{0\}$.

By the resolvent identity (2.1), then for $R_0(z)$ we have:

**Corollary 3.6.** For $k \geq 0$ and $s, s' > k + \frac{1}{2}$, we have $R^{(k)}_0(z) \in B(s, -s)$ is analytic for $z \in \Xi \setminus \{0\}$. Furthermore, the boundary values

$$R^{(k)}_0(\lambda \pm i \epsilon) = \lim_{\epsilon \downarrow 0} R^{(k)}_0(\lambda \pm i \epsilon) \in B(s, -s')$$

exist for any $\lambda \in (0, \infty)$, and the bound

$$\left\| R^{(k)}_0(z) \right\|_{B(s, -s')} = O\left( |z|^{-\frac{(2m-1)(k+1)}{2m}} \right)$$

(3.6)

holds as $z \to \infty$ in $\Xi \setminus \{0\}$.

**Lemma 3.7.** (1) Let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 2m$. Then $v R_0(0 \pm i0) v \in B(0, 0)$ are compact operators.

(2) Let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 2$. Then for $\lambda > 0$, $v R_0(\lambda \pm i0) v \in B(0, 0)$ are compact operators.

**Proof.** When $\lambda = 0$, since $v(x) (1 + |x|)^{\beta/2} \in L^\infty$ and the Hilbert-Schmidt norm of $(1 + |x|)^{-\beta/2} |x - y|^{2m-n} (1 + |y|)^{-\beta/2}$ is finite, thus $v R_0(0 \pm i0) v$ is compact in $L^2(\mathbb{R}^n)$. The second conclusion holds by the same argument as in Lemma 3.3. □
Lemma 3.8. For $H = (-\Delta)^m + V$, assume that $H$ has no positive embedded eigenvalue.

(1) Let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 2$. Then for $s, s' > 1/2$, $R_V(z) \in B(s, -s')$ is continuous for $z \in \Xi \setminus (\Sigma \cup \{0\})$. Furthermore, the boundary value

$$R_V(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} R_V(\lambda \pm i\epsilon) \in B(s, -s')$$

exists for $\lambda \in \sigma_c(H) \setminus (\Sigma \cup \{0\})$.

(2) Let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 2m$. Assume that 0 is a regular point of $H$. Then for $s, s' > m$, the function $R_V(z) \in B(s, -s')$ defined on $z \in \Xi \setminus \Sigma$ is continuous at $z = 0$.

Proof. The conclusions follow from Lemma 3.7 and the symmetric resolvent identity (2.8) provided

$$(U + vR_0(\lambda \pm i\epsilon)v)^{-1} \to (U + vR_0(\lambda \pm i0)v)^{-1} \text{ as } \epsilon \downarrow 0.$$ 

The convergence holds if and only if both limit operators $U + vR_0(\lambda \pm i0)v : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible. According to Lemma 3.7 and Fredholm’s alternative theorem, it is enough to show that $(U + vR_0(\lambda \pm i0)v)u = 0$ only admits zero solution in $L^2(\mathbb{R}^n)$. Note that $(U + vR_0(\lambda \pm i0)v)u = 0$ implies that $(H - \lambda)u = 0$. Thus $u = 0$ under the assumptions that zero is a regular point of $H$ and $H$ has no positive eigenvalue.

Theorem 3.9. For $k = 0, 1, 2, 3, \cdots$, let $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 2 + 2k$. Then for $s, s' > k + \frac{1}{2}$, $R_V^{(k)}(z) \in B(s, -s')$ is continuous for $z \in \Xi \setminus (\Sigma \cup \{0\})$. Furthermore, estimate (3.4) can be extended from $z \in \Xi \setminus [V_0, +\infty)$ to $z \in \Xi \setminus (\Sigma \cup \{0\})$, i.e. the bound

$$\left\| R_V^{(k)}(z) \right\|_{B(s, -s')} \lesssim |z|^{-\frac{(2m-1)(1+k)}{2m}}$$

holds as $z \to \infty$ in $\Xi \setminus (\Sigma \cup \{0\})$.

Proof. This is a corollary of Proposition 3.4 and Lemma 3.8.

4. Kato-Jensen type decay estimates

In this section, with the helps of lower energy asymptotic expansions, higher energy decay estimates and the limiting absorption principle, we derive the behavior of $\lambda \to 0$ and $\lambda \to \infty$ for the spectral density $dE(\lambda)$ of $H = (-\Delta)^m + V$ in $B(s, -s')$ with suitable $s, s' > 0$. Using the spectral representation theorem, we give the proof of Theorem 1.5 and the local decay estimates.

According to the asymptotic expansions of $R_V(\mu^{2m})$ in Section 2, we get the following results of spectral density $dE(\lambda)$ as $\lambda \to 0$.

Proposition 4.1. For $H = (-\Delta)^m + V$ with $|V(x)| \lesssim (1 + |x|)^{-\beta}$ for some $\beta > 0$. Let $dE(\lambda)$ be the spectral density of $H$. For $0 < \lambda \ll 1$, we obtain the following results:

I For $n > 2m$, let $\beta > n$ and $s, s' > \frac{n}{2} + 2$. If 0 is a regular point of $H$, then

$$dE(\lambda) = \lambda^{n-2m}L_1 + \lambda^{n-2m+2}L_2 + o(\lambda^{n-2m+2}),$$

where $L_1 = \text{Im}(\alpha_n)A_n$ and $L_2 = \text{Im}(\alpha_{n+1})A_{n+1}$ if $n > 4m$. If $2m < n \leq 4m$, for $j = 1, 2$:

$$L_j = \begin{cases} \text{Im}(\beta_j)B_j, & 2m < n \leq 4m \text{ and } n \text{ is odd;} \\ \text{Im}(\tau_j)C_j, & 2m < n \leq 4m \text{ and } n \text{ is even.} \end{cases}$$
(II) For \( n > 4m \), let \( \beta > n + 4 \) and \( s, s' > \frac{n}{2} + 2 \). If \( 0 \) is an eigenvalue of \( H \), then
\[
dE(\lambda) = \lambda^{n-6m} A^e_\ell + \lambda^{n-6m+2} A^e_{\ell+1} + o(\lambda^{n-6m+2}),
\]
where \( \tilde{A}_\ell^e = \text{Im}(\alpha_\ell^e A^e_\ell) \) for \( \ell = \infty \), \( \infty + 1 \).

(III) For \( n \leq 4m \) and \( n \) is odd. Let \( \beta > n+4k \) where \( 2k = 4m-n+1 \) and \( 1 \leq k \leq \left[ \frac{n}{2} \right] + 1 \). Let \( s, s' > \frac{n}{2} + 2k \), we have the following asymptotic expansions in \( B(s, -s') \):

- If \( 0 \) is the \( j \)-th kind resonance of \( H \) with \( 1 \leq j \leq k \), then
  \[
dE(\lambda) = \frac{\tilde{B}_0^j}{\lambda^{2(m-k+j)-1}} + \sum_{\ell=1}^{k-j} \frac{\tilde{B}_\ell^j}{\lambda^{2(m-k+j-\ell)-1}} + \sum_{\ell=k-j+1}^{2m-3k+3j-2} \frac{\tilde{B}_\ell^j}{\lambda^{2m-3k+3j-\ell-1}} + \tilde{B}_{2m-3k+3j-1} + o(\lambda),
  \]
  where \( \tilde{B}_\ell^j = \text{Im}(\beta_\ell^j) B_\ell^j \) for \( \ell = 0, 1, \ldots, 2m - 3k + 3j - 1 \).

- If \( 0 \) is an eigenvalue of \( H \), then
  \[
dE(\lambda) = \frac{\tilde{B}_1^{k+1}}{\lambda^{2m-1}} + \sum_{\ell=2}^{2m-1} \frac{\tilde{B}_\ell^{k+1}}{\mu^{2m-\ell}} + \tilde{B}_{2m} + o(\lambda),
  \]
  where \( \tilde{B}_\ell^{k+1} = \text{Im}(\beta_\ell^{k+1}) B_\ell^{k+1} \) for \( \ell = 0, 1, \ldots, 2m \).

(IV) For \( n \leq 4m \) and \( n \) is even. Let \( \beta > n+4k+2 \) where \( 2k = 4m-n+2 \) and \( 1 \leq k \leq \left[ \frac{n}{2} \right] + 1 \). Let \( s, s' > \frac{n}{2} + 2k + 1 \), we have the following asymptotic expansions in \( B(s, -s') \):

- If \( 0 \) is the \( j \)-th kind resonance of \( H \) with \( 1 \leq j \leq k-1 \), then
  \[
dE(\lambda) = \frac{\tilde{C}_0^j}{\lambda^{2(m-k+j)}} + \sum_{\ell=1}^{k-j} \frac{\tilde{C}_\ell^j}{\lambda^{2(m-k+j-\ell)}} + \sum_{\ell=k-j+1}^{m-k+j-1} \left( \frac{\tilde{C}_0^j}{\lambda^{2(m-k+j-\ell)}} + \frac{\ln(\lambda) \tilde{C}_1^j}{\lambda^{2(m-k+j-\ell)}} \right) + \ln(\lambda) \tilde{C}_{m-k+j,1}^j + \tilde{C}_{m-k+j,0}^j + o(\lambda^0),
  \]
  where \( \tilde{C}_{\ell,0}^j = \text{Im}(\tau_{\ell,0}^j) C_{\ell,0}^j \) for \( \ell = 0, 1, \ldots, m-k+j \) and \( \ell = 0, 1 \).

- If \( 0 \) is the \( k \)-th kind resonance of \( H \), then
  \[
dE(\lambda) = \frac{\tilde{C}_0^k}{\lambda^{2m}} + \frac{\tilde{C}_0^k}{\lambda^{2m}} + \sum_{\ell=1}^{m-1} \left( \frac{\tilde{C}_0^k}{\lambda^{2(m-\ell)}} + \frac{\tilde{C}_1^k}{\lambda^{2(m-\ell)}} + \frac{\tilde{C}_{0,2}^k}{\lambda^{2m-\ell}} \right) + \tilde{C}_{m-\ell,0}^k + (\lambda(\lambda))^{-1} \tilde{C}_{m,1}^k + (\lambda(\lambda))^{-2} \tilde{C}_{m,2}^k + o(\lambda^0),
  \]
  where \( \tilde{C}_{\ell,0}^k = \text{Im}(\tau_{\ell,0}^k) C_{\ell,0}^k \) for \( \ell = 0, 1, \ldots, m \) and \( \ell = 1, 2 \).

- If \( 0 \) is an eigenvalue of \( H \), then
  \[
dE(\lambda) = \frac{\tilde{C}_{0,1}^{k+1}}{\lambda^{2m}} + \sum_{\ell=1}^{m-1} \left( \frac{\tilde{C}_{0,1}^{k+1}}{\lambda^{2(m-\ell)}} + \frac{\tilde{C}_{0,2}^{k+1}}{\lambda^{2m}} \right) + \tilde{C}_{0,0}^{k+1} \tilde{C}_{m,1}^{k+1} + o(\lambda^0),
  \]
  where \( \tilde{C}_{\ell,0}^{k+1} = \text{Im}(\tau_{\ell,0}^{k+1}) C_{\ell,0}^{k+1} \) for \( \ell = 0, 1, \ldots, m \) and \( \ell = 0, 1 \).

Proof. Using Stone’s formula and the resolvent asymptotic expansions of \( R_V(\mu^{2m}) \) in Section 2 we can immediately obtain the above expansions. \( \square \)

For spectral density \( dE(\lambda) \) as \( \lambda \to \infty \), we have the following conclusions.
Proposition 4.2. For \( H = (-\Delta)^m + V \), let \(|V(x)| \lesssim (1 + |x|)^{-\beta} \) with some \( \beta > 2 + 2k \) and \( k \in \mathbb{N} \). Then for any \( s, s' > k + \frac{1}{2} \), the following estimate

\[
\frac{d^{k+1}}{d\lambda^{k+1}} E(\lambda) = O(\lambda^{-\frac{|2m-1|(k+1)}{2m}})
\]  

holds in \( B(s, -s') \) as \( \lambda \to \infty \).

Proof. Using Stone’s formula and Theorem 3.9, we get the above estimate. \( \square \)

As an directly application of the lower energy asymptotic expansions and higher energy decay estimates, we prove the local decay estimates and the time asymptotic expansion for the propagator \( e^{-itH} \).

Proof of Theorem 1.7. For \(|z| \to \infty\), by Proposition 4.2 we have

\[
\| \langle x \rangle^{-\sigma}(H - z)^{-1}\langle x \rangle^{-\sigma} \|_{L^2(\mathbb{R}^n)} = O(|z|^{-\frac{2m-1}{2m}}), \quad \sigma > 1/2.
\]

For \(|z| \to 0\), by the lower energy asymptotic expansion of \( R_V(z) \), we have

\[
\| \langle x \rangle^{-\sigma}(H - z)^{-1}\langle x \rangle^{-\sigma} \|_{L^2(\mathbb{R}^n)} = O(1), \quad \sigma > n/2.
\]

Then the theorem holds by the Corollary of [RS78, P.146]. \( \square \)

Theorem 4.3. For \( H = (-\Delta)^m + V(x) \) with \(|V(x)| \lesssim (1 + |x|)^{-\beta} \) for some \( \beta > 0 \). Assume that \( H \) has no positive embedded eigenvalue.

(I) For \( n > 2m \), let \( \beta > n \) and \( s, s' > \frac{n}{2} \). If \( 0 \) is a regular point of \( H \), then in \( B(s, -s') \)

\[
e^{-itH} P_{ac}(H) = |t|^{-\frac{n}{2m}} A + o(|t|^{-\frac{n}{2m}}), \quad t \to \infty
\]

where the operator \( A \in B(s, -s') \)

(II) For \( n > 4m \), let \( \beta > n + 4 \) and \( s, s' > \frac{n}{2} + 2 \). If \( 0 \) is an eigenvalue of \( H \), then in \( B(s, -s') \) we have:

\[
e^{-itH} P_{ac}(H) = |t|^{\frac{2}{2m}} B^{\sigma} + o(|t|^{\frac{2}{2m}}), \quad t \to \infty
\]

where the operator \( B^{\sigma} \in B(s, -s') \)

(III) For \( n \leq 4m \) and \( n \) is odd. Let \( \beta > n + 4k \) where \( 2k = 4m - n + 1 \) and \( 1 \leq k \leq \left[ \frac{n}{2} \right] + 1 \). Let \( s, s' > \frac{n}{2} + 2k \), then in \( B(s, -s') \) we have the following asymptotic expansions:

- If \( 0 \) is the \( j \)-th kind of resonance of \( H \) with \( 1 \leq j \leq k \), then

\[
e^{-itH} P_{ac}(H) = |t|^{\frac{2(j-k-1)}{2m}} \tilde{B}^{\tilde{j}} + o(|t|^{\frac{2(j-k-1)}{2m}}), \quad t \to \infty
\]

where the operators \( \tilde{B}^{\tilde{j}} \in B(s, -s') \)

- If \( 0 \) is an eigenvalue of \( H \), then

\[
e^{-itH} P_{ac}(H) = |t|^{-\frac{1}{2m}} \tilde{B}^{k+1} + o(|t|^{-\frac{1}{2m}}), \quad t \to \infty
\]

where the operator \( \tilde{B}^{k+1} \in B(s, -s') \)

(IV) For \( n \leq 4m \) and \( n \) is even. Let \( \beta > n + 4k + 2 \) where \( 2k = 4m - n + 2 \) and \( 1 \leq k \leq \left[ \frac{n}{2} \right] + 1 \). Let \( s, s' > \frac{n}{2} + 2k + 1 \), then in \( B(s, -s') \) we have the following asymptotic expansions:

- If \( 0 \) is the \( j \)-th kind of resonance of \( H \) with \( 1 \leq j \leq k - 1 \), then

\[
e^{-itH} P_{ac}(H) = |t|^{\frac{1}{m}} \tilde{C}^{\tilde{j}} + o(|t|^{\frac{1}{m}}), \quad t \to \infty
\]
where the operators \( \tilde{C}^j \in B(s, -s') \).

- If 0 is the \( k \)-th kind of resonance of \( H \), then
  \[
  e^{-itH} P_{ac}(H) = (\ln |t|)^{-1} \tilde{C}_1^k + (\ln |t|)^{-2} \tilde{C}_2^k + o((\ln |t|)^{-1}), \quad t \to \infty
  \]
  where the operators \( \tilde{C}_1^k, \tilde{C}_2^k \in B(s, -s') \).

- If 0 is an eigenvalue of \( H \), then
  \[
  e^{-itH} P_{ac}(H) = (\ln |t|)^{-1} \tilde{C}_1^{k+1} + o((\ln |t|)^{-1}), \quad t \to \infty
  \]
  where the operator \( \tilde{C}_1^{k+1} \in B(s, -s') \).

Proof. Let \( \chi(\lambda) \) be a smooth cutoff function with support in \( 0 < \lambda < 1 \). Then

\[
\begin{align*}
e^{-itH} P_{ac}(H) &= 2m \int_{0}^{\infty} e^{-it\lambda^{2m}}\lambda^{2m-1} E(\lambda) d\lambda \\
&= 2m \int_{0}^{\infty} e^{-it\lambda^{2m}}\lambda^{2m-1} \chi(\lambda) E(\lambda) d\lambda + 2m \int_{0}^{\infty} e^{-it\lambda^{2m}}(1 - \chi(\lambda))\lambda^{2m-1} E(\lambda) d\lambda \\
&=: I + II.
\end{align*}
\]

For term II, note that the integral is supported in \([1, \infty)\). By (1.1) we know, it is actually the Fourier transform of the \( L^1(\mathbb{R}) \) integrable function \((1 - \chi(\lambda))\lambda^{2m-1} E(\lambda)\). Thus, by the Riemann-Lebesgue’s lemma, we know the contribution of term II is \(|t|^{-k} \) for any large \( k > 0 \).

For term I, we have: for \( f(x) \in C_{0}^{\infty}(\mathbb{R}) \),

\[
\int_{0}^{\infty} e^{-i\lambda x} f(x) x^{\tau} dx \sim \sum_{j=0}^{\theta_j \lambda^{-j-1-\tau}, \ Re(\tau) > -1, (4.2)} \frac{\theta_j}{(j+1)^{\tau}} f^{(j)}(0). \]

where \( \theta_j = i^{j+\tau+1} \frac{1}{(j+1)^{\tau}} f^{(j)}(0) \). See Stein’s book [Ste93, P. 355].

On the other hand, for \( n = 4m + 2 - 2k \) with \( 1 \leq k \leq \left[ \frac{m}{2} \right] + 1 \), we need to treat such term

\[
\int_{0}^{\infty} \frac{e^{-it\lambda}}{\lambda((\ln \lambda - a)^2 + \pi^2)} d\lambda, \quad a \neq 0.
\]

However, the integral is \( O(1/\ln t) \) as \( t \to \infty \). See [Jen84].

From the above expansions of \( e^{-itH} P_{ac}(H) \) in \( B(s, -s') \), then Theorem 1.5 is a corollary of Theorem 4.3.

5. \( L^p \)-Decay Estimate

In this section, we use the Kato-Jensen type decay estimates to derive the \( L^p \)-decay estimate for \( e^{-itH} \). For the free propagator \( e^{-it(-\Delta)^m} \), by the stationary phase methods, we have (see e.g Kim, Arnold and Yao [KAY12]):

Lemma 5.1. For any \( m \geq 1 \), the kernel \( K_t(x) \) of \( e^{-it(-\Delta)^m} \) is smooth and satisfies the following pointwise estimate

\[
|K_t(x)| \lesssim |t|^{-\frac{n+|\alpha|}{2m}} (1 + |t|^{-\frac{3}{2m}}|x|)^{-\frac{(m-1)|\alpha| + |\alpha|}{2m-1}}, \quad t \neq 0.
\]
which implies that for $0 \leq |\alpha| \leq (m - 1)n$,
\[ \|D^\alpha e^{-it(-\Delta)^{m/2}}u\|_{L^\infty(R^n)} \lesssim |t|^{-\frac{n+|\alpha|}{2m}}\|u\|_{L^1(R^n)}, \quad t \neq 0. \]

In particular,
\[ \|e^{-it(-\Delta)^{m/2}}u\|_{L^\infty(R^n)} \lesssim |t|^{-\frac{n}{2m}}\|u\|_{L^1(R^n)}, \quad t \neq 0. \]

In the case that $V(x) \neq 0$, we can derive the weak estimates: $L^1 \cap L^2 \to L^\infty + L^2$-decay estimates.

**Definition 5.2.** For any measurable function $f$, if $f = f_1 + f_2$ with $f_1 \in L^2(R^n)$, $f_2 \in L^\infty(R^n)$ and satisfies
\[ \inf \{ \|f_1\|_{L^2(R^n)} + \|f_2\|_{L^\infty(R^n)} \} < \infty, \]
where the infimum takes over all the splitting of $f$. Then we denote $f \in L^2 + L^\infty(R^n)$ and $L^2 + L^\infty(R^n)$ is a Banach space with the norm
\[ \|f\|_{L^2 + L^\infty(R^n)} = \inf \{ \|f_1\|_{L^2(R^n)} + \|f_2\|_{L^\infty(R^n)} \}. \]

Note that for $f \in L^2 + L^\infty(R^n)$, since $f$ can be divided as $f = f + 0 = 0 + f$, then $\|f\|_{L^2 + L^\infty(R^n)} \leq \|f\|_{L^2(R^n)}$ and $\|f\|_{L^2 + L^\infty(R^n)} \leq \|f\|_{L^\infty(R^n)}$.

In the sequel, we need the boundness of $P_{ac}(H)$ in the weighted $L^2$ spaces. Denotes $P_{\text{disc}}(H)$ be the projection onto the subspace of discrete spectrum of $H$. Denotes $\sharp$ be the number of discrete spectrum point. Thus $P_{ac}(H) = I - P_{\text{disc}}(H)$ and $P_{\text{disc}}(H) = \sum_j \langle \cdot, e_j \rangle e_j$ where $e_j$ is the eigenvector. Denotes $N(\gamma); H$ be the number (count the multiplicity) of eigenvalues of $H$ which is less than or equal to $\gamma$. By using the Birman-Solomyak bound for operator $A_{\ell}(\alpha V) = (-\Delta)^{\ell} - \alpha V$ where $\ell, \alpha > 0$ in [BS91], then for $n > 2m$ we have
\[ N(0; (-\Delta)^m + V) \leq C(n) \int_{R^n} V_{n/2m}^*(x)dx \tag{5.1} \]
where $V_{\gamma}(x) = -\min\{0, V(x)\}$.

**Lemma 5.3.** For $H = (-\Delta)^m + V, V(x) \in L^\infty(R^n) \cap L^{n/2m}(R^n)$. Assume that $0$ is a regular point of $H$ and there are only finite embedded eigenvalues. Then for $\sigma \in R$ and any $1 \leq p \leq \infty$, we have
\[ \|(1 + |x|)^{-\sigma}P_{ac}(H)(1 + |x|)^{\sigma}f\|_{L^p(R^n)} \lesssim \|f\|_{L^p(R^n)}. \tag{5.2} \]

**Proof.** By the Birman-Solomyak bound (5.1) of $H$, we know the number $\sharp$ is finite. It is enough to show that $P_{\text{disc}}(H)$ satisfies the estimate (5.2). Since
\[ \|(1 + |\cdot|)^{-\sigma}P_{ac}(H)(1 + |\cdot|)^{\sigma}f\|_{L^p(R^n)} \]
\[ = \left\| \sum_{j=0}^{\sharp} \langle (1 + |x|)^{\sigma}f, e_j \rangle (1 + |y|)^{-\sigma}e_j \right\|_{L^p(R^n)} \]
\[ \lesssim \sum_{j=0}^{\sharp} \left( \|f\|_{L^p(R^n)} \| (1 + |x|)^{\sigma}e_j \|_{L^{p'}(R^n)} \| (1 + |y|)^{-\sigma}e_j \|_{L^p(R^n)} \right) \]
\[ \lesssim \sum_{j=0}^{\sharp} \|f\|_{L^p(R^n)} \| (1 + |x|)^{\sigma}e_j \|_{L^{p'}(R^n)} \| (1 + |y|)^{-\sigma}e_j \|_{L^p(R^n)}. \]
Furthermore, by the Theorem 14.5.2 in [Hö05], we know that eigenfunction $e_j$ satisfies

$$(1 + |x|)^N e_j(x) \in L^2(\mathbb{R}^n)$$

for all $N \in \mathbb{R}$.

Hence, by the Hölder’s inequality we know the sum in the last step is finite. □

Now, we start the proof of the $L^1 \cap L^2 - L^\infty + L^2$ decay estimates from the following lemma.

**Lemma 5.4.** For any $a > 0$ and $b > 0$, we have

$$\int_0^t \frac{ds}{(1 + |t-s|)^a(1 + |s|)^b} \lesssim \begin{cases} (1 + |t|)^{-a-b+1}, & 0 < a, b < 1, \\ (1 + |t|)^{-\min\{a, b\}}, & \text{otherwise}. \end{cases}$$

and

$$\int_0^t \frac{ds}{(1 + |\ln(t-s)|)(1 + |s|)^a} \lesssim \begin{cases} (1 + |t|)^{-a+1}(1 + |\ln t|)^{-1}, & 0 < a < 1, \\ (1 + |\ln t|)^{-1}, & a \geq 1. \end{cases}$$

Proof. The proof follows from the proof of Lemma 4.4 in [FSY18]. □

**Proof of Theorem 1.10.** Our strategy is applying the iterated Duhamel formula

$$e^{-itH}P_{ac}(H) = e^{-itH_0}P_{ac}(H) + i \int_0^t e^{-i(t-s)H_0}VP_{ac}(H)e^{i\tau H_0}ds$$

$$- \int_0^t \int_0^s e^{-i(t-s)H_0}Ve^{-i(s-\tau)H}P_{ac}(H)V e^{-i\tau H_0}d\tau ds$$

$$:= I + II + III$$

and then estimate each term of (5.5). By the definition of $L^2 + L^\infty(\mathbb{R}^n)$, we have

$$\|e^{-itH_0}P_{ac}(H)u\|_{L^2 + L^\infty(\mathbb{R}^n)} \leq \min \left\{ \|e^{-itH_0}P_{ac}(H)u\|_{L^2(\mathbb{R}^n)}, \|e^{-itH_0}P_{ac}(H)u\|_{L^\infty(\mathbb{R}^n)} \right\}.$$

Then for $0 < |t| \leq 1$, we have

$$\|e^{-itH_0}P_{ac}(H)u\|_{L^2 + L^\infty(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}.$$

And for $|t| > 1$, by Lemma 5.1, we have

$$\|e^{-itH_0}P_{ac}(H)u\|_{L^2 + L^\infty(\mathbb{R}^n)} \leq |t|^{-\frac{n}{2m}} \|u\|_{L^1(\mathbb{R}^n)}.$$

Thus for the first term $I$ we have

$$\|e^{-itH_0}P_{ac}(H)u\|_{L^2 + L^\infty(\mathbb{R}^n)} \lesssim (1 + |t|)^{-\frac{n}{2m}} \|u\|_{L^1 \cap L^2(\mathbb{R}^n)}.$$
For the second term $II$ of (5.5), we have

\[
\int_0^t \left< e^{-i(t-s)H_0} V P_{ac}(H) e^{iH_0 u} \right>_{L^2 + L^\infty} ds
\]

\[
\lesssim \int_0^t (1 + |t-s|) \frac{n}{2m} \left< V P_{ac}(H) e^{-iH_0 u} \right>_{L^1 \cap L^2} ds
\]

\[
\lesssim \int_0^t (1 + |t-s|) \frac{n}{2m} \left< \left( e - e^{-iH_0 u} \right) P_{ac}(H) e^{-iH_0 u} \right>_{L^2} ds
\]

\[
\lesssim \int_0^t (1 + |t-s|) \frac{n}{2m} \left< \left( e^{-iH_0} - 1 \right) P_{ac}(H) e^{-iH_0 u} \right>_{L^2} ds
\]

\[
\lesssim (1 + |t|) \left< e^{-iH_0} - 1 \right>_{L^1 \cap L^2}.
\]

For the third term $III$ of (5.5), we have

\[
\int_0^t \int_0^s \left< e^{-i(t-s)H_0} V e^{i(s-\tau)H} P_{ac}(H) e^{-i\tau H_0 u} \right>_{L^2 + L^\infty} d\tau ds
\]

\[
\lesssim \int_0^t \int_0^s (1 + |t-s|) \frac{n}{2m} \left< V e^{i(s-\tau)H} P_{ac}(H) e^{-i\tau H_0 u} \right>_{L^1 \cap L^2} d\tau ds
\]

\[
\lesssim \int_0^t \int_0^s (1 + |t-s|) \frac{n}{2m} \left< \left( e^{i(s-\tau)H} - 1 \right) P_{ac}(H) e^{-i\tau H_0 u} \right>_{L^2} d\tau ds
\]

\[
\lesssim \int_0^t \int_0^s (1 + |t-s|) \frac{n}{2m} \left< \left( e^{i\tau H_0} - 1 \right) P_{ac}(H) e^{-i\tau H_0 u} \right>_{L^2} d\tau ds
\]

\[
\lesssim (1 + |t|) \left< e^{i\tau H_0} - 1 \right>_{L^1 \cap L^2}.
\]

Thus we can combine the steps above to conclude the proof for the regular case of Theorem 1.10. The resonance cases hold by the same processes as to the regular case. We only need to plug into the respectively time decay rate of Kato-Jensen decay estimates instead of the regular case. \hfill \Box
6. Endpoint Strichartz estimates

In this section, we consider the following nonlinear higher-order Schrödinger equation

\[
\begin{aligned}
i \partial_t \psi &= \left( (-\Delta)^m + V \right) \psi + h(t, x), \\
\psi(0, x) &= \psi_0(x) \in L^2(\mathbb{R}^n),
\end{aligned}
\]  

(6.1)

where \( h(t, x) \) is the source term. We aim to establish the endpoint Strichartz estimates for the solution of problem (6.1). With the lack of the \( L^1 - L^\infty \) dispersive estimate of \( e^{-it((-\Delta)^m + V)} \), we will apply the Kato-Jensen type decay estimates (see Corollary 1.5) and the local decay estimates of \( e^{-it((-\Delta)^m + V)} \) to obtain the endpoint Strichartz estimates of problem (6.1).

**Definition 6.1.** For \( n > 2m, \) if \( q, r \in \mathbb{R} \) and \( 2 \leq q \leq \infty \) satisfies

\[
\frac{2m}{q} + \frac{n}{r} = \frac{n}{2},
\]

then we say \( (q, r) \) is an \( m \)-admissible pair. Note that if \( q = 2 \), then \( r = \frac{2n}{n-2m} \).

**Theorem 6.2.** Consider the nonlinear problem (6.1). Let \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) with some \( \beta > n + 4 \). Assume zero is a regular point of \( H = (-\Delta)^m + V \) and there does not exist positive embedded eigenvalue of \( H \). Then for any \( m \)-admissible pairs \( (q, r) \) and \( (\tilde{q}, \tilde{r}) \), the following estimates hold:

1. **Homogeneous Strichartz estimate**

\[
\| e^{-itH} P_{ac}(H) \psi_0 \|_{L^q L^r_t(\mathbb{R} \times \mathbb{R}^n)} \leq C(n) \| \psi_0 \|_{L^2(\mathbb{R}^n)}.
\]

(6.3)

2. **Dual homogeneous Strichartz estimate**

\[
\left\| \int_{\mathbb{R}} e^{-isH} P_{ac}(H) h(s, \cdot) ds \right\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n)} \leq C(n) \| h \|_{L^{\tilde{q}}_t L^{\tilde{r}}_x(\mathbb{R} \times \mathbb{R}^n)}.
\]

(6.4)

3. **The solution** \( \psi(x, t) \) of the problem (6.1) satisfies

\[
\| P_{ac}(H) \psi(x, t) \|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n)} \leq \| \psi_0 \|_{L^2(\mathbb{R}^n)} + \| h \|_{L^{\tilde{q}}_t L^{\tilde{r}}_x(\mathbb{R} \times \mathbb{R}^n)}.
\]

(6.5)

By Keel and Tao’s TT*-methods (see [KT98]) and \( L^1-L^\infty \) estimates of the free propagator \( e^{-it(-\Delta)^m} \), we have the following Strichartz estimates.

**Lemma 6.3.** For the free propagator \( e^{-it(-\Delta)^m} \), we have

\[
\| e^{-it(-\Delta)^m} \psi \|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| \psi \|_{L^2(\mathbb{R}^n)},
\]

(6.6)

\[
\left\| \int_{t-s}^t e^{-i(t-s)(-\Delta)^m} f(s) ds \right\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| f \|_{L^{\tilde{q}}_t L^{\tilde{r}}_x(\mathbb{R} \times \mathbb{R}^n)},
\]

(6.7)

where \( (q, r) \) and \( (\tilde{q}, \tilde{r}) \) are \( m \)-admissible pairs.

Next, we give the proof of Theorem 6.2.

**Proof of Theorem 6.2.** We divide the proof into the following three steps.

Step 1) We aim to show the homogeneous Strichartz estimate (6.3) and the dual homogeneous Strichartz estimate (6.4).
Consider the following problem
\[
\begin{aligned}
    i\partial_t \psi &= H_0 \psi + V \psi, \\
    \psi(0, \cdot) &= \psi_0 \in L^2(\mathbb{R}^n).
\end{aligned}
\] (6.8)

For the homogeneous Strichartz estimate (6.3), using Duhamel formula it is enough to show
\[
\left\| \int_0^t e^{-i(t-s)H_0} V e^{-iH} P_{ac}(H) \psi ds \right\|_{L^2_t L^q_x(\mathbb{R}^n)} \lesssim \| \psi \|_{L^2(\mathbb{R}^n)}.
\]

In fact, by (6.7) and the local decay estimates (1.6), we have
\[
\left\| \int_0^t e^{-i(t-s)H_0} V e^{-iH} P_{ac}(H) \psi ds \right\|_{L^2_t L^q_x(\mathbb{R}^n)} \lesssim \left\| V e^{-itH} P_{ac}(H) \psi \right\|_{L^2_t L^q_x(\mathbb{R}^n)} \lesssim \left\| \psi \right\|_{L^2(\mathbb{R}^n)}.
\]

Further, the dual homogeneous Strichartz estimate (6.4) follows by the $TT^*$-method.

Step 2) We aim to show the retarded Strichartz estimate (6.5). The solution $\Psi(t, x)$ of equation (6.11) satisfies
\[
P_{ac}(H) \Psi(t, x) = e^{-itH_0} P_{ac}(H) \Psi_0 - i \int_0^t e^{-i(t-s)H_0} V P_{ac}(H) \Psi(s) ds - i \int_0^t e^{-i(t-s)H_0} P_{ac}(H) h(s) ds.
\]

Then by Hölder's inequality and Step 1, we have
\[
\begin{aligned}
\left\| P_{ac}(H) \Psi(t, x) \right\|_{L^q_t L^r_x(\mathbb{R}^n)} &\lesssim \left\| e^{-itH_0} P_{ac}(H) \Psi_0 \right\|_{L^q_t L^r_x(\mathbb{R}^n)} + \left\| \int_0^t e^{-i(t-s)H_0} P_{ac}(H) h(s, \cdot) ds \right\|_{L^q_t L^r_x(\mathbb{R}^n)} \\
&+ \left\| \int_0^t e^{-i(t-s)H_0} V P_{ac}(H) \Psi(s) ds \right\|_{L^q_t L^r_x(\mathbb{R}^n)} \\
&\lesssim \left\| \Psi_0 \right\|_{L^2} + \| h(t) \|_{L^q_t L^r_x(\mathbb{R}^n)} + \left\| V P_{ac}(H) \Psi(t) \right\|_{L^2_t L^r_x(\mathbb{R}^n)} \\
&\lesssim \left\| \Psi_0 \right\|_{L^2} + \| h(t) \|_{L^q_t L^r_x(\mathbb{R}^n)} + \left\| \langle x \rangle^{-\sigma} \right\|_{L^q_t L^r_x(\mathbb{R}^n)} \left\| P_{ac}(H) \Psi(t) \right\|_{L^2_t L^r_x(\mathbb{R}^n)}.
\end{aligned}
\] (6.9)

Now we aim to show that
\[
\left\| \langle x \rangle^{-\sigma} P_{ac}(H) \Psi(t) \right\|_{L^q_t L^r_x(\mathbb{R}^n)} \lesssim \left\| \Psi_0 \right\|_{L^2(\mathbb{R}^n)} + \| h \|_{L^q_t L^r_x(\mathbb{R}^n)}.
\] (6.10)

First, by Duhamel’s formula for $\Psi$, we have
\[
\begin{aligned}
\left\| \langle x \rangle^{-\sigma} P_{ac}(H) \Psi(t) \right\|_{L^q_t L^r_x(\mathbb{R}^n)} &\lesssim \left\| \langle x \rangle^{-\sigma} e^{-itH} P_{ac}(H) \Psi_0 \right\|_{L^q_t L^r_x(\mathbb{R}^n)} + \left\| \int_0^t \langle x \rangle^{-\sigma} e^{-i(t-s)H} P_{ac}(H) h(s) ds \right\|_{L^q_t L^r_x(\mathbb{R}^n)}
\end{aligned}
\]

Then, we will finish the proof which only needs to show
\[
\begin{aligned}
\left\| \int_0^t \langle x \rangle^{-\sigma} e^{-i(t-s)H} P_{ac}(H) h(s) ds \right\|_{L^q_t L^r_x(\mathbb{R}^n)} &\lesssim \left\| \Psi_0 \right\|_{L^2(\mathbb{R}^n)} + \| h \|_{L^q_t L^r_x(\mathbb{R}^n)}.
\end{aligned}
\] (6.10)

Step 3) We show the local decay estimate of the source term (6.10).

Consider the Cauchy problem
\[
\begin{aligned}
    i\partial_t \phi &= H_0 \phi + h(t) = H\phi - V\phi + h(t), \\
    \phi(0, \cdot) &= \Psi_0.
\end{aligned}
\] (6.11)
Then Duhamel formula for the solution $\phi(t, x)$ reads

$$P_{ac}(H)\phi(t) = e^{-itH}P_{ac}(H)\Psi_0 + i \int_0^t e^{-i(t-s)H}P_{ac}(H)V\phi(s)ds - i \int_0^t e^{-i(t-s)H}P_{ac}(H)h(s)ds.$$  

(6.12)

For the left hand side of (6.12), since $\phi(t)$ is also a solution of $i\partial_t\phi = H_0\phi + h(t)$, by (6.6), (6.7) and Duhamel formula again, we have

$$\|\langle x \rangle^{-\sigma} P_{ac}(H)\phi(t)\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|\langle x \rangle^{-\sigma} P_{ac}(H)\langle x \rangle^\sigma \langle x \rangle^{-\sigma} e^{-itH_0}\Psi_0\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^n)}$$

$$+ \|\langle x \rangle^{-\sigma} P_{ac}(H)\langle x \rangle^\sigma \langle x \rangle^{-\sigma} \int_0^t e^{-i(t-s)H_0}h(s)ds\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|\Psi_0\|_{L^2(\mathbb{R}^n)} + \|\int_0^t e^{-i(t-s)H_0}h(s)ds\|_{L_t^2 L_x^{2\eta}(\mathbb{R} \times \mathbb{R}^n)}$$

$$\lesssim \|\Psi_0\|_{L^2(\mathbb{R}^n)} + \|h\|_{L_t^{2\eta} L_x^{2\eta}(\mathbb{R} \times \mathbb{R}^n)}.$$  

Here, we apply Lemma 5.3 the boundness of $P_{ac}(H)$. The local decay estimate for the first term on the right hand side of (6.12) follows from (1.6).

For the second term of the right hand side of (6.12), notice that

$$\|\int_0^t \langle x \rangle^{-\sigma} e^{-i(t-s)H}P_{ac}(H)V\phi(s)ds\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|\int_0^t \|\langle x \rangle^{-\sigma} e^{-i(t-s)H}P_{ac}(H)V\phi(s)\|_{L_x^2(\mathbb{R}^n)}ds\|_{L_t^2(\mathbb{R})}$$

$$\lesssim \|\int_0^t (1 + |t-s|)^{-\frac{n}{4}}\|\langle x \rangle^\sigma V\phi(s)\|_{L_x^2(\mathbb{R}^n)}ds\|_{L_t^2(\mathbb{R})} \lesssim \|\Psi_0\|_{L^2(\mathbb{R}^n)} + \|h\|_{L_t^{2\eta} L_x^{2\eta}(\mathbb{R} \times \mathbb{R}^n)}.$$  

\[\square\]

### 7. Absence of embedded eigenvalues

#### 7.1. Examples of existence of positive embedded eigenvalue.

In 1929, von Neumann and Wigner [Wig93] found an example of Schrödinger operator acting in $L^2(\mathbb{R}^3)$ with a spherically symmetric potential which vanishes like $O(|x|^{-1})$ at infinity such that it possesses a positive eigenvalue embedded in the continuous spectrum. On the other hand, Kato in the famous work [Kat59] also showed that $-\Delta + V$ has no positive eigenvalues if the potential $V(x)$ decays fast enough at infinity (e.g. $o(|x|^{-1})$). So the Wigner-von Neumann counterexample shows that Kato’s result is sharp in essence.

For higher order Schrödinger type operator $H = (-\Delta)^m + V$ with $m \geq 2$, we will give some higher-order examples to show that there still exists some positive eigenvalue embedded in the continuous spectrum, even for $C_0^\infty$-potential. For even $m$, if $\phi \in L^2(\mathbb{R}^n)$ is the eigenfunction of $H = (-\Delta)^m + V$ with eigenvalue +1, i.e. $(-\Delta)^m + V \phi = \phi$. If $\phi$ is strictly positive (or negative), then

$$V(x) = \phi^{-1}(x)(\phi(x) - (-\Delta)^m \phi(x)).$$  

(7.1)
Our strategy is to find a \( \phi(x) \) such that \( \phi(x) = (-\Delta)^m \phi(x) \) for \( |x| > r_0 > 0 \), then by \( (7.1) \) the potential \( V \) has compact support in \( B(0, r_0) \).

**Lemma 7.1.** Let \( \Phi(x) = \mathcal{F}^{-1}((1 + | \cdot |^2)^{-1})(x) \), then \( \Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \). Moreover, \( \Phi(x) \) is positive and \( \Phi(x) = O(|x|^{-N}) \) for any \( N > 0 \) as \( |x| \to \infty \).

Indeed, \( \Phi(x) \) is the kernel of Bessel potential \( (1 - \Delta)^{-1} \). Thus, we have

\[
\Phi(x) = \frac{1}{4\pi} \int_0^\infty e^{-|x|^2/4t} e^{-t/4\pi} t^{-n/2} dt, \quad x \in \mathbb{R}^n. \tag{7.2}
\]

In addition, if \( n \geq 3 \), then

\[
\Phi(x) = \frac{e^{-|x|}}{2(2\pi)^{(n-1)/2}\Gamma((n-1)/2)} \int_0^\infty e^{-|t| t^{(n-3)/2}} dt, \quad x \in \mathbb{R}^n. \tag{7.3}
\]

See Section V.3 in Stein’s book [Ste70] or [AS61]. Note that \( \Phi(x) = \Phi(r) \) where \( r = |x| \). For \( r > r_0 > 0 \), by the above lemma, then \((-\Delta)^m \Phi = \Phi\) if \( m \) is an even integer. For any \( \epsilon > 0 \), define a radial function \( u(r) \):

\[
u(r) = \begin{cases} 
\Phi(r), & r > r_0 + \epsilon; \\
u_0(r), & 0 \leq r \leq r_0 - \epsilon;
\end{cases} \tag{7.4}
\]

such that \( 0 < u_0(r) \in C^\infty(\mathbb{R}^+). \) Let the eigenfunction \( \phi(x) = u(|x|) \), and then \( \phi \in L^2(\mathbb{R}^3) \). By \((7.1)\), we have \( V(x) \in C^\infty_0(\mathbb{R}^n) \) and \((-\Delta + V)\phi = \phi\) for each even integer \( m \geq 2 \).

### 7.2. Proof of Theorem 1.11

In this part, we give the proof of Theorem 1.11. The proof relies on the virial identity for homogeneous operator \( h(D) + V \), where \( h \) is a homogeneous function including \((-\Delta)^s\) for \( s \in \mathbb{R}^+ \). The virial identity is useful in the area of spectral analysis.

**Lemma 7.2.** For operators \( H = h(D) + V(x) \), \( h \) is a homogeneous function with degree \( \varrho \). \( V(x) \) is a real-valued function on \( \mathbb{R}^n \). Suppose that \( V \) satisfies the following:

(i) \( V \) is \( h(D) \)-bounded with relative bound less than one;

(ii) There exists a multiplication operator \( \mathcal{V} \) on \( L^2(\mathbb{R}^n) \) with \( \mathcal{D}(V) \supset \mathcal{D}(h(D)) \), such that for all \( \phi \in \mathcal{D}(h(D)) \),

\[
s - \lim_{\theta \to 1}(\theta - 1)^{-1}(V_\theta - V)\phi(x) = \mathcal{V}\phi(x), \tag{7.5}
\]

where \( \mathcal{D}(h(D)) \) and \( \mathcal{D}(V) \) are the self-adjoint domain of \( h(D) \) and \( V \) respectively.

For any eigenfunction \( \psi \) of \( H \) with related eigenvalue \( \lambda \), that is

\[
h(D)\psi + V\psi = \lambda\psi, \quad \psi \in \mathcal{D}(h(D)).
\]

Then we have

\[
g(\psi, h(D)\psi) = (\psi, \mathcal{V}\psi) = g(\psi, (\lambda - V)\psi). \tag{7.6}
\]

**Remark 7.3.** Notice that \( \mathcal{V} \) is just \( x \cdot \nabla V \) formally. In fact, if \( (7.5) \) holds, \( \mathcal{V} \) is given by \( x \cdot \nabla V \) where the derivatives are interpreted in the sense of distributions. On the other hand, if there exists a function \( \mathcal{V}(x) \) such that

\[
\lim_{\theta \to 1}(\theta - 1)^{-1}(V_\theta(x) - V(x)) = \mathcal{V}(x) \text{ a.e. } x \in \mathbb{R}^n,
\]

then \( (7.5) \) holds.
**Proof.** For \( \theta > 0 \), \( U(\theta) \) be the unitary family defined as follows:

\[
(U(\theta)\psi)(x) = \theta^{n/2}\psi(\theta x), \quad \psi \in L^2(\mathbb{R}^n).
\]

Denote \( V_\theta(x) = V(\theta x) \), then for \( V(x) \) as a multiplication operator \( V \), we have

\[
V_\theta = U(\theta)VU^{-1}(\theta).
\]

(7.7)

Notice that for \( h(D) \), we have

\[
U(\theta)h(D)U^{-1}(\theta) = \theta^{-\theta}h(D) = h(U(\theta)DU^{-1}(\theta)).
\]

(7.8)

Indeed, for \(-i\frac{\partial}{\partial x_j} \), \( j = 1, 2, \cdots, d \), we have

\[
U(\theta)(-i\frac{\partial}{\partial x_j})U^{-1}(\theta)f(x) = \theta^{-1}(-i\frac{\partial}{\partial x_j})f(x).
\]

Thus (7.8) holds by the homogeneous of \( h \).

Since \( \psi \) is an eigenfunction of \( H = h(D) + V \) with related eigenvalue \( \lambda \), by the identities (7.7) and (7.8), we have

\[
\lambda \psi_\theta = U(\theta)\lambda \psi U^{-1}(\theta) = U(\theta)(h(D)\psi + V\psi)U^{-1}(\theta) = (\theta^{-\theta}h(D) + V_\theta)\psi_\theta,
\]

where \( \psi_\theta(x) = \psi(\theta x) \). Thus we get

\[
(\theta(D) + \theta^\theta V_\theta)\psi_\theta = \theta^\theta \lambda \psi_\theta.
\]

(7.9)

By the above equality (7.9) and \( H\psi = \lambda \psi \), we have

\[
\lambda(\theta^\theta - 1)(\psi_\theta, \psi) = \left( (\theta(D) + \theta^\theta V_\theta)\psi_\theta, \psi \right) - (\psi_\theta, (h(D) + V)\psi) = \theta^\theta(\theta^\theta \psi_\theta, \psi) - (\psi_\theta, V\psi) = (\theta^\theta - 1)(V_\theta\psi_\theta, \psi) + (\psi_\theta, (V_\theta - V)\psi).
\]

since \( V(x) \) is real valued. Thus

\[
\frac{\theta^\theta - 1}{\theta - 1}(V_\theta\psi_\theta, \psi) + \frac{1}{\theta - 1}(\psi_\theta, (V_\theta - V)\psi) = \lambda \frac{\theta^\theta - 1}{\theta - 1}(\psi_\theta, \psi).
\]

Taking the limit as \( \theta \to 1 \) yields (7.6).

**Proof of Theorem 1.11.** Since \( V \) satisfies the hypotheses of Proposition 7.2, then for any eigenfunction \( \psi \) of \( H_h = h(D) + V \), we have the following virial identity:

\[
(\psi, h(D)\psi) = \frac{1}{\theta}(\psi, V\psi).
\]

(7.10)

Note that the positivity of \( h(D) \) implies that \( \text{Ker}(h(D)) = \{0\} \).

In case (1), \( V(\gamma x) < V(x) \) with \( \gamma > 1 \) implies that \( \mathcal{V}(x) \leq 0 \). Hence the virial identity (7.10) holds only if \( \psi = 0 \).

In case (2), \( \mathcal{V} = -\nu V \), so by (7.6), we know

\[
\lambda(\psi, \psi) = \frac{1}{\theta}(\psi, V\psi) = -\nu^{-1}(\tau - \nu)(\psi, V\psi)
\]

\[
= -\nu^{-1}(\tau - \nu)(\psi, h(D)\psi).
\]

By the positivity of \( h(D) \) and \( \rho - \nu > 0 \), we conclude that \( \lambda < 0 \) if \( \psi \neq 0 \).
In case (3), by (7.10) we have
\[-a\lambda(\psi, \psi) = -a(\psi, (h(D) + V)\psi) + (a + 1)(\psi, (h(D) - \varrho^{-1}V)\psi),\]
\[= (\psi, (h(D) - \tau^{-1}(1 + a)V - aV)\psi).\]
Thus (1.8) implies that \(\lambda \leq 0\) if \(\psi \neq 0\). \(\square\)

8. Proof of asymptotic expansions and identification of resonance spaces

By the symmetric resolvent identity (2.8), we need to derive the asymptotic expansions of \(M^{-1}(\mu)\) in \(L^2(\mathbb{R}^n)\) for \(\mu\) near zero. In this section, we show the processes of deriving the expansion of \(M^{-1}(\mu)\) and identify the resonance spaces case by case.

Here we give the needed lemmas in the following.

Lemma 8.1. Let \(A\) be a closed operator and \(S\) a projection. Suppose \(A + S\) has a bounded inverse, then \(A\) has a bounded inverse if and only if
\[B \equiv S - S(A + S)^{-1}S\]
has a bounded inverse in \(S\mathcal{H}\). Furthermore,
\[A^{-1} = (A + S)^{-1} + (A + S)^{-1}SB^{-1}S(A + S)^{-1}.\]

Lemma 8.2. ([JN04, Proposition 1]) Let \(F \subset \mathbb{C}\) have zero as an accumulation point. Let \(T(z), z \in F\) be a family of bounded operators of the form
\[T(z) = T_0 + zT_1(z)\]
with \(T_1(z)\) uniformly bounded as \(z \to 0\). Suppose 0 is an isolated point of the spectrum of \(T_0\), and let \(S\) be the corresponding Riesz projection. If \(T_0S = 0\), then for sufficient small \(z \in F\) the operator \(\tilde{T}(z) : S\mathcal{H} \to S\mathcal{H}\) defined by
\[\tilde{T}(z) = \frac{1}{z}(S - S(T(z) + S)^{-1}S) = \sum_{j=0}^{\infty} (-1)^j z^j S[T_1(z)(T_0 + S)^{-1}]^{j+1}S\]
is uniformly bounded as \(z \to 0\). The operator \(T(z)\) has a bounded inverse in \(\mathcal{H}\) if and only if \(\tilde{T}(z)\) has a bounded inverse in \(S\mathcal{H}\), and in this case
\[T(z)^{-1} = (T(z) + S)^{-1} + \frac{1}{z}(T(z) + S)^{-1}S\tilde{T}(z)^{-1}S(T(z) + S)^{-1}.\]

Lemma 8.3. ([Jen80, Lemma 2.3]) Denote the Riesz potential \(I_\alpha = (-\Delta)^{-\alpha/2}\) on \(\mathbb{R}^n\) with \(0 < \alpha < n\).

(1) If \(0 < \alpha < n/2\), \(s, s' \geq 0\) and \(s + s' \geq \alpha\), then \(I_\alpha \in B(s, -s')\).
(2) If \(n/2 \leq \alpha < n\), \(s, s' > \alpha - n/2\) and \(s + s' \geq \alpha\), then \(I_\alpha \in B(s, -s')\).

In the following, we denote \(D_j = (T_j + S_{j+1})^{-1}\) where \(T_j\) and \(S_j\) see Definition 2.6.
8.1. **Proof of resolvent asymptotic expansions.** In this subsection, we give the proof of the resolvent asymptotic expansions at zero-resonance case by case.

**Proof of Theorem 2.5** $(n > 4m)$. In this case, for $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4$, we have the following expansions for $M(\mu)$ in $B(0, 0)$:

$$M(\mu) = U + vG_0v + \sum_{j=1}^{n-1} \mu^{2mj}vG_jv + \mu^{n-2m}vG_Rv + \mu^{n-2m+2}vG_{R+1}v + vE_0(\mu)v. \quad (8.5)$$

Denote

$$M(\mu) = \sum_{j=1}^{n-1} \mu^{2mj}vG_jv + \mu^{n-2m}vG_Rv + \mu^{n-2m+2}vG_{R+1}v + vE_0(\mu)v.$$

Recall that $\aleph = \frac{n}{2m}$. Theorem 2.5 holds by symmetric resolvent identity (2.8) and the following Proposition.

**Proposition 8.4.** Assume that $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4$. For $n > 4m$ and $0 < |\mu| \ll 1$, we obtain the following expansions for $(M(\mu))^{-1}$ in $B(0, 0)$:

(i) If $0$ is a regular point of $H$, then we have

$$\left( M(\mu) \right)^{-1} = T_0^{-1} + \sum_{l=1}^{n-1} \mu^{2ml} \tilde{A}_l + \tilde{\alpha}_R \mu^{n-2m} \tilde{A}_\xi + \tilde{\alpha}_{R+1} \mu^{n-2m+2} \tilde{A}_{\xi+1} + O(\mu^{n-2m+2}) \quad (8.6)$$

where the operators $\tilde{A}_l \in B(0, 0)$ and $\tilde{\alpha}_R, \tilde{\alpha}_{R+1} \in \mathbb{C} \setminus \mathbb{R}$.

(ii) If $0$ is an eigenvalue of $H$, then we have

$$\left( M(\mu) \right)^{-1} = \frac{\tilde{A}_l}{\mu^{2m}} + \sum_{l=2}^{n-1} \mu^{2m(l-1)} \tilde{A}_l + \tilde{\alpha}_R \mu^{n-6m} \tilde{A}_\xi + \tilde{\alpha}_{R+1} \mu^{n-6m+2} \tilde{A}_{\xi+1} + O(\mu^{n-6m+2}) \quad (8.7)$$

where the operators $\tilde{A}_l \in B(0, 0)$ and $\tilde{\alpha}_R, \tilde{\alpha}_{R+1} \in \mathbb{C} \setminus \mathbb{R}$. Furthermore, $\tilde{A}_l = (S_1 vG_1 vS_1)^{-1}$.

**Proof.** (i) Since

$$M(\mu) = T_0 + \mathcal{M}(\mu),$$

and $T_0$ is invertible, then $\mathcal{M}(\mu)T_0^{-1}$ is uniformly bounded with bound less than 1 for tiny $|\mu|$. Thus

$$\left( M(\mu) \right)^{-1} = T_0^{-1} (1 + \mathcal{M}(\mu)T_0^{-1})^{-1}$$

$$= T_0^{-1} - T_0^{-1} \mathcal{M}(\mu)T_0^{-1} + T_0^{-1} \left( \mathcal{M}(\mu)T_0^{-1} \right)^2 (1 + \mathcal{M}(\mu)T_0^{-1})^{-1}. \quad (8.8)$$

(ii) If $T_0$ is not invertible, then we know that zero is an eigenvalue of $H$. We use a Neumann series expansion. Using (2.11) we have

$$\left( M(\mu) + S_1 \right)^{-1} = (T_0 + S_1 + \mathcal{M}(\mu))^{-1}$$

$$= D_0 \left( I + \mathcal{M}(\mu)D_0 \right)^{-1}$$

$$= D_0 - D_0 \mathcal{M}(\mu)D_0 + D_0 \mathcal{M}(\mu)D_0^2 + O(\mu^{6m}) \quad (8.9)$$
Then for \( M_1(\mu) = S_1 - S_1(M(\mu) + S_1)^{-1}S_1 \), we have
\[
M_1(\mu) = S_1 - S_1 \left( D_0 - D_0M(\mu)D_0 + D_0(M(\mu)D_0)^2 + O(\mu^{6m}) \right) S_1
\]
\[
= S_1M(\mu)S_1 - S_1(M(\mu)D_0)^2S_1 + O(\mu^{6m})
\]
\[
:= \mu^{2m}S_1vG_1vS_1 + M_1(\mu).
\]

Since \( S_1vG_1vS_1 \) is invertible on \( S_1L^2(\mathbb{R}^n) \), see Lemma 8.5. Then we can obtain the expansions of \((M_1(\mu))^{-1}\) by Neumann series.

From Lemma 8.1 we have
\[
(M(\mu))^{-1} = (M(\mu) + S_1)^{-1} + (M(\mu) + S_1)^{-1}S_1(M(\mu))^{-1}(M(\mu) + S_1)^{-1}
\]
(8.11)
Substituting the expansions of \((M(\mu) + S_1)^{-1}\) and \((M(\mu))^{-1}\) into (8.11), we obtain (8.7).

**Lemma 8.5.** Assume that \(|V(x)| \lesssim (1 + |x|)^{-\beta}\) with some \( \beta > n + 4 \). Then we obtain
\[
\ker (S_1vG_1vS_1) = \{0\}.
\]

**Proof.** Take \( \phi \in S_1L^2(\mathbb{R}^n) \) with \( S_1vG_1vS_1\phi = 0 \). Then using (2.10), we have
\[
G_1 = \lim_{\mu \to 0} \frac{R_0(\mu^{2m}) - G_0}{\mu^{2m}},
\]
thus
\[
0 = \langle S_1vG_1vS_1\phi, \phi \rangle = \langle G_1v\phi, \phi \rangle = \lim_{\mu \to 0} \left\langle \left( \frac{R_0(\mu^{2m}) - G_0}{\mu^{2m}} \right)v\phi, v\phi \right\rangle
\]
\[
= \lim_{\mu \to 0} \frac{1}{\mu^{2m}} \int_{\mathbb{R}^n} \left( \frac{1}{|\xi|^{2m} - \mu^{2m}} - \frac{1}{|\xi|^2} \right) \hat{v}\phi(\xi)\hat{v}\phi(\xi) d\xi
\]
\[
= \lim_{\mu \to 0} \int_{\mathbb{R}^n} \frac{\hat{v}\phi(\xi)|^2}{|\xi|^{2m} - \mu^{2m}} d\xi = \int_{\mathbb{R}^n} \frac{\hat{v}\phi(\xi)|^2}{|\xi|^{4m}} d\xi = \langle G_0v\phi, G_0v\phi \rangle.
\]
Here we used the dominated convergence theorem as \( \mu \to 0 \) with chosen \( \mu \) such that \( \text{Re}(\mu^4) < 0 \) on the last equality. This implies that \( \hat{v}\phi = 0 \) and \( v\phi = 0 \). Recall that \( S_{k+1} \leq S_1 \), then \( \phi \in S_1L^2 \) which implies that \( \phi = -UvG_0v\phi \). Thus the kernel of \( S_1vG_1vS_1 \) is trivial.

**Proof of Theorem 2.8** (2m < n ≤ 4m and n odd). In this case, for \(|V(x)| \lesssim (1 + |x|)^{-\beta}\) with some \( \beta > 0 \), we have the following expansions for \( M(\mu) \) in \( B(0,0) \):
\[
M(\mu) = U + vG_0v + \tilde{c}_1\mu^{2(m-k)+1}P + \sum_{j=2}^{k} c_j\mu^{2(m+j-k)-1}vG_jv + \mu^{2m}vG_{k+1}v + vE_3(\mu)v.
\]
(8.12)

Denote
\[
\mathcal{M}(\mu) = \tilde{c}_1\mu^{2(m-k)+1}P + \sum_{j=2}^{k} c_j\mu^{2(m+j-k)-1}vG_jv + \mu^{2m}vG_{k+1}v + vE_3(\mu)v.
\]

Theorem 2.8 holds by symmetric resolvent identity (2.8) and the following Proposition.
Proposition 8.6. For $n = 4m + 1 - 2k$ with $k$ chosen as follows and $0 < |\mu| \ll 1$. Assume that $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4k$. We obtain the following expansions of $(M(\mu))^{-1}$ in $B(0, 0)$:

(i) For $1 \leq k \leq m$, if $0$ is a regular point of $H$, then we have

$$
(M(\mu))^{-1} = T_0^{-1} + \sum_{l=1}^{k} \beta_l \mu^{2(m-l-k)-1} \tilde{B}_l + \mu^{2m} \tilde{B}_{k+1} + O(\mu^{2m+})
$$

(8.13)

where the operators $\tilde{B}_l \in B(s, -s')$ with $s, s' > \frac{n}{2} + 2k$ and $\beta_l \in \mathbb{C} \setminus \mathbb{R}$. Furthermore, $\tilde{B}_0 = T_0^{-1}$

(ii) For $1 \leq k \leq \left[ \frac{m}{2} \right] + 1$, if $0$ is the $j$-th kind of resonance of $H$ with $1 \leq j \leq k$, then we have

$$
(M(\mu))^{-1} = \frac{\beta_j \mu^{2m-k-j-1}}{k^{2(m-k-j)+1}} + \sum_{l=1}^{k-j} \frac{\beta_j \mu^{2m-k-j-1}}{l^{2m-k-j-1}} + \sum_{l=k-j+1}^{2m-3k+3j-2} \frac{\beta_j \mu^{2m-k-j-1}}{l^{2m-3k+3j-1}} + O(\mu),
$$

(8.14)

where the operators $\tilde{B}_j^l \in B(s, -s')$ with $s, s' > \frac{n}{2} + 2k$ and $\beta_j \in \mathbb{C} \setminus \mathbb{R}$. Furthermore, $\tilde{B}_0^j = (S_{j+1} v G_{j+1} v S_{j})^{-1}$

(iii) For $1 \leq k \leq \left[ \frac{m}{2} \right] + 1$, if $0$ is an eigenvalue of $H$, then we have

$$
(M(\mu))^{-1} = \frac{\tilde{B}_0^{k+1}}{\mu^{2m}} + \sum_{l=1}^{2m-1} \frac{\tilde{B}_0^{k+1} \tilde{B}_0^{k+1}}{l^{2m-1}} + \tilde{B}_0^{k+1} \tilde{B}_0^{k+1} + O(\mu),
$$

(8.15)

where the operators $\tilde{B}_0^{k+1} \in B(s, -s')$ with $s, s' > \frac{n}{2} + 2k$ and $\beta_j \in \mathbb{C} \setminus \mathbb{R}$. Furthermore, $\tilde{B}_0^{k+1} = (S_{k+1} v G_{k+1} v S_{k+1})^{-1}$.

Proof. (i) The proof of (8.13) follows from the proof of (8.6).

(ii) Using (2.13), we have

$$
(M(\mu) + S_1)^{-1} = (T_0 + S_1 + M(\mu))^{-1}
$$

$$
= D_0(I + M(\mu)D_0)^{-1}
$$

$$
= D_0 - D_0 M(\mu) D_0 + D_0 (M(\mu)D_0)^{2} + O(\mu^{4(m-k)+2})
$$

(8.16)

Then for $M_1(\mu) = S_1 - S_1 (M(\mu) + S_1) S_1$, since $S_1 D_0 = D_0 S_1 = S_1$, we have

$$
M_1(\mu) = S_1 - S_1 \left(D_0 - D_0 M(\mu) D_0 + D_0 (M(\mu) D_0)^{2} + O(\mu^{4(m-k)+2})\right) S_1
$$

$$
= S_1 M(\mu) S_1 - S_1 (M(\mu) D_0)^{2} S_1 + O(\mu^{4(m-k)+2})
$$

$$
= c_1 \mu^{2(m-k)+1} S_1 P S_1 + \sum_{l=2}^{k} c_l \mu^{2(m-l-k)-1} S_1 v G_{l+1} v S_1
$$

$$
+ \mu^{2m} S_1 v G_{k+1} v S_1 - c_1 \mu^{4(m-k)+2} S_1 D_0 P D_0 P D_0 S_1 + O(\mu^{4(m-k)+2})
$$

$$
:= c_1 \mu^{2(m-k)+1} S_1 P S_1 + M_1(\mu)
$$

(8.17)
By the definition of the first kind of resonance, the operator \( T_1 = S_1 P S_1 \) is invertible on \( S_1 L^2(\mathbb{R}^n) \), then we can obtain the expansions of \( (M_1(\mu))^{-1} \) by Neumann series.

From Lemma 8.1, we also have (8.11). Substituting the expansions of \( (M(\mu) + S_1)^{-1} \) and \( (M_1(\mu))^{-1} \) into (8.11), we obtain (8.14) with \( j = 1 \).

For the second kind of resonance, we aim to derive the expansions of \( (M_1(\mu))^{-1} \). Define \( \tilde{M}_1(\mu) = \frac{M_1(\mu)}{c_1 \mu^{2(m-k)+1}} \), using (8.17) we have

\[
\tilde{M}_1(\mu) = S_1 P S_1 + \sum_{l=2}^{k} \tilde{c}_l \mu^{2l-2} S_1 v G_l v S_1 - \tilde{c}_1 \mu^{2(m-k)+1} S_1 P D_0 P S_1 + \tilde{c}_k + 1 \mu^{2k-1} S_1 v G_{k+1} v S_1 + O(\mu^{2k-1+})
\]

(8.18)

By the definition of the second kind of resonance, we know that the operator \( T_1 \) is not invertible on \( S_1 L^2(\mathbb{R}^n) \). Since \( S_2 \) is the Riesz projection onto the kernel of \( T_1 \) on \( S_1 L^2(\mathbb{R}^n) \), so \( T_1 + S_2 \) is invertible on \( S_1 L^2(\mathbb{R}^n) \). Furthermore, we can obtain

\[
(M_1(\mu) + S_2)^{-1} = (T_1 + S_2 + \tilde{M}_1(\mu))^{-1} D_1 - D_1 \tilde{M}_1(\mu) D_1 + D_1 (\tilde{M}_1(\mu) D_1)^{2} + O(\mu^{4+})
\]

(8.19)

Then for \( M_2(\mu) = S_2 - S_2(\tilde{M}_1(\mu) + S_2)^{-1} S_2 \), since \( S_2 D_1 = D_1 S_2 = S_2 \) and \( S_2 P = P S_2 = 0 \), we have

\[
M_2(\mu) = S_2 - S_2 \left( D_1 - D_1 \tilde{M}_1(\mu) D_1 + D_1 (\tilde{M}_1(\mu) D_1)^{2} + O(\mu^{4+}) \right) S_2
\]

\[
= S_2 \tilde{M}_1(\mu) S_2 - S_2 (\tilde{M}_1(\mu) D_1)^{2} S_2 + O(\mu^{4+})
\]

\[
= \tilde{c}_2 \mu^{2} S_2 v G_2 v + \sum_{l=3}^{k} \tilde{c}_l \mu^{2l-2} S_2 v G_l v S_2 + \tilde{c}_k + 1 \mu^{2k-1} S_2 v G_{k+1} v S_2 + O(\mu^{2k-1})
\]

(8.20)

Since the operator \( S_2 v G_2 v S_2 \) is invertible on \( S_2 L^2(\mathbb{R}^n) \), then we can obtain the expansions of \( (M_2(\mu))^{-1} \) by Neumann series. From Lemma 8.1, we have

\[
(M_1(\mu))^{-1} = (M_1(\mu) + S_2)^{-1} + (M_1(\mu) + S_2)^{-1} S_2 (M_2(\mu))^{-1} (M_1(\mu) + S_2)^{-1}
\]

(8.21)

Substituting the expansions of \( (M_1(\mu) + S_2)^{-1} \) and \( (M_2(\mu)) \) into (8.21), we obtain the inverse of \( \tilde{M}_1(\mu) \), then we can obtain the expansions of \( (M_1(\mu))^{-1} \). Furthermore, we can obtain (8.14) with \( j = 2 \).

By an induction process we can get the expansions of (8.14) and (8.15).

Now, we show that the kernel of the operator \( S_{k+1} v G_{k+1} v S_{k+1} \) on \( S_{k+1} L^2(\mathbb{R}^n) \) is trivial which means the iterated process stop here.

Lemma 8.7. Assume that \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) with some \( \beta > n + 4k \). Then we have

\[
\text{ker}(S_{k+1} v G_{k+1} v S_{k+1}) = \{0\}.
\]
Proof. Take $\phi \in S_{k+1}L^2(\mathbb{R}^n)$ with $S_{k+1}vG_{k+1}vS_{k+1}\phi = 0$. Then we have

$$\langle S_{k+1}vG_{k+1}vS_{k+1}\phi, \phi \rangle = \langle G_{k+1}v\phi, v\phi \rangle = 0.$$  

By the definition of $S_j$ with $1 \leq j \leq k + 1$, we also have

$$\langle v, \phi \rangle = \langle G_jv\phi, v\phi \rangle = 0.$$  

Then using (2.6), we obtain

$$G_{k+1} = \lim_{\mu \to 0} \frac{R_0(\mu^{2m}) - G_0 - \sum_{j=1}^{k} c_j\mu^{2(m-k+j)-1}G_j}{\mu^{2m}}.$$  

Thus, we have

$$0 = \langle S_{k+1}vG_{k+1}vS_{k+1}\phi, \phi \rangle = \langle G_{k+1}v\phi, v\phi \rangle$$

$$= \lim_{\mu \to 0} \left\langle \left( \frac{R_0(\mu^{2m}) - G_0 - \sum_{j=1}^{k} c_j\mu^{2(m-k+j)-1}G_j}{\mu^{2m}} \right)(\mu^{2m}) - G_0 - \sum_{j=1}^{k} c_j\mu^{2(m-k+j)-1}G_j, v\phi, v\phi \right\rangle$$

$$= \lim_{\mu \to 0} \frac{1}{\mu^{2m}} \left\langle \left( R_0(\mu^{2m}) - G_0 \right) v\phi, v\phi \right\rangle$$

$$= \lim_{\mu \to 0} \frac{1}{\mu^{2m}} \int_{\mathbb{R}^n} \left( \frac{1}{|\xi|^{2m} - \mu^{2m}} \right) v\phi(\xi)v(\xi)d\xi$$

$$= \lim_{\mu \to 0} \int_{\mathbb{R}^n} \frac{1}{|\xi|^{2m} - \mu^{2m}} \left( \sum_{j=1}^{k} c_j\mu^{2(m-k+j)-1}G_j \right) v\phi(\xi)v(\xi)d\xi$$

Here we used the dominated convergence theorem as $\mu \to 0$ with chosen $\mu$ such that $\Re(\mu^{2m}) < 0$ on the last equality. This implies that $v\phi = 0$ and $v\phi = 0$. Recall that $S_{k+1} \leq S_1$, then $\phi \in S_1L^2$ which implies that $\phi = -UvG_0v\phi$. Thus the kernel of $S_{k+1}vG_{k+1}vS_{k+1}$ is trivial. \(\square\)

**Proof of Theorem 2.9.** \(2m < n \leq 4m \text{ and } n \text{ even.}\) In this case, for $|v(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > 0$, we have the following expansions for $M(\mu)$ in $B(0,0)$:

$$M(\mu) = U + vG_0v + \tilde{d}_1\mu^{2(m-k+1)}P + \sum_{j=2}^{k-1} d_j\mu^{2(m+j-k)}vG_jv$$

$$\quad + \mu^{2m}g(\mu)vG_kv + \mu^{2m}vG_{k+1}v + vE_4(\mu)v. \quad (8.22)$$

Denote

$$\mathcal{M}(\mu) = \tilde{d}_1\mu^{2(m-k+1)}P + \sum_{j=2}^{k-1} d_j\mu^{2(m+j-k)}vG_jv + \mu^{2m}g(\mu)vG_kv + \mu^{2m}vG_{k+1}v + vE_4(\mu)v.$$  

Theorem 2.9 holds the following Proposition.

**Proposition 8.8.** For $n = 4m + 2 - 2k$ with $k$ chosen as follows and $0 < |\mu| \ll 1$. Assume $|V(x)| \lesssim (1 + |x|)^{-\beta}$ with some $\beta > n + 4k + 2$. we obtain the following expansions of $R_V(\mu^{2m})$ in $B(s,-s)$:

(i) For $1 \leq k \leq m$, if $0$ is a regular point of $H$, then we have

$$\left( M(\mu) \right)^{-1} = \tilde{C}_0 + \sum_{l=1}^{k-1} \mu^{2(m+l-k)}\tilde{C}_l + \mu^{2m}g(\mu)\tilde{C}_k + \mu^{2m}\tilde{C}_{k+1} + O(\mu^{2m+}) \quad (8.23)$$
Proof. (i) The proof of (8.23) follows from the proof of (8.6). Furthermore, \( \tilde{C}_0 = T_0^{-1} \).

(ii) For \( 1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil + 1 \), if 0 is of the \( j \)-th kind of resonance of \( H \) with \( 1 \leq j \leq k - 1 \), then we have

\[
(M(\mu))^{-1} = \frac{\tilde{c}_j(\mathcal{C}_{0,0})}{\mu^{2(m-k+j)}} + \sum_{l=1}^{k-j} \frac{\tilde{c}_j(\mathcal{C}_{l,0})}{\mu^{2(m-k+j-l)}} + \sum_{l=k-j+1}^{m-k+j-1} \left( \frac{\tilde{c}_j(\mathcal{C}_{l,0})}{\mu^{2(m-k+j-l)}} + \frac{\ln(\mu)\tilde{c}_j(\mathcal{C}_{l,1})}{\mu^{2(m-k+j-l)}} \right) + \ln(\mu)\tilde{c}_j(\mathcal{C}_{m-k+j,0}) + O(\mu^{0+}),
\]

where the operators \( \tilde{c}_j(\mathcal{C}_{l,0}) \in B(s,-s') \) with \( s,s' > \frac{n}{2} + 2k \) and \( \tilde{\tau}_{l,0}, \tilde{\tau}_{l,1} \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( \mathcal{C}_{0,0} = (S_j v G_j v S_j)^{-1} \).

(iii) For \( 1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil + 1 \), if 0 is of the \( k \)-th kind of resonance of \( H \), then we have

\[
(M(\mu))^{-1} = \frac{\tilde{c}_k(\mathcal{C}_{0,0})}{\mu^{2m}g(\mu)} + \sum_{l=1}^{k-1} \frac{\tilde{c}_k(\mathcal{C}_{l,0})}{\mu^{2(m-l)}g(\mu)} + \frac{\tilde{c}_k(\mathcal{C}_{l,1})}{\mu^{2(m-l)}g(\mu)} + \frac{\tilde{c}_k(\mathcal{C}_{l,2})}{\mu^{2(m-l)}g(\mu)} + \frac{\tilde{\tau}_{m,0}\tilde{c}_k(\mathcal{C}_{m,0}) + (g(\mu))^{-1}\tilde{\tau}_{m,1}\tilde{c}_k(\mathcal{C}_{m,1}) + (g(\mu))^{-2}\tilde{\tau}_{m,2}\tilde{c}_k(\mathcal{C}_{m,2}) + O(\mu^{0+})},
\]

where the operators \( \tilde{c}_k(\mathcal{C}_{l,0}), \tilde{c}_k(\mathcal{C}_{l,1}), \tilde{c}_k(\mathcal{C}_{l,2}) \in B(s,-s') \) with \( s,s' > \frac{n}{2} + 2k \) and \( \tilde{\tau}_{l,0}, \tilde{\tau}_{l,1}, \tilde{\tau}_{l,2} \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( \mathcal{C}_{0,1} = (S_k v G_k v S_k)^{-1} \).

(iv) For \( 1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil + 1 \), if 0 is an eigenvalue of \( H \), then we have

\[
(M(\mu))^{-1} = \frac{\tilde{c}_k+1(\mathcal{C}_{0,0})}{\mu^{2m}g(\mu)} + \sum_{l=1}^{m-k} \frac{\tilde{c}_k+1(\mathcal{C}_{l,0})}{\mu^{2(m-l)}g(\mu)} + \frac{\tilde{c}_k+1(\mathcal{C}_{l,1})}{\mu^{2(m-l)}g(\mu)} + \frac{\tilde{c}_k+1(\mathcal{C}_{l,2})}{\mu^{2(m-l)}g(\mu)} + \frac{\tilde{\tau}_{m,0}\tilde{c}_k+1(\mathcal{C}_{m,0}) + (g(\mu))^{-1}\tilde{\tau}_{m,1}\tilde{c}_k+1(\mathcal{C}_{m,1}) + (g(\mu))^{-2}\tilde{\tau}_{m,2}\tilde{c}_k+1(\mathcal{C}_{m,2}) + O(\mu^{0+})},
\]

where the operators \( \tilde{c}_k+1(\mathcal{C}_{l,0}), \tilde{c}_k+1(\mathcal{C}_{l,1}), \tilde{c}_k+1(\mathcal{C}_{l,2}) \in B(s,-s') \) with \( s,s' > \frac{n}{2} + 2k \) and \( \tilde{\tau}_{l,0}, \tilde{\tau}_{l,1}, \tilde{\tau}_{l,2} \in \mathbb{C} \setminus \mathbb{R} \). Furthermore, \( \mathcal{C}_{0,0} = (S_{k+1} v G_{k+1} v S_{k+1})^{-1} \).

Proof. (i) The proof of (8.23) follows from the proof of (8.6).

(ii) Using (2.15), we have

\[
(M(\mu) + S_1)^{-1} = (T_0 + S_1 + \mathcal{M}(\mu))^{-1} = D_0(1 + \mathcal{M}(\mu)D_0)^{-1} = D_0 - D_0\mathcal{M}(\mu)D_0 + D_0(\mathcal{M}(\mu)D_0)^2 + O(\mu^{4(m-k)+2+})
\]
Then for $M_1(\mu) = S_1 - S_1(M(\mu) + S_1)S_1$, since $S_1D_0 = D_0S_1 = S_1$, we have

$$
M_1(\mu) = S_1 - S_1\left(D_0 - D_0M(\mu)D_0 + D_0(M(\mu)D_0)^2 + O(\mu^{4(m-k)+2})\right)S_1
$$

$$
= S_1M(\mu)S_1 - S_1(M(\mu)D_0)^2S_1 + O(\mu^{4(m-k)+2})
$$

$$
= \tilde{d}_1\mu^{2(m-k)+2}S_1PS_1 + \sum_{l=2}^{k-1} d_l\mu^{2(m-l-k)}S_1vG_lvS_1 + \mu^{2m}g(\mu)S_1vG_kvS_1
$$

$$
+ \mu^{2m}S_1vG_{k+1}vS_1 - \tilde{d}_2\mu^{4(m-k)+4}S_1PD_0PS_1 + O(\mu^{4(m-k)+4})
$$

$$
:= \tilde{d}_1\mu^{2(m-k)+2}S_1PS_1 + M(\mu)
$$

By the definition of the first kind of resonance, the operator $T_1 = S_1PS_1$ is invertible on $S_1L^2(\mathbb{R}^n)$, then we can obtain the expansions of $(M_1(\mu))^{-1}$ by Neumann series.

From Lemma 8.1, we also have (8.11). Substituting the expansions of $(M(\mu) + S_1)^{-1}$ and $(M_1(\mu))^{-1}$ into (8.11), we obtain (8.24) with $j = 1$.

For the second kind of resonance, we aim to derive the expansions of $(M_1(\mu))^{-1}$. Define $\tilde{M}_1(\mu) = \frac{M_1(\mu)}{d_1\mu^{2(m-k)+2}}$, using (8.28) we have

$$
\tilde{M}_1(\mu) = S_1PS_1 + \sum_{l=2}^{k-1} \tilde{d}_l\mu^{2l-2}S_1vG_lvS_1 - \tilde{d}_1\mu^{2(m-k)+2}S_1PD_0PS_1
$$

$$
+ \tilde{d}_k\mu^{2k-2}g(\mu)S_1vG_kvS_1 + \tilde{d}_{k+1}\mu^{2k-2}S_1vG_{k+1}vS_1 + O(\mu^{2k-2})
$$

$$
:= T_1 + \tilde{\mathcal{M}}_1(\mu).
$$

By the definition of the second kind of resonance, we know that the operator $T_1$ is not invertible on $S_1L^2(\mathbb{R}^n)$. Since $S_2$ is the Riesz projection onto the kernel of $T_1$ on $S_1L^2(\mathbb{R}^n)$, so $T_1 + S_2$ is invertible on $S_1L^2(\mathbb{R}^n)$. Furthermore, we can obtain

$$
\left(\tilde{M}_1(\mu) + S_2\right)^{-1} = (T_1 + S_2 + \tilde{\mathcal{M}}_1(\mu))^{-1}
$$

$$
= D_1 - D_1\tilde{\mathcal{M}}_1(\mu)D_1 + D_1\left(\tilde{\mathcal{M}}_1(\mu)D_1\right)^2 + O(\mu^{4+})
$$

(8.30)

Then for $M_2(\mu) = S_2 - S_2(\tilde{\mathcal{M}}_1(\mu) + S_2)^{-1}S_2$, since $S_2D_1 = D_1S_2 = S_2$ and $S_2P = PS_2 = 0$, we have

$$
M_2(\mu) = S_2 - S_2\left(D_1 - D_1\tilde{\mathcal{M}}_1(\mu)D_1 + D_1\left(\tilde{\mathcal{M}}_1(\mu)D_1\right)^2 + O(\mu^{4+})\right)S_2
$$

$$
= S_2\tilde{\mathcal{M}}_1(\mu)S_2 - S_2\left(\tilde{\mathcal{M}}_1(\mu)D_1\right)^2S_2 + O(\mu^{4+})
$$

$$
= \tilde{d}_2\mu^2S_2vG_2vS_2 + \sum_{l=3}^{k-1} \tilde{d}_l\mu^{2l-2}S_2vG_lvS_2 + \tilde{d}_k\mu^{2k-2}g(\mu)S_2vG_kvS_2
$$

$$
+ \tilde{d}_{k+1}\mu^{2k-2}S_2vG_{k+1}vS_2 + O(\mu^{2k-2})
$$

$$
:= \tilde{d}_2\mu^2S_2vG_2vS_2 + \mathcal{M}_2(\mu).
$$
Since the operator $S_2 v G_2 v S_2$ is invertible on $S_2 L^2(\mathbb{R}^n)$, then we can obtain the expansions of $(M_2(\mu))^{-1}$ by Neumann series. From Lemma 8.1 we have
\[
\left(\widetilde{M}_1(\mu)\right)^{-1} = \left(\widetilde{M}_1(\mu) + S_2\right)^{-1} + \left(\widetilde{M}_1(\mu) + S_2\right)^{-1} S_2 \left(M_2(\mu)\right)^{-1} \left(\widetilde{M}_1(\mu) + S_2\right)^{-1}
\] (8.32)
Substituting the expansions of $(\widetilde{M}_1(\mu) + S_2)^{-1}$ and $(M_2(\mu))$ into (8.21), we obtain the inverse of $\widetilde{M}_1(\mu)$, then we can obtain the expansions of $(M_1(\mu))^{-1}$. Furthermore, we can obtain (8.24) when $j = 2$.

By an induction process we can get the expansions of (8.24) for all $2 \leq j \leq k - 1$, then we derive (8.25) and (8.26). What’s more, by Lemma 8.7, we know that $\ker(S_{k+1} v G_{k+1} v S_{k+1}) = \{0\}$, which means the iterated process stop here.

8.2. Proof of identification of resonance subspace. In this part, we aim to identify the resonance subspace of each kind for different dimensional cases with $n > 2m$.

**Proof of Proposition 2.10** ($n > 4m$). We first note that
\[
((\Delta)^m + V) \psi = 0 \iff (I + G_0 V) \psi = 0.
\]
First, suppose that $\phi \in S_1 L^2 \setminus \{0\}$. Then $(U + v G_0 v) \phi = 0$. Multiplying by $U$, one has
\[
\phi(x) = -U v G_0 v \phi = U v(x) \int_{\mathbb{R}^n} \frac{v(y) \phi(y)}{|x - y|^{n-2m}} dy.
\]
Accordingly, we define
\[
\psi(x) = \int_{\mathbb{R}^n} \frac{v(y) \phi(y)}{|x - y|^{n-2m}} dy \quad (= -G_0 v \phi). \quad (8.33)
\]
Since $v \phi \in L^2_\frac{n}{2}(\mathbb{R}^n) \subset L^2_\frac{n}{2+2}(\mathbb{R}^n)$, we have $\psi \in L^2(\mathbb{R}^n)$ by Lemma 8.3 (1). Further, $\phi(x) = U v(x) \psi(x)$ and
\[
\psi(x) = -G_0 v \phi(x) = -G_0 V \psi(x),
\]
which implies $(I + G_0 V) \psi(x) = 0$.

Secondly, assume $\phi(x) = U v(x) \psi(x)$ for $\psi(x)$ a non-zero distributional solution to $H \psi = 0$. It is clear that $\phi \in L^2_\frac{n}{2}(\mathbb{R}^n) \subset L^2_\frac{n}{2+2}(\mathbb{R}^n)$ and now
\[
(U + v G_0 v) \phi(x) = v(x) \psi(x) + v(x) G_0 V \psi(x) = v(x) (I + G_0 V) \psi(x) = 0.
\]
Thus showing that $\phi \in S_1 L^2(\mathbb{R}^n)$. □

For $3m \leq n \leq 4m$, there exists $N$ kinds of resonances where $N \geq 2$ by Definition 2.6. If $n$ goes down to close $3m$, then $N$ goes up to close $[m/2] + 1$. Next, we give the identity condition of each projection $S_N$ for $1 \leq N \leq [m/2] + 1$.

**Lemma 8.9.** For $1 \leq N \leq [m/2] + 1$, then $\phi \in S_{N+1} L^2$ if and only if
\[
\int_{\mathbb{R}^n} y_{a_1} y_{a_2} \cdots y_{a_{N-1}} v(y) \phi(y) dy = 0 \quad (8.34)
\]
for $a_\ell \in \{1, 2, \ldots, n\}$ with $1 \leq \ell \leq N - 1$ and $y_{a_0} = 1$. 

Proof. For $N = 1$, if $\phi \in S_2L^2 \setminus \{0\}$, by the definition of $S_2$, then

$$0 = \langle S_1P\phi, \phi \rangle = \|P\phi\|^2_{L^2},$$

thus $P\phi = 0$, that is

$$\int_{\mathbb{R}^n} v(y)\phi(y)dy = 0. \quad (8.35)$$

For $N = 2$, if $\phi \in S_3L^2 \setminus \{0\}$, then

$$0 = \langle S_2vG_2vS_2\phi, \phi \rangle = \langle G_2v\phi, v\phi \rangle$$

$$\begin{align*}
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x)|x - y|^2v(y)\phi(y)dxdy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x)(|x|^2 + |y|^2 - 2x \cdot y)v(y)\phi(y)dxdy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x)(-2x \cdot y)v(y)\phi(y)dxdy \\
&= -2\left| \int_{\mathbb{R}^n} yyv(y)\phi(y)dy \right|^2
\end{align*}$$

thus for all $j = 1, 2, \ldots, n$,

$$\int_{\mathbb{R}^n} y_jv(y)\phi(y)dy = 0. \quad (8.36)$$

Suppose that the conclusion (8.34) holds for all $1 \leq j \leq N$. If $\phi \in S_{N+1}L^2 \setminus \{0\}$, then by the multinomial theorem, we have

$$0 = \langle S_NvG_NV_S\phi, \phi \rangle = \langle G_NV\phi, v\phi \rangle$$

$$\begin{align*}
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x)|x - y|^{2N-2}v(y)\phi(y)dxdy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x)(|x|^2 - 2x \cdot y + |y|^2)^Nv(y)\phi(y)dxdy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x) \sum_{k_1+k_2+k_3=N-1} C(k)|x|^{2k_1}(-2x \cdot y)^{k_2}|y|^{2k_3}v(y)\phi(y)dxdy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x) \sum_{k_1+k_2+k_3=N-1} C(k)|x|^{2k_1} \\
&\times \sum_{a_1+\cdots+a_n=k_2} C(a)(-2)^{k_3}(x_1y_1)^{a_1}\cdots(x_ny_n)^{a_n}|y|^{2k_3}v(y)\phi(y)dxdy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x) \sum_{k_1+k_2+k_3=N-1} C(k) \sum_{a_1,\ldots,a_k=1; a_1\leq\cdots\leq a_k}^{k_2} C(a)(-2)^{k_2} \\
&\times |x|^{2k_1}(x_1 x_2 \cdots x_{a_k})(y_1 y_2 \cdots y_{a_k}) |y|^{2k_3}v(y)\phi(y)dxdy
\end{align*}$$

where $C(k) = \frac{(N-1)!}{k_1!k_2!\cdots k_n!}$ and $C(a) = \frac{k_2^a}{a_1!a_2!\cdots a_n!}$.

Note that the term

$$|x|^{2k_1} \sum_{a_1,\ldots,a_k=1; a_1\leq\cdots\leq a_k}^{k_2} (x_1 x_2 \cdots x_{a_k})$$
has actually the same form as $\sum_{a_1,\ldots,a_{k_2}=1:a_1\leq\cdots\leq a_{k_2}}(x_{a_1}x_{a_2}\cdots x_{a_{k_2}})$. Since $|x-y|^{2N-2}$ is symmetric about $x$ and $y$, by $S_{N+1} \leq S_N \leq \cdots \leq S_1$, then the terms contribute zero is the term as following $(x, y$ has the same order$)$:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x)\phi(x) \sum_{a_1,\ldots,a_{N-1}=1:a_1\leq\cdots\leq a_{N-1}} (x_{a_1}x_{a_2}\cdots x_{a_{N-1}})(y_{a_1}y_{a_2}\cdots y_{a_{N-1}})v(y)\phi(y)dxdy.$$ 

Note that the above terms come from terms like $|x|^{2a}(-2x \cdot y)^{N-1-2a}|y|^{2a}$, thus all the above terms have the same sign (+ or −). Thus Lemma 8.9 holds by the mathematical induction. □

**Proof of Proposition 2.11** (2m < n ≤ 4m and n odd). (i) First, suppose that $\phi \in S_1L^2 \setminus \{0\}$. Then $(U+vG_0^0)\phi = 0$, and multiplying by $U$, one has

$$\phi(x) = -UvG_0v\phi = Uv(x) \int_{\mathbb{R}^n} \frac{v(y)\phi(y)}{|x-y|^{n-2m}}dy.$$ 

Accordingly, we define

$$\psi(x) = \int_{\mathbb{R}^n} \frac{v(y)\phi(y)}{|x-y|^{n-2m}}dy \quad (= -G_0v\phi). \quad (8.37)$$

Since $v\phi \in L^2_2(\mathbb{R}^n) \subset L^2_2(\mathbb{R}^n)$, we have that $\psi \in W_{2m-\frac{4}{3}}(\mathbb{R}^n)$ by Lemma 8.3(2). Further $\phi(x) = Uv(x)\psi(x)$ and

$$\psi(x) = -G_0v\phi(x) = -G_0V\psi(x) \implies (I+G_0V)\psi(x) = 0.$$ 

Secondly, assume $\phi(x) = Uv(x)\psi(x)$ for $\psi(x)$ a non-zero distributional solution to $H\psi = 0$. It is clear that $\phi \in L^2_0(\frac{n}{2}-2m+\frac{n}{2}-) \subset L^2_{(n+2k-2m)-}(\mathbb{R}^n)$ and now

$$(U+vG_0v)\phi(x) = v(x)\psi(x) + v(x)G_0V\psi(x) = v(x)(I+G_0V)\psi(x) = 0.$$ 

Thus showing that $\phi \in S_1L^2(\mathbb{R}^n)$.

(ii) Assume first that $\phi \in S_2L^2(\mathbb{R}^n) \setminus \{0\}$. Since $S_2 \leq S_1$, then by Lemma 8.9 and our definition of $\psi(x)$, we have

$$\psi(x) = \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}}\right)v(y)\phi(y)dy.$$
Thus Lemma 8.3 shows that \( \psi \in W_{2m-\frac{n}{2}-1}(\mathbb{R}^n) \) as desired.

On the other hand, if \( \phi = \U \psi \) as in hypothesis, we have

\[
\psi(x) \equiv \int_{\mathbb{R}^n} \left( \frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \right) v(y)\phi(y)dy + \frac{1}{(1+|x|)^{n-2m}} \int_{\mathbb{R}^n} v(y)\phi(y)dy.
\]

The first term and \( \psi(x) \) are in \( L^2_{-(k-\frac{n}{2})}(\mathbb{R}^n) \), thus we must have that

\[
\frac{1}{(1+|x|)^{n-2m}} \int_{\mathbb{R}^n} v(y)\phi(y)dy \in L^2_{-(k-\frac{n}{2})}(\mathbb{R}^n),
\]

then

\[
\int_{\mathbb{R}^n} v(y)\phi(y)dy = 0
\]

that is \( 0 = P\phi = S_1PS_1\phi \) and \( \phi \in S_2L^2(\mathbb{R}^n) \) as desired.

(iii) For \( j = 3 \), assume first that \( \phi \in S_3L^2(\mathbb{R}^n) \setminus \{0\} \). Since \( S_3 \leq S_2 \leq S_1 \), then by Lemma 8.9, we have

\[
\psi(x) = \int_{\mathbb{R}^n} \left[ \frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \right] v(y)\phi(y)dy + \frac{1}{(1+|x|)^{n-2m+1}} 2(1+C_{3,1-m}^1) x \cdot y v(y)\phi(y)dy.
\]

Since

\[
\frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \sim \frac{C_{n-2m}^1}{(1+|x|)^{n-2m}} - \frac{2x \cdot y}{(1+|x|)^{n-2m+2}}
\]

\[
\sum_{l=0}^{n-2m-2} \frac{|x|^l}{(1+|x|)^{n-2m}|x-y|^{n-2m}} + \frac{C_{n-2m}^1}{(1+|x|)^{n-2m}} - \frac{C_{n-2m}^1}{(1+|x|)^{n-2m+1}|x-y|^{n-2m}}
\]

\[
+ \frac{|x|^{n-2m} - |x-y|^{n-2m}}{(1+|x|)^{n-2m}|x-y|^{n-2m}} - \frac{2(1+C_{3,1-m}^1) x \cdot y}{(1+|x|)^{n-2m+2}}
\]

\[
:= I + II + III
\]

(8.40)
For the term $I$ of (8.40), we have
\[ I \lesssim \frac{1}{(1 + |x|)^2|x - y|^{n-2m}}. \]

For the term $II$ of (8.40), using (8.38), we have
\[ II \lesssim \frac{1}{1 + |x|} \left| \frac{1}{|x - y|^{n-2m}} - \frac{1}{(1 + |x|)^{n-2m}} \right| \]
\[ \lesssim \frac{1 + |y|}{(1 + |x|)^2|x - y|^{n-2m}} + \frac{n-2m}{(1 + |x|)^{l+1}|x - y|^{n-2m-1}}. \]

Then for the term $III$ of (8.40), we have
\[ III \lesssim \frac{|x|^2 + 2x \cdot y + |y|^2}{(1 + |x|)^{n-2m}} - 2C_{n-1}^{\frac{1}{2}-m}|x|^{n-2m}x \cdot y - |x|^{n-2m-1} \]
\[ + \frac{2C_{n-1}^{\frac{1}{2}-m}|x|^{n-2m-3}x \cdot y}{(1 + |x|)^{n-2m}|x - y|^{n-2m-1}} + \frac{|x|^{n-2m-1}|x - y|^2}{(1 + |x|)^{n-2m}|x - y|^{n-2m}} - \frac{2(1 + C_{n-1}^{\frac{1}{2}-m})|x| \cdot y}{(1 + |x|)^{n-2m+2}} \]
\[ \lesssim \sum_{l=2}^{n-2m} \frac{(1 + |y|)^l}{(1 + |x|)^{l+1}|x - y|^{n-2m-1}} + \frac{(1 + |y|)^2}{(1 + |x|)^4|x - y|^{n-2m+1}} + \frac{(1 + |y|)^2}{(1 + |x|)^4|x - y|^{n-2m-2}}. \]

Thus Lemma 8.3 shows that $\psi \in W_{2m-\frac{n}{2} - 2}(\mathbb{R}^n)$ as desired.

For each resonance subspace $S_j L^2(\mathbb{R}^n)$, applying Lemma 8.9 then by the process which is similar to (8.38) and (8.40), one can gain the weight of $(1 + |x|)^{-\sigma}$ and $(1 + |y|)^{\sigma'}$ for some certainly $\sigma, \sigma' > 0$. Using the weight $(1 + |x|)^{-\sigma}$, we boost the weighted space $L^2_{\sigma}$ up to $L^2$. The weight $(1 + |y|)^{\sigma'}$ is absorbed by $v(y)$. Hence, following the similar strategy as for $S_2 L^2$ and $S_3 L^2$, we can get the result for $S_j L^2$ with $3 \leq j \leq k + 1$, that is if $\phi(x) \in S_j L^2(\mathbb{R}^n)$ then we have $\psi \in W_{2m-\frac{n}{2} - j}(\mathbb{R}^n)$. Particularly, for $j = k + 1$, we obtain $\psi \in L^2_{\frac{n}{2}}(\mathbb{R}^n)$ which is a subset of $L^2(\mathbb{R}^n)$. Thus the $k + 1$-th kind of “resonance” is actually eigenvalue. \(\square\)

**Proof of Proposition 2.12** \((2m < n \leq 4m \text{ and } n \text{ even})\). (i) First, suppose that $\phi \in S_1 L^2 \setminus \{0\}$. Then $(U + v G_0)\phi = 0$, and multiplying by $U$, one has
\[ \phi(x) = -U v G_0 \phi = U v(x) \int_{\mathbb{R}^n} \frac{v(y)\phi(y)}{|x - y|^{n-2m}} dy. \]

Accordingly, define
\[ \psi(x) = \int_{\mathbb{R}^n} \frac{v(y)\phi(y)}{|x - y|^{n-2m}} dy \quad (= -G_0 v \phi). \quad (8.41) \]

Since $v \phi \in L^2_{\frac{n}{2}}(\mathbb{R}^n) \subset L^2_{\frac{n}{2} +} (\mathbb{R}^n)$, then by Lemma 8.3(2) and the identity $\psi = -G_0 V \psi$, we know $\psi \in W_{2m-\frac{n}{2} -}(\mathbb{R}^n)$. Furthermore, $\phi(x) = U v(x) \psi(x)$ and
\[ \psi(x) = -G_0 v \phi(x) = -G_0 v G_0 \psi(x) \implies (I + G_0 V) \psi(x) = 0. \]

Secondly, assume $\phi(x) = U v(x) \psi(x)$ for $\psi(x)$ a non-zero distributional solution to $H \psi = 0$. It is clear that $\phi \in L^2_{\frac{n}{2} - 2m +} (\mathbb{R}^n) \subset L^2_{n+2k-2m-}(\mathbb{R}^n)$ and now
\[ (U + v G_0 v)\phi(x) = v(x)\psi(x) + v(x)G_0 V \psi(x) = v(x)(I + G_0 V)\psi(x) = 0. \]
Thus showing that $\phi \in S_1 L^2(\mathbb{R}^n)$.

(ii) For $j = 2$, assume first that $\phi \in S_2 L^2(\mathbb{R}^n) \setminus \{0\}$. Since $S_2 \leq S_1$, then by Lemma 8.9 and our definition of $\psi(x)$, we have

$$\psi(x) = \int_{\mathbb{R}^n} \left( \frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \right) v(y)\phi(y)dy.$$  

Using

$$\left| \frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \right| \lesssim \sum_{l=0}^{n-2m-1} \frac{C^l_{n-2m}|x|^l}{(1+|x|)^{n-2m}|x-y|^{n-2m}} + \left( \frac{|x|^2 + |y|^2 - 2x \cdot y}{(1+|x|)^{n-2m}|x-y|^{n-2m}} \right) \left( |x|^2 + |y|^2 - 2x \cdot y \right)_{n-2m} - |x|^{n-2m} \right) \frac{1}{(1+|x|)^{n-2m}} \int_{\mathbb{R}^n} v(y)\phi(y)dy. \quad (8.42)$$

Thus Lemma 8.9 shows that $\psi \in W_{2m-\frac{n}{2}-1}(\mathbb{R}^n)$ as desired.

On the other hand, if $\phi = Uv$, $v$ as in hypothesis, we have

$$\psi(x) = \int_{\mathbb{R}^n} \left( \frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \right) v(y)\phi(y)dy + \frac{1}{(1+|x|)^{n-2m}} \int_{\mathbb{R}^n} v(y)\phi(y)dy. \quad (8.43)$$

The first term and $\psi(x)$ are in $W_{2m-\frac{n}{2}-1}(\mathbb{R}^n)$, thus we must have that

$$\frac{1}{(1+|x|)^{n-2m}} \int_{\mathbb{R}^n} v(y)\phi(y)dy \in L^2_{-(2m-\frac{n}{2}-1)}(\mathbb{R}^n),$$

then

$$\int_{\mathbb{R}^n} v(y)\phi(y)dy = 0,$$

that is $0 = P\phi = S_1 PS_1 \phi$ and $\phi \in S_2 L^2(\mathbb{R}^n)$ as desired.

(iii) For $j = 3$, assume first that $\phi \in S_3 L^2(\mathbb{R}^n) \setminus \{0\}$. Since $S_3 \leq S_2 \leq S_1$, then by Lemma 8.9 we have

$$\psi(x) = \int_{\mathbb{R}^n} \left[ \frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \right] v(y)\phi(y)dy - \frac{n-2m}{(1+|x|)^{n-2m+1}} - \frac{2x \cdot y}{(1+|x|)^{n-2m+2}} \right] v(y)\phi(y)dy.$$

Since

$$\left| \frac{1}{|x-y|^{n-2m}} - \frac{1}{(1+|x|)^{n-2m}} \right| \lesssim \sum_{l=0}^{n-2m-2} \frac{x^l}{|x|^{n-2m}|x-y|^{n-2m}} + \frac{C^1_{n-2m}|x|^{n-2m}}{(1+|x|)^{n-2m+1}} + \frac{C^1_{n-2m}|x|^{n-2m-1}}{(1+|x|)^{n-2m+1}|x-y|^{n-2m}} \right) + \left| \frac{|x-y|^{n-2m}}{(1+|x|)^{n-2m}} - \frac{2C^1_{\frac{n}{2}-m}x \cdot y}{(1+|x|)^{n-2m+2}} \right|$$

$$:= I + II + III \quad (8.44)$$
For the term $I$ of (8.44), we have
\[
I \lesssim \frac{1}{(1 + |x|)^2|x - y|^{n-2m}}.
\]

For the term $II$ of (8.44), using (8.42), we have
\[
II \lesssim \frac{1}{1 + |x|} \cdot \left| \frac{1}{|x - y|^{n-m}} - \frac{1}{(1 + |x|)^{n-m}} \right| \lesssim \sum_{l=1}^{n-2m} \frac{(1 + |y|)^l}{(1 + |x|)(l + 1)|x - y|^{n-2m}}.
\]

Then for the term $III$ of (8.40), we have
\[
III \lesssim \left( |x|^2 + 2x \cdot y + |y|^2 \right)^{\frac{n}{2} - m} - 2C_1 \frac{|x|^{n-2m} |x - y|^2}{(1 + |x|)^{n-2m}} \lesssim \sum_{l=1}^{n-2m} \frac{(1 + |y|)^{l+1}}{(1 + |x|)^{l+1}|x - y|^{n-2m}}.
\]

Thus Lemma 8.3 shows that $\psi \in W_{2m-\frac{n}{2}}(\mathbb{R}^n)$ as desired.

For $S_j L^2(\mathbb{R}^n)$, Proposition 2.12 holds by the same argument as in Proposition 2.11 for $S_j L^2(\mathbb{R}^n)$ with $3 \leq j \leq k$. 

Acknowledgements: A. Soffer is partially supported by NSFC grant No. 11671163 and NSF grant DMS-1600749. X. H. Yao is partially supported by NSFC (No. 11771165) and the program for Changjiang Scholars and Innovative Research Team in University (IRT13066).

Part of this work was done while the second author was a visiting professor at Central China Normal University (CCNU). The authors would like to thank Dr. S. L. Huang for interesting discussions, who also obtained similar results on the absence of positive embedded eigenvalue.

References

[ABG96] Werner O. Amrein, Anne Boutet de Monvel, and Vladimir Georgescu, Co-groups, commutator methods and spectral theory of $N$-body Hamiltonians, Modern Birkhäuser Classics, Birkhäuser/Springer, Basel, 1996, [2013] reprint of the 1996 edition. MR 3136195

[Agm70] Shmuel Agmon, Lower bounds for solutions of Schrödinger equations, J. Analyse Math. 23 (1970), 1–25. MR 0276624

[Agm75] , Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218. MR 0397194

[AS61] N. Aronszajn and K. T. Smith, Theory of Bessel potentials. I, Ann. Inst. Fourier (Grenoble) 11 (1961), 385–475. MR 0143935

[BS91] M. Sh. Birman and M. Z. Solomyak, Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations, Estimates and asymptotics for discrete spectra of integral and differential equations (Leningrad, 1989–90), Adv. Soviet Math., vol. 7, Amer. Math. Soc., Providence, RI, 1991, pp. 1–55. MR 1306507

[Caz03] Thierry Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. MR 2002047
[CS01] O. Costin and A. Soffer, *Resonance theory for Schrödinger operators*, Comm. Math. Phys. **224** (2001), no. 1, 133–152, Dedicated to Joel L. Lebowitz. MR 1868995

[D’A15] Piero D’Ancona, *Kato smoothing and Strichartz estimates for wave equations with magnetic potentials*, Comm. Math. Phys. **335** (2015), no. 1, 1–16. MR 3314497

[Dav97] E. B. Davies, *Limits on $L^p$ regularity of self-adjoint elliptic operators*, J. Differential Equations **135** (1997), no. 1, 83–102. MR 1434916

[DDY14] Qingquan Deng, Yong Ding, and Xiaohua Yao, *Gaussian bounds for higher-order elliptic differential operators with Kato type potentials*, J. Funct. Anal. **266** (2014), no. 8, 5377–5397. MR 3177340

[DF08] Piero D’Ancona and Luca Fanelli, *Strichartz and smoothing estimates of dispersive equations with magnetic potentials*, Comm. Partial Differential Equations **33** (2008), no. 4-6, 1082–1112. MR 2424390

[DH98a] E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in $L^p(Ω)$*, Math. Z. **227** (1998), no. 3, 511–523. MR 1612685

[DH98b] , *Kato class potentials for higher order elliptic operators*, J. London Math. Soc. (2) **58** (1998), no. 3, 669–678. MR 1678156

[ES04] M. Burak Erdoğan and Wilhelm Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three. I*, Dyn. Partial Differ. Equ. **1** (2004), no. 4, 359–379. MR 2127577

[ES06] , *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three. II*, J. Anal. Math. **99** (2006), 199–248. MR 2279551

[FHHH82] Richard Froese, Ira Herbst, Maria Hoffmann-Ostenhof, and Thomas Hoffmann-Ostenhof, *On the absence of positive eigenvalues for one-body Schrödinger operators*, J. Analyse Math. **41** (1982), 272–284. MR 687957

[FSY18] Hongliang Feng, Avy Soffer, and Xiaohua Yao, *Decay estimates and Strichartz estimates of fourth-order Schrödinger operator*, J. Funct. Anal. **274** (2018), no. 2, 605–658. MR 3724151

[FWY18] Hongliang Feng, Zhao Wu, and Xiaohua Yao, *Time asymptotic expansions of solution for fourth-order schrödinger equation with zero resonance or eigenvalue*, https://arxiv.org/abs/1812.00223 (2018).

[GG15] Michael Goldberg and William R. Green, *Dispersive estimates for higher dimensional Schrödinger operators with threshold eigenvalues I: The odd dimensional case*, J. Funct. Anal. **269** (2015), no. 3, 633–682. MR 3350725

[GG17] , *Dispersive estimates for higher dimensional Schrödinger operators with threshold eigenvalues II: The even dimensional case*, J. Spectr. Theory **7** (2017), no. 1, 33–86. MR 3629407

[GLS16] Vladimir Georgescu, Manuel Larenas, and Avy Soffer, *Abstract theory of pointwise decay with applications to wave and Schrödinger equations*, Ann. Henri Poincaré **17** (2016), no. 8, 2075–2101. MR 3522025

[GS04] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251** (2004), no. 1, 157–178. MR 2096737

[GT18] William R. Green and Ebru Toprak, *On the fourth order schrödinger equation in four dimensions: dispersive estimates and zero energy resonance*, https://arxiv.org/abs/1810.03678 (2018).

[GV06] Michael Goldberg and Monica Visan, *A counterexample to dispersive estimates for Schrödinger operators in higher dimensions*, Comm. Math. Phys. **266** (2006), no. 1, 211–238. MR 2231971

[Hö05] Lars Hörmander, *The analysis of linear partial differential operators. II*, Classics in Mathematics, Springer-Verlag, Berlin, 2005, Differential operators with constant coefficients, Reprint of the 1983 original. MR 2108588

[HS15] I. Herbst and E. Skibsted, *Decay of eigenfunctions of elliptic PDE’s, I*, Adv. Math. **270** (2015), 138–180. MR 3286533

[HS17] , *Decay of eigenfunctions of elliptic PDE’s, II*, Adv. Math. **306** (2017), 177–199. MR 3581301
A. D. Ionescu and D. Jerison, *On the absence of positive eigenvalues of Schrödinger operators with rough potentials*, Geom. Funct. Anal. 13 (2003), no. 5, 1029–1081. MR 2024415

Arne Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions results in $L^2(\mathbb{R}^m)$, $m \geq 5$*, Duke Math. J. 47 (1980), no. 1, 57–80. MR 563367

Arne Jensen and Tosio Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J. 46 (1979), no. 3, 583–611. MR 544248

Arne Jensen and Gheorghe Nenciu, *A unified approach to resolvent expansions at thresholds*, Rev. Math. Phys. 13 (2001), no. 6, 717–754. MR 1841744

J.-L. Journé, A. Soffer, and C. D. Sogge, *Decay estimates for Schrödinger operators*, Comm. Pure Appl. Math. 44 (1991), no. 5, 573–604. MR 1105875

Tosio Kato, *Growth properties of solutions of the reduced wave equation with a variable coefficient*, Comm. Pure Appl. Math. 12 (1959), 403–425. MR 0108633

JinMyong Kim, Anton Arnold, and Xiaohua Yao, *Global estimates of fundamental solutions for higher-order Schrödinger equations*, Monatsh. Math. 168 (2012), no. 2, 253–266. MR 2984150

Alexander Komech and Elena Kopylova, *Dispersion decay and scattering theory*, John Wiley & Sons, Inc., Hoboken, NJ, 2012. MR 3015024

Markus Keel and Terence Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), no. 5, 955–980. MR 1646048

Herbert Koch and Daniel Tataru, *Carleman estimates and absence of embedded eigenvalues*, Comm. Math. Phys. 267 (2006), no. 2, 419–449. MR 2252331

S. T. Kuroda, *An introduction to scattering theory*, Lecture Notes Series, vol. 51, Aarhus Universitet, Matematisk Institut, Aarhus, 1978. MR 528757

Manuel Larenas and Avy Soffer, *Abstract theory of decay estimates: perturbed hamiltonians*, https://arxiv.org/abs/1508.04490 (2015).

E. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Comm. Math. Phys. 78 (1980/81), no. 3, 391–408. MR 603501

Minoru Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations*, J. Funct. Anal. 49 (1982), no. 1, 10–56. MR 680855

———, *High energy resolvent estimates. II. Higher order elliptic operators*, J. Math. Soc. Japan 36 (1984), no. 1, 1–10. MR 723588

Jacob S. Møller and Matthias Westrich, *Regularity of eigenstates in regular Mourre theory*, J. Funct. Anal. 260 (2011), no. 3, 852–878. MR 2737399

Michael Reed and Barry Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR 0493421

Igor Rodnianski and Wilhelm Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. 155 (2004), no. 3, 451–513. MR 2038194

Martin Schechter, *Spectra of partial differential operators*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1971, North-Holland Series in Applied Mathematics and Mechanics, Vol. 14. MR 0447834

W. Schlag, *Dispersive estimates for Schrödinger operators in dimension two*, Comm. Math. Phys. 257 (2005), no. 1, 87–117. MR 2163570

———, *Dispersive estimates for Schrödinger operators: a survey*, Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud., vol. 163, Princeton Univ. Press, Princeton, NJ, 2007, pp. 255–285. MR 2333215

Barry Simon, *On positive eigenvalues of one-body Schrödinger operators*, Comm. Pure Appl. Math. 22 (1969), 531–538. MR 0247300

———, *Harmonic analysis*, A Comprehensive Course in Analysis, Part 3, American Mathematical Society, Providence, RI, 2015. MR 3410783
DECAY ESTIMATES FOR HIGHER ORDER ELLIPTIC OPERATORS

[Sim15b] ______, Operator theory, A Comprehensive Course in Analysis, Part 4, American Mathematical Society, Providence, RI, 2015. MR 3364494

[Sim18a] ______, Tosio Kato’s work on non-relativistic quantum mechanics: part 1, Bull. Math. Sci. 8 (2018), no. 1, 121–232. MR 3775269

[Sim18b] ______, Tosio Kato’s work on non-relativistic quantum mechanics: part 2, Bull. Math. Sci. (2018), In Press.

[Ste70] Elias M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR 0290095

[Ste93] ______, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192

[SYY18] Adam Sikora, Lixin Yan, and Xiaohua Yao, Spectral multipliers, Bochner-Riesz means and uniform Sobolev inequalities for elliptic operators, Int. Math. Res. Not. IMRN (2018), no. 10, 3070–3121. MR 3805196

[Tao06] Terence Tao, Nonlinear dispersive equations, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, Local and global analysis. MR 2233925

[Wig93] Eugene Paul Wigner, The collected works of Eugene Paul Wigner. Part A. The scientific papers. Vol. I, Springer-Verlag, Berlin, 1993, With a preface by Jagdish Mehra and Arthur S. Wightman, With a biographical sketch by Mehra, and annotation by Brian R. Judd and George W. Mackey, Edited by Wightman. MR 1383096

[Yaj95] Kenji Yajima, The Wk,p-continuity of wave operators for Schrödinger operators, J. Math. Soc. Japan 47 (1995), no. 3, 551–581. MR 1331331

Hongliang Feng, School of Mathematics, Sun Yat-sen University, Guangzhou, 510275, P.R. China

E-mail address: fenghongliang@aliyun.com

Avy Soffer, School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P.R. China, On leave from Rutgers University

E-mail address: soffer@math.rutgers.edu

Zhao Wu, School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P.R. China

E-mail address: wuzhao218@yahoo.com

Xiaohua Yao, Department of Mathematics and Hubei Province Key Laboratory of Mathematical Physics, Central China Normal University, Wuhan, 430079, P.R. China

E-mail address: yaoxiaohua@mail.ccnu.edu.cn