Estimations of Order $\alpha_s^3$ and $\alpha_s^4$ Corrections to Mass-Dependent Observables

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Abstract

A simple procedure to estimate $O(\alpha_s^3)$ and $O(\alpha_s^4)$ corrections to mass-dependent observables is conjectured. The method is tested in a number of cases where the $O(\alpha_s^3)$ contribution is exactly known, and reasonable agreement is found.

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Given the great difficulty of evaluating higher-order corrections, it is very desirable to have reasonable methods to estimate their sign and magnitude. In fact, significant and interesting investigations in this subject have been carried out in the past \[1,2,3,4,5,6\]. The aim of this note is to propose a simple estimation method to treat an important class of mass-dependent observables.

We first recall salient features of the estimation methods proposed in the literature. Calling \( R(s) \) an observable depending on a single time-like kinematic variable \( s \), such as a squared centre-of-mass energy, we consider the QCD expansion

\[
R(s) = \sum_{n=0}^{\infty} R_n(s, \mu^2) a^n(\mu^2), \tag{1}
\]

where \( a(\mu^2) = \alpha_s^{(n_f)}(\mu^2)/\pi \), \( \mu \) is the renormalization scale, and \( n_f \) is the number of flavours active at that scale. By factoring out an appropriate power of \( s \), it is always possible to render \( R(s) \) dimensionless. Henceforth, we shall adopt this convention. As \( R(s) \) is renormalization-group invariant, we may choose \( \mu^2 = s \), in which case Eq. (1) becomes

\[
R(s) = \sum_{n=0}^{\infty} r_n a^n(s), \tag{2}
\]

where \( r_n = R_n(s, s) \). If \( R(s) \) does not depend on masses or other kinematical variables, the \( r_n \) are numerical constants. Estimations of higher-order corrections using optimization procedures based on the fastest apparent convergence (FAC) \[1\] and the principle of minimal sensitivity (PMS) \[2\] have been carried out in two main scenarios:

(i) Suppose that \( r_0, r_1, \) and \( r_2 \) are known. Then the FAC and PMS approaches lead to the estimates \[1,2\]

\[
\begin{align*}
 r_3^{\text{FAC}} &= r_2 \left( \frac{r_2}{r_1} + \frac{\beta_1}{\beta_0} \right), \\
 r_3^{\text{PMS}} &= r_1 \left( \frac{r_2}{r_1} + \frac{\beta_1}{2\beta_0} \right)^2,
\end{align*}
\tag{3}
\]

where

\[
\beta_0 = \frac{1}{4} \left( 11 - \frac{2}{3} n_f \right), \quad \beta_1 = \frac{1}{16} \left( 102 - \frac{38}{3} n_f \right) \tag{4}
\]

are the first two coefficients of the QCD \( \beta \) function.

(ii) Suppose that \( r_0, r_1, r_2, \) and \( r_3 \) are known. Then in both the FAC and PMS methods one finds \[1,2,3,4\]

\[
 r_4^{\text{FAC/PMS}} = r_2 \left( 3 \frac{r_3}{r_1} - 2 \frac{r_2^2}{r_1^2} - \frac{r_2 \beta_1}{2r_1 \beta_0} + \frac{\beta_2}{\beta_0} \right), \tag{5}
\]

where, in the \( \overline{\text{MS}} \) scheme, \[5\]

\[
\beta_2 = \frac{1}{64} \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right). \tag{6}
\]
If $r_3$ is not known, we may consider employing the estimates from Eq. (3). Inserting these expressions into Eq. (5), one finds

\[
\begin{align*}
r_4^{\text{FAC}} &= r_2 \left( \frac{r_2^2}{r_1^2} + \frac{5r_2\beta_1}{2r_1\beta_0} + \frac{\beta_2}{\beta_0} \right), \\
r_4^{\text{PMS}} &= r_2 \left( \frac{r_2^2}{r_1^2} + \frac{5r_2\beta_1}{2r_1\beta_0} + \frac{\beta_2}{\beta_0} + \frac{3\beta_2^2}{4\beta_0^2} \right). 
\end{align*}
\]

(7)

Formally, the FAC and PMS procedures could be applied in both Minkowskian and Euclidean spaces. However, on general grounds it is expected that the optimization procedures are more accurate when applied in Euclidean space, as one avoids the presence of physical thresholds. Accordingly, when the theoretical expansion for an observable is given Minkowskian space, it has been proposed [4] to apply the optimization procedures to an associated function defined in the Euclidean region, namely

\[
D(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{(s + Q^2)^2},
\]

(8)

with $Q^2 \geq 0$. $R(s)$ admits the inverse integral representation

\[
R(s) = \frac{1}{2\pi i} \int_{-s-i\epsilon}^{-s+i\epsilon} ds' \frac{D(s')}{s'},
\]

(9)

where $D(s')$ is the analytic continuation of $D(Q^2)$ to the complex plane. Inserting the expansion of Eq. (1) into Eq. (8), carrying out the integration, and then setting $\mu^2 = Q^2$, one finds

\[
D(Q^2) = \sum_{n=0}^{\infty} d_n a^n(Q^2),
\]

(10)

where

\[
\begin{align*}
d_0 &= r_0, & d_1 &= r_1, & d_2 &= r_2, \\
d_3 &= r_3 + \frac{\pi^2}{3} r_1^2 \beta_0^2, & d_4 &= r_4 + \pi^2 \beta_0 \left( r_2 \beta_0 + \frac{5}{6} r_1 \beta_1 \right).
\end{align*}
\]

(11)

(12)

In the Euclidean approach, one estimates $d_3$ and $d_4$ using the expressions analogous to Eqs. (3), (4), and (7) with $r_n$ replaced by $d_n$, and obtains $r_3$ and $r_4$ from Eq. (12).

At this point, we turn our attention to mass-dependent observables of the form

\[
T(s) = m^2(\mu^2) \sum_{n=0}^{\infty} T_n(s, \mu^2) a^n(\mu^2),
\]

(13)

where $m(\mu^2)$ is a running quark mass. Again, without loss of generality, we may assume that the $T_n(s, \mu^2)$ are dimensionless. Setting $\mu^2 = s$ and defining $t_n = T_n(s, s)$, we have

\[
T(s) = m^2(s) \sum_{n=0}^{\infty} t_n a^n(s).
\]

(14)
If \( T(s)/m^2(s) \) does not depend on masses or other kinematic variables, the \( t_n \) are numerical constants. The associated function defined in the Euclidean region is

\[
F(Q^2) = Q^2 \int_0^{\infty} ds \frac{T(s)}{(s + Q^2)^2}.
\]

In order to carry out the \( s \) integration in Eq. (15), we substitute in Eq. (14) the expansions

\[
a(s) = a(\mu^2) \left\{ 1 - a(\mu^2)\beta_0 \ell + a^2(\mu^2)\ell(\beta_0^2 - \beta_1) + a^3(\mu^2)\ell \left( - \frac{\beta_0^2 \ell^2}{2} + \frac{5}{2} \beta_0 \beta_1 \ell - \beta_2 \right) \right. \\
+ a^4(\mu^2)\ell \left[ \frac{\beta_0^4 \ell^3}{3} - \frac{13}{3} \beta_0^2 \beta_1 \ell^2 + 3 \left( \frac{\beta_0^2}{2} + \beta_0 \beta_2 \right) \ell - \beta_3 \right] + O(a^5\ell^5) \}.
\]

\[
m(s) = m(\mu^2) \left\{ 1 - a(\mu^2)\gamma_0 \ell + a^2(\mu^2)\ell \left[ \frac{\gamma_0}{2} (\beta_0 + \gamma_0) \ell - \gamma_1 \right] \right.
+ a^3(\mu^2)\ell \left[ \frac{-\gamma_0}{3} (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) \ell^2 + \left( \frac{\beta_1 \gamma_0}{2} + \gamma_1 (\beta_0 + \gamma_0) \right) \ell - \gamma_2 \right] \\
+ a^4(\mu^2)\ell \left[ \frac{\gamma_0}{4} (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) \ell^3 \\
- \left( \frac{\beta_0 \gamma_0}{2} + \gamma_1 (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) \right) \ell^2 \\
+ \left( \frac{\beta_0 \gamma_0}{2} + \gamma_1 \left( \beta_0 + \frac{\gamma_0}{2} \right) \right) + \gamma_2 \left( \frac{3}{2} \beta_0 \gamma_0 \right) \ell - \gamma_3 \right] + O(a^5\ell^5) \}.
\]

where \( \ell = \ln(s/\mu^2) \) and, in the \( \overline{\text{MS}} \) scheme, the coefficients of the mass anomalous dimension are

\[
\gamma_0 = 1, \quad \gamma_1 = \frac{1}{16} \left( \frac{202}{3} - \frac{20}{9} n_f \right), \\
\gamma_2 = \frac{1}{64} \left( 1249 - \frac{2216}{27} + \frac{160}{3} \zeta(3) \right) n_f - \frac{140}{81} n_f^2.
\]

Here, \( \zeta \) is Riemann’s zeta function, with value \( \zeta(3) \approx 1.20206 \). The coefficient \( \gamma_3 \) is presently unknown. For completeness, we have also presented the \( O(\alpha^4) \) term in Eq. (16), although we shall not need it here. These substitutions bring Eq. (14) into the form of Eq. (13) with coefficient functions \( T_n(s, \mu^2) \) that depend on \( s \) only through powers of \( \ell \). Inserting the expression thus obtained into Eq. (13) and using the elementary integrals

\[
Q^2 \int_0^{\infty} ds \frac{1; \ell; \ell^2; \ell^3; \ell^4}{(s + Q^2)^2} = \left\{ 1; L; \frac{\pi^2}{3}; L^2 + \frac{\pi^2}{3}; L^3 + \pi^2 L; L^4 + 2\pi^2 L^2 + \frac{7\pi^4}{15} \right\},
\]

where \( L = \ln(Q^2/\mu^2) \), one obtains an expansion of the form

\[
F(Q^2) = m^2(\mu^2) \sum_{n=0}^{\infty} F_n(Q^2, \mu^2) \alpha^n(\mu^2).
\]
Setting $\mu^2 = Q^2$, Eq. (20) becomes

$$F(Q^2) = m^2(Q^2) \sum_{n=0}^{\infty} f_n a^n(Q^2),$$  \hspace{1cm} (21)$$

where $f_n = F_n(Q^2, Q^2)$ are numerical constants. Specifically, one finds

$$f_0 = t_0, \quad f_1 = t_1, \quad f_2 = t_2 + \frac{\pi^2}{3} t_0 \gamma_0 (\beta_0 + 2 \gamma_0),$$  \hspace{1cm} (22)$$

$$f_3 = t_3 + \frac{\pi^2}{3} \{ t_1 (\beta_0 + \gamma_0) (\beta_0 + 2 \gamma_0) + t_0 [\beta_1 \gamma_0 + 2 \gamma_1 (\beta_0 + 2 \gamma_0)] \},$$  \hspace{1cm} (23)$$

$$f_4 = t_4 + \frac{\pi^2}{3} \{ t_2 (\beta_0 + \gamma_0) \left( \beta_0 + \frac{2}{3} \gamma_0 \right) + t_1 \left[ \beta_1 \left( \frac{5}{6} \beta_0 + \gamma_0 \right) + \frac{4}{3} \gamma_1 (\beta_0 + \gamma_0) \right] + t_0 \left[ \frac{\beta_2 \gamma_0}{3} + \frac{2}{3} \gamma_1 (\beta_1 + \gamma_0) + \gamma_2 \left( \beta_0 + \frac{4}{3} \gamma_0 \right) \right] \} + \frac{7\pi^4}{15} t_0 \gamma_0 (\beta_0 + \gamma_0) (\beta_0 + 2 \gamma_0) \left( \frac{\beta_0}{2} + \gamma_0 \right).$$  \hspace{1cm} (24)$$

The case of Eqs. (1) and (2), in which the $m^2(\mu^2)$ factor is not present, can be obtained from Eqs. (22)–(24) by setting $\gamma_i = 0$ ($i = 0, 1, 2$). In fact, the relations between the $f_n$ and $t_n$ then become identical to those between the $d_n$ and $r_n$ in Eqs. (11) and (12).

Our proposal is to apply the estimation procedure described before to the $f_n$ expansion of Eq. (21) and to obtain the corresponding $t_n$ coefficients via Eqs. (22) and (24). For instance, if the $t_n$ are known for $n \leq 2$, we obtain $f_n$ for $n \leq 2$ from Eq. (22) and estimate $f_3$ using the expressions analogous to Eq. (3) with $r_n$ replaced by $f_n$. The estimate for $t_3$ is then obtained from Eq. (23). If the $t_n$ are known for $n \leq 3$, we obtain $f_n$ for $n \leq 3$ from Eqs. (22) and (23), $f_4$ is estimated from the expression analogous to Eq. (3) with $r_n$ replaced by $f_n$, and $t_4$ follows from Eq. (24). If $t_3$ is not known, we may also attempt to estimate $f_4$ from the expressions analogous to Eq. (4), and $t_4$ once more from Eq. (24). This proposal essentially relates the estimation of the higher-order coefficients in the mass-dependent expansion to the previously considered mass-independent case. It should be pointed out that there is an element of arbitrariness in this approach. In Eq. (20), we have set $\mu^2 = Q^2$ and proposed to apply the optimization procedure to $\sum_{n=0}^{\infty} f_n a^n(Q^2)$, the cofactor of $m^2(Q^2)$. Had we chosen a different scale $\mu^2 \neq Q^2$ in Eq. (20), the expansion in Eq. (21) and the estimation procedure would be different. On the other hand, the choice $\mu^2 = Q^2$ seems natural and convenient, as all the logarithms in the $F_n(Q^2, \mu^2)$ vanish. In fact, this feature has an additional very useful property: it renders the analysis of $t_4$ independent of the unknown coefficient $\gamma_3$.

An interesting application is the estimation of the $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^4)$ coefficients in the evaluation of the partial width $\Gamma(H \rightarrow \text{hadrons})$ involving final-state quarks with running mass $m(\mu^2) \ll M_H$. The relevant expansion [10] is of the form of Eq. (14) with...
Table 1: Estimations of $t_3$ and $t_4$ in $\Gamma(H\to q\bar{q})$ based on the FAC and PMS optimizations of the associated function $F(Q^2)/m^2(Q^2)$, defined in the Euclidean region, and Eqs. (22)–(24). The estimation of $t_4$ employs the exact value of $t_3$.

| $n_f$ | $t_3^{\text{exact}}$ | $t_3^{\text{FAC}}$ | $t_3^{\text{PMS}}$ | $t_4^{\text{FAC/PMS}}$ |
|-------|-----------------------|---------------------|---------------------|----------------------|
| 3     | 89.156                | 75.729              | 80.206              | −945.28             |
| 4     | 65.198                | 64.956              | 68.316              | −1098.8             |
| 5     | 41.758                | 53.295              | 55.547              | −1237.4             |

Table 2: As in Table 1, but using the original function $T(s)/m^2(s)$, defined in the Minkowskian region.

| $n_f$ | $t_3^{\text{exact}}$ | $t_3^{\text{FAC}}$ | $t_3^{\text{PMS}}$ | $t_4^{\text{FAC/PMS}}$ |
|-------|-----------------------|---------------------|---------------------|----------------------|
| 3     | 89.156                | 235.82              | 240.30              | −527.81             |
| 4     | 65.198                | 211.20              | 214.56              | −748.62             |
| 5     | 41.758                | 186.67              | 188.92              | −949.39             |

\[ \sqrt{s} = M_H, \ t_0 = 1, \ t_1 = 17/3, \text{ and } t_2 \approx 35.93996 - 1.35865\ n_f, \] where $n_f$ is the number of active flavours at scale $\sqrt{s} = M_H$. The calculation assumes that there is one massive flavour, identical with that present in the final state of the reaction $H \to q\bar{q}$, and $n_f - 1$ massless ones. Table 1 compares the $t_3$ estimates of the proposed procedure, based on the FAC and PMS optimizations of $F(Q^2)/m^2(Q^2)$, with the exact result and also provides the $t_4$ estimate obtained using the exact value of $t_3$. Table 2 displays the corresponding estimations based on the optimization of the original function $T(s)/m^2(s)$ defined in the Minkowskian domain. We see that the predicted signs for $t_3$ are correct, but it is apparent that the magnitude of the estimations is much closer to the exact result in the Euclidean approach. In fact, the estimations of $t_3$ in Table 1 are fairly good: $t_3^{\text{FAC}}$ shows errors of $(-15, -0.4, +28)\%$ for $n_f = 3, 4, 5$, respectively; for $t_3^{\text{PMS}}$ the corresponding errors are $(-10, +5, +33)\%$. We also see that the $t_4$ coefficients are predicted to be large and negative.

There are two other cases in which $t_3$ is exactly known: these are the terms proportional to $m_0^2$ in the absorptive parts of the axial-vector correlators pertinent to the parton-level decays $W^+ \to cs$ and $Z \to bb$. Here, $m_c$ is evaluated with $n_f = 4$ at $\sqrt{s} = M_W$ and $m_b$ with $n_f = 5$ at $\sqrt{s} = M_Z$. In either case, it is assumed that the remaining $n_f - 1$ quarks are massless. The corresponding coefficients are $t_1 = 5/3$, $t_2 \approx -3.06004 - 0.02532\ n_f$. 

\[ \sqrt{s} = M_H, \]
Table 3: Estimations of $t_3$ and $t_4$ in the $m_q^2$ contribution to $\Gamma(W^+ \to cs)$ based on the FAC and PMS optimizations of the associated function $F(Q^2)/m^2(Q^2)$, defined in the Euclidean region, and Eqs. (22)–(24). The estimation of $t_4$ employs the exact value of $t_3$.  

| $n_f$ | $t_3^{\text{exact}}$ | $t_3^{\text{FAC}}$ | $t_3^{\text{PMS}}$ | $t_4^{\text{FAC/PMS}}$ |
|-------|------------------|------------------|------------------|------------------|
| 3     | -87.394          | -105.06          | -103.75          | -582.03          |
| 4     | -83.356          | -98.597          | -97.609          | -487.93          |
| 5     | -79.598          | -92.475          | -91.813          | -400.78          |

Table 4: As in Table 3, but for $Z \to b\bar{b}$.

| $n_f$ | $t_3^{\text{exact}}$ | $t_3^{\text{FAC}}$ | $t_3^{\text{PMS}}$ | $t_4^{\text{FAC/PMS}}$ |
|-------|------------------|------------------|------------------|------------------|
| 3     | 32.096           | 8.6895           | 11.587           | -331.80          |
| 4     | 20.311           | 4.2754           | 6.4494           | -400.40          |
| 5     | 8.6525           | -0.70475         | 0.75256          | -464.41          |

for $W^+ \to cs$ and $t_1 = 11/3, t_2 \approx 18.02329 - 0.74754 n_f$ for $Z \to b\bar{b}$. Tables 3 and 4 show the FAC and PMS estimations for these transitions in the Euclidean approach. For illustration, we also consider other values of $n_f$. In the case of $W^+ \to cs$, we see once more that the $t_3$ estimations are fairly good, with relative errors of 20% or below. Instead, in the case of $Z \to b\bar{b}$, the relative errors are large. We note, however, that in this case both the exact and estimated $t_3$ values are relatively small. In fact, a simple, way to characterize Tables 1, 3, and 4 is to say that the $t_3$ estimations have absolute errors of order 20 or below. When $t_3^{\text{exact}}$ is large, as in Tables 1 and 3, this leads to fairly accurate results.

There are some important cases in which the $t_3$ coefficients are not known. Examples include mass relations of the form

$$M_q = \mu_q \sum_{n=0}^{\infty} t_n a^n(\mu_q^2),$$  \hspace{1cm} (25)$$

where $M_q$ is the pole mass and $\mu_q = m_q(\mu_q^2)$ is the $\overline{\text{MS}}$ mass of quark $q$. In contrast to the previous applications, Eq. (25) does not depend on an external mass parameter such as $M_H, M_W$, or $M_Z$ which, in principle, can have arbitrary values independent of $m_q$. On the other hand, defining $T(s) = m_q(s) \sum_{n=0}^{\infty} t_n a^n(s)$ for arbitrary $s \geq 0$, we have the
Table 5: Estimations of $t_3$ and $t_4$ in Eq. (25) based on the FAC and PMS optimizations of the associated function $F(Q^2)/m(Q^2)$, defined in the Euclidean region, and Eqs. (27)–(29).

| $n_f$ | $t_3^{FAC}$ | $t_3^{PMS}$ | $t_4^{FAC}$ | $t_4^{PMS}$ |
|-------|------------|------------|------------|------------|
| 3     | 152.71     | 153.76     | 2083.8     | 2123.4     |
| 4     | 124.10     | 124.89     | 1544.1     | 1571.4     |
| 5     | 97.729     | 98.259     | 1091.0     | 1107.8     |
| 6     | 73.616     | 73.903     | 718.74     | 727.00     |

Table 5: Estimations of $t_3$ and $t_4$ in Eq. (25) based on the FAC and PMS optimizations of the associated function $F(Q^2)/m(Q^2)$, defined in the Euclidean region, and Eqs. (27)–(29).

mathematical identity

$$M_q = \frac{1}{2\pi i} \int_{-\mu q+i\epsilon}^{-\mu q-i\epsilon} ds' \int_{0}^{\infty} ds \frac{T(s)}{(s+s')^2}. \quad (26)$$

In analogy with the previous applications, we introduce the function $F(Q^2)$ defined by Eq. (15) in the Euclidean region $Q^2 \geq 0$. Because of their linear dependence on $m_q$, the relations between the $f_n$ and $t_n$ are different from those in Eqs. (22)–(24). We now have

$$f_0 = t_0, \quad f_1 = t_1, \quad f_2 = t_2 + \frac{\pi^2}{6} t_0 \gamma_0 (\beta_0 + \gamma_0), \quad (27)$$

$$f_3 = t_3 + \frac{\pi^2}{3} \left\{ t_1 (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) + t_0 \left[ \frac{\beta_1 \gamma_0}{2} + \gamma_1 (\beta_0 + \gamma_0) \right] \right\}, \quad (28)$$

$$f_4 = t_4 + \frac{\pi^2}{3} \left\{ t_2 \left( \beta_0 + \frac{\gamma_0}{2} \right) \left( \beta_0 + \frac{\gamma_0}{3} \right) + t_1 \left[ \frac{\beta_1}{2} \left( \frac{5}{3} \beta_0 + \gamma_0 \right) + \frac{\gamma_1}{3} (2\beta_0 + \gamma_0) \right] \right. + t_0 \left[ \frac{\beta_2 \gamma_0}{6} + \frac{\gamma_1}{3} (\beta_1 + \frac{\gamma_1}{2}) + \gamma_2 \left( \frac{\beta_0}{2} + \frac{\gamma_0}{3} \right) \right] \right\} \right.$$

$$\left. + \frac{7\pi^4}{60} t_0 \gamma_0 (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) \left( \beta_0 + \frac{\gamma_0}{3} \right) \right\} \right. + \frac{7\pi^4}{60} t_0 \gamma_0 (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) \left( \beta_0 + \frac{\gamma_0}{3} \right). \quad (29)$$

Assuming once more that $n_f - 1$ quarks are massless, we have in the case of Eq. (24) $t_0 = 1$, $t_1 = 4/3$, and $t_2 \approx 14.48476 - 1.04137 n_f$ [12]. Table 5 displays the coefficients $t_3$ and $t_4$ in Eq. (25) estimated by the FAC and PMS optimizations of the associated function $F(Q^2)/m(Q^2)$, using Eqs. (13) and (14) with $r_n$ replaced by $f_n$. The entry for $n_f = 6$ corresponding to $M_t/m_t$, where $t$ is the top quark, gives an expansion very close to that obtained in a different approach based on an optimization of the ratios $M_t/m_t(M^2_t)$ and $m_t(M^2_t)/\mu_t$ [13]. For example, if the expansion is made in powers of $a(M^2_t)$ rather than $a(\mu^2_t)$, the FAC result in Table 5 becomes

$$M_t = \mu_t \left[ 1 + \frac{4}{3} a(M^2_t) + 8.2366 a^2(M^2_t) + 79.838 a^3(M^2_t) + 835.69 a^4(M^2_t) + \mathcal{O}(a^5) \right], \quad (30)$$
while the approach of Ref. [13] leads to

\[
M_t = \mu_t \left[ 1 + \frac{4}{3} a(M_t^2) + 8.2366 a^2(M_t^2) + 76.172 a^3(M_t^2) + 797.95 a^4(M_t^2) + \mathcal{O}(a^5) \right].
\]  

(31)

It is well known that, if expressed in terms of \( \mu_t \), the QCD corrections to \( \Delta \rho_f \), the fermionic contributions to the electroweak \( \rho \) parameter, are of the form [14]

\[
\Delta \rho_f = \frac{3G_\mu \mu_t^2}{8\pi^2 \sqrt{2}} \left[ 1 - 0.19325 a(M_t^2) - 3.9696 a^2(M_t^2) + \mathcal{O}(a^3) \right],
\]  

(32)

where \( G_\mu \) is Fermi’s constant. Most of the second-order coefficient in Eq. (32), \(- 4.2072\), arises from the opening of a new channel, namely the double-triangle diagram. It is clear that at present there is no sufficient information to optimize Eq. (32). However, if one makes the reasonable assumption that the higher-order terms in Eq. (32) follow the pattern of rather small coefficients shown by the leading contributions, we can combine this result with Eqs. (30) or (31) to estimate \( t_3 \) and \( t_4 \) in \( \Delta \rho_f / M_t^2 \). Substituting Eq. (30) into Eq. (32), one finds

\[
\Delta \rho_f = \frac{3G_\mu M_t^2}{8\pi^2 \sqrt{2}} \left[ 1 - 2.8599 a(M_t^2) - 14.594 a^2(M_t^2) - 90.527 a^3(M_t^2) - 924.88 a^4(M_t^2) + \mathcal{O}(a^5) \right].
\]  

(33)

The pattern of rapidly increasing coefficients of the same sign displayed in Eqs. (30), (31), and (33) also emerges from the analysis of infrared-renormalon contributions [17]. As pointed out in Ref. [13], expansions with much better convergence properties can be obtained by optimizing the scale \( \mu \) at which \( a(\mu^2) \) is evaluated.

In summary, we have presented a procedure to estimate \( \mathcal{O}(\alpha_s^3) \) and \( \mathcal{O}(\alpha_s^4) \) corrections to a class of mass-dependent observables of the type shown in Eqs. (13) and (14). Although there are elements of arbitrariness in its construction, the proposed algorithm, based on the optimization of associated expansions in the Euclidean region, is quite simple and obviates the dependence on the unknown coefficient \( \gamma_3 \) of the mass anomalous dimension. In the cases where the \( t_3 \) coefficients are exactly known, the proposed algorithm estimates \( t_3 \) with absolute errors of order 20 or below. In two of the three cases considered, \( t_3^{\text{exact}} \) is large, and the estimations are fairly accurate, with reasonable relative errors. We have then generalized the estimation algorithm to important expansions of the form of Eq. (23), where the \( t_3 \) coefficients are so far unknown. The corresponding \( t_3 \) and \( t_4 \) estimations turn out to be close to those found in recent analyses based on alternative optimizations procedures.

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