Investigation of the kinetic equation of cascade fragmentation theory at not self-similar subdivision

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Introduction. In this work we investigate the cascade fragmentation process of material sample. This process is the particular case of the cascade kinetic processes which arise in different fields of theoretical physics (see, for example, [1]). In spite of the fact that cascade type of fragmentation is the simplest one and the fragmentation which has no the cascade character is more widespread in nature, such processes are important from the physical point of view and have been not completely studied. In particular, it is not studied the cascade fragmentation process that is not completely self-similar in contrast to the analyzed one in the classical work [2]. In the present communication, we investigate the fragmentation being not self-similar that it is analogous to the process introduced in the paper [3].

General equation. We describe the material fragmentation process by the particle density \( n(r, t) \). The value \( n(r, t) \, dr \) is the average fragment number which have their sizes in the interval \((r, r + dr)\) at the time moment \( t \). The fragmentation process is determined completely by the conditional probability density \( P(\rho, r, t) \) of the splinter formation of given fragment. More strictly, the probability of the formation of splinter that has the size in the interval \((\rho, \rho + d\rho)\) during the time interval \((t, t + dt)\) is equal to \( P(\rho, r, t) \, d\rho \, dt \) when it is occurred the decay of the fragment with the size \( r \). We introduce also the intensity \( \mu(r, t) \) of the fragment decay. It is the average number of decays of fragments with the size \( r \) in the time interval \((t, t + dt)\).

The evolution of the density \( n(r, t) \) is described in terms of values pointed out and on the base of the kinetic equation

\[
\dot{n}(r, t) = \int_{r}^{\infty} P(r, \rho, t) n(\rho, t) \, d\rho - \mu(r, t) n(r, t).
\]

In the article [4], it has been studied the fragmentation process with the scale-invariant subdivision having temporally independent conditional probability
density of transitions
\[ P(\rho, r, t) = \frac{1}{r} P(\rho/r) . \]

It has been done following the idea contained in the pioneer work [2]. At the present communication, it is considered the fragmentation process that has no the scale-invariant subdivision when the transition probability density has the form

\[ P(\rho, r, t) = P(\rho/r) \tag{2} \]
as it has been done in the paper [3]. So, the chosen subdivision scale exists since, in this case, the density \( P(\rho, r, t) \) has the physical dimensionality of the inverse length. Following to works [4], [5], we take into account the volume conservation in the fragment system

\[ V = \int_0^\infty r^3 n(r, t) dr = \text{const}. \tag{3} \]

This property gives the next relation for the intensity

\[ \mu(r, t) = \int_0^r P(\rho/r) \left( \frac{\rho}{r} \right)^3 \, d\rho = r \int_0^1 P(x)x^3 \, dx \equiv \mu r, \quad \mu = \text{const}. \tag{4} \]

Thus, we consider the next kinetic equation for the particle density

\[ \dot{n}(r, t) = \int_r^\infty P(r/\rho)n(\rho, t) d\rho - \mu r n(r, t) . \tag{5} \]

**The limit distribution.** The formula (2) permits to use the Mellin transformation

\[ M(s, t) = \int_0^\infty r^{s-1} n(r, t) dr \]

for study some solutions of the equation (5). In this case, \( M(1, t) = N(t) \) is the average fragment number at the moment \( t \); \( M(2, t) = N(t)\lambda(t) \) where
$\lambda(t)$ is the average fragment size; $M(3, t) = S(t)$ is the total fragment surface area; $M(4, t) = V = \text{const}$ is the total fragment volume.

The application of the Mellin transformation to both parts of the equation (5) leads to the equation

$$\dot{M}(s, t) = M(s + 1, t)(p(s) - \mu),$$

where $p(s) = \int_0^1 x^{s-1}P(x)dx$ and $\mu = p(4)$ according to (4). In particular,

$$M(3, t) = M(3, 0) + V(p(3) - p(4))t$$

since $M(4, t) = V = \text{const}$. The asymptotic behavior of the distribution $f(r, t)$ density at $t \to \infty$ is of interest for us. In this case, it is sufficient to use the second term $M(3, t) \sim V(p(3) - p(4))t$ only in the last formula. From equations (6) and (7), we find

$$M(2, t) \sim \frac{1}{2}V(p(3) - p(4))(p(2) - p(4))t^2 = N(t)\lambda(t),$$

$$M(1, t) \sim \frac{1}{6}V(p(3) - p(4))(p(2) - p(4))(p(1) - p(4))t^3 = N(t).$$

Therefore, it follows

$$\lambda(t) \sim \frac{M(2, t)}{M(1, t)} = \frac{3}{p(1) - p(4)} \frac{1}{t}.$$

**The analysis of the model $P(x) \sim x^\alpha$.** In general case, the study of solutions of the differential equation (6) is sufficiently difficult. However, for the probability distribution density $f(r, t) = n(r, t)/N(t)$ on fragment sizes when the variable $r$ is replaced by $\lambda(t)r$, the analysis of this equation becomes much more simple. Let us introduce the designation by $F(s, t)$ of the Mellin transformation

$$F(s, t) = \lambda(t) \int_0^\infty r^{s-1}f(\lambda(t)r, t)dr.$$
of the distribution $\lambda(t)f(\lambda(t)r, t)$ density. Changing the variables $r \Rightarrow \lambda(t)r$ in the equation (5) and computing the Mellin transformation pointed out, we obtain

$$\dot{F}(s, t) = -F(s, t)\frac{d}{dt} \ln \left( \frac{N}{\lambda} \right) - \frac{s}{\lambda} F(s, t) \dot{\lambda} + \lambda(p(s) - \mu)F(s + 1, t).$$  \hspace{1cm} (11)$$

The distribution $\lambda(t)f(\lambda(t)r, t)$ density and its Mellin transformation $F(s, t)$ are tended to $f_\infty(r)$ and $F(s)$ at $t \to \infty$, correspondingly. These limit functions are nontrivial ($\neq 0; \infty$). In this case, $\dot{F}(s, t) \to 0, \ t \to \infty$. So, the equation (11) take the form

$$0 = -F(s)\frac{d}{dt} \ln \left( \frac{N}{\lambda} \right) + \lambda(p(s) - \mu)F(s + 1) - \frac{s}{\lambda} \dot{\lambda} F(s).$$ \hspace{1cm} (12)$$

at asymptotically large values $t$. Let us find the limit distribution density $f_\infty(r)$ for the model dependence $P(x)$,

$$P(p/r) = C \left( \frac{p}{r} \right)^{\alpha}, \ \alpha > 0.$$ \hspace{1cm} (13)$$

In this case,

$$p(s) = \frac{C}{\alpha + s}, \ \mu = p(4) = \frac{C}{\alpha + 4}.$$  

Thus, the average total surface area and the average size are defined by the formulas

$$S(t) \sim \frac{CV}{(\alpha + 3)(\alpha + 4)}t, \hspace{1cm} (14)$$

$$\lambda(t) \sim \frac{(\alpha + 1)(\alpha + 4)}{C}t^{-1} \hspace{1cm} (15)$$

and the average fragment number is

$$N(t) \sim \frac{C^3V}{(\alpha + 4)^3(\alpha + 1)(\alpha + 2)(\alpha + 3)}t^3.$$  

Using (13), let us transform the equation (12),

$$F(s + 1) = \frac{\alpha + s}{\alpha + 1}F(s).$$
The solution of this functional equation which satisfies to the normalization condition $F(1, t) = 1$ is

$$F(s) = \frac{\Gamma(\alpha + s)}{\Gamma(\alpha + 1)(\alpha + 1)^{s-1}}.$$ 

Computing the inverse Mellin transformation, we find the limit distribution on fragment size density

$$f_\infty(r) = \frac{(\alpha + 1)^{\alpha+1} r^\alpha e^{-(\alpha+1)r}}{\Gamma(\alpha + 1)}.$$ 

It is in accordance with the result of the work [3] which has been obtained in the case when $\alpha$ is the integer. However, the last distribution density differs essentially from the logarithmically normal Kolmogorov law.

References

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