The Automorphisms Group and the Classification of Gradings of Finite Dimensional Associative Algebras

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Abstract. Let $A$ be an $n$-dimensional algebra over a field $k$ and $a(A)$ its quantum symmetry semigroup. We prove that the automorphisms group $\text{Aut}_{\text{Alg}}(A)$ of $A$ is isomorphic to the group $U(G(a(A)^*))$ of all invertible group-like elements of the finite dual $a(A)^o$. For a group $G$, all $G$-gradings on $A$ are explicitly described and classified: the set of isomorphisms classes of all $G$-gradings on $A$ is in bijection with the quotient set $\text{Hom}_{\text{Bialg}}(a(A), k[G])/\approx$ of all bialgebra maps $a(A) \rightarrow k[G]$, via the equivalence relation implemented by the conjugation with an invertible group-like element of $a(A)^o$.

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1. Introduction

The quantum symmetry semigroup $a(A)$ of a finite dimensional associative algebra $A$ over a field $k$, introduced by Manin [16] and Tambara [19], is used to approach two classical open problems: the description of the automorphisms group $\text{Aut}_{\text{Alg}}(A)$ of $A$ and the classification of all $G$-gradings on $A$, for an arbitrary group $G$. The description of the automorphisms group of an algebra is an old problem arising from Hilbert’s invariant theory: the first notable
result obtained was the Skolem–Noether theorem [23]. Since then a lot a literature was devoted to this problem: see for instance [7,9,22] and their list of references. The classification of all $G$-gradings on an algebra $A$ is a problem formulated by Zelmanov [25] in the case that $A = M_n(k)$, the usual matrix algebra. The problem is interesting but very difficult and has generated a lot of interest. It is also far from being solved: see [6,11,14] and their references. The similar question for the classification of all $G$-gradings on Lie algebras was also studied a lot [12].

In this paper a new approach to the two above mentioned problems is proposed using the power of a key object in quantum groups and noncommutative geometry, namely the quantum symmetry semigroup $a(A)$ [16,19] of an $n$-dimensional algebra $A$. Let $\{e_1 := 1_A, \ldots, e_n\}$ be a $k$-basis of $A$ and $\{\tau^a_{i,j} \mid i, j, s = 1, \ldots, n\}$ the structure constants of $A$. Then, $a(A)$ is the free algebra generated by $\{x_{si} \mid s, i = 1, \ldots, n\}$ and the relations:

$$
\sum_{a=1}^{n} \tau^{a}_{i,j} x_{a,u} = \sum_{s,t=1}^{n} \tau^{a}_{s,t} x_{si} x_{tj}, \quad x_{a1} = \delta_{a,1}
$$

(1)

for all $a$, $i$, $j = 1, \ldots, n$, where $\delta_{a,1}$ is the Kroneker symbol. Furthermore, $a(A)$ has a canonical bialgebra structure such that the canonical map $\eta_A : A \to A \otimes a(A)$, $\eta_A(e_i) := \sum_{s=1}^{n} e_s \otimes x_{si}$, for all $i = 1, \ldots, n$ is a right $a(A)$-comodule algebra structure on $A$ and the pair $(a(A), \eta_A)$ is the initial object of the category of all bialgebras coacting on $A$ (Theorem 2.6). The bialgebra $a(A)$ captures most of the essential information of the algebra $A$ and this idea is exploited in this paper. Theorem 3.1 provides an explicit description of a group isomorphism between the group of automorphisms of $A$ and the group of all invertible group-like elements of the finite dual $a(A)^{\circ}$:

$$
\text{Aut}_{\text{Alg}}(A) \cong U\left( G\left( a(A)^{\circ} \right) \right).
$$

Proposition 3.2 proves that for an arbitrary group $G$ there exists an explicitly described bijection between the set of all $G$-gradings on $A$ and the set of all bialgebra homomorphisms $a(A) \to k[G]$. Furthermore, all $G$-gradings on $A$ are explicitly classified and parameterized in Theorem 3.4: the set $G$-gradings($A$) of isomorphisms classes of all $G$-gradings on $A$ is in bijection with the quotient set $\text{Hom}_{\text{BiAlg}}(a(A), k[G]) / \approx$ of all bialgebra maps $a(A) \to k[G]$ via the equivalence relation implemented by the usual conjugation with an invertible group-like element of $a(A)^{\circ}$. Similar results were recently obtained for Leibniz/Lie algebras [5].

2. Preliminaries

All vector spaces, (bi)linear maps, associative algebras or bialgebras are over an arbitrary field $k$ and $\otimes = \otimes_k$. We shall denote by $\text{Alg}_k$ the category of all unital associative algebras and unital persevering morphisms. For $A \in \text{Alg}_k$
we denote by $\text{Aut}_{\text{Alg}}(A)$ the automorphisms group of $A$ and $k[G]$ denotes the usual group algebra of a group $G$. Let $G$ be a group and $A$ an algebra. We recall that a $G$-grading on $A$ is a vector space decomposition $A = \bigoplus_{\sigma \in G} A_\sigma$ such that $A_\sigma A_\tau \subseteq A_{\sigma \tau}$, for all $\sigma, \tau \in G$. Two $G$-gradings $A = \bigoplus_{\sigma \in G} A_\sigma = \bigoplus_{\sigma' \in G} A'_{\sigma'}$ on $A$ are called isomorphic if there exists $w \in \text{Aut}_{\text{Alg}}(A)$ an algebra automorphism of $A$ such that $w(A_\sigma) \subseteq A'_{\sigma}$, for all $\sigma \in G$. Since $w$ is bijective and $A$ is $G$-graded we can prove easily that the last condition is equivalent to $w(A_\sigma) = A'_{\sigma}$, for all $\sigma \in G$, which is the condition that usually appears in the literature in the classification of $G$-gradings [12]. For more details on the theory of $G$-graded rings we refer to [18].

For any bialgebra $H$ the set of group-like elements, denoted by $G(H) := \{g \in H \mid \Delta(g) = g \otimes g \text{ and } \varepsilon(g) = 1\}$, is a monoid with respect to the multiplication of $H$. We denote by $H^o$, the finite dual bialgebra of $H$, i.e.

$$H^o := \{f \in H^* \mid f(I) = 0, \text{ for some ideal } I \triangleleft H \text{ with } \dim_k(H/I) < \infty\}$$

It is well known (see for instance [20, pag. 62]) that $G(H^o) = \text{Hom}_{\text{Alg}}(H, k)$, the set of all algebra homomorphisms $H \rightarrow k$. For a later use we also recall the following well known result [17, Example 4.1.7]: let $A$ be an algebra and $G$ a group. Then there exists a bijection between the set of all $G$-gradings on $A$ and the set of all right $k[G]$-comodule algebra structures on $A$ given such that the $G$-grading on $A$ associated to a right $k[G]$-comodule algebra structures $\varrho : A \rightarrow A \otimes k[G]$ is given for any $\sigma \in G$ by:

$$A_\sigma = \{a \in A \mid \varrho(a) = a \otimes \sigma\}$$

For unexplained concepts on Hopf algebras we refer to [17,20,24] and for categorical constructions to [2].

**Tambara’s construction revised.** We shall recall the construction and the main properties of the quantum symmetry semigroup of a finite dimensional algebra $A$. The results below in this sections are well-known or have been recently proven in much more general cases [3,4]. The following is [19, Theorem 1.1]: we shall present a short proof since we prefer to give an explicit construction of the algebra $a(A)$, using generators and relations implemented by the structure constants of $A$ in the spirit of [1, Theorem 3.2].

**Theorem 2.1.** Let $A$ be a finite dimensional algebra over a field $k$. Then the functor $A \otimes - : \text{Alg}_k \rightarrow \text{Alg}_k$ has a left adjoint $a(A, -)$.

**Proof.** Assume that $\dim_k(A) = n$ and fix $\{e_1, \ldots, e_n\}$ a basis in $A$ over $k$ such that $e_1 = 1_A$, the unit of $A$. Let $\{\tau_{i,j}^s \mid i, j, s = 1, \ldots, n\}$ be the structure constants of $A$, i.e. for any $i, j = 1, \ldots, n$ we have:

$$e_i e_j = \sum_{s=1}^n \tau_{i,j}^s e_s.$$  (3)
Let $B$ be an arbitrary algebra and let $\{f_i \mid i \in I\}$ be a basis of $B$ containing $1_B$, i.e. $1_B = f_{i_0}$, for some $i_0 \in I$. For any $i, j \in I$, let $B_{i,j} \subseteq I$ be a finite subset of $I$ such that for any $i, j \in I$ we have:

$$f_i f_j = \sum_{u \in B_{i,j}} \beta_{i,j}^u f_u.$$  \hfill (4)

Let $a(A, B)$ be the free algebra generated by $\{x_{si} \mid s = 1, \ldots, n, \ i \in I\}$ and the relations:

$$\sum_{u \in B_{i,j}} \beta_{i,j}^u x_{au} = \sum_{s,t=1}^n \tau_s^a x_{si} x_{tj}, \quad x_{ai_0} = \delta_{a,1}$$ \hfill (5)

for all $a = 1, \ldots, n$ and $i, j \in I$, where $\delta_{a,1}$ is the Kroneker symbol. Using the relations (5) we can easily see that the map defined by

$$\eta_B : B \rightarrow A \otimes a(A, B), \quad \eta_B(f_i) := \sum_{s=1}^n e_s \otimes x_{si}, \quad \text{for all } i \in I.$$ \hfill (6)

is an algebra homomorphism preserving the unit. Furthermore, for any $k$-algebra $C$ the canonical map:

$$\gamma_{B,C} : \text{Hom}_{\text{Alg}}(a(A, B), C) \rightarrow \text{Hom}_{\text{Alg}}(B, A \otimes C), \quad \gamma_{B,C}(\theta) := (\text{Id}_A \otimes \theta) \circ \eta_B$$ \hfill (7)

is bijective, that is for any algebra map $f : B \rightarrow A \otimes C$ there exists a unique algebra homomorphism $\theta : A(A, B) \rightarrow C$ such that the following diagram is commutative:

$$\begin{array}{ccc}
B & \xrightarrow{\eta_B} & A \otimes a(A, B) \\
\downarrow f & & \downarrow \text{Id}_A \otimes \theta \\
A \otimes C & & 
\end{array}$$ \hfill (8)

i.e. $f = (\text{Id}_A \otimes \theta) \circ \eta_B$. The natural isomorphism (7) proves that the functor $a(A, -)$ is a left adjoint of $A \otimes -$.

By taking $C := k$ in the bijection described in (7) we obtain:

**Corollary 2.2.** Let $A$ and $B$ be two algebras such that $A$ is finite dimensional. Then the following map is bijective:

$$\gamma : \text{Hom}_{\text{Alg}}(a(A, B), k) \rightarrow \text{Hom}_{\text{Alg}}(B, A), \quad \gamma(\theta) := (\text{Id}_A \otimes \theta) \circ \eta_B.$$ \hfill (9)

**Definition 2.3.** Let $A$ be a finite dimensional algebra over a field $k$. The algebra $a(A) := a(A, A)$ is called the quantum symmetry semigroup of $A$.

The explicit description, using generators and relations, of the algebra $a(A)$ is the following: let $\{e_1, \ldots, e_n\}$ be a $k$-basis of $A$ such that $e_1 = 1_A$ and
let \( \{ \tau_{i,j}^s \mid i, j, s = 1, \ldots, n \} \) be the structure constants of \( A \). Then, \( a(A) \) is the free algebra generated by \( \{ x_{si} \mid s, i = 1, \ldots, n \} \) and the relations:

\[
\sum_{u=1}^{n} \tau_{i,j}^u x_{au} = \sum_{s,t=1}^{n} \tau_{s,t}^{a} x_{si} x_{tj}, \quad x_{a1} = \delta_{a,1}
\]

for all \( a, i, j = 1, \ldots, n \), where \( \delta_{a,1} \) is the Kroneker symbol. Furthermore, the canonical map

\[
\eta_A : A \to A \otimes a(A), \quad \eta_A(e_i) := \sum_{s=1}^{n} e_s \otimes x_{si}, \quad \text{for all } i = 1, \ldots, n
\]

is an algebra homomorphism. The bijection given by (7) applied for \( B := A \) shows that the algebra \( a(A) \) satisfies the following first universal property:

**Corollary 2.4.** Let \( A \) be a finite dimensional algebra. Then for any algebra \( C \) and any algebra homomorphism \( f : A \to A \otimes C \), there exists a unique algebra homomorphism \( \theta : a(A) \to C \) such that \( f = (\text{Id}_A \otimes \theta) \circ \eta_A \), i.e. the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & A \otimes a(A) \\
\downarrow f & & \downarrow (\text{Id}_A \otimes \theta) \\
A \otimes C & & A \otimes C
\end{array}
\]

The algebra \( a(A) \) is also a bialgebra such that \( (A, \eta_A) \) is a right \( a(A) \)-comodule algebra [19]. Moreover, we can explicitly describe the coalgebra structure on \( a(A) \) as follows:

**Proposition 2.5.** Let \( A \) be a \( k \)-algebra of dimension \( n \). Then there exists a unique bialgebra structure on \( a(A) \) such that the algebra homomorphism \( \eta_A : A \to A \otimes a(A) \) becomes a right \( a(A) \)-comodule structure on \( A \). More precisely, the comultiplication and the counit on \( a(A) \) are given for any \( i, j = 1, \ldots, n \) by

\[
\Delta(x_{ij}) = \sum_{s=1}^{n} x_{is} \otimes x_{sj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{i,j}
\]

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\]

**Proof.** Consider the algebra homomorphism \( f : A \to A \otimes a(A) \otimes a(A) \) defined by \( f := (\eta_A \otimes \text{Id}_{a(A)}) \circ \eta_A \). It follows from Corollary 2.4 that there exists a unique algebra homomorphism \( \Delta : a(A) \to a(A) \otimes a(A) \) such that \( (\text{Id}_A \otimes \Delta) \circ \eta_A = f \); that is, the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & A \otimes a(A) \\
\downarrow \eta_A & & \downarrow (\text{Id}_A \otimes \Delta) \\
A \otimes a(A) & \xrightarrow{(\eta_A \otimes \text{Id}_{a(A)})} & A \otimes a(A) \otimes a(A)
\end{array}
\]
Now, if we evaluate the diagram (14) at each $e_i$, for $i = 1, \ldots, n$ we obtain, taking into account (11), the following:

$$\sum_{t=1}^{n} e_t \otimes \Delta(x_{ti}) = (\eta_A \otimes \text{Id})(\sum_{s=1}^{n} e_s \otimes x_{si}) = \sum_{s=1}^{n} (\sum_{t=1}^{n} e_t \otimes x_{ts}) \otimes x_{si}$$

and hence $\Delta(x_{ti}) = \sum_{s=1}^{n} x_{ts} \otimes x_{si}$, for all $t, i = 1, \ldots, n$. Obviously, $\Delta$ given by this formula on generators is coassociative. In a similar fashion, applying once again Corollary 2.4, we obtain that there exists a unique algebra homomorphism $\varepsilon : a(A) \to k$ such that the following diagram is commutative:

$$\begin{array}{c}
A & \xrightarrow{\eta_A} & A \otimes a(A) \\
\downarrow & & \downarrow \text{Id}_A \otimes \varepsilon \\
\text{can} & & \text{can} \\
A \otimes k & & \end{array}$$

where $\text{can} : A \to A \otimes k$ is the canonical isomorphism, $\text{can}(x) = x \otimes 1$, for all $x \in A$. If we evaluate this diagram at each $e_t$, for $t = 1, \ldots, n$, we obtain $\varepsilon(x_{ij}) = \delta_{i,j}$, for all $i, j = 1, \ldots, n$. It can be easily checked that $\varepsilon$ is a counit for $\Delta$, thus $a(A)$ is a bialgebra. Furthermore, the commutativity of the above two diagrams implies that the canonical map $\eta_A : A \to A \otimes a(A)$ defines a right $a(A)$-comodule structure on $A$, i.e. $A$ is a right $a(A)$-comodule algebra. □

The pair $(a(A), \eta_A)$, with the coalgebra structure defined in Proposition 2.5, is called the *universal coacting bialgebra* of $A$. The reason is that it fulfills the following universal property which extends Corollary 2.4 and shows that $(a(A), \eta_A)$ is the initial object of the category of all bialgebras coacting on $A$:

**Theorem 2.6.** Let $A$ be a finite dimensional algebra. Then, for any bialgebra $H$ and any algebra homomorphism $f : A \to A \otimes H$ which makes $A$ into a right $H$-comodule there exists a unique bialgebra homomorphism $\theta : a(A) \to H$ such that the following diagram is commutative:

$$\begin{array}{c}
A & \xrightarrow{\eta_A} & A \otimes a(A) \\
\downarrow f & & \downarrow \text{Id}_A \otimes \theta \\
A \otimes H & & \end{array}$$

**Proof.** It follows from Corollary 2.4 there exists a unique algebra map $\theta : a(A) \to H$ such that diagram (16) commutes. The proof will be finished once we show that $\theta$ is a coalgebra homomorphism as well. This follows by using again the
universal property of $a(A)$. Indeed, we obtain a unique algebra homomorphism $\psi: a(A) \to H \otimes H$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & A \otimes a(A) \\
\downarrow & & \downarrow \\
A \otimes H \otimes H & \xrightarrow{\text{Id}_A \otimes \psi} & A \otimes H \otimes H
\end{array}
$$

The proof will be finished once we show that $(\theta \otimes \theta) \circ \Delta$ makes diagram (17) commutative, and this is just an elementary fact. Similarly, one can show that $\varepsilon_H \circ \theta = \varepsilon$ and the proof is now finished.

We end this section with the following two questions:

1. Are there two finite dimensional non-isomorphic algebras $A$ and $B$ such that $a(A) \cong a(B)$? (isomorphism of bialgebras)?

2. If $A$ and $B$ are two finite dimensional algebras, is it true that $a(A \otimes B) \cong a(A) \otimes a(B)$ (isomorphism of bialgebras)?

3. The Automorphisms Group and the Classification of Gradings on Algebras

The power of the quantum symmetry semigroup of an algebra $A$ is evidenced by its applications to the two open problems mentioned in the introduction. The first application is related to the description of the automorphisms group $\text{Aut}_{\text{Alg}}(A)$ of $A$.

**Theorem 3.1.** Let $A$ be a finite dimensional algebra with basis $\{e_1, \ldots, e_n\}$ and let $U\left( G\left( a(A)^o \right) \right)$ be the group of all invertible group-like elements of the finite dual $a(A)^o$. Then the map defined for any $\theta \in U\left( G\left( a(A)^o \right) \right)$ and $i = 1, \ldots, n$ by:

$$
\varphi: U\left( G\left( a(A)^o \right) \right) \to \text{Aut}_{\text{Alg}}(A), \quad \varphi(\theta)(e_i) := \sum_{s=1}^n \theta(x_{si}) e_s
$$

is an isomorphism of groups.

**Proof.** By applying Corollary 2.2 for $B := A$ it follows that the map

$$
\gamma: \text{Hom}_{\text{Alg}}(a(A), k) \to \text{End}_{\text{Alg}}(A), \quad \gamma(\theta) = (\text{Id}_A \otimes \theta) \circ \eta_A
$$

is bijective. Based on formula (11), it can be easily seen that $\gamma$ takes the form given by (18). As we mentioned in the preliminaries we have $\text{Hom}_{\text{Alg}}(a(A), k) = G\left( a(A)^o \right)$. Therefore, since $\varphi$ is the restriction of $\gamma$ to the invertible elements of the two monoids, the proof will be finished once we show that $\gamma$ is an isomorphism of monoids. We mention that the monoid structure on $\text{End}_{\text{Alg}}(A)$
is given by the usual composition of endomorphisms of the algebra \( A \), while \( G(a(A)^\circ) \) is a monoid with respect to the multiplication of the bialgebra \( a(A)^\circ \), that is the convolution product:

\[
(\theta_1 \ast \theta_2)(x_{sj}) = \sum_{t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj})
\]

(19)

for all \( \theta_1, \theta_2 \in G(a(A)^\circ) \) and \( j, s = 1, \ldots, n \). Now, for any \( \theta_1, \theta_2 \in G(a(A)^\circ) \) and \( j = 1, \ldots, n \) we have:

\[
(\gamma(\theta_1) \circ \gamma(\theta_2))(e_j) = \gamma(\theta_1)\left(\sum_{t=1}^{n} \theta_2(x_{tj})e_t\right) = \sum_{s,t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj})e_s
\]

\[
= \sum_{s=1}^{n} \left(\sum_{t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj})\right)e_s = \sum_{s=1}^{n} (\theta_1 \ast \theta_2)(x_{sj})e_s = \gamma(\theta_1 \ast \theta_2)(e_j)
\]

thus, \( \gamma(\theta_1 \ast \theta_2) = \gamma(\theta_1) \circ \gamma(\theta_2) \), and therefore \( \gamma \) respects the multiplication. We are left to show that \( \gamma \) also preserves the unit. Note that the unit 1 of the monoid \( G(a(A)^\circ) \) is the counit \( \varepsilon_a(A) \) of the bialgebra \( a(A) \) and we obtain:

\[
\gamma(1)(e_i) = \gamma(\varepsilon_a(A))(e_i) = \sum_{s=1}^{n} \varepsilon_a(A)(x_{si})e_s = \sum_{s=1}^{n} \delta_{si}e_s = e_i = \text{Id}_A(e_i)
\]

Thus we have proved that \( \gamma \) is an isomorphism of monoids and the proof is finished.

The second application of the bialgebra \( a(A) \) is related to the problem of classifying of all \( G \)-gradings on the algebra \( A \).

**Proposition 3.2.** Let \( G \) be a group and \( A \) a finite dimensional algebra. Then there exists a bijection between the set of all \( G \)-gradings on \( A \) and the set of all bialgebra homomorphisms \( a(A) \to k[G] \). The bijection is given such that the \( G \)-grading on \( A = \bigoplus_{\sigma \in G} A_{\sigma}^\theta \) associated to a bialgebra map \( \theta : a(A) \to k[G] \) is given by:

\[
A_{\sigma}^\theta := \{ x \in A \mid (\text{Id}_A \otimes \theta) \circ \eta_A(x) = x \otimes \sigma \}
\]

(20)

for all \( \sigma \in G \).

**Proof.** Applying Theorem 2.6 for the bialgebra \( H := k[G] \) yields a bijection between the set of all bialgebra homomorphisms \( a(A) \to k[G] \) and the set of all algebra homomorphisms \( f : A \to A \otimes k[G] \) which makes \( A \) into a right \( k[G] \)-comodule. The proof is now finished since the latter set is in bijective correspondence with the set of all \( G \)-gradings on \( A \) (see (2) and [17, Example 4.1.7]).

It is easy to see that, under the bijection given by Proposition 3.2, the \( G \)-grading on \( A \) associated to the trivial bialgebra map \( a(A) \to k[G] \), \( x \mapsto \varepsilon(x) \), is just the trivial grading on \( A \), that is \( A_1 := A \) and \( A_{\sigma} := 0 \), for any \( 1 \neq \sigma \in G \).
In the next step we shall classify all \(G\)-gradings on a given algebra \(A\). We need to recall one more elementary fact from Hopf algebras: if \(H\) and \(L\) are two bialgebras over a field \(k\) then the abelian group \(\text{Hom}(H, L)\) of all \(k\)-linear maps is an unital associative algebra under the convolution product \([24]\): 
\[
(\theta_1 \ast \theta_2)(h) := \sum \theta_1(h_{(1)})\theta_2(h_{(2)}),
\]
for all \(\theta_1, \theta_2 \in \text{Hom}(H, L)\) and \(h \in H\).

**Definition 3.3.** Let \(G\) be a group and \(A\) a finite dimensional algebra. Two morphisms of bialgebras \(\theta_1, \theta_2: a(A) \rightarrow k[G]\) are called conjugate and denote \(\theta_1 \approx \theta_2\), if there exists \(g \in U(G(a(A)^o))\) an invertible group-like element of the finite dual \(a(A)^o\) such that \(\theta_2 = g \ast \theta_1 \ast g^{-1}\), in the convolution algebra \(\text{Hom}(a(A), k[G])\).

We denote by \(\text{Hom}_{\text{BiAlg}}(a(A), k[G])/\approx\) the quotient set of the set of all bialgebra maps \(a(A) \rightarrow k[G]\) through the above equivalence relation and by \(\hat{\theta}\) the equivalence class of \(\theta \in \text{Hom}_{\text{BiAlg}}(a(A), k[G])\).

**Theorem 3.4.** Let \(G\) be a group, \(A\) a finite dimensional algebra and \(G\) – gradings \((A)\) the set of isomorphisms classes of all \(G\)-gradings on \(A\). Then the map 
\[
\text{Hom}_{\text{BiAlg}}(a(A), k[G])/\approx \mapsto G - \text{gradings}(A), \quad \hat{\theta} \mapsto A^{(\theta)} := \oplus_{\sigma \in G} A^{(\theta)}_{\sigma}
\]
where \(A^{(\theta)}_{\sigma} = \{x \in A | (\text{Id}_A \otimes \theta) \circ \eta_A(x) = x \otimes \sigma\}, \) for all \(\sigma \in G\), is bijective.

**Proof.** Let \(\{e_1, \ldots, e_n\}\) be a basis in \(A\). Using Proposition 3.2 we obtain that for any \(G\)-grading \(A = \oplus_{\sigma \in G} A_{\sigma}\) on \(A\) there exists a unique bialgebra map \(\theta : a(A) \rightarrow k[G]\) such that \(A_{\sigma} = A^{(\theta)}_{\sigma}\), for all \(\sigma \in G\). It remains to prove when two such \(G\)-gradings are isomorphic. Let \(\theta_1, \theta_2 : a(A) \rightarrow k[G]\) be two bialgebra maps and let \(A = A^{(\theta_1)} := \oplus_{\sigma \in G} A^{(\theta_1)}_{\sigma} = \oplus_{\sigma \in G} A^{(\theta_2)}_{\sigma} := A^{(\theta_2)}\) be the associated \(G\)-gradings. It follows from the proof of Proposition 3.2 that given a \(G\)-grading on \(A\) is equivalent (and the correspondence is bijective) to a right \(k[G]\)-comodule structure \(\rho : A \rightarrow A \otimes k[G]\) on \(A\) such that the right coaction \(\rho\) is a morphism of algebras. Moreover, the right coactions \(\rho^{(\theta_1)}\) and \(\rho^{(\theta_2)} : A \rightarrow A \otimes k[G]\) are implemented form \(\theta_1\) and \(\theta_2\) using Theorem 2.6, that is they are given for any \(j = 1, 2\) by
\[
\rho^{(\theta_j)} : A \rightarrow A \otimes k[G], \quad \rho^{(\theta_j)}(e_i) = \sum_{s=1}^{n} e_s \otimes \theta_j(x_{si})
\]
for all \(i = 1, \ldots, n\). It is well known in Hopf algebras \([17]\) that the two \(G\)-gradings \(A^{(\theta_1)}\) and \(A^{(\theta_2)}\) are isomorphic if and only if \((A, \rho^{(\theta_1)})\) and \((A, \rho^{(\theta_2)})\) are isomorphic as algebras and right \(k[G]\)-comodules, that is there exists \(w : A \rightarrow A\) an algebra automorphism of \(A\) such that \(\rho^{(\theta_2)} \circ w = (w \otimes \text{Id}_k[G]) \circ \rho^{(\theta_1)}\). We apply now Theorem 3.1: for any algebra automorphism \(w : A \rightarrow A\) there exists a unique \(g \in U(G(a(A)^o))\) an invertible group-like element
of the finite dual $a(A)^0$ such that $w = w_g$ is given for any $i = 1, \ldots, n$ by

$$w_g(e_i) = \sum_{s=1}^{n} g(x_{si}) e_s$$

(22)

Using (21) and (22) we can easily compute that:

$$(\rho^{(\theta_2)} \circ w_g)(e_i) = \sum_{a=1}^{n} e_a \otimes \left( \sum_{s=1}^{n} \theta_2(x_{as}) g(x_{si}) \right)$$

and

$$(w_g \otimes \text{Id}_{k[G]} \circ \rho^{(\theta_1)})(e_i) = \sum_{a=1}^{n} e_a \otimes \left( \sum_{s=1}^{n} g(x_{as}) \theta_1(x_{si}) \right)$$

for all $i = 1, \ldots, n$. Thus, the algebra automorphism $w_g : A \to A$ is also a right $k[G]$-comodule map if and only if

$$\sum_{s=1}^{n} g(x_{as}) \theta_1(x_{si}) = \sum_{s=1}^{n} \theta_2(x_{as}) g(x_{si})$$

(23)

for all $a, i = 1, \ldots, n$. Taking into account the formula of the comultiplication on the bialgebra $a(A)$, the equation (23) can be rewritten in a compact form as $(g \star \theta_1)(x_{ai}) = (\theta_2 \star g)(x_{ai})$, for all $a, i = 1, \ldots, n$ in the convolution algebra $\text{Hom}(a(A), k[G])$, or (since $\{x_{ai}\}_{a,i=1,\ldots,n}$ is a system of generators of $a(A)$) just as $g \star \theta_1 = \theta_2 \star g$. We also note that $g : a(A) \to k$ is an invertible group-like element, and therefore is an invertible element in the above convolution algebra.

To conclude, we have proved that two $G$-gradings $A^{(\theta_1)}$ and $A^{(\theta_2)}$ on $A$ associated to two bialgebra maps $\theta_1, \theta_2 : a(A) \to k[G]$ are isomorphic if and only if there exists $g \in U(G(a(A)^0))$ such that $\theta_2 = g \star \theta_1 \star g^{-1}$, in the convolution algebra $\text{Hom}(a(A), k[G])$, that is $\theta_1$ and $\theta_2$ are conjugated and this finished the proof. \(\square\)

Theorems 3.1 and 3.4 offer a theoretical answer to the two classic problems, reducing them to typical problems for Hopf algebras. While the explicit description of the bialgebra $a(A)$ is relatively easy (albeit using a laborious computation, eliminating the redundant relations among the $n^3$ relations (1) that define it), the really difficult part is the description of the finite dual $a(A)^0$ using generators and relations. Unfortunately, the explicit description of the finite dual of a finite generated bialgebra is a problem that has not been studied until now. It does however merit attention in the future; the first steps in this direction have been taken very recently in [8,10,13,15].

**Example 3.5.** Let $A := k[X]/(X^2)$. Then $a(k[X]/(X^2))$ is the algebra $k(x,y \mid x^2 = 0, xy + yx = 0)$ with the coalgebra structure given by:

$$\Delta(x) = x \otimes y + 1 \otimes x, \quad \Delta(y) = y \otimes y, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 1$$
Indeed, let \( \{e_1 := 1, e_2 := x \} \) be a basis of \( A \), where \( x \) is the class of \( X \) in \( A \). Then the only non-zero structure constants of the algebra \( A \) are \( \tau_{1,j}^i = \tau_{2,1}^j = 1 \), for \( j = 1, 2 \). Now, it is just a routine computation, to reduce the eight relations (10) defining the algebra \( a(k[X]/(X^2)) \) to:

\[
x_{12}^2 = 0, \quad x_{12}x_{22} + x_{22}x_{12} = 0
\]

Denoting \( x_{12} = x \) si \( x_{22} = y \) the conclusion follows (the coalgebra structure arises from Proposition 2.5). Applying Theorem 3.1 we obtain that \( \text{Aut}_{\text{Alg}}(k[X]/(X^2)) \) is in bijection with the set of all morphisms of algebras \( \vartheta : a(k[X]/(X^2)) \to k \) such that the matrix \( (\vartheta(x_{ij})) \in M_2(k) \) is invertible. Such a morphism is defined by: \( \vartheta(x_{11}) = \vartheta(1) := 1, \vartheta(x_{21}) = \vartheta(0) := 0, \vartheta(x_{12}) := \alpha \) and \( \vartheta(x_{22}) := \beta \), for some \( \alpha, \beta \in k \). Since, \( x_{12}^2 = 0 \) we obtain that \( \alpha = 0 \) and \( \beta \neq 0 \) since the matrix \( (\vartheta(x_{ij})) \) is invertible. Thus, \( \text{Aut}_{\text{Alg}}(k[X]/(X^2)) \cong k^* \).

**Example 3.6.** Let \( A := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subseteq M_2(k) \) be the algebra of upper-triangular matrices with the basis \( \{e_1 := I_2, e_2 := e_{21}, e_3 := e_{22}\} \). Then \( a(A) \) is the algebra generated by \( \{x_{ij} | i, j = 1, 2, 3\} \) and the relations:

\[
\begin{align*}
x_{11} &= 1, \quad x_{21} = x_{31} = 0, \quad x_{12}^2 = 0, \quad x_{12}x_{13} = x_{12}, \quad x_{13}x_{12} = 0 \\
x_{12}^2 &= x_{13} = 1, \quad x_{12}x_{22} + x_{22}x_{12} = -x_{22}x_{32}, \quad x_{12}x_{23} + x_{23}x_{13} + x_{22}x_{33} = x_{22} \\
x_{13}x_{22} + x_{23}(x_{12} + x_{32}) &= 0, \quad x_{13}x_{23} + x_{23}(x_{13} + x_{33}) = x_{23} \\
x_{32}x_{33} + x_{32}x_{13} + x_{12}x_{33} &= x_{32}, \quad x_{32}^2 + x_{32}x_{12} + x_{12}x_{32} = 0
\end{align*}
\]

Indeed, the only non-zero structure constants of the algebra \( A \) are:

\( \tau_{1,j}^i = \tau_{2,1}^j = \tau_{2,3}^2 = \tau_{3,3}^3 := 1 \)

for all \( j = 1, 2, 3 \). A routine computation shows that out of the 27 relations (10) defining the algebra \( a(A) \) only the above remain.

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