Differential modules with $\infty$-simplicial faces and $A_\infty$-algebras

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Abstract

In the present paper, by using the colored version of the Koszul duality, the concept of a differential module with $\infty$-simplicial faces is introduced. The homotopy invariance of the structure of a differential module with $\infty$-simplicial faces is proved. The relationships between differential modules with $\infty$-simplicial faces and $A_\infty$-algebras are established. The notion of a chain realization of a differential module with $\infty$-simplicial faces and the concept of a tensor product of differential modules with $\infty$-simplicial faces are introduced. It is proved that for an arbitrary $A_\infty$-algebra the chain realization of the tensor differential module with $\infty$-simplicial faces, which corresponds to this $A_\infty$-algebra, and the $B$-construction of this $A_\infty$-algebra are isomorphic differential coalgebras.

The concept of a chain realization of simplicial differential modules and in particular the concept of a chain realization of differential modules with simplicial faces was introduced in [1], when studying homotopy properties of $E_\infty$-coalgebras. As was showed in [1], a chain realization of the tensor differential module with simplicial faces, which is defined by an arbitrary differential algebra, is a $B$-construction of this differential algebra. By using the representation of a $B$-construction as a chain realization, in [2] was obtained that the $B$-construction of an arbitrary cocommutative Hopf algebra is an $E_\infty$-algebra, which by a suitable ”gluing” of $E_\infty$-coalgebras of chain complexes of standard simplicial simplexes is obtained. The presence of the structure $E_\infty$-coalgebra on the above considered $B$-construction implies that on cohomology this $B$-construction exist the natural action of Steenrod operations, which satisfy the Adem relations [1] and Cartan formula [3]. In particular, if we consider the Steenrod algebra as a cocommutative Hopf algebra, then we obtain the well known action of Steenrod operations on cohomology of the Steenrod algebra (see for example [4]), i.e., on the second term of the Adams spectral sequence for the stable homotopy groups of spheres.

The present paper consists of three paragraphs. In the first paragraph, the necessary notions, constructions and assertions of the Koszul duality theory for quadratic algebras [9] and the differential Lie modules theory [10] respectively are carried over to the cases of quadratic colored algebras and colored graded coalgebras. In the second paragraph, a description of the colored coalgebra $(F^i, \nabla)$ Koszul dual to the quadratic colored algebra of simplicial faces $(F, \pi)$ is given. Further, the notion of
a differential module with $\infty$-simplicial faces is introduced as a differential Lie module over a colored coalgebra $(F^i, \nabla)$ or, equivalently, as a differential module over a $\text{co-B}$-construction of this colored coalgebra. The homotopy invariance of the structure of a differential module with $\infty$-simplicial faces is established. The connection between the concepts of a differential module with $\infty$-simplicial faces and differential module with homotopy simplicial faces is considered. In the third paragraph, the concepts of a chain realization of a differential module with $\infty$-simplicial faces and tensor product of differential modules with $\infty$-simplicial faces are introduced. It is shown that every $A_{\infty}$-algebra defines the tensor differential module with $\infty$-simplicial faces. Further, for an arbitrary $A_{\infty}$-algebra, it is proved that the chain realization of its tensor differential module with $\infty$-simplicial faces and the $B$-construction of this $A_{\infty}$-algebra are isomorphic as differential coalgebras.

We proceed to precise definitions and statements. All modules and maps of modules considered in this paper are assumed to be $K$-modules and $K$-linear maps of modules, respectively, where $K$ is an arbitrary commutative ring with unity.

§ 1. Colored coalgebras Koszul dual to quadratic colored algebras

Let $I$ be a set of nonnegative integers. Each element of the set $I$ we will called a color. A colored differential module with colors from the set $I$ or, more briefly, simply a colored differential module $(X, d)$ is by definition an arbitrary family of differential graded modules $X = \{X(s, t)_m\}$, $m \in \mathbb{Z}$, $d : X(s, t) \to X(s, t)_{m-1}$ that are indexed by all pairs of elements $(s, t) \in I \times I$. In the case when colored differential module $(X, d)$ satisfies the condition $d = 0$ for all $s, t \in I$, we will say that $X$ is a colored graded module.

A map of colored differential modules $f : (X, d) \to (Y, d)$ is any family of maps of differential graded modules $f = \{f(s, t) : (X(s, t), d) \to (Y(s, t), d)\}_{s, t \in I}$. Similarly we introduce the notion of a homotopy $h : X \to Y$ between maps colored differential modules $f, g : (X, d) \to (Y, d)$. It is clearly that colored differential modules and their maps form a category in which obvious way direct sums and direct products of objects, the kernels and cokernels of morphisms, and also homology of objects are defined.

In further a differential bigraded module we will call an arbitrary differential bigraded module $(X, d)$, $X = \{X_{n,m}\}$, $n, m \in \mathbb{Z}$, $n \geq 0, d : X_{s,\bullet} \to X_{s,\bullet-1}$. For an arbitrary differential bigraded modules $(X, d)$ and $(Y, d)$ there is the colored differential module $(\text{hom}(X; Y), d)$. The elements of the module $\text{hom}(X; Y)_{s, t}$ are an arbitrary maps of graded modules $f : X_{t, \bullet} \to Y_{s, \bullet + m}$ of degree $m \in \mathbb{Z}$, and the differential $d : \text{hom}(X; Y)_{s, t} \to \text{hom}(X; Y)_{s, t-1}$ for any fixed $s, t \in I$ is given on elements $f \in \text{hom}(X; Y)_{s, t}$ by the following formula:

$$d(f) = df + (-1)^{s-t+m+1}fd : X_{t, \bullet} \to Y_{s, \bullet + m-1}.$$ 

For an arbitrary colored differential modules $(X, d)$ and $(Y, d)$, we set by definition $X(s, k)_p \otimes Y(l, t)_q = 0, k \neq l$, and define $X(s, k)_p \otimes Y(l, t)_q, k = l$, as the usual tensor product of modules.
A tensor product of the colored differential modules \((X, d)\) and \((Y, d)\) is the colored differential module \((X \otimes Y, d)\), where

\[(X \otimes Y)(s, t)_m = \bigoplus_{k,l \in I} \bigoplus_{p+q=m} X(s, k)_p \otimes Y(l, t)_q = \bigoplus_{k,l \in I} \bigoplus_{p+q=m} X(s, k)_p \otimes Y(k, t)_q,
\]

and its differential \(d : (X \otimes Y)(s, t)_\bullet \to (X \otimes Y)(s, t)_{\bullet-1}\), for an arbitrary element \(x \otimes y \in X(s, l)_p \otimes Y(l, t)_q\), is defined by the following formula:

\[d(x \otimes y) = d(x) \otimes y + (-1)^{s-l+p} x \otimes d(y).
\]

A colored differential algebra \((A, d, \pi)\) is an arbitrary colored differential module \((A, d)\) equipped with a multiplication \(\pi : A \otimes A \to A\), which is a map of colored differential modules and satisfies the associativity condition \(\pi(\pi \otimes 1) = \pi(1 \otimes \pi)\). If a colored differential algebra \((A, d, \pi)\) satisfies the condition \(d = 0\) then we will say that there is given a colored graded algebra \((A, \pi)\).

A map of colored differential algebras \(f : (A', d, \pi) \to (A'', d, \pi)\) is an arbitrary map of colored differential modules \(f : (A', d) \to (A'', d)\) that satisfies the condition \(\pi(f \otimes f) = f \pi\).

A unity of a colored differential algebra \((A, d, \pi)\) is a family of elements \(1_s = \{1_k\}, k \in I, 1_k \in A(k, k)_0\), that satisfy the condition \(\pi(1_s \otimes a) = a = \pi(a \otimes 1_t)\) for each element \(a \in A(s, t)_m\), \(s, t \in I, m \in \mathbb{Z}\).

The canonical example of a colored differential algebra is the colored differential algebra \((\text{hom}(X; X), d, \pi)\), where \((\text{hom}(X; X), d)\) is considered above colored differential module, which for any differential bigraded module \((X, d)\) is defined. A multiplication \(\pi : \text{hom}(X; X) \otimes \text{hom}(X; X) \to \text{hom}(X; X)\) of this colored differential algebra is given for an arbitrary elements \(g \in \text{hom}(X, X)(s, k)_n\) and \(f \in \text{hom}(X, X)(l, t)_m\) by the following rule:

\[
\pi(g \otimes f) = \begin{cases} 
    g f \in \text{hom}(X; X)(s, t)_{n+m}, & k = l, \\
    0 \in \text{hom}(X; X)(s, t)_{n+m}, & k \neq l,
\end{cases}
\]

where \(gf : X_{t, \bullet} \to X_{s, \bullet+n+m}\) is a composition of the maps \(f : X_{t, \bullet} \to X_{l, \bullet+k}, \cdot+m\) and \(g : X_{k, \bullet+l} \to X_{s, \bullet+n}\). It is clear that a unity of the colored differential algebra \((\text{hom}(X, X), d, \pi)\) is a family of elements \(1_s = \{1_s\}, s \in I\), where for any \(s \in I\) the element \(1_s \in \text{hom}(X, X)(s, s)_0\) is the identity map of the graded module \(X_{s, \bullet}\).

Further we will denote by \(K_I\) the colored graded module that is defined by equalities \(K_I(s, s)_m = K, m = 0, s \in I, K_I(s, s)_m = 0, m \neq 0, s \in I, K_I(s, t)_m = 0, s \neq t, m \in \mathbb{Z}\). It is clearly that the multiplication in the ring \(K\) allows regarded \(K_I\) as colored graded algebra \((K_I, \pi)\).

By using considered above the operation of a tensor product of colored graded modules, we define the usual way the notions of a tensor algebra of a graded module, a two-sided ideal of a colored graded algebra generated by a submodule of this colored graded algebra and a quotient algebra of a colored graded algebra by a two-sided ideal.

**Definition 1.1.** A colored graded algebra \((A, \pi)\) with a unity is called a quadratic colored graded algebra or, more briefly, a quadratic colored algebra if this colored graded algebra \((A, \pi)\) is isomorphic to the quotient algebra \(R = T(M)/(Q)\), where
$T(M)$ is a tensor algebra of some colored graded module $M = \{M(s, t)_m\}_{s, t \in I}$, $m \in \mathbb{Z}$, $m \geq 0$, and $(Q)$ is a two-sided ideal of the colored graded algebra $T(M)$ generated by some submodule $Q$ of the colored graded module $M \otimes M \subset T(M)$.

We note that for any quadratic colored algebra $R = T(M)/(Q)$ the composition of the obvious embedding $M \to T(M)$ and the projection $T(M) \to T(M)/(Q)$ is an embedding $M \to R = T(M)/(Q)$ of colored graded modules. In what follows, we identify the image of this embedding with the colored graded module $M$ and always assume that $M$ is a submodule of the colored graded module $R$.

A colored graded coalgebra $(C, \nabla)$ is an arbitrary colored graded module $C = \{C(s, t)_m\}_{s, t \in I}$, $m \in \mathbb{Z}$, $m \geq 0$, equipped with a comultiplication $\nabla : C \to C \otimes C$ that is a map of colored graded modules satisfies the condition $(\nabla \otimes 1)\nabla = (1 \otimes \nabla)\nabla$. A colored graded coalgebra $(C, \nabla)$ is said to be connected if $C_0 = (K_I)_0$. For a colored graded coalgebra $(C, \nabla)$ the usual way we define notions of counit $\varepsilon : C \to K_I$ and coaugmentation $\nu : K_I \to C$.

It is easy to see that a comultiplication in the colored graded coalgebra $(C, \nabla)$ with counit $\varepsilon : C \to K_I$ and coaugmentation $\nu : K_I \to C$ determines a comultiplication $\nabla : C \to C \otimes C$ in the colored graded submodule $C = \ker(\varepsilon)$, which for an arbitrary element $c \in C(s, t)$, $s, t \in I$, is defined by the formula $\nabla(c) = \nabla(c) - (1_s \otimes c + c \otimes 1_t)$. Thus, for any colored graded coalgebra $(C, \nabla)$ the colored graded coalgebra $(\overline{C}, \overline{\nabla})$ without counit is always defined.

The suspension of a colored graded module $M$ is a colored graded module $SM$ with grading defined for $s, t \in I$ by the equality $(SM)(s, t)_{m+1} = M(s, t)_m$. Let us denote the elements of $SM$ by $[x]$, where $x \in M$. For any submodule $Q$ of the colored graded module $M \otimes M$ we denote by $S^{\otimes 2}Q$ the corresponding submodule of the colored graded module $SM \otimes SM$.

**Definition 1.2.** The colored coalgebra Koszul dual to a quadratic colored algebra $R = T(M)/(Q)$ is the colored graded coalgebra $(R^! , \nabla)$ with counit $\varepsilon : R^! \to K_I$ and coaugmentation $\nu : K_I \to R^!$ defined by the following formulae:

\[
R^! = \bigoplus_{k \geq 0} (R^!(k)), \quad (R^!(0)) = K_I, \quad (R^!(1)) = SM,
\]

\[
(R^!(k)) = \bigcap_{i+2+j=k} (SM)^{\otimes i} \otimes S^{\otimes 2}Q \otimes (SM)^{\otimes j}, \quad k \geq 2, \quad i \geq 0, \quad j \geq 0,
\]

\[
\nabla(1_s) = 1_s \otimes 1_s, \quad 1_s \in (R^!(0))(s, s)_0 = K_I(s, s)_0, \quad s \in I,
\]

\[
\nabla([x_1 \otimes \ldots \otimes x_k]) = 1_s \otimes [x_1, \ldots, x_k] + [x_1, \ldots, x_k] \otimes 1_t +
\]

\[
+ \sum_{i=1}^{k-1} [x_1, \ldots, x_i] \otimes [x_{i+1}, \ldots, x_k], \quad [x_1, \ldots, x_k] \in (R^!(k))(s, t), \quad s, t \in I,
\]

\[
\varepsilon(1_s) = 1_s, \quad \varepsilon([x_1, \ldots, x_k]) = 0, \quad k \geq 1, \quad \nu(1_s) = 1_s,
\]

where $[x_1, \ldots, x_k] = [x_1] \otimes \ldots \otimes [x_k]$, $x_j \in F^!(s, t_j)$, $s_j, t_j \in I$, $1 \leq j \leq k$, $s_1 = s$, $t_k = t$, $s_p = t_{p+1}$, $1 \leq p \leq k - 1$.

Let us now consider the notion of a co-$B$-construction of a connected colored graded coalgebra with counit and coaugmentation.
A desuspension over a colored graded module \( M \) is the colored graded module \( S^{-1}M \), which for any \( s, t \in I \) is defined by the formula \((S^{-1}X)(s, t)_{m-1} = X(s, t)_m\). The elements of \( S^{-1}X \) are traditionally denoted by \([x]\), where \( x \in M \).

Let \((C, \nabla)\) be the colored graded coalgebra without counit determined by the colored graded coalgebra \((C, \nabla)\) with counit and coaugmentation. A co-\(B\)-construction of a colored graded coalgebra \((C, \nabla)\) is the colored differential algebra \((\Omega(C), d, \pi)\) defined as follows:

\[
\Omega(C) = \bigoplus_{k \geq 0} (S^{-1}C)^{\otimes k}, \quad (S^{-1}C)^{\otimes 0} = K_1,
\]

\[
d([c_1, \ldots, c_k]) = \sum_{i=1}^{k} (-1)^{i+\mu_i} [c_1, \ldots, c_{i-1}, \nabla(c_i), c_{i+1}, \ldots, c_k],
\]

\[
\nabla(c_i) = \sum c_i' \otimes c_i'', \quad \pi([c_1, \ldots, c_q] \otimes [c_{q+1}, \ldots, c_k]) = [c_1, \ldots, c_q, c_{q+1}, \ldots, c_k],
\]

where \([c_1, \ldots, c_k] = [c_1] \otimes \cdots \otimes [c_k], c_j \in \overline{C}(s_j, t_j)q_j, s_j = t_{j+1}, 1 \leq j \leq k, c_i' \in \overline{C}(s_i, t_i)_{q_i}, \mu_i = s_i - t_i' + q_i + \ldots + q_{i-1} + q_i, 1 \leq i \leq k\).

A twisting cochain from a colored graded coalgebra \((C, \nabla)\) to a colored differential algebra \((A, d, \pi)\) is defined as a map \(\varphi : C_\bullet \to A_{-1}\) of colored graded modules which has degree \(-1\) and satisfies the cochain twisting condition \(d\varphi + \varphi \nabla = 0\), where the map of colored graded modules \(\varphi \cup \varphi : C_\bullet \to A_{-2}\) of degree \(-2\) is defined by the formula \(\varphi \cup \varphi = \pi(\varphi \otimes \varphi)\nabla\).

The simplest example of a twisting cochain from any colored graded coalgebra \((C, \nabla)\) to the colored differential algebra \((\Omega(C), d, \pi)\) is the map \(\varphi^{\Omega} : C_\bullet \to \Omega(C)_{-1}\), \(\varphi^{\Omega}(c) = [c], c \in C\).

For any twisting cochain \(\varphi : C_\bullet \to A_{-1}\) from a colored graded coalgebra \((C, \nabla)\) to a colored graded algebra \((A, \pi)\) is defined the map of colored differential algebras \(\Omega(\varphi) : (\Omega(C), d, \pi) \to (A, d = 0, \pi)\) that on the generators \([c_1, \ldots, c_k]\) of the colored graded module \(\Omega(C)\) is given by the following formula:

\[
\Omega(\varphi)([c_1, \ldots, c_k]) = \pi^{(k)}(\varphi(c_1) \otimes \cdots \otimes \varphi(c_k)),
\]

where \(\pi^{(1)}\) is the identity map of the colored algebra \(A\), and for \(k \geq 2\) the map \(\pi^{(k)} = \pi(1 \otimes \pi^{(k-1)})\) is the iterated multiplication in the colored algebra \((A, \pi)\).

Let us now consider the notion of a differential module over a colored differential algebra.

A tensor product of a colored differential module \((X, d)\) and differential bigraded module \((Y, d)\) is the differential bigraded module \((X \otimes Y, d)\) for which

\[
(X \otimes Y)_{n,m} = \bigoplus_{s+t=n \land p+q=m} X(n, s)_p \otimes Y(s, q), \quad n \in I, \ m \in \mathbb{Z},
\]

and the differential \(d : (X \otimes Y)_{n,\bullet} \to (X \otimes Y)_{n,\bullet-1}\) is defined on an arbitrary element \(x \otimes y \in X(n, s)_p \otimes Y(s, q)\) by the following formula:

\[
d(x \otimes y) = d(x) \otimes y + (-1)^{n-s+p} x \otimes d(y).
\]
A left differential module \((X, d, \mu)\) over a colored differential algebra \((A, d, \pi)\) or, more briefly, a differential \(A\)-module is a differential bigraded module \((X, d)\) endowed with a left action \(\mu : A \otimes X \to X\), which is a map of differential bigraded modules of bidegree \((0, 0)\) and satisfies the condition \(\mu(\pi \otimes 1) = \mu(1 \otimes \mu)\).

Given a differential bigraded module \((X, d)\), consider the corresponding colored differential algebra \((\text{hom}(X; X), d, \pi)\). It is easy to see that endowing \((X, d)\) with the structure of a left differential module \((X, d, \mu)\) over a colored differential algebra \((A, d, \pi)\) is equivalent to specifying a map \(\tilde{\mu} : (A, d, \pi) \to (\text{hom}(X; X), d, \pi)\) of colored differential algebras. Indeed, the maps \(\mu\) and \(\tilde{\mu}\) uniquely determine each other by the formula \((\tilde{\mu}(a))(x) = \mu(a \otimes x)\), where \(a \in A, x \in X\).

Now we note that for any quadratic colored algebra \((R, \pi)\) there is the twisting cochain \(\varphi^1 : R^\ast \to R_{\ast-1}\), which is defined by the following rule:

\[
\varphi^1([x_1, \ldots, x_k]) = \begin{cases} 
  x_1, & \text{if } k = 1, \\
  0, & \text{if } k > 0,
\end{cases}
\]

where \((R^\ast, \nabla, \vartheta)\) is the colored coalgebra Koszul dual to the quadratic colored algebra \((R, \pi)\). It follows that, as mentioned above, for any quadratic colored algebra \((R, \pi)\) the map of colored differential algebras \(\Omega(\varphi^1) : (\Omega(R^\ast), d, \pi) \to (R, d = 0, \pi)\) is defined, which makes it possible to regard any differential \(R\)-module as a differential \(\Omega(R^\ast)\)-module.

To describe the homotopy properties of differential \(\Omega(R^\ast)\)-modules we first consider the colored variant of a homotopy technique of differential Lie modules over graded coalgebras [10].

Let \((C, \nabla)\) be any connected colored graded coalgebra with counit and coaugmentation.

A differential Lie module \((X, d, \psi)\) over a colored coalgebra \((C, \nabla)\) or, briefly, a differential Lie \(C\)-module is an arbitrary differential bigraded module \((X, d)\) equipped with the map of bidegree \(\psi : (C \otimes X)_{\ast, \bullet} \to X_{\ast, \bullet-1}\) of bidegree \((0, -1)\) that satisfies the following conditions:

1. \(\psi(\nu \otimes 1) = d\), where \(\nu : K_1 \to C\) is the coaugmentation of a colored graded coalgebra \((C, \nabla)\).

2. \(d(\psi) + \psi + \psi = 0\), where \(\psi : (C \otimes X)_{\ast, \bullet} \to X_{\ast, \bullet-1}\) is considered as the element \(\psi \in \text{Hom}(C \otimes X; X)_{0, -1}\) of the differential bigraded module \((\text{Hom}(C \otimes X; X), d)\) and \(\psi + \psi : (C \otimes X)_{\ast, \bullet} \to X_{\ast, \bullet-2}\) is defined by the formula \(\psi + \psi = \psi(1 \otimes \psi)(\nabla \otimes 1)\).

The same way as it was done for non-colored coalgebras in [10], it is easy to verify that the consideration of the structure of a differential Lie module \((X, d, \psi)\) over a colored graded coalgebra \((C, \nabla)\) on a differential bigraded module \((X, d)\) is equivalent to the consideration of a perturbation \(t : (C \otimes X)_{\ast, \bullet} \to (C \otimes X)_{\ast, \bullet-1}\) of the differential \(1 \otimes d : (C \otimes X)_{\ast, \bullet} \to (C \otimes X)_{\ast, \bullet-1}\) on the differential module \((C \otimes X, 1 \otimes d)\), which is a derivation of the free \(C\)-comodule \(C \otimes X\).

A morphism \(f : (X, d, \psi) \to (Y, d, \psi)\) of differential Lie modules over a colored graded coalgebra \((C, \nabla)\) is a map of bigraded modules \(f : (C \otimes X)_{\ast, \bullet} \to Y_{\ast, \bullet}\) of bidegree \((0, 0)\) that satisfies the condition \(d(f) + \psi + f + f \otimes \psi = 0\), where \(f\) is considered as the element \(f \in \text{Hom}(C \otimes X; Y)_{0, 0}\) of the differential bigraded module
(\text{Hom}(C \otimes X; X), d) \) and \( \psi \cup f : (C \otimes X)_{*, \bullet} \to Y_{*, \bullet-1} \), \( f \cup \psi : (C \otimes X)_{\bullet, *-1} \to Y_{\bullet, *-1} \)
respectively are defined by the formulae

\[
\psi \cup f = \psi(1 \otimes f)(\nabla \otimes 1), \quad f \cup \psi = f(1 \otimes \psi)(\nabla \otimes 1).
\]

The composition \( \text{gf} : (X, d, \psi) \to (X', d, \psi) \) of morphisms \( f : (X, d, \psi) \to (X', d, \psi) \)
and \( g : (X', d, \psi) \to (X'', d, \psi) \) of differential Lie modules
over a colored graded coalgebra \((C, \nabla)\) is defined as
the map \( \text{gf} : (C \otimes X)_{*, \bullet} \to (C \otimes X')_{*, \bullet} \) that
is given by the formula \( \text{gf} = g \circ f = g(1 \otimes f)(1 \otimes \nabla) \). The identity morphism \( 1_{(X, d, \psi)} : (X, d, \psi) \to (X, d, \psi) \) for every differential Lie \( \text{C-module} \ (X, d, \psi) \)
is defined as the map \( 1_{(C \otimes X), \bullet} : Y_{, \bullet} \to Y_{, \bullet} \) that
satisfies the condition \( d(h) + \psi \cup h + h \cup \psi = f - g \), where \( h \)
is considered as the element \( h \in \text{Hom}(C \otimes X; Y)_{01} \) of the
differential bigraded module \( \text{Hom}(C \otimes X; Y), d \) and maps \( \psi \cup h : (C \otimes X)_{*, \bullet} \to Y_{*, \bullet} ; h \cup \psi : (C \otimes X)_{*, \bullet} \to Y_{*, \bullet} \)
respectively are defined by the formulae

\[
\psi \cup h = \psi(1 \otimes h)(\nabla \otimes 1), \quad h \cup \psi = h(1 \otimes \psi)(\nabla \otimes 1).
\]

Let \( \eta : (X, d, \psi) \simeq (Y, d, \psi) ; \xi \) be any morphisms of differential Lie \( \text{C-modules} \)
with counit \( \varepsilon : C \to K_{I} \) and coaugmentation \( \nu : K_{I} \to C \),
and let \( \eta : (X, d, \psi) \to (X, d, \psi) \) be a homotopy between the
morphisms \( \xi \eta \) and \( 1_{X} \) of differential Lie \( \text{C-modules} \) which satisfies the conditions
\( \eta h = 0, \xi h = 0, \eta h = 0 \). Every triple \( (\eta : (X, d, \psi) \simeq (Y, d, \psi) ; \xi, h) \) of the above
form is said to be a SDR-data of differential Lie \( \text{C-modules} \).

The following assertion, which is proved in the same way as it is done in [10] for
non-colored coalgebras, establishes the homotopy invariance of the structure of a
differential Lie module over any colored graded coalgebra under homotopy
equivalences of the type of SDR-data of differential bigraded modules.

**Theorem 1.1.** Let \( (X, d, \psi) \) be a differential Lie module over a connected graded
calgebra \((C, \nabla)\) with counit \( \varepsilon : C \to K_{I} \) and coaugmentation \( \nu : K_{I} \to C \),
and let \( (\eta : (X, d, \psi) \simeq (Y, d, \psi) ; \xi, h) \) be any SDR-data of differential bigraded modules.
Then the differential bigraded module \((Y, d, \psi) \) admits the structure of a differential Lie
\( \text{C-module} \ (Y, d, \psi) \), which is defined by the formula

\[
\overline{\psi} = \varepsilon \otimes d + \eta t(1 \otimes \xi) + \eta t(1 \otimes h)(1 \otimes t)(\nabla \otimes 1)(1 \otimes \xi) + \ldots
\]

Moreover, the formulas

\[
\overline{\xi} = \varepsilon \otimes \xi + (\varepsilon \otimes h)(1 \otimes t)(\nabla \otimes 1)(1 \otimes \xi) + + (\varepsilon \otimes h)(1 \otimes t)(\nabla \otimes 1)(1 \otimes h)(1 \otimes t)(\nabla \otimes 1)(1 \otimes \xi) + \ldots;
\]

\[
\overline{\eta} = \varepsilon \otimes \eta + (\varepsilon \otimes \eta)(1 \otimes t)(\nabla \otimes 1)(1 \otimes h) + + (\varepsilon \otimes \eta)(1 \otimes t)(\nabla \otimes 1)(1 \otimes h)(1 \otimes t)(\nabla \otimes 1)(1 \otimes h) + \ldots;
\]

\[
\overline{h} = \varepsilon \otimes h + (\varepsilon \otimes h)(1 \otimes t)(\nabla \otimes 1)(1 \otimes h) + \ldots.
\]
where \( t = \psi - (\varepsilon \otimes d) \), define the SDR-data \( (\overline{\eta} : (X, d, \psi) \rightarrow (Y, d, \overline{\psi}) : \overline{\xi}, \overline{h}) \) of differential Lie \( C \)-modules.

Further the co-\( B \)-construction \( (\Omega(R^!), d, \pi) \), where \((R^!, \nabla)\) is the colored coalgebra Koszul dual to a quadratic colored algebra \((R, \pi)\), we will be denoted by \((R_\infty, d, \pi)\).

Let us consider homotopy properties of differential \( R_\infty \)-modules. It is easy to see that the introduction of the structure of a differential \( R_\infty \)-module on a differential bigraded module \((X, d)\) is equivalent to the introduction of the structure of a differential \( R^! \)-module on \((X, d)\). Indeed, every differential \( R_\infty \)-module \((X, d, \mu)\) defines the differential \( R^! \)-module \((X, d, \psi_\mu)\), where \( \psi_\mu = \mu(\varphi \otimes 1) : (R^! \otimes X)_{s_1} \rightarrow X_{s_2-1} \).

Conversely, the structure map \( \psi : (R^! \otimes X)_{s_1, 1, 1} \rightarrow X_{s_2, 1, 1-1} \) of a differential \( R^! \)-module \((X, d, \psi)\) determines the twisting cochain \( \psi : R^! \rightarrow \text{hom}(X; X) \). It twisting cochain induces the map \( \mu_\psi = \Omega(\overline{\psi}) : \Omega(R^!) = R_\infty \rightarrow \text{hom}(X; X) \) of colored differential algebras, which as said above defines the structure map \( \mu_\psi : R_\infty \otimes X \rightarrow X \) of the differential \( R_\infty \)-module \((X, d, \mu_\psi)\).

**Definition 1.3.** By an \( R_\infty \)-map \( f : (X, d, \mu) \rightarrow (Y, d, \mu) \) of differential \( R_\infty \)-modules for a given any quadratic colored algebra \((R, \pi)\) further we mean a morphism \( f : (X, d, \psi_\mu) \rightarrow (Y, d, \psi_\mu) \) of the corresponding differential \( R^! \)-modules.

Thus, an \( R_\infty \)-map \( f : (X, d, \mu) \rightarrow (Y, d, \mu) \) of differential \( R_\infty \)-modules is a map \( f : (R^! \otimes X)_{s_1, 1, 1} \rightarrow Y_{s_2, 1, 1-1} \) that satisfies the condition \( d(f) + \psi_\mu \cup f + f \cup \psi_\mu = 0 \).

**Definition 1.4.** By an \( R_\infty \)-homotopy \( h : (X, d, \mu) \rightarrow (Y, d, \mu) \) between given \( R_\infty \)-maps \( f, g : (X, d, \mu) \rightarrow (Y, d, \mu) \) of differential \( R_\infty \)-modules for any quadratic colored algebra \((R, \pi)\) we mean a homotopy \( h : (X, d, \psi_\mu) \rightarrow (Y, d, \psi_\mu) \) between morphisms \( f, g : (X, d, \psi_\mu) \rightarrow (Y, d, \psi_\mu) \) of the corresponding differential \( R^! \)-modules.

Thus, an \( R_\infty \)-homotopy \( h : (X, d, \mu) \rightarrow (Y, d, \mu) \) between given \( R_\infty \)-maps \( f, g : (X, d, \mu) \rightarrow (Y, d, \mu) \) of differential \( R_\infty \)-modules is a map \( h : (R^! \otimes X)_{s_1, 1, 1} \rightarrow Y_{s_2, 1, 1-1} \) that satisfies the condition \( d(f) + \psi_\mu \cup h + h \cup \psi_\mu = f - g \).

Suppose that \( \eta : (X, d, \mu) \rightarrow (Y, d, \mu) : \xi \) are \( R_\infty \)-maps of differential \( R_\infty \)-modules such that \( \eta \xi = 1_Y \) and \( h : (X, d, \mu) \rightarrow (X, d, \mu) \) is an \( R_\infty \)-homotopy between the \( R_\infty \)-maps \( \eta \xi \) and \( 1_X \) of differential \( R_\infty \)-modules which satisfies the conditions \( \eta h = 0, \xi h = 0, hh = 0 \). Every triple \( (\eta : (X, d, \mu) \rightarrow (Y, d, \mu) : \xi, h) \) of the above form is said to be \( R_\infty \)-SDR-data of differential \( R_\infty \)-modules.

The following theorem, which follows from Theorem 1.1, asserts the \( R_\infty \)-homotopy invariance of the structure of a differential \( R_\infty \)-module under homotopy equivalences of the type of SDR-data of differential bigraded modules.

**Theorem 1.2.** Let an arbitrary differential \( R_\infty \)-module \((X, d, \mu)\) and any SDR-data \((\eta : (X, d) \rightarrow (Y, d) : \xi, h)\) of differential bigraded modules are given. Then \((Y, d)\) can be equipped with the structure of a differential \( R_\infty \)-module \((Y, d, \mu)\) given by the formula (1.1). Moreover, there is an \( R_\infty \)-SDR-data \((\overline{\eta} : (X, d, \mu) \rightarrow (Y, d, \mu) : \overline{\xi}, \overline{h})\) of differential \( R_\infty \)-modules, which is defined by the formulas (1.2) – (1.4) and satisfies the initial conditions \( \overline{\eta}(\nu \otimes 1) = \eta, \overline{\xi}(\nu \otimes 1) = \xi, \overline{h}(\nu \otimes 1) = h \), where \( \nu : K_f \rightarrow R^! \) is coaugmentation of the colored coalgebra \((R^!, \nabla)\) Koszul dual to a quadratic colored algebra \((R, \pi)\). □

§ 2. Differential modules with \( \infty \)-simplicial faces

8
Let \((F, \pi)\) be the colored graded \(K\)-algebra whose generators are the elements \(\partial^n_i \in F(n-1, n)_0, \; n-1 \in I, \; i \in \mathbb{Z}, \; 0 \leq i \leq n\), connected by the simplicial commutation relations

\[
\partial^n_i \partial^n_j = \partial^{n-1}_j \partial^n_i, \quad i < j, \; n-1 \in I,
\]

where \(\partial_i^n \partial^n_j = \pi(\partial^{n-1}_i \otimes \partial^n_j)\). Further the colored graded algebra \((F, \pi)\) will be called the colored algebra of simplicial faces.

It is easy to see that the colored algebra of simplicial faces \((F, \pi)\) is a quadratic colored algebra. Indeed, let \(M\) be a colored graded module defined by the conditions \(M(s, t)_m = 0\) for \(s, t \in I, \; m > 0, \; M(s, t)_0 = 0\) for \((s, t) \neq (n-1, n), \; n-1 \in I, \; M(n-1, n)_0\) is a free \(K\)-module with the generators \(\partial^n_i\), where \(n-1 \in I, \; 0 \leq i \leq n.\)

In the colored graded module \(M \otimes M\) we consider the submodule \(Q\) that is defined by the conditions \(Q(s, t)_m = 0\) for \(s, t \in I, \; m > 0, \; Q(s, t)_0 = 0\) for \((s, t) \neq (n-2, n), \; n-2 \in I, \; Q(n-2, n)_0\) is a free \(K\)-module with the generators \(\partial^{n-1}_i \otimes \partial^n_j - \partial^n_j \otimes \partial^{n-1}_i\), where \(n-2 \in I, \; 0 \leq i < j \leq n\). It is clear that the colored algebra of simplicial faces \((F, \pi)\) is isomorphic to the quotient algebra \(T(M)/(Q)\), and hence \((F, \pi)\) is a quadratic colored algebra.

**Definition 2.1.** By a differential module with simplicial faces we mean an arbitrary differential module \((X, d, \mu)\) over the colored algebra of simplicial faces \((F, \pi)\). By a map \(f : (X, d, \mu) \rightarrow (Y, d, \mu)\) of differential modules with simplicial faces we mean any map \(f : (X, d) \rightarrow (Y, d)\) of differential modules that satisfies the condition \(f \mu = \mu(1_F \otimes f)\).

By a homotopy between maps \(f, g : (X, d, \mu) \rightarrow (Y, d, \mu)\) of differential modules with simplicial faces we mean any homotopy \(h : X_{i, \bullet} \rightarrow Y_{i, \bullet+1}\) between maps \(f, g : (X, d) \rightarrow (Y, d)\) of differential bigraded modules that the condition \(h \mu = \mu(1_F \otimes h)\).

It is clear that the consideration of a differential module with simplicial faces \((X, d, \mu)\) is equivalent to the consideration of a differential module \((X, d)\) equipped with a family of the maps \(\mu(\partial^n_i) = \partial_i : X_{n, \bullet} \rightarrow X_{n-1, \bullet}\), where \(0 \leq i \leq n, \; n \geq 0\), which are maps of differential modules, i.e. satisfy the condition \(d \partial_i + \partial_i d = 0\), and also satisfy the simplicial commutation relations

\[
\partial_i \partial_j = \partial_{j-1} \partial_i : X_{n, \bullet} \rightarrow X_{n-2, \bullet}, \quad 0 \leq i < j \leq n.
\]  \hspace{1cm} (2.1)

The above maps \(\partial_i\) of differential modules are referred to as simplicial operators of faces or, more briefly, simplicial faces of the differential bigraded module \((X, d)\).

It is easy to see that the consideration of a map \(f : (X, d, \partial_i) \rightarrow (Y, d, \partial_i)\) of differential modules with simplicial faces is equivalent to the consideration of a map \(f : (X, d) \rightarrow (Y, d)\) of differential modules that satisfies the condition

\[
\partial_i f = f \partial_i : X_{n, \bullet} \rightarrow Y_{n-1, \bullet}, \quad 0 \leq i \leq n.
\]  \hspace{1cm} (2.2)

The consideration of a homotopy between maps \(f, g : (X, d, \partial_i) \rightarrow (Y, d, \partial_i)\) of differential modules with simplicial faces is equivalent to the consideration of a homotopy \(h : X_{i, \bullet} \rightarrow Y_{i, \bullet+1}\) between maps \(f, g : (X, d) \rightarrow (Y, d)\) of differential bigraded modules that satisfies the condition

\[
\partial_i h + h \partial_i = 0 : X_{n, \bullet} \rightarrow Y_{n-1, \bullet+1}, \quad 0 \leq i \leq n.
\] \hspace{1cm} (2.3)
Note that a difference between the concept of a differential module with simplicial faces and the notion of a simplicial differential module, i.e. of a simplicial object in the category of differential modules [3], lies in the fact that for differential modules with simplicial faces not assumed the existence of simplicial degeneracy operators.

We now carry some necessary considerations for a introduction of the concept of a differential module with \( \infty \)-simplicial faces.

Let there be given arbitrary nonnegative integer \( n \geq 0 \). Any collection of nonnegative integers \((i_1, \ldots, i_k)\), where \( 0 \leq i_1 < \ldots < i_k \leq n \), further will be called a ordered collection. Let \( \Sigma_k \) be the symmetric group of permutations of \( k \) symbols. For any permutation \( \sigma \in \Sigma_k \) and any ordered collection \((i_1, \ldots, i_k)\) we consider the collection \((\sigma(i_1), \ldots, \sigma(i_k))\), where \( \sigma \) acts on the collection \((i_1, \ldots, i_k)\) in a standard way, i.e., permutes the numbers in this collection. For this collection \((\sigma(i_1), \ldots, \sigma(i_k))\) we define the collection \((\overline{\sigma(i_1)}, \ldots, \overline{\sigma(i_k)})\) by setting

\[
\overline{\sigma(i_s)} = \sigma(i_s) - \alpha(\sigma(i_s)), \quad 1 \leq s \leq k,
\]

\( \alpha(\sigma(i_s)) \) is the number of those elements of \((\sigma(i_1), \ldots, \sigma(i_s), \ldots, \sigma(i_k))\) on the right of \( \sigma(i_s) \) which are smaller than \( \sigma(i_s) \).

Consider the colored coalgebra \((F^\dagger, \nabla)\) Koszul dual to the colored algebra of simplicial faces \((F, \pi)\). It is easy to see that generators of the free \( K \)-module \((\Omega(F^\dagger))^{(k)}(n-k, n)_k\), \( k \geq 1, n - k \in I \), are the elements

\[
\partial_{\overline{\sigma(i_1)}} \ldots \partial_{\overline{\sigma(i_k)}} = \sum_{\sigma \in \Sigma_k} (-1)^{\text{sign}(\sigma)} [\partial_{\sigma(i_1)}^{n-k+1}] \ldots [\partial_{\sigma(i_k)}^{n}], \quad 0 \leq i_1 < \ldots < i_k \leq n,
\]

and that the equality \((F^\dagger)(k)(s,t)_m = 0 \) for \((s,t) \neq (n-k,n), m \neq k, n-k \in I \), holds.

A direct calculation shows that a comultiplication \( \nabla \) of the colored graded coalgebra \( F^\dagger \) on the generators \([\partial_{\overline{\sigma(i_1)}}^{n-k+1}] \ldots \partial_{\overline{\sigma(i_k)}}^{n}] \) of the module \((\Omega(F^\dagger))^{(k)}(n-k, n)_k\), \( k \geq 1, n - k \in I \), is given by the following formula:

\[
\nabla([\partial_{\overline{\sigma(i_1)}}^{n-k+1}] \ldots \partial_{\overline{\sigma(i_k)}}^{n}] = 1_{n-k} \otimes ([\partial_{\overline{\sigma(i_1)}}^{n-k+1}] \ldots \partial_{\overline{\sigma(i_k)}}^{n}] + ([\partial_{\overline{\sigma(i_1)}}^{n-k+1}] \ldots \partial_{\overline{\sigma(i_k)}}^{n}] \otimes 1_n + \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} (-1)^{\text{sign}(\sigma)} [\partial_{\sigma(i_1)}^{n-k+1}] \ldots [\partial_{\sigma(i_m)}^{n-k+m+1}] [\partial_{\overline{\sigma(i_m+1)}}^{n}] \ldots [\partial_{\overline{\sigma(i_k)}}^{n}],
\]

where \( I_\sigma \) is the set all partitions of the collection \((\overline{\sigma(i_1)}, \ldots, \overline{\sigma(i_k)})\) into two ordered collections \((\overline{\sigma(i_1)}, \ldots, \overline{\sigma(i_m)})\) and \((\overline{\sigma(i_{m+1})}, \ldots, \overline{\sigma(i_k)})\), \( 1 \leq m \leq k-1 \).

**Definition 2.2.** The colored differential algebra \((\Omega(F^\dagger), d, \pi)\) will be called the colored algebra of \( \infty \)-simplicial faces and denoted by \((F^\infty, d, \pi)\).

It is easy to see that generators of colored algebra \((F^\infty, \pi)\) are the elements

\[
\partial_{(i_1, \ldots, i_k)}^n = [\partial_{\overline{\sigma(i_1)}}^{n-k+1}] \ldots [\partial_{\overline{\sigma(i_k)}}^{n}] \in F^\infty(n-k,n)_{k-1}, \quad 0 \leq i_1 < \ldots < i_k \leq n, \quad n - k \in I,
\]

that connected by the relations

\[
d(\partial_{(i_1, \ldots, i_k)}^n) = \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} (-1)^{\text{sign}(\sigma)} \partial_{\sigma(i_1), \ldots, \sigma(i_m)}^{n-(k-m)} \partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)),\quad (2.4)}
\]
where \( \partial_{(\sigma(i_1),\ldots,\sigma(i_m))} \partial_{(\sigma(i_{m+1}),\ldots,\sigma(i_k))} = \pi(\partial_{(\sigma(i_1),\ldots,\sigma(i_m))} \otimes \partial_{(\sigma(i_{m+1}),\ldots,\sigma(i_k))}) \), and \( I_\sigma \) is just like that in the above formula for the comultiplication of the colored graded coalgebra \( (F^d, \nabla) \).

The following statement helps to simplify the procedure for enumerating all partitions from the set \( I_\sigma \), by which goes a summation in the formula (2.4).

**Proposition 2.1.** Let any ordered collection \((i_1, \ldots, i_k)\) and any permutation \( \sigma \in \Sigma_k \) are given. The partition \((\sigma(i_1), \ldots, \sigma(i_m))\) \( \mid \sigma(i_{m+1}), \ldots, \sigma(i_k) \) is a partition into two ordered collections if and only if the partition \((\sigma(i_1), \ldots, \sigma(i_m))\) \( \mid \sigma(i_{m+1}), \ldots, \sigma(i_k) \) also is a partition into two ordered collections.

**Proof.** Suppose that \((\sigma(i_1), \ldots, \sigma(i_m)) \mid \sigma(i_{m+1}), \ldots, \sigma(i_k) \) is a partition into two ordered collections. Let us show that \((\sigma(i_1), \ldots, \sigma(i_m)) \mid \sigma(i_{m+1}), \ldots, \sigma(i_k) \) is a partition into two ordered collections. It is easy to see that \( I_\sigma = \sigma(i_k) \) for \( m + 1 \leq t \leq k \) and hence we have \( \sigma(i_{m+1}) < \ldots < \sigma(i_k) \). Denote by \( \beta(\sigma(i_s)) \), where \( 2 \leq s \leq m \), the number of those elements of \((\sigma(i_1), \ldots, \sigma(i_m))\) on the right of \( \sigma(i_s) \) which are smaller than \( \sigma(i_s) \) and larger \( \sigma(i_{s-1}) \). It is clear that \( \beta(\sigma(i_s)) = \alpha(\sigma(i_s)) - \alpha(\sigma(i_{s-1})) \). Moreover, since \( \sigma(i_{s-1}) < \sigma(i_s) \), it follows that \( \beta(\sigma(i_s)) \leq \sigma(i_k) - \sigma(i_{s-1}) - 1 \). From this we obtain

\[
\sigma(i_{s-1}) = \sigma(i_s) - \alpha(\sigma(i_{s-1})) < \sigma(i_s) - \beta(\sigma(i_s)) - \alpha(\sigma(i_{s-1})) = \sigma(i_s)
\]

and hence we have \( \sigma(i_1) < \ldots < \sigma(i_m) \). We now show the converse. Suppose that \((\sigma(i_1), \ldots, \sigma(i_m)) \mid \sigma(i_{m+1}), \ldots, \sigma(i_k) \) is a partition into two ordered collections. Since \( \sigma(i_{s-1}) < \sigma(i_s) \), \( 2 \leq s \leq m \), it follows that \( \sigma(i_{s-1}) - \sigma(i_s) < \alpha(\sigma(i_{s-1})) - \alpha(\sigma(i_s)) \).

Now assume the opposite, i.e. assume that for some \( 2 \leq s \leq m \) the condition \( \sigma(i_{s-1}) > \sigma(i_s) \) is true. For the above \( s \) we denote by \( \gamma(\sigma(i_{s-1})) \) the number of those elements of \((\sigma(i_1), \ldots, \sigma(i_s), \ldots, \sigma(i_k))\) on the right of \( \sigma(i_{s-1}) \) which are smaller than \( \sigma(i_{s-1}) \) and larger \( \sigma(i_s) \) or equal to \( \sigma(i_s) \). It is easy to see that \( \alpha(\sigma(i_{s-1})) = \alpha(\sigma(i_s)) + \gamma(\sigma(i_{s-1})) \) and \( \gamma(\sigma(i_{s-1})) \leq \sigma(i_{s-1}) - \sigma(i_s) \). From this we obtain the condition

\[
\alpha(\sigma(i_{s-1})) - \alpha(\sigma(i_s)) = \gamma(\sigma(i_{s-1})) \leq \sigma(i_{s-1}) - \sigma(i_s),
\]

which contradicts to the condition \( \sigma(i_{s-1}) - \sigma(i_s) < \alpha(\sigma(i_{s-1})) - \alpha(\sigma(i_s)) \). Thus, we have \( \sigma(i_1) < \ldots < \sigma(i_m) \). Carrying out similar arguments in the case \( \sigma(i_{s-1}) < \sigma(i_s) \), \( m+2 \leq s \leq k \), we obtain \( \sigma(i_{m+1}) < \ldots < \sigma(i_k) \). \[ \blacksquare \]

**Definition 2.3.** By a differential module with \( \infty \)-simplicial faces we mean an arbitrary differential module \((X, d, \mu)\) over the colored algebra of \( \infty \)-simplicial faces \((F_\infty, d, \pi)\). By a morphism \( f : (X, d, \mu) \rightarrow (Y, d, \mu) \) of differential modules with \( \infty \)-simplicial faces we mean any \( F_\infty \)-map \( f : (X, d, \mu) \rightarrow (Y, d, \mu) \) of differential \( F_\infty \)-modules. By a homotopy between morphisms \( f, g : (X, d, \mu) \rightarrow (Y, d, \mu) \) of differential modules with \( \infty \)-simplicial faces we mean any \( F_\infty \)-homotopy \( h : (X, d, \mu) \rightarrow (Y, d, \mu) \) between \( F_\infty \)-maps \( f \) and \( g \) of differential \( F_\infty \)-modules. By a SDR-data of differential modules with \( \infty \)-simplicial faces we mean any \( F_\infty \)-SDR-Data of differential \( F_\infty \)-modules.

It is easy to see that the consideration of a differential module with \( \infty \)-simplicial faces \((X, d, \mu)\) is equivalent to the consideration of a differential bigraded module.
\((X, d)\) equipped with a family of module maps

\[
\tilde{\partial} = \{ \partial_{(i_1, \ldots, i_k)} = \tilde{\mu}(\partial_{i_1}^{n-k+1} \wedge \cdots \wedge \partial_{i_k}^n) : X_{n, \bullet} \to X_{n-k, \bullet+k-1} \}, \quad n \geq 0,
\]

where \(0 \leq i_1 < \ldots < i_k \leq n\), which satisfy the following relations:

\[
d(\partial_{(i_1, \ldots, i_k)}) = \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} (-1)^{\text{sign}(\sigma)+1} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))}, \quad (2.5)
\]

The above module maps \(\partial_{(i_1, \ldots, i_k)}\) will be called \(\infty\)-simplicial faces of a differential bigraded module \((X, d)\). Further, each differential module with \(\infty\)-simplicial faces \((X, d, \mu)\) will be identified with the corresponding triple \((X, d, \tilde{\partial})\).

For \(k = 1\), the relations \((2.5)\) take the form \(d(\partial_{(i)}) = 0\). This indicates that all \(\infty\)-simplicial faces \(\partial_{(i)}\) are maps of differential modules. For \(k = 2\), relations \((2.5)\) take the form

\[
d(\partial_{(i,j)}) = \partial_{(j-1)} \partial_{(i)} - \partial_{(i)} \partial_{(j)}, \quad i < j.
\]

This means that the \(\infty\)-simplicial face \(\partial_{(i,j)}\) is a homotopy between the maps of differential modules \(\partial_{(j-1)} \partial_{(i)}\) and \(\partial_{(i)} \partial_{(j)}\). Thus, \(\infty\)-simplicial faces \(\partial_{(i)}\) satisfy the simplicial commutation relations \((2.1)\) up to homotopy. For \(k = 3\), by using Proposition \(2.1\) we easily obtain that the relations \((2.5)\) take the following form:

\[
d(\partial_{(i_1, i_2, i_3)}) = -\partial_{(i_1)} \partial_{(i_2, i_3)} - \partial_{(i_1, i_2)} \partial_{(i_3)} - \partial_{(i_3-2)} \partial_{(i_1, i_2)} -
\]

\[
- \partial_{(i_2-1, i_3-1)} \partial_{(i_1)} + \partial_{(i_2-1)} \partial_{(i_1, i_3)} + \partial_{(i_1, i_3-1)} \partial_{(i_2)}, \quad i_1 < i_2 < i_3.
\]

The consideration of a morphism \(f : (X, d, \mu) \to (Y, d, \mu)\) of differential modules with \(\infty\)-simplicial faces is equivalent to the consideration of a family of module maps

\[
f_{(i_1, \ldots, i_k)} = \tilde{f}(1_n) : X_{n, \bullet} \to Y_{n, \bullet},
\]

where \(1_n = 1 \in K = (F^d)^{(0)}(n, n)_0\), \(\tilde{f}(a)(x) = f(a \otimes x)\), \(a \in F^d\), \(0 \leq i_1 < \ldots < i_k \leq n\), which satisfy the conditions

\[
d(f_{(i)}) = 0, \quad d(f_{(i_1, \ldots, i_k)}) = -\partial_{(i_1, \ldots, i_k)} f_{(i)} + f_{(i)} \partial_{(i_1, \ldots, i_k)} +
\]

\[
= \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} (-1)^{\text{sign}(\sigma)+1} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} f_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))} - f_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))}, \quad (2.6)
\]

Further, each morphism \(f : (X, d, \mu) \to (Y, d, \mu)\) of differential modules with \(\infty\)-simplicial faces will be identified with the corresponding family of maps \(\tilde{f} : (X, d, \tilde{\partial}) \to (Y, d, \tilde{\partial})\).

For \(k = 1\), the relations \((2.6)\) take the form

\[
d(f_{(i)}) = f_{(i)} \partial_{(i)} - \partial_{(i)} f_{(i)}, \quad i \geq 0.
\]
This means that the map \( f_{(i)} \) is a homotopy between the maps of differential modules \( f_{(i)} \partial_{(i)} \) and \( \partial_{(i)} f_{(i)} \). Thus, the map of differential modules \( f_{(i)} \) satisfies the condition (2.2) up to homotopy. For \( k = 2 \), the relations (2.6) take the following form:

\[
d(f_{(i,j)}) = -\partial_{(i,j)} f_{(i)} + f_{(i,j)} \partial_{(i,j)} - \partial_{(i)} f_{(j)} + \partial_{(j-1)} f_{(i)} + f_{(i)} \partial_{(j)} - f_{(j-1)} \partial_{(i)}, \quad i < j.
\]

The consideration of a homotopy \( h : (X, d, \mu) \to (Y, d, \mu) \) between morphism \( f, g : (X, d, \mu) \to (Y, d, \mu) \) of differential modules with \( \infty \)-simplicial faces is equivalent to the consideration of a family of module maps

\[
\tilde{h} = \{ h_{(i)} = \tilde{h}(1_n) : X_{n, \bullet} \to Y_{n, \bullet + 1}, \]

where \( 1_n = 1 \in K = (F^l)^{(0)}(n, n)_0, \ h(a)(x) = h(a \otimes x), \ a \in F^l, \ 0 \leq i_1 < \ldots < i_k \leq n, \) which satisfy the conditions

\[
d(h_{(i)}) = f_{(i)} - g_{(i)}, \quad d(h_{(i_1, \ldots, i_k)}) =
\]

\[
= f_{(i_1, \ldots, i_k)} - g_{(i_1, \ldots, i_k)} - \partial_{(i_1, \ldots, i_k)} h_{(i)} - h_{(i)} \partial_{(i_1, \ldots, i_k)} + \sum_{\sigma \in \Sigma_k} \sum_{\ell_{\sigma}} (-1)^{\text{sign}(\sigma) + 1} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} h_{(\sigma(i_{m+1}), \ldots, \sigma(i_{k}))} + h_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_{k}))}.
\]

Further, each homotopy \( h : (X, d, \mu) \to (Y, d, \mu) \) of differential modules with \( \infty \)-simplicial faces will be identified with the corresponding family of maps \( \tilde{h} : (X, d, \tilde{\partial}) \to (Y, d, \tilde{\partial}). \)

For \( k = 1 \), the relations (2.7) take the form

\[
d(h_{(i)}) = f_{(i)} - g_{(i)} - \partial_{(i)} h_{(i)} - h_{(i)} \partial_{(i)}, \quad i \geq 0.
\]

These equalities admit a good interpretation in the following special case. Suppose that \( f, g : (X, d, \partial_{(i)}) \to (Y, d, \partial_{(j)}) \) are any maps of differential modules with simplicial faces, which we regard as morphisms \( \tilde{f}, \tilde{g} : X \to Y \) of differential modules with \( \infty \)-simplicial faces, and let \( h : X \to Y \) be a homotopy between \( f \) and \( g \). In the case under consideration, we have \( f_{(i)} = g_{(i)} = 0 \); therefore, the relations \( d(h_{(i)}) = f_{(i)} - g_{(i)} - \partial_{(i)} h_{(i)} - h_{(i)} \partial_{(i)}, \ i \geq 0, \) can be written as the relations

\[
d(h_{(i)}) = 0 - (\partial_{(i)} h_{(i)} + h_{(i)} \partial_{(i)}), \quad i \geq 0,
\]

which means that the map \( h_{(i)} \) is a homotopy between the maps \( 0 \) and \( \partial_{(i)} h_{(i)} + h_{(i)} \partial_{(i)} \) of differential modules. Thus, in considered case the homotopy \( h_{(i)} \) between \( f_{(i)} \) and \( g_{(i)} \) satisfies the condition (2.3) up to homotopy. For \( k = 2 \), the relations (2.7) take the following form:

\[
d(h_{(i,j)}) = f_{(i,j)} - g_{(i,j)} - \partial_{(i,j)} h_{(i)} - h_{(i)} \partial_{(i,j)} - \partial_{(i)} h_{(j)} + \partial_{(j-1)} h_{(i)} - h_{(i)} \partial_{(j)} + h_{(j-1)} \partial_{(i)}), \quad i < j.
\]
Applying the theorem 1.2 to the colored algebra \((F, \pi)\) of simplicial faces we obtain the assertion, which establishes the homotopy invariance of the structure of a differential module with \(\infty\)-simplicial faces under homotopy equivalences of the type of SDR-data of differential bigraded modules.

**Theorem 2.1.** Let any differential module with \(\infty\)-simplicial faces \((X, d, \tilde{\partial})\) and an arbitrary SDR-data \((\eta : (X, d) \longrightarrow (Y, d) : \xi, h)\) of differential bigraded modules are given. Then \((Y, d)\) can be equipped with the structure of a differential module with \(\infty\)-simplicial faces \((Y, d, \tilde{\partial})\) given by the formula (1.1). Moreover, there is an \(F_{\infty}\)-SDR-data \((\tilde{\eta} : (X, d, \mu) \longrightarrow (Y, d, \mu) : \tilde{\xi}, \tilde{h})\) of differential modules with \(\infty\)-simplicial faces, which is defined by the formulas (1.2) – (1.4) and satisfies the initial conditions \(\eta(\cdot) = \eta, \xi(\cdot) = \xi, h(\cdot) = h\). ■

§ 3. The chain realization of differential modules with \(\infty\)-simplicial faces and the \(B\)-construction for \(A_{\infty}\)-algebras

For the colored algebra of \(\infty\)-simplicial faces \((F_{\infty}, d, \pi)\) the differential graded module \((F_{\infty}(m, n), d)\) we denoted by \((F_{\infty}[n], m, d)\). It is easy to see that for each \(n \geq 0\) the differential bigraded module \((F_{\infty}[n], d, F_{\infty}[n] = \{F_{\infty}[n], m, p\}, m \geq 0, p \geq 0, d : F_{\infty}[n] \longrightarrow F_{\infty}[n+1],\) equipped with the structure of a differential module with \(\infty\)-simplicial faces \(\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)} : F_{\infty}[n], m \longrightarrow F_{\infty}[n+k, m+k\_i-1]\}\), which for every element \(a \in F_{\infty}[n], \mu\_m = F_{\infty}(m, n)\_m\) are defined by the following equality:

\[
\partial_{(i_1, \ldots, i_k)}(a) = \pi(\partial_{(i_1, \ldots, i_k)}^m \otimes a), \quad \partial_{(i_1, \ldots, i_k)}^m \in F_{\infty}(m - k, m)_k - 1.
\]

It is clear that for the differential module with \(\infty\)-simplicial faces \((F_{\infty}[n], d, \tilde{\partial})\) its corresponding differential module \((F_{\infty}[n], d, \mu)\) over the colored algebra of \(\infty\)-simplicial faces \((F_{\infty}, d, \pi)\) is the free bigraded module \((F_{\infty}[n], \mu)\) over the colored graded algebra \((F_{\infty}, \pi)\) with one generator \(1_n \in F_{\infty}[n], n, 0 = F_{\infty}(n, n), 0\), and the differential \(d : F_{\infty}[n] \longrightarrow F_{\infty}[n+1]\) completely defined by the differential in \((F_{\infty}, d, \pi)\) and the equality \(d(1_n) = 0\), i.e. is defined for any elements \(\mu(a \otimes 1_n) \in F_{\infty}[n]\) by the formula \(d(\mu(a \otimes 1_n)) = \mu(a \otimes d(1_n))\).

Now suppose that an arbitrary differential module with \(\infty\)-simplicial faces \((X, d, \tilde{\partial})\) is given. Consider for any fixed element \(x \in X_{n,q}\) the map \(\overline{\pi} : F_{\infty}[n], m, \cdot \longrightarrow X_{\cdot, n+q}\) of bigraded modules, which for every element \(a \in F_{\infty}[n], m, p\) is defined by the formula

\[
\overline{\pi}(a) = (-1)^{(p+m)q} \mu(a \otimes x), \tag{3.1}
\]

where \((X, d, \mu)\) is the differential \(F_{\infty}\)-module that corresponds to the differential module with \(\infty\)-simplicial faces \((X, d, \tilde{\partial})\). By the definition of a differential in the tensor product of a colored differential module and the differential bigraded module we obtain that the map \(\overline{\pi} : F_{\infty}[n], \cdot, \cdot \longrightarrow X_{\cdot, \cdot+q}\), which considered as the element \(\overline{\pi} \in \text{Hom}(F_{\infty}[n]; X), 0, q)\) of the differential bigraded module \((\text{Hom}(F_{\infty}[n]; X), d)\), satisfies the following formula:

\[
d(\overline{\pi}) = (-1)^{q}(\overline{\pi}d(x)). \tag{3.2}
\]

This formula implies that for an arbitrary cycle \(x \in X_{n,q}\) of the differential bigraded module \((X, d)\) its corresponding map \(\overline{\pi}\) is the map of differential bigraded modules.
The formula implies that for an arbitrary element $F$ following true relation:

$$\mu(a \otimes x) = (-1)^{i+p} \mu(\pi(a \otimes x)) \pi(a \otimes x) = ax$$

which correspond to the above elements. It is easy to see that (3.1) implies the following true relation:

$$\mu(a \otimes x) = (-1)^{i+p} \mu a x. \quad (3.3)$$

In particular, if as a differential $F_{\infty}$-module $(X, d, \mu)$ is take the differential $F_{\infty}$-module $(F_{\infty}[n], d, \mu)$, then for the above element $\mu(a \otimes x) = \pi(a \otimes x) = ax$ the relation (3.3) can be written as the relation

$$\mu(a \otimes x) = (-1)^{i+p} \mu a x. \quad (3.3')$$

By using the formula (3.1) we obtain that for any element $x \in X_{n,q}$ its corresponding map $\pi : F_{\infty}[n]_{\bullet} \to X_{n,\bullet+q}$ is related to $\infty$-simplicial faces $\partial(i_1, \ldots, i_k)$ on $F_{\infty}[n]$ and on $X$ by the formula

$$\partial(i_1, \ldots, i_k) \pi = (-1)^{q} \pi \partial(i_1, \ldots, i_k). \quad (3.4)$$

This formula implies that for an arbitrary element $x \in X_{n,q}$ its corresponding map $\pi : F_{\infty}[n]_{\bullet} \to X_{n,\bullet+q}$ is completely defined by its value on the generator $1_n \in F_{\infty}[n]_{n,0}$, i.e. completely defined by the equality $\pi(1_n) = (-1)^{mq} x$.

Now, let us consider for each element $\partial(i_1, \ldots, i_k) \in F_{\infty}[n]_{n-k,k-1}$ its corresponding family $\delta(i_1, \ldots, i_k) = (1-n)^{n-k} \partial(i_1, \ldots, i_k) : F_{\infty}[n-k]_{\bullet} \to F_{\infty}[n]_{\bullet+k-1}$.

Further the maps $\delta(i_1, \ldots, i_k)$ we will be called $\infty$-cosimplicial cofaces of the family $F_{\infty}[\bullet] = \{(F_{\infty}[n], d, \tilde{\partial})\}_{n \geq 0}$ of differential modules with $\infty$-simplicial faces. By using the formulas (2.4), (3.2) and (3.3') we obtain that $\infty$-cosimplicial cofaces of the family $F_{\infty}[\bullet]$ related by following relations:

$$d(\delta(i_1, \ldots, i_k)) = \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} (-1)^{\text{sign}(\sigma) + k(m-1)} \delta(\sigma(i_{m+1}), \ldots, \sigma(i_1)) \delta(\sigma(i_1), \ldots, \sigma(i_m)), \quad (3.5)$$

where the set $I_\sigma$ is just that in the formula (2.4).

Recall [13] that by a $D_{\infty}$-module one means any bigraded module $X = \{X_{n,m}\}$, $n, m \in \mathbb{Z}$, equipped with a family $\{d^k : X_{n,\bullet} \to X_{n-k,\bullet+k-1} \mid k \in \mathbb{Z}, \; k \geq 0\}$ of module maps, which for each integer $k \geq 0$ satisfy the following relation:

$$\sum_{i+j=k} d^i d^j = 0. \quad (3.6)$$

For every differential module with $\infty$-simplicial faces $(X, d, \tilde{\partial})$, we define the family of maps $\{d^k : X_{n,\bullet} \to X_{n-k,\bullet+k-1}\}$, $k, n \in \mathbb{Z}$, $k \geq 0$, $n \geq 0$, by setting

$$d^0 = d : X_{n,\bullet} \to X_{n-1,\bullet}, \quad d^k = 0 : X_{n,\bullet} \to X_{n-k,\bullet+k-1}, \; k > n,$$

$$d^k = \sum_{0 \leq i_1 < \cdots < i_k \leq n} (-1)^{i_1 + \cdots + i_k} \partial(i_1, \ldots, i_k) : X_{n,\bullet} \to X_{n-k,\bullet+k-1}, \; k \leq n \quad (3.7)$$
Since for any permutation \( \sigma \in \Sigma_k \) of integers \( i_1 < \ldots < i_k \) there is the true equality
\[
\overline{\sigma(i_1)} + \ldots + \overline{\sigma(i_k)} = \sigma(i_1) + \ldots + \sigma(i_k) - I(\sigma),
\]
where \( I(\sigma) = \alpha(\sigma(i_1)) + \ldots + \alpha(\sigma(i_k)) \) is a number of the permutation \( \sigma \), by using \( \text{sign}(\sigma) \equiv I(\sigma) \mod(2) \) and (2.5), it is easy to check that, for maps from the family (3.7), the relations (3.6) are true.

Now consider for each differential module with \( \infty \)-simplicial faces \((X, d, \overline{\partial})\) its corresponding \( D_\infty \)-module \((X, d^k)\) and define the differential graded module \((\overline{X}, \partial)\) by setting
\[
\overline{X}_n = \bigoplus_{s+t=n} X_{s,t}, \quad \partial = \sum_{k \geq 0} d^k : \overline{X}_n \to \overline{X}_{n-1}.
\]
It is easy to see that from the relations (3.6) follows the equality \( \partial \partial = 0 \). In particular, for each differential module with \( \infty \)-simplicial faces \((F_\infty[n], d, \overline{\partial}), n \geq 0\), there is the differential module \((F_\infty[n], \partial)\).

The above considered map \( \mathfrak{T} : F_\infty[n]_{\ast, \bullet} \to X_{\ast, \bullet+q} \) that corresponds to an arbitrary element \( x \in X_{n,q} \) induces the map of graded modules \( \mathfrak{T} : F_\infty[n]_{\ast, \bullet} \to X_{\ast, \bullet+q} \) of degree \( q \), which, by the equality (3.4), satisfies the relation \( \partial(x) = d(x) \). In particular, since \( \partial(\delta^{(i_1, \ldots, i_k)}) = d(\delta^{(i_1, \ldots, i_k)}) \neq 0 \), the maps \( \delta^{(i_1, \ldots, i_k)} : F_\infty[n-k]_{\ast, \bullet} \to F_\infty[n]_{\ast+1, \bullet+1} \), which are induced by \( \infty \)-cosimplicial cofaces \( \delta^{(i_1, \ldots, i_k)} : F_\infty[n-k]_{\ast, \bullet} \to F_\infty[n]_{\ast+1, \bullet+1}, k \geq 2 \), are not maps of differential modules.

Now suppose that an arbitrary differential module with \( \infty \)-simplicial faces \((X, d, \overline{\partial})\) is given. For each fixed \( n \geq 0 \), let us consider the tensor product \((F_\infty[n]_{\ast, \bullet} \otimes X_{n, \bullet, \ast}, d)\) of differential modules \((F_\infty[n]_{\ast, \bullet}, \partial)\) and \((X_{n, \bullet, \ast}, d)\).

**Theorem 3.1.** The differential in the direct sum \( \bigoplus_{n \geq 0} (F_\infty[n]_{\ast, \bullet} \otimes X_{n, \bullet, \ast}, d) \) of differential modules induces a well defined differential in the quotient module
\[
\bigoplus_{n \geq 0} (F_\infty[n]_{\ast, \bullet} \otimes X_{n, \bullet, \ast})/\sim
\]
of the graded module \( \bigoplus_{n \geq 0} (F_\infty[n]_{\ast, \bullet} \otimes X_{n, \bullet, \ast}) \) by the equivalence relation \( \sim \), which is generated by the following relations:
\[
\delta^{(i_1, \ldots, i_k)}(a) \otimes x \sim (-1)^{s(k-1) + (n+q)k} a \otimes \partial^{(i_1, \ldots, i_k)}(x), \quad (3.8)
\]
where \( a \in F_\infty[n-k]_{s, \bullet}, x \in X_{n,q} \) are an arbitrary elements, and \( (i_1, \ldots, i_k) \) is any ordered collection.

**Proof.** It suffices to show that from the condition (3.8) follows the condition
\[
d(\delta^{(i_1, \ldots, i_k)}(a) \otimes x) \sim d((-1)^{s(k-1) + (n+q)k} a \otimes \partial^{(i_1, \ldots, i_k)}(x)).
\]
Since
\[
\partial \delta^{(i_1, \ldots, i_k)} + (-1)^k \delta^{(i_1, \ldots, i_k)} \partial = \partial(\delta^{(i_1, \ldots, i_k)}) = d(\delta^{(i_1, \ldots, i_k)}),
\]
by using the formula (3.5) we obtain
\[
d(\delta^{(i_1, \ldots, i_k)}(a) \otimes x) = \partial(\delta^{(i_1, \ldots, i_k)}(a)) \otimes x + (-1)^{s+k-1} \delta^{(i_1, \ldots, i_k)}(a) \otimes d(x) =
\]
On the other hand since \(d\partial_{(i_1,\ldots,i_k)} + \partial_{(i_1,\ldots,i_k)}d = d(\partial_{(i_1,\ldots,i_k)})\), by using the formula (2.5) we obtain
\[
d((-1)^{s-k-1}+(n+q)k)\partial(a) \otimes \partial_{(i_1,\ldots,i_k)}(x) =
\]
\[
= ((-1)^{s-k-1}+(n+q)k)\partial(a) \otimes \partial_{(i_1,\ldots,i_k)}(x) + ((-1)^{s-k-1}+(n+q)k)a \otimes d(\partial_{(i_1,\ldots,i_k)}(x)) =
\]
\[
= ((-1)^{s-k-1}+(n+q)k)\partial(a) \otimes \partial_{(i_1,\ldots,i_k)}(x) + ((-1)^{s-k-1}+(n+q)k+1)a \otimes \partial_{(i_1,\ldots,i_k)}(d(x)) +
\]
\[
+ ((-1)^{s-k-1}+(n+q)k)a \otimes \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} ((-1)^{\operatorname{sgn}(\sigma)+1}\partial(\sigma(i_1),\ldots,\sigma(i_m))) \otimes \partial(\sigma(i_{m+1},\ldots,\sigma(i_k)))(x).
\]

Now, by using the relations (3.8) we obtain
\[
((-1)^{s-k-1}+(n+q)k)\partial(a) \otimes x \sim (-1)^{s-k-1}((s-1)(k-1)+(n+q)k)\partial(a) \otimes \partial_{(i_1,\ldots,i_k)}(x) =
\]
\[
= ((-1)^{s-k-1}+(n+q)k)\partial(a) \otimes \partial_{(i_1,\ldots,i_k)}(x),
\]
\[
((-1)^{s-k-1}+(n+q)k)\partial(a) \otimes d(x) \sim ((-1)^{s-k-1}+(n+q-1)k)\partial(a) \otimes \partial_{(i_1,\ldots,i_k)}(d(x)) =
\]
\[
= ((-1)^{s-k-1}+(n+q-1)k+)a \otimes \partial_{(i_1,\ldots,i_k)}(d(x)),
\]
\[
\sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} ((-1)^{\operatorname{sgn}(\sigma)+1}\partial(\sigma(i_1),\ldots,\sigma(i_m))) \otimes \partial(\sigma(i_{m+1},\ldots,\sigma(i_k)))(x) \sim
\]
\[
\sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} ((-1)^{\operatorname{sgn}(\sigma)+1}\partial(\sigma(i_1),\ldots,\sigma(i_m))) \otimes \partial(\sigma(i_{m+1},\ldots,\sigma(i_k)))(x) \sim
\]
\[
\sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} ((-1)^{\operatorname{sgn}(\sigma)+1}\partial(\sigma(i_1),\ldots,\sigma(i_m))) \otimes \partial(\sigma(i_{m+1},\ldots,\sigma(i_k)))(x),
\]
where \(\nu = (s + m - 1)(k - m - 1) + (n + q)(k - m)\), \(\varepsilon = s(m - 1) + (n + q - 1)m\), and hence we have the relation
\[
\sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} ((-1)^{\operatorname{sgn}(\sigma)+1}\partial(\sigma(i_1),\ldots,\sigma(i_m))) \otimes x \sim
\]
\[
\sim ((-1)^{s-k-1}+(n+q)k)\partial(a) \otimes \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} ((-1)^{\operatorname{sgn}(\sigma)+1}\partial(\sigma(i_1),\ldots,\sigma(i_m))) \otimes \partial(\sigma(i_{m+1},\ldots,\sigma(i_k)))(x),
\]
since \(\nu + \varepsilon \equiv sk + (n + q)k + k(m - 1) + 1 \mod(2)\) holds. The verification of the required condition is completed. ■

**Definition 3.1.** By a chain realization of the differential module with \(\infty\)-simplicial faces \((X, d, \partial)\) we mean the differential graded module \(((X), d)\) that is defined as the differential quotient module
\[
((X), d) = \bigoplus_{n \geq 0} (F_{\infty}[n] \otimes X_{n, \bullet}, d) / \sim
\]
by the equivalence relation $\sim$, which is generated by the relations (3.8).

Consider the differential module $(|X|, d)$ in more detail. Since $F^n_\infty[n]$ is a free
$F^n_\infty$-module with the generator $1_n$, from the easily verified equality
$\delta(i_1,\ldots,i_k)(1_{n-k}) = (-1)^{(n-k)k} \partial(i_1,\ldots,i_k)(1_n)$ and the formulas (3.4), (3.8) follows that, for any elements
$a = \partial(i_1,\ldots,i_s)\cdots\partial(j_1,\ldots,j_t)(1) \in F^n_\infty[n]$ and $x \in X_{n,\bullet}$, the relation
$a \otimes x \sim (-1)^s 1_m \otimes \partial(i_1,\ldots,i_s)\cdots\partial(j_1,\ldots,j_t)(x)$ holds, where $(-1)^s$ is computed by using (3.4) and (3.8). In
particular, by using the equality $d(1_n) = 0$ we obtain that, for each element $x \in X_{n,q}$, the following relation holds:

$$
\partial(1_n) \otimes x = \sum_{0 \leq i_1 < \cdots < i_k \leq n} (-1)^{i_1 + \cdots + i_k} \partial(i_1,\ldots,i_k)(1_n) \otimes x \sim \\
\sum_{0 \leq i_1 < \cdots < i_k \leq n} (-1)^{i_1 + \cdots + i_k + n(k-1)+(q-1)k} 1_{n-k} \otimes \partial(i_1,\ldots,i_k)(x). 
$$

(3.9)

Let us consider for $(X,d,\tilde{\partial})$ its corresponding the $D_\infty$-module $(X,d^k)$, and let us
define new the $D_\infty$-module $(X,\tilde{d}^k)$ by setting
$\tilde{d}^k(x) = (-1)^{n(k-1)+(q-1)k} d^k(x)$, $k \geq 0$, $x \in X_{n,q}$. A direct check shows that $(X,\tilde{d}^k)$ really is a $D_\infty$-module, and hence for
$(X,\tilde{d}^k)$ is defined its corresponding differential module $(\overline{X},\overline{\partial})$, whose the differential
$\overline{\partial} : X_{\bullet} \to X_{\bullet-1}$, for any elements $x \in X_{n,q} \subset X_{n+q}$, is given by the formula

$$
\overline{\partial}(x) = \sum_{k \geq 0} \tilde{d}^k(x) = (-1)^n d(x) + \sum_{0 \leq i_1 < \cdots < i_k \leq n} (-1)^{i_1 + \cdots + i_k + n(k-1)+(q-1)k} \partial(i_1,\ldots,i_k)(x)
$$

and extended by linearity to all elements of the module $\overline{X}$. In what follows the elements $x \in X_{n,q} \subset X_{n+q}$ we will denote by $[x]$. We note that each equivalence
class $[a \otimes x] \in |X|$ is a linear combination of equivalence classes of the form $[1_n \otimes y]$, where
the elements $y \in X_{n,\bullet}$ are uniquely determined by the relations (3.4) and (3.8).
Consider the map of graded modules $F : |X|_{\bullet} \to \overline{X}_{\bullet}$ that, for any elements $[1_n \otimes x]$, where $x \in X_{n,q}$, is given by the equality
$F([1_n \otimes x]) = [x]$ and extended by linearity to all elements of the module $|X|$. Clear that the map $F$ is an isomorphism of graded
modules. Furthermore, from the relations (3.9) follows that $F$ is an isomorphism of differential modules $F : (|X|, d) \to (\overline{X}, \overline{\partial})$.

Now we show that each $A_\infty$-algebra defines some the differential module with $\infty$-
simplicial faces $(T(A), d, \overline{\partial})$. Recall [5] (see also [6, 2]) that an $A_\infty$-algebra $(A, d, \pi_n)$
is defined as a differential module $(A, d)$, $A = \{A_n\}, n \in \mathbb{Z}$, $n \geq 0$, $d : A_{\bullet} \to A_{\bullet-1}$, equipped with a family of maps $\{\pi_n : (A^{(n+2)}_{\bullet})_{\bullet} \to A_{\bullet+n} | n \in \mathbb{Z}, n \geq 0\}$ which, for
all integers $n \geq -1$, satisfy the following relations:

$$
d(\pi_{n+1}) = \sum_{m=0}^{n} \sum_{t=1}^{m+2} (-1)^{(t(n-m+1)+n+1)} \pi_m(1 \otimes \ldots \otimes 1 \otimes \pi_{n-m} \otimes 1 \otimes \ldots \otimes 1),
$$

(3.10)

where $d(\pi_{n+1}) = d\pi_{n+1} + (-1)^n \pi_{n+1} d$. For any $A_\infty$-algebra $(A, d, \pi_n)$, we consider
the differential bigraded module $(T(A), d)$, where $T(X) = \{T(A)_{n,m}\}, n, m \in \mathbb{Z}$,
$n \geq 0, m \geq 0$, which is defined by the equalities $T(A)_{n,m} = (A^{\otimes n})_m$, $n \geq 0, m \geq 0$,
$T(A)_{0,0} = K, T(A)_{n,0} = 0$, $n > 0$, $T(A)_{0,m} = 0$, $m > 0$, and its the differential
\[ d : T(A)_{n_0, \ast} \to T(A)_{n_0, \ast - 1} \] is the usual differential in a tensor product. We define the family of the maps \( \tilde{\partial} = \{ \tilde{\partial}^n_{(i_1, \ldots, i_k)} : T(A)_{n,q} \to T(A)_{n-k,q+k-1} \} \), where \( n \geq 0, q \geq 0 \) and \( 0 \leq i_1 < \ldots < i_k \leq n \), by setting
\[
\tilde{\partial}^n_{(i_1, \ldots, i_k)} = \begin{cases} 
(-1)^{k(q-1)}1 \otimes (j-1) \otimes \pi_{k-1} \otimes 1 \otimes (n-k-j), & \text{if } 1 \leq j \leq n - k \\
\text{and } (i_1, \ldots, i_k) = (j, j+1, \ldots, j+k-1), & \text{otherwise.} 
\end{cases} \quad (3.11)
\]

**Theorem 3.2.** Given any \( A_{\infty} \)-algebra \( (A, d, \pi_n) \), the triple \( (T(A), d, \tilde{\partial}) \) is a differential module with \( \infty \)-simplicial faces.

**Proof.** For a family of maps \( \tilde{\partial} = \{ \tilde{\partial}^n_{(i_1, \ldots, i_k)} : T(A)_{n,q} \to T(A)_{n-k,q+k-1} \} \), which are defined by the formula (3.11), we need to check the validity of the relations (2.5). First, for the maps \( \tilde{\partial}^{n+3}_{(1,2,\ldots,n+2)} = (-1)^{n+2(q-1)}\pi_{n+1} : (A \otimes (n+3))_q \to A_{q+n+1}, n \geq -1, \) we will verify that the relations (2.5) are true. Clearly, for \( n = -1, \) i.e. for \( \tilde{\partial}^2_{(1)} = (-1)^{q-1}\pi_0, \) the condition (2.5) holds, since from the equality \( d(\pi_0) = d\pi_0 - \pi_0d = 0 \) we obtain \( d(\tilde{\partial}^2_{(1)}) = d\tilde{\partial}^2_{(1)} + \tilde{\partial}^2_{(1)}d = (-1)^{q-1}d(\pi_0) = 0. \) Suppose now that \( n \geq 0. \) We note that, for any permutation \( \sigma \in \Sigma_{n+2} \) of integers \( 1, \ldots, n+2, \) the equality \( \sigma(1) = 1 \) is always true. Indeed, for the collection \( (\sigma(1), \ldots, \sigma(n+2)) \), all its numbers smaller than \( \sigma(1) \) are to the right of \( \sigma(1) \), and the quantity of such numbers is \( \alpha(\sigma(1)) = \sigma(1) - 1. \) From the condition \( \sigma(1) = 1 \) and the formula (3.11) follows that in considered case the relations (2.5) can be written as
\[
d(\tilde{\partial}^{n+3}_{(1,2,\ldots,n+2)}) = \sum_{m=0}^{n} \sum_{t=1}^{m+2} (-1)^{\text{sign}(\sigma_{t,m})+1} \tilde{\partial}^{m+2}_{(1,2,\ldots,m+1)} \tilde{\partial}^{n+3}_{(t+1,\ldots,t+n-m)}, \quad (3.12)
\]
where \( d(\tilde{\partial}^{n+3}_{(1,2,\ldots,n+2)}) = d\tilde{\partial}^{n+3}_{(1,2,\ldots,n+2)} + \tilde{\partial}^{n+3}_{(1,2,\ldots,n+2)}d, \) and \( \sigma_{t,m} \in \Sigma_{n+2} \) is a permutation of numbers \( 1, \ldots, n + 2, \) which satisfies the conditions
\[
\sigma_{t,m}(1) = 1, \quad \sigma_{t,m}(2) = 2, \ldots, \quad \sigma_{t,m}(m+1) = m+1,
\]
\[
\sigma_{t,m}(m+2) = t, \quad \sigma_{t,m}(m+3) = t+1, \ldots, \quad \sigma_{t,m}(n+2) = t+n-m.
\]
It is easy to verify that the permutation \( \sigma_{t,m} \) acts on the collection \( (1, 2, \ldots, n+2) \) by a partitioning of this collection on three blocks
\[
(1, 2, \ldots, t-1 \mid t, t+1, \ldots, t+n-m \mid t+n-m+1, \ldots, n+2)
\]
and by a permutation of places of the second and third blocks. From this follows that the number of inversions \( I(\sigma_{t,m}) \) of the permutation \( \sigma_{t,m} \) is equal to the product of the lengths of the second and third blocks of the above partition, i.e. \( I(\sigma_{t,m}) = (n-m+1)(m-t+2). \) If now multiply the left and right sides of the equality (3.10) by \( (-1)^{n+2(q-1)} \) and notice that \( I(\sigma_{t,m}) \equiv \text{sign}(\sigma_{t,m}) \mod(2), \) then by using (3.11) we obtain the relations (3.12). In an analogous way we can verify the validity of the relations (2.5) for an arbitrary maps \( \tilde{\partial}^n_{(j,j+1,\ldots,j+k-1)}, \) where \( k \geq 1 \) and \( 1 \leq j \leq n-k. \) Consider now the case of the maps \( \tilde{\partial}^n_{(i_1, \ldots, i_k)}, \) when the ordered collection \( (i_1, \ldots, i_k) \)
not a collection of the form \((j, j + 1, \ldots, j + k - 1)\), where \(k \geq 1\) and \(1 \leq j \leq n - k\). It follows from the Proposition 2.1 that in this case the relations (2.5) require a verification only for those collections \((i, \ldots, i_k)\), which have the form

\[
(j, j + 1, \ldots, j + l - 1, t, t + 1, \ldots, t + m - 1),
\]

where \(l \geq 1\), \(m \geq 1\), \(l + m = k\), \(j + l < t\), since in otherwise the left and right sides of the relations (2.5) are equal zero. For collections of the above form, the relations (2.5) can be written as the relations

\[
d(\partial^{n}_{(j, j + 1, \ldots, j + l - 1, t, t + 1, \ldots, t + m - 1)}) = 0 = -\partial^{n-m}_{(j, j + 1, \ldots, j + l - 1)}\partial^{n}_{(t, t + 1, \ldots, t + m - 1)} + (-1)^{l+m+1}\partial^{n-l}_{(t - l, t + 1 - l, \ldots, t + m - 1 - l)}\partial^{n}_{(j, j + 1, \ldots, j + l - 1)},
\]

that easy follows from the obvious equalities

\[
(1 \otimes \ldots \otimes 1 \otimes \pi_{l-1} \otimes 1 \otimes \ldots \otimes 1)(1 \otimes \ldots \otimes 1 \otimes \pi_{m-1} \otimes 1 \otimes \ldots \otimes 1) =
\]

\[
= (-1)^{(m-1)(l-1)}(1 \otimes \ldots \otimes 1 \otimes \pi_{m-1} \otimes 1 \otimes \ldots \otimes 1)(1 \otimes \ldots \otimes 1 \otimes \pi_{l-1} \otimes 1 \otimes \ldots \otimes 1),
\]

where \(l \geq 1\), \(m \geq 1\), \(j + l < t\). ■

For any \(A_\infty\)-algebra \((A, d, \pi_n)\), let us consider the chain realization \((|T(A)|, d)\) of the differential module with \(\infty\)-simplicial faces \((T(A), d, \partial)\), which corresponds to the \(A_\infty\)-algebra \((A, d, \pi_n)\). As was said above, by using the isomorphism \(F\) we can identify the differential module \((|T(A)|, d)\) and the differential module \((\overline{T(A)}, \partial)\): here \(\overline{T(A)} = \bigoplus_{n+q=m}(A^\otimes n)_q\) and the differential \(\partial: \overline{T(A)} \to \overline{T(A)}\) is given by the following formula:

\[
\partial([a_1 \otimes \ldots \otimes a_n]) = (-1)^n \sum_{i=1}^{n} (-1)^{\varepsilon}[a_1 \otimes \ldots \otimes d(a_i) \otimes \ldots \otimes a_n] +
\]

\[
+ \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} (-1)^{\frac{k(k+1)}{2} + i + n(k-1) + (k-1)[a_1 \otimes \ldots \otimes \pi_{k-1}(a_i \otimes \ldots \otimes a_{i+k}) \otimes \ldots \otimes a_n]},
\]

where \(\varepsilon = \deg(a_1) + \ldots + \deg(a_{i-1})\) and elements \([a_1 \otimes \ldots \otimes a_n] \in \overline{T(A)}\) are generators of the module \(\overline{T(A)}\).

Let us proceed to defining the structure of a differential coalgebra \((|T(A)|, d, \nabla)\) on the differential module \((|T(A)|, d)\), where \((A, d, \pi_n)\) is an arbitrary \(A_\infty\)-algebra. For this we introduce a conception of the tensor product of differential modules with \(\infty\)-simplicial faces.

**Definition 3.2.** By a tensor product \((X \otimes Y, d, \tilde{\partial})\) of the differential modules with \(\infty\)-simplicial faces \((X, d, \tilde{\partial})\) and \((Y, d, \tilde{\partial})\) we mean a tensor product \((X \otimes Y, d)\) of the differential bigraded modules \((X, d)\) and \((Y, d)\) equipped with a family of the \(\infty\)-simplicial faces \(\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)}: (X \otimes Y)_{n, \bullet} \to (X \otimes Y)_{n-k, \bullet+k-1}\}\), which are given on an arbitrary element \(x \otimes y \in X_{q,s} \otimes Y_{l,t}\) by the following rule:

\[
\partial_{(i_1, \ldots, i_k)}(x \otimes y) =
\]

\[
(\text{3.14})
\]
\[
\begin{cases}
\partial_{(i_1, \ldots, i_k)}(x) \otimes y, & 0 \leq i_1 < \ldots < i_k < q, \\
0, & 0 \leq i_1 < \ldots < i_k = q, \\
(-1)^{(k-1)q+s} x \otimes \partial_{(i_1-q, \ldots, i_k-q)}(y), & q < i_1 < \ldots < i_k \leq q + l, \\
0, & 0 \leq i_1 \leq q < i_k \leq q + l.
\end{cases}
\]

**Theorem 3.3.** A tensor product of differential modules with $\infty$ of the relations (2.14) is a differential module with $\infty$-simplicial faces.

**Proof.** For $\infty$-simplicial faces in a tensor product $(X \otimes Y, d, \partial)$ of the differential modules with $\infty$-simplicial faces $(X, d, \partial)$ and $(Y, d, \partial)$, we need to check the validity of the relations (2.5). We will check each case of the formula (3.14).

1. Suppose that $0 \leq i_1 < \ldots < i_k < q$. In this case, for any element $x \otimes y \in X_{q,s} \otimes Y_{l,t}$, we obtain

\[
(d(\partial_{(i_1, \ldots, i_k)}))(x \otimes y) = \\
= d(\partial_{(i_1, \ldots, i_k)}(x \otimes y)) + (-1)^{-k+(k-1)+1}\partial_{(i_1, \ldots, i_k)}(d(x \otimes y)) = \\
= d(\partial_{(i_1-1, \ldots, i_k)}(x) \otimes y) + \partial_{(i_1, \ldots, i_k)}(d(x) \otimes y + (-1)^{q+s}x \otimes d(y)) = \\
= d(\partial_{(i_1, \ldots, i_k)}(x)) \otimes y + (-1)^{q+s-1}\partial_{(i_1, \ldots, i_k)}(x) \otimes d(y) + \\
+ \partial_{(i_1, \ldots, i_k)}(d(x)) \otimes y + (-1)^{q+s}\partial_{(i_1, \ldots, i_k)}(x) \otimes d(y) = (d(\partial_{(i_1, \ldots, i_k)}))(x \otimes y) = \\
= \sum_{\sigma \in \Sigma_k} \sum_{I_{\sigma}} (-1)^{\text{sign}(\sigma)} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))} (x \otimes y) = \\
= \sum_{\sigma \in \Sigma_k} \sum_{I_{\sigma}} (-1)^{\text{sign}(\sigma)} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))} (x \otimes y).
\]

2. Suppose that $0 \leq i_1 < \ldots < i_k = q$. In this case, for an arbitrary element $x \otimes y \in X_{q,s} \otimes Y_{l,t}$, by the definition of $\infty$-simplicial faces in a tensor product we have

\[
(d(\partial_{(i_1, \ldots, i_k)}))(x \otimes y) = 0.
\]

Now we need check that if $0 \leq i_1 < \ldots < i_k = q$, then for any element $x \otimes y \in X_{q,s} \otimes Y_{l,t}$ the following equality holds:

\[
\sum_{\sigma \in \Sigma_k} \sum_{I_{\sigma}} (-1)^{\text{sign}(\sigma)} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))} (x \otimes y) = 0. \tag{3.15}
\]

For each summand of the left side of (3.15), we will show that the following equality holds:

\[
\partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))} (x \otimes y) = 0.
\]

Indeed, for an arbitrary summand $\partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))} (x \otimes y)$, we suppose that $\sigma \in \Sigma_k$ is a permutation such that $i_k = q \in \{\sigma(i_1), \ldots, \sigma(i_k)\}$. In this case we will show that $\sigma(i_k) = q$ is true. Assume by contradiction that we have $\sigma(i_k) < q$. From this, by using $\sigma(i_k) = \sigma(i_k)$, we obtain $\sigma(i_k) < q$. Since for the $\infty$-simplicial face $\partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))}$ the condition $0 \leq \sigma(i_{m+1}) < \ldots < \sigma(i_k) < q$ holds, we get
\[ \sigma(i_{k-p}) < q - p, \ 0 \leq p \leq k - (m + 1). \] Since \( \sigma(i_{k-p}) = \sigma(i_{k-p}) - \alpha(\sigma(i_{k-p})) \) and \( 0 \leq \alpha(\sigma(i_{k-p})) \leq p \), we obtain \( \sigma(i_{k-p}) < q, \ 0 \leq p \leq k - (m + 1) \), that contradicts the condition \( q \in \{ \sigma(m+1), \ldots, \sigma(i_k) \} \). Thus, the above assumption \( \sigma(i_k) < q \) is incorrect and hence we have \( \sigma(i_k) = q \). From this follows that in considered case, for each summand \( \partial_{(\sigma(i_1), \ldots, \sigma(i_m))}(\sigma(i_{m+1}), \ldots, \sigma(i_k))(x \otimes y) \), where \( x \otimes y \in X_{q,s} \otimes Y_{l,t} \), we obtain, by using \( \sigma(i_k) = q \), that the equality \( \partial_{(\sigma(i_1), \ldots, \sigma(i_m))}(\sigma(i_{m+1}), \ldots, \sigma(i_k))(x \otimes y) = 0 \) is true. Now, for an arbitrary summand \( \partial_{(\sigma(i_1), \ldots, \sigma(i_m))}(\sigma(i_{m+1}), \ldots, \sigma(i_k))(x \otimes y) \), we suppose that \( \sigma \in \Sigma_k \) is a permutation such that \( i_k = q \in \{ \sigma(i_1), \ldots, \sigma(i_m) \} \). In this case we will show that \( \sigma(i_m) = q - (k - m) \). Assume by contradiction that we have \( \sigma(i_m) < q - (k - m) \). From this, by using \( \sigma(i_m) = \sigma(i_m) - \alpha(\sigma(i_m)) \) and \( 0 \leq \alpha(\sigma(i_m)) \leq k - m \), we obtain \( \sigma(i_m) < q \). Since for the \( \infty \)-simplicial face \( \partial_{(\sigma(i_1), \ldots, \sigma(i_m))}(x \otimes y) \), we have \( \sigma(i_1) < \sigma(i_m) < \cdots < \sigma(i_m) < q \) holds, we get \( \sigma(i_p) < q - (k - p) \), \( 1 \leq p \leq m \). Since \( \sigma(i_p) = \sigma(i_p) - \alpha(\sigma(i_p)) \) and \( 0 \leq \alpha(\sigma(i_p)) \leq k - p \), we obtain \( \sigma(i_p) < q, \ 1 \leq p \leq m \), that contradicts the condition \( q \in \{ \sigma(i_1), \ldots, \sigma(i_m) \} \). Thus, the above assumption \( \sigma(i_m) < q - (k - m) \) is incorrect and hence we have \( \sigma(i_m) = q - (k - m) \). From this follows that in considered case, for each summand \( \partial_{(\sigma(i_1), \ldots, \sigma(i_m))}(\sigma(i_{m+1}), \ldots, \sigma(i_k))(x \otimes y) = \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))}(x \otimes y) \), where \( x \otimes y \in X_{q,s} \otimes Y_{l,t} \), we obtain, by using \( \sigma(i_m) = q - (k - m) \) and \( \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))}(x \otimes y) = \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))}(x \otimes y) = \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))}(x \otimes y) = 0 \) is true.

Suppose that \( q < i_1 < \ldots < i_k \leq q + l \). In this case, for an arbitrary element \( x \otimes y \in X_{q,s} \otimes Y_{l,t} \), we have

\[
(\partial_{(\sigma(i_1), \ldots, \sigma(i_k))})(x \otimes y) = 0.
\]
Now we need check that if $0 \leq i_1 \leq q < i_k$, then for any element $x \otimes y \in X_{q,s} \otimes Y_{l,t}$ the equality (3.15) is true. For any summand $(-1)^{\text{sign}(\sigma) + 1} \partial_{\sigma(i_1), \ldots, \sigma(i_m)} \partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes y)$ in (3.15), we suppose that $\sigma \in \Sigma_k$ is a permutation such that $i_k = q + c \in \{\sigma(i_{m+1}), \ldots, \sigma(i_k)\}$, $c = i_k - q > 0$. In this case, similar to how it was done in 2), we can show that $\sigma(i_k) = q + c$, i.e. that $\sigma(i_k) > q$. If $\sigma(i_{m+1}) \leq q$, then $(-1)^{\text{sign}(\sigma) + 1} \partial_{\sigma(i_1), \ldots, \sigma(i_m)} \partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes y) = 0$, since $\partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes y) = 0$. If $\sigma(i_{m+1}) > q$, then consider separately the cases a) $\sigma(i_k) > \sigma(i_{m+1})$ and b) $\sigma(i_k) < \sigma(i_{m+1})$. In the case a) we have $\sigma(i_j) > q$. If we assume that $i_1 = \sigma(i_1)$ for some $1 \leq l \leq k$, then obtain $\sigma(i_l) = \sigma(i_1)$. From the inequalities $\sigma(i_1) = i_1 \leq q$ and $q < \sigma(i_{m+1}) < \ldots < \sigma(i_k)$ follows that $1 \leq l \leq m$ and hence we have $\sigma(i_1) < \sigma(i_l) \leq q$. Thus, in the case a) we obtain

$$(-1)^{\text{sign}(\sigma) + 1} \partial_{\sigma(i_1), \ldots, \sigma(i_m)} \partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes y) =$$

$$(= (-1)^{\text{sign}(\sigma) + 1 + (k-m-1)q + s} \partial_{\sigma(i_1), \ldots, \sigma(i_m), \sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes \partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes y) = 0).$$

In the case b) we have $\sigma(i_1) < \ldots < \sigma(i_m) < (i_{m+1}) < \ldots < \sigma(i_k)$. From this inequalities, similar to how it was done in the proof of Proposition 2.1, we obtain the inequalities $\sigma(i_1) < \ldots < \sigma(i_m) < \sigma(i_{m+1}) < \ldots < \sigma(i_k)$, which imply that $\sigma$ is the identity permutation. If $i_m > q$, then we get $(-1)^1 \partial_{i_1, \ldots, i_m} \partial_{i_{m+1}, \ldots, i_k}(x \otimes y) = 0$, since $i_1 \leq q$. If $i_m < q$, then in the sum (3.15) we obtain the summand

$$(-1)^1 \partial_{i_1, \ldots, i_m} \partial_{i_{m+1}, \ldots, i_k}(x \otimes y) = (-1)^{(k-m-1)q + s + 1} \partial_{i_1, \ldots, i_m} \otimes \partial_{i_{m+1}, \ldots, i_k - q}.$$  

In (3.15) this summand vanishes together with $(-1)^{\text{sign}(\sigma) + 1} \partial_{i_{m+1}, \ldots, i_k - m} \partial_{i_1, \ldots, i_m}$, where $\sigma$ is a permutation, which permute the places the blocks $(i_1, \ldots, i_m)$ and $(i_{m+1}, \ldots, i_k)$. Indeed, since $\text{sign}(\sigma) = m(k - m)$, we obtain

$$(-1)^{\text{sign}(\sigma) + 1} \partial_{i_{m+1} - m, \ldots, i_k - m} \partial_{i_1, \ldots, i_m}(x \otimes y) =$$

$$(= (-1)^{m(k - m) + 1 + (k-m-1)(q-m)+s} \partial_{i_1, \ldots, i_m} \otimes \partial_{i_{m+1} - q, \ldots, i_k - q}.$$  

Thus, for $(-1)^{\text{sign}(\sigma) + 1} \partial_{\sigma(i_1), \ldots, \sigma(i_m)} \partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes y)$ in the sum (3.15), if the permutation $\sigma \in \Sigma_k$ such that $i_k = q + c \in \{\sigma(i_{m+1}), \ldots, \sigma(i_k)\}$, $c = i_k - q \geq 0$, then this summand either is equal to zero or vanishes together with another summand in this sum. In a similar way we verify that, for $(-1)^{\text{sign}(\sigma) + 1} \partial_{\sigma(i_1), \ldots, \sigma(i_m)} \partial_{\sigma(i_{m+1}), \ldots, \sigma(i_k)}(x \otimes y)$ in the sum (3.15), if $\sigma \in \Sigma_k$ such that $i_k = q + c \in \{\sigma(i_1), \ldots, \sigma(i_m)\}$, $c = i_k - q \geq 0$, then this summand either is equal to zero or vanishes together with another summand in this sum.

Thus, in all cases, for $\infty$-simplicial faces of the tensor product $(X \otimes Y, d, \tilde{d})$, the condition (2.5) holds, and hence $(X \otimes Y, d, \tilde{d})$ is a differential module with $\infty$-simplicial faces. ■

Now, by using the concept of a tensor product of differential modules with $\infty$-simplicial faces, we prove the following assertion.

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Theorem 3.4. For any $A_{\infty}$-algebra $(A, d, \pi_n)$, the chain realization $([T(A)], d)$ of the differential $F_{\infty}$-module $(T(A), d, \bar{d})$ equipped with a comultiplication $\nabla : [T(A)] \to [T(A)] \otimes [T(A)]$, which on generators is given by the formula

$$\nabla([1_n \otimes (a_1 \otimes \ldots \otimes a_n)]) =$$

$$= \sum_{i=0}^{n} (-1)^{(n-i)\varepsilon_i} [1_i \otimes (a_1 \otimes \ldots \otimes a_i)] \otimes [1_{n-i} \otimes (a_{i+1} \otimes \ldots \otimes a_n)],$$

where $\varepsilon_i = \deg(a_1) + \ldots + \deg(a_i)$, is a differential coalgebra.

Proof. It is easy to see that the comultiplication $\nabla$ is an associative. Let us show that $\nabla$ is a map of differential modules. For the differential module with $\infty$-simplicial faces $(T(A), d, \bar{d})$, we consider the differential module with $\infty$-simplicial faces $(T(A), d', \bar{d}')$, which is defined by the equalities $d' = (-1)^n d : T(A)_{n, \bullet} \to T(A)_{n, \bullet-1}$ and

$$\bar{d}' = \{ \partial'_{(i_1, \ldots, i_k)} = (-1)^{n(k-1)+q-1} \partial(i_1, \ldots, i_k) : T(A)_{n,q} \to T(A)_{n-k,q+k-1} \}. $$

Similarly to the proof of the Theorem 3.2 can be proved that the triple $(T(A), d', \bar{d}')$ is a differential module with $\infty$-simplicial faces. Direct calculations show that the family of maps

$$\tilde{\Delta} = \{ \Delta_{(i_1, \ldots, i_k)} : T(A)_{n, \bullet} \to (T(A) \otimes T(A))_{n-k, \bullet+k} \}, \quad \Delta_{(i_1, \ldots, i_k)} = 0, \quad k \geq 1,$n, \bullet \}\}

$$\Delta(i) (a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{(n-i)\varepsilon_i} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n),$$

where $\varepsilon_i = \deg(a_1) + \ldots + \deg(a_i)$, is the morphism of differential modules with $\infty$-simplicial faces

$$\tilde{\Delta} : (T(A), d', \bar{d}') \to (T(A), d', \bar{d}') \otimes (T(A), d', \bar{d}') = (T(A) \otimes T(A), d', \bar{d}).$$

From this it follows that the map of bigraded modules $\Delta(i)$ induces the map of differential modules $\Delta : (T(A), \bar{d}') \to (T(A) \otimes T(A), \bar{d}')$. Now, by using the formula (3.11), we note that the $\infty$-simplicial faces $\partial'_{(i_1, \ldots, i_k)} : T(A)_{n, \bullet} \to T(A)_{n-k, \bullet+k-1}$, where $i_1 = 0$ or $i_k = n$, satisfy the condition $\partial'_{(i_1, \ldots, i_k)} = 0$. From this it follows that the equality of differential modules $(T(A) \otimes T(A), \bar{d}') = \bar{\Delta}(T(A), \bar{d}') \otimes (T(A), \bar{d}')$ holds. By using this equality, and also by using the equality $(T(A), \bar{d}') = (T(A), \bar{d})$, we obtain the map of differential modules $\Delta : (T(A), \bar{d}) \to (T(A), \bar{d}) \otimes (T(A), \bar{d})$, which on generators is given by the following formula:

$$\Delta([a_1 \otimes \ldots \otimes a_n]) = \sum_{i=0}^{n} (-1)^{(n-i)\varepsilon_i} [a_1 \otimes \ldots \otimes a_i] \otimes [a_{i+1} \otimes \ldots \otimes a_n]. \quad (3.16)$$

Clear that $\Delta$ satisfies the condition $(\Delta \otimes 1) \Delta = (1 \otimes \Delta) \Delta$. Now, if we consider the isomorphism of differential modules $F : ([T(A)], d) \to (T(A), \bar{d})$, $F([1_n \otimes x]) = [x]$, then it is easy to see that for the map $\nabla$, which is given in the statement of this
theorem, the equality $\nabla = (F^{-1} \otimes F^{-1}) \Delta F$ holds, and hence $\nabla$ is a map of differential modules. Moreover, from this equality follows that $F$ is an isomorphism of differential coalgebras $F : ([T(A)], d, \nabla) \to (\overline{T}(A), \overline{\partial}, \Delta)$.

The differential coalgebra $(\overline{T}(A), \overline{\partial}, \Delta)$ is well known \[6\] as the $B$-construction $(B(A), d, \Delta)$ of the $A_\infty$-algebra $(A, d, \pi_n)$. Thus, up to isomorphism of differential coalgebras, the $B$-construction of the $A_\infty$-algebra $(A, d, \pi_n)$ is a chain realization of the tensor differential module with $\infty$-simplicial faces $(T(A), d, \overline{\partial})$, which is defined by this $A_\infty$-algebra $(A, d, \pi_n)$.

We conclude this paragraph by pointing out that if an $A_\infty$-algebra $(A, d, \pi_n)$ is the differential associative algebra $(A, d, \pi)$, where $\pi_0 = \pi$ and $\pi_n = 0$, $n > 0$, then $(\overline{T}(A), \overline{\partial}, \Delta)$ is usual $B$-construction $(B(A), d, \Delta)$ of the algebra $(A, d, \pi)$, and for this $B$-construction, the signs in the formulas (3.13) and (3.16) are coincide with the sings, which was given in \[2\].

References

[1] V. A. Smirnov, “Homotopy theory of coalgebras”, Izv. Akad. Nauk SSSR Ser. Mat. 49:6 (1985), 1302–1321; English transl. in Math. USSR-Izv. 27:3 (1986), 575–592.

[2] V. A. Smirnov, “Homology of $B$-constructions and of co-$B$-constructions”, Izv. Ross. Akad. Nauk Ser. Mat. 58:4 (1994), 80–96; English transl. in Russian Acad. Sci. Izv. Math. 45:1 (1995), 79–95.

[3] S. V. Lapin, “Extension of the multiplication operation in $E_\infty$-algebras to an $A_\infty$-morphism of $E_\infty$-algebras and Cartan objects in the category of May algebras”, Matem. Zametki, 2011, Vol. 89, No. 5, pp. 719–737; English transl. in Math. Notes, 2011, Vol. 89, No. 5, pp. 672–688.

[4] J. P. May, “A general algebraic approach of Steenrod operations”, Lect. Notes in Math., 168, Springer-Verlag, Berlin, 1970, 153–231.

[5] J. D. Stasheff, “Homotopy associativity of $H$-spaces I,II”, Trans.Amer. Math. Soc., 108:2 (1963), 275-312.

[6] T. V. Kadeishvili, “On the homology theory of fibered spaces,” Usp. Mat. Nauk, 35, 183–188 (1980).

[7] V. A. Smirnov, “Simplicial and Operad Methods in Algebraic Topology”, Factorial, Moscow, 2002; English transl. in Transl. Math. Monogr., Vol. 198, Amer. Math. Soc., Providence (2001).

[8] S.V. Lapin, “$D_\infty$-differential $A_\infty$-algebras and spectral sequences”, Mat. Sb. 193:1 (2002), 119–142; English transl. in Sb. Math. 193 (2002).

[9] J. L. Loday, B. Vallette, “Algebraic operads”, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer-Verlag, Berlin, 2012.
[10] V. A. Smirnov, “Lie algebras over operads and their applications in homotopy theory”, Izv. Ross. Akad. Nauk Ser. Mat. 62 (3), 121–154 (1998); English transl. in Izv. Math. 62 (3), 549–580 (1998).

[11] S. V. Lapin, “Homotopy simplicial faces and the homology of realizations of simplicial topological spaces”, Matem. Zametki, 2013, Vol. 94, No. 5, pp. 661–681; English transl. in Math. Notes, 2013, Vol. 94, No. 5, pp. 619–635.

[12] S. V. Lapin, “Homotopy properties of differential Lie modules over curved coalgebras and Koszul duality”, Matem. Zametki, 2013, Vol. 94, No. 3, pp. 354–372; English transl. in Math. Notes, 2013, Vol. 94, No. 3, pp. 335–350.

[13] S. V. Lapin, “Differential perturbations and $D_\infty$-differential modules”, Mat. Sb. 192:11 (2001), 55–76; English transl. in Sb. Math. 192:11 (2001), 1639–1659.