Unitarity and Lee-Wick prescription at one loop level in the effective Myers-Pospelov electrodynamics: The $e^+ + e^-$ annihilation

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We study perturbative unitarity in a Lorentz-symmetry-violating QED model with higher-order derivative operators in the light of the results of Lee and Wick to preserve unitarity in indefinite metric theories. Specifically, we consider the fermionic sector of the Myers-Pospelov model, which includes dimension-five operators, coupled to standard photons. We canonically quantize the model, paying attention to its effective character, and show that its Hamiltonian is stable, emphasizing the exact stage at which the indefinite metric appears and decomposes into a positive-metric sector and a negative-metric sector. Finally, we verify the optical theorem at the one-loop level in the annihilation channel of the forward-scattering process $e^+(p_2, r)+e^-(p_1, s)$ by applying the Lee-Wick prescription in which the states associated with the negative metric are left out from the asymptotic Hilbert space, but nevertheless are considered in the loop integration via the propagator.

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I. INTRODUCTION

Gravitational effects of elementary particles are expected to become significant at energy scales of the Planck mass $m_P \approx 10^{19}$ GeV. To describe the interplay between gravity and matter at these energies and to search for new physics, an effective approach has been actively exploited in the absence of a more fundamental theory. A class of gravitationally induced effects which could be observable at standard-model energies is the breakdown of Lorentz symmetry, which nevertheless is Planck-mass suppressed. Many experiments have been designed to possibly detect such weak signals, and they range from precision laboratory experiments to astrophysical observations.

The standard-model extension (SME) is an effective framework that incorporates all possible Lorentz-invariance-violating terms for matter and gravity in the Lagrangian. The breakdown of Lorentz symmetry originates from preferred directions which are believed to arise from expectation values of tensor fields in a more fundamental theory. A great number of the phenomenological searches for Lorentz symmetry violation has been codified within the framework of the SME [1–2]. Originally, it was constructed to include only renormalizable mass-dimension operators,—i.e, with dimension $d \leq 4$. Recently, a generalization of the SME incorporating higher-order derivative operators has been proposed. Such a program has been successfully implemented in the photon sector [3], the fermion sector [4], and more recently in the linearized sector of gravity [5].

The pioneering work of Myers and Pospelov focuses on Lorentz invariance violation with dimension-five operators coupled to a constant four-vector $n_{\mu}$ and having cubic dispersion relations in the lowest-order momentum expansion [9]. The Myers-Pospelov (MP) model has been studied to extract bounds upon its parameters from radiative corrections [7–10], cosmological observations [11], anisotropies [12], synchrotron radiation [13] and also to analyze stability and causality [14]. One can show that for a special choice of nonminimal SME coefficients, one arrives at the MP model. Recently, an approach to introduce higher-order Lorentz symmetry violation, which lies beyond the scope of the nonminimal SME with modified terms quadratic in the fields, has been proposed with higher-order coupling terms [15].

The interest in higher-order derivative operators in quantum field theory dates back to the work of Podolsky [16]. He considered a higher-order electrodynamics to deal with infinities arising from the introduction of point charges. Some years later, Pais and Uhlenbeck realized that these higher-order derivative terms may lead to some problems with stability [17]. The breakthrough in relation to stability came with the studies of Lee and Wick in the context of quantum field theories with an indefinite metric [18]. They give an important insight into the relation between the possible loss of unitarity and the interplay between statistics, stability and negative norm states. Recently, the Lee-Wick ideas have been applied to solve the hierarchy problem in the standard model [19], to study the spectrum of cosmological perturbations [20] and to construct a renormalization program in higher-derivative gravity [21]. Also, higher-order derivative operators have been included in quantum gravity approaches [22–23], in anisotropic regularization schemes [24], and in semiclassical gravity [25], and they arise in the study of the phenomenology of loop quantum gravity [26] and in string theory [27].

In 1969, T. D. Lee and G. W. Wick proposed a modified QED model with the advantage of being finite, but leading to an indefinite metric in Hilbert space [18]. They provide the main ideas towards the construction of an indefinite metric quantum field theory with a uni-
tary $S$ matrix. The indefiniteness of the metric of the Lee-Wick quantum electrodynamics comes from a non-Hermitian Hamiltonian, which, however, can be seen to arise from the presence of a higher-order derivative term as well [28]. Several issues regarding stability and unitarity were solved using what is now called the Lee-Wick prescription. The analysis was extended by Cutkosky using covariant perturbation theory based on Feynman diagrams [29].

The origin of the possible loss of unitarity in an indefinite metric theory can be found in the definition of the inner product. To see this, consider two arbitrary states $|\phi\rangle = \sum_i \phi_i |i\rangle$ and $|\psi\rangle = \sum_j \psi_j |j\rangle$ expanded in a basis $|i\rangle$ with $i = 1, 2, 3, \ldots$ and $\phi_i, \psi_j$ complex numbers. As in usual quantum mechanics, the inner product between the states is defined by $\langle \phi | \psi \rangle = \phi_i^* \eta_{ij} \psi_j$, where the metric $\eta = (\eta_{ij})$ is assumed to be a nonsingular Hermitian matrix and where the asterisk denotes complex conjugation. Now, however, the generalization consists to allow for an indefinite metric, such that the diagonal terms of the metric $\eta_{ij}$ can take negative values. In this way, the metric $\eta$ in the Hilbert space is not positive definite, and one may have states with a negative norm or ghosts in the theory. The extended inner product induced by the indefinite metric $\eta$ in general leads to a pseudo-unitary representation to a positive definite metric representation [30, 31]. Importantly, one can show that in an indefinite metric theory, the algebra of creation and annihilation operators completely determines the class of metrics $\eta$. For example, in a fermion system, once the Lagrangian is given, the equal-time anticommutation relations for the canonically conjugated field variables, the construction of the Hamiltonian and the derivation of the free-field propagator using two different approaches—the vacuum expectation value of the time-ordered product $\psi(x) \bar{\psi}(y)$, and the integration over momentum space. We explicitly exhibit the effective character of the model by performing the quantization in a box, which leads to the exclusion of the excitations with imaginary frequencies. We follow closely the conventions of Ref. [34].

In Sec. III, we compute the imaginary part of the amplitude for the one-loop diagram in the annihilation channel arising from the forward-scattering process $e^+ (p_2, r) + e^- (p_1, s)$. The calculation is made using slightly modified Feynman rules with respect to Ref. [35], which are stated at the beginning of this section. To this end, we calculate the amplitude $M_F$ for the corresponding graph, and we also identify the integral $J_{\mu\nu}$ that produces the discontinuity in the amplitude $M_F$, which yields the corresponding imaginary part. The contributions to such an integral are determined by the method of residues according to the appropriately defined Lee-Wick contour, which is constructed by taking the same position of the poles which yields the correct answer when the propagator is calculated by integrating in momentum space.

The discontinuities of $M_F$ arising from $J_{\mu\nu}$ are subsequently obtained, yielding some unexpected cancellations, which nevertheless are crucial to prove the validity of the optical theorem in this case. Some details in the derivation of such discontinuities are given in the Appendix. In Sec. IV we determine the amplitude $M_I$ for the process $e^+ (p_2, r) + e^- (p_1, s) \rightarrow e^+ (k_2, \bar{r}) + e^- (k_1, s)$ and calculate the sum over the momenta and spins of the final states in $|M_I|^2$ as required by the optical theorem. Unitarity is successfully verified by comparing this result with that obtained in the previous section for the imaginary part of $M_F$. We close with Sec. V which contains our conclusions and final comments.

II. MODEL DEFINITIONS

The modified QED Lagrangian we are interested in is obtained via the minimal coupling substitution in the derivative terms of the fermionic sector in the Myers and Pospelov (MP) model,

$$\mathcal{L} = \bar{\psi} (i \slashed{D} - m) \psi + g \bar{\psi} \gamma^\mu (n \cdot D)^2 \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where $D_{\mu} = \partial_{\mu} - ie A_{\mu}$ and $n_\mu$ is a constant four-vector breaking the Lorentz symmetry and chosen in the timelike direction, such that $n_\mu = (1, 0, 0, 0)$. As usual, we will quantize this extended electrodynamics in the interaction picture, and we follow the conventions of Ref. [35].
We will work in the axial gauge \((\cdot \cdot A) = 0\), such that the interaction term is given by the standard one in QED. Also, the photon properties remain the same as in QED. On the contrary, the fermion sector will be drastically modified, and we start to study its properties in the free case. The corresponding equation of motion is
\[
(i\p - m + g\gamma^0 \partial_0^2) \psi(x) = 0, 
\]
which includes higher-order time derivatives. Considering \(\psi(x) = \int \psi(p)e^{-ipx}dp\), we obtain the eigenvalue equation for the spinor field \(\psi(p)\):
\[
(\gamma^0(p_0 - gp_0^2) + \gamma^i p_i - m) \psi(p) = 0. 
\]
Taking the determinant of the above matrix yields the dispersion relation
\[
(p_0 - gp_0^2)^2 - p^2 - m^2 = 0,
\]
whose solutions are
\[
\omega_1 = \frac{1 - \sqrt{1 - 4gE(p)}}{2g} = \frac{1 - N_1}{2g}, \\
W_1 = \frac{1 + \sqrt{1 - 4gE(p)}}{2g} = \frac{1 + N_1}{2g},
\]

\[
\omega_2 = \frac{1 - \sqrt{1 + 4gE(p)}}{2g} = \frac{1 - N_2}{2g}, \\
W_2 = \frac{1 + \sqrt{1 + 4gE(p)}}{2g} = \frac{1 + N_2}{2g}. 
\]

Here \(E(p) = \sqrt{p^2 + m^2}\), \(N_1 = \sqrt{1 - 4gE(p)}\) and \(N_2 = \sqrt{1 + 4gE(p)}\). Let us observe that these functions are invariant under the change \(p \rightarrow -p\). We identify the solutions \(\omega_1\) and \(\omega_2\) with perturbations in \(g\) of the usual ones \(\pm E(p)\), while \(W_1\) and \(W_2\) correspond to the contributions of new degrees of freedom coming from higher-energy scales, which are nonperturbative in \(g\). We emphasize that \(\omega_2\) is negative, so that the energy corresponding to this on-shell particle is \(\omega_2 = -\omega_2\). The above eigenvalues satisfy the relation
\[
W_1 + \omega_1 = W_2 + \omega_2. 
\]
The following additional identities follow from the definitions \(5\) and \(6\):
\[
E(p) = \omega_1 - gw_1^2, \\
E(p) = W_1 - gW_1^2, \\
-E(p) = \omega_2 - gw_2^2, \\
-E(p) = W_2 - gW_2^2, 
\]

\[
E(p) - gw_1^2 = \omega_1 N_1, \\
E(p) - gW_1^2 = -W_1 N_1, \\
E(p) + gw_2^2 = -\omega_2 N_2, \\
E(p) + gW_2^2 = W_2 N_2. 
\]
From Eq. \(8\), we observe that \((p_0 - gp_0^2) = +E(p)\) for \(p_0 = \omega_1\) or \(p_0 = W_1\), while \((p_0 - gp_0^2) = -E(p)\) for \(p_0 = \omega_2\) or \(p_0 = W_2\). In this way, the corresponding spinors satisfy the Dirac equations
\[
(\gamma^0 E(p) + \gamma^i p_i - m) u(p) = 0, \text{ for } p_0 = \omega_1, W_1, 
\]
and
\[
(\gamma^0 E(p) + \gamma^i p_i + m) u(p) = 0, \text{ for } p_0 = \omega_2, W_2, 
\]
which correspond to the standard spinor solutions \(u^r(p)\) and \(v^r(p)\) labeled with the spin index \(r, s\). Our conventions for the completeness relations are
\[
\sum_r u^r(p)u^r(p) = \delta_{rs}, \\
\sum_r v^r(p)v^r(p) = \delta_{rs}. 
\]
while for orthogonality we have
\[
u^s(p)u^r(p) = 2E_p\delta^{sr}, \\
\bar{v}^s(p)v^r(p) = 2E_p\delta^{sr}, \\
u^s(p)v^r(\bar{-}p) = 0, \\
v^s(p)u^r(\bar{-}p) = 0. 
\]
The above relations can be equivalently written as
\[
\bar{u}^s(p)u^r(p) = 2m\delta^{sr}, \\
\bar{v}^s(p)v^r(p) = -2m\delta^{sr}, \\
u^s(p)v^r(p) = 0, \\
v^s(p)u^r(p) = 0. 
\]
Before proceeding to the quantization, let us recall the effective character of the timelike Myers-Pospelov model, which is vividly illustrated by the existence of the momentum cutoff \(|p|_{\text{max}} = \sqrt{M^2 - m^2}\), where \(M = 1/4g\) is a very high mass (\(M \gg m\)), possible of the order of the Planck mass, which suppresses the Lorentz violating effects to be consistent with the already determined experimental bounds. In fact, beyond \(|p|_{\text{max}}\), the frequencies \(\omega_1\) and \(W_1\) become imaginary thus introducing stability problems together with runaway solutions which make the quantization inconsistent.

In order to avoid such situation and to justify the effective character of the theory, we quantize the model in a cubic box of side \(L\), under the following assumptions: first we impose the standard periodic boundary conditions upon the momenta, such that
\[
p_i = \frac{2\pi}{L} n_i, \quad i = x, y, z, \quad n_i = 0, \pm 1, \pm 2, \ldots, \pm N, \ldots
\]
where the maximum value of each \(n_i\) is \(N_{\text{max}} \rightarrow \infty\). In this way the set of discrete functions \(\exp(\mathbf{p} \cdot \mathbf{x})/\sqrt{V}\) is complete and orthonormal inside the box. The second
This guarantees that

$$E(p) = p^2 + m^2 = \left(\frac{2\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2) + m^2 \leq \frac{M^2 - m^2}{3N_{\text{max}}^2} + m^2 = M^2,$$

for all values of $p$, in such a way that $\omega_1$ and $W_1$ are always real inside the box. A confirmation that the scale $M$ is the physical way of realizing the mathematical infinity in the effective model comes from the substitution of the relation \[^{(16)}\] into Eq. \[^{(15)}\] which yields

$$p_i \approx \frac{M}{\sqrt{3N_{\text{max}}}}.$$  \[^{(18)}\]

This implies $(p_i)_{\text{max}} \approx M/\sqrt{3}$. The analogous substitution in the standard box quantization would produce $(p_i)_{\text{max}} \approx \infty/\infty$. In the following we have not carried out in mathematical detail the two assumptions described above, but we have proceeded in an heuristic fashion by restricting ourselves to the effective model where we take for granted that $E(p) \leq M = 1/4q$, together with the condition that the sum over the momenta runs over the entire $p$ space. The following expressions related to the box quantization will be required:

$$V \delta_{pp'} = \int d^3x e^{i(p-p') \cdot x} = \sum_p e^{i(p \cdot x - p' \cdot x')} = \delta^3(x - x'),$$

$$\Phi(x) = \lim_{V \to \infty} \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \sum_p \frac{1}{\sqrt{V}} e^{-ip_0 x_0 + ip \cdot x} \tilde{\Phi}(p),$$

$$\lim_{V \to \infty} \frac{1}{V} F(p) = \int_{-\infty}^{+\infty} \frac{dp_0}{(2\pi)^3} F(p).$$ \[^{(19)}\]

The first relation above is just the box normalization of the plane wave, while the second one is the statement of completeness. The third relation defines the discrete Fourier transform in the box, together with the continuum one with respect to time. The last equation yields the passage to the continuum once all the calculations have been performed.

Having in mind the discretization previously defined we make the following expansion of the fields inside the box:

$$\psi(x) = \lim_{V \to \infty} \sum_{p,s} \frac{1}{\sqrt{2VE(p)}} \left( a^s(p) e^{-i\omega_1 x_0} + c^s_p e^{-iW_1 x_0} \right) \times e^{ip \cdot x} + \frac{1}{\sqrt{N_1}} \left( \bar{a}^s(p) e^{-i\omega_2 x_0} + d^s_p e^{-iW_2 x_0} \right) e^{-ip \cdot x},$$

$$\bar{\psi}(x) = \lim_{V \to \infty} \sum_{p,s} \frac{1}{\sqrt{2VE(p)}} \left( \bar{a}^s(p) e^{-i\omega_1 x_0} + c^s_p e^{-iW_1 x_0} \right) \times e^{-ip \cdot x} + \frac{1}{\sqrt{N_2}} \left( a^s(p) e^{-i\omega_2 x_0} + d^s_p e^{-iW_2 x_0} \right) e^{ip \cdot x},$$ \[^{(20)}\]

where we have $\bar{\psi}(x) = \psi^\dagger \gamma^0$, because the all frequencies are real inside the box. The commutation relations for particles and antiparticles (with frequencies $\omega_1$ and $\omega_2$, respectively) are taken to be the usual anticommutators

$$\{a_p^s, a_{p'}^q\} = \{b_p^s, b_{p'}^q\} = \delta^{sr} \delta_{pp'},$$ \[^{(21)}\]

while for the new excitations (with frequencies $W_1$ and $W_2$) we impose

$$\{c_p^s, e_{p'}^q\} = \{d_p^s, d_{p'}^q\} = -\delta^{sr} \delta_{pp'}. \[^{(22)}\]

The action of the annihilation operators on the vacuum is defined by

$$a_p^s|0\rangle = 0, \quad b_p^s|0\rangle = 0,$$

$$c_p^s|0\rangle = 0, \quad d_p^s|0\rangle = 0.$$ \[^{(23)}\]

The anticommutation relations in Eqs. \[^{(21)}\] and \[^{(22)}\] display the exact stage at which the indefinite metric decomposes into a positive-metric sector and a negative-metric sector. Of course, due to the negative sign in Eq. \[^{(22)}\], one is led to identify the particles created with the operators $c_p^s$ and $d_p^s$ as those with a negative-metric, while those created with $a_p^s$ and $b_p^s$ as particles with a positive-metric. We will show in the Sec. \[^{(11)}\] that Eqs. \[^{(21)}\] and \[^{(22)}\] are necessary in order to fulfill the equal-time anticommutation relations given in Eq. \[^{(28)}\].

The choice of vacuum in Eq. \[^{(23)}\] leads to the usual interpretation of the field $\psi(x)$ annihilating fermions with positive energy $\omega_1$ and creating antifermions with positive energy $|\omega_2|$. In addition, the field $\bar{\psi}(x)$ annihilates negative-metric fermions with positive energy $W_1$ and negative-metric antifermions with positive energy $W_2$. We will show in Section \[^{(13)}\] that this choice of vacuum leads to a Hamiltonian bounded from below.

### A. Canonical variables

Here we deal with the canonical quantization for the purely timelike MP Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial - H) \psi + g\psi^\dagger \psi^\dagger,$$ \[^{(24)}\]

which is hermitian, up to a total derivative, in virtue of the field definitions in Eq. \[^{(20)}\]. The canonically conjugated momenta to $\psi$,

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}},$$ \[^{(25)}\]

...
and to \( \psi \),

\[
\pi_\psi = \frac{\partial L}{\partial \dot{\psi}} - \frac{\partial \pi_\psi}{\partial t},
\]

are given by

\[
\pi_\psi = i\psi^\dagger - g\psi^\dagger,
\pi_\dot{\psi} = g\psi^\dagger.
\]

Starting from the relations (20), (21), and (22), and after taking the limit \( V \to \infty \), we have verified the following equal-time commutation relations:

\[
\{ \psi(x), \pi_\psi(y) \} = i\delta^3(x - y),
\]

\[
\{ \dot{\psi}(x), \pi_\psi(y) \} = i\delta^3(x - y),
\]

with the remaining ones being zero.

### B. Stability

The Hamiltonian density is obtained from the Legendre transformation \( H = \pi_\psi \dot{\psi} + \pi_{\dot{\psi}} - L \), leading to the Hamiltonian

\[
H = \int d^3x \left( -g\psi^\dagger \dot{\psi} + \psi^\dagger (-i\gamma^i \partial_i + m) \psi(x) \right).
\]

In a calculation analogous to the standard fermionic case, we have verified that the Hamiltonian can be written as

\[
H = \sum_{p,s} (\omega_1 a_p^\dagger a_p^s + \omega_2 b_p^\dagger b_p^s - W_1 \epsilon_p^s c_p^s - W_2 d_p^s d_p^s)
\]

in terms of the creation-annihilation operators. In obtaining Eq. (30), the identities (10) have been repeatedly used. Since \( \omega_2 < 0 \) and \( \omega_1 > 0 \), the above Eq. (30) provides the usual interpretation for particle and antiparticle states: \( a_p^s \) creates particles with momentum \( p \), spin component \( s \) and energy \( \omega_1(p) \) and \( b_p^s \) creates antiparticles with momentum \( p \), spin component \( s \) and energy \( |\omega_2(p)| \), both of which are observable. A similar interpretation is given for the operators \( c_p^s \) and \( d_p^s \) in terms of particles described by states with a negative metric.

Introducing the number operators for particles in states with a positive metric, \( \hat{N}_{1p} = \sum_s a_p^s a_p^s \), \( \hat{N}_{2p} = \sum_s b_p^s b_p^s \), and for particles in states with a negative metric, \( \hat{N}_{1p} = -\sum_s c_p^s c_p^s \), \( \hat{N}_{2p} = -\sum_s d_p^s d_p^s \), we can write

\[
H = \sum_p \left( \omega_1 \hat{N}_{1p} - \omega_2 \hat{N}_{2p} + W_1 \hat{N}_{1p} + W_2 \hat{N}_{2p} \right),
\]

which is clearly bounded from below after dropping the usual infinite constant. The Hamiltonian is stable and hermitian in the effective model.

### C. The propagator

Here we derive the fermion propagator \( S_F(x - y) \) for the purely timelike MP model defined by the Lagrangian (24). To verify that its four-momentum representation (where one specifies the position of the poles \( \omega_1, \omega_2, W_1, W_2 \) in the complex \( p_0 \) plane and hence the contour integration \( C_F \)) is correct, we first calculate the propagator according to its definition in terms of the vacuum expectation value \( S_F(x - y) = \langle 0 | T \{ \psi(x) \bar{\psi}(y) \} | 0 \rangle \), yielding

\[
S_F(x - y) = \theta(x_0 - y_0) \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle - \theta(y_0 - x_0) \times \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle.
\]

Without loss of generality, we can set \( y = 0 \). First, we consider \( x_0 > 0 \), and hence we need to calculate \( S_F^{(>})(x) = \langle 0 | \psi(x) \bar{\psi}(0) | 0 \rangle \). Using the expressions for the fields in Eq. (20), we find

\[
S_F^{(>})(x) = \mathcal{L} |0\rangle \left[ \sum_{p,s} \frac{1}{\sqrt{2V E(p)}} \left( \frac{\epsilon_p^s u^s(p)}{N_1} e^{-i\omega_1 x_0 + ip \cdot x} + \frac{\epsilon_p^s u^s(p)}{\sqrt{N_1}} e^{-iW_1 x_0 + ip \cdot x} \right)
+ \frac{\epsilon_p^s u^s(p)}{\sqrt{N_2}} e^{-iW_2 x_0 - ip \cdot x} \right.
+ \frac{\epsilon_p^s u^s(p)}{ \sqrt{N_1} } e^{-iW_1 x_0 + ip \cdot x} + \frac{ \epsilon_p^s u^s(p)}{ \sqrt{N_2} } e^{-iW_2 x_0 - ip \cdot x} \right] |0\rangle.
\]

The properties in Eq. (23) allow us to rewrite

\[
|0\rangle |X_p X_q\rangle = |0\rangle |X_p X_q\rangle = \pm \delta^{s_{p,q}} |p,q\rangle,
\]

where the plus sign for \( X = a, b \) and the minus sign for \( X = c, d \). This yields

\[
S_F^{(>})(x) = \mathcal{L} \sum_{p,s} \frac{1}{2VE(p)} \left( \frac{u^s(p)}{N_1} \bar{u}^s(p) e^{-i\omega_1 x_0 + ip \cdot x} - e^{-iW_1 x_0 + ip \cdot x} \right) - \frac{v^s(p)}{N_2} \bar{v}^s(p) e^{-iW_2 x_0 - ip \cdot x}.
\]
Using the completeness relation in Eq. [12], we obtain

\[ S_F^{(>)}(x) = L \sum_p \frac{1}{2VE(p)} \left( \frac{\hat{p} + m}{N_1} e^{-i\omega_1 x_0 + ip \cdot x} - e^{-iW_1 x_0 + ip \cdot x} - \frac{\hat{p} - m}{N_2} e^{-iW_2 x_0 - ip \cdot x} \right), \] (35)

which can be rewritten as

\[ S_F^{(>)}(x) = (i\hat{p} + m + g\gamma^0 \partial_0^2) L \sum_p \frac{1}{2VE(p)} \left( \frac{e^{-i\omega_1 x_0 + ip \cdot x}}{N_1} - \frac{e^{-iW_1 x_0 + ip \cdot x}}{N_1} + \frac{e^{-iW_2 x_0 - ip \cdot x}}{N_2} \right), \] (36)

where the limit \( L \to \infty \) can now be taken, yielding

\[ S_F^{(>)}(x) = (i\hat{p} + m + g\gamma^0 \partial_0^2) \int \frac{d^3p}{2E(p)(2\pi)^3} \left( \frac{e^{-i\omega_1 x_0 + ip \cdot x}}{N_1} - \frac{e^{-iW_1 x_0 + ip \cdot x}}{N_1} + \frac{e^{-iW_2 x_0 - ip \cdot x}}{N_2} \right), \] (37)

where \( P^\mu = (p_0 - g\gamma_0^0, p) \). The propagator in momentum space is

\[ S(p) = \frac{i(p^0 + m)}{P^2 - m^2}, \] (41)

which can be cast in the same form as Eq. (37):

\[ S_F^{(>)}(x) = (i\hat{p} + m + g\gamma^0 \partial_0^2) \int \frac{d^3p}{2E(p)(2\pi)^3} \frac{e^{-i\omega_2 x_0 - ip \cdot x}}{N_2}. \] (39)

Now we turn to the calculation of the propagator in momentum space. From the equation of motion [3], the Feynman propagator is

\[ S_F(x - y) = L \int \frac{dp_0}{(2\pi)^3} \sum_p \frac{1}{V} \frac{i(P^0 + m)}{P^2 - m^2} e^{-ip(x - y)}, \] (40)

\[ S(p) = \frac{i(P^0 + m)}{g^2(p_0 - (\omega_1 - i\epsilon))(p_0 - (\omega_2 + i\epsilon))(p_0 - (W_1 - i\epsilon))(p_0 - (W_2 - i\epsilon)). \] (42)

That is to say, \( \omega_1, W_1 \) and \( W_2 \) are in the lower \( p_0 \) complex plane, while \( \omega_2 \) is in the upper \( p_0 \) complex plane. This choice is shown in Fig. 1 together with the motion of the poles as momentum \( p \) increases. The arrows indicate their trajectories in the \( p_0 \) plane according to Eqs. [5] and [6]. We see that both poles \( \omega_2 \) and \( W_1 \) collapse at the value \( 1/2g \) when \( E(p) = 1/4g \) and move in opposite imaginary directions when \( E(p) > 1/4g \). Since our effective model is defined only for \( E \leq 1/4g = M \), such imaginary frequencies do not play any role in the calculation and are shown in Fig. 1 just for completeness. As \( |p| \) increases, the poles \( \omega_2 \) and \( W_2 \) move in real and opposite directions. In this way, the Feynman contour \( C_F \) is defined as the real axis with the poles located as shown in Fig. 1.

Next, we derive the propagator using the expression [42] together with the contour \( C_F \) and show that we recover the expressions (37) and (39) obtained in the previous calculation by using its definition in terms of vacuum expectation values. To this end, let us set \( y = 0 \) and consider \( x_0 > 0 \). The factor \( e^{ip_0 x_0} \) in Eq. (40) indicates that we have to close our contour from below (\( \text{Im} p_0 < 0 \),
FIG. 1: The contour of integration $C_F$ which defines the Feynman propagator. For $x_0 > 0$, it picks up the positive poles $\omega_1, W_1, W_2$, and for $x_0 < 0$, it picks up the negative pole $\omega_2$. At the energy $E = 1/4g$, the two poles $\omega_1$ and $W_1$ collide and move in opposite directions parallel to the imaginary axis as the energy increases. Nevertheless, our effective model is always valid only for $E \leq 1/4g$, so that the case of energies larger than $1/4g$ is not relevant to our calculation. The poles $\omega_2$ and $W_2$ always stay in the real axis and move in opposite directions.

thus enclosing the poles $\omega_1, W_1$ and $W_2$. This yields

$$S_F^{(+)}(x) = \frac{i}{2\pi} \sum_p \frac{1}{V} (-2\pi i) \left[ (\gamma^0 E(p) - \gamma \cdot p + m) \right.$$  

$$\times \left( \frac{e^{-i\omega_1 x_0 + \gamma \cdot p}}{g^2(\omega_1 - \omega_2, \omega_1 - W_1, \omega_1 - W_2) + e^{-iW_1 x_0 + \gamma \cdot p}} + \frac{1}{g^2(W_2 - \omega_1)(W_2 - \omega_2)(W_2 - W_1)} \right).$$  

Using the identities

$$g^2(\omega_1 - \omega_2)(\omega_1 - W_1)(\omega_1 - W_2) = 2E N_1,$$

$$g^2(W_1 - \omega_1)(W_1 - \omega_2)(W_1 - W_2) = -2E N_1,$$

$$g^2(W_2 - \omega_1)(W_2 - \omega_2)(W_2 - W_1) = 2E N_2,$$

we have

$$S_F^{(+)}(x) = \frac{i}{2\pi} \sum_p \frac{1}{V} \left[ (\gamma^0 E(p) - \gamma \cdot p + m) \right.$$  

$$\times \left( \frac{e^{-i\omega_1 x_0 + \gamma \cdot p}}{2E N_1} - \frac{e^{-iW_1 x_0 + \gamma \cdot p}}{2E N_1} \right) + (-\gamma^0 E(p) - \gamma \cdot p + m) \frac{e^{-iW_2 x_0 + \gamma \cdot p}}{2E N_2}. $$  

where we have changed $p \rightarrow -p$ in the last term. After taking the limit $V \rightarrow \infty$, the above expression reduces to

$$S_F^{(+)}(x) = (i\theta + m + g^2 \gamma^0 \frac{2}{2E \pi^3} \int \frac{d^3p}{N_1} \frac{e^{-i\omega_1 x_0 + \gamma \cdot p}}{N_1} + \frac{e^{-iW_1 x_0 + \gamma \cdot p}}{N_2}).$$  

The above expression reproduces the form of the propagator obtained in Eq. (37). When $x_0 < 0$, the $p_0$ integration is made by closing the contour from above, and we obtain

$$S_F^{(-)}(x) = \frac{i}{2\pi} \sum_p \frac{1}{V} \left[ (\gamma^0 E(p) - \gamma \cdot p + m) \right.$$  

$$\times \frac{e^{-i\omega_2 x_0 + \gamma \cdot p}}{g^2(\omega_2 - \omega_1)(\omega_2 - W_1)(\omega_2 - W_2)}. $$  

Using now the relation

$$g^2(\omega_2 - \omega_1)(\omega_2 - W_1)(\omega_2 - W_2) = -2E N_2, $$

we finally arrive at

$$S_F^{(-)}(x) = (i\theta + m + g^2 \gamma^0 \frac{2}{2E \pi^3} \int \frac{d^3p}{N_2} \frac{e^{-i\omega_2 x_0 + \gamma \cdot p}}{N_2},$$  

which reproduces Eq. (39). In this way, we have proved that the prescription (12) for the position of the poles in the complex $p_0$ plane yields the correct fermionic propagator, according to the definition in Eq. (32).

### III. Unitarity in the Myers-Pospelov Electrodynamics: One-Loop Level

Our goal is to verify the optical theorem for the simple diagram shown in Fig. 2. In the conventions of Ref. [35], the $S$ matrix is defined as

$$S \equiv 1 + iT = 1 + (2\pi)^4 \delta^4(P_i - P_f) \prod_f \left( \frac{1}{2VE_i} \right)^{1/2} \times \prod_f \left( \frac{1}{2VE_f} \right)^{1/2} (iM).$$  

The mass fermions factors $\prod_F (2m_F)^{1/2}$ appearing in the corresponding expression of Ref. [35] have been incorporated in our definitions of the fermion wave functions $u^i$ and $v^i$. Then we have the relation $u^i(p) = \sqrt{2m} U^i(p)$ and correspondingly for $v^i(p)$, which transforms the spinor completeness and orthogonality relations for $U^i(p)$ in Ref. [35] into ours, given in Eqs. (12) and (13). We take the same definition and normalization of the one-particle state as in Ref. [35], so

$$|e^-, p, \sigma \rangle = a^\dagger_\sigma |0\rangle,$$

$$|e^+, q, r \rangle = b^\dagger_\eta |0\rangle,$$  

where
In our case, the process scattering amplitude for the process $p \to p'$ with the amplitude $A$, corresponds to the last expression in Eq. (19), the required completeness relation for one-particle states is

$$\sum_{p, s} |s \rangle \langle p, s| = 1.$$

FIG. 2: The optical theorem, showing the forward-scattering process $e^+(p_1, s) + e^+(p_2, r)$ on the left-hand side and the sum over intermediate states $e^-(k_1, s) + e^+(k_2, r)$, conserving total energy $p_A$, on the right-hand side. The phase-space measure is defined by

$$\int d\Pi_{k_1} d\Pi_{k_2} = \int \left( \frac{d^3 k_1}{(2\pi)^3} \frac{1}{2E(k_1)} \right) \left( \frac{d^3 k_2}{(2\pi)^3} \frac{1}{2E(k_2)} \right) (2\pi)^4 \delta^4 (p_A - (k_1 + k_2)).$$

together with

$$(p, s|q, r) = \delta^{sr} \delta_{p, q}.$$

Then, the completeness relation for one particle states is

$$\sum_{p, s} |s \rangle \langle p, s| = 1.$$

Following standard steps and taking the limit $V \to \infty$ according to the last expression in Eq. (19), the required form of the optical theorem reads

$$2\text{Im} M(A \to A) = \sum_n \left( \prod_i \int d^3 k_i \frac{1}{(2\pi)^3 2E(k_i)} \right) \times \sum_{s_i} |M(A \to B(k_1, s_1, \ldots, k_n, s_n))|^2 \times (2\pi)^4 \delta^4 (p_A - \sum_i k_i).$$

Here $A$ denotes the initial and final processes associated with the amplitude $M(A \to A) = M_f$, each process carrying total momentum $p_A$. The term in the right-hand side $M(A \to B(k_1, s_1, \ldots, k_n, s_n)) = M_f$ denotes the scattering amplitude for the process $A$ going to the final states of $n$ particles described by $B(k_1, s_1, \ldots, k_n, s_n)$. In our case, the process $A$ corresponds to $e^- (p_1, s) + e^+ (p_2, r)$, with $p_A = p_1 + p_2$ and the process $B$ is $e^- (k_1, s) + e^+ (k_2, r)$, as shown in Fig. 2.

We have adopted the Lee-Wick prescription, where the intermediate states on the right-hand side of Eq. (54) include only the positive-metric states—i.e., $e^-$ and $e^+$ in our case—while leaving out the negative-metric states [18]. Nevertheless, the latter states may contribute to the imaginary part of the left-hand side via the propagator in the loop integral, and this mismatching could be a cause for the failure of unitarity. Still, as we show in the following, a proper choice of the so-called Lee-Wick integration contour will restore unitarity at the level of the optical theorem [54].

With the conventions indicated above, the required Feynman rules to calculate the above amplitudes are the same as those for QED in Ref. [33], except for the following changes: (i) The photon propagator is

$$D_{\mu\nu}(p) = -\frac{i}{p^2 + i\epsilon} \left[ \eta_{\mu\nu} - \frac{p_\mu n_\nu + n_\mu p_\nu}{(n \cdot p)} + p_\mu p_\nu \frac{n^2}{(n \cdot p)^2} \right],$$

which is given in the homogeneous temporal axial gauge. Nevertheless, in our case this propagator joins two conserved currents, in such a way that only the term proportional to $\eta_{\mu\nu}$ contributes. Also, we know that the Fadeev-Popov ghosts required in the case of the axial gauges decouple, so that there is no contribution coming from them. (ii) The fermion propagator is

$$S(q) = \frac{i(\gamma_0 f(q_0) + \gamma_0 q^i + m)}{g^2(q_0 - \omega_1) (q_0 - \omega_2) (q_0 - W_1) (q_0 - W_2)},$$

with the notation

$$f(q_0) = q_0 - g q_a^2,$$

in agreement with Eq. (41), where the position of the poles has already been specified in Eq. (42). Here we have contributions from the negative-metric states via the poles $W_1$ and $W_2$. (iii) The last change in the Feynman rules comes from the fermionic external lines, where we have to introduce the following replacements:

$$u^s(p) \to \frac{u^s(p)}{\sqrt{N_1(p)}}, \quad v^s(p) \to \frac{v^s(p)}{\sqrt{N_2(p)}},$$

and analogously for $\bar{u}^s(p)$ and $\bar{v}^s(p)$. As established previously, the spinors $u^s(p)$ and $v^s(p)$ are the same as in standard QED. The first relation in (58) can be directly seen from the action

$$\psi^+ (x) | e^-, q, r \rangle = \frac{1}{\sqrt{2VE(q)}} \frac{u_s(p)}{\sqrt{N_1(p)}} \times e^{-ipx} a^{s, r}_{p} a^{s, r}_{q} | 0 \rangle,$$

required in the Wick expansion of the interaction Hamiltonian. The property

$$a^{s, r}_{p} a^{s, r}_{q} | 0 \rangle = \delta^{sr} \delta_{p, q},$$

yields

$$\psi^+ (x) | e^-, q, r \rangle = \left( \frac{1}{\sqrt{2VE(q)}} \frac{u_s(q)}{\sqrt{N_1(q)}} \right) e^{-i\omega_1(q)x_0} e^{i q \cdot \mathbf{x}}.$$
The first parenthesis in the right hand side of the above equation has been factored out in the expression \([50]\) for the \(S\) matrix, and the resulting exponential contributes to the total energy-momentum conservation, with the physical energy \(\omega_1(q)\) arising from the dispersion relations. Then, only the term \(u_\gamma(q)/\sqrt{N_1(q)}\) contributes to the amplitude \((i\mathcal{M})\). A similar result can be obtained for the remaining spinors, thus validating the replacements shown in Eq. \([58]\).

The amplitude \(\mathcal{M}_F\) for the graph shown in the left-hand side of Fig. 2 is

\[
i\mathcal{M}_F = (-1) \frac{1}{N_1(p_1)N_2(p_2)} \bar{v}^\nu(p_2)(-ie\gamma^\mu)u^\mu(p_1)
\]

\[
D_{\mu\nu}(p) \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[(-ie\gamma^p)S(q-p)(-ie\gamma^{\nu})S(q)\right]
\]

\[
\times D_{\rho\sigma}(p) \bar{u}^\nu(p_1)(-ie\gamma^\sigma)v^\rho(p_2),
\]

where the minus sign comes from the fermion loop, and \(p^\mu = p_1^\mu + p_2^\mu\). Let us define the currents

\[
J_1^\mu(p_1, p_2) = \frac{1}{\sqrt{N_1(p_1)N_2(p_2)}} \bar{v}^\nu(p_2)\gamma^\mu u^\nu(p_1),
\]

\[
J_2^\mu(p_1, p_2) = \frac{1}{\sqrt{N_1(p_1)N_2(p_2)}} \bar{u}^\nu(p_1)\gamma^\mu v^\nu(p_2)
\]

\[
= [J_1^\mu(p_1, p_2)]^\ast.
\]

Due to current conservation at the ingoing and outgoing vertices, \(p_\mu J_1^\mu = 0\) and \(p_\mu J_2^\mu = 0\), the only contribution from the photon propagator to the amplitude \(\mathcal{M}_F\) arises from the term containing \(\eta_{\mu\nu}\).

In the center-of-mass frame \((p = 0)\) and using Eq. \((66)\), we can write

\[
\mathcal{M}_F = -\frac{e^4}{p^4} J_1^\nu J_2^\mu \int \frac{d^3q}{(2\pi)^3} \int_{C_{LW}} dq_0 (-i)
\]

\[
\times \text{Tr} \left[ \gamma_\mu \gamma_0 f(q_0 - p_0) + \gamma_\nu \gamma_4 q_4 + m \right] \gamma_\nu
\]

\[
\times \left( \frac{\gamma_0 f(q_0) + \gamma_4 q_4 + m}{D_{q-p}} \right) D_q,
\]

\[
= \int_{C_{LW}} dq_0 T_{\mu\nu}(p_0, q_0) I(p_0, q_0, q).
\]

It is convenient to define

\[
T_{\mu\nu}(p_0, q_0, q) \equiv \text{Tr} \left[ \gamma_\mu (\gamma_0 f(q_0 - p_0) + \gamma_4 q_4 + m) \gamma_\nu
\]

\[
\times (\gamma_0 f(q_0) + \gamma_4 q_4 + m) \right],
\]

\[
I(p_0, q_0, q) \equiv \frac{-i}{D_q D_{q-p}},
\]

\[
J_{\mu\nu}(p_0, q_0) = \int_{C_{LW}} dq_0 T_{\mu\nu}(p_0, q_0) I(p_0, q_0, q).
\]

We recall that the poles \(\omega_1\) and \(W_1\) and \(W_2\) are in the lower complex \(p_0\) plane, while \(\omega_2\) is in the upper complex \(p_0\) plane.

We define the corresponding Lee-Wick contour \(C_{LW}\), shown in Fig. 3 such that the poles \(\omega_1, W_1, W_2, \omega_1, \bar{W}_1\) and \(\bar{W}_2\) are in the lower sector, while the poles \(\omega_2\) and \(\bar{W}_2\) are in the upper sector. Then we have two ways of calculating the integral \(J_{\mu\nu}(p_0, q_0)\) by closing the Lee-Wick contour in the upper or the lower complex \(p_0\) plane.

First, closing the contour downward yields

\[
J_{\mu\nu}(p_0, q_0) = (-2\pi i) \sum_z T_{\mu\nu}(q_0, p_0, q)|_{q_0=z} [\text{Res } I(q_0, p_0, q)]_{q_0=z} = (-2\pi i)
\]

\[
\times \sum_z [T_{\mu\nu}(q_0, p_0, q)|_{q_0=z}] I_z,
\]

where \(z\) runs over the poles \(\omega_1, W_1, W_2, \omega_1, \bar{W}_1,\) and \(\bar{W}_2\) and \(\text{Res } I(q_0, \ldots)\) at the pole \(z\). Since the integral \(J_{\mu\nu}(p_0, q_0)\) in a full circle at infinity is zero because the integrand behaves as \(q_0^{-4}\) in that limit, closing the Lee-Wick contour upward and including the remaining poles \(\omega_2, \bar{W}_2,\) should yield the same result as Eq. \((71)\). In other words, we expect
where we have repeatedly used Eqs. (44) and (48). In the equations above, the eigenvalues ω₁, ω₂, W₁ and W₂ are all functions of E(q), according to the definitions (9) and (10).

In order to compute the imaginary part of $M_F$, we recall that

\[ M_F = -\frac{e^4}{p^4} J^\mu_1 J^\mu_2 \int \frac{d^3q}{(2\pi)^3} \sum_z T_{\mu\nu}(q_0, p_0, q)\bigg|_{q_0=\bar{z}} I_z, \]

and we use the relation

\[ \text{Im}(M_F(p_0 + i\epsilon)) = \frac{1}{2i} \text{Disc}(M_F(p_0 + i\epsilon) - M_F(p_0 - i\epsilon)). \]

The discontinuity in Eq. (74) arises only from each contribution $I_z, I_{\bar{z}}$. Assuming that one of them occurs at $p_0 = \alpha$, we focus on the relevant part of $I_z$, which we write as

\[ I_z = U(p_0, q, \ldots) \frac{1}{(p_0 - \alpha)}. \]

The contribution to the discontinuity will be

\[ \text{Disc}(I_z) = U(p_0 = \alpha, q) \frac{1}{2i} \times \left( \frac{1}{p_0 - \alpha + i\epsilon} - \frac{1}{p_0 - \alpha - i\epsilon} \right) = \pi U(p_0 = \alpha, q) \delta(p_0 - \alpha), \]

according to the identity

\[ \frac{\epsilon}{x^2 + \epsilon^2} = \pi \delta(x). \]

Next, we list the contributions to the discontinuity arising from each $I_z$ and leave to the Appendix the detailed derivation of the results:

\begin{align*}
\text{Disc}(I_{\omega_1}) &= \frac{i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (\omega_1 - \omega_2)) , \\
\text{Disc}(I_{\omega_2}) &= \frac{-i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (\omega_1 - \omega_2)) , \\
\text{Disc}(I_{W_1}) &= \frac{i\pi^2}{E^2(q)N_1} \delta(p_0 - (W_1 - \omega_1)) , \\
\text{Disc}(I_{W_2}) &= \frac{-i\pi^2}{E^2(q)N_1} \delta(p_0 - (W_1 - \omega_1)) , \\
\text{Disc}(I_{\bar{W}_1}) &= \frac{i\pi^2}{E^2(q)N_1} \delta(p_0 - (W_1 - \omega_1)) , \\
\text{Disc}(I_{\bar{W}_2}) &= \frac{-i\pi^2}{E^2(q)N_1} \delta(p_0 - (W_1 - \omega_1)) , \\
\text{Disc}(I_{\bar{\omega}_1}) &= 0 , \\
\text{Disc}(I_{\bar{\omega}_2}) &= 0 , \\
\text{Disc}(I_{\bar{\omega}_2}) &= \frac{i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (\omega_1 - \omega_2)) .
\end{align*}
The next step is to calculate the discontinuities in $\mathcal{M}$, from the general expression

$$
\text{Disc}(\mathcal{M}_F) = -\frac{e^4}{p^4} J_1^\mu(p_1) J_2^\nu(p_2) \int \frac{d^4q}{(2\pi)^4} \times \sum_z T_{\mu\nu}(q_0, p_0, \bm{q}) \bigg|_{q_0 = z} \text{Disc}(I_z). \tag{80}
$$

Next we concentrate on the basic ingredient

$$
U_{\mu\nu}(\bm{q}) \bigg|_z \equiv T_{\mu\nu}(q_0, p_0, \bm{q}) \bigg|_{q_0 = z} \times \text{Disc}(I_z). \tag{81}
$$

where we introduce the further notation

$$
T_{\mu\nu}(q_0, p_0, \bm{q}) = \text{Tr}(\gamma^\mu \gamma^0 f(q_0 - p_0) + \gamma^\nu \gamma^0 f(q_0 - p_0)) = F_{\mu\nu}(f(q_0 - p_0), f(q_0)),
$$

in order to emphasize the relevant variables at this stage. According to Eq. (80), together with the definitions (81) and (82), in order to calculate $\text{Disc}(\mathcal{M}_F)$, we need the combinations $F_{\mu\nu}(f(q_0 - p_0), f(q_0)) \times \text{Disc}(I_z)$. The relevant terms here are the products of $F_{\mu\nu}(f(q_0 - p_0), f(q_0))$ times the delta functions appearing in each $\text{Disc}(I_z)$ arising from Eq. (79). We provide a list of them in the following:

$$
F_{\mu\nu}(f(q_0 - p_0), f(q_0)) \bigg|_{q_0 = \omega_1} \delta(p_0 - (\omega_1 - \omega_2)) = F_{\mu\nu}(-E(\bm{q}), E(\bm{q})) \delta(p_0 - (\omega_1 - \omega_2)),
$$

$$
F_{\mu\nu}(f(q_0 - p_0), f(q_0)) \bigg|_{q_0 = \omega_1} \delta(p_0 - (W_1 - \omega_1)) = F_{\mu\nu}(E(\bm{q}), E(\bm{q})) \delta(p_0 - (W_1 - \omega_1)),
$$

$$
F_{\mu\nu}(f(q_0 - p_0), f(q_0)) \bigg|_{q_0 = W_1} \delta(p_0 - (\omega_1 - \omega_2)) = F_{\mu\nu}(E(\bm{q}), -E(\bm{q})) \delta(p_0 - (\omega_1 - \omega_2)),
$$

$$
F_{\mu\nu}(f(q_0 - p_0), f(q_0)) \bigg|_{q_0 = W_2} \delta(p_0 - (\omega_1 - \omega_2)) = F_{\mu\nu}(E(\bm{q}), -E(\bm{q})) \delta(p_0 - (\omega_1 - \omega_2)).
$$

In proving the above results, we have extensively used the relations (8). In this way, the contributions to the discontinuity of $\mathcal{M}$ are

$$
U_{\mu\nu}(\bm{q}) \bigg|_{\omega_1} = \frac{i \pi^2}{E^2(\bm{q}) N_1 N_2} F_{\mu\nu}(-E(\bm{q}), E(\bm{q})) \times \delta(p_0 - (\omega_1 - \omega_2)), \tag{84}
$$

$$
U_{\mu\nu}(\bm{q}) \bigg|_{\omega_2} = -\frac{i \pi^2}{E^2(\bm{q}) N_1 N_2} F_{\mu\nu}(E(\bm{q}), E(\bm{q})) \times \delta(p_0 - (W_1 - \omega_1)), \tag{85}
$$

$$
U_{\mu\nu}(\bm{q}) \bigg|_{W_1} = \frac{i \pi^2}{E^2(\bm{q}) N_1 N_2} F_{\mu\nu}(E(\bm{q}), E(\bm{q})) \times \delta(p_0 - (W_1 - \omega_1)), \tag{86}
$$

$$
U_{\mu\nu}(\bm{q}) \bigg|_{W_2} = -\frac{i \pi^2}{E^2(\bm{q}) N_1 N_2} F_{\mu\nu}(E(\bm{q}), E(\bm{q})) \times \delta(p_0 - (\omega_1 - \omega_2)). \tag{87}
$$

Let us observe that unexpected cancellations occur:

$$
U_{\mu\nu}(\bm{q}) \bigg|_{\omega_1} + U_{\mu\nu}(\bm{q}) \bigg|_{W_1} = 0,
$$

$$
U_{\mu\nu}(\bm{q}) \bigg|_{\omega_2} + U_{\mu\nu}(\bm{q}) \bigg|_{W_2} = 0. \tag{92}
$$

Now we are in position to calculate the final result for $\text{Disc}(\mathcal{M}_F)$, in agreement with the Lee-Wick contour $C_{\text{LW}}$...
in the center-of-mass frame. If we were to close the Lee-Wick contour from above in Eq. (70), the relation (72) tells us that nothing but the poles $\hat{s}, \hat{\omega}_2$ need to be included. Then, the corresponding contributions to $U_\mu^\nu(q)|_{\hat{s}}$ arise only from $U_\mu^\nu(q)|_{\hat{\omega}_2}$ in Eq. (91), which is equal to $U_\mu^\nu(q)|_{\omega_1}$. In this way, we have explicitly shown that the result in Eq. (93) is independent of the way in which we calculate the $q_0$ integral in Eq. (70). As mentioned previously, this is a consequence of the fact that such an integral is zero in a circle at infinity.

IV. VERIFICATION OF THE OPTICAL THEOREM

We have already calculated the left-hand side of Eq. (54) in the evaluation of the optical theorem. Now we deal with the contribution of the final states required in the right-hand side of this equation. To the order considered, we have only two-particle final states. In this way, we start by calculating the amplitude $M_F$ for the process $e^-(p_1, s) + e^+(p_2, r) \to e^-(k_1, \hat{s}) + e^+(k_2, \hat{r})$. As already stated, we apply the Lee-Wick prescription in such a way that we only consider the asymptotic states corresponding to those with a positive metric, corresponding to the frequencies $\omega_1$ and $\omega_2$. We have also defined our effective model to have $E(p) < 1/4g$, in order to deal only with real frequencies at which the Hamiltonian is stable and hermitian. We obtain

$$M_F = -\frac{i}{p^2} J^\mu_1 \left[ \bar{v}^\nu(k_2) (-ie\gamma^\mu) u^\nu(k_1) \right] \frac{1}{\sqrt{N_1(k_1) N_2(k_2)}}.$$  

(94)

where we have introduced the current defined in Eq. (63). This yields

$$|M_F|^2 = \frac{e^2}{p^2} \frac{1}{N_1(k_1) N_2(k_2)} J^\mu_1(p_1, p_2) \left[ J^\alpha_1(p_1, p_2) \right]^* \left[ \bar{v}^\nu(k_2) (e\gamma^\mu) u^\nu(k_1) \right] \bar{u}^\nu(k_1)(e\gamma_\alpha) e^\nu(k_2).$$  

(95)

where $W$ is the right-hand side of Eq. (54), which we denote by $W$,

$$W = \sum_{r, \bar{s}} \int \frac{d^3 k_1}{(2\pi)^3} \frac{1}{2E(k_1)} \frac{d^3 k_2}{(2\pi)^3} \frac{1}{2E(k_2)} \frac{1}{(2\pi)^4} \delta^4(p = k_1 + k_2) |M_F|^2,$$  

(96)

we perform the sum over the spin components $\bar{s}, \bar{r}$ with the result

$$\sum_{r, \bar{s}} |M_F|^2 = \frac{1}{p^2} \frac{1}{N_1(k_1) N_2(k_2)} J^\mu_1(p_1, p_2) \left[ J^\alpha_1(p_1, p_2) \right]^* \left[ \bar{v}^\nu(k_2) E(k_1) + \bar{u}^\nu(k_1) E(k_2) \right] \bar{u}^\nu(k_1)(e\gamma_\alpha) e^\nu(k_2).$$  

(97)

In the center-of-mass frame, we have

$$W = \frac{e^2}{p^2} \frac{1}{N_1(k_1) N_2(k_2)} J^\mu_1(p_1, p_2) \left[ J^\alpha_1(p_1, p_2) \right]^* \left[ \bar{v}^\nu(k_2) E(k_1) + \bar{u}^\nu(k_1) E(k_2) \right] \bar{u}^\nu(k_1)(e\gamma_\alpha) e^\nu(k_2).$$  

(98)

In the last step, we relabel $k_1 \to q$, and we integrate over $d^3 k_2$. In this way,

$$W = \frac{e^2}{p^2} \frac{1}{N_1(k_1) N_2(k_2)} J^\mu_1(p_1, p_2) \left[ J^\alpha_1(p_1, p_2) \right]^* \left[ \bar{v}^\nu(q) E(k_1) + \bar{u}^\nu(k_1) E(k_2) \right] \bar{u}^\nu(k_1)(e\gamma_\alpha) e^\nu(k_2).$$  

(99)

The cyclic property of the trace together with the relation $[J^\mu_1]^* = J^\nu_2$ from Eq. (63) show that $W = 2\text{Im}[M_F] = -i\text{Disc}(M_F)$, where the last expression is given in Eq. (93), thus verifying the optical theorem.

V. CONCLUSIONS AND OUTLOOK

The effective approach to quantum field theory provides a powerful tool to search for new physics beyond the standard model. In particular, the search for quantum gravity effects at low energies in the form of Lorentz
symmetry violations has become an active research area from both the phenomenological and experimental points of view. The majority of these searches have been realized by coupling constant tensors, yielding Lorentz invariance violation, with derivative operators of renormalizable mass dimension. In this way one guarantees from the beginning some crucial requirements about stability and unitarity in the effective quantum field theory.

However, the study of Lorentz symmetry violation incorporating higher-order derivative operators has attracted interest in the last few years. There are good reasons for this: (i) Bounds arising from higher-order derivative operators have been less explored experimentally as compared to those arising from renormalizable models, and (ii) Due to the increase of the number of degrees of freedom in models with higher-order derivative operators, they have the potentiality to capture higher-energy degrees of freedom associated with new physics.

Also, it is well known that the introduction of higher-order derivative operators has the advantage of smoothing ultraviolet divergences. The nonminimal SME and the Myers-Pospelov model provide the framework to develop possible effects of higher-order Lorentz invariance violation. One of the goals in this construction has been to show in the past works of Lee and Wick. A further step in generalizing this result will be to probe the optical theorem in each of the remaining one-loop Feynman diagrams that appear in the model, for the same scattering processes.

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Appendix A: THE CONTRIBUTIONS TO THE DISCONTINUITY OF $I(p_0, q)$

For each term $I_z$ or $I_\perp$, we indicate the possible contributions to the discontinuity [the choices of $p_0 = Y(|q|)$ which make each denominator zero] and analyze which such conditions can be fulfilled. On the one hand, we have

$$p_0(|p|) = \omega_1(|p|) + |\omega_2(|p|)| = \sqrt{1 + 4gE(|p|) - \sqrt{1 - 4gE(|p|)}}$$

$$= \frac{N_2(|p|) - N_1(|p|)}{2g}$$

(A1)

in the center-of-mass frame, where $p_{e^+} = -p_{e^-} = p$. On the other hand, the condition for a discontinuity to occur is that the equation $p_0(|p|) = Y(|q|)$ have a solution for $|q|$. The function $Y(|q|)$ will depend on the various combinations of the eigenvalues $\omega_1, \omega_2, W_1$ and $W_2$ defined in Eqs. (5) and (6). We have to consider only the positive contributions to $p_0$, which we discuss below, in order to further evaluate the discontinuities arising from Eq. (A1).

1. Identification of the contributions to the discontinuity

The following cases arise:

**Case 1:**

$$p_0 = \frac{N_2(q) - N_1(q)}{2g}$$

(A2)

which is directly solved by choosing $q = p$ according to Eq. (A1).

**Case 2:**

$$p_0 = \frac{N_2(q) + N_1(q)}{2g} = \frac{\sqrt{1 + 4gE(q)} + \sqrt{1 - 4gE(q)}}{2g}.$$  

(A3)
Substituting Eq. (A1) and taking the square of the resulting equation, we obtain
\[ \sqrt{1 - 16g^2E^2(p)} = - \sqrt{1 - 16g^2E^2(q)}, \tag{A4} \]
which produces a sign inconsistency, leading to no solution in this case.

**Case 3:**
\[
p_0 = \frac{N_2(q)}{g} = \frac{\sqrt{1 + 4gE(q)}}{g}. \tag{A5} \]
Replacing \( p_0 \) as before and taking the square of the resulting equation yields
\[ - \sqrt{1 - 16g^2E^2(p)} = 1 + 8gE^2(q). \tag{A6} \]
The left-hand side of the above equation is negative, while the right-hand side is positive, leading again to no solution for \( |q| \).

**Case 4:**
\[
p_0 = \frac{N_1(q)}{g} = \frac{\sqrt{1 - 4gE(q)}}{g}. \tag{A7} \]
Replacing \( p_0 \) as before and taking the square of the resulting equation yields
\[ - \sqrt{1 - 16g^2E^2(p)} = 1 - 8gE^2(q). \tag{A8} \]
Since \( 4gE^2(q) < 1 \), we still can have a solution in the region
\[ 1 < 8gE^2(q) < 2. \tag{A9} \]
In this case the right-hand side of Eq. (A8) is negative. Solving for the resulting equation, we get
\[ E^2(q) = \frac{1 + \sqrt{1 - 16g^2E^2(p)}}{8g}. \tag{A10} \]
In fact, Eq. (A10) is satisfied for the whole range of values of \( E(p) \), while for \( E(p) = 0 \), we have \( 8gE^2(q) = 1 + \sqrt{1 - 16g^2m^2} < 2 \), while for \( E(p) = E_{\text{max}} = 1/4g \) we obtain \( 8gE^2(q) = 1 \). Thus, this case will contribute to the discontinuity.

### 2. The particular cases

To compute \( \text{Disc}(I_{\omega_1}) \) from the first Eq. (73), we have the possible choices for \( p_0 \)
\[
p_0 = \omega_1 - \omega_2 = \frac{N_2 - N_1}{2g}, \tag{A11} \]
\[
p_0 = \omega_1 - W_1 = -\frac{N_1}{g} < 0, \tag{A12} \]
and
\[
p_0 = \omega_1 - W_2 = -\frac{N_1 + N_2}{2g} < 0, \tag{A13} \]
where we have used Eqs. (43) and (49). Since \( N_2 > N_1 > 0 \), the only contribution arises from the first case in Eq. (A11), which yields
\[
\text{Disc}(I_{\omega_1}) = i(2\pi)^2 \frac{\delta(-\omega_1 + \omega_2 + p_0)}{g^2E(q)N_1} \times \frac{1}{1 - p_0/(\omega_1 - W_1 - p_0)}(\omega_1 - W_2 - p_0). \tag{A14} 
\]
The delta function allows us to rewrite the denominator as
\[ \frac{-1}{g^2(\omega_2 - \omega_1)(\omega_2 - W_1)(\omega_2 - W_2)}. \tag{A15} \]
Using the relation (48) we finally arrive at
\[
\text{Disc}(I_{\omega_1}) = \frac{(i\pi)^2}{E^2(q)N_1N_2} \delta(p_0 - (\omega_1 - \omega_2)). \tag{A16} 
\]
The next calculation for \( \text{Disc}(I_{\omega_1 + p_0}) \) follows closely the previous case, so we mention only the relevant points. From Eq. (73), we read the following possible values for \( p_0 \):
\[
p_0 = - (\omega_1 - \omega_2) = -\frac{N_2 - N_1}{g} < 0, \tag{A17} \]
\[
p_0 = - (\omega_1 - W_1) = -\frac{N_1}{g}, \tag{A18} \]
\[
p_0 = - (\omega_1 - W_2) = -\frac{N_1 + N_2}{2g}, \tag{A19} \]
According to Sec. A1, only the second case in Eq. (A17) survives, yielding
\[
\text{Disc}(I_{\omega_1 + p_0}) = - \frac{i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (W_1 - \omega_1)). \tag{A20} 
\]
For \( \text{Disc}(I_{W_1}) \), the possibilities for \( p_0 \) are
\[
p_0 = W_1 - \omega_1 = \frac{N_1}{g}, \tag{A21} \]
\[
p_0 = W_1 - \omega_2 = \frac{N_1 + N_2}{2g}, \tag{A22} \]
\[
p_0 = W_1 - W_2 = \frac{N_1 - N_2}{2g} < 0. \tag{A23} \]
From Sec. A1, we conclude that the only contribution arises from the first case in Eq. (A21), which produces
\[
\text{Disc}(I_{W_1}) = \frac{i\pi^2}{E^2(q)N_1^2} \delta(p_0 - (W_1 - \omega_1)). \tag{A24} 
\]
Now, we look at \( \text{Disc}(I_{W_1^2}) \). From Eq. (73), we have the following possible values for \( p_0 \):
\[
p_0 = -(W_1 - \omega_1) = -\frac{N_1}{g} < 0, \tag{A25} \]
\[
p_0 = -(W_1 - \omega_2) = -\frac{N_1 + N_2}{2g} < 0, \tag{A26} \]
\[
p_0 = -(W_1 - W_2) = \frac{N_2 - N_1}{2g}. \tag{A27} \]
From Sec. [A1] we conclude that the only contribution arises from the third term in Eq. (A21). We are left with

$$\text{Disc}(I_{W_1}) = -\frac{i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (W_2 - W_1)).$$  \hspace{1cm} (A22)

From Eq. (7), we get $W_2 - W_1 = \omega_1 - \omega_2$ so that we can write

$$\text{Disc}(I_{W_1}) = -\frac{i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (\omega_1 - \omega_2)).$$  \hspace{1cm} (A23)

For $\text{Disc}(I_{W_2})$ the choices for $p_0$ are

$$p_0 = W_2 - \omega_1 = \frac{N_2 + N_1}{2g},$$
$$p_0 = W_2 - \omega_2 = \frac{N_2}{g},$$
$$p_0 = W_2 - W_1 = \frac{N_2 - N_1}{2g}.  \hspace{1cm} (A24)$$

The discontinuity arises only from the third contribution of the above equation, yielding

$$\text{Disc}(I_{W_2}) = \frac{i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (\omega_1 - \omega_2)).$$  \hspace{1cm} (A25)

For $\text{Disc}(I_{\overline{W}_2})$, we have

$$p_0 = -(W_2 - \omega_1) = -\frac{N_2 + N_1}{2g} < 0,$$
$$p_0 = -(W_2 - \omega_2) = -\frac{N_2}{g},$$
$$p_0 = -(W_2 - W_1) = -\frac{N_2 - N_1}{2g} < 0,  \hspace{1cm} (A26)$$

in such a way that $\text{Disc}(I_{\overline{W}_2}) = 0$.

For $\text{Disc}(I_{W_3})$, we have

$$p_0 = \omega_2 - \omega_1 = \frac{(N_1 - N_2)}{2g},$$
$$p_0 = \omega_2 - W_1 = -\frac{(N_1 + N_2)}{2g} < 0,$$
$$p_0 = \omega_2 - W_2 = -\frac{N_2}{g} < 0.$$  \hspace{1cm} (A27)

Since all the contributions are negative we conclude that $\text{Disc}(I_{W_3}) = 0$.

For $\text{Disc}(I_{\overline{W}_3})$, we have

$$p_0 = -(\omega_2 - \omega_1) = -\frac{(N_2 - N_1)}{2g},$$
$$p_0 = -(\omega_2 - W_1) = -\frac{(N_1 + N_2)}{2g},$$
$$p_0 = -(\omega_2 - W_2) = -\frac{N_2}{g} < 0.$$  \hspace{1cm} (A28)

Only the first term in the previous equations contribute, yielding

$$\text{Disc}(I_{\overline{W}_3}) = \frac{i\pi^2}{E^2(q)N_1N_2} \delta(p_0 - (\omega_1 - \omega_2)).  \hspace{1cm} (A29)$$

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