ON CONGRUENCE OF THE ITERATED FORM $\sigma^k(m) = 0 \mod m$

A PREPRINT

Zeraoulia Rafik*
Department of mathematics Batna2 University.Algeria
53, Route de Constantine. Fésdis, Batna
r.zeraoulia@univ-batna2.dz

February 22, 2021

ABSTRACT

Inspired by the question of Graeme L. Cohen and Herman J. te Riele, The Authors of [1] who they investigated a question Given $n$ is there an integer $k$ for which $\sigma^k(n) = 0 \mod n$? They did this in a 1995 paper and asserted through computation that the answer was yes for $n \leq 400$. The aim of this paper is to give a negative answer to the reverse question of Graeme L. Cohen and Herman J. te Riele such that we shall prove that there is no fixed integer $n$ for which $\sigma^k(n) = 0 \mod n$ for all iteration $k$ of sum divisor function using H.Lenstra problem for aliquot sequence at the same time we prove that there exist an integer $n$ satisfies $\sigma^k(n) = 0 \mod n$ for all odd iteration $k$ rather than that we shall prove that if $n$ is multiperfect with $L$ the lcm (the least common multiple ) of $1$+each exponents in the prime factorization of $n$, and with $L$ prime, then $n = 6$.we conclude our paper with open conjecture after attempting to show that $6$ is the only integer satisfy periodicity with small period dividing $L$

Keywords Iterative sum power divisor function · Aliquot sequence · Squarefree integers · periodic sequences

1 Introduction

In all of this paper $s(m) = \sigma(m) - m$ denote the sum of propre divisor function ,namely ,aliquot sequence and $s^0(m) = m, s^k(m) = s(s^{k-1}(m))$ the iteration of aliquot sequence .We define $\sigma$ is sum of divisor function such that:$\sigma^k(m) = m, \sigma^k(m) = \sigma(\sigma^{k-1}(m)), m \geq 1$ the iterated of sum of divisor function.

There is a great deal of literature concerning the iteration of the function $s(m) = \sigma(m) - m$, much of it concerned with whether the iterated values eventually terminate at zero, cycle or become unbounded depending on the value of $m$ see [2] and see (3,p.64) for details .Less work has been done on $\sigma$ iterate itself.Many problems on iterate of sum divisor function remain open ,proving or disproving them using numerical evidence are beyond of current technology .Some of these open problems are :

• 1) for any $n > 1, \frac{\sigma^{n+1}(n)}{\sigma(n)} \to 1$ as , $m \to \infty$
• 2) for any $n > 1, \frac{\sigma^{n+1}(n)}{\sigma(n)} \to \infty$ as , $m \to \infty$
• 3) for any $n > 1, \sigma^{1/m}(n) \to \infty$ as $m \to \infty$
• 4) for any $n > 1$,there is $m$ with $n|\sigma^m(n)$
• 5) for any $n, l > 1$,there is $m$ with $l|\sigma^m(n)$
• 6) for any $n_1, n_2 > 1$, there are $m_1, m_2$ with : $\sigma^{m_1}(n_1) = \sigma^{m_2}(n_2)$

*Teacher at high school Hamla3.Batna. (https://sites.google.com/univ-batna2.dz/zeraouliarafik05/-accueil?authuser=0, )— researcher in dynamical system and number theory
We would like to argue that the prime factorization of $m$ can be viewed as the problem of finding all prime factors of $m$. We just don't think we will see an example with fewer than a 1000 decimal digits (which isn't insanely large).

This last is the crux.

We have shown that the problem of finding prime factors of $m$ is equivalent to the question of the existence of large odd multiperfect numbers. Indeed, this question may be equivalent to the question of the existence of large odd multiperfect numbers.

Let us write $w$ for $\omega(m)$ and let us note that for a multiperfect $m$, $m_1$ will differ from $m$ by having less than $O(\log(w))$ additional prime factors, some in common with the prime factors of $m$. So the prime factorization of $m_1$ looks very much like the prime factorization of $m$.

We would like to argue that the prime factorization of $m_2$ should be quite different from that of $m_1$, because any additional powers of $q$ for $q$ a small prime dividing $n$ will affect $(q^n)$ and thus remove some prime factors. However, it is possible that there are (insanely large) odd multiperfect numbers which would be factors of $m$ in addition to the prime factors of $m$.

2 Main results

1) There is no integer $m$ for which $\sigma^k(m) = 0 \mod m$, for all iteration $k$, with $\sigma^k(m)$ is the k-fold iterate of $\sigma$ the sum divisor function.

2) If $n$ is multiperfect number with $L$ the lcm of 1+ each exponents in the prime factorisation of $n$, and with $L$ prime then $n = 6$

Note that the combination of the two above results show that the first main result is hold for odd integer $k$, Thus if $m = 6$ and $k$ is odd integer then $\sigma^k(m) = 0 \mod m$ hold.

2.1 Proof of result 1

Firstly, Note that multiperfect number indicate the same meaning with metaperfect number. $\sigma(m) = 0 \mod m$ means $m$ is a multiply perfect number. To avoid triviality we assume $m > 1$. Note that $\sigma(m)/m$ is bounded above by the product over all primes $q$ dividing $m$ of $q/(q-1)$, which is $O(\log p)$ with $p$ the largest prime divisor of $m$, and strictly less than $p$ when $p > 3$. (Indeed, it is less than $\omega(m)$, the number of distinct prime divisors of $m$, when $\omega(m) > 4$). While $\sigma(m)$ itself does not have to be multiperfect, We suspect there are finitely many numbers with $\sigma(\sigma(m))$ a multiple of $m$. In particular, let $g_0 = m_0 = m, m_n = \sigma(m_n)$, and $g_{n+1} = \gcd(g_n, m_{n+1})$. I suspect $g_3 < m$. I base this suspicion on the observation that the power of 2 exactly dividing $m_n$ appears to change between $m_n, m_{n+1}$, and $m_{n+2}$.

Let us write $w$ for $\omega(m)$ and let us note that for a multiperfect $m$, $m_1$ will differ from $m$ by having less than $O(\log(w))$ additional prime factors, some in common with the prime factors of $m$. So the prime factorization of $m_1$ looks very much like the prime factorization of $m$.

We would like to argue that the prime factorization of $m_2$ should be quite different from that of $m_1$, because any additional powers of $q$ for $q$ a small prime dividing $n$ will affect $\sigma(q^n)$ and thus remove some prime factors. However, it is possible that there are (insanely large) odd multiperfect numbers which would be factors of $m$ and not be affected by this. Indeed, this question may be equivalent to the question of the existence of large odd multiperfect numbers.

Let us look at $S(m) = \sigma(m)/m$. Letting the following products run over the distinct primes $q$ dividing $m$, we have $\prod q/(q-1) < S(m) < \prod q/(q-1)$. (And the lower bound is at least half the upper bound, so we have $S(m) \leq \prod q/(q-1)$.) As observed above, $S(m) < L(m)$ when $4 < L(m)$ and $S(m) < 2\omega(m)$ the rest of the time. So $S(m)$ is pretty small compared to $m$ and often small compared to $\log p$ where $p$ is the greatest prime factor of $m$.

Let $r_n = m_{n+1}/m_n = S(m_n)$. The assumption in the problem implies $r_n > r_0$, for if $m$ properly divides $k$ then $S(m) < S(k)$. Then $m_n > m_{n+1}$ for all $n$, since $m_n$ is an increasing sequence. We believe we can’t have both conditions hold indefinitely. However, We now switch ground on our assertion above that $g_3 < m$ always happens: We think it can, We just don’t think we will see an example with fewer than a 1000 decimal digits (which isn’t insanely large).

Here is an example which probably encourage us to think that we have high luck to have metaperfect number for all iteration $k$. The number $n = 13188979363639752997731839211623940096$ satisfies $\sigma(n) = 5n$ and since $\gcd(5, n) = 1$ $\sigma^2(m) = \sigma(5m) = 6\sigma(m) = 30m$, so at least there’s one example where $m_1$ and $m_2$ are multiples of $m$. Whether $n$ qualifies as very large indeed is a matter of taste. When $S(m)$ is coprime to $m$, we clearly have $m | m_2$ as well as $m | m_1$, and by multiplicativity of $S$ we also have $S(S(m)\cdot m) = S(S(m))\cdot S(m)$. What if $S(m)$ is not coprime to $m$? We still have

$$S(S(m)m) < S(m)S(S(m)).$$

This last is the crux. $S()$ grows slower than $\log()$, so we need to show this contradicts the growth rate implied by all $m_n$ being multiples of $m$. We may rewrite the last inequality and we join it in the context of Lenstra sharper result.

Lenstra proved in [6] that for every $k$ there is an $m$ so that $(s^k(m))$ will for simplicity be denoted by $s(m)$:

$$s^0(m) < s(m) < s^2(m) < \cdots < s^k(m).$$

Note that no need to confuse with $s(m)$ which indicate the aliquot sequence as it were defined in the introduction and $S(m)$ which indicate the ratio $\sigma(m)/m$. The sharper result which it were derived from Lenstra result, namely inequality (2) is reformulated as a Theorem in, see Theorem 1 in ([7],p.642) which states:

$$s^0(m) < s(m) < s^2(m) < \cdots < s^k(m)$$
Theorem: For every $k$ and $\delta > 0$ and for all $m$ except a sequence of density 0

\[(1 - \delta)m(\frac{s(m)}{m})^i < s^i(m) < (1 + \delta)m(\frac{s(m)}{m})^i, 1 \leq i \leq k\]  
(3)

We shall show that the inequality (1) looks like the upper bound of $s^i(n)$ in (3), (1) can be rewritten as:

\[\frac{\sigma(s(m))}{\sigma(m)} < \sigma(m) < d(\frac{d}{d - 1})^{\omega(d)}\]  
(4)

With the assumption that $m$ is not coprime to $S(m) = \frac{\sigma(m)}{m}$, this means that $S(m)$ must be an integer which means that $m|S(m)$ in the meantime $m|\sigma(m)$, thus $m$ is metaperfect number for the first iteration, let $d$ be the largest prime divisor of $\sigma(m)$ and since $m|\sigma(m)$ the inequality which it is defined in (1) become:

\[\frac{\sigma(s(m))}{s(m)} < d(1 + \delta)^{\omega(d)} < d(1 + \delta)^{\omega(d)}(\frac{\sigma(m)}{m})^{\omega(d)}\]  
(7)

We have used the standard result if $d|m$ then $(\frac{\sigma(d)}{d}) \leq (\frac{\sigma(m)}{m})$. The same instruction would work in the case of $d$ is not a prime divisor. We may let this as a short exercise for readers. We have showed in the case of $S(m)$ is not coprime to $m$ that we have the identical inequality with problem which it is proved by Lenstra, thus we have always the same upper bound as given in (3), thus lead to the proof of the first result by means there is no integer $m$ satisfy the congruence $\sigma^k(m) = 0 \mod m$, for all iteration $k$

2.2 Appendix for result 1

A more challenging problem is to ask for integers $m$ and $p$ such that for all integers $k$, $p_0 = p, p_{k+1} = \sigma(p_k)$, and $p_k = 0 \mod m$. The current problem adds the restriction that $p = m$, which implies $m$ is a metaperfect number. Since multiperfect numbers are rare, it should be hard to find a metaperfect number, a number $m$ that satisfies $\sigma^k(m) = 0 \mod m$ for all iterations of $\sigma$.

Indeed, $\sigma(m) < mw(\omega)$ for most values of $m$, so for a potentially metaperfect number to exist, we can’t depend on $\sigma(p_k)/m$ to be coprime to $m$ for very many $k$. More likely, $\sigma(p_k)/m$, if integral, will share a small factor with $m$ and further iterations of $\sigma$ will avoid certain large prime factors of $m$. This is what was observed, and what I hoped to prove and did not in the other answer.

It is an interesting side question to determine $\min_k g_k$ where $g_0 = g_0$ and $g_{k+1} = gcd(p_{k+1}, g_k)$. In particular, do the iterates of $\sigma$ encounter a square or twice a square, regardless of the starting point? If so, then the minimum is odd and likely 1. Otherwise $p$ is a seed for $m$, and $p$ might be useful in looking for multiperfect numbers which are multiples of $m$.

Cohen and te Riele investigated a weaker question: Given $n$ is there a $k$ for which $\sigma^k(n) = 0 \mod n$? They did this in a 1996 paper and asserted through computation that the answer was yes for $n \leq 400$. Their data suggest to us both that the weaker question has an affirmative answer, and that there are no metaperfect numbers or even seeds for a number.

2.3 Proof of result 2

Key idea The key idea in showing $n = 6$ is the only multiperfect number with $L$ prime is to show all prime factors of $\sigma(n)$ and thus of $n$ are 0 or 1 mod $L$ all with multiplicity $L - 1$ and thus $(L/(L - 1))^{m+1} \geq \sigma(n)/n \geq L$ where $m$ is the number of distinct prime factors of $n$ that are 1 mod $L$. This quickly leads to $L = 2$, and then showing the largest prime factor of $n$ must be at most one more than the second largest prime factor.

Proof: Note that if $L$ is prime, then the prime factors of $n$ all occur with the same multiplicity, namely $L - 1$. Thus $\sigma(n)$ is a product of terms of the form $\sigma(p^{L-1})$, which means (since $L$ is prime) that each factor of $\sigma(p^{L-1})$ is either $L$
or is a prime which is $1 \mod L$. This means $n$ and $\sigma(n)$ both have as prime factors only numbers $q$ which are either $0$ or $1 \mod L$. If $L$ is a factor of $n$, then it occurs to the $L$th power, by hypothesis. If $q = kL + 1$ is a factor of $n$, then $\sigma(q^{L-1})$ is divisible by $L$ exactly once, using standard elementary results. Thus $\sigma(n)$ is divisible by $L^m$ or by $L^{m+L}$ exactly, where $m$ counts the number of distinct prime factors of $n$ that are $1 \mod L$.

Suppose $m$ is at least as large as $L$. Then $\sigma(n)/n$ is at least $L^m/L^{L-1}$, but also $\sigma(n)/n$ is less than $(L/((L - 1))^{m+1}$, by standard inequalities for $\sigma()$. Taking logs, we have $(m + 1 - L) \log L$ is less than or equal to $(\log(\sigma(n))/n)$ which is less than $(m + 1)/((L - 1)$, or $\log L$ is less than $(1 + \frac{L}{(m+1-L)(L-1)})$ which is at most $3$ when $L = 2$, is at most $2$ when $L = 3$, and is at most $1.5$ when $L = 5$ or larger. The only solutions to the inequality with $L$ a prime are $L = 3$ and $m$ at most $4$, and $L = 2$.

If $m$ is smaller than $L$, then we have $\sigma(n)/n$ is no more than $((L/((L - 1))^L)$ which is at most $4$. But $\sigma(n)/n$ is at least $L$, since all prime factors of $\sigma(n)$ are $0$ or $1 \mod L$, so again $L$ is $2$ or $3$.

If $L$ is $3$, we now are faced with $3$ less than or equal $\sigma(n)/n$ less than $(3/2)/(7/6)^m$ for $m$ at most $4$, since we have to take the primes which are $1 \mod 3$ into account. But $(7/6)^4$ is less than $2$. So $L$ cannot be $3$.

So if $n$ is multiperfect and $L$ is prime, then $L$ must be $2$, and thus $n$ is squarefree. It is now easy to show $n$ must be $6$, for instance by considering the largest prime factor of $n$ has to be at most one more than the second largest prime factor of $n$. We leave this as exercise for readers.

### 2.4 Appendix for open conjecture

Let $r = \gcd(k, e + 1)$, and $p$ a prime. Then $\sigma_k(p^r) \equiv r^{e+1} \mod \sigma(p^r)$. Also, $r = 1$ if and only if $\sigma(p^r)$ divides $\sigma_k(p^r)$. Thus for $k$ coprime to $\tau(n)$, we have $\sigma(n)$ divides $\sigma_k(n)$. The relation also suggests that for a given $n$ the sequence $\sigma(n)$ must be periodic in $k$ with a period dividing $L$, the least common multiple of $(1 + 1$ each exponent) in the prime factorization of $n$. Once can show a nonreduced representation $\sigma_k(n) = a_k\sigma(n)/b_k$, where the $b_k$ are integers not necessarily coprime to the integers $a_k$ or to $\sigma(n)$, with the property that the $b_k$ are bounded and periodic with period $L$. This is not enough to show $\sigma_k(n)$ is periodic with small period, unfortunately.

If now $n$ is multiperfect (so $n$ divides $\sigma(n)$) we have $n$ divides $\sigma_k(n)$ for $k$ coprime to $\tau(n)$. In particular if $\tau(n)$ is a power of $2$, then $n$ divides $\sigma_k(n)$ for all odd $k > 0$.

It is still possible that $n$ can divide $\sigma_k(n)$ for $k$ not coprime to $\tau(n)$. However if $L$ is not prime, it seems likely that there will be more than one nonzero value of $\sigma_k(n)$ mod $\sigma(n)$. If this is so, it would be one ingredient in a proof that $6$ is the unique number having the titled properties, the other ingredient being that $6$ is the onlly nontrivial multiperfect number with $L$ a prime.

We botched an earlier edit which claimed that $6$ is the only known multiperfect number $n$ which satisfies $\sigma_k(n)$ mod $n = 2$. It is true, but the analysis had some flaws. However, one expects multiperfect numbers other than $1$ and $6$ to be a multiple of $4$; when $n$ satisfies $\sigma(n)$ mod $n = 0$ and $\sigma_1(n)$ mod $n = 2$, and in addition $n$ mod $4 = 0$, then all odd prime factors of $n$ except one must occur to an even multiplicity, and the remaining odd prime factor must occur to a multiplicity of $1$ mod $4$ and must be a prime that is $3$ mod $4$. While simple, these observations say a lot about $n$ and suggest that any numbers satisfying the title congruences are rare indeed, perhaps more so than odd multiperfect numbers.

### 2.5 Open Conjecture

The only integer satisfy the congruence $\sigma^k(n) = 0 \mod n$ for every odd integer $k$ and satisfy periodicity with small period is $n = 6$.

**Acknowledgements** I would like to thank Gerald R. Paseman From MO for his great contributions to this paper (Indication for proofs, Appendix) and for his encouragements to produce the best of the best for number theory field.

### References

[1] Graeme L. Cohen and Herman J. J. te Riele. Iterating the Sum of Divisors Function. *1991 Mathematics Subject Classification: 11A25, 11Y70*, the c A K Peters, Ltd. 1058458/96

[2] P. Erdos, A. Granville, C. Pomerance, and C. Spiro. On the normal behavior of the iterates of some arithmetical functions. 204 in Analytic Number Theory, Allerton Park, 1989
[3] R. K. Guy Unsolved Problems in Number Theory. *Springer, New York*, 1994.

[4] P. Erdos, A. Granville, C. Pomerance and S. Spiro on the normal behavior of the iterates of some arithmetic function printed in the united states of america. *Edited by: Brace C Berndt, Harold G Diamond, Heini Helberstam, Adolf Hildebrand*, 1990.

[5] Maier On the third iterates of the phi, and sigma functions. *Institute of Mathematics Polish Academy of Sciences, Colloquium Mathematicum*, 1984, 1985/1/49, 1/123, 130

[6] Paul Pollack and Carl Pomerance Some problems of Erdős on the sum of divisors function. *AMS Journal: Trans. Amer. Math. Soc. Ser. B* 3 (2016), 1, 26, DOI: https://doi.org/10.1090/btran/10, April 5, 2016

[7] P. Erdos On the asymptotic properties of aliquot sequences. *A MATHEMATICS OF COMPUTATION, VOLUME 30, NUMBER 135 JULY 1976, PAGES 641, 645*, 1976

[8] Lenstra Problem 6064, Amer. Math. Monthly 82 (1975), 1016. *Solution by the proposer, op. cit. 84 (1977), 380, 1977*