Lifting Markov Bases and Higher Codimension Toric Fiber Products

Johannes Rauh\textsuperscript{1}, Seth Sullivant\textsuperscript{2}

\textsuperscript{1}MPI for Mathematics in the Sciences
Inselstraße 22
04103 Leipzig
Tel.: +49 (0) 341 9959 543
jrauh@mis.mpg.de

\textsuperscript{2}NC Statue University
Box 8205
Raleigh, NC 27695
Tel.: +1 (919) 513-7445
smsulli2@ncsu.edu

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We study how to lift Markov bases and Gröbner bases along linear maps of lattices. We give a lifting algorithm that allows to compute such bases iteratively provided a certain associated semigroup is normal. Our main application is the toric fiber product of toric ideals, where lifting gives Markov bases of the factor ideals that satisfy the compatible projection property. We illustrate the technique by computing Markov bases of various infinite families of hierarchical models. The methodology also implies new finiteness results for iterated toric fiber products.

Keywords: Markov bases, toric fiber product, lifting, Gröbner bases

Contents

1 Introduction 2

2 Markov bases and Gröbner bases of lattice point problems 6
2.1 Markov bases and Gröbner bases of lattices 7
2.2 Markov bases and Gröbner bases of systems of inequalities 9
2.3 Sign-consistency and Graver bases 12
1 Introduction

Let \( B \in \mathbb{Z}^{h \times n} \) be an integer matrix and let \( \mathcal{M} \subseteq \mathbb{Z}^n \). For any \( b \in \mathbb{Z}^h \) let \( F(B, b)_{\mathcal{M}} \) be the fiber graph with vertex set \( F(B, b) = \{ v \in \mathbb{N}^n : Bv = b \} \), where vertices \( u, v \in F(B, b) \) are connected by an edge if and only if \( u - v \in \pm \mathcal{M} \). Then \( \mathcal{M} \) is called a Markov basis, if and only if all fiber graphs are connected. Elements of Markov bases are sometimes called moves, since they can be used as moves in MCMC simulations to sample from \( F(B, b) \) [1]. Alternatively, Markov bases consist of exponent vectors of a binomial generating set of the toric ideal \( I_B \) (see Theorem 4).

The best general algorithm to compute a Markov basis of a matrix is the one implemented in 4ti2 [2]. However, many matrices that appear in applications are too large, and 4ti2 cannot compute a Markov basis within a reasonable time, using a reasonable amount of memory. In these situations, one hopes for procedures that take into account the structure of the Markov basis problem and that can use that structure to build a Markov basis of a large problem from Markov bases of simpler pieces and “lifting” operations.

In this paper we study how to lift a Markov basis along a linear map. The lifting procedure generalizes similar prior constructions. For example, the algorithm implemented in 4ti2 relies on lifting Markov bases along a coordinate projection [3]. The construction used to compute a Markov basis of codimension zero toric fiber products is also an instance of lifting [4]. Similar ideas are used in [5] to relate an ideal with its preimage under a monomial ring homomorphism. We study lifting in a very general context for arbitrary matrices \( B \) and arbitrary linear maps \( \phi \). The only assumption that we have to
Figure 1: a) Consider a graph $G$ with vertex set $V \subseteq \mathbb{Z}^n$ and a linear map $\phi : \mathbb{Z}^n \to \mathbb{Z}^t$. If the image of $G$ is connected and if each $\phi$-fiber of $G$ is connected, then $G$ itself is connected. The edges in the lower graph correspond to a PF Markov basis. The vertical edges in the upper graph correspond to a kernel Markov basis. The remaining edges are lifts of the PF Markov basis. b) An illustration of the algorithm applied to the toric fiber product: The goal is to lift along the map $\xi$. This can be accomplished in two steps, by first lifting along $\phi$ and $\phi'$ and by then gluing the results.

make is that a certain affine semigroup is normal (see Section 3.1). Even if this condition is violated, in many cases it is possible to adjust our algorithm. An example is given in Section 5.2.

Our procedure allows to transform the problem of computing a Markov basis of $B$ into a series of smaller Markov basis computations. The efficiency of lifting crucially depends on the choice of the linear map. If everything goes well, it is possible to compute complicated Markov bases of large matrices inductively by iterating the lifting procedure.

The idea behind lifting is sketched in Figure 1(a): For a linear map $\phi : \mathbb{Z}^n \to \mathbb{Z}^t$ and a graph $G = (V, E)$ with $V \subseteq \mathbb{Z}^n$ define the image graph $\phi(G) = (V', E')$ by $V' = \phi(V)$ and $(x', y') \in \phi(G)$ if and only if there is $(x, y) \in E$ with $x' = \phi(x)$ and $y' = \phi(y)$. If $G$ is a fiber graph of $B$ with respect to a Markov basis, then $G$ is connected, and so is $\phi(G)$. Our approach is to turn this observation around as follows: Given a graph homomorphism $\phi$ induced by a linear map as above, if its image $\phi(G)$ is connected and if each fiber $\phi^{-1}(x) \cap G$ is connected, then $G$ is connected. Thus, our strategy is as follows: First, we find a set of moves that connects the $\phi$-fibers (that is the sets of the form $\phi^{-1}(x) \cap G$). Such a set of moves we call a kernel Markov basis, because each $\phi$-fiber has the form $G \cap (u + \ker_{\mathbb{Z}} \phi)$. Second, we find a set of moves that connects the projected fiber graphs $\phi(G)$. Such a set of moves we call a projected fiber (PF) Markov basis. Then we lift the PF Markov basis to obtain suitable moves in $\mathbb{Z}^n$. In the last step, the actual lifting step, we need to find “enough” preimages of the edges of the image graph. In Section 3 we give a general lifting algorithm of which the central step is again
a Markov basis computation.

The two steps of finding and lifting the PF Markov basis require a generalization of the notion of Markov basis beyond the one that is typically used in applications. This generalized notion is introduced in Section 2. An important special case is the notion of an inequality Markov basis, which is a set of moves that connects all generalized fibers of the form $\{u \in L : Du \geq c\}$ where $L$ is a fixed lattice and $D$ is a fixed matrix. When a certain associated semigroups is normal, the problem of finding a PF Markov basis can be solved by finding an inequality Markov basis (Section 3.1). The problem of lifting a Markov basis element can always be phrased as an inequality Markov basis problem (Section 3.2).

The main open problem of lifting is how to compute PF Markov bases in the general case, when the associated affine semigroup is not normal. If there are only finitely many holes (or the structure of the holes is sufficiently well understood), our techniques to compute PF Markov bases can easily be adapted. However, in general it can be computationally challenging to simply compute the set of holes of an affine semigroup [6].

Our lifting procedure not only works for Markov bases, but also for the related concept of Gr"obner bases. While a Markov basis connects a set of integer vectors, a Gr"obner basis allows to find minimal elements with respect to some suitable order. For this, the notion of Gr"obner bases has to be generalized in a similar way as the notion of Markov bases. To apply our lifting ideas, we need to assume that the involved orders on the fibers and the projected fibers are compatible, in a sense that is explained in detail in Section 3.

As mentioned before, the complexity of lifting crucially depends on the choice of the map $\phi$. In general, we do not know how to find a good map $\phi$ for a given Markov basis or Gr"obner basis problem, or whether such a good map exists at all. For hierarchical models, which we study in Section 5, it is natural to use marginalization maps.

Our motivation for studying the lifting procedure comes from the study of the toric fiber product [4]. Let $A \in \mathbb{Z}^{s \times t}$, and denote by $NA$ the affine semigroup generated by the columns $A$. The toric fiber product is a construction that takes two ideals $I, J$ that are homogeneous with respect to an $NA$-grading and produces another larger ideal $I \times_A J$. In this paper we focus on the case of toric ideals $IB, IB'$ associated with matrices $B, B'$. In this case, the toric fiber product $IB \times_A IB'$ is again a toric ideal of a matrix $B \times_A B'$, which is the (matrix) toric fiber product of $B$ and $B'$.

The guiding principle in the theory of toric fiber products is that the product should inherit many of the nice properties of its factors. The complexity of the toric fiber product grows with its codimension $\text{codim} A = t - \dim NA$, defined as the difference between the number of columns of $A$ and the dimension of the semigroup $NA$. If the codimension is zero, then the toric fiber product behaves nicely: For example, Markov bases of $B$ and $B'$ can be glued together to Markov bases of $B \times B'$ [4], and if the two semigroups $NB$ and $NB'$ are normal, then so is $NB \times_A B'$ [7] (the corresponding statements for ideals also hold for non-toric ideals). While the codimension one case is more complicated, still a lot can be said, and if $B, B'$ are nice enough (specifically, if $B, B'$ have slow-varying Markov bases), these can be glued together to produce a Markov
basis of $B \times_A B'$, as shown in [7]. In [7] it was also shown that for higher codimensions when the Markov bases of $B$ and $B'$ satisfy the compatible projection property, they can be glued together to produce a Markov basis of $B \times_A B'$. Although Markov bases with the compatible projection property always exist, [7] did not give an approach for constructing them.

In the present paper we use our lifting idea to develop a framework to compute compatible Markov bases directly from scratch as follows; see Figure 1(b): The $NA$-gradings induce linear projections $\phi, \phi'$ from the fiber graphs of $B, B'$ to $\mathbb{N}^t$ (where $t$ is the width of $A$). The analogous projection $\xi$ from the fiber graphs of $B \times_A B'$ to $\mathbb{N}^t$ factorizes through these two maps. We want to lift along $\xi$. The first observation is that a kernel Markov basis $\mathcal{M}_0$ of $\xi$ is given by a corresponding basis of the associated codimension-zero product (Lemma 29). A PF Markov basis can be computed if the semigroup of the associated codimension-zero product is normal (Section 4.2). Finally, instead of lifting along $\xi$ it is possible to first lift along $\phi$ and $\phi'$ and to glue the resulting Markov bases of $I$ and $J$ (Lemma 35). In the language of [7], this fact implies that the lifted Markov bases of $I$ and $J$ satisfy the compatible projection property. Our approach generalizes and allows to construct Gröbner bases of the toric fiber product.

To sum up, our strategy to compute Markov bases (or Gröbner bases) of a toric fiber product $B \times_A B'$ is as follows:

1. Find a description of the projected fibers.
2. Find a (generalized) Markov basis $\mathcal{G}$ for this description.
3. Find Markov bases $\mathcal{M}$ and $\mathcal{M'}$ of $B$ and $B'$ that lift $\mathcal{G}$.
4. Glue $\mathcal{M}$ and $\mathcal{M'}$ to obtain a Markov basis of the toric fiber product.

Our construction allows for a fairly straightforward algorithm to produce Markov bases in many instances where they were not known before. We focus in particular in this paper on constructing Markov bases for hierarchical models where our constructions allow us to give explicit new instances exhibiting concrete bounds for the finiteness results that are proven nonconstructively in [8]. On the other hand, the fact that we cannot work simply with given Markov or Gröbner bases of $B$ and $B'$ means it is difficult to predict when nice properties of $B$ and $B'$ are passed on to $B \times_A B'$. Even so, we provide some examples where a careful analysis allows us to bound degrees of Markov basis elements and prove normality using the Gröbner bases.

The paper is organized as follows. After introducing the generalized notion of Markov bases and Gröbner bases in Section 2, we describe how to lift Markov and Gröbner bases in Section 3. In Section 4 we explain the toric fiber product construction and show how to lift in this case. Our main motivating examples to study concern Markov bases of hierarchical models, and we explore these examples in detail in Section 5. Section 6 explores consequences of the general theory to producing finiteness results for Markov bases of iterated toric fiber products, which we apply to deduce finiteness results for Markov bases of hierarchical models.
2 Markov bases and Gröbner bases of lattice point problems

We introduce a notion of Markov basis and Gröbner basis for lattice point problems, generalizing the usual notions associated to integer matrices. The basic idea is that a Markov basis of a family of sets of integer vectors consists of moves that connects all these sets. The usual notion of a Markov basis of a matrix $B \in \mathbb{Z}^{h \times n}$ arises by considering the fibers of $B$, where $B$ is considered as a map $\mathbb{N}^n \to \mathbb{Z}^h$. Similarly, a Gröbner basis of a family of sets is a set of moves that allows to move towards a minimum on each of these sets, with respect to some order.

Let $\succeq$ be a preorder on $\mathbb{Z}^n$. Then $\succeq$ is a total preorder, if for all $u, v \in \mathbb{Z}^n$ either $u \succeq v$ or $v \succeq u$ (or both). A preorder $\succeq$ is additive if it is total and if $u \succeq v$ implies $u + w \succeq v + w$ for all $u, v, w \in \mathbb{Z}^n$. Our main example is the following: Let $c \in \mathbb{Q}^n$ and define $\succeq_c$ by $u \succeq_c v :\iff \langle c, u \rangle \geq \langle c, v \rangle$.

We explicitly allow $c = 0$. Although the preorder $\succeq_0$ is trivial, in the sense that $u \succeq_0 v$ holds for all $u, v \in \mathbb{Z}^n$, it is useful since it allows a unified treatment of Markov bases and Gröbner bases.

More generally, for $c_1, \ldots, c_r \in \mathbb{Q}^n$, define $\succeq_{c_1, \ldots, c_r}$ by

$$u \succeq_{c_1, \ldots, c_r} v :\iff \begin{cases} \langle c_1, u \rangle > \langle c_1, v \rangle, \\ or \langle c_1, u \rangle = \langle c_1, v \rangle and \langle c_2, u \rangle > \langle c_2, v \rangle, \\ or \langle c_1, u \rangle = \langle c_1, v \rangle and \langle c_2, u \rangle = \langle c_2, v \rangle and \langle c_3, u \rangle > \langle c_3, v \rangle, \\ \vdots \\ or \langle c_1, (u - v) \rangle = \langle c_2, (u - v) \rangle = \cdots = \langle c_{r-1}, (u - v) \rangle = 0 and \langle c_r, u \rangle \geq \langle c_r, v \rangle. \end{cases}$$

In fact, any additive preorder is of the form $\succeq_{c_1, \ldots, c_r}$. Moreover, many additive preorders that appear in practice can be approximated by preorders of the form $\succeq_c$ in a sense to be made precise later (see Remark [3]).

Fix an additive preorder $\succeq$ on $\mathbb{Z}^n$, let $F \subseteq \mathbb{Z}^n$ and let $M \subseteq \mathbb{Z}^n$. Construct a directed graph $F_M, \succeq$ with vertex set $F$ as follows: For $u, v \in F$ make an edge $u \rightarrow v$ if and only if $v - u \in \pm M$ and $u \succeq v$.

In a directed graph $G$, declare two vertices $u, v$ equivalent $u \sim v$ if there is a directed path from $u$ to $v$ and from $v$ to $u$. The equivalence classes of $G$ are called the strongly connected components of $G$. The quotient by the equivalence relation is a directed graph $G/\sim$ that does not contain directed cycles.

**Definition 1.** Let $F$ be a collection of subsets of $\mathbb{Z}^n$. The set $M \subseteq \mathbb{Z}^n$ is a Gröbner basis for $F$ with respect to $\succeq$ if for all $F \in F$ the following three conditions are satisfied:

1. $F_M, \succeq$ is weakly connected (i.e. the underlying undirected graph is connected).
2. $F_M, \succeq/\sim$ contains at most one sink.
3. Each sink in $\mathbf{F}_{M,\succeq}/\sim$ is a $\succeq$-minimum.

In the special case that $\succeq = \geq_0$ (that is, $u \geq v$ for all $u, v \in \mathbb{Z}^n$), $M$ is called a Markov basis for $\mathbf{F}$, and the condition on $\mathbf{F}_{M,\succeq}/\sim$ is equivalent to $\mathbf{F}_{M,\geq_0}$ being connected. In this case, we also write $\mathbf{F}_M$ instead of $\mathbf{F}_{M,\succeq}$. Since all edges in $\mathbf{F}_M$ are bidirected, it is convenient to think of $\mathbf{F}_M$ as an undirected graph.

For arbitrary families of subsets $\mathcal{F}$, the three conditions of Definition 1 are independent. We are mostly interested in the case that all $\mathbf{F} \in \mathcal{F}$ are finite. In this case, every connected component of $\mathbf{F}_{M,\succeq}/\sim$ has a minimum, and thus a sink. Hence, $M$ is a Gröbner basis if and only if $\mathbf{F}_{M,\succeq}/\sim$ contains precisely one sink for all $\mathbf{F} \in \mathcal{F}$. In particular, it suffices to only check the second condition of the definition. The argument remains true when $\mathbf{F}$ is possibly infinite and $\succeq$ defines a well-ordering on each $\mathbf{F}$, a case which arises in the context of Gröbner bases for lattices with respect to term orders (see Section 2.1).

The following lemma follows directly from the definition:

**Lemma 2.** Let $M$ be a Markov basis for $\mathcal{F}$. Then $M$ is a $\succeq$-Gröbner basis if and only if the following two conditions are satisfied:

1. If $u, v \in \mathbf{F}$ are both $\succeq$-minimal, then there is a path from $u$ to $v$ in $\mathbf{F}_{M,\succeq}$.
2. If $u \in \mathbf{F} \in \mathcal{F}$ is not $\succeq$-minimal in $\mathbf{F}$, then there is a path $u \rightarrow u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_r$ in $\mathbf{F}_{M,\succeq}$ with $u_r < u$.

Gröbner bases can be used to find $\succeq$-minimal elements within a set $\mathbf{F} \in \mathcal{F}$. In particular, a Gröbner basis with respect to $\succeq_\prec$ can be used to solve the integer program

$$\text{minimize } \langle \mathbf{c}, u \rangle \quad \text{subject to } u \in \mathbf{F}$$

by following the arrows towards the sink equivalence class of $\mathbf{F}$, if it exists. If there is no such sink, then the integer program is unbounded, and following the arrows gives an infinite descending sequence.

### 2.1 Markov bases and Gröbner bases of lattices

Let $\mathbf{B}$ be a $h \times n$ integer matrix and $\mathbf{F}(\mathbf{B}, b)$ denote the fiber

$$\mathbf{F}(\mathbf{B}, b) := \{ v \in \mathbb{N}^n : \mathbf{B}v = b \}.$$ 

More generally, if $\mathcal{L} \subseteq \mathbb{Z}^n$ is a lattice (that is, a subgroup of $\mathbb{Z}^n$), we can consider fibers of the form

$$\mathbf{F}^{\text{lat}}(\mathcal{L}, u) := \{ v \in \mathbb{N}^n : v \in \mathcal{L} + u \}.$$ 

This contains the fibers $\mathbf{F}(\mathbf{B}, b)$ as a subcase, since $\mathbf{F}(\mathbf{B}, Bu) = \mathbf{F}^{\text{lat}}(\ker_{\mathbb{Z}} \mathbf{B}, u)$.

The most commonly studied case of both Markov basis and Gröbner basis arises when

$$\mathcal{F} = \mathcal{F}(\mathbf{B}) := \{ \mathbf{F}(\mathbf{B}, b) : b \in \mathbb{Z}^h \}.$$
In this case, Markov bases of $\mathcal{F}$ correspond to binomial generating sets of the associated toric ideal $I_B$, and Gröbner bases of $\mathcal{F}$ correspond to Gröbner bases of $I_B$ (see Theorem 4 below). Similarly, Markov bases of $\mathcal{F}$ correspond to generating sets of lattice ideals (see Corollary 5 below).

Let $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring. Any additive preorder $\succeq$ on $\mathbb{Z}^n$ induces a preorder on the monomials in $\mathbb{K}[x]$ (denoted by the same symbol) by $x^u \succeq x^v$ if and only if $u \succeq v$. If $\succeq = \succeq_\ell$, then this preorder is called the weight order induced by $\ell$.

For any polynomial $f \in \mathbb{K}[x]$, the initial form of $f$ with respect to $\succeq$, denoted by $\text{in}_{\succeq}(f)$, is the sum of all terms $c_u x^u$ in $f$ such that $u$ is $\succeq$-maximal. For an ideal $I \subseteq \mathbb{K}[x]$, 

$$\text{in}_{\succeq}(I) := \langle \text{in}_{\succeq}(f) : f \in I \rangle.$$ 

A set of polynomials $G \subseteq I$ is a Gröbner basis for $I$ with respect to $\succeq$ if and only if 

$$\langle \text{in}_{\succeq}(g) : g \in G \rangle = \text{in}_{\succeq}(I).$$

Note that a Gröbner basis with respect to $\succeq_0$ is nothing but a generating set of $I$.

**Remark 3.** Most orders that are used in practice are term orders: An additive preorder on $\mathbb{Z}^n$ is called a term order if it is a well-ordering; that is, $x^u \succeq x^v$ and $x^v \succeq x^u$ implies $x^u = x^v$, and every set of monomials has a minimum with respect to $\succeq$. If $\succeq$ is a term order, then $\text{in}_{\succeq}(I)$ is a monomial ideal for any $I$.

For any term order $\succeq$ on $\mathbb{K}[x]$ and any ideal $I \subseteq \mathbb{K}[x]$, there exists a weight vector $\ell$ such that $\text{in}_{\succeq}(I) = \text{in}_{\succeq_\ell}(I)$; see [10, Proposition 1.11]. Hence, weight preorders can be used to approximate term orders when working with a fixed ideal.

For any subset $M \subseteq \mathbb{Z}^n$ consider the binomial ideal

$$I_M := \langle x^{m^+} - x^{m^-} : m \in M \rangle,$$

where $m = m^+ - m^-$ is the decomposition of $m$ into its positive and negative part with $\text{supp}(m^+) \cap \text{supp}(m^-) = \emptyset$. For a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$, the ideal $I_\mathcal{L}$ is called a lattice ideal. If $\mathcal{L}$ is a saturated lattice, that is, if $\mathcal{L} = \ker_\mathbb{Z} B$ for some integer matrix $B$, then $I_\mathcal{L} := I_B$ is called a toric ideal.

**Theorem 4.** [1, 10] A finite subset $M \subseteq \ker_\mathbb{Z} B$ is a Markov basis of $\mathcal{F}(B)$ if and only if $I_M = I_B$. For any additive preorder $\succeq$ on $\mathbb{Z}^n$ for which $0$ is the smallest element in $\mathbb{N}^n$, a finite subset $M \subseteq \ker_\mathbb{Z} B$ is a $\succeq$-Gröbner basis of $\mathcal{F}(B)$ if and only if $\{x^{m^+} - x^{m^-} : m \in M \}$ is a $\succeq$-Gröbner basis of $I_B$.

See Theorems 5.3 and 5.5 in [10] for a proof. Theorem 4 can easily be generalized to lattice ideals as follows:

**Corollary 5.** A finite set $M \subseteq \mathcal{L}$ is a Markov basis of $\mathcal{F}(\mathcal{L})$ if and only if $I_M = I_\mathcal{L}$. For any additive preorder $\succeq$ on $\mathbb{Z}^n$ for which $0$ is the smallest element in $\mathbb{N}^n$, a finite subset $M \subseteq \mathcal{L}$ is a $\succeq$-Gröbner basis of $\mathcal{F}(\mathcal{L})$ if and only if $\{x^{m^+} - x^{m^-} : m \in M \}$ is a $\succeq$-Gröbner basis of $I_\mathcal{L}$.

Hilbert’s basis theorem implies that finite Markov bases and Gröbner bases exist for any lattice. They can be computed using 4ti2 [2].
2.2 Markov bases and Gröbner bases of systems of inequalities

Let $D \in \mathbb{Z}^{r \times n}$ be an integer matrix and $\mathcal{L} \subseteq \mathbb{Z}^n$ a lattice. For any $c \in \mathbb{Z}^r$ and $v \in \mathbb{Z}^n$ let

$$\mathbf{F}^\text{in}(\mathcal{L}, v, D, c) := \{u \in \mathcal{L} + v : Du \geq c\},$$

and let

$$\mathcal{F}^\text{in}(\mathcal{L}, D) := \{\mathbf{F}^\text{in}(\mathcal{L}, v, D, c) : c \in \mathbb{Z}^r \text{ and } v \in \mathbb{Z}^n\}.$$ 

A Markov basis of $\mathcal{F}^\text{in}(\mathcal{L}, D)$ is called an inequality Markov basis of $\mathcal{L}$ and $D$ or just an $(\mathcal{L}, D)$-Markov basis. If $\succeq$ is an additive preorder, then a $\succeq$-Gröbner basis of $\mathcal{F}^\text{in}(\mathcal{L}, D)$ is called an $(\mathcal{L}, D, \succeq)$-Gröbner basis. Later, we often choose $\mathcal{L} = \mathbb{Z}^n$. This choice allows us to study linear inequalities over the integers. In this case, we suppress $\mathcal{L}$ from the notation.

Inequality Markov bases can be computed by relating them to Markov bases of lattices, which can be computed in practice using 4ti2. We explain this in the remainder of the section. The first step is to restrict to the case that the restriction of the matrix $D$ to the lattice $\mathcal{L}$ has rank $\dim(\mathcal{L})$.

Let $t := \dim(\mathcal{L})$. We may choose a lattice basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$ such that $\mathcal{L}$ belongs to the subspace spanned by $e_1, \ldots, e_t$. The restriction of $D$ to $\mathcal{L}$ is represented by the matrix $D_1$ which consists of the first $t$ rows of $D$. If $D_1$ has rank $t' < t$, then we may in addition choose our lattice basis in such a way that $\ker_{\mathbb{Z}} D_1$ is generated by $e_{t+1}, \ldots, e_t$. With respect to this basis, the last $t-t'$ columns of $D_1$ vanish. Let $D'$ be the submatrix consisting of the first $t'$ columns of $D$ (or $D_1$), and let $\mathcal{L}'$ be the lattice $\mathcal{L}$ considered as a sublattice of $\mathbb{Z}^t$. Then solving a system of inequalities of the form $Dv \geq c$ for $v \in \mathcal{L}$ is equivalent to solving a system of the form $D'v' \geq c$ for $v' \in \mathcal{L}'$, in the sense that

$$\{v \in \mathcal{L} : Dv \geq c\} \cong \{v' \in \mathcal{L}' : D'v' \geq c\} \times (\mathcal{L} \cap \ker_{\mathbb{Z}} D).$$

Therefore, if $\mathcal{G}' \subset \mathcal{L}'$ is an $(\mathcal{L}', D')$-Markov basis, then an $(\mathcal{L}, D)$-Markov basis is given by

$$\{(b, 0, \ldots, 0) : b \in \mathcal{G}'\} \cup \{f_{t+1}, \ldots, f_t\},$$

where $f_{t+1}, \ldots, f_t$ generate $\mathcal{L} \cap \ker_{\mathbb{Z}} D$. Conversely, any $(\mathcal{L}, D)$-Markov basis can be truncated to an $(\mathcal{L}', D')$-Markov basis. Therefore, it suffices to know how to compute inequality Markov bases in the case that the restriction of $D$ to $\mathcal{L}$ has rank $\dim(\mathcal{L})$.

If $L \in \mathbb{Z}^{n \times t}$ is a matrix such that the columns of $L$ are a lattice basis of $\mathcal{L}$, then this is the same as requiring that $\rank(DL) = t$. As a side note, observe that the same construction allows to restrict to the case that $\mathcal{L}$ has full rank.

**Lemma 6.** Let $D \subseteq \mathbb{Z}^{r \times n}$, let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a lattice, let $L$ be an $(n \times t)$-integer matrix such that the columns of $L$ are a lattice basis of $\mathcal{L}$, and assume that $\tilde{D} = DL \in \mathbb{Z}^{r \times t}$ has rank $t$.

1. If $\mathcal{G}$ is an $(\mathcal{L}, D)$-Markov basis, then $D(\mathcal{G})$ is a Markov basis of the lattice $\mathbb{Z} \tilde{D}$ spanned by the columns of $\tilde{D}$.
2. If $G'$ is a Markov basis of $\mathbb{Z}\tilde{D}$, then $D^{-1}(G') \cap \mathcal{L}$ (the inverse image of $G'$ under the linear map corresponding to $D$) is an $(\mathcal{L}, D)$-Markov basis.

Proof. It suffices to show that the fibers we want to connect by the $(\mathcal{L}, D)$-Markov basis and by the $\mathbb{Z}\tilde{D}$-Markov basis are bijective via affine maps with linear part given by $D$. By assumption, $D$ is invertible on $\mathcal{L}$. Now,

$$F^{in}(\mathcal{L}, v, D, c) = \{u \in \mathcal{L} + v : Du \geq c\} = \{Lw + v : D(Lw + v) \geq c\} \overset{(1)}{=} \{w \in \mathbb{Z}^t : \tilde{D}w \geq c - Dv\} \overset{(2)}{=} \{(w, w') \in \mathbb{Z}^t \times \mathbb{N}^r : \tilde{D}w - w' = c - Dv\} \overset{(3)}{=} \{w' \in \mathbb{N}^r : w' \in \mathbb{Z}\tilde{D} + Dv - c\} = F^{lat}(\mathbb{Z}\tilde{D}, Dv - c).
$$

The bijection (1) arises from multiplication by a left-inverse of $L$. The bijections (2) and (3) arise from the linear projections from $(w, w')$ to either the first or second coordinate. In total, the resulting map from $F^{in}(\mathcal{L}, v, D, c)$ to $F^{lat}(\mathbb{Z}\tilde{D}, Dv - c)$ is given by $u \mapsto \tilde{D}^{-1}(u - v) - c + Dv$, where $L^{-1}$ is a left-inverse of $L$. By assumption, $u - v \in \mathcal{L}$, and hence $\tilde{D}L^{-1}(u - v) = D(u - v)$. Therefore, for any values of $c$ and $v$, the bijection between $F^{in}(\mathcal{L}, v, D, c)$ and $F^{lat}(\mathbb{Z}\tilde{D}, Dv - c)$ is an affine map with linear part given by $D$. \qed

Example 7. Suppose we want to compute an inequality Markov basis of $D = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 1 \\ -2 & -1 \end{pmatrix}$, that is, we want to obtain a set of moves that connects all integer points $(y_1, y_2)$ that satisfy

$$y_1 \geq c_1, \quad y_1 + y_2 \leq c_2, \quad y_1 + y_2 \geq c_3, \quad 2y_1 + y_2 \leq c_4$$

(1)
for any $c_1, c_2, c_3, c_4$. The two columns of $D$ span a two-dimensional lattice $\mathbb{Z}D \subset \mathbb{Z}^3$. By 4ti2, a Markov basis of $\mathbb{Z}D$ is given by

$$G' = \{(1, 0, 0, -1), \ (1, -1, 0)\}.$$  

The inverse image of $G'$ under $D$ is

$$D^{-1}G' = \{(1, -1), \ (1, -2)\}.$$  

Lemma 6 (with $L = \mathbb{Z}^2$ and $\tilde{D} = D$) implies that this is an inequality Markov basis. The situation is visualized in Figure 2.

**Example 8.** Suppose we want to compute an inequality Markov bases for the following system of equations and inequalities:

$$y_1 + y_2 + y_3 = 0, \quad y_1 \geq c_1, \quad y_3 \geq c_2, \quad y_1 + y_2 \geq c_3, \quad y_1 - y_3 \leq c_4. \quad (3)$$

One possibility to study this system is to replace the first equation by the two inequalities

$$y_1 + y_2 + y_3 \geq 0 \quad \text{and} \quad y_1 + y_2 + y_3 \leq 0.$$  

This leads to a matrix $D'$ of size $6 \times 3$ and a Markov basis of cardinality $3$. Alternatively, one can observe that the first equation defines a lattice $L$, which is generated by the columns of

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}.$$  

This choice of $L$ corresponds to eliminating $y_3$ from (3) and leads to the same system of inequalities as in Example 7. The matrix $LD'$ is equal to the matrix $D$ augmented by two rows of zeros. By Lemma 6, the set

$$G = \{(1, -1, 0), \ (1, -2, 1)\} \quad (4)$$

is a Markov basis of (3).

To construct Gröbner bases of toric fiber products, we also need to find inequality Gröbner bases for the family $\mathcal{F}^m(\mathcal{L}, D)$. Such Gröbner bases can be computed from lattice Gröbner bases, following the same conversion as in the proof of Lemma 6. As above we may assume that $D$ restricted to $\mathcal{L}$ has rank $\dim(\mathcal{L})$. Otherwise, either no element of $\mathcal{F}^m(\mathcal{L}, D)$ has a minimum, or all non-empty elements of $\mathcal{F}^m(\mathcal{L}, D)$ have an infinite number of minima.

**Lemma 9.** Let $D \subseteq \mathbb{Z}^{r \times n}$, let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a lattice, let $L$ be an $(n \times t)$-integer matrix such that the columns of $L$ are a lattice basis of $\mathcal{L}$, and assume that $\tilde{D} = DL \in \mathbb{Z}^{r \times t}$ has rank $t$. Let $\succeq$ and $\succeq'$ be additive preorders on $\mathbb{Z}^n$ and $\mathbb{Z}^r$ such that for all $m_1, m_2 \in \mathbb{Z}^n$ with $m_1 - m_2 \in \mathcal{L}$,

$$m_1 \succeq m_2 \quad \text{if and only if} \quad Dm_1 \succeq' Dm_2.$$  

11
1. If $G$ is an $(L, D, \succeq)$-Gröbner basis, then $G' = D(G)$ is a $\succeq'$-Gröbner basis of $\mathbb{Z}D$.

2. If $G'$ is an $\succeq'$-Gröbner basis of $\mathbb{Z}D$, then $D^{-1}(G') \cap L$ is an $(L, D, \succeq)$-Gröbner basis.

Proof. As in the proof of Lemma 6, if $G = L^{-1}(G')$, then the two graphs $\mathbf{F}^m(L, v, D, c)_G$ and $\mathbf{F}^\text{lat}(\mathbb{Z}D, Dv - c)_{G'}$ are isomorphic as undirected graphs. The compatibility of the preorders $\succeq$ and $\succeq'$ guarantees that the edge directions point to a unique sink, if a sink exists.

2.3 Sign-consistency and Graver bases

Markov bases and Gröbner bases of lattices are related to Graver bases:

Definition 10. A pair of vectors $v, v' \in \mathbb{Z}^n$ is sign-consistent, if $v_i v'_i \geq 0$ for all $i = 1, \ldots, n$. A sum $\sum_j v_j$ with $v_j \in \mathbb{Z}^n$ is a conformal sum, if any pair $v_i, v'_i$ of summands is sign-consistent.

Let $L \subseteq \mathbb{Z}^n$ be a lattice. An element $v \in L \setminus \{0\}$ is primitive, if the following holds: If $v = v_1 + v_2$ is a conformal sum with $v_1, v_2 \in L$ then either $v_1 = 0$ or $v_2 = 0$. The set of all primitive elements is called the Graver basis of $L$. Alternatively, the Graver basis can be defined as the unique minimal subset $G_0 \subseteq L$ such that any element of $L$ can be written as a conformal sum of elements of $G_0$.

Sign-consistency is an important tool to remove redundant elements from Gröbner bases:

Lemma 11. Let $G$ be a $\succeq$-Gröbner basis of a lattice $L$. If $v, v_1, v_2 \in G \setminus \{0\}$ and if $v = v_1 + v_2$ is a conformal sum, then $G \setminus \{v\}$ is also a $\succeq$-Gröbner basis.

Proof. Suppose that $u \in \mathbf{F}(B, b)$, $u + v \in \mathbf{F}(B, b)$ with $u + v \preceq u$. Then $u + v_1, u + v_2 \in \mathbf{F}(B, b)$, and so $G$ connects $\mathbf{F}(B, b)$. Moreover, either $u + v_1 \preceq u$ or $u + v_2 \preceq u$ (or both).

The argument in the lemma shows that the Graver basis of $L$ is also a Gröbner basis for any additive preorder. In this sense, a Graver basis is a universal Gröbner basis (however, in general there may be smaller universal Gröbner bases, see [10, Chapter 4]). In particular, any minimal Gröbner basis consists of primitive vectors.

The concept of a Graver basis is tied to the coordinate hyperplanes. Therefore, there is no natural concept of an inequality Graver basis, or a Graver basis of a more general family of sets. Still, Graver bases play a role when computing Markov bases. Namely, there are some lattices for which the Markov basis is in fact a Graver basis. In such cases, to compute such a basis, it may be faster to use the program graver instead of the program markov (both programs belong to 4ti2).

Lemma 12. If a matrix $L$ is of the form $(\hat{L})$, then the Graver basis of $\mathbb{Z}L$ is a minimal Markov basis of $\mathbb{Z}L$. 
Proof. Recall that a lattice \( \mathcal{L} \) is of Lawrence type, if it consists of vectors of the form \((u, -u)\). Any lattice of Lawrence type satisfies the conclusion of the lemma (e.g. [10, Thm. 7.1]). If \( L \) has the indicated form, then \( \mathbb{Z}L \) is of Lawrence type.

### 3 Lifting Markov and Gröbner bases

As mentioned in the previous section, Markov and Gröbner bases of lattices can be computed using the software 4ti2. For larger examples, the algorithms implemented in 4ti2 may not terminate within a reasonable time. In this section we discuss an idea that allows to compute a larger Gröbner basis by lifting a Gröbner basis that lives in lower dimensions. For this idea to be useful, it is necessary to control both the smaller Gröbner basis as well as the lifting procedure. Later, we will apply the lifting procedure to the toric fiber product, where the lifting procedure can be simplified using the special structure of the product. Lifting procedures appear in special cases in [3, 11, 5].

**Definition 13.** Let \( \mathcal{F} \) be a collection of subsets of \( \mathbb{Z}^n \), let \( \phi : \mathbb{Z}^n \to \mathbb{Z}^t \) be a linear map, and let \( \succeq \) and \( \succeq' \) be two additive preorders on \( \mathbb{Z}^n \) and \( \mathbb{Z}^t \). We say that \( \succeq \) and \( \succeq' \) are compatible with \( \phi \) and \( \mathcal{F} \), if the following holds:

- For all \( u, v \in \mathcal{F} \), if \( \phi(u) \neq \phi(v) \), then \( \phi(u) \succeq' \phi(v) \) if and only if \( u \succeq v \).

In other words, \( \succeq' \) determines \( \succeq \) as soon as different fibers of \( \phi \) are involved. In particular, \( \phi \) is (weakly) monotone; that is, if \( u \succeq v \), then \( \phi(u) \succeq' \phi(v) \). As an example, the preorders \( \succeq_0 \) on \( \mathbb{Z}^n \) and \( \mathbb{Z}^t \) are always compatible.

Observe the following indirect effect: If \( u, u', v \in \mathcal{F} \) satisfy \( \phi(u) = \phi(u') \succeq' \phi(v) \), then the compatibility condition implies \( u \succeq v \succeq u \) and \( u' \succeq v \succeq u' \), and thus \( u \succeq u' \succeq u \). Therefore, in this case, there is only one unique preorder \( \succeq \) that is compatible with \( \succeq' \). This effect does not occur if \( \succeq' \) is an ordered (and not just a preorder).

If \( \mathcal{F} \) is a collection of subsets of \( \mathbb{Z}^n \), then \( \phi(\mathcal{F}) = \{ \phi(F) : F \in \mathcal{F} \} \) is the set of images of those subsets under the linear map \( \phi \).

**Definition 14.** Let \( \phi : \mathbb{Z}^n \to \mathbb{Z}^t \) be a linear map, let \( \succeq \) be an additive preorder on \( \mathbb{Z}^n \), and let \( \mathcal{F} \) be a collection of subsets of \( \mathbb{Z}^n \). A \((\mathcal{F}, \phi, \succeq)\)-lift of \( \mathcal{G} \subseteq \mathbb{Z}^t \) is a set \( \mathcal{M} \subseteq \mathbb{Z}^n \) such that for all \( F \in \mathcal{F} \) and \( v, v' \in F \) with \( v \succeq v' \) that satisfy \( \phi(v - v') \in \mathcal{G} \) there are \( m_0 \in \ker Z \phi \) and \( m \in \mathcal{M} \) such that \( v, v + m_0, v + m_0 + m \) is a \( \succeq \)-non-increasing path in \( F \) with \( \phi(v + m_0 + m) = \phi(v') \). In other words, we can move from \( v \) to \( v' \) by applying first a move \( m_0 \) from the kernel of \( \phi \), then a move \( m \) from the lift, and finally again a move from the kernel of \( \phi \). Apart from the last step, all other steps should be non-increasing.

If \( \mathcal{F} \) and \( \succeq \) are understood from the context, we simply speak of a \( \phi \)-lift.

Figure 3 illustrates the definition. We will later see that lifts exist in the setting of Gröbner bases of lattices (Section 3.2).

Lifting allows us to relate certain Gröbner bases in \( \mathbb{Z}^n \) and \( \mathbb{Z}^t \). In order to state this correspondence precisely, the following definitions are needed:
Figure 3: An illustration of the definition of a $\phi$-lift. The arrows point in the direction of $\preceq$-smaller nodes.

**Definition 15.** Let $\mathcal{F}$ be a family of subsets of $\mathbb{Z}^n$, and let $\phi: \mathbb{Z}^n \rightarrow \mathbb{Z}^t$ be a linear map. Let $\preceq$ and $\preceq'$ be additive preorders on $\mathbb{Z}^n$ and $\mathbb{Z}^t$, respectively. A $\preceq'$-Gröbner basis of $\phi(\mathcal{F})$ is called a projected fiber Gröbner basis ($\text{PF Gröbner basis}$). A kernel Gröbner basis of the lifting is a $\preceq$-Gröbner basis of the family of sets of the form $\mathcal{F} \cap (u + \ker \phi)$, for $\mathcal{F} \in \mathcal{F}$ and $u \in \mathbb{Z}^n$.

**Theorem 16.** Let $\mathcal{F}$ be a family of subsets of $\mathbb{Z}^n$, and let $\phi: \mathbb{Z}^n \rightarrow \mathbb{Z}^t$ be a linear map. Let $\preceq$ and $\preceq'$ be additive preorders on $\mathbb{Z}^n$ and $\mathbb{Z}^t$, respectively, that are compatible with $\phi$. Let $\mathcal{G}$ be a $\preceq'$-Gröbner basis, and let $\mathcal{M}_1$ be a $(\mathcal{F}, \phi, \preceq)$-lift of $\mathcal{G}$. Let $\mathcal{M}_0$ be a kernel Gröbner basis. Then $\mathcal{M}_0 \cup \mathcal{M}_1$ is a $\preceq$-Gröbner basis of $\mathcal{F}$.

**Proof of Theorem 16.** Let $\mathcal{M} := \mathcal{M}_0 \cup \mathcal{M}_1$. We want to apply Lemma 2. First, we show that $\mathcal{M}$ is a Markov basis. Let $u, v \in \mathcal{F} \in \mathcal{F}$. Since $\mathcal{G}$ is a $\preceq'$-Gröbner basis for $\phi(\mathcal{F})$, there are $g_1, \ldots, g_r \in \pm \mathcal{G}$ such that $\phi(u), \phi(u) + g_1, \ldots, \phi(u) + g_1 + \cdots + g_r = \phi(v)$ is a path in $\phi(\mathcal{F})$ from $\phi(u)$ to $\phi(v)$. Using Definition 14, this path lifts to a path in $\mathcal{F}$ from $u$ to $v$ with moves in $\pm \mathcal{M}$.

Next, we show that the two conditions of Lemma 2 are satisfied. The argument is similar, but now we need to take into account the preorders. For the first condition, let $u, v \in \mathcal{F}$ be $\preceq$-minimal in $\mathcal{F}$. Since $\phi$ is monotone, $\phi(u), \phi(v)$ are $\preceq'$-minimal in $\phi(\mathcal{F})$. Since $\mathcal{G}$ is a $\preceq'$-Gröbner basis for $\phi(\mathcal{F})$, there are $g_1, \ldots, g_r \in \pm \mathcal{G}$ such that $\phi(u), \phi(u) + g_1, \ldots, \phi(u) + g_1 + \cdots + g_r = \phi(v)$. Since $\mathcal{M}_1$ is a $(\mathcal{F}, \phi, \preceq)$-lift, we can use moves from $\mathcal{M}_0$ and $\mathcal{M}_1$ to lift this path to a non-increasing path in $\mathcal{F}$ as follows: In the first step, since $\phi(u) + g_1 \in \phi(\mathcal{F})$, there exists $\tilde{u}_1 \in \mathcal{F}$ with $\phi(\tilde{u}_1) = \phi(u) + g_1$. Let $m, m_0$ be as in Definition 14 applied to $u, \tilde{u}_1$ in place of $v, v'$. Then there is a non-increasing path from $u$ to $u + m_0$ using moves from $\mathcal{M}_0$. Adding the move $m \in \mathcal{M}_1$, we obtain a non-increasing
We discuss these three points in the special case $B$ be described as the integer kernel of the matrix $M$ with $b_i$ and $e_i$ the $i$th unit vector.

Before discussing the other two points, let us give another interpretation to $B^\phi$. Given $B$ and $\phi$ as above, let $\phi'$ be the linear map corresponding to $B^\phi$. In the following we only care about how $\phi$ acts on each fiber. Now, lifting along $\phi$ is essentially the same as
lifting along $\phi'$, since both maps have the same kernel Gröbner bases, and the projected fiber Gröbner bases are equivalent. Moreover, the linear map $\psi_B$ corresponding to $B$ factorizes through $\phi'$ (to be precise, $\psi_B$ arises from $\phi'$ by composition with a coordinate projection). Therefore, in our study of lifting, we could restrict attention to linear maps $\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}^t$ that are factors of $\psi_B$, and in this case, $B^\phi$ is nothing but a matrix that represents $\phi$.

3.1 Gröbner bases of projected fibers

Let $u \in \phi(F(B, b))$, and let $v \in F(B, b)$ such that $u = \phi(v)$. Then $B^\phi v = (b_u)$. Conversely, if $(b_u)$ lies in the affine semigroup $NB^\phi$, then $u$ lies in $\phi(F(B, b))$. In other words, descriptions of the projected fibers $\phi(F(B, b))$ can be obtained from suitable descriptions of $NB^\phi$.

Let $NB^\phi := (\mathbb{Z}B^\phi \cap \mathbb{R}_{\geq} B^\phi)$ be the normalization of $NB^\phi$. Elements of $NB^\phi \setminus NB^\phi$ are called holes. The semigroup $NB^\phi$ is normal if and only if $NB^\phi = NB^\phi$; that is, if and only if there are no holes. Normality of semigroups can be checked using the software Normaliz\cite{12}. See \cite{6} for an algorithm to compute the holes of non-normal semigroups.

Lemma 18. Let $B \in \mathbb{Z}^{h \times n}$, let $\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}^t$ be a linear map, and let

$$L = \{u \in \mathbb{Z}^t : \begin{pmatrix} 0 \\ u \end{pmatrix} \in \mathbb{Z}B^\phi\}.$$

If $NB^\phi$ is normal, then there exists an $r \times t$ integer matrix $D$ such that the following holds: For any $b \in NB$, there exists $c \in \mathbb{Z}^r$ with

$$\phi(F(B, b)) = \{u \in L + v : Du \geq c\} = F_{\text{lat}}(L, v, D, c).$$

The matrix $D$ can be obtained from an inequality description of the cone $\mathbb{R}_{\geq} B^\phi$.

Proof. If $NB^\phi$ is normal, then it is equal to the intersection of the lattice $\mathbb{Z}B^\phi$ and the polyhedral cone $\mathbb{R}_{\geq} B^\phi$. Let $(D_1 D_2)$ be a matrix such that

$$\mathbb{R}_{\geq} B^\phi = \{(b, u) \in \mathbb{R}^{n+t} : D_1 b + D_2 u \geq 0\}.$$

Hence, if $F(B, b) \neq \emptyset$, then $\phi(F(B, b)) = \{u \in L + v : D_2 u \geq -D_1 b\}$ where $v \in \mathbb{Z}^t$ is any vector such that $(b_v) \in \mathbb{Z}B^\phi$.  

By Lemma 18, if $NB^\phi$ is normal, a PF-Gröbner basis can be computed via an $(L, D)$-Gröbner basis for suitable $L$ and $D$. We demonstrate this in the next example. In general, the $(L, D)$-Gröbner basis might be larger than a minimal PF-Gröbner basis, because a PF-Gröbner basis does not need to work for all sets of the form $F_{\text{fin}}(L, v, D, c)$ for all $c \in \mathbb{Z}^r$, but it suffices if it works for those fibers where $c$ lies in the affine semigroup $-ND_1 B$.  

16
Let $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ and $\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then $B\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This matrix has rank four, and hence the kernel Markov basis is empty.

Denote the coordinates in $\mathbb{R}^5$ by $x_1, x_2, y_1, y_2, y_3$. According to Normaliz, the affine semigroup $NB^\phi$ is normal and consists of all integer solutions of

\begin{align*}
y_1 + y_2 + y_3 &= x_1, \\
y_1 &\geq 0, \\
y_1 + y_2 &\leq x_1, \\
y_1 + y_2 &\geq x_1 - \frac{1}{2}x_2, \\
2y_1 + y_2 &\leq 2x_1 - x_2.
\end{align*}

A Markov basis for these projected fibers is the same as a Markov basis in Example 8. In fact, the gray set in Figure 2 is equal to the projected fiber $\phi(F(B, (i)))$.

Even if $NB^\phi$ is not normal, the inequality description of $\mathbb{R}^5_{\geq}B^\phi$ gives valuable information about $NB^\phi$. Namely, $NB^\phi$ can be described as $NB^\phi = NB^\phi \setminus H$, where $H$ denotes the set of holes of $NB^\phi$. A similar description can be given to the projected fibers: If $(b, h) \in H$, then we call $h \in NB^\phi$ a hole of $\phi(F(B, b))$. In some instances, the set of holes is small enough that we can still find a good PF Markov basis. We will illustrate this in Section 5.2.

### 3.2 Lifting Gröbner bases of lattices

Finally, we give an algorithm for lifting for Gröbner bases of a lattice $L$; that is, we want to compute a Gröbner basis of $L$ from a PF Gröbner basis $G$. Since a union of lifts of singletons $\{g\}$ for all $g \in G$ is a lift of $G$, it suffices to know how to lift a single element. Lifting is easy if $\text{ker}_Z \phi \cap L = \{0\}$. In this case, the lift of $g$ consists of the unique element $m \in L \cap \phi^{-1}(g)$. This special case of lifting appears in [11]. In general, the problem to lift $g \in G$ can again be formulated as a Gröbner basis computation:

For any $F \in F$ and $u_1, u_2 \in \phi(F)$ with $u_2 - u_1 = g$ let

$$F_{\text{lift}}(F, \phi, u_1, u_2) := \{v \in F : \phi(v) \in \{u_1, u_2\}\} = \{v \in F : \phi(v) - u_1 \in \{0, g\}\}.$$ 

If $M$ is a Markov basis of the family

$$F_{\text{lift}}(F, \phi, g) := \{F_{\text{lift}}(F, \phi, u_1, u_2) : F \in F, u_1, u_2 \in \phi(F), u_1 - u_2 = g\},$$

then $M_g = \{m \in M : \phi(m) = g\}$ lifts $g$.

**Lemma 20.** Let $L \subseteq \mathbb{Z}^n$ be a lattice and $D$ an $r \times n$ integer matrix. Then

$$F_{\text{lift}}(F_{\text{in}}(L, D), \phi, g) \subseteq F_{\text{in}}(L_g, D_g)$$

for a suitable lattice $L_g$ and matrix $D_g$. 

---

Example 19. Let $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ and $\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This matrix has rank four, and hence the kernel Markov basis is empty.
Proposition 21. Let $\succeq, \succeq'$ be compatible additive preorders for $\phi : \mathbb{Z}^n \to \mathbb{Z}'$, let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a lattice, $D \in \mathbb{Z}^{n \times n}$, and let $\mathcal{G}$ be a PF Gröbner basis for $\mathbb{F}^m(\mathcal{L}, D)$. For each $g \in \mathcal{G}$, let $\mathcal{L}_g$, $D_g$ be as in Lemma 20, let $\mathcal{M}_g'$ be an $(\mathcal{L}_g, D_g, \succeq)$-Gröbner basis, and let $\mathcal{M}_g = \{ m \in \mathcal{M}_g' : \phi(m) = +g \}$. Then $\bigcup_{g \in \mathcal{G}} \mathcal{M}_g$ is an $(\mathbb{F}^m(\mathcal{L}, D), \phi, \succeq)$-lift of $\mathcal{G}$.

Example 22. We continue Example 19 using the Markov basis [4]. In this case, since $\ker B^0 = \{0\}$, the lifting procedure yields one lift for each of the two vectors. Hence the lifted Markov basis is

$$
\mathcal{M} = \left\{ (1, -1, 0, 0), (1, 0, -2, 1) \right\}.
$$

This is also the Markov basis that tsi2 computes when given the matrix $B$.

Less trivial examples of lifting appear in Section 5.

3.3 The codimension-one case and the slow-varying property

The complexity of projecting the fibers and of lifting crucially depends on the choice of the map $\phi$. How to find a good $\phi$ is difficult to say in general. One aspect is the dimensionality of the projected fibers.

Definition 23. The codimension of the $(\mathcal{F}, \phi, \succeq)$-lifting is defined as $\sup_{F \in \mathcal{F}} \dim(\phi(F))$, where $\dim(\phi(F))$ denotes the dimension of the affine hull of $\phi(F)$.

In the case $\mathcal{F} = \mathcal{F}(B)$ of matrices, the codimension is $\dim(\phi(\ker \mathbb{Z} B))$, in the case $\mathcal{F} = \mathcal{F}_{\text{lat}}(\mathcal{L})$ of lattices, the codimension is $\dim(\phi(\mathcal{L}))$, where in both cases $\dim$ denotes the dimension of a lattice.

In this section we focus on the codimension-one case and relate our theory to some results of [7]. Let $g \in \mathbb{Z}'$ be a generator of $\phi(\ker \mathbb{Z} B)$. In this case, the projected fibers are at most one-dimensional. For any $b \in \mathbb{N}B$ and $u_0 \in \mathbb{F}(\mathcal{B}, b)$ we have $\phi(\mathbb{F}(\mathcal{B}, b)) \subseteq u_0 + \mathbb{Z}g$.

If there are no holes, then $\phi(\mathbb{F}(\mathcal{B}, b))$ consists of consecutive elements of $u_0 + \mathbb{Z}g$; that is $\phi(\mathbb{F}(\mathcal{B}, b)) = \{ u_0 + kg : l \leq k \leq l' \}$ for some $l, l' \in \mathbb{Z}$. In this case, $\{ \pm g \}$ is a PF Gröbner basis for any additive preorder on $\mathbb{Z}'$.
Definition 24. In the codimension-one case, a Gröbner basis $\mathcal{M}$ of $\mathcal{B}$ is slow-varying with respect to $\phi$, if there exists a single vector $g \in \mathbb{Z}^t$ such that $\phi(\mathcal{M}) \subseteq \{0, \pm g\}$.

Slow-varying Markov bases are useful in the gluing procedure in the toric fiber product construction as shown in [7]. Clearly, a slow-varying Gröbner basis exists if and only if $\{\pm g\}$ is a PF Gröbner basis for any additive preorder on $\mathbb{Z}^t$. Hence:

Lemma 25. Assume that $\phi$ has codimension one with respect to $\mathcal{B}$. If $N\mathcal{B}\phi$ is normal, then there exists a slow-varying Gröbner basis for any additive preorder.

4 The toric fiber product

We now turn our attention to the toric fiber product construction. This construction involves several maps that lend themselves naturally as candidates for lifting. In Sections 4.1 to 4.3, we show how the results of the previous section help to compute Gröbner bases of toric fiber products. Section 4.4 contains an elaborate example.

We first recall the construction and fix the notation. The toric fiber product is defined for general $\mathbb{N}\mathcal{A}$-homogeneous ideals in [4]. We focus exclusively on the case of toric fiber products of toric ideals, and hence, toric fiber products of matrices.

Definition 26. Let $\mathcal{A} \in \mathbb{Z}^{s \times t}$ be an integer matrix with columns $a_1, \ldots, a_t$. Any surjection $\phi : \{n\} \to \{t\}$ induces a surjective map $\mathbb{Z}^n \to \mathbb{Z}^t$, $e_i \mapsto e_{\phi(i)}$, which we denote by $\phi$ again. Let $\mathcal{B} = (b_1, \ldots, b_n)$ be an integer matrix with $n$ columns. We say that $\mathcal{B}$ is $\mathcal{A}$-graded by $\phi$, if one of the following two equivalent statements is satisfied:

- There is a linear map $\pi : \mathbb{N}\mathcal{B} \to \mathbb{N}\mathcal{A}$ with $\pi(b_i) = a_{\phi(i)}$.
- The map $\phi : \mathbb{Z}^n \to \mathbb{Z}^t$ satisfies $\phi(\ker \mathbb{Z} \mathcal{B}) \subseteq \ker \mathbb{Z} \mathcal{A}$.

Given two matrices $\mathcal{B}, \mathcal{B}'$ that are $\mathcal{A}$-graded by two maps $\phi, \phi'$, the toric fiber product is the matrix

$$
\mathcal{B} \times_{\mathcal{A}} \mathcal{B}' := \left\{ \begin{pmatrix} b_i \\ b'_j \end{pmatrix} : \phi(i) = \phi'(j) \right\}
$$

that consists of all pairs of columns from $\mathcal{B}$ and $\mathcal{B}'$ that are mapped to the same column of $\mathcal{A}$. The codimension of this toric fiber product is equal to $\dim \ker \mathbb{Z} \mathcal{A}$.

Consider the map $\psi : \mathbb{Z}^{\mathcal{B} \times_{\mathcal{A}} \mathcal{B}'} \to \mathbb{Z}^\mathcal{B}$ that maps the unit vector $e_{i,j}$ corresponding to $(b_i, b'_j)$ to the $i$th unit vector $e_i \in \mathbb{Z}^\mathcal{B}$, and consider the corresponding map $\psi' : \mathbb{Z}^{\mathcal{B} \times_{\mathcal{A}} \mathcal{B}'} \to \mathbb{Z}^{\mathcal{B}'}$ that maps $e_{i,j}$ to $e_j \in \mathbb{Z}^{\mathcal{B}'}$. Then the following diagram commutes:
where $\xi = \phi \circ \psi = \phi' \circ \psi'$.

Let $\succeq_x, \succeq_B, \succeq_B'$ and $\succeq_A$ be additive preorders on $\mathbb{Z}^{B \times A B'}$, $\mathbb{Z}^B$, $\mathbb{Z}^{B'}$, and $\mathbb{Z}^A$, respectively, that are compatible with $\phi$, $\phi'$ and $\xi$. In general, it is not possible to require that $\psi$ and $\psi'$ are also compatible with respect to these orders. Instead, we call $\succeq_x$ compatible, if it satisfies the following weaker property:

- For any $u_1, u_2 \in \mathbb{Z}^{B \times A B'}$, if $\psi(u_1) \succeq_B \psi(u_2)$ and if $\psi'(u_1) \succeq_{B'} \psi'(u_2)$, then $u_1 \succeq_x u_2$.

For given preorders $\succeq_B, \succeq_B'$ and $\succeq_A$, a compatible preorder $\succeq_x$ on $\mathbb{Z}^{B \times A B'}$ can be constructed as follows:

\[ u_1 \succeq_x u_2 \iff \psi(u_1) \triangleright_B \psi(u_2) \]

or $\psi(u_1) \succeq_B \psi(u_2) \succeq_B \psi(u_1)$ and $\psi'(u_1) \succeq_{B'} \psi'(u_2)$.

Our goal is to compute $\succeq_x$-Gröbner bases of $\ker_{\mathbb{Z}} B \times_A B'$. We want to apply the lifting machinery from the previous section and lift along the map $\xi$. To apply Theorem 16 we need to understand the kernel Gröbner basis, the PF Gröbner basis, and we need to lift the PF Gröbner basis. This will be described in the next three sections. A key result is that we only need to compute lifts along $\phi$ and $\phi'$, which can be “glued” to produce lifts along $\xi$. The complexity of these lifts is governed by the codimension of the toric fiber product.

**Lemma 27.** The codimension of the toric fiber product $B \times_A B'$ is not less than the codimension of $(F(B \times_A B'), \xi, \succeq_x)$-liftings, $(F(B), \phi, \succeq_B)$-liftings and $(F(B'), \phi', \succeq_{B'})$-liftings.

**Proof.** This follows from the inclusion $\xi(\ker_{\mathbb{Z}} B \times_A B') \subseteq \phi(\ker_{\mathbb{Z}} B) \subseteq \ker_{\mathbb{Z}} (A)$, together with the symmetric inclusion. \qed

The results in this section are very technical. A simple example is given in Section 4.4 after presenting the theory. Larger examples that show how to apply the results of this section to hierarchical models will be given in Section 5.

### 4.1 Kernel Gröbner basis and the associated codimension zero toric fiber product

To compute the kernel Gröbner basis, we need the following definition:

**Definition 28.** Let $B, B'$ be integer matrices that are $\mathbb{N}A$-graded via maps $\phi, \phi'$ as above. The associated codimension zero toric fiber product is the matrix $B^\phi \times_A (B')^{\phi'}$, where $\hat{A}$ is the unit matrix in $\mathbb{N}^{t \times t}$ and where $B^\phi$ and $(B')^{\phi'}$ are $\mathbb{N}A$-graded using the same maps $\phi, \phi'$.

**Lemma 29.** $\ker_{\mathbb{Z}}(\xi) \cap \ker_{\mathbb{Z}}(B \times_A B') = \ker_{\mathbb{Z}}(B^\phi \times_A (B')^{\phi'})$. Hence, when lifting along $\xi$, a kernel Gröbner basis is given by a Gröbner basis of $\ker_{\mathbb{Z}}(B^\phi \times_A (B')^{\phi'})$. 

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20
Proof. Observe that \( B \times _\mathcal{A} B' \) can be identified with a submatrix of \( B^\phi \times _\mathcal{A} (B')^{\phi'} \). In fact, a sequence of row operations turns the matrix \( B^\phi \times _\mathcal{A} (B')^{\phi'} \) into the matrix with columns
\[
\begin{pmatrix}
  b_i \\
  b'_j \\
  e_{\phi(i)}
\end{pmatrix}
\]
for all \( i, j \) with \( \phi(i) = \phi'(j) \).

Clearly, the kernel of this last matrix is \( \ker Z(x) \cap \ker Z(B \times _\mathcal{A} B') \).}

Note that \( \ker \hat{A} = \{ 0 \} \), and so \( B^\phi \times _\mathcal{A} (B')^{\phi'} \) is a codimension zero toric fiber product. Computation of Markov bases and Gröbner bases of codimension zero toric fiber product was described in [4]. We review the main result here.

Let \( m \in \ker Z^B \). Then \( \phi(m) \in \ker Z(\hat{A}) = \{ 0 \} \), and so \( \phi(m^+) = \phi(m^-) \). Hence there exist maps \( \sigma_+, \sigma_- \) such that \( m = \sum_i e_{\sigma_+(i)} - \sum_i e_{\sigma_-(i)} \) and \( \phi(e_{\sigma_+(i)}) = \phi(e_{\sigma_-(i)}) \). Choose a map \( \tau \) with \( \phi'(\tau(i)) = \phi(e_{\sigma_+(i)}) \). Then
\[
\tilde{m} = \sum_i e_{\sigma_+(i), \tau(i)} - \sum_i e_{\sigma_-(i), \tau(i)}
\]
lies in the kernel of \( B^\phi \times _\mathcal{A} (B')^{\phi'} \). Call \( \tilde{m} \) a lift of \( m \). This name is justified by the fact that the set \( \text{Lifts}(m) \) of all such lifts is a \( (\mathcal{F}(B^\phi), \psi, \geq_\times ) \)-lift of \( m \). Denote by \( \text{Lifts}(\mathcal{M}) := \bigcup_{m \in \mathcal{M}} \text{Lifts}(m) \) the set of all such lifts of all \( m \in \mathcal{M} \subseteq \ker Z^B \). We can similarly define the set \( \text{Lifts}(\mathcal{M}') \), where \( \mathcal{M}' \subseteq \ker Z(B')^{\phi'} \).

A second set of moves that we need is

\[
\text{Quads} := \{ f_{i_1,i_2;j_1,j_2} : \phi(i_1) = \phi(i_2) = \phi'(j_1) = \phi'(j_2) \},
\]

where \( f_{i_1,i_2;j_1,j_2} = e_{i_1,j_1} + e_{i_2,j_2} - e_{i_1,j_2} - e_{i_2,j_1} \) and \( e_{i,j} \) is the standard unit vector in \( \mathbb{Z}^{B^\phi \times _\mathcal{A} (B')^{\phi'}} \) corresponding to \( (b_i, b'_j) \).

**Theorem 30.** [4] Suppose that \( \mathcal{M} \) and \( \mathcal{M}' \) are Markov bases for \( \ker Z^B \) and \( \ker Z(B')^{\phi'} \), respectively. Then
\[
\text{Lifts}(\mathcal{M}) \cup \text{Lifts}(\mathcal{M}') \cup \text{Quads}
\]
is a Markov basis for \( \ker Z^B \times _\mathcal{A} (B')^{\phi'} \). If, in addition, \( \mathcal{M} \) and \( \mathcal{M}' \) are Gröbner bases for compatible preorders, then, for any compatible additive preorder \( \preceq_\times \) on \( \mathbb{Z}^{B^\phi \times _\mathcal{A} (B')^{\phi'}} \), [5] is a Gröbner basis of \( \ker Z^B \times _\mathcal{A} (B')^{\phi'} \).

### 4.2 Projected fiber intersections

Next, we want to understand the geometry of the projected fibers \( \xi(\mathcal{F}(B \times _\mathcal{A} B', (b, b')) \). These have a simple relation to the projected fibers \( \phi(\mathcal{F}(B, b)) \) and \( \phi'(\mathcal{F}(B', b')) \).

**Lemma 31.** \( \xi(\mathcal{F}(B \times _\mathcal{A} B', (b, b'))) = \phi(\mathcal{F}(B, b)) \cap \phi'(\mathcal{F}(B', b')) \).
Definition 32. The projected fiber intersection (PFI) Gröbner basis of the toric fiber product is a projected fiber Gröbner basis for $\xi$.

A PFI Gröbner basis can be computed as an inequality Markov basis if the projected fibers $\xi(F(B \times_\Lambda B', (b', b))))$ can be described in terms of linear equations and inequalities. It easily follows from Lemma 31 that this is the case if the same condition holds true for the projected fibers $\phi(F(B, b))$ and $\phi'(F'(B', b'))$. Such inequality representations are easiest to obtain if $N\mathbb{B}_\phi$ and $N(B')\phi'$ are both normal. In fact, if both $N\mathbb{B}_\phi$ and $N(B')\phi'$ are normal, then $N(B \times_\Lambda B')$ is also normal. This follows from Lemma 29 and the fact that normality is preserved in codimension-zero TFPs [7, Theorem 2.5] (but not in higher codimension [13]).

Remark 33. Suppose that $B = B'$ and $\phi = \phi'$, and suppose that all projected fibers have an inequality description $\phi(F(B, b)) = \{u \in \Lambda + u_0(b) : Du \geq c(b)\}$, where the integer matrix $D$ and the lattice $\Lambda$ are independent of $b$. Then, an inequality Gröbner basis for $D$ is a PF Markov basis for $\phi$ as well as a PFI Markov basis for the toric fiber product, because

$$\phi(F(B, b)) \cap \phi(F(B, b')) = \{u \in (\Lambda + u_0(b)) \cap (\Lambda + u_0(b')) : Du \geq \max\{c(b), c(b')\}\}.$$  

4.3 Gluing $\xi$-lifts from $\phi$-lifts and $\phi'$-lifts

Finally, we show how to lift moves $g \in \mathbb{Z}^t$ along $\xi$ by gluing $\phi$-lifts and $\phi'$-lifts of $g$. Let $m \in \ker_\mathbb{Z} B$ and $m' \in \ker_\mathbb{Z} B'$ such that $\phi(m) = \phi'(m') = g$. The goal of gluing is to construct a move $\tilde{m}$ with $\psi(\tilde{m}) = m$ and $\psi'(\tilde{m}) = m'$. In general, $\tilde{m}$ will be larger than both $m$ and $m'$, but the idea is to construct $\tilde{m}$ as small as possible. The first step is to extend $m$ and $m'$ to make them compatible for gluing.

Let $v = \phi'(m'^+) - \phi(m) = \phi'(m'^+) - \phi(m^-)$. Then $\phi(m^+) + v^+ = \phi'(m'^+) + v^+$ and $\phi(m^-) + v^+ = \phi'(m'^-) + v^+$. Choose vectors $\overline{m}^+, \overline{m}^- \in \mathbb{N}^n$ and $\overline{m}'^+, \overline{m}'^- \in \mathbb{N}^n$ that satisfy $\phi(\overline{m}^+ - m^+) = \phi(\overline{m}^- - m^-) = v^+$ and $\phi(\overline{m}'^+ - m'^+) = \phi(\overline{m}'^- - m'^-) = v^-$. Since $\phi(\overline{m}^+)$ and $\phi(\overline{m}^-) = \phi'(\overline{m}^-)$, there are functions $\sigma, \sigma', \tau, \tau'$ satisfying

$$\overline{m}^+ = \sum_i e_{\sigma(i)} \quad \overline{m}^- = \sum_j e_{\tau(j)} \quad \overline{m}'^+ = \sum_i e_{\sigma'(i)} \quad \overline{m}'^- = \sum_j e_{\tau'(j)}$$
and $\phi(\sigma(i)) = \phi'(\sigma'(i))$ and $\phi(\tau(j)) = \phi'(\tau'(j))$. Then the vector
\[
\tilde{m} = \sum_i e_{a_i, \sigma'_i} - \sum_j e_{\tau'_j, \tau'_j}
\]
belongs to $\ker_{\xi}(B \times_A B')$. We call $\tilde{m}$ a glue of $m$ and $m'$. The set $\text{Glues}(m, m')$ of all glues of $m$ and $m'$ is finite, since $\phi^{-1}(v^+) \cap \mathbb{N}^n$ and $\phi'^{-1}(v^+) \cap \mathbb{N}'^n$ are finite. For any $M \subseteq \ker_{\xi}(B)$, $M' \subseteq \ker_{\xi}(B')$ denote by $\text{Glues}(M, M')$ the set of all glues of compatible elements of $M$ and $M'$. See [7] for a more detailed description of the gluing procedure.

The gluing construction has the following crucial property:

**Lemma 34.** Let $m \in \ker_{\xi}(B)$, $m' \in \ker_{\xi}(B')$ with $\phi(m) = \phi'(m')$, and let $w \in \mathbb{N}^{B \times_A B'}$. If $\psi(w) + m \geq 0$ and $\psi(w) + m' \geq 0$, then there exists $\tilde{m} \in \text{Glues}(m, m')$ with $w + \tilde{m} \geq 0$.

**Proof.** This is a restatement of Lemma 4.8 of [7].

**Lemma 35.** Let $M \subseteq \ker_{\xi}(B)$ and $M' \subseteq \ker_{\xi}(B')$ be $(\mathcal{F}(B), \phi, \geq_{B})$- and $(\mathcal{F}(B'), \phi', \geq_{B'})$-lifts of a $\geq_{A}$-Gröbner basis $\mathcal{G}$ of $\xi(\mathcal{F}(B \times_A B'))$. Then $\text{Glues}(M, M')$ is a $(\mathcal{F}(B \times_A B'), \xi, \geq_{\lambda})$-lift of $\mathcal{G}$.

**Proof.** Suppose that $w_1, w_2 \in \mathbb{N}^{B \times_A B'}$ satisfy $\xi(w_1 - w_2) = g \in \mathcal{G}$ and $(B \times_A B')(w_1 - w_2) = 0$. Then $v_1 = \psi(w_1)$ and $v_2 = \psi(w_2)$ satisfy $\phi(v_1 - v_2) = g$ and $B(v_1 - v_2) = 0$.

Since $\mathcal{G}$ lifts $\mathcal{G}$, there are $m \in M$ and $m_0, m_1 \in \ker \phi$ as in Definition 14. Similarly, $v_1' = \psi'(v_1)$ and $v_2' = \psi'(v_2)$ satisfy $\phi'(v_1' - v_2') = g$ and $B'(v_1' - v_2') = 0$, so we can find $m' \in M'$ and $m_0', m_1' \in \ker \phi'$ as in Definition 14. By Lemma 34, there are $\tilde{m}_0 \in \text{Glues}(m_0, m_0')$ and $\tilde{m} \in \text{Glues}(m, m')$ such that $w_1 + \tilde{m}_0 \geq 0$ and $w_1 + \tilde{m}_0 + \tilde{m} \geq 0$.

Let $\tilde{m}_1 := w_2 - w_1 - \tilde{m}_0 - \tilde{m}$. Then $\xi(\tilde{m}_1) = \phi(v_2 - v_1 - m_0 - m_1) = \phi(m_1) = 0$. Thus, $\xi(\tilde{m}_0) = \xi(\tilde{m}_1) = 0$. $\xi(\tilde{m}) = g$ and $(B \times_A B')\tilde{m}_0 = (B \times_A B')\tilde{m}_1 + \tilde{m}$ is non-increasing, due to our compatibility requirements. Hence the conditions of Definition 14 are verified.

The results of this section are related to the following notion from [7].

**Definition 36.** Two Markov bases $M \subseteq \ker_{\xi}(B)$, $M' \subseteq \ker_{\xi}(B')$ satisfy the compatible projection property, if the graph $\phi(\mathcal{F}(B, b)_M) \cap \phi'(\mathcal{F}(B', b')_{M'})$ is connected for all $b \in \mathbb{N}B$, $b' \in \mathbb{N}B'$. Here, $\phi(\mathcal{F}(B, b)_M)$ denotes the image of the graph $\mathcal{F}(B, b)_M$ under $\phi$, as defined in the introduction.

Theorem 4.9 in [7] says that if $M$ and $M'$ have the compatible projection property, then the union of $\text{Glues}(M, M')$ and a Markov basis of the associated codimension-zero toric fiber product is a Markov basis of $B \times_A B'$. The proof of Lemma 35 basically shows that, if $M$ and $M'$ lift a PFI Markov basis $\mathcal{G}$, then $\phi(\mathcal{F}(B, b)_M) \cap \phi(\mathcal{F}(B', b')_{M'}) = \xi(\mathcal{F}(B \times_A B', (b, b')))_{\mathcal{G}}$. Hence, in this case, $M$ and $M'$ have the compatible projection property.

The compatible projection property is weaker than the property of being lifts. Sometimes it is possible to find subsets of lifts which still satisfy the compatible projection property. In this way, a smaller Markov basis of $B \times_A B'$ can be found. For an example see Section 5.2.

We conclude this section with another result from [7]:

23
Lemma 37 (Theorem 4.2 in [7]). Let $\mathcal{B} \times_A \mathcal{B}'$ be a codimension-one toric fiber product, and let $\mathcal{M}, \mathcal{M}'$ be slow-varying Markov bases of $\mathcal{B}$ and $\mathcal{B}'$. Then $\mathcal{M}$ and $\mathcal{M}'$ satisfy the compatible projection property.

4.4 A simple example

Consider the matrix $\mathcal{B}$ and the map $\phi$ from Example 22. Then $\phi$ corresponds to the map

$$1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 2, \quad 4 \mapsto 3.$$ 

Let $\mathcal{B}' = \mathcal{B}$, and let

$$\phi' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be the map that arises from $\phi$ by switching the role of the first two coordinates in the image. The corresponding toric fiber product is

$$\mathcal{B} \times_A \mathcal{B}' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ -1 & -1 \\ -2 & -1 \\ -1 & -2 \end{pmatrix}.$$ 

Using the symmetry between $\phi$ and $\phi'$, the projected fiber intersections can be described as the set of integer solutions of inequalities of the form

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_1 + y_2 \leq c_1, \quad y_1 + y_2 \geq c_2, \quad 2y_1 + y_2 \leq c_3, \quad y_1 + 2y_2 \leq c_4,$$

corresponding to the matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 1 \\ -2 & -1 \\ -1 & -2 \end{pmatrix}.$$ 

The Markov basis of the lattice generated by the columns of $D$ contains three elements:

$$(0, 1, -1, 1, -1, -2), \quad (1, -1, 0, 0, -1, 1), \quad (1, 0, -1, 1, -2, -1).$$

The inverse images under $D$ are $(0, 1), (1, -1)$ and $(1, 0)$, and so the PF Markov basis is given by

$$G = \left\{ g_1 = (0, 1, -1), \quad g_2 = (1, -1, 0), \quad g_3 = (1, 0, -1) \right\}.$$ 

Each move in $G$ has a single $\phi$-lift, and the lifted Markov basis is

$$M = \left\{ m_1 = (0, -1, 2, -1), \quad m_2 = (1, -1, 0, 0), \quad m_3 = (1, -2, 2, -1) \right\}.$$
Both $\mathcal{G}$ and $\mathcal{M}$ are symmetric under the exchange of $y_1$ and $y_2$, and so $\mathcal{M}$ is also a $\phi'$-lift of $\mathcal{G}'$. We have
\[
\phi(m_1) = g_1 = \phi'(m_3), \quad \phi(m_2) = g_2 = \phi'(-m_2), \quad \phi(m_3) = g_3 = \phi'(m_1).
\]
In each case, one can check that there is just a single glued element:
\[
\begin{align*}
\text{Glues}(m_1, m_3) & = \{ \hat{m}_1 = (-2, 2, -1, 2, -1) \}, \\
\text{Glues}(m_2, -m_2) & = \{ \hat{m}_2 = (1, 0, -1, 0, 0) \}, \\
\text{Glues}(m_3, m_1) & = \{ \hat{m}_3 = (-1, 2, -2, 2, -1) \}.
\end{align*}
\]
Thus these three moves form a Markov basis of the TFP. In fact, it suffices to take the first two moves: Suppose that we want to apply $\hat{m}_3$: Then $x_1, x_5 \geq 1$ and $x_3 \geq 2$. Hence we can apply $\hat{m}_2$. The result has $x_1 \geq 2$ and $x_3 \geq 2$. Hence we can apply $\hat{m}_1$. But $\hat{m}_3 = \hat{m}_1 + \hat{m}_2$.

In fact, Example 12 gives the Markov basis $\{ \hat{m}_2, \hat{m}_4 \}$ with $\hat{m}_4 = (3, -2, 0, -2, 1)$. Observe that $\hat{m}_4 = -\hat{m}_1 + \hat{m}_2$, and an argument as above shows that $\{ \hat{m}_2, \hat{m}_4 \}$ is equivalent to $\{ \hat{m}_1, \hat{m}_2 \}$.

## 5 Application to Hierarchical Models

This section explores our main applications to constructing Markov bases of hierarchical models and gives two more complex examples. The Markov basis of the four-cycle is studied in Section 5.1. We collect known results about the no three-way interaction model that allow to compute kernel Gröbner bases and PFI Gröbner bases for some values of the parameter $d$. Section 5.2 contains an example of a codimension-one toric fiber product where the associated semigroup has holes. Thus, a PFI Markov basis cannot directly be computed as an inequality Markov basis, but it is not difficult to adjust our ideas. The example also illustrates that the Markov basis obtained through our algorithm is in general too large. A detailed analysis shows that not all lifted moves are necessary.

Before delving into the examples, let us fix the notation. Let $\Gamma$ be a simplicial complex with vertex set $V$, and let $d \in \mathbb{Z}_{\leq 2}$. For $F \subseteq V$ let $D_F := \prod_{j \in F}[d_j]$. For each $i = (i_j)_{j \in V} \in D_V$ let $i_F = (i_j)_{j \in F}$, be the subvector with index set $F$. Let $B_{\Gamma,d}$ be the matrix that consists of the following $\#D_V$ columns, one for each $i \in D_V$:
\[
b_i := \bigoplus_{F \in \text{facet}(\Gamma)} e_{i_F} \in \bigoplus_{F \in \text{facet}(\Gamma)} \mathbb{Z}^{D_F},
\]
where $e_{i_F}$ denotes the $i_F$th standard unit vector in $\mathbb{Z}^{D_F}$. The matrix $B_{\Gamma,d}$ is the *design matrix of the hierarchical model specified by $\Gamma$ and $d$*. If $d_i = 2$ for all $i \in V$, we speak of a *binary* hierarchical model. The fibers of $B_{\Gamma,d}$ have the following natural interpretation: For each facet $F \in \Gamma$, the linear map of $B_{\Gamma,d}$ computes the $F$-margins, and thus, the fibers
of $B_{\Gamma,d}$ correspond to sets of non-negative tensors where a certain number of margins (determined by $\Gamma$) is fixed.

The kernel of $B_{\Gamma,d}$ lies in $\mathbb{Z}^{D_{V'}}$. Elements of $\mathbb{Z}^{D_{V'}}$ are often written in tableau notation, as the difference of two matrices of indices. For example, the vector

$$2e_{111} + e_{222} - e_{112} - e_{121} - e_{211}$$

is represented in tableau notation as

$$\begin{bmatrix} 111 \\ 112 \\ 121 \\ 211 \\ 222 \end{bmatrix}$$

The matrix $B_{\Gamma,d}$ is a toric fiber product whenever $\Gamma$ is missing edges, and the associated codimension zero toric fiber product can also be described by a simplicial complex obtained by filling in the separator.

**Proposition 38** (Propositions 5.1 and 5.2 in [7]). Let $\Gamma$ be a simplicial complex on $V$. Let $V_1, V_2 \subseteq V$ such that $V = V_1 \cup V_2$, and $\Gamma = \Gamma|_{V_1} \cup \Gamma|_{V_2}$. Let $S = V \cap V'$. Then:

1. $B_{\Gamma,d} = B_{\Gamma|_{V_1}, d_{V_1}} \times B_{\Gamma|_{V_2}, d_{V_2}}$. The $B_{\Gamma|_{S}, d_{S}}$-grading of $B_{\Gamma|_{V_1}, d_{V_1}}$ and $B_{\Gamma|_{V_2}, d_{V_2}}$ is given by the marginalization maps $\phi : \mathbb{Z}^{D_{V_1}} \to \mathbb{Z}^{D_S}$ and $\phi' : \mathbb{Z}^{D_{V_2}} \to \mathbb{Z}^{D_S}$. Thus, the $B_{\Gamma|_{S}, d_{S}}$-grading of the product $B_{\Gamma,d}$ is also given by the marginalization map $\xi : \mathbb{Z}^{D_{V'}} \to \mathbb{Z}^{D_S}$.

2. Let $\tilde{\Gamma} := \Gamma \cup 2^S$. Then $B_{\Gamma|_{V_1}, d_{V_1}} \times B_{\Gamma|_{V_2}, d_{V_2}} = B_{\tilde{\Gamma}, d}$.

Normality of $NB_{\Gamma,d}$ plays an important role in easily determining a PF Markov basis. Only in certain special cases do we possess classifications of normal hierarchical models.

**Theorem 39.** [14] Let $\Gamma = [12][13][23]$ be a 3-cycle (also called “no three-way interaction model”). Then $NB_{\Gamma,d}$ is normal if and only if, up to symmetry, $d$ is one of:

$$(3, 4, 4), \quad (3, 4, 5), \quad (3, 5, 5), \quad (2, p, q), \quad (3, 3, q), \quad \text{with } p, q \in \mathbb{N}.$$  

### 5.1 The 4-cycle

In this section, we use the toric fiber product and lifting techniques to construct Markov and Gröbner bases of the 4-cycle model $\Gamma = C_4 := [12][13][24][34]$, for various values of $d$. In the case $d_1 = d_3 = 2$, Markov bases were already computed in [15] and [7]. We apply Proposition 38 with $V_1 = \{1, 2, 3\}$ and $V_2 = \{2, 3, 4\}$, so that $\Gamma_1 = [12][13]$ and $\Gamma_2 = [24][34]$; see Figure 4a). In fact, our results easily generalize to the complete bipartite graph $K_{2,N}$, which arises by iterating the toric fiber product, as detailed in Section 6.
5.1.1 The associated codimension zero product

By Proposition [38] the associated codimension zero product is the hierarchical model on the simplicial complex $\tilde{C}_4 := [12][13][23][24][34]$; see Figure 4. This is obtained by gluing the two triangles $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ along an edge. Theorem 30 can be used to construct the Markov basis in this case, provided we know the Markov bases for $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. The Markov bases of triangles are not known in general, but are simple to compute in some instances [1].

**Theorem 40.** Let $C_3 := [12][13][23]$ be a triangle, and let $d = (p, 2, r)$. For any sequences $i := i_1, \ldots, i_k \in [p]$ and $j := j_1, \ldots, j_k \in [r]$ with pairwise distinct entries, let

$$f_{i,j} := \sum_{t=1}^{k} \left( e_{i_1,1,j_t} - e_{i_2,2,j_t} + e_{i_2,2,j_{t+1}} - e_{i_1,1,j_{t+1}} \right)$$

where $j_{k+1} := j_1$. Then

$$\mathcal{M} = \{ f_{i,j} : k = 2, \ldots, \min(p, r), i := i_1, \ldots, i_k \in [p], j := j_1, \ldots, j_k \in [r] \}$$

is a Graver basis of $\ker \mathcal{B}_{C_3,d}$. In particular, $\mathcal{M}$ is a Gröbner basis for any additive preorder.

With the help of Theorems 40 and 30 it is easy to construct a Gröbner basis of $\tilde{C}_4$ when $d = (p, 2, r, q)$.

**Example 41.** Let $d = (p, 2, 3, q)$. A Gröbner basis for $\ker \mathcal{B}_{\tilde{C}_4, d}$ consists of the moves

$$\begin{bmatrix} a_1bce_1 \\ a_2bce_2 \end{bmatrix} - \begin{bmatrix} a_1bce_2 \\ a_2bce_1 \end{bmatrix}, \quad \begin{bmatrix} a_11c_1e_1 \\ a_22c_1e_2 \\ a_12c_2e_4 \end{bmatrix} - \begin{bmatrix} a_1c_1e_1 \\ a_2c_1e_2 \\ a_1c_2e_3 \end{bmatrix}, \quad \begin{bmatrix} a_11c_1e_1 \\ a_22c_1e_2 \\ a_12c_2e_4 \end{bmatrix} - \begin{bmatrix} a_11c_1e_1 \\ a_22c_1e_2 \\ a_12c_2e_4 \end{bmatrix},$$

$$\begin{bmatrix} a_{11}c_1e_1 \\ a_{12}e_2 \\ a_{13}e_3 \end{bmatrix} - \begin{bmatrix} a_{11}c_1e_1 \\ a_{12}e_2 \\ a_{13}e_3 \end{bmatrix}, \quad \begin{bmatrix} a_{11}c_1e_1 \\ a_{12}e_2 \\ a_{13}e_3 \end{bmatrix} - \begin{bmatrix} a_{11}c_1e_1 \\ a_{12}e_2 \\ a_{13}e_3 \end{bmatrix},$$

where $a, a_1, a_2, \ldots, a_6 \in [p], b \in [2], c, c_1, c_2 \in [3]$ and $e, e_1, e_2, \ldots, e_6 \in [q]$. This Gröbner basis works for any choice of preorders $\succeq_X$, $\succeq_B$, $\succeq_{B'}$, $\succeq_A$ that satisfy the compatibility conditions from Section 4.

5.1.2 The projected fibers

Next we describe a PF Gröbner basis, associated with the projection

$$\phi : \mathbb{Z}^{D_{V_1}} \to \mathbb{Z}^{D_{23}}, \quad e_{i_1,i_2,i_3} \to e_{i_2,i_3}.$$
to the missing [23] margin. We first compute a description of the projected fibers \( \phi(\mathcal{B}_{F_{1,d}}) \). We are mostly interested in the case where we can find an inequality description. Then, according to Remark 33, an inequality Gröbner basis for this inequality description will be a PFI Gröbner basis. We assume that \( \mathbb{N}\mathcal{B}_{C_{3,d}} \) is normal (Theorem 39). The facets of \( \mathbb{R}_{\geq} \mathcal{B}_{C_{3,d}} \) are well-studied. In the case \( d = (p, 2, r) \) the result is [10]:

**Proposition 42.** Let \( d = (p, 2, r) \). The cone \( \mathbb{R}_{\geq} \mathcal{B}_{C_{3,d}} \) is the solution to the following system of inequalities:

\[
\begin{align*}
y_{ij}^{12} & \geq 0, \quad y_{ik}^{13} \geq 0, \quad y_{jk}^{23} \geq 0, \\
y_{ij}^{1} - y_{ij}^{12} & \geq 0, \quad y_{j}^{2} - y_{jk}^{23} \geq 0, \quad y_{k}^{3} - y_{ik}^{13} \geq 0, \\
y_{ij}^{1} - y_{ik}^{13} & \geq 0, \quad y_{j}^{2} - y_{ij}^{12} \geq 0, \quad y_{k}^{3} - y_{jk}^{23} \geq 0, \\
y_{ij}^{1} - y_{j}^{1} + y_{ij}^{12} & \geq 0, \quad y_{i}^{0} - y_{ij}^{1} - y_{k}^{3} + y_{ik}^{13} \geq 0, \quad y_{j}^{0} - y_{j}^{2} - y_{k}^{3} + y_{jk}^{23} \geq 0 \\
\sum_{i \in A, k \in B} y_{ik}^{13} + \sum_{i \in A} (y_{ij}^{12} - y_{i}^{1}) + \sum_{k \in B} (y_{jk}^{23} - y_{k}^{3}) - y_{2}^{2} + p^{0} & \geq 0 \\
\sum_{i \in A, k \in B} y_{ik}^{13} - \sum_{i \in A} (y_{ij}^{12} + y_{i}^{1}) - \sum_{k \in B} (y_{jk}^{23} + y_{k}^{3}) + y_{2}^{2} & \geq 0 \\
- \sum_{i \in A, k \in B} y_{ik}^{13} + \sum_{i \in A} (y_{ij}^{12} - y_{i}^{1}) - \sum_{k \in B} (y_{jk}^{23} - y_{k}^{3}) - y_{2}^{2} & \geq 0 \\
- \sum_{i \in A, k \in B} y_{ik}^{13} - \sum_{i \in A} (y_{ij}^{12} - y_{i}^{1}) + \sum_{k \in B} (y_{jk}^{23} - y_{k}^{3}) - y_{2}^{2} & \geq 0.
\end{align*}
\]

Here, \( y_{i}^{F} \) is the coordinate corresponding to the unit vector \( \epsilon_{i}^{F} \) corresponding to the \( F \)-marginal taking the value \( i \).

The projection \( \phi \) onto the [23] marginal amounts to setting all of the variables in the inequalities that appear in the [12][13] model to fixed numbers and looking at the induced inequality system on the other variables. In particular, the only indeterminates that do not appear in [12][13] are the indeterminates \( y_{jk}^{23} \). Using the relations \( y_{23}^{23} = y_{k}^{3} - y_{1k}^{23} \) and \( y_{23}^{23} = y_{2}^{2} - \sum_{k=1}^{r-1} y_{23}^{23} \) we can eliminate all \( y_{23}^{23} \) and \( y_{23}^{23} \) and restrict attention to the indeterminates \( y_{ik}^{13} \) with \( k \in \{1, \ldots, r - 1\} \). The linear forms constraining these coordinates are all of the form \( \sum_{k \in B} y_{1k}^{23} \) for some \( B \subseteq \{1, \ldots, r - 1\} \). Hence we have to solve the following problem:

**Problem 43.** Fix an integer \( t \). For each \( u, l \in \mathbb{Z}^{2[t]} \) let

\[
S(u, l) := \left\{ x \in \mathbb{Z}^{t} : l_{A} \leq \sum_{i \in A} x_{i} \leq u_{A} \text{ for all } A \subseteq [t] \right\}.
\]

We wish to find inequality Markov bases for the collection \( \mathcal{F}_{t} := \{ S(u, l) : u, l \in \mathbb{Z}^{2[t]} \} \).

By Lemma 12 a Markov basis for Problem 43 is also a Gröbner basis with respect to any additive preorder. Note that the inequality system in Problem 43 does not
depend on $p$. Hence, if we solve Problem 43 for some $t$, we have a PF Markov basis of $\Gamma_1 = [12][13]$ for all triples $(p, 2, t + 1)$ for all $p$. The resulting polytopes whose integer points we are trying to connect are called generalized permutahedra [17].

For $t = 1$, the solution is trivial, a Markov basis consists of two moves $\{ \pm 1 \}$. For $t = 2$, by Example 8, the Markov basis consists of six moves $\{ \pm (1, 0), \pm (0, 1), \pm (1, -1) \}$. Note, however, that for the purposes of lifting, we should really consider this as part of the $2 \times (t + 1)$ matrix, whose row and column sums are equal to zero. Hence, we must complete these vectors to $2 \times (t + 1)$ matrices with this property. The Markov basis for $t = 1$ becomes $\pm (1 -1 0)$. For $t = 2$, up to the natural $\mathbb{Z}_2 \times S_3$ symmetry, the inequality Markov basis consists of a single move $(1 -1 0)$.

We computed the Markov bases for various values of $t$ using 4ti2. Table 1 summarizes our results, classifying the elements in the Markov basis up to symmetry. In the table, the moves are already converted into the form of $2 \times (t + 1)$-tables in which we need them later as a PF Markov basis. A Markov basis of $\mathcal{F}_t$ can be obtained by dropping the second row and the last column.

We do not know a general solution to Problem 43 and we think it will be an interesting challenge to try to find a general form for the inequality Markov basis in this case.

### 5.1.3 Lifting the IPF Gröbner basis

Next, supposing that we have solved Problem 43 for $t = r - 1$, we explain how to lift along the map $\phi$.

| $r = t + 1$ | new moves |
|------------|------------|
| 2          | $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ |
| 3          | $\emptyset$ |
| 4          | $\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$ |
| 5          | $\begin{pmatrix} 2 & 1 & -1 & -1 & -1 \\ -2 & -1 & 1 & 1 & 1 \end{pmatrix}$ |
| 6          | $\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 & -1 & -1 \\ -2 & -2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & -2 & -2 & -1 \\ -3 & -1 & -1 & 2 & 2 & 1 \end{pmatrix}$ |

Table 1: The PF Markov bases for various values of $r$ up to symmetry. Each Markov basis contains the moves from the previous rows padded with columns of zeros. For example, the Markov basis of $r = 3$ consists of $(1 -1 0)$, $(1 0 -1)$ and $(0 1 -1)$. |
Proposition 44. Let $b$ be a $2 \times r$ matrix belonging to a PF Gröbner basis of $\mathcal{F}(\mathcal{B}_{\Gamma_1})$. There is a lifting of $b$ along $\phi$ to $(p, 2, r)$-arrays in which the combinatorial types are in bijections with directed acyclic multigraphs with vertex set $[r]$ such that for each vertex $i \in [r]$, outdeg$(i) - \text{indeg}(i) = b'_i$.

The bijection in the proposition is as follows: associate to such a multigraph $G$, and a collection of elements $a_{ij} \in [p]$, one for each edge $i \rightarrow j \in E(G)$, the move

$$\sum_{i \rightarrow j \in E(G)} (e_{a_{ij},1,i} + e_{a_{ij},2,j} - e_{a_{ij},1,j} - e_{a_{ij},2,i}).$$

Proof. If we remove the restriction that $G$ does not contain directed cycles, the set of all such moves produced contains all vectors in $\ker \mathcal{B}_{\Gamma_1,(p,2,r)}$ that project to $b$. If a graph $G = (V, E)$ has a directed cycle $C \subseteq E$, then each corresponding move can be conformally decomposed into a lift of $b$ that corresponds to the directed multigraph $(V, E \setminus C)$ and an element of $\ker \mathcal{B}_{\mathcal{G}_3,(p,2,r)}$ corresponding to the multigraph $(V, C)$. By Lemma 11, moves that possess such a conformal decomposition are redundant.

Example 45. Let $r = 3$. Up to symmetry the PF Gröbner basis contains a single move $b = \left( \begin{smallmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right)$ for which $b' = (1, -1, 0)$. There are two acyclic directed multigraphs that satisfy the prescribed indegree and outdegree conditions, the graph with a single edge $1 \rightarrow 2$ and the graph with two edges $1 \rightarrow 3$ and $3 \rightarrow 2$. The corresponding lifts in this case are, in tableau notation,

$$\begin{bmatrix} a_{1c_1} \\ a_{2c_2} \end{bmatrix} - \begin{bmatrix} a_{1c_2} \\ a_{2c_1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11c_1} \\ a_{12c_3} \\ a_{21c_3} \\ a_{22c_2} \end{bmatrix} - \begin{bmatrix} a_{11c_3} \\ a_{12c_2} \\ a_{21c_2} \\ a_{22c_3} \end{bmatrix}, \quad \text{for } a, a_1, a_2 \in [p], c_1, c_2, c_3 \in [3].$$

Finally, we need to glue the lifts coming from $\Gamma_1 = [12][13]$ and $\Gamma_2 = [24][34]$. Lemma 35 tells us to calculate $\text{Glues}(m, m')$ for all pairs of lifts $m, m'$ of the same element $g$ in the PF Gröbner basis. However, it can happen that some such glues are not actually needed in the resulting Gröbner basis and can be eliminated, as the following example shows.

Example 46. For $r = 3$ consider the glued move

$$\begin{bmatrix} a_{111c_1} \\ a_{123c_1} \\ a_{213c_2} \\ a_{222c_2} \end{bmatrix} - \begin{bmatrix} a_{13c_1} \\ a_{12c_2} \\ a_{23c_2} \end{bmatrix} \in \text{Glues} \left\{ \begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix} - \begin{bmatrix} a_{13} \\ a_{13} \\ a_{22} \end{bmatrix}, \quad \begin{bmatrix} 11c_1 \\ 23c_1 \\ 13c_2 \end{bmatrix} - \begin{bmatrix} 13c_1 \\ 21c_1 \\ 12c_2 \end{bmatrix} \right\}.$$

This move is the conformal decomposition of two degree 2 moves (which are themselves glue moves) namely

$$\begin{bmatrix} a_{111c_1} \\ a_{123c_1} \end{bmatrix} - \begin{bmatrix} a_{13c_1} \\ a_{12c_2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{213c_2} \\ a_{222c_2} \end{bmatrix} - \begin{bmatrix} a_{12c_2} \\ a_{23c_2} \end{bmatrix}.$$

By Lemma 11 the move does not appear in a minimal Gröbner basis.
Figure 5: Gluing two copies of \( \tilde{C}_4 \). The dashed edges are the additional edges of the codimension zero product.

Applying the glue construction to all different pairs and throwing out the bad combination in Example 46 produces the following general result.

\[ \text{Figure 5: Gluing two copies of } \tilde{C}_4. \text{ The dashed edges are the additional edges of the codimension zero product.} \]

**Theorem 47.** Let \( \Gamma = C_4 := [12][13][24][34] \) and \( d = (p, 2, 3, q) \), and let \( \succeq \) be as in Section 4. A \( \succeq \)-Gröbner basis of \( B_{C_4,d} \) consists of the moves from Example 47 (from the associated codimension zero product) together with the necessary glue moves:

\[
\begin{pmatrix}
a_1c_1e \\
a_2c_2e
\end{pmatrix} - \begin{pmatrix}
a_1c_2e \\
a_2c_1e
\end{pmatrix}, \quad
\begin{pmatrix}
a_1c_1e_1 \\
a_2c_2e_2
da_1c_2e_3 \\
a_2c_1e_3
\end{pmatrix} - \begin{pmatrix}
a_1c_2e_1 \\
a_2c_1e_1
da_1c_3e_2 \\
a_2c_3e_2
\end{pmatrix}, \quad
\begin{pmatrix}
a_1c_1e \\
a_2c_2e
\end{pmatrix} - \begin{pmatrix}
a_1c_1e_1 \\
a_2c_2e_1
da_1c_1e_2 \\
a_2c_1e_2
\end{pmatrix}
\]

where \( a, a_1, a_2, a_3 \in [p] \), \( c_1, c_2, c_3 \in \{1, 2, 3\} \), and \( e, e_1, e_2, e_3 \in [q] \).

This example provides us with an explicit instance of the finiteness stabilization of the independent set theorem of [8]. In particular, because of the moves coming from the codimension zero product, we see that the Markov basis stabilizes up to symmetry when \( p = q = 6 \).

The fact that the Gröbner basis in Theorem 47 is square-free implies that the semigroup \( \mathbb{N}B_{C_4} \) is normal for \( d = (p, 2, 3, q) \), see [10] Proposition 13.15. More generally, iterating the argument (see Section 6 below) shows that the semigroup \( \mathbb{N}B_{K_{2,N}} \) of the complete bipartite graph \( K_{2,N} \) is normal for \( d_1 = 2, d_2 = 3 \). As mentioned before, in general, the toric fiber product does not preserve normality [13].

**5.2 Example: \( K_4 \) minus an edge**

In this section, we consider the problem of constructing a Markov basis for the complexes obtained from \( \tilde{C}_4 := [12][13][23][24][34] \) with \( d = (2, 2, 2, 2) \), by gluing multiple copies of \( \tilde{C}_4 \) together along the “missing edge” [14]; see Figure 5. This example serves two purposes: First, it demonstrates how our results can be adjusted in the presence of holes in the projected fibers, if the structure of the holes is nice enough. A more complex example in which the projected fibers have holes (the complete bipartite graph \( K_{3,N} \)) is discussed at the end of Section 4. Second, it illustrates that our procedure does not in general yield a minimal Markov basis. The main focus of this section lies on understanding the projected fibers. Therefore, we do not write out the final Markov basis explicitly, but we give a Markov basis of \( \tilde{C}_4 \) that satisfies the compatible projection property in Proposition 50.

Gluing binary hierarchical models along a missing edge is a codimension one toric fiber product; see [7] for further examples. If the associated codimension zero semigroup
were normal, the Markov basis of $\ker_Z B_{C_4}$ would be slow-varying (Lemma 25) and we could directly apply the results of [11] to construct a Markov basis (Lemma 37).

The associated codimension-zero complex of $\tilde C_4$ is equal to the complete graph $K_4 := [12][13][14][23][24][34]$ (Proposition 38). The Markov basis of a complete graph is empty (since $\ker_Z B_{K_4} = \{0\}$), and so the kernel Markov basis consists of only quadratic moves (Theorem 30).

Next, we compute a PFI Markov basis. The result is the following:

**Lemma 48.** A PFI Markov basis for gluing binary copies of $\tilde C_4$ along the missing edge is given by (in tableau notation and in tensor notation, respectively)

$$ G = \left\{ \begin{bmatrix} 00 \\ 11 \end{bmatrix}, \begin{bmatrix} 00 \\ 01 \end{bmatrix}, \begin{bmatrix} 01 \\ 00 \end{bmatrix}, \begin{bmatrix} 01 \\ 10 \end{bmatrix}, \begin{bmatrix} 10 \\ 01 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} +1 & -1 \end{bmatrix}, \begin{bmatrix} +2 & -2 \end{bmatrix} \right\}. $$

To prove this result, we need to study $\mathbb{N} B_{K_4,d}$. This semigroup is not normal, and so we cannot compute a PF Markov basis directly as an inequality Markov basis. We will prove Lemma 38 after describing the single hole.

**Proposition 49.** With $d = (2, 2, 2, 2)$, the semigroup $\mathbb{N} B_{K_4,d}$ has a single hole $\mathbf{1}$, which has a one in each component (that is, all pair margins are equal to one).

**Proof.** We follow the algorithm of [9]. Direct computation with Normaliz yields that there is exactly one Hilbert basis element of the normalization of $\mathbb{N} B_{K_4,d}$ which is not in $\mathbb{N} B_{K_4,d}$, namely the vector $\mathbf{1}$. That means that $\mathbf{1}$ is the unique fundamental hole of $\mathbb{N} B_{K_4,d}$. Any other hole of $\mathbb{N} B_{K_4,d}$ must be of the form $\mathbf{1} + f$ for some nonzero $f \in \mathbb{N} B_{K_4,d}$. Hence, it suffices to check whether $\mathbf{1} + f \in \mathbb{N} B_{K_4,d}$ for each generator $f$ of $\mathbb{N} B_{K_4,d}$. By symmetry, we can check this for any single generator, say $f = B_{K_4,d} e_{0000}$ equals the first column of $B_{K_4,d}$. In this case $\mathbf{1} + f$ consists of all pair margins equal to $\left( \frac{1}{4} \right)$. The table

$$ v := e_{0001} + e_{0010} + e_{0100} + e_{1000} + e_{1111} $$

has these pair margins (i.e. $\mathbf{1} + f = B_{K_4,d} v$), and so $\mathbf{1} + f$ is not a hole.

**Proof of Lemma 38.** $A$ has codimension one, and $\phi(\ker_Z B_{C_4}) = Zg$ is generated by the single move $g := \left( \begin{array}{c} -1 \\ -1 \\ 1 \end{array} \right)$. Every projected fiber $F$ without hole is of the form $\{u_0 + kg : l \leq k \leq l\}$. Thus, the move $g$ suffices to connect $\phi(F)$.

Since $\mathbf{1}$ is the unique hole of $\mathbb{N} B_{K_4,d}$, there is a single fiber $F(B_{C_4,d}, \mathbf{1})$ that has a hole. The fiber $F(B_{C_4,d}, \mathbf{1})$ consists of the following four tables:

$$ \begin{align*}
\begin{bmatrix}
0000 \\
0001 \\
1011 \\
1101 \\
0111
\end{bmatrix},
\begin{bmatrix}
0001 \\
1010 \\
1100 \\
0111 \\
1010
\end{bmatrix},
\begin{bmatrix}
1000 \\
0011 \\
0101 \\
1110 \\
0101
\end{bmatrix},
\begin{bmatrix}
1001 \\
0010 \\
0100 \\
1111 \\
0100
\end{bmatrix}.
\end{align*} $$

32
The projected fiber consists of the corresponding [14]-marginals:

\[ \phi(F(B_{C_4,d}, 1)) = \left\{ \begin{array}{l}
007 \\
017 \\
117
\end{array} , \begin{array}{l}
001 \\
011 \\
111
\end{array} \right\} = \left\{ \begin{array}{l}
(20) \\
(02)
\end{array} \right\} . \]

Thus, to connect \( \phi(F(B_{C_4,d}, 1)) \), we need the move \( 2g = (\frac{1}{2} \frac{-1}{2}) \).

The set \( \{ g, 2g \} \) also connects all intersections of projected fibers: If the intersection only involves projected fibers without holes, then it is connected by \( g \). Otherwise, the intersection is a subset of the two-element set \( \phi(F(B_{C_4,d}, 1)) \) and thus connected by \( 2g \).

Next, we want to lift the PFI Markov basis. We will not follow the lifting procedure of Section 3.2 in detail, but we will just compute enough lifts to ensure the compatible projection property. First, we compute a Markov basis of \( C_4 \), then we compare it with the lifts. The graph \( C_4 \) was already studied in Section 5.1. As in Example 41, we obtain a Markov basis of \( ker Z B_{C_4,d} \) from Theorems 30 and 40:

\[ \mathcal{M}_1 = \left\{ \begin{array}{l}
[0ab0] - [0ab1] \\
\left[1 0 0 1b\right] \left[0 1 1 0b\right] \left[1 1 1 0d\right] \left[0 0 0 a\right] \left[1 1 1 c\right] \left[0 0 1 d\right]
\end{array} \right\} \]

Under \( \phi \) this Markov basis projects onto the set \( \{0\} \cup \pm G \). In particular, \( \mathcal{M}_1 \) is not slow-varying. One can show that \( \mathcal{M}_1 \) lifts the first element \( g = (\frac{1}{2} \frac{-1}{2}) \) of \( G \) using the algorithm from Section 3.2. However, \( \mathcal{M}_1 \) does not lift the second element \( 2g \). In fact, the lift of \( 2g \) computed according to Section 3.2 contains 75 binomials, among them the elements of \( 2\mathcal{M} \) of degree up to eight. The following result shows that it suffices to work with lifts of degree at most four. As it turns out, these additional lifts are sums of two elements from \( \mathcal{M}_1 \) of degree two.

**Proposition 50.** The set of moves

\[ \mathcal{M} = \mathcal{M}_1 \cup \left\{ \begin{array}{l}
[0ab0] - [0ab1] \\
\left[1 0 0 1b\right] \left[0 1 1 0b\right] \left[1 1 1 0d\right]
\end{array} \right\} ; a, b, c, d \in \{0, 1\} \]

is a Markov basis of \( B_{C_4,d} \) that satisfies the compatible projection property.

**Proof.** Suppose that \( b \neq 1, b' \neq 1 \). The projected fibers \( \phi(F(B_{C_4,d}, b)) \) and \( \phi(F(B_{C_4,d}, b')) \) have no holes. As shown in the proof of Lemma 48, the intersection \( \phi(F(B_{C_4,d}, b)) \cap \phi(F(B_{C_4,d}, b')) \) is connected by \( g \). Since \( \mathcal{M}_1 \) lifts \( g \),

\[ \phi(F(B_{C_4,d}, b), \mathcal{M}_1) \cap \phi(F(B_{C_4,d}, b'), \mathcal{M}_1) = \phi(F(B_{C_4,d}, b)) \cap \phi(F(B_{C_4,d}, b')) \{g\} \]

is also connected. Therefore, the compatible projection property is satisfied for this intersection.
It remains to study intersections of the form \( \phi(F(B_{C_4,d}, 1)) \cap \phi(F(B_{C_4,d}, b')) \) involving the hole. If \( b' = 1 \), then \( \phi(F(B_{C_4,d}, 1), M_1) \cap \phi(F(B_{C_4,d}, b'), M_1) = \phi(F(B_{C_4,d}, 1), M_1) \) is connected, since \( M_1 \) is a Markov basis of \( B_{C_4,d} \). So assume that \( b' \neq 1 \). Furthermore, we may assume that this intersection is non-empty. Then \( F(B_{C_4,d}, b') \) consists of elements with the same [14]-margins as 1 and with total entry sum equal to four. Therefore, any lift of 2g that connects two elements of \( F(B_{C_4,d}, b') \) has degree at most four.

As \( b' \neq 1 \), the fiber \( F(B_{C_4,d}, b') \) has no hole. Suppose that there exist \( v_1, v_2 \in F(B', b') \) such that \( m = v_1 - v_2 \) is one of the degree four moves in \( M_1 \). Since \( v_1 \) and \( v_2 \) are also of degree four, \( v_1 = m^+ \) and \( v_2 = m^- \). One can check that in this case no other move of \( M_1 \) can be applied to \( v_1 \) or \( v_2 \), and so \( F(B_{C_4,d}, b') = \{v_1, v_2\} \). Hence, \( \phi(F(B_{C_4,d}, 1)) \cap \phi(F(B_{C_4,d}, b')) = \emptyset \). Therefore, whenever \( \phi(F(B_{C_4,d}, 1)) \cap \phi(F(B_{C_4,d}, b')) \) is not empty, then \( F(B_{C_4,d}, b') \) is connected by the quadratic moves in \( M_1 \).

The intersection \( \phi(F(B_{C_4,d}, 1)) \cap \phi(F(B_{C_4,d}, b')) \) consists of at most two points. If it consists of one point, then it is connected. Otherwise, if it consists of two points \( u_{-1}, u_{-1} \), then \( u_{-1} - u_{-1} = \pm 2g \), and \( \phi(F(B_{C_4,d}, b')) \) contains \( u_{-1}, u_{-1} \) and \( u_{0} = \frac{1}{2}(u_{-1} + u_{-1}) \). Any path from \( u_{+1} \) to \( u_{-1} \) in \( \phi(F(B_{C_4,d}, b'), M_1) \) passes through \( u_{0} \). To go from \( u_{+1} \) to \( u_{-1} \) directly, it suffices to add to \( M_1 \) all sums of two quadratic moves in \( M_1 \). \( \square \)

Using the results of Section 4, the Markov basis in Proposition 50 can be glued with itself to compute a Markov basis of the solid graph on the right hand side of Figure 5 (together with the result of the computation of the Markov basis of \( B_{K_4,d} \)). In fact, as discussed in Section 4, any number of copies of \( C_4 \) can be glued at the missing edge [14]. The gluing procedure is straightforward, so we do not describe it in detail here.

### 6 Finiteness results for iterated toric fiber products

Forming the Markov basis of the toric fiber product can lead to moves of larger degree than any of the moves in any of the Markov bases that went into the construction. However, we will show that, no matter how many factors are involved in an iterated toric fiber product over the same base \( A \), if the degrees of the PFI Markov basis stabilize and all other Markov bases have bounded degree, then there exists a bound on the degree of the glued moves. To prove this, we need to be precise about what is meant by iterated toric fiber product and stabilization.

The toric fiber product \( B \times_A B' \) is again \( A \)-graded in a natural way, by the map \( \xi \). If \( B'' \) is another \( A \)-graded integer matrix, then

\[
(B \times_A B') \times_A B'' = B \times_A (B' \times_A B'').
\]

In fact, our algorithm easily generalizes to the following related algorithm which is symmetric in the three matrices \( B \), \( B' \) and \( B'' \): Let \( G \) be a Markov basis of the family of sets

\[
\{\phi(F(B, b)) \cap \phi'(F(B', b')) \cap \phi''(F(B'', b'')) : b \in \mathbb{N}B, b' \in \mathbb{N}B', b'' \in \mathbb{N}B''\}
\]

34
Then a Markov basis of $B \times_A B' \times_A B''$ is given by the union of a Markov basis of the associated codimension-zero product $\hat{B} \times_A \hat{B'} \times_A \hat{B''}$ and the set

$$\bigcup_{g \in G} \text{Glues}(\text{Lifts}_\phi(g), \text{Lifts}_{\phi'}(g), \text{Lifts}_{\phi''}(g)),$$

where

$$\text{Glues}(f, g, h) := \text{Glues}(f, \text{Glues}(g, h)) = \text{Glues}(\text{Glues}(f, g), h).$$

Similarly, we can define the toric fiber powers $\times_A^r B$.

For any $v \in \mathbb{Z}^n$ let $\deg(v) := \max\{|v^+|_1, |v^-|_1\}$ (then $\deg(v)$ equals the degree of the binomial $x^{v^+} - x^{v^-}$ corresponding to $v$; see Theorem 3). For $\mathcal{M} \subseteq \mathbb{Z}^n$ let $\deg(\mathcal{M}) := \sup\{\deg(v) : v \in \mathcal{M}\}$. For any family $\mathcal{F}$ of subsets of $\mathbb{Z}^n$ the Markov degree $\text{mardeg}(\mathcal{F})$ is the minimum of $\deg(\mathcal{M})$ where $\mathcal{M}$ ranges over all Markov bases $\mathcal{M}$ of $\mathcal{F}$. For a matrix $B$ we define $\text{mardeg}(B) := \text{mardeg}(\mathcal{F}(B))$. Our key lemma to obtain bounds on the Markov degrees of iterated toric fiber products is the following:

**Lemma 51.** Let $B_1, \ldots, B_r$ be integer matrices with $A$-gradings $\phi_1, \ldots, \phi_r$, and consider lifts $m_1 \in \ker A B_1$, $\ldots$, $m_r \in \ker A B_r$ of the same move $g \in \mathbb{Z}^A$. Then the degree of any glued move $\tilde{m} \in \text{Glues}(m_1, \ldots, m_r)$ is bounded by

$$\deg(\tilde{m}) \leq \deg(g) + \deg \left( \max_{i=1,\ldots,r} (\phi_i(m_i^+) - g^+) \right).$$

Here, $\max_{i=1,\ldots,r}(\phi_i(m_i^+) - g^+)$ is a vector obtained by taking the maximum in each coordinate over all the vectors $\phi_i(m_i^+) - g^+$.

**Proof.** First, let $r = 2$, and let $\tilde{m}$ be a glue of $m$ and $m'$. In the notation of Section 4.3

$$\xi(\tilde{m}^+) = \phi(m^+) + \phi'(m'^+) = \phi(m^+) + v^+. $$

Checking each component, one sees that

$$\phi(m^+) + v^+ = \max \{\phi(m^+), \phi'(m'^+)\},$$

where max denotes the component-wise maximum. Using induction, one sees that

$$\xi^r(\tilde{m}^+) - g^+ = \max_{i=1,\ldots,r} (\phi_i(m_i^+) - g^+),$$

where $\xi^r$ denotes the natural map $B_1 \times_A B_2 \times_A \cdots \times_A B_r \to A$. Since $\xi^r(\tilde{m}) = g$ and since $\xi^r$ preserves the degree of positive vectors, $\deg(\tilde{m}^+) - \deg(g^+) = \deg(\tilde{m}^-) - \deg(g^-)$ and

$$\deg(\tilde{m}) = \max\{\deg(\tilde{m}^+), \deg(\tilde{m}^-)\} = \deg(g) + \deg(\tilde{m}^+) - \deg(g^+)$$

$$= \deg(g) + \deg(\xi^r(\tilde{m}^+)) - \deg(g^+) = \deg(g) + \deg(\xi^r(\tilde{m}^+) - g^+),$$

where the last equality uses that $\xi^r(\tilde{m}^+) \geq g^+$ component-wise. \qed

35
As an example, we apply Lemma 51 to prove the following result:

**Theorem 52.** Let $G$ be a PFI Markov basis of the set of all intersected projected fibers

$$\left\{ \cap_{i=1}^r \phi(F(B, b_i)) : r \in \mathbb{N}, b_i \in \mathbb{N}B \right\}.$$ 

If $G$ is finite, then there is a constant $C > 0$ such that $\text{mardeg}(\times_r A B) \leq C$ for any $r > 0$.

**Proof.** Let $\hat{M}$ be a Markov basis of $B^\phi$. The degree of the Markov basis of the associated codimension-zero toric fiber product is bounded by $\max\{2, \deg(\hat{M})\}$. To prove the statement, it remains to find a bound for the glued moves that is independent of $r$. Such a bound is given by Lemma 51. \qed

The proof of Theorem 52 is constructive in the sense that a Markov basis of the toric fiber powers can be obtained explicitly by following the constructions discussed in this paper. In the same way, a numerical value for the constant $C$ can be computed explicitly. The same remark holds for the other results of this section.

**Corollary 53.** Let $B$ be an $A$-graded integer matrix such that $\mathbb{N}B^\phi$ is normal. Then $\sup_{r \in \mathbb{N}} \text{mardeg}(\times_r A B)$ is finite.

**Proof.** Let $D$ be an integer matrix such that for all $b \in \mathbb{N}B$ there exists $c = c(b)$ such that

$$\phi(F(B, b)) = \{ v \in \mathbb{Z}^A : Dv \geq c \}.$$ 

This implies that for all $b_1, \ldots, b_r \in \mathbb{N}B$,

$$\phi(F(B, b_1)) \cap \cdots \cap \phi(F(B, b_r)) = \{ v \in \mathbb{Z}^A : Dv \geq \max_{i=1}^r c(b_i) \}.$$ 

This shows that an inequality Markov basis of $D$ is a finite PFI Markov basis that works for any toric fiber power $\times_r A B$. Therefore, we can apply Theorem 52. \qed

Lemma 51 can be applied to more general situations. The crucial point is that there needs to be a single finite Markov basis. For example, Corollary 53 holds true if there are only finitely many holes. As further examples, we mention the following result:

**Theorem 54.** Let $B_1, \ldots, B_s$ be $A$-graded matrices such that the semigroups $\mathbb{N}B_1^\phi, \ldots, \mathbb{N}B_s^\phi$ are normal. Then there is a constant $C \in \mathbb{N}$ such that

$$\text{mardeg} \left( \times_{r_1} A B_1 \times_{r_2} A B_2 \times_{r_3} A B_3 \cdots \times_{r_s} A B_s \right) \leq C$$ 

for all $r_1, \ldots, r_s \in \mathbb{N}$.

The same ideas can be applied in the specific situation of hierarchical models, taking advantage of the situations where we know that the semigroup of the associated codimension-zero product is normal. For example:
Corollary 55. Consider the complete bipartite graph $K_{2,N}$ with $2+N$ vertices. For each $k \in \mathbb{N}$, there is a constant $C(k) \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ and $d \in \mathbb{N}^N$ with $d_1 = 2$ and $d_2 = k$, $	ext{mardeg}(B_{K_{2,N},d}) \leq C(k)$.

Proof. $K_{2,N}$ is obtained by gluing $N$ paths of 3 nodes on the pair of end-points of the missing edge. With our conditions on $d$, each such path corresponds to a hierarchical model of $K_{3,1}$ with parameters by Theorem 39. More precisely, by Proposition 42, the projected fibers have an inequality description that is independent of $d$. Therefore, in this situation, there exists a finite inequality Markov basis $G$ (any solution of Problem 43 with $t = d_2 - 1$) that can be used as a PFI Markov basis, independent of $r$ and the choice of $d_3, \ldots, d_{2+N}$.

Proposition 44 gives a combinatorial description of the lifts. In particular, there is a finite number of combinatorial types of lifts. This finite number is independent of $d$. Moreover, if $m$ lifts $g$, then the quantity $\phi(m^+) - g^+$ only depends on the combinatorial type of the lift. Therefore, there is a constant $d^*(g)$ with

$$\deg \left( \max_{m \in \text{Lifts}(g)} (\phi(m^+) - g^+) \right) \leq d^*(g),$$

and this bound is again independent of $d$. By Lemma 51, the degree of any glued move is upper bounded by $\max_{g \in G} \deg(g) + d^*(g)$. Therefore, the statement follows as in the proof of Theorem 52.

Note that results from [8] imply a finiteness result of this type for any fixed $N$. The novelty of Corollary 55 is that a bound holds regardless of $N$.

It is a nontrivial problem to determine the number $C(k)$ from Corollary 55. For $k = 2$, a Markov basis was explicitly calculated in [15], and the result there implies that $C(2) = 4$. Careful reasoning about the lifting procedure for the PF Markov basis that is described in Proposition 44 can be used to produce bounds on $C(k)$ in other instances. For example, it is not difficult to show that $C(3) = 6$. We do not know the growth rate of $C(k)$.

The conditions on $d$ in the statement of Corollary 55 are chosen such that all factors arising in the toric fiber product have normal semigroups. We conjecture that this assumption is not necessary; i.e. we conjecture that there is a function $C(d_1, d_2) \in \mathbb{N}$ such that $\deg(B_{K_{2,N},d}) \leq C(d_1, d_2)$. More generally, we formulate the following conjecture:

Conjecture 56. Let $B_1, \ldots, B_s$ be arbitrary $A$-graded matrices. Then there is a constant $C \in \mathbb{N}$ such that

$$\text{mardeg} \left( \left( \bigotimes_{A} B_1 \right) \times_A \left( \bigotimes_{A} B_2 \right) \times_A \ldots \times_A \left( \bigotimes_{A} B_s \right) \right) \leq C$$

for all $r_1, \ldots, r_s \in \mathbb{N}$.

As an example we bound the Markov basis of the complete bipartite graph $K_{3,N}$ with binary nodes. $K_{3,N}$ can be obtained by gluing $N$ three-stars; see Figure 6(a). For brevity, we just summarize the main results here and refer to [18] for the details.
Figure 6: a) Gluing three three-stars to obtain $K_{3,3}$. b) The associated codimension-zero factor $K_4$.

The Markov basis of the associated codimension-zero product arises by lifting moves from the Markov basis of the codimension-zero factor $K_4$; see Figure 6(b). The Markov basis of $K_4$ has 20 elements of degrees four and six. As shown in [18, Section 5], the moves that arise by gluing these elements are redundant in view of the moves that arise by lifting the PFI Markov basis.

To understand the projected fibers, we need to understand the semigroup of $K_4$. Following the algorithm from [6] we find the following: This semigroup is not normal. Even worse, it has infinitely many holes. Fortunately, within the projected fibers the holes are vertices. Therefore, the holes can be separated from their projected fibers by linear inequalities. To be precise, there are two linear forms $l_1, l_2$ with the following property: If $h$ is a hole of a projected fiber $\phi(F)$, then $l_i(h) < \min\{l_i(u) : u \in \phi(F)\}$ for some $i$. This allows to give an inequality description of the projected fibers. The corresponding inequality Markov basis can be used as a PFI Markov basis.

The inequality Markov basis consists of 16 moves of degrees in four symmetry classes, two of degree two and two of degree four. As shown in [18, Section 4], lifting increases the degree by two. Using Lemma 51 one can see that gluing different lifts of the same move leads to moves of degree at most 12. In fact, for any lift $m$ of $g$, the tableau $\phi(m^+) - g^+$ is square-free, that is, it does not contain two identical rows. Therefore, for any glued lift $\tilde{m}$ of $g$, the tableau $\phi(\tilde{m}^+) - g^+$ will also be square free, and hence of degree at most 8. Therefore, $\deg(\tilde{m}) \leq \deg(g) + 8 \leq 12$. In total, we obtain the following result:

**Theorem 57.** For any $N$, the Markov degree of the binary hierarchical model of the complete bipartite graph $K_{3,N}$ is at most 12.

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