Nonlinear Matter Spectra in Growing Neutrino Quintessence

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Abstract

We investigate the nonlinear power spectra of density perturbations and acoustic oscillations in growing neutrino quintessence. In this scenario, the neutrino mass has a strong dependence on the quintessence field. The induced coupling stops the evolution of the field when the neutrinos become nonrelativistic, and triggers the transition to the accelerating phase of the cosmological expansion. At redshifts around five, the neutrino fluctuations are still linear and acoustic oscillations are present in the neutrino power spectrum, induced by the acoustic oscillations in the baryonic and dark-matter sectors. The neutrino perturbations become nonlinear at redshifts around three. The mode coupling generated by the nonlinearities erases the oscillations in the neutrino spectrum at some redshift above two. There is a potential danger that at later times the influence of the gravitational potentials induced by the neutrino inhomogeneities could erase the oscillations from the baryonic and dark-matter spectra, making the scenario incompatible with observations. For the scenario to be viable, the neutrino-induced gravitational potentials in the range of baryonic acoustic oscillations should not grow to average values much larger than $10^{-4}$. The magnitude of the expected potentials is still not known reliably, as the process of structure formation is poorly understood in growing neutrino quintessence.

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1 Introduction

A popular extension of the quintessence scenario [1] assumes the presence of a coupling between the dark-energy and dark-matter sectors [2]. This assumption provides an extended framework in which one may hope to address the coincidence problem, i.e. the reason behind the comparable present contributions from dark matter and dark energy to the total energy density. In a variation of this scenario, characterized as growing neutrino quintessence, the interaction with dark energy is shifted from the dark matter to the cosmological neutrino sector [3]. The neutrino-dark energy coupling $\beta$ can be large, and generate a force substantially stronger than the standard gravitational interaction \cite{4}. As a result, even if the neutrinos contribute only a small fraction to the total energy density, they may have a significant effect on the cosmological evolution \cite{3}. Their effect becomes important when the neutrino mass stops being negligible and the neutrinos become nonrelativistic. This happens at a redshift $z_{nr} \sim 5 - 10$, with the exact value depending on the particular model. At lower redshifts, the presence of the neutrinos forces the quintessence field to stop evolving, so that its potential acts as an effective cosmological constant, whose value is related to the present neutrino mass.

The presence of an additional force in the neutrino sector, which is stronger than gravity by a factor $\sim \beta^2$, has profound implications for the evolution of cosmological neutrino perturbations. As soon as neutrinos become nonrelativistic ($z < z_{nr}$), the perturbations in their energy density start to grow, with a characteristic timescale that is shorter by a factor $\sim \beta^{-2}$ relative to the one characterizing standard gravitational collapse. Static solutions of Einstein’s equations are known, which describe neutrino lumps held together by the force mediated by the quintessence field. Such configurations may be the endpoint of the collapse process \cite{5}. The analysis of the growth of neutrino perturbations beyond linear level is complicated and the final stages are not well understood \cite{5}. The gravitational potential induced by large-scale structures in the neutrino sector, which can have a size of 100 Mpc or more, can affect the propagation of photons and influence the cosmic microwave background (CMB) through the integrated Sachs-Wolfe (ISW) effect. An analysis based on linear perturbation theory leads to the conclusion that models with a strong neutrino-dark energy coupling are excluded \cite{7}. However, an extrapolation of linear growth is strongly misleading, as it would imply completely unrealistic neutrino-induced cosmological gravitational potentials \cite{8}. In particular, the effects of backreaction and virialization are not accounted for in a linear treatment. If these are sufficiently strong, the neutrino structures may not be as dense as the extrapolation of the linear analysis would indicate. The resulting gravitational potentials may be sufficiently small so as not to affect the CMB significantly \cite{8}. As a general rule of thumb, potentials larger than $\sim 10^{-5}$ at length scales of 100 Mpc or more give too strong an effect on the CMB.

In this work we study the power spectra of dark-matter and neutrino perturbations in the scenario of growing neutrino quintessence. The growth of perturbations can be used in order to constrain the scenario through comparison with the observed large-scale structure. The most prominent feature of the baryonic and dark-matter spectrum is a series of peaks and valleys, characterized as baryon acoustic oscillations (BAO). They originate in the period of recombination, and correspond to sound waves in the relativistic plasma of that epoch. The characteristic length scale of BAO is around 100 Mpc. Even in the standard $\Lambda$CDM scenario, the exact form of the dark-matter power spectrum at such scales is not easy to compute precisely, because of the failure of linear perturbation theory to describe reliably the growth of the corresponding fluctuations under gravitational collapse. At length scales below about 10 Mpc, the evolution is highly non-linear, so that only numerical N-body simulations can capture the dynamics of the formation of galaxies and clusters of galaxies. However, fluctuations with length scales of around 100 Mpc fall within the mildly non-linear regime, for which analytical methods have been developed. We focus on scales in the range 50–200 Mpc, within which BAO are visible. In growing neutrino quintessence the neutrino power spectrum displays a much faster growth and overtakes the dark-matter spectrum at redshifts below $z \sim 4$. The nonlinear corrections become very large, so that analytical methods become unreliable even in the BAO range at redshifts near $z = 0$. Our aim is to explore the range of validity of the analytical methods and investigate the form of the spectra within this range.

The various analytical methods \cite{9}–\cite{15} that have been developed in order to go beyond linear perturbation theory essentially amount to resumptions of subsets of perturbative diagrams of arbitrarily high order, in a way analogous to the renormalization group (RG). We follow the approach of \cite{13}, named time-RG or TRG, which uses time as the flow parameter that describes the evolution of physical quantities, such as the spectra. The method has been applied to $\Lambda$CDM and quintessence.
cosmologies [19], allowing for a possible coupling of dark energy to dark matter [17], or a variable equation of state [18]. It has also been used for the study of models with neutrinos of constant mass [19]. A comparative analysis of several analytical methods, using N-body simulations as a reference, has been carried out in ref. [20] for ΛCDM cosmology. The study demonstrates that TRG remains accurate at the 1-2% level over the whole BAO range at all redshifts.

In the following section we summarize the formalism we use. In section 3 we present the results of the numerical integration of the evolution equations for the spectra. Finally, in section 4 we give our conclusions.

2 Neutrinos coupled to dark energy

2.1 Evolution equations for the perturbations

We assume that the energy density of the Universe receives significant contributions from three components: a) nonrelativistic matter, in which we group standard baryonic matter (BM) and cold dark matter (CDM); b) massive neutrinos, whose equation of state varies during the cosmological evolution; c) a slowly varying, classical scalar field \( \phi \), named cosmon [1], whose contribution to the energy density is characterized as dark energy (DE). We also allow for a direct coupling between the neutrinos and the cosmon field. In particular, we assume that the mass \( m_\nu \) of the neutrinos has a dependence on \( \phi \), and define the coupling as \( \beta(\phi) = -d \ln m_\nu(\phi)/d\phi \). In the present paper we consider a model with constant \( \beta \). The equation of motion of the cosmon field takes the form

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu
u} \frac{\partial \phi}{\partial x^\nu} \right) = -\frac{dU}{d\phi} + \beta(\phi) \left( T_\nu \right)_\mu. \tag{2.1}
\]

At early times, when the neutrinos are relativistic and their energy-momentum tensor is traceless, they are effectively decoupled from the cosmon field. Only when the neutrinos become nonrelativistic the cosmon-neutrino coupling \( \beta \) becomes effective, leading to energy exchange between the neutrino and dark-energy sectors. We normalize all dimensionful quantities, such as the cosmon field, with respect to the reduced Planck mass \( M = (8\pi G)^{-1/2} \). This is equivalent to setting \( M = 1 \).

We are interested in the more recent stages of the cosmological evolution (redshifts \( z \lesssim 10 \)). In the scenario we consider, the neutrinos become nonrelativistic at such redshifts. In the remaining of the section, in which we develop the formalism for the treatment of the nonlinear corrections to the power spectra, we assume that the neutrino pressure vanishes. We approximate the metric as

\[
d\tau^2 = a^2(\tau) \left[ (1 + 2\Phi(\tau, \vec{x})) d\tau^2 - (1 - 2\Phi(\tau, \vec{x}) \right] d\vec{x} d\vec{x}. \tag{2.2}
\]

We assume that the Newtonian potential \( \Phi \) is weak: \( \Phi \ll 1 \). Also we decompose the cosmon field \( \phi \) as \( \phi(\tau, \vec{x}) = \hat{\phi}(\tau) + \delta\phi(\tau, \vec{x}) \), with \( \delta\phi \ll 1 \). In general, \( \hat{\phi} = O(1) \) in units of \( M \). Finally, we decompose the density as \( \rho(\tau, \vec{x}) = \bar{\rho}(\tau) + \delta\rho(\tau, \vec{x}) \), while we neglect the pressure. A self-consistent expansion scheme can be obtained if we assume the hierarchy of scales: \( \Phi, \delta\phi \ll |\delta\vec{v}| \ll \delta\rho/\bar{\rho} \lesssim 1 \). Such a hierarchy is predicted by the linear analysis for subhorizon perturbations with momenta \( k \gg H = \dot{a}/a \). For subhorizon perturbations, it is consistent to make the additional assumption that the spatial derivatives of \( \Phi, \delta\phi \) dominate over the time derivatives. The predictions of the linear analysis allow us to make a more quantitative statement. We assume that a spatial derivative acting on \( \Phi, \delta\phi \) or \( \delta\vec{v} \) increases the position of that quantity in the hierarchy by one level. In this sense \( \nabla^2 \Phi \) is comparable to \( H\delta\vec{v} \), while \( \nabla^2 \Phi \) is comparable to \( H^2 \delta\rho/\bar{\rho} \).

With the above assumptions, one can derive the equations that describe the evolution of the Universe. The details of the calculation are given in [17]. The evolution of the homogeneous background is described by

\[
H^2 = \frac{1}{3} \left[ a^2 \sum_{i=1,2} \rho_i + \frac{1}{2} \dot{\phi}^2 + a^2 U(\bar{\phi}) \right] \equiv \frac{1}{3} a^2 \rho_{\text{tot}} \tag{2.3a}
\]

\[
\ddot{\phi} + 3H\dot{\phi} = -\beta_i \dot{\rho}_i \tag{2.3b}
\]

\[
\ddot{\rho}_i + 3H\dot{\rho}_i = -\beta_i \dot{\rho}_i \tag{2.3c}
\]

3
where we have defined \( \rho_{\text{tot}} \equiv \sum_i \bar{\rho}_i + \hat{\phi}^2/(2a^2) + U(\hat{\phi}) \). The values \( i = 1, 2 \) correspond to neutrinos and CDM+BM, respectively. For the neutrinos we consider a constant coupling \( \beta_1 = \beta \), while we assume that the CDM+BM sector does not couple to the cosmic field: \( \beta_2 = 0 \).

We describe the perturbations in terms of the comnon perturbation \( \delta \phi \), the gravitational potential \( \Phi \), the density perturbations \( \delta \rho_i \) and the velocity fields \( v_i \). We have two Poisson equations

\[
\nabla^2 \delta \phi = - a^2 \sum_i \beta_i \delta \rho_i \equiv -3 \sum_i \beta_i H^2 \Omega_i \delta_i, \quad (2.4a)
\]

\[
\nabla^2 \Phi - \frac{1}{2} a^2 \sum_i \delta \rho_i \equiv \frac{3}{2} H^2 \sum_i \Omega_i \delta_i, \quad (2.4b)
\]

with \( \Omega_i(\tau) \equiv a^3/(3H^2) \), and the continuity and Euler equations

\[
\dot{\rho}_i + 3H\delta \rho_i + \nabla [(\rho_i + \delta \rho_i)\delta \nu_i] = - \beta_i \delta \phi \delta \rho_i, \quad (2.5a)
\]

\[
\dot{\delta} \nu_i + (H - \beta_i \delta \phi) \delta \nu_i + (\delta \nu_i \cdot \nabla) \delta \nu_i = - \nabla \Phi + \beta_i \delta \phi. \quad (2.5b)
\]

The inspection of eqs. (2.4a), (2.4b) reveals a potential shortcoming of the assumed hierarchy for large \( \beta \). From the Euler equation one concludes that the scalar quantity comparable to the gravitational potential is actually \( \beta \delta \phi \). From eqs. (2.4a), (2.4b) we can see that \( \beta \delta \phi \) is typically larger than \( \Phi \) by a factor \( \sim \beta^2 \). This limits the range of validity of a hierarchy in which \( \beta \delta \phi \) and \( \Phi \) are treated on equal footing. Furthermore, eqs. (2.4a), (2.4b) implicitly neglect effects arising from a difference between the local neutrino mass, as given by the local value of the cosmon field, and the average cosmological neutrino mass, as given by the background value \( \bar{\phi} \). Including higher orders in \( \beta \delta \phi \) would increase considerably the algebraic complexity of the evolution equations.

### 2.2 Evolution equations for the power spectra

The evolution equations are expressed in their most useful form in terms of the density contrasts \( \delta_i \equiv \delta \rho_i/\bar{\rho}_i \ll 1 \) and the divergence of the velocity field \( \theta_i(k, \tau) \equiv \nabla \cdot \delta \nu_i(k, \tau) \). For the Fourier transformed quantities, we obtain from eq. (2.5a)

\[
\delta_i(k, \tau) + \theta_i(k, \tau) + \int d^3k_1 d^3k_2 \delta D(k - k_1 - k_2) \tilde{\alpha}(k_1, k_2) \theta_i(k_1, \tau) \delta_i(k_2, \tau) = 0, \quad (2.6)
\]

where

\[
\tilde{\alpha}(k_1, k_2) = \frac{k_1 \cdot (k_1 + k_2)}{k_1^2 k_2^2}. \quad (2.7)
\]

Eqs. (2.5b), (2.4a), (2.4b) give

\[
\begin{align*}
\dot{\theta}_i(k, \tau) + (H - \beta_i \delta \phi) \theta_i(k, \tau) &+ \frac{3H^2}{2} \sum_j \Omega_j (2\beta_j + 1) \delta_j(k, \tau) \\
&+ \int d^3k_1 d^3k_2 \delta D(k - k_1 - k_2) \tilde{\beta}(k_1, k_2) \theta_i(k_1, \tau) \theta_i(k_2, \tau) = 0,
\end{align*} \quad (2.8)
\]

where

\[
\tilde{\beta}(k_1, k_2) = \frac{(k_1 + k_2)^2 k_1 \cdot k_2}{2k_1^2 k_2^2}. \quad (2.9)
\]

We define the quadruplet

\[
\begin{pmatrix}
\varphi_1(k, \eta) \\
\varphi_2(k, \eta) \\
\varphi_3(k, \eta) \\
\varphi_4(k, \eta)
\end{pmatrix} = e^{-\eta} \begin{pmatrix}
\delta_c(k, \eta) \\
\theta_c(k, \eta) \\
\theta_m(k, \eta) \\
\delta_m(k, \eta)
\end{pmatrix}, \quad (2.10)
\]
where $\eta = \ln(a(t)/a_0)$, with $a_0$ the scale factor at some convenient time, at which we define the initial conditions. (We normalize the scale factor so that $a_0 = 1$ today.) The index $\nu$ refers to neutrinos and $m$ to the CDM+BM sector. This allows us to bring eqs. (2.12), (2.13) in the form [2] [12] [13]

$$\partial_\eta \varphi_\nu(k, \eta) + \Omega_{ab\nu}\varphi_\nu(k, \eta) = e^\eta \gamma_{abc}(k, -k_1, -k_2)\varphi_b(k_1, \eta)\varphi_c(k_2, \eta).$$

(2.11)

The indices $a, b, c$ take values $1, \ldots, 4$. The values 1, 2 characterize neutrino density and velocity perturbations, while 3, 4 refer to CDM+BM quantities. Repeated momenta are integrated over, while repeated indices are summed over. The functions $\gamma_\nu$ that determine effective vertices, are analogous to those employed in [12] [13]. The non-zero components are

$$\begin{align*}
\gamma_{121}(k, k_1, k_2) &= \frac{\tilde{a}(k, k_2)}{2}\delta_D(k + k_1 + k_2) = \gamma_{112}(k, k_2, k_1) \\
\gamma_{222}(k, k_1, k_2) &= \frac{\tilde{b}(k, k_2)}{2}\delta_D(k + k_1 + k_2) \\
\gamma_{341}(k, k_3, k_4) &= \frac{\tilde{a}(k, k_3)}{2}\delta_D(k + k_3 + k_4) = \gamma_{334}(k, k_3, k_4) \\
\gamma_{444}(k, k_3, k_4) &= \tilde{b}(k, k_3, k_4)\delta_D(k + k_3 + k_4).
\end{align*}$$

(2.12)

The $\Omega$-matrix is

$$\Omega(\eta) = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-\frac{3}{2}\Omega_{\nu}(2\beta^2 + 1) & 2 - \beta\phi' + \frac{H'}{H} & -\frac{3}{2}\Omega_m & 0 \\
0 & 0 & 1 & -1 \\
-\frac{3}{2}\Omega_{\nu} & 0 & -\frac{3}{2}\Omega_m & 2 + \frac{H'}{H}
\end{pmatrix},$$

(2.13)

where a prime denotes a derivative with respect to $\eta$.

The next step is to derive evolution equations for the power spectra. The spectra and bispectra are defined as

$$\begin{align*}
\langle \varphi_\nu(k, \eta)\varphi_\nu(q, \eta) \rangle &\equiv \delta_D(k + q)P_{ab}(k, \eta) \\
\langle \varphi_\nu(k, \eta)\varphi_\nu(q, \eta)\varphi_\nu(p, \eta) \rangle &\equiv \delta_D(k + q + p)B_{abc}(k, q, p, \eta).
\end{align*}$$

(2.14)

The essential approximation that we have to make in order to obtain a closed system of equations is a truncation of the four-point function which appears in the evolution equation for the trispectrum. In this way we obtain [13] [17]

$$\begin{align*}
\partial_\eta P_{ab}(k, \eta) &= -\Omega_{ac}P_{cb}(k, \eta) - \Omega_{bc}P_{ac}(k, \eta) \\
+ e^\eta \int d^3q[\gamma_{ac}(k, -q, q - k)P_{bd}(k, -q, q - k) \\
+ \gamma_{bd}(k, -q, q - k)P_{ac}(k, -q, q - k) \\
+ \gamma_{cd}(q - k, k, -q)P_{ac}(k, \eta)P_{bd}(q, \eta)],
\end{align*}$$

(2.15)

$$\begin{align*}
\partial_\eta B_{abc}(k, -q, q - k) &= -\Omega_{ad}B_{dce}(k, -q, q - k) - \Omega_{bd}B_{ace}(k, -q, q - k) - \Omega_{cd}B_{ade}(k, -q, q - k) \\
+ 2e^\eta \int d^3q[\gamma_{ade}(k, -q, q - k)P_{bde}(q, \eta)P_{ac}(k, \eta) \\
+ \gamma_{bde}(-q, q - k, k)P_{ade}(k, -q, \eta)P_{ac}(k, \eta) \\
+ \gamma_{cde}(q - k, k, -q)P_{ade}(k, \eta)P_{bde}(q, \eta)].
\end{align*}$$

(2.16)

3 Numerical analysis

3.1 Approximations

The presence of two massive species (neutrinos and CDM+BM) complicates the structure of the equations compared to the case where they are treated as a single fluid, discussed in [13]. The full
system of eqs. (2.15), (2.16) contains 74 equations, namely, 10 for the power spectra and 64 for the bispectra, compared to the 11 equations of the single-matter case [13]. An accurate calculation also requires the discretization of the $k$-space with at least 500 points. Taking into account that the bispectra depend on three external momenta, it is apparent that the necessary computing power is significant. In order to make the numerical integration of the evolution equations feasible, we have to make additional approximations.

As will be apparent in the following, the method we are employing remains valid only within the mildly nonlinear regime. Strong nonlinearities and backreaction effects, as well as the process of virialization, are not accounted for in our treatment. In the scenario we are considering, the strongest nonlinearities appear in the neutrino sector, while the CDM+BM spectrum remains within the linear regime. It is, therefore, a good approximation to neglect the nonlinear terms in the evolution equations for the CDM+BM spectrum. In practice, this means that we may set the vertices $\gamma_{334}, \gamma_{343}, \gamma_{444}$, defined in eq. (2.12), equal to zero.

We derive an approximate solution of eqs. (2.15), (2.16) in the following way:

- At the first stage, we integrate the full system of equations setting all the vertices $\gamma_{abc}$ equal to zero. This reproduces the linear spectra.

- Subsequently, we integrate the 11 equations for $P_{ab}, B_{abc}$ with $a, b = 1, 2$. Notice that these involve only the vertices $\gamma_{112}, \gamma_{121}, \gamma_{222}$. All spectra and bispectra appearing in these equations with an index 3 or 4, because of the nonzero entries $\Omega_{23}$ and $\Omega_{41}$ entries of eq. (2.14), are approximated by their linear solutions derived at the previous stage.

The above procedure can reproduce the modifications in the neutrino spectrum induced by mode coupling at the nonlinear level. Our main result will be the effect of this process on the BAO in the spectrum. Our analysis is performed at redshifts above $z \sim 2$, at which the only significant nonlinearities appear in the neutrino sector.

### 3.2 Results

We consider a model in which the cosmon field has a potential $V(\phi) = M^4 \exp(-\alpha \phi)$, with $\alpha = 10$. As we mentioned earlier, we normalize dimensionful quantities with respect to the Planck mass $M = 1$. The coupling between the neutrinos and the cosmon field is taken to be constant. This is equivalent
Figure 2: The neutrino density power spectrum $P_{11}(k, \eta)$ (solid lines) and the CDM+BM density spectrum $P_{33}(k, \eta)$ (dotted lines) at redshifts $z = 4.70, 4.08, 3.04, 2.77, 2.69, 2.60$ (starting from below).

to assuming that the neutrino mass is $m_\nu(\phi) = \tilde{m}_\nu \exp(-\beta \phi)$. In our calculation we consider three
degenerate neutrinos and use $\beta = -52$. The constant $\tilde{m}_\nu$ and the present value of the background field
$\phi$ are chosen such that the present neutrino mass is $m_\nu = 2.1$ eV. The evolution of the cosmological
background for $z < z_{nr} = 8$ is shown in fig. 1. We have approximated the neutrinos as nonrelativistic
during this period. We depict the fractional energy density in neutrinos (solid), CDM+BM (dashed)
and dark energy (dotted). We also plot the acceleration parameter $q = a\ddot{a}/\dot{a}^2$ (dot-dashed). The
cosmological expansion becomes accelerating at a redshift $z \approx 1$. The present fractional contributions
of neutrinos, CDM+BM and the cosmon field to the total energy density are $\Omega_\nu = 0.13, \Omega_m = 0.23,
\Omega_\phi = 0.64$, respectively. The spatial curvature is assumed to vanish, while the current expansion
rate is $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$.

We start the integration of the evolution equations for the spectra at a redshift $z_{nr} = 8$. The
neutrinos are already nonrelativistic at this time in our model. The initial conditions at $z_{nr}$ are
obtained through the implementation of the background and linear-perturbation equations in the
code CAMB [21], generalized for the case of mass-varying neutrinos. We assume that the primordial
spectrum is scale invariant with spectral index close to 1. In the background evolution, we take into
account the transition of neutrinos from being relativistic at high redshifts to nonrelativistic near $z_{nr}$.
We point out that the neutrinos are essentially decoupled from the cosmon field during the time that
they are relativistic (see eq. (2.1)). We perform a careful analysis of the evolution of the neutrino
perturbations by solving the Boltzmann equation. In this way we obtain the neutrino and matter
spectra at $z_{nr}$. The mixed spectra are obtained as the geometrical averages of the pure ones, i.e.

$$P_{12}(k, \eta_{nr}) \simeq \sqrt{P_{11}(k, \eta_{nr})P_{22}(k, \eta_{nr})}, \quad P_{13}(k, \eta_{nr}) \simeq \sqrt{P_{11}(k, \eta_{nr})P_{33}(k, \eta_{nr})},$$

(3.1)

eq (3.1)

etc, consistently with the expectation in the linear regime.

The evolution of the spectra at redshifts below $z_{nr}$ is linear to a good approximation down to
$z \sim 3$. Around this time the neutrino spectrum has grown sufficiently for the nonlinear corrections
to affect the evolution. This is demonstrated in fig. 2 where we depict the neutrino density power
spectrum $P_{11}(k, \eta)$ (solid lines) and the CDM+BM density spectrum $P_{33}(k, \eta)$ (dotted lines) at various
redshifts. The solid lines, starting from below, correspond to $z = 4.70, 4.08, 3.04, 2.77, 2.69, 2.60$.
The dotted lines correspond to the same redshifts, but they almost coincide. We point out that the
actual density spectra are obtained from those depicted in fig. 2 by multiplication with a factor
$\exp(2\eta) = (a(\tau)/a_i)^2 = 81(1 + z)^{-2}$, where we have taken $z_i = z_{nr} = 8$ in our calculation.
This factor arises because of the definitions (2.10), (2.14), while $\eta = \ln[a(\tau)/a_i]$, with $a_i$ the scale factor
at some convenient time, at which we define the initial conditions. In our calculation $a_i = 1/9$. The
spectra are given in units of \((\text{Mpc}/h)^3\), consistently with their definition (2.14). They are also smaller by a factor \((2\pi)^3\) compared to the definition in CAMB.

At high redshifts the neutrino density spectrum is very suppressed because of free-streaming, as the neutrinos are relativistic over the entire depicted momentum range. It is apparent from fig. 2 that, as soon as the neutrinos become nonrelativistic, their spectrum grows very fast because of the additional attractive force generated by the neutrino-dark energy coupling. This force is \(2\beta^2 \approx 5400\) times stronger than gravity. The neutrino density spectrum at \(z = 4.70, 4.08, 3.04\) is within the linear regime, while the nonlinearities become apparent at \(z = 2.77, 2.69, 2.60\): starting from large \(k\), the neutrino spectrum grows faster than the linear analysis would predict. The matter spectrum retains its shape, as its evolution is described by linear theory to a very good approximation for the whole range of redshifts depicted in fig. 2.

The most interesting feature of the spectra in fig. 2 is their oscillatory behavior. The BAO in the matter sector originate in the period of recombination, and correspond to sound waves in the relativistic plasma of that epoch. The corresponding oscillations in the neutrino spectrum develop as soon as the neutrinos become nonrelativistic and start falling in the gravitational potentials generated by the CDM+BM inhomogeneities. The corresponding process takes place within the linear regime and is fast, as the CDM+BM power spectrum exceeds that of neutrinos by several orders of magnitude. The amplitude of oscillations in the neutrino spectrum starts diminishing as soon as the nonlinear corrections become important. Again the transition is fast, and the oscillations are hardly visible at \(z = 2.60\). This phenomenon has its origin in the mode coupling induced by the nonlinearities. It is observed within the \(\Lambda\)CDM model as well: near \(z = 0\) the higher peaks in the matter spectrum disappear and only the first two or three survive when the nonlinear corrections are taken into account. In the model at hand, the nonlinearities in the neutrino sector are very strong, because of the rapid growth of the spectrum. The oscillations disappear at a redshift above \(z \sim 2\).

A crucial question is whether the disappearance of oscillations from the neutrino sector induces their elimination from the CDM+BM sector as well. We address this issue in the following section. An important quantity in the context of this discussion is the gravitational potential induced by the inhomogeneities. We can have an estimate of its magnitude at various scales by starting from the Poisson equation (2.15) for the gravitational potential and defining an associated power spectrum as

\[
\Delta^2_\Phi(k, \eta) = 4\pi k^3 \rho_m \left( \frac{3}{2} \frac{H^2}{k^2} \right)^2 \left( \Omega^2_\nu P_{11} + 2\Omega_\nu \Omega_m P_{12} + \Omega^2_m P_{22} \right).
\]

In fig. 3 we plot \(\Delta_\Phi\) as a function of \(k\) for redshifts \(z = 4.70, 4.08, 3.04, 2.77, 2.69, 2.60\). At \(z = 4.70\) the potential is mainly generated by the CDM+BM inhomogeneities, while for \(z = 2.60\)
the main contribution at large $k$ comes from the neutrino inhomogeneities. This is apparent in the change of shape of the function $\Delta_\Phi(k)$. Already at this stage the potential takes values in the region $[10^{-6} - 10^{-5}]$ in the BAO range. At $z \lesssim 2.6$ the neutrinos dominate the gravitational potential, so that its magnitude depends on the growth of inhomogeneities in this sector. We point out that $k$ is the comoving momentum, so that the physical length scale is related to $k/a(\eta)$.

The form of the spectra at redshifts below $z \simeq 2.6$ cannot be obtained within the scheme we are employing, as the validity of TRG lies within the mildly nonlinear regime. At its present development, the scheme cannot account for the process of virialization and formation of bound structures. As a result, inhomogeneities tend to grow without limit, and the power spectrum diverges. The integration of eqs. (2.15), (2.16) below $z = 2.6$ results in an increase of the spectrum much faster than the one predicted by the linear treatment.

4 Conclusions

The numerical analysis of the previous section has led to some concrete conclusions:

- The neutrino power spectrum is subdominant to the CDM+BM spectrum at high redshifts because of free-streaming during the period that neutrinos are relativistic. The neutrino mass grows during the cosmological evolution of the cosmon field. As soon as the neutrinos become nonrelativistic ($z \simeq 8$ in our model), their spectrum grows rapidly and overtakes the CDM+BM spectrum.

- During the period that the neutrinos are nonrelativistic and the evolution linear ($8 \gtrsim z \gtrsim 3$), the neutrino spectrum develops oscillations induced by the BAO in the CDM+BM sector. The oscillations in the neutrino sector are erased by the mode coupling generated by the nonlinearities when these become significant ($3 \gtrsim z \gtrsim 2.6$).

- At high redshifts ($z \gtrsim 3$) the gravitational potentials in the BAO range are dominated by CDM+BM inhomogeneities. At lower redshifts, the neutrino inhomogeneities start dominating and they determine the potentials. At the time when the perturbations in the neutrino sector become highly nonlinear ($z \simeq 2.6$ in our model), the potentials take values in the region $[10^{-6} - 10^{-5}]$ in the BAO range.

The issue that cannot be resolved by our analysis is whether the BAO persist in the CDM+BM sector at redshifts below the one at which the neutrino sector becomes highly nonlinear. An extrapolation of our results below $z = 2.6$ within a linear treatment would indicate that the potentials induced by the neutrino inhomogeneities grow rapidly extremely large, so that their influence would erase the BAO from the CDM+BM sector already at redshifts above $z = 2$. Such a conclusion would make growing neutrino quintessence incompatible with observations. However, the linear increase is strongly misleading, as the neutrino-induced gravitational potential would exceed the extreme value resulting from all neutrinos within the horizon being concentrated at a single point [3]. Unfortunately, our present approximation cannot account for a slowing down of the growth as compared to the linear approximation.

Nevertheless, an important lesson can be learned from a linear extrapolation. One finds that, at the time when the oscillatory behavior disappears from the CDM+BM density spectrum, the gravitational potentials in the BAO range take values in the region $[10^{-3} - 10^{-2}]$. A significant reduction of the amplitude of oscillations, at the 10–20% level, corresponds to potentials in the region $[10^{-4} - 10^{-3}]$. The same critical orders of magnitude of the gravitational potential persist even if the neutrino spectrum is modelled artificially to grow at a much slower rate than the one predicted by an extrapolation of our results: the disappearance of BAO at some redshift between $z \sim 2$ and $z \sim 0$ is linked to the presence of gravitational potentials $\sim 10^{-4}$ or larger. It has been established already that the appearance of such potentials would make the predictions of growing neutrino quintessence for the CMB incompatible with its measured spectrum, because of a very strong ISW effect [3].

We expect our qualitative conclusions to remain valid for all variations of the scenario of growing neutrino quintessence (different forms of the potential $U(\phi)$ and the coupling $\beta(\phi)$). The strong influence of the neutrino sector on the background evolution requires a large coupling to the cosmon field. In turn, this generates a strong long-range force between neutrinos. This causes the neutrino perturbations to become nonlinear early in the cosmological evolution, and the oscillations to disappear from their density power spectrum. The influence on the CDM+BM sector and the observable
BAO depends on the final size of nonlinear inhomogeneities in the neutrino sector and the respective gravitational potentials. Similar considerations would also apply to other models of mass-varying neutrinos [4], if the neutrinos influence the cosmological evolution considerably.

Our analysis of density spectra in growing neutrino quintessence sets a rough quantitative bound on the typical gravitational potentials generated by the neutrino inhomogeneities: they should not become much larger than $10^{-4}$ in the BAO region. Otherwise, the amplitude of BAO will be reduced or completely eliminated. As has been discussed in [6, 8], the formation of structures in growing neutrino quintessence is not well understood. It is possible that highly nonlinear structures or virialized objects start forming early, and their backreaction prevents the neutrino power spectrum from growing very large. It is conceivable that the gravitational potentials at large scales remain sufficiently small for the CMB spectrum and the BAO to remain largely unaffected. However, a better understanding of structure formation in the scenario of growing neutrino quintessence is required in order to resolve this issue.

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