DEFORMATIONS OF THE CLASSICAL $\mathcal{W}$-ALGEBRAS ASSOCIATED TO $D_n, E_6$ AND $G_2$

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1. INTRODUCTION.

The purpose of this paper is the computation of the Poisson brackets in the deformed $W$-algebras $W_q(\mathfrak{g})$, where $\mathfrak{g}$ is of type $D_n, E_6$ or $G_2$. Let us first briefly recall some facts about two main descriptions of $W$-algebras.

The first description is via the Drinfeld-Sokolov reduction. Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{n}$ be its nilpotent subalgebra. Let $M = \mu^{-1}(f)/L\mathfrak{n}$, where $\mu : \hat{\mathfrak{g}}^* \to L\mathfrak{n}^*$ is the momentum map and $f \in L\mathfrak{n}^*$ is a certain character [2]. Then $M$ is a Poisson manifold, and the ordinary $W$-algebra $W(\mathfrak{g})$ is the Poisson algebra of functions on $M$. The manifold $M$ can be identified with the space of certain differential operators. For example if $\mathfrak{g} = \mathfrak{sl}_n$ then these operators are of the form

$$\partial^n + a_{n-2}\partial^{n-2} + \cdots + a_0,$$

and the Poisson structure under consideration is called the second Gelfand-Dickey bracket [9]. Recently in [8, 11] a $q$-deformation of $W$-algebras was obtained by considering the space of $q$-difference operators.

Let us now recall the second description of $W$-algebras. It was proved by B. Feigin and E. Frenkel [3, 4] that as Poisson algebras $W(\mathfrak{g}^L)$ is isomorphic to the center $Z(\hat{\mathfrak{g}})$ of the completion of the universal enveloping algebra $\hat{U}(\mathfrak{g})_{-h^\vee}$ of $\mathfrak{g}$ at the critical level $-h^\vee$, where $h^\vee$ is the dual Coxeter number and $\mathfrak{g}^L$ is the Langlands dual of the algebra $\mathfrak{g}$. This description was used in [5] to produce a $q$-deformed $W$-algebras $W_q(\mathfrak{g})$, where $\mathfrak{g}$ is of the classical type, and to compute the Poisson structure in the $A_n$ case. In the cases $B_n$ and $C_n$ some Poisson brackets were computed in [6].

It is convenient to study $W$-algebras via the Miura transformation. For ordinary $W$-algebras it can be defined as follows (see [3, 4, 5]). One considers the Wakimoto homomorphism from $\hat{U}(\mathfrak{g})$ to the tensor product of some Heisenberg algebra and some commutative algebra
$H(\mathfrak{g})$ - algebra of functionals on some hyperplane in the dual space to the Heisenberg subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The restriction of this map to the center gives the homomorphism $Z(\hat{\mathfrak{g}}) \to H(\mathfrak{g})$, composition of which with the isomorphism $W(\mathfrak{g}) \simeq Z(\hat{\mathfrak{g}})$ on the left is just the Miura transformation [2, 5].

The $q$-deformed version of this picture is the Wakimoto realization of $\hat{U}_q(\mathfrak{g})$ in the tensor product of a certain Heisenberg algebra and some Heisenberg-Poisson algebra $H_q(\mathfrak{g})$. The restriction to the center $Z_q(\hat{\mathfrak{g}})$ of $\hat{U}_q(\mathfrak{g})$ gives the $q$-deformed Miura transformation $Z_q(\mathfrak{g}) \to H_q(\mathfrak{g})$. The image is called the $q$-$W$-algebra $W_q(\mathfrak{g})$. Thus, in order to describe deformed $W$-algebras we have to describe the Heisenberg-Poisson algebra $H_q(\mathfrak{g})$ and the generators of $W_q(\mathfrak{g})$. In [5] E. Frenkel and N. Reshetikhin did this in the $A_n$ case using the explicit formulas for the Wakimoto realization [1]. Motivated by these results they gave a conjectural description of $H_q(\mathfrak{g})$ for general $\mathfrak{g}$ (see Sect. 11 of [5] and the next section) and of $W_q(\mathfrak{g})$ for $\mathfrak{g}$ of classical series. The key element of this conjecture was that the formulas for the generators of the deformed $W$-algebra coincide with the formulas for the eigenvalues of the corresponding transfer-matrices obtained by analytic Bethe Ansatz (see Conjecture 1 of [5]). In order to verify this conjecture, one has to check that the Poisson brackets between the generators of $W_q(\mathfrak{g})$, constructed in this way, close among themselves. This had been done in [5] for the $A_n$ series and in [6, 7] for the $B_n$ and $C_n$ series. However the question remained open for other series.

In this paper we study the case when $\mathfrak{g}$ is of the type $D_n$, $E_6$ or $G_2$. We exhibit the generators and relations of the algebra $H_q(\mathfrak{g})$ explicitey and compute the Poisson brackets between them. Next, we construct the generators of the $W$-algebra $W_q(\mathfrak{g})$, following the conjecture of [5] that they should coincide with the corresponding formulas for the eigenvalues of transfer-matrices (see [10]). Finally, we compute the Poisson brackets between them.

The paper is organized as follows. In section 2 we describe the algebra $H_q(\mathfrak{g})$ via the generators and relations. Sections 3, 4 and 5 are devoted to the cases of $D_n$, $E_6$ and $G_2$ respectively. Each of these sections is divided in three subsections. In the first one we define matrices which are used in the construction of $H_q(\mathfrak{g})$. In the second subsection we describe the new set of generators of $H_q(\mathfrak{g})$ which is convenient for the computation of the Poisson brackets. Then we axiomatically define the generators of the corresponding $W$-algebra. In the last subsection we compute the Poisson brackets between the generators of $H_q(\mathfrak{g})$ and $W_q(\mathfrak{g})$. 
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2. Heisenberg-Poisson algebras.

In this section we describe the algebra $H_q(g)$. The matrices $M, \tilde{M}$ and $D$ will be explicitly given in the next sections. It should be noted that $\tilde{M}$ is the deformation of twice the symmetrized Cartan matrix of the corresponding algebra. The presentation in this section follows [6, 7].

Let $g$ be either $D_n, E_6$ or $G_2$. Let $\hat{U}_q(\hat{g})$ be the completion of the quantum universal enveloping algebra $U_q(\hat{g})$ of $g$ as defined in [5].

We consider a Heisenberg-Poisson algebra $H_q(g)$ with generators $a_i[n], 1 \leq i \leq \text{rank}(g)$ and relations:

$$\{a_i[n], a_j[m]\} = \tilde{M}_{ij}(q^n)\delta_{n,-m}.$$  

There is unique set of "dual" generators $y_i[n]$ such that

$$\{y_i[n], a_j[m]\} = D_{ij}(q^n)\delta_{n,-m}.$$  

Then $y_i[n]$ satisfy

$$\{y_i[n], y_j[m]\} = M_{ij}(q^n)\delta_{n,-m}.$$  

Let’s form the generating series:

$$Y_i(z) = q^{2(\rho,\omega_i)} \exp \left( - \sum_{m \in \mathbb{Z}} y_i[m] z^{-m} \right)$$

They satisfy the following relations:

$$\{Y_i(z), Y_j(w)\} = \mathcal{M}_{ij} \left( \frac{w}{z} \right) Y_i(z) Y_j(w),$$

where

$$\mathcal{M}_{ij}(x) = \sum_{m \in \mathbb{Z}} M_{ij}(q^m)x^m.$$  

The coefficients of the generating functions $Y_i(z)$ generate the algebra $H_q(g)$. In the next sections we will introduce the new generating functions $\Lambda_i(z)$ which also have generators as coefficients. Finally the generating functions whose coefficients generate $W_q(g)$ will be denoted $T_i(z)$, where $1 \leq i \leq \text{rank of the Cartan subalgebra of } g)$. In the $D_n$ [6] and $G_2$ cases all $T_i(z)$’s can be constructed explicitly, whereas in the case of $E_6$ we explicitely construct only $T_1(z)$. 

3. Matrices. Consider the matrices $M(t), D(t), \tilde{M}(t)$ defined as follows. Let $M(t) = (M_{ij}(t)), 1 \leq i, j \leq n,$ where
\[
M_{ij}(t) = \frac{(t_{\min(i,j)} - t_{\min(i,j)})(t_{\max(i,j)} - t_{\max(i,j)})}{(t^n - t^{(n-1)})},
\]
\[
M_{ni}(t) = M_{n-1,i}(t) = \frac{(t^n - t^{(n-1)})}{(t - t^{(n-1)})}, 1 \leq i, j \leq n - 2,
\]
\[
M_{n,n-1}(t) = \frac{(t^n - t^{(n-2)})}{(t + t^{(n-1)})(t^{(n-1)} + t^{(n-2)})},
\]
\[
M_{n-1,n-1}(t) = M_{nn}(t) = \frac{(t^n - t^n)}{(t + t^{(n-1)})(t^{(n-1)} + t^{(n-2)})}.
\]

Let $D(t) = (t - t^{-1}) \cdot I_n,$ where $I_n$ is the $n \times n$ identity matrix. Then
\[
\tilde{M}(t) = D(t)M(t)^{-1}D(t) =
\]
\[
\begin{pmatrix}
t^2 - t^{-2} & -(t - t^{-1}) & \cdots & 0 & 0 & 0 \\
-(t - t^{-1}) & t^2 - t^{-2} & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & t^2 - t^{-2} & -(t - t^{-1}) & -(t - t^{-1}) \\
0 & 0 & \cdots & -(t - t^{-1}) & t^2 - t^{-2} & 0 \\
0 & 0 & \cdots & -(t - t^{-1}) & 0 & t^2 - t^{-2}
\end{pmatrix},
\]
which is a $t$–deformation of twice the Cartan matrix of $D_n.$

3.2. Generators. Introduce as in [5]

\[
\lambda_i(z) = Y_i(zq^{-i+1})Y_{i-1}(zq^{-i}), \quad i = 1, \ldots, n - 2,
\]
\[
\lambda_{n-1}(z) = Y_{n}(zq^{-n+2})Y_{n-1}(zq^{-n+2})Y_{n-2}(zq^{-n+1}),
\]
\[
\lambda_n(z) = Y_{n-1}(zq^{-n+2})Y_{n-1}(zq^{-n}),
\]
\[
\lambda_{n+1}(z) = Y_n(zq^{-n+2})Y_{n-1}(zq^{-n}),
\]
\[
\lambda_{n+2}(z) = Y_{n-2}(zq^{-n+1})Y_{n-1}(zq^{-n})Y_{n}(zq^{-n}),
\]
\[
\lambda_{2n-i+1}(z) = Y_{i-1}(zq^{-2n+i+2})Y_{i}(zq^{-2n+i+1}) \quad i = 1, \ldots, n - 2,
\]
where $Y_0(z) = 1.$

Remark. The relations between $Y_i(z)$ and the functions $Q_i(u), 1 \leq i \leq n,$ which appear in [10] in the formulas for the eigenvalues of the transfer matrices of the $D_n^{(1)}$ model are as follows:

\[
Y_i(zq^m) = \frac{Q_i(u + \frac{m+1}{2}\eta)}{Q_i(u + \frac{m-1}{2}\eta)}
\]
Let

\[ T_1(z) = \sum_{i=1}^{2n} \Lambda_i(z). \]
\[ T_2(z) = \sum_{(i,j) \in S} \Lambda_i(z) \Lambda_j(zq^2), \]

where the set \( S \) consists of pairs \((i, j)\) such that either \( i < j \) or \((i, j) = (n + 1, n)\).

\[ \{ \Lambda_i(z), \Lambda_i(w) \} = M_{11} \left( \frac{w}{z} \right) \Lambda_i(z) \Lambda_i(w), \]
\[ \{ \Lambda_i(z), \Lambda_j(w) \} = M_{11} \left( \frac{w}{z} \right) \Lambda_i(z) \Lambda_j(w) + \left( \delta \left( \frac{w}{zq^2} \right) - \delta \left( \frac{w}{z} \right) \right) 
+ \left( \delta \left( \frac{w}{zq^{2n-2i}} \right) \delta_{i+j,2n+1} - \delta \left( \frac{w}{zq^{2n-2i-2}} \right) \delta_{i+j,2n+1} \right) 
\times \Lambda_i(z) \Lambda_j(w), \]

if \( i < j \).

\[ \{ T_1(z), T_1(w) \} = M_{11} \left( \frac{w}{z} \right) T_1(z) T_1(w) 
+ \delta \left( \frac{w}{zq^2} \right) T_2(z) - \delta \left( \frac{wq^2}{z} \right) T_2(w) 
+ \delta \left( \frac{w}{zq^{2n-2}} \right) - \delta \left( \frac{w^{2n-2}}{z} \right). \]
4. \textbf{E}_6 \text{ Case.}

4.1. \textbf{Matrices.} Consider the matrices $M(t), D(t), \tilde{M}(t)$ defined as follows. Let $M(t) = (M_{ij}(t)), 1 \leq i, j \leq 6, \text{ } M_{ij}(t) = M_{ji}(t)$, where

$$M_{11}(t) = M_{55}(t) = \frac{(t - t^{-1})(t^8 - t^{-8})}{(t^6 + t^{-6})(t^3 - t^{-3})}$$

$$M_{12}(t) = M_{45}(t) = \frac{(t - t^{-1})(t^5 - t^{-5})(t^2 + t^{-2})}{(t^6 + t^{-6})(t^3 - t^{-3})}$$

$$M_{22}(t) = M_{44}(t) = \frac{(t^4 - t^{-4})(t^5 - t^{-5})}{(t^6 + t^{-6})(t^3 - t^{-3})}$$

$$M_{13}(t) = M_{26}(t) = M_{46} = M_{35} = \frac{(t^4 - t^{-4})}{(t^6 + t^{-6})}$$

$$M_{23}(t) = M_{34}(t) = \frac{(t^4 - t^{-4})(t + t^{-1})}{(t^6 + t^{-6})}$$

$$M_{33}(t) = \frac{(t^3 - t^{-3})(t + t^{-1})(t^2 + t^{-2})}{(t^6 + t^{-6})}$$

$$M_{16}(t) = M_{56}(t) = \frac{(t - t^{-1})(t^2 + t^{-2})}{(t^6 + t^{-6})}$$

$$M_{36}(t) = \frac{(t^3 - t^{-3})(t^2 + t^{-2})}{(t^6 + t^{-6})}$$

$$M_{66}(t) = \frac{(t^4 - t^{-4})(t^3 + t^{-3})}{(t + t^{-1})(t^6 + t^{-6})}$$

$$M_{14}(t) = M_{25}(t) = \frac{(t^2 - t^{-2})(t^4 - t^{-4})}{(t^6 + t^{-6})(t^3 - t^{-3})}$$

$$M_{24}(t) = \frac{(t^2 - t^{-2})(t^4 - t^{-4})(t + t^{-1})}{(t^6 + t^{-6})(t^3 - t^{-3})}$$

$$M_{15}(t) = \frac{(t - t^{-1})(t^4 - t^{-4})}{(t^6 + t^{-6})(t^3 - t^{-3})}.$$ 

Let $D(t) = (t - t^{-1}) \cdot I_6$, where $I_6$ is the $6 \times 6$ identity matrix. Then

$$\tilde{M}(t) = D(t)M(t)^{-1}D(t) =$$

$$\begin{pmatrix}
  t^2 - t^{-2} & t^{-1} - t & 0 & 0 & 0 & 0 \\
  t^{-1} - t & t^2 - t^{-2} & t^{-1} - t & 0 & 0 & 0 \\
  0 & t^{-1} - t & t^2 - t^{-2} & t^{-1} - t & 0 & t^{-1} - t \\
  0 & 0 & t^{-1} - t & t^2 - t^{-2} & t^{-1} - t & 0 \\
  0 & 0 & 0 & t^{-1} - t & t^2 - t^{-2} & 0 \\
  0 & 0 & t^{-1} - t & 0 & 0 & t^2 - t^{-2}
\end{pmatrix}$$
is a $t$-deformation of twice the Cartan matrix of $E_6$.

4.2. **Generators.** Introduce

\[
\begin{align*}
\Lambda_1(z) &= Y_1^{-1}(zq^{-8})Y_2(zq^{-7})Y_3(zq^{-8})Y_6(zq^{-7}), \\
\Lambda_2(z) &= Y_1^{-1}(zq^{-8})Y_2(zq^{-7})Y_6(zq^{-9}), \\
\Lambda_3(z) &= Y_1^{-1}(zq^{-8})Y_3(zq^{-6})Y_4(zq^{-7}), \\
\Lambda_4(z) &= Y_1^{-1}(zq^{-8})Y_4(zq^{-5})Y_5(zq^{-6}), \\
\Lambda_5(z) &= Y_2^{-1}(zq^{-9})Y_3(zq^{-8})Y_6(zq^{-9}), \\
\Lambda_6(z) &= Y_2^{-1}(zq^{-9})Y_6(zq^{-7}), \\
\Lambda_7(z) &= Y_3^{-1}(zq^{-10})Y_4(zq^{-9}), \\
\Lambda_8(z) &= Y_4^{-1}(zq^{-11})Y_5(zq^{-10}), \\
\Lambda_9(z) &= Y_1(zq^{-6})Y_2^{-1}(zq^{-7})Y_3(zq^{-6})Y_4^{-1}(zq^{-7}), \\
\Lambda_{10}(z) &= Y_1(zq^{-6})Y_2^{-1}(zq^{-7})Y_4(zq^{-5})Y_5^{-1}(zq^{-6}), \\
\Lambda_{11}(z) &= Y_1(zq^{-6})Y_3^{-1}(zq^{-8})Y_6(zq^{-7}), \\
\Lambda_{12}(z) &= Y_1(zq^{-6})Y_6^{-1}(zq^{-9}), \\
\Lambda_{13}(z) &= Y_2(zq^{-5})Y_3^{-1}(zq^{-6})Y_4(zq^{-5})Y_5^{-1}(zq^{-6}), \\
\Lambda_{14}(z) &= Y_2(zq^{-5})Y_4^{-1}(zq^{-7}), \\
\Lambda_{15}(z) &= Y_3(zq^{-4})Y_5^{-1}(zq^{-6})Y_6^{-1}(zq^{-5}), \\
\Lambda_{16}(z) &= Y_5^{-1}(zq^{-6})Y_6(zq^{-3}), \\
\Lambda_{17}(z) &= Y_1^{-1}(zq^{-8})Y_5(zq^{-4}), \\
\Lambda_{18}(z) &= Y_1(zq^{-6})Y_2^{-1}(zq^{-7})Y_5(zq^{-4}), \\
\Lambda_{19}(z) &= Y_2(zq^{-5})Y_3^{-1}(zq^{-6})Y_5(zq^{-4}), \\
\Lambda_{20}(z) &= Y_3(zq^{-4})Y_4^{-1}(zq^{-5})Y_5(zq^{-4})Y_6^{-1}(zq^{-5}), \\
\Lambda_{21}(z) &= Y_4^{-1}(zq^{-5})Y_5(zq^{-4})Y_6(zq^{-3}), \\
\Lambda_{22}(z) &= Y_4(zq^{-3})Y_6^{-1}(zq^{-5}), \\
\Lambda_{23}(z) &= Y_5^{-1}(zq^{-3})Y_4(zq^{-3})Y_6(zq^{-3}), \\
\Lambda_{24}(z) &= Y_2^{-1}(zq^{-3})Y_3(zq^{-2}), \\
\Lambda_{25}(z) &= Y_1^{-1}(zq^{-2})Y_2(zq^{-1}), \\
\Lambda_{26}(z) &= Y_1(z), \\
\Lambda_{27}(z) &= Y_5^{-1}(zq^{-12}).
\end{align*}
\]

**Remark.** The relations between $Y_i(z)$ and the functions $Q_i(u), 1 \leq i \leq 6$, which appear in [10] in the formulas for the eigenvalues of the
transfer matrices of the $E_6^{(1)}$ model are as follows:

$$Y_i(zq^m) = \frac{Q_{\sigma(i)}(u - (m + 1)\eta)}{Q_{\sigma(i)}(u - (m - 1)\eta)},$$

where $\sigma$ is a permutation $(1)(2)(3)(456)$.

Let

$$T_1(z) = \sum_{i=1}^{27} \Lambda_i(z).$$

4.3. Poisson brackets.

$$\{\Lambda_i(z), \Lambda_j(w)\} = M_{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_j(w) + (\text{sum of } \delta\text{- functions})\Lambda_i(z)\Lambda_j(w).$$

$$\{T_1(z), T_1(w)\} = M_{11}\left(\frac{w}{z}\right)T_1(z)T_1(w) + \delta\left(\frac{wq^2}{z}\right)T_2(z) - \delta\left(\frac{w}{zq^2}\right)T_2(w) + \delta\left(\frac{wq^8}{z}\right)T_5(zq^4) - \delta\left(\frac{w}{zq^8}\right)T_5(wq^4),$$

where $T_5(z)$ is the $W$-algebra generator corresponding to the fifth fundamental weight (which is dual to the first one in our notation).

5. $G_2$ case.

5.1. Matrices. Consider the matrices $M(t), D(t)$ and $\widetilde{M}(t)$ defined as follows. Let $M(t) = (M_{ij}(t)), 1 \leq i, j \leq 2$, where

$$M_{22}(t) = \frac{(t^3 - t^{-3})(t + t^{-1})(t^2 + t^{-2})}{t^6 + t^{-6}},$$

$$M_{11}(t) = \frac{(t^3 + t^{-3})(t - t^{-1})(t^2 + t^{-2})}{t^6 + t^{-6}},$$

$$M_{12}(t) = M_{21}(t) = \frac{(t^3 - t^{-3})(t^2 + t^{-2})}{t^6 + t^{-6}}.$$

Let

$$D(t) = \begin{pmatrix} t - t^{-1} & 0 \\ 0 & t^3 - t^{-3} \end{pmatrix}.$$

Then

$$\widetilde{M}(t) = D(t)M(t)^{-1}D(t) = \begin{pmatrix} t^2 - t^{-2} & -(t^3 - t^{-3}) \\ -(t^3 - t^{-3}) & t^6 - t^{-6} \end{pmatrix}$$

is a $t$-deformation of the symmetrized Cartan matrix of $G_2$. 
5.2. Generators. Introduce

\[ \Lambda_1(z) = Y_1(z), \]
\[ \Lambda_2(z) = Y_1^{-1}(zq^{-2})Y_2(zq^{-1}), \]
\[ \Lambda_3(z) = Y_1(zq^{-4})Y_1(zq^{-6})Y_2^{-1}(zq^{-7}), \]
\[ \Lambda_4(z) = Y_1(zq^{-4})Y_1^{-1}(zq^{-8}), \]
\[ \Lambda_5(z) = Y_1^{-1}(zq^{-6})Y_1^{-1}(zq^{-8})Y_2(zq^{-5}), \]
\[ \Lambda_6(z) = Y_1(zq^{-10})Y_2^{-1}(zq^{-11}), \]
\[ \Lambda_7(z) = Y_1^{-1}(zq^{-12}). \]

Remark. The relations between \( Y_i(z) \) and the functions \( Q_i(u), i = 1, 2 \), which appear in [10] in the formulas for the eigenvalues of the transfer matrices of the \( G_{2}^{(1)} \) model are as follows:

\[ Y_1(zq^m) = \frac{Q_1(u + \frac{13+m}{3} \eta)}{Q_1(u + \frac{11+m}{3} \eta)} \]
\[ Y_2(zq^m) = \frac{Q_2(u + \frac{15+m}{3} \eta)}{Q_2(u + \frac{9+m}{3} \eta)} \]

Let

\[ T_1(z) = \sum_{i=1}^{7} \Lambda_i(z). \]

\[ T_2(z) = \sum_{i=2}^{7} \Lambda_1(z)\Lambda_i(zq^2) + \sum_{i=2}^{6} \Lambda_i(z)\Lambda_7(zq^2) + \left( \text{sum of } \delta \text{-functions} \right) \Lambda_i(z)\Lambda_7(zq^2). \]

5.3. Poisson brackets.

\[ \{ \Lambda_i(z), \Lambda_j(w) \} = M_{11} \left( \frac{w}{z} \right) \Lambda_i(z)\Lambda_j(w) \]
\[ + \text{(sum of } \delta \text{-functions}) \Lambda_i(z)\Lambda_j(w). \]

\[ \{ T_1(z), T_1(w) \} = M_{11} \left( \frac{w}{z} \right) T_1(z) T_1(w) \]
\[ + \delta \left( \frac{w}{zq^2} \right) T_2(z) - \delta \left( \frac{wq^3}{z} \right) T_2(w) \]
\[ + \delta \left( \frac{w}{zq^8} \right) T_1(zq^4) - \delta \left( \frac{wq^8}{z} \right) T_1(wq^4) \]
\[ + \delta \left( \frac{w}{zq^{12}} \right) - \delta \left( \frac{wq^{12}}{z} \right). \]

There is an obvious
Proposition 5.1. In the cases $E_6$ and $G_2$ if we replace $Y_i^\epsilon(zq^n)$ by $Y_i^{-\epsilon}(zq^{-n})$ in $T_1(z)$ then we obtain $T_\alpha(zq^{12})$, where $T_\alpha(z)$ corresponds to the dual root (i.e. $\alpha = 5$ in $E_6$ case and $1$ in $G_2$ case).  

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