Open & Closed vs. Pure Open String Disk Amplitudes

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Abstract

We establish a relation between disk amplitudes involving \( N_o \) open and \( N_c \) closed strings and disk amplitudes with only \( N_o + 2N_c \) open strings. This map, which represents a sort of generalized KLT relation on the disk, reveals important structures between open & closed and pure open string disk amplitudes: it relates couplings of brane and bulk string states to pure brane couplings.

On the string world-sheet this becomes a non-trivial monodromy problem, which reduces the disk amplitude of \( N_o \) open and \( N_c \) closed strings to a sum of many color ordered partial subamplitudes of \( N_o + 2N_c \) open strings. This sum can be further reduced to a sum over \((N_o+2N_c-3)!\) subamplitudes of \( N = N_o + 2N_c \) open strings only. Hence, the computation of disk amplitudes involving open and closed strings is reduced to computing these subamplitudes in the open string sector.

In this sector we find a string theory generalization and proof of the Kleiss–Kuijf and Bern–Carrasco–Johansson relations: All order \( \alpha' \) identities between open string subamplitudes are derived, which reproduce these field-theory relations in the limit \( \alpha' \rightarrow 0 \). These identities allow to reduce the number of independent subamplitudes of an open string \( N \)-point amplitude to \((N-3)\)! This number is identical to the dimension of a minimal basis of generalized Gaussian hypergeometric functions describing the full \( N \)-point open string amplitude.
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1. Introduction

The famous Kawai, Lewellen, Tye (KLT) relation allows to express pure closed string tree–level (sphere) amplitudes as sum of squares of pure open string tree–level (disk) amplitudes [1]. This way at tree–level e.g. graviton scattering amplitudes are expressed by gluon amplitudes. More precisely, a graviton amplitude can be expressed as a sum of squares of partial color ordered gluon amplitudes (multiplied by some sin–factors). In the low–energy effective action this duality leads to drastic simplifications of gravitational interactions. In more technical terms the absence of interactions of left– and right–moving world–sheet fields of the closed string allows to factorize any closed string tree–level amplitude into products of disk amplitudes of left–moving fields and right–moving fields.

When considering tree–level amplitudes of both open and closed strings the interacting string world–sheet is a disk. In that case the left– and right–moving world–sheet fields of the closed string do interact and the full amplitude cannot be factorized into sums of squares of pure open string disk amplitudes. This is the generic situation in string theories with D–branes with brane and bulk fields. In the presence of D–branes boundary conditions have to be imposed resulting in an interaction of left– and right–moving fields.

On the other hand, in this work we show concretely how disk amplitudes of open and closed strings are computed and how these amplitudes can be expressed in terms of pure open string disk amplitudes. This way any amplitude involving open and closed strings is mapped to pure open string amplitudes. By this e.g. a disk amplitude involving both brane and bulk fields is related to an amplitude of only brane fields. Hence this map gives a dictionary between mixed couplings involving brane and bulk fields to pure brane couplings. As a consequence disk amplitudes involving both members from gauge multiplets and members from the supergravity multiplet are related to pure amplitudes involving only members from gauge multiplets. By using these arguments the string expansion w.r.t. to flat background may be reduced to fluctuations in the open string sector on the D–brane world–volume. The corresponding fermionic couplings may simply be obtained by acting with the space–time supersymmetry currents on the relevant correlators in the open string sector [2]. At any rate, we believe that explicit map of (tree–level) couplings of brane and bulk fields to pure brane couplings may have some deeper insight into the dynamics of D–branes, cf. also [3].

In practice the map between disk amplitudes of open and closed strings and pure open string disk amplitudes is much more involved than in the KLT case due to the additional mixing between left– and right–moving fields. On the string world–sheet this becomes a non–trivial monodromy problem, which reduces the disk amplitude of $N_o$ open and $N_c$ closed strings to a sum of many color ordered partial subamplitudes of $N_o + 2N_c$ open strings (supplemented with phases and sin–factors). By explicitly deforming the underlying contours we considerably reduce the number of terms in the expression. This step is
equivalent to finding relations between partial subamplitudes of open string amplitudes. Hence, the problem of computing disk amplitudes of open and closed strings is reduced to finding relations between pure open string amplitudes such, that the sum over color ordered partial subamplitudes of $N_o + 2N_c$ open strings can be reduced. In this work we will find, that for open strings there are many more identities between color ordered subamplitudes beyond the usual cyclic, reflection and parity symmetries. These new string theory relations between open string subamplitudes, which are derived in this work, allow to eventually reduce the full disk amplitude of $N_o$ open and $N_c$ closed strings to a minimal number of open string disk amplitudes, namely $(N_o + 2N_c - 3)!$.

In field–theory such relations are known as Kleiss–Kuijf, Bern–Carrasco–Johansson and dual Ward identities. However we find a string theory generalization and proof of these relations: All order $\alpha'$ identities between open string subamplitudes are derived, which reproduce these field–theory relations in the limit $\alpha' \to 0$. These identities allow to reduce the number of independent subamplitudes of an open string $N$–point amplitude to $(N-3)!$. This number is identical to the dimension of a minimal basis of generalized Gaussian hypergeometric functions describing the full $N$–point open string amplitude [4,5,2]

When relating disk amplitudes of open and closed strings to pure open string disk amplitudes the open string subamplitudes generically do not yet appear as world–sheet integrals in canonical form, i.e. with integrations along the segment $[0,1]$, which would give a direct translation to (generalized) Euler integrals. We explicitly map the corresponding world–sheet integrals to open string amplitudes given in their canonical form by (generalized) Euler integrals (along the segment $[0,1]$).

In the past the computation of disk amplitudes of open and closed strings has been accomplished explicitly only for very simple and restricted cases, i.e. low number of specific external states, see Refs. [6,7,8,9], [10,11,12], and [13,14,15]. In this work we generalize these results and present the formalism and tools to compute disk amplitudes for any number of open and closed strings.

The organization of this article is as follows. In Section 2 after presenting the world–sheet and space–time tools to compute disk amplitudes of $N_o$ open and $N_c$ closed strings we express the latter as sum over $N_o+2N_c$ open string color ordered partial subamplitudes. Furthermore, we explicitly present the generic complex world–sheet integrals for the cases $(N_o, N_c) = (2, 1), (3, 1), (2, 2), (4, 1), (0, 3)$ and $(3, 2)$. In Section 3 we demonstrate, how a disk amplitudes involving $N_o$ open and $N_c$ closed strings is reduced to a sum over color ordered open string $N_o + 2N_c$–point amplitudes. We present explicit results for the cases $(N_o, N_c) = (2, 1), (3, 1), (2, 2), (4, 1), (0, 3)$ and $(3, 2)$ by expressing the results in terms of generalized Euler integrals describing 4, 5, 6– and 7–point open string scattering, respectively. In Section 4 we derive (string) relations between partial subamplitudes involving $N$ open strings. These relations allow to express all partial amplitudes in terms of a minimal basis of $(N-3)!$ subamplitudes. For $N = N_o+2N_c$ these relations are then used to simplify
the sum over partial amplitudes as it appears for disk amplitudes involving $N_o$ open and $N_c$ closed strings. In Section 5 we explicitly compute disk amplitudes of open and closed strings and show in detail, how they are mapped to pure open string disk amplitudes thus giving a dictionary of couplings of brane and bulk fields to pure brane couplings. In the appendix we compute complex world–sheet integrals by splitting them into holomorphic and anti–holomorphic pieces.

2. Disk scattering from D–branes

In the following we shall discuss tree–level disk scattering from D–branes. The world–sheet diagram of a string $S$–matrix describing the interaction of open and closed strings at (open string) tree–level can be conformally mapped to a surface with one boundary. According to the Riemann mapping theorem the latter is equivalent to the unit disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Eventually with the Möbius transformation $z \to i \frac{1+z}{1-z}$ the unit disk $D$ can be conformally mapped to the upper (complex) half–plane $H_+ = \{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\}$. The string states, which correspond to asymptotic states in the string $S$–matrix formulation, are created through vertex operators. In theories with D–branes massless (charged) fields as e.g. gauge vectors or open string moduli originate from open string excitations living on the D–brane world–volume. Hence the boundary of the disk diagram is attached to the D–brane world–volume. On the other hand, the massless closed string or bulk fields representing e.g. the graviton, dilaton field and closed (geometric) string moduli live in the bulk and are inserted in the bulk of the disk.

2.1. Disk scattering of open and closed strings

The generic expression for the superstring disk amplitude involving $N_o$ massless open and $N_c$ massless closed string states is:

$$A(N_o, N_c) = \sum_{\sigma \in S_{N_o-1}} V_{\text{CKG}}^{-1} \left( \int_{I_{\sigma}} \prod_{j=1}^{N_o} dx_j \prod_{i=1}^{N_c} \int_{H_+} d^2 z_i \right) \left( \prod_{j=1}^{N_o} : V_o(x_j) : \prod_{i=1}^{N_c} : V_c(\overline{z}_i, z_i) : \right).$$

(2.1)

The open string vertex operators $V_o(x_i)$ are inserted at the positions $x_i$ on the boundary of the disk. The latter are integrated along the boundary of $H_+$, subject to the integration region $I_{\sigma}$. On the other hand, the closed string vertex operators $V_c(\overline{z}_i, z_i)$ are inserted at points $z_i$ inside the disk, cf. the next Figure:
The sum runs over all \((N_o - 1)\)! cyclic inequivalent orderings \(\sigma \in S_{N_o - 1} = S_{N_o}/\mathbb{Z}_{N_o}\) of the \(N_o\) open string vertex operators along the boundary of the disk. Each permutation \(\pi\) gives rise to an integration region \(\mathcal{I}_\sigma = \{x \in \mathbb{R} \mid x_1 < x_{\sigma(2)} < \ldots < x_{\sigma(N_o)}\}\). The open string vertex operators \(V(x_i)\) contain Chan–Paton factors \(T^a\) carrying the gauge degrees of freedom of the open string state. Depending on the ordering \(\mathcal{I}_\sigma\) of the vertex operator positions we obtain \((N_o - 1)\)! partial amplitudes with the group factor \(\text{Tr}(T_1 T_{\sigma(2)} \ldots T_{\sigma(N_o)})\). Furthermore, in Eq. (2.1) the factor \(V_{CKG}\) accounts for the volume of the conformal Killing group of the disk after choosing the conformal gauge. It will be canceled by fixing three vertex positions on the disk and introducing the respective \(c\)-ghost correlator.

The upper half plane \(H_+\) may be obtained from the full complex plane \(\mathbb{C}\) representing the sphere through a \(\mathbb{Z}_2\) identification \(z \simeq \bar{z}\). It is convenient to perform the computations in the double cover, \(i.e.\) in the complex plane \(\mathbb{C}\), and extend the definition of these fields to the entire complex plane \(\mathbb{C}\) by taking into account the interaction between left–moving and right–moving closed string fields [7]. Hence to evaluate the disk–integration in (2.1) over the closed string vertex positions \(z_i\) it is convenient to go to the double–cover and take into account the mixing of left– and right–moving fields. This is described in more detail in the next Subsection. An other way of arguing stems from the fact, that in (2.1) only vertex operators, which are invariant under the world–sheet parity \(\Omega: z \leftrightarrow \bar{z}\), enter. Hence, in the following we use vertex operators invariant under the world–sheet parity and integrate their positions \(z_i\) over the whole complex plane \(\mathbb{C}\).

2.2. Open and closed string vertex operators and their disk interactions

The vertex operator for the gauge vector \(A^a\), which comprises the massless bosonic Neveu–Schwarz (NS) open string mode, is in the \((-1)\)-ghost picture given by

\[
V^{(-1)}_{A^a}(z, \xi, p) = g_A T^a e^{-\phi(z)} \xi^\mu \psi_\mu(z) e^{ip\rho X^\rho(z)},
\]

while in the zero–ghost picture it takes the form:

\[
V^{(0)}_{A^a}(z, \xi, p) = \frac{g_A}{(2\alpha')^{1/2}} T^a \xi_\mu \left[ i\partial X^\mu + 2\alpha' (p\psi) \psi_\mu \right] e^{ip\rho X^\rho(z)}.
\]
On the other hand, the vertex operator for gaugino $\chi^a$, which gives rise to the massless Ramond (R) open string mode is given by

$$V_{\chi}^{(-1/2)}(z,u,p) = g_\chi T^a e^{-\frac{i}{g} \phi(z)} u^a S_\alpha(z) e^{ip_\mu X^\mu(z)},$$

in the $(-1/2)$-ghost picture, in the above definitions $T^a$ are the Chan–Paton factors accounting for the gauge degrees of freedom of the two open string ends. Furthermore, the on–shell constraints $p^2 = 0$, $\not{q} = 0$ are imposed. The vertex operator for massless bosonic NSNS closed string modes is given by

$$V_{G}^{(-1,1)}(\bar{z},z,\epsilon,q) = g E_{\mu\nu} e^{-\frac{1}{g} \phi(\bar{z})} e^{\phi(z)} \psi^{\mu}(\bar{z}) \psi^{\nu}(z) e^{iq_\mu X^\mu(\bar{z},z)},$$

$$V_{G}^{(0,0)}(\bar{z},z,\epsilon,q) = -\frac{2g}{\alpha'} E_{\mu\nu} [i\partial X^\mu + \frac{\alpha'}{2} (q\psi) \psi^{\mu}(\bar{z})] [i\partial X^\nu + \frac{\alpha'}{2} (q\psi) \psi^{\nu}(z)] e^{iq_\mu X^\mu(\bar{z},z)},$$

in the $(-1,1)$ and $(0,0)$ ghost picture, respectively. The polarization tensor $E_{\mu\nu}$, which is subject to the on–shell conditions $E_{\mu\nu} q^\nu = 0 = E_{\mu\nu} q^\nu$ and $q^2 = 0$, is symmetric for the graviton and dilaton field and anti–symmetric for anti–symmetric tensor field. For further details, see also [18]. The vertex operator of the $n + 1$–form RR field strengths $F_{n+1}$ is given by [19]

$$V_{F_{n+1}}^{(-1/2,-1/2)}(\bar{z},z,f,q) = \frac{g_0}{(n+1)!} F_{\mu_0...\mu_n} (P^+ \Gamma^{\mu_0}...\Gamma^{\mu_n}) \alpha_\beta \times e^{-\frac{1}{g} \phi(\bar{z})} e^{-\frac{1}{g} \phi(z)} S_\alpha(z) S_\beta(\bar{z}) e^{iq_\mu X^\mu(\bar{z},z)},$$

with the ten $32 \times 32$ $\Gamma$–matrices $\Gamma^\mu$, the spin fields $S_\alpha$, $S_\beta$ and the chiral projection operator $P^+ = \frac{1}{2}(I_{32} + \Gamma_{11})$ in $D = 10$. The $n + 1$–tensor $F_{\mu_0...\mu_n}$ is the Fourier transform of the Ramond $n$–form potential $c_n$, i.e. $F_{\mu_0...\mu_n} = i(n+1) q_{[\mu_0} c_{\mu_1...\mu_n]}$, with $c_{\mu_2...\mu_n} q^n = 0$ and $q^2 = 0$.

The open string vertex operators $V_{\phi}(x_i)$ involve the holomorphic fields $X^\mu, \psi^\nu, S_\alpha, \phi$ whose interactions are described by the usual correlators

$$\langle X^\mu(z_1) X^\nu(z_2) \rangle = -2\alpha' g^{\mu\nu} \ln(z_1 - z_2), \quad \langle \psi^\mu(z_1) \psi^\nu(z_2) \rangle = \frac{g^{\mu\nu}}{z_1 - z_2},$$

$$\langle S_\alpha(z_1) S_\beta(z_2) \rangle = \frac{C_{\alpha\beta}}{(z_1 - z_2)^{5/4}}, \quad \langle \phi(z_1) \phi(z_2) \rangle = -\ln(z_1 - z_2).$$

---

1 The open string vertex couplings are $g_\phi = (2\alpha')^{1/2} g_{YM}$, $g_\chi = (2\alpha')^{1/2} \alpha^{1/4} g_{YM}$ and $g_A = (2\alpha')^{1/2} g_{YM}$ for the scalar, gaugino and vector, respectively [16]. The gauge coupling $g_{YM}$ can be expressed in terms of the ten–dimensional gauge coupling $g_{10}$ and the dilaton field $\phi_{10}$ through the relation $g_{YM} = g_{YM} e^{\phi_{10}/2}$. Finally, the closed string coupling $g_c$ is given by $g_c = \frac{4\alpha' e^{\phi_{10}}}{2\pi} [16]$. In this work we consider open and closed strings scattering off one stack $a$ of D–branes. Hence the overall normalization of all disk amplitudes has to be supplemented by the factor $C_D = \frac{1}{2g_D^{pa} a^2}$ [17].
on the sphere \( C \). Here \( g^{\mu \nu} \) is the background metric and \( C_{\alpha \beta} \) the \( D = 10 \) charge conjugation matrix with non-vanishing entries only for spinor indices of opposite chirality. The holomorphic fields \( X^\mu, \psi^\nu, S_\alpha, \phi \) are defined on the upper half plane \( \mathbf{H}_+ \). At the boundary of \( \mathbf{H}_+ \) boundary conditions are imposed for these fields:

\[
\begin{align*}
(g_{\mu \nu} + 2\pi \alpha' f_{\mu \nu}) \partial X^\nu(z) &= (g_{\mu \nu} - 2\pi \alpha' f_{\mu \nu}) \bar{\partial} X^\nu(\bar{z}) , \\
(g_{\mu \nu} + 2\pi \alpha' f_{\mu \nu}) \psi^\nu(z) &= (g_{\mu \nu} - 2\pi \alpha' f_{\mu \nu}) \bar{\psi}^\nu(\bar{z}) .
\end{align*}
\]

The matrix \( f_{\mu \nu} \) depends on whether Dirichlet, Neumann or mixed boundary conditions are imposed at the open string end points attached to the \( Dp \)-brane. Mixed open string boundary conditions are specified by a non-trivial background flux \( Dp \) on the sphere \( C \).

The matrix \( D \) is given by [20,21,10]:

\[
D = -g^{-1} + 2 \left( g + 2\pi \alpha' f \right)^{-1} .
\]

The matrix \( M \) may be obtained from the relations \( \Gamma_{\alpha \beta} = D^{\mu \nu} (M^{-1})^{\nu \gamma} M_{\alpha \gamma} \) and \( C_{\alpha \gamma} M_{\beta \delta} C^{\gamma \delta} = C_{\alpha \beta} \), which follow by considering the OPEs \( \psi^\mu(z)S_\alpha(w) \) and \( S_\alpha(z)S_\beta(w) \), respectively, cf. Ref. [9] for more details.

In order to cancel the total background ghost charge of \(-2\) on the disk, the vertices in the correlator (2.1) have to be chosen in the appropriate ghost picture. The correlator of vertex operators in the integrand of (2.1) is evaluated by performing all possible Wick contractions and by using the Greens functions as (2.7) and (2.9).

The amplitude (2.1) involves an omnipresent correlator of a product of exponentials of space–time bosonic fields \( X^\mu \)

\[
\begin{align*}
\langle \prod_{j=1}^{N_o} :e^{ip_{j\mu} X^\mu(x_j)} : \prod_{i=1}^{N_c} :e^{iq_{i\mu} X^\mu(\bar{z}_i,z_i)} : \rangle &= (2\pi)^{10} \delta \left( \sum_{j=1}^{N_o} p_j + \sum_{i=1}^{N_c} q_i \right) \\
&\times \prod_{j_1 < j_2} N_o \left| x_{j_1} - x_{j_2} \right|^{2\alpha' p_{j_1 \mu} p_{j_2 \mu}} \prod_{i=1}^{N_c} \left| z_i - \bar{z}_i \right|^{\alpha' q_{i \mu} q_{i \mu}} \\
&\times \prod_{j=1}^{N_o} \prod_{i=1}^{N_c} \left| x_j - z_i \right|^{2\alpha' p_{j \mu} q_i} \prod_{i_1 < i_2}^{N_c} \left| z_{i_1} - z_{i_2} \right|^{\alpha' q_{i_1 \mu} q_{i_2 \mu}} \left| z_{i_1} - \bar{z}_{i_2} \right|^{\alpha' q_{i_1 \mu} D q_{i_2 \mu}} .
\end{align*}
\]

(2.11)
subject to momentum conservation

$$\sum_{j=1}^{N_o} p_j + \sum_{i=1}^{N_c} q_{i\parallel} = 0$$  \hspace{1cm} (2.12)

along the longitudinal brane directions, with $q_{i\parallel} = \frac{1}{2}(q_i + Dq_i)$.

In the remainder of this article we shall work with the choice $\alpha' = \alpha'_{\text{closed}} \equiv 2$ for both type I and type II strings. In order to correctly accommodate this choice in the open sector of 2.

string amplitudes the momenta $p_i$ in the open string vertex operator has to be doubled, i.e. the operator $V_o(z, 2p)$ is used in the amplitudes. This has the consequence, that in open string amplitudes the momenta $p_i$ of the open strings appear with an additional factor of 2.

To summarize, after working out all Wick contractions in the double cover $C$ the amplitude (2.1) assumes the generic form

$$\mathcal{A}(N_o, N_c) = \sum_{\sigma \in S_{N_o}/\mathbb{Z}_{N_o}} V_{\text{CKG}}^{-1} \left( \int_{I_{\sigma}} \prod_{j=1}^{N_o} dx_j \prod_{i=1}^{N_c} \int_C d^2 z_i \right) \sum_I \mathcal{K}_I$$

$$\times \prod_{j_1 < j_2}^{N_o} |x_{j_1} - x_{j_2}|^{4p_{j_1}p_{j_2}} (x_{j_1} - x_{j_2})^{n_{j_1j_2}} \prod_{i=1}^{N_c} |z_i - \bar{z}_i|^{2q_{i\parallel}} (z_i - \bar{z}_i)^{r_{i\parallel}}$$

$$\times \prod_{j=1}^{N_o} \prod_{i=1}^{N_c} |x_j - z_i|^{4p_iq_i} (x_j - z_i)^{m_{ij}} (x_j - \bar{z}_i)^{\bar{m}_{ij}} \prod_{i_1 < i_2}^{N_c} |z_{i_1} - z_{i_2}|^{2q_{i_1i_2}} (z_{i_1} - z_{i_2})^{r_{i_1i_2}} (\bar{z}_{i_1} - z_{i_2})^{\bar{r}_{i_1i_2}}$$

$$:= \sum_{\sigma \in S_{N_o}/\mathbb{Z}_{N_o}} \sum_I \mathcal{K}_I \mathcal{A}^I(1, \sigma(2), \ldots, \sigma(N_o); N_o + 1, \ldots, N_o + N_c) ,$$

(2.13)

with some integers $n_{ij}^I, m_{ij}^I, \bar{m}_{ij}^I, r_{ij}^I, \bar{r}_{ij}^I, \bar{r}_{ij}^I \in \mathbb{Z}$ referring to the kinematical factor $\mathcal{K}_I$.

For massless external states these numbers must obey:

$$\sum_{k<j}^{N_o} n_{k,j}^I + \sum_{j<k}^{N_o} n_{j,k}^I + \sum_{i=1}^{N_o} (m_{ij}^I + \bar{m}_{ij}^I) + 2 = 0 \quad , \quad j = 1, \ldots, N_o ,$$

$$r_{ii}^I + \sum_{j=1}^{N_o} m_{ij}^I + \sum_{k<i}^{N_o} (r_{ki}^I + \bar{r}_{ki}^I) + \sum_{i<k}^{N_o} (r_{ik}^I + \bar{r}_{ik}^I) + 2 = 0 \quad , \quad i = 1, \ldots, N_c ,$$

$$r_{ii}^I + \sum_{j=1}^{N_o} \bar{m}_{ij}^I + \sum_{k<i}^{N_o} (\bar{r}_{ki}^I + \bar{r}_{ki}^I) + \sum_{i<k}^{N_o} (\bar{r}_{ik}^I + \bar{r}_{ik}^I) + 2 = 0 \quad , \quad i = 1, \ldots, N_c .$$

(2.14)

Note, that these integers and the kinematical factor $\mathcal{K}_I$ do not depend on the ordering $\sigma$.  

9
2.3. Splitting the complex world-sheet integrals

After performing all Wick contractions the amplitude (2.1) boils down to a product of various polynomials in differences of the open and closed string positions $x_k$ and $z_j, \bar{z}_j$, cf. (2.13). To compute the integral over these positions we write it as an integral over holomorphic and anti–holomorphic coordinates following the method proposed in [1]. After introducing the parameterization $z_j = z_{1j} + i z_{2j}, \ j = 1, \ldots, N_c$ the integrand becomes an analytic function in $z_{2j}$. We then deform the $z_{2j}$–integral along the real axis $\text{Im}(z_{2j}) = 0$ to the pure imaginary axis $\text{Re}(z_{2j}) = 0$, i.e. $iz_{2j} \in \mathbb{R}$. This way, the variables

$$\xi_j = z_{1j} + i z_{2j} \equiv z_j, \quad \eta_j = z_{1j} - i z_{2j} \equiv \bar{z}_j, \quad j = 1, \ldots, N_c \quad (2.15)$$

become real quantities, i.e. $\xi_j, \eta_j \in \mathbb{R}$. We may concentrate on one term of the sum (2.13), i.e. one particular subamplitude, in the following denoted by $A^I_\sigma(N_o, N_c) := A^I(1, \sigma(2), \ldots, \sigma(N_o); N_o + 1, \ldots, N_o + N_c)$ referring to one specific kinematics $K_I$. With the Jacobian $\det \frac{\partial (z_{1j}, z_{2j})}{\partial (\xi_j, \eta_j)} = \left( \frac{i}{2} \right)^{N_c}$, after fixing the position of the first open string vertex at $x_1 = -\infty$ and fixing two other open string positions subject to the ordering $I_\sigma$ for a given kinematics $K_I$ one subamplitude of (2.1) or (2.13) may then be written

$$A^I_\sigma(N_o, N_c) = \left( \frac{i}{2} \right)^{N_c} \left( \int_{I_\sigma} \prod_{l=2}^{N_o-2} dx_l \prod_{i,j=1}^{N_c} \int_{-\infty}^{\infty} d\xi_i \int_{-\infty}^{\infty} d\eta_j \right) \Pi(x_l, \xi_i, \eta_j)$$

$$\times \prod_{2 \leq l_1 < l_2}^{N_o} |x_{l_1} - x_{l_2}|^{2p_{12}} |x_{l_1} - x_{l_2}|^{n_{12}} \prod_{i=1}^{N_c} |\xi_i - \eta_i|^{2q_{i}} (\xi_i - \eta_i)^{-r_{i}}$$

$$\times \prod_{l=2}^{N_o} \prod_{i=1}^{N_c} |x_l - \xi_l|^{2p_{l}} |x_l - \eta_l|^{2p_{l}} \prod_{i,j=1}^{N_c} |\xi_i - \xi_j|^{q_{ij}} |\eta_i - \eta_j|^{q_{ij}} |\xi_i - \eta_j|^{q_{ij}} |\eta_i - \xi_j|^{q_{ij}}$$

$$\times (\xi_i - \xi_j)^{r_{ij}} (\eta_i - \eta_j)^{r_{ij}} (\xi_i - \eta_j)^{r_{ij}} (\eta_i - \xi_j)^{r_{ij}} \quad (2.16)$$

as an integral over $N_o + 2N_c - 3$ real positions $x_l, \xi_i, \eta_j$ with the phase factor:

$$\Pi(x_l, \xi_i, \eta_j) = e^{2\pi i p_{ij} \theta[-(x_l-\xi_l)(x_l-\eta_l)]} e^{i\pi q_{ij} \theta[-(\xi_i-\xi_j)(\eta_i-\eta_j)]}$$

$$\times e^{i\pi q_{ij} \theta[-(\xi_i-\eta_j)(\eta_i-\xi_j)]} e^{2\pi i q_{ij} \theta(\eta_i-\xi_i)} \quad (2.17)$$

In (2.16) the phase factor $\Pi(x_l, \xi_i, \eta_j)$ accounts for the correct branch of the integrand. Note, that this phase is independent on the kinematical structure $K_I$ and the integers as the branching is caused by the kinematic invariants only.
The amplitude (2.16) may be interpreted as a disk amplitude involving $N_o + 2N_c$ open strings with the following $N_o + 2N_c$ open string vertex operator positions $z_r$:

\[
\begin{align*}
    z_1 &= -\infty , & z_l &= x_l , & l &= 2, \ldots, N_o , \\
    z_{N_o+2i-1} &= \xi_i , & z_{N_o+2i} &= \eta_i , & i &= 1, \ldots, N_c .
\end{align*}
\]

(2.18)

For the sequel we introduce the $N_o + 2N_c$ open string momenta $k_r$

\[
\begin{align*}
    k_i &= p_i , & i &= 1, \ldots, N_o \\
    k_{N_o+2j-1} &= \frac{1}{2} D q_j , & k_{N_o+2j} &= \frac{1}{2} q_j , & j &= 1, \ldots, N_c ,
\end{align*}
\]

(2.19)

which fulfill the massless condition $k_r^2 = 0$ for $D = 1$. In terms of these momenta, the momentum conservation (2.12) reads:

\[
\sum_{r=1}^{N_o+2N_c} k_r = 0 .
\]

(2.20)

The (open string) kinematic invariants $\hat{s}_{ij} = \alpha'(k_i + k_j)^2$ become ($\alpha' = 2$):

\[
\hat{s}_{ij} = 4 k_i k_j .
\]

(2.21)

The expression (2.16) is by far not the final expression\footnote{A variant of (2.16) (for the simpler case $D = 1, q^2_{i\parallel} = 0$ and $n, m, \overline{m}, r, \overline{r}, \tilde{r}, \overline{\tilde{r}} = 0$) has been recently given in [22].}. The position integrals in $\xi_i$ and $\eta_j$ take into account all possible open string orderings along the real axis. The set of these positions $\xi_i, \eta_j$ has also a relative ordering w.r.t. to the open string positions $x_l$. In other words, the amplitude (2.16) decomposes into a sum of ordered partial amplitudes $A(1, \Sigma(2), \ldots, \Sigma(N_o + 2N_c))$ involving $N_o + 2N_c$ open strings, with $\Sigma \in S_{N_o+2N_c}/\mathbb{Z}_{N_o+2N_c} \setminus S_{N_o}/\mathbb{Z}_{N_o}$ subject to the given ordering $\sigma$ of the $N_o$ open strings. As we shall see, there exist many non–trivial relations between those partial amplitudes and the expression (2.16) may be written in terms of a basis of a minimal set of partial amplitudes. Indeed, the number of terms in the expression (2.16) may be considerably reduced by deforming the contours in the complex $\eta_j$–planes. This procedure is mathematically equivalent to finding relations between various partial amplitudes and expressing (2.16) in terms of a minimal set. The method of deforming the contours in the complex $\eta_j$–planes and reducing (2.16) to a minimal set is somewhat similar as the method in [1], however much more involved due to the mixing of the holomorphic and anti–holomorphic coordinates $\xi_i$ and $\eta_j$ showing up in the phase (2.17). Eventually, the open string world–sheet integrals, which appear in (2.16) as partial amplitudes, should be mapped to canonical form given by (generalized) Euler integrals along the segment $[0,1]$. This program will be pursued in Section 3. Apart from [1] some earlier work on handling complex (sphere) integrals may be found in Refs. [23]. However, these integrals are much simpler than (2.13) due to the absence of the additional mixing of holomorphic and anti–holomorphic coordinates, which causes additional branchings in the integrand. Hence these reference are not of any help.
2.4. Higher–point disk amplitudes with open and closed strings

After having presented the vertex operators, their interactions and the general form of a disk amplitude of \(N_o\) open and \(N_c\) closed strings, given in (2.13) and (2.16) we are now prepared to compute general disk amplitudes (2.1). In the remainder of this Subsection we shall discuss the six cases \((N_o, N_c) = (2, 1), (3, 1), (2, 2), (4, 1), (0, 3)\) and \((3, 2)\). We shall be concerned with the general case in Section 3.

2.4.1. Three–point disk amplitudes with two open and one closed string

For an amplitude of two open and one closed string in (2.1) we have one partial amplitude with group ordering \(\text{Tr}(T_1 T_2)\). Due to \(\text{PSL}(2, \mathbb{R})\) invariance on the disk we may fix three vertex positions. A convenient choice is

\[
x_1 = -\infty, \quad x_2 = 1, \quad \bar{z} = -ix, \quad z = ix,
\]

with \(x \in \mathbb{R}^+\), cf. also Subsection 2.4.3. This choice implies the \(c\)–ghost contribution:

\[
\langle c(x_1) c(x_2) \tilde{c}(\bar{z}) \rangle = (x_1 - x_2) (x_1 - \bar{z}) (x_2 - \bar{z}) = x_\infty^2 (1 + ix).
\]

Therefore, the partial amplitude of (2.1) becomes:

\[
A(1, 2; 3) = x_\infty^2 \int_0^\infty dx \ (1 + ix) \ \langle : V_o(x_\infty) : : V_o(1) : : V_c(-ix, ix) : \rangle.
\]

For this three–point process to give a result, which does not vanish on–shell, the closed string momentum \(q\) has to have also a non–vanishing direction \(q_\perp \neq 0\) transverse to the D–brane world–volume, with \(q = q_\parallel + q_\perp\). For the choice (2.22) the correlator (2.11) assumes the form

\[
\langle e^{2ip_1 \mu X^\mu (-\infty)} e^{2ip_2 \mu X^\mu (1)} e^{iq_\mu X^\mu (-ix, ix)} \rangle = (2x)^s \ (1 + ix)^t \ (1 - ix)^t, \quad (2.25)
\]

with the kinematic invariants

\[
s = 4 \ p_1 p_2, \quad t = u = 2 \ p_1 q = 2 \ p_2 q = -2 \ p_1 p_2 = -q_\parallel^2, \quad (2.26)
\]

i.e. \(s = -2t\) [7,10]. After performing all Wick contractions, for each kinematics \(K_I\) the expression (2.24) generically reduces to the form

\[
A^I(1, 2; 3) = 2 \int_0^\infty dx \ (2x)^{s-2-2n_1^I} \ (1 + ix)^{t+n_1^I} \ (1 - ix)^{t+n_1^I} = 2^{s-2-2n_1^I} B\left(\frac{s}{2} - t + \frac{1}{2}, \frac{s}{2} - \frac{1}{2} - n_1^I\right), \quad (2.27)
\]

\[^3\text{An alternative choice is: } x_1 = -x, \ x_2 = x, \ \bar{z} = -i, \ z = i, \text{ with } x \in \mathbb{R}_+ [8].\]
with some integer \( n^I \).

The result (2.27) originates from the more general class\(^4\) of complex integrals

\[
I(\alpha_0, \alpha_1, \alpha_2) = \int_0^\infty dx \ x^{\alpha_0} (x - i)^{\alpha_1} (x + i)^{\alpha_2} = i \ e^{\frac{1}{2}i\pi(\alpha_0 + \alpha_1 + \alpha_2)}
\]

\[
\times \left\{ B(1 + \alpha_0 + \alpha_1, -1 - \alpha_0 - \alpha_1 - \alpha_2) \ 2F_1 \left[ \begin{array}{c} -\alpha_1, -1 - \alpha_0 - \alpha_1 - \alpha_2 \\ -\alpha_0 - \alpha_1 \end{array} \right] ; -1 \right\}
\]

\[
- e^{i\pi(\alpha_0 + \alpha_1)} B(1 + \alpha_0, -1 - \alpha_0 - \alpha_1) \ 2F_1 \left[ \begin{array}{c} -\alpha_2, 1 + \alpha_0 \\ 2 + \alpha_0 + \alpha_1 \end{array} \right] ; -1 \right\},
\]

(2.29)

with \( \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R} \), constrained by analyticity to \( \text{Re}(\alpha_0) > -1 \), \( \text{Re}(\alpha_0 + \alpha_1 + \alpha_2) < -1 \). The integral (2.29) along the positive real axis may be derived from a closed contour \( C_0 + C_\infty \) in the complex \( x \)-plane. This contour, which in Fig. 2 is drawn in blue, consists of one large \( C_\infty \) and one small circle \( C_0 \). The latter encircles the point \( x = 0 \) clockwise. Except the point \( x = 0 \) all singularities of the integrand are inside of the large circle \( C_\infty \).

![Fig. 2 Complex x–plane and contour integrations.](image)

---

\(^4\) An often encountered case is \( \alpha_0 = \delta - 1 \), \( \alpha_1 = \alpha - \delta \), \( \alpha_2 = -\alpha - \delta \), for which we have [12,10]:

\[
I(\delta - 1, \alpha - \delta, -\alpha - \delta) = \sqrt{\pi} 2^{-\delta} e^{-i\pi\frac{\delta}{2}} \frac{\Gamma \left( \frac{\delta}{2} \right) \Gamma \left( \frac{1}{2} + \frac{\delta}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{\delta}{2} - \frac{\alpha}{2} \right) \Gamma \left( \frac{1}{2} + \frac{\delta}{2} + \frac{\alpha}{2} \right)}.
\]

(2.28)
The circle $C_{\infty}$ may be deformed to infinity, where the integrand approaches zero, and hence gives a zero contribution. On the other hand, the full contour $C_0 + C_{\infty}$ may be deformed to the two loops $C_1$ and $C_2$ in Fig. 2 drawn in red: The loop\(^5\) $C_1 \int_{0}^{(i+)}$ starts at $x = 0$, encircles once the point $x = i$ and returns to its starting point. The second loop $C_2 \int_{-i\infty}^{(-i+)}$ starts at $x = -i\infty$ encircles once the point $x = -i$ and returns to its starting point. After inspecting the other contours in the complex $x$–plane we find\(^6\)

$$
\int_{+\infty}^{(0-)} dx \, x^{0} \, (x - i)^{\alpha_1} (x + i)^{\alpha_2} = 2i \sin[\pi(\alpha_0 + \alpha_1)] (-1)^{\alpha_0} I(\alpha_0, \alpha_1, \alpha_2)
$$

$$
= -2 \, e^{\pi(\alpha_0 + \alpha_1)} \sin(\pi \alpha_1) \int_{0}^{1} dx \, x^{\alpha_0} (1 - x)^{\alpha_1} (1 + x)^{\alpha_2}
$$

$$
+ 2 \, \sin(\pi \alpha_2) \int_{-\infty}^{-1} dx \, (-x)^{\alpha_0} (1 - x)^{\alpha_1} (-1 - x)^{\alpha_2},
$$

which proves (2.29). Alternatively, the result (2.29) may be written in the following form

$$
I(\alpha_0, \alpha_1, \alpha_2) = -i \, e^{\frac{1}{2} \pi(\alpha_0 + \alpha_1 + \alpha_2)} e^{\pi(\alpha_0 + \alpha_1 + \alpha_2)} \times \left\{ B(1 + \alpha_0 + \alpha_2, -1 - \alpha_0 - \alpha_1 - \alpha_2) \, {}_{2}F_{1} \left[ \begin{array}{c} -\alpha_2, -1 - \alpha_0 - \alpha_1 - \alpha_2 \\ -\alpha_0 - \alpha_2 \end{array} \right] \right\},
$$

which may be derived by deforming the full contour $C_0 + C_{\infty}$ differently. In this case the contour $C_1$ starts at $x = i\infty$, encircles once the point $x = i$ and returns to its starting point, while the contour $C_2$ starts at $x = 0$, encircles once the point $x = -i$ and returns to the origin. Hence we have:

$$
\int_{+\infty}^{(0-)} dx \, x^{\alpha_0} \, (x - i)^{\alpha_1} (x + i)^{\alpha_2} = 2i \sin[\pi(\alpha_0 + \alpha_2)] (-1)^{\alpha_0} I(\alpha_0, \alpha_1, \alpha_2)
$$

$$
= -2 \, e^{-i\pi(\alpha_0 + \alpha_2)} \sin(\pi \alpha_2) \int_{0}^{0} dx \, (-x)^{\alpha_0} (1 - x)^{\alpha_1} (1 + x)^{\alpha_2}
$$

$$
+ 2 \, \sin(\pi \alpha_1) \int_{1}^{\infty} dx \, x^{\alpha_0} (x - 1)^{\alpha_1} (1 + x)^{\alpha_2}.
$$

\(^5\) The integral $\int_{z}^{(w+)} dx \, f(x)$ defines an integral taken along a contour $C$, which starts at a point $z$ in the complex $x$–plane, encircles the point $w$ once counter–clockwise and returns to its starting point. All singularities of the integrand except $x = w$ are outside of $C$.

\(^6\) Here $(-z)^{\alpha_0} := e^{\alpha_0 \ln(-z)}$ is defined through the principal value of the logarithm.
Furthermore, the result (2.29) may also be derived by using (3.197.3) of [24]:

\[ I(\alpha_0, \alpha_1, \alpha_2) = -ie^{\frac{1}{2}i\pi(\alpha_0+\alpha_1+\alpha_2)}e^{i\pi(\alpha_0+\alpha_1)} \times B(1+\alpha_0, -1-\alpha_0-\alpha_1-\alpha_2) \, _2F_1 \left[ \begin{array}{c} 1+\alpha_0, -\alpha_2; \\ -\alpha_1-\alpha_2 \end{array} \right]. \] (2.31)

With Eq. 2.10(4) of [25] this becomes Eq. (2.29).

After having discussed the results (2.29) and (2.30) for the general complex integral \( I(\alpha_0, \alpha_1, \alpha_2) \), we are now prepared to write the result (2.27) in a different form. For \( \alpha_0 = -2\alpha_2 - 2 = -2t - 2n_1 - 2 \) and \( \alpha_1 = \alpha_2 = t + n_1 = u + n_1 \) the two expressions (2.29) and (2.30) may be combined to give (2.27):

\[ A(1, 2; 3) = \frac{i}{4} \left\{ e^{i\pi(t+n_1)} B(s - 2n_1 - 1, u + n_1 + 1) + B(u + n_1 + 1, t + n_1 + 1) \right. \\
+ e^{i\pi(t+n_1)} B(t + n_1 + 1, s - 2n_1 - 1) \right\}. \] (2.32)

Indeed, with the identity

\[ e^{i\pi(s+n_1)} B(s + n_1 + 1, u - n_1 - n_2 - 1) + B(t + n_2 + 1, s + n_1 + 1) \\
+ e^{-i\pi(t+n_2)} B(t + n_2 + 1, u - n_1 - n_2 - 1) = 0, \quad s + t + u = 0, \quad n_i \in \mathbb{Z} \] (2.33)

we may prove the equivalence of (2.27) and (2.32). Above we have dropped the kinematical index \( I \).

### 2.4.2. Three open strings and one closed string

For an amplitude of three open and one closed string in (2.1) we have two different partial amplitudes. Due to \( PSL(2, \mathbb{R}) \) invariance on the disk we may fix three vertex positions. For the partial amplitude with group ordering \( \text{Tr}(T_1T_2T_3) \) a convenient choice is

\[ x_1 = x_\infty := -\infty, \quad x_2 = 0, \quad x_3 = 1. \] (2.34)

This choice implies the \( c \)-ghost contribution:

\[ \langle c(x_1)c(x_2)c(x_3) \rangle = (x_1 - x_2) (x_1 - x_3) (x_2 - x_3) = x_\infty^2. \] (2.35)

For this choice in the amplitude (2.1) we are left over with one integration of the closed string position \( z \) over the complex plane \( \mathbb{C} \). Therefore, the partial amplitude (2.1) becomes:

\[ A(1, 2, 3; 4) = \langle c(-\infty)c(0)c(1) \rangle \int_\mathbb{C} d^2z \langle : V_\circ(-\infty) : : V_\circ(0) : : V_\circ(1) : : V_c(z, \bar{z}) : \rangle. \] (2.36)

---

The partial amplitude with the group ordering \( \text{Tr}(T_1T_3T_2) \) may be simply obtained by interchanging of the second and third open string.
For the choice \((2.34)\) the correlator \((2.11)\) assumes the form

\[
\langle e^{2ip_1 \mu X^\mu(-\infty)} e^{2ip_2 \mu X^\mu(0)} e^{2ip_3 \mu X^\mu(1)} e^{iq_\nu X^\nu(\overline{z}; z)} \rangle = z^t \overline{z}^s \ (1-z)^t \ (1-\overline{z})^s \ |z-\overline{z}|^{2q_\parallel^2}, \tag{2.37}
\]

with the kinematic invariants:

\[
s = 2 \ p_3 q \ , \ t = 2 \ p_2 q \ , \ u = 2 \ p_1 q . \tag{2.38}
\]

From \((2.12)\) it follows: \(s + t + u = -2 \ q_\parallel^2\). Furthermore, we have: \(p_1p_2 = \frac{s}{2} + \frac{q_\parallel^2}{2}, \ p_2p_3 = \frac{t}{2} + \frac{q_\parallel^2}{2}\) and \(p_1p_3 = \frac{u}{2} + \frac{q_\parallel^2}{2}\). After performing all Wick contractions the amplitude \((2.36)\) boils down to

\[
A^f(1, 2, 3; 4) = G^{(\alpha^f)} \left[ t + n_1^f, s + m_1^f \right] , \\
\left[ t + n_2^f, s + m_2^f \right] ,
\]

for any kinematics \(K^f\) with four integers \(n_1^f, m_1^f\) and \(\alpha^f \in \mathbb{R}\) and the integral

\[
G^{(\alpha)} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] := \int_C d^2z \ z^{\lambda_1} \ \overline{z}^{\lambda_2} \ (1-z)^{\gamma_1} \ (1-\overline{z})^{\gamma_2} \ |z-\overline{z}|^{\tilde{\alpha}} \ (z-\overline{z})^{\tilde{\alpha}} \\
= \pi \ \lambda_2 \ \frac{\Gamma(1 + \gamma_1) \ \Gamma(-1 - \alpha - \lambda_2 - \gamma_2) \ \Gamma(2 + \alpha + \lambda_1 + \lambda_2)}{\Gamma(1 - \lambda_2) \ \Gamma(-\alpha - \gamma_2) \ \Gamma(3 + \alpha + \lambda_1 + \lambda_2 + \gamma_1)} \\
\times \ \binom{\begin{array}{c} -\gamma_2, 1 + \lambda_2, 1 + \gamma_1 \\ -\alpha - \gamma_2, 3 + \alpha + \lambda_1 + \lambda_2 + \gamma_1 \end{array}}{1} + \pi e^{i\pi(m_1 + m_2)} \\
\times \ \alpha \ \frac{\Gamma(1 + \gamma_1) \ \Gamma(-1 - \alpha - \lambda_2 - \gamma_2) \ \Gamma(-2 - \alpha - \lambda_1 - \lambda_2 - \gamma_1 - \gamma_2)}{\Gamma(1 - \alpha) \ \Gamma(-1 - \alpha - \lambda_1 - \lambda_2 - \gamma_2) \ \Gamma(-\lambda_2 - \gamma_2)} \\
\times \ \binom{\begin{array}{c} -\gamma_2, -1 - \alpha - \lambda_2 - \gamma_2, -\alpha - 2 - \lambda_1 - \lambda_2 - \gamma_1 - \gamma_2 \\ -\lambda_2 - \gamma_2, -1 - \alpha - \lambda_1 - \lambda_2 - \gamma_2 \end{array}}{1}, \tag{2.40}
\]

with

\[
\lambda_i = t + n_i \ , \ \gamma_i = s + m_i \ , \ \alpha = \tilde{\alpha} + \tilde{\alpha} , \tag{2.41}
\]

and \(n_i, m_i, \tilde{\alpha} \in \mathbb{Z}\). Here \(\tilde{\alpha}\) denotes the non–integer part of \(\alpha\), \(i.e. \ \tilde{\alpha} = 2q_\parallel^2\). The integral \((2.40)\) is evaluated in Appendix A. A special case arises for \(\tilde{\alpha} = 0\), \(G^{(0)} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] := V \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] ,\)

with \([26,1]::

\[
V \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] := \int_C d^2z \ z^{\lambda_1} \ \overline{z}^{\lambda_2} \ (1-z)^{\gamma_1} \ (1-\overline{z})^{\gamma_2} = \pi \ \frac{\Gamma(1 + \lambda_1) \ \Gamma(1 + \gamma_1)}{\Gamma(2 + \lambda_1 + \gamma_1)} \ \frac{\Gamma(-1 - \lambda_2 - \gamma_2)}{\Gamma(-\lambda_2) \ \Gamma(-\gamma_2)} . \tag{2.42}
\]
2.4.3. Two open strings and two closed strings

For an amplitude of two open and two closed strings in (2.1) we have one partial amplitude with group ordering \( \text{Tr}(T_1T_2) \). The conformal killing group \( \text{PSL}(2, \mathbb{R}) \) of the disk allows to fix three vertex positions. On the double cover \( \mathbb{C} \), with an appropriate \( \text{PSL}(2, \mathbb{C}) \) transformation we may fix two positions on the boundary and the real part of a closed string modulus. A convenient choice is \((z_1 := x_1, \ z_2 := x_2)\)

\[
z_1 = z_\infty := -\infty \ , \ z_2 = 1 \ , \ \bar{z}_3 = -ix \ , \ z_3 = ix \ , \ \bar{z}_4 = \bar{z} \ , \ z_4 = z \ , \quad (2.43)
\]

with \( z \in \mathbb{H}_+ \) and \( x \in \mathbb{R}_+ \). Three arbitrary points \( w_1, w_2 \in \mathbb{R} \) and \( w_3 \in \mathbb{C} \) may be mapped to the points \( z_1, z_2, z_3 \) of (2.43) by the following \( \text{PSL}(2, \mathbb{C}) \) transformation

\[
P = \begin{bmatrix} w_{12} & w_{13} & w_{23} (1 - ix) \end{bmatrix}^{-1/2} \begin{bmatrix} -w_{12} + ix & w_3 & w_{12} - ix \ w_{23} & -w_1 \end{bmatrix} \in \text{PSL}(2, \mathbb{C}) ,
\]

(2.44)

with \( w_i \rightarrow \frac{p_{11} w_i + p_{13}}{p_{21} w_i + p_{22}} \) and \( w_{ij} = w_i - w_j \). The transformation (2.44) must map boundary points \( w \in \mathbb{R} \) onto boundary points. This requirement\(^8\) yields the following relation between \( w_1, w_2, w_3 \) and \( x \)

\[
x = \frac{\text{Im}(w_3) (w_1 - w_2)}{|w_3 - \frac{1}{2} (w_1 + w_2)|^2 - \frac{1}{4} (w_1 - w_2)^2} \in \mathbb{R} ,
\]

(2.45)

which constrains (2.44) to be an element of \( \text{PSL}(2, \mathbb{R}) \). Indeed with (2.45) the matrix (2.44) becomes:

\[
P = |w_{13}| w_{12}^{1/2} \left[ \left| w_3 - \frac{1}{2} (w_1 + w_2) \right|^2 - \frac{1}{4} w_{12}^2 \right]^{1/2} \times \begin{bmatrix} \frac{1}{2} (w_3 + \bar{w}_3) - w_1 \ w_{12} \\
|w_3 - \frac{1}{2} (w_1 + w_2)|^2 - \frac{1}{4} w_{12}^2 \ -w_1 \left[ \left| w_3 - \frac{1}{2} (w_1 + w_2) \right|^2 - \frac{1}{4} w_{12}^2 \right] \end{bmatrix} \in \text{PSL}(2, \mathbb{R}) .
\]

Depending on the ordering of the points \( w_1, w_2 \) along the boundary and the values of \( \text{Im}(w_3) \), the number \( x \) may take positive or negative real values, respectively, \( \text{cf.} \) the next Figure.

---

\(^8\) We wish to stress, that this condition rules out the more symmetric choice: \( z_1 = -\infty \), \( z_2 = 0 \), \( \bar{z}_3 = -ix \), \( z_3 = ix \), \( \bar{z}_4 = \bar{z} \), \( z_4 = z \), which would give the empty solution for \( x \).
In the double cover the closed string position $z$ is integrated over the full complex plane $C$ and the $x$–integration goes from $-\infty$ to $\infty$. The choice (2.43) implies the $c$–ghost contribution:

$$\langle c(z_1)c(z_2)\tilde{c}(\overline{z}_3) \rangle = (z_1 - z_2) (z_1 - \overline{z}_3) (z_2 - \overline{z}_3) = (1 + ix) z^2_{\infty}.$$  \hfill (2.46)

Eventually, the partial amplitude (2.1) becomes:

$$A(1, 2; 3, 4) = \int_{-\infty}^{\infty} dx \, \langle c(-\infty)c(1)c(ix) \rangle \times \int_{C} d^2 z \, \langle : V_o(-\infty) : : V_o(1) : : V_c(-ix, ix) : : V_c(z, z) : \rangle .$$  \hfill (2.47)

For the choice (2.43) the correlator (2.11) becomes

$$\langle e^{2ip_1 \mu X^\mu (-\infty)} e^{2ip_2 \mu X^\mu (1)} e^{iq_1 \mu X^\mu (-ix, ix)} e^{iq_2 \mu X^\mu (z, z)} \rangle = (1 + ix)^u (1 - ix)^u (1 - z)^t (1 - \overline{z})^t (z + ix)^s/2 (z - ix)^s/2 (z - \overline{z})^s/2 (\overline{z} + ix)^s/2 ,$$  \hfill (2.48)

with the kinematic invariants:

$$u = 2 \, p_2 q_1 = 2 \, p_1 q_2 , \quad t = 2 \, p_2 q_2 = 2 \, p_1 q_1 , \quad s = 2 \, q_1 q_2 = 2 \, p_1 p_2 .$$  \hfill (2.49)

Of course, with (2.12), we have: $s + t + u = 0$. After performing all Wick contractions for each kinematics $K_I$ the amplitude (2.47) boils down to

$$A^I(1, 2; 3, 4) = W^{(\kappa^I, \alpha^I_0)} \left[ u + n^I_0 , t + n^I_1 , \frac{1}{2}s + n^I_3 , \frac{1}{2}s + n^I_5 \right] ,$$  \hfill (2.50)
with the ten integers \( m_i, n_i, \kappa, \alpha_0 \in \mathbb{Z} \) and the integrals of the following type\(^9\)

\[
W^{(\kappa, \alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] = \int_{-\infty}^{\infty} dx \ x^{\alpha_0} \ (1 + ix)^{\alpha_1} \ (1 - ix)^{\alpha_2} \ I^{(\kappa)} \left[ \begin{array}{c} \lambda_1, \gamma_1, \beta_1 \\ \lambda_2, \gamma_2, \beta_2 \end{array} \right] (ix), \quad (2.54)
\]

with

\[
I^{(\kappa)} \left[ \begin{array}{c} \lambda_1, \gamma_1, \beta_1 \\ \lambda_2, \gamma_2, \beta_2 \end{array} \right] (ix) = \int_{C} d^2z \ (1 - z)^{\lambda_1} \ (1 - \overline{z})^{\lambda_2} \ (z - ix)^{\gamma_1} \ (z + ix)^{\beta_1} \times (\overline{z} + ix)^{\gamma_2} \ (\overline{z} - ix)^{\beta_2} \ (z - \overline{z})^{\kappa},
\]

and the assignments

\[
\begin{align*}
\alpha_1 &= u + n_0, & \lambda_1 &= t + n_1, & \gamma_1 &= \frac{1}{2} s + n_3, & \beta_1 &= \frac{1}{2} s + n_5, \\
\alpha_2 &= u + m_0, & \lambda_2 &= t + n_2, & \gamma_2 &= \frac{1}{2} s + n_4, & \beta_2 &= \frac{1}{2} s + n_6,
\end{align*}
\]

and the integers \( m_i, n_i, \kappa, \alpha_0 \in \mathbb{Z} \) subject to the analyticity condition (2.14):

\[
\alpha_0 + \alpha_1 + \alpha_2 + \lambda_1 + \lambda_2 + \gamma_1 + \gamma_2 + \beta_1 + \beta_2 + \kappa + 4 = 0. \quad (2.57)
\]

\(^9\) Alternatively, instead of the choice (2.43) we could have chosen

\[
z_1 = z_\infty := -y, \quad z_2 = y, \quad z_3 = -i, \quad z_3 = i, \quad \overline{z}_4 = \overline{w}, \quad z_4 = w, \quad (2.51)
\]

with \( y \in \mathbb{R}, \ w \in \mathbb{C} \) in lines of Footnote 3. For this case we find:

\[
\langle e^{2ip_1 \mu x^\mu (-y)} e^{2ip_2 \mu x^\mu (y)} e^{iq_1 \mu x^\mu (-i)} e^{iq_2 \mu x^\mu (\overline{w}, w)} \rangle = (2y)^{2s} \ (y + i)^{t+u} \ (y - i)^{t+u} \times (w - i)^{s/2} \ (w + i)^{s/2} \ (\overline{w} - i)^{s/2} \ (\overline{w} + i)^{s/2} \ (y - w)^{t} \ (y - \overline{w})^{t} \ (y + w)^{u} \ (y + \overline{w})^{u}.
\]

However, with the transformation

\[
z = \frac{2y}{1 - y^2} \ \frac{1 - yw}{y + w}, \quad x = -\frac{2y}{1 - y^2} \quad (2.53)
\]

we may identify the two choices (2.43) and (2.51), \textit{i.e.} Eqs. (2.48) and (2.52) become identical. Furthermore, the transformation (2.53) gives the alternative representation for \( W \):

\[
W^{(\kappa, \alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] = 2^{-4} \int_{-\infty}^{\infty} dy \ \int_{C} d^2w \ (w - \overline{w})^{\kappa} \ (2y)^{\alpha_0 + \gamma_1 + \gamma_2 + \beta_1 + \beta_2 + \kappa} \times (y + i)^{2\alpha_1 + \lambda_1 + \lambda_2 + \gamma_2 + \beta_1 + \beta_2 + \kappa} \times (w - i)^{\gamma_1} \ (w + i)^{\beta_1} \ (\overline{w} - i)^{\beta_2} \ (\overline{w} + i)^{\gamma_2} \times (y - w)^{\lambda_1} \ (y - \overline{w})^{\lambda_2} \ (y + w)^{-\lambda_1 - \gamma_1 - \beta_1 - \kappa} \ (y + \overline{w})^{-\lambda_2 - \gamma_2 - \beta_2 - \kappa},
\]

subject to the condition (2.57).
Clearly, the complex integral (2.55) fulfills:

$$I^{(\kappa)} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (-x) = I^{(\kappa)} \left[ \frac{\lambda_1, \beta_1, \gamma_1}{\lambda_2, \beta_2, \gamma_2} \right] (x).$$ (2.58)

The integrand of (2.1) is invariant under the parity symmetries $\bar{z}_3 \leftrightarrow z_3$ and $\bar{z}_4 \leftrightarrow z_4$ acting on the positions of the closed string vertex operators. After the fixing (2.43) in Eq. (2.47) these symmetries become the operations $x \to -x$ and $\bar{z} \leftrightarrow z$, respectively. Moreover, the integral (2.54) shares the following symmetry

$$W^{(\kappa, \alpha_0)} \left[ \frac{\alpha_1, \lambda_1, \gamma_1, \beta_1}{\alpha_2, \lambda_2, \gamma_2, \beta_2} \right] = (-1)^{\alpha_0 + \alpha_1 + \alpha_2 + \lambda_1 + \lambda_2}$$

$$\times W^{(\kappa, \alpha_0)} \left[ \frac{2 + \kappa + \alpha_1 + \lambda_1 + \lambda_2 + \beta_1 + \gamma_2}{2 + \kappa + \alpha_2 + \lambda_1 + \lambda_2 + \gamma_1 + \beta_2} \right]$$

$$\left[ -2 - \kappa - \lambda_1 - \gamma_1 - \beta_1, 2 + \kappa + \alpha_1 + \lambda_1 + \lambda_2 + \beta_1 + \gamma_2, \beta_2, \gamma_1 \right]$$

$$\left[ -2 - \kappa - \lambda_2 - \gamma_2 - \beta_2, 2 + \kappa + \alpha_2 + \lambda_1 + \lambda_2 + \gamma_1 + \beta_2, \beta_1, \gamma_2 \right].$$ (2.59)

which exchanges the invariants $t$ and $u$ in the exponents (2.56). It may be proven by performing the following change of variables

$$z \to \frac{w + y^2}{w - 1}, \quad x \to y$$

in the integrand of the l.h.s. of (2.59). Furthermore, for (2.57) the integral (2.54) enjoys the following identity:

$$W^{(\kappa, \alpha_0)} \left[ \frac{\alpha_1, \lambda_1, \gamma_1, \beta_1}{\alpha_2, \lambda_2, \gamma_2, \beta_2} \right] = 2^{\kappa - \alpha_0} (-1)^{\kappa + \lambda_1 + \lambda_2}$$

$$\times W^{(\alpha_0, \kappa)} \left[ \frac{-2 - \kappa - \lambda_1 - \gamma_1 - \beta_1, 2 + \kappa + \alpha_1 + \lambda_1 + \lambda_2 + \beta_1 + \gamma_2, \beta_2, \gamma_1}{-2 - \kappa - \lambda_2 - \gamma_2 - \beta_2, 2 + \kappa + \alpha_2 + \lambda_1 + \lambda_2 + \gamma_1 + \beta_2, \beta_1, \gamma_2} \right].$$ (2.60)

This relation may be proven by first performing the change of variables

$$x \to \frac{z_2}{1 - z_1}, \quad z_1 \to -\frac{z_1}{1 - z_1}, \quad z_2 \to \frac{x}{1 - z_1},$$

with $z_1 = \frac{1}{2}(z + \bar{z})$ and $z_2 = \frac{1}{2}(z - \bar{z})$ in the integrand of the l.h.s. of (2.60) and then applying (2.59). The relation (2.60) proves to be useful for converting negative powers of $(z - \bar{z})$ into negative powers of $x$, e.g.:

$$W^{(-2, 0)} \left[ \frac{u - 1, t, \frac{s}{2}, \frac{s}{2}}{u - 1, t, \frac{s}{2}, \frac{s}{2}} \right] = \frac{1}{4} \ W^{(0, -2)} \left[ \frac{u, t - 1, \frac{s}{2}, \frac{s}{2}}{u, t - 1, \frac{s}{2}, \frac{s}{2}} \right].$$ (2.61)

Let us now discuss the evaluation of the integral (2.54). To compute the integral (2.55) over the complex $z$–plane we split it up into holomorphic and anti–holomorphic contour integrals along the method proposed in [1]. After introducing the parameterization
$z = z_1 + iz_2$ the integrand may be considered as an analytic function in $z_2$. We then deform the $z_2$–integral along the real axis $\text{Im}(z_2) = 0$ to the pure imaginary axis $\text{Re}(z_2) = 0$, i.e. $iz_2 \in \mathbb{R}$. This way, the variables $\xi = z_1 + iz_2 \equiv z$, $\eta = z_1 - iz_2 \equiv \bar{z}$ become real quantities, i.e. $\xi, \eta \in \mathbb{R}$. Similarly, we deform the $x$–integration from the real axis to a contour along the pure imaginary axis, i.e. $\rho = ix$ becomes real. With the Jacobian $\frac{\partial(x,z_1,z_2)}{\partial(\rho,\xi,\eta)} = \frac{1}{2}$ we arrive at:

$$W^{(\kappa,\alpha_0)}[\alpha_1,\lambda_1,\gamma_1,\beta_1,\alpha_2,\lambda_2,\gamma_2,\beta_2] = \frac{1}{2} \int_{-\infty}^{\infty} d\rho \ |1 + \rho|^{\hat{\alpha}_1} |1 - \rho|^{\hat{\alpha}_2} \rho^{\alpha_0} (1 + \rho)^{\alpha_0} (1 - \rho)^{\mu_0} \times$$

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \ |1 - \xi|^{\hat{\lambda}_1} |\xi - \rho|^{\hat{\gamma}_1} |\xi + \rho|^{\hat{\beta}_1} \times |1 - \eta|^{\hat{\lambda}_2} |\eta + \rho|^{\hat{\gamma}_2} |\eta - \rho|^{\hat{\beta}_2} (\xi - \eta)^{\kappa} \Pi(\rho,\xi,\eta) \times$$

$$(1 - \xi)^{\mu_1} (\xi - \rho)^{\mu_3} (\xi + \rho)^{\mu_5} (1 - \eta)^{\mu_2} (\eta + \rho)^{\mu_4} (\eta - \rho)^{\mu_6} .$$

(2.62)

The quantities with a hat refer to their non–integer part. In (2.62) the phase factor $\Pi(\rho,\xi,\eta)$ following from (2.17) accounts for the correct branch of the integrand. The phases are analyzed in Appendix B. Eventually, in the integrand the latter are accommodated by choosing the respective contours in the complex $\eta$– and $\rho$–plane. More precisely, for a given pair of $\rho, \xi \in \mathbb{R}$ we may consider the $\eta$–integral as an integration in the complex $\eta$–plane and the phases $\Pi(\rho,\xi,\eta)$ give rise to the integration contours in the complex $\eta$–plane as shown in Appendix B. The result for (2.50) is presented in Subsection 3.4.

2.4.4. Four open strings and one closed string

For an amplitude of four open and one closed string in (2.1) in total we have six partial amplitudes $A(1,\sigma(2),\sigma(3),\sigma(4);5)$, with $\sigma \in S_3$ permuting the open strings. Due to $\text{PSL}(2,\mathbb{R})$ invariance on the disk we may fix three vertex positions. A convenient choice is ($z_i := x_i, \ i = 1,\ldots,4$)

$$z_1 = -\infty , \ z_2 = 1 , \ z_3 = -x , \ z_4 = x , \ z_5 = z ,$$

(2.63)

with $z \in \mathbb{H}_+$ and $x \in \mathbb{R}$. After analytic continuation in $x$ this setup becomes similar to the setup (2.43), relevant for two open and two closed strings. In the double cover the closed string position $z$ is integrated over the full complex plane $\mathbb{C}$. The choice (2.63) implies the $c$–ghost contribution:

$$\langle c(z_1)c(z_2)c(z_3) \rangle = (z_1 - z_2) (z_1 - z_3) (z_2 - z_3) = -(1 + x) z_\infty^2 .$$

(2.64)
Eventually, the partial amplitude (2.1) becomes:

$$A(1, \sigma(2), \sigma(3), \sigma(4); 5) = \int_{\mathcal{I}_\sigma} dx \, \langle c(-\infty)c(-x)c(1) \rangle$$

$$\times \int_{\mathcal{C}} d^2 z \, \langle : V_o(-\infty) : : V_o(1) : : V_o(-x) : : V_o(x) : : V_c(z) : \rangle .$$

(2.65)

For the choice (2.63) the integration range $\mathcal{I}_\sigma$ in the real variable $x$ is related to the specific ordering $\sigma$ as follows:

$$A(1, 3, 4, 2; 5) \simeq A(1, 2, 4, 3; 5) : \mathcal{I}_{\sigma_1} = \{ x \in \mathbb{R} \mid 0 < x < 1 \} ,$$

$$A(1, 4, 3, 2; 5) \simeq A(1, 2, 3, 4; 5) : \mathcal{I}_{\sigma_2} = \{ x \in \mathbb{R} \mid -1 < x < 0 \} ,$$

$$A(1, 4, 2, 3; 5) \simeq A(1, 3, 2, 4; 5) : \mathcal{I}_{\sigma_3} = \{ x \in \mathbb{R} \mid -\infty < x < -1 \cup 1 < x < \infty \} .$$

(2.66)

The correlator (2.11) becomes for the choice (2.63):

$$\langle e^{2i p_1 \mu X^\mu(-\infty)} e^{2i p_2 \mu X^\mu(1)} e^{2i p_3 \mu X^\mu(-x)} e^{2i p_4 \mu X^\mu(x)} e^{i q \mu X^\mu(z)} \rangle$$

$$= |2x|^{4p_3 p_4} |1 + x|^{4p_2 p_3} |1 - x|^{4p_2 p_4} (1 - z)^{2p_2 q} (1 - \overline{z})^{2p_2 q}$$

$$\times (x + z)^{2p_3 q} (x + \overline{z})^{2p_3 q} (x - z)^{2p_4 q} (x - \overline{z})^{2p_4 q} |z - \overline{z}|^{2q^2} .$$

(2.67)

We have the five kinematic invariants:

$$s_1 = 4 \, p_1 p_2 , \quad s_2 = 4 \, p_2 p_3 , \quad s_3 = 4 \, p_3 p_4 , \quad s_4 = 2 \, p_4 q , \quad s_5 = 2 \, p_1 q .$$

(2.68)

Furthermore, we have $p_1 p_3 = -\frac{s_1}{4} - \frac{s_3}{4} + \frac{q^2}{4}$, $p_1 p_4 = -\frac{s_2}{4} - \frac{s_4}{4} - \frac{s_5}{4} - \frac{q^2}{4}$, $p_2 p_4 = -\frac{s_1}{4} - \frac{s_3}{4} + \frac{q^2}{4}$, and $p_2 q = -\frac{s_1}{4} + \frac{s_3}{4} - \frac{s_5}{4} - \frac{q^2}{4}$, $p_3 q = -\frac{s_1}{4} - \frac{s_3}{4} - \frac{s_5}{4} - \frac{q^2}{4}$. After performing all Wick contractions for a given kinematics $\mathcal{K}$ the amplitude (2.65) boils down to

$$A_{\mathcal{K}}^I(1, \sigma(2), \sigma(3), \sigma(4); 5) = W_{\sigma}^{(\kappa, \alpha_0)} \left[ \begin{array}{c} \alpha_1^I, \lambda_1^I, \gamma_1^I, \beta_1^I \\ \alpha_2^I, \lambda_2^I, \gamma_2^I, \beta_2^I \end{array} \right],$$

(2.69)

and the integrals of the following type

$$W_{\sigma}^{(\kappa, \alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] := 2^{\alpha_0} \int_{\mathcal{I}_\sigma} dx \, |x|^{\alpha_0} |1 + x|^{\alpha_1} |1 - x|^{\alpha_2}$$

$$\times x^{m_0} (1 + x)^{m_1} (1 - x)^{m_2} I^{(\kappa)} \left[ \begin{array}{c} \lambda_1, \gamma_1, \beta_1 \\ \lambda_2, \gamma_2, \beta_2 \end{array} \right] (x) ,$$

(2.70)

with

$$I^{(\kappa)} \left[ \begin{array}{c} \lambda_1, \gamma_1, \beta_1 \\ \lambda_2, \gamma_2, \beta_2 \end{array} \right] (x) = \int_{\mathcal{C}} d^2 z \, (1 - z)^{\lambda_1} (1 - \overline{z})^{\lambda_2} (z - x)^{\gamma_1} (\overline{z} - x)^{\gamma_2}$$

$$\times (z + x)^{\beta_1} (\overline{z} + x)^{\beta_2} |z - \overline{z}|^{\kappa} |z - \overline{z}|^{\kappa} ,$$

(2.71)
the assignments
\[ \alpha_0 = s_3 + m_0, \quad \alpha_1 = s_2 + m_1, \quad \alpha_2 = -s_2 - s_3 + 2s_5 + m_2 + 2q^2, \]
\[ \lambda_1 = -\frac{s_1}{2} + \frac{s_3}{2} - s_5 - q^2 + n_1, \quad \lambda_2 = -\frac{s_1}{2} + \frac{s_3}{2} - s_5 - q^2 + n_2, \]
\[ \gamma_1 = s_4 + n_3, \quad \gamma_2 = s_4 + n_4, \]
\[ \beta_1 = \frac{s_1}{2} - \frac{s_3}{2} - s_4 - q^2 + n_5, \quad \beta_2 = \frac{s_1}{2} - \frac{s_3}{2} - s_4 - q^2 + n_6, \]
\[ \kappa = 2q^2 + \tilde{\kappa}, \] (2.72)
and the integers \( m_i, n_i, \tilde{\kappa} \in \mathbb{Z} \) subject to the analyticity condition (2.14):
\[ \alpha_0 + \alpha_1 + \alpha_2 + \lambda_1 + \lambda_2 + \gamma_1 + \gamma_2 + \beta_1 + \beta_2 + \kappa + 4 = 0. \] (2.73)

The complex integral (2.70) fulfills:
\[ I^{(\kappa)} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (-x) = I^{(\kappa)} \left[ \frac{\lambda_1, \beta_1, \gamma_1}{\lambda_2, \beta_2, \gamma_2} \right] (x). \] (2.74)

This relation is used in Appendix C.

Let us now discuss the evaluation of the integral (2.70). To compute the integral
(2.71) over the complex \( z \)-plane we split it up into holomorphic and anti-holomorphic
contour integrals along the method proposed in [1]. After introducing the parameterization
\( z = z_1 + iz_2 \) the integrand may be considered as an analytic function in \( z_2 \). We then deform
the \( z_2 \)-integral along the real axis \( \text{Im}(z_2) = 0 \) to the pure imaginary axis \( \text{Re}(z_2) = 0 \), i.e.
\( iz_2 \in \mathbb{R} \). This way, the variables \( \xi = z_1 + iz_2 \equiv z, \eta = z_1 - iz_2 \equiv \bar{z} \) become real quantities,
i.e. \( \xi, \eta \in \mathbb{R} \). With the Jacobian \( \text{det} \frac{\partial(z_1, z_2)}{\partial(z, \bar{z})} = \frac{i}{2} \) we arrive at:
\[ W^{\kappa, \alpha_0}_{\sigma} \left[ \frac{\alpha_1, \lambda_1, \gamma_1, \beta_1}{\alpha_2, \lambda_2, \gamma_2, \beta_2} \right] = \frac{i}{2} \int_{\mathbb{R}} dx \left| 2x^{\alpha_0} \right| \left| 1 + x^{\lambda_1} \right| \left| 1 - x^{\lambda_2} \right|^{m_0} \left| (1 + x)^{m_1} \right| \left| (1 - x)^{m_2} \right|
\times \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left| 1 - \xi^{-\lambda_1} \right| \left| \xi - x^{-\lambda_1} \right| \left| \xi + x^{\beta_1} \right|
\times \left| 1 - \eta^{-\lambda_2} \right| \left| \eta - x^{-\lambda_2} \right| \left| \eta + x^{\beta_2} \right| \left| \xi - \eta^{-\beta_2} \right| \left| \xi + \eta^{\beta_2} \right|^{\kappa} \Pi(x, \xi, \eta)
\times (1 - \xi)^{n_1} (\xi - x)^{n_3} (\xi + x)^{n_5} (1 - \eta)^{n_2} (\eta - x)^{n_4} (\eta + x)^{n_6}. \] (2.75)

The quantities with a hat refer to their non-integer part. In (2.75) the phase factor
\( \Pi(x, \xi, \eta) \) following from (2.17) accounts for the correct branch of the integrand. In this
phase factor the variable \( x \) enters as a parameter. The phases are analyzed in Appendix
C. Eventually, in the integrand the latter are accommodated by choosing the respective
contours in the complex \( \eta \)-plane for a given range \( x \in \mathbb{R} \). More precisely, for a given pair
of \( x, \xi \in \mathbb{R} \) we may consider the \( \eta \)-integral as an integration in the complex \( \eta \)-plane and
the phases \( \Pi(x, \xi, \eta) \) give rise to the integration contours in the complex \( \eta \)-plane as shown
in Appendix C. The result for (2.69) is presented in Subsection 3.5.
2.4.5. Three closed strings

In this Subsection we discuss the disk amplitude (2.1) of three closed strings \( A(1, 2, 3) \). With a \( PSL(2, \mathbb{R}) \) transformation two arbitrary points \( w_1, w_2 \in \mathbb{H}_+ \) can be mapped to the two special points \( z_1 = i, z_2 = ix_\pm \) along the positive imaginary axis, with

\[
x_\pm = \frac{\text{Re}(w_1 - w_2)^2 + \text{Im}(w_1)^2 + \text{Im}(w_2)^2 \mp |w_1 - w_2| |w_1 - \bar{w}_2|}{2 \text{ Im}(w_1) \text{ Im}(w_2)} \in \mathbb{R}^+ \tag{2.76}
\]

and \( 0 < x_+ < 1 \) and \( x_- > 1 \). Therefore, this map allows for the following choice of two closed string vertex positions

\[
\tau_1 = -i, \quad z_1 = i , \quad \tau_2 = -ix , \quad z_2 = ix \tag{2.77}
\]

with \( 0 < x < 1 \), see also [7]. This choice implies the \( c \)-ghost contribution:

\[
\langle c(\tau_1)c(z_1)c(\tau_2) \rangle = 2i (1 - x^2). \tag{2.78}
\]

Hence, in the double cover of \( \mathbb{H}_+ \) the disk amplitude (2.1) of three closed strings becomes

\[
A(1, 2, 3) = 2 \int_{-1}^{1} dx (1 - x^2) \int d^2 z \langle : V_c(-i, i) : V_c(-ix, ix) : V_c(\tau, z) : \rangle + h.c. \tag{2.79}
\]

For this three–point process to give a result, which does not vanish on-shell, the closed string momenta \( q_i \) have to have also a non-vanishing directions \( q_{\perp i} \neq 0 \) transverse to the D-brane worldvolume, with \( q_i = q_{\parallel i} + q_{\perp i}, \quad i = 1, 2, 3 \) and momentum conservation (2.12):

\[
q_{\parallel 1} + q_{\parallel 2} + q_{\parallel 3} = 0. \tag{2.80}
\]

For the choice (2.77) the correlator (2.11) becomes \( (z_3 \equiv z) \):

\[
\langle e^{i q_1, u_{\tau_1}(\tau_1, z_1)} e^{i q_2, u_{\tau_2}(\tau_2, z_2)} e^{i q_3, u_{\tau_3}(\tau_3, z_3)} \rangle = 2^2 q_{\parallel 1}^2 + 2 q_{\parallel 2}^2 + 2 q_{\parallel 3}^2 \left| x \right|^2 \left| q_{\parallel 2} \right|^2 \left| z - \tau \right|^2 \left| q_{\parallel 3} \right|^2 \times \left| 1 - x \right|^2 q_{\parallel 2} q_2 \left| 1 + x \right|^2 q_{\parallel 1} D q_2 \left| i - z \right|^2 q_{\parallel 3} q_3 \left| i - \tau \right|^2 q_{\parallel 1} D q_3 \left| z - ix \right|^2 q_{\parallel 2} q_3 \left| z + ix \right|^2 q_{\parallel 2} D q_3 . \tag{2.81}
\]

We have the following six kinematic invariants

\[
s_1 = 2 q_{\parallel 1}^2 = q_{\parallel 1} D q_1 , \quad s_2 = q_{\parallel 1} D q_2 , \quad s_3 = 2 q_{\parallel 2}^2 = q_{\parallel 2} D q_2 , \quad s_4 = q_{\parallel 2} D q_3 , \quad s_5 = 2 q_{\parallel 3}^2 = q_{\parallel 3} D q_3 , \quad s_6 = q_{\parallel 1} D q_3 \tag{2.82}
\]

to describe the scattering process. From (2.80) and (2.82) we find:

\[
q_{12} = \frac{1}{2} (-s_1 - s_3 + s_5) = s_2 ,
q_{23} = \frac{1}{2} (s_1 - s_3 - s_5) = s_4 ,
q_{13} = \frac{1}{2} (-s_1 + s_3 - s_5) = s_6 . \tag{2.83}
\]
After performing all Wick contractions and including the ghost correlator (2.78) for any kinematics $K_I$ the amplitude (2.79) reduces to

$$(1 + e^{i\pi s_0}) A_I(1, 2, 3) = (-1)^{m_3 + n_6 + n_k}$$

\begin{equation}
\times \left\{ W^{(\kappa', \alpha_0', \alpha_4')} [\alpha_1', \lambda_1', \gamma_1', \beta_1', \epsilon_1'] + W^{(\kappa', \alpha_0', \alpha_4')} [\alpha_2', \lambda_1', \gamma_1', \beta_1', \epsilon_1'] \right\},
\end{equation}

and the complex integral of the following type

\begin{equation}
W^{(\kappa, \alpha_0, \alpha_3)} [\alpha_1, \lambda_1, \gamma_1, \beta_1, \epsilon_1] := 2^{1+2\alpha_0+\alpha_3} \int_{-1}^{1} dx |x|^{\tilde{\alpha}_3} |1 + x|^{\tilde{\alpha}_1} |1 - x|^{\tilde{\alpha}_2}
\end{equation}

\begin{equation}
\times x^{m_3} (1 + x)^{1 + m_1} (1 - x)^{1 + m_2} I^{(\kappa)} \left[ \begin{array}{c}
\lambda_1, 
\gamma_1, 
\beta_1, 
\epsilon_1 \\
\alpha_2, 
\lambda_2, 
\gamma_2, 
\beta_2, 
\epsilon_2
\end{array} \right] (x),
\end{equation}

with

\begin{equation}
I^{(\kappa)} \left[ \begin{array}{c}
\lambda_1, 
\gamma_1, 
\beta_1, 
\epsilon_1 \\
\lambda_2, 
\gamma_2, 
\beta_2, 
\epsilon_2
\end{array} \right] (x) = \int_{C} d^2 z (1 - z)^{\lambda_1} (1 - \overline{z})^{\lambda_2} (1 + z)^{\gamma_1} (1 + \overline{z})^{\gamma_2}
\end{equation}

\begin{equation}
\times (z - x)^{\beta_1} (\overline{z} - x)^{\beta_2} (z + x)^{\epsilon_1} (\overline{z} + x)^{\epsilon_2} |z + \overline{z}|^{\kappa} (z + \overline{z})^{\bar{\kappa}},
\end{equation}

the assignments

$$\begin{aligned}
\alpha_0 &= s_1 + m_0, & \alpha_3 &= s_3 + m_3, & \kappa &= s_5 + \bar{\kappa}, \\
\alpha_1 &= 2s_2 + m_1, & \alpha_2 &= -s_1 - s_3 + s_5 - 2s_2 + m_2, \\
\lambda_1 &= s_6 + n_1, & \lambda_2 &= s_6 + n_2, \\
\gamma_1 &= \frac{1}{2} (-s_1 + s_3 - s_5) - s_6 + n_3, & \gamma_2 &= \frac{1}{2} (-s_1 + s_3 - s_5) - s_6 + n_4, \\
\beta_1 &= s_4 + n_5, & \beta_2 &= s_4 + n_6, \\
\epsilon_1 &= \frac{1}{2} (s_1 - s_3 - s_5) - s_4 + n_7, & \epsilon_2 &= \frac{1}{2} (s_1 - s_3 - s_5) - s_4 + n_8,
\end{aligned}$$

and the integers $m_i, n_i, \bar{\kappa} \in \mathbb{Z}$ subject to the analyticity conditions (2.14):

$$\begin{aligned}
2 \alpha_0 + \alpha_1 + \alpha_2 + \lambda_1 + \lambda_2 + \gamma_1 + \gamma_2 + 4 &= 0, \\
2 \alpha_3 + \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \epsilon_1 + \epsilon_2 + 4 &= 0, \\
\lambda_1 + \gamma_1 + \beta_1 + \epsilon_1 + \kappa + 2 &= 0, \\
\lambda_2 + \gamma_2 + \beta_2 + \epsilon_2 + \kappa + 2 &= 0.
\end{aligned}$$

Note, that we have the following relation:

$$I^{(\kappa)} \left[ \begin{array}{c}
\lambda_1, 
\gamma_1, 
\beta_1, 
\epsilon_1 \\
\alpha_2, 
\lambda_2, 
\gamma_2, 
\beta_2, 
\epsilon_2
\end{array} \right] (-x) = I^{(\kappa)} \left[ \begin{array}{c}
\lambda_1, 
\gamma_1, 
\epsilon_1, 
\beta_1 \\
\lambda_2, 
\gamma_2, 
\epsilon_2, 
\beta_2
\end{array} \right] (x).$$

(2.89)

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Let us now discuss the evaluation of the integral (2.85). To compute the integral (2.86) over the complex $z$–plane as before we split it up into holomorphic and anti–holomorphic contour integrals. With the Jacobian $\det \left( \frac{\partial (z_1, z_2)}{\partial (\xi, \eta)} \right) = \frac{i}{2}$ we arrive at:

$$W^{(\kappa, \alpha_0, \alpha_3)} \left[ \frac{\alpha_1, \lambda_1, \gamma_1, \beta_1, \epsilon_1}{\alpha_2, \lambda_2, \gamma_2, \beta_2, \epsilon_2} \right] = 2^{\alpha_0} i \int_{-1}^{1} dx \left| 2x \right|^{\alpha_3} \left| 1 + x \right|^{\alpha_1} \left| 1 - x \right|^{\alpha_2}$$

$$\times (2x)^{m_3} (1 + x)^{1 + m_1} (1 - x)^{1 + m_2}$$

$$\times \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left| 1 - \xi \right|^{\lambda_1} \left| 1 + \xi \right|^{\gamma_1} \left| \xi - x \right|^{\beta_1} \left| \xi + x \right|^{\epsilon_1}$$

$$\times \left| 1 - \eta \right|^{\lambda_2} \left| 1 + \eta \right|^{\gamma_2} \left| \eta - x \right|^{\beta_2} \left| \eta + x \right|^{\epsilon_2} \left| \xi + \eta \right|^{\kappa} \Pi(x, \xi, \eta)$$

$$\times (1 - \xi)^{n_1} (1 + \xi)^{n_3} (\xi - x)^{n_5} (\xi + x)^{n_7}$$

$$\times (1 - \eta)^{n_2} (1 + \eta)^{n_4} (\eta - x)^{n_6} (\eta + x)^{n_8} (\xi + \eta)^{\hat{\kappa}}.$$

The hatted quantities refer to their non–integer part, while a tilde on them denotes their integer part, i.e. $\kappa = \hat{\kappa} + \tilde{\kappa}$, with $\hat{\kappa} = s_5$, etc. In (2.90) the phase factor $\Pi(x, \xi, \eta)$ following from (2.17) accounts for the correct branch of the integrand. The coordinate $x$ enters this phase factor as a parameter. For a given range of $x \in \mathbb{R}$ the phases, which are analyzed in Appendix D, are accommodated in the integrand by choosing the respective contours in the complex $\eta$–plane. More precisely, for a given pair of $x, \xi \in \mathbb{R}$ we may consider the $\eta$–integral as an integration in the complex $\eta$–plane and the phases $\Pi(x, \xi, \eta)$ give rise to the integration contours in the complex $\eta$–plane as shown in Appendix D. The result for (2.84) is presented in Subsection 3.6.

2.4.6. Three open strings and two closed strings

For an amplitude of four open and one closed string in (2.1) in total we consider the partial amplitudes $A(1, 2, 3; 4, 5)$. Due to $PSL(2, \mathbb{R})$ invariance on the disk we may fix the three open string vertex positions as (2.34):

$$x_1 = -\infty \ , \ \ x_2 = 0 \ , \ \ x_3 = 1 , \quad (2.91)$$

for the two closed strings. For this choice in the amplitude (2.1) we are left with two complex integrations of the two closed string positions $z_1, z_2 := w$, with $z_1, z_2 \in \mathbb{H}_+$. In the double cover the closed string positions $z, w$ are integrated over the full complex plane $\mathbb{C}$. The choice (2.91) implies the $c$–ghost contribution (2.46):

$$\langle c(x_1) c(x_2) c(x_3) \rangle = (x_1 - x_2) (x_1 - x_3) (x_2 - x_3) = z_\infty^2 . \quad (2.92)$$
Eventually, the partial amplitude (2.1) becomes:
\[ A(1, 2, 3; 4, 5) = \langle c(-\infty)c(0)c(1) \rangle \times \int_C d^2z_1 \int_C d^2z_2 \langle :V_o(-\infty) :V_o(0) :V_o(1) :V_c(\bar{z}_1, z_1) :V_c(\bar{z}_2, z_2) : \rangle . \]
(2.93)

For the choice (2.63) the correlator (2.11) becomes:
\[ \langle e^{2ip_1\mu X^\mu(-\infty)} e^{2ip_2\mu X^\mu(0)} e^{iq_1\mu X^\mu(1)} e^{iq_2\mu X^\mu(\bar{z}_2, z_2)} \rangle \\
= z_1^{2p_2q_1} \bar{z}_1^{2p_2q_1} (1 - z_1)^{2p_3q_1} (1 - \bar{z}_1)^{2p_3q_1} \times z_2^{2p_2q_2} \bar{z}_2^{2p_2q_2} (1 - z_2)^{2p_3q_2} (1 - \bar{z}_2)^{2p_3q_2} \\
\times (z_1 - z_2)^{q_1q_2} (\bar{z}_1 - \bar{z}_2)^{q_1q_2} (z_1 - \bar{z}_2)^{q_1q_2} (\bar{z}_1 - z_2)^{q_1q_2} . \]
(2.94)

We have the five kinematic invariants:
\[ s_1 = 4p_1p_2 , s_2 = 4p_2p_3 , s_3 = 2p_3q_1 , s_4 = q_1q_2 , s_5 = 2p_1q_2 . \]
(2.95)

Furthermore, we have\[ p_1p_3 = -\frac{s_3}{4} - \frac{s_4}{4} + s_4 , p_1q_1 = \frac{s_3}{4} - s_4 - \frac{s_5}{4} , p_2q_1 = -\frac{s_3}{4} - \frac{s_5}{4} + \frac{s_4}{2} , \]
and\[ p_2q_2 = -\frac{s_1}{4} + \frac{s_3}{4} - \frac{s_4}{2} , p_3q_2 = \frac{s_1}{4} - \frac{s_4}{2} - s_4 . \]After performing all Wick contractions for any kinematics \( K \), the amplitude (2.93) boils down to
\[ A^I(1, 2, 3; 4, 5) = J^{(\kappa_1, \kappa_2)} \left[ \lambda_1^{(I)}, \gamma_1^{(I)}, \bar{\lambda}_1^{(I)}, \bar{\gamma}_1^{(I)}; \delta_1^{(I)}, \bar{\delta}_1^{(I)} \right] . \]
(2.96)

In (2.96) the class of two–dimensional complex integrals is given by
\[ J^{(\kappa_1, \kappa_2)} \left[ \lambda_1, \gamma_1, \bar{\lambda}_1, \bar{\gamma}_1, \delta_1, \bar{\delta}_1 \right] = \int_C d^2z_1 \int_C d^2z_2 z_1^{\lambda_1} \bar{z}_1^{\gamma_1} (1 - z_1)^{\gamma_1} (1 - \bar{z}_1)^{\gamma_1} (z_1 - \bar{z}_1)^{\kappa_1} \times z_2^{\bar{\lambda}_1} \bar{z}_2^{\bar{\gamma}_1} (1 - z_2)^{\bar{\gamma}_1} (1 - \bar{z}_2)^{\bar{\gamma}_1} (z_1 - \bar{z}_2)^{\kappa_2} \times (z_1 - z_2)^{\delta_1} (\bar{z}_1 - \bar{z}_2)^{\delta_2} (z_1 - \bar{z}_2)^{\bar{\delta}_1} (\bar{z}_1 - z_2)^{\bar{\delta}_2} , \]
(2.97)

with the assignments
\[ \lambda_1 = -\frac{s_2}{2} - s_3 + s_5 + n_1 , \quad \lambda_2 = -\frac{s_3}{2} - s_3 + s_5 + n_2 , \]
\[ \gamma_1 = s_3 + n_3 , \quad \gamma_2 = s_3 + n_4 , \]
\[ \bar{\lambda}_1 = -\frac{s_1}{2} + s_3 - s_5 + n_5 , \quad \bar{\lambda}_2 = -\frac{s_4}{2} + s_3 - s_5 + n_6 , \]
\[ \bar{\gamma}_1 = \frac{s_3}{2} - s_3 - 2s_4 + n_7 , \quad \bar{\gamma}_2 = \frac{s_4}{2} - s_3 - 2s_4 + n_8 , \]
\[ \delta_1 = s_4 + n_9 , \quad \delta_2 = s_4 + n_{10} , \]
\[ \bar{\delta}_1 = s_4 + n_{11} , \quad \bar{\delta}_2 = s_4 + n_{12} . \]
(2.98)
and the integers integers \( n_i, \kappa_i \in \mathbb{Z} \). Again, we define the non-integer part of these quantities by putting a hat on them.

Let us now discuss the evaluation of the integral (2.97). We split it up into holomorphic and anti-holomorphic contour integrals and apply the methods described in the previous cases. After introducing the parameterization \( z_i = z_{1i} + i z_{2i}, \ i = 1, 2 \) the integrand may be considered as an analytic function in \( z_{21} \) and \( z_{22} \). We then deform the \( z_{2i} \)-integrals along the real axis \( \text{Im}(z_{2i}) = 0 \) to the pure imaginary axis \( \text{Re}(z_{2i}) = 0 \), \( i.e. \ i z_{2i} \in \mathbb{R} \). This way, the variables \( \xi_i = z_{1i} + i z_{2i} \equiv z_i, \ \eta_i = z_{1i} - i z_{2i} \equiv \bar{z_i} \) become real quantities, \( i.e. \ \xi_i, \eta_i \in \mathbb{R}, \ i = 1, 2 \). With the Jacobian \( \det \frac{\partial (z_{1i}, z_{2i})}{\partial (\xi_i, \eta_i)} = \left( \frac{i}{2} \right)^2 \) we arrive at:

\[
J^{(\kappa_1, \kappa_2)} \left[ \lambda_1, \gamma_1, \tilde{\lambda}_1, \tilde{\gamma}_1, \delta_1, \tilde{\delta}_1 \right. \\
\lambda_2, \gamma_2, \tilde{\lambda}_2, \tilde{\gamma}_2, \delta_2, \tilde{\delta}_2 \right] = -\frac{1}{4} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \Pi(\xi_1, \xi_2, \eta_1, \eta_2) \\
\times |\xi_1|^{\lambda_1} |\eta_1|^{\lambda_2} |1 - \xi_1|^{\gamma_1} |1 - \eta_1|^{\gamma_2} \\
\times |\xi_2|^{\tilde{\lambda}_1} |\eta_2|^{\tilde{\lambda}_2} |1 - \xi_2|^{\tilde{\gamma}_1} |1 - \eta_2|^{\tilde{\gamma}_2} \\
\times |\xi_1 - \xi_2|^{\delta_1} |\eta_1 - \eta_2|^{\delta_2} |\xi_1 - \eta_2|^{\delta_1} |\eta_1 - \xi_2|^{\delta_2} \\
\times \xi_1^{n_1} \eta_1^{n_2} (1 - \xi_1)^{n_3} (1 - \eta_1)^{n_4} (\xi_1 - \eta_1)^{\kappa_1} \\
\times \xi_2^{n_5} \eta_2^{n_6} (1 - \xi_2)^{n_7} (1 - \eta_2)^{n_8} (\xi_2 - \eta_2)^{\kappa_2} \\
\times (\xi_1 - \xi_2)^{n_9} (\eta_1 - \eta_2)^{n_{10}} (\xi_1 - \eta_2)^{n_{11}} (\eta_1 - \xi_2)^{n_{12}}.
\]

(2.99)

In (2.99) the phase factor \( \Pi(\xi_1, \xi_2, \eta_1, \eta_2) \) following from (2.17) accounts for the correct branch of the integrand. In the integrand these phases are accommodated by choosing the respective contours in the complex \( \eta_1, \eta_2 \)-planes for given \( \xi_1, \xi_2 \). The result for (2.96) is presented in Subsection 3.7.

3. Disk amplitudes of open & closed strings vs. pure open string disk amplitudes

In this Section we establish the relation between a disk amplitude with \( N_o \) open & \( N_c \) closed strings to a disk amplitude of \( N_o + 2 N_c \) open strings. By reducing disk amplitudes of open & closed strings to disk amplitudes involving only open strings the task of computing those amplitudes reduces to the problem of computing pure open string amplitudes. This map reveals important relations between open & closed string disk amplitudes and pure open string disk amplitudes. For the latter a great deal of detailed results exists [27,4,28,5,2,29], in contrast to disk amplitudes of open and closed strings due to the more involved integrations over the full complex plane.
3.1. Basis of complex contour integrals and open string subamplitude relations

For each open string ordering $\sigma$ in the amplitude (2.16) the phase factor $\Pi(x_l, \xi_i, \eta_j)$, which is given in (2.17), accounts for the correct branch of the integrand. For any given ordering $\Sigma$ of the insertions $x_l, \xi_i, \eta_j$ this phase is fixed and independent on the points $x_l, \xi_i, \eta_j$. Hence the amplitude (2.16) decomposes into a sum over partial ordered $N_o+2N_c$ open string amplitudes times a specific phase factor $\Pi(\Sigma)$ determined by the ordering of the points $x_l, \xi_i, \eta_j$:

$$A_\sigma(N_o, N_c) = \left(\frac{i}{2}\right)^{N_c} \sum_{\Sigma \in \mathcal{P}} \Pi(\Sigma) \ A(1, \Sigma(2), \ldots, \Sigma(N_o+2N_c)),$$

(3.1)

with:

$$\mathcal{P} = \{\Sigma \in S_{N_o+2N_c}/Z_{N_o+2N_c} \mid \Sigma(i_l) = \sigma(l), l = 2, \ldots, N_o; 1 < i_2 < \ldots < i_{N_o} \leq N_o+2N_c\}.$$

The factor $\Pi(\Sigma)$ is the phase factor from (2.17) for the insertions $x_l, \xi_i, \eta_j$ with the ordering $\Sigma$ of the $N_o+2N_c$ open string positions. Note, that the kinematical factors $K_I$ and the integers $n_{ij}^I$ have no effect on this phase. Hence, the following discussion holds for any chosen kinematics. This is how we have dropped the index $I$ in (3.1).

For a given (initial) ordering $\sigma$ of the $N_o$ open string positions $x_l$ the sum runs over terms accounting for all possible orderings of the points $\xi_i, \eta_j$ related to the closed string positions. For a given ordering of the positions $x_k, \xi_i \in \mathbb{R}$ we consider the $\eta_j$–integrals in (2.16) as an integration in the complex $\eta_j$–plane and the phase $\Pi(x_l, \xi_i, \eta_j) \equiv \Pi(\Sigma)$ gives rise to the corresponding integration contours in the complex $\eta_j$–plane. Then the amount of terms in the sum (3.1) may be drastically reduced by deforming the contour integrals in the complex $\eta_j$–planes. This procedure is equivalent to finding relations among the partial amplitudes $A(1, \Sigma(1), \ldots, \Sigma(N_o+2N_c))$ and express the sum (3.1) in terms of a minimal basis. Stated differently, identities for higher open string amplitudes may be generated by deforming and relating the complex integrals appearing in (2.16) or (3.1). This concept is pursued in Section 4.

The positions of the open string points $x_l$ relative to the point $\xi_j$ determine the contour in the $\eta_j$–plane around these points. Similarly, the positions of the closed string points $\xi_i$ relative to the point $\xi_j$ determine the contour in the $\eta_j$–plane around the points $\eta_j = \eta_i$, cf. next Figure.
Analyzing the phases (2.17) related to the polynomials \((x_l - \xi_j)^{2p_l q_j}\) and \((x_l - \eta_j)^{2p_l q_j}\) it follows, that in the complex \(\eta_j\)-plane the points \(\eta_j = x_l\) are encircled above for \(\xi_j > x_l\) and below for \(\xi_j < x_l\). Similarly, analyzing the phases (2.17) related to the polynomials \((\xi_i - \xi_j)^{q_i q_j}\) and \((\eta_i - \eta_j)^{q_i q_j}\) it follows, that the points \(\eta_j = \eta_i\) are encircled above for \(\xi_j > \xi_i\) and below for \(\xi_j < \xi_i\). Hence, the point \(\eta_i = \eta_j\) is always avoided by the contours in the complex \(\eta_i, \eta_j\)-planes. In contrast to the closed string case on the sphere [1] in Eq. (2.16) the \(\xi_i\) and \(\eta_j\) integrations do not decouple due to the mixed polynomials \((\eta_i - \xi_j)^{q_i D_q j}\), \((\xi_i - \eta_j)^{q_j D_q i}\) and \((\xi_i - \eta_i)^{2q_i^2}\). The polynomials \((\eta_i - \xi_j)^{q_i D_q j}\), \((\xi_i - \eta_j)^{q_j D_q i}\) give rise to additional branchings in the complex \(\eta_j\)-plane. For \(\xi_j > \eta_i\) in the \(\eta_j\)-plane the point \(\eta_j = \xi_i\) is avoided above and for \(\xi_j < \eta_i\) the latter is avoided below, cf. next Figure.

Furthermore, the polynomials \((\xi_j - \eta_j)^{2q_j^2}\) implying the difference of the two closed string points \(\xi_j\) and \(\eta_j\) give rise to additional branchings in the complex \(\eta_j\)-plane. Hence an additional contour appears at the points \(\eta_j = \xi_j\) in the complex \(\eta_j\)-plane, cf. next Figure.
Let us consider the contour in the complex $\eta_j$–plane, with $j \in \{1, \ldots, N_c\}$. If for this $j$ the following inequalities holds

$$\xi_j < x_l, \quad \xi_j < \eta_i, \quad \xi_i, \quad i \in \{1, \ldots, \hat{j}, \ldots, N_c\},$$

in the complex $\eta_j$–plane all the contours avoid the points $\eta_j = x_i, \xi_i, \xi_j, \eta_i$ below and hence they can be deformed away to infinity. Therefore in that case, there is no contribution to the sum (3.1), see the next Figure.

In the sum (3.1) the contour in Fig. 7 sums up $N_o + 2N_c - 1$ subamplitude contributions, which give a vanishing contribution. By changing the positions $\eta_i, \xi_i$ such that (3.3) stays fulfilled, we find more vanishing contributions to (3.1). Furthermore, by repeating this analysis in different complex $\eta$–planes further vanishing contributions are discovered.

On the other hand, if (3.3) is not met, there a contributions to (3.1) from non–vanishing contours in all complex $\eta$–planes. Their analysis in the complex $\eta$–planes is quite tedious due to the additional branchings described in Fig. 5 and Fig. 6. An illustrative example is shown in the next Figure. We consider the integration region $x_{\sigma(2)} < \xi_j < \xi_k < x_{\sigma(3)} < \ldots < x_{\sigma(N_o)} < \xi_i, \xi_i < \eta_i$ and $\eta_j < \eta_k < \eta_i$. The contour in the $\eta_j$–plane can be deformed to give a contribution from $-\infty < \eta_j < x_{\sigma(2)}$, the contour in the $\eta_k$–plane can be deformed to give a contribution from $\xi_i < \eta_i < \infty$, while the contour in the $\eta_k$–plane can be deformed to give a contribution for the range $\eta_j < \eta_k < \xi_j$:

$$\sin(2\pi p_{\sigma(2)}q_j) \sin(2\pi q_j^2) \left\{ \sin(\pi q_j) A(\Sigma_1) + \sin[\pi(q_kq_j + 2q_kp_{\sigma(2)})] A(\Sigma_2) \right\}. \quad (3.4)$$
Here, the integrals $A(\Sigma)$ represent open string subamplitudes referring to the following ordering $\Sigma$ of the $N_o + 2N_c$ open string positions:

$$\Sigma_1 : -\infty < \eta_j < x_{\sigma(2)} < \eta_k < \xi_j < \xi_k < x_{\sigma(3)} < \ldots < x_{\sigma(N_o)} < \xi_i < \eta_i < \infty ,$$

$$\Sigma_2 : -\infty < \eta_j < \eta_k < x_{\sigma(2)} < \xi_j < \xi_k < x_{\sigma(3)} < \ldots < x_{\sigma(N_o)} < \xi_i < \eta_i < \infty .$$

The piece (3.4) comprises into the sum (3.1).

From the previous discussion it is evident, how to treat the general case $(N_o, N_c)$ and how to reduce the sum (3.1). For the four cases $(N_o, N_c) = (3, 1), (2, 2), (4, 1)$ and $(0, 3)$ the complex splitting (2.16) and the phase (2.17) give rise to the phase structure displayed in Table 1 of Appendix A, Tables 2,3 of Appendix B, Tables 4,5 of Appendix C, and Table 6 of Appendix D, respectively. The corresponding contours in the complex $\eta-$plane are displayed in the figures attached there. To outline the methods described above in the following we shall explicitly work out the six cases $(N_o, N_c) = (2, 1), (3, 1), (2, 2), (4, 1), (0, 3)$, and $(3, 2)$.

### 3.2. Two open & one closed string versus four open strings on the disk

In this Subsection we establish the relation between a disk amplitude with two open & one closed string, given in (2.27), to a disk amplitude of four open strings. On the other
hand, the general expression of a (partial ordered) four open string amplitude is given by the Gaussian hypergeometric function

\[
A(1, 2, 3, 4) := \int_0^1 dx \ (2x)^{\hat{s}_1 + \hat{n}_1} \ (1 + x)^{\hat{s}_2 + \hat{n}_2} \ (1 - x)^{\hat{s}_3 + \hat{n}_3} \\
= 2^{\hat{s}_1 + \hat{n}_1} B(\hat{s}_1 + \hat{n}_1 + 1, \hat{s}_3 + \hat{n}_3 + 1) \ _2F_1 \left[ \begin{array}{c} \hat{s}_1 + \hat{n}_1 + 1, -\hat{s}_2 - \hat{n}_2 \\ 2 + \hat{s}_1 + \hat{s}_3 + \hat{n}_1 + \hat{n}_3 + 2 \\ -1 \end{array} \right]
\]

(3.6)

with the three integers \( \hat{n}_i \in \mathbb{Z} \), the three kinematic invariants \( \hat{s}_i = \alpha'(k_i + k_{i+1})^2 \), subject to the cyclic identification \( i + 4 \equiv i \) and the momenta \( k_i \) of the four external open strings. The three invariants \( \hat{s}_i \), which fulfill \( \hat{s}_1 + \hat{s}_2 + \hat{s}_3 = 0 \) are related to the invariants (2.21) as follows:

\[
\hat{s}_1 = \hat{s}_{12} = 4k_1k_2, \quad \hat{s}_2 = \hat{s}_{13} = 4k_1k_3, \quad \hat{s}_3 = \hat{s}_{14} = 4k_1k_4.
\]

The representation (3.6) refers to the partial ordering (1234) of the four open string vertex operators along the boundary of the disk. Furthermore, we have the constraint \( \hat{n}_1 + \hat{n}_2 + \hat{n}_3 + 2 = 0 \). This allows to write (3.6) as follows:

\[
A(1, 2, 3, 4) = \frac{1}{2} \int_0^1 dx \ x^{\hat{s}_1 + \hat{n}_1} \ (1 - x)^{\hat{s}_3 + \hat{n}_3} = \frac{1}{2} B(\hat{s}_1 + \hat{n}_1 + 1, \hat{s}_3 + \hat{n}_3 + 1).
\]

(3.7)

On the other hand, after some manipulations involving relations between Gamma–functions, the amplitude (2.27) involving two open and one closed string may be cast into:

\[
A(1; 2; 3) = -\frac{1}{2} \sin(\pi t) B(s - 2n_1 - 1, t + n_1 + 1).
\]

(3.8)

In (3.8) the Euler beta function may be considered as world–sheet integral (3.7) describing a (partial ordered) four open string disk amplitude. More precisely the latter, which originates from (3.6) corresponds to the choice of vertex operator positions

\[
z_1 = -\infty, \ z_2 = 1, \ z_3 = -x, \ z_4 = x,
\]

(3.9)

with \(-1 < x < 0\). With the identifications

\[
\hat{s}_1 = s, \quad \hat{s}_2 = \hat{s}_3 = t, \\
\hat{n}_1 = -2 - 2n_1, \quad \hat{n}_2 = \hat{n}_3 = n_1,
\]

(3.10)

i.e. \( \hat{s}_1 = -2\hat{s}_2 \), the two Euler beta functions in (3.7) and (3.8) match and we have the following relation

\[
A(1, 2; 3) = -\sin(\pi t) A(1, 2, 3, 4)
\]

(3.11)
between the amplitude $A(1, 2, 3)$ of two open and one closed string and an amplitude involving four open strings $A(1, 2, 3, 4)$. According to (2.19) and (2.26) the relations (3.10) are fulfilled by the following assignment of the four open string momenta:

$$k_1 = p_1, \ k_2 = p_2, \ k_3 = \frac{1}{2} Dq, \ k_4 = \frac{1}{2} q. \quad (3.12)$$

Finally, the alternative expression (2.32) may be cast into a sum of four open string amplitudes:

$$A(1, 2; 3) = \frac{i}{2} \left\{ e^{i\pi t} \ A(1, 4, 3, 2) + A(1, 3, 2, 4) + e^{i\pi t} \ A(1, 3, 4, 2) \right\}. \quad (3.13)$$

The equivalence of the two expressions (3.11) and (3.13) can be shown by using the relations

$$A(1, 3, 2, 4) = -2 \cos(\pi t) \ A(1, 3, 4, 2), \quad A(1, 3, 4, 2) = A(1, 4, 3, 2), \quad (3.14)$$

which are a consequence of (2.33). See also Subsection 4.3.1.

### 3.3. Three open & one closed string versus five open strings on the disk

In this Subsection we establish the relation between a disk amplitude with three open & one closed string to a disk amplitude of five open strings. The generic expression of a disk amplitude $A(1, 2, 3; 4)$ involving three open and closed strings is given in (2.39) and (2.40). On the other hand, the general expression of a (partial ordered) five open string amplitude is given by the Gaussian hypergeometric function [4,5]

$$A(1, 2, 3, 4, 5) = \int_0^1 dx \int_0^1 dy \ x^{\tilde{s}_2+n_1} y^{\tilde{s}_5+n_2} (1-x)^{\tilde{s}_3+n_3} (1-y)^{\tilde{s}_4+n_4} (1-xy)^{\tilde{s}_1-\tilde{s}_3-\tilde{s}_4+n_5}, \quad (3.15)$$

with the momenta $k_i$ of the five external open strings and the five kinematic invariants $\tilde{s}_i = \alpha'(k_i + k_{i+1})^2$, subject to the cyclic identification $i + 5 \equiv i$. For $\alpha' = 2$ the five invariants $\tilde{s}_i$ can be related to (2.21) as follows:

$$\tilde{s}_1 = \tilde{s}_{12}, \ \tilde{s}_2 = \tilde{s}_{23}, \ \tilde{s}_3 = \tilde{s}_{34}, \ \tilde{s}_4 = \tilde{s}_{45}, \ \tilde{s}_5 = \tilde{s}_{51}. \quad (3.16)$$

The above representation refers to the partial ordering (12345) of the five open string vertex operators along the boundary of the disk.

Our task is to express (2.40) as a sum over integrals of the type (3.15). By writing $z = \xi + i\eta$ and deforming the $\eta$–integration contour the integral (2.40) may be written as an integral with $z$ and $\bar{z}$ treated as independent real variables, cf. Appendix A. With this information the expression (2.40) may be cast into the form

$$G^{(\alpha)} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] = \sin(\pi \tilde{\lambda}_2) \ A(1, 5, 2, 4, 3) + \sin(\pi \tilde{\alpha}) \ A(1, 2, 3, 4, 5), \quad (3.17)$$

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with the two integrals:

\[
A(1, 5, 2, 4, 3) := (-1)^{n_2} \int_0^1 d\xi \int_{-\infty}^0 d\eta \, \xi^{\lambda_1} (1 - \xi)^{\gamma_1} (-\eta)^{\lambda_2} (1 - \eta)^{\gamma_2} (\xi - \eta)^{\alpha}, \\
A(1, 2, 3, 4, 5) := (-1)^{m_1+m_2+\tilde{\alpha}} \int_1^{\infty} d\xi \int_1^{\infty} d\eta \, \xi^{\lambda_1} (\xi - 1)^{\gamma_1} \eta^{\lambda_2} (\eta - 1)^{\gamma_2} (\eta - \xi)^{\alpha}.
\]

(3.18)

Quantities with a hat refer to the non–integer parts of the parameter introduced in (2.41), i.e. \( \alpha = \tilde{\alpha} + \bar{\alpha} \), \( \lambda_i = \tilde{\lambda}_i + \lambda_i \), \( \gamma_i = \tilde{\gamma}_i + \gamma_i \), with \( \tilde{\lambda}_i = t \), \( \tilde{\gamma}_i = s \), \( \bar{\alpha} = 2q^2 \parallel \) and \( n_i, m_i, \bar{\alpha} \in \mathbb{Z} \).

With (3.17) any disk amplitude involving three open and one closed string (2.36) can be written as:

\[
A(1, 2, 3; 4) = \sin(\pi t) \, A(1, 5, 2, 4, 3) + \sin(2\pi q^2 \parallel) \, A(1, 2, 3, 4, 5).
\]

(3.19)

The two real integrals (3.18) appear as world–sheet integrals describing (partial ordered) five open string disk amplitudes. Indeed, the integrals (3.18) correspond to partial ordered amplitudes of a five open string disk amplitude with the the choice of vertex operator positions

\[
z_1 = -\infty, \ z_2 = 0, \ z_3 = 1, \ z_4 = \xi, \ z_5 = \eta,
\]

(3.20)

with \( \xi, \eta \in \mathbb{R} \). More precisely, the expressions (3.18) describe the following ordering of vertex positions:

\[
A(1, 5, 2, 4, 3): \ 0 < \xi < 1, \ -\infty < \eta < 0 \\
\Longleftrightarrow \ z_1 < z_5 < z_2 < z_4 < z_3,
\]

A(1, 2, 3, 4, 5): \ 1 < \xi < \infty, \ \xi < \eta < \infty \\
\Longleftrightarrow \ z_1 < z_2 < z_3 < z_4 < z_5.
\]

(3.21)

According to (2.19) we have the assignment:

\[
k_1 = p_1, \ k_2 = p_2, \ k_3 = p_3, \ k_4 = \frac{1}{2} q, \ k_5 = \frac{1}{2} Dq.
\]

(3.22)

As a consequence from (2.38) and (2.19) we have

\[
\hat{s}_1 = 2 \ s + 2 \ q^2 \parallel, \ \hat{s}_2 = 2 \ u + 2 \ q^2 \parallel, \ \hat{s}_3 = s, \ \hat{s}_4 = 2 \ q^2 \parallel, \ \hat{s}_5 = u.
\]

(3.23)

with \( \hat{s}_1 + \hat{s}_2 = 2\hat{s}_3 + 2\hat{s}_4 + 2\hat{s}_5 \).
Finally after some coordinate transformations the two partial amplitudes (3.18) may be brought into the canonical form (3.15), subject to the choice (3.23). Indeed, with

$$
\xi \rightarrow \frac{(1-x)}{y}, \quad \eta \rightarrow -\frac{x}{1-xy}
$$

(3.24)

the integral $A(15243)$ of (3.18) becomes:

$$
A(1,5,2,4,3) = (-1)^{n_2} \int_0^1 dx \int_0^1 dy \ x^{\lambda_2} \ y^{\lambda_1+\lambda_2+\alpha+1} \ (1-x)^{\lambda_1} \ (1-y)^{\gamma_1} \times (1-xy)^{-\lambda_1-\lambda_2-\gamma_1-\gamma_2-\alpha-3}
$$

$$
= (-1)^{n_2} \frac{\Gamma(1+\lambda_1) \Gamma(1+\lambda_2) \Gamma(1+\gamma_1) \Gamma(\lambda_1+\lambda_2+\alpha+2)}{\Gamma(2+\lambda_1+\lambda_2) \Gamma(\lambda_1+\lambda_2+\gamma_1+\alpha+3)} \times {}_3F_2\left[\begin{array}{c}
1+\lambda_2, \ 2+\alpha+\lambda_1+\lambda_2, \ 3+\alpha+\lambda_1+\lambda_2+\gamma_1+\gamma_2 \\
2+\lambda_1+\lambda_2, \ 3+\alpha+\lambda_1+\lambda_2+\gamma_1
\end{array}\right].
$$

(3.25)

On the other hand, with the transformations

$$
\xi \rightarrow \frac{1}{x}, \quad \eta \rightarrow \frac{1}{xy}
$$

(3.26)

the integral $A(12345)$ of (3.18) assumes the form:

$$
A(1,2,3,4,5) = (-1)^{m_1+m_2+\tilde{\alpha}} \int_0^1 dx \int_0^1 dy \ x^{-\lambda_1-\lambda_2-\gamma_1-\gamma_2-\alpha-3} \ y^{-\lambda_2-\gamma_2-\alpha-2} \ (1-x)^{\gamma_1} \times (1-xy)^{\gamma_2}
$$

$$
= (-1)^{m_1+m_2+\tilde{\alpha}} \frac{\Gamma(1+\alpha) \Gamma(1+\gamma_1) \Gamma(-1-\alpha-\lambda_2-\gamma_2) \Gamma(-2-\alpha-\lambda_1-\lambda_2-\gamma_1-\gamma_2)}{\Gamma(-\lambda_2-\gamma_2) \Gamma(-1-\alpha-\lambda_1-\lambda_2-\gamma_2)} \times {}_3F_2\left[\begin{array}{c}
-\gamma_2, -1-\alpha-\lambda_2-\gamma_2, -2-\alpha-\lambda_1-\lambda_2-\gamma_1-\gamma_2 \\
-\lambda_2-\gamma_2, -1-\alpha-\lambda_1-\lambda_2
\end{array}\right].
$$

(3.27)

Eventually, with (2.41) the subamplitudes (3.25) and (3.27) reduce to:

$$
A(1,5,2,4,3) = (-1)^{n_2} \int_0^1 dx \int_0^1 dy \ x^{t+n_2} \ y^{2t+\alpha+1+n_1+n_2} \ (1-x)^{t+n_1} \ (1-y)^{s+m_1} \times (1-xy)^{2u-\alpha-3-n_1-n_2-m_1-m_2},
$$

$$
A(1,2,3,4,5) = (-1)^{m_1+m_2+\tilde{\alpha}} \int_0^1 dx \int_0^1 dy \ x^{2u-\alpha-3-n_1-n_2-m_1-m_2} \ y^{u-\alpha-2-n_2-m_2} \times (1-x)^{s+m_1} \ (1-y)^{\alpha} \times (1-xy)^{s+m_2}.
$$

(3.28)
3.4. Two open & two closed strings versus six open strings on the disk

In this Subsection we establish the relation between a disk amplitude with two open & two closed strings to a disk amplitude of six open strings. The generic expression of a disk amplitude $A(1, 2; 3, 4)$ involving two open and two closed strings is given in (2.50) and (2.54). On the other hand, the general expression of a (partial ordered) six open string amplitude is given by the generalized Euler integral [4,5]

$$A(1, 2, 3, 4, 5, 6) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \ x^{\hat{s}_2} y^{\hat{t}_2} z^{\hat{s}_6} (1-x)^{\hat{s}_3} (1-y)^{\hat{s}_4} (1-z)^{\hat{s}_5}$$

$$\times (1-xy)^{\hat{t}_1-\hat{s}_3-\hat{s}_4} (1-yz)^{\hat{t}_1-\hat{s}_4-\hat{s}_5} (1-xyz)^{\hat{s}_1+\hat{s}_4-\hat{t}_1-\hat{t}_3},$$

(3.29)

with the six momenta $k_i$ of the six external open strings and the nine kinematic invariants $\hat{s}_i = \alpha'(k_i + k_{i+1})^2$ and $\hat{t}_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2$, subject to the cyclic identification $i + 6 \equiv i$. For $\alpha' = 2$ the nine invariants $\hat{s}_i$ can be related to (2.1) as follows:

$$\hat{s}_1 = \hat{s}_{12} , \hat{s}_2 = \hat{s}_{23} , \hat{s}_3 = \hat{s}_{34} , \hat{s}_4 = \hat{s}_{45} , \hat{s}_5 = \hat{s}_{56} , \hat{s}_6 = \hat{s}_{61} ,$$

$$\hat{t}_1 = \hat{s}_{12} + \hat{s}_{23} + \hat{s}_{13} , \hat{t}_2 = \hat{s}_{23} + \hat{s}_{24} + \hat{s}_{34} , \hat{t}_3 = \hat{s}_{12} + \hat{s}_{26} + \hat{s}_{16} .$$

(3.30)

The above representation refers to the partial ordering (123456) of the six open string vertex operators along the boundary of the disk. The integrals (3.29) integrate to triple hypergeometric functions. The latter belong to the class of multiple Gaussian hypergeometric functions [4].

Our task is to express (2.54) as a sum over integrals of the type (3.29). This is achieved by converting the complex integration in $z$ into two real integrals by splitting the complex $z$–integral up into holomorphic and anti–holomorphic contour integrals. In addition, the $x$–integration along the real axis is converted to an integration along the imaginary axis by analytic continuation of the integrand, given in Eq. (2.62). This procedure is performed explicitly in the Appendix B and allows to cast the expression (2.62) into the form

$$W^{(\kappa, \alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] = 2^{-\alpha_0} i$$

$$\times \left\{ e^{i\pi(\hat{\gamma}_2 + \hat{\beta}_2)} \left[ \sin(\pi \hat{\gamma}_2) A(1, 6, 3, 5, 4, 2) + \sin(\pi \hat{\beta}_2) A(1, 6, 4, 5, 3, 2) \right] \right.$$

$$- \sin(\pi \hat{\lambda}_2) \left[ e^{i\pi \hat{\gamma}_2} A(1, 3, 5, 4, 2, 6) + A(1, 3, 4, 5, 2, 6) \right.$$  

$$\left. + e^{i\pi \hat{\beta}_2} A(1, 4, 5, 3, 2, 6) + A(1, 4, 3, 5, 2, 6) \right]$$

$$+ \sin(\pi \hat{\beta}_2) A(1, 3, 5, 2, 4, 6) + \sin(\pi \hat{\gamma}_2) A(1, 6, 3, 2, 5, 4) \right\} + R ,$$

(3.31)
with the set of five integrals (cf. Appendix B.1 and B.2)

\[
A(1, 6, 3, 5, 4, 2) = 2^{\alpha_0} (-1)^{n_3+n_4+n_6} \int_0^1 d\rho \int_{-\rho}^\rho d\xi \int_0^{-\rho} d\eta \ \rho^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (1 - \eta)^{\lambda_2} (\rho - \xi)^{\gamma_1} (-\rho - \eta)^{\gamma_2} (\rho + \xi)^{\beta_1} (\rho - \eta)^{\beta_2} (\xi - \eta)^{\kappa}
\]

\[
A(1, 3, 5, 4, 2, 6) = 2^{\alpha_0} (-1)^{n_2+n_3+\kappa} \int_0^1 d\rho \int_{-\rho}^\rho d\xi \int_0^{\rho} d\eta \ \rho^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (\eta - 1)^{\lambda_2} (\rho - \xi)^{\gamma_1} (\rho + \xi)^{\beta_1} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa}
\]

\[
A(1, 3, 4, 5, 2, 6) = 2^{\alpha_0} (-1)^{n_2+\kappa} \int_0^1 d\rho \int_{-\rho}^\rho d\xi \int_0^{\infty} d\eta \ \rho^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (\eta - 1)^{\lambda_2} (\rho - \xi)^{\gamma_1} (\rho + \xi)^{\beta_1} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa}
\]

\[
A(1, 3, 5, 2, 4, 6) = 2^{\alpha_0} (-1)^{m_0+n_2+n_3+\kappa} \int_0^\infty d\rho \int_{-\rho}^\rho d\xi \int_0^{\rho} d\eta \ \rho^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (\eta - 1)^{\lambda_2} (\rho - \xi)^{\gamma_1} (\rho + \xi)^{\beta_1} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa}
\]

\[
A(1, 6, 3, 2, 5, 4) = 2^{\alpha_0} (-1)^{m_0+n_1+n_3+n_4+n_6} \int_0^\infty d\rho \int_{-\rho}^\rho d\xi \int_0^{-\rho} d\eta \ \rho^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (\xi - 1)^{\lambda_1} (1 - \eta)^{\lambda_2} (\rho - \xi)^{\gamma_1} (\rho + \xi)^{\beta_1} (\rho - \eta)^{\beta_2} (\xi - \eta)^{\kappa}
\]

(3.32)

and the three integrals (cf. Appendix B.3):

\[
A(1, 6, 4, 5, 3, 2) = 2^{\alpha_0} (-1)^{n_4+n_5+\alpha_0} \int_{-1}^0 d\rho \int_{-\rho}^{\rho} d\xi \int_0^{-\rho} d\eta \ (-\rho)^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (1 - \eta)^{\lambda_2} (\xi - \rho)^{\gamma_1} (-\rho - \eta)^{\gamma_2} (-\rho - \xi)^{\beta_1} (\rho - \eta)^{\beta_2} (\xi - \eta)^{\kappa}
\]

\[
A(1, 4, 5, 3, 2, 6) = 2^{\alpha_0} (-1)^{n_2+n_5+n_6+\kappa+\alpha_0} \int_{-1}^0 d\rho \int_{-\rho}^{\rho} d\xi \int_0^{\rho} d\eta \ (-\rho)^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (\eta - 1)^{\lambda_2} (\xi - \rho)^{\gamma_1} (\rho + \xi)^{\gamma_2} (-\rho - \xi)^{\beta_1} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa}
\]

\[
A(1, 4, 3, 5, 2, 6) = 2^{\alpha_0} (-1)^{n_2+\kappa+\alpha_0} \int_{-1}^0 d\rho \int_{-\rho}^{\rho} d\xi \int_0^{\rho} d\eta \ (-\rho)^{\alpha_0} (1 + \rho)^{\alpha_1} (1 - \rho)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (\eta - 1)^{\lambda_2} (\xi - \rho)^{\gamma_1} (\rho + \xi)^{\gamma_2} (\rho + \xi)^{\beta_1} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa}
\]

(3.33)

Furthermore, in (3.31) there may be a contribution from a residuum \( R \) contributing in the case \( \alpha \leq -1 \), cf. Appendix B.

The eight real integrals (3.32) and (3.33) appear as world–sheet integrals describing (partial ordered) six open string disk amplitudes. Indeed, the expressions (3.32) correspond to partial ordered amplitudes of a six open string disk amplitude with the following choice of vertex operator positions

\[
z_1 = -\infty \ , \ z_2 = 1 \ , \ z_3 = -\rho \ , \ z_4 = \rho \ , \ z_5 = \xi \ , \ z_6 = \eta \ , \quad (3.34)
\]
with ξ, η, ρ ∈ \( \mathbb{R} \). More precisely, the expressions (3.32) describe the following ordering of vertex positions

\[
A(1, 6, 3, 5, 4, 2) : \quad 0 < \rho < 1 , \quad -\rho < \xi < \rho , \quad -\infty < \eta < -\rho
\]
\[\iff z_1 < z_6 < z_3 < z_5 < z_4 < z_2 , \]
\[
A(1, 3, 5, 4, 2, 6) : \quad 0 < \rho < 1 , \quad -\rho < \xi < \rho , \quad 1 < \eta < \infty
\]
\[\iff z_1 < z_3 < z_5 < z_4 < z_2 < z_6 , \]
\[
A(1, 3, 4, 5, 2, 6) : \quad 0 < \rho < 1 , \quad \rho < \xi < 1 , \quad 1 < \eta < \infty
\]
\[\iff z_1 < z_3 < z_5 < z_4 < z_2 < z_6 , \]  
(3.35)
\[
A(1, 3, 5, 2, 4, 6) : \quad 1 < \rho < \infty , \quad -\rho < \xi < 1 , \quad \rho < \eta < \infty
\]
\[\iff z_1 < z_3 < z_5 < z_4 < z_2 < z_6 , \]
\[
A(1, 6, 3, 2, 5, 4) : \quad 1 < \rho < \infty , \quad 1 < \xi < \rho , \quad -\infty < \eta < -\rho
\]
\[\iff z_1 < z_6 < z_3 < z_2 < z_5 < z_4 , \]

and:

\[
A(1, 6, 4, 5, 3, 2) : \quad -1 < \rho < 0 , \quad \rho < \xi < -\rho , \quad -\infty < \eta < \rho
\]
\[\iff z_1 < z_6 < z_4 < z_5 < z_3 < z_2 , \]
\[
A(1, 4, 5, 3, 2, 6) : \quad -1 < \rho < 0 , \quad \rho < \xi < -\rho , \quad 1 < \eta < \infty
\]
\[\iff z_1 < z_4 < z_5 < z_3 < z_2 < z_6 , \]  
(3.36)
\[
A(1, 4, 3, 5, 2, 6) : \quad -1 < \rho < 0 , \quad -\rho < \xi < 1 , \quad 1 < \eta < \infty
\]
\[\iff z_1 < z_4 < z_3 < z_5 < z_2 < z_6 . \]

According to (2.19) we have the assignment:

\[
k_1 = p_1 , \quad k_2 = p_2 , \quad k_3 = \frac{1}{2} q_1 , \quad k_4 = \frac{1}{2} q_1 , \quad k_5 = \frac{1}{2} q_2 , \quad k_6 = \frac{1}{2} q_2 . \]  
(3.37)

As a consequence from (2.49) and (2.19) we have:

\[
\hat{s}_1 = 2 s , \quad \hat{s}_2 = \hat{s}_6 = u , \quad \hat{s}_3 = 0 , \quad \hat{s}_5 = 0 , \quad \hat{s}_4 = \frac{s}{2} , \quad \hat{t}_1 = \hat{t}_3 = s , \quad \hat{t}_2 = 2 u . \]  
(3.38)

The specific choice (3.37) of the six lightlike external momenta \( k_i \) is shown in the next Figure. The polygon in Fig. 9 specifies the scattering configuration of six open strings. We will add some comments on this configuration in Section 5.
Not all of the eight integrals (3.32) and (3.33) are independent. Indeed, after partial integration some relations may be found, e.g.:

\[
\sin(\pi s) \left[ A(1, 6, 3, 5, 4, 2) - A(1, 6, 4, 5, 3, 2) \right] = \sin(\pi t) \left[ A(1, 3, 5, 4, 2, 6) - A(1, 4, 5, 3, 2, 6) \right],
\]
\[
\sin\left(\frac{\pi s}{2}\right) \left[ A(1, 3, 5, 2, 4, 6) + A(1, 6, 3, 2, 5, 4) \right] - \sin(\pi t) \left[ A(1, 3, 4, 5, 2, 6) + A(1, 4, 3, 5, 2, 6) \right] = \cos\left(\frac{\pi s}{2}\right) \sin(\pi t) \left[ A(1, 3, 5, 4, 2, 6) + A(1, 4, 5, 3, 2, 6) \right] - \sin\left(\frac{\pi s}{2}\right) \cos(\pi s) \left[ A(1, 6, 3, 5, 4, 2) + A(1, 6, 4, 5, 3, 2) \right].
\]

These two relations allow to simplify the result (3.31) for the relevant case (3.38). This allows to cast the final result for the two open and two closed string disk amplitude into the following form

\[
\mathcal{A}(1, 2; 3, 4) = 2 \sin\left(\frac{\pi s}{2}\right) \sin(\pi s) A(1, 6, 3, 5, 4, 2) - 2 \sin\left(\frac{\pi s}{2}\right) \sin(\pi t) A(1, 3, 5, 4, 2, 6),
\]

(3.40)

with the two partial six open string amplitudes \(A(1, 2, 4, 5, 3, 6)\) and \(A(1, 3, 5, 4, 2, 6)\) given in (3.32). An alternative expression for (3.40) is:

\[
\mathcal{A}(1, 2; 3, 4) = 2 \sin\left(\frac{\pi s}{2}\right) \sin(\pi s) A(1, 4, 5, 3, 6, 2) - 2 \sin\left(\frac{\pi s}{2}\right) \sin(\pi u) A(1, 5, 3, 6, 2, 4).
\]

(3.41)
Comparing the two results (3.40) and (3.41) makes manifest the symmetry of the amplitude $A(1,2;3,4)$ under exchanging the labels 3 and 4, i.e. permuting the two closed strings.

Finally, after some coordinate transformations the partial amplitudes $A(1,2,3,4,5,6)$ and $A(1,3,5,2,4,6)$ may be brought into the canonical form (3.29), subject to the choice (3.38). Indeed with

$$\rho \to \frac{1 - yz}{1 + yz}, \quad \xi \to 1 - \frac{2y}{1 + yz}, \quad \eta \to 1 - \frac{2}{x(1 + yz)} \quad (3.42)$$

for the condition (2.57) the integral $A(163542)$ of (3.32) becomes:

$$A(1, 6, 3, 5, 4, 2) = \frac{1}{2} (-1)^{n_3 + n_4 + n_6} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ x^{-2 - \kappa - \lambda_2 - \gamma_2 - \beta_2} (1 - x)^{\gamma_2}$$

$$\times y^{1 + \alpha_2 + \lambda_1 + \gamma_1} (1 - y)^{\beta_1} z^{\alpha_2} (1 - z)^{\gamma_1} (1 - xy)^{\kappa} (1 - yz)^{\alpha_0} (1 - xyz)^{\beta_2} \quad (3.43)$$

On the other hand, with the transformations

$$\rho \to \frac{y}{2 - y}, \quad \xi \to -\frac{y(1 - 2x)}{2 - y}, \quad \eta \to \frac{2 - yz}{z(2 - y)} \quad (3.44)$$

for the condition (2.57) the integral $A(135426)$ of (3.32) assumes the form:

$$A(1, 3, 5, 4, 2, 6) = (-1)^{n_2 + n_3 + \kappa} \frac{1}{2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ x^{\beta_1} (1 - x)^{\gamma_1} y^{1 + \alpha_0 + \gamma_1 + \beta_1}$$

$$\times (1 - y)^{\alpha_2} z^{-2 - \lambda_2 - \gamma_2 - \beta_2 - \kappa} (1 - z)^{\lambda_2} (1 - xy)^{\lambda_1} (1 - yz)^{\beta_2} \quad (3.45)$$

With the choice (2.56) these two integrals (3.43) and (3.45) down to the expressions:

$$A(1, 6, 3, 5, 4, 2) = -\frac{1}{2} (-1)^{n_3 + n_4 + n_6} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ x^{u - \kappa - 2 - n_2 - n_4 - n_6} (1 - x)^{\frac{t}{2} + n_4}$$

$$\times y^{-\frac{s}{2} + 1 + m_0 + n_1 + n_3} (1 - y)^{\frac{s}{2} + n_5} z^{u + m_0} (1 - z)^{\frac{t}{2} + n_3} (1 - xy)^{\kappa}$$

$$\times (1 - yz)^{\alpha_0} (1 - xyz)^{\frac{t}{2} + n_6},$$

$$A(1, 3, 5, 4, 2, 6) = \frac{1}{2} (-1)^{n_2 + n_3 + \kappa} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ x^{\frac{t}{2} + n_5} (1 - x)^{\frac{s}{2} + n_3} y^{s + 1 + \alpha_0 + n_3 + n_5}$$

$$\times (1 - y)^{u + m_0} z^{u - \kappa - 2 - n_2 - n_4 - n_6} (1 - z)^{t + n_2} (1 - xy)^{t + n_1} (1 - yz)^{\frac{t}{2} + n_6} \quad (3.46)$$

In superstring amplitudes the complex integral (2.54), whose final result is given in (3.31) and (3.40), typically exhibits double or single poles in the kinematic invariants $s, t$ and $u$. E.g. the integral

$$W^{(0,0)} \left[ \frac{u - 1, t - 1, \frac{s}{2}, \frac{s}{2}}{u - 1, t - 1, \frac{s}{2}, \frac{s}{2}} \right] = \int_{-\infty}^{\infty} dx \ |1 - ix|^{2u - 2} \ \int_{\mathbb{C}} d^2 z \ |1 - z|^{2t - 2} |z - i x|^s |z + i x|^s \quad (3.47)$$
has poles for $z, \overline{z} \to 1$. With (3.40) we find:

$$W^{(0,0)} \left[ \frac{u-1, t-1, s}{u-1, t-1, \frac{s}{2}, \frac{s}{2}} \right] = 2\pi^2 \left( \frac{1}{t} + \frac{1}{u} + \zeta(3) s^2 + \frac{1}{4} \zeta(4) s^3 \right) + \ldots . \quad (3.48)$$

The first term stems\(^{10}\) from the limit $z \to 1$ in the integrand. According to (2.59) the integral (3.47) is invariant under the exchange of $t$ and $u$. This behaviour is manifest in the expansions (3.48). On the other hand, there also appear world–sheet integrals without poles in the kinematic invariants $s, t$ and $u$. E.g. the integral

$$W^{(0,0)} \left[ \frac{u, t, s}{u, t, \frac{s}{2} - 1, \frac{s}{2} - 1} \right] = \int_{-\infty}^{\infty} dx \frac{|1-ix|^{2u}}{|1-z|^2} \int_{C} d^2z \left| z-ix \right|^{s-2} \left| z+ix \right|^{-2} \quad (3.49)$$

has possible poles for $z \to \pm ix$, which are cancelled after performing the remaining $x$–integration. With (3.40) we find:

$$W^{(0,0)} \left[ \frac{u, t, \frac{s}{2} - 1, \frac{s}{2} - 1}{u, t, \frac{s}{2} - 1, \frac{s}{2} - 1} \right] = -4\pi^2 \left[ 1 + s + s^2 + \frac{1}{4} \zeta(2) (t^2 + 3tu + u^2) \right] + \ldots . \quad (3.50)$$

According to (2.59) the integral (3.49) is invariant under the exchange of $t$ and $u$. Again, this behaviour is manifest in the expansions (3.50).

### 3.5. Four open & one closed string versus six open strings on the disk

In this Subsection we establish the relation between a disk amplitude with four open & one closed string to a disk amplitude of six open strings. The generic expression of a (color ordered) disk amplitude $A(1, \sigma(2), \sigma(3), \sigma(4); 5)$ involving four open and one closed string is given in (2.69) and (2.70). On the other hand, the general expression of a (partial ordered) six open string amplitude is given in (3.29).

Our task is to express (2.70) as a sum over integrals of the type (3.29). This is achieved by converting the complex integration into two real integrals by splitting the complex $z$–integral up into holomorphic and anti–holomorphic contour integrals. This procedure is performed in the Appendix C. Depending on the range of the real parameter $x$ we obtain three integrals of the type (3.29). Eventually, we may cast the expressions for the three

---

\(^{10}\) This may be deduced from: $\int_{|z|<\epsilon} d^2z \frac{1}{|z|^{-2+k_i k_j}} = \frac{2\pi}{k_i k_j} \ln|k_i k_j|^{-1}$ [30].
partial amplitudes (2.70) into the following form

\[
W_{\sigma_1}^{(\kappa,\alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] = \sin(\pi \beta_2) A(1,6,3,5,4,2) + \sin(\pi \kappa) A(1,3,4,2,5,6) \\
+ \sin(\pi \kappa) A(1,3,4,5,6,2) + \sin[\pi(\kappa + \lambda_2)] A(1,3,4,5,2,6),
\]

\[
W_{\sigma_2}^{(\kappa,\alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] = \sin(\pi \gamma_2) A(1,6,4,5,3,2) + \sin(\pi \kappa) A(1,4,3,2,5,6) \\
+ \sin(\pi \kappa) A(1,4,3,5,6,2) + \sin[\pi(\kappa + \lambda_2)] A(1,4,3,5,2,6),
\]

\[
W_{\sigma_3}^{(\kappa,\alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] = \sin(\pi \beta_2) A(1,6,3,5,2,4) + \sin(\pi \kappa) A(1,3,2,4,5,6) \\
+ \sin(\pi \kappa) A(1,3,2,5,6,4) + \sin[\pi(\kappa + \gamma_2)] A(1,3,2,5,4,6),
\]

(3.51)

with the four integrals for \( \sigma_1 \)

\[
A(1,6,3,5,4,2) = 2^{\alpha_0} (-1)^{n_3+n_4+n_6} \int_0^1 dx \int_x^{-x} d\xi \int_{-\infty}^{-x} d\eta \ x^{\alpha_0} (1+x)^{\alpha_1} (1-x)^{\alpha_2} \\
\times (1-\xi)^{\lambda_1} (1-\eta)^{\lambda_2} (x-\xi)^{\gamma_1} (x-\eta)^{\gamma_2} (\xi+x)^{\beta_1} (-x-\eta)^{\beta_2} (\xi-\eta)^{\kappa},
\]

\[
A(1,3,4,5,2,6) = 2^{\alpha_0} (-1)^{n_2+\kappa} \int_0^1 dx \int_{-1}^{1} d\xi \int_{-\infty}^{\infty} d\eta \ x^{\alpha_0} (1+x)^{\alpha_1} (1-x)^{\alpha_2} \\
\times (1-\xi)^{\lambda_1} (1-\eta)^{\lambda_2} (\xi-x)^{\gamma_1} (\eta-x)^{\gamma_2} (\xi+x)^{\beta_1} (\eta+x)^{\beta_2} (\eta-\xi)^{\kappa},
\]

\[
A(1,3,4,5,6,2) = 2^{\alpha_0} (-1)^{\kappa} \int_0^1 dx \int_0^{1} d\xi \int_{-\infty}^{\infty} d\eta \ x^{\alpha_0} (1+x)^{\alpha_1} (1-x)^{\alpha_2} \\
\times (1-\xi)^{\lambda_1} (1-\eta)^{\lambda_2} (\xi-x)^{\gamma_1} (\eta-x)^{\gamma_2} (\xi+x)^{\beta_1} (\eta+x)^{\beta_2} (\eta-\xi)^{\kappa},
\]

\[
A(1,3,4,2,5,6) = 2^{\alpha_0} (-1)^{n_1+n_2+\kappa} \int_0^1 dx \int_1^{\infty} d\xi \int_{-\xi}^{\infty} d\eta \ x^{\alpha_0} (1+x)^{\alpha_1} (1-x)^{\alpha_2} \\
\times (\xi-1)^{\lambda_1} (\xi-x)^{\gamma_1} (\xi+x)^{\beta_1} (\eta-1)^{\lambda_2} (\eta-x)^{\gamma_2} (\eta+x)^{\beta_2} (\eta-\xi)^{\kappa},
\]

(3.52)

the four integrals for \( \sigma_2 \)

\[
A(1,6,4,5,3,2) = 2^{\alpha_0} (-1)^{n_4+n_5+n_6+m_0} \int_{-1}^{0} dx \int_{-x}^{x} d\xi \int_{-\infty}^{\infty} d\eta \ (-x)^{\alpha_0} (1+x)^{\alpha_1} (1-x)^{\alpha_2} \\
\times (1-\xi)^{\lambda_1} (1-\eta)^{\lambda_2} (\xi-x)^{\gamma_1} (x-\eta)^{\gamma_2} (-x-\xi)^{\beta_1} (-x-\eta)^{\beta_2} (\xi-\eta)^{\kappa},
\]

\[
A(1,4,3,5,2,6) = 2^{\alpha_0} (-1)^{n_2+m_0+\kappa} \int_{-1}^{0} dx \int_{-x}^{1} d\xi \int_{-\infty}^{\infty} d\eta \ (-x)^{\alpha_0} (1+x)^{\alpha_1} (1-x)^{\alpha_2} \\
\times (1-\xi)^{\lambda_1} (\eta-1)^{\lambda_2} (\xi-x)^{\gamma_1} (\eta-x)^{\gamma_2} (\xi+x)^{\beta_1} (\eta+x)^{\beta_2} (\eta-\xi)^{\kappa},
\]

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the four integrals for $\kappa$

\[
A(1, 4, 3, 5, 6, 2) = 2^{\alpha_0} (-1)^{m_0 + \tilde{\kappa}} \int_{-1}^{0} dx \int_{-x}^{1} d\xi \int_{-x}^{1} d\eta (-x)^{\alpha_0} (1 + x)^{\alpha_1} (1 - x)^{\alpha_2} \\
\times (1 - \xi)^{\lambda_1} (1 - \eta)^{\lambda_2} (\xi - x)^{\gamma_1} (\eta - x)^{\gamma_2} (\xi + x)^{\beta_1} (\eta + x)^{\beta_2} (\eta - \xi)^{\kappa},
\]

\[
A(1, 4, 3, 2, 5, 6) = 2^{\alpha_0} (-1)^{n_1 + n_2 + m_0 + \tilde{\kappa}} \int_{-1}^{0} dx \int_{-x}^{1} d\xi \int_{-x}^{1} d\eta (-x)^{\alpha_0} (1 + x)^{\alpha_1} (1 - x)^{\alpha_2} \\
\times (\xi - 1)^{\lambda_1} (\xi - x)^{\gamma_1} (\xi + x)^{\beta_1} (\eta - 1)^{\lambda_2} (\eta - x)^{\gamma_2} (\eta + x)^{\beta_2} (\eta - \xi)^{\kappa},
\]

(3.53)

Note, that the quantities with a hat are the non–integer parts of the variables (2.72). In addition, we have introduced $\kappa = \tilde{\kappa} + \bar{\kappa}$, with $\tilde{\kappa} = 2 q_{||}^2$, cf. also Appendix C.

The twelve real integrals (3.52), (3.53) and (3.54) appear as world–sheet integrals describing (partial ordered) six open string disk amplitudes. Indeed, the expressions (3.52) correspond to partial ordered amplitudes of a six open string disk amplitude with the following choice of vertex operator positions

\[
z_1 = -\infty, \quad z_2 = 1, \quad z_3 = -x, \quad z_4 = x, \quad z_5 = \xi, \quad z_6 = \eta,
\]

(3.55)

with $\xi, \eta, x \in \mathbb{R}$. In the following we shall explicitly work out the case $\kappa \in \mathbb{Z}$, i.e. $\tilde{\kappa} = 2 q_{||}^2 = 0$ and $\kappa = \bar{\kappa}$. In this case only the first lines of (3.51) contribute. Then expressions
(3.52) describe the following ordering of vertex positions:

\[
\begin{align*}
\sigma_1 & : \ A(1, 6, 3, 5, 4, 2) : \quad 0 < x < 1 , \ -x < \xi < x , \ -\infty < \eta < -x \\
& \quad \iff z_1 < z_6 < z_3 < z_5 < z_4 < z_2 , \\
A(1, 3, 4, 5, 2, 6) : \quad 0 < x < 1 , \ x < \xi < 1 , \ 1 < \eta < \infty \\
& \quad \iff z_1 < z_3 < z_4 < z_5 < z_2 < z_6 , \\
\sigma_2 & : \ A(1, 6, 4, 5, 3, 2) : \quad -1 < x < 0 , \ x < \xi < -x , \ -\infty < \eta < x \\
& \quad \iff z_1 < z_6 < z_4 < z_5 < z_3 < z_2 , \\
A(1, 4, 3, 5, 2, 6) : \quad -1 < x < 0 , \ -x < \xi < 1 , \ 1 < \eta < \infty , \\
& \quad \iff z_1 < z_4 < z_3 < z_5 < z_2 < z_6 , \\
\sigma_3 & : \ A(1, 6, 3, 5, 2, 4) : \quad 1 < x < \infty , \ -x < \xi < 1 , \ -\infty < \eta < -x \\
& \quad \ -\infty < x < -1 , \ 1 < \xi < -x , \ -x < \eta < \infty \\
& \quad \iff z_1 < z_6 < z_3 < z_5 < z_2 < z_4 , \\
A(1, 3, 2, 5, 4, 6) : \quad 1 < x < \infty , \ 1 < \xi < x , \ x < \eta < \infty \\
& \quad \ -\infty < x < -1 , \ x < \xi < 1 , \ -\infty < \eta < x \\
& \quad \iff z_1 < z_3 < z_2 < z_5 < z_4 < z_6 .
\end{align*}
\]

According to (2.19) we have the assignment:

\[
k_1 = p_1 , \quad k_2 = p_2 , \quad k_3 = p_3 , \quad k_4 = p_4 , \quad k_5 = \frac{1}{2} q , \quad k_6 = \frac{1}{2} q .
\] (3.57)

As a consequence from (2.68) and (2.19) we have:

\[
\begin{align*}
\hat{s}_1 &= s_1 , \quad \hat{s}_2 = s_2 , \quad \hat{s}_3 = s_3 , \quad \hat{s}_4 = s_4 , \quad \hat{s}_5 = 0 , \quad \hat{s}_6 = s_5 , \\
\hat{t}_1 &= 2 s_4 , \quad \hat{t}_2 = 2 s_5 , \quad \hat{t}_3 = \frac{1}{2} s_1 + \frac{1}{2} s_3 .
\end{align*}
\] (3.58)

Finally after some coordinate transformations the set of partial amplitudes (3.52) may be brought into the canonical form (3.29), subject to the choice (3.58). Indeed, the following transformations

\[
\begin{align*}
A(1, 6, 3, 5, 4, 2) : & \quad x \to -1 + \frac{2}{1 + y z} , \quad \xi \to 1 - \frac{2 y}{1 + y z} , \quad \eta \to 1 - \frac{2}{x (1 + y z)} , \\
A(1, 3, 4, 5, 2, 6) : & \quad x \to \frac{x y}{2 - x y} , \quad \xi \to \frac{(2 - x) y}{2 - x y} , \quad \eta \to \frac{2 - x y z}{z (2 - x y)} , \\
A(1, 6, 4, 5, 3, 2) : & \quad x \to 1 - \frac{2}{1 + y z} , \quad \xi \to 1 - \frac{2 y}{1 + y z} , \quad \eta \to 1 - \frac{2}{x (1 + y z)} ,
\end{align*}
\] (3.59)
acting on their integrands bring for the case (2.73) the corresponding integrals (3.52) into

\[
A(1, 4, 3, 5, 2, 6) : \quad x \to -\frac{xy}{2 - xy}, \quad \xi \to \frac{(2 - x)}{2 - y}, \quad \eta \to \frac{2 - xyz}{z (2 - xy)};
\]

\[
A(1, 6, 3, 5, 2, 4) = A(1, 4, 2, 5, 3, 6) : \quad x \to \frac{1}{1 - 2 y}, \quad \xi \to \frac{1 - 2 y}{1 - 2 y}, \quad \eta \to -\frac{2 - y}{x (1 - 2 y)};
\]

\[
A(1, 3, 2, 5, 4, 6) = A(1, 6, 4, 5, 2, 3) : \quad x \to -\frac{1}{1 - 2 x}, \quad \xi \to \frac{1 - 2 y}{1 - 2 y}, \quad \eta \to -\frac{2 - z}{z (1 - 2 x)};
\]
with the six partial amplitudes given in (3.60). The second amplitude $A(1,4,3,2;5)$ follows from the first amplitude $A(1,3,4,2;5)$ by permuting the open string labels 3 and 4 and replacing $s_2 \to -s_2 - s_3 + 2s_5$, $s_4 \to \frac{s_1}{2} - \frac{s_4}{2} - s_4$. Furthermore, the third amplitude $A(1,4,2,3;5)$ is obtained from the second by relabeling the open strings 2 and 3 and performing $s_1 \to -s_1 - s_2 + 2s_4$, $s_3 \to -s_2 - s_3 + 2s_5$.

3.6. *Three closed strings versus six open strings on the disk*

In this Subsection we establish the relation between a disk amplitude with three closed strings to a disk amplitude of six open strings. The generic expression of a disk amplitude $A(1,2,3)$ of three closed strings is given in (2.84) and (2.85). On the other hand, the general expression of a (partial ordered) six open string amplitude can be found in (3.29).

Our task is to express (2.85) as a sum over integrals of the type (3.29). This is achieved by converting the complex integration into two real integrals (2.90) by splitting the complex $z$–integral up into holomorphic and anti–holomorphic contour integrals. This procedure is performed in the Appendix D. The contributions of the four contours $I_a, I_d, I_{c_1}$ and $I_{c_2}$, given in Eqs. (D.3) and (D.4), give

$$0 < x < 1: \sin(\pi s_4) \ A(1,2,5,4,6,3) - \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} + s_6 \right) \right] A(1,3,4,6,2,5),$$

$$-1 < x < 0: \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} - s_4 \right) \right] A(1,2,5,3,6,4) - \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} + s_6 \right) \right] A(1,4,3,6,2,5)$$

(3.62)

as a result of applying the amplitude relations:

$$\sin(\pi s_5) \ A(1,3,4,2,6,5) - \sin(\pi s_6) \ A(1,5,2,4,6,3) - \sin(\pi s_6) \ A(1,5,2,4,3,6)$$

$$- \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} + s_6 \right) \right] A(1,3,4,6,2,5) = 0,$$

$$\sin(\pi s_5) \ A(1,4,3,2,6,5) - \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} - s_4 + s_6 \right) \right] A(1,5,2,3,6,4) - \sin(\pi s_6) \ A(1,5,2,3,4,6) - \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} + s_6 \right) \right] A(1,4,3,6,2,5) = 0.$$  

(3.63)

The terms (3.62) combine with the contributions (D.5) from the contours $I_{b_1}$ and $I_{b_2}$ into:

$$W^{(\kappa, \alpha_0, \alpha_3)} \left[ \alpha_1, \lambda_1, \gamma_1, \beta_1, \epsilon_1 \right]$$

$$\left[ \alpha_2, \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right]$$

$$= \sin(\pi s_4) \ A(1,2,5,4,6,3) + \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} - s_4 \right) \right] A(1,2,5,3,6,4)$$

$$- e^{i\pi s_5} \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} + \frac{s_5}{2} + s_6 \right) \right] \left[ A(1,3,4,6,2,5) + A(1,4,3,6,2,5) \right],$$

$$- e^{\frac{1}{3}i\pi (s_1 - s_3 + s_5)} \sin(\pi s_5) \left[ A(1,3,4,6,5,2) + A(1,4,3,6,5,2) \right].$$  

(3.64)
Eventually, the amplitude (2.84) can be expressed in terms of a six–dimensional basis of partial ordered six open string amplitudes (3.29) as:

\[
A(1, 2, 3) = \left\{ \sin(\pi s_4) \ A(1, 3, 6, 4, 5, 2) + \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} - s_4 \right) \right] A(1, 4, 6, 3, 5, 2) \right.

+ \sin \left[ \pi \left( \frac{s_1}{2} + \frac{s_3}{2} - \frac{s_5}{2} - s_6 \right) \right] \left[ A(1, 3, 4, 6, 2, 5) + A(1, 4, 3, 6, 2, 5) \right] \\
+ \left( \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} \right) \right] - \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} + \frac{s_5}{2} \right) \right] \right) \times \left[ A(1, 3, 4, 6, 5, 2) + A(1, 4, 3, 6, 5, 2) \right] \right\} (-1)^{m_3 + n_6 + n_8}.
\]

(3.65)

In (3.65) the basis of six integrals (cf. Appendix D.1. and D.2.) is given by:

\[
A(1, 3, 6, 4, 5, 2) = 2^{1+\alpha_0+\alpha_3} (-1)^{n_5} \int_0^1 dx \int_{-x}^x d\xi \int_{-x}^x d\eta \ x^{\alpha_3} (1 + x)^{1+\alpha_1} (1 - x)^{1+\alpha_2} \\
\times (1 - \xi)^{\alpha_1} (1 + \xi)^{\gamma_1} (x - \xi)^{\beta_1} (\xi + x)^{\epsilon_1} \\
\times (1 - \eta)^{\alpha_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa}
\]

\[
A(1, 3, 4, 6, 2, 5) = 2^{1+\alpha_0+\alpha_3} (-1)^{n_2+n_5+n_7} \int_0^1 dx \int_{-x}^x d\xi \int_{-x}^x d\eta \ x^{\alpha_3} (1 + x)^{1+\alpha_1} (1 - x)^{1+\alpha_2} \\
\times (1 - x)^{1+\alpha_2} (1 - \xi)^{\alpha_1} (1 + \xi)^{\gamma_1} (x - \xi)^{\beta_1} (-x - \xi)^{\epsilon_1} \\
\times (\eta - 1)^{\alpha_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa}
\]

\[
A(1, 3, 4, 6, 5, 2) = 2^{1+\alpha_0+\alpha_3} (-1)^{n_5+n_7} \int_0^1 dx \int_{-\xi}^x d\xi \int_{-\xi}^x d\eta \ x^{\alpha_3} (1 + x)^{1+\alpha_1} (1 - x)^{1+\alpha_2} \\
\times (1 - \xi)^{\alpha_1} (1 + \xi)^{\gamma_1} (x - \xi)^{\beta_1} (-\xi - x)^{\epsilon_1} \\
\times (1 - \eta)^{\alpha_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa}
\]

\[
A(1, 4, 6, 3, 5, 2) = 2^{1+\alpha_0+\alpha_3} (-1)^{n_0+n_7} \int_{-1}^0 dx \int_{-x}^x d\xi \int_{-x}^x d\eta \ (-x)^{\alpha_3} (1 + x)^{1+\alpha_1} \\
\times (1 - x)^{1+\alpha_2} (1 - \xi)^{\alpha_1} (1 + \xi)^{\gamma_1} (\xi - x)^{\beta_1} (-\xi - x)^{\epsilon_1} \\
\times (1 - \eta)^{\alpha_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa}
\]

\[
A(1, 4, 3, 6, 2, 5) = 2^{1+\alpha_0+\alpha_3} (-1)^{n_0+n_2+n_5+n_7} \int_{-1}^0 dx \int_{-1}^x d\xi \int_{-1}^x d\eta \ (-x)^{\alpha_3} (1 + x)^{1+\alpha_1} \\
\times (1 - x)^{1+\alpha_2} (1 - \xi)^{\alpha_1} (1 + \xi)^{\gamma_1} (x - \xi)^{\beta_1} (-\xi - x)^{\epsilon_1} \\
\times (\eta - 1)^{\alpha_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa}
\]

\[
A(1, 4, 3, 6, 5, 2) = 2^{1+\alpha_0+\alpha_3} (-1)^{n_0+n_5+n_7} \int_{-1}^0 dx \int_{-\pi}^x d\xi \int_{-\pi}^x d\eta \ (-x)^{\alpha_3} (1 + x)^{1+\alpha_1} \\
\times (1 - x)^{1+\alpha_2} (1 - \xi)^{\alpha_1} (1 + \xi)^{\gamma_1} (x - \xi)^{\beta_1} (-\xi - x)^{\epsilon_1}
\]

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According to (2.19) we have the assignment:

\[ \times (1 - \eta)^{\lambda_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa} \]. \quad (3.66) 

The six real integrals (3.66) appear as world–sheet integrals describing (partial ordered) six open string disk amplitudes. Indeed, the expressions (3.66) correspond to partial ordered amplitudes of a six open string disk amplitude with the following choice of vertex operator positions

\[ z_1 = -1 \, , \, z_2 = 1 \, , \, z_3 = -x \, , \, z_4 = x \, , \, z_5 = \eta \, , \, z_6 = -\xi \] \quad (3.67)

with \( \xi, \eta \in \mathbb{R} \) and \(-1 < x < 1 \). More precisely, the expressions (3.66) describe the following ordering of vertex positions

\[
A(1, 3, 6, 4, 5, 2) : \quad 0 < x < 1 \, , \, -x < \xi < x \, , \, x < \eta < 1 \\
\iff z_1 < z_3 < z_6 < z_4 < z_5 < z_2 , \\
A(1, 3, 4, 6, 2, 5) : \quad 0 < x < 1 \, , \, -1 < \xi < -x \, , \, 1 < \eta < \infty \\
\iff z_1 < z_3 < z_4 < z_6 < z_2 < z_5 , \\
A(1, 3, 4, 6, 5, 2) : \quad 0 < x < 1 \, , \, -1 < \xi < -x \, , \, -\xi < \eta < 1 \\
\iff z_1 < z_3 < z_4 < z_6 < z_5 < z_2 , \\
A(1, 4, 6, 3, 5, 2) : \quad -1 < x < 0 \, , \, x < \xi < -x \, , \, -x < \eta < 1 \\
\iff z_1 < z_4 < z_6 < z_3 < z_5 < z_2 , \\
A(1, 4, 3, 6, 2, 5) : \quad -1 < x < 0 \, , \, -1 < \xi < x \, , \, 1 < \eta < \infty \\
\iff z_1 < z_4 < z_3 < z_6 < z_2 < z_5 , \\
A(1, 4, 3, 6, 5, 2) : \quad -1 < x < 0 \, , \, -1 < \xi < x \, , \, -\xi < \eta < 1 \\
\iff z_1 < z_4 < z_3 < z_6 < z_5 < z_2 .
\]

(3.68)

According to (2.19) we have the assignment:

\[ k_1 = \frac{1}{2} Dq_1 \, , \, k_2 = \frac{1}{2} q_1 \, , \, k_3 = \frac{1}{2} Dq_2 \, , \, k_4 = \frac{1}{2} q_2 \, , \, k_5 = \frac{1}{2} Dq_3 \, , \, k_6 = \frac{1}{2} q_3 . \quad (3.69) \]

As a consequence from (2.82) and (2.21) we have:

\[ \hat{s}_i = \bar{s}_i \, , \, i = 1, \ldots , 6 , \]

\[ \hat{t}_1 = \frac{1}{2} (s_1 - s_3 + s_5) \, , \, \hat{t}_2 = \frac{1}{2} (-s_1 + s_3 + s_5) \, , \, \hat{t}_3 = \frac{1}{2} (s_1 + s_3 - s_5) . \quad (3.70) \]

Eventually, after some coordinate transformations the set of partial amplitudes (3.66) may be brought into the canonical form (3.29), subject to the choice (3.70).

\[
A(1, 3, 6, 4, 5, 2) = \frac{1}{2} (-1)^{\rho_3} \int_0^1 dx \int_0^1 dy \int_0^1 dz \, x^{\beta_1} (1 - x)^{\epsilon_1} \, y^{1 + \alpha_0 + \lambda_2 + \gamma_2} (1 - y)^{\beta_2} \\
\times z^{\alpha_0} (1 - z)^{\lambda_2} (1 - xy)^{\kappa} (1 - yz)^{\gamma_1} (1 - xyz)^{\gamma_1},
\]

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subject to the conditions (2.88).
Furthermore, by comparing (3.65) and (3.72) the symmetry under the permutation 1 ↔ λ2 ↔ 3

The two expressions (3.65) and (3.72) are manifest symmetric under the permutation



subject to the conditions (2.88).
Instead of (3.65) we may also put the amplitude (2.84) into the form:

\[ A(1, 2, 3) = \left\{ \sin(\pi s_4) A(1, 2, 3, 6, 4, 5) + \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} - s_4 \right) \right] A(1, 2, 4, 6, 3, 5) + \sin(\pi s_6) \left[ A(1, 5, 2, 3, 4, 6) + A(1, 5, 2, 4, 3, 6) \right] \right. \]

\[ + \left( \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} \right) \right] - \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} + \frac{s_5}{2} \right) \right] \right) \]

\[ \times \left[ A(1, 2, 3, 4, 6, 5) + A(1, 2, 4, 3, 6, 5) \right] \left\} (-1)^{m_3 + n_6 + n_8} . \]

The two expressions (3.65) and (3.72) are manifest symmetric under the permutation 3 ↔ 4, with (β_i ↔ ε_i, α_1 ↔ α_2)

\[ s_2 \rightarrow \frac{1}{2} (-s_1 - s_3 + s_5) - s_2 , \quad s_4 \rightarrow \frac{1}{2} (s_1 - s_3 - s_5) - s_4 . \]

Furthermore, by comparing (3.65) and (3.72) the symmetry under the permutation 1 ↔ 2, with (λ_i ↔ γ_i, α_1 ↔ α_2)

\[ s_2 \rightarrow \frac{1}{2} (-s_1 - s_3 + s_5) - s_2 , \quad s_6 \rightarrow \frac{1}{2} (-s_1 + s_3 - s_5) - s_6 \]

can be exhibited. Finally, the symmetry 5 ↔ 6, with (λ_i ↔ γ_i, β_i ↔ ε_i)

\[ s_4 \rightarrow \frac{1}{2} (s_1 - s_3 - s_5) - s_4 , \quad s_6 \rightarrow \frac{1}{2} (-s_1 + s_3 - s_5) - s_6 \]
can be proven by applying some relations between partial amplitudes. Moreover, it can be shown, that the result (3.65) is invariant under the exchange (12) ↔ (34), with

\[ s_1 \leftrightarrow s_3 , \quad s_4 \leftrightarrow s_6 , \]

(3.76)

and the exchange (34) ↔ (56), with:

\[ s_2 \leftrightarrow s_6 , \quad s_3 \leftrightarrow s_5 . \]

(3.77)

Finally, let us discuss the special case \( q_{31} = 0 \). For this case the invariants (2.82) simplify to \( s_4 = \frac{1}{4}(s_1 - s_3) \), \( s_5 = 0 \) and \( s_6 = \frac{1}{4}(-s_1 + s_3) \), i.e. the assignments (2.87) become: \( \hat{\kappa} = 0 \), \( \hat{\lambda}_i = \hat{\gamma}_i = \frac{1}{4}(-s_1 + s_3) \), \( \hat{\beta}_i = \hat{\epsilon}_i = \frac{1}{4}(s_1 - s_3) \) and \( \hat{\lambda}_i + \hat{\beta}_i = 0 \). The four contours giving rise to the integrals \( I_a, I_{c_2}, I_{b_1} \) and \( I_{b_2} \) of Appendix D do not contribute, while the two contours corresponding to the integrals \( I_d \) and \( I_{c_1} \) amount to:

\[
A(1, 2, 3) = \sin \left[ \frac{\pi}{4}(s_1 - s_3) \right] \left\{ A(1, 3, 6, 4, 5, 2) + A(1, 4, 6, 3, 5, 2) \right. \\
- \left. A(1, 3, 4, 6, 2, 5) - A(1, 4, 3, 6, 2, 5) \right\} ,
\]

(3.78)

with the four partial ordered six open string amplitudes \( A(1, 3, 6, 4, 5, 2) \), \( A(1, 4, 6, 3, 5, 2) \), \( A(1, 3, 4, 6, 2, 5) \) and \( A(1, 4, 3, 6, 2, 5) \) given in (3.66) or in canonical form in (3.71). Note, that for this case we have the identities

\[
A(1, 6, 3, 4, 2, 5) = A(1, 3, 4, 6, 2, 5) , \quad A(1, 6, 4, 3, 2, 5) = A(1, 4, 3, 6, 2, 5)
\]

(3.79)

relating the two contour integrals \( I_d \) and \( I_{b_1} \).

3.7. Three open & two closed strings versus seven open strings on the disk

In this Subsection we establish the relation between a disk amplitude with three open & two closed strings to a disk amplitude of seven open strings. The generic expression of a disk amplitude \( A(1, 2, 3; 4, 5) \) involving three open and two closed strings is given in (2.96) and (2.97). On the other hand, the general expression of a (partial ordered) seven open string amplitude is given by the generalized Euler integral [4,2]

\[
A(1, 2, 3, 4, 5, 6, 7) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \, x^{\hat{s}_2} y^{\hat{t}_2} z^{\hat{t}_6} w^{\hat{s}_7} (1 - x)^{\hat{s}_3} (1 - y)^{\hat{s}_4} (1 - z)^{\hat{s}_5} \\
\times (1 - w)^{\hat{s}_6} (1 - xy)^{\hat{t}_3 - \hat{s}_3 - \hat{s}_4} (1 - yz)^{\hat{t}_4 - \hat{s}_4 - \hat{s}_5} (1 - xzw)^{\hat{t}_1 - \hat{t}_4 - \hat{t}_7} \\
\times (1 - zw)^{\hat{t}_5 - \hat{s}_5 - \hat{s}_6} (1 - yzw)^{\hat{t}_5 - \hat{t}_4 - \hat{t}_7} (1 - xzw)^{\hat{t}_1 - \hat{t}_4 - \hat{t}_7} ,
\]

(3.80)
with the seven momenta \( k_i \) of the seven external open strings and the 14 kinematic invariants \( \tilde{s}_i = \alpha'(k_i + k_{i+1})^2 \) and \( \hat{t}_j = \alpha'(k_j + k_{j+1} + k_{j+2})^2 \), subject to the cyclic identification \( i + 7 \equiv i \). For \( \alpha' = 2 \) the nine invariants \( \tilde{s}_i \) can be related to (2.21) as follows:

\[
\begin{align*}
\tilde{s}_1 &= \tilde{s}_{12} , \quad \tilde{s}_2 = \tilde{s}_{23} , \quad \tilde{s}_3 = \tilde{s}_{34} , \quad \tilde{s}_4 = \tilde{s}_{45} , \quad \tilde{s}_5 = \tilde{s}_{56} , \quad \tilde{s}_6 = \tilde{s}_{67} , \quad \tilde{s}_7 = \tilde{s}_{71} , \\
\hat{t}_1 &= \hat{t}_{12} + \tilde{s}_{23} + \tilde{s}_{13} , \quad \hat{t}_2 = \hat{t}_{23} + \tilde{s}_{24} + \tilde{s}_{34} , \quad \hat{t}_3 = \hat{t}_{34} + \tilde{s}_{35} + \tilde{s}_{45} , \\
\hat{t}_4 &= \hat{t}_{45} + \tilde{s}_{46} + \tilde{s}_{56} , \quad \hat{t}_5 = \hat{t}_{56} + \tilde{s}_{57} + \tilde{s}_{67} , \quad \hat{t}_6 = \hat{t}_{67} + \tilde{s}_{67} + \tilde{s}_{71} , \quad \hat{t}_7 = \hat{t}_{12} + \tilde{s}_{17} + \tilde{s}_{27} .
\end{align*}
\]

(3.81)

The above representation refers to the partial ordering \((1234567)\) of the seven open string vertex operators along the boundary of the disk. The integrals (3.80) integrate to multiple Gaussian hypergeometric functions [4].

Our task is to express (2.97) as a sum over integrals of the type (3.80). This is achieved by converting the two complex integrations in \( z_1 \) and \( z_2 \) into four real integrals by splitting the two complex integrals up into holomorphic and anti–holomorphic contour integrals, cf. Eq. (2.99). The expression (2.99) decomposes into a sum of partial ordered amplitudes describing a seven open string disk amplitude with the following choice of vertex operator positions

\[
z_1 = -\infty , \quad z_2 = 0 , \quad z_3 = 1 , \quad z_4 = \xi_1 , \quad z_5 = \eta_1 , \quad z_6 = \xi_2 , \quad z_7 = \eta_2 ,
\]

(3.82)

with \( \xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R} \).

According to (2.19) we have the assignment:

\[
k_1 = p_1 , \quad k_2 = p_2 , \quad k_3 = p_3 , \quad k_4 = \frac{1}{2} q_1 , \quad k_5 = \frac{1}{2} q_1 , \quad k_6 = \frac{1}{2} q_2 , \quad k_7 = \frac{1}{2} q_2 .
\]

(3.83)

As a consequence from (2.95) and (2.19) we have:

\[
\tilde{s}_1 = s_1 , \quad \tilde{s}_2 = s_2 , \quad \tilde{s}_3 = s_3 , \quad \tilde{s}_4 = 0 , \quad \tilde{s}_5 = s_4 , \quad \tilde{s}_6 = 0 , \quad \tilde{s}_7 = s_5 , \\
\hat{t}_1 = 4 s_4 , \quad \hat{t}_2 = \frac{1}{2} s_2 + s_5 , \quad \hat{t}_3 = 2 s_3 , \quad \hat{t}_4 = \hat{t}_5 = 2 s_4 , \quad \hat{t}_6 = 2 s_5 , \quad \hat{t}_7 = \frac{1}{2} s_1 + s_3 .
\]

(3.84)

Eventually, the amplitude (2.96) can be expressed as sum over partial ordered disk amplitudes involving seven open strings with integrals of the type (3.80):

\[
A(1, 2, 3; 4, 5) = -2 \sin \left( \pi \left( \frac{s_1}{2} - s_3 + s_5 \right) \right) \sin(\pi s_3) \left[ A(1, 5, 3, 4, 6, 2, 7) + A(1, 4, 3, 5, 6, 2, 7) \right] \\
+ 2 \sin \left( \frac{\pi s_3}{2} \right) \sin(\pi s_4) \left[ A(1, 5, 6, 3, 2, 4, 7) - A(1, 5, 6, 2, 3, 4, 7) \right] \\
+ 2 \sin \left( \pi \left( \frac{s_3}{2} + s_3 - s_5 \right) \right) \sin \left[ \pi \left( \frac{s_1}{2} - s_3 - 2s_4 \right) \right] \times \left[ A(1, 4, 2, 5, 6, 3, 7) + A(1, 5, 2, 4, 6, 3, 7) \right] + \ldots .
\]

(3.85)
4. Open string subamplitude relations

In the previous Section we have seen that any disk amplitude of $N_o$ open and $N_c$ closed strings decomposes into a sum over many color ordered $N_o + 2N_c$-point open string partial subamplitudes, cf. (3.1). In the previous Section we have derived relations between those subamplitudes by explicitly deforming the contour integrals in the complex $\eta_i$-planes. In this Section we develop a different method of deriving relations between subamplitudes of open string amplitudes.

We derive (string) relations between partial subamplitudes involving $N$ open strings. For $N = N_o + 2N_c$ and the condition (2.19) these relations can be used to simplify (3.1).

4.1. String theory generalization of Kleiss–Kuijf and Bern–Carrasco–Johansson relations

At tree–level with states all in the adjoint representation the full $N$–gluon amplitude $A$ can be decomposed as:

$$A(1, 2, \ldots, N) = g_{YM}^{N-2} \sum_{\sigma \in S_{N-1}} \text{Tr}(T^{a_1}T^{a_2(2)} \ldots T^{a_{\sigma(N)}}(N)) A(1, \sigma(2), \ldots, \sigma(N)) ,$$ (4.1)

with $S_{N-1} = S_N/\mathbb{Z}_N$ and $A(1, 2, \ldots, N)$ the tree–level color–ordered $N$–leg partial amplitude (helicity subamplitude). The $T^a$ are color–group generators encoding the color of each external states. The sum is over all $(N-1)!$ cyclic inequivalent permutations of external states, which is equivalent to all permutations with the first state kept fixed. The whole kinematics, i.e. helicities and polarizations are encoded in the partial amplitudes $A(1, 2, \ldots, N)$. By construction the latter are independent on color indices.

In (4.1) the $(N-1)!$ subamplitudes are not all independent. In fact, in addition to cyclic symmetries by applying reflection and parity symmetries

$$A(1, 2, \ldots, N) = (-1)^N A(N, N-1, \ldots, 2, 1)$$ (4.2)

we may reduce the number of independent partial amplitudes from $(N-1)!$ to $\frac{1}{2} (N-1)!$. So far these results are just inherited from the properties of the group traces. In addition, in field–theory these results are a consequence of studying the sum of Feynman diagrams which contribute to each subamplitude, while in string theory they follow from the properties of the string world–sheet. Hence these relations (4.2) hold both in field and string theory.

Moreover, in field theory there holds the dual Ward identity [31]

$$A_{FT}(1, 2, 3, \ldots, N-1, N) + A_{FT}(2, 1, 3, \ldots, N-1, N) + A_{FT}(2, 3, 1, \ldots, N-1, N)$$
$$+ \ldots + A_{FT}(2, 3, \ldots, N-1, 1, N) = 0 ,$$

(4.3)
which is not satisfied by the string subamplitudes. Furthermore, Kleiss and Kuijf have found a new set of relations, which allows to express all subamplitudes in terms of a minimal basis of \((N - 2)!\) elements \([32,33]\). Recently, in \([34]\] further relations have been derived which allow to boost this number from \((N - 2)!\) elements to \((N - 3)!\) independent basis elements. The proof of these relations is completely based on a field theory derivation \([33,34]\).

In the following, we demonstrate, that also in string theory relations between various subamplitudes may be derived. These relations, which are different than the field–theory relations, may be considered as the string theory upgrade of the dual Ward identity (4.3), Kleiss–Kuijf and Bern–Carrasco–Johansson relations, which then hold to all orders in \(\alpha'\). These identities also allow to reduce the number of independent subamplitudes to \((N - 3)!\). Clearly, in the field–theory limit our relations boil down to the Kleiss–Kuijf and Bern–Carrasco–Johansson relations. However, it should be stressed that the full string theory amplitude generically do not fulfill neither dual Ward nor Kleiss–Kuijf nor Bern–Carrasco–Johansson relations, however they do fulfill modified relations, which boil down to the former in the field–theory limit. A prominent example is the photon–decoupling identity or subcyclic property:

\[
\sum_{\sigma \in S_{N-1}} A_{FT}(1, \sigma(2), \sigma(3), \ldots, \sigma(N)) = 0 ,
\]

which is not fulfilled by the full string amplitude. Hence, for \(\alpha' \to 0\) our string theory relations provide a prove of the Kleiss–Kuijf and Bern–Carrasco–Johansson relations for any color group and for arbitrary space–time dimension.

### 4.2. World–sheet derivation of open string subamplitude relations

In this section by applying world–sheet string techniques we derive new algebraic identities between subamplitudes relevant for string theory. Recall, that the expression for partial amplitude \(A(1, \ldots, N)\) has the general form (with \(\alpha' = 2\))

\[
A(1, \ldots, N) = V_{CKG}^{-1} \int_{z_1 < \ldots < z_N} \left( \prod_{j=1}^{N} dz_j \right) \sum_{\mathcal{K}_I} \prod_{i<j}^{N} |z_i - z_j|^\hat{s}_{ij} (z_i - z_j)^n_{ij} ,
\]

cf. (2.13) for the case \(N_c = 0, N_o = N\). Here, the integers \(n_{ij}^I\) refer to the kinematical factor \(\mathcal{K}_I\) and the invariants \(\hat{s}_{ij}\) are defined in (2.21). The field theory identity (4.3) has the following generalization in string theory \([35]\):

\[
A(1, 2, \ldots, N) + e^{i\pi \hat{s}_{12}} A(2, 1, 3, \ldots, N - 1, N) + e^{i\pi (\hat{s}_{12} + \hat{s}_{13})} A(2, 3, 1, \ldots, N - 1, N) \\
+ \ldots + e^{i\pi (\hat{s}_{12} + \hat{s}_{13} + \ldots + \hat{s}_{1N-1})} A(2, 3, \ldots, N - 1, 1, N) = 0 .
\]

This relation can be derived by analytically continuing the \(z_1\)–integration in (4.5) to the whole complex plane and integrating \(z_1\) along the contour integral depicted in the next Figure.
The $z_1$–integration along the real axis gives a phase factor each time when encircling one open string vertex position $z_j$, $j = 2, \ldots, N$. On the other hand, the semicircle can be deformed to infinity. There the integrand behaves as $z_1^{-2h_1}$ with $h_1$ the conformal weight of the vertex operator $V_o(z_1)$. Since we consider massless open strings, we have $h_1 = 1$ and thus no contribution from the semicircle. Of course, since the partial amplitudes $A(1, \ldots, N)$ are real we obtain two relations from (4.6).

Note, that the integers $n_{ij}^I$ in (4.5) do not enter in the derivation of the result (4.6) as the branching is caused by the factors $|z_i - z_j|^{s_{ij}}$ only. However, the factors $(z_i - z_j)^{n_{ij}^I}$ do not affect the phases. The same is true for the kinematical factors $K_I$. More precisely, to each given kinematics $\mathcal{K}_I$ a specific set of integers $n_{ij}^I$ is assigned, but the latter are independent on the ordering $\sigma$, since the contractions of all fields yielding the factors $(z_i - z_j)^{n_{ij}^I}$ are taken w.r.t. to one specific ordering for any partial amplitude. It is only the integration region $\mathcal{I}_\sigma$, which changes for each subamplitude (4.5). Hence, the kinematical factors $K_I$ and the integers $n_{ij}^I$ have no effect on the result (4.6). Essentially the derivation of the subamplitude relations could be restricted to Neveu–Schwarz states only as the amplitudes involving Ramond states may be simply obtained from the latter by using supersymmetric Ward identities [2].

In the field–theory limit $\alpha' \rightarrow 0$, i.e. $e^{i\pi s_{ij}} \rightarrow 1$, the expression (4.6) boils down to the dual Ward identity (4.3). Of course, by analytically continuing any other position $z_j$, $j = 2, \ldots, N$ we obtain further relations. This way including permutations we can generate $N!$ complex relations of the type (4.6), which allow for expressing all subamplitudes in terms of a minimal basis. In fact, these relations allow for a complete reduction\footnote{On the day when submitting this project for publication the letter [36] has appeared, with results similar to what we have found in this Subsection 4.2. However, prior to this publication our results have been made already public in various seminars, cf. e.g. [37].} of the $(N-1)!$ string subamplitudes of an open string $N$–point amplitude to a minimal basis of

$$\nu_o(N) = (N - 3)!$$  \hspace{1cm} (4.7)
subamplitudes just like in field–theory. This number is identical to the dimension of a minimal basis of generalized Gaussian hypergeometric functions describing the full \( N \)-point open string amplitude [4,5,2].

Note, that the proof of the relation (4.6) and its permutations do not rely on any kinematic properties of the subamplitudes, on the amount of supersymmetry or the space–time dimension. Moreover, they hold for any type of massless string states both from the NS and R sector. Hence, these relations hold in any space–time dimension \( D \), for any amount of supersymmetry and any gauge group.

4.3. Examples

In this Subsection we discuss some examples and show how the solutions of the string relations of the type (4.6) reproduce the known field–theory relations in their \( \alpha' \to 0 \)–limit.

4.3.1. \( N = 4 \)

The subamplitudes are of the form (3.7). Cyclicity reduces the number of subamplitudes to \((4 – 1)! = 6\). Starting at the relation (4.6)

\[
A(1, 2, 3, 4) + e^{i\pi s} A(2, 1, 3, 4) + e^{i\pi(s+t)} A(2, 3, 1, 4) = 0 \quad (4.8)
\]

and its permutations give rise to 24 relations, which allow to express all six subamplitudes in terms of one \((\nu_0(4) = 1)\). As basis we choose the subamplitude \( A(1,2,3,4) \) and obtain the result:

\[
\frac{A(1,2,4,3)}{A(1,2,3,4)} = \frac{\sin(\pi u)}{\sin(\pi t)} , \quad \frac{A(1,3,2,4)}{A(1,2,3,4)} = \frac{\sin(\pi s)}{\sin(\pi t)} . \quad (4.9)
\]

As a result these relations allow to express all six partial amplitudes in terms of one, say \( A(1,2,3,4) \):

\[
A(1,4,3,2) = A(1,2,3,4) , \\
A(1,2,4,3) = A(1,3,4,2) = \frac{\sin(\pi u)}{\sin(\pi t)} A(1,2,3,4) , \\
A(1,3,2,4) = A(1,4,2,3) = \frac{\sin(\pi s)}{\sin(\pi t)} A(1,2,3,4) . \quad (4.10)
\]

Clearly, in the field–theory limit the relations (4.9) simply reduce to the well–known identities [34]:

\[
\frac{A_{FT}(1,2,4,3)}{A_{FT}(1,2,3,4)} = \frac{u}{t} , \quad \frac{A_{FT}(1,3,2,4)}{A_{FT}(1,2,3,4)} = \frac{s}{t} . \quad (4.11)
\]

and the subcyclic property

\[
A_{FT}(1,2,3,4) + A_{FT}(1,3,4,2) + A_{FT}(1,4,2,3) = 0
\]

can be proven.
4.3.2. \( N = 5 \)

The subamplitudes are of the form (3.15). Cyclicity, reflection and parity transformations (4.2) reduce the number of subamplitudes to \( \frac{1}{2}(5 - 1)! = 12 \). However, our new relations of the type (4.6) allow for a reduction of this number to a minimal basis of two \((\nu_o(5) = 2)\). Starting at the relation (4.6)

\[
A(1, 2, 3, 4, 5) + e^{i\pi s_1} A(2, 1, 3, 4, 5) + e^{i\pi (s_4 - s_2)} A(2, 3, 1, 4, 5) + e^{-i\pi s_5} A(2, 3, 4, 1, 5) = 0
\]

(4.12)

and its permutations give 120 relations, which allow to express all twelve or 24 subamplitudes in terms of two. The five kinematic invariants are defined in (3.16). As basis we choose the two subamplitudes \( A(1, 2, 3, 4, 5) = -A(1, 5, 4, 3, 2) \) and \( A(1, 3, 2, 4, 5) = -A(1, 5, 4, 2, 3) \). For completeness we write down all 24 subamplitudes:

\[
A(1, 2, 5, 4, 3) = -A(1, 3, 4, 5, 2) = \sin[\pi (\hat{s}_3 - \hat{s}_1 - \hat{s}_5)]^{-1}
\]

\[
\times \{ \sin[\pi (\hat{s}_3 - \hat{s}_5)] A(1, 2, 3, 4, 5) + \sin[\pi (\hat{s}_2 + \hat{s}_3 - \hat{s}_5)] A(1, 3, 2, 4, 5) \} ,
\]

(4.13)

\[
A(1, 3, 4, 2, 5) = -A(1, 5, 2, 4, 3) = \sin[\pi (\hat{s}_3 - \hat{s}_1 - \hat{s}_5)]^{-1}
\]

\[
\times \{ \sin(\pi \hat{s}_1) A(1, 2, 3, 4, 5) + \sin[\pi (\hat{s}_1 + \hat{s}_2)] A(1, 3, 2, 4, 5) \} ,
\]

\[
A(1, 2, 4, 5, 3) = -A(1, 3, 5, 4, 2) = \sin[\pi (\hat{s}_1 - \hat{s}_3 - \hat{s}_4)]^{-1}
\]

\[
\times \{ \sin(\pi \hat{s}_3) A(1, 2, 3, 4, 5) + \sin[\pi (\hat{s}_2 + \hat{s}_3)] A(1, 3, 2, 4, 5) \} ,
\]

(4.14)

\[
A(1, 4, 2, 3, 5) = -A(1, 5, 3, 2, 4) = -\sin[\pi (\hat{s}_1 - \hat{s}_3 - \hat{s}_4)]^{-1} \sin[\pi (\hat{s}_2 - \hat{s}_4 - \hat{s}_5)]^{-1}
\]

\[
\times \{ \sin(\pi \hat{s}_1) \sin(\pi \hat{s}_3) A(1, 2, 3, 4, 5) + \sin[\pi (\hat{s}_1 + \hat{s}_5)] A(1, 3, 2, 4, 5) \} ,
\]

\[
A(1, 4, 3, 2, 5) = -A(1, 5, 2, 3, 4) = -\sin[\pi (\hat{s}_1 - \hat{s}_3 - \hat{s}_5)]^{-1} \sin[\pi (\hat{s}_2 - \hat{s}_4 - \hat{s}_5)]^{-1}
\]

\[
\times \{ \sin[\pi (\hat{s}_1 + \hat{s}_2 - \hat{s}_4)] A(1, 2, 3, 4, 5) + \sin(\pi \hat{s}_1) \sin[\pi (\hat{s}_2 + \hat{s}_3 - \hat{s}_5)] A(1, 3, 2, 4, 5) \} ,
\]

(4.15)
\[ A(1, 2, 5, 3, 4) = -A(1, 4, 3, 5, 2) \]
\[ = \sin[\pi(\hat{s}_1 - \hat{s}_3 - \hat{s}_4)]^{-1} \sin[\pi(\hat{s}_3 - \hat{s}_1 - \hat{s}_5)]^{-1} \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{s}_5)]^{-1} \times \left\{ -\frac{1}{4} \left( \sin[\pi(\hat{s}_1 - \hat{s}_2 - \hat{s}_3)] - \sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{s}_3)] + \sin[\pi(\hat{s}_1 + \hat{s}_2 + \hat{s}_3)] \right) \right. \]
\[ + \sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{s}_3 - 2\hat{s}_4)] + \sin[\pi(-\hat{s}_1 + \hat{s}_2 + \hat{s}_3 - 2\hat{s}_5)] - \sin[\pi(\hat{s}_1 + \hat{s}_2 + \hat{s}_3 - 2\hat{s}_4 - 2\hat{s}_5)]) \right) \right. \]
\[ + \left. \sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{s}_4)] \sin[\pi(\hat{s}_2 + \hat{s}_3 - \hat{s}_5)] \sin[\pi(\hat{s}_4 + \hat{s}_5)] \right) \right. \]
\[ \left. A(1, 3, 2, 4, 5) \right\}, \quad (4.16) \]
\[ A(1, 3, 5, 2, 4) = -A(1, 4, 2, 5, 3) \]
\[ = \sin[\pi(\hat{s}_1 - \hat{s}_3 - \hat{s}_4)]^{-1} \sin[\pi(\hat{s}_3 - \hat{s}_1 - \hat{s}_5)]^{-1} \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{s}_5)]^{-1} \times \left\{ \sin(\pi\hat{s}_1) \sin(\pi\hat{s}_3) \sin[\pi(\hat{s}_4 + \hat{s}_5)] \right. \]
\[ \left. \times \left[ -\frac{1}{4} \left( \sin[\pi(\hat{s}_1 + \hat{s}_2 + \hat{s}_3 + \hat{s}_4 + \hat{s}_5)] - \sin[\pi(\hat{s}_1 + \hat{s}_2 + \hat{s}_3 - \hat{s}_4 - \hat{s}_5)] \right) \right. \right. \]
\[ \left. + \sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{s}_3 - \hat{s}_4 - \hat{s}_5)] - \sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{s}_3 + \hat{s}_4 + \hat{s}_5)] \right) \right. \]
\[ - \sin[\pi(\hat{s}_1 - \hat{s}_2 - \hat{s}_3 + \hat{s}_4 + \hat{s}_5)] - \sin[\pi(-\hat{s}_1 + \hat{s}_2 + \hat{s}_3 + \hat{s}_4 - \hat{s}_5)] \right) \right\} \right. \]
\[ \left. A(1, 3, 2, 4, 5) \right\}, \quad (4.17) \]

Again, the field–theory limit of these relations, which is simply obtained by replacing the \( \sin \)–function by its argument boils down to a system of identities, which solves the Kleiss–Kuijf and Bern–Carrasco–Johansson identities. E.g. the full string amplitudes fulfill the following relations:

\[ \sin[\pi(\hat{s}_2 - \hat{s}_4)] \] \( A(1, 2, 3, 4, 5) + \{ \sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{s}_4)] - \sin(\pi\hat{s}_1) \} \right) \]
\[ A(1, 3, 4, 5, 2) \]
\[ + \sin[\pi(\hat{s}_2 - \hat{s}_4)] \] \( A(1, 4, 5, 2, 3) + \{ \sin(\pi\hat{s}_5) + \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{s}_5)] \} \right) \]
\[ A(1, 5, 2, 3, 4) = 0 \],
\[ [\sin(\pi\hat{s}_1) + \sin(\pi\hat{s}_5)] \] \( A(1, 2, 3, 4, 5) + \sin[\pi(\hat{s}_1 + \hat{s}_5)] \]
\[ A(1, 3, 4, 5, 2) \]
\[ + \{ \sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{s}_4)] - \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{s}_5)] \} \right) \]
\[ A(1, 4, 5, 2, 3) \]
\[ + \sin[\pi(\hat{s}_1 + \hat{s}_5)] \] \( A(1, 5, 2, 3, 4) = 0 \),
\[ (4.18) \]

Clearly, in the field theory limit, these two relations boil down to the the subcyclic identity [34]:

\[ A_{FT}(1, 2, 3, 4, 5) + A_{FT}(1, 3, 4, 5, 2) + A_{FT}(1, 4, 5, 2, 3) + A_{FT}(1, 5, 2, 3, 4) = 0 \]. \( (4.19) \)

In these lines also the Kleiss–Kuijf relation

\[ A_{FT}(1, 2, 3, 5, 4) + A_{FT}(1, 2, 3, 4, 5) + A_{FT}(1, 2, 4, 3, 5) + A_{FT}(1, 4, 2, 3, 5) = 0 \], \( (4.20) \)
the Bern–Carrasco–Johansson relations

\[ A_{FT}(1,3,4,2,5) \hat{s}_{13} \hat{s}_{24} + \hat{s}_{12} \hat{s}_{45} A_{FT}(1,2,3,4,5) - \hat{s}_{14} (\hat{s}_{24} + \hat{s}_{25}) A_{FT}(1,4,3,2,5) = 0 , \]
\[ A_{FT}(1,2,4,3,5) \hat{s}_{35} \hat{s}_{24} + \hat{s}_{14} \hat{s}_{25} A_{FT}(1,4,3,2,5) - \hat{s}_{45} (\hat{s}_{12} + \hat{s}_{24}) A_{FT}(1,2,3,4,5) = 0 , \]
\[ A_{FT}(1,4,2,3,5) \hat{s}_{35} \hat{s}_{24} + \hat{s}_{12} \hat{s}_{45} A_{FT}(1,2,3,4,5) - \hat{s}_{25} (\hat{s}_{14} + \hat{s}_{24}) A_{FT}(1,4,3,2,5) = 0 , \]
\[ A_{FT}(1,3,2,4,5) \hat{s}_{13} \hat{s}_{24} + \hat{s}_{14} \hat{s}_{25} A_{FT}(1,4,3,2,5) - \hat{s}_{12} (\hat{s}_{24} + \hat{s}_{45}) A_{FT}(1,2,3,4,5) = 0 , \]

and

\[ A_{FT}(1,2,4,3,5) \hat{s}_{24} - (\hat{s}_{14} + \hat{s}_{45}) A_{FT}(1,2,3,4,5) - \hat{s}_{14} A_{FT}(1,2,3,5,4) = 0 . \]

(4.22)

can be trivially deduced.

4.3.3. \( N = 6 \)

The subamplitudes are of the form (3.29). Cyclicity, reflection and parity transformations (4.2) reduce the number of subamplitudes to \( \frac{1}{2}(6 - 1)! = 60 \). However, our new relations of the type (4.6) allow for a reduction of this number to a minimal basis of six \( (\nu_6(6) = 6) \). Starting at the relation (4.6)

\[
A(1,2,3,4,5,6) + e^{i\pi s_1} A(2,1,3,4,5,6) + e^{i\pi (\hat{\ell}_1 - \hat{s}_2)} A(2,3,1,4,5,6) \\
+ e^{i\pi (\hat{s}_5 - \hat{s}_2)} A(2,3,4,1,5,6) + e^{-i\pi s_6} A(2,3,4,5,1,6) = 0
\]

(4.23)

and its permutations give 720 relations, which allow to express all 60 subamplitudes in terms of six. The nine kinematic invariants are defined in (3.30). We choose the six subamplitudes \( A(1,2,3,4,5,6) \), \( A(1,2,3,4,6,5) \), \( A(1,2,3,5,4,6) \), \( A(1,2,4,3,5,6) \), \( A(1,2,4,3,6,5) \) and \( A(1,2,4,5,3,6) \) as basis and derive the relations

\[
A(1,2,3,5,6,4) = -\sin[\pi (\hat{s}_2 + \hat{s}_5 - \hat{\ell}_1 - \hat{\ell}_2)]^{-1} \{ \sin[\pi (\hat{s}_2 - \hat{\ell}_2)] A(1,2,3,4,5,6) \\
+ \sin[\pi (\hat{s}_2 + \hat{s}_3 - \hat{\ell}_2)] A(1,2,4,3,5,6) \\
+ \sin[\pi (\hat{s}_2 - \hat{s}_4 - \hat{\ell}_2)] A(1,2,3,5,4,6) \} ,
\]
\[
A(1,2,3,6,4,5) = -\sin[\pi (\hat{s}_2 + \hat{s}_4 - \hat{s}_6 - \hat{\ell}_1)]^{-1} \{ \sin[\pi (\hat{s}_2 - \hat{s}_6)] A(1,2,3,4,5,6) \\
+ \sin[\pi (\hat{s}_2 + \hat{s}_3 - \hat{s}_6)] A(1,2,4,3,5,6) \\
+ \sin[\pi (\hat{s}_2 - \hat{s}_5 - \hat{s}_6)] A(1,2,3,4,6,5) \\
+ \sin[\pi (\hat{s}_2 + \hat{s}_3 - \hat{s}_5 - \hat{s}_6)] A(1,2,4,3,6,5) \\
+ \sin[\pi (\hat{s}_2 - \hat{s}_4 - \hat{s}_6 + \hat{\ell}_3)] A(1,2,4,5,3,6) \} ,
\]
\[
A(1,2,5,3,4,6) = \sin[\pi (\hat{s}_3 + \hat{s}_6 - \hat{\ell}_2 - \hat{\ell}_3)]^{-1} \{ -\sin[\pi (\hat{s}_6 - \hat{\ell}_2)] A(1,2,3,4,5,6) \\
- \sin[\pi (\hat{s}_5 + \hat{s}_6 - \hat{\ell}_2)] A(1,2,3,4,6,5) \\
+ \sin[\pi (\hat{s}_4 - \hat{s}_6 + \hat{\ell}_2)] A(1,2,3,5,4,6) \} ,
\]

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\[ A(1, 2, 5, 3, 6, 4) = \sin[\pi(\hat{s}_2 + \hat{s}_5 - \hat{t}_1 - \hat{t}_2)]^{-1} \sin[\pi(\hat{s}_3 + \hat{s}_6 - \hat{t}_2 - \hat{t}_3)]^{-1} \times \left\{ \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{t}_2)] \sin[\pi(\hat{s}_6 - \hat{t}_2)] A(1, 2, 3, 4, 5, 6) + \sin[\pi(\hat{s}_2 + \hat{s}_3 - \hat{s}_4 - \hat{t}_2)] \sin[\pi(\hat{s}_6 - \hat{t}_2)] A(1, 2, 4, 3, 5, 6) + \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{t}_2)] \sin[\pi(\hat{s}_5 + \hat{s}_6 - \hat{t}_2)] A(1, 2, 3, 4, 6, 5) + \sin[\pi(\hat{s}_2 + \hat{s}_3 - \hat{s}_4 - \hat{t}_2)] \sin[\pi(\hat{s}_5 + \hat{s}_6 - \hat{t}_2)] A(1, 2, 4, 3, 6, 5) - \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{t}_2)] \sin[\pi(\hat{s}_4 - \hat{s}_6 + \hat{t}_2)] A(1, 2, 3, 5, 4, 6) + \sin(\pi \hat{s}_4) \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{s}_6 + \hat{t}_3)] A(1, 2, 4, 5, 3, 6) \right\}, \\
A(1, 3, 2, 4, 5, 6) = -\sin[\pi(\hat{s}_1 + \hat{s}_2 - \hat{t}_1)] \left\{ \sin[\pi(\hat{s}_1 - \hat{t}_1)] A(1, 2, 3, 4, 5, 6) + \sin[\pi(\hat{s}_1 - \hat{s}_3 - \hat{t}_1)] A(1, 2, 4, 3, 5, 6) + \sin[\pi(\hat{s}_1 + \hat{s}_4 - \hat{t}_1 - \hat{t}_3)] A(1, 2, 4, 5, 3, 6) \right\}, \\
A(1, 4, 2, 3, 5, 6) = \sin[\pi(\hat{s}_2 + \hat{s}_5 - \hat{t}_1 - \hat{t}_2)]^{-1} \left\{ -\sin[\pi(\hat{s}_5 - \hat{t}_1)] A(1, 2, 3, 4, 5, 6) - \sin[\pi(\hat{s}_4 + \hat{s}_5 - \hat{t}_1)] A(1, 2, 3, 5, 4, 6) + \sin[\pi(\hat{s}_3 - \hat{s}_5 + \hat{t}_1)] A(1, 2, 4, 3, 5, 6) \right\}, \\
A(1, 5, 4, 6, 3, 2) = \sin[(\hat{s}_2 + \hat{s}_4 - \hat{s}_6 - \hat{t}_1)]^{-1} \left\{ -\sin[\pi(\hat{s}_2 - \hat{s}_6)] A(1, 2, 3, 4, 5, 6) - \sin[\pi(\hat{s}_2 + \hat{s}_3 - \hat{s}_6)] A(1, 2, 4, 3, 5, 6) - \sin[\pi(\hat{s}_2 - \hat{s}_5 - \hat{s}_6)] A(1, 2, 3, 4, 6, 5) - \sin[\pi(\hat{s}_2 + \hat{s}_3 - \hat{s}_5 - \hat{s}_6)] A(1, 2, 4, 3, 6, 5) - \sin[\pi(\hat{s}_2 - \hat{s}_4 - \hat{s}_6 + \hat{t}_3)] A(1, 2, 4, 5, 3, 6) \right\}, \\
\vdots \\
\]

### 4.4. Open string subamplitude relations and amplitudes of open and closed strings

In this Subsection we explicitly work out the sum (3.1) by taking into account the the phase $\Pi(\Sigma)$ given in Eq. (2.17). As already described in Eq. (3.2) for a given ordering $\sigma$ of open strings this sum gives $\nu(N_o, N_c)$ partial ordered amplitudes of $N_o + 2N_c$ open strings. However, by using the open string subamplitude relations derived above we are able to reduce the number of terms drastically. As discussed in Subsection 3.1 this manipulation is equivalent to deforming the complex contours and reduce to the independent contours as performed in Section 3. The open string subamplitude relations for $N$ open strings from above can be applied by imposing on them the momentum constraint (2.19). Since for a disk amplitude of $N$ open strings the number of independent subamplitudes is given by $\nu_o(N) = (N - 3)!$ this means that for a disk amplitude of $N_o$ open strings and $N_c$ closed strings the final result can be written in terms of a basis of $\nu_o(N_o + 2N_c) = (N_o + 2N_c - 3)!$ subamplitudes. Hence in (3.1) the number $\nu(N_o, N_c)$ of subamplitudes can be reduced to

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\( (N_o + 2N_c - 3)! \) subamplitudes. In other words the final result for the a disk amplitude of \( N_o \) open strings and \( N_c \) closed strings can be written in terms of

\[
\tilde{\nu}(N_o, N_c) = (N_o + 2N_c - 3)!
\]  

(4.25)

partial amplitudes of a pure open string disk amplitude involving \( N_o + 2N_c \) open strings. For special configurations of the closed string momenta as \( q_i^2 = 0 \) this number is further reduced due to additional relations within the basis of subamplitudes. In fact, in Section 3 we have seen, that:

\[
\tilde{\nu}_0(3, 1) \mid_{\text{special} \ config.} = 1 \ , \ \tilde{\nu}_0(2, 2) \mid_{\text{special} \ config.} = 2 \ , \ \tilde{\nu}_0(4, 1) \mid_{\text{special} \ config.} = 4 , \ldots .
\]  

(4.26)

4.4.1. Three open and one closed string

According to (3.2) for this case in the sum (3.1) we expect \( \nu(3, 1) = 12 \) partial ordered amplitudes of five open strings. The phases (2.17) give

\[
\Pi(\xi, \eta) = \begin{cases} 
  e^{i\pi\hat{\gamma}_2} , & (1 - \xi) (1 - \eta) < 0 , \\
  e^{i\pi\lambda_2} , & \xi \eta < 0 , \\
  e^{i\pi\hat{\alpha}} , & \xi < \eta , 
\end{cases}
\]  

(4.27)

with \( \hat{\lambda}_2, \hat{\gamma}_2, \hat{\alpha} \) given in (2.41) and referring to the non–integer parts, i.e. \( \hat{\lambda}_2 = t, \hat{\gamma}_2 = s, \hat{\alpha} = 2q^2 \). With (4.27) the sum (3.1) becomes:

\[
A(1, 2, 3; 4) = A(1, 5, 4, 2, 3) + e^{i\pi\hat{\alpha}} A(1, 4, 5, 2, 3) + e^{i\pi\hat{\alpha}} e^{i\pi\hat{\lambda}_2} A(1, 4, 2, 5, 3) \\
+ e^{i\pi\hat{\alpha}} e^{i\pi\hat{\lambda}_2} e^{i\pi\hat{\gamma}_2} A(1, 4, 2, 3, 5) + e^{i\pi\hat{\lambda}_2} A(1, 5, 2, 4, 3) + A(1, 2, 5, 4, 3) \\
+ e^{i\pi\hat{\alpha}} A(1, 2, 4, 5, 3) + e^{i\pi\hat{\alpha}} e^{i\pi\hat{\gamma}_2} A(1, 2, 4, 3, 5) + e^{i\pi\hat{\lambda}_2} e^{i\pi\hat{\gamma}_2} A(1, 5, 2, 3, 4) \\
+ e^{i\pi\hat{\gamma}_2} A(1, 2, 5, 3, 4) + A(1, 2, 3, 5, 4) + e^{i\pi\hat{\alpha}} A(1, 2, 3, 4, 5) .
\]  

(4.28)

The phase factor (4.27) is in correspondence to the phases displayed in Table 1. With (3.23) the solutions (4.13)–(4.17) boil down to:

\[
A(1, 3, 4, 2, 5) = -A_1 \ , \ A(1, 5, 4, 3, 2) = -A_2 \ , \\
A(1, 5, 3, 4, 2) = -A(1, 2, 4, 3, 5) \\
= -\frac{1}{2} \frac{1}{\sin(\pi s)} \left\{ \sin[\pi(2t + \hat{\alpha})] \cos(\pi t) \right\} A_1 - 2 \frac{\sin(\pi \hat{\alpha}) \sin[\pi(2u + \hat{\alpha})]}{\sin(2\pi t)} A_2 ,
\]

\[
A(1, 4, 3, 5, 2) = -A(1, 2, 5, 3, 4) = -\frac{1}{2} \frac{1}{\sin(\pi s)} \frac{1}{\sin(\pi u)} \left\{ \frac{\sin[\pi(u + \hat{\alpha})] \sin[\pi(2t + \hat{\alpha})]}{\cos(\pi t)} \right\} A_1 \\
- \frac{2}{\sin(\pi \hat{\alpha})} \frac{\sin[\pi(2u + \hat{\alpha})]}{\sin(2\pi t)} \frac{\sin[\pi(s - t)]}{\sin(\pi t)} A_2 .
\]
$A(1, 5, 2, 3, 4) = -A(1, 4, 3, 2, 5)$

\[
= \frac{1}{2} \frac{\sin(\pi u)}{\sin(\pi s)} \left\{ \frac{\sin[\pi(2t + \hat{\alpha})]}{\cos(\pi t)} A_1 - 2 A_2 \sin(\pi t) \sin[\pi(2s + \hat{\alpha})] \right\},
\]

$A(1, 4, 2, 3, 5) = -A(1, 5, 3, 2, 4)$

\[
= \frac{1}{2} \frac{\sin(\pi(s + \hat{\alpha}) \sin[\pi(2t + \hat{\alpha})]}{\cos(\pi t)} A_1
+ 2 \frac{\sin(\pi t \sin[\pi(2s + \hat{\alpha}) \sin[\pi(t - u)]]}{\sin(2\pi t)} A_2,
\]

$A(1, 2, 4, 5, 3) = -A(1, 3, 5, 4, 2)$

\[
= -\frac{1}{2} \frac{\sin(\pi(s - t))}{\cos(\pi t)} A_1 - 2 \frac{\sin[\pi(2u + \hat{\alpha})] \sin[\pi(s + \hat{\alpha})]}{\sin(2\pi t)} A_2,
\]

$A(1, 2, 5, 4, 3) = -A(1, 3, 4, 5, 2)$

\[
= -A(1, 3, 2, 5, 4)
= \frac{1}{2} \frac{1}{\sin(\pi u)} \left\{ \frac{\sin[\pi(s - t)]}{\cos(\pi t)} A_1 + 2 \frac{\sin[\pi(u + \hat{\alpha})] \sin[\pi(2s + \hat{\alpha})]}{\sin(2\pi t)} A_2 \right\},
\]

$A(1, 4, 5, 3, 2) = -A(1, 2, 3, 5, 4)$

\[
= -\frac{1}{4} \frac{\sin(\pi s) \sin(\pi u) \cos(\pi t)}{\sin(\pi s) \sin(\pi u) \cos(\pi t)}
\times \left\{ 2 \sin(2\pi t) \sin[\pi(2t + \hat{\alpha})] A_1 + [\cos(2\pi t) - \cos(2\pi s) + 2 \sin(\pi u)^2] A_2 \right\},
\]

$A(1, 5, 4, 2, 3) = -A(1, 3, 2, 4, 5)$

\[
= -A(1, 3, 2, 5, 3)
= -\frac{1}{4} \frac{1}{\sin(\pi s) \sin(\pi u) \cos(\pi t)}
\times \left\{ [\cos(2\pi s) + \cos(2\pi u) - 2 \cos(\pi t)^2] A_1
+ 2 \frac{\sin[\pi(s + \hat{\alpha})] \sin[\pi(2s + \hat{\alpha})]}{\sin(\pi t)} A_2 \right\},
\]

(4.29)

with $A_1 = A(1, 5, 2, 4, 3)$ and $A_2 = A(1, 2, 3, 4, 5)$. After inserting these solutions into (4.28) we find (with $s + t + u = -\hat{\alpha}$):

\[
A(1, 2, 3, 4) = 2i \sin(\pi \lambda_2) A(1, 5, 2, 4, 3) + 2i \sin(\pi \hat{\alpha}) A(1, 2, 3, 4, 5). \quad (4.30)
\]

For $\hat{\alpha} = 0$ this result has also been derived by the same methods in [38]. However, here we have considered the most general case involving $\alpha \notin \mathbb{Z}$, i.e. $\hat{\alpha} \neq 0$. For this case the phases (4.27) have to be considered, cf. Appendix A. As a result, in this case the full amplitude is expressed by two independent partial amplitudes $A(1, 5, 2, 4, 3)$ and $A(1, 2, 3, 4, 5)$ in agreement with the counting formula (4.25).
4.4.2. Two open and two closed strings

According to (3.2) for this case in the sum (3.1) we expect \( \nu(2, 2) = 120 \) partial ordered amplitudes of six open strings. The phases (2.17) give

\[
\Pi(\rho, \xi, \eta) = \begin{cases} 
  e^{i\pi \hat{\gamma}_2}, & (\xi - \rho) (\eta + \rho) < 0, \\
  e^{i\pi \hat{\beta}_2}, & (\xi + \rho) (\eta - \rho) < 0, \\
  e^{i\pi \hat{\lambda}_2}, & (1 - \xi) (1 - \eta) < 0, \\
  e^{i\pi \alpha_2}, & |x| > 1, 
\end{cases} \tag{4.31}
\]

with \( \hat{\omega}_2, \hat{\lambda}_2, \hat{\gamma}_2, \hat{\beta}_2 \) given in (2.56) and referring to the non–integer parts, *i.e.* \( \hat{\omega}_2 = u, \hat{\lambda}_2 = t, \hat{\gamma}_2 = \hat{\beta}_2 = \frac{\pi}{2} \). With (4.31) the sum (3.1) becomes:

\[
\mathcal{A}(1, 2; 3, 4) = [ A(1, 6, 5, 3, 4, 2) + A(1, 5, 6, 3, 4, 2) ] + e^{i\pi \hat{\gamma}_2} A(1, 5, 3, 6, 4, 2) \\
+ e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 5, 3, 4, 6, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 5, 3, 4, 2, 6) + e^{i\pi \hat{\beta}_2} A(1, 6, 3, 5, 4, 2) \\
+ e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} [ A(1, 3, 6, 5, 4, 2) + A(1, 3, 5, 6, 4, 2) ] + e^{i\pi \hat{\gamma}_2} A(1, 3, 5, 4, 6, 2) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} A(1, 3, 5, 4, 2, 6) + e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 6, 3, 4, 5, 2) + e^{i\pi \hat{\beta}_2} A(1, 3, 6, 4, 5, 2) \\
+ [ A(1, 3, 4, 6, 5, 2) + A(1, 3, 4, 5, 6, 2) ] + e^{i\pi \hat{\lambda}_2} A(1, 3, 4, 5, 2, 6) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\gamma}_2} A(1, 6, 3, 4, 2, 5) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} A(1, 3, 6, 4, 2, 5) + e^{i\pi \hat{\lambda}_2} A(1, 3, 4, 6, 2, 5) \\
+ [ A(1, 3, 4, 2, 6, 5) + A(1, 3, 4, 2, 5, 6) ] + [ A(1, 6, 5, 4, 3, 2) + A(1, 5, 6, 4, 3, 2) ] \\
+ e^{i\pi \hat{\beta}_2} A(1, 5, 4, 6, 3, 2) + e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 5, 4, 3, 6, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 5, 4, 3, 2, 6) \\
+ e^{i\pi \hat{\gamma}_2} A(1, 6, 4, 5, 3, 2) + e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} [ A(1, 4, 6, 5, 3, 2) + A(1, 4, 5, 6, 3, 2) ] \\
+ e^{i\pi \hat{\beta}_2} A(1, 4, 5, 3, 6, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} A(1, 4, 5, 3, 2, 6) + e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 6, 4, 3, 5, 2) \\
+ e^{i\pi \hat{\gamma}_2} A(1, 4, 6, 3, 5, 2) + [ A(1, 4, 6, 3, 5, 2) + A(1, 4, 3, 5, 6, 2) ] \\
+ e^{i\pi \hat{\lambda}_2} A(1, 4, 3, 5, 2, 6) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\gamma}_2} A(1, 6, 4, 3, 2, 5) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} A(1, 4, 6, 3, 2, 5) \\
+ e^{i\pi \hat{\lambda}_2} A(1, 4, 3, 6, 2, 5) + [ A(1, 4, 3, 2, 6, 5) + A(1, 4, 3, 2, 5, 6) ] \\
+ e^{i\pi \alpha_2} \left\{ [ A(1, 6, 5, 3, 2, 4) + A(1, 5, 6, 3, 2, 4) ] + e^{i\pi \hat{\gamma}_2} A(1, 5, 3, 6, 2, 4) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} A(1, 5, 3, 2, 6, 4) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 5, 3, 2, 4, 6) + e^{i\pi \hat{\beta}_2} A(1, 6, 3, 5, 2, 4) \\
+ e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\gamma}_2} [ A(1, 3, 6, 5, 2, 4) + A(1, 3, 5, 6, 2, 4) ] + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 3, 5, 2, 6, 4) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} A(1, 3, 5, 2, 4, 6) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} A(1, 6, 3, 2, 5, 4) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} A(1, 3, 6, 2, 5, 4) \\
+ e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\gamma}_2} [ A(1, 3, 2, 6, 5, 4) + A(1, 3, 2, 5, 6, 4) ] + e^{i\pi \hat{\gamma}_2} A(1, 3, 2, 5, 4, 6) \right\}
\]
\[ + e^{i\pi\lambda_2} e^{i\pi\gamma_2} e^{i\pi\beta_2} A(1, 6, 3, 2, 4, 5) + e^{i\pi\lambda_2} e^{i\pi\beta_2} A(1, 3, 6, 2, 4, 5) + e^{i\pi\beta_2} A(1, 3, 2, 6, 4, 5) + [ A(1, 3, 2, 4, 6, 5) + A(1, 3, 2, 4, 5, 6) ] \].

(4.32)

The phase factor (4.31) is in correspondence to the phases displayed in Tables 2 and 3. With (3.38) the solution (4.24) boils down to:

\[ A(1, 2, 3, 5, 6, 4) = \sin(\pi t)^{-1} \{ \sin(\pi u) A(1, 2, 3, 4, 5, 6) + \sin(\pi u) A(1, 2, 4, 3, 5, 6) \right. \]

\[ - \sin \left[ \pi \left( \frac{s}{2} + t \right) \right] A(1, 2, 3, 5, 4, 6) \right\}, \]

(4.33)

\[ A(1, 2, 3, 6, 4, 5) = A(1, 2, 4, 5, 3, 6) \]

\[ A(1, 2, 5, 3, 4, 6) = \sin(\pi t)^{-1} \{ \sin(\pi u) \left[ A(1, 2, 3, 4, 5, 6) + A(1, 2, 3, 4, 6, 5) \right] \right. \]

\[ + \sin \left[ \pi \left( \frac{s}{2} + u \right) \right] A(1, 2, 3, 5, 4, 6) \right\}, \]

\[ A(1, 2, 5, 3, 6, 4) = \sin(\pi t)^{-2} \{ \sin \left( \frac{\pi s}{2} \right)^2 A(1, 2, 4, 5, 3, 6) \right. \]

\[ - \sin(\pi u) \sin \left[ \pi \left( \frac{s}{2} + t \right) \right] \left[ A(1, 2, 3, 4, 5, 6) + A(1, 2, 3, 4, 6, 5) \right] \right. \]

\[ - \sin(\pi u) \sin \left[ \pi \left( \frac{s}{2} + t \right) \right] \left[ A(1, 2, 4, 3, 5, 6) + A(1, 2, 4, 3, 6, 5) \right] \right. \]

\[ + \sin \left[ \pi \left( \frac{s}{2} + t \right) \right]^2 A(1, 2, 3, 5, 4, 6) \right\}, \]

(4.34)

\[ A(1, 3, 2, 4, 5, 6) = \sin(\pi t)^{-1} \{ \sin \left( \frac{\pi s}{2} \right) A(1, 2, 4, 5, 3, 6) \right. \]

\[ + \sin(\pi s) \left[ A(1, 2, 3, 4, 5, 6) + A(1, 2, 4, 3, 5, 6) \right] \right\}, \]

(4.35)

\[ A(1, 4, 2, 3, 5, 6) = \sin(\pi t)^{-1} \{ \sin \left( \frac{\pi s}{2} \right) A(1, 2, 3, 5, 4, 6) \right. \]

\[ + \sin(\pi s) \left[ A(1, 2, 3, 4, 5, 6) + A(1, 2, 4, 3, 5, 6) \right] \right\}, \]

\[ A(1, 5, 4, 6, 3, 2) = A(1, 2, 4, 5, 3, 6) \],

\[ A(1, 2, 4, 6, 3, 5) = A(1, 2, 3, 5, 4, 6) \],

\[ A(1, 5, 3, 6, 4, 2) = A(1, 2, 3, 5, 4, 6) \],

\]

and the whole result (4.32) becomes (3.31) or (3.40):

\[ A(1, 2, 3, 4) = 4 \sin \left( \frac{\pi s}{2} \right) \sin(\pi s) A(1, 6, 3, 5, 4, 2) - 4 \sin \left( \frac{\pi s}{2} \right) \sin(\pi t) A(1, 3, 5, 4, 2, 6) \].

(4.36)

Note, that this expression is different than the one given in [38].

4.4.3. Four open and one closed string

According to (3.2) for a given open string ordering \( \sigma \), described in (2.66), in the sum (3.1) we expect \( \nu(4, 1) = 20 \) partial ordered amplitudes of six open strings. The open
string coordinate \( x \) represents an ordering coordinate for the phases (2.17)

\[
\Pi(x, \xi, \eta) = \begin{cases} 
    e^{i\pi \lambda_2}, & (1 - \xi) (1 - \eta) < 0, \\
    e^{i\pi \gamma_2}, & (\xi - x) (\eta - x) < 0, \\
    e^{i\pi \beta_2}, & (\xi + x) (\eta + x) < 0,
\end{cases}
\]  

(4.35)

with \( \lambda_2, \gamma_2, \beta_2 \) given in (2.72) and referring to the non-integer parts, i.e. \( \lambda_2 = -\frac{1}{2}s_1 + \frac{1}{2}s_3 - s_5, \quad \gamma_2 = s_4, \quad \beta_2 = \frac{1}{2}s_1 - \frac{1}{2}s_3 - s_4 \). We have worked out the general case \( \kappa \notin \mathbb{Z} \) in Subsection 3.5 and Appendix C. Here we shall restrict to the case \( \kappa \in \mathbb{Z} \), i.e. no branching from the term \( (z - \overline{z})^\kappa \) is considered. For the ordering \( \sigma_1 \) the sum (3.1) decomposes as:

\[
A(1, 3, 4, 2; 5) = [ A(1, 6, 5, 3, 4, 2) + A(1, 5, 6, 3, 4, 2) ] + e^{i\pi \lambda_2} A(1, 5, 3, 6, 4, 2) \\
+ e^{i\pi \gamma_2} e^{i\pi \beta_2} A(1, 5, 3, 4, 6, 2) + e^{i\pi \lambda_2} e^{i\pi \gamma_2} e^{i\pi \beta_2} A(1, 5, 3, 4, 2, 6) \\
+ e^{i\pi \beta_2} A(1, 6, 3, 5, 4, 2) + [ A(1, 3, 6, 5, 4, 2) + A(1, 3, 5, 6, 4, 2) ] \\
+ e^{i\pi \gamma_2} A(1, 3, 5, 4, 6, 2) + e^{i\pi \lambda_2} e^{i\pi \gamma_2} A(1, 3, 5, 4, 2, 6) \\
+ e^{i\pi \gamma_2} e^{i\pi \beta_2} A(1, 6, 3, 4, 5, 2) + e^{i\pi \gamma_2} A(1, 3, 6, 4, 5, 2) \\
+ [ A(1, 3, 4, 6, 5, 2) + A(1, 3, 4, 5, 6, 2) ] + e^{i\pi \lambda_2} A(1, 3, 4, 5, 2, 6) \\
+ e^{i\pi \lambda_2} e^{i\pi \beta_2} e^{i\pi \gamma_2} A(1, 6, 3, 4, 2, 5) + e^{i\pi \lambda_2} e^{i\pi \gamma_2} A(1, 3, 6, 4, 2, 5) \\
+ e^{i\pi \lambda_2} A(1, 3, 4, 6, 2, 5) + [ A(1, 3, 4, 2, 6, 5) + A(1, 3, 4, 2, 5, 6) ] .
\]  

(4.36)

With the subamplitude relations (4.24) subject to (3.58)

\[
A(1, 3, 6, 4, 2, 5) = -cs_1 \frac{sn_4}{\sin(\pi s_4)} A(1, 3, 4, 5, 2, 6) - cs_1 \sin(\pi s_5) A(1, 5, 3, 4, 2, 6) \\
- \frac{sn_4}{\sin(\pi s_4)} [ A(1, 5, 2, 4, 3, 6) + A(1, 5, 3, 4, 2, 6) ] ,
\]

\[
A(1, 2, 4, 3, 5, 6) = -A(1, 2, 4, 3, 6, 5) - cs_2 \frac{sn_1}{\sin(\pi s_4)} A(1, 5, 3, 4, 2, 6) \\
- cs_2 \sin(\pi s_4) A(1, 2, 4, 5, 3, 6) ,
\]

\[
A(1, 2, 4, 5, 6, 3) = -A(1, 2, 4, 6, 5, 3) - \frac{sn_4}{\sin(\pi s_4)} A(1, 2, 4, 5, 3, 6) \\
+ \frac{cs_1 sn_1^2}{\sin(\pi s_4)} A(1, 3, 4, 5, 2, 6) + \frac{cs_1 sn_1}{\sin(\pi s_4)} \frac{\sin(\pi s_5)}{\sin(\pi s_4)} A(1, 5, 3, 4, 2, 6) ,
\]

\[
A(1, 2, 4, 6, 3, 5) = A(1, 2, 4, 5, 3, 6) + \frac{sn_4}{\sin(\pi s_4)} [ A(1, 5, 2, 4, 3, 6) - A(1, 5, 3, 4, 2, 6) ] ,
\]

\[
A(1, 2, 5, 4, 3, 6) = -cs_2 \left[ sn_2 A(1, 2, 4, 5, 3, 6) - \sin(\pi s_5) A(1, 5, 2, 4, 3, 6) \right] ,
\]

\[
A(1, 2, 5, 4, 6, 3) = \sin(\pi s_4)^{-1} \left[ sn_2 A(1, 2, 4, 5, 3, 6) - sn_1 A(1, 3, 4, 5, 2, 6) \\
- \sin(\pi s_5) A(1, 5, 2, 4, 3, 6) \right] ,
\]

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\[ A(1, 2, 5, 6, 4, 3) = -A(1, 3, 4, 5, 6, 2) - \frac{cs_2 \, sn_2^2}{\sin(\pi s_4)} \, A(1, 2, 4, 5, 3, 6) \]
\[ + \frac{sn_3}{\sin(\pi s_4)} \, A(1, 3, 4, 5, 2, 6) + \frac{cs_2 \, sn_2 \, \sin(\pi s_5)}{\sin(\pi s_4)} \, A(1, 5, 2, 4, 3, 6) , \]
\[ A(1, 2, 6, 4, 3, 5) = -cs_2 \, sn_2 \, A(1, 2, 4, 5, 3, 6) - \frac{cs_2 \, sn_1 \, sn_2}{\sin(\pi s_4)} \, A(1, 5, 2, 4, 3, 6) \]
\[ + \frac{sn_3}{\sin(\pi s_4)} \, A(1, 5, 3, 4, 2, 6) , \]
\[ A(1, 3, 4, 6, 2, 5) = A(1, 3, 4, 5, 2, 6) + \frac{sn_2}{\sin(\pi s_4)} \, [ \, A(1, 5, 2, 4, 3, 6) - A(1, 5, 3, 4, 2, 6) \, ] \, , \]
\[ A(1, 3, 5, 4, 6, 2) = \sin(\pi s_4)^{-1} \, [ \, sn_2 \, A(1, 2, 4, 5, 3, 6) - sn_1 A(1, 3, 4, 5, 2, 6) \]
\[ - \sin(\pi s_5) \, A(1, 5, 3, 4, 2, 6) \, ] \, , \]
\[ A(1, 3, 4, 2, 5, 6) = -A(1, 3, 4, 2, 6, 5) - cs_1 \, sn_2 \, A(1, 5, 3, 4, 2, 6) \]
\[ + cs_1 \, \sin(\pi s_4) \, A(1, 3, 4, 5, 2, 6) \, , \]
\[ A(1, 3, 5, 4, 2, 6) = -cs_1 \, [ \, sn_1 \, A(1, 3, 4, 5, 2, 6) + \sin(\pi s_5) \, A(1, 5, 3, 4, 2, 6) \, ] \, , \]
\[(4.37)\]

with
\[
sn_1 = \sin \left[ \frac{\pi}{2} \left( s_1 - s_3 + 2s_5 \right) \right] , \quad sn_2 = \sin \left[ \frac{\pi}{2} \left( s_1 - s_3 - 2s_4 \right) \right] ,
\]
\[
sn_3 = cs_1^{-1} = \sin \left[ \frac{\pi}{2} \left( s_1 - s_3 - 2s_4 + 2s_5 \right) \right] , \quad sn_4 = cs_2^{-1} = \sin \left[ \frac{\pi}{2} \left( s_1 - s_3 \right) \right] ,
\]
\[(4.38)\]

and (4.2) we arrive at:
\[ \mathcal{A}(1, 3, 4, 2; 5) = 2i \, \sin(\pi \hat{\beta}_2) \, \mathcal{A}(1, 2, 4, 5, 3, 6) + 2i \, \sin(\pi \hat{\lambda}_2) \, \mathcal{A}(1, 3, 4, 5, 2, 6) \, . \]
\[(4.39)\]

For the ordering \( \sigma_2 \) the sum (3.1) gives:
\[ \mathcal{A}(1, 4, 3, 2; 5) = [ \, A(1, 6, 5, 4, 3, 2) + A(1, 5, 6, 4, 3, 2) \, ] + e^{i\pi \hat{\gamma}_2} \, A(1, 5, 4, 6, 3, 2) \]
\[ + e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} \, A(1, 5, 4, 3, 6, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} \, A(1, 5, 4, 3, 2, 6) \]
\[ + e^{i\pi \hat{\gamma}_2} \, A(1, 6, 4, 5, 3, 2) + [ \, A(1, 4, 6, 5, 3, 2) + A(1, 4, 5, 6, 3, 2) \, ] \]
\[ + e^{i\pi \hat{\beta}_2} \, A(1, 4, 5, 3, 6, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} \, A(1, 4, 5, 3, 2, 6) \]
\[ + e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} \, A(1, 6, 4, 3, 5, 2) + e^{i\pi \hat{\beta}_2} \, A(1, 4, 6, 3, 5, 2) \]
\[ [ \, A(1, 4, 3, 6, 5, 2) + A(1, 4, 3, 5, 6, 2) \, ] + e^{i\pi \hat{\lambda}_2} \, A(1, 4, 3, 5, 2, 6) \]
\[ + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\gamma}_2} \, A(1, 6, 4, 3, 2, 5) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} \, A(1, 4, 6, 3, 2, 5) \]
\[ + e^{i\pi \hat{\lambda}_2} \, A(1, 4, 3, 6, 2, 5) + [ \, A(1, 4, 3, 2, 6, 5) + A(1, 4, 3, 2, 5, 6) \, ] \, . \]
\[(4.40)\]
With the subamplitude relations (4.24) subject to (3.58) and (4.2) we arrive at:

\[
A(1,4,3,2;5) = 2i \sin(\pi \gamma_2) A(1,2,3,4,5,6) + 2i \sin(\pi \lambda_2) A(1,4,3,5,2,6) .
\] (4.41)

Finally for the ordering \(\sigma_3\) the sum (3.1) becomes:

\[
A(1,4,2,3;5) = [ A(1,6,5,3,2,4) + A(1,5,6,3,2,4) ] + e^{i\pi \beta_2} A(1,5,3,6,2,4) \\
+ e^{i\pi \lambda_2} e^{i\pi \beta_2} A(1,5,3,2,6,4) + e^{i\pi \lambda_2} e^{i\pi \gamma_2} e^{i\pi \beta_2} A(1,5,3,2,4,6) \\
+ e^{i\pi \beta_2} A(1,6,3,5,2,4) + [ A(1,3,6,5,2,4) + A(1,3,5,6,2,4) ] \\
+ e^{i\pi \lambda_2} A(1,3,5,2,6,4) + e^{i\pi \lambda_2} e^{i\pi \gamma_2} A(1,3,5,2,4,6) \\
+ e^{i\pi \lambda_2} e^{i\pi \beta_2} A(1,6,3,2,5,4) + e^{i\pi \lambda_2} A(1,3,6,2,5,4) \\
+ [ A(1,3,2,6,5,4) + A(1,3,2,5,6,4) ] + e^{i\pi \gamma_2} A(1,3,2,5,4,6) \\
+ e^{i\pi \lambda_2} e^{i\pi \gamma_2} e^{i\pi \beta_2} A(1,6,3,2,4,5) + e^{i\pi \lambda_2} e^{i\pi \gamma_2} A(1,3,6,2,4,5) \\
+ e^{i\pi \gamma_2} A(1,3,2,6,4,5) + [ A(1,3,2,4,6,5) + A(1,3,2,4,5,6) ] .
\] (4.42)

With the subamplitude relations (4.24) subject to (3.58) and (4.2) we arrive at:

\[
A(1,4,2,3;5) = 2i \sin(\pi \beta_2) A(1,4,2,5,3,6) + 2i \sin(\pi \gamma_2) A(1,3,2,5,4,6) .
\] (4.43)

The case \(x < -1\) gives exactly the same expression.

According to the results (4.39), (4.41) and (4.43) we might conclude that the basis is six–dimensional for this case, in agreement with (4.25). However, due to (3.58) some momenta are restricted. Hence, we should expect a smaller basis. In fact, for (3.58) two of the six partial amplitudes may be expressed by a basis of four:

\[
\sin[\pi(s_2 - 2s_4 - 2s_5)] \sin\left[\frac{\pi}{2}(s_1 - s_3 - 2s_4)\right] A(1,4,2,5,3,6) \\
= -\sin[\pi(s_1 - s_3 - 2s_4)] \sin(\pi s_4) A(1,2,3,5,4,6) \\
- \sin[\pi(s_1 + s_2 - s_3 - 2s_4)] \sin(\pi s_4) A(1,3,2,5,4,6) \\
+ \sin\left[\frac{\pi}{2}(s_1 - s_3 - 2s_4)\right] \sin[\pi(s_3 + 2s_4)] A(1,2,4,5,3,6) \\
- \sin(\pi s_3) \sin\left[\frac{\pi}{2}(s_1 - s_3 + 2s_5)\right] A(1,3,4,5,2,6),
\]

\[
\sin[\pi(s_2 - 2s_4 - 2s_5)] \sin\left[\frac{\pi}{2}(s_1 - s_3 + 2s_5)\right] A(1,4,3,5,2,6) \\
= \sin\left[\frac{\pi}{2}(s_1 - s_3 - 2s_4)\right] \sin[\pi(s_2 + s_3 - 2s_5)] A(1,2,4,5,3,6) \\
- \sin(\pi s_4) \sin[\pi(s_1 - s_3 + 2s_5)] A(1,3,2,5,4,6) \\
- \sin(\pi s_4) \sin[\pi(s_1 - s_2 - s_3 + 2s_5)] A(1,2,3,5,4,6) \\
- \sin\left[\frac{\pi}{2}(s_1 - s_3 + 2s_5)\right] \sin[\pi(s_2 + s_3 - 2s_4 - 2s_5)] A(1,3,4,5,2,6).
\] (4.44)
This confirms (4.26). The phase factor (4.35) is in correspondence to the phases displayed in Tables 4 and 5.

4.4.4. Three closed strings

Although we have worked out this case in full generality in Subsection 3.6 and Appendix D, we find it instructive to also work out the sum (3.1). According to (3.2) in the sum (3.1) we expect \( \nu(0, 3) = 40 \) partial ordered amplitudes of six open strings. The open string coordinate \( x \) represents an ordering coordinate for the phases (2.17)

\[
\Pi(x, \xi, \eta) = \begin{cases} 
    e^{i\pi\lambda_2}, & (1 - \xi) (1 - \eta) < 0, \\
    e^{i\pi\gamma_2}, & (1 + \xi) (1 + \eta) < 0, \\
    e^{i\pi\beta_2}, & (\xi - x) (\eta - x) < 0, \\
    e^{i\pi\epsilon_2}, & (\xi + x) (\eta + x) < 0, \\
    e^{i\pi\kappa}, & \xi + \eta < 0,
\end{cases} \tag{4.45}
\]

with \( \lambda_2, \gamma_2, \beta_2, \epsilon_2, \kappa \) defined in (2.87) and referring to their non–integer parts, i.e. \( \lambda_2 = s_6, \gamma_2 = \frac{1}{2}(-s_1 + s_3 - s_5) - s_6, \beta_2 = s_4, \epsilon_2 = \frac{1}{2}(s_1 - s_3 - s_5) - s_4 \) and \( \kappa = s_5 \). With (4.45) for \( 0 < x < 1 \) the sum decomposes (3.1) as:

\[
W(1, 2, 3)_+ = e^{i\pi\lambda_2} e^{i\pi\gamma_2} e^{i\pi\beta_2} e^{i\pi\epsilon_2} A(6, 5, 1, 3, 4, 2) + e^{i\pi\lambda_2} e^{i\pi\beta_2} e^{i\pi\epsilon_2} A(6, 1, 5, 3, 4, 2) \\
+ e^{i\pi\lambda_2} e^{i\pi\beta_2} A(6, 1, 3, 5, 4, 2) + e^{i\pi\lambda_2} A(6, 1, 3, 4, 5, 2) + A(6, 1, 3, 4, 2, 5) \\
+ e^{i\pi\gamma_2} e^{i\pi\kappa} A(1, 5, 3, 4, 2, 6) + e^{i\pi\gamma_2} e^{i\pi\epsilon_2} e^{i\pi\kappa} A(1, 3, 5, 4, 2, 6) \\
+ e^{i\pi\beta_2} e^{i\pi\gamma_2} e^{i\pi\epsilon_2} e^{i\pi\kappa} A(1, 3, 4, 5, 2, 6) + e^{i\pi\lambda_2} e^{i\pi\beta_2} e^{i\pi\gamma_2} e^{i\pi\epsilon_2} e^{i\pi\kappa} A(1, 3, 4, 2, 5, 6) \\
+ e^{i\pi\lambda_2} e^{i\pi\beta_2} e^{i\pi\gamma_2} e^{i\pi\epsilon_2} A(1, 3, 4, 2, 6, 5) + e^{i\pi\beta_2} e^{i\pi\epsilon_2} e^{i\pi\kappa} A(1, 5, 6, 3, 4, 2) \\
+ e^{i\pi\beta_2} e^{i\pi\epsilon_2} A(1, 6, 5, 3, 4, 2) + e^{i\pi\beta_2} A(1, 6, 3, 5, 4, 2) + A(1, 6, 3, 4, 5, 2) \\
+ e^{i\pi\lambda_2} A(1, 6, 3, 4, 2, 5) + e^{i\pi\beta_2} e^{i\pi\kappa} A(1, 5, 3, 6, 4, 2) + e^{i\pi\kappa} A(1, 3, 5, 6, 4, 2) \\
+ A(1, 3, 6, 5, 4, 2) + e^{i\pi\beta_2} A(1, 3, 6, 4, 5, 2) + e^{i\pi\lambda_2} e^{i\pi\beta_2} A(1, 3, 6, 4, 2, 5) \\
+ e^{i\pi\kappa} A(1, 5, 3, 4, 6, 2) + e^{i\pi\epsilon_2} e^{i\pi\kappa} A(1, 3, 5, 4, 6, 2) \\
+ e^{i\pi\beta_2} e^{i\pi\epsilon_2} e^{i\pi\kappa} A(1, 3, 4, 5, 6, 2) + e^{i\pi\beta_2} e^{i\pi\epsilon_2} A(1, 3, 4, 6, 5, 2) \\
+ e^{i\pi\lambda_2} e^{i\pi\beta_2} e^{i\pi\epsilon_2} A(1, 3, 4, 6, 2, 5). \tag{4.46}
\]

With the subamplitude relations (4.24) subject to (3.70), e.g.

\[
A(5, 6, 1, 3, 4, 2) + e^{i\pi\kappa} A(5, 1, 3, 4, 2, 6) + e^{i\pi\beta_2} e^{i\pi\gamma_2} e^{i\pi\epsilon_2} e^{i\pi\kappa} A(5, 1, 6, 3, 4, 2) \\
+ e^{i\pi\gamma_2} e^{i\pi\epsilon_2} e^{i\pi\kappa} A(5, 1, 3, 6, 4, 2) + e^{i\pi\gamma_2} e^{i\pi\kappa} A(5, 1, 3, 4, 6, 2) = 0, \tag{4.47}
\]

\[68\]
Eq. (4.46) can be brought into the form:

\[
W(1, 2, 3)_+ = 2i \left\{ -\sin(\pi s_5) A(1, 3, 4, 2, 6, 5) + \sin(\pi s_6) A(1, 5, 2, 4, 3, 6) \\
+ \sin(\pi s_4) A(1, 2, 5, 4, 6, 3) + \sin[\pi (s_4 + s_6)] A(1, 5, 2, 4, 6, 3) \\
- e^{-i\pi(\gamma_2 + \lambda_2)} \sin(\pi s_5) \left[ A(1, 3, 4, 6, 5, 2) + e^{i\pi \hat{\lambda}_2} A(1, 3, 4, 6, 2, 5) \right] \right\}.
\]  

(4.48)

On the other hand, for \(-1 < x < 0\) we obtain:

\[
W(1, 2, 3)_- = e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} A(6, 5, 1, 4, 3, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} A(6, 1, 5, 4, 3, 2) \\
+ e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\epsilon}_2} A(6, 1, 4, 5, 3, 2) + e^{i\pi \hat{\lambda}_2} A(6, 1, 4, 3, 5, 2) + A(6, 1, 4, 3, 2, 5) \\
+ e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} A(4, 1, 4, 3, 5, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} e^{i\pi \hat{\kappa}} A(1, 4, 3, 2, 5, 6) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\epsilon}_2} A(1, 4, 3, 2, 6, 5) + e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} e^{i\pi \hat{\kappa}} A(1, 5, 6, 4, 3, 2) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\epsilon}_2} A(1, 6, 5, 4, 3, 2) + e^{i\pi \hat{\epsilon}_2} A(1, 6, 4, 5, 3, 2) + A(1, 6, 4, 3, 5, 2) \\
+ A(1, 4, 6, 5, 3, 2) + e^{i\pi \hat{\epsilon}_2} A(1, 4, 6, 3, 5, 2) + e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\epsilon}_2} A(1, 4, 6, 3, 2, 5) \\
+ e^{i\pi \hat{\kappa}} A(1, 5, 4, 3, 6, 2) + e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} A(1, 4, 5, 3, 6, 2) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\epsilon}_2} e^{i\pi \hat{\kappa}} A(1, 4, 3, 5, 6, 2) + e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} A(1, 4, 3, 6, 5, 2) \\
+ e^{i\pi \hat{\lambda}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} A(1, 4, 3, 6, 2, 5) \right).
\]  

(4.49)

With the subamplitude relations (4.24) subject to (3.70), e.g.

\[
A(5, 6, 1, 4, 3, 2) + e^{i\pi \hat{\kappa}} A(5, 1, 4, 3, 2, 6) + e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\epsilon}_2} e^{i\pi \hat{\kappa}} A(5, 1, 6, 4, 3, 2) \\
+ e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\beta}_2} e^{i\pi \hat{\epsilon}_2} A(5, 1, 4, 6, 3, 2) + e^{i\pi \hat{\gamma}_2} e^{i\pi \hat{\epsilon}_2} A(5, 1, 4, 3, 6, 2) = 0,
\]  

(4.50)

Eq. (4.49) reduces to:

\[
W(1, 2, 3)_- = 2i \left\{ -\sin(\pi s_5) A(1, 4, 3, 2, 6, 5) + \sin(\pi s_6) A(1, 5, 2, 3, 4, 6) \\
+ \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} - s_4 \right) \right] A(1, 2, 5, 3, 6, 4) \\
+ \sin \left[ \pi \left( \frac{s_1}{2} - \frac{s_3}{2} - \frac{s_5}{2} - s_4 + s_6 \right) \right] A(1, 5, 2, 3, 6, 4) \\
- e^{-i\pi(\gamma_2 + \lambda_2)} \sin(\pi s_5) \left[ A(1, 4, 3, 6, 5, 2) + e^{i\pi \hat{\lambda}_2} A(1, 4, 3, 6, 2, 5) \right] \right\}.
\]  

(4.51)
Eventually, the two expressions (4.48) and (4.51) sum up to the result (3.64):

\[
W^{(\kappa,\alpha_0,\alpha_3)}_{\alpha_1,\lambda_1,\gamma_1,\beta_1,\epsilon_1,\alpha_2,\lambda_2,\gamma_2,\beta_2,\epsilon_2} = W(1,2,3) + W(1,2,3) .
\] (4.52)

The phase factor (4.45) is in correspondence to the phases displayed in Table 6. Finally as a remark for \( x > 1 \) one obtains exactly the same expression (4.48) as for \( 0 < x < 1 \), while the case \( x < -1 \) yields (4.51).

5. Couplings of brane & bulk string states vs. pure brane couplings

In this Section we explicitly compute disk amplitudes involving open and closed strings and express these amplitudes as pure open string disk amplitudes. This provides the map between couplings of brane & bulk string states onto pure brane couplings. We would like to point out, that disk amplitudes involving only closed strings are somewhat special. For these amplitudes the pole structure is completely inherited from the pure open string amplitudes. As a consequence these amplitudes furnish a divergence due to a dilaton tadpole.

Essentially the relation between couplings of brane & bulk string states to pure brane couplings could be restricted to Neveu–Schwarz fields only as the couplings involving Ramond fields may be simply obtained from the latter by using supersymmetric Ward identities [2].

5.1. Two brane & one bulk field versus four brane fields

The three–point amplitude (2.24) involving two open strings and one closed string is identical to the four–point amplitude of involving four open strings after suitable identification of polarizations and momenta. This fact has been anticipated at the level of the kinematics in [8]. However, in this Subsection for completeness and as illustrative example we want to derive the full relation: We present the three–point amplitude involving two vectors (2.3) and one massless closed string field (2.5) and equate it with the appropriate four–point open superstring amplitude with four vectors (2.2) and (2.3).

The amplitude under consideration follows from the expression (2.24)

\[
\mathcal{A}(1,2;3) = x_\infty^2 \int_{-\infty}^{\infty} dx \ (1 + ix) \times \langle : V_{A_1}^{(0)} (x_\infty, \zeta_1, 2p_1) : : V_{A_2}^{(0)} (1, \zeta_2, 2p_2) : : V_G^{(-1,-1)} (-ix, ix, \epsilon, q) :</rangle ,
\] (5.1)

with the two open string vertex operators \( V_o \) given in (2.3) and the closed string vertex in (2.5). With (2.37) and (2.26) the full amplitude may be cast into the form (2.27)

\[
\mathcal{A}(1,2;3) = 2^{s-1} B \left( \frac{1}{2}, \frac{1}{2} - t \right) K(1,2,3) ,
\] (5.2)
with the kinematical factor:

\[
K(1, 2, 3) = \{ t \ (\zeta_1 \zeta_2) \ \text{tr}(\epsilon D) - 4 \ t \ (\zeta_1 \epsilon D \zeta_2) + \frac{1}{2} \ \text{tr}(\epsilon D) \ (\zeta_1 k_3) \ (\zeta_2 k_4) + 2 \ (\zeta_1 \zeta_2) \ (k_1 \epsilon D k_2) \\
- \ [ \ (\zeta_1 \epsilon D p_1) \ (\zeta_2 q) + (\zeta_2 \epsilon D p_2) \ (\zeta_1 q) + (\zeta_1 \epsilon D p_1) \ (\zeta_2 D q) + (\zeta_2 \epsilon D p_2) \ (\zeta_1 D q) ] \\
+ \ [ \ (\zeta_1 \epsilon D k_3) \ (\zeta_2 k_1) + (\zeta_2 \epsilon D k_3) \ (\zeta_1 k_2) ] \}.
\]

(5.3)

Note, that we have worked with a symmetric tensor \( \epsilon \) taking into account the degrees of freedom of the graviton and dilaton field. This is why the kinematical factor appears to be symmetric in the indices 3 and 4.

On the other hand, the (partial ordered) four open superstring amplitude involving four vectors (2.2) and (2.3)

\[
A(1, 2, 3, 4) = x_1^2 \int_1^\infty dx \ (1 + x) \\
\times \langle : V^{(0)}_{A^+}(x_\infty, \xi_1, 2k_1) : : V^{(0)}_{A^+}(1, \xi_2, 2k_2) : : V^{(-1)}_{A^+}(-x, \xi_3, 2k_3) : : V^{(-1)}_{A^+}(x, \xi_4, 2k_4) : \rangle
\]

assumes the form

\[
A(1234) = \frac{1}{s_2} \ B(\tilde{s}_1, \tilde{s}_3) \ K(\xi_1, k_1; \xi_2, k_2; \xi_3, k_3; \xi_4, k_4),
\]

(5.5)

with \( \tilde{s}_1 = 4k_1 k_2 \), \( \tilde{s}_2 = 4k_1 k_3 \), \( \tilde{s}_3 = 4k_1 k_4 \) and the kinematical factor [39]:

\[
K(\xi_1, k_1; \xi_2, k_2; \xi_3, k_3; \xi_4, k_4) = \tilde{s}_2 \tilde{s}_3 \ (\xi_1 \xi_2) \ (\xi_3 \xi_4) + \tilde{s}_1 \tilde{s}_2 \ (\xi_1 \xi_4) \ (\xi_2 \xi_3) + \tilde{s}_1 \tilde{s}_3 \ (\xi_1 \xi_3) \ (\xi_2 \xi_4) \\
+ \tilde{s}_1 \ [(\xi_1 \xi_3)(\xi_2 k_3)(\xi_4 k_1) + (\xi_1 \xi_4)(\xi_2 k_4)(\xi_3 k_1) + (\xi_2 \xi_3)(\xi_1 k_3)(\xi_4 k_2) + (\xi_2 \xi_4)(\xi_1 k_4)(\xi_3 k_2)] \\
+ \tilde{s}_2 \ [(\xi_1 \xi_2)(\xi_3 k_2)(\xi_4 k_1) + (\xi_1 \xi_4)(\xi_2 k_1)(\xi_3 k_4) + (\xi_2 \xi_3)(\xi_1 k_2)(\xi_4 k_3) + (\xi_3 \xi_4)(\xi_1 k_4)(\xi_2 k_3)] \\
+ \tilde{s}_3 \ [(\xi_1 \xi_2)(\xi_3 k_1)(\xi_4 k_2) + (\xi_1 \xi_3)(\xi_2 k_1)(\xi_4 k_3) + (\xi_2 \xi_4)(\xi_1 k_2)(\xi_3 k_4) + (\xi_3 \xi_4)(\xi_1 k_3)(\xi_2 k_4)].
\]

(5.6)

With the assignment of momenta (3.12) and

\[
\xi_1 = \xi_1 , \quad \xi_2 = \xi_2 , \quad \xi_3 \otimes \xi_4 = \epsilon D ,
\]

we have the following relation between the amplitude (5.2) of two open and one closed string and the four open string partial amplitude (5.5)

\[
A(1, 2; 3) = \sin(\pi t) \ A(1, 2, 3, 4),
\]

(5.8)

in agreement with the proposition (3.11). Note, that the relation (5.8) holds for both NS and R states as external states.
5.2. Two closed strings versus four brane fields

A correspondence between a disk amplitude of two closed strings and a disk amplitude of four open strings has been observed in Refs. [9,7]. We briefly exhibit this result to show the difference to the relation (5.8).

We consider the disk amplitude

\[ A(1, 2) = \int_{0}^{1} dy \, (1 - y^2) \langle V_G^{(-1,-1)}(-iy, iy, \epsilon_1, q_1) : V_G^{(0,0)}(-i, i, \epsilon_2, q_2) : \rangle , \]  

(5.9)

with the two closed string vertex operators given in (2.5). From momentum conservation (2.12) we have \( q_1 + q_2 = 0 \), with \( q = \frac{1}{2}(q + Dq) \). The kinematic invariants are \( s = 2q_1^2 = 2q_2^2 \), \( t = q_1 q_2 \) and \( u = q_1 Dq_2 \). The amplitude (5.9) can be related to the four open string amplitude \( A(1234) \), given in (5.5), as

\[ A(1, 2) = A(1, 2, 3, 4) , \]  

(5.10)

subject to the identifications [9,7]

\[ k_1 = \frac{1}{2}Dq_1 \quad , \quad k_2 = q_1 \quad , \quad k_3 = \frac{1}{2}Dq_2 \quad , \quad k_4 = \frac{1}{2}q_2 , \]  

\[ \hat{s}_1 = s \quad , \quad \hat{s}_2 = t \quad , \quad \hat{s}_3 = u , \]  

(5.11)

and:

\[ \xi_1 \otimes \xi_2 = \epsilon_1 D \quad , \quad \xi_3 \otimes \xi_4 = \epsilon_2 D . \]  

(5.12)

In the double cover with the relations (4.10) the result (5.10) becomes:

\[ A(1, 2) = \left( 1 + \frac{\sin(\pi u)}{\sin(\pi t)} \right) A(1, 2, 3, 4) , \]  

(5.13)

Note, that in contrast to the equation (5.8), in which a sin–factor enters, in the relation (5.10) an amplitude of only closed strings \( A(1, 2) \) is directly equated with an amplitude of four open strings. As a consequence, the singularity structure of the disk amplitude of two closed strings is described by the four open string amplitude. A mathematical explanation for this behaviour follows from the fact, that for the choice \( z_1 = iy \) and \( z_2 = i \) of the closed string vertex positions all factors of complex \( i \) drop out in the correlator (5.9). Hence, the integral (5.9) becomes a real integral of the type (3.7) and no analytic continuation producing sin–factors is necessary. Furthermore, in the case of three closed strings on the disk only one complex coordinate has to be analytically continued. As a consequence only one sin–factor appears in the relation (3.65) to six open string amplitudes.
5.3. Three open & one closed strings versus five brane fields

The four-point amplitude (2.47) involving three open strings and one closed string is identical to the five-point amplitude of five open strings after suitable identification of polarizations and momenta. In this Subsection we want to present the four-point amplitude involving three vectors (2.3) and one massless closed string Ramond field (2.6) and equate it with the appropriate five-point open superstring amplitude with five vectors (2.2) and (2.3).

The amplitude under consideration follows from the expression (2.36)
\[
A(1, 2, 3; 4) = x_\infty^2 \times \int_C d^2 z \langle V_{A_0}^{(0)}(x_\infty, \zeta_1, 2p_1) \ V_{A_2}^{(0)}(0, \zeta_2, 2p_2) \ V_{A_3}^{(-1)}(1, \zeta_3, 2p_3) \ V_F^{(-1, -1)}(z, z, f, q) \rangle,
\]
with the three open string vertex operators \(V_o\) given in (2.2) and (2.3) and the Ramond closed string vertex (2.6). After performing the Wick contractions, applying the Greens functions from Section 2, with Eqs. (2.37), (2.38) and with the internal correlator \[40\]
\[
\langle \Lambda_i(z_1) \Lambda_i(z_2) \Sigma^j(z, \bar{z}) \rangle = (M_j + \overline{M}_j) \ \frac{\delta^{i_1 i_2}}{z_12(z - \bar{z})^{\frac{3}{4}}} \left[ (z_1 - z)(z_2 - \bar{z}) \right]^{1/2},
\]
the whole amplitude (5.14) may be written in terms of a single function \(F\)
\[
A(1, 2, 3; 4) = \frac{i}{\sqrt{2}} \text{Tr}(T^{a_1} T^{a_2} T^{a_3}) (M_j + \overline{M}_j) \ \epsilon_{\lambda_1 \lambda_2 \mu_3 \mu} \ P_1^{\lambda_1} P_2^{\lambda_2} \ s_3^{\mu_3} \ f^\mu \ F,
\]
with the function:
\[
F = \frac{1}{2} \left( G^{(-1)} \begin{bmatrix} t - 1, s - 1 \\ t, s \end{bmatrix} - G^{(-1)} \begin{bmatrix} t - 1, s \\ t, s - 1 \end{bmatrix} \right) = \frac{1}{2} V \begin{bmatrix} t - 1, s - 1 \\ t, s - 1 \end{bmatrix}.
\]

To cast the amplitude (5.14) into the short form (5.16) the following identities
\[
u \left( G^{(-1)} \begin{bmatrix} t - 1, s - 1 \\ t, s \end{bmatrix} + G^{(-1)} \begin{bmatrix} t - 1, s \\ t + 1, s - 1 \end{bmatrix} \right) = -t \left( G^{(-1)} \begin{bmatrix} t - 1, s \\ t, s - 1 \end{bmatrix} + G^{(-1)} \begin{bmatrix} t - 1, s - 1 \\ t, s \end{bmatrix} \right)
\]
\[
= -s \ G^{(-1)} \begin{bmatrix} t - 1, s - 1 \\ t + 1, s - 1 \end{bmatrix} + (1 - s) \ G^{(-1)} \begin{bmatrix} t, s - 1 \ t, s - 1 \end{bmatrix}
\]
\[
12 \ We perform the computation for some \(D = 4\) superstring compactification with \(f^\mu\) being the vector field strength of a scalar with some internal index \(j\) \[40\]. Furthermore, the polarizations \(\zeta_1, \zeta_2\) of the first two vectors are aligned w.r.t. to some internal directions \(i_1\) and \(i_2\).
have been used. These relations may be proven by inserting the explicit result (A.17) and using various hypergeometric functions identities.

On the other hand, from the five open superstring amplitude involving three vectors (2.2) and (2.3) and two gauginos (2.4) we may extract the partial ordered amplitude

$$A(1, 5, 2, 4, 3) = x^2 \int_0^1 d\xi \int_0^0 d\eta \langle : V_{A_1}^{(0)}(x_\infty, \xi_1, 2k_1) : V_{A_2}^{(0)}(1, \xi_2, 2k_2) : \rangle \times : V_{A_3}^{(-1)}(1, \xi_3, 2k_3) : \langle : V_{\chi}^{(-1/2)}(\xi, u_4, 2k_4) : V_{\chi}^{(-1/2)}(\eta, \bar{v}_5, 2k_5) : \rangle.$$  

(5.19)

After performing all contractions and using (3.25) the subamplitude (5.19) becomes

$$A(1, 5, 2, 4, 3) = \epsilon_{1\lambda_1 2\mu_3 \mu} k_1^{\lambda_1} k_2^{\lambda_2} \xi_3^{\mu_3} (u_4^{\alpha} \sigma_{\alpha \beta}^{\mu} \bar{v}_5^{\beta}) \int_0^1 d\xi \int_0^0 d\eta \xi^{-s_2 - s_3 + s_5 - 1} (1 - \xi)^{s_3 - 1}$$

$$\times (-\eta)^{-s_1 + s_3 - s_5} (1 - \eta)^{-s_1 - s_3 - s_5 - 1} (\xi - \eta)^{-s_1 - s_3 - s_5 - 1}$$

$$\times \Gamma(-s_2 - s_3 + s_5) \Gamma(1 - s_1 + s_3 - s_5) \Gamma(s_3) \Gamma(1 - s_1 - s_2 + s_4)$$

$$\times 3F_2 \left[ \frac{1 - s_1 + s_3 - s_5, 1 - s_1 + s_2 + s_4, -s_2}{1 - s_1 + s_2 + s_3 + s_4} \right]$$

$$\times \epsilon_{1\lambda_1 2\mu_3 \mu} k_1^{\lambda_1} k_2^{\lambda_2} \xi_3^{\mu_3} (u_4^{\alpha} \sigma_{\alpha \beta}^{\mu} \bar{v}_5^{\beta}),$$

(5.20)

with the five invariants \( \hat{s}_i \) given in (3.16).

With the assignment of momenta (3.22), (3.23), and (note \( \sigma^{\mu \alpha \beta} \sigma_{\alpha \beta}^{\nu} = -2\delta^{\mu \nu} \))

$$\xi_1 = \zeta_1, \quad \xi_2 = \zeta_2, \quad \xi_3 = \zeta_3, \quad u_4^{\alpha} \otimes \bar{v}_5^{\beta} = f_\mu \sigma^{\mu \alpha \beta},$$

(5.21)

we have the following relation between the amplitude (5.14) of three open and one closed string and the five open string partial amplitudes (5.20)

$$A(1, 2, 3; 4) = \sin(\pi t) A(1, 5, 2, 4, 3),$$

(5.22)

in agreement with the proposition (3.19).

The function \( F \) encodes the \( \alpha' \)-dependence of the string \( S \)-matrix. It is given by

$$F = \frac{1}{s} - \frac{\zeta(3)}{4} tu - \frac{\zeta(5)}{32} tu (u^2 - st) + \ldots.$$  

(5.23)

5.4. Two open & two closed strings versus six brane fields

The four–point amplitude (2.47) involving two open strings and two closed strings is identical to the six–point amplitude of six open strings after suitable identification of polarizations and momenta. In this Subsection we want to present the four–point amplitude
involving two vectors (2.3) and two massless closed string fields (2.5) and equate it with the appropriate six–point open superstring amplitude with six vectors (2.2) and (2.3).

The amplitude under consideration follows from the expression (2.47)

\[ A(1, 2; 3, 4) = \int_{-\infty}^{\infty} dx \int_{C} d^2 z \langle : V_{A_{1}}^{(0)} (x_1, \zeta_1, 2p_1) : V_{A_{02}}^{(0)} (1, \zeta_2, 2p_2) : V_{G_{1}}^{(0,0)} (-ix, ix, \epsilon_1, q_1) : V_{G_{2}}^{(-1,-1)} (\bar{z}, z, \epsilon_2, q_2) : \rangle, \]

with the two open string vertex operators \( V_o \) given in (2.3) and the two closed string vertices (2.5). With (2.48) and (2.49) the full amplitude may be cast\(^{13}\) into the form (2.50)

\[ A(1, 2; 3, 4) = F_1 K_1(1, 2, 3, 4) + F_2 K_2(1, 2, 3, 4), \]

with the two kinematical factors

\[ K_1(1, 2, 3, 4) = tu (\zeta_1 \zeta_2) \left\{ \begin{array}{l} \text{tr}(\epsilon_1 \epsilon_2) + \frac{2}{t} (p_1 \epsilon_1 \epsilon_2 p_2) + \frac{2}{u} (p_2 \epsilon_1 \epsilon_2 p_1) \\ + \frac{1}{u^2} (p_2 \epsilon_1 p_2) (p_1 \epsilon_2 p_1) + \frac{1}{t^2} (p_1 \epsilon_1 p_1) (p_2 \epsilon_2 p_2) + \frac{2}{tu} (p_1 \epsilon_1 p_2) (p_1 \epsilon_2 p_2) \end{array} \right\}, \]

\[ K_2(1, 2, 3, 4) = (\zeta_1 \zeta_2) \left\{ \begin{array}{l} (p_1 \epsilon_1 \epsilon_2 p_1) + (p_2 \epsilon_1 \epsilon_2 p_2) - \frac{u}{t} (p_1 \epsilon_1 \epsilon_2 p_2) - \frac{t}{u} (p_2 \epsilon_1 \epsilon_2 p_1) \\ + \frac{1}{u} [(p_1 \epsilon_1 p_2) (p_1 \epsilon_2 p_1) + (p_2 \epsilon_1 p_2) (p_1 \epsilon_2 p_2)] \\ + \frac{1}{t} [(p_1 \epsilon_1 p_1) (p_1 \epsilon_2 p_2) + (p_1 \epsilon_1 p_2) (p_2 \epsilon_2 p_2)] \\ - \frac{t}{u^2} (p_2 \epsilon_1 p_2) (p_1 \epsilon_2 p_1) - \frac{u}{t^2} (p_1 \epsilon_1 p_1) (p_2 \epsilon_2 p_2) - \left( \frac{1}{t} + \frac{1}{u} \right) (p_1 \epsilon_1 p_2) (p_1 \epsilon_2 p_2) \end{array} \right\}, \]

and the complex integrals, cf. Eq. (2.54):

\[ F_1 = \int_{-\infty}^{\infty} dx \, x^2 (1 + ix)^{u-1} (1 - ix)^{u-1} \]
\[ \times \int_{C} d^2 z (1 - z)^t (1 - \overline{z})^t (z - ix)^{\frac{t}{2} - 1} (\overline{z} - ix)^{\frac{t}{2} - 1} (z + ix)^{\frac{t}{2} - 1} (\overline{z} + ix)^{\frac{t}{2} - 1}, \]

\[ F_2 = \frac{u}{2} \int_{-\infty}^{\infty} dx \, (1 + ix)^{u-1} (1 - ix)^{u-1} \]
\[ \times \int_{C} d^2 z (z + \overline{z}) (1 - z)^t (1 - \overline{z})^t (z - ix)^{\frac{t}{2} - 1} (\overline{z} - ix)^{\frac{t}{2} - 1} (z + ix)^{\frac{t}{2} - 1} (\overline{z} + ix)^{\frac{t}{2} - 1}. \]

\(^{13}\) We perform the computation for a setup with the polarizations \( \zeta_1, \zeta_2 \) of the two vectors aligned orthogonal w.r.t. the D–brane world–volume. Hence these states describe D–brane positions.
The amplitude (5.24) is completely specified by the basis of two generalized hypergeometric functions \( F_1, F_2 \) and the two kinematical factors \( K_1, K_2 \). Since the two functions \( F_1, F_2 \) and the kinematics \( K_1, K_2 \) are invariant under the exchange \( t \leftrightarrow u, \epsilon_1 \leftrightarrow \epsilon_2 \) cf. also (2.59), this symmetry is manifest in the amplitude (5.25).

According to (3.31) and (3.40) the two integrals can be evaluated and written in terms of two six open string partial amplitudes (3.29)

\[
\begin{align*}
F_1 &= \sin \left( \frac{\pi s}{2} \right) \sin(\pi t) \ A_1(163542) - \sin \left( \frac{\pi s}{2} \right) \sin(\pi t) \ A_1(135426), \\
F_2 &= \sin \left( \frac{\pi s}{2} \right) \sin(\pi t) \ A_2(163542) - \sin \left( \frac{\pi s}{2} \right) \sin(\pi t) \ A_2(135426),
\end{align*}
\]

with the four world-sheets integrals:

\[
\begin{align*}
A_1(163542) &= -\frac{1}{8} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ x^{\frac{u}{2}} (1-x)^{\frac{y}{2}} y^{-\frac{z}{2}} (1-y)^{\frac{z}{2}} (1-xz)^{\frac{r}{2}} \\
&\quad \times \frac{1}{(1-x) y (1-y) z (1-z) (1-xyz)}, \\
A_1(135426) &= -\frac{1}{8} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ (1-x)^{\frac{u}{2}} y^{\frac{r}{2}} (1-y)^{\frac{r}{2}} z^{\frac{r}{2}} (1-z)^{\frac{r}{2}} \\
&\quad \times (1-x) (1-xz) \ \frac{y}{x (1-x) (1-y) (1-yz)},
A_2(163542) &= \frac{u}{8} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ x^{\frac{u}{2}} (1-x)^{\frac{y}{2}} y^{-\frac{z}{2}} (1-y)^{\frac{z}{2}} z^{\frac{r}{2}} (1-z)^{\frac{r}{2}} \\
&\quad \times \left\{ \frac{1}{x (1-x) y (1-y) z (1-z)} - \frac{1}{(1-x) (1-y) (1-xyz)} \\
&\quad - \frac{1}{x (1-y) (1-z) (1-xyz)} + \frac{1}{(1-x) y z (1-z) (1-xyz)} \right\}, \\
A_2(135426) &= \frac{u}{8} \int_0^1 dx \int_0^1 dy \int_0^1 dz \ (1-x)^{\frac{u}{2}} y^{\frac{r}{2}} (1-y)^{\frac{r}{2}} z^{\frac{r}{2}} (1-z)^{\frac{r}{2}} \\
&\quad \times (1-x) (1-xz) \ \frac{y}{x y (1-x) (1-yz)} \ \left\{ \frac{1}{x y z} + \frac{1}{1-yz} \right\}.
\end{align*}
\]

On the other hand, from the six open superstring amplitude involving six vectors (2.2) and (2.3) we may extract the two (partial ordered) amplitudes

\[
A(1, 6, 3, 5, 4, 2) = x_{\infty}^2 \int_0^1 d\rho \ (1+\rho) \int_{-\rho}^\rho d\xi \int_{-\infty}^{-\rho} d\eta \ A^{(0)}(x_{\infty}, \zeta_1, 2k_1) : V^{(0)}_{A_{\alpha}}(1, \zeta_2, 2k_2) : \\
&\quad \times : V^{(1)}_{A_{\alpha}}(-\rho, \zeta_3, 2k_3) : : V^{(1)}_{A_{\alpha}}(\rho, \zeta_4, 2k_4) : : V^{(0)}_{A_{\alpha}}(\xi, \zeta_5, 2k_5) : : V^{(0)}_{A_{\alpha}}(\eta, \zeta_6, 2k_6) :,
\]

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$$A(1, 3, 5, 4, 2, 6) = x_7^2 \int_0^1 d\rho \ (1 + \rho) \int_{-\rho}^\rho d\xi \int_{-\infty}^\infty d\eta \ \langle : V_{A_2}^{(0)}(x_\infty, \zeta_1, 2k_1) : \rangle V_{A_2}^{(0)}(1, \zeta_2, 2k_2) \rangle \times V_{A_2}^{(-1)}(-\rho, \zeta_3, 2k_3) \rangle : \rangle V_{A_2}^{(0)}(\xi, \zeta_5, 2k_5) \rangle : \rangle V_{A_2}^{(0)}(\eta, \zeta_6, 2k_6) \rangle \ .$$

The final result for the two partial amplitudes (5.31) can be written as

$$A(1, 6, 3, 5, 4, 2) = K_1 A_1(163542) + K_2 A_2(163542) \ ,$$
$$A(1, 3, 5, 4, 2, 6) = K_1 A_1(135426) + K_2 A_2(135426) \ ,$$

with the kinematical factors $K_1, K_2$ given in (5.26) and the four six open string disk integrals:

$$A_1(163542) = -2^{s_3} \int_0^1 d\rho \int_{-\rho}^\rho d\xi \int_{-\infty}^\infty d\eta \ \rho^{s_3+2} (1 + \rho)^{s_3-1} \ (1 - \rho)^{-s_2-s_3-t_2-1} \times (1 - \xi)^{s_3+s_6-t_2-t_3} \ (1 - \eta)^{-s_1-s_6+t_3} \ (\rho - \xi)^{s_4-1} \ (\rho - \eta)^{s_1+s_4-t_1-t_3-1} \times (\rho + \xi)^{-s_3-s_4+t_3-1} \ (\eta - \xi)^{s_5} \ ,$$

$$A_1(135426) = -2^{s_3} \int_0^1 d\rho \int_{-\rho}^\rho d\xi \int_{-\infty}^\infty d\eta \ \rho^{s_3+2} \ (1 + \rho)^{s_3-1} \ (1 - \rho)^{-s_2-s_3-t_2-1} \times (1 - \xi)^{s_3+s_6-t_2-t_3} \ (\eta - 1)^{-s_1-s_6+t_3} \ (\rho - \xi)^{s_4-1} \ (\rho + \eta)^{s_1+s_4-t_1-t_3-1} \times (\rho + \xi)^{-s_3-s_4+t_3-1} \ (\eta - \xi)^{s_5} \ ,$$

$$A_2(163542) = 2^{s_3-1} s_6 \int_0^1 d\rho \int_{-\rho}^\rho d\xi \int_{-\infty}^\infty d\eta \ \rho^{s_3} \ (1 + \rho)^{s_3-1} \ (1 - \rho)^{-s_2-s_3+t_2-1} \times (1 - \xi)^{s_3+s_6-t_2-t_3} \ (1 - \eta)^{-s_1-s_6+t_3} \ (\rho - \xi)^{s_4-1} \ (\rho - \eta)^{s_1+s_4-t_1-t_3-1} \times (\rho + \xi)^{-s_3-s_4+t_3-1} \ (\eta - \xi)^{s_5} \ ,$$

$$A_2(135426) = 2^{s_3-1} s_6 \int_0^1 d\rho \int_{-\rho}^\rho d\xi \int_{-\infty}^\infty d\eta \ \rho^{s_3} \ (1 + \rho)^{s_3-1} \ (1 - \rho)^{-s_2-s_3+t_2-1} \times (1 - \xi)^{s_3+s_6-t_2-t_3} \ (\eta - 1)^{-s_1-s_6+t_3} \ (\rho - \xi)^{s_4-1} \ (\rho + \eta)^{s_1+s_4-t_1-t_3-1} \times (\rho + \xi)^{-s_3-s_4+t_3-1} \ (\eta - \xi)^{s_5} \ .$$

Clearly, with (3.38) and the transformations (3.59) and (3.44) the two sets of expressions (5.29), (5.30) and (5.33), (5.34) may be identified.

With the assignment of momenta (3.37) and

$$\xi_1 = \xi_1 \ , \ \xi_2 = \xi_2 \ , \ \xi_3 \otimes \xi_4 = \epsilon_1 \ , \ \xi_5 \otimes \xi_6 = \epsilon_2 ,$$

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we have the following relation between the amplitude (5.24) of two open and two closed
strings and the pair of six open string partial amplitudes (5.31)

\[
\mathcal{A}(1, 2; 3, 4) = \sin \left( \frac{\pi s}{2} \right) \sin(\pi s) \ A(1, 6, 3, 5, 4, 2) - \sin \left( \frac{\pi s}{2} \right) \sin(\pi t) \ A(1, 3, 5, 4, 2, 6),
\]

(5.36)
in agreement with the proposition (3.40).

The \(\alpha'\)–expansion of the amplitude (5.1) is completely determined by the \(\alpha'\)–
expansions of the two functions (5.27)

\[
F_1 = -\frac{1}{s} - \frac{1}{4} \zeta(2) \ s + \frac{1}{16} \ (3 \ s^2 + 2 \ tu) \ \zeta(3) - \frac{1}{64} \ (19 \ s^3 - 5 \ stu) \ \zeta(4) \\
+ \frac{1}{64} \ s^2 \ (3 \ s^2 + 5 \ tu) \ \zeta(2) \ \zeta(3) + \frac{3}{768} \ (45 \ s^4 + 22 \ s^2 \ tu + 8 \ t^2 u^2) \ \zeta(5) + \ldots,
\]

\[
F_2 = \frac{1}{4} \ zeta(2) \ tu - \frac{3}{16} \ stu \ \zeta(3) + \frac{1}{64} \ (19 \ s^2 tu - 4 \ t^2 u^2) \ \zeta(4) \\
- \frac{1}{128} \ stu \ (6 \ s^2 + 8 \ tu) \ \zeta(2) \ \zeta(3) - \frac{5}{256} \ stu \ (9 \ s^2 - 4 \ tu) \ \zeta(5) + \ldots.
\]

(5.37)
Note, that the integrals (5.29) and (5.30) allow for at most a triple pole, which is converted
to a single pole by the two \(\sin\)–factors in (5.27). The lowest (field–theory) order of the
amplitude (2.1) is given by the first term of (5.25), \(i.e.\ A(1, 2; 3, 4)|_{\alpha'0} = K_1/s\). This order
has been already determined in [14]. On the other hand, the NLO string correction at
\(\alpha'^2\) stems from both \(K_1\) and \(K_2\). Note, that this order gives non–trivial gravitational
corrections to the D–brane world volume couplings at the order \(\alpha'^2\).

6. Concluding remarks

We have made explicit\(^{14}\) the link between disk amplitudes of open & closed strings and
amplitudes of pure open strings. This map represents a sort of generalized KLT relation
on the disk. It relates a disk amplitude of \(N_o\) open and \(N_c\) closed strings to a sum of disk
amplitudes involving \(N_o + 2N_c\) open strings only. After analytic continuation the complex
world–sheet integrations over the closed string positions \(z_i\) become real integrals along the

\(^{14}\) The general expressions on the disk world–sheet are given in Eqs. (2.16), (2.18) and (2.19),
while the integrated amplitudes are presented in Eqs. (3.11), (3.19), (3.40), (3.61), (3.65) and
(3.85), respectively.
boundary of the disk. This way each closed string state is divided up into two (interacting) open string fields at real positions \( \eta_i, \xi_i \):

\[
V_{\text{closed}}(\xi_i, \eta_i) \simeq V_{\text{open}}(\eta_i) V_{\text{open}}(\xi_i).
\]

The correspondence between open & closed strings and amplitudes of pure open strings is made on the string world–sheet. Therefore this map holds for amplitudes in any space–time dimensions and gives relations between couplings of brane & bulk string states vs. pure brane couplings. Some illustrative examples for these relations, which have been derived in this work, are:

\[
\mathcal{A}(1, 2) = A(1, 2, 3, 4),
\]
\[
\mathcal{A}(1, 2; 3) = \sin(\pi t) \ A(1, 2, 3, 4),
\]
\[
\mathcal{A}(1, 2, 3; 4) = \sin(\pi t) \ A(1, 5, 2, 4, 3),
\]
\[
\mathcal{A}(1, 2; 3, 4) = \sin\left(\frac{\pi s}{2}\right) \sin(\pi s) \ A(1, 6, 3, 5, 4, 2) - \sin\left(\frac{\pi s}{2}\right) \sin(\pi t) \ A(1, 3, 5, 4, 2, 6),
\]
\[
\mathcal{A}(1, 2, 3; 4, 5) = \sin(\pi s_4) \ A(1, 6, 4, 5, 3, 2) - \sin \pi \left(\frac{s_1}{2} - \frac{s_3}{2} + s_5\right) \ A(1, 4, 3, 5, 2, 6),
\]
\[
\mathcal{A}(1, 2, 3) = \sin \pi \left(\frac{s_1}{4} - \frac{s_3}{4}\right) \ \{ \ A(1, 3, 6, 4, 5, 2) + A(1, 4, 6, 3, 5, 2) - A(1, 3, 4, 6, 2, 5) - A(1, 4, 3, 6, 2, 5) \ \},
\]

Here, the open/closed disk amplitudes are denoted by \( \mathcal{A}(1, \ldots, N_o; N_o + 1, \ldots, N_o + N_c) \), while the partial ordered open string amplitudes are given by \( A(1, \ldots, N_o + 2N_c) \) and permutations thereof.

This sort of generalized KLT relation on the disk gives rise to direct relations between gauge and gravitational amplitudes and pure gauge amplitudes. At the level of the effective action the map (6.1), \textit{i.e.} the correspondence between disk amplitudes of open & closed strings and pure open string amplitudes reveals important relations between brane and bulk couplings and pure brane couplings, \textit{cf.} Section 5. Note, that this map is different than in the KLT case, where tree–level couplings of closed strings are mapped to squares of open string couplings. Hence this map does not directly translate into a map at the level of couplings in a Lagrangian.

Formally for a disk amplitude of two vectors and two massless Neveu–Schwarz closed string states we obtain

\[
\langle A_{\mu_1}(x_1) \ A_{\mu_2}(x_2) \ G_{\mu_3\mu_4}(\bar{z}_1, z_1) \ G_{\mu_5\mu_6}(\bar{z}_2, z_2) \rangle \ \xi^{\mu_1} \ \xi^{\mu_2} \ \xi^{\mu_3} \ \xi^{\mu_4} \ \xi^{\mu_5} \ \xi^{\mu_6},
\]

\[
\sim \langle A_{\mu_1}(x_1) \ A_{\mu_2}(x_2) \ A_{\mu_3}(x_3) \ A_{\mu_4}(x_4) \ A_{\mu_5}(x_5) \ A_{\mu_6}(x_6) \rangle \ \xi^{\mu_1} \ \xi^{\mu_2} \ \xi^{\mu_3} \ \xi^{\mu_4} \ \xi^{\mu_5} \ \xi^{\mu_6},
\]

\[
6.3,
\]

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with the identifications (5.35) and assignment of momenta (3.37). For a special choice of polarizations the first amplitude of (6.3) has been computed in Subsection 5.3. For a disk amplitude of two vectors and two massless Ramond $p$– and $q$–forms the correspondence

\begin{align*}
\langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) F_{\alpha\beta}(\vec{z}_1, z_1) F_{\gamma\delta}(\vec{z}_2, z_2) angle \\
\times \zeta^{\mu_1} \zeta^{\mu_2} f^{\mu_1...\mu_p}_1 \left( P^+ \Gamma^{[\mu_0}...\Gamma^{\mu_p]} \right)^{\alpha\beta} f^{\nu_0...\nu_q}_2 \left( P^+ \Gamma^{[\nu_0}...\Gamma^{\nu_q]} \right)^{\gamma\delta} \\
\simeq \langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) \chi_{\alpha}(x_3) \chi_{\beta}(x_4) \chi_{\gamma}(x_5) \chi_{\delta}(x_6) \rangle \xi^{\mu_1} \xi^{\mu_2} u^\alpha v^\beta u^\gamma v^\delta 
\end{align*}

(6.4)
can be established.

After compactification the relations (6.3) and (6.4) give rise to various relations between open and closed string moduli and pure open string moduli. These identities establish non–trivial relations between disk correlators involving internal open & closed conformal fields of open and closed string vertex operators and correlators involving only internal open conformal fields of open string vertex operators [40]. This way disk amplitudes involving open and closed string moduli are mapped to amplitudes of only open string moduli [40]. See also Refs. [41] for a possible relation.

With the techniques developed in this work any disk amplitude with an arbitrary number of open and closed strings can be computed. It should be possible to derive similar relations as the one we have found at the disk tree–level for other Riemann surfaces with boundaries. E.g. at the one–loop–level a cylinder amplitude of open & closed strings should be expressible as sum of cylinder amplitudes involving only open strings.

Finally, we find it interesting to mention, that the polygon Fig. 9 relevant to the scattering of two open and two closed strings appears in a different context [42]. The polygon made of the sequence of lightlike segments specifies the scattering configuration of six gluons with self–crossing in the $T$–dual coordinates. The polygon defines the boundary conditions for the string world–sheet relevant for the leading exponential behaviour of the six–point scattering amplitude in field–theory. The polygon contains the kinematic information about the momenta of the six gluons. Furthermore, the same polygon as Fig. 9 specifies the scattering configuration of two gluons and two quarks in the $T$–dual coordinates [43] and the configuration space is a subspace of the one of the six–gluon worldsheet with crossings.

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Appendix A. Complex world–sheet integrals \( G^{(\alpha)}[\lambda_1, \gamma_1]_{\lambda_2, \gamma_2} \)

In this appendix we compute the complex integral (2.40)

\[
G^{(\alpha)}[\lambda_1, \gamma_1]_{\lambda_2, \gamma_2} := \int_{\mathbb{C}} d^2z \ z^{\lambda_1} \bar{z}^{\lambda_2} (1 - z)^{\gamma_1} (1 - \bar{z})^{\gamma_2} (z - \bar{z})^\alpha ,
\]

for \( \alpha \in \mathbb{R} \). In analogy to the procedure applied in Subsection 2.3.3 we split the complex integral into holomorphic and anti–holomorphic pieces by rotating the \( z_2 \)–integration from the real axis to the pure imaginary axis, \( i.e. \ z_2 \in \mathbb{R} \). This way, the variables \( \xi = z_1 + iz_2 \) and \( \eta = z_1 - iz_2 \) become two real independent quantities, \( i.e. \xi, \eta \in \mathbb{R} \) and (A.1) becomes

\[
G^{(\alpha)}[\lambda_1, \gamma_1]_{\lambda_2, \gamma_2} = \frac{1}{2} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \ |\xi|^\hat{\lambda}_1 |1 - \xi|^\hat{\gamma}_1 |\eta|^\hat{\gamma}_2 |1 - \eta|^\hat{\lambda}_2 |\xi - \eta|^{\hat{\alpha}} \times \xi^{n_1} (1 - \xi)^{m_1} \eta^{n_2} (1 - \eta)^{m_2} (\xi - \eta)^{\hat{\alpha}} \Pi(\xi, \eta) ,
\]

with the decomposition \( \lambda_i = \hat{\lambda}_i + n_i, \gamma_i = \hat{\gamma}_i + m_i, \alpha = \hat{\alpha} + \tilde{\alpha} \) separating integer and non–integer parts and a phase factor \( \Pi \) given in the Table 1.

| \( (\xi, \eta) \) | \( \eta < \xi \) | \( \xi < \eta < 0 \) | \( 0 < \eta < 1 \) | \( \eta > 1 \) | total phase |
|----------------|----------------|----------------|----------------|----------------|----------------|
| (i) \( \xi < 0 \) | 1 \( e^{i\pi \hat{\alpha}} \) | \( e^{i\pi (\hat{\alpha} + \lambda_2 + \gamma_2)} \) | \( \sigma_\lambda \) |
| (ii) \( 0 < \xi < 1 \) | \( e^{i\pi \lambda_2} \) | 1 \( e^{i\pi \hat{\alpha}} \) | \( e^{i\pi (\hat{\alpha} + \gamma_2)} \) | \( 1 \) |
| (iii) \( \xi > 1 \) | \( e^{i\pi (\lambda_2 + \gamma_2)} \) | \( e^{i\pi \gamma_2} \) | 1 \( e^{i\pi \hat{\alpha}} \) | \( \sigma_\gamma \) |

| \( (\xi, \eta) \) | \( \eta < 0 \) | \( 0 < \eta < \xi \) | \( \xi < \eta < 1 \) | \( \eta > 1 \) | total phase |
|----------------|----------------|----------------|----------------|----------------|----------------|
| (i) \( \xi < 0 \) | 1 \( e^{i\pi \hat{\alpha}} \) | \( e^{i\pi (\hat{\alpha} + \lambda_2 + \gamma_2)} \) | \( \sigma_\lambda \) |
| (ii) \( 0 < \xi < 1 \) | \( e^{i\pi \lambda_2} \) | 1 \( e^{i\pi \hat{\alpha}} \) | \( e^{i\pi (\hat{\alpha} + \gamma_2)} \) | \( 1 \) |
| (iii) \( \xi > 1 \) | \( e^{i\pi (\lambda_2 + \gamma_2)} \) | \( e^{i\pi \gamma_2} \) | 1 \( e^{i\pi \hat{\alpha}} \) | \( \sigma_\gamma \) |

Table 1: Phases \( \Pi(\xi, \eta) \) along the integration region \( (\xi, \eta) \).

The phases \( \sigma_\lambda, \sigma_\gamma \) are defined as: \( \sigma_\lambda = e^{i\pi(n_1 + n_2)}, \sigma_\gamma = e^{i\pi(m_1 + m_2)} \).

A.1. Case \( \alpha = -2 \)

Let us first consider the case \( \alpha = -2 \). For this case, we find

\[
G^{(-2)}[\lambda_1, \gamma_1]_{\lambda_2, \gamma_2} = - \sin(\pi \gamma_2) \frac{\Gamma(1 + \lambda_1) \Gamma(1 - \lambda_2 - \gamma_2) \Gamma(1 + \gamma_1) \Gamma(1 + \gamma_2)}{\Gamma(2 + \lambda_1 + \gamma_1) \Gamma(2 - \lambda_2)} \times \frac{1}{3} \binom{1 + \lambda_1 - \lambda_2 - \gamma_2}{2 + \lambda_1 + \gamma_1 - 2 - \lambda_2} - \pi (\lambda_1 \gamma_2 - \lambda_2 \gamma_1) \times \left[ \frac{\Gamma(\gamma_1 + \gamma_2) \Gamma(\lambda_1 + \lambda_2)}{\Gamma(1 + \lambda_1 + \lambda_2 + \gamma_1 + \gamma_2)} + \sigma_\gamma \frac{\Gamma(\gamma_1 + \gamma_2) \Gamma(-\lambda_1 - \lambda_2 - \gamma_1 - \gamma_2)}{\Gamma(1 - \lambda_1 - \lambda_2)} \right] ,
\]

Obviously, the result is symmetric under the exchange of \( \lambda_i \leftrightarrow \gamma_i \).

After analyzing the structure of the contour integrals, displayed in the next two figures, we find that only the case \( (ii) \) contributes to (A.2). Furthermore, there is a contribution from the residuum at \( \eta = \xi \) of case \( (iii) \).
We may deform the contour in the complex $\eta$–plane and integrate along the real $\eta$–axis from $-\infty$ until 0:

$$G_{ii}^{(-2)} \left[ \lambda_1, \gamma_1 \atop \lambda_2, \gamma_2 \right] = \sin(\pi \lambda_2) \int_0^1 d\xi \, \xi^{\lambda_1} \left( 1 - \xi \right)^{\gamma_1} \int_0^\infty d\eta \, \eta^{\lambda_2} \left( 1 + \eta \right)^{\gamma_2} \left( \xi + \eta \right)^{-2} \quad (A.4)$$

After writing $(\xi + \eta)^{-2} = \xi^{-2} (1 + \eta/\xi)^{-2}$ the $\eta$–integration is performed using the integral 3.259 (3) of [24]. Then the identity 9.131 (2) of [24] allows to write the hypergeometric function $\, _2F_1[1 - \xi]$ as a sum of two hypergeometric functions $\, _2F_1[\xi]$ with argument $\xi$. Finally the $\xi$–integral is performed with 7.512 (12) of [24] and (A.4) gives:

$$G_{ii}^{(-2)} \left[ \lambda_1, \gamma_1 \atop \lambda_2, \gamma_2 \right] = -\pi \frac{\Gamma(1 + \gamma_1) \Gamma(1 + \lambda_1) \Gamma(1 - \gamma_2 - \lambda_2)}{\Gamma(-\gamma_2) \Gamma(2 + \lambda_1 + \gamma_1) \Gamma(2 - \lambda_2)} \, _3F_2 \left[ 1 + \lambda_1, 1 - \lambda_2 - \gamma_2, 2; 1 \atop 2 + \lambda_1 + \gamma_1, 2 - \lambda_2 \right]$$

$$- \pi \lambda_2 \frac{\Gamma(1 + \gamma_1) \Gamma(1 + \lambda_2)}{\Gamma(1 + \lambda_1 + \lambda_2 + \gamma_1)} \, _3F_2 \left[ \lambda_1 + \lambda_2, -\gamma_2, 1 + \lambda_2; 1 \atop 1 + \lambda_1 + \lambda_2 + \gamma_1, \lambda_2 \right]. \quad (A.5)$$
Alternatively, we may integrate from 1 until $\infty$ including the residuum at $\eta = \xi$:

\[
G_{ii}^{(-2)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] = \sin(\pi \gamma_2) \int_0^1 d\xi \, \xi^{\lambda_1} (1 - \xi)^{\gamma_1} \int_1^{\infty} d\eta \, \eta^{\lambda_2} (\eta - 1)^{\gamma_2} (\eta - \xi)^{-2} \]

\[+ \int_0^1 d\xi \, \xi^{\lambda_1} (1 - \xi)^{\gamma_1} \int_{\eta = \xi}^{\infty} d\eta \, \eta^{\lambda_2} (\eta - 1)^{\gamma_2} (\xi - \eta)^{-2} \]

\[= -\pi \frac{\Gamma(1 + \gamma_1) \Gamma(1 + \lambda_1) \Gamma(1 - \gamma_2 - \lambda_2)}{\Gamma(-\gamma_2) \Gamma(2 + \lambda_1 + \gamma_1) \Gamma(2 - \lambda_2)} \, {}_3F_2 \left[ 1 + \lambda_1, 1 - \lambda_2 - \gamma_2, 2; 1 \right] + \pi (\lambda_1 \gamma_2 - \lambda_2 \gamma_1) \frac{\Gamma(\lambda_1 + \lambda_2) \Gamma(\gamma_1 + \gamma_2)}{\Gamma(1 + \lambda_1 + \lambda_2 + \gamma_1 + \gamma_2)} . \tag{A.6}
\]

The residuum of (A.6) is evaluated by taking into account the expansions:

\[
\eta^{\lambda_2} = \xi^{\lambda_2} + \lambda_2 \xi^{\lambda_2 - 1} (\eta - \xi) + \ldots ,
\]

\[
(1 - \eta)^{\gamma_2} = (1 - \xi)^{\gamma_2} - \gamma_2 (1 - \xi)^{\gamma_2 - 1} (\eta - \xi) + \ldots . \tag{A.7}
\]

It is straightforward to show, that the two expressions (A.5) and (A.6) are indeed identical and yield the first two terms of (A.3). Finally, for the case (iii) the residuum at $\eta = \xi$ gives:

\[
R_{iii}^{(-2)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] = \frac{1}{2} \sigma_\gamma \int_1^{\infty} d\xi \, \xi^{\lambda_1} (\xi - 1)^{\gamma_1} \int_{\eta = \xi}^{\infty} d\eta \, \eta^{\lambda_2} (\eta - 1)^{\gamma_2} (\xi - \eta)^{-2} \]

\[= \pi \sigma_\gamma (\lambda_1 \gamma_2 - \lambda_2 \gamma_1) \frac{\Gamma(\gamma_1 + \gamma_2) \Gamma(1 - \lambda_1 - \gamma_2)}{\Gamma(1 + \lambda_1 + \gamma_1 + \gamma_2)} , \tag{A.8}
\]

which is the last term of (A.3).

To conclude, the total result for (A.3) is the sum of (A.6) and (A.8), i.e.

\[
G^{(-2)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] = G_{ii}^{(-2)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] + R_{iii}^{(-2)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right]
\]

and combines into (A.3).

By partial integration we may deduce some relations between the two integrals $G^{(-2)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right]$ and $V \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right]$. The latter function has been introduced in (2.42). E.g. the following relation may be proven:

\[
G^{(-2)} \left[ \frac{\lambda, \gamma}{\lambda, \gamma} \right] = -\frac{1}{2} \left( \lambda \, V \left[ \frac{\lambda - 1, \gamma}{\lambda - 1, \gamma} \right] + \gamma \, V \left[ \frac{\lambda, \gamma - 1}{\lambda, \gamma - 1} \right] \right) .
\]

For the case $\gamma_i = \gamma$, $\lambda_i = \lambda$ Eq. (A.3) reduces to:

\[
G^{(-2)} \left[ \frac{\lambda, \gamma}{\lambda, \gamma} \right] := \int_{C} d^2 z \, |z|^{2\lambda} |1 - z|^{2\gamma} \frac{1}{(z - \overline{z})^2} = \frac{\pi}{2} \frac{\Gamma(\lambda) \Gamma(\gamma) \Gamma(-\gamma - \lambda)}{\Gamma(-\lambda) \Gamma(-\gamma) \Gamma(\gamma + \lambda)} , \tag{A.9}
\]

which may be used to prove the previous relation explicitly.
A.2. General case $\alpha \neq -2$

In this subsection we shall compute (A.3) for general $\alpha \in \mathbb{R}$. After introducing the parameterization $\xi = z_1 + iz_2 \equiv z$, $\eta = z_1 - iz_2 \equiv \overline{z}$ the integral (A.3) may be written as (A.2) with the phase factor $\Pi$ given in the Table 3. After analyzing the structure of the contour integrals we find that the two cases $(ii)$ and $(iii)$ contribute to (A.2).

For case $(ii)$ we may deform the contour in the complex $\eta$–plane to the left hand side and integrate along the real $\eta$–axis from $-\infty$ until 0. Alternatively, as we shall demonstrate, we may also deform the contour to the right hand side and integrate along the real $\eta$–axis from $\xi$ until $\infty$, see the next Figure.
Hence the total contributions from (ii) is:

$$G^{(\alpha)}_{ii} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] = \sin(\pi \lambda_2) \int_0^1 d\xi \frac{\xi^{\lambda_1}}{(1 - \xi)^{\gamma_1}} \int_0^\infty d\eta \frac{\eta^{\lambda_2}}{(1 + \eta)^{\gamma_2}} (\xi + \eta)^{\alpha}.$$  \hspace{1cm} \text{(A.10)}$$

After writing \((\xi + \eta)^{-\alpha} = \xi^{-\alpha}(1 + \eta/\xi)^{-\alpha}\) we proceed as described below (A.4) and obtain

$$G^{(\alpha)}_{ii} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] = \sin(\pi \lambda_2) \frac{\Gamma(1 + \alpha + \lambda_2)\Gamma(1 + \gamma_1)\Gamma(1 + \lambda_1)\Gamma(-\alpha - 1 - \gamma_2 - \lambda_2)}{\Gamma(-\gamma_2)\Gamma(2 + \lambda_1 + \gamma_1)} \times \binom{3F_2}{1 + \lambda_1, -\alpha - 1 - \lambda_2 - \gamma_2, -\alpha, 2 + \lambda_1 + \gamma_1, -\alpha - \lambda_2; 1} + \sin(\pi \lambda_2) \frac{\Gamma(1 + \gamma_1)\Gamma(1 + \lambda_1)\Gamma(-\alpha - 1 - \lambda_2)}{\Gamma(-\alpha)(3 + \alpha + \lambda_1 + \lambda_2 + \gamma_1)} \times \binom{3F_2}{2 + \alpha + \lambda_1 + \lambda_2, -\gamma_2, 1 + \lambda_2, 3 + \alpha + \lambda_1 + \lambda_2 + \gamma_1, 2 - \alpha + \lambda_2; 1}$$

which gives the first two lines of (2.40) and reduces to (A.5) for \(\alpha = -2\). Alternatively, we may integrate from \(\xi\) until 1 with phase factor \(e^{i\pi \hat{\alpha}}\) and from 1 until \(\infty\) with phase factor \(e^{i\pi \hat{(\alpha + \gamma_2)}}\):

$$G^{(\alpha)}_{ii} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] = \sin(\pi \alpha) \int_0^1 d\xi \frac{\xi^{\lambda_1}}{(1 - \xi)^{\gamma_1}} \int_0^1 d\eta \frac{\eta^{\lambda_2}}{(1 + \eta)^{\gamma_2}} (\eta - \xi)^{\alpha} + \sin[\pi(\gamma_2 + \alpha)] \int_0^1 d\xi \frac{\xi^{\lambda_1}}{(1 - \xi)^{\gamma_1}} \int_1^\infty d\eta \frac{\eta^{\lambda_2}}{(1 + \eta)^{\gamma_2}} (\eta - \xi)^{\alpha}$$

$$= \sin[\pi(\gamma_2 + \alpha)] \frac{\Gamma(1 + \lambda_1)\Gamma(1 + \gamma_1)\Gamma(1 + \gamma_2)\Gamma(-1 - \alpha - \lambda_2 - \gamma_2)}{\Gamma(2 + \lambda_1 + \gamma_1)\Gamma(-\alpha - \lambda_2)} \times \binom{3F_2}{1 + \lambda_1, -\alpha - 1 - \lambda_2 - \gamma_2, -\alpha, 2 + \lambda_1 + \gamma_1, -\alpha - \lambda_2; 1} + \sin(\pi \alpha) \frac{\Gamma(1 + \lambda_1)\Gamma(1 + \gamma_1)\Gamma(1 + \lambda_1)\Gamma(2 + \alpha + \gamma_1 + \lambda_2)}{\Gamma(2 + \alpha + \gamma_2)\Gamma(3 + \alpha + \lambda_1 + \gamma_1 + \lambda_2)} \times \binom{3F_2}{-\lambda_2, 1 + \gamma_2, 2 + \alpha + \gamma_1 + \lambda_2, 3 + \alpha + \lambda_1 + \gamma_1 + \lambda_2, 2 + \alpha + \gamma_2, 3 + \alpha + \lambda_1 + \gamma_1 + \lambda_2; 1}.$$  \hspace{1cm} \text{(A.12)}$$

It is straightforward to show, that both expressions (A.12) and (A.11) are indeed identical. Furthermore, for \(\alpha = -2\) both expressions reduce to (A.5) and (A.6), respectively.
Again, for $\alpha \in \mathbb{Z}$ the second term of (A.12) may be interpreted as the residuum at $\eta = \xi$. With the expansions

\begin{align*}
\eta^{\lambda_2} &= \xi^{\lambda_2} \sum_{n=0}^{\infty} (-1)^n \xi^{-n} \frac{(-\lambda_2, n)}{(1, n)} (\eta - \xi)^n, \\
(1 - \eta)^{\gamma_2} &= (1 - \xi)^{\gamma_2} \sum_{n=0}^{\infty} (1 - \xi)^{-n} \frac{(-\gamma_2, n)}{(1, n)} (\eta - \xi)^n, 
\end{align*}
(A.13)

we compute the residuum at $\eta = \xi$

\begin{align*}
\frac{1}{2} \int_0^1 d\xi \xi^{\lambda_1} (1 - \xi)^{\gamma_1} \int_{\eta=\xi} d\eta \eta^{\lambda_2} (1 - \eta)^{\gamma_2} (\xi - \eta)^\alpha \\
&= -\pi \frac{\Gamma(1 + \lambda_1 + \lambda_2) \Gamma(2 + \alpha + \gamma_1 + \gamma_2) \Gamma(-1 - \alpha - \gamma_2)}{\Gamma(-\alpha) \Gamma(-\gamma_2) \Gamma(3 + \alpha + \lambda_1 + \lambda_2 + \gamma_1 + \gamma_2)} \\
&\times {}_3F_2 \left[ \begin{array}{c}
-\lambda_2, 2 + \alpha + \gamma_1 + \gamma_2, 1 + \alpha \\
2 + \alpha + \gamma_2, -\lambda_1 - \lambda_2
\end{array} ; 1 \right],
\end{align*}

which indeed agrees with the second term of (A.12) and reduces to the second term of (A.6) for $\alpha = -2$.

Finally for the case (iii) we may deform the contour in the complex $\eta$–plane and integrate along the real $\eta$–axis from $\xi$ until $\infty$, see the next Figure.

![Deformed contour of the case (iii).](image)

Fig. 16 Deformed contour of the case (iii).

Hence the total contributions from (iii) is:

\begin{align*}
G^{(\alpha)}_{\text{iii}} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] &= \sin(\pi \alpha) \sigma_{\gamma} \int_1^\infty d\xi \xi^{\lambda_1} (\xi - 1)^{\gamma_1} \int_{\xi}^\infty d\eta \eta^{\lambda_2} (\eta - 1)^{\gamma_2} (\eta - \xi)^\alpha \\
&= \sin(\pi \alpha) \sigma_{\gamma} {}_3F_2 \left[ \begin{array}{c}
-\gamma_2, -1 - \alpha - \lambda_2 - \gamma_2, -\alpha - 2 - \lambda_1 - \lambda_2 - \gamma_1 - \gamma_2 \\
-\lambda_2 - \gamma_2, -1 - \alpha - \lambda_1 - \lambda_2 - \gamma_2
\end{array} ; 1 \right] \\
&\times \frac{\Gamma(1 + \alpha) \Gamma(1 + \gamma_1) \Gamma(-1 - \alpha - \lambda_2 - \gamma_2) \Gamma(-2 - \alpha - \lambda_1 - \lambda_2 - \gamma_1 - \gamma_2)}{\Gamma(-1 - \alpha - \lambda_1 - \lambda_2 - \gamma_2) \Gamma(-\lambda_2 - \gamma_2)}.
\end{align*}
(A.14)
Note, that for the case $\alpha = -2$, the above expression gives exactly the residuum (A.8):

$$G^{(-2)}_{iii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] = R^{(-2)}_{iii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right].$$

For $\alpha \in \mathbb{Z}$, with (A.13) and

$$(\eta - 1)^{\gamma_2} = (\xi - 1)^{\gamma_2} \sum_{n=0}^{\infty} (\xi - 1)^{-n} \frac{(-\gamma_2, n)}{1, n} (\xi - \eta)^n$$  \hspace{1cm} (A.15)

we may compute the general expression for the residuum at $\eta = \xi$ in the case of (iii):

$$R^{(\alpha)}_{iii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] = \frac{1}{2} \sigma \int_{1}^{\infty} d\xi \; \xi^{\lambda_1} (\xi - 1)^{\gamma_1} \int_{\eta=\xi}^{\infty} d\eta \; \eta^{\lambda_2} (\eta - 1)^{\gamma_2} (\xi - \eta)^\alpha$$

$$= \pi \sigma \frac{\Gamma(-\alpha - \gamma_2) \Gamma(1 + \alpha + \gamma_2) \Gamma(2 + \alpha + \gamma_1 + \gamma_2) \Gamma(-2 - \alpha - \lambda_1 - \lambda_2 - \gamma_1 - \gamma_2)}{\Gamma(-\alpha) \Gamma(-\gamma_2) \Gamma(-\lambda_1 - \lambda_2) \Gamma(2 + \alpha + \gamma_2)} ,$$

$$\times 3F_2 \left[ \begin{array}{c} -\lambda_2, 1 + \alpha, 2 + \alpha + \gamma_1 + \gamma_2 \\ -\lambda_1 - \lambda_2, 2 + \alpha + \gamma_2 \end{array} ; 1 \right]$$  \hspace{1cm} (A.16)

which reduces to the expression (A.8) for $\alpha = -2$ and agrees with (A.14) for $\alpha \in \mathbb{Z}$, i.e.

$$G^{(\alpha)}_{iii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] = R^{(\alpha)}_{iii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right], \; \alpha \in \mathbb{Z}.$$  \hspace{1cm} (A.17)

To conclude, the total result for (2.40) is the sum of (A.12) and (A.14), i.e.

$$G^{(\alpha)} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] = \begin{cases} G^{(\alpha)}_{ii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] + R^{(\alpha)}_{iii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right], \; \alpha \in \mathbb{Z}, \\ G^{(\alpha)}_{ii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right] + G^{(\alpha)}_{iii} \left[ \begin{array}{c} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{array} \right], \; \alpha \notin \mathbb{Z}, \end{cases}$$  \hspace{1cm} (A.17)

which combines into (2.40).
A.3. Relations

It is straightforward to prove the following relations

\[
\begin{align*}
G^{(a)} \left[ \frac{\lambda_1 + 1, \gamma_1}{\lambda_2, \gamma_2} \right] - G^{(a)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2 + 1, \gamma_2} \right] &= G^{(a+1)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right], \\
G^{(a)} \left[ \frac{\lambda_1 + 2, \gamma_1}{\lambda_2, \gamma_2} \right] + G^{(a)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2 + 2, \gamma_2} \right] - 2 G^{(a)} \left[ \frac{\lambda_1 + 1, \gamma_1}{\lambda_2 + 1, \gamma_2} \right] &= G^{(a+2)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right], \\
G^{(a)} \left[ \frac{\lambda_1, \gamma_1 + 1}{\lambda_2, \gamma_2} \right] - G^{(a)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2 + 1} \right] &= -G^{(a+1)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right], \\
G^{(a)} \left[ \frac{\lambda_1, \gamma_1 + 1}{\lambda_2, \gamma_2} \right] + G^{(a)} \left[ \frac{\lambda_1 + 1, \gamma_1}{\lambda_2, \gamma_2} \right] &= G^{(a)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right], \\
G^{(a)} \left[ \frac{\lambda_1, \gamma_1 + 1}{\lambda_2, \gamma_2 + 1} \right] + G^{(a)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2 + 1, \gamma_2} \right] &= G^{(a)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right], \\
G^{(a)} \left[ \frac{\lambda_1 + 1, \gamma_1}{\lambda_2, \gamma_2} \right] - G^{(a)} \left[ \frac{\lambda_1, \gamma_1}{\lambda_2 - 1, \gamma_2} \right] &= -G^{(a+1)} \left[ \frac{\lambda_1 - 1, \gamma_1}{\lambda_2 - 1, \gamma_2} \right],
\end{align*}
\]

for generic \( \alpha \in \mathbb{R} \). Furthermore, we may prove the identity

\[
G^{(a)} \left[ \frac{\lambda - 1, \gamma}{\lambda, \gamma} \right] = -\frac{\alpha}{\lambda} G^{(a-1)} \left[ \frac{\lambda, \gamma}{\lambda, \gamma} \right] + \frac{\gamma}{\lambda} G^{(a)} \left[ \frac{\lambda, \gamma - 1}{\lambda, \gamma} \right],
\]

which follows from partial integration.

A.4. Alternative derivation for \( \lambda_i = \lambda, \; \gamma_i = \gamma \)

For \( \lambda_i = \lambda, \; \gamma_i = \gamma \) there is yet another way to prove (2.40): With

\[
\Gamma(-\lambda)^{-1} \int_0^\infty d\xi \; \xi^{-\lambda - 1} e^{-\xi|z|^2} = |z|^{2\lambda},
\]

we may write the integral \( G^{(a)}[\lambda, \gamma] \)

\[
G^{(a)} \left[ \frac{\lambda, \gamma}{\lambda, \gamma} \right] = (2i)^\alpha \left[ 1 + (-1)^\alpha \right] \Gamma(-\lambda)^{-1} \Gamma(-\gamma)^{-1} \int_{-\infty}^{\infty} dz_1 \int_0^\infty dz_2 \; z_2^\alpha
\]

\[
\times \int_0^\infty d\xi \int_0^\infty d\eta \; \xi^{-\lambda - 1} \eta^{-\gamma - 1} e^{-\xi(z_1^2 + z_2^2) - \eta(1-z_1^2+z_2^2)},
\]

which gives:

\[
G^{(a)} \left[ \frac{\lambda, \gamma}{\lambda, \gamma} \right] = \frac{\sqrt{\pi}}{2} (2i)^\alpha \left[ 1 + (-1)^\alpha \right] \frac{\Gamma \left( \frac{1}{2} + \frac{\alpha}{2} \right)}{\Gamma(-\lambda) \Gamma(-\gamma)}
\]

\[
\times \int_0^\infty d\xi \int_0^\infty d\eta \; \xi^{-\lambda - 1} \eta^{-\gamma - 1} (\xi + \eta)^{-1 - \frac{\alpha}{2}} e^{-\frac{\xi}{1+\eta}}.
\]
With the change of coordinates $\eta = \frac{x}{1-x} \xi$ and afterwards $\xi = \frac{y}{x}$ (cf. [26]), and performing the $x$– and $y$–integrations we arrive at:

$$G^{(\alpha)} \left[ \frac{\lambda, \gamma}{\lambda, \gamma} \right] = \frac{\sqrt{\pi}}{2} (2i)^\alpha \left[ 1 + (-1)^\alpha \right] \frac{\Gamma \left( \frac{1}{2} + \frac{\alpha}{2} \right)}{\Gamma(-\lambda) \Gamma(-\gamma)} \times \frac{\Gamma \left( 1 + \frac{\alpha}{2} + \gamma \right)}{\Gamma \left( 2 + \alpha + \lambda + \gamma \right)} \frac{\Gamma \left( -1 - \frac{\alpha}{2} - \lambda - \gamma \right)}{\Gamma \left( 1 + \frac{\alpha}{2} + \lambda \right)} \frac{\Gamma \left( 1 + \frac{\alpha}{2} + \lambda + \gamma \right)}{\Gamma \left( 2 + \alpha + \lambda + \gamma \right)} \times \Gamma \left( 1 + \frac{\alpha}{2} + \gamma \right) \Gamma \left( -1 - \frac{\alpha}{2} - \lambda - \gamma \right) \Gamma \left( 1 + \frac{\alpha}{2} + \lambda \right).$$

It is straightforward to show, that for $\lambda_i = \lambda, \gamma_i = \gamma$ Eq. (2.40) boils down to the above result.

**Appendix B. Complex world–sheet integral $W^{(\kappa, \alpha_0)} \left[ \frac{\alpha_1, \lambda_1, \gamma_1, \beta_1}{\alpha_2, \lambda_2, \gamma_2, \beta_2} \right]$**

In this Appendix we shall accomplish the computation of the complex integral (2.54). It has been argued in Subsection 2.4.3, that this amounts to consider the integral (2.62)

$$W^{(\kappa, \alpha_0)} \left[ \frac{\alpha_1, \lambda_1, \gamma_1, \beta_1}{\alpha_2, \lambda_2, \gamma_2, \beta_2} \right] = \frac{1}{2} \int_{-\infty}^{\infty} d\rho \ |\rho|^{\alpha_0} \ |1 + \rho|^{\tilde{\alpha}_1} \ |1 - \rho|^{\tilde{\alpha}_2} \times (1 + \rho)^{n_0} (1 - \rho)^{m_0} I^{(\kappa)} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (\rho),$$

with:

$$I^{(\kappa)} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (\rho) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \ |1 - \xi|^{\tilde{\lambda}_1} \ |\xi - \rho|^{\tilde{\gamma}_1} \ |\xi + \rho|^{\tilde{\beta}_1} \times |1 - \eta|^{\tilde{\lambda}_2} \ |\eta + \rho|^{\tilde{\gamma}_2} \ |\eta - \rho|^{\tilde{\beta}_2} \ |\xi - \eta|^{\kappa} \Pi(\rho, \xi, \eta) \times (1 - \xi)^{n_1} (\xi - \rho)^{n_3} (\xi + \rho)^{n_5} (1 - \eta)^{n_2} (\eta + \rho)^{n_4} (\eta - \rho)^{n_6}.$$

Here we have defined the non–integer part of the parameter (2.56) by putting hat. In the following we shall first analyze the phase factor $\Pi(\rho, \xi, \eta)$, which has been introduced in (B.2). In what follows we have to discuss the three cases (1) $0 < \rho < 1$, (2) $-1 < \rho < 0$, and (3) $\rho > 1$, separately. We restrict to the case $\kappa \in \mathbb{Z}$, i.e. no branching arises from the term $(\xi - \eta)^{\kappa}$

**B.1. Case $0 < \rho < 1$**

For this case the analysis of the phase factor $\Pi(\rho, \xi, \eta)$ is summarized in Table 2.
Table 2: Phases $\Pi(\rho, \xi, \eta)$ along the integration region $(\xi, \eta)$ for $0 < \rho < 1$.

We have introduced the total phases $\sigma_\lambda := e^{i\pi(n_1+n_2)}$, $\sigma_\gamma := e^{i\pi(n_3+n_4)}$ and $\sigma_\beta := e^{i\pi(n_5+n_6)}$. The different phase structures in the complex $\eta$–plane are shown in the next four figures. More precisely, these figures display the way, how to integrate in the complex $\eta$–plane to take into account the phases of Table 2.

Fig. 17 The complex $\eta$–plane and the contour integrals for the two cases (i) and (ii).

Fig. 18 The complex $\eta$–plane and the contour integrals for the two cases (iii) and (iv).
After analyzing the structure of the contour integrals we find that the two cases (ii) and (iii) contribute to (B.2). For case (ii) we may deform the contour in the complex \( \eta \)-plane into two pieces. One piece along the real \( \eta \)-axis from \(-\infty\) to \(-\rho\) and reverse taking into account the phase factors \( e^{-i\pi \hat{\gamma}_2} \) and \( e^{i\pi \hat{\gamma}_2} \), respectively. The second piece goes along the real \( \eta \)-axis from \(+1\) to \(\infty\) and reverse taking into account the phase factors \( e^{i\pi \hat{\lambda}_2} \) and \( e^{-i\pi \hat{\lambda}_2} \), respectively, see the next Figure.

\[
\sigma_{\gamma} \quad e^{-i\pi \hat{\gamma}_2} \quad \xi \quad e^{-i\pi \hat{\lambda}_2} \quad +1
\]

\[-\rho < \xi < \rho\]

**Fig. 19** Deformed contours of the case (ii).

Hence the two contributions \( I_a \) and \( I_{c_1} \) from (ii) give

\[
I_a^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1 \big| \lambda_2, \gamma_2, \beta_2 \right](\rho) = \sin(\pi \gamma_2) \ e^{i\pi(\gamma_2+\beta_2)} \ \sigma_{\gamma} \ \int_{-\rho}^{\rho} d\xi \ \int_{-\infty}^{-\rho} d\eta \ (1 - \xi)\lambda_1 \ (\rho - \xi)\gamma_1 \ (\rho + \xi)\beta_1 \\
\times (1 - \eta)\lambda_2 \ (-\rho - \eta)\gamma_2 \ (-\rho - \eta)\beta_2 \ (\xi - \eta)\kappa ,
\]

\[
I_{c_1}^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1 \big| \lambda_2, \gamma_2, \beta_2 \right](\rho) = -\sin(\pi \lambda_2) \ e^{i\pi(\gamma_2+\kappa)} \ \sigma_{\gamma} \ \int_{-\rho}^{\rho} d\xi \ \int_{1}^{\infty} d\eta \ (1 - \xi)\lambda_1 \ (\rho - \xi)\gamma_1 \ (\rho + \xi)\beta_1 \\
\times (\eta - 1)\lambda_2 \ (\eta + \rho)\gamma_2 \ (\eta - \rho)\beta_2 \ (\eta - \xi)\kappa ,
\]

\( \text{(B.3)} \)

respectively. On the other hand, for case (iii) we may first deform the whole contour in the complex \( \eta \)-plane and integrate along the real \( \eta \)-axis from \(+1\) until \(\infty\) and reverse respecting the phase factors \( e^{-i\pi \hat{\lambda}_2} \) and \( e^{i\pi \hat{\lambda}_2} \), respectively, see the next Figure.

\[
1 \quad \xi \quad +1 \quad e^{i\pi \hat{\lambda}_2} \quad e^{-i\pi \hat{\lambda}_2}
\]

\( \rho < \xi < 1 \)

**Fig. 20** Deformed contour of the case (iii).
Therefore, the contribution $I_{c_2}$ from (iii) gives:

$$
I_{c_2}^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1 \atop \lambda_2, \gamma_2, \beta_2 \right] (\rho) = -\sin(\pi \lambda_2) \left(-1\right)^{\kappa} \int_{-\rho}^{\rho} \int_{-\infty}^{1} d\xi \: \int_{-\infty}^{1} d\eta \: (1 - \xi)^{\lambda_1} (\xi - \rho)^{\gamma_1} (\xi + \rho)^{\beta_1} \\
\times (\eta - 1)^{\lambda_2} (\eta + \rho)^{\gamma_2} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa}.
$$

(B.4)

B.2. Case $-1 < \rho < 0$

For this case the analysis of the phase factor $\Pi(\rho, \xi, \eta)$ boils down to the previous Subsection B.1. By using the exchange symmetry (2.58) we can apply the previous results with the following regrouping $\gamma_i \leftrightarrow \beta_i$ and $\alpha_1 \leftrightarrow \alpha_2$.

Hence the two contributions $I_a'$ and $I_{c_1}'$ from (ii) yield

$$
I_{a_1}^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1 \atop \lambda_2, \gamma_2, \beta_2 \right] (\rho) = \sin(\pi \beta_2) \: e^{i \pi (\gamma_2 + \beta_2)} \: \sigma_\beta \: \int_{-\rho}^{\rho} \int_{-\infty}^{-\rho} d\xi \: \int_{-\infty}^{1} d\eta \: (1 - \xi)^{\lambda_1} (\xi - \rho)^{\gamma_1} (-\rho - \xi)^{\beta_1} \\
\times (1 - \eta)^{\lambda_2} (-\rho - \eta)^{\gamma_2} (\rho - \eta)^{\beta_2} (\xi - \eta)^{\kappa},
$$

$$
I_{c_1}^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1 \atop \lambda_2, \gamma_2, \beta_2 \right] (\rho) = -\sin(\pi \lambda_2) \: e^{i \pi (\beta_2 + \kappa)} \: \sigma_\beta \: \int_{-\rho}^{\rho} \int_{1}^{\infty} d\xi \: \int_{1}^{\infty} d\eta \: (1 - \xi)^{\lambda_1} (\xi - \rho)^{\gamma_1} (-\rho - \xi)^{\beta_1} \\
\times (\eta - 1)^{\lambda_2} (\rho + \eta)^{\gamma_2} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa},
$$

(B.5)

respectively. Furthermore, the contribution from case (iii) gives:

$$
I_{c_2}^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1 \atop \lambda_2, \gamma_2, \beta_2 \right] (\rho) = -\sin(\pi \lambda_2) \left(-1\right)^{\kappa} \int_{-\rho}^{\rho} \int_{1}^{\infty} d\xi \: \int_{1}^{\infty} d\eta \: (1 - \xi)^{\lambda_1} (\xi - \rho)^{\gamma_1} (\rho + \xi)^{\beta_1} \\
\times (\eta - 1)^{\lambda_2} (\rho + \eta)^{\gamma_2} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa}.
$$

(B.6)

B.3. Case $\rho > 1$

For this case the analysis of the phase factor $\Pi(\rho, \xi, \eta)$ is summarized in Table 3.
Table 3: Phases $\Pi(\rho, \xi, \eta)$ along the integration region $(\xi, \eta)$ for $\rho > 1$.

The different phase structures in the complex $\eta$–plane are shown in the next four figures.

Fig. 21 The complex $\eta$–plane and the contour integrals for the two cases (i) and (ii).

Fig. 22 The complex $\eta$–plane and the contour integrals for the two cases (iii) and (iv).

After analyzing the structure of the contour integrals we find that the two cases (ii) and (iii) contribute to (B.2). For case (ii) we may deform the contour in the complex $\eta$–plane and integrate along the real $\eta$–axis from $\rho$ to $\infty$ and reverse taking into account the phase factors $e^{-i\pi\beta_2}$ and $e^{i\pi\beta_2}$, respectively. On the other hand in the case of (iii) we may deform
the contour and integrate along the real \( \eta \)-axis from \(-\infty\) to the point \(-\rho\), encircling \(-\rho\) and moving backwards to \(-\infty\), see the next Figure.

\[
-\rho < \xi < 1 \\
1 < \xi < \rho
\]

Fig. 23 Deformed contours of the two cases (ii) and (iii).

Hence the contributions \( I_{b_1} \) and \( I_{b_2} \) from (ii) and (iii) give

\[
I_{b_1}^{(\kappa)} \left[ \begin{array}{c} \lambda_1, \gamma_1, \beta_1 \\ \lambda_2, \gamma_2, \beta_2 \end{array} \right] (\rho) = \sigma_\gamma \sin(\pi \beta_2) e^{i\pi(\lambda_2 + \gamma_2 + \beta_2 + \kappa)} \int_{-\rho}^{1} d\xi \int_{\rho}^{\infty} d\eta (1 - \xi)^{\lambda_1} (\rho - \xi)^{\gamma_1} (\rho + \xi)^{\beta_1} \\
\times (\eta - 1)^{\lambda_2} (\rho + \eta)^{\gamma_2} (\eta - \rho)^{\beta_2} (\eta - \xi)^{\kappa},
\]

\[
I_{b_2}^{(\kappa)} \left[ \begin{array}{c} \lambda_1, \gamma_1, \beta_1 \\ \lambda_2, \gamma_2, \beta_2 \end{array} \right] (\rho) = \sigma_\lambda \sigma_\gamma \sin(\pi \gamma_2) e^{i\pi(\lambda_2 + \gamma_2 + \beta_2)} \int_{1}^{\rho} d\xi \int_{-\infty}^{\rho} d\eta (\xi - 1)^{\lambda_1} (\rho - \xi)^{\gamma_1} (\rho + \xi)^{\beta_1} \\
\times (1 - \eta)^{\lambda_2} (-\rho - \eta)^{\gamma_2} (\rho - \eta)^{\beta_2} (\xi - \eta)^{\kappa},
\]

(B.7)

respectively.

Finally, for the case \(-\infty < \rho < -1\) we obtain exactly the same contributions.

**B.4. Total contribution in complex \( \rho \)-plane**

So far, we have reduced the \( \xi \) and \( \eta \) integrations from (B.2) to contour integrals in the complex \( \eta \)-plane. These integrals, which are given in Eqs. (B.3), (B.4), (B.5), (B.6) and (B.7) depend analytically on the complex variable \( \rho \). Now we have to analyze the phase structure in the complex \( \rho \)-plane and properly take into account all phases. The analysis in the complex \( \rho \)-plane is identical to the discussions in Subsections 2.4.1 and 3.2. After some work and putting all parts together we find the final result given in (3.31).

**B.5. Special cases**

In this Subsection we discuss some interesting special cases for the complex integral \( W^{(\kappa, \alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, \gamma_1, \beta_1 \\ \alpha_2, \lambda_2, \gamma_2, \beta_2 \end{array} \right] \).
\( \lambda_1, \lambda_2 = 0 \):

For this case the integral (2.55) can be reduced to the integral (2.42). The whole integral (2.54) factorizes into an integral of the type (2.29) and (2.42):

\[
W^{(0, \alpha_0)} \left[ \begin{array}{c} \alpha_1, 0, \gamma_1, \beta_1 \\ \alpha_2, 0, \gamma_2, \beta_2 \end{array} \right] = 2^{-\alpha_0} V \left[ \begin{array}{c} \beta_1, \gamma_1 \\ \beta_2, \gamma_2 \end{array} \right] \int_0^\infty dx \ (2x)^{2+\alpha_0+\gamma_1+\gamma_2+\beta_1+\beta_2} (1+ix)^{\alpha_1} (1-ix)^{\alpha_2}. 
\]

On the other hand, for \( \lambda_1, \lambda_2, \kappa = 0 \) the non–vanishing terms in (3.31) are:

\[
2^{-\alpha_0} e^{i\pi(\gamma_2+\beta_2)} \sin(\pi \gamma_2) A(1, 6, 3, 5, 4, 2) \rightarrow 2^{-\alpha_0} e^{i\pi(\gamma_2+\beta_2)} V \left[ \begin{array}{c} \beta_1, \gamma_1 \\ \beta_2, \gamma_2 \end{array} \right] 
\times \int_0^1 d\rho \ (2\rho)^{2+\alpha_0+\gamma_1+\gamma_2+\beta_1+\beta_2} (1+\rho)^{\alpha_1} (1-\rho)^{\alpha_2},
\]

\[
2^{-\alpha_0} \left[ \sin(\pi \beta_2) A(1, 3, 5, 2, 4, 6) + \sin(\pi \gamma_2) A(1, 6, 3, 2, 5, 4) \right] \rightarrow 2^{-\alpha_0} V \left[ \begin{array}{c} \beta_1, \gamma_1 \\ \beta_2, \gamma_2 \end{array} \right] 
\times \int_1^\infty d\rho \ (2\rho)^{2+\alpha_0+\gamma_1+\gamma_2+\beta_1+\beta_2} (1+\rho)^{\alpha_1} (\rho-1)^{\alpha_2},
\]

\[
2^{-\alpha_0} e^{i\pi(\gamma_2+\beta_2)} \sin(\pi \beta_2) A(1, 6, 4, 5, 3, 2) \rightarrow 2^{-\alpha_0} e^{i\pi(\gamma_2+\beta_2)} V \left[ \begin{array}{c} \beta_1, \gamma_1 \\ \beta_2, \gamma_2 \end{array} \right] 
\times \int_{-1}^0 d\rho \ (-2\rho)^{2+\alpha_0+\gamma_1+\gamma_2+\beta_1+\beta_2} (1+\rho)^{\alpha_1} (1-\rho)^{\alpha_2}. 
\]

(B.8)

The three expressions in (B.9) combine to (B.8). This can be verified by making use of the relation (3.13) and (2.27).

\( \gamma_1, \gamma_2 = 0 \):

Again for this case the complex \( z \)–integral (2.55) can be reduced to the integral (2.42). The whole integral (2.54) factorizes into an integral of the type (2.29) and (2.42):

\[
W^{(0, \alpha_0)} \left[ \begin{array}{c} \alpha_1, \lambda_1, 0, \beta_1 \\ \alpha_2, \lambda_2, 0, \beta_2 \end{array} \right] = V \left[ \begin{array}{c} \lambda_1, \beta_1 \\ \lambda_2, \beta_2 \end{array} \right] \int_0^\infty dx \ x^{\alpha_0} (1+ix)^{1+\alpha_1+\lambda_1+\beta_1} (1-ix)^{1+\alpha_2+\lambda_2+\beta_2}. 
\]

On the other hand, for \( \gamma_1, \gamma_2, \kappa = 0 \) the non–vanishing terms in (3.31) are:

\[
-2^{-\alpha_0} \sin(\pi \lambda_2) [ e^{i\pi \gamma_2} A(1, 3, 5, 4, 2, 6) + A(1, 3, 4, 5, 2, 6) ] \rightarrow -2^{-\alpha_0} V \left[ \begin{array}{c} \lambda_1, \beta_1 \\ \lambda_2, \beta_2 \end{array} \right] 
\times \int_0^1 d\rho \ (2\rho)^{\alpha_0} (1+\rho)^{1+\alpha_1+\lambda_1+\beta_1} (1-\rho)^{1+\alpha_2+\lambda_2+\beta_2},
\]

(B.10)
\[ 2^{-\alpha_0} \sin(\pi \beta_2) A(1, 3, 5, 2, 4, 6) \to 2^{-\alpha_0} V \left[ \frac{\lambda_1, \beta_1}{\lambda_2, \beta_2} \right] \]
\[ \times \int_1^\infty d\rho (2\rho)^{\alpha_0} (1 + \rho)^{1+\alpha_1+\lambda_1+\beta_1} (\rho - 1)^{1+\alpha_2+\lambda_2+\beta_2}, \]
\[-2^{-\alpha_0} \sin(\pi \lambda_2) A(1, 4, 3, 5, 2, 6) \to -2^{-\alpha_0} V \left[ \frac{\lambda_1, \beta_1}{\lambda_2, \beta_2} \right] \]
\[ \times \int_{-1}^0 d\rho (-2\rho)^{\alpha_0} (1 + \rho)^{1+\alpha_1+\lambda_1+\beta_1} (1 - \rho)^{1+\alpha_2+\lambda_2+\beta_2}, \]
\[ 2^{-\alpha_0} \left[ e^{i\pi(\gamma_2+\beta_2)} \sin(\pi \beta_2) A(1, 6, 4, 5, 3, 2) - \sin(\pi \lambda_2) e^{i\pi \beta_2} A(1, 4, 5, 3, 2, 6) \right] \to 0. \]

Again, the three expressions in (B.9) combine to (B.10). This can be verified by making use of the relation (3.13) and (2.27).

**\( \beta_1, \beta_2 = 0 \):**

Again for this case the integral (2.55) can be reduced to the integral (2.42). The whole integral (2.54) factorizes into an integral of the type (2.29) and (2.42):

\[ W^{(0, \alpha_0)} \left[ \frac{\alpha_1, \lambda_2, \gamma_1, 0}{\alpha_2, \lambda_2, \gamma_2, 0} \right] = V \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] \int_0^\infty dx x^{\alpha_0} (1 + ix)^{1+\alpha_1+\lambda_2+\gamma_2} (1 - ix)^{1+\alpha_2+\lambda_1+\gamma_1}. \]

(B.12)

On the other hand, for \( \beta_1, \beta_2, \kappa = 0 \) the non–vanishing terms in (3.31) are:

\[ 2^{-\alpha_0} e^{i\pi(\gamma_2+\beta_2)} \sin(\pi \gamma_2) A(1, 6, 3, 5, 4, 2) - 2^{-\alpha_0} \sin(\pi \lambda_2) e^{i\pi \gamma_2} A(1, 3, 5, 4, 2, 6) \to 0, \]
\[-2^{-\alpha_0} \sin(\pi \lambda_2) A(1, 3, 4, 5, 2, 6) \to -2^{-\alpha_0} V \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] \]
\[ \times \int_0^1 d\rho (2\rho)^{\alpha_0} (1 + \rho)^{1+\alpha_1+\lambda_2+\gamma_2} (1 - \rho)^{1+\alpha_2+\lambda_1+\gamma_1}, \]
\[ 2^{-\alpha_0} \sin(\pi \gamma_2) A(1, 6, 3, 2, 5, 4) \to 2^{-\alpha_0} V \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] \]
\[ \times \int_1^\infty d\rho (2\rho)^{\alpha_0} (1 + \rho)^{1+\alpha_1+\lambda_2+\gamma_2} (\rho - 1)^{1+\alpha_2+\lambda_1+\gamma_1}, \]
\[-2^{-\alpha_0} \sin(\pi \lambda_2) [ e^{i\pi \beta_2} A(1, 4, 5, 3, 2, 6) + A(1, 4, 3, 5, 2, 6) ] \to -2^{-\alpha_0} V \left[ \frac{\lambda_1, \gamma_1}{\lambda_2, \gamma_2} \right] \]
\[ \times \int_{-1}^0 d\rho (-2\rho)^{\alpha_0} (1 + \rho)^{1+\alpha_1+\lambda_2+\gamma_2} (1 - \rho)^{1+\alpha_2+\lambda_1+\gamma_1}. \]

The three expressions in (B.9) combine to (B.12). This can be verified by making use of the relation (3.13) and (2.27).
Appendix C. Complex world–sheet integral \( W^{'(κ,α)}_σ \left[ \frac{α_1, λ_1, γ_1, β_1}{α_2, λ_2, γ_2, β_2} \right] \)

In this Appendix we shall accomplish the computation of the complex integral (2.70). It has been argued in Subsection 2.4.4, that this amounts to consider the integral (2.75)

\[
W^{'(κ,α)}_σ \left[ \frac{α_1, λ_1, γ_1, β_1}{α_2, λ_2, γ_2, β_2} \right] = \frac{i}{2} \int^{∞}_{-∞} dx \left| 2x \right|^α_0 \left| 1 + x \right|^α_1 \left| 1 - x \right|^α_2 \\
\times (2x)^m_0 \left( 1 + x \right)^m_1 \left( 1 - x \right)^m_2 I^{(κ)} \left[ \frac{λ_1, γ_1, β_1}{λ_2, γ_2, β_2} \right] (x),
\]

(C.1)

with:

\[
I^{(κ)} \left[ \frac{λ_1, γ_1, β_1}{λ_2, γ_2, β_2} \right] (x) = \int^{∞}_{-∞} dξ \int^{∞}_{-∞} dη \left| 1 - ξ \right|^λ_1 \left| ξ - x \right|^γ_1 \left| ξ + x \right|^β_1 \\
\times \left| 1 - η \right|^λ_2 \left| η - x \right|^γ_2 \left| η + x \right|^β_2 \Pi(ξ, η) \\
\times (1 - ξ)^n_1 (ξ - x)^n_3 (ξ + x)^n_5 (1 - η)^n_2 (η - x)^n_4 (η + x)^n_6 (ξ - η)^κ.
\]

(C.2)

Again, quantities with a hat refer to the non–integer part of the parameter (2.72). Furthermore we write κ = ̂κ + ̃κ separating the non–integer part ̂κ. In (C.1) the integration w.r.t. the real variable \( x \) is determined by the range \( I_σ \). For a given range \( I_σ \) the phase factor \( Π(ξ, η) \), which has been introduced in (C.2), is determined by the variables \( ξ \) and \( η \). Hence in the following for a given \( x \in I_σ \) we shall analyze the phase factor \( Π(ξ, η) \). In what follows we have to discuss the three cases (1) \( 0 < x < 1 \), (2) \( -1 < x < 0 \), and (3) \( -∞ < x < -1 ∪ 1 < x < ∞ \), separately.

C.1. Case \( 0 < x < 1 \)

According to (2.66) this case describes the ordering σ₁ of the four open strings along the real \( x \)–axis. The values of the phase \( Π(ξ, η) \) are displayed in the next Table.

| \((ξ, η)\) | \(η < ξ\) | \(ξ < η < -x\) | \(-x < η < x\) | \(x < η < 1\) | \(η > 1\) | total phase |
|---|---|---|---|---|---|---|
| (i) | \(ξ < -x\) | 1 | \(e^{iπκ}\) | \(e^{iπ(κ+β_2)}\) | \(e^{iπ(κ+β_2+γ_2)}\) | \(e^{iπ(κ+β_2+γ_2+λ_2)}\) | \(σγσβ\) |

| \((ξ, η)\) | \(η < -x\) | \(-x < η < ξ\) | \(ξ < η < x\) | \(x < η < 1\) | \(η > 1\) | total phase |
|---|---|---|---|---|---|---|
| (ii) | \(-x < ξ < x\) | \(e^{iπβ_2}\) | 1 | \(e^{iπκ}\) | \(e^{iπ(κ+γ_2)}\) | \(e^{iπ(κ+γ_2+λ_2)}\) | \(σγ\) |

| \((ξ, η)\) | \(η < -x\) | \(-x < η < ξ\) | \(x < η < 1\) | \(ξ < η < 1\) | \(η > 1\) | total phase |
|---|---|---|---|---|---|---|
| (iii) | \(x < ξ < 1\) | \(e^{iπ(γ_2+β_2)}\) | \(e^{iπγ_2}\) | 1 | \(e^{iπκ}\) | \(e^{iπ(κ+λ_2)}\) | 1 |

| \((ξ, η)\) | \(η < -x\) | \(-x < η < x\) | \(x < η < 1\) | \(1 < η < ξ\) | \(η > ξ\) | total phase |
|---|---|---|---|---|---|---|
| (iv) | \(ξ > 1\) | \(e^{iπ(λ_2+γ_2+β_2)}\) | \(e^{iπ(λ_2+γ_2)}\) | \(e^{iπλ_2}\) | 1 | \(e^{iπκ}\) | \(σλ\) |

**Table 4:** Phases \( Π(ξ, η) \) along the integration region \( (ξ, η) \) for \( 0 < x < 1 \).
We have introduced the total phases \( \sigma_\lambda := e^{i\pi(n_1+n_2)} \), \( \sigma_\gamma := e^{i\pi(n_3+n_4)} \) and \( \sigma_\beta := e^{i\pi(n_5+n_6)} \).

The different phase structures in the complex \( \eta \)-plane are shown in the next four figures. More precisely, these figures display the way, how to integrate in the complex \( \eta \)-plane to take into account the phases of Table 4.

**Fig. 24** The complex \( \eta \)-plane and the contour integrals for the two cases (i) and (ii).

**Fig. 25** The complex \( \eta \)-plane and the contour integrals for the two cases (iii) and (iv).

After analyzing the structure of the contour integrals we find that the three cases (ii), (iii) and (iv) contribute to (C.2). Furthermore, in the two cases (iii) and (iv) there are contributions from a residuum at \( \eta = \xi \). For case (ii) we may deform the contour in the complex \( \eta \)-plane and integrate along the real \( \eta \)-axis from \(-\infty\) to \(-x\) and reverse taking into account the phase factors \( e^{-i\pi \hat{\beta}_2} \) and \( e^{i\pi \hat{\beta}_2} \), respectively. On the other hand, in the case of (iv) we may deform the contour and integrate along the real \( \eta \)-axis from \( \xi \) until \( \infty \) and reverse taking into account the corresponding phases \( e^{-i\pi \hat{\kappa}} \) and \( e^{i\pi \hat{\kappa}} \), respectively, see the next Figure.
$-x < \xi < x$

\[ \eta > 1 \]

\[ \xi > 1 \]

**Fig. 26** Deformed contours of the two cases (ii) and (iv).

Hence the contributions from (ii) and (iv) give:

\[
I^{(\kappa)}_{\sigma_1,ii} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (x) = \sin(\pi \beta_2) \sigma_\gamma \int_x^1 d\xi \int_{-\infty}^{-x} d\eta \left(1 - \xi\right)^{\lambda_1} \left(\xi - x\right)^{\gamma_1} \left(\xi + x\right)^{\beta_1} \times (1 - \eta)^{\lambda_2} \left(-\eta + x\right)^{\gamma_2} \left(-\eta - x\right)^{\beta_2} \left(\eta -\xi\right)^{\kappa},
\]

\[
I^{(\kappa)}_{\sigma_1,iv} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (x) = \sin(\pi \kappa) \sigma_\lambda \int_1^\infty d\xi \int_\xi^\infty d\eta \left(\xi - 1\right)^{\lambda_1} \left(\xi - x\right)^{\gamma_1} \left(\xi + x\right)^{\beta_1} \times (\eta - 1)^{\lambda_2} \left(\eta - x\right)^{\gamma_2} \left(\eta + x\right)^{\beta_2} \left(\eta - \xi\right)^{\kappa}.
\]

\[ \text{(C.3)} \]

On the other hand, for case (iii) we may first deform the whole contour in the complex \(\eta\)-plane and integrate along the real \(\eta\)-axis from \(\xi\) until \(\infty\) and reverse respecting the phase factors at \(\eta = \xi\) and \(\eta = 1\), respectively, see the next Figure.

\[ x < \xi < 1 \]

**Fig. 27** Deformed contour of the case (iii).

Therefore, the contribution from (iii) gives:

\[
I^{(\kappa)}_{\sigma_1,iii} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (x) = \sin[\pi (\kappa + \lambda_2)] \int_x^1 d\xi \int_1^\infty d\eta \left(1 - \xi\right)^{\lambda_1} \left(\xi - x\right)^{\gamma_1} \left(\xi + x\right)^{\beta_1} \times (\eta - 1)^{\lambda_2} \left(\eta - x\right)^{\gamma_2} \left(\eta + x\right)^{\beta_2} \left(\eta - \xi\right)^{\kappa}.
\]

\[ \text{(C.4)} \]

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\[ + \sin(\pi \kappa) \int_{-x}^{x} d\xi \int_{-\infty}^{\infty} d\eta \ (1 - \xi)^{\lambda_1} (\xi - x)^{\gamma_1} (\xi + x)^{\beta_1} \times (1 - \eta)^{\lambda_2} (\eta - x)^{\gamma_2} (\eta + x)^{\beta_2} (\eta - \xi)^{\kappa}. \]

After taking into account the \( x \)-integration of (C.1) the integrals (C.3) and (C.4) give rise to (3.31).

\( \text{C.2. Case } -1 < x < 0 \)

According to (2.66) this case describes the ordering \( \sigma_2 \) of the four open strings along the real \( x \)-axis. For this case the analysis of the phase factor \( \Pi(x, \xi, \eta) \) boils down to the previous Subsection C.1. By using the exchange symmetry (2.74) we can apply the previous results with the following regrouping \( \gamma_i \leftrightarrow \beta_i \) and \( \alpha_1 \leftrightarrow \alpha_2 \).

Hence the two contributions \( I_{ii} \) and \( I_{iv} \) from the cases (i\(i\)) and (i\(v\)) yield

\[ I^{(\kappa)}_{\sigma_2,ii} \left[ \lambda_1, \gamma_1, \beta_1 \right] \left[ \lambda_2, \gamma_2, \beta_2 \right] (x) = \sin(\pi \gamma_2) \sigma_{\beta} \int_{-x}^{x} d\xi \int_{-\infty}^{\infty} d\eta \ (1 - \xi)^{\lambda_1} (\xi - x)^{\gamma_1} (-x - \xi)^{\beta_1} \times (1 - \eta)^{\lambda_2} (x - \eta)^{\gamma_2} (-x - \eta)^{\beta_2} (\xi - \eta)^{\kappa}, \]

\( (C.5) \)

Furthermore the contribution from case (iii) gives:

\[ I^{(\kappa)}_{\sigma_2,iii} \left[ \lambda_1, \gamma_1, \beta_1 \right] \left[ \lambda_2, \gamma_2, \beta_2 \right] (x) = \sin(\pi \kappa) \sigma_{\lambda} \int_{-x}^{x} d\xi \int_{-\infty}^{\infty} d\eta \ (1 - \xi)^{\lambda_1} (\xi - x)^{\gamma_1} (\xi + x)^{\beta_1} \times (\eta - 1)^{\lambda_2} (\eta - x)^{\gamma_2} (\eta + x)^{\beta_2} (\eta - \xi)^{\kappa}. \]

\( (C.6) \)

After taking into account the \( x \)-integration of (C.1) the integrals (C.5) and (C.6) give rise to (3.31).
C.3. Case $x > 1$ and $x < -1$

According to (2.66) this case describes the ordering $\sigma_3$ of the four open strings along the real $x$–axis. For $x > 1$ the values of the phase $\Pi(x, \xi, \eta)$ are displayed in the next Table.

| $(\xi, \eta)$ | $\eta < \xi$ | $-x < \eta < x$ | $1 < \eta < x$ | $\eta > x$ | total phase |
|---------------|---------------|-----------------|-----------------|------------|-------------|
| (i) $\xi < -x$ | $1$            | $e^{i\pi \hat{\kappa}}$ | $e^{i\pi (\hat{\kappa} + \hat{\beta}_2)}$ | $e^{i\pi (\hat{\kappa} + \hat{\beta}_2 + \hat{\gamma}_2)}$ | $\sigma_\gamma \sigma_\beta$ |

| $(\xi, \eta)$ | $\eta < -x$ | $-x < \eta < \xi$ | $1 < \eta < x$ | $\eta > x$ | total phase |
|---------------|---------------|-----------------|-----------------|------------|-------------|
| (ii) $-x < \xi < 1$ | $e^{i\pi \beta_2}$ | $1$            | $e^{i\pi \hat{\kappa}}$ | $e^{i\pi (\hat{\kappa} + \hat{\gamma}_2)}$ | $\sigma_\gamma$ |

| $(\xi, \eta)$ | $\eta < -x$ | $-x < \eta < 1$ | $1 < \eta < x$ | $\eta > x$ | total phase |
|---------------|---------------|-----------------|-----------------|------------|-------------|
| (iii) $1 < \xi < x$ | $e^{i\pi (\lambda_2 + \hat{\beta}_2)}$ | $e^{i\pi \lambda_2}$ | $1$            | $e^{i\pi \hat{\kappa}}$ | $e^{i\pi (\hat{\kappa} + \hat{\gamma}_2)}$ | $\sigma_\lambda \sigma_\gamma$ |

| $(\xi, \eta)$ | $\eta < -x$ | $-x < \eta < 1$ | $1 < \eta < x$ | $\eta > x$ | total phase |
|---------------|---------------|-----------------|-----------------|------------|-------------|
| (iv) $\xi > x$ | $e^{i\pi (\lambda_2 + \hat{\beta}_2 + \hat{\gamma}_2)}$ | $e^{i\pi (\lambda_2 + \hat{\gamma}_2)}$ | $e^{i\pi \hat{\gamma}_2}$ | $1$            | $e^{i\pi \hat{\kappa}}$ | $\sigma_\lambda$ |

Table 5: Phases $\Pi(x, \xi, \eta)$ along the integration region $(\xi, \eta)$ for $x > 1$.

Again, the different phase structures in the complex $\eta$–plane are shown in the next four figures.

Fig. 28 The complex $\eta$–plane and the contour integrals for the two cases (i) and (ii).

Fig. 29 The complex $\eta$–plane and the contour integrals for the two cases (iii) and (iv).
After analyzing the structure of the contour integrals we find that the three cases (\(ii\)), (\(iii\)) and (\(iv\)) contribute to (C.2). Furthermore, in the two cases (\(iii\)) and (\(iv\)) there are contributions from a residuum at \(\eta = \xi\). For case (\(ii\)) we may deform the contour in the complex \(\eta\)–plane and integrate along the real \(\eta\)–axis from \(-\infty\) to \(-x\) and reverse taking into account the phase factors \(e^{-i\pi\beta_2}\) and \(e^{i\pi\beta_2}\), respectively. On the other hand, in the case of (\(iv\)) we may deform the contour and integrate along the real \(\eta\)–axis from \(\xi\) until \(\infty\) and reverse taking into account the corresponding phases \(e^{-i\pi\kappa}\) and \(e^{i\pi\kappa}\), respectively, see the next Figure.

**Fig. 30** Deformed contours of the two cases (\(ii\)) and (\(iv\)).

Hence the contributions from (\(ii\)) and (\(iv\)) give:

\[
I^{(\alpha)}_{\sigma_3,ii} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (x) = \sin(\pi\beta_2) \sigma_\gamma \int_{-x}^{-\infty} d\xi \int_{-x}^{1} d\eta \ (1 - \xi)^{\lambda_1} (x - \xi)^{\gamma_1} (\xi + x)^{\beta_1} \\
\times (1 - \eta)^{\lambda_2} (x - \eta)^{\gamma_2} (-\eta - x)^{\beta_2} (\xi - \eta)^{\kappa},
\]

\[
(C.7)
\]

\[
I^{(\alpha)}_{\sigma_3,iv} \left[ \frac{\lambda_1, \gamma_1, \beta_1}{\lambda_2, \gamma_2, \beta_2} \right] (x) = \sin(\pi\kappa) \sigma_\lambda \int_{x}^{\infty} d\xi \int_{x}^{\infty} d\eta \ (\xi - 1)^{\lambda_1} (\xi - x)^{\gamma_1} (\xi + x)^{\beta_1} \\
\times (\eta - 1)^{\lambda_2} (\eta - x)^{\gamma_2} (\eta + x)^{\beta_2} (\eta - \xi)^{\kappa}.
\]

On the other hand, for case (\(iii\)) we may first deform the whole contour in the complex \(\eta\)–plane and integrate along the real \(\eta\)–axis from \(\xi\) until \(\infty\) and reverse respecting the phase factors at \(\eta = \xi\) and \(\eta = x\), respectively, see the next Figure.
Therefore, the contribution from (iii) gives:

\[
I_{\sigma_3, \text{iii}}^{(\alpha)} \left[ \lambda_1, \gamma_1, \beta_1, \lambda_2, \gamma_2, \beta_2 \right] (x) = \sin[\pi(\kappa + \gamma_2)] \sigma_\lambda \sigma_\gamma \int_1^x d\xi \int_x^\infty d\eta \ (\xi - 1)^{\lambda_1} (x - \xi)^{\gamma_1} (\xi + x)^{\beta_1} \\
\times (\eta - 1)^{\lambda_2} (\eta - x)^{\gamma_2} (\eta + x)^{\beta_2} (\eta - x)^\kappa \\
+ \sin(\pi\kappa) \sigma_\lambda \sigma_\gamma \int_1^x d\xi \int_x^{\xi} d\eta \ (\xi - 1)^{\lambda_1} (x - \xi)^{\gamma_1} (\xi + x)^{\beta_1} \\
\times (\eta - 1)^{\lambda_2} (x - \eta)^{\gamma_2} (\eta + x)^{\beta_2} (\eta - x)^\kappa .
\]

(C.8)

The case \( x < -1 \) gives the analogous expressions. After taking into account the \( x \)-integration of (C.1) the integrals (C.7) and (C.8) give rise to (3.61).

Appendix D. Complex world–sheet integral \( W^{(\kappa, \alpha_0, \alpha_3)} \left[ \alpha_1, \lambda_1, \gamma_1, \beta_1, \epsilon_1 \middle| \alpha_2, \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] \)

In this Appendix we undertake the computation of the complex integral (2.86). It has been argued in Subsection 2.4.5, that this amounts to consider the integral (2.90)

\[
W^{(\kappa, \alpha_0, \alpha_3)} \left[ \alpha_1, \lambda_1, \gamma_1, \beta_1, \epsilon_1 \middle| \alpha_2, \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] = 2^{\alpha_0} i \int_{-1}^1 dx \ |2x|^{\alpha_3} |1 + x|^{\alpha_1} |1 - x|^{\alpha_2} \\
\times (2x)^{m_3} (1 + x)^{1+m_1} (1 - x)^{1+m_2} I^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1, \epsilon_1 \middle| \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] (x),
\]

(D.1)

with:

\[
I^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1, \epsilon_1 \middle| \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] (x) = \int_{-\infty}^\infty d\xi \int_{-\infty}^\infty d\eta \ |1 - \xi|^{\lambda_1} |1 + \xi|^{\gamma_1} |\xi - x|^{\beta_1} |\xi + x|^{\epsilon_1} \\
\times |1 - \eta|^{\lambda_2} |1 + \eta|^{\gamma_2} |\eta - x|^{\beta_2} |\eta + x|^{\epsilon_2} |\xi + \eta|^{\kappa} \Pi(x, \xi, \eta) \\
\times (1 - \xi)^{n_1} (1 + \xi)^{n_3} (\xi - x)^{n_5} (\xi + x)^{n_7} \\
\times (1 - \eta)^{n_2} (1 + \eta)^{n_4} (\eta - x)^{n_6} (\eta + x)^{n_8} (\xi + \eta)^{\kappa} .
\]

(D.2)
Again, hatted quantities refer to the non–integer part of the parameter (2.87). Furthermore we write \( \kappa = \hat{\kappa} + \tilde{\kappa} \) separating the non–integer part \( \hat{\kappa} \). For a given \( x \in \mathbb{R} \) the phase factor \( \Pi(x, \xi, \eta) \), which has been introduced in (D.2), is determined by the variables \( \xi \) and \( \eta \). Hence in the following for the range \(-1 < x < 1\) we shall analyze the phase factor \( \Pi(x, \xi, \eta) \). In what follows we have to discuss the two cases (1) \( 0 < x < 1 \), and (2) \(-1 < x < 0\), separately.

**D.1. Case \( 0 < x < 1 \)**

For this case the values of the phase \( \Pi(x, \xi, \eta) \) are displayed in the next Table.

| \((\xi, \eta)\) | \(\eta < -1\) | \(-1 < \eta < -x\) | \(-x < \eta < x\) | \(x < \eta < 1\) | \(1 < \eta < -\xi\) | \(\eta > -\xi\) | total phase |
|---------------|---------------|----------------|----------------|----------------|----------------|----------------|-------------|
| \((i)\) \(\xi < 1\) | \(e^{i\pi \hat{\kappa}}\) | \(e^{i\pi (\gamma_2 + \hat{\kappa})}\) | \(e^{i\pi (\gamma_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\gamma_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \gamma_2 + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \gamma_2 + \hat{\kappa})}\) | \(\sigma_\gamma \sigma_\beta \sigma_\epsilon\) |

| \((\xi, \eta)\) | \(\eta < -1\) | \(-1 < \eta < -x\) | \(-x < \eta < x\) | \(x < \eta < -\xi\) | \(-\xi < \eta < 1\) | \(\eta > 1\) | total phase |
|---------------|---------------|----------------|----------------|----------------|----------------|---------|-------------|
| \((ii)\) \(-1 < x < x\) | \(e^{i\pi (\gamma_2 + \hat{\kappa})}\) | \(e^{i\pi (\gamma_2 + \hat{\kappa})}\) | \(e^{i\pi (\xi + \hat{\kappa})}\) | \(e^{i\pi (\beta_3 + \hat{\kappa})}\) | \(e^{i\pi (\beta_3 + \hat{\kappa})}\) | | \(\sigma_\beta\) |

| \((\xi, \eta)\) | \(\eta < 1\) | \(-1 < \eta < -x\) | \(-x < \eta < -\xi\) | \(-\xi < \eta < 1\) | \(x < \eta < 1\) | \(\eta > 1\) | total phase |
|---------------|---------------|----------------|----------------|----------------|----------------|---------|-------------|
| \((iii)\) \(-x < \xi < x\) | \(e^{i\pi (\gamma_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\gamma_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \beta_2 + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \beta_2 + \hat{\kappa})}\) | | |

| \((\xi, \eta)\) | \(\eta < -1\) | \(-1 < \eta < -x\) | \(-x < \eta < -\xi\) | \(-\xi < \eta < 1\) | \(1 < \eta < -\xi\) | \(\eta > -\xi\) | total phase |
|---------------|---------------|----------------|----------------|----------------|----------------|----------------|-------------|
| \((iv)\) \(x < \xi < 1\) | \(e^{i\pi (\gamma_2 + \beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\beta_2 + \xi + \hat{\kappa})}\) | | \(\sigma_\lambda\) |

| \((\xi, \eta)\) | \(\eta < -1\) | \(-1 < \eta < -x\) | \(-x < \eta < -\xi\) | \(-\xi < \eta < 1\) | \(1 < \eta < -\xi\) | \(\eta > -\xi\) | total phase |
|---------------|---------------|----------------|----------------|----------------|----------------|----------------|-------------|
| \((v)\) \(\xi > 1\) | \(e^{i\pi (\lambda_2 + \gamma_2 + \beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \gamma_2 + \beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \beta_2 + \xi + \hat{\kappa})}\) | \(e^{i\pi (\lambda_2 + \beta_2 + \xi + \hat{\kappa})}\) | | | |

**Table 6: Phases \( \Pi(x, \xi, \eta) \) along the integration region \((\xi, \eta)\) for \(0 < x < 1\).**

We have introduced the total phases \( \sigma_\lambda := e^{i\pi (n_1 + n_2)} \), \( \sigma_\gamma := e^{i\pi (n_3 + n_4)} \), \( \sigma_\beta := e^{i\pi (n_5 + n_6)} \), and \( \sigma_\epsilon := e^{i\pi (n_7 + n_8)} \). The different phase structures in the complex \( \eta \)–plane are shown in the next five figures. More precisely, these figures display the way, how to integrate in the complex \( \eta \)–plane to take into account the phases of Table 6.

**Fig. 32** *The complex \( \eta \)–plane and the contour integrals for the two cases (i) and (ii).*
After analyzing the structure of the contour integrals we find that the four cases (i) – (iv) contribute to (D.2). For case (i) we may deform the contour in the complex \( \eta \)-plane and integrate along the real \( \eta \)-axis from \(-\xi\) to \(\infty\) and reverse taking into account the phase factors \(e^{-i\pi \hat{\kappa}}\) and \(e^{i\pi \hat{\kappa}}\), respectively, cf. the left diagram of the next figure.

On the other hand, in the case of (iv) we may deform the contour and integrate along the real \( \eta \)-axis from \(+1\) to \(\infty\) and reverse taking into account the corresponding phases \(e^{-i\pi \hat{\kappa}}\)
and $e^{i\pi \hat{\lambda}_2}$. Hence the contributions from (i) and (iv) give:

$$I_a^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1, \epsilon_1, \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] (x) = \sin[\pi(\lambda_2 + \gamma_2 + \beta_2 + \epsilon_2)] \sigma_\gamma \sigma_\beta \sigma_\epsilon$$

$$\times \int_{-\infty}^{-1} d\xi \int_{-\infty}^{\infty} d\eta \ (1-\xi)^{\lambda_1} (-1-\xi)^{\gamma_1} (x-\xi)^{\beta_1} (-\xi-x)^{\epsilon_1}$$

$$\times (\eta-1)^{\lambda_2} (1+\eta)^{\gamma_2} (\eta-x)^{\beta_2} (\eta+x)^{\epsilon_2} (\xi+\eta)^{\kappa},$$

$$I_a^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1, \epsilon_1, \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] (x) = \sin(\pi \lambda_2) \int_{-x}^{x} \int_{1}^{1} d\xi \int_{1}^{\infty} d\eta \ (1-\xi)^{\lambda_1} (1+\xi)^{\gamma_1} (\xi-x)^{\beta_1} (\xi+x)^{\epsilon_1}$$

$$\times (\eta-1)^{\lambda_2} (1+\eta)^{\gamma_2} (\eta-x)^{\beta_2} (\eta+x)^{\epsilon_2} (\xi+\eta)^{\kappa}. \quad \text{(D.3)}$$

On the other hand, for case (iii) we may deform the whole contour in the complex $\eta$–plane and integrate along the real $\eta$–axis from $x$ until $\infty$ and reverse respecting the phase factors $e^{i\pi \hat{\beta}_2}$ at $\eta = x$ and $e^{i\pi(\lambda_2+\hat{\beta}_2)}$, respectively, see the next Figure.

![Deformed contour of the case (iii).](image)

Therefore, the two contributions from (iii) give:

$$I_c^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1, \epsilon_1, \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] (x) = \sin(\pi \beta_2) \sigma_\beta \int_{-x}^{x} \int_{1}^{1} d\xi \int_{1}^{\infty} d\eta \ (1-\xi)^{\lambda_1} (1+\xi)^{\gamma_1} (\xi-x)^{\beta_1} (\xi+x)^{\epsilon_1}$$

$$\times (1-\eta)^{\lambda_2} (1+\eta)^{\gamma_2} (\eta-x)^{\beta_2} (\eta+x)^{\epsilon_2} (\xi+\eta)^{\kappa},$$

$$I_c^{(\kappa)} \left[ \lambda_1, \gamma_1, \beta_1, \epsilon_1, \lambda_2, \gamma_2, \beta_2, \epsilon_2 \right] (x) = \sin[\pi(\lambda_2 + \beta_2)] \sigma_\beta \int_{-x}^{x} \int_{1}^{\infty} d\xi \int_{1}^{\infty} d\eta \ (1-\xi)^{\lambda_1} (1+\xi)^{\gamma_1} (x-\xi)^{\beta_1}$$

$$\times (\xi+x)^{\epsilon_1} (\eta-1)^{\lambda_2} (1+\eta)^{\gamma_2} (\eta-x)^{\beta_2} (\eta+x)^{\epsilon_2} (\xi+\eta)^{\kappa}. \quad \text{(D.4)}$$

Finally, for case (ii) we may deform the whole contour in the complex $\eta$–plane to a contour from $\eta = -\xi$ until infinity, going back to $\eta = -\xi$ and encircling the latter point clockwise, see the next Figure.
In the region $-\xi < \eta < 1$ we are left with the phase $e^{i\pi(\hat{\kappa} + \hat{\beta}_2 + \hat{\epsilon}_2)}$. The latter receives an additional phase factor $e^{i\pi \hat{\lambda}_2}$ after passing the point $\eta = 1$ into the region $1 < \eta < \infty$. Therefore, the total contribution to (ii) is

$$I_b^{(\kappa)} \left[ \frac{\lambda_1, \gamma_1, \beta_1, \epsilon_1}{\lambda_2, \gamma_2, \beta_2, \epsilon_2} \right] (x) = -\sin(\pi \kappa) \sigma_\beta \sigma_\epsilon e^{\pi i (\lambda_2 + \beta_2 + \epsilon_2 + \kappa)}$$

$$\times \int_{-1}^{-\xi} d\xi \int_{1}^{\infty} d\eta \ (1 - \xi)^{\lambda_1} (1 + \xi)^{\gamma_1} (x - \xi)^{\beta_1} (-\xi - x)^{\epsilon_1}$$

$$\times (\eta - 1)^{\lambda_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa} ,$$

$$I^{(\kappa)}_{b_2} \left[ \frac{\lambda_1, \gamma_1, \beta_1, \epsilon_1}{\lambda_2, \gamma_2, \beta_2, \epsilon_2} \right] (x) = -\sin(\pi \kappa) \sigma_\beta \sigma_\epsilon e^{\pi i (\beta_2 + \epsilon_2 + \kappa)}$$

$$\times \int_{-1}^{-\xi} d\xi \int_{1}^{\infty} d\eta \ (1 - \xi)^{\lambda_1} (1 + \xi)^{\gamma_1} (x - \xi)^{\beta_1}$$

$$\times (-\xi - x)^{\epsilon_1} (1 - \eta)^{\lambda_2} (1 + \eta)^{\gamma_2} (\eta - x)^{\beta_2} (\eta + x)^{\epsilon_2} (\xi + \eta)^{\kappa} .$$

(D.5)

**D.2. Case $-1 < x < 0$**

The case $-1 < x < 0$ can be inferred from the previous case $0 < x < 1$ by using the relation (2.89). As a result for the case under consideration the phase structure $\Pi(x, \xi, \eta)$ of (D.2) is obtained from Table 6 by interchanging $-x$ and $x$ and permuting $\hat{\epsilon}_i$ and $\hat{\beta}_i$, i.e. $\hat{\epsilon} \leftrightarrow \hat{\beta}$.
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