ON CHEBYSHEV POLYNOMIALS in the COMPLEX PLANE

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Abstract. The estimates of the uniform norm of the Chebyshev polynomials associated with a compact set \( K \) in the complex plane are established. These estimates are exact (up to a constant factor) in the case where \( K \) consists of a finite number of quasiconformal curves or arcs. The case where \( K \) is a uniformly perfect subset of the real line is also studied.

1. Introduction and main results

Let \( K \subset \mathbb{C} \) be a compact set in the complex plane \( \mathbb{C} \) with a connected complement \( \Omega := \overline{\mathbb{C}} \setminus K \), where \( \overline{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \). We assume that \( \text{cap}(K) > 0 \), where \( \text{cap}(K) \) denotes the logarithmic capacity of \( K \) (see [22] - [24]). Denote by \( T_n(z) = T_n(z, K) \), \( n \in \mathbb{N} := \{ 1, 2, \ldots \} \) the \( n \)-th Chebyshev polynomial associated with \( K \), i.e., \( T_n(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_0, c_k \in \mathbb{C} \), is the (unique) monic polynomial which minimizes the supremum norm \( ||T_n||_K := \sup_{z \in K} |T_n(z)| \) among all monic polynomials of the same degree.

It is well-known (see, for example, [23, Theorem 5.5.4 and Corollary 5.5.5]) that

\[
||T_n||_K \geq \text{cap}(K)^n \quad \text{and} \quad \lim_{n \to \infty} ||T_n||_K^{1/n} = \text{cap}(K).
\]

We are interested in estimates from above for the quantity

\[
t_n(K) := \frac{||T_n||_K}{\text{cap}(K)^n}.
\]

We refer the reader to [25]-[27], [29]-[32], [34], [8], [4] and many references therein for a comprehensive survey of this subject.

First, let \( K \) consist of disjoint closed connected sets (continua) \( K^j, j = 1, 2, \ldots, m \), i.e.,

\[
K = \bigcup_{j=1}^m K^j; \quad K^j \cap K^k = \emptyset \quad \text{for} \quad j \neq k; \quad \text{diam}(K^j) > 0.
\]

Here

\[
\text{diam}(S) := \sup_{z, \zeta \in S} |z - \zeta|, \quad S \subset \mathbb{C}.
\]

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Theorem 1  Under the above assumptions,

\[(1.2)\quad t_n(K) = O(\log n) \quad \text{for } n \to \infty.\]

If more information is known about the geometry of \(K\), (1.2) can be improved, for example, in the following way. A Jordan curve \(L \subset \mathbb{C}\) is called quasiconformal (see [1] or [16, p. 100]) if for every \(z_1, z_2 \in L\),

\[(1.3)\quad \text{diam}(L(z_1, z_2)) \leq \beta_L|z_2 - z_1|,\]

where \(L(z_1, z_2)\) is the smaller subarc of \(L\) between \(z_1\) and \(z_2\), a constant \(\beta_L > 1\) depends only on \(L\). Any subarc of a quasiconformal curve is called a quasiconformal arc.

Theorem 2  Let each \(K_j\) in (1.1) be either a quasiconformal arc or a closed Jordan domain bounded by a quasiconformal curve. Then

\[(1.4)\quad t_n(K) = O(1) \quad \text{for } n \to \infty.\]

The estimate (1.4) was proved by other methods in [34] and recently in [31] (for sufficiently smooth \(\partial K^j\)), in [32] (for piecewise sufficiently smooth \(\partial K^j\)), and in [4] (for quasismooth in the sense of Lavrentiev \(\partial K^j\)).

The question whether (1.4) does hold for a general continuum seems to be still open. In the Oberwolfach meeting (see [14] or [20, p. 365]) Pommerenke asked about an example for a continuum \(K\) such that the sequence \(\{t_n(K)\}\) is unbounded. It is mentioned in [20, p. 365] that “D. Wrase in Karlsruhe has shown that an example constructed by J. Clunie [9] for a different purpose has the required property”. But we could not find the proof of this result.

Moreover, in the case where \(K\) is a continuum, one of the major sources for estimates of \(t_n(K)\) are Faber polynomials \(F_n(z) = F_n(z, K)\) associated with \(K\) (see [25], [27]). Gaier [11, Theorem 2], using the same example by Clunie, [9] has shown that there exist a continuum \(K^*\) bounded by a quasiconformal curve with \(\text{cap}(K^*) = 1\), a positive constant \(\alpha\) and an infinite set \(\Lambda \subset \mathbb{N}\) such that for the (monic) polynomial \(F_n(z) = F_n(z, K^*)\) we have

\[\|F_n\|_{K^*} > n^\alpha, \quad n \in \Lambda.\]

Note that the first result of this kind (without the restriction on \(K^*\) to be a quasidisk) was proved by Pommerenke [19].

Hence, Theorem 1 and Theorem 2 reveal the essential difference between the Chebyshev and the Faber polynomials. It is worth pointing out that the case of multiply connected \(\Omega\) presents a more delicate problem (see for example [34]).
Let now $K \subset \mathbb{R}$, where $\mathbb{R}$ is the real line, consist of an infinite number of components. According to [13, Theorem 4.4] in this case $\{t_n(K)\}$ can increase faster than any sequence $\{t_n\}$ satisfying $t_n \geq 1$ and $\lim_{n \to \infty} (\log t_n)/n = 0$. Therefore, in order to have particular bounds for $t_n(K)$ some additional assumptions on $K$ are needed. We assume that $K$ is uniformly perfect, which according to Beardon and Pommerenke [6] means that there exists a constant $0 < \gamma_K < 1$ such that for $z \in K$,

$$K \cap \{\zeta : \gamma_K r \leq |z - \zeta| \leq r\} \neq \emptyset, \quad 0 < r < \text{diam}(K).$$

The classical Cantor set is an example of a uniformly perfect set. Pommerenke [21] has shown that uniformly perfect sets can be described using a density condition in terms of the logarithmic capacity. Namely, $K$ is uniformly perfect if and only if there exists a constant $0 < \lambda_K < 1$ such that for $z \in K$,

$$(1.5) \quad \text{cap}(K \cap \{\zeta : |\zeta - z| \leq r\}) \geq \lambda_K r, \quad 0 < r < \text{diam}(K).$$

Note that sets satisfying (1.5) play a significant role in the solution of the inverse problem of the constructive theory of functions of a complex variable. We refer to [28] where they are called $c$-dense sets. Other interesting properties of the uniformly perfect sets can be found in [12, pp. 343–345].

**Theorem 3** For a uniformly perfect set $K \subset \mathbb{R}$ there exists a constant $c = c(K) > 0$ such that

$$t_n(K) = O(n^c) \quad \text{for } n \to \infty. \quad (1.6)$$

Following Carleson [7] we say that a compact set $K \subset \mathbb{R}$ is homogeneous if there is a constant $\eta_K > 0$ such that for all $x \in K$,

$$|K \cap (x - r, x + r)| \geq \eta_K r, \quad 0 < r < \text{diam}(K).$$

Here, $|S|$ is the linear measure (length) of a (Borel) set $S \subset \mathbb{C}$ (see [22, p. 129]). The Cantor sets of positive length are examples of homogeneous sets (see [18, p. 125]). Recently Christiansen, Simon, and Zinchenko [8] have shown that for the homogeneous subsets of the real line the term $O(n^c)$ in (1.6) can be replaced by $O(1)$. It is worth pointing out that there is a principal difference between the above mentioned classes of compact sets, i.e., $K$ is the Parreau-Widom set in the case of the homogeneous $K \subset \mathbb{R}$ and it is not, in general, the Parreau-Widom set in the case of the uniformly perfect $K$. See [8] for more details.

In what follows, we use the convention that $c, c_1, \ldots$ denote positive constants (different in different sections) that are either absolute or they depend only on $K$; otherwise, the dependence on other parameters is explicitly stated. For the nonnegative functions $a$ and $b$ we write $a \preceq b$ if $a \leq cb$, and $a \asymp b$ if $a \preceq b$ and $b \preceq a$ simultaneously.
We also use the additional notation
\[ d(z, S) := \text{dist}\left(\{z\}, S\right) := \inf_{\zeta \in S} |z - \zeta|, \quad z \in \mathbb{C}, S \subset \mathbb{C}. \]

2. The basic potential-theoretic functions

Let \( K \) be as in (1.1). Following Widom [34], we extend the concept of Faber polynomials to the case of compact sets with the finite number of connected components. Since in [34] all \( \partial K_j \) are sufficiently smooth curves, we need to add some purely technical details. Denote by \( g_{\Omega}(z, z_0), z, z_0 \in \Omega \), the Green function for \( \Omega \) with pole at \( z_0 \). It has a multiple-valued harmonic conjugate \( \tilde{g}_{\Omega}(z, z_0) \).

Thus, the analytic function
\[ \Phi_{\Omega}(z, z_0) := \exp\left(g_{\Omega}(z, z_0) + i\tilde{g}_{\Omega}(z, z_0)\right) \]
is also multiple-valued. We write \( g_{\Omega}(z), \tilde{g}_{\Omega}(z), \) and \( \Phi_{\Omega}(z) \) in the case \( z_0 = \infty \).

Let \( g_{\Omega}(z) := 0, \quad z \in K, \)
\[ K_s := \{ z \in \mathbb{C} : g(z) \leq s \}, \quad \Omega_s := \overline{\mathbb{C}} \setminus K_s, \quad s > 0. \]

Then for \( z \in \Omega_s \),
\[ g_{\Omega_s}(z) = g_{\Omega}(z) - s, \quad \Phi_{\Omega_s}(z) = e^{-s}\Phi_{\Omega}(z). \]

For \( n \in \mathbb{N} \), if \( \Phi_{\Omega}(z)^n \) is single-valued in \( \Omega_{1/n^2} \), we set
\[ W_n(z) := \Phi_{\Omega}(z)^n, \quad z \in \Omega_{1/n^2}. \]

If \( \Phi_{\Omega}(z)^n \) is multiple-valued in \( \Omega_{1/n^2} \), then according to [34, pp. 159, 211] there exist \( q \leq m - 1 \) points \( z_{1,n}, \ldots, z_{q,n} \in \Omega_{1/n^2} \) such that the function
\[ W_n(z) := \Phi_{\Omega}(z)^n \prod_{l=1}^{q} \Phi_{\Omega_{1/n^2}}(z, z_{l,n})^{-1}, \quad z \in \Omega_{1/n^2}, \]
is single-valued in \( \Omega_{1/n^2} \). Moreover, all \( z_{l,n} \) lie in the convex hull of \( K_{1/n^2} \).

In both cases we consider the entire function
\[ F_n(z) := \frac{1}{2\pi i} \int_{C_n} \frac{W_n(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C}, \]
where \( C_n \subset \Omega_{1/n^2} \setminus \{\infty\} \) is a Jordan curve, oriented in the positive direction, containing \( K_{1/n^2} \) and \( z \) in its interior.
Since all points $z_{l,n}$ are in the convex hull of $K_1$, by the symmetry property of the Green function, we obtain

$$\prod_{l=1}^{q} \Phi_{1/n^2}(\infty, z_{l,n}) = \exp \left( \sum_{l=1}^{q} g_{1/n^2}(z_{l,n}) \right) \leq \exp \left( \sum_{l=1}^{q} g_{1}(z_{l,n}) \right) \leq 1.$$ 

For $z \in \Omega_{1/n^2} \setminus \{\infty\}$, let $C'_n \subset \Omega_{1/n^2} \setminus \{\infty\}$ be any Jordan curve, oriented in the positive direction, containing $K_{1/n^2}$ in its interior and $z$ in its exterior. Since by the Cauchy formula

$$F_n(z) = W_n(z) + \frac{1}{2\pi i} \int_{C'_n} \frac{W_n(\zeta)}{\zeta - z} d\zeta,$$

we see that $F_n(z) = \alpha_n z^n + \ldots$ is a polynomial with the property

$$|\alpha_n| = \lim_{z \to \infty} \frac{|F_n(z)|}{z^n} = \lim_{z \to \infty} \frac{|W_n(z)|}{z^n}$$

(2.2) $$\geq \text{cap}(K)^{-n} \prod_{l=1}^{q} \left| \Phi_{1/n^2}(z_{l,n}) \right|^{-1} \geq \text{cap}(K)^{-n}.$$ 

Now let $K$ consist of one component, i.e., $m = 1$ and let $\Phi : \Omega \to D^* := \{w : |w| > 1\}$ be the Riemann conformal mapping with $\Phi(\infty) = \infty, \Phi'(\infty) > 0$. We follow a technique of [10, Chapter IX], [5, p. 387] and for $k, n \in \mathbb{N}, k \geq 2$, consider the Dzjadyk polynomial kernel

$$K_{1,k,n}(\zeta, z), \quad \zeta \in \Omega \setminus \{\infty\}, z \in K,$$

which is a polynomial with respect to $z$ of degree $(k + 3)k(n - 1) - 1$ with continuous coefficients depending on $\zeta$.

According to [5, p. 389, Theorem 2.4] we have

$$\left| \frac{1}{\zeta - z} - K_{1,k,n}(\zeta, z) \right| \leq c_1 \frac{||\tilde{\zeta}_{1/n} - \zeta||^k}{|\zeta - z|(|\tilde{\zeta}_{1/n} - z|)^k} \left( 1 + \frac{\zeta - z}{\tilde{\zeta}_{1/n} - z} \right)^k,$$

where $c_1 = c_1(K, k)$ and

$$\tilde{\zeta}_{\delta} := \Phi^{-1}_1((1 + \delta)\Phi(\zeta)), \quad \delta > 0.$$ 

Let $w := \Phi(\zeta), \tilde{w}_{1/n} := w(1 + 1/n)$. A straightforward calculation shows that for $n > 32^2$ and $|w| \geq 1 + 32/n$, we have

$$\frac{|\tilde{w}_{1/n} - w|}{|\tilde{w}_{1/n} - 1|} < \frac{1}{32}. $$
Therefore, [5, p. 23, Lemma 2.3] implies
\[
\left| \frac{\tilde{\zeta}_{1/n} - \zeta}{\tilde{\zeta}_{1/n} - z} \right| \leq 16 \left| \frac{\tilde{w}_{1/n} - w}{\tilde{w}_{1/n} - 1} \right| < \frac{1}{2},
\]
i.e.,
\[
\left| \frac{1}{\zeta - z} - K_{1,1,k,n}(\zeta, z) \right| \leq c_2 \frac{d(\zeta, K)^k}{|\zeta - z|^{k+1}}, \quad c_2 = c_2(K, k).
\]

We summarize our reasoning as follows. Given \( k \in \mathbb{N} \), there exist sufficiently large constants \( n_0 = n_0(k) \) and \( c_3 = c_3(k) \) such that for any integer \( n > n_0 \) and \( \zeta \) with \( |\Phi_{\Omega}(\zeta)| - 1 \geq c_3/n \), there exists a polynomial
\[
p_{n,k,\zeta,K}(z) = \sum_{l=0}^{n} a_{l,k,K}(\zeta) z^l,
\]
where \( a_{l,k,K} \) are continuous functions of \( \zeta \), satisfying
\[
\left| \frac{1}{\zeta - z} - p_{n,k,\zeta,K}(z) \right| \leq c_2 \frac{d(\zeta, K)^k}{|\zeta - z|^{k+1}}, \quad z \in K.
\]

Indeed, to get (2.4) we can take
\[
p_{n,k,\zeta,K}(z) := K_{1,1,k,N}(\zeta, z), \quad N := \left\lfloor \frac{n}{k(k + 3)} \right\rfloor.
\]

Furthermore, by virtue of (2.3), for \( \zeta \in \Omega \) with \( c_4 \leq |\Phi_{\Omega}(\zeta)| - 1 \leq c_5 \) we have
\[
\left| \frac{1}{\zeta - z} - p_{n,k,\zeta,K}(z) \right| \leq c_6 \frac{d(\zeta, K)^k}{n^k}, \quad z \in K, c_6 = c_6(c_4, c_5, K, k).
\]

Let \( K \) now be as in (1.1) with \( m > 1 \). Denote by \( r_K > 0 \) any fixed number such that \( K_{r_K} \) consists of exactly \( m \) components, i.e.,
\[
K_{r_K} = \bigcup_{j=1}^{m} K_{r_K}^j, \quad K^j \subset K_{r_K}^j.
\]

Let \( \Omega^j := \overline{\mathcal{C}} \setminus K^j \). The maximum principle for the appropriate linear combination of harmonic functions \( g_{\Omega} \) and \( \log |\Phi_{\Omega}| \) in \( K_{r_K}^j \setminus K^j \) shows that
\[
g_{\Omega}(\zeta) \leq \log |\Phi_{\Omega}(\zeta)|, \quad \zeta \in K_{r_K}^j \setminus K^j.
\]

For sufficiently large \( v \in \mathbb{N} \), \( \zeta \in K_{r_K}^j \setminus K^j \) with \( |\Phi_{\Omega}(\zeta)| - 1 \geq c_3/v \), and \( z \in K^l, l = 1, \ldots m \), by virtue of (2.4) and (2.5), applied for the continuum \( K^l \), we have
\[
\left| \frac{1}{\zeta - z} - p_{v,k,\zeta,K^l}(z) \right| \leq c_7 \begin{cases}
\frac{d(\zeta, K^j)^k}{|\zeta - z|^{k+1}} & \text{if } l = j, \\
\frac{n^{-k}}{n} & \text{if } l \neq j.
\end{cases}
\]

(2.7)
Here \( c_7 := c_2 + c_6 \).

For \( \zeta \) as in (2.7) and \( l = 1, \ldots, m \), consider functions

\[
h_l(z) := \begin{cases} 
1 & \text{if } z \in K^l, \\
0 & \text{if } z \in K \setminus K^l,
\end{cases}
\]

\[
f_{\zeta,l}(z) := \frac{h_l(z)}{\zeta - z}, \quad z \in K,
\]

so that

\[
\frac{1}{\zeta - z} = \sum_{l=1}^{m} f_{\zeta,l}(z), \quad z \in K.
\]

Since \( h_l \) can be extended analytically to \( K_{r_K} \), by the Walsh approximation theorem [33, pp. 75-76] there is \( u_0 = u_0(K) \in \mathbb{N} \), such that for any integer \( u > u_0 \), there exists a polynomial \( q_{u,l} \in P_u \) satisfying

\[
(2.8) \quad || h_l - q_{u,l} ||_K \leq e^{-ur_K}.
\]

For sufficiently large \( n \) and \( \zeta \in K_{\alpha} \setminus K_{\beta} \) with \( |\Phi_{\Omega}(\zeta)| - 1 \geq c^*/n \geq c_3/v \), where \( v \) and the constant \( c^* \) are to be chosen later, consider the polynomial

\[
t_{u,v,k,\zeta,l} := q_{u,l}p_{v,k,\zeta,K^l} \in P_{u+v}.
\]

Let

\[
R_K := \max_{1 \leq j \leq m} || \log |\Phi_{\Omega}| ||_{\partial K}.
\]

Since for \( z \in K^p, p = 1, \ldots, m \),

\[
|f_{\zeta,l}(z) - t_{u,v,k,\zeta,l}(z)| \leq \begin{cases} 
|f_{\zeta,l}(z) - p_{v,k,\zeta,K^l}(z)| + |p_{v,k,\zeta,K^l}(z)||h_l(z) - q_{u,l}(z)| & \text{if } p = l, \\
|p_{v,k,\zeta,K^l}(z)||h_l(z) - q_{u,l}(z)| & \text{if } p \neq l,
\end{cases}
\]

by (2.7), (2.8), and the Bernstein-Walsh lemma (see [33, p. 77] or [24, p. 153]), we obtain the following estimates:

if \( l = j \), then

\[
(2.9) \quad |f_{\zeta,l}(z) - t_{u,v,k,\zeta,l}(z)| \leq \begin{cases} 
c_7 \frac{d(\zeta, K^j)^k}{|\zeta - z|^{k+1}} + \frac{c_7 + 1}{d(\zeta, K^j)} e^{-ur_K} & \text{if } p = l, \\
\frac{c_7 + 1}{d(\zeta, K^j)} e^{vR_K - ur_K} & \text{if } p \neq l;
\end{cases}
\]
if $l \neq j$, then

$$|f_{\zeta,l}(z) - t_{u,v,k,\zeta,l}(z)|$$

(2.10)

$$\leq \begin{cases} 
\frac{c_7}{n^k} + c_8e^{-ur_K} & \text{if } p = l, \\
\frac{c_7d(\zeta,K^j)^k}{|z - \zeta|^{k+1}} + \frac{c_7 + 1}{d(\zeta,K^j)}e^{-ur_K} & \text{if } p = j, \\
c_8e^{vR_K - ur_K} & \text{if } p \neq l, p \neq j.
\end{cases}$$

Therefore, for the polynomial

$$t_{u,v,k,\zeta} := \sum_{l=1}^{m} t_{u,v,k,\zeta,l} \in P_{u+v}$$

according to (2.9) and (2.10) for $\zeta$ as in (2.7) and $z \in K^p$, we obtain:

if $p = j$, then

$$\left| \frac{1}{\zeta - z} - t_{u,v,k,\zeta}(z) \right|$$

(2.11)

$$\leq m \left( \frac{c_7d(\zeta,K^j)^k}{|z - \zeta|^{k+1}} + \frac{c_7 + 1}{d(\zeta,K^j)}e^{-ur_K} \right);$$

if $p \neq j$, then

$$\left| \frac{1}{\zeta - z} - t_{u,v,k,\zeta}(z) \right| \leq \frac{c_7 + 1}{d(\zeta,K^j)}e^{vR_K - ur_K}
+ \frac{c_7}{n^k} + c_8e^{-ur_K} + (m - 2)c_8e^{vR_K - ur_K}.$$  

(2.12)

Let

$$u := \left\lfloor \frac{2R_K(n - 1)}{2R_K + r_K} \right\rfloor, \quad v := \left\lfloor \frac{r_K(n - 1)}{2R_K + r_K} \right\rfloor$$

Note that $v \geq n/c_9$. To be sure that (2.11) and (2.12) hold we need to have

$c_3/v \leq c^*/n$ which dictates the choice $c^* := c_3c_9$.

Thus, using the L"owner inequality (see [5, p. 359, Corollary 2.5]), $d(\zeta,K^j) \geq c_{10}/n^2, c_{10} = c_{10}(K,k)$, we obtain a polynomial

$$s_{n-1,k,\zeta} := t_{u,v,k,\zeta} \in P_{n-1}$$

satisfying, by virtue of (2.11) and (2.12), for $\zeta \in K^j_{r_K} \setminus K^j$ with $|\Phi_{\Omega_j}(\zeta)| - 1 \geq c^*/n$, where $c^* = c^*(K,k)$ and $n > n_1 = n_1(K,k)$, the inequality

$$\left| \frac{1}{\zeta - z} - s_{n-1,k,\zeta}(z) \right| \leq c_{11} \begin{cases} 
\frac{d(\zeta,K^j)^k}{|z - \zeta|^{k+1}} & \text{if } z \in K^j, \\
\frac{c_7}{n^k} & \text{if } z \in K \setminus K^j,
\end{cases}$$

(2.13)
where \( c_{11} = c_{11}(K, k) \).

3. Chebyshev polynomials for a system of continua

We start with the proof of the following estimate.

**Lemma 1** Let \( K \) be as in (1.1). Then for \( k \in \mathbb{N} \),

\[
(3.1) \quad t_n(K) \leq c_1 \sum_{j=1}^{m} \left| \int_{L_{c^*/n,j}} \frac{d(\zeta, K^j)^k}{|\zeta - s_{j+1}|} \right|_{K^j}, \quad n \geq n_1,
\]

where \( c^* \) and \( n_1 \) are the constants from (2.13), \( c_1 = c_1(K, k) \), and

\[
L_{\delta,j} = \{ \zeta \in \Omega^j : |\Phi_{\Omega^j}(\zeta)| = 1 + \delta \}, \quad \delta > 0.
\]

**Proof.** Let \( F_n \) be defined by (2.1). By our assumption \( n \) is so large that the curves \( S_{n,j} := L_{c^*/n,j} \subset \Omega_{1/n^2} \) are mutually disjoint. Let \( S_n := \bigcup_{j=1}^{m} S_{n,j} \).

By [5, p. 23, Lemma 2.3], for \( \zeta \in \Omega^j, \; w := \Phi_{\Omega^j}(\zeta), \) and \( \Psi_{\Omega^j} := \Phi_{\Omega^j}^{-1} \), we have

\[
(3.2) \quad |\Psi'_{\Omega^j}(w)| \asymp \frac{d(\zeta, K^j)}{|w| - 1}.
\]

Therefore,

\[
|S_{n,j}| = \int_{|w|=1+c^*/n} |\Psi'_{\Omega^j}(w)||dw| \asymp \frac{n}{c^*} \int_{|w|=1+c^*/n} d(\Psi_{\Omega^j}(w), K^j)|dw| \leq cn, \quad c = c(K, k).
\]

By the Cauchy formula

\[
F_n(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{S_{n,j}} W_n(\zeta) \frac{1}{\zeta - z} d\zeta, \quad z \in K.
\]

We can certainly assume that \( k > 1 \). Consider polynomial \( F^*_n(z) = \alpha_n z^n + \ldots \in \mathbb{P}_n \) defined as follows

\[
F^*_n(z) := \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{S_{n,j}} W_n(\zeta) \left( \frac{1}{\zeta - z} - s_{n-1,k,\zeta}(z) \right) d\zeta, \quad z \in K,
\]

where \( s_{n-1,k,\zeta} \in \mathbb{P}_{n-1} \) satisfies (2.13).
Since by (2.6), for $\zeta \in S_n$,
\[ |W_n(\zeta)| \leq |\Phi_\Omega(\zeta)|^n = \exp(ng_\Omega(\zeta)) \leq c_2, \]
where $c_2 = c_2(K, k)$, according to (2.13) and (3.3), for $z \in K^j$, we obtain
\[ |F^*_n(z)| \leq c_3 \left( \int_{S_{n,j}} \frac{d(\zeta, K^j)^k}{|\zeta - z|^{k+1}} |d\zeta| + n^{-k} \sum_{l=1, l \neq j}^m |S_{n,l}| \right) \]
\[ \leq c_4 \int_{S_{n,j}} \frac{d(\zeta, K^j)^k}{|\zeta - z|^{k+1}} |d\zeta|, \]
where $c_l = c_l(K, k), l = 3, 4$.

Making use of (2.2) and the obvious inequality $t_n(K) \cap(K)^n \leq \|F^*_n\|_K/|\alpha_n|$ we finally obtain (3.1).

Proof of Theorem 1. Changing the variable in the integrals from (3.1) and using (3.2), for sufficiently large $n$, we obtain
\[ \left\| \int_{L_{c^*/n,j}} \frac{d(\zeta, K^j)|d\zeta|}{|\zeta - \cdot|^2} \right\|_{K^j} \]
\[ \leq \frac{1}{n} \left\| \int_{|w|=1+c^*/n} \left( \frac{|\Psi'_\Omega(w)|}{|\Psi_\Omega(w) - \cdot|} \right)^2 |d\zeta| \right\|_{K^j}. \]

Furthermore, since by [27, Chapter IX, §4, Lemma 3],
\[ \left\| \int_{|w|=1+c^*/n} \left( \frac{|\Psi'_\Omega(w)|}{|\Psi_\Omega(w) - \cdot|} \right)^2 |d\zeta| \right\|_{K^j} \leq n \log n, \]
the inequalities (3.1) (with $k = 1$) and (3.4) imply (1.2).

Theorem 2 is a particular case of a more general result which we describe below. Let $K$ consist of one component, i.e., $m = 1$, and let $\Omega$ be a John domain which can be defined as follows (see [22, p. 98]). For a crosscut $\gamma \subset \Omega \setminus \{\infty\}$ of $\Omega$ let $H(\gamma)$ be a bounded component of $\Omega \setminus \gamma$. We say that $\gamma$ is a circular crosscut if $\gamma \subset \Omega \cap C(z, r)$ for some $z \in \partial \Omega = \partial K, r > 0$, and $z \in H(\gamma)$. Here $C(z, r) := \{\zeta : |\zeta - z| = r\}$. Then $\Omega$ is a John domain if there exists a constant $\lambda_\Omega > 1$ such that for any circular crosscut $\gamma$ of $\Omega$,
\[ \text{diam}(H(\gamma)) \leq \lambda_\Omega |\gamma|. \]
By virtue of (1.3) the complement of a quasiconformal arc as well as the unbounded Jordan domain with a quasiconformal boundary both are John domains.

According to (3.5) the function $\Psi_\Omega$ has a continuous extension to $\overline{D^2}$ which we denote by the same letter $\Psi_\Omega$. Next, we assume that $\partial K$ is piecewise quasiconformal, i.e., there exist

$$\theta_1 < \theta_2 < \ldots < \theta_p < \theta_{p+1} := \theta_1 + 2\pi, \quad p \geq 2$$

such that each $J_l := \Psi_\Omega(J'_l), l = 1, \ldots, p$, where $J'_l := \{e^{i\theta} : \theta_l \leq \theta \leq \theta_{l+1}\}$ is a quasiconformal arc.

Let

$$z_l := \Psi_\Omega(e^{i\theta_l}), \quad \Gamma'_l := \{re^{i\theta_l} : r \geq 1\}, \quad \Gamma_l := \Psi_\Omega(\Gamma'_l),$$

$$\Omega'_l := \{re^{i\theta} : \theta_l < \theta < \theta_{l+1}, r > 1\}, \quad \Omega_l := \Psi_\Omega(\Omega'_l).$$

By [3, Lemma 2],

$$(3.6) \quad |\zeta - z_l| \leq d(\zeta, K), \quad \zeta \in \Gamma_l.$$  

Moreover, according to [3, (4.14)],

$$(3.7) \quad |\Gamma_l(\zeta_1, \zeta_2)| \leq |\zeta_2 - \zeta_1|, \quad \zeta_1, \zeta_2 \in \Gamma_l.$$  

Here for any arc or unbounded curve $\Gamma \subset \mathbb{C}$ and $\zeta_1, \zeta_2 \in \Gamma$, we denote by $\Gamma(\zeta_1, \zeta_2)$ the bounded subarc of $\Gamma$ between these points.

Thus, by virtue of (3.6) and (3.7), the curve $L^*_l := \partial \Omega_l = \Gamma_l \cup J_l \cup \Gamma_{l+1}$ satisfies

$$\text{diam}(L^*_l(\zeta_1, \zeta_2)) \leq |\zeta_2 - \zeta_1|, \quad \zeta_1, \zeta_2 \in L^*_l,$$

i.e., by the Ahlfors criterion (see [16, p. 100]), $L^*_l$ is quasiconformal. Since by the same Ahlfors criterion $\partial \Omega'_l = \Gamma'_l \cup J'_l \cup \Gamma'_{l+1}$ is also quasiconformal, the restriction of $\Phi_{\Omega}$ to $\Omega_l$ can be extended to a $Q_l$-quasiconformal homeomorphism $\Phi_l : \mathbb{C} \to \mathbb{C}$ with some $Q_l \geq 1$ (see [16, p. 98]).

The following result describes the distortion properties of $\Phi_l$ and the inverse mapping $\Phi_l^{-1}$ which both are $Q$-quasiconformal with $Q := \max_{l=1,\ldots,p} Q_l$.

**Lemma 2** ([5, p. 29]) Let $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a $Q$-quasiconformal mapping, $Q \geq 1$, with $F(\infty) = \infty$. Let $\zeta_k \in \mathbb{C}, w_k := F(\zeta_k), k = 1, 2, 3$, be such that $|w_1 - w_2| \leq c_5 |w_1 - w_3|$. Then $|\zeta_1 - \zeta_2| \leq c_6 |\zeta_1 - \zeta_3|$ and, in addition,

$$\frac{1}{c_7} \frac{|w_1 - w_3|^{1/Q}}{|w_1 - w_2|} \leq \frac{|\zeta_1 - \zeta_3|}{|\zeta_1 - \zeta_2|} \leq c_7 \frac{|w_1 - w_3|}{|w_1 - w_2|}^Q,$$

where $c_j = c_j(c_5, Q), j = 6, 7$. 

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We claim that for \( z \in \partial K \setminus J_l \),
\[
(3.8) \quad d(z, J_l) \leq d(z, \Omega_l).
\]
Indeed, let \( z'_l \in \partial \Omega_l \) be such that \( |z - z'_l| = d(z, \Omega_l) \). The nontrivial case arises when \( z'_l \not\in J_l \), i.e., \( z'_l \in \Gamma_k \) for \( k = l \) or \( k = l + 1 \). Then by (3.6) we obtain
\[
d(z, J_l) \leq |z - z_k| \leq |z - z'_l| + |z'_l - z_k| \leq |z - z'_l| = d(z, \Omega_l),
\]
which yields (3.8).

For \( z \in \partial K \), denote by \( z^*_l \) any point of \( J_l \) with the property \( |z - z^*_l| = d(z, J_l) \). We claim that
\[
(3.9) \quad |\zeta - z^*_l| \leq |\zeta - z|, \quad \zeta \in \Omega_l, \ z \in \partial K \setminus J_l.
\]
Indeed, by (3.8),
\[
|\zeta - z^*_l| \leq |\zeta - z| + |z - z^*_l| \leq |\zeta - z| + d(z, \Omega_l) \leq 2|\zeta - z|
\]
and (3.9) follows.

For \( \zeta \in \Omega \setminus \{\infty\} \) denote by \( \zeta_K := \Psi_\Omega(\Phi_\Omega(\zeta)/|\Phi_\Omega(\zeta)|) \) the “projection” of \( \zeta \) on \( K \). As an immediate application of Lemma 2, for \( \zeta \in \Omega_l \) and \( z \in J_l \), we have
\[
\frac{d(\zeta, K)}{|\zeta - z|} \leq \left| \frac{\zeta - \zeta_K}{\zeta - z} \right| \leq \left( \frac{|\Phi_\Omega(\zeta)| - 1}{|\Phi_\Omega(\zeta) - \Phi_\Omega(z)|} \right)^{1/Q}.
\]

Now let \( K \) be as in (1.1). We assume that each \( \Omega^j \) is a John domain and each \( \partial K^j \) is piecewise quasiconformal, i.e., each \( \partial K^j \) consists of \( p_j \) quasiconformal arcs \( J_{i,j}, l = 1, \ldots, p_j \) as described above. Let \( \Phi_{i,j} \) be the appropriate quasiconformal homeomorphism of \( C \) which is conformal in a subdomain \( \Omega^j_i \) of \( \Omega^j \) with \( J_{i,j} \subset \partial \Omega^j_i \). For \( z \in \partial K^j \), denote by \( z^*_{i,j} \) the nearest to \( z \) point of \( J_{i,j} \) and let \( w_{i,j} := \Phi_{i,j}(z^*_{i,j}), J'_{i,j} := \Phi_{i,j}(J_{i,j}) \).

According to (3.2), Lemma 2 with \( F = \Psi_\Omega^j \) restricted to \( \Phi_\Omega^j(\Omega^j_1) \) and the triplet of points \( \tau, \tau/|\tau|, w_{i,j} \), as well as (3.9), for \( z \in \partial K^j, s = c^*/n, \) and sufficiently large \( n \), we obtain
\[
\int_{L_{s,j}} \frac{d(\zeta, K)^k|d\zeta|}{|\zeta - z|^{k+1}} = \sum_{l=1}^{p_j} \int_{L_{s,j} \cap \Omega^j_i} \frac{d(\zeta, K)^k|d\zeta|}{|\zeta - z|^{k+1}} \leq \sum_{l=1}^{p_j} \int_{L_{s,j} \cap \Omega^j_i} \frac{d(\zeta, K)^k|d\zeta|}{|\zeta - z^*_{i,j}|^{k+1}}
\]
\[
\leq \sum_{l=1}^{p_j} \frac{1}{s} \int_{|\tau| = 1 + s, \tau/|\tau| \in J'_{i,j}} \frac{|\Psi_\Omega^j(\tau) - \Psi_\Omega^j(\tau/|\tau|)|}{|\Psi_\Omega^j(\tau) - \Psi_\Omega^j(w_{i,j})|}^{k+1} |d\tau|
\]
\[
\leq \sum_{l=1}^{p_j} \frac{1}{s} \int_{|\tau| = 1 + s} \frac{s^{(k+1)/Q}|d\tau|}{|\tau - w_{i,j}|^{(k+1)/Q}} \leq 1
\]
if we fix $k$ satisfying $k + 1 > Q$.

Comparing the last estimate with Lemma 1 we obtain the following statement.

**Theorem 4** Let $K$ be as in (1.1). Assume that each $\Omega^j$ is a John domain and each $\partial K^j$ is piecewise quasiconformal. Then (1.4) holds.

This theorem yields Theorem 2.

4. Chebyshev polynomials for uniformly perfect sets

We introduce some definitions and notations from geometric function theory. Let $K \subset \mathbb{R}$ be a uniformly perfect set satisfying

$$K \subset I := [-1, 1], \pm 1 \in K \neq I. \tag{4.1}$$

The open (with respect to $\mathbb{R}$) set $I \setminus K$ consists of either a finite number $N \geq 1$ or an infinite number $N = \infty$ of disjoint open intervals, i.e.,

$$I \setminus K = \bigcup_{j=1}^{N} (\alpha_j, \beta_j),$$

where $(\alpha_j, \beta_j) \cap (\alpha_k, \beta_k) = \emptyset$ for $j \neq k$.

It follows immediately from (1.5) that $\Omega$ is regular (for the Dirichlet problem), see [23], [24], i.e., $g_\Omega$ extends continuously to $K$ and $g_\Omega(x) := 0$, $x \in K$. Moreover, the Green function satisfies

$$g_\Omega(\zeta) \leq c_1 d(\zeta, K)^\alpha, \quad \zeta \in \mathbb{C}, \tag{4.2}$$

where constants $c_1$ and $\alpha$ could depend only on $\lambda_K$ from (1.5), see [15, Lemma 4.1] or [12, p. 119].

We need the Levin conformal mapping which can be defined as follows (for details, see [17], [2]). Consider the univalent in the upper half-plane $\mathbb{H} := \{z : \Im z > 0\}$ function

$$\phi(z) = \phi(z, K) := \pi + i \left( \int_K \log(z - \zeta) d\mu(\zeta) - \log \text{cap}(K) \right), \quad z \in \mathbb{H},$$

where $\mu = \mu_K$ is the equilibrium measure for $K$. It maps $\mathbb{H}$ onto a vertical half-strip with $N$ slits parallel to the imaginary axis, i.e., the domain

$$\Sigma_K := \{w : 0 < \Re w < \pi, \Im w > 0\} \setminus \bigcup_{j=1}^{N} [u_j, u_j + iv_j]. \tag{4.3}$$
where $0 < u_j = u_j(K) < \pi$ and $v_j = v_j(K) > 0$.

The continuous extension of $\phi$ to $\overline{H}$ satisfies the following boundary correspondence

$$
\phi(\infty) = \infty, \quad \phi((-\infty, -1]) = \{w : \Re w = 0, \Im w \geq 0\},
$$

$$
\phi([1, \infty)) = \{w : \Re w = \pi, \Im w \geq 0\}, \quad \phi(K) = [0, \pi],
$$

$$
\phi([\alpha_j, \beta_j]) = [u_j, u_j + iv_j], \quad j = 1, \ldots, N.
$$

Note that in the last relation each point of $[u_j, u_j + iv_j]$ has two preimages on $[\alpha_j, \beta_j]$.

The crucial fact is that $\phi$ satisfies

$$
g_{\Omega}(z) = \Im \{\phi(z)\}, \quad z \in \overline{H}. \tag{4.4}
$$

For a horizontal crosscut $\gamma$ of $\Sigma_K$, i.e., an interval $\gamma = (a + ib, c + ib) \subset \Sigma_k$ with endpoints on $\partial \Sigma_K$, denote by $h(\gamma)$ its ”height”, that is, $h(\gamma) := b$.

**Lemma 3** Any horizontal crosscut $\gamma$ of $\Sigma_K$ with the property $h(\gamma) \leq \sup_j v_j$ satisfies

$$
h(\gamma) \leq c_2|\gamma|, \quad c_2 = c_2(\lambda_K). \tag{4.5}
$$

**Proof.** For convenience, let $u_{-1} := 0$ and $u_0 := \pi$. Let $\gamma = (u_j + ih(\gamma), u_k + ih(\gamma))$ and $R := \{w = u + iv : u_j < u < u_k, 0 < v < h(\gamma)\}$. Denote by $\Gamma'$ the family of crosscuts of $\Sigma_K \cap R$ which join $(u_j, u_k)$ to $\gamma$ and let $\Gamma^*$ be the family of crosscuts of the rectangle $R$ which join its horizontal boundary intervals. We refer to [1], [16], [12] for the basic properties of the module of a family of curves and arcs (such as conformal invariance, comparison principle, composition laws, etc.) We use these properties without further citation.

For the modules of $\Gamma'$ and $\Gamma^*$ we have

$$
m(\Gamma') \leq m(\Gamma^*) = \frac{|\gamma|}{h(\gamma)}. \tag{4.6}
$$

At the same time, we claim that for the module of $\Gamma := \phi^{-1}(\Gamma')$ the estimate

$$
m(\Gamma) \geq c_3, \quad c_3 = c_3(\lambda_K) \tag{4.7}
$$

holds.

Indeed, without loss of generality, we assume that $j, k \geq 1$ and $\beta_j - \alpha_j \leq \beta_k - \alpha_k$. The other particular cases may be handled in much the same way. Denote by $\Gamma_1$ the family of all crosscuts of

$$
G_1 := \{z = \alpha_j + re^{i\theta} : \beta_j - \alpha_j < r < 2(\beta_j - \alpha_j), 0 < \theta < \pi\}.
$$
which join $F_1 := K \cap [\beta_j, 2\beta_j - \alpha_j]$ with $[3\alpha_j - 2\beta_j, 2\alpha_j - \beta_j]$. Since $\Gamma_1$ is “fewer and longer” than $\Gamma$, the comparison principle yields

\[(4.8) \quad m(\Gamma_1) \leq m(\Gamma).\]

Note that by (1.5),

\[(4.9) \quad \text{cap}(F_1) \geq \lambda_K (\beta_j - \alpha_j).\]

Consider the conformal mapping of $G_1$ onto

\[G_2 := \{ w = re^{i\theta} : r_0 < r < 1, 0 < \theta < \pi \}, \quad r_0 := \exp \left( -\frac{\pi^2}{\log 2} \right),\]

given by the function

\[w = f(z) := \exp \left( \frac{i\pi}{\log 2} \frac{z - \alpha_j}{\beta_j - \alpha_j} \right)\]

with the boundary correspondence

\[f([\beta_j, 2\beta_j - \alpha_j]) = \{ w = e^{i\theta} : 0 \leq \theta \leq \pi \}, \quad f([3\alpha_j - 2\beta_j, 2\alpha_j - \beta_j]) = \{ w = r_0 e^{i\theta} : 0 \leq \theta \leq \pi \}.\]

Since for $\beta_j \leq x_1 < x_2 \leq 2\beta_j - \alpha_j$,

\[|f(x_2) - f(x_1)| \geq \frac{x_2 - x_1}{2(\beta_j - \alpha_j)},\]

by the Fekete-Szeg"{o} Theorem (see [23, p. 153]) and (4.9) for the set $F_2 := f(F_1)$ we have

\[\text{cap}(F_2) \geq \frac{\text{cap}(F_1)}{2(\beta_j - \alpha_j)} \geq \lambda_K.\]

Furthermore, let $\Gamma_2 := f(\Gamma_1)$ and denote by $\Gamma_3$ the family of all crosscuts of the annulus $\{ \tau : r_0 < |\tau| < 1 \}$ which join $F_3 := F_2 \cup \overline{F_2}$, where $\overline{F_2} := \{ \tau : |\tau| = r_0 \}$, with the circular boundary component $\{ \tau : |\tau| = r_0 \}$. By the symmetry principle $m(\Gamma_3) = 2m(\Gamma_2)$. Now we apply Pfluger’s theorem (see [22, p. 212]) to obtain

\[m(\Gamma_3) \geq \pi \left( \log \frac{1 + r_0}{\sqrt{r_0} \text{cap}(F_3)} \right)^{-1}\]

\[\geq \pi \left( \log \frac{1 + r_0}{\sqrt{r_0} \text{cap}(F_2)} \right)^{-1} \geq \pi \left( \log \frac{2(1 + r_0)}{\sqrt{r_0} \lambda_K} \right)^{-1} =: 2c_3.\]

Therefore, the conformal invariance of the module yields

\[m(\Gamma_1) = m(\Gamma_2) = \frac{1}{2} m(\Gamma_3) \geq c_3,\]
which together with (4.8) implies (4.7).

At last, by virtue of the conformal invariance of the module, as well as (4.6) and (4.7), we have (4.5) with
\[ c_2 := c_3^{-1}. \]

\[ \square \]

Let now \( 1 \leq N < \infty \). According to [34], for \( n \in \mathbb{N} \), either \( \Phi_{\Omega}(z)^n \) is single-valued or it is multiple-valued. In the first case, we set \( W_n(z) := \Phi_{\Omega}(z)^n \) and in the second case there exist \( q \leq N \) points \( x_{1,n}, \ldots, x_{q,n} \in I \setminus K \), such that
\[ W_n(z) := \Phi_{\Omega}(z) \prod_{l=1}^{q} \Phi_{\Omega}(z, x_{l,n})^{-1}, \quad z \in \Omega, \]
is single-valued in \( \Omega \). According to [34, pp. 159, 211] each complementary interval \((\beta_j, \alpha_{j+1})\) cannot have more than one point from \( \{x_{l,n}\} \).

Let polynomials \( F_n(z) = F_n(z, K) \) be defined as in Section 2, i.e.,
\[ F_n(z) := \frac{1}{2\pi i} \int_{C_n} \frac{W_n(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C}, \]
where \( C_n \subset \Omega \setminus \{\infty\} \) is a Jordan curve, oriented in the positive direction, containing \( K \) and \( z \) in its interior.

By the Cauchy formula, for \( z \in \Omega \setminus \{\infty\} \) and sufficiently small \( t > 0 \), we have
\[ F_n(z) = W_n(z) + \frac{1}{2\pi i} \int_{\tilde{K}_t} \frac{W_n(\zeta)}{\zeta - z} d\zeta, \]
where \( \tilde{K}_t := \{\zeta \in \Omega : d(\zeta, K) = t\} \) consists of \( N + 1 \) disjoint curves each surrounding exactly one component of \( K \).

Passing to the limit, we obtain for \( z \in \Omega \) with \( |z| < 2 \),
\[ |F_n(z)| \leq |\Phi_{\Omega}(z)|^n + \frac{1}{2\pi} \lim_{t \to 1^+} \int_{\tilde{K}_t} \frac{|\Phi_{\Omega}(\zeta)|^n}{|\zeta - z|} |d\zeta| \]
\[ \leq e^{ng_{\Omega}(z)} + \frac{1}{\pi} \int_{I \setminus D(z, d(z, K))} \frac{|d\zeta|}{|\zeta - z|} \]
\[ \leq e^{ng_{\Omega}(z)} + |\log d(z, K)|. \]

(4.10)

Here,
\[ D(z, r) := \{\zeta : |\zeta - z| < r\}, \quad z \in \mathbb{C}, \quad r > 0. \]

According to (4.2) and (4.10), for \( z \) with the property \( g_{\Omega}(z) = 1/n \), we have the inequality
\[ |F_n(z)| \leq c_4 \log(n + 1), \quad c_4 = c_4(\lambda_K), \]

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which by the maximum principle for $F_n$ in $K_{1/n}$ is also true for $z \in K$.

Note that $F_n(z) = \alpha_n z^n + \ldots$, where as in (2.2)
$$|\alpha_n| = \lim_{z \to \infty} \left| \frac{F_n(z)}{z^n} \right| = \lim_{z \to \infty} \left| \frac{W_n(z)}{z^n} \right|$$
$$= \text{cap}(K)^{-n} \exp \left( -\sum_{l=1}^{q} g_{\Omega}(x_{l,n}) \right).$$

Therefore, by (4.4),
$$t_n(K) \leq \left\| \frac{F_n}{|\alpha_n|} \right\|_K \text{cap}(K)^{-n} \leq c_4 \log(n + 1) \exp(V(K)),$$
where
$$V(K) := \sum_{j=1}^{N} v_j$$
and $v_j = v_j(K)$ are defined by (4.3).

**Proof of Theorem 3.** Applying linear transformation if necessary we always can assume that $K$ satisfies (4.1). By virtue of Theorem 2, the only nontrivial case arises when $K$ consists of infinitely many components. Consider
$$K_n^* := I \cap \{ z \in \mathbb{C} : g_{\Omega}(z) \leq 1/n \}, \quad n \in \mathbb{N}.$$  

It is worth pointing out that $K_n^*$ is uniformly perfect with $\lambda(K_n^*) = \lambda(K)$. Moreover, by Lemma 3, $K_n^*$ consists of $N + 1 = N(K, n) + 1 \leq c_5 n$ disjoint closed intervals and
$$\text{cap}(K) \leq \text{cap}(K_n^*) \leq \text{cap}(\{ z \in \mathbb{C} : g_{\Omega}(z) \leq 1/n \}) = e^{1/n} \text{cap}(K).$$

Let $F_n(z) = F_n(z, K_n^*)$ be the Faber-Widom polynomial as above (constructed for $K_n^*$ instead of $K$). Denote by $v_{n,j} := v_j(K_n^*)$, $j = 1, \ldots, N$, the quantities $v_j$ defined by (4.3) for $K_n^*$ instead of $K$. Note that
$$\max_{1 \leq j \leq N} v_{n,j} \leq \sup_{1 \leq j < \infty} v_j(K) = c_6.$$

For sufficiently large $n$, consider the sets
$$\Lambda_0 := \left\{ j : v_{n,j} \leq \frac{1}{n} \right\},$$
$$\Lambda_k := \left\{ j : \frac{2^{k-1}}{n} < v_{n,j} \leq \frac{2^k}{n} \right\}, \quad k = 1, \ldots, k_0 := \lfloor \log_2(nc_6) \rfloor + 1.$$
Since the number of elements in $\Lambda_0$ is at most $c_5 n$ and by Lemma 3 the number of elements in $\Lambda_k$ is at most $c_7 n 2^{-k}$, we obtain

$$V(K^n) = \sum_{k=0}^{k_0} \sum_{j \in \Lambda_k} v_{n,j} \leq c_5 + c_7 k_0 \leq c_8 \log n.$$  

Therefore, by (4.11) and (4.12)

$$t_n(K) \leq t_n(K^*_n) \frac{\text{cap}(K^*_n)^n}{\text{cap}(K)^n} \leq n^{c_8} \log n,$$

which implies (1.6).

\[\square\]

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