Role of wavefront and velocity distribution with magnetic lens on matter-wave diffraction in near-field regime

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Abstract. Near-field diffraction has been widely studied and applied in many research fields including optics, quantum mechanics and sensor technology. High accuracy sensors, for instance, have been developed based on the Talbot effect, i.e., the self-imaging effect, due to the high sensitivity of the near-field diffraction pattern. Since the visibility of the diffraction pattern depends intensively on velocity distribution and wavefront of the incoming wave, the Talbot-Lau effect is generally used to improve the spatial coherence of the incident wave and hence a better fringe contrast. This research aims to investigate the influences of the wavefront spreading out radially involving the Gaussian velocity distribution on the diffraction pattern of a matter-wave which are key points in design optimization and development of tools for improving fringe quality. The probability density to find the particles in the Talbot-Lau interference pattern can be obtained by employing Feynman path integrals. The theoretical simulation results show that the alteration of wavefront leads to changes in fringe contrast and the width of fringe peaks, whereas the velocity distribution affects only the fringe visibility.

1. Introduction
Near-field diffraction including the Talbot effect or self-imaging effect of monochromatic light has been extensively studied in classical optics [1,2]. Since the Talbot length depends on the wavelength of the incident wave, laser radiation which is highly coherent provides high-visibility diffraction patterns [3]. Matter waves, by contrast, are non-monochromatic and have low directionality. Hence, their fringe contrast drops. However, the coherence of an incident wave could be improved by employing Talbot-Lau effect [4]. With an additional grating, the visibility of the interference pattern increases. Several applications such as interferometer [5], surface profilometer [6] and displacement sensor [7] have been developed based on the Talbot and Talbot-Lau effects. In this paper, we present the study of the propagation of a matter-wave (electron beam) that diffracts through a parallel-gratings system in the near-field regime. The role of wavefront and velocity distribution on the probability density distribution of the matter-wave interference pattern are numerically investigated based on Talbot-Lau interferometer [8]. Finally, the simulation results and the comparison between the numerical and experimental results will be presented and discussed.
2. Theory and method

The experimental setup used in this research consists of a couple of identical gratings \((G_a, G_b)\) and a magnetic lens with a focal length \(F\) as shown in figure 1. The magnetic lens is included in the system to manipulate the fringe pattern in order to make it observable in experiments.

![Figure 1](image-url)

**Figure 1.** Matter-wave of electron diffracts through the grating \(G_a\) and \(G_b\) at time \(t_0\) and \(t_1\), respectively. At time \(t_2\), the wave passes through the region of a magnetic lens with a focal length \(F\), and then moves to a particle detector.

For theoretical consideration, we assume the initial wave for particles falling onto \(G_a\) at \(t_0 = 0\) is in the form

\[
\psi_0(x_0, z_0, t_0) = X_0(x_0, t_0 = 0)Z_0(z_0, t_0 = 0)
\]

\[= C_x \exp \left( -\frac{x_0^2}{\beta_x^2} + ik_0 x_0 \right) C_z \exp \left( -\frac{z_0^2}{\beta_z^2} + ik_0 z_0 \right)
\]

(1)

Where \(C_x, C_z\) and \(\beta_x, \beta_z\) denote the normalization factor and the Gaussian radius for the individual function, respectively. The wave vector is \(k_0 = 2\pi / \lambda_0\), where the de Broglie wavelength of the particle mass \(m\) moving with an initial speed \(v_0\) is \(\lambda_0 = h/mv_0\). Nevertheless, the wave vector \(k_y = k_0 \sin \theta\) in the \(x\)-component has to be taken into account due to the spatial incoherence of the beam [3]. After diffracting through the grating \(G_a\), the wave function \(\psi_0\) is transformed into

\[
\psi'_0(x_0, z_0, t_0) = T_{G_a}(x_0)\psi_0(x_0, z_0, 0) = \sum_{n=-\infty}^{\infty} A_n \exp \left\{ in \frac{2\pi}{d} x_0 \right\} X_0(x_0, 0)Z_0(z_0, 0),
\]

(2)

where \(T_{G_a}(x_0)\) is the transmission function according to the periodicity of the grating period \(d\) along \(x_0\)-axis. The Fourier component \(A_n = \sin(n\pi f) / n\pi\) depends on the open fraction \(f\), which yields a slit window of \(fd\) (figure 1), for instance, \(f = 0.5\) for a binary grating.

At time \(t_1\), letting \(\psi_1\) denote that the wave has propagated via a distance \(L_1\), the corresponding wave function is given by

\[
\psi_1(x_1, z_1, t_1) = \int_{-\infty}^{\infty} dx_0 K_{p_{\psi_1}}(x_1, t_1; x_0, 0)T_{G_a}(x_0)X_0(x_0, 0) \int_{-\infty}^{\infty} dz_0 K_{p_{\psi_2}}(z_1, t_1; z_0, 0)Z_0(z_0, 0)
\]

(3)
with
\[ K_{\text{free}}(q_b, t_b; q_a, t_a) = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}} \exp \left( \frac{im(q_b - q_a)^2}{2\hbar(t_b - t_a)} \right); q = x, z \] (4)

is the Feynman propagator for a free particle [9].

When the wave propagates through the grating \( G_b \), the wave function \( \psi'_1(x, z, t_1) \) transforms into
\[ \psi'_1(x, z, t_1) = T_{G_b}(x) \psi'_1 = \sum_{n=0}^{\infty} A_n \exp \left( in \frac{d}{d} x \right) \psi'_1(x, z, t_1). \] (5)

In analogy with equation (3), the wave function propagating to \( z = z_2 \) during the time interval \( t_2 - t_1 \) can be determined as
\[ \psi_2(x_2, z_2, t_2) = \int dx \int dz_1 K_{\text{free}}(x_2, t_2; x_1, t_1) K_{\text{free}}(z_2, t_2; z_1, t_1) \psi'_1(x_1, z_1, t_1). \] (6)

The integrals can be done analytically and the wave function can be separated in terms of
\[ \psi_2(x_2, z_2, t_2) = C X_2(x_2, t_2) Z_2(z_2, t_2), \] (7)
where constant factors which are independent of the interference pattern have been omitted into the factor \( C \) [10]. In the limit of large \( \beta_x \), in the near-field region, the wave function \( K_2(x_2, t_2) \) is reduced to
\[ X_2(x_2, t_2) = \sum_{n, m} A_n A_m \exp \{ (k_0 + k_d(n + m)) x_2 \}
+ \frac{i\hbar k_d}{2m} (2k_0 + k_d(2n + m)) \left( \frac{2m}{k_d} \right), \] (8)
with \( k_d = 2\pi / d \). The wave function \( Z_2(z_2, t_2) \) is in the form of Gaussian wave packet with constant group velocity \( v_0 = \hbar k_0 / m \) along the \( z \)-axis
\[ Z_2(z_2, t_2) = \exp \left\{ -\frac{1}{\beta_x^2 \gamma_x^2} (z_2 - v_0 t_2)^2 + \frac{ik_0}{\gamma_x} \left( z_2 - v_0 t_2 \right) \right\} \], (9)
where \( \gamma_x = 1 + (2i\hbar t_2 / m\beta_x^2) \).

Subsequently, at the time \( t_2 \) in the magnetic lens realm, we employ a lens transmission function \( T_F(x_2) \) with focal length \( F \) [11]. The wave function \( \psi'_2 \) is transformed into
\[ \psi'_2(x_2, z_2, t_2) = T_F(x_2) \psi_2(x_2, z_2, t_2) = \exp \left( \frac{ik_0 x_2^2}{2F} \right) \psi_2(x_2, z_2, t_2). \] (10)
Lastly, the propagation from \( (x_2, z_2) \) to \( (x_3, z_3) \) during the time interval \( t_3 - t_2 \) is then given by
\[ \psi_3(x_3, z_3, t_3) = \int dx \int dz K_{\text{free}}(x_3, t_3; x_2, t_2) K_{\text{free}}(z_3, t_3; z_2, t_2) \psi'_2(x_2, z_2, t_2). \] (11)

The modulus square of \( \psi_3 \) is interpreted as the probability for detecting particle at \( (x_3, z_3) \), letting \( \psi_3^* \psi_3 = c' \langle X_3 X_3 \rangle \langle Z_3 Z_3 \rangle \) when \( c' \) represents the corresponding normalization factor. By the exact integrations on the above equation, we obtain longitudinal probability density \( Z_3' Z_3 \) in the form of moving Gaussian wave packet function with average position corresponding to the classical motion of a free particle along the \( z \)-axis. Thus, the specific times \( t_4, t_5, t_6 \) are associated with the distances \( L_1, L_2, L_3 \) by \( L_1 = v_0 t_1 \), \( L_2 = v_0 (t_2 - t_1) \) and \( L_3 = v_0 (t_3 - t_2) \) [12]. Therefore, the transverse probability density \( X_3 X_3 \) can be expressed as
whereas the velocity distribution affects mainly on the fringe visibility. the width of significantly lower than that of a plane wave. As can be seen in figure 2(c), the fringe contrast drops and the fringe peak of the incoming wave is improved. The fringe quality of the radial wave is consistent with the fringe of the monochromatic wave (see figure 2(b)). However, the visibility of the fringe decreases with increasing distance \( L' \). Next, the effect of wavefront of the monochromatic wave on the interference pattern is numerically investigated. Although the spatial coherence of the incoming wave is improved due to the Talbot-Lau effect, the fringe quality of the radial wave is significantly lower than that of a plane wave. As can be seen in figure 2(c), the fringe contrast drops and the width of the fringe peak increases. When effects of both the velocity distribution and wavefront are taken into account, the diffraction patterns as shown in figure 2(d) are obtained. It seems that the wavefront spreading out radially causes a broadening of the fringe peak and a fringe contrast reduction, whereas the velocity distribution affects mainly on the fringe visibility.

\[
|X_3|^2 = \sum_{n_1,n_2,m_1,m_2=\infty} A_{n_1} A_{n_2} A_{m_1} A_{m_2} \exp\left\{ \frac{2\pi i(n_1 + n_2 - m_1 - m_2) x_3}{M d} \right\} \exp\left\{ \frac{i}{M L_T} \left[ k_\theta d (M L_1 (m_1 - n_1) + (M L_2 + L_3) (m_1 + m_2 - n_1 - n_2)) + \pi (M L_1 (m_1^2 - n_1^2) + (M L_2 + L_3) ((m_1 + m_2)^2 - (n_1 + n_2)^2)) \right] \right\},
\]

where \( M = 1 + (L_3 / F) \) and \( L_T = d^2 / \lambda_0 \) is the Talbot length with the de Broglie wavelength \( \lambda_0 \). In case of a coherent beam without the magnetic lens (\( \theta \to 0 \) or \( k_\theta \to 0 \) and \( F \to \infty \) or \( M \to 1 \)), equation (12) becomes

\[
|X_3|^2_{\theta \to 0} = \sum_{n_1,n_2,m_1,m_2=\infty} A_{n_1} A_{n_2} A_{m_1} A_{m_2} \exp\left\{ \frac{2\pi i(n_1 + n_2 - m_1 - m_2) x_3}{d} \right\} \exp\left\{ i \pi \left[ \frac{L_1}{L_T} (m_1^2 - n_1^2) + \frac{(L_2 + L_3)}{L_T} ((m_1 + m_2)^2 - (n_1 + n_2)^2) \right] \right\}.
\]

According to the above equation, the grating self-image with period \( d \) appears when \( L_1 \) and \( L_2 + L_3 \) equal to integer multiples of the Talbot length, so-called Talbot effect [3,8]. Comparing with equation (12), \( M \) can be directly explained as the magnification due to the magnetic lens that increases the self-images period to \( M d \) [11]. Nevertheless, the velocity distribution in the electron beam and the spatially incoherent beam of the incident wave has to be taken into account. For the velocity distribution, we use the distribution function in the form of Gaussian distribution of velocity around \( v_{\text{avg}} \) [13]. The spatially incoherence of the beam will be included by a sum over all independent plane waves due to the variety of wave vector \( k_\theta \) [9]. Hence, the collective probability with the velocity distribution is given by

\[
I(x_3,L_1,L_2,L_3) = \sum_{n_1,n_2,m_1,m_2=\infty} \sum_{\theta=\infty} \exp\left\{ -\frac{(v_0 - v_{\text{avg}})^2}{2\Delta v^2} \right\} |X_3|^2,
\]

where \( \Delta v \) represents the full width at half maximum of the Gaussian distribution function [10].

3. Results and discussion

First, the diffraction pattern of a monochromatic plane wave propagating through a couple of identical gratings with \( f = 0.3 \) placed \( 2L_T \) apart from each other is considered. Here, the lens is excluded and the distance between the grating \( G_b \) and the observed fringe is labelled as \( L' \). Self-imaging as shown in figure 2(a) occurs at the distance \( L' = N L_T \) behind the grating \( G_b \), where \( N \) is integer. In case of a plane wave with velocity spread of 7.5% or \( \Delta v = 0.075v_{\text{avg}} \) [10], the position and the width of the fringe peak remain consistent with the fringe of a monoeenergetic wave (see figure 2(b)). However, the visibility of the fringe decreases with increasing distance \( L' \). Next, the effect of wavefront of the monochromatic wave on the interference pattern is numerically investigated. Although the spatial coherence of the incoming wave is improved due to the Talbot-Lau effect, the fringe quality of the radial wave is significantly lower than that of a plane wave. As can be seen in figure 2(c), the fringe contrast drops and the width of the fringe peak increases. When effects of both the velocity distribution and wavefront are taken into account, the diffraction patterns as shown in figure 2(d) are obtained. It seems that the wavefront spreading out radially causes a broadening of the fringe peak and a fringe contrast reduction, whereas the velocity distribution affects mainly on the fringe visibility.
The experiment reported in this paper was performed using the home-made Talbot-Lau setup as described in reference [14]. The electrons emitted from a thermionic source are accelerated by acceleration voltage of 400 V before entering the grating system consisting of 2 identical gratings \( G_s \) and \( G_h \) with \( f = 0.3 \) and \( d = 257 \text{nm} \). The distance between the grating is \( 2L_T \) \( (L_1 = 2L_T) \). A magnetic lens is placed at a distance of \( 2L_T \) behind the grating \( G_h \) \( (L_2 = 2L_T) \). The diffraction pattern at the position of the magnetic lens is magnified by 21,000 times on the particle detector which is located 20 mm away from the lens \( (L_3 = 20 \text{mm}) \). In case of a radial matter-wave, we found that the velocity spread of about 2.5% and the lens with focal length of 952 \text{nm} provides a best-fit curve to the experimental data (see figure 3).

**Figure 2.** Diffraction patterns of (a) a monochromatic plane wave, (b) a plane wave with velocity spread of 7.5% (c) a monoenergetic radial wave (d) a radial wave with velocity spread of 7.5% at different distance \( L' \) behind the grating \( G_h \).
Figure 3. Comparison between experimental result (red curve) and simulation (black curve) of a radial wave with velocity spread of about 2.5%.

4. Conclusion
We present the study of the effects of wavefront and velocity distribution on matter-wave diffraction in near field regime. The probability density of finding electrons for Talbot-Lau interference pattern is obtained by applying Feynman path integrals. The theoretical simulation has been fitted to the experimental result by adjusting the velocity distribution of the incoming wave and focal length of the magnetic lens.

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