HILBERT SERIES OF CERTAIN JET SCHEMES OF DETERMINANTAL VARIETIES

SUDHIR R. GHORPADE, BOYAN JONOV, AND B. A. SETHURAMAN

Abstract. We consider the affine variety $Z_{m,n}^{2,2}$ of first order jets over $Z_{m,n}^{m,n}$, where $Z_{m,n}^{m,n}$ is the classical determinantal variety given by the vanishing of all $2 \times 2$ minors of a generic $m \times n$ matrix. When $2 < m \leq n$, this jet scheme $Z_{m,n}^{m,n}$ has two irreducible components: a trivial component, isomorphic to an affine space, and a nontrivial component that is the closure of the jets supported over the smooth locus of $Z_{m,n}^{m,n}$. This second component is referred to as the principal component of $Z_{m,n}^{m,n}$; it is, in fact, a cone and can also be regarded as a projective subvariety of $P^{2mn-1}$. We prove that the degree of the principal component of $Z_{m,n}^{m,n}$ is the square of the degree of $Z_{m,n}^{m,n}$ and more generally, the Hilbert series of the principal component of $Z_{m,n}^{m,n}$ is the square of the Hilbert series of $Z_{m,n}^{m,n}$. As an application, we compute the $a$-invariant of the principal component of $Z_{m,n}^{m,n}$ and show that the principal component of $Z_{m,n}^{m,n}$ is Gorenstein if and only if $m = n$.

1. Introduction

Let $F$ be an algebraically closed field and $m, n, r$ be integers with $1 \leq r \leq m \leq n$. Let $Z_{r}^{m,n}$ denote the affine variety in $A_{F}^{mn}$ defined by the vanishing of all $r \times r$ minors of an $m \times n$ matrix whose entries are independent indeterminates over $F$. Equivalently, $Z_{r}^{m,n}$ is the locus of $m \times n$ matrices over $F$ of rank $< r$. This is a classical and well-studied object and a number of its properties are known. For example, we know that $Z_{r}^{m,n}$ is irreducible, rational, arithmetically Cohen-Macaulay and projectively normal. Moreover the multiplicity of $Z_{r}^{m,n}$ (at its vertex, since $Z_{r}^{m,n}$ is evidently a cone) or equivalently, the degree of the corresponding projective subvariety of $P_{F}^{2mn-1}$ is given by the following elegant formula (cf. [1, Rem. 20.18 and 20.19] or [13, Cor. 6.2]; see also [19] for an alternative proof and [2, Ch. 2, §4] or [16, p. 352] for an alternative approach and a different formula):

$$e(Z_{r}^{m,n}) = \det_{1 \leq i,j \leq r-1}(\binom{m + n - i - j}{m-i}).$$

(1)

More generally, the Hilbert series of $Z_{r}^{m,n}$ (or more precisely, of the corresponding projective subvariety of $P_{F}^{2mn-1}$) is also known and is explicitly given by

$$\sum_{k \geq 0} h_{k} t^{k} \left(\frac{1}{(1-t)^{d}} - 1\right),$$

(2)

where $d = (r - 1)(m + n - r + 1)$ is the dimension of $Z_{r}^{m,n}$ (as an affine variety), and the coefficients $h_{k}$ are given by sums of binomial determinants as follows:

$$h_{k} = \sum_{k_{1} + \cdots + k_{r-1} = k} \det_{1 \leq i,j \leq r-1}(\binom{m-i}{k_{i}} \binom{n-j}{k_{i} + i - j}).$$

For a proof of this formula, we refer to [14] (see also [11] and [8]). Using this (cf. [8]), or otherwise (cf. [37]), it can be shown that $Z_{r}^{m,n}$ is Gorenstein if and only if $m = n$. Moreover, one can also show that the $a$-invariant of the (homogeneous)
coordinate ring of $\mathbb{Z}_r^{m,n}$ [which, by definition, is the least degree of a generator of its graded canonical module] is $n(1 - r)$; see, e.g., [13] or [11] Thm. 4.

We now turn to jet schemes, which have been of much recent interest due in large part to Nash’s suggestion [33] that jet schemes should give information about singularities of the base; see, e.g., [22] [31] [10]. If $\mathcal{Z}$ is a scheme of finite type over $\mathbb{F}$ and $k$ a positive integer, then a $(k - 1)$-jet on $\mathcal{Z}$ is a morphism $\text{Spec} \mathcal{F}[t]/(t^k) \to \mathcal{Z}$. The set of $(k - 1)$-jets on $\mathcal{Z}$ forms a scheme of finite type over $\mathbb{F}$, denoted $\mathcal{J}_{k-1}(\mathcal{Z})$ and called the $(k - 1)$th jet scheme of $\mathcal{Z}$. A little more concretely, suppose $\mathcal{Z}$ is the affine scheme $\text{Spec} S/I$ defined by the ideal $I = (f_1, \ldots, f_s)$ in the polynomial ring $S = \mathbb{F}[X_1, \ldots, X_N]$. Consider independent indeterminates $t$ and $X^{(i)} (i = 1, \ldots, N$ and $\ell = 0, \ldots, k - 1)$ over $\mathbb{F}$ and the corresponding polynomial ring $S^{(\ell)}$ in the $Nk$ variables $X^{(i)}_j$. For each $j = 1, \ldots, s$, the polynomial $f_j (X_0^{(0)} + tX_1^{(1)} + \cdots + t^{k-1}X_j^{(k-1)}, \ldots, X_N^{(0)} + tX_N^{(1)} + \cdots + t^{k-1}X_N^{(k-1)})$ is of the form $f_j^{(0)} + t^{j(k)} + \cdots + t^{k-1}f_j^{(k-1)}$ modulo $(t^k)$ for unique $f_j^{(\ell)} \in S^{(\ell)}$ ($0 \leq \ell < k$). $\mathcal{J}_{k-1}(\mathcal{Z})$ is then the affine scheme $\text{Spec} S^{(k)}/I'$, where $I'$ is the ideal generated by all $f_j^{(\ell)}$, $1 \leq j \leq s$, $0 \leq \ell < k$. (Often in the literature, authors conflate the algebraic set in $\mathbb{A}^{Nk}$ consisting of the zeros of the polynomials $f_j^{(\ell)}$ with $\mathcal{J}_{k-1}(\mathcal{Z})$ itself. This is generally harmless, especially when considering topological properties such as components, since the points of this algebraic set correspond bijectively with the set of closed points of $\mathcal{J}_{k-1}(\mathcal{Z})$ as $\mathbb{F}$ is algebraically closed, and the set of closed points of an affine scheme is dense in the scheme. See [28] Ch. 2, Rem. 3.49 for instance.)

When $\mathcal{Z}$ is smooth of dimension $d$, the jet scheme $\mathcal{J}_{k-1}(\mathcal{Z})$ is known to be smooth of dimension $kd$. In general, $\mathcal{J}_{k-1}(\mathcal{Z})$ can have multiple irreducible components, and these include a principal component that corresponds to the closure of the set of closed points of $\mathcal{J}_{k-1}(\mathcal{Z})$. This principal component is irreducible of codimension $r/k$. Its irreducible components are usually quite complicated and interesting. In fact, very little seems to be known about the structure of these components and their numerical invariants such as multiplicities. For example, even when $\mathcal{Z}$ is a monomial scheme such as the one given by $X_1X_2\cdots X_r = 0$, where $r \leq N$, determining the irreducible components and the multiplicity of $\mathcal{J}_{k-1}(\mathcal{Z})$ appears to require some effort; see, e.g., [17] and [38]. Irreducible components of jet schemes of toric surfaces are discussed in [30], while the irreducibility of jet schemes of the commuting matrix pairs scheme is discussed in [34]. In a more recent work [6], the Hilbert series of arc spaces (that are, in a sense, limits of $k$th jet schemes as $k \to \infty$) of seemingly simple objects such as the double line $y^2 = 0$ are shown to have connections with the Rogers-Ramanujan identities.

Now determinantal varieties such as $\mathbb{Z}_r^{m,n}$ above are natural examples of singular algebraic varieties and it is not surprising that the study of their jet schemes has been of considerable interest. This was done first by Kosir and Sethuraman in [22] and [23] (see also Yuen [39]). To describe the related results, henceforth we fix positive integers $r, k, m, n$ with $r \leq m \leq n$, and let $\mathbb{Z}_r^{m,n}$ denote the $(k - 1)$th jet scheme on $\mathbb{Z}_r^{m,n}$. It was shown in [22] that $\mathbb{Z}_r^{m,n}$ is irreducible of codimension $k(n - m + 1)$ when $r = m$, and if $r < m$, then it can have $\geq 1 + \lfloor k/2 \rfloor$ irreducible components with equality when $r = 2$ or $k = 2$. A more unified result has recently been obtained by Docampo [9] who shows that $\mathbb{Z}_r^{m,n}$ has exactly $k + 1 - [k/r]$ irreducible components. At any rate, the best understood case with multiple components is $\mathbb{Z}_2^{m,n}$, where $2 \leq m \leq n$. In this case $\mathbb{Z}_2^{m,n} = Z_0 \cup Z_1$, where $Z_1$ is isomorphic to $\mathbb{A}^{mn}$ while $Z_0$ is the principal component which is the closure of the jets supported over the smooth points of the base variety $\mathbb{Z}_2^{m,n}$. Here it will be convenient to consider $2mn$ indeterminates denoted $x_{i,j}, y_{i,j}$ for $1 \leq i \leq m, 1 \leq j \leq n$, and...
and the corresponding polynomial ring $R = \mathbb{F}[x_{i,j}, y_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n]$. Also let $I = I_{2,n}^{m,n}$ and $I_0$ denote, respectively, the ideals of $R$ corresponding to the jet scheme $Z_{2,n}^{m,n}$ and its principal component $Z_0$. In [23], it was shown that both $I$ and $I_0$ are homogeneous radical ideals of $R$ (so that $I_0$ is prime) and moreover, their Gröbner bases were explicitly determined. The leading term ideal $\text{LT}(I_0)$ with respect to this Gröbner basis is generated by squarefree monomials and hence $R/\text{LT}(I_0)$ is the Stanley-Reisner ring of a simplicial complex $\Delta_0$. Jonov [20] subsequently studied this simplicial complex. He showed that $\Delta_0$ is shellable and thus deduced that $R/I_0$ is Cohen-Macaulay. (This last result was independently obtained by Smith and Weyman as well in [35], using their geometric technique for computing syzygies.) Jonov also found a formula for the multiplicity of $R/I_0$, namely,

$$e(R/I_0) = \sum_{i=1}^{m} \sum_{j=1}^{n} \det \left( \binom{i+n-2}{j-2} \binom{m}{i-1} \binom{m}{n-j} \binom{m+j-3}{m-2} \right).$$

Equation (3) above is the starting point of the present paper. We first show that the right side of this equation simplifies remarkably to yield the pretty result

$$e(R/I_0) = \left( m + n - 2 \right)^2 = e(Z_{2,n}^{m,n}).$$

(this was already mentioned in [20], Remark 2.8 by way of a remark). Next, we proceed to determine the Hilbert series of $R/I_0$ or of the principal component $Z_0$. We use the well-known connections between the Hilbert series of $R/I_0$, that of $R/\text{LT}(I_0)$, and the shelling of the facets of the simplicial complex $\Delta_0$ obtained in [20]. With some effort we are led to an initial formula for the Hilbert series of $R/I_0$, which is enormously complicated and involves multiple sums of products of binomials in the same vein as the right side of (3). But we persist with the combinatorics and are eventually rewarded with the main result of this paper. Namely, just like the multiplicity, the Hilbert series of $R/I_0$ is precisely the square of the Hilbert series of the base determinantal variety $Z_{2,n}^{m,n}$. As a corollary of this, we are able to determine the $a$-invariant of $R/I_0$ and the Hilbert series of its graded canonical module. Moreover, we show that, as in the case of classical determinantal varieties, $Z_0$ is Gorenstein if and only if $m = n$.

The proofs given here are completely elementary, but highly combinatorial and rather intricate. Heuristically, it appears to us that up to some flat deformation (such as the Gröbner deformation of $I_0$ to $\text{LT}(I_0)$), which preserves the Hilbert series, the coordinate ring of the principal component (suitably deformed) should look like the tensor product of the coordinate ring of the base (similarly deformed) with itself. (This would reflect the fact that at the smooth points, the base variety locally looks like its tangent space.) It would follow then that the Hilbert series of the principal component is the square of that of $Z_{2,n}^{m,n}$. We emphasize that this is only heuristics (with all of its ever-present dangers); nevertheless, we suspect that analogous results relating the Hilbert series of the principal component to that of the base scheme should hold more generally for all $Z_{n,k}^{m,n}$, and possibly also for jet schemes over a wider class of affine base schemes. We do not know how to prove this, and leave it open for investigation.

2. Binomials and Lattice Paths

In this section we collect some preliminaries concerning binomial coefficients, alterations of summations, and lattice paths. These will be useful in the sequel.
2.1. Binomials. To begin with, let us recall that the binomial coefficient \( \binom{a}{s} \) is defined for any integer parameters \( s, a \) (and with the standard convention that the empty product is taken as 1) as follows:

\[
\binom{a}{s} = \begin{cases} 
\frac{s(s-1)\cdots(s-a+1)}{a!} & \text{if } a \geq 0, \\
0 & \text{if } a < 0.
\end{cases}
\]

In fact, this definition makes sense not only for any \( s \in \mathbb{Z} \) but also for \( s \) in any overring of \( \mathbb{Z} \) and in particular, \( s \) can be an indeterminate over \( \mathbb{Q} \) in which case \( \binom{a}{s} \) is a polynomial in \( s \) of degree \( a \), provided \( a \geq 0 \). Now let \( s, a \in \mathbb{Z} \). Note that

\[
\binom{a}{s} = 0 \iff \text{either } a < 0 \text{ or } a > s \geq 0.
\]

(4)

One has to be careful with the validity of some of the familiar identities; for example,

\[
\binom{s}{a} = \binom{s}{s-a} \iff \text{either } s \geq 0 \text{ or } s < a < 0,
\]

(5)

whereas some standard identities such as the Pascal triangle identity or its alternative equivalent version below are valid for arbitrary integer parameters:

\[
\binom{s}{a-1} + \binom{s}{a} = \binom{s+1}{a} \quad \text{and} \quad \binom{s+a}{a} + \binom{s+a}{a+1} = \binom{s+a+1}{a+1}.
\]

(6)

The equivalence of the two identities above follows from the simple fact below, which is also valid for arbitrary integer parameters:

\[
\binom{s+a}{a} = (-1)^a \binom{-s-1}{a}, \quad \text{that is,} \quad \binom{s}{a} = (-1)^a \binom{a-s-1}{a}.
\]

(7)

We now record some basic facts, which are often used in later sections. Proofs are easy and are briefly outlined for the sake of completeness.

**Lemma 1.** For any \( e, s, t \in \mathbb{Z} \) with \( s \leq t \), we have

\[
\sum_{s<d \leq t} \binom{d}{e} = \binom{t+1}{e+1} - \binom{s+1}{e+1}.
\]

Proof. Induct on \( t-s \), using the first identity in (6) to rewrite \( \binom{t+1}{e+1} \). \( \square \)

The following result is a version of the so-called Chu-Vandermonde identity.

**Lemma 2.** For any \( s, t, \alpha, \beta \in \mathbb{Z} \), we have

\[
\sum_{j \in \mathbb{Z}} \binom{s+j}{\alpha+j} \binom{t}{\beta-j} = \binom{s+t}{\alpha+\beta},
\]

(8)

and

\[
\sum_{j \in \mathbb{Z}} \binom{s+a+j}{\alpha+j} \binom{t+\beta-j}{\beta-j} = \binom{s+t+a+\beta+1}{\alpha+\beta+1},
\]

(9)

where in view of (4), the summation on the left in (8) as well as in (9) is essentially finite in the sense that all except finitely many summands are zero.

Proof. Let \( X \) be an indeterminate over \( \mathbb{Q} \). Use the binomial theorem, namely, \( (1+X)^d = \sum_{i=0}^{\infty} \binom{d}{i} X^i \), which is valid in the formal power series ring \( \mathbb{Q}[[X]] \) for any \( d \in \mathbb{Z} \), and compare the coefficients of \( X^{\alpha+\beta} \) on the two sides of the identity \( (1+X)^s (1+X)^t = (1+X)^{s+t} \) to obtain (8). Now (8) and (7) imply (9). \( \square \)
2.2. Alterations of Summations. As in (8) and (9) above, we will often deal with summations that are essentially finite, by which we mean that the parameters in the sum range over an infinite set, but the summation is zero for all except finitely many values of parameters, and so the summation is, in fact, finite. It is, however, very useful that the parameters range freely over a seemingly infinite set so that useful alterations such as the ones listed below can be readily made. These are too obvious to be stated as lemmas and proved formally. But for ease of reference, we record below some elementary transformations of essentially finite summations. In what follows, $f : \mathbb{Z}^2 \to \mathbb{Q}$ will denote a rational-valued function of two integer parameters with the property that the support of $f$, namely, the set $\{(s_1, s_2) \in \mathbb{Z}^2 : f(s_1, s_2) \neq 0\}$ is finite or more generally, it is diagonally finite, i.e., for each $k \in \mathbb{Z}$, the set $\{(s_1, s_2) \in \mathbb{Z}^2 : s_1 + s_2 = k$ and $f(s_1, s_2) \neq 0\}$ is finite. In this case, for any $\nu \in \mathbb{Z}$ and any $\alpha, \beta \in \mathbb{Z}$ such that $\alpha + \beta = \nu$, we have

$$\sum_{s_1 + s_2 = k - \nu} f(s_1, s_2) = \sum_{t_1 + t_2 = k} f(t_1 - \alpha, t_2 - \beta),$$

where writing $s_1 + s_2 = k - \nu$ below the first summation indicates that the sum is over all $(s_1, s_2) \in \mathbb{Z}^2$ satisfying $s_1 + s_2 = k - \nu$. A similar meaning applies for the second summation and in fact, for all such summations appearing in the sequel. Since the “diagonal condition” $t_1 + t_2 = k$ is symmetric, we also have

$$\sum_{t_1 + t_2 = k} f(t_1, t_2) = \sum_{t_1 + t_2 = k} f(t_2, t_1).$$

Thus, for example, using (10) and (11), we find

$$\sum_{t_1 + t_2 = k} f(t_1, t_2) = \sum_{t_1 + t_2 = k} f(t_2 + 1, t_1 - 1) = \sum_{t_1 + t_2 = k} f(t_1 + 1, t_2 - 1).$$

2.3. Lattice Paths. Let $A = (a, a')$ and $E = (e, e')$ be points in the integer lattice $\mathbb{Z}^2$. By a lattice path from $A$ to $E$ we mean a finite sequence $L = (P_0, P_1, \ldots, P_t)$ of points in $\mathbb{Z}^2$ with $P_0 = A$, $P_t = E$ and $P_i - P_{i-1} = (1, 0)$ or $(0, 1)$ for $i = 1, \ldots, t$.

The lattice path $L$ can and will be identified with its point set $\{P_j : 0 \leq j \leq t\}$; indeed, $L$ is obtained by simply arranging the elements of this set in a lexicographic order. The point $A = P_0$ is called the initial point of $L$ while $E = P_t$ is called the end point of $L$. We say that a point $P_j$ is a NE-turn of the lattice path $L$ if $0 < j < t$ and $P_j - P_{j-1} = (0, 1)$ while $P_{j+1} - P_j = (1, 0)$. Note that a lattice path is also determined by its NE-turns.

In more intuitive terms, a lattice path consists of vertical or horizontal steps of length 1, and a NE-turn is simply a North-East turn. For example, a lattice path from $A = (1, 1)$ to $E = (4, 5)$ may be depicted as in Figure 1, and it may be noted that the points $(1, 2)$ and $(2, 4)$ are its NE-turns.

![Figure 1. A lattice path from $A = (1, 1)$ to $E = (4, 5)$](image-url)
If we let $\mathcal{P}(A \rightarrow E)$ denote the set of lattice paths from $A = (a, a')$ to $E = (e, e')$ and for any $k \in \mathbb{Z}$, let $\mathcal{P}_k(A \rightarrow E)$ denote the subset of $\mathcal{P}(A \rightarrow E)$ consisting of lattice paths with exactly $k$ NE-turns, then it is easily seen that

$$|\mathcal{P}(A \rightarrow E)| = \binom{e - a + e' - a'}{e - a} \quad \text{and} \quad |\mathcal{P}_k(A \rightarrow E)| = \binom{e - a}{k} \binom{e' - a'}{k}.$$ (12)

where as usual, for a finite set $\mathcal{P}$, we denote by $|\mathcal{P}|$ the cardinality of $\mathcal{P}$. Given any two $d$-tuples $A = (A_1, \ldots, A_d)$ and $E = (E_1, \ldots, E_d)$ of points in $\mathbb{Z}^2$, by a lattice path from $A$ to $E$ we mean a $d$-tuple $L = (L_1, \ldots, L_d)$, where $L_r$ is a lattice path from $A_r$ to $E_r$, for $1 \leq r \leq d$. We call $L$ to be nonintersecting if no two of the paths $L_1, \ldots, L_d$ have a point in common. We say that $L$ has $k$ NE-turns if the total number of NE-turns in the $d$ paths $L_1, \ldots, L_d$ is $k$. The set of nonintersecting lattice paths from $A = (A_1, \ldots, A_d)$ to $E = (E_1, \ldots, E_d)$ will be denoted by $\mathcal{P}(A \rightarrow E)$ or simply by $\mathcal{P}(A \rightarrow E)$, and its subset consisting of nonintersecting lattice paths with exactly $k$ NE-turns will be denoted by $\mathcal{P}_k(A \rightarrow E)$ or simply by $\mathcal{P}_k(A \rightarrow E)$.

**Proposition 3.** Let $d$ be a positive integer and let $A_r = (a_r, a'_r)$ and $E_r = (e_r, e'_r)$, $r = 1, \ldots, d$, be points in $\mathbb{Z}^2$. Also let $A = (A_1, \ldots, A_d)$ and $E = (E_1, \ldots, E_d)$.

(i) Suppose

$$a_1 \leq \cdots \leq a_d, \quad e_1 \leq \cdots \leq e_d \quad \text{and} \quad a'_1 \geq \cdots \geq a'_d, \quad e'_1 \geq \cdots \geq e'_d. $$

Then the number of nonintersecting lattice paths from $A$ to $E$ is equal to

$$\det \left( \frac{e_j - a_i + e'_j - a'_i}{e_j - a_i} \right)_{1 \leq i, j \leq d}.$$ (13)

(ii) Let $k \in \mathbb{Z}$ and suppose

$$a_1 \leq \cdots \leq a_d, \quad e_1 < \cdots < e_d \quad \text{and} \quad a'_1 > \cdots > a'_d, \quad e'_1 \geq \cdots \geq e'_d. $$

Then the number of nonintersecting lattice paths from $A$ to $E$ with exactly $k$ NE-turns is equal to

$$\sum_{k_1 + \cdots + k_d = k} \det \left( \frac{e_j - a_i + i - j}{k_i + i - j} \frac{e'_j - a'_i - i + j}{k_i} \right)_{1 \leq i, j \leq d}.$$ (14)

Part (i) of the above proposition is due to Gessel and Viennot [12, Thm. 1] although some of the ideas can be traced back to Chaundy [7], Karlin and McGregor [21], and Lindström [27]. The statement here is a little more general than that of [12] and a proof can be found, for example, in [15 § 3] or [25 § 2.2]. Part (ii) was proved independently by Modak [29], Krattenthaler [24] and Kulkarni [26] (see also [14]), although the hypothesis in [29] and [26] on the coordinates of the initial and the end points is slightly more restrictive than in (ii) above where we follow [24] Thm. 1]. The following consequence is frequently used in Section 4.

**Corollary 4.** For any $a, b, c, d, s \in \mathbb{Z}$ with $a < c$ and $b \geq d$, the cardinality of $\mathcal{P}_s ((1, 2) \rightarrow (a, b), (1, 1) \rightarrow (c, d))$ is given by

$$\sum_{s_1 + s_2 = s} \binom{a - 1}{s_1} \binom{b - 2}{s_2} \binom{c - 1}{s_1} \binom{d - 1}{s_2} - \binom{a}{s_2 + 1} \binom{b - 2}{s_1 - 1} \binom{c - 2}{s_2} \binom{d - 1}{s_2}. $$

**Proof.** This is just a special case of part (ii) of Proposition 3. \hfill \Box
3. Multiplicity

As in the Introduction, we fix in the remainder of this paper an algebraically closed field \( \mathbb{F} \) and integers \( m, n \) with \( 2 < m \leq n \). Also let \( x_{i,j}, y_{i,j} \), where \( 1 \leq i \leq m, 1 \leq j \leq n \), be independent indeterminates over \( \mathbb{F} \). Denote by \( V_\varepsilon \) the set \( \{ x_{i,j} : 1 \leq i \leq m \) and \( 1 \leq j \leq n \} \) of the “\( x \)-variables”, and by \( V_y \) a similar set of the “\( y \)-variables”. Let \( V = V_\varepsilon \cup V_y \) and let \( R = \mathbb{F}[V] \) be the corresponding polynomial ring in \( 2mn \) variables; also, let \( R_x = \mathbb{F}[V_\varepsilon] \) and \( R_y = \mathbb{F}[V_y] \) be the corresponding polynomial rings in \( mn \) variables. By the support of a monomial \( F \) in \( R \), denoted \( \text{supp}(F) \), we mean the subset of \( V \) consisting of the variables appearing in \( F \). Clearly a monomial \( F \) in \( R \) can be uniquely written as

\[
F = F_x F_y \quad \text{where} \quad F_x, F_y \text{ are monomials with } F_x \in R_x \text{ and } F_y \in R_y
\]

and moreover, \( F \) is squarefree if and only if both \( F_x \) and \( F_y \) are squarefree. Note that squarefree monomials can be identified with their supports, and in particular, faces of a simplicial complex \( \Delta \) with vertex set \( V \) can be viewed as squarefree monomials in \( R \). With this in view, we may not distinguish between a squarefree monomial and its support, and we may sometimes write \( x_{i,j} \in G \) rather than \( x_{i,j} \mid G \) when \( G \) is a squarefree monomial in \( R \) and \( x_{i,j} \) is a variable appearing in it. A monomial \( G \) in \( R_x \) will be called a lattice path monomial in \( R_x \) if there is a positive integer \( t \) and variables \( x_{i_1,j_1}, \ldots, x_{i_t,j_t} \) in \( V_\varepsilon \) such that

\[
G = \prod_{s=1}^{t} x_{i_s,j_s} \quad \text{with} \quad (i_s - i_{s-1}, j_s - j_{s-1}) = (1,0) \text{ or } (0,1) \text{ for } 1 < s \leq t.
\]

In this case \( G \) is called a lattice path monomial from \( x_{i_1,j_1} \) to \( x_{i_t,j_t} \), and we will refer to \( x_{i_t,j_t} \) as the leader of \( G \) and denote it by \( \mu(G) \). Note that \( \mu(G) = x_{i_t,j_t} \) depends only on \( G \) (and not on the given ordering of the variables appearing in it) since \((i_1,j_1)\) is lexicographically the least among the pairs \((i,j)\) for which \( x_{i,j} \in \text{supp}(G) \).

A variable \( x_{i_1,j_1} \in \text{supp}(G) \) will be called an ES-turn of \( G \) if \( 1 < s < t \), \( i_s = i_{s-1} \), and \( j_s = j_{s-1} \). Analogously, a variable \( x_{i_t,j_s} \in \text{supp}(G) \) will be called a SE-turn of \( G \) if \( 1 < s < t \), \( j_s = j_{s-1} \), and \( i_s = i_{s+1} \). Moreover, we will call a variable \( x_{i_1,j_s} \in \text{supp}(G) \) the midpoint of a segment in \( G \) if \( 1 < s < t \) and either \( i_{s-1} = i_s = i_{s+1} \) (horizontal segment) or \( j_{s-1} = j_s = j_{s+1} \) (vertical segment). It may be noted that a variable \( x_{i_1,j_s} \) with \( 1 < s < t \) is either an ES-turn or a SE-turn or the midpoint of a segment in \( G \).

Evidently, lattice path monomials in \( R_x \) correspond to lattice paths in the sense of \[2.3\] if we turn the \( m \times n \) rectangular matrix \((x_{i,j})\) to the left by 90° and identify the variable \( x_{i,j} \) with the lattice point \((i,j)\). In this way leaders correspond to initial points while ES-turns correspond to NE-turns. Lattice path monomials in \( R_y \) together with their leaders, ES-turns, SE-turns, and midpoints of segments are similarly defined (and similarly identified with lattice paths in the sense of \[2.3\]).

We have noted in the Introduction that a Gröbner basis (w.r.t. reverse lexicographic order on monomials with the \( 2mn \) variables arranged suitably) of the ideal \( I \) of the variety \( Z_{2,m,n}^{m,n} \) of first-order jets over \( Z_{2,m,n}^{m,n} \) as well as of the ideal \( I_0 \) of the principal component \( Z_0 \) of \( Z_{2,m,n}^{m,n} \) was determined by Košir and Sethuraman \[23\]. As a consequence, one can write down the generators of the leading term ideal of \( Z_0 \) (cf. \[20\] Prop. 1.1), say \( \text{LT}(I_0) \), and deduce that \( R/\text{LT}(I_0) \) is the Stanley-Reisner ring of a simplicial complex \( \Delta_0 \) with \( V \) as its set of vertices. A precise description of the facets of \( \Delta_0 \) was given by Jonov \[20\] § 2 and we recall it below.

**Proposition 5.** A squarefree monomial \( F \), decomposed as in \[15\] above, is a facet of \( \Delta_0 \) if and only if there is a unique \((i,j) \in \mathbb{Z}^2 \) with \( 1 \leq i \leq m, 1 \leq j \leq n \) such that \((i,j) \neq (m,n) \) and \( F_x \) is a lattice path monomial from \( x_{i,j} \) to \( x_{m,n} \), whereas
Figure 2. Lattice path monomials $F_x$ and $F_y = F^U_y F^L_y$ in Proposition 5

$F_y = F^U_y F^L_y$ where $F^U_y$ is a lattice path monomial from $y_{1,1}$ to $y_{i,n}$, $F^L_y$ is a lattice path monomial from $y_{2,1}$ to $y_{m,j}$, and the supports of $F^U_y$ and $F^L_y$ are disjoint.

The lattice path monomials $F_x$ and $F_y = F^U_y F^L_y$ are illustrated in Figure 2 by the corresponding “paths” in rectangular matrices.

Using Proposition 5 together with the first identity in (12) and part (i) of Proposition 3, Jonov showed that the simplicial complex $\Delta_0$ is pure (i.e., all its facets have the same dimension) and deduced the dimension and the formula stated in the Introduction for the multiplicity of the coordinate ring $R/\mathcal{I}_0$ of $Z_0$.

Corollary 6. The (Krull) dimension of $R/\mathcal{I}_0$ is $2(m+n-1)$ and the multiplicity of $R/\mathcal{I}_0$ is given by (3).

Now here is the pretty result about the multiplicity that was alluded to in the Introduction. Namely, that the formula (3) admits a remarkable simplification.

Theorem 7. The multiplicity of $R/\mathcal{I}_0$ is given by

$$e(R/\mathcal{I}_0) = \left(\frac{m+n-2}{m-1}\right)^2.$$  \hfill (17)

Proof. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\Delta_{i,j}$ denote the $2 \times 2$ determinant in (3). Observe that if $(i,j) = (m,n)$, then $\Delta_{i,j} = 0$. Thus, by expanding this determinant and rearranging the summands, we can write

$$e(R/\mathcal{I}_0) = \sum_{i=1}^{m} \left(\begin{array}{c} i+n-2 \\ i-1 \end{array}\right) \sum_{j=1}^{n} S_{i,j} - \sum_{i=1}^{m} \left(\begin{array}{c} i+n-3 \\ i-2 \end{array}\right) \sum_{j=1}^{n} T_{i,j},$$

where for $1 \leq i \leq m$ and $1 \leq j \leq n$, we have put

$$S_{i,j} = \left(\begin{array}{c} m+n-i-j \\ m-i \end{array}\right) \left(\begin{array}{c} m+j-3 \\ m-2 \end{array}\right) \text{ and } T_{i,j} = \left(\begin{array}{c} m+n-i-j \\ m-i \end{array}\right) \left(\begin{array}{c} m+j-2 \\ m-1 \end{array}\right).$$

Rewriting $S_{i,j}$ using (5) and then noting that the resulting product is zero if $j < 1$ or $j > n$, thanks to (4), we see from equation (9) in Lemma 2 that

$$\sum_{j=1}^{n} S_{i,j} = \sum_{j} \left(\begin{array}{c} m+n-i-j \\ n-j \end{array}\right) \left(\begin{array}{c} m+j-3 \\ j-1 \end{array}\right) = \left(\begin{array}{c} 2m+n+i-2 \\ n-1 \end{array}\right)$$

for each $i = 1, \ldots, m$. In a similar manner,

$$\sum_{j=1}^{n} T_{i,j} = \sum_{j} \left(\begin{array}{c} m+n-i-j \\ n-j \end{array}\right) \left(\begin{array}{c} m+j-2 \\ j-1 \end{array}\right) = \left(\begin{array}{c} 2m+n+i-1 \\ n-1 \end{array}\right)$$

for each $i = 1, \ldots, m$.
for each \( i = 1, \ldots, m \). It follows that \( e(R/I_0) \) is given by the telescoping sum

\[
e(R/I_0) = \sum_{i=1}^{m} (a_i - a_{i-1}) \quad \text{where} \quad a_i := \binom{i + n - 2}{i - 1} \binom{2m + n - i - 2}{n - 1} \quad (0 \leq i \leq m).
\]

Since \( a_0 = 0 \) and \( a_m = \binom{m+n-2}{m-1}^2 \), we obtain the desired result. \( \square \)

It may be noted that in view of [1] and [17], the multiplicity of the principal component \( Z_0 \) is precisely the square of the multiplicity of the base variety \( Z_2^{m,n} \).

### 4. Hilbert series

Let us begin by recalling that a shelling of a pure simplicial complex \( \Delta \) is a linear ordering \( F_1, \ldots, F_e \) of its facets such that for all positive integers \( i, j \) with \( j < i \leq e \), there exist some \( v \in F_i \setminus F_j \) and some positive integer \( k < i \) such that \( F_i \setminus F_k = \{ v \} \). Given such a shelling and any \( t \in \{ 1, \ldots, e \} \), we let

\[
c(F_t) = \{ v \in F_t : \text{there exists } s < t \text{ such that } F_t \setminus F_s = \{ v \} \}.
\]

Elements of \( c(F_t) \) will be referred to as the corners of \( F_t \). It may be noted that \( c(F_t) \) is nonempty if and only if \( t > 1 \). Recall also that a simplicial complex \( \Delta \) is said to be shellable if it is pure and it has a shelling. The following result is well-known (cf. [3 Thm. 6.3]).

**Proposition 8.** Let \( \Delta \) be a shellable simplicial complex and let \( R_{\Delta} \) denote its Stanley-Reisner ring. Then

(i) \( R_\Delta \) is Cohen-Macaulay and its (Krull) dimension \( \dim R_\Delta \) is \( 1 + \dim \Delta \).

(ii) Suppose \( d = \dim R_\Delta \) and \( F_1, \ldots, F_e \) is a shelling of \( \Delta \). Then the Hilbert series of \( R_\Delta \) is given by

\[
\sum_{j \geq 0} \frac{b_j z^j}{(1 - z)^d} \quad \text{where} \quad h_j = |\{ t \in \{ 1, \ldots, e \} : |c(F_t)| = j \}| \text{ for } j \geq 0.
\]

In [20], Jonov showed that the simplicial complex \( \Delta_0 \) mentioned in the previous section is shellable and concluded using part (i) of Proposition 8 that the coordinate ring of \( R/I_0 \) of the principal component \( Z_0 \) of \( Z_2^{m,n} \) is Cohen-Macaulay. We shall now proceed to use part (ii) of Proposition 5 to determine the Hilbert series of \( R/I_0 \). We will use the notation and terminology introduced at the beginning of Section 3. Further, we introduce the following “anti-lexicographic” linear order on \( V_x \), i.e., on the \( x \)-variables. For any \( x_{a,b}, x_{c,d} \in V_x \), define

\[
x_{a,b} < x_{c,d} \iff \text{either } a > c \text{ or } a = c \text{ and } b > d.
\]

Given a lattice path monomial \( G \) as in [16], the spread of \( G \), denoted \( \text{sp}(G) \), is the set of variables that are on or below the corresponding lattice path; more precisely,

\[
\text{sp}(G) = \{ x_{a,b} : i_s \leq a \leq m \text{ and } 1 \leq b \leq j_s \text{ for some } s = 1, \ldots, t \}.
\]

The notion of spread is defined for lattice path monomials in \( R_y \) in exactly the same manner. It may be observed that if \( G, H \) are lattice path monomials (both in \( R_x \) or both in \( R_y \)), then the condition \( \text{sp}(G) \subseteq \text{sp}(H) \) means, roughly speaking, that \( H \) is to the right of \( G \); moreover, if \( \mu(G) = \mu(H) \) and \( \text{sp}(G) = \text{sp}(H) \), then we must have \( G = H \).

Notice that the lattice path monomials \( F_{y}^{ij} \) and \( F_{x}^{k} \) of Proposition 5 have the property that \( \text{sp}(F_{y}^{ij}) \subseteq \text{sp}(F_{x}^{k}) \).

Following [20], we now define a partial order on the facets of \( \Delta_0 \).
Definition 9. For any facets $P, Q$ of $\Delta_0$ with decompositions $P = P_x P_y^L P_y^U$ and $Q = Q_x Q_y^L Q_y^U$ as in Proposition 3, define $P < Q$ if one of the following four conditions hold: (i) $\mu(P_x) < \mu(Q_x)$, (ii) $\mu(P_x) = \mu(Q_x)$ and $\text{sp}(P_x) \subset \text{sp}(Q_x)$, (iii) $P_x = Q_x$ and $\text{sp}(P_y^L) \subset \text{sp}(Q_y^L)$, (iv) $P_x = Q_x$, $P_y^U = Q_y^U$, and $\text{sp}(P_y^L) \subset \text{sp}(Q_y^L)$.

The next result is a consequence of [20, Thm. 3.2] and its proof.

Proposition 10. The relation $<$ in Definition 9 defines a partial order and any extension of it to a total order on the facets of $\Delta_0$ gives a shelling of $\Delta_0$.

The terminology of ES-turns can be extended from lattice path monomials to facets of $\Delta_0$ as follows. For any facet $F$ of $\Delta_0$ having a decomposition $F = F_x F_y^L F_y^U$ as in Proposition 3 by an ES-turn of $F$ we shall mean an ES-turn of either $F_x$ or $F_y^L$ or $F_y^U$. It turns out that the corners of a facet of $\Delta_0$ are essentially its ES-turns or the leader of its $x$-component. There are, however, some subtleties involved and a precise relation is given below.

Lemma 11. Let $F$ be a facet of $\Delta_0$ and $F = F_x F_y^L F_y^U$ be its decomposition as in Proposition 3. Also let $v \in V$ be a vertex of $\Delta_0$. Then we have the following.

(i) If $v \in c(F)$, then either $v = \mu(F_x)$ or $v$ is an ES-turn of $F$. In particular, $x_{m,n} \notin c(F)$ and $y_{m,n} \notin c(F)$.

(ii) If $\mu(F_x) = x_{i,j}$ with $(i, j) \neq (m, n - 1)$, then $\mu(F_x) \in c(F)$. Moreover, $x_{m,n-1} \notin c(F)$.

(iii) If $v$ is an ES-turn of $F_x$, then $v \in c(F)$.

(iv) If $v$ is an ES-turn of $F_y^L$ or of $F_y^U$, then $v \in c(F)$, except when $v$ is an ES-turn of $F_y^L$ such that $v = y_{1,2}$ or when $v$ is an ES-turn of $F_y^U$ such that $v = y_{m-1,j+1}$ and $\mu(F_x) = x_{m,j}$ for some $j < n$.

Proof. (i) Let $P = P_x F_y^L F_y^U$ be a facet of $\Delta_0$ such that $F \setminus F = \{v\}$ and $F > P$. The latter implies that one of the four possibilities in Definition 9 must arise. First, suppose $\mu(P_x) < \mu(F_x)$. Then $\mu(F_x)$ is a vertex of $F$ that is smaller than $\mu(P_x)$ in the standard lexicographic order, and hence $\mu(F_x) \notin P_x$; consequently, $v = \mu(F_x)$, and we are done. Now suppose $\mu(P_x) = \mu(F_x)$ and $\text{sp}(P_x) \subset \text{sp}(F_x)$. Then $P_x \neq F_x$ and hence $F_x \setminus P_x = \{v\}$. Note that since $\mu(P_x)$ and $x_{m,n}$ are in $P_x$, the vertex $v$ is an ES-turn, SE-turn, or the midpoint of a segment of $F_x$. In case it is the midpoint of a segment of $F_x$, the other two vertices in that segment must be in $P_x$ and since $P_x$ is a lattice path monomial, we see that $v \in P_x$, which is a contradiction. Also if $v = x_{k,l}$ (say) is a SE-turn of $F_x$, then $x_{k-1,l}$ and $x_{k,l+1}$ must be in $F_x$ and hence in $P_x$. But then $P_x$ must contain $x_{k-1,l+1}$, which is a contradiction since $x_{k-1,l+1} \notin \text{sp}(F_x)$. It follows that $v$ is an ES-turn of $F_x$. Next, suppose $P_x = F_x$ and $\text{sp}(P_y^L) \subset \text{sp}(F_y^L)$. Then $F_y^L \setminus P_y^L = \{v\}$. Since $\mu(P_x) = \mu(F_x)$, in view of Proposition 3 we see that the initial and the terminal variables of $P_y^L$ and $F_y^L$ coincide and so $v$ is neither of these. Arguing as in the preceding case, we can rule out the possibilities that $v$ is an ES-turn or the midpoint of a segment of $F_y^L$. Hence $v$ is an ES-turn of $F_y^U$. In a similar manner, we see that if $P_x = F_x$, $P_y^L = F_y^L$, and $\text{sp}(P_y^U) \subset \text{sp}(F_y^U)$, then $v$ is an ES-turn of $F_y^U$. Thus (i) is proved.

(ii) Let $\mu(F_x) = x_{i,j}$ with $(i, j) \neq (m, n - 1)$. Then either $x_{i,j+1} \in F_x$ or $x_{i,j+1} \in F_x$. First, suppose $x_{i,j+1} \in F_x$. We define a new facet $P$ as follows. Let $P_x = F_x \setminus \{x_{i,j}\}$, and $P_y^L = F_y^L \cup \{y_{m,j+1}\}$. To define $P_y^U$, we take $P_y^U = F_y^U$ and $y_{m,j+1} \notin F_y^U$. If $y_{m,j+1} \notin F_y^U$, then this must mean that $i = m$, and hence $j < n - 1$. We therefore define $P_y^U = (F_y^L \cup F_y^U) \setminus \{y_{m-1,j+2}\}$. Observe that $P = P_x P_y^L P_y^U$ is a facet of $\Delta_0$ and since $\mu(P_x) < \mu(F_x)$, we have $P < F$. It follows that $\mu(F_x) \in c(F)$. Next, suppose $x_{i+1,j} \in F_x$. We first assume that $(i, j) \neq (m - 1, n)$. Now define a new facet $P$ as follows. First, we let $P_x = F_x \setminus \{x_{i,j}\}$. If $y_{i+1,n} \notin F_y^L$, then we
let \( P^y_x = F^y_x \cup \{y_{i+1,n}\} \) and \( P^y_k = F^k_y \). If \( y_{i+1,n} \in F^k_y \), then \( j \) must equal \( n \). If now \( i \leq m - 2 \), then we let \( P^y_x = F^y_x \setminus \{y_{i+1,n}\} \cup \{y_{i+2,n-1}\} \). We are left with the special case \( i = m - 1, j = n \). Here we let \( P^y_x = \{x_{m,n-1}, x_{m,n}\}, \) \( P^y_u = F^y_u \cup \{y_{m,n}\} \), and \( P^y_k = F^k_y \setminus \{y_{m,n}\} \). In all three cases, it is easy to verify that \( P = P^y_x P^y_u P^y_k \) is a facet of \( \Delta_0 \) such that \( F \setminus P = \{x_{i,j}\} \) and \( P < F \). Consequently, \( \mu(F_x) \in c(F) \). Finally, we show that \( x_{m,n-1} \notin c(F) \). Assume, on the contrary, that there is a facet \( P \) of \( \Delta_0 \) such that \( F \setminus P = \{x_{m,n-1}\} \). By (i) above, \( \mu(F) = x_{m,n-1} \), because there can be no ES-turn at \( x_{m,n-1} \). In view of Proposition \( \Box \) \( P \) must contain at least one variable other than \( x_{m,n-1} \), and since \( x_{m,n-1} \notin c(P) \), it follows that \( x_{m,n-1} \notin P \). This forces \( \mu(F_x) < \mu(P_x) \), which violates the fact that \( P < F \). Thus (ii) is proved.

(iii) Let \( v = x_{k,l} \) be an ES-turn of \( F_x \). Define \( P_x = F_x \setminus \{x_{k,l}\} \cup \{x_{k+1,l-1}\} \) and \( P_y = F_y \). It is clear that \( P = P_x P_y \) is a facet of \( \Delta_0 \) such that \( P < F \) and \( F \setminus P = \{v\} \). This proves (iii).

(iv) First, suppose \( v = y_{k,l} \) is an ES-turn of \( F^y_x \). Then \( k < m \) and \( l > 1 \). Define \( P_x = F_x \setminus \{y_{k,l}\} \cup \{y_{k+1,l-1}\} \). It is easy to see that \( \mu(F) = x_{m,n-1} \), \( v = y_{k,l} \), and \( P^y_x = F^y_x \setminus \{y_{k,l}\} \cup \{y_{k+1,l-1}\} \), whereas in case \( y_{k+1,l-1} \) is in \( F^y_y \) and also \( k < m - 1 \) and \( l > 2 \), we define \( P_x = F_x \setminus \{y_{k,l}\} \cup \{y_{k+1,l-1}\} \), and \( P^y_k = F^k_y \setminus \{y_{k+1,l-1}\} \cup \{y_{k+2,l-2}\} \). We verify that in both the cases, \( P = P_x P_y P^y_k \) is a facet of \( \Delta_0 \) such that \( P < F \) and \( F \setminus P = \{v\} \).

When \( l = 2 \), it is easy to see that \( v = y_{k,2} \) can be an ES-turn of \( F^y_x \) only when \( k = 1 \) lest \( F^y_x \) and \( F^y_k \) intersect at \( y_{k,1} \). We now show that \( y_{1,2} \) is not a corner of \( F \).

Suppose that \( P = P_x P_y P^y_k \) is a facet of \( \Delta_0 \) such that \( F \setminus P = \{v\}, v = y_{1,2} \) and \( F > P \). By Proposition \( \Box \) \( P_y \) must start at \( y_{1,1} \) and \( P^y_k \) must start at \( y_{2,1} \). For \( P_y \) to avoid \( v = y_{1,2} \), it must be the case that \( P_y^k \) contains \( y_{2,1} \). This contradicts the fact that \( P_x^y \) and \( F^y \) do not intersect.

We are left with the situation where \( k = m - 1 \) and \( v = y_{k,l} \) is an ES-turn of \( F^y_x \) and moreover, \( y_{m,l-1} \in F^y_y \). Now since \( F^y_y \) has an ES-turn at \( y_{m-1,l} \), we see that \( l > 1 \) and both \( y_{m-1,l-1} \) and \( y_{m,l} \) are in \( F^y_y \). In particular, \( y_{m,1} \notin F^y_y \) and since \( y_{m,l-1} \in F^y_y \), in view of Proposition \( \Box \) it follows that \( F^y_y \) ends at \( y_{m-1,l} \), while \( F^y_y \) ends at \( y_{m,n} \) and also that \( \mu(F_x) = x_{m,n-1} \). Now if there were a face \( P = P_x P_y P^y_k \) of \( \Delta_0 \) such that \( F \setminus P = \{v\} \) and \( F > P \), then \( P_x = F_x \) and \( P^y_k = F^k_y \), whereas \( F^y_y \setminus P^y_y = \{y_{m-1,l}\} \). But then \( P^y_y \) is a lattice path monomial that contains both \( y_{m-1,l-1} \) and \( y_{m,l} \) and does not contain \( y_{m-1,l-1} \); so it must contain \( y_{m-1,l} \). This is a contradiction since \( y_{m-1,l} \in F^y_y \setminus P^y_y \) and the monomials \( P^y_y \) and \( P^y_k \) have no variable in common. This completes the proof.

For any integers \( i,j,k \) with \( k \geq 0, 1 \leq i \leq m \) and \( 1 \leq j \leq n \), we define \( C^i_{j,k} \) to be the number of facets \( F = F_x F_y \) of \( \Delta_0 \) such that \( \mu(F) = x_{i,j} \) and \( F \) has exactly \( k \) ES-turns that are in \( c(F) \). Now a useful consequence of Lemma \( \Box \) is the following.

**Corollary 12.** The Hilbert series of the coordinate ring \( R/I_0 \) of the principal component \( Z_0 \) of \( \mathcal{Z}^{m,n} \) is given by

\[
\frac{\sum_{k \geq 0} h_k z^k}{(1 - z)^2(m+n-1)},
\]  

(18)
it follows from Corollary 6 and part (ii) of Proposition 8 that the Hilbert series of a simplicial complex such that

\[ \text{LT}(I) \]

is leading term ideal of \( I \). It is well-known that the (Krull) dimension as well as the Hilbert series of \( R/\text{LT}(I) \) is precisely the simplicial complex such that \( R/\text{LT}(I) \) is the Stanley-Reisner ring of \( \Delta_0 \). Thus, it follows from Corollary 6 and part (ii) of Proposition 8 that the Hilbert series of \( R/\Delta_0 \) is given by (18), where \( h_0 = 1 \) and for \( k \geq 1 \),

\[ h_k = C^k_{m,n-1} + \sum_{(i,j) \neq (m,n-1)} C^k_{i,j} - 1, \quad \text{for } 1 \leq j \leq n \text{ with } (i,j) \neq (m,n-1) \text{ and } (i,j) \neq (m,n). \]

where the sum is over all pairs \((i,j)\) of integers satisfying \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) with \((i,j) \neq (m,n-1)\) and \((i,j) \neq (m,n)\).

Proof. It is well-known that the (Krull) dimension as well as the Hilbert series of \( R/\Delta_0 \) coincides with that of \( R/\text{LT}(I_0) \) (see, e.g., [3, §3]), where \( \text{LT}(I_0) \) denotes the leading term ideal of \( I_0 \) as in [23] and [20, Prop. 1.1]. Now \( \Delta_0 \) is precisely the simplicial complex such that \( R/\text{LT}(I_0) \) is the Stanley-Reisner ring of \( \Delta_0 \). Thus, it follows from Corollary 6 and part (ii) of Proposition 8 that the Hilbert series of \( R/\Delta_0 \) is given by (18), where \( h_0 = 1 \) and for \( k \geq 1 \),

\[ h_k = |\{F : F \text{ a facet of } \Delta_0 \text{ with } |c(F)| = k\}|. \]

Partitioning the facets \( F = F_x F_y \) in the above set in accordance with the values of \( \mu(F_x) \) and noting from Proposition 5 that \( \mu(F_x) \neq (m,n) \), and then applying Lemma 11 we obtain the desired result. \( \square \)

We have seen in Section 3 that lattice path monomials can be related to lattice paths in the sense of [23] if we rotate to the left by 90° and identify the variable \( x_{i,j} \) with the point \((i,j)\) of \( \mathbb{Z}^2 \). Also recall that for any \((a,a'),(e,e') \in \mathbb{Z}^2 \) and \( s \in \mathbb{Z} \), we denote by \( \mathcal{P}_s((a,a') \rightarrow (e,e')) \) the set of lattice paths from \((a,a')\) to \((e,e')\) with \( s \) NE-turns. Likewise, if \((a_1,a_1'),(e_i,e'_i) \in \mathbb{Z}^2 \) for \( i = 1,2 \) and \( s \in \mathbb{Z} \), then by \( \mathcal{P}_s((a_1,a_1') \rightarrow (e_1,e'_1), (a_2,a_2') \rightarrow (e_2,e'_2)) \) we denote the set of pairs \((L_1,L_2)\) of nonintersecting lattice paths such that \( L_i \) is from \((a_i,a'_i)\) to \((e_i,e'_i)\) for \( i = 1,2 \), and the paths \( L_1 \) and \( L_2 \) together have exactly \( s \) NE-turns. Evidently, these sets are empty (and hence of cardinality 0) when \( s < 0 \).

Lemma 13. Let \( s, i, j \in \mathbb{Z} \) with \( s \geq 0 \), \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

(i) If \( i \neq m \), then

\[ C^s_{i,j} = \sum_{s_1+s_2=s} |\mathcal{P}_{s_1}((i,j) \rightarrow (m,n))| |\mathcal{P}_{s_2}((1,2) \rightarrow (i,n), (1,1) \rightarrow (m,j))|, \]

where the sum is over pairs \((s_1,s_2)\) of nonnegative integers with \( s_1 + s_2 = s \).
(ii) If \(1 < j < n - 1\), then
\[
C_{m,j}^s = \sum_{p=1}^{m-2} \sum_{q=j+1}^{n-1} |\mathcal{P}_{s-1} ((1,2) \to (p,q), (1,1) \to (m,j))| + \sum_{p=1}^{m-2} |\mathcal{P}_{s-1} ((1,2) \to (p,j), (1,1) \to (m,j))| + |\mathcal{P}_{s} ((1,2) \to (m-1,j), (1,1) \to (m,j))|.
\]

(iii) \(C_{m,1}^s = \binom{n-2}{s} \binom{m-1}{s}\) and
\[
C_{m,n-1}^s = \sum_{p=1}^{m-2} |\mathcal{P}_{s-1} ((1,2) \to (p,n-1), (1,1) \to (m,n-1))| + |\mathcal{P}_{s} ((1,2) \to (m-1,n-1), (1,1) \to (m,n-1))|.
\]

**Proof.** Let \(i, j \in \mathbb{Z}\) with \(1 \leq i \leq m\), \(1 \leq j \leq n\), and \((i,j) \neq (m,n)\). By a \(90^\circ\) rotation to the left, we see from Proposition 3 that the facets \(F = F_z F_y\) of \(\Delta_0\) with \(\mu(F_z) = x_{i,j}\) are in one-to-one correspondence with the triples \((L, L_1^*, L_2^*)\) of lattice paths, where \(L\) is from \((i,j)\) to \((m,n)\), while \(L_1^*\) is from \((1,1)\) to \((i,n)\) and \(L_2^*\) is from \((2,1)\) to \((m,j)\), and moreover \(L_1^*, L_2^*\) are nonintersecting. We will now modify \(L_1, L_2\) slightly keeping in mind the hypothesis in Corollary 2. To this end, first note that \((1,2) \in L_1^*\) since \(2 < m \leq n\). Thus if we let \(L_1 := L_1^* \setminus \{(1,1)\}\) and \(L_2 := L_2^* \cup \{(1,1)\}\), then \((L_1^*, L_2^*)\) and \((L_1, L_2)\) are pairs of nonintersecting lattice paths that determine each other and have exactly the same NE-turns, except that if \(L_1^*\) had a NE turn at \((1,2)\), then \(L_1\) will not have a NE turn at \((1,2)\). Note though, that by Lemma 11 (iv), \(y_{1,2}\) is not a corner of any facet, and this switch will therefore not affect the count of corners. Consequently, the facets \(F = F_z F_y\) of \(\Delta_0\) with \(\mu(F_z) = x_{i,j}\) are in one-to-one correspondence with \(\mathcal{P}((i,j) \to (m,n)) \times \mathcal{P}((1,2) \to (i,n), (1,1) \to (m,j))\). The lattice paths \(L\) and \((L_1, L_2)\) corresponding to the components \(F_z\) and \((F^U_y, F^L_y)\) of the facet \(F = F_z F_y\) are illustrated in Figure 3; these may be compared with Figure 2 that depicts the facets \(\Delta_1\) of \(\Delta\). From this, we readily obtain the formula in (i).

(ii) Suppose \(i = m\) and \(1 < j < n - 1\). Then for a facet \(F = F_z F_y\) of \(\Delta_0\) with \(\mu(F_z) = x_{m,j}\), the lattice path \(L\) corresponding to \(F_z\) is from \((m,j)\) to \((m,n)\) and evidently, this has no NE-turns. Now consider the pair \((L_1, L_2)\) in \(\mathcal{P}((1,2) \to (i,n), (1,1) \to (m,j))\) corresponding to \((F^U_y, F^L_y)\). Suppose the last NE-turn of \(L_1\) is at \((p,q+1)\). Note that if \(q < j\), then we must have \((m,j) \in L_1\), which contradicts the fact that \(L_1, L_2\) are nonintersecting. Thus \(1 \leq p \leq m - 1\) and \(j \leq q < n\). Moreover, if \(q = j\), then by part (iv) of Lemma 3, we see that either \(p \leq m - 2\) or the NE-turn \((p,q+1)\) is not in \(c(F)\). It follows that \(L_1\) can be replaced by its truncation \(\hat{L}_1\), which is a lattice path from \((1,2)\) to \((p,q)\) such that \(\hat{L}_1\) and \(L_2\) are nonintersecting. Moreover, the number of NE-turns of \(\hat{L}_1\) in \(c(F)\) are exactly one less than the number of NE-turns of \(L_1\) in \(c(F)\), except when \((p,q) = (m-1,j)\) in which case they are the same. Thus by varying \((p,q)\) over an appropriate range, we obtain the formula in (ii).

(iii) If \((i,j) = (m,1)\) and \(F = F_z F_y\) is a facet of \(\Delta_0\) with \(\mu(F_z) = x_{m,1}\), then the path \(L\) corresponding to \(F_z\) as well as the path \(L_2\) corresponding to \(F^L_y\) have no NE-turns. Moreover, every NE-turn of the path \(L_1 \in \mathcal{P}((1,2) \to (m,n))\) corresponding
to $F_y^U$ is necessarily in $c(F)$, thanks to Lemma 11. Thus, in view of (12), we see that $C_{m,1}^s = \binom{n-2}{s} \binom{m-1}{s}$. Finally, if $(i,j) = (m,n-1)$, then arguing as in (ii) above, we see that for a facet $F = F_xF_y$ of $\Delta_0$ with $\mu(F_x) = x_{m,n-1}$, the lattice path $L$ corresponding to $F_x$ has no NE-turns and the last NE-turn of the lattice path $L_1$ corresponding to $F_y^U$ must be $(p,n)$ for some $p = 1, \ldots, m-1$. Moreover, by Lemma 11 this turn is counted as a corner (i.e., $x_{p,n} \in c(F)$) if and only if $p < m - 1$. Thus upon replacing $L_1$ by its truncation up to $(p,n-1)$, we obtain the desired formula for $C_{m,n-1}^s$ in (iii). □

We can already use the results obtained thus far to write down an explicit formula for the Hilbert series of the graded ring $R/I_0$ corresponding to $\mathcal{Z}_0$. Indeed, it suffices to combine Corollary 12, Lemma 13, and Corollary 4. However the resulting formula is much too complicated and we will instead use results in Section 2 for simplifying various terms in (19) so as to eventually arrive at an elegant formula for (18).

**Lemma 14.** Let $k$ be a positive integer. Then $C_{m,n-1}^k = \sum_{t_1+t_2=k} \binom{m-2}{t_1} \binom{n-2}{t_2} - \binom{m}{t_1+1} \binom{n}{t_2+1} - \binom{m-2}{t_1+1} \binom{n-2}{t_2+1}$.  

Proof. For $s \in \mathbb{Z}$, let $f(s) := \binom{n-1}{s} \binom{n-2}{s}$ and $g(s) := \binom{m-2}{s} \binom{n-2}{s}$. By Corollary 4,

$$\sum_{p=1}^{m-2} |\mathcal{P}_{k-1} ((1,2) \to (p,n-1), (1,1) \to (m,n-1))| = \sum_{s_1+s_2=k-1} \binom{m-3}{s_1} \binom{n-3}{s_2} f(s_2) - \binom{m-1}{s_1+1} \binom{n-1}{s_2+1} g(s_1)$$

$$= \sum_{s_1+s_2=k-1} \binom{m-2}{s_1+1} \binom{n-3}{s_2} f(s_2) - \binom{m-1}{s_1+1} \binom{n-3}{s_2+1} g(s_1)$$

where the penultimate equality follows from Lemma 1 since $\binom{0}{s_1+1} = 0 = \binom{1}{s_2+2}$ for $s_1, s_2 \geq 0$ and also since $\binom{n-1}{s_1} f(s_2) = 0 = \binom{n-3}{s_2} g(s_1)$ if $s_1 < 0$ or $s_2 < 0$, while the last equality follows by altering the summations (twice!) as in (10). On the other hand, by Corollary 4 $|\mathcal{P}_k ((1,2) \to (m-1,n-1), (1,1) \to (m,n-1))|$ is equal to

$$\sum_{t_1+t_2=k} \binom{m-2}{t_1} \binom{n-3}{t_2} f(t_2) - \binom{m-1}{t_1+1} \binom{n-3}{t_2+1} g(t_1).$$

Now combining (20) and (21) using (9), and then using part (iii) of Lemma 13, we obtain the desired result. □

**Lemma 15.** Let $k$ be a positive integer. Then $\sum_{i=1}^{m-1} \sum_{j=1}^{n} C_{i,j}^{k-1}$ is equal to

$$\sum_{t_1+t_2=k} \binom{m}{t_1+1} \binom{n}{t_1} \binom{m-2}{t_2+1} \binom{n-2}{t_2}. $$

□
Proof. Using (12) and part (i) of Lemma 13, we see that \( \sum_{i=1}^{m-1} \sum_{j=1}^{n} C_{i,j}^{k-1} \) equals
\[
\sum_{i=1}^{m-1} \sum_{j=1}^{n} \sum_{k_1+k_2=k-1} (m-i) \binom{n-j}{k_1} \binom{n}{k_1} |P_{k_2}((1,2) \rightarrow (i,n),(1,1) \rightarrow (m,j))|
\]
Applying Corollary 4 and then suitably interchanging summations and noting that
\[
\text{and where the simplified expressions for } N_{1,2} \text{ follow by rewriting each of the summands in } N_{1} \text{ and } N_{2} \text{ using (9), invoking (4) (noting that } k_1, s_1, s_2 \geq 0, \text{ and then applying (7) for suitable values of } \alpha, \beta \text{ and } \gamma. \text{ A similar simplification is possible in } M_{1} \text{ and } M_{2} \text{ if we add and subtract the term corresponding to } i = m, \text{ and in view of (4), this is only necessary if } k_1 = 0. \text{ Thus,}
\]
\[
M_{1} = \left( \frac{m}{k_1 + s_1 + 1} \right) - \delta_{0,k_1} \left( \frac{m-1}{s_1} \right) \quad \text{and} \quad M_{2} = \left( \frac{m+1}{k_1 + s_2 + 2} \right) - \delta_{0,k_1} \left( \frac{m}{s_2 + 1} \right),
\]
where \( \delta \) is the Kronecker delta. Substituting the simplified values of \( M_{1}, N_{1}, M_{2}, N_{2} \) in (22), and letting \( A(s_1, s_2) := (m-1, s_1, s_2) \) and \( B(s_1, s_2) := (m-2, s_1, s_2) \) for \( s_1, s_2 \in \mathbb{Z} \), we see that (22) is of the form \( E_{3} = S_{3} \), where
\[
E_{3} = \sum_{k_1,s_1,s_2=k-1, s_1,s_2 \geq 0} \left( \frac{m}{k-s_2} \right) \left( \frac{n}{k-s_1} \right) A(s_1, s_2) - \left( \frac{m+1}{k-s_1+1} \right) \left( \frac{n}{k-s_2} \right) B(s_1, s_2)
\]
and \( S_{3} \) is the part where the Kronecker delta is nonzero, i.e.,
\[
S_{3} = \sum_{s_1+s_2=k-1} \left( \frac{m}{s_1+1} \right) \left( \frac{n}{s_1+1} \right) B(s_1, s_2) - \left( \frac{m-1}{s_1} \right) \left( \frac{n}{s_1+1} \right) A(s_1, s_2).
\]
Altering the summation as in (10), we see that \( S_{3} \) can be written as
\[
\sum_{t_1+t_2=k} \left( \frac{m}{t_1+1} \right) \left( \frac{n}{t_1+1} \right) \left( \frac{m-2}{t_2-1} \right) \left( \frac{n-2}{t_2-1} \right) - \left( \frac{m-1}{t_1} \right) \left( \frac{n}{t_2-1} \right) \left( \frac{m-1}{t_2-1} \right) \left( \frac{n-2}{t_1} \right).
\]
On the other hand, in view of (4) and (11), we can write
\[
E_{3} = \sum_{\ell=0}^{k-1} \sum_{s_1+s_2=\ell} \left( \frac{m}{k-s_1} \right) \left( \frac{n}{k-s_2} \right) A(s_2, s_1) - \left( \frac{m+1}{k-s_1+1} \right) \left( \frac{n}{k-s_2} \right) B(s_1, s_2).
\]
By (6), \( (m+1, k-s_1+1) = (m, k-s_1) + (m, k-(s_1-1)) \) and using this to split the second summand in \( E_{3} \) into two parts and combining one of the parts with the first summand in \( E_{3} \) and then applying (6) once again, we see that
\[
E_{3} = \sum_{\ell=0}^{k-1} \sum_{s_1+s_2=\ell} f(s_1, s_2) - f(s_1-1, s_2),
\]
where
where \( f(s_1, s_2) := \binom{m}{k-s_1} \binom{n}{k-s_2} \binom{m-2}{s_1} \binom{n-2}{s_2} \) for \( s_1, s_2 \in \mathbb{Z} \). Now in view of (10), we find that \( E_3 \) is given by the telescoping sum
\[
E_3 = \sum_{\ell=0}^{k-1} F_\ell - F_{\ell-1}, \quad \text{where for } \ell \in \mathbb{Z}, \ F_\ell := \sum_{s_1+s_2=\ell} f(s_1, s_2).
\]
From the definition of \( f \), we see that \( F_{-1} = 0 \), and thus \( E_3 = F_{k-1} \), i.e.,
\[
E_3 = \sum_{s_1+s_2=k-1} \binom{m}{k-s_1} \binom{n}{k-s_2} \binom{m-2}{s_1} \binom{n-2}{s_2}.
\]
Now we can replace \( k - s_1, k - s_2 \) by \( s_2 + 1, s_1 + 1 \), respectively, in the above summand, and then alter the summation using (10) to obtain
\[
E_3 = \sum_{t_1+t_2=k} \binom{m}{t_2} \binom{n}{t_1+1} \binom{m-2}{t_2} \binom{n-2}{t_1}.
\]
Finally by adding (24) and (23) termwise and using (6), we obtain the desired formula for \( E_3 + S_3 \), i.e., for \( \sum_{i=1}^{m-1} \sum_{j=1}^{n} C_{i,j}^{k-1} \).

**Lemma 16.** Let \( k \) be a positive integer. Then \( \sum_{j=1}^{n-2} C_{m,j}^{k-1} \) is equal to
\[
\sum_{t_1+t_2=k} \binom{m-1}{t_1} \binom{n-2}{t_2} \binom{m-1}{t_1} \binom{n-2}{t_2} - \binom{m}{t_2+1} \binom{m-2}{t_2} \binom{n-2}{t_1} \binom{n-2}{t_1}.
\]
**Proof.** The desired result is easily verified when \( n = 3 \) and so we assume that \( n > 3 \). For \( j, s \in \mathbb{Z} \), let \( f_j(s) := \binom{m-1}{j} \binom{1}{s} \) and \( g_j(s) := \binom{m-2}{j} \binom{1}{s} \). In view of parts (iii) and (ii) of Lemma 13 together with [4] and Corollary 4 we see that
\[
C_{m,1}^{k-1} = \binom{n-2}{k-1} \binom{m-1}{k-1}, \quad \text{and } \sum_{j=2}^{n-2} C_{m,j}^{k-1} = S_4 + S_5 + S_6,
\]
where
\[
S_4 = \sum_{j=2}^{n-2} \sum_{p=1}^{m-2} \sum_{q=j+1}^{n-2} \sum_{s_1, s_2 \geq 0} \binom{p-1}{s_1} \binom{q-2}{s_2} f_j(s_2) - \binom{p}{s_2+1} \binom{q-2}{s_2} g_j(s_1),
\]
\[
S_5 = \sum_{j=2}^{n-2} \sum_{p=1}^{m-2} \sum_{q=j+1}^{n-2} \binom{p-1}{s_1} \binom{j-2}{s_2} f_j(s_2) - \binom{p}{s_2+1} \binom{j-2}{s_2} g_j(s_1),
\]
\[
S_6 = \sum_{j=2}^{n-2} \sum_{p=1}^{m-2} \binom{m-2}{s_1} \binom{j-2}{s_1} f_j(s_2) - \binom{m-1}{s_2+1} \binom{j-2}{s_2} g_j(s_1).
\]
Interchanging \( s_1 \) and \( s_2 \) in the second summand for \( S_6 \) as in (11), we can write
\[
S_6 = \sum_{s_1+s_2=k-1} \lambda(s_1, s_2) \left\{ \binom{m-2}{s_1} \binom{m-1}{s_2} - \binom{m-2}{s_1+1} \binom{m-2}{s_2-1} \right\},
\]
where for \( s_1, s_2 \in \mathbb{Z} \), we have let \( \lambda(s_1, s_2) := \sum_{j=2}^{n-2} \binom{j-2}{s_1} \binom{j-2}{s_2} \). Next, by Lemma 1
\[
\sum_{p=1}^{m-2} \binom{p-1}{s_1} = \binom{m-2}{s_1+1} \quad \text{and} \quad \sum_{p=1}^{m-2} \binom{p}{s_2+1} = \binom{m-1}{s_2+2} \quad \text{for } s_1, s_2 \geq 0.
\]
Consequently, by interchanging summations and rearranging terms, we find

$$S_5 = \sum_{j=2}^{n-2} \sum_{s_1+s_2=k-2 \atop s_1, s_2 \geq 0} \left( \frac{m-2}{s_1+1} \right) \left( \frac{j-2}{s_1} \right) f_j(s_2) - \left( \frac{m-1}{s_2+2} \right) \left( \frac{j-2}{s_2} \right) g_j(s_1)$$

$$= \sum_{s_1+s_2=k-2} \lambda(s_1, s_2) \left\{ \left( \frac{m-2}{s_1+1} \right) \left( \frac{m-1}{s_2} \right) - \left( \frac{m-1}{s_1+2} \right) \left( \frac{m-2}{s_2} \right) \right\}$$

$$= \sum_{s_1+s_2=k-1} \lambda(s_1-1, s_2) \left\{ \left( \frac{m-2}{s_1} \right) \left( \frac{m-1}{s_2} \right) - \left( \frac{m-1}{s_1+1} \right) \left( \frac{m-2}{s_2} \right) \right\} \quad (27)$$

where the penultimate equality follows from (4) and (11) by interchanging $s_1$ and $s_2$ in the second summand of the preceding formula, while the last equality follows from (10). Now using (6) we easily see that for any $s_1, s_2 \in \mathbb{Z}$,

$$\lambda(s_1 - 1, s_2) + \lambda(s_1, s_2) = \nu(s_1, s_2), \quad \text{where} \quad \nu(s_1, s_2) := \sum_{j=2}^{n-2} \left( \frac{j-1}{s_1} \right) \left( \frac{j-1}{s_2} \right).$$

Hence we can combine (27) and (26) to obtain

$$S_5 + S_6 = \sum_{s_1+s_2=k-1} \nu(s_1, s_2) \left\{ \left( \frac{m-2}{s_1} \right) \left( \frac{m-1}{s_2} \right) - \left( \frac{m-1}{s_1+1} \right) \left( \frac{m-2}{s_2} \right) \right\}. \quad (28)$$

It remains to consider $S_4$ or rather $C_{m,1}^{k-1} + S_4$. This is a little more complicated, but it can be handled using arguments similar to those in the proof of Lemma 15 as follows. First, by interchanging summations and using Lemma 1, we find

$$S_4 = \sum_{j=2}^{n-2} \sum_{s_1+s_2=k-2 \atop s_1, s_2 \geq 0} \left( \frac{m-1}{s_1+1} \right) \theta(s_1) f_j(s_2) - \left( \frac{m}{s_2+2} \right) \theta(s_2) g_j(s_1)$$

where for $s \in \mathbb{Z}$, we have let $\theta(s) := \binom{n-2}{s+1} - \binom{j-1}{s+1}$. Now observe that if $s_1 < 0$ or $s_2 < 0$, then $\theta(s_1) f_j(s_2) = 0 = \theta(s_2) g_j(s_1)$. Thus, we may drop the condition $s_1, s_2 \geq 0$ in the above expression for $S_4$, and then alter each of the two summations over $(s_1, s_2)$ using (10) to write

$$S_4 = \sum_{j=2}^{n-2} \sum_{s_1+s_2=k-1} \left( \frac{m-1}{s_1} \right) \theta(s_1 - 1) f_j(s_2) - \left( \frac{m}{s_2+1} \right) \theta(s_2 - 1) g_j(s_1).$$

Next, we collate the terms involving $j$ and bring the summation over $j$ inside, and note that by Lemma 1, $\sum_{j=2}^{n-2} \binom{j-1}{s+1} = \binom{n-2}{s+1} - \delta_{0,s}$ for any $s \geq 0$. This yields

$$S_4 = \sum_{s_1+s_2=k-1} \left( \frac{m-1}{s_1} \right) \left( \frac{n-2}{s_1} \right) \left( \frac{m-1}{s_2} \right) \left( \frac{n-2}{s_2+1} - \delta_{0,s} \right)$$

$$- \left( \frac{m}{s_2+1} \right) \left( \frac{n-2}{s_2} \right) \left( \frac{m}{s_1} \right) \left( \frac{n-2}{s_1+1} - \delta_{0,s_1} \right)$$

$$- \left( \frac{m}{s_1} \right) \left( \frac{n-1}{s_2} \right) \nu(s_1, s_2) + \left( \frac{m}{s_2+1} \right) \left( \frac{m-2}{s_1-1} \right) \nu(s_1, s_2).$$

Since $\binom{m-2}{s_1-1} = 0$ when $s_1 = 0$, the only contribution of the terms involving Kronecker delta is when $s_2 = 0$, and it is $-\binom{m-2}{k-1} \binom{k-2}{k-1}$, i.e., precisely $-C_{m,1}^{k-1}$. It follows
that \( C_{m,n-1}^{k-1} + S_1 = S_4^* + E_4 \), where

\[
S_4^* = \sum_{s_1+s_2=k-1} \left( \begin{array}{c} m-1 \\ s_1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ s_1 \\ \end{array} \right) \left( \begin{array}{c} m-1 \\ s_2 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ s_2+1 \\ \end{array} \right) - \left( \begin{array}{c} m \\ s_2+1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ s_2 \\ \end{array} \right) \left( \begin{array}{c} m-2 \\ s_1-1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ s_1+1 \\ \end{array} \right)
\]

and

\[
E_4 = \sum_{s_1+s_2=k-1} \nu(s_1, s_2) \left\{ \left( \begin{array}{c} m \\ s_2+1 \\ \end{array} \right) \left( \begin{array}{c} m-2 \\ s_1-1 \\ \end{array} \right) - \left( \begin{array}{c} m-1 \\ s_1 \\ \end{array} \right) \left( \begin{array}{c} m-1 \\ s_2 \\ \end{array} \right) \right\}
\]

where the last equality follows by interchanging \( s_1 \) and \( s_2 \) while noting that \( \nu \) is symmetric in \( s_1, s_2 \).

Now combining (28) and (29), and then making an easy calculation using (6), we see that

\[
E_4 + S_5 + S_6 = \sum_{s_1+s_2=k-1} \nu(s_1, s_2) \left\{ \left( \begin{array}{c} m-1 \\ s_1 \\ \end{array} \right) \left( \begin{array}{c} m-2 \\ s_2-1 \\ \end{array} \right) - \left( \begin{array}{c} m-2 \\ s_1 \\ \end{array} \right) \left( \begin{array}{c} m-1 \\ s_2 \\ \end{array} \right) \right\} = 0,
\]

where the last equality follows by interchanging \( s_1 \) and \( s_2 \) in one of the summations above. Thus \( \sum_{j=1}^{n-2} C_{m,j}^{k-1} = S_4^* \). Finally, using (10), we readily see that \( S_4^* \) is precisely the desired formula in the statement of the lemma.

The results of Lemmas 14 and 16 can be combined.

**Corollary 17.** Let \( k \) be a positive integer. Then \( C_{m,n-1}^k + \sum_{j=1}^{n-2} C_{m,j}^{k-1} \) is equal to

\[
\sum_{t_1+t_2=k} \left( \begin{array}{c} m-1 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} m-1 \\ t_2 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_2 \\ \end{array} \right) - \left( \begin{array}{c} m-1 \\ t_2 \\ \end{array} \right) \left( \begin{array}{c} m-1 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_2 \\ \end{array} \right).
\]

**Proof.** Consider the formula for \( \sum_{j=1}^{n-2} C_{m,j}^{k-1} \) given by Lemma 16. This is a difference of two summations over \( (t_1, t_2) \in \mathbb{Z}^2 \) with \( t_1 + t_2 = k \). Alter the first of these summations by interchanging \( t_1 \) and \( t_2 \), while putting \( (t_2+1) = (m-1) + (t_2+1) \) in the second summation to split it into two summations. Then using (6), we readily see that the formula for \( \sum_{j=1}^{n-2} C_{m,j}^{k-1} \) becomes

\[
\sum_{t_1+t_2=k} \left( \begin{array}{c} m-2 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} m-1 \\ t_2 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_2 \\ \end{array} \right) - \left( \begin{array}{c} m-1 \\ t_2 \\ \end{array} \right) \left( \begin{array}{c} m-2 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} n-2 \\ t_2 \\ \end{array} \right).
\]

This can be added termwise, using (6) once again, with the formula for \( C_{m,n-1}^k \) given by Lemma 14 to obtain the desired result.

We are now ready for our main theorem.

**Theorem 18.** The Hilbert series of \( R/I_0 \) is given by

\[
\left( \frac{\sum_{\nu=0}^{m-1} \nu(z-\nu)^{m+n-1}}{(1-z)^{m+n-1}} \right)^2.
\]

**Proof.** First note that (30) is of the form \( (1-z)^{-2(m+n-1)} \sum_{k=0}^{2m-2} h_k^* z^k \), where

\[
h_k^* = \sum_{t_1+t_2=k} \left( \begin{array}{c} m-1 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} n-1 \\ t_1 \\ \end{array} \right) \left( \begin{array}{c} m-1 \\ t_2 \\ \end{array} \right) \left( \begin{array}{c} n-1 \\ t_2 \\ \end{array} \right)
\]

for \( k \in \mathbb{Z} \).
On the other hand, by Corollary 12, we see that the Hilbert series of $R/I_0$ is given by $(1 - z)^{-2(m+n-1)} \sum_{k \geq 0} h_k z^k$, where $h_0 = 1$ and

$$h_k = \left( C_{m,n-1}^k + \sum_{j=1}^{n-2} C_{m,j}^{k-1} \right) + \sum_{i=1}^{m-1} \sum_{j=1}^{n} C_{i,j}^{k-1} \quad \text{for } k \geq 1. \quad (32)$$

It is clear that $h_0^k = 1 = h_0$ and so it suffices to show that $h_k^k = h_k$ for all $k \geq 1$. In view of Corollary 17 and Lemma 15, this is equivalent to showing that

$$\sum_{t_1+t_2=k} P_1(t_1, t_2) - P_2(t_1, t_2) + P_3(t_1, t_2) - P_4(t_1, t_2) - P(t_1, t_2) = 0 \quad \text{for } k \geq 1,$$

where $P_i(t_1, t_2)$ for $i = 1, \ldots, 4$ and $P(t_1, t_2)$ are the relevant summands, namely,

$$P_1(t_1, t_2) := \binom{m-1}{t_1} \binom{n-2}{t_2}, \quad P_2(t_1, t_2) := \binom{m}{t_1+1} \binom{n-1}{t_2}, \quad P_3(t_1, t_2) := \binom{m}{t_1} \binom{n-1}{t_2+1}, \quad P_4(t_1, t_2) := \binom{m}{t_1} \binom{n-2}{t_2},$$

and $P(t_1, t_2) := \binom{m-1}{t_1} \binom{n-1}{t_2}$ for $t_1, t_2 \in \mathbb{Z}$. To this end, we will make an extensive use of alterations as in (10) and (11); more specifically, the fact that

$$\sum_{t_1+t_2=k} f(t_1, t_2) = \sum_{t_1+t_2=k} f(t_2, t_1) = \sum_{t_1+t_2=k} f(t_1+1, t_2-1) = \sum_{t_1+t_2=k} f(t_2+1, t_1-1)$$

for any $f : \mathbb{Z}^2 \to \mathbb{Q}$ with finite support and any $k \in \mathbb{Z}$. Now fix any integer positive integer $k$ and any $(t_1, t_2) \in \mathbb{Z}^2$ with $t_1 + t_2 = k$. Observe that

$$P_3(t_1 - 1, t_2 + 1) - P_4(t_1, t_2) = \binom{m-1}{t_2} \binom{n-1}{t_1} \binom{m-1}{t_2} \binom{n-2}{t_1},$$

Using (6) twice, we may substitute $\binom{n-2}{t_1} + \binom{n-2}{t_2}$ for $\binom{n}{t_1}$ in the RHS of the above identity to obtain

$$-P_2(t_1, t_2) + P_3(t_1 - 1, t_2 + 1) - P_4(t_2, t_1) = Q_1(t_1, t_2) + Q_2(t_1, t_2),$$

where

$$Q_1(t_1, t_2) := \binom{m-1}{t_2+1} \binom{n-2}{t_1-1} \binom{n-2}{t_2} \text{ and } Q_2(t_1, t_2) := \binom{m-1}{t_2+1} \binom{n-1}{t_1-1} \binom{n-1}{t_2}.$$

Finally, observe that $P_1(t_1, t_2) + Q_1(t_1 + 1, t_2 - 1) + Q_2(t_2 + 1, t_1 - 1) = P(t_1, t_2).$ This yields the desired result. $\square$

It may be noted that in view of (2) and (30), the Hilbert series of the principal component $Z_0$ is precisely the square of the Hilbert series of the base variety $Z_{2,m,n}$, and as such, Theorem 7 could be deduced as a consequence of Theorem 13.

As an application of Theorem 13, we will now compute the $a$-invariant of the coordinate ring $R/I_0$ of the principal component $Z_0$ of $Z_{2,m,n}$ and determine when $Z_0$ is Gorenstein. Recall that if $A$ is a finitely generated, positively graded Cohen-Macaulay algebra over a field, then $A$ admits a graded canonical module $\omega_A$ and the $a$-invariant of $A$ is defined as the negative of the least degree of a generator of $\omega_A$. If the Hilbert series of $A$ is given by $H_A(z) = h(z)/(1 - z)^d$, where $d = \dim A$ and $h(z) \in \mathbb{Q}[z]$ with $h(1) \neq 0$, then the $a$-invariant of $A$ is the order of pole of $H_A(z)$ at infinity, viz., $- (d - \deg h(z))$. Moreover, the Hilbert series of $\omega_A$ is given by $H_{\omega_A}(z) = (-1)^d H_A(z^{-1})$. As a general reference for these notions and results, one may consult [4], especially Sections 3.6 and 4.4. The following result is an analogue of a theorem of Gräbe [13] (see also [14] Thm. 4) for classical determinantal varieties which says that if $1 \leq r \leq m \leq n$, then the $a$-invariant of the coordinate ring of $Z_{r,m,n}$ is $-(r-1)n$. 

Corollary 19. The a-invariant of R/I₀ is equal to −2n and the Hilbert series of the graded canonical module of R/I₀ is given by
\[
\left( \sum_{e=0}^{m-1} \binom{m-1}{e} \binom{n-1}{e} z^{m+n-e-1} \right)^2 \left( 1 - z \right)^{m+n-1}.
\] (33)

Proof. We know from [20, Thm. 1.2] that A = R/I₀ is Cohen-Macaulay and it is obviously a finitely generated, positively graded \( \mathbb{F} \)-algebra. Moreover, by Theorem [18], the Hilbert series of A is given by \( h_0(z)/(1 - z)^{2(m+n-1)} \), where
\[
h_0(z) = \left( \sum_{e=0}^{m-1} \binom{m-1}{e} \binom{n-1}{e} z^e \right)^2.
\]

Since \( 2 \leq m \leq n \), we see that \( h_0(z) \) is a polynomial in \( z \) of degree \( 2(m - 1) \) with leading coefficient \( (n-1)^2 \) and all other coefficients nonnegative integers; in particular, \( h_0(1) \neq 0 \). Hence the a-invariant of \( A = R/I₀ \) is \( 2(m-1) - 2(m+n-1) = -2n \) and also that the Hilbert series of \( \omega_A \) is given by (33). \( \Box \)

The following result is an analogue of a theorem of Svanes [37] (see also [8]) for classical determinantal varieties which says that for any \( r \geq 1 \), the coordinate ring of \( \mathbb{Z}^m \) is Gorenstein if and only if \( m = n \).

Corollary 20. The coordinate ring \( R/I₀ \) of \( \mathbb{Z}_0 \) is Gorenstein if and only if \( m = n \).

Proof. By [20, Thm. 1.2] and [23, Prop. 3.3], \( A = R/I₀ \) is a Cohen-Macaulay domain. Hence from a well-known result of Stanley [36, Thm. 4.4] (see also [1, Cor. 4.6]), we see that \( A \) is Gorenstein if and only if \( H_A(z) = (-1)^d z^a H_A(z^{-1}) \) for some \( a \in \mathbb{Z} \). Moreover, in this case the integer \( a \) is necessarily the a-invariant of \( A \). Thus, from Corollary [19] we see that \( R/I₀ \) is Gorenstein if and only if
\[
\left( \sum_{e=0}^{m-1} \binom{m-1}{e} \binom{n-1}{e} z^e \right)^2 = \left( \sum_{e=0}^{m-1} \binom{m-1}{e} \binom{n-1}{e} z^{m+n-e-1} \right)^2.
\]

Since both the polynomials inside the square brackets on the two sides of the above equality have positive leading coefficients, it follows that \( R/I₀ \) is Gorenstein if and only if \( \binom{n-1}{e} = \binom{n-1}{m-1-e} \) for all \( e = 0, 1, \ldots, m-1 \). Since \( 1 < m-1 \leq n-1 \), the latter clearly holds if and only if \( m = n \). \( \Box \)

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Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India.

E-mail address: srg@math.iitb.ac.in

Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA 93106, USA.

E-mail address: boyan@math.ucsb.edu

Department of Mathematics, California State University Northridge, Northridge, CA 91330, USA.

E-mail address: al.sethuraman@csun.edu