The proof of a conjecture on largest Laplacian and signless
Laplacian H-eigenvalues of uniform hypergraphs∗

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Abstract

Let $A(G)$, $L(G)$ and $Q(G)$ be the adjacency tensor, Laplacian tensor and signless Laplacian tensor
of uniform hypergraph $G$, respectively. Denote by $\lambda(T)$ the largest $H$-eigenvalue of tensor $T$. Let $H$ be a uniform hypergraph, and $H'$ be obtained from $H$ by inserting a new vertex with degree one in each edge. We prove that $\lambda(Q(H')) \leq \lambda(Q(H))$. Denote by $G^k$ the $k$th power hypergraph of an ordinary graph $G$ with maximum degree $\Delta \geq 2$. We will prove that $\{\lambda(Q(G^k))\}$ is a strictly decreasing sequence, which imply Conjecture 4.1 of Hu, Qi and Shao in [4]. We also prove that $\lambda(Q(G^k))$ converges to $\Delta$ when $k$ goes to infinity. The definiton of $k$th power hypergraph $G^k$ has been generalized as $G^k,s$. We also prove some eigenvalues properties about $A(G^k,s)$, which generalize some known results. Some related results about $L(G)$ are also mentioned.

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1 Introduction

Let $G$ be an ordinary graph, and $A(G)$ be the adjacency matrix of $G$. We denote the set $\{1, 2, \cdots, n\}$ by $[n]$. Hypergraph is a natural generalization of ordinary graph (see [1]). A hypergraph $G = (V(G), E(G))$ on $n$ vertices is a set of vertices, say $V(G) = \{1, 2, \cdots, n\}$ and a set of edges, say $E(G) = \{e_1, e_2, \cdots, e_m\}$, where $e_i = \{i_1, i_2, \cdots, i_l\}$, $i_j \in [n]$, $j = 1, 2, \cdots, l$. If $|e_i| = k$ for any $i = 1, 2, \cdots, m$, then $G$ is called a $k$-uniform hypergraph. In particular, the 2-uniform hypergraphs are exactly the ordinary graphs. For a vertex $v \in V(G)$ the degree $d_G(v)$ is defined as $d_G(v) = |\{e_i : v \in e_i \in E(G)\}|$. Vertex with degree one is called pendent vertex in this paper.

An order $k$ dimension $n$ tensor $T = (T_{i_1i_2\cdots i_k}) \in \mathbb{C}^{n \times n \times \cdots \times n}$ is a multidimensional array with $n^k$ entries, where $i_j \in [n]$ for each $j = 1, 2, \cdots, k$.

To study the properties of uniform hypergraphs by algebraic methods, adjacency matrix, signless Laplacian matrix and Laplacian matrix of graph are generalized to adjacency tensor, signless Laplacian tensor and Laplacian tensor of uniform hypergraph.

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1
**Definition 1** [9] [11]. Let $G = (V(G), E(G))$ be a $k$-uniform hypergraph on $n$ vertices. The adjacency tensor of $G$ is defined as the $k$-th order $n$-dimensional tensor $A(G)$ whose $(i_1 \cdots i_k)$-entry is:

$$(A(G))_{i_1 \cdots i_k} = \begin{cases} 1 & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\ 0 & \text{otherwise}. \end{cases}$$

Let $D(G)$ be a $k$-th order $n$-dimensional diagonal tensor, with its diagonal entry $D_{ii}$, the degree of vertex $i$, for all $i \in [n]$. Then $Q(G) = D(G) + A(G)$ is the signless Laplacian tensor of the uniform hypergraph $G$, and $L(G) = D(G) - A(G)$ is the Laplacian tensor of the uniform hypergraph $G$.

The following general product of tensors, was defined in [12] by Shao, which is a generalization of the matrix case.

**Definition 2** Let $A \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_k}$ and $B \in \mathbb{C}^{n_2 \times n_3 \times \cdots \times n_{k+1}}$ be order $m \geq 2$ and $k \geq 1$ tensors, respectively. The product $AB$ is the following tensor $C$ of order $(m-1)(k-1) + 1$ with entries:

$$C_{i_1 \alpha_1 \cdots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m \in [n_2]} A_{i_1 i_2 \cdots i_m} B_{i_2 \alpha_1 \cdots \alpha_{m-1}},$$

where $i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n_3] \times \cdots \times [n_{k+1}]$.

Let $T$ be an order $k$ dimension $n$ tensor, let $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension $n$. Then by (1) $Tx$ is a vector in $\mathbb{C}^n$ whose $i$th component is as the following

$$(Tx)_i = \sum_{i_2, \ldots, i_k = 1}^n T_{i_1 \cdots i_k} x_{i_2} \cdots x_{i_k}.$$  

Let $x^{[k]} = (x_1^k, \ldots, x_n^k)^T$. Then (see [2] [11]) a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $T$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$Tx = \lambda x^{[k-1]},$$

and in this case, $x$ is called an eigenvector of $T$ corresponding to eigenvalue $\lambda$.

An eigenvalue of $T$ is called an H-eigenvalue, if there exists a real eigenvector corresponding to it ([11]). In this paper we will focus on the largest H-eigenvalue of tensor $T$, denoted by $\lambda(T)$.

The concept of power hypergraphs was introduced in [4].

**Definition 3** Let $G = (V(G), E(G))$ be an ordinary graph. For every $k \geq 2$, the $k$th power of $G$, $G^k := (V(G^k), E(G^k))$ is defined as the $k$-uniform hypergraph with the edge set

$$E(G^k) := \{e \cup \{i_{e,1}, \ldots, i_{e,k-2}\} \mid e \in E(G)\}$$

and the vertex set

$$V(G^k) := V(G) \cup \{\cup_{e \in E(G)} \{i_{e,1}, \ldots, i_{e,k-2}\}\}.$$  

For convenience here $G^2 = G$. In [4], the $k$-th power of path, and cycle is called loose path, and loose cycle, respectively. Denote by $S_m$ the star with $m$ edges. The $k$-th power of star is called sunflower in [4], or hyperstar in [10].

**Definition 4** [4] Let $G = (V, E)$ be a $k$-uniform hypergraph. If there is a disjoint partition of the vertex set $V$ as $V = V_0 \cup V_1 \cup \cdots \cup V_d$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_d| = k - 1$, and $E = \{V_0 \cup V_i \mid i \in [d]\}$, then $G$ is called a sunflower. The degree $d$ of the vertex in $V_0$, which is called the heart, is the size of the sunflower. Denote by $S^k_d$ the $k$-uniform sunflower of size $d$.  

2
For even $k$, when $G$ is a cycle or star, Hu, Qi and Shao proved that $\{\lambda(Q(G^k))\}$ is a strictly decreasing sequence in [1]. They believed that it is true for any graph $G$, see Conjecture 4.1 of [1]. This phenomena was also observed in [16] when $G$ is a path (namely, $G^k$ is a loose path).

**Conjecture 5** [2] Let $G$ be an ordinary graph, $k = 2r$ be even and $G^k$ be the $k$-th power hypergraph of $G$. Then $\{\lambda(L(G^k)) = \lambda(Q(G^k))\}$ is a strictly decreasing sequence.

For $t \geq 1$ let $tS_k^1$ be $t$ disjoint union of $S_k^1$. We may point out that when $G = tS_k^2$, we have $\lambda(L(G^k)) = \lambda(Q(G^k)) = 2$ for any $k \geq 2$. Namely, in this case Conjecture [5] is false.

Let $H$ be a uniform hypergraph and $H \neq tS_k^1$, and $H'$ be obtained from $H$ by inserting a new pendant vertex in each edge. In Section 3, we will prove that $\lambda(Q(H')) < \lambda(Q(H))$ in Theorem [10]. So for any ordinary graph $G \neq tS_1$ (maximum degree $\Delta \geq 2$), $\{\lambda(Q(G^k))\}$ is a strictly decreasing sequence, which affirm Conjecture [5] for $\lambda(Q(G^k))$. We also determine the value $\lim_{k \to \infty} \lambda(Q(G^k))$ in Theorem [20].

For an ordinary graph $G$, the definition for $k$-th power hypergraph $G^k$ has been generalized by Khan and Fan in [7].

**Definition 6** Let $G = (V, E)$ be an ordinary graph. For any $k \geq 3$ and $1 \leq s \leq k/2$. For each $v \in V$ (and $e \in E$), let $V_v$ (and $V_e$) be a new vertex set with $s$ (and $k - 2s$) elements such that all these new sets are pairwise disjoint. Then the generalized power of $G$, denoted by $G^{k,s}$, is defined as the $k$-uniform hypergraph with the vertex set

$$V(G^{k,s}) = \left(\bigcup_{v \in V} V_v\right) \bigcup \left(\bigcup_{e \in E} V_e\right)$$

and edge set

$$E(G^{k,s}) = \{V_v \cup V_v \cup V_e : e = \{u, v\} \in E\}.$$

If $s = 1$, then $G^{k,s}$ is exactly the $k$-th power hypergraph $G^k$. The eigenvalues properties about $A(G^k)$ was discussed in [17], and $A(G^{k,k/2})$ was discussed in [7] and [8]. In Section 4, we will prove some eigenvalues properties about $A(G^{k,s})$, which generalize some known results.

## 2 Auxiliary results for nonnegative tensors and H-spectrum of hypergraphs

In [8], the weak irreducibility of nonnegative tensors was defined. It was proved in [8] and [15] that a $k$-uniform hypergraph $G$ is connected if and only if its adjacency tensor $A(G)$ (and so $Q(G)$) is weakly irreducible.

Let $T$ be a $k$th-order $n$-dimensional nonnegative tensor. The spectral radius of $T$ is defined as (see [10], [7] and [8])

$$\rho(T) = \max\{|\mu| : \mu \text{ is an eigenvalue of } T\}.$$ 

Part of Perron-Frobenius theorem for nonnegative tensors is stated in the following for reference.

**Theorem 7** [2] Let $T$ be a nonnegative tensor. Then we have the following statements.

1. $\rho(T)$ is an eigenvalue of $T$ with a nonnegative eigenvector $x$ corresponding to it;
2. If $T$ is weakly irreducible, then $x$ is positive, and for any eigenvalue $\mu$ with nonnegative eigenvector, $\mu = \rho(T)$ holding;
3. The nonnegative eigenvector $x$ corresponding to $\rho(T)$ is unique up to a constant multiple.

In virtue of (1) of Theorem [7], the largest H-eigenvalue of $A(G)$ (or $Q(G)$) is exactly the spectral radius of $A(G)$ (or $Q(G)$). For weakly irreducible nonnegative $T$ of order $k$, the positive eigenvector $x$ with $||x||_k = 1$ corresponding to $\rho(T)$ (i.e., the largest H-eigenvalue) is called the principal eigenvector of $T$ in this paper.
Lemma 8 Suppose that $T$ is a weakly irreducible nonnegative tensor of order $k$. If there exists a nonnegative vector $y$ such that $Ty \leq \mu y^{k-1}$ and $(Ty)_i < \mu y_i^{k-1}$ holding for some $i$, then $\lambda(T) < \mu$.

The H-spectrum of a real tensor $T$, denoted by $Hspec(T)$, is defined to be the set of distinct H-eigenvalues of $T$ [13]. Namely,

$$Hspec(T) = \{ \mu \mid \mu \text{ is an H-eigenvalue of } T \}.$$  

Lemma 9 Let $G = \bigcup_{i=1}^{t} G_i$, where $G_i$ is a connected uniform hypergraph. Then

$$Hspec(L(G)) = \bigcup_{i=1}^{t} Hspec(L(G_i)), \quad (4)$$

and so

$$\lambda(L(G)) = \max_{1 \leq i \leq t} \{ \lambda(L(G_i)) \}.$$  

Proof Without loss of the generality, we may assume that the vertices of $G$ are ordered in such a way that if $i < j$, then any vertex in $G_i$ precedes any vertex in $G_j$. Let $x$ be a column vector of dimension $|V(G)|$. We write $x$ in the following block form

$$x = (x_1^T, x_2^T, \ldots, x_t^T)^T, \quad (5)$$

where $x_i$ is a column vector corresponding to the vertices of $G_i$. Then it is not difficult to see that

$$L(G)x = ((L(G_1)x_1)^T, (L(G_2)x_2)^T, \ldots, (L(G_t)x_t)^T)^T. \quad (6)$$

Now we prove Eq.(4). If $\lambda \in Hspec(L(G))$ with a real eigenvector $x$ as in (5), where $x_j \neq 0$. Then by $L(G)x = \lambda x^{k-1}$ and Eq.(6) we have

$$L(G_j)x_j = \lambda x_j^{k-1}.$$  

Thus

$$\lambda \in Hspec(L(G_j)) \subseteq \bigcup_{i=1}^{t} Hspec(L(G_i)). \quad (7)$$

On the other hand, if $\lambda \in \bigcup_{i=1}^{t} Hspec(L(G_i))$, say, $\lambda \in Hspec(L(G_j))$ for some $1 \leq j \leq t$ with a real eigenvector $x_j$. Take $x_i = 0$ for all $i \neq j$ and take $x$ as in (5). Then by Eq.(6) we can verify that $L(G)x = \lambda x^{k-1}$, thus $\lambda \in Hspec(L(G))$. Combining these two aspects, we obtain (4).

Particularly, we have

$$\lambda(L(G)) = \max_{1 \leq i \leq t} \{ \lambda(L(G_i)) \}.$$  

Similarly, we may prove that these results are also true for $Q(G)$ and $A(G)$. \qedsymbol
3 Largest H-eigenvalue of signless Laplacian tensor of $G^k$

In this section we will prove that for any graph $G$ with maximum degree $\Delta \geq 2$, $\{\lambda(Q(G^k))\}$ is a strictly decreasing sequence. First we will prove a more general result by constructing a new vector and using Lemma 8.

**Theorem 10** Let $H$ be a $k$-uniform $(k \geq 2)$ hypergraph, and $H'$ be obtained from $H$ by inserting a new pendant vertex in each edge. Then $\lambda(Q(H')) \leq \lambda(Q(H))$, equality holding if and only if $H = tS^k_1$ for some $t$.

**Proof** If $H = tS^k_1$ for some $t$, then $\lambda(Q(H')) = \lambda(Q(H)) = 2$. We suppose that $H \neq tS^k_1$ for any $t$.

Denote by $E(H) = \{e_1, e_2, \ldots, e_m\}$, and let $v_i$ be the new pendant vertex inserted in $e_i$ for any $1 \leq i \leq m$, i.e., $V(H') = V(H) \cup \{v_1, v_2, \ldots, v_m\}$.

(1). First suppose that $H$ is a connected, so $Q(H)$ is weakly irreducible. Let $x$ be the principal eigenvector to $\lambda(Q(H))$, namely, $Q(H)x = \lambda(Q(H))x^{[k-1]}$.

Now we construct a new vector $y$ (of dimension $|V(H')|$) from $x$ by adding $m$ components. If $w \in V(H)$, set $y_w = x_w$; if $w = v_i$, i.e., $w$ is a new pendant vertex inserted in $e_i$, set $y_w = \min\{x_u \mid u \in e_i\}$.

Now we will show $Q(H')y \leq \lambda(Q(H))y^{[k]}$.

For any vertex $w \in V(H)$ we have

\[
(Q(H')y)_w = d_H(w)y^k_w + \sum_{\{w,v_{t_1},v_{t_2},\ldots,v_{t_k}\} \in E(H')} y_{v_{t_1}}y_{v_{t_2}}\cdots y_{v_{t_k}}
\leq d_H(w)x^k_w + \sum_{\{w,v_{t_1},v_{t_2},\ldots,v_{t_k}\} \in E(H')} x_{v_{t_1}}x_{v_{t_2}}\cdots x_{v_{t_k}}
\]

\[
= x_w [d_H(w)x^{k-1}_w + \sum_{\{w,v_{t_1},v_{t_2},\ldots,v_{t_k}\} \in E(H')} x_{v_{t_1}}x_{v_{t_2}}\cdots x_{v_{t_k}}]
\]

\[
= x_w (Q(H)x)_w
= x_w \lambda(Q(H))x^{k-1}_w
= \lambda(Q(H))x^k_w
= \lambda(Q(H))y^k_w.
\]

Ineq. (4) is due to the fact $y_{v_i} \leq x_w$. Furthermore, if $x_w > y_{v_i}$, namely, $x_w > \min\{x_u \mid u \in e_i\}$ for some edge $e_i$ containing $w$, Inequality (4) becomes strict.

For $w = v_i$ for some $1 \leq i \leq m$, i.e., $w$ is a new pendant vertex inserted in $e_i$, we suppose $e_i = \{w_1, w_2, \ldots, w_k\}$ and $x_{w_1} = \min\{x_{w_1}, x_{w_2}, \ldots, x_{w_k}\}$. Then we have

\[
(Q(H')y)_w = y^k_w + y_{w_1}y_{w_2}\cdots y_{w_k}
= x^k_{w_1} + x_{w_1}x_{w_2}\cdots x_{w_k}
\leq x_{w_1} [d_H(w_1)x^{k-1}_{w_1} + x_{w_2}\cdots x_{w_k}]
\leq x_{w_1} (Q(H)x)_{w_1}
= \lambda(Q(H))x^k_{w_1}
= \lambda(Q(H))y^k_{w_1}.
\]
Ineq. (5) is due to the fact $d_H(w_1) \geq 1$. Furthermore, if $d_H(w_1) > 1$, Ineq. (5) becomes strict. So we have proved $Q(H')y \leq \lambda(Q(H))y^{[k]}$.

If there exists $\{w, w'\} \subseteq e_i$ for some $e_i \in E(H)$ such that $x_w > x_{w'}$, then $x_w > \min\{x_u \mid u \in e_i\}$, and then Ineq. (4) becomes strict. Now we suppose all the vertices in each edge have the equal corresponding component in $x$. Furthermore, since $H$ is connected, we know that all the components of $x$ are equal, thus $H$ is regular. We suppose $H$ is a $d$-regular. Since $H \neq S^k_t$, we have $d \geq 2$. Now for some edge $e_i$, set $x_{w_1} = \min\{x_u \mid u \in e_i\}$, then $d_H(w_1) = d > 1$, and Ineq. (5) becomes a strict one.

Thus we have proved that $(Q(H')y)_i < \lambda(Q(H))y^{[k]}$ for some $i$, therefore, $\lambda(Q(H')) < \lambda(Q(H))$ by Lemma 8.

(2) If $H = H_1 \cup H_2 \cup \cdots \cup H_t$, where $H_i$ is a connected component of $H$ for $1 \leq i \leq t$ and $t \geq 2$, then

$$H' = H'_1 \cup H'_2 \cup \cdots \cup H'_t.$$ 

Since $H \neq tS^k_1$, we may suppose that $H_i \neq S^k_1$ for $1 \leq i \leq t'$, and $t' \geq 1$. Then for $1 \leq i \leq t'$ we have $\lambda(Q(H'_i)) < \lambda(Q(H_i))$ by the above arguments. It is obvious that $\lambda(Q(H')) > \lambda(Q(H'_i))$ and $\lambda(Q(H'_i')) > \lambda(Q(S^{k+1}_i))$ hold. By Lemma 9 we have

$$\lambda(Q(H')) = \max_{1 \leq i \leq t'} \{\lambda(Q(H'_i))\} < \max_{1 \leq i \leq t'} \{\lambda(Q(H_i))\} = \lambda(Q(H)).$$

The proof is completed.

By Theorem 10 we have the following result for $\{\lambda(Q(G^k))\}$, which affirm Conjecture 5 for $Q(G^k)$.

**Theorem 11** Let $G$ be an ordinary graph with maximum $\Delta \geq 2$. When $k \geq 2$ we have $\lambda(Q(G^{k+1})) < \lambda(Q(G^k))$.

The notion of odd-bipartite even-uniform hypergraphs was introduced in [5].

**Definition 12** [5] Let $k$ be even and $G = (V, E)$ be a $k$-uniform hypergraph. It is called odd-bipartite if either it is trivial (i.e., $E = \emptyset$) or there is a disjoint partition of the vertex set $V$ as $V = V_1 \cup V_2$ such that $V_1, V_2 \neq \emptyset$ and every edge in $E$ intersects $V_1$ with exactly an odd number of vertices.

For even uniform odd-bipartite hypergraph, the following result was proved in [6] (see Theorem 5.8 of [6]), or in [12] (see Theorem 2.2 of [12]).

**Lemma 13** [6] [12] Let $G$ be a connected even uniform odd-bipartite hypergraph. Then $\lambda(L(G)) = \lambda(Q(G))$.

In fact Lemma 13 is also true for general even uniform odd-bipartite hypergraph $G$, see Lemma 14.

**Lemma 14** Let $G$ be an even uniform odd-bipartite hypergraph. Then $\lambda(L(G)) = \lambda(Q(G))$.

**Proof** By Lemma 13 we only need to consider the case that $G$ is not connected. Set $G = \bigcup_{i=1}^t G_i$, where $G_i$ is a connected component of $G$ for $1 \leq i \leq t$ and $t \geq 2$. Since $G$ is even uniform and odd-bipartite, each $G_i$ is connected even uniform and odd-bipartite. Thus by Lemma 13 we have $\lambda(L(G_i)) = \lambda(Q(G_i))$ ($i = 1, \cdots, t$), and moreover, by Lemma 9 we have

$$\lambda(L(G)) = \max_{1 \leq i \leq t} \{\lambda(L(G_i))\} = \max_{1 \leq i \leq t} \{\lambda(Q(G_i))\} = \lambda(Q(G)).$$

The proof is completed. □
Remark 15  We can mention here that the condition $\lambda(L(G)) = \lambda(Q(G))$ does not imply that $G$ is an even uniform odd-bipartite hypergraph. In fact take $G = G_1 \cup G_2$, where $G_1$ is not odd-bipartite and $G_2$ is a sunflower with size $\Delta$ satisfying

$$\Delta > \lambda(Q(G_1)) \geq \lambda(L(G_1)).$$

From Proposition 3.2 of [6] and Lemma 13 we know that

$$\lambda(L(G_2)) = \lambda(Q(G_2)) > \Delta.$$

Then $G$ is not odd-bipartite (since $G_1$ is not), but we have

$$\lambda(L(G)) = \max_{1 \leq i \leq 2} \{\lambda(L(G_i))\} = \lambda(L(G_2)) = \lambda(Q(G_2)) = \max_{1 \leq i \leq 2} \{\lambda(Q(G_i))\} = \lambda(Q(G)).$$

Obviously, when $k$ is even and $k \geq 4$, the $k$-th power hypergraph $G^k$ is odd-bipartite. Then Lemma 14 and Theorem 11 imply that Conjecture 8 is true for $L(G^k)$.

Theorem 16  Let $G = (V, E)$ be an ordinary graph with maximum $\Delta \geq 2$, $k = 2r$ be even and $G^k$ be the $k$-power hypergraph of $G$. Then $\{\lambda(L(G^k)) = \lambda(Q(G^k))\}$ is a strictly decreasing sequence.

The value $\lim_{k \to \infty} \lambda(Q(G^k))$ was determined for a regular graph $G$ by Zhou et al. in [17].

Lemma 17  [77] For any $d$-regular graph $G$ with $d \geq 2$, we have $\lim_{k \to \infty} \lambda(Q(G^k)) = d$.

Now we will prove that when $k$ goes to infinity, $\lambda(Q(S_d^k))$ converges to $d$, the maximum degree of $S_d^k$.

Lemma 18  When $k \geq 2, d \geq 2$ we have $\lim_{k \to \infty} \lambda(Q(S_d^k)) = d$.

Proof  Write $\lambda_k = \lambda(Q(S_d^k))$ for short. Let $x$ be the principal eigenvector of $Q(S_d^k)$ corresponding to $\lambda_k$. Let $a$ be the component of $x$ corresponding to the heart of $S_d^k$. By the symmetry of the pendent vertices in the same edge, we see that they have the same component in $x$. Furthermore, by the uniqueness of $x$ (see (3) of Theorem 7), we see that all the pendent vertices in $S_d^k$ have the same component in $x$, say $b$. Then $\lambda_k$ satisfies the following equations

$$\begin{cases} \lambda_k a^{k-1} = da^{k-1} + db^{k-1}, \\
\lambda_k b^{k-1} = b^{k-1} + ab^{k-2}. \end{cases}$$

By eliminations of $a$ and $b$, we obtain

$$(\lambda_k - d)(\lambda_k - 1)^{k-1} - d = 0.$$ 

Set

$$f_k(\lambda) = (\lambda - d)(\lambda - 1)^{k-1} - d,$$ 

then $\lambda_k$ is the largest real root of the equation $f_k(\lambda) = 0$.

Particularly,

$$(\lambda_{k+1} - d)(\lambda_{k+1} - 1)^{k} = d.$$ 

Since $f_k(d) = -d < 0$, and $\lim_{k \to +\infty} f_k(\lambda) = +\infty$, we have $\lambda_k > d$.

So $\{\lambda_k\}$ is a strictly decreasing sequence and $\lambda_k > d$ when $d \geq 2$. Thus $\lim_{k \to \infty} \lambda_k$ exists. From

$$(\lambda_k - d)(\lambda_k - 1)^{k-1} = d,$$ 

we have

$$\lim_{k \to \infty} \lambda_k - d = \frac{d}{\lim_{k \to \infty}(\lambda_k - 1)^{k-1}} = 0,$$ 

thus $\lim_{k \to \infty} \lambda_k = d$, i.e., $\lim_{k \to \infty} \lambda(Q(S_d^k)) = d$ holds. \hfill \Box

To determine the value $\lim_{k \to \infty} \lambda(Q(G^k))$ for a general graph $G$, we first cite a result just for a graph due to Kőnig.
Lemma 19 [9] Every graph G of maximum degree \( \Delta \) is an induced subgraph of some \( \Delta \)-regular graph.

Theorem 20 Let \( G \) be an ordinary graph with maximum degree \( \Delta \geq 2 \). Then we have \( \lim_{k \to \infty} \lambda(Q(G^k)) = \Delta \).

Proof We have proved that \( \{\lambda(Q(G^k))\} \) is a strictly decreasing sequence by Theorem 11. Obviously, \( \lambda(Q(G^k)) > 0 \) for any \( k \geq 2 \). Thus \( \lim_{k \to \infty} \lambda(Q(G^k)) \) exists.

It is known that if \( F' \) is a sub-hypergraph of \( F \), then \( \lambda(Q(F')) \leq \lambda(Q(F)) \) (see Proposition 4.5 in [4]). Since \( G \) has maximum degree \( \Delta \), \( G \) contains star \( S_{\Delta} \) as a sub-graph, and then \( G^k \) contains the sunflower \( S_{\Delta}^k \) as a sub-hypergraph. Thus \( \lambda(Q(G^k)) \geq \lambda(Q(S_{\Delta}^k)) \). Furthermore Lemma 18 implies that \[ \lim_{k \to \infty} \lambda(Q(G^k)) \geq \lim_{k \to \infty} \lambda(Q(S_{\Delta}^k)) = \Delta. \]

On the other hand, by Lemma 19 \( G \) is a subgraph of some \( \Delta \)-regular graph \( F \). Then \( G^k \) is a sub-hypergraph of \( F^k \). Thus \( \lambda(Q(G^k)) \leq \lambda(Q(F^k)) \). Furthermore by Lemma 17 we have \[ \lim_{k \to \infty} \lambda(Q(G^k)) \leq \lim_{k \to \infty} \lambda(Q(F^k)) = \Delta. \]

So we obtain \[ \lim_{k \to \infty} \lambda(Q(G^k)) = \Delta. \]

\( \square \)

4 Largest H-eigenvalue of adjacency tensor of \( G^{k,s} \)

In [17], it was proved that \( \lambda(A(G^k)) = \lambda(A(G))^k \); in [7] it was proved that \( \lambda(A(G^{k,k/2})) = \lambda(A(G)) \). By using the technique provided in [17], we will prove a general case.

Theorem 21 If \( \mu \neq 0 \) is an eigenvalue of the adjacency matrix \( A(G) \) of graph \( G \), then \( \mu^{\frac{2k}{\mu}} \) is an eigenvalue of the adjacency tensor \( A(G^{k,s}) \). Moreover \( \lambda(A(G^{k,s})) = \lambda(A(G))^\frac{k}{s} \).

Proof Suppose that \( x \) is an eigenvector of the eigenvalue \( \mu \neq 0 \) of \( A(G) \). As shown in Definition 6 for any edge \( e = \{u, v\} \) denote by \( V_u \cup V_v \) the corresponding edge of \( G^{k,s} \).

Now we construct a new vector \( y \) (of dimension \( |V(G^{k,s})| \)) from \( x \) by adding components. Set \[ y_w = \left\{ \begin{array}{ll} (x_v)^{\frac{k}{s}} & \text{if } w \in V_v \text{ for some } v, \\ (\mu^{-1}x_u x_v)^{\frac{k}{s}} & \text{if } w \in V_v \text{ for some edge } e = \{u, v\}. \end{array} \right. \]

Now we will show \( A(G^{k,s})y = \mu^{\frac{2k}{\mu}}y^{[k-1]} \) holding.

For any \( w \in V_v \) for some \( v \), by the formula \[ \sum_{\{u,v\} \in E(G)} x_u = \mu x_v, \]

we have

\[ (A(G^{k,s})y)_w = \sum_{\{u,v\} \in E(G)} (x_v)^{\frac{2(k-1)}{s}} (x_u)^{\frac{2k}{s}} (\mu^{-1}x_u x_v)^{\frac{k-2}{s}} = \mu^{\frac{2k}{\mu} - 1} (x_v)^{\frac{k-2}{s}} \sum_{\{u,v\} \in E(G)} x_u \]

\[ = \mu^{\frac{2k}{\mu}} (x_v)^{\frac{2(k-1)}{s}} = \mu^{\frac{2k}{\mu}} y_w^{k-1}. \]
For \( w \in V \) for any edge \( e = \{u, v\} \), we have
\[
(A(G^{k,s})y)_w = (x_u)^{2k} (x_v)^{2k} (\mu^{-1} x_u x_v)^{k-s-1} \mu^{2k} (\mu^{-1} x_u x_v)^{k-s-1} \mu^{2k} = \mu^{2k-s} y^{k-1}_w.
\]

Hence \( \mu^{2k-s} \) is an eigenvalue of \( A(G^{k,s}) \) with eigenvector \( y \).

If \( G \) is connected and \( \mu = \lambda(A(G)) \), then we may choose \( x \) as a positive eigenvector of \( \lambda(A(G)) \) by Perron-Frobenius theorem for irreducible nonnegative matrix. In this case \( y \) is a positive eigenvector of the eigenvalue \( \lambda(A(G))^{2k} \) of tensor \( A(G^{k,s}) \). In virtue of (2) of Theorem\(^7\) (or see Lemma 15 of \[17\]), we have
\[
\lambda(A(G^{k,s})) = \lambda(A(G))^{2k}.
\]

If \( G = G_1 \cup G_2 \cup \cdots \cup G_t \), where \( G_i \) is a connected component of \( G \) for \( 1 \leq i \leq t \) and \( t \geq 2 \), then by Lemma\(^8\)
\[
\lambda(A(G^{k,s})) = \max_{1 \leq i \leq t} \{ \lambda(A(G_i^{k,s})) \} = \max_{1 \leq i \leq t} \{ \lambda(A(G_i))^{2k} \} = \lambda(A(G))^{2k}.
\]

The proof is completed. \( \Box \)

Take \( s = 1 \), or \( s = k/2 \) for even \( k \), and noting that \( G^k = G^{k,1} \), we have Corollary\(^22\) and Corollary\(^23\)

**Corollary 22** \[7\] If \( \mu \neq 0 \) is an eigenvalue of the adjacency matrix \( A(G) \) of graph \( G \), then \( \mu^{2k} \) is an eigenvalue of the adjacency tensor \( A(G^{k}) \) of the hypergraph \( G^{k} \). Moreover \( \lambda(A(G^{k})) = \lambda(A(G))^{2k} \).

**Corollary 23** \[7\] Let \( G \) be a connected ordinary graph, and let \( x > 0 \) be vector defined on \( V(G) \). Let \( x > 0 \) be a vector defined on \( V(G^{k,k/2}) \) such that \( x_u = x_u^{2k} \) for each vertex \( u \in V \). Then \( x \) is an eigenvector of \( A(G) \) corresponding to \( \lambda(A(G)) \) if and only if \( x \) is an eigenvector of \( A(G^{k,k/2}) \) corresponding to \( \lambda(A(G^{k,k/2})) \). Hence \( \lambda(A(G^{k,k/2})) = \lambda(A(G)) \).

**Corollary 24** Let \( G \) be an ordinary graph with maximum \( \Delta \geq 2 \), \( s \geq 1 \) be a fixed integer. Then \( \lambda(A(G^{k+1,s})) < \lambda(A(G^{k,s})) \), and \( \lim_{k \to \infty} \lambda(A(G^{k,s})) = 1 \).

**References**

[1] C. Berge, Hypergraph. Combinatorics of Finite sets, third edition, North-Holland, Amsterdam, 1973.

[2] K.C. Chang, K. Pearson, T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6(2008)507-520.

[3] S. Friedland, A. Gaubert, L. Han, Perron-Frobenius theorems for nonnegative multilinear forms and extensions, Linear Algebra Appl., 438(2013)738-749.

[4] S. Hu, L. Qi, J. Shao, Cored hypergraphs and their Laplacian eigenvalues, Linear Algebra Appl., 439 (2013) 2980-2998.

[5] S. Hu, L. Qi, The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph, Discrete Appl. Math. 169 (2014) 140-151.

[6] S. Hu, L. Qi, J. Xie, The largest Laplacian and signless Laplacian II-eigenvalues of a uniform hypergraph, Linear Algebra Appl., 469 (2015) 1-27.

[7] M. Khan, Y. Fan, On the spectral radius of a class of non-odd-bipartite even uniform hypergraphs, Linear Algebra Appl. (2015), doi: 10.1016/j.laa.2015.04.005.
[8] M. I., Khan, Y. Z., Fan, The least H-eigenvalue of generalized power hypergraphs. arXiv preprint [arXiv:1504.03839] 2015.

[9] D. König, Theorie der endlichen und unendlichen Graphen, Akademische Verlagsgesellschaft, Leipzig, 1936

[10] H. Li, Y. Shao, L. Qi, The external spectral radii of k-uniform supertrees, to appear in: Journal of Combinatorial Optimization, [arXiv:1405.7257v1], May 2014.

[11] L. Qi, H⁺-eigenvalue of Laplacian and signless Laplacian tensors, Commun. Math. Sci. 12(2014)1045-1064.

[12] J. Shao, A general product of tensors with applications, Linear Algebra Appl., 439(2013)2350-2366.

[13] J. Shao, H. Shan, B. Wu, Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs, to appear in: Linear and Multilinear Algebra, arXiv: 1403.4845.

[14] Q. Yang, Y. Yang, Further results for Perron-Frobenius Theorem for nonnegative tensors II, Siam J Matrix Anal. Appl., 32(4)(2011)1236-1250.

[15] Y. Yang, Q. Yang, On some properties of nonegative weakly irreducible tensors, arXiv: 1111.0713 v3, 2011.

[16] J. Yue, L. Zhang, M. Lu, Largest adjacency, signless Laplacian, and Laplacian H-eigenvalues of loose paths, to appear in: Frontiers of Mathematics in China.

[17] J. Zhou, L. Sun, W. Wang, C. Bu, Some spectral properties of uniform hypergraphs, the electronic journal of combinatorics 21(4) (2014), #P4.24