Ground state alternative for \( p \)-Laplacian with potential term

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Abstract

Let \( \Omega \) be a domain in \( \mathbb{R}^d \), \( d \geq 2 \), and \( 1 < p < \infty \). Fix \( V \in L^\infty_{\text{loc}}(\Omega) \). Consider the functional \( Q \) and its Gâteaux derivative \( Q' \) given by

\[
Q(u) := \int_\Omega (|\nabla u|^p + V|u|^p) \, dx,
\]

\[
\frac{1}{p} Q'(u) := -\nabla \cdot (|\nabla u|^{p-2} \nabla u) + V|u|^{p-2} u.
\]

If \( Q \geq 0 \) on \( C_0^\infty(\Omega) \), then either there is a positive continuous function \( W \) such that \( \int W|u|^p \, dx \leq Q(u) \) for all \( u \in C_0^\infty(\Omega) \), or there is a sequence \( u_k \in C_0^\infty(\Omega) \) and a function \( v > 0 \) satisfying \( Q'(v) = 0 \), such that \( Q(u_k) \to 0 \), and \( u_k \to v \) in \( L^p_{\text{loc}}(\Omega) \). In the latter case, \( v \) is (up to a multiplicative constant) the unique positive supersolution of the equation \( Q'(u) = 0 \) in \( \Omega \), and one has for \( Q \) an inequality of Poincaré type: there exists a positive continuous function \( W \) such that for every \( \psi \in C_0^\infty(\Omega) \) satisfying \( \int \psi v \, dx \neq 0 \) there exists a constant \( C > 0 \) such that

\[ C^{-1} \int W|u|^p \, dx \leq Q(u) + C \left| \int \psi v \, dx \right|^p \]

for all \( u \in C_0^\infty(\Omega) \).

As a consequence, we prove positivity properties for the quasilinear operator \( Q' \) that are known to hold for general subcritical resp. critical second-order linear elliptic operators.

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1 Introduction

Positivity properties of quasilinear elliptic equations, in particular those with the $p$-Laplacian term in the principal part, have been extensively studied over the recent decades (see for example [2, 3, 4, 6, 7, 11, 12, 15, 16, 21, 25, 33, 38] and the references therein). The objective of the present paper is to study general positivity properties of such equations defined on general domains in $\mathbb{R}^d$. We generalize some results, obtained for $p = 2$ in [32], to the case of $p \in (1, \infty)$. In particular, we extend to the case of the $p$-Laplacian the dichotomy of [32] obtained for nonnegative Schrödinger operators which states that either the associated quadratic form has a weighted spectral gap or the operator admits a unique ground state.

Some of the proofs in this paper (in particular, the uniqueness of a global positive supersolution for a critical operator) seem to be new even for the previously studied case $p = 2$.

Fix $p \in (1, \infty)$, a domain $\Omega \subseteq \mathbb{R}^d$, and a potential $V \in L^\infty_{\text{loc}}(\Omega)$. We denote by $\Delta_p(u) := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ the $p$-Laplacian. Throughout this paper we assume that

$$Q(u) := \int_\Omega (|\nabla u|^p + V|u|^p) \, dx \geq 0 \quad (1.1)$$

for all $u \in C^\infty_0(\Omega)$.

**Definition 1.1.** We say that a function $v \in W^{1,p}_{\text{loc}}(\Omega)$ is a (weak) solution of the equation

$$\frac{1}{p} Q'(v) := -\Delta_p(v) + V|v|^{p-2}v = 0 \quad \text{in } \Omega, \quad (1.2)$$

if for every $\varphi \in C^\infty_0(\Omega)$

$$\int_\Omega (|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi + V|v|^{p-2}v \varphi) \, dx = 0. \quad (1.3)$$

We say that a positive function $v \in C^1_{\text{loc}}(\Omega)$ is a positive supersolution of the equation (1.2) if for every nonnegative $\varphi \in C^\infty_0(\Omega)$

$$\int_\Omega (|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi + V|v|^{p-2}v \varphi) \, dx \geq 0. \quad (1.4)$$

**Remark 1.2.** It is well-known that any weak solution of (1.2) admits Hölder continuous first derivatives, and that any nonnegative solution of (1.2) satisfies the Harnack inequality [10, 36, 37].
Definition 1.3. We say that the functional $Q$ has a \textit{weighted spectral gap} in $\Omega$ (or $Q$ is \textit{strictly positive} in $\Omega$) if there is a positive continuous function $W$ in $\Omega$ such that
\[ Q(u) \geq \int_{\Omega} W|u|^p \, dx \quad \forall u \in C_0^\infty(\Omega). \] (1.5)

Definition 1.4. We say that a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ is a \textit{null sequence}, if $u_k \geq 0$ for all $k \in \mathbb{N}$, and there exists an open set $B \subset \Omega$ (i.e., $\overline{B}$ is compact in $\Omega$) such that $\int_B |u_k|^p \, dx = 1$, and
\[ \lim_{k \to \infty} Q(u_k) = \lim_{k \to \infty} \int_{\Omega} (|\nabla u_k|^p + V|u_k|^p) \, dx = 0. \] (1.6)

We say that a positive function $v \in C^1_{\text{loc}}(\Omega)$ is a \textit{ground state} of the functional $Q$ in $\Omega$ if $v$ is an $L^p_{\text{loc}}(\Omega)$ limit of a null sequence. If $Q \geq 0$, and $Q$ admits a ground state in $\Omega$, we say that $Q$ is \textit{degenerately positive} in $\Omega$.

Definition 1.5. The functional $Q$ is \textit{nonpositive} in $\Omega$ if it takes negative values on $C_0^\infty(\Omega)$.

The main result of the present paper reads as follows:

\textbf{Theorem 1.6.} Let $\Omega \subseteq \mathbb{R}^d$ be a domain, $V \in L^\infty_{\text{loc}}(\Omega)$, and $p \in (1, \infty)$. Suppose that the functional $Q$ is nonnegative. Then

(a) $Q$ has either a weighted spectral gap or a ground state.

(b) If the functional $Q$ admits a ground state $v$, then $v$ satisfies (1.2).

(c) The functional $Q$ admits a ground state if and only if (1.2) admits a unique positive supersolution.

(d) If $Q$ has a ground state $v$, then there exists a positive continuous function $W$ in $\Omega$, such that for every $\psi \in C_0^\infty(\Omega)$ satisfying $\int \psi \, dx \neq 0$ there exists a constant $C > 0$ such that the following inequality holds:
\[ C^{-1} \int_{\Omega} W|u|^p \, dx \leq Q(u) + C \left| \int_{\Omega} \psi u \, dx \right|^p \quad \forall u \in C_0^\infty(\Omega). \] (1.7)

Theorem 1.6 extends [32, Theorem 1.5] that deals with the linear case $p = 2$. 

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The outline of the present paper is as follows. The next section gives preliminary results concerning positive solutions of the equation $Q'(u) = 0$. In particular, it introduces the generalized Picone’s identity \[2, 3\] which is a crucial tool in our study. Section 3 is devoted to the proof of Theorem 1.6. In Section 4, we study, using Theorem 1.6, criticality properties of the functional $Q$ along the lines of criticality theory for second-order linear elliptic operators \[27, 28\].

In Section 5 we prove, for $1 < p \leq d$, the existence of a unique (up to a multiplicative constant) positive solution of the equation $Q'(u) = 0$ in $\Omega \setminus \{x_0\}$ which has a minimal growth in a neighborhood of infinity in $\Omega$. The proof of the above result relies on an unpublished lemma of L. Véron (Lemma 5.1) concerning the exact asymptotic behavior of a singular positive solution of the equation $Q'(u) = 0$ in a punctured neighborhood of $x_0$. We thank Professor Véron for kindly supplying the proof of this key result. The study of positive solutions which has a minimal growth in a neighborhood of infinity in $\Omega$ leads us to another characterization of strict positivity in terms of these solutions.

In Section 6, we pose a number of open problems suggested by the results of the present paper. The Appendix contains a new energy estimate for the functional $Q$ valid for $p > 2$. This estimate leads to an alternative proof of Lemma 3.2 for the case $p > 2$.

2 Positive solutions and Picone identity

Let $v > 0$, $v \in C^1_{\text{loc}}(\Omega)$, and $u \geq 0$, $u \in C_0^\infty(\Omega)$. Denote

$$R(u, v) := |\nabla u|^p - \nabla \left( \frac{u^p}{v^{p-1}} \right) \cdot |\nabla v|^{p-2}\nabla v,$$  \hspace{1cm} (2.1)

and

$$L(u, v) := |\nabla u|^p + (p - 1) \frac{u^p}{v^p} |\nabla v|^p - p \frac{u^{p-1}}{v^{p-1}} |\nabla u| |\nabla v|^{p-2}\nabla v.$$  \hspace{1cm} (2.2)

Then the following (generalized) Picone identity holds \[2, 3\]

$$R(u, v) = L(u, v).$$  \hspace{1cm} (2.3)

Write $L(u, v) = L_1(u, v) + L_2(u, v)$, where

$$L_1(u, v) := |\nabla u|^p + (p - 1) \frac{u^p}{v^p} |\nabla v|^p - p \frac{u^{p-1}}{v^{p-1}} |\nabla u||\nabla v|^{p-1},$$  \hspace{1cm} (2.4)
and
\[ L_2(u, v) := p \frac{u^{p-1}}{v^{p-1}} |\nabla v|^{p-2}(|\nabla u||\nabla v| - \nabla u \cdot \nabla v) \geq 0. \] (2.5)

From the obvious inequality \( t^p + (p-1) - pt \geq 0 \), we also have that \( L_1(u, v) \geq 0 \). Therefore, \( L(u, v) \geq 0 \) in \( \Omega \).

Let \( v \in C^1_\text{loc}(\Omega) \) be a positive solution (resp. supersolution) of (1.2). Using (2.3) and (1.3) (resp. (1.4)), we infer that for every \( u \in C^\infty_0(\Omega) \), \( u \geq 0 \),
\[ Q(u) = \int_\Omega L(u, v) \, dx \geq 0, \quad \text{resp.} \quad Q(u) \geq \int_\Omega L(u, v) \, dx \geq 0 \] (2.6)

For any smooth subdomain \( \Omega' \subset \Omega \) consider the variational problem
\[ \lambda_{1,p}(\Omega') := \inf_{u \in W^{1,p}_0(\Omega')} \frac{\int_{\Omega'}(|\nabla u|^p + V|u|^p) \, dx}{\int_{\Omega'} |u|^p \, dx}. \] (2.7)

It is well-known that for such a subdomain, (2.7) admits (up to a multiplicative constant) a unique minimizer \( \varphi \) [12, 17]. Moreover, \( \varphi \) is a positive solution of the quasilinear eigenvalue problem
\[ \begin{cases} 
Q'(\varphi) = \lambda_{1,p}(\Omega')|\varphi|^{p-2}\varphi & \text{in } \Omega', \\
\varphi = 0 & \text{on } \partial\Omega'. 
\end{cases} \] (2.8)

\( \lambda_{1,p}(\Omega') \) and \( \varphi \) are called the principal eigenvalue and eigenfunction of the operator \( Q' \), respectively.

The following theorem was proved by J. García-Melián, and J. Sabina de Lis [17] (see also [2, 3]).

**Theorem 2.1.** Assume that \( \Omega \subset \mathbb{R}^d \) is a bounded \( C^{1+\alpha} \)-domain, \( 0 < \alpha < 1 \), and suppose that \( V \in L^\infty(\Omega) \). Then the following assertions are equivalent

(i) \( Q' \) satisfies the maximum principle: If \( u \) is a solution of the equation
\[ Q'(u) = f \geq 0 \text{ in } \Omega \] with some \( f \in L^\infty(\Omega) \), and satisfies \( u \geq 0 \) on \( \partial\Omega \), then \( u \) is nonnegative in \( \Omega \).

(ii) \( Q' \) satisfies the strong maximum principle: If \( u \) is a solution of the equation \( Q'(u) = f \geq 0 \) in \( \Omega \) with some \( f \in L^\infty(\Omega) \), and satisfies \( u \geq 0 \) on \( \partial\Omega \), then \( u > 0 \) in \( \Omega \).

(iii) \( \lambda_{1,p}(\Omega) > 0 \).
(iv) For some $0 \leq f \in L^\infty(\Omega)$ there exists a positive strict supersolution $v$ satisfying $Q'(v) = f$ in $\Omega$, and $v = 0$ on $\partial \Omega$.

(iv') There exists a positive strict supersolution $v$ satisfying $Q'(v) \approx 0$ in $\Omega$, such that $v \in C^{1+\alpha}(\partial \Omega)$ and $f \in L^\infty(\Omega)$.

(v) For each nonnegative $f \in C^\alpha(\Omega) \cap L^\infty(\Omega)$ there exists a unique weak nonnegative solution of the problem $Q'(u) = f$ in $\Omega$, and $u = 0$ on $\partial \Omega$.

We shall need also the following comparison principle of J. García-Melián, and J. Sabina de Lis [17, Theorem 5].

Theorem 2.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,\alpha}$, $0 < \alpha \leq 1$. Assume that $\lambda_{1,p}(\Omega) > 0$ and let $u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying $Q'(u_i) \in L^\infty(\Omega)$, $u_i|_{\partial \Omega} \in C^{1+\alpha}(\partial \Omega)$, where $i = 1, 2$. Suppose further that the following inequalities are satisfied

\[
\begin{cases}
Q'(u_1) \leq Q'(u_2) & \text{in } \Omega, \\
Q'(u_2) \geq 0 & \text{in } \Omega, \\
u_1 \leq u_2 & \text{on } \partial \Omega, \\
u_2 \geq 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.9)

Then

\[u_1 \leq u_2 \quad \text{in } \Omega.\]

The following theorem generalizes the well-known Allegretto-Piepenbrink theorem (see [8, Theorem 2.12] and the references therein).

Theorem 2.3. Let $Q$ be a functional of the form (1.1). Then the following assertions are equivalent

(i) The functional $Q$ is nonnegative on $C^\infty_0(\Omega)$.

(ii) Equation (1.2) admits a global positive solution.

(iii) Equation (1.2) admits a global positive supersolution.

Proof. (i) ⇒ (ii): Assume that $Q \geq 0$ on $C^\infty_0(\Omega)$. Then $Q$ is nonnegative on $C^\infty_0(\Omega')$ for any smooth bounded domain $\Omega' \Subset \Omega$. Fix an exhaustion $\{\Omega_N\}_{N=1}^\infty$ of $\Omega$ (i.e., a sequence of smooth, relatively compact domains such that $x_0 \in \Omega_1$, $\text{cl}(\Omega_N) \subset \Omega_{N+1}$ and $\bigcup_{N=1}^\infty \Omega_N = \Omega$). By the strict monotonicity of $\lambda_{1,p}(\Omega)$ as a function of $\Omega$ [2, Theorem 2.3], it follows that $\lambda_{1,p}(\Omega_N) > 0$
for all \( N \geq 1 \). Let \( f_N \in C_0^\infty(\Omega_N \setminus \Omega_{N-1}) \) be a nonnegative nonzero function. By Theorem 2.1, there exists a unique positive solution of the problem

\[
Q'(u_N) = f_N \quad \text{in} \quad \Omega_N, \quad u_N = 0 \quad \text{on} \quad \partial \Omega_N.
\]

Set \( v_N(x) := u_N(x)/u_N(x_0) \). By Harnack’s inequality and elliptic regularity \([34, 37]\), it follows that \( \{v_N\} \) admits a subsequence which converges locally uniformly to a positive solution \( v \) of the equation \( Q'(u) = 0 \) in \( \Omega \).

(ii) \( \Rightarrow \) (iii) is trivial.

(iii) \( \Rightarrow \) (i): Suppose that (1.2) admits a global positive supersolution \( v \) in \( \Omega \). If \( u \in C_0^\infty(\Omega) \) is a nonnegative function, then by (2.6) \( Q(u) \geq 0 \). Let \( u \in W_{loc}^{1,p}(\Omega) \) be a nonnegative function with compact support. By taking a sequence \( \{u_k\} \subset C_0^\infty(\Omega) \) of nonnegative functions such that \( u_k \to u \) in \( W_{loc}^{1,p}(\Omega) \), we infer that \( Q(u) \geq 0 \). Thus, \( Q \geq 0 \) on the cone of all \( W_{loc}^{1,p}(\Omega) \) nonnegative functions with compact support in \( \Omega \). Since \( Q(u) = Q(|u|) \) on \( C_0^\infty(\Omega) \), it follows that \( Q \) is nonnegative on \( C_0^\infty(\Omega) \).

3 Proof of Theorem 1.6

We shall start with the following two lemmas. For \( B \subset \Omega \) let

\[
c_B := \inf_{u \in C_0^\infty(\Omega)} Q(u) = \inf_{0 \leq u \in C_0^\infty(\Omega)} \frac{\int_B |u|^p \, dx}{\int_B |u|^p \, dx}.
\] (3.1)

**Lemma 3.1.** If for every open set \( B \subset \Omega \), \( c_B > 0 \), then there exists a \( W \in C(\Omega) \), \( W > 0 \), such that

\[
Q(u) \geq \int_\Omega W(x)|u(x)|^p \, dx \quad \forall u \in C_0^\infty(\Omega).
\] (3.2)

In other words, \( Q \) has a weighted spectral gap with the weight \( W \).

**Proof.** Let \( \{B_j\}_{j=1}^\infty \) be a locally finite covering of \( \Omega \) by open balls \( B_i \subset \Omega \), and let \( \{\chi_j\} \) be a locally finite partition of unity on \( \Omega \) subordinated to this covering. Set \( C_j := \min\{c_{B_j}, 1\} \). Then

\[
2^{-j}Q(u) \geq 2^{-j}c_{B_j} \int_{B_j} |u|^p \, dx \geq 2^{-j}C_j \int_\Omega \chi_j |u|^p \, dx \quad \forall u \in C_0^\infty(\Omega).
\] (3.3)

By summation (3.3) over \( j \in \mathbb{N} \) and by interchanging the order of summation and integration we obtain (3.2) with \( W(x) := \sum_{j=1}^\infty 2^{-j}C_j \chi_j(x) \). \( \square \)
Lemma 3.2. If there exists a nonempty open set $B \subseteq \Omega$ such that $c_B = 0$, then $Q$ admits a ground state.

Proof. Fix a positive (super)solution $v \in C^1(\Omega)$. Since $c_B = 0$, there exists a sequence $\{u_k\} \subset C_0^\infty(\Omega)$, $u_k \geq 0$, such that $\int_B |u_k|^p \, dx = 1$ and $Q(u_k) \to 0$. Let $\omega \subseteq \Omega$ be an open connected set containing $B$.

Step 1. Let $\omega' \subset \omega$. By (2.6), and since $L(u_k, v) \geq 0$, we have that
\[
\int_{\omega'} |\nabla u_k|^p \, dx \leq o(1) + C \int_{\omega'} u_k^{p-1} |\nabla u_k| \, dx,
\]
with the constant $C = C(\omega)$ independent of $\omega'$. Invoking Young’s inequality, we arrive at
\[
\int_{\omega'} |\nabla u_k|^p \, dx \leq o(1) + \frac{1}{2} \int_{\omega'} |\nabla u_k|^p \, dx + C \int_{\omega'} u_k^p \, dx.
\]
Therefore,
\[
\int_{\omega'} |\nabla u_k|^p \, dx \leq C(\omega) \int_{\omega'} u_k^p \, dx + o(1).
\]

Step 2. Let
\[
\omega_0 := \{x \in \omega : \exists \rho(x) \in (0, d(x, \Omega \setminus \omega)), \sup_k \int_{B_\rho(x)} |u_k|^p \, dx < \infty\}.
\]

Since $\int_B |u_k|^p \, dx = 1 < \infty$, $B \subset \omega_0$. Moreover, $\omega_0$ is an open set. Indeed, let $x \in \omega_0$, $B_{\rho(x)}(x) \subset \omega_0$. Then for every point $y \in B_{\rho(x)}(x)$ there is a $\rho(y) > 0$ such that $B_{\rho(y)}(y) \subset B_{\rho(x)}(x)$. Therefore, $\int_{B_{\rho(y)}} |u_k|^p \, dx \leq \int_{B_{\rho(x)}} |u_k|^p \, dx$. Consequently $y \in \omega_0$, and $\omega_0$ is open.

Let us show now that $\omega_0$ is a relatively closed set in $\omega$. We shall use the following version of Poincaré inequality
\[
\int_{B_1(0)} |u|^p \, dx \leq C \int_{B_1(0)} |\nabla u|^p \, dx + C_r \int_{B_1(0)} u \, dx \quad \forall r \in (0, 1), \ \forall u \in C^\infty(\mathbb{R}^d),
\]
which follows for example from [39, Theorem 4.2.1]. It easily follows from (3.8) that for every \( \epsilon > 0 \) and \( \rho > 0 \) there exist \( \delta_\epsilon > 0 \) and \( C(\epsilon, \rho) \) such that for every \( x \in \omega, \delta \in (0, \min\{\delta_\epsilon, d(x, \partial \omega)\}) \), and \( u \in C_0^\infty(\Omega) \),

\[
\int_{B_\delta(x)} |u|^p \, dx \leq \epsilon \int_{B_\delta(x)} |\nabla u|^p \, dx + C(\epsilon, \rho) \int_{B_\rho(x)} |u|^p \, dx. \tag{3.9}
\]

Let \( x_j \in \omega_0 \), \( x_j \to x_0 \in \omega \). Let \( \epsilon < (C(\omega))^{-1/2} \), where \( C(\omega) \) is the constant in (3.6). Let \( \delta_\epsilon > 0 \) be as in (3.9) and fix \( \delta \in (0, \min\{\delta_\epsilon, d(x, \partial \omega)\}) \). Finally, choose \( j \) such that \( |x_0 - x_j| \leq \frac{\delta}{2} \). Then, with \( \rho = \rho(x_j) \), (3.6) and (3.9) imply

\[
\frac{1}{2} C(\omega)^{-1} \int_{B_\delta(x_j)} |\nabla u_k|^p \, dx \leq C(\epsilon, \rho(x_j)) \int_{B_\rho(x_j)} |u_k|^p \, dx + o(1). \tag{3.10}
\]

The right hand side of (3.10) is bounded in \( k \) by the definition of \( \omega_0 \). Thus \( \int_{B_\delta(x_j)} |\nabla u_k|^p \, dx \) is bounded, and by (3.9), \( \int_{B_\delta(x_j)} |u_k|^p \, dx \) is also bounded. Since, by the choice of \( j \), \( B_{\delta/2}(x_0) \subset B_\delta(x_j) \), it follows that \( \int_{B_{\delta/2}(x_0)} (|\nabla u_k|^p + |u_k|^p) \, dx \) is bounded and consequently, \( x_0 \in \omega_0 \), and \( \omega_0 \) is also relatively closed.

Since \( \omega \) is connected and \( \omega_0 \neq \emptyset \), it follows that \( \omega_0 = \omega \) and \( \{u_k\} \) is bounded in \( L_{loc}^p(\Omega) \). Invoking again (3.6) it follows that \( \{u_k\} \) is bounded in \( W_{loc}^{1,p}(\Omega) \).

**Step 3.** Consider now a weakly convergent renamed subsequence \( u_k \rightharpoonup u \) in \( W_{loc}^{1,p}(\Omega) \). Let \( \omega \Subset \Omega \) be a smooth domain, and set \( Q^\omega(u) := \int_\omega L(u, v) \, dx \). We claim that the functional \( Q^\omega(u) \) is weakly lower semicontinuous in \( W_1^{1,p}(\omega) \).

Indeed, the functionals \( \int_\omega |\nabla u|^p \, dx \) and \( \int_\omega (p-1) \frac{|\nabla u|^p}{|u|^p} |\nabla v|^p \, dx \) are weakly lower semicontinuous in \( W_1^{1,p}(\omega) \) since their Lagrangians \( (\mathcal{L}(q, z, x) = |q|^p \) and \( \mathcal{L}(q, z, x) = (p - 1) \frac{|q|^p}{|u(x)|^p} |\nabla v(x)|^p \), resp.) are convex functions of \( q \). So, it suffices to show that the functional

\[
J^\omega(u) := \int_\omega \frac{u^{p-1}}{u_{p-1}} \nabla u \cdot |\nabla v|^{p-2} \nabla v \, dx \tag{3.11}
\]

is weakly continuous on any sequence \( \{u_k\} \) satisfying \( u_k \rightharpoonup u \) in \( W_1^{1,p}(\omega) \).

Indeed,

\[
J^\omega(u_k) - J^\omega(u) = \int_\omega |\nabla v|^{p-2} |\nabla v| \nabla u_k \cdot (u_k^{p-1} - u_{p-1}) \, dx + \\
\int_\omega \nabla (u_k - u) \cdot \nabla |\nabla v|^{p-2} |\nabla v|^p u_k \, dx. \tag{3.12}
\]
Consider the first term of the right hand side of (3.12). Since \( u_k \to u \) in \( W^{1,p}(\omega) \), it follows by the compactness of the local Sobolev imbeddings that (up to a subsequence) \( u_k \to u \) in \( L^p(\omega) \). Then, for a renamed subsequence, there exists a \( U \in L^p(\omega) \), such that \( 0 \leq u_k \leq U \) and \( u_k \to u \) a.e. in \( \omega \). Therefore, \( u_k^{p-1} \leq U^{p-1} \in L^{p'}(\omega) \), where \( p' := p/(p-1) \) is the conjugate exponent of \( p \). Consequently, 

\[
|u_k^{p-1} - u^{p-1}|^{p'} \leq C(U^p + u^p) \in L^1(\omega).
\] (3.13)

Hence by Hölder’s inequality and Lebesgue’s dominated convergence theorem,

\[
\left| \int_\omega |v|^{p-2}v^{1-p} \nabla u_k(u_k^{p-1} - u^{p-1}) \, dx \right| \leq C\|\nabla u_k\|_p\|u_k^{p-1} - u^{p-1}\|_{p'} \to 0.
\] (3.14)

Consider the functional

\[
\Phi(w) := \int_\omega \nabla w \cdot \nabla |\nabla v|^{p-2}v^{1-p}u^{p-1} \, dx.
\]

Note \( |\nabla v|^{p-1}v^{1-p}u^{p-1} \in L^{p'}(\omega) \). Therefore, Hölder’s inequality implies that \( \Phi \) is a continuous functional on \( W^{1,p}(\omega) \). Hence, by the definition of weak convergence, the second term of the right hand side of (3.12) converges to zero.

We conclude that \( 0 \leq Q^\omega(u) \leq \lim inf Q^\omega(u_k) = 0 \). Moreover, \( \int_B u^p \, dx = 1 \). Now we repeat the argument of [2]. Since \( Q^\omega(u) = 0 \) for every subdomain \( \omega \subset \Omega \) containing \( B \), it follows that \( L_1(u, v) = 0 \) and \( L_2(u, v) = 0 \). Recall that \( f(t) = tp + p - 1 - pt \) is a nonnegative function on \( \mathbb{R}_+ \) which attains its zero minimum only at \( t = 1 \). Therefore, \( L_1(u, v) = 0 \) implies that \( u^{-1}|\nabla u| = v^{-1}|\nabla v| \). On the other hand, \( L_2(u, v) = 0 \) implies that \( \nabla u \) is parallel to \( \nabla v \). Hence, \( u = cv \), where \( c > 0 \). The value of \( c \) is determined by the condition \( \int_B (cv)^p \, dx = 1 \). Therefore \( cv \) is the limit of every weakly convergent subsequence of \( \{u_k\} \). It follows that the original sequence \( u_k \to cv \) in \( L^p_{\text{loc}}(\Omega) \).

\( \square \)

**Remark 3.3.** The argument in Step 3 of the proof of Lemma 3.2 shows that the functional \( Q \) has some weakly lower semicontinuity properties, as Proposition 3.5 below demonstrates.

First, we need to extend the definition of the functional \( Q \) from \( C_0^\infty(\Omega) \) to a larger set in \( W^{1,p}_{\text{loc}}(\Omega) \).
Definition 3.4. Let $v$ be a positive solution of the equation $Q'(u) = 0$ in $\Omega$. We define a functional $Q_v : W^{1,p}_{\text{loc}}(\Omega) \to [0, +\infty]$ by

$$Q_v(u) := \begin{cases} \int_{\Omega} L(|u|, v) \, dx & \text{if this integral is finite,} \\ \infty & \text{otherwise.} \end{cases}$$

Also, for $u \in W^{1,p}_{\text{loc}}(\Omega)$ and any $\omega \Subset \Omega$ define $Q^\omega_v(u) := \int_{\omega} L(|u|, v) \, dx$.

We have

Proposition 3.5. Let

$$D := \{ u \in W^{1,p}_{\text{loc}}(\Omega) \mid Q_v(u) < \infty, \text{ and } \exists \{u_k\} \subset \bigcup_{\omega \Subset \Omega} W^{1,p}_0(\omega) \text{ s.t.} u_k \to u \text{ in } W^{1,p}_{\text{loc}}(\Omega), \text{ and } Q_v(u_k) \to Q_v(u) \}. \quad (3.15)$$

Then the functional $Q$ on $C_0^\infty(\Omega)$ admits an extension to the set $D$ given by $Q(u) := Q_v(u)$. This extension is independent of the positive solution $v$. Moreover, the functional $Q$ is continuous on $W^{1,p}_0(\omega) \subset D$ for every $\omega \Subset \Omega$, and is weakly lower semicontinuous in the following sense:

$$u_k, u \in D, u_k \rightharpoonup u \text{ in } W^{1,p}_{\text{loc}}(\Omega), \sup_{k \in \mathbb{N}} Q(u_k) < \infty \Rightarrow \liminf_{k \to \infty} Q(u_k) \geq Q(u). \quad (3.16)$$

Note that the functional $Q^{1/p}$ is generally not a norm (see the discussion prior to Problem 6.1). If $Q^{1/p}$ is a norm and $Q$ is strictly positive, then the set $D$ is a closure of $C_0^\infty(\Omega)$ in that norm and so it is a Banach space. In this case, the condition in (3.16) is equivalent to weak convergence in $D$, and $Q$ is weakly lower semicontinuous as a monotone increasing function of the norm.

Proof. The functionals $Q$ and $Q_v$ obviously admit continuous extensions to $W^{1,p}_0(\omega)$ for every $\omega \Subset \Omega$. Since $Q$ and $Q_v$ are even and coincide on non-negative $C_0^\infty(\Omega)$-functions, their respective extensions to $W^{1,p}_0(\omega)$ are equal. Consequently, the set $D$ is independent of $v$. Moreover, the functional $Q_v$ evaluated on $D$ is independent of $v$ and thus defines the extension of $Q$.

Let $u_k \rightharpoonup u$ in $W^{1,p}_{\text{loc}}(\Omega)$, $u_k, u \in D$, and $Q_v(u_k) \leq C$ for some $C > 0$. Assume first that $u_k \geq 0$. Then $Q_v(u_k) \geq Q^\omega_v(u_k)$ for every open $\omega \Subset \Omega$. Step 3 of Lemma 3.2 implies that $\liminf Q_v(u_k) \geq \liminf Q^\omega_v(u_k) \geq Q^\omega_v(u)$. Since $\omega$ is arbitrary, we have $\liminf Q_v(u_k) \geq Q_v(u) = Q(u)$.
Let now remove the restriction $u_k \geq 0$. Let $\omega \subset \Omega$ be a smooth domain. Due to the compactness of the imbedding of $W^{1,p}(\omega)$ into $L^p(\omega)$, $u_k \to u$ a.e. on a renamed subsequence, and $|u_k| \leq U$ with some $U \in L^p_{\text{loc}}(\Omega)$. Consequently $|u_k| \to |u|$ in $L^p_{\text{loc}}(\Omega)$. Since $|u_k|$ is bounded with respect to the seminorms of $W^{1,p}_{\text{loc}}(\Omega)$, $|u_k| \to |u|$ in $W^{1,p}_{\text{loc}}(\Omega)$. Since the functional $Q_v$ is even, by the previous argument it follows that

$$\liminf_{k \to \infty} Q_v(u_k) = \liminf_{k \to \infty} Q_v(|u_k|) \geq Q_v(|u|) = Q_v(u) = Q(u),$$

and the proposition is proved. \hfill \Box

Proof of Theorem 1.6. Part (a) follows from Lemma 3.1 and Lemma 3.2. To prove (b) and (c), observe that from Lemma 3.2 it follows that for each positive supersolution $v$ of (1.2), any null sequence $\{u_k\}$ converges to a constant multiple of $v$. This implies that all positive supersolutions of (1.2) are scalar multiples of the same function. On the other hand, if $Q$ admits a weighted spectral gap with a weight $W$, then by Theorem 2.3, the equation $\frac{1}{p}Q'(u) - W|u|^{p-2}u = 0$ admits a positive solution $v$ in $\Omega$. So, $v$ is a strictly positive supersolution of the equation $Q'(u) = 0$ in $\Omega$. In addition, by Theorem 2.3, the equation $Q'(u) = 0$ admits a positive solution $w$ in $\Omega$. Clearly $v \neq w$, and the equation $Q'(u) = 0$ admits two linearly independent positive subsolutions in $\Omega$.

It remains to prove (d). First we claim that for every open set $B \subset \Omega$ there is a strictly positive continuous function $W$ such that

$$\int_\Omega W|u|^p \, dx \leq Q(u) + \int_B |u|^p \, dx \quad \forall u \in C^\infty_0(\Omega). \quad (3.17)$$

Indeed, denote the functional in the right hand side of (3.17) by $\bar{Q}$. Clearly, $\bar{Q} \geq 0$ on $C^\infty_0(\Omega)$. Suppose that $\bar{Q}$ admits a null sequence $\{u_k\}$, then $\{u_k\}$ is a null sequence of $Q$, which implies $\int_B |u_k|^p \, dx \to \int_B |v|^p \, dx$, where $v$ is a ground state for $Q$. Consequently, $\liminf \bar{Q}(u_k) > 0$, which contradicts the definition of $\{u_k\}$. Therefore, part (a) of the present theorem implies that $\bar{Q}$ admits a weighted spectral gap.

Therefore, in order to prove (1.7), it suffices to show that for some open $B \subset \Omega$,

$$\int_B |u|^p \, dx \leq C \left( Q(u) + \left| \int_\Omega u \psi \, dx \right|^p \right) \quad \forall u \in C^\infty_0(\Omega). \quad (3.18)$$
Suppose that (3.18) fails. Then there is a sequence \( \{u_k\} \subset C_0^\infty(\Omega) \) such that \( \int_B |u_k|^p \, dx = 1 \), \( Q(u_k) \to 0 \) and \( \int_\Omega u_k \psi \, dx \to 0 \). Since \( \{u_k\} \) is a null sequence it converges in \( L^p_{\text{loc}}(\Omega) \) to \( v \), where \( v > 0 \) is the ground state of \( Q \). Then \( \int_\Omega u_k \psi \, dx \to \int_\Omega v \psi \, dx \neq 0 \), and we arrive at a contradiction.

For \( u \in C_0^\infty(\Omega) \), we define

\[
\tilde{Q}(u) := \begin{cases} 
Q(u) & \text{if } \tilde{Q} \text{ has a weighted spectral gap}, \\
Q(u) + C \left| \int_\Omega \psi u \, dx \right|^p & \text{if } \tilde{Q} \text{ has a ground state},
\end{cases}
\]

where \( C \) is the constant in (1.7).

**Proposition 3.6.** For any \( C > 0 \), the set \( S := \{ u \in C_0^\infty(\Omega) \mid u \geq 0, \tilde{Q}(u) \leq C \} \) is bounded in \( W^{1,p}_{\text{loc}}(\Omega) \) and therefore, it is relatively compact in \( L^p_{\text{loc}}(\Omega) \).

**Proof.** Let \( u \in S \) and let \( v \) be a positive solution of (1.2). Let \( \omega \in \Omega \) be an open set. Then by (2.6), we have that

\[
\int_\omega (|\nabla u|^p + |u|^p) \, dx \leq \int_\omega (L(u,v) + C|u|^{p-1} |\nabla u| + |u|^p) \, dx \leq C + C \int_\omega |u|^{p-1} |\nabla u| \, dx + \int_\omega |u|^p \, dx.
\]

In light of Young’s inequality we obtain

\[
\int_\omega (|\nabla u|^p + |u|^p) \, dx \leq C + \frac{1}{2} \int_\omega |\nabla u|^p \, dx + C \int_\omega |u|^p \, dx.
\]

Consequently,

\[
\int_\omega (|\nabla u|^p + |u|^p) \, dx \leq C + C \int_\omega |u|^p \, dx. \tag{3.19}
\]

By (1.5) or (1.7) and the definition of \( S \), the right hand side of (3.19) is uniformly bounded in \( S \), and therefore \( S \) is a bounded set in \( W^{1,p}_{\text{loc}}(\Omega) \). \( \square \)

### 4 Criticality theory

In this section we prove several positivity properties of the functional \( Q \) along the lines of criticality theory for second-order linear elliptic operators [27, 28].
Recall that $Q$ is said to be strictly positive in $\Omega$ if $Q$ has a weighted spectral gap on $C_0^\infty(\Omega)$. The functional $Q$ is degenerately positive in $\Omega$ if $Q \geq 0$ on $C_0^\infty(\Omega)$ and $Q$ admits a ground state in $\Omega$. The functional $Q$ is nonpositive in $\Omega$ if it takes negative values on $C_0^\infty(\Omega)$. For $V \in L^\infty_{\text{loc}}(\Omega)$, we denote in the present section

$$Q_V(u) := \int_\Omega (|\nabla u|^p + V|u|^p) \, dx$$

(4.1)

to emphasize the dependence of $Q$ on the potential $V$.

**Proposition 4.1.** Let $V_i \in L^\infty_{\text{loc}}(\Omega)$ and suppose that $V_2 \succeq V_1$. If $Q_{V_1} \geq 0$ on $C_0^\infty(\Omega)$, then $Q_{V_2}$ is strictly positive in $\Omega$, and if $Q_{V_2}$ is degenerately positive in $\Omega$, then $Q_{V_1}$ is nonpositive in $\Omega$.

**Proof.** Obviously,

$$Q_{V_2}(u) = Q_{V_1}(u) + \int (V_2 - V_1)|u|^p \, dx \geq 0 \quad \forall u \in C_0^\infty(\Omega).$$

Suppose that $Q_{V_1}$ has a null sequence $\{u_k\}$ with a ground state $v$, such that $u_k \to v$ in $L^p_{\text{loc}}(\Omega)$. Evaluating the limit of $Q_{V_2}(u_k)$, we have by Fatou’s lemma,

$$0 = \lim_{k \to \infty} Q_{V_2}(u_k) \geq \liminf_{k \to \infty} \int (V_2 - V_1)|u_k|^p \, dx \geq \int (V_2 - V_1)|v|^p \, dx > 0$$

(4.2)

and we arrive at a contradiction.

**Proposition 4.2.** Let $\Omega_1 \subset \Omega_2$ be domains in $\mathbb{R}^d$ such that $\Omega_2 \setminus \overline{\Omega_1} \neq \emptyset$. Let $Q_V$ be defined on $C_0^\infty(\Omega_2)$.

1. If $Q_V \geq 0$ on $C_0^\infty(\Omega_2)$, then $Q_V$ is strictly positive in $\Omega_1$.

2. If $Q_V$ is degenerately positive in $\Omega_1$, then $Q_V$ is nonpositive in $\Omega_2$.

**Proof.** 1. If $Q_V$ is strictly positive in $\Omega_2$, then the first assertion is trivial. Suppose that $Q_V$ is degenerately positive in $\Omega_2$, and let $v$ be the ground state of $Q_V$ in $\Omega_2$. Take $\psi \in C_0^\infty(\Omega_2 \setminus \overline{\Omega_1})$ such that $\int_{\Omega_2} v\psi \, dx \neq 0$.

Due to (1.7) restricted to $u \in C_0^\infty(\Omega_1)$, we conclude that $Q_V$ is strictly positive in $\Omega_1$.

2. Assume that $Q_V \geq 0$ on $C_0^\infty(\Omega_2)$. Then by the first part, $Q_V$ is strictly positive in $\Omega_1$, which is a contradiction.

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Proposition 4.3. Let $V_0, V_1 \in L^\infty_{\text{loc}}(\Omega)$, $V_0 \neq V_1$. For $t \in \mathbb{R}$ we denote
\[ Q_t(u) := tQ_{V_1}(u) + (1 - t)Q_{V_0}(u), \tag{4.3} \]
and suppose that $Q_{V_i} \geq 0$ on $C_0^\infty(\Omega)$ for $i = 0, 1$.

Then the functional $Q_t \geq 0$ on $C_0^\infty(\Omega)$ for all $t \in [0, 1]$. Moreover, if $V_0 \neq V_1$, then $Q_t$ is strictly positive in $\Omega$ for all $t \in (0, 1)$.

Proof. The first assertion is immediate. To prove the second assertion, assume first that at least one of $Q_{V_0}, Q_{V_1}$, say, $Q_{V_0}$, is strictly positive with a weight $W_0$. Then for $0 \leq \tau < 1$, the functional $Q_\tau$ is strictly positive with the weight $(1 - \tau)W_0$.

Assume now that both $Q_{V_0}$ and $Q_{V_1}$ are degenerately positive with ground states $v_0, v_1$, respectively, and assume that for some $\tau \in (0, 1)$, $Q_\tau$ has a null sequence $\{u_k\}$ and a ground state $v_\tau$.

Note that $v_\tau$ is not a multiple of $v_0$ or of $v_1$ since $V_0 \neq V_1$. Then there exist $\psi_i \in C_0^\infty(\Omega)$, $i = 0, 1$, such that
\[ \int_\Omega \psi_i v_i \, dx \neq 0, \quad \text{and} \quad \int_\Omega \psi_i v_\tau \, dx = 0 \quad i = 0, 1. \tag{4.4} \]

By (1.7), for $i = 0, 1$ there exist a continuous function $W_i > 0$ in $\Omega$, and a constant $C_i > 0$, such that
\[ \int_\Omega W_i |u|^p \, dx \leq Q_i(u) + C_i \left( \int_\Omega u \psi_i \, dx \right)^p \quad \forall u \in C_0^\infty(\Omega). \tag{4.5} \]

Let $W_\tau := \tau W_1 + (1 - \tau)W_0$. Then
\[ \int_\Omega W_\tau |u|^p \, dx \leq Q_\tau(u) + C_1 \tau \left( \int_\Omega u \psi_1 \, dx \right)^p + C_0 (1 - \tau) \left( \int_\Omega u \psi_0 \, dx \right)^p \quad \forall u \in C_0^\infty(\Omega). \tag{4.6} \]

Substituting $u = u_k$ and passing to the limit, taking into account that $u_k \to v_\tau$ in $L^p_{\text{loc}}(\Omega)$ as well as (4.4) and Fatou’s lemma, we have
\[ 0 < \int_\Omega W_\tau |v_\tau|^p \, dx \leq C_1 \tau \left( \int_\Omega v_\tau \psi_1 \, dx \right)^p + C_0 (1 - \tau) \left( \int_\Omega v_\tau \psi_0 \, dx \right)^p = 0, \tag{4.7} \]
and we arrive at a contradiction. Therefore, $Q_\tau$ does not admit a ground state, and by Theorem 1.6 the functional $Q_\tau$ is strictly positive in $\Omega$. \qed
**Proposition 4.4.** Let $Q_V$ be a strictly positive functional in $\Omega$. Consider $V_0 \in L^\infty(\Omega)$ such that $V_0 \not\equiv 0$ and $\text{supp} \, V_0 \subseteq \Omega$. Then there exist $\tau_+ > 0$ and $-\infty \leq \tau_- < 0$ such that $Q_{V+\tau_+V_0}$ is strictly positive in $\Omega$ for $t \in (\tau_-, \tau_+)$, and $Q_{V+\tau_+V_0}$ is degenerately positive in $\Omega$.

**Proof.** If $Q_V$ has a weighted spectral gap in $\Omega$ with a continuous weight $W$, then $Q_{V+\tau_+V_0}$ satisfies the inequality
\[
\int_{\Omega} (W + tV_0)|u|^p \, dx \leq Q_{V+\tau_+V_0}(u) \quad \text{on } C_0^\infty(\Omega).
\] (4.8)

The weight $W + tV_0$ is strictly positive in $\Omega$ for $|t|$ small, since $W$ is a strictly positive continuous function and $V_0$ is a bounded function with compact support. By Proposition 4.3, the set of $t \in \mathbb{R}$, for which $Q_{V+\tau_+V_0}$ is strictly positive, is an interval. Moreover, this interval does not extend to $+\infty$.

Let $\tau_+$ be the right endpoint of this interval. Obviously $Q_{V+\tau_+V_0}$ is non-negative on $C_0^\infty(\Omega)$. If, on the other hand, the functional $Q_{V+\tau_+V_0}$ is strictly positive in $\Omega$, then by the preceding argument, there exists $\delta > 0$ such that the functional $Q_{V+(\tau_++\delta)V_0}$ is strictly positive in $\Omega$, which contradicts the definition of $\tau_+$.

**Proposition 4.5.** Let $Q_V$ be a degenerately positive functional in $\Omega$, and let $v$ be the corresponding ground state. Consider $V_0 \in L^\infty(\Omega)$ such that $\text{supp} \, V_0 \subseteq \Omega$. Then there exists $0 < \tau_+ \leq \infty$ such that $Q_{V+\tau_+V_0}$ is strictly positive in $\Omega$ for $t \in (0, \tau_+)$ if and only if
\[
\int_{\Omega} V_0|v|^p \, dx > 0.
\] (4.9)

**Proof.** Suppose that there exists $t > 0$ such that $Q_{V+\tau_+V_0}$ is strictly positive in $\Omega$. Then there exists $W \in C(\Omega)$, $W > 0$, such that
\[
Q_V(u) + t \int_{\Omega} V_0|u|^p \, dx \geq \int_{\Omega} W|u|^p \, dx \quad \forall u \in C_0^\infty(\Omega).
\] (4.10)

Let $\{u_k\}$ be a null sequence for the functional $Q_V$, and let $v > 0$ be the ground state of $Q_V$ which is the $L^p_{\text{loc}}(\Omega)$ limit of $\{u_k\}$. By (4.10) and Fatou’s
Thus, (4.9) is satisfied.

Suppose that (4.9) holds true, but for any \( t > 0 \) the functional \( Q_{V + tV_0} \) is nonpositive in \( \Omega \). Therefore, for any \( t > 0 \) there exists \( u_t \in C_0^\infty(\Omega) \) such that

\[
Q_{V}(u_t) + t \int_{\Omega} V_0|u_t|^p \, dx < 0.
\]

(4.11)

Clearly, we may assume that \( u_t \geq 0 \). Since \( Q_{V}(u_t) \geq 0 \) it follows that

\[
\int_{\Omega} V_0|u_t|^p \, dx < 0.
\]

(4.12)

In particular, \( \text{supp}(u_t) \cap \text{supp}(V_0) \neq \emptyset \). Therefore, we may assume that

\[
\int_{\text{supp}(V_0)} |u_t|^p \, dx = 1.
\]

It follows that

\[
\lim_{t \to 0} t \int_{\Omega} V_0|u_t|^p \, dx = 0,
\]

(4.13)

and by the nonnegativity of \( Q_{V} \) and (4.11),

\[
0 \leq \liminf_{t \to 0} Q_{V}(u_t) \leq \limsup_{t \to 0} Q_{V}(u_t) \leq 0.
\]

(4.14)

It follows that \( \{u_t\} \) is a null sequence, and therefore, \( u_t \to v \) in \( L_p^\text{loc}(\Omega) \) as \( t \to 0 \), where \( v \) is the corresponding ground state of \( Q_{V} \). Using a standard argument similar to (3.13) we have (for a subsequence)

\[
\lim_{t \to 0} \int_{\Omega} V_0|u_t|^p \, dx = \int_{\Omega} V_0|v|^p \, dx.
\]

(4.15)

Combining (4.9), and (4.12) and (4.15), we obtain

\[
0 < \int_{\Omega} V_0|v|^p \, dx = \lim_{t \to 0} \int_{\Omega} V_0|u_t|^p \, dx \leq 0,
\]

(4.16)

which is a contradiction.

Remark 4.6. An alternative proof of Proposition 4.5 can be derived from the (nonsymmetric) technique in [30].
5 Minimal growth

In this section we study the existence of positive solutions of the equation $Q'(u) = 0$ of minimal growth in a neighborhood of infinity in $\Omega$, and obtain a new characterization of strict positivity in terms of these solutions.

Throughout this section we assume that $1 < p \leq d$. Therefore, for any $x_0 \in \Omega$, any positive solution $v$ of the equation $Q'(u) = 0$ in a punctured neighborhood of $x_0$ has either a removable singularity at $x_0$, or

$$v(x) \asymp \begin{cases} |x - x_0|^{\alpha(d,p)} & p < d, \\ -\log|x - x_0| & p = d, \end{cases} \text{ as } x \to x_0,$$

where $\alpha(d,p) := (p-d)/(p-1)$ [34, 35, 38]. Here $f \asymp g$ means that $c \leq f/g \leq C$, where $c$ and $C$ are positive constants. In particular, in the nonremovable case,

$$\lim_{x \to x_0} v(x) = \infty. \quad (5.2)$$

**Lemma 5.1** (L. Véron, private communication). Assume that $1 < p \leq d$, and let $x_0 \in \mathbb{R}^d$ be fixed. Suppose that $v$ is a positive solution of the equation $Q'(u) = 0$ in a punctured neighborhood of $x_0$ which has a nonremovable singularity at $x_0$. Then

$$v(x) \asymp \begin{cases} |x - x_0|^{\alpha(d,p)} & p < d, \\ -\log|x - x_0| & p = d, \end{cases} \text{ as } x \to x_0,$$

where $f \sim g$ means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = C$$

for some positive constant $C$.

**Remark 5.2.** The asymptotics (5.3) has been proved for $p = 2$ in [18], for $1 < p \leq d$ and $V = 0$ in [23, Theorem 2.1], and also in some other cases in [19]. The proof below uses a technique involving a scaling argument together with a comparison principle that has been used for example in [5].

**Proof.** Assume that $1 < p < d$, the proof for $p = d$ needs some minor modifications, and is left to the reader. Without loss of generality, we assume also that $x_0 = 0$. 

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Since \( V \in L^\infty_{\text{loc}}(\Omega) \), the solution \( v \) satisfies \( v(x) \preccurlyeq |x|^{\alpha(d,p)} \). Let

\[
c := \limsup_{x \to 0} \frac{v(x)}{|x|^{\alpha(d,p)}} = \lim_{n \to \infty} \frac{v(x_n)}{|x_n|^{\alpha(d,p)}},
\]  

(5.4)

and set \( \mu(x) := c|x|^{\alpha(d,p)} \). Define

\[
v_n(x) := |x_n|^{-\alpha(p,d)} v(|x_n|x),
\]

where \( x_n \to 0 \) is defined by (5.4).

Note that in an arbitrarily large punctured ball

\[
C^{-1} \mu(x) \leq v_n(x) \leq C \mu(x)
\]

for all \( n \) large enough, and in such a ball \( v_n \) is a positive solution of the quasilinear elliptic equation

\[
-\Delta_p v_n(x) + |x_n|^p V(x/|x_n|) v_n^{p-1}(x) = 0.
\]

Since \( \{v_n\} \) is locally bounded and bounded away from zero in any punctured ball, a standard elliptic argument implies that there is a subsequence of \( \{v_n\} \) that converges to a positive singular solution \( U \) of the limiting equation \( -\Delta_p U = 0 \) in the punctured space. Since \( U \simeq \mu \) in the punctured space, it follows that \( U \) tends to zero at infinity. On the other hand, [23, Theorem 2.1], implies that \( U(x) \sim \mu(x) \) as \( x \to 0 \). Hence we can apply the comparison principle (Theorem 2.2), and compare the functions \( U \) and \( \mu \) on arbitrarily large balls, to obtain that \( U = \mu \). This implies that

\[
\lim_{n \to \infty} \|v(x)/\mu(x) - 1\|_{L^\infty(|x|=|x_n|)} = 0.
\]  

(5.5)

In other words, \( v \) is almost equal to \( \mu \) on a sequence of concentric spheres converging to 0.

In order to prove that \( v \) is almost equal to \( \mu \) uniformly in the sequence of the concentric annuli \( A_n := \{|x_n| \leq |x| \leq |x_{n+1}|\} \), we construct two radial perturbations of \( \mu \). Let \( \mu_-(x) := \mu(x) - \delta|x|^a \) and \( \mu_+(x) := \mu(x) + \delta|x|^a \) (for some \( a > (p-d)/(p-1) \)). It turns out that \( \mu_- \) (resp., \( \mu_+ \)) is a radial subsolution (resp., supersolution) of the equation \( Q'(u) = 0 \) near the origin, and therefore using the comparison principle in the annulus \( A_n \) and (5.5), it follows that

\[
\lim_{r \to 0} \|v(x)/\mu(x) - 1\|_{L^\infty(|x|=|r|)} = 0.
\]
Definition 5.3. Let $K$ be a compact set in $\Omega$. A positive solution of the equation $Q'(u) = 0$ in $\Omega \setminus K$ is said to be a positive solution of minimal growth in a neighborhood of infinity in $\Omega$, if for any compact set $K_1$ in $\Omega$, with a smooth boundary, satisfying $\text{int}(K_1) \supset K$, and any positive supersolution $v \in C^0((\Omega \setminus K_1) \cup \partial K_1)$ of the equation $Q'(u) = 0$ in $\Omega \setminus K_1$, the inequality $u \leq v$ on $\partial K_1$ implies that $u \leq v$ in $\Omega \setminus K_1$. A positive solution $u$ of the equation $Q'(u) = 0$ in $\Omega$, which has minimal growth in a neighborhood of infinity in $\Omega$ is called a global minimal solution of the equation $Q'(u) = 0$ in $\Omega$.

The following result is an extension to the $p$-Laplacian of the corresponding result of S. Agmon concerning positive solutions of real linear second-order elliptic operators [1].

Theorem 5.4. Suppose that $1 < p \leq d$, and $Q$ is nonnegative on $C_0^\infty(\Omega)$. Then for any $x_0 \in \Omega$ the equation $Q'(u) = 0$ has (up to a multiple constant) a unique positive solution $v$ in $\Omega \setminus \{x_0\}$ of minimal growth in a neighborhood of infinity in $\Omega$.

Moreover, $v$ is either a global minimal solution of the equation $Q'(u) = 0$ in $\Omega$, or $v$ has a nonremovable singularity at $x_0$.

Proof. Take $x_0 \in \Omega$, and consider an exhaustion $\{\Omega_N\}_{N=1}^\infty$ of $\Omega$ (as in the proof of Theorem 2.3). Fix $N \geq 1$, and denote $\Omega_{N,k} := \Omega_N \setminus B(x_0,1/k)$.

Let $\{f_k\}$ be a sequence of nonzero nonnegative smooth functions such that for each $k \geq 2$, the function $f_k$ is supported in $B(x_0,2/k) \setminus B(x_0,1/k)$.

Recall that $\lambda_{1,p}(\Omega_{N,k}) > 0$ for all $N,k \geq 1$. By Theorem 2.1 there exists a unique positive solution of the problem

$$
\begin{cases}
Q'(u_{N,k}) = c_k f_k & \text{in } \Omega_{N,k}, \\
u_{N,k} = 0 & \text{on } \partial \Omega_{N,k}, \\
u_{N,k}(x_1) = 1,
\end{cases}
$$

(5.6)

where $x_1 \neq x_0$ is a fixed point in $\Omega_1$ and $c_k > 0$. By Harnack’s inequality and elliptic regularity, it follows that $\{u_{N,k}\}$ admits a subsequence which converges locally uniformly in $\Omega_N \setminus \{x_0\}$ to a positive solution $G_N(\cdot,x_0)$ of the equation $Q'(u) = 0$ in $\Omega_N \setminus \{x_0\}$. Moreover, $G_N(\cdot,x_0) = 0$ on $\partial \Omega_N$, and $G_N(x_1,x_0) = 1$.

Since $\lambda_{1,p}(\Omega_N) > 0$, and it is the unique eigenvalue with a positive Dirichlet eigenfunction, it follows that $G_N(\cdot,x_0)$ has a nonremovable singularity at $x_0$. 

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Recall that (5.3) holds true with \( v(\cdot) = G_N(\cdot, x_0) \). It is convenient to normalize \( G_N \) in the traditional way, so that

\[
\lim_{x \to x_0} \frac{G_N(x, x_0)}{|x - x_0|^\alpha(p,d)} = \frac{p - 1}{d - p} |S^{d-1}|^{-1/(p-1)} \quad p < d, \\
\lim_{x \to x_0} \frac{G_N(x, x_0)}{- \log |x - x_0|} = |S^{d-1}|^{-1/(d-1)} \quad p = d,
\]

(5.7)

where \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \). Using the comparison principle (Theorem 2.2), and (5.3), it follows that \( G_N(\cdot, x_0) \) is the unique function with the above properties. Hence, \( G_N(\cdot, x_0) \) might be called the (Dirichlet) positive \( p \)-Green function of the functional \( Q \) in \( \Omega_N \) with a pole at \( x_0 \).

By the comparison principle (Theorem 2.2) and (5.3), it follows that the sequence \( \{G_N(\cdot, x_0)\} \) is nondecreasing as a function of \( N \), and therefore it converges locally uniformly in \( \Omega \setminus \{x_0\} \) either to a positive function \( G(\cdot, x_0) \) or to infinity.

In the first case \( G(\cdot, x_0) \) is a positive solution of the equation \( Q'(u) = 0 \) in \( \Omega \setminus \{x_0\} \) and has the asymptotic behavior (5.3) near \( x_0 \). We call \( G(\cdot, x_0) \) the minimal positive \( p \)-Green function of the functional \( Q \) in \( \Omega \) with a pole at \( x_0 \).

In the second case, we consider the normalized sequence

\[
v_N(x) := \frac{G_N(x, x_0)}{G_N(x_1, x_0)} \quad n = 1, 2, \ldots
\]

By Harnack’s inequality and elliptic regularity, it follows that \( \{v_N\} \) admits a subsequence which converges locally uniformly to a positive solution \( v \) of the equation \( Q'(u) = 0 \) in \( \Omega \setminus \{x_0\} \).

Assume that \( v \) has a nonremovable singularity at \( x_0 \). Therefore, for each \( N \geq 1 \) we obtain by the comparison principle and (5.7) that

\[
G_N(x, x_0) \leq C v(x) \quad \forall x \in \Omega \setminus \{x_0\}
\]

for some \( C > 0 \) independent of \( N \). But this contradicts our assumption that \( G_N \to \infty \) as \( N \to \infty \).

Note that \( G(\cdot, x_0) \) (in the first case) and \( v \) (in the second case) are limits of a sequence of positive solutions that for \( \delta > 0 \) are uniformly bounded on \( \partial B(x_0, \delta) \), and take zero boundary condition on \( \partial \Omega_N \). Therefore, by the comparison principle, \( G \) and \( v \) are positive solutions in \( \Omega \setminus \{x_0\} \) of minimal
growth in a neighborhood of infinity in Ω. In particular, v is a global minimal positive solution of the equation \( Q'(u) = 0 \) in Ω.

Using again the comparison principle and (5.3), it follows that such a solution is unique.

The next theorem demonstrates that a global minimal positive solution of the equation \( Q'(u) = 0 \) in Ω is a ground state.

**Theorem 5.5.** Assume that \( 1 < p \leq d \) and that \( Q_v \geq 0 \) on \( C_0^\infty(\Omega) \). Then \( Q_v \) is degenerately positive in \( \Omega \) if and only if the equation \( Q'(u) = 0 \) admits a global minimal positive solution in \( \Omega \).

**Proof.** Assume that \( Q_v \) is strictly positive and assume that there exists a global minimal positive solution \( v \) of the equation \( Q'(u) = 0 \) in \( \Omega \). By Proposition 4.4 there exists a nonzero nonnegative function \( V_1 \in C_0^\infty(\Omega) \) with \( \text{supp} \ V_1 \subset B(x_0, \delta) \) for some \( \delta > 0 \), such that \( Q_{V - V_1} \) is strictly positive in \( \Omega \). Therefore, in light of Theorem 2.3 there exists a positive solution \( v_1 \) of the equation \( Q_{V - V_1}'(u) = 0 \) in \( \Omega \).

Clearly, \( v_1 \) is a positive supersolution of the equation \( Q_{V}'(u) = 0 \) in \( \Omega \) which is not a solution. On the other hand, \( v \) is a positive solution of the equation \( Q'(u) = 0 \) in \( \Omega \) which has minimal growth in a neighborhood of infinity in \( \Omega \). Therefore, there exists \( \varepsilon > 0 \) such that \( \varepsilon v \leq v_1 \) in \( \Omega \). Define

\[
\varepsilon_0 := \max\{\varepsilon > 0 \mid \varepsilon v \leq v_1 \text{ in } \Omega\}.
\]

Clearly \( \varepsilon_0 v \leq v_1 \) in \( \Omega \). Consequently, there exist \( \delta_1, \delta_2 > 0 \) and \( x_1 \in \Omega \) such that

\[
(1 + \delta_1)\varepsilon_0 v(x) \leq v_1(x) \quad x \in B(x_1, \delta_2).
\]

Hence, by the definition of minimal growth, we have

\[
(1 + \delta_1)\varepsilon_0 v(x) \leq v_1(x) \quad x \in \Omega \setminus B(x_1, \delta_2),
\]

and thus \( (1 + \delta_1)\varepsilon_0 v \leq v_1 \) in \( \Omega \), which is a contradiction to the definition of \( \varepsilon_0 \).

Assume that \( Q \) admits a positive minimal \( p \)-Green function \( G(\cdot, x_0) \) in \( \Omega \). We need to prove that \( Q \) is strictly positive.

Consider an exhaustion \( \{\Omega_N\}_{N=1}^\infty \) of \( \Omega \) such that \( x_0 \in \Omega_1 \) and \( x_1 \in \Omega \setminus \Omega_1 \). Fix a nonzero nonnegative function \( f \in C_0^\infty(\Omega_1) \). By Theorem 2.3 there exists a unique positive solution of the Dirichlet problem

\[
\begin{align*}
Q'(u_N) &= f \quad \text{in } \Omega_N, \\
u_N &= 0 \quad \text{on } \partial \Omega_N.
\end{align*}
\] (5.8)
By the comparison principle (Theorem 2.2), \( \{u_N\} \) is an increasing sequence. Suppose that \( \{u_N(x_1)\} \) is bounded. Then \( u_N \to u \), where \( u \) satisfies the equation \( Q'(u) = f \geq 0 \) in \( \Omega \). Since \( u \) is a positive supersolution of the equation \( Q'(u) = 0 \) in \( \Omega \) which is not a solution, Theorem 1.6 implies that \( Q \) is strictly positive.

Suppose that \( u_N(x_1) \to \infty \). Then \( v_N(x) := u_N(x)/u_N(x_1) \) solves the problem

\[
\begin{align*}
Q'(v_N) &= \frac{f(x)}{u_N(x_1)^{p-1}} \quad \text{in } \Omega_N, \\
v_N &= 0 \quad \text{on } \partial \Omega_N, \\
v_N(x_1) &= 1.
\end{align*}
\]

By Harnack’s inequality, and the comparison principle,

\[
v_N \approx G_N(\cdot, x_0) \quad \text{in } \Omega_N \setminus \Omega_1.
\]

By a standard elliptic argument, we may extract a subsequence of \( \{v_N\} \) that converges to a positive supersolution \( v \) of the equation \( Q'(u) = 0 \) in \( \Omega \). Recall that \( G_N(\cdot, x_0) \to G(\cdot, x_0) \). Hence, (5.10) implies that \( v \approx G(\cdot, x_0) \) in \( \Omega \setminus \Omega_1 \), and in particular, \( v \) is a positive solution in \( \Omega \setminus \Omega_1 \) of minimal growth in a neighborhood of infinity in \( \Omega \). Note that \( v \neq cG(\cdot, x_0) \) since \( v \) is not singular at \( x_0 \). Since the equation \( Q'(u) = 0 \) does not admit a global minimal solution in \( \Omega \), it follows that \( v \) satisfies \( Q'(u) \geq 0 \) in \( \Omega \), and by Theorem 1.6, \( Q \) is strictly positive in \( \Omega \).

\[\square\]

6 Open problems

We conclude the paper with a number of open problems suggested by the above results which are left for future investigation. All these questions are already resolved when \( p = 2 \).

The first problem (Problem 6.1) deals with the weakly lower semicontinuity and convexity of the functional \( Q \). As was shown in Proposition 3.5, \( Q \) is weakly lower semicontinuous in a limited sense, a property which is closely related to convexity. However, \( Q \) is not necessarily convex even if \( Q \geq 0 \) on \( C^\infty_0(\Omega) \); for \( p > 2 \) see the elementary one-dimensional counterexample at the end of [9], and also the proof of Theorem 7 in [17], for \( p < 2 \) see [14], Example 2. Note also that Proposition 3.5 does not assert that the domain \( D \) defined by (3.15) is weakly closed.
Problem 6.1. Does $Q$ have a natural extension to a weakly closed set where it is weakly lower semicontinuous? Under what conditions a nonnegative functional $Q$ of the form (1.1) is convex?

Clearly $Q$ is convex when $V \geq 0$ or $p = 2$. The convexity of $Q$ gives rise to an energy space for the form $Q$ that would generalize the space $\mathcal{D}^{1,p}$, similarly to the known case $p = 2$ [32]. Indeed, if $Q$ is a nonnegative convex functional on $C_0^\infty(\Omega)$, then it follows that $Q^{\frac{1}{2}}$ is a norm on $C_0^\infty(\Omega)$. Moreover, by Theorem 1.6 if $Q$ has a weighted spectral gap in $\Omega$, then the completion of $C_0^\infty(\Omega)$ with respect to this norm is continuously imbedded into $L^p_{\text{loc}}(\Omega)$. On the other hand, if $Q$ has a ground state $v$ in $\Omega$, then $v$ belongs to the equivalence class of 0 in this completion, and there is no continuous imbedding of the completion even into $\mathcal{D}'(\Omega)$. However, due to (1.7), the completion of $C_0^\infty(\Omega)$ with respect to the norm induced by the right hand side of (1.7) is continuously imbedded into $L^p_{\text{loc}}(\Omega)$.

Problem 6.2. Do the results of this paper extend to quasilinear functionals of the form

$$Q^A(u) := \int_\Omega (|A(x)\nabla u|^p + V|u|^q) \, dx,$$

where $A$ is a strictly positive definite matrix and $1 < p \leq q < \infty$?

We note that Picone-type identity for the case $A(x) = a(x)I$ was established in [22].

Problem 6.3. Generalize the results of Section 5 to the case $d < p < \infty$.

Problem 6.4. Let $\Omega_1 \subsetneq \Omega$ be domains in $\mathbb{R}^d$. Suppose that $Q$ is strictly positive in $\Omega_1$. Show that there exists an open domain $\Omega_1 \subsetneq \Omega_2 \subset \Omega$ such that $Q_V$ is strictly positive in $\Omega_2$.

Problem 6.4 was studied in [31] under the assumption $p = 2$, and stronger statements were proved.

Problem 6.5. Let $\Omega = \mathbb{R}^d$ and assume that $Q = Q_V$ are strictly positive in $\Omega$, for $j = 1, 2$. For $y \in \mathbb{R}^d$ denote $V_y(x) := V_1(x) + V_2(x - y)$. Show that under suitable decay conditions on $V_j$, there exists $R > 0$ such that for every $y \in \mathbb{R}^d \setminus B_R(0)$ the functional $Q_{V_y}$ is strictly positive in $\Omega$.

This phenomenon has been proved for $p = 2$ in [24, 26, 36] for Schrödinger operators and in [29] for the non-selfadjoint case.
Appendix: Energy inequality

The following inequality, established for \( p \geq 2 \), estimates the functional \( Q \) from below by an expression that leads to an alternative proof of Lemma 3.2 for the case \( p \geq 2 \).

Lemma A.1. Assume that \( p \geq 2 \). Let \( v \in C^1(\Omega) \) be a positive solution of the equation \( Q'(u) = 0 \) in \( \Omega \), and let \( u \in W^{1,p}_{\text{loc}}(\Omega) \), \( u \geq 0 \), supp \( u \) \( \subset \Omega \). Then

\[
Q(u) \begin{cases} 
\geq Q_1(u) + Q_2(u) & \text{if } p > 2, \\
= Q_1(u) & \text{if } p = 2.
\end{cases}
\]  

(A.1)

where

\[
Q_1(u) := \frac{2}{p} \int_{\Omega} |\nabla v|^{p-2} v^2 |\nabla \left( \frac{u}{v} \right)^\frac{p}{2} |^2 \, dx, \quad \text{and} \quad Q_2(u) := \int_{\{\nabla v = 0\}} |\nabla u|^p \, dx.
\]

Proof. Since \( p \geq 2 \), the obvious inequality \( t^p + (p-1) - pt \geq (p-1)(t-1)^2 \) implies

\[
L_1(u,v) \geq (p-1) \left( \frac{\nabla u}{u} - \frac{\nabla v}{v} \right)^2 \left( \frac{u}{v} \right)^p |\nabla v|^{p-2} v^2,
\]  

(A.2)

where \( L_1 \) is defined by (2.4). We use the identity

\[
\left| \nabla \left( \frac{u}{v} \right)^\frac{p}{2} \right|^2 = \left( \frac{p}{2} \right)^2 \left( \frac{u}{v} \right)^p \left| \nabla u - \frac{\nabla u}{u} \right|^2.
\]  

(A.3)

Substitution of (A.3) into (A.2) and using the identity

\[
\left( \frac{\nabla u}{u} - \frac{\nabla v}{v} \right)^2 - \left| \frac{\nabla u}{u} - \frac{\nabla v}{v} \right|^2 = 2 \frac{\nabla u \cdot \nabla v - |\nabla u||\nabla v|}{uv}
\]  

(A.4)

gives

\[
L_1(u,v) \geq (p-1) \left( \frac{2}{p} \right)^2 |\nabla v|^{p-2} v^2 \left| \nabla \left( \frac{u}{v} \right)^\frac{p}{2} \right|^2 + 2(p-1)(\nabla u \cdot \nabla v - |\nabla u||\nabla v|) \left( \frac{u}{v} \right)^{p-1} |\nabla v|^{p-2},
\]

which is the same as
\[ L_1(u, v) \geq (p - 1) \left( \frac{2}{p} \right)^2 |\nabla v|^{p-2} v^2 \left| \nabla \left[ \left( \frac{u}{v} \right)^{\frac{p}{2}} \right] \right|^2 - 2 \frac{p - 1}{p} L_2(u, v), \]  
where \( L_2 \) is defined by (2.5). Since \( 1 \geq 2/(p^2 - p) \) for \( p \geq 2 \), (A.5) implies that

\[ L(u, v) \geq \frac{2}{p} |\nabla v|^{p-2} v^2 \left| \nabla \left[ \left( \frac{u}{v} \right)^{\frac{p}{2}} \right] \right|^2. \]  

Note that for \( p = 2 \) we have

\[ L(u, v) = v^2 |\nabla \left( \frac{u}{v} \right)|^2 \]  

(see [32, Lemma 2.4]). Moreover, on the critical set \( \{ x \in \Omega \mid |\nabla v(x)| = 0 \} \), we have \( L(u, v) = |\nabla u|^p \). Therefore for \( p > 2 \) we have

\[ L(u, v) \geq \frac{2}{p} |\nabla v|^{p-2} v^2 \left| \nabla \left[ \left( \frac{u}{v} \right)^{\frac{p}{2}} \right] \right|^2 + 1_{\{|\nabla v| = 0\}} |\nabla u|^p. \]

Integrating the latter inequality over \( \Omega \), we arrive at (A.1).

**Remark A.2.**

1. Note that (A.1) is based on the pointwise inequality (A.6).
2. Note that for \( p = 2 \) we have \( Q(u) = Q_1(u) \) (see for example [32, Lemma 2.4]). The set of all critical points of a positive solution of the linear equation \(-\Delta u + Vu = 0\) in \( \Omega \) has studied in [20].

**Alternative proof of Lemma 3.2.** Assume that \( p \geq 2 \). By (A.1), if \( Q(u_k) \to 0 \), then \( Q_1(u_k) \to 0 \) and \( Q_2(u_k) \to 0 \). Since \( c_B = 0 \), there exists a sequence \( u_k \in C_0^{\infty}(\Omega) \), \( u_k \geq 0 \), such that \( \int_B u_k^p = 1 \) and \( Q(u_k) \to 0 \). Repeating the first two steps of the proof of Lemma 3.2, we deduce that \( \{u_k\} \) is bounded in \( W^{1,p}(\omega) \) for every \( \omega \in \Omega, \omega \supset B \).

Consider now a weakly convergent renamed subsequence \( u_k \rightharpoonup u \) in \( W^{1,p}(\omega) \). Let

\[ Q_1^\omega(w) := \frac{2}{p} \int_\omega |\nabla v|^{p-2} v^2 \left| \nabla \left[ \left( \frac{w}{v} \right)^{\frac{p}{2}} \right] \right|^2 dx, \quad \text{and} \quad Q_2^\omega(w) := \int_{\{\nabla v = 0\} \cap \omega} |\nabla w|^p dx. \]

Since \( Q_1^\omega \) and \( Q_2^\omega \) are continuous convex functionals on \( W^{1,p}(\omega) \), they are weakly lower semicontinuous, and therefore,

\[ Q_1^\omega(u) \leq \lim_{k \to \infty} Q_1^\omega(u_k) = 0 \quad \text{and} \quad Q_2^\omega(u) \leq \lim_{k \to \infty} Q_2^\omega(u_k) = 0. \]
Consequently, $\nabla [(u/v)^p] = 0$ almost everywhere in $\omega \setminus \{\nabla v = 0\}$ and $\nabla u = 0$ in $\omega \cap \{\nabla v = 0\}$. Note that if $\nabla u(x) = 0$ and $\nabla v(x) = 0$, then $\nabla \left[ (u(x)/v(x))^p \right] = 0$. Thus, $\nabla \left[ (u/v)^p \right] = 0$ a.e. in $\omega$, and since it holds for every $\omega$ containing $B$, it follows that $u/v = \text{constant}$ a.e. in $\omega$. By the compact Sobolev imbedding on $B$, $\int_B u^p \, dx = \lim \int_B u_k^p \, dx = 1$, and therefore, $u = cv$, where $c^{-p} = \int_B v^p \, dx$. Note that any subsequence of $\{u_k\}$ has a subsequence converging to $cv$ with the same $c$. From the compactness of the local Sobolev imbedding, it follows that $u_k \rightarrow cv$ in $L^p_{\text{loc}}(\Omega)$. In other words, $\{u_k\}$ is a null sequence.

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