Tunable crossed Andreev reflection and elastic co-tunneling between quantum dots

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The mesoscopic system of two separate quantum dots connected by a common superconductor has been studied extensively for Cooper pair splitting, and is predicted to be the building block of an emulated Kitaev chain. The physical mechanisms underlying these phenomena include crossed Andreev reflection (CAR) and elastic co-tunneling (ECT) between quantum dots. In this work, we consider subgap Andreev bound states in a semiconductor-superconductor nanowire for mediating interdot couplings. We find that the strengths of CAR and ECT have a sensitive but different dependence on the physical parameters of the Andreev bound states, e.g., chemical potential, spin-orbit coupling and Zeeman spin splitting. Thus, the strengths as well as the ratio of CAR and ECT become highly tunable by controlling an external gate or by varying an externally applied magnetic field. In addition, unlike coupling through a metallic superconductor, the interdot couplings no longer decay exponentially with the dot distance. We also propose how to extract the strengths of CAR and ECT experimentally. Our findings will be useful to maximizing the efficiency of Cooper pair splitting as well as to implementing an emulated Kitaev chain at the sweet spot.

Introduction.—Two spatially separated quantum dots connected by a common superconductor is a widely used mesoscopic system for exploring various types of fundamental physics. The phenomenon of Cooper pair splitting has been investigated both theoretically [1–6] and experimentally [7–23], aiming to achieve nonlocal quantum entanglement between electrons [24, 25]. In the context of topological superconductivity, a dot-superconductor array can become an effective Kitaev chain under appropriate conditions, hosting Majorana zero modes at the ends of the array [26–28]. These Majorana modes are non-Abelian anyons which can implement topological quantum computation [29–35].

The key physics here is the superconductor-mediated crossed Andreev reflection (CAR) and elastic co-tunneling (ECT) between quantum dots, where an incoming electron is reflected as an outgoing hole or electron nonlocally. However, when mediated by the quasiparticle continuum in a metallic superconductor, the interdot couplings are strongly suppressed: it has been shown that the strength of CAR decays exponentially with the interdot distance on the scale of the superconducting coherence length [1, 5], in addition to a power-law decay on the scale of the Fermi wavelength [1, 3, 5]. The latter suppression is mitigated in a one-dimensional superconductor, such as a proximitized nanowire [36]. Moreover, in general it is hard to experimentally vary the strength of quasiparticle continuum mediated CAR and ECT. Therefore the optimal parameter regimes for the desired devices are usually not reached, e.g., CAR ≫ ECT for Cooper pair splitting, and CAR ≈ ECT for Kitaev chain emulation [26–28, 37]. These two major challenges, i.e., strong suppression of the effective coupling strengths and lack of tunability, hinder further progresses in the experimental investigations.

In this paper, we address the two challenges at once by considering subgap Andreev bound states (ABSs) in a short semiconductor-superconductor nanowire [see Fig. 1(a) for schematic]. We find that ABSs can mediate highly tunable CAR and ECT between quantum dots and their ratio can be tuned to an arbitrary value, e.g., by controlling the chemical potential of the ABS via an external gate. In particular, we explicitly show the dependences of CAR and ECT on the chemical potential, spin-orbit coupling and Zeeman spin splitting of the ABS for different dot spin channels. We also propose how to extract the strengths of the interdot CAR and ECT from resonant current in a three-terminal junction. These findings do not rely on the microscopic details of the ABS wavefunctions.

Model.—The Hamiltonian for the dot-superconductor-
dot system, as illustrated in Fig. 1(a), is

$$\begin{align*}
H &= H_S + H_D + H_{SD}, \\
H_S &= E_1 \gamma_1^d \gamma_1^\dagger + E_2 \gamma_2^d \gamma_2^\dagger, \\
H_D &= \varepsilon d_{l\eta}^d d_{l\eta} + \varepsilon \varepsilon d_{r\sigma}^d d_{r\sigma}, \\
H_{SD} &= -t_l c_{lx\eta}^d d_{l\eta} - t_r c_{rx\sigma}^d d_{r\sigma} + \text{H.c.}. \\
\end{align*}$$

(1)

Here $H_S$ is the Hamiltonian for the semiconductor-superconductor hybrid nanowire with an induced s-wave pairing. In the short-wire limit where the level spacing is larger than the superconducting gap, we only consider two normal states closest to the semiconductor Fermi energy (which form a degenerate Kramers’ pair in the presence of time-reversal invariance). With a finite amount of induced pairing, the pair of normal states become two ABSs defined as

$$\gamma_1 = \sum_{x,s=\uparrow,\downarrow} [u_i(x)c_{xs} + v_i(x)c_{xs}^\dagger],$$

where the wavefunctions and excitation energies are obtained by solving the Bogoliubov-de Gennes (BdG) equation

$$\begin{align*}
H_{\text{BdG}}(x)(u_i, v_i)^\dagger &= E_i(u_i, v_i)^\dagger. \\
\end{align*}$$

In order to make our findings as generic as possible, we do not assume any specific form for $h_{\text{BdG}}(x)$ in this work. Instead, we focus on its symmetry properties, e.g., time-reversal symmetry and spin-rotation symmetry. $H_D$ describes the quantum dots located near the opposite ends of the nanowire. With a large charging energy and a weak magnetic field, each quantum dot accommodates only one spin-polarized energy level near the Fermi energy, with levels denoted by $d_{l\eta}$ and $d_{r\sigma}$, respectively. Here the dot spins are either up or down ($\eta, \sigma = \uparrow, \downarrow$), and they have a common polarization axis, which can be realized by applying a global magnetic field. $H_{SD}$ describes the spin-conserved electron tunneling between the dot levels and the ends of the hybrid nanowire at $x = x_{l,r}$.

**ABS-mediated CAR and ECT.**—In the weak tunneling limit where the dot-superconductor tunneling $t_{l,r}$ is smaller than the induced gap $\Delta$, we can apply a Schrieffer-Wolf transformation to obtain an effective Hamiltonian for the coupled quantum dots. That is, $H_{\text{eff}} = H_D + H_{\text{interdot}}$, with

$$H_{\text{interdot}} = -PH_{SD}(1 - P)H_{SD}P + O(t^3_{l,r}/\Delta^2)$$

$$= -\Gamma_{\text{CAR}} d_{l\eta}^d d_{r\eta} - \Gamma_{\text{ECT}} d_{l\sigma}^d d_{r\sigma} + \text{H.c..}$$

(2)

Here $P$ is the projection operator onto the ground state of the uncoupled dot-superconductor system, and $\Gamma_{\text{CAR}}$, $\Gamma_{\text{ECT}}$ are the ABS-mediated CAR and ECT between the two spin-polarized dot levels, with

$$\begin{align*}
\Gamma_{\text{CAR}} &= \frac{t_{l} t_{r}}{\Delta} \sum_{m=1,2} \frac{u_m(x\eta)^* u_m(x\sigma) - u_m(x\sigma)^* u_m(x\eta)}{E_m/\Delta}, \\
\Gamma_{\text{ECT}} &= \frac{t_{l} t_{r}}{\Delta} \sum_{m=1,2} \frac{u_m(x\eta)^* u_m(x\sigma) - u_m(x\sigma)^* u_m(x\eta)}{E_m/\Delta}.
\end{align*}$$

(3)

Physically, the interdot CAR is induced by a coherent process where two electrons from a Cooper pair tunnel into two separate quantum dots via the virtual ABS excitations [see schematic in Fig. 1(b)]. In contrast, for ECT a single electron hops from one dot to the other via the ABSs [see schematic in Fig. 1(c)]. Equation (3) is the most general expression. Note that a finite interdot coupling requires supports of the ABS wavefunctions on both ends of the hybrid nanowire. In what follows we will analyze the dependence of the interdot coupling strengths on the physical parameters in the system. We define $P_{\eta\sigma}^{\text{CAR}} = |\Gamma_{\eta\sigma}\Delta/(t_{l,r})|^2$ with $\eta$ = CAR or ECT to characterize the strengths. As we will see, $P_{\eta\sigma}^{\text{CAR}}$ is proportional to the experimentally measurable current $I_{\eta\sigma}^\text{DC}$.

**Energy and angle dependence.**—We first consider a time-reversal invariant hybrid nanowire. Physically, this corresponds to a situation where the Zeeman splitting from the global magnetic field is negligible compared to the spin-orbit strength. The excitation energies of the two ABSs are then degenerate if $\xi_n = \mu$ being the normal-state energy. The grey line denotes the excitation energy $E_m/\Delta = \sqrt{2\varepsilon^2 + \Delta}$ of the ABS (for better visual effect we shift $E_m/\Delta \rightarrow E_m/\Delta - 1/2$). (b) Angle dependence of $P_{\eta\sigma}^{\text{CAR}}$ for a time-reversal invariant nanowire with spin-orbit interaction. $f(\theta) = p^2 + q^2 \cos^2 \theta$ for CAR-$\uparrow\downarrow$ and ECT-$\uparrow\uparrow$, $g(\theta) = q^2 \sin^2 \theta$ for CAR-$\uparrow\downarrow$ and ECT-$\uparrow\uparrow$. $\theta$ is the angle between the spin-orbit field and the dot spin polarization axis. Here we choose $p^2 = 0.2$ and $q^2 = 0.1$ for curve plotting.
shows that $P^a$ has a separable dependence on the energy $\xi_n$ and on the wavefunctions $\psi_n, \psi_{\pi}$ of the ABs. In particular, the energy dependence is universal because it only depends on the BCS coherence factors. As shown in Fig. 2(a), $C_0(z)$ for CAR has a single peak centered at $z = 0$ ($\xi_n = 0$) and decays as $z^{-4}$ at large $|z|$, while $E_0(z)$ for ECT has double peaks located at $z = \pm 1$, and decays as $z^{-2}$ at large $|z|$. Interestingly, the ECT profile has a dip at $z = 0$ due to the destructive interference between two virtual paths with a relative $\pi$-phase shift. The wavefunction part in Eq. (4) gives an overall prefactor to the energy-dependence profiles. Time-reversal invariance of the hybrid nanowire guarantees $\psi_{\pi}(x\sigma) = T\psi_n(x\sigma)$ with $T = -i\sigma_x K$, which gives the following symmetry relations between different dot-spin channels

$$P^a_{\uparrow\uparrow} = P^a_{\downarrow\downarrow}, \quad P^a_{\uparrow\downarrow} = P^a_{\downarrow\uparrow},$$

(5)

for $a = \text{CAR}$ or ECT.

If the spin-orbit field is the only or the dominant spinful field in the hybrid nanowire and its direction is constant throughout the wire, we can explicitly write the spinors of the normal-state wavefunctions as $\psi_n(x\sigma) = \phi_n(x)(\cos(\theta/2), \sin(\theta/2))^\top$ and $\psi_{\pi}(x\sigma) = T\psi_n(x\sigma)$, where $\phi_n(x)$ is the spatial wavefunction and $\theta$ is the angle between the spin-orbit field and the spin polarization axis in quantum dots. Here the dot spin axis is fixed along $\sigma_z$ and the spin-orbit field is rotating in the $xz$-plane without loss of generality. Plugging the wavefunctions into Eq. (4), we obtain

$$P^\text{CAR}_{\uparrow\uparrow} = C_0(z) \cdot f(\theta), \quad P^\text{EKT}_{\uparrow\uparrow} = E_0(z) \cdot f(\theta),$$

$$P^\text{CAR}_{\uparrow\downarrow} = C_0(z) \cdot g(\theta), \quad P^\text{EKT}_{\uparrow\downarrow} = E_0(z) \cdot g(\theta),$$

(6)

where $f(\theta) = p^2 + q^2 \cos^2 \theta, \quad g(\theta) = q^2 \sin^2 \theta$ with $p = \Re[\phi_n(x)\phi_n^*(x)]$, $q = \Im[\phi_n(x)\phi_n^*(x)]$, $P^a$ in the other channels are related by Eq. (5). As shown in Fig. 2(b), $P^a$ has a sinusoidal dependence on the angle. In particular, CAR-$\uparrow\uparrow$ and ECT-$\uparrow\uparrow$ are more favorable channels, because $f(\theta) \geq p^2$. While in the CAR-$\uparrow\downarrow$ and ECT-$\uparrow\downarrow$ channels, $P^a$ vanishes at $\theta = 0$ or $\pi$, i.e., equal-spin CAR and opposite-spin ECT processes are not allowed when the total dot-superconductor-dot system has spin conservation. Thereby, in order to have finite CAR and ECT simultaneously in a particular dot spin channel, it is crucial to have a finite spin-orbit field in the hybrid segment, with its direction not aligned with the dot spin axis.

**Effect of Zeeman spin splitting.**—We now consider the effect of induced Zeeman splitting in the hybrid segment. This relaxes the assumption of time-reversal invariance in the nanowire. It also provides an additional experimentally accessible parameter to tune the profiles of CAR and ECT, and allows for an additional comparison between experiment and theory. The Zeeman field direction is parallel to the spin-polarization axis in dots, because we have assumed a globally applied magnetic field in the system. We also assume the field strength $E_Z$ to be weak, i.e., $E_Z$ is less than the spin-orbit interaction and the superconducting gap. Under these assumptions, we can project the Zeeman field along the spin-orbit axis, such that the ABS energies are now split as $E_{1,2} = \sqrt{\xi_n^2 + \Delta^2} \pm E_Z \cos \theta$, while the ABS wavefunctions remain the same as those in the time-reversal invariant scenario. Thereby, the interdot CAR becomes

$$P^\text{CAR}_{\uparrow\uparrow}(E_Z) = P^\text{CAR}_{\downarrow\downarrow}(E_Z) = C_0(z) \cdot q^2 \sin^2 \theta,$$

$$P^\text{EKT}_{\uparrow\uparrow}(E_Z) = P^\text{EKT}_{\downarrow\downarrow}(E_Z) = C_0(z) \cdot (p^2 + q^2 \cos^2 \theta),$$

(7)

where $C_0(z) = (z^2 + 1 - \delta^2)^{-2}$ is the modified energy-dependence profile for CAR with $\delta = E_Z \cos \theta / \Delta < 1$. Figures 3(a) and 3(b) show the energy dependence of $P^\text{CAR}$ for different values of $E_Z/\Delta$. Both $P^\text{CAR}$ and $P^\text{EKT}$ increase with $E_Z$, with their profiles remaining symmetric about $z = 0$. The interdot ECT are

$$P^\text{EKT}_{\uparrow\uparrow}(E_Z) = P^\text{EKT}_{\downarrow\downarrow}(-E_Z) = \frac{(z - \delta \cos \theta)^2 p^2 + (z \cos \theta - \delta)^2 q^2}{(z^2 + 1 - \delta^2)^2},$$

$$P^\text{EKT}_{\uparrow\downarrow}(E_Z) = P^\text{EKT}_{\downarrow\uparrow}(E_Z) = \frac{z^2 q^2 + \delta^2 p^2}{(z^2 + 1 - \delta^2)^2} \sin^2 \theta.$$  

(8)

As shown in Fig. 3(c), the profile of $P^\text{EKT}$ becomes asymmetric when $E_Z > 0$, with one peak being lifted and the
other suppressed. According to the symmetry relation in Eq. (8), the profile of $I^\text{ECT}_{\mu l}(E_Z)$ would become asymmetric in the opposite way. On the other hand, $P^\text{ECT}_{\mu l}$ remains symmetric at a finite Zeeman field, with the dip at $z = 0$ being elevated to a finite value, as shown in Fig. 3(d).

Extracting $\Gamma^\sigma$ experimentally.—To reach the optimal parameter regime for the desired application, it is necessary to be able to extract the strengths of the effective interdot couplings experimentally. For this purpose, we propose a three-terminal junction, where two quantum dots are now connected with two external normal electrodes, respectively, as shown by the schematics in Fig. 4. The strengths of $\Gamma^\text{CAR}$ and $\Gamma^\text{ECT}$ can be extracted from measuring the resonant current under appropriate conditions.

Our considerations and calculations follow the same spirit as those in Refs. [1, 2], which focused on CAR current for Cooper pair splitting in a similar setup. Compared to the previous works, the differences made in our calculations include: (1) We now consider ABSs instead of quasiparticle continuum in the superconducting segment. (2) Spin-orbit interaction in the hybrid segment breaks spin conservation. (3) Dot spins are polarized. (4) We generalize the calculations to ECT scenarios.

The total Hamiltonian for the three-terminal junction, as shown in Fig. 4, is $H_{\text{tot}} = H + H_L + H_{DL}$. $H$ is the dot-superconductor-dot system introduced by Eq. (1). $H_L = \sum_k (\varepsilon_k - \mu_l) a_{kq}^\dag a_{kq} + (\varepsilon_k - \mu_r) a_{rko}^\dag a_{rko}$ are the normal leads, which are conventional Fermi liquids with electrons filled up to the Fermi energy $\mu_{l,r}$. $H_{DL} = \sum_k \left(-t_{l} d_{l}^\dag a_{lkq} - t_{r} d_{r}^\dag a_{rko}\right)$ describes the dot-lead tunneling. The relevant parameter regime for generating resonant current is [1, 2]

$$\Delta, U > \delta \mu > \Gamma_{DL}, k_B T,$$

$$\Gamma_{DL} > \Gamma_{SD}. \quad \varepsilon_l, \varepsilon_r \approx \mu_S.$$ (9)

Here $\delta \mu$ is the applied bias voltage, with $\delta \mu = \mu_S - \mu_{l,r} > 0$ for generating CAR current, and $\delta \mu/2 = \mu_r - \mu_S = \mu_S - \mu_l > 0$ for ECT current, as shown schematically in Figs. 4(a) and 4(b). The bias voltage needs to be smaller than the induced gap $\Delta$ in the nanowire as well as the charging energy $U$ in the dots, which would suppress such undesired processes as local Andreev reflection and inelastic co-tunneling. On the other hand, the bias voltage window needs to be large enough in order to include the full width of the broadened dot states, i.e., $\delta \mu > \Gamma_{DL} = \pi v|t_0|^{2} + |t_{\nu}|^{2}$ with $\nu$ being the lead density of states. The dot-lead coupling should be stronger than the superconductor-dot coupling $\Gamma_{DL} > \Gamma_{SD} = t_{l}t_{r}/\Delta$, such that the interdot tunneling process has the shortest time scale and thus maintain coherence. Dot energies need to be tuned close to the superconducting Fermi energy to make dot levels on resonance. Once all these criteria are met, resonant current will flow between source and drain leads.

The currents are calculated using the rate equation altogether with the $T$-matrix approach [1, 2, 38]. Calculation details are in the supplemental material [39], and we only show the results here. When $\mu_S > \mu_{l,r}$, Cooper pairs from the superconducting lead would split into two electrons, which flow to two separate normal leads, respectively, giving the following spin-selective CAR current

$$I^\text{CAR}_{\eta \sigma} = \frac{e}{h} \frac{\Gamma_{DL}^{2}}{(\varepsilon_l + \varepsilon_r)^{2} + \Gamma_{DL}^{2}} \cdot |I^\text{CAR}_{\eta \sigma}|^{2} \frac{\Gamma_{DL}}{\Gamma_{DL}}.$$ (10)

with $\Gamma^\text{CAR}_{\eta \sigma}$ being the interdot CAR defined in Eq. (3).

As shown in Fig. 4(c), in the $(\varepsilon_l, \varepsilon_r)$-plane this current has a Breit-Wigner resonance form with its broadening width being the dot-lead coupling $\Gamma_{DL}$ and the maximum value reached when $\varepsilon_l = -\varepsilon_r$. On the other hand, in exactly the same setup but with a different bias voltage: $\mu_l < \mu_S < \mu_r$, now a single electron flows from one to the other normal lead, giving the spin-selective ECT current

$$I^\text{ECT}_{\eta \sigma} = \frac{e}{h} \frac{\Gamma_{DL}^{2}}{(\varepsilon_l - \varepsilon_r)^{2} + \Gamma_{DL}^{2}} \cdot |I^\text{ECT}_{\eta \sigma}|^{2} \frac{\Gamma_{DL}}{\Gamma_{DL}}.$$ (11)

where $\Gamma^\text{ECT}_{\eta \sigma}$ is defined in Eq. (3). The ECT current has the same Breit-Wigner form, but now assumes the maximum value when $\varepsilon_l = \varepsilon_r$, as shown in Fig. 4(d).

Equations (10) and (11) indicate that resonant current is proportional to the square of the corresponding interdot coupling strength. Thus, experimentally one can extract the coupling strengths using the formula $\Gamma^\alpha = \sqrt{I^\text{max}_{\alpha DL} h/e}$, where $\Gamma_{DL}$ is read off from the resonance broadening width in gate voltage times the lever arm, $\varepsilon_l$.
and $f_{\text{max}}^a$ is the current value along $\varepsilon_l = -\varepsilon_r$ for CAR and $\varepsilon_l = \varepsilon_r$ for ECT.

Discussion.—We have shown that effective interdot couplings of CAR and ECT can be mediated by ABSs in a short hybrid nanowire and that their ratio can be controlled by the ABS chemical potential, giving an experimental knob for tuning the physical system into the optimal parameter regime as demanded. In particular, one can enhance the Cooper pair splitting efficiency to nearly one by tuning the ABS energy to where the ECT profile drops to zero [e.g., $z = 0$ in Fig. 2(a)], or adjust the emulated Kitaev chain to its sweet spot by tuning the ABS energy to where ECT and CAR profiles cross each other [e.g., $z \approx \pm 1$ in Fig. 2(a) when $f(\theta) \approx g(\theta)$]. In practice, this tuning protocol can be implemented by controlling the electrostatic gate near the semiconductor-superconductor segment, eliminating the need of non-collinear magnetic fields \cite{27}. This makes our proposal especially appealing, since all the necessary ingredients, i.e., spin-polarized quantum dots \cite{40}, gated hybrid nanowire with spin-orbit interaction \cite{41, 42}, are within reach of existing materials and technologies.

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I. BDG HAMILTONIAN OF THE HYBRID SEGMENT

In the presence of time-reversal invariance and induced superconducting pairing being s-wave, Anderson’s theorem guarantees the following form of BdG Hamiltonian in the terms of Kramers’ pair:

\[ H_S = \xi_n (a_n^\dagger a_n + a_{\pi}^\dagger a_{\pi}) + \Delta (a_n^\dagger a_{\pi}^\dagger + a_{\pi} a_n) \]

\[ = \begin{pmatrix} a_n^\dagger \\ a_{\pi}^\dagger \end{pmatrix}^\dagger \begin{pmatrix} \xi_n & \Delta \\ \Delta^\ast & -\xi_n \end{pmatrix} \begin{pmatrix} a_n \\ a_{\pi} \end{pmatrix}, \]

where \( a_n, a_{\pi} \) denote the time-reversed normal states with wavefunctions \( \psi_n \) and \( \psi_{\pi} = -i\sigma_y \psi_n^\ast \). Thus the eigenenergies are

\[ E_{1,2} = E_n = \sqrt{\xi_n^2 + \Delta^2}, \]

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and the BdG eigenfunctions are

\[
\gamma_1 = u_0 a_n + v_0 a_n^\dagger, \\
\gamma_2 = u_0 a_n^\dagger - u_0 a_n.
\]  

(S-3)

Here \( \xi_n = \varepsilon_n - \mu \) is the normal-state energy relative to the chemical potential, \( u_0^2 = 1 - v_0^2 = 1/2 + \xi_n/E_n \) are BCS coherence factors. We further assume that spin-orbit field is the only or the dominant spinful field in the hybrid segment, thus the normal-state wavefunctions can be written as

\[
\psi_n(x\sigma) = \phi_n(x) \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix},
\]

(S-4)

where \( \theta \) is the angle between the spin-orbit field and the dot spin axis.

We now introduce a Zeeman field \( H_Z = E_Z \int dx [c_1^\dagger(x)c_1(x) - c_1^\dagger(x)c_1(x)] \) in the hybrid segment, with its direction parallel to the dot spin axis. In the weak-field limit, i.e., \( E_Z < E_{so}, \Delta \), the effect can be treated perturbatively by projecting it onto the eigenfunctions in the presence of time-reversal invariance. In particular,

\[
\begin{align*}
(H_Z)_{nn} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_n^*(x,\sigma)(\sigma_\sigma')(\sigma_\sigma')^\dagger \psi_n(x,\sigma') = E_Z \cos \theta, \\
(H_Z)_{n\pi} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_n^*(x,\sigma)(\sigma_\sigma')(\sigma_\pi)^\dagger \psi_n(x,\sigma') = -E_Z \cos \theta, \\
(H_Z)_{\pi n} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_n^*(x,\sigma)(\sigma_\pi)(\sigma_{\pi'})^\dagger \psi_n(x,\sigma') = E_Z \sin \theta \times \int [\phi(x)]^2 dx, \\
(H_Z)_{\pi\pi} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_n^*(x,\sigma)(\sigma_\pi)(\sigma_\pi')^\dagger \psi_n(x,\sigma') = E_Z \sin \theta \times \int [\phi^*(x)]^2 dx.
\end{align*}
\]

(S-5)

In the strong spin-orbit interaction regime with the spin-orbit length being smaller than the nanowire length, the phase of \( \phi(x) \) oscillate fast, making \( |\int [\phi(x)]^2 dx| \ll 1 \). We can thus neglect \( (H_Z)_{n\pi} \) and \( (H_Z)_{\pi n} \). Now the BdG Hamiltonian of the hybrid segment becomes

\[
H_S = \begin{pmatrix} a_n^\dagger & a_n \end{pmatrix}^\dagger \begin{pmatrix} \xi_n + E_Z \cos \theta & \Delta \\ \Delta & -\xi_n + E_Z \cos \theta \end{pmatrix} \begin{pmatrix} a_n^\dagger \\ a_n \end{pmatrix}.
\]

(S-6)

The eigenenergies are now split as

\[
E_{1,2} = \sqrt{\xi_n^2 + \Delta^2} \pm E_Z \cos \theta,
\]

(S-7)

and the BdG eigenfunctions remain the same as the time-reversal invariant case because the effective Zeeman term is an identity in the Nambu space.

**II. RESONANT CAR AND ECT CURRENT**

In this section, we give details of how we calculate the resonant current in the normal-dot-superconductor-dot-normal junction. The methods we use include the T-matrix approach and the rate equation, which are standard for resonant current calculations [1, 2] in such mesoscopic systems. In terms of the parameter regime for generating resonant current, we consider

\[
\Delta, U > \delta \mu > \Gamma_{DL}, k_B T, \quad \Gamma_{DL} > \Gamma_{SD}, \quad \varepsilon_l \approx \varepsilon_r \approx \mu_S,
\]

(S-8)

which applies to both CAR and ECT.

**A. Crossed Andreev reflection**

We first consider the CAR current, where two electrons pass from the superconductor via the virtual dot states to two different leads. The whole tunneling process can be decomposed into two main parts. In the first part, a Cooper
pair breaks up, where one electron tunnels from superconductor to one dot level, leaving behind a quasiparticle excitation $E_m > \Delta$ in superconductor. Almost simultaneously the second electron of the Cooper pair tunnels from superconductor to the other dot level before the first electron escapes from the dot to the electrode, because the relevant time scale is $\hbar/\Delta < \hbar/\Gamma_{DL}$. Tunneling back to the superconductor is unlikely because $\Gamma_{SD} < \Gamma_{DL}$. The amplitude for the transition from the initial to the final state is thus

$$\langle f | T(\varepsilon_i) | i \rangle \approx \langle f | T(0) | i \rangle = \langle f | T_2 | DD \rangle \langle DD | T_1 | i \rangle,$$

(S-9)

where $T(0) = T(\varepsilon_i = 0)$. The initial state is $| i \rangle = | 0_S \rangle | 0_D \rangle | \mu_l \rangle$, where the superconducting is in its ground state with no quasiparticle excitations, both dots levels are vacant and the normal leads are Fermi liquids filled up to its relevant time scale is $\hbar \Gamma_{DL}$. Thus using the second-order perturbation theory we immediately have

$$\langle DD | T_1 | i \rangle = \frac{1}{i \eta - (\varepsilon_1 + \varepsilon_2)} \cdot \Gamma_{\eta \sigma}^{CAR},$$

(S-11)

with

$$\Gamma_{\eta \sigma}^{CAR} = \frac{t_l t_r}{\Delta} \sum_{m=1,2} \frac{u_m(l\eta) v_m^*(r\sigma) - u_m(r\sigma) v_m^*(l\eta)}{E_m/\Delta}.\quad$$

(S-12)

Here the spin in the dots are polarized in the $\sigma_r$ direction. Since $\varepsilon_1 \approx \varepsilon_r \approx \mu_S = 0$, the energy denominator diverges as $1/\eta$, indicating that the tunneling between dots and leads is resonant. Thus for the second part of the tunneling process, we must include the tunnel Hamiltonian to all orders, i.e.,

$$T_2 = H_{DL} \sum_{n=0}^{\infty} \left( \frac{1}{i \eta - H_{DL}} \right)^{2n+1},$$

(S-13)

and thus the transition amplitude for the second part is

$$\langle f | T_2 | DD \rangle = \left\{ \langle pq | H_{DL1} | Dq \rangle \langle Dq | \sum_{m=0}^{\infty} \left( \frac{1}{i \eta - H_{DL1}} \right)^{2m} | Dq \rangle \langle Dq | \frac{1}{i \eta - H_{DL2}} | DD \rangle 
+ \langle pq | H_{DL2} | pD \rangle \langle pD | \sum_{m=0}^{\infty} \left( \frac{1}{i \eta - H_{DL2}} \right)^{2m} | pD \rangle \langle pD | \frac{1}{i \eta - H_{DL1}} | DD \rangle \right\} \times \langle DD | \sum_{n=0}^{\infty} \left( \frac{1}{i \eta - H_{DL}} \right)^{2n} | DD \rangle.$$

(S-14)

Among all the above terms, the geometrical summation are calculated in a similar manner, and here we only focus on $\langle DD | \sum_{n=0}^{\infty} \left( \frac{1}{i \eta - H_{DL}} \right)^{2n} | DD \rangle$, which is

$$\langle DD | \sum_{n=0}^{\infty} \left( \frac{1}{i \eta - H_{DL}} \right)^{2n} | DD \rangle \approx 1 + \sum_{n=1}^{\infty} \left( \langle DD | \left( \frac{1}{i \eta - H_{DL}} \right)^2 | DD \rangle \right)^n = \frac{1}{1 - \langle DD | \left( \frac{1}{i \eta - H_{DL}} \right)^2 | DD \rangle},$$

(S-15)

with

$$\langle DD | \left( \frac{1}{i \eta - H_{DL}} \right)^2 | DD \rangle = \frac{1}{i \eta - (\varepsilon_1 + \varepsilon_2)} \left( \frac{\Gamma_{DL2}}{2\pi} \int d\varepsilon_k \frac{1}{i \eta - (\varepsilon_1 + \varepsilon_k)} + \frac{\Gamma_{DL1}}{2\pi} \int d\varepsilon_k \frac{1}{i \eta - (\varepsilon_2 + \varepsilon_k)} \right) = \frac{1}{\varepsilon_1 + \varepsilon_2 - i \eta} (\text{Re} \Sigma + i \Gamma_{DL}),$$

(S-16)
where \( \text{Re} \Sigma = \ln(E_{c1}/E_{c2}) \) is a logarithmic divergence with the cutoff energy in the conduction band in the normal lead, and \( \Gamma_{DL} = \Gamma_{DLl} + \Gamma_{DLr} \). Note that here we assume that the dominant process is \( |DD\rangle \rightarrow |pD\rangle \rightarrow |DD\rangle \) or \( |DD\rangle \rightarrow |Dq\rangle \rightarrow |DD\rangle \), with \( |pD\rangle \) denoting that the left electron is in the lead and the right electron in the dot. So the geometrical summation is

\[
\left\langle DD \right| \sum_{n=0}^{\infty} \left( \frac{1}{} - H_{0} \right)^{2n} \left| DD \right\rangle = \frac{\varepsilon_{1} + \varepsilon_{2} - i\eta}{\varepsilon_{1} + \varepsilon_{2} - i\Gamma_{DL}}.
\]

(S-17)

The key finding here is that the divergent denominator now becomes the numerator and the new denominator has a finite imaginary part which is proportional to the dot-lead coupling. Finally the transition amplitude for the second part is

\[
\langle f|T_{2}|DD \rangle = \left( t'_{l} \frac{\varepsilon_{1} + \varepsilon_{q} - i\eta}{\varepsilon_{1} + \varepsilon_{q} - i\Gamma_{DL}l} t'_{r} \right) \left( t'_{l} \frac{\varepsilon_{2} + \varepsilon_{p} - i\eta}{\varepsilon_{2} + \varepsilon_{p} - i\Gamma_{DL}r} \right) \frac{\varepsilon_{1} + \varepsilon_{2} - i\eta}{\varepsilon_{1} + \varepsilon_{2} - i\Gamma_{DL}}.
\]

(S-18)

Plugging it into the formula for CAR current, we have

\[
I_{\text{CAR}} = \frac{e}{h} \int d\varepsilon_{p} \nu_{2} \int d\varepsilon_{q} |\langle pq|T(0)|i\rangle|^{2} \delta(\varepsilon_{p} + \varepsilon_{q})
\]

\[
= \frac{e}{h} \Gamma_{DLl} \Gamma_{DLr} |\Gamma_{\eta\sigma}^{\text{CAR}}|^{2} \int d\varepsilon_{p} \frac{1}{[(\varepsilon_{1} - \varepsilon_{p})^{2} + \Gamma_{DLl}^{2}][(\varepsilon_{2} + \varepsilon_{p})^{2} + \Gamma_{DLr}^{2}]}
\]

\[
= \frac{e}{h} \frac{\Gamma_{DL}}{(\varepsilon_{1} + \varepsilon_{2})^{2} + \Gamma_{DL}^{2}} |\Gamma_{\eta\sigma}^{\text{CAR}}|^{2}.
\]

(S-19)

So the CAR current has the form of a Breit-Wigner resonance profile, which assumes its maximum value at \( \varepsilon_{1} + \varepsilon_{2} = 0 \).

### B. Elastic co-tunneling

For the ECT current, a single electron passes from the lead with higher chemical potential via the dot and superconductor states to the other lead with lower chemical potential. Here we make three assumptions in our derivation. First, when calculating the transition rate \( W_{ij} \) between particular initial and final state, we assume that both normal leads are vacant. Second, the Fermi-Dirac distribution will be taken into account only in the final step for \( \rho_{i} \). Third, when the chemical potential in leads are equal, the current flowing in the opposite directions cancel with each other. Under these assumptions, we first calculate the transition rate, focusing on the scenario of a single electron passing from the right lead to the left lead. The total tunneling process can be separated into three parts.

\[
\langle p|T(\varepsilon_{i} = \varepsilon_{q})|q\rangle = \langle p|T_{3}|1\rangle(1)|T_{2}|2\rangle(2)|T_{1}|q\rangle,
\]

(S-20)

where \( |l\rangle \) or \( |r\rangle \) means an electron is in the left or right dot. For the first step, we need to include the resonant tunneling between dot and lead, such that

\[
\langle r|T_{1}|q\rangle = \left( \sum_{n=0}^{\infty} \frac{1}{i\eta - H_{0}} H_{DLr} \right)^{2n} \langle r \left| \frac{1}{i\eta - H_{0}} H_{DLr} \right| q \rangle = \frac{t'_{l}}{\varepsilon_{r} - \varepsilon_{q} - i\Gamma_{DLr}}.
\]

(S-21)

And

\[
\langle l|T_{2}|r\rangle = \left( \frac{1}{i\eta - H_{SD}} \right)^{2} \langle l \left| \frac{1}{i\eta - H_{SD}} \right| r \rangle = \frac{\Gamma_{\eta\sigma}^{\text{ECT}}}{i\eta - (\varepsilon_{1} - \varepsilon_{q})},
\]

(S-22)
with
\[ \Gamma_{\eta \sigma}^{\text{ECT}} = \frac{t_l t_r}{\Delta} \sum_{m=1,2} \frac{u_m(l\eta) u_m^*(r\sigma) - v_m(r\sigma) v_m^*(l\eta)}{E_m/\Delta}. \] (S-23)

For the third step, we have
\[ \langle p | T_3 | 1 \rangle = \langle p | H_{DLl} | l \rangle \left( \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_0} H_{DLl} \right)^2 n \right) \left| l \right\rangle \\
= t'_{l} \frac{\varepsilon_l - \varepsilon_q - i\eta}{\varepsilon_l - \varepsilon_q - i\Gamma_{DLl}}. \] (S-24)

Therefore we have the transition amplitude and the transition rate to be
\[ \langle p | T | q \rangle = \frac{t'_l t'_r \Gamma_{\eta \sigma}^{\text{ECT}}}{(\varepsilon_l - \varepsilon_q - i\Gamma_{DLl})(\varepsilon_r - \varepsilon_q - i\Gamma_{DLr})}, \]
\[ W_{pq} = 2\pi |\langle p | T | q \rangle|^2 \delta(\varepsilon_p - \varepsilon_q). \] (S-25)

The ECT current now is
\[ I = \frac{e}{\hbar} \sum_{f,i} W_{fi} \rho_i \]
\[ = \frac{e}{\hbar} \nu_l \int d\varepsilon_p \nu_r \int_{-\delta\mu/2}^{\delta\mu/2} d\varepsilon_q |\langle p | T | q \rangle|^2 \delta(\varepsilon_p - \varepsilon_q) \]
\[ = \frac{e}{\hbar} \int_{-\delta\mu/2}^{\delta\mu/2} d\varepsilon_q \left[ (\varepsilon_l - \varepsilon_q)^2 + \frac{\gamma_{DLl}^2 \gamma_{DLr}^2 |\Gamma_{\eta \sigma}^{\text{ECT}}|^2}{(\varepsilon_r - \varepsilon_q)^2 + \Gamma_{DLr}^2} \right] \]
\[ = \frac{e}{\hbar} \frac{\Gamma_{DLl}}{(\varepsilon_l - \varepsilon_r)^2 + \Gamma_{DLl}^2} \left[ |\Gamma_{\eta \sigma}^{\text{ECT}}|^2 \right]. \] (S-26)

Note that the integral of the outgoing electron energy \( \varepsilon_p \) disappear because of the energy conservation. The integral window of the incoming electron energy \( \varepsilon_q \) is because when \( \delta\mu = 0 \) the system is in equilibrium and left-flowing and right-flowing currents cancel and \( I = 0 \). So the net current is due to the window of the biased voltage. The ECT current is also in the form of the Breit-Wigner resonance, and assumes its maximum value at \( \varepsilon_l = \varepsilon_r \).