Weighted Counting of Matchings in Unbounded-Treewidth Graph Families

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Abstract

We consider a weighted counting problem on matchings, denoted PrMatching(G), on an arbitrary fixed graph family G. The input consists of a graph G ∈ G and of rational probabilities of existence on every edge of G, assuming independence. The output is the probability of obtaining a matching of G in the resulting distribution, i.e., a set of edges that are pairwise disjoint. It is known that, if G has bounded treewidth, then PrMatching(G) can be solved in polynomial time. In this paper we show that, under some assumptions, bounded treewidth in fact characterizes the tractable graph families for this problem. More precisely, we show intractability for all graph families G satisfying the following treewidth-constructibility requirement: given an integer k in unary, we can construct in polynomial time a graph G with treewidth at least k. Our hardness result is then the following: for any treewidth-constructible graph family G, the problem PrMatching(G) is intractable. This generalizes known hardness results for weighted matching counting under some restrictions that do not bound treewidth, e.g., being planar, 3-regular, or bipartite; it also answers a question left open in [1]. We also obtain a similar lower bound for the weighted counting of edge covers.

This is the full version of the article appearing in MFCS’22, containing complete proofs.

1 Introduction

Many complexity results on computational problems rely on a study of fundamental graph patterns such as independent sets, vertex covers, edge covers, matchings, cliques, etc. In this paper we specifically study counting problems for such patterns, and for the most part focus on counting the matchings: given an input graph G, we wish to count how many edge subsets of G are a matching, i.e., each vertex has at most one incident edge.

Our goal is to address an apparent gap between the existing intractability and tractability results for counting matchings and similar patterns. On the one hand, counting the matchings is known to be #P-hard, and hardness is known even when the input graph is restricted in certain ways, e.g., being planar, being 3-regular, or being bipartite [20, 15, 29, 28]. On the other hand, some restrictions can make the problem tractable, e.g., imposing that the input graphs have bounded treewidth [6, 1], because matchings can be described in monadic...
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second-order logic. But this does not settle the complexity of the problem; could there be other restrictions on graphs that makes it tractable to count matchings or other patterns?

This paper answers this question in the negative, for a weighted version of counting problems: we show that, at least for matchings and edge covers, and under a technical assumption on the graph family, the weighted counting problem is intractable if we do not bound the treewidth of the input graphs. Thus, treewidth is the right parameter to ensure tractability. Our weighted counting problems are of the following form: we fix a graph family \( G \) (e.g., 3-regular graphs, graphs of treewidth \( \leq 2 \)), we are given as input a graph \( G \) of \( G \) along with an independent probability of existence for each edge, and the goal is to compute the probability in this distribution of the subsets of edges of \( G \) which have a certain property, e.g., they are a matching, they are an edge cover. Note that the class \( G \) restricts the shape of the graphs, but the edge probabilities are arbitrary — and indeed there are known tractability results when we restrict the graphs and probabilities to be symmetric [7]. Our paper shows the hardness of these problems when \( G \) is not of bounded treewidth; the specific technical assumption on \( G \) is that one can effectively construct graphs of \( G \) having arbitrarily high treewidth, i.e., the treewidth-constructible requirement from [1] (cf. Definition 2.2):

\[ \textbf{Result 1.} \text{ Let } G \text{ be an arbitrary family of graphs which is treewidth-constructible. Then the problem, given a graph } G = (V, E) \text{ of } G \text{ and rational probability values } \pi(e) \text{ for every edge of } G, \text{ of computing the probability of a matching in } G \text{ under } \pi, \text{ is } \#P\text{-hard under ZPP reductions.} \]

We obtain an analogous result for edge covers. Thus, as bounded-treewidth makes the problems tractable, our results imply that treewidth characterizes the tractable graph families for these problems — for weighted counting, and assuming treewidth-constructibility. We leave open the complexity of unweighted counting, and of weighted counting on graph families that have unbounded treewidth but satisfy weaker requirements than treewidth-constructibility, e.g., being strongly unbounded poly-logarithmically [18, 14].

The paper is devoted to showing Result 1 (with the proofs of technical claims deferred to the appendix). At a high level, we use the standard technique of reducing from the \#P-hard problem of counting matchings on a 3-regular planar graph \( G \) [28], using the randomized polynomial-time grid minor extraction result of [11] as in [1]. However, the big technical challenge is to reduce the counting of matchings of \( G \) to the problem of computing the probability of a matching on the arbitrary subdivision \( G' \) of \( G \) that we extract. For this, we use the classical interpolation method, where we design a linear equation system relating the matchings to the result of polynomially many oracle calls on \( G' \), with different probability assignments; and we argue that the matrix is invertible. After the preliminaries (Section 2), we present this proof, first in the case where \( G' \) is a 6-subdivision of \( G \) (Section 3), and then when it is a \( n \)-subdivision, i.e., when all edges are subdivided to the same length \( n \) (Section 4). These special cases already pose some difficulties, most of which are solved by adapting techniques by Dalvi and Suciu [13]; e.g., to show invertibility, we study the Jacobian determinant of the mapping associating edge probabilities to the probability of matchings on paths with fixed endpoints, and we borrow a technique from [13] to effectively construct suitable rational edge probabilities.

The main novelties of this work are in Section 5, where we extend the proof to the general case: \( G' \) is a subdivision of \( G \), and different edges of \( G \) may be subdivided in \( G' \) to different lengths. To obtain the equation system, we show that we can assign probabilities on short paths so that they “behave” like long paths. Proving this stand-alone emulation result (Proposition 5.2) was the main technical obstacle; the proof is by solving a system of equations involving the Fibonacci sequence. It also introduces further complications, e.g., dealing
with numerical error (because the resulting probabilities are irrational), and distinguishing even-length and odd-length subdivisions. After concluding the proof of Result 1 in Section 5, we adapt it in Section 6 to edge covers.

Related work. Our work follows a line of results that show the intractability of some problems on any “sufficiently constructible” unbounded-treewidth graph family. Kreutzer and Tazari [18] (see also [14]) show that there are formulas in an expressive formalism (MSO2) that are intractable to check on any subgraph-closed unbounded treewidth graph family that is closed under taking subgraphs and satisfies a requirement of being strongly unbounded poly-logarithmically. This was extended in [1] to the weighted counting problem, this time for a query in first-order logic, with a different hardness notion (#P-hardness under randomized reductions), and under the stronger requirement of treewidth-constructibility. Our focus here is to show that the hardness of weighted counting already holds for natural and well-studied graph properties, e.g., “being a matching”; this was left as an open problem in [1].

For such weak patterns, lower bounds were shown in [1] and [2] on the size of tractable representations: for any graph $G$ of bounded degree having treewidth $k$, any so-called d-SDNNF circuit representing the matchings (or edge covers) of $G$ must have exponential size in $k$. However, this does not imply that the problems are intractable, as some tractable counting algorithms do not work via such circuit representations (e.g., the one in [13]). Thus, our hardness result does not follow from this size bound, but rather complements it.

The necessity of bounded treewidth has also been studied for graphical models [10] and Bayesian networks [19]. Specifically, [19] shows the intractability of inference in a Bayesian network as a function of the treewidth (but without otherwise restricting the class of network), and [10] restricts the shape of the graphical model but allows arbitrary “potential functions” (whereas we assume independence across edges). There are also necessity results on treewidth for the problem of counting the homomorphisms between two structures in the CSP context [12]; but this has no clear relationship to our problems, where we do (weighted) counting of the substructures that have a certain form (e.g., are matchings).

Note that, unlike our problem of weighted counting of matchings, the problem of finding a matching of maximal weight in a weighted graph is tractable on arbitrary graphs, using Edmond’s blossom algorithm [23].

2 Preliminaries

We write $\mathbb{N}^+$ for $\mathbb{N}\setminus\{0\}$, and for $n \in \mathbb{N}^+$ we write $[n]$ the set $\{0, \ldots, n-1\}$. We write $\mathbb{R}$ the real numbers and $\mathbb{Q}$ the rational numbers. Recall that decimal fractions are rational numbers that can be written as a fraction $a/10^k$ of an integer $a$ and a power of ten $10^k$.

Reductions and complexity classes. Recall that #P is the class of counting problems that count the number of accepting paths of a nondeterministic polynomial-time Turing machine. A problem $P_1$ is #P-hard if every problem $P_2$ of #P reduces to $P_1$ in polynomial time; following Valiant [21, 22], we use here the notion of Turing reductions, i.e., $P_2$ can be solved in polynomial time with an oracle for $P_1$. We specifically study what we call #P-hardness under zero-error probabilistic polynomial-time (ZPP) reductions. To define these, we define a randomized algorithm as an algorithm that has access to an additional random tape. We say that a decision problem is in ZPP if there is a randomized algorithm that (always) runs in polynomial time on the input instance, and returns the correct answer on the instance (i.e., accepting or rejecting) with some constant probability, and otherwise
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returns a special failure value. The probabilities are taken over the draws of the contents of the random tape. The exact value of the acceptance probability is not important, because we can make it exponentially small by simply repeating the algorithm polynomially many times. Going beyond decision problems, a ZPP algorithm is a randomized algorithm that runs in polynomial time but may return a special failure value with some constant probability. A ZPP (Turing) reduction from a problem $P_1$ to a problem $P_2$ is then a ZPP algorithm having access to an oracle for $P_2$ that takes an instance of problem $P_1$, runs in polynomial time, returns the correct output (for $P_1$) with some constant probability, and returns the special failure value otherwise. Again, the failure probability can be made arbitrarily small by invoking the reduction multiple times. A problem $P_2$ is then said to be $\#P$-hard under ZPP reductions if any $\#P$-hard problem $P_1$ has a ZPP reduction to it. We will implicitly rely on the fact that we can show $\#P$-hardness under ZPP reductions by reducing in ZPP from any problem which is $\#P$-hard (under Turing reductions); see Appendix A for details.

**Graphs and problem studied.** A finite undirected graph $G = (V, E)$ consists of a finite set $V$ of vertices (or nodes) and of a set $E$ of edges of the form $\{x, y\}$ for $x, y \in V$ with $x \neq y$. A graph family $F$ is a (possibly infinite) set of graphs. For $v \in V$, we write $\mathcal{E}_G(v)$ for the set of edges that are incident to $v$. Recall that a matching of $G$ is a set of edges $M \subseteq E$ that do not share any vertices, i.e., for every $e, e' \in M$ with $e \neq e'$ we have $e \cap e' = \emptyset$; or equivalently, we have $|\{\mathcal{E}_G(v) \cap M\}| \leq 1$ for all $v \in V$. For a graph family $F$, we write $\#\text{Matching}(F)$ the problem of counting the matchings for graphs in $F$: the input is a graph $G \in F$, and the output is the number of matchings of $G$, written $\#\text{Matching}(G)$.

We study a weighted version of $\#\text{Matching}$, defined on probabilistic graphs. A probabilistic graph is a pair $(G, \pi)$ where $G = (V, E)$ is a graph and $\pi : E \rightarrow [0, 1]$ maps every edge $e$ of $H$ to a probability value $\pi(e)$. The probabilistic graph $(G, \pi)$ defines a probability distribution on the set of subsets $E'$ of $E$, where each edge $e \in E$ is in $E'$ with probability $\pi(e)$, assuming independence across edges. Formally, the probability of each subset $E'$ is:

$$\Pr_{G, \pi}(E') := \prod_{e \in E'} \pi(e) \times \prod_{e \in E \setminus E'} (1 - \pi(e)).$$

Given a probabilistic graph $(G, \pi)$, the probability of a matching in $G$ under $\pi$, denoted $\Pr_{\text{matching}}(G, \pi)$, is the probability of obtaining a matching in the distribution. Formally:

$$\Pr_{\text{matching}}(G, \pi) := \sum_{\text{matching } M \text{ of } G} \Pr_{G, \pi}(M).$$

In particular, if $\pi$ maps every edge to the probability $1/2$, then we have $\Pr_{\text{matching}}(G, \pi) = \#\text{Matching}(G)/2^{|E|}$. For a graph family $F$, we will study the problem $\Pr_{\text{Matching}}(F)$ of computing the probability of a matching: the input is a probabilistic graph $(G, \pi)$ where $G \in F$ and $\pi$ is an arbitrary function with rational probability values, and the output is $\Pr_{\text{matching}}(G, \pi)$. Note that $F$ only specifies the graph $G$ and not the probabilities $\pi$, in particular $\pi$ can give probability 0 to edges, which amounts to removing them.

**Treewidth and topological minors.** Treewidth is a parameter mapping any graph $G$ to a number $\text{tw}(G)$ intuitively describing how far $G$ is from being a tree. We omit the formal definition of treewidth (see [27]), as we only rely on the following extraction result: given any planar graph $H$ of maximum degree 3, and a graph $G$ of sufficiently high treewidth, it is possible (in randomized polynomial time) to find $H$ as a topological minor of $G$. We now define this.
The degree of a node $v$ in $H = (V_H, E_H)$ is simply $|\mathcal{E}_H(v)|$. We say that $H$ is 3-regular if every vertex has degree 3, and call $H$ planar if it can be drawn on the plane without edge crossings, in the usual sense [26]. Given $H$ and $\eta: E_H \to \mathbb{N}^+$, the $\eta$-subdivision of $H$, written $\text{Sub}(H, \eta)$, is the graph obtained from $H$ by replacing every edge $e = \{x, y\}$ by a path of length $\eta(e)$, whose end vertices are identified with $x$ and $y$, all intermediate vertices being fresh across all edges. We abuse notation and write $\text{Sub}(H, i)$ for $i \in \mathbb{N}_+$ to mean $\text{Sub}(H, \eta_i)$ for $\eta_i$ the constant-$i$ function. Note that $\text{Sub}(H, 1) = H$. A subgraph of a graph $G = (V_G, E_G)$ is a graph $(V_G', E_G')$ where $E_G' \subseteq E_G$ and $V_G' \subseteq V_G$ such that each edge $e \in E_G'$ for each edge $e \in E_G$. The graph $H = (V_H, E_H)$ is a topological minor of the graph $G = (V_G, E_G)$ if there is a function $\eta: E_H \to \mathbb{N}^+$ such that there is an isomorphism $f$ from the subdivision $\text{Sub}(H, \eta) = (V_H', E_H')$ to some subgraph $G' = (V_G', E_G')$ of $G$, i.e., a bijection $f: V_H' \to V_G'$, such that for every $x, y \in V_H'$ we have $\{x, y\} \in E_H'$ if and only if $f(x), f(y) \in E_G'$.

We can now state the extraction result that we use, which follows from the work of Chekuri and Chuzhoy [11]:

> **Theorem 2.1** [Direct consequence of [11], see, e.g., [1], Lemma 4.4]. There exists $c \in \mathbb{N}$ and a ZPP algorithm\(^1\) that, given as input a planar graph $H = (V_H, E_H)$ of maximum degree 3 and another graph $G$ with $\text{tw}(G) \geq |V_H|^c$, computes a subgraph $G'$ of $G$, a function $\eta: V_H \to \mathbb{N}^+$, and an isomorphism from $\text{Sub}(H, \eta)$ to $G'$ (witnessing that $H$ is a topological minor of $G$).

Our intractability result will apply to graph families where large treewidth graphs can be efficiently found, which we formalize as treewidth-constructibility like in [1]:

> **Definition 2.2.** A graph family $\mathcal{F}$ is treewidth-constructible if there is a polynomial-time algorithm that, given an integer $k$ written in unary\(^2\), outputs a graph $G \in \mathcal{F}$ with $\text{tw}(G) \geq k$.

**Kronecker products and Vandermonde matrices.** To simplify notation, we will work with matrices indexed with arbitrary finite sets (not necessarily ordered). Given two finite sets $I, J$ of same cardinality, we write $\mathbb{R}^{I,J}$ (resp., $\mathbb{Q}^{I,J}$) the set of matrices with real values (resp., rational values) whose rows are indexed by $I$ and columns by $J$. When $A \in \mathbb{R}^{I,J}$ and $(i,j) \in I \times J$, we write $a_{i,j}$ the corresponding entry. We recall that the inverse of an invertible matrix $M$ with entries in $\mathbb{Q}$ also has entries in $\mathbb{Q}$ and can be computed in polynomial time in the encoding size of $M$.

Given two matrices $A \in \mathbb{R}^{I,J}$ and $B \in \mathbb{R}^{K,L}$, the Kronecker product of $A$ and $B$, denoted $A \otimes B$, is the matrix $C \in \mathbb{R}^{I \times K, J \times L}$ defined by $c_{(i,k),(j,l)} := a_{i,j} \times b_{k,l}$ for $(i,j,k,l) \in I \times J \times K \times L$. Recall that $A \otimes B$ is invertible if and only if both $A$ and $B$ are. For $n \in \mathbb{N}^+$ and $(p_0, \ldots, p_{n-1}) \in \mathbb{R}^n$, we denote by $V(p_0, \ldots, p_{n-1})$ the Vandermonde matrix with coefficients $(p_0, \ldots, p_{n-1})$, i.e., the matrix in $\mathbb{R}^{[n],[n]}$ whose $(i,j)$-th entry is $p_j^i$. Recall that this matrix is invertible if and only if the $p_0, \ldots, p_{n-1}$ are pairwise distinct.

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\(^1\) The randomized algorithm from [11] is indeed a ZPP algorithm because the output that it returns (namely, a prospective embedding of a grid as a topological minor of the input graph) can be verified in (deterministic) polynomial time. Hence, we can always detect when the algorithm has failed, and then return the special failure value.

\(^2\) Note that the existence of such an algorithm for $k$ written in unary would be implied by the same claim but with $k$ given in binary. In other words, the existence of an algorithm for $k$ given in unary is a weaker requirement. This is simply because, given an integer in unary, we can convert it in PTIME to an integer in binary.
3 Proof When Every Subdivision Has Length 6

Towards showing our main result (Result 1), we first show in this section a much simpler result: counting the matchings of a graph $G$ reduces to counting the probability of a matching on the graph where each edge is subdivided into a path of length 6. We use similar techniques to previous work, in particular Greenhill [15] and Dalvi and Suciu [13], but present them in detail because we will adapt them in the rest of the paper. Formally, in this section, we show:

> **Proposition 3.1.** For any graph family $\mathcal{F}$, the problem $\texttt{#Matching}(\mathcal{F})$ reduces in polynomial time to $\texttt{PrMatching}$ where $\mathcal{G} = \{\text{Sub}(H, 6) \mid H \in \mathcal{F}\}$.

Let $H = (V, E)$ be a graph in $\mathcal{F}$ for which we wish to count the number of matchings, with $m := |E|$. Let us start by fixing for the remainder of this section an arbitrary orientation $\overrightarrow{H}$ of $H$ obtained by choosing some orientation of the edges, i.e., $\overrightarrow{H} = (V, \overrightarrow{E})$ is a directed graph where for every edge $(x, y) \in E$ we add exactly one of $(x, y)$ or $(y, x)$ in $\overrightarrow{E}$. The high-level idea of the reduction is then the following. First, using $\overrightarrow{H}$, we define some sets $S_{\tau}$, based on 4-tuples $\tau \in [m + 1]^4$, such that the number of matchings of $H$ can be computed from the cardinalities $|S_{\tau}|$. Second, we argue that these cardinalities can be connected to the results of oracle calls for the $\texttt{PrMatching}$ problem by a system of linear equations. Third, we argue that the matrix of this system can be made invertible. We now detail these three steps.

**Step 1: Defining the sets $S_{\tau}$ and linking them to matchings.** We define a selection function of the graph $H$ as a function $\mu$ that maps each vertex $x \in V$ to at most one incident edge, i.e., to a subset of $E_H(x)$ of size at most one. We will partition the set of selection functions by counting the number of edges of each type that each selection function has, as defined next. Given a selection function $\mu$, consider each edge $e = (x, y)$ of $\overrightarrow{H}$. The edge $e$ can have one of four types: letting $b$ be 1 if $\mu(x)$ selects $e$ (i.e., $\mu(x) = \{(x, y)\}$) and 0 otherwise (i.e., $(x, y) \notin \mu(x)$), and letting $b'$ be 1 if $\mu(y)$ selects $e$ and 0 otherwise, we say that $e$ has type $bb'$ with respect to (w.r.t.) $\mu$. We now define the sets $S_{\tau}$ as follows.

**Definition 3.2.** For a 4-tuple $\tau \in [m + 1]^4$, indexed in binary, let $S_{\tau} \subseteq S$ be the set of the selection functions $\mu$ such that, for all $b, b' \in \{0, 1\}$, precisely $\tau_{bb'}$ edges have type $bb'$ w.r.t. $\mu$.

Observe that $S_{\tau}$ is empty unless $\tau_{00} + \tau_{01} + \tau_{10} + \tau_{11} = m$. We can then easily connect the cardinalities $|S_{\tau}|$ to the number of matchings of $H$ as follows (see Appendix B):

**Fact 3.3.** We have that $\texttt{#Matching}(H) = \sum_{\tau \in [m + 1]^4} |S_{\tau}|$.

**Step 2: Recovering the $|S_{\tau}|$ from oracle calls.** We now explain how to use the oracle for $\texttt{PrMatching}$ to compute in polynomial time all the values $|S_{\tau}|$, allowing us to conclude via Fact 3.3. We will invoke the oracle on $(m + 1)^4$ probabilistic graphs, denoted $H_6(\kappa)$ for $\kappa \in [m + 1]^4$, as defined next. To this end, let us consider $(m + 1)^4$ 4-tuples of probability values, written $\rho_{\kappa} = (\rho_{\kappa,00}, \rho_{\kappa,01}, \rho_{\kappa,10}, \rho_{\kappa,11}) \in [0, 1]^4$ for $\kappa \in [m + 1]^4$; the precise choice of these values will be explained in Step 3. For $\kappa \in [m + 1]^4$, we then define $H_6(\kappa)$ to be the probabilistic graph $(H_6, \pi_{\kappa})$ where $H_6 := \text{Sub}(H, 6)$ is the 6-subdivision of $H$ and the probabilities $\pi_{\kappa}$ are defined as follows. For every directed edge $(x, y)$ of $\overrightarrow{H}$, the subdivision $H_6$ contains an (undirected) path between $x$ and $y$, and we define $\pi_{\kappa}$ on this path as follows:

$$x \xrightarrow{1/2} v_1 \xrightarrow{\rho_{\kappa,00}} v_2 \xrightarrow{\rho_{\kappa,01}} v_3 \xrightarrow{\rho_{\kappa,10}} v_4 \xrightarrow{\rho_{\kappa,11}} v_5 \xrightarrow{1/2} y$$
We now introduce some notation for the probability of matchings in paths of length 4. We write $\Pi_4(\rho_\kappa)$ the probability of having a matching in the 4-edge path with successive probabilities $\rho_{\kappa,00}, \rho_{\kappa,01}, \rho_{\kappa,10}, \rho_{\kappa,11}$. The value can be explicitly computed as a polynomial in the values $\rho_{b,b'}$, e.g., using Equation (1). Accordingly, we will also use $\Pi_4$ as a polynomial with real variables, i.e., $\Pi_4(\chi)$ for a 4-tuple $\chi$ of real values (which may not be in $[0,1]$). We also define variants of these definitions that account for the two surrounding edges, i.e., those with probability 1/2: for $b, b' \in \{0,1\}$, write $\Pi_4^{b b'}(\rho_\kappa)$ to denote the probability of having a matching in the same 4-edge path but when adding an edge incident to the first vertex with probability 1 if $b = 1$, and adding an edge incident to the last vertex with probability 1 if $b' = 1$. Equivalently, $\Pi_4^{b b'}(\rho_\kappa)$ is the probability of obtaining a matching where we further require if $b = 1$ that the edge with probability $\rho_{\kappa,00}$ is not taken, and if $b' = 1$ that the edge with probability $\rho_{\kappa,11}$ is not taken. The values $\Pi_4^{b b'}(\rho_\kappa)$ are also explicitly computable as polynomials, and again we also see $\Pi_4^{b b'}$ as a polynomial with real variables. To simplify notation, for $b, b' \in \{0,1\}$ and $\kappa \in [m+1]^4$ let us write $\Lambda_{\kappa,bb'} := \Pi_4^{b b'}(\rho_\kappa)$.

We then show that the probability of a matching in the subdivided graph $H_6$ can be obtained by first summing over the possible edge type cardinalities $\tau$, and then regrouping the edges of the same type by noticing that the matchings corresponding to the selection functions in the set $S_\tau$ all have the same probability. Namely, we show (cf. Appendix C):

**Fact 3.4.** For each $\kappa \in [m+1]^4$, we have:

$$2^{2m} \cdot \Pr_{\text{matching}}(H_6(\kappa)) = \sum_{\tau \in [m+1]^4} |S_\tau| \times (\Lambda_{\kappa,00})^{\tau_{00}} \times (\Lambda_{\kappa,01})^{\tau_{01}} \times (\Lambda_{\kappa,10})^{\tau_{10}} \times (\Lambda_{\kappa,11})^{\tau_{11}}.$$

Now, let us write $c_\kappa := \Pr_{\text{matching}}(H_6(\kappa))$ the value returned by the oracle call on $H_6(\kappa)$, and let $C$ be the vector of these oracle answers. Let $S$ be the vector $|S_\tau|$ of the values that we wish to compute. Both these vectors are indexed by $[m+1]^4$. Observe that the equation above defines a system of linear equations $VS = C$ with $V \in \mathbb{R}^{[m+1]^4 \times [m+1]^4}$ defined by

$$v_{\kappa,\tau} := 2^{-2m} \times (\Lambda_{\kappa,00})^{\tau_{00}} \times (\Lambda_{\kappa,01})^{\tau_{01}} \times (\Lambda_{\kappa,10})^{\tau_{10}} \times (\Lambda_{\kappa,11})^{\tau_{11}}.$$

Therefore, if we can choose 4-tuples of probability values $\rho_\kappa$ that make $V$ invertible, we would be able to recover all $|S_\tau|$ values from the oracle answers $C$, from which we could compute the number of matchings of $H$ using Fact 3.3. This is what we do next.

**Step 3: Making $V$ invertible.** We now explain how to choose in polynomial time $(m+1)^4$ 4-tuples $\rho_\kappa$ of rational probability values, for $\kappa \in [m+1]^4$, such that $V$ is invertible. To this end, consider the matrix $U$ defined like $V$ except that each 4-tuple $\rho_\kappa$ is replaced by a 4-tuple of variables $\chi_\kappa = (\chi_{\kappa,00}, \chi_{\kappa,01}, \chi_{\kappa,10}, \chi_{\kappa,11})$. Each cell $m_{\kappa,\tau}$ of $U$ is then a polynomial $P_\tau$ in the 4 variables $\chi_{\kappa,bb'}$ for $b, b' \in \{0,1\}$; in particular, note that the polynomial only depends on the column $\tau$, whereas the variables $\chi_{\kappa,bb'}$ only depend on the row $\kappa$. We can then find suitable values $\rho_\kappa$ using a technique introduced by Dalvi and Suciu [13] (see Appendix D):

**Proposition 3.5 (From Proposition 8.44 of [13]).** Fix $k \in \mathbb{N}$, let $(x_i)_{i \in I}$ be $k$-tuples of real variables indexed by a finite set $I$, let $(P_j)_{j \in J}$ be polynomials in $k$ variables indexed by a finite set $J$, and consider the matrix $M$ indexed by $I \times J$ such that $m_{i,j} = P_j(x_i)$ for all $(i,j) \in I \times J$. Assume that $\det(M)$ is not the null polynomial. There is an algorithm that runs in polynomial time in $M$ and finds $|I|$ $k$-tuples of decimal fractions $(a_{i,j})_{i,j \in I}$ with values in $[0,1]$ such that the matrix obtained by substituting each $x_i$ by $a_i$ in $M$ is invertible.

If $\det(U)$ is not the null polynomial, we can invoke this result with $k = 4$ and $I = J = [m+1]^4$ on the matrix $U$, which gives us in polynomial time the desired rational probability values $\rho_\kappa$ (namely, the $a_i$ from the proposition) and concludes the proof of Proposition 3.1.
Hence, the only remaining point is to argue that \( \det(U) \) is not the null polynomial (in the \( \chi_k \)). To this end, let us study the mapping \( \xi : \mathbb{R}^4 \to \mathbb{R}^4 \), defined as follows, with \( \chi \) denoting a 4-tuple of real variables: \( \xi(\chi) := (\Pi_{0}^{0}(\chi), \Pi_{1}^{0}(\chi), \Pi_{0}^{1}(\chi), \Pi_{1}^{1}(\chi)) \). For a 4-tuple of reals \( \rho \), we call the mapping \( \xi \) invertible around point \( \rho \) if there is \( \epsilon > 0 \) such that the \( \epsilon \)-neighborhood around \( \xi(\rho) \), i.e., the set \( \{ \alpha \in \mathbb{R}^4 \mid |\alpha_{bb'} - \xi(\rho)_{bb'}| \leq \epsilon \text{ for each } b, b' \in \{0,1\} \} \), is included in the image of \( \xi \). We conclude by showing two claims:

> **Fact 3.6.** The mapping \( \xi \) is invertible around some point.

**Proof.** By the inverse function theorem [25], if the Jacobian determinant of \( \xi \) at a point is not null, then \( \xi \) is invertible around that point. Recall that the Jacobian determinant of \( \xi \) is the determinant of the Jacobian matrix of \( \xi \), which is the \( 4 \times 4 \) matrix \( J_{\xi} \) whose entry at cell \(( (b_1, b_2), (b'_1, b'_2)) \) is \( \frac{\partial \xi_{bb'}}{\partial \chi_{b_1b'_1} \chi_{b_2b'_2}} \). We explicitly compute \( \det(J) \) with the help of SageMath, showing that it is not the null polynomial (see Appendix E).

> **Fact 3.7.** If \( \xi \) is invertible around some point \( \rho \), then \( \det(U) \) is not the null polynomial.

**Proof.** The invertibility of \( \xi \) around \( \rho \) implies that there exist, for each \( b, b' \in \{0,1\} \), a set of \( m + 1 \) distinct values \( \Psi_{bb'} := \{ \psi_{bb',0}, \ldots, \psi_{bb',m-1} \} \) such that the Cartesian product \( \Psi := \times_{b,b' \in \{0,1\}} \Psi_{bb'} \) is included in the \( \epsilon \)-neighborhood of \( \xi(\rho) \). Let us index the \((m+1)^4\) 4-tuples of \( \Psi \) as \( \psi_{\kappa} \) for \( \kappa \in [m+1]^4 \), i.e., \( \psi_{\kappa} = (\psi_{00,\kappa_{00}}, \psi_{01,\kappa_{01}}, \psi_{10,\kappa_{10}}, \psi_{11,\kappa_{11}}) \). Using invertibility, let \( \alpha_{\kappa} \) be a preimage of each \( \psi_{\kappa} \), i.e., \( \xi(\alpha_{\kappa}) = \psi_{\kappa} \) for all \( \kappa \in [m+1]^4 \). But then observe that, for this choice of \( \chi_k \) (i.e., substituting the \( \chi_k \) by the \( \alpha_k \)), each cell \( u_{\kappa,\tau} \) of the matrix \( U \) becomes:

\[
 u_{\kappa,\tau} = 2^{-2m} \times (\psi_{00,\kappa_{00}})^{\tau_{00}} \times (\psi_{01,\kappa_{01}})^{\tau_{01}} \times (\psi_{10,\kappa_{10}})^{\tau_{10}} \times (\psi_{11,\kappa_{11}})^{\tau_{11}}.
\]

Thus, \( U \) is the Kronecker product of four Vandermonde matrices \( U_{bb'} \) for \( b, b' \in \{0,1\} \), where \( U_{bb'} \) is \( \mathcal{V}(\psi_{bb',0}, \ldots, \psi_{bb',m-1}) \). As the \( \Psi_{bb'} \) consist of pairwise distinct values, these Vandermonde matrices are invertible, and their Kronecker product \( U \) also is.

## 4 Proof When All Subdivisions Have the Same Length \( \geq 7 \)

We now prove a variant of Proposition 3.1 where all edges of the initial graph are subdivided the same number of times (at least 7). Given a graph \( H \) and integer \( K > 0 \), we write \( G_K \) to mean \( \text{Sub}(H, K) \). In this section we show:

> **Proposition 4.1.** Fix an integer \( K \geq 7 \). Then, for any graph family \( \mathcal{F} \), the problem \#\text{Matching}(\mathcal{F}) reduces in polynomial time to \#\text{PrMatching}(\mathcal{G}) \), where \( \mathcal{G} = \{ H_K \mid H \in \mathcal{F} \} \).

To prove this, we follow the same strategy as for Proposition 3.1. The first step — the definition of the \( S_{\tau} \) — is strictly identical; for \( m \) the number of edges of \( H \), we fix again an orientation \( \overrightarrow{H} \) of \( H \), and denote \( S_{\tau} \) for \( \tau \in [m+1]^4 \) the \((m+1)^4\) sets of selection functions defined from \( \overrightarrow{H} \) as in Definition 3.2. In particular, Fact 3.3 still holds. Now, we will again construct \((m+1)^4\) probabilistic graphs, denoted \( H_K(\kappa) \) for \( \kappa \in [m+1]^4 \), such that, letting \( c_k := \text{Pr}_{\text{matching}}(H_K(\kappa)) \), the \([S_{\tau}] \) and the \( c_k \) form a linear system of equations \( \mathbf{V} \mathbf{S} = \mathbf{C} \). We will then again use the Jacobian technique to argue that the determinant of this matrix is not the null polynomial, and complete the proof using Proposition 3.5 to compute in polynomial time rational values that make \( \mathbf{V} \) have rational entries and be invertible. The difference with Section 3 is in the construction of the probabilistic graphs \( H_K(\kappa) \), and in the Jacobian determinant. Before we start, we need to extend the notation from Section 3.
Probabilistic path graphs. For \( n \in \mathbb{N}^+ \) we denote by \( P_n \) the path of length \( n \), i.e., \( P_n = (\{v_0, \ldots, v_n\}, E) \) where \( E = \{(v_i, v_{i+1}) \mid 0 \leq i < n-1\} \). For \( \rho \in [0,1]^n \), we let \( \Pi_n(\rho) \) be the probabilistic graph where each edge \( \{v_i, v_{i+1}\} \) of \( P_n \) has probability \( \rho_i \). We write \( \Pi_n(\rho) \) the probability of a matching in \( P_n(\rho) \). For \( b, b' \in \{0, 1\} \), we write \( \Pi_n^{bb'}(\rho) \) to denote \( \Pi_{n+2}(b, \rho, b') \), i.e., the probability of a matching in \( P_n(\rho) \) where we add an edge to the left if \( b = 1 \) and add an edge to the right if \( b' = 1 \). In particular \( \Pi_n^{00}(\rho) = \Pi_n(\rho) \). We call the quadruple of values \( \Pi_n^{bb'}(\rho) \) for \( b, b' \in \{0, 1\} \) the behavior of the path \( P_n(\rho) \). Each \( \Pi_n^{bb'}(\rho) \) is a polynomial in the probabilities \( \rho \), and thus we also see \( \Pi_n^{bb'}(\rho) \) as a polynomial with real variables as in Section 3. We will use the following two lemmas. The first one expresses the behavior of the concatenation of two paths as a function of the behavior of each path (cf. Appendix F):

- Lemma 4.2. Let \( n,n' \in \mathbb{N}^+ \) and \( \rho \in [0,1]^n \), \( \rho' \in [0,1]^{n'} \) be tuples of probability values. Then, for every \( b, b' \in \{0, 1\} \), we have:

\[
\Pi_n^{bb'}(\rho, \rho') = (\Pi_n^{00}(\rho) \times \Pi_n^{b'}(\rho')) + (\Pi_n^{11}(\rho) \times \Pi_n^{0b'}(\rho')) - (\Pi_n^{10}(\rho) \times \Pi_n^{1b'}(\rho')).
\]

The second lemma expresses the values \( \Pi_n^{bb'}(1/2, \ldots, 1/2) \) in terms of the Fibonacci sequence. Recall that this is the integer sequence defined by \( f_0 := 0 \), \( f_1 := 1 \), and \( f_n := f_{n-1} + f_{n-2} \) for all \( n \in \mathbb{N}^+ \), and that this sequence satisfies Cassini’s identity [24], which says that \( f^2 = f_{n+1}f_{n-1} + (-1)^{n+1} \) for every \( n \in \mathbb{N}^+ \). We have (cf. Appendix G):

- Lemma 4.3. For all \( n \in \mathbb{N}^+ \), \( b, b' \in \{0, 1\} \), we have \( \Pi_n^{bb'}(1/2, \ldots, 1/2) = \frac{f_{n+2} - (-1)^n}{2^n} \).

Proving Proposition 4.1. Let us now build the graphs \( H_K(\kappa) \). As before, consider \((m+1)^4 \) 4-tuples of probability values \( \rho_\kappa = (\rho_{\kappa,00}, \rho_{\kappa,01}, \rho_{\kappa,10}, \rho_{\kappa,11}) \) for \( \kappa \in [m+1]^4 \), to be chosen later. Each graph \( H_K(\kappa) \) has \( H_K \) as its underlying graph, and for every directed edge \((x, y) \in E\), we set the probabilities on the corresponding undirected path in \( H_K \) as follows:

\[
x \xrightarrow{1/2} v_1 \xrightarrow{\rho_{\kappa,00}} v_2 \xrightarrow{\rho_{\kappa,01}} v_3 \xrightarrow{\rho_{\kappa,10}} v_4 \xrightarrow{\rho_{\kappa,11}} v_5 \xrightarrow{1/2} v_6 \xrightarrow{1/2} \cdots \xrightarrow{1/2} v_{K-1} \xrightarrow{1/2} y
\]

Note that this is like in Section 3, but giving probability \( 1/2 \) to the \( N := K - 6 \) extra edges on the path. For \( b, b' \in \{0, 1\} \) we write again \( \Lambda_{\kappa,bb'} := \Pi_n^{bb'}(\rho_\kappa) \) the behavior of the 4-path with probabilities \( \rho_\kappa \), and we define the behavior \( \Upsilon_{\kappa,bb'} := \Pi_K^{bb'}(\rho_\kappa, 1/2, \ldots, 1/2) \) of the path depicted above without the first and last edges. Note that with Lemma 4.2 and Lemma 4.3, we can then express the \( \Upsilon_{\kappa,bb'} \) as a function of the \( \Lambda_{\kappa,bb'} \) and of the Fibonacci numbers:

- Fact 4.4. We have \( \Upsilon_{\kappa,bb'} = 2^{-N} \times (\Lambda_{\kappa,b0} \times f_{N+1-b'} + \Lambda_{\kappa,b1} \times f_{N-b'}) \) for \( b, b' \in \{0, 1\} \).

Studying the graphs \( H_K(\kappa) \), by the same reasoning as for Fact 3.4, we can easily show:

\[
2^{2m} \times \Pr_{\text{matching}}(H_K(\kappa)) = \sum_{\tau \in [m+1]^4} |S_\tau| \times (\Upsilon_{\kappa,00})^{\tau_{00}} \times (\Upsilon_{\kappa,01})^{\tau_{01}} \times (\Upsilon_{\kappa,10})^{\tau_{10}} \times (\Upsilon_{\kappa,11})^{\tau_{11}}.
\]

This is again a system of linear equations \( VS = C \) with \( V \in \mathbb{R}^{[m+1]^4 \times [m+1]^4} \), where \( v_{\kappa,\tau} := 2^{-6m} \times (\Upsilon_{\kappa,00})^{\tau_{00}} \times (\Upsilon_{\kappa,01})^{\tau_{01}} \times (\Upsilon_{\kappa,10})^{\tau_{10}} \times (\Upsilon_{\kappa,11})^{\tau_{11}} \). To show that we can compute in polynomial time 4-tuples of rational probability values \( \rho_\kappa \) for \( \kappa \in [m+1]^4 \) that make \( V \) have rational entries and be invertible, we reason as in Section 3. Specifically, we study the Jacobian determinant of the mapping \( \xi_N : \chi \mapsto (\Pi_{K-2}^{00}(\chi, 1/2, \ldots, 1/2), \Pi_{K-2}^{11}(\chi, 1/2, \ldots, 1/2), \Pi_{K-2}^{10}(\chi, 1/2, \ldots, 1/2), \Pi_{K-2}^{01}(\chi, 1/2, \ldots, 1/2)) \), where \( \chi \) is a 4-tuple of real variables. We show that this determinant is not the null polynomial. To do this, starting from the Jacobian \( J_\xi \) of Section 3, using Fact 4.4 and Cassini’s identity, and using the fact that the determinant is multilinear and alternating, we obtain (cf. Appendix H):
Fact 4.5. We have: \( \det(J_{S_N}) = 2^{-4N} \times \det(J_{S}) \).

Hence, \( \det(J_{S_N}) \) is not the null polynomial and, as in Section 3, we can use Proposition 3.5 to complete the proof of Proposition 4.1 (cf. Appendix D).

5 Proof for Arbitrary Subdivisions

In this section we finally prove our main result (Result 1), which we re-state here:

Theorem 5.1. Let \( G \) be an arbitrary family of graphs which is treewidth-constructible. Then \( \Pr_{\text{Matching}}(G) \) is \#P-hard under ZPP reductions.

We will reduce from the problem of counting matchings in 3-regular planar graphs of which is \#P-hard by \[28\]. Our reduction will be similar to that of Section 4, with the major issue that the various edges of the input graph can now be subdivided to different lengths.

The proof consists of five steps. In step 1, we show a general result allowing us to assign probabilities to a path of length 4 so as to “emulate” the behavior of any long path of even length. We then revisit the proof of the previous section. Step 2 extracts the input graph \( H \) from the treewidth-constructible family. Step 3 relates the number of matchings of \( H \) to cardinalities similar to those of the previous section, but taking the parities of the subdivisions into account. Step 4 then explains how to conclude using emulation. Last, step 5 works around the issue that the probabilities of Step 1 could be irrational, by explaining how we can conclude with sufficiently precise approximations. We now detail these steps.

Step 1: Emulating long even paths. We start by presenting the main technical tool, namely, how to emulate long paths of even length by paths of length 4.

Proposition 5.2 (Emulation result). There exist closed-form expressions, denoted \( p(i), q(i), r(i), s(i) \), such that for every even integer \( i \geq 4 \) the following hold:

(A) the expressions evaluate to well-defined probability values, i.e., we have \( 0 \leq p(i), q(i), r(i), s(i) \leq 1 \); and

(B) the path of length \( i \) with probabilities \( p(i), q(i), r(i), s(i) \) behaves like a path of length \( i \) with probabilities \( 1/2 \), i.e., \( \Pi_i^{b'}(p(i), q(i), r(i), s(i)) = \Pi_i^{b'}(1/2, \ldots, 1/2) \) for all \( b, b' \in \{0, 1\} \).

Further, each of these expressions is of the form \( P \sqrt{Q} \) where \( P, Q, R \) are polynomials in the Fibonacci numbers \( f_{i-1} \) and \( f_{i-2} \) and in \( 2^{-i} \), with rational coefficients.

Proof sketch. The result is simple to state, but we did not find an elegant way to show it. Our proof consists of four steps: (i) rewriting condition (B) into a simpler equivalent system of equations (using Lemma 4.3), (ii) proving that any solution of that system must be in \( (0, 1) \), (iii) exhibiting closed-form expressions that satisfy the system, found with the help of SageMath; and (iv) verifying that these expressions are well-defined. See Appendix J.

Remark 5.3. As \( \Pi_i^{b'}(1/2, \ldots, 1/2) \) is symmetric, one would expect the closed-form expressions to satisfy \( p(i) = s(i) \) and \( q(i) = r(i) \). However, surprisingly, numerical evaluation (already for \( i = 6 \)) shows that our solution does not have this property.

Note that, in holographic literature, graphs may be multigraphs (i.e., can have multiple edges between two nodes) — see [17]. However, inspecting the proof of [28], we see that the graphs are in fact simple.
Remark 5.4. It is necessary to require that $i$ is even, as otherwise Proposition 5.2 demonstrably does not hold. In fact, we can prove that, more generally, the behavior of a probabilistic path inherently depends on the parity of its length (cf. Appendix I). This is why we will distinguish even-length and odd-length subdivisions in the sequel.

Step 2: Choosing the graph in $G$. Let $H = (V, E)$ be the input to the reduction, i.e., the 3-regular planar graph for which we want to compute $\#\text{Matching}(H)$, and let $m := |E|$. We first build the graph $H_{10} = \text{Sub}(H, 10)$, writing $H_{10} = (V_{10}, E_{10})$ and we compute $k := |V_{10}|^c$ where $c$ is the constant from Theorem 2.1. Notice that $H_{10}$ is a planar graph of maximum degree 3, and that the size of $k$ in unary is polynomial in (the encoding size of) $H$. Intuitively, this initial subdivision in 10 will ensure that we have enough room for our probabilistic gadgets. Now, we use the treewidth-constructibility of $G$ to build in polynomial time a graph $G = (V_G, E_G) \in G$ such that tw($G$) $\geq k$, and using Theorem 2.1 we compute in ZPP a subgraph $G'$ of $G$ with a subdivision $\eta_{10} : E_{10} \to \mathbb{N}^+$ of $H_{10}$ and an isomorphism from $\text{Sub}(H_{10}, \eta_{10})$ to $G'$. This gives us a subdivision $\eta : E \to \mathbb{N}^+$ of $H$ and an isomorphism $f$ from $\text{Sub}(H, \eta)$ to $G'$, with the initial subdivision ensuring that $\eta(e) \geq 10$ for each $e \in E$.

Step 3: Defining the new sets $S_{\tau, \tau'}$ and linking them to matchings. As before, fix an orientation $\overline{H}$ of $H$. We call an edge $e$ of $H$ even if $\eta(e)$ is even, and odd otherwise. For $\tau, \tau' \in [m + 1]^4$, both indexed in binary, we define $S_{\tau, \tau'}$ to be the set of selection functions $\mu$ of $H$ such that, for $b, b' \in \{0, 1\}$, precisely $\tau_{bb'}$ even edges $e$ of $H$ have type $bb'$ w.r.t. $\mu$, and precisely $\tau_{bb'}$ odd edges $e$ of $H$ have type $bb'$ w.r.t. $\mu$. Then, as in Section 3, we have:

$$\#\text{Matching}(H) = \sum_{\tau, \tau' \in [m + 1]^4} |S_{\tau, \tau'}|.$$  (3)

Step 4: Describing the probabilistic graphs and obtaining the system. To complete the definition of the reduction, let us build the $(m + 1)^8$ probabilistic graphs on which we want to invoke the oracle, denoted $G(\kappa, \kappa')$ for $\kappa, \kappa' \in [m + 1]^4$. Let $K := \max_{\eta(e) \text{ is even}} \sum_{e \in E} \eta(e)$ and $K' := \max_{\eta(e) \text{ is odd}} \sum_{e \in E} \eta(e)$ and $N := K - 6$ and $N' := K' - 6$. The underlying graph of $G(\kappa, \kappa')$ is $G$, every edge $e \in E_G$ that is not in $G'$ is assigned probability zero, and we explain next what is the probability associated to the edges that are in $G'$. Consider $2 \times (m + 1)^4$ 4-tuples of probability values $\rho_\kappa = (\rho_{\kappa,00,0}, \rho_{\kappa,01,0}, \rho_{\kappa,10,0}, \rho_{\kappa,11,0})$ and $\rho_\kappa' = (\rho_{\kappa',00,0}, \rho_{\kappa',01,0}, \rho_{\kappa',10,0}, \rho_{\kappa',11,0})$ for $\kappa, \kappa' \in [m + 1]^4$, to be chosen later. For every directed edge $(x, y) \in \overline{H}$, let $\gamma := \eta((x, y))$ be the length to which it is subdivided in $G'$. Letting $f(x), v_1, \ldots, v_{\gamma-1}, f(y)$ be the corresponding path in $G'$, we set the probabilities of the $\gamma$ edges along that path as follows:

- If $\gamma$ is even (illustrated in Figure 1):
  - $1/2, \rho_{\kappa,00,0}, \rho_{\kappa,01,0}, \rho_{\kappa,10,0}, \rho_{\kappa,11,0}$ for the first 5 edges,
  - $p(N - \gamma + 10), q(N - \gamma + 10), r(N - \gamma + 10), s(N - \gamma + 10)$ for the next four edges,
  - $1/2$ for the remaining $\gamma - 9$ edges.

- If $\gamma$ is odd:
  - $1/2, \rho_{\kappa',00,0}, \rho_{\kappa',01,0}, \rho_{\kappa',10,0}, \rho_{\kappa',11,0}$ for the first 5 edges,
  - $p(N' - \gamma + 10), q(N' - \gamma + 10), r(N' - \gamma + 10), s(N' - \gamma + 10)$ for the next four edges,
  - $1/2$ for the remaining $\gamma - 9$ edges.

We know that $N - \gamma + 10$ (resp., $N' - \gamma + 10$) is an even integer when $\gamma$ is even (resp., when $\gamma$ is odd); and it is $\geq 4$ by definition of $K$ (resp., of $K'$). Thus, using Proposition 5.2 and then Lemma 4.2, we know that the path that we defined behaves exactly like the
path $P_K(1/2, \rho_{0,0}, \rho_{0,1}, \rho_{1,0}, \rho_{1,1}, 1/2, \ldots, 1/2)$ if $\gamma$ is even, and exactly like the path $P_{K'}(1/2, \rho'_{0,0}, \rho'_{0,1}, \rho'_{1,0}, \rho'_{1,1}, 1/2, \ldots, 1/2)$ if $\gamma$ is odd (see again Figure 1).

We have now managed to ensure that all paths for even edges (resp., for odd edges) behave as if they had been subdivided to length $K$ (resp., to length $K'$). We continue the proof as in the previous section, except that we distinguish odd and even edges. Specifically, for $b, b' \in \{0,1\}$, we write as in the previous section $\Upsilon_{K,bb'} := \Pi_{K-2}^b(\rho_{0,1}, \ldots, 1/2)$ and $\Upsilon_{K',bb'} := \Pi_{K'-2}^{b'}(\rho'_{0,1}, \ldots, 1/2)$. Using the same reasoning as for Equation 2, we obtain:

$$2^{2m} \times \operatorname{Pr}_{\text{matching}}(G(\kappa, \kappa')) = \sum_{\tau, \tau' \in [m+1]^4} |\mathcal{S}_\tau, \tau'| \times (\Upsilon_{\kappa,00})^{\tau_00} \times (\Upsilon_{\kappa,01})^{\tau_01} \times (\Upsilon_{\kappa,10})^{\tau_{10}} \times (\Upsilon_{\kappa,11})^{\tau_{11}} \times (\Upsilon'_{\kappa',00})^{\tau_{00}'} \times (\Upsilon'_{\kappa',01})^{\tau_{01}'} \times (\Upsilon'_{\kappa',10})^{\tau_{10}'} \times (\Upsilon'_{\kappa',11})^{\tau_{11}'} \times (\Upsilon_{\kappa,\tau}) \times (\Upsilon_{\kappa',\tau'}), \tag{4}$$

i.e., we obtain a system of linear equations $\Gamma \mathbf{S} = \mathbf{C}$ with $\mathbf{S}$ the vector of the desired values $|\mathcal{S}_\tau, \tau'|$, with $\mathbf{C}$ the vector of the oracle answers $\operatorname{Pr}_{\text{matching}}(G(\kappa, \kappa'))$, and with $\Gamma \in \mathbb{R}^{[m+1]^4 \times [m+1]^4}$, whose entries are given according to the above equation. But notice that we have $\Gamma = \mathbf{V} \otimes \mathbf{V}'$, with $v_{\kappa,\tau} := 2^{-m} \times (\Upsilon_{\kappa,00})^{\tau_00} \times (\Upsilon_{\kappa,01})^{\tau_01} \times (\Upsilon_{\kappa,10})^{\tau_{10}} \times (\Upsilon_{\kappa,11})^{\tau_{11}}$ and $v'_{\kappa',\tau'} := 2^{-m} \times (\Upsilon'_{\kappa',00})^{\tau_{00}'} \times (\Upsilon'_{\kappa',01})^{\tau_{01}'} \times (\Upsilon'_{\kappa',10})^{\tau_{10}'} \times (\Upsilon'_{\kappa',11})^{\tau_{11}'}$. Since $\mathbf{V}$ and $\mathbf{V}'$ share no variables and are identical up to renaming variables, to argue that there exist 4-tuples of probabilistic values $\rho_{\kappa}$ and $\rho'_{\kappa'}$ for $\kappa, \kappa' \in [m+1]^4$ that make $\Gamma$ invertible, it is enough to know that the Jacobian determinant of the mapping $\xi_N$ is not identically null, as we showed in the previous section (Fact 4.5). Thus, we can again use Proposition 3.5 to compute in polynomial time $2 \times (m+1)^4$ 4-tuples of rational probability values $\rho_{\kappa}$ and $\rho'_{\kappa'}$ such that the matrices $\mathbf{V}$ and $\mathbf{V}'$, hence $\Gamma$, are invertible (cf. Appendix D). By Equation 4, $\Gamma$ has rational entries, and its inverse $\Gamma^{-1}$ also does and is computable in polynomial time.

**Step 5: Using decimal fractions approximations.** The last issue is that we cannot really obtain $\mathbf{C}$ via oracle calls, because the graphs $G(\kappa, \kappa')$ may have irrational edge probabilities, namely, the $p(i), q(i), r(i), s(i)$. We now argue that we can still recover the $\mathbf{C}$, so that we can compute $\mathbf{S} = \Gamma^{-1} \mathbf{C}$ and conclude. To do this, we first observe that $\mathbf{C}$ is in fact a vector of decimal fractions, as the graphs $G(\kappa, \kappa')$ emulate a graph where the probabilities are decimal fractions; further, we can bound the number of decimal places of its values to $[m \times (\max(N, N') + 1)] \times z$, with $z$ the maximal number of decimal places of a decimal fraction in $\rho_{\kappa}, \rho'_{\kappa'}$. Second, we show how to compute decimal fraction approximations $p(i), q(i), r(i), s(i)$ of the $p(i), q(i), r(i), s(i)$, in polynomial time in the desired number of places, using the form that they have according to Proposition 5.2. Third, we argue that when invoking the oracles on the graphs where we replace $p(i), q(i), r(i), s(i)$ by $p(i), q(i), r(i), s(i)$, then the error on
the answer is bounded as a function of that of the approximations, so that we can recover \( C \)
exactly if the approximations were sufficiently precise. See Appendix K for detailed proofs.

6 Result for Edge Covers

Having shown Result 1, we now explain how to adapt its proof to obtain our analogous
results for edge covers. We only sketch the argument, and refer to Appendix L for more
details. Recall that an edge cover of a graph \( G = (V, E) \) is a set of edges \( S \subseteq E \) such
that \( V = \bigcup_{e \in S} e \). Given a probabilistic graph \( (G, \pi) \), we define \( \text{Pr}_{\text{EdgeCover}}(G, \pi) \) to be the
sum of the probabilities of all edge covers in the probability distribution induced by \( \pi \), and
define \( \text{PrEdgeCover}(F) \) for a graph family \( F \) to be the corresponding computational problem.

We first note that, in this context, the strict analogue of Result 1 does not hold. Indeed, take some treewidth-constructible graph family \( G \), and consider the graph family \( G' \) obtained
from \( G \) as follows: for every graph \( G \in G \), we add to \( G' \) the graph that is obtained from \( G \)
by attaching a dangling edge with a fresh vertex to every node of \( G \). The family \( G' \) is still
treewidth-constructible, but \( \text{PrEdgeCover}(G') \) is now tractable as it is easy to see that the
edge covers of a graph in \( G' \) are precisely the edge subsets where all dangling edges are kept.

To avoid this, let us assume that \( G \) is closed under taking subgraphs, i.e., if \( G \in G \) and \( G' \)
is a subgraph of \( G \), then \( G' \in G \). We then have:

- **Theorem 6.1.** Let \( G \) be an arbitrary family of graphs which is treewidth-constructible and
closed under taking subgraphs. Then \( \text{PrEdgeCover}(G) \) is \#P-hard under ZPP reductions.

This is proved like Result 1, with the following modifications. We reduce from counting
edge covers (instead of matchings) on 3-regular planar graphs: this is hard by [9], even on
simple graphs [4, Appendix D]. We now define a selection function \( \mu \) to map each vertex \( x \in V \)
to at least one incident edge, and we define the types and the sets \( S_{\tau, \tau'} \) as before, via an
arbitrary orientation of the graph \( H \). We obtain the number of edge covers of \( H \) from the
quantities \( |S_{\tau, \tau'}| \) exactly as in Equation 3. We redefine \( \Pi_{\text{bb}}^b(\rho) \) to be the probability of
an edge cover in a path of length \( n \) with probabilities \( \rho \) on the edges and with endpoint
constraints given by \( b, b' \) as before. Lemma 4.3 then becomes \( \Pi_{\text{bb}}^b(1/2, \ldots, 1/2) = \frac{f_{b+b'} n}{2} \),
i.e., the role of \( b, b' \) is “reversed”. Analogous versions of Lemma 4.2 and of Proposition 5.2
still hold, so the relevant Jacobian determinants are still non-identically null. We take
the graph \( G \in G \) again via the topological minor extraction result, but this time directly
extracting \( \text{Sub}(H, \eta) \in G \) as \( G \) is subgraph-closed. The rest of the proof is identical.

We point out that the situation is different for perfect matchings. Indeed, using a weighted
variant of the FKT algorithm [8, Chapter 4], the weighted counting of perfect matchings is
polynomial-time over the class of planar graphs, which is treewidth-constructible.

We conclude by leaving open two directions for future work. The first one would be to
obtain the same kind of lower bounds when the probabilities annotate the nodes instead of the
edges, that is, studying the corresponding weighted counting problems for, e.g., independent
sets, vertex covers, or cliques. We believe that the corresponding result should hold and do
not expect any surprises. The second question would be to show our hardness results in the
unweighted case, e.g., unweighted counting of matchings, assuming that the graph family is
subgraph-closed. This appears to be much more challenging, as our current proof crucially
relies on the ability to use arbitrary probability values.
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We formally show the result about hardness under ZPP reduction claimed in the preliminaries:

\textbf{Claim A.1.} If $P_1$ is a problem which is #P-hard, and $P_1$ has a ZPP-reduction to a problem $P_2$, then $P_2$ is #P-hard under ZPP reductions.

\textbf{Proof.} Let $P_0$ be a #P-hard problem, and let us show that $P_0$ has a ZPP-reduction to $P_2$. We will do so by composing the two reductions. The only subtle point is that the reductions are Turing reductions, so we must control the overall probability of failure knowing that each reduction to the $P_2$-oracle may fail. Specifically, given an instance $I$ to $P_0$, perform the polynomial-time reduction using an oracle to $P_1$, and evaluate the calls to $P_1$ by performing the ZPP-reduction to $P_2$ and using the oracle to $P_2$. Let $n$ be the total number of calls of the ZPP-reduction to $P_2$: it is a polynomial in $|I|$. Let $c$ be the failure probability of the ZPP-reduction, and let $k$ be sufficiently large so that $nc^k \leq c$; this is polynomial in $n$. When performing the ZPP-reduction to $P_2$, repeat each call $k$ times. This guarantees that, by the union bound, the total probability of failure is at most $nc^k \leq c$, i.e., it is at most a constant; so indeed we have defined a ZPP reduction from $P_0$ to $P_2$.

Following this claim, to prove Result 1, we will establish #P-hardness under ZPP reductions simply by giving a ZPP reduction from a #P-hard problem.

\section*{B Proof of Fact 3.3}

We prove Fact 3.3, whose statement we recall:

\textbf{Fact 3.3.} We have that $\#\text{Matching}(H) = \sum_{\tau \in [m+1]^*} |S_\tau|$.

Indeed, let us show that there is a bijection between the matchings of $H$ and the set $\bigcup_{\tau_0 = \tau_{10} = 0}^{\tau_0 \in [m+1]^*} S_\tau$: as the $S_\tau$ are pairwise disjoint, this is enough to prove the claim. If $M$ is a matching of $H$, let $\mu_M$ be the selection function of $H$ that assigns $\emptyset$ to every node that is not in any edge of $M$, and that assigns $\{e\}$ to every node that is in $e$ for some edge $e \in M$. It is easy to see that $\mu_M$ is a well-defined selection function (because $S$ is a matching), that it is in some $S_\tau$ with $\tau_0 = \tau_{10} = 0$, and that the mapping $M \mapsto \mu_M$ is injective. Furthermore, any selection function $\mu$ that is in some $S_\tau$ with $\tau_0 = \tau_{10} = 0$ can be obtained as $\mu_M$ for some matching $M$ of $H$: take $M := \{x, y\}$, $\mu_M(y) = \mu(y) \cap \mu(x)$, i.e., $\tau_1$ is the cardinality of the matching and $\tau_0 = m - \tau_{11}$ is the cardinality of the complement of the matching. Hence we indeed have a bijection between the matchings of $H$ and $\bigcup_{\tau_0 = \tau_{10} = 0}^{\tau_0 \in [m+1]^*} S_\tau$. 


We prove Fact 3.4, whose statement we recall here:

Fact 3.4. For each $\kappa \in [m+1]^{4}$, we have:

$$2^{2m} \times \Pr_{\text{matching}}(H_{0}(\kappa)) = \sum_{\tau \in [m+1]^{4}} |S_{\tau}| \times (\Lambda_{\kappa,00})^{\tau_{00}} \times (\Lambda_{\kappa,01})^{\tau_{01}} \times (\Lambda_{\kappa,10})^{\tau_{10}} \times (\Lambda_{\kappa,11})^{\tau_{11}}.$$

Indeed, for a matching $M_{6}$ of $H_{6}$, consider the selection function $\mu_{M_{6}}$ defined as follows: for every directed edge $(x, y)$ of $\overrightarrow{H}$, letting $x - v_{1} - v_{2} - v_{3} - v_{4} - v_{5} - y$ be the corresponding path in $H_{6}$, we add $\{x, y\}$ to $\mu_{M_{6}}(x)$ if the first edge of that path is in $M_{6}$, and we add $\{x, y\}$ to $\mu_{M_{6}}(y)$ if the last edge of that path is in $M_{6}$. Observe that $\mu_{M_{6}}$ is indeed a selection function, and that it is in $S_{\tau}$ for exactly one 4-tuple $\tau \in [m+1]^{4}$. We now use the definition of $\Pr_{\text{matching}}(H_{0}(\kappa))$, i.e., Equation (1), and split the sum according to the 4-tuple $\tau$ for which the selection function $\mu_{M_{6}}$ is in $S_{\tau}$, and then split again according to the specific $\mu \in S_{\tau}$ to which the selection function is equal, as follows.

$$\Pr_{\text{matching}}(H_{0}(\kappa)) = \sum_{\text{matching } M_{6} \text{ of } H_{6}} \Pr_{H_{6}(\kappa)}(M_{6}) = \sum_{\tau \in [m+1]^{4}} \sum_{\mu \in S_{\tau}} \Pr_{H_{0}(\kappa)}(M_{6}). \quad (5)$$

But, by definition of $H_{0}(\kappa)$ and of the $\Lambda_{\kappa,bb'}$, and because the edges are independent, for a selection function $\mu \in S_{\tau}$ we have

$$\sum_{\text{matching } M_{6} \text{ of } H_{6} \text{ s.t. } \mu_{M_{6}} = \mu} \Pr_{H_{0}(\kappa)}(M_{6}) = \frac{1}{2^{2m}} \times \left( \prod_{\{x, y\} \in E} \Lambda_{\kappa,00} \right) \left( \prod_{\{x, y\} \text{ s.t. } (x, y) \notin \mu(x) \cup \mu(y)} \Lambda_{\kappa,01} \right) \times \left( \prod_{\{x, y\} \in E} \Lambda_{\kappa,10} \right) \left( \prod_{\{x, y\} \text{ s.t. } (x, y) \notin \mu(x) \cup \mu(y)} \Lambda_{\kappa,11} \right) = \frac{1}{2^{2m}} \times (\Lambda_{\kappa,00})^{\tau_{00}} \times (\Lambda_{\kappa,01})^{\tau_{01}} \times (\Lambda_{\kappa,10})^{\tau_{10}} \times (\Lambda_{\kappa,11})^{\tau_{11}}.$$
Then, for every 

We prove Lemma 4.2, whose statement we recall here.

In our context, we only use this result with $k = 4$ and $I = J = [m+1]^4$. For Section 3, the polynomials $P_r$ are those mentioned in the “Making $V$ invertible” paragraph. For Section 4 the polynomials $P_r$ are this time given according to Equation 2. For Section 5, we use it independently for the two matrices $V$ and $V'$, which are both identical (up to renaming of variables) to the matrix from Section 4.

### E Explicit Computation of the Jacobian Determinant

By explicit computation with the help of SageMath (see jacobian-pqrs.ipynb in supplementary material [5]) we obtain that $\det(J_k) = \chi_{00}\chi_{11}^2(1 - \chi_{00})(1 - \chi_{01})^2(1 - \chi_{11})^3$. This evaluates to $\frac{1}{128}$ for $\chi_{00} = \chi_{11} = 1/2$, $\chi_{01} = \chi_{10} = 0$.

### F Proof of Lemma 4.2

We prove Lemma 4.2, whose statement we recall here.

- **Lemma 4.2.** Let $n, n' \in \mathbb{N}^+$ and $\rho \in [0,1]^n$, $\rho' \in [0,1]^{n'}$ be tuples of probability values. Then, for every $b, b' \in \{0,1\}$, we have:

  
  $$\Pi^{bb'}_{n+n'}(\rho, \rho') = \Pi^{00}_{n}(\rho) \times \Pi^{10}_{n'}(\rho') + \Pi^{01}_{n}(\rho) \times \Pi^{11}_{n'}(\rho') - \Pi^{11}_{n}(\rho) \times \Pi^{10}_{n'}(\rho')$$

  
  The intuition for the result is that the edge subsets that are matchings are those where the $(n + 1)$-th edge is not kept, plus those where the $n$-th edge is not kept, minus those where the $n$-th and $(n+1)$-th edge were not kept (as these were counted twice).

  We now give the formal proof. We prove the claim for $b = b' = 0$, the other cases being similar. Let us see $P_{n+n'}$ as the concatenation of the two paths $P_n$ and $P_{n'}$, where $P_n$ and $P_{n'}$ are disjoint except for the connecting node, and define the following sets and quantities, for $b, b' \in \{0,1\}$:

  - $M^{bb'}_{n+n'}$ is the set of matchings $M$ of $P_{n+n'}$ such that the $n$-th edge is in $M$ iff $b = 1$ and the $(n+1)$-th edge is in $M$ iff $b' = 1$ (in particular $M^{11}_{n+n'}$ is empty); and $\alpha^{bb'}$ is the probability of $M^{bb'}_{n+n'}$ in $P_{n+n'}(\rho, \rho')$.

  - $M^{bb}_{n}$ is the set of matchings of $P_n$ such that, if $b = 1$ then the $n$-th edge is not in $M$; in particular, $\Pi^{00}_{n}(\rho)$ is the probability of $M^{bb}_{n}$ in $P_n(\rho)$

  - $M^{bb}_{n'}$ is the set of matchings of $P_{n'}$ such that, if $b = 1$ then the first edge is not in $M$; in particular, $\Pi^{01}_{n'}(\rho')$ is the probability of $M^{bb}_{n'}$ in $P_{n'}(\rho')$.

  Observe that the set of matchings of $P_{n+n'}$ is the disjoint union of $M^{00}_{n+n'}$, $M^{01}_{n+n'}$ and $M^{10}_{n+n'}$, so that we have

  $$\Pi^{00}_{n+n'}(\rho, \rho') = \alpha^{00} + \alpha^{01} + \alpha^{10}. \quad (6)$$

  Now, if $S$ is a set of sets of edges of $P_n$ and $S'$ a set of sets of edges of $P_{n'}$, define $S \cup S' := \{M \cup M' \mid M \in S \text{ and } M' \in S'\}$. Then, observe that

  - $M^{*0}_{n} \cup M^{1*}_{n'} = M^{00}_{n+n'} \cup M^{10}_{n+n'}$; and
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\[ M_{n+1}^* \cap M_{n+1}^0 = M_{n+1}^{00} \cup M_{n+1}^{01}; \text{ and} \]
\[ M_{n+1}^* \cap M_{n+1}^1 = M_{n+1}^{00}. \]

The above right-hand side unions being disjoint, this implies that (1):

\[ \Pr(M_{n}^* \cap M_{n}^\prime) = \alpha^{00} + \alpha^{10}; \text{ and} \]
\[ \Pr(M_{n}^* \cap M_{n}^\prime) = \alpha^{00} + \alpha^{01}; \text{ and} \]
\[ \Pr(M_{n}^* \cap M_{n}^\prime) = \alpha^{00}. \]

where the probability distribution of \( \Pr \) is that of \( P_{n+1}(\rho, \rho') \). Now, using the fact that the edges are independent, notice that \( \Pr(M_{n}^* \cap M_{n}^\prime) = \Pi_{n}^{00}(\rho) \times \Pi_{n}^{00}(\rho') \). Combining this observation together with (1) and Equation (6) concludes.

G Proof of Lemma 4.3

Here we prove Lemma 4.3, whose statement we recall.

**Lemma 4.3.** For all \( n \in \mathbb{N}^+ \), \( b, b' \in \{0, 1\} \), we have \( \Pi_{n}^{bb'}(1/2, \ldots, 1/2) = \frac{f_{n+2-\delta(n')}}{2^{n-1}} \).

We first show the claim for \( b = b' = 0 \) by induction on \( n \). Recall that we have \( \Pi_{n}^{00}(1/2, \ldots, 1/2) = \Pi_{n}(1/2, \ldots, 1/2). \) We have \( \Pi_{1}(1/2) = 1 \) and \( f_{3}/2 = 1 \), and \( \Pi_{2}(1/2) = \frac{3}{4} \) and \( f_{4}/2 = \frac{1}{2} \). For \( n > 2 \), we have:

\[ \Pi_{n}^{00}(1/2, \ldots, 1/2) = \Pi_{n}(1/2, \ldots, 1/2) = \frac{1}{2} \Pi_{n-1}(1/2, \ldots, 1/2) + \frac{1}{2} \Pi_{n-2}(1/2, \ldots, 1/2) \]
\[ = \frac{1}{2} \Pi_{n-1}^{00}(1/2, \ldots, 1/2) + \frac{1}{2} \Pi_{n-2}^{00}(1/2, \ldots, 1/2). \]

Using the induction hypothesis, we have:

\[ \Pi_{n}^{00}(1/2, \ldots, 1/2) = \frac{1}{2} \left( \frac{f_{n+1}}{2^n} + \frac{1}{2} \right) \frac{f_{n-1}}{2^{n-1}}. \]

Hence:

\[ \Pi_{n}^{00}(1/2, \ldots, 1/2) = \frac{1}{2^n} (f_{n+1} + f_{n}). \]

Now, the definition of the Fibonacci sequence concludes.

Second, we show the claim for arbitrary \( b, b' \). It is clear that \( \Pi_{n}^{01}(1/2, \ldots, 1/2) = \frac{1}{2} \Pi_{n-1}^{01}(1/2, \ldots, 1/2) \) because the last edge must be absent, and then the condition on the \( n-1 \) remaining edges is the same as the condition on \( P_{n-1}(1/2, \ldots, 1/2). \) For the same reason, \( \Pi_{n}^{10}(1/2, \ldots, 1/2) = \frac{1}{2} \Pi_{n-1}^{10}(1/2, \ldots, 1/2) \), and \( \Pi_{n}^{11}(1/2, \ldots, 1/2) = \frac{1}{2^2} \Pi_{n-2}(1/2, \ldots, 1/2). \)

Hence, we obtain the claimed equalities.

H Proof of Fact 4.5

We prove Fact 4.5, whose statement we recall:

**Fact 4.5.** We have: \( \det(J_{\xi N}) = 2^{-4N} \times \det(J_{\xi}) \).

Let us first consider the Jacobian \( J_{\xi N} \), which is as follows:
same determinants. This gives:

\[
J_{\xi N} = \begin{pmatrix}
\frac{\partial Y_{x,00}}{\partial X_{00}} & \frac{\partial Y_{x,00}}{\partial X_{01}} & \frac{\partial Y_{x,00}}{\partial X_{10}} & \frac{\partial Y_{x,00}}{\partial X_{11}} \\
\frac{\partial Y_{x,01}}{\partial X_{00}} & \frac{\partial Y_{x,01}}{\partial X_{01}} & \frac{\partial Y_{x,01}}{\partial X_{10}} & \frac{\partial Y_{x,01}}{\partial X_{11}} \\
\frac{\partial Y_{x,10}}{\partial X_{00}} & \frac{\partial Y_{x,10}}{\partial X_{01}} & \frac{\partial Y_{x,10}}{\partial X_{10}} & \frac{\partial Y_{x,10}}{\partial X_{11}} \\
\frac{\partial Y_{x,11}}{\partial X_{00}} & \frac{\partial Y_{x,11}}{\partial X_{01}} & \frac{\partial Y_{x,11}}{\partial X_{10}} & \frac{\partial Y_{x,11}}{\partial X_{11}} \\
\end{pmatrix}
\]

And recall from Section 3 that we have:

\[
J_{\xi} = \begin{pmatrix}
\frac{\partial A_{x,00}}{\partial X_{00}} & \frac{\partial A_{x,00}}{\partial X_{01}} & \frac{\partial A_{x,00}}{\partial X_{10}} & \frac{\partial A_{x,00}}{\partial X_{11}} \\
\frac{\partial A_{x,01}}{\partial X_{00}} & \frac{\partial A_{x,01}}{\partial X_{01}} & \frac{\partial A_{x,01}}{\partial X_{10}} & \frac{\partial A_{x,01}}{\partial X_{11}} \\
\frac{\partial A_{x,10}}{\partial X_{00}} & \frac{\partial A_{x,10}}{\partial X_{01}} & \frac{\partial A_{x,10}}{\partial X_{10}} & \frac{\partial A_{x,10}}{\partial X_{11}} \\
\frac{\partial A_{x,11}}{\partial X_{00}} & \frac{\partial A_{x,11}}{\partial X_{01}} & \frac{\partial A_{x,11}}{\partial X_{10}} & \frac{\partial A_{x,11}}{\partial X_{11}} \\
\end{pmatrix}
\]

Let us call \( L_{00}, L_{01}, L_{10}, L_{11} \) the lines of the above, so that we write it:

\[
J_{\xi} = \begin{pmatrix}
L_{00} \\
L_{01} \\
L_{10} \\
L_{11}
\end{pmatrix}
\]

Now, using Fact 4.4, we can express the lines of the matrix \( J_{\xi N} \) as linear combinations of the \( L_{b,b'} \), thus:

\[
J_{\xi N} = \frac{1}{24N} \begin{pmatrix}
L_{00}f_{N+1} + L_{01}f_{N} \\
L_{00}f_{N} + L_{01}f_{N-1} \\
L_{10}f_{N+1} + L_{11}f_{N} \\
L_{10}f_{N} + L_{11}f_{N-1}
\end{pmatrix}
\]

We now use two properties of the determinant:

- It is multilinear, so that the determinant of the above matrix can be expressed as the sum of the determinants of the 16 matrices obtained by choosing one term in each row
- It is alternating: the terms where the same \( L_{b,b'} \) occurs twice (even with different coefficients) have a determinant of zero

Thus, we can write \( \det(J_{\xi N}) \) as:

\[
\det(J_{\xi N}) = \frac{1}{24N} \left( f_{N+1}^2f_{N-1}^2 \det \begin{pmatrix}
L_{00} \\
L_{01} \\
L_{10} \\
L_{11}
\end{pmatrix} + f_{N+1}f_{N-1}f_{N}^2 \det \begin{pmatrix}
L_{00} \\
L_{01} \\
L_{11} \\
L_{10}
\end{pmatrix}
\right) + f_{N+1}f_{N-1}f_{N}^2 \det \begin{pmatrix}
L_{00} \\
L_{01} \\
L_{10} \\
L_{11}
\end{pmatrix} + f_{N}^4 \det \begin{pmatrix}
L_{00} \\
L_{01} \\
L_{11} \\
L_{10}
\end{pmatrix}
\]

Note that the determinant in the first term is that of \( J_{\xi} \). As for the others, we use the fact that swapping two rows of a matrix multiplies the determinant by \(-1\), to obtain the same determinants. This gives:

\[
\det(J_{\xi N}) = \frac{f_{N+1}^2f_{N-1}^2}{24N} - \frac{2f_{N+1}f_{N-1}f_{N}^2}{24N} + \frac{f_{N}^4}{24N} \times \det(J_{\xi})
\]
We now observe that:
\[ f_{N+1}^2 f_{N-1}^2 - 2f_{N+1} f_{N-1} f_N^2 + f_N^4 = (f_{N+1} f_{N-1} - f_N^2)^2 \]

By Cassini’s identity, the right-hand side evaluates to 1, so that we have the claimed equality.

**Proof for Remark 5.4**

In this section we prove the following proposition, which justifies that we need to redefine the parameters \( S_r \) to differentiate between even-length and odd-length subdivisions (cf. Remark 5.4). Note that this is not necessary for the proof of our main result and can safely be skipped; we only include it for completeness.

> Proposition I.1. Let \( n \in \mathbb{N}^+ \) and \( \rho \in [0, 1]^n \), and let us consider the “determinant-style” quantity

\[ D = \Pi_{n}^{01}(\rho) \times \Pi_{n}^{00}(\rho) - \Pi_{n}^{00}(\rho) \times \Pi_{n}^{11}(\rho). \]

Then \( D \) is null if and only if one of the \( \rho_i \) is 0 or 1. Otherwise, \( D \) is positive if \( n \) is even, and negative if \( n \) is odd.

The proof is by induction on \( n \). We note that similar reasoning appears in [3, Lemma 7.2].

For \( n = 1 \), we have \( \Pi_{n}^{00}(\rho_0) = 1 \) and \( \Pi_{n}^{00}(\rho_0) = 1 - \rho_0 \) for \( b, b' \in \{0, 1\} \) such that \((b, b') \neq (0, 0)\). Thus \( D = (1 - \rho_0)^2 - 1 + \rho_0 = -\rho_0(1 - \rho_0) \), which is zero if \( \rho_0 = 0 \) or \( \rho_0 = 1 \) and negative otherwise.

For \( n = 2 \), we have:
- \( \Pi_{2}^{00}(\rho_0, \rho_1) = 1 - \rho_0 \rho_1 \)
- \( \Pi_{2}^{00}(\rho_0, \rho_1) = 1 - \rho_0 \)
- \( \Pi_{2}^{01}(\rho_0, \rho_1) = 1 - \rho_1 \)
- \( \Pi_{2}^{11}(\rho_0, \rho_1) = (1 - \rho_0)(1 - \rho_1) \)

Thus \( D = (1 - \rho_0)(1 - \rho_1) - (1 - \rho_0 \rho_1)(1 - \rho_0)(1 - \rho_1) = (1 - \rho_0)(1 - \rho_1)\rho_0 \rho_1 \). This is zero if one of \( \rho_0 \) or \( \rho_1 \) is equal to 0 or 1, and positive otherwise.

We now reason by induction. For brevity we omit the arguments of \( \Pi \) and use \( \Pi_{n+1}^{00} \) for \( \Pi_{n+1}^{00}(\rho) \) and \( \Pi_{n+2}^{00} \) for \( \Pi_{n+2}^{00}(\rho_1, \ldots, \rho_{n+2}) \). We also write for brevity \( \overline{p} = 1 - p \) for \( p \in [0, 1] \).

For \( n > 2 \), we have:
- \( \Pi_{n}^{11} = \frac{\overline{\rho_0} \rho_{n-1} \Pi_{n-2}^{10}}{\Pi_{n-2}^{02}} \) because the possible edge subsets with an added edge to the right are those with an added edge to both ends, plus those where the \( \rho_0 \) edge is kept (and the \( \rho_{n-1} \) edge is not kept)
- \( \Pi_{n}^{00} = \Pi_{n}^{11} + \rho_0 \overline{\rho_{n-1}} \Pi_{n-2}^{10} \) (symmetrically)
- \( \Pi_{n}^{00} = \Pi_{n}^{11} + \Pi_{n}^{10} - \Pi_{n}^{11} + \rho_0 \rho_{n-1} \Pi_{n-2}^{11} \) because the possible edge subsets with no added edges are those where the \( \rho_0 \) edge must not be kept, plus those where the \( \rho_{n-1} \) edge must not be kept, minus those where the \( \rho_0 \) and \( \rho_{n-1} \) edge must both not be kept (double counts), plus those where the \( \rho_0 \) and \( \rho_{n-1} \) edges must both be kept. Expanding the definitions of \( \Pi_{n}^{01} \) and \( \Pi_{n}^{10} \), this rewrites to: \( \Pi_{n}^{00} = \Pi_{n}^{11} + \rho_0 \overline{\rho_{n-1}} \Pi_{n-2}^{10} + \overline{\rho_0} \rho_{n-1} \Pi_{n-2}^{01} + \rho_0 \rho_{n-1} \Pi_{n-2}^{11} \)

Let us compute \( D \):

\[ D = (\Pi_{n}^{11} + \overline{\rho_0} \rho_{n-1} \Pi_{n-2}^{10})(\Pi_{n}^{11} + \overline{\rho_0} \rho_{n-1} \Pi_{n-2}^{01}) - \Pi_{n}^{11} (\Pi_{n}^{11} + \overline{\rho_0} \rho_{n-1} \Pi_{n-2}^{10} + \rho_0 \rho_{n-1} \Pi_{n-2}^{01} + \rho_0 \rho_{n-1} \Pi_{n-2}^{11}) \]
Many terms simplify, leaving:

\[ \Delta = \rho_0 \rho_\Delta \rho_{n-2} \rho_{n-1} \prod_{n=1}^{\infty} - \prod_{n=2}^{\infty} \rho_0 \rho_{n-1} \prod_{n=2}^{\infty} \]

Expanding the definition of \( \prod_{n=1}^{\infty} \) and factoring yields:

\[ \Delta = \rho_0 \rho_{n-1} \rho_\Delta \prod_{n=2}^{\infty} \left( \frac{\prod_{n=1}^{\infty} - \prod_{n=2}^{\infty}}{\rho_0 \rho_{n-1} \prod_{n=2}^{\infty}} \right) \]

We recognize the expression of the determinant-style expression for \( n = 2 \). Thus, \( \Delta \) is zero if that expression is zero, i.e., by induction, one of the \( \rho_1, \ldots, \rho_{n-2} \) is 0 or 1; or if one of \( \rho_0, \rho_{n-1} \) is 0 or 1. Thus \( \Delta \) is zero if one of the \( \rho_0, \ldots, \rho_{n-1} \) is 0 or 1. Otherwise the sign of \( \Delta \) is that of the expression for \( n = 2 \), so the induction hypothesis concludes.

### J Proof of the Emulation Result (Proposition 5.2)

In this section we prove Proposition 5.2. We recall its statement for the reader’s convenience:

**Proposition 5.2 (Emulation result).** There exist closed-form expressions, denoted \( p(i), q(i), r(i), s(i) \), such that for every even integer \( i \geq 4 \) the following hold:

(A) the expressions evaluate to well-defined probability values, i.e., we have \( 0 \leq p(i), q(i), r(i), s(i) \leq 1 \); and

(B) the path of length \( 4 \) with probabilities \( p(i), q(i), r(i), s(i) \) behaves like a path of length \( i \) with probabilities \( 1/2 \), i.e., \( \Pi_{n=1}^{4b} (p(i), q(i), r(i), s(i)) = \Pi_{n=1}^{4b} (1/2, \ldots, 1/2) \) for all \( b \leq 0, 1 \).

Further, each of these expressions is of the form \( \frac{P \pm \sqrt{Q}}{R} \) where \( P, Q, R \) are polynomials in the Fibonacci numbers \( f_i \) and \( f_{i-2} \) and in \( 2^{-i} \), with rational coefficients.

We prove Proposition 5.2 in the rest of this appendix section. We fix once and for all the even integer \( i \geq 4 \). To prove the result, we first give some general-purpose inequality lemmas about the Fibonacci sequence in Appendix J.1 which we use in several places. Then we prove Proposition 5.2 in four steps, corresponding to the following four subsections.

First, in Section J.2 we derive a system of equations, denoted (E), that is equivalent to (B). We then we prove in Section J.3 that any tuple of real numbers \( (p, q, r, s) \) that is a solution to (E) must be in \( (0, 1)^4 \). In Section J.4 we use SageMath to help us find symbolic expressions that satisfy system (E). Last, in Section J.5, we show that these expressions are indeed well-defined. Putting it all together gives us Proposition 5.2.

#### J.1 Inequality Lemmas on the Fibonacci Sequence

Let us first prove the general-purpose results on the Fibonacci sequence. We will use Binet’s formula, a closed-form expression for \( f_n \) given by \( f_n = \frac{\varphi^n - (-\varphi)^n}{\sqrt{5}} \), where \( \varphi := \frac{1 + \sqrt{5}}{2} \approx 1.61 \) is the golden ratio. Looking at the statement of the next three lemmas, notice that it is clear that these inequalities are true asymptotically, since \( f_n \approx \frac{\varphi^n}{\sqrt{5}} \) when \( n \) goes to infinity; however, we need to prove that the inequalities hold when starting from some specific values. To this end, we will use the following two trivialities:

**Fact J.1.** For all \( \alpha > 0 \) and \( n \geq n_\alpha := \frac{\log \alpha}{\log 2 - \log \varphi} \), we have \( 2^n \geq \alpha \times \varphi^n \).

**Proof.** We have \( n_\alpha (\log 2 - \log \varphi) \geq \log \alpha \), noting that \( \log 2 - \log \varphi > 0 \) because \( \varphi < 2 \). Thus, we have \( n \log 2 \geq \log \alpha + n \log \varphi \). As the exponential is an increasing function, we obtain the claimed inequality.

and
Fact J.2. For all \( n \geq 2 \) we have
\[
\frac{1}{2} \frac{\varphi^n}{\sqrt{5}} \leq f_n \leq \frac{3}{2} \frac{\varphi^n}{\sqrt{5}}.
\]

Proof. Let us first show that we have
\[
\frac{\varphi^n}{\sqrt{5}} - \frac{1}{2} \leq f_n \leq \frac{\varphi^n}{\sqrt{5}} + \frac{1}{2}.
\]  
(7)

To show this, let us consider the quantity
\[
\log \left( \frac{\varphi}{\sqrt{5}} \right) \leq \log f_n \leq \log \left( \frac{\varphi}{\sqrt{5}} \right) + \frac{1}{2}.
\]

The exponential is an increasing function, so we can exponentiate and get:
\[
(\varphi - 1)^n \leq \frac{\sqrt{5}}{2}.
\]

The left-hand-side is clearly positive because \( \varphi - 1 > 0 \), so:
\[
|\left(\varphi - 1\right)^n| \leq \frac{\sqrt{5}}{2}.
\]

Thus:
\[
\frac{|\left(\varphi - 1\right)^n|}{\sqrt{5}} \leq \frac{1}{2}.
\]

Binet's formula allows us to get Equation 7.

Now \( \frac{\varphi^n}{\sqrt{5}} \geq 1 \) for \( n \geq 2 \), hence \( \frac{1}{2} \leq \frac{1}{2} \frac{\varphi^n}{\sqrt{5}} \), and therefore
\[
\frac{1}{2} \frac{\varphi^n}{\sqrt{5}} \leq f_n \leq \frac{3}{2} \frac{\varphi^n}{\sqrt{5}}
\]
when \( n \geq 2 \), just as claimed.

Indeed, these will allow us to show, for each inequality that we want to prove to be true for all \( n \geq m \), that it holds for all \( n \geq n_\alpha \) for some \( n_\alpha \), and then to check by direct computation that the inequality is also true for all \( n \in [m, n_\alpha] \). The script where these computations are done can be found as Fibonacci-inequalities.ipynb in the supplementary material [5]. Let us proceed.

Lemma J.3. We have \( 2^n \geq \frac{f_n^2}{f_{n-2}} \) for all \( n \geq 4 \).

Proof. By Fact J.2, the following inequality is true for \( n \geq 4 \):
\[
\frac{f_n^2}{f_{n-2}} \leq \frac{\left(\frac{3}{2} \frac{\varphi^n}{\sqrt{5}}\right)^2}{\frac{1}{2} \frac{\varphi^{n-2}}{\sqrt{5}}} = \frac{9\varphi^2}{2\sqrt{5}} \times \varphi^n.
\]

We now use Fact J.1 with \( \alpha := \frac{9\varphi^2}{2\sqrt{5}} \), and obtain \( n_\alpha \approx 7.8 \). Hence for \( n \geq 8 \) we have indeed \( \frac{f_n^2}{f_{n-2}} < 2^n \). We prove that this is also the case for \( n \in [4, 7] \) by direct computation.
\textbf{Lemma J.4.} We have $2^n \geq \frac{f_{n-1}f_{n+1}}{f_{n-2}f_n}$ for all $n \geq 4$.

\textbf{Proof.} By Fact J.2, the following inequality is true for $n \geq 4$:

$$\frac{f_{n-1}f_{n+1}}{f_{n-2}f_n} \leq \left(\frac{1}{2} \frac{\phi^{n+1}}{\sqrt{5}}\right)^2 \left(\frac{1}{2} \frac{\phi^n}{\sqrt{5}}\right) = \frac{27\phi^2}{2\sqrt{5}} \times \phi^n.$$

We now use Fact J.1 with $\alpha := \frac{27\phi}{2\sqrt{5}}$, and obtain $n_\alpha \approx 10.7$. Hence for $n \geq 11$ we have indeed $\frac{f_{n-1}f_{n+1}}{f_{n-2}f_n} < 2^n$. We prove that this is also the case for $n \in [4, 10]$ by direct computation. 

\textbf{Lemma J.5.} We have $2^n \geq 10 \times \frac{f_n^5}{f_{n-2}}$ for $n \geq 48$.

\textbf{Proof.} By Fact J.2, the following inequality is true for $n \geq 4$:

$$10 \times \frac{f_n^5}{f_{n-2}^3} \leq 10 \times \left(\frac{3}{2} \frac{\phi^6}{\sqrt{5}}\right)^5 \left(\frac{1}{2} \frac{\phi^n}{\sqrt{5}}\right) = \frac{1215 \phi^8}{\sqrt{5}} \times \phi^n.$$

We now use Fact J.1 with $\alpha := \frac{1215 \phi^8}{\sqrt{5}}$ and obtain $n_\alpha \approx 47.9$. Hence for $n \geq 48$ we have indeed $10 \times \frac{f_n^5}{f_{n-2}} \leq 2^n$.

\section*{J.2 Step (i): An equivalent system}

In this section we derive a system of equations that is equivalent to (B).

For brevity, we write $\Pi_{bb}'$ for $\Pi_{bb}'(p, q, r, s)$, and write $\bar{p}$ for $1 - p$ (and similarly for $q, r$ and $s$). Note that we can explicitly express $\Pi_{40}'$ as a sum of products of the $p, q, r, s$ and of the $\bar{p}, \bar{q}, \bar{r}, \bar{s}$, intuitively corresponding to the edge subsets of the graph $P_4(p, q, r, s)$ that are a matching, and the same is true for $\Pi_{bb}'$ by removing the terms involving $p$ (if $b = 1$) and those involving $s$ (if $b' = 1$).

We start by writing out explicitly system (B), which by Lemma 4.3 consists of the following four equations:

(B) :

$$\Pi_{40}^0 = \frac{f_{i+2}}{2^i}$$

$$\Pi_{40}^1 = \frac{f_{i+1}}{2^i}$$

$$\Pi_{40}^{10} = \frac{f_{i+1}}{2^i}$$

$$\Pi_{40}^{11} = \frac{f_i}{2^i}$$

We claim that for any tuple $(p, q, r, s)$ of real numbers, this tuple satisfies (B) if and only if it satisfies the following system, denoted (E).
Weighted Counting of Matchings in Unbounded-Treewidth Graph Families

(E) :

\[
pq\bar{s} = \frac{f_{i-1}}{2^i}
\]  
(12)

\[
s\bar{r}p = \frac{f_{i-1}}{2^i}
\]  
(13)

\[
q\bar{r} = \frac{1}{f_{i-1}}
\]  
(14)

\[
\bar{p}\bar{s} = \frac{f_{i-1}^2}{2^i f_{i-2}}
\]  
(15)

We prove each implication in turn in the next two sections. Note that we actually only need the direction \((E) \Rightarrow (B)\) for the proof of Proposition 5.2, but we prove the equivalence for completeness.

J.2.1 (E) implies (B)

We prove here that (E) implies (B).

Getting Equation (11). Observe that we have:

\[
\Pi_{4}^{11} = \bar{p}\bar{s}(1 - qr).
\]  
(16)

This is by explicit computation on \(P_4(p, q, r, s)\): the edge subsets that are a matching must have the first and last edge missing, and cannot have both remaining edges present.

Now, we use Equation (15) and Equation (14) to substitute \(\bar{p}\bar{s}\) and \(qr\), and obtain:

\[
\Pi_{4}^{11} = \frac{f_{i-1}^2}{2^i f_{i-2}} \left(1 - \frac{1}{f_{i-1}^2}\right).
\]

Let us bring to the same denominator. We get:

\[
\Pi_{4}^{11} = \frac{f_{i-1}^2}{2^i f_{i-2}} \times \frac{f_{i-1}^2 - 1}{f_{i-1}^2}.
\]

We use Cassini’s identity, remembering that \(i\) is even so \(i - 1\) is odd, and get:

\[
\Pi_{4}^{11} = \frac{f_{i-1}^2}{2^i f_{i-2}} \times \frac{f_i f_{i-2}}{f_{i-1}^2}.
\]

Simplifying, we get Equation (11).

Getting Equations (9) and (10). Next, observe that we have:

\[
\Pi_{4}^{01} = \Pi_{4}^{11} + pq\bar{s}.
\]  
(17)

Indeed, the edge subsets of \(P_4\) that are a matching and have the last edge missing are those that are a matching and have the first and last edge missing, corresponding to \(\Pi_{4}^{11}\), plus the ones where the last edge is missing and the first edge is not missing. The latter implies that the second edge must be missing, corresponding to the term \(pq\bar{s}\).

Now, we have just shown that \(\Pi_{4}^{11} = \frac{f_i}{2^i}\), and we have \(pq\bar{s} = \frac{f_{i-1}}{2^i}\) by Equation (12). So we have:

\[
\Pi_{4}^{01} = \frac{f_i}{2^i} + \frac{f_{i-1}}{2^i}.
\]
By definition of the Fibonacci sequence \((f_{i+1} = f_i + f_{i-1})\), we obtain Equation (9).

Equation (10) is obtained symmetrically, using Equation (13) and the fact that:

\[
\Pi_4^{10} = \Pi_4^{11} + s\tilde{r}\tilde{p}.
\]  

(18)

**Getting Equation (8).** Last, we have:

\[
\Pi_4^{00} = \Pi_4^{11} + ps\tilde{q}\tilde{r} + p\tilde{q}\tilde{s} + s\tilde{r}\tilde{p}.
\]  

(19)

This is because the edge subsets of \(P_4\) that are matchings are those where both the first and last edge are missing, corresponding to \(\Pi_1^{11}\), plus those where the first and last edge are present (corresponding to \(ps\tilde{q}\tilde{r}\) because the other edges must be missing), plus those where the first edge is present and the last one missing (corresponding to \(p\tilde{q}\tilde{s}\) because the second edge must be missing), plus those where the last edge is present and the first one is missing (corresponding analogously to \(s\tilde{r}\tilde{p}\)).

By Equations (11), (12), and (13), this means that

\[
\Pi_4^{00} = \frac{f_i}{2^i} + ps\tilde{q}\tilde{r} + \frac{f_{i-1}}{2^i} + \frac{f_{i-1}}{2^i}.
\]  

(20)

We now need to get rid of the term \(ps\tilde{q}\tilde{r}\). But let us now multiply Equations (12), (13) and (14). We obtain

\[
p\tilde{q}r\tilde{s}s = \frac{1}{2^{2i}}.
\]

This implies that

\[
ps\tilde{q}\tilde{r} = \frac{1}{2^{2i}(p\tilde{s})(q\tilde{r})},
\]

and using Equations (14) and (15) we get

\[
ps\tilde{q}\tilde{r} = \frac{f_{i-2}}{2^i}.
\]

We now inject the above into (20) and obtain

\[
\Pi_4^{00} = \frac{f_i}{2^i} + \frac{f_{i-2}}{2^i} + \frac{f_{i-1}}{2^i} + \frac{f_{i-1}}{2^i} = \frac{f_{i+2}}{2^i},
\]

where the last line is obtained by applying three times the definition of the Fibonacci sequence.

We have thus obtained Equation (8).

**J.2.2 (B) implies (E)**

In this section we prove that (B) implies (E). We point out again that this is not strictly necessary for the proof of Proposition 5.2 and can safely be skipped; we only include this for completeness.
Getting Equations (12) and (13). By Equations (9) and (10), we have $\Pi_4^{01} = \Pi_4^{10}$. Hence by Equations (17) and (18):

$$pq\bar{s} = s\bar{r}\bar{p}.$$ 

Calling this quantity $Q$, Equations (17) and (18) rewrite to:

$$\Pi_4^{01} = \Pi_4^{10} = Q + \Pi_4^{11}.$$ 

Now, by system (B) and by the definition of the Fibonacci sequence, we have

$$\Pi_4^{01} = \frac{f_{i+1}}{2^i} = \frac{f_i}{2^i} + \frac{f_{i-1}}{2^i} = \Pi_4^{11} + \frac{f_{i-1}}{2^i}.$$ 

This implies that $Q = pq\bar{s} = s\bar{r}\bar{p} = \frac{f_{i+1}}{2^i}$, i.e., we have Equations (12) and (13).

Getting Equation (14). Let us now compute the “determinant-style” expression $\Pi_4^{01}\Pi_4^{10} - \Pi_4^{00}\Pi_4^{11}$. Combining Equations (17), (18), and (19), we have:

$$\Pi_4^{01}\Pi_4^{10} - \Pi_4^{00}\Pi_4^{11} = (\Pi_4^{11} + pq\bar{s})(\Pi_4^{11} + s\bar{r}\bar{p}) - (\Pi_4^{11} + ps\bar{q}r + pq\bar{s} + s\bar{r}\bar{p})\Pi_4^{11}.$$ 

This simplifies, leaving us with:

$$\Pi_4^{01}\Pi_4^{10} - \Pi_4^{00}\Pi_4^{11} = pq\bar{s}r\bar{p} - ps\bar{q}r\Pi_4^{11}.$$ 

We now inject (16) to obtain

$$\Pi_4^{01}\Pi_4^{10} - \Pi_4^{00}\Pi_4^{11} = pq\bar{s}r\bar{p} - ps\bar{q}r\Pi_4^{11}(1 - qr).$$ 

This simplifies again, and we obtain, after reordering of terms:

$$\Pi_4^{01}\Pi_4^{10} - \Pi_4^{00}\Pi_4^{11} = pqrpq\bar{r}s.$$ 

On the other hand, by system (B) we have that

$$2^i(\Pi_4^{01}\Pi_4^{10} - \Pi_4^{00}\Pi_4^{11}) = f_{i+1}^2 - f_{i+2}f_i.$$ 

Now, remembering that $i$ is even, we have by Cassini’s identity that $f_{i+2}^2 = f_if_{i+2} + 1$. Therefore, we obtain that

$$pqrpq\bar{r}s = \frac{1}{2^i}. \quad (21)$$

Continuing, notice that

$$pqrpq\bar{r}s = (pq\bar{s})(s\bar{r}\bar{p})qr = qr\frac{f_{i+1}^2}{2^i}$$

by Equations (12) and (13). Together with (21), we obtain $qr = \frac{1}{f_{i+1}}$, that is, Equation (14).
Getting Equation (15). We know that $\Pi_{11}^{11} = \bar{p}s(1 - qr)$, and $\Pi_{12}^{11} = \frac{f_i}{2}$ by Equation (11). Hence, we have

$$\bar{p}s = \frac{f_i}{2(1 - qr)}.$$

We now use (14) and Cassini’s identity to simplify $1 - qr$, and we obtain $\bar{p}s = \frac{f_i^2 - 1}{2f_i - 2}$, that is, Equation (15). This concludes the proof that system (B) and system (E) are equivalent.

J.3 Step (ii): Membership in $(0, 1)$

We now prove that any tuple of real numbers $(p, q, r, s)$ that satisfies system (E) must be in $(0, 1)^4$. Clearly, looking at the equations of (E), none of $p, q, r, s$ can be equal to 0 or to 1. Let us then consider three possible ranges for each variable that cover all the possibilities: the variable is either in $(-\infty, 0)$, or it is in $(0, 1)$, or it is in $(1, \infty)$. Observe that each equation of (E) tells us something about the possible ranges of $p, q, r, s$, by considering that all expressions are positive: for instance, Equation (14) implies that $q < 0$ if $r \in (-\infty, 0)$, and Equation (15) that $p < 0$ if $r \in (1, \infty)$. We use a helper script to analyze the $3^4 = 81$ possible ranges of $p, q, r, s$, shown in Algorithm 1. This algorithm as a Python script can be found as `prune-ranges.py` in supplementary material.

**Algorithm 1** Small helper script to prune out possible ranges of $p, q, r$ and $s$.

**Input:** Nothing.

**Output:** Prints out the possible ranges that are not discarded by simple considerations on the sign of expressions in system (E).

```python
/* We can use $-0.5$, $0.5$ and $1.5$ as representatives of the three candidate ranges for each variable */
for (p, q, r, s) in \{-0.5, 0.5, 1.5\}^4 do
    if p(1 - q)(1 - s) < 0 then // By Equation (12)
        continue;
    end
    if s(1 - r)(1 - p) < 0 then // By Equation (13)
        continue;
    end
    if qr < 0 then // By Equation (14)
        continue;
    end
    if (1 - p)(1 - s) < 0 then // By Equation (15)
        continue;
    end
    print((p, q, r, s));
/* This outputs $(-0.5, 1.5, 1.5, -0.5), (1.5, 1.5, 1.5, 1.5), (0.5, -0.5, -0.5, 0.5),$ $(-0.5, 1.5, 0.5, 0.5), (0.5, 0.5, 0.5, 0.5), and (0.5, 0.5, 0.5, 0.5). */
```

Only 6 ranges survive this script. The last one of them is $(0, 1)^4$, and we will show that the five first ranges are in fact impossible, by chasing down inequalities from (E) until we reach contradictions.
First two cases. In the first two cases, we have \( q, r \in (1, \infty) \). But this is clearly not possible by Equation (14), since \( \frac{1}{f_{i-1}} < 1 \) (given that \( i \geq 4 \)).

Third case. The third case to discard is when \( p, s \in (0, 1) \) and \( q, r \in (-\infty, 0) \). Call this assumption \( (\dagger) \). Observe then that \( \bar{q} > 1 \), and multiply this inequality by \( p \) and by \( \bar{s} \) (which are both \( > 0 \)) to obtain

\[
\bar{q} \bar{p} \bar{s} > \bar{p} \bar{s}.
\]

Now, by (12), this implies:

\[
\frac{f_{i-1}}{2^1} > \bar{p} \bar{s}.
\]

Hence \( \bar{s} < \frac{\bar{p}}{\bar{s}} \times \frac{f_{i-1}}{2^1} \). Multiply by \( \bar{p} \) (which is \( > 0 \)) and use (15) to get

\[
\frac{f_i}{2^i} = \bar{p} \bar{s} < \bar{p} \frac{f_{i-1}}{2^i}
\]

and so \( \bar{p} > \frac{f_{i-1}}{f_{i-2}} \). As \( p > 0 \), we then have:

\[
\bar{p} > p \frac{f_{i-1}}{f_{i-2}}.
\]

Recalling that \( p = 1 - \bar{p} \), we have:

\[
\bar{p} > (1 - \bar{p}) \frac{f_{i-1}}{f_{i-2}}.
\]

Rearranging terms, we obtain:

\[
\bar{p} > \frac{f_{i-1}}{f_{i-2}} \frac{1}{1 + \frac{f_{i-1}}{f_{i-2}}}.
\]

We can simplify by multiplying the numerator and denominator by \( f_{i-2} \) and using the definition of the Fibonacci sequence, to get:

\[
\bar{p} > \frac{f_{i-1}}{f_i}.
\]

(22)

This in turn implies, multiplying by \( \bar{s} \) (which is \( > 0 \)) and by (15) again, that

\[
\frac{f_i^2}{2^i f_{i-2}} = \bar{p} \bar{s} > \bar{s} \times \frac{f_{i-1}}{f_i}
\]

so that

\[
\frac{f_i f_{i-1}}{2^i f_{i-2}} > \bar{s}.
\]

Now, using \( \bar{s} = 1 - s \) and reordering terms, we get that

\[
s > \frac{2^i f_{i-2} - f_i f_{i-1}}{2^i f_{i-2}}.
\]
But $2^i f_{i-2} > f_i f_{i-1}$ by Lemma J.3 (using $f_i > f_{i-1}$), and $s > 0$ by (†), hence we have

$$\frac{1}{s} < \frac{2^i f_{i-2}}{2^i f_{i-2} - f_i f_{i-1}}. \quad (23)$$

Now, we multiply (22) by $s$ and $\bar{\tau}$ (which are $> 0$) to obtain

$$\frac{f_{i-1}}{2^i} = \bar{\rho} s \bar{\tau} > s \bar{\tau} \times \frac{f_{i-1}}{f_i},$$

hence $\bar{\tau} < \frac{1}{2} \times \frac{f_{i-1}}{f_i}$, and thanks to (23) we get

$$\bar{\tau} < \frac{f_i f_{i-2}}{2^i f_{i-2} - f_i f_{i-1}},$$

implying

$$r > \frac{2^i f_{i-2} - f_i f_{i-1} - f_i f_{i-2}}{2^i f_{i-2} - f_i f_{i-1}} = \frac{2^i f_{i-2} - f_i^2}{2^i f_{i-2} - f_i f_{i-1}},$$

where the last equality is using the definition of the Fibonacci sequence. This last expression is positive according to Lemma J.3. But $r$ is supposed to be negative by (‡), a contradiction.

**Fourth case.** The fourth case to discard is when $s \in (-\infty, 0)$, $p, q \in (0, 1)$ and $r \in (1, \infty)$. Call again this assumption (†). We start by multiplying $r > 1$ by $q$, which is positive by (‡), to get, with (14),

$$\frac{1}{f^2_{i-1}} = qr > q,$$

hence

$$\frac{1}{f^2_{i-1}} > 1 - \bar{q},$$

so that reordering terms

$$\bar{q} > 1 - \frac{1}{f^2_{i-1}} = \frac{f^2_{i-1} - 1}{f^2_{i-1}},$$

and by Cassini’s identity we get:

$$\bar{q} > \frac{f_i f_{i-2}}{f^2_{i-1}}.$$

We multiply this last inequality by $p$ and $\bar{s}$, which are positive by (‡), and obtain, from (12),

$$\frac{f_{i-1}}{2^i} = \bar{q} p \bar{s} > p \bar{s} \times \frac{f_i f_{i-2}}{f^2_{i-1}}.$$

Therefore $\bar{s} < \frac{1}{p} \times \frac{f^2_{i-1}}{2^i f_{i-2} f_i}$. Multiply by $\bar{p}$, which is positive by (‡), and use (15) to obtain

$$\frac{f^2_{i-1}}{2^i f_{i-2}} = \bar{p} \bar{s} < \frac{\bar{p}}{p} \times \frac{f^3_{i-1}}{2^i f_{i-2} f_i},$$
which simplifies to \( \bar{\gamma} > \frac{f_{i-1}}{f_{i+1}} \). As \( p > 0 \), and by similar reasoning as in the previous case, we get that \( \bar{p} > \frac{f_{i-1}}{f_{i+1}} \). Multiply by \( \bar{s} \) and use (15) again to obtain

\[
\frac{f_{i-1}^2}{2^i f_{i-2}} = \bar{p} \bar{s} > \bar{s} \times \frac{f_i}{f_{i+1}},
\]

hence

\[
\bar{s} < \frac{f_{i-1}^2 f_{i+1}}{2^i f_{i-2} f_i}
\]

and, using \( \bar{s} = 1 - s \) and reordering, we get:

\[
s > \frac{2^i f_{i-2} f_i - f_{i-1}^2 f_{i+1}}{2^i f_{i-2} f_i}.
\]

But this last expression is positive by Lemma J.4, whereas \( s \) is negative according to (†), a contradiction.

**Fifth case.** The last case to discard is when \( p \in (-\infty, 0) \), \( q \in (1, \infty) \) and \( r, s \in (0, 1) \). But this case is symmetrical to the fourth case, so we are done.

### J.4 Step (iii): Satisfying the System

We use SageMath to obtain a solution. The code can be found in the Jupyter notebook obtain-solution-and-check-sigma-small-N.ipynb in supplementary material [5]. Writing \( T := 1/2^i \) and \( F_k := f_{i+k} \), we define:

\[
P := 2 F_{-1} F_{-2}^2 + 2 (F_{-1}^2 - 1) F_{-2},
\]

\[
Q := 2 F_{-1}^2 F_{-2} - 2 (F_{-1}^4 + F_{-1}^3 F_{-2}) T
\]

\[
A := 2 F_{-1} F_{-2}^2,
\]

\[
\Xi := F_{-1}^2 F_{-2} - (F_{-1}^4 + 2 F_{-1}^3 F_{-2} + F_{-1}^2 F_{-2}^2) T
\]

\[
\Theta := F_{-1}^2 T - F_{-2}
\]

\[
C_0 := (F_{-1}^4 - 2 F_{-1}^2 + 1) F_{-2}^2
\]

\[
C_1 := 2 \left( (F_{-1}^2 + F_{-2}) F_{-2}^2 - 2 (F_{-1}^5 - F_{-1}^3) F_{-2}^2 + (F_{-1}^5 - 2 F_{-1}^4 + F_{-1}^2) F_{-2} \right)
\]

\[
C_2 := F_{-1}^5 + 4 F_{-1}^3 F_{-2}^2 + F_{-1}^4 F_{-2} - 2 F_{-1}^6 + F_{-1}^4
\]

\[
+ 2(3F_{-1}^6 - F_{-1}^4) F_{-2}^2 + 4(F_{-1}^7 - F_{-1}^5) F_{-2}
\]

\[
\Sigma := C_0 - C_1 T + C_2 T^2.
\]

> **Remark J.6.** These expressions could be simplified, using the properties of the Fibonacci sequence. Nevertheless, we leave them as-is in this section so that it is easier to see that they match the ones in the notebook, and to keep the notebook as clean as possible. We will simplify some of these in the next section.

Finally, we pose

\[
p(i) := (A + \Xi + \Theta + \sqrt{\Sigma})/P
\]

\[
q(i) := (\Xi - \Theta + \sqrt{\Sigma})/Q
\]

\[
r(i) := (\Xi - \Theta - \sqrt{\Sigma})/Q
\]

\[
s(i) := (A + \Xi + \Theta - \sqrt{\Sigma})/P
\]
One can then check by executing the notebook that these expressions satisfy system (E), when we look at $F'_1$ and $F'_2$ as symbolic variables. Note that one can understand how this verification can be performed; when computing the left-hand sides we obtain fractions whose numerator is a polynomial $P_1$ in $T$ and the $F_k$, plus such a polynomial $P_2$ times a square root of such a polynomial, divided by such a polynomial. Up to multiplying by the denominator, showing the identities amounts to checking that the polynomials $P_1$ and $P_2$ are correct, which can be done by expanding them and checking that the monomials are correct.

We point, however, that the resulting expressions are only symbolic expressions, and that we do not know a priori if they are well-defined, that is, if $\Sigma$ is always non-negative and if the denominators are not null. We prove in the next section that this is the case, using properties of the Fibonacci sequence.

**J.5 Step (iv): Checking Well-Definedness**

We now prove that the expressions $p(i), q(i), r(i), s(i)$ are well-defined, i.e., that $\Sigma$ is non-negative and that the denominators $P$ and $Q$ are never null. This will effectively conclude the proof of Proposition 5.2. Remember that $i$ is an even integer greater than 4.

**Checking that $P$ is not null.** By Cassini’s identity we have $F_{-1}^2 = F_{-2} F_0 + 1$, and so we get

\[
P = 2 F_{-1} F_{-2}^2 + 2 F_{-2} F_0
= 2 F_{-2}^2 (F_{-1} + F_0)
= 2 F_{-2} F_1,
\]

hence $P$ is clearly not null.

**Checking that $Q$ is not null.** We have:

\[
Q = 0
\iff 2 F_{-2} F_{-2}^2 = 2 T (F_{-1}^4 + F_{-1}^3 F_{-2})
\iff F_{-2} = T F_{-1} (F_{-1} + F_{-2})
\iff 2^i = \frac{F_{-1} F_0}{F_{-2}}
\]

But $2^i > \frac{F_{-1} F_0}{F_{-2}}$ according to Lemma J.3 (using $F_0 > F_{-1}$), so $Q$ cannot be null.

**Checking that $\Sigma$ is non-negative.** Recall that $\Sigma = C_0 - C_1 T + C_2 T^2$, where $T$ does not occur in any of the $C_i$. First, we simplify $C_0$, using Cassini’s equality:

\[
C_0 = (F_{-1}^2 - 2 F_{-2}^2 + 1) F_{-2}^2
= F_{-2}^2 (F_{-1}^2 - 1)^2
= F_{-2}^2 (F_{-2} F_0)^2
= F_{-1}^2 F_0^2.
\]

Next, we crudely upper bound $C_1$, again using Cassini’s identity, and the monotonicity
of the Fibonacci sequence.

\[
C_1 = 2 \left( (F_4^4 + F_2^2)F_3^2 + 2 (F_4^2 - F_2^2)F_2^2 + (F_6^6 - 2F_4^4 + F_2^2)F_0^2 \right)
\]

\[
= 2 \left( (F_4^2 + F_2^2)F_3^2 + 2 F_2^2 F_3^2 (F_2^2 - 1) + F_2 F_3 (F_2^2 - 1)^2 \right)
\]

\[
= 2 \left( (F_{-1}^2 + F_0^2)F_{-1}^2 + 2 F_{-2}^2 F_{-1}^2 F_0 + F_{-2}^2 F_0^2 F_0^2 \right)
\]

\[
\leq 2 \left( (F_0^0 + F_0^0)F_0^0 + 2 F_0 F_0^3 F_0^0 + F_0^0 F_0^2 F_0^2 \right)
\]

\[
= 2 (F_0^0 + 4F_0^3)
\]

\[
= 10F_0^7.
\]

We then show that \(C_2 \geq 0\) as follows:

\[
C_2 = F_{-1}^8 + 4F_{-1}^5 F_{-2}^3 + F_{-1}^3 F_{-2}^4 - 2F_{-1}^6 + F_{-1}^4
\]

\[
+ 2 (3F_{-1}^6 - F_{-1}^4)F_{-2}^2 + 4(F_7^7 - F_5^5)F_{-2}
\]

\[
= F_{-1}^8 (F_{-2}^2 - 2) + 4F_{-1}^5 F_{-2}^2 + F_{-1}^3 F_{-2}^4 + F_{-1}^4
\]

\[
+ 2F_{-2}^2 F_{-1}^4 (3F_{-2}^2 - 1) + 4F_{-2}^2 F_{-1}^5 (F_{-2}^2 - 1)
\]

But clearly \(F_{-2}^2 \geq 2\) (since \(i \geq 4\) and \(F_3 = 2\)), hence \(C_2 \geq 0\) indeed.

But then, observe that this implies

\[
\Sigma \geq F_{-2}^4 F_0^7 - 10F_0^7 \times T.
\]

This last term is non-negative if and only if \(2^i \geq 10 \times \frac{F_0^7}{F_{-2}^2}\), and we prove this to be the case in Lemma J.5 for \(i \geq 48\). We complete the proof by checking by direct computation that \(\Sigma \geq 0\) for all even integers between 4 and 47 (obtain-solution-and-check-sigma-small-N.ipynb in supplementary material [5]).

## K Proofs for the Precision Argument

In this section we give the proof details for step 5 of Section 5. Recall that our goal is to determine the result \(C\) of all oracle calls on the graphs \(G(\kappa, \kappa')\), because then we could recover \(S\) as \(\Gamma^{-1}C\) since we can compute the inverse of \(\Gamma\) in polynomial time. The problem is that we cannot actually invoke the oracle on the graphs \(G(\kappa, \kappa')\), because some of the edge probabilities are non-rational, namely, the \(p(i), q(i), r(i), s(i)\).

To work around this issue, the first step is to argue that we can compute decimal fraction approximations of the \(p(i), q(i), r(i), s(i)\), in polynomial time in the number of desired decimal places. This easily follows from their expressions of the form \(\frac{P \pm \sqrt{Q}}{R}\) with \(P, Q, R\) polynomials in quantities that we can compute exactly.

> **Lemma K.1.** Given an even integer \(i \geq 4\) and number \(z\) of decimal places, both written in unary, we can compute in polynomial time in \(i\) and \(z\) four decimal fractions \(0 \leq p(i), q(i), r(i), s(i) \leq 1\) such that \(|p(i) - p(i)| \leq 2^{-z}\) and similarly for \(q(i), r(i), s(i)\).

**Proof.** First note that we can compute exactly the Fibonacci numbers, the value \(2^i\), and polynomials in these values, as exact rationals. Now, we recall from Proposition 5.2 that the expressions for \(p(i), q(i), r(i),\) and \(s(i)\) are a sum of such polynomials and of \(\sqrt{\Sigma}\), where \(\Sigma\) is such a polynomial, divided by such polynomials (\(P\) or \(Q\)). So the problem boils down to
approximating expressions of the form $X/Y$ where $X$, $Y$, and $\Sigma$ are computed as exact rationals.

We know that there is some value $l$ polynomial in $i$ such that $|Y| > 2^{-l}$, because $Y$ is computed in polynomial time, i.e., the number of decimal places of $Y$ must be polynomial in $i$. Let us approximate $\sqrt{\Sigma}$ to have error at most $2^{-(l+z)}$, which we can do in polynomial time in $l$ and $z$ (see, e.g., [16]). The absolute error in the result is then at most $2^{-l}$, because we know $X/Y$ exactly and we know $\sqrt{\Sigma}/Y$ up to $2^{-z}$ since the error in $\sqrt{\Sigma}$ is at most multiplied by $2^i$.

We now point out that the oracle result $C$ that we wish to obtain is in fact a vector of decimal fractions, and that we can bound its number of decimal places. This is easy to notice if we consider the graph $\text{Sub}(H, \eta')$ where $\eta'$ subdivides each edge $e$ to $N$ or $N'$ depending on the parity of $\eta(e)$. Indeed, proposition 5.2 then ensures that the oracle result $C$ on the “ideal” graphs $G(\kappa, \kappa')$ is the same result that we would obtain on the graph $\text{Sub}(H, \eta')$ with probabilities set as in the lower part of Figure 1, i.e., not using the $p(i), q(i), r(i), s(i)$. Now, as all the probabilities on that graph are decimal fractions, the answer $C$ is in fact itself a vector of decimal fractions. Further, we can bound the number of decimal places of its components to $[m \times (\max(N, N') + 10)] \times z$ with $z$ the maximal number of decimal places of a decimal fraction in $p_e, \rho_e$, which is polynomial. This uses the following immediate result:

**Lemma K.2.** Let $(G, \pi)$ be a probabilistic graph with $\pi$ mapping each edge $e \in E$ to a probability $\pi(e)$ which is a decimal fraction with at most $z$ decimal places. Then the number of decimal places of $\Pr_{\text{matching}}(G, \pi)$ is at most $mz$, where $m$ is the number of edges of $m$.

**Proof.** The answer to $\Pr_{\text{matching}}(G, \pi)$ is a sum over edge subsets of $(G, \pi)$, so it suffices to show the result for each edge subset. Now, the probability of an edge subset is a product of $m$ which are either edge probabilities $\pi(e)$ or their complement $1-\pi(e)$. Both of these have at most $z$ decimal places, so the product has at most $mz$ decimal places, which concludes.

Now, the last step is to argue that we can recover exactly the oracle result $C$ by invoking the oracles on graphs with approximations of the non-rational probabilities. Let $z' := m \times (\max(N, N') + 10) \times z$ be the maximal number of decimal places of a component of $C$. Let $z'' := z' + 2m + 1$. Use Lemma K.1 to compute, for each $0 \leq i \leq \max(N, N')$, decimal fraction approximations $\hat{p}(i), \hat{q}(i), \hat{r}(i), \hat{s}(i)$ of $p(i), q(i), r(i), s(i)$ which are accurate to $z''$ binary places. Let $\hat{G}(\kappa, \kappa')$ be the graph defined like $G(\kappa, \kappa')$ but with the probabilities $p(i), q(i), r(i), s(i)$ replaced by $\hat{p}(i), \hat{q}(i), \hat{r}(i), \hat{s}(i)$. We call the oracle on these graphs, and obtain a vector $\hat{C}$. The only missing ingredient is to bound the error on each component of $\hat{C}$ relative to $C$. This follows from an easy variant of Lemma K.2:

**Lemma K.3.** Let $(G, \pi)$ with $G = (V, E)$ be a probabilistic graph with $\pi$ mapping each edge $e \in E$ to a real value, and let $\hat{\pi}$ mapping each edge $e$ to a decimal fraction $\hat{\pi}(e)$. Then we have: $|\Pr_{\text{matching}}(G, \pi) - \Pr_{\text{matching}}(G, \hat{\pi})| \leq 2^{2m} \max_{e \in E} |\hat{\pi}(e) - \pi(e)|$.

**Proof.** Let $W$ be the set of all edge subsets of $G$ which are matchings. Write $\delta(e) := \hat{\pi}(e) - \pi(e)$ for $e \in E$. We have:

$$\Pr_{\text{matching}}(G, \hat{\pi}) - \Pr_{\text{matching}}(G, \pi) = \sum_{E' \in W} \prod_{e \in E'} \hat{\pi}(e) \prod_{e \notin E'} (1 - \hat{\pi}(e)) - \Pr_{\text{matching}}(G, \pi)$$

Thus, injecting $\delta(e)$:

$$\Pr_{\text{matching}}(G, \hat{\pi}) - \Pr_{\text{matching}}(G, \pi) = \sum_{E' \in W} \prod_{e \in E'} (\pi(e) + \delta(e)) \prod_{e \notin E'} [(1 - \pi(e)] - \delta(e)] - \Pr_{\text{matching}}(G, \pi)$$
Let us bound the absolute value of the left-hand side to conclude. Note that, in the right-hand side, we can expand each term of the sum to obtain $2^m$ terms, one of which is the corresponding term of $\Pr_{\text{matching}}(G, \pi)$, the others all having some $\delta(e)$ as a factor, and the other quantities in the product are between $-1$ and $1$ because this is the case of the values of $\pi(e)$, of $\hat{\pi}(e)$, and of their differences $\delta(e)$. Hence, using the triangle inequality we obtain

$$\left| \Pr_{\text{matching}}(G, \hat{\pi}) - \Pr_{\text{matching}}(G, \pi) \right| \leq \sum_{E' \in W} 2^m \max_{e \in E} |\hat{\pi}(e) - \pi(e)|,$$

and the result follows immediately.

Thus, the result $\hat{C}$ of the oracle calls is such that each component has error at most $2^{-(\varepsilon'+1)}$ relative to $C$. Thus, truncating them to $\varepsilon'$ decimal places matches the exact value of $C$. Thus, we can recover the values $C$ and conclude the proof.

**Proof of Theorem 6.1**

In this section we give more details on the proof of Proposition 6.1 by explaining how the proof of Sections 3–5 and of the relevant lemmas and propositions is modified.

Using the new definition of the $\Pi_{n}^{00}$ as given in Section 6, Lemma 4.3 becomes:

> **Lemma L.1.** For all $n \in \mathbb{N}^+$, $b, b' \in \{0, 1\}$, we have $\Pi_{n}^{bb'}(1/2, \ldots, 1/2) = \frac{f_{n+b+b'}}{2^n}$.

**Proof.** We first prove it for $b = b' = 1$, by induction on $n$. For $n = 1$, we have $\Pi_{1}^{11}(1/2) = 1$, and $f_{1}/2^1 = 1$. For $n = 2$ we have $\Pi_{1}^{11}(1/2, 1/2) = 3/4$, and $f_{2}/2^2 = 3/4$. Now, assuming the claim holds for $1 \leq i \leq n - 1$, $n \geq 3$, let us look at $\Pi_{n}^{11}(1/2, \ldots, 1/2)$. By case analysis on the first edge, we have:

$$\Pi_{n}^{11}(1/2, \ldots, 1/2) = \frac{1}{2} \Pi_{n-1}^{11}(1/2, \ldots, 1/2) + \frac{1}{2} \Pi_{n-2}^{11}(1/2, \ldots, 1/2),$$

and the induction hypothesis together with the definition of the Fibonacci sequence concludes.

Now, it is clear that $\Pi_{n}^{01}(1/2, \ldots, 1/2) = \frac{1}{2} \Pi_{n-1}^{11}(1/2, \ldots, 1/2)$, because the first edge must be present, and then the condition on the $n - 1$ remaining edges is the same as the condition of $P_{n-1}(1/2, \ldots, 1/2)$. Similarly we have $\Pi_{n}^{00}(1/2, \ldots, 1/2) = \frac{1}{2} \Pi_{n-1}^{11}(1/2, \ldots, 1/2)$ and $\Pi_{n}^{01}(1/2, \ldots, 1/2) = \frac{1}{2} \Pi_{n-2}^{11}(1/2, \ldots, 1/2)$, so the claim holds.

Lemma 4.2 becomes:

> **Lemma L.2.** Let $n, n' \in \mathbb{N}^+$ and $\rho \in [0, 1]^n$, $\rho' \in [0, 1]^{n'}$ be tuples of probability values. Then, for every $b, b' \in \{0, 1\}$ we have

$$\Pi_{n+n'}^{bb'}(\rho, \rho') = (\Pi_{n}^{bb}(\rho) \times \Pi_{n'}^{bb}(\rho')) + (\Pi_{n}^{b1}(\rho) \times \Pi_{n'}^{0b}(\rho')) - (\Pi_{n}^{0b}(\rho) \times \Pi_{n'}^{bb}(\rho')).$$

**Proof sketch.** We only sketch the proof, as a formal proof similar to that of Lemma 4.2 is easy to obtain from this. The possible edge subsets in $P_{n+n'}$ that are edge covers are those where the $n$-th edge is kept, plus those where the $(n+1)$-th edge is kept, minus those where both the $n$-th and $(n+1)$-th edge were kept (as these were counted twice).

Next, we argue that the analogue of Proposition 5.2 still holds. To this end, let us inspect the system of Equation (B) from this proposition, where the $\Pi_{n}^{00}$ are now defined using edge covers. We first look at $\Pi_{4}^{00}(p, q, r, s)$. By explicit computation we have

$$\Pi_{4}^{00}(p, q, r, s) = ps(1 - qr).$$
Indeed, the edge subsets of $P_4$ that are an edge cover must have the first and last edge present, and cannot have both middle edges absent. Notice then that this is the same as the “old” $\Pi_{11}^{bb}(p, q, r, s)$ (for matchings), but switching $p, q, r, s$ into $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ (see Equation 16).

We now look at $\Pi_{01}^{bb}(p, q, r, s)$. We have

$$\Pi_{01}^{bb}(p, q, r, s) = \Pi_{00}^{bb}(p, q, r, s) + prs.$$  \hfill (24)

Indeed, the edge subsets of $P_4$ that are an edge cover and have the first edge present are those that are an edge cover and have the first and last edge present, corresponding to $\Pi_{00}^{bb}(p, q, r, s)$, plus the ones where the first edge is present and the last edge is missing. The latter implies that the third edge must be present, corresponding to the term $prs$. Notice then that this is the same as the old $\Pi_{10}^{bb}(p, q, r, s)$, but switching $p, q, r, s$ into $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ (see Equation 18).

Similarly, the new $\Pi_{10}^{bb}(p, q, r, s)$ is equal to the old $\Pi_{01}^{bb}(p, q, r, s)$, but switching $p, q, r, s$ into $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ (Equation 17).

In the same manner, we easily can show that the new $\Pi_{11}^{bb}(p, q, r, s)$ is equal to the old $\Pi_{00}^{bb}(p, q, r, s)$, but switching $p, q, r, s$ into $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ (Equation 19).

Therefore, using Lemma L.1, the new System (B) can be obtained from the old one by simply switching $p, q, r, s$ into $\bar{p}, \bar{q}, \bar{r}, \bar{s}$, hence we can take the solutions to the new system to be $(1 - p(i), 1 - q(i), 1 - r(i), 1 - s(i))$, where $p(i), q(i), r(i), s(i)$ is the solution to the old system: indeed, this still satisfies condition (A), and the obtained solutions are also of the prescribed form.

Last, we point out that the relevant Jacobian determinants are again not the null polynomials: this follows directly from the previous remarks on how the $\Pi_{bb}^{bb}(p, q, r, s)$ have been changed, and from the fact that the determinant is alternating. For instance, the new $\det(J)$ is then the “old” $\det(J)$, but again swapping $\chi_{00}, \chi_{01}, \chi_{10}, \chi_{11}$ by $1 - \chi_{00}, 1 - \chi_{01}, 1 - \chi_{10}, 1 - \chi_{11}$: indeed, when we derive the new $\Pi_{bb}^{bb}(\chi)$ by $\chi_{b1}^{b1}, \chi_{b2}^{b2}$, only the sign changes. Since it changes in every cell and the matrix is $4 \times 4$, it does not change globally. An inspection of the rest of the proof reveals that it still works.