Thin film flow over a bump on an inclined channel

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Abstract. Lubrication theory is applied to model thin film flow on an inclined channel. The flow is uniform, but it is disturbed by a bump so that it produces a surface wave that is formulated into a single equation of the fluid thickness. Since the model is strongly non-linear, a numerical method is chosen for solving the equation. The effect of the bump can be seen by generating the wave at the surface. The wave increases the amplitude, splits, and propagates. For the long run, we obtain a steady form of the surface. The profile depends on the bump.

1. Introduction
A thin film flow on an inclined channel is considered. A uniform fluid flows down and disturbed by a bump, so that surface wave is generated and it propagates. This phenomenon is modeled based on the equation of continuity and the equation of momentum. But since the fluid thickness is much smaller than the fluid length, the second equation can be simplified into the theory of lubrication, see for example, in Acheson [1]. The equations together with the boundary conditions are then used to construct a single equation, similar to the problem in Wiryanto and Febrianti [2], but the bottom topography is involved.

Similar problem for the flat bottom has been done by King et al. [3]. The model was in the form of an integral-different equation. The surface boundary condition was applied to thin airfoil theory, such as in van Dyke [4]. They worked for steady thin film flow but involving a fast upward airflow above the film. As the solution, a periodic surface wave was obtained analytically and numerically. They also found that the wave is typically rolling waves, such as obtained by Merkin and Needham [5], Rifky Fauzan and Wiryanto [6] occurring in the flow down sluices of canals, as the effect of the frictional resistance of the rough sluice bed.

Since the problem here is strongly non-linear, a numerical approach is proposed. A finite difference method is chosen for solving the model, similar to Wiryanto and Febrianti [2]. In some works by Wiryanto [7-9] the method has analyzed the stability so that we can apply the method here and the wave generation can be observed. In those papers, Wiryanto, and Wiryanto and Febrianti worked on wave propagation at the thin film without a bump. We found that part of the wave splitting and propagating. Wiryanto & Mungkasi [10, 11] obtained similar phenomena for shallow water model formulated into Boussinesq equations, so did Magdalena and Wiryanto [12]. Their models are derived based on the fluid having relatively not thin compared to the horizontal scale, such as wavelength.

2. Methods
2.1. Wave generation
We consider a thin fluid flowing down on an inclined channel. The sketch of the flow is illustrated in Figure 1. The coordinates are chosen Cartesian with horizontal $x$-axis along the bottom of the
channel and the vertical $y$–axis perpendicular to the other. A bump is on the bottom given as $y = d(x)$ and disturbing the uniform flow generating surface wave presenting as $y = h(x,t)$. The flow in the thin fluid layer is taken to be governed by the lubrication equations, which are written as

$$
\begin{align*}
\frac{u_x + v_x}{u_y + \mu u_{yy} + \rho g \sin \theta} &= 0 \\
-p_x - p_y - \rho g \cos \theta &= 0
\end{align*}
$$

(1)

All fluid quantities are written in the conventional notation. These equations are to be solved subject to the conditions

i. $u = v = 0$ on $y = d(x)$

ii. $u_y = 0$ on $y = h(x,t)$

iii. $p = 0$ on $y = h(x,t)$

iv. $h_t + h_x u - v = 0$ on $y = h(x,t)$

The fluid is non slip at the bottom expressed in condition (i), and the other three conditions are on the surface presenting no shear stress (ii), we define the pressure at the surface as the reference (iii), and the kinematic condition is given in (iv).

Figure 1. Sketch of coordinates and flow

Following Wiryanto & Febrianti [2], Equations (1) and boundary conditions (i)-(iv) are readily integrated and the resulting single equation put in the form:

$$
 h_t + \frac{\rho g}{3\mu} (h - d)^3 (\sin \theta - h_x \cos \theta) \big|_x = 0
$$

(2)

Equation (2) is a non-linear equation for the fluid layer height. In the case of $d = 0$, no bump on the bottom, Equation (2) gives to the model solved by Wiryanto & Febrianti [2], any constant solution satisfies the equation. Therefore in the previous work, the model was used to observe the wave propagation, the initial condition was required. On the contrary, in (2) the wave expressed by the $h$ is generated by $d$ as the external force.

3. Result and Discussion

In solving (2), we first discretize the space and time. We use notation $i$ for space and $n$ for time, also step $\Delta x$ and $\Delta t$. We then integrate (2) concerning $x$ to reduce the order of the equation. The integral of the $h_t$ is approximated by the midpoint, and it is followed by forwarding time approximation

$$
 h_t \big|_{x_{i+\frac{1}{2}}} \Delta x + \frac{\rho g}{3\mu} (h - d)^3 (\sin \theta - h_x \cos \theta) \big|_{x_i}^{x_{i+1}} = 0
$$

After shifting the index, the finite difference equation is written in explicit form
\[ h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x} \left[ h dR^3 \left( \frac{\rho}{\mu} dh R gc + \frac{\rho}{\mu} gs \right) - h dL^3 \left( \frac{\rho}{\mu} dh L gc + \frac{\rho}{\mu} gs \right) \right] \]

where

\[ h dR = \frac{(h_{i+1} - d_{i+1}) + (h_i - d_i)}{2}, \quad h dL = \frac{(h_{i} - d_{i}) + (h_{i-1} - d_{i-1})}{2} \]

\[ dhR = \frac{h_{i+1} - h_i}{\Delta x}, \quad dhL = \frac{h_i - h_{i-1}}{\Delta x}, \quad gc = g \cos \theta, \quad gs = g \sin \theta \]

The scheme requires initial condition \( h(x, 0) \) and boundary conditions at the left and right fluid domain \( h(0, t) \) and \( h(100, t) \). We provide \( h(x, 0) = 0.2 \) indicating the uniform flow without disturbance. The left condition is given \( h(0, t) = 0.2 \). This position is assumed far from the disturbance, and the flow remains uniform. Meanwhile, the right condition is absorbed, formulated by linear extrapolation from the calculation in the flow domain.

Most of our calculation uses \( \Delta x = 0.1 \) and \( \Delta t = 0.01 \). The fluid domain is chosen \( x \in [0,100] \) and steps time until \( t = 1200 \), and the scheme is stable. In this paper, we give two types of a bump to illustrate the wave generation at the thin film. The first result is plotted in Figure 2. The uniform flow is disturbed by the bump

\[ d(x) = \begin{cases} 
0.05 \sin(0.1\pi(x - 20)), & 20 \leq x \leq 40 \\
0, & 0 < x < 20 \cup 40 < x < 100 
\end{cases} \]

So that surface wave is generated by growing the amplitude followed by splitting and propagating in the right direction. In its propagation, the wave deforms into lower amplitude and wider shape. That phenomenon is animated in Figure 2 as the plot of \( h(x, t) \) the numerical solution of (2). In this result, we calculate (2) for angle \( \theta = 5^\circ \) and \( \rho = 1, \mu = 1 \). In Figure 3 we show a plot of animation \( h(x, t) \) obtained from different bump, i.e.

\[ d(x) = \begin{cases} 
0.05 \left( \text{sech}(0.5(x - 30)) \right)^2, & 20 \leq x \leq 40 \\
0, & 0 < x < 20 \cup 40 < x < 100 
\end{cases} \]

Similar surface wave is generated, with different amplitude.

**Figure 2.** Plot wave \( h(x, t) \) generated by \( d(x) = 0.05 \sin(0.1\pi(x - 20)) \) for \( x \in [20,40] \) and zero for outside of the interval
Figure 3. Plot wave $h(x, t)$ generated by $d(x) = 0.05 \left( \text{sech}(0.5(x - 30)) \right)^2$ for $x \in [20, 40]$ and zero for outside of that interval

For other values of the physical quantities, the numerical solution of (2) is also obtained. Similar animation can be performed. The profile above the bump relatively does not change. It indicates as a steady solution and proportional to the bump. The difference that can be observed is the splitting wave propagating to the right. When the angle is increased, the wave propagates faster than the smaller angle. It is in contrast for $\mu$, and smaller $\mu$ produces a wave that propagates faster and also larger amplitude.

4. Conclusion
The effect of a bump to a uniform flow has been modeled into a single equation based on lubrication theory. The model was then solved numerically to simulate the wave generation: growing, splitting, and propagating. The numerical method was a finite difference method that was successful in performing the wave generation. We found that the profile of the wave depends on the bump disturbing the flow. Meanwhile, the characteristic of the wave motion relates to the physical quantity of the fluid and also the angle of the channel inclination

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