Output Feedback Tracking Control for a Class of Uncertain Systems subject to Unmodeled Dynamics and Delay at Input

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Abstract

Besides parametric uncertainties and disturbances, the unmodeled dynamics and time delay at the input are often present in practical systems, which cannot be ignored in some cases. This paper aims to solve output feedback tracking control problem for a class of nonlinear uncertain systems subject to unmodeled high-frequency gains and time delay at the input. By the additive state decomposition, the uncertain system is transformed to an uncertainty-free system, where the uncertainties, disturbance and effect of unmodeled dynamics plus time delay are lumped into a new disturbance at the output. Sequently, additive state decomposition is used to decompose the transformed system, which simplifies the tracking controller design. To demonstrate the effectiveness, the proposed control scheme is applied to three benchmark examples.

Index Terms

Additive state decomposition, tracking, input delay, unmodeled dynamics, output feedback, nonlinear systems.

I. INTRODUCTION

Tracking control of an uncertain system is a challenging problem. Most of research mainly focuses on systems subject to parametric uncertainties and additive disturbances [1]-[3]. Also, some research focuses on systems subject to uncertainties at the input, such as backlash, dead zone or other nonlinearities [4]-[5]. It is well known that unmodeled dynamics and time delay at the input are also often present in practical systems. For example, the unmodeled dynamics and time delay at the input often exist in flight control systems [6]-[8]. These uncertainties at the input may produce a significant degradation in the tracking performance or even cause problems.
instability if not dealt with properly. In the literature, there are some academic examples to demonstrate that uncertainties at the input cannot be ignored in some cases. For example, in [9], the authors constructed a simple example, later known as Rohrs’ example, to show that conventional adaptive control algorithms lose their robustness in the presence of unmodeled dynamics. Also, some control algorithms may lose their robustness in the presence of input delay, see for example the repetitive control example considered in [10]. Therefore, it is important to explicitly consider unmodeled dynamics and time delay at the input in the controller design.

In this paper, the output feedback tracking control problem is investigated for a class of single-input single-output (SISO) nonlinear systems subject to mismatching parametric uncertainty, mismatching additive disturbances, unmodeled high-frequency gains and time delay at the input. Before introducing our main idea, some accepted control methods in the literature to handle uncertainties are briefly reviewed. A nature way is to estimate all of the unknown parameters, then compensate for them. In [11], the tracking problem for a linear system subject to unknown parameters and the unknown input delay was considered, where both the parameters and input delay were estimated by the proposed method. However, this method cannot handle unparameterized uncertainties such as unmodeled high-frequency gains. The second way is to design adaptive control with robustness against unmodeled dynamics and time delay at the input. In [12], the Rohrs’ example and the two-cart example, which are tracking problems for uncertain linear systems subject to unmodeled dynamics and time delay at the input respectively, were revisited by the $L_1$ adaptive control. In [13], the authors analyzed that their proposed method is robust against time delay at the input. The third way is to convert a tracking problem to a stabilization problem by the idea of internal model principle [14], if disturbances or desired trajectories are limited to a special case. In [15], the problem of set point output tracking of an uncertain linear system with multiple delays in both the state and control vectors was considered. There also exist other methods to handle uncertainties. However, some of them such as high-gain feedback cannot be applied to the considered system directly as they rely on rapid changing control signal to attenuate uncertainties and disturbance. After passing unmodeled high-frequency gains or time delay at the input, the rapid changing control signal will be distorted a lot which will affect the feedback and then may destabilize the system. This explains why high-gain feedback is often avoided in practice.

Compared with these existing literature, the problem studied in this paper is more general
since not only the uncertainties at the input but also the output feedback and mismatching are considered. For output feedback, the state needs to be estimated which is difficult mainly due to the uncertainties and disturbances in the state equation. Even if parameters and disturbance can be estimated, it is also difficult to compensate for mismatching uncertain parameters and disturbance directly. To tackle these difficulties, two new mechanisms are adopted in this paper. First, the input is redefined to make it smooth and bounded to handle uncertainties at input. As a consequence, the effect of unmodeled high-frequency gains and time delay at the input is always bounded. And then, to handle estimate and mismatching problem, the input-redefinition system is transformed to an uncertainty-free system, which is proved to be input-output equivalent with the aid of the additive state decomposition\textsuperscript{1} [16]. All mismatching uncertainties, mismatching disturbance and effect of unmodeled dynamics plus time delay are lumped into a new disturbance at the output. An observer is then designed for the transformed system to estimate the new state and the new disturbance. Next, the transformed system is ‘additively’ decomposed into two independent subsystems in charge of corresponding subtasks, namely the tracking (including rejection) subtask and the input-realization subtask. Then one can design controller for each subtask respectively, and finally combines them to achieve the original control task. Three benchmark examples are given to demonstrate the effectiveness of the proposed control scheme.

The additive state decomposition is a decomposition scheme also proposed in our previous work [17], where the additive state decomposition is used to transform output feedback tracking control for systems with measurable nonlinearities and unknown disturbances and then to decompose it into three simpler problems. This hence makes a challenging control problem tractable. In this paper, a different control problem is investigated by using additive state decomposition. Correspondingly, the transform and decomposition are different. The major contributions of this paper are: i) a tracking control scheme proposed to handle mismatching parametric uncertainty, mismatching additive disturbances, unmodeled high-frequency gains and time delay at the input; ii) a model transform proposed to lump various uncertainties together; iii) additive state decomposition in the controller design, especially in how to handle saturation term.

This paper is organized as follows. In Section II, the problem formulation is given and

\textsuperscript{1}In this paper we have replaced the term “additive decomposition” in [16] with the more descriptive term “additive state decomposition”.
the additive state decomposition is introduced briefly first. In Section III, input is redefined and the input-redefinition system is transformed to an uncertainty-free system in sense of input-output equivalence. Sequently, controller design is given in Section IV. In Section V, two-cart example is revisited by the proposed control scheme. Section VI concludes this paper.

II. PROBLEM FORMULATION AND ADDITIVE STATE DECOMPOSITION

A. Problem Formulation

Consider a class of SISO nonlinear systems as follows:

\[ \dot{x} = f(t, x, \theta) + bu_\xi + d, \\
\text{ } y = c^T x. \tag{1} \]

Here \( b \in \mathbb{R}^n \) and \( c \in \mathbb{R}^n \) are constant vectors, \( \theta(t) \in \mathbb{R}^m \) belongs to a given compact set \( \Omega \subseteq \mathbb{R}^m \), \( x(t) \in \mathbb{R}^n \) is the state vector, \( y(t) \in \mathbb{R} \) is the output, \( d(t) \in \mathbb{R}^n \) is a bounded disturbance vector, and \( u_\xi(t) \in \mathbb{R} \) is the control subject to an unmodeled high-frequency gain and a time delay as follows:

\[ u_\xi(s) = H(s) e^{-\tau s} u(s) \tag{2} \]

where \( H(s) \) is an unknown stable proper transfer function with \( H(0) = 1 \) representing the unmodeled high-frequency gain at the input and \( \tau \in \mathbb{R} \) is the input delay. It is assumed that only \( y \) is available from measurement. The desired trajectory \( r(t) \in \mathbb{R} \) is known a priori, \( t \geq 0 \). In the following, for convenience, we will drop the notation \( t \) except when necessary for clarity.

For system (1), the following assumptions are made.

**Assumption 1.** The function \( f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) satisfies \( f(t, 0, \theta) \equiv 0 \), and is bounded when \( x \) is bounded on \([0, \infty)\). Moreover, for given \( \theta \in \Omega \), there exist positive definite matrices \( P \in \mathbb{R}^{n \times n} \) and \( Q \in \mathbb{R}^{n \times n} \) such that

\[ P \partial_x f(t, x, \theta) + \partial_x^T f(t, x, \theta) P \leq -Q, \forall x \in \mathbb{R}^n, \tag{3} \]

where \( \partial_x f \triangleq \frac{\partial f}{\partial x} \in \mathbb{R}^{n \times n} \).

**Definition 1** [18]. The \( L_1 \) gain of a stable proper SISO system is defined \( \|G(s)\|_{L_1} = \int_0^\infty |g(t)| dt \), where \( g(t) \) is the impulse response of \( G(s) \).
Assumption 2. There exists a known stable proper transfer function \( C(s) \) with \( C(0) = 1 \) such that \( \| C(s)(H(s) - 1) \|_{L_1} \leq \epsilon_H, \| sC(s) \|_{L_1} \leq \epsilon_r \), where \( \epsilon_H, \epsilon_r \in \mathbb{R} \) are positive real.

Under Assumptions 1-2, the objective here is to design a tracking controller \( u \) such that \( y \to r \) with a good tracking accuracy, i.e., \( y - r \) is ultimately bounded by a small value.

Remark 1. From Assumption 1, since \( f(t, x, \theta) = \partial_x f(t, x + \mu x, \theta) x, \mu \in (0, 1) \) by the Taylor expansion. Consequently, the system \( \dot{x} = f(t, x, \theta) \) is exponentially stable by (3). In practice, many systems are stable themselves or they can be stabilized by output feedback control. The following three benchmark systems all satisfy Assumption 1.

Example 1 (Rohrs’ Example). Consider the Rohrs’ example system as follows [9]:

\[
y(s) = \frac{2}{s + 1} \frac{229}{s^2 + 30s + 229} u(s). \tag{4}
\]

The nominal system is assumed to be \( y(s) = \frac{2}{s+3} u(s) \) here. In this case, the system (4) can be formulated into (1) as

\[
\dot{x} = -(3 + \theta) x + 2u_x, x(0) = 1
\]

\[
y = x
\]

where the parameter \( \theta = -2 \) is assumed unknown and \( H(s) = \frac{229}{s^2 + 30s + 229}, \tau = 0. \) It is easy to see that Assumption 1 is satisfied. Choose \( C(s) = \frac{1}{2s+1} \). Then Assumption 2 is satisfied with \( \epsilon_H = 0.12 \) and \( \epsilon_r = 1. \)

Example 2 (Nonlinear). Consider a simple nonlinear system as follows [19]:

\[
\dot{x} = -x - (1 + \theta) x^3 + u(t - \tau) + d, x(0) = 1
\]

\[
y = x
\]

where \( x, y, u, d \in \mathbb{R} \), the parameter \( \theta(t) = 0.2 \sin(0.1t + 1) \), the input delay \( \tau = 0.1 \) and \( d(t) = 0.5 \sin(0.2t) \) are assumed unknown. The system (6) can be formulated into (1) with \( f(t, x, \theta) = -x - (1 + \theta) x^3 \) and \( H(s) = 1, \tau = 0.1. \) It is easy to verify \( \partial_x f(t, x, \theta) = -1 - 3(1 + \theta) x^2 \leq -1 \). Therefore, Assumption 1 is satisfied. Let \( C(s) = \frac{1}{2s+1}. \) Then Assumption 2 is satisfied with \( \epsilon_H = 0 \) and \( \epsilon_r = 1. \)

Remark 2. The Rohrs’ example system in Example 1 is proposed to demonstrate that conventional adaptive control algorithms developed at that time lose their robustness in the presence of unmodeled dynamics [9]. For the tracking problem in Example 2, there exist robustness issues by using exact feedback linearization [19]. Compared with the system in [19], the input delay is added in (6) to make system worse. The two benchmark examples
tell us that the uncertainties either on the system parameters or at the input cannot be ignored in practice when design a tracking controller, even if the original systems are stable. This is also the initial motivation of this paper.

B. Additive State Decomposition

In order to make the paper self-contained, additive state decomposition \[16\] is introduced briefly here. Consider the following ‘original’ system:

\[
f(t, \dot{x}, x) = 0, \quad x(0) = x_0
\] (7)

where \(x \in \mathbb{R}^n\). We first bring in a ‘primary’ system having the same dimension as (7), according to:

\[
f_p(t, \dot{x}_p, x_p) = 0, \quad x_p(0) = x_{p,0}
\] (8)

where \(x_p \in \mathbb{R}^n\). From the original system (7) and the primary system (8) we derive the following ‘secondary’ system:

\[
f(t, \dot{x}, x) - f_p(t, \dot{x}_p, x_p) = 0, \quad x(0) = x_0
\] (9)

where \(x_p \in \mathbb{R}^n\) is given by the primary system (8). Define a new variable \(x_s \in \mathbb{R}^n\) as follows:

\[
x_s \triangleq x - x_p.
\] (10)

Then the secondary system (9) can be further written as follows:

\[
f(t, \dot{x}_s + \dot{x}_p, x_s + x_p) - f_p(t, \dot{x}_p, x_p) = 0, \quad x_s(0) = x_0 - x_{p,0}.
\] (11)

From the definition (10), we have

\[
x(t) = x_p(t) + x_s(t), \quad t \geq 0.
\] (12)

**Remark 3.** By the additive state decomposition, the system (7) is decomposed into two subsystems with the same dimension as the original system. In this sense our decomposition is “additive”. In addition, this decomposition is with respect to state. So, we call it “additive state decomposition”.

As a special case of (7), a class of differential dynamic systems is considered as follows:

\[
\dot{x} = f(t, x), \quad x(0) = x_0, \quad y = h(t, x)
\] (13)
where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Two systems, denoted by the primary system and (derived) secondary system respectively, are defined as follows:

$$\dot{x}_p = f_p (t, x_p) , x_p (0) = x_{p,0}$$
$$y_p = h_p (t, x_p)$$

(14)

and

$$\dot{x}_s = f (t, x_p + x_s) - f_p (t, x_p) , x_s (0) = x_0 - x_{p,0},$$

$$y_s = h (t, x_p + x_s) - h_p (t, x_p)$$

(15)

where $x_s \triangleq x - x_p$ and $y_s \triangleq y - y_p$. The secondary system (15) is determined by the original system (13) and the primary system (14). From the definition, we have

$$x(t) = x_p (t) + x_s (t), y(t) = y_p (t) + y_s (t), t \geq 0.$$ 

(16)

III. INPUT REDEFINITION AND MODEL TRANSFORMATION

Since $H (s)$ is the unmodeled high-frequency gain and $\tau$ is the input delay, the control signal should be smooth (low-frequency signal) so that it will maintain its original form as far as possible after passing $H (s) e^{-\tau s}$. Otherwise, the control signal will be distorted a lot. This explains why high-gain feedback in practice is often avoided. For such a purpose, the input is redefined to make control signal smooth and bounded first. This makes the effect of $H (s) e^{-\tau s}$ under control, i.e., the effect will be predicted and bounded.

A. Input Redefinition

Redefine the input as follows:

$$u (s) = C (s) [\sigma_a (v) (s)]$$

where $v \in \mathbb{R}$ is the redefined control input and $\sigma_a : \mathbb{R} \rightarrow [-a, a]$ is a saturation function defined as $\sigma_a (x) \triangleq \text{sign}(x) \min ([|x|, a])$. Then $u_\xi$ is written as

$$u_\xi (s) = H (s) e^{-\tau s} C (s) [\sigma_a (v) (s)]$$

$$= C (s) [\sigma_a (v) (s)] + \xi (s)$$

(17)

where $\xi (s) = C (s) (H (s) e^{-\tau s} - 1) [\sigma_a (v) (s)]$ represents the effect of the unmodeled high-frequency gain and the time delay. The function $\xi (s)$ can be further written as

$$\xi (s) = C (s) (H (s) - 1) e^{-\tau s} [\sigma_a (v) (s)] + C (s) (e^{-\tau s} - 1) [\sigma_a (v) (s)].$$

(18)
From the definition of $\sigma_a$, we have $\sup_{-\infty<x<\infty} |\sigma_a(x)| \leq a$. In this paper $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. By Assumption 2, $\xi$ is bounded as follows:

$$
\sup_{t \geq 0} |\xi(t)| \leq \|C(s)(H(s) - 1)\|_{\mathcal{L}^1} a + \|sC(s)\|_{\mathcal{L}^1} \sup_{t \geq 0} \left\{ \frac{(e^{-\tau s} - 1)}{s} \right\} \left\| \mathcal{L}^{-1} \left\{ \left( \sigma_a(v(s) \right) \right\} \right\|
$$

$$
\leq \varepsilon_H a + \varepsilon_\tau \sup_{t \geq 0} \int_{t}^{t+\tau} \sigma_a(v(\lambda)) d\lambda
$$

$$
\leq (\varepsilon_H + \tau \varepsilon_\tau) a \tag{19}
$$

where $\xi(t) = \mathcal{L}^{-1}(\xi(s))$. The input redefinition makes $\xi$ bounded not matter what the redefined control input $v$ is. Therefore, the redefined control input $v$ can be designed freely. According to input redefinition above, the controller (2) is rewritten as

$$
u_\xi = u + \xi. \tag{20}$$

Here $u(t) = \mathcal{L}^{-1}(C(s)[\sigma_a(v)(s)])$ can be written in the form of state equation as follows

$$\dot{z} = A_z z + b_z \sigma_a(v)$$

$$u = c^T_z z + d_z \sigma_a(v) \tag{21}$$

where the vectors and matrices are compatibly dimensioned depending on $C(s)$. Substituting (20) into the system (1) results in

$$\dot{x} = f(t, x, \theta) + bu + dh, \quad x(0) = x_0$$

$$y = c^T x \tag{22}$$

where $dh = d + \xi$. The system (22) with the redefined controller (21) is called as the input-redefinition system here.

**B. Model Transformation**

The unknown parameter $\theta$ and the unknown disturbances $d$ are not appear in “matching” positions for the control input, i.e., $\theta$ and $d$ do not appear like $b(u_\xi + \theta^T x + d)$. Therefore, in a general system except for one dimensional system, the unknown uncertainties cannot be often compensated for directly. Even if $\theta$ and $d$ satisfy the “matching condition”, it is also difficult to compensate for since the state $x$ is unknown. To tackle this difficulty, we first transform the input-redefinition system (22) to an uncertainty-free system, which is proved to be input-output equivalent with the aid of the additive state decomposition as stated in Theorem 1. Before proving the theorem, the following lemma is needed.
Lemma 1. Consider the following system

\[ \dot{x} = f(t, x + z, \theta) - f(t, z, \theta) + \rho \]  \hspace{1cm} (23)

where \( \rho(t) \in \mathbb{R}^n \) is bounded. Under Assumption 1, the solutions of (23) satisfy

\[ \| x(t) \| \leq \beta (\| x(t_0) \|, t - t_0) + \gamma \sup_{t_0 \leq s \leq t} \| \rho(s) \| \]  \hspace{1cm} (24)

where \( \beta \) is a class \( KL \) function \([20, p.144]\) and \( \gamma = 2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P) \lambda_{\min}(Q)} \).

Proof. By the Taylor expansion, the function \( f(t, x + z, \theta) \) can be written as

\[ f(t, x + z, \theta) = f(t, z, \theta) + \partial_x f(t, x + z + \mu x, \theta) x \]

where \( \mu \in (0, 1) \). Then the system (23) can be rewritten as

\[ \dot{x} = \partial_x f(t, x + x + \mu x, \theta) x + \rho. \]  \hspace{1cm} (25)

Choose Lyapunov function \( V = x^T P x \). By Assumption 1, the derivative of \( V \) along (25) satisfies

\[ \dot{V} \leq -\lambda_{\min}(Q) \| x \|^2 + \lambda_{\max}(P) \| x \| \| \rho \| \leq -\frac{1}{2} \lambda_{\min}(Q) \| x \|^2, \; \forall \| x \| \geq 2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \| \rho \|. \]

By Theorem 4.19 \([20, p.176]\), we can conclude this proof. \( \square \)

With Lemma 1 in hand, we have

Theorem 1. Under Assumption 1, there always exists an estimate of \( \theta \), namely \( \hat{\theta} \in \Omega \), such that the system (22) is input-output equivalent to the following system:

\[ \begin{align*}
\dot{x}_{\text{new}} &= f(t, x_{\text{new}}, \hat{\theta}) + bu, x_{\text{new}}(0) = 0 \\
y &= c^T x_{\text{new}} + d_{\text{new}}.
\end{align*} \]  \hspace{1cm} (26)

Here \( x_{\text{new}} \) and \( d_{\text{new}} \) satisfy

\[ \begin{align*}
\| x - x_{\text{new}} \| &\leq \beta (\| x_0 \|, t - t_0) + \gamma \sup_{t_0 \leq s \leq t} \| d_{\hat{\theta}}(s) \| \\
\| d_{\text{new}} \| &\leq \| c \| \beta (\| x_0 \|, t - t_0) + \| c \| \gamma \sup_{t_0 \leq s \leq t} \| d_{\hat{\theta}}(s) \|
\end{align*} \]  \hspace{1cm} (27)

where \( \beta \) is a class \( KL \) function, \( \gamma = 2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P) \lambda_{\min}(Q)} \) and \( d_{\hat{\theta}} = f(t, x_{\text{new}}, \theta) - f(t, x_{\text{new}}, \hat{\theta}) + d_h. \)
Proof. In the following, additive state decomposition is utilized to decompose the system (22) first. Consider the system (22) as the original system and choose the primary system as follows:

\[
\dot{x}_p = f(t, x_p, \hat{\theta}) + bu, \quad x_p(0) = 0
\]
\[
y_p = c^T x_p.
\]  
(28)

Then the secondary system is determined by the original system (22) and the primary system (28) with the rule (15) that

\[
\dot{x}_s = f(t, x_p + x_s, \theta) - f(t, x_p, \hat{\theta}) + d_h, \quad x_s(0) = x_0
\]
\[
y_s = c^T x_s.
\]  
(29)

According to (16), we have \(x = x_p + x_s\) and \(y = y_p + y_s\). Consequently, we can get an uncertainty-free system as follows

\[
\dot{x}_p = f(t, x_p, \hat{\theta}) + bu, \quad x_p(0) = 0
\]
\[
y = c^T x_p + y_s
\]

where \(u\) and \(y\) are the same to those in (22). Let \(x_p = x_{\text{new}}\) and \(d_{\text{new}} = y_s\). We can conclude that the system (22) is input-output equivalent to (26). Next, we will prove that (27) is satisfied. The system (29) can be rewritten as

\[
\dot{x}_s = f(t, x_p + x_s, \theta) - f(t, x_p, \theta) + d_{\tilde{\theta}}, \quad x_s(0) = x_0
\]
\[
y_s = c^T x_s
\]  
(30)

where \(d_{\tilde{\theta}} = f(t, x_p, \theta) - f(t, x_p, \hat{\theta}) + d_h\). Then, by Lemma 1, we have

\[
\|x(t) - x_{\text{new}}(t)\| = \|x_s(t)\| \leq \beta (\|x_0\|, t - t_0) + \gamma \sup_{t_0 \leq s \leq t} \|d_{\tilde{\theta}}(s)\|
\]
\[
\|d_{\text{new}}(t)\| \leq \|c\| \|x_s(t)\| \leq \|c\| \beta (\|x_0\|, t - t_0) + \|c\| \gamma \sup_{t_0 \leq s \leq t} \|d_{\tilde{\theta}}(s)\|.
\]

□

For the uncertainty-free transformed system (26), we design an observer to estimate \(x_{\text{new}}\) and \(d_{\text{new}}\), which is stated in Theorem 2.

Theorem 2. Under Assumption 1, an observer is designed to estimate state \(x_{\text{new}}\) and \(d_{\text{new}}\) in (26) as follows

\[
\dot{\hat{x}}_{\text{new}} = f(t, \hat{x}_{\text{new}}, \hat{\theta}) + bu, \quad \hat{x}_{\text{new}}(0) = 0
\]
\[
\hat{d}_{\text{new}} = y - c^T \hat{x}_{\text{new}}.
\]  
(31)
Then $\hat{x}_{\text{new}} \equiv x_{\text{new}}$ and $\hat{d}_{\text{new}} \equiv d_{\text{new}}$.

Proof. Subtracting (31) from (26) results in

$$\dot{\hat{x}}_{\text{new}} = \partial_x f(t, x_{\text{new}} + \hat{x}_{\text{new}} + \mu x_{\text{new}}, \hat{\theta})\hat{x}_{\text{new}}, \hat{x}_{\text{new}}(0) = 0,$$

where $\mu \in (0, 1)$ and $\hat{x}_{\text{new}} \triangleq x_{\text{new}} - \hat{x}_{\text{new}}$. Then $\hat{x}_{\text{new}} \equiv 0$. This implies that $\hat{x}_{\text{new}} \equiv x_{\text{new}}$.

Consequently, by the relation $y = c^T x_{\text{new}} + d_{\text{new}}$ in (26), we have $\hat{d}_{\text{new}} \equiv d_{\text{new}}$. □

Remark 4. By (21), the control signal $u$ is always bounded. Therefore, by Lemma 1, the state $x_{\text{new}}$ is always bounded. Consequently, by (27), $d_{\text{new}}$ is always bounded as well. It is interesting to note that the new state $x_{\text{new}}$ and disturbance $d_{\text{new}}$ in the transformed system (26) can be observed directly rather than asymptotically or exponentially. This will facilitate the analysis and design later.

Example 3 (Rohrs’ Example, Example 1 Continued). According to input redefinition above, the Rohrs’ example system (5) can be rewritten as follows:

$$\dot{x} = -(3 + \theta) x + 2u + (d + 2\xi)$$
$$y = x$$

where $\sup_{t \geq 0} |\xi(t)| \leq 0.12a$, and $u$ is generated by $\dot{z} = -0.5z + 0.5\sigma_a(v), u = z$. Then, according to (26), the uncertainty-free transformed system of (5) is

$$\dot{x}_{\text{new}} = -(3 + \hat{\theta}) x_{\text{new}} + 2u$$
$$y = x_{\text{new}} + d_{\text{new}}$$

(32)

where $\hat{\theta}$ will be specified later.

Example 4 (Nonlinear, Example 2 Continued). According to input redefinition above, the nonlinear system (6) can be rewritten as follows:

$$\dot{x} = -x - (1 + \theta) x^3 + u + (d + \xi)$$
$$y = x$$

where $\sup_{t \geq 0} |\xi(t)| \leq 0.1a$, and $u$ is generated by $\dot{z} = -0.5z + 0.5\sigma_a(v), u = z$. Then, according to (26), the uncertainty-free transformed system of (6) is

$$\dot{x}_{\text{new}} = -x_{\text{new}} - (1 + \hat{\theta}) x_{\text{new}}^3 + u, x_{\text{new}}(0) = 0$$
$$y = x_{\text{new}} + d_{\text{new}}$$

(33)

where $\hat{\theta}$ will be specified later.
IV. CONTROLLER DESIGN

In this section, the transformed system (26) is ‘additively’ decomposed into two independent subsystems in charge of corresponding subtasks. Then one can design controller for each subtask respectively, and finally combines them to achieve the original control task.

A. Additive State Decomposition of Transformed System

Currently, based on the new transformed system (26), the objective is to design a tracking controller $u$ such that $y \rightarrow r$ with a good tracking accuracy, i.e., $y - r$ is ultimately bounded by a small value. While, $u$ is realized by (21). According to this fact, the transformed system (26) is ‘additively’ decomposed into two independent subsystems in charge of corresponding subtasks, namely the tracking (including rejection) subtask and the input-realization subtask. This is shown in Fig.1.

![Additive state decomposition flow](image)

Consider the transformed system (26) as the original system. According to the principle above, we choose the primary system as follows:

$$
\dot{x}_p = f(t, x_p, \dot{\theta}) + bu_p, x_p(0) = 0
$$
$$
y_p = c^T x_p + d_{\text{new}}.
$$

(34)

Then the secondary system is determined by the original system (26) and the primary system (34) with the rule (15), and we can obtain that

$$
\dot{x}_s = f(t, x_p + x_s, \dot{\theta}) - f(t, x_p, \dot{\theta}) + b(u - u_p), x_s(0) = 0
$$
$$
y_s = c^T x_s.
$$

(35)
According to (16), we have

$$x_{\text{new}} = x_p + x_s \quad \text{and} \quad y = y_p + y_s.$$  \hfill (36)

The strategy here is to assign the tracking (including rejection) subtask to the primary system (34) and the input-realization subtask to the secondary system (35). It is clear from (34)-(36) that if the controller \( u_p \) drives \( y_p \to r \) in (34) and \( u \) drives \( y_s \to 0 \) in (35), then \( y \to r \) as \( t \to \infty \). The benefit brought by the additive state decomposition is that the controller \( u \) will not affect the tracking and rejection performance since the primary system (34) is independent of the secondary system (35). Since the states \( x_p \) and \( x_s \) are unknown except for addition of them, namely \( x_{\text{new}} \), an observer is proposed to estimate \( x_p \) and \( x_s \).

**Remark 5.** Although the proposed additive state decomposition gives clear how to decompose a system, it still leaves a freedom to choose the primary system. By the additive state decomposition, the transformed system (26) can be also decomposed into a primary system

\[
\begin{align*}
\dot{x}_p &= Ax_p + b u_p, \quad x_p(0) = 0 \\
y_p &= c^T x_p + d_{\text{new}}
\end{align*}
\]  \hfill (37)

and the derived secondary system

\[
\begin{align*}
\dot{x}_s &= f \left(t, x_p + x_s, \hat{\theta}\right) - Ax_p + b \left(u_\xi - u_p\right), \quad x_s(0) = x_0 \\
y_s &= c^T x_s
\end{align*}
\]  \hfill (38)

where \( A \in \mathbb{R}^{n \times n} \) is an arbitrary constant matrix. Therefore, there is an infinite number of decompositions. The principle here is to derive the secondary system with an equilibrium point close to zero as far as possible. If so, the problem for the secondary system is only a stabilization problem, which is easier compared with a tracking problem. In (35), \( x_s = 0 \) is an equilibrium point of \( \dot{x}_s = f \left(t, x_p + x_s, \hat{\theta}\right) - f \left(t, x_p, \hat{\theta}\right) \), whereas in (38), \( x_s = 0 \) is not an equilibrium point of \( \dot{x}_s = f \left(t, x_p + x_s, \hat{\theta}\right) - Ax_p \). This is why we choose the primary system as (34) not (37). From the mention above, a good additive state decomposition often depends on a concrete problem.

**Theorem 3.** Under Assumption 1, suppose that an observer is designed to estimate state \( x_p \) and \( x_s \) in (34)-(35) as follows:

\[
\begin{align*}
\dot{\hat{x}}_p &= f \left(t, \hat{x}_p, \hat{\theta}\right) + b u_p, \quad \hat{x}_p(0) = 0 \\
\dot{\hat{x}}_s &= x_{\text{new}} - \hat{x}_p.
\end{align*}
\]
Then \( \hat{x}_p \equiv x_p \) and \( \hat{x}_s \equiv x_s \).

**Proof.** Similar to the proof of Theorem 2. \( \square \)

So far, we have transformed the original system to an uncertainty-free system, in which the new state and the new disturbance can be estimated directly. And then, decompose the transformed system into two independent subsystems in charge of corresponding subtasks. In the following, we are going to investigate the controller design with respect to the two decomposed subtasks respectively.

B. Problem for Tracking (including Rejection) Subtask

**Problem 1.** For (34), design a controller

\[
 u_p = u^r (t, x_p, r, d_{\text{new}})
\]

such that \( y_p \rightarrow r + B(\delta_r) \) as \( t \rightarrow \infty \), meanwhile keeping the state \( x_p \) bounded, where \( \delta_r \in \mathbb{R} \).

**Remark 6 (on Problem 1).** Since \( y_p = c^T x_p + d_{\text{new}} \), Problem 1 can be also considered to design \( u_p \) such that \( c^T x_p \rightarrow r - d_{\text{new}} \). Here, the difference between \( r \) and \( d_{\text{new}} \) should be clarified. The reference \( r \) is often known a priori, i.e., \( r(t + T) \) is known at the time \( t \), where \( T > 0 \). Moreover, its derivative is often given or can be obtained by analytic methods.

 Whereas, the new disturbance \( d_{\text{new}} \) only can be obtained at the time \( t \) whose derivative only can be obtained by numerical methods. By recalling (27), the new disturbance \( d_{\text{new}} \) finally depends on \( d \) and \( \xi \) as \( t \rightarrow \infty \). In practice, low frequency band is often dominant in the reference signal and disturbance. Therefore, from a practical point of view, we can also modify the tracking target, namely \( r - d_{\text{new}} \).

For example, let \( r - d_{\text{new}} \) pass a low-pass filter to obtain its major component. If the major component of \( r - d_{\text{new}} \) belongs to a fixed family of functions of time, Problem 1 can also be considered as an output regulation problem [21].

\[\begin{align*}
2B(\delta) & \triangleq \{x \in \mathbb{R} | |x| \leq \delta\}; \\
\text{the notation } x(t) \rightarrow B(\delta) & \text{ means } \min_{y \in B(\delta)} \|x(t) - y\| \rightarrow 0; \\
B(\delta_1) + B(\delta_2) & \triangleq \{x + y | x \in B(\delta_1), y \in B(\delta_2)\}
\end{align*}\]
C. Problem for Input-Realization Subtask

As shown in Fig. 1, the input realization subtask aims to make \( y_s \to 0 \). Let us investigate the secondary system (35). By Lemma 1, we have

\[
\|x_s(t)\| \leq \beta (\|x_s(t_0)\|, t - t_0) + \gamma \|b\| \sup_{t_0 \leq s \leq t} \|u(s) - u_p(s)\|.
\]

(41)

This implies that \( y_s \to \gamma \|b\| \|c\| B(\delta_s) \) as \( u \to u_p + B(\delta_s) \), where \( \delta_s \in \mathbb{R} \). It is noticed that \( u \) only can be realized by (21). Therefore, problem for input-realization subtask can be stated as follows:

**Problem 2.** Given a signal \( u_p \), design a controller \( v = v^s(t, u_p) \) for (21) such that \( u \to u_p + B(\delta_s) \) as \( t \to \infty \).

This is also a tracking problem but with a saturation constraint. Here we give a solution to the Problem 2. The main difficult is how to handle the saturation in (21). Here, additive state decomposition will be used again. Taking (21) as the original system, we choose the primary system as follows

\[
\begin{align*}
\dot{z}_p &= A_z z_p + b_z v \\
u_{zp} &= c^T_z z_p + d_z v
\end{align*}
\]

(42)

Then the secondary system is determined by the original system (21) and the primary system (42) with the rule (15), and we can obtain that

\[
\begin{align*}
\dot{z}_s &= A_z z_s + b_z (\sigma_a(v) - v) \\
u_{zs} &= c^T_z z_p + d_z (\sigma_a(v) - v)
\end{align*}
\]

(43)

According to (16), we have \( z = z_p + z_s \) and \( u = u_{zp} + u_{zs} \). The benefit brought by the additive state decomposition is that the controller saturation will not affect the primary system (42). Moreover, the controller \( v \) can be designed only based on the primary system (42), where the controller \( v \) uses the state \( z_p \) not \( z \). So, the strategy here is to design \( v = v^s(t, u_p) \) in (42) to drive \( u_{zp} \to u_p \) as \( t \to \infty \) and neglect the secondary system (43). Since \( v^s(t, u_p) \) is bounded, the state of the secondary system (43) will be bounded as well. If \( \sigma_a(v^s(t, u_p)) - v^s(t, u_p) \to 0 \) as \( t \to \infty \), then \( u_{zs} \to 0 \) as \( t \to \infty \). Consequently, \( u \to u_p \) as \( t \to \infty \). For (42), the transfer function from \( v \) to \( u_{zp} \) is \( u_{zp}(s) = C(s) v(s) \). If \( C(s) \) is designed to be minimum phase, an easy way is to design \( v \) to be

\[
v(s) = C^{-1}(s) u_p(s).
\]

(44)
The design will make the signal $\sigma_a(v)$ close to the idea one, meanwhile maintaining the signal $\sigma_a(v)$ smooth as far as possible. By recalling (18), it will make the effect of the unmodeled high-frequency gain and the time delay $\xi$ smaller.

D. Controller Integration

With the solutions of the two problems in hand, we can state

**Theorem 4.** Under Assumptions 1-2, suppose i) Problems 1-2 are solved; ii) the controller for system (1) (or (26)) is designed as

Observer:

$$
\dot{\hat{x}}_{\text{new}} = f(t, \hat{x}_{\text{new}}, \hat{\theta}) + bu, \hat{x}_{\text{new}}(0) = 0,
$$

$$
\dot{\hat{x}}_p = f(t, \hat{x}_p, \hat{\theta}) + bu_p, \hat{x}_p(0) = 0,
$$

$$
\hat{d}_{\text{new}} = y - c^T \dot{\hat{x}}_{\text{new}}
$$

(45)

Controller:

$$
u_p = u^r(t, \hat{x}_p, r, \hat{d}_{\text{new}}), v = v^s(t, u_p)
$$

$$
\dot{z} = A_z z + b_z \sigma_a(v), u = c_z^T z + d_z \sigma_a(v)
$$

(46)

Then the output of system (1) (or (26)) satisfies that $y \rightarrow r + B(\delta_r + \gamma \|b\| \|c\| \delta_s)$ as $t \rightarrow \infty$, meanwhile keeping all states bounded. In particular, if $\delta_r + \delta_s = 0$, then the output in system (1) (or (26)) satisfies that $y \rightarrow r$ as $t \rightarrow \infty$.

**Proof.** It is easy to follow the proof in Theorems 2-3 that the observer (45) will make

$$
\hat{x}_{\text{new}} \equiv x_{\text{new}}, \hat{d}_{\text{new}} \equiv d_{\text{new}}, \hat{x}_p \equiv x_p.
$$

(47)

Suppose that Problem 1 is solved. By (40) and (47), the controller $u_p = u^r(t, \hat{x}_p, r, \hat{d}_{\text{new}})$ can drive $y_p \rightarrow r + B(\delta_r)$ as $t \rightarrow \infty$ in (34). Suppose that Problem 2 is solved. By (47), the controller $v = v^s(t, u_p)$ can drive $u \rightarrow u_p + B(\delta_s)$ as $t \rightarrow \infty$ in (35). Further by (41), we have $y_s \rightarrow B(\gamma \|b\| \|c\| \delta_s)$. Since $y = y_p + y_s$, we have $y \rightarrow r + B(\delta_r + \gamma \|b\| \|c\| \delta_s)$. □

**Example 5 (Rohrs’ Example, Example 3 Continued).** According to (34), the primary system of linear system (32) can be rewritten as follows:

$$
\dot{x}_p = -(3 + \hat{\theta})x_p + 2u_p
$$

$$
y = x_p + d_{\text{new}}
$$

January 26, 2012 DRAFT
Design $u_p = \frac{1}{T}(2 + \dot{\theta})x_p + r + \dot{r} - d_{\text{new}} - \dot{d}_{\text{new}}$. Then the system above becomes $\dot{e}_p = -e_p$, where $e_p = y_p - r$. Therefore, $y_p \to r$ as $t \to \infty$. According to (44), $v$ is designed as $v^*(t, u_p) = 2\dot{u}_p + u_p$. Here $\dot{u}_p$ and $\dot{d}_{\text{new}}$ are approximated by $\dot{d}_{\text{new}} \approx L^{-1}(\frac{s}{0.1s+1}d_{\text{new}}(s))$ and $\dot{u}_p \approx L^{-1}(\frac{s}{0.1s+1}u_p(s))$, respectively. Suppose $\dot{\theta} = 0$ and given $r = 0.5$ and $r = 0.5\sin(0.2t)$, respectively. Driven by the resulting controller (46), the simulation result is shown in Fig.2.

Example 6 (Nonlinear Example 4 Continued). According to (34), the primary system of nonlinear system (33) can be rewritten as follows:

\[ \dot{x}_p = -x_p - (1 + \dot{\theta})x_p^3 + u_p, \quad x_p(0) = 0 \]

\[ y_p = x_p + d_{\text{new}}. \]

Design $u_p = (1 + \dot{\theta})x_p^3 + r + \dot{r} - d_{\text{new}} - d_{\text{new}}$. Then the system above becomes $\dot{e}_p = -e_p$, where $e_p = y_p - r$. Therefore, $y_p \to r$ as $t \to \infty$. According to (44), $v^*(t, u_p)$ is designed as $v^*(t, u_p) = 2\dot{u}_p + u_p$. Here the derivative of $u_p$ and $d_{\text{new}}$ are approximated by $\dot{d}_{\text{new}} \approx L^{-1}(\frac{s}{0.1s+1}d_{\text{new}}(s))$ and $\dot{u}_p \approx L^{-1}(\frac{s}{0.1s+1}u_p(s))$, respectively. Suppose $\dot{\theta} = 0$ and given $r = 0.5$ and $r = 0.5\sin(0.2t)$, respectively. Driven by the resulting controller (46), the simulation result is shown in Fig.3.

Remark 7. The derivative of $d_{\text{new}}$ and $u_p$ can be also obtained by differentiator technique [23], [24]. It is interesting to note that $\dot{\theta}$ is different from $\theta$, but $y \to r$ with a good tracking accuracy. This is one major advantage of this proposed control scheme. Moreover, all the unknown parts such as $\theta$, $d$ and the effect of $H(s)e^{-st}$ are treated as a lumped disturbance $d_{\text{new}}$. This can explain why the proposed scheme can handle many uncertainties.

V. TWO-CART EXAMPLE

The two-cart mass-spring-damper example was originally proposed as a benchmark problem for robust control design [12], [22]. Next, we will revisit the two-cart example by the proposed control scheme.

The two-cart system is shown in Fig.4. The states $x_1(t)$ and $x_2(t)$ represent the absolute position of the two carts, whose masses are $m_1$ and $m_2$ respectively; $k_1, k_2$ are the spring constants, and $b_1, b_2$ are the damping coefficients; $d(t)$ is a disturbance force acting on the mass $m_2$; $u(t)$ is the control force subject to an unmodeled high-frequency gain and a time delay, which acts upon the mass $m_1$. The parameter $m_1 = 1$ is known, whereas the following parameters $m_2 = 2$, $k_1 = 0.8$, $k_2 = 0.5$, $b_1 = 1.3$, $b_2 = 0.9$ are assumed unknown. The unmodeled high-frequency gain and a time delay is assumed to be $H(s)e^{-ts} = \frac{229}{s^2 + 30s + 229}e^{-0.1s}$. 

January 26, 2012 DRAFT
The disturbance force $\zeta(t)$ is modeled as a first-order (colored) stochastic process generated by driving a low-pass filter with continuous-time white noise $\varepsilon(s)$, with zero-mean and unit intensity, i.e. $\Xi = 1$, as follows $\zeta(s) = \frac{0.1}{s+0.1} \varepsilon(s)$.

The overall state-space representation is formulated into (1) as follows:

$$\dot{x} = A(\theta)x + bu + d$$
$$y = c^T x$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, A(\theta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_2}{m_2} & \frac{k_1+k_2}{m_2} & \frac{b_2}{m_2} & \frac{b_1+b_2}{m_2} \end{bmatrix}, d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{m_2} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \theta = \begin{bmatrix} m_1 & m_2 & k_1 & k_2 & b_1 & b_2 \end{bmatrix}.$$
Then, according to (26), the uncertainty-free transformed system of (48) is
\[
\dot{x}_{new} = A(\hat{\theta})x_{new} + bu, \ x_{new}(0) = 0
\]
\[
y = c^T x_{new} + d_{new}.
\]
where
\[
d_{new} = c^T e^{A(\theta)t}x_0 + \int_0^t c^T e^{A(\theta)(t-s)}[d(s) + \xi(s) + A(\theta) x_{new}(s) - A(\hat{\theta}) x_{new}(s)]ds.
\]

According to (34), the primary system of (49) can be rewritten as follows:
\[
\dot{x}_p = A(\hat{\theta})x_p + bu_p, \ x_p(0) = 0
\]
\[
y_p = c^T x_p + d_{new}.
\]
(50)

The transfer function from \(u_p\) to \(y_p\) in (50) is \(G_{yu}(s)\), which is a minimum phase. Thus, \(u_p\) can be designed as \(u_p(s) = G_{yu}^{-1}(s)(r - d_{new}(s))\), which can drive \(y_p \rightarrow r\). The Problem 1 is solved. Furthermore, according to (44), redefined input \(v\) is designed as \(v^s(t, u_p) = \mathcal{L}^{-1}\left[C^{-1}(s) G_{yu}^{-1}(s)(r - d_{new}(s))\right]\). To realize the control, \(v^s(t, u_p)\) is approximated to be \(v^s(t, u_p) = \mathcal{L}^{-1}\left(Q(s) C^{-1}(s) G_{yu}^{-1}(s)(r - d_{new}(s))\right)\).

(51)

where \(Q(s)\) is a fifth-order low-pass filter to make the compensator physically realizable (the order of denominator is greater than or equal to that of numerator). For simplicity, \(Q(s)\) is chosen to be \(Q(s) = \prod_{k=1}^{5} (\frac{1}{10k} + 1)\) here. The Problem 2 is solved. Therefore, according to (45)-(46), the controller for the two-cart system is designed as follows:
\[
\dot{x}_{new} = A(\hat{\theta})x_{new} + bu, \ x_{new}(0) = 0, \ d_{new} = y - c^T \dot{x}_{new}
\]
\[
\dot{z} = -0.5z + 0.5\sigma_a(v^s(t, u_p)), \ u = z
\]
(52)

where \(v^s(t, u_p)\) is given by (51) and \(\sigma\) is chosen to be 1 here.

To show the effectiveness, the proposed controller (52) is applied to three cases:

Case 1: \(\hat{\theta} = \theta\)

Case 2: \(\hat{\theta} = [1 \ 1 \ 1 \ 0.9 \ 1.5 \ 1]^T\)

Case 3: \(\theta = \hat{\theta} = [1 \ 1 \ 1 \ 0.9 \ 1.5 \ 1]^T\).

Case 1 implies the parameters are known exactly. Case 2 implies the parameters are unknown. While, Case 3 implies the parameters are changed to be a specified one. The simulations are shown in Figs. 5-7. The proposed controller achieves a good tracking accuracy. Moreover, it is seen that the response in Cases 2-3 is faster than that in Case 1. And, the tracking accuracy in Cases 1,3 is better than that in Case 2. So, Case 2 is a tradeoff between Case 1 and Case 3.
Remark 8. The simulations show that the proposed controller can handle the case that the estimate parameters are different from the true parameters. Moreover, the response is similar to that of the model with the estimate parameters. This implies that the proposed controller in fact achieves the results similar to model reference adaptive control. However, unlike model reference adaptive control, unknown parameters are not estimated and changed directly.

Remark 9. If the considered system is parameterized but there exist many uncertain parameters, then an adaptive control often needs corresponding number of estimators, i.e., corresponding number of integrators. This will cause parameters converging to true values with very slow rate or cannot converge to true values if without persistent excitation. Whereas, in the proposed control, five uncertain parameters and disturbance are lumped into the disturbance $d_{\text{new}}$, which can be estimated directly.

VI. CONCLUSIONS

Output tracking control for a class of uncertain systems subject to unmodeled dynamics and time delay at input is considered. Our main contribution lies on the presentation of a new decomposition scheme, named additive state decomposition, which not only transforms the uncertain system to be an uncertainty-free system but also simplifies the controller design. The proposed control scheme is with the following two salient features. (i) The proposed control scheme can handle mismatching uncertainties and mismatching disturbance. Moreover, it can achieve a good tracking performance without exact parameters. (ii) The proposed control scheme has considered many uncertainties. In the presence of these uncertainties, the closed-loop system is still stable when incorporating the proposed controller. Three benchmark examples are given to show the effectiveness of the proposed control scheme.

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Fig. 2. Output of the Rohrs’ example system

Fig. 3. Output of the nonlinear system
Fig. 4. The two-cart system

Fig. 5. Output of the two-cart system in Case 1
Fig. 6. Output of the two-cart system in Case 2

Fig. 7. Output of the two-cart system in Case 3