ON THE BETTI NUMBERS OF A LOOP SPACE

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Abstract. Let $A$ be a special homotopy $G$-algebra over a commutative unital ring $k$ such that both $H(A)$ and $\text{Tor}_i^A(k,k)$ are finitely generated $k$-modules for all $i$, and let $\tau_i(A)$ be the cardinality of a minimal generating set for the $k$-module $\text{Tor}_i^A(k,k)$. Then the set $\{\tau_i(A)\}$ is unbounded if and only if $\tilde{H}(A)$ has two or more algebra generators. When $A = C^*(X;k)$ is the simplicial cochain complex of a simply connected finite CW-complex $X$, there is a similar statement for the "Betti numbers" of the loop space $\Omega X$. This unifies existing proofs over a field $k$ of zero or positive characteristic.

1. Introduction

Let $Y$ be a topological space, let $k$ be a commutative ring with identity, and assume that the $i$th-cohomology group $H^i(Y;k)$ of $Y$ is finitely generated as a $k$-module. We refer to the cardinality of a minimal generating set of $H^i(Y;k)$, denoted by $\beta_i(Y)$, as the generalized $i$th-Betti number of $Y$.

Theorem 1. Let $X$ be a simply connected space. If $H^*(X;k)$ is finitely generated as a $k$-module and $H^*(\Omega X;k)$ has finite type, then the set of generalized $i$th-Betti numbers $\{\beta_i(\Omega X;k)\}$ is unbounded if and only if $\tilde{H}^*(X;k)$ has at least two algebra generators.

Theorem 1 was proved by Sullivan [11] over fields of characteristic zero and by McCleary [8] over fields of positive characteristic. However, Theorem 1 is a consequence of the following more general algebraic fact: Let $A' = \{A'_i\}, i \geq 0$, with $A'_0 = \mathbb{Z}$, $A'_1 = 0$, be a torsion free graded abelian group endowed with a homotopy $G$-algebra (hga) structure. Then for $A = A' \otimes_{\mathbb{Z}} k$ we have the following theorem whose proof appears in Section 4:

Theorem 2. Assume that $H^*(A)$ is finitely generated as a $k$-module and that $\text{Tor}_i^A(k,k)$ has finite type. Let $\tau_i(A)$ denote the cardinality of a minimal generating set of $\text{Tor}_i^A(k,k)$. Then the set $\{\tau_i(A)\}$ is unbounded if and only if $\tilde{H}^*(X;k)$ has at least two algebra generators.

Let $C^*(X;k) = C^*(\text{Sing}^1 X;k)/C^{>0}(\text{Sing} X;k)$ in which $\text{Sing}^1 X \subset \text{Sing} X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard $n$-simplex $\Delta^n$ to the base point $x$ of $X$. To deduce Theorem 2 from Theorem 1 set $A = C^*(X;k)$, and apply Proposition 2 below together with the filtered hga model $(RH(A), d_h) \rightarrow A$ of $A$ (a special case of the filtered Hirsch algebra [9]). Let $BA$ denote the bar construction of $A$. When $\tilde{H}(A)$ has at least

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two algebra generators, we construct two infinite sequences in the filtered model
and take all possible \(\sim_1\)-products of their components to detect a submodule of \(H^*(BA)\) at least as large as the polynomial algebra \(\mathbb{k}[x, y]\).

Each of the sequences mentioned above can be thought of as generalizations of
an infinite sequence (\(\infty\)-implications of its first component) introduced by Browder [1]. Indeed, this work arose after writing down these special sequences in the hga
resolution of a commutative graded algebra (cga) over the integers via formulas
(3.2)–(3.4) below, at which point we realized that their construction mimics that
of Massey symmetric products defined by Kraines [7] (see also [9]). In general, a
sequence formed from Massey symmetric products is closely related to the one ob-
tained from \(A_\infty\)-operations in an \(A_\infty\)-algebra defined by Stasheff [10] by restricting
to the same variables in question. When a differential graded algebra (dga) \(A\) is
free as a \(k\)-module, the sequence of \(A_\infty\)-operations on the homology \(H(A)\) was
constructed by Kadeishvili [5].

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2. Some preliminaries and conventions

We adopt the notations and terminology of [9]. We fix a ground ring \(k\) with
identity, a primary example of which is the integers \(\mathbb{Z}\). Let \(\mathbb{Z}_k \subset \mathbb{Z}\) be the subset
defined by
\[
\mathbb{Z}_k = \{ \lambda \in \mathbb{Z} | \lambda \cdot k \to k, \lambda \to \lambda k, \text{ is injective} \}.
\]
Let \(\mu \in \mathbb{Z} \setminus \mathbb{Z}_k\) denote the smallest integer such that \(\mu k = 0\) for all \(\kappa \in k\). Thus if
\(\mu = 0\), \(\mathbb{Z}_k = \mathbb{Z} \setminus 0\) (e.g. \(k\) is a field of characteristic zero).

A (positively) graded algebra \(A\) is 1-reduced if \(A^0 = k\) and \(A^1 = 0\). For a general
definition of an homotopy Gerstenhaber algebra (hga) \((A, d, \{E_{p,q}\}_{p \geq 0, q = 0, 1}\) see
[3, 4, 6]. The defining identities for an hga are the following: Given \(k \geq 1,
\begin{align*}
dE_{k,1}(a_1, \ldots, a_k; b) &= \sum_{i=1}^k (-1)^{e_{i-1}} E_{k,1}(a_1, \ldots, da_i, \ldots, a_k; b) \\
&\quad + \sum_{i=1}^{k-1} (-1)^{\sum_{j=1}^i \epsilon_j} E_{k,1}(a_1, \ldots, a_i b, \ldots, a_k; b) \\
&\quad + \sum_{i=1}^{k-1} (-1)^{\sum_{j=1}^i \epsilon_j + |a_i|} E_{k,1}(a_1, \ldots, a_{i+1}, \ldots, a_k; b) \\
&\quad + (-1)^{|a_i|} a_i^{-1} E_{k-1,1}(a_2, \ldots, a_k; b),
\end{align*}
\tag{2.1}
\]
and
\begin{align*}
E_{k,1}(a_1, \ldots, a_k; b \cdot c) &= \sum_{i=0}^k (-1)^{|b|} (a_i^b + c_i^b) E_{i,1}(a_1, \ldots, a_i; b) \cdot E_{k-i,1}(a_{i+1}, \ldots, a_k; c) \\
\end{align*}
\tag{2.2}
and
\begin{align*}
\sum_{\substack{k_1 + \cdots + k_p = k \\
1 \leq p \leq k + 1}} (-1)^{\epsilon} E_{p,1} \left(E_{k_1,j_1}(a_1, \ldots, a_{k_1}; b_1'), \ldots, E_{k_p,j_p}(a_{k_{p-1}+1}, \ldots, a_k; b_p'); c\right) \\
&= E_{k,1} (a_1, \ldots, a_k; E_{k-1,1}(b_1, \ldots, b_k; c)),
\end{align*}
\tag{2.3}
for \(b_i' \in \{1, b_1, \ldots, b_k\}, \epsilon = \sum_{i=1}^p (|b_i'| + 1)(\epsilon_{k_i}^a + \epsilon_{k_i}^b), b_i' \neq 1,
\begin{align*}
\epsilon_{i}^a &= |a_1| + \cdots + |a_i| + i.
\end{align*}
A morphism \( f : A \to A' \) of hga’s is a dga map \( f \) commuting with all \( E_{k,1} \).

**Remark 1.** Note that we do not use axiom (2.3) in the sequel.

Below we review the notion of an hga resolution of a cga as a special Hirsch algebra (the existence of such a resolution is proved in [9]). Given a cga \( H \), its hga resolution is a multiplicative resolution

\[
\rho : (R^*H^*, d) \to H^*, \quad RH = T(V), \quad V = (\mathcal{V}),
\]

endowed with an hga structure

\[
E_{k,1} : RH \otimes k \otimes RH \to RH, \quad k \geq 1,
\]

together with a decomposition of \( V \) such that \( V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*} \), where \( \mathcal{E}^{*,*} = \{ \mathcal{E}_{p,q}^{*,*} \} \) is distinguished by an isomorphism of modules

\[
E_{k,1} : \otimes_{r=1}^k R^*H^k \otimes V^{j,t} \cong \mathcal{E}_{k,1}^s-k-t \subset V^{k-s,t}, \quad (s,t) = \left( \sum_{r=1}^k i_r + j, \sum_{r=1}^k k_r + \ell \right).
\]

Furthermore, if \( H \) is a \( \mathbb{Z} \)-algebra, its hga resolution \( (RH, d) \) is automatically endowed with two operations \( \cup \) and \( \ast \). The first operation \( \cup \) appears because each cocycle \( a \sim a \in \mathcal{E}_{1,1} \cap R^{-1}H^2 \), where \( a \in R^0H^2 \), is killed by some element in \( R^{-2}H^2 \), denoted by \( a \cup_{\mathbb{Z}} a \). The second operation arises from the non-commutativity of \( \sim \)-product in the usual way, and satisfies Steenrod’s formula for the \( \ast \)-cochain operation. These two operations are related to each other by the initial relations \( a \sim a \sim a = 2a \cup_{\mathbb{Z}} a \), and \( a \sim a \sim b = a \cup_{\mathbb{Z}} b \), \( a \neq b \in U \) with \( \langle U \rangle = U \). Note also that \( a \sim b = a \cup b = 0 \) for \( a \in U \) of odd degree. In general, \( U = T \oplus N \), with an element of \( T \) given by \( a_1 \cup_{\mathbb{Z}} \cdots \cup_{\mathbb{Z}} a_n, a_1 \in U, n \geq 2 \). The action of the resolution differential \( d \) on elements of \( T \) such that \( da = 0 \) is

\[
d(a_1 \cup_{\mathbb{Z}} \cdots \cup_{\mathbb{Z}} a_n)
= \sum_{i,j} \frac{(-1)^{a_{i1} + \cdots + a_{ik} + a_{j1} + \cdots + a_{jk}}(a_{i1} \cup_{\mathbb{Z}} a_{j1} \cup_{\mathbb{Z}} \cdots \cup_{\mathbb{Z}} a_{ik}, a_{j1} \cup_{\mathbb{Z}} \cdots \cup_{\mathbb{Z}} a_{jk})}{i < j \}
\]

where we sum over all unshuffles \( i:j = (i_1 < \cdots < i_k ; j_1 < \cdots < j_l) \) of \( u \) with \( (a_{i1}, \ldots, a_{ik}) = (a_{j1}, \ldots, a_{jl}) \) if and only if \( i = i' \) and \( j = j' \) denotes \( E_{k,1} \). In particular, for \( a_1 = \cdots = a_n = a = a^{i,j} \) and \( n \geq 2 \) we get \( da^{i,j,n} = \sum_{k+\ell = n} a^{i,j,k} \cup_{\mathbb{Z}} a^{i,j,\ell}, k, \ell \geq 1 \). And in general \( d(a \sim b) = nd(a \cup b), n \geq 1 \).

An hga resolution \( (RH, d) \) is minimal if

\[
d(U) \subset \mathcal{E} + \mathcal{D} + \kappa \cdot V
\]

where \( \mathcal{D}^{*,*} \subset R^*H^* \) denotes the submodule of decomposables \( RH^+ \). \( RH^+ \) and \( \kappa \in k \) is non-invertible; For example, \( \kappa \in \mathbb{Z} \setminus \{-1, 1\} \) when \( k = \mathbb{Z} \) and \( \kappa = 0 \) when \( k \) is a field.

Let \( K = \{ K^j \} \) with \( K^j = \{ a \in V^{-1,j} \mid da = \lambda b, \lambda \neq \pm 1, b \in V^{0,j} \} \). Note that a general form of a relation in (minimal) \( (RH, d) \) starting by variables \( v_i \in K \cup V^{0,*} \) is

\[
du = \sum_{s \geq 1} \lambda_s P_s(v_1, \ldots, v_r) + \lambda v, \quad \lambda \neq \pm 1, \lambda_s \neq 0, r_s \geq 1,
\]

\[
u \in \bigcup_{i \geq 1} V^{-i,*}, \quad v \in \bigcup_{i \geq 1} V^{-i,*} \setminus K,
\]

where \( P_s \) is a polynomial in \( \{ v_i \} \).
where \( P_s(v_1, \ldots, v_r) \) is a monomial in \( D^{*+} \subset R^*H^* \).

Let \( A \) be an hga and let \( \rho : (RH, d) \to H \) be an hga resolution. A filtered hga model of \( A \) is an hga quasi-isomorphism

\[
f : (RH, d_h) \to (A, d_A)
\]

in which

\[
d_h = d + h, \quad h = h^2 + \cdots + h^r + \cdots, \quad h^r : R^p H^q \to R^{p+r} H^{q-r+1}.
\]

The equality \( d_h^2 = 0 \) implies the sequence of equalities

\[
dh^2 + h^2d = 0, \quad dh^3 + h^3d = -h^2h^2, \quad dh^4 + h^4d = -h^2h^3, \quad\ldots,
\]

and \( h \) is referred to as a perturbation of \( d \). The map \( h^r|_{R^{r-1}H} : R^{r-1}H \to R^rH \), \( r \geq 2 \), denoted by \( h^{r+} \), is referred to as the transgressive component of \( h \). The fact that the perturbation \( h \) acts as a derivation on elements of \( E \) implies \( h^r|_E = 0 \). For the existence of the filtered model see [9].

In the sequel, \( A' \) denotes a 1-reduced torsion free hga over \( \mathbb{Z} \), while \( A \) denotes the tensor product hga \( A' \otimes_{\mathbb{Z}} k \). Denote also \( H = H^*(A') \) and \( H_k = H^*(A) \). Assume \((RH, d)\) is minimal and let \( RH_k = RH \otimes_{\mathbb{Z}} k \); in particular, \( RH_k = T(V_k) \) for \( V_k = V \otimes_{\mathbb{Z}} k \). When \( k \) is a field of characteristic zero, \( \rho \otimes 1 : RH_k \to H \otimes_{\mathbb{Z}} k = H_k \) is an hga resolution of \( H_k \), which is not minimal when \( \text{Tor } H \neq 0 \). In general, given a filtered model \((RH, d_h)\) of \( A' \), we obtain an hga model

\[
f \otimes 1 : (RH_k, d_h \otimes 1) \to (A, d_A).
\]

for \((A, d_A)\). Denote \( \tilde{V}_k = s^{-1}(V_k^{>0}) \oplus k \) and define the differential \( \tilde{d}_h \) on \( \tilde{V}_k \) by the restriction of \( d + h \) to \( \tilde{V}_k \) and obtain the cochain complex \((\tilde{V}_k, \tilde{d}_h)\).

Since the map \( f \otimes 1 \) is in particular a homology isomorphism (by the universal coefficient theorem), the following two propositions follow immediately from the results in [2] and the standard isomorphisms \( H^*(BA, d_{BA}) \approx \text{Tor}^A(k, k) \) and \( H^*(BC^*(X; k), d_{BC}) \approx H^*(\Omega X; k) \).

**Proposition 1.** There are isomorphisms

\[
H^*(\tilde{V}_k, \tilde{d}_h) \approx H^*(B(RH_k), d_{B(RH_k)}) \approx H^*(BA, d_{BA}) \approx \text{Tor}^A(k, k).
\]

And for \( A = C^*(X; k) \) we obtain:

**Proposition 2.** There are isomorphisms

\[
H^*(\tilde{V}_k, \tilde{d}_h) \approx H^*(BC^*(X; k), d_{BC}) \approx H^*(\Omega X; k).
\]

Given \((RH, d)\) and \( x, c \in V \) with \( dx, dc \in D + \lambda V \), \( \lambda \neq 1 \), let \( \eta_{x,c} \) denote an element of \( E_{>1,1} \) such that

\[
x \rightsquigarrow_1 c := \eta_{x,c} + x \rightsquigarrow_1 c
\]

satisfies \( d(x \rightsquigarrow_1 c) \in D + \lambda V \). For example, if \( dx \in \lambda V \), then \( \eta_{x,c} = 0 \), and if \( dx = \sum a_i b_i + \lambda v \) with \( da_i, db_i \in \lambda V \), then \( \eta_{x,c} = \sum (-1)^{|a_i|} E_{2,1}(a_i, b_i; c) \). In general, \( \eta_{x,c} \) can be found as follows: Let \( j : B(RH) \to \overline{R^*H} \to \overline{V} \) be the canonical projection used by the proof of the first isomorphism in Proposition 1 and choose \( y \in B(RH) \) so that \( j(y) = \tilde{x} \) and \( j \mu_E(y; c) = \tilde{\eta}_{x,c} + \overline{x \rightsquigarrow_1 c} \), where the product \( \mu_E : B(RH) \otimes B(RH) \to B(RH) \) is determined by the hga structure on \( RH \).

The following proposition is simple but useful. Let \( D_k \subset RH \) be a subset defined by \( D_k = D \) for \( \mu = 0 \) and

\[
D_k = \{ u + \lambda v | u \in D, v \in V, \lambda \text{ is divisible by } \mu \} \quad \text{for } \mu \geq 2.
\]
Definition 1. An element \( x \in V \) with \( d_h x \in D + \lambda V, \lambda \neq 1 \), is \( \lambda \)-homologous to zero, denoted by \([x]_\lambda = 0\), if there are \( u, v \in V \) and \( z \in D \) such that
\[
d_h u = x + z + \lambda v;
\]
x is weakly homologous to zero when \( v = 0 \) above.

Proposition 3. Let \( c \in V \) and \( d_h c \in D_k \). If \( d_h c \) has a summand component \( ab \in D \) such that \( a, b \in V, d_h a, d_h b \in D_k \), both \( a \) and \( b \) are not weakly homologous to zero, then \( c \) is also not weakly homologous to zero.

Proof. The proof is straightforward using the equality \( d_h^2 = 0 \).

In particular, for \( k = \mathbb{Z} \), under hypotheses of the proposition if \([a], [b] \neq 0\), then \([c] \neq 0 \) in \( H^*(V, d_h) \).

Note that over a field \( k \), Proposition 3 reflects the obvious fact that \( x \in H^*(\Omega X; k) \) is non-zero whenever some \( x' \otimes x'' \neq 0 \) in \( \Delta x = \sum x' \otimes x'' \).

3. Formal \( \infty \)-implication sequences

Let \( x \) be an element of a Hopf algebra over a finite field. In [1], W. Browder introduced the notion of \( \infty \)-implications (of an infinite sequence) associated with \( x \) in the Hopf algebra. The following can be thought of as a generalization of this: Let \( x^{-1} p \) denote the (right most) \( p \)-th-power of \( x \) with respect to \( \sim_1 \)-product with the convention that \( x^{-1} 1 = x \).

Definition 2. Let \( x \in V^k, k \geq 2, d_h x \in D_k \). A sequence \( x = \{x(i)\}_{i \geq 0} \) is a formal \( \infty \)-implication sequence (f.i.s.) of \( x \) if

(i) \( x(0) = x, x(i) \in V^{(i+1)k-i}, \) and \( x(i) \) is not \( \mu \)-homologous to zero for all \( i \);
(ii) Either \( x(i) = x^{-1(i+1)} \) or \( x(i) \) is resolved from the following relation in the filtered hga model \((RH, d_h)\):
\[
d_h b(i) = x^{-1(i+1)} + z(i) + \mu' x(i), \quad b(i) \in V, z(i) \in D, \mu' \text{ is divisible by } \mu.
\]

We are interested in the existence of an f.i.s. for an odd dimensional \( x \in V \).

Proposition 4. Let \( x \in V \) be of odd degree with \( d_h x \in D_k \) such that \( x \) is not \( \mu \)-homologous to zero. For \( \mu \geq 2 \), assume, in addition, there is no relation \( d_h u = \mu x \mod D \), some \( u \in V \). Then \( x \) has an f.i.s. \( x = \{x(i)\}_{i \geq 0} \).

Proof. Suppose we have constructed \( x(i) \) for \( 0 \leq i < n \). If \( x^{-1(n+1)} \) is not \( \mu \)-homologous to zero, set \( x(n) = x^{-1(n+1)} \); otherwise, there is the relation \( d_h u = x^{-1(n+1)} + z + \mu' v \) for some \( u, v \in V, z \in D \) and \( \mu' \) divisible by \( \mu \). Using (2.1)–(2.2), one can easily establish the fact that \( dx^{-1(n+1)} \) contains a summand component of the form \(-\sum k+\ell=n+1 \binom{n+1}{k} x^{-1k} x^{-1\ell}, k, \ell \geq 1 \). We have that \( v \neq 0 \) in the aforementioned relation since Proposition 3 (applied for \( c = x^{-1(n+1)} \) and \( a \cdot b = -\binom{n+1}{k} x^{-1k} x^{-1\ell}, \) some \( k \)). Clearly, \( d_h v = -\frac{1}{\mu'} d_h (x^{-1(n+1)} + z) \in D \); Assuming \( \mu' \) to be maximal \( v \) is not \( \lambda \)-homologous to zero. Set \( x(n) = v \) and \( b(n) = u, z(n) = z \) to obtain (3.1) for \( i = n \).
Thus, for $\mu = 0$ (when $k$ is a field of characteristic zero, for example) $x = \{x^{-1(n+1)}\}_{n \geq 0}$.

**Remark 2.** 1. The restriction on $x$ in Proposition 4 that no relation $dk(u) = \mu x$ mod $D$ exists is essential. A counterexample is provided by the exceptional group $F_4$: Let $A = C^* (BF_4; Z_3)$ be the cochain complex of the classifying space $BF_4$. Then we have the relation $du = 3x$ in $(RH, d)$ corresponding to the Bockstein cohomology homomorphism $\delta x = x_9$ on $H^*(BF_4; Z_3)$ (in the notation of [13]), but the element $x(2)$ does not exist (see [9] for more details).

2. Note that if $du = \mu x$ in Proposition 4 but $[u][x] \neq 0 \in H_k$, then one can modify the proof of the proposition to show that $x$ again has an f.i.s. $\{x(i)\}_{i \geq 0}$. Note that in the above example we just have $[u][x] = 0 \in H_{Z_3} = H^*(BF_4; Z_3)$.

3. The existence of $\infty$-implications of $x$ in [1] uses both the $\sim$-product and the Pontrjagin product in the loop space (co)homology. In our case each component of the sequence $x$ is determined by item (ii) of Definition 2 in which the first case modifies the proof of the proposition to show that $x$ again has an f.i.s. $\{x(i)\}_{i \geq 0}$. In particular, primitivity of $x$ required in [1] is not issue for the existence of $\infty$-implications of $x$.

In certain cases, a given odd dimensional $b \in V$ rises to an infinite sequence $b = \{b_i\}_{i \geq 0}$ with $b = b_0$ in the hga resolution $(RH, d)$. These sequences are built by explicit formulas and include also the case $du = \lambda b$, i.e., when the hypothesis of Proposition 4 formally fails (see, for example, Case I of the proof of Proposition 5 below). Namely, we have the following cases:

(i) For $b \in V^{b,0}$ and $[b]^2 = 0 \in H$ (i.e., there exists $b_1 \in V^{-1,1}$ with $db_1 = b^2$; e.g. $b_1 = ab + \frac{\lambda-1}{2} b \sim b$ for $da = \lambda b$ with $\lambda$ odd, some $a \in V^{-1,1}$), $b = \{b_i\}_{i \geq 0}$ is given by

\[
db = \sum_{i+j=n-1} b_i b_j
\]

and satisfies the following relation with $c_i \in V$

\[
dc = -(-1)^n ((n+1)b_n + b_0 \sim b_{n-1}) + \sum_{i+j=n-1} (-1)^i (c_i b_i - b_i c_j),
\]

(iii) For $b \in V^{b,0}$ and $[b]^2 \neq 0 \in H$ (and $b_1 = b \sim b$), $b = \{b_i\}_{i \geq 0}$ is given by

\[
db = \sum_{i+j=2k-1} b_i b_j, \quad \db_{2k+1} = \sum_{i+j=k} (2b_{2i} b_{2j} + b_{2i-1} b_{2j+1}),
\]

and satisfies the following relation with $c_i \in V$ (below $c_1 = 0$)

\[
dc = -(2k+1)b_{2k} - b_0 \sim b_{2k-1} + \sum_{i+j=k} 2 (c_{2j-1} b_{2i} - b_{2i} c_{2j-1})
\]

\[
- \sum_{i+j=k} (c_{2j} b_{2i-1} - b_{2i-1} c_{2j}),
\]

\[
dc_{2k+1} = (k+1)b_{2k+1} + b_0 \sim b_{2k} + \sum_{i+j=2k} (-1)^j (c_i b_i - b_i c_j),
\]

$\forall k \geq 1$.
(iii) For \( b \in V^{-1,*} \) and \( db = \mu c, \mu \geq 2, c \in V^{0,*} \) (below \( \omega_0 := c \)), \( b = \{b_i\}_{i \geq 0} \) is given by
\[
(3.4) \quad db_n = \sum_{i+j=n-1} b_i b_j + \mu c_n,
\]
\[
e_c = -\omega_0 \sim_1 b_{n-1} - \sum_{i+j=n-1 \atop i \geq 1; j \geq 0} (-1)^i \omega_i \sim_1 b_j - (-1)^n \omega_n, \quad n \geq 1
\]
and satisfies the following relation with \( c_i \in V \)
\[
dc_1 = 2b_1 + b_0 \sim_1 b_0 + \mu \omega_0 \cup_2 b_0,
\]
\[
dc_n = -(-1)^n ((n + 1)b_n + b_0 \sim_1 b_{n-1}) + \sum_{i+j=n-1} (-1)^i (c_j b_i - b_i c_j) + \mu a_n,
\]
\[
a_n = \sum_{i+j=n-2} (-1)^i ((\omega_i \cup_2 b_0) \sim_1 b_j + \omega_i \sim_1 c_{j+1}) + \omega_n \cup_1 b_0,
\]
\[
d\omega_k = \sum_{i+j=k-1} \mu \omega_i \sim_1 \omega_j, \quad \omega_k = \mu^k \omega_0 \cup_{k+1} ^2, \quad k \geq 1, \quad n \geq 2.
\]

For example, in view of Proposition 2, the formulas above are enough to calculate the loop space cohomology algebra with coefficients in \( k \) for Moore spaces, i.e., the CW-complexes obtained by attaching an \((n+1)\)-cell to the \( n \)-sphere \( S^n \) by a map \( S^n \rightarrow S^n \) of degree \( \mu \).

3.1. Odd dimensional element \( l(a) \). Given \( m \geq 2 \), let \( H(A) \) be finitely generated as a \( k \)-module with \( H^i(A) = 0 \) for \( i > m \). Let \( Z_k \) be the subset of \( RH \) defined by
\[
Z_k = Z_k' + Z_k'' + D_k,
\]
\[
Z_k' = \{ v \in V \mid du = \lambda v, \quad u \in V, \quad \lambda \in Z_k \}
\]
and
\[
Z_k'' = \{ v \in V \mid v = \lambda u, \quad u \in V, \quad \lambda \in Z \setminus Z_k \}.
\]
Given \( x \in V \) with \( d_h x = w \in Z_k, w = w' + w'' + z \), define
\[
\check{x} = \frac{l.c.m. (\lambda'; \mu)}{\lambda''} (\lambda' x - u), \quad du = \lambda' w', \quad w'' = \lambda'' v'',
\]
to obtain \( d_h \check{x} \in D_k \).

Regarding \((2.5)\), define also the following subsets \( K^*_\mu, K^*_0 \subset V^{-1,*} \) with \( K^*_\mu \subset K^* \) as
\[
K^*_\mu = \{ a \in K \mid \lambda \text{ is divisible by } \mu \},
\]
\[
K_0 = \{ u \in V^{-1,*} \setminus E \mid du \in D^{0,*} \},
\]
and assign to a given even dimensional element \( a \in V^{0,*} \cup K^*_\mu \) an odd dimensional element \( l(a) \in V \) with \( dl(a) \in D_k \) as follows. If \( a \in V^{0,*} \), let \( l(a) \in K_0 \) be an element such that \( dl(a) = a^h \), where \( k \geq 2 \) is chosen to be the smallest. If \( a \in K^*_\mu \) with \( da = \lambda b \) consider the relation
\[
(3.5) \quad du_1 = -a^2 + \lambda v_1, \quad dv_1 = \frac{1}{\lambda} d(a^2), \quad u_1 \in V^{-3,*}, \quad v_1 \in V^{-2,*},
\]
and the perturbation \( hu_1 = h^2u_1 + h^3u_1 \). When \( hu_1 \in Z_k \), set \( l(a) = \bar{u}_1 \), while when \( h^3u_1 \notin Z_k \), consider \( u_1 = h^3u_1\varepsilon \), the component of \( h^3u_1 \) in \( V^{0,*} \), and define \( l(a) \) as \( l(u_1) \). When \( h^2u_1 \notin Z_k \), and \( h^3u_1 \in Z_k \), choose the smallest \( n > 1 \) such that there is the relation

\[
du_n = -a h^2u_{n-1} + \lambda v_n, \quad dv_n = \frac{1}{\lambda}d(ah^2u_{n-1}), \quad u_n \in V^{-3,*}, \quad v_n \in V^{-2,*},
\]

with \( h^2u_n \in Z_k \).

(The inequality \((n + 1)|a| > m\) guarantees the existence of such a relation, since \( h^2u_1 \in D + K_\mu \), while \( K_\mu^j = 0 \) for \( j > m \) in the minimal \( V \subset RH \).) Then set \( l(a) = \bar{u}_n \) for \( h^3u_n \in Z_k \); otherwise, define \( l(a) \) as \( l(u_1) \) for \( u_n = h^3u_n|_{V^{0,*}} \).

4. PROOF OF THEOREM 2

The proof of the theorem relies on the two basic propositions below in which the condition that \( H(A) \) has at least two algebra generators is treated in two specific cases.

**Proposition 5.** Let \( H_k \) be a finitely generated \( k \)-module with \( \mu \geq 2 \). If \( \tilde{H}_k \) has at least two algebra generators and \( \tilde{H}_Q \) is either trivial or has a single algebra generator, there are two sequences of odd degree elements \( x_k = \{x(i)\}_{i \geq 0} \) and \( y_k = \{y(j)\}_{j \geq 0} \) in \( V_k \) whose degrees form arithmetic progressions such that all \( \bar{x}(i), \bar{y}(j) \) are \( d_h \)-cochains in \( V_k \) and the classes \( \{s^{-1}(x(i) \sim_1 y(j))\}_{i,j \geq 0} \) are linearly independent in \( H(V_k, d_h) \).

**Proof.** The hypotheses of the proposition imply that \( K_\mu \) defined in subsection 3.1 above is non-empty; also by the restriction on \( H_Q \), relation (2.5) reduces to

\[
da = \lambda b^m, \quad \lambda \neq 0, \quad m \geq 1, \quad (\lambda, m) \neq (1, 1), \quad b \in V^{0,*}
\]

for \( a \in V^{-1,*} \) to be of the smallest degree.

In the three cases below, we exhibit two odd dimensional elements \( x, y \in V \setminus E \) that fail to be \( \mu \)-homologous to zero.

Case I. Let \( a \in K_\mu \) be of the smallest degree in \( K_\mu \cup K_0 \) with \( da = \lambda b^m \) and let \(|a|\) be even. Consider the element \( l(a) \). If it is not \( \lambda \)-homologous to zero, set \( x = l(a) \); otherwise, we must have relation (2.5) in which \( v_1 = a \) for some \( i \) and \( hu \in Z_k \) with \(|u| < |l(a)| \), \( u \in \bigcup_{i \geq 1} V^{-1,*} \setminus E \). By (2.5) choose \( u \) to be of the smallest degree with \( hu \in Z_k, u \neq u_i, a_1 \), where \( u_i \) is given by (3.4), (3.6) and \( da_1 = -ab + \lambda b_1, \quad db_1 = b^2 \).

Set \( x = \bar{u} \) for \(|u|\) odd. If \(|u|\) is even and \( u \in \bigcup_{i \geq 1} V^{-1,*} \setminus E \) set \( x = \bar{v} \); if \( u \in K_0 \) and \( du \) contains an odd dimensional \( v_i \in V^{0,*} \) with \(|v_i| \neq 0 \in H_Q \), set \( x = v_i \); otherwise, for each monomial \( P_s(v_1, ..., v_r) \) choose a variable \( v_i \) with a relation \( du_i = \mu_i v_i \) (for example, we can choose \( v_i \) to be odd dimensional for all \( s \)). Let \( \lambda \) be the smallest integer divisible by all \( \mu_i \), and replace \( v_i \) by \( \hat{\lambda} v_i \) to detect a new relation in \((RH, d)\) given again by (2.5):

\[
dw = \sum_{1 \leq s \leq n} \frac{\lambda_s \lambda}{\mu_i} P_s(v_1, ..., v_{i-1}, u_i, v_{i+1}, ..., v_r) + \lambda u, \quad \lambda_s \in Z_k, \quad w \in V^{-2,*}.
\]

Hence, \(|w|\) is odd, and set \( x = \bar{w} \) for \( h^2w \in Z_k \). If \( h^2w \notin Z_k \) we have the following two subcases:

(1) Assume there exists \( v \in K_\mu \) with \( dv = \lambda h^2w \). If \([\bar{v}]_\lambda \neq 0 \), set \( x = v \); otherwise we have a relation \( dh^2u' = v + z + \lambda' v' \), some \( u', v' \in V, \quad z \in D \). Clearly,
\[ h^{tr} v' = \frac{1}{\lambda} h^2 w \mod D, \] and set \( x = \frac{1}{\lambda} w + v' \). Note that \( x \) is not \( \lambda \)-homologous to zero since the component \( \frac{1}{\lambda} u \) in \( dx \).

(ii) Assume \( [h^2 w] \neq 0 \in H_Q \). When \( r_s > 1 \) for all \( s \), choose a variable \( v_j \) different from \( v_i \) in \( P_s(v_1, ..., v_{r_s}) \) to form \( w' \) entirely analogously to \( w \), and then find \( x \) similarly to the above unless \([h^2 w'] \neq 0 \in H_Q \), in which case set \( x = \alpha w + \beta w' \), some \( \alpha, \beta \in \mathbb{Z} \). When \( k = \{ s \in \mathbb{Z} | r_s = 1 \} \) \( \neq \emptyset \), i.e., \( P_s(v_1, ..., v_{r_s}) = v_1^{2m_s+1} := v_k^{2m_s+1}, m_s \geq 1 \), |v_s| is odd for \( s \in k \) (in particular, \( \mu_s \) is even, since \([v_a]^2 = 0 \in H \) for \( \mu_s \) odd; c.f. (3.2)), then

\[
du' = \begin{cases} 
\sum_{s \in k} \frac{1}{t} (v_s \sim v_s) v_s^{2m_s-1} \\
+ \sum_{s \notin k} \frac{1}{t} P_s(v_1, ..., v_{j-1}, u_j, v_{j+1}, ..., v_s) + \lambda u, \quad k \neq u, \\
\sum_{s \in k} \lambda_s (v_s \sim v_s) v_s^{2m_s-1} + 2u, \quad k = u
\end{cases}
\]

with \( u' \in V_r \), and by considering \( h^2 u' \) we find \( x \) as in item (i).

To find \( y \), consider \( b \) and the associated sequence \( b = \{ b_i \} \) given by (3.2) or (3.3). If \( bb_i \in \mathbb{Z}_k \) for all \( i \), set \( y = b \) and \( y = \{ b_i \}_{i \geq 0} \). If \( bb \notin \mathbb{Z}_k \), consider the smallest \( p \) such that \( h^p bb \notin \mathbb{Z}_k \). Consider \( t_p = h^p bb | V_{r_p} \), and if \( \{ l(t_p) \} \neq 0 \), set \( y = l(t_p) \); if \( \{ l(t_p) \} \neq 0 \) and \( \alpha h^3 u_i + \beta h^3 b_p = 0 \), \( \alpha, \beta \in \mathbb{Z} \), for some \( u_i \) from (3.3)–(3.3), set \( y = \alpha u_i + \beta b_p \); otherwise, we obtain \( l(t_p) \in K_0 \) different from \( l(a) \) above; consequently, we must have another relation in \((RH, d)\) given by (2.4) in which \( v_i = t_p \) for some \( i \) and \( hu \in \mathbb{Z}_k \) with \(| u | < | l(t_p) | \), and then \( y \) is found similarly to \( x \).

Case II. Let \( a \in K_\mu \) be of the smallest degree in \( K_\mu \cup K_0 \) with \( da = \lambda b \) and let \(|a| \) be odd. Set \( x = a \). Consider \( l(b) \in K_0 \), and then \( y \) is found as in Case I.

Case III. Let \( a \in K_0 \) be of smallest degree in \( K_\mu \cup K_0 \) with \( da = \lambda b^m, m \geq 2 \), and \([b] \neq 0 \in H_Q \). Set

\[
x = \begin{cases} 
b, & |b| \text{ is odd} \\
a, & |b| \text{ is even}
\end{cases}
\]

To find \( y \) consider the following two subcases:

(i) Assume \( \lambda \in \mathbb{Z} \setminus \mathbb{Z}_k \). When both \(|a| \) and \(|b| \) are odd, set \( y = a \); otherwise, either \(|a| \) or \(|b| \) is even, in which case consider \( l(\tilde{a}) \) or \( l(b) \) respectively, and then \( y \) is found as in Case I.

(ii) Assume \( \lambda \in \mathbb{Z}_k \). Since \( K_\mu \neq \emptyset \), this subcase reduces either to Case I or to Case II.

Finally, having found the elements \( x \) and \( y \) in Cases I-III, consider the f.i.s. \( x \) and \( y \) in \( V \) and the induced sequences \( x_k = \{ x(i) \}_{i \geq 0} \) and \( y_k = \{ y(j) \}_{j \geq 0} \) in \( V_k \). Then the both sequences \( x_k \) and \( y_k \) consist of \( d_h \)-cocycles in \( V_k \) whose degrees form an arithmetic progression respectively. Thus, we obtain that \([x_k], [y_k] \subset H(V_k, d_h)\) are sequences of non-trivial classes. Moreover, they are linearly independent and \( \{ s^{-1}(x(i) \sim y(j)) \}_{i,j \geq 0} \) is the sequence of linearly independent classes in \( H(V_k, d_h) \) as required.

\[ \square \]

Before proving the second basic proposition we need the following auxiliary statement. Given a cochain complex \((C^*, d)\) over \( \mathbb{Q} \), let \( S_C(T) = \sum_{n \geq 0} (\dim_{\mathbb{Q}} C^n) T^n \) and \( S_{H(C)}(T) = \sum_{n \geq 0} (\dim_{\mathbb{Q}} H^n(C)) T^n \) be the Poincaré series. As usual, we write
\[ \sum_{n \geq 0} a_n T^n \leq \sum_{n \geq 0} b_n T^n \] if and only if \( a_n \leq b_n \). The following proposition can be thought of as a modification of Propositions 3 and 4 in [12] for the non-commutative case.

**Proposition 6.** Given an element \( y \in V_q \) of total degree \( K_y \geq 2 \) such that \( d_h(\bar{y}) = 0 \), let \( y \bar{V}_q \subset \bar{V}_q \) be a subcomplex (additively) generated by the expressions \( \{ \bar{y} = s^{-1} y, s^{-1}(y - 1 v) \} \). Then

\[
(4.1) \quad S_{H(\bar{V}_q/y \bar{V}_q)}(T) \leq (1 + T^{k-1})S_{H(\bar{V}_q)}(T).
\]

**Proof.** Consider the inclusion of cochain complexes \( \bar{s}^k \bar{V}_q \rightarrow \bar{V}_q \) defined for \( 1 \in \bar{Q} = (\bar{s}^k \bar{V}_q)^k \) by \( \bar{v}(1) = \bar{y} \), and for \( \bar{s}^k(\bar{v}) \in (\bar{s}^k \bar{V}_q)^{>k}, v \in V_q^{>1} \), by \( \bar{v}(\bar{s}^k(\bar{v})) = s^{-1}(y - 1 v) \). Then \( \bar{v}(\bar{s}^k \bar{V}_q) = y \bar{V}_q \) and there is the short exact sequence of cochain complexes

\[
0 \rightarrow \bar{s}^k \bar{V}_q \rightarrow \bar{V}_q \rightarrow \bar{V}_q/y \bar{V}_q \rightarrow 0.
\]

Consider the induced long exact sequence

\[
\cdots \rightarrow H^{n-k}(\bar{V}_q) \xrightarrow{H^n} H^n(\bar{V}_q) \rightarrow H^n(\bar{V}_q/y \bar{V}_q) \rightarrow H^{n-k+1}(\bar{V}_q) \rightarrow \cdots.
\]

Let \( I = \oplus I_n \), where \( I_n = \text{Im}(H^n) \), \( n \geq 0 \), and form the exact sequence

\[
0 \rightarrow I_n \rightarrow H^n(\bar{V}_q) \rightarrow H^n(\bar{V}_q/y \bar{V}_q) \rightarrow H^{n-k+1}(\bar{V}_q) \rightarrow I_{n+1} \rightarrow 0.
\]

Since \( I_0 = 0 \), we have

\[
\sum_{n \geq 0} (\dim Q I_n + \dim Q I_{n+1}) T^n = \frac{(1 + T)S_I(T)}{T}.
\]

Now apply the Euler-Poincaré lemma for the above exact sequence to obtain the equality

\[
\frac{(1 + T)S_I(T)}{T} - S_{H(\bar{V}_q)}(T) + S_{H(\bar{V}_q/y \bar{V}_q)}(T) - T^{k-1}S_{H(\bar{V}_q)}(T) = 0.
\]

Consequently,

\[
S_{H(\bar{V}_q/y \bar{V}_q)}(T) = (1 + T^{k-1})S_{H(\bar{V}_q)}(T) - \frac{(1 + T)S_I(T)}{T},
\]

and since \( S_I(T) \geq 0 \), we get (4.1) as required. \( \square \)

**Proposition 7.** Let \( H_k \) be a finitely generated \( \mathbb{k} \)-module. If \( H_q \) has at least two algebra generators and \( A_q = A' \otimes \mathbb{Q} \), the set \( \left\{ \tau_i(A_q) = \dim Q \text{Tor}_i A_q(\mathbb{Q}, \mathbb{Q}) \right\} \) is unbounded.

**Proof.** Consider the first two generators \( a_i \in V_q^{*-1,*} \) with \( da_i \in D^{0,*}, i = 1, 2 \). We have two cases:

(i) Both \( |a_1| \) and \( |a_2| \) are odd. Set \( x = a_1 \) and \( y = a_2 \). Then both \( \bar{x} \) and \( \bar{y} \) are \( \bar{d}_h \)-cocycles and the classes \( [\bar{x}] \) and \( [\bar{y}] \) are non-trivial in \( H(\bar{V}_q, \bar{d}_h) \). Consequently, the classes

\[
(4.2) \quad \left\{ s^{-1} \left( x^{-1} - 1 \ y^{-1} \right) \right\}_{i,j \geq 1}
\]

are linearly independent in \( H(\bar{V}_q, \bar{d}_h) \).

(ii) Either \( |a_1| \) or \( |a_2| \) is even. Denote the (smallest) even dimensional generator by \( a \) and consider \( da \). Then for \( a, (2.5) \) reduces to

\[
da = uv, \ u \in V_q^{0,2k+1} \quad \text{and} \quad v \in R^0 H_q^{2\ell}, \ \text{some} \ k, \ell \geq 1.
\]
There are the following induced relations in \((RH_0, d)\):
\[
\begin{align*}
db &= -u(a + u \cdot v) - au, & b &\in V_0^{-2(2k+\ell+1)} \quad \text{and} \\
dc &= -u(v \cdot a + (u \cup v)v + u(v \cup 2v)) - a^2 + bv, & c &\in V_0^{-3(4k+\ell)+2}.
\end{align*}
\]
Thus we have \(hc = h^2c + h^3c\), and in particular, \(dh^2c = h^2b \cdot v\). Consider the following two cases:

1. Assume \(hc \in D\). Set \(x = u, y = c\), and obtain linearly independent classes in \(H(V_0, d_h)\) by formula (4.2).
2. Assume \(hc \notin D\). Let \((\bar{W}, d_{W}) = (V_0/\bar{C}, \bar{d}_W)\), where \(C \subset V_0\) is a subcomplex (additively) generated by the expressions \(hc\) and \(hc \cdot z\) for \(z \in V_0\). Define \(\bar{x}\) and \(\bar{y}\) as the projections of the elements \(u\) and \(c\) from \(V_0\) under the quotient map \(V_0 \rightarrow V_0/\bar{C}\), respectively. Then \(\bar{x}\) and \(\bar{y}\) are \(d_W\)-cocycles in \(\bar{W}\). Once again apply formula (4.2) to obtain linearly independent classes in \(H(\bar{W}, \bar{d}_W)\). Finally, Proposition 6 implies that \(S_{H(\bar{W})}(T) \leq S_{H(V_0)}(T)\), and an application of Proposition 11 completes the proof.

4.1. **Proof of Theorem 2**. In view of Proposition 11, the proof reduces to the examination of the \(k\)-module \(H(V_0, d_h)\). If \(\tilde{H}_k\) has a single algebra generator \(a\), then the set \(\{\tau_i(A)\}\) is bounded since \(\tau_i(A) = 1\). For example, this can be seen from the fact that \(H(V_0, d_h)\) is generated by a single sequence induced by (3.2) or by (3.3), where \(x = a\) or \(x = l(a)\) for \(|a|\) odd or even respectively, and by \(\cdot\)-products of its components. If \(\tilde{H}_k\) has at least two algebra generators, then the proof follows from Propositions 5 and 7.

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