Random Banach spaces.
The Limitations of the Method

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We shall study the properties of typical $n$–dimensional subspaces of $l^N_{\infty} = (\mathbb{R}^N, \|\cdot\|_{\infty})$, or equivalently, typical $n$–dimensional quotients of $l^N_1 = (\mathbb{R}^N, \|\cdot\|_1)$, where the meaning what is typical and what is not is defined in terms of the Haar measure $\mu_{n,N}$ on the Grassmann manifold $G_{n,N}$ of all $n$–dimensional subspaces of $\mathbb{R}^N$.

In [Gl.2], Gluskin proved that a “typical” $n$–dimensional subspace $E$ of $l^N_{\infty}$ enjoys the property

$$\|P\| \geq \frac{ck}{\sqrt{n \log n}},$$

for every projection $P : E \to E$, with $\min\{\text{rank } P, \text{rank } (Id - P)\} = k$, where $c$ is a numerical constant. In particular, if $k \geq n^{\alpha}$, $\alpha > \frac{1}{2}$, then no projection $P$ on $E$ with both rank $P$ and corank $P$ greater than $k$ can be “well” bounded. Several other results,[Sz.1],[Sz.2],[Ma.1],[Ma.2] showed that a “typical” proportional (i.e. $\dim E \approx \beta N$ for some “fixed” $\beta \in (0, 1)$) subspace $E$ of $l^N_{\infty}$ has the property that every “well” bounded operator on $E$ is indeed a “small” perturbation of a multiple of the identity $\lambda \text{Id}_E$. However, the estimates on the distance between $T$ and $\lambda \text{Id}_E$ have been done in terms of the geometry of $\mathbb{R}^N$ rather than $E$ itself. In this note, we obtain the estimates on the distance between $T$ and $\lambda \text{Id}_E$ in intrinsic terms of the geometry of $E$, namely, in terms of the Gelfand numbers of $T - \lambda \text{Id}_E$ (Sections 2 and 3). On the other hand, we show in Section 4, that if $k \leq n^{1/2}$ then a “typical” $n$–dimensional subspace $E$ of $l^N_{\infty}$ (for any $N \geq n$) contains a $k$–dimensional well-complemented subspace $G$ isomorphic to $l^p_k$ with either $p = 2$ or $p = \infty$ and therefore admits operators which are “fairly” far away from the line $\{ \lambda \text{Id}_E \}_{\lambda \in \mathbb{R}}$.

We shall employ the standard notation of local theory of Banach spaces as used in e.g. [F-L-M]. For basics on multivariate Gaussian random variables the reader is referreded to [T].

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∗Part of this research was done while this author has been visiting the Case Western Reserve University. Supported in part by a grant from KBN (Poland).
†Supported in part by a grant from the National Science Foundation (U.S.A.).
1 Generic subspaces. Equivalent Gaussian approach

Let $P_n$ for $n \in \mathbb{N}$ be a sequence of properties of $n$-dimensional Banach spaces and let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function. We shall say that $P = \{P_n\}_{n \in \mathbb{N}}$ is a generic property of $n$-dimensional subspaces of $l_\infty^n$ where $N = f(n)$ iff for every $n \in \mathbb{N}$ we have

$$\mu_{n,N}\{E \in G_{n,N} \mid E \text{ satisfies } P_n\} \geq 1 - \varepsilon^n,$$

for some $\varepsilon \in (0,1)$, $\varepsilon$ independent of $n$. In the sequel, we shall say that a generic $n$-dimensional subspace of $l_\infty^n$ has a property $P = \{P_n\}_{n \in \mathbb{N}}$ rather then that the property $P$ is a generic one. Since the dual of an $n$-dimensional subspace $E$ of $l_\infty^n$ is the quotient $F = l_\infty^n/E^\perp$, where $l_\infty^n = (\mathbb{R}^N, \| \cdot \|)$ and $E^\perp$ is the orthogonal complement of $E$ in $\mathbb{R}^N$, the notions of generic properties of quotients of $l_1^n$ and generic subspaces of $l_1^n$ can be defined. E.g. a generic quotient of $l_1^n$ satisfies a property $P$ iff the corresponding (via duality) geometric generic subspace of $l_\infty^n$ satisfies the dual property $P^*$.

In the context of quotients it is more convenient to consider an equivalent approach.

Let $g$ be an $\mathbb{R}^n$-valued Gaussian vector distributed according to the $N(0,n^{-1}\text{Id}_{\mathbb{R}^n})$ law (i.e., the covariance matrix of $g$ is $n^{-1}\text{Id}_{\mathbb{R}^n}$, or the coordinates of $g$ are i.i.d. $N(0,1/n)$ random variables). Let $g_1, g_2, \ldots , g_N$ be independent copies of $g$ and $\Gamma$ an $n \times N$ matrix whose columns are $g_1, g_2, \ldots , g_N$; alternatively, $\Gamma = \Gamma(\omega)$ can be described as a Gaussian $n \times N$ matrix with i.i.d. $N(0,1/n)$ entries. If we think of $\Gamma$ as of a (random) linear map from $\mathbb{R}^N$ to $\mathbb{R}^n$, $l_1^n/\ker \Gamma$ is a random quotient of $l_1^n$. By the rotational invariance of the Gaussian measure, this model is “measure theoretically” equivalent to the one described at the beginning of this section and based on the Haar measure on the Grassmannian. Still equivalently, we may consider the random norm on $\mathbb{R}^n$, whose unit ball is a random absolute convex body

$$B = B(\omega) = \text{absconv} \{g_j : j = 1,2,\ldots , N\} = \Gamma(B_1^N),$$

where $B_1^n$ is the unit ball of $l_1^n$.

In the sequel we will need some basic facts about Gaussian vectors, Gaussian matrices and “Gaussian bodies”. The first lemma is an elementary consequence of the formulae for Gaussian density in $\mathbb{R}^d$.

**Lemma 1.1** If $g$ is a Gaussian random variable with distribution $N(0,d^{-1}\text{Id}_{\mathbb{R}^d})$, then

(i) $\mathbb{E} \| g \|^2 = 1$

(ii) $\mathbb{P} (\| g \|^2 \geq \lambda) \leq \exp (-d\lambda^2/8)$ for $\lambda \geq 2$

(iii) $\mathbb{P} (\| g \|^2 \leq t) \leq \left(t e^{1/2}\right)^d$

(iv) $\mathbb{P} \left(\frac{1}{2} \leq \| g \|^2 \leq 2\right) \geq 1 - \exp (-c_0d)$,

where $c_0$ is a universal constant.

The next lemma can be derived from the first one using a standard $\varepsilon$-net argument (cf. [Sz.3], Lemma 2.8). Another, more precise argument (giving e.g. $c > \frac{1}{4}$ and $C < 2$) can be found in [Si].
Lemma 1.2 Let $k \leq \frac{1}{2}N$ and $\Lambda$ be an $N \times k$ matrix with all i.i.d. Gaussian $N(0, 1)$ entries. Then
\[ P\left( c \| x \|_2 \leq k^{-1/2} \| \Lambda x \|_2 \leq C \| x \|_2 \text{ for every } x \in \mathbb{R}^k \right) \geq 1 - \exp(-c_1d), \]
where $c, C, c_1$ are universal constants.

We have an immediate

Corollary 1.3 If $k$ and $N$ are as in Lemma 1.2, $g_1, g_2, \ldots, g_N$ are i.i.d. Gaussian random variables with distribution $N(0, k^{-1}Id_{\mathbb{R}^k})$ and $B = B(\omega) = \text{absconv} \{g_1, g_2, \ldots, g_N\}$, then
\[ P\left( B \supset c\sqrt{k^{-1/2}D} \right) \geq 1 - \exp(-c_1d), \]
where $D$ stands for the Euclidean unit ball in $\mathbb{R}^k$.

The last lemma gives more precise information about the random bodies $B(\omega) \subset \mathbb{R}^k$ (cf. [Gl.3],[Gl.4]).

Lemma 1.4 If $g_1, g_2, \ldots, g_N, B$ are as in Corollary 1.3 and $2^k \geq N \geq 2k$, then
\[ (i) \quad P\left( B \supset c'\sqrt{\log N/k} \right) \geq 1 - \exp(-c_2k) \]
\[ (ii) \quad P\left( \left( \frac{\text{vol } B}{\text{vol } D} \right)^{1/k} \leq C' \sqrt{\log N/k} \right) \geq 1 - \exp(-c_2k), \]
where $c, C', c_2 > 0$ are universal constants.

Remark The volume estimate from Lemma 1.4 (ii) actually holds for any $B = \text{absconv} \{x_1, x_2, \ldots, x_N\}$ as long as we control
\[ \max \{ \| x_j \|_2 : j = 1, 2, \ldots, N \}, \]
which in our case we do by Lemma 1.1 (iv) (see [Ca-P] or [Gl.4]).

2 The proportional case.

Our starting point is the following result ([Ma.2], Proposition 2.3).

Theorem 2.1 There is a numerical constant $c > 0$ such that for every $n \geq 2$ there is a norm $\| \cdot \|_{X_n}$ on $\mathbb{R}^n$ such that
\[ (i) \quad X_n = (\mathbb{R}^n, \| \cdot \|) \text{ is isometrically isomorphic to a quotient of } l_1^{2n}, \]
\[ (ii) \quad \| x \|_2 \leq \| x \|_{X_n} \leq \| x \|_1 \text{ for every } x \in \mathbb{R}^n, \]
\[ (iii) \quad \text{for every } T \in L(\mathbb{R}^n) \text{ there are } \lambda_T \in \mathbb{R}, V_T \in L(\mathbb{R}^n) \text{ and a linear subspace } \]
\[ E_T \subset \mathbb{R}^n \text{ with } \dim E_T > \frac{2n}{8}, \text{ such that } \]
\[ a) \quad V_T = T + \lambda_T Id_{\mathbb{R}^n}, \]
\[ b) \quad | \lambda_T | \leq c \| T \|_{X_n}, \]
\[ c) \quad \| V_T |E_T \| \leq cn^{-\frac{1}{2}} \| T \|_{X_n}. \]
In fact, Theorem 2.1 holds for “sort of” generic $n$-dimensional quotients of $l_1^n$ (cf. [Ma.1]). However in order to adapt the above result to our present setting, we need to make a couple of observations. First, the condition (ii), which is not crucial for our purposes, has to be superseded by the properties listed in Corollary 1.3 and Lemma 1.4 (the condition of type (ii) may be, moreover, achieved, up to a universal constant and restricted to the span of, say, the first $n/2$ unit vectors $e_1, e_2, \ldots, e_{n/2}$; see [Ma.-T]). Next (and more importantly), we have to point out that in all constructions leading to Theorem 2.1–like statements ([G1.1], [G1.2], [Ma.1], [Ma.2], [Ma.-T], [Sz.1], [Sz.2] etc.), “generic” had a somewhat different (and slightly less natural) meaning. We take here an opportunity to present a remark which rectifies this problem. What happens is that when ensuring the condition (iii) from Theorem 2.1, we need to work with the set
\[
\mathcal{T} = \{ T \in L(\mathbb{R}^n) : \|T\|_{X_n} \leq 1 \}
\]
or, more specifically, with nets of $\mathcal{T}$ in the $l_1^N$ metric. Now we have ($hs(\cdot)$ is the Hilbert-Schmidt norm)

If $X, Y$ are quotients of $l_1^N$, endowed with the canonical inner product and $T : X \to Y$ verifies $\|T : X \to Y\| \leq 1$, then $hs(T) \leq N^{1/2}$.

The above statement is shown by estimating $hs(T) = hs(T^*) = \pi_2(T^*)$ by the $\pi_2$-norm of (a restriction of) the formal identity $Id : l_\infty^N \to l_2^N$, which is $N^{1/2}$. If $N = 2n$, it can be shown in a standard way that $\mathcal{T}' = \{ T \in L(\mathbb{R}^n) : hs(T) \leq N^{1/2} \}$ admits a $\delta$-net in the $\|\cdot\|_2^N$ metric which is of cardinality not exceeding $(C/\delta)^{n^2}$, where $C$ is a numerical constant, and this can be easily incorporated into existing proofs of Theorem 2.1-like statements. Unfortunately, we do not see how to handle in the same “unified” way e.g. the casse considered in Theorem 3.1.

We now prove the following.

**Theorem 2.2** There is a numerical constant $K > 0$ such that if $X_n$ is the quotient space from Theorem 2.1 then, for every operator $T \in L(\mathbb{R}^n)$, we have
\[
\inf \{ c_k(T - \lambda Id_{\mathbb{R}^n}) : \lambda \in \mathbb{R} \} \leq Kn^{-\frac{1}{2}}\|T\|_{X_n}.
\]

We recall here that, for an operator $u : X \to Y$, the $k$th Gelfand number of $u$ is defined by
\[
c_k(u) = \inf \{ \|u|Z\| : Z \subset X, \text{ codim } Z < k \}.
\]

Because of a well-known duality relation between the Gelfand numbers and the so-called “Kolmogorov numbers” $d_k(\cdot)$ (namely $d_k(u^*) = c_k(u)$), one can also state our results in terms of the latter ones. Note a slight abuse of notation; in the above and in what follows we pretend that $\frac{n}{2}$ and similar expressions are integers.

Observe that Theorem 2.2 is the best possible. This follows either from Corollary 4.3 below or from the fact that, for an $n$-dimensional “generic” quotient of $l_1^N$, (for any $N > n$), and for a “generic” element of $O(n)$, the left hand side of the inequality in Theorem 2.2 is of order $n$ while the right hand side–at most of order $n^{1/2}$. In fact, any $n$-dimensional normed space can be represented on $\mathbb{R}^n$ so that the last remark holds.
**Proof of Theorem 2.2** Obviously, it is enough to prove the theorem for every operator $T \in L(\mathbb{R}^n)$ satisfying
\[
\inf \{ c_n^2(T - \lambda \text{Id}_{\mathbb{R}^n}) | \lambda \in \mathbb{R} \} = 1.
\] (2.1)
To this end, fix such an operator $T$. It is well known, [Gl.1], [Sz.2], that
\[
\text{vol}(B_{X^n}) \leq \left( \frac{c_1}{n} \right)^n
\]
and
\[
B_{X^n} \supset \frac{1}{\sqrt{n}} B_n^2.
\]
Hence, by [Sz.-T], we infer that there exists an $\frac{3n}{4}$-dimensional subspace, say $E$, of $\mathbb{R}^n$ such that
\[
B_{X^n} \cap E \subset \frac{c_2}{\sqrt{n}} B_{E}^2. \tag{2.2}
\]

**Claim.** For every $\lambda \in \mathbb{R}^n$ the operator
\[
T_\lambda |E = (T - \lambda \text{Id}_{\mathbb{R}^n}) |E \tag{2.3}
\]
has at least $n/4$ s-numbers greater than or equal to $1/c_2$.

Indeed, if this was not the case then, by (2.2), we would have for some $\lambda_0 \in \mathbb{R}$
\[
\frac{1}{c_2} > \| T_{\lambda_0} |E_0 : (E_0, n^{-\frac{1}{2}} \| \cdot \|_2) \rightarrow (\mathbb{R}^n, n^{-\frac{1}{2}} \| \cdot \|_2) \| \geq \\
\frac{1}{c_2} \| T_{\lambda_0} |E_0 : (E_0, \| \cdot \|_{X^n}) \rightarrow (\mathbb{R}^n, \| \cdot \|_{X^n}) \|, \tag{2.4}
\]
where $E_0$ is an $\frac{n}{2}$-dimensional subspace such that $\| T_{\lambda_0} |E_0 \|_2 < 1/c_2$. But (2.4) implies that
\[
c_n^2(T - \lambda_0 \text{Id}_{\mathbb{R}^n}) < 1,
\]
a contradiction with (2.1), which concludes the proof of the claim.

In particular, the Claim yields that for every $\lambda \in \mathbb{R}$ the operator $T - \lambda \text{Id}_{\mathbb{R}^n}$ has at least $n/4$ s-numbers greater than or equal to $1/c_2$, which means that for every subspace $F \subset \mathbb{R}^n$ with $\dim F \geq \frac{7n}{8}$ we have
\[
\| T - \lambda \text{Id}_{\mathbb{R}^n} |F \|_2 \geq \frac{1}{c_2}.
\]
Hence, by Theorem 2.1 (iii), c), we infer that
\[
1 \leq c_2 n^{-\frac{1}{2}} \| T \|_{X^n}
\]
which implies $\| T \|_{X^n} \geq Kn^{\frac{1}{4}}$, where $K = (cc_2)^{-1}$. This concludes the proof of the theorem. \[\square\]
3 The $l^1$-type estimates.

For a Banach space $X = (\mathbb{R}^n, \| \cdot \|_B)$ we set

$$M^*_B = \int_{S^{n-1}} \| x \|_B^* d\mu(x),$$

where $d\mu$ stands for the normalized Lebesque measure on the unit sphere $S^{n-1}$ and $\| \cdot \|_B^*$ denotes the dual norm to $\| \cdot \|_B$; this is the "average width" of $B$.

In the sequel we shall need the following fact which can be found in [P–T], Theorem 1 (cf. [Pi.2], Theorem 1.3).

**Fact I** There exist a numeric constant $C > 1$ such that for every symmetric convex body $B \subset \mathbb{R}^n$ and for every $k = 1, 2, \ldots, n-1$ there exists a subspace $E_k \subset \mathbb{R}^n$ with $\text{codim } E_k = k$ such that

$$B \cap E_k \subset CM^*_B \sqrt{\frac{n}{k}} \cap E_k.$$

Recall that for an operator $T \in L(\mathbb{R}^n)$ we say that $T \in M_n(\alpha, \beta)$, where $\alpha, \beta > 0$ iff there is a linear subspace $F \subset \mathbb{R}^n$ with $\dim F \geq \alpha$ such that

$$\|P_F \perp T x\|_2 \geq \beta \|x\|_2$$
for every $x \in F$,

(wher $P_F \perp$ denotes the orthogonal projection onto $F^\perp$). Also, for $\gamma > 0$, we denote

$$\tilde{M}_n(\gamma) = \bigcup_{k=1}^{n/2} M_n(k, \gamma/k).$$

The following fact has been proved in [Sz.2]

**Fact II** A generic $n$-dimensional quotient $X_n$ of $l_1^n$ enjoys the property

$$\|T\|_{X_n} \geq \frac{c_1 \gamma}{\sqrt{n \log n}}$$
for every $T \in \tilde{M}_n(\gamma)$.

From Fact I and Fact II we deduce

**Theorem 3.1** A generic $n$-dimensional quotient $X_n$ of $l_1^n$ has the property that for every $T \in L(\mathbb{R}^n)$ we have

$$\inf \{ \sum_{i=1}^{n} c_i (T - \lambda I_{\mathbb{R}^n}) \mid \lambda \in \mathbb{R} \} \leq c n^{2/3} \sqrt{\log^3 n} \|T\|_{X_n},$$

where $c$ is a numerical constant.
Remark. It is imaginable that one could strengthen the above inequality to get $O(n^{1/2} \|T\|_{X_n})$ on the right hand side; $o(n^{1/2}) \|T\|_{X_n}$ is impossible, see the comments following Theorem 2.2.

Proof. In order to simplify the notation we shall assume that $n = 2^k$ for some $k \in \mathbb{N}$. Let $X_n$ be a generic $n$–dimensional quotient of $l^1_n$. Obviously, it suffices to prove that

$$\sum_{i=1}^{n} c_i(T) \leq cn^{2/3} \sqrt{\log^3 n} \|T\|_{X_n},$$

for every $T \in L(\mathbb{R}^n)$ satisfying $\text{tr} T = 0$. To this end, fix $T \in L(\mathbb{R}^n)$ such that $\text{tr} T = 0$ and

$$\sum_{i=1}^{n} c_i(T) = n.$$

Then, there is $i \leq k$ such that

$$2^{i-1}c_{2^i} \geq \frac{n}{\log n}.$$  

(3.6)

If $2^i \leq n^{2/3}$ then we have

$$\|T\|_{X_n} \geq c_{2^i} \geq \frac{2n^{1/3}}{\log n},$$

which combined with (3.6), by a standard homogeneity argument, yields (3.3) and we are done.

Thus, assume that $2^i > n^{2/3}$. It is well known (cf. Corollary [13]) that $n^{-1/2}D \subset B_{X(E)}$ and that $M^*_X \leq c_2 \sqrt{n^{-1} \log n}$, (see e.g. [F-L-M]). Thus, applying Fact [11] we infer that there exists a linear subspace $F_{2i-1} \subset \mathbb{R}^n$ with codim $F_{2i-1} = 2^{i-1}$ such that

$$\frac{1}{\sqrt{n}} B^n_2 \cap F_{2i-1} \subset B_{X_n} \cap F_{2i-1} \subset c_2 C \sqrt{\frac{\log n}{2^{i-1}}} B^n_2 \cap F_{2i-1}.$$  

(3.8)

Claim. The operator $T|_{F_{2i-1}}$ has at least $2^{i-1}$ $s$-numbers greater than or equal to

$$(c_2C)^{-1} \frac{n}{2^{i-1} \log^3 n}.$$  

Indeed, assume to the contrary that there exists a linear subspace $F \subset F_{2i-1}$ with codim $F = 2^i$ such that

$$\|T|F : (F, \| \cdot \|_2) \rightarrow (\mathbb{R}^n, \| \cdot \|_2) \| < (c_2C)^{-1} \frac{n}{2^{i-1} \log^3 n}. $$

(3.9)

Then, by (3.8) and (3.7) we have

$$(c_2C)^{-1} \frac{n}{2^{i-1} \log^3 n} > \|T|F : (F, n^{-1/2} \| \cdot \|_2) \rightarrow (\mathbb{R}^n, n^{-1/2} \| \cdot \|_2) \| =$$

$$(c_2C)^{-1} \frac{2^{i-1}}{n \log n} \|T|F : (F, c_2 C \sqrt{\frac{\log n}{2^{i-1}}} \| \cdot \|_2) \rightarrow (\mathbb{R}^n, n^{-1/2} \| \cdot \|_2) \| \geq$$
\[(c_2 C)^{-1} \sqrt{\frac{2^{i-1}}{n \log n}} \| T \| F : (F, \| \cdot \|_{x_n}) \to (\mathbb{R}^n, \| \cdot \|_{x_n}) \geq (3.10)\]

\[(c_2 C)^{-1} \sqrt{\frac{2^{i-1}}{n \log n}} c_2(T) \geq (c_2 C)^{-1} \sqrt{\frac{2^{i-1}}{n \log n}} \frac{n}{2^{i-1} \log n} = (c_2 C)^{-1} \sqrt{\frac{n}{2^{i-1} \log^3 n}},\]

a contradiction which concludes the proof of the claim.

Now, observe that if the median s-number of T

\[s_{n/2}(T) < \frac{1}{2} (c_2 C)^{-1} \sqrt{\frac{n}{2^{i-1} \log n}}\]

then, by [Ma.1], Lemma 2.6, we obtain that

\[T \in \tilde{M}_n \left( \frac{1}{32} (c_2 C)^{-1} \sqrt{\frac{2^i}{\log n}} \right),\]

while if

\[s_{n/2}(T) \geq \frac{1}{2} (c_2 C)^{-1} \sqrt{\frac{n}{2^{i-1} \log n}}\]

then, by [Ma.1], Theorem 3.1, we infer that

\[T \in \tilde{M}_n \left( \frac{c_3}{2} (c_2 C)^{-1} \sqrt{\frac{n^3}{2^{i-1}}} \right),\]

where \(c_3 < 1\) is the constant from Theorem 3.1 in [Ma.1]. In the first case Fact II yields

\[\| T \|_{x_n} \geq \frac{c_1}{32} (c_2 C)^{-1} \sqrt{\frac{2^i}{\log n}} \geq \frac{c_1}{32} (c_2 C)^{-1} \sqrt{\frac{n^{2/3}}{\log n}},\]

while in the second case, by Fact II, we get

\[\| T \|_{x_n} \geq \frac{c_1 c_3}{2} (c_2 C)^{-1} \sqrt{\frac{n}{\log^3 n}}.\]

thus, by (3.6), in both cases we obtain

\[\sum_{i=1}^{n} c_i(T) \leq 32 c_2 C (c_1 c_3)^{-1} n^{2/3} \sqrt{\log^3 n} \| T \|_{x_n},\]

which proves (3.5) with \(c = 32 c_2 C (c_1 c_3)^{-1}\) and completes the proof of the theorem.

\[\square\]
4 The positive statements.

In this section we prove several “positive” statements about existence of “nontrivial” operators on generic Banach spaces, which will show that the results of the preceding section are “essentially” optimal (cf. Cor. 4.3). Results similar to some of the presented below were obtained independently by Gluskin [Gl.5]. The first of these statements will also show that, for generic Banach spaces, the following conjecture due to Pisier [Pi.1] (and usually referred to as the “dichotomy conjecture”) holds.

There exist \( C > 1 \) and a function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \lim_{\lambda \to \infty} f(\lambda) = \infty \), such that if \( E \subset l_\infty^N \), then there exists \( F \subset E \), \( \dim F = k \geq f(\dim E / \log N) \) verifying \( d(l_k^\infty, F) \leq C \).

At the time of this writing the conjecture is open for general \( E \). However, we have

**Proposition 4.1** There exist universal constants \( C, c > 0 \) such that if \( E \) is a generic quotient of \( l_1^N \), \( \dim E = d \) (resp. a generic subspace of \( l_\infty^N \)), then there is a subspace \( F \subset E \), \( \dim F = k \geq c \min\{d/2, d/\log N\} \) verifying

(i) \( d(l_k^1, F) \leq C \) (resp. \( d(l_k^\infty, F) \leq C \))

(ii) \( F \) is \( C \)-complemented in \( E \).

**Proposition 4.2** If, in the notation of Proposition 4.1, \( N \geq d^2 \), then there is \( G \subset E \), \( \dim G = h \geq \min\{c \log N, d\} \), satisfying

(i) \( d(l_h^2, G) \leq C \) (resp. \( d(l_h^\infty, G) \leq C \))

(ii) \( G \) is \( C \)-complemented in \( E \).

**Remark** If, in Proposition 4.2, one assumes that \( N > d^{1+\alpha} \) for \( \alpha \in (0, 1) \), one gets \( G \) which is \( C/\sqrt{\alpha} \) complemented in \( E \); for arbitrary \( N \) and \( d \) we get \( C \frac{\log N}{\log N/d} \) complementation.

**Corollary 4.3** A “generic” \( d \)-dimensional quotient of \( l_1^N \) contains a \( C \)-complemented subspace \( C \)-isomorphic to \( l_p^k \), \( k \geq c\sqrt{d} \), either for \( p = 1 \) or \( p = 2 \) (resp. a “generic” subspace of \( l_\infty^N \) contains such a subspace with either \( p = \infty \) or \( p = 2 \)).

**Proof of Corollary 4.3** If \( \log N < d^{1/2} \), we use Prop. 4.1 to get a \( C \)-complemented subspace \( C \)-isomorphic to \( l_1^k \) (resp. \( l_\infty^k \)). If \( \log N > d^{1/2} \) (hence \( N > d^2 \)), use Prop 4.2 to get a \( C \)-complemented Hilbertian subspace.

**Proof of Proposition 4.1** We will prove the statement for a random quotient \( Q \); the “subspace” variant follows by duality.

Recall that \( g_j = Qe_j \), \( j = 1, 2, \ldots, N \) are independent Gaussian vectors with distribution \( N(0, d^{-1}Id_{\mathbb{R}^d}) \) and that the unit ball \( B \) of our random quotient is absconv\{\( g_1, g_2, \ldots, g_N \}\}. Clearly, we can assume that \( N \leq \exp\{cd\} \). It then follows from Lemma 1.1 (iv) that

\[
P\left( \frac{1}{2} \leq \| g_j \| \leq 2 \text{ for } j = 1, 2, \ldots, N \right) \geq 1 - \exp(-c_1d)
\]  

(4.11)
provided that $c$ is chosen to satisfy $c \leq c_0/2$, where $c_0$ is the constant from Lemma 1.1. Moreover, if $k \leq d/2$ and $A \subset \{1, 2, \ldots, N\}$ with $|A| = k$, then (cf. Lemma 1.2)
\[
\| \sum_{j \in A} t_j g_j \| \geq c_2 \left( \sum_{j \in A} |t_j|^2 \right)^{1/2} \quad \text{for all choices of scalars } (t_j)_{j \in A} \tag{4.12}
\]
with the similar probability as in (4.11). In fact, since \( \binom{N}{k} < \left( \frac{Ne}{k} \right)^k \), (4.12) happens for all such $A$ with the same estimate on the probability as in (4.11) provided $k \log \frac{N}{k} \leq c_3 d$. In particular, this happens if $k \leq c \frac{d}{\log N}$ (we do not use this fact here).

In the next step we shall show that, for fixed $A \subset \{1, 2, \ldots, N\}$ with $|A| \leq k = c \min \{ d^{1/2}, \frac{d}{\log N} \}$, and $E = \{ g_j \mid j \in A \}$, we have
\[
\mathbb{P} \left( \| P_E g_j \| \leq k^{1/2} \mid j \not\in A \right) \geq 1 - \exp \left( -c_4 \frac{d}{k} \right) \tag{4.13}
\]

Observe that (4.12) and (4.13) imply the conclusion of the Proposition 4.1 with $C = c_2^{-1}$. To this end, note that the operator $u : l_1^k \to E$ sending $\{ e_1, e_2, \ldots, e_k \}$ into $\{ g_j \mid j \in A \}$ is of norm 1, while $\| u^{-1} P_E \| \leq c_2^{-1}$ (notice that (4.12) implies that $\| \cdot \|_B \leq c_2^{-1} k^{1/2} \| \cdot \|_2$ on $E$).

To prove (4.13), assume for simplicity that $A = \{1, 2, \ldots, k\}$. For fixed $\{ g_1, g_2, \ldots, g_k \}$, and hence fixed $E$, $\tilde{g}_j = P_E g_j$, $j = k + 1, k + 2, \ldots, N$ are independent $E$–valued Gaussian vectors with $N(0, \frac{1}{d} \mathbb{I}_E)$ distribution. In particular, $\mathbb{E} \| \tilde{g}_j \|_2^2 = \frac{k}{d}$ and, by Lemma 1.1 (ii),
\[
\mathbb{P} \left( \| \tilde{g}_j \| \geq k^{1/2} \right) \leq \exp \left( -\frac{k}{8} \left( \frac{k^{1/2}}{(k/d)^{1/2}} \right)^2 \right) = \exp \left( -\frac{d}{8k} \right),
\]
(we used the fact that $k \leq cd^{1/2}$, and hence $k^{1/2} \geq c^{-1}(\frac{k}{d})^{1/2} \geq 2(\frac{k}{d})^{1/2}$). Note a slight abuse of notation; in fact the expectation and the probability above are conditional on $\{ g_j \mid j \leq k \}$. To deduce (4.13), we need to know that $N \exp \left( -\frac{d}{8k} \right)$ is small. This happens e.g. when $k \leq c \frac{d}{10 \log N}$ and can be forced by the proper choice of $c$.

\[\square\]

**Proof of Proposition 4.2** Clearly, it is enough to prove the Proposition for quotients of $l_1^N$. The variant for subspaces of $l_\infty^N$ will follow by duality.

As follows from Lemma 1.4 (i), the unit ball $B$ of a generic $d$–dimensional quotient of $l_1^N$ contains a Euclidean ball $D_r$ with radius $r = c' \sqrt{\log (N/d) d}$. On the other hand, by Lemma 1.4 (ii),
\[
\left( \frac{|B|}{|D_r|} \right)^{1/2} \leq C'',
\]
and hence the so–called volume ratio of $B$ with respect to $D_r$ remains bounded by a universal numerical constant. In particular, (cf. Sz-T), this implies that, say, for $k \leq d/2$, and for a generic $k$–dimensional subspace $G$ and some universal constant $C$,
\[
G \cap D_r \subset G \cap B \subset C (G \cap D_r),
\]

10
which means that $G$ considered as a subspace of $B$ is $C$–Euclidean.

To conclude the proof we will show that if $P$ is the orthogonal projection onto the generic $k$–dimensional subspace $G$ of $\mathbb{R}^N$ and $k \geq \log N$, then

$$P(B) \subset C_1\sqrt{\frac{k}{d}}(G \cap D),$$

(4.14)

where $D$ denotes the Euclidean unit ball in $\mathbb{R}^n$. This will suffice, since if $k \simeq \log N$ and if $C$ is large enough then $C_1\sqrt{\frac{k}{d}} \leq Cc'\sqrt{\log \frac{N/d}{d}}$ (remember that $N \geq d^2$ and therefore $\log N \leq 2\log \frac{N}{d}$). Thus, (4.14) implies $P(B) \subset CD_r$. To prove (4.14) observe that for a fixed $G$ (and hence fixed $P$) and for $j \in \{1, 2, \ldots, N\}$, $Pg_j$ is a Gaussian random vector with distribution $N(0, \sqrt{\frac{k}{d}}Id_G)$. Therefore, by Lemma [1.1] (ii),

$$\mathbf{P}\left(\|Pg_j\|_2 \leq \sqrt{\frac{k}{d}}\right) \geq 1 - \exp\left(-\frac{1}{8}\lambda^2k\right)$$

for $\lambda \geq 2$. Choosing $\lambda$ sufficiently large and using the fact that $k \geq \log N$ we can obtain a similar estimate for $\mathbf{P}\left(\|P_{g_j}\|_2 \leq \sqrt{\frac{k}{d}}\mid j = 1, 2, \ldots, N\right)$. Finally, by the rotational invariance of the joint distribution of $g_j$’s and the Fubbini theorem we deduce that a “generic” $B$, a “generic” $P$ and $G$ satisfy (4.14).

\[\square\]
References

[Ca.–P] Carl, B. & Pajor A. Gelfand numbers of operators with values in a Hilbert space. *Invent. Math.* 94 (1988), 479–504.

[Gl.1] Gluskin, E. D., The diameter of the Minkowski compactum is roughly equal to $n$. *Func. Anal. i Priloz.*, 15 (1) (1981), 72–73 (in Russian).

[Gl.2] Gluskin, E. D., Finite dimensional analog of a space without basis. *Dokl. AN SSSR* 261 (5) (1981), 1046–1050 (in Russian)

[Gl.3] Gluskin, E. D., Oktaeder is poorly approximable by random subspaces. *Func. Anal. i Priloz.*, 20 (1) (1986), 14–20 (in Russian)

[Gl.4] Gluskin, E. D., Extremal properties of orthogonal parallelograms and their applications to the geometry of Banach spaces. *Mat. Sbor.* 136 (178) (1988), 85–96 (in Russian)

[Gl.5] Gluskin, E. D., Personal communications.

[F-L-M] Figiel, T., Lindenstrauss, J., & Milman, V. D., The dimension of almost spherical sections of convex bodies. *Acta Math.* 139 (1977), 56–94.

[Ma.1] Mankiewicz, P., Factoring the identity operator on a subspace of $l_n^\infty$. *Studia Math.*, 95 (1989), 134–139.

[Ma.2] Mankiewicz, P., Subspace mixing properties of operators in $\mathbb{R}^n$ with applications to Gluskin spaces. *Studia Math.*, 88 (1988), 51–67.

[Ma.–T] Mankiewicz, P. & Tomczak–Jaegermann, N., Random subspaces and quotients of finite dimensional Banach spaces. *Preprint. Odense University* (1998).

[P–T] Pajor, A. & Tomczak–Jaegermann, N., Subspaces of small codimensions of finite dimensional Banach spaces. *Proc. AMS*, 97 (1986), 637-642.

[Pi.1] Pisier, G., Remarques sur un resultat non publie de B. Maurey. *Seminaire d’Analyse Fonctionelle, Ecole Polytechnique*, (1980–1981) exp. 5.

[Pi.2] Pisier, G., A new approach to several results of V. Milman. *J. Reine Angew. Math.*, 393 (1989), 115–131.

[Si.] Silverstein, J., The smallest eigenvalue of a large dimensional Wishart matrix. *Ann. Probab.*, 13 (1985), 1364–1368.

[Sz.1] Szarek, S. J., On finite dimensional basis problem with an appendix on nets of Grassmann manifolds. *Acta Math.* 151 (1983), 153–179.

[Sz.2] Szarek, S. J., On the existence and uniqueness of complex structure and spaces with few operators. *Trans. Amer. Math. Soc.*, 293 (1986), 339–353.
[Sz.3] Szarek, S. J., Spaces with large distance to $l^n_\infty$ and random matrices. *Amer. J. Math.* 112 (1990), 899–942.

[Sz-T] Szarek, S. J. & Tomczak-Jaegermann, N., On nearly Euclidean decomposition for some classes of Banach spaces. *Comp. Math.*, 40 (1980), 367–385.

[T] Tong, Y., Probability Inequalities in Multivariate distributions. *Academic Press*, New York.