Calculation of the invariant characteristics of forced oscillations of a beam with longitudinal compression and it’s phenomenological model

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Abstract. This paper is devoted to the comparison of the invariant characteristics of experimental data and solutions of the phenomenological model. Set of experimental data was obtained in the process of registering oscillations of a beam with longitudinal compression. In the course of the experiment excitation of oscillations is performed by exposure of alternating magnetic field on a magnet placed on the loose end. Depending on the frequency of external stimulus both chaotic and periodic oscillations were obtained. Based on the analysis of experimental arrays a phenomenological model was developed. To verify it’s adequacy, a number of invariant numerical characteristics were calculated, including the frequency spectrum, correlation dimension, Lyapunov exponents, and $\beta$-statentropy. Based on the comparison of the characteristics obtained for the experimental data and for the mathematical model, it is concluded that it is qualitatively corresponds with experiment.

1. Introduction
Mechanical and electro-mechanical systems with chaotic behavior have long been in the sphere of keen interest of both theorists and creators of experimental devices with chaotic behavior. In addition to that beam oscillatory systems with several minima of potential under the influence of external periodic forces have been actively studied recently. Rather considerable literature is devoted to magnetoelastic beams with Euler instability, we refer only to works [1, 2, 3] and the bibliography in them.

2. Description of the experiment
2.1. Beam structure
The beam under consideration was implemented on the basis of a pair of steel strips connected at the loose ends and compressed in the longitudinal direction with a strained kevlar thread. At the loose end there is an aluminium foil shutter and a small SmCo5 magnet fixed on it, which is affected by an alternating magnetic field, their mass can be neglected, since it is rather small. The whole structure is bolted to a massive ($\approx 5$ kg) steel plate, which fixes the entire installation (see fig. 1, a more detailed description is contained in [4]).
2.2. Phase portraits of oscillations

Depending on the frequency of external stimulus, a large set of data was obtained. This set contains both ordered periodic and chaotic oscillations. It can be noted that the transition to chaos is carried out by period doubling. To reconstruct the attractor \((x_n, x_{n+\tau})\), a set of phase portraits was built, some of which are shown in Fig. 2, where \(\tau\) is equal to the index corresponding to the first zero or the first value of the autocorrelation function nearest to zero.

![Phase portraits of oscillations](image)

**Figure 2.** Phase portraits of the reconstructed from experimental data attractor

3. Phenomenological model

3.1. Construction of the mathematical model of the experiment

To construct the phenomenological model of the experiment described, several assumptions were made (see also descriptions of the construction of similar models in [5, 6, 7]). First of all, the movement of the material point in the force field was considered, and not the movement of the beam. Thus, the potential function must have the following properties: it must have a maximum at zero \((x = 0)\) deviation and two minima at \(x = \pm x_0\). This means that the equilibrium position at \(x = 0\) is unstable, and the equilibrium at \(x = \pm x_0\) is locally stable. The simplest realization of such potential is given by a fourth-degree polynomial

\[
U(x) = Ax^4 - Bx^2.
\]

Putting down the equation of the balance of forces, we have the equation

\[
m\ddot{x} = F = F_{\text{pot}} + F_{\text{diss}} + F_{\text{ext}} = -\frac{dU}{dx} - \eta\dot{x} + F_{\text{ext}},
\]

where \(m\) is the mass, \(F_{\text{pot}}\) is the potential force, \(F_{\text{diss}}\) is the friction force and \(F_{\text{ext}}\) is the external force. Given the type of potential (1), we have

\[
F_{\text{pot}} = -4Ax^3 + 2Bx = -4Ax(x^2 - x_0^2),
\]

and \(x = \pm x_0 = \pm \sqrt{B/(2A)}\) are points of stable equilibrium. Whereas we assume that the external force is given by the formula \(F_{\text{ext}} = F_0 \cos(\omega t)\), thereby, the equation (2) is written as
\[ \ddot{x} = -\eta m^{-1} \dot{x} - 4Am^{-1}(x^2 - x_0^2) + F_0 m^{-1} \cos(\omega t). \]  
(3)

For the variable \( x \) we select a new scale \( y = xx_0 \), then

\[ \ddot{y} = -\eta m^{-1} \dot{y} - 4Ax_0^2 m^{-1} y(y^2 - 1) + F_0 m^{-1} x_0^{-1} \cos(\omega t). \]  
(4)

Consider oscillations near one of the points \( y = \pm 1 \). By replacing \( y = 1 + \xi \) or \( y = -1 + \xi \) and assuming external forces to be absent, we linearize the resulting equation at zero. As a result, the equation of small free oscillations near the equilibrium points will look like

\[ \ddot{\xi} = -\eta m^{-1} \dot{\xi} - 8Ax_0^2 m^{-1} \xi. \]

In this case the oscillation frequency is equal to \( \omega_0^2 = 8Ax_0^2 m^{-1} \). This allows us to normalize the time \( \tau = \omega_0 t \), as the result we obtain the Duffing equation with external influence:

\[ \ddot{y} + \alpha \dot{y} + \frac{1}{2} y(y^2 - 1) = f \cos(\Omega \tau), \]  
(5)

where \( \alpha = \eta \omega_0 m^{-1} \) is the loss parameter, \( f = F_0 (mx_0 \omega_0^2)^{-1} \) is the parameter of the intensity of external influence, \( \Omega = \omega_0^{-1} \) is the frequency of the external impact, normalized on the frequency of small free oscillations.

### 3.2. Selection of parameters of the phenomenological model

It is known that, when the parameters in the equation (5) change, the Feigenbaum scenario of transition to chaos through a cascade of period doubling bifurcations can be realized (see for example [5]). This being the case, the following method was proposed for selecting the parameters of the problem (5). For fixed values of the parameters \( \alpha \) and \( f \) of the equation (5) such that when \( \Omega \) changes, there is a Feigenbaum scenario of transition to chaos, the frequency \( \Omega \), at which the first period doubling occurred, and the interval \( \Omega \), in which (5) has a chaotic attractor is determined. Further, the parameters \( \alpha \) and \( f \) are chosen in such a way that the ratio of the length of the chaotic interval to the first doubling is close to the experimental data. It appeared that, the most similar pattern to the experiment is observed at the values of the parameters \( \alpha = 0.988 \) and \( f = 0.745 \). With these values, the first doubling of the period occurs approximately at \( \Omega \approx 0.71 \), and chaotic oscillations occur in the range from \( \Omega \approx 0.76 \) to \( \Omega \approx 0.854 \) (see fig. 3). It is convenient to trail the doubling bifurcations and chaotic oscillations of the (5) model by the values of the Lyapunov indices (Fig. 4) calculated on the attractor (an algorithm from [8, 9] is used).

![Figure 3. Phase portraits of the attractor of the phenomenological model (5)](image-url)
4. Comparison of the invariant characteristics of the experimental system and the phenomenological model

To effectively compare the qualitative behavior of the solutions of the phenomenological model (5) and experimental data, it is required to find some invariant characteristics, which could be found in both cases. Comparing these indicators, it is possible to choose the parameters of the problem (5) so that they are close to the corresponding characteristics from the experiment. It is important to note that a direct comparison of the chaotic solution of the equation (5) and experimental data cannot give close results due to the fact that the trajectories of the chaotic regime (5) in the case of the positivity of the highest Lyapunov exponent exponentially scatter.

The following parameters were chosen as invariant characteristics by which the behavior of the experimental and model systems will be compared: the spectral density function, the correlation integral and the correlation dimension, and finally, the $\beta$-statentropy.

4.1. Calculation of spectral density functions

One of the standard methods for assessing the characteristics of signals is the spectral analysis of the received data sets. For data corresponding to different frequencies of external influence in the experiment, the frequency spectra were calculated on the basis of the fast Fourier transform. Also, for the equation (5) with the above values of the parameters, the data sets were formed and the frequency spectra were also calculated for them. The resulting spectral density functions are shown in Fig. 5 and 6. Comparison of these functions shows that they correspond to each other, but the presence in the experimental data of the random component does not make it possible to use this method to the full extent.

4.2. Calculation of the correlation dimension

Another important invariant characteristic of an attractor is its correlation dimension $D_c$. The definition of this characteristic is based on the calculation of the correlation integral (see [10])

$$C(\delta) = \frac{1}{n^2} \sum_{i,j} \theta(\delta - |y_i - y_j|), \quad \text{where} \quad \theta(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}, \quad (6)$$
Figure 6. Frequency spectrum of model data

Figure 7. Dependence of the correlation dimension on the frequency of external influence

4.3. Calculation of the \( \beta \)-statentropy

Numerical analysis of the experimental data and the model equation showed that the most preferable invariant characteristic of the attractor, which is reliably calculated in both cases is the \( \beta \)-statentropy. In accordance with the algorithm described in [11, 12], the \( \beta \)-statentropy is calculated using the following formulas:

\[
\eta_n^{(k)}(\beta) = \begin{cases} 
\frac{\beta r_n^{(k)}(\beta)}{k(r_n^{(k+1)}(\beta) - r_n^{(k)}(\beta))}, & \beta \neq 0 \\
\frac{1}{k(r_n^{(0)}(0) - r_n^{(k+1)}(0))}, & \beta = 0
\end{cases}
\]

where

\[
r_n^{(k)}(\beta) = \frac{1}{n} \sum_{j=1}^{n} \gamma \left( \min_{i \neq j} \rho(\xi_i, \xi_j) \right),
\]

\[
\gamma(t) = \begin{cases} 
t^\beta, & \beta \neq 0 \\
-\ln t, & \beta = 0
\end{cases}
\]

Here \( \xi_j, j = 1, \ldots, n \) are specially processed elements of the data array, \( \min_{i \neq j} \rho(\xi_i, \xi_j) \) is the \( k \)-th minimum distance between points \( \xi_j \) and \( \xi_i \). The most illustrative results were obtained using the formula (7) with \( \beta = 0, k = 0 \). Table 1 summarizes the results for selected experimental data arrays and their corresponding solutions of the (5) equation. Chaotic oscillations correspond to positive values of \( \eta \), and ordered (in this case, periodic) values correspond to zero. It is easy to see that the entropy estimation for experimental and model data is in coherence with each other.
Table 1. Results of $\beta$-statentropy calculation.

| Frequency (Hz) | $\eta$  | $\Omega$ | $\eta$  |
|---------------|---------|----------|---------|
| 8.9286        | 0.1387  | 0.7600   | 0.1054  |
| 8.6806        | 0.1379  | 0.7750   | 0.1248  |
| 8.4459        | 0.1371  | 0.7900   | 0.1365  |
| 8.2237        | 0.137   | 0.8040   | 0.139   |
| 8.0128        | 0.1349  | 0.8180   | 0.1343  |
| 7.8125        | 0.1327  | 0.8530   | 0.1146  |
| 7.6220        | 0.1073  | 0.8602   | 0.0828  |

5. Conclusion
In conclusion, note that a large amount of data obtained in the course of the full scale experiment allowed to propose a phenomenological model of the process. Comparison of the invariant numerical characteristics of the experimental data and the solution of the phenomenological model showed that it adequately reflects the basic properties of the experimental system. The most successful research tool is the calculation of the $\beta$-statentropy, which gives most fitting results for the experiment and the model.

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References
[1] Moon F C, Holmes P J 1979 A magnetoelastic strange attractor *Journal of Sound and Vibration* 65:2 275 - 296
[2] Tam J I, Holmes P 2014 Revisiting a magneto-elastic strange attractor *Journal of Sound and Vibration* 333:6 1767 - 1780
[3] Kumar K A, Ali S F, Arockiarajan A 2017 Magneto-elastic oscillator: Modeling and analysis with nonlinear magnetic interaction *Journal of Sound and Vibration* 397 265 - 284
[4] Glyzin S D, Lokhanin M V, Sirotin D M Invariant Characteristics of Forced Oscillations of a Beam with Longitudinal Compression *Modeling and Analysis of Information Systems* 25:1 54 - 62. (In Russian)
[5] Ueda Y 1978 Randomly transitional phenomena in the system governed by Duffings equation, *Nagoya University Institute of Plasma Physics*, Japan, 1978, IPPJ - 341.
[6] Kapitanov D V, Ovchinnikov V F, Smirnov L V The dynamics of an axially loaded elastic bar after loss of stability *Problems of strength and plasticity*, 76:3 205 - 216 (in Russian)
[7] Sararin V A About chaotic behaviour of an electrostatic pendulum at parametrical influence *Bulletin of Perm University. Series: Physics* 27-28 18 - 23 (in Russian)
[8] Wolf A, Swift J B, Swinney H L, Vastano J A 1985 Determining Lyapunov exponents from a time series *Physica D: Nonlinear Phenomena* 16 285 - 317
[9] Glyzin D S, Glyzin S D, Kolesov A Yu and Rozov N Kh 2005. The Dynamic Renormalization Method for Finding the Maximum Lyapunov Exponent of a Chaotic Attractor *Differential Equations* 41:2 284 - 289
[10] Grassberger P, and Procaccia I 1983 Measuring the strangeness of strange attractors *Physica D: Nonlinear Phenomena* 9(1) 189 - 208.
[11] Timofeev E A 2005 A Consistent Estimator of the Entropy of Measures and Dynamical Systems *Math. Notes* 77:6 831 - 842
[12] Timofeev E A 2006 Invariants of measures admiting statistical estimates *St. Petersburg Math. J.* 17:3 527-551.