ON NILPOTENT LEIBNIZ SUPERALGEBRAS

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Abstract. The aim of this work is to present the first problems that appear in
the study of nilpotent Leibniz superalgebras. These superalgebras and so the
problems, will be considered as a natural generalization of nilpotent Leibniz
algebras and Lie superalgebras.

2000 MSC: 17A32, 17B30.
Key-Words: Lie superalgebras, Leibniz superalgebras, nilindex.

1. Introduction

The notion of Leibniz superalgebras was firstly introduced in [1], although graded
Leibniz algebra was considered before in work [8]. As Leibniz algebras are a gener-
alization of Lie algebras [9], then many of the features of Leibniz superalgebras are
generalization of Lie superalgebras.

The study of nilpotent Leibniz algebras [1], [3], [4] shows that many nilpo
tent properties of Lie algebras can be extended for nilpotent Leibniz algebras. The
results of nilpotent Leibniz algebras may help us to study nilpotent Leibniz super-
algebras. However, nilpotent Leibniz superalgebras turn out more complex than
nilpotent Lie superalgebras.

This is the frame of our work: nilpotent Leibniz superalgebras. In the case of
Leibniz algebras appears the notion of zero-filiform algebra, this notion does not
exist in Lie algebras. This algebra has maximal nilindex. In Leibniz superalgebras
we offer the analogue of zero-filiform Leibniz algebras, that is zero-filiform Leibniz
superalgebras, $ZF^{n,m}$. But not all of zero-filiform Leibniz superalgebras have max-
imal nilindex, there is only one in particular pair of dimensions: $(n,n)$, $(n,n+1)$,
as could be seen in theorem 3.9.

Before to studying general classes of Leibniz superalgebras (zero-filiform and
filiform Leibniz superalgebras) we had to solve the problem of finding a suitable
basis; a so-called adapted basis, see theorems 3.5 and 3.8. The function $f(n,m)$,
defined as the maximal nilindex for Lie superalgebras of type $(n,m)$, is always
$\leq n + m - 1$ [10] but the same function $f(n,m)$ on Leibniz superalgebras can
be $n + m$. Also, this value is only obtained for the special zero-filiform Leibniz
superalgebras that appear in the particular pair of dimensions mentioned above.
By applying direct sum with $\mathbb{C}$ to these special Leibniz superalgebras we obtain
that $f(n+1,n) = 2n$ and $f(n,n+2) = 2n + 1$, see theorem 3.11. Maximal nilindex
determination is an open problem for the general case.

This work was supported by the PAICYT, FQM143 of the Junta de Andalucía (Spain),
by INTAS (Ref. Nr. 04-83-3035) and by Junta de Extremadura-Consejería de Infraestructuras y
Desarrollo Tecnológico and Feder (Ref. Nr. 3PR05A074).
Analogously as for Lie superalgebras [10] we will refer the nilpotent Leibniz superalgebras of type \((n, m)\), with nilindex \(f(n, m)\), as maximal class Leibniz superalgebras. We will denote the variety of these Leibniz superalgebras as \(\mathcal{M}^{n,m}\).

In this work we have obtained many results concerning nilindex and the function maximal nilindex \(f(n, m)\); we have found many of superalgebras with open orbits which determine irreducible components of the variety of nilpotent Leibniz superalgebras, and we have obtained the relative position of the subvarieties \(\mathcal{M}^{n,m}\) and \(\mathcal{Z}F^{n,m}\) in some cases. Also we conjecture that there exists one unique Leibniz superalgebra of type \((n+1, n)\) with nilindex equal to \(2n\) (conjecture 2).

In this paper, most of classification proofs are omitted because of they are very laborious and they do not apport any new idea.

2. Preliminaries

The vector space \(V\) is said to be \(\mathbb{Z}_2\)-graded if it admits a decomposition in direct sum, \(V = V_0 \oplus V_1\); An element \(X\) of \(V\) is called homogeneous of degree \(\gamma\), \(\gamma \in \mathbb{Z}_2\), if it is an element of \(V_\gamma\). In particular, the elements of \(V_0\) (resp. \(V_1\)) are also called even (resp. odd).

Let \(V = V_0 \oplus V_1\) and \(W = W_0 \oplus W_1\) be two \(\mathbb{Z}_2\)-graded vector spaces. A linear mapping \(f : V \to W\) is said to be homogeneous of degree \(\gamma\), \(\gamma \in \mathbb{Z}_2\), if \(f(V_\alpha) \subseteq W_{\alpha+\gamma(mod2)}\) for all \(\alpha \in \mathbb{Z}_2\). In particular, if the linear mapping is homogeneous of degree 0 is said to be a homomorphism of the two \(\mathbb{Z}_2\)-graded vector spaces. Now it is clear how we define an isomorphism or an automorphism of \(\mathbb{Z}_2\)-graded vector spaces.

We say that two Leibniz superalgebras, \(L_1\) and \(L_2\), are isomorphic if there exists a \(\mathbb{Z}_2\)-graded vector space isomorphism, \(\varphi : L_1 \to L_2\), satisfying \(\varphi([X, Y]) = [\varphi(X), \varphi(Y)]\) for all \(X, Y\) of \(L_1\). Then \(\varphi\) is called an isomorphism of Leibniz superalgebras and it is always assumed to be consistent with \(\mathbb{Z}_2\)-graduations; that is, they are homogeneous linear mappings of degree zero.

**Definition 2.1.** A \(\mathbb{Z}_2\)-graded vector space \(L = L_0 \oplus L_1\) is called a Leibniz superalgebra if it is equipped with a product \([-,-]\) which satisfies the following conditions:

\[
[L_\alpha, L_\beta] \subseteq L_{\alpha + \beta(mod2)} \text{ for all } \alpha, \beta \in \mathbb{Z}_2
\]

\[
[x, [y, z]] = [[x, y], z] - (-1)^{\alpha\beta}[[x, z], y] \text{ graded Leibniz identity}
\]

for all \(x \in L, y \in L_\alpha, z \in L_\beta, \alpha, \beta \in \mathbb{Z}_2\).

Note that if a Leibniz superalgebra \(L\) satisfies the identity \([x, y] = -(-1)^{\alpha\beta}[y, x]\) for all \(x \in L_\alpha, y \in L_\beta\), then the graded Leibniz identity is reduced to the following graded Lie identity:

\[
(-1)^{\alpha\gamma}[x, [y, z]] + (-1)^{\alpha\beta}[y, [z, x]] + (-1)^{\beta\gamma}[z, [x, y]] = 0.
\]

Therefore Leibniz superalgebras are a generalization of Lie superalgebras.

If we denote by \(R_X\) the right multiplication operator, i.e. \(R_X : L \to L\), then the graded Leibniz identity can be expressed in the following form:

\[
R_{[X,Y]} = R_Y R_X - (-1)^{\alpha\beta} R_X R_Y \quad (1)
\]

where \(X \in L_\alpha, Y \in L_\beta\).
We denote by $R(L)$ the set of all right multiplication operators. It is not difficult to prove that $R(L)$ with the multiplication defined by:
\[
<R_a, R_b> := R_a R_b - (-1)^{\alpha \beta} R_b R_a
\]  
for $R_a \in R(L)_\alpha$, $R_b \in R(L)_\beta$, becomes a Lie superalgebra.

In order to provide an example of non-Lie Leibniz superalgebra, we can consider an associative superalgebra, $A = A_0 \oplus A_1$, and a linear mapping $D : A \to A$ satisfying the condition:
\[
D(a(Db)) = Da Db = D((Da)b)
\]
for all $a, b \in A$ and define a new multiplication over the underlying $\mathbb{Z}_2$-graded vector space, $<, >$, by:
\[
<a, b>_D := a(Db) - (-1)^{\alpha \beta} D(b)a
\]
for $a \in A_\alpha, b \in A_\beta$. Then $A$ equipped with multiplication $<,>$ becomes a Leibniz superalgebra, which in general is not a Lie superalgebra.

The descending central sequence of a Leibniz superalgebra $L = L_0 \oplus L_1$ is defined by $C^0(L) = L$, $C^{k+1}(L) = [C^k(L), L]$ for all $k \geq 0$. If $C^k(L) = 0$ for some $k$, the Leibniz superalgebra is called nilpotent. The smallest natural number $k$ such as $C^k(L) = 0$ is called the nilindex of $L$.

We denote by $\text{Nil}^{n,m}$ the variety of nilpotent Leibniz superalgebras $L = L_0 \oplus L_1$ with $\dim L_0 = n$, $\dim L_1 = m$; and by $\text{Leib}^{n,m}$ denote the variety of Leibniz superalgebras. The above property $C^k(L) = 0$ can be realized via finite numbers of polynomial relations on structure constants and therefore, this set forms a subvariety of the variety $\text{Leib}^{n,m}$.

For Leibniz superalgebras we have the following analogue of Engel’s theorem:

**Theorem 2.2. (Engel’s theorem)** \[^{[1]}\] A finite dimensional Leibniz superalgebra $L$ is nilpotent if and only if the operators $R_X$ are nilpotent for all $X \in L$.

If we take an homogeneous basis $\{X_0, ..., X_{n-1}, Y_1, ..., Y_m\}$ for $L$ ($L \in \text{Leib}^{n,m}$), the superalgebra is completely determined by:
\[
[X_i, X_j] = \sum_{k=1}^{n-1} C^k_{ij} X_k, \quad [X_i, Y_j] = \sum_{k=1}^{m} D^k_{ij} Y_k,
\]
\[
[Y_i, X_j] = \sum_{k=1}^{m} E^k_{ij} Y_k, \quad [Y_i, Y_j] = \sum_{k=1}^{n-1} F^k_{ij} X_k,
\]
where $\{C^k_{ij}, D^k_{ij}, E^k_{ij}, F^k_{ij}\}$ are structure constants. These structure constants verify the restrictions obtained by the graded Leibniz identity \[^{[1]}\].

Let $V = V_0 \oplus V_1$ be the underlying vector space of $L$, $L = L_0 \oplus L_1 \in \text{Leib}^{n,m}$ and let $G(V)$ be the group of the invertible linear mappings of the form $f = f_0 + f_1$ such that $f_0 \in GL(n, C)$ and $f_1 \in GL(m, C)$ ($G(V) = GL(n, C) \oplus GL(m, C)$). The action of $G(V)$ on $\text{Leib}^{n,m}$ induces an action on the Leibniz superalgebras variety: two laws $\lambda_1$ and $\lambda_2$ are isomorphic, if there exists a linear mapping $f$, $f = f_0 + f_1 \in G(V)$, such that
\[
\lambda_2(X, Y) = f^{-1}_{\alpha + \beta}(\lambda_1(f_0(X), f_\beta(Y))) \quad \text{for all} \quad X \in V_\alpha, Y \in V_\beta.
\]
We denote by $O(\lambda)$ the orbit of $\lambda$ corresponding to this action.
The superalgebras with open orbits in $N^{n,m}$ are called rigid. The closures of these open orbits give irreducible components of the variety $N^{n,m}$. Then, the fact of finding such algebras is crucial for the description of the variety $N^{n,m}$.

The description of the variety of any class of algebras or superalgebras is a difficult problem. Different papers, for example [2], [5], [7], [11] are concerning the applications of algebraic groups theory to the description of the variety of Lie algebras.

**Definition 2.3.** For a Leibniz superalgebra $L = L_0 \bigoplus L_1$ we define the set $Z(L)$, $Z(L) = \{X \in L : [L, X] = 0\}$ which will be called the right annihilator of $L$.

It is easy to see that $Z(L)$ is a two-sided ideal of $L$ and $[X, X] \in Z(L)$ for any $X \in L_0$, this notion is good and compatible with the right annihilator in Leibniz algebras. If we consider $I = \text{ideal} < [X, Y] + (-1)^{\beta \gamma}[Y, X] : X \in L_\alpha, Y \in L_\beta>$, then $I \subseteq Z(L)$.

**Definition 2.4.** For a Leibniz superalgebra $L$ we define the sets

\[
L(L) = \{X \in L : [X, L] = 0\},
\]

\[
\text{Cent}(L) = \{X \in L : [X, L] = [L, X] = 0\}
\]

which are called left annihilator and center of $L$, respectively.

We can extract a result of Leibniz algebras [2] and apply it to Leibniz superalgebras: for any $s, r \in \mathbb{N}$ the following subsets of $\text{Leib}^{n,m}$ are closed relatively to the Zariski topology:

1. $\{\mu \in \text{Leib}^{n,m} | \dim \mu^s \leq r\}$
2. $\{\mu \in \text{Leib}^{n,m} | \dim \mu \geq s\}$
3. $\{\mu \in \text{Leib}^{n,m} | \dim \mu^s \geq r\}$
4. $\{\mu \in \text{Leib}^{n,m} | \dim \text{Cent}(\mu) \geq s\}$

where $\mu^s = C^s(\mu)$. Hence, a superalgebra $\mu$ does not belong to $\text{clO}(\lambda)$ if one of the following conditions holds:

1. $\dim \lambda^s < \dim \mu^s$ for some $s$,
2. $\dim \lambda > \dim \mu$,
3. $\dim \mu^s > \dim \mu$,
4. $\dim \text{Cent}(\lambda) > \dim \text{Cent}(\mu)$.

3. Details in Leibniz Superalgebras

Let $L = L_0 \bigoplus L_1$ be a nilpotent Leibniz superalgebra with $\dim L_0 = n$ and $\dim L_1 = m$. From (2) we have that $R(L)$ is a Lie superalgebra, in particular $R(L_0)$ is a Lie algebra. As $L_1$ has $L_0$-module structure we can consider $R(L_0)$ as a subset of $GL(V)$, where $V$ is vector space corresponding to $L_1$. So, we have a Lie algebra formed by nilpotent endomorphisms of $V$. And applying the Engel’s theorem [6] we have existence of subspaces of $V$:

\[
V_0 \subseteq V_1 \subseteq V_2 \subseteq ... \subseteq V_m = V,
\]

with $\dim(V_i) = i$ and $R(L_0)(V_{i+1}) \subseteq V_i$.

We define two new descending sequences, $C^k(L_0), C^k(L_1)$ as follows: $C^0(L_i) = L_i, C^{k+1}(L_i) = [C^k(L_i), L_0]$ for $k \geq 0, i \in \{0,1\}$. Analogously as for Lie superalgebras [10], if $L = L_0 \bigoplus L_1$ is a nilpotent Leibniz superalgebra, then $L$ has
super-nilindex or s-nilindex \((p, q)\) if the following conditions hold:

\[(C^{p-1}(L_0))(C^{q-1}(L_1)) \neq 0, \quad C^p(L_0) = C^q(L_1) = 0\]

We have for Lie superalgebras the invariant called characteristic sequence that can be naturally extended for Leibniz superalgebras. Thus, we have the following definition.

**Definition 3.1.** For an arbitrary element \(X \in L_0\), the operator \(R_X\) is a nilpotent endomorphism of space \(L_i\), where \(i \in \{0, 1\}\). We denote by \(g_z(X)\) descending sequence of dimensions of Jordan blocks of \(R_X\). We define the invariant of a Leibniz superalgebra \(L\) as follows:

\[g_z(L) = \left( \max_{X \in L_0 - [L_0, L_0]} g_z(X), \max_{\tilde{X} \in L_0 - [L_0, L_0]} g_z(\tilde{X}) \right),\]

where \(g_z\) is the lexicographic order.

The couple \(g_z(L)\) is called characteristic sequence of Leibniz superalgebra \(L\).

We denote by \(N^{n,m}_{p,q}\) the subset of the set \(N^{n,m}\), with s-nilindex \((k_0, k_1)\), where \(k_0 \leq p\) and \(k_1 \leq q\).

**Lemma 3.2.** The set \(N^{n,m}_{p,q}\) is a subvariety of the variety \(N^{n,m}\).

**Proof.** The proof of this lemma is evident, because the set \(N^{n,m}_{p,q}\) can be realized via finite numbers of polynomial equations of the structure constants. \(\square\)

**Definition 3.3.** A Leibniz superalgebra \(L\), \(L \in N^{n,m}\), is called zero-filiform if its s-nilindex is \((n, m)\).

We denote by \(ZF^{n,m}\) the set of zero-filiform Leibniz superalgebras.

**Remark 3.4.**

1) If \(L = L_0 \bigoplus L_1\) is a zero-filiform Leibniz superalgebra then from [3] we have that \(L_0\) is a zero-filiform Leibniz algebra.

2) Since \(ZF^{n,m} = N^{n,m} \setminus (N^{n,m}_{n-1,m} \cup N^{n,m}_{n,m-1})\), then \(ZF^{n,m}\) is an open set in Zariski topology.

3) Note that zero-filiform Leibniz superalgebra \(L\) of type \((n, m)\) can be realized as superalgebra with \(g_z(L) = (n|m)\).

Before to studying general classes of Leibniz (super)algebras, it is useful to solve the problem of finding a suitable basis; a so-called adapted basis. This question is not trivial even for Lie superalgebras and it is difficult to demonstrate the general existence of such a basis for Leibniz superalgebras. Particularly, we prove in the following theorem that there always exists an adapted basis for the class of zero-filiform Leibniz superalgebras.

**Theorem 3.5.** If \(L = L_0 \bigoplus L_1 \in ZF^{n,m}\), then there exists an adapted basis of \(L\), namely \(\{X_0, X_1, ..., X_{n-1}, Y_1, Y_2, ..., Y_m\}\), with \(\{X_0, X_1, ..., X_{n-1}\}\) a basis of \(L_0\) and \(\{Y_1, Y_2, ..., Y_m\}\) a basis of \(L_1\), such that:

\[
[X_i, X_0] = X_{i+1}, \quad 0 \leq i \leq n - 2, \quad [X_{n-1}, X_0] = 0,
[Y_j, X_0] = Y_{j+1}, \quad 1 \leq j \leq m - 1, \quad [Y_m, X_0] = 0.
\]

Moreover \([Y_j, X_k] = 0\) for \(1 \leq j \leq m\) and \(1 \leq k \leq n - 1\), and the omitted products of \(L_0 = \langle X_0, X_1, ..., X_{n-1} \rangle\) vanish.
Proof: As \( L = L_0 \bigoplus L_1 \) is a zero-filiform Leibniz superalgebra, then \( L_0 \) is a zero-filiform Leibniz algebra. Thus from \(^3\) we have an adapted basis for \( L_0 : \{ X_0, X_1, \ldots, X_{n-1} \} \) with \( [X_i, X_0] = X_{i+1} \), \( 0 \leq i \leq n - 2 \), \( [X_{n-1}, X_0] = 0 \), \( [X_i, X_k] = 0 \) for \( 0 \leq i \leq n - 1 \), and \( 1 \leq k \leq n - 1 \).

As we stated at the beginning of this section we have:

\[
0 \subset V_1 \subset \ldots \subset V_m \text{ with } \dim(V_{i+1}/V_i) = 1,
\]

where each \( V_i \) is the vector space of generators: \( \{ Y_1, Y_2, \ldots, Y_i \} \), \( V_i = \langle Y_1, Y_2, \ldots, Y_i \rangle \), with \( [V_{i+1}, L_0] = V_i \).

So, \( [V_1, L_0] = 0 \) and then \( [Y_1, X_0] = 0 \) \( \forall i \). As \( [V_2, L_0] = V_1 \) we have that exists a non-null scalar namely \( \lambda_2 \) such that \( [Y_2, X_0] = \lambda_2 Y_1 \). By induction, it is possible to prove that there exists a set of non-null scalars \( \{ \lambda_2, \lambda_3, \ldots, \lambda_m \} \) and vectors \( \{ X_{i_2}, X_{i_3}, \ldots, X_{i_m} \} \subset \{ X_0, X_1, \ldots, X_{n-1} \} \) verify

\[
[Y_k, X_{i_k}] = \lambda_k Y_{k-1} + \Psi_k(Y_{k-2}, \ldots, Y_1), \quad 2 \leq k \leq m,
\]

(3)

where \( \Psi_k(v_1, v_2, \ldots, v_k) \) represents a linear combination of the vectors \( \{ v_1, v_2, \ldots, v_k \} \).

Using the graded Leibniz identity we can assert that \( i_2 = i_3 = \cdots = i_m = 0 \). In fact, if there exists \( i_k \in \{ 1, \ldots, n - 1 \} \) we have

\[
[Y_k, X_{i_k}] = [Y_k, [X_{i_k-1}, X_0]] = \left( [Y_k, X_{i_k-1}], X_0 \right) = \left( [Y_k, X_0], X_{i_k-1} \right)
\]

But from (3) we obtain that \( Y_{k-1} \in V_{k-2} \) which is a contradiction with the definition of the subspaces \( V_i \).

Thus, we have the following expression for the basis vectors

\[
\begin{align*}
[Y_1, X_j] &= 0, & \forall j \\
[Y_2, X_0] &= \lambda_2 Y_1, & \lambda_2 \neq 0 \\
[Y_3, X_0] &= \lambda_3 Y_2 + \Psi_3(Y_1), & \lambda_3 \neq 0 \\
[Y_i, X_0] &= \lambda_i Y_{i-1} + \Psi_i(Y_{i-2}, \ldots, Y_1), & \lambda_i \neq 0, \ 4 \leq i \leq m
\end{align*}
\]

Using the change of basis

\[
\begin{cases}
X'_i = X_i, & 0 \leq i \leq n - 1 \\
Y'_m = Y_m \\
Y'_{m-j} = [Y'_{m-j+1}, X_0], & 1 \leq j \leq m
\end{cases}
\]

and namely \( Y_j = Y'_{m-j+1} \) for \( 1 \leq j \leq m \) we obtain \( [Y_j, X_0] = Y_{j+1} \) with \( 1 \leq j \leq m - 1 \) and \( [Y_m, X_0] = 0 \). Only rest to prove \( [Y_j, X_k] = 0 \) for \( 1 \leq j \leq m \) and \( 1 \leq k \leq n - 1 \) for to conclude the proof.

Using graded Leibniz identity for the vectors \( (Y_j, X_0, X_0) \) with \( 1 \leq j \leq m \), we obtain that \( [Y_j, X_1] = 0 \). It is easily seen by induction in \( k \) and using graded Leibniz identity for the vectors \( (Y_j, X_{k-1}, X_0) \) that \( [Y_j, X_k] = 0 \) for \( 1 \leq j \leq m \) and \( 1 \leq k \leq n - 1 \) which concludes the proof.

\( \square \)

**Definition 3.6.** A Leibniz superalgebra of \( N^{n,m} \) is called filiform if its \( s \)-nilindex is \( (n-1,m) \).

**Remark 3.7.**

1) If \( L = L_0 \bigoplus L_1 \) is a filiform Leibniz superalgebra then from \(^3\) we have that \( L_0 \) is a filiform Leibniz algebra.
2) Note that a filiform Leibniz superalgebra $L$ of type $(n, m)$ can be realized as a Leibniz superalgebra with $\text{gz}(L) = (n - 1, 1)m$.

We denote by $F^{n,m}$ the set of all filiform Leibniz superalgebras.

The next theorem shows that in the class of filiform Leibniz superalgebras it is also possible to assert about existence of an adapted basis.

**Theorem 3.8.** Let $L = L_0 \bigoplus L_1$ be a filiform Leibniz superalgebra, $L \in F^{n,m}$. Then there exists a basis $\{X_0, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_m\}$ of $L$ such that $L$ can be expressed in one of the following of laws:

\[
\begin{align*}
(I) : & \quad [X_i, X_0] = X_{i+1}, \quad 1 \leq i \leq n - 2, \\
& \quad [Y_j, X_0] = Y_{j+1}, \quad 1 \leq j \leq m - 1, \\
& \quad [Y_m, X_0] = 0, \\
& \quad [Y_j, X_k] = 0, \quad 1 \leq j \leq m, \; 2 \leq k \leq n - 1, \\
& \quad [X_0, X_0] = X_2, \\
& \quad [X_0, X_1] = \alpha_3 X_3 + \cdots + \alpha_{n-2} X_{n-2} + \theta X_{n-1} \\
& \quad [X_i, X_1] = \alpha_3 X_{i+2} + \cdots + \alpha_{n-i} X_{n-1}, \quad 1 \leq i \leq n - 3 \\
(II) : & \quad [X_i, X_0] = X_{i+1}, \quad 2 \leq i \leq n - 2, \\
& \quad [Y_j, X_0] = Y_{j+1}, \quad 1 \leq j \leq m - 1, \\
& \quad [Y_m, X_0] = 0, \\
& \quad [Y_j, X_k] = 0, \quad 1 \leq j \leq m, \; 2 \leq k \leq n - 1, \\
& \quad [X_0, X_0] = X_2, \\
& \quad [X_0, X_1] = \beta_3 X_3 + \cdots + \beta_{n-1} X_{n-1} \\
& \quad [X_i, X_1] = \gamma_i X_{n-1} \\
& \quad [X_i, X_1] = \beta_3 X_{i+2} + \cdots + \beta_{n-i} X_{n-1}, \quad 2 \leq i \leq n - 3 \\
(III) : & \quad [X_i, X_0] = -[X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq n - 2, \\
& \quad [X_i, X_j] = -[X_j, X_i] = \text{lin} < X_{i+j+1}, X_{i+j+2}, \ldots, X_n >, \quad 1 \leq i, j \leq n - 1, \\
& \quad [Y_j, X_0] = Y_{j+1}, \quad 1 \leq j \leq m - 1
\end{align*}
\]

In (I) and (II) the omitted products of $L_0 = < X_0, X_1, \ldots, X_{n-1} >$ vanish.

**Proof.** If $L_0$ is a non-Lie filiform Leibniz algebra, then from [3] we have that there exists a basis of $L_0 : \{X_0, X_1, \ldots, X_{n-1}\}$ such that $L_0$ can be expressed as the even products (i.e. the products $[X_i, X_j]$) of (I) or as the even products of (II). Then applying a similar reasoning as in theorem [5,6] we obtain the existence of vectors of $L_1 : \{Y_1, \ldots, Y_m\}$ which satisfy the multiplications (I) and (II) of the theorem.

If $L_0$ is a filiform Lie algebra, then using the even products from [11] and using the mentioned reasoning for $\{Y_1, \ldots, Y_m\}$ way we obtain the family (III). 

In [11] we have the following result

**Theorem 3.9.** Let $L$ be a $n$-dimensional Leibniz superalgebra with maximal index of nilpotency. Then $L$ is isomorphic to one of the two following non isomorphic algebras:

\[
[e_i, e_1] = e_{i+1} \quad 1 \leq i \leq n - 1,
\]
\[
[e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1, \quad [e_i, e_2] = 2e_{i+2}, \quad 1 \leq i \leq n - 2,
\]

where omitted products are zero.

Seeing the process of the proof of this theorem we observe that the Leibniz superalgebra
\[
[e_i, e_1] = e_{i+1} \quad 1 \leq i \leq n - 1
\]
is really a split Leibniz superalgebra, i.e. is a Leibniz algebra with all the basis elements \(e_i\) even ones. So, this case is not interesting for our study. However, the Leibniz superalgebra
\[
[e_i, e_1] = e_{i+1} \quad 1 \leq i \leq n - 1, \quad [e_i, e_2] = 2e_{i+2} \quad 1 \leq i \leq n - 2
\]
has even part all the \(e_i\) with \(i\) even, and the odd part is constituted by all the \(e_i\) with \(i\) odd. That is, \(L = L_0 \bigoplus L_1 = \langle e_2, e_4, \ldots > \bigoplus < e_1, e_3, \ldots >\), thus there are two possibilities:

- if \(n\) is even, then we have \(L = \langle e_2, e_4, \ldots, e_n > \bigoplus < e_1, e_3, \ldots, e_{n-1} >\) so, \(\dim(L_0) = \dim(L_1)\).

- if \(n\) is odd, then we have \(L = \langle e_2, e_4, \ldots, e_n > \bigoplus < e_1, e_3, \ldots, e_{n} >\) so, \(\dim(L_1) = 1 + \dim(L_0)\).

As for Lie superalgebras [10], the function that gives the maximal nilindex for each pair of dimensions \(n\) and \(m\) (dimensions of the even and odd parts, respectively) will be noted by \(f(n, m)\). Thus we have the following important theorems for our study.

**Theorem 3.10.** \(f(n, m)\) is equal to \(n + m\) if and only if \(m = n\) or \(m = n + 1\).

By applying direct sum with \(\mathbb{C}\) to the non-split superalgebras of the theorem 3.9 we obtain the following result.

**Theorem 3.11.** \(f(n + 1, n) = 2n\) and \(f(n, n + 2) = 2n + 1\).

At this point, for the rest of possibilities of the pair \((n, m)\) the function \(f(n, m)\) is unknown.

Since the non-split superalgebras of the theorem 3.9 are zero-filiform Leibniz superalgebras, in the following sections we will start considering the set \(ZF^{n+1,m}\).

Moreover, in the next sections we will study Leibniz superalgebras with the dimension of the odd part up to three and generic dimension of the even part, and generic dimension of the odd part and two-dimensional even part.

4. **Leibniz Superalgebras with Two-Dimensional Odd Part**

We will consider the case \(n = 1\) separately.
4.1. Case $n = 1$.

**Lemma 4.1.** Let $L$ be any Leibniz superalgebra $L \in ZF^{2,2}$. Then it is isomorphic to one of the following Leibniz superalgebras, pairwise non-isomorphic, that can be expressed in an adapted basis $\{X_0, X_1, Y_1, Y_2\}$ by

$$
\mu_1^\alpha = \begin{cases}
[X_0, X_0] = X_1, \\
Y_1, X_0 = Y_2, \\
X_0, Y_1 = \alpha Y_2, \quad \alpha \in \mathbb{C} \\
Y_1, Y_1 = X_1
\end{cases}
\mu_2 = \begin{cases}
[X_0, X_0] = X_1 \\
Y_1, X_0 = Y_2 \\
[X_0, Y_1] = \frac{1}{2} Y_2 \\
Y_1, Y_1 = X_0 \\
Y_2, Y_1 = X_1
\end{cases}
$$

*Proof.* By using a generic change of basis, along with the graded Leibniz identity we obtain the lemma. \(\square\)

**Remark 4.2.** By using the change of basis $e_1 = Y_1$, $e_2 = X_0$, $e_3 = \frac{1}{2} Y_2$, $e_4 = \frac{1}{2} X_1$, it is easy to see that $\mu_2$ is a superalgebra of the theorem 3.9 for the case $(2, 2)$.

**Proposition 4.3.** $f(2, 2) = 4(= n + m)$

*Proof.* Is a corollary of the above lemma. \(\square\)

**Proposition 4.4.**

$$M^{2,2} = O(\mu_2)$$

*Proof.* According to the above lemma it is sufficient to prove that if $L \in N^{2,2}$ with nilindex 4, then $L \in ZF^{2,2}$. In fact, if $L \notin ZF^{2,2}$ then we have two possible cases:

Case 1. $L = L_0 + L_1$, with $L_0$ abelian. In this case $L$ is a Lie superalgebra and then it will have nilindex $< 4$, see [10].

Case 2. $L = L_0 + L_1$, with $L_0$ zero-nilform Leibniz algebra. In this case as $L \notin ZF^{2,2}$ then $Y_2 \notin C^1(L)$ ($\{Y_1, Y_2\}$ basis of $L_1$) which leads to nilindex $< 4$. \(\square\)

**Proposition 4.5.** The orbit $O(\mu_2)$ is a Zariski open subset of $N^{2,2}$

*Proof.* It is a consequence of

$$M^{n+1,m} = N^{n+1,m} - N_f^{n+1,m}-1$$

\(\square\)

4.2. Case $n > 1$.

**Lemma 4.6.** Let $L$ be any zero-nilform Leibniz superalgebra $L \in ZF^{n+1,2}$, with $n \geq 2$. Then it is isomorphic to one of the following Leibniz superalgebras, pairwise non-isomorphic, that can be expressed in an adapted basis $\{X_0, X_1, \ldots, X_n, Y_1, Y_2\}$ by

$$
\mu_1^\alpha = \begin{cases}
[X_0, X_0] = X_1, \\
Y_1, X_0 = Y_2, \\
X_0, Y_1 = \alpha Y_2, \quad \alpha \in \mathbb{C} \\
Y_1, Y_1 = X_1
\end{cases}
\mu_2 = \begin{cases}
[X_0, X_0] = X_1 \\
Y_1, X_0 = Y_2 \\
[X_0, Y_1] = \frac{1}{2} Y_2 \\
Y_1, Y_1 = X_0 \\
Y_2, Y_1 = X_1
\end{cases}
$$

*Proof.* By using a generic change of basis, along with the graded Leibniz identity we obtain the lemma. \(\square\)
Now we consider different cases:

**with the restrictions:**

verifying the graded Leibniz identity. This fact leads to

\[
\begin{align*}
\mu_1 &= \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq n-1 \\
[Y_1, X_0] = Y_2 \\
[X_0, Y_1] = \alpha Y_2, & \alpha \in \mathbb{C} \\
[Y_1, Y_1] = X_n 
\end{cases} \\
\mu_2 &= \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq n-1 \\
[Y_1, X_0] = Y_2 \\
[Y_1, Y_1] = X_{n-1} \\
[Y_2, Y_1] = X_n 
\end{cases} \\
\mu_3 &= \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq n-1 \\
[Y_1, X_0] = Y_2 \\
[X_0, Y_1] = -Y_2, \\
[Y_1, Y_1] = X_{n-1} \\
[Y_1, Y_2] = X_n 
\end{cases} 
\]

**Remark 4.7.** All the above Leibniz superalgebras have nilindex \( n + 1 \) (= \( n + 1 \) + \( m - 2 \)), two units less than the total dimension of the Leibniz superalgebras.

**Proof.** The family of \( \mathcal{Z}F^{n+1,2} \), with \( n \geq 2 \), can be expressed, in an adapted basis \( \{ X_0, X_1, \ldots, X_n, Y_1, Y_2 \} \), by

\[
\begin{align*}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n-1 \\
[Y_1, X_0] &= Y_2 \\
[X_0, Y_1] &= \alpha Y_2 \\
[Y_1, Y_1] &= \sum_{i=0}^{n} b_{i1}^i X_i \\
[Y_1, Y_2] &= \sum_{i=0}^{n} b_{i2}^i X_i \\
[Y_2, Y_1] &= \sum_{i=0}^{n} b_{21}^i X_i \\
[Y_2, Y_2] &= b_{22}^n X_n 
\end{align*}
\]

verifying the graded Leibniz identity. This fact leads to

\[
\begin{align*}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n-1 \\
[Y_1, X_0] &= Y_2 \\
[X_0, Y_1] &= \alpha Y_2 \\
[Y_1, Y_1] &= b_{11}^{n-1} X_{n-1} + b_{11}^n X_n \\
[Y_1, Y_2] &= b_{12}^n X_n \\
[Y_2, Y_1] &= b_{21}^n X_n \\
[Y_2, Y_2] &= b_{22}^n X_n 
\end{align*}
\]

with the restrictions:

\[
\begin{align*}
ab_{21}^n &= 0 \\
(a + 1)b_{12}^n &= 0 \\
b_{12}^n - b_{11}^{n-1} + b_{21}^n &= 0 
\end{align*}
\]

Now we consider different cases:

- Case 1. \( b_{11}^n = 0 \).
  - Case 1.1. \( a = 0 \).
than \( n \), and it will be isomorphic to the following Leibniz superalgebra that can be expressed with

\[ \text{dim} \]



Theorem 4.9. Let \( L \) be any filiform (non Lie) Leibniz superalgebra \( L = L_0 \oplus L_1 \) with \( \text{dim}(L_0) = n + 1 \), \( n > 1 \) and \( \text{dim}(L_1) = 2 \). If \( L \) has nilindex \( n + 2 \), then \( n = 2 \) and it will be isomorphic to the following Leibniz superalgebra that can be expressed...
in an adapted basis \( \{X_0, X_1, X_2, Y_1, Y_2\} \) by

\[
R^{3,2} = \begin{cases} 
[X_1, X_0] = X_2 \\
[X_0, X_0] = X_2 \\
[X_0, Y_1] = \frac{1}{2}Y_2 \\
[X_1, Y_1] = \frac{1}{2}Y_2 \\
[Y_1, X_0] = Y_2 \\
[Y_1, Y_1] = X_0 \\
[Y_2, Y_1] = X_2 
\end{cases}
\]

**Corollary 4.10.**

\[ f(3,2) = 4, \]
\[ f(n + 1, 2) = n + 1, \text{ if } n \geq 3 \]

**Corollary 4.11.**

\[ \mathcal{M}^{3,2} \not\subset \mathcal{Z}^{3,2} \]

**Corollary 4.12.**

\[ \mathcal{Z}^{n+1,2} \subset \mathcal{M}^{n+1,2}, \quad n \geq 3 \]

**Proof of the theorem.** For \( n > 2 \), by using the graded Leibniz identity it is easy to see that any filiform Leibniz (no Lie) superalgebra \( L = L_0 \oplus L_1 \) with adapted basis \( \{X_0, X_1, X_2, X_3, \ldots, X_n, Y_1, Y_2\} \) verifies that \( [Y_1, Y_1] \in < X_2, \ldots, X_n > \), which implies that \( L \) will have always nilindex \( n \). For \( n = 2 \), all the cases are as described below except for one case in which it is possible that \( X_0 \in < [Y_1, Y_1] > \). This exception corresponds to the two-parametric family

\[
\begin{cases} 
[X_1, X_0] = X_2 \\
[X_0, X_0] = X_2 \\
[X_0, Y_1] = \lambda Y_2 \\
[X_1, Y_1] = \lambda Y_2 \\
[Y_1, X_0] = 2\lambda Y_2 \\
[Y_1, Y_1] = 2\lambda\beta X_0 \\
[Y_2, Y_1] = \beta X_2 
\end{cases}
\]

with \( \lambda\beta \neq 0 \). If we apply the change of scale \( \{X'_0 = \frac{1}{4}X_0, X'_1 = \frac{1}{2}X_1, X'_2 = \frac{1}{4}X_2, Y'_1 = \frac{1}{2\sqrt{\lambda\beta}}Y_1, Y'_2 = \frac{\sqrt{\lambda}}{2\sqrt{\beta}}Y_2\} \) to the two-parametric family, we obtain \( R \). This concludes the proof.

\[ \square \]

5. **Leibniz Superalgebras with three-dimensional odd part**

In this section and in the rest, most of classification proofs are omitted because of they are very laborious and they do not apport any new idea.
5.1. Case $n = 1$.

**Lemma 5.1.** Let $L$ be any Leibniz superalgebra $L \in \mathbb{Z}^{2,3}$. Then, it is isomorphic to one of the following Leibniz superalgebras, pairwise non-isomorphic, that can be expressed in an adapted basis $\{X_0, X_1, Y_1, Y_2, Y_3\}$ by

$$
\mu_1 = \begin{cases}
[X_0, X_0] = X_1, \\
[Y_1, X_0] = Y_2, \\
[Y_2, X_0] = Y_3, \\
[Y_1, Y_1] = X_1
\end{cases} 
\quad \mu_2 = \begin{cases}
[X_0, X_0] = X_1 \\
[Y_1, X_0] = Y_2 \\
[Y_2, X_0] = Y_3 \\
[Y_1, Y_1] = X_1
\end{cases}
$$

$$
\mu_3 = \begin{cases}
[X_0, X_0] = X_1 \\
[Y_1, X_0] = Y_2 \\
[Y_2, X_0] = Y_3 \\
[X_0, Y_1] = -Y_2 \\
[X_0, Y_2] = -Y_3 \\
[Y_1, Y_1] = X_1
\end{cases} 
\quad \mu_4 = \begin{cases}
[X_0, X_0] = X_1 \\
[Y_1, X_0] = Y_2 \\
[Y_2, X_0] = Y_3 \\
[X_0, Y_1] = -Y_2 + Y_3 \\
[X_0, Y_2] = -Y_3 \\
[Y_1, Y_1] = X_1
\end{cases}
$$

$$
\mu_5 = \begin{cases}
[X_0, X_0] = X_1 \\
[Y_1, X_0] = Y_2 \\
[Y_2, X_0] = Y_3 \\
[X_0, Y_1] = -Y_2 \\
[X_0, Y_2] = -Y_3 \\
[Y_1, Y_3] = -X_1 \\
[Y_2, Y_2] = X_1 \\
[Y_3, Y_1] = -X_1
\end{cases} 
\quad \mu_6 = \begin{cases}
[X_0, X_0] = X_1 \\
[Y_1, X_0] = Y_2 \\
[Y_2, X_0] = Y_3 \\
[X_0, Y_1] = \frac{1}{2} Y_2 \\
[X_1, Y_1] = \frac{1}{2} Y_3 \\
[Y_1, Y_1] = X_0 \\
[Y_2, Y_1] = X_1
\end{cases}
$$

*Proof.* By using a generic change of basis, along with the graded Leibniz identity we obtain the lemma. □

**Remark 5.2.** It is easily to see that $\mu_6$ is the superalgebra (in theorem for the case $(2, 3)$, if we only consider the change of basis: $e_1 = Y_1, e_2 = X_0, e_3 = \frac{1}{2} Y_2, e_4 = \frac{1}{2} X_1, e_5 = Y_3$.

**Proposition 5.3.**

$$f(2, 3) = 5$$

*Proof.* Is a corollary of the above lemma. □

**Proposition 5.4.**

$$\mathcal{M}^{2,3} = O(\mu_6)$$

*Proof.* Is a corollary of the above lemma. □

**Proposition 5.5.** $O(\mu_5), O(\mu_6)$ are Zariski open subsets of $N^{2,3}$. 
5.2. Case n = 2.

We are going to give two lemmas which will be useful in this section and in the remaining ones.

**Lemma 5.6.** Let L = L₀ ⊕ L₁ be any Leibniz superalgebra with s-nilindex (n + 1, 3) or (n, 3). Then, if we call \{X₀, X₁, . . . , Xₙ, Y₁, Y₂, Y₃\} an adapted basis of L, it will verify

\[ [Yᵢ, Yⱼ] \in Cⁿ⁻⁶+i+j(L₀) \quad \text{for all } i, j \]

**Proof.** Using the graded Leibniz identity we obtain that

\[ [Y₃, Y₃] = [[Y₂, Y₃], X₀] = [[Y₁, Y₃], X₀] = [[[Y₃, Y₁], X₀], X₀] = \frac{1}{2} [[[Y₂, Y₂], X₀], X₀] = \frac{1}{3} [[[Y₁, Y₂], X₀], X₀] = \frac{1}{3} [[[Y₂, Y₁], X₀], X₀], X₀] = \]

which leads to the lemma. □

**Lemma 5.7.** Let L = L₀ ⊕ L₁ be any Leibniz superalgebra with s-nilindex (n + 1, 3) and n ≥ 2. Then, if we denote \{X₀, X₁, . . . , Xₙ, Y₁, Y₂, Y₃\} an adapted basis of L, it will verify

\[ [Yᵢ, Yⱼ] \in C^{k-2}(L₀) \quad i + j = k, \quad 3 ≤ k ≤ 5 \]

\[ [Y₃, Y₃] \in C^3(L₀) ∩ Z(L₀) \]

**Proof.** Using the graded Leibniz identity we obtain that

\[ [Y₁, Y₂] + [Y₂, Y₁] = [[Y₁, Y₁], X₀] \in C¹(L₀) \]

Furthermore \[ [Y₁, Y₂], [Y₂, Y₁] \in C¹(L₀) \]. In fact, if \( b_{12}^0 \neq 0 \) then by the above equality we have that \( b_{12}^0 = -b_{21}^0 \) (\( b_{12}^0 \) and \( b_{21}^0 \) are respectively the coefficients of \( X₀ \) in \( [Y₁, Y₂] \) and \( [Y₂, Y₁] \)). By considering the products

\[ [X₁, [Y₁, Y₂]] = [[X₁, Y₁], Y₂] + [[X₁, Y₂], Y₁] = b_{12}^0 X₂ + γX₃ + . . . \]

\[ [X₁, [Y₂, Y₁]] = [[X₁, Y₂], Y₁] + [[X₁, Y₁], Y₂] = -b_{12}^0 X₂ + βX₃ + . . . \]

we obtain that \( b_{12}^0 = 0 \) which is a contradiction. Thus, \( [Y₁, Y₂], [Y₂, Y₁] \in C¹(L₀) \).

Analogously, it can be proved that \( [Y₂, Y₂], [Y₁, Y₃], [Y₃, Y₁] \in C²(L₀) \). Finally by considering \( [Y₃, Y₂] = [[Y₃, Y₁], X₀], [Y₂, Y₃] = [[Y₁, Y₂], X₀] \) and \( [Y₃, Y₃] = [[Y₃, Y₂], X₀] \) we have the lemma. □
Lemma 5.8. Let $L$ be any Leibniz superalgebra $L \in ZF^{3,3}$. Then it is isomorphic to one of the following Leibniz superalgebras, pairwise non-isomorphic, that can be expressed in an adapted basis $\{X_0, X_1, X_2, Y_1, Y_2, Y_3\}$ by

$$
\mu_1 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
Y_1, Y_2 = X_2 \\
Y_2, Y_1 = -X_2
\end{cases} \quad \mu_2 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
Y_1, Y_1 = X_2 \\
Y_1, Y_2 = X_2 \\
Y_2, Y_1 = -X_2
\end{cases}
$$

$$
\mu_3 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
Y_1, Y_1 = X_1 \\
Y_1, Y_2 = \alpha X_2 \quad \alpha \in \mathbb{C} \\
Y_2, Y_1 = (1 - \alpha) X_2
\end{cases} \quad \mu_4 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
Y_0, Y_1 = Y_3 \\
Y_1, Y_1 = X_2
\end{cases}
$$

$$
\mu_5 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
X_0, Y_1 = Y_3 \\
Y_1, Y_1 = \alpha X_2 \quad \alpha \in \mathbb{C} \\
Y_2, Y_1 = -X_2
\end{cases} \quad \mu_6 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
Y_1, Y_1 = X_1 \\
Y_1, Y_2 = \alpha X_2 \quad \alpha \in \mathbb{C} \\
Y_2, Y_1 = (1 - \alpha) X_2
\end{cases}
$$

$$
\mu_7 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
X_0, Y_1 = \alpha Y_2 \quad \alpha \in \mathbb{C} \\
X_1, Y_1 = \alpha Y_3 \\
Y_1, Y_1 = X_2
\end{cases} \quad \mu_8 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
X_0, Y_2 = -Y_3 \\
Y_1, Y_1 = X_1 \\
Y_1, Y_2 = X_2
\end{cases}
$$

$$
\mu_9 = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
X_0, Y_1 = -Y_2 + Y_3 \\
X_0, Y_2 = -Y_3 \\
Y_1, Y_1 = X_1 \\
Y_1, Y_2 = X_2
\end{cases} \quad \mu_{10} = \begin{cases}
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 1 \\
Y_j, X_0 = Y_{j+1} & 1 \leq j \leq 2 \\
X_0, Y_2 = -Y_3 \\
Y_1, Y_3 = -X_2 \\
Y_2, Y_2 = X_2 \\
Y_3, Y_1 = -X_2
\end{cases}
$$
Remark 5.9. It is easily to see that $\mu_{12}$ is the Leibniz superalgebra of the theorem 5.9 for the case $(3, 3)$.

Proposition 5.10.

$$f(3, 3) = 6$$

Proof. Is a corollary of the above lemma. □

Proposition 5.11.

$$\mathcal{M}^{3, 3} = O(\mu_{12})$$

Proof. Is a corollary of the above lemma. □

Proposition 5.12. $O(\mu_{11}), O(\mu_{12})$ are Zariski open subsets of $N^{3, 3}$.

Proof. Since $\dim C^3(\mu_i) = 0$ for $1 \leq i \leq 9$ and $\dim C^3(\mu_{10}) = \dim C^3(\mu_{11}) = 1$ and $\dim C^3(\mu_{12}) = 3$, we have that $O(\mu_{12})$ is an open subset of $N^{3, 3}$ and $O(\mu_{11}) \not\subset \cup_{1 \leq i \leq 9} clO(\mu_i)$. Note that $O(\mu_{10}) \subset clO(\mu_{11})$, in fact if we take $f_t = f_{t_{10}} + f_{t_{11}}$, where $f_{t_{10}}(X_0) = t^{-1}X_0$, $f_{t_{10}}(X_1) = t^{-2}X_1$, $f_{t_{10}}(X_2) = t^{-3}X_2$, and $f_{t_{11}}(Y_1) = t^{-\frac{1}{2}}Y_1$, $f_{t_{11}}(Y_2) = t^{-2}Y_2$, $f_{t_{11}}(Y_3) = t^{-\frac{3}{2}}Y_3$, we obtain that if $t \to 0$ then $O(\mu_{10}) \subset clO(\mu_{11})$.

Since $\dim Cent(\mu_{12}) = 4$ and $\dim Cent(\mu_{11}) = 2$, we have that $O(\mu_{11}) \not\subset clO(\mu_{12})$. Thus $O(\mu_{11}) \not\subset \cup_{1 \leq i \leq 12, i \neq 11} clO(\mu_i)$, that is, $O(\mu_{11})$ is an open subset of $N^{3, 3}$ □

5.3. Case $n = 3$.

Lemma 5.13. Let $L$ be any Leibniz superalgebra $L \in ZF^{4, 3}$. Then, it is isomorphic to one of the following Leibniz superalgebras that can be expressed in an adapted basis $\{X_0, X_1, X_2, X_3, Y_1, Y_2, Y_3\}$ by
\[ \mu_1 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = -Y_2 + Y_3 \\
[X_0, Y_2] = -Y_3 \\
[Y_1, Y_1] = X_3 
\end{cases} \]

\[ \mu_2 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = -Y_2 \\
[X_0, Y_2] = (-1 - \alpha)Y_3 & \alpha \in \mathbb{C} \\
[X_1, Y_1] = \alpha Y_3 \\
[Y_1, Y_1] = X_3 
\end{cases} \]

\[ \mu_3 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = -Y_2 \\
[X_0, Y_2] = (-1 - \alpha)Y_3 & \alpha \in \mathbb{C} \\
[X_1, Y_1] = \alpha Y_3 \\
[Y_1, Y_1] = X_2 \\
[Y_1, Y_2] = X_3 
\end{cases} \]

\[ \mu_4 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = -Y_2 + Y_3 \\
[X_0, Y_2] = (-1 - \alpha)Y_3 & \alpha \in \mathbb{C} \\
[X_1, Y_1] = \alpha Y_3 \\
[Y_1, Y_1] = X_2 \\
[Y_1, Y_2] = X_3 
\end{cases} \]

\[ \mu_5 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = Y_3 \\
[Y_1, Y_1] = X_3 
\end{cases} \]

\[ \mu_6 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[Y_1, Y_1] = X_2 \\
[Y_2, Y_1] = X_3 
\end{cases} \]

\[ \mu_7 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = Y_3 \\
[Y_1, Y_1] = X_3 \\
[Y_2, Y_1] = X_2 
\end{cases} \]

\[ \mu_8 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = \alpha Y_2 & \alpha \in \mathbb{C} \\
[X_1, Y_1] = \alpha Y_3 \\
[Y_1, Y_1] = X_3 
\end{cases} \]

\[ \mu_9 = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = -Y_2 \\
[X_0, Y_2] = -Y_3 \\
[Y_1, Y_1] = X_3 \\
[Y_2, Y_2] = -X_3 \\
[Y_3, Y_1] = X_3 
\end{cases} \]

\[ \mu_{10} = \begin{cases} 
[X_i, X_0] = X_{i+1}, & 0 \leq i \leq 2 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq 2 \\
[X_0, Y_1] = -Y_2 \\
[X_0, Y_2] = -Y_3 \\
[Y_1, Y_1] = X_2 \\
[Y_1, Y_3] = X_3 \\
[Y_2, Y_2] = -X_3 \\
[Y_3, Y_1] = X_3 
\end{cases} \]
By using a generic change of basis, along with the graded Leibniz identity

Remark 5.14. All the above zero-filiform Leibniz algebras have nilindex 4, three units less than the total dimension of the superalgebra.

Proposition 5.15. There exists one non empty subset $J$ of $\mathbb{C}$ such that the set $\operatorname{cl}(\cup_{\alpha \in J} O(\mu_{11}^\alpha))$ is an irreducible component of $\mathcal{N}^{4,3}$

Proof. For $1 \leq i \leq 13$, $i \neq 9, 10, 11$ the invariant $\dim Z(\mu_i)$ is higher or equal to 4, but for $i = 9, 10, 11$ it is less than 4. And $\mu_9 \subset \operatorname{cl}O(\mu_{10})$, $\mu_{10} \subset \operatorname{cl}O(\mu_{11})$, in fact, if we take the following maps

$$f_i = f_{i0} + f_{i1}, \text{ where } f_{i0}(X_0) = tX_0, \quad f_{i0}(X_1) = t^2X_1, \quad f_{i0}(X_2) = t^3X_2,$$

$$f_{i1}(X_3) = t^4X_3, \text{ and } f_{i1}(Y_1) = tY_1, \quad f_{i1}(Y_2) = t^2Y_2, \quad f_{i1}(Y_3) = t^3Y_3,$$

and correspondingly

$$g_t = g_{t0} + g_{t1}, \text{ where } g_{t0}(X_0) = tX_0, \quad g_{t0}(X_1) = t^2X_1, \quad g_{t0}(X_2) = t^3X_2,$$

$$g_{t1}(X_3) = t^4X_3, \text{ and } g_{t1}(Y_1) = tY_1, \quad g_{t1}(Y_2) = t^2Y_2, \quad g_{t1}(Y_3) = t^3Y_3,$$

we obtain that if $t \to 0$ then $\mu_9 \subset \operatorname{cl}O(\mu_{10})$, $\mu_{10} \subset \operatorname{cl}O(\mu_{11})$.

We have that exists such subset $J$ of $\mathbb{C}$ stated in the proposition. □

The question now is if there exists any Leibniz superalgebra of nilindex higher than 4 that is the maximal nilindex maximal for the zero-filiform Leibniz superalgebras. When we search this superalgebra in filiform Leibniz superalgebras (the only remain family of Leibniz superalgebras that can arise nilindex 6, one unit smaller than the total dimension) appears the following proposition.

Proposition 5.16. Let $L$ be any filiform (non Lie) Leibniz superalgebra $L = L_0 \oplus L_1$ with $\dim(L_0) = 4$, and $\dim(L_1) = 3$. If $L$ has nilindex 6, then it will be isomorphic to the following Leibniz superalgebra that can be expressed in an
adapted basis \( \{X_0, X_1, X_2, X_3, Y_1, Y_2, Y_3\} \) by

\[
R^{4,3} = \begin{cases} 
[X_1, X_0] = X_2 \\
[X_2, X_0] = X_3 \\
[X_0, X_0] = X_2 \\
[X_0, Y_1] = \frac{b_{11}^0}{2(b_{11}^0 + b_{11}^1)} Y_2 - \frac{b_{11}^2 b_{11}^0}{2(b_{11}^0 + b_{11}^1)^2} Y_3 \\
[X_1, Y_1] = \frac{b_{11}^0}{2(b_{11}^0 + b_{11}^1)} Y_2 - \frac{b_{11}^2 b_{11}^0}{2(b_{11}^0 + b_{11}^1)^2} Y_3 \\
[X_2, Y_1] = \frac{b_{11}^0}{2(b_{11}^0 + b_{11}^1)} Y_3 \\
[Y_1, X_0] = Y_2 \\
[Y_2, X_0] = Y_3 \\
[Y_1, Y_3] = b_{11}^0 X_0 + b_{11}^1 X_1 + b_{11}^2 X_2 + b_{11}^3 X_3 \\
[Y_2, Y_1] = (b_{11}^0 + b_{11}^1)X_2 + b_{11}^2 X_3 \\
[Y_3, Y_1] = (b_{11}^0 + b_{11}^1)X_3 
\end{cases}
\]

with the restrictions \( b_{11}^0 \neq 0 \) and \( b_{11}^1 \neq -b_{11}^0 \). By applying the change of basis

\[
\begin{align*}
X_0' &= b_{11}^0 X_0 + b_{11}^1 X_1 + b_{11}^2 X_2 + b_{11}^3 X_3 \\
X_1' &= (b_{11}^0 + b_{11}^1)X_1 + b_{11}^2 X_2 + b_{11}^3 X_3 \\
X_2' &= b_{11}^0 (b_{11}^0 + b_{11}^1)X_2 + b_{11}^1 b_{11}^1 X_3 \\
X_3' &= (b_{11}^0)^2 (b_{11}^0 + b_{11}^1)X_3 \\
Y_1' &= Y_1 \\
Y_2' &= b_{11}^0 Y_2 \\
Y_3' &= (b_{11}^0)^2 Y_3
\end{align*}
\]

\( R^{4,3} \) is obtained.

**Proposition 5.17.**

\[ f(4, 3) = 6 \]

**Proof.** Is a corollary of the above proposition.
Proposition 5.18.

\( \mathcal{M}^{4,3} \not\subset ZF^{4,3} \)

Proof. Is a corollary of the proposition 5.16 \(\Box\)

5.4. Case \( n \geq 4 \).

Lemma 5.19. Let \( L \) be any Leibniz superalgebra \( L \in ZF^{n+1,3} \), with \( n \geq 4 \). Then it is isomorphic to one of the following Leibniz superalgebras, pairwise non-isomorphic, that can be expressed in an adapted basis \( \{X_0, X_1, X_2, X_3, X_4, \ldots, X_n, Y_1, Y_2, Y_3\} \) by

\[
\begin{align*}
\mu_1 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= -Y_2 + Y_3 \\
[X_0, Y_2] &= -Y_3 \\
[Y_1, Y_1] &= X_n 
\end{array} \right. \\
\mu_2 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= -Y_2 \\
[X_0, Y_2] &= (-1 - \alpha)Y_3 & \alpha \in \mathbb{C} \\
[X_1, Y_1] &= \alpha Y_3 \\
[Y_1, Y_1] &= X_{n-1} \\
[Y_1, Y_2] &= X_n 
\end{array} \right. \\
\mu_3 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= -Y_2 \\
[X_0, Y_2] &= (-1 - \alpha)Y_3 & \alpha \in \mathbb{C} \\
[X_1, Y_1] &= \alpha Y_3 \\
[Y_1, Y_1] &= X_{n-1} \\
[Y_1, Y_2] &= X_n 
\end{array} \right. \\
\mu_4 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= Y_3 \\
[Y_1, Y_1] &= X_n 
\end{array} \right. \\
\mu_5 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= Y_3 \\
[Y_1, Y_1] &= X_n 
\end{array} \right. \\
\mu_6 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[Y_1, Y_1] &= X_{n-1} \\
[Y_2, Y_1] &= X_n 
\end{array} \right. \\
\mu_7 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= Y_3 \\
[Y_1, Y_1] &= X_{n-1} \\
[Y_2, Y_1] &= X_n 
\end{array} \right. \\
\mu_8 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= \alpha Y_2 & \alpha \in \mathbb{C} \\
[X_1, Y_1] &= \alpha Y_3 \\
[Y_1, Y_1] &= X_n 
\end{array} \right. \\
\mu_9 &= \left\{ \begin{array}{ll}
[X_i, X_0] &= X_{i+1}, & 0 \leq i \leq n - 1 \\
[Y_j, X_0] &= Y_{j+1}, & 1 \leq j \leq 2 \\
[X_0, Y_1] &= \alpha Y_2 \\
[X_1, Y_1] &= \alpha Y_3 \\
[Y_1, Y_1] &= X_n 
\end{array} \right. 
\end{align*}
\]
we obtain the result.

By using a generic change of basis, along with the graded Leibniz identity we obtain the result. \[ \Box \]
Remark 5.20. All the above zero-filiform Leibniz algebras have nilindex $n + 1$, three units smaller than the total dimension of the superalgebra.

Theorem 5.21. Let $L$ be an arbitrary (non Lie) Leibniz filiform superalgebra with $gz(L) = (n, 1|3)$ and $n \geq 4$. Then $L$ always has nilindex $n$.

Proof. Is a consequence of the above lemma, of the lemma 5.6, and of grade 1 Leibniz identity. □

Theorem 5.22. $M^{n+1,3} \notin ZF^{n+1,3}$, if $n = 4$

Proof. Consider $R^{4,3} \oplus \{X_4\} \notin ZF^{5,3}$ with nilindex 5. □

Conjecture 1. $f(n + 1, 3) = n + 1$, if $n \geq 4$

$M^{n+1,3} \subset ZF^{n+1,3}$, if $n \geq 5$

6. Leibniz Superalgebras with two-dimensional even part

According to the precedent cases, now the first subcase to consider is $m = 4$. In these cases, that is $n = 2$ and $m \geq 4$, in order to prove the following proposition we use the condition of nilpotency, graded Leibniz identity and generic changes of basis (isomorphisms). However, in this case the complexity is higher than the other cases.

Proposition 6.1. Let $L$ be any Leibniz superalgebra $L \in ZF^{2,m}$, with $m \geq 4$. Then it is isomorphic to one of the following Leibniz superalgebras, pairwise non-isomorphic, that can be expressed in an adapted basis $\{X_0, X_1, Y_1, Y_2, Y_3, Y_4, \ldots, Y_m\}$ by

$$\mu_k = \begin{cases} [X_0, X_0] = X_1 \\ [Y_i, X_0] = Y_{i+1} & 1 \leq i \leq m - 1 \\ [X_0, Y_j] = -Y_{j+1} & 1 \leq j \leq m - 1 \\ [Y_i, Y_j] = (-1)^{i+1}X_1 & 1 \leq i, j \leq m, \ i + j = 2k + 2 \end{cases}$$

with $1 \leq k \leq \lceil \frac{m-1}{2} \rceil$.

$$\mu_k = \begin{cases} [X_0, X_0] = X_1 \\ [Y_i, X_0] = Y_{i+1} & 1 \leq i \leq m - 1 \\ [X_0, Y_1] = -Y_2 + Y_m \\ [X_0, Y_j] = -Y_{j+1} & 2 \leq j \leq m - 1 \\ [Y_i, Y_j] = (-1)^{i+1}X_1 & 1 \leq i, j \leq m, \ i + j = 2k + 2 - 2\lfloor \frac{m}{2} \rfloor \end{cases}$$

with $\lfloor \frac{m}{2} \rfloor \leq k \leq m - 2$. 
Theorem 6.2. When $m$ is odd ($m \geq 4$), then $O(\mu_{m-1})$ is an open subset of $ZF^{2,m}$.

Proof. The present proof is evident, because all the above Leibniz superalgebras for $m$ even have nilindex $m$, but for $m$ odd, $\mu_{m-1}$ will have maximal nilindex $m+1$. □

Lemma 6.3. Let $L = L_0 \oplus L_1$ be an arbitrary Leibniz filiform superalgebra with $gz(L) = (1,1|4)$. Then $L_0$ is the abelian algebra and $L$ will have nilindex 4.

Proof. The above lemma is a consequence of lemma 5.6, of the graded Leibniz identity and of the condition of being nilpotent. □

Proposition 6.4.

$f(2,4) = 5$

Proof. Consider $\mu_0 \oplus <Y_4>$, where $\mu_0$ is the zero-filiform Leibniz superalgebra of maximal nilindex for the case $n = 2$ and $m = 3$ (see Lemma 6.1). □

We have as corollaries the following proposition and theorem.

Proposition 6.5.

$\mathcal{M}^{2,4} \notin ZF^{2,4}$

Theorem 6.6.

$f(2,m) = m + 1$, if $m$ is odd and $m \geq 5$

$f(2,m) = m$, if $m$ is even and $m \geq 6$

Theorem 6.7. If $m \geq 5$, then

$\mathcal{M}^{2,m} \notin ZF^{2,m}$

Proof. For $m$ odd we can consider the Lie superalgebra $K^{2,m}$ (see [10]) that has nilindex $m + 1$ and it is not included in $ZF^{2,m}$. For $m$ even we can consider $K^{2,m-1} \oplus \mathbb{C}$ with nilindex $m$. □
7. Conjecture

Conjecture 2.
If \( n + 1 \) and \( n \) are the dimensions of the even part and the odd part, respectively, the only non split Leibniz superalgebra of type \((n + 1, n)\) and nilindex \( 2n \) is the following filiform Leibniz superalgebra. The law of this superalgebra can be expressed, in an adapted basis \( \{ X_0, X_1, \ldots, X_n, Y_1, Y_2, \ldots, Y_n \} \), by

\[
R^{n+1,n} = \begin{cases} 
[X_i, X_0] = X_{i+1} & 1 \leq i \leq n - 1 \\
[X_0, X_0] = X_2 \\
[X_0, Y_1] = \frac{1}{2}Y_2 \\
[X_i, Y_1] = \frac{1}{2}Y_{i+1} & 1 \leq i \leq n - 1 \\
[Y_j, X_0] = Y_{j+1} & 1 \leq j \leq n - 1 \\
[Y_1, Y_1] = X_0 \\
[Y_i, Y_1] = X_i & 1 \leq i \leq n 
\end{cases}
\]

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