Surplus Solid Angle
as an Imprint of Hořava-Lifshitz Gravity

Sung-Soo Kim
Physique Théorique et Mathématique
Université Libre de Bruxelles and International Solvay Institutes,
ULB-C.P. 231, B-1050 Bruxelles, Belgium
sungsoo.kim@ulb.ac.be

Taekyung Kim and Yoonbai Kim
Department of Physics, BK21 Physics Research Division, and Institute of Basic Science,
Sungkyunkwan University, Suwon 440-746, Korea
pojawd, yoonbai@skku.edu

Abstract
We consider the electrostatic field of a point charge coupled to Hořava-Lifshitz gravity and find an exact solution describing the space with a surplus (or deficit) solid angle. Although, theoretically in general relativity, a surplus angle is hardly to be obtained in the presence of ordinary matter with positive energy distribution, it seems natural in Hořava-Lifshitz gravity. We present the sudden disappearance and reappearance of a star image as an astrophysical effect of a surplus angle. We also consider matter configurations of all possible power law behaviors coupled to Hořava-Lifshitz gravity and obtain a series of exact solutions.
1 Introduction

Recently, a power-counting renormalizable and unitarity-keeping, ultraviolet (UV) complete theory of gravity was proposed [1, 2, 3] to equip space and time with an anisotropic scaling in a Lifshitz fixed point [4, 5]. In its infrared (IR) fixed point, the general coordinate invariance is accidentally arisen and the general relativity is assumed to be recovered. In this Hořava-Lifshitz (HL) gravity, subsequent studies have been done in various directions. Specifically they cover studies of cosmology [6, 7], black hole physics and their thermodynamics [8, 9, 10, 11], and other related subjects [12]. There have also been some discussions on the field theories in the Lifshitz fixed point and flat space-time [5, 13]. In order to survive as a viable quantum gravity consistent with observation, there remain numerous issues and challenges to be settled down in HL gravity of which the list includes smooth IR limit vs the detailed balance condition, the Hamiltonian dynamics in the presence of the lapse function with spatial dependence (the projectability condition), perturbative renormalizability, strong coupling, and propagating modes including the degree of scalar graviton, etc. [14, 15, 16, 17, 18].

An attractive subject is to find a testable evidence given as a unique characteristic of HL gravity, distinguishable from the properties of GR. Although HL gravity is assumed to reproduce GR as its IR theory, it may be intriguing to ask which effect of HL gravity in the UV regime is most likely to be detected through astronomical or astrophysical observation. Following the usual strategy as is often done in GR, one may look for a static solution of HL theory, employ it as a gravitational background, and investigate possible effects to a test particle such as change of positions of the stars, which is ascribed to HL gravity, not to GR.

In this paper we consider an electrostatic field of a point charge as matter, and obtain spherically symmetric solutions which describe a space with either a surplus or deficit solid angle. The result of a surplus solid angle from an ordinary matter with positive energy density is contrasted with a genuine feature of GR in which it can usually be materialized by the source of negative mass or energy. We propose the sudden disappearance and reappearance of stars in front of an observer as evidence of a resultant surplus solid angle of the HL gravity theory. Because of this qualitatively different nature between HL gravity and GR, a single observation of a surplus (solid) angle can possibly support or rule out HL gravity. Although we could not determine the specific value of a surplus solid angle yet, it must be one of strong candidates which awaits observed data as a possible astrophysical candidate to test HL gravity.

The paper is organized as follows. In Sec. 2 we review the spherically symmetric static vacuum solutions. A surplus solid angle coming from the solution of HL gravity coupled to the electrostatic field of a point charge is discussed in Sec. 3. We study the matters with various power-behavior long tails and obtain exact solutions including charged black hole solutions in Sec. 4. We then conclude with further discussion.
2 Spherically Symmetric Vacuum Solutions

HL gravity [1] is a newly proposed nonrelativistic gravity based on anisotropic scaling between time and space

\[ t \rightarrow \ell^2 t, \quad x^i \rightarrow \ell x^i, \tag{2.1} \]

with \( z = 3 \) to make the theory in (1+3) dimensions power counting renormalizable. The metric can be written in the ADM decomposition

\[ ds^2 = -N^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right). \tag{2.2} \]

The action is given by

\[ S_{HL} = \int dt d^3 x \sqrt{g} N (\mathcal{L}_{IR} + \mathcal{L}_{UV}), \tag{2.3} \]

where \( \mathcal{L}_{IR} \) contains the quadratic or lower derivative terms, and \( \mathcal{L}_{UV} \) the higher derivative terms

\[
\mathcal{L}_{IR} = \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 \Lambda}{8 (1 - 3 \lambda)} (R - 3 \Lambda), \tag{2.4}
\]

\[
\mathcal{L}_{UV} = -\frac{\kappa^2}{2 w^4} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu w^2}{2} R^{ij} \right) + \frac{\kappa^2 \mu^2 (1 - 4 \lambda)}{32 (1 - 3 \lambda)} R^2. \tag{2.5}
\]

Here the extrinsic curvature is

\[
K_{ij} \equiv \frac{1}{2N} \left( g_{ij} - \nabla_i N_j - \nabla_j N_i \right), \quad K = g^{ij} K_{ij}, \quad K^{ij} = g^{ik} g^{jl} K_{kl} \tag{2.6}
\]

with \( g_{ij} \equiv \frac{\partial g_{ij}}{\partial t} \) and \( \nabla_i \) covariant derivative with respect to \( g_{ij} \). \( R \) and \( R_{ij} \) are three-dimensional scalar curvature and Ricci tensor, and the Cotton tensor is given by

\[
C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}} \nabla_k \left( R^j_l - \frac{1}{4} R \delta^j_l \right). \tag{2.7}
\]

The gravity action (2.3) is invariant under a restricted class of diffeomorphism, foliation-preserving diffeomorphism. General coordinate invariance is regarded as an accidental symmetry in the low energy scale with the choice of \( \lambda = 1 \). There are seven independent terms in the action (2.3), yet the parameters in HL gravity are five, \( \kappa, \lambda, \Lambda, w, \mu \) thanks to the detailed balance. When the IR action (2.4) is directly compared to the (1+3)-dimensional Einstein-Hilbert action

\[ T \text{aking the Einstein limit as an IR limit is actually nontrivial. Quantization procedure of consideration [15, 16] occasionally restricts the usage of the space-dependent lapse function (2.2) and the detailed balance condition. We will also discuss in the subsequent sections that the long distance limit of spherically symmetric solutions of HL gravity does not lead to those of Einstein gravity [17, 18]. However, for this classical static problem, we will keep the detailed balance condition to make the problem tractable.} \]
with speed of light $c$, Newton’s gravitational constant $G$, and the effective cosmological constant $\Lambda_E$, three parameters can be determined in addition to $\lambda = 1$,

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3\lambda}}, \quad 16\pi G = \frac{\kappa^2}{2c}, \quad \Lambda_E = \frac{3}{2} \Lambda.$$  \hspace{1cm} (2.8)

In the presence of matter fields, the matter action takes the form

$$S_M = \int dt d^3x \sqrt{g} N \mathcal{L}_M(N, N_i, g_{ij}).$$  \hspace{1cm} (2.9)

We now restrict ourselves to the static solutions with spherical symmetry. Let us choose a reference frame and introduce spherical coordinates $(t, r, \theta, \phi)$ with a static metric

$$ds^2 = -B(r)e^{2\delta(r)}dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (2.10)

The projectability condition is not taken into account in this work, although it is necessary especially for quantization of the HL gravity [11]. In order to fulfill it automatically it is convenient to use Painlevé-Gullstrand type coordinates [14], which will be presented elsewhere [19].

Since all the components of Cotton tensor vanish, $C^{ij} = 0$ under this metric, we can write the action (2.3) in a simple form

$$S_{HL} = \frac{\pi \kappa^2 \mu^2}{2(3\lambda - 1)} \int dt \int dr e^\delta \times \left\{(1 - 3\lambda) \left[ \tilde{B}^2 + 2\left( \frac{\tilde{B}}{r} + \frac{\tilde{B}'}{2} \right)^2 \right] - (1 - 4\lambda) \left( \frac{\tilde{B}}{r} + \tilde{B}' \right)^2 + 2\Lambda r \left( \frac{\tilde{B}}{r} + \tilde{B}' \right) + 3\Lambda^2 r^2 \right\},$$  \hspace{1cm} (2.11)

where we introduced $\tilde{B} = B - 1$. The equations of motion are then readily obtained by varying the action with respect to the metric functions

$$\left[ (\lambda - 1) \tilde{B}' - \frac{2\lambda}{r} \tilde{B} - 2\Lambda r \right] \delta' + (\lambda - 1) \tilde{B}'' - \frac{2(\lambda - 1)}{r^2} \tilde{B} = \frac{8(1 - 3\lambda)r^2}{\kappa^2 \mu^2} \frac{\partial \mathcal{L}_M}{\partial B},$$  \hspace{1cm} (2.12)

$$\left[ (1 - 3\lambda) \tilde{B}^2 + 2\left( \frac{\tilde{B}}{r} + \frac{\tilde{B}'}{2} \right)^2 \right] - (1 - 4\lambda) \left( \frac{\tilde{B}}{r} + \tilde{B}' \right)^2 + 2\Lambda r \left( \frac{\tilde{B}}{r} + \tilde{B}' \right) + 3\Lambda^2 r^2 = \frac{8(1 - 3\lambda)r^2}{\kappa^2 \mu^2} \left( \mathcal{L}_M + \frac{\partial \mathcal{L}_M}{\partial \delta} \right),$$  \hspace{1cm} (2.13)

where we included the matter action for future purpose

$$S_M = 4\pi \int_{-\infty}^{\infty} dt \int_0^{\infty} dr r^2 e^\delta \mathcal{L}_M(B, \delta).$$  \hspace{1cm} (2.14)

It was claimed that $\lambda$ flows to unity in the low energy regime and HL gravity reduces to the Einstein theory with a negative cosmological constant as long as the higher order curvature
terms, $\mathcal{O}(R^2, R^2, \ldots)$, are neglected. In the Eqs. (2.12) and (2.13), these correspond to insertion of $\lambda = 1$ and neglect of the terms quadratic in the metric functions, $\tilde{B}^2$ and $\tilde{B} \delta$. Then, without matter $\mathcal{L}_M = 0$, the equations reduce to the Einstein equations and then yield anti-de Sitter (AdS) Schwarzschild solution with integration constants $\delta_0$ and $M$,

$$
\frac{d \delta}{dr} = 0, \quad \implies \delta(r) = \delta_0 = 0,
$$

(2.15)

$$
\frac{d}{dr} (r B) = 1 - \frac{3 \Lambda r^2}{2}, \quad \implies B(r) = -\frac{\Lambda r^2}{2} + 1 - \frac{M}{r}.
$$

(2.16)

The known exact solutions to (2.12) and (2.13) are as follows [8, 11]. The generic solution is $B(r) = 1 - \Lambda r^2$, which is independent of $\lambda$. For $\lambda \geq 1/3$ and $\lambda \neq 1$, there exists another solution

$$
B(r) = 1 - \Lambda r^2 + B_0 r^{2 \lambda \pm \sqrt{2(3 \lambda - 1)}}/\lambda, \quad \delta(r) = \frac{1 + 3 \lambda \mp 2 \sqrt{(3 \lambda - 1)}}{1 - \lambda} \ln(r/r_0),
$$

(2.17)

and, for $\lambda = 1$,

$$
B(r) = 1 - \Lambda r^2 \pm \sqrt{c_1 r}, \quad \delta(r) = \delta_0 = 0,
$$

(2.18)

where $B_0$ and $c_1$ are integration constants. There are also some special solutions depending on the value of $\lambda$, see the appendix.

All the known exact solutions consistently show that they cannot reproduce the AdS Schwarzschild solution (2.15) and (2.16) in the long distance limit. After substituting $\lambda = 1$, the linear Einstein equations (2.15) and (2.16) are obtained from the nonlinear equations (2.12) and (2.13) without matter, by utilizing $|\Lambda r| \gg |\tilde{B}'| \sim |\tilde{B}|/r$. This assumption is not consistent with the leading long distance behavior of $B(r)$ in (A-1). Therefore, the nonlinear terms cannot be neglected even in a long distance limit, and it may support possible mismatch between the IR limit of HL gravity and the Einstein theory in the context of classical solutions.

The solution for $\lambda = 1$ is of interest. For the lower minus sign in (2.18), it is easy to see that the number of horizons varies from zero to two depending on the relation between $c_1$ and $\Lambda$. To see physical singularity in the spatial sector, we examine the three-dimensional analogue of the Kretschmann invariant

$$
R^{ijkl}R_{ijkl} = 4R_{ij}R^{ij} - R^2 = \frac{2}{r^2} \left[ B'^2 + 2 \left( \frac{B - 1}{r} \right)^2 \right],
$$

(2.19)

where we used a three-dimensional identity, $R_{ijkl} = g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik} - \frac{1}{2} (g_{ik}g_{jl} - g_{il}g_{jk})R$. The $\lambda = 1$ solution (2.18) then has singularity, $R^{ijkl}R_{ijkl} = \frac{2(2 + 12\Lambda^2)}{r^2} + 12\Lambda^2$, unlike the first solution independent of $\lambda$. For sufficiently large $r$, we have a constant piece of the curvature, whose value is different from the constant curvature of the AdS Schwarzschild solution.

We note that the absence of singularity does not guarantee that other observers do not see singularity at all. By the same token, although a singularity is obtained from (2.19), it is subtle
whether the singularity still appears as a singularity to other observers. Since time reversal, time translation, parity, and spacial rotation are symmetries of the system, observers who are connected by these symmetries surely see the singularity. However, if we take into account that boost is not a symmetry, then it is not clear how the singularity is observed to those who are in a relative motion.

We close this section with a remark. In this nonrelativistic HL gravity with speed of light larger than that in general relativity, horizons and curvature singularities can be dependent upon the choice of reference frames, and therefore the meaning of these is different from those in general relativity with general coordinate invariance.

3 Electrostatic Field and Surplus (Deficit) Solid Angle

The action of U(1) gauge theory in the Lifshitz fixed point with $z = 2$ is constructed in flat space and time by using detailed balance [3],

$$S_{U(1) UV} = \int dt d^3x \left[ \frac{1}{2} E_i^2 - \frac{1}{8g^2} (\partial_i F_{ik})(\partial_j F_{jk}) \right],$$

(3.1)

where $E_i = F_{i0}$ is electric field and $F_{ij} = \partial_i A_j - \partial_j A_i$. This vector field theory possesses neither (Galilean) boost symmetry nor electromagnetic duality even in the absence of external sources.

In IR regime, both Lorentz symmetry and duality are accidentally arisen, and the action is

$$S_{U(1) IR} = -\frac{1}{4} \int dt d^3x F_{\mu\nu} F^{\mu\nu},$$

(3.2)

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Along the flow line from the UV fixed point to the IR fixed point, generic action is assumed to take the following form,

$$S_{U(1)} = \int dt d^3x \left[ \frac{1}{2} E_i^2 - F_{U(1)}(F_{ij}, \partial_i F_{jk}) \right],$$

(3.3)

where the potential can contain all possible gauge-invariant combinations of $F_{ij}$ and $\partial_i F_{jk}$ as long as it satisfies $F_{U(1)} = -\frac{1}{8g^2} (\partial_i F_{ik})(\partial_j F_{jk})$ at the Lifshitz fixed point and $F_{U(1)} \rightarrow -\frac{1}{4} F_{ij}^2$ in the IR limit. If the theory in the UV limit (3.1) flows to the IR limit (3.2) keeping the detailed balance, the form of the action in the intermediate scale (3.3) is restricted. Although we discussed a specific anisotropic scaling of $z = 2$, the above discussion can straightforwardly be generalized to the cases of arbitrary $z$ (see the $z = 3$ case given in Ref. [7]).

Unlike the magnetic potential terms which change and become complicated along the flow line, the kinetic term of electric field is unaltered. For this reason, we consider only the electrostatic field due to a point object of electric charge $q_e$ at the origin, and the magnetic potential will be discussed in the following section.
Using the same static metric (2.10), we find that the equations of motion are given in terms of nonvanishing components
\[
\frac{\partial L_M}{\partial B} = 0, \quad L_M + \frac{\partial L_M}{\partial \delta} = -\frac{1}{2} e^{-2\delta} F_{r0}^2 = -\frac{1}{32\pi^2 r^4},
\]
where we used
\[
E_r = F_{r0} = \frac{e^\delta q_0}{4\pi r^2}.
\]
In the limit of Einstein theory, the Eqs. (2.12) and (2.13) reduce to
\[
dr \frac{d\delta}{dr} = 0, \quad \rightarrow \quad \delta(r) = \delta_0 = 0,
\]
\[
dr (rB) = 1 - \frac{3\Lambda}{2} r^2 - \frac{1 - 3\lambda}{8\pi^2 \kappa^2 \mu^2 \Lambda r^2}, \quad \rightarrow \quad B(r) = -\frac{M}{r} + 1 - \frac{\Lambda}{2} r^2 + \frac{1 - 3\lambda}{8\pi^2 \kappa^2 \mu^2 \Lambda r^2}.
\]
It follows from (2.8) that \(\Lambda < 0\) leads to \(\lambda > 1/3\) which gives the following asymptotic behaviors
\[
\lim_{r \to \infty} B(r) \sim \frac{|\Lambda|}{2} r^2 \to \infty \quad \text{and} \quad \lim_{r \to 0} B(r) \sim \frac{1 - 3\lambda}{8\pi^2 \kappa^2 \mu^2 \Lambda r^2} \to \infty,
\]
and thus the maximum number of horizon is four. Let us restrict ourselves to the limit \(\Lambda \to 0\) keeping \(M > 0\) and \(\kappa^2 \mu^2 \Lambda\) finite. When \(\lambda > 1/3\) and \(\Lambda \to 0^-\), one then finds that horizons are given as
\[
\begin{cases}
\begin{align*}
r_H &= \frac{1}{2} \left[ M \pm \sqrt{M^2 - \frac{(1 - 3\lambda)q_0^2}{2\pi^2 \kappa^2 \mu^2 \Lambda}} \right], & \text{when} & M^2 > \frac{(1 - 3\lambda)q_0^2}{2\pi^2 \kappa^2 \mu^2 \Lambda} \\
\end{align*}
\end{cases}
\]
\[
\begin{cases}
\begin{align*}
r_H &= \frac{M}{2}, & \text{when} & M^2 = \frac{(1 - 3\lambda)q_0^2}{2\pi^2 \kappa^2 \mu^2 \Lambda} \\
\end{align*}
\end{cases}
\]
\[
\begin{cases}
\begin{align*}
\text{no horizon}, & \text{when} & M^2 < \frac{(1 - 3\lambda)q_0^2}{2\pi^2 \kappa^2 \mu^2 \Lambda}.
\end{align*}
\end{cases}
\]
(3.8)
The first case reproduces the AdS Reissner-Nordström black hole and the second case is its extremal limit. For \(\lambda = 1/3\), the coefficient of \(1/r^2\) term in (3.7) vanishes and thus leads to the limit of Schwarzschild black hole. When \(\lambda > 1/3\) and \(\Lambda \to 0^+\), only one horizon is formed at
\[
r_H = \frac{1}{2} \left[ M + \sqrt{M^2 - \frac{(1 - 3\lambda)q_0^2}{2\pi^2 \kappa^2 \mu^2 \Lambda}} \right].
\]
(3.9)
By solving the full Eqs. (2.12) and (2.13) including the matter (3.4), we obtain an exact solution for \(1/3 \leq \lambda < 1/2\),
\[
B(r) = 1 \pm \Delta_{q,\lambda} - \Lambda r^2, \quad \delta(r) = \frac{1 - \lambda}{\lambda} \ln(r/r_0),
\]
(3.10)
where

$$\Delta_{q_e, \lambda} = \sqrt{\frac{1 - 3\lambda}{2\lambda - 1}} \frac{|q_e|}{2\pi\kappa\mu}. \quad (3.11)$$

In the absence of electric charge $q_e = 0$, the solution reproduces a vacuum solution (2.18) as expected. Unlike the vacuum solutions, there is no special solution for $\lambda = 1$. Notice that the electrostatic field couples to HL gravity gives a constant contribution $\pm \Delta_{q_e, \lambda}$ in (3.10), unlike the $1/r^2$ contribution of (3.7) in Einstein gravity. This may lead to a surprising consequence that is not expected in Einstein gravity. Indeed, via the following rescaling

$$dt \to (1 \pm \Delta_{q_e, \lambda})^{\frac{1}{4\lambda}} dt, \quad dr \to \sqrt{1 \pm \Delta_{q_e, \lambda}} \ dr, \quad (3.15)$$

one can rewrite the metric with (3.10) as

$$ds^2 = -(1 - \Lambda r^2) \frac{(1 - \lambda)}{\lambda} dt^2 + \frac{dr^2}{1 - \Lambda r^2} + r^2 (1 \pm \Delta_{q_e, \lambda}) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.16)$$

which describes a space with a surplus (or deficit) solid angle.

When $0 < \Delta_{q_e, \lambda} < 1$, the lower minus sign gives an AdS space with a deficit solid angle $4\pi \Delta_{q_e, \lambda}$ [20, 21]. The upper plus sign, on the other hand, yields an AdS space with a surplus solid angle; in the limit of vanishing $\Lambda$, the area of a sphere of radius $r$ is not $4\pi r^2$, but larger [see Fig. 1-(a)].

### 3.1 Effects of Surplus Solid Angles

It is instructive to consider its development figures, to understand geometry with a surplus angle. For convenience, let us consider flat space and time limit, $\Lambda \to 0$, and set $\theta = \pi/2$. In Fig. 1-(b),

$$B(r) = 1 - \Lambda r^2 \pm \sqrt{\Delta_{q_e,1}} \ r^2, \quad \delta(r) = \delta_0 = 0 \quad (3.12)$$

with $\Delta_{q_e,1} = \sqrt{8(q^2 + p^2)/(\kappa\mu g)^2}$ and an integration constant $c_1$. The difference is that the authors of [10] used analytic continuations to make the cosmological constant positive whereas we have not. This results in overall sign difference in the gravity sector. On the other hand, the matter sector in [10] needs some modifications. To obtain solutions, one should solve equations of motion for metric functions and a vector field simultaneously. Hence, the matter part of the Lagrangian density in (14) of Ref. [10] should be corrected as

$$\frac{N}{r^2 \sqrt{f}} (q^2 + p^2) \to \frac{N}{r^4 \sqrt{f}} \left( \frac{f}{N^2} F_{\theta\phi}^2 r^2 \right) \quad (3.13)$$

$$= \frac{N}{r^4 \sqrt{f}} (q^2 - p^2), \quad (3.14)$$

and one then varies the action with respect to $N$ and $f$ in (3.13) instead of those in (3.14).
the left figure is $\mathbb{R}^2$ (white color) plane with a cut of a half straight line connected to a point charge at the apex A, and a pie-slice (grey color) on the right is a surplus piece. To make a surplus angle like Fig. 1(a), one inserts a pie-slice into the cut and glues the lines of the same color together.

![Diagram](image)

Figure 1: The sudden disappearance of a star image due to a surplus solid angle: (a) Folded geometry of an excess cone representing $\theta = \frac{\pi}{2}$ of three-dimensional flat space with surplus angle $4\pi \Delta \varphi, \lambda = \frac{\pi}{2}$, (b) Its development figures where the surplus solid angle region is given by grey area.

We then investigate possible effects of the surplus solid angle based on the development figures. We introduce a singularity at the apex A, a star S, and three observers scattered at different points O, O', and O'' but connected by spatial rotations along the angle $\phi$ as shown in Fig. 1. Imagine that two light rays in green, L1 and L2, are emitted from a star and reach to observers. The L1 passes through the left side of the apex A and arrives at the observer O, the L2 reaches to the observer O'' after passing the right side of the apex A. The observers O and O'' look very close to each other in the development figure, all of the observers at O, O', O'', are actually separated apart shown in Fig. 1(a). Suppose that the singularity source at the apex A is opaque as charged stars usually do, then it is very likely that the third observer O' in the grey colored development figure cannot observe any light from the star S. It means that any observer at the grey colored region cannot see the image of the star at S.

We may also see the effects of a surplus angle in the following way. Imagine this time that a star is slowly moving from O through O' to O'', and there is a static observer at S tracking down the trajectory of the star. As the star once passes through the red line, the star image may disappear for a while to an observer at S and reappear at a distant point.

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3 A charged object can be bright, but the color of the object at A can be assumed to be different from that of the star S as usual. An important point is that this grey colored region due to the surplus angle, where the star image disappears, does not depend on the size of the opaque object at A.
If we imagine a drastic case that a charged astronomical object is suddenly created (or annihilated) and such event happens at the apex A, then all the stars in the entire grey region disappear (or appear) in front of the observer at S in the twinkling of an eye. This sudden disappearance of a star image can be a testable astronomical or astrophysical effect of the surplus solid angle.

In the aforementioned argument we used only the straight propagation property of light in flat space and the speed of light does not affect the obtained result. To confirm, we comment on how speed of light changes by checking lightlike geodesics. The Lagrangian \( L \) for the trajectory of light associated with the metric (3.16) reads

\[
L = -(1 - \Lambda r^2) (r/r_0)^{2(1-\lambda)} \dot{t}^2 + \frac{\dot{r}^2}{1 - \Lambda r^2} + r^2 (1 \pm \Delta_{q_e,\lambda}) (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2),
\]  

(3.17)

where the dot is the derivative with respect to affine parameter. This yields two constants of motion, \( E = 2(1 - \Lambda r^2) (r/r_0)^{2(1-\lambda)} \dot{t} > 0 \) and \( \ell = 2r^2 (1 \pm \Delta_{q_e,\lambda}) \sin^2 \theta \dot{\phi} \). Since we are interested in the light radially emitted from a star S, there no angular momentum, \( \ell = 0 \), and then the azimuth angle is also a constant of motion. Choosing \( \theta = \pi/2 \) and \( \dot{\theta} = 0 \), we find that for the light with vanishing Lagrangian \( (L = 0) \),

\[
\dot{r} = \pm \frac{E}{2(r/r_0)^{1-\lambda}}, \quad (1/3 \leq \lambda < 1/2),
\]

(3.18)

which shows that the speed of light is \( \dot{r}_0 = E/2 \) by placing S at \( r_0 \), and more importantly that it diverges as light approaches the apex A and decreases when traveling away from the apex.

When \( \Delta_{q_e,\lambda} \geq 1 \) with the lower minus sign, the metric function (3.10) is rewritten as

\[
B(r) = -\Lambda r^2 - M \quad \text{with} \quad M = \Delta_{q_e,\lambda} - 1 \geq 0.
\]

(3.19)

A horizon is at \( r_H = \sqrt{M/(-\Lambda)} \), and the metric takes the same form as that of (1+2)-dimensional BTZ black hole of mass \( M \) which is now proportional to the electric charge \( |q_e| \). Thus, both a resultant surplus (or deficit) solid angle and mass \( M \) are a quantity that is proportional to the absolute value of electric charge \( |q_e| \). It means that the origin of those observable quantities in a given frame is not an arbitrary integration constant, but an externally given charge. In Einstein gravity, mass and (electric) charge are independently observed black hole charges. In a given frame these two quantities which appear in the solution (3.19) of HL gravity for various \( \lambda \), hardly seem to be distinguished in these classical configurations.

In Einstein gravity with the linear equation of \( B(r) \) (3.7), the deficit solid angle is formed by a global monopole of which energy density is proportional to \( 1/r^2 \) for large \( r \) [20]. In order to obtain a surplus solid angle in the context of Einstein theory, we need exotic negative energy density everywhere, e.g., a \( \delta \)-function like distribution of antimatter with negative mass in (1+2) dimensions, and linearly divergent negative energy distribution (like a global monopole) in (1+3) dimensions. But these violate the positive energy theorem. In HL gravity, however, both the surplus and deficit solid angles are naturally obtained from ordinary electrostatic field of which
energy density is always positive. This implies that astrophysical observation of a sudden jump of a star image due to a surplus solid angle may become an imprint of HL gravity.

Although the realistic value of surplus (or deficit) solid angle tempts us as

$$4\pi \Delta q_{e,\lambda} = \sqrt{\frac{\Lambda_E}{3(2\lambda - 1)} \frac{16\pi G}{c} |q_e|}$$

by using (2.8), the obtained solution (3.10) is not connected to the AdS charged black hole solution of Einstein theory (3.6)–(3.7) in the long distance limit, and thus a direct usage of the values of $c, G, \Lambda_E$ in the present universe seems not to be natural.

It is also worth noting that one can easily find a singularity at the origin, inversely proportional to the fourth order of the radial coordinate as expected,

$$R^{ijkl} R_{ijkl} = 4 \left( \frac{\Delta q_{e,\lambda}}{r^2} \pm \Lambda \right)^2 + 8\Lambda^2,$$

where (3.10) is used. This singularity is due to the divergent electrostatic field (3.5) from a point electric charge, not directly from the $\delta$-function-like singularity of the point charged particle itself. In nonrelativistic HL gravity an observer seems possibly to hit the singularity.

### 4 Monopoles of Anisotropic Scalings

In the previous section we dealt with the kinetic term composed of electric field of net charge. In this section we consider static gravitating monopoles which are supported from the potential. The potential terms in HL gravity allow spatial derivatives whose form depends on the anisotropic scaling and changes along the flow from the UV fixed point to the IR limit.

Before we examine the derivatives and potentials for monopoles, we first explore possible profile of the monopole configurations. For both global monopoles of O(3) linear sigma model and magnetic monopoles of U(1) gauge theory in the HL type field theories, the long distance behavior of the Lagrangian density must exhibit power-law behavior as $L_M \sim 1/r^n$ irrespective of the value of $z$. To be specific, we consider

$$\frac{\partial L_M}{\partial B} \approx 0, \quad L_M + \frac{\partial L_M}{\partial \delta} \approx -\frac{\alpha}{r^n}, \quad (n = 0, 1, 2, ...),$$

where $\alpha$ is a constant to be determined by the explicit form of matter Lagrangian and the monopole configurations of interest. In particular, we are interested in non-negative integer $n$, otherwise the matter distribution diverges at spatial infinity, and positive $\alpha$ to ensure positivity of the matter energy density.$^4$ It follows from (2.12) and (2.13) that exact solutions are given as follows. When

$^4$If we use $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}}$ as the energy-momentum tensor of matter fields, we obtain the energy density...
$n \neq 6$ and $\frac{1}{3} \leq \lambda < \frac{1}{3} + \frac{2(n-6)^2}{3n^2}$, we find

$$B(r) = 1 - \Lambda r^2 \pm \Delta_{\alpha,\lambda} r^{4-n}, \quad \delta(r) = \frac{n(n-6)(\lambda-1)}{2[4+n(\lambda-1)]} \ln(r/r_0), \quad (4.2)$$

where

$$\Delta_{\alpha,\lambda} = \sqrt{\frac{8(1-3\lambda)\alpha}{\left\{n \left[1 + \frac{n(\lambda-1)}{8}\right] - 3\right\} \kappa^2 \mu^2}}. \quad (4.3)$$

When $n = 6$, solution exists only when $\lambda = 1/3$. It follows from $(2.12)$ and $(2.13)$ that $\lambda = 1/3$ yields vanishing matter contributions and thus the solution is the same as a vacuum solution with $\lambda = 1/3$. We also find special solutions for $\lambda = 1$,

$$B(r) = 1 - \Lambda r^2 \pm \sqrt{\Delta_{\alpha,1} r^{4-n} + c_1 r}, \quad \delta(r) = \delta_0 = 0, \quad (n \neq 3), \quad (4.4)$$

$$B(r) = 1 - \Lambda r^2 \pm \frac{16\alpha}{\kappa^2 \mu^2} r \ln r + c_1 r, \quad \delta(r) = \delta_0 = 0, \quad (n = 3) \quad (4.5)$$

with an integration constant $c_1$. Note that the $n = 4$ solutions in $(4.2)$ and $(4.5)$ correspond respectively to $(3.10)$ and $(3.12)$ of the electrostatic field discussed in the previous section. Since the Lagrangian density $(4.1)$ is singular only at the origin for any positive $n$, the spatial Kretschmann invariant has singularity at $r = 0$ for all the solutions of $n > 0$,

$$R_{ijkl}R^{ijkl} = \begin{cases} 4 \left[\left(\frac{n^2}{8} - n + 3\right)\Delta_{\alpha,\lambda} r^{-n} \pm (n-6)\Lambda \Delta_{\alpha,\lambda} r^{n/2} + 3\Lambda^2\right] \quad \text{for } (4.2) \\
\frac{8}{4r} \left[\frac{c_1 + (4-n)\Delta_{\alpha,1} r^{3-n}}{\Delta_{\alpha,1} r^{4-n}} \mp \Lambda\right] + 4 \left[\frac{\sqrt{c_1 r + \Delta_{\alpha,1} r^{4-n}}}{r^2} \mp \Lambda\right] \quad \text{for } (4.4). \quad (4.6) \\
\frac{8}{4r} \left[\frac{c_1 + \frac{16\alpha}{\kappa^2 \mu^2}(1 + \ln r)}{\Delta_{\alpha,1} r^{4-n}} \mp \Lambda\right] + 4 \left[\frac{\sqrt{c_1 r + \frac{16\alpha}{\kappa^2 \mu^2} r \ln r}}{r^2} \mp \Lambda\right] \quad \text{for } (4.5) \end{cases}$$

As previously discussed for $n = 4$, the above solution $(4.4)$ is unphysical for $n > 3$ because the corresponding $\alpha$ becomes negative. On the other hand, the solution of lower $n$ ($n = 0, 1, 2$) is

for the spherically symmetric matter distribution and the metric $(2.10)$.

$$-T^0_0 = -2\pi \left(L_M + \frac{\partial L_M}{\partial \phi}\right) \quad \text{with} \quad \frac{\partial L_M}{\partial B} = 0.$$ 

Non-negativity of the energy density distribution from the matter fields of our interest requires non-negative $\alpha$ in $(4.1)$. We may not accept positive $\alpha$ to get a physical solution except for the case of a negative cosmological constant ($n = 0$), otherwise it leads to negative energy density in the entire space given in $(r, \theta, \phi)$ coordinates. Since the time-derivative terms in the action are quadratic, the aforementioned $-T^0_0$ is the same as canonical Hamiltonian density by Legendre transform.
obtained from positive $\alpha$. For (4.4) with $c_1 \geq 0$, the solution is well defined in entire $r$ and no more singularity except $r = 0$. When $c_1 < 0$ and $n < 3$, $B(r)$ in (4.4) is real for $r \geq r_n = \left(\frac{-c_1}{\Delta_{n,1}}\right)^{\frac{1}{3-n}}$ and the second line of (4.6) is singular at $r = r_n$. When $c_1 < 0$ and $n > 3$, $B(r)$ in (4.4) is real for $r \leq r_n$ and the second line of (4.6) is singular at $r = r_n$. For (4.5) with positive $\alpha$ and arbitrary $c_1$, $B(r)$ in (4.4) is real for $r \geq r_3 = \exp\left(-\frac{c_1}{\beta_{10}}\right)$ and the fourth line of (4.6) becomes singular at $r = r_3$.

For (4.2), there is no horizon for the upper plus sign, but the number of horizons change from zero to two for the lower minus sign as

$$
\begin{cases}
    r_{H\pm} = -\frac{1}{2\Lambda} \left( \bar{v}_{\lambda} \pm \sqrt{\bar{v}_{\lambda}^2 + 4\Lambda} \right), & \text{when } \bar{v}_{\lambda}^2 > -4\Lambda \\
    r_{eH} = -\frac{\bar{v}_{\lambda}}{2\Lambda}, & \text{when } \bar{v}_{\lambda}^2 = -4\Lambda \\
    \text{no horizon}, & \text{when } \bar{v}_{\lambda}^2 < -4\Lambda
\end{cases}
$$

(4.7)

where

$$
\bar{v}_{\lambda} = \sqrt{\frac{3\lambda - 1}{3 - \lambda}} \frac{4\nu}{\kappa \mu}.
$$

(4.8)

In a similar fashion, it is easy to see from (4.4) that no horizon exits for the upper plus sign, while horizons for the lower minus sign exist and their structure is given as

$$
\begin{cases}
    \text{two horizons,} & \text{when } c_1 > \frac{1}{r_{eH}} - \left(2\Lambda + \bar{v}_{\lambda}^2\right)r_{eH} + \Lambda^2 r_{eH}^3 \\
    \frac{\sqrt{\bar{v}_{\lambda}^2 + 2\Lambda + \sqrt{\bar{v}_{\lambda}^4 + 4\bar{v}_{\lambda}^2\Lambda + 16\Lambda^2}}}{6\Lambda^2}, & \text{when } c_1 = \frac{1}{r_{eH}} - \left(2\Lambda + \bar{v}_{\lambda}^2\right)r_{eH} + \Lambda^2 r_{eH}^3 \\
    \text{no horizon}, & \text{when } c_1 < \frac{1}{r_{eH}} - \left(2\Lambda + \bar{v}_{\lambda}^2\right)r_{eH} + \Lambda^2 r_{eH}^3
\end{cases}
$$

(4.9)

In the limit of Einstein gravity with the matter (4.1), we have

$$
B(r) = 1 - \frac{\Lambda}{2}r^2 - \frac{M}{r} + \frac{4\alpha(1 - 3\lambda)}{(n - 3)\kappa^2 \mu^2 \Lambda} r^{2-n}, \quad \delta(r) = \tilde{\delta}_0 = 0
$$

(4.10)

which does not correspond to the solutions in HL gravity (4.2)–(4.5) in the long distance limit. The matter contribution in HL gravity goes as $r^{2-n}$ while that of Einstein gravity follows $r^{2-n}$ as shown in (4.10). This noticeable difference arises form the nonlinear terms in the Eqs. (2.12) and (2.13), originated from higher order curvature terms.

Let us now investigate field configurations that give the matter distribution of power-law behavior (4.1). We explore a few possible field configurations in detail and also discuss their energy distributions mostly at long distance limit.

Because of electromagnetic duality of the Maxwell theory (3.2), the magnetic multipoles in the IR regime ($z = 1$) of the HL U(1) gauge theory share the same pole structure with those of the electric field.
At the Lifshitz UV fixed point, the theory has an anisotropic scaling $z$ and magnetic potential in the $U(1)$ theory takes the form which is built out of the detailed balance,

$$
-\frac{1}{4} F_{ij} F^{ij} \quad \text{IR} \rightarrow \text{UV} = \begin{cases} 
-\frac{1}{8g_2^2} \left( (\nabla^2)^{\frac{z}{2} - 1} F_{ij} \right)^2 & \text{for odd } z \\
\frac{1}{8g_2^2} \left( (\nabla^2)^{\frac{z}{2}} \partial_i F_{ij} \right)^2 & \text{for even } z
\end{cases}.
$$

(4.11)

The magnetostatic field of a magnetic monopole of charge $q_M$ is obtained by solving the Bianchi identity with a $\delta$-function singularity at the origin, $\epsilon_{ijk} \partial_i F_{jk} = q_M \delta(3)(\vec{x})$. By insertion of the monopole field $\epsilon_{ijk} F_{jk} = q_M x^i/4\pi r^3$ into the Lagrangian density of $U(1)$ gauge theory with anisotropic scaling $z$, we read distribution of (4.11) with $n = 2(z + 1)$. The results are summarized in the second and third columns of Table 1.

| $n$ | objects and scaling |
|-----|---------------------|
| 0   | cosmological constant |
| 2   | monopole             |
| 4   | $z = 1$              |
| 6   | $z = 2$              |
| 8   | $z = 3$              |
| 10  | $z = 4$              |
| $\vdots$ | $\vdots$         |
| $n$ | $z = \frac{n}{2} - 1$ |

Table 1. Electric, magnetic, and global monopoles in field theories with various scaling $z$.

We also consider other gravitating topological soliton, global monopole, in the context of HL gravity. A scalar field $\phi^a$ ($a = 1, 2, 3$) in the Lifshitz fixed point with an anisotropic scaling $z$ (2.1) takes the following UV action in the background metric (2.2) with vanishing shift function

$$
S_{O(3)\text{UV}} = \int dt d^3x \sqrt{gN} \left( -\frac{1}{2N^2} \phi^{a2} - V_{\text{UV}} \right),
$$

(4.12)

where the potential in the UV regime is obtained by the detailed balance as

$$
V_{\text{UV}}(\phi^a, \partial_i \phi^a, ...) = \begin{cases} 
\frac{1}{8\kappa_2^2} \left[ \partial_i (\nabla^2)^{\frac{z}{2} - 1} \phi^a \right]^2 & \text{for odd } z \\
\frac{1}{8\kappa_2^2} \left[ (\nabla^2)^{\frac{z}{2}} \phi^a \right]^2 & \text{for even } z
\end{cases}.
$$

(4.13)
Spacetime symmetry in HL gravity assumes to be suddenly enhanced in the IR limit, i.e., the full diffeomorphism symmetry is restored and the invariant light speed is recovered, \( x^0 = ct = t \) in our unit system. The IR action is supposedly given by

\[
S_{O(3)_{IR}} = \int d^4x \sqrt{-g} \left( -\frac{g^{00}}{2} \partial_0 \phi^a \partial_0 \phi^a - V_{IR} \right),
\]

(4.14)

where \( g^{00} = 1/N^2 \) and the potential assumes to be of ordinary quadratic spatial derivatives and of a quartic order self-interactions. The simplest potential that gives rise to global monopoles is

\[
V_{IR}(\phi^a, \partial_i \phi^a, ...) = -\frac{g^{ij}}{2} \partial_i \phi^a \partial_j \phi^a - \frac{\lambda_M}{4} (\phi^2 - v^2)^2, \quad \phi^2 \equiv \phi^a \phi^a.
\]

(4.15)

This has a global \( O(3) \) symmetry, which is spontaneously broken to \( O(2) \).

In the intermediate regime or along the flow line from the Lifshitz UV fixed point to the relativistic IR fixed point, one expects that kinetic term has the same form

\[
S_{O(3)} = \int dt d^3x \sqrt{g} N \left( -\frac{1}{2N^2} \dot{\phi}^a \dot{\phi}^a - V \right),
\]

(4.16)

but the potential \( V(\phi^a, \partial_i \phi^a, ...) \) should be connected to \( V_{UV} \) and \( V_{IR} \) at each UV and IR limit. It is not clear whether the detailed valance can be attained along with the flow.

We now attempt to find possible sources for (4.1) from the \( O(3) \) linear sigma model with various anisotropic scaling \( z \). A constant contribution, the \( n = 0 \) case in (4.1), is easily identified as either the symmetric vacuum \( \phi = 0 \) in (4.15) or the core of global monopole \( r \leq 1/(\sqrt{\lambda_M} v) \) with \( \phi(r) \approx 0 \), which yields

\[
\frac{\partial \mathcal{L}_M}{\partial B} = 0, \quad \mathcal{L}_M + \frac{\partial \mathcal{L}_M}{\partial \delta} = -\frac{\lambda_M}{4} v^4.
\]

(4.17)

This gives a positive contribution to the cosmological constant (see also Table. 1), and then one is advised to introduce the net cosmological constant

\[
\Lambda_{v, \lambda} = \Lambda + \frac{2(1 - 3\lambda)\lambda_M v^4}{3\Lambda_\kappa^2 \mu^2}.
\]

(4.18)

The corresponding solution in Einstein theory (2.15)–(2.16) remains unchanged except \( \Lambda_{v, \lambda} \) instead of \( \Lambda \). In HL gravity, nontrivial solutions are

\[
B(r) = 1 - \Lambda v^2 \pm \sqrt{\Lambda(\Lambda - \Lambda_{v, \lambda})} r^2,
\]

(4.19)

which agrees with (4.2) for \( n = 0 \).

It is worth pointing out that when the net cosmological constant \( \Lambda_{v, \lambda} \) vanishes, the vacuum solution of Einstein theory with vanishing integration constant reproduces flat space as expected.
On the other hand, in HL gravity, we additionally obtain a nonconstant solutions describing an AdS space due to nonlinearity

\[ B(r) = 1 - 2\Lambda r^2. \]  

(4.20)

To explore other possible configurations, we choose a hedgehog ansatz

\[ \phi^a = \hat{r}^a \phi(r) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \phi(r), \]  

(4.21)

to write

\[ S_{\text{O(3)IR}} = 4\pi \int_{-\infty}^{\infty} dt \int_{0}^{\infty} drr^2 e^\delta \left[ -\frac{B}{2} \phi'^2 - \frac{\phi^2}{r^2} - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right], \]  

(4.22)

which yields

\[ \frac{\partial L_M}{\partial B} = -\frac{1}{2} \phi'^2, \]  

(4.23)

\[ L_M + \frac{\partial L_M}{\partial \delta} = -\frac{B}{2} \phi'^2 - \frac{\phi^2}{r^2} - \frac{\lambda}{4} (\phi^2 - v^2)^2. \]  

(4.24)

We impose the boundary conditions of the scalar field as, by requiring single-valuedness of the field at the monopole position and finite energy at spatial infinity,

\[ \phi(0) = 0, \quad \phi(\infty) = v. \]  

(4.25)

A configuration satisfying both boundary conditions can be chosen such that

\[ \phi(r) = \begin{cases} 0, & \text{for } r \leq \frac{1}{\sqrt{\lambda} v}, \\ v, & \text{for } r > \frac{1}{\sqrt{\lambda} v}, \end{cases} \]  

(4.26)

which is one that the scalar field has vacuum expectation value \( v \) outside the monopole core \( r > 1/(\sqrt{\lambda} v) \). Near the vacuum, the first term in (4.24) is negligible, and thus the leading term approximation for the right-hand side of (4.23)–(4.24) leads to

\[ \frac{\partial L_M}{\partial B} \approx 0, \quad L_M + \frac{\partial L_M}{\partial \delta} \approx -\frac{v^2}{r^2}, \]  

(4.27)

which corresponds to the case \( n = 2 \) and \( \alpha = v^2 \). Then the spatial metric \( B(r) \) for a gravitating global monopole is given by either (4.2) or (4.4).

We now discuss energy configurations near the Lifshitz fixed point. If we suppose a global monopole of the ansatz (4.21), satisfying the same boundary conditions (4.25), then we may read its energy in flat space and time from (4.12)–(4.13)

\[ E \approx \begin{cases} \frac{\pi}{2\kappa^2} \int_{0}^{\infty} dr r^2 \left[ \frac{d}{dr} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{1}{r^2} \right) \right]^{\frac{z-1}{2}} \phi^2, & \text{for odd } z \\ \frac{\pi}{2\kappa^2} \int_{0}^{\infty} dr r^2 \left[ \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{1}{r^2} \right) \phi \right]^2, & \text{for even } z \end{cases} \]  

(4.28)
For large \( r \), the leading term may also be \( v^2/r^2 \) term in the parenthesis, and then the leading energy density behaves \( O(1/r^{2z}) \) at long distance limit. Comparing it to (4.1), we have \( n = 2z \) and \( \alpha \approx v^{2z} > 0 \), which seems true even in curved space and time governed by the metric (2.10) (see also the fourth column in Table. 1). An intriguing property of the \( z > 3/2 \) global monopole is possible finiteness of its energy in flat space and time because Derrick’s theorem is not applied due to the higher derivative terms for anisotropic scaling.

To be consistent with Hořava-Lifshitz gravity with \( z = 3 \), we may take into account the scalar field theory with the same scaling behavior, of which spatial derivative term may be sixth order, \( \partial_i \partial^2 \phi \partial_i \partial^2 \phi \). The expected leading behavior of energy density of a global monopole for large \( r \) is of order of \( 1/r^6 \), which is beyond the present scope and should be dealt through further study. Studying in detail global solitons is an attractive future research direction of the field theories in the Lifshitz fixed point.

For an odd or arbitrary real positive \( n \), we may need field theories in the Lifshitz fixed point of fractional or arbitrary real anisotropic scaling \( z \). These field theories may not be easy to deal with due to nonlocality, and so they may be realized in field theories at their UV fixed point.

5 Conclusion and Discussion

We have considered HL gravity with the detailed valance, coupled with matters of power-law behaviors \( 1/r^n \), and shown corresponding exact solutions. When both positivity of the matter energy density and converging matter contributions at spatial infinity are required, the solutions exit only for a limited range of \( \lambda \). Some cases, \( n > 4 \), do not allow solutions for \( \lambda = 1 \), which is assumed to the limit to which \( \lambda \) flows in the IR limit. We also compared the long distance limit of the obtained metric \( B(r) \) with GR limit where the higher order terms in the action are neglected from the beginning. We consistently found that if we keep the detailed balance, then two limits do not coincide with each other, which may imply that the detailed valance condition is to be modified to yield GR in the IR limit.

The most intriguing feature is that when HL gravity couples to the electric field of a point charge, it naturally gives rise to a surplus solid angle, which is hardly realized in GR unless exotic negative energy density is introduced. As presented in Sec. 3, a surplus solid angle depends not only on the parameters \( \kappa \mu \) but also on \( \lambda \). The valid range of \( \lambda \) for this case does not include \( \lambda = 1 \). This indicates that a surplus angle cannot emerge in the IR limit, but is likely formed in the intermediate or UV regime where \( \kappa \) and \( \mu \) can be unconstrained. Hence, this surplus angle is a unique feature of HL gravity that may be used as a test of HL gravity.

We presented one of the possible effects of a surplus angle, sudden disappearance (or appearance) of a star when an observer travels through a surplus area in a folded geometry. Considering the repulsive nature among the same charges, realization of such object with accumulated charges seems unlikely. However, once it was formed in early universe, it is likely inherited and its remnant might be yet residing in the present universe. Since the HL solution of surplus solid angle
is not connected to the GR solution of AdS charged black hole, we could not obtain a numerical value of the surplus angle but an observation of surplus angle definitely supports HL gravity as an astrophysical evidence.

It should be stressed that since there is no boost symmetry, a singularity seen in a given (static) frame does not guarantee that other observers will see the singularity, especially for those who are in a relative motion to the static observer. This may alter the effects of surplus angles. The sudden disappearance that we have described is based on a source and observer in static configuration. When either one is moving, such effects should be reexamined with care, although surplus angle is a global property.

**Appendix**

We now analyze (2.12) and (2.13) in more detail. Consider asymptotic behaviors of the solutions to (2.12)–(2.13). For sufficiently large \( r \) at asymptotic region, we assume the divergence of \( B(r) \) arises as a power behavior. Then some computation with the Eq. (2.13) falls the leading behavior into two classes,

\[
B(r) \approx \begin{cases}
(\infty-i) & -\Lambda r^2 \\
(\infty-ii) & B_{\infty} r^{p_+} \quad \text{for } \lambda > 1
\end{cases},
\]

where the coefficient \( B_{\infty} \) is an undetermined constant and

\[
p_+ = \frac{2\lambda + \sqrt{2(3\lambda - 1)}}{\lambda - 1}.
\]

The long distance behavior in the first case (\( \infty-i \)) coincides with the leading IR behavior in (2.16) but the coefficient is changed from \(-\Lambda/2\) to \(-\Lambda\). Since \( p_+ > 2 \) for \( \lambda > 1 \), the long distance behavior in the second case (\( \infty-ii \)) implies a possibility of new solution due to higher order (and derivative) terms.

By the higher order terms for UV completion (2.5), one expects that the short distance behavior is severely distorted from \(-M/r\) term in (2.16). The allowed powers for various \( \lambda \) are categorized as

\[
B(r) \approx \begin{cases}
(0-i) & 1 \\
(0-ii) & c_+^2 \\
(0-iii) & B_{0-} r^{p_-} \quad \text{or } B_{0+} r^{p_+} \\
(0-iv) & B_{0+} r^{p_+}
\end{cases},
\]

where \( B_{0\pm} \) are undetermined, \( p_+ \) is given in (A-1), and \( p_- \) is

\[
p_- = \frac{2\lambda - \sqrt{2(3\lambda - 1)}}{\lambda - 1}.
\]
The known exact static vacuum solutions can be summarized and classified as how they connect two asymptotes with various values of $\lambda$. The simplest solution independent of $\lambda$ is obtained by adding unity and the cosmological constant term, $B(r) = 1 - \Lambda r^2$ and arbitrary $\delta(r)$, which connects ($\infty-i$) in (A-1) and ($0-i$) in (A-3). For $\lambda \geq 1/3$ and $\lambda \neq 1$, one obtains other solutions, $B(r) = 1 - \Lambda r^2 + B_{\pm} r^{p_{\pm}}$ and $\delta(r) = \frac{1+3\lambda^2}{1-\lambda}-2\sqrt{2(3\lambda-1)} \ln(r/r_0)$, which connect one of ($\infty-i$)–($\infty-ii$) and one among ($0-i$) and ($0-iii$)–($0-iv$). If we take the limit $\lambda = 1/3$ then $p_+ = p_- = -1$, which is nothing but AdS Schwarzschild solution with twice the cosmological constant. We note that the case $\lambda = 1/3$ in fact allows $B(r) = 1 - \Lambda r^2 + \frac{M}{r}$ and arbitrary $\delta(r)$ since (2.12) vanishes in this limit.

A particular value of $\lambda$ also gives special solutions: For $\lambda = \frac{1}{2}$ the solution is of the same form as the first solution independent of $\lambda$, but allows any integration constant $c_1$, $B(r) = -\Lambda r^2 + c_1^2$ and $\delta(r) = \ln(r/r_0)$, which connects ($\infty-i$) and ($0-ii$). For $\lambda = 1$, one obtains another solution connecting ($\infty-i$) and ($0-i$),

$$B(r) = 1 - \Lambda r^2 \pm \sqrt{c_1 r} , \quad \delta(r) = \delta_0 = 0$$  \hspace{1cm} (A-5)

with an integration constant $c_1$. We can see that all the solutions above span possible boundary behaviors in (A-1) and (A-3).

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