The geometric realization of a simplicial Hausdorff space is Hausdorff

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Abstract

It is shown that the thin geometric realization of a simplicial Hausdorff space is Hausdorff. This proves a famous claim by Graeme Segal that the thin geometric realisation of a simplicial k-space is a k-space.

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1 Introduction

1.1 The main problem

In one of his many landmark papers [3], Graeme Segal introduced the geometric realization functor for simplicial spaces, which he called the “thin” realization functor in the subsequent article [2]. He claimed that the thin geometric realization of a simplicial space which is compactly-generated Hausdorff must be compactly-generated Hausdorff. However, while it is essentially obvious that the geometric realization of a simplicial compactly-generated space must be compactly-generated, the Hausdorff property is a whole different matter since cocartesian squares are implicit in the definition of the thin geometric realization and they are known to behave badly with respect to separation axioms. At the time of [3], Segal’s claim was thought to be dubious and no convincing proof of it ever appeared in the litterature. This difficulty brought some to turn away from k-spaces and work with weak-Hausdorff compactly-generated spaces instead. It is much easier indeed to show that the geometric realization of a compactly-generated weak-Hausdorff space is itself compactly-generated weak Hausdorff (this was done by Gaunce Lewis in the appendix of his PhD thesis). In the following pages, we will prove that Segal was right after all!
1.2 Definitions and notation

In this paper, we will use the French notation for the sets of integers: \( \mathbb{N} \) will denote the set of natural numbers (i.e. non-negative integers), and \( \mathbb{N}^* \) the one of positive integers. Recall the simplicial category \( \Delta \) whose objects are the ordered sets \([n] = \{0, 1, \ldots, n\}\) for \( n \in \mathbb{N} \) and whose morphisms are the non-decreasing maps, with the obvious compositions and identities. All the morphisms are composites of morphisms of two types, namely the face morphisms

\[
\delta^k_i : \begin{cases} [k] & \to [k + 1] \\
j < i & \mapsto j \\
j \geq i & \mapsto j + 1
\end{cases}
\]

for \( k \in \mathbb{N} \) and \( i \in [k + 1] \), and the degeneracy morphisms

\[
\sigma^k_i : \begin{cases} [k] & \to [k - 1] \\
j < i & \mapsto j \\
j \geq i & \mapsto j - 1
\end{cases}
\]

for \( k \in \mathbb{N}^* \) and \( i \in [k - 1] \). See [1] for a comprehensive account.

There is (covariant) functor \( \Delta^* : \Delta \to \text{Top} \) which sends \([n] \) to the \( n \)-simplex \( \Delta^n := \{(t_i)_{0 \leq i \leq n} \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1\} \), and any morphism \( \delta : [n] \to [m] \) to

\[
\Delta^*(\delta) : \begin{cases} \Delta^n & \to \Delta^m \\
(t_i)_{0 \leq i \leq n} & \mapsto \left( \sum_{i \in \delta^{-1}(j)} t_i \right)_{0 \leq j \leq m}.
\end{cases}
\]

A simplicial space is a contravariant functor \( \Delta \to \text{Top} \). Given such a functor, we set \( A_n := A([n]) \) for any \( n \in \mathbb{N} \). For \( k \in \mathbb{N} \), we will write \( d^k_i := A(\delta^k_i) \) for \( i \in [k + 1] \) (the face maps of \( A \)), and \( s^k_i := A(\sigma^k_i) \) for \( i \in [k - 1] \) (the degeneracy maps of \( A \)). When no confusion is possible, we will simply write \( \delta_i \) instead of \( \delta^k_i \), \( \sigma_i \) instead of \( \sigma^k_i \), \( d_i \) instead of \( d^k_i \) and \( s_i \) instead of \( s^k_i \). If \( \delta \) is a morphism in \( \Delta \), we will also write \( \delta^* \) instead of \( A(\delta) \).

For \( n \in \mathbb{N} \), a point \( x \in A_n \) is said to be degenerate when in the image of some \( s_i \).

**Definition 1.1.** The thin geometric realization of a simplicial space \( A \), denoted by \( |A| \), is the quotient space of \( \prod_{n \in \mathbb{N}} A_n \times \Delta^n \) under the relations \( (x, \Delta^*(\delta)[y]) \sim (A(\delta)[x], y) \), for \( (m, n) \in \mathbb{N}^2 \), \( x \in A_m \), \( y \in \Delta_n \) and \( \delta \in \text{Hom}_\Delta([n], [m]) \).

For every \( n \in \mathbb{N} \), we thus have a natural map \( \pi_n : A_n \times \Delta^n \to |A| \).

Our simple aim here is to prove the following theorem:
Theorem 1.1. Let $A$ be a simplicial space and assume that $A_n$ is Hausdorff for each $n \in \mathbb{N}$. Then $|A|$ is Hausdorff.

The proof, although very technical, has a very straightforward basic strategy: we will give a general construction of “flexible” open neighborhoods for the points of $|A|$ (see Section 2 for the construction and Section 3 for the proof of openness), and then show that those neighborhoods may be used to separate points (Section 4). In the rest of the paper, $A$ denotes an arbitrary simplicial space (no separation assumption will be made until Section 4).

2 Constructing open subsets in a geometric realization

In the whole section, we fix an integer $n \in \mathbb{N}$, a non-degenerate simplex $x \in A_n$ and a point $\alpha \in \Delta^n \setminus \partial \Delta^n$. Our goal is to give a general construction of non-trivial open neighborhoods of $\pi_n(x, \alpha)$ in $|A|$. This will be done by constructing, for every $k \in \mathbb{N}$, an open subset $V_k$ of $A_k \times \Delta^k$ such that $(x, \alpha) \in V_n$ and the family $(V_k)_{k \in \mathbb{N}}$ is compatible, i.e. for every morphism $f : [k] \to [k']$ in $\Delta$:

$$\forall (y, \beta) \in A_{k'} \times \Delta^k, (y, f_*(\beta)) \in V_{k'} \iff (f^*(y), \beta) \in V_k;$$

in this case $\bigcup_{k \in \mathbb{N}} \pi_k(V_k)$ is an open subset of $|A|$ which contains $\pi_n(x, \alpha)$ and its inverse image by $\pi_k$ is $V_k$ for each $k \in \mathbb{N}$.

Remark 1. Since every morphism in the simplicial category is a composite of face and degeneracy morphisms, a family $(V_k)_{k \in \mathbb{N}}$ is compatible if and only if it satisfies the following two sets of properties:

$$\forall k \in \mathbb{N}^*, \forall (y, \beta) \in A_k \times \Delta^{k-1}, \forall i \in [k], (y, (d_i)_*(\beta)) \in V_k \iff (d_i(y), \beta) \in V_{k-1}. \quad (1)$$

$$\forall k \in \mathbb{N}, \forall (y, \beta) \in A_k \times \Delta^{k+1}, \forall i \in [k], (y, (\sigma_i)_*(\beta)) \in V_k \iff (s_i(y), \beta) \in V_{k+1}. \quad (2)$$

2.1 Suitable families of open subsets of the $A_k$’s

Our starting point is the following very basic lemma on simplicial sets:

Lemma 2.1. Let $x \in A_n$ be a non-degenerate simplex. Let $\sigma : [N] \to [n]$ and $\tau : [N] \to [m]$ be epimorphisms. Then the following conditions are equivalent:

(i) The simplex $\sigma^*(x)$ belongs to $\tau^*(A_m)$.

(ii) There is an epimorphism $\rho : [m] \to [n]$ such that $\sigma = \rho \circ \tau$.

(iii) $\forall (i, j) \in [N]^2$, $\tau(i) = \tau(j) \Rightarrow \sigma(i) = \sigma(j)$. 

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Proof. The only non-trivial statement is that (i) implies (ii). Assume then that
\( \sigma^*(x) = \tau^*(y) \) for some \( y \in A_m \). We may choose a section \( \delta : [n] \to [N] \) of \( \sigma \), hence \( (\tau \circ \delta)^*(y) = (\sigma \circ \delta)^*(x) = x \). If \( \tau \circ \delta \) were not one-to-one, we would be able to decompose it as \( \tau \circ \delta = \sigma' \circ \delta' \) for some \( i \) and some morphism \( \sigma' \), which would yield \( x \in s_i(A_{m-1}) \), contradicting the fact that \( x \) is non-degenerate. \( \square \)

**Definition 2.1.** Let \( N \geq n \) be an integer and \( x \in A_n \) be a non-degenerate simplex. A family \( (U_\sigma)_{\sigma : [N] \to [n]} \) is called \( x \)-admissible when:

(i) The set \( U_\sigma \) is an open neighborhood of \( \sigma^*(x) \) in \( A_N \) for every epimorphism \( \sigma : [N] \to [n] \).

(ii) For every \( \sigma : [N] \to [n] \) and every \( \tau : [N] \to [m] \), one has

\[
U_\sigma \cap \tau^*(A_m) \neq \emptyset \iff (\exists \rho : [m] \to [n]) : \sigma = \rho \circ \tau.
\]

**Remark 2.** Let \( \sigma : [N] \to [k] \) and \( \tau : [N] \to [m] \). By the previous lemma, \( \sigma^*(x) \) does not belong to the union of all \( \tau^*(A_m) \), for \( N \geq m \), where \( \tau \) ranges over the set of all epimorphisms from \( [N] \) for which no epimorphism \( \rho \) satisfies \( \sigma = \rho \circ \tau \). Assuming that, for every \( \sigma \), we may find an open neighborhood \( U_\sigma \) of \( \sigma^*(x) \) which is disjoint from this union, then the family \( (U_\sigma)_{\sigma : [N] \to [n]} \) is obviously \( x \)-admissible.

**Example 1.** Assume that \( A_k \) is Hausdorff for every \( k \in \mathbb{N} \). Let \( \sigma : [N] \to [k] \). Then, for every epimorphism \( \tau : [N] \to [m] \) such that no \( \rho \) satisfies \( \sigma = \rho \circ \tau \), the subset \( \tau^*(A_m) \) is closed in \( A_N \) since it is a retract of \( A_N \), hence the (finite) union of all such subsets is closed in \( A_N \) and we may choose \( U_\sigma \) as its complementary subset in \( A_N \) (and any open neighborhood of \( \sigma^*(x) \) in this complementary subset will also do).

Throughout the rest of the section, we set an \( x \)-admissible family \( (U_\sigma)_{\sigma : [N] \to [n]} \) which we first extend as follows: given \( k \leq N \) and \( \sigma : [k] \to [n] \), we set

\[
U_\sigma := \bigcap_{\tau : [N] \to [k]} (\tau^*)^{-1}(U_{\sigma \circ \tau}) \subset A_k.
\]

The following properties then generalize the axioms defining an \( x \)-admissible family:

**Lemma 2.2.** Let \( \sigma : [k] \to [n] \) with \( k \leq N \). Then:

(i) One has \( \sigma^*(x) \in U_\sigma \).

(ii) For every \( \sigma' : [k'] \to [k] \) with \( k' \leq N \), one has \( (\sigma')^*(U_\sigma) \subset U_{\sigma \circ \sigma'} \).

(iii) For every \( \tau : [k] \to [k'] \), the condition \( U_\sigma \cap \tau^*(A_{k'}) \neq \emptyset \) is equivalent to

\[
(\exists \rho : [k'] \to [n]) : \rho \circ \tau = \sigma.
\]

**Proof.** (i) Indeed \( \tau^*(\sigma^*(x)) = (\sigma \circ \tau)^*(x) \in U_{\sigma \circ \tau} \) for every \( \tau : [N] \to [k] \).
(ii) Let \( x' \in U_{\sigma} \) and \( \tau : [N] \to [k'] \). Then \( \tau^*(\sigma'(x')) = (\sigma' \circ \tau)^*(x') \in U_{\sigma \circ \sigma' \circ \tau} \). Thus \( (\sigma')^*(x') \in U_{\sigma \circ \sigma'} \).

(iii) Let \( x \in A_k \) such that \( \tau^*(x) \in U_{\sigma} \). Choose an arbitrary \( \sigma' : [N] \to [k] \).
Then \( (\tau \circ \sigma')^*(x) = (\sigma')^*(\tau^*(x)) \in U_{\sigma \circ \sigma'} \). It follows from axiom (ii) that some \( \rho : [k'] \to [n] \) satisfies \( \sigma \circ \sigma' = \rho \circ \tau \circ \sigma' \), hence \( \sigma = \rho \circ \tau \) since \( \sigma' \) is onto. The converse is trivial. 

\( \square \)

2.2 The open subsets \( U(\sigma) \)

**Definition 2.2.** For \( \sigma : [k] \to [n] \), set 
\[ I_{\sigma} := \{ \delta : [k'] \to [k] \mid \text{s.t. } \sigma \circ \delta : [k'] \to [n] \text{ and } k' \leq N \} \]
and 
\[ U(\sigma) := \bigcap_{\delta \in I_{\sigma}} (\delta^*)^{-1}(U_{\sigma \circ \delta}) \subset A_k. \]

Clearly, \( I_{\sigma} \) is non-empty and better, for every \( (i, j) \in [k]^2 \) such that \( \sigma(i) \neq \sigma(j) \), there is some \( \delta \in I_{\sigma} \) with \( i \) and \( j \) in its range.

The \( U(\sigma) \) sets have the following main properties:

**Proposition 2.3.** Let \( \sigma : [k] \to [n] \). Then:

(a) The set \( U(\sigma) \) is an open neighborhood of \( \sigma^*(x) \) in \( A_k \).

(b) One has \( U(\sigma) \subset U_{\sigma} \) whenever \( k \leq N \).

(c) For every \( \delta : [i] \to [k] \) such that \( \sigma \circ \delta \) is onto, one has \( \delta^*(U(\sigma)) \subset U_{\sigma \circ \delta} \).

(d) For every \( \tau : [k'] \to [k] \), one has \( \tau^*(U(\sigma)) \subset U(\sigma \circ \tau) \).

(e) For every \( \tau : [k] \to [k'] \), the condition \( \tau^*(A_{k'}) \cap U(\sigma) \neq \emptyset \) is equivalent to the existence of some \( \rho : [k'] \to [n] \) such that \( \sigma = \rho \circ \tau \).

**Proof.** (a) trivially derives from statement (i) in Lemma 2.2 and the definition of \( I_{\sigma} \).

(b) is obvious since \( \text{id}[k] \in I_{\sigma} \) whenever \( k \leq N \).

(c) Let \( x' \in U(\sigma) \) and \( \delta' \in I_{\sigma \circ \delta} \). Then \( \delta \circ \delta' \in I_{\sigma} \) and so \( (\delta')^*(\delta^*(x)) = (\delta \circ \delta')^*(x') \in U_{\sigma \circ \delta \circ \delta'} \). Thus \( \delta^*(x) \in U(\sigma \circ \delta) \).

(d) Let \( x \in U(\sigma) \). Let \( \delta : [i] \to [k'] \) in \( I_{\sigma \circ \tau} \). We decompose \( \tau \circ \delta = \delta' \circ \tau' \) where \( \delta' \) is a monomorphism and \( \tau' \) an epimorphism. Since \( \sigma \circ \tau \circ \delta = \sigma \circ \delta' \circ \tau' \) is an epimorphism, \( \sigma \circ \delta' \) also is, hence \( (\delta')^*(x) \in U_{\sigma \circ \delta'} \) (notice that the domain of \( \delta' \) is \( [j] \) for some \( j \leq i \leq N \) since \( \tau' \) is onto). By statement (iii) in Lemma 2.2 it follows that \( (\tau')^*(\delta')^*(x)) \in U_{\sigma \circ \delta' \circ \tau'} \), hence \( \delta^*(\tau^*(x)) \in U_{\sigma \circ \delta} \). Therefore \( \tau^*(x) \in U(\sigma \circ \tau) \).
onto morphism $g$ from $k$ morphisms. To avoid any confusion with the simplicial category, a morphism such morphism corresponds an epimorphism $\sigma$.

Let us write $\forall \in \Gamma'$ corresponds a unique $\Gamma'$ such that $\tau^*(x) \in U(\sigma)$. Let $\delta \in I_\sigma$ with domain $[i]$. Then $(\tau \circ \delta)^*(x) \in U_{\sigma \circ \delta}$. Decompose then $\tau \circ \delta = \delta' \circ \tau'$ where $\delta'$ is a monomorphism and $\tau'$ an epimorphism. Then $(\tau')^*((\delta')^*(x)) \in U_{\sigma \circ \delta}$, hence statement (iii) in Lemma 2.2 shows that $\forall (y, z) \in [i]'^2$, $\tau'(y) = \tau'(z) \Rightarrow (\sigma \circ \delta)(y) = (\sigma \circ \delta)(z)$. Since $\delta'$ is one-to-one, this yields $\tau(y') = \tau(z') \Rightarrow \sigma(y') = \sigma(z')$ for every $y'$ and $z'$ in the range of $\delta$. Let finally $(y, z) \in [k]'^2$ such that $\sigma(y) \neq \sigma(z)$. By a previous remark, there is some $\delta \in I_\sigma$ the range of which contains $y$ and $z$. Hence $\tau(y) \neq \tau(z)$.

\[\square\]

2.3 The category $\Gamma'$

The relation $\leq$ on $\mathcal{P}(\mathbb{N})$ defined by

$$\forall A \leq B \iff (A = B \text{ or } \sup A < \inf B)$$

yields a structure of poset on $\mathcal{P}(\mathbb{N})$. We define the category $\Gamma'$ as the one with the same objects as $\Delta$ and for which, for any $(k, k') \in \mathbb{N}^2$, the morphisms from $[k]$ to $[k']$ are the increasing maps $\mathcal{P}([k]) \to \mathcal{P}([k'])$ which respect disjoint unions and map non-empty sets to non-empty sets, with the obvious composition of morphisms. To avoid any confusion with the simplicial category, a morphism $f$ from $[k]$ to $[k']$ in $\Gamma'$ will be written $f : [k] \to [k']$.

A morphism $f : [k] \to [k']$ in $\Gamma'$ is called onto when $f([k]) = [k']$. To any such morphism corresponds an epimorphism $\sigma : [k'] \to [k]$ in $\Delta$ defined by: $\forall i \in [k']$, $i \in f(\sigma(i))$. Conversely, to every epimorphism $f : [k'] \to [k]$ in $\Gamma$ corresponds a unique $\Gamma'(f) : [k] \to [k']$ defined by $\forall A \in \mathcal{P}([k])$, $\Gamma'(f)(A) = f^{-1}(A)$. Clearly, we have just defined reciprocal bijections between the set of epimorphisms from $[k']$ to $[k]$ in $\Delta$ and the set of onto morphisms from $[k]$ to $[k']$ in $\Gamma'$.

**Notation 2.3.** Let $f : [k] \to [k']$ and $\delta : [k'] \hookrightarrow [k''']$. Define then $\delta_*(f) : [k] \to [k''']$ by

$$\forall A \subset [k], \ \delta_*(f)(A) = \delta(f(A)).$$

Then every $f : [k] \to [k']$ clearly has a unique decomposition as $f = \delta_*(g)$ for an onto morphism $g : [k] \to [k''']$ and a monomorphism $\delta : [k'''] \hookrightarrow [k']$ in $\Delta$. We set:

$$\text{red}(f) := g \text{ and } \sup(f) := \delta.$$

2.4 The family $(W(f, \varepsilon))$ of open subsets of the simplicies

Let us write $\alpha = (t_0(\alpha), \ldots, t_n(\alpha))$, hence $t_i(\alpha) > 0$ for every $i \in [n]$.

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1Notice that $\emptyset$ is the minimum element of $\mathcal{P}(\mathbb{N})$ for $\leq$ with the usual convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. 

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Definition 2.4. A family \((I_{i,j})_{0 \leq i < j \leq n}\) of open intervals of \([0, +\infty[\) with compact closure in \([0, +\infty[\) is called \(\alpha\)-admissible when \(\lambda_{i,j} := \frac{t_j(\alpha)}{t_i(\alpha)}\) belongs to \(I_{i,j}\) for every pair \((i, j) \in [n]^2\) such that \(i < j\).

Clearly, such a family exists, and we may choose one for the rest of the section. Let then \(\varepsilon \in ]0, 1[\). For any \(f : [n] \to [k]\), we define \(W(f, \varepsilon) \subset \Delta^k\) as the subset consisting of the points \(\beta = (t_0, \ldots, t_k) \in \Delta^k\) for which
\[
\forall i \in [n], \sum_{p \in f(i)} t_p > 0 \quad \forall (i, j) \in [n]^2, i < j \Rightarrow \frac{\sum_{p \in f(j)} t_p}{\sum_{p \in f(i)} t_p} \in I_{i,j} \quad \text{and} \quad \sum_{p \in f([n])} t_p > 1-\varepsilon.
\]

2.5 Completing the construction of \(U_\varepsilon\)

2.5.1 The open sets \(U(f)\)

Given an onto morphism \(f : [n] \to [k]\), thus corresponding to an epimorphism \(\sigma : [k] \to [n]\), we set \(U(f) := U(\sigma) \subset A_k\). For an arbitrary \(f : [n] \to [k]\), we set
\[
U(f) := (\sup(f)^*)^{-1}(U(\text{red}(f))) \subset A_k.
\]

Obviously, \(U(f)\) is an open subset of \(A_k\).

2.5.2 Ordering the morphisms of \(\Gamma'\)

Let \(f : [k] \to [k']\) and \(i \in [k']\). Set
\[
\#f(i) := \begin{cases} 
\#f(\{j\}) & \text{whenever } i \in f(\{j\}) \\
0 & \text{when } i \notin f([k]).
\end{cases}
\]

Assume now that \(i < k'\). If \(\#f(i) = 0\) and \(\#f(i + 1) \geq 1\), we define \(f^{+i}\) by:
\[
f^{+i}(j) = \begin{cases} 
\{f(\{j\}) \cup \{i\} & \text{when } i + 1 \in f(\{j\}) \\
\{f(\{j\}) & \text{otherwise.}
\end{cases}
\]
If \( \#f(i) \geq 1 \) and \( \#f(i + 1) = 0 \), we define \( f^{+i} \) by:

\[
f^{+i}(i) = \begin{cases} f(i) \cup \{i + 1\} & \text{when } i \in f(i) \\ f(i) & \text{otherwise.} \end{cases}
\]

In any case, \( f^{+i} \) is obtained from \( f \) by attaching \( i \) or \( i + 1 \) to an adjacent set of the form \( f(i) \}. We denote by \( R \) the binary relation defined on \( \text{Hom}_{\Gamma}([k],[k']) \) by \( f \) for every \( i \) and every \( f \) for which \( f^{+i} \) is defined. We then define \( \leq \) as the pre-order relation generated by \( R \). Actually, this is an order relation on \( \text{Hom}_{\Gamma}([k],[k']) \). Consider indeed the order relation \( \subset \) on \( \text{Hom}_{\Gamma}([k],[k']) \) defined by \( f \subset g \iff (\forall j \in [k], f(j) \subset g(j)) \).

Notice that, whenever \( f^{+i} \) is defined, \( f \subset f^{+i} \). It follows that \( f \leq g \Rightarrow f \subset g \) for every \( f \) and \( g \) in \( \text{Hom}_{\Gamma}([k],[k']) \), hence \( \text{Hom}_{\Gamma}([k],[k']), \leq \) is a poset (its maximal elements are the onto morphisms). The opposite order relation will be denoted by \( \geq \).

**Remark 3.** Notice that \( \leq \) is strictly stronger than \( \subset \). For example, for \( f : [0] \Rightarrow [3] \) which maps \( \{0\} \) to \( \{0, 4\} \), and \( g : [0] \Rightarrow [4] \) which maps \( \{0\} \) to \( \{0, 2, 4\} \), one obviously has \( f \subset g \) whilst the statement \( f \leq g \) is false (notice that \( f \subset g \) is an irreducible chain for \( \subset \) whereas no \( i \) satisfies \( g = f^{+i} \)).

### 2.5.3 The definition of \( U_\varepsilon \)

Let \( \varepsilon \in [0, 1[ \) and \( k \in \mathbb{N} \). Set then

\[
U_{k,\varepsilon} := \bigcup_{f : [n] \Rightarrow [k]} \left( U(f) \times \bigcap_{g \geq f} W(g, \varepsilon) \right)
\]

which is clearly an open subset of \( A_k \times \Delta^k \). Set also

\[
U'_{k,\varepsilon} := \bigcup_{f : [n] \Rightarrow [k]} U(f) \times W(f, \varepsilon)
\]

and notice that \( U_{k,\varepsilon} \subset U'_{k,\varepsilon} \).

For an arbitrary \( \varepsilon \in [0, 1[ \), we finally define:

\[
U_\varepsilon := \bigcup_{k \in \mathbb{N}} \pi_k(U_{k,\varepsilon}) \subset |A|.
\]

For every \( \varepsilon \in [0, 1[ \), one has \( \alpha \in W(id_{[m]}, \varepsilon) \) and \( x \in U(id_{[m]}) \), hence \( (x, \alpha) \in U_{m,\varepsilon} \) since \( id_{[m]} \) is maximal. This shows that \( [(x, \alpha)] \in U_\varepsilon \). In the next section, we will show that \( U_\varepsilon \) is an open subset of \( |A| \) by proving that the family \( (U_{k,\varepsilon})_{k \in \mathbb{N}} \) is compatible.
3 The proof that $U_\varepsilon$ is an open subset of $|A|$

Here, we will prove the following proposition:

**Proposition 3.1.** Let $\varepsilon \in ]0,1[$. Then:

(a) For every $k \in \mathbb{N}^*$, $(y, \beta) \in A_k \times \Delta^{k-1}$ and $i \in [k]$:

\[(y, (\delta_i)_*(\beta)) \in U_{k,\varepsilon} \iff (d_i(y), \beta) \in U_{k-1,\varepsilon}.
\]

(b) For every $k \in \mathbb{N}$, $(x', \alpha') \in A_k \times \Delta^{k+1}$ and $i \in [k]$:

\[(g, (\sigma_i)_*(\beta)) \in U_{k,\varepsilon} \iff (s_i(y), \beta) \in U_{k+1,\varepsilon}.
\]

This has the following immediate corollary as explained in the introduction of Section 2.

**Corollary 3.2.** For every $\varepsilon \in ]0,1[$, the subset $U_\varepsilon$ is open in $|A|$ and $\forall k \in \mathbb{N}$, $U_{k,\varepsilon} = \pi_{k-1}(U_\varepsilon)$.

3.1 One last notation

Given $f : [k] \Rightarrow [k']$, and $i \in [k'-1]$ such that $\# f(i) \neq 1$, we define $f_{-i} : [k] \Rightarrow [k'-1]$ by

\[\forall j \in [k], f_{-i}({\{j\}}) = \sigma_i(f({\{j\}}) \setminus \{i\}).\]

If $\# f(k') \neq 1$, we define $f_{-k'} : [k] \Rightarrow [k'-1]$ by

\[\forall j \in [k], f_{-k'}({\{j\}}) = f({\{j\}}) \setminus \{k'\}.
\]

Obviously $(\delta_i)^*(f_{-i}) = f$ when $\# f(i) = 0$. Furthermore, if $f \leq g$ and $f_{-i}$ is defined, then $g_{-i}$ is defined. The following results are then straightforward:

**Lemma 3.3.** Let $f : [k] \Rightarrow [k']$ and $g : [k] \Rightarrow [k']$ together with some $i \in [k']$ such that $\# f(i) \neq 1$. Then

\[f \leq g \Rightarrow f_{-i} \leq g_{-i}.
\]

**Lemma 3.4.** Let $f : [k] \Rightarrow [k']$ and $g : [k] \Rightarrow [k'-1]$ together with some $i \in [k']$ such that $\# f(i) \neq 1$. Assume $f_{-i} \leq g$.

- If $\# g(i) = \# g(i-1) = 0$, then $f \leq (\delta_i)_*(g)$.
- If $\# g(i) > 0$ and $\# g(i-1) = 0$, then $f \leq (\delta_i)_*(g)^{+1}$.
- If $\# g(i) = 0$ and $\# g(i-1) > 0$, then $f \leq (\delta_i)_*(g)^{+(i-1)}$.
- If $\# g(i) > 0$ and $\# g(i-1) > 0$, then $f \leq (\delta_i)_*(g)^{+(i-1)}$ or $f \leq (\delta_i)_*(g)^{+i}$.

**Lemma 3.5.** Let $f : [k] \Rightarrow [k'-1]$ and $i \in [k'-1]$ such that $\# f(i) > 0$. Let $g \geq (\delta_i)_*(f)^{+1}$. Then $g_{-i} \geq f$. 9
3.2 Proof of statement (a) in Proposition 3.1

We fix an arbitrary pair \((y, \beta) \in A_k \times \Delta^{k-1}\) and an arbitrary integer \(i \in [k]\).

Assume first that there exists some \(f : [n] \Rightarrow [k]\) such that
\[
(y, (\delta_i)_*(\beta)) \in U(f) \times \bigcap_{g \geq f} W(g, \varepsilon).
\]

Hence \((\delta_i)_*(\beta) \in W(f, \varepsilon)\) which yields \(#_f(i) \neq 1\). The rest of the proof essentially rests upon the following claim:
\[
(d_i(y), \beta) \in U(f_{-1}) \times \bigcap_{g \geq f_{-1}} W(g, \varepsilon).
\]

- We first show that \(d_i(U(f)) \subset U(f_{-1})\). Let \(y \in U(f)\).
  - Assume \(#_f(i) = 0\). Obviously \(\text{red } f_{-1} = \text{red } f\) and \(\delta_i \circ \text{sup}(f_{-1}) = \text{sup}(f)\), hence \(\text{sup}(f_{-1})^*(d_i(y)) \in U(\text{red}(f_{-1}))\) i.e. \(d_i(y) \in U(f_{-1})\).
    Notice conversely that, for an arbitrary \(z\), the condition \(d_i(z) \in U(f_{-1})\) implies \(z \in U(f)\).
  - Assume \(#_f(i) \geq 2\). Denote respectively by \(\sigma : [k'] \rightarrow [n]\) and \(\sigma' : [k'-1] \rightarrow [n]\) the epimorphisms corresponding to \(\text{red}(f)\) and \(\text{red}(f_{-1})\).
    Then there is some \(j\) such that \(\sigma \circ \delta_j = \sigma'\) and the square
    \[
    \begin{array}{ccc}
    [k'-1] & \xrightarrow{\text{sup}(f_{-1})} & [k-1] \\
    \delta_j & \downarrow & \delta_i \\
    [k'] & \xrightarrow{\text{sup}(f)} & [k]
    \end{array}
    \]
    is commutative. However \(d_j(\text{sup}(f)^*(y)) \in U(\sigma')\) since \(\text{sup}(f)^*(y) \in U(\sigma)\) and \(\delta_j \in I_\sigma\) (cf. statement (c) in Proposition 2.3). It follows that \(\text{sup}(f_{-1})^*(d_i(y)) \in U(\sigma')\) hence \(d_i(y) \in U(f_{-1})\).

- Let now \(g \geq f_{-1}\). We wish to prove that \(\beta \in W(g, \varepsilon)\).
  - If \(#_g(i) = #_g(i-1) = 0\), then the definition of the \(W(h, \eta)\)'s obviously yield that the condition \((\delta_i)_*(\beta) \in W((\delta_i)_*(g), \varepsilon)\) implies \(\beta \in W(g, \varepsilon)\), whilst Lemma 3.3 shows \(f \leq (\delta_i)_*(g)\).
  - If \(#_g(i) > 0\) and \(#_g(i-1) = 0\), then \((\delta_i)_*(\beta) \in W((\delta_i)_*(g)^{i+}, \varepsilon)\) clearly implies \(\beta \in W(g, \varepsilon)\), whilst Lemma 3.4 shows \(f \leq (\delta_i)_*(g)^{i+}\).
  - If \(#_g(i) = 0\) and \(#_g(i-1) > 0\), then \((\delta_i)_*(\beta) \in W((\delta_i)_*(g)^{i-1}, \varepsilon)\) clearly implies \(\beta \in W(g, \varepsilon)\), whilst Lemma 3.4 shows \(f \leq (\delta_i)_*(g)^{i-1}\).
  - Either \(#_g(i-1) > 0\) and \(#_g(i) > 0\). Then both conditions \((\delta_i)_*(\beta) \in W((\delta_i)_*(g)^{i+}, \varepsilon)\) and \((\delta_i)_*(\beta) \in W((\delta_i)_*(g)^{i-1}, \varepsilon)\) clearly imply that \(\beta \in W(g, \varepsilon)\), whilst one has \(f \leq (\delta_i)_*(g)^{i-1}\) or \(f \leq (\delta_i)_*(g)^{i+}\).
In any case, this shows \( \beta \in W(g, \varepsilon) \), hence \( \beta \in \bigcap_{g \geq f_i} W(g, \varepsilon) \).

Conversely, assume \((d_i(y), \beta) \in U(f) \times \bigcap_{g \geq f} W(g, \varepsilon)\) for some \( f : [n] \Rightarrow [k - 1] \).

It was proven earlier that if \( \#_y(i) = 0 \), then, for every \( z \), one has \( z \in U(f) \) if and only if \( d_i(z) \in U(f) \). Applying this to \((\delta_i)_*(f)\) yields \( y \in U((\delta_i)_*(f)) \) since \((\delta_i)_*(f) = f\). On the other hand, for any \( g \geq (\delta_i)_*(f) \), Lemma 5.3 shows \( g_{-i} \geq (\delta_i)_*(f)_{-i} = f \) hence \( \beta \in W(g_{-i}, \varepsilon) \) and \((\delta_i)_*(\beta) \in W(g, \varepsilon)\) by using the definition of the \( W(h, \eta) \)'s. We deduce that

\[
(y, (\delta_i)_*(\beta)) \in U((\delta_i)_*(f)) \times \bigcap_{g \geq (\delta_i)_*(f)} W(g, \varepsilon),
\]

which finishes the proof of point (a) in Proposition 3.1.

**Remark 4.** A similar strategy of proof shows that for every \( k \in \mathbb{N}^*, \varepsilon \in ]0, 1[; i \in [k] \) and \((y, \beta) \in A_k \times \Delta^{k-1}:

\[
(y, (\delta_i)_*(\beta)) \in U_{n, \varepsilon}^k \iff (d_i(y), \beta) \in U_{n-1, \varepsilon}^k.
\]

### 3.3 Proof of statement (b) in Proposition 3.1

Let \((y, \beta) \in A_k \times \Delta^{k+1} \), and \( i \in [k] \).

- Assume that \((y, (\sigma_i)_*(\beta)) \in U(f) \times \bigcap_{g \geq f} W(g, \varepsilon)\) for some \( f : [n] \Rightarrow [k] \).
  - Assume \( \#_f(i) = 0 \). We may write \((d_{i+1}(s_i(y)), (\sigma_i)_*(\beta)) \in U(f) \times \bigcap_{g \geq f} W(g, \varepsilon)\) and \((d_i(s_i(y)), (\sigma_i)_*(\beta)) \in U(f) \times \bigcap_{g \geq f} W(g, \varepsilon)\) since \((\delta_i)_*(f) = (\delta_{i+1})_*(f)\). Statement (a) then yields \((s_i(y), (\sigma_i \circ \delta_{i+1})_*(\beta)) \in U((\delta_{i+1})_*)(f) \times \bigcap_{g \geq (\delta_{i+1})_*}(f) W(g, \varepsilon)\) and \((s_i(y), (\sigma_i \circ \delta_i)_*(\beta)) \in U((\delta_i)_*(f)) \times \bigcap_{g \geq (\delta_i)_*(f)} W(g, \varepsilon)\). However \((\delta_i)_*(f) = (\delta_{i+1})_*)(f)\) since \( \#_f(i) = 0 \). Thus \( s_i(y) \in U((\delta_i)_*(f))\), \((\sigma_i \circ \delta_{i+1})_*(\beta) \in \bigcap_{g \geq (\delta_i)_*(f)} W(g, \varepsilon)\) and \((\sigma_i \circ \delta_i)_*(\beta) \in \bigcap_{g \geq (\delta_i)_*(f)} W(g, \varepsilon)\). It follows that \( \beta \in \bigcap_{g \geq (\delta_i)_*(f)} W(g, \varepsilon) \) and finally \((s_i(y), \beta) \in U((\delta_i)_*(f)) \times \bigcap_{g \geq (\delta_i)_*(f)} W(g, \varepsilon)\).
  - Assume \( \#_f(i) \geq 1 \). Set then \( f' := (\delta_i)_*(f)^{++i} \) and let \( g \geq f' \). Then Lemma 5.3 shows \( g_{-i} \geq f \). From \((\sigma_i)_*(\beta) \in W(g_{-i}, \varepsilon)\) immediately follows \( \beta \in W(g, \varepsilon)\) by using the definition of the \( W(h, \eta) \)'s. Furthermore, should we let \( \sigma \) and \( \sigma' \) denote the epimorphisms which respectively correspond to red\((f)\) and red\((f')\), then there exists some \( j \) such
that $\sigma \circ \sigma_j = \sigma'$ and the square

$$
\begin{array}{ccc}
[k' + 1] & \xrightarrow{\sup(f')} & [k + 1] \\
\downarrow \sigma_j & & \downarrow \sigma_i \\
[k'] & \xrightarrow{\sup(f)} & [k]
\end{array}
$$

is commutative. Since $\sup(f)^*y \in U(\text{red}(f))$, one has $s_j(\sup(f)^*y) \in U(\text{red}(f))\sigma_i$, hence $\sup(f)^*(s_i(y)) \in U(\text{red}(f'))$. Therefore, $(s_i(y) , \beta) \in U(f') \times \bigcap_{g \geq f} W(g, \varepsilon)$.

Conversely, assume that $(s_i(y), \beta) \in U(f) \times \bigcap_{g \leq f} W(g, \varepsilon)$ for some $f : [n] \Rightarrow [k + 1]$.

* Assume that $i \not\in f([n])$ and $i + 1 \not\in f([n])$. Then $\#f(i) = 0$ hence $y = d_i(s_i(y)) \in U(f_{i-1})$ by the proof of statement (a). Let $g \geq f_{i-1}$. If $\#_g(i) = 0$, then $(\delta_i)_*(g) \geq f$ hence $\beta \in W((\delta_i)_*(g), \varepsilon)$ yields $(\sigma_i)_*(\beta) \in W(g, \varepsilon)$. If $\#_g(i) \geq 1$, then $(\delta_i)_*(g)^{+\varepsilon} \geq f$ and $\beta \in W((\delta_i)_*(g)^{+\varepsilon}, \varepsilon)$ yields $(\sigma_i)_*(\beta) \in W(g, \varepsilon)$. In any case, we have shown that $(y, (\sigma_i)_*(\beta)) \in U(f_{i-1}) \times \bigcap_{g \leq f_{i-1}} W(g, \varepsilon)$.

* Assume $i$ is in $f([n])$ and $i + 1$ is not. Then $\sup(f_{-(i+1)}) = \sigma_i \circ \sup(f)$ and $\text{red}(f) = \text{red}(f_{-(i+1)})$, hence $\sup(f_{-(i+1)})^*(y) = \sup(f)^*(s_i(y)) \in U(\text{red}(f))$ i.e. $y \in U(f_{-(i+1)})$. Let $g \geq f_{-(i+1)}$. Then $(\delta_i)_*(g)^{+\varepsilon} \geq f$, whilst $\beta \in W(g, \varepsilon)$ since $s_i(\beta) \in W((\delta_i)_*(g)^{+\varepsilon}, \varepsilon)$. It follows that $(y, (\sigma_i)_*(\beta)) \in U(f_{-(i+1)}) \times \bigcap_{g \geq f_{-(i+1)}} W(g, \varepsilon)$.

* If $i + 1$ is in $f([n])$ and $i$ is not, a similar proof as the above one shows that $(y, (\sigma_i)_*(\beta)) \in U(f_{i-1}) \times \bigcap_{g \geq f_{i-1}} W(g, \varepsilon)$.

* Assume $i$ and $i + 1$ belong to $f([n])$. We claim that $\{i, i + 1\} \subset f\{j\}$ for some $j \in [m]$. Assuming this holds, then $y = d_i(s_i(y)) \in U(f_{i-1})$ since $\#f(i) \geq 2$, whilst, for every $g \geq f_{i-1}$, one has $(\delta_i)_*(g)^{+\varepsilon} \geq f$, therefore $\beta \in W((\delta_i)_*(g)^{+\varepsilon}, \varepsilon)$ i.e. $(\sigma_i)_*(\beta) \in W(g, \varepsilon)$. This shows that $(y, (\sigma_i)_*(\beta)) \in U(f_{i-1}) \times \bigcap_{g \geq f_{i-1}} W(g, \varepsilon)$.

Let us finally prove the above claim. Assume indeed that no $j \in [m]$ satisfies $\{i, i + 1\} \subset f\{j\}$, and let $\sigma : [k'] \Rightarrow [n]$ be the epimorphism corresponding to $\text{red}(f)$. Then, for some one-to-one morphism $\delta : [k] \Rightarrow [k']$ and for some $j' \in [k' - 1]$, the square

$$
\begin{array}{ccc}
[k'] & \xrightarrow{\sup(f')} & [k] \\
\downarrow \sigma_j & & \downarrow \sigma_i \\
[k'] & \xrightarrow{\delta} & [k]
\end{array}
$$

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is commutative with \( \sigma(j') \neq \sigma(j' + 1) \). Then \((\sigma_i \circ \sup(f))^*(y) \in U(\text{red}(f)) \) yields \( s_{j'}(\delta^*(y)) \in U(\sigma) \), which contradicts property (e) of Proposition 2.3 applied to \( \tau = \sigma_{j'} \).

In any case, we have shown that \((y, (\sigma_i)_*(\beta)) \in U_{\kappa, \varepsilon} \).

This finishes the proof of Proposition 3.1.

Remark 5. One might wonder whether we could have avoided all those technicalities, especially the introduction of \( \leq \), and why not simply define \( U_{\kappa, \varepsilon} \) as \( \bigcup_{f \in [n] \Rightarrow [k]} U(f) \times W(f, \varepsilon) \) (which was our first idea). The trouble is that the last points in the proof of statement (b) of Proposition 3.1 miserably fail if such a definition is considered.

4 Application to simplicial Hausdorff spaces

In this section, we assume that \( A \) is a simplicial Hausdorff space. We arbitrarily choose two pairs \((x, \alpha) \) and \((y, \beta) \) such that \( x \in A_n \) is non-degenerate, \( y \in A_m \) is non-degenerate, \( \alpha \in \Delta^n \setminus \partial \Delta^n \) and \( \beta \in \Delta^m \setminus \partial \Delta^m \). We also pick an arbitrary integer \( N \geq \max(n, m) \). Example 1 yields that we may choose an \( x \)-admissible family \((U_\sigma)_{\sigma : [N] \Rightarrow [n]} \) and a \( y \)-admissible family \((V_\sigma)_{\sigma : [N] \Rightarrow [n]} \). Using the procedure of Sections 2.2 and 2.1 we recover two families \((U(f)) \) and \((V(g)) \).

We also choose an \( \alpha \)-admissible family \((I_{i,j}) \) of intervals and a \( \beta \)-admissible family of intervals. For every \( \varepsilon \in [0, 1] \), we obtain respective families \((W(f, \varepsilon))_{f : [n] \Rightarrow [k]} \) and \((T(f, \varepsilon))_{f : [n] \Rightarrow [k]} \) corresponding to \((I_{i,j}) \) and to \((J_{k,l}) \).

For every \( k \in \mathbb{N} \), the procedure of Section 2.5.3 yields subsets \( U_{k, \varepsilon} \) and \( U'_{k, \varepsilon} \) of \( A_k \times \Delta^k \) from \((U(f)) \) and \((W(f, \varepsilon)) \), and subsets \( V_{k, \varepsilon} \) and \( V'_{k, \varepsilon} \) of \( A_k \times \Delta^k \) from \((V(g)) \) and \((T(g, \varepsilon)) \). This yields open subsets \( U_{\varepsilon} \) and \( V_{\varepsilon} \) of \( |A| \).

4.1 A sufficient condition for disjointness

Proposition 4.1. Assume that for every \( k \in \mathbb{N} \), every onto \( f : [n] \Rightarrow [k] \), every onto \( g : [m] \Rightarrow [k] \), and every \( \varepsilon \in [0, 1] \), one has

\[
\left( U(f) \times W(f, \varepsilon) \right) \cap \left( V(g) \times T(g, \varepsilon) \right) = \emptyset.
\]

Then there exists an \( \eta \in [0, 1] \) such that \( U_{\eta} \cap V_{\eta} = \emptyset \).

Proof. Notice first that, for every \( 0 < \varepsilon \leq \eta < 1 \) and every \( k \in \mathbb{N} \), one has \( U_{k, \varepsilon} \subset U'_{k, \varepsilon} \subset U'_{k, \eta} \) and \( V_{k, \varepsilon} \subset V'_{k, \varepsilon} \subset V'_{k, \eta} \). It thus suffices to provide some \( \eta \in [0, 1] \) such that \( U'_{k, \eta} \cap V'_{k, \eta} = \emptyset \). There are two major steps in the proof of the existence of \( \eta \):

1. By a finite induction process, we will show that there is some \( \eta \in [0, 1] \) such that

   \[
   \forall k \leq 2(n + 2)(m + 2), \quad U'_{k, \eta} \cap V'_{k, \eta} = \emptyset.
   \]
We may thus write
\[ U = \bigcup_{\varepsilon} \bigcup_{\varepsilon} V_{\varepsilon, \xi} \]
for every \( \varepsilon \). Let now \( M \) be an integer such that \( 0 \leq M < 2(n+2)(m+2) \) and there is some \( \varepsilon_M \in [0,1[ \) such that \( U_{\varepsilon_M} \cap V_{\varepsilon_M} = \emptyset \) whenever \( k \leq M \).

By another induction process, we will show that such an \( \eta \) necessarily satisfies:
\[ \forall k \geq 2(n+2)(m+2), \quad U'_{k, \eta} \cap V'_{k, \xi} = \emptyset. \]

We first consider the case \( k = 0 \). If \( n > 0 \) (resp. \( m > 0 \)) then \( U'_{k, \xi} = \emptyset \) (resp. \( V'_{k, \xi} = \emptyset \)). If \( m = n = 0 \), then \( U'_{0, \xi} = U_{0, \xi} \) and \( V'_{0, \xi} = V_{0, \xi} \) are clearly disjoint for every \( \varepsilon \). Let now \( M \) be an integer such that \( 0 \leq M < 2(n+2)(m+2) \) and there is some \( \varepsilon_M \in [0,1[ \) such that \( U_{k, \varepsilon_M} \cap V_{k, \varepsilon_M} = \emptyset \) whenever \( k \leq M \).

Let \( f : [n] \to [M+1] \) and \( g : [m] \to [M+1] \). Clearly, we may assume that either \( f \) or \( g \) is not onto and use a reductio ad absurdum by assuming that
\[ (U(f) \times W(f, \varepsilon)) \cap (V(g) \times T(g, \varepsilon)) \neq \emptyset \]
for every \( \varepsilon \in ]0,1[ \). We may then choose some \( y \in U(f) \cap V(g) \) and, for every \( i \in \mathbb{N}^* \), some \( \alpha_i \in W(f, \frac{1}{i}) \cap T(g, \frac{1}{i}) \). Since
\[ W(f, \varepsilon_M) \cap T(g, \varepsilon_M) \]
is closed in \( \Delta^{M+1} \), and therefore compact, the sequence \( \alpha_n \) has an adherence value \( \overline{\alpha} \in W(f, \varepsilon_M) \cap T(g, \varepsilon_M) \). If \( f \) is not onto, then picking some \( j \in [n] \setminus f([n]) \), we see that \( t_j(\alpha_n) \leq \frac{1}{n} \) for any \( n \in \mathbb{N}^* \), hence \( t_j(\overline{\alpha}) = 0 \). In any case, we see that \( \overline{\alpha} \in \partial \Delta^{M+1} \).

We may thus write \( \overline{\alpha} = \delta_i^*(\beta) \) for some \( i \) and some \( \beta \in \Delta^M \). Therefore \( (y, (\delta_i)^*(\beta)) \in U'_{M+1, \varepsilon_M} \cap V'_{M+1, \varepsilon_M} \), which implies, using Remark 14, that \((\delta_i(y), \beta) \in U'_{M+1, \varepsilon_M} \cap V'_{M+1, \varepsilon_M} = \emptyset \). This is a contradiction, hence the existence of some \( \varepsilon_{f, g} \in [0,1[ \) such that
\[ (U(f) \times W(f, \varepsilon_{f, g})) \cap (V(g) \times T(g, \varepsilon_{f, g})) = \emptyset. \]

Setting \( \varepsilon_{f, g} = \max(\varepsilon_M, \varepsilon_{f, g}) \), and then \( \varepsilon_{M+1} = \min\{ \varepsilon_{f, g}, \text{ s.t. } f \text{ and } g \text{ are not both onto} \} \), we see that \( \varepsilon_{M+1} \leq \varepsilon_M \), and \( U'_{M+1, \varepsilon_{M+1}} \cap V'_{M+1, \varepsilon_{M+1}} = \emptyset \), hence
\[ \forall k \leq M+1, \quad U'_{k, \varepsilon_{M+1}} \cap V'_{k, \varepsilon_{M+1}} = \emptyset. \]
This finite induction process yields some \( \eta \in ]0,1[ \) such that
\[ \forall k \leq 2(n+2)(m+2), \quad U'_{k, \eta} \cap V'_{k, \eta} = \emptyset. \]

We now move on to step (2). Let then \( M \geq 2(n+2)(m+2) \) such that \( U'_{M, \eta} \cap V'_{M, \eta} = \emptyset \). Let \( f : [n] \to [M+1] \) and \( g : [m] \to [M+1] \). We will prove that
\[ (U(f) \times W(f, \eta)) \cap (V(g) \times T(g, \eta)) = \emptyset \]
by a reductio ad absurdum. Assume that there is some \( (\gamma, \gamma) \in (U(f) \times W(f, \eta)) \cap (V(g) \times T(g, \eta)) \).

- Assume that there exists \( i \in [n] \) and \( j \in [m] \) such that \( \#(f([i]) \cap g([j])) \geq 2 \), and choose two distinct elements \( k < l \) in \( f([i]) \cap g([j]) \). Define \( \gamma' \in \Delta^M \) by:
\[
 t_p(\gamma') = \begin{cases} 
 t_p(\gamma) & \text{for } p < k \\
 t_k(\gamma) + t_l(\gamma) & \text{for } p = k \\
 t_p(\gamma) & \text{for } k < p < l \\
 t_{p+1}(\gamma) & \text{for } l \leq p \leq M.
\end{cases}
\]

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Proposition 4.1 are satisfied. We will need to tackle separately the case of the open subset $x$ and then show that the various admissible families that were used in the construction $\text{Lemma 4.2.}$

Let $\forall \alpha \in \mathbb{N}$, $\mathbb{N} \ni \alpha$. Theorem 1.1 is now within our reach. We now assume ($4.2$ Finishing the proof of Theorem 1.1)

This shows that $\forall (\alpha, \beta) \ni \mathbb{N}$. Hence $\forall (\alpha, \beta) \ni \mathbb{N}$ and Remark yields the contradiction $d(z, \gamma') \ni U^{\prime}_{M}, \eta \cap V^{\prime}_{M}, \eta = \emptyset$.

Assume finally that $f([n]) \cup g([m]) = [M + 1]$, e.g. $\#(f([n])) \geq \#(g([m]))$, hence $\#(f([n])) \geq (n + 2)(m + 2)$ and we may find some $i \ni [n]$ such that $\#(f([i])) = m + 3$. Since we have assumed that $\forall j \ni [m], \#(f([i])) \leq 1$, we deduce that $\#(f([i]) \cap g([j])) \geq 2$.

We then choose two distinct elements $k < l$ of $f([i]) \cap g([m])$ and define $\gamma' \in \mathbb{N}$ by:

$$t_p(\gamma') = \begin{cases} t_k(\gamma) & \text{for } p < k \\ 1/t_k(\gamma) & \text{for } p = k \\ t_p(\gamma) & \text{for } k < p < l \\ t_{p+1}(\gamma) & \text{for } l \leq p \leq N. \end{cases}$$

Then $(\delta_\gamma)^*(\gamma') \ni \mathbb{N}$. Hence $(\delta_\gamma)^*(\gamma') \in U^{\prime}_{M}, \eta \cap V^{\prime}_{M}, \eta$ and Remark yields the final contradiction $d(z, \gamma') \in U^{\prime}_{M}, \eta \cap V^{\prime}_{M}, \eta = \emptyset$.

This shows that $U^{\prime}_{M}, \eta \cap V^{\prime}_{M}, \eta = \emptyset$. This induction process then proves that $\forall k \ni \mathbb{N}, U^{\prime}_{k, \eta} \cap V^{\prime}_{k, \eta} = \emptyset$ and it follows that $U^{\prime}_{0} \cap V^{\prime}_{0} = \emptyset$. $\blacksquare$

### 4.2 Finishing the proof of Theorem 1.1

Theorem is now within our reach. We now assume $(x, \alpha) \neq (y, \beta)$. We will then show that the various admissible families that were used in the construction of the open subset $U_{\alpha}$ and $V_{\alpha}$ may be carefully chosen so that the assumptions of Proposition are satisfied. We will need to tackle separately the case $x \neq y$ and the case $x = y$. The following basic lemma on simplicial sets will prove essential:

**Lemma 4.2.** Let $N \geq \max(m, n)$. 

(i) If \( x \neq y \), then \( \sigma^*(x) \neq \tau^*(y) \) for every \( \sigma : [N] \rightarrow [n] \) and \( \tau : [N] \rightarrow [m] \).

(ii) If \( x = y \), then \( \sigma^*(x) \neq \tau^*(y) \) for every distinct \( \sigma : [N] \rightarrow [n] \) and \( \tau : [N] \rightarrow [m] \).

Proof. Assume there is some \( \sigma : [N] \rightarrow [n] \) and some \( \tau : [N] \rightarrow [m] \) such that \( \sigma^*(x) = \tau^*(y) \). Applying Lemma 2.1 to both \( x \) and \( y \), we easily find that \( m = n \) and \( \sigma = \tau \). However, \( \sigma^* \) is one-to-one since \( \sigma \) has a section in \( \Delta \). Hence \( x = y \), which proves both statements in the lemma.

4.2.1 The case \( x \neq y \)

Since \( A_{m+n} \) is Hausdorff, the above lemma and the method of Example 1 shows we may choose the families \( (U_\sigma)_{\sigma : [N] \rightarrow [n]} \) and \( (V_\tau)_{\tau : [N] \rightarrow [m]} \) so that

\[ \forall (\sigma, \tau), \ U_\sigma \cap V_\tau = \emptyset. \]

The following lemma will then show that we may use Proposition 4.1 which will complete the case \( x \neq y \):

Lemma 4.3. (a) For every \( k \leq m + n \) and every \( \sigma : [k] \rightarrow [n] \) and \( \tau : [k] \rightarrow [m] \), one has \( U_\sigma \cap V_\tau = \emptyset \).

(b) For every \( \sigma : [k] \rightarrow [n] \) and \( \tau : [k] \rightarrow [m] \), one has \( U(\sigma) \cap V(\tau) = \emptyset \).

Proof. (a) Choose an arbitrary \( \sigma' : [n+m] \rightarrow [k] \). Then the assumptions show that \((\sigma')^*(U_\sigma \cap V_\tau) \subset U_{\sigma'\sigma} \cap V_{\tau\sigma'} = \emptyset \).

(b) If \( k \leq m + n \), then \( U(\sigma) \subset U_\sigma \), and \( V(\tau) \subset V_\tau \) hence we may readily use (a). In the case \( k \geq m + n \), we proceed by onward induction. Let \( k \geq m + n \) such that \( U(\sigma) \cap V(\tau) = \emptyset \) for every \( \sigma : [k] \rightarrow [n] \) and \( \tau : [k] \rightarrow [m] \).

Let then \( \sigma : [k+1] \rightarrow [n] \) and \( \tau : [k+1] \rightarrow [m] \). We set \( J_\sigma := \{ i : [k+1] : i \in [k+1] \} \) and define \( J_\tau \) accordingly. Then \# \( J_\sigma \geq k + 2 - n \) and \# \( J_\tau \geq k + 2 - m \). Thus \# \( J_\sigma + \# J_\tau > k + 2 \), and so \( J_\sigma \cap J_\tau \neq \emptyset \).

Choose \( i \in J_\sigma \cap J_\tau \). Point (c) in Proposition 2.3 and the induction assumption then show that \( d_i(U(\sigma) \cap V(\tau)) \subset U(\sigma \circ \delta_i) \cap V(\tau \circ \delta_i) = \emptyset \), hence \( U(\sigma) \cap V(\tau) = \emptyset \).

4.2.2 The case \( x = y \)

We now set \( N := (n+1)^2 \). Again, Lemma 4.2 and the method of Example 1 show we may choose the \( x \)-admissible family \((U_\sigma)_{\sigma : [(n+1)^2] \rightarrow [n]} \) so that \( U_\sigma \cap U_\tau = \emptyset \) whenever \( \sigma \neq \tau \).

For \( i \in [n] \), set \( \delta(i) := t_i(\alpha) - t_i(\beta) \). Since \( \alpha \neq \beta \), and \( \sum_{0 \leq i \leq n} \delta(i) = 0 \), we may find two indices \( (k, l) \in [n]^2 \) such that \( \delta(k) < 0 < \delta(l) \), e.g. with \( k < l \) if not, we
simply switch \((x, \alpha)\) and \((y, \beta)\). Clearly, we may choose our \(\alpha\)-admissible family \((I_{i,j})\) and our \(\beta\)-admissible family \((J_{i,j})\) so that \(\overline{I_{k,l}} \cap \overline{J_{k,l}} = \emptyset\). Obviously
\[
\forall \varepsilon \in ]0, 1[\subseteq \overline{W(\text{id}_{[n]}, \varepsilon)} \cap \overline{T(\text{id}_{[n]}, \varepsilon)} = \emptyset.
\]
Proposition 4.1 may then be used thanks to the next lemma, and this will complete both the case \(x = y\) and the proof of Theorem 4.1.

**Lemma 4.4.** Let \(\varepsilon \in ]0, 1[\).

(a) For every \(k \leq (n+1)^2\), every onto morphism \(f : [n] \Rightarrow [k]\) and every onto morphism \(g : [n] \Rightarrow [k]\), one has
\[
f \neq g \Rightarrow U(f) \cap U(g) = \emptyset
\]
(b) For every \(k \in \mathbb{N}\), every onto morphism \(f : [n] \Rightarrow [k]\) and every onto morphism \(g : [n] \Rightarrow [k]\),
\[
(U(f) \times \overline{W(f, \varepsilon)}) \cap (U(g) \times \overline{T(g, \varepsilon)}) = \emptyset
\]

**Proof.** (a) Obviously, it suffices to prove that \(U_\sigma \cap U_\tau = \emptyset\) whenever \(\sigma \neq \tau\). Let \(\sigma : [k] \rightarrow [n]\) and \(\tau : [k] \rightarrow [n]\), with \(\sigma \neq \tau\). Choose some \(\sigma' : [(n+1)^2] \rightarrow [k]\). Then \((\sigma')^* (U_\sigma \cap U_\tau) \subseteq U_{\sigma \circ \sigma'} \cap U_{\tau \circ \sigma'}\). However, if \(\sigma \circ \sigma' = \tau \circ \sigma'\), then \(\sigma = \tau\). Thus \(\sigma \circ \sigma' \neq \tau \circ \sigma'\) and we deduce from the hypothesis that \(U_{\sigma \circ \sigma'} \cap U_{\tau \circ \sigma'} = \emptyset\), which yields \(U_\sigma \cap U_\tau = \emptyset\).

(b) Let \(k \leq (n+1)^2\). Clearly \(\overline{W(f, \varepsilon)} \cap \overline{T(f, \varepsilon)} = \emptyset\) given the construction of the families \((I_{i,j})\) and \((J_{i,j})\). Then either \(f \neq g\) and \(U(f) \cap U(g) = \emptyset\), or \(f = g\) and \(\overline{W(f, \varepsilon)} \cap \overline{T(g, \varepsilon)} = \emptyset\) since \(\overline{k,l} \cap J_{k,l} = \emptyset\). In any case, the claimed property is proven.

For the case \(k \geq (n+1)^2\), we proceed by onward induction. Let \(k \geq (n+1)^2\) such that the claimed result holds for every pair of onto morphisms from \([n]\) to \([k]\) in \(\Gamma'\). Let \(f : [n] \Rightarrow [k+1]\) and \(g : [n] \Rightarrow [k+1]\) be onto morphisms, and \(\sigma : [k+1] \rightarrow [n]\) and \(\tau : [k+1] \rightarrow [n]\) the epimorphisms in \(\Delta\) associated to them.

Assume that \(#f(i) \cap #g(j)\) \(\leq 1\) for every \((i, j) \in [n]^2\). Then \(#f(i)\) \(\leq n+1\) for every \(i\) (since \(g\) is onto), and \(k+1 = #f([n]) \leq (n+1)^2\) which contradicts the fact that \(f\) is onto. Hence we may find a pair \((i, j) \in [n]^2\) and some \(l \in [k+1]\) such that \(\{i, j, l\} \subseteq \{i\} \cap g(\{j\})\). Assume finally that there is some \((z, \gamma)\) in \((U(f) \times \overline{W(f, \varepsilon)}) \cap (U(g) \times \overline{T(g, \varepsilon)})\). Notice then that
\[
(d_l(z), (\sigma_l)_\ast(\gamma)) \in (U(f_{-l}) \times \overline{W(f_{-l}, \varepsilon)}) \cap (U(g_{-l}) \times \overline{T(g_{-l}, \varepsilon)}).
\]

Indeed, it is obvious on the one hand that \((\sigma_l)_\ast(\gamma) \in \overline{W(f_{-l}, \varepsilon)} \cap \overline{T(g_{-l}, \varepsilon)}\); on the other hand \(\sigma \circ \delta_l\) and \(\tau \circ \delta_l\) clearly are the epimorphisms respectively associated to the onto morphisms \(f_{-l}\) and \(g_{-l}\), hence point (c) in Lemma 2.3 shows \(d_l(z) \in U(f_{-l}) \cap U(g_{-l})\). Finally, the contradiction comes from the induction hypothesis since \(f_{-l}\) and \(g_{-l}\) are both onto.

\(\square\)
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