BAROCLINIC INSTABILITY IN STELLAR RADIATION ZONES

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ABSTRACT

Surfaces of constant pressure and constant density do not coincide in differentially rotating stars. Stellar radiation zones with baroclinic stratification can be unstable. Instabilities in radiation zones are of crucial importance for angular momentum transport, mixing of chemical species, and, possibly, for magnetic field generation. This paper performs linear analysis of baroclinic instability in differentially rotating stars. Linear stability equations are formulated for differential rotation of arbitrary shape and then solved numerically for rotation nonuniform in radius. As the differential rotation increases, \( r^{-1} \) and \( g \)-modes of initially stable global oscillations transform smoothly into growing modes of baroclinic instability. The instability can therefore be interpreted as stability loss to \( r^{-1} \) and \( g \)-modes excitation. Regions of stellar parameters where \( r^{-1} \) or \( g \)-modes are preferentially excited are defined. Baroclinic instability sets at a very small differential rotation of below 1\%. The characteristic time of instability growth is about 1000 rotation periods. Growing disturbances possess kinetic helicity. Magnetic field generation by the turbulence resulting from baroclinic instability in differentially rotating radiation zones is therefore possible.

Key words: dynamo – hydrodynamics – instabilities – stars: rotation

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1. INTRODUCTION

Stratification in not too rapidly rotating stars is close to spherically symmetric. Disregarding deviations from this symmetry, only two possibilities for mutual orientation of the gradients of pressure and entropy are possible. Both possibilities of parallel and antiparallel orientation are realized in the convection and radiation zones of stars, respectively. Stratification in radiation zones is believed to be stable because radial displacements are opposed by buoyancy. The stabilizing effect of subadiabatic stratification is, however, obvious for strictly parallel gradients of entropy and pressure only. Even a slight deviation from barotropic stratification can provoke an instability (Tassoul 2000, chapter 3).

Figure 1 illustrates the origin of instability expected for radiation zones with baroclinic stratification. Displacements in narrow cones between isentropic and isobaric surfaces are potentially unstable. Projections of the displacements in the directions of the entropy and pressure gradients have the same sign. Therefore, the buoyancy force supports the displacements. Instability can develop at the expense of gravitational energy (Shibahashi 1980).

Baroclinicity, in turn, naturally results from nonuniform rotation. If angular velocity varies with the cylindrical coordinate \( z \) along the rotation axis, the centrifugal force is not conservative so that the pressure force per unit mass, \( \rho^{-1} \nabla P \), should not be conservative either, \( \nabla \rho \times \nabla P \neq 0 \), in order to balance the centrifugal force. Baroclinic instability can be expected for rotating radiation zones with \( z \)-dependent angular velocity.

The study of baroclinic instability in an astrophysical context has a long history. Goldreich & Schubert (1967) found the condition \( d \Omega/dz = 0 \) as necessary for the stability of rotating radiation zones. Shibahashi (1980) noticed the significance of the relation between differential rotation and baroclinicity for stability. These and related studies (Acheson 1978; Spruit & Knobloch 1984; Korycansky 1991) considered local stability. The spatial scale of disturbances was assumed to be small in comparison with the scale height. This paper concerns stability against disturbances that are global in horizontal dimensions. The radial scale of disturbances is still assumed to be short. Subadiabatic stratification in radiation zones precludes mixing on large radial scales. A similar approach was applied to (barotropic) instabilities in the solar tachocline (Charbonneau et al. 1999; Gilman et al. 2007; Kitchatinov & Rüdiger 2009) to find that the dominating modes of the instabilities are global in horizontal dimensions. We shall see that the same is true of baroclinic instability. It was suggested in Kitchatinov (2013; hereafter K13) that growing modes of baroclinic instability correspond to global Rossby waves (\( r \)-modes) and internal gravity waves (\( g \)-modes). This paper confirms this expectation by following the dependence of eigenmodes on differential rotation. As the differential rotation and the resulting baroclinicity increase from zero, \( r^{-1} \) and \( g \)-modes transform smoothly into unstable eigenmodes. Baroclinic instability can therefore be interpreted as stability loss to excitation of \( r^{-1} \) and \( g \)-modes. The regions in parametric space, where a specific mode dominates, will be identified.

Differential rotation is known to affect gravitoinertial waves in an essential way (Ando 1985; Lee & Saio 1993; Mathis 2009; Baruteau & Rieutord 2013; Alvan et al. 2013). We consider this influence, taking into account the deviation of stratification from barotropy, and find that some of the modified \( r^{-1} \) and \( g \)-modes are amplified while some others are damped because of baroclinicity induced by the differential rotation.

Unstable modes possess kinetic helicity, \( u \cdot (\nabla \times u) \neq 0 \). Helicity is important for the generation of global magnetic fields (cf. Moffatt 1978 or a more recent discussion of helicity effects by Brandenburg & Subramanian 2005). The instability can therefore be relevant to the origin of magnetic fields of stellar radiation zones.

Section 2 provides the mathematical formulation of the problem. Linear stability equations are formulated for differential rotations of arbitrary shape. Section 3 presents and discusses the results. The earlier hypothesis (K13) that the growing modes of baroclinic instability can be understood as modified \( r^{-1} \) and
2. LINEAR STABILITY PROBLEM

2.1. Background Equilibrium

The background equilibrium is assumed to be axially symmetric about the rotation axis. Hydrodynamic stability is considered. The steady motion equation then reads as follows:

\[(V \cdot \nabla)V = -\frac{1}{\rho} \nabla P - \nabla \psi,\]  

where \(\psi\) is the gravity potential, other notations are standard, and the effect of viscosity on the global flow is neglected. The principal motion in the radiation zone is rotation,

\[V = e_\theta r \sin \theta \Omega.\]  

In this equation, the usual spherical coordinates \((r, \theta, \phi)\) are used, \(e_\theta\) is the azimuthal unit vector, and \(\Omega\) is the angular velocity.

The meridional flow in subadiabatically stratified radiation zones is small. The characteristic time of meridional circulation in the solar radiation zone exceeds the age of the Sun (Tassoul 2000). Nevertheless, the meridional motion gives important condition to the balance of meridional forces,

\[r \sin \theta \frac{\partial \Omega^2}{\partial z} = -\frac{1}{\rho^2} (\nabla \rho \times \nabla P)_\phi,\]  

where \(\partial \rho / \partial z = \cos \theta \partial \rho / \partial r - r^{-1} \sin \theta \partial \rho / \partial \theta\) is a gradient along the rotation axis. Condition (3) can be obtained by taking the azimuthal component of curled Equation (1). The centrifugal force in rotational motion is conservative only if the angular velocity does not vary with the cylindrical coordinate \(z\). The left-hand side of Equation (3) accounts for the nonconservative part of centrifugal force. This force alone would drive a vortical meridional flow. In the radiation zone of a star, however, the force is balanced by buoyancy accounted for by the right-hand side of Equation (3). The equilibrium is baroclinic: the surfaces of constant pressure and entropy do not coincide. The surfaces of constant pressure and entropy \(s = c_s \ln(P) - c_p \ln(\rho)\) differ as well:

\[\nabla \rho \times \nabla P = -\frac{\rho}{c_p} \nabla s \times \nabla P = -\frac{\rho^2}{c_p} \nabla s \times g^*,\]  

where \(g^* = -\nabla \psi + r \sin \theta \Omega (e_\phi \times \Omega)\) is the effective gravity. The instability considered in this paper is caused by the baroclinicity of the stratification, not by the rotationally induced deviation from spherical symmetry. We neglect the deviation of isobaric surfaces from spheres but keep the baroclinicity as it is defined by Equation (4),

\[\frac{\partial s}{\partial \theta} = 2g^{-1}c_pr \sin \theta \Omega \left( \cos \theta r \frac{\partial \Omega}{\partial r} - \sin \theta \frac{\partial \Omega}{\partial \theta} \right),\]  

where \(g\) is gravity. For the case of shellular rotation where \(\Omega\) depends only on \(r\), the expression for the meridional gradient of entropy simplifies further to

\[\frac{\partial s}{\partial \theta} = -2q g^{-1}c_pr \Omega^2 \sin \theta \cos \theta,\]  

where

\[q = -\frac{r}{\Omega} \frac{d\Omega}{dr}\]  

is the normalized shear of the radial differential rotation.
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\[ \hat{L} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \] (9)

is the angular part of the Laplacian operator. Dimensionless variables are used. Physical variables can be restored from normalized entropy (S), toroidal (W), and poloidal (V) flow potentials by using Equation (8) and the following relations:

\[ s' = -\frac{i c_p N^2}{g k} S, \quad p_u = (\Omega_0^2/k)V, \quad T_u = \Omega_0^2 W, \] (10)

where \( \Omega_0 \) is the characteristic value of angular velocity.

The mathematical formulation of the stability problem reduces to the eigenvalue problem for a set of three ordinary differential equations with latitude as the independent variable. The equations for toroidal flow,

\[ (\hat{\omega} - m\hat{\Omega})(\hat{L}W) = -i \frac{\epsilon_v}{\lambda^2} (\hat{L}W) - m \frac{\partial^2(1 - \mu^2\hat{\Omega})}{\partial \mu^2} W \]

\[ + \frac{\partial ((1 - \mu^2\hat{\Omega})}{\partial \mu}(\hat{L}V) \]

\[ + \frac{\partial^2((1 - \mu^2\hat{\Omega})}{\partial \mu^2}(1 - \mu^2) \frac{\partial V}{\partial \mu}, \] (11)

and poloidal flow,

\[ (\hat{\omega} - m\hat{\Omega})(\hat{L}V) = -i \frac{\epsilon_v}{\lambda^2} (\hat{L}V) - \hat{\lambda}^2(\hat{L}S) \]

\[ + 2m \left( \frac{\partial (\mu \hat{\Omega})}{\partial \mu} V + (1 - \mu^2) \frac{\partial \Omega}{\partial \mu} \frac{\partial V}{\partial \mu} \right) \]

\[ - 2\mu \hat{\Omega}(\hat{L}W) - 2(1 - \mu^2) \frac{\partial (\mu \hat{\Omega})}{\partial \mu} \frac{\partial W}{\partial \mu} - 2m^2 \frac{\partial \hat{\Omega}}{\partial \mu} W, \] (12)

are identical to that of the barotropic stability analysis (Kitchatinov & Rüdiger 2009). They are not modified by allowing for baroclinicity. In these equations, \( \hat{\omega} = \omega/\Omega_0 \) is the normalized eigenvalue, \( \hat{\Omega} = \Omega/\Omega_0 \) is the normalized angular velocity, \( \mu = \cos \theta \),

\[ \hat{\lambda} = \frac{N}{\Omega_0 kr} \] (13)

is the key parameter controlling the effect of subadiabatic stratification (the \( \hat{\lambda} \) can also be understood as the normalized radial wavelength), and \( N \) is the buoyancy frequency:

\[ N^2 = \frac{g}{c_p} \frac{\partial s}{\partial r}. \] (14)

Finite diffusion is included in Equations (11) and (12) via the parameters

\[ \epsilon_v = \frac{\nu N^2}{\Omega_0^3 r^2}, \quad \epsilon_\chi = \frac{\chi N^2}{\Omega_0^3 r^2}, \] (15)

where \( \nu \) is the viscosity and \( \chi \) is the thermal diffusivity.

In the case of baroclinic stratification, entropy disturbances are produced not only by radial displacements but also by meridional motions. This effect is included in the entropy equation,

\[ (\hat{\omega} - m\hat{\Omega})S = -i \frac{\epsilon_v}{\lambda^2} S + \hat{L}V - \frac{i \Omega_0}{\lambda N} \]

\[ \times \left( \mu r \frac{\partial \hat{\Omega}^2}{\partial r} + (1 - \mu^2) \frac{\partial \hat{\Omega}^2}{\partial \mu} \right) \left( mW - (1 - \mu^2) \frac{\partial V}{\partial \mu} \right), \] (16)

via its second line. Equation (5) was used to express baroclinicity in terms of the angular velocity gradient when deriving Equation (16). In the case of shellular rotation, \( \partial \Omega/\partial \theta = 0 \), the entropy equation simplifies to

\[ (\hat{\omega} - m\hat{\Omega})S = -i \frac{\epsilon_v}{\lambda^2} S + \hat{L}V + i \frac{Q}{\lambda} \mu \hat{\Omega}^2 \left( mW - (1 - \mu^2) \frac{\partial V}{\partial \mu} \right), \] (17)

where

\[ Q = 2q \frac{\Omega_0}{N}, \] (18)

and \( q \) is the shear parameter of Equation (7).

All computations in this paper are performed for the case of shellular rotation. One can put \( \hat{\Omega} = 1 \) in this case. It was found convenient, however, to leave \( \hat{\Omega} \) as a parameter of the equation system for the following reason. The \( \Omega_0 \) used for normalization purposes can formally have any value not necessarily equal to the local angular velocity. Then, the dependence on rotation rate can be studied by varying \( \hat{\Omega} \). In this way, it will be shown in particular that g-modes of oscillations in nonrotating fluid (\( \hat{\Omega} = 0 \)) transform smoothly into unstable modes of baroclinic instability as the rotation rate increases gradually to \( \hat{\Omega} = 1 \).

The stability problem with Equations (11), (12), and (17) was solved numerically. The eigenfunctions for the given azimuthal wavenumber \( m \) were expanded in the series of Legendre polynomials,

\[ S = e^{i(m\phi - \omega t)} \sum_{l=\text{max}(|m|, 1)}^{K} S_l p_l^{|m|}(\mu), \] (19)

and similarly for \( V \) and \( W \). This gives the matrix equation for coefficients \( S_l, V_l, \) and \( W_l \) of the expansion. The expansion (19) with 100 harmonix \( (K = \text{max}(|m|, 1) + 99) \) provides sufficient numerical resolution. The actual number of equations in the system is not about 3K but half as many because the complete system of equations splits into two independent subsystems governing modes of different types of equatorial symmetry (cf. Section 2.3). Computations were performed for the azimuthal wavenumbers \(|m| \leq 10\). Modes of higher \( m \) are a matter of local theory.

Two types of numerical codes were used. One of them uses a standard routine of the EISPACK library to compute the eigenvalue of the most rapidly growing mode, i.e., the largest \( \lambda(\hat{\omega}) \). Another type of solvers used the inverse iteration method (also known as the “fixed-point method”) to find the eigenvalue closest to a seed eigenvalue and the corresponding eigenmode. This method is convenient for following the dependence of an eigenvalue on the problem parameters and for finding the structure of the eigenmode of a given eigenvalue.

Computations were performed with the following values of the dissipation parameters of Equation (15),

\[ \epsilon_\chi = 10^{-4}, \quad \epsilon_v = 2 \times 10^{-10}. \] (20)
characteristic of the upper radiation zone of the Sun. Dependence on thermal conductivity has been discussed in K13. With the dissipation parameters fixed, only two controlling parameters remain variable: the normalized radial scale $\lambda$ of Equation (13) and rotational shear $Q$ of Equation (18).

2.3. Symmetry Properties

As is usual for global stability problems in spherical geometry (Rüdiger et al. 2013), there are two types of modes of different equatorial symmetry. Symmetric modes have the entropy disturbances and also $u_r$ and $u_\phi$ components of velocity perturbations symmetric about the equator. $u_\phi$ is antisymmetric for these modes. In terms of the flow potentials, this reads: $V(\mu) - V(-\mu)$, $W(\mu) = -W(-\mu)$, and $\Sigma(\mu) = S(-\mu)$. The symmetry can be understood as being defined relative to the mirror reflection about the equatorial plane. The notations $\Sigma$ and $\Sigma_m$ will be used for symmetric and antisymmetric modes, respectively, where $m$ is the azimuthal wavenumber. The antisymmetric modes with $V(\mu) = -V(-\mu)$, $W(\mu) = W(-\mu)$, and $\Sigma(\mu) = -S(-\mu)$ have entropy disturbances as well as $u_r$ and $u_\phi$ antisymmetric about the equator but symmetric $u_\theta$.

A more significant property of the system of Equations (11), (12), and (17) is its symmetry relative to the transformation

$$\quad (q, m, \hat{\omega}, W, V, S) \rightarrow (-q, -m, -\hat{\omega}^*, W^*, -V^*, S^*), \quad (21)$$

where the upper star means a complex conjugate. The symmetry rule in particular means that every mode with an azimuthal wavenumber $m$ growing for rotational shear $q$ has a counterpart mode for the shear of opposite sense, $-q$, growing at the same rate and having the azimuthal wavenumber $-m$. Therefore, if the results for a certain sign of $q$ are known, they also define the results for the rotational shear of opposite sense. Subrotation ($q > 0$) is expected for differential rotation caused by spin-down of solar-type stars. The sign of $q$ is, however, uncertain when differential rotation results from nonuniform compression/evaporation in expanding stars. This paper reports the results for positive $q$, i.e., for angular velocity increasing inward.

It may be noted that the symmetry rule of Equation (21) can be generalized to an arbitrary—not necessarily radial—differential rotation. In this case, the transformation $q \rightarrow -q$ in Equation (21) should be replaced by $\partial \Omega / \partial z \rightarrow -\partial \Omega / \partial z$ keeping the latitudinal differential rotation unchanged, i.e., the radial shear should be transformed as follows: $\mu r \partial \Omega / \partial r \rightarrow -\mu r \partial \Omega / \partial r - 2(1 - \mu^2) \partial \Omega / \partial \mu$.

The presence of the parameter $q$ of the background equilibrium in Equation (21) means that stability properties depend on the sign of the azimuthal wavenumber $m$: stability for given $q$ depends on the sign of $m$. Such a dependence means that a certain handedness is inherent to the system and the unstable modes can possess helicity (Rüdiger et al. 2012). The absolute value of helicity is indefinite in the linear stability problem. The relative kinetic helicity,

$$H_{rel} = \frac{\langle |u| \cdot (\nabla \times |u|) \rangle}{\langle k u^2 \rangle}, \quad (22)$$

can, however, be defined. The angular brackets in this equation mean the azimuthal averaging:

$$\langle u \cdot (\nabla \times u) \rangle = \frac{1}{2\pi} \int_0^{2\pi} u \cdot (\nabla \times u) d\phi. \quad (23)$$

For axisymmetric modes, it can be understood as phase averaging (linear solutions are defined up to the phase multiplier $e^{i\phi}$). The overline in Equation (23) means the full-sphere averaging:

$$\overline{u^2} = \frac{1}{2} \int_{-1}^{1} \langle (u^2) \rangle d\mu. \quad (24)$$

The relative helicity (22) written in terms of the flow potentials reads

$$H_{rel} = \frac{1}{\sin^2 \theta v^2} \left[ \Re \left( mW + \sin \theta \frac{\partial V}{\partial \theta} \right) \Im \left( mV + \sin \theta \frac{\partial W}{\partial \theta} \right) \right], \quad (25)$$

where $\overline{v^2} = \overline{u^2} / (\Omega^2 r^2)$ is the normalized kinetic energy. Equation (25) shows that only axisymmetric modes have to combine toroidal and poloidal flows to possess helicity. Nonaxisymmetric toroidal or poloidal flows alone can be helical.

2.4. Two Modes of Stable Oscillations

Solutions for particular limiting cases are helpful in interpreting the results of subsequent computations. Two families of stable oscillations can be found in the case of zero dissipation ($\nu = \chi = 0$).

1. In nonrotating fluid ($\hat{\Omega} = 0$), the eigenvalue equations simplify to read

$$\hat{\phi} \hat{L} V = -\hat{\lambda}^2 \hat{L} S, \quad \hat{\phi} S = \hat{L} V, \quad \hat{\phi} \hat{L} W = 0. \quad (26)$$

Apart from the trivial solution of steady toroidal vortices ($\hat{\phi} = S = V = 0$ and $W$ is an arbitrary function), there is a family of poloidal ($W = 0$) oscillations:

$$\hat{\omega} = \pm \hat{\lambda} \sqrt{l(l + 1)}, \quad l = 1, 2, \ldots \quad (27)$$

The spectrum (27) depends on $\Omega_0$ because of normalization only. The expression for dimensional frequencies,

$$\omega = \pm \frac{N}{kr} \sqrt{l(l + 1)}, \quad l = 1, 2, \ldots \quad (28)$$

shows that these are the internal gravity waves, or $g$-modes.

2. Another family of low-frequency ($\omega \sim \Omega$) oscillations can be found for uniform rotation ($q = 0$) and strongly subadiabatic stratification ($\hat{\lambda} \gg 1$ or $N \gg \Omega kr$). Equation (12) in the leading order in $\hat{\lambda}$ gives $S = 0$. Then, Equation (17) yields $V = 0$, and Equation (11) provides the spectrum of purely toroidal oscillations:

$$\hat{\phi} = m \hat{\Omega} \left( 1 - \frac{2}{l(l + 1)} \right), \quad m = \pm 1,$n

$$\pm 2, \ldots, \quad l = |m|, |m| + 1, |m| + 2, \ldots \quad (29)$$

These are $r$-modes or Rossby waves—global vortices drifting in the corotating frame in counterrotation direction.
3. RESULTS AND DISCUSSION

3.1. Baroclinic Instability as Stability Loss to $g$- and $r$-modes Excitation

All of the unstable modes are oscillatory. The overstability was interpreted as stability loss to excitation of $g$- and $r$-modes in K13. Now, we confirm this hypothesis by computations. The computations show that the $g$- and $r$-modes transform continuously into modes of baroclinic instability as the differential rotation is varied smoothly from zero to a sufficiently large value.

In case of $g$-modes, the shear parameter $Q$ of Equation (18) and $\tilde{\lambda}$ of Equation (13) were fixed and the normalized rotation rate $\tilde{\Omega}$ was varied from zero to one. Figure 2 shows trajectories of several eigenvalues, which initially (for $\tilde{\Omega} = 0$) belong to $g$-modes of Equation (27), on the plane of growth rate $\gamma = 2\pi \Im(\tilde{\omega})$ and frequency $\Re(\tilde{\omega}) - m\tilde{\Omega}$ in corotating frame. In nonrotating fluid ($\tilde{\Omega} = 0$), the frequencies do not depend on the azimuthal wavenumber $m$. The “initial” eigenvalues have minor decay rates $\gamma < 0$ because of the finite diffusion of Equation (20). Rotation brakes the degeneracy in the azimuthal wavenumber because of the Coriolis acceleration. As rotation rate grows, the modes attain considerable growth or decay rates dependent on $m$. Their frequencies are changed mildly by rotation. The velocity field of these modes is dominated by the poloidal flow. The kinetic energy of the disturbances is the sum of kinetic energies of their poloidal and toroidal parts,

$$v^2 = v_p^2 + v_t^2 = \frac{1}{4} \sum_l l(l+1)(|V_l|^2 + |W_l|^2).$$  (30)
It is \( \vec{v}_r^2 \gg \vec{v}_t^2 \) for all modes of Figure 2. The unstable modes in rotating fluid remain close though not identical to the original g-modes. The name ”g-modes” is kept for these rotationally modified almost poloidal oscillations.

All but one mode of Figure 2 growing at \( \hat{\Omega} = \hat{1} \) are the most rapidly growing modes of their symmetry type. For example, the axisymmetric mode of panel (b) of this figure is the most rapidly growing A0 mode and the mode with \( m = 2 \) of panel (c) is the most rapidly growing S2 mode of baroclinic instability. There only exception is the \( m = -1 \) mode (A-1) of panel (d).

The most rapidly growing A-1 is an r-mode.

A similar plot for r-modes is shown in Figure 3. These modes do not exist without rotation. In this case, therefore, rotation rate is fixed, \( \hat{\Omega} = 1 \), and differential rotation increased gradually from zero to a finite value. In the case of uniform rotation, the r-modes with positive and negative m are physically identical. Differential rotation brakes the identity so that the modes with positive m decay and those with negative m grow exponentially. Drift rates (frequencies) are not changed by shellular differential rotation. The flow in unstable disturbances of Figure 3 is dominated by its toroidal part, \( \vec{v}_r^2 \gg \vec{v}_t^2 \). The disturbances remain close, although not identical, to the Rossby vortices. The name ”r-modes” is therefore kept for these low-frequency almost toroidal eigenmodes.

It may be noted that the correlation of entropy and radial velocity, \( \frac{G}{u_r} \equiv \frac{\int u_r S}{\int S^2} \) is positive for all growing modes. For decaying disturbances, the correlation is predominantly negative (it can be \( G > 0 \) for the slowly decaying modes very close to the instability border). The instability is therefore fed by release of gravitational energy, as it should be the case for the baroclinic instability (Figure 1).

3.2. Stability Maps

Figure 4 shows the lines separating regions of stability and instability on the plane of two basic parameters measuring differential rotation and radial scale of disturbances. Different lines correspond to the modes of different equatorial and axial symmetries. The marginal stability lines are shown only for those modes, which require the smallest differential rotation for excitation in some range of radial wavelengths. As the wavelength decreases, modes with increasingly larger |m| have the lowest threshold value of differential rotation for the onset of the instability.

It is \( N/\Omega > 300 \) in the upper radiation zone of the Sun. Accordingly, the computations were performed for \( \hat{\lambda} \lesssim 30 \) so that \( kr \gg 10 \) remains at least moderately large. The dashed line in Figure 4 represents the g-mode A0. The full lines in the regions of their minima belong to r-modes. Kinks in the full lines signify transitions from r- to g-modes with increasing radial wavelength. The g-modes dominate on the relatively long radial wavelengths side of the figure, and r-modes—at relatively short wavelengths.

Even a very small differential rotation with \( Q \sim 10^{-6} \) can provoke instability. With \( N/\Omega \sim 10^3 \) this corresponds to very small rotational shear of \( q \sim 10^{-3} \). The critical shear decreases with increasing rotation rate.

Figure 5 shows the lines of constant growth rate \( \gamma = 2\pi \dot{\omega} \). The growth rates are normalized so that the amplitude of excitation increases by a factor of \( e^\gamma \) in one revolution of a star. The figure shows maximum growth rates among unstable modes. This maximum belongs to r- or g-modes depending on the position on the plot. The dotted line on the stability map shows the border between the regions where r- or g-modes grow most rapidly. The wavy shape of the lines at relatively small \( \hat{\lambda} \) is caused by the switching of the maximum growth rates to increasingly high azimuthal wavenumbers as \( \hat{\lambda} \) decreases (Figure 4).

The growth rates are small. The e-folding times are, however, short compared to timescales of stellar evolution. Even in a slowly rotating star like the Sun, the e-folding time for a radial differential rotation of 1% is about 10,000 yr. This time decreases in proportion to \( \Omega^{-2} \) in faster rotators. The e-folding time may be as short as one year in the infant Sun arriving on the main sequence with a rotation period of about one day and having appreciable differential rotation in its radiation core.
yet decoupled from convective envelope (Hartmann & Noyes 1987).

Our computations do not help to decide which modes are preferred by baroclinic instability. The r- and g-modes have comparable growth rates. The threshold differential rotation for onset of g-modes instability decreases with increasing \( \lambda \). The assumption of short radial scales does not allow us to see whether the decrease saturates for sufficiently large radial scales. The analysis should be global not only in horizontal dimensions but in radius also to decide about this. Here, we meet a rare case where instability in the subadiabatically stratified radiation zone can be global in radius (cf. however, Bonanno & Urpin 2013a, 2013b).

### 3.3. Instability Patterns and Kinetic Helicity

Figure 6 shows patterns of flow and entropy disturbances for unstable S1 and S-1 modes, which are g- and r-mode, respectively. The patterns look similar in spite of the difference in the modes nature. Radial velocity and entropy patterns in both modes are phase-shifted so that the sign of their correlation \( G \) of Equation (31) cannot be estimated by eye. The correlation is nevertheless positive, as it should be for baroclinic instability: \( G = 2.17 \times 10^{-5} \) and \( G = 3.67 \times 10^{-3} \) for g- and r-modes, respectively. The smallness of the correlation explains the small growth rates. The larger correlation for r-mode does not lead to larger growth rate because the poloidal part of the flow in this mode is small.

This figure and Figures 2 and 3 show that the stability properties depend on the sign of the azimuthal wavenumber \( m \). This means that baroclinic rotating fluids possess certain handedness. The unstable modes can therefore be expected to possess helicity. Figure 7 shows that both modes of Figure 6 are indeed helical. The relative helicity is not small. It is mainly contributed to by the scalar product of horizontal velocity and vorticity. Radial velocity and vorticity are both small for these modes, which are horizontally global but short-scaled in radius.

### Figure 5.
Lines of constant growth rates on the plane of normalized differential rotation \( Q \) (18) and radial wavelength \( \lambda \) (13). The isolines show the maximum growth rates among unstable modes. The dotted line is the watershed between the regions where r- or g-modes are growing most rapidly. (A color version of this figure is available in the online journal.)

### Figure 6.
Left panel: streamlines of toroidal flow, isocontours of radial velocity and entropy disturbances for unstable S1 mode. Full (dotted) lines show clockwise (counterclockwise) circulation for toroidal flow and positive (negative) levels for radial velocity and entropy. This is g-mode with \( \nu^2/Q^2 = 34.6 \) and growth rate \( \gamma = 4.93 \times 10^{-4} \). Right panel: the same for S-1 mode, which is r-mode with \( \nu^2/Q^2 = 1.93 \times 10^{-4} \) and growth rate \( \gamma = 2.94 \times 10^{-4} \). The modes were computed for \( Q = 10^{-3} \) and \( \lambda = 3 \).
The helicity profiles are antisymmetric about the equator as it should be the case with an effect of Coriolis force. Helicity of unstable g-modes is predominantly positive in the northern hemisphere (K13). Latitudinal profiles of r-modes helicity are less regular with the tendency to be negative in the northern hemisphere. Both types of modes, however, show global helicity patterns.

Kinetic helicity (of either sign) is known to be important for dynamos (Brandenburg & Subramanian 2005). Baroclinic instability may therefore have some bearing on the stellar radiation zone’s magnetism. Solar-type stars have appreciable differential rotation in their radiation cores in about initial 100 Myr of their main-sequence life (Pinsonneault et al. 1989). This time is long compared to characteristic growth time of the instability. It suffices for the helical baroclinic modes to generate internal magnetic fields in the young stars, similar to helical convection of their envelopes.

Radiation zones of solar-type stars are deep beneath the outer convective envelopes and not accessible to observation. More massive stars have external radiation zones. Differential rotation can be present in young stars of this type because of nonuniform contraction to the main sequence. Alecian et al. (2013) observed a rapid (several years) change in a magnetic topology of the almost entirely radiative pre-main sequence star HD 190073. They interpreted this change as a manifestation of the start of a dynamo operation in a newly born convective core. An alternative interpretation could be a “baroclinic dynamo” in the radiative envelope. Observations of Hubrig et al. (2009) hint at a decline in magnetic fields in the Herbig Ae/Be stars with age. They interpreted the decline as indication of a dynamo mechanism that decays with age. In relation to the possibility of dynamo by baroclinic instability, it is tempting to know whether g- and r-modes of global oscillations presumably excited by the instability can be observed on Herbig stars.

Another widely discussed possibility for magnetic field generation in radiation zones is the Tayler–Spruit dynamo driven jointly by differential rotation and instability of the toroidal magnetic field (Spruit 2002). The difference with baroclinic instability, however, is that Tayler (1973) instability of the toroidal field does not produce net helicity. More specifically, the instability properties are symmetric relative to the change of sign of the azimuthal wavenumber: the unstable modes with $m$ of opposite sign have equal growth rates and helicities of opposite sense (Rüdiger et al. 2012) so that a fluctuating helicity with zero mean can be expected. Direct numerical simulations of Tayler instability by Zahn et al. (2007) did not show a dynamo.

4. CONCLUSIONS

Linear analysis does not permit the determination of the eventual state to which instability leads. It may be noted, however, that the correlation of radial velocity and entropy was positive for all unstable modes in our computations. The instability therefore reduces baroclinicity to produce slight deviation from the thermal wind balance of Equation (5). Any such deviation results in a (weak) meridional flow that reacts back on stratification and differential rotation to reestablish the balance. In this way, the instability can indirectly affect the differential rotation. Another possibility is the angular momentum transport by turbulence resulting from the instability. Turbulence can be expected in view of the vast variety of baroclinic instability modes. Turbulence in radiation zones, irrespective of its origin, is highly anisotropic with predominance of horizontal motions, $u_r^2/u_\theta^2 \sim \Omega^2/(r^2 N^4) \ll 1$, where $r$ is the turbulence correlation time (Kitchatinov & Brandenburg 2012). Turbulence can transport angular momentum to reduce the differential rotation. This transport is not by the eddy viscosity only. Anisotropic turbulence produces nondiffusive fluxes of angular momentum (Rüdiger 1989) so that redistribution of angular momentum proceeds much faster than eddy diffusion of chemical species. Because the threshold value of differential rotation for the onset of baroclinic instability is very low (Figure 4), the instability may lead to an almost uniform rotation.

Baroclinic instability may be relevant to the origin of magnetic fields in radiation zones of stars. Convective instability in rotating stars is known to be capable of generating magnetic fields. The kinetic helicity of rotating convection plays a key role in the dynamo process. Baroclinic instability is helical as well (Figure 7). It can drive a (transient) dynamo in the radiation zone until the differential rotation declines.

The instability excites two families of physically different modes—the Rossby waves and internal gravity waves. Judging from Figure 5, g-modes have slightly larger growth rates for given rotational shear compared to r-modes. If g-modes are preferentially excited, the helicity pattern positive in the northern hemisphere and antisymmetric about the equator can be expected from the instability (Figure 7).

Differential rotation is not the only possible cause of baroclinicity. The influence of a magnetic field is another possibility (Mathis & Zahn 2005; Strugarek et al. 2011). Equation (3) for the background equilibrium modifies taking into account the axisymmetric toroidal field, $B$, to read

$$r \sin \theta \left\{ \frac{\partial\Omega_\lambda}{\partial z} - \Omega^2 \right\} = \frac{1}{\rho^2} (\nabla \rho \times \nabla P^*)_\phi, \quad (32)$$

where $\Omega_\lambda = B/(\sqrt{4\pi \rho r \sin \theta})$ is the angular Alfvén frequency, and $P^* = P + B^2/(8\pi)$ is the total pressure. In addition to nonconservative centrifugal force, magnetic tension can also produce baroclinicity. This raises the question of whether magnetically induced baroclinic instability is possible. The question is complicated by the presence of another magnetic instability (Tayler 1973). However, Bonanno & Urpin (2012) took the influence of the toroidal magnetic field on stratification into account to find that the field is always unstable.
irrespective of whether criteria for Tayler (1973) instability are satisfied. Baroclinic instability has a bearing not only on stars. Tassoul & Tassoul (1983) pointed to this instability as a possible source of turbulence in accretion disks. More recently, this possibility was studied by Klahr & Bodenheimer (2003) and Nelson et al. (2013). The stratorotational instability of a Couette flow (Shalybkov & Rudiger 2005) is most likely of baroclinic type as well.

Computations in this paper were performed for radial differential rotation. Weak meridional flow in early-type stars leads to rotation laws dependent on latitude (Espinosa Lara & Rieutord 2013). The study of baroclinic instability for differential rotation including dependence on latitude can therefore be noted as a perspective for future work.

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