Characterizations of regular semi-groups by interval-valued $Q$-fuzzy ideals with thresholds $(\alpha, \beta)$

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Abstract
This paper expose a study on characterizations of regular semi-groups by Interval-valued $Q$-fuzzy ideals with thresholds $(\alpha, \beta)$. Further, characterizations of regular semi-groups by interval-valued $Q$-fuzzy interior (bi,quasi) ideals with thresholds $(\alpha, \beta)$ are also discussed.

Keywords
Regular semi-group, interval-valued $Q$-fuzzy subset with thresholds $((\alpha, \beta))$, interval-valued $Q$-fuzzy sub semi-group with thresholds $((\alpha, \beta))$, interval-valued $Q$-fuzzy interior ideal with thresholds $((\alpha, \beta))$, interval-valued $Q$-fuzzy left(right, two-sided) ideal with thresholds $((\alpha, \beta))$, interval-valued $Q$-fuzzy bi-ideal with thresholds $((\alpha, \beta))$, interval-valued $Q$-fuzzy generalized bi-ideal with thresholds $((\alpha, \beta))$, interval-valued $Q$-fuzzy quasi-ideal with thresholds $((\alpha, \beta))$.

AMS Subject Classification
Primary 03E72; Secondary 16Y30.

1. Introduction
The important concept of a fuzzy set introduced by Zadeh in 1965 (see [15]) has opened up keen insights and applications in wide range of scientific fields. Rosenfeld introduced the concept of fuzzy groups [12]. Among others, fuzzy semi-groups were introduced by Kuroki [10]. A theory of fuzzy sets on ordered semi-groups has been recently developed (see[4,5,6,8,9]). Fuzzy sets in ordered semi-groups were first studied by Kehayopulu and Tsingelis in [5], then they defined fuzzy analogies for several notations, which have proved useful in the theory of ordered semi-group. In [7], they have discussed fuzzy bi-ideals in ordered semi-groups and they discuss fuzzy interior ideals in ordered semi-group in [9]. Fuzzy semi-groups were generalized in two folds: fuzzy ordered semi-groups and fuzzy ternary semi-groups. Since ordered semi-groups are useful for computer science, especially in theory of automata and formal language, fuzzy ordered semi-group has been extensively studied (see [1,2,3,5,6]). Interval-valued fuzzy subsets were proposed thirty years ago as a natural extension of fuzzy sets by L. A. Zadeh [15]. In [15], Zadeh also constructed a method of approximate inference using his interval-valued fuzzy subsets. In [11], AL. Narayanan and T. Manikantan introduced the notions of interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semi-groups. M. Shabir and Israr Ali Khan [13] have studied about interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in ordered semi-groups. Thillaigovindan and V. Chinnadurai[14] initiated some study on Interval-valued fuzzy generalized bi-ideals. This paper characterize the ordered semi-groups in terms of interval-valued $Q$-fuzzy left (right, interior and bi-)ideals.
2. Preliminaries

Definition 2.1. An interval-valued Q-fuzzy subset \( \mathfrak{A} \) in a universe \( X \) of the form
\[
\mathfrak{A}(y, q) = \begin{cases} 
\mathfrak{T} \in D(0, 1) & \text{if } (y, q) = (x, q) \\
\emptyset & \text{if } (y, q) \neq (x, q)
\end{cases}
\]
is said to be an interval-valued Q-fuzzy point with support \( (x, q) \) and value \( T \) and is denoted by \( (x, q)_{\mathfrak{T}} \). An interval-valued Q-fuzzy point \( (x, q)_{\mathfrak{T}} \) is said to belong to \( (\mathfrak{A}, q, \mathfrak{B}) \) with \( q \)-quasi-coincident with an interval-valued Q-fuzzy set \( \mathfrak{A} \) written \( (x, q)_{\mathfrak{T}} \in \mathfrak{A}(\text{resp. } (x, q)_{\mathfrak{T}} \in \mathfrak{A}(\text{resp. } (x, q)_{\mathfrak{T}} \in \mathfrak{A}) \) means that \( (x, q)_{\mathfrak{T}} \in \mathfrak{A} \) or \( (x, q)_{\mathfrak{T}} \in \mathfrak{A} \). To say that \( (x, q)_{\mathfrak{T}} \mathfrak{A} \) means that \( (x, q)_{\mathfrak{T}} \mathfrak{A} \) does not hold. For every two interval-valued Q-fuzzy subsets \( \mathfrak{A} \) and \( \mathfrak{B} \) of \( S \), \( \mathfrak{A} \leq \mathfrak{B} \) means that, for all \( x \in S \), \( \mathfrak{A}(x) \leq \mathfrak{B}(x) \). The symbols \( \mathfrak{A} \wedge \mathfrak{B} \) and \( \mathfrak{A} \vee \mathfrak{B} \) will mean the following interval-valued Q-fuzzy subset of \( S \)
\[
(\mathfrak{A} \wedge \mathfrak{B})(x, q) = \mathfrak{A}(x, q) \wedge \mathfrak{B}(x, q)
\]
\[
(\mathfrak{A} \vee \mathfrak{B})(x, q) = \mathfrak{A}(x, q) \vee \mathfrak{B}(x, q).
\]
for all \( x \in S \), \( q \in Q \).

3. Some Results on Characterizations of Regular Semi-groups by Interval-valued Q-fuzzy Ideals with Thresholds \((\alpha, \beta)\)

In this section interval-valued Q-fuzzy left (right, interior, generalized, quasi) ideals with thresholds \((\alpha, \beta)\) are defined and some useful results are unraveled.

Definition 3.1. An interval-valued Q-fuzzy subset \( \mathfrak{A} \) of a semi-group \( S \) is called an interval-valued Q-fuzzy sub-semigroup with thresholds \((\alpha, \beta)\) of \( S \), where \( \alpha < \beta \) and \( \alpha, \beta \in D[0, 1] \), if it satisfies the following condition for all \( x, y, z \in S \), \( q \in Q \)
\[
\max \left\{ \mathfrak{A}(x, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(x, q), \mathfrak{A}(y, q), \beta \right\}.
\]

Definition 3.2. An interval-valued Q-fuzzy subset \( \mathfrak{A} \) of a semi-group \( S \) is called an interval-valued Q-fuzzy left (right) ideal with thresholds \((\alpha, \beta)\) of \( S \), where \( \alpha < \beta \) and \( \alpha, \beta \in D[0, 1] \), if it satisfies the following condition for all \( x, y, z \in S \), \( q \in Q \)
\[
\max \left\{ \mathfrak{A}(x, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(y, q), \beta \right\}.
\]

An interval-valued Q-fuzzy subset \( \mathfrak{A} \) of a semi-group \( S \) is called an interval-valued Q-fuzzy ideal with thresholds \((\alpha, \beta)\) of \( S \) if it is both an interval-valued Q-fuzzy left ideal and interval-valued Q-fuzzy right ideal with thresholds \((\alpha, \beta)\) of \( S \).

Definition 3.3. An interval-valued Q-fuzzy subset \( \mathfrak{A} \) of a semi-group \( S \) is called an interval-valued Q-fuzzy interior ideal with thresholds \((\alpha, \beta)\) of \( S \), where \( \alpha < \beta \) and \( \alpha, \beta \in D[0, 1] \), if it satisfies the following conditions for all \( x, a, y \in S \), \( q \in Q \)
\[
(1) \max \left\{ \mathfrak{A}(xy, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(x, q), \mathfrak{A}(y, q), \beta \right\},
\]
\[
(2) \max \left\{ \mathfrak{A}(ax, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(a, q), \beta \right\}.
\]

Definition 3.4. An interval-valued Q-fuzzy subset \( \mathfrak{A} \) of a semi-group \( S \) is called an interval-valued Q-fuzzy bi-ideal with thresholds \((\alpha, \beta)\) of \( S \), where \( \alpha < \beta \) and \( \alpha, \beta \in D[0, 1] \), if it satisfies the following conditions for all \( x, y, z \in S \), \( q \in Q \)
\[
(1) \max \left\{ \mathfrak{A}(xy, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(x, q), \mathfrak{A}(y, q), \beta \right\},
\]
\[
(2) \max \left\{ \mathfrak{A}(xyz, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(x, q), \mathfrak{A}(z, q), \beta \right\}.
\]

Definition 3.5. An interval-valued Q-fuzzy subset \( \mathfrak{A} \) of a semi-group \( S \) is called an interval-valued Q-fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\) of \( S \), where \( \alpha < \beta \) and \( \alpha, \beta \in D[0, 1] \), if it satisfies the following condition
\[
\max \left\{ \mathfrak{A}(xyz, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(x, q), \mathfrak{A}(z, q), \beta \right\}.
\]

Theorem 3.6. Let \( \alpha, \beta \in D[0, 1] \) and \( \alpha < \beta \) and \( \mathfrak{A} \) be a nonzero interval-valued Q-fuzzy sub-semi-group with thresholds \((\alpha, \beta)\) of \( S \). Then, the set \( \mathfrak{A} = \left\{ x \in S | \mathfrak{A}(x) > \alpha \land q \in Q \right\} \) is a semi-group of \( S \).

Proof: Let \( x, y \in \mathfrak{A} \). Then \( \mathfrak{A}(x) \geq \alpha \) and \( \mathfrak{A}(y) \geq \alpha \).
Since \( \mathfrak{A} \) is an interval-valued Q-fuzzy sub-semi-group with thresholds \((\alpha, \beta)\) of \( S \), we have
\[
\max \left\{ \mathfrak{A}(x, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(x, q), \mathfrak{A}(y, q), \beta \right\} > \min \left\{ \mathfrak{A}, \beta \right\} = \alpha.
\]
This implies that \( \mathfrak{A}(x, q) \geq \alpha \). So \( xy \in \mathfrak{A} \).
Thus, \( \mathfrak{A} \) is a sub-semi-group of \( S \).

Theorem 3.7. Let \( \alpha, \beta \in D[0, 1] \) and \( \alpha < \beta \) and \( \mathfrak{A} \) be a nonzero interval-valued Q-fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\) of \( S \). Then the set \( \mathfrak{A} = \left\{ x \in S | \mathfrak{A}(x) > \alpha \land q \in Q \right\} \) is a generalized bi-ideal of \( S \).

Proof: Let \( x, z \in \mathfrak{A} \). Then \( \mathfrak{A}(x) \geq \alpha \) and \( \mathfrak{A}(z) \geq \alpha \).
Since \( \mathfrak{A} \) is an interval-valued Q-fuzzy generalized bi-ideals with thresholds \((\alpha, \beta)\) of \( S \), we have
\[
\max \left\{ \mathfrak{A}(xyz, q), \alpha \right\} \geq \min \left\{ \mathfrak{A}(x, q), \mathfrak{A}(z, q), \beta \right\} > \min \left\{ \mathfrak{A}, \beta \right\} = \alpha.
\]
This implies that \( \mathfrak{A}(xyz) > \alpha \). So \( xz \in \mathfrak{A} \).
Thus, \( \mathfrak{A} \) is a generalized bi-ideal of \( S \).

Theorem 3.8. Let \( \alpha, \beta \in D[0, 1] \) and \( \alpha < \beta \) and \( \mathfrak{A} \) be a nonzero interval-valued Q-fuzzy generalized bi-ideal with thresholds \((\alpha, \beta)\) of \( S \). Then the set \( \mathfrak{A} = \left\{ x \in S | \mathfrak{A}(x) > \alpha \land q \in Q \right\} \) is a bi-ideal of \( S \).

Proof: Follows from Theorem 3.6 and Theorem 3.7.
Theorem 3.9. Let $\alpha, \beta \in D[0,1]$ and $\alpha < \beta$ and $\lambda$ be a noninterval value function $Q$-fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta)$ of $S$.
Then, the set $\lambda_{\alpha} = \{x \in S | \lambda(x, q) > \alpha, \forall q \in Q\}$ is a left (resp. right) ideal of $S$.

proof: Follows from Theorem 3.6 and Theorem 3.7.

Theorem 3.10. Let $L$ be a left (resp. right) ideal of $S$ and let $\lambda$ be an interval-valued $Q$-fuzzy subset in $S$ such that
\[
\lambda(x, q) \geq \beta \quad \text{if} \quad x \in L \text{ and } q \in Q
\]
\[
\lambda(x, q) \leq \alpha \quad \text{if} \quad x \in S \setminus L \text{ and } q \in Q
\]
Then, $\lambda$ is an interval-valued $Q$-fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta)$ of $S$.

Proof: (1) On the contrary suppose that there exist $x, y \in S$, $q \in Q$ such that
\[
\max \{\lambda(xy, q), \alpha\} < \min \{\lambda(y, q), \beta\}
\]
Then, $y \in L$ and $xy \notin L$. This is a contradiction to the fact that $L$ is a left ideal of $S$. So,
\[
\max \{\lambda(xy, q), \alpha\} \geq \min \{\lambda(y, q), \beta\}
\]
Thus, $\lambda$ is an interval-valued $Q$-fuzzy left ideal with thresholds $(\alpha, \beta)$ of $S$.

Similarly, we can prove the following theorems. $\square$

Theorem 3.11. Let $A$ be a sub-semi-group of $S$ and let $\lambda$ be an interval-valued $Q$-fuzzy subset in $S$ such that
\[
\lambda(x, q) \geq \beta \quad \text{if} \quad x \in A \text{ and } q \in Q
\]
\[
\lambda(x, q) \leq \alpha \quad \text{if} \quad x \in S \setminus A \text{ and } q \in Q
\]
Then, $\lambda$ is an interval-valued $Q$-fuzzy semigroup with thresholds $(\alpha, \beta)$ of $S$.

\proof $\square$

Theorem 3.12. Let $B$ be a generalized bi-ideal of $S$ and let $\lambda$ be an interval-valued $Q$-fuzzy subset in $S$ such that
\[
\lambda(x, q) \geq \beta \quad \text{if} \quad x \in B \text{ and } q \in Q
\]
\[
\lambda(x, q) \leq \alpha \quad \text{if} \quad x \in S \setminus B \text{ and } q \in Q
\]
Then, $\lambda$ is an interval-valued $Q$-fuzzy generalized bi-ideal with thresholds $(\alpha, \beta)$ of $S$.

\proof $\square$

Theorem 3.13. Let $B$ be a bi-ideal of $S$ and let $\lambda$ be an interval-valued $Q$-fuzzy subset in $S$ such that
\[
\lambda(x, q) \geq \beta \quad \text{if} \quad x \in B \text{ and } q \in Q
\]
\[
\lambda(x, q) \leq \alpha \quad \text{if} \quad x \in S \setminus B \text{ and } q \in Q
\]
Then, $\lambda$ is an interval-valued $Q$-fuzzy bi-ideal with thresholds $(\alpha, \beta)$ of $S$.

\proof $\square$

Definition 3.14. Let $\lambda$ and $\mu$ be interval valued $Q$-fuzzy subsets of $S$.

Then, define $\lambda \lambda_{\alpha} \mu$ as
\[
\left(\lambda \lambda_{\alpha} \mu\right)(x, q) = \begin{cases} \lor \{\lambda(y, q) \land \mu(z, q) \land \beta\} \lor \alpha, & \text{if } \exists y, z \in S, q \in Q \text{ such that } x = yz \\
\alpha, & \text{otherwise} \end{cases}
\]

Theorem 3.15. If $\lambda$ is an interval-value $Q$-fuzzy left ideal and $\mu$ is an interval-value $Q$-fuzzy right ideal with thresholds $(\alpha, \beta)$ of a semi-group $S$, then $\lambda \lambda_{\alpha} \mu$ is an interval-value $Q$-fuzzy two-sided ideal with thresholds $(\alpha, \beta)$ of $S$.

\proof Let $x, y \in S, q \in Q$. Then,
\[
\left(\lambda \lambda_{\alpha} \mu\right)(y, q) \land \beta = \left[\lor \{\lambda(i, q) \land \mu(j, q) \land \beta\} \lor \alpha\right] \land \beta
\]
\[
= \left[\lor \{\lambda(i, q) \land \beta \land \mu(j, q) \lor \alpha\}\right] \land \beta.
\]
If $i = j$ then $xy = x(ij) = (xi)j$. Since $\lambda$ is an interval-value $Q$-fuzzy left ideal with thresholds $(\alpha, \beta)$, so by definition $\max\{\lambda(xi, q), \alpha\} \geq \min\{\lambda(i, q), \beta\}$.
Thus, $\left(\lambda \lambda_{\alpha} \mu\right)(y, q) \land \beta$.

\proof $\square$
(Since each \( \overline{x}_i \) is an interval-valued Q-fuzzy left ideal of \( S \), so \( \max\{\overline{x}_i(xy, q), \overline{a}\} \geq \min\{\overline{x}_i(y, q), \overline{b}\} \) for all \( i \in I \). Thus,
\[
\left( \bigwedge_{i \in I} \overline{x}_i \right)(xy, q) \vee \overline{a} = \left[ \bigwedge_{i \in I} \left( \overline{x}_i(xy, q) \right) \right] \vee \overline{a} = \bigwedge_{i \in I} \left( \overline{x}_i(xy, q) \right) \vee \overline{a} \geq \bigwedge_{i \in I} \left( \overline{x}_i(y, q) \wedge \overline{b} \right) = \left( \bigwedge_{i \in I} \overline{x}_i \right)(y, q) \wedge \overline{b}.
\]
Hence, \( \bigwedge_{i \in I} \overline{x}_i \) is an interval-valued Q-fuzzy left ideal with threshold \( (\overline{a}, \overline{b}) \) of \( S \).

Similarly, we can prove that intersection of interval-valued Q-fuzzy right ideals with thresholds \( (\overline{a}, \overline{b}) \) of a semi-group \( S \) is an interval-valued Q-fuzzy right ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \). Thus intersection of interval-valued Q-fuzzy two-sided ideals with thresholds \( (\overline{a}, \overline{b}) \) of a semi-group \( S \) is an interval-valued Q-fuzzy two-sided ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \). □

**Lemma 3.17.** Let \( \overline{x} \) and \( \overline{p} \) be interval-valued Q-fuzzy left ideals with thresholds \( (\overline{a}, \overline{b}) \) of \( S \). Then, \( \overline{x} \wedge \overline{p} \) is an interval-valued Q-fuzzy left ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \), where \( \overline{x} \wedge \overline{p} \) is defined as
\[
\left( \overline{x} \wedge \overline{p} \right)(x, q) = \max \left\{ \min \{\overline{x}(x, q), \overline{p}(x, q), \overline{b}\}, \overline{a} \right\}
\]
**Proof:** Let \( x, y \in S \) and \( q \in Q \). Then,
\[
\left( \overline{x} \wedge \overline{p} \right)(x, q) \vee \overline{a} = \left\{ \left( \overline{x}(xy, q) \wedge \overline{p}(xy, q) \wedge \overline{b} \right) \right\} \vee \overline{a} = \left( \overline{x}(xy, q) \wedge \overline{p}(xy, q) \wedge \overline{b} \right) \vee \overline{a} \geq \left( \right. \overline{x}(y, q) \wedge \overline{b} \left. \right) \vee \overline{a} = \left( \right. \overline{x}(y, q) \wedge \overline{p}(y, q) \wedge \overline{b} \left. \right) \vee \overline{a} = \left( \overline{x} \wedge \overline{p} \right)(y, q) \vee \overline{a} \geq \left( \overline{x} \wedge \overline{p} \right)(y, q) \wedge \overline{b}.
\]
Thus, \( \overline{x} \wedge \overline{p} \) is an interval-valued Q-fuzzy left ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \).

It is clear that every interval-valued Q-fuzzy bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of a semi-group \( S \) is an interval-valued Q-fuzzy generalized bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \). The next example shows that the interval-valued Q-fuzzy generalized bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \) is not necessarily an interval-valued Q-fuzzy bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \). □

**Example 3.18.** Consider the semi-group \( S = \{a, b, c, d\} \) and \( Q \) be any non-empty set
\[
\begin{array}{cccc}
\cdot & a & b & c \\
\hline
a & a & a & a \\
b & a & a & a \\
c & a & a & b \\
d & a & a & b \\
\end{array}
\]
Let \( \overline{x} \) be an interval-valued Q-fuzzy subset of \( S \) such that \( \overline{x}(a, q)=[0.4,0.5], \overline{x}(b, q)=[0.1,0.2], \overline{x}(c, q)=[0.2,0.3], \overline{x}(d, q)=[0,0] \). Then, \( \overline{x} \) is an interval-valued Q-fuzzy generalized bi-ideal with thresholds \( (\overline{a} = 0.1, \overline{b} = 0.5) \) of \( S \).

Because \( \max\{\overline{x}(xy, q), 0.1\} = \overline{x}(a, q) \vee 0.1 = [0.4, 0.5] \geq \overline{x}(x, q) \wedge \overline{x}(y, q) \wedge 0.5 \),

But \( \overline{x} \) is not an interval-valued Q-fuzzy bi-ideal with thresholds \( (\overline{a} = 0.1, \overline{b} = 0.5) \) of \( S \). Because \( \overline{x}(cc, q) \vee 0.1 = \overline{x}(b, q) = [0, 0.2] \neq [0.2, 0.3] = \overline{x}(c, q) \wedge \overline{x}(c, q) \vee 0.5 \).

**Lemma 3.19.** Every interval-valued Q-fuzzy generalized bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of a regular semi-group \( S \) is an interval-valued Q-fuzzy bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \).

**Proof:** Let \( \overline{x} \) be any interval-valued Q-fuzzy generalized bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \) and let \( a, b \) be any elements of \( S \). Then, there exists an element \( x \in S \) such that \( b = bx \). Thus, we have \( \overline{x}(ab, q) \vee \overline{a} = \overline{x}(a(bx), q) \vee \overline{a} = \overline{x}(a(bx), q) \vee \overline{a} \geq \min \left\{ \overline{x}(a, q), \overline{x}(b, q) \right\} \). This shows that \( \overline{x} \) is an interval-valued Q-fuzzy semi-group with thresholds \( (\overline{a}, \overline{b}) \) of \( S \) and so \( \overline{x} \) is an interval-valued Q-fuzzy bi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \). □

**Definition 3.20.** An interval-valued Q-fuzzy subset \( \overline{x} \) of a semi-group \( S \) is called an interval-valued Q-fuzzy quasi-ideal with thresholds \( (\overline{a}, \overline{b}) \) of \( S \), if it satisfies, \( \max\{\overline{x}(x, q), \overline{a}\} \geq \min\left\{ \overline{x}(\sigma \overline{S}(\overline{x}) (x, q), \overline{S}(\sigma \overline{X}(x)) \right\}. \) Where \( \overline{S} \) is an interval-valued Q-fuzzy subset of \( S \) mapping every element of \( S \times Q \) on \( T \).
Theorem 3.21. Let $\overline{\lambda}$ be an interval-valued Q-fuzzy quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of a semi-group $S$.

Then, the set $\overline{\lambda}_{\mathcal{S}} = \{ x \in S | \overline{\lambda}(x,q) \geq \overline{\alpha}, \forall q \in Q \}$ is quasi-ideal of $S$.

Proof: In order to show that $\overline{\lambda}_{\mathcal{S}}$ is a quasi-ideal of $S$, we have to show that $\overline{S}\overline{\lambda}_{\mathcal{S}} \cap \overline{\lambda}_{\mathcal{S}} S \subseteq \overline{\lambda}_{\mathcal{S}}$. Let $a \in \overline{S}\overline{\lambda}_{\mathcal{S}}$ and $b \in \overline{\lambda}_{\mathcal{S}} S$. This implies that $a \in \overline{S}\overline{\lambda}_{\mathcal{S}}$ and $b \in \overline{\lambda}_{\mathcal{S}} S$. If $a = bx$ and $a = y$ for some $x, y \in S$ and $x, y \in S$. Thus, $\overline{\lambda}(x,q) > \overline{\alpha}$ and $\overline{\lambda}(y,q) > \overline{\alpha}$. Now, max($\overline{\lambda}(a,q), \overline{\alpha}$) $\geq$ min\$i=1,j=1$ \{ $\overline{\lambda}(i,q) \wedge \overline{\lambda}(j,q) \wedge \overline{\beta}$ \} $\vee \overline{\alpha}$

Since $\overline{S} \wedge (x,q) = \overline{\lambda}(x,q) \wedge \overline{\beta}$ $\vee \overline{\alpha}$

Similarly, $\overline{S} \wedge (x,q) \geq \overline{\alpha}$. Thus we have max($\overline{\lambda}(a,q), \overline{\alpha}$) $\geq$ min\$i=1,j=1$ \{ $\overline{\lambda}(i,q) \wedge \overline{\lambda}(j,q) \wedge \overline{\beta}$ \} $\vee \overline{\alpha}$

Thus, $\overline{\lambda}(a,q) > \overline{\alpha}$. Hence, $\overline{\lambda}$ is a quasi-ideal of $S$.

Remark 3.22. Every i-v Q-fuzzy quasi-ideal of $S$ is an i-v Q-fuzzy quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

Lemma 3.23. A non-empty subset $T$ of a semi-group $S$ is a quasi-ideal of $S$ if and only if the characteristic function $\overline{\lambda}$ is an interval-valued Q-fuzzy quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

Proof: Suppose $T$ is a quasi-ideal of $S$. Let $\overline{\lambda}$ be the characteristic function of $T$. If $x \in S$, $x \notin T$, then $\overline{\lambda}(x,q) = \overline{\alpha}$ and so min\$i=1,j=1$ \{ $\overline{\lambda}(i,q) \wedge \overline{\lambda}(j,q) \wedge \overline{\beta}$ \} $\vee \overline{\alpha}$

If $x \in T$, then max($\overline{\lambda}(x,q), \overline{\alpha}$) $\geq$ min\$i=1,j=1$ \{ $\overline{\lambda}(i,q) \wedge \overline{\lambda}(j,q) \wedge \overline{\beta}$ \} $\vee \overline{\alpha}$

Hence, $\overline{\lambda}$ is a Q-fuzzy quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$.

Conversely, assume that $\overline{\lambda}$ is an interval-valued Q-fuzzy quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$. Let $a \in T S \cap S T$. Then, there exist $b, c \in S$ and $x, y \in T$ such that $a = xb$ and $a = cy$. Then, we have

$(\overline{\lambda} \wedge (x,q) = \overline{\alpha}$

Thus, we have $\overline{\lambda}(a,q) = \overline{\alpha}$.

Lemma 3.24. The characteristic function $\overline{\lambda}$ is an interval-valued Q-fuzzy left ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ if and only if $L$ is a left ideal of $S$.

Proof: Let $\overline{\lambda}$ be an interval-valued Q-fuzzy left ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$. Let $y \in S$, then $\overline{\lambda}(y,q) = \overline{\alpha}$. Since $\overline{\lambda}$ is an interval-valued Q-fuzzy left ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$, so $\overline{\lambda}(y,q) = \overline{\beta}$. Since $\overline{\alpha} < \overline{\beta}$, so $\overline{\lambda}(y,q) \geq \overline{\beta}$. Thus, $\overline{\lambda}(T, q) = \overline{\alpha}$, which implies that $\overline{\lambda} \in T$. Hence, $\overline{\lambda}$ is an interval-valued Q-fuzzy left ideal with thresholds $(\overline{\alpha}, \overline{\beta})$.

Similarly, the characteristic function $\overline{\lambda}$ is an interval-valued Q-fuzzy right ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ if and only if $R$ is a right ideal of $S$. Hence, it follows that characteristic function $\overline{\lambda}$ is an interval-valued Q-fuzzy two-sided ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of $S$ if and only if $I$ is a two-sided ideal of $S$.
\( (\overline{x} \circ \overline{\beta}) (x, q) \geq \min \left\{ (\overline{x} \circ \overline{\alpha}) (x, q), (\overline{x} \circ \overline{\mu}) (x, q) \right\} \).

Thus, \( \overline{x} \) is an interval-valued Q-fuzzy quasi-ideal with thresholds \( (\overline{\alpha}, \overline{\beta}) \) of \( S \).

Similarly, we can show that every interval-valued Q-fuzzy right ideal with thresholds \( (\overline{\alpha}, \overline{\beta}) \) of \( S \) is an interval-valued Q-fuzzy quasi-ideal with thresholds \( (\overline{\alpha}, \overline{\beta}) \) of \( S \).

**Lemma 3.26.** Every interval-valued Q-fuzzy quasi-ideal with thresholds \( (\overline{\alpha}, \overline{\beta}) \) of \( S \) is an interval-valued Q-fuzzy bi-ideal with thresholds \( (\overline{\alpha}, \overline{\beta}) \) of \( S \).

**Proof:** Suppose that \( \overline{x} \) is an interval-valued Q-fuzzy quasi-ideal with thresholds \( (\overline{\alpha}, \overline{\beta}) \) of a semi-group \( S \). Now,
\[
\overline{x}(xy, q) \vee \overline{\alpha} = \min \left\{ \overline{x}(xy, q) \wedge (\overline{\alpha} \circ \overline{\beta}) (xy, q), \overline{x}(xy, q) \wedge (\overline{\beta} \circ \overline{\alpha}) (xy, q) \right\}.
\]

Also, for all \( x, y \in S \),
\[
\overline{x}(xy, q) \wedge \overline{\alpha} = \min \left\{ \overline{x}(xy, q) \wedge (\overline{\beta} \circ \overline{\alpha}) (xy, q), \overline{x}(xy, q) \wedge (\overline{\alpha} \circ \overline{\beta}) (xy, q) \right\}.
\]

Thus, \( \overline{x} \) is an interval-valued Q-fuzzy bi-ideal with thresholds \( (\overline{\alpha}, \overline{\beta}) \) of \( S \). \( \square \)

**Example 3.28.** Consider the semi-group \( S = \{0, a, b, c\} \) and \( Q \) be any non-empty set

|   | a | b | c |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 |
| b | 0 | 0 | 0 |
| c | 0 | 0 | a |

Let \( \overline{x} \) be an interval-valued Q-fuzzy subset of \( S \) such that \( \overline{x}(0, q) = [0.7, 0.8], \overline{x}(a, q) = [0.4, 0.5], \overline{x}(b, q) = [0.6, 0.7], \overline{x}(c, q) = [0.0, 0.1] \). Then, \( \overline{x} \) is an interval-valued Q-fuzzy right ideal with thresholds \( (\overline{\alpha} = \overline{0.3}, \overline{\beta} = \overline{0.5}) \) of \( S \), which is not an interval-valued Q-fuzzy two-sided ideal with thresholds \( (\overline{\alpha} = \overline{0.3}, \overline{\beta} = \overline{0.5}) \) of \( S \), that is it is not an interval-valued Q-fuzzy two-sided ideal with thresholds \( (\overline{\alpha} = \overline{0.3}, \overline{\beta} = \overline{0.5}) \) of \( S \).

**Definition 3.29.** Let \( \overline{x} \) be an interval-valued Q-fuzzy subset of a semi-group \( S \),

We define the \( \overline{x} \) as \( \overline{x}(x, q) = \{ \overline{x}(x, q) \wedge \overline{\beta} \} \vee \overline{\alpha} \).

**Lemma 3.30.** Let \( \overline{x} \) and \( \overline{\mu} \) be interval valued Q-fuzzy subsets of a semi-group \( S \). Then the following hold.

1. \( (\overline{x} \wedge \overline{\mu}) \) \( \overline{\Pi} = (\overline{x} \overline{\Pi} \overline{\mu}) \overline{\Pi} \)
2. \( (\overline{x} \vee \overline{\mu}) \) \( \overline{\Pi} = (\overline{x} \overline{\Pi} \overline{\mu}) \overline{\Pi} \)
3. \( (\overline{x} \circ \overline{\mu}) \) \( \overline{\Pi} \geq (\overline{x} \overline{\Pi} \overline{\mu}) \overline{\Pi} \)

If every element \( x \) of \( S \) is expressible as \( x = bc \), then
\[
(\overline{x} \circ \overline{\mu}) \overline{\Pi} = (\overline{x} \overline{\Pi} \overline{\mu}) \overline{\Pi}.
\]

**Proof:** For all \( a \in S \) and \( q \in Q \)

1. \( (\overline{x} \wedge \overline{\mu}) \) \( \overline{\Pi} (a, q) = (\overline{x}(a, q) \wedge \overline{\mu}(a, q) \wedge \overline{\beta}) \vee \overline{\alpha} \)

2. \( (\overline{x} \vee \overline{\mu}) \) \( \overline{\Pi} (a, q) = (\overline{x}(a, q) \vee \overline{\mu}(a, q) \wedge \overline{\beta}) \wedge \overline{\alpha} \)

(2) \( (\overline{x} \vee \overline{\mu}) \) \( \overline{\Pi} (a, q) = (\overline{x} \overline{\Pi} \overline{\mu}) (a, q) \).
Lemma 3.32. Let $A$ and $B$ be non-empty subsets of a semi-group $S$ and $Q$ be any non-empty set. Then, the following properties hold.

1. $A \subseteq B$ if and only if $(C_A)^\overline{\beta}_B \subseteq (C_B)^\overline{\beta}_B$.
2. $(C_A \wedge C_B)^\overline{\beta}_B = (C_{A \cap B})^\overline{\beta}_B$.
3. $(C_A \vee C_B)^\overline{\beta}_B = (C_{A \cup B})^\overline{\beta}_B$.
4. $(C_A \circ B)^\overline{\beta}_B = (C_{A \cup B})^\overline{\beta}_B$.

Proof: (1) Suppose $A, B \subseteq S$ and $(C_A)^\overline{\beta}_B \subseteq (C_B)^\overline{\beta}_B$. Let $a \in A$ and $q \in Q$. Then, $(C_A)^\overline{\beta}_B(a, q) = \overline{\beta}$. Since $(C_A)^\overline{\beta}_B \subseteq (C_B)^\overline{\beta}_B$, so $\overline{\beta} = (C_A)^\overline{\beta}_B \subseteq (C_B)^\overline{\beta}_B$. Which implies that $(C_B)^\overline{\beta}_B(a, q) = \overline{\beta}$, so $a \in B$. Thus, $A \subseteq B$.

Conversely, let $A, B \subseteq S$ such that $A \subseteq B$. On the contrary suppose that there exist $x \in S$ such that $(C_A)^\overline{\beta}_B(x, q) > (C_B)^\overline{\beta}_B(x, q)$. Then, $(C_A)^\overline{\beta}_B(x, q) = \overline{\beta}$ and $(C_B)^\overline{\beta}_B(x, q) = \alpha$, which implies that $x \in A$ and $x \notin B$. So, $A \not\subseteq B$, which is a contradiction. Thus, $(C_A)^\overline{\beta}_B \subseteq (C_B)^\overline{\beta}_B$.

(2) Let $a$ be any element of $S$. Suppose $a \in A \cap B$. It implies that $a \in A$ and $a \in B$. So

$$(C_A \wedge C_B)^\overline{\beta}_B(a, q) = [(C_A \wedge C_B)(a, q) \wedge \overline{\beta}] \vee \alpha = [(C_A(a, q) \wedge C_B(a, q)) \wedge \overline{\beta}] \vee \alpha = [(C_A(a, q) \wedge C_B(a, q) \wedge \overline{\beta}) \wedge \overline{\beta}] \vee \alpha = (\overline{\beta} \wedge \overline{\beta}) \wedge \overline{\beta} \wedge \alpha = \alpha \wedge \overline{\beta} = \overline{\beta} = (C_{A \cap B})^\overline{\beta}_B(a, q).$$

If $a \notin A \cap B$, then it implies $a \notin A$ or $a \notin B$. So,

$$(C_A \wedge C_B)^\overline{\beta}_B(a, q) = [(C_A \wedge C_B)(a, q) \wedge \overline{\beta}] \vee \alpha = [(C_A(a, q) \wedge C_B(a, q)) \wedge \overline{\beta}] \vee \alpha = (\overline{\beta} \wedge \overline{\beta}) \wedge \overline{\beta} \wedge \alpha = 0 \wedge \overline{\beta} = \overline{\beta} = (C_{A \cap B})^\overline{\beta}_B(a, q).$$

(3) Let $a \in A \cup B$, it implies that $a \in A$ or $a \in B$. So,

$$(C_A \vee C_B)^\overline{\beta}_B(a, q) = [(C_A \vee C_B)(a, q) \wedge \overline{\beta}] \vee \alpha = [(C_A(a, q) \vee C_B(a, q)) \wedge \overline{\beta}] \vee \alpha = (\overline{\beta} \wedge \overline{\beta}) \wedge \overline{\beta} \wedge \alpha = \overline{\beta} \wedge \overline{\beta} \wedge \alpha = \overline{\beta} = (C_{A \cup B})^\overline{\beta}_B(a, q).$$

If $a \notin A \cup B$, then it implies $a \notin A$ and $a \notin B$. So,

$$(C_A \vee C_B)^\overline{\beta}_B(a, q) = [(C_A \vee C_B)(a, q) \wedge \overline{\beta}] \vee \alpha = [(C_A(a, q) \vee C_B(a, q)) \wedge \overline{\beta}] \vee \alpha = (\overline{\beta} \wedge \overline{\beta}) \wedge \overline{\beta} \wedge \alpha = \overline{\beta} \wedge \overline{\beta} \wedge \alpha = \overline{\beta} = (C_{A \cup B})^\overline{\beta}_B(a, q).$$

(4) Let $a$ be any element of $S$. Suppose $a \in AB$. Then $a = xy$ for some $x \in A$ and $y \in B$. Thus, we have
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$$(\mathcal{C}_A \circ \mathcal{C}_B)(a, q) = \bigvee_{a=uv} \left((\mathcal{C}_A(u, q) \wedge \mathcal{C}_B(v, q) \wedge \overline{b}) \vee \overline{a}\right)$$

\[= \left(\overline{b} \wedge \mathcal{C}_B(v, q) \wedge \overline{b}\right) \vee \overline{a}\]

\[= \overline{b} \vee \overline{a}\]

and so $(\mathcal{C}_A \circ \mathcal{C}_B)(a, q) = \overline{b}$.

Since $a \notin AB$, $(\mathcal{C}_A \circ \mathcal{C}_B)(a, q) = \overline{b}$. So, $(\mathcal{C}_A \circ \mathcal{C}_B)(a, q) = (\mathcal{C}_A \circ \mathcal{C}_B)(a, q)$. Now, if $a \notin AB$, then $a \neq xy$, for all $x \in A$ and $y \in B$. If $a = uv$ for some $u, v \in S$, then we have

$$(\mathcal{C}_A \circ \mathcal{C}_B)(a, q) = \bigvee_{a=uv} \left((\mathcal{C}_A(u, q) \wedge \mathcal{C}_B(v, q) \wedge \overline{b}) \vee \overline{a}\right)$$

\[= \overline{a}\]

Thus, $(\mathcal{C}_A \circ \mathcal{C}_B)(a, q) = (\mathcal{C}_A \circ \mathcal{C}_B)(a, q)$. \hspace{1cm} \square

**Lemma 3.33.** The lower part of the interval-valued characteristic function $(\mathcal{C}_L P) \overline{a}$ is an interval-valued $Q$-fuzzy left ideal with thresholds $(\overline{a}, \overline{b})$ of $S$ if and only if $L$ is a left ideal of $S$.

**Proof:** Let $L$ be a left ideal of $S$. Then, by Theorem 3.11 $(\mathcal{C}_L P) \overline{a}$ is an interval-valued $Q$-fuzzy left ideal with thresholds $(\overline{a}, \overline{b})$ of $S$.

Conversely, assume that $(\mathcal{C}_L P) \overline{a}$ is an interval-valued $Q$-fuzzy left ideal with thresholds $(\overline{a}, \overline{b})$ of $S$. Let $y \in L$. Then, $(\mathcal{C}_L P)(y, q) = \overline{b}$. Since $(\mathcal{C}_L P) \overline{a}$ is an interval-valued $Q$-fuzzy left ideal with thresholds $(\overline{a}, \overline{b})$ of $S$, so $(\mathcal{C}_L P)(y, q) \wedge \overline{a} \geq (\mathcal{C}_L P)(y, q) \wedge \overline{b}$. Since $\overline{a} < \overline{b}$, so $(\mathcal{C}_L P)(y, q) \wedge \overline{b} = \overline{b}$, which implies that $(\mathcal{C}_L P)(y, q) = \overline{b}$.

Hence, $y \in (\mathcal{C}_L P) \overline{a}$. Thus, $L$ is a left ideal of $S$.

Similarly, we can prove that the interval-valued characteristic function $(\mathcal{C}_R P) \overline{a}$ is an interval-valued $Q$-fuzzy right ideal with thresholds $(\overline{a}, \overline{b})$ of $S$ if and only if $R$ is a right ideal of $S$. Thus, an interval-valued characteristic function $(\mathcal{C}_R P) \overline{a}$ is an interval-valued $Q$-fuzzy two-sided ideal with thresholds $(\overline{a}, \overline{b})$ of $S$ if and only if $T$ is a two-sided ideal of $S$. \hspace{1cm} \square

**Lemma 3.34.** Let $T$ be a non-empty subset of a semi-group $S$. Then, $T$ is a quasi-ideal of $S$ if and only if an interval-valued characteristic function $(\mathcal{C}_T P) \overline{a}$ is an interval-valued $Q$-fuzzy quasi-ideal with thresholds $(\overline{a}, \overline{b})$ of $S$.

**Proof:** Suppose $T$ is a quasi-ideal of $S$. Let $(\mathcal{C}_T P) \overline{a}$ be an interval-valued characteristic function of $T$. Let $x \in S$. If $x \notin T$, then $x \notin ST$ or $x \notin TS$, then $(\mathcal{S} \circ (\mathcal{C}_T P) \overline{a})(x, q) = \overline{a}$ and so max $\left\{((\mathcal{C}_T P) \circ \mathcal{S})(a, q), (\mathcal{S} \circ (\mathcal{C}_T P) \overline{a})(x, q)\right\}$

\[= \overline{a} = (\mathcal{C}_T P)(x, q) \wedge \overline{a}\]

If $x \in T$, then

\[\max\left\{((\mathcal{C}_T P) \circ \mathcal{S})(a, q), (\mathcal{S} \circ (\mathcal{C}_T P) \overline{a})(x, q)\right\} = \overline{a} = (\mathcal{C}_T P)(x, q) \wedge \overline{a}\]

Hence, $(\mathcal{C}_T P) \overline{a}$ is an interval-valued $Q$-fuzzy quasi-ideal with thresholds $(\overline{a}, \overline{b})$ of $S$.

Conversely, assume that $(\mathcal{C}_T P) \overline{a}$ is an interval-valued $Q$-fuzzy quasi-ideal with thresholds $(\overline{a}, \overline{b})$ of $S$. Let $a \in TS \cap ST$. Then, there exist $b, c \in S$ and $x, y \in T$ such that $a = xb$ and $a = cy$. Then

\[((\mathcal{C}_T P) \circ \mathcal{S})(a, q) = \bigvee_{a=xy} \left((\mathcal{C}_T P)(x, q) \wedge TS(b, q) \wedge \overline{b}\right) \vee \overline{a}\]

\[= \overline{b} \vee \overline{a}\]

$\overline{a}$. Similarly, $(\mathcal{S} \circ (\mathcal{C}_T P) \overline{a})(a, q) = \overline{a}$. Hence, $(\mathcal{C}_T P) \overline{a}$ is a quasi-ideal of $S$. \hspace{1cm} \square

**Theorem 3.35.** For a semi-group $S$ the following conditions are equivalent.

1. $S$ is regular.
2. $(\mathcal{X} \wedge \mathcal{P} P) = (\mathcal{X} \circ \mathcal{P} P)$ for every interval-valued $Q$-fuzzy right ideal $\mathcal{X}$ and every interval-valued $Q$-fuzzy left ideal $\mathcal{P}$ with thresholds $(\overline{a}, \overline{b})$ of $S$.

**Proof:** First assume that (1) holds. Let $\mathcal{X}$ be an interval valued $Q$-fuzzy right ideal and $\mathcal{P}$ an interval valued $Q$-fuzzy left ideal with thresholds $(\overline{a}, \overline{b})$ of $S$. Let $a \in S$, we have for all $q \in Q$

\[\mathcal{X} \circ \mathcal{P}(a, q) = \bigvee_{a=xy} \left(\left(\mathcal{X}(y, q) \wedge \mathcal{P}(x, q) \wedge \overline{b}\right) \vee \overline{a}\right)\]

\[= \bigvee_{a=xy} \left(\left(\mathcal{X}(y, q) \wedge \mathcal{P}(x, q) \wedge \overline{b}\right) \vee \overline{a}\right)\]

\[\leq \bigvee_{a=xy} \left(\left(\left(\mathcal{X}(y, q) \wedge \overline{a}\right) \wedge \left([-\mathcal{P}(z, q) \wedge \overline{b}\right) \vee \overline{a}\right)\right)\]

\[= \left(\mathcal{X}(y, q) \wedge \overline{a}\right) \wedge \left([-\mathcal{P}(z, q) \wedge \overline{b}\right) \vee \overline{a}\]

\[= \left(\mathcal{X}(a, q) \wedge \mathcal{P}(a, q) \wedge \overline{b}\right) \vee \overline{a}\]

\[= \mathcal{X} \wedge \mathcal{P}(a, q) \wedge \overline{b}\]

\[= \mathcal{X} \wedge \mathcal{P}(a, q) \wedge \overline{b}\]

So, $(\mathcal{X} \wedge \mathcal{P} P) \leq (\mathcal{X} \circ \mathcal{P} P)$. Since $S$ is regular and $a \in S$, so there exists an element $x \in S$ such that $a = axa$. So,

\[= \bigvee_{a=xy} \left(\left(\mathcal{X}(y, q) \wedge \mathcal{P}(x, q) \wedge \overline{b}\right) \vee \overline{a}\right)\]

\[\geq \left(\mathcal{X}(a, q) \wedge \mathcal{P}(a, q) \wedge \overline{b}\right) \vee \overline{a}\]

\[\geq \left(\mathcal{X}(a, q) \wedge \mathcal{P}(a, q) \wedge \overline{b}\right) \vee \overline{a}\]

\[\geq \left(\mathcal{X}(a, q) \wedge \mathcal{P}(a, q) \wedge \overline{b}\right) \vee \overline{a}\]

\[\geq \left(\mathcal{X}(a, q) \wedge \mathcal{P}(a, q) \wedge \overline{b}\right) \vee \overline{a}\]

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greater than or equal to \((\overline{X}(a,q) \land \overline{B}) \land [\overline{M}(a,q) \land \overline{B})] \lor \overline{\alpha}
\]
\[
= \{\overline{X}(a,q) \land \overline{M}(a,q) \land \overline{B}\} \lor \overline{\alpha}
\]
\[
= (\overline{X} \land \overline{M}(a,q)) (a,q).
\]

So, \((\delta \circ \overline{X} \circ \overline{M}) \geq (\overline{X} \land \overline{M}(a,q))\). Thus \((\delta \circ \overline{X} \circ \overline{M}) = (\overline{X} \land \overline{M}(a,q))\) and so (1) implies (2).

(2) implies (1) : assume that (2) holds. Let \(R\) and \(L\) be right ideal and left ideal of \(S\), respectively. In order to see that \(R \cap L = RL\) holds. Let \(a\) be any element of \(R \cap L\). Then, by Lemma 3.33, interval-valued Q-fuzzy characteristic functions \((\overline{C}(R))_{\overline{P}}\) and \((\overline{C}(L))_{\overline{P}}\) of \(R\) and \(L\) are interval-valued Q-fuzzy right ideal and interval-valued Q-fuzzy left ideal with thresholds \((\overline{\alpha}, \overline{\beta})\) of \(S\), respectively. Thus, we have

\[
(\overline{C}(R))_{\overline{P}}(a,q) = (\overline{C}(L))_{\overline{P}}(a,q) \quad \text{by Lemma 3.32}
\]
\[
= (\overline{C}(R \cap L))_{\overline{P}}(a,q) \quad \text{by (1)}
\]
\[
= (\overline{C}R \cap \overline{L})_{\overline{P}}(a,q) \quad \text{by Lemma 3.32}
\]
\[
= \overline{\beta}.
\]

So, \(a \in RL\), which implies that \(R \cap L \subseteq RL\). Thus, \(R \cap L = RL\). Hence, it follows from Theorem (For a semi-group \(S\) the following condition are equivalent.

(1) \(S\) is regular.
(2) \(R \cap L = RL\) for every right ideal \(R\) and every left ideal \(L\) of \(S\).
(3) ASA = \(A\) for every quasi-ideal \(A\) of \(S\).) that \(S\) is regular and so (2) implies (1).

**Theorem 3.36.** For a semi-group \(S\), the following conditions are equivalent.

(1) \(S\) is regular.
(2) \(((\delta \circ \overline{X} \circ \overline{M})_{\overline{P}} \leq (\delta \circ \overline{X} \circ \overline{M})_{\overline{P}}\) for every interval-valued Q-fuzzy right ideal \(\overline{X}\), every interval-valued Q-fuzzy generalized bi-ideal \(\overline{X}\) and every interval-valued Q-fuzzy left ideal \(\overline{M}\) with thresholds \((\overline{\alpha}, \overline{\beta})\) of \(S\).
(3) \(((\delta \circ \overline{X} \circ \overline{M})_{\overline{P}} \leq (\delta \circ \overline{X} \circ \overline{M})_{\overline{P}}\) for every interval-valued Q-fuzzy right ideal \(\overline{X}\), every interval-valued Q-fuzzy bi-ideal \(\overline{X}\) and every interval-valued Q-fuzzy left ideal \(\overline{M}\) with thresholds \((\overline{\alpha}, \overline{\beta})\) of \(S\).
(4) \(((\delta \circ \overline{X} \circ \overline{M})_{\overline{P}} \leq (\delta \circ \overline{X} \circ \overline{M})_{\overline{P}}\) for every interval-valued Q-fuzzy right ideal \(\overline{X}\), every interval-valued Q-fuzzy quasi-ideal \(\overline{X}\) and every interval-valued Q-fuzzy left ideal \(\overline{M}\) with thresholds \((\overline{\alpha}, \overline{\beta})\) of \(S\).

**Proof:** (1) \(\Rightarrow\) (2) : Let \(\overline{X}, \overline{M}\) and \(\overline{\alpha}\) be interval-valued Q-fuzzy right ideal, interval-valued Q-fuzzy generalized bi-ideal, and any interval-valued Q-fuzzy left ideal with thresholds \((\overline{\alpha}, \overline{\beta})\) of \(S\), respectively. Let \(a\) be any element of \(S\). Since \(S\) is regular, there exists an element \(x \in S\) such that \(a = axa\).
Theorem 3.37. For a semi-group $S$, the following conditions are equivalent.

1. $S$ is regular.
2. $\overline{X}^{\pi_p} \subseteq (\overline{X} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{X})$ for every interval-valued $Q$-fuzzy generalized bi-ideal $\overline{X}$ with thresholds $(\overline{\pi_p}, \overline{\pi_p})$ of $\overline{X}$.
3. $\overline{X}^{\pi_p} \subseteq (\overline{X} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{X})$ for every interval-valued $Q$-fuzzy bi-ideal $\overline{X}$ with thresholds $(\overline{\pi_p}, \overline{\pi_p})$ of $\overline{X}$.
4. $\overline{X}^{\pi_p} \subseteq (\overline{X} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{X})$ for every interval-valued $Q$-fuzzy quasi-ideal $\overline{X}$ with thresholds $(\overline{\pi_p}, \overline{\pi_p})$ of $\overline{X}$.

Proof: (1) $\Rightarrow$ (2): Let $\overline{X}$ be an interval-valued $Q$-fuzzy generalized bi-ideal with thresholds $(\overline{\pi_p}, \overline{\pi_p})$ of $S$ and let $a$ be any element of $S$. Since $S$ is regular, there exists an element $x \in S$ such that $a = axa$. Hence, we have

$$\overline{X} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{X} \supseteq (\overline{X} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{X}) \supseteq (\overline{X} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{X}).$$

Thus, $\overline{X}^{\pi_p} \subseteq (\overline{X} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{X})$.

(2) $\Rightarrow$ (3): Let $T$ be any quasi-ideal of $S$. Then we have

$$T \subseteq \overline{T} \subseteq \overline{T} \cdot \overline{S} \subseteq T \cdot \overline{S} \subseteq \overline{T} \cdot \overline{S} \subseteq T \cdot \overline{S} \subseteq T,$$

where $T$ is an interval-valued $Q$-fuzzy quasi-ideal with thresholds $(\overline{\pi_p}, \overline{\pi_p})$ of $S$. Now, by assumption and Lemma 3.32, we have

$$\overline{C_T}^{\pi_p} \subseteq \overline{C_T} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{C_T} = (\overline{C_T} \cdot \overline{\pi_p} \cdot \overline{\pi_p} \cdot \overline{C_T})^{\pi_p}.$$
Theorem 3.39. For a semi-group $S$, the following conditions are equivalent.

1. $S$ is regular.
2. $\overline{\lambda} \cup \overline{\mu} \leq (\overline{\lambda} \circ_{\overline{p}} \overline{\mu})(a,q)$ for every interval-valued Q-fuzzy quasi-ideal $\overline{\lambda}$ and every interval-valued Q-fuzzy left ideal $\overline{\mu}$ with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ of $S$.
3. $\left(\overline{\lambda} \cup \overline{\mu}\right) \leq (\overline{\lambda} \circ_{\overline{p}} \overline{\mu})$ for every interval-valued Q-fuzzy bi-ideal $\overline{\lambda}$ and every interval-valued Q-fuzzy left ideal $\overline{\mu}$ with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ of $S$.
4. $\left(\overline{\lambda} \cup \overline{\mu}\right) \leq (\overline{\lambda} \circ_{\overline{p}} \overline{\mu})$ for every interval-valued Q-fuzzy generalized bi-ideal $\overline{\lambda}$ and every interval-valued Q-fuzzy left ideal $\overline{\mu}$ with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ of $S$.

Proof: (1) $\Rightarrow$ (4): Let $\overline{\lambda}$ and $\overline{\mu}$ be any interval-valued Q-fuzzy generalized bi-ideal and any interval-valued Q-fuzzy left ideal with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ of $S$, respectively. Let $a$ be any element of $S$. Then, there exists an element $x \in S$ such that $a = axa$. Thus we have

$$\left(\overline{\lambda} \circ_{\overline{p}} \overline{\mu}\right)(a,q) = \bigvee_{a'=y_{\overline{p}}} \left(\overline{\lambda}(y_{\overline{p}}) \cup \overline{\mu}(z_{\overline{p}}) \cup \overline{\alpha}\right)$$

which is equal to

$$= \left(\overline{\lambda}(a,q) \cup \overline{\mu}(a,q) \cup \overline{\alpha}\right) \cup \overline{\beta} \cup \overline{\alpha}$$

So, $\left(\overline{\lambda} \circ_{\overline{p}} \overline{\mu}\right) \geq (\overline{\lambda} \cup \overline{\mu})$. Now, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1): Let $\overline{\lambda}$ be an interval-valued Q-fuzzy right ideal and $\overline{\mu}$ be an interval-valued Q-fuzzy left ideal with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ of $S$. Since every interval-valued Q-fuzzy right ideal with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ of $S$ is an interval-valued Q-fuzzy quasi-ideal with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ of $S$, so $\left(\overline{\lambda} \circ_{\overline{p}} \overline{\mu}\right) \geq (\overline{\lambda} \cup \overline{\mu})$. Thus, by Theorem 3.35 that $S$ is regular.

4. Conclusion

In recent years interval-valued fuzzy sets and Q-fuzzy sets have been studied by several researchers [5,14,17,21,22,24,25,27,28,29,31]. In this consequence, the study reveals some important aspects of regular, intra-regular and ordered semi-group over interval-valued Q-fuzzy sets. Further, characterization of regular semi-groups by interval-valued fuzzy ideals with thresholds $\left(\overline{\alpha}, \overline{\beta}\right)$ are provided.

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