Finite-temperature phase transitions in $\nu = 2$ bilayer quantum Hall systems

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In this paper, the influence of an in-plane magnetic field $B_\parallel$ on the finite-temperature phase transitions in $\nu = 2$ bilayer quantum Hall systems are examined. It is found that there can exist two types of finite-temperature phase transitions. The first is the Kosterlitz-Thouless (KT) transitions, which can have an unusual non-monotonic dependence on $B_\parallel$; the second type originates from the crossing of energy levels and always increases with $B_\parallel$. Based on these results, we point out that the threshold temperature observed in the inelastic light scattering experiments cannot be the KT transition temperature, because the latter shows a totally different $B_\parallel$-dependence as compared with the experimental observation. Instead, it should be the level-crossing temperature, which we found agrees with the $B_\parallel$-dependence observed. Moreover, combining the knowledge of these two transition temperatures, a complete finite-temperature phase diagram is presented.

73.40.Hm, 73.20.Dx, 75.30.Kz

Recent theoretical works predict that, besides the fully spin polarized ferromagnetic phase (F) and the paramagnetic symmetric or spin singlet (S) phase, a novel canted antiferromagnetic (C) phase can exist in the filling factor $\nu = 2$ bilayer quantum Hall (QH) systems. In such a C phase, the electron spins in each layer are tilted away from the external magnetic field direction due to the competition between ferromagnetic ordering and singlet ordering. Encouraging experimental evidence in support of the C phase has recently emerged through inelastic light scattering spectroscopy and transport measurements. In particular, it is observed for certain samples in the inelastic light scattering experiments that there is a threshold temperature $T_{SDE}$ below which the spin conserved spin-density excitation (SDE) mode ($\omega_0$ mode) seems to lose all spectral weight. Because the $U(1)$ planar spin rotational symmetry is spontaneously broken in the C phase, there should be a finite-temperature Kosterlitz-Thouless (KT) transition with a characteristic energy scale which is about the vortex-antivortex binding energy. It is claimed that the observed threshold temperature is the predicted KT transition temperature $T_{KT}$ in the C phase.

While the predicted value of the KT transition temperature ($T_{KT} \approx 1.8$ K in the Hartree-Fock theory) is reasonably close to that of the threshold temperature ($T_{SDE} \approx 0.52$ K) in the inelastic light scattering experiment under normal magnetic fields, which seems to support the identification between these two temperature scales, we point out that such an interpretation meets trouble in the tilted magnetic field experiment for two reasons. First, it is found in Ref. that the threshold temperature rises as the parallel magnetic field $B_\parallel$ increases. Nevertheless, we notice that (i) the sample used in the experiment is located near the F-C phase boundary in the quantum phase diagram (see the inset in Fig.); and (ii) an in-plane magnetic field effectively moves a sample even closer to the F-C phase boundary. Hence the symmetry-breaking order parameter and therefore $T_{KT}$ should be reduced (c.f. Figs. 8 and 11 of Ref. [3]), rather than enhanced, when $B_\parallel$ increases. Thus the physical content of these two characteristic temperatures should not be the same. Second, it is questionable to regard the observed disappearance of the $\omega_0$ mode at the threshold temperature as the transition to the C phase, if one is reminded that the spectral weight of the $\omega_0$ mode (which does not involve any spin flip) is also greatly suppressed in the F phase, where almost all spin-up (down) states are occupied (empty).

When temperature $T \gtrsim T_{KT}$ for the systems near the F-C quantum phase boundary, it is expected that, although the expectation value of the in-plane spin component vanishes and the $U(1)$ planar spin rotational symmetry is restored, the spin component $S_z$ along the direction of the external magnetic field may still be nonzero. That is, these systems at $T \gtrsim T_{KT}$ should behave somewhat like the F phase at finite temperatures. (See Fig. for our finite-temperature phase diagram.) Thus it needs higher temperatures for these systems to loss all their spin polarizations, such that the $\omega_0$ mode can be observed.

In this paper, the finite-temperature phase transitions in $\nu = 2$ bilayer quantum Hall systems are investigated. As discussed above, one had not yet reached the correct theoretical understanding for the reported threshold temperature. Hence we focus our attention on solving this issue. Since the aforementioned arguments are quite general, the same qualitative results should be obtained irrespective of which kind of approximation methods being employed. For simplicity, we use the Hartree-Fock approximation in the following. We show that the KT transition temperature in the C phase can have an unusual non-monotonic dependence on the tilted angle of the applied magnetic field. That is, $T_{KT}$ can either rise or fall as $B_\parallel$ is turned on, depending on whether the samples are initially located near the C-S or the F-C phase boundary in the quantum phase diagram. By using the sample
parameters in the tilted field experiment, we show that $T_{KT}$ decreases as the tilted angle increases. Thus the KT transition scenario do fail to explain the $B_{\parallel}$-dependence of the threshold temperature in Ref.\[4\]. Instead, in order to link with the observed threshold temperature, we propose another characteristic temperature $T_X$ caused by the crossing of energy levels, since its variation with respect to $B_{\parallel}$ agrees qualitatively with the reported threshold temperature. Based on the dependence of $T_{KT}$ and $T_X$ on the tunneling-induced symmetric-antisymmetric energy gap, a complete finite-temperature phase diagram.

$$c_A = \frac{\Delta_{SAS}^2[(\Delta_{SAS}^2 - \Delta_z^2)^2 - (2U_{-}\Delta_z^2)](2U_+ - \Delta_{SAS})^2 - (\Delta_{SAS}^2 - \Delta_z^2)^2]}{(2U_-)^2(\Delta_{SAS}^2 - \Delta_z^2)^4},$$  

$$c_E = \frac{(2U_-\Delta_z^2 - (\Delta_{SAS}^2 - \Delta_z^2)^2)[\Delta_{SAS}^2(\Delta_{SAS}^2 - \Delta_z^2)^2 - \Delta_{SAS}^2(2U_-\Delta_z^2)]}{(2U_-)^2(\Delta_{SAS}^2 - \Delta_z^2)^4}.$$  

Here $\Delta_{SAS}$ is the tunneling-induced symmetric-antisymmetric energy separation, $\Delta_z$ is the Zeeman energy, and $U_\pm = (U_A - U_E)/2$ with $U_{A/E} = \sum_p V_{A/E}(p, 0)$ the fractional transforms of the intralayer/interlayer Coulomb interaction. The matrix elements $V_{A/E}(p_1, p_2)$ of the intralayer/interlayer Coulomb interaction are

$$V_{A/E}(p_1, p_2) = \frac{1}{\Omega} \sum_q v_{A/E}(q)\delta_{p_1, q_1}e^{-qQp_2l^2}/e^{i\varepsilon q_{Xp_2}l^2},$$  

where $\Omega$ is the area of the system, $v_{A/E}(q) = (2\pi^2/eq)F_A(q, b)$ and $v_{E}(q) = v_A(q)F_E(q, b)e^{-\varrho_0}$ are the Fourier transforms of the intralayer and the interlayer Coulomb interaction potentials. $\varepsilon$ is the dielectric constant of the system, and $d$ is the interlayer separation. We have also included the finite-thickness correction by introducing the form factor $F_A(E, q, b)$ in the intralayer/interlayer Coulomb potential, where $F_A(q, b) = 2\cos^{-1}(1 - e^{-\varrho_0}/b^2q^2); F_E(q, b) = \sum_{1}^2\cos^{-1}(q_0/2)/b^2q^2$, and $b$ is the width of a quantum well. Since we know $\rho_s$ exactly within the microscopic Hartree-Fock approximation, the KT transition temperature can be easily determined. As shown in Refs.\[2,3\], the KT transition temperature along with the symmetry-breaking order parameter drops continuously to zero as the phase boundaries are approached from within the C phase.

Now we consider the tilted magnetic field case, where a parallel magnetic field $B_{\parallel}$ and a perpendicular field $B_{\perp}$ both appear with the tilted angle $\Theta = \tan^{-1}(B_{\parallel}/B_{\perp})$. The effect of the parallel magnetic field $B_{\parallel}$ on $T_{KT}$ can be incorporated by the following replacements\[5,6\]:

$$\Delta_{SAS} \rightarrow \Delta_{SAS}e^{-Q^2l^2/4},$$

$$\Delta_z \rightarrow \Delta_z\sqrt{1 + (B_{\parallel}/B_{\perp})^2},$$

$$V_E(p_1, p_2) \rightarrow \tilde{V}_E(p_1, p_2) = V_E(p_1, p_2)e^{\pm i\varrho_0p_2l^2},$$

with $Q = B_{\parallel}d/B_{\perp}l^2$ and the magnetic length $l = \sqrt{\hbar c/EB_{\perp}}$. In Fig.\[3\] we show the transition temperature $T_{KT}$ as a function of the tilted angle $\Theta$ for some typical sample parameters. Since it is relatively easy to tune $\Delta_{SAS}$ in fabrication, we vary its value with other system parameters being fixed. Three possible situations are depicted in Fig.\[4\]: (i) if the system begins in the C phase and near the C-S phase boundary (triangle), then $T_{KT}$ (and $\rho_s$) grows as $B_{\parallel}$ is turned on; (ii) if the system is in the C phase and near the F-C phase boundary (cross), then $T_{KT}$ (and $\rho_s$) is reduced as $B_{\parallel}$ is turned on; (iii) when the system lies between the two phase boundaries (circle), $T_{KT}$ (and $\rho_s$) can have an unusual nonmonotonic dependence on $B_{\parallel}$, that is, it can increase and then decrease as $B_{\parallel}$ increases. We find that the enhancement (suppression) of $T_{KT}$ as $\Theta$ increases can be large for the system near the C-S (F-C) phase boundary. However, when the system lies between the two phase boundaries, the magnitude of $T_{KT}$ may have roughly the same value. Since the crossed line is the predicted $T_{KT}$ for the sample studied in Ref.\[4\], which has a decreasing dependence on $\Theta$, the experimentally observed enhancement of $T_{SDE}$ can not be explained by the result of $T_{KT}$.

Motivated by the above results, we look for another characteristic temperature above which the spectral weight of the $\omega_0$ mode indeed becomes significant. As mentioned before, for the systems near the F-C phase boundary in the quantum phase diagram, a nonvanishing spin polarization is possible when $T \gtrsim T_{KT}$. Consequently, the mean-field Hamiltonian of the self-consistent Hartree-Fock theory at finite temperatures, which takes this fact into account, reduces to

$$H_{HF} = -\frac{\Delta_{SAS} + \delta_{SAS}}{2} \sum_{\tau, k, \sigma} \tau c_{\tau, k, \sigma}^\dagger c_{\tau, k, \sigma} - \frac{\Delta_z + \delta_z}{2} \sum_{\tau, k, \sigma} \sigma c_{\tau, k, \sigma}^\dagger c_{\tau, k, \sigma},$$

where $\delta_{SAS} = (U_E/2)\sum_{\tau, \sigma} \tau f(E_{\tau, \sigma})$ and $\delta_z = (U_A/2)\sum_{\tau, \sigma} \sigma f(E_{\tau, \sigma})$. Here the Landau gauge is as-
assumed, and $c^+_{\tau,k}\sigma$ creates an electron at the lowest Landau level orbital $k$ in the symmetric ($\tau = 1$) or the anti-symmetric ($\tau = -1$) subbands with spin $\sigma/2$ ($\sigma = \pm 1$). The thermal averages $\langle c^+_{\tau,k}\sigma c_{\tau,k}\sigma \rangle = f(E_{\tau,\sigma})$, where $f(E)$ is the Fermi-Dirac distribution function and the energy eigenvalues of this mean-field Hamiltonian are
\[ E_{\tau,\sigma} = -\frac{\tau}{2}(\Delta_{\text{SAS}} + \delta_{\text{SAS}}) - \frac{\sigma}{2}(\Delta_z + \delta_z). \] (7)

We assume that the translational symmetry is not broken, thus these expectation values have no intra-Landau level dependence. Since we consider only the case of $T > T_{KT}$, the order parameters for the C phase are dropped. Moreover, because of the symmetry in the energy levels, the chemical potential is fixed at zero for all temperatures. We see that the four energy levels for the non-interacting electrons are shifted by the self-consistent mean fields, $\Delta_{\text{SAS}}$ and $\delta_z$, both of which have temperature dependence.

For $T > T_{KT}$, if we assume that the effective Zeeman energy, $\Delta_z + \delta_z$, is initially larger than the effective symmetric/anti-symmetric energy gap, $\Delta_{\text{SAS}} + \delta_{\text{SAS}}$, the crossing of energy levels can occur at a higher temperature $T = T_X > T_{KT}$, because the mean field $\delta_z$ is a monotonic decreasing function of $T$. Thus at $T = T_X$, where level crossing occurs, one has
\[ \Delta_{\text{SAS}} + \delta_{\text{SAS}} = \Delta_z + \delta_z, \] (8)

\[ \frac{1}{2} \sum_{\tau,\sigma} \sigma f(E_{\tau,\sigma}) = \frac{1}{2} \sum_{\tau,\sigma} \tau f(E_{\tau,\sigma}) = f(E_{+1,+1}) - \frac{1}{2} \] (9)

By solving Eqs. (8) and (9) with Eq. (7), the level-crossing temperature $T_X$ can be determined. Combining the knowledge of the KT transition temperature and a complete phase diagram at finite temperatures can be obtained as shown in Fig. 2, where the phase boundaries for the tilted angle $\Theta = 30^\circ$ are also presented. In the F phase, the planar spins are thermally randomized but $\langle S_z \rangle$ remains nonzero. This finite-temperature phase diagram indeed confirms our previous arguments. Note that for $\Delta_{\text{SAS}}$ slightly larger than $0.23 e^2/\ell$, the C phase directly transits to the S phase at finite temperatures, in which $\langle S_z \rangle = 0$. It can be seen that an in-plane magnetic field moves the finite-temperature phase boundaries to the right, due to the effective modification of the sample parameters given in Eq. (8). With a fixed $\Delta_{\text{SAS}}$, the change of the transition temperatures for a tilted magnetic field with $\Theta = 30^\circ$ can be read out directly from Fig. 2. This finite-temperature phase diagram indicates that $T_X$ is an increasing function of $\Theta$. By using Eq. (8), the tilted-field dependence of $T_X$ is explicitly shown in Fig. 2. The values of $\Delta_{\text{SAS}}$ are chosen such that the corresponding samples can undergo both C$\rightarrow$F and F$\rightarrow$S phase transitions with rising temperatures (see Fig. 2), even though only the F$\rightarrow$S transition temperatures $T_X$ are plotted. In general, it takes a higher $T_X$ for a sample with a smaller $\Delta_{\text{SAS}}$ to transit to the S phase, which is reasonable since the F phase becomes more stable for a larger ratio of $\Delta_z/\Delta_{\text{SAS}}$. The in-plane magnetic field always elevates $T_X$, in contrast to its effect on $T_{KT}$, which is more complicated as shown in Fig. 1. The result shows that for systems near the $T = 0$ F-C phase boundary (say, for the cross symbol in the inset of Fig. 1), the level-crossing temperature $T_X$ indeed increases with $B_\parallel$, and should be identified as the experimentally observed $T_{\text{SDE}}$.

Before closing this paper, some remarks are in order. First, we would like to comment that the threshold temperature in the experiment is $T_{\text{SDE}} \approx 0.5$ K, which is considerably lower than the calculated $T_X \approx 17$ K using the actual experimental sample parameters (see Fig. 2 for $\Delta_{\text{SAS}} = 0.1 e^2/\ell$). Therefore, the present theory is not quantitatively satisfying. The quantum fluctuations neglected in the mean-field theory should lower the calculated level-crossing temperature $T_X$ and reduce this discrepancy. Although the above analysis is crude, it provides a starting point for interpreting the enhancement of the threshold temperature in Ref. [3]. Second, we notice that, for the sample in the transport experiment (say, the sample with the total density $n_i = 0.7 \times 10^{11}$ cm$^{-2}$ at the balanced point), which is initially located in the C phase and near the $T = 0$ F-C phase boundary, its activation energy decreases as the tilted angle increases from zero. We suggest that the energy scale set by $T_{KT}$ (i.e. the vortex-antivortex binding energy) may be related to this activation energy, since they have similar $B_\parallel$-dependence.

In conclusion, we have investigated the dependence of phase transition temperatures, $T_{KT}$ and $T_X$, on the in-plane magnetic field and demonstrated that it is $T_X$, rather than $T_{KT}$, that agrees qualitatively with the experimentally reported threshold temperature. We have also obtained a finite-temperature phase diagram of the bilayer systems based on the Hartree-Fock approximation. A verification of these two different phase transitions awaits experimental measurements to probe the C phase more directly at lower temperatures. For example, the heat capacity measurements should show power law temperature dependence in the C phase (see Fig. 12 of Ref. [3]) because of the existence of the gapless Goldstone mode due to spontaneous symmetry breaking. Once that being achieved, it would be quite interesting for future experiments to confirm the predicted non-monotonic $B_\parallel$-dependence of $T_{KT}$.

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In contrast, the \( \omega_0 \) mode has a maximum spectral weight in the S phase, where there are as many spin-up (down) empty states as there are spin-up (down) electrons at low temperatures. For example, starting from the effective boson theory, we get the same conclusions. As discussed in Refs. [1, 2], the finite-temperature renormalizations for \( T_KT \) can be much larger in the vicinity of the C-S phase boundary such that this estimation can then no longer be used. Since we are interested mainly in the systems near the F-C phase boundary, our discussions may be still valid.

Here we assume that two quantum wells are arbitrarily narrow, and the effect of the in-plane magnetic field on orbital degrees of freedom is ignored.
FIG. 1. $k_B T_{KT}$ as a function of the tilted angle $\Theta$ (in unit of degree) of the applied magnetic field with a fixed $B_\perp$. The energy unit is the intralayer Coulomb energy $e^2/\epsilon l$. The Zeeman energy caused by $B_\perp$ is $\Delta_z = 0.008$. The interlayer separation is $d = 1.45$ and the layer thickness is $b = 1.0$. Crosses, circles, and triangles correspond to $\Delta_{SAS} = 0.1$, 0.2, and 0.4, respectively. Their locations in the $\Delta_z - \Delta_{SAS}$ quantum phase diagram calculated by the Hartree-Fock theory with $\Theta = 0^\circ$ are shown in the inset. Notice that the cross symbol represents the experimental sample of Ref.[9], and its location is very close to the F-C phase boundary in the quantum phase diagram.

FIG. 2. $\nu=2$ bilayer phase diagrams at finite temperatures. The energy unit is $e^2/\epsilon l$. The sample parameters, $\Delta_z$, $d$, and $b$ are the same as in Fig. 1. The continuous lines are the phase boundaries for a perpendicular magnetic field. The dotted lines are for a magnetic field tilted by 30 degrees.

FIG. 3. $k_B T_X$ as a function of the tilted angle $\Theta$ (in unit of degree) of the applied magnetic field with a fixed $B_\perp$. The energy unit is $e^2/\epsilon l$. Continuous, dotted, and dashed lines correspond to $\Delta_{SAS} = 0.062$, 0.15, and 0.23, respectively. The sample parameters, $\Delta_z$, $d$, and $b$ are the same as in Fig. 1.