On surfaces with \( p_g = 2, \ q = 1 \) and \( K^2 = 5 \)

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Abstract

We consider minimal surfaces of general type with \( p_g = 2, \ q = 1 \) and \( K^2 = 5 \). We provide a stratification of the corresponding moduli space \( \mathcal{M} \) and we give some bounds for the number and the dimensions of its irreducible components.

0 Introduction

Recently there has been considerable interest in understanding the geometry of irregular surfaces of general type. Although the classification of such surfaces is still far from being achieved, their study has produced in the last years a considerable amount of results, see for instance the survey papers [BCP06] and [MePa09].

Minimal surfaces of general type satisfy the classical inequalities:

\[ \chi(\mathcal{O}_S) := p_g - q + 1 \geq 1, \]
\[ K^2_S \geq 2p_g \text{ if } S \text{ is irregular (Debarre’s inequality)}, \]
\[ K^2_S \leq 9\chi(\mathcal{O}_S) \text{ (Miyaoka–Yau inequality)}. \]

If \( S \) is irregular and \( K^2_S = 2\chi \), then it follows \( q = 1 \). In this case the Albanese map \( f: S \rightarrow \text{Alb}(S) \) is a genus 2 fibration whose fibres are all 2-connected. The corresponding classification was given by Catanese ([Cat81]) for \( K^2_S = 2 \), and by Horikawa ([Hor82]) in the general case.

The study of irregular surfaces with \( K^2_S = 2\chi + 1 \) was started by Catanese and Ciliberto in [CaCi91] and [CaCi93]. They investigated the case \( \chi = 1 \), i.e., \( p_g = q = 1 \) and \( K^2_S = 3 \), proving that for this class of surfaces the genus \( g \) of the fibre of the Albanese map can be either 2 or 3. They also described all surfaces with \( q = 3 \) and started the classification of surfaces with \( g = 2 \), which was later completed by Catanese and Pignatelli in [CaPi06], by using a structure theorem for genus 2 fibrations which is proven in the same work.

For \( \chi \geq 2 \) the situation is far more complicated and not yet thoroughly studied. In this paper we consider the case \( \chi = 2 \), and we investigate the surfaces whose numerical invariants are

\[ K^2_S = 5, \ p_g = 2, \ q = 1. \]

By a result of Horikawa, given any irregular minimal surface of general type with \( 2\chi \leq K^2 < \frac{8}{3}\chi \), its Albanese map \( f: S \rightarrow \text{Alb}(S) \) is a genus 2 fibration over a smooth curve of genus \( q \). Then in our case we have a genus 2 fibration \( f: S \rightarrow B \) over an elliptic curve \( B \).

We can therefore use the results of Horikawa-Xiao and those of Catanese-Pignatelli in order to construct our surfaces and describe their moduli space. In fact, we first study the rank 2 vector bundle \( V_1 := f_\ast \omega_S \), distinguishing the two cases where \( V_1 \) is either decomposable or indecomposable. Then we divide the problem in various subcases, according to the behaviour of \( V_2 := f_\ast \omega^2_S \), and for each subcase we study the corresponding stratum of the moduli space \( \mathcal{M} \). By Riemann-Roch and [Cle05], at a point \( [S] \in \mathcal{M} \) we have

\[ \dim_{[S]} \mathcal{M} \geq 10\chi(\mathcal{O}_S) - 2K^2_S + p_g = 12, \]

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hence, to understand the irreducible components of \( M \), we have to consider only those strata whose dimension is greater than or equal to 12.

Our main results are

**Theorem 0.1.** Let \( M' \) be the subspace of \( M \) corresponding to surfaces such that \( V_1 \) is decomposable. There is a stratification into unirational algebraic subsets:

\[
M' = M_I \cup M_{IIa} \cup M_{IIb} \cup M_{IIIa} \cup M_{IIIb} \cup M_{IVa} \cup M_{IVb} \cup M_{V_{gen}} \cup M_{V,2},
\]

where \( M_{IIc}, M_{IVa}, M_{IVb} \) and \( M_{IVc} \) have dimension \( \leq 11 \), so they can be disregarded in the determination of the irreducible components, while:

- \( M_I \) is nonempty, irreducible, of dimension at most 13;
- \( M_{IIa}, M_{IIb}, M_{IIIa}, M_{IIIb} \) have dimension at most 12;
- \( M_{V_{gen}} \) is non-empty, of dimension 11;
- \( M_{V,2} \) is a generically smooth, irreducible component of dimension 12.

**Theorem 0.2.** Let \( M'' \) be the subspace of \( M \) corresponding to surfaces such that \( V_1 \) is indecomposable. There is a stratification

\[
M'' = M_{VI} \cup M_{VIIa} \cup M_{VIIb},
\]

where the strata \( M_{VIIa} \) and \( M_{VIIb} \) have dimension \( \leq 11 \), while \( M_{VI} \) has dimension at most 12.

Using Theorems 0.1 and 0.2 and some easy additional arguments, one can prove the following

**Corollary 0.3.** The moduli space \( M \) of minimal surfaces of general type with \( p_g = 2 \), \( q = 1 \) and \( K^2 = 5 \) is unirational and contains at least 2 irreducible components. Moreover, the dimension of each irreducible component is either 12 or 13, and there is at most one component of dimension 13.

Of course, it would be interesting to exactly describe all irreducible components of \( M \) and also to understand how their closures intersect, but we will not try to develop this point here.

Now let us explain how this paper is organized.

In Section 1 we present some preliminaries, and we set up notation and terminology. In particular we recall Atiyah’s classification of vector bundles over an elliptic curve and Horikawa’s and Catanese–Pignatelli’s approaches to the study of genus 2 fibrations.

In Section 2 we investigate the structure and the possible splitting types of the vector bundles \( V_1 = f_1^* \omega_S \) and \( V_2 = f_2^* \omega_S^2 \).

Finally, Section 3 deals with the study of the moduli space \( M \).

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# 1 Preliminaries

## 1.1 Vector bundles over an elliptic curve

The classification of vector bundles of an elliptic curve was given in [At57]. Here we just recall the results needed in order to make this paper self-contained, and we refer the reader to Atiyah’s paper for further details. Let \( B \) be an elliptic curve and let \( o \) be the identity
element in the group law of $B$. If $\tau \in B$, we set $E_\tau(1, 1) := \mathcal{O}_B(\tau)$ and for all $r \geq 2$ we denote by $E_\tau(r, 1)$ the unique indecomposable rank $r$ vector bundle on $B$ defined recursively by the short exact sequence

$$0 \rightarrow \mathcal{O}_B \rightarrow E_\tau(r, 1) \rightarrow E_\tau(r - 1, 1) \rightarrow 0.$$  

Moreover, we set $F_1 := \mathcal{O}_B$ and for all $r \geq 2$ we denote by $F_r$ the unique indecomposable rank $r$ vector bundle on $B$ defined recursively by the short exact sequence

$$0 \rightarrow \mathcal{O}_B \rightarrow F_r \rightarrow F_{r-1} \rightarrow 0.$$

**Proposition 1.1.** [At57] (i) For all $L \in \text{Pic}^0(B)$ and for all $r \geq 2$ we have

$$h^0(E_\tau(r, 1) \otimes L) = 1, \quad h^1(E_\tau(r, 1) \otimes L) = 0.$$  

Moreover every indecomposable rank $r$ vector bundle $V$ on $B$ such that $\deg V = 1$ is isomorphic to $E_\tau(r, 1) \otimes L$ for some $L \in \text{Pic}^0(B)$.

(ii) For all $L \in \text{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ we have

$$h^0(F_r \otimes L) = h^1(F_r \otimes L) = 0,$$

whereas $h^0(F_r) = h^1(F_r) = 1$. Moreover every indecomposable rank $r$ vector bundle $V$ on $B$ such that $\deg V = 0$ is isomorphic to $F_r \otimes L$ for a unique $L \in \text{Pic}^0(B)$.

By using Proposition 1.1 we can prove

**Proposition 1.2.** Let $V$ be a rank $3$ vector bundle on $B$, such that $\det V = \mathcal{O}_B(\tau)$ for some $\tau \in B$. Then the following holds.

(i) If $h^1(V \otimes L) = 0$ for all $L \in \text{Pic}^0(B)$, then $V = E_\tau(3, 1)$.

(ii) If $h^1(V \otimes L) = 0$ for all $L \in \text{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ and $h^1(V) = 1$, then either $V = E_\tau(2, 1) \otimes \mathcal{O}_B$ or $V = F_2 \otimes \mathcal{O}_B(\tau)$.

(iii) If $h^1(V \otimes L) = 0$ for all $L \in \text{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ and $h^1(V) = 2$, then $V = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$.

**Proof.** (i) Assume $h^1(V \otimes L) = 0$ for all $L \in \text{Pic}^0(B)$. If $V$ is indecomposable, then $V = E_\tau(3, 1)$ by Atiyah’s classification. Suppose now that $V = W \oplus M$, where $W$ is indecomposable of rank $2$ and $M$ is a line bundle. By our assumptions on the cohomology of $V$, it follows $0 \leq \deg M \leq 1$. If $\deg M = 0$, then $h^1(V \otimes M^{-1}) = 1$ yields a contradiction. If $\deg M = 1$, then $\deg W = 0$, hence $W = F_2 \otimes L$ for some $L \in \text{Pic}^0(B)$. It follows $h^1(V \otimes L^{-1}) = 1$, again a contradiction. Finally, suppose that $V = L_1 \oplus L_2 \oplus L_3$, where the $L_i$ are line bundles. We must have $\deg L_i \geq 0$, hence we may assume $\deg L_1 = 0$, $\deg L_2 = 0$, $\deg L_3 = 1$; therefore we have $h^1(V \otimes L_1^{-1}) \geq 1$, a contradiction. This concludes the proof of part (i).

(ii) Since $h^1(V) = 1$, the vector bundle $V$ cannot be indecomposable. Suppose that $V = W \oplus M$, where $W$ is indecomposable of rank $2$ and $M$ is a line bundle; as before, we have $0 \leq \deg M \leq 1$. If $\deg M = 0$ we have $\deg W = 1$, hence $h^1(M) = h^1(V) = 1$. It follows $M = \mathcal{O}_B$ and $V = E_\tau(2, 1) \oplus \mathcal{O}_B$. If $\deg M = 1$ we have $\deg W = 0$, since $h^1(V) = 1$, the only possibility is $V = F_2 \oplus \mathcal{O}_B(\tau)$. Finally, suppose that $V = L_1 \oplus L_2 \oplus L_3$, where the $L_i$ are line bundles. Taking $L \in \text{Pic}^0(B) \setminus \{\mathcal{O}_B\}$, we have $h^1((L_1 \oplus L_2 \oplus L_3) \otimes L) = 0$, hence $\deg L_i \geq 0$; on the other hand $\deg V = 1$, hence, as before, we may assume $\deg L_1 = 0$, $\deg L_2 = 0$, $\deg L_3 = 1$; moreover $L_1 \otimes L \neq \mathcal{O}_B$ and $L_2 \otimes L \neq \mathcal{O}_B$ for all $L \in \text{Pic}^0(B) \setminus \{\mathcal{O}_B\}$. Hence we obtain $V = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$, so $h^1(V) = 2$, a contradiction. This concludes the proof of part (ii).

(iii) Since $h^1(V) = 2$, arguing as before we see that $V = L_1 \oplus L_2 \oplus L_3$, where the $L_i$ are line bundles. Moreover $h^1(V \otimes L) = 0$ for all $L \in \text{Pic}^0(B)$ implies $\deg L_i \geq 0$. So we may assume $\deg L_1 = 0$, $\deg L_2 = 0$, $\deg L_3 = 1$, which implies $V = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$. This concludes the proof of part (iii).
Remark 1.3. A similar result holds if one replaces $\text{Pic}^0(B) \setminus \{O_B\}$ with $\text{Pic}^0(B) \setminus \{M\}$, for any $M \in \text{Pic}^0(B)$.

Proposition 1.4. (i) Set $W := E_r(2, 1)$. Then we have

$$S^2W = \bigoplus_{i=1}^3 L_i(\tau), \quad S^3W = W(\tau) \oplus W(\tau),$$

where the $L_i$ are the three non-trivial 2-torsion line bundles on $B$.

(ii) $S^{-1}F_2 = F_r$, for all $r \geq 2$.

Proof. (i) If $\tau = o$, see [Al57] p. 438-439]. The general case follows since, by Proposition 1.1 we have $E_r(2, 1) = E_{o}(2, 1) \odot L$, where $L$ is any line bundle on $B$ such that $L^{\otimes 2} = O_B(\tau - o)$.

(ii) See [Al57] Theorem 9.

1.2 Structure theorems for genus 2 fibrations

1.2.1 Horikawa’s method

The following approach to genus 2 fibrations was introduced by Horikawa in [Hor77]; see also [XK55] §1 for further details. Let $f : S \to B$ be a relatively minimal genus 2 fibration over a smooth curve $B$ of genus $b$, set $V_1 := f_*\omega_{S/B}$ and let $\pi_1 : \mathbb{P}(V_1) \to B$ be the associated $\mathbb{P}^1$-bundle. Let us consider the relative canonical map $\phi : S \to \mathbb{P}(V_1)$, whose indeterminacy locus is contained in the fibres of $f$ which are not 2-connected. After composing with a finite number of blow-ups, we can extend $\phi$ to a generically finite, degree 2 morphism $\tilde{\phi} : \tilde{S} \to \mathbb{P}(V_1)$; let $B$ be the branch divisor of $\tilde{\phi}$. There exists a divisor $F \in \text{Pic}(\mathbb{P}(V_1))$ such that $2F = B$, so we can consider the double cover $S' \to \mathbb{P}(V_1)$ branched at $B$, and it is no difficult to see that there exists a birational morphism $\tilde{S} \to S'$. The Néron Severi group of $\mathbb{P}(V_1)$ is generated by $C_0$ and $\Gamma$, that are the classes of $O_{\mathbb{P}(V_1)}(1)$ and of a fiber, respectively; since $B_1' = 6$, it follows that $B = 6C_0 + \pi^1_1 \alpha$, for some $\alpha \in \text{Pic}(B)$. After applying a finite number of elementary transformations to the pair $(\mathbb{P}(V_1), B)$, we obtain that $B$ has only the following types of singularities, defined when $k \geq 1$:

(0) a double point or a simple triple point;
(II$\_k$) a fibre $\Gamma$ plus two triple points on it (hence these are quadruple points of $B$); each of these triple points is $(2k-1)$-fold or 2-fold;
(III$\_k$) two triple points on a fibre, each of these is 2k-fold or $(2k + 1)$-fold;
(IV$\_k$) a fibre $\Gamma$ plus a $(4k - 2)$ or a $(4k - 1)$-fold triple point on it which has a contact of order 6 with $\Gamma$;
(V$\_k$) a $4k$ or $(4k + 1)$-fold triple point $x$ which has a contact of order 6 with the fibre through $x$;
(V) a fibre $\Gamma$ plus a quadruple point $x$ on $\Gamma$, which a blow-up in $x$ results in a double point in the proper transform of $\Gamma$.

We recall that a $k$-fold triple point is a triple point that results in a simple triple point after $k - 1$ blow-ups. Let us denote by $\nu(\ast)$ the number of fibres of type $\ast$.

Theorem 1.5. [Hor77] The following equality holds:

$$K_S^2 = 2p_a(S) - 4 + 6b + \sum_k \left( (2k - 1) (\nu(\text{II}_k) + \nu(\text{III}_k)) + 2k (\nu(\text{I}_k) + \nu(\text{IV}_k)) \right) + \nu(\text{V}).$$

4
1.2.2 Catanese-Pignatelli’s method

Now we recall Catanese-Pignatelli approach to genus 2 fibrations, which roughly speaking consists in considering the relative bicanonical map instead of the canonical one. We closely follow the treatment given in [CaPi06] and [Pi09], referring the reader to those papers for further details. For any relatively minimal genus 2 fibration \( f : S \rightarrow B \), we can consider the rank 3 vector bundle \( V_2 := f_* \omega^2_{S|B} \) and the corresponding \( \mathbb{P}^2 \)-bundle \( \pi_2 : \mathbb{P}(V_2) \rightarrow B \). Therefore we can associate to the fibration \( f \) the 5-tuple \((B, V_1, \tau, \xi, w)\), where

- \( B \) is the base curve;
- \( V_1 = f_* \omega^2_{S|B} \);
- \( \tau \) is an effective divisor on \( B \) of degree \( K_2^2 - 6(b - 1) - 2\chi(\mathcal{O}_S) \), corresponding to the fibres of \( f \) which are not 2-connected;
- \( \xi \) is an element of \( \text{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2V_1)/\text{Aut}_B(\mathcal{O}_\tau) \) giving the short exact sequence
  \[
  0 \rightarrow S^2V_1 \xrightarrow{\sigma_2} V_2 \rightarrow \mathcal{O}_\tau \rightarrow 0,
  \]
  where \( \sigma_2 \) is the natural map induced by the tensor product of canonical sections of the fibres of \( f \); then \( \sigma_2 \) yields a rational map \( \mathbb{P}(V_1) \dashrightarrow \mathbb{P}(V_2) \) (the relative version of the 2-Veronese embedding \( \mathbb{P}^1 \dashrightarrow \mathbb{P}^2 \) birational onto a conic bundle \( \mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \pi_2^*(\det V_1)|^{-2} \)). More precisely, if \( x_0, x_1 \) are generators for the stalk of \( V_1 \), then the equation of \( \mathcal{C} \) is locally given by
  \[
  \sigma_2(x_0^2)\sigma_2(x_1^2) - (\sigma_2(x_0x_1))^2 = 0. 
  \]
- \( w \in \mathbb{P}H^0(B, \tilde{\mathcal{A}}_6) \), where \( \tilde{\mathcal{A}}_6 := A_6 \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2} \) and \( A_6 \) is given by the following short exact sequence:
  \[
  0 \rightarrow (\det V_1)^2 \otimes V_2 \xrightarrow{i_3} S^2V_2 \rightarrow A_6 \rightarrow 0.
  \]
  Here the map \( i_3 \) is locally defined as follows: if \( x_0, x_1 \) are generators for the stalk of \( V_1 \) and \( y_0, y_1, y_2 \) are generators for the stalk of \( V_2 \), then
  \[
  i_3((x_0 \wedge x_1)^{\otimes 2} \otimes y_1) := \sigma_2(x_0^2)\sigma_2(x_1^2)y_1 - \sigma_2(x_0x_1)^2y_1.
  \]

The relative bicanonical map, which is always a morphism, induces a factorization of the fibration \( f \) as
\[
S \xrightarrow{r} X \xrightarrow{\psi} \mathcal{C} \xrightarrow{\pi_2|_B} B,
\]
where \( r \) is a contraction of \((-2)\)-curves to Rational Double Points, and \( \psi \) is a finite double cover. The element \( w \in \mathbb{P}H^0(\tilde{\mathcal{A}}_6) = |\mathcal{O}_C(6) \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}| \) corresponds to the divisorial part \( \Delta \) of the branch locus of \( \psi \). In fact, the branch locus of \( \psi \) consists of a disjoint union \( \Delta \cup \mathcal{P} \), where \( \mathcal{P} \subset \text{Sing}(\mathcal{C}) \) is a finite set of points in natural bijection with \( \text{supp}(\tau) \). Notice that \( A_6 \) is the quotient of \( S^2V_2 \) by the subbundle of the relative cubics vanishing on \( \mathcal{C} \); geometrically, this reflects the fact that, in general, not all the divisors in \( |\mathcal{O}_C(6) \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}| \) can be written as the complete intersection of \( \mathcal{C} \) with a relative cubic \( \mathcal{G} \in |\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}| \). Finally, observe that if
\[
0 \rightarrow G_1 \rightarrow G_2 \rightarrow \mathcal{A}_6 \rightarrow 0
\]
is the short exact sequence obtained by tensoring \( \mathcal{E} \) with \( (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2} \), we obtain
\[
h^0(\mathcal{A}_6) \leq h^0(G_2) - h^0(G_1) + h^1(G_1).
\]
We call \((B, V_1, \tau, \xi, w)\) the associate 5-pie of the fibration \( f : S \rightarrow B \).

**Theorem 1.6.** Assume that we have a 5-pie \((B, V_1, \tau, \xi, w)\) as before, such that the following (open) conditions are satisfied:

- \( B \) is a curve of genus 2.
- \( V_1 \) is a vector bundle of rank 3.
- \( \tau \) is an effective divisor on \( B \) of degree \( K_2^2 - 6(b - 1) - 2\chi(\mathcal{O}_S) \).
- \( \xi \) is an element of \( \text{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2V_1)/\text{Aut}_B(\mathcal{O}_\tau) \).
- \( w \in \mathbb{P}H^0(B, \tilde{\mathcal{A}}_6) \).
- \( \mathcal{A}_6 \) is a vector bundle of rank 3.
- \( \mathcal{C} \) is a conic bundle.
- \( \mathcal{P} \subset \text{Sing}(\mathcal{C}) \) is a finite set of points.
- \( \text{supp}(\tau) \) is finite.
- \( h^0(\mathcal{A}_6) \leq h^0(G_2) - h^0(G_1) + h^1(G_1) \).
Proposition 2.1. the conic bundle $C$ has only Rational Double Points as singularities;

(P2) the curve $\Delta$ has only simple singularities, where “simple” means that the germ of the double cover of $C$ branched on it has at most a Rational Double Point.

Then there exists a unique relatively minimal genus 2 fibration $f : S \rightarrow B$ having the above as associate 5-ple. Moreover, the surface $S$ has the following invariants:

$$\chi(O_S) = \deg V_1 + (b - 1),$$

$$K_S^2 = 2\deg V_1 + \deg \tau + 8(b - 1).$$

2 Surfaces of general type with $p_g = 2$, $q = 1$ and $K^2 = 5$

2.1 The sheaf $V_1$

Let $S$ be a minimal surface of general type with $p_g = 2$, $q = 1$ and $K^2 = 5$. Its Albanese variety $B := \text{Alb}(S)$ is an elliptic curve, and its Albanese map $f : S \rightarrow B$ is a genus 2 fibration ([Hor82 Theorem 3.1]). Notice that since $B$ is elliptic then $\omega_B = \omega_S$. By Theorem 1.6 we have $\deg(\tau) = 1$, i.e. $\tau$ is a point of $B$. The genus 2 fibration contains exactly one singular fibre, which comes from a singularity of $B$. Notice that since $B$ is a minimal surface of general type with $p_g = 2$, $q = 1$ and $K^2 = 5$, its Albanese map $f$ is the canonical resolution of the singularities of a degree 2 double cover of $C$ branched on it having at most a Rational Double Point.

Let now $E_1$ be a rank 1 subsheaf of maximal degree of $V_1 = f_*\omega_S$; then there is a short exact sequence

$$0 \rightarrow E \rightarrow V_1 \rightarrow F \rightarrow 0$$

such that $F$ is locally free and $\deg F \geq 0$, see [Fu98]: moreover one clearly has $1 \leq h^0(E) \leq h^0(V_1) = 2$. Setting $e := \deg E - \deg F$, by [X85 Théorème 2.1 p.16] there are exactly two possibilities:

- $\deg F = 1$, $\deg E = 1$, $e = 0$
- $\deg F = 2$, $\deg E = 0$, $e = 2$.

**Proposition 2.1.** (i) If $e = 0$ then (up to translations) either $V_1 = O_B(p) \oplus O_B(2o - p)$ for some $p \neq B$ or $V_1 = F_2(\eta)$, where $\eta \in E$ is a $2$-torsion point.

(ii) If $e = 2$ then $V_1 = O_B(D) \oplus L$, where $D$ is an effective divisor of degree 2 on $B$ and $L \in \text{Pic}^0(B)$ is a non-trivial, torsion line bundle. This case occurs if and only if the canonical map $\phi|_S$ of $S$ factors through $f$.

**Proof.** (i) If $e = 0$, up to a translation we may assume $E = O_B(p)$, $F = O_B(2o - p)$, for some $p \in B$. If $D \neq E$, then $\text{Ext}^1(F, E) = 0$ and we obtain $V_1 = O_B(p) \oplus O_B(2o - p)$.

If $F = E$, then $\text{Ext}^1(F, E) = \mathbb{C}$. In that case $2o = 2p$, so any non-trivial extension class corresponds to $V_1 = F_2(\eta)$, where $2\eta \in [2o]$.

(ii) If $e = 2$ then $\deg E = 2$, hence $E = O_B(D)$ for some effective divisor $D$ on $B$. We have $h^0(E) = 2$ and $h^1(E) = 0$, so $h^0(V_1) = h^0(E) + h^0(F)$, which implies $h^0(F) = 0$. Then $F$ is a non-trivial, degree zero line bundle. Since $\text{Ext}^1(F, E) = 0$, it follows $V_1 = O_B(D) \oplus F$, and Simpson’s results ([Sim93]) imply that $F$ is a non-trivial torsion line bundle on $B$. The last assertion follows from [X85 Théorème 5.1 p.71].

**Proposition 2.2.** The case $e = 2$ does not occur.

**Proof.** If $e = 2$, then $S$ would be the canonical resolution of the singularities of a degree 2 cover of $\mathbb{P}(V_1) = \mathbb{P}(O_B(D) \oplus L)$. Since $V_1$ is decomposable, we can take global coordinates on the fibres of $\pi_1 : \mathbb{P}(V_1) \rightarrow B$, namely

$$x_0 \in H^0(O_{\mathbb{P}(V_1)}(1) \otimes \pi^*_1 O_B(-D)), \quad x_1 \in H^0(O_{\mathbb{P}(V_1)}(1) \otimes \pi^*_1 L^{-1}).$$
Putting $M = \mathcal{O}_B(D)$, we obtain $x_0^ix_1^j \in H^0(\mathcal{O}_{\mathbb{P}(V_1)}(i + j) \otimes \pi_1^*M^{-i} \otimes \pi_1^*L^{-j})$. Since $B'$ is algebraically equivalent to $6c_0 - 3\Gamma$, we have $B' \in |H^0(\mathcal{O}_{\mathbb{P}(V_1)}(6) \otimes \pi_1^*T^{-1})|$ for a suitable degree 3 line bundle $T$ on $B$, so the equation of $B'$ can be written as

$$
\sum_{i+j=6} a_{ij}x_0^ix_1^j = 0, \quad (6)
$$

where $a_{ij} \in H^0(\mathbb{P}(V_1), \pi_1^*(T^{-1} \otimes M^{i} \otimes L^{j}))$. In particular $a_{06} = a_{15} = 0$, so $x_2^6$ divides the left-hand side of (6). Hence $B'$ is non-reduced, a contradiction.

Propositions 2.1 and 2.2 imply the following

**Corollary 2.3.** Let $S$ be a minimal surface of general type with $p_g = 2$, $q = 1$, $K_S^2 = 5$. Then the canonical map of $S$ does not factor through the Albanese fibration.

### 2.2 The sheaf $V_2$

#### 2.2.1 The case where $V_1$ is decomposable

If $V_1$ is decomposable then Propositions 2.1 and 2.2 yield $V_1 = \mathcal{O}_B(p) \oplus \mathcal{O}_B(2o - p)$, so we have $S^2V_1 = \bigoplus_{i=1}^3 P_i$, where $P_1 = \mathcal{O}_B(2p)$, $P_2 = \mathcal{O}_B(2o)$, $P_3 = \mathcal{O}_B(4o - 2p)$. Fix a section $f_0 \in H^0(\mathcal{O}_B(\tau)) \setminus \{0\}$; applying the functor $\text{Hom}(-, S^3V_1)$ to the exact sequence

$$
0 \longrightarrow \mathcal{O}_B(-\tau) \longrightarrow \mathcal{O}_B(o) \longrightarrow \mathcal{O}_\tau \longrightarrow 0
$$

we obtain

$$
\text{Ext}^1(\mathcal{O}_\tau, S^2V_1) = \bigoplus_{i=1}^3 \frac{H^0(P_i(\tau - o))}{H^0(P_i(-o))} \cong \mathbb{C}^3, \quad (7)
$$

that is $\text{Ext}^1(\mathcal{O}_\tau, S^2V_1)$ can be identified with the space of global sections of $\bigoplus H^0(P_i(\tau - o))$, modulo the subspace of sections vanishing in $\tau$. For any $(f_1, f_2, f_3) \in \bigoplus H^0(P_i(\tau - o))$, we denote by $(f_1, f_2, f_3)$ its image in $\text{Ext}^1(\mathcal{O}_\tau, S^2V_1)$. Arguing as in [CaPi06, p.1032], this implies that $V_2 = f_\ast \omega_S^2$ is the cokernel of a short exact sequence

$$
0 \longrightarrow \mathcal{O}_B(-\tau) \longrightarrow \mathcal{O}_B(o) \oplus \bigoplus_{i=1}^3 P_i \longrightarrow V_2 \longrightarrow 0, \quad (8)
$$

where the injective map $i$ is given by $i(f_0, f_1, f_2, f_3)$.

**Remark 2.4.** If we choose the map $i'$ given by $i'(f_0, f_1 + f_0g_1, f_2 + f_0g_2, f_3 + f_0g_3)$, with $g_i \in H^0(P_i(-o))$, we obtain a commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_B(-\tau) \longrightarrow & \mathcal{O}(o) \oplus \bigoplus_{i=1}^3 P_i & \longrightarrow & V_2' & \longrightarrow & 0 \\
\downarrow & & & \mathbb{C}^3 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_B(-\tau) \longrightarrow & \mathcal{O}(o) \oplus \bigoplus_{i=1}^3 P_i & \longrightarrow & V_2 & \longrightarrow & 0
\end{array}
$$

where the matrix $M$ is given by

$$
\begin{pmatrix}
1 & g_1 & g_2 & g_3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

Hence $V_2' \cong V_2$, so the isomorphism class of $V_2$ only depends on $(f_1, f_2, f_3)$.

Notice that $V_2$ is a vector bundle if and only if $f_1$, $f_2$, $f_3$ do not vanish simultaneously in $\tau$, that is if and only if $\xi = (f_1, f_2, f_3)$ is not the trivial extension class. Let $m$ be the cardinality of the set $\{i | f_i = 0\}$; hence $0 \leq m \leq 2$. Now we give the description of $V_2$ in the different cases.
Proposition 2.5. Assume $V_1 = \mathcal{O}_B(p) \oplus \mathcal{O}_B(2o - p)$. Then there are precisely the following possibilities:

(I) $m = 0$, $\mathcal{O}_B(4o - 4p) \neq \mathcal{O}_B$, $V_2(-2o) = E_r(3, 1)$

(IIa) $m = 0$, $\mathcal{O}_B(4o - 4p) = \mathcal{O}_B$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = F_2(2o - 2p) \oplus \mathcal{O}_B(\tau)$

(IIb) $m = 0$, $\mathcal{O}_B(4o - 4p) = \mathcal{O}_B$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_r(2, 1) \oplus \mathcal{O}_B$

(IIc) $m = 1$, $\mathcal{O}_B(4o - 4p) = \mathcal{O}_B$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2o - 2p) \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau + 2p - 2o)$

(IIIa) $m = 1$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_{\tau + 2o - 2p}(2, 1) \oplus \mathcal{O}_B(2p - 2o)$

(IIIb) $m = 1$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_{\tau + 2o - 2p}(2, 1) \oplus \mathcal{O}_B(2o - 2p)$

(IIIC) $m = 1$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_r(2, 1) \oplus \mathcal{O}_B$

(Iva) $m = 2$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2p - 2o) \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau + 2o - 2p)$

(Ivb) $m = 2$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2o - 2p) \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau + 2o - 2p)$

(Ivc) $m = 2$, $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2p - 2o) \oplus \mathcal{O}_B(2o - 2p) \oplus \mathcal{O}_B(\tau)$

(V) $0 \leq m \leq 2$, $\mathcal{O}_B(2o - 2p) = \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$.

Proof. The proof is not difficult, but one needs to consider several cases; for the reader’s convenience, we will write it in detail. Let $L \in \text{Pic}^0(B)$; tensoring the exact sequence (5) with $L(-2o)$ we obtain

$$0 \rightarrow L(-o - \tau) \rightarrow L(-o) \oplus L(2p - 2o) \oplus L \oplus L(2o - 2p) \rightarrow V_2(-2o) \otimes L \rightarrow 0, \quad (9)$$

which in turn induces a linear map in cohomology

$$\alpha : H^1(L(-o - \tau)) \rightarrow H^1(L(-o) \oplus L(2p - 2o) \oplus L \oplus L(2o - 2p))$$

such that $H^1(V_2(-2o) \otimes L)$ is isomorphic to the cokernel of $\alpha$. Notice that $\text{det} V_2(-2o) = \mathcal{O}_B(\tau)$. The first component of $\alpha$ is always surjective, since it is induced by the short exact sequence

$$0 \rightarrow L(-o - \tau) \rightarrow L(-o) \rightarrow \mathcal{O}_r \rightarrow 0,$$

therefore if $L \notin \{\mathcal{O}_B(2o - 2p), \mathcal{O}_B, \mathcal{O}_B(2p - 2o)\}$ the map $\alpha$ is surjective and $H^1(V_2(-2o) \otimes L) = 0$. Taking the dual of $\alpha$, we obtain the map

$$\alpha^* : H^0(L^*(o) \oplus L^*(2o - 2p) \oplus L^* \oplus L^*(2p - 2o)) \rightarrow H^0(L^*(o + \tau)),$$

which is given by $(f_0, f_1, f_2, f_3)$; moreover $H^1(V_2(-2o) \otimes L)^*$ is isomorphic to $\ker \alpha^*$. If $\mathcal{O}_B(2o - 2p) = \mathcal{O}_B$, then $\alpha^*$ is injective for all $L \in \text{Pic}^0(B) \setminus \{\mathcal{O}_B\}$, whereas for $L = \mathcal{O}_B$ it has a 2-dimensional kernel; by using Proposition 1.2 we conclude that $V_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$, so we are in case (V). Therefore we may assume $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$. Since $\alpha^*$ is injective unless $L \in \{\mathcal{O}_B(2o - 2p), \mathcal{O}_B, \mathcal{O}_B(2p - 2o)\}$, we have just to consider these three cases.

If $L = \mathcal{O}_B(2o - 2p)$ we obtain

$$h^1(V_2(-2o) \otimes L) = \begin{cases} 0 & \text{if } \mathcal{O}_B(4o - 4p) \neq \mathcal{O}_B \text{ and } f_3 \neq 0; \\ 1 & \text{if } \mathcal{O}_B(4o - 4p) = \mathcal{O}_B \text{ and } f_3 = 0; \\ 1 & \text{if } \mathcal{O}_B(4o - 4p) = \mathcal{O}_B \text{ and } f_3 = 0; \\ 2 & \text{if } \mathcal{O}_B(4o - 4p) = \mathcal{O}_B \text{ and } f_3 = 0. \\ \end{cases}$$

Analogously, if $L = \mathcal{O}_B(2p - 2o)$ we obtain

$$h^1(V_2(-2o) \otimes L) = \begin{cases} 0 & \text{if } \mathcal{O}_B(4o - 4p) \neq \mathcal{O}_B \text{ and } f_3 \neq 0; \\ 1 & \text{if } \mathcal{O}_B(4o - 4p) = \mathcal{O}_B \text{ and } f_3 = 0; \\ 1 & \text{if } \mathcal{O}_B(4o - 4p) = \mathcal{O}_B \text{ and } f_3 = 0; \\ 2 & \text{if } \mathcal{O}_B(4o - 4p) = \mathcal{O}_B \text{ and } f_3 = 0. \\ \end{cases}$$
Finally, if $L = \mathcal{O}_B$ we obtain

$$h^1(V_2(-2\alpha) \otimes L) = \begin{cases} 0 & \text{if } \bar{f}_2 \neq 0; \\ 1 & \text{if } \bar{f}_2 = 0. \end{cases}$$

Now we observe that if $\bar{f}_1 = 0$ then $P_i(-2\alpha)$ is a direct summand of $V_2(-2\alpha)$, and we analyze the different possibilities.

Assume first $\mathcal{O}_B(4\alpha - 4\beta) \neq \mathcal{O}_B$. In this case there exist exactly $m$ line bundles $L$ such that $H^1(V_2(-2\alpha) \otimes L) \neq 0$. By a straightforward application of Proposition 1.2 and Remark 1.3 we obtain cases (I), (IIa), (IIb), (IIc), (Iva), (IVb), (IVc).

Now assume $\mathcal{O}_B(4\alpha - 4\beta) = \mathcal{O}_B$. Then the only new possibilities are:

- $\bar{f}_i \neq 0$ for all $i$, that is $m = 0$; then $H^1(V_2(-2\alpha) \otimes L)$ is trivial for all $L \in \text{Pic}^0(B)$, except in the case $L = \mathcal{O}_B(2\alpha - 2\beta) = \mathcal{O}_B(2p - 2\alpha)$ where it is 1-dimensional. By Proposition 1.2 and Remark 1.3 this is either (Ia) or (IIb).

- $\bar{f}_1 \neq 0, \bar{f}_2 = 0, \bar{f}_3 \neq 0$; then $H^1(V_2(-2\alpha) \otimes L)$ is trivial for all $L \in \text{Pic}^0(B)$, except in the cases $L = \mathcal{O}_B(2\alpha - 2\beta)$ and $L = \mathcal{O}_B$ where it is 1-dimensional; this is (Ii).

The proof is now complete. \qed

2.2.2 The case where $V_1$ is indecomposable

If $V_1$ is indecomposable, then $V_1 = F_2(\eta)$, where $\eta$ is a 2-torsion point, so Proposition 1.3 yields $S^2V_1 = F_3(2\alpha)$. Arguing as in Subsection 2.2.1, we obtain

$$\text{Ext}^1(\mathcal{O}_\tau, S^2V_1) = \frac{H^0(F_3(o + \tau))}{H^0(F_3(o))} \cong \mathbb{C}^3,$$

(10)

that is $\text{Ext}^1(\mathcal{O}_\tau, S^2V_1)$ can be identified with the space of global sections of $F_3(o + \tau)$, modulo the subspace of sections vanishing in $\tau$. For any $v \in H^0(F_3(o + \tau))$, we will denote by $\bar{v}$ its image in $\text{Ext}^1(\mathcal{O}_\tau, S^2V_1)$. Now let us fix a section $f_0 \in H^0(\mathcal{O}_B(\tau)) \setminus \{0\}$. Then $V_2$ is the cokernel of a short exact sequence

$$0 \to \mathcal{O}_B(o - \tau) \xrightarrow{i} \mathcal{O}_B(o) \oplus F_3(2\alpha) \to V_2 \to 0,$$

(11)

where the injective map $i$ is given by $i(f_0, v)$. Notice that $V_2$ is a vector bundle if and only if $v$ does not vanish in $\tau$, that is if and only if $\xi := \bar{v}$ is not the trivial extension class. We can now give a more precise description of $V_2$.

**Proposition 2.6.** Assume $V_1 = F_2(\eta)$, where $\eta \in E$ is a 2-torsion point. Then we have the following possibilities:

(VI) $V_2(-2\alpha) = E_\tau(3, 1)$

(VIIa) $V_2(-2\alpha) = F_2 \oplus \mathcal{O}_B(\tau)$

(VIIb) $V_2(-2\alpha) = E_\tau(2, 1) \oplus \mathcal{O}_B$.

Moreover, for a general choice of $\xi \in \text{Ext}^1(S^2V_1, \mathcal{O}_\tau)$ only (VI) occurs.

**Proof.** Let $L \in \text{Pic}^0(B)$; tensoring the exact sequence (11) with $L(-2\alpha)$ we obtain

$$0 \to L(-o - \tau) \to L(-o) \oplus (F_3 \otimes L) \to V_2(-2\alpha) \otimes L \to 0,$$

(12)

which in turn induces a linear map in cohomology

$$\alpha: H^1(L(-o - \tau)) \to H^1(L(-o)) \oplus H^1(F_3 \otimes L)$$

such that $H^1(V_2(-2\alpha) \otimes L)$ is isomorphic to the cokernel of $\alpha$. As in the proof of Proposition 2.5, the first component of $\alpha$ is always surjective. If $L \neq \mathcal{O}_B$ then $H^1(F_3 \otimes L) = 0$ (see Proposition 1.1); consequently, $\alpha$ is surjective and $H^1(V_2(-2\alpha) \otimes L) = 0$. We must now investigate what happens for $L = \mathcal{O}_B$. Let $v \in \text{Hom}(\mathcal{O}_B(-o - \tau), F_3)$ $\cong H^0(F_3(o + \tau))$, and let $Q$ be the cokernel of the corresponding map $v: \mathcal{O}_B(-o - \tau) \to F_3$. 
Claim 2.7. For a general choice of \( v \), we have
\[
Q = \mathcal{O}_B(q) \oplus \mathcal{O}_B(o + \tau - q)
\]
for some \( q \in B \). Moreover, \( Q = \mathcal{O}_B \oplus \mathcal{O}_B(o + \tau) \) if and only if \( \text{im} v \subset W \), where \( W \) is the unique subbundle of \( F_3 \) isomorphic to \( F_2 \), see [A157] p.433.

Proof. Since \( F_3(o + \tau) \) is globally generated, for a general choice of \( v \) the sheaf \( Q \) is locally free. If \( Q \) were indecomposable then \( Q = F_2(u) \), where \( u \in B \) is such that \( \mathcal{O}_B(2u) = \mathcal{O}_B(o + \tau) \). Since \( F_v \) is self-dual, by taking duals we obtain the exact sequence
\[
0 \to F_2(-u) \to F_3 \to \mathcal{O}_B(o + \tau) \to 0.
\]
By composing it with the injective morphism \( \mathcal{O}_B(-u) \to F_2(-u) \) induced by the section of \( F_2 \), we conclude that \( \mathcal{O}_B \) is a sub–vector bundle of \( F_3(u) \), but this is a contradiction, since every section of \( F_3(u) \) vanishes in \( u \) (see [CaSch02] Section 5, p.108); thus \( Q \) must be decomposable. Moreover, we have \( Q \cong \mathcal{O}_B \oplus \mathcal{O}_B(o + \tau) \) if and only if there exists a surjective map \( F_3 \to \mathcal{O}_B \) whose kernel contains \( v \). But such a kernel is exactly \( W \), so we are done.

In order to complete the proof of Proposition 2.6, let us take a general \( v \in H^0(F_3(o + \tau)) \). We must then study the exact sequence
\[
0 \to \mathcal{O}_B(-o - \tau) \to F_3 \xrightarrow{j} \mathcal{O}_B(q) \oplus \mathcal{O}_B(o + \tau - q) \to 0,
\]
and in particular the map \( \beta \) induced in cohomology as follows:
\[
H^0(\mathcal{O}_B(q) \oplus \mathcal{O}_B(o + \tau - q)) \to H^1(\mathcal{O}_B(-o - \tau))) \xrightarrow{\beta} H^1(F_3) \to 0. \tag{13}
\]
Dualizing (13), using Serre duality and exploiting the isomorphism \( F_3^* \cong F_3 \) we obtain
\[
0 \to H^0(F_3) \xrightarrow{\beta^*} H^0(\mathcal{O}_B(o + \tau)) \to H^1(\mathcal{O}_B(-o - \tau) \oplus \mathcal{O}_B(-o - \tau + q)),
\]
hence \( \text{im} \beta^* \) can be identified with \( \langle s_q \rangle \), the line generated by the unique non-zero section \( s_q \in H^0(\mathcal{O}_B(o + \tau)) \) such that \( s_q(q) = 0 \). Now, looking at sequence (12) for \( L = \mathcal{O}_B \), we see that \( \alpha \) is dual to
\[
\alpha^*: H^0(\mathcal{O}_B(o)) \oplus H^0(F_3) \xrightarrow{(f_0, \beta^*)} H^0(\mathcal{O}_B(o + \tau)),
\]
so the image of \( \alpha^* \) is the subspace spanned by \( s_o \) and \( s_q \). Since \( v \) is general we have \( o \neq q \), hence \( s_o \) and \( s_q \) are linearly independent sections in \( H^0(\mathcal{O}_B(o + \tau)) \) and this implies that \( \alpha^* \) is an isomorphism. Consequently, \( \alpha \) is also an isomorphism and for a general choice of \( \xi = \dim V \) we obtain \( h^1(V_2(-2o)) = 0 \). For some special choice of \( v \in H^0(F_3(o + \tau)) \) it may happen that \( \alpha^* \) has a 1-dimensional kernel, consequently, \( \alpha \) has a 1-dimensional cokernel and \( h^1(V_2(-2o)) = 1 \). Therefore we can apply Proposition 2.6 concluding the proof of Proposition 2.6.

3 The moduli space

Let \( M \) be the moduli space of minimal surfaces of general type \( S \) with \( p_g(S) = 2, q(S) = 1 \) and \( K_S^2 = 5 \). We write \( M = M' \cup M'' \), where \( M' \) corresponds to surfaces such that \( V_1 \) is decomposable and \( M'' \) corresponds to surfaces such that \( V_1 \) is indecomposable.

Definition 3.1. We stratify \( M' \) and \( M'' \) as
\[
M' = M_3 \cup M_{IIa} \cup \cdots \cup M_V,
\]
\[
M'' = M_{VI} \cup M_{VIa} \cup M_{VIIb},
\]
according to the decomposition type for \( V_2 \), as in Propositions 2.5 and 2.6.
Now we want to estimate the dimensions of these strata. By Catanese-Pignatelli's structure theorem for genus 2 fibrations, we can consider a surjective map $\Phi: D \to M$, where $D$ is the set of admissible 5–tuples $(B, V_1, \tau, \xi, w)$ which give surfaces with our numerical invariants and belonging to a given stratum. Therefore in each case the dimension of the stratum is less than or equal to the dimension of $D$.

Moreover, we will see that each strata can be parametrized via a unirational family; therefore $M$ itself is unirational.

**Remark 3.2.** In order to compute the exact dimension of each strata of the moduli space, we must compute the dimension of the corresponding parameter space $D$, and then subtract from the result the dimension of the general fibre of $\Phi$. Such a fibre will correspond to the orbit of the action of certain automorphism groups over our construction data.

Locally around the point $[S] \in M$, the coarse moduli space $M$ is analytically isomorphic to the quotient of the base $T$ of the Kuranishi family by the finite group $\text{Aut}(S)$. Hence

$$h^1(S, T_S) \geq \dim_{[S]} M \geq h^1(S, T_S) - h^2(S, T_S) = 10\chi(O_S) - 2K_S^2 = 10.$$ 

When $g = 1$ one obtains the better lower bound $10\chi(O_S) - 2K_S^2 + p_g = 12$, see [Ran95] and [Oeb95]. So in our case we have

$$h^1(S, T_S) \geq \dim_{[S]} M \geq 12.$$ 

This implies that those strata whose dimension is less than 12 can be disregarded for the determination of the irreducible components of $M$.

For further application, let us describe a method that can be used in order to estimate $h^1(S, T_S)$, see [Pi00]. There is an exact sequence

$$0 \to \omega_S \to \Omega^1_S \otimes \omega_S \to \omega_S^{\otimes 2} \to \mathcal{O}_{\text{Crit}(f)}(\omega_S^{\otimes 2}) \to 0,$$

where $f: S \to B := \text{Alb}(S)$ is the Albanese map of $S$. Setting $\mathcal{F} := (\Omega^1_S \otimes \omega_S)/\omega_S$, we get

$$0 \to \mathcal{F} \to \omega_S^{\otimes 2} \to \mathcal{O}_{\text{Crit}(f)}(\omega_S^{\otimes 2}) \to 0.$$ 

Therefore

$$2 = h^0(S, \omega_S) \leq h^0(S, \Omega^1_S \otimes \omega_S) \leq h^0(S, \omega_S) + h^0(S, \mathcal{F}) = 2 + h^0(S, \mathcal{F}), \ (14)$$

and by the Serre duality $h^2(S, T_S) = h^0(S, \Omega^1_S \otimes \omega_S)$. Finally,

$$0 \to H^0(S, \mathcal{F}) \to H^0(S, \omega_S^{\otimes 2}) \to H^0(S, \omega_S^{\otimes 2} \otimes \mathcal{O}_{\text{Crit}(f)}) \to 0$$

implies that $H^0(S, \mathcal{F})$ is the vector space given by the bicanonical curves of $S$ passing through $\text{Crit}(f)$.

Let us start by studying $M'$. We have $\mathcal{O}_B(p) \oplus \mathcal{O}_B(2o - p) \cong \mathcal{O}_B(q) \oplus \mathcal{O}_B(2o - q)$ if and only if either $p = q$ or $p + q \in [2o]$; therefore, when $p$ varies in $B$, the vector bundle $V_1$ varies into a 1-dimensional family isomorphic to $\mathbb{P}^1$.

**Proposition 3.3.** The stratum $M_1$ is nonempty, irreducible, of dimension at most 13.

**Proof.** Set $W := E_r(3, 1)$; then $V_2 = W(2o)$ and we have a short exact sequence

$$0 \to W(2o - 2\tau) \to S^3W(2o - 2\tau) \to \tilde{A}_6 \to 0,$$

see (3) and (4). By [CaC93] Section 1] we obtain

$$h^0(W(2o - 2\tau)) = 1, \quad h^1(W(2o - 2\tau)) = 0, \quad h^0(S^3W(2o - 2\tau)) = 10,$$

hence $h^0(\tilde{A}_6) = 9$. We have 1 parameter for $B$, 1 parameter for $V_1$, 2 parameters for $\xi$, 1 parameter for $\tau$ and 8 parameters from $\mathbb{P}H^0(\tilde{A}_6)$. Therefore $M_1$ has dimension at most 13, and it is irreducible since it can be parametrized via an irreducible family.
Now let us show that it is non-empty. For the sake of simplicity, we assume \( \tau = o \) and we write \( \pi: \mathbb{P}(W) \longrightarrow B \) and \( \pi_2: \mathbb{P}(V_2) \longrightarrow B \) for the projective bundles associated to \( W \) and \( V_2 \), respectively. There is an isomorphism of projective bundles \( \psi: \mathbb{P}(W) \longrightarrow \mathbb{P}(V_2) \) such that

\[
\psi^* \mathcal{O}_{\mathbb{P}(V_2)}(1) \cong \mathcal{O}_{\mathbb{P}(W)}(1) \otimes \pi^* \mathcal{O}_B(2o).
\]

The projective bundle \( \mathbb{P}(W) \) can be identified with \( \text{Sym}^3 B \), see for instance [CaCi93]. For all \( x \in B \), set:

\[
D_x = \{x + x_2 + x_3 \mid x_2, x_3 \in B\},
\]

\[
F_x = \{x_1 + x_2 + x_3 \mid x_1 \oplus x_2 \oplus x_3 = x\}.
\]

Then \( D_o \) is the divisor class of \( \mathcal{O}_{\mathbb{P}(W)}(1) \), and (15) implies that

\[
\mathcal{O}_{\mathbb{P}(V_2)}(1) = \mathcal{O}_{\mathbb{P}(V_2)}(D_o + 2F_o).
\]

Thus \( C \in |\mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \pi_2^* (\text{det}(V_1))^{-2}| = |2D_o + 4F_o - 4F_o| = |2D_o| \).

Let now \( \varphi: \tilde{B} \longrightarrow B \) be an isogeny of degree 3, and set \( G := \ker(\varphi) \cong \mathbb{Z}_3 \). If we write

\[
\varphi^{-1}(o) = \{\tilde{o}, \tilde{a}, \tilde{b}\},
\]

we have \( G = \langle t_\tilde{a}^* \rangle \), where \( t_\tilde{a}^* \) is the translation by \( \tilde{a} \).

By [AlG57] there exists a line bundle \( L \in \text{Pic}(\tilde{B}) \) of degree 1 such that

\[
\varphi_* L = W
\]

and moreover

\[
\varphi^* \varphi_* L = \varphi^* E_r(3, 1) = \mathcal{O}_{\tilde{B}}(\tilde{o}) \oplus t_\tilde{a}^* \mathcal{O}_{\tilde{B}}(\tilde{o}) \oplus (t_\tilde{b}^* \tilde{o}^2) \mathcal{O}_{\tilde{B}}(\tilde{o}) = \mathcal{O}_{\tilde{B}}(\tilde{o}) \oplus \mathcal{O}_{\tilde{B}}(\tilde{a}) \oplus \mathcal{O}_{\tilde{B}}(\tilde{b}).
\]

see [Is05] Theorem 2.2. Let us define \( \tilde{E} := \varphi^*(W \otimes \mathcal{O}_B(2o)) \); since the divisor \( 2\tilde{a} + 2\tilde{b} \) is linearly equivalent to \( 4\tilde{o} \), equation (17) yields

\[
\tilde{E} = \varphi^* W \otimes \mathcal{O}_{\tilde{B}}(2\tilde{o} + 2\tilde{a} + 2\tilde{b})
\]

\[
= \mathcal{O}_{\tilde{B}}(3\tilde{o} + 2\tilde{a} + 2\tilde{b}) \otimes \mathcal{O}_{\tilde{B}}(2\tilde{o} + 3\tilde{a} + 2\tilde{b}) \otimes \mathcal{O}_{\tilde{B}}(2\tilde{o} + 2\tilde{a} + 3\tilde{b})
\]

\[
= \mathcal{O}_{\tilde{B}}(7\tilde{o}) \oplus \mathcal{O}_{\tilde{B}}(6\tilde{o} + \tilde{a}) \oplus \mathcal{O}_{\tilde{B}}(6\tilde{o} + \tilde{b}).
\]

From the commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(\tilde{E}) & \stackrel{\Phi}{\longrightarrow} & \mathbb{P}(V_2) \\
\pi_2 \downarrow & & \downarrow \pi_2 \\
\tilde{B} & \longrightarrow & B
\end{array}
\]

it follows

\[
\Phi_* \Phi^* \mathcal{O}_{\mathbb{P}(V_2)}(D_o) = \mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes \mathcal{O}_{\mathbb{P}(\tilde{E})}
\]

\[
= \mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes (\mathcal{O}_{\mathbb{P}(V_2)} \oplus \mathcal{L} \oplus \mathcal{L}^2)
\]

\[
= \mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes (\mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes \mathcal{L} \otimes (\mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes \mathcal{L}^2),
\]

where \( \mathcal{L} \) is the 3–torsion line bundle inducing the étale \( \mathbb{Z}_3 \)-cover \( \Phi: \mathbb{P}(\tilde{E}) \longrightarrow \mathbb{P}(V_2) \). By (19) we see that

\[
\Phi^* \mathcal{O}_{\mathbb{P}(V_2)}(D_o) = \Phi^*(\mathcal{O}_{\mathbb{P}(V_2)}(1) \otimes \pi_2^* \mathcal{O}_B(-2o))
\]

\[
= \mathcal{O}_{\mathbb{P}(\tilde{E})}(1) \otimes \tilde{\pi}_2^* \mathcal{O}_{\tilde{B}}(-6\tilde{o}).
\]
Let $y_0$, $y_1$ and $y_2$ be global coordinates on the fibers of $\tilde{\pi}_2$, namely
\[
y_0 \in H^0(\mathcal{O}_{\tilde{E}}(1) \otimes \pi^*_2 \mathcal{O}_B(-7\tilde{o})) \quad y_1 \in H^0(\mathcal{O}_{\tilde{E}}(1) \otimes \pi^*_2 \mathcal{O}_B(-6\tilde{o} - \tilde{a})) \quad y_2 \in H^0(\mathcal{O}_{\tilde{E}}(1) \otimes \pi^*_2 \mathcal{O}_B(-6\tilde{o} - \tilde{b})).
\]

We have $h^0(\Phi^* \mathcal{O}_B(2D_o)) = 3$ and a general section of $\Phi^* \mathcal{O}_B(2D_o)$ can be written as
\[
\sigma = \lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2,
\]
where $\lambda_0 \in H^0(\pi^*_2 \mathcal{O}_B(\tilde{o}))$, $\lambda_1 \in H^0(\pi^*_2 \mathcal{O}_B(\tilde{a}))$ and $\lambda_2 \in H^0(\pi^*_2 \mathcal{O}_B(\tilde{b}))$.

Then a straightforward computation shows that we can choose the $y_i$ so that the action of $t^*_a \in G$ on the $y_i$ is given by
\[
t^*_a: \begin{cases} 
y_0 \mapsto y_1 
y_1 \mapsto y_2 
y_2 \mapsto y_0.
\end{cases}
\]
(18)
Therefore $t^*_a \sigma = (t^*_a \lambda_0)y_1 + (t^*_a \lambda_1)y_2 + (t^*_a \lambda_2)y_0$, so $\sigma$ is $G$–invariant if and only if $t^*_a \lambda_0 = \lambda_1$, $t^*_a \lambda_1 = \lambda_2$ and $t^*_a \lambda_2 = \lambda_0$. Since $(t^*_a)^2 = t^*_b$, this is equivalent to require $\lambda_1 = t^*_a \lambda_0$ and $\lambda_2 = t^*_b \lambda_0$. So a general invariant section of $\Phi^* \mathcal{O}_B(2D_o)$ is given by
\[
\lambda y_0 + (t^*_a \lambda)y_1 + (t^*_b \lambda)y_2,
\]
where $\lambda \in H^0(\mathcal{O}_{\tilde{B}}(\tilde{o}))$.

Now a general section of $\Phi^* \mathcal{O}_B(2D_o)$ is of the form:
\[
\sigma = \sum_{i+j+k=2} \lambda_{ijk} y_0^i y_1^j y_2^k = \lambda_{200} y_0^2 + \lambda_{020} y_0^2 + \lambda_{002} y_2^2 + \lambda_{110} y_0 y_1 + \lambda_{101} y_0 y_2 + \lambda_{011} y_1 y_2,
\]
where the $\lambda_{ijk}$ are sections of pullbacks of suitable line bundles on $\tilde{B}$.

By (18), $t^*_a$ acts on $\sigma$ as
\[
t^*_a \sigma = (t^*_a \lambda_{200})y_0^2 + (t^*_a \lambda_{020})y_0^2 + (t^*_a \lambda_{002})y_2^2 + (t^*_a \lambda_{110})y_0 y_1 + (t^*_a \lambda_{101})y_0 y_2 + (t^*_a \lambda_{011})y_1 y_2,
\]
so $\sigma$ is $G$–invariant if and only if
\[
\begin{align*}
\lambda_{200} &= t^*_a \lambda_{200} \\
\lambda_{020} &= t^*_a \lambda_{020} \\
\lambda_{002} &= t^*_a \lambda_{002} \\
\lambda_{110} &= t^*_a \lambda_{110} \\
\lambda_{101} &= t^*_a \lambda_{101} \\
\lambda_{011} &= t^*_b \lambda_{011}.
\end{align*}
\]
Hence a general invariant section of $\Phi^* \mathcal{O}_B(2D_o)$ can be written as
\[
\lambda y_0^2 + (t^*_a \lambda)y_0^2 + (t^*_b \lambda)y_2^2 + \mu y_0 y_1 + (t^*_b \mu)y_0 y_2 + (t^*_a \mu)y_1 y_2,
\]
with $\lambda \in H^0(\mathcal{O}_{\tilde{B}}(2\tilde{o}))$, $\mu \in H^0(\mathcal{O}_{\tilde{B}}(\tilde{o} + \tilde{a}))$.

Denoting by $\tilde{\rho} \in \tilde{B}$ any of the points in $\varphi^{-1}(p)$, the short exact sequence (11) lifts to
\[
0 \longrightarrow \mathcal{O}_{\tilde{B}}(6\tilde{\rho}) \oplus \mathcal{O}_{\tilde{B}}(6\tilde{\rho}) \oplus \mathcal{O}_{\tilde{B}}(12\tilde{o} - 6\tilde{\rho}) \xrightarrow{\tilde{\sigma}_2} \tilde{E} \longrightarrow \mathcal{O}_{\tilde{B}+\tilde{a}+\tilde{b}} \longrightarrow 0.
\]
(20)
Taking global coordinates $\tilde{x}_0$, $\tilde{x}_1$ on the fibres of $\varphi^* \mathcal{V}_1 = \mathcal{O}_{\tilde{B}}(3\tilde{\rho}) \oplus \mathcal{O}_{\tilde{B}}(6\tilde{o} - 3\tilde{\rho})$, the map $\tilde{\sigma}_2$ is given by
\[
\begin{cases}
\tilde{\sigma}_2(\tilde{x}_0^2) = a_{00} y_0 + a_{01} y_1 + a_{02} y_2 \\
\tilde{\sigma}_2(\tilde{x}_0 \tilde{x}_1) = a_{10} y_0 + a_{11} y_1 + a_{12} y_2 \\
\tilde{\sigma}_2(\tilde{x}_1^2) = a_{20} y_0 + a_{21} y_1 + a_{22} y_2.
\end{cases}
\]
where
\[
\begin{align*}
a_{00} &\in H^0(\pi^*_2 \mathcal{O}_B(7\tilde{o} - 6\tilde{\rho})) \\
a_{01} &\in H^0(\pi^*_2 \mathcal{O}_B(6\tilde{o} - 6\tilde{\rho} + \tilde{a})) \\
a_{02} &\in H^0(\pi^*_2 \mathcal{O}_B(6\tilde{o} - \tilde{b})) \\
a_{10} &\in H^0(\pi^*_2 \mathcal{O}_B(6\tilde{o} - \tilde{b} + \tilde{a})) \\
a_{11} &\in H^0(\pi^*_2 \mathcal{O}_B(6\tilde{o} + \tilde{b})) \\
a_{12} &\in H^0(\pi^*_2 \mathcal{O}_B(\tilde{a})) \\
a_{20} &\in H^0(\pi^*_2 \mathcal{O}_B(6\tilde{\rho} - 5\tilde{o})) \\
a_{21} &\in H^0(\pi^*_2 \mathcal{O}_B(6\tilde{\rho} - 6\tilde{o} + \tilde{a})) \\
a_{22} &\in H^0(\pi^*_2 \mathcal{O}_B(6\tilde{\rho} - \tilde{b})).
\end{align*}
\]
Let us consider now the conic bundle $\tilde{C} \subset \mathbb{F}(\tilde{E})$ given by

$$(a_{00}y_0 + a_{01}y_1 + a_{02}y_2)(a_{20}y_0 + a_{21}y_1 + a_{22}) - (a_{10}y_0 + a_{11}y_1 + a_{12}y_2)^2 = 0.$$ 

If we choose

$$a_{01} = t_4^*a_{00}, \quad a_{02} = t_5^*a_{00}, \quad a_{11} = t_4^*a_{10}, \quad a_{12} = t_5^*a_{10}, \quad a_{21} = t_5^*a_{20}, \quad a_{22} = t_4^*a_{20}$$

the equation of $\tilde{C}$ is $G$-invariant, hence of the form \[10\]; in fact, we have

$$\lambda = a_{00}a_{20} - a_{10}^2, \quad \mu = a_{00}(t_4^*a_{20}) + (t_5^*a_{00})a_{20} - 2a_{10}(t_5^*a_{10}).$$

We claim that, for a general choice of $a_{00}$, $a_{10}$, $a_{20}$, the only singularities of $\tilde{C}$ are three rational double points of type $A_1$, lying over the three points $\tilde{o}$, $\tilde{a}$, $\tilde{b}$. Since $\tilde{\sigma}_2$ is of maximal rank outside these points, and since they form an orbit for the $G$-action, it is sufficient to check that the fibre over $\tilde{o}$ has a node (which will be automatically a point of type $A_1$ for $\tilde{C}$). In a neighborhood of this fibre, set

$$u_0 := a_{00}(\tilde{o})y_0 + a_{01}(\tilde{o})y_1 + a_{02}(\tilde{o})y_2, \quad u_1 := a_{10}(\tilde{o})y_0 + a_{11}(\tilde{o})y_1 + a_{12}(\tilde{o})y_2, \quad u_2 := a_{20}(\tilde{o})y_0 + a_{21}(\tilde{o})y_1 + a_{22}(\tilde{o})y_2.$$ 

Since $\tilde{\sigma}_2$ drops rank in $\tilde{o}$, we can find $c_0, c_2 \in \mathbb{C}$ such that $u_1 = c_0u_0 + c_2u_2$; then a local equation of the fibre of $\tilde{C}$ over $\tilde{o}$ is given by

$$u_0u_2 - (c_0u_0 + c_2u_2)^2 = 0. \tag{21}$$

Since for a general choice of $a_{00}$, $a_{10}$, $a_{20}$ (i.e. for a general choice of $c_0$, $c_2$) the quadratic form \[21\] splits into two distinct linear forms, our claim is proven.

Therefore the image of $\tilde{C}$ in $\mathbb{P}(V_2)$ is a conic bundle $C$ with a unique singular point of type $A_1$, lying over the point $o \in B$. Moreover, by construction, $C$ is the conic bundle associated with the map $\tilde{\sigma}_2: S^2V_1 \to V_2$, so condition (P1) of Theorem \[1.6\] is satisfied.

The relative cubic $G$ belongs to the linear system $|O_{\mathbb{P}(V_2)}(3) \otimes \pi_2^*O_B(-4\tau - 2\tau)| = |3D_o + 6F_o - 6F_o| = |3D_o|$. By \cite{CaM93}, the linear system $|3D_o|$ is base point free, hence its restriction to $C$ is base point free too. This implies that a general complete intersection of the form $G \cap C$ is smooth and does not contain the unique singular point of $C$. Thus condition (P2) is also satisfied, and consequently $M_1$ is not empty.

**Proposition 3.4.** The stratum $M_{IIa}$ has dimension at most 12.

**Proof.** In case (IIa) we have $O_B(4\tau - 4p) = O_B$, so there are no parameters for $V_1$. The vector bundle $\tilde{A}_o$ fits into the short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \tilde{A}_o \longrightarrow 0,$$

where

$$G_1 = F_2(2\tau - 2\tau) \oplus O_B(2\tau), \quad G_2 = S^3F_2(2\tau - 2\tau) \oplus S^2F_2(2\tau - \tau) \oplus F_2(2\tau) \oplus O_B(2\tau + \tau).$$

By Proposition \[1.3\] we have $S^2F_2 = F_3$, $S^3F_2 = F_4$. Now there are two possibilities.

- $O_B(2\tau - 2\tau) \neq O_B$. In this case

$$h^0(G_1) = 1, \quad h^1(G_1) = 0, \quad h^0(G_2) = 10,$$

hence $h^0(\tilde{A}_o) = h^0(G_2) - h^0(G_1) = 9$. We have 1 parameter for $B$, 2 parameters for $\xi$, 1 parameter for $\tau$ and 8 parameters from $PH^0(\tilde{A}_o)$. 

14
• $O_B(2p - 2\tau) = O_B$. In this case

$$h^0(G_1) = 2, \quad h^1(G_1) = 1, \quad h^0(G_2) = 11,$$

hence $h^0(\tilde{A}_6) \leq 10$ by \cite{CaPi06}. We have 1 parameter for $B$, 2 parameters for $\xi$, no parameters for $\tau$ and $V_1$ and at most 9 parameters from $\mathbb{P}H^0(\tilde{A}_6)$.

Summing up, we conclude that $\mathcal{M}_{IIa}$ has dimension at most 12. $\square$

**Proposition 3.5.** The stratum $\mathcal{M}_{IIб}$ has dimension at most 12.

**Proof.** Set $W = E_4(2, 1)$; then $V_2(-2\omega) = W \oplus O_B$ and tensoring the exact sequence \cite{CaPi06} with $O_B(-6\omega)$ we obtain

$$0 \longrightarrow W \oplus O_B \longrightarrow (S^3W \oplus S^3W) \oplus (W \oplus O_B) \longrightarrow A_6(-6\omega) \longrightarrow 0. \tag{22}$$

Arguing as in \cite[Lemma 6.14]{CaPi06}, we see that the second component of the map $i_3$ is actually the identity, hence the exact sequence \eqref{eq:splitting} splits, giving

$$\tilde{A}_6 = A_6(-4\omega - 2\tau) = (S^3W \oplus S^3W)(2\omega - 2\tau).$$

By Proposition \cite{CaPi06} this in turn implies

$$\tilde{A}_6 = \left(W \oplus W \oplus \bigoplus_{i=1}^3 L_4\right)(2\omega - \tau),$$

hence $h^0(\tilde{A}_6) = 9$. We have 1 parameter for $B$, no parameters for $V_1$, 2 parameters for $\xi$, 1 parameter for $\tau$ and 8 parameters from $\mathbb{P}H^0(\tilde{A}_6)$. Therefore $\mathcal{M}_{IIб}$ has dimension at most 12.

The fact that it is nonempty can be proven as in case $\mathcal{M}_I$ (using an isogeny of degree 2 instead of 3); the details are left to the reader. $\square$

**Proposition 3.6.** The stratum $\mathcal{M}_{IIc}$ has dimension at most 11.

**Proof.** In case (IIC) we have $O_B(2\omega - 2p) = O_B(2p - 2\omega)$, with $O_B(2\omega - 2p) \neq O_B$, and the map $\sigma_2$ has the form

$$\sigma_2 : O_B(2p) \oplus O_B(2\omega) \oplus O_B(4\omega - 2p) \longrightarrow O_B(2p) \oplus O_B(2\omega) \oplus O_B(4\omega - 2p + \tau).$$

Take global coordinates

$$x_0 \in H^0(O_{\mathbb{P}(V_1)}(1) \otimes \pi_1^*O_B(-p)), \quad x_1 \in H^0(O_{\mathbb{P}(V_1)}(1) \otimes \pi_1^*O_B(-2\omega + p))$$

on the fibres of $\mathbb{P}(V_1)$ and, similarly, global coordinates $y_0, y_1, y_2$ on the fibres of $\mathbb{P}(V_2)$. With respect to these coordinates, $\sigma_2$ is given by

$$\begin{cases} 
\sigma_2(x_0^2) = a_{00}y_0 + a_{02}y_2 \\
\sigma_2(x_0x_1) = a_{11}y_1 + a_{12}y_2 \\
\sigma_2(x_1^2) = a_{20}y_0 + a_{22}y_2
\end{cases},$$

where $a_{00}, a_{11}, a_{20} \in \mathbb{C}, a_{02}, a_{22} \in H^0(O_B(\tau)), a_{12} \in H^0(O_B(2\omega - 2p))$. Therefore the equation of the conic bundle $C \subset \mathbb{P}(V_2)$ is

$$(a_{00}y_0 + a_{02}y_2)(a_{20}y_0 + a_{22}y_2) - (a_{11}y_1 + a_{12}y_2)^2 = 0.$$ 

Moreover, since the rank of $\sigma_2$ drops exactly at the point $\tau$, it follows $a_{11} \neq 0$. This means that the coefficient of the term $y_1^2$ is a non-zero constant, hence the same argument of \cite[Lemma 3.5]{Pi09} shows that exact sequence \cite{CaPi06} splits. Therefore we obtain

$$\tilde{A}_6 = O_B(2p - 2\tau) \oplus O_B(4\omega - 2p + \tau) \oplus O_B(4p - 2\omega - 2\tau) \oplus O_B(2p - \tau) \oplus O_B(4\omega - 2p) \oplus O_B(6\omega - 4p) \oplus O_B(2\omega - \tau).$$
where

\[ \tilde{H} \]

Hence

There are several possibilities.

Then we have

\[ h \]

obtain

Proof. Case (IIIb) is obtained from case (IIIa) by considering 2\( o \) instead of \( p \); this shows that the corresponding strata coincide. So it is sufficient to consider case (IIIa); set

\[ W := E_{\tau+2o-2p}, \quad L := \mathcal{O}_B(2p-2o). \]

Then we have

\[ V_2(-2o) = W \oplus L \quad \text{and tensoring the exact sequence} \]

so

\[ \tilde{A}_6 = A_6(-4o-2\tau) \]

fits into the short exact sequence

\[ 0 \rightarrow G_1 \rightarrow G_2 \rightarrow \tilde{A}_6 \rightarrow 0, \]

where

\[ G_1 = (W \oplus L)(2o-2\tau), \quad G_2 = (S^3W \oplus (S^2W \oplus L) \oplus (W \otimes L^2) \oplus L^3)(2o-2\tau). \]

There are several possibilities.

- \( L(2o-2\tau) \neq \mathcal{O}_B, \quad L^3(2o-2\tau) \neq \mathcal{O}_B. \) In this case

\[ h^0(G_1) = 1, \quad h^1(G_1) = 0, \quad h^0(G_2) = 10, \]

hence \( h^0(\tilde{A}_6) = 9. \) We have 1 parameter for \( B \), 1 parameter for \( V_1 \), 1 parameter for \( \xi \), 1 parameter for \( \tau \) and 8 parameters from \( \mathbb{P}H^0(\tilde{A}_6) \).

- \( L(2o-2\tau) \neq \mathcal{O}_B, \quad L^3(2o-2\tau) = \mathcal{O}_B. \) In this case

\[ h^0(G_1) = 1, \quad h^1(G_1) = 0, \quad h^0(G_2) = 11, \]

hence \( h^0(\tilde{A}_6) = 10. \) We have 1 parameter for \( B \), 1 parameter for \( V_1 \), 1 parameter for \( \xi \), no parameters for \( \tau \) and 9 parameters from \( \mathbb{P}H^0(\tilde{A}_6) \).

- \( L(2o-2\tau) = \mathcal{O}_B, \quad L^3(2o-2\tau) \neq \mathcal{O}_B. \) We have

\[ h^0(G_1) = 2, \quad h^1(G_1) = 1, \quad h^0(G_2) = 10, \]

hence \( h^0(\tilde{A}_6) \leq 9 \) by \( \text{(5)} \). We have 1 parameter for \( B \), 1 parameter for \( V_1 \), 1 parameter for \( \xi \), no parameters for \( \tau \) and at most 8 parameters from \( \mathbb{P}H^0(\tilde{A}_6) \).

- \( L(2o-2\tau) = \mathcal{O}_B, \quad L^3(2o-2\tau) = \mathcal{O}_B. \) Notice that this implies \( L^2 = \mathcal{O}_B \), so there are no parameters for \( V_1 \). We obtain

\[ h^0(G_1) = 2, \quad h^1(G_1) = 1, \quad h^0(G_2) = 11, \]

hence \( h^0(\tilde{A}_6) \leq 10 \) by \( \text{(5)} \). We have 1 parameter for \( B \), 1 parameter for \( \xi \), no parameters for \( \tau \) and at most 9 parameters from \( \mathbb{P}H^0(\tilde{A}_6) \).

Summing up, we conclude that the dimension of the stratum \( \mathcal{M}_{IIIa} = \mathcal{M}_{IIIb} \) is at most 12.

\[ \square \]
Proposition 3.8. The stratum $\mathcal{M}_{\text{IVC}}$ has dimension at most 12.

Proof. As in the proof of Proposition 3.5, $h^0(\tilde{A}_6) = 9$. We have 1 parameter for $B$, 1 parameters for $V_{\xi}$, 1 parameter for $\xi$, 1 parameter for $\tau$ and 8 parameters from $\mathbb{P}H^0(\tilde{A}_6)$. Therefore $\mathcal{M}_{\text{IVC}}$ has dimension at most 12.

Proposition 3.9. The strata $\mathcal{M}_{\text{IVA}}, \mathcal{M}_{\text{IVB}}$ have dimension at most 11.

Proof. The proof is the same as in case (IIc); the details are left to the reader.

Proposition 3.10. The stratum $\mathcal{M}_{\text{IVC}}$ has dimension at most 11.

Proof. In case (IVc) the vector bundles $G_1, G_2$ in exact sequence (4) are as follows:

\[
G_1 = O_B(2p - 2\tau) \oplus O_B(4p - 2p - 2\tau) \oplus O_B(2p - \tau), \]
\[
G_2 = O_B(6p - 4p - 2\tau) \oplus O_B(8p - 6p - 2\tau) \oplus O_B(2p + \tau) \oplus O_B(2p - 2\tau) \oplus O_B(4p - 2p - 2\tau) \oplus O_B(4p - 4p - 2\tau) \oplus O_B(2p - \tau).
\]

A tedious but elementary analysis of all possibilities, together with inequality (5), shows that the number of parameters involved in the construction never exceeds 11. Hence $\mathcal{M}_{\text{IVC}}$ has dimension at most 11.

Now let us write $\mathcal{M}_V = \mathcal{M}_{V, \text{gen}} \cup \mathcal{M}_{V, 2}$, where $\mathcal{M}_{V, 2}$ consists of surfaces with $O_B(2p - 2\tau) = O_B$ and $\mathcal{M}_{V, \text{gen}}$ is the rest.

Proposition 3.11. $\mathcal{M}_{V, \text{gen}}$ and $\mathcal{M}_{V, 2}$ are both non-empty.

Proof. In case (V) we have $O_B(2p - 2p) = O_B$, hence the map $\sigma_2 : S^2V_1 \rightarrow V_2$ has the form

\[
\sigma_2 : O_B(2p)^3 \longrightarrow O_B(2p)^2 \oplus O_B(2p + \tau).
\]

Recall that for a general choice of $\sigma_2$ we have $\tilde{f}_i \neq 0$ for all $i \in \{1, 2, 3\}$. Take coordinates $x_0, x_1$ on the fibres of $V_1$ and $y_0, y_1, y_2$ on $V_2$; with respect to these coordinates, $\sigma_2$ is given by

\[
\begin{align*}
\sigma_2(x^2_0) &= a_{00}y_0 + a_{01}y_1 + a_{02}f_0 y_2 \\
\sigma_2(x_0 x_1) &= a_{10}y_0 + a_{11}y_1 + a_{12}f_0 y_2 \\
\sigma_2(x^2_1) &= a_{20}y_0 + a_{21}y_1 + a_{22}f_0 y_2,
\end{align*}
\]

where $a_{ij} \in \mathbb{C}$ and $f_0 \in H^0(\mathcal{O}_B(\tau))$. Moreover, since the rank of $\sigma_2$ drops precisely at the point $\tau$, it follows $\det(a_{ij}) \neq 0$.

Therefore the global equation of the relative conic $\mathcal{C} \subset \mathbb{P}(V_2)$ is

\[
(a_{00}y_0 + a_{01}y_1 + a_{02}f_0 y_2)(a_{20}y_0 + a_{21}y_1 + a_{22}f_0 y_2) - (a_{10}y_0 + a_{11}y_1 + a_{12}f_0 y_2)^2 = 0.
\]

Notice that at least one of the coefficient of $y_0^2$, $y_1^2$ or $y_0 y_1$ in the equation of $\mathcal{C}$ is not zero, otherwise $y_2^2$ divides the equation of $\mathcal{C}$. Since each of these coefficients is a non-zero constant, by the argument in [Pf09, Lemma 3.5] one sees that in any case the exact sequence (3) splits. Therefore we obtain

\[
\tilde{A}_6 = O_B(2p - 2\tau)^2 \oplus O_B(2p - \tau)^2 \oplus O_B(2p)^2 \oplus O_B(2p + \tau),
\]

so

\[
h^0(\tilde{A}_6) = \begin{cases}
11 & \text{if } O_B(2p - 2\tau) = O_B, \\
9 & \text{otherwise}.
\end{cases}
\]

Choosing $a_{02} = a_{22} = a_{10} = a_{11} = 0$, $a_{00} = a_{01} = a_{20} = a_{12} = 1$, $a_{21} = -1$, the equation of $\mathcal{C}$ becomes

\[
y_0^2 - y_1^2 - f_0 y_2^2 = 0.
\]
Hence $C$ has a unique singular point (of type $A_1$), namely the point $P$ with homogeneous coordinates $[0 : 0 : 1]$ lying on the fibre over $\tau$; in particular, condition $(P_1)$ of Theorem 1.6 is satisfied. Since $\mathcal{G}$ splits, the curve $\Delta$ defined by the section $w \in H^0(\mathcal{A}_0)$ is cut by a relative cubic $\mathcal{G} \in |\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes \pi^* \mathcal{O}_B(-4o - 2\tau)|$; let us write the equation of $\mathcal{G}$ as
\[
\sum_{i+j+k=3} b_{ijk} y_0^i y_1^j y_2^k = 0,
\]
where $b_{ijk} \in H^0(\mathbb{P}(V_2), \pi^* \mathcal{O}_B(2o + (k - 2)\tau))$. If $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$ then all the coefficients of $\mathcal{G}$ are generically non-zero; one checks that in this case the linear system $|\mathcal{G}|$ in $\mathbb{P}(V_2)$ is base-point free, hence the linear system $|\Delta|$ in $C$ is base-point free too; by Bertini theorem, we conclude that for a general choice of $\Delta$ condition $(P_2)$ is also satisfied, hence $\mathcal{M}_{V,2}$ is non-empty.

If $\mathcal{O}_B(2o - 2\tau) \neq \mathcal{O}_B$, then $b_{300} = b_{210} = b_{120} = b_{030} = 0$. So the relative cubic $\mathcal{G}$ splits as $\mathcal{G} = \mathcal{H} \cup \mathcal{G}'$, where $\mathcal{H}$ is the relative hyperplane $\{y_2 = 0\}$ and $\mathcal{G}'$ is the relative conic
\[
b_{201} y_0^2 + b_{111} y_0 y_1 + b_{102} y_0 y_2 + b_{021} y_1^2 + b_{012} y_1 y_2 + b_{003} y_2^2 = 0.
\]
Consequently, $\Delta$ splits as $\Delta = \mathcal{H}_C \cup \Delta'$, where $\mathcal{H}_C = \mathcal{H} \cap C$ and $\Delta' = \mathcal{G}' \cap C$. The sections $b_{201}, b_{102}, b_{111}$ all vanish at the same point, namely the unique point $q \in B$ such that $\mathcal{O}_B(2o - \tau) = \mathcal{O}_B(q)$; notice that $q \neq \tau$. Hence the base locus of $|\mathcal{G}'|$ is the line $y_2 = 0$ in the fibre $\pi^{-1}(q)$, and this in turn implies that the base locus of $|\Delta'|$ in $C$ are the two points $P_1 = [1 : 1 : 0]$ and $P_2 = [1 : -1 : 0]$ on the fibre of $C$ over $q$. Now let us make a general choice of the coefficients in (23). Then $\Delta$ does not contain the unique singular point of $C$; moreover, a standard local computation together with Bertini theorem show that

- $\Delta'$ is smooth;
- $\Delta'$ and $\mathcal{H}_C$ intersect transversally at $P_1$ and $P_2$.

So condition $(P_2)$ is satisfied and $\mathcal{M}_{V,\text{gen}}$ is non-empty.

Let us compute now the dimensions of $\mathcal{M}_{V,2}$ and $\mathcal{M}_{V,\text{gen}}$.

**Proposition 3.12.** $\mathcal{M}_{V,2}$ has dimension 12, whereas $\mathcal{M}_{V,\text{gen}}$ has dimension 11. Moreover, $\mathcal{M}_{V,2}$ is a generically smooth, irreducible component of $\mathcal{M}$.

**Proof.** We first compute the dimension of the parameter space $\mathcal{D}$ in each case. If $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$, we have 1 parameter for $B$, 2 parameters for $\xi$ and 10 parameters from $\mathbb{P}H^0(\mathcal{A}_0)$; otherwise we have 1 parameter for $B$, 2 parameters for $\xi$, 1 parameter from $\tau$ and 8 parameters from $\mathbb{P}H^0(\mathcal{A}_0)$. Therefore $\mathcal{M}_{V,2}$ has dimension at most 13, whereas $\mathcal{M}_{V,\text{gen}}$ has dimension at most 12.

By Remark 3.2 we have now to find the dimension of the general fibre of $\Phi: \mathcal{D} \to \mathcal{M}$, and for this we have to consider the action of certain automorphism groups over our data.

Observe first that in both cases we can forget the action of $\text{Aut}(B)$, since we have fixed a point of $B$ by choosing $\det V_1 = \mathcal{O}_B(2o)$. So we have only to consider the action of $\text{Aut}(V_1) \times \text{Aut}(V_2)$.

We are therefore reduced to solve the following problem: given an admissible 5-tuple $(B, V_1, t, \xi, w)$, corresponding to the genus 2 fibration $f: S \to B$, we must find the dimension of the subvariety $Z \subset \text{Aut}(V_1) \times \text{Aut}(V_2)$ given by the pairs $(\phi_1, \phi_2)$ which make the following diagram commuting:
\[
\begin{array}{c}
0 \rightarrow S^2 V_1 \xrightarrow{\sigma_2} V_2 \xrightarrow{\sigma_2} \mathcal{O}_T \rightarrow 0 \\
\xrightarrow{S^2 \phi_1} \xrightarrow{\phi_2} \xrightarrow{\phi_2} \xrightarrow{0}
\end{array}
\]

In fact, the dimension of the of the fibre $\Phi^{-1}([S])$ is given by $\dim Z - 1$. Geometrically, this expresses the fact that the points in such a fibre are in 1-to-1 correspondence with the family of automorphisms of the projective bundle $\mathbb{P}(V_2)$ fixing the conic bundle $C$. 

18
In fact, in the general case we can assume that such a fibre has equation $y_0(y_0 + f_0y_2) - y_1^2 = 0$. (25)

In fact, in the general case $C$ has a nodal fibre over the point $\tau$; without loss of generality we can assume that such a fibre has equation $y_1(y_0 - y_1) = 0$, so that the conic bundle has the form $(y_0 + a_0f_0y_2)(y_1 + a_{12}f_0y_2) - (y_1 + a_{12}f_0y_2)^2 = 0$. Now the claim follows by using the linear change of coordinates

$$y'_0 := y_0 + a_0f_0y_2, \quad y'_1 := y_1 + a_{12}f_0y_2, \quad y'_2 := (a_{22} - a_{12})y_2.$$

Therefore, in order to compute the dimension of the general fibre of $\Phi$, we may assume that the matrix associated with $\sigma_2 : S^2V_1 \to V_2$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & f_0 \end{pmatrix}.$$ 

Let now $\phi_1 \in \text{Aut}(V_1)$, given by $\phi_1(x_0) = ax_0 + cx_1$ and $\phi_1(x_1) = bx_0 + dx_1$, $a, b, c, d \in \mathbb{C}$. Then the action of $S^2\phi_1$ on $S^2V_1$ is expressed by the matrix

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$ 

On the other hand, the general $\phi_2 \in \text{Aut}(V_2)$ is given by

$$\begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31}f_0 & b_{32}f_0 & b_{33} \end{pmatrix},$$

where $b_{ij} \in \mathbb{C}$. Hence, imposing that the diagram (21) commutes, by straightforward computations one finds that any pair $(\phi_1, \phi_2) \in Z$ is either of the form

$$\phi_1 = \begin{pmatrix} a & a \\ c & -a \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} a^2 & 0 \\ 2ac + c^2 & -a^2 \\ c^2f_0 & acf_0 & a(a + c) \end{pmatrix};$$

or of the form

$$\phi_1 = \begin{pmatrix} a & 0 \\ c & a + c \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} a^2 & 0 \\ 2ac + c^2 & (a + c)^2 \\ c^2f_0 & (a + c)f_0 & a(a + c + 1) \end{pmatrix}.$$ 

It follows that $Z \subset \text{Aut}(V_1) \times \text{Aut}(V_2)$ is a subvariety of dimension 2. Consequently, the general fibre of $\Phi$ has dimension 1; this means that the dimension of $\mathcal{M}_{V,2}$ equals 12, whereas the dimension of $\mathcal{M}_{V,\text{gen}}$ equals 11.

Now we want to prove that $\mathcal{M}_{V,2}$ is an irreducible component of $\mathcal{M}$. In order to do this, we will show that $h^1(S, T_S) = 12$ for a general $S \in \mathcal{M}_{V,2}$. Since dim $\mathcal{M}_{V,2} = 12$, this will also prove that this component is generically smooth.

The condition $h^1(S, T_S) \leq 12$ is equivalent to $h^2(S, T_S) = h^0(S, \Omega_S^1 \otimes \omega_S) \leq 2$. By Remark 5.2 it is therefore enough to prove that $h^0(F) = 0$, where $F := (\Omega_S^1 \otimes \omega_S)/\omega_S$ or, equivalently, that there are no bicanonical curves of $S$ containing the 0-dimensional scheme $\text{Crit}(f)$.

By the results in Subsection 1.2.2, the Albanese fibration $f : S \to B$ factors as the composition of the conic bundle $C \to B$ and a finite double cover $\psi : S \to C$ branched on the node of $C$ and on a smooth curve $\Delta$ not passing through the node.
Let us study the 0-dimensional scheme $\text{Crit}(f)$. Since all the fibres of $C$ are reduced, the critical points of $f$ must be fixed by the involution of $S$. The isolated fixed point is the preimage of the node of $C$, and it is critical for $f$. The other critical points of $f$ are the points of $S$ whose images in $C$ are the ramification points for the map $\Delta \to B$.

As before, we can choose $C$ of equation $y_i^3 - y_i^2 + f_{0i} y_i^2 = 0$, and the curve $\Delta$ is defined as the complete intersection of $C$ with a relative cubic $G \in |O_C(3) \otimes O_B(-4a - 2\tau)|$. Since $O_B(2\alpha - 2\tau) = O_B$, we can choose $G$ of equation

$$ay_i^3 + by_i^2 + \lambda y_i^2 = 0,$$

where $a, b \in \mathbb{C}$ and $\lambda \in H^0(\mathbb{P}(V_2), \pi_* O_B(3\tau))$, see (23). The node $P$ of $C$ is the point with homogeneous coordinates $[0 : 0 : 1]$ lying on the fibre over $\tau$, and $\text{Crit}(\Delta \to \mathbb{P})$ is defined by

$$\text{rank} \left( \begin{array}{cc} y_0 & -y_1 \\ ay_0^2 & by_1^2 \end{array} \right) f_0^2 y_2 \lambda y_2^2 \leq 1.$$ 

This is obviously equivalent to set equal 0 all the minors of order 2. So we must solve the system of equations

$$\begin{cases} bgy_0^2 + ay_0^2 y_1 = 0 \\ \lambda y_0^2 - a f_0^2 y_2 y_1 = 0 \\ \lambda y_1 y_2^2 + by_1^2 y_2 = 0, \end{cases}$$

that is

$$\begin{cases} y_0 y_1 (by_1 + ay_0) = 0 \\ y_0 y_2 (\lambda y_2 - a f_0^2 y_0) = 0 \\ y_1 y_2 (\lambda y_2 + b f_0^2 y_1) = 0. \end{cases}$$

This yields

$$\begin{cases} y_0 = y_1 = 0 \} \cup \{ y_0 = y_2 = 0 \} \cup \{ y_0 = \lambda y_2 + b f_0^2 y_1 = 0 \} \cup \\
\{ y_1 = y_2 = 0 \} \cup \{ y_1 = \lambda y_2 - a f_0^2 y_0 = 0 \} \cup \\
\{ y_2 = by_1 + ay_0 = 0 \} \cup \{ \lambda y_2 + b f_0^2 y_1 = \lambda y_2 - a f_0^2 y_0 = by_1 + ay_0 = 0 \}. \end{cases}$$

Let us compute, in each case, the solutions in $C$:

- $\{ y_0 = y_1 = 0 \}$ In this case, because $f_0(\tau) = 0$, the unique solution in $C$ is the point $P$.
- $\{ y_0 = y_2 = 0 \}$ By looking at the equation of $C$ we have also that $y_1 = 0$, and this is impossible. So in this case there are no solutions.

- $\{ y_0 = \lambda y_2 + b f_0^2 y_1 = 0 \}$ We must solve

$$\begin{cases} y_0 = \lambda y_2 + b f_0^2 y_1 = 0 \\ y_0^2 - y_1^2 + f_0^2 y_2 = 0, \end{cases}$$

that is

$$\begin{cases} y_0 = 0 \\ \lambda y_2 + b f_0^2 y_1 = 0 \\ (-y_1 + f_0 y_2)(y_1 + f_0 y_2) = 0, \end{cases}$$

which gives

$$\begin{cases} y_0 = 0 \\ y_1 = f_0 y_2 \\ y_2(\lambda + b f_0^2) = 0 \} \cup \\
\{ y_0 = 0 \\ y_1 = -f_0 y_2 \\ y_2(\lambda - b f_0^2) = 0. \end{cases}$$

Since $y_2 \neq 0$ the solutions are the three points $[0 : f_0(\rho_1) : 1]$ lying on the fibres over $\rho_1$, where $\rho_1 + \rho_2 + \rho_3 = \text{div}(\lambda + b f_0^2)$, and the three points $[0 : -f_0(\rho_1') : 1]$ lying on the fibres over $\rho_1'$, where $\rho_1' + \rho_2 + \rho_3 = \text{div}(\lambda - b f_0^2)$.

- $\{ y_1 = y_2 = 0 \}$ The equation of $C$ also gives $y_0 = 0$, which is impossible; so in this case there are no solutions.

- $\{ y_1 = \lambda y_2 - a f_0^2 y_0 = 0 \}$ The computations are the same as in the case $\{ y_0 = \lambda y_2 + b f_0^2 y_1 = 0 \}$. The solutions are the three points $[-\sqrt{-1} f_0(\varepsilon_1) : 0 : 1]$ lying on the fibres over $\varepsilon_1$, where $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \text{div}(\lambda + a\sqrt{-1} f_0^2)$, and the three points $[\sqrt{-1} f_0(\varepsilon_1) : 0 : 1]$ lying on the fibres over $\varepsilon_1'$, where $\varepsilon_1' + \varepsilon_2' + \varepsilon_3' = \text{div}(\lambda - a\sqrt{-1} f_0^2)$.

- $\{ y_2 = by_1 + ay_0 = 0 \}$ From the equation of $C$ it follows that for a generic choice of $a$ and $b$ we must have $y_0 = y_1 = y_2 = 0$, which is impossible. So in this case there are no solutions.
\[ \lambda y_2 + bf_0^2 y_1 = \lambda y_2 - af_0^2 y_0 = by_1 + ay_0 = 0 \] In this case we find six points, three on the curve
\[
\begin{align*}
  b y_1 + a y_0 &= 0 \\
  c y_0 + b f_0 y_2 &= 0
\end{align*}
\]
and three on the curve
\[
\begin{align*}
  b y_1 + a y_0 &= 0 \\
  c y_0 - b f_0 y_2 &= 0,
\end{align*}
\]
where \(-c^2 = b^2 - a^2\). In general, \(a, b\) and \(c\) are nonzero and, in such a case, the solutions are the three points \([-bf_0(\sigma_i) : af_0(\sigma_i) : c] \) lying on the fibres over \(\sigma_i\), where \(\sigma_1 + \sigma_2 + \sigma_3 = \text{div}(c\lambda + af_0^2)\) and the three points \([bf_0(\sigma'_i) : -af_0(\sigma'_i) : c] \) lying on the fibres over \(\sigma'_i\), where \(\sigma'_1 + \sigma'_2 + \sigma'_3 = \text{div}(c\lambda - af_0^2)\).

Summing up, for a general \(S \in \mathcal{M}_{V,2}\) the 0-dimensional scheme \(\text{Crit}(f)\) consists precisely of 19 distinct points. One is the preimage \(Q := \psi^{-1}(P)\) of \(P\) in \(S\), and the others correspond to the singularities of eighteen 2-connected nodal curves, as in the following picture:

\[
\begin{array}{c}
\text{Sing}(C) \\
\text{Crit}(\Delta \rightarrow B) \\
\tau \\
B
\end{array}
\]

Notice that this agrees with the Zeuthen–Segre formula
\[
19 = \chi_{\text{top}}(S) = \chi_{\text{top}}(B)\chi_{\text{top}}(F) + \sum \chi_{\text{top}}(F_p) - \chi_{\text{top}}(F) = \sum \chi_{\text{top}}(F_p) - \chi_{\text{top}}(F),
\]
where the sum runs over the singular fibres of \(f\). Thus for a general \(S \in \mathcal{M}_{V,2}\), the Albanese map has exactly 19 singular fibres.

Since the linear system \(|2K_S|\) is the pullback via the relative bicanonical map of the linear system \(|\mathcal{O}_{\mathbb{P}(V^2)}(1)|\), we must now compute the dimension of the vector space of elements in \(H^0(\mathcal{O}_{\mathbb{P}(V^2)}(1))\) which contain \(\text{Crit}(f)\).

Let us consider the six curves
\[
\begin{align*}
A_1: \quad &\begin{cases} 
  y_0 = 0 \\
  y_1 - f_0 y_2 = 0
\end{cases}, & A_2: \quad &\begin{cases} 
  y_0 = 0 \\
  y_1 + f_0 y_2 = 0
\end{cases}, \\
B_1: \quad &\begin{cases} 
  y_1 = 0 \\
  y_0 - \sqrt{-f_0} y_2 = 0
\end{cases}, & B_2: \quad &\begin{cases} 
  y_0 = 0 \\
  y_0 + \sqrt{-f_0} y_2 = 0
\end{cases}, \\
C_1: \quad &\begin{cases} 
  by_1 + ay_0 = 0 \\
  cy_0 + bf_0 y_2 = 0
\end{cases}, & C_2: \quad &\begin{cases} 
  by_1 + ay_0 = 0 \\
  cy_0 - bf_0 y_2 = 0
\end{cases}
\end{align*}
\]

Each curve contains \(Q\) and three other points of \(\text{Crit}(f)\) as in the following picture:
The Néron–Severi group $\text{NS}(\mathbb{P}(V_2))$ is generated by $H$ and $\Psi$, where $H$ is the class of $\mathcal{O}_{\mathbb{P}(V_2)}(1)$ and $\Psi$ is the class of a fibre.

Let $Y$ be an element of $|\mathcal{O}_{\mathbb{P}(V_2)}(1)|$ containing $\text{Crit}(f)$. Thus $Y$ contains 4 points in each curve $A_j, B_j, C_j, j = 1, 2$. Since the numerical class of these curves is $(H - 2\Psi)^2$, we have

$$H(H - 2\Psi)^2 = H(H^2 - 2H\Psi) = H^3 - 4H\Psi = 7 - 4 = 3$$

and so, by Bézout theorem, $Y$ contains all the curves $A_j, B_j, C_j$. Let us write the equation of $Y$ as $\alpha_0 + \beta y_1 + \gamma y_2 = 0$, where $\alpha, \beta \in H_0(\pi^* 2\mathcal{O}_{\mathcal{B}}(2\omega))$ and $\gamma \in H_0(\pi^* 2\mathcal{O}_{\mathcal{B}}(2\omega + \tau))$.

By imposing that $Y$ contains $A_1$, we find

$$\beta f_0 y_2 + \gamma y_2 \equiv 0,$$

which implies $\gamma = -\beta f_0$. By imposing that $Y$ contains $A_2$, we find

$$-\beta f_0 y_2 + \gamma y_2 \equiv 0,$$

which implies $\gamma = \beta f_0$. It follows $\gamma = \beta = 0$, hence $Y$ has equation $\alpha_0 y_2 = 0$. Similarly, by imposing that $Y$ contains both $B_1$ and $B_2$, we obtain that $Y$ is of the form $2\beta y_1 = 0$. Thus $Y \equiv 0$, i.e.

$$\text{Ker}[H^0(\omega_S^2) \longrightarrow H^0(\mathcal{O}_{\text{Crit}(f)}(\omega_S^2))] = 0,$$

which implies $h^1(T_S) = 12$. This shows that $\mathcal{M}_{V, 2}$ is a generically smooth, irreducible component of $\mathcal{M}$ of dimension 12.

Finally, we consider the strata belonging to $\mathcal{M}''$. The surfaces belonging to these strata satisfy $V_1 = F_2(\eta)$, where $\eta$ is a 2–torsion point, hence $V_1$ will not play any role in the computation of parameters.

**Proposition 3.13.** The stratum $\mathcal{M}_{V_1}$ has dimension at most 12.

**Proof.** Set $W := E_2(3, 1)$; then we have a short exact sequence

$$0 \longrightarrow W(2\omega - 2\tau) \longrightarrow S^3 W(2\omega - 2\tau) \longrightarrow \tilde{A}_0 \longrightarrow 0.$$

By [CaC93, Section 1] we obtain

$$h^0(W(2\omega - 2\tau)) = 1, \quad h^1(W(2\omega - 2\tau)) = 0, \quad h^0(S^3 W(2\omega - 2\tau)) = 10,$$

hence $h^0(\tilde{A}_0) = 9$. We have 1 parameter for $B$, 2 parameters for $\xi$, 1 parameter for $\tau$ and 8 parameters from $\mathbb{P} H^0(\tilde{A}_0)$. Therefore either $\mathcal{M}_{V_1}$ has dimension at most 12.

**Proposition 3.14.** The stratum $\mathcal{M}_{V_{1a}}$ has dimension at most 11.
Proof. In this case \( V_2(-2\alpha) = F_2 \oplus \mathcal{O}_B(\tau) \), and \( \xi \) belongs to a family which is at most 1-dimensional, see Proposition 2.6. The vector bundle \( \tilde{A}_6 \) fits into a short exact sequence

\[
0 \rightarrow G_1 \rightarrow G_2 \rightarrow \tilde{A}_6 \rightarrow 0,
\]

where

\[
G_1 = (F_2 \oplus \mathcal{O}_B(\tau))(2\alpha - 2\tau), \quad G_2 = (F_4 \oplus F_3(\tau) \oplus F_2(2\tau) \oplus \mathcal{O}_B(3\tau))(2\alpha - 2\tau).
\]

We distinguish two cases.

(i) \( \mathcal{O}_B(2\alpha - 2\tau) \neq \mathcal{O}_B \). We obtain

\[
h^0(G_1) = 1, \quad h^1(G_1) = 0, \quad h^0(G_2) = 10,
\]

therefore \( h^0(\tilde{A}_6) = 9 \). We have 1 parameter for \( B \), at most one parameter for \( \xi \), one parameter for \( \tau \) and 8 parameters from \( \mathbb{P}H^0(\tilde{A}_6) \).

(ii) \( \mathcal{O}_B(2\alpha - 2\tau) = \mathcal{O}_B \). We obtain

\[
h^0(G_1) = 2, \quad h^1(G_1) = 1, \quad h^0(G_2) = 11,
\]

hence \( h^0(\tilde{A}_6) \leq 10 \), see \( [5] \). We have 1 parameter for \( B \), at most one parameter for \( \xi \), no parameters for \( \tau \) and at most 9 parameters from \( \mathbb{P}H^0(\tilde{A}_6) \).

It follows that \( \mathcal{M}_{\text{VIIa}} \) has dimension at most 11.

**Proposition 3.15.** The stratum \( \mathcal{M}_{\text{VIIb}} \) has dimension at most 11.

**Proof.** In this case \( \xi \) belongs to a family which is at most 1-dimensional. Set \( W = E_1(2, 1) \); then \( V_2(-2\alpha) = W \oplus \mathcal{O}_B \) and tensoring the exact sequence \( \tau_{\mathcal{O}_B} \) with \( \mathcal{O}_B(-4\alpha - 2\tau) \) we obtain

\[
0 \rightarrow (W \oplus \mathcal{O}_B)(2\alpha - 2\tau) \rightarrow (\mathcal{O} \oplus [S^3 W \oplus S^2 W] \oplus (W \oplus \mathcal{O}_B))(2\alpha - 2\tau) \rightarrow \tilde{A}_6 \rightarrow 0. \tag{26}
\]

Arguing as in \([CaPi06] \) Lemma 6.14, we see that the second component of the map \( i_3 \) is the identity, hence the exact sequence \( \tau_{\mathcal{O}_B} \) splits, giving

\[
\tilde{A}_6 = (S^3 W \oplus S^2 W)(2\alpha - 2\tau).
\]

By Proposition 1.4 this in turn implies

\[
\tilde{A}_6 = (W \oplus W \oplus \bigoplus_{i=1}^3 L_i)(2\alpha - \tau),
\]

hence \( h^0(\tilde{A}_6) = 9 \). We have 1 parameter for \( B \), at most 1 parameter for \( \xi \), 1 parameter for \( \tau \) and 8 parameters from \( \mathbb{P}H^0(\tilde{A}_6) \). Therefore \( \mathcal{M}_{\text{VIIb}} \) has dimension at most 11.

Summing up, we have the following

**Corollary 3.16.** The moduli space \( \mathcal{M} \) of minimal surfaces of general type with \( p_g = 2 \), \( q = 1 \) and \( K^2 = 5 \) is unirational and contains at least 2 irreducible components. Moreover, the dimension of each irreducible component is either 12 or 13, and there is at most one component of dimension 13.

**Proof.** Notice first that \( \mathcal{M}_{V, \text{gen}} \) is not contained in the closure of \( \mathcal{M}_{V, 2} \), since in the former case \( \tau \) is a general point, whereas in the latter \( \tau \) is a 2-torsion point. So \( \mathcal{M} \) contains at least two irreducible components, namely \( \mathcal{M}_{V, 2} \) and the component containing \( \mathcal{M}_{V, \text{gen}} \). Moreover there is at most one component of dimension 13, namely \( \mathcal{M}_1 \).

It would be desirable to exactly describe all irreducible components of \( \mathcal{M} \) and to understand how their closures intersect, but we will not try to develop this point here.
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