Entanglement of Collectively Interacting Harmonic Chains: An effective Two-Dimensional System

R.G. Unanyan1,2, M. Fleischhauer2 and D. Bruß1

1 Institut für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany
2 Fachbereich Physik, Technische Universität Kaiserslautern, 67663, Kaiserslautern, Germany

We study the ground-state entanglement of one-dimensional harmonic chains that are coupled to each other by a collective interaction as realized e.g. in an anisotropic ion crystal. Due to the collective type of coupling, where each chain interacts with every other one in the same way, the total system shows critical behavior in the direction orthogonal to the chains while the isolated harmonic chains can be gapped and non-critical. We derive lower and most importantly upper bounds for the entanglement, quantified by the von Neumann entropy, between a compact block of oscillators and its environment. For sufficiently large size of the subsystems the bounds coincide and show that the area law for entanglement is violated by a logarithmic correction.

Presently there is a growing interest in the interrelation between entanglement and ground-state properties of many-body lattice models. For a number of spin systems a strict correspondence between the absence of criticality, the presence of an energy gap, and an area law for the entanglement was established. The latter states that the entanglement of a compact sub-set of lattice sites with the rest of the system, measured by the von Neumann entropy, scales with the surface area of the sub-set. For critical spin systems it was shown that an additional logarithmic correction to the area law emerges.

A similar relation between criticality and entanglement was suggested for harmonic lattice models. In an area law was established for harmonic lattice models in arbitrary dimensions with nearest-neighbor coupling which have a gaped spectrum. For finite-range couplings in one dimension a one-to-one correspondence between the validity of the area law and non-criticality was established, and logarithmic corrections were derived for critical systems.

Although the relation between criticality and entanglement area law seems rather universal, there are a number of examples where this relation does not hold. Until now there is no general understanding of the conditions for the validity of an entropy area law in particular in higher dimensions. In the present paper we discuss a specific gapless oscillator model with dimension larger than one, for which an exact asymptotic expression for the entropy can be obtained. Due to the collective nature of the interactions in one spatial direction the system is critical and thus a violation of the area law is expected. We here derive a lower and, most importantly, an upper bound for the entropy and in this way obtain an exact form of the correction term to the area law.

Let us consider a set of parallel harmonic chains (see Fig.1) each containing \( n_x \) oscillators, with \( n_x \to \infty \) in the thermodynamic limit. We will refer to the direction parallel to the chains as \( x \)-axis, and to the orthogonal direction as \( y \)-axis. The number of parallel chains is denoted as \( n_y \), again with \( n_y \to \infty \) in the thermodynamic limit. The oscillators are described by the canonical variables \((q_i,p_i)\), where \( i = 1,2,...,N (N = n_x n_y) \) is a collective index that labels the oscillator. We adopt the following notation: \( i = 1,...,n_x \) correspond to the oscillators in the first chain with growing \( x \) coordinate, \( i = n_x + 1,...,2n_x \) corresponds to oscillators in the second chain and so on.

We consider a quadratic Hamiltonian of the form

\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i,j=1}^{N} V_{ij} q_i q_j, \tag{1}
\]

with a coupling matrix \( V \). We are interested only in a translationally invariant coupling, i.e. we assume that the matrix elements of \( V \) depend only on the difference of the \( x \) coordinates and the difference of the \( y \) coordinates. Hence \( V \) is a block Toeplitz matrix. For oscillator systems with a quadratic coupling of the form of eq. the ground state

\[
\Psi_0 (\mathbf{q}) \sim \exp \left( -\frac{1}{2} \frac{1}{|\mathbf{q}|} \right) \tag{2}
\]

and all its properties, as e.g. the correlation length in position or momentum space, are determined by the square root of \( V \), where \( \mathbf{q} = (q_1,q_2,...,q_N) \) is the vector of position variables. The ground state can easily be determined if \( V \) is the square of another matrix, which we assume to be again a Toeplitz matrix,

\[
V = Z^2/n_y. \tag{3}
\]

The factor \( 1/n_y \) is chosen such that the matrix elements of \( V \) remain finite in the limit \( N \to \infty \). Assuming \( Z \) to be a Toeplitz matrix guarantees that the coupling \( V \) is a Toeplitz matrix as well. We furthermore consider \( Z \) to be of the block-matrix form

\[
Z = \begin{bmatrix}
\Lambda & Q & Q & \cdots & Q \\
Q & \Lambda & Q & \cdots & Q \\
Q & Q & \Lambda & \cdots & Q \\
\cdots & \cdots & \cdots & \cdots & Q \\
Q & Q & Q & \cdots & \Lambda
\end{bmatrix}. \tag{4}
\]

The elements of \( Z \) are \( n_x \times n_x \) matrices and characterize according to eq. correlations. The diagonal elements...
of $Z$ describe correlations within one chain, i.e. in $x$ direction, the off-diagonal elements describe correlations between the chains. $A$ and $Q$ are both assumed to be Toeplitz matrices of finite range, i.e. their matrix elements $\Lambda_k$ and $Q_k$, where $\Lambda_k \equiv \Lambda_{|k-j|} = \langle j|\Lambda|j \rangle$, vanish exactly for $k \geq R$. The finite range of $\Lambda$ and $Q$ ensures that the interaction $V$ is of finite range within the chains, while the form of $Z$ implies that $V$ is constant orthogonal to the chains. We assume furthermore that $\Lambda$, $Q$ and $\Lambda - Q$ are positive definite matrices. A simple calculation shows that the ground state of $N$ harmonic oscillators (see Fig.1) is gapless. It should be noted however that the collective nature of the interactions is not sufficient for a gapless spectrum of the Hamiltonian.

Since all off-diagonal elements of $Z$ are identical, correlations between oscillators do not depend on their distance in $y$ direction and the total system is critical irrespective of the correlation properties within the chains. Thus one expects that the entropy area law is broken. In fact one can easily find a lower bound to the entropy by the following simple argument: Let us consider a partition of the set of $N$ oscillators into a compact sub-system I with $N_0 = l_x l_y$ and a sub-system II with $N - N_0$ oscillators (see Fig.1). If we now consider harmonic chains in $y$– rather than in $x$– direction, the “$y$”-chains couple to each other with finite-range interaction $\Lambda$ (see Fig.1b). We thus have reason to assume that $S \geq l_y l_y$, where $S_0$ is the entropy of a single “$y$”-chain. Since the coupling within the chain is now collective ($Q$), the “$y$”-chain itself is critical and its entropy scales as $S_0 \sim \ln l_y$. Thus $S \geq l_y \ln l_y$, which in the thermodynamic limit $\{l_x, l_y\} \to \infty$ is much larger than surface area $2(l_x + l_y)$. While it is easy to see that the area law is broken, it is non-trivial to find an upper bound to the entropy and the exact form of the correction term. This will be done in the following.

Using the spectral representation of $V$, the correlation matrices $V^{1/2}$ and $V^{-1/2}$ can be decomposed as

$$V^{1/2} = \left[ (\Lambda - Q) \otimes 1_y + n_y Q \otimes P_{n_y,n_y} \right] / \sqrt{n_y},$$

and

$$V^{-1/2} = \left\{ (\Lambda - Q)^{-1} \otimes 1_y + \right.$$}

$$+ \left[ (\Lambda - Q + n_y Q)^{-1} - (\Lambda - Q)^{-1} \right] \otimes P_{n_y,n_y} \left\} \sqrt{n_y},$$

where $1_y$ is the unity matrix of size $n_y \times n_y$ and $P_{nm} = |P_{nm}|/|\rho_{nm}|$ is the projector onto the (in general non-normalized) vector

$$|P_{nm}| = \frac{1}{\sqrt{n}} (1,1,\ldots,1)^T.$$

Following Refs. [2, 3, 4, 9], the von-Neumann entropy or the entropy of entanglement of the two compact parts I and II can be calculated from a decomposition of $V^{1/2}$ into the two subsystems. To this we express $V^{1/2}$ and $V^{-1/2}$ in a block form according to the two sub-systems by proper reordering of rows and columns

$$V^{1/2} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad V^{-1/2} = \begin{bmatrix} D & E \\ E^T & F \end{bmatrix}.$$ (7)

Here $A$ and $D$ are $N_0 \times N_0$ matrices describing correlations within sub-system I, $C$ and $F$ are $(N - N_0) \times (N - N_0)$ matrices describing correlations within sub-system II, and the matrices $B$ and $E$ describe the correlations between them. The entropy of entanglement is then given by the eigenvalues $\mu_i \geq 1$ of the matrix product $A \cdot D$ [2]:

$$S = \sum_{i=1}^{N_0} f(\sqrt{\mu_i}),$$

$$f(x) = \frac{x+1}{2} \ln \frac{x+1}{2} - \frac{x-1}{2} \ln \frac{x-1}{2}.$$ (9)

Despite the simplicity of its form, expression [8] cannot be explicitly evaluated in general. Due to the special interaction matrix the eigenvalues can however be evaluated in the thermodynamic limit:

From the spectral decomposition of $V^{1/2}$, eq.[5], one easily finds that the subsystem matrices read

$$A = [A_0 \otimes 1_y + (A_1 - A_0) \otimes P_{n_y,l_y}] \sqrt{n_y},$$

$$D = [D_0 \otimes 1_y + n_y D_1 \otimes P_{n_y,l_y}] / \sqrt{n_y},$$

where $A_0, A_1$ and $D_0, D_1$ are $l_y \times l_y$ principal submatrices of $(\Lambda - Q)^{-1}, (\Lambda - Q + n_y Q)^{-1}$, and $(\Lambda - Q), Q$ respectively. For large $n_y$ one has

$$A \cdot D \approx (A_0 \cdot D_0) \otimes 1_y + n_y (A_0 \cdot D_1) \otimes P_{n_y,l_y},$$

Here we have used that $P_{n_y,l_y}^2 = l_y/n_y P_{n_y,l_y}$, which scales as $1/n_y$ for fixed $l_y$ and is thus negligible in the thermodynamic limit. Furthermore $P_{n_y,l_y}$ has one nonzero
eigenvalue $l_y/n_y$, which vanishes in the thermodynamic limit ($l_y$ fixed and $n_y \to \infty$), and ($l_y - 1$) zero eigenvalues. Thus the $l_x/l_y$ eigenvalues of $A \cdot D$ can be decomposed into two sets. The first set consists of the $l_x$ eigenvalues of $A_0 \cdot D_0$ each of which occurs ($l_y - 1$)

Here and in the following $\alpha_k(X)$ denotes the $k$th eigenvalues of the matrix $X$. The total number of these eigenvalues is $l_x(l_y - 1)$. The second set consists of the $l_x$ eigenvalues of ($A_0 \cdot D_0 + l_y (A_0 \cdot D_1)$)

Expression (13) for the second set of eigenvalues can be simplified using Lidskii’s theorem [10] which states: Let $X$ and $Y$ be $M$-dimensional Hermitian matrices. Moreover let $\alpha_k(X)$, $\alpha_k(Y)$ and $\alpha_k(X - Y)$, $k = 1, ..., M$ be the eigenvalues of $X$, $Y$ and $X - Y$ respectively in ascending order $\{\alpha_1(X) < \alpha_2(X) < ... < \alpha_M(X)\}$. Then there exist numbers $w_{kj} \geq 0$, $(k, j = 1, ..., M)$, such that 

Equation (14) implies that for sufficiently large $l_y$ the eigenvalues of the matrix $A_0 \cdot D_0 + l_y (A_0 \cdot D_1)$ are

An upper bound to the entropy can be found by evaluating the sum over the eigenvalues (12) and (13) in eq. (15) separately

Taking into account eq. (12) one recognizes that $S_1$ is apart from a prefactor ($l_y - 1$) formally equivalent to the von-Neumann entropy of a linear oscillator chain of length $l_x$ with interaction $V = (\Lambda - Q)^2$

Since $\Lambda - Q$ was assumed to be strictly positive, the interaction $V$ has only nonzero eigenvalues and thus corresponds to a gapped oscillator chain. As shown in [4,8] the entropy of such a linear chain saturates in the thermodynamic limit, i.e. it becomes independent on the length $l_x$ of the chain. Thus we have in the thermodynamic limit

To obtain an upper bound to $S_2$ we use the inequality $F(x) < 1 - \ln 2 + \ln x$. This yields with eq. (15)

To further evaluate the last term we make use of the convexity of the logarithm together with the arithmetic mean inequality

where we have used $\sum_k w_{kj} = 1$ in the last step.

We now have to evaluate the remaining logarithm. For this we make use of the fact that $\Lambda$ and $Q$ are regular (i.e. strictly positive) Toeplitz matrices. Because of this, their elements can be obtained from the non-negative spectral functions $\lambda(\theta)$ and $q(\theta)$ [11] $\Lambda_k = \frac{1}{2\pi} \int_0^{2\pi} \lambda(\theta) \exp[-ik\theta] \, d\theta$, 

$Q_k = \frac{1}{2\pi} \int_0^{2\pi} q(\theta) \exp[-ik\theta] \, d\theta$. Since we have assumed above that also $\Lambda - Q$ is strictly positive, the functions $\lambda(\theta)$, $q(\theta)$ are strictly positive and $\lambda(\theta) > q(\theta)$. In addition, we require also that $(\lambda(\theta) - q(\theta))^{\pm 1}$ and $q(\theta)$ have bounded derivatives of second order. As a consequence one finds (see [11] page 221)

which is a constant independent on $l_x$. Thus the desired upper bound to the entropy for sufficiently large $l_x, l_y$ is:

where $c_1, c_2$ are some constants independent of the size of the subsystem.

A lower bound to the entropy can be found from the
inequality $F(x) \geq \ln x$. This yields, with eq. (15),

$$S \geq \frac{(l_y - 1)}{2} \sum_{k=1}^{l_x} \ln \left[ \alpha_k (A_0 \cdot D_0) \right]$$

$$+ \frac{t_x}{2} \ln(l_y) + \frac{1}{2} \sum_{k=1}^{l_x} \ln \left( \sum_{j=1}^{l_x} w_{kj} \alpha_j (A_0 \cdot D_1) \right).$$

Making use of Jensen’s inequality for concave functions

$$\ln \left( \sum_{j} \alpha_j \right) \geq \sum_{j} \ln(\alpha_j)$$

and $\sum_{k} w_{kj} = 1$ we find

$$S \geq \frac{(l_y - 1)}{2} \sum_{k=1}^{l_x} \ln \left[ \alpha_k (A_0 \cdot D_0) \right]$$

$$+ \frac{t_x}{2} \ln(l_y) + \frac{1}{2} \sum_{j=1}^{l_x} \ln \left( \alpha_j (A_0 \cdot D_1) \right).$$

To evaluate the sums over the logarithms we employ Szegő’s theorem [11] for determinants of a Toeplitz matrices $T$. The theorem states: for sufficiently large $l_x$

$$\ln \left( \det(T) \right) \approx q_0 l_x + \sum_{k=1}^{\infty} k |q_k|^2,$$

for regular spectral function $q(\theta)$. Here $q_k$ is Fourier coefficients of $\ln q(\theta)$. Since moreover

$$\sum_{j} \ln \left( \alpha_j (A_0 \cdot D_1) \right) = \ln \left( \prod_{j} \alpha_j (A_0 \cdot D_1) \right) = \ln \left[ \det(A_0) \det(D_1) \right],$$

we eventually find the lower bound

$$S \geq a_1 l_x + a_2 l_y + \frac{t_x}{2} \ln(l_y).$$

Here $a_1, a_2$ are constants independent of the size of the subsystem and we have ignored an unimportant constant term.

By combining the two estimates (22) and (26) one finds

$$c_1 l_x + c_2 l_y + \frac{t_x}{2} \ln(l_y) \geq S \geq a_1 l_x + a_2 l_y + \frac{t_x}{2} \ln(l_y).$$

Since both sides of this inequality have the same functional form, $S$ approaches for large $l_x, l_y$ the asymptotic value

$$S \approx \frac{t_x}{2} \ln(l_y), \quad l_x, l_y \gg 1. \quad (27)$$

This is the main result of our paper. It shows that the entropy area law is violated for a set of harmonic chains, which for themselves have a gaped spectrum and are non-critical but become gapless by a collective interaction between the chains. Both upper and lower bound to the entropy attain the same logarithmic correction term to the area law.

A physical system that can be approximated by the model studied here is an anisotropic ion crystal. In such a system the Coulomb-interaction in the direction of the small lattice constant can in first approximation be considered as collective, while the one in an orthogonal direction is of finite range.

In conclusion, we derived an exact asymptotic expression for the entanglement entropy of a critical system of interacting oscillators in more than one dimension. We found that similar to one-dimensional systems [6] the entanglement area law is violated by a logarithmic correction proportional to the surface area in the critical direction. To our knowledge the system of collectively interacting harmonic strings considered here, which is approximately realized e.g. in an anisotropic ion crystal, is the first nontrivial example of a critical two-dimensional system for which the correction to the area law can explicitly be calculated.

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