Solution of nonlinear mixed integral equation via collocation method basing on orthogonal polynomials

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1. Introduction

The presence of integral equations in various sciences and their fields of applications led to different techniques of solutions. For example, in the contact problem (Abdou et al., [1, 2]), fluid mechanics (Haggag and Dosqiyas [3]), displacement problems in mechanics, and the theory of elasticity (Aleksandrovsk and Covalence [4], Konstanda [5], Georgiadis and Panos [6]), and in laser theory (Gao, et al., [7]). Due to the tremendous development in the various basic sciences, we find many analytical and numerical methods that have helped to solve many of these problems. One of these important methods that helped to solve some problems in different spaces, is the method of orthogonal polynomials; see Abdou et al., [8, 9]. Hafez and Yousri in [10], applied the Legendre-Chebyshev collocation method for solving two-dimensional integral equations of linear Volterra-Fredholm types. Nemati, et al. in [11], used the Legendre polynomials method to discuss the solution of a class of two-dimensional nonlinear Volterra integral equation of the second kind. Moreover, Tohid and Samadi in [12], used the orthogonal polynomials method via Legendre polynomials to discuss the solution of the nonlinear Volterra integral equation with a continuous kernel. Khader, et al. in [13, 14] used the Chebyshev polynomials method and Legendre polynomials method, respectively to solve Ricatiti, logistic and delay differential equations with variable coefficients. Al-Bugami in [15], applied Trapezoidal and Simpson methods to obtain the numerical solution of the integral equation in the two-dimensional problem in a surface crack layer. In [16], Bakhshayesh used the Galerkin method to discuss the approximate solution of Volterra integral equations with discontinuous kernel and with convolution kernel. Elzaki and Alamri in [17], used the homotopy perturbation method and the Adomian decomposition method to discuss some kinds of nonlinear integral equations. While, in [18] Almousa and Ismail used the same two numerical methods to obtain the numerical solution of Volterra-Fredholm integral equation in two-dimensional. For quadratic integral equations, Bassem and Alalyani in [19], used Chebyshev polynomials method for solving a quadratic integral equation with a continuous kernel. While, in [20] Abdou, et al. used the Chebyshev polynomial method to solve numerically, quadratic integral equation with discontinuous continuous. Almasieh and Mele used, in [21] Hybrid function method to solve a nonlinear integral equation of the Fredholm type. Brezinski and Zalglia in [22], applied the extrapolation method for obtaining the numerical solution of a nonlinear Fredholm integral equation. Katani in [23], used the quadrature method to discuss the solution of Fredholm integral equations of the second kind.

Consider the NMIE:

$$\mu(x)\Phi(x,t) = y(f(x,t),\Phi(x,t)) + \lambda_1 \int_0^1 k(x,y)\Phi(y,t)dy + \lambda_1 \int_0^1 \int_0^1 G(t,\tau)k(x,y)\Phi(y,\tau)d\eta d\tau$$

(1)

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https://doi.org/10.1016/j.heliyon.2022.e11827
Received 1 June 2022; Received in revised form 4 October 2022; Accepted 16 November 2022

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This formula is considered in the space \( L_2[0, 1] \times C[0, T], \ T < 1 \), where the continuous kernels \( k(x, y) \) and \( G(t, \tau) \) are considered in position and time, respectively. To discuss the existence of a unique solution of equation (1), rewrite it in the integral operator form
\[
W^* \Phi = \gamma (f(x,t), \Phi(x,t)) + K^\Phi + U^\Phi
\]  
(2)

where
\[
K^\Phi = \lambda_1 \int_0^1 k(x,y)\Phi(y;t)dy, \\
U^\Phi = \lambda_2 \int_0^1 G(t, \tau)k(x,y)\Phi(y; \tau)d\tau d\tau
\]  
(3)

Hence, using equation (2) with the aid of equation (3), we have
\[
\mu(x)\Phi(x;t) = W^* \Phi(x,t)
\]  
(4)

Equation (4) represents the integral operator form of NMIE (1).

We assume the following conditions:

(i) The kernel \( k(x, y) \in C([0, 1] \times [0, 1]) \), and satisfies the discontinuity condition
\[
\left[ \int_0^1 \int_0^1 |k(x,y)|^2 \right]^{\frac{1}{2}} = c, \quad (c \text{ is constant})
\]
(ii) The kernel of the time \( G(t, \tau) \in C([0, T] \times C[0, T]), 0 \leq \tau \leq t \leq T < 1 \), and satisfies the condition \( |G(t, \tau)| \leq N, \forall t \in [0, T], (N \text{ is constant}) \)
(iii) The given function \( \gamma (f(x,t), \Phi(x,t)) \) with its partial derivatives with respect to \( x, t \) are continuous in \( L_2[0, 1] \times C[0, T] \) and satisfies
\[
\|\gamma (f, \Phi)\| \leq |f(x,t)| + \delta |\Phi(x;t)|; \|f(x,t)\| = M, \quad \delta, M \text{ are constants}
\]

In the remaining part of this paper, we discuss the existence of a unique solution of the NMIE of the third kind in the space of integration \( L_2[0, 1] \times C[0, T], \ T < 1 \). Then, using a collocation numerical method based on HPs and LPs, we have, in each case, a nonlinear algebraic system. The convergence of each system is considered. Many examples for the first, second and third NMIE are solved, numerically, using Mable 18. The error estimate for each example, is computed.

2. Special cases

From the NMIF (1) many important and special cases can be derived
(2.1) If \( \gamma (f(x,t), \Phi(x;t)) = f(x,t) \), we have
\[
\mu(x)\Phi(x;t) = f(x, t)
\]
\[
+ \lambda_1 \int_0^1 k(x,y)\Phi(y;t)dy + \lambda_2 \int_0^1 G(t, \tau)k(x,y)\Phi(y; \tau)d\tau d\tau
\]  
(5)

Equation (5) represents MIE of the first kind, for \( \mu(x) = 0 \), of the second kind for \( \mu(x) = \text{constant} \neq 0 \), and of the third kind for \( \mu(x) \neq 0 \forall x \in [0, 1] \). Many authors discussed the solution of equation (5) using different methods, see (Wang and Wang [24], Mirzaee, and Hoseini[25]).

(2.2) If \( \gamma (f(x,t), \Phi(x;t)) = 0 \), we have
\[
\mu(x)\Phi(x;t) = \lambda_1 \int_0^1 k(x,y)\Phi(y;t)dy
\]
\[
+ \lambda_2 \int_0^1 G(t, \tau)k(x,y)\Phi(y; \tau)d\tau d\tau, \quad (\mu(x) \neq 0)
\]  
(6)

The importance of the integral equation (6) due to its appearing in the problems of mathematical physics, especially when the position kernel has a singular term in the form of Cauchy kernel, Carleman function, or logarithmic kernel. Therefore, we find that many of the spectral relationships were deduced from (6), and they were also, used to obtain the solutions of the mixed integral equation of the second type, (see Paripour and Kamyar [26], Abdou, et al., [27]).

(2.3) If \( \lambda_1 = 0 \), we have
\[
\mu(x)\Phi(x;t) = \gamma (f(x,t), \Phi(x;t)) + \lambda_2 \int_0^1 G(t, \tau)k(x,y)\Phi(y; \tau)d\tau d\tau
\]  
(7)

Equation (7) represents a NMIE of first, second and third kind of Volterra- Fredholm type, according to the values of \( \mu(x) \), (see Alhazmi [28]).

3. Basic equations and famous relations

Special functions play a prominent role in solving many mathematical problems, especially in finding approximate solutions to them. These functions have varied according to the definition areas of the functions used. Therefore, we find that most of the authors used Legendre and Chebyshev polynomials in solving integral equation, especially when the domain of integration is \( x \in [0, 1] \), or \( x \in [-1, 1] \), (Abdou and Basheen [29]). While, the Hermite and Laguerre polynomials are usually used to solve problems on semi-infinite \( (x \in [0, \infty]) \), and finite domains \( (x \in [-\infty, \infty]) \), (see Siyyam [30]). In applying the Hermite and Laguerre polynomials in the domain \( x \in [0, 1] \), we consider the kernel of integral equation has a value at \( x \in [0, 1] \) and vanishes at \( x \in (-\infty, 0), \ x \in (0, \infty) \).

In this section, some important relationships and properties of both the Hermite and Laguerre polynomials are presented. The basic rules of how to use the collocation method to solve integral equations were also mentioned.

1- Hermite Rodrigues’ formula
\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.
\]
2- Hermite recurrence relations
\[
H_{n+1}(x) = 2xH_n(x) - nH_{n-1}(x); \quad H_1(x) = 2xH_0(x)
\]
3- Hermite orthogonal relation
\[
\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = 2^n \sqrt{\pi} \delta_{nm}, \quad (\delta_{nm} = 1(n = m), \ \delta_{nm} = 0(n \neq m))
\]
4- Laguerre Rodrigues’ formula
\[
L_n(x) = e^{-x} \frac{d^n}{dx^n} (xe^{-x})
\]
5- Laguerre recurrence relations
\[
(n + 1)L_{n+1}(x) = (2n + 1 - x)L_n(x) - nL_{n-1}(x)
\]
6- Laguerre orthogonal relation
\[
\int_{0}^{\infty} e^{-x} L_n(x)L_m(x)dx = \delta_{nm}, \quad (\delta_{nm} = 1(n = m), \ \delta_{nm} = 0(n \neq m))
\]
7. Principal base of collocation method

Basing the approximation method of the FIE with continuous kernel,

$$\varphi(x) = f(x) + \lambda \int_a^b k(x, y) \varphi(y) dy.$$  \hspace{1cm} (8)

in the form

$$\varphi(x) = \sum_{k=1}^m c_k \psi_k(x).$$  \hspace{1cm} (9)

Using equation (9) in equation (8), we get

$$\sum_{k=1}^m c_k \psi_k(x) = f(x) + \lambda \int_a^b k(x, y) \sum_{k=1}^m c_k \psi_k(y) dy + E(x, c_1, c_2, ..., c_m).$$  \hspace{1cm} (10)

Where $\psi_1(x), \psi_2(x), ..., \psi_m(x)$ are the linearly independent $m$ functions on and $E(x, c_1, c_2, ..., c_m)$ is the error that depend on $x; x \in [a, b]$. Depending upon the way that the coefficients $c_i, i = 1, 2, ..., m$, are chosen, this is method insists that the error vanishes at the distinct point $x_i, i = 1, 2, ..., m$, such that $a = x_1 < ... < x_m = b$ and $h = (b - a) < N$. So equation (10) becomes

$$\sum_{k=1}^m c_k \psi_k(x_i) = f(x_i) + \lambda \int_a^b k(x, y) \sum_{k=1}^m c_k \psi_k(y) dy, \quad i = 1, ..., m.$$  \hspace{1cm} (11)

The last equation (11) represents a linear system in unknown constants that will be determined. Then, using the result in equation (9) to obtain the unknown function $\varphi(x)$ approximately.

4. The normality and continuity of the integral operator

The integral operator of equation (2) takes the form:

$$\|W \Phi\| \leq \|K \Phi\| + \|U\| \Phi\|$$  \hspace{1cm} (12)

where

$$\|K \Phi\| \leq |\lambda| \left[ \max_{0 \leq t \leq T} \left( k^2(x,y)dxdy \right)^{\frac{1}{2}} \right] \|\psi(x,y)\| \leq |\lambda| c \|\Phi\|$$

and

$$\|G \Phi\| \leq |\lambda| \left[ \max_{0 \leq t \leq T} \left( k^2(x,y)dxdy \right)^{\frac{1}{2}} \right] \|\psi(x,y)\| \leq |\lambda| c N \|\Phi\| T$$

Hence, the inequality (12), yields

$$\|W \Phi\| \leq (c |\lambda| + |\lambda| c NT) \|\Phi\| \leq a \|\Phi\|, \quad a = |\lambda| c (1 + NT).$$  \hspace{1cm} (13)

From inequality (13), we deduce that the integral operator $W \Phi$ is bounded.

$$\|W \Phi\| \leq a, \quad a = |\lambda| c (1 + NT), \quad |\lambda| = \max \{\lambda_1, \lambda_2\}$$

For the continuity, we assume the two functions $\Phi_1(x, t), \Phi_2(x, t)$ represent two different solutions of (2). Hence, we have

$$\|W (\Phi_1 - \Phi_2)\| \leq \lambda_1 \left[ \int_a^b k(x, y) \Phi_1(y, t) - \Phi_2(y, t) dy \right] + |\lambda| \left[ \int_a^b \int_0^1 G(t, r)k(x, y) \Phi_1(y, r) - \Phi_2(y, r) dy dr \right]$$

Using the conditions (i) and (ii) and applying the Cauchy-Schwarz inequality, we have

$$\|W (\Phi_1 - \Phi_2)\| \leq a |\lambda| c (1 + NT).$$  \hspace{1cm} (14)

The result of inequality (14) tends to the continuity of the operator $W$. If $a < 1$, hence, $W$ is a contraction operator. Then, by Banach fixed point theorem the nonlinear mixed integral equation has a unique solution.

4.1. Convergence and stability of solution

Lemma 3. Besides the conditions (i)-(iii), the infinite series $\sum_{n=0}^{\infty} \psi_n(x, t)$ is uniformly convergent to a continuous solution function $\Phi(x, t)$.

Proof. We construct the sequence of functions $\Phi_n(x, t)$ as

$$\mu(x)\Phi_n(x, t) = \gamma(f(x, t), \Phi_n(x, t)) + \int_0^1 \int_0^1 k(x, y)\Phi_{n-1}(y, t) dy \mathrm{d}r + \int_0^1 \int_0^1 G(t, r)k(x, y)\Phi_{n-1}(t, r) dy \mathrm{d}r$$

\hspace{1cm} (15)

For ease of manipulation, introduce

$$\psi_n(x, t) = \Phi_n(x, t) - \Phi_{n-1}(x, t), \quad \|\psi_n(x, t)\| = \|\mu(x)\Phi_n(x, t) - \gamma(f(x, t), \psi_n(x, t))\|.$$  \hspace{1cm} (16)

Where the unknown function of equation (15) takes the form

$$\Phi_n(x, t) = \sum_{n=0}^{\infty} \psi_n(x, t), \quad n = 1, 2, ..., \text{ (17)}$$

Equation (17) represents the approximate solution of the NMIE (1).

Using the properties of the modulus and then, with the aid of equation (16), we have

$$\|\psi_n(x, t)\| \leq \|\psi_{n-1}(x, t)\| + \|\mu(x)\| \leq \|\psi_{n-1}(x, t)\| \leq M.$$  \hspace{1cm} (18)

The above formula can adapt in the form

$$\|\psi_n(x, t)\| \leq \|\psi_{n-1}(x, t)\| + \|\mu(x)\| \leq M.$$  \hspace{1cm} (19)

Using the conditions (i), (ii), and mathematical induction method, we get

$$\|\psi_n(x, t)\| \leq (1 + NT) \|\mu(x)\| \leq M.$$  \hspace{1cm} (20)

The inequality (18) tells us its bound makes the sequence $\{\psi_n(x, t)\}$ converges and then, the sequence $\{\Phi_n(x, t)\}$ converges. Hence, the infinite series

$$\Phi(x, t) = \sum_{n=0}^{\infty} \psi_n(x, t), \quad \forall t \in [0, T].$$

is uniformly convergent since the terms $\psi_n(x, t)$ are dominated by $\eta^i$.

5. Collocation method basing on Hermite and Laguerre Polynomials

For the NMIE (1), we divide the time interval $[0, T]$ into $M$ subintervals with length $\ell = T/H; H$ can be even or odd, where in $t = t_m, r = t_n, 0 < m, n < H$, to obtain

$$\|W (\Phi_1 - \Phi_2)\| \leq a |\lambda| c (1 + NT).$$  \hspace{1cm} (14)
\[
\mu(x)\Phi(x; t_m) = \gamma(f(x; t_m), \Phi(x; t_m)) + \lambda_1 \int_{0}^{1} k(x, y)\Phi(y; t_m)dy
\]
\[+ \lambda_2 \sum_{n=0}^{m} \omega_n G(t_m, t_n) \int_{0}^{1} k(x, y)\Phi(y; t_n)dy + R(h^{m+1}_m),
\]
\[\ell_n(n = 0, m, \ell / 2(0 < n < m) \quad (19)\]

The term \(R(h^{m+1}_m)\), in equation (19), represents the error resulting from
time division \(h_m = \max_{0 \leq j \leq m} \epsilon_n, \quad \epsilon_n = t_{n+1} - t_n\). Note that \(p\) depends on
the number of differences. For example, for the third order \(i = 3\), the error is \(R(\epsilon^3)\).

The previous system of equation (19) can be resolved using collocation
method. For this, equation (19) is based on approximating the
solution using the partial sum:
\[S(x; t_m) = \sum_{k=0}^{H} c_k(t_m)\psi_k(x), \quad m = 0, 1, \ldots, H \quad (20)\]

for \(H\) independent linear function \(\psi_1, \psi_2, \ldots, \psi_H\) in the interval \([0, 1]\).

If we substitute for the approximate solution of equation (20) in
equation (19) instead of \(\Phi(x; t_m)\), it will produce an error \(E(x, c_0(t_i), c_1(t_i),
\ldots, c_{H}(t_i))\) that depends on \(x\) and \(t\) and the transactions are optional \(c_0(t_i), c_1(t_i), \ldots, c_{H}(t_i)\). Then, we have
\[\mu(x)S(x; t_m) = \gamma(f(x; t_m), S(x; t_m))
\]
\[+ \lambda_1 \int_{0}^{1} k(x, y)S(y; t_m)dy
\]
\[+ \lambda_2 \sum_{n=0}^{m} \omega_n G(t_m, t_n) \int_{0}^{1} k(x, y)S(y; t_n)dy
\]
\[+ E(x, c_0(t_i), c_1(t_i), \ldots, c_{H}(t_i)) + R(h^{m+1}_m) \quad (21)\]
The formula (21) represents the error after applied using the using collocation
method.

We divide the position interval \([0, 1]\) into \(L\) subintervals with length \(h = 1/L; L\) can be even or odd, where \(x = x_i, \quad y = y_j, \quad 0 < i, j < L\). In this
manner, we have
\[\mu(x_s)S(x_s; t_m) = \gamma(f(x_s; t_m), S(x_s; t_m))
\]
\[+ \lambda_1 \int_{0}^{1} k(x_s, y)S(y; t_m)dy
\]
\[+ \lambda_2 \sum_{n=0}^{m} \omega_n G(t_m, t_n) \int_{0}^{1} k(x_s, y)S(y; t_n)dy
\]
\[+ E(x_s, c_0(t_i), c_1(t_i), \ldots, c_{H}(t_i)) \quad (22)\]

To determine the coefficients \(c_0(t_i), c_1(t_i), \ldots, c_{H}(t_i)\), of the approximate
solution of equation (21) we use the linear independent functions \(\psi_0(x), \psi_1(x), \ldots, \psi_H(x)\) instead of \(S(x; t_m)\) in equation (20) such that the
error \(E(x, c_0(t_i), c_1(t_i), \ldots, c_{H}(t_i))\) vanishes. The system of equation (22) can be solved using recurrence relations.

5.1. Collocation method with Hermite Polynomials

Assume the unknown approximate function \(S(x; t_m)\) and the known function \(f(x; t_m)\) respectively, take the forms
\[S(x; t_m) = \sum_{k=0}^{H} c_k(t_m)H_k(x) = \sum_{k=0}^{H} c_k(t_m)H_k(x); \quad m = 0, 1, \ldots, H \quad (23)\]
\[f(x; t_m) = \sum_{k=0}^{H} f_k(t_m)H_k(x) = \sum_{k=0}^{H} f_k(t_m)H_k(x), \quad (24)\]

where, \(H_k(x)\) is the Hermite function of order \(k, c_{k,m}\) are constants will be determined and the constants of the given function \(f_{k,m}\) can be
calculating from the following relation
\[f_{k,m} = \frac{1}{2^{\ell} \ell! \sqrt{\pi(\ell)}} \int_{-\infty}^{\infty} e^{-x^2} f(x)H_k(x) dx, \quad (m, k = 0, 1, \ldots, H). \quad (25)\]

Equation (24) represents the integral formula to calculate the known
coefficients for the known function, after applying the orthogonal polynomials
formula.

Then, substituting from equation (23) into equation (22), we obtain
the residual error form
\[R_{k,m} = \mu_1 \sum_{n=0}^{H} c_k(t_m)H_{n} - \gamma(f_{k,m}, \sum_{k=0}^{H} c_k(t_m)H_{k,i})
\]
\[+ \lambda_1 \sum_{n=0}^{H} \omega_n \sum_{k=0}^{m} c_k(t_m)H_{k,i} \int_{0}^{1} k(x, y)H_{k,i}dy
\]
\[+ \lambda_2 \sum_{n=0}^{m} \omega_n c_k(t_m)G_{m,n} \int_{0}^{1} k(x, y)H_{k,i}dy \quad (26)\]

Where in equation (25), \(R_{k,m}\) are the error of order \((L \times H)\), and
vanishes at \(i\) points of position and \(m\) points of time, i.e. \(R_{k,m} = 0\) at
\[0 \leq x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_L = 1; \quad 0 \leq t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_H = T. \quad (27)\]

5.2. Collocation with Laguerre Polynomials

For using Laguerre polynomials, we assume
\[\sum_{k=0}^{H} \sum_{m=0}^{H} c_k(t_m)H_{k,i} - \gamma(f_{k,m}, \sum_{k=0}^{H} c_k(t_m)H_{k,i})
\]
\[+ \lambda_1 \sum_{n=0}^{H} c_k(t_m)H_{k,i} \int_{0}^{1} k(x, y)H_{k,i}dy
\]
\[+ \lambda_2 \sum_{n=0}^{m} \omega_n c_k(t_m)G_{m,n} \int_{0}^{1} k(x, y)H_{k,i}dy \quad (28)\]

In equation (28), \(R_{k,m}\) represents the error of order \((L \times H)\), and
vanishes at \(i\) points of position and \(m\) points of time, i.e. \(R_{k,m} = 0\) at
\[0 \leq x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_L = 1; \quad 0 \leq t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_H = T. \quad (29)\]

6. Applications and numerical results

Application 1.1. Consider the NMIE of the third kind
\[\mu(x)\Phi(x; t) = f(x; t) + \Phi^2(x; t)
\]
\[- \lambda_1 \int_{0}^{1} x^2 \Phi(x; t)dy
\]
\[+ \lambda_2 \int_{0}^{1} \frac{1}{t^2} x^2 \Phi(x; t)dy \quad (E.S.\Phi(x; t) = x^2) \quad (30)\]
Where, $\lambda_1$, $\lambda_2$ are constants, describing the kind of the materials used. The kernel of position is $k(x, y) = x^2 y^2$, while the kernel of time is $G(t, r) = r^2 t$. The given function $f(x; t)$ represents the basic surface of the material, while $\Phi(x; t)$ is the unknown function. The integral equation (29) will be computed, using Mable 18 at time $t \in [0, 0.1]$. Table 1 is computed when $\mu(x) = x^2$, $\lambda_1 = 0.01$, $\lambda_2 = 0.3$, to have

$$f(x; t) = x^2 t^2 - x^2 t^3 + 0.01 \int_0^1 x^2 y^2 dy + 0.3 \int_0^1 r^2 r^3 x^2 y^3 dy dr$$

In this case, the NMIE of the third kind with exact solution $\Phi(x; t) = x^2 t^2$, becomes

$$x^2 \Phi(x; t) = [x^4 t^2 - x^4 t^3 + 0.002 x^2 t^2 + 0.15 x^2 t^3]$$

$$+ \Phi^2(x; t) - 0.01 \int_0^1 x^2 y^2 \Phi(y; t) dy$$

$$- 0.3 \int_0^1 \int_0^1 r^2 r^3 x^2 y^2 \Phi(y; r) dr dy dr$$

In application 1.1, Table 1, Fig. 1-a, and Fig. 1-b, describe the numerical results of solving nonlinear mixed integral equation of the third kind and its corresponding error. Here, we use collection method via the spectral relationships of Hermite polynomials (Fig. 1-a) and Laguerre polynomials (Fig. 1-b) at $\mu = x^2$, $0 \leq t \leq 0.1$, $\lambda_1 = 0.01$, $\lambda_2 = 0.3$.

**Application 1.2:** Consider NMIE of the third kind when $\mu(x) = x$, $\lambda_1 = 0.3$, $\lambda_2 = 0.3$.

$$x \Phi(x; t) = f(x; t) + \Phi^3(x; t)$$

$$- 0.3 \int_0^1 x^2 y^2 \Phi(y; t) dy$$

$$- 0.33 \int_0^1 t r^2 x^2 y^2 \Phi(y; r) dr dy dr \quad (E.S. \Phi(x; t) = t^2 e^x)$$  (30)

The integral equation (30) is calculating using Mable 18 through the time $t \in [0, 0.6]$.

In application 1.2 the results of using collocation methods via Hermite polynomials and Laguerre polynomials for solving the NMIE (30) of the third kind, when $\mu = x$, $0 \leq t \leq 0.6$, $\lambda_1 = 0.3$, $\lambda_2 = 0.33$ are described in Table 2, Fig. 1-a and Fig. 1-b, respectively.

**Application 2.1:** Consider the nonlinear mixed integral equation of the second kind $\mu(x) = 1$, $\lambda_1 = \lambda_2 = 0.03$.

$$\Phi(x; t) = f(x; t) + \Phi^4(x; t) - 0.03 \int_0^1 x^2 y^2 \Phi(y; t) dy$$

$$- 0.03 \int_0^1 t r^2 x^2 y^2 \Phi(y; r) dr dy dr \quad (E.S. \Phi(x; t) = t^2 e^x)$$  (31)
The integral equation (31) is calculating with the corresponding error, using Mable 18 at time \( t \in [0, 0.01] \).

The results of application 2.1 for solving the NMIE (31) of the second kind after using collocation methods via Hermite polynomials and Laguerre polynomials, when \( \mu = 1, 0 \leq t \leq 0.01, \lambda_1 = \lambda_2 = 0.03 \) are described in Table 3, Fig. 3-a and Fig. 3-b, respectively.

Application 2.2 (MIE of the second kind): Consider

\[
0.5\Phi(x; t) = f(x; t) + 0.3\Phi^3(x; t) - 0.033 \int_0^1 x^2 \Phi(y; t) dy
\]

The nonlinear mixed integral equation of the second kind (32) is computed at \( \mu = 0.5 \), and at the same time of equation (31) \( t = [0, 0.01] \).

To show the effect of changing the constant \( \mu \) on the integral equation.

Table 3 contains numerical results of collocation method via Ps and LPs at \( \mu = 1, 0 \leq t \leq 0.01, \lambda_1 = \lambda_2 = 0.03 \).

Table 3.

| x   | Exact Solution | Hermite Polynomials | Error of Hermite | Laguerre Polynomials | Error of Laguerre |
|-----|---------------|---------------------|------------------|----------------------|-------------------|
| 0.0 | 0.0           | 0.0                 | 0.0              | 0.0                  | 0.0               |
| 0.1 | 0.00003567212 | 0.00003567212       | 0.0              | 0.00003567212        | 0.00003567212    |
| 0.2 | 0.0002718281  | 0.0002718281        | 0.0              | 0.0002718281         | 0.0002718281    |
| 0.3 | 0.0004816879  | 0.0004816879        | 0.0              | 0.0004816879         | 0.0004816879    |
| 0.4 | 0.0007389056  | 0.0007389056        | 0.0              | 0.0007389056         | 0.0007389056    |
| 0.5 | 0.0012182493  | 0.0012182493        | 0.0              | 0.0012182493         | 0.0012182493    |
| 0.6 | 0.0018085536  | 0.0018085536        | 0.0              | 0.0018085536         | 0.0018085536    |
| 0.7 | 0.0033115451  | 0.0033115451        | 0.0              | 0.0033115451         | 0.0033115451    |
| 0.8 | 0.0054598150  | 0.0054598150        | 0.0              | 0.0054598150         | 0.0054598150    |
| 0.9 | 0.0090017131  | 0.0090017131        | 0.0              | 0.0090017131         | 0.0090017131    |
| 1.0 | 0.0148413154  | 0.0148413154        | 0.0              | 0.0148413154         | 0.0148413154    |

\[
-0.05 \int_0^1 \int_0^t r^2 x^2 \Phi(y; t) dy dr = 0.3 \Phi^3(x; t) - 0.033 \int_0^1 x^2 \Phi(y; t) dy + f(x; t)
\]

Table 4 contains numerical results of collocation method via Ps and LPs at \( \mu = 0.5, 0 \leq t \leq 0.01, \lambda_1 = 0.033, \lambda_2 = 0.05 \).
Fig. 3. Fig. 3-a and Fig. 3-b describe the relation between exact solution and numerical solution of NMEI of the second kind using HPs (Fig. 3-a) and LPs (Fig. 3-a) at $\mu = 1$, $0 \leq t \leq 0.01$, $\lambda_1 = \lambda_2 = 0.03$.

Fig. 4. Fig. 4-a and Fig. 4-b contain the relation between exact solution and numerical solution using HPs (Fig. 4-a) and LPs (Fig. 4-b), for NMIE of the second kind at $\mu = 0.5$, $0 \leq t \leq 0.01$.

Table 4. In Table 4, we have numerical and error results of collocation method via HPs and LPs at $\mu = 0.5$, $0 \leq t \leq 0.01$.

| x    | Exact Solution | Hermite Polynomial | Error of Hermite | Laguerre Polynomial | Error of Laguerre |
|------|----------------|--------------------|------------------|--------------------|------------------|
| 0.0  | 0.0            | 0.0                | 0.0              | 0.0                | 0.0              |
| 0.1  | 0.0164872127   | 0.0164872127       | 2.492 x 10^-30   | 0.0164872127       | 3.451 x 10^-31   |
| 0.2  | 0.0271828182   | 0.0271828182       | 1.92 x 10^-30    | 0.0271828182       | 4.871 x 10^-30   |
| 0.3  | 0.0448168907   | 0.0448168907       | 1.57 x 10^-30    | 0.0448168907       | 1.471 x 10^-30   |
| 0.4  | 0.0738905609   | 0.0738905609       | 1.21 x 10^-30    | 0.0738905609       | 2.950 x 10^-30   |
| 0.5  | 0.1218249396   | 0.1218249396       | 6.22 x 10^-29    | 0.1218249396       | 4.922 x 10^-30   |
| 0.6  | 0.200853692    | 0.200853692        | 1.12 x 10^-29    | 0.200853692        | 7.389 x 10^-30   |
| 0.7  | 0.3311545195   | 0.3311545195       | 1.61 x 10^-28    | 0.3311545195       | 1.035 x 10^-29   |
| 0.8  | 0.5459815003   | 0.5459815003       | 2.18 x 10^-28    | 0.5459815003       | 1.380 x 10^-29   |
| 0.9  | 0.90017113130  | 0.90017113130      | 2.84 x 10^-28    | 0.90017113130      | 1.775 x 10^-29   |
| 1.0  | 1.4841315910   | 1.4841315910       | 3.58 x 10^-28    | 1.4841315910       | 2.219 x 10^-29   |
In the integral
\[ \lambda_1 = 0 \]
\[ \lambda_2 = 0.02 \]
\[ \mu_0, 0 \leq t \leq 0.1 \]

**Fig. (5-a)** & **Fig. (5-b)**

Application 3: Consider the nonlinear mixed integral equation of the first kind \( \mu = 0, \lambda_1 = 0.01, \lambda_2 = 0.02 \).

\[
\int_0^1 xy^2 \Phi(y; t) \, dy = 0.01 \int_0^1 xy^2 \Phi(y; t) \, dy \\
+ 0.02 \int_0^1 \int_0^1 r^2 xy^2 \Phi(y; t) \, dy \, dr, 0 \leq t \leq 0.1 \\
+ (E.S. \Phi(x; t) = r^2 x^2)
\]

(33)

In application 3 the results of using collocation methods via Hermite polynomials and Laguerre polynomials for solving the mixed nonlinear integral equation (33) of the first kind, when \( \mu = 0, 0 \leq t \leq 0.1, \lambda_1 = 0.01, \lambda_2 = 0.02 \) are described in Table 5, Fig. 5-a and Fig. 5-b.

7. Conclusion

In this paper, we consider NMIEs of the third kind with continuous kernels; where the convergence of the solution is discussed. In the numerical results we use the collocation method and represent the approximate solution in the form of two Hermite and Laguerre polynomials and the corresponding error is computed. We noticed from the numerical results that:

1. In NMIE of the third kind the error is minimal at small times, which reach 0.01. While the error increases relatively with the increase of time. It is also noticed that the behavior of the NMIE of the second type follows the same behavior of the third type. In the case of NMIE of the first kind, we find that the error, even at very small times, is much higher than that of the third or second kind. These results may lead us to report that Hermite and Laguerre polynomials are not preferred to be used to solve NMIE of the first kind.

2. For the NMIE of the third kind in Table 1, Fig. 1-a and Fig. 1-b and at \( \mu = x^2, 0 \leq t \leq 0.1, \lambda_1 = 0.01, \lambda_2 = 0.3, \) the error of collocation method via the two polynomials is decreasing in the period of \( 0.1 \leq x \leq 0.2 \), and is increasing throughout the period of \( 0.3 \leq x \leq 1 \). While, in Table 2 Fig. 2-a and Fig. 2-b and at \( \mu = x, \lambda_1 = 0.3, \lambda_2 = 0.33, 0 \leq t \leq 0.6, \) the error has stability decreasing in the period of \( 0.1 \leq x \leq 0.6, \) and is increasing throughout the period of \( 0.7 \leq x \leq 1 \) for HPs and LPs.

3. For the NMIE of the second kind in Table 3, Fig. 3-a and Fig. 3-b at \( \mu = 1, 0 \leq t \leq 0.01, \lambda_1 = \lambda_2 = 0.03, \) the error of HPs and LPs has stability decreasing in the period of \( 0.1 \leq x \leq 0.6, \) and is increasing throughout \( 0.7 \leq x \leq 1 \). In addition, in Table 4, Fig. 4-a and Fig. 4-b at \( \mu = 0.5, \lambda_1 = 0.033, \lambda_2 = 0.05, 0 \leq t \leq 0.01, \) the error of HPs and LPs has stability decreasing in the period of \( 0.1 \leq x \leq 0.3, \) and is increasing throughout \( 0.4 \leq x \leq 1 \).

4. For the NMIE of the first kind in Table 5, Fig. 5-a and Fig. 5-b at \( \mu = 0, \lambda_1 = 0.01, \lambda_2 = 0.02, 0 \leq t \leq 0.1, \) the error, after using the two polynomials, is decreasing in the period of \( 0.1 \leq x \leq 0.4, \) and is increasing throughout the period of \( 0.5 \leq x \leq 1 \).

Future work: We will try to solve the previous NMIE (1), using Chebyshev and Legendre polynomials. And a general comparison between the two results of the work will hold.
Declarations

Author contribution statement

All authors listed have significantly contributed to the development and the writing of this article.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

Data included in article/supp. material/referenced in article.

Declaration of interests statement

The authors declare no conflict of interest.

Acknowledgements

The author thanks the reviewers for their abundant effort and valuable comments for this research, which made it in a good shape for the reader.

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