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A class of semipositone $p$-Laplacian problems with a critical growth reaction term

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Abstract: We prove the existence of ground state positive solutions for a class of semipositone $p$-Laplacian problems with a critical growth reaction term. The proofs are established by obtaining crucial uniform $C^{1,\alpha}$ a priori estimates and by concentration compactness arguments. Our results are new even in the semilinear case $p = 2$.

Keywords: critical semipositone $p$-Laplacian problems, ground state positive solutions, concentration compactness, uniform $C^{1,\alpha}$ a priori estimates

MSC: Primary 35B33, Secondary 35J92, 35B09, 35B45

1 Introduction

Consider the $p$-superlinear semipositone $p$-Laplacian problem

$$
\begin{cases}
-\Delta_p u = u^{q-1} - \mu & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, $p < q < p^*$, $\mu > 0$ is a parameter, and $p^* = Np/(N-p)$ is the critical Sobolev exponent. The scaling $u \mapsto \mu^{1/(q-1)} u$ transforms the first equation in (1.1) into

$$
-\Delta_p u = \mu^{(q-p)/(q-1)} \left( u^{q-1} - 1 \right),
$$

so in the subcritical case $q < p^*$, it follows from the results in Castro et al.[1] and Chhetri et al.[2] that this problem has a weak positive solution for sufficiently small $\mu > 0$ when $p > 1$ (see also Unsurangie [3], Allegretto et al.[4], Ambrosetti et al.[5], and Caldwell et al.[6] for the case when $p = 2$). On the other hand, in the critical case $q = p^*$, it follows from a standard argument involving the Pohozaev identity for the $p$-Laplacian (see Guedda and Véron [7, Theorem 1.1]) that problem (1.1) has no solution for any $\mu > 0$ when $\Omega$ is star-shaped. The purpose of the present paper is to show that this situation can be reversed by the addition of lower-order terms, as was observed in the positone case by Brézis and Nirenberg in the celebrated paper [8]. However, this extension to the semipositone case is not straightforward as $u = 0$ is no longer a subsolution, making it much harder to find a positive solution as was pointed out in Lions [9]. The positive solutions that we obtain here are ground states, i.e., they minimize the energy among all positive solutions.

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We study the Brézis-Nirenberg type critical semipositone \( p \)-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + u^{p^*-1} - \mu \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \lambda, \mu > 0 \) are parameters. Let \( W^{1,p}_0(\Omega) \) be the usual Sobolev space with the norm given by

\[
||u||^p = \int_\Omega |\nabla u|^p \, dx.
\]

For a given \( \lambda > 0 \), the energy of a weak solution \( u \in W^{1,p}_0(\Omega) \) of problem (1.2) is given by

\[
I_\mu(u) = \int_\Omega \left( \frac{|\nabla u|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^{p^*}}{p^*} + \mu u \right) \, dx,
\]

and clearly all weak solutions lie on the set

\[
\mathcal{N}_\mu = \left\{ u \in W^{1,p}_0(\Omega) : u > 0 \text{ in } \Omega \text{ and } \int_\Omega |\nabla u|^p \, dx = \int_\Omega \left( \lambda u^p + u^{p^*} - \mu u \right) \, dx \right\}.
\]

We will refer to a weak solution that minimizes \( I_\mu \) on \( \mathcal{N}_\mu \) as a ground state. Let

\[
\lambda_1 = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}
\]

be the first Dirichlet eigenvalue of the \( p \)-Laplacian, which is positive. We will prove the following existence theorem.

**Theorem 1.1.** If \( N \geq p^2 \) and \( \lambda \in (0, \lambda_1) \), then there exists \( \mu^* > 0 \) such that for all \( \mu \in (0, \mu^*) \), problem (1.2) has a ground state solution \( u_\mu \in C^{1,1}(\bar{\Omega}) \) for some \( \alpha \in (0, 1) \).

The scaling \( u \mapsto \mu^{-1/(p^*-p)} u \) transforms the first equation in the critical semipositone \( p \)-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + \mu \left( u^{p^*-1} - 1 \right) \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

into

\[
-\Delta_p u = \lambda u^{p-1} + u^{p^*-1} - \mu^{(p^*-1)/(p^*-p)},
\]

so as an immediate corollary we have the following existence theorem for problem (1.4).

**Theorem 1.2.** If \( N \geq p^2 \) and \( \lambda \in (0, \Lambda_1) \), then there exists \( \mu^* > 0 \) such that for all \( \mu \in (0, \mu^*) \), problem (1.4) has a ground state solution \( u_\mu \in C^{1,\alpha}(\bar{\Omega}) \) for some \( \alpha \in (0, 1) \).

We would like to emphasize that Theorems 1.1 and 1.2 are new even in the semilinear case \( p = 2 \).

The outline of the proof of Theorem 1.1 is as follows. We consider the modified problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + u^{p^*-1} - \mu f(u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( f \) is a nonlinear term.
where \( u_+(x) = \max \{ u(x), 0 \} \) and

\[
    f(t) = \begin{cases} 
        1, & t \geq 0 \\
        1 - |t|^{p-1}, & -1 < t < 0 \\
        0, & t \leq -1.
    \end{cases}
\]

Weak solutions of this problem coincide with critical points of the \( C^1 \)-functional

\[
    I_\mu(u) = \int_\Omega \left( \frac{|
abla u|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^{p^*}}{p^*} \right) \, dx + \mu \left[ \int_{\{u < 0\}} u \, dx + \mu \int_{\{-1 < u < 0\}} \left( u - \frac{|u|^{p-1} u}{p} \right) \, dx \left( 1 - \frac{1}{p} \right) |\{u \leq -1\}| \right], \quad u \in W^{1,p}_0(\Omega),
\]

where \(|\cdot|\) denotes the Lebesgue measure in \( \mathbb{R}^N \). Recall that \( I_\mu \) satisfies the Palais-Smale compactness condition at the level \( c \in \mathbb{R} \), or the (PS)\(_c\) condition for short, if every sequence \( \{ u_j \} \subset W^{1,p}_0(\Omega) \) such that \( I_\mu(u_j) \to c \) and \( I'_\mu(u_j) \to 0 \), called a (PS)\(_c\) sequence for \( I_\mu \), has a convergent subsequence. As we will see in Lemma 2.1 in the next section, it follows from concentration compactness arguments that \( I_\mu \) satisfies the (PS)\(_c\) condition for all

\[
    c < \frac{1}{N} S^{N/p} \left( 1 - \frac{1}{p} \right) \mu \, |\Omega|,
\]

where \( S \) is the best Sobolev constant (see (2.1)). First we will construct a mountain pass level below this threshold for compactness for all sufficiently small \( \mu > 0 \). This part of the proof is more or less standard. The novelty of the paper lies in the fact that the solution \( u_\mu \) of the modified problem (1.5) thus obtained is positive, and hence also a solution of our original problem (1.2), if \( \mu \) is further restricted. Note that this does not follow from the strong maximum principle as usual since \(-\mu f(0) < 0\). This is precisely the main difficulty in finding positive solutions of semipositone problems (see Lions [9]). We will prove that for every sequence \( \mu_j \to 0 \), a subsequence of \( u_{\mu_j} \) is positive in \( \Omega \). The idea is to show that a subsequence of \( u_{\mu_j} \) converges in \( C^1_0(\overline{\Omega}) \) to a solution of the limit problem

\[
    \begin{aligned}
        -\Delta_p u &= \lambda u^{p-1} + u^{p^*-1} \quad \text{in } \Omega \\
        u &> 0 \quad \text{in } \Omega \\
        u &= 0 \quad \text{on } \partial \Omega.
    \end{aligned}
\]

This requires a uniform \( C^{1,a}(\overline{\Omega}) \) estimate of \( u_{\mu_j} \) for some \( a \in (0, 1) \). We will obtain such an estimate by showing that \( u_{\mu_j} \) is uniformly bounded in \( W^{1,p}_0(\Omega) \) and uniformly equi-integrable in \( L^p(\Omega) \), and applying a result of de Figueiredo et al.[10]. The proof of uniform equi-integrability in \( L^p(\Omega) \) involves a second (nonstandard) application of the concentration compactness principle. Finally, we use the mountain pass characterization of our solution to show that it is indeed a ground state.

**Remark 1.3.** Establishing the existence of solutions to the critical semipositone problem

\[
    \begin{aligned}
        -\Delta_p u &= \mu \left( u^{p-1} + u^{p^*-1} - 1 \right) \quad \text{in } \Omega \\
        u &> 0 \quad \text{in } \Omega \\
        u &= 0 \quad \text{on } \partial \Omega
    \end{aligned}
\]

for small \( \mu \) remains open.
2 Preliminaries

Let

\[
S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\mu \left( \frac{1}{p} \int_{\Omega} |u|^p \, dx \right)^{p/p'}}
\]

be the best constant in the Sobolev inequality, which is independent of \(\Omega\). The proof of Theorem 1.1 will make use of the following compactness result.

**Lemma 2.1.** For any fixed \(\lambda, \mu > 0\), \(I_\mu\) satisfies the (PS)_c condition for all

\[
c < \frac{1}{N} S^{N/p} - \left(1 - \frac{1}{p}\right) \mu |\Omega|.
\]

**Proof.** Let \((u_j)\) be a (PS)_c sequence. First we show that \((u_j)\) is bounded. We have

\[
I_\mu(u_j) = \int_{\Omega} \left( \frac{|\nabla u_j|^p}{p} - \frac{\lambda |u_j|^{p-1} u_j}{p} \right) \, dx + \mu \left[ \int_{\{u_j \leq 0\}} u_j \, dx \right] + \mu \left[ \int_{\{u_j > 0\}} \left(1 - \frac{1}{p}\right) u_j \, dx \right] = c + o(1)
\]

and

\[
I'_\mu(u_j)v = \int_{\Omega} \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{p-1} v - u_j^{p-1} v \right) \, dx + \mu \left[ \int_{\{u_j \leq 0\}} v \, dx \right] + \mu \left[ \int_{\{u_j > 0\}} \left(1 - |u_j|^{p-1}\right) v \, dx \right] = o(1) \|v\| \quad \forall v \in W_0^{1,p}(\Omega).
\]

Taking \(v = u_j\) in (2.4), dividing by \(p\), and subtracting from (2.3) gives

\[
\frac{1}{N} \int_{\Omega} u_j^{p'} \, dx \leq c + \left(1 - \frac{1}{p}\right) \mu |\Omega| + o(1) \left(\|u_j\| + 1\right),
\]

and it follows from this, (2.3), and the Hölder inequality that \((u_j)\) is bounded in \(W_0^{1,p}(\Omega)\).

Since \((u_j)\) is bounded, so is \((u_{j+})\), a renamed subsequence of which then converges to some \(v \geq 0\) weakly in \(W_0^{1,p}(\Omega)\), strongly in \(L^q(\Omega)\) for all \(q \in [1, p^*)\) and a.e. in \(\Omega\), and

\[
|\nabla u_{j+}|^p \, dx \xrightarrow{w^*} \kappa, \quad u_{j+}^{p'} \, dx \xrightarrow{w^*} v
\]

in the sense of measures, where \(\kappa\) and \(v\) are bounded nonnegative measures on \(\overline{\Omega}\) (see, e.g., Folland [11]). By the concentration compactness principle of Lions [12, 13], then there exist an at most countable index set \(I\) and points \(x_i \in \overline{\Omega}, i \in I\) such that

\[
\kappa \geq |\nabla v|^p \, dx + \sum_{i \in I} \xi_i \delta_{x_i}, \quad v = v^{p'} \, dx + \sum_{i \in I} v_i \delta_{x_i},
\]

(2.7)
where \( \kappa_i, v_i > 0 \) and \( v_i^{p'/p} \leq \kappa_i/S \). We claim that \( I = \emptyset \). Suppose by contradiction that there exists \( i \in I \). Let \( \varphi : \mathbb{R}^N \to [0,1] \) be a smooth function such that \( \varphi(x) = 1 \) for \( |x| \leq 1 \) and \( \varphi(x) = 0 \) for \( |x| \geq 2 \). Then set

\[
\varphi_{i,\rho}(x) = \varphi\left( \frac{x-x_i}{\rho} \right), \quad x \in \mathbb{R}^N
\]

for \( i \in I \) and \( \rho > 0 \), and note that \( \varphi_{i,\rho} : \mathbb{R}^N \to [0,1] \) is a smooth function such that \( \varphi_{i,\rho}(x) = 1 \) for \( |x-x_i| \leq \rho \) and \( \varphi_{i,\rho}(x) = 0 \) for \( |x-x_i| \geq 2\rho \). The sequence \( (\varphi_{i,\rho} u_i) \) is bounded in \( W_0^{1,p}(\Omega) \) and hence taking \( \nu = \varphi_{i,\rho} u_i \) in (2.4) gives

\[
\int_{\Omega} \left( \varphi_{i,\rho} |\nabla u_i|^p + u_i |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi_{i,\rho} - \lambda \varphi_{i,\rho} u_i^p - \varphi_{i,\rho} \nu^p + \mu \varphi_{i,\rho} u_i \right) dx = o(1). \tag{2.8}
\]

By (2.6),

\[
\int_{\Omega} \varphi_{i,\rho} |\nabla u_i|^p dx \to \int_{\Omega} \varphi_{i,\rho} dx, \quad \int_{\Omega} \varphi_{i,\rho} u_i^p dx \to \int_{\Omega} \varphi_{i,\rho} dx.
\]

Denoting by \( C \) a generic positive constant independent of \( j \) and \( \rho \),

\[
\left| \int_{\Omega} \left( u_i |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi_{i,\rho} - \lambda \varphi_{i,\rho} u_i^p + \mu \varphi_{i,\rho} u_i \right) dx \right| \leq C \left( \frac{1}{\rho} + \mu \right) \left( I_j^{1/p} + I_j \right),
\]

where

\[
I_j := \int_{\Omega \cap B_{2\rho}(x_i)} u_i^p dx \to \int_{\Omega \cap B_{2\rho}(x_i)} v^p dx \leq C \rho^p \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^p dx \right)^{p/p'},
\]

So passing to the limit in (2.8) gives

\[
\int_{\Omega} \varphi_{i,\rho} dx - \int_{\Omega} \varphi_{i,\rho} dx \leq C \left[ (1 + \mu \rho) \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^p dx \right)^{1/p'} + \int_{\Omega \cap B_{2\rho}(x_i)} v^p dx \right].
\]

Letting \( \rho \to 0 \) and using (2.7) now gives \( \kappa_i \leq v_i \), which together with \( v_i > 0 \) and \( v_i^{p'/p} \leq \kappa_i/S \) then gives \( v_i \geq S^{N/p} \). On the other hand, passing to the limit in (2.5) and using (2.6) and (2.7) gives

\[
v_i \leq N \left[ c + \left( 1 - \frac{1}{p} \right) \mu |\Omega| \right] < S^{N/p}
\]

by (2.2), a contradiction. Hence \( I = \emptyset \) and

\[
\int_{\Omega} u_i^p dx \to \int_{\Omega} v^p dx. \tag{2.9}
\]

Passing to a further subsequence, \( u_j \) converges to some \( u \) weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^q(\Omega) \) for all \( q \in [1, p') \), and a.e. in \( \Omega \). Since

\[
|u_j^{p-1} (u_j - u)| \leq u_j^{p-1} + u_j^{p-1} |u| \leq \left( 2 - \frac{1}{p'} \right) u_j^{p-1} + \frac{1}{p'} |u|^{p'}
\]

by Young’s inequality,

\[
\int_{\Omega} u_j^{p-1} (u_j - u) dx \to 0
\]

by (2.9) and the dominated convergence theorem. Then taking \( \nu = u_j - u \) in (2.4) gives

\[
\int_{\Omega} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla (u_j - u) dx \to 0,
\]

so \( u_j \to u \) in \( W_0^{1,p}(\Omega) \) for a renamed subsequence (see, e.g., Perera et al.[14, Proposition 1.3]).
The infimum in (2.1) is attained by the family of functions

$$u_\varepsilon(x) = \frac{C_{N,p} \varepsilon^{(N-p)/p^*}}{(\varepsilon + |x|^{(p-1)/(N-p)})^p}, \quad \varepsilon > 0$$

when $\Omega = \mathbb{R}^N$, where the constant $C_{N,p} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} u_\varepsilon^p \, dx = S^{N/p}.$$

Without loss of generality, we may assume that $0 \in \Omega$. Let $r > 0$ be so small that $B_{2r}(0) \subset \Omega$, take a function $\psi \in C^\infty_0(B_{2r}(0), [0, 1])$ such that $\psi = 1$ on $B_r(0)$, and set

$$\tilde{u}_\varepsilon(x) = \psi(x) u_\varepsilon(x), \quad v_\varepsilon(x) = \frac{\tilde{u}_\varepsilon(x)}{\left(\int_{\Omega} \tilde{u}_\varepsilon^p \, dx\right)^{1/p^*}},$$

so that $\int_{\Omega} v_\varepsilon^p \, dx = 1$. Then we have the well-known estimates

$$\int_{\Omega} |\nabla v_\varepsilon|^p \, dx \leq S + C \varepsilon^{(N-p)/p}, \quad (2.10)$$

$$\int_{\Omega} v_\varepsilon^p \, dx \geq \begin{cases} \frac{1}{C} \varepsilon^{p-1}, & N > p^2, \\ \frac{1}{C} \varepsilon^{p-1} |\log \varepsilon|, & N = p^2, \end{cases} \quad (2.11)$$

where $C = C(N, p) > 0$ is a constant (see, e.g., Drábek and Huang [15]).

## 3 Proof of Theorem 1.1

First we show that $I_\mu$ has a uniformly positive mountain pass level below the threshold for compactness given in Lemma 2.1 for all sufficiently small $\mu > 0$. Let $v_\varepsilon$ be as in the last section.

**Lemma 3.1.** There exist $\mu_0, \rho, c_0 > 0, R > \rho$, and $\beta < \frac{1}{N} S^{N/p}$ such that the following hold for all $\mu \in (0, \mu_0)$:

(i) $\|u\| = \rho \Rightarrow I_\mu(u) \geq c_0$,

(ii) $I_\mu(t v_\varepsilon) \leq 0$ for all $t \geq R$ and $\varepsilon \in (0, 1]$,

(iii) denoting by $\Gamma = \{ y \in C([0, 1], W^{1,p}_0(\Omega)) : y(0) = 0, \ y(1) = R v_\varepsilon \}$ the class of paths joining the origin to $R v_\varepsilon$,

$$c_0 \leq c_\mu := \inf_{y \in \Gamma} \max_{u \in y([0,1])} I_\mu(u) \leq \beta - \left(1 - \frac{1}{p}\right) \mu |\Omega|$$

for all sufficiently small $\varepsilon > 0$,

(iv) $I_\mu$ has a critical point $u_\mu$ at the level $c_\mu$.

**Proof.** By (1.3) and (2.1),

$$I_\mu(u) \geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^p - \frac{S^{p^*/p}}{p^*} \|u\|^{p^*} - \left(1 - \frac{1}{p}\right) \mu |\Omega|,$$

and (i) follows from this for sufficiently small $\rho, c_0, \mu > 0$ since $\lambda < \lambda_1$. 

Since \( \nu_\varepsilon \geq 0 \),
\[
I_\mu(t\nu_\varepsilon) = \frac{t^p}{p} \int_{\Omega} \left( |\nabla \nu_\varepsilon|^p - \lambda \nu_\varepsilon^p \right) \, dx - \frac{t^p}{p} + \mu t \int_{\Omega} \nu_\varepsilon \, dx
\]
for \( t \geq 0 \). By the Hölder’s and Young’s inequalities,
\[
\mu t \int_{\Omega} \nu_\varepsilon \, dx \leq \mu t |\Omega|^{1-1/p} \left( \int_{\Omega} \nu_\varepsilon^p \, dx \right)^{1/p} \leq C_\lambda \mu^{p/(p-1)} + \frac{\lambda t^p}{2^p} \int_{\Omega} \nu_\varepsilon^p \, dx,
\]
where
\[
C_\lambda = \left( 1 - \frac{1}{p} \right) \left( \frac{2}{\lambda} \right)^{1/(p-1)} |\Omega|,
\]
so
\[
I_\mu(t\nu_\varepsilon) \leq \frac{t^p}{p} \int_{\Omega} \left( |\nabla \nu_\varepsilon|^p - \frac{\lambda}{2} \nu_\varepsilon^p \right) \, dx - \frac{t^p}{p} + C_\lambda \mu^{p/(p-1)}.
\]

Then by (2.10) and for \( \varepsilon, \mu \in (0, 1) \),
\[
I_\mu(t\nu_\varepsilon) \leq (S + C) \frac{t^p}{p} - \frac{t^p}{p^*} + C_\lambda,
\]
from which (ii) follows for sufficiently large \( R > \rho \).

The first inequality in (3.1) is immediate from (i) since \( R > \rho \). Maximizing the right-hand side of (3.2) over \( t \geq 0 \) gives
\[
c_\mu \leq \frac{1}{N} \left( \int_{\Omega} \left( |\nabla \nu_\varepsilon|^p - \frac{\lambda}{2} \nu_\varepsilon^p \right) \, dx \right)^{N/p} + C_\lambda \mu^{p/(p-1)},
\]
and (2.10) and (2.11) imply that the integral on the right-hand side is strictly less than \( S \) for all sufficiently small \( \varepsilon > 0 \) since \( N > p^2 \) and \( \lambda > 0 \), so the second inequality in (3.1) holds for sufficiently small \( \mu > 0 \).

Finally, (iv) follows from (i)–(iii), Lemma 2.1, and the mountain pass lemma (see Ambrosetti and Rabinowitz [16]).

Next we show that \( u_\mu \) is uniformly bounded in \( W_0^{1,p}(\Omega) \) and uniformly equi-integrable in \( L^p(\Omega) \), and hence also uniformly bounded in \( C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \) by de Figueiredo et al. [10, Proposition 3.7], for all sufficiently small \( \mu \in (0, \mu_0) \).

**Lemma 3.2.** There exists \( \mu^* \in (0, \mu_0] \) such that the following hold for all \( \mu \in (0, \mu^*) \):

(i) \( u_\mu \) is uniformly bounded in \( W_0^{1,p}(\Omega) \),

(ii) \( \int \frac{|u_\mu|^p}{p} \, dx \to 0 \) as \( |E| \to 0 \), uniformly in \( \mu \),

(iii) \( u_\mu \) is uniformly bounded in \( C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \).

**Proof.** We have
\[
I_\mu(u_\mu) = \int_{\Omega} \left( \frac{|\nabla u_\mu|^p}{p} - \frac{\lambda u_\mu^p}{p} - \frac{u_\mu^{p^*}}{p^*} \right) \, dx + \mu \left[ \int_{\{u_\mu > 0\}} u_\mu \, dx \right.
\]
\[
\left. + \int_{\{-1 < u_\mu < 0\}} \left( u_\mu - \frac{|u_\mu|^{p-1} u_\mu}{p} \right) \, dx - \left( 1 - \frac{1}{p} \right) |\{u_\mu \leq -1\}| \right] = c_\mu
\]
(3.3)
and
\[
\mathcal{I}_\mu^*(u_\mu) v = \int_\Omega \left( |\nabla u_\mu|^{p-2} \nabla u_\mu \cdot \nabla v - \lambda u_\mu^{p-1} v - u_\mu^{p+1} v \right) dx + \mu \left[ \int_{\{u_\mu > 0\}} v \, dx \right] + \int_{\{-1 < u_\mu < 0\}} \left( 1 - |u_\mu|^{p-1} \right) v \, dx = 0 \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.4}
\]

Taking \( v = u_\mu \) in (3.4), dividing by \( p \), and subtracting from (3.3) gives
\[
\frac{1}{N} \int_\Omega u_\mu^p \, dx \leq c_\mu + \left( 1 - \frac{1}{p} \right) \mu \, |\Omega| \leq \beta
\tag{3.5}
\]
by (3.1), and (i) follows from this, (3.4) with \( v = u_\mu \), and the Hölder inequality.

If (ii) does not hold, then there exist sequences \( \mu_j \to 0 \) and \( (E_j) \) with \( |E_j| \to 0 \) such that
\[
\lim_{E \to E_j} \int |u_{\mu_j}|^p \, dx > 0. \tag{3.6}
\]

Since \( (u_{\mu_j}) \) is bounded by (i), so is \( (u_{\mu_j}) \), a renamed subsequence of which then converges to some \( u \geq 0 \) weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^q(\Omega) \) for all \( q \in [1, p^* \} \) and a.e. in \( \Omega \), and
\[
|\nabla u_{\mu_j}|^p \, dx \overset{w^*}{\to} \kappa, \quad u_{\mu_j}^p \, dx \overset{w^*}{\to} v
\tag{3.7}
\]
in the sense of measures, where \( \kappa \) and \( v \) are bounded nonnegative measures on \( \overline{\Omega} \). By Lions [12, 13], then there exist an at most countable index set \( I \) and points \( x_i \in \overline{\Omega}, i \in I \) such that
\[
\kappa \geq |\nabla v|^p \, dx + \sum_{i \in I} \kappa_i \delta_{x_i}, \quad v = v^p \, dx + \sum_{i \in I} v_i \delta_{x_i},
\tag{3.8}
\]
where \( \kappa_i, v_i > 0 \) and \( v_i^{p/p^*} \leq \kappa_i/S \). Suppose \( I \) is nonempty, say, \( i \in I \). An argument similar to that in the proof of Lemma 2.1 shows that \( \kappa_i \leq v_i \), so \( v_i \geq S^{N/p} \). On the other hand, passing to the limit in (3.5) with \( \mu = \mu_j \) and using (3.7) and (3.8) gives \( v_i \leq N \beta < S^{N/p} \), a contradiction. Hence \( I = \emptyset \) and
\[
\int_\Omega u_{\mu_j}^p \, dx \to \int_\Omega v^p \, dx.
\]

As in the proof of Lemma 2.1, a further subsequence of \( (u_{\mu_j}) \) then converges to some \( u \in W_0^{1,p}(\Omega) \), and hence also in \( L^p(\Omega) \), and a.e. in \( \Omega \). Then
\[
\int_{E_j} |u_{\mu_j}|^p \, dx \leq \int_\Omega \left( |u_{\mu_j}|^p - |u|^p \right) \, dx + \int_{E_j} |u|^p \, dx \to 0,
\]
contradicting (3.6).

Finally, (iii) follows from (i), (ii), and de Figueiredo et al. [10, Proposition 3.7].

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We claim that \( u_\mu \) is positive in \( \Omega \), and hence a weak solution of problem (1.2), for all sufficiently small \( \mu \in (0, \mu_*). \) It suffices to show that for every sequence \( \mu_j \to 0 \), a subsequence of \( u_{\mu_j} \) is
positive in $\Omega$. By Lemma 3.2 (iii), a renamed subsequence of $u_{\mu_j}$ converges to some $u$ in $C_0^1(\overline{\Omega})$. We have

$$I_{\mu_j}(u_{\mu_j}) = \int_\Omega \left( \frac{|\nabla u_{\mu_j}|^p}{p} - \frac{\lambda u^p_{\mu_j}}{p} - \frac{u^{p_+}}{p^+} \right) dx + \mu_j \left[ \int_{\{u_{\mu_j} > 0\}} u_{\mu_j} - \int_{\{-1 < u_{\mu_j} < 0\}} u_{\mu_j} \right] + \frac{|u_{\mu_j}|^{p-1}|u_{\mu_j}|}{p} dx - (1 - \frac{1}{p}) \left[ \{u_{\mu_j} \leq -1\} \right] = c_{\mu_j} \geq c_0$$

by (3.1) and

$$I_{\mu_j}(u_{\mu_j}) v = \int_\Omega \left( (|\nabla u_{\mu_j}|^{p-2} \nabla u_{\mu_j} \cdot \nabla v - \lambda u_{\mu_j}^{p-1} v - u_{\mu_j}^{p-1} v) dx + \mu_j \left[ \int_{\{u_{\mu_j} > 0\}} v dx \right] + \frac{|u_{\mu_j}|^{p-1}|u_{\mu_j}|}{p} dx - (1 - \frac{1}{p}) \left[ \{u_{\mu_j} \leq -1\} \right] = c_{\mu_j} \geq c_0$$

and passing to the limits gives

$$\int_\Omega \left( \frac{|\nabla u|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^{p_+}}{p^+} \right) dx \geq c_0$$

and

$$\int_\Omega \left( (|\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda u^{p-1} v - u^{p-1} v) dx = 0 \forall v \in W_0^{1,p}(\Omega),$$

so $u$ is a nontrivial weak solution of the problem

$$\begin{cases}
-\Delta_p u = \lambda u^{p-1} + u^{p-1} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Then $u > 0$ in $\Omega$ and its interior normal derivative $\partial u/\partial \nu > 0$ on $\partial \Omega$ by the strong maximum principle and the Hopf lemma for the $p$-Laplacian (see Vázquez [17]). Since $u_{\mu_j} \to u$ in $C_0^1(\overline{\Omega})$, then $u_{\mu_j} > 0$ in $\Omega$ for all sufficiently large $j$.

It remains to show that $u_{\mu_j}$ minimizes $I_{\mu}$ on $\mathcal{N}_{\mu_j}$ when it is positive. For each $w \in \mathcal{N}_{\mu_j}$, we will construct a path $y_w \in \Gamma$ such that

$$\max_{u \in y_w([0,1])} I_{\mu_j}(u) = I_{\mu_j}(w).$$

Since

$$I_{\mu_j}(u_{\mu_j}) = c_{\mu_j} \leq \max_{u \in y_w([0,1])} I_{\mu}(u)$$

by the definition of $c_{\mu_j}$, the desired conclusion will then follow. First we note that the function

$$g(t) = I_{\mu_j}(tw) = \frac{t^p}{p} \int_\Omega (|\nabla w|^p - \lambda w^p) dx + \frac{t^{p^+}}{p^+} \int_\Omega w^{p^+} dx + \mu t \int_\Omega w dx, \quad t \geq 0$$

has a unique maximum at $t = 1$. Indeed,

$$g'(t) = t^{p-1} \int_\Omega (|\nabla w|^p - \lambda w^p) dx - t^{p^+ - 1} \int_\Omega w^{p^+} dx + \mu \int_\Omega w dx$$

$$= (t^{p-1} - t^{p^+ - 1}) \int_\Omega (|\nabla w|^p - \lambda w^p) dx + (1 - t^{p^+ - 1}) \mu \int_\Omega w dx$$

and
since $w \in \mathbb{N}_\mu$, and the last two integrals are positive since $\lambda < \lambda_1$ and $w > 0$, so $g'(t) > 0$ for $0 \leq t < 1$, $g'(1) = 0$, and $g'(t) < 0$ for $t > 1$. Hence
\[
\max_{t \leq 0} I_{\mu}(t w) = I_{\mu}(w) > 0
\]
since $g(0) = 0$. In view of Lemma 3.1 (ii), now it suffices to observe that there exists $\tilde{R} > \max \{1, R\}$ such that
\[
I_{\mu}(\tilde{R} u) = \frac{\tilde{R}^p}{p} \int_\Omega (|\nabla u|^p - \lambda u^p) \, dx - \frac{\tilde{R}^{p^*}}{p^*} \int_\Omega u^{p^*} \, dx + \mu \tilde{R} \int_\Omega u \, dx \leq 0
\]
for all $u$ on the line segment joining $w$ to $v_\varepsilon$ since all norms on a finite dimensional space are equivalent.

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References

[1] Alfonso Castro, Djairo G. de Figueiredo, and Emer Lopera. Existence of positive solutions for a semipositone $p$-Laplacian problem. *Proc. Roy. Soc. Edinburgh Sect. A*, 146(3):475–482, 2016.

[2] M. Chhetri, P. Drábek, and R. Shivaji. Existence of positive solutions for a class of $p$-Laplacian superlinear semipositone problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(5):925–936, 2015.

[3] S. Unsurangie. *Existence of a solution for a wave and elliptic Dirichlet problem*. PhD thesis, University of North Texas, Denton, 1988.

[4] W. Allegretto, P. Nistri, and P. Zecca. Positive solutions of elliptic nonpositone problems. *Differential Integral Equations*, 5(1):95–101, 1992.

[5] A. Ambrosetti, D. Arcoya, and B. Buffoni. Positive solutions for some semi-positone problems via bifurcation theory. *Differential Integral Equations*, 7(3-4):655–663, 1994.

[6] Scott Caldwell, Alfonso Castro, Ratnasingham Shivaji, and Sumalee Unsurangsie. Positive solutions for classes of multi-parameter elliptic semipositone problems. *Electron. J. Differential Equations*, pages No. 96, 10 pp. (electronic), 2007.

[7] Mohammed Guedda and Laurent Véron. Quasilinear elliptic equations involving critical Sobolev exponents. *Nonlinear Anal.*, 13(8):879–902, 1989.

[8] Haïm Brézis and Louis Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[9] P.-L. Lions. On the existence of positive solutions of semilinear elliptic equations. *SIAM Rev.*, 24(4):441–467, 1982.

[10] Djairo G. de Figueiredo, Jean-Pierre Gossez, and Pedro Ubilla. Local “superlinearity” and “sublinearity” for the $p$-Laplacian. *J. Funct. Anal.*, 257(3):721–752, 2009.

[11] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[12] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[13] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana*, 1(2):45–121, 1985.

[14] Kanishka Perera, Ravi P. Agarwal, and Donal O’Regan. *Morse theoretic aspects of $p$-Laplacian type operators*, volume 161 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[15] Pavel Drábek and Yin Xi Huang. Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbb{R}^N$ with critical Sobolev exponent. *J. Differential Equations*, 140(1):106–132, 1997.

[16] Antonio Ambrosetti and Paul H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Functional Analysis*, 14:349–381, 1973.

[17] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.