Measuring velocity ratios
with correlation functions

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Abstract

We show how to determine the ratio of the transverse velocity of
a source to the velocity of emitted particles, using split–bin
correlation functions. The technique is to measure $S_2$ and $S_2^\phi$.

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subtract the contributions from the single–particle distribution, and take the ratio as the bin size goes to zero. We demonstrate the technique for two cases: each source decays into two particles, and each source emits a large number of particles.

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1 Introduction

There has been considerable study of correlation functions in high energy and nuclear physics. However, with the exception of source size measurements using identical particle (Hanbury–Brown/Twiss) interferometry [1], the connection between measurements of correlation functions and the physics of particle collisions is tenuous. For example, it is possible to measure the size of correlation sources (the number of pions produced) in high energy collisions [2], but no other source characteristics are currently identifiable. This is unfortunate, as the source size is anomalously large in ultra–relativistic nuclear collisions [2, 3] (the sources decay into at least fifteen pions each [4]), and the nature of these sources has not yet been determined.

In this article, we propose to measure the velocities of correlation sources, the objects that make up the intermediate structure of collisions, by using the rapidity, $y$, and azimuthal angle, $\phi$, correlation functions [5]

$$S_2(\delta y; \Delta Y) = \frac{\langle \rho^{(2)} \rangle_{\delta y}}{\langle \rho^{(2)} \rangle_{\Delta Y}},$$

(1)

and

$$S_2^\phi(\delta y; \Delta Y) = \frac{\langle \rho^{(2)} \rangle_{\phi \delta y}}{\langle \rho^{(2)} \rangle_{\phi \Delta Y}},$$

(2)

where measurements are made over a window from $Y_0$ to $Y_0 + \Delta Y$. Here

$$\langle \rho^{(2)} \rangle_x = \sum_{i=1}^{\Delta Y/x} \int_{Y_0+(i-1)x}^{Y_0+i x} dy_1 \int_{Y_0+(i-1/2)x}^{Y_0+(i+1/2)x} dy_2 \rho^{(2)} \frac{\Delta Y/4}{x \Delta Y/4},$$

(3)
and

\[
\langle \rho^{(2)} \rangle_x^\phi = \sum_{i=1}^{\Delta Y/x} \frac{\int_{Y_0+(i-1)x}^{Y_0+ix} dy_1 \int_0^\pi d\phi_1^{\text{lab}} \int_{Y_0+(i-1)x}^{Y_0+ix} dy_2 \int_\pi^{2\pi} d\phi_2^{\text{lab}} \rho^{(2)}}{x \Delta Y} ,
\]

(4)

where \( \Delta Y \) is an integer multiple of \( x \). Split–bin correlation functions are useful for measuring correlations with maximum statistics (without re–use of data). \( S_2 \) is closely related to the standard two–particle correlation function:

\[
S_2(\delta y; \Delta Y) \simeq \frac{\rho^{(2)}(0, \delta y/2)}{\rho^{(2)}(0, \Delta Y/2)} .
\]

(5)

For large \( \Delta Y \), the correlation functions are almost independent of \( \Delta Y \), but in practice it is impossible to make correlation measurements in particle collisions without using a large bin for reference, as the amount of produced matter fluctuates from event to event.

The proposed technique is to measure the ratio

\[
R(\Delta Y) = \frac{S_2^\phi(0; \Delta Y) - \Sigma_2^\phi(\Delta Y)}{S_2(0; \Delta Y) - \Sigma_2(\Delta Y)} .
\]

(6)

Here

\[
\Sigma_2(\Delta Y) = \frac{\Delta Y}{4} \int_{Y_0}^{Y_0+\Delta Y} dy \rho^2(y)
\]

(7)

and

\[
\Sigma_2^\phi(\Delta Y) = \frac{\Delta Y}{4} \int_{Y_0}^{Y_0+\Delta Y} dy \int_0^\pi d\phi_1^{\text{lab}} \int_0^\pi d\phi_2^{\text{lab}} \rho(y, \phi_1^{\text{lab}}) \rho(y, \phi_2^{\text{lab}})
\]

(8)

are the values of \( S_2 \) and \( S_2^\phi \) from the single–particle distributions, \( \rho(y) \) and \( \rho(y, \phi^{\text{lab}}) \). In the case of a flat distribution \([\rho(y, \phi^{\text{lab}}) = \text{const.}]\),
\( \Sigma_2 = \Sigma_2^0 = 1 \). It is also possible to measure emission velocities using standard two–particle interferometry techniques; however, these measurements are somewhat more difficult than the technique proposed here, as they require particle identification. The technique proposed here has the advantage of being useable even in the absence of particle identification.

We assume isotropic emission and show that, in the non–relativistic limit, \( R \) depends only on the velocity ratio, \( v_s/v_\pi \), where \( v_s \) is the source transverse velocity, and \( v_\pi \) is the velocity of emitted particles in the source rest frame. In section \( \text{2} \), we calculate \( R \) for the case of a resonance that decays isotropically into two particles. In section \( \text{3} \), we calculate \( R \) for isotropic decay of a thermal source, finding that the shape is not very different from the first case. Finally, we summarize our results and discuss applications and future work in section \( \text{4} \).

## 2 Two–particle decay

Suppose that we have an event with a single resonance that decays into two particles, and a flat background of \( N - 2 \) particles in a window of rapidity \( \Delta Y \). The background contribution to the two–particle density is then constant:

\[
\rho_0^{(2)}(y_1, \phi_1^{\text{lab}}, y_2, \phi_2^{\text{lab}}) = \frac{(N - 2) (N - 3)}{(4 \pi^2 \Delta Y^2)}.
\]  

\( \text{(9)} \)
Adding the contribution from combining one background particle with one particle from the resonance, we obtain the total uncorrelated signal,

$$
\rho_u^{(2)}(y_1, \phi_{1}^{\text{lab}}, y_2, \phi_{2}^{\text{lab}}) = \left[ N (N - 1) - 2 \right] / (4 \pi^2 \Delta Y^2). \tag{10}
$$

Given the resonance azimuthal angle, $\phi_r$, and the resonance rapidity, $y_r$, we calculate the laboratory coordinates, $y$ and $\phi^{\text{lab}}$, as functions of the center-of-momentum coordinates, $\theta$ and $\phi$.

$$
y = \tanh^{-1} \left( \frac{(1 - v_s^2)^{1/2} v_\pi \cos \theta}{1 + v_s v_\pi \sin \theta \sin \phi} \right) + y_r, \tag{11}
$$

$$
\phi^{\text{lab}} = \tan^{-1} \left( \frac{v_s + v_\pi \sin \theta \sin \phi}{(1 - v_s^2)^{1/2} v_\pi \sin \theta \cos \phi} \right) + \pi H(-\cos \phi) + \phi_r, \tag{12}
$$

where

$$
H(x) = \begin{cases} 
0 & x \leq 0, \\
1 & x > 0.
\end{cases} \tag{13}
$$

[We use standard high–energy units, with $c = 1$. In deriving eq. (12), we assume that $-\pi/2 \leq \tan^{-1} x < \pi/2$.] We assume that $y_r$ and $\phi_r$ are distributed randomly, with flat distributions. The resonance contribution to $\rho^{(2)}$ is simplest in the resonance rest frame:

$$
\rho_r^{(2)}(\theta_1, \phi_1, \theta_2, \phi_2) = \frac{\sin \theta_1 \sin \theta_2 \delta (\cos \theta_1 + \cos \theta_2) \delta (|\phi_1 - \phi_2| - \pi)}{2\pi}, \tag{14}
$$

where $\delta$ is the Dirac $\delta$–function.

The resonance contribution to $S_2(\delta y; \Delta Y)$ is

$$
S_{2,r}(\delta y; \Delta Y) = \frac{\Delta Y \int_0^{\Delta Y} dy_r \int_{\delta y/2}^{\delta y/2} dy_2 \int_0^{2\pi} d\phi_2 \rho_r^{(2)}(y_1, \phi_1, y_2, \phi_2)}{\delta y^2 h^2 \Delta Y^2 \rho_u^{(2)}(y_1, y_2)}, \tag{15}
$$

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where
\[ \rho_t^{(2)}(y_1, \phi_1, y_2, \phi_2) = \left| \frac{\partial \theta_1}{\partial y_1} \right| \left| \frac{\partial \theta_2}{\partial y_2} \right| \rho_t^{(2)}(\theta_1, \phi_1, \theta_2, \phi_2). \] (16)

As we assume flat distributions, the contribution from all bins is the same, so we use only the bin from \( y = 0 \) to \( \delta y \), and take \( Y_0 = 0 \). The contribution at \( \delta y = 0 \) is
\[ S_{2,r}(0; \Delta Y) = \lim_{\delta y \to 0} \frac{4 \Delta Y \left[ \int_{\delta y/4}^{\delta y/2} dy_r \int_0^{2y_r - \delta y/2} dy_1 + \int_{\delta y/2}^{3\delta y/4} dy_r \int_{2y_r - \delta y}^{\delta y/2} dy_1 \right]}{\delta y^2 \left[ N(N - 1) - 2 \right] \left[ (1 - v_s^2)^{1/2} v_\pi \right]} \], (17)
\[ = \frac{\Delta Y}{2 \left[ N(N - 1) - 2 \right] \left( 1 - v_s^2 \right)^{1/2} v_\pi}. \] (18)

Evaluating \( S_{2,\phi} \) in the same manner, we obtain
\[ S_{2,\phi}(0; \Delta Y) = \frac{\Delta Y \int_{\phi_{\text{lab}}}^{2\pi} d\phi_r \int_{\phi_{\text{lab}} = 0}^{2\pi} d\phi_1 \int_{\phi_{\text{lab}} = \pi}^{2\pi} d\phi_2 \delta(\phi_1 - \phi_2 - \pi)}{2 \pi^2 \left[ N(N - 1) - 2 \right] \left( 1 - v_s^2 \right)^{1/2} v_\pi}, \] (19)

where
\[ g(x) = \begin{cases} x & 0 \leq x < \pi, \\ 2\pi - x & \pi \leq x < 2\pi. \end{cases} \] (21)

In the limit \( \Delta Y \to \infty \),
\[ S_2 = \Sigma_2 + S_{2,r}, \quad S_2^\phi = \Sigma_2^\phi + S_{2,\phi}. \] (22)

Our ratio is then
\[ R = \frac{1}{\pi^2} \int_{\phi = 0}^{2\pi} d\phi g(\phi) \left( \phi_{\text{lab}}(\phi) - \phi_{\text{lab}}(\phi + \pi) \right), \] (23)
Note that this result is independent of the size of the background, \( N \)! In fact, if we allow the presence of more than one resonance, we find that \( S_{2,r} \) and \( S_{2,r}^\phi \) are both proportional to the number of sources, \( n_s \), so \( R \) is also independent of the number of resonances.

For the case of non–relativistic resonance decay, eq. (12) reduces to

\[
\phi_{\text{lab}} = \tan^{-1} \left( \frac{v_s/v_\pi + \sin \phi}{\cos \phi} \right) + \pi H (-\cos \phi) + \phi_r, \tag{24}
\]

as we can take \( \cos \theta = 0 \). It is clear from eqs. (23) and (24) that, in the non–relativistic limit, \( R \) depends only on the ratio \( v_s / v_\pi \). In Fig. 1, we show \( R \) for \( v_\pi = c/2 \) and \( v_\pi = c \), along with the value in the non–relativistic limit.

In both the relativistic and non–relativistic cases, \( R(0) = 2 \); this is an identity, due to the different topologies of \( \phi \) and \( y \) (\( \phi = 2\pi \) is equivalent to \( \phi = 0 \), while \( y = \delta y \) is not equivalent to \( y = 0 \)). The behavior as \( v_\pi \to 1 \) is analytically intractable, but from Fig. 1 it is clearly not too different from the non–relativistic case. The limit \( v_s \to 1 \) can be taken analytically, yielding

\[
R = \frac{2 (1 - v_s^2)^{1/2}}{\pi^2} \ln \left[ \frac{v_s + v_\pi}{v_s - v_\pi} \right]. \tag{25}
\]

The limit of large \( v_s \) differs only by a factor \( \gamma_s = (1 - v_s^2)^{1/2} \) in the relativistic and non–relativistic cases.

### 3 Thermal sources

A typical example of a source that emits a large number of particles is a thermal source, such as a large droplet. Imagine that we have a single
source, as before, but that the source emits \( n_\pi \) particles, where \( n_\pi \gg 1 \).

The two–particle density from this source is

\[
\rho^{(2)}_r(\theta_1, \phi_1, \theta_2, \phi_2) = \frac{n_\pi (n_\pi - 1) \sin \theta_1 \sin \theta_2}{16 \pi^2}.
\] (26)

The \( \delta \)–function constraints are absent, as for large \( n_\pi \) we can essentially ignore conservation of energy and momentum.

Again moving to the resonance rest frame, and taking \( \Delta Y \to \infty \), we obtain

\[
S_{2,\pi}(0) \propto n_\pi (n_\pi - 1) \int_{-\infty}^{\infty} dy \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \left| \frac{\partial \theta_1}{\partial y} \right| \left| \frac{\partial \theta_2}{\partial y} \right| \sin \theta_1 \sin \theta_2 ,
\] (27)

and

\[
S^\phi_{2,\pi}(0) \propto 4n_\pi (n_\pi - 1) \int_{-\infty}^{\infty} dy \int_{\phi_1^{\text{lab}}=0}^{\pi} d\phi_1 \int_{\phi_2^{\text{lab}}=\pi}^{2\pi} d\phi_2 \left| \frac{\partial \theta_1}{\partial y} \right| \left| \frac{\partial \theta_2}{\partial y} \right| \sin \theta_1 \sin \theta_2.
\] (28)

Taking the ratio as before, we find

\[
R = \frac{2 \int_{-\infty}^{\infty} dy \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \left| \frac{\partial \theta_1}{\partial y} \right| \left| \frac{\partial \theta_2}{\partial y} \right| \sin \theta_1 \sin \theta_2 g \left( |\phi_1^{\text{lab}} - \phi_2^{\text{lab}}| \right)}{\pi \int_{-\infty}^{\infty} dy \left[ \int_0^{2\pi} d\phi \left| \frac{\partial \theta}{\partial y} \right| \sin \theta \right]^{2}}.
\] (29)

Note that \( R \) is independent of \( n_\pi \) as well as being independent of \( N \!\!\!\!' \). In fact, \( R \) is even independent of the number of sources, as before.

We cannot evaluate the relativistic denominator for the thermal source, but we can take the non–relativistic limit as before. We obtain

\[
R = \frac{1}{4 \pi^3} \int_0^{\pi} d\theta \sin \theta \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 g \left( |\phi_1^{\text{lab}} - \phi_2^{\text{lab}}| \right) ,
\] (30)

where

\[
\phi^{\text{lab}} = \tan^{-1} \left( \frac{v_\pi/v_s + \sin \theta \sin \phi}{\sin \theta \cos \phi} \right) + \pi H (-\cos \phi) + \phi_r.
\] (31)
It is again clear that, in the non–relativistic limit, $R$ depends only on $v_s / v_\pi$.

In Fig. 2, we display $R$ as in the case of two–particle decay. We find now that $R(0) = 1$ and $R(\infty) = 0$, so the normalization is clearly different from that obtained for two–particle decay. Because the curves are different, there will be some dependence of the measured velocity on the source size. If the sources are known to emit many pions, as is the case in ultra–relativistic nuclear collisions, this dependence is weak. However, any velocity measurement using the proposed technique depends somewhat on knowledge of the source size.

4 Conclusions

We have demonstrated a new procedure for measuring the ratio of a source transverse velocity to the velocity of emitted particles, using the split–bin correlation functions $S_2$ and $S_2^\phi$. We showed the technique for two simple cases: two–body decay, and unconstrained decay. In both cases, the technique gives a measurement that is independent of the number of sources, and the background size. In addition, we find that the measurement is independent of the source size for large sources. We have considered somewhat simple cases for pedagogical purposes, in order to illustrate our technique in a simple, model–independent framework. The specific issue of contamination by dynamical effects should be addressed separately.

Similar techniques could be used to measure many other quantities, so the potential usefulness of this line of enquiry is enormous. For example,
comparison of energy and particle–number correlation functions could provide a relatively simple way to measure the energy distribution of minijets in ultra–relativistic nuclear collisions. These techniques are probably of the most use in high multiplicity collisions, where individual sources often cannot be resolved. We hope that our results will encourage other researchers to investigate new uses for correlation functions.

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Figure captions

1. $R$ vs. $\tanh \left( \frac{\gamma_s}{v_s} / v_\pi \right)$ for two-particle decay.

2. $R$ vs. $\tanh \left( \frac{\gamma_s}{v_s} / v_\pi \right)$ for a thermal source.
Two-Particle Decay

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Graph showing the decay of two particles with different velocities $v_\pi$.}
\end{figure}
Thermal Source

\[ R \]

\[ \tanh\left(\frac{v_s \gamma_s}{v_\pi}\right) \]

- \( v_\pi = 1 \) (dotted line)
- \( v_\pi = 1/2 \) (dashed line)
- \( v_\pi = 0 \) (solid line)

Figure 2