Suboptimal Stabilization of Unknown Nonlinear Systems via Extended State Observers*

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Abstract—This paper introduces a locally optimal stabilizer for multi-input multi-output autonomous nonlinear systems of any order with totally unknown dynamics. The control scheme proposed in this paper lies at the intersection of the active disturbance rejection control (ADRC) and the state-dependent Riccati equation (SDRE) technique. It is shown that using an extended state observer (ESO), a state-dependent coefficient matrix for the nonlinear system is obtainable which is used by the SDRE technique to construct a SDRE+ESO controller. As the SDRE technique is not guaranteed to be globally asymptotically stable, for systems with known linearization at the equilibrium, an algorithmic method is proposed for an approximated estimation of its region of attraction (ROA). Then, it is shown that the global asymptotic stability is achievable using a switching controller constructed by the SDRE+ESO method and ADRC for inside and outside the estimated ROA, respectively.

I. INTRODUCTION

The active disturbance rejection control (ADRC) has demonstrated powerful features in dealing with uncertainties according to its use of high-gain observers capable of estimating the total dynamics of the system as an extended state variable [1], [2]. However, ADRC still may not be a proper solution for many systems according to its demand of control effort emanating from its feedback linearization nature. This paper proposes a suboptimal solution for this problem inspiring from the state-dependent Riccati equation (SDRE) control technique. The proposed scheme comprises the advantages of the ADRC as well as the SDRE method that makes it applicable for unknown systems without demanding high computational capabilities or offline heavy simulations.

The SDRE technique has been widely used in the literature since 1960s as a tool for suboptimal stabilization of nonlinear systems [3], [4] and showed its effectiveness for many engineering applications. The suboptimal control scheme constructed by SDRE is appealing due to its simplicity by direct implementation of the LQR scheme for nonlinear systems. The main idea behind the SDRE scheme is to express a nonlinear system \( \dot{x} = f(x) \) as \( \dot{x} = A(x)x \) such that \( A(x)x = f(x) \) where \( A(x) \) is called the state-dependent coefficient (SDC) matrix. The SDC matrix is not unique and several researches have been carried out on its existence [5] and methods for its selection and the way it effects the system response [6]–[8]. Although SDRE does not guarantee a globally asymptotically stable (GAS) closed-loop response for nonlinear systems, its effectiveness has demonstrated by simulations. It is shown that SDRE control method, under mild conditions, is locally optimal and locally stable [9], [10], therefore, there exists a region of attraction (ROA) and there are methods for its calculation [11], [12]. It is provable that for systems where the closed-loop SDC matrix, \( A_{CL} \), happens to be globally symmetric (which comprises the scalar systems), the SDRE scheme provides a GAS response [13]. Also, it is shown that if \( \exp(A_{CL}t) \) is globally bounded by a positive definite matrix then the closed-loop system under an SDRE scheme is GAS [14].

In this paper an SDRE scheme is proposed for a special type of unknown nonlinear systems in which the SDC matrix as well as the state variables are estimated through an ESO, which is called a SDRE+ESO controller. In spite of the ESO capability of estimating the whole model of the system, unlike the previous SRDE schemes, the presented method in this study does not need any modeling for the nonlinear system (in fact the only thing that is needed is the sign of the system input). Therefore, the proposed scheme provides a suboptimal stabilizer for unknown nonlinear systems which is implementable in real time with a very low computational effort. The state-of-the-art proposed methods for the determination of the ROA need additional information about the system which may not be available for an unknown system. Accordingly, a method for an approximate estimation of the ROA is presented for those systems with known linearization at the zero equilibrium (which is the case for a wide range of systems such as robots or spacecraft in a proximity operation). The proposed approximated ROA is not a real domain of attraction for the closed-loop system, but in combination with a switching controller constructed by SDRE+ESO and ADRC provides the GAS for the closed-loop system while it is an optimal stabilizer when approaching zero equilibrium.

II. PRELIMINARIES

First, basic definitions and formulations are reviewed in this section. Next, the control scheme is introduced and finally, the essential assumptions are presented.

A. Notation

Let \( \mathbb{R}^{m \times n} \) and \( \mathbb{R}^{n} \) denote the space of real \( m \times n \) matrices and \( n \)-dimensional vectors, respectively. Also, let \( \mathbb{N} \) denote the set of all natural number. The \( i \)th element of a vector \( v \) and the \( ij \)th element of a matrix \( M \) are referred to by scalars \( v(i) \) and \( M(i,j) \), respectively. An \( m \times n \) matrix with all elements equal to \( a \) is shown as \( [a]_{m \times n} \). The \( n \)-dimensional identity matrix is denoted by \( I_{n} \). The Hadamard

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\( A \)
\( x \)
\( f \)
\( A(x) \)
\( f(x) \)
\( A_{CL} \)
\( \exp(A_{CL}t) \)
\( A_{CL} \)
\( I \)
\( m \)
\( n \)
\( R \)
\( R^{m \times n} \)
\( R^{n} \)
\( \mathbb{N} \)
\( v(i) \)
\( M(i,j) \)
\( [a]_{m \times n} \)
\( I_{n} \)
and Kronecker products are denoted by $\otimes$ and $\otimes$, respectively. Operator $\otimes$ for vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ reads $\mathbb{R}^{nxm} \ni C = a \otimes b \Rightarrow C_{(i,j)} = a(i)/b(j)$.

The derivative of a matrix $M \in \mathbb{R}^{mxn}$ with respect to a vector $v \in \mathbb{R}^r$ is defined as $\partial M/\partial v = [\partial M/\partial v(1), \ldots, \partial M/\partial v(n)] \in \mathbb{R}^{mxnr}$.

**B. System definition**

In this paper an autonomous $k$-th order, $n$-dimensional system is considered as follows:

$$
\begin{cases}
\dot{x}_i = x_{i+1} : & i = 1, \ldots, k-1 \\
\dot{x}_k = f(x) + G(x)u \\
y = x_1
\end{cases}
$$

(1)

where $x = [x_1^T, \ldots, x_k^T]^T \in \mathbb{R}^{kn}$ and $y \in \mathbb{R}^n$ denote the state and the output of the system. The continuous differentiable vector valued function $f(\cdot) : \mathbb{R}^{kn} \rightarrow \mathbb{R}^n$ defines the system dynamics, such that $f(x) = 0$ if $x = 0$, which is considered generally unknown in this study. At least, the sign of the matrix valued function $G(\cdot) : \mathbb{R}^{kn} \rightarrow \mathbb{R}^{nxn}$ is known (the minimum information on how the control input is affecting the system dynamics) and it is always invertible.

System (1) is expressible as:

$$
\dot{x} = A(x)x + \tilde{B}(x)u
$$

(2)

where $A(\cdot) \in \mathbb{R}^{nxkn}$ and $\tilde{B}(\cdot) \in \mathbb{R}^{nxn}$ are

$$
A(x) = \begin{bmatrix} 0 & \mathbb{I}_{(k-1)n} \\ \mathbb{F}(x) \end{bmatrix}, \tilde{B}(x) = \begin{bmatrix} 0 \\ G(x) \end{bmatrix}
$$

(3)

in which $\hat{G}(x)$ is our best estimation of the real $G(\cdot)$ such that satisfies $\text{sgn} \left( x^T \hat{G}(x) x \right) = \text{sgn} \left( x^T G(x) x \right) \forall x \in \mathbb{R}^{kn} - \{0\}$ (having the same sign for $G(x)$ and $\hat{G}(x)$ is essential since obviously a sign change may cause a divergence in the closed-loop response). Therefore, when $G(x) = \hat{G}(x)$, then, as is considered in the previous literature, $F(x)$ is an SDC matrix for $f(x)$, i.e. $F(x)x = f(x)$. On the other hand, when $G(x) \neq \hat{G}(x)$, then, $F(x)$ is an SDC matrix for $f(x) + \delta G(x)u$ defining $\delta G(x) = G(x) - \hat{G}(x)$, i.e. $F(x)x = f(x) + \delta G(x)u$, and its construction is supposed to be same as the previous form as long as $u$ is a function of $x$. The SDC matrix is not unique, for instance, $F(x) + E(x)$ is another SDC matrix if $E(x)x = 0$.

**C. State and SDC matrix observation**

Consider the nonlinear ESO in the following general form in order to estimate the value of $x(t)$ as well as the extended state variable $x_{k+1} = \hat{x}_k - \hat{G}(x)u$ [15]:

$$
\begin{cases}
\dot{\hat{x}}_i = \hat{x}_{i+1} + e_i(y - \hat{x}_1) : & i = 1, \ldots, k-1 \\
\dot{\hat{x}}_k = \hat{x}_{k+1} + e_k(y - \hat{x}_1) + \hat{G}(\hat{x})u \\
\hat{x}_{k+1} = e_{k+1}(y - \hat{x}_1)
\end{cases}
$$

(4)

where $e_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for $i = 1, \ldots, k+1$ are (linear or nonlinear) functions to be determined.

**Remark 1:** The extended state $\hat{x}_{k+1}$ is supposed to estimate the value of $\hat{x}_k - \hat{G}(x)u$. In fact, when $G(\cdot) = \hat{G}(\cdot)$, the extended state $\hat{x}_{k+1}$ converges to the value of $f(\hat{x})$. On the other part, when $G(\cdot) \neq \hat{G}(\cdot)$, the extended state $\hat{x}_{k+1}$ estimates the value of $f(\hat{x}) + \delta G(\hat{x})u$.

In addition to the ESO formulated above, we consider the following observation to estimate the value of $A(x)$:

$$
\hat{A}(x) = \frac{[0]_{(k-1)nxn} \mathbb{I}_{(k-1)n}}{F(x)}
$$

(5)

in which $\hat{F}(x)$ should be estimated according to the outputs of ESO expressed by (4). A proper formula for $\hat{F}(x)$ should satisfy the following properties summarized in a remark:

**Remark 2:** The considered formula for the SDC matrix $\hat{F}(x)$ should (1) satisfy $\hat{x}_{n+1} = \hat{F}(x)\hat{x}$; (2) have bounded solutions for any value of $\hat{x}$; (3) contain all elements of $\hat{x}$ in order to (as a rule of thumb) increase the ROA.

Accordingly, consider the following form:

$$
\hat{F}(x) = W(x) \otimes (\hat{x}_{k+1} \otimes \hat{x})
$$

(6)

where $W(\cdot) : \mathbb{R}^{kn} \rightarrow \mathbb{R}^{nxkn}$ should be chosen such that above-mentioned properties of Remark 2 are satisfied.

1) **Continuous SDC matrix:** Matrix valued function $W(\hat{x})$ can be constructed as follows for any bounded $\rho_i \neq 0$:

$$
W_{i,j}(\hat{x}) = \begin{cases}
\rho_i \rho_{i,j}(\hat{x})/|\mathcal{J}_1| & \text{if } j \in \mathcal{J}_1 \\
(1 - \rho_i \rho_{i,j}(\hat{x}))/|\mathcal{J}_2| & \text{if } j \in \mathcal{J}_2
\end{cases}
$$

(7)

The sets $\mathcal{J}_1, \mathcal{J}_2 \subset \mathbb{N}$ (which are generally bounded) so that $\mathcal{J}_1 \cup \mathcal{J}_2 = \{1, \ldots, kn\}, \mathcal{J}_1, \mathcal{J}_2 \neq \emptyset, \mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$ (8)

in which $\rho_i(\hat{x})$ and $1 - \rho_i(\hat{x})$ as well as their partial derivatives should go to zero faster than their denominators when they are used in constructing $\hat{F}(x)$. The following example proposes a formula (which is inspired from the fact that $\lim_{x \rightarrow 0} \exp(-\alpha|x|/x) = 0$ and $\lim_{x \rightarrow 0} (1 - \exp(-\beta|x|))/x = \pm \beta$ for any $\alpha, \beta > 0$) for the evaluation of $\rho_i(\hat{x})$:

**Example 1:** The following function can be used in (7) for systems with $n \geq 2$ considering $\mathcal{J}_1 \equiv \mathcal{J} = \{i, i+n, \ldots, i+(k-1)n\}$ for all $\omega > 0$:

$$
\rho_i(\hat{x}) = \exp \left( -\infty \prod_{j \in \mathcal{J}, (i,k,n]} |\hat{x}(j)| \prod_{j \in \mathcal{J}, (i,k,n]} |\hat{x}(j)| \right)
$$

(9)

**Example 2:** For a second-order scalar system ($k = 2, n = 1$) we have:

$$
\hat{F}(x) = \hat{x}_3 \left[ \exp(-\hat{x}_2/\hat{x}_1) - 1 \exp(-\hat{x}_2/\hat{x}_1) \right]
$$

(10)

2) **Discontinuous SDC matrix:** Accordingly, to prevent singularity problems, the following switching form can be considered corresponding to (6) for all $i = 1, \ldots, n$ and any bounded $\rho_i \neq 0$:

$$
W_{i,j}(\hat{x}) = \begin{cases}
\rho_i \hat{x}_j & \text{if } j = j^* \\
(1 - \rho_i \hat{x}_j)/w_i(j) & \text{if } j \neq j^*
\end{cases}
$$

(11)

where $w_i(j) \in \mathbb{R}$ satisfies $\sum_{j=1,j \neq j^*}^{kn} w_i(j) = kn - 1$ such that

$$
j^* \equiv j^*(\hat{x}) = \text{argmin}_{j \in (1, \ldots, kn)} |\hat{x}(j)|
$$

(12)
Example 3: For a second-order scalar system \((k = 2, n = 1)\) we have:
\[
\tilde{F}(\tilde{x}) = \begin{cases} \tilde{x}_1 [1 - \tilde{x}_1/\tilde{x}_2] & \text{if } \tilde{x}_1 < \tilde{x}_2 \\
\tilde{x}_1 [(1 - \tilde{x}_2)/\tilde{x}_1] & \text{if } \tilde{x}_1 > \tilde{x}_2 
\end{cases}
\]

\(D. \text{ Control Scheme}\)

We consider a switching controller in the form of SDRE technique equipped by ESO for \(x \in \Omega\) and in the form of ADRC for \(x \notin \Omega\) in which \(\Omega \subseteq \mathbb{R}^{kn}\) is the ROA or an estimation of that without instability effects:
\[
u = \begin{cases}
\nu_0 & \text{if } t < \tau \quad \text{(Start-up)} \\
\nu_{in}(\tilde{x}) & \text{if } \tilde{x} \in \Omega \quad \text{(SDRE + ESO)} \\
\nu_{out}(\tilde{x}) & \text{if } \tilde{x} \notin \Omega \quad \text{(ADRC)}
\end{cases}
\]
\[
u_{in}(\tilde{x}) = -K_{in}(\tilde{x})\tilde{x}
\]
\[
u_{out}(\tilde{x}) = -\hat{G}^{-1}(\tilde{x})(K_{out}\tilde{x} + \tilde{x}_{k+1})
\]
and \(\nu_0\) is some bounded controller to be determined (e.g. \(\nu_0 = 0\)) for the start-up phase at which the ESO is not sufficiently converged. The gain matrices
\[
K_{in}(\tilde{x}) = R^{-1}\hat{B}_T(\tilde{x})P_{in}(\tilde{x})
\]
\[
K_{out} = R^{-1}B_0^TP_{out}
\]
are found by solving the following equations for \(P_{in}(\tilde{x})\) and \(P_{out}\), respectively:
\[
\hat{A}_T(\tilde{x})P_{in}(\tilde{x}) + P_{in}(\tilde{x})\hat{A}(\tilde{x}) = -P_{in}(\tilde{x})\hat{B}(\tilde{x})R^{-1}\hat{B}_T(\tilde{x})P_{in}(\tilde{x}) + Q = 0
\]
\[
A_0^TP_{out} + P_{out}A_0 - P_{out}B_0R^{-1}B_0^TP_{out} + Q = 0
\]
where the pair \(\{A_0, B_0\}\) defines a chain integrator system corresponding to the size of \((1)\) as:
\[
A_0 = \left[ \begin{array}{c|c}
0 & I_{(k-1)n} \\
0 & I_n 
\end{array} \right], \quad B_0 = \left[ \begin{array}{c}
0 \\
I_n 
\end{array} \right]
\]
and symmetric positive-definite matrices \(Q \in \mathbb{R}^{kn \times kn}\) and \(R \in \mathbb{R}^{n \times n}\) are associated with the cost function \(J = 0.5 \int_0^\infty (x^TQx + u^TR_0u)dt\). Since the global asymptotic stability cannot be generally guaranteed for a closed-loop system with \(\nu_{in}\), the ADRC is used as \(\nu_{out}\) to ensure the stability outside \(\Omega\). However, for an unknown system, obtaining \(\Omega\) may be impossible. In this case, if only the first derivative of \(f(x)\) at the origin, \((df/dx)_{x=0}\), is available, then we can obtain some approximate estimations. In this study we propose the necessary condition for \(\Omega\) to be a ROA (inspired from the results of \([12]\)), which is the satisfaction of the Lyapunov function \(V(x) = 0.5x^TPx\). Therefore, one may consider:
\[
\tilde{\Omega} \approx \{\tilde{x} \in \mathbb{R}^{kn} : \tilde{x}^TP(\tilde{x}_{k+1} - \tilde{B}(\tilde{x})K_{in}(\tilde{x}))\tilde{x} < 0 \}
\]
with some \(P\) satisfying the following inequality:
\[
J_{CL}(0)^T P + PJ_{CL}(0) < 0
\]
in which \(J_{CL}(0)\) is the Jacobian matrix of the closed-loop system under \(u = u_{in}\) at \(x = 0\). The following steps are considered for obtaining the value of \(J_{CL}(0)\) as an algorithm that can be done once offline:

Algorithm 1 (Closed-loop Jacobian): This algorithm obtains the Jacobian of the closed-loop system described by system \((1)\) under controller \(u = u_{in}\) defined by \((15)\). It is supposed that we already know the values of \(\partial f/\partial x\) as well as \(G(x)\) (or similarly \(B(x)\)) at \(x = 0\). Then, follow these steps:

1) Construct the following matrix:
\[
J(0) = \left[ \begin{array}{c}
0_{(k-1)n \times n} I_{(k-1)n} \\
\partial f(x)/\partial x_{x=0} 
\end{array} \right]
\]

2) Find \(P_{in}(0)\) by solving the following equation:
\[
J^T(0)P_{in}(0) + P_{in}(0)J(0) - P_{in}(0)B(0)R^{-1}B^T(0)P_{in}(0) + Q = 0
\]

3) Substitute the calculated matrices in the following equation and obtain \(J_{CL}(0)\):
\[
J_{CL}(0) = J(0) + B(0)R^{-1}B^T(0)P_{in}(0)
\]

E. Assumptions

In the rest of the paper we frequently refer to the following assumptions:

Assumption 1 (Open-loop system): Suppose:
1) All partial derivatives of \(f(x)\) exists at all \(x \in \mathbb{R}^{kn}\) and \(t \in [0, \infty)\).
2) There exists positive scalars of \(\alpha_i\) and \(\beta_i\) for \(i = 1, \ldots, k\) such that \(\|f\| + \|\partial f/\partial x\| \leq \alpha_0 + \sum_{i=1}^{k} \alpha_i \|x_i\|^{\beta_i}\) for all \(x \in \mathbb{R}^{kn}\).

Assumption 2 (SDC matrix): Suppose the respective pairs \(\{\hat{A}(\tilde{x}), \hat{B}(\tilde{x})\}\) and \(\{\hat{A}(\tilde{x}), Q^{1/2}\}\) are globally pointwise controllable and observable SDC parameterizations of the nonlinear system \((1)\) for all \(\tilde{x} \in \tilde{\Omega}\).

Assumption 3 (ROA): Suppose \(\tilde{\Omega}\) in \((14)\) is considered small enough such that if \(\tilde{x} \in \tilde{\Omega}\), then \(x\) lies inside the ROA.

Assumption 4 (Closed-loop system): Suppose there exists a positive scalar \(\gamma\) such that the closed-loop solution of the deterministic system satisfies \(\|u\| + \|x\| < \gamma\) for all \(t \in [0, \infty)\) for all initial conditions in \(\mathbb{R}^{kn}\).

Assumption 5 (Observer): Under Assumptions 1 and 4, the ESO formulated in \((4)\) is sufficiently convergent which means that for some arbitrarily small \(a_i > 0\) there exists a \(\tau_i \in [0, \infty)\) such that for any \(t \in [\tau_i, \infty)\), \(\|\tilde{x}_i(t) - x_i(t)\| < a_i\) and there exists a \(b_i > 0\) such that for \(t \in [0, \tau_i)\), \(\|\tilde{x}_i(t) - x_i(t)\| < b_i\).

Assumption 5 is not a strong assumption while linear and nonlinear ESOs proposed in \([2]\) can satisfy the condition by adjusting some scalars. The first part of Assumption 5 states that the ESO converges to an arbitrary bound after \(\tau = \max\{\tau_1, \ldots, \tau_{k+1}\}\) and the second part makes sure that the ESO error is bounded before this time.
III. MAIN RESULTS

This section presents the main results and proofs about the optimality and stability of the proposed control scheme. First, we need some proves about the SDC matrices proposed in the previous section which are presented alongside some needed lemmas.

Lemma 1 (7)): An SDC representation \( A(x) \) exists for a nonlinear function \( f(x) \) such that \( A(x)x = f(x) \) holds, the pair \( \{A(x), \hat{B}(x)\} \) is globally pointwise controllable, and the pair \( \{A(x), Q^{1/2}\} \) is globally pointwise observable if and only if \( f(x) \) is linearly independent from \( x \) and \( Q^{1/2}x \neq 0 \) for all \( x \in \mathbb{R} - \{0\} \).

Proposition 1 (SDC matrix–existence): If \( Q > 0 \), then, a representation of the form (1) exists for the nonlinear system (1) such that the pairs \( \{\hat{A}(\hat{x}), \hat{B}(\hat{x})\} \) and \( \{A(\hat{x}), Q^{1/2}\} \) are globally pointwise controllable and globally pointwise observable, respectively.

Proof: According to Lemma 1, if \( f(x) = \lambda x \) (linear dependency) and at the same time \( Q^{1/2}x = \hat{x} \), then there is no matrix \( \hat{A}(\hat{x}) \) for \( f(x) \) to satisfy \( \{\hat{A}(\hat{x}), Q^{1/2}\} \) being globally pointwise observable. Since for an unknown system the linear independency cannot be guaranteed, therefore it is sufficient for the solvability to satisfy \( Q > 0 \) (i.e. \( Q^{1/2}x \neq 0 \)) in order to prevent an unobservable situation as well as a negative definite weight from the cost function.

Proposition 2 (SDC matrix–controllability): Assume that for all \( x \in \mathbb{R}^n \), \( \text{rank}(\hat{G}(x)) = n \) (or equivalently \( \text{rank}(\hat{B}(x)) = n \)). Then, the pair \( \{\hat{A}(\hat{x}), \hat{B}(\hat{x})\} \) is globally pointwise controllable.

Proof: Consider the following matrix which is needed to be full rank according to the Kalman’s law of controllability, \( C = \left[ \hat{B}(\hat{x}), \hat{A}(\hat{x})\hat{B}(\hat{x}), \ldots, \hat{A}^{n-1}(\hat{x})\hat{B}(\hat{x}) \right] \), the rank of the controllability matrix for system (2) obeys the inequality \( \text{rank}(C) \geq \text{rank}(H(\mathbb{I}_k \otimes \hat{G}(\hat{x}))) \) and \( \{\hat{A}(\hat{x}), \hat{B}(\hat{x})\} \) where \( H \in \mathbb{R}^{kn \times kn} \) is constructed by \( k^2 \) blocks of square \( n \times n \) matrices:

\[
H = \begin{bmatrix}
0_n & 0_n & 0_n & \cdots & 0_n \\
0_n & 0_n & 0_n & \cdots & 0_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0_n & I_n & H_k & \cdots & H_2 \\
I_n & H_k & H_{k-1} & \cdots & H_2 \\
\end{bmatrix}
\] (27)

in which defining \( \hat{F}(\hat{x}) = \left[ \hat{F}_1(\hat{x}), \ldots, \hat{F}_k(\hat{x}) \right] \), matrix \( H_i \) is considered as a function of \( \hat{F}_j(\hat{x}) \) for \( j = i, \ldots, k \):

\[
H_i = \begin{cases}
\hat{F}_k(\hat{x}) & \text{if } i = k \\
\hat{F}_{k-1}(\hat{x}) + \hat{F}_k(\hat{x}) & \text{if } i = k - 1 \\
\hat{F}_{k-2}(\hat{x}) + \hat{F}_{k-1}(\hat{x}) + \hat{F}_k(\hat{x}) & \text{if } i = k - 2 \\
\vdots & \text{if } i = 2 \\
\hat{F}_0(\hat{x}) & \text{if } i = 1 \\
0 & \text{if } i = 0
\end{cases}
\] (28)

Therefore, since obviously \( \text{rank}(H) \geq kn \) and \( \text{rank}(\mathbb{I}_k \otimes \hat{G}(\hat{x})) = \text{rank}(\hat{F}(\hat{x})) \), if \( \text{rank}(\hat{F}(\hat{x})) = n \), we have \( \text{rank}(C) \geq \text{rank}(H(\mathbb{I}_k \otimes \hat{G}(\hat{x}))) \geq kn \) which means that \( \text{rank}(C) = kn \).

Lemma 2: Function \( p_1(\cdot) : D_p \rightarrow \mathbb{R} \) expressed in (9) has the domain \( D_p = \mathbb{R}^{kn} - \{\hat{x} \in \mathbb{R}^n : \exists t \in \mathbb{N}, \hat{x}(t) = 0\} \) and for all \( \hat{x} \in D_p \), and \( t \in \mathbb{R} \), there exist \( \sigma_1, \sigma_2 > 0 \) such that \( \|p_1(\hat{x})/\hat{x}_a(\hat{x})\| \leq \sigma_1 \cos \hat{f}(i, k, n) \) and \( \|1 - p_1(\hat{x})/\hat{x}_a(\hat{x})\| \leq \sigma_2 \) if \( f \notin J(i, k, n) \).

Proof: The statements are true as long as their limits at \( \hat{x}(t) \rightarrow 0^+ \) exist. According to the exponential naïve of \( p_1(\hat{x}) \), it always has a faster approach toward zero in comparison to a polynomial \( \hat{x}_a(\hat{x}) \). We have \( \lim_{\hat{x}(t) \rightarrow 0^+} p_1(\hat{x})/\hat{x}_a(\hat{x}) = 0^+ \) if \( f \in J(i, k, n) \) and \( -\infty < \lim_{\hat{x}(t) \rightarrow 0^+} (1 - p_1(\hat{x})/\hat{x}_a(\hat{x}) < \infty \) if \( f \notin J(i, k, n) \) since the elements of \( \hat{x} \) are supposed to be nonzero.

Proposition 3 (SDC matrix–boundedness): Suppose Assumptions 1, 4, and 5 are satisfied. Then, the norm of the SDC matrix proposed by (5) and the norm of its derivative with respect to \( \hat{x}(i) \) for \( i = 1, \ldots, k \) are bounded if:

1. A continuous SDC matrix is constructed by (6), (7), (9) and there exists a set of \( \sigma_i > 0 \) such that \( |\hat{x}_a(i)(t)| \geq \sigma_i \) for all \( t \in [0, \infty) \) and \( i = 1, \ldots, kn \).
2. A discontinuous SDC matrix is constructed by (6), (11), (12), and problem (12) has a unique solution for all \( t \in [0, \infty) \).

Proof: According to Assumption 5 the output of the designed ESO (\( \hat{x} \) as well as \( \hat{x}_{n+1} \)) is always bounded as long as the real state variables of the system is finite which is the case as long as Assumption 4 and 5 are satisfied. Therefore, both continuous and discontinuous SDC matrices have finite norms unless the denominator is zero which will not occur according to the conditions specified above by items (1) and (2). For the derivatives we prove the boundedness of \( ||\hat{F}(\hat{x})/\hat{x}_a(i)|| \) for all \( l = 1, \ldots, kn \). First, consider item (1) and note that (considering (6)–(9)) the elements of \( \hat{F}(\hat{x}) \) can be summarized as \( \hat{F}(\hat{x}) = \hat{x}_{k+1}(i)p_1(\hat{x})/\hat{x}_a(i) \) and \( \hat{F}_i(\hat{x}) = \hat{x}_a(i)(1 - p_1(\hat{x}))/\hat{x}_a(i) \) if \( j \in J(i, k, n) \) and \( f \notin J(i, k, n) \), respectively, where \( \pi(i) = -\ln(p_1(\hat{x})) \) which is defined according to (9) as \( \pi(i) = \varpi(\prod_{(x,J,i,k,n)}(\hat{F}(\hat{x}))/\prod_{\hat{x} \in J(i,k,n)}(\hat{F}(\hat{x})) \). Therefore, we have four conditions for the derivative of \( \hat{F}(\hat{x}) \) with respect to the elements of \( \hat{x} \) (without loss of generality suppose \( \varpi = 1 \)):

\[
\frac{\partial \hat{F}_i(\hat{x})}{\partial \hat{x}_a(i)} = \begin{cases}
\frac{\hat{x}_{k+1}(i)\pi(i)-\pi(i)}{\hat{x}_a(i)} & \text{if condition(i)} \\
\frac{\hat{x}_{k+1}(i)\pi(i)\pi(i)}{\hat{x}_a(i)} & \text{if condition(ii)} \\
\frac{\hat{x}_{k+1}(i)\pi(i)\pi(i)-1}{\hat{x}_a(i)} & \text{if condition(iii)} \\
\frac{\hat{x}_{k+1}(i)\pi(i)\pi(i)}{\hat{x}_a(i)} & \text{if condition(iv)}
\end{cases}
\] (29)

where conditions (i)–(iv) corresponds to the satisfaction of “\( j = l \in J(i, k, n) \)”, “\( j, l \in J(i, k, n) \), \( j \neq l \)”, “\( j, l \in J(i, k, n) \), \( j \neq l \)”, and “\( j, l \in J(i, k, n) \), \( j \neq l \)”, respectively. According to Lemma 2, equation (29) has bounded values, therefore, item (1) is proved. For item (2), following the definition stated in (11) and (12), one can obtain the following statement:

\[
\text{If } \exists j \in \mathbb{N}, \hat{x}(j) = 0 \Rightarrow j^* = j
\] (30)
Therefore, according to the above statement and the way $\tilde{F}(\tilde{x})$ is defined by (6) and (11) we have:

$$\exists j \in \mathbb{N}, \mathcal{E}(j) = 0 \Rightarrow \exists \sigma > 0 : \| \mathcal{E}_{\tilde{x},j}(\tilde{x}) \| \leq \sigma \quad (31)$$

since the zero is eliminated from the denominator. Hence, item (2) is proved as well.

**Lemma 3** ([10]): Suppose the pair $\{A(x), B(x)\}$ defined by (2) is globally pointwise controllable and $Q$ is selected such that the pair $\{A(x), Q^{1/2}\}$ is globally pointwise observable. Also, suppose the SDC matrix $A(x)$ along with its derivatives with respect to all $x$ elements are bounded. Then:

1) The SDRE nonlinear regulator produces a closed-loop solution which is locally asymptotically stable.
2) The SDRE nonlinear feedback solution and its associated state and co-state trajectories satisfy the first necessary condition for optimality of the nonlinear regulator problem.
3) As the state is driven asymptotically to zero, the SDRE nonlinear feedback solution and its associated state and co-state trajectories approach the second necessary condition for optimality of the nonlinear regulator problem at a quadratic rate.

**Lemma 4:** Under Assumption 1, the closed-loop system constructed by (1), (4), and (14) satisfies the condition of Assumption 4.

**Proof:** First consider the fact that $u_{in}$ and $u_{out}$ are functions of $x$. Therefore, any uncertainty in the $G(x)$ does not affect the boundedness of the model as long as the real and the estimated values of $G(x)$ are finite. According to the results of Lemma 3 the closed-loop response of SDRE in its ROA satisfy the boundedness of the input and state. For the deterministic system outside the ROA, it is simply verifiable that $f(x) + G(x)u_{out}$ is just a linear quadratic regulator (LQR) for a $k$th-order chain integrator system at which both $u$ and $x$ are bounded. Thus, the closed-loop response of the proposed controllers for a deterministic system (i.e. $\tilde{x} = x$) always has bounded state and control input vectors.

**Theorem 1 (Controller–stability and optimality):** Suppose Assumptions 1, 2, and 3 as well as Assumption 5 are satisfied. The closed-loop system constructed by (1), (4), and (14) is locally optimal and GAS if a continuous (formulated in (6), (7), and (9)) or discontinuous (formulated in (6), (11), and (12)) SDC matrix is used such that item (1) or (2) of Proposition 3 holds, respectively.

**Proof:** First consider the deterministic case. According to Lemma 3–item (1), a ROA exists where inside that the system is asymptotically stable. Outside the ROA the controller acts as a LQR applied to a chain of integrators which is again asymptotically stable. The system is locally optimal in the ROA according to Lemma 3–item (2) if the controllability and observability conditions as well as the boundedness of the SDC matrices are satisfied. According to Assumption 2 the respective pairs are controllable and observable (which is feasible according to Proposition 1 and already satisfied for a wide range of systems according to Proposition 3). The boundedness of the proposed SDC matrices is proved in Proposition 2.

The estimation is the output of an ESO. Therefore, since the observer can be designed sufficiently convergent independent of the control input behavior (as long as the control input is bounded as is stated in Assumption 4 which is satisfied without a need of additional consideration according to Lemma 4), it does not affect the whole stability of the system (note that the scalar $\tau$ can be set equal to $\max\{\tau_1, \cdots, \tau_{k-1}\}$ after all state variables are converged). Moreover, the estimation tolerance does not impose instabilities caused by switching since Assumption 3 holds.

Before presenting the result about using the approximated estimation of $\Omega$, expressed by (25), we need to prove that Algorithm 1 is true. Hence, consider the following lemmas:

**Lemma 5:** For two matrices $M$ and $N$ with consistent dimensions supposing $x \in \mathbb{R}^{kn}$ we have $\partial(M + N)/\partial x = \partial M/\partial x + \partial N/\partial x$ and $\partial(MN)/\partial x = (\partial M/\partial x)(I_{kn} \otimes N) + M(\partial N/\partial x)$.

**Lemma 6:** Suppose $F(x)$ is a SDC representation for $f(x)$ such that $f(x) = F(x)x$ and $f(x) = 0$ if $x = 0$. Then, we have $\partial f(x)/\partial x = F(0)$ at $x = 0$.

**Proof:** According to the definition and using Lemma 5, we have $\partial f(x)/\partial x = (\partial F(x)/\partial x)(I_{kn} \otimes x) + F(x)$, then the prove is complete by substituting $x = 0$.

**Lemma 7:** For a system with a known Jacobian at the origin, Algorithm 1 gives the Jacobian of the closed-loop system constructed by (1) with $u_{in}$ as a control input.

**Proof:** We need to calculate the derivative of the following equation

$$\dot{x} = \begin{bmatrix} 0 & I_{(k-1)n} \\ f(x) \end{bmatrix} + \tilde{B}(x)R^{-1}\tilde{B}^T(x)P_{in}(x)x \quad (22)$$

with respect to $x \in \mathbb{R}^{kn}$. According to Lemma 5 we have:

$$J_{CL}(x) = J(x) + \tilde{B}(x)R^{-1}\tilde{B}^T(x)\partial P_{in}(x)/\partial x (I_{kn} \otimes x) + \tilde{B}(x)R^{-1}\tilde{B}^T(x)P_{in}(x) \quad (33)$$

Then, substituting $x = 0$, equality (26) is proved. Finally, equation (25) is justified by considering Lemma 6.

**Proposition 4 (ROA–Approximate estimation):** For a system with a known Jacobian at the origin, under conditions of Theorem 1, if $\Omega$ is approximately estimated by (22), then the closed-loop system constructed by (1), (4), and (14) is also locally optimal and globally asymptotically stable, but may need more than one switching between $u_{in}$ and $u_{out}$.

**Proof:** The control law formulated by (14) switches between two asymptotically stable closed-loop systems with candidate Lyapunov functions of $V_{in} = 0.5x^TP_{in}x$ and $V_{out} = 0.5x^TP_{out}x$. According to Theorem 2.1 in [16] switching between two controllers with $V_{in} < 0$ and $V_{out} < 0$ corresponding to the above-mentioned Lyapunov functions, result in an asymptotically stable controller. The number of switching is one if the approximately estimated ROA is exactly a real ROA. Since it is not necessarily the case, more than one switching may be required.
IV. NUMERICAL EXAMPLE

Consider an inverted pendulum system expressible as:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{g}{T} \sin(x_1) - b x_2 + \frac{1}{T} \cos(x_1)u \\
\end{align*}
\]  \hspace{1cm} (34)

where \( x_1 \equiv \theta \) is the pendulum angle, \( x_2 \equiv \dot{\theta} \) is its angular velocity, and \( y = x_1 \). The gravitational constant, damping coefficient of the hinge, and length of the massless link are denoted by \( g, b, \) and \( l, \) respectively. The physics of the system is unknown for the user. The only available knowledge about the system is its order and dimension alongside the fact that a positive \( u \) causes a negative \( \dot{x}_2 \). Therefore, \( \tilde{G}(x) = \text{sgn}(\cos(x_1)) \), \( n = 1 \), and \( k = 2 \) are all we know about the system. Consider the following linear ESO for system (34):

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 + 3(y - \hat{x}_2)/\epsilon \\
\dot{\hat{x}}_2 &= \hat{x}_3 + 3(y - \hat{x}_2)/\epsilon^2 + \text{sgn}(\cos(x_1))u \\
\dot{\hat{x}}_3 &= (y - \hat{x}_1)/\epsilon^3
\end{align*}
\]  \hspace{1cm} (35)

which is stable for \( \epsilon \ll 1 \) [2]. System (34) can be expressed in the form of (2),(3) in which the matrix \( \tilde{F}(\hat{x}) \) is constructed exactly as is shown in (10) or (13) which are referred to by continuous or discontinuous SDC matrices, respectively. The linearized form of system (34) at the zero equilibrium should be determined as follows if the closed-loop response needed to be GAS.

\[
J(0) = \begin{bmatrix} 0 & 0 \\ g/l & -b \end{bmatrix}, \quad B(0) = \begin{bmatrix} 0 \\ 1/l \end{bmatrix}
\]  \hspace{1cm} (36)

For the simulations the weighting matrices are considered \( Q = I_2 \) and \( R = 1 \). The initial condition in the simulation are \( x_1 = 45 \) deg and \( x_2 = 5 \) deg/s. The system parameters are \( l = 2.5, b = 10, \) and \( g = 9.81 \). The initial estimated values are considered \( \hat{x}_1 = x_1 + 10^{-6}, \hat{x}_2 = x_2 - 10^{-6}, \) and \( \hat{x}_3 = (g/l) \sin(x_1) - bx_2 \). These values imply that the ESO is assumed to be already sufficiently converged and a start-up phase is not required anymore. Inequality (23) is solved as the Lyapunov function \( J_{CL}(0)TP + PJ_{CL}(0) + 10^{-6}I_2 = 0 \).

Two control schemes are used for stabilization of system (34) in the region where \(-\pi/2 < x_1 < \pi/2\) and compared by conventional ADRC: The SDRE + ESO which is formulated as \( u_{in} = -K_{in}\hat{x} \) in (15) for the whole space assuming that \( \Omega = \mathbb{R} \); and the switching controller \( u \) expressed by (14). The former has no guarantee for being GAS (e.g. the dashed red plot in Fig. 1) but it is shown that if it happens to be GAS, then its corresponding control effort will be more efficient in comparison to the latter (e.g. the solid red plot in Fig. 1) which is less efficient but more safe since it is proved to be GAS (e.g. the solid/dashed blue plots in Fig. 1).

V. CONCLUSIONS

A suboptimal SDRE-based scheme has proposed for stabilization of unknown systems where the SDC matrix has estimated by the use of an ESO. It has shown that the proposed SDRE+ESO alongside a special form of ADRC are able to globally stabilize an unknown nonlinear system by a switching rule such that the closed-loop response converges to the optimal solution.

Fig. 1: The closed-loop response of the cost function \( J(t) \) to the proposed controllers. The gray area shows the response under ADRCs with different constant state feedback gains.

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