Generalized geometric structures on complex and symplectic manifolds

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Abstract

On a smooth manifold $M$, generalized complex (generalized paracomplex) structures provide a notion of interpolation between complex (paracomplex) and symplectic structures on $M$.

Given a complex manifold $(M, j)$, we define six families of distinguished generalized complex or paracomplex structures on $M$. Each one of them interpolates between two geometric structures on $M$ compatible with $j$, for instance, between totally real foliations and Kähler structures, or between hypercomplex and $\mathbb{C}$-symplectic structures. These structures on $M$ are sections of fiber bundles over $M$ with typical fiber $G/H$ for some Lie groups $G$ and $H$. We determine $G$ and $H$ in each case.

We proceed similarly for symplectic manifolds. We define six families of generalized structures on $(M, \omega)$, each of them interpolating between two structures compatible with $\omega$, for instance, between a $\mathbb{C}$-symplectic and a para-Kähler structure (aka bi-Lagrangian foliation).

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1 Introduction

Generalized complex geometry arose from the work [19] of Nigel Hitchin. It encompasses complex and symplectic geometry as its extremal special cases. In this section, we recall from the seminal work [15] the definitions and basic facts on generalized complex structures, and on generalized paracomplex structures (from [23]).

Let $M$ be a smooth manifold (by smooth we mean of class $C^\infty$; all the objects considered will belong to this class). The extended tangent bundle is the vector bundle $\mathcal{T}M = TM + TM^*$ over $M$. A canonical split pseudo-Riemannian structure on $\mathcal{T}M$ is defined by

$$b(u + \sigma, v + \tau) = \tau(u) + \sigma(v),$$

for smooth sections $u + \sigma, v + \tau$ of $\mathcal{T}M$. The Courant bracket of these sections [7] is given by

$$[u + \sigma, v + \tau] = [u, v] + \mathcal{L}_u \tau - \mathcal{L}_v \sigma - \frac{1}{2} d (\tau (u) - \sigma (v)),$$

where $\mathcal{L}$ denotes the Lie derivative.

A paracomplex structure $r$ on the smooth manifold $M$ is a smooth tensor field of type $(1, 1)$ on $M$ satisfying $r^2 = \text{id}$ such that the eigendistributions of $r$ associated to the eigenvalues $1$ and $-1$ are integrable and have the same dimension [9]. Among all the

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equivalent definitions of a complex structure \( j \) on \( M \) we choose the following: It is a smooth tensor field of type \((1,1)\) on \( M \) satisfying \( j^2 = -\text{id} \) such that the eigendistributions of \( j \) in \( TM \otimes \mathbb{C} \) associated to the eigenvalues \( i \) and \(-i\) are involutive (for the \( \mathbb{C}\)-bilinear extension of the Lie bracket).

A real linear isomorphism \( S \) with \( S^2 = \lambda \text{id}, \lambda = \pm 1 \), is called split if it has exactly two eigenspaces (of the complexification of the vector space, if \( \lambda = -1 \)) with the same dimension; this is always the case if \( \lambda = -1 \).

For \( \lambda = \pm 1 \), let \( S \) be a smooth section of \( \text{End} (TM) \) satisfying

\[
S^2 = \lambda \text{id}, \ S \text{ is split and skew-symmetric for } \lambda
\]

and such that the set of smooth sections of the \( \pm \sqrt{\lambda} \)-eigenspace of \( S \) is closed under the Courant bracket (if \( \lambda = -1 \), this means as usual closedness under the \( \mathbb{C}\)-linear extension of the bracket to sections of the complexification of \( TM \)). Then, for \( \lambda = -1 \) (respectively, \( \lambda = 1 \)), \( S \) is called a generalized complex (respectively, generalized paracomplex) structure on \( M \). Notice that in [23] the latter is not required to be split.

We also need the notion of \((+)\)-generalized paracomplex structure \( S \). It is the same as a generalized paracomplex structure, but closedness under the Courant bracket is required only for sections of the \( 1 \)-eigendistribution of \( S \).

As far as we know, Izu Vaisman [22], was the first one to consider generalized complex and paracomplex structures simultaneously in a systematic way.

2 Generalized geometric structures on complex manifolds

2.1 Geometric structures compatible with \( j \)

Let \((M,j)\) be a complex manifold. We consider the following well-known integrable geometric structures on \( M \) compatible with \( j \). The reason of the nomenclature integrable \((\lambda,0)\)- or \((0,\ell)\)-structures will become apparent in Theorem 2.4.

Integrable \((1,0)\)-structure or complex product structure on \((M,j)\). It is given by a paracomplex structure \( r \) on \( M \) with \( rj = -jr \). Then \((M,j,r)\) is a complex product manifold [2], also called para-hypercomplex [4, 11] or neutral hypercomplex manifold [20, 12].

Integrable \((-1,0)\)-structure or hypercomplex structure on \((M,j)\). It is given by a complex structure \( r \) on \( M \) which is \( j \)-antilinear, that is, \( rj = -jr \).

Integrable \((0,1)\)-structure or pseudo-Kähler structure on \((M,j)\). It is given by a symplectic form \( \omega \) on \( M \) for which \( j \) is skew-symmetric. If \( g \) denotes the pseudo-Riemannian metric given by \( g(u,v) = \omega(ju,v) \), then \((M,g,j)\) is pseudo-Kähler.

Integrable \((0,-1)\)-structure or \( \mathbb{C}\)-symplectic structure on \((M,j)\). It is given by a symplectic form \( \omega \) on \( M \) for which \( j \) is symmetric. If \( \theta \) denotes the two-form given by \( \theta(u,v) = \omega(ju,v) \), then \( \Omega = \omega - i\theta \) is a \( \mathbb{C}\)-symplectic structure on \( M \).

We also have \((+)+\)-integrable \((1,0)\)-structure or totally real foliation of \((M,j)\). It is given by a tensor field \( r \) of type \((1,1)\) on \( M \) with \( r^2 = \text{id} \) and \( rj = -jr \), such that the \( 1 \)-eigensection \( D \) of \( r \) is an integrable distribution. Then \( D \oplus jD = TM \) holds and the leaves of \( D \) are totally real submanifolds of \( M \).
Recall that for a hypercomplex or a complex product structure \((j, r)\), \(jr\) turns out to be split and integrable (see [2]). Also, if \(j\) is a complex structure on \(M\) and \(\omega\) is a closed 2-form on \(M\) for which \(j\) is symmetric, then \(\Omega\) (or equivalently \(\theta\)) as above is closed. Notice that hypercomplex and \(\mathbb{C}\)-symplectic manifolds have even complex dimension.

2.2 Slash structures on \((M, j)\)

**Definition 2.1** Let \((M, j)\) be a complex manifold. For \(\ell = \pm 1\), let \(J_\ell\) be the complex structure on the real vector bundle \(TM\) over \(M\) given by

\[
J_\ell = \begin{pmatrix} j & 0 \\ 0 & \ell j^* \end{pmatrix}
\]

Notice that \(J_{-1}\) is a generalized complex structure on \(M\), but \(J_1\) is not, since it is not skew-symmetric for \(b\). Indeed, for all sections \(u + \sigma, v + \tau\) of \(TM\),

\[
b(J_\ell(u + \sigma), v + \tau) = b(ju + \ell j^* \sigma, v + \tau) = \tau(ju) + \ell \sigma(jv)
\]

Now we introduce six families of generalized geometric structures on \((M, j)\) interpolating between some of the structures listed in the previous subsection.

**Definition 2.2** Let \((M, j)\) be a complex manifold. Given \(\lambda = \pm 1\) and \(\ell = \pm 1\), a generalized complex structure \(S\) (for \(\lambda = -1\)) or a generalized paracomplex structure \(S\) (for \(\lambda = 1\)) on \(M\) is said to be an **integrable** \((\lambda, \ell)\)-structure on \((M, j)\) if

\[SJ_\ell = -J_\ell S\]

Analogously, given \(\ell = \pm 1\), a \((+)\)-generalized paracomplex structure \(S\) on \(M\) is said to be a \((+)\)-**integrable** \((1, \ell)\)-structure on \((M, j)\) if \(SJ_\ell = -J_\ell S\).

We call \(S_j(\lambda, \ell)\) the set of all integrable \((\lambda, \ell)\)-structures on \((M, j)\), and \(S_j^+(1, \ell)\) the set of all \((+)\)-integrable \((1, \ell)\)-structures. An element of \(S_j(-1, -1)\) may be called, for instance, a hypercomplex / \(\mathbb{C}\)-symplectic structure on \((M, j)\). That suggests the name slash structures for these structures on \(M\).

Given a bilinear form \(c\) on a real vector space \(V\), let \(c^\flat \in \text{End}(V, V^*)\) be defined by \(c^\flat(u)(v) = c(u, v)\). The form \(c\) is symmetric (respectively, skew-symmetric) if and only if \((c^\flat)^* = c^\flat\) (respectively, \((c^\flat)^* = -c^\flat\)).

**Example 2.3** If \(r\) and \(\omega\) are integrable \((\lambda, 0)\)- and \((0, \ell)\)-structures on \((M, j)\), respectively, then

\[
R = \begin{pmatrix} r & 0 \\ 0 & -r^* \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & \lambda (\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix}
\]

belong to \(S_j(\lambda, \ell)\).

The following simple theorem justifies the terminology introduced in the section and includes the notion of interpolation. See comments on this concept in subsection 2.5.
Theorem 2.4 Let \((M, j)\) be a complex manifold. For \(\lambda = \pm 1, \ell = \pm 1,\) integrable \((\lambda, \ell)\)-structures on \((M, j)\) interpolate between integrable \((\lambda, 0)\)- and \((0, \ell)\)-structures on \((M, j)\), that is, if
\[
R = \begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & p \\ \omega^\ast & 0 \end{pmatrix}
\]
belong to \(S_j(\lambda, \ell)\), then \(r\) and \(\omega\) are integrable \((\lambda, 0)\)- and \((0, \ell)\)-structures on \((M, j)\), respectively.

Also, for \(\ell = \pm 1,\) \((+)\)-integrable \((1, \ell)\)-structures interpolate between \((+)\)-integrable \((1, 0)\)- and integrable \((0, \ell)\)-structures on \((M, j)\).

**Proof.** We call \(q = \omega^\ast\). It is well known from \([15]\) and \([23]\) that if \(R\) and \(Q\) as above are both generalized complex (respectively, paracomplex) structures, then \(r\) is a complex (respectively, paracomplex) structure on \(M\) and \(\omega\) is a closed 2-form. Also, that \(t = -r^\ast\) and \(p = -q^{-1}\) (respectively, \(p = q^{-1}\)).

Now, since \(R\) and \(Q\) anti-commute with \(J_\ell\), one has that \(rj = -jr\) and \(qj = -\ell j^\ast q\). This means that \(q(ju)(v) = -\ell q(u)(jv)\) for all vector fields \(u, v\) on \(M\). Hence, \(\omega(ju, v) = -\ell \omega(u, jv)\) for all \(u, v\) and so \(j\) is symmetric or skew-symmetric for \(\omega\), depending on whether \(\ell = -1\) or \(\ell = 1\). Thus, \(r\) and \(\omega\) are integrable \((\lambda, 0)\)- and \((0, \ell)\)-structures, respectively.

Suppose that \(\lambda = 1\) and \(R\) is a \((+)\)-integrable \((1, \ell)\)-structure, that is, the 1-eigensection \(\{u + \sigma \mid ru = u, r^\ast \sigma = -\sigma\}\) of \(R\) is involutive with respect to the Courant bracket. It is easy to see from the arguments in \([23]\) that the 1-eigensection of \(r\) is integrable (even if the \((-1)\)-eigensection of \(R\) is not). Finally, standard arguments show that if the 1-eigensection of \(Q\) as above is Courant involutive, then so is also its \((-1)\)-eigensection. Hence, \(Q\) is a generalized paracomplex structure and thus \(\omega\) is an integrable \((0, \ell)\)-structure on \(M\).

**Remark 2.5** a) The choice of five compatible geometric structures on \((M, j)\) was strongly conditioned by Courant involutivity. For instance, we have not considered anti-Kähler structures \(g\) on \((M, j)\), i.e. pseudo-Riemannian metrics \(g\) for which \(j\) is symmetric and parallel \([5]\), since we have not been able to relate the integrability condition (that \(j\) be parallel with respect to the Levi-Civita connection of \(g\)) to the Courant bracket.

b) We believe that the search for nontrivial examples of each of the twelve generalized structures on complex or symplectic manifolds can contribute, in same cases, to a better understanding of these manifolds in general. And also in particular, in the same way, for example, that generalized complex structures shed light on the geometry of nil- and solvamifolds \([6, 3]\).

### 2.3 A signature associated to integrable \((1, 1)\)-structures on \((M, j)\)

**Proposition 2.6** Let \(S\) be an integrable \((1, 1)\)-structure on a complex manifold \((M, j)\) of complex dimension \(m\). Then the form \(\beta_S\) on \(TM\) defined by \(\beta_S(x, y) = b(SJ_+x, y)\) is symmetric and has signature \((2n, 4m - 2n)\) for some integer \(n\) with \(0 \leq n \leq 2m\).

**Proof.** The form \(\beta_S\) is symmetric since \(S\) and \(J_+\) anti-commute and are skew-symmetric and symmetric for \(b\) (see \([11]\)), respectively.

One has that \((SJ_+)^2 = \text{id}\). For \(\delta = \pm 1\), let \(D_\delta\) be the \(\delta\)-eigensection of \(SJ_+\). One verifies that \(J_+ (D_\delta) = D_\delta\), so \(D_+\) and \(D_-\) have both dimension \(2m\).

For \(\delta = \pm 1\) let \(b^\delta := b|_{D_\delta \times D_\delta}\) and \(\beta^\delta := \beta_S|_{D_\delta \times D_\delta}\). One computes \(b(D_+, D_-) = 0\); in particular, by the orthogonality lemma (2.30 in \([17]\), \(b^\delta\) is nondegenerate. Suppose that
$b^+$ has signature $(n, 2m - n)$. Hence $b^-$ has signature $(2m - n, n)$ ($b$ is split). On the other hand, one computes also that $b^\delta = \delta b^\delta$. Therefore the signature of $\beta_S$ is $(2n, 4m - 2n)$, as desired.

**Definition 2.7** An integrable $(1, 1)$-structure $S$ on $(M, j)$ as above is called an **integrable $(1, 1; n)$-structure**, and we write $\text{sig}(S) = n$. If $\beta_S$ is split (or equivalently, $n = m$), by the next proposition, the $(1, 1; n)$-structure is called a (complex product)/(split Kähler) structure on $(M, j)$.

**Proposition 2.8** a) Let $r$ be an integrable $(1, 0)$-structure on $(M, j)$, that is, a complex product structure on $M$ compatible with $j$. Then

$$R = \begin{pmatrix} r & 0 \\ 0 & -r^* \end{pmatrix}$$

is a $(1, 1; n)$-structure on $(M, j)$ if and only if $n = m$.

b) Let $\omega$ be an integrable $(0, 1)$-structure on $(M, j)$. Then

$$Q = \begin{pmatrix} 0 & (\omega^b)^{-1} \\ \omega^b & 0 \end{pmatrix}$$

is a $(1, 1; n)$-structure on $(M, j)$ if and only if the pseudo-Kähler metric $g(u, v) = \omega(ju, v)$ on $M$ has signature $(n, 2m - n)$.

**Proof.** a) Since $rj = -jr$, we compute

$$\beta_R(u + \sigma, v + \tau) = \tau(rju) + \sigma(rjv).$$

Now, $rj$ squares to the identity and is split (its $(-1)$- and $1$-eigensections are interchanged by $j$). Then, locally, there exists a basis $\{u_1, \ldots, u_{2m}\}$ of $TM$ such that $rj(u_i) = u_i$ for $1 \leq i \leq m$ and $rj(u_i) = -u_i$ for $m < i \leq 2m$. Let $\{\alpha_1, \ldots, \alpha_{2m}\}$ be the dual basis. Analyzing the signs of $\beta_R(u_i + \alpha_i, u_i + \alpha_i)$ and $\beta_R(u_i - \alpha_i, u_i - \alpha_i)$, one concludes that $\beta_R$ is split, and this yields (a).

b) One computes

$$\beta_Q(u + \sigma, v + \tau) = \omega(ju, v) + \tau((\omega^b)^{-1}j^*\sigma) = g(u, v) + h(\sigma, \tau),$$

where the symmetric form $h$ on $T^*M$ is defined by the last equality. Now,

$$(\omega^b)^* \omega(z, w) = h(\omega^b z, \omega^b w) = \omega^b(w)(\omega^b)^{-1}j^*\omega^b(z) =$$

$$= -\omega^b(w)((\omega^b)^{-1}\omega^b(jz)) = -\omega^b(w)(jz) =$$

$$= \omega(jz, w) = g(z, w),$$

since for an integrable $(0, 1)$-structure $\omega$ on $(M, j)$, $j$ is skew-symmetric for $\omega$, that is, $j^*\omega^b = -\omega^b j$. Therefore, if $\phi : TM \oplus TM \to TM$ is defined by $\phi(u, z) = (u, \omega^b z)$, then

$$\phi^*\beta_Q((u, z), (v, w)) = g(u, v) + g(z, w).$$

This implies the assertion of (b), since $\phi^*\beta_Q$ and $\beta_Q$ have the same signature. \qed
2.4 The associated bundles over \((M,j)\)

Let \(L\) denote the Lorentz numbers \(a + eb, \varepsilon^2 = 1\). Let \(V\) be a vector space over \(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{L} \) or \(\mathbb{H}\), where \(\mathbb{H} = \mathbb{C} + j\mathbb{C}\) are the quaternions (we consider right vector spaces over \(\mathbb{H}\)). Recall from \([17]\) that an \(\mathbb{R}\)-bilinear map \(C : V \times V \to \mathbb{F}\) satisfying \(C (x\lambda, y\mu) = \lambda C (x, y) \mu\) for any \(\lambda, \mu \in \mathbb{F}\) and \(x, y \in V\) is called Hermitian (respectively, anti-Hermitian) if \(C (x, y) = C (y, x)\) (respectively, \(C (x, y) = - C (y, x)\)) for all \(x, y \in V\). Also, a Hermitian form on a vector space \(V\) over \(\mathbb{F} \neq \mathbb{L}\) is said to be split if it has \(\mathbb{F}\)-signature \((n, n)\), where \(2n = \dim_{\mathbb{F}} V\). The \(\mathbb{L}\)-signature does not make sense, since \(\varepsilon \varepsilon = -1\).

Generalized complex structures on a \((2n)\)-dimensional manifold \(N\) are sections of a bundle over \(N\) with typical fiber \(O (2n, 2n) / U (n, n)\) \([15]\). In the same way, generalized paracomplex structures on an \(m\)-dimensional manifold \(N\) are sections of a bundle over \(N\) with typical fiber \(O (m, m) / Gl (m, \mathbb{R})\), since \(Gl (m, \mathbb{R})\) is the \(\mathbb{L}\)-unitary group (Section 3 in \([15]\)). Theorem 2.9 below presents analogous statements for integrable \((\lambda, \ell)\)-structures on a complex manifold \((M, j)\).

Let \(O (m, m)\) and \(Sp (m, \mathbb{R})\) be the groups of automorphisms of a split symmetric and skew-symmetric form on \(\mathbb{R}^{2m}\), respectively. Let \(SO^+ (2m)\) and \(Sp (n, n)\) \((m = 2n)\) be the groups of automorphisms of an anti-Hermitian (respectively, a split Hermitian) form on \(\mathbb{H}^m\). In \([17]\) they are called \(SK (m, \mathbb{H})\) and \(HU (n, n)\), respectively.

**Theorem 2.9** Let \((M, j)\) be a complex manifold of complex dimension \(m\). Then, integrable \((\lambda, \ell)\)- or \((1, 1; n)\)-structures on \((M, j)\) are smooth sections of a fiber bundle over \(M\) with typical fiber \(G/H\), according to the following table \((m = 2k\text{ in the case }\lambda = \ell = -1)\).

| \(\lambda\) | \(\ell\) | \(n\) | \(G\) | \(H\) |
|---|---|---|---|---|
| 1 | 1 | \(n\) | \(O (2m, \mathbb{C})\) | \(O (n, 2m - n)\) |
| 1 | -1 | \(-\) | \(U (m, m)\) | \(Sp (m, \mathbb{R})\) |
| -1 | 1 | \(-\) | \(O (2m, \mathbb{C})\) | \(SO^+ (2m)\) |
| -1 | -1 | \(-\) | \(U (2k, 2k)\) | \(Sp (k, k)\) |

**Corollary 2.10** A complex manifold admitting a hypercomplex / \(\mathbb{C}\)-symplectic structure has even complex dimension.

Before proving the theorem we introduce some notation and present a proposition. Now we work at the algebraic level. We fix \(p \in M\) and call \(E = T_p M\). By abuse of notation, in the rest of the subsection we write \(b\) and \(J\) instead of \(b_p\) and \((J_p)_p\), omitting the subindex \(p\). Also, we sometimes identify \((1, -1) = (+, -), etc.\)

Let \(\sigma (\lambda, \ell)\) denote the set of all \(S \in \text{End}_{\mathbb{R}} (E)\) satisfying

\[S^2 = \lambda \text{id}, S \text{ is split, skew-symmetric for } b \text{ and } SJ = -J S.\]

Note that \((E, J)\) is a vector space over \(\mathbb{C}\) via \((a + ib) x = ax + J b x)\).

**Proposition 2.11** For \(\ell = \pm 1\), let \(b_\ell : E \times E \to \mathbb{C}\) be defined by

\[b_\ell (x, y) = b (x, y) - i b (J x, y).\]

Then \(b_-\) is split \(\mathbb{C}\)-Hermitian and \(b_+\) is \(\mathbb{C}\)-bilinear symmetric (with respect to \(J_-, J_+\), respectively).

Also, if \(S \in \text{End}_{\mathbb{R}} (E)\) satisfies \(S^2 = \lambda \text{id}\), then \(S \in \sigma (\lambda, \ell)\) if and only if

\[b_\ell (S x, S y) = -\lambda b_\ell (x, y)\]

for any \(x, y \in E\).
Proof. First notice that $T \in \text{End}_\mathbb{R} (\mathbb{E})$ with $T^2 = \mu$ id is symmetric or skew-symmetric for $b$ if and only if
\[
 b(Tx, Ty) = \pm b(x, T^2y) = \pm \mu b(x, y)
\]
for all $x, y$. Using (1) together with (3) with $T = J_\ell$ and $\mu = \ell$, it is easy to check that
\[
 ib_\ell(x, y) = b_\ell(x, J_\ell y) = \ell b_\ell(J_\ell x, y)
\]
for all $x, y$. Also, it follows immediately from the definitions that $b_\ell(x, y) = b_\ell(y, x)$ or $b_\ell(x, y) = b_\ell(y, x)$ for all $x, y$, depending on whether $\ell = 1$ or $\ell = -1$, respectively. Besides, $b_-$ is split since $b = \Re b_-$ is split. Thus, the first assertion is true.

Now we prove the second assertion. Suppose first that $S \in \sigma(\lambda, \ell)$. Since $S$ anti-commutes with $J_\ell$, we compute (using (3) with $T = S$ and $\mu = \lambda$)
\[
 b_\ell(Sx, Sy) = b(Sx, Sy) - ib(Sx, J_\ell Sy) = -\lambda b(x, y) + ib(Sx, SJ_\ell y) = -\lambda b(x, y) - ib(x, J_\ell y) = -\lambda(b(x, y) + ib(x, J_\ell y)) = -\lambda b(x, y).
\]
Conversely, suppose that $S^2 = \lambda$ id and (2) holds. By (3) with $T = S$ and $\mu = \lambda$, $S$ is skew-symmetric for $b = \Re b_\ell$. Now we compute
\[
 b_\ell(x, SJ_\ell y) = \lambda b_\ell(S^2x, SJ_\ell y) = \lambda(\lambda b_\ell(Sx, J_\ell y)) = -i \lambda b_\ell(x, S^2y) = i\lambda b_\ell(x, Sy) = -b_\ell(x, J_\ell Sy).
\]
Since $b_\ell$ is nondegenerate, $S$ anti-commutes with $J_\ell$. This implies, in particular, that if $\lambda = 1$, then $J_\ell(D_+) = D_-$, where $D_\pm$ is the $(\pm 1)$-eigenspace of $S$. Hence, $S$ is split. Therefore, $S \in \sigma(\lambda, \ell)$. \hfill \qed

The core of the arguments in the proofs of Theorems 2.9 and 3.10 are essentially from 1.6 in [21], except those involving the Lorentz numbers. We put them in context and complete details (write in coordinates, choose particular presentations, prove the transitivity of the actions).

We use the notation and the standard forms of inner products of the book [17]. In particular, Hermitian and anti-Hermitian forms differ from those in [21] by conjugation. We resort repeatedly to the Basis Theorem ([17], 4.2). For inner products on $\mathbb{E}$-vector spaces we refer to [18] (where Lorentz numbers are called double numbers and denoted by $\mathbb{D}$).

Proof of Theorem 2.9. For $\ell = \pm 1$, by the first assertion in Proposition 2.11 and the Basis Theorem, there exist complex linear coordinates $\phi_\ell^{-1} = (z, w) : (\mathbb{E}, J_\ell) \rightarrow \mathbb{C}^{2m}$ such that $B_\ell := \phi_\ell^*b_\ell$ have the forms
\[
 B_-((z, w), (z', w')) = z^t z' - w^t w' \quad \text{and} \quad B_+(Z, Z') = Z^t Z',
\]
where $z, w, z', w' \in \mathbb{C}^m, Z, Z' \in \mathbb{C}^{2m}$ are column vectors and the superscript $t$ denotes transpose.

Let $\Sigma(\lambda, \ell)$ be the subset of $\text{End}_{\mathbb{R}} (\mathbb{C}^{2m})$ corresponding to $\sigma(\lambda, \ell)$ via the isomorphism $\phi_\ell$. By the second statement of Proposition 2.11, $U(m, m)$ and $O(2m, \mathbb{C})$ (the Lie groups preserving $B_-$ and $B_+$, respectively) act by conjugation on $\Sigma(\lambda, -), \Sigma(\lambda, +)$, respectively.
In what follows, for each case $(\lambda, \ell) \neq (1, 1)$ we present a particular real isomorphism $S$ of $\mathbb{C}^{2m}$ and show, using the second statement of Proposition 2.11 that $S$ belongs to $\Sigma (\lambda, \ell)$ (actually, we write down the computation only for $\lambda = 1 = -\ell$, the other being analogous). Then we check that the group $G$ associated to $(\lambda, \ell)$ in the table acts transitively on $\Sigma (\lambda, \ell)$, with isotropy subgroup the corresponding group $H$ in the table. In this way, one concludes that $\Sigma (\lambda, \ell)$ may be identified with $G/H$, as desired. The case $(1, 1; n)$ is dealt with similarly.

**Case $(+, -)$:** Let $S \in \text{End}_\mathbb{R} (\mathbb{C}^{2m})$ be defined by $S (z, w) = (\overline{w}, \overline{z})$. We use the second statement of Proposition 2.11 to show that $S$ belongs to $\Sigma (+, -)$. Clearly, $S^2 = \text{id}$ and also

$$B_-(S(z, w), S(z', w')) = B_-(\overline{(w, z)}, \overline{(w', z')}) = w'\overline{w} - z'\overline{z} = -B_-(z, w), (z', w').$$

Now let $V$ be the 1-eigenspace of $S$, that is, $V = \{ (z, \overline{z}) \mid z \in \mathbb{C}^m \} \cong \mathbb{R}^{2m}$. One has $V \oplus iV = \mathbb{C}^{2m}$ and verifies that $\alpha := -iB_\mid_{V \times V}$ is a symplectic form on $V$. Indeed, one computes $\alpha((z, \overline{z}), (z', \overline{z'})) = 2(z'y - y'z)$ if $z = x + iy$ and $z' = x' + iy'$.

Given $A \in Sp(V, \alpha)$, the map $\tilde{A}$ defined by $\tilde{A}(X + iY) = AX + iAY$, for $X, Y \in V$, is in $U(m, m)$. This gives an inclusion of $Sp(m, \mathbb{R}) \cong Sp(V, \alpha)$ into $U(m, m)$.

Now we check that the isotropy subgroup $H$ at $S$ of the action of $U(m, m)$ on $\Sigma (+, -)$ is $Sp(V, \alpha)$. Assume that $A \in Sp(V, \alpha)$. Clearly, $AS = SA (S|_V = \text{id}_V)$. Hence, $\tilde{A}$ commutes with $S$, since $S$ is anti-linear. Then $\tilde{A} \in H$. Conversely, if $L \in U(m, m)$ commutes with $S$, then $L$ preserves $V$ and so $L = \tilde{A}$ for some $A \in Sp(V, \alpha)$.

It remains to show that the action is transitive. Let $T \in \Sigma (+, -)$ and let $W$ be the 1-eigenspace of $T$. One verifies, using (2), that $\theta = -iB\mid_{W \times W}$ is a symplectic form on $W$. Let $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ be vectors in $W$ such that $\theta (X_s, Y_t) = 2\delta_{st}$ and $\theta (X_s, X_t) = \theta (Y_s, Y_t) = 0$ for all $s \leq t$. Let $F : V \to W$ be the linear transformation with $F(e_s, e_s) = X_s, F(i\epsilon_t - i\epsilon_t) = Y_t$, where $\{e_1, \ldots, e_m\}$ is the canonical basis of $\mathbb{R}^m$. Then $F$ extends $\mathbb{C}$-linearly to $\tilde{F} \in U(m, m)$ such that $T = \tilde{F}S\tilde{F}^{-1}$. Therefore, $\Sigma (+, -)$ can be identified with $U(m, m)/Sp(m, \mathbb{R})$, as desired.

**Case $(-, -)$:** Any $S \in \Sigma (-, -)$ gives $\mathbb{C}^{2m}$ the structure of a right $\mathbb{H}$-vector space via $Z(u + jv) = uZ + v(SZ)$ $(Z \in \mathbb{C}^{2m}, u, v \in \mathbb{C})$. Given $S \in \Sigma (-, -)$, let

$$C(Z, Z') = B_-(Z, Z') - B_-(Z, SZ').$$

By Lemma 2.72 in [17] (using (2) and the fact that $uj = jv$ for all $u \in \mathbb{C}$), $C$ is an $\mathbb{H}$-Hermitian form, which is split since $B_-$ is so. In particular $m$ is even, say, $m = 2k$. Now, $L \in U(m, m)$ commutes with $S$ if and only if $L$ is an isometry for $C$. Hence, the isotropy subgroup at $S$ of the action of $U(m, m)$ is $Sp(k, k)$. The action is transitive: If $T$ is another element of $\Sigma (-, -)$, then one has another $\mathbb{H}$-structure on $\mathbb{E}$ and can define $C_T$ in the same way as $C$. By the Basis Theorem, they are isometric. There exists an $\mathbb{H}$-linear isometry $F : (\mathbb{E}, C) \to (\mathbb{E}, C_T)$, which satisfies $F \in U(m, m)$ and $T = FSF^{-1}$. Therefore, $\Sigma (-, -)$ can be identified with $U(m, m)/Sp(k, k)$, as desired.

We give an example of $S \in \Sigma (-, -)$: Write $z = (z_1, z_2), w = (w_1, w_2)$, with $z_s, w_t \in \mathbb{C}^k$ and define $S \in \text{End}_\mathbb{R} (\mathbb{C}^{2k})$ by $S(z_1, z_2, w_1, w_2) = (-\overline{z_2}, \overline{z_1}, -\overline{w_2}, \overline{w_1})$.

**Case $(+, +)$:** Let $S \in \text{End}_\mathbb{R} (\mathbb{C}^{2m})$ be defined by $S(z, w) = (i\overline{z}, -i\overline{w})$, for $z \in \mathbb{C}^n, w \in \mathbb{C}^{2m-n}$, which belongs to $\Sigma (+, +; n)$. In fact, one uses the second statement of Proposition 2.11 to show that $S \in \Sigma (+, +)$ and computes

$$ReB_+(S(iz, iw), (z', w')) = Re (\overline{z'}z' - \overline{w'}w').$$
which is a real symmetric form of signature $(2n, 4m)$. This implies that $S \in \Sigma (+, +; n)$, since $b = \Re b \pm$. Let $V$ be the 1-eigenspace of $S$, that is,

$$V = \{(1 + i)x, (1 - i)y \mid x \in \mathbb{R}^n, y \in \mathbb{R}^{2m-n}\} \cong \mathbb{R}^{2m}.$$

Then $V \oplus iV = \mathbb{C}^{2m}$. One verifies that $g := -i \, B_{+}|_{V \times V}$ is a real symmetric form on $V$ of signature $(n, 2m - n)$. Indeed, one computes

$$g\left(\left((1 + i)x, (1 - i)y\right), \left((1 + i)x', (1 - i)y'\right)\right) = 2 \left(x'x' - y'y\right).$$

Given $A \in O(V, g)$, then $\tilde{A} (X + iY) = AX + iAY \ (X, Y \in V)$ satisfies $\tilde{A} \in O(2m, \mathbb{C})$. This gives an inclusion of $O(n, 2m - n)$ in $O(2m, \mathbb{C})$.

We check that the isotropy subgroup at $S$ is $O(V, g)$: Since $S$ is anti-linear, $S$ commutes with $\tilde{A}$ for any $A \in O(V, g)$. Besides, if $L \in O(2m, \mathbb{C})$ commutes with $S$, then $L$ preserves $V$ and so $L = \tilde{A}$ for some $A \in O(V, g)$.

Now we see that the action is transitive. Let $T \in \Sigma (+, +, n)$ and let $W$ be the 1-eigenspace of $T$. Then $h := -i \, B_{+}|_{W \times W}$ is a real symmetric form on $W$ of signature $(n, 2m - n)$. In fact, if $Tu = u$ and $Tv = v$,

$$h(u, v) = -iB_{+}(u, v) = -iB_{+}(Tu, Tv) = iB_{+}(u, v) = h(u, v)$$

and also $\Re B_{+}(Ti u, v) = \Re B_{+}(-iT u, Tv) = h(u, v)$. Let $v_1, \ldots, v_{2m}$ be a basis of $W$ such that $h(v_s, v_s) = 2$ if $s \leq n$, $h(v_s, v_s) = -2$ if $s > n$ and $h(v_s, v_t) = 0$ for all $s \neq t$. Let $F : V \to W$ be the linear transformation with $F((1 + i)e_s) = v_s$ if $s \leq n$ and $F((1 - i)e_s) = v_s$ if $s > n$. Then $F$ extends linearly to $\tilde{F} \in O(2m, \mathbb{C})$ such that $T = \tilde{F}SF^{-1}$.

**Case $(-, +)$:** Let $S \in \text{End}_{\mathbb{R}}(\mathbb{C}^{2m})$ be defined by $S(z, w) = (-\overline{w}, \overline{z})$, which belongs to $\Sigma (-, +)$. Notice that $(\mathbb{C}^{2m}, S)$ is a right $\mathbb{H}$-vector space via $Z(z + jw) = Zz + (SZ)w$.

Let $C(Z, Z') = B_{+}(SZ, Z') - jB_{+}(Z, Z')$. Then $C$ is skew $\mathbb{H}$-Hermitian. Now, $L \in O(2m, \mathbb{C})$ commutes with $S$ if and only if $L$ is an isometry for $C$. Hence, the isotropy subgroup at $S$ of the action of $O(2m, \mathbb{C})$ is $SO^*(2m)$. The action is transitive: If $T$ is another element of $\Sigma (-, +)$, then one has another $\mathbb{H}$-structure on $E$ compatible with $j$ and can define $C_T$ in the same way as $C$. By the Basis Theorem, they are isometric. Then there exists an $\mathbb{H}$-linear isometry $F : (E, C) \to (E, C_T)$ satisfying $F \in O(2m, \mathbb{C})$ and $T = FSF^{-1}$.  

### 2.5 Slash structures and the notion of interpolation

Generalized complex geometry on smooth manifolds generalizes complex and symplectic structures and simultaneously provides a notion of interpolation between them.

In our opinion, integrable $(\lambda, \ell)$-structures on complex manifolds are good generalizations of integrable $(\lambda, 0)$- or $(0, \ell)$-structures, but for the sake of simplicity, we have presented a rather weak definition of interpolation, which in some cases is not what one would expect from that concept, but (again in our view) in most cases is appropriate.

In the papers devoted to generalized complex structures the notion of interpolation is not made explicit; there is no need of doing so, because their existence on a smooth manifold $M$ implies the existence of almost complex and almost symplectic structures on $M$. In contrast, on a complex manifold $M$ with odd complex dimension there may exist an integrable $(-1, 1)$-structure (for instance a $0, (1)$-structure, i.e. a pseudo-Kähler structure), but there cannot exist $(-1, 0)$-structures on $M$ (even non-integrable ones), since
these require $M$ to have even complex dimension. The only other slash structures that are
defective in this sense are $(1, -1)$-integrable structures on $M$ with odd complex dimension,
since $M$ carries a compatible $\mathbb{C}$-symplectic structure only if its complex dimension is even.

On the other hand, a stronger possible notion of interpolation on a complex manifold
$M$ could require that, pointwise (or equivalently, at the linear algebra level on the extended
tangent space at a fixed point of $M$), $(\lambda, 0)$- and $(0, \ell)$-structures are in the same orbit of
the group $G$ as in Theorem 2.9. The signature makes this fail for $(1, 1)$-structures. Indeed,
by that theorem and Proposition 2.8 a pseudo Kähler structure on $M$ is pointwise in the
same orbit as a complex product structure (both compatible with $j$) only if it is split. We
have this situation for no other slash structure; in particular, pointwise, hypercomplex and
pseudo-Kähler structures on $M$ of any signature (if existing) are in the same $G$-orbit.

2.6 $B$-fields preserving slash structures on $(M, j)$
Let $\omega$ be a closed two-form on $M$ and let $B_\omega$ be the vector bundle isomorphism of $\mathbb{T}M$
defined by $B_\omega (u + \sigma) = u + \sigma + \omega^j(u)$, which is called a $B$-field transformation. It is well-
known that $B_\omega$ is an isometry for $b$ and preserves generalized complex and paracomplex
structures (acting by conjugation $S \mapsto B_\omega \cdot S = B_\omega \circ S \circ B_{-\omega}$).

**Proposition 2.12** Let $(M, j)$ be a complex manifold and let $\omega$ be a closed two-form on $M$. If $j$ is symmetric for $\omega$, then $B_\omega$ preserves integrable $(\lambda, 1)$- and $(1, 1; n)$-structures
on $M$. Also, if $j$ is skew-symmetric for $\omega$, then $B_\omega$ preserves integrable $(\lambda, -1)$-structures
on $M$.

For instance, a compatible Kähler form $\omega$ on $(M, j)$ provides a $B$-field transformation
of hypercomplex / $\mathbb{C}$-symplectic structures on $(M, j)$, but in general $\omega$ does not need to
be nondegenerate.

**Proof.** Let $\omega$ be as in the statement of the proposition. To see that $B_\omega$ preserves integrable
$(\lambda, \ell)$-structures on $M$, it suffices to show that $B_\omega$ commutes with $J_\ell$, or equivalently, that
$\omega^j \cdot j = \ell j^* \cdot \omega^j$. That is, $j$ is symmetric or skew-symmetric for $\omega$, depending on whether $\ell = 1$
or $\ell = -1$, which is true by hypothesis.

Now, let $S$ be an integrable $(1, 1; n)$-structure on $M$. Since $B_\omega$ commutes with $J_+$ and
is an isometry for $b$, one computes $\beta_{B_\omega \cdot S} = B_{\omega}^* \omega \beta_S$, and so $\beta_{B_\omega \cdot S}$ and $\beta_S$ have the same
signature. Thus, $B_\omega \cdot S$ is an integrable $(1, 1; n)$-structure. \hfill \Box

3 Generalized geometric structures on symplectic manifolds

3.1 Geometric structures compatible with $\omega$
Let $(M, \omega)$ be a symplectic manifold. We consider the following geometric structures on $M$
compatible with $\omega$.

**Integrable $(1, 0)$-structure or bi-Lagrangian foliation of $(M, \omega)$** [16] [13]. It is given
by a paracomplex structure $r$ on $M$ which is skew-symmetric for $\omega$. Then the leaves of the
eigendistributions of $r$ are complementary Lagrangian submanifolds. This structure is also called
para-Kähler [14] or Kähler $\mathbb{L}$-manifold [18].

**Integrable $(-1, 0)$-structure or pseudo-Kähler structure on $(M, \omega)$**. It is given by
a complex structure $j$ on $M$ which is skew-symmetric for $\omega$. If $g$ denotes the pseudo-
Riemannian metric on $M$ given by $g(x, y) = \omega(jx, y)$, then $(M, g, j)$ is pseudo-Kähler.
**Integrable** $(0, 1)$-structure or $\mathbb{L}$-symplectic structure on $(M, \omega)$. It is given by a symplectic form $\theta$ on $M$ such that the tensor field $A$ given by $\theta^\flat = \omega^\flat \circ A$ satisfies $A^2 = \text{id}$ and is split and symmetric for $\omega$. Then $\Omega = \omega + \varepsilon \theta$ is an $\mathbb{L}$-symplectic structure on $M$ ($TM$ is a vector space over $\mathbb{L}$ via $(a + b\varepsilon)u = au + \varepsilon Av$). This structure may be also called a bi-symplectic foliation on $(M, \omega)$. See Proposition 3.1 below.

**Integrable** $(0, -1)$-structure or $\mathbb{C}$-symplectic structure on $(M, \omega)$. It is given by a symplectic form $\theta$ on $M$ such that the tensor field $A$ given by $\theta^\flat = \omega^\flat \circ A$ satisfies $A^2 = -\text{id}$ and is symmetric for $\omega$. Then $\Omega = \omega - i\theta$ is a $\mathbb{C}$-symplectic structure on $M$.

We also have

(+)-integrable $(1, 0)$-structure or Lagrangian foliation of $(M, \omega)$ with a Lagrangian Ehresmann connection. It is given by a tensor field $r$ of type $(1, 1)$ on $M$ with $r^2 = \text{id}$ which is skew-symmetric for $\omega$, such that the $1$-eigensection $D_+^\omega$ of $r$ is an integrable distribution. Then $M \rightarrow M/D_+$ is a Lagrangian foliation with $D_-$ (the $(-1)$-eigensection of $r$) a Lagrangian Ehresmann connection.

All these structures compatible with $\omega$ are well-known except possibly the $\mathbb{L}$-symplectic ones. In the literature we have found an example in the recent paper [10]: If $\sigma_1$ and $\sigma_2$ are as in Theorem A in that article, then one can check that $\sigma_1 + \varepsilon \sigma_2$ is $\mathbb{L}$-symplectic. As it is the case for $\mathbb{C}$-symplectic structures, if $(M, \omega)$ admits an integrable $\mathbb{L}$-symplectic structure, then $\dim M$ is a multiple of 4.

**Proposition 3.1** Let $(M, \omega)$ be a symplectic manifold. Suppose that the closed two-form $\theta$ on $M$ determines an $\mathbb{L}$-symplectic structure on $M$. Then, for $\delta = \pm 1$, the $\delta$-eigendistributions $D_\delta$ of the tensor field $A = (\omega^\flat)^{-1} \circ \theta^\flat$ are integrable and the restriction of $\omega$ to the leaves of both foliations is nondegenerate; in particular, the leaves are symplectic.

Conversely, suppose that $M$ has two complementary foliations $D_\delta$ ($\delta = \pm 1$) with equal dimensions and $\omega|_{D_+ \times D_-}$ is nondegenerate, and define the tensor field $A$ on $M$ of type $(1, 1)$ by $A|_{D_\pm} = \delta \text{id}|_{D_\pm}$. Then $\theta^\flat = \omega^\flat \circ A$ determines an $\mathbb{L}$-symplectic structure on $M$.

**Proof.** First, we check that $D_\delta$ are involutive for $\delta = \pm 1$. Since $\omega$ is nondegenerate, it suffices to see that

$$\omega(A[u, v], z) = \delta \omega([u, v], z)$$

(4)

for any locally defined vector fields $u, v, z$ on $M$, with $u, v$ local sections of $D_\delta$. Now, the left hand side equals

$$\theta([u, v], z) = u\theta(v, z) - v\theta(u, z) + z\theta(u, v) + \theta([u, z], v) - \theta([v, z], u)$$

$$= u\omega(Av, z) - v\omega(Au, z) + z\omega(Au, v) + \omega([u, z], Av) - \omega([v, z], Au)$$

(we have used that $\theta$ is closed and $A$ is symmetric for $\omega$). Since $Au = \delta u$, $Av = \delta v$ and $\omega$ is closed, this is the same as the right hand side of (4), as desired. Also, one computes that $\omega(D_+, D_-) = 0$. Hence the form $\omega$ restricted to $D_\pm$ is nondegenerate. Similar arguments yield the converse. \qed

### 3.2 Slash structures on $(M, \omega)$

**Definition 3.2** Let $(M, \omega)$ be a symplectic manifold. For $k = -1, k = 1$ let $I_k$ be the generalized complex, respectively generalized paracomplex, structure on $M$ given by

$$I_k = \begin{pmatrix} 0 & k(\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix}.$$
**Definition 3.3** Let \((M, \omega)\) be a symplectic manifold. Given \(\lambda = \pm 1\) and \(\ell = \pm 1\), a generalized complex structure \(S\) (for \(\lambda = -1\)) or a generalized paracomplex structure \(S\) (for \(\lambda = 1\)) on \(M\) is said to be an **integrable** \((\lambda, \ell)\)-**structure** on \((M, \omega)\) if

\[
SI_M = I_M S \quad \text{and} \quad I_M S \text{ is split.} \tag{5}
\]

The condition of \(SI_M\) being split is empty if \(\ell = -1\), since \((SI_{-1})^2 = - id\).

In the same way, a \((+)-\)generalized paracomplex structure \(S\) on \(M\) is said to be a **\((+)-integrable** \((1, \ell)\)-**structure** on \((M, \omega)\) if \(SI_\ell = I_\ell S\) and \(SI_\ell\) is split.

We call \(\mathcal{S}_\omega (\lambda, \ell)\) the set of all integrable \((\lambda, \ell)\)-structures on \((M, \omega)\), and \(\mathcal{S}_\omega^+ (1, \ell)\) the set of all \((+)-\)integrable \((1, \ell)\)-structures.

**Example 3.4** If \(r\) and \(\theta\) are integrable \((\lambda, 0)\)- and \((0, \ell)\)-structures on \((M, \omega)\), respectively, then easy computations show that

\[
R = \begin{pmatrix} r & 0 \\ 0 & -r^* \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & \lambda(\theta^b)^{-1} \\ \theta^a & 0 \end{pmatrix}
\]

belong to \(\mathcal{S}_\omega (\lambda, \ell)\). We only comment that \(I_\lambda Q\) is split since it consists of the blocks \(\lambda A\) and \(\lambda A^*\), where \(A\) is the split tensor field associated to \(\theta\) as in the definition of integrable \((0, 1)\)-structures above. For this, see the end of the proof of Theorem 3.5.

The following simple theorem justifies the terminology introduced in the previous subsection and includes the notion of interpolation. See the comment at the end of the section.

**Theorem 3.5** Let \((M, \omega)\) be a symplectic manifold. For \(\lambda = \pm 1, \ell = \pm 1\), integrable \((\lambda, \ell)\)-structures on \((M, \omega)\) interpolate between integrable \((\lambda, 0)\)- and \((0, \ell)\)-structures on \((M, \omega)\), that is, if

\[
R = \begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & p \\ \theta^a & 0 \end{pmatrix}
\]

belong to \(\mathcal{S}_\omega (\lambda, \ell)\), then \(r\) and \(\theta\) are integrable \((\lambda, 0)\)- and \((0, \ell)\)-structures on \((M, \omega)\), respectively.

Also, for \(\ell = \pm 1\), \((+)-\)integrable \((1, \ell)\)-structures interpolate between \((+)-\)integrable \((1, 0)\)- and \((0, \ell)\)-structures on \((M, j)\).

**Proof.** The first paragraph of the proof of Theorem 2.4 applies, in particular \(t = -r^*\), \(\theta\) is a closed 2-form and \(p = \lambda(\theta^b)^{-1}\), and also \(r\) is a complex or paracomplex structure on \(M\) depending on whether \(\lambda = -1\) or \(\lambda = 1\).

Suppose first that \(R\) as above commutes with \(I_M\). Hence, \(-r^*(\omega^b) = \omega^b r\), or equivalently, \(\omega(u, rv) = -\omega(ru, v)\) for all vector fields \(u, v\). That is, \(r\) is skew-symmetric for \(\omega\), as desired.

Now suppose that \(QI_M = I_M Q\). Since \(p = \lambda(\theta^b)^{-1}\), we have that

\[
\lambda(\theta^b)^{-1}\omega^b = \lambda\ell(\omega^b)^{-1}\theta^b.
\]

Calling \(A = (\omega^b)^{-1}\theta^b\), which is a tensor field of type \((1, 1)\) on \(M\), the expression above yields \(A^{-1} = \ell A\), or equivalently, \(A^2 = \ell id\).
Now we verify that $A$ is symmetric for $\omega$, i.e., $\omega(Au,v) = \omega(u,Av)$, or equivalently, $\omega^b(Au)(v) = \omega^b(u)(Av)$ for all vector fields $u,v$ on $M$. This is the same as $\omega^bA = A^*\omega^b$, which is true since $A^* = (\theta^b)^*(\omega^b)^{-1} = (-1)^2\theta^b(\omega^b)^{-1}$ ($\theta$ and $\omega$ are both skew-symmetric).

It remains only to show that $A$ is split if $\ell = 1$. By hypothesis, the matrix $I_\lambda Q = \lambda \text{diag}(A,A^*)$ is split ($A^{-1} = A$). Since the dimensions of the 1-eigenspaces of $A$ and $A^*$ coincide, $A$ must be split.

The last statement is true by the same reasons as in Theorem 2.4. □

### 3.3 Slash structures on $(M,\omega)$ in classical terms

**Proposition 3.6** An integrable $(\lambda,\ell)$-structure $S$ on a symplectic manifold $(M,\omega)$ has the form

$$S = \begin{pmatrix} A & \lambda B (\omega^b)^{-1} \\ \omega^b B & -A^* \end{pmatrix},$$

where $A$ and $B$ are endomorphisms of $TM$ satisfying

$$\lambda A^2 + \ell B^2 = \text{id}, \quad AB + BA = 0, \quad \omega^bA = -A^*\omega^b$$

and, for $\ell = 1$, that the following matrix (which squares to the identity) is split.

$$\begin{pmatrix} B & A \\ \lambda A & B \end{pmatrix}.$$ (7)

**Proof.** For a bilinear map $\pi : V^* \times V^* \to \mathbb{R}$, let $\pi^\sharp : V^* \to V$ be defined by $\eta(\pi^\sharp(\xi)) = \pi(\xi,\eta)$, for all $\xi,\eta \in V^*$.

Since $S$ is a generalized complex (for $\lambda = -1$) or paracomplex structure (for $\lambda = 1$), by [8] (see also [22]) one has

$$S = \begin{pmatrix} A & \pi^\sharp \\ \theta^b & -A^* \end{pmatrix},$$

where $\theta$ and $\pi$ are skew-symmetric, and $A$ satisfies

$$A^2 + \pi^\sharp \theta^b = \lambda \text{id}, \quad \theta^bA = A^*\theta^b, \quad \text{and} \quad \pi^\sharp A^* = A\pi^\sharp.$$ (9)

Now, using the first equation in (5) we have that

$$\pi^\sharp \omega^b = \lambda \ell (\omega^b)^{-1} \theta^b \quad \text{and} \quad \omega^bA = -A^*\omega^b.$$ (10)

Putting $B = (\omega^b)^{-1} \theta^b$, we have $\pi^\sharp = \lambda \ell B(\omega^b)^{-1}$ and so (6) holds. Besides, the second equations in (9) and (10) yield $AB + BA = 0$. Notice that, in particular, $A^2 + B^2 = (A+B)^2$. The last assertion corresponds to the fact that $SI_\lambda$ must be split if $\ell = 1$, and follows from the fact that an easy computation shows that $\lambda \phi^{-1}SI_\lambda \phi$ equals (7), where $\phi : TM \oplus TM \to TM$ is defined by $\phi(u,v) = (u,\omega^b v)$. □

M. Crainic obtained in [8] (see also [22]) conditions on $A,\theta$ and $\pi$ for $S$ as in (8) to be Courant integrable. One can deduce conditions on $A$ and $B$ as in (6) for the integrability of $S$. Since the resulting expressions are simpler, but still quite involved, we do not write them down.
3.4 A signature associated to integrable \((-1, 1)\)-structures on \((M, \omega)\)

**Proposition 3.7** Let \(S\) be an integrable \((-1, 1)\)-structure on a symplectic manifold \((M, \omega)\) of dimension \(2m\). Then the form \(\beta_S\) on \(TM\) defined by \(\beta_S(x, y) = b(I_S x, y)\) is symmetric and has signature \((4n, 4m - 4n)\) for some integer \(n\) with \(0 \leq n \leq m\).

**Proof.** The form \(\beta_S\) is symmetric since \(S\) and \(I_\) are skew-symmetric for \(b\). One has that \((I_S)^2 = \text{id}\). For \(\delta = \pm 1\), let \(D_\delta\) be the \(\delta\)-eigensection of \(I_S\). Since \(I_S\) is required to be split, \(D_+\) and \(D_-\) have both dimension \(2n\).

One computes \(b(D_+, D_-) = 0\). For \(\delta = \pm 1\) let \(b^\delta := b|_{D_\delta \times D_\delta}\) and \(\beta^\delta := \beta_S|_{D_\delta \times D_\delta}\). By the orthogonality lemma (2.30 in [17]), \(b^\delta\) is nondegenerate. One computes also \(b^\delta = b^\delta\). Now, since \(I_-\) is an isometry for \(b\) and preserves \(D_\delta\), \(\beta^\delta = \beta^\delta\) has signature \((2n, 2m - 2n)\) for some integer \(0 \leq n \leq m\). Then \(\beta^-\) has signature \((2m - 2n, 2n)\) \((b\text{ is split})\). Therefore, \(\beta^-\) has signature \((2n, 2m - 2n)\) and the signature of \(\beta_S\) is \((4n, 4m - 4n)\). \(\square\)

**Definition 3.8** An integrable \((-1, 1)\)-structure \(S\) on \((M, \omega)\) as above is called an integrable \((-1, 1; n)\)-structure, and we write \(\text{sig}(S) = n\). If \(m = 2n\), by the next proposition, the \((-1, 1; n)\)-structure is called a (split Kähler) \(/ L\)-symplectic structure on \((M, \omega)\).

**Proposition 3.9** a) Let \(j\) be an integrable \((-1, 0)\)-structure on \((M, \omega)\). Then

\[
R = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}
\]

is a \((-1, 1; n)\)-structure on \((M, \omega)\) if and only if the pseudo-Kähler metric \(g(u, v) = \omega(ju, v)\) on \(M\) has signature \((2n, 2m - 2n)\).

b) Let \(\theta\) be an integrable \((0, 1)\)-structure on \((M, \omega)\). Then

\[
Q = \begin{pmatrix} 0 & -(\theta^\circ)^{-1} \\ \theta^\circ & 0 \end{pmatrix}
\]

is a \((-1, 1; n)\)-structure on \((M, \omega)\) if and only if \(m = 2n\).

**Proof.** a) One computes

\[
\beta_R (u + \sigma, v + \tau) = \omega(ju, v) + \tau (\omega^b)^{-1} j^* \sigma = g(u, v) + h(\sigma, \tau),
\]

where the symmetric form \(h\) on \(T^*M\) is defined by the last equality. Now,

\[
(\omega^b)^* h(z, w) = h(\omega^b z, \omega^b w) = -\omega^b(w)(jz) = \omega(jz, w) = g(z, w),
\]

since for an integrable \((-1, 0)\)-structure \(j\) on \((M, \omega)\), \(j\) is skew-symmetric for \(\omega\). Therefore, if \(\phi : TM \oplus TM \to TM\) is the isomorphism defined at the end of the proof of Proposition 3.6 then

\[
\phi^* \frac{\partial}{\partial z} ((u, z), (v, w)) = g(u, v) + g(z, w).
\]

This implies the assertion of (a), since \(\phi^* \beta_R\) and \(\beta_R\) have the same signature.

b) As in the definition of integrable \(L\)-symplectic structure, we call \(A = (\omega^b)^{-1} \theta^\circ\). We compute

\[
\beta_Q (u + \sigma, v + \tau) = -\tau (A u) - \sigma (A v).
\]

We have used that \(\theta^\circ (\omega^b)^{-1} = A^*\) (since \(\theta\) and \(\omega\) are skew-symmetric) and that \(A^{-1} = A\). Since \(A\) is split, locally, there exists a basis \(\{u_1, \ldots, u_{2m}\}\) of \(TM\) such that \(A u_i = u_i\) for \(1 \leq i \leq m\) and \(A u_i = -u_i\) for \(m < i \leq 2m\). Let \(\{\alpha_1, \ldots, \alpha_{2m}\}\) be the dual basis. Analyzing the signs of \(\beta_Q (u_i + \alpha_i, u_i + \alpha_i)\) and \(\beta_Q (u_i - \alpha_i, u_i - \alpha_i)\), one concludes that \(\beta_Q\) is split, and this yields (b). \(\square\)
3.5 The associated homogeneous bundles over \((M,\omega)\)

Now, as we did in the complex case, we work at the algebraic level. We fix \(p \in M\) and call \(E = T_p M\). By abuse of notation, in the rest of the section we write \(b\) and \(I_k\) instead of \(b_p\) and \((I_k)_p\), omitting the subindex \(p\).

**Theorem 3.10** Let \((M,\omega)\) be a symplectic manifold of dimension \(2m\). Then, integrable \((\lambda,\ell)\)- or \((-1,1; n)\)-structures on \((M,\omega)\) are smooth sections of a fiber bundle over \(M\) with typical fiber \(G/H\), according to the following table.

| \(\lambda\) | \(\ell\) | \(\text{sig}\) | \(G\) | \(H\) |
|---|---|---|---|---|
| 1 | 1 | - | \(GL(2m, \mathbb{R})\) | \(GL(m, \mathbb{R}) \times GL(m, \mathbb{R})\) |
| 1 | -1 | - | \(U(m, m)\) | \(GL(m, \mathbb{C})\) |
| -1 | 1 | \(n\) | \(U(m, m)\) | \(U(n, m - n) \times U(m - n, n)\) |
| -1 | -1 | - | \(GL(2m, \mathbb{R})\) | \(GL(m, \mathbb{C})\) |

Before proving the theorem we introduce some notation and present a proposition. Let \(\sigma(\lambda,\ell)\) denote the set of all \(S \in \text{End}_\mathbb{R}(E)\) satisfying

\[
S^2 = \lambda \text{id}, \ \text{S is split, skew-symmetric for b, and } SI_\lambda = I_\lambda S \text{ is split.}
\]

Note that \((E, I_k)\) is a vector space over \(\mathbb{C}\) (respectively, \(\mathbb{L}\)) for \(k = -1\) (respectively, \(k = 1\)). The notion of \(\mathbb{L}\)-Hermitian forms [18] is analogous to the one of \(\mathbb{C}\)-Hermitian forms (see the beginning of Subsection 2.4).

**Proposition 3.11** Let \(b_- : E \times E \to \mathbb{C}\) and \(b_+ : E \times E \to \mathbb{L}\) be defined by

\[
b_-(x, y) = b(x, y) - ib(x, I_-y) \quad \text{and} \quad b_+(x, y) = b(x, y) + \varepsilon b(x, I_+y).
\]

Then \(b_-\) is split \(\mathbb{C}\)-Hermitian and \(b_+\) is \(\mathbb{L}\)-Hermitian (with respect to \(I_-, I_+\), respectively).

Also, if \(S \in \text{End}_\mathbb{R}(E)\) satisfies \(S^2 = \lambda \text{id}\) and \(I_\lambda S\) is split, then \(S \in \sigma(\lambda,\ell)\) if and only if

\[
b_\lambda(x, y) = -\lambda b_\lambda(x, y)
\]

for any \(x, y \in E\).

**Proof.** We call \(\epsilon_1 = \varepsilon\) and \(\epsilon_{-1} = i\) (in particular, \(\epsilon_k^2 = k\)). First, for \(k = \pm 1\), one has to show that

\[
\epsilon_kb_k(x, y) = b_k(x, I_ky) = -b_k(I_kx, y) \quad \text{and} \quad b_k(x, y) = b_k(y, x)
\]

for all \(x, y\). This follows easily from the definitions and the fact that \(I_k\) is skew-symmetric for \(b\). Also, \(b_-\) is split since \(b = \text{Re } b_-\) is split.

Now we prove the second assertion. Suppose first that \(S \in \sigma(\lambda,\ell)\). We call \(k = \lambda\ell\). Since \(S\) commutes with \(I_k\), we compute (using [3] with \(T = S\) and \(\mu = \lambda\))

\[
b_k(Sx, Sy) = b(Sx, Sy) + k\epsilon_kb(Sx, I_ky) = -\lambda b(x, y) + k\epsilon_kb(Sx, SI_ky) = -\lambda b(x, y) - \lambda k\epsilon_kb(x, I_ky) = -\lambda b_k(x, y).
\]
Conversely, suppose that $S^2 = \lambda \text{id}$, $SI_k$ is split and $\mu = \lambda$, $S$ is skew-symmetric for $b = \text{Re} \ b_k$. Now, for $k = \pm 1$, we compute

$$
k(x, SI_k y) = \lambda b_k (S^2 x, SI_k y) = \lambda (-\lambda) b_k (Sx, y) = -\epsilon_k b_k (Sx, y) = -\epsilon_k \lambda (-\lambda) b_k (x, S y) = b_k (x, I_k S y).
$$

Since $b_k$ is nondegenerate, $S$ commutes with $I_k$. Therefore, $S \in \sigma (\lambda, \ell)$. $
$

**Proof of Theorem 3.10.** We follow the same scheme as in the proof of Theorem 2.9. We suppose first that $\lambda \ell = -1$. By the first assertion in Proposition 3.11, there exist complex linear coordinates $(\phi_-)^{-1} = (z, w) : (\mathbb{E}, I_-) \to \mathbb{C}^{2m}$ such that $B_- := (\phi_-)^* b_-$ is given by

$$
B_- ((z, w), (z', w')) = \overline{z'} w' + \overline{w} z',
$$

which is equivalent to the standard split Hermitian form $z' w' - \overline{w} z'$. Let $\Sigma (\lambda, \ell)$ be the subset of $\text{End}_\mathbb{C} (\mathbb{C}^{2m})$ corresponding to $\sigma (\lambda, \ell)$ via the isomorphism $\phi_-$. Clearly $U(m, m)$ acts on $\Sigma (+, -)$ and $\Sigma (-, +)$ by conjugation.

**Case** $(+, -)$: Let $S \in \text{End}_\mathbb{C} (\mathbb{C}^{2m})$ be defined by $S(z, w) = (z, -w)$. Using the second assertion of Proposition 3.11 one verifies that $S$ belongs to $\Sigma (+, -)$ (since $\ell = -1$, there is no need to check that $iS$ is split). For $\delta = \pm 1$, let $V'_{\delta}$ be the $\delta$-eigenspace of $S$, that is,

$$
V'_{\delta} = \{(z, 0) \mid z \in \mathbb{C}^m\} \quad \text{and} \quad V_{\delta} = \{(0, z) \mid z \in \mathbb{C}^m\}.
$$

Given $A \in \text{Gl} (m, \mathbb{C})$, if $\tilde{A} (z, w) = (Az, (A^t)^{-1} w)$, then $\tilde{A} \in U(m, m)$. This provides an inclusion of $\text{Gl} (m, \mathbb{C})$ into $U(m, m)$. Let $H$ be the isotropy subgroup at $S$. For $A \in \text{Gl} (m, \mathbb{C})$, clearly $\tilde{A}$ commutes with $S$ and so $\tilde{A} \in H$. Besides, if $L \in U(m, m)$ commutes with $S$, then $L$ preserves $V_+$ and $V_-$. Hence $L(z, w) = (Az, Bw)$ for some $A, B \in \text{Gl} (m, \mathbb{C})$. Now, $B^{-1} = \overline{A^t}$ since $L$ is an isometry for $B_-$, so $L = \tilde{A}$. Therefore $H = \text{Gl} (m, \mathbb{C})$.

The action is transitive: Let $T \in \Sigma (+, -)$ and for $\delta = \pm 1$ let $W'_{\delta}$ be the $\delta$-eigenspace of $T$. By (11), $W'_{\delta}$ is isotropic for $B_-$. Let $\beta : W'_{+} \to (W'_-)^*$ be given by $\beta (u) (v) = B_- (\bar{u}, v)$, which is an isomorphism of vector spaces over $\mathbb{C}$. Let $u_1, \ldots, u_m$ be a basis of $W'_+$ over $\mathbb{C}$ and let $v_1, \ldots, v_m$ be the basis of $W_-$ dual to $\beta (\overline{w}_s), s = 1, \ldots, m$. Let $F : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ be given by $F (e_s, 0) = u_s$ and $F (0, e_s) = v_s$. Then $F \in \text{U}(m, m)$ and $T = FSF^{-1}$. So the action is transitive.

**Case** $(-, +; n)$: Write $z = (z_1, z_2), w = (w_1, w_2)$, with $z_1, w_1 \in \mathbb{C}^n, z_2, w_2 \in \mathbb{C}^{m-n}$, $0 \leq n \leq m$. Let $S \in \text{End}_\mathbb{C} (\mathbb{C}^{2m})$ be defined by

$$
S(z_1, z_2, w_1, w_2) = (-iw_1, iw_2, -iz_1, iz_2).
$$

We have that $S^2 = -\text{id}$ and $iS (z_1, z_2, w_1, w_2) = (w_1, -w_2, z_1, -z_2)$. For $\delta = \pm 1$, the $\delta$-eigenspace of $iS$ is

$$
V'_{\delta} = \{(z, \delta r (z)) \mid z \in \mathbb{C}^m\} \cong \mathbb{C}^m.
$$

where $r (z_1, z_2) = (z_1, -z_2)$ for $z_1 \in \mathbb{C}^n, z_2 \in \mathbb{C}^{m-n}$. Hence, $iS$ is split. One computes that $S$ is an isometry for $B_-$. Then, the second assertion of Proposition 3.11 implies that $S$ belongs to $\Sigma (-, +)$. Now, it turns out that

$$
\text{Re} B_- (iS(z_1, z_2, w_1, w_2), (z'_1, z'_2, w'_1, w'_2)) = \text{Re} \left(\overline{w'_1} w'_1 - \overline{w'_2} w'_2 + \overline{z'_1} z'_1 - \overline{z'_2} z'_2\right),
$$

16
which is a real inner product on $\mathbb{C}^{2m}$ of signature $(4n, 4m - 4n)$. Therefore, $S \in \Sigma (-, +; n)$.

One verifies that $\beta^\delta := B_-|_{V_\delta \times V_\delta}$ is $\mathbb{C}$-Hermitian with Hermitian signature $(n, m - n)$ for $\delta = 1$ and $(m - n, n)$ for $\delta = -1$. There is an obvious isomorphism $\psi_\delta : \mathbb{C}^m \to V_\delta$, $\psi_\delta (z) = (z, \delta \bar{r} (z))$. Given $A \in U(n, m - n)$ and $B \in U(m - n, n)$, the map $(A, B) \mapsto \alpha_{A,B}$ defines an inclusion of $U(n, m - n) \times U(m - n, n)$ into $U(m, m)$, where $\alpha_{A_1, A_2} x = \psi_\delta A_\delta (\psi_\delta)^{-1} x$ if $x \in V_\delta$.

Now suppose that $\alpha$ is in the isotropy subgroup at $S$ of the action of $U(m, m)$, or equivalently, that $\alpha$ is in $U(m, m)$ and commutes with $S$. Hence, $\alpha$ preserves $V_\delta$ for $\delta = \pm 1$. Then, $\alpha$ must have the form $\alpha_{A,B}$ as above.

It remains to show that the action is transitive. Let $T \in \Sigma (-, +; n)$ and for $\delta = \pm 1$ let $W_\delta$ be the $\delta$-eigenspace of $i T$ (it is a complex subspace, since it is the $(-\delta i)$-eigenspace of $T$). By [11], one has that $B_- (W_+, W_-) = 0$, and so $\gamma^\delta := B_-|_{W_\delta \times W_\delta}$ is a nondegenerate $\mathbb{C}$-Hermitian form on $W_\delta$. Since $T \in \Sigma (-, +; n)$, $\gamma^+$ and $\gamma^-$ have Hermitian signature $(n, m - n)$ and $(m - n, n)$, respectively. One uses the Basis Theorem to see that there exists $F \in U(m, m)$ such that $T = FSF^{-1}$. Therefore, $\Sigma (-, +; n)$ can be identified with $U(m, m) / (U(n, m - n) \times U(m, n, n))$, as desired.

Now assume that $\lambda \ell = 1$. By Proposition [5.11] there exist Lorentz linear coordinates $\phi^\pm_{-1} : \mathbb{E} \to \mathbb{L}^{2m}$, such that $B_+ := \phi^+_+ b_+$ has the form

\[ B_+ (Z, Z') = Z' Z', \]

where $Z, Z' \in \mathbb{L}^{2m}$. Let $\Sigma (\lambda, \ell)$ be the subset of $\text{End}_L (\mathbb{L}^{2m})$ corresponding to $\sigma (\lambda, \ell)$ via the isomorphism $\phi_+^{-1}$.

Let $e = (1 - \varepsilon)/2, \bar{e} = (1 + \varepsilon)/2$, which are null Lorentz numbers forming a basis of $L$. On has $e^2 = e, \bar{e}^2 = \bar{e}, e\bar{e} = 0$ and $e\bar{e} = -e, \bar{e}e = \bar{e}$.

By [15] (Section 3), the group $G$ of transformations preserving $B_+$ (that is, $L$-unitary transformations) is isomorphic to $\text{Gl} (2m, \mathbb{R})$; more precisely, any element of $G$ has the form $A \in \text{Gl} (2m, \mathbb{R})$, where

\[ A(xe + y\bar{e}) = (Ax)e + ((A^t)^{-1}y)\bar{e} \] (12)

for all $x, y \in \mathbb{R}^{2m}$. Clearly $\text{Gl} (2m, \mathbb{R})$ acts by conjugation on $\Sigma (+, +)$ and $\Sigma (-, -)$.

**Case** $(+, +)$: Let $S \in \text{End}_L (\mathbb{L}^{2m})$ be defined by $S(xe + y\bar{e}) = r(x)e - r(y)\bar{e}$, where $x, y \in \mathbb{R}^{2m}$ and $r(x_1, x_2) = (x_1, -x_2)$, with $x_i \in \mathbb{R}^{m}$ (in particular, $r^2 = \text{id}$ and $r$ is split). Hence $\varepsilon S(xe + y\bar{e}) = -r(x)e - r(y)\bar{e}$. Both $S$ and $\varepsilon S$ square to the identity and are split, as required ($I_+ \rightarrow \text{corresponds to multiplication by } \varepsilon \text{ in } \mathbb{L}^{2m}$). We compute

\[ B_+ (S(xe + y\bar{e}), S(x' e + y' \bar{e})) = -r(y)' r(x') e - r(x) \bar{e} \]

\[ = -B_+ (xe + y\bar{e}, x' e + y' \bar{e}), \]

since $r^t r = \text{id}$. Therefore $S \in \Sigma (+, +)$. The isotropy subgroup of the action of $\text{Gl} (2m, \mathbb{R})$ at $S$ consists of the maps $A$ as in [12], where $A(x_1, x_2) = (ax_1, bx_2)$ for some $a, b \in \text{Gl} (m, \mathbb{R})$, hence, it can be identified with $\text{Gl} (m, \mathbb{R}) \times \text{Gl} (m, \mathbb{R})$.

Now, we see that the action is transitive. Let $T \in \Sigma (+, +)$. Since $T$ is $L$-linear, $T(xe + y\bar{e}) = f(x)e + g(y)\bar{e}$ for some linear endomorphisms $f, g$ of $\mathbb{R}^{2m}$. The condition that $T^2 = \text{id}$ implies that $f^2 = g^2 = \text{id}$. Suppose that $f$ and $g$ have signatures $(k, 2m - k)$ and $(l, 2m - l)$, respectively. Since both $T$ and $\varepsilon T(xe + y\bar{e}) = -f(x)e + g(y)\bar{e}$ are split by hypothesis, we have that $k + l = 2m$ and $2m - k + l = 2m$. Hence $k = l = m$ and so $f$
and \( g \) are split. Then \( f \) is conjugate to \( r \) in \( Gl(2m, \mathbb{R}) \), say \( f = crc^{-1} \) with \( c \in Gl(2m, \mathbb{R}) \). Besides, an easy computation using that \( T \) is an anti-isometry for \( B_{+} \) yields \( g = -(f^t)^{-1} \). Therefore \( T = CSC^{-1} \) with \( C(xe + ye) = c(x)e + (c^t)^{-1}(y)e \), as desired.

**Case \((-,-):** Let \( S \in \text{End}_{\mathbb{L}}(\mathbb{L}^{2m}) \) be defined by \( S(xe + ye) = j(x)e + j(y)e \), where \( x, y \in \mathbb{R}^{2m} \) and \( j(x_1, x_2) = (-x_2, x_1) \), with \( x_i \in \mathbb{R}^m \) (in particular, \( j^2 = -\text{id} \) and \( j^tj = \text{id} \)). Computations analogous to those of the case \((+,-)\) show that \( S \in \Sigma(-,-) \) and that the isotropy subgroup of the action of \( Gl(2m, \mathbb{R}) \) at \( S \) consists of the maps \( \hat{A} \) as in (12), where \( A \) commutes with \( j \), that is, \( A \in Gl(m, \mathbb{C}) \) via the canonical identification of \( (\mathbb{R}^{2m}, j) \) with \( \mathbb{C}^m \). Also, transitivity of the action follows from similar arguments as in the case \((+,+)\). \( \square \)

Finally, we comment on the strength of the notion of interpolation for slash structures on symplectic manifolds, in analogy with Subsection 2.5 for complex manifolds. Suppose that the dimension of the symplectic manifold \( M \) is \( m = 2n \). If \( n \) is odd there may exist integrable \((\lambda, \ell)\)-structures (for instance \((\lambda, 0)\)-structures, i.e. pseudo-Kähler structures or bi-Lagrangian foliations compatible with \( \omega \)), but there cannot exist \((0, \ell)\)-structures on \( M \) \((\ell = \pm 1; \text{even not integrable ones}) \), since these require \( n \) to be even.

Moreover, by Theorem 3.10 and Proposition 3.9, pointwise, a \((-1, 0)\)-structure on \( M \) (i.e. a pseudo-Kähler structure on \( M \) compatible with \( \omega \)) is in the same \( G \)-orbit as a \((0, 1)\)-structure on \( M \) \((G \text{ as in that theorem}) \) only if the pseudo-Kähler structure is split. We have this type of shortcoming for no other slash structure on \((M, \omega)\); in particular, pointwise, \( \mathbb{C}\)-symplectic and pseudo-Kähler structures on \( M \) of any signature (if existing) are in the same \( G \)-orbit.

Most of the structures considered on complex and symplectic manifolds have been extensively studied. In the bibliography we refer mainly to those which are less known or have aroused special interest lately.

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