ON COMPACT 3-MANIFOLDS WITH NONNEGATIVE SCALAR CURVATURE WITH A CMC BOUNDARY COMPONENT

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ABSTRACT. We apply the Riemannian Penrose inequality and the Riemannian positive mass theorem to derive inequalities on the boundary of a class of compact Riemannian 3-manifolds with nonnegative scalar curvature. The boundary of such a manifold has a CMC component, i.e. a 2-sphere with positive constant mean curvature; and the rest of the boundary, if nonempty, consists of closed minimal surfaces. A key step in our proof is the construction of a collar extension that is inspired by the method of Mantoulidis-Schoen [12].

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we are interested in a compact Riemannian 3-manifold $\Omega$ with nonnegative scalar curvature, with boundary $\partial \Omega$, such that $\partial \Omega$ has a component $\Sigma_o$ that is a topological 2-sphere with positive mean curvature. When $\partial \Omega \setminus \Sigma_o \neq \emptyset$, we assume that $\partial \Omega \setminus \Sigma_o$ is the unique, closed minimal surface (possibly disconnected) in $\Omega$, i.e. there are no other closed minimal surfaces in $\Omega$. In this case, we denote $\partial \Omega \setminus \Sigma_o$ by $\Sigma_h$. In a relativistic context, such an $\Omega$ represents a finite body in a time-symmetric initial data set, surrounding the apparent horizon modeled by $\Sigma_h$.

Motivated by the quasi-local mass problem (cf. [18]), we want to understand the effect of nonnegative scalar curvature and the existence of $\Sigma_o$ on the boundary geometry of $\Sigma_o$. To be more precise, let $g$ denote the induced metric on $\Sigma_o$ and $H$ be the mean curvature of $\Sigma_o$ in $\Omega$. We want to understand the restriction imposed by the scalar curvature and the horizon boundary $\Sigma_h$ on the pair $(g, H)$.

A special case of this question was studied in [15]. It was proved in [15] that

$$(\Sigma_o, g) \text{ is a round sphere } \Rightarrow \sqrt{\frac{|\Sigma_o|}{16\pi}} \left[ 1 - \frac{1}{16\pi |\Sigma_o|} \left( \int_{\Sigma_o} H d\sigma \right)^2 \right] \geq \sqrt{\frac{|\Sigma_h|}{16\pi}},$$

where $|\Sigma_o|$, $|\Sigma_h|$ are the area of $\Sigma_o$, $\Sigma_h$, respectively, and $d\sigma$ denotes the area element on $\Sigma_o$. The left side of the above inequality closely resembles the Hawking mass [7] of $\Sigma_o$ in $\Omega$, given by

$$m_H(\Sigma_o) = \sqrt{\frac{|\Sigma_o|}{16\pi}} \left[ 1 - \frac{1}{16\pi} \int_{\Sigma_o} H^2 \ d\sigma \right].$$

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The Hawking mass functional $m_H(\cdot)$ played a key role in Huisken and Ilmanen’s proof of the Riemannian Penrose inequality (cf. [2, 9]) when the horizon is connected. In particular, by the results in [9], if a weak solution $\{\Sigma_t\}$ consisting of connected surfaces to the inverse mean curvature flow with initial condition $\Sigma_h$ exists in $\Omega$ and if $\Sigma_o$ happens to be a leaf in $\{\Sigma_t\}$, then one would have $m_H(\Sigma_o) \geq \sqrt{\frac{\text{area}(\Sigma_h)}{16\pi}}$.

In general, without imposing suitable conditions on $\Sigma_o$, one should not expect to have $m_H(\Sigma_o) \geq \sqrt{\frac{\text{area}(\Sigma_h)}{16\pi}}$ since $m_H(\Sigma_o)$ may even fail to be positive. On the other hand, if a 2-surface is a stable constant mean curvature (CMC) surface in a 3-manifold with nonnegative scalar curvature, Christodoulou and Yau [4] showed that its Hawking mass is always nonnegative.

In this paper, we consider an $\Omega$ in which $\Sigma_o$ is a CMC surface. We have

**Theorem 1.1.** Let $\Omega$ be a compact, orientable, Riemannian 3-manifold with boundary $\partial\Omega$. Suppose $\partial\Omega$ is the disjoint union of $\Sigma_o$ and $\Sigma_h$ such that

(a) $\Sigma_o$ is a topological 2-sphere with constant mean curvature $H_o > 0$;
(b) $\Sigma_h$, which may have multiple components, is a minimal surface; and
(c) there are no other closed minimal surfaces in $\Omega$.

Suppose $\Omega$ has nonnegative scalar curvature and the induced metric $g$ on $\Sigma_o$ has positive Gauss curvature. There exists a quantity $0 < \eta(g) \leq \infty$, uniquely determined by $(\Sigma_o, g)$ and invariant under scaling of $g$, such that if

$$W := \frac{1}{16\pi} \int_{\Sigma_o} H_o^2 d\sigma < \eta(g),$$

then

$$\sqrt{\frac{\text{area}(\Sigma_h)}{16\pi}} \leq \left[ \frac{W}{\eta(g) - W} \right]^{\frac{1}{2}} \sqrt{\frac{\text{area}(\Sigma_o)}{16\pi}} + m_H(\Sigma_o).$$

(1.1)

Here $\eta(g) = \infty$ if $g$ is a round metric. In this case, (1.1) reduces to $\sqrt{\frac{\text{area}(\Sigma_h)}{16\pi}} \leq m_H(\Sigma_o)$.

Theorem 1.1 has the following analogue when $\partial\Omega = \Sigma_o$.

**Theorem 1.2.** Let $\Omega$ be a compact, Riemannian 3-manifold with nonnegative scalar curvature, with boundary $\Sigma_o$. Suppose $\Sigma_o$ is a topological 2-sphere with constant mean curvature $H_o > 0$. Suppose the induced metric $g$ on $\Sigma_o$ has positive Gauss curvature. Let $\eta(g)$ be the scaling invariant of $(\Sigma_o, g)$ stated in Theorem 1.1. If

$$W := \frac{1}{16\pi} \int_{\Sigma_o} H_o^2 d\sigma < \eta(g),$$

then

$$\left[ \frac{W}{\eta(g) - W} \right]^{\frac{1}{2}} \sqrt{\frac{\text{area}(\Sigma_o)}{16\pi}} + m_H(\Sigma_o) \geq 0.$$
The quantity \( \eta(g) \) measures how far \( g \) is different from a round metric on \( \Sigma_o \). We will give its precise definition in Section 4. For now we give a few remarks on Theorems 1.1 and 1.2.

**Remark 1.1.** For a fixed \( \delta \in (0, 1) \), it is proved in Proposition 4.1 that

\[
\eta(g) \geq C \frac{C}{||g - g_o||_{C^0,\delta}(\Sigma_o)}
\]

for some positive constant \( C \) independent on \( g \). In particular, \( \eta(g) \) tends to \( \infty \) as \( g \) approaches \( g_o \) in the \( C^2,\delta \)-norm. On the other hand, given an \( \Omega \) in Theorem 1.1, by Shi and Tam’s result [20, Theorem 1] (or more precisely by their proof), one has

\[
\int_{\Sigma_o} H_o d\sigma < \int_{\Sigma_o} H_E d\sigma,
\]

where \( H_o \) is the mean curvature of the isometric embedding of \( \Sigma_o \) in \( \mathbb{R}^3 \). Consequently,

\[
W < \omega(g) := \frac{1}{16\pi |\Sigma_o|} \left( \int_{\Sigma_o} H_E d\sigma \right)^2.
\]

Therefore, the condition \( W < \eta(g) \) is automatically satisfied if \( \omega(g) \leq \eta(g) \). By (1.3), this is true if \( g \) is \( C^2,\delta \)-close to a round metric.

**Remark 1.2.** Given an \( \Omega \) in Theorem 1.2, one knows \( W < \eta(g) \) always holds if \( g \) is \( C^2,\delta \)-close to a round metric for the reason explained in Remark 1.1. Therefore, inequality (1.2) is true for any CMC surface \( \Sigma \) bounding a compact 3-manifold with nonnegative scalar curvature, provided the induced metric on \( \Sigma \) is sufficiently round. This may be compared with the result of Christodoulou and Yau [4] which gives \( m_H(\Sigma) \geq 0 \) under the extrinsic curvature condition.

**Remark 1.3.** On an asymptotically flat 3-manifold \( M \), there exist foliations by CMC spheres near infinity (cf. [10, 13, 8, 5, 16]). For instance, Nerz [16] obtained the existence and uniqueness of such a foliation without assuming asymptotic symmetry conditions. Let \( \{\Sigma_\sigma\}_{\sigma > \sigma_0} \) be a foliation of CMC spheres near infinity of \( M \) and suppose \( \partial M \) consists of outermost minimal surfaces. Let \( \Omega_\sigma \) be the region bounded by \( \Sigma_\sigma \) and \( \partial M \). Let \( g_\sigma \) be the induced metric on \( \Sigma_\sigma \). If \( M \) is \( C^2,\delta \)-asymptotically flat with decay rate \( \tau > \frac{1}{2} \), it follows from Nerz’s work (cf. [16, Proposition 4.4]) that, upon pulling-back to \( S^2 \), the rescaled metric \( \tilde{g}_\sigma := \sigma^{-2} g_\sigma \) satisfies

\[
||\tilde{g}_\sigma - g_\star||_{C^2,\delta}(S^2) \leq C \sigma^{-\tau}
\]

for some fixed round metric \( g_\star \) of area \( 4\pi \) and a constant \( C \) independent on \( \sigma \). Thus, along \( \{\Sigma_\sigma\} \), \( W = 1 + O(\sigma^{-\tau}) \) while \( \eta(g_\sigma) \to \infty \) by (1.3). Hence, Theorem 1.1 is applicable to \( \Omega_\sigma \) for large \( \sigma \). However, our estimate of \( \eta(g) \) in (1.3) is not strong enough to imply \( \left[ \frac{W}{\eta(g_\sigma)} \right]^{1/2} \sqrt{\frac{|\Sigma_\sigma|}{16\pi}} \to 0 \) along \( \{\Sigma_\sigma\} \). If this could be shown, then one

\[\text{We thank Christopher Nerz for explaining this estimate along the CMC foliation.}\]
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would recover the Riemannian Penrose inequality by taking limit of (1.1) since the Hawking mass \( m_\mathcal{H}(\Sigma_\sigma) \) approaches to the ADM mass \([1]\) along \( \{\Sigma_\sigma\} \).

When \( \partial \Omega = \Sigma_o \cup \Sigma_h \), we have another result separate from Theorem 1.1.

**Theorem 1.3.** Let \( \Omega \) be a compact, orientable, Riemannian 3-manifold with boundary \( \partial \Omega \). Suppose \( \partial \Omega \) is the disjoint union of \( \Sigma_o \) and \( \Sigma_h \) such that

(a) \( \Sigma_o \) is a topological 2-sphere with constant mean curvature \( H_o > 0 \);
(b) \( \Sigma_h \), which may have multiple components, is a minimal surface; and
(c) there are no other closed minimal surfaces in \( \Omega \).

Suppose \( \Omega \) has nonnegative scalar curvature and the induced metric \( g \) on \( \Sigma_o \) has positive Gauss curvature. There exist constants \( 0 < \beta_g \leq 1 \) and \( \alpha_g \geq 0 \), determined by \( (\Sigma_o, g) \), such that if

\[
W := \frac{1}{16\pi} \int_{\Sigma_o} H_o^2 d\sigma < \frac{\beta_g}{1 + \alpha_g},
\]

then

\[
\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \left[ \left( \frac{\alpha_g W}{\beta_g - (1 + \alpha_g)W} \right)^{\frac{1}{2}} + 1 \right] m_\mathcal{H}(\Sigma_o).
\]

(1.4)

If \( g \) is a round metric, one can take \( \beta_g = 1 \) and \( \alpha_g = 0 \). In this case, (1.4) reduces to

\[
\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq m_\mathcal{H}(\Sigma_o).
\]

**Remark 1.4.** Similar to \( \eta(g) \), the constants \( \alpha_g \) and \( \beta_g \) also measure how far \( g \) is different from a round metric. By the proof of Proposition 4.1 in Section 4, one can take \( \alpha_g \to 0 \) and \( \beta_g \to 1 \) as \( g \) approaches a round metric. As a result, suppose \( \Omega \) is normalized so that \( |\Sigma_o| = 4\pi \) and the mean curvature constant \( H_o \) satisfies \( H_o < 2 \), then the condition \( W < \frac{\beta_g}{1 + \alpha_g} \) is always met if \( g \) is sufficiently round.

Now we outline the idea of the proof of Theorems 1.1–1.3. When the intrinsic metric \( g \) on \( \Sigma_o \) is round, Theorems 1.1 and 1.3 follow from [15] and Theorem 1.2 follows from [14, 20]. Thus, the major case to prove is when \( g \) is not a round metric. In this case, our proof is inspired by the work of Mantoulidis-Schoen [12]. Suppose \( (\Sigma_o, g) \) is not isometric to a round sphere, we want to construct a collar extension \( (N, \gamma) \) of \( \Omega \), where \( N = [0, 1] \times \Sigma_o \) and \( \gamma \) is a suitably chosen metric, such that

a) \( \gamma \) has nonnegative scalar curvature;

b) the induced metric from \( \gamma \) on \( \Sigma_o := \{0\} \times \Sigma_o \) agrees with \( g \), and the mean curvature of \( \Sigma_0 \) in \( (N, \gamma) \) equals the mean curvature \( H_o \) of \( \Sigma_o \) in \( \Omega \); and

c) the induced metric from \( \gamma \) on \( \Sigma_1 := \{1\} \times \Sigma_o \) is a round metric, and the Hawking mass of \( \Sigma_1 \) in \( (N, \gamma) \) is suitably controlled by the pair \( (g, H_o) \).

We then attach \( (N, \gamma) \) to \( \Omega \) (see figure 1) to obtain a manifold \( \hat{\Omega} \) whose (outer) boundary \( \Sigma_1 \) is a round sphere with constant mean curvature. Though \( \hat{\Omega} \) may not be smooth across \( \Sigma_o \), conditions a) and b) above ensure that the result in [15], which
itself was proved using the Riemannian Penrose inequality \cite{2, 9} can be applied to $\hat{\Omega}$ to obtain
\[
(1.5) \quad m_H(\Sigma_1) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}.
\]
(If $\Sigma_h = \emptyset$, we apply the positive mass theorem \cite{19, 21} instead to have $m_H(\Sigma_1) \geq 0$.) This, combined with c), then implies the inequalities in Theorems 1.1 – 1.3.

Remark 1.5. It is worth mentioning that the method described above indeed reveals information of the boundary component $\Sigma_0$ in the non-CMC case as well. Without assuming that $\Sigma_0$ is a CMC surface, Theorems 1.1 – 1.3 remain true if one let $H_0 = \min_{\Sigma_0} H$ in the expressions of $W$ and $m_H(\Sigma_0)$. With such a choice of $H_0$, the mean curvature of $\Sigma_0$ in $\Omega$, which is $H$, dominates the mean curvature of $\Sigma_0$ in $(N, \gamma)$ which is the constant $H_0$ (cf. figure 1 above). Therefore, by employing the techniques in \cite{14}, one knows \[(1.5) \quad (or \quad m_H(\Sigma_1) \geq 0)\] still holds on $\hat{\Omega}$.

This paper is organized as follows. In Section 2, we construct a suitable collar extension of $\Sigma_0$. In Section 3, we combine the collar extension and the Riemannian Penrose inequality (or the Riemannian positive mass theorem) to draw conclusions on $\partial \Omega$. In Section 4, we give the definition and estimate of $\eta(g)$ and prove Theorems 1.1 – 1.3. A comparison between inequalities (1.1) and (1.4) is included in an appendix.

2. Collar extensions

In this section, we let $\{g(t)\}_{t \in [0, 1]}$ be a fixed, smooth path of metrics on $\Sigma = S^2$, satisfying
\[
(2.1) \quad K(g(t)) > 0,
\]
where $K(\cdot)$ denotes the Gauss curvature of a metric, and
\begin{equation}
(2.2) \quad \text{tr}_{g(t)}g'(t) = 0
\end{equation}
for all $t \in [0, 1]$, where $\text{tr}_{g(t)}(\cdot)$ is taking trace on $(\Sigma, g(t))$. Let $|\Sigma|_{g(t)}$ be the area of $(\Sigma, g(t))$ which is a constant by (2.2). Let $r_o > 0$ be the corresponding constant given by
\begin{equation}
(2.3) \quad |\Sigma|_{g(t)} = 4\pi r_o^2.
\end{equation}
We will be interested in a metric $\gamma$ on $N = [0, 1] \times \Sigma$ of the form
\begin{equation*}
\gamma = A^2 dt^2 + E(t)g(t),
\end{equation*}
where $A > 0$ is a constant and $E(t) > 0$ is a function. To make a suitable choice of $E(t)$, we consider part of a spatial Schwarzschild metric
\begin{equation}
(2.4) \quad \gamma_m = \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_*
\end{equation}
of mass $m \leq \frac{1}{2} r_o$ defined on $[r_o, \infty) \times S^2$. Here $g_*$ denotes the standard metric on $S^2$ of area $4\pi$. We emphasize that we do allow $m$ to be negative in (2.4).

Making a change of variable
\begin{equation*}
s = \int_{r_o}^{r} \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} dr,
\end{equation*}
we rewrite $\gamma_m$ as
\begin{equation*}
\gamma_m = ds^2 + u_m^2(s) g_*,
\end{equation*}
where $s \in [0, \infty)$ and $u_m(s) = r(s)$ which satisfies
\begin{equation}
(2.5) \quad u_m(0) = r_o, \quad u'_m(s) = \left(1 - \frac{2m}{u_m(s)}\right)^{\frac{1}{2}}, \quad u''_m(s) = \frac{m}{u_m(s)^2}.
\end{equation}
Given any constants $A > 0$ and $k \geq 0$, we define
\begin{equation}
(2.6) \quad E(t) = r_o^{-2} u_m^2(Ak).
\end{equation}
With such a choice of $E(t)$, the mean curvature $H(t)$ of $\Sigma_t := \{t\} \times \Sigma$ with respect to $\gamma$ is
\begin{equation}
(2.7) \quad H(t) = A^{-1} E^{-1} E' = 2k u_m^{-1} \left(1 - \frac{2m}{u_m}\right)^{\frac{1}{2}}
\end{equation}
by (2.2) and (2.5). The Hawking mass, $m_H(\Sigma_t)$, of $\Sigma_t$ in $(N, \gamma)$ is
\begin{equation}
(2.8) \quad m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|h(t)}{16\pi}} \left[1 - \frac{1}{16\pi} \int_{\Sigma_t} H(t)^2 d\sigma_{h(t)}\right] = \frac{1}{2} u_m(Ak)(1 - k^2) + mk^2,
\end{equation}
where $h(t) := E(t)g(t)$ and $d\sigma_{h(t)}$ is the area element on $(\Sigma_t, h(t))$. 


Next we consider the scalar curvature of $\gamma$, denoted by $R(\gamma)$. Direct calculation gives

$$R(\gamma) = 2K(h) + A^{-2} \left[ -\text{tr}_h h'' - \frac{1}{4}(\text{tr}_h h')^2 + \frac{3}{4}|h'|^2_h \right],$$

where, by (2.2),

$$\text{tr}_h h' = 2E^{-1}E',$$

$$|h'|^2_h = E^{-2} \left[ 2(E')^2 + E^2|g'|_g^2 \right],$$

$$\text{tr}_h h'' = 2E^{-1}E'' + \text{tr}_g g'',$$

and

$$0 = \left[ (\text{tr}_g g')' \right] = \text{tr}_g g'' - |g'|_g^2.$$

Hence,

$$R(\gamma) = E^{-1}2K(g) + A^{-2} \left[ -\frac{1}{4}|g'|_g^2 - 2E^{-1}E'' + \frac{1}{2}E^{-2}(E')^2 \right].$$

Plugging in $E(t) = r_o^{-2}u_m^2(Akt)$ and using (2.5), we have

$$A^{-2} \left[ -2E^{-1}E'' + \frac{1}{2}E^{-2}(E')^2 \right]$$

$$= k^2 \left[ -2u_m^2(u_m')^2 - 4u_m^{-1}u_m'' \right]$$

$$= k^2 \left[ -2u_m^2 \left( 1 - \frac{2m}{u_m} \right) - 4u_m^{-3}m \right]$$

$$= -k^2 2u_m^{-2}.$$

Therefore, it follows from (2.9) and (2.10) that

$$R(\gamma) = r_o^2 u_m 2K(g) - k^2 2u_m^2 - \frac{1}{4}A^{-2}|g'|_g^2$$

$$= 2u_m^{-2} \left[ r_o^2 K(g) - k^2 - u_m A^{-2} \frac{1}{8}|g'|_g^2 \right].$$

Now we define two quantities associated to the path $\{g(t)\}_{t \in [0,1]}$:

$$\beta := \min_{t \in [0,1], x \in \Sigma} r_o^2 K(g(t))(x)$$

and

$$\alpha := \max_{t \in [0,1], x \in \Sigma} \frac{1}{4}|g'|_g^2(t, x).$$

Clearly, $\alpha = 0$ if and only if $\{g(t)\}_{t \in [0,1]}$ is a constant path. Moreover, by the Gauss-Bonnet theorem and (2.3),

$$\int_{\Sigma} r_o^2 K(g(t))d\sigma_{g(t)} = 4\pi r_o^2 = \int_{\Sigma} 1d\sigma_{g(t)}, \forall t.$$
Therefore,
\[(2.15) \quad \beta \leq 1, \text{ and } \beta = 1 \iff r_o^2 K(g(t))(x) = 1, \forall t, x.\]

In terms of $\beta$ and $\alpha$, it follows from (2.11) that
\[(2.16) \quad R(\gamma) \geq 2u_m^{-2} \left[ \beta - k^2 - \frac{1}{2} u_m^2 A^{-2} \alpha \right].\]

To further estimate $R(\gamma)$, we consider the cases of $m < 0$ and $m \geq 0$ separately.

**Case 1:** $m < 0$. In this case, (2.5) and the fact $u_m(s) \geq r_o$ imply
\[(2.17) \quad u_m'(s) \leq \left( 1 - \frac{2m}{r_o} \right)^\frac{1}{2},\]
and therefore
\[(2.18) \quad u_m(s) \leq r_o + \left( 1 - \frac{2m}{r_o} \right)^\frac{1}{2} s.\]

Hence, by (2.16) and (2.18),
\[(2.19) \quad R(\gamma) \geq 2u_m^{-2} \left\{ \beta - k^2 - \frac{1}{2} \left[ 1 - \frac{2m}{r_o} \right]^\frac{1}{2} kt + r_o A^{-1} \right\}^2 \alpha \]
\[\geq 2u_m^{-2} \left\{ \beta - k^2 - \left[ 1 - \frac{2m}{r_o} \right]^2 k^2 + (r_o A^{-1})^2 \alpha \right\} .\]

**Case 2:** $m \geq 0$. In this case, (2.5) implies $u_m'(s) \leq 1$ and
\[(2.20) \quad u_m(s) \leq r_o + s.\]

Therefore, by (2.16) and (2.20),
\[(2.21) \quad R(\gamma) \geq 2u_m^{-2} \left[ \beta - k^2 - \frac{1}{2} (kt + r_o A^{-1})^2 \alpha \right] \]
\[\geq 2u_m^{-2} \left[ \beta - k^2 - (k^2 + r_o^2 A^{-2}) \alpha \right] .\]

We are led to the following proposition.

**Proposition 2.1.** Given a smooth path of metrics $\{g(t)\}_{t \in [0,1]}$ on $\Sigma$ satisfying (2.1) and (2.2), let $r_o$, $\beta$, and $\alpha$ be the constants defined by (2.3), (2.12), and (2.13), respectively. Suppose $\alpha > 0$, i.e. $\{g(t)\}_{t \in [0,1]}$ is not a constant path. Let $m \leq \frac{1}{2} r_o$ and $k \geq 0$ be two constants satisfying
\[(2.22) \quad \beta - \left[ 1 + \left( 1 - \frac{2m}{r_o} \right) \alpha \right] k^2 > 0, \text{ if } m < 0\]
or
\[(2.23) \quad \beta - (1 + \alpha) k^2 > 0, \text{ if } m \geq 0.\]
Let \( A_o > 0 \) be the constant given by

\[
A_o = r_o \left[ \frac{\alpha}{\beta - \left[ 1 + \left( 1 - \frac{2m}{r_o} \right) \alpha \right] k^2} \right]^{\frac{1}{2}} \quad \text{if } m < 0
\]

or

\[
A_o = r_o \left[ \frac{\alpha}{\beta - (1 + \alpha)k^2} \right]^{\frac{1}{2}}, \quad \text{if } m \geq 0.
\]

Let \( u_m(s) \) be the function defined by \((2.5)\). Then, for any constant \( A \geq A_o \), the metric

\[
\gamma = A^2 dt^2 + r_o^{-2} u_m^2(Akt) g(t)
\]
on \( N = [0, 1] \times \Sigma \) satisfies

(i) \( R(\gamma) \geq 0 \), where \( R(\gamma) \) is the scalar curvature of \( \gamma \);

(ii) the induced metric on \( \Sigma_0 := \{0\} \times \Sigma \) is \( g(0) \), and the mean curvature of \( \Sigma_0 \) is \( H(0) = 2kr_o^{-1}(1 - \frac{2m}{r_o})^{\frac{1}{2}} \); and

(iii) \( \Sigma_t := \{t\} \times \Sigma \) has positive constant mean curvature for each \( t \) and its Hawking mass is

\[
m_H(\Sigma_t) = \frac{1}{2} \left[ u_m(Akt) - r_o \right] (1 - k^2) + m_H(\Sigma_0).
\]

Proof. (i) is a direct corollary of \((2.19)\) and \((2.21)\). (ii) follows from \((2.7)\) and the fact \( u_m(0) = r_o \). (iii) is implied by \((2.7)\) and \((2.8)\). \(\square\)

Remark 2.1. In Proposition \(2.1\) one indeed has \( R(\gamma) > 0 \) on \([0, 1] \times \Sigma \). This is because in both \((2.19)\) and \((2.21)\), the second inequality is a strict inequality unless \( t = 1 \).

Now suppose \( g(1) \) is a round metric and \( g(0) \) is not round, then \( r_o^2 K(g(1)) = 1 \) and \( \beta < 1 \) by \((2.15)\). Thus, by \((2.11)\), the inequality in \((2.16)\) is strict at \( t = 1 \). Therefore, in this case, \( R(\gamma) > 0 \) everywhere on \( N \).

Remark 2.2. When \( \alpha = 0 \), by \((2.16)\), it suffices to require \( \beta \geq k^2 \) for \( \gamma \) to have \( R(\gamma) \geq 0 \). In particular, if \( \{g(t)\}_{t \in [0, 1]} \) consists of a fixed round metric and \( k^2 = \beta = 1 \), then \( \gamma \) reduces to the Schwarzschild metric \( \gamma_m \).

3. Application

In this section, we let \( \Omega \) be a compact Riemannian 3-manifold with the following properties:

- \( \Omega \) has nonnegative scalar curvature;
- \( \partial \Omega \) is the disjoint union of \( \Sigma_o \) and \( \Sigma_h \), where \( \Sigma_o \) is a topological 2-sphere and \( \Sigma_h \), if nonempty, is the unique, closed minimal surface (possibly disconnected) in \( \Omega \);
- the mean curvature of \( \Sigma_o \) in \( \Omega \) is a positive constant \( H_o \); and
- there exists a smooth path of metrics \( \{g(t)\}_{t \in [0, 1]} \) on \( \Sigma := \Sigma_o \) satisfying \((2.1)\) and \((2.2)\) such that \( g(0) = g \), which is the induced metric on \( \Sigma \) from \( \Omega \), and \( g(1) \) is a round metric.
We will apply a suitable collar extension constructed in Proposition 2.1 and the Riemannian Penrose inequality (or the positive mass theorem) to draw information on the geometry of $\Sigma_o$.

First, we consider a result obtained by applying Proposition 2.1 with parameters $m < 0$. In this case, we impose a condition

$$\left( \frac{1}{4} H_o^2 r_o^2 \right) \alpha < \beta$$

on $\Sigma_o$, where $r_o$ is the area radius of $(\Sigma_o, g)$ and $\beta, \alpha$ are the constants, associated to the path $\{g(t)\}_{t \in [0,1]}$, defined in (2.12), (2.13), respectively.

**Theorem 3.1.** If (3.1) holds, then

$$\beta - \left[ 1 + \left( 1 - \frac{2m}{r_o} \right) \alpha \right] k^2 > 0.$$ 

**Proof.** If $\alpha = 0$, then $g$ is a round metric. In this case, the claim reduces to $m \mu (\Sigma_o) \geq \sqrt{\frac{\left| \Sigma_h \right|}{16\pi}}$, which follows from [15, Theorem 1]. Therefore, it suffices to consider the case $g$ is not round, i.e. $\alpha > 0$.

We will construct a suitable metric $\gamma$ on $N = \Sigma \times [0,1]$ and attach $(N, \gamma)$ to $\Omega$ along $\Sigma_o$. To do so, note that (3.1) implies there are constants $m < 0$ satisfying

$$\beta - \frac{1}{4} H_o^2 r_o^2 \alpha - \frac{1}{4} H_o^2 r_o^2 \left( 1 - \frac{2m}{r_o} \right)^{-1} > 0.$$ 

For any such an $m$, define

$$k = \frac{1}{2} H_o r_o \left( 1 - \frac{2m}{r_o} \right)^{-\frac{1}{2}}.$$ 

Then (3.3) gives

$$\beta - \left[ 1 + \left( 1 - \frac{2m}{r_o} \right) \alpha \right] k^2 > 0.$$ 

Now let

$$A_o = r_o \left[ \frac{\alpha}{\beta - \left[ 1 + \left( 1 - \frac{2m}{r_o} \right) \alpha \right] k^2} \right]^{\frac{1}{2}}$$ 

and consider the metric

$$\gamma = A_o^2 dt^2 + r_o^{-2} u_m^2 (A_o kt) g(t)$$

on $N$. Let $\Sigma_t := \{t\} \times \Sigma$. It follows from (3.5), (3.6) and Proposition 2.1 that $(N, \gamma)$ has nonnegative scalar curvature, each $\Sigma_t$ has positive constant mean curvature, the
induced metric from \( \gamma \) on \( \Sigma_0 \) agrees with \( g \), the mean curvature \( H(0) \) of \( \Sigma_0 \) equals \( H_o \), and the Hawking mass of \( \Sigma_1 \) in \( (N, \gamma) \) and the Hawking mass of \( \Sigma_o \) in \( \Omega \) are related by

\[
(3.8) \quad m_H(\Sigma_1) = \frac{1}{2} [u_m(A_o k) - r_o] (1 - k^2) + m_H(\Sigma_o).
\]

Now we glue \( (N, \gamma) \) and \( \Omega \) along their common boundary component \( \Sigma_0 = \Sigma_o \) to obtain a Riemannian manifold \( \hat{\Omega} \). The metric \( \hat{g} \) on \( \hat{\Omega} \) is Lipschitz across \( \Sigma_o \) and smooth everywhere else; it has nonnegative scalar curvature away from \( \Sigma_o \); and the mean curvature of \( \Sigma_o \) from both sides in \( \hat{\Omega} \) agree. Moreover, \( \partial \hat{\Omega} = \Sigma_h \cup \Sigma_1 \) where \( \Sigma_1 \) is isometric to a round sphere and has constant mean curvature. Therefore, applying the mollification method used in [14, 15] which smooths out the corner of \( \hat{g} \) at \( \Sigma_o \), we know that [15, Theorem 1] applies to \( \hat{\Omega} \) to give

\[
(3.9) \quad m_H(\Sigma_1) \geq \sqrt{\frac{\lvert \Sigma_h \rvert}{16\pi}}.
\]

(A more precise and direct way to derive (3.9) is as follows. Since \( \Sigma_1 \) is both round and having constant mean curvature, we can again attach to \( \hat{\Omega} \), along \( \Sigma_1 \), a manifold \( N_{\infty} = (\{r_1, \infty\} \times S^2, \gamma_m) \) with \( 4\pi r_1^2 = |\Sigma_1|, \gamma_m \) given by (2.4) and \( m = m_H(\Sigma_1) \). Indeed, \( N_{\infty} \) is the region that is exterior to a rotationally symmetric sphere with area \( |\Sigma_1| \) in the spatial Schwarzschild manifold whose mass is \( m_H(\Sigma_1) \). We denote the resulting manifold by \( \hat{M} \), which consists of three pieces \( \Omega, N \) and \( N_{\infty} \). The metric on \( \hat{M} \) satisfies the mean curvature matching condition across both \( \Sigma_o \) and \( \Sigma_1 \). Therefore, one can repeat the same proof in [15], starting from Lemma 3 on page 278 and ending at equation (47) on page 280, to conclude that the Riemannian Penrose inequality still holds on such an \( \hat{M} \), which proves (3.9).

To proceed, we note that (3.8) and (3.9) imply

\[
(3.10) \quad \frac{1}{2} [u_m(A_o k) - r_o] (1 - k^2) + m_H(\Sigma_o) \geq \sqrt{\frac{\lvert \Sigma_h \rvert}{16\pi}}.
\]

By (3.5) and (2.15),

\[
(3.11) \quad k^2 < \beta \leq 1,
\]

and, by (2.18),

\[
(3.12) \quad u_m(A_o k) - r_o \leq \left(1 - \frac{2m}{r_o}\right)^{\frac{1}{2}} A_o k = \frac{1}{2} H_o r_o A_o.
\]

Therefore, (3.10) – (3.12) imply

\[
(3.13) \quad \frac{1}{4} H_o r_o A_o (1 - k^2) + m_H(\Sigma_o) \geq \sqrt{\frac{\lvert \Sigma_h \rvert}{16\pi}}.
\]
where
\[
\frac{1}{4} H_o r_o A_o = \frac{1}{2} r_o \left[ \frac{1}{2} r_o^2 \frac{H_o^2 r_o^2 \alpha}{\left( \beta - \frac{1}{4} H_o^2 r_o^2 \alpha \right) - k^2} \right]^{\frac{1}{2}}.
\]

In summary, we have proved
\[
\frac{1}{2} r_o \left[ \frac{1}{2} H_o^2 r_o^2 \frac{\alpha}{\left( \beta - \frac{1}{4} H_o^2 r_o^2 \alpha \right) - k^2} \right]^{\frac{1}{2}} (1 - k^2) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}
\]
for any \( m < 0 \) satisfying (3.3).

To obtain a result that does not involve \( m \) or \( k \), we can let \( m \to -\infty \) and (3.4) shows
\[
\lim_{m \to -\infty} k = 0.
\]

It follows from (3.15) and (3.16) that
\[
\frac{1}{2} r_o \left[ \frac{1}{2} H_o^2 r_o^2 \frac{\alpha}{\left( \beta - \frac{1}{4} H_o^2 r_o^2 \alpha \right) - k^2} \right]^{\frac{1}{2}} + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}},
\]
which proves the theorem. \( \square \)

**Remark 3.1.** If \( \Sigma_h = \emptyset \), i.e., if \( \Omega \) is merely a compact 3-manifold with nonnegative scalar curvature, with boundary \( \partial \Omega = \Sigma_o \), then, replacing the Riemannian Penrose inequality by the Riemannian positive mass theorem in the proof, one has \( m_H(\Sigma_1) \geq 0 \) (cf. [14, 20]). In this case, the result becomes
\[
\frac{1}{2} r_o \left[ \frac{1}{2} H_o^2 r_o^2 \frac{\alpha}{\left( \beta - \frac{1}{4} H_o^2 r_o^2 \alpha \right) - k^2} \right]^{\frac{1}{2}} + m_H(\Sigma_o) \geq 0.
\]

Next, we consider a corresponding result obtained by applying Proposition 2.1 with parameters \( m \geq 0 \). In this case, we assume a condition
\[
\frac{1}{4} H_o^2 r_o^2 < \frac{\beta}{1 + \alpha}.
\]

**Theorem 3.2.** Suppose (3.19) holds. Given any constant \( m \in [0, \frac{1}{2} r_o) \) satisfying
\[
\frac{1}{4} H_o^2 r_o^2 < \frac{\beta}{1 + \alpha} \left( 1 - \frac{2m}{r_o} \right),
\]
define
\[
k = \frac{1}{2} H_o r_o \left( 1 - \frac{2m}{r_o} \right)^{-\frac{1}{2}}, \quad A_o = r_o \left[ \frac{\alpha}{\beta - (1 + \alpha) k^2} \right]^{\frac{1}{2}}.
\]

Then
\[
\frac{1}{2} A_o k (1 - k^2) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}.
\]
In particular, if one chooses \( m = 0 \), then

\[
\left( \frac{\alpha \left( \frac{1}{4} H_o^2 r_o^2 \right)}{\beta - (1 + \alpha) \left( \frac{1}{4} H_o^2 r_o^2 \right)} \right)^{\frac{1}{2}} m_H(\Sigma_o) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}},
\]

and consequently

\[
\left( \frac{\frac{1}{4} H_o^2 r_o^2}{\beta - (1 + \alpha) \left( \frac{1}{4} H_o^2 r_o^2 \right)} \right)^{\frac{1}{2}} m_H(\Sigma_o) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}.
\]

**Proof.** Again, it suffices to assume \( \alpha > 0 \). By (3.20) and (3.21),

\[
\beta - (1 + \alpha) k^2 = \beta - (1 + \alpha) \frac{1}{4} H_o^2 r_o^2 \left( 1 - \frac{2m}{r_o} \right)^{-1} > 0.
\]

Consider the metric

\[
\gamma = A_o^2 dt^2 + r_o^{-2} u_m(A_o k t) g(t)
\]

on \( N = [0, 1] \times \Sigma \). Let \( \Sigma_t := \{ t \} \times \Sigma \). It follows from (3.21), (3.24) and Proposition 2.1 that \( (N, \gamma) \) has nonnegative scalar curvature, the induced metric from \( \gamma \) on \( \Sigma_0 \) agrees with \( g \), the mean curvature \( H(0) \) of \( \Sigma_0 \) equals \( H_o \), and the Hawking mass of \( \Sigma_1 \) in \( (N, \gamma) \) and the Hawking mass of \( \Sigma_0 \) in \( \Omega \) are related by

\[
m_H(\Sigma_1) = \frac{1}{2} \left[ u_m(A_o k) - r_o \right] \left( 1 - k^2 \right) + m_H(\Sigma_o).
\]

Attaching \( (N, \gamma) \) to \( \Omega \), we have

\[
m_H(\Sigma_1) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}
\]

by the reason explained in the proof of Theorem 3.1. It follows from (3.25) and (3.26) that

\[
\frac{1}{2} \left[ u_m(A_o k) - r_o \right] \left( 1 - k^2 \right) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}.
\]

Again, since \( \beta \leq 1 \), (3.24) implies \( k^2 < 1 \). Also, (2.20) shows

\[
u_m(A_o k) - r_o \leq A_o k.
\]

Therefore, (3.27) implies

\[
\frac{1}{2} A_o k \left( 1 - k^2 \right) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}},
\]

where

\[
A_o k = r_o \left[ \frac{\alpha k^2}{\beta - (1 + \alpha) k^2} \right]^\frac{1}{2}.
\]
Thus, we have proved

\[
\frac{1}{2} r_o \left[ \frac{\alpha k^2}{\beta - (1 + \alpha) k^2} \right]^{\frac{1}{2}} \left( 1 - k^2 \right) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_H|}{16\pi}}
\]

for any \( m \in [0, \frac{1}{2} r_o) \) satisfying (3.20).

To obtain a result that does not involve \( m \) or \( k \), we can take \( m = 0 \). In this case, \( k = \frac{1}{2} H_o r_o \), and (3.30) becomes

\[
\left[ \frac{\alpha \frac{1}{4} H_o^2 r_o^2}{\beta - (1 + \alpha) \frac{1}{4} H_o^2 r_o^2} \right]^{\frac{1}{2}} m_H(\Sigma_o) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_H|}{16\pi}},
\]

which proves (3.22). Inequality (3.23) follows from (3.22) simply by the fact \( \frac{\alpha}{1 + \alpha} \leq 1 \). This completes the proof. \( \square \)

Remark 3.2. In the derivation of Theorems 3.1 and 3.2, besides taking \( m = -\infty \) and \( m = 0 \), one can minimize the first term in (3.15) and (3.30), subject to the constraint \( m \) satisfies (3.25) and (3.20), respectively. We leave this calculation in Appendix A.

Remark 3.3. If \( g \) is not a round metric, i.e. \( \alpha > 0 \), the collar \((N, \gamma)\) that we attached to \( \Omega \) indeed has strictly positive scalar curvature by Remark 2.1. Therefore, by the rigidity statement of the Riemannian Penrose inequality, one naturally would expect that inequalities in (3.10) and (3.27) are indeed strict. Therefore, equalities in Theorems 3.1 and 3.2 should hold only if \( \alpha = 0 \), i.e. when \( g \) is a round metric on \( \Sigma_o \). However, we do not have a rigorous proof of this claim.

4. Definition of \( \eta(g) \)

In this section, we define the quantity \( \eta(g) \) and prove Theorems 1.1 – 1.3. Given a metric \( g \) with positive Gauss curvature on \( \Sigma = S^2 \), let \( \{h(t)\}_{t \in [0,1]} \) denote a smooth path of metrics on \( \Sigma \) such that

(i) \( h(0) \) is isometric to \( g \) and \( h(1) \) is a round metric;
(ii) \( h(t) \) has positive Gauss curvature, i.e. \( K(h(t)) > 0 \), \( \forall \ t \); and
(iii') \( |\Sigma|_{h(t)} = |\Sigma|_g \), i.e. the area of \( \Sigma, h(t) \) is a constant, \( \forall \ t \).

There are various ways to construct such a path. For instance, one may apply the uniformization theorem to write \( g = e^{2w} g_o \) for some function \( w \) and a round metric \( g_o \), and to define \( h(t) = e^{2(1-t)w} g_o \) (cf. [17]), followed by an area normalization.

Given such a path \( \{h(t)\}_{t \in [0,1]} \), applying the proof of Lemma 1.2 in [12] to \( \{h(t)\}_{t \in [0,1]} \), one can construct a new path of metrics \( \{g(t)\}_{t \in [0,1]} \), satisfying (i) and (ii), with \( h(t) \) replaced by \( g(t) \), together with the following property that is stronger than (iii'):

(iii) \( \frac{d}{dt} |d\sigma_{g(t)}| = 0 \), or equivalently \( \text{tr}_{g(t)} g'(t) = 0 \), \( \forall \ t \). Here \( d\sigma_{g(t)} \) is the area form of \( g(t) \).

We include this construction of \( \{g(t)\}_{t \in [0,1]} \) by Mantoulidis and Schoen in the lemma below for the purpose of later obtaining estimates on \( \eta(g) \).
Lemma 4.1 ([12]). Given \( \{h(t)\}_{t \in [0,1]} \) satisfying (i), (ii) and (iii') above, there exists \( \{g(t)\}_{t \in [0,1]} \) satisfying (i), (ii) and (iii).

**Proof.** Let \( \nabla_{h(t)}, \Delta_{h(t)} \) denote the gradient, the Laplacian on \((\Sigma, h(t))\), respectively. Given a 1-parameter family of diffeomorphisms \( \{\phi_t\} \) on \( \Sigma \), define \( g(t) := \phi_t^*(h(t)) \). Then

\[
\begin{align*}
g'(t) &= \phi_t^*(h'(t)) + \phi_t^*(L_X h(t)), \\
\text{tr}_{g(t)} g'(t) &= \phi_t^* \left( \text{tr}_{h(t)} (h'(t) + L_X h(t)) \right),
\end{align*}
\]

where \( X = X(x,t) \) is the vector field satisfying \( \frac{d}{dt} \phi_t = X(\phi_t, t) \). Thus, to satisfy (iii), it suffices to demand \( \text{tr}_{h(t)} L_X h(t) = -\text{tr}_{h(t)} h'(t) \), i.e.

\[
\text{div}_{h(t)} X = -\frac{1}{2} \text{tr}_{h(t)} h'(t).
\]

A way to pick such an \( X \) is to assume \( X = \nabla_{h(t)} u \) for some function \( u = u(x,t) \) satisfying

\[
\Delta_{h(t)} u = -\frac{1}{2} \text{tr}_{h(t)} h'(t) \text{ and } \int_{\Sigma} u \, d\sigma_{h(t)} = 0.
\]

Since

\[
\int_{\Sigma} \text{tr}_{h(t)} h'(t) d\sigma_{h(t)} = 0
\]

by (iii'), \((4.4)\) has a unique solution \( u \) that depends smoothly on \( t \) whenever \( h(t) \) is smooth on \( t \). This finishes the proof. \( \Box \)

Given any smooth path \( \{g(t)\}_{t \in [0,1]} \) with properties (i), (ii) and (iii), let

\[
\beta_{\{g(t)\}} := \min_{t \in [0,1], x \in \Sigma} \frac{1}{4\pi} |\Sigma| g(t)(x)
\]

and

\[
\alpha_{\{g(t)\}} := \max_{t \in [0,1], x \in \Sigma} \frac{1}{4} |g'(t,x)|^2,
\]

where \( |g'(t,x)|^2 \) denotes the square norm of \( g'(t) \) with respect to \( g(t) \).

**Definition 4.1.** Given a metric \( g \) with positive Gauss curvature on \( \Sigma = S^2 \), define

\[
\eta(g) := \sup_{\{g(t)\}} \frac{\beta_{\{g(t)\}}}{\alpha_{\{g(t)\}}},
\]

where the supremum is taken over all paths \( \{g(t)\}_{t \in [0,1]} \) satisfying (i), (ii) and (iii). Similarly, one may also define

\[
\kappa(g) := \sup_{\{g(t)\}} \frac{\beta_{\{g(t)\}}}{1 + \alpha_{\{g(t)\}}},
\]
We next estimate \( \eta(g) \) and \( \kappa(g) \) satisfy
\[
0 < \eta(g) \leq \infty \quad \text{and} \quad 0 < \kappa(g) \leq 1,
\]
where the second inequality follows from (2.15). Moreover, for constant \( c > 0 \), it is straightforward to check that
\[
\eta(c^2 g) = \eta(g) \quad \text{and} \quad \kappa(c^2 g) = \kappa(g).
\]
If \( g = g_o \) is a round metric, by taking \( \{ g(t) \} \) to be a constant path, one has \( \alpha_{(g(t))} = 0 \) and \( \beta_{(g(t))} = 1 \), hence
\[
\eta(g_o) = \infty \quad \text{and} \quad \kappa(g_o) = 1.
\]
Below, we give a lower bound of \( \eta(g) \) and \( \kappa(g) \) for \( g \) that is close to a round metric.

**Proposition 4.1.** Let \( g_o \) be the standard metric of area \( 4\pi \) on \( \Sigma = S^2 \). There exists a constant \( \epsilon_0 > 0 \) such that if \( ||g - g_o||_{C^{2,\delta}(\Sigma)} < \epsilon_0 \), then
\[
\eta(g) \geq \frac{C}{||g - g_o||_{C^{0,\delta}(\Sigma)}} \quad \text{and} \quad \kappa(g) \geq 1 - C||g - g_o||_{C^{2,\delta}(\Sigma)}.
\]
Here \( C \) is some positive constant that is independent on \( g \) and \( ||\cdot||_{C^{k,\delta}(\Sigma)} \) is the \( C^{k,\delta} \) norm on \( (\Sigma, g_o) \) for an integer \( k \geq 0 \) and a constant \( \delta \in (0, 1) \).

**Proof.** Given any \( \epsilon > 0 \), let \( U_\epsilon \) be the set of metrics \( g \) satisfying \( ||g - g_o||_{C^{2,\delta}(\Sigma)} < \epsilon \). First, choose a small \( \epsilon_0 \) so that elements in \( U_{\epsilon_0} \) all have positive Gauss curvature.

Given any \( g \in U_{\epsilon_0} \), let \( \tau = g - g_o \). Then \( ||\tau||_{C^{2,\delta}(\Sigma)} < \epsilon_0 \). For each \( t \in [0, 1] \), define \( \tilde{h}(t), a(t) \) and \( h(t) \), respectively by
\[
\tilde{h}(t) = g_o + (1 - t)\tau, \quad |\Sigma|_{\tilde{h}(t)} = a(t)|\Sigma|_g, \quad h(t) = a^{-1}(t)\tilde{h}(t).
\]
Then \( |\Sigma|_{\tilde{h}(t)} = a^{-1}(t)|\Sigma|_g = |\Sigma|_g \). Hence, \( \{ h(t) \}_{t \in [0, 1]} \) is a path satisfying properties (i), (ii) and (iii'). Moreover,
\[
||\tilde{h}(t) - g_o||_{C^{2,\delta}(\Sigma)} \leq ||\tau||_{C^{2,\delta}(\Sigma)}, \quad |a(t) - 1| \leq C_1||\tau||_{C^{2,\delta}(\Sigma)},
\]
and
\[
||h(t) - g_o||_{C^{2,\delta}(\Sigma)}
\]
\[
= ||a^{-1}(t)(1 - t)\tau + (a^{-1}(t) - 1)g_o||_{C^{2,\delta}(\Sigma)}
\]
\[
\leq C_2||\tau||_{C^{2,\delta}(\Sigma)}.
\]
Here and below, \( C_1, C_2, \ldots \) always denote constants that do not depend on \( \tau \) and \( t \).

Now let \( \{ g(t) \}_{t \in [0,1]} \) be the path of metrics constructed from \( \{ h(t) \}_{t \in [0,1]} \) in the proof of Lemma 4.1. It follows from (4.10) and the fact \( g(t) = \phi_t^*(h(t)) \) that
\[
\beta_{(g(t))} = \frac{|\Sigma|_g}{4\pi} \min_{t \in [0,1], x \in \Sigma} K(h(t))(x) \geq 1 - C_3||\tau||_{C^{2,\delta}(\Sigma)}.
\]
We next estimate \( \alpha_{(g(t))} \). By (4.1), \( g'(t) = \phi_t^*(H(t)) \), where
\[ H(t) = h'(t) + L_X h(t). \]
Hence, \( |g'|^2_g = \phi^*_t(|H|^2_h) \). Therefore,

\[
\alpha_{\{g(t)\}} = \max_{t \in [0,1], x \in \Sigma} \frac{1}{4} |H|^2_h(t, x) \\
(4.12) \\
\leq \max_{t \in [0,1], x \in \Sigma} \frac{1}{2} \left[ |h'|^2_h + |L_X h(t)|^2_h \right] (t, x).
\]

Plugging in \( X = \nabla_{h(t)} u \), we have

\[
L_X h(t) = 2\nabla^2_{h(t)} u,
(4.13)
\]

where \( \nabla^2_{h(t)} \) denotes the Hessian on \((\Sigma, h(t))\). By (4.14) and (4.10) and the standard linear elliptic estimates, we have

\[
||u||_{C^{2,\delta}(\Sigma)} \leq C_4 ||\text{tr}_{h(t)} h'(t)||_{C^{0,\delta}(\Sigma)}.
(4.14)
\]

Therefore, by (4.13) and (4.14),

\[
|L_X h(t)|_h \leq C_5 ||\text{tr}_{h(t)} h'(t)||_{C^{0,\delta}(\Sigma)}.
(4.15)
\]

It follows from (4.12) and (4.15) that

\[
\alpha_{\{g(t)\}} \leq \max_{t \in [0,1], x \in \Sigma} \frac{1}{2} |h'|^2_h(t, x) + \max_{t \in [0,1]} ||\text{tr}_{h(t)} h'(t)||^2_{C^{0,\delta}(\Sigma)}.
(4.16)
\]

By (4.8), we have

\[
\text{tr}_{h(t)} h'(t) = -2a^{-1} a' - \text{tr}_{h(t)} \tau,
(4.17)
\]

\[
|h'|^2_h = 2a^{-2}(a')^2 + |\tau|^2_h + 2a^{-1} a' \text{tr}_{h(t)} \tau,
(4.18)
\]

\[
a'(t) = -\frac{1}{2|\Sigma|_g} \int_{\Sigma} \text{tr}_{h(t)} \tau d\sigma_{h(t)}.
(4.19)
\]

Thus, by (4.9) and (4.17) - (4.19), we have

\[
|h'|^2_h \leq C_7 ||\tau||^2_{C^{0,\delta}(\Sigma)} \quad \text{and} \quad ||\text{tr}_{h(t)} h'(t)||^2_{C^{0,\delta}(\Sigma)} \leq C_8 ||\tau||^2_{C^{0,\delta}(\Sigma)}.
(4.20)
\]

Finally, by (4.16) and (4.20), we conclude

\[
\alpha_{\{g(t)\}} \leq C_9 ||\tau||^2_{C^{0,\delta}(\Sigma)}.
(4.21)
\]

Estimate (4.17) then follow readily from (4.11) and (4.21). \( \Box \)

We now give the proof of Theorems 1.1 - 1.3.

**Proof of Theorems 1.1 and 1.2** It suffices to assume that \( g \) is not a round metric. Let \( \{g^{(j)}(t)\}_{t \in [0,1], j = 1, 2, \ldots} \) be a sequence of path of metrics, satisfying (i), (ii) and (iii), such that

\[
\frac{\beta_{\{g^{(j)}(t)\}}}{\alpha_{\{g^{(j)}(t)\}}} \to \eta(g), \quad \text{as} \quad j \to \infty.
\]

Suppose \( \mathcal{W} < \eta(g) \), then

\[
\mathcal{W} < \frac{\beta_{\{g^{(j)}(t)\}}}{\alpha_{\{g^{(j)}(t)\}}} \quad \text{for large} \quad j.
\]
For these $j$, by Theorem 3.1 and Remark 3.1,
\[
\frac{1}{2} r_o \left[ \frac{W}{\alpha_{\{g^{(j)}(t)\}}^{-1} \beta_{\{g^{(j)}(t)\}} - W} \right]^{\frac{1}{2}} + m_h(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}, \quad \text{when } \Sigma_h \neq \emptyset
\]
and
\[
\frac{1}{2} r_o \left[ \frac{W}{\alpha_{\{g^{(j)}(t)\}}^{-1} \beta_{\{g^{(j)}(t)\}} - W} \right]^{\frac{1}{2}} + m_h(\Sigma_o) \geq 0, \quad \text{when } \partial \Omega = \Sigma_o.
\]
Taking $j \to \infty$, Theorems 1.1 and 1.2 follow. \hfill \square

**Proof of Theorem 1.3.** Assume that $g$ is not a round metric. Pick any path $\{g(t)\}_{t \in [0,1]}$ used in Section 3 and choose $\alpha_g, \beta_g$ to be $\alpha, \beta$ associated to that path, respectively. Theorem 1.3 then follows directly from (3.22) in Theorem 3.2. \hfill \square

It would be desirable to improve Theorem 1.3 in a way that Theorem 1.1 is proved from Theorem 3.1. However, due to the fact that (3.22) involves both $\frac{\beta}{1+\alpha}$ and $\frac{\alpha}{1+\alpha}$, we can only replace $\frac{\beta}{1+\alpha}$ by $\kappa(g)$ at the expense of giving up $\frac{\alpha}{1+\alpha}$. We record the following theorem.

**Theorem 4.1.** Let $\Omega$ be a compact, orientable, Riemannian 3-manifold with boundary $\partial \Omega$. Suppose $\partial \Omega$ is the disjoint union of $\Sigma_o$ and $\Sigma_h$ such that

(a) $\Sigma_o$ is a topological 2-sphere with constant mean curvature $H_o > 0$;
(b) $\Sigma_h$, which may have multiple components, is a minimal surface; and
(c) there are no other closed minimal surfaces in $\Omega$.

Suppose $\Omega$ has nonnegative scalar curvature and the induced metric $g$ on $\Sigma_o$ has positive Gauss curvature. Let $0 < \kappa(g) \leq 1$ be the scaling invariant of $(\Sigma_o, g)$ defined in Definition 4.1. If
\[
W := \frac{1}{16\pi} \int_{\Sigma_o} H_o^2 d\sigma < \kappa(g),
\]
then
\[
\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \left[ \left( \frac{W}{\kappa(g) - W} \right)^{\frac{1}{2}} + 1 \right] m_h(\Sigma_o).
\]

**Proof.** If $g$ is round, we have $\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq m_h(\Sigma_o)$, in particular (4.22) holds. So we assume that $g$ is not a round metric. Similar to the proof of Theorem 1.1 above, let $\{g^{(j)}(t)\}_{t \in [0,1]}$, $j = 1, 2, \ldots$, be a sequence of path of metrics, satisfying (i), (ii) and (iii), with
\[
\frac{\beta_{\{g^{(j)}(t)\}}}{1 + \alpha_{\{g^{(j)}(t)\}}} \to \kappa(g), \quad \text{as } j \to \infty.
\]
Suppose $W < \kappa(g)$, then
\[
W < \frac{\beta_{\{g^{(j)}(t)\}}}{1 + \alpha_{\{g^{(j)}(t)\}}}, \quad \text{for large } j.
\]
For these \(j\), by (3.23) in Theorem 3.2,

\[
\left(\frac{\mathcal{W}}{\frac{\beta(g(t))}{1 + \alpha(g(t))}} - \mathcal{W}\right)^{\frac{1}{2}} m_H(\Sigma_o) + m_H(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}.
\]

Taking \(j \to \infty\), Theorem 4.1 follows. \(\Box\)

To end this paper, we remark that, besides employing the construction of Mantoulidis and Schoen in Lemma 4.1, there are other methods to obtain \(\{g(t)\}_{t \in [0,1]}\) satisfying (i), (ii) and (iii) used in Definition 4.1. For instance, one may apply Hamilton’s modified Ricci flow \([6]\) on closed surfaces. Using results from \([6, 3]\), Lin and Sormani \([11]\) introduced a concept of asphericity mass for a CMC surface normalized to have area \(4\pi\) and used it to obtain upper bounds of the surface’s Bartnik mass. It would be interesting to understand the relation between \(\eta(g)\) or \(\kappa(g)\) and the asphericity mass since they are all determined solely by the intrinsic metric on the surface. It is also conceivably possible that the modified Ricci flow \([6]\) may be used to obtain refined estimates of \(\eta(g)\) and \(\kappa(g)\). We leave these for interested readers.

**APPENDIX A**

In this appendix, we give the calculation, stated in Remark 3.2, which minimizes the left side of (3.15) and (3.30), subject to the condition \(m\) satisfies (3.3) and (3.20), respectively.

We first consider the context of Theorem 3.2. Suppose \(\alpha > 0\). Let \(\mathcal{W} = \frac{1}{4}H_o^2 r_o^2\) and define

\[(A.1) \quad \kappa := \frac{\beta}{1 + \alpha} \in (0, 1).\]

Condition (3.19) becomes \(\mathcal{W} < \kappa\) and the constraint (3.20) is

\[(A.2) \quad \mathcal{W} < \kappa \left(1 - \frac{2m}{r_o}\right), \quad m \in \left[0, \frac{1}{2} r_o\right).\]

The quantity that we want to minimize is

\[(A.3) \quad \Phi := \frac{1}{2} r_o \left[\frac{\alpha k^2}{\beta - (1 + \alpha) k^2}\right]^{\frac{1}{2}} (1 - k^2) = \frac{1}{2} r_o \left(\frac{\alpha}{1 + \alpha}\right)^{\frac{1}{2}} \left(\frac{x}{k - x}\right)^{\frac{1}{2}} (1 - x)\]

where \(x := k^2 = \mathcal{W} \left(1 - \frac{2m}{r_o}\right)^{-1}\). In terms of \(x\), the constraint (A.2) translates into

\[(A.4) \quad \mathcal{W} \leq x < \kappa.\]
The solution to this calculus problem can be derived by considering

\[(A.5) \quad f(x) := \left(\frac{x}{\kappa - x}\right) (1 - x)^2,\]

whose derivative is \(f'(x) = \frac{(1-x)}{(\kappa-x)^2} (2x^2 - 3\kappa x + \kappa).\) We therefore have

**Theorem 3.2'** In the setting of Theorem 3.2, suppose \(\alpha > 0\) and let \(\kappa\) be given by \((A.1)\). Then

\[(A.6) \quad \min_{W \leq x < \kappa} \Phi(x) + m_H(\Sigma_o) \geq \sqrt{\frac{\left|\Sigma_h\right|}{16\pi}},\]

where

a) if \(\kappa \leq \frac{8}{9}\) or if \(\kappa > \frac{8}{9}\) and \(x_2 := \frac{3\kappa + \sqrt{9\kappa^2 - 8\kappa}}{4} \leq W\), then
   \[\min_{W \leq x < \kappa} \Phi(x) = \Phi|_{x=W};\]

b) if \(\kappa > \frac{8}{9}\) and \(x_1 := \frac{3\kappa - \sqrt{9\kappa^2 - 8\kappa}}{4} \leq W < x_2\), then
   \[\min_{W \leq x < \kappa} \Phi(x) = \Phi|_{x=x_2};\]

c) if \(\kappa > \frac{8}{9}\) and \(W < x_1\), then
   \[\min_{W \leq x < \kappa} \Phi(x) = \min \{\Phi|_{x=W}, \Phi|_{x=x_2}\}.\]

In particular, since \(\Phi|_{x=x_2}\) is determined only by \(\alpha\) and \(\beta\), one has

\[\min_{W \leq x < \kappa} \Phi(x) = \Phi|_{x=W}\]

for small \(W\).

Here

\[\Phi|_{x=W} = \Phi|_{m=0} = \left[\frac{\alpha \frac{1}{4} H_o^2 r_o^2}{\beta - (1 + \alpha) \frac{1}{4} H_o^2 r_o^2}\right]^\frac{1}{2} m_H(\Sigma_o),\]

and \(x_1, x_2 \in (0, \kappa)\) are the roots to \(2x^2 - 3\kappa x + \kappa = 0\).

Next we consider the context of Theorem 3.1. Suppose \(\alpha > 0\). Define

\[(A.7) \quad b := \beta - \alpha W \in (0, 1),\]

where \(W = \frac{1}{4} H_o^2 r_o^2\). The condition \((3.1)\) becomes \(b > 0\) and the constraint \((3.3)\) is

\[(A.8) \quad b > W \left(1 - \frac{2m}{r_o}\right)^{-1}, \quad m < 0,\]

respectively.
The quantity that we want to minimize is
\[ \Psi := \frac{1}{2} r_o \left[ \frac{\alpha \mathcal{W}}{r_o} \right] \frac{1}{2} (1 - k^2) \]
where \( x := k^2 = \mathcal{W} \left( 1 - \frac{2m}{r_o} \right)^{-1} \). There are two cases to consider when interpreting the constraint. If \( b < \mathcal{W}, \) (A.8) translates into
\[ 0 < x < b. \]
If \( \mathcal{W} \leq b, \) (A.8) translates into
\[ 0 < x < \mathcal{W}. \]
In either case, the solution to this calculus problem can be derived by considering
\[ \tilde{f}(x) := \left( \frac{1}{b-x} \right) (1-x)^2, \]
whose derivative is
\[ \tilde{f}'(x) = \frac{(1-x)}{(b-x)^2} [x - (2b - 1)]. \]
We therefore have
\[ \text{Theorem 3.1'} \quad \text{In the setting of Theorem 3.1, suppose } \alpha > 0 \text{ and let } b \text{ be given by (A.7).} \]
(1) If \( b < \mathcal{W}, \) then
\[ \min_{0 < x < b} \Psi + m_H(\Sigma_o) \geq \sqrt{\frac{\lvert \Sigma_h \rvert}{16\pi}}, \]
where
a) if \( b \leq \frac{1}{2}, \) min\(_{0 < x < b} \Psi = \Psi |_{x=0+}; \)
b) if \( b > \frac{1}{2}, \) min\(_{0 < x < b} \Psi(x) = \Psi |_{x=2b-1}. \)
(2) If \( \mathcal{W} \leq b, \) then
\[ \min_{0 < x < \mathcal{W}} \Psi + m_H(\Sigma_o) \geq \sqrt{\frac{\lvert \Sigma_h \rvert}{16\pi}}, \]
where
a) if \( b \leq \frac{1}{2}, \) min\(_{0 < x < \mathcal{W}} \Psi = \Psi |_{x=0+}; \)
b) if \( \frac{1}{2} < b < \frac{1+b}{2}, \) min\(_{0 < x < \mathcal{W}} \Psi(x) = \Psi |_{x=2b-1}; \)
c) if \( b \geq \frac{1+b}{2}, \) min\(_{0 < x < \mathcal{W}} \Psi(x) = \Psi |_{x=\mathcal{W}-}. \)
Here
\[ \Psi |_{x=0+} := \lim_{x \to 0+} \Psi = \lim_{m \to -\infty} \Psi = \frac{1}{2} r_o \left[ \frac{\frac{1}{4} H_o^2 r_o^2 \alpha}{\beta - \frac{1}{4} H_o^2 r_o^2 \alpha} \right]^{\frac{1}{2}} \]
and

\[
\Psi_{|x\to W^-} := \lim_{x\to W^-} \Psi = \lim_{m\to 0} \Psi = \left[ \frac{\alpha^2}{2} \frac{H^2}{\gamma o} \right]^{\frac{1}{2}} m_H(\Sigma_o).
\]

It follows from Theorem 3.1’ and Theorem 3.2’ (2) that, if \( W < \frac{\beta}{1+\alpha} \), there are cases, depending on \( W, \alpha \) and \( \beta \), in which the optimal values of \( \Phi \) and \( \Psi \) both occur at \( m = 0 \) and they agree.

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