A Note on Derived Geometric Interpretation of Classical Field Theories

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Abstract

In this note, we would like to provide a conceptual introduction to the interaction between derived geometry and physics based on the formalism that has been heavily studied by Kevin Costello ([CGV1], [CGV2]). Main motivations of our current attempt are as follows: (i) to provide a brief introduction to derived algebraic geometry ([Toen], [Lur2], [An]), which can be, roughly speaking, thought of as a higher categorical refinement of an ordinary algebraic geometry, (ii) to understand how certain derived objects naturally appear in a theory describing a particular physical phenomenon and give rise to a formal mathematical treatment, such as redefining a perturbative classical field theory (or its quantum counterpart) by using the language of derived algebraic geometry ([CoS] appendix A), and (iii) how the notion of factorization algebra together with certain higher categorical structures come into play to encode the structure of so-called observables in those theories by employing certain cohomological/homotopical methods in [CGV1] and [CGV2]. Adopting such a heavy and relatively enriched language allows us to formalize the notion of quantization and observables in a quantum field theory as well. The following serves as an introductory material and consists of underlying mathematical treatment for each task in an expository manner.

Acknowledgment. This note, which can be thought of as a periodic research report, serves as an introductory survey on certain mathematical structures encoding the essence of Costello’s approach to derived-geometric formulation of field theories and the structure of observables in an expository manner. Materials we present here are very well-known to the experts and as a disclaimer they are not meant to provide neither original nor new results (nor a complete reference list) related to either of the subjects mentioned above. But we hope that the material we present herein provides a brief introduction and a naïve guideline to the existing literature for non-experts who may wish to learn the subject. I am very grateful to Ali Ulaş Özgür Kişisel and Bayram Tekin for their enlightening, fruitful and enjoyable conversations during our regular research meetings that essentially lead the preparation of this note.

Contents

1 Introduction 1
2 Factorization algebras and the structure of observables 4
3 Derived formulation of field theories 6
  3.1 Why does the term "derived" emerge? 6
  3.2 Why does "stacky" language come in? 8
4 Recasting some examples 11

1 Introduction

To make the first touch with physics and realize where derived geometry comes into play, we shall discuss certain notions and structures in a rather intuitive manner, such as derived critical locus and the symplectic structure ([Vezz], [Cal]) on this derived object. Afterwards, we shall investigate the derived interpretation of a field theory: Together with the Lagrangian formalism, one can realize, for instance, a classical field theory on a smooth manifold $M$ as a sheaf of derived stack (of the derived critical locus of the action functional) on $M$ since it can be described as a formal

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moduli problem ([CGV2], [Lur]) cut out by a system of PDEs determined by the corresponding Euler-Lagrange equations governing the system under consideration.

Before discussing the derived version of the definition, we first recall how to define a naïve and algebro-geometric version of a classical field theory ([Mnev]) in Lagrangian formalism:

**Definition 1.0.1.** A classical field theory on a manifold \( M \) consists of the following data:

(i) the space \( \mathbb{F}_M \) of fields of the theory defined to be the space \( \Gamma(M, \mathcal{F}) \) of sections of a particular sheaf \( \mathcal{F} \) on \( M \),

(ii) the action functional \( \mathcal{S} : \mathbb{F}_M \rightarrow k \) (\( \mathbb{R} \) or \( \mathbb{C} \)) that captures the behavior of the system under consideration.

Furthermore, if we want to describe a quantum system, as a third component we need to introduce (iii) the path integral quantization formalism ([Gw], [Mnev], [Hon]) associated to the classical system governed by the action functional.

**Remark 1.0.1.** In order to encode the dynamics of the system in a well-established manner, we need to study the critical locus \( \text{crit}(\mathcal{S}) \) of \( \mathcal{S} \). One can determine \( \text{crit}(\mathcal{S}) \) by employing variational techniques for the functional \( \mathcal{S} \) and that leads to define \( \text{crit}(\mathcal{S}) \) to be the space of solutions to the Euler-Lagrange equations modulo gauge equivalences. Therefore, a classical field theory can be thought of as a study of the moduli space of solutions to E-L equations.

**Definition 1.0.2.** A classical field theory on a manifold \( M \) is called scalar (gauge or \( \sigma \)-model resp.) if \( \mathbb{F}_M \) is defined to be \( C^\infty(M) \) (the space \( \mathcal{A} \) of all \( G \)-connections on a principal \( G \)-bundle over \( M \) or the space \( \text{Maps}(M, N) \) of smooth maps from \( M \) to \( N \) for some fixed target manifold \( N \) respectively.)

**Example 1.0.1.** ([KW]) In accordance with the above definitions we consider the underlying theory (given as a \( \sigma \)-model) for a classical free particle of mass \( m \) moving in \( \mathbb{R}^n \) together with a certain potential energy \( V : \mathbb{R}^n \rightarrow \mathbb{R} \): Let \( \mathbb{F}_M := \text{Maps}(M, \mathbb{R}^n) \) for \( M := [0, 1] \) (in that case \( \mathcal{F} \) is just the trivial bundle on \( M \)), and the action functional

\[
\mathcal{S}(q) := \int_{[0,1]} \left( \frac{m||\dot{q}||^2}{2} - V(q) \right) \quad \text{for all } q : [0,1] \rightarrow \mathbb{R}^n.
\]

Then the corresponding Euler-Lagrange equation becomes

\[
m\ddot{q} = -\nabla V(q),
\]

which is indeed the Newton’s equation of motion.

**Example 1.0.2.** Consider a classical free particle (of unit mass) moving in a Riemannian manifold \( N \) without any potential energy: Set \( \mathbb{F}_M := \text{Maps}(M, N) \) with \( M := [0,1] \). Let \( f \in \text{Maps}(M, N) \) be a smooth path in \( N \), and the action functional given by

\[
\mathcal{S}(f) := \frac{1}{2} \int_{[0,1]} ||\dot{f}||^2,
\]

which is called the energy functional in Riemannian geometry (cf. [Pet] ch.5). Then the corresponding Euler-Lagrange equations in a local chart \( x = (x^j)_{j=1,...,\text{dim}N} \) for \( N \) are given as

\[
\dddot{f}^k + \Gamma^k_{ij} \dddot{f}^i \dot{f}^j = 0 \quad \text{for } k = 1,2,...,\text{dim}N,
\]

where \( f^k \) denotes local component of \( f \), i.e, \( f^k := x^k \circ f \), and \( \Gamma^k_{ij} := \Gamma^k_{ij}(f(t)) \) is the Christoffel symbol for each \( i,j,k \). These equations are indeed the geodesic equations in Riemannian geometry.

**Example 1.0.3.** ([Mnev]) Consider the theory with free scalar massive fields. Let \( M \) be a Riemannian manifold and set \( \mathbb{F}_M := C^\infty(M) \). Let \( \phi \in \mathbb{F}_M \), then we define the action functional governing the theory as

\[
\mathcal{S}(\phi) := \int_M \left( \frac{||d\phi||^2}{2} - \frac{m^2}{2} \phi^2 \right).
\]

The corresponding E-L equation in this case reads as

\[
(\Delta + m^2)\phi = 0.
\]
Example 1.0.4. Consider the $SU(2)$ Chern-Simons gauge theory ([Wit3]) on a closed, orientable 3-manifold $X$ (we may consider, in particular, an integral homology 3-sphere for some technical reasons [Rub]) as a non-trivial prototype example for a $3$-TQFT formalism in the sense of Atiyah [A] (for a complete mathematical treatment of the subject, see [Mnev], [Hon]). Main ingredients of this structure are encoded by the theory of principal $G$-bundles in the following sense: Let $P \rightarrow X$ be a principal $SU(2)$-bundle on $X$, $\sigma \in \Gamma(U,P)$ a local trivializing section given schematically as

$$
\begin{array}{ccc}
P \bullet SU(2) & \rightarrow & P \\
\sigma & \downarrow & \\
X & \xrightarrow{\pi} & 
\end{array}
$$

(1.7)

Note that when $G = SU(2)$, $P$ is a trivial principal bundle over $X$, i.e. $P \cong X \times SU(2)$ compatible with the bundle structure, and hence there exists a globally defined nowhere vanishing section $\sigma \in \Gamma(X,P)$. Assume $\omega$ is a Lie algebra-valued connection one-form on $P$. Let $A := \sigma^*\omega$ be its representative, i.e. the Lie algebra-valued connection 1-form on $X$, called the Yang-Mills field. Then the theory consists of the space $\mathcal{F}_X$ of fields, which is defined to be the infinite-dimensional space $\mathcal{A}$ of all $SU(2)$-connections on a principal $SU(2)$-bundle over $X$, i.e. $\mathcal{A} := \Omega^1(X) \otimes g$ (in that case $\mathcal{F}$ is the "twisted" cotangent bundle $T^*X \otimes g$), and the Chern-Simons action functional $CS : \mathcal{A} \rightarrow S^1$ given by

$$
CS(A) := \frac{k}{4\pi} \int_X \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A), \quad k \in \mathbb{Z},
$$

(1.8)

together with the gauge group $G = \text{Map}(X,SU(2))$ acting on the space $\mathcal{A}$ as follows: For all $g \in G$ and $A \in \mathcal{A}$, we set

$$
g \circ A := g^{-1} \cdot A \cdot g + g^{-1} \cdot dg.
$$

(1.9)

The corresponding Euler-Lagrange equation in this case turns out to be

$$
F_A = 0,
$$

(1.10)

where $F_A = dA + A \wedge A$ is the $g$-valued curvature two-form on $X$ associated to $A \in \Omega^1(X) \otimes g$. Furthermore, under the gauge transformation, the curvature 2-form $F_A$ behaves as follows:

$$
F_A \mapsto g \circ F_A := g^{-1} \cdot F_A \cdot g \quad \text{for all } g \in G.
$$

(1.11)

Note that the moduli space $\mathcal{M}_{\text{flat}}$ of flat connections, i.e. $A \in \mathcal{A}$ with $F_A = 0$, modulo gauge transformations emerges in many other areas of mathematics, such as topological quantum field theory, low-dimensional quantum invariants (e.g. for 3-manifolds and knots [Wit3]) or (infinite dimensional) Morse theory (i.e. Floer’s Instanton homology theory, [Rub], [DFK], [Flo]).

Definition 1.0.1 above can be re-stated by using the language in [CGV2] (ch. 3) as follows: In a classical field theory one can make a reasonable measurement only on those fields which are the solutions to the Euler-Lagrange equations of the action functional describing the system. Therefore, measurements or observables are those functions defined on the space $\mathcal{E}L$ of solutions to the Euler-Lagrange equations, and hence by adopting the Lagrangian formalism a classical field theory can be thought of as the study of the critical locus of the action functional as indicated in Remark 1.0.1. In other words, a classical field theory can be realized as a formal moduli problem (in the sense of [Lur]) cut out by a system of PDEs determined by the corresponding Euler-Lagrange equations. Note that instead of a naïve moduli we consider it as a derived moduli problem (the reason will be discussed below) and we would like to understand the local classical observables $\mathcal{O}^{cl}(U)$ for all open subset $U \subseteq M$ as well. Therefore, the assignment

$$
U \xrightarrow{\mathcal{E}L} \mathcal{E}L(U)
$$

(1.12)

can be realized as a sheaf of the derived spaces of solutions to the E-L equations (or sheaf of derived stacks), and hence the space $\mathcal{O}^{cl}(U)$ of classical observables on $U$ is defined as

$$
\mathcal{O}^{cl}(U) := \mathcal{O}_{\mathcal{E}L(U)}
$$

(1.13)
where \( \mathcal{O}_{\mathcal{L}(U)} \) denotes the algebra of functions on the formal moduli space \( \mathcal{L}(U) \). In the language of derived schemes/stacks, derived objects are locally modeled on commutative differential graded algebras (\( cdgas \)), and hence in our case \( \text{Obs}^{cl}(U) \) can naturally be realized as a certain \( cdga \). Moreover, the space \( \text{Obs}^{cl}(U) \) is the dual space of \( \mathcal{L}(U) \) for each open subset \( U \subseteq M \), and hence the assignment
\[
U \xrightarrow{\text{Obs}^{cl}} \text{Obs}^{cl}(U)
\]
gives rise to a certain co-sheaf which will be discussed in a rather succinct and naïve way below.

## 2 Factorization algebras and the structure of observables

Costello’s main motivation in [CGV1] and [CGV2] to study factorization algebras associated to a perturbative QFT is to generalize the deformation quantization approach to quantum mechanics developed by Kontsevich [Kon]. In other words, deformation quantization essentially encodes the nature of observables in one-dimensional quantum field theories and factorization algebra formalism provides an \( n \)-dimensional generalization of this approach. To be more precise, recall that observables in classical mechanics and those in corresponding quantum mechanical system can be described in the following way: Let \( (M, \omega) \) be a symplectic manifold (thought of as a phase space) and the space \( A^{cl} \) of classical observables on \( M \) defined as the space \( C^{\infty}(M) \) of smooth functions on \( M \). Then \( A^{cl} \) forms a Poisson algebra with respect to the Poisson bracket \( \{ \cdot, \cdot \} \) on \( C^{\infty}(M) \) given by
\[
\{ f, g \} := -w(X_f, X_g) = X_f(g) \quad \text{for all } f, g \in C^{\infty}(M),
\]
where \( X_f \) is the Hamiltonian vector field associated to \( f \) defined implicitly as
\[
\iota_{X_f} \omega = df.
\]
Here, \( \iota_{X_f} \omega \) denotes the contraction of a 2-form \( \omega \) with the vector field \( X_f \) in the sense that
\[
\iota_{X_f} \omega (\cdot) := \omega(X_f, \cdot).
\]
Employing canonical/geometric quantization formalism (cf. [Hon], [Bla], [Wo], [Br]), the notion of quantization boils down to the study of representation theory of classical observables in the sense that one can construct the quantum Hilbert space \( \mathcal{H} \) and a Lie algebra homomorphism \(^1\)
\[
\mathcal{Q} : (C^{\infty}(M), \{ \cdot, \cdot \}) \longrightarrow (\text{End}(\mathcal{H}), \{ \cdot, \cdot \})
\]
together with the Dirac’s quantum condition: \( \forall f, g \in C^{\infty}(M) \) we have
\[
[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar \mathcal{Q}(\{ f, g \})
\]
where \( \{ \cdot, \cdot \} \) denotes the usual commutator on \( \text{End}(\mathcal{H}) \).

In accordance with the above set-up, while the classical observables form a Poisson algebra, the space \( A^{cl} \) of quantum observables forms an associative algebra which is related to classical one by the quantum condition 2.5. Deformation quantization, in fact, serves as a mathematical treatment that captures this correspondence, i.e. the deformation of a commutative structure to a non-commutative one, for a general Poisson manifolds. Factorization algebras, on the other hand, are algebrao-geometric objects which are manifestly described in the language analogous to that of (co-)sheaves (for a complete discussion see [CGV1] Ch. 3 or [Gin]) as follows:

**Definition 2.0.1.** A prefactorization algebra \( \mathcal{F} \) on a manifold \( M \) consists of the following data:

- For each open subset \( U \subseteq M \), a cochain complex \( \mathcal{F}(U) \).
- For each open subsets \( U \subseteq V \) of \( M \), a cochain map \( \iota_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \).
- For any finite collection \( U_1, ..., U_n \) of pairwise disjoint open subsets of \( V \subseteq M \), \( V \) open in \( M \), there is a morphism
  \[
  \iota_{U_1,...,U_n,V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V)
  \]
together with certain compatibility conditions:

\(^1\) A Lie algebra homomorphism \( \beta : g \rightarrow \mathfrak{h} \) is a linear map of vector spaces such that \( \beta([X,Y]_g) = [\beta(X), \beta(Y)]_\mathfrak{h} \). Keep in mind that, one can easily suppress the constant \( *-i\hbar \) in 2.5 into the definition of \( \mathcal{Q} \) such that the quantum condition 2.5 becomes the usual compatibility condition that a Lie algebra homomorphism satisfies.
i. The invariance under the action of symmetric group $S_n$ permuting the ordering of the collection $U_1, \ldots, U_n$ in the sense that

$$t_{U_1, \ldots, U_n, V} = t_{t_U(U_1), \ldots, t_U(U_n), V}$$

for any $\sigma \in S_n$. \hfill (2.7)

That is, the morphism $t_{U_1, \ldots, U_n, V}$ is independent of the ordering of open subsets $U_1, \ldots, U_n$, but it depends only on the family $\{U_i\}$.

ii. The associativity condition in the sense that if $U_{i_1} \cap \cdots \cap U_{i_k} \subset V$ and $V \cap \cdots \cap V_k \subset W$ where $U_{i_j}$ (resp. $V_j$) are pairwise disjoint open subsets of $V$ (resp. $W$) with $W$ open in $M$, then the following diagram commutes.

$$\begin{array}{ccc}
\bigotimes_{i=1}^k \bigotimes_{j=1}^{n_i} F(U_{ij}) & \rightarrow & \bigotimes_{i=1}^k F(V_i) \\
\downarrow & & \downarrow \\
F(W) & & 
\end{array}$$

With this definition in hand, a prefactorization algebra behaves like a co-presheaf except the fact that we use tensor product instead of a direct sum of cochain complexes. Furthermore, we can define so-called a factorization algebra once we impose a certain local-to-global condition (aka gluing axiom) on a prefactorization algebra analogous to the one imposed on presheaves (for details see [Gin]).

Factorization algebras in fact serve as $n$-dimensional counterparts to those objects realized in deformation quantization formalism. In particular, one recovers the observables in classical/quantum mechanics when restricts to the case $n = 1$ (for a readable discussion see [CGV1] ch. 1). Furthermore, in a typical gauge theory, holonomy observables, namely Wilson line operators, can be formalized in terms of such objects. For instance, the ones discussed in [Wit3] related to Witten’s Knot invariants arising from the analysis of certain partition functions in three-dimensional $SU(2)$-Chern-Simons theory. In such approach to perturbative quantum field theories, the quantum observables in these type of theories form a factorization algebra which turns out to be related to the (commutative) factorization algebra of associated classical observables in the following sense:

Theorem 2.0.1. (Weak quantization Theorem [CGV1]): For a classical field theory and a choice of BV quantization,

1. The space $\text{Obs}^q$ of quantum observables forms a factorization algebra over the ring $\mathbb{R}[[\hbar]]$.

2. $\text{Obs}^{cl} \cong \text{Obs}^q \mod \hbar$ as a homotopy equivalences where $\text{Obs}^{cl}$ denotes the associated factorization algebra of classical observables.

Note that the above theorem is just a part of the story, and it is indeed weak in a way that it is not able to capture the data related to Poisson structures on the space of observables. To provide a correct $n$-dimensional analogue of deformation quantization approach, we need to refine the notion of classical field theory in such a way that the richness of this new set-up become visible. This is where derived algebraic geometry comes into play.

As we discussed above, the space of classical observables forms a (commutative) factorization algebra and this allows us to employ certain cohomological methods encoding the structure of observables in a given theory in the following sense (cf. [CGV1] ch. 1): Factorization algebra $\text{Obs}^{cl}$ of observables can be realized as a particular assignment analogous to co-sheaf of cochain complexes as mentioned above. That is, for each $U \subseteq M \text{Obs}^{cl}(U)$ has a $\mathbb{Z}$-graded structure

$$\text{Obs}^{cl}(U) = \bigoplus_{i \in \mathbb{Z}} \text{Obs}^{cl}_i(U) : \cdots \rightarrow \text{Obs}^{cl}_{-1}(U) \rightarrow \text{Obs}^{cl}_0(U) \rightarrow \text{Obs}^{cl}_1(U) \rightarrow \cdots$$

together with suitable connecting homomorphisms $d_i : \text{Obs}^{cl}_i(U) \rightarrow \text{Obs}^{cl}_{i+1}(U)$ for each $i$. The corresponding cohomology groups $H^i(\text{Obs}^{cl}(U))$ encodes the structure of observables as follows:

- "Physically meaningful" observables are the closed ones with cohomological degree 0, i.e., $\mathcal{O} \in \text{Obs}^{cl}_0(U)$ with $d_0 \mathcal{O} = 0$. (and hence $[\mathcal{O}] \in H^0(\text{Obs}^{cl}(U))$.)
• $H^1(\text{Obs}^{cl}(U))$ contains anomalies, i.e., obstructions for classical observables to be lifted to the quantum level. In a gauge theory, for instance, there exists certain classical observables respecting gauge symmetries such that they do not admit any lift to quantum observables respecting gauge symmetries. This behaviour is indeed encoded by a non-zero element in $H^1(\text{Obs}^{cl}(U))$

• $H^n(\text{Obs}^{cl}(U))$ with $n < 0$ can be interpreted as symmetries, higher symmetries of observables etc. through higher categorical arguments.

• $H^n(\text{Obs}^{cl}(U))$ with $n > 1$ has no clear physical interpretation.

3 Derived formulation of field theories

Together with the derived interpretation of a classical field theory outlined in [CGV2], one can employ a number of mathematical techniques and notions naturally appearing in derived algebraic geometry. For instance, we may consider a classical field theory as the study of the derived critical locus ([Vezz], [Call]) of the action functional since it can be considered as a formal moduli problem in the sense indicated above. Indeed, passing to the derived moduli space of solutions corresponds to Batalin-Vilkovisky formalism for a classical field theory which will be briefly discussed below. In derived algebraic geometry, any formal moduli problem arising as the derived critical locus admits a symplectic structure of cohomological degree $-1$. (cf. [Call]) This observation is crucial and it ensures the existence of a symplectic structure on the space $\text{Obs}^{cl}$ of classical observables. In the language of derived algebraic geometry, therefore, we have the following definition ([CGV2] ch. 3):

**Definition 3.0.1.** A (perturbative) classical field theory is a formal elliptic moduli problem equipped with a symplectic form of cohomological degree $-1$.

Equivalently, one has the following definition:

**Definition 3.0.2.** ([CoS] Appendix, [CGV2] ch. 3) Let $M$ be the space of fields (e.g. a finite dimensional smooth manifold, algebraic variety or a smooth scheme) for some base manifold $X$, i.e. $M := \mathbb{F}_X$, and $S : M \to k$ a smooth action functional on $M$. A (perturbative) classical field theory is a sheaf of derived stack (of the derived critical locus $\text{dcrit}(S)$ of the action functional $S$) on $M$ equipped with a symplectic form of cohomological degree $-1$.

We intend to unpack Definition 3.0.2 in a well-established manner as follows:

3.1 Why does the term "derived" emerge?

We may first discuss naïve or underived realization of a classical field theory in the language of intersection theory. Let $M$ denote the space of fields on a base manifold $X$ as in Definition 3.0.2. Assume $M$ is a finite dimensional manifold. As indicated in the Remark 1.0.1, a classical field theory can be considered as the study of the critical locus $\text{crit}(S) \subset M$ of the action functional $S$ on $M$. However, computing certain path integrals perturbatively around classical solutions to the E-L equations is usually pathological if the critical points are degenerate. To avoid such pathologies, one can employ a certain trick so-called the Batalin-Vilkovisky formalism which, roughly speaking, consists of adding certain fields, such as ghosts, anti-fields etc..., to the functional ([Call]). This problem, on the other hand, can be formulated in the language of intersection theory as follows: We define $\text{crit}(S)$ to be the intersection of the graph $G(dS) \subset T^*M$ of $dS \in \Omega^1(M)$ and the zero-section of the cotangent bundle $T^*M$ over $M$ (cf. [CGV2] ch. 5). That is,

$$\text{crit}(S) := G(dS) \cap M.$$  \hfill (3.1)

As in [Vezz], by adopting algebro-geometric language (see [Va] ch. 9), $\text{crit}(S)$ can be described in terms of a fibered product $M \times_{T^*M} M$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\text{crit}(S) := M \times_{T^*M} M & \longrightarrow & M \\
\downarrow & & \downarrow 0 \\
M & \rightarrow & T^*M \\
\end{array}
$$  \hfill (3.2)
Even if $M$ is a smooth manifold, for instance, the intersection $M \times_{T^*M} M$ would be highly pathological, and hence an object $(\text{crit}(S), \mathcal{O}_{\text{crit}(S)})$ generically fails to live in the same category, i.e. intersection would not define a manifold at all (e.g. non-transverse intersection of two submanifold is not a submanifold in general). In fact, pathologies arising from the degeneracy of critical points correspond to pathological intersection in the above sense.

If we employ, however, derived set-up and introduce the derived geometric counterpart of a smooth manifold, namely derived manifold, then we can circumvent the non-existence problem for a fibered product. It follows from the fact that the theory of derived schemes is equivalent to that of dg-schemes ([Cio], [Arin]) in the case of characteristic zero, one can work with a dg-scheme $(X, \mathcal{O}_X)$ for which the structure sheaf $\mathcal{O}_X$ is a sheaf of commutative differential graded algebras. Furthermore, one can show that the category (or the correct terminology would be the ∞-category) of derived manifold admits the fibered product. Therefore, this leads to following motivation behind the use of "derived" formulation in Definition 3.0.2:

One has to enlarge and re-design the notion of category together with new enlarged-objects in a way that the intersection of any two such objects always lives in the enlarged version of a category.

This requires to re-organize the local model for the intersection of ringed spaces as follows: As in [CGV2], instead of naïve intersection determined algebraically by

$$\mathcal{O}_{\text{crit}(S)} := \mathcal{O}_{G(dS)} \otimes_{O_{T^*M}} \mathcal{O}_M,$$

we introduce the derived version as

$$\mathcal{O}_{\text{der}(S)} := \mathcal{O}_{G(dS)} \otimes_{O_{T^*M}}^{L} \mathcal{O}_M.$$

where $\cdot \otimes_{O_{T^*M}}^{L} \cdot$ denotes the derived tensor product.

A digression on the definition of $\cdot \otimes_{O_{T^*M}}^{L} \cdot$ : (cf. [Lur3] ch. 0) Let $R$ be a commutative ring, $B$ a $R$-module. Then derived tensor product $\cdot \otimes_{R}^{L} B$ arises from the construction of left-derived functor associated to the right-exact functor (cf. [Va] ch. 23)

$$\cdot \otimes_{R} B : \text{Mod}_R \to \text{Mod}_R.$$ (3.5)

Let $A$, $B$ be two commutative algebras over $R$. Then the definition of $A \otimes_{R}^{L} B$ naturally appears in the construction of the $i^{th}$ Tor groups $\text{Tor}_R^i(A, B)$ given by the $i^{th}$ homology of the tensor product complex $(P_{\bullet} \otimes_R B, d')$:

$$\cdots \to P_2 \otimes_R B \to P_1 \otimes_R B \xrightarrow{d'} P_0 \otimes_R B \to 0$$ (3.6)

where $P_{\bullet}$ is a projective resolution of $A$ equipped with a differential $d$ such that $(P_{\bullet}, d)$ becomes a commutative dg-algebra over $R$ and $d' := d \otimes_R \text{id}_B$. Since $B$ is a commutative $R$-algebra, the tensor product complex inherits the structure of a commutative dg-algebra over $R$ as well, and we denote this tensor product complex by $A \otimes_{R}^{L} B$. That is, we set

$$A \otimes_{R}^{L} B := (P_{\bullet} \otimes_R B, d').$$ (3.7)

Remark 3.1.1. The resulting commutative dg-algebra $A \otimes_{R}^{L} B$ is independent of the choice of $(P_{\bullet} \otimes_R B, d')$ up to quasi-isomorphism. The end of digression.

If we go back the local model discussion and structure of algebra of functions for the derived tensor product, then for each open subset $U \subseteq M$ we have

$$\mathcal{O}_{\text{der}(S)}(U) := \mathcal{O}_{G(dS)}(U) \otimes_{O_{T^*M}(U)}^{L} \mathcal{O}_M(U)$$ (3.8)

where RHS corresponds to the tensor product complex of dg-algebra as above. Together with this local model, $(\text{crit}(S), \mathcal{O}_{\text{crit}(S)})$ becomes a dg-scheme with its structure sheaf $\mathcal{O}_{\text{crit}(S)}$ being the sheaf of commutative dg $k$-algebras such that $\mathcal{O}_{\text{der}(S)}$ can be manifestly given as a Koszul resolution of $\mathcal{O}_M$ as a module over $O_{T^*M}$:

$$\mathcal{O}_{\text{der}(S)} : \cdots \to \Gamma(M, \wedge^2 TM) \to \Gamma(M, TM) \xrightarrow{\text{ad}} \mathcal{O}_M \to 0$$ (3.9)
where $\Gamma(M, \wedge^i TM)$ is the space of polyvector fields of degree $i$ (or $i$-vector fields) and $\iota_S$ denotes the contraction with $dS$ in the sense that for any 1-vector field $X \in \Gamma(M, TM)$ we define

$$\iota_S(X) := dS(X) = XS.$$  

Then, extending to $i$-vector fields by linearity, we set

$$O_{\text{dcrit}(S)} := \left( \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \Gamma(M, \wedge^i TM), \ i \iota_S \right).$$  

### Remark 3.1.2.

$(\text{dcrit}(S), O_{\text{dcrit}(S)})$ admits further derived structure; namely, a symplectic form of cohomological degree $-1$ (see [PTVV] corollary 2.11). The description of this structure, however, is beyond the scope of the current discussion. For the construction, we refer to [PTVV]. You may also see [Brav] or [BJ] for an accessible presentation of PTVV’s shifted symplectic geometry.

### Remark 3.1.3.

Existence of such a derived geometric structure will be crucial when we discuss the notion of quantization for n-dimensional classical field theories. Indeed, this new structure is really what we need and it leads n-dimensional generalization of what we have already had in the case of quantization of classical mechanics. Recall that observables in classical mechanics with a phase space $(X, \omega)$ forms a Poisson algebra with respect to the Poisson bracket $\{ \cdot, \cdot \}$ on $C^\infty(X)$. By using canonical/geometric quantization formalism ([Br], [Hon], [Bla], [Wol]) the notion of quantization boils down to the study of representation theory of classical observables in the sense that one can construct the quantum Hilbert space $\mathcal{H}$ and a certain Lie algebra homomorphism as outlined in the beginning of Section 2.

### 3.2 Why does "stacky" language come in?

Studying the critical locus of the action functional $S$ is just one part of the story, and we already observe that in order to avoid the degenerate critical points one requires to introduce the notion of derived intersection and hence the derived critical locus which is well-behaved than the usual one. For a more complete discussion, see [Neu] or [Vezz2]. Other part of the story is related to moduli nature of the problem. Indeed, one requires to quotient out by symmetries while studying the solution space of the E-L equations, but the quotient space might be highly pathological as well. For instance, the action of the gauge group $G$ on a manifold $X$ may not be free, and hence the resulting quotient $X/G$ would not be a manifold, but it can be realized as an orbifold $[X//G]$, which is indeed a particular stack, given by the orbifold quotient ([BSS] and [Neu] provide further examples and details).

**A digression on a moduli problem and stacks.** A moduli problem is a problem of constructing a classifying space (or a moduli space $M$) for certain geometric objects (such as manifolds, algebraic varieties, vector bundles etc...) up to their intrinsic symmetries. The wish-list for a "fine" moduli space $M$ is as follows (see [BenZ] for an accessible overview and [Neu] for a rather complete treatment):

1. $M$ is supposed to serve as a parameter space in a sense that there must be a one-to-one correspondence between the points of $M$ and the set of isomorphism classes of objects to be classified:

$$\{\text{points of } M\} \leftrightarrow \{\text{isomorphism classes}\}$$  

2. The existence of universal classifying object.

In the language of category theory, a moduli problem can be formalized as a certain functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \text{Sets}$$  

which is called the moduli functor where $\mathcal{C}^{op}$ is the opposite category of the category $\mathcal{C}$ and $\text{Sets}$ is the category of sets. In order to make the argument more transparent, we take $\mathcal{C}$ to be the category $\text{Sch}$ of $k$-schemes. Note that for each scheme $U \in \text{Sch}$, $\mathcal{F}(U)$ is the set of isomorphism classes parametrized by $U$, and for each morphism $f : U \rightarrow V$ of schemes, we have a morphism $\mathcal{F}(f) : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ of sets. Together with the above formalism, the existence of a fine moduli space corresponds to the representability of the moduli functor $\mathcal{F}$ in the sense that

$$\mathcal{F} = \text{Hom}_{\text{Sch}}(\cdot, M) \text{ for some } M \in \text{Sch}.$$  

If this is the case, then we say that $\mathcal{F}$ is represented by $M$.
In many cases, however, the moduli functor is not representable in the category $\text{Sch}$ of schemes. This is essentially where the notion of stack comes into play. The notion of stack which can be thought of as a first instance such that the ordinary notion of category no longer suffices to define such an object. To make sense of this new object in a well-established manner and enjoy the richness of this new structure, we need to introduce a higher categorical notion, namely a 2-category ([Neu], [Stk]). The theory of stacks, therefore, employs higher categorical techniques and notions in a way that it provides a mathematical treatment for the representability problem by re-defining the moduli functor as a stack, a particular groupoid-valued pseudo-functor with local-to-global properties,

$$\mathcal{X} : \text{C}^{\text{op}} \longrightarrow \text{Grpds}$$

(3.15)

where $\text{Grpds}$ denotes the 2-category of groupoids with objects being categories $\mathcal{C}$ in which all morphisms are isomorphism (these sorts of categories are called groupoid), 1-morphisms being functors $F : \mathcal{C} \rightarrow \mathcal{D}$ between groupoids, and 2-morphisms being the natural transformations $\psi : F \Rightarrow F'$ between two functors.

**Remark 3.2.1.** In order to make sense of local-to-global (or "glueing") type arguments, one requires to introduce an appropriate notion of topology on a category $\mathcal{C}$. Such a structure is manifestly given in [Neu] and called the Grothendieck topology $\tau$. Furthermore, a category $\mathcal{C}$ equipped with a Grothendieck topology $\tau$ is called site. Note that if we have a site $\mathcal{C}$, then we can define a sheaf on $\mathcal{C}$ in a well-established manner as well. This essentially leads the functor of points-type approach to defining a scheme $X$ in the following sense: Given a scheme $X$, one can define a sheaf (on the category $\text{Sch}$ of schemes) by using the Yoneda functor $\text{Hom}_{\text{Sch}}(\cdot, X)$ as

$$\underline{X} : \text{Sch}^{\text{op}} \longrightarrow \text{Sets}$$

(3.16)

where $\underline{X} := \text{Hom}_{\text{Sch}}(\cdot, X)$. This is indeed a sheaf by the theorem of Grothendieck ([Neu]).

**Remark 3.2.2.** Any 1-category (i.e. the usual category) can be realized as a 2-category in which there exists no non-trivial higher structures, i.e. 2-morphisms in a 1-category are just identities.

**Remark 3.2.3.** By using 2-categorical version of the Yoneda lemma, so-called 2-Yoneda lemma [Neu], one can show that the moduli functor $\mathcal{X}$ turns out to be representable in the 2-category $\text{Stks}$ of stacks. As in the case of introduction of derived intersection, we enlarge the category with certain non-trivial higher structures in a way that the moduli problem become representable in this enhanced-version even if it was not in the first place. The price we have to pay is to adopt higher categorical dictionary leading the change in the level of abstraction in a way that objects under consideration become rather counter-intuitive. Indeed, stacks and 2-categories serve as a motivating/prototype conceptual examples before introducing the notions like $\infty$-categories, derived schemes, higher stacks and derived stacks ([Vezz2]).

The end of digression.

Now, one can define the moduli functor $\mathcal{E}\mathcal{L}$ corresponding to a given classical field theory as

$$\mathcal{E}\mathcal{L} : \text{C}^{\text{op}} \longrightarrow \text{Sets}, \ U \mapsto \mathcal{E}\mathcal{L}(U),$$

(3.17)

where $\mathcal{E}\mathcal{L}(U)$ is the set of isomorphism classes of solutions to the E-L equations over $U$. More precisely, $\mathcal{E}\mathcal{L}(U)$ is the moduli space $\text{E}\text{L}(U)/\mathcal{G}$ of solutions to the E-L equations modulo gauge transformation $\mathcal{G}$. But, as we discuss above, the quotient space might be pathological in general and it fails to live in the same category. In other words, the moduli functor $\mathcal{E}\mathcal{L}$ in general is not representable in $\mathcal{C}$. In order to circumvent the problem, we introduce the "stacky" version of $\mathcal{E}\mathcal{L}$ as the quotient stack

$$[\mathcal{E}\mathcal{L}/\mathcal{G}] : \text{C}^{\text{op}} \longrightarrow \text{Grpds}, \ U \mapsto [\mathcal{E}\mathcal{L}/\mathcal{G}](U),$$

(3.18)

where $[\mathcal{E}\mathcal{L}/\mathcal{G}](U)$ is the groupoid of solutions to the E-L equations over $U$. Even if this explains the emergence of stacky language in Definition 3.0.2 in a rather intuitive way, the discussion above is just the tip of the iceberg and is still too naïve to capture the notion of derived stack, and hence we need further notions in order to enjoy the richness of Definition 3.0.2, such as the formal neighborhood of a point in a derived scheme/stack, a formal moduli problem, $\mathcal{L}_\infty$ algebras etc. For an expository introduction to derive stacks, see [Vezz2].
Revisiting Definition 3.0.2. Let \( M \) be the space of fields (e.g. a finite dimensional smooth manifold, algebraic variety or a smooth scheme) for some base manifold \( X \) and \( S \) a smooth action functional on \( M \). We define a perturbative classical field theory on \( M \) to be the sheaf \( \mathcal{E}L \) of derived stacks on \( M \) as follows: To each open subset \( U \) of \( M \), one assigns

\[
U \mapsto \mathcal{E}L(U) \in \text{dStk}
\]

where \( \text{dStk} \) denotes the \( \infty \)-category of derived stacks and \( \mathcal{E}L(U) \) is given in the functor of points formalism as

\[
\mathcal{E}L(U) : \text{cdga}_k^{-\infty} \to s\text{Sets} \quad \text{or} \quad \infty - \text{Grpds}
\]

where \( \text{cdga}_k^{-\infty} \) and \( s\text{Sets} \) (\( \infty - \text{Grpds} \)) denote the category of commutative differential graded \( k \)-algebras in non-positive degrees, and \( \infty \)-category of simplicial sets (\( \infty \)-groupoids) respectively, and \( \mathcal{E}L(U)(R) \) is the simplicial set of solutions to the defining relations (i.e. EL-equations) with values in \( R \). In other words, the points of \( \mathcal{E}L(U) \) form an \( \infty \)-groupoid.

As discussed above, in order to circumvent certain pathologies we work with the derived moduli space of solutions instead of the naïve one. Furthermore, we also intend to capture the perturbative behavior of the theory, and hence this derived moduli space is defined as a formal moduli problem ([Lur])

\[
\mathcal{E}L(U) : \text{dgArt}_k \to s\text{Sets}
\]

where \( \text{dgArt}_k \) the category of dg artinian algebras, where morphisms are simply maps of dg commutative algebras (cf. [CGV2] Appendix A).

Remark 3.2.4. In order to remember the perturbative behavior around the solution \( p \in \mathcal{E}L(U) \), we employ the notion of formal neighborhood of a point (cf. [CGV2] Appendix A) for which it is more suited to make use of dg artinian algebras as a local model for the scheme structure instead of the usual commutative \( k \)-algebras to keep track infinitesimal directions assigned to a point \( p \) (small thickenings of a point). That is, the scheme structure, informally speaking, is locally modeled on a kind of nilpotent commutative dg-algebras such that the structure consists of points with infinitesimal directions attached to them. Furthermore, every formal moduli functor can be manifested by using the language of \( \mathcal{L}_\infty \) algebras in the sense of [Lur].

A digression on \( \mathcal{L}_\infty \) algebras and the Maurer-Cartan functor. Informally speaking, a \( \mathcal{L}_\infty \) algebra \( \mathcal{E} \) can be considered as a certain dg Lie algebra endowed with a sequence \( \{ l_n \} \) of multilinear maps of (cohomological) degree \( 2 - n \) as

\[
l_n : \mathcal{E}^\otimes n \to \mathcal{E},
\]

which are called \( n \)-bracket with \( n = 1, 2, \ldots \) such that each bracket satisfies certain graded anti-symmetry condition and \( n \)-Jacobi rule (for a complete definition see [CGV2], App. A).

Remark 3.2.5. When \( n = 1 \) or 2, from axioms of \( \mathcal{L}_\infty \) algebra, one can recover the usual differential and the Lie bracket denoted by \( l_1 := d \) and \( l_2 := [\cdot, \cdot] \).

A first natural example of \( \mathcal{L}_\infty \) algebras. Let \( M \) be a smooth manifold and \( \mathfrak{g} \) a Lie algebra. Then there exists a natural \( \mathcal{L}_\infty \) algebra (which will be central and appear in the context of gauge theories) given as follows:

\[
\mathcal{E} := \Omega^*(M) \otimes \mathfrak{g},
\]

where the only non-zero multilinear maps are \( l_1 := d_{dR} \) and \( l_2 := [\cdot, \cdot] \) given by

\[
[\alpha \otimes X, \beta \otimes Y] := \alpha \wedge \beta \otimes [X,Y]_{\mathfrak{g}}.
\]

Definition 3.2.1. For a \( \mathcal{L}_\infty \) algebra \( \mathcal{E} \), the Maurer-Cartan (MC) equation is given as

\[
\sum_{n=1}^{\infty} \frac{1}{n!} l_n(\alpha^{\otimes n}) = 0
\]

where \( \alpha \) is an element of degree 1.

Note that when we reconsider the case \( \mathcal{E} := \Omega^*(M) \otimes \mathfrak{g} \), the MC equation reduces to

\[
dA + \frac{1}{2}[A,A] = 0 \quad \text{where} \quad A \in \Omega^1(M) \otimes \mathfrak{g}.
\]
Definition 3.2.2. For a $\mathcal{L}_\infty$ algebra $\mathcal{E}$, the Maurer-Cartan functor $\mathcal{MC}_\mathcal{E}$ associated to $\mathcal{E}$ is defined as

$$\mathcal{MC}_\mathcal{E} : \mathfrak{dgArt}_k \to sSets, \quad (A, m) \mapsto \mathcal{MC}_\mathcal{E}(A)$$

where $n$-simplices of the simplicial set $\mathcal{MC}_\mathcal{E}(A)$ (for an introduction to simplicial sets, see [CGV1] App. A) are solutions (of cohomological degree 1) to the MC equation in the triple complex $\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)$. For a more concrete treatment to the notions like double/triple complexes and their total complexes, see [Stk], ch. 12. Here $m$ is the unique maximal ideal such that $m^k = 0$ for some $k$, and $\Delta^n$ denotes the $n$-simplex in $\mathbb{R}^{n+1}$ given as a set

$$\Delta^n := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1 \text{ and } 0 \leq x_k \leq 1 \text{ for all } k\}.$$  

Theorem 3.2.1. ([Lur]) Every formal moduli problem is represented by the Maurer-Cartan functor $\mathcal{MC}_\mathfrak{g}$ of an $\mathcal{L}_\infty$ algebra $\mathfrak{g}$. More precisely, there exists an equivalence of $\infty$-categories

$$\mathfrak{dgla}_k \sim \text{Moduli}_k \subset \text{Fun}(\mathfrak{dgArt}_k, sSets)$$

where $\mathfrak{dgla}_k$ and $\text{Moduli}_k$ denote $\infty$-categories of differential graded Lie algebras over $k$ and formal moduli problems over $k$ respectively with $k$ being a field of characteristic zero.

The end of digression.

4 Recasting some examples

Now we are in place of summarizing what we have done so far and provide a kind of a recipe to motivate constructions encoding the derived re-interpretation of a classical field theory.

i. Employing the above approaches, describing a classical field theory boils down to the study of the moduli space $\mathcal{E}L$ of solutions to the Euler-Lagrange equations (and hence the critical locus of action functional) which in fact corresponds to a certain moduli functor.

ii. As stressed above, a moduli functor, however, would not be representable in general due to certain pathologies, such as the existence of degenerate critical points or non-freeness of the action of the symmetry group acting on the space of fields. In order to avoid pathologies of these kinds (and to capture the perturbative behavior at the same time), one requires to adopt the language of derived algebraic geometry, and hence one needs to replace the naïve notion of moduli problem by so-called a formal moduli problem in the sense of Lurie as discussed above.

iii. Formal moduli problem $\mathcal{F}$, on the other hand, are unexpectedly tractable notion (thanks to Lurie’s theorem 3.2.1 stated above) in the sense that understanding $\mathcal{F}$, at the end of the day, boils down to finding a suitable (local) $\mathcal{L}_\infty$ algebra $\mathcal{E}$ such that $\mathcal{F}$ can be represented by the Maurer-Cartan functor $\mathcal{MC}_\mathcal{E}$ associated to $\mathcal{E}$.

iv. Having obtained appropriate $\mathcal{L}_\infty$ algebra $\mathcal{E}$, analyze the structure of $\mathcal{E}$ so as to encode the aspects of the theory.

Revisiting Example 1.0.3. Consider a free scalar massless field theory on a Riemannian manifold $M$ with space of field being $C^\infty(M)$ and the action functional governing the theory as

$$S(\phi) := \int_M \phi \Delta \phi.$$  

The corresponding E-L equation in this case turns out to be

$$\Delta \phi = 0,$$

and hence the moduli space $EL$ of solutions to the E-L equations is the moduli space of harmonic functions

$$\{\phi \in C^\infty(M) : \Delta \phi = 0\}.$$
Now, having employed the derived enrichment $\mathcal{E}L$ of $EL$ as described above, we need to find a suitable $L_\infty$ algebra $E$ whose Maurer-Cartan functor $MC_E$ represents the formal moduli problem $\mathcal{E}L$. The answer is as follows: We define $E$ to be the two-term complex (for detailed construction see [CGV1] ch. 2 or [CGV2] ch. 4) concentrated in degree 0 and 1

$$E : C^\infty(M) \xrightarrow{\Delta} C^\infty(M)[-1],$$  \hspace{1cm} (4.4)

equipped with a sequence $\{l_n\}$ of multilinear maps where $l_1 := \Delta$ and $l_i = 0$ for all $i \neq 1$. The Maurer-Cartan equation, on the other hand, turns out to be

$$\Delta \phi = 0,$$  \hspace{1cm} (4.5)

and hence the set of 0-simplices of the simplicial set $MC_E(A)$ for $A$ ordinary Artinian algebra is given as

$$\{ \phi : \Delta \phi = 0 \}$$  \hspace{1cm} (4.6)
as desired. For further details and interpretation of other simplices, see [CGV2], ch. 4.

**Revisiting Example 1.0.4.** We shall revisit $SU(2)$ Chern-Simons gauge theory on a closed, orientable 3-manifold $X$. As before, let $P \to X$ be a principal $SU(2)$-bundle on $X$, its Lie algebra $g := su(2)$, and $A \in A := \Omega^1(X) \otimes g$ the Lie algebra-valued connection 1-form on $X$ together with the Chern-Simons action functional $CS : A \to S^1$ given by

$$CS(A) := \frac{k}{4\pi} \int_X \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad k \in \mathbb{Z},$$  \hspace{1cm} (4.7)

together with the gauge group $G = \text{Map}(X, SU(2))$ acting on the space $A$ as usual. The corresponding E-L equation in this case turns out to be

$$F_A := dA + \frac{1}{2}[A, A] = 0,$$  \hspace{1cm} (4.8)

where $F_A$ is the curvature two-form on $X$ associated to $A$, and hence the moduli space $M_{flat}$ of flat connections, i.e. $A \in A$ with $F_A = 0$, modulo gauge transformations is

$$\{ [A] \in \Omega^1(X) \otimes g : dA + A \wedge A = 0 \}.$$  \hspace{1cm} (4.9)

As before, having introduced derived counterpart $\mathcal{E}L_{flat}$ of $M_{flat}$, we define the suitable $L_\infty$ algebra $E$ encoding the formal moduli problem as follows:

$$E := \Omega^*(X) \otimes g[1],$$  \hspace{1cm} (4.10)

where the only non-zero multilinear maps are $l_1 := d \otimes$ and $l_2 := [\cdot, \cdot]$ given as in 3.24. Notice that the Maurer-Cartan equation in this case becomes

$$dA + \frac{1}{2}[A, A] = 0,$$  \hspace{1cm} (4.11)

and hence the corresponding the Maurer-Cartan functor $MC_E$ yields the desired result (for a complete treatment see [CGV2] ch. 4, [Costello 3] ch. 5, or [Costello 4]). Furthermore, as stressed in [Costello 3], the space of all fields associated to the theory are encoded by $L_\infty$ algebra $E$ in the Batalin-Vilkovisky formalism as follows:

- The space of degree $-1$ fields, so-called *ghosts*, corresponds to the space
  $$\Omega^0(X) \otimes g = \text{Map}(X, SU(2)) = G.$$  \hspace{1cm} (4.12)

- The space of degree 0 fields, so-called *fields*, corresponds to the space
  $$\Omega^1(X) \otimes g.$$  \hspace{1cm} (4.13)

- The space of degree 1 fields, so-called *anti-fields*, corresponds to the space
  $$\Omega^2(X) \otimes g.$$  \hspace{1cm} (4.14)

- The space of degree 2 fields, so-called *anti-ghosts*, corresponds to the space
  $$\Omega^3(X) \otimes g.$$  \hspace{1cm} (4.15)
References

[An] Anel M., *The Geometry of Ambiguity: An Introduction to the Ideas of Derived Geometry* (2018).

[A] M. Atiyah, *Topological Quantum Field Theory*. Publications Mathématiques de l’IHÉS, Volume 68 (1988), p. 175-186.

[AB] M. Atiyah and R. Bott, *The Yang-Mills Equations over Riemann Surfaces*. Phil. Trans. R. Soc. Lond. A 1983 308, 523-615.

[Arin] Arinkin D., Caldararu A., Hablicsek M., *Formality of derived intersections and the orbifoldHKR isomorphism*. arXiv preprint arXiv:1412.5233 (2014).

[BSS] Benini, M., Schenkel, A., and Schreiber, U. (2018). *The Stack of Yang–Mills Fields on Lorentzian Manifolds*. Communications in Mathematical Physics, 359(2), 765-820.

[BenZ] Ben-Zvi, D.D., 2008. Moduli spaces. The Princeton Companion to Mathematics"(T. Gowers, J. Barrow-Green, I. Leader, eds.), Princeton University Press, Princeton, NJ, pp.408-419.

[Bla] Blau M., *Symplectic Geometry and Geometric Quantization* (1992). Available at http://www.blau.itp.unibe.ch/Lecturenotes.html

[Brav] Brav, C., Bussi, V. and Joyce, D., 2013. A’Darboux theorem for derived schemes with shifted symplectic structure. arXiv preprint arXiv:1305.6302.

[BJ] Borisov, D. and Joyce, D., 2017. Virtual fundamental classes for moduli spaces of sheaves on Calabi–Yau four-folds. Geometry & Topology, 21(6), pp.3231-3311.

[Br] Brian C. Hall, *Quantum Theory for Mathematicians*. Graduate Texts in Mathematics book series (GTM, volume 267), doi: 10.1007/978-1-4614-7116-5

[Cal] Calaque D., *Three Lectures on derived symplectic geometry and topological field theories*. Indagationes Mathematicae 25, no. 5 (2014): 926-947.

[Cio] Ciocan-Fontanine, I., Kapranov, M., *Derived Quot schemes*, Ann. Sci. Ecole Norm. Sup. (4) 34 (2001), no. 3, 403-440

[CGV1] Costello K. & Gwilliam O., *Factorization Algebras in quantum field theory*. Vol. 1. Cambridge University Press, 2016.

[CGV2] Costello K. & Gwilliam O., *Factorization Algebras in quantum field theory*. Vol. 2. Draft available from author’s website as http://people.mpim-bonn.mpg.de/gwilliam/vol2may8.pdf

[CoS] Costello, K. and Scheimbauer, C., *Lectures on mathematical aspects of (twisted) supersymmetric gauge theories*. In Mathematical aspects of quantum field theories 2015 (pp. 57-87). Springer, Cham.

[Cos3] Costello, K., 2011. *Renormalization and effective field theory (No. 170)*. American Mathematical Soc. Publ. Math. IHES 36, (1969), 75-110.

[CoS4] Costello, Kevin J. *Renormalisation and the Batalin-Vilkovisky formalism*. arXiv preprint arXiv:0706.1533 (2007). number 1764 of the series Lecture Notes in Mathematics.

[DFK] S. Donaldson, M. Furuta , D. Kotschick, *Floer Homology Groups in Yang-Mills Theory* (Cambridge Tracts in Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511543098.

[Flo] Floer, A., *An instanton invariant for three manifolds*. Comm. Math. Phys. Volume 118, Number 2 (1988), 215-240

[Gin] Ginot, G. (2015). *Notes on factorization algebras, factorization homology and applications*. In Mathematical aspects of quantum field theories (pp. 429-552). Springer, Cham.

[Gw] Gwilliam O., *Factorization algebras and free field theories*. Available at http://people.mpim-bonn.mpg.de/gwilliam/thesis.pdf

[Hon] Honda K., *Lecture notes for MATH 655: Topological Quantum Field Theory*. Available at http://www.math.ucla.edu/~honda/. 10.1007/978-1-4757-2261-1
