Gauss-Bonnet boson stars

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We construct boson stars in (4+1)-dimensional Gauss-Bonnet gravity. We study the properties of the solutions in dependence on the coupling constants and investigate these in detail. While the “thick wall” limit is independent of the value of the Gauss-Bonnet coupling, we find that the spiraling behaviour characteristic for boson stars in standard Einstein gravity disappears for large enough values of the Gauss-Bonnet coupling. Our results show that in this case the scalar field cannot have arbitrarily high values at the center of the boson star and that it is hence impossible to reach the “thin wall” limit. Moreover, for large enough Gauss-Bonnet coupling we find a unique relation between the mass and the radius (qualitatively similar to those of neutron stars) which is not present in the Einstein gravity limit.

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I. INTRODUCTION

Solitons are localized and finite energy solutions that are in general non-singular and stable and as such can be seen as models for particles. In the context of relativistic field theories, solitons are typically classified into two distinct groups, namely “topological solitons” and “non-topological solitons”. Topological solitons \(^{[1]}\) require degenerate vacuum states. The topological character of the field is represented by an integer which is called the topological charge. Topological solitons result (in most cases) from a spontaneous symmetry breaking. Non-topological solitons, on the other hand, arise in field theories with unbroken continuous symmetries. Examples of non-topological solitons are \(Q\)-balls \(^{[2–4]}\). These are made out of scalar fields prevented from collapse by Heisenberg’s uncertainty principle and repulsive self–interaction. They carry a non-vanishing Noether charge \(Q\) that is globally conserved due to the global \(U(1)\) symmetry of the model. \(Q\) can e.g. be interpreted as particle number \(^{[2]}\). As such, \(Q\)-balls have been constructed in (3 + 1)-dimensional models with non-renormalizable scalar field potential \(^{[5–7]}\), but also appear in supersymmetric extensions of the Standard Model \(^{[8–10]}\). These supersymmetric \(Q\)-balls have been considered as possible candidates for baryonic dark matter \(^{[11]}\) and their astrophysical implications have been discussed \(^{[12]}\).

The self-gravitating counterparts of \(Q\)-balls, so-called “boson stars” have also been discussed extensively \(^{[13–19]}\). Since the discovery of an elementary scalar particle in nature \(^{[20]}\) it is by itself interesting to study these type of objects. Even if observations would exclude them to exist they are described by relatively simple equations and could hence act as simple toy models for a wide range of objects such as particles, compact stars, e.g. neutron stars and even centers of galaxies \(^{[21]}\).

In this paper we are interested in boson stars in the context of (4+1)-dimensional Gauss-Bonnet theory which appears naturally in the low energy effective action of quantum gravity models \(^{[22]}\). An important property of Gauss-Bonnet gravity is that its spectrum does not include new propagating degrees of freedom besides gravitation. In (3 + 1) dimensions, the Gauss-Bonnet term is a total derivative so it only contributes to the field equations when it is coupled to a dilaton field. Here, we are only interested in the effect of Gauss-Bonnet gravity and will hence study these objects in the minimal number of dimensions in which the term does not become a total derivative. In \(^{[23]}\) compact stars consisting of a perfect fluid have been studied in modified
models of gravity, including Einstein-Gauss-Bonnet-dilaton gravity in \((3 + 1)\) dimensions. Let us remark that boson stars are not made out of a perfect fluid since the diagonal spatial components of the energy-momentum tensor are not all equal.

The interest in studying boson stars in higher dimensions is also supported by several other arguments. First of all, higher dimensions appear in attempts to find a quantum description of gravity as well as in unified models. Examples are Kaluza-Klein theories and String Theory. For black holes it became clear that many of their properties in \((3+1)\) dimensions do not extend to higher dimensions. It is therefore natural to consider other localized objects in higher dimensions such as boson stars and see which influence the number of dimensions has on their properties. \(Q\)-balls and boson star solutions of the full system of coupled non-linear equations in \((4 + 1)\)-dimensional asymptotically flat space-time have been investigated in \([24–26]\) and it was indeed found that the behaviour of the solutions depends crucially on the number of spatial dimensions.

Also, if we add a negative cosmological constant we obtain boson stars in \((4 + 1)\)-dimensional Anti-de Sitter (AdS) space-time and according to the AdS/CFT correspondence \([27, 28]\) this theory would correspond to a \((3 + 1)\)-dimensional strongly coupled conformal field theory on the boundary of AdS. As such boson stars have been suggested to be the dual of condensates of glueballs \([29]\) and their properties in \(d + 1\) dimensions have been investigated \([24, 25]\).

Finally, it appears technically simpler to rotate objects in higher dimensions. In the case of \((3+1)\) dimensions rotating boson stars are axially symmetric \([5–7]\), so we need to solve partial differential equations, while in \((4 + 1)\) dimensions we have two planes of rotation. If we choose the two angular momenta associated with the two orthogonal planes equal to each other the symmetry of the solutions can be enhanced and “only” ordinary differential equations need to be solved \([26]\).

Our paper is organized as follows: in Section II we give the model, equations of motion as well as the definition of mass, charge and radius of our solutions. In Section III we discuss our numerical results, while in Section IV we conclude.

II. THE MODEL AND THE EQUATIONS

In this paper we study boson stars in 5-dimensional Gauss-Bonnet gravity. The action reads:

\[
S = \int d^5x \sqrt{-g} \left( R + \alpha \left( R^{MNKL} R_{MNKL} - 4 R^{MN} R_{MN} + R^2 \right) + 16\pi G_5 L_{\text{matter}} \right),
\]

where \(\alpha\) is the Gauss–Bonnet coupling and \(\alpha = 0\) corresponds to Einstein gravity. \(G_5\) is Newton’s constant in 5 dimensions which is connected to the 5-dimensional Planck mass \(M_{\text{pl,5}}\) by \(G_5 = M_{\text{pl,5}}^3\). \(L_{\text{matter}}\) denotes the matter Lagrangian of the complex valued scalar field \(\psi\) which reads:

\[
L_{\text{matter}} = - (\partial_\mu \psi)^* \partial^\mu \psi - U(\psi)
\]

where the scalar potential reads

\[
U(\psi) = m^2 \eta_{\text{susy}}^2 \left( 1 - \exp \left( -\frac{\psi^2}{\eta_{\text{susy}}^2} \right) \right).
\]

This potential can be developed into a series as follows

\[
U(\psi) = m^2 |\psi|^2 - \frac{m^2 |\psi|^4}{2 \eta_{\text{susy}}^2} + \frac{m^2 |\psi|^6}{6 \eta_{\text{susy}}^4} + O(|\psi|^8).
\]

In the following, we will use the potential up to 6th order in \(\psi\). We have checked that the results are qualitatively similar, however, it turns out that the numerics at critical points is easier with a \(\psi^6\) potential.

The gravity equations are obtained from the variation of the action with respect to the metric fields and read

\[
G_{MN} + \frac{\alpha}{2} H_{MN} = 8\pi G_5 T_{MN}, M, N = 0, 1, 2, 3, 4.
\]
where $H_{MN}$ is given by
\[
H_{MN} = 2 \left( R_{MABC} R_{N}^{ABC} - 2 R_{MANB} R^{AB} - 2 R_{MA} R_{N}^{A} + R R_{MN} \right) - \frac{1}{2} g_{MN} \left( R^2 - 4 R_{AB} R^{AB} + R_{ABCD} R^{ABCD} \right), \quad A, B, C = 0, 1, 2, 3, 4 ,
\]
and $T_{MN}$ is the energy-momentum tensor
\[
T_{MN} = g_{MN} \mathcal{L} - 2 \frac{\partial \mathcal{L}}{\partial g^{MN}} = -g_{MN} \left[ \frac{1}{2} g^{KL} \left( \partial_K \psi^* \partial_L \psi + \partial_L \psi^* \partial_K \psi \right) + \partial_M \psi^* \partial_N \psi + \partial_N \psi^* \partial_M \psi \right].
\]
The variation of the action with respect to the matter field leads to the Klein-Gordon equation which reads
\[
\left( \Box - \frac{\partial U}{\partial |\psi|^2} \right) \psi = 0.
\]

The matter Lagrangian $\mathcal{L}_{matter}$ is invariant under the global U(1) transformation
\[
\psi \rightarrow \psi e^{i \chi} .
\]
As such the locally conserved Noether current $j^M$, $M = 0, 1, ..., 4$ associated to this symmetry is given by
\[
j^M = -\frac{i}{2} \left( \psi^* \partial^M \psi - \psi \partial^M \psi^* \right) \quad \text{with} \quad j^M_0 = 0 .
\]
The globally conserved Noether charge $Q$ of the system then reads
\[
Q = -\int d^4 x \sqrt{-g} j^0 .
\]

**A. Ansatz and equations of motion**

We choose the following Ansatz for the metric:
\[
ds^2 = -N(r) \frac{A^2(r) dt^2}{4} + \frac{1}{N(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 + \sin^2 \theta \sin^2 \varphi d\chi^2 \right)
\]
where $N$ and $A$ are functions of $r$ only and $\theta, \varphi, \chi$ are angular coordinates. We further choose
\[
N(r) = 1 - \frac{2n(r)}{r^2} ,
\]
such that $n(\infty)$ will determine the gravitational mass of the solution. For the scalar field we choose
\[
\psi(r, t) = f(r) e^{i \omega t} ,
\]
where $\omega$ is the internal frequency. In general, boson star solutions exist only in a limited parameter range of $\omega$. This parameter range depends on the choice of potential. For our potential we have $\omega \in [0 : 1]$.

We introduce the following rescalings
\[
r \rightarrow \frac{r}{m} , \quad \omega \rightarrow m \omega , \quad \psi \rightarrow \eta_{susy} \psi , \quad n \rightarrow n/m^2 , \quad \alpha \rightarrow \alpha / \sqrt{m}
\]
and find that the equations depend only on the dimensionless coupling constants $\alpha$ and
\[
\kappa = 8 \pi G_5 \eta_{susy}^2 .
\]
The equations of motion are

\[ A' = \frac{2\kappa r^3 (A^2 N f'^2 + \omega^2 f^2)}{3AN^2 (2\alpha - 2\alpha N + r^2)}, \]

\[ N'(r) = 2r \left( \frac{1 - N}{r^2 + 2\alpha(1 - N)} \right) - \frac{2}{3} \frac{\kappa r^3 (1 - e^{-f^2})NA^2 + \omega^2 f^2 + N^2 A^2 f'^2}{r^2 + 2\alpha(1 - N)}, \]

for the metric functions and

\[ (r^3 AN f')' = r^3 A \left( \frac{1}{2} \frac{\partial U}{\partial f} - \frac{\omega^2 f}{NA^2} \right) \]

for the matter field function. Here and in the following the prime will denote the derivative with respect to \( r \).

These equations have to be solved numerically subject to appropriate boundary conditions. We want to construct globally regular solutions with finite energy. At the origin we hence require

\[ f'(0) = 0, \quad n(0) = 0. \]

Moreover, the scalar field function falls off exponentially with

\[ f(r >> 1) \sim \frac{1}{r^2} \exp \left( -\sqrt{1 - \omega^2 r} \right) + ... \]

and we hence require \( f(\infty) = 0 \), while we choose \( A(\infty) = 1 \) (any other choice would just result in a rescaling of the time coordinate).

### B. Mass, charge and radius

As in most cases considered so far the scalar field function falls off exponentially in our case. Hence, there are different possibilities to define the “radius” of our boson star solutions. Let us remark that models with a V-shaped potential have been considered \[30, 31\] that possess compact boson stars with a well-defined outer radius (very similar to those of “standard stars”)\[32–34\]. Here we follow \[35\] and define the radius of the boson star as an averaged radial coordinate

\[ R = \frac{2\pi^2}{Q} \int_0^\infty dr \, r^3 \frac{\omega f^2}{AN}. \]

Furthermore, the explicit expression for the Noether charge reads

\[ Q = 2\pi^2 \int_0^\infty dr \, r^3 \frac{\omega f^2}{AN}. \]

For \( \kappa = 0 \) we have \( A \equiv 1 \) and \( n \equiv 0 \). Then the mass \( M \) corresponds to the integral of the energy density \( \epsilon = -T^0_0 \) and reads

\[ M = 2\pi^2 \int_0^\infty dr \, r^3 \left( N\phi'^2 + \frac{\omega^2 \phi^2}{N} + U(\phi) \right). \]

For \( \kappa \neq 0 \) the same procedure would lead to the following integral

\[ M_I = 2\pi^2 \int_0^\infty dr \, r^3 A \left( N\phi'^2 + \frac{\omega^2 \phi^2}{N} + U(\phi) \right). \]
In the following, we will refer to the mass as to the “inertial mass”. Using the equation of motion (18) this reads

\[ M_I = \frac{6\pi^2}{\kappa} \int_0^\infty dr A n'. \]  

(26)

Clearly, if \( A \equiv 1 \) the inertial mass would be given in terms of \( n(\infty) \). However, in general, we will have \( A \neq 0 \). We hence define the gravitational mass to be given by the asymptotic behaviour of the metric function \( M_G \sim n(r \to \infty)/\kappa \). As shown in [36] this procedure can also be applied in the case of Gauss-Bonnet gravity.

C. Corresponding black hole solutions

The equations of motion (17) - (19) possess black hole solutions for \( f \equiv 0 \) which implies \( A \equiv 1 \). The metric function \( N(r) \) is then given by:

\[ N(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{8\alpha M}{r^4}} \right), \]

(27)

where \( M \) is an integration constant that corresponds to the mass of the solution. This is the Boulware-Deser solution [37]. Note that for \( \alpha \to 0 \) this becomes \( N(r) = 1 - 2M/r^2 \), which is the Tangherlini-Schwarzschild solution in \((4 + 1)\) dimensions [38]. This solution has an event horizon at

\[ r_h = \sqrt{2M - \alpha}. \]  

(28)

The gravitational mass of these black hole solutions is given by \( M_G = \lim_{r \to \infty} n(r)/\kappa = M/\kappa \).

![Graphs showing charge Q as function of ω for different values of α and κ](image)

(a) \( κ = 0.02 \) (b) \( κ = 0.05 \)

FIG. 1: We give the charge \( Q \) as function of \( ω \) for different values of \( \alpha \) and \( κ = 0.02 \) (left) and \( κ = 0.05 \) (right), respectively.

III. NUMERICAL RESULTS

The solutions to the coupled system of nonlinear differential equations are only known numerically. We have solved these equations using the ODE solver COLSYS [39]. The solutions have relative errors on the order of \( 10^{-6} - 10^{-10} \).
In Fig\[1\] and Fig\[2\] we give the charge $Q$ and the gravitational mass $M_G$, respectively, as function of $\omega$ for two different values of $\kappa$ and a range of values of the Gauss-Bonnet parameter $\alpha$. We observe that the maximal value $\omega_{\text{max}}$ up to where the boson stars exist does not depend on the value of $\alpha$ and $\kappa$. This is not surprising since $\omega \to \omega_{\text{max}}$ is the thick wall limit with $f(r) \approx 0$ over all space. Hence, details of the gravity model do not matter in this limit. The thin wall limit, on the other hand, is strongly influenced by the choice of the Gauss-Bonnet coupling. This is clearly seen in Fig[1] and Fig[2]. For small values of $\alpha$ we find that the behaviour is similar to the $\alpha = 0$ limit of Einstein gravity.

The solutions exist down to a minimal value of frequency, $\omega_{\text{min}}$, and from there a second branch of solutions exists extending backwards in $\omega$ up to a critical value $\omega_{\text{cr}}$ where a third branch of solutions and consecutively a spiraling behaviour appears such that the charge $Q$ and the mass $M_G$ are higher on the third branch as compared to the second branch. When increasing the Gauss-Bonnet coupling $\alpha$, we find that this spiraling behaviour disappears for $\alpha$ large enough. The actual value of $\alpha$ where this happens is hard to determine precisely, but all our numerical results indicate that the larger $\kappa$ the larger we have to choose $\alpha$ to see the spiraling behaviour disappear. For $\alpha$ increasing we then find that the solutions on the second branch exist up to a critical value $\omega_{\text{cr}}$ and that from there a third branch extends backwards in $\omega$, but this time the charge and mass of the solutions on the third branch is lower than that of the solutions on the second branch. Hence we observe that the “spiral unfolds”. Finally, we find that increasing $\alpha$ even further we are left with only one branch of solutions.

To understand this pattern in more detail, we have also plotted the value of the scalar field function at the origin, $\phi(0)$, as function of $\omega$. Our results for several values of $\alpha$ are given in Fig[3]. For small $\alpha$, we find the spiraling behaviour characteristic for boson stars in Einstein gravity with $\phi(0) \to \infty$ at the center of the spiral in Fig[1] or Fig[2]. This latter case corresponds to the thin wall limit with $f(r)$ strongly peaked at $r = 0$. For large values of $\alpha$ we find that the range of values of $f(0)$ is limited. While $\omega$ tends to smaller and smaller values $f(0)$ stays nearly constant. The larger $\alpha$ the smaller is this nearly constant value of $f(0)$. In other words: as soon as the Gauss-Bonnet coupling is large enough we cannot construct “thin-wall” boson stars.

Let us remark that the numerical calculations become very tedious at the turning points of the curves and it is difficult to determine the exact values of $\omega_{\text{min}}$ and $\omega_{\text{cr}}$. However, we have plotted our existing results in order to understand the qualitative pattern of the solutions. In Fig[4] we give the value of $\omega_{\text{min}}$ as well as $\omega_{\text{cr}}$, which corresponds to the value of $\omega$ where the second and third branch join, as function of $\alpha$ for two different values of $\kappa$. We find that $\omega_{\text{min}}$ decreases continuously with $\alpha$, while $\omega_{\text{cr}}$ shows a more complicated behaviour. First the value of $\omega_{\text{cr}}$ increases, but can of course not increase without bound since $\omega = 1$ is the limiting value for $\omega$ up to where solutions can exist. Then $\omega_{\text{cr}}$ shows an oscillating behaviour for intermediate values of $\alpha$ and finally decreases.

For large enough $\alpha$ we find that $\omega_{\text{cr}}$ becomes equal to $\omega_{\text{min}}$ such that only one branch of solutions exists.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{We give the gravitational mass $M_G$ as function of $\omega$ for different values of $\alpha$ and $\kappa = 0.02$ (left) and $\kappa = 0.05$ (right), respectively.}
\end{figure}
FIG. 3: We give the central value $f(0)$ as function of $\omega$ for different values of $\alpha$ and $\kappa = 0.02$ (left) and $\kappa = 0.05$ (right), respectively.

FIG. 4: We show $\omega_{\text{min}}$ as well as $\omega_{\text{cr}}$ (which corresponds to the value of $\omega$ where the second and third branch of solutions join) as function of $\alpha$ for two different values of $\kappa$.

In order to have an idea of the sizes of Gauss-Bonnet boson stars in comparison to their Einstein counterparts, we have also computed the radius of these objects using (22). In Fig.5 we shown the value of $R$ in dependence on $\omega$ for different values of $\alpha$ and two values of $\kappa$. We find that the range of radii possible first increases with increasing $\alpha$ with the largest range of possible $R$ at $\alpha \approx 0.3$, but that then the range of possible $R$ is narrowing again until it becomes smaller than in the Einstein gravity case for $\alpha \gtrsim 2$.

A. Physical values and comparison to neutron stars

Here we would like to briefly comment on the physical quantities of boson stars and the relation to neutron stars which we stated above might be well “toy-modeled by the former. First of all, let us make a connection
FIG. 5: We give the radius $R$ as function of the frequency $\omega$ for different values of $\alpha$ and $\kappa = 0.02$ (left) and $\kappa = 0.05$ (right), respectively.

FIG. 6: We give the gravitational mass $M_G$ as function of the radius $R$ for different values of $\alpha$ and $\kappa = 0.02$ (a) and $\kappa = 0.05$ (b), respectively. We compare this with the radius-mass relation for the corresponding black hole solutions existing for $f = 0$. In this latter case, the gravitational mass of the black hole is given in terms of the radius of the event horizon $r_h$ as follows $M_G = M/\kappa = (r_h^2 + \alpha)/(2\kappa)$.

between the dimensionless quantities used in our paper and the physical values (indicated by a subscript “phys” in the following). The mass and radius are

$$M_{\text{phys}} = 10^{27}\text{kg} \left(6.67 \cdot 10^{-11}\text{Nm}^2/\text{kg} \cdot \text{s}^2\right) \left(\frac{m}{10^{-7}\text{eV}}\right) M_G \cdot \kappa,$$

(29)

and

$$R_{\text{phys}} = 10^{-3}\text{km} \left(\frac{10^{-7}\text{eV}}{m}\right) R,$$

(30)

where $m$ is the mass of the scalar field in eV and we have taken into account that Newton’s constant $G_5$ in (4+1) dimensions might have a different value than the one in (3+1) dimensions. As indicated in Fig.6 we find
that the Gauss-Bonnet boson stars are very dense and exceed e.g. the density of black holes. In some sense this
is not surprising, because our boson stars are non-compact objects without a definite radius beyond which the
energy density and pressure become strictly zero. Hence different definitions of the radius can be considered.
However, it is interesting that the mass-radius relation becomes unique for a sufficiently high Gauss-Bonnet
coupling in the sense that a given radius uniquely determines the mass of the boson star. This is different in
the Einstein-Hilbert case, where a spiraling behaviour leads to several boson star solutions with different masses
existing for one given value of the radius. Neutron stars possess (in most cases) a unique relation between the
mass and the radius that has to be determined using the equation of state (see e.g. [40, 41] and references therein).
Interestingly, the qualitative features of the mass-radius relation of neutron stars seem to resemble
some of the mass-radius relations we obtain for large enough values of α (compare e.g. Fig. 5 in this paper
and Fig.2 in [40]). In [23] perfect fluid stars in (3+1)-dimensional Einstein-Gauss-Bonnet-dilaton theory were
considered. The mass-radius diagram looks similar to what we find.

IV. SUMMARY AND OUTLOOK

We have constructed Gauss-Bonnet boson stars in (4+1)-dimensional space-time and have investigated their
properties depending on the coupling constants in the model. Since we were aiming at studying the sole effect
of the Gauss-Bonnet term without taking into account a dilaton (4+1) dimensions is the lowest number of
dimensions where this higher gravity term has an effect. We find that a qualitative change happens when the
Gauss-Bonnet parameter α is large enough. While in Einstein-Hilbert gravity the solutions reach a “thin wall”
limit with the scalar field at the center of the star diverging, we cannot reach a “thin wall” limit for α large
enough. Furthermore, the spiraling behaviour characteristic for boson stars in Einstein gravity disappears for α
large enough and a unique mass-radius relation for the Gauss-Bonnet boson stars is found. Surely, if we want to
compare this to astrophysical objects and promote the idea that boson stars could act as toy models for neutron
stars, we would have to study these effects in (3+1) dimensions. However, this would require an additional
scalar field, the dilaton. It would then be interesting to see whether the qualitative features of the mass-radius
relation that resemble those of “real” neutron stars will still be present. In [23] this was investigated for perfect
fluid stars and constraints on the Gauss-Bonnet parameter were obtained.

It would also be interesting to extend our results to asymptotically Anti-de Sitter (AdS) space-time. In
the context of the AdS/CFT correspondence [23, 28] planar boson stars in AdS have been interpreted as Bose-
Einstein condensates of glueballs [24, 29]. However, these solutions have only been considered in Einstein gravity
which corresponds to the large N limit on the Quantum Field Theory side. Hence it would be interesting to
take higher order curvature corrections into account. Our model would than holographically describe a strongly
coupled Quantum Field theory away from $N = \infty$ on a 3-sphere. This is currently under investigation.

Finally, it would be interesting to study the rotating counterparts of our solutions. In (3+1) dimensions,
rotating boson stars are necessarily axially symmetric and partial differential equations have to be solved. In
(4+1) dimensions two orthogonal planes of rotation exist. If the two angular momenta associated to these
planes would be chosen equal, the symmetry of the system can be enhanced and “only” ordinary differential
equations need to be solved [26].

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