DYNAMIC PRICING OF NETWORK GOODS IN DUOPOLY MARKETS WITH BOUNDEDLY RATIONAL CONSUMERS

HAIYING LIU
Business School of Central South University
Changsha, Hunan 410083, China
and
School of Accountancy
 Hunan University of Finance and Economics
Changsha, Hunan 410205, China

XINXING LUO, WENJIE BI,
YUEMING MAN AND KOK LAY TEO
Business School of Central South University
Changsha, Hunan 410083, China
and
Department of Mathematics and Statistics
Curtin University, Australia

(Communicated by Ada Che)

Abstract. In this paper, we present a dynamic pricing model for two firms selling products displaying network effects for which consumers are with bounded rationality. We formulate this model in the form of differential games and derive the open-loop equilibrium prices for the firms. Then, we show the existence and uniqueness of such open-loop equilibrium prices. The model is further extended to the case with heterogeneous network effects. Their steady-state prices obtained are compared. A numerical example is solved and the results obtained are used to analyze how the steady-state prices and market shares of both firms are influenced by the cost, price sensitivity and the network effects of the products.

1. Introduction. With a rapid advancement of Internet technology and social media, more and more goods and services have displayed network effects such as, online video games, APP software, telecommunication services, social network sites and group buying deals. The consumer’s utility obtained from a good or service increases as more consumers use the same good (Cabral [6], Laussel and Resende [26]). Take the example of online games, when a consumer chooses the online Android game (such as https://play.google.com/store/apps/category/GAME), he/she will tend to consider the number of people having purchased or downloaded each game, as a more popular game tends to have better player community and post-sale service. Meanwhile, online game is an experience goods, meaning that the consumer

2010 Mathematics Subject Classification. Primary: 91A80; Secondary: 91A23.
Key words and phrases. Dynamic pricing, network effect, bounded rationality, duopoly market, differential games.

This work is supported by NSFC NO.91646115/71371191, 71210003, 712221061, PSFC NO.2015JJ2194 and 2015CX010.

1 Corresponding Author Email: beenjoy@126.com

429
will know the quality of the goods only after using it. Moreover, there are consumer
segmentations for an online game, such as experts, amateurs and beginners. They
have different level of sensitivity to network effects. So, the utility of a consumer
purchasing an online game depends on his/her preference gained from learning, the
price and the heterogeneous network effects. We will model this utility in Section
3. Recently, a rapid growth in information technology has enabled many firms to
monitor the demand of their products in real time and adjust prices dynamically
in response to the changes in demand patterns (Levin, McGill, and Nediak [27]).
Many retailers are now re-examining their pricing policies and exploring means of
implementing dynamic pricing approaches that are more responsive to consumer
demand (Elmaghraby and Keskinocak [18], Herbon [22], Khouja [25]). Hence, it is
of great significance for firms to incorporate network effects and consumer’s learning
behavior into their pricing policies in order to attract more consumers and hence
attain more profits. Nowadays, many pricing models of network goods are assumed
to be in a monopolistic market. In this way, explicit consideration of competitors
can be avoided. However, pure monopolistic market or pure competitive market
is rarely observed in reality. So the analysis of oligopolistic, especially duopolistic
dynamic pricing of network goods is required.

The remainder of the paper is organized as follows: Section 2 provides a brief
review of related literatures. In Section 3, the basic model is formulated. We
analyze the equilibrium of our model in Section 4. We then extend it to the case
with heterogeneous network effects in Section 5. We perform numerical examples
in Section 6 and make some useful concluding remarks in Section 7.

2. Literature review. Following seminal works by Farrell and Saloner [19], and
Katz and Shapiro [24], network effects have been extensively studied in the econom-
ic literature (Damiano and Hao [11], Ellison and Fudenberg [16], and see Economides
[15] for a more detailed review). Broadly speaking, there are two kinds of
network effects, global network effects and local network effects. The main differ-
ence is which one causes the consumer’s utility to increase. The former one means
that a consumer’s utility for a good depends on the total consumption while the
later one means that a consumer gains his/her utility if his/her neighbour purchases
the same good (Bloch and Querou [5], Candogan, Bimpikis, and Ozdaglar [8]). In
our model, we consider only global network effects. Monopoly dynamic pricing
models for network goods have been studied in the literature (Cabral, Salant, and
Woroch [7]). For example, Bensaid and Lesne [4] consider pricing for a monopolist
with goods that display positive network externalities and show that subperfect
game equilibrium prices go up as time passes. Dhebar and Oren [12] study the opti-
mal dynamic pricing of a new product or service in the presence of network effects.
This work maximizes the present value of the monopolist’s profit where the dynam-
ics is the demand for network access. However, our model maximizes the discounted
total profits of both firms. More recently, Radner, Radunskaya, and Sundararajan
[29] consider dynamic pricing of network goods when facing boundedly rational
consumers, which is in contrast with the standard notion of a rational-expectation
equilibrium. In their model, only a random fraction of consumers, called inattentive
consumers (Gabaix [20] [21]), pays attention to the price change in each period.

However, pure monopoly rarely exists in real market. So optimal pricing under
competition has received considerable attention recently (Ellison, Fudenberg and
Mbhis [17], Xu and Cai [31], Zhang, Bai and Tang [32]). Furthermore, Anderson
et al. [1] study an oligopolistic price competition and derive the symmetric Nash equilibrium that all firms charge the same price for their products with network effects. And they use the multinomial logit (MNL) model to determine market shares. Doganoglu [13], Mitchell and Skrzypacz [28] model network effects by assuming that the consumer's utility is an increasing function of past market shares and derive the Markov perfect equilibrium for this infinite-period game. Cabral [6] considers a duopolistic dynamic price competition of network goods where the rate of consumers choosing to leave the network is replaced with a constant hazard rate. They derive symmetric equilibrium and prove its existence and uniqueness. Also, they show the monotonicity of the pricing function, meaning that larger networks lead to higher prices.

The articles most relevant to our work are Cabral [6], Chintagunta and Rao [10], and Du, Cooper, and Wang [14]. Chintagunta and Rao [10] study dynamic pricing competition between two firms by formulating a differential game model. The open-loop price paths over time are derived. Du, Cooper, and Wang [14] consider the optimal pricing of multi-products with network effects in both homogeneous products and heterogeneous products settings under monopoly market.

Unlike Du, Cooper, and Wang [14], our paper considers an infinite period dynamic pricing of a single product with network effects under duopoly competition. Moreover, we build a continuous time differential game model, where consumers are with reinforcement learning behavior. On the other hand, Du, Cooper, and Wang [14] study a discrete time optimal model. Hence, the problem formulation, methodology and conclusions in our paper are quite distinct from those reported in the paper by Du, Cooper, and Wang [14]. For the paper by Chintagunta and Rao [10], they do not consider network effects, which is the main focus of our paper. Caplin and Nalebuu [9] provide general conditions under which there exist a pure strategy price equilibrium in oligopoly market with MNL model of product differentiation. However, they also do not consider network effects and learning behavior. Cabral [6] considers dynamic pricing competition with network effects. However, it is assumed that consumer’s preference for network goods is an exogenous random variable, which is fixed once it is determined. It is very different from our work, which involves the dynamics of consumer’s preference updating by using reinforcement learning rule.

In this paper, the consumer’s utility for purchasing product is assumed to be a linear function of the product’s network effects, price and consumer’s preference. The network sensitivity parameter in the subsequent section stands for the strength of that product’s network effects (i.e., how much is the increase of the consumer’s utility when one more person purchases the same network good?). We first consider a homogeneous case in which all consumers have the same network sensitivity parameter. We then derive the open-loop equilibrium steady-state prices for both firms. It clearly shows that the stronger the network effects of the product are, the higher the prices should be charged by the firms. The market demand for each product is determined by the classical logit model. Thus, the market demand evolves dynamically.

The main contribution of the paper is stated below. We consider the dynamic pricing of network goods in a competitive setting while most of the works in the existing literature are focused on the monopolistic market. Furthermore, the dynamic pricing of network goods in a competitive setting, where consumer’s learning behavior are with bounded rationality. It is then extended to the case with heterogeneous network effects, which is more in line with reality.
3. Model formulation.

3.1. Consumers utility for purchasing network goods. We consider a dynamic pricing model of network goods in which two firms compete over an infinite horizon. Denote the two firms in the market as Firm 1 and Firm 2. They each sell a single product that displays network effects, i.e., consumer’s utility gained from purchasing the product is increased as more people use the same product (such as cell phone call). Each of the two firms announces the price $p_i(t)$ for its product at time $t$, $i = 1, 2$. We assume that there are lots of consumers in the market and each one of them chooses only a unit of one firm’s product at each time.

The consumer’s utility for purchasing Firm $i$’s product is assumed to be a linear function of the product’s price, consumer’s preference and the network effects ($i = 1, 2$) as in Du, Cooper, and Wang [14],

$$u_i = \sigma_i + \beta_i \xi_i(t) - \gamma_i p_i(t) + \alpha_i x_i(t) + \varepsilon_i(t).$$

(1)

Table 1 lists all the notations used in the paper.

| Notation | Description |
|----------|-------------|
| $p_i(t)$ | the price of Firm $i$’s product at time $t$ |
| $q_i(t)$ | the probability of a consumer purchases Firm $i$’s product |
| $\xi_i(t)$ | consumer’s time-varying preference for Firm $i$’s product |
| $\beta_i$ | the coefficient of the time-varying preference for Firm $i$’s product |
| $\gamma_i$ | the price sensitivity parameter for Firm $i$’s product |
| $x_i(t)$ | market’s total demand of Firm $i$’s product at time $t$ |
| $\alpha_i$ | the network effects sensitivity parameter for Firm $i$’s product |
| $\dot{\xi}_{is}(t)$ | the rate of change of $\xi_i(t)$ with respect to time $t$ |
| $\varepsilon_i(t)$ | the stochastic utility gained by the consumer for purchasing Firm $i$’s product at time $t$ |
| $\Psi_s$ | the fraction of consumers in segment $s$ |
| $q_is(t)$ | the probability of purchasing Firm $i$’s product for consumers in segment $s$ |
| $\xi_is(t)$ | the time varying preference of Firm $i$’s product for consumers in segment $s$ |

In Eq. (1), $\sigma_i + \beta_i \xi_i(t)$ denotes the total preference of each consumer for Firm $i$’s product at time $t$. The term $\alpha_i x_i(t)$ is the utility that the consumer gains from network effects for purchasing Firm $i$’s product. In this section, we assume that the increase of each consumer’s utility is the same when one more person purchases the same network goods (i.e., there is only one network sensitivity parameter for Firm $i$’s product). The sensitivity of the network effects is different between the two firms due to the heterogeneity of these two kinds of products (i.e., $\alpha_1 \neq \alpha_2$). In Section 4, we will consider the heterogeneous case, where the sensitivity of the network effects differs among consumers. For each $i = 1, 2$, let $\varepsilon_i(t)$ denote the stochastic utility gained by the consumer at time $t$ for purchasing Firm $i$’s product. Each of them is assumed to be i.i.d.

3.2. Brand choice probabilities. The network effects of the product will affect the probability of consumer choosing it. Moreover, the larger the sensitivity of the network effects for the product is (i.e., the larger the magnitude of $\alpha_i$), the more likely the consumers will choose it. Many probability functions, such as logit function, power function and probit function, can model the choice probability...
mentioned above. However, the logit function is widely used in the market research on making brand choices with risk and uncertainty (Anderson, de Palma, and Thisse\textsuperscript{2}, Ben-Akiva and Lerman\textsuperscript{3}). Hence, we use the logit model in this paper. Furthermore, since we assume that the network effects are homogeneous among all consumers, their choice probabilities for firms’ products are the same, so the probability that a consumer purchases Firm 1’s product is

\[ q_1(t) = \frac{\exp(\sigma_1 + \beta_1 \xi_1(t) - \gamma_1 p_1(t) + \alpha_1 x_1(t))}{\sum_{j=1}^2 \exp(\sigma_j + \beta_j \xi_j(t) - \gamma_j p_j(t) + \alpha_j x_j(t))}. \]

On the other hand, the probability that he/she chooses to purchase Firm 2’s product is

\[ q_2(t) = 1 - q_1(t). \]

For simplicity of analysis, the total number of consumers in the market is normalized to be 1, so the total market demand of Firm i’s product is \( x_i(t) = q_i(t) \) (\( i = 1, 2 \)). Then the probability for consumers choosing Firm 1’s product can be rewritten as:

\[ q_1(t) = \frac{\exp(\sigma_1 + \beta_1 \xi_1(t) - \gamma_1 p_1(t) + \alpha_1 q_1(t))}{\sum_{j=1}^2 \exp(\sigma_j + \beta_j \xi_j(t) - \gamma_j p_j(t) + \alpha_j q_j(t))}. \]

### 3.3. The dynamics of consumer’s preference and dynamic pricing model.

In this paper, we assume that network goods are experience goods such as online games and telecommunication services. Here, it is assumed that consumers are boundedly rational, meaning that they can only learn whether or not a brand is of high quality through repeated consumption experience. In other words, consumers are reinforcement learners; they update their preference according to the following rule (Hopkins\textsuperscript{23}),

\[ \xi_i(t + 1) = 1 * q_i(t) + (1 - \delta) \xi_i(t). \]

This updating rule is in discrete time, \( 1 * q_i(t) \) is the expectation value of purchasing Firm i’s product and \( \delta \) is a forgetting (or decay) parameter. In continuous time, consumer’s preference \( \xi_i(t) \) for network goods evolves according to the following dynamic system,

\[ \dot{\xi}_i(t) = q_i(t) - \delta \xi_i(t). \]

where \( \delta \) is the decay rate for Firm i such that \( 0 < \delta < 1 \). The consumer’s initial preference for Firm i’s goods \( \xi_i(0) = 1/2 \) (typically it would be the average market share of Firm i as in Chintagunta and Rao\textsuperscript{10}). This preference updating system is widely used and is intuitively in line with reality. It aggregates preferences among consumers in the level of individuals (see Chintagunta and Rao\textsuperscript{10} for more detailed explanation).

Firm i wants to choose a price path \( p_i(t) \) to maximize its total discounted profit in the long run. We assume that Firm i produces the network goods at a constant marginal cost, \( c_i \). Therefore, the dynamic pricing problem can now be formulated as the following optimization problem (\( i = 1, 2 \)),

\[ \max J_i = \int_0^\infty e^{-rt} q_i(t)(p_i(t) - c_i) \, dt \]

s.t. \( \dot{\xi}_1(t) = q_1(t) - \delta \xi_1(t), \)

\[ \dot{\xi}_2(t) = q_2(t) - \delta \xi_2(t), \]

\( \xi_1(0) = 1/2, \xi_2(0) = 1/2, \)
where \( r \) denotes the discount factor. In order to prove the existence and uniqueness of Nash equilibrium of our dynamic pricing model, the following assumptions are assumed to be satisfied:

**Assumption 1.** The following assumptions hold for all \( t \). (we drop “\( t \)” for ease of exposition.)

a. For each \( i = 1, 2 \), the strategy set of \( p_i \) is nonempty and convex.

b. For each \( i = 1, 2 \), Firm \( i \)'s instantaneous profit \( q_i(p_i - c_i) \) is uniformly bounded, i.e., \( \sup_{p_i} q_i(p_i - c_i) < +\infty \).

4. **Equilibrium analysis.**

4.1. **Steady-state equilibrium prices.** In this subsection, we will analyze the steady-state equilibrium prices. First, we have the following lemma.

**Lemma 1.** For each \( i = 1, 2 \), Firm \( i \)'s instantaneous profit \( q_i(p_i - c_i) \) is quasiconcave in \( p_i \).

**Proof.** Taking the first derivative of \( q_i(p_i - c_i) \) with respect to \( p_i \) yields

\[
\frac{\partial q_i(p_i - c_i)}{\partial p_i} = q_i + (p_i - c_i) \frac{\partial q_i}{\partial p_i} = q_i \gamma_i \left( 1 - \frac{q_j}{1 - (\alpha_i + \alpha_j) q_i q_j} (p_i - c_i) \right)
\]

Let \( k_i = [1 - \gamma_i \frac{q_j}{1 - (\alpha_i + \alpha_j) q_i q_j} (p_i - c_i)] \). Then, there are two cases to consider:

a. \( p_i > 0, k_i \neq 0 \). Note that \( k_i \) is continuous and \( \lim_{p_i \to \infty} k_i = -\infty \). Thus, it follows that \( k_i < 0 \) for all \( p_i > 0 \), which means that \( q_i(p_i - c_i) \) is strictly decreasing in \( p_i \). Therefore, \( q_i(p_i - c_i) \) is quasiconcave.

b. There exists a \( \tilde{p_i} \) such that \( k_i(\tilde{p_i}) = 0 \). Since \( k_i(\tilde{p_i}) = 0 \), \( \frac{\partial q_i(p_i - c_i)}{\partial p_i} = 0 \) for \( p_i = \tilde{p_i} \). Now, by taking the second derivative of \( q_i(p_i - c_i) \) with respect to \( p_i \) evaluated at \( \tilde{p_i} \), we obtain

\[
\frac{\partial^2 q_i(p_i - c_i)}{\partial p_i^2} = \frac{\partial^2 q_i}{\partial p_i^2} \bigg|_{p_i = \tilde{p_i}} + \frac{\partial k_i}{\partial p_i} \bigg|_{p_i = \tilde{p_i}} = 0 + \frac{\partial k_i}{\partial p_i} \bigg|_{p_i = \tilde{p_i}} = 0.
\]

Thus, \( \tilde{p_i} \) is a local maximum of \( q_i(p_i - c_i) \) and there exists no interior minimum for \( p_i > 0 \). Hence, \( \tilde{p_i} \) must be unique. Consequently, Firm \( i \)'s instantaneous profit \( q_i(p_i - c_i) \) increases in \( p_i \in [0, \tilde{p_i}) \) and decreases in \( (\tilde{p_i}, +\infty) \). Thus, it is concave. 

\( \square \)

**Theorem 1.** There exists a unique open-loop equilibrium in this dynamic pricing model.

**Proof.** According to Theorem 2.1 in Vives (30), it follows that, if, for each \( i = 1, 2 \), the strategy set is nonempty, convex and compact, and Firm \( i \)'s instantaneous profit \( q_i(p_i - c_i) \) is continuous with respect to both firms' prices and quasiconcave with respect to its own price, then the unique Nash equilibrium exists. Here, we assume that the firm can charge a price from \([0, +\infty)\) for its network goods, which is clearly not compact. However, \( q_i(p_i - c_i) \) is assumed to be uniformly bounded in Assumption 1. This assumption allows us to restrict Firm \( i \)'s price to lie in a nonempty and compact set. Moreover, \( q_i(p_i - c_i) \) is continuous and quasiconcave in \( p_i \). Hence, there exists a unique open-loop equilibrium for this dynamic pricing model. 

\( \square \)
Now, we calculate the equilibrium strategies under the condition that Firm 1 knows its competitor’s (i.e. Firm 2’s) price strategy and Vice Versa (Open-Loop Equilibrium). To find the equilibrium steady-state price for this game, we formulate the current value Hamiltonian for Firm 1 (the same for Firm 2)

\[ H_1 = q_2(p_1 - c_1) + h_1(q_1 - \delta \xi_1) + m_1(q_2 - \delta \xi_2), \]

where \( h_1 \) and \( m_1 \) are the co-state variables. When the steady state price is reached, the following three conditions must be satisfied.

\[
\frac{\partial H_1}{\partial p_1} = 0, \\
\dot{h}_1 = rh_1 - \frac{\partial H_1}{\partial \xi_1}, \\
\dot{m}_1 = rm_1 - \frac{\partial H_1}{\partial \xi_2}.
\]

Hence, we have (detailed analysis is given in Appendix)

\[
p_1^* = c_1 - \frac{\delta \xi_1^*}{\gamma_1 A^*} - (h_1 - m_1), \\
(r + \delta)h_1 + (p_1 - c_1)\beta_1 A^* + (h_1 - m_1)\beta_2 A^* = 0, \\
(r + \delta)m_1 - (p_1 - c_1)\beta_2 A^* - (h_1 - m_1)\beta_1 A^* = 0,
\]

where \( A^* = \frac{\delta^2 \xi_1^* \xi_2^*}{(\alpha_1 + \alpha_2)\delta ^2 \xi_1^* \xi_2^* - 1} \) and “*” denotes the steady state values.

**Theorem 2.** The equilibrium steady-state prices for Firm 1 and Firm 2 are

\[
p_1^* = c_1 + \xi_1^* \frac{\delta}{\gamma_1} \left[ \frac{1}{\delta^2 \xi_1^* \xi_2^*} - \left( \frac{\beta_1 + \beta_2}{r + \delta} + (\alpha_1 + \alpha_2) \right) \right], \tag{5}
\]

\[
p_2^* = c_2 + \xi_2^* \frac{\delta}{\gamma_2} \left[ \frac{1}{\delta^2 \xi_1^* \xi_2^*} - \left( \frac{\beta_1 + \beta_2}{r + \delta} + (\alpha_1 + \alpha_2) \right) \right], \tag{6}
\]

respectively.

**Proof.** See Appendix. \(\square\)

From Eq. (5) and Eq. (6), it is clear that \( \frac{\partial p_i^*}{\partial \xi_k} < 0 \) (for \( \alpha_k > 0 \), \( k = 1, 2 \)). This means that the stronger the network effects perceived by the consumers purchasing both firms’ products in the market (i.e., the higher the magnitude of \( \alpha_1 + \alpha_2 \)), the lower the price should be charged by the firms when other parameters remain unchanged. The intuition is that consumers’ utility with stronger network effects will be higher than that with lower network effects when there are more people to share the same goods. Thus, to gain more profit under strong network effects, the firms should lower the price to encourage more consumers to purchase the goods. The increment of the profits achieved by increasing demand will exceed the losses caused by decreasing price. Moreover, a firm, which reaches the steady state of a higher consumer’s preference, should charge a higher steady state price if the marginal costs of both firms are assumed to be the same (i.e., when \( \xi_i^* > \xi_j^* \), we have \( p_i^* > p_j^* \)).

Next, we wish to identify the differences between the optimal prices when the two firms behave myopically, and charge the steady-state prices mentioned above. Let \( p_1^{m}, p_2^{m} \) denote the myopic optimal prices charged by Firm 1 and Firm 2, respectively (here and below, the superscript “m” is used to denote “myopic”).
By letting the firm’s discount factor be approaching infinity, we obtain the myopic optimal prices for both firms as given below:

\[ p_i^m = c_1 + \xi_1^m \frac{\delta}{\gamma_1} \left[ \frac{1}{\beta_1^m} - (\alpha_1 + \alpha_2) \right], \]  
(7)

\[ p_i^2 = c_2 + \xi_2^m \frac{\delta}{\gamma_2} \left[ \frac{1}{\beta_2^m} - (\alpha_1 + \alpha_2) \right]. \]  
(8)

These two equations show that although the firms are myopic, they still consider the products network effects of both firms when making optimal pricing strategies. Furthermore, they set the myopic optimal prices higher for their products than the prices when they try to maximize the long-term profits (i.e., \( p_i^m > p_i^*, i = 1, 2 \)).

Note that the existence and uniqueness of the Nash equilibrium in oligopoly market with the logit demand function have been proved by Caplin and Nalebuff [9].

5. Heterogeneous network effects. In this section, we will compare the equilibrium steady-state prices for the two firms in the context of heterogeneous network effects, where network sensitivity differs among consumers. Suppose that there exist \( S \) segments in the market denoted by \( \alpha_{is}, s = 1, 2, \ldots, S; \ i = 1, 2 \). The main difference when compared with previous case is on their heterogeneous network sensitivities among segments. To proceed, it is assumed that the network sensitivity parameter is the same for consumers in the same segment. Let \( \alpha \) denote the network sensitivity parameter of consumers in segment \( s \) and let \( \Psi_s \) be the fraction of consumers in segment \( s \) such that \( \sum_{s=1}^{S} \Psi_s = 1 \). Thus, for \( i = 1, 2 \), the probability of purchasing Firm \( i \)'s product for consumers in segment \( s \) is

\[ q_{is}(t) = \frac{\exp(\sigma_i + \beta_i \xi_{is}(t) - \gamma_i \psi_i(t) + \alpha_{is} \sigma_i(t))}{\sum_{j=1}^{S} \exp(\sigma_j + \beta_j \xi_{js}(t) - \gamma_j \psi_j(t) + \alpha_{js} \sigma_j(t))}, \]

and the evolution of their preference on Firm \( i \)'s product is

\[ \dot{\xi}_{is}(t) = q_{is}(t) - \delta \xi_{is}(t). \]

The total market demand for Firm \( i \)'s product is:

\[ x_i(t) = \sum_{s=1}^{S} \Psi_s q_{is}. \]

Hence, the optimization problem for Firm \( i \) is to maximize its discounted profit

\[ J_i = \int_{0}^{\infty} e^{-rt} \sum_{s=1}^{S} \Psi_s q_{is}(p_i(t) - c_i) \, dt \]

subject to

\[ \dot{\xi}_{is}(t) = q_{is}(t) - \delta \xi_{is}(t), \]

\[ \dot{\xi}_{is}(t) = q_{is}(t) - \delta \xi_{is}(t), \]

\[ q_{is}(t) + q_{2s}(t) = 1, \quad s = 1, 2, \ldots, S. \]

**Theorem 3.** For each \( i = 1, 2 \), the equilibrium steady-state price for Firm \( i \)'s product with heterogeneous consumers network sensitivities is

\[ p_i^* = \frac{\sum_{s=1}^{S} \Psi_s q_{is} \dot{\xi}_{is}^*}{\gamma_1 \left[ \left( \sum_{s=1}^{S} \Psi_s W_s^* \right)^2 - (\beta_1 + \beta_2) \sum_{s=1}^{S} \Psi_s W_s^* \right]}, \]

where \( W_s^* = \frac{\sigma_{is}(\alpha_{is} + \alpha_{ij} + \alpha_{ij} + \alpha_{ij} + \alpha_{ij}) \xi_{is} \xi_{is}^*}{\xi_{is} \xi_{is}^* + 1}. \)

**Proof.** See Appendix.
When $\Psi_1 = 1, \Psi_2 = \Psi_3 = \ldots = \Psi_s = 0$, we obtain the equilibrium steady-state price:
\[
p^*_i = c_i + \xi^*_i \frac{\delta}{\gamma_i} \left[ \frac{1}{\beta \xi^*_i \xi^*_2} - \left( \frac{\beta_1 + \beta_2}{r + \delta} + (\alpha_1 s + \alpha_2 s) \right) \right],
\]
where $i = 1, 2$. Clearly, these results are same as those obtained for the case with homogeneous network sensitivity (see Eq. (5) and Eq. (6)).

6. Numerical examples. To illustrate the results of our model, the following numerical example is considered. We suppose that there are two Android online game companies offering, respectively, Racing Game, named Real Racing 2 and INDY 500 Arcade Racing (data of these two Games is available in the website [https://play.google.com/store/apps/category/GAME](https://play.google.com/store/apps/category/GAME)). Consumers visit the website, and check the number of downloads or reviews of these games, or even try them, then decide which to buy. These two firms, in order to attract consumers to buy their goods, may charge a very low price or offer some basic function free first. Then they will charge more when their Game becomes popular or the new version is released or the consumer needs advanced function. So for these firms, it is a dynamic pricing problem with network effects under competition and the utility of consumers follows our model (1).

We first examine how the ratio of the two costs, the two price sensitivity parameters, the time-varying preference and the two network effects can affect the profits, equilibrium prices, and market shares. Then a comparison study is carried out showing the differences between myopic optimal prices and equilibrium steady-state prices. Finally, the equilibrium steady-state prices of both firms with heterogeneous network sensitivity parameters will be compared with the case having no heterogeneous network sensitivity parameters.

Without lose of generality, we set $r = 0.95, \delta = 0.9, \sigma_1 = \sigma_2 = 1$, and let Firm 2’s parameters fixed, i.e. $c_2 = 1, \gamma_2 = 0.8, \beta_2 = 0.9$ and $\alpha_2 = 0.95$. Then we let Firm 1’s parameters, such as $c_1, \gamma_1, \beta_1$ and $\alpha_1$, be changed from 0.1 to 1 in an increment of 0.01 while keeping other parameters the same as those of Firm 2. The comparison results are shown in Figure 1 to Figure 12.

First, we should point out that, from Figure 1 to Figure 12, it is easy to see that if all the above parameters for the two firms are the same, then the two firms act like one firm sharing the same equilibrium prices, market shares and profits.

Figure 1 shows that the equilibrium prices of both firms decrease with respect to the ratio $c_1 / c_2$ of the network effects (see the explanation given in Section 4). Moreover, Figure 1 and Figure 2 show that when $\alpha_1 < \alpha_2$, we have $p^*_1 < p^*_2$, and $q^*_1 < q^*_2$. It suggests that Firm 2 has a “monopoly” power because the network effects of its goods are stronger. So even Firm 2 charges a higher price, its market share is still higher. For Firm 1, since its network effects are lower, it should charge a lower price than Firm 2 so as to attract more market share as shown in Figure 2.

As shown in Figure 3, when the ratio $c_1 / c_2$ of the cost of the two firms increases, $p^*_1$ increases but $p^*_2$ decreases first and then increases. If $c_1 < c_2$, then $p^*_1 < p^*_2$. Figure 4 shows an opposite where Firm 1’s market share decreases while Firm 2’s market share increases with respect to $c_1$. If $c_1 < c_2$, then $q^*_1 < q^*_2$. It is easy to understand the behavior of Firm 1: when the cost is low, it should charge a lower price to lure more consumers so as to attract a higher market share. This agrees with our intuition. However, Firm 2’s behavior is a little bit different. Since Firm 2’s cost is high ($c_2 = 1$) and Firm 1’s cost is low ($c_1 < c_2$), Firm 2 should charge
a higher price so as to gain some profit. Consequently, it will have a lower market share as shown in Figure 4. On the other hand, when \( c_1 \) is increased, meaning that Firm 2’s cost is decreased when compared with Firm 1, then Firm 2’s equilibrium price should be decreased such that more market share can be attained. Moreover, From Figure 3, it is observed that when the cost of Firm 1 becomes high enough (for example \( c_1 = 0.7 \) such that \( \frac{c_1}{c_2} = 0.7 \)), it loses the advantage of being low cost. In this situation, Firm 2 can increase its price to gain more profit.

Figure 5 and Figure 6 illustrate that the equilibrium prices of both the firms are decreased with respect to the ratio \( \beta_1 \beta_2 \). Moreover, it is observed that when \( \beta_1 < \beta_2 \), then \( p_1^* < p_2^* \), and \( q_1^* < q_2^* \). The reason is similar to the case when the network effects for both of the firms have positive effects on the utility of consumers.

In Figure 7, it is easy to observe that when Firm 2’s price sensitivity parameter \( \gamma_2 \) is fixed, the effect of \( \gamma_1 \) on its equilibrium price is almost negligible. On the other hand, if \( \gamma_1 < \gamma_2 \), then \( p_1^* < p_2^* \). However, Firm 1’s price is decreased with respect to the ratio \( \gamma_1 \gamma_2 \). From Figure 8, it is clearly seen that Firm 1’s market share is decreased with \( \frac{\beta_1}{\beta_2} \) and Firm 2’s market share is increased with it.
Figure 5. The relationship between $p_i^*$ and $\beta_1$ when $\beta_2 = 0.9$

Figure 6. The relationship between $q_i^*$ and $\beta_1$ when $\beta_2 = 0.9$

Figure 7. The relationship between $p_i^*$ with $\gamma_1$ when $\gamma_2 = 0.8$

Figure 8. The relationship between $q_i^*$ with $\gamma_1$ when $\gamma_2 = 0.8$

Figure 9. The relationship between $J_i^*$ and $\alpha_1$ when $\alpha_2 = 0.95$

Figure 10. The relationship between $J_i^*$ and $c_1$ when $c_2 = 1$. 
Figure 9 to Figure 12 show that the firms’ profits will decrease when the ratio of the network effects, the time-varying preference and the price sensitivity are increased. The firm with higher network effects and preference sensitivity will get a higher profit when compared to that with lower network effects and preference sensitivity. However, the situation will be opposite when the firm has higher cost and price sensitivity. It is not surprising to see that the trend is similar to that of the market share $q_i$ and that of the equilibrium price $q_i$ as shown in Figure 1 to Figure 8. This is because the profit depends largely on the market share and equilibrium price.

Figure 13 depicts the differences between myopic optimal price and equilibrium steady-state price of Firm 1. It can be seen that Firm 1 will charge a higher price if it behaves myopically as we discussed in Section 4. Meanwhile, its myopic optimal price and equilibrium price are decreased with respect to the network effects.

Finally, we wish to compare the equilibrium steady-state prices for cases with and without heterogeneous network sensitivity parameters. As in the previous numerical study, the values of the parameters remain unchanged, except the network effects sensitivity parameters of both firms. For simplicity of analysis, we consider a symmetric duopoly case and set $\alpha_i = \alpha_j = \alpha_{i1} = \alpha_{j1} = 0.5, \Psi_1 = \Psi_2 = 0.5, \alpha_{i2} = \alpha_{j2}$.
In these settings, the equilibrium steady-state prices for both firms with homogeneous network effects sensitivities are the same and equal to 3.0275. Figure 14 shows a comparison between the equilibrium steady-state prices for cases with and without heterogeneous network effects sensitivities. It can be clearly seen that the equilibrium steady state prices charged by both firms for heterogeneous case are always greater than those for the homogeneous case. It conforms to the rules observed in Figure 1, where a firm should set a lower price when its network effects sensitivity is increased.

7. Conclusions. In this paper, we consider dynamic pricing policies for two firms producing products that display network effects in duopoly markets. We used the logit demand function along with the network effects sensitivity parameters and the dynamics of consumers preference to characterize this model. We first considered the case that consumer’s network effects sensitivity parameters are homogeneous. Then, we proved the existence and uniqueness of the open-loop Nash equilibrium for this dynamic pricing model and obtained the steady-state prices for both firms. By analyzing the influence of network effects sensitivity on the firm’s steady state price, we concluded that the firm with a higher network effects sensitivity parameter may charge a higher price than its competitor, because a high network effects of their products may give the firm the “monopoly power” (i.e., its market share will increase although the firm charges a higher price to its consumers). We then extended the basic model to the case with heterogeneous network effects and explore its impact on the equilibrium steady-state prices of both firms. Our numerical examples showed that the heterogeneous steady-state price was higher than the homogeneous one.

In this paper, we consider the firms in duopoly markets. However, there are lots of firms who sell products with network effects properties. So future research may consider oligopoly or fully competitive firms in the market. Moreover, we have not derived the feedback equilibrium solutions for firms’ dynamic pricing policies. It is an interesting question yet to be solved.

Appendix.

Proof of Theorem 2.

Proof. The objective function for Firm 1 is:

$$J_1 = \int_0^\infty e^{-rt}q_1(t)(p_1(t) - c_1)dt$$

subject to

$$\dot{\xi}_1(t) = q_1(t) - \delta \xi_1(t),$$

$$\dot{\xi}_2(t) = q_2(t) - \delta \xi_2(t),$$

where the market share of Firm 1 at time t is:

$$q_1(t) = \frac{\exp(\sigma_1 + \beta_1 \xi_1(t) - \gamma_1 p_1(t) + \alpha_1 q_1(t))}{\sum_{j=1}^{2} \exp(\sigma_j + \beta_j \xi_j(t) - \gamma_j p_j(t) + \alpha_j q_j(t))}$$

(10)

In the following, we drop the argument “t” for ease of exposition. The current-value Hamiltonian for Firm 1 is given by

$$H_1 = q_1(p_1 - c_1) + h_1(q_1 - \delta \xi_1) + m_1(q_2 - \delta \xi_2).$$
The necessary conditions for an open-loop Nash equilibrium are:

\[
\frac{\partial H_1}{\partial p_1} = 0, \\
\dot{h}_1 = r h_1 - \frac{\partial H_1}{\partial \xi_1}, \\
\dot{m}_1 = r m_1 - \frac{\partial H_1}{\partial \xi_2}
\]

where \( \frac{\partial q_1}{\partial p_1} = \frac{\partial q_1}{\partial \xi_1} + h_1 \frac{\partial q_1}{\partial p_1} + m_1 \frac{\partial q_2}{\partial p_1} = 0 \).

Substituting Eq. (16) and Eq. (17) into Eq. (13)-Eq. (15) gives

\[
\dot{h}_1 = r h_1 - [(p_1 - c_1) \frac{\partial q_1}{\partial \xi_1} + h_1 \frac{\partial q_1}{\partial \xi_1} - h_1 \beta + m_1 \frac{\partial q_2}{\partial \xi_1}],
\]

\[
\dot{m}_1 = r m_1 - [(p_1 - c_1) \frac{\partial q_1}{\partial \xi_2} + h_1 \frac{\partial q_1}{\partial \xi_2} - m_1 \beta + m_1 \frac{\partial q_2}{\partial \xi_2}].
\]

Let

\[
F(q_1, p_1, \xi_1, \xi_2) = \exp(\sigma_1 + \beta_1 \xi_1 - \gamma_1 p_1 + \alpha_1 q_1) - q_1
\]

where \( q_1 + q_2 = 1 \).

According to the implicit function theorem,

\[
\frac{\partial q_1}{\partial p_1} = -\frac{F_{p_1}}{F_{q_1}}, \quad \frac{\partial q_1}{\partial \xi_1} = -\frac{F_{\xi_1}}{F_{q_1}}, \quad \frac{\partial q_1}{\partial \xi_2} = -\frac{F_{\xi_2}}{F_{q_1}},
\]

\[
F_{q_1} = (\alpha_1 + \alpha_2) q_1 q_2 - 1, \quad F_{p_1} = -\gamma_1 q_1 q_2
\]

\[
F_{\xi_1} = \beta_1 q_1 q_2, \quad F_{\xi_2} = -\beta_2 q_1 q_2
\]

Hence,

\[
\frac{\partial q_1}{\partial p_1} = \gamma_1 A, \quad \frac{\partial q_1}{\partial \xi_1} = -\beta_1 A, \quad \frac{\partial q_1}{\partial \xi_2} = \beta_2 A
\]

where \( A = \frac{q_1 q_2}{(\alpha_1 + \alpha_2) q_1 q_2 - 1} \).

Similarly, we obtain

\[
\frac{\partial q_2}{\partial p_1} = -\gamma_1 A, \quad \frac{\partial q_2}{\partial \xi_1} = \beta_1 A, \quad \frac{\partial q_2}{\partial \xi_2} = -\beta_2 A
\]

Substituting Eq. (16) and Eq. (17) into Eq. (13)-Eq. (15) gives

\[
\frac{\partial H_1}{\partial p_1} = q_1 + (p_1 - c_1) \gamma_1 A + h_1 \gamma_1 A - m_1 \gamma_1 A = 0,
\]

\[
\dot{h}_1 = r h_1 - [(p_1 - c_1) \beta_1 A - h_1 \beta_1 A - h_1 \beta + m_1 \beta_1 A]
\]

\[
= (r + \delta) h_1 + (p_1 - c_1) \beta_1 A + (h_1 - m_1) \beta_1 A
\]

\[
\dot{m}_1 = r m_1 - [(p_1 - c_1) \beta_2 A + h_1 \beta_2 A - m_1 \beta_2 A - m_1 \beta_2 A]
\]

\[
= (r + \delta) m_1 - (p_1 - c_1) \beta_2 A - (h_1 - m_2) \beta_2 A
\]

From Eq. (18), we have

\[
p_1 - c_1 = -\frac{q_1}{\gamma_1 A} - (h_1 - m_1)
\]

So

\[
\dot{h}_1 = (r + \delta) h_1 - \frac{q_1}{\gamma_1 A} \beta_1 A - (h_1 - m_1) \beta_1 A + (h_1 - m_1) \beta_1 A
\]

\[
= (r + \delta) h_1 - \frac{q_1}{\gamma_1} \beta_1
\]
\[ \dot{m}_1 = (r + \delta)h_1 + \frac{q_1}{\gamma_1} \beta_2 A + (h_1 - m_1)\beta_2 A - (h_1 - m_1)\beta_2 A \]  
\[ = (r + \delta)m_1 + \frac{q_1}{\gamma_1} \beta_2 \]  
Taking differentiating of Eq. \((18)\) with respect to time gives  
\[ \frac{1}{\gamma_i} \dot{q}_1((\alpha_1 + \alpha_2)q_2^2 - 1) + \dot{p}_1 + (\dot{h}_1 - \dot{m}_1) = 0 \]  
Substituting \(\dot{h}_1, \dot{m}_1\), given, respectively, by Eq. \((21)\) and Eq. \((22)\), into Eq. \((23)\), we obtain  
\[ \frac{1}{\gamma_1} \dot{q}_1((\alpha_1 + \alpha_2)q_2^2 - 1) - (\beta_1 + \beta_2)q_1 + \dot{p}_1 + (r + \delta)(h_1 - m_1) = 0 \]  
From Eq. \((18)\), it follows that  
\[ h_1 - m_1 = -\frac{q_1}{\gamma_1 A} - (p_1 - c_1) \]  
Therefore, Eq. \((23)\) can be rewritten as:  
\[ \dot{p}_1 - (r + \delta)(p_1 - c_1) + \frac{1}{\gamma_1} \dot{q}_1((\alpha_1 + \alpha_2)q_2^2 - 1) - (\beta_1 + \beta_2 + (r + \delta)(\alpha_1 + \alpha_2))q_1 + \frac{r + \delta}{q_2} = 0 \]  
Similarly, we have  
\[ \dot{p}_2 - (r + \delta)(p_2 - c_2) + \frac{1}{\gamma_2} \dot{q}_2((\alpha_1 + \alpha_2)q_1^2 - 1) - (\beta_1 + \beta_2 + (r + \delta)(\alpha_1 + \alpha_2))q_2 + \frac{r + \delta}{q_1} = 0 \]  
Eq. \((24)\) together with Eq. \((25)\) characterizes the equilibrium price paths for the two firms over time.

Now, by setting \(\dot{p}_1 = \dot{p}_2 = \dot{q}_1 = \dot{q}_2 = \dot{\xi}_1 = \dot{\xi}_2 = 0\), we obtain the equilibrium steady-state prices as given below.

\[ - (r + \delta)(p_1^* - c_1) + \frac{1}{\gamma_1} \frac{r + \delta}{q_2^*} - (\beta_1 + \beta_2 + (r + \delta)(\alpha_1 + \alpha_2))q_1^* = 0, \]  
\[ - (r + \delta)(p_2^* - c_2) + \frac{1}{\gamma_2} \frac{r + \delta}{q_1^*} - (\beta_1 + \beta_2 + (r + \delta)(\alpha_1 + \alpha_2))q_2^* = 0, \]  
\[ q_1^* = \delta \xi_1^*, \quad q_2^* = \delta \xi_2^* \]  
where “*” denote the steady state values.

Moreover, from Eq. \((10)\), we have

\[ q_1^* = \frac{\exp(\sigma_1 + \beta_1 \xi_1^* - \gamma_1 p_1^* + \alpha_1 q_1^*)}{\Sigma_{j=1}^r \exp(\sigma_j + \beta_j \xi_j^* - \gamma_j p_j^* + \alpha_j q_j^*)} \]  
\[ q_1^* + q_2^* = 1 \]  
Now, we can solve the system of Eq. \((26)\)-Eq. \((30)\) for \(p_1^*, p_2^*, q_1^*, q_2^*, \xi_1^*, \xi_2^*\). In particular,

\[ p_1^* = c_1 + \xi_1^* \frac{\delta}{\gamma_1} \left[ \frac{1}{\delta^2 \xi_1^* \xi_2^*} - (\frac{\beta_1 + \beta_2}{r + \delta} + (\alpha_1 + \alpha_2)) \right], \]  
\[ p_2^* = c_2 + \xi_2^* \frac{\delta}{\gamma_2} \left[ \frac{1}{\delta^2 \xi_1^* \xi_2^*} - (\frac{\beta_1 + \beta_2}{r + \delta} + (\alpha_1 + \alpha_2)) \right]. \]
**Proof of Theorem 3.**

*Proof.* The objective function for Firm 1 is:

\[
J_1 = \int_0^\infty e^{-rt} \sum_{s=1}^\infty \psi_s q_{1s}(p_1(t) - c_1) dt
\]

subject to 2S state equations

\[
\dot{\xi}_{1s}(t) = q_{1s}(t) - \delta \xi_{1s}(t), \quad \dot{\xi}_{2s}(t) = q_{2s}(t) - \delta \xi_{2s}(t), \quad s = 1, 2, ...S
\]

where the market share of Firm 1 at time \(t\) is:

\[
q_{1s}(t) = \frac{\exp(\sigma_1 + \beta_1 \xi_{1s}(t) - \gamma_1 p_1(t) + \alpha_1 \sum_{s=1}^S \psi_s q_{1s}(t))}{\sum_{j=1}^S \exp(\sigma_j + \beta_j \xi_{js}(t) - \gamma_j p_j(t) + \alpha_j \sum_{s=1}^S \psi_s q_{js}(t))}
\]

In the following, we drop the argument “\(t\)” for ease of exposition. The current-value Hamiltonian for Firm 1 is given by

\[
H_1 = \sum_{s=1}^S \psi_s q_{1s}(p_1 - c_1) + \sum_{s=1}^S h_{1s}(q_{1s} - \delta \xi_{1s}) + \sum_{s=1}^S m_{1s}(q_{2s} - \delta \xi_{2s})
\]

The necessary conditions for an open-loop Nash equilibrium in the steady state are:

\[
\frac{\partial H_1}{\partial p_1} = 0 \tag{33}
\]

\[
\dot{\xi}_{1s} = \begin{cases} r h_{1s} - \frac{\partial H_1}{\partial q_{1s}} = 0, & s = 1, 2, ...S, \end{cases} \tag{34}
\]

\[
\frac{\partial H_1}{\partial p_1} = \sum_{s=1}^S \psi_s q_{1s} + (p_1 - c_1) \sum_{s=1}^S \psi_s q_{1s} - \sum_{s=1}^S h_{1s}(q_{1s} - \delta \xi_{1s}) + \sum_{s=1}^S m_{1s}(q_{2s} - \delta \xi_{2s}) = 0, \tag{35}
\]

\[
\dot{\xi}_{2s} = \begin{cases} r h_{2s} - \frac{\partial H_1}{\partial q_{2s}} = 0, & s = 1, 2, ...S, \end{cases} \tag{36}
\]

\[
\dot{m}_{1s} = \begin{cases} r m_{1s} - [(p_1 - c_1)\psi_s \frac{\partial q_{1s}}{\partial q_{1s}} + h_{1s}\frac{\partial q_{1s}}{\partial \xi_{1s}} - h_{1s}\gamma_1 - m_{1s}\frac{\partial q_{2s}}{\partial \xi_{1s}}] = 0, & s = 1, 2, ...S, \end{cases} \tag{37}
\]

Let

\[
F(q_{1s}, p_1, \xi_{1s}, \xi_{2s}) = \frac{\exp(\sigma_1 + \beta_1 \xi_{1s}(t) - \gamma_1 p_1(t) + \alpha_1 \sum_{s=1}^S \psi_s q_{1s}(t)) - q_{1s}}{\sum_{j=1}^S \exp(\sigma_j + \beta_j \xi_{js}(t) - \gamma_j p_j(t) + \alpha_j \sum_{s=1}^S \psi_s q_{js}(t))}
\]

where \(q_{1s} + q_{2s} = 1\). According to the implicit function theorem,

\[
\frac{\partial q_{1s}}{\partial p_1} = -F_{q_1s} F_{q_1s}, \quad \frac{\partial q_{1s}}{\partial \xi_{1s}} = -F_{q_1s} F_{q_1s}, \quad \frac{\partial q_{1s}}{\partial \xi_{2s}} = -F_{q_1s} F_{q_1s}
\]

\[
F_{q_1s} = \psi_s (\alpha_1 + \alpha_2) q_{1s} q_{2s} - 1, \quad F_{p_1} = -\gamma_1 q_{1s} q_{2s}, \quad F_{\xi_{1s}} = \beta_1 q_{1s} q_{2s}, \quad F_{\xi_{2s}} = -\beta_2 q_{1s} q_{2s}
\]

Hence,

\[
\frac{\partial q_{1s}}{\partial p_1} = \gamma_1 W_s, \quad \frac{\partial q_{1s}}{\partial \xi_{1s}} = -\beta_1 W_s, \quad \frac{\partial q_{1s}}{\partial \xi_{2s}} = \beta_2 W_s \tag{38}
\]

where \(W_s = \frac{1}{\psi_s (\alpha_1 + \alpha_2) q_{1s} q_{2s} - 1}\).

Similarly, we obtain

\[
\frac{\partial q_{2s}}{\partial p_1} = -\gamma_1 W_s, \quad \frac{\partial q_{2s}}{\partial \xi_{1s}} = \beta_1 W_s, \quad \frac{\partial q_{2s}}{\partial \xi_{2s}} = -\beta_2 W_s \tag{39}
\]

Substituting Eq. (38) and Eq. (39) into Eq. (33)-Eq. (35) gives

\[
\frac{\partial H_1}{\partial p_1} = \sum_{s=1}^S \psi_s q_{1s} + (p_1 - c_1) \sum_{s=1}^S \psi_s q_{1s} + \sum_{s=1}^S h_{1s} \gamma_1 W_s + \sum_{s=1}^S m_{1s}(-\gamma_1 W_s) = 0,
\]
which implies that
\[
\sum_{s=1}^{S} \Psi_{s} q_{1s} + \gamma_{1}(p_{1} - c_{1}) \sum_{s=1}^{S} \Psi_{s} W_{s} + \gamma_{1} \sum_{s=1}^{S} h_{1s} W_{s} = 0 \tag{40}
\]

\[
\dot{h}_{1s} = rh_{1s} - [(p_{1} - c_{1})\Psi_{s}(-\beta_{1} W_{s}) + h_{1s}(-\beta_{1} W_{s}) - h_{1s}\delta + m_{1s}\beta_{1} W_{s}] = 0
\]

Thus,
\[
(r + \delta)h_{1s} + \beta_{1}(p_{1} - c_{1})\Psi_{s} W_{s} + \beta_{1} h_{1s} W_{s} - m_{1s}\beta_{1} W_{s} = 0, s = 1, 2, ..., S. \tag{41}
\]

\[
m_{1s} = rm_{1s} - [(p_{1} - c_{1})\Psi_{s}\beta_{2} W_{s} + h_{1s}\beta_{2} W_{s} - m_{1s}\delta + m_{1s}(-\beta_{2} W_{s})] = 0,
\]

and hence
\[
(r + \delta)m_{1s} - \beta_{2}(p_{1} - c_{1})\Psi_{s} W_{s} - \beta_{2} h_{1s} W_{s} + m_{1s}\beta_{2} W_{s} = 0, s = 1, 2, ..., S. \tag{42}
\]

Combining Eq. (41) and Eq. (42), we have
\[
h_{1s} W_{s} = -\frac{\beta_{1}(p_{1} - c_{1})\Psi_{s} W_{s}^{2}}{r + \delta + (\beta_{1} + \beta_{2})W_{s}}, \tag{43}
\]
\[
m_{1s} W_{s} = -\frac{\beta_{2}(p_{1} - c_{1})\Psi_{s} W_{s}^{2}}{r + \delta + (\beta_{1} + \beta_{2})W_{s}}. \tag{44}
\]

Substituting Eq. (43) and Eq. (44) into Eq. (40) gives
\[
p_{1} = c_{1} - \frac{\sum_{s=1}^{S} \Psi_{s} q_{1s}}{\gamma_{1} \sum_{s=1}^{S} \Psi_{s} W_{s} - (\beta_{1} + \beta_{2}) \sum_{s=1}^{S} \Psi_{s} W_{s}^{2} + (\beta_{1} + \beta_{2}) W_{s}},
\]

where \(W_{s} = \Psi_{s}^{(\alpha_{1s} + \alpha_{2s})} q_{1s}^{\alpha_{2s}} r_{1s}^{\alpha_{2s}} 1 \).

In the steady-state, we have
\[
\dot{\xi}_{1s}(t) = \dot{q}_{1s} - \dot{\xi}_{1s} = 0, \quad \dot{\xi}_{2s}(t) = \dot{q}_{2s} - \dot{\xi}_{2s} = 0, \quad s = 1, 2, ..., S.
\]

Hence, the steady state equilibrium price for Firm 1’ network goods is
\[
p_{1}^{*} = c_{1} - \frac{\delta \sum_{s=1}^{S} \Psi_{s} \xi_{1s}^{*}}{\gamma_{1} \sum_{s=1}^{S} \Psi_{s} W_{s}^{*} - (\beta_{1} + \beta_{2}) \sum_{s=1}^{S} \Psi_{s} W_{s}^{2} + (\beta_{1} + \beta_{2}) W_{s}^{*}},
\]

where \(W_{s}^{*} = \Psi_{s}^{(\alpha_{1s} + \alpha_{2s})} q_{1s}^{\alpha_{2s}} r_{1s}^{\alpha_{2s}} 1 \). Similarly, the steady state equilibrium price for Firm 2’ network goods is
\[
p_{2}^{*} = c_{2} - \frac{\delta \sum_{s=1}^{S} \Psi_{s} \xi_{2s}^{*}}{\gamma_{2} \sum_{s=1}^{S} \Psi_{s} W_{s}^{*} - (\beta_{1} + \beta_{2}) \sum_{s=1}^{S} \Psi_{s} W_{s}^{2} + (\beta_{1} + \beta_{2}) W_{s}^{*}},
\]

where \(\Psi_{1} = 1, \Psi_{2} = \Psi_{3} = ... = \Psi_{S} = 0,\)

\[
p_{1}^{s} = c_{1} + \xi_{1s}^{*} \frac{\delta}{\gamma_{1}} \left[ \frac{1}{\delta^{2} \xi_{1s} \xi_{2s}^{*}} - \left( \frac{\beta_{1} + \beta_{2}}{\alpha_{1s} + \alpha_{2s}} \right) \right],
\]
\[
p_{2}^{s} = c_{2} + \xi_{2s}^{*} \frac{\delta}{\gamma_{2}} \left[ \frac{1}{\delta^{2} \xi_{1s} \xi_{2s}^{*}} - \left( \frac{\beta_{1} + \beta_{2}}{\alpha_{1s} + \alpha_{2s}} \right) \right]
\]

**Acknowledgments.** We would like to thank the referees and the associate editor for their valuable comments and suggestions. This research is supported by National Natural Science Foundation of China (Grant nos. 71371191, 71210003, and 712221061), PSFC NO.2015JJ2194 and Innovation Driven Plan of Central South University NO. 2015CX010.
REFERENCES

[1] S. P. Anderson and A. D. Palma, Multiproduct firms: A nested logit approach, The Journal of Industrial Economics, 40 (1992), 261–276.
[2] S. P. Anderson, A. D. Palma and J. F. Thisse, Discrete Choice Theory of Product Differentiation, MIT press, 1992.
[3] M. E. Ben-Akiva and S. R. Lerman, Discrete Choice Analysis: Theory and Application to Travel Demand, MIT press, 1985.
[4] B. Bensaid and J. P. Lesne, Dynamic monopoly pricing with network externalities, International Journal of Industrial Organization, 14 (1996), 837–855.
[5] F. Bloch and N. Quéré, Pricing in social networks, Games and economic behavior, 80 (2013), 243–261.
[6] L. Cabral, Dynamic price competition with network effects, The Review of Economic Studies, 78 (2011), 83–111.
[7] L. Cabral, D. J. Salant and G. A. Woroch, Monopoly pricing with network externalities, International Journal of Industrial Organization, 17 (1999), 199–214.
[8] O. Candogan, K. Bimpikis and A. Ozdaglar, Optimal pricing in networks with externalities, Operations Research, 60 (2012), 883–905.
[9] A. Caplin and B. Nalebuff, Aggregation and imperfect competition: On the existence of equilibrium, The Econometric Society, 59 (1991), 25–50.
[10] P. K. Chintagunta and V. R. Rao, Pricing strategies in a dynamic duopoly: A differential game model, Management Science, 42 (1996), 1501–1514.
[11] E. Damiano and L. Hao, Competing matchmaking, Journal of the European Economic Association, 6 (2008), 789–818.
[12] A. Dhebar and S. S. Oren, Optimal dynamic pricing for expanding networks, Marketing Science, 4 (1985), 336–351.
[13] T. Doganoglu, Dynamic price competition with consumption externalities, Netnomics, 5 (2003), 43–69.
[14] C. Du, W. L. Cooper and Z. Wang, Optimal Pricing for a Multinomial Logit Choice Model with Network Effects, 2014. Available at SSRN 2477548: http://ssrn.com/abstract=2477548
[15] N. Economides, The economics of networks, International journal of industrial organization, 14 (1996), 673–699.
[16] G. Ellison and D. Fudenberg, Knife-edge or plateau: When do market models tip?, The Quarterly Journal of Economics, 118 (2003), 1249–1278.
[17] G. Ellison, D. Fudenberg and M. Möbius, Competing auctions, Journal of the European Economic Association, 2 (2004), 30–66.
[18] W. Elmaghraby and P. Keskinocak, Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions, Management Science, 49 (2003), 1287–1309.
[19] J. Farrell and G. Saloner, Standardization, compatibility, and innovation, The RAND Journal of Economics, 16 (1985), 70–83.
[20] X. Gabaix, A sparsity-based model of bounded rationality, The Quarterly Journal of Economics, 129 (2014), 1661–1710.
[21] X. Gabaix, Sparse dynamic programming and aggregate fluctuations, manuscript, 2013.
[22] A. Herbon, Dynamic pricing vs. acquiring information on consumers’ heterogeneous sensitivity to product freshness, International Journal of Production Research, 52 (2014), 918–933.
[23] E. Hopkins, Adaptive learning models of consumer behavior, Journal of economic behavior & organization, 64 (2007), 348–368.
[24] M. L. Katz and C. Shapiro, Network externalities, competition, and compatibility, The American economic review, 75 (2014), 424–440.
[25] M. J. Khourja, Optimal ordering, discounting, and pricing in the single-period problem, International Journal of Production Economics, 65 (2000), 201–216.
[26] D. Laussel and J. Resende, Dynamic price competition in aftermarket with network effects, Journal of Mathematical Economics, 50 (2014), 106–118.
[27] Y. Levin, J. McGill and M. Nediak, Dynamic pricing in the presence of strategic consumers and oligopolistic competition, Management Science, 55 (2008), 32–46.
[28] M. F. Mitchell and A. Skrzypacz, Network externalities and long-run market shares, Economic Theory, 29 (2006), 621–648.
[29] R. Radner, A. Radunskaya and A. Sundararajan, Dynamic pricing of network goods with boundedly rational consumers, *Proceedings of the National Academy of Sciences*, 111 (2014), 99–104.

[30] X. Vives, *Oligopoly Pricing: Old Ideas and New Tools*, MIT press, 2001.

[31] X. L. Xu and X. Q. Cai, Price and delivery-time competition of perishable products: Existence and uniqueness of Nash equilibrium, *Journal of Industrial and Management Optimization*, 4 (2008), 843–859.

[32] J. X. Zhang, Z. Y. Bai and W. S. Tang, Optimal pricing policy for deteriorating items with preservation technology investment, *Journal of Industrial and Management Optimization*, 10 (2014), 1261–1277.

Received April 2015; 1st revision August 2015; 2nd revision November 2015.

E-mail address: lisabarry@126.com
E-mail address: star@mail.csu.edu.cn
E-mail address: beenjoy@126.com
E-mail address: manyueming@qq.com
E-mail address: K.L.Teo@curtin.edu.au