ON THE SZEGŐ METRIC

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Abstract. We introduce a new biholomorphically invariant metric based on Fefferman’s invariant Szegő kernel and investigate the relation of the new metric to the Bergman and Carathéodory metrics. A key tool is a new absolutely invariant function assembled from the Szegő and Bergman kernels.

1. Introduction

In this paper we introduce the Szegő metric, which is defined similarly to the Bergman metric using the Szegő kernel instead of the Bergman kernel. The well-known Szegő kernel $S(z, \zeta)$ is a reproducing kernel for $H^2(\partial \Omega)$ (the closure in $L^2(\partial \Omega)$ of the set of holomorphic functions that are continuous up to the boundary); thus

$$f(z) = \int_{\partial \Omega} S(z, \zeta) f(\zeta) \, d\sigma_E(\zeta), \quad \forall f \in H^2(\partial \Omega)$$

where $\sigma_E$ stands for the Euclidean surface measure on $\partial \Omega$. The problem with this definition though is that, unlike the volume measure on $\Omega$, the Euclidean surface measure is not transformed nicely under a biholomorphic mapping. To resolve this issue, Fefferman introduced the Fefferman surface area measure, $\sigma_F$ (p. 259 of [11]).

We define the Szegő metric using the Szegő kernel with respect to the Fefferman surface area measure. Hence it is invariant under biholomorphic mappings.

In section 2, we provide background information on the Fefferman surface measure and define the Szegő metric. In section 3, we introduce a biholomorphically invariant function $SK_\Omega(z, w)$ which serves to compare the Bergman and Szegő kernels and then proceed to use this function to derive a number of asymptotic results relating the Szegő and Bergman metrics. In section 4, we show that the Szegő metric is always greater than or equal to the Carathéodory metric. In section 5 we show that there is no universal upper bound or positive lower bound for the ratio of the Szegő and Bergman metrics.

Standing assumption. We assume throughout this paper that $\Omega = \{\rho < 0\} \subset \subset \mathbb{C}^n$ is a strongly pseudoconvex domain with $C^\infty$ boundary. (We note however that the Szegő kernel and metric discussed in this paper will be naturally interpretable on many other domains; transformation laws such as Propositions 1, 2, 3 and Theorem 1 below will hold with additional hypotheses on $\Phi$ as needed.)

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2. Background

Let $H^2(\Omega)$ be the closure in $L^2(\partial \Omega)$ of $A(\Omega) = \mathcal{O}(\Omega) \cap C(\overline{\Omega})$. Then there exists a sesqui-holomorphic Szegő kernel $S(z, \cdot)$ such that

$$f(z) = \int_{\partial \Omega} S(z, \zeta) f(\zeta) \, d\sigma_F(\zeta), \quad \forall f \in H^2(\Omega)$$

where $\sigma_F$ is the Fefferman measure defined as follows:

$$d\sigma_F \wedge d\rho = c_n \sqrt{-\det \left( \begin{array}{cc} 0 & \rho_k \\ \rho_j & \rho_j \rho_k \end{array} \right)} \, dV$$

or equivalently

$$d\sigma_F = c_n \sqrt{-\det \left( \begin{array}{cc} 0 & \rho_k \\ \rho_j & \rho_j \rho_k \end{array} \right)} \frac{d\sigma_E}{\|d\rho\|},$$

where $\sigma_E$ is the usual Euclidean surface measure and $\rho_j = \frac{\partial \rho}{\partial z_j}$, $\rho_j \rho_k = \frac{\partial \rho}{\partial z_j \partial \overline{z}_k}$.

Note that the surface measure $\sigma_F$ does not depend on the choice of the defining function $\rho$; one can check this letting $\tilde{\rho} = \rho h$, where $h > 0$ is a smooth function, and calculating $d\sigma_F$ with $\tilde{\rho}$.

Remark 1. The constant $c_n$ used above is a dimensional constant which was left unspecified in [11] but has been assigned different values later for convenience in different contexts: for example, $c_n = 2^{2n/(n+1)}$ in [1] and $c_n = 1$ in [14].

Proposition 1. Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Then we have

$$\int_{\partial \Omega_2} |f|^2 \, d\sigma_{F,\Omega_2} = \int_{\partial \Omega_1} |f \circ \Phi|^2 \, \det J_C \Phi \frac{2^{n+1}}{n+1} \, d\sigma_{F,\Omega_1},$$

where $\sigma_{F,\Omega_j}$ denotes the Fefferman measure on $\Omega_j$ for $j = 1, 2$ and $J_C \Phi$ is the complex Jacobian matrix of $\Phi$.

Proof. Recall that $\Phi$ extends to a diffeomorphism between $\overline{\Omega}_1$ and $\overline{\Omega}_2$ [9].

Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping and $\Omega_2 = \{ \rho < 0 \}$. Then we have

$$d\sigma_{F,\Omega_1} \wedge d(\rho \circ \Phi) = c_n \sqrt{-\det \left( \begin{array}{cc} 0 & (\rho \circ \Phi)_{\overline{k}} \\ (\rho \circ \Phi)_j & (\rho \circ \Phi)_{j\overline{k}} \end{array} \right)} \, dV_{\Omega_1}.$$ 

Since

$$\left( \begin{array}{cc} 0 & (\rho \circ \Phi)_{\overline{k}} \\ (\rho \circ \Phi)_j & (\rho \circ \Phi)_{j\overline{k}} \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & J_C \Phi \end{array} \right) \left( \begin{array}{cc} 0 & \rho_k \\ \rho_j & \rho_j \rho_k \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & J_C \Phi \end{array} \right),$$

we get

$$\det \left( \begin{array}{cc} 0 & (\rho \circ \Phi)_{\overline{k}} \\ (\rho \circ \Phi)_j & (\rho \circ \Phi)_{j\overline{k}} \end{array} \right) = \det \left( \begin{array}{cc} 0 & \rho_k \\ \rho_j & \rho_j \rho_k \end{array} \right) \det J_C \Phi |^2.$$
Therefore we have
\[
d\sigma_F^{\partial\Omega_1} \wedge d(\rho \circ \Phi) = c_n |\det J_C \Phi|^{2/(n+1)} \cdot \sqrt{n+1} \left| - \det \begin{pmatrix} 0 & \rho_k \\ \rho_j & \rho_{jk} \end{pmatrix} \right| \det J_K \Phi^{-1}^2 \ dV_{\Omega_2} \\
= c_n |\det J_C \Phi|^{2/(n+1)} \cdot |\det J_C \Phi|^{-2} \cdot \sqrt{n+1} \left| - \det \begin{pmatrix} 0 & \rho_k \\ \rho_j & \rho_{jk} \end{pmatrix} \right| \ dV_{\Omega_2} \\
= |\det J_C \Phi|^{-2n/(n+1)} d\sigma_F^{\partial\Omega_2} \wedge d\rho
\]
and it follows that \(d\sigma_F^{\partial\Omega_2}\) pulls back to \(|\det J_C \Phi|^{2n/(n+1)} d\sigma_F^{\partial\Omega_1}\). 

**Proposition 2.** Let \(\Phi : \Omega_1 \to \Omega_2, \ \Omega_1, \Omega_2 \subset \mathbb{C}^n\) be a biholomorphic mapping. Assume there exists a well-defined holomorphic branch of \((\det J_C \Phi(z))^{n/(n+1)}\) on \(\Omega_1\). Then we have
\[
(2.2) \quad S_{\Omega_1}(z, w) = S_{\Omega_2}(\Phi(z), \Phi(w)) (\det J_C \Phi(z))^{n/(n+1)} \left(\frac{\det J_C \Phi(w)}{\det J_C \Phi(z)}\right)^{n/(n+1)},
\]
where \(S_{\Omega_1}(z, w)\) is the Szegő kernel on \(\Omega_j\) for \(j = 1, 2\).

**Proof.** It is obvious that the right hand side of \((2.2)\) is anti-holomorphic with respect to \(w\), so it will suffice to show that it also satisfies the reproducing property.

Let \(f \in H^2(\Omega_1)\). Then we get
\[
\int_{\partial\Omega_1} S_{\Omega_2}(\Phi(z), \Phi(w)) (\det J_C \Phi(z))^{n/(n+1)} \left(\frac{\det J_C \Phi(w)}{\det J_C \Phi(z)}\right)^{n/(n+1)} f(w) \ d\sigma_F^{\partial\Omega_1}(w) \\
= (\det J_C \Phi(z))^{n/(n+1)} \int_{\partial\Omega_2} S_{\Omega_2}(\Phi(z), \tilde{w}) \left(\frac{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}\right)^{n/(n+1)} \cdot \ f(\Phi^{-1}(\tilde{w})) \ d\sigma_F^{\partial\Omega_2}(\tilde{w}).
\]
Note that \(\left(\frac{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}\right)^{n/(n+1)} \ d\sigma_F^{\partial\Omega_2}(\tilde{w})\) is holomorphic with respect to \(\tilde{w}\) since we have
\[
\left(\frac{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}\right)^{n/(n+1)} |\det J_C \Phi(\Phi^{-1}(\tilde{w}))|^{2n/(n+1)} \\
= \left(\frac{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}{\det J_C \Phi(\Phi^{-1}(\tilde{w}))}\right)^{n/(n+1)} |\det J_C \Phi(\Phi^{-1}(\tilde{w}))|^{-2n/(n+1)} \\
= \left(\det J_C \Phi(\Phi^{-1}(\tilde{w}))\right)^{-n/(n+1)}.
\]
Hence we obtain
\[
\int_{\partial\Omega_1} S_{\Omega_2}(\Phi(z), \Phi(w)) (\det J_C \Phi(z))^{n/(n+1)} \left(\frac{\det J_C \Phi(w)}{\det J_C \Phi(z)}\right)^{n/(n+1)} f(w) \ d\sigma_F^{\partial\Omega_1}(w) \\
= (\det J_C \Phi(z))^{n/(n+1)} \left(\det J_C \Phi(\Phi^{-1}(\Phi(z)))\right)^{-n/(n+1)} f(\Phi^{-1}(\Phi(z))) \\
= f(z)
\]
as required. 

\[\square\]
**Definition 1.** We define the Szegő metric on $\Omega$ at $z$ in the direction $\xi$, $F_S^\Omega(z, \xi)$, as follows:

$$F_S^\Omega(z, \xi) = \left( \sum_{j,k=1}^{n} \frac{\partial^2 \log S_\Omega(z, \xi \xi_k)}{\partial z_j \partial \bar{z}_k} \xi_j \xi_k \right)^{1/2}.$$  

**Remark 2.** Note that one can write $S_\Omega(z, w) = \sum_{\alpha} \phi_\alpha(z) \overline{\phi_\alpha(w)}$ where the $\phi_\alpha$'s form an orthonormal basis of $H^2(\partial \Omega)$. Hence $S_\Omega(z, z)$ is a positive strongly plurisubharmonic function, ensuring that $F_S^\Omega(z, \xi)$ is a genuine Kähler metric. The orthonormal expansion may also be used to show that

$$F_S^\Omega(z, \xi) \geq \gamma_\Omega |\xi|$$

for some positive constant $\gamma_\Omega$.

**Remark 3.** Note that $F_S^\Omega(z, \xi)$ does not depend on the choice of the dimensional constant $c_n$ discussed in Remark 1.

**Proposition 3.** The Szegő metric is invariant under biholomorphic mappings satisfying the hypotheses of Proposition 2, i.e., if $\Phi : \Omega_1 \to \Omega_2$ is such a mapping and $z \in \Omega_1$, $\xi \in T_z \Omega_1$, then

$$F_S^\Omega_1(z, \xi) = F_S^\Omega_2(\Phi(z), J_C \Phi(z) \xi).$$

**Proof.** From (2.2), we have

$$S_{\Omega_1}(z, z) = S_{\Omega_2}(\Phi(z), \Phi(z)) |\det J_C \Phi(z)|^{2n/(n+1)}.$$  

Hence we have

$$\log S_{\Omega_1}(z, z) = \log S_{\Omega_2}(\Phi(z), \Phi(z)) + \frac{n}{n + 1} \left[ \log (\det J_C \Phi(z)) + \log \left( \frac{\det J_C \Phi(z)}{\det J_C \Phi(z)} \right) \right].$$

Let $\Phi(z) = w$. Then

$$\sum_{j,k} \frac{\partial^2 \log S_{\Omega_1}(z, z)}{\partial z_j \partial \bar{z}_k} \xi_j \xi_k = \sum_{j,k} \sum_{l,m} \frac{\partial^2 \log S_{\Omega_2}(\Phi(z), \Phi(z))}{\partial w_l \partial \bar{w}_m} \frac{\partial w_l}{\partial z_j} \frac{\partial \bar{w}_m}{\partial \bar{z}_k} \xi_j \xi_k$$

$$= \sum_{l,m} \frac{\partial^2 \log S_{\Omega_2}(w, w)}{\partial w_l \partial \bar{w}_m} (J_C \Phi(z) \xi)_l \left( J_C \Phi(z) \xi \right)_m. \quad \Box$$

2.1. The Szegő metric on the unit ball. Let $\mathbb{B}^n = \{ \rho = |z|^2 - 1 < 0 \} \subset \mathbb{C}^n$. Then

$$\det \begin{pmatrix} 0 & \rho_k \\ \rho_j & \rho_j \rho_k \end{pmatrix} = -1 \quad \text{on} \ \partial \mathbb{B}^n.$$  

Hence $d\sigma_F^{\partial \mathbb{B}^n} = \frac{\pi}{2} d\sigma_E^{\partial \mathbb{B}^n}$ for $S = \{ |z|^2 = 1 \} \subset \mathbb{C}^n$ and the Szegő kernel for the unit ball in $\mathbb{C}^n$ is given by

$$S(z, \zeta) = \frac{1}{c_n} \frac{(n-1)!}{\pi^n} \frac{1}{(1 - z \cdot \overline{\zeta})^n}. \quad (2.4)$$

One can rewrite (2.4) as follows:

$$\int_{\partial \mathbb{B}^n} S(z, \zeta) f(\zeta) d\sigma_F(\zeta) = \int_{\partial \mathbb{B}^n} \frac{(n-1)!}{\pi^n} \frac{1}{(1 - z \cdot \overline{\zeta})^n} f(\zeta) d\sigma_F(\zeta) = f(z), \quad \forall f \in H^2(\partial \mathbb{B}^n).$$
If we calculate the Szegő metric for $B^n$ at the origin, we get
\[
\log S(z, z) = \log \left( \frac{(n-1)!}{c_n \cdot 2\pi^n} \right) - n \log(1 - |z|^2),
\]
and
\[
\partial^2 \log S(z, z) \bigg|_{z=0} = \begin{cases} 
\frac{n \overline{z}_j z_k}{(1 - |z|^2)^2} & j \neq k \\n\frac{n |z_j|^2}{(1 - |z|^2)^2} + n \frac{1}{(1 - |z|^2)} & j = k
\end{cases}.
\]
Hence we have
\[
(2.5) \quad F_S^{B^n}(0, \xi) = \sqrt{n} |\xi|.
\]

**Remark 4.** Note that the Bergman metric on the unit ball in $\mathbb{C}^n$ evaluated at the origin is given as
\[
(2.6) \quad F_B^{B^n}(0, \xi) = \sqrt{n + 1} |\xi|
\]
and the Kobayashi or Carathéodory metric on the unit ball in $\mathbb{C}^n$ at the origin is given as
\[
(2.7) \quad F_K^{B^n}(0, \xi) = F_C^{B^n}(0, \xi) = |\xi|.
\]
Since all four metrics are invariant under the automorphism group of $B^n$ which acts transitively on $B^n$, relations between the metrics at the origin will propagate throughout $B^n$. In particular, from (2.5), (2.6) and (2.7) we obtain
\[
(2.8) \quad F_S^{B^n}(z, \xi) = \sqrt{n} F_C^{B^n}(z, \xi) = \sqrt{n} F_K^{B^n}(z, \xi) = \sqrt{\frac{n}{n+1}} F_B^{B^n}(z, \xi), \quad \forall z \in B^n.
\]

3. An Invariant Function and Some Boundary Asymptotics

**Theorem 1.** Let
\[
SK_{\Omega}(z, w) = \frac{S_{\Omega}(z, w)^{n+1}}{K_{\Omega}(z, w)^n},
\]
where $S_{\Omega}$ and $K_{\Omega}$ are the Szegő and Bergman kernels on $\Omega$. Then $SK_{\Omega}(z, w)$ is invariant under biholomorphic mappings satisfying the hypotheses of Proposition 2, i.e., if $\Phi : \Omega_1 \rightarrow \Omega_2$ is such a mapping then we have
\[
SK_{\Omega_1}(z, w) = SK_{\Omega_2}(\Phi(z), \Phi(w)).
\]

**Proof.** It is a well-known fact (see for example section 6.1 in [7]) that
\[
(3.2) \quad K_{\Omega_1}(z, w) = (\det J_C \Phi(z)) K_{\Omega_2}(\Phi(z), \Phi(w)) \left( \det \overline{J_C \Phi(w)} \right).
\]
Hence from (2.2) and (3.2), we get
\[
\frac{S_{\Omega_1}(z, w)^{n+1}}{K_{\Omega_1}(z, w)^n} = \frac{S_{\Omega_2}(\Phi(z), \Phi(w))^{n+1} (\det J_C \Phi(z))^n (\det \overline{J_C \Phi(w)})^n}{K_{\Omega_2}(\Phi(z), \Phi(w))^n (\det J_C \Phi(z))^n (\det \overline{J_C \Phi(w)})^n} = \frac{S_{\Omega_2}(\Phi(z), \Phi(w))^{n+1}}{K_{\Omega_1}(\Phi(z), \Phi(w))^n}.
\]
Remark 5. One can easily calculate $SK_{\mathbb{B}^n}(z, z)$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$, using (2.4) and the well known formula

$$K_{\mathbb{B}^n}(z, w) = \frac{1}{\pi^n (1 - z \cdot \overline{w})^{n+1}}$$

for the Bergman kernel on the unit ball to obtain

$$SK_{\mathbb{B}^n}(z, z) = \frac{(n - 1)!}{c_n} (n \pi)^n, \quad \forall z \in \mathbb{B}^n.$$

For the remainder of this section we assume that the defining function $\rho$ for $\Omega$ has been chosen to satisfy Fefferman’s approximate Monge–Ampère equation

$$- \det \begin{pmatrix} 0 & \rho \nabla \\ \rho_j & \rho_{jk} \end{pmatrix}_{1 \leq j, k \leq n} = 1 + O \left( |\rho|^{n+1} \right)$$

(see [10] – we could also use the not-completely-smooth exact solution to this equation [6, 17]).

We set $r = -\rho$; thus $r > 0$ in $\Omega$.

We have the following asymptotic expansions of the Bergman and Szegő kernels (see [9, 12, 14] and additional references cited in these papers, but the material we are quoting is set forth especially clearly in section 1.1 and Lemma 1.2 from [15]):

$$K_\Omega(z, z) = \begin{cases} \frac{n!}{\pi^n r^{n+1}} + \frac{(n-1)! q_\Omega}{c_{n-1} r^{n-2}} + O \left( \frac{1}{r^{n-2}} \right), & n \geq 3 \\ \frac{2}{c_3 r^3} + 3 \tilde{q}_\Omega \cdot \log r + O \left( 1 \right), & n = 2 \end{cases}$$

$$S_\Omega(z, z) = \begin{cases} \frac{(n-1)!}{c_{n-1} r^{n-2}} + \frac{(n-3)! q_\Omega}{c_{n-3} r^{n-4}} + O \left( \frac{1}{r^{n-3}} \right), & n \geq 4 \\ \frac{2}{c_3 r^3} + \frac{2}{c_2 r^2} + \frac{\mu_1 + \frac{\tilde{q}_\Omega}{c_2}}{r^2} \cdot \log r + O \left( r \right), & n = 2 \end{cases}$$

where $\mu_1 \in C^\infty(\overline{\Omega})$ and $q_\Omega$ and $\tilde{q}_\Omega$ are certain local geometric boundary invariants – in terms of Moser’s normal form [8] we have $q_\Omega = \frac{2}{3 \pi} \left\| A_{22}^0 \right\|^2$ for $n \geq 3$ and $\tilde{q}_\Omega = -\frac{8}{\pi^2} A_{14}^0$ for $n = 2$. Moreover, $r^{n+1}K_\Omega(z, z) \in C^{n+1-\epsilon}(\overline{\Omega})$ and $r^nS_\Omega(z, z) \in C^{\max\{n, 3\}-\epsilon}(\overline{\Omega})$ for each $\epsilon > 0$. (The remainder terms are equal to a power of $r$ times a first-degree polynomial in $\log r$ with coefficients in $C^\infty(\overline{\Omega})$; later in this section the remainder terms have a similar structure but with higher degree in $\log r$.)

Combining these results we obtain the following.

**Theorem 2.** The function $SK_\Omega(z, z)$ satisfies

$$SK_\Omega(z, z) \in \begin{cases} C^{n-\epsilon}(\overline{\Omega}), & n \geq 3 \\ C^{4-\epsilon}(\overline{\Omega}), & n = 2 \end{cases}$$
with asymptotics

\[ SK_{\Omega}(z, z) = \begin{cases} \frac{(n-1)!}{c_n^n} \frac{1}{(n\pi)^n} + \frac{(n-3)!}{c_n^n} \frac{3q_{\Omega}}{n} r^2 + O(r^3), & n \geq 4 \\
\frac{2}{c_3^3(3\pi)} \frac{2}{r^2} + \frac{9q_{\Omega}}{c_3^3} + O(r^3 \log r), & n = 3 \\
\frac{2}{c_2^2\pi^2} + \mu_2 r^2 + \mu_3 r^4 \log r + \frac{3\pi^2 q_{\Omega}}{c_2^2} r^6 \log^2 r + O(r^6 \log r), & n = 2 \end{cases} \]

for \( z \) close to the boundary, where \( \mu_2, \mu_3 \in C^\infty(\Omega) \).

We will use this result to examine the relation between the Bergman and Szegő metrics. It will be helpful to introduce the quantity

\[ E(z, \xi) = (n+1) \left( F_{S}^{\Omega}(z, \xi) \right)^2 - n \left( F_{B}^{\Omega}(z, \xi) \right)^2. \]

**Theorem 3.** For \( n \geq 3 \) the following hold.

(a) \( E \in C^{n-2-\epsilon}(T\Omega) \).

(b) There are constants \( 0 < m_{\Omega} < M_{\Omega} < \infty \) so that

\[ m_{\Omega} F_{S}^{\Omega}(z, \xi) \leq F_{B}^{\Omega}(z, \xi) \leq M_{\Omega} F_{S}^{\Omega}(z, \xi) \]

on \( T\Omega \).

(c) \( E(z, \xi) = 0 \) when \( z \in \partial\Omega \) and \( \xi \) lies in the maximal complex subspace of \( T_{z}\partial\Omega \).

(d) \( E(z, \xi) \equiv 0 \) on all of \( T(\partial\Omega) \) if and only if the boundary is locally spherical.

(e) If \( \Omega \) is simply connected then \( E(z, \xi) \equiv 0 \) on \( T\Omega \) if and only if \( \Omega \) is biholomorphic to the ball.

**Proof.** We start by noting that

\[ E(z, \xi) = \sum_{j,k=1}^{n} \frac{\partial^2 \left( \log SK_{\Omega}(z, z) \right)}{\partial z_j \partial \overline{z}_k} \xi_j \overline{\xi}_k. \]  

Then (a) follows from the smoothness result in Theorem 2. Statement (b) then follows from (a) and (2.3) along with the Bergman version of (2.3).

For \( z \in \partial\Omega \) we use (3.3) and Theorem 2 to conclude that

\[ E(z, \xi) = \frac{6\pi^n q_{\Omega}}{(n-1)(n-2)} \sum_{j,k=1}^{n} r_j r_k \xi_j \overline{\xi}_k \]

and thus \( E(z, \xi) = 0 \) when \( \sum_{j=1}^{n} r_j \xi_j = 0 \), verifying (c). From the same computation we see that \( E \) will vanish on all of \( T(\partial\Omega) \) if and only if the invariant \( q_{\Omega} \) vanishes identically, so from Corollary 2.5 in [5] it follows that (d) holds.

The “if” half of (e) follows from (2.8) and the invariance properties. The “only if” half follows from (d) along with Theorem C in [7] (see also [18] and section 8 of [4]).

For \( n = 2 \) we have instead the following result.

**Theorem 4.** For \( n = 2 \) the following hold.

(a) \( E \in C^{2-\epsilon}(T\Omega) \).
There are constants $0 < m_\Omega < M_\Omega < \infty$ so that
\[ m_\Omega F_\Omega^\Omega(z, \xi) \leq F_\Omega^\Omega(z, \xi) \leq M_\Omega F_\Omega^\Omega(z, \xi) \]
on $T\Omega$.
(c) If $E \in C^4(T\overline{\Omega})$ then the boundary is locally spherical.
(d) If $\Omega$ is simply connected then $E \in C^4(T\overline{\Omega})$ if and only if $\Omega$ is biholomorphic to the ball, in which case we in fact have $E(z, \xi) \equiv 0$ on $T\overline{\Omega}$.

**Proof.** We need to explain part (c), everything else falling into place as before.

If $E \in C^4(T\overline{\Omega})$ then the $r^6 \log^2 r$ term from the expansion in Theorem 2 must disappear, forcing $\tilde{q}_\Omega \equiv 0$. Using an argument of Burns appearing as Theorem 3.2 in Graham's paper [12] along with the previously cited material from [15] we obtain revised expansions

\[
\begin{align*}
K_\Omega(z, z) &= \frac{2}{\pi^2 r^3} + \mu_4 + 9 q_\Omega^* r \log r + O(r) \\
S_\Omega(z, z) &= \frac{1}{c_2 r^2} + \mu_5 + \frac{1}{c_2} q_\Omega^* r^2 \log r + O(r^2) \\
SK_\Omega(z, z) &= \frac{1}{c_2^2 r^2} + \mu_6 - \frac{3}{c_2^2} q_\Omega^* r^4 \log r + O(r^4) \\
E(z, \xi) &= \mu_7 - 72 q_\Omega^* r^2 \log r \sum_{j,k=1}^n r_j r_k \xi_j \xi_k + O(r^2),
\end{align*}
\]

where $q_\Omega^* = \frac{8}{15\pi^2} |A_{24}|^2$ and $\mu_4, \mu_5, \mu_6 \in C^\infty(\Omega)$, $\mu_7 \in C^\infty(T\overline{\Omega})$. Our smoothness assumption on $E$ now forces $q_\Omega^* \equiv 0$ and this in turn implies that the boundary is spherical. (We note that by Proposition 1.9 in [12], the condition $\tilde{q}_\Omega \equiv 0$ alone does not guarantee that the boundary is spherical.)

For the sake of completeness we also record the corresponding results in one dimension.

**Theorem 5.** For $n = 1$ the following hold.

(a) $E \in C^\infty(T\overline{\Omega})$.
(b) There are constants $0 < m_\Omega < M_\Omega < \infty$ so that
\[ m_\Omega F_\Omega^\Omega(z, \xi) \leq F_\Omega^\Omega(z, \xi) \leq M_\Omega F_\Omega^\Omega(z, \xi) \]
on $T\Omega$.
(c) If $\Omega$ is simply connected then $E(z, \xi) \equiv 0$.

**Proof.** (a) follows from Theorem 23.2 in [2] and the well-known fact that $r S_\Omega(z, z)$ and $r^2 K_\Omega(z, z)$ are in $C^\infty(\overline{\Omega})$ and are nowhere vanishing on $\overline{\Omega}$. Statement (b) follows from (a) as in the proof of Theorem 3 above.

(c) follows from (2.8), invariance properties and the Riemann mapping theorem.  

4. Comparison with the Carathéodory metric

In this section we discuss the comparison between the Carathéodory and Szegő metrics and show that the Szegő metric is always greater than or equal to the Carathéodory metric. The proof follows the same method that was used to show that the Bergman metric is greater than or equal to the Carathéodory metric in [13].

We define the Carathéodory metric on a domain \( \Omega \subset \mathbb{C}^n \) at \( p \in \Omega \) in the direction \( \xi \in \mathbb{C}^n \), as

\[
F^\Omega_C(p, \xi) = \sup \left\{ \left( \sum_{j=1}^{n} \left| \frac{\partial \phi(p)}{\partial z_j} \xi_j \right|^2 \right)^{1/2} : \phi \in \mathcal{O}(\Omega, \Delta), \phi(p) = 0 \right\},
\]

where \( \mathcal{O}(\Omega, \Delta) \) denotes the set of holomorphic mappings from \( \Omega \) to \( \Delta \), the unit disc in \( \mathbb{C} \).

**Theorem 6.** The Szegő metric is greater than or equal to the Carathéodory metric.

**Proof.** One can show that

\[
(F^\Omega_S(p, \xi))^2 = \sup \left\{ |\xi g(p)|^2 : g \in H^2(\partial \Omega), g(p) = 0, \|g\|_{L^2(\partial \Omega)} = 1 \right\}
\]

using the Hilbert space method. Refer to Theorem 6.2.5 in [16] for further details.

Let \( p \in \Omega \). We have

\[
\|S(\cdot, p)\|_{L^2(\partial \Omega)}^2 = \|S(p, \cdot)\|_{L^2(\partial \Omega)}^2 = \int_{\partial \Omega} \frac{S(p, \zeta) S(p, \zeta) d\sigma_F(\zeta)}{S(p, \zeta)} = S(p, p) = S(p, p).
\]

Let \( \phi : \Omega \rightarrow \Delta \) be a holomorphic function with \( \phi(p) = 0 \). Define a holomorphic function \( g : \Omega \rightarrow \Delta \) as follows:

\[
g(z) = \frac{S(z, p)}{\sqrt{S(p, p)}} \phi(z).
\]

Then \( \|g\|_{L^2(\partial \Omega)} \leq 1 \) and \( g(p) = 0 \). Hence from (4.1) we get

\[
(F^\Omega_S(p, \xi))^2 \geq \frac{|\xi g(p)|^2}{S(p, p)} = \frac{S(p, p)^2 |\xi \phi(p)|^2}{S(p, p)^2} = |\xi \phi(p)|^2.
\]

Therefore we get \( F^\Omega_S(p, \xi) \geq F^\Omega_C(p, \xi) \). \( \square \)

**Remark 6.** This argument works on any smoothly bounded pseudoconvex domain where the Szegő metric is defined.

**Remark 7.** The equation (2.8) shows that the inequality \( F^\Omega_S(p, \xi) \geq F^\Omega_C(p, \xi) \) is sharp even in some cases where \( F^\Omega_C(p, \xi) > 0 \), whereas we have \( F^\Omega_B(p, \xi) \geq F^\Omega_C(p, \xi) \) if \( F^\Omega_C(p, \xi) > 0 \) [16].
5. Comparison with the Bergman metric

In this section we carry out some computations on annuli to show that the constants $m_\Omega$ and $M_\Omega$ in Theorem 5 must depend on $\Omega$.

**Theorem 7.** There are no constants $0 < m < M < \infty$ independent of $\Omega$ with the property that

$$m F_\Omega^S(z, \xi) \leq F_\Omega^B(z, \xi) \leq M F_\Omega^S(z, \xi)$$

on $T\Omega$.

**Proof.** The results of Proposition 4 below show that

$$F_\Omega^B(\sqrt{r}, 1) / F_\Omega^B(\sqrt{r}, 1) \to \infty$$

and

$$F_\Omega^B(\sqrt{r}, 1) / F_\Omega^B(\sqrt{r}, 1) \to 0$$

as $r \to 0$, where $\Omega_r = \{ z \in \mathbb{C} : r < |z| < 1 \}$. \hfill \Box

**Proposition 4.** Let $\Omega_r = \{ r < |z| < 1 \} \subset \mathbb{C}$ and $r \in (0, 1)$. We have

$$\lim_{r \to 0} \frac{F_\Omega^B(\sqrt{r}, 1)}{\sqrt{\log(1/r)}} = 2 \quad \text{and} \quad \lim_{r \to 0} \sqrt{r} \cdot F_\Omega^B(\sqrt{r}, 1) = \frac{1}{2}.$$  

Also,

$$\lim_{r \to 0} \frac{F_\Omega^B(\sqrt{r}, 1)}{\sqrt{\log(1/r)}} = \sqrt{2} \quad \text{and} \quad \lim_{r \to 0} F_\Omega^B(\sqrt{r}, 1) = 1.$$  

**Proof.** On the boundary of a planar domain, the Fefferman measure is $c_1^2 ds$, where $ds$ denotes the element of arclength. In view of Remark 3 we may set $c_1 = 2$ so that $d\sigma_F = ds$.

The Szegő and Bergman spaces of $\Omega_r$ admit orthonormal bases $\{a_n(r)z^n\}_{n \in \mathbb{Z}}$ and $\{b_n(r)z^n\}_{n \in \mathbb{Z}}$ with $a_n(r)$ and $b_n(r) \geq 0$; thus $B_r(z, \zeta) = \sum_{n \in \mathbb{Z}} (b_n(r))^2 z^n \zeta^n$ and

$$S_r(z, \zeta) = \sum_{n \in \mathbb{Z}} (a_n(r))^2 z^n \zeta^n.$$  

One can calculate $a_n(r)$ and $b_n(r)$ as follows:

$$\int_{\partial \Omega_r} |a_n(r)z^n|^2 ds = \int_{|z|=r} r^{2n} |a_n(r)|^2 ds + \int_{|z|=1} |a_n(r)|^2 ds$$

$$= |a_n(r)|^2 2\pi (r^{2n+1} + 1) = 1,$$

hence

$$|a_n(r)|^2 = \frac{1}{2\pi(1 + r^{2n+1})}, \quad n \in \mathbb{Z}.$$
Also we have
\[
\int_{\Omega_r} |b_n(r)z^n|^2 \, dA = \int_0^{2\pi} \int_r^1 |b_n(r)|^2 t^{2n} \, dt \, d\theta = |b_n(r)|^2 2\pi \frac{1}{2n+2} (1 - r^{2n+2}) = 1, \quad n \neq -1,
\]
\[
\int_{\Omega_r} |b_{-1}(r)z^{-1}|^2 \, dA = \int_0^{2\pi} \int_r^1 |b_{-1}(r)|^2 \frac{1}{t} \, dt \, d\theta = |b_{-1}(r)|^2 2\pi \ln(1/r) = 1,
\]
and so
\[
|b_n(r)|^2 = \begin{cases} \frac{n+1}{\pi} \cdot \frac{1}{1-r^{2n+2}}, & n \neq -1, \\ \frac{1}{2\pi \ln(1/r)}, & n = -1. \end{cases}
\]

Let $B_r(z, \zeta)$ and $S_r(z, \zeta)$ be the Bergman and Szegő kernel on $\Omega_r$ respectively and $z \in \Omega_r$. We have
\[
(F_{B}^{\Omega_r}(z, 1))^2 = \frac{\partial^2 \log B_r(z, z)}{\partial z \partial \overline{z}} = \frac{B_r(z, z) \cdot (B_r(z, z))_z - (B_r(z, z))_{\overline{z}}^2}{(B_r(z, z))^2} = \frac{\beta_0(z, r) \cdot \beta_2(z, r) - |\beta_1(z, r)|^2}{(\beta_0(z, r))^2},
\]
where
\[
\beta_0(z, r) = B_r(z, z), \quad \beta_1(z, r) = (B_r(z, z))_z, \quad \text{and} \quad \beta_2(z, r) = (B_r(z, z))_{\overline{z}}.
\]
We also get
\[
(F_{S}^{\Omega_r}(z, 1))^2 = \frac{\alpha_0(z, r) \cdot \alpha_2(z, r) - |\alpha_1(z, r)|^2}{(\alpha_0(z, r))^2},
\]
where
\[
\alpha_0(z, r) = S_r(z, z), \quad \alpha_1(z, r) = (S_r(z, z))_z, \quad \text{and} \quad \alpha_2(z, r) = (S_r(z, z))_{\overline{z}}.
\]
Let us calculate $\alpha_j(r^q, r)$ for $j = 0, 1, 2$, $q > 0$ and estimate $F_{S}^{\Omega_r}(r^q, 1):
\[
2\pi \alpha_0(r^q, r) = \sum_{n \in \mathbb{Z}} \frac{1}{(1 + r^{2n+1})^{2nq}},
\]
\[
2\pi \alpha_1(r^q, r) = \sum_{n \in \mathbb{Z}} \frac{1}{(1 + r^{2n+1})} \cdot n \cdot r^{(2n-1)q},
\]
\[
2\pi \alpha_2(r^q, r) = \sum_{n \in \mathbb{Z}} \frac{1}{(1 + r^{2n+1})} \cdot n^2 \cdot r^{2(n-1)q}.
\]
Note that

\[ 2\pi \alpha_0(\sqrt{r}, r) = \frac{2}{1 + \sqrt{r}} + \frac{2r}{1 + r^3} + O(r^2), \]
\[ 2\pi \alpha_1(\sqrt{r}, r) = \frac{-1}{\sqrt{r}(1 + r)} - \frac{\sqrt{r}}{1 + r^3} + O(r^{3/2}), \]
\[ 2\pi \alpha_2(\sqrt{r}, r) = \frac{1}{r(1 + r)} + \frac{5}{1 + r^3} + O(r), \]

and that

\[ 2\pi \alpha_0(\sqrt{5\sqrt{r}}, r) = \frac{1}{1 + \sqrt{r}} + \frac{2}{1 + r^3} + O(r^2), \]
\[ 2\pi \alpha_1(\sqrt{5\sqrt{r}}, r) = \frac{-1}{\sqrt{5\sqrt{r}(1 + r)}} - \frac{\sqrt{5\sqrt{r}}}{1 + r^3} + O(r^{3/2}), \]
\[ 2\pi \alpha_2(\sqrt{5\sqrt{r}}, r) = \frac{1}{r(1 + r)} + \frac{5}{1 + r^3} + O(r), \]

which one can verify easily using the comparison test with the geometric series. Therefore we get

\[ \lim_{r \to 0} r \cdot (F_{S}^{\Omega}(\sqrt{r}, 1))^2 = \frac{1}{4}, \quad \text{and} \quad \lim_{r \to 0} (F_{S}^{\Omega}(\sqrt{r}, 1))^2 = 1. \]

One can calculate \( \beta_j(r^q, r) \)'s for \( j = 1, 2, 3 \) and estimate \( F_{B}^{\Omega}(r^q, 1) \) in a similar way:

\[ \pi \beta_0(r^q, r) = \left( \sum_{n \in \mathbb{Z} \setminus \{1\}} \frac{(n + 1)}{1 - r^{2n+2}} n^{2} \right) + \frac{1}{2r^{2} \log (1/r)}, \]
\[ \pi \beta_1(r^q, r) = \left( \sum_{n \in \mathbb{Z} \setminus \{1\}} \frac{(n + 1)}{1 - r^{2n+2}} n \cdot r^{(2n-1)} \right) - \frac{1}{2r^{3} \log (1/r)}, \]
\[ \pi \beta_2(r^q, r) = \left( \sum_{n \in \mathbb{Z} \setminus \{1\}} \frac{(n + 1)}{1 - r^{2n+2}} n^{2} r^{(2n-2)} \right) + \frac{1}{2r^{4} \log (1/r)}. \]

We have

\[ \pi \beta_0(\sqrt{r}, r) = \frac{1}{2r \log (1/r)} + \frac{2}{1 - r^2} + O(r), \]
\[ \pi \beta_1(\sqrt{r}, r) = -\frac{1}{2r^{3/2} \log (1/r)} - \frac{2}{\sqrt{r}(1 - r^2)} + O(\sqrt{r}), \]
\[ \pi \beta_2(\sqrt{r}, r) = \frac{1}{2r^2 \log (1/r)} + \frac{4}{r(1 - r^2)} + O(1), \]
and
\[
\pi \beta_0(\sqrt[r]{r}, r) = \frac{1}{2r^{2/5} \log (1/r)} + \frac{1}{1 - r^2} + O \left( r^{2/5} \right),
\]
\[
\pi \beta_1(\sqrt[r]{r}, r) = -\frac{1}{2r^{3/5} \log (1/r)} + \frac{2r^{1/5}}{1 - r^4} + O \left( r^{3/5} \right),
\]
\[
\pi \beta_2(\sqrt[r]{r}, r) = \frac{1}{2r^{4/5} \log (1/r)} + \frac{2}{1 - r^4} + O \left( r^{2/5} \right).
\]
Therefore we get
\[
\lim_{r \to 0} \frac{(F^\Omega_{\mathcal{B}}(\sqrt[r]{r}, 1))^2}{\log (1/r)} = 4 \quad \text{and} \quad \lim_{r \to 0} \frac{(F^\Omega_{\mathcal{B}}(\sqrt[r]{r}, 1))^2}{\log (1/r)} = 2.
\]

\[\square\]

Remark 8. We note that the Szegő and Bergman kernels of \( \Omega_r \) can be written in closed form in terms of elliptic functions (see for example [3]) though that is not particularly helpful for the computations above.

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