HOLOMORPHIC FOLIATION ASSOCIATED WITH A SEMI-POSITIVE CLASS OF NUMERICAL DIMENSION ONE

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Abstract. Let $X$ be a compact Kähler manifold and $\alpha$ be a class in the Dolbeault cohomology class of bidegree $(1,1)$ on $X$. When the numerical dimension of $\alpha$ is one and $\alpha$ admits at least two smooth semi-positive representatives, we show the existence of a family of real analytic Levi-flat hypersurfaces in $X$ and a holomorphic foliation on a suitable domain of $X$ along whose leaves any semi-positive representative of $\alpha$ is zero. As an application, we give the affirmative answer to [K2, Conjecture 2.1] on the relation between the semi-positivity of the line bundle $[Y]$ and the analytic structure of a neighborhood of $Y$ for a smooth connected hypersurface $Y$ of $X$.

1. Introduction

Let $X$ be a connected compact Kähler manifold and $\alpha \in H^{1,1}(X,\mathbb{R}):= H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{R})$ be a class such that $\#\text{SP}(\alpha) > 1$ and $\text{nd}(\alpha) = 1$, where $\text{SP}(\alpha)$ is the set of all $C^\infty$-smooth $d$-closed semi-positive $(1,1)$-forms on $X$ which represents the class $\alpha$, and $\text{nd}(\alpha) := \max\{k \in \{0,1,2,\ldots, \dim X\} \mid \alpha^\wedge k \neq 0 \text{ in } H^{k,k}(X,\mathbb{C})\}$. For such a class $\alpha$, we denote by $K_\alpha$ the closed subset of $X$ defined by

$$K_\alpha := \bigcap_{\theta \in \text{SP}(\alpha)} \bigcap_{\psi \in \text{PSH}^\infty(X,\theta)} \{x \in X \mid (d\psi)_x = 0\},$$

where $\text{PSH}^\infty(X,\theta)$ is the set of all the $\theta$-plurisubharmonic functions of class $C^\infty$ for a $C^\infty$-smooth $(1,1)$-form $\theta$ on $X$: i.e. $\text{PSH}^\infty(X,\theta) := \{\psi : X \to \mathbb{R} : C^\infty \mid \theta + \sqrt{-1} \partial \bar{\partial} \psi \geq 0\}$. Note that it follows from the $\partial \bar{\partial}$-lemma that $\text{PSH}^\infty(X,\theta) \not\supset \mathbb{R}$ holds for $\theta \in \text{SP}(\alpha)$ and $K_\alpha \subset X$, since $\#\text{SP}(\alpha) > 1$. For such $X$ and $\alpha$, we show the following:

**Theorem 1.1.** Let $X$ and $\alpha$ be as above. Then there uniquely exists a non-singular holomorphic foliation $\mathcal{F}_\alpha$ on $X \setminus K_\alpha$ of complex codimension 1 such that $i^*_L \theta = 0$ for any $\theta \in \text{SP}(\alpha)$ and any leaf $L$ of $\mathcal{F}_\alpha$, where $i_L : L \to X$ is the holomorphic immersion.

We investigate how large can $\mathcal{F}_\alpha$ be analytically extended by classifying the connected components of $K_\alpha$ from the viewpoint of the existence of an $\mathcal{F}_\alpha$-adaptive function in the following sense on a suitable neighborhood: We say that a continuous function $h : \overline{W} \to [-\infty, +\infty]$ on the closure of a domain (connected open subset) $W$ of $X$ is $\mathcal{F}_\alpha$-adaptive if $h|_W$ is a $\mathbb{R}$-valued non-constant pluriharmonic function, $h|_{W \setminus K_\alpha}$ is $\mathcal{F}_\alpha$-leafwise constant, and the preimage $h^{-1}\{\max_{\overline{W}} h, \min_{\overline{W}} h\}$ coincides with the boundary $\partial W$ of $W$, where the topology of $[-\infty, +\infty]$ is the one such that $[-\infty, +\infty]$ is homeomorphic to the interval $[0,1] \subset \mathbb{R}$.

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Definition 1.2. A connected component \(K'\) of \(K_\alpha\) is said to be an essential component if there does not exist a connected open neighborhood \(W\) of \(K'\) in \(X\) such that \(W \cap K_\alpha\) is a relatively compact subset of \(W\) and that there exists an \(\mathcal{F}_\alpha\)-adapative function on \(W\). The union of all the essential components of \(K_\alpha\) is denoted by \(K^{\text{ess}}_\alpha\).

Our second main result is the following:

Theorem 1.3. Let \(X\) and \(\alpha\) be as above. Then the holomorphic foliation \(\mathcal{F}_\alpha\) on \(X \setminus K_\alpha\) as in Theorem 1.1 can be extended to \(X \setminus K^{\text{ess}}_\alpha\) as a (maybe singular) holomorphic foliation. Moreover, one of the following holds:

Case I: There exists a surjective holomorphic map \(\Phi: X \rightarrow R\) to a compact Riemann surface \(R\) and a Kähler class \(\alpha_R\) of \(R\) such that \(\alpha = \Phi^* \alpha_R\). In this case, \(\mathcal{F}_\alpha\) is the foliation defined by the fibration \(\Phi\), \(K^{\text{ess}}_\alpha = \emptyset\), \(K_\alpha\) is included in the set of all the critical points of \(\Phi\), and the set of all the singular points \((\Phi^{-1}(p))_{\text{sing}}\) of the (set-theoretical) fiber \(\Phi^{-1}(p)\) is included in \(K_\alpha\) for any point \(p \in R\).

Case II: Not in the case I and \(K^{\text{ess}}_\alpha = \emptyset\). In this case, there exist an open covering \(\{U_1, U_2\}\) of \(X\) consisting of two domains and \(\mathcal{F}_\alpha\)-adapative functions \(h_j: \overline{U_j} \rightarrow [-\infty, +\infty]\) for each \(j = 1, 2\) such that, on each connected component \(W\) of \(U_1 \cap U_2\), there exist constants \(a_W, b_W \in \mathbb{R}\) such that \(h_2 = a_W h_1 + b_W\) holds on \(W\). The foliation \(\mathcal{F}_\alpha\) is defined on \(X\), its tangent bundle is perpendicular to \(\partial h_j\) on \(U_j\), and \(K_\alpha \cap U_j = \{x \in U_j \mid (dh_j)_x = 0\}\) holds for \(j = 1, 2\).

Case III: \(K^{\text{ess}}_\alpha \neq \emptyset\). In this case, \(X \setminus K_\alpha\) is a domain of \(X\) and there exists an \(\mathcal{F}_\alpha\)-adapative function \(h_\alpha: \overline{X \setminus K^{\text{ess}}_\alpha} \rightarrow [-\infty, +\infty]\). The tangent bundle of the foliation \(\mathcal{F}_\alpha\) on \(X \setminus K^{\text{ess}}_\alpha\) is perpendicular to \(\partial h_\alpha\), and \(K_\alpha \setminus K^{\text{ess}}_\alpha = \{x \in X \setminus K^{\text{ess}}_\alpha \mid (dh_\alpha)_x = 0\}\) holds.

As a corollary, one has the following:

Theorem 1.4. Let \(X\) be a connected compact Kähler manifold. Assume that there exists a \((1, 1)\)-class \(\alpha \in H^{1,1}(X, \mathbb{R})\) with \#SP(\(\alpha\)) > 1 and \(\text{nd}(\alpha) = 1\). Then \(X\) admits uncountably many compact Levi-flat hypersurfaces of class \(C^\omega\) (i.e., real analytic).

Let \(Y\) be a non-singular hypersurface of \(X\) such that the normal bundle \(N_{Y/X} = [Y]_Y\) is unitary flat (i.e., \(N_{Y/X} \in H^1(Y, U(1))\), where \(U(1) := \{t \in \mathbb{C} \mid |t| = 1\}\), where \([Y]\) is the holomorphic line bundle on \(X\) which corresponds to the divisor \(Y\). Note that the first Chern class \(c_1([Y])\) of \([Y]\) satisfies \(\text{nd}(c_1([Y])) = 1\) in this case. Our motivation comes from the study of the relation between the semi-positivity of \([Y]\) (i.e., non-emptiness of \(\text{SP}(c_1([Y]))\)) and the complex analytic structure of a neighborhood of \(Y\). In [K2, Conjecture 2.1], we conjectured that \([Y]\) is semi-positive if and only if the pair \((Y, X)\) is of class \((\beta')\) or \((\beta'')\) in the classification of Ueda [U]. The following corollary, which follows from [K3, Theorem 1.4] and the argument in the proof of Theorem 1.4, gives an affirmative answer to [K2, Conjecture 2.1] when \(Y\) is non-singular.

Corollary 1.5. Let \(X\) be a connected compact Kähler manifold and \(Y \subset X\) be a non-singular connected hypersurface such that \(N_{Y/X}\) is unitary flat. Then \([Y]\) is semi-positive if and only if there exists a neighborhood \(V\) of \(Y\) such that \([Y]|_V\) is unitary flat: i.e., there exists a non-singular holomorphic foliation on \(V\) which has \(Y\) as a leaf along which the holonomy is \(U(1)\)-linear.
Note that Ohsawa pointed out in [O, Remark 5.2] that Corollary 1.5 for a surface $X$ can be shown by combining [K3, Theorem 1.4] and Siu's solution [Si] of Grauert–Riemenschneider's conjecture (Kählerness assumption is not needed in his proof). Note also that this kind of results can be regarded as a generalization of Brunella's theorem [B] for the blow-up of the projective plane at general nine points. See §6 and §7 for details.

The foliation $\mathcal{F}_\alpha$ is constructed by considering the eigenvectors which belong to the eigenvalue zero of each element of $\text{SP}(\alpha)$, or equivalently, by considering the Monge–Ampère foliation for each element $\psi \in \text{PSH}_\infty(X, \theta)$. In §3, we will show that $\sqrt{-1} \partial \bar{\partial} \psi = g_\psi \cdot \sqrt{-1} \partial \psi \wedge \bar{\partial} \psi$ holds for an $\mathcal{F}_\alpha$-leafwise constant function $g_\psi$ on a suitable domain of $X$ essentially by a linear-algebraic arguments. When $g_\psi$ is a non-constant function on some level set of $\psi$, we show that the situation is in Case I. When $g_\psi$ is constant on any level set of $\psi$, $g_\psi = \chi \circ \psi$ holds on a domain for some real function $\chi$. By considering a solution $G$ of a suitable ordinary differential equation concerning on $\chi$, one can see that $h_0 := G \circ \psi$ is a pluriharmonic function. In this case, we show that the situation is either Case II or III by considering the analytic continuation of $h_0$.

The organization of the paper is as follows. In §2, we collect some fundamental facts and known results. In §3, we investigate the relation between the level sets of a non-constant element of $\text{PSH}_\infty(X, \theta)$ and Monge–Ampère foliation for an element $\theta \in \text{SP}(\alpha)$. The main result in this section is Theorem 3.5, in which we classify the situation into two cases (a) and (b) according to the constantness or non-constantness of $g_\psi$ on each level set of $\psi$, where $\psi$ and $g_\psi$ are the functions as above. Here we also show Theorem 1.4. In §4, we investigate the cases (a) and (b) in Theorem 3.5 as a preliminary step for the proof of main results. In §5, we show Theorems 1.1 and 1.3. In §6, we investigate the case where the class $\alpha$ is the first Chern class of the line bundle $[Y]$ for a hypersurface $Y$ on $X$. Here we show Corollary 1.5. In §7, we give some examples.

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## 2. Preliminaries

### 2.1. On the set $\text{PSH}_\infty(X, \theta)$

Let $X$ be a compact Kähler manifold and $\theta$ be a $d$-closed $C^\infty$ $(1, 1)$-form on $X$.

**Lemma 2.1.** Let $\varphi$ and $\psi$ be elements of $\text{PSH}_\infty(X, \theta)$.

(i) Let $\chi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class $C^\infty$. Assume that $\chi$ is non-decreasing in each variable, convex, and that $\chi(p + r, q + r) = \chi(p, q) + r$ for any $p, q, r \in \mathbb{R}$. Then the function $\chi(\varphi, \psi) : x \mapsto \chi(\varphi(x), \psi(x))$ is also an element of $\text{PSH}_\infty(X, \theta)$.

(ii) The function $\log(e^\varphi + e^\psi) : x \mapsto \log(e^{\varphi(x)} + e^{\psi(x)})$ is an element of $\text{PSH}_\infty(X, \theta)$. 


See also [GZ, Proposition 2.3] for the assertion (ii). Note that, in [GZ], this assertion is proven directly by using the formulae

\[
\partial \log(e^\varphi + e^\psi) = \frac{e^\varphi \partial \varphi + e^\psi \partial \psi}{e^\varphi + e^\psi}
\]

and

\[
\sqrt{-1} \partial \bar{\partial} \log(e^\varphi + e^\psi) = \frac{e^\varphi \sqrt{-1} \partial \varphi + e^\psi \sqrt{-1} \partial \psi}{e^\varphi + e^\psi} + \frac{e^\varphi + e^\psi \sqrt{-1} \partial (\varphi - \psi) \wedge \bar{\partial} (\varphi - \psi)}{(e^\varphi + e^\psi)^2}.
\]

**Proof of Lemma 2.1.** (i) It is enough to show that, for each point \(x \in X\), there exists a neighborhood \(B\) of \(x\) such that \(\chi(\varphi, \psi)|_B\) satisfies \(\theta|_B + \sqrt{-1} \partial \bar{\partial} \chi(\varphi, \psi)|_B \geq 0\). Take \(B\) so that it is a small open ball in a coordinates system around \(x\). Then, as \(H^1(B, \mathcal{O}_B) = 0\) by Oka’s vanishing, one can easily deduce the existence of a function \(f\) on \(B\) such that \(\theta|_B = \sqrt{-1} \partial \bar{\partial} f\). As \(\varphi\) and \(\psi\) are elements of \(\text{PSH}^\infty(X, \theta)\), both \(f + \varphi\) and \(f + \psi\) are plurisubharmonic on \(B\). Therefore, by the assumption, \(f + \chi(\varphi, \psi) = \chi(f + \varphi, f + \psi)\) is also plurisubharmonic on \(B\), which proves the assertion.

(ii) The assertion follows by considering \(\chi(s, t) := \log(e^s + e^t)\).

Next, let us see some fundamental properties of the set \(\text{PSH}^\infty(X, \theta)\) when \(\theta \in \text{SP}(\alpha)\) for a class \(\alpha \in H^{1,1}(X, \mathbb{R})\) with \(\text{nd}(\alpha) = 1\).

**Lemma 2.2.** Let \(X\) be a compact Kähler manifold of dimension \(n\) and \(\alpha \in H^{1,1}(X, \mathbb{R})\) be a class such that \(\text{nd}(\alpha) = 1\). Take \(\theta \in \text{SP}(\alpha)\).

(i) For \(\varphi, \psi \in \text{PSH}^\infty(X, \theta)\), it holds that \((\theta + \sqrt{-1} \partial \bar{\partial} \varphi) \wedge (\theta + \sqrt{-1} \partial \bar{\partial} \psi) \equiv 0\).

(ii) For \(\varphi \in \text{PSH}^\infty(X, \theta)\), \(\theta \wedge \partial \bar{\partial} \varphi \equiv 0\) and \((\theta + \sqrt{-1} \partial \bar{\partial} \varphi)^2 := (\theta + \sqrt{-1} \partial \bar{\partial} \varphi)^\wedge 2 \equiv 0\) hold.

(iii) For \(\varphi, \psi \in \text{PSH}^\infty(X, \theta)\), it holds that \(\partial \varphi \wedge \bar{\partial} \psi \equiv 0\).

**Proof.** (i) Take a Kähler form \(\omega\) of \(X\). As

\[
\int_X (\theta + \sqrt{-1} \partial \bar{\partial} \varphi) \wedge (\theta + \sqrt{-1} \partial \bar{\partial} \psi) \wedge \omega^{n-2} = 0
\]

by the assumption \(\text{nd}(\alpha) = 1\), it follows from the semi-positivity of the forms \(\theta + \sqrt{-1} \partial \bar{\partial} \varphi\) and \(\theta + \sqrt{-1} \partial \bar{\partial} \psi\) that \((\theta + \sqrt{-1} \partial \bar{\partial} \varphi) \wedge (\theta + \sqrt{-1} \partial \bar{\partial} \psi) \equiv 0\) (More precisely, take a suitable coordinates \((z^1, z^2, \ldots, z^n)\) around each point such that both \(\omega\) and \(\theta + \sqrt{-1} \partial \bar{\partial} \varphi\) are represented by diagonal matrices at the point and apply Lemma 3.1).

(ii) Note that \(\theta^2 \equiv 0\) follows from the assertion (i) (consider the case where \(\varphi \equiv \psi \equiv 0\)). The assertion (ii) follows from (i) by considering the cases \((\varphi, \psi) = (0, \varphi)\) and \((\varphi, \psi) = (\varphi, \psi)\).

(iii) The assertion (iii) follows from (i) and (ii).

2.2. **On the set \(K_\alpha\).** In this subsection, we show the following:

**Lemma 2.3.** Let \(X\) be a compact Kähler manifold and \(\alpha \in H^{1,1}(X, \mathbb{R})\) be a class such that \(\text{nd}(\alpha) = 1\) and \#\(\text{SP}(\alpha) > 1\). Then, for any \(\theta \in \text{SP}(\alpha)\), it follows that

\[
K_\alpha = \bigcap_{\psi \in \text{PSH}^\infty(X, \theta)} \{x \in X \mid (d\psi)_x = 0\}.
\]
Proof. Denote by $K_\theta$ the set in the right hand side. As the inclusion $K_\alpha \subset K_\theta$ is clear by definition, here we show the other inclusion. Take $x \in K_\theta$, $\theta' \in \text{SP}(\alpha)$, and $\psi \in \text{PSH}^\infty(X, \theta')$. By the $\partial\bar{\partial}$-lemma, there exists a function $f$ on $X$ such that $\theta' = \theta + \sqrt{-1}\partial\bar{\partial}f$. By the regularity theorem, one has that $f$ is of class $C^\infty$. Thus both the functions $f$ and $f + \psi$ are elements of $\text{PSH}^\infty(X, \theta)$. As $x$ is an element of $K_\theta$, one has that $(d\psi)_x = (df)_x = 0$, from which it follows that $x \in K_\alpha$, since $\theta'$ and $\psi$ are arbitrary.

2.3. Fundamental observation on $\mathcal{F}_\alpha$-adaptive function. Let $X$ be a compact complex manifold of dimension $n$. In this subsection, we consider a domain $W$ of $X$ and a continuous function $h$ : $\overline{W} \rightarrow [\infty, +\infty]$ such that $h|_W$ is a non-constant pluriharmonic function and that $h^{-1}(\{\max_{\overline{W}}h, \min_{\overline{W}}h\}) = \partial W$. Note that $h$ is $\mathcal{F}_\alpha$-adaptive in the sense of §1 if $\mathcal{F}_\alpha$ as in Theorem 1.1 actually exists and it is $\mathcal{F}_\alpha$-leafwise constant. Note also that the image $h(W)$ coincides with the open interval $(\min_{\overline{W}}h, \max_{\overline{W}}h)$. First let us show the following:

Lemma 2.4. The function $h|_W : W \rightarrow (\min_{\overline{W}}h, \max_{\overline{W}}h)$ is proper.

Proof. It is sufficient to show that the preimage $K := h^{-1}([a, b])$ of the closed interval $[a, b] \subset (\min_{\overline{W}}h, \max_{\overline{W}}h)$ is compact, since a closed subset of a compact set is also compact (Consider $a := \min I$ and $b := \max I$ for a compact subset $I \subset (\min_{\overline{W}}h, \max_{\overline{W}}h)$). As $h^{-1}(\{\max_{\overline{W}}h, \min_{\overline{W}}h\}) = \partial W$, it is clear that $\overline{K} \cap \partial W = \emptyset$, where the closure is taken by regarding $K$ as a subset of $X$. From this and the closedness of $K$ as a subset of $W$, it follows that $K$ is a compact subset of a compact space $X$. Therefore $K$ is compact.

By the real-analyticity of a pluriharmonic function, it follows from [SS] that the set $\mathcal{C}$ of all the critical values of $h|_W$ is discrete. Note that $\#\mathcal{C} < \infty$ holds for example if there exists real numbers $a$ and $b$ such that $\min_{\overline{W}}h < a < b < \max_{\overline{W}}h$ and that $h$ is submersion on $h^{-1}(\{\min_{\overline{W}}h, a\}) \cup h^{-1}(\{b, \max_{\overline{W}}h\})$.

Sort the elements of $\mathcal{C}$ as $\mathcal{C} = \{c_j\}_{j \in \mathbb{N}}^M (N \in \mathbb{Z} \cup \{-\infty\}, M \in \mathbb{Z} \cup \{+\infty\})$ so that $c_j < c_k$ holds if $j < k$. For the interval $I_j := (c_j, c_{j+1})$, it follows from the properness of $h|_{h^{-1}(I_j)}$ and Ehresmann’s lemma that $h^{-1}(I_j)$ is homeomorphic to the product $h^{-1}(r_j) \times I_j$, where $r_j \in I_j$ is a regular value. From this, one has that $h^{-1}(I_j)$ is connected if and only if each fiber of $h|_{h^{-1}(I_j)}$ is connected. Note that the fiber $h^{-1}(r_j)$ is a compact Levi-flat hypersurface of $X$.

In this paper, we often consider the holomorphic foliation $\mathcal{F}$ on $W$ defined by $T_\mathcal{F} = (\partial h)^\perp \subset T_W$ for the pair $(W, h)$, where $T_\mathcal{F}$ is the holomorphic tangent bundle of $\mathcal{F}$ and $T_W$ is the holomorphic tangent bundle of $W$. In other word, $\mathcal{F}$ is the foliation which is defined by letting $\{F = \text{constant}\}$ be the local defining functions of leaves around a point $x \in W$, where $F$ is a holomorphic function defined on a neighborhood of $x$ whose real part $\text{Re}F$ coincides with $h$ (Note that $dF = 2\partial h$). We regard the set $\mathcal{F}_{\text{sing}} := \{x \in W \mid (dh)_x = 0\}$ as the singular part of the foliation $\mathcal{F}$ (though $\mathcal{F}$ may be regarded as a non-singular holomorphic foliation not only on $X \setminus \mathcal{F}_{\text{sing}}$ but also on the regular part of an irreducible component of $\mathcal{F}_{\text{sing}}$ of codimension 1). For this foliation $\mathcal{F}$, we show the following:

Lemma 2.5. Let $W, h$, and $\mathcal{F}$ be as above, and $\varphi : W \rightarrow \mathbb{R}$ be a function of class $C^\infty$ such that $\varphi|_{W \setminus \mathcal{F}_{\text{sing}}}$ is $\mathcal{F}$-leafwise constant. Then $\varphi$ is $\mathcal{F}$-leafwise constant also on $W$ in
the following sense: For each $x_0 \in F_{\text{sing}}$ and a holomorphic function $F: B \to \mathbb{C}$ on a sufficiently small neighborhood $B$ of $x_0$ such that $\Re F = h|_B$, there exists a constant $w \in \mathbb{C}$ such that $B \cap F_{\text{sing}}$ is included in the preimage $F^{-1}(w)$ and $\varphi|_{F^{-1}(w)}$ is constant. Moreover, for $x_0, B$ and $F$ as above, the following holds if $x_0$ is a regular point of a irreducible component of $B \cap F_{\text{sing}}$ of codimension 1 by shrinking $B$ if necessary: There exists a coordinate system $w = (w^1, w^2, \ldots, w^n)$ of $B$ with $x_0 = (0, 0, \ldots, 0)$ and an integer $m \in \mathbb{Z}_{>1}$ such that $F(w) = (w^1)^m + F(x_0)$ and that $\varphi(w) = \chi(w^1)$ holds on $B$ for some function $\chi: \varphi(B) \to \mathbb{R}$.

**Proof.** Let $x_0, B$, and $F$ be as in the statement. By changing $h$ and $F$ by adding some constant, we may assume $h(x_0) = F(x_0) = 0$ without loss of generality. Set $S := B \cap F_{\text{sing}}$ and let $S = \bigcup_{\mu=1}^M S^\mu$ be the irreducible decomposition of $S$. By shrinking $B$ if necessary, we may assume that each $S^\mu$ includes the point $x_0$. It holds that $d(F|_{S^\mu})_{\text{reg}} = (dF|_{S^\mu})_{\text{reg}} \equiv 0$ holds on the regular part $(S^\mu)_{\text{reg}}$ of $S^\mu$ for each $S^\mu$ by the definition of $S$. Thus if follows that $F|_{S^\mu}$ is a constant function, since there exists an open dense connected subset of $S^\mu$ which is included in $(S^\mu)_{\text{reg}}$ by local parametrization theorem [D, 4.19]. Therefore, one has that $F|_S \equiv 0$ since $F(x_0) = 0$: i.e. $S \subset F^{-1}(0)$.

Next let us consider the analytic set $A := F^{-1}(0)$. Let $A = \bigcup_{\nu=1}^N A^\nu$ be the irreducible decomposition. We may assume that each $A^\nu$ includes the point $x_0$ again by shrinking $B$ if necessary. Take a point $y_0 \in (A^\nu)_{\text{reg}}$. As $A^\nu$ is a hypersurface, one can take holomorphic coordinates $(w^1, w^2, \ldots, w^n)$ of a small neighborhood $B'$ of $y_0$ in $B$ such that $y_0 = (0, 0, \ldots, 0)$ and $B' \cap A = B' \cap A^\nu = \{w^1 = 0\}$. We may assume that there exists $m \in \mathbb{Z}_{>0}$ such that $F|_{B'} = (w^1)^m$. By considering the continuous function

$$B' \ni y \mapsto \max \left\{ \left| \frac{\partial \varphi}{\partial w^2}(y) \right|, \left| \frac{\partial \varphi}{\partial w^3}(y) \right|, \ldots, \left| \frac{\partial \varphi}{\partial w^n}(y) \right| \right\} \in \mathbb{R},$$

it follows from the $F$-leafwise constantness of $\varphi|_{B' \cap A^\nu}$ that $\varphi|_{B' \cap A^\nu}$ is constant. Therefore, again from local parametrization theorem [D, 4.19], one can deduce that $\varphi|_A$ is constant, which proves the first half half of the statement.

Finally we show the latter half of the statement. In this case, we may assume that $M = N = 1$, $A_1 = S_1$, and that $x_0 = y_0$ for $y_0$ as above. We also may assume that $B = B'$. Take the coordinates $(w^1, w^2, \ldots, w^n)$ as above. As $2\partial h = dF = m(w^1)^{m-1} \cdot dw^1$, it follows that $m > 1$. The existence of a function $\chi$ as in the statement follows from the fact that each leaf of $F|_B$ is defined by $\{w^1 = \text{constant}\}$. \hfill $\square$

3. Level sets of a non-constant element of $\text{PSH}^\infty(X, \theta)$ and Monge–Ampère foliation

Let $X$ be a connected compact Kähler manifold, $\alpha \in H^{1,1}(X, \mathbb{R})$ a class with $\text{nd}(\alpha) = 1$ and $\#\text{SP}(\alpha) > 1$, $\theta$ an element of $\text{SP}(\alpha)$, and $\psi$ be an element $\text{PSH}^\infty(X, \theta) \setminus \mathbb{R}$.

For an open subset $U \subset X$ such that $(\theta + \sqrt{-1} \partial \bar{\partial} \psi)_x \neq 0$ holds for any point $x \in U$, denote by $\mathcal{F}(\theta, \psi, U)$ the foliation on $U$ whose tangent space $T_{\mathcal{F}(\theta, \psi, U), x}$ at a point $x \in U$ coincides with the set of all the eigenvectors which belongs to the eigenvalue 0 of $(\theta + \sqrt{-1} \partial \bar{\partial} \psi)_x$. In another word, for a small (simply connected Stein) neighborhood
B of each point of U, \( F(\theta, \psi, U)|_B \) is the Monge–Ampère foliation of \( \sqrt{-1} \partial \overline{\partial} (\varphi_0 + \psi) \), where \( \varphi_0 \) is a function on B such that \( \sqrt{-1} \partial \overline{\partial} \varphi_0 = \theta|_B \). Note that each leaf of \( F(\theta, \psi, U) \) is a holomorphically immersed complex submanifolds of U ([So], see also [BK], note also that here we used Lemma 2.2 (ii)). In this section, we investigate the relation between the leaves of \( F(\theta, \psi, U) \) for a suitable U and level sets of \( \psi \). In what follows, we use the following notation: Let \( I_\psi \) be the image \( \psi(X) \), which is a bounded closed connected interval, \( (I_\psi)_{reg} \) the set of all the regular values of \( \psi \), and \( U_\psi \) be the preimage \( \psi^{-1}((I_\psi)_{reg}) \). Note that the Lebesgue measure of \( I_\psi \setminus (I_\psi)_{reg} \) is zero by Sard’s theorem.

In this section, we often use the following fundamental fact.

**Lemma 3.1.** Let \( A = (a_{jk})_{j,k=1}^n \) be a Hermitian matrix of order n. Assume that \( A \) is positive semi-definite and that \( a_{jj} = 0 \) for each \( j \in \{2,3,\ldots,n\} \). Then it holds that \( a_{jk} = 0 \) for any \( j \) and \( k \) with \( (j,k) \neq (0,0) \).

**Proof.** For \( j, k \in \{1,2,\ldots,n\} \) with \( j < k \), consider a submatrix

\[
A_{jk} := \begin{pmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk} \end{pmatrix}.
\]

As \( A_{jk} \) is also positive semi-definite, one has that the determinant \( \det A_{jk} \) is non-negative. From this and the assumption, it follows that \( -|a_{jk}|^2 \geq 0 \), which proves the lemma. \( \square \)

### 3.1 Observation on a neighborhood of a point \( x_0 \) such that \((d\psi)_{x_0} \neq 0 \) and \((\theta + \sqrt{-1} \partial \overline{\partial} \psi)_{x_0} \neq 0 \).

In this subsection, we observe on a neighborhood of a point \( x_0 \in X \) such that \((d\psi)_{x_0} \neq 0 \) and \((\theta + \sqrt{-1} \partial \overline{\partial} \psi)_{x_0} \neq 0 \). Take a small neighborhood \( B_0 \) of \( x_0 \) such that \((d\psi)_x \neq 0 \) and \((\theta + \sqrt{-1} \partial \overline{\partial} \psi)_x \neq 0 \) holds on each \( x \in B_0 \). By shrinking \( B_0 \) if necessary, we may assume that the leaf \( Y_0 \) of \( F(\theta, \psi, B_0) \) which contains \( x_0 \) is a (holomorphically embedded) complex submanifold of \( B_0 \). Again by shrinking \( B_0 \), one can take coordinates \((z^1, z^2, \ldots, z^n)\) of \( B_0 \) such that \( x_0 = (0,0,\ldots,0) \) and \( Y_0 = \{z^1 = 0\} \). First we show the following:

**Lemma 3.2.** For \( x_0 \in X \) as above, the following holds:

(i) There exists a constant \( a \in \mathbb{C} \) such that \((d\psi)_{x_0} = a(dz^1)_{x_0} \).

(ii) There exists a constant \( f_{\theta, \psi}(x_0) \in \mathbb{R}_{\geq 0} \) such that \( \theta_{x_0} = f_{\theta, \psi}(x_0) \cdot (\sqrt{-1} \partial \psi \wedge \overline{\partial} \psi)_{x_0} \).

(iii) There exists a constant \( g_{\psi}(x_0) \in \mathbb{R} \) such that \((\sqrt{-1} \partial \overline{\partial} \psi)_{x_0} = g_{\psi}(x_0) \cdot (\sqrt{-1} \partial \psi \wedge \overline{\partial} \psi)_{x_0} \).

**Proof.** Consider the function \( \hat{\psi} := \log(1 + e^\psi) \), which is an element of \( \text{PSH}^\infty(X, \theta) \) by Lemma 2.1. It follows from the equation (2) that

\[
\theta + \sqrt{-1} \partial \overline{\partial} \hat{\psi} = \frac{1}{1 + e^\psi} \cdot \theta + \frac{e^\psi}{1 + e^\psi} \cdot (\theta + \sqrt{-1} \partial \overline{\partial} \psi) + \frac{e^\psi}{(1 + e^\psi)^2} \cdot \sqrt{-1} \partial \psi \wedge \overline{\partial} \psi.
\]

From Lemma 2.2 and the semi-positivity of each of the terms of the right hand side of the equation above, it follows that

\[
\theta_{x_0} \wedge (\sqrt{-1} dz^1 \wedge \overline{dz^1})_{x_0} = (\sqrt{-1} \partial \psi \wedge \overline{\partial} \psi)_{x_0} \wedge (\sqrt{-1} d\psi \wedge \overline{d\psi})_{x_0} = 0
\]

holds, since \((\theta + \sqrt{-1} \partial \overline{\partial} \psi)_{x_0} = A \cdot (\sqrt{-1} d\psi \wedge \overline{d\psi})_{x_0} \) for some \( A > 0 \), which follows from the definition of \( Y_0 \). By considering the wedge product of the form \( \bigwedge_{k \in \{2,3,\ldots,n\} \setminus \{j\}} \sqrt{-1} dz^k \wedge \overline{dz^k} \) and each of the terms of the equation (3) for \( j = 2,3,\ldots,n \), one can deduce from
Lemma 3.1 that both $\theta_{x_0}$ and $(\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi)_{x_0}$ are perpendicular to the form $(\sqrt{-1}dz^1 \wedge \bar{dz}^1)_{x_0}$, which proves the assertion (i) and (ii). The assertion (iii) follows from (i), (ii), and the equation $(\theta + \sqrt{-1}\partial\psi)_{x_0} = A \cdot (\sqrt{-1}dz^1 \wedge \bar{dz}^1)_{x_0}$. \hfill \Box

Let $f_{\theta,\psi}: B_0 \to \mathbb{R}_{\geq 0}$ and $g_{\psi}: B_0 \to \mathbb{R}$ be the functions such that

$$\theta|_{B_0} = f_{\theta,\psi} \cdot \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi$$

and

$$\sqrt{-1}\partial\bar{\partial}\psi|_{B_0} = g_{\psi} \cdot \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi$$

hold, whose existence at $x \in B_0$ is assured by Lemma 3.2 for $x_0 = x$. As

$$f_{\theta,\psi} = \frac{\text{tr}_\omega \theta}{\text{tr}_\omega \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi}, \quad \text{and} \quad g_{\psi} = \frac{\text{tr}_\omega \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi}{\text{tr}_\omega \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi}$$

holds for a Kähler form $\omega$ on $X$, both $f_{\theta,\psi}$ and $g_{\psi}$ are of class $C^\infty$.

Next we show the following lemma on the leafwise constantness of some functions on $B_0$.

**Lemma 3.3.** The following holds:

(i) For any element $\varphi \in \text{PSH}^\infty(X,\theta)$, there exists a function $F_{\varphi,\psi}: B_0 \to \mathbb{C}$ such that $\partial\varphi = F_{\varphi,\psi} \cdot \partial\psi$ holds on $B_0$ and the function $|F_{\varphi,\psi}|$ is $\mathcal{F}(\theta,\psi,B_0)$-leafwise constant. Especially, $\varphi|_{B_0}$ is $\mathcal{F}(\theta,\psi,B_0)$-leafwise constant.

(ii) Both the functions $f_{\theta,\psi}$ and $g_{\psi}$ are $\mathcal{F}(\theta,\psi,B_0)$-leafwise constant.

**Proof.** As our choice of a point $x_0$ is arbitrary, it is sufficient to show the assertions at $x_0$ or along $Y_0$.

The first half of the assertion (i) follows by applying the same argument as in the proof of Lemma 3.2 to the function $\hat{\varphi} := \log(1 + e^\varphi)$ and $(\theta + \sqrt{-1}\partial\bar{\partial}\varphi) \wedge (\theta + \sqrt{-1}\partial\bar{\partial}\psi)$. Note that it also follows from the same argument at the same time that the forms $\theta + \sqrt{-1}\partial\bar{\partial}\varphi$ (and thus $\sqrt{-1}\partial\bar{\partial}\varphi$) are pointwisely parallel to the form $\sqrt{-1}dz^1 \wedge \bar{dz}^1$ (Note also that the assertion is clear at a point $x$ such that $(\partial\varphi)_x = 0$ by letting $F_{\varphi,\psi}(x) := 0$). From now on, we show the $\mathcal{F}(\theta,\psi,B_0)$-leafwise constantness of the function $|F_{\varphi,\psi}|$, where $F_{\varphi,\psi}$ is the function as in the statement (Note that the last part of the assertion (i) follows from the first half of the assertion since one obtains from this that $d\varphi$ is zero along the leaves of $\mathcal{F}(\theta,\psi,B_0)$). As $\sqrt{-1}\partial\bar{\partial}\varphi = |F_{\varphi,\psi}|^2 \cdot \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi$, one has that

$$-\partial\varphi \wedge (\sqrt{-1}\partial\bar{\partial}\varphi) = \left(\partial|F_{\varphi,\psi}|^2\right) \wedge \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi - |F_{\varphi,\psi}|^2 \cdot g_{\psi} \cdot \partial\psi \wedge (\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi).$$

As the left hand side is pointwisely parallel to the form $dz^1 \wedge \sqrt{-1}dz^1 \wedge \bar{dz}^1(\equiv 0)$ and the second term of the right hand side is equal to zero, one has that $(|F_{\varphi,\psi}|^2) \wedge \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \equiv 0$, which proves the constantness of the function $|F_{\varphi,\psi}||_{Y_0}$.

The assertion (ii) can also be shown by the same argument: for a function $G := f_{\theta,\psi}$ or $G := g_{\psi}$, one has that

$$0 \equiv \partial(\sqrt{-1}G \wedge \partial\psi \wedge \bar{\partial}\psi) = \partial G \wedge \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi - G \cdot g_{\psi} \cdot \partial\psi \wedge \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi$$

since both $\theta$ and $\sqrt{-1}\partial\bar{\partial}\psi$ is $\partial$-closed. As the second term of the right hand side is zero, one has that $G|_{Y_0}$ is constant, which proves the lemma. \hfill \Box
3.2. Observation on $U_{\psi}$. In this subsection, let us consider the foliation $\mathcal{F}(\theta, \psi)$ on $U_{\psi}$, which is defined by $\mathcal{F}(\theta, \psi) := \mathcal{F}(\theta, \hat{\psi}, U_{\psi})$ for the function $\hat{\psi} := \log(1 + e^{\psi})$. Note that $\hat{\psi} \in \text{PSH}^{\infty}(X, \theta)$ by Lemma 2.1. Note also that it follows from the equation (1) that $(I_{\psi})_{\text{reg}} = \{\log(1 + e^{r}) \mid r \in (I_{\psi})_{\text{reg}}\}$, and from the equation (2) that $(\theta + \sqrt{-1}\partial\bar{\partial}\hat{\psi})_{x} \neq 0$ holds on each point $x \in U_{\psi}$. It also follows from the equation (2) and Lemma 3.2 (ii) that $T_{\mathcal{F}(\theta, \psi)} = (\partial\hat{\psi})^{-1} \subset T_{\psi}$. Therefore $\mathcal{F}(\theta, \psi)$ is a $C^\infty$ foliation of real codimension 2.

On a neighborhood $U$ of each point $x \in U_{\psi}$ such that $(\theta + \sqrt{-1}\partial\bar{\partial}\hat{\psi})_{x} \neq 0$, it holds that $\mathcal{F}(\theta, \psi, U) = \mathcal{F}(\theta, \hat{\psi}, U)$, since

$$\theta + \sqrt{-1}\partial\bar{\partial}\hat{\psi} = \left(\frac{1}{1 + e^{\psi}} \cdot f_{\theta, \psi} + \frac{e^{\psi}}{1 + e^{\psi}} \cdot (f_{\theta, \psi} + g_{\psi}) + \frac{e^{\psi}}{(1 + e^{\psi})^2} \right) \cdot \sqrt{-1}\partial\psi \wedge \overline{\partial}\psi$$

is pointwisely parallel to $\theta + \sqrt{-1}\partial\bar{\partial}\hat{\psi} = (f_{\theta, \psi} + g_{\psi}) \cdot \sqrt{-1}\partial\psi \wedge \overline{\partial}\psi$ on a neighborhood of such a point $x$, where $f_{\theta, \psi}$ and $g_{\psi}$ are the functions on a neighborhood of $x$ as in the previous subsection.

For this foliation $\mathcal{F}(\theta, \psi)$ on $U_{\psi}$, we show the following:

**Proposition 3.4.** The following holds:

(i) For any $r \in (I_{\psi})_{\text{reg}}$, $\psi^{-1}(r)$ is the union of some leaves of $\mathcal{F}(\theta, \psi)$.

(ii) For any $\hat{\theta} \in \text{SP}(\alpha)$, $\hat{\theta}$ is zero along each leaf of $\mathcal{F}(\theta, \psi)$: i.e. $i_{\mathcal{L}}\hat{\theta} \equiv 0$ holds for any leaf $\mathcal{L}$ of $\mathcal{F}(\theta, \psi)$, where $i_{\mathcal{L}} : \mathcal{L} \to X$ is the holomorphic immersion.

**Proof.** (i) By Lemma 3.2 (i), the function $\psi$ is $\mathcal{F}(\theta, \psi)$-leafwise constant. Therefore, for each leaf $\mathcal{L}$ of $\mathcal{F}(\theta, \psi)$ and for each $r \in (I_{\psi})_{\text{reg}}$, either $\mathcal{L} \cap \psi^{-1}(r) = \emptyset$ or $\mathcal{L} \subset \psi^{-1}(r)$. holds.

(ii) By the $\partial\bar{\partial}$-lemma, there exists a function $\varphi \in \text{PSH}(X, \theta)$ such that $\hat{\theta} = \theta + \sqrt{-1}\partial\bar{\partial}\varphi$. The form $\theta$ is zero along the leaves of $\mathcal{F}(\theta, \psi)$ by Lemma 3.2 (ii) and $\varphi$ is $\mathcal{F}(\theta, \psi)$-leafwise constant by Lemma 3.3 (i), from which the assertion follows.

Now we can state the main result of this section.

**Theorem 3.5.** The foliation $\mathcal{F}(\theta, \psi)$ is a non-singular holomorphic foliation on $U_{\psi}$. Moreover, either (a) or (b) holds:

(a) There exists a surjective holomorphic map $\Phi : X \to R$ to a compact Riemann surface $R$ whose leaves are connected such that $\mathcal{F}(\theta, \psi)$ coincides with the restriction of the foliation on $X$ whose leaves are the fibers of $\Phi$. In this case, $\psi = \psi_{R} \circ \Phi$ for some function $\psi_{R} : R \to \mathbb{R}$.

(b) For any $r \in (I_{\psi})_{\text{reg}}$ and any connected component $A$ of the preimage $\psi^{-1}(r)$, there does not exist a non-constant $\mathcal{F}(\theta, \psi)$-leafwise constant $\mathbb{R}$-valued function on $A$ of class $C^\infty$. In this case, for any connected component $U$ of $U_{\psi}$, there exists an $\mathcal{F}(\theta, \psi)$-adaptive function $h : \overline{U} \to [-\infty, +\infty]$ such that $h|_{U} = \chi_{U} \circ \psi|_{U}$ for some strictly increasing function $\chi_{U} : \psi(U) \to \mathbb{R}$ of class $C^\infty$.

Theorem 1.4 follows from this Theorem 3.5 as follows:

**Proof of Theorem 1.4.** In the case (a) of Theorem 3.5, the preimage of almost all Jordan loops in $R$ of class $C^\omega$ are real analytic compact Levi-flat hypersurfaces of $X$. In the case
(b) of Theorem 3.5, \( h^{-1}(r) \) is a real analytic compact Levi-flat hypersurface of \( X \) for any \( r \in \text{Image } \chi_U \).

**Proof of Theorem 3.5.** By Lemmata 3.2 and 3.3, there exist \( F(\theta, \psi) \)-leafwise constant functions \( \hat{f} := f_{\theta, \hat{\psi}}: U_\psi \to \mathbb{R}_{\geq 0} \) and \( \hat{g} := g_{\theta}: U_\psi \to \mathbb{R} \) of class \( C^\infty \) such that \( \theta = \hat{f} \cdot \sqrt{-1} \partial \hat{\psi} \wedge \overline{\partial} \hat{\psi} \) and \( \sqrt{-1} \partial \hat{\psi} = \hat{g} \cdot \sqrt{-1} \partial \hat{\psi} \wedge \overline{\partial} \hat{\psi} \) holds on \( U_\psi \).

First, we consider the case where, for any \( r \in (I_\psi)_{\text{reg}} \) and any connected component \( A \) of the preimage \( \psi^{-1}(r) \), there does not exist a non-constant \( F(\theta, \psi) \)-leafwise constant \( \mathbb{R} \)-valued function on \( A \) of class \( C^\infty \). Take a connected component \( U \) of \( U_\psi \). As \( \hat{\psi}|_U \) is a proper submersion, it follows from Ehresmann’s lemma that \( U \cap \hat{\psi}^{-1}(r) \) is connected for any \( r \in \hat{\psi}(U) \), from which one has that there exists a function \( G_U: (a, b) \to \mathbb{R} \) such that \( \hat{g}|_U = G_U \circ \hat{\psi} \) holds, where \( (a, b) := \hat{\psi}(U) \), by the assumption. Take a non-constant function \( \chi: (a, b) \to \mathbb{R} \) such that

\[
\chi'(t) = \exp \left( - \int_{\frac{a}{\sqrt{b}}}^t G_U(s) \, ds \right).
\]

Then, it follow from the equation \( \chi''(t) = -G_U(t) \cdot \chi'(t) \) that \( \sqrt{-1} \partial \hat{h} = 0 \) holds for the function \( h := \chi \circ \hat{\psi} \) on \( U \), since \( \sqrt{-1} \partial \hat{h} = (\chi' \circ \hat{\psi}) \cdot \sqrt{-1} \partial \hat{\psi} + (\chi'' \circ \hat{\psi}) \cdot \sqrt{-1} \partial \hat{\psi} \wedge \overline{\partial} \hat{\psi} \). As the function function \( h: \overline{U} \to [-\infty, +\infty] \) is clearly \( F(\theta, \psi) \)-adaptive by construction, the assertion \( (b) \) holds in this case by letting \( \chi_U(t) := \chi(\log(1 + e^t)) \). Note that, as is clear by the pluriharmonicity of \( h \), \( F(\theta, \psi) \) is a non-singular holomorphic foliation in this case.

Next, we consider the case where there exist \( r \in (I_\psi)_{\text{reg}} \) and a connected component \( A \) of the preimage \( \psi^{-1}(r) \) such that there exists a non-constant \( F(\theta, \psi) \)-leafwise constant \( \mathbb{R} \)-valued function \( \tilde{g} \) on \( A \) of class \( C^\infty \). In this case, it follows form the following Lemma 3.6 that there exist a leaf \( Y \) of \( F(\theta, \psi) \) and a surjective holomorphic map \( \Phi: X \to R \) to a compact Riemann surface \( R \) whose fibers are connected such that \( Y \) is a fiber of \( \Phi \). In what follows, we show the existence of the function \( \psi_R \) on \( R \) as in the assertion \( (a) \). Note that, if such a function \( \psi_R \) exists, then it follows from \( d\psi = \Phi^*d\psi_R \) that \( \Phi \) and \( \psi_R \) has no critical point on \( U_\psi \) and \( \Phi(U_\psi) \), respectively, from which one obtains by using Lemma 3.2 \( (i) \) that \( F(\theta, \psi) \) coincides with the foliation on \( U_\psi \) whose leaves are the fibers of \( \Phi \).

Now it is sufficient to show that \( \psi|_Z \) is constant for a fiber \( Z \) of \( \Phi \). Assuming that it is not the case (and thus \( d\psi|_{Z_{\text{reg}}} \neq 0 \)), we prove it by contradiction. If \( d\psi|_{Z_{\text{reg}}} \neq 0 \), it follows from the equation \( (2) \) that \( (\theta + \sqrt{-1} \partial \hat{\psi})|_{Z_{\text{reg}}} \neq 0 \). From this and the semi-positivity of \( \theta + \sqrt{-1} \partial \hat{\psi} \), one has that

\[
(\alpha.c_1([Z]).\{\omega\})^{n-2} = \int_Z (\theta + \sqrt{-1} \partial \hat{\psi})|_Z \wedge \omega|_Z^{n-2} > 0
\]

holds for a Kähler form \( \omega \) of \( X \). On the other hand, it follows

\[
(\alpha.c_1([Y]).\{\omega\})^{n-2} = \int_Y \theta|_Y \wedge \omega|_Y^{n-2} = 0
\]

from Proposition 3.4 \( (ii) \), since \( Y \) is a leaf of \( F(\theta, \psi) \). As both \( Y \) and \( Z \) are fibers of \( \Phi \), these lead to the contradiction. \( \square \)
Lemma 3.6. Assume that there exist \( r \in (I_\psi)_{\text{reg}} \) and a connected component \( A \) of the preimage \( \psi^{-1}(r) \) such that there exists a non-constant \( F(\theta, \psi) \)-leafwise constant \( \mathbb{R} \)-valued function \( \tilde{g} \) on \( A \) of class \( C^\infty \). Then there exists a leaf \( Y \) of \( \mathcal{F}(\theta, \psi) \) and a surjective holomorphic map \( \Phi : X \to R \) to a compact Riemann surface \( R \) whose fibers are connected such that \( Y \) is a fiber of \( \Phi \).

Proof. By Sard’s theorem, there exists an open non-empty connected interval \( J \subset \mathbb{R} \) which is included in the set of all the regular values of the function \( \tilde{g} : A \to \mathbb{R} \) (Note that the set of all the critical values of \( \tilde{g} \) is closed since the map \( \tilde{g} \), which is a map from a compact space to a Hausdorff space, is a closed map). Take different three points \( p_1, p_2, \) and \( p_3 \) from \( J \), and set \( Y_j := \tilde{g}^{-1}(p_j) \) for \( j = 1, 2, 3 \). As each \( p_j \) is a regular value, \( Y_j \) is a real submanifold of \( A \). Therefore, as \( \tilde{g} \) is \( \mathcal{F}(\theta, \psi) \)-leafwise constant and each leaf of \( \mathcal{F}(\theta, \psi) \) is a holomorphically immersed complex submanifold, it follows that \( Y_j \) is (holomorphically embedded image of) a complex submanifold of \( X \) for \( j = 1, 2, 3 \). Moreover, by applying Ehresmann’s lemma to the map \( \tilde{g}^{-1}(J) \), one obtains that \( Y_j \) is homotopic to \( Y_3 \), from which it follows that the line bundle \( L := [Y_2 - Y_3] \) on \( X \) is topologically trivial. Note that \( L|_{Y_3} \) is holomorphically trivial by the construction.

Now we may assume that the restriction map \( H^1(X, \mathcal{O}_X) \to H^1(Y_1, \mathcal{O}_{Y_1}) \) is injective, since otherwise the natural map \( X \to \text{Alb}(X)/\text{Alb}(Y_1) \) induced by the Albanese map defines a fibration such that \( Y_1 \) is a fiber by [CLPT, Proposition 2.7] (see also [N]). As the natural map \( \text{Pic}^0(X) \to \text{Pic}^0(Y_1) \) is a finite covering of the image in this case, one has that there exists a positive integer \( m \) such that \( L^m := L^{\otimes m} \) is the holomorphically trivial line bundle on \( X \). Thus one obtains a fibration \( X \to \mathbb{P}^1 =: R \) to the projective line by considering the complete linear system \( |L^m| \) of which \( Y_2 \) and \( Y_3 \) are fibers. Finally one can modify the fibration so that its fibers are connected by considering a branched finite covering of \( R \), which proves the lemma. \( \square \)

4. Observation in Cases (a) and (b) in Theorem 3.5

Let \( X \) be a connected compact Kähler manifold and \( \alpha \in H^{1,1}(X, \mathbb{R}) \) be a class with \( \text{nd}(\alpha) = 1 \) and \( \# \text{SP}(\alpha) > 1 \). In this section, we investigate the foliation \( \mathcal{F}(\theta, \psi) \) and the domain \( U_\psi \) for \( \theta \in \text{SP}(\alpha) \) and \( \psi \in \text{PSH}^\infty(X, \theta) \setminus \mathbb{R} \) in the following two cases: First in the case where the assertion (a) of Theorem 3.5 holds for some \( \theta \in \text{SP}(\alpha) \) and \( \psi \in \text{PSH}^\infty(X, \theta) \setminus \mathbb{R} \), next in the case where the assertion (b) of Theorem 3.5 holds for any \( \theta \in \text{SP}(\alpha) \) and any \( \psi \in \text{PSH}^\infty(X, \theta) \setminus \mathbb{R} \).

From this section, we often use the following:\footnote{We first showed Lemma 4.1 only when \( X \) is either a surface or a projective manifold by using Hodge index theorem and the hard Lefschetz theorem (see [V, Theorem 6.33] and [V, Theorem 6.25] for example). The proof in the general case is taught by Prof. Valentino Tosatti.}

Lemma 4.1. Let \( X \) be a compact complex manifold of dimension \( n \), \( \theta \) an element of \( \text{SP}(\alpha) \), \( \omega \) a Kähler form of \( X \), and \( \hat{\theta} \) be a \( C^\infty \)ly smooth d-closed semi-positive \((1,1)\)-form. Assume that both \( \{\theta \wedge \theta\} \) and \( \{\hat{\theta} \wedge \hat{\theta}\} \) are zero in \( H^{2,2}(X, \mathbb{C}) \) and that \( \int_X \hat{\theta} \wedge \omega^{n-1} = \int_X \theta \wedge \omega^{n-1} \) holds. Then \( \hat{\theta} \) is an element of \( \text{SP}(\alpha) \).
**Proof.** It is sufficient to show that \( \{ \eta \} = 0 \in H^{1,1}(X, \mathbb{R}) \) for the form \( \eta := \theta - \hat{\theta} \). Consider the bilinear form \( Q \) defined by

\[
Q(\beta, \gamma) := \int_X \beta \wedge \overline{\gamma} \wedge \omega^{n-2}.
\]

As \( \int_X \eta \wedge \omega^{n-1} = 0 \) and \( Q(\eta, \eta) = 0 \) by the assumption, the lemma follows from [DN, Theorem A] (See also the argument in the end of [TZ, Remark 3.4]). \( \square \)

4.1. **Observation when the assertion (a) holds for some \((\theta, \psi)\).** First, let us investigate the case where the assertion (a) of Theorem 3.5 holds for some \( \theta \in \text{SP}(\alpha) \) and \( \psi \in \text{PSH}^\infty(X, \theta) \setminus \mathbb{R} \). Take such \( \theta \) and \( \psi \). Let \( \Phi: X \to R \) and \( \psi_R: R \to \mathbb{R} \) be those as in the assertion (a). Then one can show the following:

**Proposition 4.2.** Let \( X, \alpha, \theta, \psi, \Phi \), and \( R \) be as above. Then the following holds:

(i) There exists a Kähler class \( \alpha_R \) of \( R \) such that \( \alpha = \Phi^* \alpha_R \).

(ii) Any element of \( \text{SP}(\alpha) \) is zero along each fiber of \( \Phi \).

(iii) The set \( K_\alpha \) is included in the set of all the critical points of \( \Phi \).

(iv) For any \( p \in R \), the set of all the singular points \( (\Phi^{-1}(p))_{\text{sing}} \) of the (set-theoretical) fiber \( \Phi^{-1}(p) \) is included in \( K_\alpha \).

**Proof.** (i) Fix a Kähler form \( \omega_R \) of \( R \) and a non-constant non-negative function \( \rho: R \to \mathbb{R}_{\geq 0} \) of class \( C^\infty \) such that the support \( \text{Supp} \rho \) of \( \rho \) is included in \( \Phi(U_\psi) \). Then, as \( \hat{\theta} := \Phi^*(\rho \cdot \omega_R) \) satisfies \( d\hat{\theta} = 0, \hat{\theta} \wedge \hat{\theta} = 0 \), and \( \hat{\theta} \wedge \theta = 0 \) (the last equation follows from Lemma 3.2 (ii)), it follows from Lemma 4.1 that we may assume \( \hat{\theta} \in \text{SP}(\alpha) \) by replacing \( \rho \) with \( A \cdot \rho \) for a suitable positive constant \( A \). Therefore it follows that \( \alpha = \Phi^* \alpha_R \) for the class \( \alpha_R := \{ \rho \cdot \omega_R \} \). Note that \( \alpha_R \) is a Kähler class of \( R \) since \( \int_R \rho \cdot \omega_R > 0 \) holds.

(ii) Take a Kähler form \( \omega_R \) of \( R \) such that \( \{ \omega_R \} = \alpha_R \). Then, by the \( \partial \overline{\partial} \)-lemma, any element \( \theta \) of \( \text{SP}(\alpha) \) can be written as \( \theta = \Phi^* \omega_R + \sqrt{-1} \partial \overline{\partial} \varphi \) by using some element \( \varphi \in \text{PSH}^\infty(X, \Phi^* \omega_R) \). As \( \Phi^* \omega_R \) is zero along each fiber, \( \varphi \) is plurisubharmonic along each fiber of \( \Phi \), from which one can deduce that \( \varphi \) is \( \Phi \)-fiberwise constant by the maximum principle. Therefore \( \hat{\theta} \) is also zero along the fibers.

(iii) Let \( x \in X \) be a regular point of \( \Phi \). Take a Kähler form \( \omega_R \) of \( R \) such that \( \{ \omega_R \} = \alpha_R \) and a function \( f: R \to \mathbb{R} \) of class \( C^\infty \) such that \( (df)_{\Phi(x)} \neq 0 \). As \( \omega_R > 0 \) and \( X \) is compact, \( \varepsilon \in \text{PSH}^\infty(R, \omega_R) \) holds for a sufficiently small positive constant \( \varepsilon \). Thus, by letting \( \theta' := \Phi^* \omega_R \in \text{SP}(\alpha) \), one has that \( \varphi := \varepsilon \cdot (f \circ \Phi) \) is an element of \( \text{PSH}^\infty(X, \theta') \). As \( (d\varphi)_x \neq 0 \) by construction, it follows that \( x \notin K_\alpha \), from which the assertion holds.

(iv) Take a point \( x \in X \setminus K_\alpha \). It follows from Lemma 2.3 that there exists an element \( \varphi \in \text{PSH}^\infty(X, \theta') \) such that \( (d\varphi)_x \neq 0 \), where \( \theta' = \Phi^* \omega_R \) is the form as above. Note that \( \varphi \) is plurisubharmonic along each fiber of \( \Phi \) (since \( \theta' \) is fiberwise zero), from which one can deduce that \( \varphi \) is \( \Phi \)-fiberwise constant by the maximum principle. As we have seen in §3, one can consider the foliation \( \mathcal{F}(\theta', \mathcal{F}, B) \) for the function \( \tilde{\varphi} := \log(1 + e^\varphi) \) and a small neighborhood \( B \) of \( x \). Let \( Y \subset B \) be the leaf of \( \mathcal{F}(\theta', \mathcal{F}, B) \) which passes through the point \( x \). As \( \varphi \) is \( \Phi \)-fiberwise constant and \( \varphi|_y \) is constant by Lemma 3.2 (i), it follows that \( Y \subset \Phi^{-1}(p) \) for some \( p \in R \) by shrinking \( B \) if necessary. Therefore, it follows that \( x \) is a non-singular point of the fiber \( \Phi^{-1}(p) \). \( \square \)
As is clear by the argument in the proof of Proposition 4.2 (ii), we need to define the foliation $\mathcal{F}_\alpha$ so that its leaves are the fibers of $\Phi$. Note that, once we adopt such definition of $\mathcal{F}_\alpha$, it easily follows that $K_{\alpha}^s = \emptyset$ by considering the $\mathcal{F}_\alpha$-adaptive function $h_R \circ \Phi$ on $X$, where, for two different regular values $p$ and $q$ of $\Phi$, $h_R$ is a harmonic function on $R \setminus \{p, q\}$ such that $h_R(w) \to -\infty$ holds as $w \to p$ and that $h_R(w) \to +\infty$ holds as $w \to q$.

4.2. Observation when the assertion (b) holds for any $(\theta, \psi)$. Next, let us investigate the case where the assertion (b) of Theorem 3.5 holds for any $\theta \in \text{SP}(\alpha)$ and any $\psi \in \text{PSH}^\infty(X, \theta) \setminus \mathbb{R}$. In this case, the following Condition (\bigcirc) holds:

**Condition** (\bigcirc): For any $\theta \in \text{SP}(\alpha)$, $\psi \in \text{PSH}^\infty(X, \theta) \setminus \mathbb{R}$, $r \in (I_\psi)_{\text{reg}}$, and any connected component $A$ of the preimage $\psi^{-1}(r)$, there does not exist a non-constant $\mathcal{F}(\theta, \psi)$-leafwise constant $\mathbb{R}$-valued function on $A$ of class $C^\infty$. For any such $\theta$ and $\psi$, there exists an $\mathcal{F}(\theta, \psi)$-adaptive function $h : \mathcal{U} \to [-\infty, +\infty]$ such that $h|_U = \chi_U \circ \psi|_U$ for some strictly increasing function $\chi_U : \psi(U) \to \mathbb{R}$ of class $C^\infty$ for any connected component $U$ of $\Psi$. \hfill \Box

In what follows, we use the following notation for $\theta \in \text{SP}(\alpha)$: Let $\mathcal{U}(\theta)$ be the set of all triples $(\psi, U, J)$ such that $\psi \in \text{PSH}^\infty(X, \theta) \setminus \mathbb{R}$, $J$ is a connected open interval included in $(I_\psi)_{\text{reg}}$, and $U$ is a connected component of $\psi^{-1}(r)$. Let $\mathcal{U}(\theta)$ be the set of all triples $(\psi, U, J) \in \mathcal{U}(\theta)$ such that $J$ is a relatively compact connected open interval included in $(I_\psi)_{\text{reg}}$. For $(\psi, U, J) \in \mathcal{U}(\theta)$, we denote by $\mathcal{H}(\psi, U, J)$ the set of all the pluriharmonic functions $\tilde{h}$ on $U$ such that $h = \chi \circ \psi|_U$ holds for some function $\chi : J \to \mathbb{R}$.

We also use the notation $\mathcal{H}^*(\psi, U, J) := \mathcal{H}(\psi, U, J) \setminus \mathbb{R}$ for each $(\psi, U, J) \in \mathcal{U}(\theta)$. Note that the set $\mathcal{H}^*(\psi, U, J)$ is non-empty by Condition (\bigcirc). First we show the following:

**Lemma 4.3.** Let $(\psi, U, J)$ be an element of $\mathcal{U}(\theta)$ and $h = \chi_U \circ \psi|_U$ be (the restriction of) a pluriharmonic function as in Condition (\bigcirc). Then it holds that $\mathcal{H}(\psi, U, J) = \{c_1h + c_2 \mid c_1, c_2 \in \mathbb{R}\}$ and $\mathcal{H}^*(\psi, U, J) = \{c_1h + c_2 \mid c_1, c_2 \in \mathbb{R}, c_1 \neq 0\}$. Especially, any element of $\mathcal{H}^*(\psi, U, J)$ is the restriction of an $\mathcal{F}(\theta, \psi)$-adaptive function as in Condition (\bigcirc).

**Proof.** Take a pluriharmonic functions $\tilde{h}$ on $U$ such that $\tilde{h} = \chi \circ \psi|_U$ holds for some function $\chi : J \to \mathbb{R}$, and $(a)$ restriction of) an $\mathcal{F}(\theta, \psi)$-adaptive function $h|_U = \chi_U \circ \psi|_U$ as in Condition (\bigcirc). As $\chi_U$ is strictly increasing, $\chi_U$ is bijective to the image. Therefore one has that $\tilde{h} = \chi \circ \chi_U^{-1} \circ h$ holds on $U$. As $\sqrt{-1}\partial\bar{\partial}h \equiv 0$ and $\sqrt{-1}\partial\bar{\partial}\tilde{h} \equiv 0$, it holds that $(\tilde{\chi} \circ \chi_U^{-1})^{\nu} \circ \tilde{h} \cdot \sqrt{-1}\partial\bar{\partial}h \wedge \bar{\partial}h \equiv 0$. Thus one has that $(\tilde{\chi} \circ \chi_U^{-1})^{\nu} \equiv 0$, from which it follows that $\tilde{\chi} \circ \chi_U^{-1}$ is a polynomial of degree at most one. \hfill \Box

4.2.1. Comparison with another element of PSH$^\infty(X, \theta)$. Let $\theta$ be an element of SP$(\alpha)$. In this subsection, we consider two elements $\varphi$ and $\psi$ of PSH$^\infty(X, \theta)$.

**Lemma 4.4.** Assume that Condition (\bigcirc) holds and that $U_\varphi \cap U_\psi \neq \emptyset$. Let $U$ be a connected component of $U_\varphi \cap U_\psi$ and $J$ be the image $\psi(U)$. Then the following holds:

(i) $(\psi, U, J)$ is an element of $\mathcal{U}(\theta)$.

(ii) $\varphi|_U = \chi \circ \psi|_U$ holds for some strictly increasing or strictly decreasing function $\chi : J \to \mathbb{R}$.
(iii) \((\varphi, U, \chi(J))\) is an element of \(\mathcal{U}(\theta)\).

(iv) The foliation \(\mathcal{F}(\theta, \psi)|_U\) coincides with \(\mathcal{F}(\theta, \varphi)|_U\).

**Proof.** Let \(W\) be the connected component of \(\psi^{-1}(J)\) which includes \(U\). As \(W \subset U_{\varphi}\), it follows from Lemma 3.3 (i) that \(\varphi = \mathcal{F}(\theta, \psi)|_W\) is leafwise constant. By **Condition** \((\bigtriangledown)\), it holds that \(\varphi|_{W \cap \psi^{-1}(r)}\) is constant for each \(r \in J\), from which it follows that there exists a function \(\chi: J \to \mathbb{R}\) such that \(\varphi|_W = \chi \circ \psi|_W\) holds.

First, let us show that \(\chi'(r) \neq 0\) for each \(r \in J\). Take \(r \in J\). As \(J\) is the image of \(U\) by \(\psi\), there exists a point \(x \in U\) such that \(\psi(x) = r\). As \(x \in U_{\varphi} \cap U_{\psi}\), one has that \((d\varphi)_x \neq 0\) and \((d\psi)_x \neq 0\). Thus it follows from the equation \((d\varphi)_x = \chi'(r) \cdot (d\psi)_x\) that \(\chi'(r) \neq 0\). Therefore one has that \(\chi\) is strictly increasing or strictly decreasing.

Again by the equation \(d\varphi = (\chi \circ \psi) \cdot d\psi\) on \(W\), it follows that \(W \subset U_{\varphi}\), from which one has that \(U = W\). The assertions (i), (ii), and (iii) follows from these observations. The assertion (iv) follows from the fact that \(T_{\mathcal{F}(\varphi, \psi)} = (\partial \psi)\) and \(T_{\mathcal{F}(\theta, \varphi)} = (\partial \varphi)\) holds on \(U\) (by Lemma 3.2 (i)) and the equation \(d\varphi = (\chi \circ \psi) \cdot d\psi\) on \(U\).

4.2.2. Observation on foliation adaptive functions. Let \(\theta\) be an element of \(\text{SP}(\alpha)\). In this subsection, we investigate foliation adaptive functions under the assumption that **Condition** \((\bigtriangledown)\) holds. As a preliminary, first we show the following:

**Lemma 4.5.** When **Condition** \((\bigtriangledown)\) holds, the following holds for \((\psi, U, I) \in \mathcal{U}(\theta), h \in \mathcal{H}(\psi, U, I)\), and \(J := h(U)\):

(i) \(\theta|_U = (\rho \circ h) \cdot \sqrt{-1} \partial h \wedge \bar{\partial} h\) holds for some function \(\rho: J \to \mathbb{R}_{\geq 0}\).

(ii) For any \(\varphi \in \text{PSH}^\infty(X, \theta)\), there exists a function \(\chi: J \to \mathbb{R}\) such that \(\chi'' \geq -\rho\) and \(\varphi|_U = \chi \circ h\) hold, where \(\rho\) is the function as in the assertion (i).

(iii) For any \(\varphi \in \text{PSH}^\infty(X, \theta)\) with \(U \subset U_{\varphi}\), \((\varphi, U, \varphi(U)) \in \mathcal{U}(\theta)\) and \(\mathcal{H}(\varphi, U, \varphi(U)) = \mathcal{H}(\psi, U, I)\) hold.

**Proof.** The assertion (i) follows from Lemma 3.3 (ii), **Condition** \((\bigtriangledown)\), and Lemma 4.3. The assertion (ii) follows from Lemma 3.3 (i), **Condition** \((\bigtriangledown)\), and Lemma 4.3. Let us show the assertion (iii). From Lemma 4.4 (iii), it follows that \((\varphi, U, \varphi(U)) \in \mathcal{U}(\theta)\). Take an element \(\tilde{h} \in \mathcal{H}(\varphi, U, \varphi(U))\). By Lemma 4.3, there exists a function \(F: \varphi(U) \to \mathbb{R}\) such that \(\tilde{h} = F \circ \varphi|_U\). Thus one has \(\tilde{h} = (F \circ \chi) \circ h\), where \(\chi\) is the function as in the assertion (i). As both \(h\) and \(h\) is pluriharmonic and \(\sqrt{-1} \partial h(F \circ \chi) \circ h = ((F \circ \chi)'' \circ h) \cdot \sqrt{-1} \partial h \wedge \bar{\partial} h\), one has that \(F \circ \chi\) is a polynomial of at most degree one, from which the assertion follows.

In what follows, we often consider the following configuration:

**Configuration** \((\sharp)\): A domain \(W\) of a connected compact Kähler manifold \(X\) satisfies the following conditions: The boundary \(\partial W\) consists of two connected components \(H_+\) and \(H_-\), and there exist open neighborhoods \(W_{\pm}\) of \(H_{\pm}\) in \(X\) and \(\varphi_{\pm} \in \text{PSH}^\infty(X, \theta)\) for some \(\theta \in \text{SP}(\alpha)\) such that \(U_{\pm} := W_{\pm} \cap W\) and \((a_{\pm}, b_{\pm}) := \varphi_{\pm}(U_{\pm})\) satisfies \(\overline{U_+} \cap \overline{U_-} = \emptyset\), \((\varphi_{\pm}, U_{\pm}, (a_{\pm}, b_{\pm})) \in \mathcal{U}(\theta)\), \(H_- = \varphi^{-1}_-(b_-) \cap \overline{U_-}\), and \(H_+ = \varphi^{-1}_+(b_+) \cap \overline{U_+}\). \(\square\)
For such \( X, W, H_\pm, \) and \( \varphi_\pm: U_\pm \to (a_\pm, b_\pm) \) as in **Configuration (i)**, we have the following:

**Lemma 4.6.** Assume that Condition (\( \forall \)) holds. Let \( X, W, H_\pm, \) and \( \varphi_\pm: U_\pm \to (a_\pm, b_\pm) \) be as in **Configuration (i)**. Then either (i) or (ii) holds for \( V := X \setminus \overline{W} \).

(i) \( V \) is connected and there exists a continuous function \( h_V: \overline{U_+} \cup V \cup U_- \to \mathbb{R} \) which is a non-constant pluriharmonic function on the interior of \( \overline{U_+} \cup V \cup U_- \) such that \( h_V|_{U_\pm} \in \mathcal{H}^\ast(\varphi_\pm, U_\pm, (a_\pm, b_\pm)) \).

(ii) \( V \) consists of two connected components \( V^+ \) and \( V^- \) such that \( \partial V^- = H_- \) and \( \partial V^+ = H_+ \), and there exists a continuous function \( h_W: \overline{W} \to \mathbb{R} \) such that \( h_W|_{U_\pm} \in \mathcal{H}^\ast(\varphi_\pm, U_\pm, (a_\pm, b_\pm)) \).

**Proof.** Denote by \( \widetilde{V} \) the interior of \( \overline{U_+} \cup V \cup U_- \). First let us note that it follows by considering Ehresmann’s lemma for \( \varphi_\pm|_{U_\pm} \) and Mayer–Vietoris sequence for the covering \( \{\widetilde{V}, W\} \) of \( X \) that either of the following holds: \( V \) is connected, or \( V \) consists of two connected components \( V^+ \) and \( V^- \) such that \( \partial V^- = H_- \) and \( \partial V^+ = H_+ \).

Let \( \overline{U_\pm} \) be the connected component of \( \varphi_\pm^{-1}(I_{\varphi_\pm})_{\text{reg}} \) which includes \( U_\pm \). Denote by \( (c_\pm, d_\pm) \) the interval \( \varphi_\pm(\overline{U_\pm}) \). Note that \( c_\pm < a_\pm < b_\pm < d_\pm \) holds, since \( (\varphi_\pm, U_\pm, (a_\pm, b_\pm)) \in \mathcal{U}_\varepsilon(\theta) \). Take an element \( h_\pm \in \mathcal{H}^\ast(\varphi_\pm, U_\pm, (c_\pm, d_\pm)) \) as in Condition (\( \forall \)). Set \( a'_\pm := h_\pm(\varphi_\pm^{-1}(a_\pm) \cap \overline{U_\pm}), b'_\pm := h_\pm(\varphi_\pm^{-1}(b_\pm) \cap \overline{U_\pm}), c'_\pm := \inf h_\pm, \) and \( d'_\pm := \sup h_\pm \). Then, for a function \( \rho_\pm: (c'_\pm, d'_\pm) \to \mathbb{R}_{\geq 0} \) of class \( C^\infty \) such that \( \text{Supp} \rho_\pm = [a'_\pm, b'_\pm] \), it follows from Lemma 4.7 below that

\[
\theta_\pm := \left\{ \begin{aligned}
\rho_\pm \circ h_\pm & \cdot \sqrt{-1} \partial h_\pm & \wedge \overline{\partial h_\pm} & & \text{on } \overline{U_\pm}, \\
0 & & & & \text{on } X \setminus \overline{U_\pm}
\end{aligned} \right.
\]

is an element of \( \text{SP}(\alpha) \) by replacing \( \rho_\pm \) with \( A_\pm \cdot \rho_\pm \) for some \( A_\pm > 0 \) if necessary. Take a function \( f: X \to \mathbb{R} \) such that \( \theta_\pm = \theta_- + \sqrt{-1} \partial \overline{\partial} f \), whose existence is assured by the \( \partial \overline{\partial} \)-lemma. Note that \( \forall f \in \text{PSH}^\infty(\theta_\pm) \). It follows from Lemma 4.5 (ii) that \( f|_{\overline{U_\pm}} = \chi_\pm \circ h_\pm \) for some function \( \chi_\pm: (c'_\pm, d'_\pm) \to \mathbb{R} \). As \( \varphi_\pm|_{U_\pm} \equiv 0 \) by construction, one can deduce from the equation \( \sqrt{-1} \partial \overline{\partial} f|_{\overline{U_\pm}} = \sqrt{-1} \partial \overline{\partial} (\chi_\pm \circ h_\pm) \) that \( \chi_\pm'' = \pm \rho_\pm \) holds on a neighborhood of \( [a'_\pm, b'_\pm] \).

First, let us consider the case where \( \chi_-|_{(a'_- - \varepsilon, a'_-)} = A \) is not a constant function for any small positive constant \( \varepsilon \). In this case, consider the function \( h_V := f|_{V} \). As \( \theta_\pm = \theta_- + \sqrt{-1} \partial \overline{\partial} f \) and \( V \cap \text{Supp } \theta_\pm = \emptyset, \) \( h_V \) is a pluriharmonic function on \( V \). By the assumption on \( \chi_-|_{(a'_- - \varepsilon, a'_-)} \), \( h_V \) is non-constant. Therefore \( V \) is connected, since otherwise the contradiction occurs by considering the maximum principle for \( h_V|_{V^-} \). (Note that, as \( f|_{\overline{U_-}} = \chi_- \circ h_- \), \( f|_{\partial V^-} = f|_{H_-} \) is constant.) As \( \chi'_\pm \equiv 0 \) on the complement of \( [a'_\pm, b'_\pm] \), one can easily extend \( h_V \) by using elements of \( \mathcal{H}(\varphi_\pm, U_\pm, (a_\pm, b_\pm)) \) to construct a non-constant pluriharmonic function on \( \widetilde{V} \) (by Lemma 4.3), from which one has that the assertion (i) holds in this case.

Next, let us consider the case where \( \chi_-|_{(a'_- - \varepsilon, a'_-)} \equiv A \) holds for a small positive constant \( \varepsilon \) and a constant \( A \in \mathbb{R} \). Note that \( \chi_- \) is a strictly decreasing function on \( (b'_-, b'_- + \varepsilon) \) in this case, since \( \chi''_\pm = -\rho_- \) on \( [a'_-, b'_-] \) and \( \chi''_\pm|_{(a'_- - \varepsilon, a'_-)} \equiv 0 \). Let us consider the function
\( h_W := f|_{W_0} \), where \( W_0 := W \setminus U_+ \cup U_- \). As \( \theta_+ = \theta_- + \sqrt{-1} \partial \bar \partial f \) and \( W_0 \cap \text{Supp} \theta_\pm = \emptyset \), \( h_W \) is a pluriharmonic function on \( W_0 \). The non-constantness of \( \chi_-|_{(U_-, b'_- + \varepsilon)} \) implies that \( h_W \) is non-constant. As \( \chi_+'' \equiv 0 \) on the complement of \([a'_+, b'_+]\), one can easily extend \( h_W \) by using some elements of \( H(\varphi_\pm, U_\pm, (a_\pm, b_\pm)) \) to construct a non-constant pluriharmonic function on \( W \). Finally, let us show that \( V \) is not connected. Assume that \( V \) is connected. Then, as \( f|_V \) is pluriharmonic and \( f \equiv A \) on \( \{x \in U_- \mid a'_- - \varepsilon < h_+(x) < a'_+\}(\subset V) \), it follows from the identity theorem that \( f|_V \equiv A \). Thus \( \chi' \equiv 0 \) on \([b'_-, b'_+] + \delta\) for a small positive number \( \delta \), from which it follows that \( \chi_+ \) is a strictly decreasing function on \((a'_+ - \delta, a'_+)\) for small \( \delta > 0 \), since \( \chi''_+ = \rho_+ \) on \([a'_+, b'_+] \) and \( \chi'_+|_{[b'_-, b'_+] + \delta} \equiv 0 \). Therefore, from these observation on \( \chi_\pm \) and the maximum principle for the pluriharmonic function \( f|_{W_0} \), one has that the maximum value of \( f|_W \) is attained along \( H_- \) and the minimum value of \( f|_W \) is attained along \( H_+ \), which contradicts to the maximum principle for the non-constant pluriharmonic function \( f|_{W_0} \) since \( f|_{H_\pm} \equiv A \).

**Lemma 4.7.** Let \( \theta \) be an element of \( \text{SP}(\alpha, (\varphi, U, (c, d))) \) an element of \( U(\theta) \), and \( h \) be an element of \( \mathcal{H}^*(\varphi, U, (c, d)) \). For a function \( \rho: (c, d) \to \mathbb{R}_{\geq 0} \) of class \( C^\infty \) whose support is relatively compact in \((c, d)\), there exists a positive constant \( A \) such that \( A \cdot \theta_\rho \in \text{SP}(\alpha) \), where

\[
\theta_\rho := \begin{cases} 
(\rho \circ h) \cdot \sqrt{-1} \partial h \wedge \bar \partial h & \text{on } U \\
0 & \text{on } X \setminus U
\end{cases}
\]

**Proof.** Note that \( d\theta_\rho \equiv 0 \) and \( \theta_\rho \wedge \theta_\rho \equiv 0 \) by definition. As it follows from Lemma 4.5 (i) that \( \theta \wedge \theta_\rho \equiv 0 \), the lemma follows from Lemma 4.1 by letting

\[
A := \frac{\int_X \theta \wedge \omega^{\dim X - 1}}{\int_X \theta_\rho \wedge \omega^{\dim X - 1}}
\]

for a suitable Kähler form \( \omega \) of \( X \).

Moreover, we have the following:

**Lemma 4.8.** Assume that Condition \((\bigtriangledown)\) holds. Let \( X, W, H_\pm, \) and \( \varphi_\pm: U_\pm \to (a_\pm, b_\pm) \) be as in **Configuration** (2).

(i) Assume that the assertion (i) of Lemma 4.6 holds. Let \( \mathcal{F}_V \) be the foliation on \( V := X \setminus \overline{W} \) which is defined by \( T_{\mathcal{F}_V} = (\partial h_V|_V)^\perp \). Then \( h_V \) is \( \mathcal{F}_V \)-adaptive and \( K_\alpha \cap V = \{x \in V \mid (dh_V)_x = 0\} \). In this case, any element of \( \text{SP}(\alpha) \) is zero along each leaf of \( \mathcal{F}_V|_{V \setminus K_\alpha} \).

(ii) Assume that the assertion (ii) of Lemma 4.6 holds. Let \( \mathcal{F}_W \) be the foliation on \( W \) which is defined by \( T_{\mathcal{F}_W} = (\partial h_W|_W)^\perp \). Then \( h_W \) is \( \mathcal{F}_W \)-adaptive and \( K_\alpha \cap W = \{x \in W \mid (dh_W)_x = 0\} \). In this case, any element of \( \text{SP}(\alpha) \) is zero along each leaf of \( \mathcal{F}_W|_{W \setminus K_\alpha} \).

**Proof.** Here we only show the assertion (ii), since (i) is shown by the same argument. In what follows, we assume that the assertion (ii) of Lemma 4.6 holds and use the notation in the proof of Lemma 4.6. Note that \( \mathcal{F}_W|_{W \cap \overline{U}_\pm} \) coincides with the foliation \( \mathcal{F}(\theta, \varphi_\pm)|_{W \cap \overline{U}_\pm} \), since \( (\partial h_\pm)^\perp = (\partial \varphi_\pm)^\perp \) holds on \( T_{\overline{U}_\pm} \) by Lemma 4.3. Note also that \( h_W \) is clearly \( \mathcal{F}_W \)-adaptive by definition of \( \mathcal{F}_W \).

Take an element \( \hat{\theta} \in \text{SP}(\alpha) \). The \( \mathcal{F}_W \)-leafwise triviality of \( \hat{\theta} \) on \( W \cap \overline{U}_\pm \) follows from Proposition 3.4 (ii). On a neighborhood \( B_0 \) of each point \( x_0 \) of \( \{x \in W_0 \mid (dh_W)_x \neq 0\} = \)
{x ∈ W_0 | (df)_x ≠ 0}, it follows from Lemma 3.2 (i) that \( \mathcal{F}_W|_{B_0} = \mathcal{F}(\theta_-, log(1 + e^f), B_0) \) (Recall that \( h_W|_{W_0} = f|_{W_0} \) and \( f \in \text{PSH}^\infty(X, \theta_-) \)). Thus one has that \( \hat{\theta} = \mathcal{F}_W|_{B_0} \)-leafwise trivial by Lemma 3.2, 3.3 (i) and the \( \partial\overline{\partial} \)-lemma.

Therefore, it is sufficient to show that \( W_0 \cap K_\alpha = \{ x \in W_0 \mid (dh_W)_x = 0 \} \). As the inclusion \( W_0 \cap K_\alpha \subset \{ x \in W_0 \mid (dh_W)_x = 0 \} = \{ x \in W_0 \mid (df)_x = 0 \} \) simply follows from the definition of \( K_\alpha \), we will show the opposite inclusion in what follows.

Take a point \( x_0 \in W_0 \) such that \( (dh_W)_{x_0} = 0 \). Let \( B \) be a small open neighborhood of \( x_0 \). Let \( F \) be a holomorphic function on \( B \) such that \( \text{Re} F = h_W|_B \) holds. By adding constants to \( h_W \) and \( F \) if necessary, we may assume that \( h_W(x_0) = F(x_0) = 0 \). We also assume that \( B \) is small enough so that any connected component of \( F^{-1}(0) \) and any connected component of \( S := \{ x \in B \mid (df)_x = 0 \} \) contain the point \( x_0 \). Note that \( S \subset F^{-1}(0) \) (see Lemma 2.5).

We show \( x_0 \in K_\alpha \) by contradiction, by assuming that there exists \( \varphi \in \text{PSH}^\infty(X, \theta_-) \) such that \( (d\varphi)_{x_0} ≠ 0 \) (Here we used Lemma 2.3). By shrinking \( B \) if necessary, we may assume that \( \varphi \) has no critical point in \( B \). By applying Lemma 3.3 (i) for \( (\theta_-, log(1 + e^f)) \) and \( f \in \text{PSH}^\infty(X, \theta_-) \), one has that there exists a function \( G : B \to \mathbb{C} \) such that \( \partial h_W = G \partial \varphi \) and that \( |G| = \mathcal{F}(\theta_-, log(1 + e^f)), B \)-leafwise constant. As \( S = \{ x \in B \mid |G(x)| = 0 \} \), \( S \) is the union of some leaves of \( \mathcal{F}(\theta_-, log(1 + e^f)), B \). As \( S \) is a connected analytic subset of \( B \) and each leaf of \( \mathcal{F}(\theta_-, log(1 + e^f)), B \) is a complex submanifold of \( B \) of codimension 1, it follows that \( x_0 \) is a regular point of \( S \) and \( \text{dim}(S, x_0) = \text{dim}(X) - 1 \). By applying Lemma 3.3 (i) for \( (\theta_-, log(1 + e^f)) \) and \( \varphi \in \text{PSH}^\infty(X, \theta_-) \), one has that \( \varphi \) is \( \mathcal{F}_W|_{B \setminus S} \)-leafwise constant, since \( \mathcal{F}_W|_{B \setminus S} = \mathcal{F}(\theta_-, log(1 + e^f)), B \) by Lemma 3.2 (i). Thus, by Lemma 2.5, there exist a coordinate system \( w = (w^1, w^2, \ldots, w^n) \) of \( B \) with \( x_0 = (0, 0, \ldots, 0) \) and an integer \( m \in \mathbb{Z}_{≥ 1} \) such that \( F(w) = (w^1)^m \) and that \( \varphi(w) = \chi(w^1) \) holds on \( B \) for some function \( \chi : w^1(B) \to \mathbb{R} \). Note that \( \chi \) is of class \( C^\infty \), since \( \varphi \) is \( C^\infty \).

In what follows, we show the existence of a function \( \overline{\chi} : (−\delta, \delta) \to \mathbb{R} \) of class \( C^\infty \) for a positive number \( \delta \) such that \( \chi(w^1) = \overline{\chi}(\text{Re}(w^1)^m) \). Note that, if such a function exists, then it holds that \( \varphi = \overline{\chi} \circ h_W \), which proves the lemma since the calculation \( (d\varphi)_{x_0} = \overline{\chi}'(0) \cdot (dh_W)_{x_0} = 0 \) contradicts to the assumption \( (d\varphi)_{x_0} ≠ 0 \).

As we have seen in §2.3, any element of \( (−\delta, 0) \cup (0, \delta) \) is a regular value of \( h_W \) for some positive number \( \delta \). Set \( \varepsilon := \sqrt{\varepsilon} \). By shrinking \( B \), we may assume that \( B = \{(w^1, w^2, \ldots, w^n) \mid |w^j| < \varepsilon \text{ for all } j = 1, 2, \ldots, n\} \). For \( r \in (I_f)_{reg} \cap (−\delta, \delta) \), denote by \( A_r \) the connected component of \( f^{-1}(r) \) which intersects \( B \). By Condition (\( \triangle \)), \( \varphi|_{A_r} \) is constant for any \( r \in (I_f)_{reg} \cap (−\delta, \delta) \). As \( A_r \cap B = \{ w \in B \mid \text{Re}(w^1)^m = r \} \), it follows that \( \chi(w^1) = \chi(\zeta_m \cdot w^1) \) holds on a dense subset of \( \Delta_\varepsilon := \{ w^1 \in \mathbb{C} \mid |w^1| < \varepsilon \} \), where \( \zeta_m := \exp(2\pi \sqrt{-1}/m) \). As \( \chi \) is continuous, one has that \( \chi \) is invariant under the rotation by \( \zeta_m \). Therefore there exists a continuous function \( \eta : \Delta_\varepsilon \to \mathbb{R} \) such that \( \chi(w^1) = \eta((w^1)^m) \), where \( \Delta_\delta := \{ \xi \in \mathbb{C} \mid |\xi| < \delta \} \). Again by the same argument, one can show that there exists a continuous function \( \overline{\chi} : (−\delta, \delta) \to \mathbb{R} \) such that \( \eta(\xi) = \overline{\chi}(\text{Re } \xi) \). By construction, one has \( \chi(w^1) = \overline{\chi}(\text{Re}(w^1)^m) \). As \( \chi \) is of class \( C^\infty \) and the map \( \Delta_\delta \setminus \{0\} \ni w^1 \mapsto \text{Re}(w^1)^m \in (−\delta, \delta) \) is surjective submersion of class \( C^\infty \), \( \overline{\chi} \) is also smooth. \( \square \)
5. Proof of main theorems

Let $X$ be a connected compact Kähler manifold. For a class $\alpha \in H^{1,1}(X,\mathbb{R})$ with $\text{nd}(\alpha) = 1$ and $\#\text{SP}(\alpha) > 1$, we show Theorems 1.1 and 1.3.

5.1. Outline of the proof. As we have seen in §4.1, the assertions in Case I of Theorem 1.3 holds when the assertion $(a)$ of Theorem 3.5 holds for some $\theta \in \text{SP}(\alpha)$ and $\psi \in \text{PSH}^\infty(X,\theta) \setminus \mathbb{R}$. Theorem 1.1 is clear in this case. Therefore it is sufficient to show the theorems by assuming Condition $(\triangledown)$.

In what follows, we assume Condition $(\triangledown)$ and use the notation in §4.2. By Lemma 4.6, either of the following two conditions holds:

- **Condition $(\clubsuit)$:** There exist $\theta \in \text{SP}(\alpha)$ and $(\psi, U, (a, b)) \in \mathcal{U}_c(\theta)$ such that $X \setminus \overline{U}$ is connected.
- **Condition $(\diamondsuit)$:** For any $\theta \in \text{SP}(\alpha)$ and any $(\psi, U, (a, b)) \in \mathcal{U}_c(\theta)$, $X \setminus \overline{U}$ consists of two connected components $V^+$ and $V^-$ such that $\partial V^- = \psi^{-1}(a) \cap \overline{U}$, $\partial V^+ = \psi^{-1}(b) \cap \overline{U}$.

First let us consider the case where **Condition $(\clubsuit)$** holds. Fix $\theta \in \text{SP}(\alpha)$ and $(\psi, U, (a, b)) \in \mathcal{U}_c(\theta)$ such that $V := X \setminus \overline{U}$ is connected. Take an element $(\psi, \tilde{U}, (c, d)) \in \mathcal{U}(\theta)$ such that $U \subset \overline{U}$ and $c < a < b < d$, and an element $h_U \in \mathcal{H}^*(\psi, \mathcal{U}, (c, d))$. Note that $\{\tilde{U}, V\}$ is a covering of $X$ by two domains. Define a holomorphic foliation $\mathcal{F}_0$ on $X$ by letting

$$T_{\mathcal{F}_0} = \begin{cases} (\partial h_U)_{\perp} & \text{on } \tilde{U} \\ (\partial h_V)_{\perp} & \text{on } V \end{cases},$$

where $h_U$ is the function as in Lemma 4.6 $(i)$. Then it follows from Lemma 4.3 and Lemma 4.8 $(i)$ that the assertions in Case II of Theorem 1.3 holds in this case (by Proposition 3.4 $(ii)$ and Lemma 4.8 $(i)$). Note that $K_\alpha \cap \tilde{U} = \emptyset$ since $\tilde{U} \subset \psi^{-1}((I_{\psi_0})_{\text{reg}})$.

Therefore, it is sufficient to show that the assertions in Case III of Theorem 1.3 holds in the case where **Condition $(\diamondsuit)$** holds. Fix an element $\theta_0 \in \text{SP}(\alpha)$, $(\psi_0, W_0, (c_0, d_0)) \in \mathcal{U}(\theta_0)$ such that $W_0$ is a connected component of $\psi_0^{-1}((I_{\psi_0})_{\text{reg}})$, and $h_0 \in \mathcal{H}^*(\psi_0, W_0, (c_0, d_0))$. By Lemma 4.3, we may assume that $h_0 = \chi_0 \circ \psi_0$ for a strictly increasing function $\chi_0 : (c_0, d_0) \to \mathbb{R}$ by replacing $h_0$ with $-h_0$ if necessary. Fix also real numbers $a_0$ and $b_0$ such that $c_0 < a_0 < b_0 < d_0$. By **Condition $(\diamondsuit)$**, the complement $X \setminus \{x \in W_0 \mid a_0 \leq \psi_0(x) \leq b_0\}$ consists of two connected components, say $V_0^\pm$, such that $\partial V_0^- = \{x \in W_0 \mid \psi_0(x) = a_0\}$ and $\partial V_0^+ = \{x \in W_0 \mid \psi_0(x) = b_0\}$ hold. In what follows, we denote by $W_0^\pm$ the domain $V_0^\pm \cup W_0$.

In the following three subsections, we will construct such a function $h_0$ as in Theorem 1.3 by considering the analytic continuation of $h_0$ along the following three steps by assuming that **Condition $(\triangledown)$** and **Condition $(\diamondsuit)$** hold.

**Step 1:** For each element $(\psi,U, I) \in \mathcal{U}(\theta_0)$, we construct a function $h_U : W_0^- \to \mathbb{R} \cup \{-\infty\}$ such that $h_U|_{W_0} = h_0$ and that $h_U$ is pluriharmonic on a domain of $X$
which contains both \( W_0 \) and \( U \). The construction of \( h_U \) depends only on \( U \) and is independent on \( \psi \).

**Step 2:** Define a function \( h_-: W_0^- \to \mathbb{R} \cup \{-\infty\} \) by \( h_-(x) := \inf \{ h_U(x) \mid (\psi, U, I) \in \mathcal{U}(\theta_0) \} \), and show that \( h_- \) is pluriharmonic on \( W_0^- \setminus M^- \), where \( M^- = \{ x \in W_0^- \mid h_-(x) = \min_{W_0^-} h_- \} \).

**Step 3:** Construct \( h^+: W_0^+ \to \mathbb{R} \cup \{+\infty\} \) and \( M^+ \subset W_0^+ \) in the same manner, and define \( h_\alpha \) by patching \( (W_0^+, h_\pm) \). Show that \( M^+ \cup M^- = K^\text{ess} \).

We often use the following topological lemma:

**Lemma 5.1.** Assume that **Condition** \( \Diamond \) holds. Let \( U_1, U_2, \ldots, U_N \) be relatively compact domains of \( W_0^- \) such that \( (\psi_j, U_j, \psi_j(U_j)) \in \mathcal{U}_c(\theta_j) \) holds for some \( \theta_j \in \text{SP}(\alpha) \) and \( \psi_j \in \text{PSH}^\infty(X, \theta_j) \) for \( j = 1, 2, \ldots, N \). Assume that \( U_j \cap U_k = \emptyset \) if \( j \neq k \). Then \( W_0^- \setminus (U_1 \cup U_2 \cup \cdots \cup U_N) \) consists of \( N+1 \) connected components \( L_0, L_1, \ldots, L_N \). Moreover, by changing the indexes if necessary and letting \( H_j^+ \) the connected components of \( \partial U_j \), it holds that \( \partial L_0 = \partial W_0^- \cup H_1^+ \), \( \partial L_1 = H_1^- \cup H_2^+ \), \( \partial L_2 = H_2^- \cup H_3^+ \), \ldots, \( \partial L_{N-1} = H_{N-1}^- \cup H_N^+ \), and \( \partial L_N = H_N^- \).

**Proof.** Denote by \( L \) the complement \( W_0^- \setminus (U_1 \cup U_2 \cup \cdots \cup U_N) \). By considering the covering of \( X \) by \( U_j \)'s and a suitable neighborhood of \( L \), it follows from Mayer–Vietoris sequence that the rank of \( H_0(L, \mathbb{Z}) \) is at most \( N + 1 \). Then the assertion can be easily shown from **Condition** \( \Diamond \) by the induction on \( N \).

5.2. **Step 1.** Let \( (\psi, U, I) \) be an element of \( \mathcal{U}(\theta_0) \). In this subsection, we construct a function \( h_U: W_0^- \to \mathbb{R} \cup \{-\infty\} \) such that \( h_U|_{W_0} = h_0 \) and that \( h_U \) is pluriharmonic on a domain of \( X \) which contains both \( W_0 \) and \( U \).

5.2.1. The construction of \( h_U \) when \( U \cap (V_0^- \setminus W_0) = \emptyset \). Assume that \( U \cap (V_0^- \setminus W_0) = \emptyset \). In this case, we denote by \( U_{\text{outside}} \) the closed set \( W_0^- \setminus W_0 \) and define \( h_U: W_0^- \to \mathbb{R} \cup \{-\infty\} \) by

\[
 h_U := \begin{cases} 
 h_0 & \text{on } W_0 \\
 \inf_{W_0} h_0 & \text{on } U_{\text{outside}}^c. 
\end{cases}
\]

Note that this construction of \( h_U \) depends only on \( U \) and is independent on \( \psi \).

5.2.2. The construction of \( h_U \) when \( U \cap (V_0^- \setminus W_0) \neq \emptyset \) and \( U \cap W_0 \neq \emptyset \). Assume that \( U \cap (V_0^- \setminus W_0) \neq \emptyset \) and \( U \cap W_0 \neq \emptyset \). By Lemma 4.4, the connectedness of \( U \), and **Condition** \( \Diamond \), one has that \( U \cap W_0 = \{ x \in W_0 \mid c_0 < \psi_0(x) < \ell \} \) for some \( \ell \in (c_0, d_0] \).

Take an element \( h_1 \in \mathcal{H}^*(\psi, U, I) \). By applying Lemma 4.4 (ii) to \( U \cap W_0 \) and Lemma 4.3 to \( h_0|_{U \cap W_0} \) and \( h_1|_{U \cap W_0} \), one has that we may assume \( h_0 = h_1 \) holds on \( U \cap W_0 \) by replacing \( h_1 \) with \( c_1 h_1 + c_2 \) for a suitable constants \( c_1, c_2 \in \mathbb{R} \). In this case, we denote by \( U_{\text{outside}}^c \) the closed set \( W_0^- \setminus (U \cup W_0) \) and define \( h_U: W_0^- \to \mathbb{R} \cup \{-\infty\} \) by

\[
 h_U := \begin{cases} 
 h_0 & \text{on } W_0 \\
 h_1 & \text{on } U \\
 \inf_U h_1 & \text{on } U_{\text{outside}}^c. 
\end{cases}
\]
Note that, by Lemma 4.3, this construction of \( h_U \) depends only on \( U \) and is independent on \( \psi \).

5.2.3. The construction of \( h_U \) when \( U \cap (V_0^- \setminus \overline{W_0}) \neq \emptyset \) and \( U \cap W_0 = \emptyset \). Assume that \( U \cap (V_0^- \setminus \overline{W_0}) \neq \emptyset \) and \( U \cap W_0 = \emptyset \). In this case, it follows from Lemma 5.1 that the complement \( W_0^- \setminus U \) consists of two connected components. Among them, there uniquely exists a component whose boundary coincides with a connected component of \( \partial U \), which we denote by \( U_{\text{outside}}^c \). It follows from Lemma 4.6 (ii) that there exists a non-constant pluriharmonic function \( h_1 \) on \( W_0^- \setminus U_{\text{outside}}^c \) such that \( h_1|_{W_0} \in \mathcal{H}^*(\psi_0, W_0, (c_0, d_0)) \) and that \( h_1|_{U} \in \mathcal{H}^*(\psi, U, I) \). From Lemma 4.3, it follows that we may assume that \( h_1|_{W_0} = h_0 \) by replacing \( h_1 \) with \( c_1 h_1 + c_2 \) for a suitable constants \( c_1, c_2 \in \mathbb{R} \). In this case, we define \( h_U: W_0^- \rightarrow \mathbb{R} \cup \{ -\infty \} \) by

\[
h_U := \begin{cases} 
  h_1 & \text{on } W_0^- \setminus U_{\text{outside}}^c \\
  \inf_{W_0^\setminus U_{\text{outside}}^c} h_1 & \text{on } U_{\text{outside}}^c 
\end{cases}
\]

Note that, as it is clear by applying the identity theorem to \( h_U|_{W_0} \), this construction of \( h_U \) depends only on \( U \) and is independent on \( \psi \).

5.3. Step 2. First, let us check the following:

**Lemma 5.2.** Let \((\psi, U, I)\) be an element of \( \mathcal{U}(\theta_0) \). Then the following holds for \( r \in (\min_{W_0^-} h_U, \sup_{W_0^-} h_U) \):

(i) \( W_0^- \setminus h_U^{-1}(r) \) consists of two connected components \( \{ h_U < r \} \) and \( \{ h_U > r \} \).

(ii) If \( r \) is a regular value of \( h_U \), \( h_U^{-1}(r) \) is connected.

**Proof.** (i) As \( \partial \{ x \in W_0^- \setminus U_{\text{outside}}^c \mid h_U(x) < r \} = \partial U_{\text{outside}}^c \cup h_U^{-1}(r) \) and \( h_U \) takes the value \( \min_{W_0^-} h_U \) at any point of \( \partial U_{\text{outside}}^c \) and \( r \) at any point of \( h_U^{-1}(r) \), one has that any connected component of \( \{ x \in W_0^- \setminus U_{\text{outside}}^c \mid h_U(x) < r \} \) touches both \( \partial U_{\text{outside}}^c \) and \( h_U^{-1}(r) \) by the maximum principle for the non-constant pluriharmonic function \( h_U|_{W_0^- \setminus U_{\text{outside}}^c} \), from which one can easily deduce that \( \{ h_U^{-1} < r \} \) is connected. As one can show the connectedness of \( \{ h_U^{-1} > r \} \) by the same argument, the assertion holds.

(ii) Take a small positive number \( \varepsilon \) such that \( (r - \varepsilon, r + \varepsilon) \) is included in the set of all the regular values of \( h_U \). By the argument in \( \S 2.3 \), \( h_U^{-1}((r - \varepsilon, r + \varepsilon)) \) is homeomorphic to the product \( h_U^{-1}(r) \times (r - \varepsilon, r + \varepsilon) \).

First let us consider the case where \( r + \varepsilon > \inf_{W_0^-} h_0 \). In this case, it follows from \( \chi_0' > 0 \) and Ehresmann’s lemma for \( \psi_0|_{W_0} \) that \( A := h_0^{-1}(r + \varepsilon) \cap W_0 \) is connected. Let \( W \) be the connected component of \( W_0^- \setminus A \) such that \( \partial W = A \cup \partial U_{\text{outside}}^c \). By the maximum principle for \( h_U|_{W} \), one has that \( A = h_0^{-1}(r + \varepsilon) \), from which the assertion follows.

When \( r - \varepsilon < \sup_U h_U \), one can show the assertion by the same argument as in the case where \( r + \varepsilon > \inf_{W_0^-} h_0 \). Therefore, by replacing \( \varepsilon \) with a smaller number if necessary, it is sufficient to show the assertion by assuming that \( U \subset W_0^- \setminus W_0 \) and that \( \sup_U h_U < r - \varepsilon < r + \varepsilon < \inf_{W_0^-} h_0 \) (Here recall that \( \sup_U h_U < \inf_{W_0^-} h_0 \) holds in this case by the argument in the proof of Lemma 4.6). As \( h_U^{-1}((r - \varepsilon, r + \varepsilon)) \) is homeomorphic to the product \( h_U^{-1}(r) \times (r - \varepsilon, r + \varepsilon) \), the assertion is reduced to the connectedness of
Take a connected component $W$ of $h_U^{-1}((r - \varepsilon, r + \varepsilon))$. By Lemma 4.3 and the argument in the proof of Lemma 4.6, one can take constants $c_1, c_2 \in \mathbb{R}$, $\theta_\pm \in \text{SP}(\alpha)$ such that $\text{Supp} \, \theta_- \subset U$ and $\text{Supp} \, \theta_+ \subset W_0$, and an element $f \in \text{PSH}^\infty(X, \theta_-)$ such that $h_U = c_1 f + c_2$ holds on $W$. As $(f, W, f(W))$ is an element of $\mathcal{U}(\theta_-)$ by replacing $\varepsilon$ with a smaller number if necessary, it follows from Condition (\$) that the complement $W_0 \setminus W$ consists of two connected components.

Assume that $h_U^{-1}((r - \varepsilon, r + \varepsilon))$ has another component $W'$. Then it follows from the assertion (i) and a simple topological argument that $\{h_U \neq r\} \cup W'$ is connected, which contradicts to the non-connectedness of $W_0 \setminus W$. Therefore one has that $h_U^{-1}((r - \varepsilon, r + \varepsilon)) = W$ is connected.

Now let us define a function $h_- : W_0^- \to \mathbb{R} \cup \{-\infty\}$ by $h_-(x) := \inf\{h_U(x) \mid (\psi, U, I) \in \mathcal{U}(\theta_0)\}$, and the set $M^-$ by $M^- := \bigcap\{U^c_{\text{outside}} \mid (\psi, U, I) \in \mathcal{U}(\theta_0)\}$ (Though apparently this definition of $M^-$ maybe different from that in $5.1$, we will soon show the equivalence between these definitions).

We show the following:

**LEMMA 5.3.** For elements $(\psi, U, I)$ and $(\varphi, W, J)$ of $\mathcal{U}(\theta_0)$, the following holds:

(i) $W_0^- \setminus M^-$ is connected.

(ii) Either $U^c_{\text{outside}} \subset W^c_{\text{outside}}$ or $W^c_{\text{outside}} \subset U^c_{\text{outside}}$ holds.

(iii) $h_- h$ holds on $W_0^- \setminus U^c_{\text{outside}}$.

(iv) $h_- W_0^- \setminus M^-$ is a non-constant pluriharmonic function.

(v) $M^- = \{x \in W_0^- \mid h_-(x) = \min_{W_0^-} h_-\}$.

**Proof.** (i) Note that $W_0^- \setminus U^c_{\text{outside}}$ is connected by the definition of $U^c_{\text{outside}}$. The assertion holds just by a simple topological argument.

(ii) The assertion is clear by Lemma 4.4 when $U \cap \emptyset \neq \emptyset$. When $U \cap W = \emptyset$, the assertion follows from Lemma 5.1.

(iii) It is sufficient to show that $h_U \leq h_W$ holds on $W_0^- \setminus U^c_{\text{outside}}$ for an element $(\varphi, W, J) \in \mathcal{U}(\theta_0)$. As this inequality easily follows from Lemma 4.4 (and the identity theorem for $h_U|_{W_0}$ and $h_W|_{W_0}$) when $U \cap W \neq \emptyset$, in what follows we assume that $U \cap W = \emptyset$. By the assertion (ii), Either $U^c_{\text{outside}} \subset W^c_{\text{outside}}$ or $W^c_{\text{outside}} \subset U^c_{\text{outside}}$ holds. First let us consider the case $U^c_{\text{outside}} \subset W^c_{\text{outside}}$. As both the functions $h_U$ and $h_W$ are pluriharmonic functions on a domain $W_0^- \setminus U^c_{\text{outside}}$, which coincide with $h_0$ on $W_0$, it follows from the identity theorem that $h_U = h_W$ on $W_0^- \setminus U^c_{\text{outside}}$.

Next we consider the case $U^c_{\text{outside}} \subset W^c_{\text{outside}}$. From the same argument as above, it follows that $h_U = h_W$ holds on $W_0^- \setminus W^c_{\text{outside}}$. Therefore, by Lemma 5.2 and the definitions of $h_U$ and $h_W$, $h_W|_{W_0^- \setminus U^c_{\text{outside}}}$ coincides with the function $x \mapsto \max\{h_U(x), A\}$, where $A$ is the value of $h_W$ along $\partial W^c_{\text{outside}}$. Therefore $h_U \leq h_W$ holds on $W_0^- \setminus U^c_{\text{outside}}$.

(iv) Let $x$ be a point of $W_0^- \setminus M^-$. Then, by the definition of $M^-$, $x$ is included in $W_0^- \setminus U^c_{\text{outside}}$ for some $(\psi, U, I) \in \mathcal{U}(\theta_0)$. As it follows from the assertion (iii) that $h_-$ is a non-constant pluriharmonic function on a neighborhood of $x$, the assertion holds.

(v) First we show the inclusion $\{x \in W_0^- \mid h_-(x) = \min_{W_0^-} h_-\} \subset M^-$. Take a point $x \in W_0^- \setminus M^-$. Then there exists $(\psi, U, I) \in \mathcal{U}(\theta_0)$ such that $x \in W_0^- \setminus U^c_{\text{outside}}$. By the assertion (iii), (iv) and the maximum principle, it holds that $h_-$ does not attain the minimum value at $x$. Next we show the opposite inclusion. Take a point $x \in M^-$. 21
Fix a point $y \in W_0^-$. By the definition of $M^-$ and $h_U$, $h_U(x) \leq h_U(y)$ holds for any $(\psi, U, I) \in \mathcal{U}(\theta_0)$. Thus one obtains that $h_-(x) \leq h_-(y)$ holds, from which the assertion follows.

\begin{proof}
We show the following:

**Lemma 5.4.** The set $M$ is included in $K_\alpha$.

\begin{proof}
We show that any point $x \in M$ is a point of $K_\alpha$. We may assume that $x \in M^-$, since the proof for the case $x \in M^+$ can be done by the same argument as in this case. Assuming $x \notin K_\alpha$, we show the assertion by construction. By Lemma 2.3, there exists $\psi \in \text{PSH}^\infty(X, \theta)$ such that $(d\psi)_x \neq 0$. As $r := \psi(x)$ is neither the maximum or minimum of $\psi$, it follows from Sard’s theorem that $r - \varepsilon_1$ and $r + \varepsilon_2$ are regular values of $\psi$ for (generic) small positive numbers $\varepsilon_1$ and $\varepsilon_2$. Let $D$ be the connected component of $\psi^{-1}(r - \varepsilon_1, r + \varepsilon_2)$ which includes $x$, and $H_1, H_2, \ldots, H_N$ be the connected components of the boundary $\partial D$. As $(d\psi)_x \neq 0$, we may assume that $N > 1$ by replacing $\varepsilon_1, \varepsilon_2$ with smaller ones if necessary. As both $r - \varepsilon_1$ and $r + \varepsilon_2$ are regular values, one can take small open neighborhoods $W_j$ of each $H_j$ for $j = 1, 2, \ldots, N$ such that $U_j := W_j \cap D$ satisfies $(\psi, U_j, \psi(U_j)) \in \mathcal{U}(\theta_0)$. We assume that $x \notin U_j$ by shrinking $U_j$’s if necessary. It follows from the connectedness of $D$ and Lemma 5.1 that $N = 2$. By Lemma 5.3 (ii), we may assume $(U_1)_{\text{outside}}^c \subset (U_2)_{\text{outside}}^c$. Again by Lemma 5.1, it follows from $U_1 \cup U_2 \subset D$ that $(D \setminus (U_1 \cup U_2)) \cap (U_1)_{\text{outside}}^c = \emptyset$, which contradicts to $x \in M^-$. \hfill \Box

Define a holomorphic foliation $\mathcal{F}_\alpha$ on $X \setminus M$ by letting $T_{\mathcal{F}_\alpha} = (\partial h_\alpha)^\perp$. Note that, by Lemma 5.4, $\mathcal{F}_\alpha$ is defined especially on $X \setminus K_\alpha$. Note that $h_\alpha_{\vert_{X \setminus M}}$ is clearly an $\mathcal{F}_\alpha$-adaptive function.

**Lemma 5.5.** The following holds:

(i) $K_\alpha \setminus M = \{ x \in X \setminus M \mid (dh_\alpha)_x = 0 \}$.

(ii) Any element of $\text{SP}(\alpha)$ is zero along each leaf of $\mathcal{F}_\alpha_{\vert_{X \setminus M}}$.

\begin{proof}
We will show the lemma only on $W_0^-$, since one can run the same argument on $W_0^+$. Take $(\psi, U, I) \in \mathcal{U}(\theta_0)$. Then, as $h_U_{\vert W_0^- \setminus U_{\text{outside}}^c}$ is a function such as $h_W$ in Lemma 4.6 (ii) for $W = W_0^- \setminus U_{\text{outside}}^c$, the assertions (i) and (ii) on $W_0^- \setminus U_{\text{outside}}^c$ (i.e. $K_\alpha \cap (W_0^- \setminus U_{\text{outside}}^c) = \{ x \in W_0^- \setminus U_{\text{outside}}^c \mid (dh_\alpha)_x = 0 \}$ and the $\mathcal{F}_\alpha_{\vert_{X \setminus U_{\text{outside}}^c}}$-leafwise triviality of elements of $\text{SP}(\alpha)$) follows from Lemma 4.8, which proves the lemma. \hfill \Box

Finally, we show the following:

**Lemma 5.6.** The set $M$ coincides with $K_\alpha^\text{ess}$. 

\textbf{Proof.} As $h_{\alpha}|_{X \setminus M}$ is an $\mathcal{F}_\alpha$-adaptive function, $K^\text{ess}_\alpha \subset M$ holds. In what follows, we show that $x \in K^\text{ess}_\alpha$ for any point $x \in M$. Assuming $x \notin K^\text{ess}_\alpha$, we will prove it by contradiction.

As it follows from Lemma 5.4 that $x \in K_\alpha$, this assumption implies that, for the connected component $K'$ of $K_\alpha$ which contains $x$, there exists a connected open neighborhood $W$ of $K'$ in $X$ such that $W \cap K_\alpha$ is a relatively compact subset of $W$ and that there exists an $\mathcal{F}_\alpha$-adaptive function $h: \overline{W} \to [-\infty, +\infty]$. Let $c := \min_{\overline{W}} h$ and $d := \max_{\overline{W}} h$. As $W \cap K_\alpha$ is a relatively compact subset of $W$, $h(W \cap K_\alpha)$ is a relatively compact subset of $(c, d)$. Therefore there exist real numbers $a$ and $b$ such that $c < a < b < d$ and that both $h^{-1}((c, a)) \cap K_\alpha$ and $h^{-1}((b, d)) \cap K_\alpha$ are empty. Note that then it follows that $x \in h^{-1}((a, b))$.

As is followed from the arguments in §2.3, $h$ is a pluriharmonic proper submersion on $h^{-1}((a - \varepsilon, a)) \cup h^{-1}((b, b + \varepsilon))$ for a small positive number $\varepsilon$. Denote by $D_\varepsilon$ the connected component of $h^{-1}((a - \varepsilon, b + \varepsilon))$ which contains $x$. Let $H_1, H_2, \ldots, H_N$ be the connected components of the boundary $\partial D_\varepsilon$. By Lemma 5.7 below, it follows by replacing $\varepsilon$ with a smaller number that $h^{-1}_{\alpha}|_{H_j} \equiv r_j$ for some $r_j \in \mathbb{R}$ for each $j = 1, 2, \ldots, N$. Note that $h_{\alpha}^{-1}(r_j) = H_j$ holds, since $h_{\alpha}^{-1}(r_j)$ is connected by Lemma 5.2 (ii) and Lemma 5.3 (iii).

When $N = 1$, $\partial D_\varepsilon$ is connected and thus $h|_{\partial D_\varepsilon}$ is constant, which contradicts to the maximum principle since $h|_{D_\varepsilon}$ is a non-constant pluriharmonic function. Thus one has $N > 1$. It follows from the connectedness of $D_\varepsilon$ and Lemma 5.2 (i) that $N = 2$ and $D_\varepsilon \subset h_{\alpha}^{-1}((r_1, r_2))$. Therefore one have that there exists an open neighborhood of $x$ on which $h_{\alpha}$ is a non-constant pluriharmonic, which contradicts to the assumption $x \in M$ by Lemma 5.3 (v).

\textbf{Lemma 5.7.} Let $a, b, \varepsilon$, and $h: \overline{W} \to [-\infty, +\infty]$ be as in the proof of Lemma 5.6. Then, for any $r \in (a - \varepsilon, a) \cup (b, b + \varepsilon)$ and for any connected component $A$ of $h^{-1}(r)$, $h_{\alpha}|_A$ is constant.

\textbf{Proof.} Assume that there exists a connected component $A$ of $h^{-1}(r)$ such that $h_{\alpha}|_A$ is not constant for some $r \in (a - \varepsilon, a) \cup (b, b + \varepsilon)$. As $h$ is $\mathcal{F}_\alpha$-leafwise constant, one can run the same argument as in the proof of Lemma 3.6 to obtain a leaf $Y \subset A$ of $\mathcal{F}_\alpha$ and a surjective holomorphic map $\Phi: X \to R$ to a compact Riemann surface $R$ such that $Y$ is a fiber of $\Phi$. Take a Kähler form $\omega_R$ of $R$ and set $\theta := \Phi^* \omega_R$. Then it clearly holds that $d\theta \equiv 0$ and $\theta \wedge \theta \equiv 0$. Moreover, as $\{\theta\} = B \cdot c_1([Y])$ holds (since $Y$ is a fiber of $\Phi$) and $\theta|_Y \equiv 0$ (by Lemma 5.5 (ii)), it holds that $\{\theta \wedge \theta_0\} = 0 \in H^{2, 0}(X, \mathbb{C})$. Therefore one obtains from Lemma 4.1 that we may assume $\theta \in \text{SP}(\alpha)$ by multiplying some positive constant to $\omega_R$ if necessary.

Let $\psi: R \to \mathbb{R}$ be a function of class $C^\infty$. Then $\delta \cdot \psi \in \text{PSH}^\infty(R, \omega_R)$ holds for sufficiently small positive $\delta$, since $\omega_R > 0$. Take a regular value $r$ of $\delta \cdot \psi$. Then $\varphi := \delta \cdot \psi \circ \Phi$, which is clearly an element of $\text{PSH}^\infty(X, \theta)$, clearly satisfies the condition that any connected component of $\varphi^{-1}(r)$ admits a non-constant $\mathcal{F}(\theta, \varphi)$-leafwise constant function of class $C^\infty$, which contradicts to Condition (\$). \hfill \Box

5.5. \textbf{End of the proof.} By the arguments in §5.1 and the previous subsection, one can construct a foliation $\mathcal{F}_\alpha$ as in Theorem 1.1 on the domain $X \setminus K^\text{ess}_\alpha$ in each of the cases.
In §5.1, we saw that it is sufficient for proving Theorem 1.3 to show the assertion in Case III in this theorem by assuming Condition (\(\bigcirc\)) and Condition (\(\bigodot\)), which is done in the previous subsection.

The uniqueness of \(\mathcal{F}_\alpha\) is clear in Case I. In the other cases, \(\mathcal{F}_\alpha|_W\) is unique for a domain \(W\) such that \((\psi, W, \psi(W)) \in \mathcal{U}(\theta_0)\) for some \(\psi \in \text{PSH}^\infty(X, \theta_0)\). Thus the uniqueness is shown by considering the identity theorem by regarding a holomorphic foliation on \(X \setminus K_{\alpha}^{\text{ess}}\) as a holomorphic section of the projective bundle \(\mathbb{P}(T_X \setminus K_{\alpha}^{\text{ess}}) \to X \setminus K_{\alpha}^{\text{ess}}\) by considering the tangent bundle of the foliation, which proves Theorem 1.1. \(\square\)

6. Semi-positivity of the line bundle associated with an effective divisor with flat normal bundle and Ueda’s classification

Let \(X\) be a connected compact Kähler manifold of dimension \(n\), and \(D\) be an effective divisor of \(X\) whose support is smooth or has only simple normal crossing singularities (for simplicity). Assume \([D]|_X\) is unitary flat line bundle for any \(\lambda\), where \(D = \sum_{\lambda} m_\lambda Z_\lambda\) is the irreducible decomposition \((m_\lambda \in \mathbb{Z}_{\geq 0}, \lambda \subset X\) is a reduced irreducible hypersurface). Then the class \(\alpha := 2\pi c_1([D])\) satisfies \(\text{nd}(\alpha) = 1\), since

\[
(\alpha^2, \eta) = 2\pi \cdot \sum_{\lambda} m_\lambda \int_{Z_\lambda} \sqrt{-1} \Theta_h \wedge \eta = 0
\]

holds for any \(d\)-closed \((n-2, n-2)\)-form \(\eta\) of class \(C^\infty\), where \(h\) is a Hermitian metric of \([D]\) and \(\Theta_h\) is the Chern curvature tensor of \(h\).

In what follows, assume that \([D]\) is semi-positive: i.e. \(\text{SP}(\alpha) \neq \emptyset\). Take a \(C^\infty\) Hermitian metric \(h\) on \([D]\) such that \(\theta := \sqrt{-1} \Theta_h \in \text{SP}(\alpha)\). Let us consider the function \(F: X \setminus |D| \to \mathbb{R}\), where \(|D|\) is the support \(\bigcup_{\lambda} Z_\lambda\) of \(D\), defined by \(F := -\log |s_D|^2_h\) for the canonical section \(s_D \in H^0(X, [D])\). Then, by a simple calculation, one has that the function \(\psi = \log(1 + e^F) - F\) is an element of \(\text{PSH}^\infty(X, \theta)\). Note that

\[
\psi = \log(1 + |s_D|^2_h) = |s_D|^2_h + o(|s_D|^2_h)
\]

holds as a point approaches to \(|D|\), from which it follows that \(|D| = \{\psi = 0\}\) and that, for a sufficiently small positive number \(\varepsilon\), any point of \(\{0 < \psi < \varepsilon\}\) is a regular point of \(\psi\).

By using this function \(\psi\), we can show Corollary 1.5 as follows.

**Proof of Corollary 1.5.** Consider the case where \(D = Y\) is a non-singular connected hypersurface. When \(N_{Y/X}^m\) is holomorphically trivial for some positive integer \(m\), the assertion has already been shown in [K3, Theorem 1.1 (i)]. Assume that \(N_{Y/X}^m\) is not holomorphically trivial for any positive integer \(m\). In this case, by [K3, Theorem 1.4], it is sufficient to show the existence of a connected open neighborhood \(\Omega\) of \(Y\) such that \(\partial \Omega\) is a compact Levi-flat hypersurface of class \(C^2\). When the assertion (b) of Theorem 3.5 holds, we can show the existence of \(\Omega\) by letting \(\Omega := \{\psi < \varepsilon\}\) for a sufficiently small positive number \(\varepsilon\). From now on, assuming that the assertion (a) of Theorem 3.5 holds, we will prove the assertion by contradiction. If \(\Phi(Y) = R, Y\) intersects all the fibers of \(\Phi\), which contradicts to the fact that there exits a fiber of \(\Phi\) included in \(\{\psi = \varepsilon\}\) for a small positive number \(\varepsilon\), which follows from the argument in the proof of Theorem 3.5. Thus one has
that \( \Phi(Y) = \{ p \} \) holds for a point \( p \in R \). Therefore \( N^m_{Y/X} \) is holomorphically trivial, where \( m \) is the integer such that \( \Phi^m \{ p \} = mY \) holds as divisors, which contradicts to the assumption. \( \square \)

Observation 6.1. Consider the case where \( D = Y \) is a non-singular connected hypersurface. By Theorem 3.5 for \( (\theta, \psi) \), it follows by the observation above that \( F_\alpha \) is a non-singular holomorphic foliation on \( \{ 0 < \psi < \varepsilon \} \). Moreover, as is clear by considering the Monge–Ampère foliation for \( \sqrt{-1}\partial\bar{\partial}\psi \), \( F_\alpha \) can be \( C^\infty \)-smoothly extended to \( \{ \psi < \varepsilon \} \) by adding \( Y \) as a leaf. Therefore it follows from \([K3, \text{Lemma 4.4}]\) that \( F_\alpha \) can be extended to \( \{ \psi < \varepsilon \} \) as a non-singular holomorphic foliation. In \([K3]\), we considered such a foliation and showed the linearizability of the holonomy along \( Y \) by applying Pérez-Marco’s theory \([P]\). We expect that the same argument makes sense even when \( |D| \) admits only “mild” singularities in some sense, however it seems that more precise study on the holonomy along singular leaves is needed for realizing such an argument.

Assume that \( D = Y \) is a non-singular connected hypersurface. Then, if \([Y]\) is semi-positive, \#\( \text{SP}(\alpha) > 1 \) holds since \( \text{PSH}^\infty(X, \theta) \neq \mathbb{R} \). Thus it follows that either of the assertions in Case I, II, or III of Theorem 1.3 holds. As is seen in the proof of Corollary 1.5, \( (X, \alpha) \) is in Case I if \( N^m_{Y/X} \) is holomorphically trivial for some positive integer \( m \) by \([K3, \text{Theorem 1.1 (i)}]\). Let us consider the case where \( N^m_{Y/X} \) is not holomorphically trivial for any positive integer \( m \). In this case, it follows from Corollary 1.5 that one can choose a Hermitian metric \( h \) of \([Y]\) such that \( \theta = \sqrt{-1}\Theta_h \) is identically zero on a neighborhood \( V \) of \( Y \): i.e. \( h|_V \) is a flat metric on \([Y]|_V \). In this case, the function \( F = -\log |s_D|^2_h \) is pluriharmonic on \( V \setminus Y \). As it follows from the argument in the proof of Corollary 1.5 that any \( F_\alpha \)-leafwise constant function on a neighborhood of \( Y \) is \( \psi \)-fiberwise constant, one has the following by the same argument as in the proof of Lemma 4.3 or Lemma 4.5 (iii): for a sufficiently small number \( \varepsilon \), any pluriharmonic function \( h \) on \( \{ 0 < \psi < \varepsilon \} \) satisfies \( h = c_1F|_{\{ 0 < \psi < \varepsilon \}} + c_2 \) for constants \( c_1, c_2 \in \mathbb{R} \). Thus \( Y \subset K_\alpha^{\text{ess}} \), from which it follows that \( (X, \alpha) \) is in Case III.

7. Examples

Here we give some examples.

7.1. Non semi-positive case. The condition \( nd(\alpha) = 1 \) does not imply the existence of a semi-positive representative in general. Indeed, as is followed from Corollary 1.5, for a connected non-singular hypersurface \( Y \) of a connected compact Kähler manifold \( X \) with unitary flat normal bundle, \( \alpha := c_1([Y]) \) satisfies \( nd(\alpha) = 1 \) and \( \text{SP}(\alpha) = \emptyset \) if \([Y]|_V \) is not unitary flat for any neighborhood \( V \) of \( Y \): i.e. the pair \((Y, X)\) is of class \((\alpha)\) or \((\gamma)\) in Ueda’s classification \([U]\). Serre’s example gives a typical concrete example, see also \([DPS, \text{Example 1.7}]\) and \([K1]\).

7.2. Suspension construction. Many interesting examples can be constructed by considering the suspension in the following manner. Let \( Z \) be a connected complex manifold and \( F \) be a connected compact Riemann surface. Fix a representation \( \rho: \pi_1(Z, \ast) \to \text{Aut}(F) \) of the fundamental group of \( Z \), where \( \text{Aut}(F) \) is the group of holomorphic automorphisms of \( F \). Then one can construct a complex manifold \( X \) which has a locally trivial
F-bundle structure over Z whose monodromy coincides with ρ by letting \( X := F \times \tilde{Z} / \sim_\rho \), where \( \tilde{Z} \) is the universal covering of Z and \( \sim_\rho \) is the relation such that \((w, z) \sim_\rho (w', z')\) holds for \( w, w' \in F \) and \( z, z' \in \tilde{Z} \) if and only if there exists an element \( \gamma \in \pi_1(Z, *) \) such that \( w' = \rho(\gamma)(w) \) and \( z' = \gamma \cdot z \). Assume that there exists a Kähler class \( \alpha_F \) of F which is invariant by any element of Image ρ. Then it is clear that the class \( \Pr_1^* \alpha \) of \( F \times \tilde{Z} \) induces a class \( \alpha \) of X such that \( \nu(\alpha) = 1 \), where \( \Pr_1 : F \times \tilde{Z} \to F \) is the first projection.

In this subsection, we give some examples of such \((X, F, Z, \rho, \alpha)\).

**Example 7.1.** Consider the case where both \( Z \) and \( F \) are an elliptic curves. For simplicity, we assume that \( F \cong \mathbb{C}/\langle 1, \sqrt{-1} \rangle \). Consider the representation \( \rho : \langle 1, \tau \rangle \to \text{Aut}(F) \) defined by \( \rho(1) = \text{id}_F \) and \( \rho(\tau) \) is the parallel transformation induced by \( + (a + b\sqrt{-1}) : \mathbb{C} \to \mathbb{C} \), where \( \tau \) is the modulus of \( Z \), \( \text{id}_F \) is the identity map, and \( a, b \in \mathbb{R} \). Note that \( X \) is a complex torus \( \mathbb{C}^2_{(w,z)}/\Lambda \) in this case, where \( \Lambda \) is the lattice

\[
\left\langle \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} \sqrt{-1} \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} a + b\sqrt{-1} \\ \tau \end{array} \right) \right\rangle.
\]

Let \( \theta \) be the form \( \sqrt{-1} dw \wedge d\bar{w} \) on \( X \) and \( \alpha \in H^{1,1}(X, \mathbb{R}) \) be the class which is represented by \( \theta \). Let \( \mathcal{F}_\alpha \) be the foliation by curves which is determined by the eigenvectors which belongs to the eigenvalue zero of \( \theta \). As is easily seen, each leaf of \( \mathcal{F}_\alpha \) is locally defined by \( \{w = \text{constant}\} \), from which it follows that \( \mathcal{F}_\alpha \) is a non-singular holomorphic foliation.

First, let us consider the case where both \( a \) and \( b \) are rational. In this case, \( R := \mathbb{C}_w/\langle 1, \sqrt{-1}, a + b\sqrt{-1} \rangle \) is an elliptic curve. By considering the morphism \( \Phi : X \to R \) which is induced by the projection \( \mathbb{C}^2_{(w,z)} \to \mathbb{C}_w \), it follows that \#SP(\alpha) > 1 and \((X, \alpha)\) is in Case I of Theorem 1.3 in this case.

Next, let us consider the case where \( a \) or \( b \) is irrational and \( a = qb \) or \( b = qa \) holds for some rational number \( q \). For simplicity, here we restrict ourselves to the case \( a = 0 \) and \( b \in \mathbb{R} \setminus \mathbb{Q} \). In this case, consider the function \( \pi : X \to \mathbb{R}/\mathbb{Z} \) induced by the function \( \mathbb{C}^2_{(w,z)} \ni (w,z) \mapsto \text{Re} w \in \mathbb{R} \). Then, for any \( t \in \mathbb{R}/\mathbb{Z} \), the fiber \( H_t := \pi^{-1}(t) \) is a real analytic compact Levi-flat hypersurface of X such that any leaf \( \mathcal{L} \subset H_t \) of \( \mathcal{F}_\alpha \) is dense in \( H_t \). Note that any \( \mathcal{F}_\alpha \)-leafwise constant continuous function on \( H_t \) is constant in this case. Take a non-constant function \( \psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) of class \( C^\infty \). Then \( \varphi := \varepsilon \cdot (\psi \circ \pi) \) is an element of \( \text{PSH}^\infty(X, \theta) \) for a sufficiently small positive number \( \varepsilon \), from which it follows that \#SP(\alpha) > 1. As is followed from \( K_\alpha = \emptyset \) (or the fact that the complement of any \( H_t \) is connected), \((X, \alpha)\) is in Case II of Theorem 1.3 in this case.

Finally, let us consider the case where \( a = rb \) or \( b = ra \) holds for some irrational number \( r \). As each leaf of \( \mathcal{F}_\alpha \) is dense in \( X \), it follows that \( \text{SP}(\alpha) = \{\theta\} \) in this case.

**Example 7.2.** Consider the case where \( F \) is the projective line \( \mathbb{P}^1 \). Here let us consider one of the simplest cases: The case where \( Z \cong \mathbb{C}/\langle 1, \tau \rangle \) is an elliptic curve and the representation \( \rho : \langle 1, \tau \rangle \to \text{Aut}(\mathbb{P}^1) \) is the one which is defined by \( \rho(1) = \text{id}_{\mathbb{P}^1} \) and \( \rho(\tau) = U \) for some unitary matrix \( U \). Note that \( X \) is the ruled surface \( \mathbb{P}(E_\rho) \) in this case, where \( E_\rho \) is the unitary flat vector bundle on \( Z \) of rank 2 which corresponds to the representation \( \rho \). In this case, consider the fiberwise Fubini–Study form \( \theta := \frac{1}{2\pi i} \partial \overline{\partial} \log(|x|^2 + |y|^2) \) on \( X \), where \([x:y]\) is a homogeneous coordinate of a fiber. Note that \( \theta \) represents the first Chern class \( \alpha \) of the relative \( O_{\mathbb{P}^1}(1) \)-bundle. Let \( \mathcal{F}_\alpha \) be the foliation by curves which is
determined by the eigenvectors which belongs to the eigenvalue zero of $\theta$. As is easily observed, each leaf of $\mathcal{F}_\alpha$ is locally defined by $\{[x; y] = \text{constant}\}$. Therefore $\mathcal{F}_\alpha$ is a non-singular holomorphic foliation. By choosing $(x, y)$ suitably, we may assume that $U$ corresponds to the unitary rotation $w \mapsto \lambda \cdot w$ for some $\lambda \in U(1)$, where $w := x/y$ is the non-homogeneous coordinate.

When $\lambda^m = 1$ for some positive integer $m$, $\mathcal{F}_\alpha$ coincides with the foliation associated with the fibration $\Phi: X \to \mathbb{P}^1$ which is defined by $\Phi(w, z) := w^m$. Thus $(X, \alpha)$ is in Case I.

When $\lambda^m \neq 1$ for any positive integer $m$, consider the zero section $Y_0$ defined by $\{w = 0\}$ and the $\infty$-section $Y_\infty$ defined by $\{w = \infty\}$ of the ruled surface $X$. As neither $N_{Y_0/X}$ nor $N_{Y_\infty/X}$ is holomorphically trivial for any positive integer $m$, it follows from the observation in the previous section that $(X, \alpha)$ is in Case III. In this example, it follows by considering the function $h_\alpha: X \to [-\infty, +\infty]$ defined by $h_\alpha(w, z) := \log |w|$ that $K_\alpha = K^\text{ess}_\alpha = Y_0 \cup Y_\infty$ holds. \hfill \Box

7.3. The blow-up of $\mathbb{P}^2$ at nine points and K3 surfaces constructed by gluing.

Here we give an example of a compact Kähler surface $X$ and a class $\alpha \in H^{1,1}(X, \mathbb{R})$ with $\operatorname{nd}(\alpha) = 1$ and $\#\operatorname{SP}(\alpha) > 1$ which is in Case III such that the foliation $\mathcal{F}_\alpha$ on $X \setminus K^\text{ess}_\alpha$ never can be holomorphically extended to $X$.

Let $C$ be a smooth cubic of a projective plane $\mathbb{P}^2$. Take nine points $p_1, p_2, \ldots, p_9 \in C$. Denote by $X$ the blow-up of $\mathbb{P}^2$ at these nine points, and by $Y$ the strict transform of $C$. Note that $[−Y]$ coincides with the canonical bundle $K_X$ in this example. Set $\alpha := c_1([Y])$. Note that, if $N_{Y/X}^m$ is holomorphically trivial for some positive integer $m$, it is classically known that $X$ admits a structure as an elliptic surface which has $Y$ as a fiber, from which it is clear that $(X, \alpha)$ is in Case I.

In what follows, we consider the case where $N_{Y/X}^m$ is not holomorphically trivial for any positive integer $m$. It is known that $\operatorname{SP}(\alpha) \neq \emptyset$ for almost all choice of nine points $(p_1, p_2, \ldots, p_9) \in C^9$ in the measure sense (Under the Diophantine condition, see [A], [B], [U], see also [K2]. Note that we may assume that $X \setminus Y$ includes no compact curve by choosing generic nine points. Note also that Theorem 1.4 is a generalization of Brunella’s theorem [B, Theorem 1.1 (i)] for such an example). As we have seen in the previous section, $(X, \alpha)$ is in Case III and $Y \subset K^\text{ess}_\alpha$ in this case. Moreover, by the argument we mentioned in Observation 6.1, it follows that the foliation $\mathcal{F}_\alpha$ can be holomorphically extended to $(X \setminus K^\text{ess}_\alpha) \cup Y$ by adding $Y$ as a leaf. By [B, Proposition 8], such a holomorphic foliation never can be holomorphically extended to whole $X$.

Let $(X, \alpha)$ be as above. By the argument in the last of the previous section, we have that $h_\alpha = -\log |s_Y|^2_h$ holds on a neighborhood $V$ of $Y$ by replacing $h_\alpha$ with $c_1 h_\alpha + c_2$ for some $c_1, c_2 \in \mathbb{R}$, where $s_Y \in H^0(X, [Y])$ is the canonical section and $h$ is a flat metric of $[Y]|_V$, whose existence is assured by [B, Theorem 1.1 (i)] (or Theorem 1.4). Denote by $M^-$ the complement $K^\text{ess}_\alpha \setminus Y$. Note that that $h_\alpha(x) \to \inf_{X \setminus K^\text{ess}_\alpha} h_\alpha$ holds as $x \to \partial M^-$, since $h_\alpha$ is $\mathcal{F}_\alpha$-adapative. The following question is one of the biggest motivation of the present paper:

**Question 7.3.** Does it hold that $\inf_{X \setminus K^\text{ess}_\alpha} h_\alpha = -\infty$?
If \( \inf_{X \setminus K^{\text{ess}}} h_{\alpha} = -\infty \), the function \( h_{\alpha} \) can be extended to \( X \setminus Y \) by letting \( h_{\alpha}|_{M^-} \equiv -\infty \). In this case, one has that \( M^- \) is a pluripolar set since \( M^- = \{ x \in X \setminus Y \mid h_{\alpha}(x) = -\infty \} \) and \( h_{\alpha} = \inf_{A \in \mathbb{R}} h_{\alpha} \) is a plurisubharmonic function on \( X \setminus Y \), where \( h_{\alpha} \) is the plurisubharmonic function on \( X \setminus Y \) defined by \( h_{\alpha}(x) := \max \{ h_{\alpha}(x), A \} \) for \( A \in \mathbb{R} \).

Note that one can also construct an example of \((X, \alpha)\) in Case III such that \( K^{\text{ess}}_{\alpha} \) includes no compact curve by considering the gluing construction of a K3 surface \([KU]\). Again, let \( C \) be an smooth cubic of a projective plane \( \mathbb{P}^2 \). Take eighteen points \( p_1^\pm, p_2^\pm, \ldots, p_9^\pm \in C \). Denote by \( X^+ \) the blow-up of \( \mathbb{P}^2 \) at the nine points \( p_1^+, p_2^+, \ldots, p_9^+ \), and by \( X^- \) the blow-up at \( p_1^-, p_2^-, \ldots, p_9^- \). Denote by \( Y^\pm \) the strict transforms of \( C \) in \( X^\pm \). Fix an isomorphism \( g: Y^+ \to Y^- \). We may assume that neither of \( X^\pm \setminus Y^\pm \) admits compact curve and both \([Y^\pm]\) are semi-positive by choosing eighteen points generically. In \([KU]\), we showed by using [A] that there exist neighborhoods \( W^\pm \) of \( Y^\pm \) in \( X^\pm \) such that one can construct a K3 surface \( \tilde{X} \) by holomorphically gluing \( X^+ \setminus W^+ + X^- \setminus W^- \). Take an element \( \theta \in \text{SP}(c_1([Y^+])) \) whose support is included in \( U \cap (X^+ \setminus W^+) \) for a sufficiently small neighborhood \( U \) of \( \partial W^+ \) in \( X^+ \). Denote also by \( \theta \) the semi-positive \((1,1)\)-form on \( \tilde{X} \) which coincides with \( \theta \) on \( X^+ \setminus W^+ \) and is zero on the complement of \( X^+ \setminus W^+ \). Then, for the class \( \alpha := \{ \theta \} \) of \( \tilde{X} \), it follows from the arguments in the construction of \( h_{\alpha} \) that \( K^{\text{ess}}_{\alpha} \) is the union of \( M^- \)'s of \((X^\pm, c_1([Y^\pm]))\).

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