EXPLICIT IRREDUCIBLE REPRESENTATIONS OF THE IWAHORI-HECKE ALGEBRA OF TYPE $F_4$

ARUN RAM AND D. E. TAYLOR

ABSTRACT. A general method for computing irreducible representations of Weyl groups and Iwahori-Hecke algebras was introduced by the first author in [8]. In that paper the representations of the algebras of types $A_n$, $B_n$, $D_n$ and $G_2$ were computed and it is the purpose of this paper to extend these computations to $F_4$. The main goal here is to compute irreducible representations of the Iwahori-Hecke algebra of type $F_4$ by only using information in the character table of the Weyl group.

1. Introduction

In his thesis [6] P.N. Hoefsmit wrote down explicit irreducible representations of the Iwahori-Hecke algebras $HA_{n-1}$, $HB_n$, and $HD_n$, of types $A_{n-1}$, $B_n$, and $D_n$, respectively. Hoefsmit’s thesis was never published and H. Wenzl [9] independently discovered these representations in the type $A_{n-1}$ case. The irreducible representations of Hoefsmit are analogues of the “seminormal” representations of the Weyl groups of types $A_{n-1}$, $B_n$ and $D_n$ which were written down by A. Young [10]. The Iwahori-Hecke algebras depend on parameters $p$ and $q$ and one can recover the representations of Young by setting $p$ and $q$ equal to 1 in Hoefsmit’s representations.

In this paper we shall extend Hoefsmit’s result and determine explicit realizations of all the irreducible representations of the Iwahori-Hecke algebra $HF_4$. The matrix entries of these representations are well defined when $p = q = 1$ and, when one sets $p = q = 1$, our representations specialize to give explicit realizations of all the irreducible representations of the Weyl group of type $F_4$. The final results are tabulated in the last section of this paper. Our numbering scheme for the irreducible characters of $HF_4$ follows Gekk [5].

In analogy with Hoefsmit, our representations of $HF_4$ are in “seminormal” form with respect to the chain of subalgebras

$$HF_4 \supseteq HB_3 \supseteq HA_2 \supseteq HA_1,$$

1991 Mathematics Subject Classification. 20F55, 20F28.

The research of the first author was supported in part by an Australian Research Council Research Fellowship, National Science Foundation grant DMS-9622985, and the Mathematical Sciences Research Institute. Research at MSRI is supported in part by NSF grant DMS-9022140.
which means that we choose the irreducible representations $\varphi^k$ of $HF_4$ so that, for all $h \in HB_3$, the matrices $\varphi^k(h)$ are block diagonal matrices where the blocks $\varphi^\mu(h)$ are determined by the irreducible representations $\varphi^\mu$ of $HB_3$. Similarly, for all $h \in HA_2$, the matrices $\varphi^k(h)$ are block diagonal where the blocks are determined by the irreducible representations $\varphi^\lambda(h)$ of $HA_2$. In this way we construct the irreducible representations of $HF_4$ inductively, by using the branching rules for restricting representations from $HF_4$ to $HB_3$ and from $HB_3$ to $HA_2$. These branching rules can be calculated easily from the character tables of the corresponding Weyl groups.

The matrix entries of our representations are rational functions in the variables $p$ and $q$. These rational functions are quotients of polynomials in $\mathbb{Z}[p, q, p^{-1}, q^{-1}]$ and the denominators contain only the polynomials

$$[2]_p, [2]_q, [2]_{pq}, [2]_{pq^{-1}}, [2]_{p^2}, [2]_{p^2q}, [3]_p, \text{ and } [3]_q,$$

where $[2]_x = x + x^{-1}$ and $[3]_x = x^2 + 1 + x^{-2}$. This means that our representations are well defined over any field $F$ such that $p, q \in F$ and none of the polynomials in (1.1) are equal to 0.

The research in this paper was begun while the first author was visiting Sydney University on an Australian Research Council Research fellowship. Arun Ram thanks Sydney University and especially G. Lehrer for their wonderful hospitality during the year that he spent in Sydney. The first author is also grateful for support from a postdoctoral fellowship at the Mathematical Sciences Research Institute in Berkeley where the writing of this paper was completed.

2. Preliminaries

Let $p$ and $q$ be indeterminates. The Iwahori-Hecke algebra $HF_4$ is the associative algebra with 1 over the field $\mathbb{C}(p, q)$ generated by $T_1, T_2, T_3, T_4$ with relations

$$T_1T_2T_1 = T_2T_1T_2,$$
$$T_3T_4T_3 = T_4T_3T_4,$$
$$T_2T_3T_2T_3 = T_3T_2T_3T_2,$$
$$T_iT_j = T_jT_i, \quad \text{if } j \neq i \pm 1,$$
$$T_i^2 = (p - p^{-1})T_i + 1, \quad \text{for } i = 1, 2,$$
$$T_i^2 = (q - q^{-1})T_i + 1, \quad \text{for } i = 3, 4.$$

This is the Iwahori-Hecke algebra corresponding to the Weyl group $WF_4$. The Weyl group $WF_4$ is generated by $s_1, s_2, s_3, s_4$ which satisfy the same relations as the $T_i$ except with $p = q = 1$. Let $HA_1$, $HA_2$, and $HB_3$ be the subalgebras of $HF_4$ such
that

\[ HA_1 \] is generated by \( T_1 \),
\[ HA_2 \] is generated by \( T_1 \) and \( T_2 \), and
\[ HB_3 \] is generated by \( T_1 \), \( T_2 \) and \( T_3 \).

These are the Iwahori-Hecke algebras corresponding to the Weyl groups \( WA_1 = \langle s_1 \rangle \), \( WA_2 = \langle s_1, s_2 \rangle \) and \( WB_3 = \langle s_1, s_2, s_3 \rangle \), respectively.

Our goal in this paper is to compute explicit representations of \( HF_4 \) using only the information in the character tables of the Weyl groups \( WA_1, WA_2, WB_3 \) and \( WF_4 \). We shall use the following notations.

(a) \( d_\lambda \) will denote the dimension of the irreducible representation indexed by \( \lambda \);
(b) \( \chi^\lambda \) will denote the character of the irreducible representation of the Weyl group \( W \) indexed by \( \lambda \);
(c) \( \text{Id}_\lambda \) will denote the \( d_\lambda \times d_\lambda \) identity matrix;
(d) \( T_w, w \in W \), will denote the usual basis of the Iwahori-Hecke algebra \( H \) given by \( T_w = T_{i_1} \cdots T_{i_k} \) if \( w = s_{i_1} \cdots s_{i_k} \) is a reduced word for \( w \).
(e) If \( A \) and \( B \) are matrices then \( A \oplus B \) and \( A \otimes B \) will denote the standard operations of direct sum and tensor product of matrices.

We shall need the following well known facts:

FACT 1. The irreducible representations of the Iwahori-Hecke algebra are indexed in the same way as the corresponding Weyl group. Thus,

(a) The irreducible representations of \( HF_4 \) are indexed by \( k \in \{1, 2, \ldots, 25\} \) (in the same manner as in [5] and in the same order as in the table on p. 412 of [2]).
(b) The irreducible representations of \( HB_3 \) are indexed by pairs of partitions \( (\alpha, \beta) \) such that \(|\alpha| + |\beta| = 3\).
(c) The irreducible representations of \( HA_2 \) are indexed by partitions \( \lambda \) of 3.
(d) The irreducible representations of \( HA_1 \) are indexed by partitions \( \gamma \) of 2.

FACT 2. [2] §10.11. The dimension of an irreducible Iwahori-Hecke algebra representation is the same as that of the corresponding representation of the Weyl group and the branching rules for Iwahori-Hecke algebras are the same as for the corresponding Weyl groups. Thus the branching rules for the inclusions \( HF_4 \supseteq HB_3 \supseteq HA_2 \) can be calculated directly from the character tables of the corresponding Weyl groups. We have tabulated these branching rules in Tables 4.2 and 4.3.

FACT 3. [2] §10.9 and [4] (9.21). Let \( H \) be an Iwahori-Hecke algebra and let \( W \) be the corresponding Weyl group. If \( \lambda \) is an index for an irreducible representation of the Iwahori-Hecke algebra \( H \) then the minimal central idempotent corresponding to \( \lambda \) can be written in the form

\[ z_\lambda = \sum_{w \in W} z_w^\lambda T_w, \]
where \( z_\mu^\lambda \in \mathbb{C}(p, q) \) are elements which are well defined when \( p = q = 1 \). Furthermore, at \( p = q = 1 \),

\[
(2.1) \quad z_\lambda \big|_{p=q=1} = \frac{\chi^\lambda(1)}{|W|} \sum_{w \in W} \chi^\lambda(w^{-1}) w,
\]

where \( \chi^\lambda \) is the character of the irreducible representation of \( W \) indexed by \( \lambda \).

**FACT 4.** Let \( H \) be an Iwahori-Hecke algebra and let \( W \) be the corresponding Weyl group. Let \( R \) be the root system corresponding to \( W \) and let

- \( r_s = \) a reflection in a short root,
- \( r_l = \) a reflection in a long root,
- \( N_s = \) the number of positive short roots in \( R \), and
- \( N_l = \) the number of positive long roots in \( R \).

If there is only one root length then we declare all roots to be short. For each \( \lambda \) indexing an irreducible representation of \( H \) let \( \chi^\lambda \) be the character of the corresponding irreducible representation of the Weyl group and define

\[
(2.2) \quad c(\lambda) = \frac{\chi^\lambda(w_0)}{|W|} \frac{\chi^\lambda(r_s)}{\chi^\lambda(1)} q^{c(\lambda, l)}
\]

where

\[
c(\lambda, s) = \frac{N_s \chi^\lambda(r_s)}{\chi^\lambda(1)} \quad \text{and} \quad c(\lambda, l) = \frac{N_l \chi^\lambda(r_l)}{\chi^\lambda(1)}.
\]

Let \( \varphi^\lambda \) be a realization of the irreducible representation indexed by \( \lambda \) and let \( \text{Id}_\lambda \) be the \( d_\lambda \times d_\lambda \) identity matrix, where \( d_\lambda \) is the dimension of \( \varphi^\lambda \). Then we have the following result [7], [5], [8]:

(a) If \( w_0 \) is central in \( W \) then \( \varphi^\lambda(T_{w_0}) = c(\lambda) \text{Id}_\lambda \).

(b) If \( w_0 \) is not central in \( W \) then \( \varphi^\lambda(T_{w_0}^2) = c(\lambda)^2 \text{Id}_\lambda \).

### 3. Seminormal representations

We shall compute the irreducible representations of \( HF_4 \) inductively: the representations of \( HA_1 \) are one dimensional and one can immediately write them down, then we compute irreducible representations of \( HA_2 \), then \( HB_3 \), and finally \( HF_4 \). At each step we use the information from the previous cases since we construct the representations such that upon restriction to any of these subalgebras they are in block diagonal form with diagonal blocks determined by the previous calculations. The irreducible representations of \( HA_2 \) are easy to derive and the irreducible representations of \( HB_3 \) can be derived in a similar fashion to the way that we complete the calculations for \( HF_4 \) below. Thus, in our description below we shall assume that the irreducible representations of \( HA_1 \), \( HA_2 \), and \( HB_3 \) are already known and we shall describe how to obtain the irreducible representations of \( HF_4 \). The irreducible
“seminormal” representations of $HA_1$, $HA_2$, and $HB_3$ are tabulated in Section 4 below.

Let $k$ be an index for an irreducible representation of $HF_4$. The branching rule

$$\varphi^k \downarrow_{HB_3} \cong \varphi^{\mu(1)} \oplus \varphi^{\mu(2)} \oplus \cdots \oplus \varphi^{\mu(\ell)}$$

describing the restriction of representations of $HF_4$ to $HB_3$ can be computed from the character table of the corresponding Weyl groups. We shall say that the irreducible representation $\varphi^k$ of $HF_4$ is in \textit{seminormal form} if

$$\varphi^k(h) = \varphi^{\mu(1)}(h) \oplus \varphi^{\mu(2)}(h) \oplus \cdots \oplus \varphi^{\mu(\ell)}(h),$$

for all $h \in HB_3$. (3.1)

We require the two sides of (3.1) to be equal as matrices.

We shall compute irreducible representations of $HF_4$ which are in seminormal form.

Assuming that the irreducible representations of $HB_3$ are known, the seminormal condition implies that to determine the irreducible representations of $HF_4$ it is only necessary to determine the matrices $\varphi^k(T_4)$ for each $k$.

Suppose that $\varphi^k$ and $\psi^k$ are two solutions to this problem, i.e. $\varphi^k$ and $\psi^k$ are both realizations of the irreducible representation of $HF_4$ indexed by $k$ and we have

$$\varphi^k(h) = \psi^k(h) = \varphi^{\mu(1)}(h) \oplus \varphi^{\mu(2)}(h) \oplus \cdots \oplus \varphi^{\mu(\ell)}(h),$$

for all $h \in HB_3$. Then there is a matrix $P \in GL(d_k)$, where $d_k$ is the dimension of $\varphi^k$, such that $P \varphi^k(h)P^{-1} = \psi^k(h)$, for all $h \in HF_4$. By Schur’s lemma this matrix is unique up to constant multiples. On the other hand we have

$$P(\varphi^{\mu(1)}(h) \oplus \varphi^{\mu(2)}(h) \oplus \cdots \oplus \varphi^{\mu(\ell)}(h))P^{-1} = P\varphi^k(h)P^{-1} = \psi^k(h)$$

$$= \varphi^{\mu(1)}(h) \oplus \varphi^{\mu(2)}(h) \oplus \cdots \oplus \varphi^{\mu(\ell)}(h),$$

for all $h \in HB_3$. By inspection of the table of branching rules from $HF_4$ to $HB_3$ one sees that the summands $\varphi^{\mu(i)}$ are all distinct irreducible representations of $HB_3$. Hence, Schur’s lemma implies that

$$P = p_1 \text{Id}_{\mu(1)} \oplus p_2 \text{Id}_{\mu(2)} \oplus \cdots \oplus p_\ell \text{Id}_{\mu(\ell)},$$

where the $p_i$ are nonzero constants. Replacing $P$ by $p_1^{-1}P$ we may suppose that $p_1 = 1$. Conversely, any choice of $p_i \neq 0$, $p_1 = 1$, in the equation (3.2) defines a matrix $P$ such that $P\varphi^kP^{-1}$ is a seminormal representation. Thus we have the following result.

\textbf{Proposition 3.3.} If $\varphi^k$ is in seminormal form then the matrix $\varphi^k(T_4)$ is determined up to the choice of $\ell - 1$ free parameters where $\ell$ is the number of irreducible summands in $\varphi^k$ on restriction to $HB_3$. 
Let \( w_{0,1}, w_{0,2}, w_{0,3}, \) and \( w_{0,4} \) be the longest elements in the Weyl groups \( WA_1, \) \( WA_2, WB_3, \) and \( WF_4, \) respectively. Define elements
\[
\begin{align*}
D_1 &= T_{w_{0,1}} = T_1, \\
D_2 &= T_{w_{0,2}}^2 = (T_1T_2T_1)^2, \\
D_3 &= T_{w_{0,3}} = (T_3T_2T_1)^3, \\
D_4 &= T_{w_{0,4}} = (T_4T_{w_{0,3}}T_{w_{0,2}})^3T_{w_{0,2}}^{-2}.
\end{align*}
\]
(3.4)
in \( HF_4. \)

**Lemma 3.5.** If \( \varphi^k \) is in seminormal form then the matrices \( \varphi^k(D_j) \) are uniquely determined, for all \( 1 \leq k \leq 25, 1 \leq j \leq 4. \)

**Proof.** This follows from Fact 4. \( \square \)

The matrices \( \varphi^k(D_j) \) are tabulated in 5.4.

Let \( \sigma \) be a permutation matrix such that
\[
\sigma \varphi^k(h)\sigma^{-1} = \varphi^{(1^2)}(h) \oplus \cdots \oplus \varphi^{(1^2)}(h)
\]
\[
\oplus \varphi^{(2^1)}(h) \oplus \cdots \oplus \varphi^{(2^1)}(h)
\]
\[
= \bigoplus_{\lambda \vdash 3} \varphi^\lambda(h)^{\otimes m_\lambda}
\]
for all \( h \in HA_2. \) The constant \( m_\lambda \) is the number of times the matrix \( \varphi^\lambda(h) \) appears. Since \( \sigma \varphi^k(T_4)\sigma^{-1} \) commutes with all of the matrices in (3.6), it follows from Schur’s lemma that
\[
\sigma \varphi^k(T_4)\sigma^{-1} = (T^k_{(3)} \otimes \text{Id}_{(3)}) \oplus (T^k_{(21)} \otimes \text{Id}_{(21)}) \oplus (T^k_{(1^3)} \otimes \text{Id}_{(1^3)})
\]
\[
= \bigoplus_{\lambda \vdash 3} T^k_{\lambda} \otimes \text{Id}_\lambda,
\]
where, for each \( \lambda, \) \( T^k_{\lambda} \) is an \( m_\lambda \times m_\lambda \) matrix and \( \text{Id}_\lambda \) is the \( d_\lambda \times d_\lambda \) identity matrix. Note that
\[
T^k_{\lambda} \otimes \text{Id}_\lambda = \sigma \varphi^k(z_\lambda T_4)\sigma^{-1},
\]
where \( z_\lambda \) is the minimal central idempotent in \( HA_2 \) corresponding to \( \lambda. \) We can use the same method to write
\[
\sigma \varphi^k(D_3)\sigma^{-1} = \bigoplus_{\lambda \vdash 3} D^k_{\lambda} \otimes \text{Id}_\lambda,
\]
where \( D_3 = T_{w_{0,3}} \), as given in (3.4).

To determine the matrices \( \varphi^k(T_4) \) it is sufficient to determine the matrices \( T^k_{\lambda}. \) The matrices \( D^k_{\lambda} \) are completely determined by Lemma 3.5 and can easily be determined
from the tables in 5.4. The relations $(T_4 D_3)^3 = D_4 D_2^2$ and the relation $T_4^2 = (q - q^{-1})T_4 + 1$ imply that

\[(3.8) \quad (T_\lambda^k D_\lambda^k)^3 = c(k) c(\lambda)^2 \text{Id}_{m_\chi} \quad \text{and} \quad (T_\lambda^k)^2 = (q - q^{-1})T_\lambda^k + \text{Id}_{m_\chi},\]

where $c(k)$ and $c(\lambda)$ are the constants given in equation (2.2).

### 3.1. Determining the diagonal entries of $\varphi^k(T_4)$

We shall determine the diagonal entries of the matrices the matrices $T_\lambda^k$ by determining the traces of the matrices $T_\lambda^k (D_\lambda^k)^{-2}, \ T_\lambda^k (D_\lambda^k)^{-1}, \ T_\lambda^k, \ T_\lambda^k D_\lambda^k,$ and $T_\lambda^k (D_\lambda^k)^2$.

**Proposition 3.9.** Fix an index $k$ for an irreducible representation of $HF_4$ and let $\lambda$ be an index for an irreducible representation of $HA_2$. Let $T_\lambda^k, D_\lambda$ and $z_\lambda$ be as above and let $\chi^k$ and $\chi^\lambda$ be the irreducible characters of the Weyl groups $WF_4$ and $WA_2$ which correspond to $k$ and $\lambda$, respectively. Let $c(k)$ and $c(\lambda)$ be the constants defined in (2.2). Then

\[(a) \quad \text{Tr}(T_\lambda^k) = \frac{1}{12} \sum_{w \in WA_2} \chi^\lambda(w^{-1}) \big((q - q^{-1})\chi^k(w) + (q + q^{-1})\chi^k(ws_4)\big),\]

\[(b) \quad \text{Tr}(T_\lambda^k D_\lambda^k) = \frac{\chi^k(w_{0,4}) c(k) c(\lambda)^{3/2}}{6} \sum_{w \in WA_2} \chi^\lambda(w^{-1}) \chi^k(ws_4 w_{0,3}),\]

\[(c) \quad \text{Tr}((T_\lambda^k D_\lambda^k)^2) = \frac{c(k)^{3/2} c(\lambda)^{3/2}}{6} \sum_{w \in WA_2} \chi^\lambda(w^{-1}) \chi^k(w(s_4 w_{0,3})^2).\]

**Proof.** (a) From the second equation in (3.8) we have that each eigenvalue of $T_\lambda^k$ is either $q$ or $-q^{-1}$ and consequently $\text{Tr}(T_\lambda^k) = t_1 q - t_2 q^{-1}$ for some positive integers $t_1$ and $t_2$. These constants are determined as follows. Using (3.7) we get that

\[t_1 - t_2 = \text{Tr}(T_\lambda^k) \Big|_{p=q=1} = \frac{1}{\chi^\lambda(1)} \text{Tr}(T_\lambda^k \otimes \text{Id}_\chi) \Big|_{p=q=1}\]

\[= \frac{1}{\chi^\lambda(1)} \text{Tr}(\sigma \varphi^k(z_\lambda T_4) \sigma^{-1}) \Big|_{p=q=1} = \frac{1}{\chi^\lambda(1)} \text{Tr}((\varphi^k(z_\lambda T_4)) \Big|_{p=q=1}.\]

Then we use (2.1) to obtain

\[t_1 - t_2 = \frac{1}{\chi^\lambda(1)} \chi^k \left( \frac{\chi^\lambda(1)}{6} \sum_{w \in WA_2} \chi^\lambda(w^{-1}) ws_4 \right) = \frac{1}{6} \sum_{w \in WA_2} \chi^\lambda(w^{-1}) \chi^k(w s_4).\]
If $\text{Id}_k^\lambda$ is the identity matrix of the same dimension as $T_k^\lambda$ then

$$t_1 + t_2 = \text{Tr}(\text{Id}_k^\lambda)|_{p=q=1} = \frac{1}{\chi^\lambda(1)} \text{Tr}(\text{Id}_k^\lambda \otimes \text{Id}_\lambda)|_{p=q=1}$$

$$= \frac{1}{\chi^\lambda(1)} \text{Tr}(\varphi^k(z_\lambda)|_{p=q=1} = \frac{1}{6} \sum_{w \in W_{A_2}} \chi^\lambda(w^{-1})\chi^k(w).$$

These two equations determine $t_1$ and $t_2$ and thus $\text{Tr}(T_k^\lambda)$ is determined.

(b) It follows from Fact 4 and the first equation in (3.8) that the eigenvalues of $T_k^\lambda D_k^\lambda$ are all of the form $\omega^i c(\lambda)^{\frac{2}{3}} c(k)^{\frac{1}{3}}$ where $\omega$ is a primitive cube root of unity. Hence

$$\text{Tr}(T_k^\lambda D_k^\lambda) = \eta c(\lambda)^{\frac{2}{3}} c(k)^{\frac{1}{3}}$$

for some constant $\eta \in \mathbb{C}$. By setting $p$ and $q$ equal to 1 we have

$$\eta \chi^k(w_{0,4}) = \text{Tr}(T_k^\lambda D_k^\lambda)|_{p=q=1} = \frac{1}{\chi^\lambda(1)} \text{Tr}(T_k^\lambda D_k^\lambda \otimes \text{Id}_\lambda)|_{p=q=1}$$

$$= \frac{1}{\chi^\lambda(1)} \text{Tr}(\varphi^k(z_\lambda T_4 T_{w_{0,3}})|_{p=q=1}$$

as in (3.1). Using (2.1) we get

$$\eta \chi^k(w_{0,4}) = \frac{1}{6} \sum_{w \in W_{A_2}} \chi^\lambda(w^{-1})\chi^k(ws_4 w_{0,3}).$$

The proof of (c) is similar to that of (b) once one notes that Fact 4 and the first equation in (3.8) imply that the eigenvalues of the matrix $(T_k^\lambda D_k^\lambda)^2$ are all of the form $\omega^{2i} c(\lambda)^{\frac{2}{3}} c(k)^{\frac{2}{3}}$. □

**Lemma 3.10.** Given matrices $T$ and $D$ such that $T^2 = (q - q^{-1})T + \text{Id}$ and $(TD)^3 = c \text{Id}$ where $c$ is a constant, we have

(a) $\text{Tr}(TD^{-1}) = (q - q^{-1}) \text{Tr}(D^{-1}) + c^{-1} \text{Tr}((TD)^2)$.  
(b) $\text{Tr}(TD^2) = c \text{Tr}(D^{-1}) - (q - q^{-1}) \text{Tr}((TD)^2)$.  
(c) $\text{Tr}(TD^{-2}) = (q - q^{-1}) \text{Tr}(D^{-2}) + c^{-1}(q - q^{-1}) \text{Tr}(TD) + c^{-1} \text{Tr}(D)$.  

**Proof.** (a) Writing the given equations in the form $T = (q - q^{-1}) \text{Id} + T^{-1}$ and $(TD)^{-1} = c^{-1}(TD)^2$, we have

$$\text{Tr}(TD^{-1}) = (q - q^{-1}) \text{Tr}(D^{-1}) + \text{Tr}(T^{-1}D^{-1})$$

$$= (q - q^{-1}) \text{Tr}(D^{-1}) + c^{-1} \text{Tr}(TD^{-1}TD).$$
(b) Similarly, from the fact that $T^{-2} = \text{Id} - (q - q^{-1})T^{-1}$,
\[
\text{Tr}(TD^2) = \text{Tr}(DTD) = c \text{Tr}(T^{-1}D^{-1}T^{-1}) = c \text{Tr}(T^{-2}D^{-1}) \\
= c \text{Tr}(D^{-1}) - c(q - q^{-1}) \text{Tr}(T^{-1}D^{-1}) \\
= c \text{Tr}(D^{-1}) - (q - q^{-1}) \text{Tr}(TD^2).
\]

(c) \[
\text{Tr}(TD^{-2}) = (q - q^{-1}) \text{Tr}(D^{-2}) + \text{Tr}(T^{-1}D^{-2}) \\
= (q - q^{-1}) \text{Tr}(D^{-2}) + \text{Tr}(D^{-1}T^{-1}D^{-1}) \\
= (q - q^{-1}) \text{Tr}(D^{-2}) + c^{-1} \text{Tr}(TDT) \\
= (q - q^{-1}) \text{Tr}(D^{-2}) + c^{-1} \text{Tr}(T^2D) \\
= (q - q^{-1}) \text{Tr}(D^{-2}) + c^{-1}(q - q^{-1}) \text{Tr}(TD) + c^{-1} \text{Tr}(D).
\]

Assume that $T^k_\lambda$ has dimension at most 5 and write $D^k_\lambda = \text{diag}(d_1, d_2, \ldots, d_r)$. The diagonal entries of $D^k_\lambda$ are determined by Proposition 3.5 and one can check directly that these diagonal entries are always all distinct. Let $S$ be a subset of \{1, 2, \ldots, r\} \setminus \{i\}$ such that $S$ and its complement have at most 2 elements. Then the diagonal entries of $T^k_\lambda$ are given by
\[
(T^k_\lambda)_{ii} = \text{Tr}(TE_{ii}) \quad \text{where, for each } 1 \leq i \leq r,
\]
\[
E_{ii} = \left( \prod_{j \in S, j \neq i} \frac{D^k_\lambda - d_j}{d_i - d_j} \right) \left( \prod_{j \not\in S, j \neq i} \frac{(D^k_\lambda)^{-1} - d_j^{-1}}{d_i^{-1} - d_j^{-1}} \right).
\]

These values can be evaluated explicitly by expanding $E_{ii}$ in terms of $(D^k_\lambda)^j$ and using Lemma 3.10 and Proposition 3.9 to evaluate the traces $\text{Tr}(T^k_\lambda(D^k_\lambda)^j)$.

Formula (3.11) suffices for computing the diagonal entries of the matrices $T^k_\lambda$, and thus of the matrices $\varphi^k(T_\lambda)$, for all $k$ except $k = 25$. The matrix $T^{25}_{(21)}$ has dimension 6 and formula (3.11) is not applicable. The diagonal entries of the matrix $\varphi^{25}(T_4)$ are computed as follows. Since the matrices $T^{25}_{(13)}$ and $T^{25}_{(3)}$ are each of dimension two we use formula (3.11) to determine their diagonal entries. By Lemma 3.10 and Proposition 3.9 we can determine the traces of the matrices
\[
T^{25}_{(21)}(D^{25}_{(21)})^{-2}, \quad T^{25}_{(21)}(D^{25}_{(21)})^{-1}, \quad T^{25}_{(21)}, \quad T^{25}_{(21)}D^{25}_{(21)}, \quad T^{25}_{(21)}(D^{25}_{(21)})^2,
\]
and these traces give five linear relations that the diagonal entries of $T^{25}_{(21)}$ must satisfy. Finally, we use the formula
\[
0 = \text{Tr}(\varphi^{25}(T_4T_3T_2T_1)) = \sum_i \varphi^{25}(T_4)_{ii}\varphi^{25}(T_3)_{ii}\varphi^{25}(T_2)_{ii}\varphi^{25}(T_1)_{ii}
\]
to determine the diagonal entries of $\varphi^{25}(T_4)$ completely. This last formula is a consequence of the following lemma.

**Lemma 3.12.** (a) $\text{Tr}(\varphi^{25}(T_4T_3T_2T_1)) = 0$.
(b) The diagonal entries of the matrix $\varphi^k(T_4T_3T_2T_1)$ satisfy

$$\varphi^k(T_4T_3T_2T_1)_{ii} = \varphi^k(T_4)_{ii}\varphi^{25}(T_3)_{ii}\varphi^k(T_2)_{ii}\varphi^{25}(T_1)_{ii}.$$

**Proof.** (a) Since the Coxeter number for the Weyl group $WF_4$ is 12 (see [1]) we have that $(T_4T_3T_2T_1)^6 = T_{w_{0,4}}$. Then it follows from Fact 4 that the eigenvalues of the matrix $\varphi^{25}(T_4T_3T_2T_1)$ must be of the form $\omega^j c(25)$ where $\omega$ is a primitive 6th root of unity and $c(25)$ is the constant given in (2.2). It follows that $\text{Tr}(\varphi^{25}(T_4T_3T_2T_1)) = \eta c(25)$ for some constant $\eta$. Then, from the character table of the Weyl group $WF_4$, we have

$$\eta = \eta c(25)|_{p=q=1} = \text{Tr}(\varphi^{25}(T_4T_3T_2T_1))|_{p=q=1} = \chi^{25}(s_4s_3s_2s_1) = 0.$$

(b) Let $h \in HB_3$. Let $z_\lambda$, $\lambda \vdash 3$, be the minimal central idempotents in $HA_2$. Since $\varphi^k$ is in seminormal form the matrices $\varphi^k(z_\lambda)$ are diagonal matrices with 1’s and 0’s on the diagonal, their sum is the identity matrix and they are mutually orthogonal. It follows that there is a single partition $\lambda$ such that

$$\varphi^k(T_4h)_{ii} = \varphi^k(z_\lambda T_4h)_{ii} = (\varphi^k(z_\lambda T_4)\varphi^k(z_\lambda h))_{ii}.$$

It follows from the seminormal condition and the fact that the branching rules for restricting representations from $HF_4$ to $HB_3$ are multiplicity free that the matrix $\varphi^k(z_\lambda h)$ is a diagonal matrix. Thus the diagonal entries of the matrix $\varphi^k(T_4h)$ satisfy

$$\varphi^k(T_4h)_{ii} = \varphi^k(T_4)_{ii}\varphi^k(h)_{ii},$$

for all $h \in HB_3$. Since the irreducible representations of $HB_3$ and $HA_2$ that we are using are also chosen to be in seminormal form, their representations also satisfy a similar identity. The result then follows by induction. \hfill $\square$

### 3.2. Computing the off-diagonal entries of $\varphi^k(T_4)$.

**Proposition 3.13.** Let $T = (t_{ij})$ and $D = \text{diag}(d_1, d_2, \ldots, d_r)$ be $r \times r$ matrices such that the diagonal entries of $D$ are distinct, none of the entries $t_{ij}$ of $T$ are 0, and

$$T^2 = (q - q^{-1})T + \text{Id} \quad \text{and} \quad (TD)^3 = c\text{Id},$$
for some constant $c$. For distinct indices $i,j,k$ define $u_{ij} = u_{ji} = t_{ij}t_{ji}$ and $v_{ijk} = t_{ij}t_{jk}/t_{ik}$. Then

(a) \[ \sum_{j \neq i} u_{ij} = -t_{ii}^2 + (q - q^{-1})t_{ii} + 1, \]

(b) \[ \sum_{j \neq i} u_{ij}d_j = -t_{ii}^2d_i + ct_{ii}d_i^2 - c(q - q^{-1})d_i^2, \]

(c) \[ \sum_{j \neq i,k} v_{ijk} = -t_{ii} - t_{kk} + (q - q^{-1}), \quad \text{for } i \neq k, \]

(d) \[ \sum_{j \neq i,k} v_{ijk}d_j = -t_{ii}d_i - t_{kk}d_k + cd_i^{-1}d_k^{-1}, \quad \text{for } i \neq k. \]

**Proof.** Equations (a) and (b) are obtained by comparing the $(i,i)$ entries on each side of the matrix equations $T^2 = (q - q^{-1})T + q$ and $TDT = cD^{-1}T^{-1}D^{-1}$. Equations (c) and (d) are obtained by comparing the $(i,k)$ entries.

The equations

(3.14) \[ v_{ijk} = \frac{v_{1ij}v_{1jk}v_{1ki}}{u_{ik}}, \quad v_{1ji} = \frac{u_{ij}}{v_{1ij}}, \quad t_{ij} = \frac{t_{1j}v_{1ij}}{t_{1i}}, \]

imply that all of the values in Proposition 3.13 are determined once we know $u_{ij}$ and $v_{1ij}$ for $i < j$ and $t_{ii}$ for $1 < i \leq r$.

In view of the relations (3.8) we may apply Proposition 3.13 to the matrices $T_k^\lambda$ and $D_k^\lambda$. Equations (a) and (b) of Proposition 3.13 give

- 4 equations in the single variable $u_{12}$ when $\dim(T_2^\lambda) = 2$,
- 6 equations in the 3 variables $u_{ij}$ when $\dim(T_3^\lambda) = 3$,
- 8 equations in the 6 variables $u_{ij}$ when $\dim(T_4^\lambda) = 4$,
- 12 equations in the 15 variables $u_{ij}$ when $\dim(T_5^\lambda) = 6$.

These equations are sufficient to determine the products $u_{ij} = t_{ij}t_{ji}$ for all $T_k^\lambda$ except $T_{25}^{(21)}$ where we have $\dim(T_{25}^{(21)}) = 6$.

If $\dim(T_k^\lambda) = 2, 3, 4$ or 4 we use the linear equations in (a) and (b) of Proposition 3.13 to solve for the $u_{ij}$. Then we use the equations in (3.14) to write the equations in (c) and (d) of Proposition 3.13 in terms of the variables $t_{1i}$ and $v_{1ij}$, $i < j$. After doing this we are able to use the subset of the resulting equations which are linear in the $v_{1ij}$ to uniquely determine the values of the $v_{1ij}$, $i < j$. This determines the $T_k^\lambda$ up to the choice of the $t_{1i}$. Finally, the equations resulting from the condition $T_3T_4T_3 = T_4T_3T_4$ force certain relations between the $T_k^\lambda$ for fixed $k$ and different $\lambda$. For each fixed $k$ we picked out a few nice equations resulting from this condition to determine the
$T^k_\lambda$ completely for all $\lambda$. This completely determined the representations $\varphi^k$ for all $k$ except $k = 25$.

The case of $\varphi^{25}$ is slightly more complex. We used the same methods as above to determine the matrices $T^{25}_\lambda$ in terms of the variables $t_{ij}$ for each $\lambda$ except $T^{25}_{(21)}$. In the case of $T^{25}_{(21)}$ we have $\dim(T^{25}_{(21)}) = 6$ and the system of 12 equations obtained from (a) and (b) of Proposition 3.13 is a rank 11 system in the 15 unknowns $u_{ij}$. These linear equations can be used to write 11 of the $u_{ij}$ variables in terms of the other 4. Next we chose the nicest equations resulting from (c) and (d) of Proposition 3.13 and the condition $T_4 T_3 T_4 = T_3 T_4 T_3$ and used Maple [3] to solve these equations. These equations are quite nontrivial and we found that we needed to choose these equations carefully in order to stay within the bounds of the capability of Maple. In this way we determined the matrices $T^{25}_\lambda$, for all $\lambda$, and thus determined $\varphi^{25}$ completely.

4. Branching rules

4.1. The branching rules from $HA_2$ to $HA_1$.

| $\varphi^\lambda$ | dim | Restriction to $HA_1$ |
|-------------------|-----|-----------------------|
| $\varphi^{(3)}$   | 1   | $\varphi^{(2)}$       |
| $\varphi^{(2,1)}$ | 2   | $\varphi^{(2)} \oplus \varphi^{(1^2)}$ |
| $\varphi^{(1^3)}$ | 1   | $\varphi^{(1^2)}$     |

4.2. The branching rules from $HB_3$ to $HA_2$.

| $\varphi^\mu$ | dim | Restriction to $HA_2$ |
|----------------|-----|-----------------------|
| $\varphi^{(3),\emptyset}$ | 1   | $\varphi^{(3)}$     |
| $\varphi^{(1^3),\emptyset}$ | 1   | $\varphi^{(1^3)}$  |
| $\varphi^{\emptyset,(3)}$ | 1   | $\varphi^{(3)}$     |
| $\varphi^{\emptyset,(1^3)}$ | 1   | $\varphi^{(1^3)}$  |
| $\varphi^{(21),\emptyset}$ | 2   | $\varphi^{(21)}$   |
| $\varphi^{\emptyset,(21)}$ | 2   | $\varphi^{(21)}$   |
| $\varphi^{(2),(1)}$ | 3   | $\varphi^{(3)} \oplus \varphi^{(21)}$ |
| $\varphi^{(1^2),(1)}$ | 3   | $\varphi^{(21)} \oplus \varphi^{(1^3)}$ |
| $\varphi^{(1),(2)}$ | 3   | $\varphi^{(3)} \oplus \varphi^{(21)}$ |
| $\varphi^{(1),(1^2)}$ | 3   | $\varphi^{(21)} \oplus \varphi^{(1^3)}$ |
4.3. The branching rules from $HF_4$ to $HB_3$. The bands in this table separate the orbits of the group of field automorphisms $\langle \alpha_p, \alpha_q \rangle$, see 5.4.

| $\varphi^k$ | dim | Restriction to $HB_3$ |
|-------------|-----|-----------------------|
| $\varphi^1$ | 1   | $\varphi^{(3),0}$     |
| $\varphi^2$ | 1   | $\varphi^{(1^3),0}$   |
| $\varphi^3$ | 1   | $\varphi^{0,(3)}$     |
| $\varphi^4$ | 1   | $\varphi^{0,(1^3)}$   |
| $\varphi^5$ | 2   | $\varphi^{(21),0}$    |
| $\varphi^6$ | 2   | $\varphi^{0,(21)}$    |
| $\varphi^7$ | 2   | $\varphi^{(3),0} \oplus \varphi^{0,(3)}$ |
| $\varphi^8$ | 2   | $\varphi^{(1^3),0} \oplus \varphi^{0,(1^3)}$ |
| $\varphi^9$ | 4   | $\varphi^{(21),0} \oplus \varphi^{0,(21)}$ |
| $\varphi^{10}$ | 9   | $\varphi^{(3),0} \oplus \varphi^{(21),0} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1),(2)}$ |
| $\varphi^{11}$ | 9   | $\varphi^{(21),0} \oplus \varphi^{(1^3),0} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(1^2)}$ |
| $\varphi^{12}$ | 9   | $\varphi^{(2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{0,(3)}$ |
| $\varphi^{13}$ | 9   | $\varphi^{(1^2),(1)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{0,(21)} \oplus \varphi^{0,(1^3)}$ |
| $\varphi^{14}$ | 6   | $\varphi^{(1^2),(1)} \oplus \varphi^{(1),(2)}$ |
| $\varphi^{15}$ | 6   | $\varphi^{(2),(1)} \oplus \varphi^{(1),(1^2)}$ |
| $\varphi^{16}$ | 12  | $\varphi^{(2),(1)} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{(1),(1^2)}$ |
| $\varphi^{17}$ | 4   | $\varphi^{(3),0} \oplus \varphi^{(2),(1)}$ |
| $\varphi^{18}$ | 4   | $\varphi^{(1^3),0} \oplus \varphi^{(1^2),(1)}$ |
| $\varphi^{19}$ | 4   | $\varphi^{(1),(2)} \oplus \varphi^{0,(3)}$ |
| $\varphi^{20}$ | 4   | $\varphi^{(1),(1^2)} \oplus \varphi^{0,(1^3)}$ |
| $\varphi^{21}$ | 8   | $\varphi^{(21),0} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1^2),(1)}$ |
| $\varphi^{22}$ | 8   | $\varphi^{(1),(2)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{0,(21)}$ |
| $\varphi^{23}$ | 8   | $\varphi^{(3),0} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{0,(3)}$ |
| $\varphi^{24}$ | 8   | $\varphi^{(1^3),0} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{0,(1^3)}$ |
| $\varphi^{25}$ | 16  | $\varphi^{(21),0} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{0,(21)}$ |
5. Seminormal Representations for $HA_1$, $HA_2$, $HB_3$, and $HF_4$

5.1. The Iwahori-Hecke algebra $HA_1$. The irreducible representations $\varphi^\lambda$ of $HA_1$ are indexed by the partitions $\lambda$ of 2 and we have

$$\varphi^{(2)}(T_1) = (p) \quad \text{and} \quad \varphi^{(12)}(T_1) = (-p^{-1}).$$

5.2. The Iwahori-Hecke algebra $HA_2$. The irreducible representations $\varphi^\lambda$ of $HA_2$ are indexed by the partitions of 3 and can be given explicitly by

$$\varphi^{(3)}(T_1) = (p), \quad \varphi^{(3)}(T_2) = (p),$$
$$\varphi^{(21)}(T_1) = \text{diag}(p, -p^{-1}), \quad \varphi^{(21)}(T_2) = M_2(p, \alpha),$$
$$\varphi^{(13)}(T_1) = (-p^{-1}), \quad \varphi^{(13)}(T_2) = (-p^{-1}),$$

where

$$M_2(p, \alpha) = -\frac{1}{[2]_p} \begin{pmatrix} p^{-2} & \alpha([2]_p - 1) \\ \frac{1}{\alpha}([2]_p + 1) & -p^2 \end{pmatrix}$$

and $[2]_p = p + p^{-1}$. The variable $\alpha$ is a free parameter, see Lemma 3.3.

5.3. The Iwahori-Hecke algebra $HB_3$. The irreducible representations $\varphi^\mu = \varphi^{\alpha, \beta}$ of $HB_3(p^2, q^2)$ are indexed by pairs of partitions $\mu = (\alpha, \beta)$ such that $|\alpha| + |\beta| = 3$. Let $\text{diag}(A, B, \ldots, C)$ denote the block diagonal matrix with the matrices $A, B, \ldots, C$ in order on the diagonal. Then, using the notation

$$[2]_x = x + x^{-1}, \quad [3]_x = x^2 + 1 + x^{-2}, \quad \text{and} \quad [0]_x = x - x^{-1},$$

irreducible seminormal representations of the Iwahori-Hecke algebra $HB_3$ can be given explicitly as follows:

$$\varphi^{(3),0}(T_1) = (p), \quad \varphi^{(3),0}(T_2) = (p), \quad \varphi^{(3),0}(T_3) = (q),$$
$$\varphi^{(12),0}(T_1) = (-p^{-1}), \quad \varphi^{(12),0}(T_2) = (-p^{-1}), \quad \varphi^{(12),0}(T_3) = (q),$$
$$\varphi^{0,(3)}(T_1) = (p), \quad \varphi^{0,(3)}(T_2) = (p), \quad \varphi^{0,(3)}(T_3) = (-q^{-1}),$$
$$\varphi^{0,(13)}(T_1) = (-p^{-1}), \quad \varphi^{0,(13)}(T_2) = (-p^{-1}), \quad \varphi^{0,(13)}(T_3) = (-q^{-1}),$$
$$\varphi^{(21),0}(T_1) = \text{diag}(p, -p^{-1}), \quad \varphi^{(21),0}(T_2) = M_2(p, 1), \quad \varphi^{(21),0}(T_3) = \text{diag}(q, q),$$
$$\varphi^{0,(21)}(T_1) = \text{diag}(p, -p^{-1}), \quad \varphi^{0,(21)}(T_2) = M_2(p, 1), \quad \varphi^{0,(21)}(T_3) = \text{diag}(-q^{-1}, -q^{-1}).$$
\[ \varphi^{(1)}_{(1)}(T_1) = \text{diag}(p, p, -p^{-1}), \quad \varphi^{(2)}_{(1)}(T_1) = \text{diag}(M_2(p, 1)), \]
\[ \varphi^{(1)}_{(2)}(T_1) = \text{diag}(p, p, -p^{-1}), \quad \varphi^{(2)}_{(2)}(T_1) = \text{diag}(M_2(p, 1), -p^{-1}), \]
\[ \varphi^{(1)}_{(1,2)}(T_1) = \text{diag}(p, -p^{-1}, -p^{-1}), \quad \varphi^{(2)}_{(1,2)}(T_1) = \text{diag}(M_2(p, 1), -p^{-1}). \]

\[ \varphi^{(1)}_{(1)}(T_3) = \text{diag}(M_{(1), (2)}, q), \quad \varphi^{(2)}_{(1)}(T_3) = \text{diag}(M_{(2), (1)}, q), \]
\[ \varphi^{(1)}_{(2)}(T_3) = \text{diag}(M_{(1,2), (1)}), \quad \varphi^{(2)}_{(2)}(T_3) = \text{diag}(M_{(1,2), (1)}), \]
\[ \varphi^{(1)}_{(1,2)}(T_3) = \text{diag}(-q^{-1}, M_{(1,2), (1)}), \quad \varphi^{(2)}_{(1,2)}(T_3) = \text{diag}(-q^{-1}, M_{(1,2), (1)}), \]

where

\[
M_{(2), (1)} = \frac{1}{[3]_p} \begin{pmatrix}
q + p^{-2}[0]_q & -[2]_p[2]_{p/q} \\
-[2]_p^2q & -q^{-1} + p^2[0]_q
\end{pmatrix},
\]
\[
M_{(1,2), (1)} = \frac{1}{[3]_p} \begin{pmatrix}
-q^{-1} + p^{-2}[0]_q & -[2]_p^2/q \\
-[2]_p[2]_{p^2/q} & q + p^2[0]_q
\end{pmatrix},
\]
\[
M_{(1), (2)} = \frac{1}{[3]_p} \begin{pmatrix}
-q^{-1} + p^{-2}[0]_q & [2]_p[2]_{pq} \\
[2]_{p^2/q} & q + p^2[0]_q
\end{pmatrix},
\]
\[
M_{(1), (1,2)} = \frac{1}{[3]_p} \begin{pmatrix}
q + p^{-2}[0]_q & [2]_p^2q \\
[2]_p[2]_{p/q} & -q^{-1} + p^2[0]_q
\end{pmatrix}.
\]

5.4. The Iwahori-Hecke algebra $HF_4$. Let $\alpha_p$ be the automorphism of $\mathbb{Q}(p, q)$ which fixes $q$ and sends $p$ to $-p^{-1}$. Similarly, let $\alpha_q$ be the automorphism which fixes $p$ and sends $q$ to $-q^{-1}$. These field automorphisms act on the entries of the matrices $\varphi^\lambda(T_i)$ and thereby permute the representations $\varphi^\lambda$. The representation resulting from the application of a field automorphism to a representation in seminormal form may no longer be seminormal. In order to bring the representation back to seminormal form it may be necessary to conjugate by a permutation matrix $\pi$. The orbits of the irreducible representations under the action of $\alpha_p$ and $\alpha_q$ and the permutations $\pi$ for conjugating back to seminormal form are given in the following table. If $\varphi$ is
a representation of $HF_4$ and $\pi$ is a permutation then we shall let $\pi \circ \varphi$ denote the representation determined by $(\pi \circ \varphi)(h) = \pi \varphi(h)\pi^{-1}$, for all $h \in HF_4$.

| $\varphi^k$ | Orbit of $\langle \alpha_p, \alpha_q \rangle$ |
|------------|---------------------------------|
| $\varphi^1$ | $\varphi^2 = \alpha_p\varphi^1$, $\varphi^3 = \alpha_q\varphi^1$, and $\varphi^4 = \alpha_p\alpha_q\varphi^1$ |
| $\varphi^5$ | $\varphi^6 = \alpha_q\varphi^5$ |
| $\varphi^7$ | $\varphi^8 = \alpha_p\varphi^7$ |
| $\varphi^{10}$ | $\varphi^{11} = \pi_{11} \circ (\alpha_p\varphi^{10})$, where $\pi_{11} = (1, 3)(4, 6)(7, 9)$ $\varphi^{12} = \pi_{12} \circ (\alpha_q\varphi^{10})$, where $\pi_{12} = (1, 7)(2, 8)(3, 9)$ $\varphi^{13} = \pi_{13} \circ (\alpha_p\alpha_q\varphi^{10})$, where $\pi_{13} = (1, 9)(2, 8)(3, 7)(4, 6)$ |
| $\varphi^{14}$ | $\varphi^{15} = \pi_{15} \circ (\alpha_p\varphi^{14})$, where $\pi_{15} = (1, 3)(4, 6)$ |
| $\varphi^{17}$ | $\varphi^{18} = \pi_{18} \circ (\alpha_p\varphi^{17})$, where $\pi_{18} = (2, 4)$ $\varphi^{19} = \pi_{19} \circ (\alpha_q\varphi^{17})$, where $\pi_{19} = (1, 4, 3, 2)$ $\varphi^{20} = \pi_{20} \circ (\alpha_p\alpha_q\varphi^{17})$, where $\pi_{20} = (1, 4)(2, 3)$ |
| $\varphi^{21}$ | $\varphi^{22} = \pi_{22} \circ (\alpha_q\varphi^{21})$, where $\pi_{22} = (1, 7, 5, 3)(2, 8, 6, 4)$ |
| $\varphi^{23}$ | $\varphi^{24} = \pi_{24} \circ (\alpha_p\varphi^{23})$, where $\pi_{24} = (2, 4)(5, 7)$ |

Let $w_{0, 1}$, $w_{0, 2}$, $w_{0, 3}$ and $w_{0, 4}$ be the longest elements in the Weyl groups $WA_1$, $WA_2$, $WB_3$ and $WF_4$, respectively. Let

$$
D_1 = T_{w_{0, 1}} = T_1,
$$

$$
D_2 = T_{w_{0, 2}}^2 = (T_1T_2T_1)^2,
$$

$$
D_3 = T_{w_{0, 3}} = (T_3T_2T_1)^3,
$$

$$
D_4 = T_{w_{0, 4}} = (T_4T_{w_{0, 3}})^3T_{w_{0, 2}}^{-2}
$$

in $HF_4$. The following tables give the values of $\varphi^k(D_j)$, for one representative from each equivalence class of representations. The rest of the matrices $\varphi^k(D_j)$ are easily obtained by applying the automorphisms $\alpha_p$ and $\alpha_q$ and conjugating by a permutation $\pi$ as indicated in 5.4 above.

$$
\varphi^1(T_1) = (p),
\varphi^1(T_{w_{0, 2}}^2) = (p^6),
\varphi^1(T_{w_{0, 3}}) = (p^6q^6),
\varphi^1(T_{w_{0, 4}}) = (p^{12}q^{12}),
$$

$$
\varphi^5(T_1) = \text{diag}(p, -p^{-1}),
\varphi^5(T_{w_{0, 2}}^2) = \text{Id},
\varphi^5(T_{w_{0, 3}}) = q^3\text{ Id},
\varphi^5(T_{w_{0, 4}}) = q^{12}\text{ Id},
$$
\begin{align*}
\psi^7(T_1) &= p \Id, & \psi^9(T_1) &= \text{diag}(p, -p^{-1}, p, -p^{-1}), \\
\psi^7(T_{w_0}^2) &= p^6 \Id, & \psi^9(T_{w_0}^2) &= \Id, \\
\psi^7(T_{w_0,3}) &= \text{diag}(p^6 q^3, -p^6 q^{-3}), & \psi^9(T_{w_0,3}) &= \text{diag}(q^3, q^3, -q^{-3}, -q^{-3}), \\
\psi^7(T_{w_0,4}) &= p^{12} \Id, & \psi^9(T_{w_0,4}) &= \Id, \\
\psi^{10}(T_1) &= \text{diag}(p, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}), & \\
\psi^{10}(T_{w_0,2}) &= \text{diag}(p^6, 1, 1, p^6, 1, 1), & \\
\psi^{10}(T_{w_0,3}) &= \text{diag}(p^6 q^3, q^3, -p^2 q, -p^2 q, -p^2 q, -p^2 q, p^2 q^{-1}, p^2 q^{-1}, p^2 q^{-1}), \\
\psi^{10}(T_{w_0,4}) &= p^4 q^4 \Id, \\
\psi^{14}(T_1) &= \text{diag}(p, -p^{-1}, -p^{-1}, p, p, -p^{-1}), & \\
\psi^{14}(T_{w_0,2}) &= \text{diag}(1, 1, p^{-6}, p^6, 1, 1), & \\
\psi^{14}(T_{w_0,3}) &= \text{diag}(-p^{-2} q, -p^{-2} q, -p^{-2} q, p^2 q^{-1}, p^2 q^{-1}, p^2 q^{-1}), \\
\psi^{14}(T_{w_0,4}) &= \Id, \\
\psi^{16}(T_1) &= \text{diag}(p, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}), & \\
\psi^{16}(T_{w_0,2}) &= \text{diag}(p^6, 1, 1, 1, 1, p^{-6}, p^6, 1, 1, 1, 1, p^{-6}), & \\
\psi^{16}(T_{w_0,3}) &= \text{diag}(-p^2 q, -p^2 q, -p^2 q, -p^{-2} q, -p^{-2} q, -p^{-2} q, p^2 q^{-1}, p^2 q^{-1}, p^2 q^{-1}, \\ & p^{-2} q^{-1}, p^{-2} q^{-1}, p^{-2} q^{-1}), \\
\psi^{16}(T_{w_0,4}) &= \Id, \\
\psi^{17}(T_1) &= \text{diag}(p, p, p, -p^{-1}), & \\
\psi^{17}(T_{w_0,2}) &= \text{diag}(p^6, p^6, 1, 1), & \\
\psi^{17}(T_{w_0,3}) &= \text{diag}(p^6 q^3, -p^2 q, -p^2 q), \\
\psi^{17}(T_{w_0,4}) &= -p^6 q^6 \Id, \\
\psi^{21}(T_1) &= \text{diag}(p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, -p^{-1}), & \\
\psi^{21}(T_{w_0,2}) &= \text{diag}(1, 1, p^6, 1, 1, 1, 1, p^{-6}), & \\
\psi^{21}(T_{w_0,3}) &= \text{diag}(q^3, q^3, -p^2 q, -p^2 q, -p^2 q, -p^2 q, -p^2 q, -p^2 q), \\
\psi^{21}(T_{w_0,4}) &= -q^6 \Id,
\end{align*}
\[ \varphi^{23}(T_1) = \text{diag}(p, p, p, -p^{-1}, p, -p^{-1}, p), \]
\[ \varphi^{23}(T_{w_0,2}) = \text{diag}(p^6, p^6, 1, 1, p^6, 1, 1, p^6), \]
\[ \varphi^{23}(T_{w_0,3}) = \text{diag}(p^6q^3, -p^2q, -p^2q, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}, p^6q^{-3}), \]
\[ \varphi^{23}(T_{w_0,4}) = -p^6 \text{Id}, \]

\[ \varphi^{25}(T_1) = \text{diag}(p, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}), \]
\[ \varphi^{25}(T_{w_0,2}) = \text{diag}(1, 1, p^6, 1, 1, 1, p^6, p^6, 1, 1, 1, p^6, 1, 1), \]
\[ \varphi^{25}(T_{w_0,3}) = \text{diag}(q^3, q^3, -p^2q, -p^2q, -p^2q, -p^2q, -p^2q, -p^2q, -p^2q, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}, -p^2q^{-1}, p^2q^{-1}, -p^2q^{-1}, -p^2q^{-1}, -p^2q^{-1}, -p^2q^{-1}, -p^2q^{-1}), \]
\[ \varphi^{25}(T_{w_0,4}) = -\text{Id}. \]

Using the methods described in the previous sections we have produced matrices \( \varphi^k(T_1) \) giving the 25 irreducible representations \( \varphi^k \) of \( HF_4 \). The following tables give the values of \( \varphi^k(T_i) \) for one representative from each equivalence class of representations. The rest of the matrices \( \varphi^k(T_i) \) are obtained by applying the automorphisms \( \alpha_p \) and \( \alpha_q \) and conjugating by a permutation \( \pi \) as indicated in 5.4 above.

We shall use the notations

\[ [2]_x = x + x^{-1}, \quad [3]_x = x^2 + 1 + x^{-2}, \quad \text{and} \quad [0]_x = x - x^{-1}, \]

and the notation

\[ \varphi^k(T_i)^{[a_1, a_2, \ldots, a_r]} \]

will denote the \( r \times r \) submatrix of \( \varphi^k(T_i) \) which is formed by the intersection of the \( a_1, \ldots, a_r \)th rows and columns. The notation \( \text{diag}(A, B, \ldots, C) \) will denote the block diagonal matrix with the matrices \( A, B, \ldots, C \) in order along the diagonal. The matrix \( M_2(x, y) \) will be as given in 5.2, the matrices \( M_{\alpha, \beta} \) are as given in 5.3 and the variables \( \alpha, \beta, \xi, \theta, \) and \( \eta \) are free parameters, see Lemma 3.3. Any entries of the matrices \( \varphi^k(T_i) \) which are not given explicitly below are taken to be 0.

The representations \( \varphi^1 \) and \( \varphi^5 \):

\[ \varphi^1(T_1) = (p), \quad \varphi^5(T_1) = \text{diag}(p, -p^{-1}), \]
\[ \varphi^1(T_2) = (p), \quad \varphi^5(T_2) = M_2(p, 1), \]
\[ \varphi^1(T_3) = (q), \quad \varphi^5(T_3) = \text{diag}(q, q), \]
\[ \varphi^1(T_4) = (q), \quad \varphi^5(T_4) = \text{diag}(q, q). \]
The representations $\varphi^7$ and $\varphi^9$.

$\varphi^7(T_1) = \text{diag}(p, p)$, 
$\varphi^9(T_1) = \text{diag}(p, -p^{-1}, p, -p^{-1})$, 
$\varphi^7(T_2) = \text{diag}(p, p)$, 
$\varphi^9(T_2) = \text{diag}(M_2(p, 1), M_2(p, 1))$, 
$\varphi^7(T_3) = \text{diag}(q, -q^{-1})$, 
$\varphi^9(T_3) = \text{diag}(q, q, -q^{-1}, q^{-1})$, 
$\varphi^7(T_4) = M_2(q, \alpha)$, 
$\varphi^9(T_4)^{[1, 3]} = \varphi^9(T_4)^{[2, 4]} = M_2(q, \alpha)$.

The representation $\varphi^{10}$.

$\varphi^{10}(T_1) = \text{diag}(p, p, -p^{-1}, p, p, -p^{-1}, p, p, -p^{-1})$, 
$\varphi^{10}(T_2) = \text{diag}(p, M_2(p, 1), p, M_2(p, 1), p, M_2(p, 1))$, 
$\varphi^{10}(T_3) = \text{diag}(q, q, M(2, 1), q, M(1, 2), q)$, 
$\varphi^{10}(T_4)^{[1, 4, 7]} = M_{10}$, 
$\varphi^{10}(T_4)^{[2, 5, 8]} = \varphi^{10}(T_4)^{[3, 6, 9]} = N_{10}$,

where

$$M_{10} = \frac{1}{[2]_q[2]_p^2q} \begin{pmatrix} p^{-2}q^{-1}[2]_q[0]_q & -[2]_q[2]_p^2q^2\xi\eta^{-1} & -[2]_q[2]_p^2q^2\xi \\ -[2]_p^2q^{-1}\eta\xi^{-1} & [2]_p^2q + p^2q[2]_q[0]_q & -[2]_p^2q^{-1}\eta \\ -[2]_p^2q\eta^{-1} & -[2]_p^2q\eta^{-1} & q^2[2]_p^2q \end{pmatrix}$$

and

$$N_{10} = \frac{1}{[2]_q[2]_{pq^{-1}}} \begin{pmatrix} pq^{-1}[2]_q[0]_q & -[2]_q[2]_pq^{-2}\theta\eta^{-1} & -[2]_q[2]_pq^{-2}\theta \\ -[2]_pq\eta\theta^{-1} & [2]_pq^{-1} + p^{-1}q[2]_q[0]_q & -[2]_pq\eta \\ -[2]_pq^{-1}\theta^{-1} & -[2]_pq^{-1}\theta^{-1} & q^2[2]_pq^{-1} \end{pmatrix}.$$
The representation $\varphi^{14}$.

\[ \varphi^{14}(T_1) = \text{diag}(p, -p^{-1}, -p^{-1}, p, p, -p^{-1}), \]
\[ \varphi^{14}(T_2) = \text{diag}(M_2(p, 1), -p^{-1}, p, M_2(p, 1)), \]
\[ \varphi^{14}(T_3) = \text{diag}(q, M_{(12,1)}, M_{(1,2)}, -q^{-1}), \]
\[ \varphi^{14}(T_4)^{[3]} = -q^{-1}, \]
\[ \varphi^{14}(T_4)^{[4]} = q, \]
\[ \varphi^{14}(T_4)^{[1,5]} = M_{14}, \]
\[ \varphi^{14}(T_4)^{[2,6]} = M_{14}, \]

where

\[ M_{14} = \frac{1}{[2]_{p^2 q^{-1}}} \begin{pmatrix}
1 + p^2 q^{-1}[0]_q & -[3]_p \alpha \\
(1 - [2]_{p^2 q^{-2}}) \alpha^{-1} & -1 + p^{-2} q[0]_q
\end{pmatrix}. \]
The representation $\varphi^{16}$.

$\varphi^{16}(T_1) = \text{diag}(p, p, -p^{-1}, p, -p^{-1}, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, -p^{-1}),$
$\varphi^{16}(T_2) = \text{diag}(p, M_2(p, 1), M_2(p, 1), -p^{-1}, p, M_2(p, 1), M_2(p, 1) - p^{-1}),$
$\varphi^{16}(T_3) = \text{diag}(M(2, 1), q, q, M(1^2, 1), M(1, 2), -q^{-1}, -q^{-1}, M(1, 2)),$
$\varphi^{16}(T_4)^{[1, 7]} = M_{16}(\xi),$
$\varphi^{16}(T_4)^{[6, 12]} = M_{16}(\eta),$
$\varphi^{16}(T_4)^{[2, 4, 8, 10]} = \varphi^{16}(T_4)^{[3, 5, 9, 12]} = N_{16},$

where

$$M_{16}(\alpha) = \frac{1}{[2]_q} \begin{pmatrix} 1 + q^{-1}[0]_q & -3\alpha \\ -[3]_q [2]_q^{-1}/\alpha [3]_q & -1 + q[0]_q \end{pmatrix},$$

$$N_{16} = \frac{1}{[2]_p [2]_q} \begin{pmatrix} f_{16}(p, q) & \frac{3[2]_p [2]_q [2]_q^{-1}/\xi \theta}{[2]_p [2]_q} & \frac{3[2]_p [2]_q [2]_q^{-1}/\xi \theta}{[2]_p [2]_q} & \frac{3[2]_p [2]_q [2]_q^{-1}/\xi \theta}{[2]_p [2]_q} \\ \frac{[3]_q [2]_p [2]_q^{-1}/p [\eta]}{[2]_p [2]_q \xi \theta} & -f_{16}(-p^{-1}, q) & \frac{3[2]_p [2]_q [2]_q^{-1}/\eta}{[2]_p [2]_q \xi \theta} & \frac{3[2]_p [2]_q [2]_q^{-1}/\eta}{[2]_p [2]_q} \\ \frac{[3]_q [2]_q [2]_p [\theta]}{[2]_p [2]_q [2]_q^{-1}/\xi} & \frac{[3]_q [2]_p [2]_q [\xi]}{[2]_p [2]_q [2]_q^{-1}/\eta} & -f_{16}(p, -q^{-1}) & \frac{3[2]_p [2]_q [2]_q^{-1}/\eta}{[2]_p [2]_q [2]_q^{-1}/\xi} \\ \frac{3[2]_p [2]_q [2]_q^{-1}/[2]_q [\xi \theta]}{[2]_p [2]_q [2]_q^{-1}/\xi} & \frac{[3]_q [2]_p [2]_q [\xi]}{[2]_p [2]_q [2]_q^{-1}/\eta} & \frac{3[2]_p [2]_q [2]_q^{-1}/\eta}{[2]_p [2]_q [2]_q^{-1}/\xi} & f_{16}(-p^{-1}, -q^{-1}) \end{pmatrix},$$

and

$$f_{16}(x, y) = \frac{-2x/y + xy + 1/xy - 1/xy^3 - y/x - 1/x^3 y^3 + y/x^3}{[2]_x^2 y} [2]_x^2 y.$$

The representation $\varphi^{17}$.

$\varphi^{17}(T_1) = \text{diag}(p, p, -p^{-1}),$
$\varphi^{17}(T_2) = \text{diag}(p, p, M_2(p, 1)),$
$\varphi^{17}(T_3) = \text{diag}(q, M(2, 1), q),$
$\varphi^{17}(T_4) = \text{diag}(M_{17}, q, q),$

where

$$M_{17} = \frac{1}{[2]_p [2]_q} \begin{pmatrix} 1 + p^{-2} [0]_q^{-1} & -[3]_p \alpha \\ (1 - [2]_p [2]_q \alpha)^{-1} & -1 + p^2 q [0]_q \end{pmatrix}.$$
The representation $\varphi^{21}$.

\[
\varphi^{21}(T_1) = \mathrm{diag}(p, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, -p^{-1}),
\]
\[
\varphi^{21}(T_2) = \mathrm{diag}(M_2(p, 1), p, M_2(p, 1), M_2(p, 1), -p^{-1}),
\]
\[
\varphi^{21}(T_3) = \mathrm{diag}(q, q, M_{(2,1)}, q, q, M_{(1,2)}),
\]
\[
\varphi^{21}(T_4)^{[3]} = \varphi^{21}(T_4)^{[8]} = q,
\]
\[
\varphi^{21}(T_4)^{[1,4,6]} = \varphi^{21}(T_4)^{[2,5,7]} = M_{21},
\]

where

\[
M_{21} = \frac{1}{[2]_p[2]_{pq}[2]_{p/q}} \begin{pmatrix}
(q[2]_p^2 + q^{-2}[0]_q)[2]_p & -[2]_q[2]_p \xi \eta^{-1} & -[2]_q[2]_p \xi \\
-3[2]_p[2]_{pq} \xi \eta^{-1} & (p^{-1}q[2]_p[0]_q + 1)[2]_{pq} & -3[2]_p[2]_{pq} \eta \\
-3[2]_p[2]_{pq} \xi^{-1} & -3[2]_p[2]_{pq} \eta^{-1} & (pq[2]_p[0]_q + 1)[2]_{pq}
\end{pmatrix}.
\]

The representation $\varphi^{23}$.

\[
\varphi^{23}(T_1) = \mathrm{diag}(p, p, p, -p^{-1}, p, p, -p^{-1}, p),
\]
\[
\varphi^{23}(T_2) = \mathrm{diag}(p, p, M_2(p, 1), p, M_2(p, 1), p),
\]
\[
\varphi^{23}(T_3) = \mathrm{diag}(q, M_{(2,1)}, q, M_{(1,2)}, -q^{-1}, -q^{-1}),
\]
\[
\varphi^{23}(T_4)^{[3,6]} = \varphi^{23}(T_4)^{[4,7]} = M_2(q, \eta/([2]_q - 1)\theta),
\]
\[
\varphi^{23}(T_4)^{[1,2,5,8]} = M_{23},
\]

where

\[
M_{23} = \frac{1}{[2]_q} \begin{pmatrix}
f_{23}(p, q) & [2]_{pq} \xi & [2]_{pq}[2]_{pq}^2 \xi & [2]_{pq} \xi \\
[3]_q[2]_p \eta & [2]_{pq} \xi & [2]_{pq}[2]_{pq} \xi & [2]_{pq} \xi \\
[3]_q[2]_p \xi & (2q^2 - 1)[2]_q \eta & [2]_{pq} \xi & [2]_{pq} \xi \\
[3]_q[2]_p \xi^2 & -3[2]_q[2]_{pq} \eta & [2]_{pq} \xi & -f_{23}(p, -1/q)
\end{pmatrix},
\]

where

\[
f_{23}(x, y) = \frac{y^4 - x^4y^2 - x^2y^2 - 1}{x^3y^4[2]_{xy}[2]_{x^2y}}
\]

and

\[
g_{23}(x, y) = \frac{x^4y^6 - x^4y^2 + x^2y^6 - x^2y^4 + x^2y^2 + y^6 + y^2 - 1}{xy[2]_{x/y}[2]_{x^2y}}.
\]
The representation $\varphi^{25}$.

$$
\varphi^{25}(T_1) = \text{diag}(p, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}, p, -p^{-1}),
$$

$$
\varphi^{25}(T_2) = \text{diag}(M_2(p, 1), p, M_2(p, 1), M_2(p, 1), -p^{-1}, p, M_2(p, 1), M_2(p, 1), -p^{-1}, M_2(p, 1)),
$$

$$
\varphi^{25}(T_3) = \text{diag}(q, q, M_{(2,1)}, q, q, M_{(1,2)}, -q^{-1}, -q^{-1}, M_{(1,2)}, -q^{-1}, -q^{-1}),
$$

$$
\varphi^{25}(T_4)_{[3,9]} = M_2(q, \alpha/([2]_q - 1)\eta),
$$

$$
\varphi^{25}(T_4)_{[8,14]} = M_2(q, \beta/([2]_q - 1)\theta),
$$

$$
\varphi^{25}(T_4)_{[1,4,6,10,12,15]} = \varphi^{25}(T_4)_{[2,5,7,11,13,16]} = M_{25},
$$
where

\[ M_{25} = \frac{1}{[2]_q} \begin{pmatrix} f_{25}(p, q) & -[2]_{p^2/q} \xi & -[2]_{p^2 q \xi} \\ -2[3]_p[2]_{p^2/q^2} \alpha & [2]_{pq}[2]_{p^2/q^2} & [2]_{pq}[2]_{p/q^2} \alpha \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \beta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} \beta \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \beta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} \beta \\ -2[3]_p[2]_{p^2 q^2} [3]_q \eta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} [3]_q \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} [3]_q \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \eta \\ -2[3]_p[2]_{p^2 q^2} [3]_q \theta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} [3]_q \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} [3]_q \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \theta \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} [3]_q \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \theta \\ -2[3]_p[2]_{p^2 q^2} \eta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \eta \\ -2[3]_p[2]_{p^2 q^2} \theta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \theta \\ -2[3]_p[2]_{p^2 q^2} \eta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \theta \\ -2[3]_p[2]_{p^2 q^2} \theta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \theta \\ -2[3]_p[2]_{p^2 q^2} \eta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \eta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \theta \\ -2[3]_p[2]_{p^2 q^2} \theta & [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} \\ [2]_p[2]_{pq}[2]_{p^2 q^2} \xi & \frac{-2[3]_p[2]_{p^2 q^2} \theta}{[2]_p[2]_{pq}[2]_{p^2 q^2} \xi} & [3]_p[2]_{p^2 q^2} [3]_q \theta \end{pmatrix} \]

and where

\[ f_{25}(x, y) = -\frac{x^4 y^2 + x^2 - x^2 y^4 + y^2}{x^2 y^4 [2]_{xy} [2]_{x/y}} \],

and

\[ g_{25}(x, y) = -\frac{x^6 y^4 - x^4 y^6 - x^4 y^2 + x^4 + x^4 y^4 + x^2 y^2 + x^2 - x^2 y^6 + y^2 - y^6}{x^4 y^4 [2]_{x/y} [2]_{x^2 y}}. \]
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Arun Ram, Department of Mathematics, Princeton University, Princeton, NJ 08544-1000
*E-mail address:* rama.math.princeton.edu

D. E. Taylor, School of Mathematics and Statistics, University of Sydney, NSW 2006 Australia
*E-mail address:* don.maths.su.oz.au