ON COHOMOLOGICAL HALL ALGEBRAS OF QUIVERS : YANGIANS

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Abstract. We consider the cohomological Hall algebra $Y^1$ of a Lagrangian substack of the moduli stack of representations of the preprojective algebra of an arbitrary quiver $Q$, and its actions on the cohomology of quiver varieties. We conjecture that $Y^1$ is equal, after a suitable extension of scalars, to the Yangian $Y$ introduced by Maulik and Okounkov, and we construct an embedding $Y^1 \subseteq Y$, intertwining the respective actions of $Y^1$ and $Y$ on the cohomology of quiver varieties.

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1. Introduction

Nakajima associated to each quiver $Q$ and pair of dimension vectors $(v,w)$ of $Q$ a symplectic resolution

$$\pi : \mathcal{M}(v,w) \to \mathcal{M}_0(v,w)$$
where \( M(v, w) \) is a smooth quasi-projective symplectic variety and \( M_\emptyset(v, w) \) is a (in general singular) affine variety. The varieties \( M(v, w) \) and \( M_\emptyset(v, w) \) have many remarkable geometric properties and have played a very important role in geometric representation theory in the past twenty years (see e.g. the introduction to [21]). In particular, when the quiver \( Q \) carries no edge loops and thus can be regarded as an orientation of the generalized Dynkin diagram of a Kac-Moody algebra \( g_Q \), Nakajima constructed an action of \( g_Q \) on the space

\[
L_w = \bigoplus_v \text{H}_{\text{top}}(\mathcal{L}(v, w)),
\]

where \( \mathcal{L}(v, w) = \pi^{-1}(0) \) is the Lagrangian quiver variety, see [14]. The resulting module is identified with the integrable irreducible highest weight module of highest weight \( \sum_i w_i\Lambda_i \) where the \( \Lambda_i \)'s are the fundamental weights of \( g_Q \). When the quiver \( Q \) is of finite type, Nakajima constructed a representation of the quantum affine algebra of \( g_Q \) on the space

\[
\bigoplus_w K^{G(w) \times \mathbb{C}^\times}(\mathcal{L}(v, w)).
\]

The resulting module is called a universal standard module, and it is a geometric analog of the global Weyl modules. A cohomological version of this construction, due to Varagnolo [22], yields an action of the Yangian of \( g_Q \) on the space

\[
\bigoplus_v \text{H}_{*}^{G(w) \times \mathbb{C}^\times}(\mathcal{L}(v, w)),
\]

and the resulting module is again the universal standard module. Finding a similar interpretation for arbitrary quivers is harder. Nakajima’s and Varagnolo’s actions do extend to the case of arbitrary quivers with no edge loops, but the precise nature of the algebra which acts or the structure of the resulting module are not well understood beyond the cases of finite, affine or Jordan quivers.

There are two main approaches to the problem of constructing and understanding symmetry algebras acting on the cohomology of Nakajima quiver varieties for general quivers. One approach is due to Maulik and Okounkov [12], who construct, using the RTT formalism and ideas from symplectic geometry, an algebra \( Y \), called a Yangian, which acts on the Borel-Moore homology groups

\[
F_w = \bigoplus_v \text{H}_{*}^{G(w) \times T}(\mathcal{M}(v, w))
\]

for any quiver \( Q \) and dimension vector \( w \). Here \( T \) is a natural torus acting by rescaling the edges of \( Q \). A different approach is developed in the series of papers [18], [19], [20] in the particular case of quivers with only one vertex (and partially extended in [21], [24], [25] to the case of arbitrary quivers), in which we defined another algebra \( Y \), called the Cohomological Hall algebra or the K-theoretic Hall algebra of the quiver \( Q \), acting on \( F_w \) or it’s K-theoretical analogue. This construction is based on the geometry of the cotangent of the moduli stacks of representations of quivers. The algebra \( Y \) is in some sense the
largest algebra which acts on the homology of all quiver varieties by means of some Hecke correspondences.

The aim of this paper is to compare the algebras $\mathcal{Y}$ and $Y$ as well as their respective actions on the spaces $F_w$ for an arbitrary quiver $Q$ and dimension vector $w$, see Theorem B below. In order to state our results more precisely, we need to introduce some notations and recall some results from [21] and [12].

**Cohomological Hall algebras.** Let $v$ be a dimension vector and let $\text{Rep}(\mathbb{C}Q,v)/G(v)$ be the moduli stack of complex representations of $Q$ of dimension $v$. Let $\Pi$ be the preprojective algebra of $Q$ and $\text{Rep}(\Pi,v)/G(v)$ be the moduli stack of complex representations of $\pi$ of dimension $v$. In [2] we defined a Lagrangian substack $\Lambda^1(v)/G(v)$ of $\text{Rep}(\Pi,v)/G(v)$, by using some semi-nilpotency condition. See also [1]. The torus $T$ acts on $\text{Rep}(\Pi,v)/G(v)$ and $\Lambda^1(v)/G(v)$. We refer to [21] for a list of geometric properties of $\Lambda^1(v)/G(v)$. Set

$$Y^1 = \bigoplus_v H^*_G(v)^\times T(\Lambda^1(v)), \quad Y = \bigoplus_v H^*_G(v)^\times T(\text{Rep}(\Pi,v)).$$

Motivated by the analogy with Yangians, we consider an extension $Y^1$ of $Y^1$ by adding a loop Cartan part equal to $Y(0) = H^*_G(\infty)^\times T$. Let $k$ be the $T$-equivariant cohomology ring $H^*_T$ of the point and $K$ be its fraction field. We write

$$Y^1_K = Y^1 \otimes_k K, \quad Y^1 = Y^1 \otimes_k K.$$

We set

$$\Lambda_{(v)} = \{0\} \times \text{Rep}(\mathbb{C}Q^*,v).$$

This is always an irreducible component of $\Lambda^1(v)$ in case $v$ is supported on a subquiver without oriented cycles. We say that an imaginary vertex is elliptic or isotropic if it carries a single 1-loop and hyperbolic if it carries more than one 1-loop. We denote by $I^I$, $I^e$ and $I^h$ respectively the real, elliptic and hyperbolic vertices of $I$. Let $\{\delta_i ; i \in I\}$ be the basis of delta functions in $\mathbb{Z}^I$. The important properties of $Y$, $Y^1$ and $Y$ proved in [21] are summarized below.

**Theorem A.** For any $\# = 0, 1$, we have the following.

(a) There is an associative $\mathbb{Z} \times \mathbb{N}^I$-graded $k$-algebra structure on $Y^\#$.

(b) There is a representation of $Y^\#$ on $F_w$ for each $w$.

(c) The diagonal action of $Y^1$ on $\bigoplus_w F_w$ is faithful.

(d) There are $K$-algebra isomorphisms $Y^1_K \simeq Y_K$.

(e) The $K$-algebra $Y_K$ is generated by the subspaces $H^G(v)^\times T(\Lambda_{(v)}) \otimes_k K$ where $v$ ranges among the following set of dimension vectors

$$\{\delta_i ; i \in I^I \cup I^e\} \cup \{l\delta_i ; i \in I^h, l \in \mathbb{N}\}.$$

The $K$-algebra $Y_K$ is generated by $Y(0)$ and the collection of fundamental classes

$$\{[\Lambda_{(\delta_i)}] ; i \in I^I \cup I^e\} \cup \{[\Lambda_{(l\delta_i)}] ; i \in I^h, l \in \mathbb{N}\}.$$
This implies that $Y_1$ coincides with the $k$-algebra constructed by Varagnolo when $Q$ has no imaginary vertex. If $Q$ has no hyperbolic vertex then the same is true after extension of scalars to $K$.

**Maulik-Okounkov Yangians.** In [12], the authors defined and studied another associative algebra acting on the Borel-Moore homology of Nakajima quiver varieties associated to an arbitrary quiver $Q$. Their construction, which stems from ideas in symplectic geometry, hinges on the notion of stable envelope to produce a quantum $R$-matrix, and then on the RTT formalism to define an associative $\mathbb{Z} \times \mathbb{Z}$-graded Lie algebra $g = g_Q$. If $Q$ is of finite type, then $g_Q$ is the semisimple Lie algebra associated with $Q$, and $Y_Q$ is the Yangian of the same type. In general, the $k$-algebra $Y_Q$ is a deformation of the enveloping algebra of the current algebra $g[\mathbb{C}]$. Their construction provides triangular decompositions

$$Y = Y_+ \otimes Y_0 \otimes Y_-, \quad g = g_+ \oplus g_0 \oplus g_-.$$

**Main result and conjecture.** In this paper, we compare $Y$ with the positive half $Y_+$. Set $Y_{+,K} = Y_+ \otimes_k K$. We make the following conjecture in Remark 4.3.

**Conjecture.** There is a unique $K$-algebra isomorphism $Y_{+,K} \simeq Y_K$ (up to some central elements) which intertwines the actions of $Y_{+,K}$ and $Y_K$ on $F_w \otimes_k K$ for any $w$.

The main result of this paper is one half of the above conjecture. It is proved in Theorem 4.1.

**Theorem B.** There is a unique embedding $Y_K \subset Y_{+,K}$ which intertwines the actions of $Y_{+,K}$ and $Y_K$ on $F_w \otimes_k K$ for any $w$.

By Theorem A(e), the proof of the above theorem boils down to checking that certain generalized Hecke correspondences corresponding to generators of $Y_1$ occur with a non zero and constant coefficient in a suitable stable envelope. Again, generalized Hecke correspondences associated to real, elliptic or hyperbolic vertices behave in very different ways and we have to treat each case separately. The case of real vertices was already considered in [11].

**Examples of COHAs and Yangians.** Here we collect the few instances for which the algebras $Y$ and $Y_1$ are explicitly known.

Assume first that $Q$ is a quiver of finite type, and let $g$ be the corresponding finite dimensional simple Lie algebra. The $r$-matrix used to define the Yangian $Y$ was explicitly calculated in [11]. It was shown there that $Y_K$ coincides with the positive half of Varagnolo’s algebra which is the standard Yangian $Y^+(g)$ of $g$. The $K$-algebra $Y^+_K$ coincides with Varagnolo’s algebra as well, and thus our main conjecture is true here. Note also that since $\text{Rep}(\Pi, v) = \Lambda^1(v)$ we have $Y_1 = Y$.

Now, let $Q$ be the Jordan quiver. In that case the $r$-matrix has been calculated in [12] and the Yangian $Y$ is identified with the positive half of the affine Yangian of $g\mathfrak{l}(1)$.
This algebra was defined as the subalgebra of $Y$ generated by $Y(0)$ and $H^{G(1)\times T}_{*}(\text{Rep}(\Pi, 1))$. By Theorem A(e) above, it is equal to the whole of $Y$. Hence in this case also our conjecture is verified. This is the simplest example for which $Y^1 \neq Y$.

Next, let $Q$ be an affine quiver and $g$ be the associated affine Kac-Moody algebra. As far as we know, the Yangian $Y$ has not been fully described in this case. As there are no imaginary vertices, the algebra $Y_1$ coincides with Varagnolo’s algebra, which is known to admit a surjective map from the Yangian $Y^+(g)$. Comparing graded dimensions using [21, thm A(c)] one should be able to check that $Y_1$ is in fact equal to $Y^+(g)$. The Kac-Moody algebra $g$ is the central extension of the loop algebra of a finite dimensional simple Lie algebra $g_0$. Then, the Yangian $Y(g)$ is a deformation of the enveloping algebra of $g_0$ where the central element $c_{l,m}$ belongs to the imaginary root space $l\delta_s + m\delta_t$. In this case the algebras $Y$ and $Y_+$ are both strictly bigger than the positive half of $U(g[u])$.

Not much is known for wild quivers, except for the fact that when $Q$ has no imaginary vertex there is always a surjective map from the Yangian $Y^+(g)$ of the Kac-Moody algebra $g$ to $Y_K$, and that $Y_K$ is strictly larger than $U(g[t])$. This last fact comes from the existence of imaginary roots and hence non-constant Kac polynomials.

**Okounkov’s conjecture.** Let us finish this introduction by mentioning another motivation for this work. In [17], A. Okounkov conjectured that the graded dimensions of the root spaces $g[\alpha]$ of $g$ are, after a suitable grading shift, precisely given by the Kac polynomial $A_\alpha(t)$. This is a vast generalization of the famous Kac conjecture, proved in [10], stating that for quivers with no 1-cycles the multiplicity of the root $\alpha$ in the Kac-Moody algebra $g_Q$ is equal to the constant term $A_\alpha(0)$. Indeed, one expects to have an isomorphism $g_Q \simeq g_Q[0]$, where the grading in $g_Q$ is counted from middle dimension up. In other words, the Lie algebra $g_Q$ should be a graded extension of $g_Q$, whose character is conjecturally given by the full Kac polynomials rather than their constant terms. As proved in [21], The Poincaré polynomials $P(\Lambda^1(v), q) = \sum_i \text{dim}_Q(H^{G(1)\times T}_{2i}(\Lambda^1(v))) q^i$ of $\Lambda^1(v)$ are determined by the formula

$$
\sum_v P(\Lambda^1(v), q) q^{(v,v)+\text{rk}(T)} z^v = (1 - q^{-1})^{-\text{rk}(T)} \text{Exp}\left( \sum_v \frac{A^1(q^{-1})}{1 - q^{-1}} z^v \right)
$$

where $A^1(t)$ is the nilpotent Kac polynomial and Exp is the plethystic exponential. This implies a variant of the above conjecture: the graded character of the cohomological Hall algebras $Y_1$ are encoded by the full nilpotent Kac polynomials $A^1_{\alpha}(t)$. A formula similar to (1.1) but with $\text{Rep}(\Pi, v)$ in place of $\Lambda^1(v)$ can be deduced from [13] and the purity of $\text{Rep}(\Pi, v)$ proved in [6]; it involves the usual Kac polynomial $A(q)$. Note that our construction does not directly provide a construction of an underlying Lie algebra. See some recent work of Davison and Meinhardt in that direction [7]. We expect the discrepancy between the various types of Kac polynomials involved to correspond to different gradings.
on the same Lie algebra. Indeed, the $k$-algebras $Y$ and $Y^1$ are different integral forms inside the $K$-algebra $Y_K$. Note also that we have $A_\alpha(1) = A_\alpha^1(1)$.

**Plan of the paper.** Section 2 contains some recollections on quiver varieties and some results of [21] pertaining to the geometry of the stacks $\Lambda^1(v)$. Section 3 provides the definitions of Maulik-Okounkov’s Yangians and of the cohomological Hall algebras. Our main theorem is proved in Section 4. This is done by explicitly realizing the action on $F_w$ of the generators for $Y^1$ given in [21, thm B] as some convolution operators with explicit generalized Hecke correspondences, by showing that these Hecke correspondences occur in the stable envelope of [12], and, finally, by using the faithfulness of the action of $Y^1$ on $\bigoplus_w F_w$.

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2. Reminder on quiver varieties

We’ll use the same notation as in [21], to which we refer for more details. To facilitate the reading, we recall some of them here.

2.1. Generalities.

All schemes considered here will be reduced. By a variety we’ll mean a reduced scheme over the field \( k = \mathbb{C} \). A subvariety \( X \) of a symplectic manifold \( M \) is isotropic if the restriction of the symplectic form to the smooth locus of \( X \) vanishes. Let \( M^{\text{op}} \) denote the manifold \( M \) with the opposite symplectic form.

Given an algebraic variety \( X \), let \( H^*(X) \) be the Borel-Moore homology group (= the locally finite singular homology group) of \( X \). Now, assume that \( X \) is equipped with the action of a linear algebraic group \( G \). We’ll always assume that \( X \) is quasi-projective and that we have fixed a very ample \( G \)-linearized line bundle over \( X \). Thus, the variety \( X \) embeds equivariantly in a projective space with a linear \( G \)-action. Let \( H^*_G(X) \) and \( H^*_G(X) \) be the equivariant Borel-Moore homology and the equivariant cohomology group of \( X \) with rational coefficients. We’ll abbreviate \( H^*_G = H^*_G(\bullet) \).

The quotient stack of a \( G \)-variety \( X \) by \( G \) is the Artin stack associated with the groupoid \( G \times X \to X \times X \). We’ll denote it by the symbol \( X/G \). We’ll use the symbol \( [X/G] \) for the fundamental class of \( X/G \) and the symbol \( X/\!/G \) for the categorical quotient.

2.2. Quivers.

Let \( Q = (I, \Omega) \) be a finite quiver with set of vertices \( I \) and set of arrows \( \Omega \). Let \( Q^* = (I, \Omega^*) \) be the opposite quiver, with the set of arrows \( \Omega^* = \{ h^* ; h \in \Omega \} \) where \( h^* \) is the arrows obtained by reversing the orientation of \( h \). The double quiver is \( \overline{Q} = (I, \overline{\Omega}) \) with \( \overline{\Omega} = \Omega \sqcup \Omega^* \). We set \( \epsilon(h) = 1 \) if \( h \in \Omega \) and \( -1 \) if \( h \in \Omega^* \). Let \( h', h'' \) be the source and the goal in \( I \) of an arrow \( h \in \Omega \). For each vertex \( i \) let \( \Omega_{ij} \) be the set of arrows in \( \Omega \) from \( i \) to \( j \). Put \( \overline{\Omega}_{ij} = \overline{\Omega}_{ij} \cup (\overline{\Omega}_{ji})^* \). If \( i \neq j \) we write

\[
q_i = |\Omega_{ii}|, \quad q_{ij} = |\Omega_{ij}|, \quad \tilde{q}_{ij} = |\overline{\Omega}_{ij}|, \quad q = (q_{ij} ; i, j \in I),
\]

Fix a tuple \( v = (v_i ; i \in I) \) in \( \mathbb{Z}^I \). The Ringel bilinear form \( \langle \bullet, \bullet \rangle \) on \( \mathbb{Z}^I \) is given by

\[
\langle v, w \rangle = v \cdot w - \sum_{h \in \Omega} v_h w_{h^*}, \quad v \cdot w = \sum_{i \in I} v_i w_i.
\]

Let \( (v, w) = \langle v, w \rangle + \langle w, v \rangle \) be the Euler bilinear form. To avoid confusions we may write \( (v, w)_Q \) for \( (v, w) \). For each \( i \in I \) let \( \delta_i \in \mathbb{Z}^I \) denote the delta function at the vertex \( i \). Set

\[
d_v = v \cdot v - \langle v, v \rangle / 2.
\]

Let \( kQ \) be the path algebra of \( Q \). A dimension vector of \( Q \) is a tuple \( v \in \mathbb{N}^I \). Let \( k^v \) denote the \( I \)-graded vector space \( k^v = \bigoplus_{i \in I} k^{v_i} \). We may abbreviate \( V = k^v \) and \( V_i = k^{v_i} \).
Let $\text{Rep}(kQ, v)$ be the set of representations of $kQ$ in $V$, with its natural structure of affine $k$-variety. Consider the symplectic vector space 
$$R(v) = \text{Rep}(k\bar{Q}, v) = \text{Rep}(kQ, v) \oplus \text{Rep}(kQ^*, v).$$
An element of $R(v)$ is a pair $\bar{x} = (x, x^*)$ where $x = (x_h; h \in \Omega)$ belongs to $\text{Rep}(kQ, v)$ and $x^* = (x_h; h \in \Omega^*)$ belongs to $\text{Rep}(kQ^*, v)$. The algebraic group $G(v) = \prod_{i \in I} GL(v_i)$ acts by conjugation on $R(v)$, preserving the symplectic form. Let $\mathfrak{g}(v)$ be the Lie algebra of $G(v)$ and $\mu : R(v) \to \mathfrak{g}(v)$ be the moment map, which is given by 
$$\mu(\bar{x}) = [x, x^*] = \sum_{h \in \Omega} [x_h, x_h^*].$$
We define $M(v) = \mu^{-1}(0)$. The preprojective algebra is defined as 
$$\Pi = k\bar{Q}/\langle \mu(\bar{x}) \mid \bar{x} \in R(v) \rangle.$$
Consider the group 
$$G_\Omega = \prod_{i \in I} SP(2q_i) \times \prod_{i \neq j} GL(q_{ij}) \times G_{\text{dil}}.$$ 
The second product is over all pairs $(i, j)$ in $I \times I$ such that $i \neq j$ and $G_{\text{dil}} = \mathbb{G}_m$. The group $G_\Omega$ acts on $R(v)$ so that the factor $G_{\text{dil}}$ acts by dilatation on the summand $\text{Rep}(kQ, v)^*$. Let $T \subset G_\Omega$ be a torus of the form 
$$T = T_{\text{sp}} \times T_{\text{dil}}, \quad T_{\text{sp}} \subseteq (\mathbb{G}_m)^\Omega, \quad T_{\text{dil}} \subseteq G_{\text{dil}}$$
where a tuple $(z_h, z)$ acts by $z_h$ on $x_h$ and by $z/z_h$ on $x_h^*$ for all $h \in \Omega$. We’ll assume that the torus $T$ contains a one parameter subgroup which scales all the quiver data by the same scalar, so we have $T_{\text{dil}} \neq \{1\}$. Let $h$ be the character of $T$ which is trivial on $T_{\text{sp}}$ and coincides with the identity on $T_{\text{dil}}$.

2.3. **The semi-nilpotent variety.**

2.3.1. Fix an increasing flag $W$ of $I$-graded vector spaces in $V$
$$W = (\{0\} = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_r = V).$$
Then, we consider the closed subset $\Lambda_W$ of $M(v)$ given by 
$$\Lambda_W = \{ \bar{x} \in M(v) \mid x(W_p) \subseteq W_{p-1}, x^*(W_p) \subseteq W_p \}.$$ 
Up to conjugacy by an element of $G(v)$, the flag $W$ is completely determined by the sequence of dimension vectors 
$$\nu_1 = \dim(W_1/W_0), \ldots, \nu_r = \dim(W_r/W_{r-1}).$$
The tuple $\nu = (\nu_1, \ldots, \nu_r)$ is a composition of $v$, i.e., it is a tuple of dimension vectors with sum $v$. We’ll write $\nu \vdash v$. We’ll say that the flag $W$ is of type $\nu$. Then, we define 
$$\Lambda_\nu = G(v) \cdot \Lambda_W \subseteq M(v),$$
where the dot denotes the $G(v)$-action on $M(v)$. We’ll say that a composition $\nu \vdash v$ is restricted if each part is concentrated in a single vertex. The strongly semi-nilpotent variety is the closed subset of $M(v)$ given by
\begin{equation}
\Lambda^1(v) = \bigcup_{\nu} \Lambda_{\nu},
\end{equation}
where $\nu$ runs over the set of all restricted compositions of $v$. The $G(v) \times T$-action on $M(v)$ yields a $G(v) \times T$-action on $\Lambda^1(v)$. In [21, thm. 3.2] we have proved the following.

**Theorem 2.1.**

(a) $\Lambda^1(v)$ is a closed Lagrangian subvariety of $R(v)$, of dimension $d_v$.
(b) $H^{G(v) \times T}_*(\Lambda^1(v))$ is pure and even.
(c) $H^*_G(v) \times T(\Lambda^1(v))$ is free as an $H_T^*$-module.

**Remark 2.2.** Let $(v)$ be the composition of $v$ with only one term. Then, we have
$$\Lambda_{(v)} = \{0\} \times \text{Rep}(kQ^*, v).$$

2.3.2. Let us consider in more details the case of the quiver with vertex set $\{i\}$ and $q_i$ loops. Then, the dimension vector $v$ is an integer.

First, assume that $q_i = 1$. Then $Q$ is the Jordan quiver and $\Lambda^1(v)$ is the set of commuting pairs $(x, x^*)$ in $g(v) \times g(v)$ with $x$ nilpotent. We identify $g(v) \times g(v)$ with $g(v) \times g(v)^*$ via the trace map. Then, the irreducible components of $\Lambda^1(v)$ are the closure of the conormal bundles to the nilpotent $G(v)$-orbits in $g(v)$.

Next, assume that $q_i > 1$. Then, by [21, prop. 3.8] the variety $\Lambda_{\nu}$ is irreducible and Lagrangian in $R(v)$ for each $\nu$ and the set of irreducible components of $\Lambda^1(v)$ is
\begin{equation}
\text{Irr}(\Lambda^1(v)) = \{\Lambda_{\nu} ; \nu \vdash v\}.
\end{equation}

2.4. Quiver varieties.

2.4.1. The symplectic vector space of representations of dimension vectors $v$, $w$ of the framed quiver associated with $\bar{Q}$ is
$$R(v, w) = R(v) \oplus \text{Hom}_I(w, v) \oplus \text{Hom}_I(v, w),$$
where $\text{Hom}_I(v, w)$ is the set of $I$-graded $k$-linear maps $V \to W$ and
$$V = k^v, \quad W = k^w, \quad V_i = k^{v_i}, \quad W_i = k^{w_i}.$$ The algebraic group $G(v) \times G(w) \times T$ acts on $R(v, w)$. The symplectic form is homogeneous of weight $h$ under the $G(v) \times G(w) \times T$-action. The $G(v)$-action is symplectic and admits the moment map
$$\mu : R(v, w) \to g(v), \quad (\bar{x}, \bar{a}) \mapsto [x, x^*] + aa^*$$
where we write
\[
\bar{x} = (x, x^*) \in R(v),
\bar{a} = (a, a^*) \in \text{Hom}_I(w, v) \oplus \text{Hom}_I(v, w).
\]

Let \( \mathcal{M}_0(v, w) \) be the categorical quotient by \( G(v) \) of the zero set

\[ M(v, w) = \mu^{-1}(0). \]

It is affine, reduced, irreducible, singular in general and \( G(w) \times T \)-equivariant. Given a character \( \theta \) of \( G(v) \) we consider the space of semi-invariants of weight \( \theta \)

\[ k[M(v, w)]^\theta \subseteq k[M(v, w)]. \]

We have the \( G(w) \times T \)-equivariant projective morphism

\[ \pi_\theta : \mathcal{M}_\theta(v, w) = \text{Proj} \left( \bigoplus_{n \in \mathbb{N}} k[M(v, w)]^{\theta^n} \right) \to \mathcal{M}_0(v, w). \]

The Hilbert-Mumford criterion implies that \( \mathcal{M}_\theta(v, w) \) is the geometric quotient by \( G(v) \) of an open subset \( \mathcal{M}_\theta(v, w) \) of \( M(v, w) \) consisting of the \( \theta \)-semistable representations. Replacing everywhere \( M(v, w) \) by \( R(v, w) \) we define an open subset \( R_\theta(v, w) \) such that \( \mathcal{M}_\theta(v, w) = R_\theta(v, w) \cap M(v, w) \). If \( \theta \) is generic then any semistable pair \((\bar{x}, \bar{a})\) is stable and the variety \( \mathcal{M}_\theta(v, w) \) is smooth, symplectic, of dimension

\[ d_{v,w} = 2v \cdot w - (v, v)_{Q}. \]

The character \( \theta = s \) given by \( s(g) = \prod_{i \in I} \det(g_i)^{-1} \) is generic. A representation \( (\bar{x}, \bar{a}) \) in \( R(v, w) \) is \((s-\)semistable if and only if it does not admit any nonzero subrepresentation whose dimension vector belongs to \( \mathbb{N}^I \times \{0\} \). We abbreviate

\[ \mathcal{M}(v, w) = \mathcal{M}_s(v, w), \quad \pi = \pi_s. \]

Let \([\bar{x}, \bar{a}]\) be the image in \( \mathcal{M}(v, w) \) of the tuple \((\bar{x}, \bar{a}) \in M_s(v, w)\).

It may be useful to realize the quiver varieties as moduli spaces of representations of some preprojective algebra. More precisely, consider the quiver \( \tilde{Q} = (\tilde{I}, \tilde{\Omega}) \) obtained from \( Q \) by adding one new vertex \( \infty \) and \( w_i \) arrows from \( \infty \) to the vertex \( i \) for all \( i \in I \). For each \( v \in \mathbb{N}^I \) set

\[ \tilde{v} = v + \delta_\infty \text{ if } w \neq 0, \text{ and } \tilde{v} = v \text{ else}. \]

If \( w \neq 0 \) we’ll identify \( G(v) \) with \( PGL(\tilde{v}) \). Then, the varieties \( \mathcal{M}(v, w) \) or \( \mathcal{M}_0(v, w) \) may be viewed as moduli spaces of (stable, semisimple) modules of dimension \( \tilde{v} \) over the preprojective algebra \( \Pi \) of \( \tilde{Q} \).
2.4.2. For any closed point \( t \) of the center \( \mathfrak{z}(v) \) of \( \mathfrak{g}(v) \), the variety

\[ M(v, w)_t = \mu^{-1}(t) \]

gives rise to a projective morphism

\[ \pi : M(v, w)_t \to M_0(v, w)_t. \]

We’ll call \( M(v, w)_t \) or \( M_0(v, w)_t \) a deformed quiver variety. Given a generic line \( \mathbb{A}^1 \subseteq \mathfrak{z}(v) \), we may consider the families of varieties

\[ M(v, w)_{\mathbb{A}^1} \to \mathbb{A}^1, \quad M_0(v, w)_{\mathbb{A}^1} \to \mathbb{A}^1. \]

If \( M(v, w) \) is non empty, then \( M(v, w)_t \) and \( M_0(v, w)_t \) are also non empty for each \( t \neq 0 \) and we get a projective morphism \( \pi : M(v, w)_{\mathbb{A}^1} \to M_0(v, w)_{\mathbb{A}^1} \) over \( \mathbb{A}^1 \) which is an isomorphism above \( \mathbb{G}_m \).

2.4.3. If the representation \( z \in M(v, w) \) is semisimple then we can decompose it into its simple constituents

\[ z = z_1^{d_1} \oplus z_2^{d_2} \oplus \cdots \oplus z_s^{d_s}, \]

where the \( z_i \)’s are non-isomorphic simples. If \( u_r = (v_r, w_r) \) is the dimension vector of \( z_r \), i.e., if \( z_r \) is a simple representation in \( M(v_r, w_r) \), then we say that \( z \) has the representation type

\[ \tau = (d_1, u_1; d_2, u_2; \ldots; d_s, u_s). \]

If \( w \neq 0 \) and \( z \) is stable, then there is a unique integer \( r \) such that \( w_r = w \) and \( w_{r'} = 0 \) for all \( r' \neq r \), hence we may assume that the representation type of \( z \) has the following form

\[ (1, v_1, w; d_2, v_2; \ldots; d_s, v_s), \]

where \((v_1, v_2, \ldots, v_s) \vdash v \) and the tuples \((d_2, v_2), \ldots, (d_s, v_s)\) are only defined up to a permutation. Let \( RT(v, w) \) be the set of all representation types of dimension \((v, w)\). Let

\[ M_0(\tau) \subseteq M_0(v, w) \]

be the set of semisimple representations with representation type equal to \( \tau \). Since \( M_0(v, w) \) is irreducible, there is a unique representation type \( \kappa_{v, w} \) such that \( M_0(\kappa_{v, w}) \) is a dense open subset of \( M_0(v, w) \). We’ll call it the generic representation type of \( RT(v, w) \). Given two representation types

\[ \tau = (1, v_1, w; d_2, v_2; \ldots; d_s, v_s) \in RT(v, w), \]
\[ \kappa = (1, u_1, z; e_2, u_2; \ldots; e_r, u_r) \in RT(u, z), \]

we define their sum by

\[ \tau \oplus \kappa = (1, v_1 + u_1, w + z; d_2, v_2; \ldots; d_s, v_s; e_2, u_2; \ldots; e_r, u_r). \]

If \( \sum_{t=1}^r v_t \leq v \) and \( \sum_{t=1}^r w_t \leq w \), the direct sum of quiver representations yields a closed embedding \( \oplus : \prod_{t=1}^r M_0(v_t, w_t) \to M_0(v, w) \) such that

\[ M_0(\tau) \oplus M_0(\kappa) \subseteq M_0(\tau \oplus \kappa). \]
For each representation type $\tau$ we write
\[
M(\tau) = \rho_0^{-1}(M_0(\tau)),
\]
\[
M_\sigma(\tau) = M(\tau) \cap M_\sigma(v, w),
\]
\[
\mathcal{M}(\tau) = M_\sigma(\tau) / / G(v).
\]
If $\mathcal{M}(\tau) \neq \emptyset$ then the map $\pi$ restricts to a locally trivial fibration $\mathcal{M}(\tau) \to M_0(\tau)$.

2.4.4. The results of this section apply to arbitrary deformed quiver varieties. To simplify the exposition we'll deal only with ordinary ones. Let $\gamma$ be a cocharacter of $G(w) \times T$. Composing it with the $G(w) \times T$-action we get a $\mathbb{G}_m$-action on $\mathcal{M}(v, w)$. We want to describe the fixed point locus $\mathcal{M}(v, w)^\bullet$ and the Bialynicki-Birula attracting variety 
\[
\mathcal{L}(v, w)^\bullet = \{ z \in \mathcal{M}(v, w) ; \exists \lim_{t \to 0} t \cdot z \}.
\]
To do this, fix a cocharacter $\rho$ of $G(v)$. Since the $G(v)$-action on $M_\sigma(v, w)$ commutes with the $G(w) \times T$-action, we can view the product $\gamma \rho$ as a cocharacter of $G(v) \times G(w) \times T$.

Let $L$ be the centralizer of $\rho$ in $G(v)$ and set 
\[
P = \{ g \in G(v) ; \exists \lim_{t \to 0} \rho(t) g \rho(t)^{-1} \},
\]
\[
U = \{ g \in G(v) ; \lim_{t \to 0} \rho(t) g \rho(t)^{-1} = 1 \},
\]
\[
M[\rho] = \{ z \in M_\sigma(v, w) ; \gamma \rho(t) \cdot z = z , \forall t \},
\]
\[
L[\rho] = \{ z \in M_\sigma(v, w) ; \lim_{t \to 0} \gamma \rho(t) \cdot z \in M[\rho] \}.
\]
Then, we have 
\[
\mathcal{M}[\rho] := (G(v) \cdot M[\rho]) / / G(v) = M[\rho] / / L,
\]
\[
\mathcal{L}[\rho] := (G(v) \cdot L[\rho]) / / G(v) = L[\rho] / / P.
\]
The set $\mathcal{M}[\rho]$ is a sum of connected components of $\mathcal{M}(v, w)^\bullet$. We have 
\[
\mathcal{L}[\rho] = \{ z \in \mathcal{M}(v, w) ; \lim_{t \to 0} t \cdot z \in \mathcal{M}[\rho] \}
\]
and 
\[
\mathcal{M}(v, w)^\bullet = \bigsqcup_\rho \mathcal{M}[\rho], \quad \mathcal{L}(v, w)^\bullet = \bigsqcup_\rho \mathcal{L}[\rho].
\]

2.4.5. Set 
\[
L_\delta^1(v, w) = R_\delta(v, w) \cap (A^1(v) \times \{0\} \times \text{Hom}_I(v, w)).
\]
The Lagrangian quiver variety is the geometric quotient 
\[
\mathcal{L}^1(v, w) = L_\delta^1(v, w) / / G(v).
\]
We have closed embeddings 
\[
L_\delta^1(v, w) \subset M_\delta(v, w), \quad \mathcal{L}^1(v, w) \subset \mathcal{M}(v, w).
\]
The following is proved in [2, prop. 3.1, 3.2].
Proposition 2.3.
(a) There is a $\mathbb{G}_m$-action $\bullet$ on $\mathcal{M}(v,w)$ such that $L^1(v,w) = L(v,w)^\bullet$.
(c) $L^1(v,w)$ is a closed Lagrangian subvariety of $\mathcal{M}(v,w)$.

2.5. Homology of quiver varieties.

Set $\mathbb{k} = H^*_T$ and $\mathbb{k}[w] = H^*_{G(w) \times T}$. Let $K$ be the fraction field of $\mathbb{k}$. We’ll abbreviate
\[ \otimes = \otimes_\mathbb{k}, \quad \text{Hom} = \text{Hom}_\mathbb{k}, \quad (\bullet)^\vee = \text{Hom}(\bullet, \mathbb{k}). \]
We may view $h$ as a non zero element of $\mathbb{k}$ of degree 2. Let $F_w$ be the $\mathbb{Z}^I \times \mathbb{Z}$-graded $\mathbb{k}[w]$-module given by
\[ F_w = \bigoplus_{v \in \mathbb{N}^I} F_w(v), \quad F_w(v) = \bigoplus_{k \in \mathbb{Z}} F_w(v,k), \quad F_w(v,k) = H^*_{k+2d_{v,w}}(\mathcal{M}(v,w)). \]
Let $|w\rangle$ be the fundamental class of $\mathcal{M}(0,w)$ in $F_w$. Let $A_w$ be the $\mathbb{Z}^I \times \mathbb{Z}$-graded $\mathbb{k}[w]$-algebra given by
\[ A_w = \bigoplus_{v_2 \in \mathbb{Z}^I} A_w(v_2) = \bigoplus_{v_2 \in \mathbb{Z}^I} \bigoplus_{k_2 \in \mathbb{Z}} A_w(v_2,k_2) \subseteq \text{End}_{\mathbb{k}[w]}(F_w), \]
where $A_w(v_2,k_2)$ consists of all $\mathbb{k}[w]$-linear endomorphisms of $F_w$ which are homogeneous of degree $(v_2,k_2)$. By [15], the $\mathbb{k}[w]$-module $F_w(v)$ is free of finite rank. Write
\[ F_w^\vee = \text{Hom}_{\mathbb{k}[w]}(F_w, \mathbb{k}[w]) = \prod_v F_w(v)^\vee. \]
Then, we have
\[ A_w = \prod_v \bigoplus_u F_w(u) \otimes_{\mathbb{k}[w]} F_w(v)^\vee \]
Let $A_w^f \subset A_w$ be the set of finite rank endomorphisms of $F_w$, which is equal to
\[ A_w^f = F_w \otimes_{\mathbb{k}[w]} F_w^\vee. \]
Under the convolution, any cycle in $H^*_{G(w) \times T}(\mathcal{M}(u,w) \times \mathcal{M}(v,w))$ which is proper over $\mathcal{M}(u,w)$ gives rise to a $\mathbb{k}[w]$-linear operator $F_w(v) \rightarrow F_w(u)$, see [4] for details.

3. Definition of $\mathcal{Y}$ and $\mathcal{Y}^1$

The aim of this section is to review the constructions of Maulik and Okounkov [12] and [21]. The sections 3.1, 3.2 and 3.4 are a reminder from [12] to which we refer for more details. Section 3.5 is a reminder of [21].

3.1. The stable envelope.
3.1.1. Fix dimension vectors $v$, $w$, $w_1$, $w_2$ such that $w = w_1 + w_2$. We’ll abbreviate

$$G_{sp} = G(w_1) \times G(w_2) \times T_{sp}, \quad G = G_{sp} \times T_{dil}, \quad A = \gamma(G_m)$$

where $\gamma$ is the cocharacter of $G(w_1) \times G(w_2)$ given by

$$\gamma(t) = 1_{w_1} \oplus t^{-1}1_{w_2}.$$  

The group $G$ acts on the varieties $\mathcal{M}(v, w)$ and $\mathcal{M}_0(v, w)$. Let $\diamondsuit$ be the $G_m$-action on $\mathcal{M}(v, w)$ associated with the cocharacter $\gamma$. Let $\mathcal{M}(v, w)^{\diamondsuit}$ be the fixed points locus and $\mathcal{L}(v, w)^{\diamondsuit}$ be the attracting set

$$\mathcal{L}(v, w)^{\diamondsuit} = \{ x \in \mathcal{M}(v, w); \exists \lim_{t \to 0} t \circ x \}.$$  

Let $\mathcal{M}_0(v, w)^{\diamondsuit}$ and $\mathcal{L}_0(v, w)^{\diamondsuit}$ be the fixed points locus and the attracting set in $\mathcal{M}_0(v, w)$. As $\pi$ is proper, we have $\mathcal{L}(v, w)^{\diamondsuit} = \pi^{-1}(\mathcal{L}_0(v, w)^{\diamondsuit})$. Hence, there is a composed homomorphism

$$\mathcal{L}(v, w)^{\diamondsuit} \to \mathcal{L}_0(v, w)^{\diamondsuit} \to \mathcal{M}_0(v, w)^{\diamondsuit}.$$  

Consider the (reduced) fiber product given by

$$\mathfrak{Z}(v, w) = \mathcal{L}(v, w)^{\diamondsuit} \times_{\mathcal{M}_0(v, w)^{\diamondsuit}} \mathcal{M}(v, w)^{\diamondsuit}.$$  

Since the $\diamondsuit$-action preserves the symplectic form of $\mathcal{M}(v, w)$, the fixed point locus $\mathcal{M}(v, w)^{\diamondsuit}$ admits also a natural symplectic form. Then $\mathfrak{Z}(v, w)$ is a Lagrangian closed subvariety of the symplectic manifold $\mathcal{M}(v, w)^{\text{op}} \times \mathcal{M}(v, w)^{\diamondsuit}$. The stable envelope is the $H^*_G$-linear map

$$H^*_G(\mathcal{M}(v, w)^{\diamondsuit}) \to H^*_G(\mathfrak{Z}(v, w))$$

given by the convolution with a $G$-equivariant Lagrangian cycle

$$\text{stab} \in H^*_G(\mathcal{M}(v, w)^{\text{op}} \times \mathcal{M}(v, w)^{\diamondsuit})$$

which is supported on $\mathfrak{Z}(v, w)$ and proper over $\mathcal{M}(v, w)$.

3.1.2. Let $\text{Irr}(\mathcal{M}(v, w)^{\diamondsuit})$ be the set of connected components of $\mathcal{M}(v, w)^{\diamondsuit}$. For each $X \in \text{Irr}(\mathcal{M}(v, w)^{\diamondsuit})$, the restriction of the $G_{sp}$-equivariant Euler class of the normal bundle $N_X/\mathfrak{m}$ to any point of $X$ is $(-1)^{\text{codim}_{\mathfrak{m}} X}/2$ times a square in $H^*_G$. A choice of a square root

$$\varepsilon_X \in H^*_{G_{sp}}$$

of this square for each component $X$ is called a polarization. Since $\mathcal{M}(v, w)$ is a quiver variety, it is the Hamiltonian reduction of the cotangent bundle to an $G_{sp}$-manifold. Therefore, it admits a canonical polarization, and any other polarization differs from it by a collection of signs. In other words, we’ll view $\varepsilon_X$ either as a class in $H^*_{G_{sp}}$ or as a sign according to the context.
3.1.3. Define the varieties $M(v, w)^*_{\mathbf{A}^1}$, $L(v, w)^*_{\mathbf{A}^1}$ and $\mathfrak{F}(v, w)_{\mathbf{A}^1}$ as above, by replacing $M(v, w)$, $M_0(v, w)$ by $M(v, w)_{\mathbf{A}^1}$, $M_0(v, w)_{\mathbf{A}^1}$ at each step. There is a (non continuous) map

$$\nu : L(v, w)^*_{\mathbf{A}^1} \to M(v, w)^*_{\mathbf{A}^1}$$

given by the limit $t \to 0$ in $[3.8]$. Since $M(v, w)_{G_m}$ and $M_0(v, w)_{G_m}$ are non empty isomorphic affine varieties, we have an isomorphism $M_0(v, w)^*_{G_m} \simeq M(v, w)^*_{G_m}$. Hence, for each component $X \in \text{Irr}(M(v, w)^*_{G_m})$ the set $\nu^{-1}(X)$ is closed in $M(v, w)_{G_m}$ and $\mathfrak{F}(v, w)_{G_m}$ is isomorphic to the closed subvariety of $M(v, w)_{G_m}$ given by

$$\nu^{-1}(M(v, w)^*_{G_m}) = \bigcup_X \nu^{-1}(X).$$

We consider the $G$-invariant cycle in $\mathfrak{F}(v, w)_{G_m}$ given by

$$\text{stab}_{G_m} = \bigcup_X \varepsilon_X [\nu^{-1}(X) \times_X X].$$

The inclusion $\{0\} \subset \mathbf{A}^1$ gives rise to a specialization map to $H^G_*(M(v, w)^{\text{op}} \times M(v, w)^{\text{op}})$ as in [4, §2.6.30], [8, §11.1]. Let us denote it by the symbol $\lim_0$. It yields a Lagrangian cycle

$$\text{stab} = \lim_0 \text{stab}_{G_m}.$$

3.1.4. Given dimension vectors $v_1$, $v_2$ such that $v = v_1 + v_2$, let $M[v_1, v_2]$ and $L[v_1, v_2]$ be the subsets of $M(v, w)^*$ and $L(v, w)^*$ attached to the cocharacter $\rho$ of $G(v)$ given by

$$\rho(t) = 1_{v_1} \oplus t^{-1}1_{v_2}.$$  

We have

$$M(v, w)^* = \bigsqcup_{v_1 + v_2 = v} M[v_1, v_2], \quad M[v_1, v_2] = M(v_1, w_1) \times M(v_2, w_2),$$

$$L(v, w)^* = \bigsqcup_{v_1 + v_2 = v} L[v_1, v_2], \quad L[v_1, v_2] = \nu^{-1}(M[v_1, v_2]),$$

$$\mathfrak{F}(v, w) = \bigsqcup_{v_1 + v_2 = v} \mathfrak{F}[v_1, v_2], \quad \mathfrak{F}[v_1, v_2] = \mathfrak{F}(v, w) \cap \left( M(v, w) \times M[v_1, v_2] \right).$$

Note that $M_0(v, w)^*$ is not the disjoint union of the varieties $M_0(v_1, w_1) \times M_0(v_2, w_2)$. Let $|v|$ denote the sum of the entries of the dimension vector $v$. We have

$$M[v_1, v_2] \cap L[u_1, u_2] \neq \emptyset \iff (|v_2| \leq |u_2| \text{ and } v_1 + v_2 = u_1 + u_2).$$

If this holds we may also write

$$M[v_1, v_2] \leq M[u_1, u_2].$$

Given $u_1$, $u_2$ with $v = u_1 + u_2$ we abbreviate

$$M[u_1, u_2 ; v_1, v_2] = M[u_1, u_2] \times M[v_1, v_2].$$
The varieties above admit deformed version by replacing $\mathcal{M}(v,w)$ by $\mathcal{M}(v,w)_{A^1}$ at each step. We'll denote them by $\mathfrak{M}[v_1,v_2]_{A^1}$, $\mathfrak{L}[v_1,v_2]_{A^1}$, etc. Consider the equivariant Lagrangian cycle

\begin{equation}
\text{stab}[v_1,v_2] = \varepsilon_{v_1,v_2} \lim_0 [\mathfrak{M}[v_1,v_2][G_m] \in H^G_* (\mathcal{M}(v,w) \times \mathcal{M}[v_1,v_2])],
\end{equation}

where $\varepsilon_{v_1,v_2} = \varepsilon_{\mathfrak{M}(v_1,v_2)}$. To avoid any confusion we may write

\begin{align*}
\mathfrak{M}[v_1,v_2 ; w_1,w_2] &= \mathfrak{M}[v_1,v_2], \\
\mathfrak{L}[v_1,v_2 ; w_1,w_2] &= \mathfrak{L}[v_1,v_2], \\
\text{stab}[v_1,v_2 ; w_1,w_2] &= \text{stab}[v_1,v_2].
\end{align*}

3.2. Residues.

3.2.1. Since the group $A$ acts trivially on the fixed points locus $\mathcal{M}(v,w)^\circ$, there is a canonical increasing filtration on the space $H^G_k (\mathcal{M}(v,w)^\circ \times \mathcal{M}(v,w)^\circ)$ whose $d$-th term is

$$\bigoplus_{\ell \leq d} H^G_{A^\ell} \cap H^G_{k+\ell} (\mathcal{M}(v,w)^\circ \times \mathcal{M}(v,w)^\circ).$$

The associated graded is a graded $H^*_A$-module whose degree $d$ term is equal to

$$H^d_A \otimes H^G_{k+d} (\mathcal{M}(v,w)^\circ \times \mathcal{M}(v,w)^\circ).$$

There is a canonical $\mathbb{Z}$-graded ring isomorphism

$$k[\varpi] = H^*_A \times T, \quad \deg(\varpi) = 2$$

where $\varpi$ is identified with the first Chern class of the linear character of $A$. The inclusion

$$i : \mathcal{M}(v,w)^\circ \times \mathcal{M}(v,w)^\circ \subset \mathcal{M}(v,w) \times \mathcal{M}(v,w)^\circ$$

is an l.c.i. morphism. Hence the pullback $i^*$ in equivariant Borel-Moore homology is well defined. Given connected components $\mathcal{X}, \mathcal{Y}$ of $\mathcal{M}(v,w)^\circ$ such that $\mathcal{X} < \mathcal{Y}$, we denote by $\bullet|_{\mathcal{X} \times \mathcal{Y}}$ the restriction to $\mathcal{X} \times \mathcal{Y}$. By [12 prop. 4.8.2], there is a unique $G/A$-equivariant Lagrangian cycle $\mathbf{r}_{\mathcal{X} \times \mathcal{Y}}$ on $\mathcal{X}^{\text{op}} \times \mathcal{Y}$ such that

\begin{equation}
\text{stab}|_{\mathcal{X} \times \mathcal{Y}} = h \varpi^{\text{codim}_{\mathcal{X}}/2-1} \cap \mathbf{r}_{\mathcal{X} \times \mathcal{Y}} \mod H^\leq_A \times H^G_{\mathcal{X} \times \mathcal{Y}}. 
\end{equation}

We'll call it the residue of the cycle $\text{stab}$. It is supported on the closed subset

$$(\mathcal{X} \times \mathcal{Y}) \cap (\mathcal{M}(v,w)^\circ \times \mathfrak{g}_{\mathfrak{m}}(v,w) \mathcal{M}(v,w)^\circ).$$

In particular, it is proper over $\mathcal{X}$. We'll also write $\mathbf{r}_{\mathcal{X} \times \mathcal{Y}}$ for its image in $H^*_A^{\text{sp}}(\mathcal{X} \times \mathcal{Y})$. 

3.2.2. By [12] lem. 3.4.2], for any $G_{sp}$-invariant Lagrangian cycle $C$ in $\mathcal{M}(v, w)^{op} \times \mathcal{M}(v, w)^{op}$ there is a unique $G_{sp}/A$-equivariant Lagrangian cycle $\text{res}(C)$ in $\mathcal{M}(v, w)^{op} \times \mathcal{M}(v, w)^{op}$, called the Lagrangian residue of $C$, such that

- $\text{res}(C)$ is supported on $C \cap (\mathcal{M}(v, w)^{op} \times \mathcal{M}(v, w)^{op})$,
- $C|_{X \times Y} = \varepsilon_X \cap \text{res}(C)|_{X \times Y}$ modulo $H_A^{\text{codim}_A X/2} \otimes H_{G_{sp}/A}(X \times Y)$,

where $X, Y \in \text{Irr}(\mathcal{M}(v, w)^{op})$ and $\varepsilon_X \in H_G^{*}$ is the polarization. The residue above is not the same as the Lagrangian residue of $\text{stab}$. Indeed, the characterization of the stable envelope in [12] thm. 3.3.4] implies that the off-diagonal terms of the Lagrangian residue of $\text{stab}$ are zero.

3.3. Hecke correspondences and residues.

Let $v_1, v_2, v, w_1, w_2, \gamma, \rho$ be as above. Fix an $I$-graded subspace $V_1 \subseteq V$ with dimension vector $v_1$. Let $P$ be the corresponding standard parabolic subgroup of $G(v)$ and $L = G(v_1) \times G(v_2)$ be the standard Levi subgroup of $P$. Fix $I$-graded linear isomorphisms $k^{v_1} \simeq V_1$ and $k^{v_2} \simeq V/V_1$.

3.3.1. The geometric quotient

\begin{equation}
\text{H}[v_1, v_2; w] = H[v_1, v_2; w]/P, \\
\text{H}[v_1, v_2; w] = \{ z \in M_{\gamma}(v, w); z(V_1 \oplus W) \subseteq V_1 \oplus W \}
\end{equation}

is called a Hecke correspondence. Given a locally closed subset $S \subseteq M(v_2)$ we set

\begin{equation}
\text{h}[v_1, S; w] = H[v_1, S; w]/P, \\
\text{H}[v_1, S; w] = \{ (\bar{x}, \bar{a}) \in \text{H}[v_1, v_2; w]; \bar{x}_2 \in S \},
\end{equation}

where $\bar{x}_2$ is the representation of $\Pi$ on $V/V_1$ equal to $z$ modulo $V_1 \oplus W$. If no confusion is possible we abbreviate

\begin{equation}
\text{H}[v_1, S] = H[v_1, S; w], \quad \text{h}[v_1, S] = h[v_1, S; w].
\end{equation}

By [21] lem. 3.14, prop. 3.15], the Hecke correspondence $h[v_1, S; w]$ is an isotropic subvariety of $\mathcal{M}(v, w)^{op} \times \mathcal{M}(v_1, w)$ if $S$ is any isotropic subvariety of the symplectic vector space $R(v_2)$ and it is a closed Lagrangian local complete intersection if $S = \Lambda_{(v_2)}$.

3.3.2. By [21] prop. 3.13], the assignment $z \mapsto (z, z|_{V_1 \oplus W})$ is a closed embedding

\begin{equation}
h[v_1, v_2; w] \subseteq \mathcal{M}(v, w) \times \mathcal{M}(v_1, w)
\end{equation}

which is projective over $\mathcal{M}(v, w)$. Consider the composed map

\begin{equation}
h[v_1, v_2; w_1] \rightarrow \mathcal{M}_0(v_2, 0) \subset \mathcal{M}_0(v_2, w_2).
\end{equation}

We define the generalized Hecke correspondence to be the (reduced) fiber product

$$g h[v_1, v_2; w_1, w_2] = h[v_1, v_2; w_1] \times_{\mathcal{M}_0(v_2, w_2)} \mathcal{M}(v_2, w_2).$$

By (3.11), there is a closed embedding

$$gh[v_1, v_2; w_1, w_2] \subseteq \mathcal{M}[v, 0] \times_{\mathcal{M}_0(v, w)} \mathcal{M}[v_1, v_2].$$
There is also an isomorphism
\[(3.13) \quad gh[v_1, v_2 ; w_1, w_2] = GH[v_1, v_2]//P \times L,\]
where \(GH[v_1, v_2]\) is the set of triples
\[(3.14) \quad (z, z'_1, z'_2) \in H[v_1, v_2 ; w_1] \times M_s(v_1, w_1) \times M_s(v_2, w_2)\]
satisfying the following conditions
- \(z|_{V_1 \oplus W_1} \simeq z'_1\) as \(\tilde{\Pi}\)-modules,
- \(a'_2 = 0,\)
- \(\pi(\bar{x}_2) \simeq \pi(\bar{x}'_2)\) as \(\Pi\)-modules.

Here, we write \(z = (\bar{x}, \bar{a}),\) \(z'_1 = (\bar{x}'_1, \bar{a}'_1)\) and \(\bar{x}_2 = \bar{x} \mod V_1\) in \(M(v_2).\) Further, the symbol \(\pi\) denotes the semisimplification of \(\Pi\)-modules.

More generally, for each locally closed subset \(S \subset M(v_2)\) and for each variety \(T\) over \(\mathcal{M}_0(v_2, w_2),\) we consider the (reduced) fiber product
\[(3.15) \quad gh[v_1, S ; w_1, T] = gh[v_1, v_2 ; w_1, T] \times_{\mathcal{M}_0(v_2, w_2)} T.\]
We may abbreviate
\[(3.16) \quad gh[v_1, v_2] = gh[v_1, v_2 ; w_1, w_2],\]
\[(3.17) \quad gh[v_1, S ; w_1, w_2] = gh[v_1, S ; w_1, \mathcal{M}(v_2, w_2)],\]
\[(3.18) \quad gh[v_1, v_2] = gh[v_1, v_2 ; w_1, w_2].\]

3.3.3. We’ll also need the following geometric quotient
\[(3.15) \quad p[v_1, v_2] = P[v_1, v_2]/P,\]
\[(3.16) \quad P[v_1, v_2] = \{z \in M_s(v, w) : z(V_1 \oplus W_1) \subseteq V_1 \oplus W_1\}.\]
From the proof of [21, prop. 3.13], we deduce that the assignment \(z \mapsto (z|_{V_1 \oplus W_1})\) is a closed embedding
\[(3.16) \quad p[v_1, v_2] \subseteq \mathcal{M}(v, w) \times \mathcal{M}(v_1, w_1).\]
There is also an obvious map
\[(3.16) \quad p[v_1, v_2] \to \mathcal{M}_0(v_2, w_2)\]
which yields the (reduced) fiber product
\[(3.16) \quad gp[v_1, v_2] = p[v_1, v_2] \times_{\mathcal{M}_0(v_2, w_2)} \mathcal{M}(v_2, w_2).\]
Using (3.16) we get a closed embedding
\[(3.17) \quad gp[v_1, v_2] \subseteq \mathcal{M}(v, w) \times \mathcal{M}[v_1, v_2].\]
We also have an isomorphism
\[(3.18) \quad gp[v_1, v_2] = GP[v_1, v_2]/P \times L,\]
where $GP[v_1, v_2]$ is the set of triples
\[(z, z'_1, z'_2) \in P[v_1, v_2] \times M_s(v_1, w_1) \times M_s(v_2, w_2)\]
satisfying the following conditions
- $z|_{V_1 \oplus W_1} \simeq z'_1$ as a $\Phi$-module,
- $\pi(z \mod V_1 \oplus W_1) \simeq \pi(z'_2)$ as $\Pi$-modules on $V \oplus W_1 \oplus W_1$.

From (3.19), (3.13), (3.15) and (3.18) we deduce that
\[(3.19)\]
\[gp[v_1, v_2] = \{ p[v_1, v_2] \cap (M[v, 0] \times M(v, 1)) \}, \]
\[gh[v_1, v_2] = \{ p[v_1, v_2] \cap M[v, v], v_1, v_2 \}. \]

3.3.4. We can now consider the residue of the stable envelope. Given dimension vectors $u_1, u_2$ with $v = u_1 + u_2$ and $|w| < |v|$, we abbreviate
\[(3.20)\]
\[r[u_1, u_2] = r_{M[u_1, u_2; v_1, v_2]}. \]

To avoid any confusion we may also write
\[r[u_1, u_2] = r[u_1, u_2 ; v_1, v_2]. \]

For a future use, let us focus on the case $u_1 = v$ and $u_2 = 0$. Then, we have
\[r[v, 0; v_1, v_2] \in H_{mid}^G/A (M[v, 0; v_1, v_2]), \]
where the degree $mid$ is given by
\[(3.21)\]
\[mid = d_{v, w_1} + d_{v, w_1} + d_{v, w_2}. \]

It is an equivariant Lagrangian cycle of the symplectic manifold
\[M[v, 0; v_1, v_2] = M(v, w_1)^{op} \times M(v_1, w_1) \times M(v_2, w_2). \]

**Proposition 3.1.**

(a) $r[v, 0; v_1, v_2]$ is supported on the closed subset $gh[v_1, v_2]$ of $M[v, 0; v_1, v_2]$.

(b) $\text{stab}[v_1, v_2]$ is supported on the closed subset $gp[v_1, v_2]$ of $M(v, w) \times M[v_1, v_2]$.

**Proof.** Replacing everywhere quiver varieties by deformed ones we define varieties $p[v_1, v_2]_{A^1}$ and $gp[v_1, v_2]_{A^1}$ such that $p[v_1, v_2]$ and $gp[v_1, v_2]$ are the fibers at 0 of $p[v_1, v_2]_{A^1}$ and $gp[v_1, v_2]_{A^1}$. Then, we have a closed embedding
\[(3.22)\]
\[gp[v_1, v_2]_{A^1} \subseteq M(v, w)_{A^1} \times M[v_1, v_2]_{A^1}. \]

Now, for each representation $z \in R(v, w)$ preserving the subspace $V_1 \oplus W_1$ of $V \oplus W$, let $z_1$ be the restriction of $z$ to $V_1 \oplus W_1$ and $z_2$ be the induced representation on the quotient space $(V \oplus W)/(V_1 \oplus W_1)$. Then, by [21 §3.7.4] we have
\[(3.23)\]
\[\mathcal{L}[v_1, v_2]_{A^1} = \mathcal{L}[v_1, v_2]_{A^1} // P, \]
\[L[v_1, v_2]_{A^1} = \{ z \in p[v_1, v_2]_{A^1}; z_2 \in M_s(v_2, w_2) \}. \]
Therefore, we have an obvious inclusion
\begin{equation}
\mathcal{L}[v_1, v_2]_{\mathcal{A}^1} \subseteq \mathfrak{p}[v_1, v_2]_{\mathcal{A}^1},
\end{equation}
hence a chain of inclusions
\begin{equation}
\mathcal{L}[v_1, v_2]_{G_m} \times \mathfrak{m}_{G_m} \mathcal{M}[v_1, v_2]_{G_m} \subseteq \mathfrak{g}\mathfrak{p}[v_1, v_2]_{G_m} \subseteq \mathfrak{g}\mathfrak{p}[v_1, v_2]_{\mathcal{A}^1}.
\end{equation}
By (3.21) the cycle stab\((v_1, v_2)\) is supported on the closure of the left hand side of (3.25) in \(\mathcal{M}(v, w)_{\mathcal{A}^1} \times \mathcal{M}[v_1, v_2]_{\mathcal{A}^1}\). Since (3.22) is a closed embedding, the cycle stab\([v_1, v_2]\) is supported on the set \(\mathfrak{g}\mathfrak{p}[v_1, v_2]\), proving part (b). From part (b) we deduce that the cycle \(r[v, 0; v_1, v_2]\) is supported on \(\mathfrak{g}\mathfrak{p}[v_1, v_2] \cap \mathcal{M}[v, 0; v_1, v_2]\).
Thus, part (a) follows from (3.19). \(\square\)

3.4. Definition of \(\mathfrak{Y}\).

3.4.1. Let \(v, w_1, w_2, \gamma\) be as above. Recall that
\begin{align*}
A_{w_1} \otimes A_{w_2} &\subset \text{End} \left( \bigoplus_{v_1, v_2} H^G_{w_1} \times T \left( \mathcal{M}(v_1, w_1) \otimes_k H^G_{w_2} \times T \left( \mathcal{M}(v_2, w_2) \right) \right) \right) \\
&\subset \text{End} \left( \bigoplus_{v_1, v_2} H^G_{w_1} \left( \mathcal{M}[v_1, v_2] \right) \right).
\end{align*}
Maulik and Okounkov have defined an \(R\)-matrix which is a formal series in \((A_{w_1} \otimes A_{w_2})[[\varpi^{-1}]]\) of the form
\begin{equation}
R_{w_1, w_2}(\varpi) = 1 + h \sum_{l > 0} R_{w_1, w_2; l} \varpi^{-l}.
\end{equation}
The \(R\)-matrix is homogenous of degree zero relatively to the \(\mathbb{Z} \times \mathbb{Z}\)-grading. The classical \(R\)-matrix is the first Fourier coefficient \(r_{w_1, w_2} = R_{w_1, w_2; 1}\). We have the decomposition
\begin{equation}
r_{w_1, w_2} = r_{w_1, w_2, -} + r_{w_1, w_2, 0} + r_{w_1, w_2, +}
\end{equation}
with
\begin{align*}
r_{w_1, w_2, -} &= \prod_{\mathcal{Y} \subset \gamma} \mathfrak{r}_{\mathcal{A} \times \mathcal{Y}}, \\
r_{w_1, w_2, 0} &= \prod_{\mathcal{Y} \subset \gamma} \mathfrak{r}_{\mathcal{Y} \times \mathcal{Y}}, \\
r_{w_1, w_2, +} &= \prod_{\mathcal{Y} \subset \gamma} \mathfrak{r}_{\mathcal{Y} \times \mathcal{Y}},
\end{align*}
where \(\mathcal{X}, \mathcal{Y}\) are connected components of \(\mathfrak{M}(v, w)^\circ\) and \(\mathfrak{r}_{\mathcal{X}, \mathcal{Y}}\) is a class in \(H^G_{\mathfrak{A}}(\mathcal{X} \times \mathcal{Y})\) which can be viewed as an operator \(H^G_{\mathfrak{A}}(\mathcal{Y}) \rightarrow H^G_{\mathfrak{A}}(\mathcal{X})\) via the convolution product. The partial order on the set \(\text{Irr}(\mathfrak{M}(v, w)^\circ)\) is as in (3.7). The first term in (3.26) is given by \(r_{\mathfrak{M}[u_1, u_2], \mathfrak{M}[v_1, v_2]} = r[u_1, u_2; v_1, v_2; u_1, w_2]\) for each dimension vectors \(u_1, u_2, v_1, v_2\) such that \(u_1 = u_2 = v_1 = v_2 = v\) and \(\mathfrak{M}[u_1, u_2] \subset \mathfrak{M}[v_1, v_2]\). It is viewed as a linear operator in \(A_{w_1}(u_1 - v_1) \otimes A_{w_2}(u_2 - v_2)\).
3.4.2. Now, we choose
\[ w_1 = w, \quad w_2 = \delta_i, \quad G = G(w) \times G(\delta_i) \times T = G(w) \times A \times T, \]
where \( A \) is as in \([3.1]\). Then, we define
\[ F_i = H^*_e(\mathfrak{M}(\delta_i)), \quad F_i^\vee = \text{Hom}(F_i, \mathbb{k}), \quad A^f_i = F_i \otimes F_i^\vee. \]
Taking the trace over \( F_i \) yields a \( k \)-linear map \( A^f_i \to k \).

Taking the formal variable \( u \) to the first equivariant Chern class of the linear character of \( G(\delta_i) \) yields isomorphisms \( k[u] \simeq k[\delta_i], F_i[u] \simeq F_\delta_i, A^f_i[u] \simeq A^f_\delta_i \) and
\[ k[w][u] \to H^*_e = k[w][w]. \]
Hence, if \( l > 0, v \in \mathbb{Z}^l \) and \( m \in A^f_i(v) \), the trace over \( F_i \) yields an element
\[ \text{tr}_{F_i}((1 \otimes m) R_{w,\delta_i; l}) \in A_w(v). \]
So, for each polynomial \( m(u) \) in \( A^f_i[u] \), there is a unique element \( E(m(u)) \) in \( \prod_w A_w \) which acts on \( F_w \) as the operator
\[ E(m(u), w) = \text{Res}_{u=\infty} \text{tr}_{F_i}((1 \otimes m(u)) R_{w,\delta_i}(u))/\hbar. \]

**Definition 3.2.** Let \( \mathbb{Y}_Q \) be the \( \mathbb{Z}^l \times \mathbb{Z} \)-graded \( k \)-subalgebra of \( \prod_w A_w \) generated by the elements \( E(m(u)) \) as \( m(u) \) runs over \( A^f_i[u] \).

There is a surjective \( \mathbb{Z}^l \times \mathbb{Z} \)-graded \( k \)-algebra homomorphism
\[ E : \text{Tensor Algebra} \left( \bigoplus_i A^f_i[u] \right) / \sim \to \mathbb{Y}_Q, \quad m(u) \mapsto E(m(u)), \]
where \( \sim \) is the two-sided ideal generated by the RTT=TTR relations \([12]\) (5.10)].

3.4.3. Since \( G_{sp}/A = G(w) \times T_{sp}, \) we have
\[ r[u_1, u_2; v_1, v_2] \in H^*_e(G(w) \times T_{sp})(\mathfrak{M}[u_1, u_2; v_1, v_2]). \]
Any element \( m \in A^f_i \) can be viewed as a constant polynomial in \( u \), yielding an operator \( e(m) = E(m) \) which acts on \( F_w \) as the operator
\[ e(m; w) = \text{tr}_{F_i}((1 \otimes m) r_{w,\delta_i}) \in A_w. \]
Let \( \mathfrak{g}_Q \) be the \( \mathbb{k} \)-submodule of \( \mathbb{Y}_Q \) spanned by the elements \( e(m) \) above. The commutator is a Lie bracket on \( \mathbb{Y}_Q \) for which \( \mathfrak{g}_Q \) is a Lie subalgebra. The Yang-Baxter equation implies that the map \( m \mapsto e(m) \) is a surjective \( \mathbb{Z}^l \times \mathbb{Z} \)-graded Lie algebra homomorphism
\[ e : \bigoplus_i A^f_i \to \mathfrak{g}_Q, \]
such that the Lie bracket on the left hand side is given by
\[ [m_j, m_i] = (\text{tr}_{F_j} \otimes 1)([r_{\delta_i,\delta_\ell}, m_j \otimes m_i]), \quad \forall m_i \in A^f_i, \quad \forall m_j \in A^f_j. \]
Let $\mathcal{V}_i$ and $\mathcal{W}_i$ be the universal and the tautological $G(w) \times T$-equivariant vector bundles on $\mathcal{M}(v, w)$, which are given by
\begin{equation}
(3.28) \quad \mathcal{V}_i = M_\ast(v, w) \times_{G(v)} V_i, \quad \mathcal{W}_i = \mathcal{M}(v, w) \times W_i.
\end{equation}
Let $\psi_{i,l}$ and $\phi_{i,l}$ be the operators in $\prod_w A_w$ given by the cap product with the $T$-equivariant cohomology class $ch_i(V_i)$ and $ch_i(W_i)$. The following is proved in \cite{12}:

1. $g_Q(v)$ is a free $\mathbb{Z}$-graded $k$-submodule of finite rank,
2. $g_Q = g_- \oplus g_0 \oplus g_+$ where
   \[ g_- = \bigoplus_{v<0} g_Q(v), \quad g_0 = g_Q(0), \quad g_+ = \bigoplus_{v>0} g_Q(v) \]
   and $v > 0$ if and only if $v \in \mathbb{N}^I$ and $v \neq 0$.
3. there are unique elements $h_{i,l}$ and $z_{i,l}$ in $Y_Q$ which act on $F_w$ via the operator of cap product with the classes $\psi_{i,l}$ and $\phi_{i,l}$. The element $z_{i,l}$ is central. We have a triangular decomposition $Y_Q \simeq Y_- \otimes Y_0 \otimes Y_+$ such that the $k$-algebra $Y_\pm$ is generated by $g_\pm$ and the adjoint action of $\{h_{i,l}\}$. We also set $Y_\rangle = Y_0 \otimes Y_+$.

Unless there is a risk of confusion we will simply write $\mathbb{Y}_Q = \mathbb{Y}$ and $g_Q = g$. We'll also write $\mathbb{Y}_{\rangle,K} = \mathbb{Y}_\rangle \otimes K$, $\mathbb{Y}_{+,K} = \mathbb{Y}_+ \otimes K$, etc.

3.5. Definition of $Y^1$.

3.5.1. Let $Y^1$ be the $\mathbb{Z}^I \times \mathbb{Z}$-graded $k$-module given by
\[ Y^1 = \bigoplus_v Y^1(v), \quad Y^1(v) = \bigoplus_k Y^1(v, k), \quad Y^{1}(v, k) = H^{G(v) \times T}_{k+2d_v}(\Lambda^1(v)). \]
For each dimension vector $v$ the $k$-module $Y^1(v)$ has a canonical $k[v]$-module structure given by the cap-product $\cap$. We have defined a $k$-algebra structure on $Y^1$ in \cite{21} §5.1.

3.5.2. Let $k(\infty)$ be MacDonald’s ring of symmetric functions with coefficients in $k$. It is the free commutative $k$-algebra generated by the set of power sum polynomials $\{p_l; l > 0\}$. It carries a comultiplication $\Delta$, a counit $\eta$, an antipode $S$ and the $\mathbb{Z}$-grading such that the element $p_l$ has the degree $2l$. We define $Y(0) = k(\infty)^{\otimes I}$. For each vertex $i \in I$ and each integer $l > 0$, let $p_{i,l}$ be the element of $Y(0)$ given by
\[ p_{i,l} = (1 \otimes \cdots \otimes 1 \otimes p_l \otimes 1 \otimes \cdots \otimes 1)/l!, \]
where $p_l$ is at the $i$-th spot. Restricting a representation of $G(\infty^I)$ to the subgroup $G(v)$, yields a $k$-algebra homomorphism
\[ Y(0) \to k[v], \quad \rho \mapsto \rho_v. \]
Let $U_i$ be the tautological $G(v) \times T$-equivariant vector bundle on $\Lambda^1(v)$, which is given by
\[ U_i = \Lambda^1(v) \times V_i, \]
with the obvious $G(v) \times T$-action. There is a $Y(0)$-action $\oplus$ on $Y^1$ such that the element $p_{i,l}$ acts via the cap product with the class $\text{ch}_l(U_i)$. For each elements $x \in Y(0)$ and $y, z \in Y^1$ we have

$$x \otimes (y \ast z) = \sum (x_1 \otimes y) \ast (x_2 \otimes z).$$

We deduce that $Y^1$ is a $Y(0)$-module $\mathcal{Z} \otimes \mathbb{Z}$-graded algebra. The COHA is the smash product

$$Y^1 = Y(0) \bowtie Y^1.$$ 

It is a free $\mathbb{N}I \times (-\mathbb{N})$-graded $k$-algebra such that $Y^1(v, k) = Y^1(v, k)$ if $v \neq 0$ and $Y^1(0, k)$ is the degree $k$ part of $Y(0)$. We set $Y_K = Y^1 \otimes K$.

3.5.3. Let $F_w$ be the $\mathbb{Z}I \times \mathbb{Z}$-graded $k[w]$-module given by

$$F_w = \bigoplus_{v \in \mathbb{N}I} F_w(v), \quad F_w(v) = \bigoplus_{k \in \mathbb{Z}} F_w(v, k), \quad F_w(v, k) = H^{G(w) \times T}_{k+2d_{w,v}}(\mathcal{M}(v, w)).$$

We set $|w| = [\mathcal{M}(0, w)]$. The following is proved in [21, prop. 5.19].

**Proposition 3.3.**

(a) The $\mathbb{Z}I \times \mathbb{Z}$-graded $k$-algebra $Y^1$ acts on the $\mathbb{Z}I \times \mathbb{Z}$-graded $k[w]$-module $F_w$.

(b) The action on the element $|v|$ yields an injective map $Y^1(v) \to F_v(v)$.

(c) The representation of $Y^1$ in $\bigoplus_w F_w$ is faithful.

(d) The action of $Y^1(v)$ on $|w|$ yields a $k[w]$-linear map $Y^1(v) \otimes k[w] \to F_w(v)$ whose image is the pushward of $H^{G(w) \times T}_v(\mathcal{L}^1(v, w))$ by the inclusion $\mathcal{L}^1(v, w) \subset \mathcal{M}(v, w)$.

3.5.4. Next, given any integer $l > 0$, let $x_{i,l}$ be the element in $Y^1$ given by $x_{i,l} = [\Lambda_{l,\delta_i}]$ if $q_i > 0$, and $x_{i,l} = \delta_{l,1} [\Lambda_{l,\delta_i}]$ if $q_i = 0$. We set

$$I^r = \{i \in I ; q_i = 0\},$$

$$I^s = \{i \in I ; q_i = 1\},$$

$$I^h = \{i \in I ; q_i > 1\}.$$ 

Then, the following is proved in [21, thm. 5.18].

**Proposition 3.4.**

(a) The $k$-algebra $Y^1$ is generated by the subset

$$Y(0) \cup \{x_{i,1} ; i \in I^r\} \cup \{x_{i,l} ; l > 0, i \in I^s \cup I^h\}.$$

(b) The $K$-algebra $Y_K$ is generated by the subset

$$(Y(0) \otimes K) \cup \{x_{i,1} \otimes 1 ; i \in I^r \cup I^s\} \cup \{x_{i,l} \otimes 1 ; l > 0, i \in I^h\}.$$
3.5.5. Assume that $v_2 = l \delta_i$. Set
\[
q_i > 0, \quad l > 0 \Rightarrow C(v_1, v_2; w) = h[v_1, \Lambda_{(v_2)}; w],
\]
\[
q_i = 0, \quad l = 1 \Rightarrow C(v_1, v_2; w) = h[v_1, v_2; w],
\]
\[
q_i = 0, \quad l > 1 \Rightarrow C(v_1, v_2; w) = \emptyset.
\]

Let $\mathcal{M}(w)$ be the disjoint sum of all $\mathcal{M}(v, w)$’s and $\mathcal{C}_{i,l}(w)$ be the disjoint sum of all $C(v_1, v_2; w)$’s where $v, v_1, v_2$ are as above. We’ll view $\mathcal{C}_{i,l}(w)$ as a closed $G(w) \times T$-invariant subvariety of the symplectic manifold $\mathcal{M}(w)^{\text{op}} \times \mathcal{M}(w)$ which is proper over $\mathcal{M}(w)$. Hence, it acts by convolution on $F_w$, yielding a $k[w]$-linear operator in $A_w$. Therefore, for each $i \in I$ and $l \in \mathbb{Z}_{>0}$ the family of correspondences $\mathcal{C}_{i,l}(w)$ defines an element
\[
\mathcal{C}_{i,l} \in \prod_w A_w.
\]

The following is proved in [21, prop. 5.20, thm. 5.22].

**Proposition 3.5.**

(a) $x_{i,l}$ acts on $F_w$ via the operator $\mathcal{C}_{i,l}(w)$.

(b) $Y^1$ is isomorphic to the $k$-subalgebra of $\prod_w A_w$ generated by
\[
\{\psi_{i,l}; l > 0, i \in I\} \cup \{\mathcal{C}_{i,1}; i \in I^r\} \cup \{\mathcal{C}_{i,l}; l > 0, i \in I^e \cup I^h\}.
\]

**Remark 3.6.** Assume that $v_2 = l \delta_i$ and $q_i > 1$. By [21, prop. 3.16], the Hecke correspondence $h[v_1, \Lambda_{(v_2)}; w]$ is either Lagrangian and irreducible or empty for any $v \equiv v_2$.

4. **Comparison of $Y_K$ and $Y_{\geq K}$**

We’ll use the same notation and assumptions as in the previous section.

4.1. **The main theorem.**

We’ll prove the following.

**Theorem 4.1.** Assume that the torus $T$ contains a one parameter subgroup which scales all the quiver data by the same scalar. Then, there is a $K$-algebra embedding $Y_K \subset Y_{\geq K}$ which intertwines the representations of $Y_{\geq K}$ and $Y_K$ in $F_w \otimes K$ given in \S 3.4 for each dimension vector $w$. The inclusion $Y_K \subset Y_{\geq K}$ restricts to an embedding $Y_K \subset Y_{+,K}$.

**Proof.** By Proposition 3.5 and the definition of $Y$, we can view $Y_K$ and $Y_{\geq K}$ as $K$-subalgebras of $\prod_w A_w \otimes K$. More precisely $Y_K$ is generated by the set
\[
\{\psi_{i,l}; i \in I, l \in \mathbb{N}\} \cup \{\mathcal{C}_{i,1}; i \in I^r \cup I^e\} \cup \{\mathcal{C}_{i,l}; l > 0, i \in I^h\},
\]
and $Y_{\geq} \otimes K$ by the set
\[
\{\psi_{i,l}; i \in I, l \in \mathbb{N}\} \cup \bigcup_{i \in I} e(A_{\delta_i}).
\]
Therefore, the theorem is a consequence of the following proposition which we will prove in the next two sections.

**Proposition 4.2.** We have

\[ (l = 1, i \in I^{r} \cup I^{e}) \quad \text{or} \quad (l > 0, i \in I^{h}) \Rightarrow \mathcal{C}_{i,l} \in e(A_i(l\delta_i)). \]

\[ \square \]

**Remark 4.3.** We conjecture that the \( K \)-algebra \( Y_{\geq K} \) is indeed generated by \( Y_{K} \) and the central elements \( \phi_{i,l} \).

4.2. **Generalized Hecke correspondences and \( R \)-matrices.**

4.2.1. To prove Proposition 4.2, we need more details on the classical \( R \)-matrix. Let \( w_1, w_2, w \) are as in (3.27) and \( v_2 = l\delta_i, \quad l \in \mathbb{N}, \quad i \in I^{h} \).

Since \( q_i > 1 \), the generic representation type in \( RT(v_2,0) \) is \( \kappa_{v_2,0} = (1, v_2) \).

We abbreviate

\[ S(v_2) = M(\kappa_{v_2,0}), \quad h[v_1, S(v_2)] = h[v_1, S(v_2) ; w], \quad gh[v_1, S(v_2)] = gh[v_1, S(v_2) ; w, \delta_i]. \]

By Proposition 3.1, the cycle \( r[v,0 ; v_1, v_2] \) is supported on the generalized Hecke correspondence \( gh[v_1, v_2] \). The goal of this section is to prove the following technical result.

**Proposition 4.4.**

(a) \( gh[v_1, S(v_2)] \) is open in \( gh[v_1, v_2] \).

(b) \( gh[v_1, S(v_2)] \simeq h[v_1, S(v_2)] \times \mathbb{P}^{l-1} \).

(c) \( gh[v_1, S(v_2)] \) has a unique top dimensional component, which is of dimension \( \text{mid}/2 \).

(d) \( r(v,0 ; v_1, v_2)|_{gh[v_1, S(v_2)]} \) is a non zero multiple of the fundamental class of the top dimensional irreducible component.

4.2.2. **Proof of (a), (b), (c).** Part (a) is easy. For (b), note that if a tuple \( z_2 = (\bar{a}_2, \bar{a}_2) \) in \( M_+ (v_2, \delta_i) \) represents a point of the stratum \( \mathcal{M}_{0} (\kappa_{v_2,0}) \), then, since \( a_2 \neq 0 \) and since the representation \( z_2 \) must have a constituent of dimension \( \delta_i \), we deduce that \( a_2 = 0 \). Now, let us prove part (c). For each \( \tau \in RT(v,w) \) and \( \tau_1 \in RT(v_1, w) \) we write

\[ h(\tau ; \tau_1) = h[v_1, S(v_2)] \cap (\mathcal{M}(\tau) \times \mathcal{M}(\tau_1)), \]

\[ gh(\tau ; \tau_1) = gh[v_1, S(v_2)] \cap (\mathcal{M}(\tau) \times \mathcal{M}(\tau_1) \times \mathcal{M}(v_2, \delta_i)). \]

Part (c) follows from the following lemma.

**Lemma 4.5.**

(a) \( gh(\tau ; \tau_1) \neq \emptyset \Rightarrow \tau \leq \tau_1 \oplus \kappa_{v_2,0} \).

(b) \( \dim gh(\tau ; \tau_1) \leq \text{mid}/2 \).
(c) \( \dim \mathfrak{h}(\tau; \tau_1) = \text{mid}/2 \iff \tau = \kappa_{v_1, w} \oplus \kappa_{v_2, 0} \text{ and } \tau_1 = \kappa_{v_1, w}. \)
(d) \( \mathfrak{h}(\kappa_{v_1, w} \oplus \kappa_{v_2, 0}; \kappa_{v_1, w}) \) has a unique top-dimensional irreducible component.

**Proof.** Part (a) is obvious. We’ll prove (b), (c), (d) simultaneously. The proof goes along lines similar to that of [21, prop. 3.16]. Consider the natural projection

\[ p : H^0[v_1, v_2] \to M_s(v_1, w) \times M(v_2). \]

First, assume that \( v_2 \) is not the dimension vector of any constituent of \( \tau_1 \). Hence, we have \( \mathfrak{h}(\tau; \tau_1) \neq \emptyset \) if and only if \( \tau = \tau_1 \oplus \kappa_{v_2, 0} \). Then, we have

\[ \Hom_{\mathfrak{H}}(N_1, N_2) = \{0\}, \quad \forall (N_1, N_2) \in M_s(\tau_1) \times M(\kappa_{v_2, 0}). \]

Here we view \( N_1 \) and \( N_2 \) as \( \mathfrak{H} \)-modules as in [21, §3.7.2]. By [21, lem. 3.19], the variety \( p^{-1}(M_s(\tau_1) \times M(\kappa_{v_2, 0})) \) is smooth over \( M_s(\tau_1) \times M(\kappa_{v_2, 0}) \) with connected fibers of dimension

\[ v_1 \cdot v_2 - (v_1 + \delta_\infty \cdot v_2)_Q. \]

Fix an \( I \)-graded subspace \( V_1 \subset V \) of dimension \( v_1 \) and let \( P \subset G(v) \) be its stabilizer. The \( P \)-action on \( H^0[v_1, v_2] \) is free. Therefore, we have

\[ \dim \mathfrak{h}(\tau_1 \oplus \kappa_{v_2, 0}; \tau_1) = -(v_1, v_2)_Q + w \cdot v_2 + \dim \mathfrak{M}(\tau_1) + 2(q_i - 1)t^2 + 1. \]

A short computation yields

\[ \text{mid}/2 - \text{codim} \mathfrak{M}(v_1, w) \mathfrak{M}(\tau_1) = -(v_1, v_2)_Q + w \cdot v_2 + \dim \mathfrak{M}(\tau_1) + 2(q_i - 1)t^2 + l. \]

We deduce that

\[ \dim \mathfrak{h}(\tau_1 \oplus \kappa_{v_2, 0}; \tau_1) = \text{mid}/2 - \text{codim} \mathfrak{M}(v_1, w) \mathfrak{M}(\tau_1) - l + 1. \]

Hence, Proposition 4.4 (b) yields

\[ \dim \mathfrak{h}(\tau_1 \oplus \kappa_{v_2, 0}; \tau_1) = \text{mid}/2 - \text{codim} \mathfrak{M}(v_1, w) \mathfrak{M}(\tau_1). \]

Further, the set \( \mathfrak{h}(\tau_1 \oplus \kappa_{v_2, 0}; \tau_1) \) is irreducible, because both \( M_s(\tau_1) \) and \( M(\kappa_{v_2, 0}) \) are irreducible. This proves the lemma in this case.

Next let us assume that \( \tau_1 \) does contain the dimension vector \( v_2 \). Let

\[ U = \{(\bar{x}_1, \bar{a}_1), \bar{x}_2) \in M_s(\tau_1) \times M(\kappa_{v_2, 0}) \mid \Hom_{\mathfrak{H}}((\bar{x}_1, \bar{a}_1), \bar{x}_2) = 0\}, \]

\[ Z = (M_s(\tau_1) \times M(\kappa_{v_2, 0})) \setminus U. \]

Note that \( U \) is a dense open subset of \( M_s(\tau_1) \times M(\kappa_{v_2, 0}) \). By the same argument as above,

\[ \dim(p^{-1}(U)/P \times \mathbb{P}^{l-1}) = \text{mid}/2 - \text{codim} \mathfrak{M}(v_1, w) \mathfrak{M}(\tau_1) \]

and \( p^{-1}(U)/P \) is again irreducible. From equations (4.2) and (4.3) we see that the lemma will be proved once we show that, for any \( \tau_1 \) as above, we have

\[ \dim(p^{-1}(Z)/P \times \mathbb{P}^{l-1}) < \text{mid}/2. \]

We will treat the case when the dimension vector \( v_2 \) occurs once in \( \tau_1 \), with multiplicity \( k \geq 1 \). The case when the dimension vector \( v_2 \) occurs more than once is similar and is left to the reader.
Then, we have dim Hom_H(N_1, N_2) ≤ k for any (N_1, N_2) ∈ Z and [21, lem. 3.19] yields

\[ \dim(p^{-1}(Z) / P) \leq \dim Z - \dim L - (v_1 + \delta_\infty, v_2)_Q + k. \]

There can be a nonzero morphism from N_1 ∈ M(τ_1) to N_2 ∈ M(κ_{v_2,0}) only if the simple constituent of N_1 of dimension v_2 is isomorphic to N_2. Therefore, we have

\[ \dim Z \leq \dim M_s(\tau_1) + \dim M(\kappa_{v_2,0}) - \dim M_0(\kappa_{v_2,0}) \]

and thus

\[ \dim Z - \dim L \leq \dim M(\tau_1) - 1. \]

Next, by [21 (3.19)] we have

\[ \dim M(\tau_1) \leq \dim M(v_1, w)/2 + \dim M_0(\tau_1)/2. \]

Therefore, we have

\[ \text{codim}_{M(v_1, w)} M(\tau_1) \geq \dim M(v_1, w)/2 - \dim M_0(\tau_1)/2. \]

Using (4.1, 4.5), (4.6) and (4.7), we deduce that (4.4) is implied by the following inequality

\[ 1 + 2(q_i - 1)\ell^2 + \dim M(v_1, w)/2 - \dim M_0(\tau_1)/2 \geq k. \]

Finally, we prove (4.8). It is obvious if k = 1, so we may assume that k > 1. Let \( \tau'_1 \) be the representation type obtained from \( \tau_1 \) by replacing \( (k, v_2) \) by \( (1, v_2), (1, v_2), \ldots, (k \text{ times}) \). We have

\[ \dim M_0(\tau'_1) = (k - 1)\left(2 + 2(q_i - 1)\ell^2\right) + \dim M_0(\tau_1). \]

Since \( q_i > 1 \), this implies that

\[ 1 + 2(q_i - 1)\ell^2 + \dim M_0(\tau'_1)/2 - \dim M_0(\tau_1)/2 > k. \]

Since \( \tau_1 \) and \( \tau'_1 \) have the same dimension type, they are either both in the image of \( \pi \) or both outside. In the latter case, the set gh(τ; τ_1) is empty and there is nothing to prove. In the former case, we have dim \( M(v_1, w) \geq \dim M_0(\tau'_1) \) and the inequality (4.8) follows.

\[ \square \]

4.2.3. Proof of (d). We concentrate now on the proof of Proposition [4.4(d)]. For degree reasons and Propositions [3.1, 4.4(c)], the restriction of the cycle [v, 0 ; v_1, v_2] to gh[v_1, S(v_2)] is a multiple of the fundamental class of the top dimensional irreducible component. We must prove it is not zero. We'll abbreviate

\[ L = L(v, w + \delta_i), \quad M = M(v, w + \delta_i), \quad M_0 = M_0(v, w + \delta_i), \quad Z = Z(v, w + \delta_i). \]

Then, for each \( k = 0, 1, \ldots, l \) we consider the (reduced) fiber product

\[ R_k = L[v_1 + k\delta_i, v_2 - k\delta_i] \times M_0[v_1, v_2] \subset Z, \]

and the closure \( \overline{R} \) in \( Z \) of the subset of \( R_0 \) given by

\[ R = L[v_1, v_2] \times M[v_1, v_2]. \]
Then $\mathcal{R}, \mathcal{R}_1, \ldots, \mathcal{R}_l$ are Lagrangian cycles in $\mathcal{M}^\op \times \mathcal{M}[v_1, v_2]$ such that $\mathcal{R}_k \subset \mathcal{R}_k \cup \cdots \cup \mathcal{R}_l$ for each $k$. By [12 prop. 3.5.1] we have

$$\text{stab}[v_1, v_2] = C_0 + C_1 + \cdots + C_l$$

where $C_0 = \varepsilon_{v_1, v_2}(\mathcal{R})$ and $C_k$ is a Lagrangian cycle in $\mathcal{M}^\op \times \mathcal{M}[v_1, v_2]$ supported on $\mathcal{R}_k$ for each $k$. By Proposition 3.1, the restriction of the cycle $\text{stab}[v_1, v_2]$ to $\mathcal{M}[v, 0; v_1, v_2]$ is supported on the generalized Hecke correspondence $gh[v_1, v_2] \subseteq \mathcal{M}[v, 0] \times_{\mathcal{M}_0} \mathcal{M}[v_1, v_2]$.

**Lemma 4.6.** If $0 < k < l$ then $\dim (\mathcal{R}_k \cap gh[v_1, S(v_2)]) < \text{mid}/2$.

**Proof.** We’ll prove that $\dim (\mathcal{R}_k \cap gh[v_1, S(v_2)]) < \text{mid}/2$. Consider the constructible subset $X_k \subset gh[v_1, S(v_2)]$ of pairs $(z, z_1)$ for which there exist simple $\bar{\Pi}$-modules $S_k, S_l$ of respective dimension $k\delta_i, l\delta_i$ such that $\text{Hom}_{\bar{\Pi}}(z, S_k) \neq 0$ and $\text{Hom}_{\bar{\Pi}}(z, S_l) \neq 0$. We have

$$(z, z_1, z_2) \in \mathcal{R}_{l-k} \cap gh[v_1, S(v_2)] \Rightarrow (z, z_1) \in X_k.$$ 

So, we must show that

$$(4.9) \quad \text{codim}_{h_{v_1, v_2}} X_k > 0.$$ 

Let $Y \subset gh[v_1, S(v_2)]$ be the dense open subset of representations $(z, z_1)$ for which all non-rigid simple factors of $z$ occur with multiplicity one. It is enough to prove that

$$(4.10) \quad \text{codim}_{Y} X_k \cap Y > 0.$$ 

Let $(z, z_1) \in X_k \cap Y$ and let $S_k, S_l$ be the corresponding simple $\bar{\Pi}$-modules. There is a short exact sequence

$$(4.11) \quad 0 \rightarrow z_1 \rightarrow z \rightarrow S_l \rightarrow 0$$

and since $\text{Hom}(z, S_k) \neq 0$ and $\text{Hom}(S_l, S_k) = 0$ there is an exact sequence

$$(4.12) \quad 0 \rightarrow z_2 \rightarrow z_1 \rightarrow S_k \rightarrow 0.$$ 

From the exact sequence

$$(4.13) \quad 0 \longrightarrow \text{Hom}(z, S_k) \longrightarrow \text{Hom}(z_1, S_k) \xrightarrow{\partial} \text{Ext}^1(S_l, S_k)$$

and the fact that $\text{Hom}(z, S_k) = \text{Hom}(z_1, S_k) = \mathbb{C}$ we deduce that $\text{Im}(\partial) = 0$. Hence, the following exact sequence splits

$$(4.14) \quad 0 \rightarrow S_k \rightarrow z/z_2 \rightarrow S_l \rightarrow 0.$$ 

Consider the set $h \subset \mathcal{M}(v, w) \times \mathcal{M}(v_1, w) \times \mathcal{M}(v_1 - k\delta_i, w)$ consisting of the triples $(z, z_1, z_2)$ of stable $\bar{\Pi}$-modules such that $z_2 \subset z_1 \subset z$. The restriction of the map

$$\rho: h \rightarrow gh[v_1, v_2], \ (z, z_1, z_2) \mapsto (z, z_1)$$
to $\rho^{-1}(Y)$ is finite. Indeed, any simple $\tilde{\Pi}$-module $S_k$ of dimension $k\delta_i$ is non-rigid, because

$$\dim \text{Ext}^1_{\tilde{\Pi}}(S_k, S_k) = 2 - (k\delta_i, k\delta_i)_Q = 2 + 2k^2(q_i - 1) > 0$$

by [21, prop. 3.1]. Therefore, for any pair $(z, z_1) \in Y$ the $\tilde{\Pi}$-module $z_1$ only has finitely many simple factors of dimension $k\delta_i$ and they all occur with multiplicity one. Thus, Lemma 4.6 will be proved once we show that there exists $Z \subset \rho^{-1}(Y)$ such that

$$\rho(Z) \supset X_k \cap Y \text{ and codim}_{\rho^{-1}(Y)} Z > 0.$$ 

Let $P \subset G(k\delta_i + l\delta_i)$ be the standard parabolic subgroup of type $(k\delta_i, l\delta_i)$ and let $\mathfrak{p}$ be its Lie algebra. Consider the stack

$$\mathcal{S} = M(k\delta_i + l\delta_i) \cap \mathfrak{p}^{2n}/P.$$ 

Let $U$ be the open substack of $\mathcal{S} \times \mathcal{M}(v_1 - k\delta_i, w)$ consisting of triples $(x, y, z_2)$ such that all non-rigid simples occuring in $z_2 \oplus x \oplus y/x$ have multiplicity one. The set $\rho^{-1}(Y)$ is the open substack of $\mathfrak{h}$ consisting of triples $(z, z_1, z_2)$ for which all non-rigid simples occuring in $z$ have multiplicity one. As all simples of dimension in $\mathbb{N}\delta_i$ are non-rigid, an argument in all points parallel to [21, lem. 3.19] shows that the map

$$\kappa : \rho^{-1}(Y) \to U, \quad (z, z_1, z_2) \mapsto (z_1/z_2, z/z_2, z_2)$$

is a stack vector bundle. Moreover, the constructible substack $\mathcal{S^s}$ parametrizing pairs $(S_k \subset S_k \oplus S_l)$ with $S_k, S_l$ simple, is of strictly positive codimension, because

$$\dim \text{Ext}^1(S_k, S_k) = -(l\delta_i, k\delta_i) = 2(q_i - 1)kl > 0.$$ 

Therefore, the following set satisfies the conditions in (4.15)

$$Z = \kappa^{-1}((\mathcal{S^s} \times \mathcal{M}(v_1 - k\delta_i, w)) \cap U).$$

Now, let us come back to the proof of Proposition 4.4(d). From the lemma, we deduce that the restrictions of the cycles $C_1, C_2, \ldots, C_{t-1}$ to $\mathcal{M}[v, 0; v_1, v_2]$ do not contribute to $r[v, 0; v_1, v_2]$. Further, by (3.9) we have

$$\text{stab}[v_1, v_2]_{\mathcal{M}[v, 0; v_1, v_2]} \in H_A^{2lvi - 2} \cap H_{G/A}^G(\mathcal{M}[v, 0; v_1, v_2]).$$

Let $Z$ denote the restriction of the cycle $C_0 + C_l$ to $\mathcal{M}[v, 0; v_1, v_2]$. We deduce that

$$Z \in H_A^{2lvi - 2} \cap H_{G/A}^G(\mathcal{M}[v, 0; v_1, v_2])$$

and the cycle $r[v, 0; v_1, v_2]$ is the symbol of the class $Z$ in

$$H_A^{2lvi - 2} \oplus H_{G/A}^G(\mathcal{M}[v, 0; v_1, v_2]).$$

Further, the class $Z$ is supported on the generalized Hecke correspondence $\mathfrak{gh}[v_1, v_2]$ and it is enough to prove that its restriction to the open set $\mathfrak{gh}[v_1, S(v_2)]$ belongs to

$$H_A^{2lvi - 2} \cap H_{G/A}^G(\mathfrak{gh}[v_1, S(v_2)]) \setminus H_A^{2lvi - 2} \cap H_{G/A}^G(\mathfrak{gh}[v_1, S(v_2)]).$$
We define the $G$-equivariant vector subbundle $E$ standard Levi subgroup. Finally, we abbreviate where $gh$ is supported on $l \in Z$.

First, we concentrate on the cycle $C_l$. The variety $R_l$ is a closed subset of $\mathcal{M} \times V$ and an affine space bundle over the Lagrangian subvariety $\mathcal{M}[v, 0] \times_{\mathfrak{g} \mathfrak{h}} \mathcal{M}[v, v_2] \subset \mathcal{M}[v, 0]^{op} \times \mathcal{M}[v, v_2]$.

By Proposition 3.1 we can assume that the cycle $C_l$ is supported in $R_l \cap \mathfrak{g} \mathfrak{p}[v_1, v_2]$. The obvious map $R_l \to \mathcal{M}[v, 0; v_1, v_2]$ yields a $G$-equivariant affine space bundle (4.17)

$$R_l \cap \mathfrak{g} \mathfrak{p}[v_1, v_2] \to \mathfrak{g} \mathfrak{h}[v_1, v_2]$$

and the cycle $C_l$ is the pull-back of a Lagrangian cycle of $\mathcal{M}[v, 0]^{op} \times \mathcal{M}[v_1, v_2]$ supported in $\mathfrak{g} \mathfrak{h}[v_1, v_2]$. Therefore, the cycle

$$Z_l = C_l|_{\mathcal{M}[v, 0; v_1, v_2]}$$

is supported on $\mathfrak{g} \mathfrak{h}[v_1, v_2]$ and its restriction to the open subset $\mathfrak{g} \mathfrak{h}[v_1, S(v_2)]$ is a rational multiple of the class (4.18)

$$\text{eu}(N/v_l) \cap [\text{top}[v_1, v_2]].$$

Here $\nu_l$ is the $G$-equivariant vector subbundle of $N$ equal to the relative tangent bundle of (4.17). Set $B_l = \text{GH}[v_1, v_2] \times E^*$. Then, from (4.17) we deduce that $\nu_l$ is first projection

$$(B_l \times \text{Hom}_I(w_2, v_1)) / P \times L \to \text{GH}[v_1, v_2] / P \times L$$

where $G \times P$ acts on $\text{Hom}_I(w_2, v)$ in the obvious way and $L$ acts trivially on $\text{Hom}_I(w_2, v)$.

Next, we consider the cycle $C_0$. We have an inclusion $\overline{R} \subseteq \mathfrak{g} \mathfrak{p}[v_1, v_2]$ and $C_0$ is a multiple of the fundamental class of $\overline{R}$. Hence, the cycle

$$Z_0 = C_0|_{\mathcal{M}[v, 0; v_1, v_2]}$$

is supported on $\mathfrak{g} \mathfrak{h}[v_1, v_2]$. Set

$$B_0 = \{(z, z_1', z_2', \tilde{a}_2) \in \text{GH}[v_1, v_2] \times T^* E ; (a_2')^*(a_2) = 0, a_2' \in \mathbb{C}(a_2')^* \}$$

with

$$z_2' = (\tilde{x}_2', \tilde{a}_2'), \quad a_2' = (a_2', (a_2')^*) \quad \tilde{a}_2 = (a_2, a_2').$$

We define the $G$-equivariant vector subbundle $\nu_0 \subset N$ as the obvious projection

$$(B_0 \times \text{Hom}_I(w_2, v_1)) / P \times L \to \mathfrak{g} \mathfrak{h}[v_1, v_2].$$

Consider the open subsets of $\mathfrak{g} \mathfrak{p}[v_1, v_2]$ and $\mathfrak{g} \mathfrak{h}[v_1, v_2]$ given by

$$(4.19) \quad \mathfrak{g} \mathfrak{p}[v_1, v_2]^0 = \{(z, z_1', z_2') \in \mathfrak{g} \mathfrak{p}[v_1, v_2]; \tilde{x}_2 \text{ is simple, } \text{Hom}_I(z_1, z_2) = 0\} / P \times L,$n

$\mathfrak{g} \mathfrak{h}[v_1, v_2]^0 = \{(z, z_1', z_2') \in \mathfrak{g} \mathfrak{h}[v_1, v_2]; \tilde{x}_2 \text{ is simple, } \text{Hom}_I(z_1, z_2) = 0\} / P \times L$,

where $z = (\tilde{x}, \tilde{a})$, $z_2 = (\tilde{x}_2, \tilde{a}_2)$ and $z_1 = z|_{V_1 \oplus W_1}$, $z_2 = z \mod V_1 \oplus W_1.$
Note that we have
\[ gh[v_1, v_2] = gp[v_1, v_2] \cap M[v, 0; v_1, v_2]. \]

We claim that we have \( gh[v_1, S(v_2)] \subseteq \mathcal{R} \) and that the restriction of the fundamental class of \( \mathcal{R} \cap gp[v_1, v_2] \) to \( gh[v_1, v_2] \) is equal to
\[ (4.20) \quad \text{eu}(N/\nu_0) \cap [\text{top}[v_1, v_2]^o], \quad \text{top}[v_1, v_2]^o = \text{top}[v_1, v_2] \cap gh[v_1, v_2]^o. \]

Comparing (4.16) and (4.20), we deduce that the restriction of the cycle \( Z \) to the open subset \( gh[v_1, v_2]^o \) is of the form \( \alpha \cap [\text{top}[v_1, v_2]^o] \), for some class \( \alpha \in H^2_G(gh[v_1, v_2]) \) which is a \( \mathbb{Q} \)-linear combination of the equivariant Euler classes \( \text{eu}(N/\nu_0) \) and \( \text{eu}(N/\nu_l) \).

Now, let \( \mathcal{L} \) be the \( G \)-equivariant line bundle on \( gh[v_1, v_2] \) whose fiber at the point represented by the tuple \((z, z_1', z_2') \in GH[v_1, v_2] \) is the line in \( E \) spanned by \((a_2')^*\). The obvious projection \( T^*E \to E^* \) yields a \( G \)-equivariant vector bundle homomorphism \( \nu_0 \to \nu_l \) which fits in an exact sequence
\[ 0 \to \mathcal{L} \to \nu_0 \to \nu_l \to h \otimes \mathcal{L}^{-1} \to 0. \]

From (4.18) we deduce that \( \alpha \) is a non zero rational multiple of the class \( h \text{eu}(N/\nu_0 + \nu_l) \).

To finish the proof, it is enough to observe that the restriction of the class \( \text{eu}(N/\nu_0 + \nu_l) \) to \( H^2_A(gh[v_1, v_2]) \) is non zero.

To prove the claim, set
\[ F = \{(a_2, a_2') \in T^*E \times T^*(E \setminus \{0\}); a_2 a_2' = a_2'(a_2')^*\}. \]

By [21, lem. 3.19], the assignment \( f : (z, z_1', z_2') \mapsto (a_2, a_2') \) gives rise to the following commutative diagram
\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{f} & F \\
\downarrow & & \downarrow \\
gh[v_1, v_2] & \xrightarrow{f'} & \{0\} \times E \setminus \{0\}.
\end{array}
\]

whose horizontal maps are smooth and \( P \times L \)-equivariant. We have
\[ f(\mathcal{R} \cap gp[v_1, v_2]^o) = \{(a_2, a_2') \in T^*(E \setminus \{0\}) \times T^*(E \setminus \{0\}); \exists t \in C, a_2' = ta_2, a_2^* = t(a_2')^*\}. \]

Hence the map \( f \) restricts to a smooth map
\[ \mathcal{R} \cap gp[v_1, v_2]^o \to \{(a_2, a_2') \in T^*E \times T^*(E \setminus \{0\}); (a_2')^*(a_2) = 0, a_2^* \in \mathbb{C}(a_2')^*\}. \]

Comparing the fibers of \( f \) and \( f' \) gives the result.

\[ \square \]

4.3. Proof of Proposition 4.2

Recall that \( w_1 = w, w_2 = \delta_i \) and \( v = v_1 + v_2 \) with \( v_2 = l\delta_i \) and \( l \) a positive integer.
4.3.1. Proof of Proposition 4.2. First, assume that $q_i > 1$. Consider the irreducible component of the Lagrangian quiver variety $\mathcal{Q}^1(v_2, \delta_i)$ given by
\[
\mathcal{C} = \{(x, \tilde{a}) \in \mathcal{M}(v_2, \delta_i) ; x = a = 0\}.
\]

Note that
\[
\mathcal{C} = \{(x, \tilde{a}) \in R_4(v_2, \delta_i) ; x = a = 0\}/G(v_2),
\]
hence it is smooth of dimension $d_{v_2, \delta_i}/2$. Let $\mathcal{C}_0$ be the image of $\mathcal{C}$ by $\pi$. The closed embedding $i : \mathcal{C} \to \mathcal{M}(v_2, \delta_i)$ and the projection $\pi : \mathcal{C} \to \mathcal{C}_0$ yield the map
\[
\phi = \pi_* \circ i^* : H^T_{\ast}(\mathcal{M}(v_2, \delta_i)) \to H^T_{\ast-d_{v_2, \delta_i}}(\mathcal{C}_0).
\]

Since the torus $T$ contains a one parameter subgroup which scales all the quiver data by the same scalar, the $T$-fixed points locus in $\mathcal{C}_0$ is $\{0\}$. Therefore, the pushforward by the closed embedding $\{0\} \to \mathcal{C}_0$ is an isomorphism $K \to H^T_{\ast}(\mathcal{C}_0) \otimes K$. Let $\psi$ be the inverse map. The composed map $\psi \circ \phi$ belongs to $F_1(v_2)^\vee \otimes K$. The same argument as in [21, prop. 5.2] implies that $F_1(v_2) \otimes K$ is $K$ times the $T$-equivariant homology of a smooth projective variety, hence the pushforward to a point yields a nondegenerate pairing
\[
(\bullet, \bullet) : F_1(v_2) \otimes F_1(v_2) \otimes K \to F_1(0) \otimes K = K.
\]

Write $\psi \circ \phi = (m, \bullet)$ for some element
\[
(4.21) \quad m \in \text{Hom}(F_1(0), F_1(v_2)) \otimes K \subset A_1^T(v_2) \otimes K.
\]

The restriction of the $K$-linear map $e(m ; w)$ to $F_w(v_1) \otimes K$ is an operator
\[
e(v_1, m ; w) \in \text{Hom}_{k[w]}(F_w(v_1), F_w(v)) \otimes K.
\]

For each composition $\nu$ of $v_2$, the fundamental class of the Hecke correspondence $h[v_1, \Lambda_\nu ; w]$ can be viewed as a $k[w]$-linear operator $F_w(v_1) \to F_w(v)$ by convolution.

Lemma 4.7. We have the following equality in $\text{Hom}_{k[w]}(F_w(v_1), F_w(v)) \otimes K$
\[
e(v_1, m ; w) = \sum_{\nu \vdash v_2} a_\nu(v, w) [h[v_1, \Lambda_\nu, w]], \quad a_\nu(v, w) \in \mathbb{Q}.
\]

Proof. Under the Kunneth isomorphism, the maps $\phi$ and $\psi$ give linear maps
\[
\phi : H^G_{\ast}(\mathfrak{M}[v, 0 ; v_1, v_2]) \to H^G_{\ast-d_{v_2, \delta_i}}(\mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w) \times \mathcal{C}_0),
\]
\[
\psi : H^G_{\ast}(\mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w) \times \mathcal{C}_0) \otimes K \to H^G_{\ast}(\mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w)) \otimes K.
\]

By Proposition 3.1 the class $\phi(r[v, 0 ; v_1, v_2])$ is supported on the variety $gh[v_1, v_2 ; w, \mathcal{C}_0]$. The projection along the third component $\mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w) \times \mathcal{C}_0 \to \mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w)$ gives an isomorphism
\[
gh[v_1, v_2 ; w, \mathcal{C}_0] = gh[v_1, \Lambda_1^T(v_2) ; w, \mathcal{C}_0] \simeq h[v_1, \Lambda_1(v_2) ; w] = [h[v_1, \Lambda_\nu; w, \mathcal{C}_0] ; \nu \vdash v_2].
\]

By (2.2) and Remark 3.6, the set of irreducible components of $\mathfrak{g}h[v_1, v_2 ; w, \mathcal{C}_0]$ is
\[
\{gh[v_1, \Lambda_\nu; w, \mathcal{C}_0] ; \nu \vdash v_2 \}. 
\]
Further, the variety \( \mathfrak{g}h[v_1, v_2; w, \mathcal{C}_0] \) is pure dimensional of dimension
\[
(mid - d_{v_2, \delta_i})/2 = (d_{v, w} + d_{v_1, w})/2.
\]
Since the cycle \( r[v, 0; v_1, v_2] \) has the degree equal to \( mid \), the class \( \phi(r[v, 0; v_1, v_2]) \) has the degree equal to \( mid - d_{v_2, \delta_i} \). So, there are rational numbers \( a_\nu(v, w) \) such that
\[
\phi(r[v, 0; v_1, v_2]) = \sum_{\nu \models v_2} a_\nu(v, w) [\mathfrak{g}h[v_1, \Lambda_\nu; w, \mathcal{C}_0]],
\]
(4.22)

hence we have
\[
\psi\phi(r[v, 0; v_1, v_2]) = \sum_{\nu \models v_2} a_\nu(v, w) [h[v_1, \Lambda_\nu; w]].
\]

Now, Proposition 4.2 in the case \( q_i > 1 \) follows by induction from [21, prop. 3.22] and the next proposition.

**Proposition 4.8.** We have

(a) \( a_{\nu_2}(v, w) \neq 0 \) for each \( v, w \),
(b) \( a_\nu(v, w) \) does not depend on \( v, w \) for each \( \nu \).

Finally, assume that \( q_i \leq 1 \). By Proposition 3.4, we may assume that \( l = 1 \). We have
\[
\mathcal{C}(v_1, v_2; w) = h[v_1, \Lambda_{\nu_2}; w].
\]
Hence, we can define the element \( m \) as in (4.21). Then, the same argument as in Lemma 4.7 and Proposition 4.8 yields
\[
e(v_1, m; w) = [\mathcal{C}(v_1, v_2; w)].
\]
So the convolution with the correspondence \( \mathcal{C}_{i, 1} \) belongs to \( e(A_{\delta_1}^i(\delta_2)) \). Proposition 4.2 is proved.

4.3.2. **Proof of Proposition 4.8(a).** Write
\[
\mathcal{C}^o = \{[\bar{x}, \bar{a}] \in \mathcal{C} : \bar{x} \in S(v_2)\}, \quad \mathcal{C}^o_0 = \pi(\mathcal{C}^o).
\]
Note that we have
\[
\mathfrak{g}h[v_1, S(v_2); w, \mathcal{C}] = \mathfrak{g}h[v_1, v_2; w, \mathcal{C}^o].
\]
Hence, we have the following fiber diagram
\[
\begin{array}{c}
\mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w) \times \mathcal{C}_0 \xrightarrow{i} \mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w) \times \mathcal{C} \xrightarrow{\iota} \mathfrak{M}[v, 0; v_1, v_2] \\
\mathfrak{g}h[v_1, v_2; w, \mathcal{C}] \xrightarrow{\beta} \mathfrak{g}h[v_1, v_2] \\
\mathfrak{M}(v, w) \times \mathfrak{M}(v_1, w) \times \mathcal{C}^o_0 \xrightarrow{\pi} \mathfrak{g}h[v_1, v_2; w, \mathcal{C}^o] \xrightarrow{f} \mathfrak{g}h[v_1, S(v_2)].
\end{array}
\]
Let \( g^! \) and \( f^! \) denote the refined pullback morphisms \((i, g)^!\) and \((i, f)^!\). See [21] §2.3.5 for more details. Since the cycle \( r[v; 0; v_1, v_2] \) is supported on \( gh[v_1, v_2] \), the proper base change implies that

\[
(i^* \alpha_*(r[v; 0; v_1, v_2])) = \alpha_*(g^!(r[v; 0; v_1, v_2])),
\]

from which we deduce that

\[
\phi(r[v; 0; v_1, v_2]) = \pi_* \alpha_*(g^!(r[v; 0; v_1, v_2])).
\]

So, the restriction of the class \( \phi(r[v; 0; v_1, v_2]) \) by the open embedding \( \mathcal{M}(v, w) \times \mathcal{M}(v_1, w) \times C^0 \subset \mathcal{M}(v, w) \times \mathcal{M}(v_1, w) \times C^0 \) is the pushforward by \( \pi \) of the class

\[
\beta_!^* g^!(r[v; 0; v_1, v_2]) = f^! \beta_!^* (r[v; 0; v_1, v_2]).
\]

First, let us compute the later. The map \( i \) is a regular embedding of codimension \( d_{v_2, \delta_i}/2 \). Consider the open subset \( gh[v_1, v_2] \) introduced in (4.19), and set \( gh[v_1, \Lambda_{(v_2)}; w, C^0] \) as above, we get a Cartesian square

\[
\begin{array}{ccc}
gh[v_1, \Lambda_{(v_2)}; w, C^0] & \xrightarrow{f} & gh[v_1, v_2]^0 \\
\downarrow & & \downarrow \\
\mathcal{M}(v_1, w_1) \times C^0 & \xrightarrow{\tau_{v_2}} & \mathcal{M}(v_1, w) \times \mathcal{M}(\tau_{v_2}),
\end{array}
\]

where the vertical maps are smooth and \( \tau_{v_2} \in RT(v_2, \delta_i) \) is the representation type \((1, 0, \delta_i; 1, v_2)\). We have

\[
\dim C^0 = d_{v_2, \delta_i}/2 = (q_i - 1)^2 + l, \quad \dim \mathcal{M}(\tau_{v_2}) = 2(q_i - 1)^2 + l + 1.
\]

So, up to restricting the map \( f \) to the open subsets above, we may assume that it is a regular embedding of codimension \( d_{v_2, \delta_i}/2 - l + 1 \) and the excess intersection formula yields

\[
f^! \beta_!^* (r[v; 0; v_1, v_2]) = eu(N) \cap f^! \beta_!^* (r[v; 0; v_1, v_2])
\]

for some vector bundle \( N \) of rank \( l - 1 \). Note that \( \Lambda_{v} \cap S(v_2) = \emptyset \) for each composition \( \nu \neq (v_2) \). Therefore, we have

\[
(4.23) \quad gh[v_1, \Lambda_{v}; w, C^0] = \emptyset, \quad \forall \nu \neq (v_2).
\]

We deduce that

\[
(4.24) \quad f^! \beta_!^* (r[v; 0; v_1, v_2]) = eu(N) \cap [gh[v_1, \Lambda_{(v_2)}; w, C^0]].
\]

Next, the argument in the proof of Proposition 4.4 implies that there is an isomorphism \( C^0 = C_0^0 \times \mathbb{P}^{l-1} \) which identifies the map \( \pi : C^0 \to C_0^0 \) with the first projection. We deduce that the pushforward of the class \( [4.23] \) by the map \( \pi \) is a nonzero multiple of the fundamental class of \( gh[v_1, \Lambda_{(v_2)}; w, C_0^0] \).
Now, from (4.23) we get that
\[ \mathfrak{gh}[v_1, \Lambda_{\nu}; w, \mathcal{C}_0^\circ] = \emptyset, \quad \forall \nu \neq (v_2). \]
Thus, by (4.22) the restriction of the class \( \phi([v, 0; v_1, v_2]) \) to \( \mathcal{M}(v, w) \times \mathcal{M}(v_1, w) \times \mathcal{C}_0^\circ \) is
\[ a_{(v_2)}(v, w)[\mathfrak{gh}[v_1, \Lambda_{(v_2)}; w, \mathcal{C}_0^\circ]]. \]
We deduce that \( a_{(v_2)}(v, w) \neq 0. \)

4.3.3. Proof of Proposition 4.8 (b), step 1. Assume that \( v_1 = 0 \) and \( v_2 = v \). Fix dimension vectors \( w_1 \) and \( w_0 \) with \( w_1 \geq w_0 \) and set \( N = v \cdot (w_1 - w_0) \). Let us prove that
\[ a_{v}(v, w_1) = a_{w}(v, w_0), \quad \forall \nu \vdash v. \]
For each \( \epsilon = 0, 1 \) we write \( G_{\epsilon} = G(w_\epsilon) \times G(\delta_1) \times T \) and
\[
\mathcal{L}_{\epsilon, \Lambda^1} = \mathcal{L}[0, v; w_\epsilon, \delta_1]_{\Lambda^1}, \quad \mathcal{Z}_{\epsilon, \Lambda^1} = \mathcal{Z}[0, v; w_\epsilon, \delta_1]_{\Lambda^1}, \\
h_{\epsilon, \Lambda^1} = h[0, v; w_\epsilon]_{\Lambda^1}, \quad \mathfrak{gh}_{\epsilon, \Lambda^1} = \mathfrak{gh}[0, v; w_\epsilon, \delta_1]_{\Lambda^1}, \\
p_{\epsilon, \Lambda^1} = p[0, v; w_\epsilon, \delta_1]_{\Lambda^1}.
\]
Hence, we have
\[
p_{\epsilon, \Lambda^1} = P_{\epsilon, \Lambda^1} \parallel G(v), \\
P_{\epsilon, \Lambda^1} = \{ (\bar{x}, \bar{a}) \in M_\epsilon(v, w_\epsilon + \delta_1)_{\Lambda^1} ; a(W_\epsilon) = 0 \}.
\]
Fix a surjective morphism of \( I \)-graded vector spaces
\[ p : W_1 \to W_0. \]
The composition with \( p \) yields an \( \Lambda^N \)-torsor
\[ p : P_{1, \Lambda^1} \to M(v, w_0 + \delta_1)_{\Lambda^1}. \]
Consider the open subset of \( p_{1, \Lambda^1} \) given by
\[ p_{1, \Lambda^1}^\circ = p^{-1}(M_\epsilon(v, w_0 + \delta_1)_{\Lambda^1} \parallel G(v)). \]
Then, the map \( p \) yields an \( \Lambda^N \)-torsor
\[ p : p_{0, \Lambda^1}^\circ \to p_{0, \Lambda^1}. \]
The map (4.24) gives an inclusion \( \mathcal{L}_{1, \Lambda^1} \subseteq p_{0, \Lambda^1}^\circ \). Since
\[ Z_{\epsilon, \mathcal{G}_m} = \mathcal{L}_{\epsilon, \mathcal{G}_m} \times_{\mathcal{M}(v, \delta_1)_{\mathcal{G}_m}} \mathcal{M}(v, \delta_1)_{\mathcal{G}_m}, \]
we get the following fiber diagram
\[
\begin{array}{ccc}
Z_{1, \mathcal{G}_m} & \rightarrow & p_{0, \Lambda^1}^\circ \times \mathcal{M}(v, \delta_1)_{\Lambda^1} \\
\downarrow p & & \downarrow p \\
Z_{0, \mathcal{G}_m} & \rightarrow & p_{0, \Lambda^1} \times \mathcal{M}(v, \delta_1)_{\Lambda^1} \\
\end{array}
\]
where the vertical maps are \( \Lambda^N \)-torsors and \( p_{\epsilon}, p_{\epsilon}^\circ \) are the fibers at 0.
Now, we have the following general fact.

Lemma 4.9. Let $p : X_{\mathbb{A}^1} \to Y_{\mathbb{A}^1}$ be a $T$-equivariant smooth morphism of $\mathbb{A}^1$-schemes which are $T$-equivariant locally trivial fibrations over $\mathbb{G}_m$. Let $X, Y$ be the fibers at 0. For any $T$-equivariant cycle $Z$ in $Y_{\mathbb{G}_m}$ we have the equality of $T$ equivariant cycles in $X$

$$p^* \lim_0 Z = \lim_0 p^*(Z)$$

Proof. Set $Z = \sum_i n_i [Z_i]$ where $Z_i$ is a $T$-invariant closed subvariety of $Y_{\mathbb{G}_m}$ and $n_i \in \mathbb{Q}$. Then, taking the Zarisky closure in $Y_{\mathbb{A}^1}$ and in $X_{\mathbb{A}^1}$ we get

$Z = \sum_i n_i [Z_i], \quad p^*(Z) = \sum_i n_i [p^{-1}(Z_i)].$

We have

$$p^* \lim_0 Z = p^* i^*(Z), \quad \lim_0 p^*(Z) = j^* p^*(Z),$$

where $i, j$ are the regular embeddings $Y \subset Y_{\mathbb{A}^1}$ and $X \subset X_{\mathbb{A}^1}$. The functoriality of Gysin morphisms with respect to smooth pull back yields $p^* i^* = j^* p^*$. Thus, the claim follows from the smoothness of $p$ and the equality $p^*(Z) = p^*(\overline{Z})$, see [9, thm. 2.3.10].

Let us abbreviate

$$\text{stab}_\epsilon = \text{stab}[0, v ; w_\epsilon, \delta_i], \quad r_\epsilon = r[v, 0; 0, v ; w_\epsilon, \delta_i].$$

From (3.20) we deduce that

(4.27) \hspace{1cm} \text{stab}_\epsilon = \pm \lim_0 [\mathfrak{M}_{v, \mathbb{G}_m}].

It is a $G_\epsilon$-equivariant cycle supported on the subset of $\mathfrak{M}(v, w_\epsilon + \delta_i) \times \mathfrak{M}(v, \delta_i)$ given by

$$\mathfrak{g}_\epsilon := p_\epsilon \times_{\mathfrak{M}(v, \delta_i)} \mathfrak{M}(v, \delta_i).$$

Let $\text{stab}_1|_{\mathfrak{g}_\epsilon}$ be the restriction of $\text{stab}_1$ to the open subset

$$\mathfrak{g}_\epsilon := p_\epsilon \times_{\mathfrak{M}(v, \delta_i)} \mathfrak{M}(v, \delta_i).$$

By Lemma 4.9 we have

(4.28) \hspace{1cm} \text{stab}_1|_{\mathfrak{g}_\epsilon} = p^*(\text{stab}_0).

Now, let us consider the $G(w_\epsilon) \times T$-equivariant cycle $r_\epsilon$. Let

$$i_\epsilon : \mathfrak{M}(v, w_\epsilon) \times \mathfrak{M}(v, \delta_i) \to \mathfrak{M}(v, w_\epsilon + \delta_i) \times \mathfrak{M}(v, \delta_i)$$
be the obvious embedding. Recall that \( \text{res}_\epsilon \) is the residue along \( i_\epsilon \) of the cycle \( \text{stab}_\epsilon \). By (3.19) we have the following fiber diagram

\[
\begin{array}{c}
\mathcal{M}(v, w_1) \times \mathcal{M}(v, \delta_i) \xrightarrow{\iota_1} \mathcal{M}(v, w_1 + \delta_i) \times \mathcal{M}(v, \delta_i) \\
\downarrow^p \downarrow^p \\
\mathfrak{g}_0 \circ \mathcal{M}(v, w_1) \times \mathcal{M}(v, \delta_i) \xrightarrow{\iota_2} \mathcal{M}(v, w_1 + \delta_i) \times \mathcal{M}(v, \delta_i) \xrightarrow{\iota_3} \mathcal{M}(v, w_1 + \delta_i) \times \mathcal{M}(v, \delta_i) \\
\end{array}
\]

(4.29)

The cycle \( r_\epsilon \) is supported on \( \mathfrak{g}_\epsilon \) and it is characterized by the following relation

\[
(i_\epsilon)^*(\text{stab}_\epsilon) = e\mu \cap r_\epsilon \text{ modulo } H^*_i \mathcal{M}(v, w_1) \times T(\mathfrak{g}_\epsilon),
\]

where \( e\mu \) is a square root of the Euler class of the normal bundle to \( i_\epsilon \). See (3.2) for more details. Since the map \( p \) is smooth, using this characterization and the formula (4.28), we deduce that the restriction \( r_1|_{\mathfrak{g}_1} \) of \( r_1 \) to the open subset \( \mathfrak{g}_1 \) of \( \mathfrak{g}_0 \) above is given by

\[
r_1|_{\mathfrak{g}_1} = p^*(r_0).
\]

Note that \( r_1 \) is a \( G(w_1) \times T \)-cycle while \( r_0 \) is \( G(w_0) \times T \)-equivariant. Hence we must first apply to \( r_1 \) the functoriality of equivariant cohomology relatively to any embedding \( G(w_0) \subset G(w_1) \) which is compatible with the flag \( W_1 \to W_0 \) in (4.26).

Finally, by Lemma 4.7 the class \( e(m, v, w_\epsilon) \) is supported on the closed subset \( \mathfrak{h}_\epsilon \) of \( \mathcal{M}(v, w_\epsilon) \) and it decomposes as the sum

\[
e(m, v, w_\epsilon) = \sum_{\nu \vdash v} a_\nu(m, v, w_\epsilon) \left[ \mathfrak{h}[0, \Lambda_\nu; w_\epsilon] \right].
\]

From (3.19) we may consider the open subset \( \mathfrak{h}_1^* = \mathfrak{h}_1 \cap \mathfrak{p}_1^* \) of \( \mathfrak{h}_1 \). From (4.30) we deduce that the restriction of \( e(m, v, w_\epsilon) \) to \( \mathfrak{h}_1^* \) is equal to \( p^*(e(m, v, w_0)) \). This implies the claim (4.25), because for all composition \( \nu \) of \( v \) we have

\[
\mathfrak{h}_1^* \cap \mathfrak{h}_1[0, \Lambda_\nu; w_1] \neq \emptyset.
\]

4.3.4. Proof of Proposition 4.3.4, step 2. Fix dimension vectors \( v, w \). Assume that \( v = v_1 + v_2 \) with \( v_2 = l \delta_i \). Let us prove that

\[
a_\nu(v, w) = a_\nu(v_2, w), \quad \forall \nu \vdash v_2.
\]

Set \( v_0 = 0 \). For each \( \epsilon = 0, 1 \) we write

\[
\begin{align*}
\mathcal{L}_{\epsilon, \mathfrak{h}^1} &= \mathcal{L} [v_\epsilon, v_2; w, \delta_i]_{\mathfrak{h}^1}, \\
\mathfrak{h}_\epsilon &= \mathfrak{h} [v_\epsilon, v_2; w], \\
p_{\epsilon, \mathfrak{h}^1} &= p [v_\epsilon, v_2; w, \delta_i]_{\mathfrak{h}^1}, \\
P_{\epsilon, \mathfrak{h}^1} &= P [v_\epsilon, v_2; w, \delta_i]_{\mathfrak{h}^1}.
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}_{\epsilon, \mathfrak{h}^1} &= \mathcal{L} [v_\epsilon, v_2; w, \delta_i]_{\mathfrak{h}^1}, \\
\mathfrak{h}_\epsilon &= \mathfrak{h} [v_\epsilon, v_2; w], \\
p_{\epsilon, \mathfrak{h}^1} &= p [v_\epsilon, v_2; w, \delta_i]_{\mathfrak{h}^1}, \\
P_{\epsilon, \mathfrak{h}^1} &= P [v_\epsilon, v_2; w, \delta_i]_{\mathfrak{h}^1}.
\end{align*}
\]
Fix $V_1, V_2, W$ and $P$ as in the previous sections. Set
\[ P'_{0, A^1} = \{ z \in M(v_2, w + \delta_1); z(W) = 0 \} = M_0(v_2, \delta_1)_{A^1} \times \text{Hom}_I(V_2, W). \]
Then, we have $P_{0, A^1} = P'_{0, A^1} \cap M_s(v_2, w + \delta_1)_{A^1}$. There is an obvious map
\[ p : P_{1, A^1} \to P'_{0, A^1}. \]
We define
\[ p^\vee_{1, A^1} = p^{-1}(P_{0, A^1}) \parallel P. \]
Since $p_{0, A^1}$ is the categorical quotient of $P_{0, A^1}$ by $G(v_2)$, the map $p$ factors to a map
\[ p : p^\vee_{1, A^1} \to p_{0, A^1}. \]
This map may not be smooth. To remedy this, we will restrict it to a suitable open subset of $p^\vee_{1, A^1}$. For each representations $z_1 = (\overline{\bar{x}}, \overline{\bar{a}})$ and $z_2 = (\overline{\bar{x}}, \overline{\bar{a}})$ we define
\[ U(z_1, z_2) = \{ \varphi \in \text{Hom}_k Q(\overline{\bar{x}}, \overline{\bar{a}}); a_2^* \circ \varphi = 0 \}, \]
\[ U = \{ (z_1, z_2) \in \mathcal{M}(v_1, w)_{A^1} \times_A (M(v_2, \delta_1)_{A^1} / G(v_2)); U(z_1, z_2) = 0 \}. \]
So $U$ is non-empty and open. Consider the obvious map
\[ q = (q_1, q_2) : p_{1, A^1} \to \mathcal{M}(v_1, w)_{A^1} \times_A (M(v_2, \delta_1)_{A^1} / G(v_2)); z \mapsto (z_1, z_2). \]
Then, we have a non-empty open subset of $p^\vee_{1, A^1}$ given by
\[ p^\vee_{1, A^1} = \{ z \in p^\vee_{1, A^1}; q(z) \in U \}. \]
Let $p_0, p_1$ be the fibers at 0 of $p^\vee_{1, A^1}, p_{1, A^1}$. Note that $p^\vee_1$ intersects all strata $\mathfrak{h}(v_1, A_v; w)$ in $p_1$. Indeed, for a generic element $z \in \mathfrak{h}(v_1, A_v; w)$ we have $\text{Hom}_k Q(\overline{\bar{x}}, \overline{\bar{a}}) = 0$, where $z_1 = (\overline{\bar{x}}, \overline{\bar{a}}) = z|_{V_1 \oplus W}$ and $(\overline{\bar{x}}, \overline{\bar{a}})$ is the endomorphism of $V/V_1$ induced by $z$.

**Lemma 4.10.**

(a) The obvious inclusion $\mathfrak{L}_{1, A^1} \subset p_{1, A^1}$ maps into $p^\vee_{1, A^1}$.

(b) The map $p : p^\vee_{1, A^1} \to p_{0, A^1}$ is smooth.

**Proof.** The inclusion $\mathfrak{L}_{1, A^1} \subset p^\vee_{1, A^1}$ is obvious. Next, note that for each $z \in p_{1, A^1}$ such that $z_2$ is stable, we have $q(z) \in U$. Indeed, the image of any $\varphi \in U(q(z))$ is a $\overline{\bar{x}}$-stable subspace of $V_2$ which is contained in $\text{Ker}(a_2^*)$. This implies the part (a). We now turn to (b). The map $p$ is the composition of the chain of morphisms
\[ p^\vee_{1, A^1} \xrightarrow{(q_1, p)} \mathcal{M}(v_1, w)_{A^1} \times_A p_{0, A^1} \xrightarrow{pr_2} p_{0, A^1}, \]
where $pr_2$ is the projection on the second factor. The map $pr_2$ is a smooth map because $\mathcal{M}(v_1, w)_{A^1} \to A^1$ is smooth. We claim that the restriction of $(q_1, p)$ to $p^\vee_{1, A^1}$ is an affine fibration over its image. Indeed, the fiber of $(q_1, p)$ over a pair
\[ ((\overline{\bar{x}}, \overline{\bar{a}}), (\overline{\bar{x}}, \overline{\bar{a}}, v)) \in \mathcal{M}(v_1, w)_{A^1} \times_A P'_{0, A^1} \]
is identified with the zero set of the affine map

\[ \mu : \bigoplus_{h \in \Omega} \text{Hom}(V_{2,h'}, V_{1,h''}) \oplus \text{Hom}(k, V_{1,i}) \to \text{Hom}(V_{2,i}, V_{1,i}) \]

defined by

\[ (y, u) \mapsto \sum_h (x_1 h y_h + y_h x_{2,h} - y_h x_{2, h} - x_1 h y_h) + ud_{2,i} + a_{1,i} v. \]

Identifying \( \text{Hom}(V_{2,j}, V_{1,i})^* \) with \( \text{Hom}(V_{1,i}, V_{2,i}) \) via the trace pairing, it is easy to see that

\[ \text{Im}(\mu - \mu(0)) \cap U(z_1, z_2) = 0. \]

It follows that the restriction of \((q_1, p)\) to \( p_{1,A^1}^o \) is an affine fibration over the open subset \( U' = \pi^{-1}(U) \) where

\[ \pi : \mathcal{M}(v_1, w)_{A^1} \times_{A^1} p_{0,A^1} \to \mathcal{M}(v_1, w)_{A^1} \times_{A^1} (M(v_2, \delta_i)_{A^1} / G(v_2)) \]

is the natural map. As a consequence the map \( p : p_{1,A^1}^o \to p_{0,A^1} \), being the composition of the affine fibration \((q_1, p) : p_{1,A^1}^o \to U' \), the open embedding \( U' \to \mathcal{M}(v_1, w)_{A^1} \times_{A^1} p_{0,A^1} \)

and the projection \( pr_2 : \mathcal{M}(v_1, w)_{A^1} \times_{A^1} p_{0,A^1} \to p_{0,A^1} \), is a smooth map. \( \square \)

We have the following fiber diagram

\[ \begin{array}{ccc}
\mathcal{Z}_{1,G_m} & \supset \mathcal{Z}_{0,G_m} & \\
\downarrow p & \downarrow p & \\
\mathcal{Z}_{1,G_m} & \supset \mathcal{Z}_{0,G_m} & \\
\end{array} \]

where the vertical maps are smooth and the left inclusions are given by (3.24). Write

\[ \text{stab}_\varepsilon = \text{stab}[v_\varepsilon, v_2; w, \delta_i], \quad \text{r}_\varepsilon = \text{r}[v, 0; v_\varepsilon, v_2; w, \delta_i]. \]

Consider the cycle

\[ \text{stab}_\varepsilon = \pm \lim_0 [\mathcal{Z}_{\varepsilon,G_m}] \]

which is supported on the set

\[ \text{gp}_\varepsilon = p_\varepsilon \times \mathcal{M}(v, \delta_i) \mathcal{M}(v, \delta_i). \]

From Lemma 4.9 we deduce that the restriction of \( \text{stab}_1 \) to the open subset

\[ \text{gp}_1^o := p_1^o \times \mathcal{M}(v_2, \delta_i) \mathcal{M}(v, \delta_i) \]

is equal to

\[ \text{stab}_1 |_{\text{gp}_1^o} = p^* (\text{stab}_0). \]

Taking the residue along the \( A \)-fixed points locus

\[ \mathcal{M}(v, w) \times \mathcal{M}(v_\varepsilon, w) \times \mathcal{M}(v_2, \delta_i) \subseteq \mathcal{M}(v, w + \delta_i) \times \mathcal{M}(v, w) \times \mathcal{M}(v_2, \delta_i) \]

of the cycle \( \text{stab}_\varepsilon \), we get the cycle \( \text{r}_\varepsilon \) which is supported on \( \text{gh}_\varepsilon \). Using (3.19) we define

\[ \text{gh}_1^o = \text{gh}_1 \cap \text{gp}_1^o. \]
Then, we deduce that the restriction of $r_1$ to $\mathfrak{g}h_1^\diamond$ is equal to $p^*(r_0)$.

Now, let $m \in F_i(v_2) \otimes K$ be as above. Then, we get

$$e(m, v, w) = \sum_{\nu \models v_2} a_\nu(v, w)[h_1[v_1, \Lambda_\nu; w]],$$

$$e(m, v_2, w) = \sum_{\nu \models v_2} a_\nu(v_2, w)[h_0[v_0, \Lambda_\nu; w]].$$

Therefore, the restriction of $e(m, v_2, w)$ to

$$h_1^\diamond := h_1 \cap p_1^\diamond,$$

see (3.19), is equal to $p^*(e(m, v_2, w))$. Since

$$h_1^\diamond \cap h_1[v_1, \Lambda_\nu; w] \neq \emptyset$$

for each composition $\nu$, claim (4.31) is proved. This finishes the proof of Proposition 4.8.
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