On the Lebesgue measure, Hausdorff dimension, and Fourier
dimension of sums and products of subsets of Euclidean space

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Abstract

We investigate the Lebesgue measure, Hausdorff dimension, and Fourier dimension
of sets of the form $R Y + Z$, where $R \subseteq (0, \infty)$ and $Y, Z \subseteq \mathbb{R}^d$. Most notable, for each
$\alpha \in [0, 1]$ and for each non-empty set $Y \subseteq \mathbb{R}^d$, we prove the existence of a compact set
$R \subseteq (0, \infty)$ such that $\dim_H(R) = \dim_F(R) = \alpha$ and $\dim_F(RY) \geq \min\{1, \dim_F(R) + \dim_F(Y)\}$. This work is a contribution to the set of problems which study the measure
and dimension of images of subsets of Euclidean space under Lipschitz maps.

1 Introduction

Notation

We use $\dim_H(\cdot), \dim_F(\cdot)$, and $\text{supp}(\cdot)$ to denote Hausdorff dimension, Fourier dimension,
and support (respectively) for sets and measures. Definitions and basic properties are in
Section 2. The expression $X \precsim Y$ means $X \leq CY$ for some positive constant $C$ whose
precise value is irrelevant in the context. The expression $X \approx Y$ means $X \precsim Y$ and $Y \precsim X$.

Motivation and Results

In this paper, we study the Lebesgue measure, Hausdorff dimension, and Fourier dimension
of sets in $\mathbb{R}^d$ of the form

$$RY + Z = \bigcup_{(r,z) \in R \times Z} (rY + z),$$

where $R \subseteq (0, \infty)$ and $Y, Z \subseteq \mathbb{R}^d$ are non-empty sets. (We assume throughout that the
sets $R, Y, Z$ are compact in order to guarantee that the set $RY + Z$ is measurable).
Intuition suggests that the Hausdorff dimension of $RY + Z$ will be at least the “aggregate” dimension of $R$, $Y$, and $Z$, and (moreover) that $RY + Z$ will have positive Lebesgue measure whenever the “aggregate” dimension of $R$, $Y$, and $Z$ exceeds $d$.

This intuition comes from the well-studied case where $Y$ equals $S$, the unit sphere in $\mathbb{R}^d$, and we begin by discussing this case. (Note that $rS + z$ is the sphere with radius $r$ and center $z$.) It is well-known that $\dim_H(S) = \dim_F(S) = d - 1$ (see, for example, [11], [18]). Here are two trivial results about $RS + Z$:

(i) If $Z = \{z_0\}$, then $\dim_H(RS + Z) = \dim_H(R) + d - 1$.
(ii) If $Z$ contains a line segment, then $RS + Z$ has positive Lebesgue measure.

The following deep theorem is due to Wolff ([17], see Corollary 3) when $d = 2$ and Oberlin [13] when $d \geq 3$.

**Theorem 1.1** (Wolff, Oberlin). Let $K \subseteq (0, \infty) \times \mathbb{R}^d$ be a non-empty compact set. If $\dim_H(K) > 1$, then

$$\bigcup_{(r, z) \in K} (rS + z)$$

has positive Lebesgue measure.

Mitsis [12] previously obtained the special case of Theorem 1.1 when $d \geq 3$ and the set of centers $\{z : (r, z) \in K\}$ is assumed to have dimension greater than 1. We note that an alternative proof of Mitsis’ result using the technology of spherical maximal operators in a fractal setting follows as an immediate consequence of the work of Krause, Iosevich, Sawyer, Taylor, and Uriarte-Tuero [9]. Simon and the Taylor [15] studied the critical case of Theorem 1.1 when $\dim_H(K) = 1$ and $d = 2$. In particular, they showed that if $Z$ is a 1-set in the plane, then $S + Z$ has zero measure if and only if $Z$ is irregular (for instance, it follows that $S + Z$ has zero measure when $Z$ is the four-corner Cantor set). In [16], Simon and Taylor considered the interior of sets of the form $S + Z$. The application of such results includes the study of pinned distance sets (see the description of the Falconer distance problem in [11]).

Our first theorem is an analog of Theorem 1.1 where the unit sphere $S$ is replaced by an arbitrary set $Y$. It conforms with the intuition that $RY + Z$ will have positive Lebesgue measure whenever the “aggregate” dimension of $R$, $Y$, and $Z$ exceeds $d$.

**Theorem 1.2.** Let $R \subseteq (0, \infty)$ and $Y, Z \subseteq \mathbb{R}^d$ be non-empty compact sets. If

$$\max \{\dim_F(RY) + \dim_H(Z), \dim_H(RY) + \dim_F(Z)\} > d,$$
then
\[ RY + Z = \bigcup_{(r,z) \in R \times Z} (rY + z) \]

has positive Lebesgue measure.

In case \( Y = S \), Theorem 1.1 implies Theorem 1.2. Indeed, if \( Y = S \) and the hypotheses of Theorem 1.2 hold, then the hypotheses of Theorem 1.1 hold. In case \( Y = S \) and \( K = R \times Z \), we do not know if Theorem 1.2 implies Theorem 1.1. We suspect it does not. We intend to resolve this in future work.

The proof of Theorem 1.2 is actually much easier than the proof of Theorem 1.1. Wolff obtained Theorem 1.1 for \( d = 2 \) as a corollary of a localized \( L^p \) estimate on functions with Fourier support near the light cone, the proof of which involves delicate bounds on circle tangencies. Oberlin and Mitsis obtained their versions of Theorem 1.1 for \( d \geq 3 \) by way of estimates for spherical averaging operators. In contrast, our proof of Theorem 1.2 is extremely short and uses only elementary Fourier analysis and geometric measure theory.

It should be noted that Theorem 1.1 can be generalized to any appropriately curved \((d-1)\)-dimensional surface [7]. However, there is no hope of replacing the sphere in Theorem 1.1 by an arbitrary \((d-1)\)-dimensional set, as the following example shows.

**Example 1.3.** Let \( Y \) be the intersection of ball centered at the origin and a cone through the origin. Then \( RY \) is of the same form whenever \( R \) contains a non-zero point. Let \( R \) be a compact interval, let \( Z \subseteq \mathbb{R}^d \) be a compact set with \( \dim_H(Z) \in (0,1) \), and let \( K = R \times Z \). Then \( \dim_H(K) \geq \dim_H(R) + \dim_H(Z) > 1 \), but \( \bigcup_{(r,z) \in K} (rY + z) = RY + Z \) has Hausdorff dimension \( d - 1 + \dim_H(Z) \), hence has Lebesgue measure 0.

We turn now from Lebesgue measure to Hausdorff dimension. When \( Y \) equals \( S \), the unit sphere in \( \mathbb{R}^d \), Oberlin [14] proved the following theorem.

**Theorem 1.4 (Oberlin).** Let \( K \subseteq (0, \infty) \times \mathbb{R}^d \) be non-empty compact sets. If \( \dim_H(K) \in [0, (d-1)/2) \), then
\[
\dim_H \left( \bigcup_{(r,z) \in K} (rS + z) \right) \geq \dim_H(K) + \dim_H(S).
\]

When \( d = 2 \), Theorem 1.4 covers only the range \( \dim_H(K) \in [0, 1/2) \). When \( d = 3 \), Theorem 1.4 misses the endpoint \( \dim_H(K) = 1 \). In dimension \( d = 2 \), the following theorem (which Oberlin [14] attributes to Wolff [17]) says the desired conclusion is obtained in the range \( \dim_H(K) \in [0, 1) \) under an additional hypothesis.
Theorem 1.5 (Wolff). Let $K \subseteq (0, \infty) \times \mathbb{R}^2$ be non-empty compact sets. If $\dim_H(\{z : (r, z) \in K\}) \in [0, 1)$, then
\[
\dim_H \left( \bigcup_{(r, z) \in K} (rS + z) \right) \geq \dim_H(K) + \dim_H(S).
\]

Our second theorem is the Hausdorff dimension version of Theorem 1.2. It conforms with the intuition that the Hausdorff dimension of $RY + Z$ will be at least the “aggregate” dimension of $R$, $Y$, and $Z$.

Theorem 1.6. Let $R \subseteq (0, \infty)$ and $Y, Z \subseteq \mathbb{R}^d$ denote non-empty compact sets. If
\[
\max \{\dim_F(RY) + \dim_H(Z), \dim_H(RY) + \dim_F(Z)\} \leq d,
\]
then
\[
\dim_H(RY + Z) \geq \max \{\dim_F(RY) + \dim_H(Z), \dim_H(RY) + \dim_F(Z)\}.
\]

In case $Y = S$ and $d \geq 3$, Theorem 1.4 implies Theorem 1.6. We do not know if the converse is true, but we suspect not, and we intend to resolve this in future work.

Theorem 1.6 applies to the case $Y = S$ and $d = 2$ in situations not covered by Theorem 1.4 or Theorem 1.5. However, when $Y = S$, $d = 2$, $K = R \times Z$, and $\dim_H(\{z : (r, z) \in K\}) = \dim_H(Z) \in [0, 1)$, Theorem 1.5 implies Theorem 1.6. Again, we do not know if the converse is true, but we suspect not, and we intend to resolve this in future work.

As with the positive Lebesgue measure results, the proof of Theorem 1.6 is much simpler than the proofs of Theorems 1.4 and 1.5. Indeed, the proof of Theorem 1.6 is nearly identical to that of Theorem 1.2.

While Theorem 1.2 applies to arbitrary sets $Y$, it should be noted that Theorem 1.4 and Theorem 1.5 can be generalized to any appropriately curved $(d - 1)$-dimensional surface [7]. However, it is not possible to replace the sphere in Theorem 1.4 or Theorem 1.5 by an arbitrary $(d - 1)$-dimensional set. An appropriate modification of Example 1.3 illustrates this.

To apply Theorem 1.2 or Theorem 1.6, it is useful to have a convenient lower bound for $\dim_F(RY)$. We record two basic lower bounds here.

Lemma 1.7. If $R \subseteq \mathbb{R}$ contains a non-zero point and $Y \subseteq \mathbb{R}^d$, then $RY$ contains a dilate of $Y$, hence $\dim_F(RY) \geq \dim_F(Y)$. 


Lemma 1.8. If $R$ and $Y$ are compact subsets of $(0, \infty)$, then $\dim_F(RY) \geq \dim_H(R) + \dim_H(Y) - 1$.

The proof of Lemma 1.7 is trivial. Lemma 1.8 can be proved by straightforwardly adapting the proof of Theorem 7 of Bourgain [1].

We now consider stronger lower bounds on $\dim_F(RY)$. In general, it is not true that $\dim_F(RY) \geq \min\{d, \dim_F(R) + \dim_F(Y)\}$.

Example 1.9. Let $R \subseteq \mathbb{R}$ be a compact set with $\dim_F(R) > 0$, and let $Y$ be a compact subset of a $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$ (i.e., of a hyperplane through the origin). Then $RY$ is of the same form as $Y$, and so $\dim_F(RY) = \dim_F(Y) = 0 < \dim_F(R)$.

However, if $Y$ is the unit sphere $S$ in $\mathbb{R}^d$, we have the following theorem.

Theorem 1.10. For every non-empty compact set $R \subseteq (0, \infty)$, we have $\dim_F(RS) \geq \dim_F(R) + \dim_F(S)$.

Note that Theorem 1.10 can readily be extended from the sphere to other smooth hypersurfaces satisfying appropriate curvature conditions.

We wonder if there are other sets in $\mathbb{R}^d$ like the sphere. More precisely, we wonder if the following conjecture is true.

Conjecture 1.11. For every $\beta \in [0, d]$, there exists a compact set $Y \subseteq \mathbb{R}^d$ such that $\dim_H(Y) = \dim_F(Y) = \beta$ and such that the following property holds: For every non-empty compact set $R \subseteq (0, \infty)$, we have $\dim_F(RY) \geq \min\{d, \dim_F(R) + \dim_F(Y)\}$.

We can also consider the following dual form of Conjecture 1.11.

Conjecture 1.12. For every $\alpha \in [0, 1]$, there exists a compact set $R \subseteq (0, \infty)$ such that $\dim_H(R) = \dim_F(R) = \alpha$ and such that the following property holds: For every compact set $Y \subseteq \mathbb{R}^d$ which contains a non-zero point, we have $\dim_F(RY) \geq \min\{d, \dim_F(R) + \dim_F(Y)\}$.

We are able to prove the following non-uniform version of this conjecture in one dimension. It is the last (and deepest) of our main results.

Theorem 1.13. For every $\alpha \in [0, 1]$ and every compact set $Y \subseteq \mathbb{R}$ which contains a non-zero point, there exists a compact set $R \subseteq (0, \infty)$ such that $\dim_H(R) = \dim_F(R) = \alpha$ and $\dim_F(RY) \geq \min\{1, \dim_F(R) + \dim_F(Y)\}$.
The proof of Theorem 1.13 is inspired by a construction of Salem sets due to Laba and Pramanik [10]. See also [4, 5, 6, 2] where generalizations of the Laba and Pramanik construction were used to show the sharpness of fractal Fourier restriction theorems.

As an immediate Corollary of Theorems 1.2 and 1.13, we have

**Corollary 1.14.** Suppose $Y, Z \subseteq \mathbb{R}$ are compact sets such that $Y$ contains a non-zero point and $\dim_F(Y) + \dim_H(Z) > 0$. For every $\alpha \in [0, 1]$ such that $\dim_F(Y) + \dim_H(Z) > 1 - \alpha$, there exists a compact set $R \subseteq (0, \infty)$ such that $\dim_H(R) = \dim_F(R) = \alpha$ and $RY + Z$ has positive Lebesgue measure.

The proofs of Theorem 1.10 and Theorem 1.13 use the following measure construction.

**Definition 1.15.** Given a finite Borel measure $\mu$ on $\mathbb{R}$ and a finite Borel measure $\nu$ on $\mathbb{R}^d$, we define the measure $\mu \cdot \nu$ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f(z) d(\mu \cdot \nu)(z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} f(ry) d\mu(r) d\nu(y).$$

Note $\mu \cdot \nu$ is a finite Borel measure with $(\mu \cdot \nu)(\mathbb{R}^d) = \mu(\mathbb{R}) \nu(\mathbb{R}^d)$ and $\text{supp}(\mu \cdot \nu) = \text{supp}(\mu) \text{supp}(\nu)$.

The proofs of Theorems 1.2, 1.6, 1.10, and 1.13 are given in Sections 3, 4, 5, and 6, respectively.

## 2 Hausdorff Measure, Hausdorff Dimension, and Fourier Dimension

In this section, we review the basics of Hausdorff measure, Hausdorff dimension, and Fourier dimension. As general references, see [3] and [11].

For each $s \geq 0$, the $s$-dimensional Hausdorff measure of a set $A \subseteq \mathbb{R}^d$ is defined to be

$$\mathcal{H}^s(A) = \sup_{\delta > 0} \left\{ \sum_{i=1}^{\infty} (\text{diam}(A_i))^s : A \subseteq \bigcup_{i=1}^{\infty} A_i, \text{diam}(A_i) < \delta \right\}.$$

Note that $\mathcal{H}^s$ is technically not a measure, but it is an outer measure. Note also that $\mathcal{H}^d$ is a constant multiple of the Lebesgue outer measure.

The support of a Borel measure $\mu$ on $\mathbb{R}^d$, denoted $\text{supp}(\mu)$, is defined to be the intersection of all closed sets $F$ with $\mu(\mathbb{R}^d \setminus F) = 0$. For each $A \subseteq \mathbb{R}^d$, let $\mathcal{M}(A)$ be the set of all non-zero finite Borel measures on $\mathbb{R}^d$ with compact support contained in $A$. 

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Lemma 2.1 (Frostman’s Lemma). Let $A$ be a Borel subset of $\mathbb{R}^d$ and let $s \geq 0$. Then $\mathcal{H}^s(A) > 0$ if and only if there exists $\mu \in \mathcal{M}(A)$ such that
\[
\mu(B(x, r)) \lesssim r^s \quad \text{for every open ball } B(x, r) \subseteq \mathbb{R}^d. \tag{2.1}
\]
For $s \geq 0$, the $s$-energy of a Borel measure $\mu$ on $\mathbb{R}^d$ is defined to be
\[
I_s(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^{-s} d\mu(x) d\mu(y).
\]

Lemma 2.2. Let $A$ be a Borel subset of $\mathbb{R}^d$ and let $s \geq 0$.

(a) If $\mu \in \mathcal{M}(A)$ and $I_s(\mu) < \infty$, then $\mathcal{H}^s(A) = \infty$.

(b) If $\mathcal{H}^s(A) > 0$, then there exists $\mu \in \mathcal{M}(A)$ such that $I_t(\mu) < \infty$ for all $t > 0$.

The Hausdorff dimension of a set $A \subseteq \mathbb{R}^d$ is defined to be
\[
\dim_H(A) = \sup \{ s \geq 0 : \mathcal{H}^s(A) > 0 \},
\]
and, if $A$ is a Borel set, we have the formulas
\[
\dim_H(A) = \sup \{ s \geq 0 : \text{ (2.1) for some } \mu \in \mathcal{M}(A) \}
\]
\[
= \sup \{ s \geq 0 : I_s(\mu) < \infty \text{ for some } \mu \in \mathcal{M}(A) \}.
\]

The Hausdorff dimension of a non-zero finite Borel measure $\mu$ on $\mathbb{R}^d$ is defined to be
\[
\dim_H(\mu) = \sup \left\{ s \geq 0 : \liminf_{r \to 0} \frac{\ln \mu(B(x, r))}{\ln r} \geq s \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d \right\},
\]
and we have the equality
\[
\dim_H(\mu) = \inf \left\{ \dim_H(A) : A \subseteq \mathbb{R}^d \text{ is Borel and } \mu(A) > 0 \right\}.
\]

Fourier analysis enters the picture via the following formula: For every $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $0 < s < d$,
\[
I_s(\mu) = c(d, s) \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{-d-s} d\xi,
\]
where $c(d, s)$ is a positive constant depending only on $d$ and $s$. The Fourier dimension of a set $A \subseteq \mathbb{R}^d$ is defined to be
\[
\dim_F(A) = \sup \left\{ 0 \leq s \leq d : \sup_{\xi \in \mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^s < \infty \text{ for some } \mu \in \mathcal{M}(A) \right\}.
\]
The Fourier dimension of a non-zero finite Borel measure \( \mu \) on \( \mathbb{R}^d \) is defined to be

\[
\dim_F(\mu) = \sup \left\{ 0 \leq s \leq d : \sup_{\xi \in \mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^s < \infty \right\}.
\]

For every Borel set \( A \subseteq \mathbb{R}^d \) and every non-zero finite Borel measure \( \mu \) on \( \mathbb{R}^d \), we have

\[
\dim_F(A) \leq \dim_H(A) \quad \text{and} \quad \dim_F(\mu) \leq \dim_H(\mu).
\]

### 3 Proof of Theorem 1.2

Assume \( \dim_F(RY) + \dim_H(Z) \geq \dim_F(RY) + \dim_F(Z) \). The proof in the opposite case is similar. Assume \( \dim_F(RY) + \dim_H(Z) > d \). Choose \( 0 \leq \alpha < \dim_F(RY) \) and \( 0 \leq \beta < \dim_H(Z) \) such that \( \alpha + \beta = d \). Choose Borel probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \) such that \( \text{supp}(\mu) \subseteq RY \), \( \text{supp}(\nu) \subseteq Z \), \( \sup_{\xi \in \mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^\alpha < \infty \), and \( I_\beta(\nu) < \infty \). Then \( \mu \ast \nu \) is a Borel probability measure with support contained in \( RY + Z \) and

\[
\int |\hat{\mu} \ast \nu(\xi)|^2 d\xi = \int |\hat{\mu}(\xi)|^2 |\hat{\nu}(\xi)|^2 d\xi \leq \int |\xi|^{(d-\alpha)-d} |\hat{\nu}(\xi)|^2 d\xi = I_{d-\alpha}(\nu) = I_\beta(\nu) < \infty.
\]

Since \( \mu \ast \nu \) is in \( L^2 \), \( \mu \ast \nu \) has an \( L^2 \) density with respect to Lebesgue measure. So the support of \( \mu \ast \nu \) has positive Lebesgue measure, hence \( RY + Z \) has positive Lebesgue measure.

### 4 Proof of Theorem 1.6

Assume \( \dim_F(RY) + \dim_H(Z) \geq \dim_F(RY) + \dim_F(Z) \). The proof in the opposite case is similar. Choose any \( s \in (0, d) \) such that \( \dim_F(RY) + \dim_H(Z) > s \). Choose \( 0 \leq \alpha < \dim_F(RY) \) and \( 0 \leq \beta < \dim_H(Z) \) such that \( \alpha + \beta = s \). Choose Borel probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \) such that \( \text{supp}(\mu) \subseteq RY \), \( \text{supp}(\nu) \subseteq Z \), \( \sup_{\xi \in \mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^\alpha < \infty \), and \( I_\beta(\nu) < \infty \). Then \( \mu \ast \nu \) is a Borel probability measure with support contained in \( RY + Z \).
and
\[
I_s(\mu * \nu) = \int |\hat{\mu} * \hat{\nu}(\xi)|^2 |\xi|^{-d} d\xi \\
= \int |\hat{\mu}(\xi)||\hat{\nu}(\xi)|^2 |\xi|^{-d} d\xi \\
\preceq \int |\xi|^{(s-a)-d} |\hat{\nu}(\xi)|^2 d\xi \\
= I_{s-a}(\nu) = I_\beta(\nu) < \infty.
\]
Since \( I_s(\mu * \nu) < \infty \), the support of \( \mu * \nu \) has infinite \( s \)-dimensional Hausdorff measure. Therefore \( RY + Z \) has infinite \( s \)-dimensional Hausdorff measure, hence \( \dim_H(RY + Z) \geq s \). By our choice of \( s \), we conclude that \( \dim_H(RY + Z) \geq \dim_F(RY) + \dim_H(Z) \).

5 Proof of Theorem 1.10

Let \( \sigma \) be the surface measure on the unit sphere \( S \subseteq \mathbb{R}^d \). The following asymptotic is well-known. It may be proved by elementary properties of the Fourier transform and known asymptotics of Bessel functions (see, for example, \([11]\)), or by the stationary phase method (see, for example, \([18]\)). For all sufficiently large \( \xi \in \mathbb{R}^d \),
\[
\hat{\sigma}(\xi) = 2|\xi|^{-(d-1)/2} \cos \left( 2\pi \left( |\xi| - \frac{d-1}{8} \right) \right) + O \left( |\xi|^{-(d+1)/2} \right) \\
= |\xi|^{-(d-1)/2} \left( c_d e^{2\pi i |\xi|} + \overline{c_d} e^{-2\pi i |\xi|} \right) + O \left( |\xi|^{-(d+1)/2} \right),
\]
where \( c_d = e^{-\pi i (d-1)/4} \).

Let \( R \subseteq (0, \infty) \). Let \( \mu \in \mathcal{M}(R) \) be arbitrary. Choose \( a, b > 0 \) such that \( \text{supp}(\mu) \subseteq [a, b] \subseteq (0, \infty) \). Define \( \mu_0 \) by \( d\mu_0(r) = r^{(d-1)/2} d\mu(r) \). Then \( \text{supp}(\mu_0) = \text{supp}(\mu) \), \( \mu_0 \in \mathcal{M}(R) \), and \( \mu_0 \cdot \sigma \in \mathcal{M}(RS) \). Furthermore, for all sufficiently large \( \xi \in \mathbb{R}^d \),
\[
\hat{\mu_0 - \sigma}(\xi) = \int_a^b \hat{\sigma}(r\xi) d\mu_0(r) \\
= |\xi|^{-(d-1)/2} \left( c_d \int_a^b r^{-(d-1)/2} e^{2\pi i r \xi} d\mu_0(r) + \overline{c_d} \int_a^b r^{-(d-1)/2} e^{-2\pi i r \xi} d\mu_0(r) \right) \\
+ O \left( |\xi|^{-(d+1)/2} \int_a^b r^{-(d+1)/2} d\mu_0(r) \right) \\
= |\xi|^{-(d-1)/2} \left( c_d \hat{\mu}(\xi) + \overline{c_d} \hat{\mu}(-\xi) \right) + O \left( |\xi|^{-(d+1)/2} \right).
\]
Therefore \( \dim_F(\mu_0 \cdot \sigma) \geq \dim_F(\mu) + d - 1 \) and (consequently) \( \dim_F(RS) \geq \dim_F(R) + d - 1 \).
6 Proof of Theorem 1.13

Theorem 1.13 follows immediately from the following theorem in terms of measures.

**Theorem 6.1.** For every $\alpha \in [0, 1]$ and let $\nu$ be a non-zero finite Borel measure on $\mathbb{R}$ with compact support not containing 0. There is a Borel probability measure $\mu$ on $\mathbb{R}$ such that $\text{supp}(\mu) \subseteq [1, 2]$, $\dim_H(\mu) = \dim_F(\mu) = \alpha$, and $\dim_F(\mu \cdot \nu) \geq \min \{1, \dim_F(\mu) + \dim_F(\nu)\}$.

The rest of this section is devoted to proving Theorem 6.1.

6.1 Proof of Theorem 6.1: General Construction

Let $\nu$ be as in the statement of the theorem. If $\alpha = 0$, then taking $\mu$ to be a point mass gives the desired result. Assume $\alpha \in (0, 1]$.

For every $n \in \mathbb{Z}_{>0}$, we use the notation $[n] = \{0, 1, \ldots, n - 1\}$ For sequences $(t_j)_{j=1}^\infty$ and $(n_j)_{j=1}^\infty$ of positive integers, we use the notation $T_j = t_1 \cdots t_j$ and $N_j = n_1 \cdots n_j$. We also use the empty product convention, so that $T_0 = N_0 = 1$.

Fix an integer $n_* \geq 2$. Fix sequences $(t_j)_{j=1}^\infty$ and $(n_j)_{j=1}^\infty$ of positive integers such that, for all $j \in \mathbb{Z}_{>0}$, we have $2 \leq n_j \leq n_*$, $t_j \leq n_j$, and $T_j \approx N_j^\alpha$.

We recursively define two families of sets: $\{A_j : j \in \mathbb{Z}_{\geq 0}\}$ and $\{B_{j+1,a} : j \in \mathbb{Z}_{\geq 0}, a \in A_j\}$. Define $A_0 = \{1\}$. Assuming that $A_j$ has been defined for a fixed $j \in \mathbb{Z}_{\geq 0}$, for each $a \in A_j$ choose a set $B_{j+1,a} \subseteq N_{j+1}^{-1}([n_{j+1}])$ such that $|B_{j+1,a}| = t_{j+1}$. Later we make a specific choice for the sets $B_{j+1,a}$, but for now the sets $B_{j+1,a}$ are arbitrary. Define $A_{j+1} = \bigcup_{a \in A_j} (a + B_{j+1,a})$. Note that this recursive definition implies that $A_j \subseteq [1, 2)$ and $|A_j| = T_j$ for all $j \in \mathbb{Z}_{\geq 0}$.

What are the sets $A_j$ and $B_{j+1,a}$? They are sets of endpoints in the following Cantor set construction. Start with the interval $[1, 2]$. Divide it into $n_1$ intervals of length $1/N_1$, keep $t_1$ of these intervals, and discard the rest. For each of the kept intervals, we do the following: Divide it into $n_2$ intervals of length $1/N_2$, keep $t_2$ of these intervals, and discard the rest. This gives, in total, $T_2$ intervals of length $1/N_2$. Continuing in this way, at the $j$-th stage we have $T_j$ intervals of length $1/N_j$. The set of left endpoints of these intervals is $A_j$. For each of these intervals, we do the following: Divide it into $n_{j+1}$ intervals of length $1/N_{j+1}$, keep $t_{j+1}$ of these intervals, and discard the rest. If $a$ is the left endpoint of the interval we started with, the set of left endpoints of the intervals kept is $a + B_{j+1,a}$. The union of all the sets $a + B_{j+1,a}$ (as $a$ ranges over $A_j$) is $A_{j+1}$. The Cantor set constructed
is
\[ \bigcap_{j=0}^{\infty} \bigcup_{a \in A_j} [a, a + N_j^{-1}] \]

For each \( j \in \mathbb{Z}_{\geq 0} \), define \( \mu_j \) to be the probability measure whose density with respect to Lebesgue measure on \( \mathbb{R} \) is
\[ \mu_j = \frac{N_j}{T_j} \sum_{a \in A_j} 1_{[a, a + N_j^{-1}]} \]

Note that we have abused notation by using the same symbol for a measure and its density; we will continue to do this. For each \( j \in \mathbb{Z}_{\geq 0} \),
\[ \text{supp}(\mu_j) = \bigcup_{a \in A_j} [a, a + N_j^{-1}] \] \hspace{1cm} (6.1)

For every \( j, k \in \mathbb{Z}_{\geq 0} \) with \( j \leq k \) and for every \( a \in A_j \), we easily verify that
\[ \mu_k([a, a + N_j^{-1}]) = T_j^{-1} \] \hspace{1cm} (6.2)

**Lemma 6.2.** The sequence \((\mu_j)_{j=0}^{\infty}\) converges weakly (i.e., in distribution) to a probability measure \( \mu \).

**Proof.** For each \( j \in \mathbb{Z}_{\geq 0} \), let \( F_j \) be the cumulative distribution function of \( \mu_j \), i.e., \( F_j(t) = \mu_j((-\infty, t]) \) for all \( t \in \mathbb{R} \). Let \( t \in \mathbb{R} \). If \( t \leq \min A_j \), then \( F_j(t) = F_{j+1}(t) = 0 \). Now assume \( t \geq \min A_j \). Let \( a(t) \) be the largest element of \( A_j \) such that \( a(t) \leq t \). Since \( F_j(a) = F_{j+1}(a) \) for each \( a \in A_j \), we have
\[ |F_{j+1}(t) - F_j(t)| = |\mu_{j+1}((a(t), t]) - \mu_j((a(t), t])| \]
\[ \leq \mu_{j+1}([a(t), a(t) + N_j^{-1}]) + \mu_j([a(t), a(t) + N_j^{-1}]) = \frac{2}{T_j} \]
where the last equality used (6.2). Since
\[ \sum_{j=0}^{\infty} T_j^{-1} \approx \sum_{j=0}^{\infty} N_j^{-\alpha} \leq \sum_{j=0}^{\infty} 2^{-j\alpha} < \infty, \]
it follows that \((F_j)_{j=0}^{\infty}\) is a uniformly convergent sequence of continuous cumulative distribution functions. Therefore the limit \( F \) is a continuous cumulative distribution function of a Borel probability measure \( \mu \) on \( \mathbb{R} \). Hence \((\mu_j)_{j=0}^{\infty}\) converges weakly to \( \mu \). \( \square \)

**Lemma 6.3.** The support of \( \mu \) is
\[ \text{supp}(\mu) = \bigcap_{j=0}^{\infty} \text{supp}(\mu_j) = \bigcap_{j=0}^{\infty} \bigcup_{a \in A_j} [a, a + N_j^{-1}] \]

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Proof. The second equality is immediate from (6.1). For the first inequality, we consider \( \subseteq \) and \( \supseteq \) separately.

\( \subseteq \): We prove the contrapositive. Suppose \( x \in \mathbb{R} \setminus \text{supp}(\mu_{j_0}) \) for some \( j_0 \in \mathbb{Z}_{\geq 0} \). Since \( \mathbb{R} \setminus \text{supp}(\mu_{j_0}) \) is open, there is an open ball \( B(x, r_0) \) contained in \( \mathbb{R} \setminus \text{supp}(\mu_{j_0}) \). Since \((\text{supp}(\mu_j))_{j=0}^{\infty}\) is a decreasing sequence of sets, \( B(x, r_0) \) is contained in \( \mathbb{R} \setminus \text{supp}(\mu_j) \) for every \( j \geq j_0 \). Thus \( \mu_j(B(x, r_0)) = 0 \) for every \( j \geq j_0 \). Choose a continuous function \( \phi : \mathbb{R} \to \mathbb{R} \) such that \( 1_{B(x, r_0/2)} \leq \phi \leq 1_{B(x, r_0)} \). Then, since \( \mu_j \to \mu \) weakly, we have \( \mu(B(x, r_0/2)) \leq \int \phi d\mu = \lim_{j \to \infty} \int \phi d\mu_j \leq \lim_{j \to \infty} \mu_j(B(x, r_0)) = 0 \). So \( \mu(B(x, r_0/2)) = 0 \). It follows that \( x \in \mathbb{R} \setminus \text{supp}(\mu) \).

\( \supseteq \): Let \( x \in \bigcap_{j=0}^{\infty} \text{supp}(\mu_j) \). Let \( B(x, r) \) be any open ball centered at \( x \). Choose \( j \) large enough that \( N_j^{-1} < r/2 \). Since \( x \in \text{supp}(\mu_j) \), we have \( \mu_j(B(x, r/2)) > 0 \), hence \( B(x, r/2) \) intersects \([a, a + N_j^{-1}]\) for some \( a \in A_j \). Therefore \( B(x, r) \) contains \([a, a + N_j^{-1}]\). Choose a continuous function \( \phi : \mathbb{R} \to \mathbb{R} \) such that \( 1_{[a, a + N_j^{-1}]} \leq \phi \leq 1_{B(x, r)} \). By (6.2), \( T_j^{-1} = \mu_k([a, a + N_j^{-1}]) \leq \int \phi d\mu_k \) for all \( k \geq j \). Then, since \( \mu_k \to \mu \) weakly, we have \( T_j^{-1} \leq \lim_{k \to \infty} \int \phi d\mu_k = \int \phi d\mu \leq \mu(B(x, r)) \). So \( \mu(B(x, r)) > 0 \). Since \( B(x, r) \) was arbitrary, it follows that \( x \in \text{supp}(\mu) \).

The measure \( \mu \) is the so-called natural measure on the Cantor set \( \bigcap_{j=0}^{\infty} \bigcup_{a \in A_j} [a, a + N_j^{-1}] \).

**Lemma 6.4.** \( \dim_H(\mu) = \alpha \).

**Proof.** For every \( \epsilon > 0 \) and \( j \in \mathbb{Z}_{\geq 0} \),

\[
H^{\alpha+\epsilon}(\text{supp}(\mu)) \leq \sum_{a \in A_j} H^{\alpha+\epsilon}([a, a + N_j^{-1}]) = T_j N_j^{-\alpha-\epsilon} \approx N_j^{-\epsilon}.
\]

Letting \( j \to \infty \), we see that \( H^{\alpha+\epsilon}(\text{supp}(\mu)) = 0 \) for all \( \epsilon > 0 \), hence \( \dim_H(\text{supp}(\mu)) \leq \alpha \). This implies \( \dim_H(\mu) \leq \alpha \).

To show that \( \dim_H(\mu) \geq \alpha \), it suffices to show that

\[
\mu(I) \lesssim |I|^\alpha
\]

for every interval \( I \) in \( \mathbb{R} \), where \( |I| \) denotes the diameter of \( I \). Let \( I \) be any interval in \( \mathbb{R} \). If \( |I| > 1 \), then

\[
\mu(I) \leq \mu(\mathbb{R}) = 1 \leq |I|^\alpha
\]

since \( \mu \) is a probability measure. Now suppose \( |I| \leq 1 \). Choose \( j \in \mathbb{Z}_{\geq 0} \) such that \( N_j^{-1} \leq |I| \leq N_j^{-1} \). Assume \( I \) intersects \( \text{supp}(\mu) \) (otherwise \( \mu(I) = 0 \)). Then \( I \) intersects
an interval \([a, a + N_j^{-1}]\) for some \(a \in A_j\). Since \(|I| \leq N_j^{-1}\) and \(A_j \subseteq N_j^{-1}[N_j]\), there are at most two such intervals; call them \(J_1\) and \(J_2\). Therefore

\[
\mu(I) = \mu(I \cap J_1) + \mu(I \cap J_2) \leq \mu(J_1) + \mu(J_2) = \frac{2}{T_j} \approx \frac{1}{N_{j+1}^\alpha} \leq |I|^{\alpha}
\]

\[\square\]

### 6.2 Proof of Theorem 6.1: Fourier Decay

We now work towards proving that the sets \(B_{j+1,a}\) can be chosen so that \(\dim_F(\mu) = \alpha\) and \(\dim_F(\mu \cdot \nu) \geq \min\{1, \dim_F(\mu) + \dim_F(\nu)\}\). The idea is to recognize that, if we choose the sets \(B_{j+1,a}\) randomly, the differences \(\mu_{j+1}(\xi) - \hat{\mu}_j(\xi)\) and \(\mu_{j+1} \cdot \nu(\xi) - \hat{\mu}_j \cdot \nu(\xi)\) can be written as sums of finitely many independent random variables. Then we use Hoeffding’s large deviation inequality to show there is a choice of the sets \(B_{j+1,a}\) which makes these differences small. Finally, we use telescoping sum and geometric series arguments to deduce the desired decay estimates for \(\hat{\mu}\) and \(\hat{\mu} \cdot \nu\).

Let \(j \in \mathbb{Z}_{\geq 0}\). We write the densities of \(\mu_j\) and \(\mu_{j+1}\) in more convenient forms. By partitioning \([0, N_j^{-1}]\) into intervals of length \(N_j^{-1}\), we see that

\[
\mu_j = \left(\frac{N_{j+1}}{T_j}\right) \left(\frac{1}{n_{j+1}}\right) \sum_{a \in A_j} \sum_{b \in N_{j+1}^{-1}[n_{j+1}]} 1_{a + b + [0, N_{j+1}^{-1}]}. 
\]

Since \(A_{j+1} = \bigcup_{a \in A_j} (a + B_{j+1,a})\), we also have

\[
\mu_{j+1} = \left(\frac{N_{j+1}}{T_j}\right) \left(\frac{1}{t_{j+1}}\right) \sum_{a \in A_j} \sum_{b \in B_{j+1,a}} 1_{a + b + [0, N_{j+1}^{-1}]}.
\]

For each \(a \in A_j\), \(b \in B_{j+1,a}\), and \(\xi \in \mathbb{R}\), define

\[
I(a, b, j, \xi) = \int_{[0,1]} e^{-2\pi i (\xi/N_{j+1}) (a N_{j+1} + b N_{j+1} + x)} dx, \quad (6.3)
\]

\[
J(a, b, j, \xi) = \int_{\mathbb{R}} \int_{[0,1]} e^{-2\pi i (\xi/N_{j+1}) (a N_{j+1} + b N_{j+1} + x)} y dx dy, \quad (6.4)
\]

\[
X_a(j, \xi) = \frac{1}{t_{j+1}} \sum_{b \in B_{j+1,a}} I(a, b, j, \xi) - \frac{1}{n_{j+1}} \sum_{b \in N_{j+1}^{-1}[n_{j+1}]} I(a, b, j, \xi), \quad (6.5)
\]

\[
Y_a(j, \xi) = \frac{1}{t_{j+1}} \sum_{b \in B_{j+1,a}} J(a, b, j, \xi) - \frac{1}{n_{j+1}} \sum_{b \in N_{j+1}^{-1}[n_{j+1}]} J(a, b, j, \xi). \quad (6.6)
\]
It follows that, for all $\xi \in \mathbb{R}$,
\[
\hat{\mu}_{j+1}(\xi) - \hat{\mu}_j(\xi) = \frac{1}{I_j} \sum_{a \in A_j} X_a(j, \xi) \tag{6.7}
\]
\[
\hat{\nu}_{j+1}(\xi) - \hat{\nu}_j(\xi) = \frac{1}{I_j} \sum_{a \in A_j} Y_a(j, \xi). \tag{6.8}
\]

By direct calculation, we have

**Lemma 6.5.** For each $j \in \mathbb{Z}_{j \geq 0}$, $a \in A_j$, $b \in B_{j+1,a}$, and $\xi \in \mathbb{R}$, we have
\[
|I(a, b, j, \xi)| \leq \min \{1, N_{j+1}/|\xi|\}, \tag{6.9}
\]
\[
|X_a(j, \xi)| \leq 2 \min \{1, N_{j+1}/|\xi|\}. \tag{6.10}
\]

Define $g : [0, \infty) \to [0, \infty)$ by
\[
g(x) = (1 + x)^{-1/2} + \sup \{ |\widehat{\nu}(tx)| : t \in \mathbb{R}, |t| \geq 1 \}
\]

The following properties of $g$ are straightforward to verify.

**Lemma 6.6.**

(i) $g$ is non-increasing

(ii) For all $\xi \in \mathbb{R}$, $|\widehat{\nu}(\xi)| \leq g(|\xi|)$.

(iii) For all $0 \leq \beta \leq 1$, $\sup_{\xi \in \mathbb{R}} |\widehat{\nu}(\xi)|(1 + |\xi|)^{\beta/2} < \infty$ iff $\sup_{\xi \in \mathbb{R}} g(|\xi|)(1 + |\xi|)^{\beta/2} < \infty$.

Recall that $\text{supp}(\nu)$ is compact and does not contain 0. Choose a Schwartz function $\phi : \mathbb{R} \to \mathbb{C}$ such that $\phi(y) = 1/y$ for all $y \in \text{supp}(\nu)$.

**Lemma 6.7.** For all $x \in \mathbb{R}$,
\[
|(\widehat{\phi \nu})(x)| \lesssim g(\frac{1}{2}|x|).
\]

**Proof.** Write
\[
|(\widehat{\phi \nu})(x)| = |(\widehat{\phi * \nu})(x)| \leq \int_{\mathbb{R}} |\widehat{\phi}(y)||\widehat{\nu}(|x - y|)dy.
\]

Note $\{ y : \frac{1}{2}|x| \geq |x - y| \} \subset \{ y : \frac{1}{2}|x| \leq |y| \}$, and bound the integral by the sum of integrals over $R_1 = \{ y : \frac{1}{2}|x| \leq |x - y| \}$ and $R_2 = \{ y : \frac{1}{2}|x| \leq |y| \}$. For the integral over $R_1$, use (i) and (ii) of Lemma 6.6. For the integral over $R_2$, note $|\widehat{\phi}(y)|^{1/2} \lesssim g(|y|)$ for all $y \in \mathbb{R}$ (because $\phi$ is Schwartz), then use (i) of Lemma 6.6. \qed
Lemma 6.8. There is a constant $C_0 > 0$ such that for each $j \in \mathbb{Z}_{\geq 0}$, $a \in A_j$, $b \in B_{j+1,a}$, and $\xi \in \mathbb{R}$, we have

$$|J(a, b, j, \xi)| \leq C_0 g(\frac{1}{2} |x|) \min \{1, N_{j+1}/|\xi|\},$$  \hspace{1cm} (6.11)

$$|Y_a(j, \xi)| \leq 2C_0 g(\frac{1}{2} |x|) \min \{1, N_{j+1}/|\xi|\}.$$  \hspace{1cm} (6.12)

Proof. Integrating $y$ in (6.4) shows that

$$J(a, b, j, \xi) = \int_{[0,1]} \hat{\nu}((\xi/N_{j+1})(aN_{j+1} + bN_{j+1} + x))dx$$

Since $|((\xi/N_{j+1})(aN_{j+1} + bN_{j+1} + x)| \geq |\xi|$ for each $x \in [0, 1]$, (i) and (ii) of Lemma 6.6 give

$$|J(a, b, j, \xi)| \leq g(|\xi|).$$

On the other hand, integrating $x$ in (6.4) shows that

$$J(a, b, j, \xi) = \int_{\mathbb{R}} e^{-2\pi i((\xi/N_{j+1})(aN_{j+1} + bN_{j+1}))} e^{-2\pi i(\xi/N_{j+1})y} - \frac{1}{2\pi i((\xi/N_{j+1})y} d\nu(y).$$

Multiplying by $-2\pi i((\xi/N_{j+1})$, using that $\phi(y) = 1/y$ for all $y \in \text{supp}(\nu)$, and integrating in $y$ gives

$$-2\pi i((\xi/N_{j+1})J(a, b, j, \xi)$$

$$= \hat{\phi}((\xi/N_{j+1})(aN_{j+1} + bN_{j+1} + 1)) - \hat{\phi}((\xi/N_{j+1})(aN_{j+1} + bN_{j+1})).$$

Since $|((\xi/N_{j+1})(aN_{j+1} + bN_{j+1} + x)| \geq |\xi|$ for each $x \in [0, 1]$, Lemma 6.7 and (i) of Lemma 6.6 give

$$|J(a, b, j, \xi)| \lesssim \frac{1}{\pi} \left( \frac{N_{j+1}}{|\xi|} \right) g\left( \frac{1}{2} |\xi| \right).$$

We need the following fact about averages over random subsets.

Lemma 6.9. Let $t \leq n$ be positive integers. Let $A$ be a finite set of size $n$, and let $F : A \to \mathbb{C}$. Let $B_t$ be the collection of all size $t$ subsets of $A$, and let $B$ be a set chosen uniformly at random from $B_t$. Then

$$\mathbb{E} \left( \frac{1}{t} \sum_{x \in B} F(x) \right) = \frac{1}{n} \sum_{x \in A} F(x).$$
Proof. There are \( \binom{n}{t} \) sets in \( B_t \). For each \( x \in A \), there are \( \binom{n-1}{t-1} \) sets in \( B_t \) that contain \( x \). Therefore

\[
\mathbb{E} \left( \frac{1}{t} \sum_{x \in B} F(x) \right) = \frac{1}{n} \cdot \frac{1}{t} \sum_{B \in B_t} \sum_{x \in B} F(x) = \frac{1}{n} \cdot \frac{1}{t} \sum_{x \in A} \left( \binom{n-1}{t-1} \right) F(x) = \frac{1}{n} \sum_{x \in A} F(x).
\]

We need the following version of Hoeffding’s inequality for complex-valued random variables.

**Lemma 6.10.** Suppose \( Z_1, \ldots, Z_t \) are independent complex-valued random variables satisfying \( \mathbb{E}(Z_i) = 0 \) and \( |Z_i| \leq c \) for \( i = 1, \ldots, t \), where \( c \) is a positive constant. For all \( u > 0 \),

\[
\mathbb{P} \left( \left| \frac{1}{t} \sum_{i=1}^{t} Z_i \right| \geq cu \right) \leq 4 \exp \left( -\frac{1}{4} tu^2 \right).
\]

**Proof.** Apply the standard Hoeffding inequality to the real and imaginary parts of \( Z_1, \ldots, Z_t \).

Define \( d_0 \) by

\[
d_0^{-1} = \max \{ \text{diam}(\text{supp}(\mu)), \text{diam}(\text{supp}(\mu \cdot \nu)) \}. \tag{6.13}
\]

Fix a real number

\[
\zeta_0 > \sum_{k \in \mathbb{Z}} \frac{2}{1 + d_0^2 |k|^2}.
\]

**Lemma 6.11.** It is possible to choose the sets \( B_{j+1,a} \) such that for every \( j \in \mathbb{Z}_{\geq 0} \) and \( \xi \in d_0 \mathbb{Z} \), we have

\[
|\hat{\mu}_{j+1}(\xi) - \hat{\mu}_j(\xi)| \leq 2 T_j^{-1/2} \ln^{1/2}(4 \zeta_0 (1 + |\xi|^2)) \min \{1, N_{j+1}/|\xi|\}, \tag{6.14}
\]

\[
|\hat{\mu}_{j+1} \nu(\xi) - \hat{\mu}_j \nu(\xi)| \leq 2 C_0 T_j^{-1/2} \ln^{1/2}(4 \zeta_0 (1 + |\xi|^2)) g(\frac{1}{2}|\xi|) \min \{1, N_{j+1}/|\xi|\}, \tag{6.15}
\]

where \( C_0 \) is the constant from Lemma 6.8, and \( d_0 \) is defined in (6.13).

**Proof.** Fix \( j \in \mathbb{Z}_{\geq 0} \) and assume a set \( A_j \subseteq [1,2] \) satisfying \( |A_j| = T_j \) is given. To simplify notation in what follows, we write \( N = N_{j+1}, n = n_{j+1}, \) and \( t = t_{j+1} \). For each \( a \in A_j \), suppose we choose \( B_{j+1,a} \) independently and uniformly at random from the collection of all
size $t$ subsets of $N^{-1}[n]$. Now fix $\xi \in \mathbb{R}$. Then \{\(X_a(j, \xi) : a \in A_j\) is a set of independent complex-valued random variables, and the same is true of \(\{Y_a(j, \xi) : a \in A_j\}\). Moreover, for each $a \in A_j$, we find that $\mathbb{E}(X_a(j, \xi)) = 0$ by applying Lemma 6.9 with $F(b) = I(a, b, j, \xi)$ for $b \in N^{-1}[n]$. Likewise, we find that $\mathbb{E}(Y_a(j, \xi)) = 0$ by applying Lemma 6.9 with $F(b) = J(a, b, j, \xi)$. We also have the bounds on $|X_a(j, \xi)|$ and $|Y_a(j, \xi)|$ from Lemma 6.5 and Lemma 6.8, respectively. Define

$$u_{j, \xi} = \sqrt{T_j^{-1} \ln (4\zeta_0(1 + |\xi|^2))}$$  \hspace{1cm} (6.16)

Let $E'(\xi)$ be the event that $\left| \frac{1}{T_j} \sum_{a \in A_j} X_a(j, \xi) \right| \geq 2u_{j, \xi} \min \{1, N/|\xi|\}$. Let $E''(\xi)$ be the event that $\left| \frac{1}{T_j} \sum_{a \in A_j} Y_a(j, \xi) \right| \geq 2C_0 u_{j, \xi} g(\frac{1}{2} |\xi|) \min \{1, N/|\xi|\}$. For each $\xi \in \mathbb{R}$, Lemma 6.10 gives

$$\mathbb{P}(E'(\xi)) \leq 4 \exp \left( -\frac{1}{4} T_j u_{j, \xi}^2 \right) = \frac{1}{\zeta_0(1 + |\xi|^2)}$$

and

$$\mathbb{P}(E''(\xi)) \leq 4 \exp \left( -\frac{1}{4} T_j u_{j, \xi}^2 \right) = \frac{1}{\zeta_0(1 + |\xi|^2)}.$$ 

Therefore

$$\mathbb{P}(\overline{(E'(d_0k))^c} \text{ and } (E''(d_0k))^c \text{ for all } k \in \mathbb{Z}) \geq 1 - \sum_{k \in \mathbb{Z}} \left( \mathbb{P}(E'(d_0k)) + \mathbb{P}(E''(d_0k)) \right)$$

$$\geq 1 - \sum_{k \in \mathbb{Z}} \frac{2}{\zeta_0(1 + d_0^2 |k|^2)} > 0.$$ 

In light of (6.7) and (6.8), this implies there is some choice of the sets $B_{j+1,a} (a \in A_j)$ such that (6.15) and (6.14) for every $\xi \in d_0 \mathbb{Z}$.

**Lemma 6.12.** With the sets $B_{j+1,a}$ chosen as in Lemma 6.11,

$$|\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\alpha/2} \ln^{1/2}(4\zeta_0(1 + |\xi|^2)) \quad \forall \xi \in \mathbb{R},$$  \hspace{1cm} (6.17)

$$|\hat{\mu} \cdot \nu(\xi)| \lesssim (1 + |\xi|)^{-\alpha/2} g\left(\frac{1}{2} |\xi|\right) \ln^{1/2}(4\zeta_0(1 + |\xi|^2)) \quad \forall \xi \in \mathbb{R}.$$  \hspace{1cm} (6.18)

**Proof.** We prove only (6.18), as the proof of (6.17) is similar and simpler. We begin by making two reductions.
First, by a standard argument (see Kahane [8, pp.252-253]), we only need to prove (6.18) for \( \xi = d_0k \in d_0\mathbb{Z} \), where \( d_0 \) is as in (6.13). For the second reduction, note that for every \( 0 \neq \xi \in \mathbb{R} \), we have

\[
\hat{\mu_0} \cdot \hat{\nu}(\xi) = \int_{\mathbb{R}} \int_{1}^{2} e^{-2\pi iy\xi} dx d\nu(y) = \int_{\mathbb{R}} \frac{e^{-2\pi iy\xi} - e^{-2\pi iy(2\xi)}}{-2\pi iy\xi} d\nu(y)
\]

\[
= \int_{\mathbb{R}} \frac{e^{-2\pi iy\xi} - e^{-2\pi iy(2\xi)}}{-2\pi iy\xi} \varphi(y) d\nu(y) = \frac{1}{-2\pi i\xi} \left( \hat{\varphi}(\xi) - \hat{\varphi}(2\xi) \right).
\]

By Lemma 6.7 and (i) of Lemma 6.6, \(|\hat{\mu_0} \cdot \hat{\nu}(\xi)| \lesssim (1 + |\xi|)^{-1} g \left( \frac{1}{2}|\xi| \right)\) for all \( 0 \neq \xi \in \mathbb{R} \). The same inequality holds when \( \xi = 0 \) by direct calculation. Therefore, by the triangle inequality, we just need to prove (6.18) with the left-hand side replaced by \(|\hat{\mu} \cdot \hat{\nu}(\xi) - \hat{\mu_0} \cdot \hat{\nu}(\xi)|\).

Lemma 6.2 says \( \mu_j \to \mu \) weakly. Then the dominated convergence theorem shows that \( \mu_j \cdot \nu \to \mu \cdot \nu \) weakly. Therefore, for every \( \xi \in \mathbb{R} \), \( \hat{\mu} \cdot \hat{\nu}(\xi) = \lim_{j \to \infty} \hat{\mu_j} \cdot \hat{\nu}(\xi) \), hence

\[
|\hat{\mu} \cdot \hat{\nu}(\xi) - \hat{\mu_0} \cdot \hat{\nu}(\xi)| \leq \sum_{j=0}^{\infty} |\mu_{j+1} \cdot \nu(\xi) - \mu_j \cdot \nu(\xi)|.
\]

If \( \xi = 0 \), each term of the sum above is zero, by direct calculation. Now assume \( \xi = d_0k \in d_0\mathbb{Z} \), \( \xi \neq 0 \). By Lemma 6.11, the sum above is

\[
\leq 2C_0 g \left( \frac{1}{2}|\xi| \right) \ln^{1/2} \left( 4\zeta_0 (1 + |\xi|^2) \right) \left( \sum_{j:N_j+1>|\xi|} T_j^{-1/2} + \sum_{j:N_j+1 \leq |\xi|} T_j^{-1/2} N_{j+1} \right)
\]

To estimate the last two sums, recall that \( 2 \leq n_j \leq n \), \( N_j = n_1 \cdots n_j \), \( T_0 = N_0 = 1 \), and \( T_j \approx N_j^\alpha \) for all \( j \in \mathbb{Z}_{\geq 0} \). This leads to \( T_j \approx N_{j+1}^\alpha \) and \( N_{j+i}/|\xi| \geq 2^{-1} N_{j+1}/|\xi| \) for all \( i,j \in \mathbb{Z}_{\geq 0} \). Thus the first sum is

\[
\approx |\xi|^{-\alpha/2} \sum_{j:N_j+1>|\xi|} (N_{j+1}/|\xi|)^{-\alpha/2} \leq |\xi|^{-\alpha/2} \sum_{j=0}^{\infty} 2^{-j} \leq |\xi|^{-\alpha/2}.
\]

And the second sum is

\[
\approx \sum_{j:N_j+1 \leq |\xi|} N_{j+1}^{-\alpha/2} \frac{N_{j+1}}{|\xi|} \leq |\xi|^{-\alpha/2} \sum_{j=0}^{\infty} 2^{-j} \leq |\xi|^{-\alpha/2}.
\]

\( \Box \)

**Lemma 6.13.** With the sets \( B_{j+1,\alpha} \) chosen as in Lemma 6.11, \( \dim_F(\mu) = \alpha \) and \( \dim_F(\mu \cdot \nu) \geq \min \{1, \dim_F(\mu) + \dim_F(\nu)\} \).

**Proof.** Lemma 6.12 and Lemma 6.4 imply \( \alpha \leq \dim_F(\mu) \leq \dim_H(\mu) = \alpha \). Lemma 6.12 and (iii) of Lemma 6.6 imply \( \dim_F(\mu \cdot \nu) \geq \dim_F(\mu) + \dim_F(\nu) \). \( \Box \)
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