ON THE NONEXISTENCE OF SMITH-TODA COMPLEXES

LEE S. NAVE

Abstract. Let $p$ be a prime. The Smith-Toda complex $V(k)$ is a finite spectrum whose $BP$-homology is isomorphic to $BP_*/(p, v_1, \ldots, v_k)$. For example, $V(-1)$ is the sphere spectrum and $V(0)$ the mod $p$ Moore spectrum. In this paper we show that if $p > 5$, then $V((p+3)/2)$ does not exist and $V((p+1)/2)$, if it exists, is not a ring spectrum. The proof uses the new homotopy fixed point spectral sequences of Hopkins and Miller.

1. Introduction

Let $p$ be a prime and recall the Brown-Peterson spectrum $BP$. It is a $p$-local ring spectrum with coefficient ring $BP_* \cong \mathbb{Z}_p[v_1, v_2, \ldots]$, where $v_i$ is the $i$th Hazewinkel generator in degree $2(p^i - 1)$. If $X$ is a spectrum, $BP_* X$ is a comodule over the Hopf algebroid $BP_*/BP_*$. (See [Rav86] for details.) Smith [Smi70] and Toda [Tod71] considered the existence of finite spectra $V(k)$ with $BP_* V(k) \cong BP_*/I_{k+1}$ (as $BP_*$-modules, hence as $BP_* BP$-comodules), where $I_{k+1} = (p, v_1, \ldots, v_k)$. For example, $V(-1) = S^0$ and $V(0) = M_p$, the mod $p$ Moore spectrum.

The construction of $V(1)$ is originally due to Adams [Ada66]. Smith constructed $V(2)$ for $p > 3$ and Toda constructed $V(3)$ for $p > 5$. Furthermore, these results are sharp. The first negative results were obtained by Toda, who showed that $V(1)$ cannot be constructed when $p = 2$ and likewise for $V(2)$ when $p = 3$. Later, Ravenel [Rav86, 7.5.1] showed that $V(3)$ does not exist when $p = 5$.

Recently, Hopkins, Mahowald, and Miller established the nonexistence of $V(p-2)$ for all $p > 3$. In this paper, we prove

Theorem 1.1. If $p > 5$, $V((p+3)/2)$ does not exist. If $V((p+1)/2)$ exists, it is not a ring spectrum.

The proof uses consequences of new work of Hopkins and Miller, not yet published, which we will briefly describe. An account of this work is given in [Rez98].

Throughout this paper, we restrict ourselves to the category of $p$-local spectra. We call a spectrum $p$-compact if it is the $p$-localization of a finite spectrum. (The $p$-compact spectra comprise the small objects in the $p$-local stable homotopy category, in the sense of [HPS97]). We write $S^0$ for the $p$-local sphere spectrum.

First we must recall the relevant parts of Morava’s theory ([Mor83], see also [Dev93]). Fix $n \geq 1$, set $q = p^n$, and let $E_{n*} \equiv W_{F_q}[[u_1, \ldots, u_{n-1}][u, u^{-1}]]$, with grading $|u| = -2$ and $|u_i| = 0$. Here $W_{F_q}$ denotes the Witt vectors with coefficients
in $\mathbb{F}_q$. $E_{n*}$ is a graded $BP_*$-algebra via the map $\theta : BP_* \to E_{n*}$ given by

$$\theta(v_i) = \begin{cases} u^{1-p^i}u_i, & i < n \\ u^{1-p^i}, & i = n \\ 0, & i > n. \end{cases}$$

We will write $v_i$ for $\theta(v_i)\in E_{n*}$. By the Landweber exact functor theorem [Lan76], $E_{n*} \otimes_{BP_*} (-)$ is a homology theory and we write $E_n$ for the representing spectrum. If $X$ is a $p$-compact spectrum, Morava's theory provides an action of $S_n$ on $E_{n*}X$, where $S_n$ is the $n$th Morava stabilizer group. Furthermore, $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ acts on $E_{n*}X$ via its action on $W\mathbb{F}_q$, making $E_{n*}X$ into a $G_n$-module, where $G_n \equiv S_n \rtimes \text{Gal}$. In fact, this action is induced by an action (in the homotopy category) of $G_n$ on $E_n$ ([Dev97], see discussion p. 767). Morava's change of rings theorem, in conjunction with [MS95], leads to a spectral sequence

$$H^*_c(G_n; E_{n*}X) \implies \pi_*L_{K(n)}X,$$

where $L_{K(n)}$ denotes localization with respect to $K(n)$, the $n$th Morava $K$-theory spectrum. This spectral sequence resembles a homotopy fixed point spectral sequence, but with two crucial differences. First, the group $G_n$ is not discrete. In fact, it is profinite and the $E_2$ term involves continuous cohomology. Second, $G_n$ acts on $E_n$ only up to homotopy. To form homotopy fixed point spectra in the usual way requires that the action be “on the nose”.

Nevertheless, Hopkins and Miller have shown that if $G \subset G_n$ is a finite subgroup one may form the homotopy fixed point spectrum $E^{h\overline{G}}_n$ and if $X$ is a $p$-compact spectrum there is a spectral sequence

$$H^*(G; E_{n*}X) \implies \pi_*(E^{h\overline{G}}_n \wedge X).$$

Subsequent work of Devinatz and Hopkins, in preparation, has shown that this can be done for any closed subgroup $G \subset G_n$, provided one works with continuous cohomology. More precisely, the spectral sequence is the $K(n)$-local $E^n_{\text{Gal}}$-based Adams spectral sequence for $E^{h\overline{G}}_n \wedge X$, where $E^n_{\text{Gal}}$ is the Landweber exact spectrum with coefficient ring $\mathbb{Z}_p[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$.

This construction is natural in $G$ and agrees with the usual spectral sequence when $G = S_n$. Also, the spectral sequence is multiplicative when $X$ is a ring spectrum. A suitable geometric boundary theorem holds as well: Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a cofiber sequence of $p$-compact spectra with $E_{n*}(h) = 0$. The resulting short exact sequence

$$0 \to E_{n*}X \to E_{n*}Y \to E_{n*}Z \to 0$$

gives rise to a connecting homomorphism $H^*_c(G; E_{n*}Z) \xrightarrow{\delta} H^*_{c+1}(G; E_{n*}X)$. Suppose $z \in \pi_* (E^{h\overline{G}}_n \wedge Z)$ is detected by $\overline{z} \in H^*_c(G; E_{n*}Z)$. Then $\delta(\overline{z})$ is a permanent cycle and $h(z) \in \pi_{*+1}(E^{h\overline{G}}_n \wedge X)$ is detected by $\delta(z)$, or else $h(z)$ is in higher filtration.

For the rest of the paper, we take $p$ odd and $n = p-1$. In this case, $S_n$ has exactly one maximal finite subgroup $G$, unique up to conjugation, namely $G \equiv \mathbb{Z}/p \times \mathbb{Z}/n^2$, with

$$e \mapsto (p, 1).$$

This construction is natural in $G$.
where the action of the right factor on the left is given by
\[ \mathbb{Z}/n^2 \rightarrow \mathbb{Z}/n \cong \text{Aut}(\mathbb{Z}/p). \]
(See \[Hew95\], \[Hew\], also \[Mil\].) If \( X \) is a \( p \)-compact spectrum, we refer to the spectral sequence
\[ H^*(G; E_{n*}X) \Longrightarrow \pi_*(E_n^hG \wedge X) \]
as the spectral sequence for \( X \).

2. Preliminaries

We begin with a discussion of the spectral sequence for \( S^0 \). These results are due to Hopkins and Miller. Because they have yet to appear in print, we include some details for the reader’s convenience. Full proofs will also appear in the author’s thesis. We remind the reader that \( n = p - 1, q = p^n \), and \( G \cong \mathbb{Z}/p \times \mathbb{Z}/n^2 \).

Our interest in this work began when Hopkins showed us how to compute the \( E_2 \) term of this spectral sequence. To wit,
\[ H^*(G; E_{n*}) \cong \mathbb{F}_q[\Delta^\pm](\alpha)[\beta], \]
where \( |\Delta| = (0,2pn^2), |\alpha| = (1,2n), \) and \( |\beta| = (2,2pn) \). (The cohomological degree is given first.) Actually, this presentation ignores much of \( H^0(G; E_{n*}) \).

More precisely, \( H^0(G; E_{n*})/p \) has a splitting with \( \mathbb{F}_q[\Delta^\pm] \) as a direct summand. With this splitting, the projection \( H^0(G; E_{n*}) \rightarrow \mathbb{F}_q[\Delta^\pm] \) gives \( \mathbb{F}_q[\Delta^\pm](\alpha)[\beta] \) the structure of an \( H^0(G; E_{n*}) \)-algebra, which is isomorphic to \( H^*(G; E_{n*}) \) in positive cohomological degrees.

The calculation of \( H^*(G; E_{n*}) \) involves choosing coordinates for \( E_{n*} \) on which the action of \( G \) is easily described. Let \( m = (p, u_1, \ldots, u_{n-1}) \) be the maximal ideal of \( E_{n*} \). There are elements \( w, w_1, \ldots, w_{n-1} \in E_{n*} \) with
\[ w \equiv u \mod (p,m^2) \]
\[ w_i \equiv u_i \mod (p,u_1,\ldots,u_{i-1},m) \]
and \( G \) action given by
\[ \sigma(w) = w + wu_{n-1}, \]
\[ \sigma(wu_i) = uw_i + wu_{i-1}, \quad \text{for } 2 \leq i \leq n-1, \]
\[ \tau(w) = \eta w, \]
and the relation \( (1 + \sigma + \cdots + \sigma^{p-1})w = 0 \), where \( \sigma \) generates \( \mathbb{Z}/p \), \( \tau \) generates \( \mathbb{Z}/n^2 \), and \( \eta \) is a primitive \( n^2 \) root of unity.

To obtain this representation of \( E_{n*} \) requires the following formulae for the action of \( G \) on the generators \( u, u_i \), which we also record for later use. Recall that an element \( g \in S_n \) has a unique expression of the form \( g = \sum_{j=0}^\infty a_jS_j \), where \( a_j \in W\mathbb{F}_q, a_j^q = a_j, \) and \( a_0 \) is a unit. We have \[DH95\] Proposition 3.3 and Theorem 4.4]
\[ g(u) \equiv a_0u + a_{n-1}^\chi uu_1 + \cdots + a_1^{n-1}uu_{n-1} \mod (p,m^2) \]
and
\[ g(uu_i) \equiv a_0^\chi uu_i + \cdots + a_{i-1}^{n-1}uu_1 \mod (p,m^2), \]
where $\chi$ denotes the Frobenius automorphism of $W\mathbb{F}_q$. Now it turns out that 
\[ \sigma = \sum_{j=0}^{\infty} a_j S^j \text{ with } a_0 \equiv 1 \text{ mod } p \text{ and } a_1 \in (W\mathbb{F}_q)^\times. \]
Also, $\tau = \eta$. Then for $1 \leq i \leq n$, 
\[
\begin{align*}
\sigma(uu_i) &\equiv uu_i + c_i uu_{i-1} \mod (p, u_1, \ldots, u_{i-2}, m^2) \\
\tau(uu_i) &\equiv \eta^p uu_i \mod (p, m^2),
\end{align*}
\]
where $c_i \in (W\mathbb{F}_q)^\times$ and we make the conventions that $u_0 = p$ and $u_n = 1$.

To get the desired representation of $E_{n*}$, Hopkins and Miller start with $s = (1-\sigma)(v_1)/p \in E_{n*}$. From the definition of the action of $S_n$ on $E_{n*}$, it is clear that if $s = t_i(\sigma^{-1})$, where we write $t_i$ for the image of $t_i$ under the map of Hopf algebroids
\[(BP_*, BP_*) \to (E_{n*}^{\text{Gal}}, \text{Map}_c(S_n, E_{n*}^{\text{Gal}})).\]
This map is a composition of several Hopf algebroid maps. By unraveling the definitions of these maps in [Dev93, (5.2)], it is clear that if $g \in S_n$ is given by $g = \sum_{j=0}^{\infty} a_j S^j$, then $t_i(g) \equiv a_0^{-1} a_i u^{i-p} \mod m$.

In particular, $s \equiv c u^{1-p} \mod m$, where $c \in (W\mathbb{F}_q)^\times$. Let
\[ t = \sum_{j=0}^{p-1} \sigma^j(u) \quad \text{and} \quad w = \frac{1}{\eta^{p^2}} \sum_{j=1}^{n^2} \eta^{-j} \tau^j(t). \]
Then $w \equiv cu \mod (p, m^2)$, $(1 + \sigma + \cdots + \sigma^{p-1})w = 0$, and $\tau(w) = \eta w$. Finally, let $w_1 = (\sigma - 1)^{n-1}(w)$.

The differentials are determined by the Toda differential in the Adams-Novikov spectral sequence (ANSS) and the nilpotence of $\beta_1 \in \pi_1 S^0$ as follows. There is a map from the ANSS to the spectral sequence for $S^0$,
\[ \text{Ext}_{BP_*BP}(BP_*, BP_*) \to H^*(G; E_{n*}). \]
By [Rav78], this map sends $\alpha_1$ to $\alpha$, $\beta_1$ to $\beta$, and $\beta_{p/p}$ to $\Delta \beta$. (We will omit the phrase “up to multiplication by a unit” from this discussion.) Because $\alpha_1$ and $\beta_1$ are permanent cycles, so are $\alpha$ and $\beta$.

Let $K$ denote the kernel of the projection $H^0(G; E_{n*}) \to \mathbb{F}_q[\Delta^\pm]$. Because $\beta \cdot K = 0$, $d_{2p-1}$ vanishes on $K$. Also, from the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1 \beta_1^p$ and the evident sparseness in our spectral sequence, we conclude that
\[ d_{2p-1}(\Delta) = \alpha \beta^{p-1}. \]
These facts and the multiplicative structure of the spectral sequence completely determine $d_{2p-1}$.

For degree reasons the next possible differential is $d_{2n^2+1}$. Since $\beta_1^{p^{n+1}} = 0$ in $\pi_1 S^0$ and $\beta^{p^{n+1}}$ is hit by some differential in our spectral sequence. The last differential which could do this is $d_{2n^2+1}$ and therefore
\[ d_{2n^2+1}(\Delta^{p-1} \alpha) = \beta^{n^2+1}. \]
Again, $d_{2n^2+1}$ vanishes on $K$. This determines $d_{2n^2+1}$ completely and the spectral sequence collapses after this point for degree reasons, hence

**Proposition 2.1.** If $s$ is odd, then $E_{\infty}^{s,t} = 0$ unless $1 \leq s \leq 2n - 1$ and 
\[ t \equiv 2n + (s-1)p + 2pn^2x \mod 2p^2n^2, \quad \text{where } x \not\equiv -1 \mod p. \]
If $s > 0$ is even, then $E_{\infty}^{s,t} = 0$ unless $2 \leq s \leq 2n^2$ and
$$t \equiv \text{spn} \mod 2p^2n^2.$$ 

We also need a handle on the zero line. Using the coordinates $w, w_i$ mentioned above, it is easy to show that

**Proposition 2.2.** $H^0(G; E_{n^*})$ is concentrated in degrees $t \equiv 0 \mod 2n$.

Finally, we will need the following partial description of $H^*(G; E_{n*}V(1))$, which is a simple consequence of (2.1).

**Proposition 2.3.** Let $A$ be the graded abelian group given by
$$A_m = H^{1,2pn+2m^2m}(G; E_{n*}/I_2),$$
made into an $\mathbb{F}_2[\Delta^\pm]$-module via the map $E_{n*} \to E_{n*}/I_2$. Then $A \cong \Sigma^{2pn}\mathbb{F}_2[\Delta^\pm]$.

3. **Proof of Theorem 1.1**

Suppose $V(k)$ exists. It is well known that the condition $BP_*V(k) \cong BP_*/I_{k+1}$ is equivalent to $H_*(V(k); \mathbb{F}_p)$ being isomorphic to the exterior algebra $E(\tau_0, \ldots, \tau_k)$ as a comodule over the mod $p$ dual Steenrod algebra. This point of view allows one to produce $V(k-1)$ as a skeleton of $V(k)$ and exhibit $V(k)$ as the cofiber of a map $f : \Sigma^{\nu_k}V(k-1) \to V(k-1)$ inducing multiplication by $v_k$ in $BP$-homology. (The notation $V(k-1)'$ is meant to distinguish this spectrum from $V(k-1)$—they need not be isomorphic.)

Recall that in the Hopf algebroid structure of $BP_*BP$, $v_k$ is invariant mod $I_k$, i.e., $\eta_n(v_k) \equiv v_k \mod I_k$. Then $v_k \in E_{n*}V(k-1) \cong E_{n*}/I_k$ is fixed by all of $S_n$. In particular, $v_k^j \in H^0(G; E_{n*}V(k-1))$ for all $j \geq 0$.

The map $f$ induces a map (of degree $|v_k|$) from the spectral sequence for $V(k-1)'$ to the spectral sequence for $V(k-1)$, which is multiplication by $v_k$ on $H^0$. In particular, if $v_k^j$ is a permanent cycle in the spectral sequence for $V(k-1)'$, then $v_k^{j+1}$ is a permanent cycle in the spectral sequence for $V(k-1)$.

Fix $p > 5$ and let $m = (p + 3)/2$. We show that if $V(m-1)$ exists, then in the spectral sequence for $V(m-1)$, $v_m$ is a permanent cycle and $v_m^2$ is not. Then clearly $V(m-1)$ cannot be a ring spectrum, and by the discussion above, $V(m)$ cannot exist. In order to simplify the notation, we will ignore the distinction between $V(k-1)'$ and $V(k-1)$. Since our arguments depend only on properties of the spectral sequence for $S^0$, which is unique in the $p$-local category, no generality is lost.

Before proceeding, we need to establish the vanishing of $H^{s,t}(G; E_{n*}/I_k)$ and $\pi_r(E_n^{hG}\wedge V(k))$ for various values of $s, t, r,$ and $k$. We are not able to calculate these groups directly for arbitrary values of $k$. Instead, we use a brute force approach which reduces the computation to the case $k = -1$.

**Lemma 3.1.** If $d \in \mathbb{Z}$ satisfies
$$d \equiv 2pmy \mod 2pn^2$$
where $y \not\equiv 1 \mod n$, then
$$H^{1,d}(G; E_{n*}/I_k) = 0.$$ 

If, in addition, $k \not\equiv p - 1$, then
$$H^{2,d-2n}(G; E_{n*}/I_k) = 0.$$
Proof. If \( k \geq p, E_{n*}/I_k = 0 \) so the result is trivial. Thus assume \( k < p \). We prove first that \( H^{1,d}(G; E_{n*}/I_k) = 0 \). Using the long exact sequences in cohomology arising from the short exact sequences of \( E_{n*} \)-modules

\[
0 \to E_{n*}/I_i \to E_{n*}/I_{i+1} \to 0,
\]

it suffices to show that \( H^{s,t}(G; E_{n*}) = 0 \) for all \( (s,t) \) with \( 1 \leq s \leq k + 1 \) and

(3.1)

\[
t = d - \sum_{j=1}^{s-1} |v_{ij}|
\]

where \( 0 \leq i_1 < i_2 < \cdots < i_{s-1} \leq k - 1 \).

First, suppose \( s \) is even. By (3.1), \( H^{s,t}(G; E_{n*}) = 0 \) unless \( t \equiv spn \mod 2pn^2 \). Substituting this into (3.1) and reducing mod \( p \) leads to \( s = 2 \) and \( i_1 = 0 \). In this case, working mod \( 2 \) leads to \( y = \equiv 1 \mod n \), contrary to our hypothesis on \( y \).

Now suppose \( s \) is odd. Again by (3.1), \( H^{s,t}(G; E_{n*}) = 0 \) unless \( t \equiv 2n + (s-1)pn \mod 2pn^2 \). Substituting this into (3.1) and reducing mod \( p \) leads to an equation in \( s \) which, keeping in mind all the constraints involved, has no solution.

Similarly, we reduce the proof that \( H^{2,d-2n}(G; E_{n*}/I_k) = 0 \) to showing that \( H^{s,t}(G; E_{n*}) = 0 \) for all \( (s,t) \) with \( 2 \leq s \leq k + 2 \) and

(3.2)

\[
t = d - 2n - \sum_{j=1}^{s-2} |v_{ij}|
\]

where \( 0 \leq i_1 < i_2 < \cdots < i_{s-2} \leq k - 1 \).

As before, we proceed by reducing (3.2) mod \( p \). This time there are no solutions for \( s \), assuming as we are that \( k < p - 1 \).

\[\square\]

Lemma 3.2. Let \( m < p - 1 \). If \( d \in \mathbb{Z} \) satisfies

\[
d \equiv 2n + 2pm + 2p^2ny \mod 2p^2n^2,
\]

where \( y \neq 0 \mod n \), and \( V(m) \) exists, then

\[
\pi_{d-1}(E^h_n \wedge V(m)) = 0.
\]

Proof. Using the long exact sequences in homotopy arising from the cofiber sequences

\[
\Sigma^{[i]} V(i-1) \to V(i-1) \to V(i),
\]

it suffices to show that in the spectral sequence for \( S^0, E_{n*}^{s,t} = 0 \) for all \( (s,t) \) with

(3.3)

\[
t - s = d - 1 - k - \sum_{j=1}^{k} |v_{ij}|
\]

where \( 0 \leq k \leq m + 1 \) and \( 0 \leq i_1 < i_2 < \cdots < i_k \leq m \).

First, suppose \( s = 0 \). Reducing (3.3) mod \( 2n \), we have \( t \equiv -(k + 1) \). By Proposition 2.2, \( E_{n*}^{0,t} = 0 \).

Next, suppose \( s \) is odd. By Proposition 2.2, we may take \( 1 \leq s \leq 2n - 1 \) and \( t \equiv 2n + (s-1)pn \mod 2pn^2 \). Substituting this into (3.3) and reducing mod \( 2n \), we are led to \( s = k + 1 \). Substituting this into (3.3) and reducing mod \( p \) yields \( k = 0 \) or \( k = 1 \) (if \( i_1 = 0 \)). The second case cannot occur because \( s \) is odd, therefore we may take \( k = 0 \) and \( s = 1 \). In this case, (3.3) becomes \( t = d \). Reducing this
mod \(2pm^2\), we are led to \(y \equiv -1 \mod n\). It follows that \(t \equiv 2n - 2pm^2 \mod 2p^2n^2\). By Proposition 3.1, \(E_{\infty}^{1,1} = 0\).

Finally, suppose \(s > 0\) is even. In this case, we may take \(2 \leq s \leq 2n^2\) and \(t \equiv spn \mod 2p^2n^2\). Working mod \(2n\) as above, we get \(s = 2nl + k + 1\), where \(0 \leq l < n\). Note that \(k\) must be odd. Substituting these equations for \(s\) and \(t\) into (3.3) and reducing mod \(p\) leads to \(l = k - 1\) (if \(i_1 > 0\)) or \(l = k - 2\) (if \(i_1 = 0\)).

To finish the proof, we need to reduce (3.3) mod \(2p^2n\), which is a bit unwieldy. There are four cases:

\[
0 \leq l \leq n - 1
\]

\[
t - s = 2n + 2pm - (k + 1) - \begin{cases} (2pn + 2n)k, & i_1 > 1 \\ (2pn + 2n)(k - 1) + 2n, & i_1 = 1 \\ (2pn + 2n)(k - 1), & i_1 = 0 \text{ and } i_2 > 1 \\ (2pn + 2n)(k - 2) + 2n, & i_1 = 0 \text{ and } i_2 = 1. \end{cases}
\]

For each case, we substitute \(t = spn\), \(s = 2nl + k + 1\), and either \(l = k - 1\) (first two cases) or \(l = k - 2\) (last two cases) and solve for \(k\). The first and third cases yield \(k \equiv -1 \mod 2p\), which is not possible. The second and fourth cases yield \(k \equiv 1 \mod 2p\), i.e., \(k = 1\). In the fourth case, this is not possible, because we are assuming that \(i_2 = 1\) in that case, so \(k > 1\). This leaves the second case, with \(k = 1\). Substituting \(t = spn\), \(s = 2nl + k + 1\), \(l = k - 1\), and \(k = 1\) into (3.3) and reducing mod \(2p^2n^2\) leads to \(y \equiv 0 \mod n\), contrary to our hypothesis on \(y\).

\[\square\]

Proposition 3.3. Let \(p > 5\) and \(k \leq p - 1\). If \(V(k - 1)\) exists, then \(v_k\) is a permanent cycle in the spectral sequence for \(V(k - 1)\).

Proof. The cases \(k = 1, 2\) and \(3\) are immediate, because \(v_1\), \(v_2\), and \(v_3\) are permanent cycles in the corresponding Adams-Novikov spectral sequences. So let \(k \geq 4\). There are many steps to the proof, so we summarize the argument first.

Consider the connecting homomorphisms

\[H^* (G; E_nV(i)) \overset{\delta_{k+1}}{\longrightarrow} H^{*+1} (G; E_nV(i) - 1)\]

arising from the cofiber sequences

\[\Sigma^{v_i} V(i-1) \overset{v_i}{\longrightarrow} V(i-1) \longrightarrow V(i) \overset{h_{i+1}}{\longrightarrow} \Sigma^{v_i+1} V(i-1).\]

Let

\[S^0 \overset{f}{\longrightarrow} V(1) \overset{g}{\longrightarrow} V(k - 2)\]

factor the inclusion of the bottom cell into \(V(k - 2)\). We will show that

\[\delta_k(v_k) = g_* (f_* (\Delta^m) \delta_3(v_3)),\]

where \(m \equiv 0 \mod p\), and that this element is nontrivial. Since \(\Delta^m\) and \(v_3\) are permanent cycles in the spectral sequences for \(S^0\) and \(V(2)\), respectively, and \(V(1)\) is a ring spectrum \(\text{Sm}_{[7]}\), it follows that \(\delta_3(v_k)\) is a permanent cycle.

So \(\delta_k(v_k)\) detects an element \(x \in \pi_{|v_k|-1} (E_n^{hG} \wedge V(k - 2))\). By Lemma 3.2, \(\pi_{|v_k|-1} (E_n^{hG} \wedge V(k - 2)) = 0\), so \(x\) is in the image of the map

\[\pi_{|v_k|} (E_n^{hG} \wedge V(k - 1)) \overset{h_k}{\longrightarrow} \pi_{|v_k|-|v_k|-1} (E_n^{hG} \wedge V(k - 2)),\]

say \(x = h_k(y)\). Note that \(y\) must have filtration zero, i.e., \(y \in H^0 (G; E_nV(k - 1))\).
Now \( \delta_k(y - v_k) = 0 \), so \( y - v_k \) pulls back to \( H^0(G; E_{n*} V(k - 2)) \). We then show that the map \((g \circ f)_*\) is onto in this bidegree and that everything in \( H^{0,|v_k|}(G; E_{n*}) \) is a permanent cycle. Therefore \( y - v_k \) is a permanent cycle, so \( v_k \) is a permanent cycle.

**Step 1:** \( \delta_k(v_k) \neq 0 \). Otherwise, \( v_k \) is in the image of the map

\[
H^0(G; E_{n*} V(k - 2)) \longrightarrow H^0(G; E_{n*} V(k - 1)),
\]
i.e., \( v_k \) lifts to a fixed point \( \gamma \in u^*F_q[[u_{k-1}, \ldots, u_{n-1}]] \), where \( s = 1 - p^k \). If \( \mu \in u^*F_q[[u_{k-1}, \ldots, u_{n-1}]] \) is a monomial, we write \( \mu \in \gamma \) if \( \mu \) appears as a term (up to multiplication by a unit) when we express \( \gamma \) as a sum of monomials.

We claim that \( u^*u_k \not\in \gamma \), which would then contradict \( \delta_k(v_k) = 0 \). From (2.2) we have the formula

\[
(\sigma - 1)(u^*u_i) \equiv c_iu^*u_{i-1} \mod (p, u_1, \ldots, u_{i-2}, m^2).
\]
Suppose \( u^* \in \gamma \). Since \( (\sigma - 1)\gamma = 0 \) and \( u^*u_{n-1} \in (\sigma - 1)u^* \), there must be another monomial \( \mu \in \gamma \) with \( u^*u_{n-1} \in (\sigma - 1)\mu \). But by the formula above, this isn’t possible, hence \( u^* \not\in \gamma \). Iterating this procedure yields \( u^*u_i \not\in \gamma \) for \( k \leq i \leq n \).

**Step 2:** The map

\[
H^{1,|v_k| - |v_{k-1}|}(G; E_{n*} V(1)) \xrightarrow{g_*} H^{1,|v_k| - |v_{k-1}|}(G; E_{n*} V(k - 2))
\]
is onto. It suffices to show that

\[
H^{2,|v_k| - |v_{k-1}| - |v_m|}(G; E_{n*} V(m - 1)) = 0
\]
for \( m = 2, \ldots, k - 2 \), which follows from Lemma 3.3.

**Step 3:** By Step 2, there is an element \( y \in H^1(G; E_{n*} V(1)) \) which maps to \( \delta_k(v_k) \in H^1(G; E_{n*} V(k - 2)) \). Then \( y = f_*(\Delta^m)\delta_3(v_{m}) \), where \( m \equiv 0 \mod p \). This follows from Proposition 2.3 after checking that the degrees work out, i.e.,

\[
|\delta_k(v_k)| \equiv |\delta_3(v_{m})| \equiv 2pm + 2pm^2 \mod 2p^2n^2,
\]
and showing that \( \delta_3(v_{m}) \neq 0 \), which follows exactly as in Step 1.

**Step 4:** The map

\[
(g \circ f)_* : H^{0,|v_k|}(G; E_{n*}) \longrightarrow H^{0,|v_k|}(G; E_{n*} V(k - 2))
\]
is surjective. For this, it suffices to show that

\[
H^{1,|v_k| - |v_{m}|}(G; E_{n*} V(m - 1)) = 0
\]
for \( m = 0, 1, \ldots, k - 2 \), which follows from Lemma 3.3.

**Step 5:** Everything in \( H^{0,|v_k|}(G; E_{n*}) \) is a permanent cycle. Indeed, for degree reasons these elements are in the kernel \( K \) of the projection \( H^0(G; E_{n*}) \to F_q[\Delta^+] \). As we discussed in the previous section, everything in \( K \) is a permanent cycle.

**Proposition 3.4.** Let \( m = (p + 3)/2 \). If \( V(m - 1) \) exists, then \( v_m^2 \) supports a differential in the spectral sequence for \( V(m - 1) \).

**Proof.** By Lemma 3.3:

\[
\pi_2|v_m| - |v_k| - 1(E_n^h G \wedge V(k - 1)) = 0
\]
for \( k = 3, 4, \ldots, m - 1 \). It follows that if \( v_{m}^{2} \) is a permanent cycle, then \( v_{m}^{2} \) is in the image of the map
\[
H^{0,2|v_{m}|}(G; E_{n}^{*}V(2)) \longrightarrow H^{0,2|v_{m}|}(G; E_{n}^{*}V(m - 1)).
\]
We complete the proof by showing that this is not possible.

Set \( s = 2(1 - p^{m}) \), i.e., \( v_{m}^{2} = u^{s}u_{m}^{2} \). Suppose \( \gamma \in H^{0,-2s}(G; E_{n}^{*}V(2)) \) and let
\[
A_{n} = \{ u^{s}u_{3}, u^{s}u_{4}, \ldots, u^{s}u_{n-1}, u^{s} \}
\]
and
\[
A_{j} = \{ u^{s}u_{i}u_{j} : 2m - j \leq i \leq j \}
\]
for \( j = m, m + 1, \ldots, n - 1 \). In the language of Step 1 of the previous proof, we claim that each of these sets is disjoint from \( \gamma \). This will complete the proof, as \( A_{m} = \{ u^{s}u_{m}^{2} \} \). This is precisely where the hypothesis \( m = (p + 3)/2 \) appears.

As in the previous proof, we have \( u^{s}u_{i} \not\in \gamma \) for \( 4 \leq i \leq n \). For \( i = 3 \), note that \( \tau(u^{s}u_{3}) \equiv \eta^{s+1} + p \cdot u^{s}u_{3} \mod (p,m^{2}) \), by (2.2). It is easily verified that \( s - 1 + p^{3} \equiv (1 - p) \mod (p-1)^{2} \), which shows that \( u^{s}u_{3} \not\in \gamma \), again by (2.2). (Recall that \( \eta \) is an \( n \) root of unity.) Thus the claim is established for \( j = n \).

Now let \( j < n \). For each \( i \) with \( 2m - j - 1 \leq i \leq j \), set
\[
B_{j,i} = \{ u^{s}u_{k}u_{j} \in A_{j} : k > i \}.
\]
From (2.2) we have \( u^{s}u_{i-1}u_{j} \in (\sigma - 1)u^{s}u_{i}u_{j} \). Then if \( u^{s}u_{i}u_{j} \in \gamma \), there must be another monomial \( \mu \in \gamma \) with \( u^{s}u_{i-1}u_{j} \in (\sigma - 1)\mu \). We claim that then
\[
\mu \in B_{j,i} \cup A_{j+1} \cup \cdots \cup A_{n}.
\]
To see this, write \( \mu = u^{s}u_{a}u_{b} \) with \( a \leq b \). It follows easily from (2.2) that \( b \geq j \) and \( a \geq i - 1 \). If \( b > j \), then
\[
2m - b \leq 2m - j - 1 \leq i - 1 \leq a,
\]
hence \( \mu \in A_{b} \). If \( b = j \), then in fact from (2.2) we have \( a \geq i \). The case \( a = i \) is ruled out by the assumption that \( \mu \not= u^{s}u_{i}u_{j} \), so \( \mu \in B_{j,i} \), as claimed.

Suppose inductively that \( A_{k} \) is disjoint from \( \gamma \) for \( k = j + 1, \ldots, n \). Suppose also that \( B_{j,i} \cap \gamma = \emptyset \) for some \( i \) with \( 2m - j - 1 \leq i \leq j \). (This is trivially true when \( i = j \), as \( B_{j,j} = \emptyset \).) Now by the previous paragraph, if \( u^{s}u_{i}u_{j} \in \gamma \), then there exists \( \mu \in \gamma \cap (B_{j,i} \cup A_{j+1} \cup \cdots \cup A_{n}) \). This is a contradiction, so \( u^{s}u_{i}u_{j} \not\in \gamma \), i.e., \( B_{j,i} \cap \gamma = \emptyset \). By downward induction on \( i \), it follows that \( A_{j} \) is disjoint from \( \gamma \).

By downward induction on \( j \), we obtain \( A_{m} \cap \gamma = \emptyset \), as desired.

References

[Ada66] J. F. Adams, On the groups \( J(X) \), IV, Topology 5 (1966), 21–71.

[Dev95] Ethan S. Devinatz, Morava’s change of rings theorem, The Cech centennial (Boston, MA, 1993) (Providence, RI), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 83–118.

[Dev97] Ethan S. Devinatz, Morava modules and Brown-Comenetz duality, Amer. J. Math. 119 (1997), no. 4, 741–770.

[DH95] Ethan S. Devinatz and Michael J. Hopkins, The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts, Amer. J. Math. 117 (1995), no. 3, 669–710.

[Hew] Thomas Hewett, Normalizers of finite subgroups of division algebras over local fields, unpublished, available on the web at the author’s home page: http://www.math.princeton.edu/~hewett.

[Hew95] Thomas Hewett, Finite subgroups of division algebras over local fields, J. Algebra 173 (1995), no. 3, 518–548.
[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. 128 (1997), no. 610, x+114.

[Lan76] Peter S. Landweber, "Homological properties of comodules over $MU_\ast MU$ and $BP_\ast BP$", Amer. J. Math. 98 (1976), no. 3, 591–610.

[Mil] Haynes Miller, *Notes on algebra*, unpublished notes.

[Mor85] Jack Morava, "Noetherian localisations of categories of cobordism comodules", Ann. of Math. (2) 121 (1985), no. 1, 1–39.

[MS95] Mark Mahowald and Hal Sadofsky, $v_n$ telescopes and the Adams spectral sequence, Duke Math. J. 78 (1995), no. 1, 101–129.

[Rav78] Douglas C. Ravenel, "The non-existence of odd primary Arf invariant elements in stable homotopy", Math. Proc. Cambridge Philos. Soc. 83 (1978), no. 3, 429–443.

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, Fla., 1986.

[Rez98] Charles Rezk, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations (Providence, RI), Contemp. Math., vol. 220, Amer. Math. Soc., Providence, RI, 1998, pp. 313–366.

[Smi70] Larry Smith, *On realizing complex bordism modules. Applications to the stable homotopy of spheres*, Amer. J. Math. 92 (1970), 793–856.

[Tod67] Hirosi Toda, "An important relation in homotopy groups of spheres", Proc. Japan Acad. 43 (1967), 839–842.

[Tod68] Hirosi Toda, "Extended $p$-th powers of complexes and applications to homotopy theory", Proc. Japan Acad. 44 (1968), 198–203.

[Tod71] Hirosi Toda, *On spectra realizing exterior parts of the Steenrod algebra*, Topology 10 (1971), 53–65.

University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195–4350

E-mail address: nave@math.washington.edu