INTRODUCTION

Unconventional shale gas reservoirs have been important sectors of global resources.\(^1\)\(^-\)\(^3\) Shale gas reservoirs are being developed with horizontal wells and multistage hydraulic fracturing.\(^4\)\(^-\)\(^7\) It is generally believed that transient flow lasts a very long time in the plays. However, some well test data from horizontal shale gas wells have been observed in a pseudo-steady flow state after a few years of production.\(^8\)\(^-\)\(^10\) Gas production forecast under pseudo-steady flow condition is realistic in these cases.

The shale gas reservoir could be thought of as a constant pressure matrix system because of the low permeability in the early development stage.\(^11\) On this basis, the shale gas flow is considered as transient flow. Several production prediction models of shale gas wells for transient flow have been established.\(^12\)\(^-\)\(^18\) Skin factor was added in transient flow model in dual porosity medium, and dynamic production data of shale gas reservoir was analyzed by Bello.\(^19\) Yue Chenjun\(^20\) established a transient flow model and obtained an analytic expression of dimensionless bottom hole pseudo-pressure and production in the Laplace space. For steady flow conditions, a few production prediction models were developed.\(^21\)\(^-\)\(^25\) Lin Yuan et al\(^26\) obtained a semi analytical model for steady flow in shale gas fractured horizontal wells using the conformal transformation method. The adsorption and desorption of shale gas were considered. The expression of shale gas...
diffusion was introduced based on the productivity equation of fractured horizontal wells in conventional gas reservoir. To some extent, this model had the form of binomial deliverability equation of shale gas. It can be seen that the production prediction model of the shale gas is relatively mature under transient flow and steady flow conditions. In view of pseudo-steady flow conditions, investigations are very limited. Michael Morgan et al. proposed a method for predicting the production of tight and shale gas reservoirs combined with the material balance equation. This method corrected the time by adopting the material balance pseudo-time function. Based on the characteristics of shale gas reservoirs, the material balance equation was modified by the addition of adsorption. The prediction equation was derived suitable for quasi steady flow and linear flow, but more parameters are needed to use this model.

A number of transient and steady flow equations have been presented for productivity analysis of shale gas well productivity. However, there are very few equations for productivity analysis under pseudo-steady flow conditions, and analytical modeling of productivity under pseudo-steady flow conditions has not been found. In this paper, an analytical model is established for predicting the performance of horizontal shale gas wells under pseudo-steady flow conditions. The model is derived by flow coupling in the matrix and fractures. A good match is confirmed between actual gas well production and model-calculated production for the wells in a North American shale gas field. The equations of the analytical model take a very simple form. They are very practical for reservoir engineers to perform production forecast and completion engineers to optimize fracturing design.

2 | MATHEMATICAL MODEL

Figure 1 shows a sketch of a multi-fractured horizontal well. Fluid flow into one wing of a fracture is illustrated in Figure 2. Dimension of a fracture wing is demonstrated in Figure 3.

An analytical model was derived to predict the productivity of horizontal shale gas wells under pseudo-steady flow conditions. The derivation of the model is detailed in Appendix. Resultant equations are summarized in this section.

The derived well productivity equation is expressed as

$$Q_{sc} = \frac{9.15 \times 10^{-8} n k m h t}{z T \mu g S^{1/2} \sqrt{c}} \left( \frac{p - p_{w}}{1 - c^{2} S^{1/2}} \right)^{1/3}$$

where $Q_{sc}$ = Total gas production rate of horizontal shale gas well in standard conditions, MMscf/d, $n$ = Fracture number, dimensionless, $k_m$ = Permeability of matrix, md, $h_t$ = Fracture height, ft, $\bar{p}$ = Average reservoir pressure, psia, $p_w$ = Bottom hole pressure, psia, $z$ = Gas compressibility factor, dimensionless, $T$ = Formation temperature, °F, $\mu_g$ = Gas viscosity, cp, $S^{1/2}$ = Distance between drainage boundary and fracture face, ft, $\phi$ = Porosity, dimensionless, $S_w$ = Water saturation, dimensionless.

$$c = \frac{4 k_m}{w t k_f S^{1/2}}$$

where $w_t$ = Fracture width, ft, $k_f$ = Permeability of fracture, md.

The average reservoir pressure is given by Equation (3).

$$\bar{p} = p_e - \frac{p_e - p_w}{3 \mu_f \sqrt{c}} \left( 1 - e^{-\sqrt{c^2}} \right)$$

where $p_e$ = Pressure at drainage boundary, psia.

The equation for pressure distribution in the matrix is expressed as

$$p_m(x, y) = p_e - \frac{p_e - p_w}{S^{1/2}} \left( y - \frac{c^2}{2 S^{1/2}} \right)$$

where $p_m$ is the pressure in the matrix, psia; $x$ is the distance from fracture tip, ft; $y$ is the distance in the direction perpendicular to fracture face, ft.

The equation for pressure distribution in the fracture is given by Equation (5)

$$p_f(x) = p_e - \left( p_e - p_w \right) e^{\sqrt{c}(x-x_i)}$$

where $p_f(x)$ is the fracture pressure at point $x$, psia.
FIGURE 2 An enlarged view of a fracture

FIGURE 3 The geometric parameters of the fracture

3 CASE STUDY

Figures 4 and 5 show the log-log plot of well 8-2 and well 12-2 in the Fuling Shale gas reservoir published by Pang et al.3 Obvious pseudo-linear flow and pseudo-steady state flow are observed in the log-log plot, the slopes of pseudo-linear and pseudo-steady flow respectively are 1/2 and 1.

Figure 6 shows the log-log diagnostic plot in a North American shale gas field published by George Stewart,8 where the x-coordinate is the normalized material balance pseudo-time, and the y-coordinate is the \( \Delta m(p)/Q \) and the derivative of \( \Delta m(p)/Q \) indicates plateaux both for a linear and for a PSSS flow regime. Figure 7 shows the material balance plot based on \( t_e(MB - 1) \) (pseudo-time) published by George Stewart,8 which exhibits a good straight line section indicating pseudo-steady flow depletion.

To validate the accuracy of the model, two wells are chose, including a well in North American field and 12-2 well in Fuling shale gas reservoir of Sichuan field of China.

For a horizontal shale gas well in the North American field, Figure 8 shows the rate of production and pressure over time for the horizontal shale gas well published...
by George Stewart. The pseudo-steady flow occurs at 2621 hours, that is 109 days. Here are the basic parameters of the well, shale gas, and reservoir: the horizontal well length, \( L = 3000 \) ft; the average fracture half length, \( x_f = 555 \) ft; the average fracture height, \( h_f = 166 \) ft; the average fracture width, \( w_f = 0.088 \) cm; a fracture spacing, 75 ft; the fracture numbers, 40; the original volume factor of gas, \( B_{gi} = 0.0035 \); \( \phi = 0.065 \); matrix permeability, \( k_m = 0.0642 \) \( \mu \)d; the initial reservoir pressure, \( p_i = 4858 \) psia; and the reservoir temperature, \( T = 150 \) ℉. After 110 days of gas production, the bottom hole pressure, \( p_{wf} = 904.16 \) psia and the actual gas production rate, 3.05 MMscf/d. The gas production rate of the horizontal shale gas well was calculated with Equation (1), and the result is 3.13 MMscf/d. The history match of the actual gas production rates and model-calculation gas production rates is shown in Figure 9. A good match is observed between the actual gas well production and model-calculated production.

For 12-2 horizontal shale gas well in Fuling shale gas reservoir of Sichuan field of China, Figure 10 shows the rate of production and pressure over time for 12-2 well. The pseudo-steady flow occurs at 1383 hours, that is 58 days. Here are the basic parameters of the well, shale gas, and reservoir: the horizontal well length, \( L = 3937 \) ft; the average fracture half length, \( x_f = 381 \) ft; the average fracture height, \( h_f = 99 \) ft; the fracture numbers, 15; the original volume factor of gas, \( B_{gi} = 0.0035 \); \( \phi = 0.065 \); matrix permeability, \( k_m = 0.0025 \) md; the initial reservoir pressure, \( p_i = 5468 \) psia; and the reservoir temperature, \( T = 80 \) ℉. After 58 days of gas production, the bottom hole pressure, \( p_{wf} = 2397.2 \) psia and the actual gas production rate, 5.73 MMscf/d. The gas production rate of the horizontal shale gas well was calculated with Equation (1), and the result is 5.48 MMscf/d. The history match of the actual gas production rates and model-calculation gas production rates are shown in Figure 11. A good match is observed between the actual gas well production and model-calculated production.
4 | SENSITIVITY ANALYSIS

Using basic parameters of the well in the North American field, the matrix permeability value was changed, and the effect of fracture conductivity on the well dimensionless productivity was calculated with Equation (1) and is presented in Figure 12. The dimensionless productivity is defined by $Q_{sc}/Q_{omax}$, where $Q_{omax}$ is the gas productivity under the maximum fracture conductivity condition.

Figure 12 indicates that well productivity increases with the increase of fracture conductivities for different permeability values, but not proportionally; the rate of productivity increase gets smaller after fracture conductivities reach a certain quality. Meanwhile, it can be seen from the three curves in Figure 12 that the different matrix permeability corresponds to the different reasonable fracture conductivity, and the higher the matrix permeability is, the greater the reasonable fracture conductivity. When $k_m$ is 0.642μd, 0.0642μd, and 0.00642μd, respectively, the rate of productivity increase gets smaller after fracture conductivities reach about $240 \times 10^{-3} \text{md-ft}$, $160 \times 10^{-3} \text{md-ft}$, and $50 \times 10^{-3} \text{md-ft}$, respectively. Therefore, for the matrix permeability is 0.642 μd, 0.0642 μd, and 0.00642 μd, the values of reasonable dimensionless fracture conductivity are $240 \times 10^{-3} \text{md-ft}$, $160 \times 10^{-3} \text{md-ft}$, and $50 \times 10^{-3} \text{md-ft}$, respectively. In the process of fracturing for a well, in order to increase production and save fracturing costs, there is no need to generate high-conductivity fractures, the reasonable fracture conductivity should be determined based on economic evaluations in the real production process.

5 | CONCLUSIONS

An analytical model was derived in this work to predict productivity of multi-fractured shale gas wells. The following conclusions are drawn in this study.

1. Production rates calculated by the analytical productivity model are in good agreement with the actual gas wells production rates of the wells in a North American field and Sichuan gas field of China.
2. Sensitivity analysis with the analytical productivity model shows that gas production rate increases nonlinearly with fracture conductivity, that the different matrix permeability corresponds to the different reasonable fracture conductivity, and the higher the matrix permeability is, the greater the reasonable fracture conductivity. In the process of fracturing for a well, in order to increase production and save fracturing costs, there is no need to generate high-conductivity fractures, the reasonable fracture conductivity should be determined based on economic evaluations.
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APPENDIX

Derivation of productivity of multi-fractured shale gas wells under pseudo-steady flow conditions

ASSUMPTIONS

1. Pseudo-steady flow conditions (Pseudo-steady flow conditions is defined that the curves of pressure distribution with different time is a set of parallel curves).
2. Linear flow in the matrix.
3. Linear-across flow in the fracture.
4. Darcy flow dominates in the fracture.

Gas that depends on elastic expansion energy flows from the matrix into the fracture under pseudo-steady flow conditions. Compression coefficient of gas is defined.

\[ C_g = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right) \]  \hspace{1cm} (A1)

where \( C_g \) is the isothermal compression coefficient of gas, \( \text{lb/psia} \).

Then

\[ C_g V \frac{\partial p}{\partial t} = -\frac{\partial V}{\partial t} = -q(x) \]  \hspace{1cm} (A2)

and

\[ V = \phi h S_{1/2} \Delta x, \]  \hspace{1cm} (A3)

where \( \phi \) is the porosity, dimensionless.

\( \frac{\partial p}{\partial t} = \frac{q(x)}{C_g V} = -\frac{q(x)}{C_g \phi h S_{1/2} \Delta x} \) is constant \hspace{1cm} (A4)

and fundamental differential equation is

\[ \frac{\partial^2 p}{\partial y^2} = \frac{\phi \mu_g C_g}{k_m} \frac{\partial p}{\partial t} \]  \hspace{1cm} (A5)

where \( \mu_g \) is the gas viscosity, \( \text{cp} \); \( k_m \) is the matrix permeability, \( \text{md} \).

Substituting Equation (A4) into Equation (A5) yields:

\[ \frac{\partial^2 p}{\partial y^2} = \frac{\phi \mu_g C_g}{k_m} \left( -\frac{q(x)}{C_g \phi h S_{1/2} \Delta x} \right) = -\frac{q(x) \mu_g}{k_m h S_{1/2} \Delta x} \]  \hspace{1cm} (A6)

Integrated Equation (A6), a solution to Equation (A6) is obtained

\[ \frac{\partial p}{\partial y} \bigg|_{y=S_{1/2}} = 0 \]  \hspace{1cm} (A7)

where \( c_1(x) \) is an integration constant and can be determined using the boundary condition expressed by Equation (A8) as

\[ \left( \frac{\partial p}{\partial y} \right) \bigg|_{y=S_{1/2}} = 0 \]  \hspace{1cm} (A8)

so

\[ c_1(x) = \frac{q(x) \mu_g}{k_m h \Delta x} \]  \hspace{1cm} (A9)

Substituting \( c_1(x) = \frac{q(x) \mu_g}{k_m h \Delta x} \) into Equation (A9) yields

\[ \frac{\partial p}{\partial y} = \frac{q(x) \mu_g}{k_m h \Delta x} \left( 1 - \frac{y}{S_{1/2}} \right) \]  \hspace{1cm} (A10)

Separating variables, Equation (A10) is changed

\[ \int dp = \frac{q(x) \mu_g}{k_m h \Delta x} \int \left( 1 - \frac{y}{S_{1/2}} \right) dy \]  \hspace{1cm} (A11)

A solution to Equation (A11) is obtained,

\[ p = \frac{q(x) \mu_g}{k_m h \Delta x} \left( y - \frac{y^2}{2S_{1/2}} \right) + c_2(x) \]  \hspace{1cm} (A12)

where \( c_2(x) \) is an integration constant and can be determined using the boundary condition expressed by Equation (A13) as

\[ p \bigg|_{y=0} = p_l \]  \hspace{1cm} (A13)

where \( p_l \) is the fracture pressure, psia. So

\[ c_2(x) = p_l(x) \]  \hspace{1cm} (A14)

Substituting \( c_2(x) = p_l(x) \) into Equation (A12) yields

\[ p = p_l(x) + \frac{q(x) \mu_g}{k_m h \Delta x} \left( y - \frac{y^2}{2S_{1/2}} \right) \]  \hspace{1cm} (A15)

When \( y = S_{1/2} \), \( p = p_e \), then

\[ q(x) = \frac{2k_m h \Delta x}{\mu_g S_{1/2}} (p_e - p_l(x)) \]  \hspace{1cm} (A16)

And \( v(x) \) is the velocity of y-direction in the matrix, which can be expressed as

\[ v(x) = \frac{q(x)}{A} = \frac{q(x)}{h \Delta x} = \frac{2k_m}{\mu_g S_{1/2}} (p_e - p_l(x)) \]  \hspace{1cm} (A17)

The flow quantity of bi-wing fractures gas in the fracture at point \( x \) may be determined based on \( v(x) \)

\[ Q(x) = 2 \int_0^x v(x) h dx = 4 \int_0^x \frac{k_m h}{\mu_g S_{1/2}} (p_e - p_l(x)) \]  \hspace{1cm} (A18)

The average width of the fracture is \( w_f \). Darcy velocity \( v_f(x) \) in the fracture can be gained by dividing Equation (A18) by the cross sectional area of the fracture

\[ v_f(x) = \frac{Q(x)}{w_f h} \]  \hspace{1cm} (A19)

Applying Darcy’s in the fracture,

\[ v_f(x) = -\frac{k_f}{\mu_g} \frac{dp_l(x)}{dx} \]  \hspace{1cm} (A20)
and Equation (A19) equals Equation (A20)
\[
\frac{Q(x)}{w_i h} = -\frac{k_i}{\mu_g} \frac{dp_i(x)}{dx} \tag{A21}
\]

Using Equation (A18) to solve Equation (A21) yields
\[
\frac{dp_i(x)}{dx} = -\frac{4k_m}{w_i k_i S_{1/2}} \int_0^x (p_e - p_i(x)) \, dx \tag{A22}
\]

Differentiation of Equation (A22) gives
\[
\frac{d^2p_i(x)}{dx^2} = -\frac{4k_m}{w_i k_i S_{1/2}} (p_e - p_i(x)) \tag{A23}
\]

where \(p_d\) is the pressure drop in the fracture at point \(x\), psia; and \(c\) is a dimensionless variable describing the contrast between matrix and fracture conductivities.

Substituting Equation (A24) and Equation (A25) into Equation (A23) yields
\[
\frac{d^2p_d}{dx^2} = cp_d \tag{A26}
\]

Boundary Conditions. The first boundary condition is expressed as
\[
\left( \frac{dp_d}{dx} \right)_{x = 0} = 0 \tag{A27}
\]

The second boundary condition is expressed as
\[
p_d \bigg|_{x = x_i} = p_i^a = p_e - p_w \tag{A28}
\]

where \(p_i^a\) is the pressure drop in wellbore, psia.

Solution. Let
\[
p_d' = \frac{dp_d}{dx} \tag{A29}
\]

Substituting Equation (A30) into Equation (A26) yields
\[
p_d' \frac{dp_d'}{dp_d} = cp_d \tag{A31}
\]

By separation of variables, a solution to Equation (A31) yields
\[
1 \frac{dp_d'}{dp_d} = 1 \frac{dp_d^2}{c^2} + c_2 \tag{A32}
\]

Using the boundary condition of Equation (A27), \(c_2\) can be obtained
\[
c_2 = 0 \tag{A33}
\]

Substituting Equation (A33) into Equation (A32) yields
\[
p_d' = \sqrt{cp_d} \tag{A34}
\]

Substituting Equation (A32) into Equation (A26) yields
\[
\frac{dp_d}{dx} = \sqrt{cp_d} \tag{A35}
\]

Integrating Equation (A35) yields
\[
\ln (p_d) = \sqrt{cp_d} + c_3 \tag{A36}
\]

Using the boundary condition of Equation (A28), \(c_3\) can be obtained
\[
c_3 = \ln (p_i^a) - \sqrt{cp_d} \tag{A37}
\]

Substituting Equation (A37) into Equation (A36) yields
\[
\ln \left( \frac{p_d}{p_i^a} \right) = \sqrt{c} (x - x_i) \tag{A38}
\]

or
\[
p_d = p_i^a e^{\sqrt{c}(x-x_i)} \tag{A39}
\]

Substituting Equation (A24) and Equation (A28) into Equation (A39), the equation of pressure drawdown distribution in the fracture is expressed as
\[
\frac{p_e - p_i(x)}{\sqrt{cp_i}} = e^{\sqrt{c}(x-x_i)} \tag{A40}
\]

The equation for pressure distribution in the fracture is expressed as
\[
\frac{p_i(x) - p_i(x_i)}{\sqrt{cp_i}} = e^{\sqrt{c}(x-x_i)} \tag{A41}
\]

Substituting Equation (A40) into Equation (A18) yields
\[
Q(x) = \frac{4k_m h}{\mu_g S_{1/2}} \int_0^x (p_e - p_w) e^{\sqrt{c}(x-x_i)} \, dx \tag{A42}
\]

which can be integrated, resulting in the following inflow performance relationship:
\[
Q(x) = \frac{4k_m h}{\mu_g S_{1/2} \sqrt{c}} \left( p_e - p_w \right) \left( 1 - e^{-\sqrt{c}(x-x_i)} \right) \tag{A43}
\]

Then
\[
Q = 2Q(x) = \frac{4k_m h}{\mu_g S_{1/2} \sqrt{c}} \left( p_e - p_w \right) \left( 1 - e^{-\sqrt{c}(x-x_i)} \right) \tag{A44}
\]

where \(Q\) is gas flow rate for both wings of a fracture.

Substituting Equation (A16) and Equation (A41) into Equation (A15) and rearranging, pressure distribution in the matrix, \(p_m(x, y)\), can be expressed as
\[
p_m(x, y) = \frac{p_e - (p_e - p_w) e^{\sqrt{c}(x-x_i)}}{\frac{x}{\delta_{y_1}}} e^{\sqrt{c}(x-x_i)} \cdot \left( y - \frac{x^2}{\delta_{y_1}^2} \right) \tag{A45}
\]

The average reservoir pressure, \(\bar{p}\), may be taken as the average pressure in the matrix owing to small volume of fractures:
Substituting Equation (A45) into Equation (A46) and integrating the solution to Equation (A46) yields

\[
\bar{p} = \frac{1}{\phi h} \left[ \int_0^{S_{1/2}} \frac{p \, dy}{\phi h x_f S_{1/2}} \right]
\]  

(A46)

The total flow rate of gas into \( n \) fractures is expressed as

\[
Q_i = \frac{8k_m h}{\mu_g S_{1/2} \sqrt{c}} \frac{(\bar{p} - p_w)}{\left( \frac{1}{1 - e^{-\sqrt{c}S_{1/2}}} - \frac{1}{3S_{1/2}} \right)}
\]  

(A48)

When Darcy's units are converted to U.S. field units, Equation (A49) becomes

\[
Q = nQ_i = \frac{8k_m h}{\mu_g S_{1/2} \sqrt{c}} \frac{(\bar{p} - p_w)}{\left( \frac{1}{1 - e^{-\sqrt{c}S_{1/2}}} - \frac{1}{3S_{1/2}} \right)}
\]  

(A49)

Substituting Equation (A50) into Equation (A52), the productivity equation of the standard condition is given:

\[
Q_{sc} = \frac{n k_m h}{zT \mu_g S_{1/2} \sqrt{c}} \frac{(\bar{p}^2 - p_w^2)}{\left( \frac{1}{1 - e^{-\sqrt{c}S_{1/2}}} - \frac{1}{3S_{1/2}} \right)}
\]  

(A53)

where the permeability is in md, viscosity is in cp, pressures are in psia, lengths are in ft, and fluid flow rate is in cubic feet per day in reservoir condition (cf/d).

Using gas state equation

\[
\left[ \frac{\bar{p} + p_w}{2} \right] \frac{Q}{zT} = \frac{p_{sc}Q_{sc}}{z_{sc}T_{sc}}
\]  

(A51)

where \( p_{sc} \) = Standard state pressure, psia, \( T_{sc} \) = Standard state temperature, °F.

Substituting \( T_{sc} = 60°F, P_{sc} = 16.645 \text{psia}, Z_{sc} = 1.0, \) and \( Q_{sc} \) in MMscf/d into Equation (A51) gives

\[
Q_{sc} = 1.80 \times 10^{-6} \left( \frac{(\bar{p} + p_w)}{zT} \right)
\]  

(A52)