D e celerat i ng and accel erat i ng back-reaction of 
vacuum to the U ni verse expansion.

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A b s t r a c t

Back-reaction of the massless scalar field vacuum to the Universe expansion is considered. An automatic renormalization procedure based on the equations of motion instead of the Friedmann equation is used to avoid the cosmological constant problem. It is found, that the vacuum tends to decelerate an expansion of Universe if the conformal acceleration of Universe is equal to zero. In contrast, the vacuum acts as the true cosmological constant in Universe expanding with the present day acceleration. Estimation for third derivative of the Universe scale factor with respect to conformal time is presented.

K ey w or ds: Univers e accelerated expansion, renormalization of the vacuum energy

1 Introduction

It is generally well-known, that the vacuum cosmological constant calculated from the momentum cutting at the Planck level is much greater than the observed one (this problem has a long history, see [1,2,3], for instance). Thus, one needs to renormalize mean value of the vacuum energy of Universe or, more precisely, to renormalize the Friedmann equation in order to provide a reasonable value of the cosmological constant.

On the one hand, the problem stimulates a search of the new kinds of matter, which would emulate the cosmological constant [4,5,6,7,8,9]. On the other hand, the issue is a deeper understanding of the vacuum at non-stationary

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and curved space-times. Considerable successes in the renormalization program, as applied to the cosmological constant problem was reached both earlier \([10,11]\) and at present \([12,13]\). However, one has noted that the aim of renormalizations (treated in the QED style) is to exclude the ultraviolet (UV) cutoff from final result. While after advance in the string theory, it is possible to guess the UV cutoff at the Planck level. Thus, renormalizations involving the UV cutoff and providing a reasonable value of the cosmological constant are acceptable in principle.

It is already mentioned, that the UV cutoff for the direct vacuum energy counting results in the cosmological constant value, which is enormously large \([1,2,3]\). However, as it was shown in Ref. \([14]\), the cosmological constant can be proportional to the squared Hubble constant if one uses the square root from the dispersion of the vacuum energy fluctuations instead of the vacuum energy itself. Calculations of these fluctuations in the different ways and with the different argumentation have been presented \([15,16,17,18,19,20,21]\). Nevertheless, identification of the cosmological constant source with the vacuum energy fluctuations is still ungrounded at the fundamental level.

As it will be shown below, the true value of the cosmological constant can be obtained by consideration of another quantity, this is the mean value of difference of the potential and kinetic energies of field oscillators in the vacuum state. Under the UV cutoff, this quantity turns out to be much smaller than the sum of these energies appearing in an ordinary reasoning concerning the cosmological constant problem. The corresponding renormalization procedure is based on the quasi-classical second-order equation of motion for the Universe scale factor instead of the Friedmann equation. This procedure can be considered as a quasi-classical limit of the quantization scheme for equations of motion suggested in \([22,23]\) to describe a dynamics of the quantum Universe, which violates the Hamiltonian constraint \(H = 0\) (i.e. the Friedmann equation).

As a result of such renormalization procedure, it is possible to connect the observed value of the Universe acceleration with the present day Hubble constant and with the vacuum state of a massless scalar field.

In addition, this allows predicting the third derivative of the Universe scale factor, which can be observable within the next few years.
Let us begin with the Einstein action for a single-component real scalar field:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( \frac{1}{2} \frac{\partial^2}{\partial t^2} - R + \frac{1}{2} \right) \left( \frac{1}{2} \right)^2 m^2 \rho^2; \tag{1}$$

and consider a homogeneous, isotropic and at Universe with the metric:

$$ds^2 = a^2(\sigma) N^2(\sigma) d\sigma^2. \tag{2}$$

Taking into account an inhomogeneity of the scalar field, one can write:

$$S = V \int d^4x \sqrt{-g} \left( \frac{1}{2} \frac{a^{02}}{N} \frac{\partial}{\partial t} \frac{a^{00}}{N} + \frac{1}{2} a^{2k2k} + \frac{1}{2} a^{2m2k} \right); \tag{3}$$

where $M_p = \frac{\hbar}{4\pi G}$ is the Planck mass and $V$ is the three-dimensional volume. Further, we shall fix the gauge by the choice of $N = 1$. Thus, we shall use the conformal time $\gamma$ which is connected with the cosmic time by relation $\gamma = a(\sigma)$. Prime means the differentiation over $\sigma$, and $k$ is the Fourier-transform of $(r) = \int_k k e^{ikr}$. Redefinitions $a^2 = V$ and $m^2 = \frac{1}{V} \sum_{k} k^2 a^2$ allow omitting the volume $V$ in intermediate calculations. Corresponding Hamiltonian is

$$H = \frac{p_a^2}{2M^2} + \frac{x}{2a^2} a^{2k2k} + \frac{1}{2} a^{2m2k}; \tag{4}$$

where momentum is connected with the velocities by the relations $a^{0k} = \frac{p_a^k}{M^2}$ and $a^{00} = \frac{p_a^{00}}{M^2}$. In terms of velocities the Hamiltonian is:

$$H = \frac{1}{2} M^2 \left( a^{02} \right) + \frac{1}{2} a^{2k2k} + \frac{1}{2} a^{2m2k}.$$ \tag{5}

Equations of motion can be obtained by varying the action (3) or from the Hamiltonian (4) by the means of the Poisson brackets [23]:

$$\begin{align*}
\frac{\partial}{\partial \sigma} a^{0k} + \left( k^2 + a^2 m^2 \right) \frac{\partial}{\partial \sigma} a^{00} + 2 \frac{\partial}{\partial \sigma} \frac{a^{00}}{a^{2k2k}} a^{2m2k} = 0; \tag{6}
M^2 a^{0k} + \frac{1}{a} a^{2k2k} \frac{\partial}{\partial \sigma} a^2 m^2 = 0. \tag{7}
\end{align*}$$

It should be noted, that, besides, there is the Friedmann equation $H = 0$ arising from variation of the action (3) over $N$. Thus, at a classical level, one may add to Eq. (7) the equation $H = 0$ multiplied by some coefficient (here $H$ is given by Eq. (5)). Our postulate is that at a quantum level, one has to use the
equation of motion for the Universe scale factor containing exact difference of the potential and kinetic energies. In the general case of a massive scalar field, one cannot build such an equation. However, it is possible for a massless scalar field. Moreover, in the conformal time, it is not necessary to add additional terms to Eq.(7) in order to provide a mutual difference of the potential and kinetic energies of a scalar field, since it already contains their exact difference in the case of $m = 0$.

3 Quasiclassical picture

Let us consider Universe scale factor as a classical quantity in the equations:

$$\hat{\phi}^0_k + k^2 \hat{\phi}^0_k + 2\frac{a^0}{a} \hat{\phi}^0_k = 0;$$

$$M^2_p a^0 + \frac{1}{a} X a^2 < \phi^0_k \phi^0_k > - a^2 k^2 < \phi^0_k \phi^0_k > = 0;$$

and use the mean values $< \phi^0_k \phi^0_k >$ and $< \phi^0_k \phi^0_k >$ in the second equation. According to Eq. (6), a quantum massless scalar field evolves at a classical background. For the stationary space-time $a = \text{const}$, the "masses" of the field oscillators (which are $a^2$) are constant. According to the virial theorem, in the ground state, the mean value of the potential energy is equal to that of the kinetic one. Consequently, there exists the natural solution $a = \text{const}$ of Eqs. (8,9) with the scalar field oscillators in a ground state.

Quantization of the scalar field [10]

$$\hat{\phi}(r) = \sum_k \hat{a}^+_{k} \phi_k(r) e^{ikr} + \hat{a}_{k} \phi_k(r) e^{ikr}$$

leads to the operators of creation and annihilation with the commutation rules $[\hat{a}_k; \hat{a}_k^+] = 1$. The complex functions $\phi_k(r)$ are $\phi_k(r) = \frac{1}{a^{2k}} e^{ikr}$ for $a = \text{const}$. In the general case they are solutions of the equation

$$0^0_k + k^2 k^0_k + 2\frac{a^0}{a} k^0_k = 0$$

and satisfy to the relation [10,25]:

$$a^2 (\phi_k^0 k^0_k k^0_k) = i.$$  

Fourier components of the scalar field can be written in the form

$$\hat{\phi}_k = \frac{1}{V} \int \hat{\phi}(r) e^{ikr} d^3r = \hat{a}^+_{k} \phi_k(r) + \hat{a}_{k} \phi_k(r);$$
The corresponding momentums are:

\[ \hat{p}_k = a^2 ( \hat{a}_k^0 \hat{a}_k^0 )^0_k = a^2 ( \hat{a}_k^+ \hat{0}_k + \hat{0}_k \hat{a}_k^0 ) : \]  

(14)

This gives the ordinary commutation relations \[ [\hat{p}_k; \hat{a}_k] = i. \]

Let us assume that the field is in the vacuum state, then

\[ a^2 < 0j^0_k \hat{a}_k^0 | 0 \hat{p}_k^0 \hat{a}_k^0 | k > = k^2 a^2 < 0j^0_k \hat{a}_k^0 | 0 \hat{p}_k^0 \hat{a}_k^0 | k > = a^2 ( \hat{a}_k^0 \hat{0}_k + \hat{0}_k \hat{a}_k^0 ) = \Theta_k; \]  

(15)

where \[ k = \frac{1}{2} a^2 ( \hat{a}_k^0 + \hat{0}_k \hat{a}_k^0 ). \]

It should be noted that, besides the considered Heisenberg picture of the field oscillators, the Schrodinger picture exists, too. In this picture, the operators \[ \hat{p}_k \] and \[ \hat{a}_k \] do not depend on the time, but the states of the field oscillators do depend (see [25] and references therein). Eigenstates of the Emakov-Lew is invariant can be build in such a representation [26,27], and the lowest eigenstate is the vacuum state.

Now we turn to consideration of the vacuum back-reaction to the Universe scale factor evolution. The simplest way of that is to calculate quantity \[ \hat{p}_k^0 \hat{a}_k^0 \] for the different kinds of dependence \[ a \] on (the Universe expansion law). For instance, let us take \[ a( ) \] in the form

\[ a( ) = a_0 (1 + \frac{\dot{H}}{a_0} )^{-1} ; \]  

(16)

to satisfy \[ a(0) \] \[ a_0 \] and \[ H \] \[ a_0^0=a_0 \text{at } = 0, \] which will be identified with the present moment of time. The conformal Hubble parameter \[ H \] is connected with the ordinary one as \[ H = \frac{\dot{a}}{a} = \frac{1}{3} \dot{H} = \frac{\dot{a}^2}{a^2}, \] where dot is the derivative over the cosmic time. Two values of the parameter: \[ = 1 \] and \[ = 1=2 \] are of particular interest.

In the first case, the conformal acceleration of Universe \[ a^0 \] equals to zero. Dependence of the scale factor on the cosmic time has the form: \[ a(t) = a_0 \frac{1 + \frac{\dot{H}}{a_0} t}{a_0}. \]

In the second case, the parameter \[ = 1=2 \] gives the conformal acceleration \[ \frac{\dot{a}^2}{a^2} = 3=2, \] which corresponds to \[ \frac{\dot{a}^2}{a^2} = \frac{\dot{a}^2}{a^2} \frac{1}{1=2} \] in the cosmic time units and is close to the observational data \[ \frac{\dot{a}^2}{a^2} = 0.55 \] [28]. Dependence of the scale factor on the cosmic time reads as \[ a(t) = a_0 \frac{1 + \frac{\dot{H}}{a_0} t}{a_0} \text{ for } = 1=2. \]

For dependence \[ a( ) \] in the form of Eq. (16), Eqs. (11), (12) are satisfied by

\[ k( ) = \frac{\sqrt{3} - \frac{\dot{H}}{a_0} H}{2a_0} \exp \frac{ik}{H} \left( 1 + \frac{1}{2} \frac{\dot{H}}{H} \right) (1 + \frac{\dot{H}}{H})^{-1=2} (1=2) \frac{k}{H} + \frac{k}{H} + \frac{k}{H} + \frac{k}{H} ; \]  

(17)
where $H_n^{(2)}(x) = J_n(x)$ $i\gamma(x)$ is the Hankel function of the second kind. When $H \rightarrow 0$, the function $k(x)$ tends to $\frac{1}{a_0} e^{ik}$. For the case of $k = 1$ calculation of $k$ gives

$$k = \frac{H}{2k(1 + \frac{1}{H})};$$

(18)

According to the relation

$$a^0 a = \frac{1}{M_p^2 V_k} x^0_k;$$

(19)

(dependence on the volume $V$ is restored), which follows from Eqs. (9,15), the vacuum back-reaction is given by

$$\frac{1}{M_p^2 V_k} x^0_k = \frac{1}{(2)^3 M_p^2 V_k} Z_{k_{m_{ax}}^0_k} k_{m_{in}} 0 d^3k = \frac{H^2 (k_{m_{ax}}^2 k_{m_{in}}^2)}{8 M_p^2 (1 + \frac{1}{H})^2};$$

(20)

One can see, that the vacuum causes a deceleration with the negative acceleration parameter $\frac{a^0 a}{a^0 a \frac{k_{m_{ax}}^2}{8} a_0 M_p}$ at the present time $t = 0$. Natural cutting of the physical momentums $k_{m_{ax}} = a_0$ at the level of $k_{m_{ax}}^0 = a_0$, $M_p$ results in $a^0 a = 1$. Let us remind that the starting assumption was an absence of acceleration $a^0 = 0$, given by Eq. (16) with $k = 1$. Thus, Universe with the zero conformal acceleration could exist only in a presence (besides vacuum) of some substance (like Einstein's cosmological constant) compensating the decelerating action of vacuum.

Let us turn to the case of Universe expanding with the observed acceleration corresponding to $k = 1$. Calculation of $k$ leads to

$$k = \frac{3 H^3}{2 k^3 (2 H)^3} \frac{H}{k (2 H)^3};$$

(21)

where $2 (1 ; 2 = H)$. Note, that these $k$ satisfy to Eq. (A.4) derived in Appendix, when the appropriate $a(\cdot )$ is considered.

Finally, for the vacuum back-reaction one can obtain:

$$\frac{1}{M_p^2 V_k} x^0_k = \frac{1}{(2)^3 M_p^2 V_k} Z_{k_{m_{ax}}^0_k} k_{m_{in}} 0 d^3k = \frac{H^2 (k_{m_{ax}}^2 k_{m_{in}}^2)}{4 M_p^2 (2 H)^4} + 9 H^4 \ln \frac{k_{m_{ax}}}{k_{m_{in}}};$$

(22)
This means that at \( t = 0 \) the acceleration parameter is positive:

\[
\frac{a^{(2)} a}{a^2} = \frac{k_{\text{max}}^2}{16 \, 2a_0^2 M_p^2};
\]

(23)

and is an order of \( a^0 \) \( 3=2 \) under cutting \( k_{\text{max}} = a_0, 4 \, M_p \, 3=2. \) This value corresponds to the initial assumptions. Thus, the observed acceleration of Universe can be caused by a pure vacuum of the massless scalar field under natural cutting of the momentums at the Planck level. Note, that in absence of lowest cutting is negligible and \( k_{\text{min}} \) can be chosen as \( k_{\text{min}} = H \).

It should be emphasized, that in our consideration the cosmological constant is derived not from the vacuum energy dispersion, but from the mean value of difference of the potential and kinetic energies in the vacuum state of the field oscillators.

In the near future, measurements of the third derivative of the Universe scale factor should be available [29]. It is possible to predict the value of parameter \( \frac{a^{(2)} a^0}{a^2 a^2} = 1 + \frac{a^{(2)} a^0}{a^2 a^2} \) at the present time \( t = 0 \):

\[
\frac{a^{(2)} a^0}{a a^2} = \frac{a}{a^0} \frac{a^{(2)} a^0}{a^2 a^2} = \frac{1}{H} \frac{k^2}{k^2} \frac{0}{0} = \frac{4}{4} \frac{k_{\text{max}}^2}{k_{\text{max}}^2} \frac{k_{\text{min}}^2}{k_{\text{min}}^2} + \frac{18 H^2}{9 H^2} \ln \left( \frac{k_{\text{max}}}{k_{\text{min}}} \right) - 1;
\]

(24)

where Eq. (19) was used. On the other hand, the dependence of \( a \) on \( t \) given by Eq. (16) for \( t = 1=2 \) leads to

\[
\frac{a^{(2)} a^0}{a a^2} = 3;
\]

(25)

The difference between Eqs. (24) and (25) arises because, in fact, we used the iteration procedure: i) assumption of the Universe expansion given by Eq. (16) allows ii) calculating the vacuum back-reaction. As the next step, iii) correction of the \( a(t) \) dependence is required. However, we restrict ourself to the steps i) and ii), because this allows a completely analytical consideration. In the general case, self-consistent system of equations (A.2), (A.4) (see Appendix) has to be solved.

Above, only massless scalar particles are under consideration, though one can guess that an analogous consideration can be extended to include the photon and graviton vacuum s. The question arises: what is with the vacuum of the massive particles (including in anton) to do? One can guess, that some renormalization scheme relevant immediately to the Lagrangian can be developed for this aim because the Lagrangian always contains the exact difference of the kinetic and potential energies both for massless and for massive particles.
It is worth to be noted, that only one kind of the particles has been taken into account in the estimation (24). However, it still can be applied to the real Universe. This results from its approximate independence on the upper cut, and thus its independence on the variety of the particles.

4 Conclusion

It has been shown, that the present acceleration of Universe can result from the back-reaction of the massless scalar field vacuum if the momentum cutting lies at the Planck level. The main assumption is that the renormalization procedure for the vacuum energy of the massless scalar field is based on the equation of motion for the Universe scale factor instead of the Friedmann equation. The equation of motion has the form, where the kinetic energy is subtracted from the potential one. The procedure allows the rough estimation for the parameter \( \frac{a^{\infty}}{a^{0}} = 2 \), where \( a^{0} \) is the third derivative of the Universe scale factor over the conformal time.

A Self-consistent system of equations for arbitrary state of the scalar field

It is possible to write down the system of equations for an arbitrary (not vacuum) state of the field oscillators. Let us denote mean values \( \langle \hat{k} \rangle \), \( \langle \hat{k}^2 \rangle \) and \( \langle \hat{k}^3 \rangle \). Taking the derivatives and using the relations \( \langle \hat{k} \rangle = a^2 \langle \hat{k} \rangle^0 \), \( \langle \hat{k}^2 \rangle^0 = a^2 \langle \hat{k}^2 \rangle^0 + 2a^2 \langle \hat{k}^2 \rangle^0 = a^2 \langle \hat{k}^0 \rangle + 2a \langle \hat{k}^0 \rangle = a^2 k^2 \langle \hat{k} \rangle \) lead to the equations:

\[
\begin{align*}
\langle \hat{k} \rangle^0 &= \frac{2}{a^2} \langle \hat{k} \rangle^2, \\
\langle \hat{k}^0 \rangle &= 2a^2 k^2 \langle \hat{k} \rangle^2, \\
\langle \hat{k}^2 \rangle^0 &= a^2 k^2 \langle \hat{k} \rangle^2 + \frac{k}{a^2} \quad (A.1)
\end{align*}
\]

With the equation for the scale factor:

\[
M_p^2 a^{\infty} + \frac{1}{a^2} \frac{\dot{a}}{a} = 0; \quad (A.2)
\]

the above equations give the closed system.
Eqs. (A.1) have the same form as that for the single time-dependent oscillator [30]. There exists the integral of motion of (A.1):

$$k \cdot k = \text{const} > 1 = 4 + x_k^2 \cdot k + p_k^2 \cdot k \quad (x_k p_k + x_k p_k) \cdot k; \quad (A.3)$$

where \( x_k = <k > \), \( p_k = <k > \) are the mean values of a scalar field and a momentum, correspondingly, which satisfy to the equations of motion \( x_k^2 = p_k = a^2 \) and \( p_k^0 = a^2 k^2 x_k \). Right hand side of (A.3) is also integral of motion.

The inequality (A.3) represents the uncertainty relation [30].

One can exclude \( k \) and \( k \) from Eqs. (A.1) and obtain the single equation

$$\frac{\text{d} \Phi}{\text{d} k} = (4k^2 \cdot k + \frac{\text{d} \Phi}{\text{d} k}) \frac{a_0^2}{a_0} \frac{a^0}{a} + 4 \frac{a_0^2}{a^2} \cdot k^2 \cdot a_0; \quad (A.4)$$

where \( k \) and \( k \) are expressed through \( k \):

$$k = \frac{\omega}{4k^2 a_0^2 a} + \frac{a}{2k^2 a^2} \frac{a_0}{a}; \quad k = \frac{a^3}{4a^2} + \frac{1}{2} \frac{a^2}{a_0} \frac{k^2}{a^2} \frac{a^3}{a_0} \cdot (A.5)$$

For \( x_k = 0, p_k = 0 \), one can rewrite the uncertainty relation in the form

$$(4k^2 \cdot k + \frac{\omega}{k})^2 \frac{a^2}{16a^0} + k^2 \cdot a^0 \cdot k > 1 = 4; \quad (A.6)$$

Eqs. (A.4) and (A.2) have the additional integral of motion:

$$M \cdot p \cdot a^2 + \frac{a}{2a^0} \cdot X \cdot 4k^2 \cdot k + \frac{\omega}{a_k} = \text{const} \quad (A.7)$$

which is the Friedmann equation when the right-hand side is equal to zero. However, the Friedmann equation (Hamiltonian constraint) is violated during a quantum epoch [22,23]. This violation, conserved at present leads to the non-zero right-hand side constant.

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