ON CELLULAR-COMPACT SPACES

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Abstract. As it was introduced by Tkachuk and Wilson in [6], a topological
space $X$ is cellular-compact if for any cellular, i.e. disjoint, family $\mathcal{U}$ of non-
empty open subsets of $X$ there is a compact subspace $K \subset X$ such that
$K \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

In this note we answer several questions raised in [6] by show ing that
(1) any first countable cellular-compact $T_2$ space is $T_3$, and so its cardinality
is at most $\mathfrak{c} = 2^\omega$;
(2) $\text{cov}(\mathcal{M}) > \omega_1$ implies that every first countable and separable cellular-
compact $T_2$ space is compact;
(3) if there is no $S$-space then any cellular-compact $T_3$ space of countable
spread is compact;
(4) $\text{MA}_{\omega_1}$ implies that every point of a compact $T_2$ space of countable spread
has a disjoint local $\pi$-base.

1. Introduction

A topological space $X$ is said to be $\kappa$-cellular-compact if for any cellular family $\mathcal{U}$
of open subsets of $X$ with $|\mathcal{U}| = \kappa$ there is a compact $K \subset X$ such that $K \cap U \neq \emptyset$
for each $U \in \mathcal{U}$. $X$ is cellular-compact iff it is $\kappa$-cellular-compact for all cardinals $\kappa$.

In [6, Theorem 4.13] the authors proved that the cardinality of a first countable
cellular-compact $T_2$ space does not exceed the cardinality of the continuum and
asked the natural question if this result can be extended to $T_2$ spaces:

Question 5.1 Let $X$ be a cellular-compact first countable $T_2$ space. Is it true that
$|X| \leq 2^\omega$?

In [1] a partial answer was given to this question by showing that this statement
can be extended to the class of Urysohn spaces. In Theorem 2.7 we give the full
affirmative answer by proving that, somewhat surprisingly, any first countable and
cellular-compact $T_2$ space is actually $T_3$.

In [6], under CH, a non-compact Tychonov cellular-compact space was con-
structed which is both first countable and separable. Consequently, the authors
raised the following question:

Question 5.2 Does there exist a model of ZFC in which every Tychonov cellular-
compact space that is both separable and first countable is compact?

Our Theorem 3.1 gives an affirmative answer to this question by showing that
the assumption $\text{cov}(\mathcal{M}) > \omega_1$, or equivalently $\text{MA}_{\omega_1}$ (countable), implies that every
cellular-compact separable and first countable $T_2$ space is compact.

The following two closely related questions were also raised in [6].

Question 5.3 Let $X$ be a hereditarily separable cellular-compact space. Must $X$ be
compact?

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2. When $T_2$ implies $T_3$ for cellular-compact spaces

Let us recall that a subset $Y \subseteq X$ is called strongly discrete (SD) iff there is a disjoint neighborhood assignment $u : Y \to \tau(X)$. Equivalently, SD sets are the ranges of choice functions on cellular families. While every infinite set in a $T_2$ space has an infinite discrete subset, there are $T_2$ spaces in which some infinite sets have no infinite SD subset, see e.g. the example in § 4 of [2]. Below we introduce a property of subsets that will allow us to circumvent this inconvenient phenomenon in general $T_2$ spaces.

**Definition 2.1.** Let $X$ be any topological space. A subset $Y \subseteq X$ is called fluffy (in $X$) iff there is a cellular family $\{U_y : y \in Y\} \subseteq \tau(X)$ such that $y \in U_y$ for each $y \in Y$.

Clearly, every SD set is fluffy. Now, it turns out that every infinite set in any $T_2$ space has an infinite fluffy subset, but to prove that we need the following simple auxiliary result.

**Lemma 2.2.** Assume that $U$ is an open set in a topological space $X$ and the point $p \notin \text{int} U$. If and $A$ and $B$ are disjoint open sets in $X$ then

$$p \notin \text{int}(U \cup A) \cap \text{int}(U \cup B).$$

**Proof.** Assume, on the contrary, that $p \in V = \text{int}(U \cup A) \cap \text{int}(U \cup B)$. Then $p \notin \text{int} U$ implies $p \in W = V \setminus U \neq \emptyset$, while $W \cap B = \emptyset$ implies $W \subseteq \text{int} A \cap \text{int} B$. This, however, contradicts that $\text{int} A \cap \text{int} B = \emptyset$ because $A$ and $B$ are disjoint open sets.

**Theorem 2.3.** Every infinite set $Y$ in a $T_2$ space $X$ has an infinite fluffy subset.

**Proof.** By induction on $n < \omega$ we shall define open sets $U_n$ and points $y_n \in Y \cap \overline{U_n}$ such that the family $\{U_n : n \in \omega\}$ is cellular and for each $n$ the following holds:

$$\text{The set } Y_n = Y \setminus \text{int} \left( \bigcup_{k < n} U_k \right) \text{ is infinite.} \tag{\ast(n)}$$

Note that we trivially have $Y_0 = Y$, hence $\ast(0)$ holds.

Now, assume that we have defined $U_k$ and $y_k$ for all $k < n$ satisfying $\ast(n)$. Pick distinct points $a$ and $b$ from $Y_n \setminus \{y_k : k < n\}$ and fix an open neighborhood $A$ of $a$ and an open neighborhood $B$ of $b$ with $A \cap B = \emptyset$. Then we may apply Lemma 2.2 with $U = \bigcup_{k < n} U_k$ for all $p \in Y_n$ to obtain that

$$Y_n = [Y_n \setminus \text{int} \overline{U \cup A}] \cup [Y_n \setminus \text{int} \overline{U \cup B}].$$

By symmetry, we may thus assume that $Y_n \setminus \text{int} \overline{U \cup A}$ is infinite.
We then put $y_n = a$ and $U_n = A \setminus \bigcup_{k < n} U_k$. We have to show that $y_n = a \in \overline{U_n}$. If $a \notin \bigcup_{k < n} U_k$ then we even have $a \in U_n$. If $a \in \bigcup_{k < n} U_k$ then $a$ is a boundary point of $\bigcup_{k < n} U_k$ and hence of $X \setminus \bigcup_{k < n} U_k$ as well. But then $a$ is a boundary point of $A \setminus \bigcup_{k < n} U_k$ as well because $A$ is a neighborhood of $a$. It is also obvious from this definition that

$$Y_{n+1} = Y_n \setminus \text{int} \left( \bigcup_{k < n} U_k \cup A \right),$$

is infinite, hence $*(n+1)$ is satisfied as well. This completes our induction and yields the infinite fluffy subset \{y_n : n \in \omega\} of $Y$. \hfill \square

We shall use fluffy sets to show that first countable cellular-compact $T_2$ spaces are $T_3$, as was promised in the introduction. Actually, something weaker than first countability will suffice for that, namely having \emph{countable closed pseudocharacter}.

Let us recall, see e.g. \[3\], that the closed pseudocharacter $\psi_c(p, X)$ of a point $p$ in a space $X$ is defined by

$$\psi_c(p, X) = \min \{|U| : U \subset \tau(X) \text{ and } \{x\} = \bigcap_{U \in \mathcal{U}} U = \bigcap_{U \notin \mathcal{U}} \overline{U} \}.$$  

Of course, if $X$ is $T_2$ then $\psi_c(p, X)$ is defined for all $p \in X$ and then so is

$$\psi_c(X) = \sup \{\psi_c(p, X) : p \in X\},$$

the closed pseudocharacter of $X$. Note that $p$ is an isolated point in $X$ iff $\psi_c(p, X) = 0$. Also, it is obvious that $\psi_c(p, X) \leq \chi(p, X)$ for all $p \in X$, hence $\psi_c(X) \leq \chi(X)$. Finally, if $\psi_c(p, X) = \omega$ then there is a sequence $\{V_n : n \in \omega\}$ of open neighborhoods of $p$ witnessing this which is also decreasing.

\begin{lemma}
Assume that $X$ is a $T_2$ space and $p \in X$ with $\psi_c(p, X) = \omega$, moreover $\{V_n : n \in \omega\}$ is a decreasing sequence of open neighborhoods of $p$ witnessing this.

(i) If $a_n \in V_n \setminus \{p\}$ for all $n \in \omega$ then \{a_n : n \in \omega\} \subseteq \{p\}.

(ii) If $A \subseteq G \subseteq \tau(X)$ and $p \in A'$ then there is a cellular family $\{U_n : n \in \omega\}$ of open sets such that $U_n \subseteq G \cap V_n$ and $A \cap U_n \neq \emptyset$ for each $n \in \omega$.

Proof. (i) Assume that \{a_n : n \in \omega\} is given with $a_n \in V_n \setminus \{p\}$ for all $n \in \omega$. Then \{a_n : n \in \omega\} \subseteq V_n and so \{a_n : n \in \omega\}' = \{a_n : m \geq n\}' \subseteq V_n$ for all $n$, hence

$$\{a_n : n \in \omega\}' \subseteq \bigcap_{n \in \omega} V_n = \{p\}.$$

(ii) The required sequence $\{U_n : n \in \omega\}$ together with points $a_n \in A \cap U_n$ will be constructed by induction on $n$ in such a way that $U_n \subseteq V_n \cap G$ and $p \notin \overline{U_n}$ for all $n < \omega$. Assume that we have constructed $\{U_k : k < n\}$, then $p \in W_n = V_n \setminus \bigcup_{k < n} U_k$, hence $p \in (A \cap W_n)'$. We may then pick a point $a_n \in (A \cap W_n) \setminus \{p\}$ and an open neighborhood $U_n$ of $a_n$ such that $U_n \subseteq G \cap W_n$ and $p \notin \overline{U_n}$. So the inductive step can be carried out and the family $\{U_n : n \in \omega\}$ with the required properties is constructed. \hfill \square

\begin{definition}
The space $X$ is said to have property $(a_n)$ (resp. $(a_n^{-})$) iff every SD-subset of $X$ of size $\kappa$ has compact closure (resp. has a complete accumulation point).

\begin{lemma}
Assume that $X$ is a $T_2$ space with property $(a_n^-)$ and $Y \subset X$ is fluffy such that $\psi_c(y, X) = \omega$ for each $y \in Y$. Then there is an SD subset $D$ of $X$ with $|D| = |Y|$ for which $Y \subset \overline{D}$. Moreover, if $X$ is also $|Y|$-cellular-compact then $D$ can be chosen so that $\overline{D}$, and hence $\overline{Y}$ as well, is compact.

\end{definition}
Proof. Since any finite subset of a $T_2$ space is SD, we may assume that $Y$ is infinite. Now, fix a cellular family \( \{ U_y : y \in Y \} \subset \tau(\mathbb{X}) \) witnessing that $Y$ is fluffy. By Lemma 2.4 for each $y \in Y$ there is a cellular family $U_y = \{ U_{y,n} : n < \omega \} \subset U_y$ such that if $a_{y,n} \in U_{y,n}$ for each $n < \omega$ then $\{ a_{y,n} : n \in \omega \} \not\subset \emptyset$, hence $y \in \{ a_{y,n} : n \in \omega \}$. Clearly, then $D = \{ a_{y,n} : y \in Y, n \in \omega \}$ is as required.

If, in addition, $X$ is $[Y]$-cellular-compact then there is a compact subset $K$ of $X$ such that $K \cap U_{y,n} \not\subset \emptyset$ for all $y \in Y$ and $n \in \omega$, hence all the points $a_{y,n}$ may be chosen from $K$, and then $\overline{D} \subset K$. \hfill \Box

We are now ready to present the main result of this section.

Theorem 2.7. Let $X$ be a $T_2$ space of countable closed pseudocharacter and assume that the closure of every countable SD subset of $X$ is countably compact. Then $X$ is countably compact, $T_3$, and first countable.

Proof. By Theorem 2.3 any infinite subset of $X$ has a countably infinite fluffy subset, hence the countable compactness of $X$ follows if we show that every countably infinite fluffy set $Y \subset X$ has an accumulation point in $X$. If $Y$ contains infinitely many isolated points then this is immediate from our assumption because any set of isolated points is SD. So, we may assume that $\psi_y(y,X) = \omega$ for each $y \in Y$. But our assumption clearly implies that $X$ possesses property $(a^-_\omega)$, hence in this case we may apply Lemma 2.6 to conclude that even $\overline{Y}$ is countably compact.

Next, to show that $X$ is $T_3$, consider a closed set $F \subset X$ and a point $p \not\in F$. Of course, we may assume that $p$ is not isolated. Then we have $\psi_y(p,X) = \omega$ and we have a strictly decreasing sequence $\{ V_n : n \in \omega \}$ of open neighborhoods of $p$ such that $\bigcap_{n \in \omega} \overline{V_n} = \{ p \}$. We claim that there is an $n \in \omega$ for which $F \cap V_n = \emptyset$, hence $F$ and $p$ do have disjoint neighborhoods.

Assume, on the contrary that $F \cap V_n \not\subset \emptyset$ for each $n$. Then, we may clearly pick distinct points $x_n \in F \cap \overline{V_n}$ and by the countable compactness of $X$ the infinite set $\{ x_n : n \in \omega \}$ has an accumulation point $x \in F$. This, however contradicts part (i) of Lemma 2.3.

Finally, it is well-known that in a countably compact $T_3$ space every point of countable pseudocharacter actually has countable character, hence $X$ is indeed first countable. \hfill \Box

Clearly, if $X$ has property $(a^-_\omega)$ then $X$ is $\kappa$-cellular-compact. If, on the other hand, $X$ is $T_2$ and $\psi_y(X) \leq \omega$ then the reverse of this implication also holds. In fact, by Lemma 2.4 then $\kappa$-cellular-compact implies that every fluffy subset of $X$ of size $\kappa$ has compact closure. In particular, if $X$ is $\omega$-cellular-compact then it satisfies all the assumptions of Theorem 2.7.

3. When cellular-compact implies compact

In [6] the authors raised several questions if cellular-compact spaces with certain properties are necessarily compact. For instance, they constructed with the help of CH a first countable and separable cellular-compact Tychonov space that is not compact and asked if this can be done in ZFC. Our next result gives a negative answer to this question.

Theorem 3.1. If $\text{cov}(\mathcal{M}) > \omega_1$ then every first countable and separable cellular-compact $T_2$ space is compact.

We actually start with giving a preparatory result. We recall that a $T_2$ space $X$ is called $\pi$-regular if every non-empty open set in $X$ includes a non-empty regular closed set. This property is clearly weaker than regularity.
We also recall that $\text{cov}(\mathcal{M}) > \mu$, the statement that the covering number of the meager ideal $\mathcal{M}$ on the reals is $> \mu$, is equivalent with $M_{\mu}(\text{countable})$, as we shall use it in this form.

**Lemma 3.2.** Assume that $X$ is a $\pi$-regular space of countable $\pi$-weight with $\mathcal{A}$ a fixed (countable) $\pi$-base of $X$, moreover $H$ is a nowhere dense subset of $X$ such that $|H| < \text{cov}(\mathcal{M})$ and

$$\sup\{\chi(h,X) : h \in H\} < \text{cov}(\mathcal{M}).$$

Then $\mathcal{A}$ has a disjoint subfamily $\mathcal{B}$ that is simultaneously a local $\pi$-base for all points $h \in H$. Consequently, if $X$ is also cellular-compact then $\overline{H}$ is compact.

**Proof.** Let us consider the countable partial order $\mathcal{P} = \langle P, \supseteq \rangle$, where

$$P = \{p \in [\mathcal{A}]^{<\omega} : p \text{ is disjoint and } \overline{p} \cap \overline{H} = \emptyset\}.$$ 

For any open set $U$ we let

$$D_U = \{p \in P : \exists V \in p \; V \subset U\}.$$ 

**Claim.** For every open set $U$ with $U \cap H \neq \emptyset$ the family $D_U$ is dense in $\mathcal{P}$.

Indeed, consider an arbitrary $p \in P$. Then $U \cap H \neq \emptyset$ implies that the open set $W_0 = U \setminus \overline{p} \neq \emptyset$. Since $H$ is nowhere dense, $W_1 = W_0 \setminus \overline{H} \neq \emptyset$ as well, hence there is a non-empty open $S$ such that $\overline{S} \subset W_1$ because $X$ is $\pi$-regular. Now, if $V \in \mathcal{A}$ is such that $V \subset S$ then $q = p \cup \{V\} \in D_U$ and $q \supset p$, and the claim is verified.

For each $h \in H$ let $U(h)$ be an open neighborhood base of $h$ in $X$ of size $\chi(h,X)$ and set $U = \bigcup\{U(h) : h \in H\}$. Our assumptions then imply $|U| < \text{cov}(\mathcal{M})$, while the Claim implies that $D_U$ is dense in $\mathcal{P}$ for each $U \in U$. Then there is a filter $G$ in $\mathcal{P}$ that intersects the dense set $D_U$ for each $U \in U$. Clearly, then $B = \bigcup G$ is as required.

Also, if a set $S$ intersects all members of $B$ then obviously we have $H \subset \overline{S}$, in particular, $\overline{S}$ is compact. 

**Proof of Theorem 3.1.** First we shall prove the result with the additional assumption that $X$ is crowded, i.e. has no isolated points. Let us next observe that $X$ is actually $T_3$ by Theorem 2.7, moreover $X$ has countable $\pi$-weight being separable and first countable, so we can apply Lemma 3.2 to $X$.

To see that $X$ is compact, we shall use the fact, see e.g. Lemma 1.7 in [4], that if every free sequence in $X$ has compact closure then $X$ is compact. Now, as $X$ is crowded, every discrete subspace, in particular every free sequence is nowhere dense in $X$. But Lemma 3.2 now implies that every nowhere dense subset of $X$ of cardinality $\leq \omega_1$ has compact closure, while in a first countable compact space every free sequence is countable. These observations together imply that every free sequence in $X$ is countable. But then we can conclude that every free sequence in $X$ has compact closure and hence $X$ is compact.

In the general case, in which $X$ may have isolated points, let $I$ denote the set of all isolated points of $X$. Then we know that $\overline{I}$ is compact because $X$ is cellular-compact. But then $Y = X \setminus \overline{I}$ is a regular closed set in $X$ which clearly inherits from $X$ separability, first countability, and the cellular-compact property. Moreover $Y$ has no isolated points, hence $Y$ is compact, and so $X$ is the union of two compact subsets.

**Proof of Theorem 5.3** (resp. 5.4) of [6] asks if a cellular-compact space that is hereditarily separable (resp. has countable spread) is actually compact. Of course, the affirmative answer to 5.4 implies the same for 5.3. The authors of [6] failed to raise the same type of question for the closely related class of hereditarily Lindelöf
spaces. Our next goal is to show that in this class cellular-compact does imply compact. Interestingly, this then implies that the affirmative answer to 5.4 is consistent, at least for $T_3$ spaces.

**Theorem 3.3.** Every hereditarily Lindelöf and cellular-compact $T_2$ space is compact.

**Proof.** According to 2.10 of [3], any hereditarily Lindelöf $T_2$ space has countable closed pseudocharacter, hence if it is cellular-compact then it is countably compact by our Theorem 2.7. But countably compact Lindelöf spaces are compact. □

It is well-known that if a space $X$ of countable spread is not hereditarily Lindelöf then it has a hereditarily separable subspace $Y$ that is not hereditarily Lindelöf. So, if $X$ is $T_3$ then $Y$ is an S-space.

**Corollary 3.4.** Any non-compact but cellular-compact $T_3$ space of countable spread has an S-subspace. So, if there is no S-space then every cellular-compact $T_3$ space of countable spread is compact.

Of course, it is known that there are models of ZFC in which no S-space exits, for instance PFA implies this. The final result of this section shows that for locally compact $T_2$ spaces already $MA_{\omega_1}$ suffices to get the same result. Note that it is consistent with $MA_{\omega_1}$ that S-spaces exist, see e.g. [5].

**Corollary 3.5.** $MA_{\omega_1}$ implies that every locally compact and cellular-compact $T_2$ space $X$ of countable spread is compact.

**Proof.** Assume, on the contrary, that $X$ is not compact, then by the previous corollary $X$ has an S-subspace, let $K$ be the one point compactification of $X$. Then clearly $K$ also has countable spread, and consequently has countable tightness. But $MA_{\omega_1}$ implies that a compact space of countably tightness has no S-subspace by a theorem of Szentmiklóssy [5]. We have reached a contradiction. □

It is an immediate consequence of this result and of Theorem 3.13 of [6] that, under $MA_{\omega_1}$, every point of a compact $T_2$ space of countable spread has a disjoint local $\pi$-base. Thus $MA_{\omega_1}$ implies a consistent affirmative answer to Question 5.8 of [6] that asked this for hereditarily separable compacta.

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