On Robin’s criterion for the Riemann Hypothesis

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Abstract: Robin’s criterion says that the Riemann Hypothesis is equivalent to
\[ \forall n \geq 5041, \quad \frac{\sigma(n)}{n} \leq e^\gamma \log_2 n, \]
where \(\sigma(n)\) is the sum of the divisors of \(n\), \(\gamma\) represents the Euler–Mascheroni constant, and \(\log_i\) denotes the \(i\)-fold iterated logarithm. In this note we get the following better effective estimates:
\[ \forall n \geq 3, \quad \frac{\sigma(n)}{n} \leq e^\gamma \log_2 n + \frac{0.3741}{\log_2 n}. \]
The idea employed will lead us to a possible new reformulation of the Riemann Hypothesis in terms of arithmetic functions.

Keywords: Primorial number, Robin’s inequality, Riemann Hypothesis.

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1 Introduction and statement of results

As usual, let \((p_k)_{k \geq 1}\) denote the increasing sequence of prime numbers, and let \(N_k\) be the primorial integer of index \(k\), the product of its \(k\) first terms. The Riemann Hypothesis (RH) claims that the
nontrivial zeros of zeta function \( \zeta(s) = \sum_{n \geq 1} n^{-s} \) are located on the critical line \( \Re(s) = \frac{1}{2} \). Several equivalent formulations of RH appeared, but the one which interests us here is that in terms of arithmetic functions, here we cite the first papers of Gronwall [8], Nicolas [11] and Robin [13], followed by, for instance, Akbary [1], Caveney et al. [6] and Lagarias [10].

Robin in his paper [13] asserted that RH is equivalent to
\[
\forall n \geq 5041, \quad \sigma(n) \leq e^\gamma n \log_2 n,
\]
with \( \sigma(n) \) denotes the sum of divisors function, \( \gamma \) the Euler–Mascheroni constant, and \( \log_i \) the \( i \)-fold iterated logarithm. This assertion is based on the known following formula (see [9]):
\[
\frac{\sigma(n)}{n} = (1 + o(1)) e^\gamma \log_2 n.
\]

In this note, we intend to join the authors who have attempted to closely determine the \( o \)-term in the formula (2). The best upper bound of the normalized of the sum of divisors function is also given by Robin [13] which proved, unconditionally, that
\[
\forall n \geq 3, \quad \frac{\sigma(n)}{n} \leq e^\gamma \log_2 n + \frac{0.6483 \log_2 n}{\log_2 n}.
\]

We propose the following result:

**Theorem 1.1.** For every integer \( n \geq 3 \), we have
\[
\frac{\sigma(n)}{n} \leq e^\gamma \log_2 n + \frac{0.3741 \log_2 n}{\log_2 n}.
\]

This improves considerably Robin’s upper bound. In parallel, we study another form of upper bound than that exposed in the theorem above, since it is completely expressed in terms of \( K(x) \), the primorial counting function which, see Balazard [4], is approximately \( \frac{\log x}{\log_2 x} \). We conclude that:

**Theorem 1.2.** If \( K(n) \) is the number of primorial integers not exceeding \( n \), then
\[
\forall n \geq 30, \quad \frac{\sigma(n)}{n} \leq e^\gamma \left( \log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} + \frac{1}{20 \log_2^2 K(n)} \right).
\]

This leads us to examine a conjecture upon which we stumbled:

**Conjecture 1.** The Riemann Hypothesis is equivalent to
\[
\forall n \geq 205, \quad \frac{\sigma(n)}{n} \leq e^\gamma \left( \log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right).
\]

See Section 4 for more background on this conjecture. The main ingredient of this paper is the recent version of the upper bound of the product over primes \( \prod_{p \leq x} \frac{p}{p-1} \), thanks to the paper of the third author in [7], as a consequence of the new estimates of Chebyshev’s summatory functions also exposed in [7]. Although there are some updates, such improvements have negligible influence on the final results. Finally, we indicate that \( e \) represents Napier’s constant, \( p \) a prime number, and with this technique, obtaining better approximations is closely linked with progress on extending the known zero-free region of the Riemann zeta-function.
2 Preliminary lemmas

The primorial counting function $K(x)$ is not known in the literature. We begin by showing some basic properties (for a more extended study, see the recent paper of the authors [3]). For each real $x \geq 1$, the integer $K(x)$ can be defined by $\max \{ k \in \mathbb{N}^*, \, N_k \leq x \}$. In the following lemma, we prove that for a given $x \geq 1$, the primorial $N_{K(x)}$ represent the smallest integer less than $x$ whose decomposition into prime numbers is the longest. Here $\omega(n)$ denotes the number of prime distinct divisors of $n$.

**Lemma 2.1.** For every real number $x \geq 1$, we have

$$K(x) = \max_{1 \leq n \leq x} \omega(n).$$

Furthermore, for any integer $n \leq x$ with $\omega(n) = K$, we have $N_K \leq n$.

**Proof.** As $N_k \leq n \leq N_{k+1}$ means that $\omega(n) \leq k$ and $K(n) = k$, hence $\omega(n) \leq K(n)$ in any interval $[N_k, N_{k+1}]$, which implies that

$$\max_{1 \leq n \leq x} \omega(n) = \max_{1 \leq n < N_{K+1}} \omega(n) = K.$$

Let $q_1 q_2 \cdots q_K$ be an integer less than $x$ with $q_1 < q_2 < \cdots < q_K$ prime numbers. For $K = 1$ it is obvious that $q_1 \geq p_1$. Now, assuming $q_i \geq p_i$ for $i < K$, it is necessary that $q_K \geq p_K$, otherwise $q_K < q_{K-1}$. □

**Lemma 2.2.** We have, when $x \geq 8$, the following inequalities:

$$\log_2 x < K(x) \leq \log x.$$

**Proof.** From the definition of $K(x)$, by taking the logarithm, we can also write the following:

$$K(x) = \max \{ k \in \mathbb{N}^*, \, \theta(p_k) \leq \log x \},$$

where $\theta$ denotes the Chebyshev function. So, by recalling the inequality $\theta(p_k) \geq k$ given in Robin [12] valid once $k \geq 3$, one easily deduces that

$$K(x) \leq \max \{ k \in \mathbb{N}^*, \, k \leq \log x \} \leq \log x, \, \forall x \geq N_3,$$

which is also valid for $8 \leq x < N_3$. For the second, a short induction on $k$ is necessary. For all $k \geq 1$, we have $N_k < e^{e^{k-1}}$. Indeed, the case $k = 1$ is obvious, and the fact that $\forall k \geq 1, \, p_{k+1} < N_k$ (according to Euclid’s proof of the infinity of primes) implies that

$$N_{k+1} = N_k p_{k+1} < e^{e^{k-1}} N_k < e^{2e^{k-1}} < e^{e^{k-1}} = e^{e^k}.$$

So, by taking the logarithm, one gets that for all $x \geq e$:

$$\log_2 x < \log_2 N_{K+1} < K(x).$$

We conclude the proof using computer verifications for the small values. In relation to $\pi(x)$ the prime counting function, we can also mention that

$$\log_2 x < K(x) \leq \log x < \pi(x).$$

□
Lemma 2.3. Let $\delta = 1.000081$. We have, when $x \geq 210$:

$$K(x) \geq \frac{1}{\delta} \frac{\log x}{x}.$$

Proof. Recalling the following estimates given in [14]:

$$\theta(x) < \delta x, \forall x > 1 \quad \text{and} \quad \pi(x) \geq \frac{x}{\log x}, \forall x \geq 17,$$

one reaches successively, for every real $x \geq e^{17\delta}$, that

$$K(x) \geq \max \{k \in \mathbb{N}^*, \delta p_k \leq \log x\} = \pi \left( \frac{\log x}{\delta} \right) \geq \frac{1}{\delta} \frac{\log x}{x}.$$

A computer check handles the cases $210 \leq x < e^{17\delta}$. \hfill \Box

Now, for $f$ a decreasing function greater than $1$ on $(1, \infty)$, we consider the following sequence

$$\mathcal{L}(n) = \prod_{p|n} f(p), \forall n > 1.$$

The term $\mathcal{L}(n)$ for the function $f(x) = \frac{x}{x-1}$ is only $\frac{n}{\varphi(n)}$, where $\varphi(n)$ denotes the Euler totient function, and $\mathcal{L}(n)$ is $\frac{\Psi_t(n)}{n}$ when $f(x) = 1 + 1/x + \cdots + 1/x^{t-1}$, $t \geq 2$, where $\Psi_t(n)$ is the generalized Dedekind psi function. We have the following Lemmas

Lemma 2.4. For every real number $x \geq 2$, the following equality

$$\max_{1 < n \leq x} \mathcal{L}(n) = \prod_{p \leq PK(x)} f(p)$$

holds.

Proof. To determine the maximum of $\mathcal{L}(n)$, when $n$ range over all integers less than or equal to $x$, we first use the fact that $f$ is greater than $1$ since this places the maximum at the class of the integers whose number of prime divisors is the largest. Then, as $f$ is also strictly decreasing, the maximum must have the smallest prime numbers in its decomposition. However, according to the previous lemma, we can clearly specify that, it is only true for $N_{K(x)}$, i.e.,

$$\max_{1 < n \leq x} \mathcal{L}(n) = \mathcal{L}(N_{K(x)}).$$

Finally, as $p | N_k$ is equivalent to $p \leq p_k$, the lemma follows. \hfill \Box

Remark 1. When $f$ is strictly increasing and greater than $1$ on $(1, \infty)$, the maximum of $\mathcal{L}(n)$ is reached at an integer $q_1 \cdots q_{K(x)}$, where at least one of $q_i$ is a prime number greater than $p_i$.

In the following lemma, we leave the generalization and show, through a simpler proof, a result concerning the order of the Euler function.

Lemma 2.5. We have

$$\limsup_{n \to +\infty} \frac{n}{e^{\gamma} \varphi(n) \log_2 n} = 1.$$
Proof. From the previous lemma and the definition of $K(n)$, we deduce that

$$\frac{\mathcal{L}(n)}{\log_2 n} \leq \frac{\mathcal{L}(N_{K(n)})}{\log_2 N_{K(n)}}.$$ 

So, our limit becomes as follows:

$$\limsup_{n \to +\infty} \frac{\mathcal{L}(n)}{e^\gamma n \log_2 n} = \lim_{k \to +\infty} \frac{\mathcal{L}(N_k)}{e^\gamma \log_2 N_k}.$$ 

In particular, when $f(x) = \frac{x}{x-1}$, one obtains according to Mertens’ theorem that

$$\mathcal{L}(N_k) = \prod_{p \leq p_k} \frac{p}{p - 1} \sim e^\gamma \log p_k,$$

as $k \to +\infty$. Thus, the lemma follows by recalling that

$$\log_2 N_k = \log(\theta(p_k)) \sim \log p_k,$$

using the Prime Number Theorem. \qed

Every proof containing explicit results requires at some point or another a digital verification of the property obtained on the finite number of cases that remain. In our case, we need to compute the values of $\sigma(n) e^{\gamma n \log_2 n}$ for fairly large $n$. We will use the result of Briggs [5], where he checked Robin’s inequality up to $10^{10}$.

**Lemma 2.6** (Briggs). Robin’s criterion holds, for $5040 < n \leq 10^{10}$. We end this section by mentioning the following recent explicit bounds of $\theta(x)$ and $\prod_{p \leq x} (1 - \frac{1}{p})$.

**Lemma 2.7** (Dusart). The following estimates hold

$$\theta(x) \geq x \left(1 - \frac{0.01}{\log^3 x}\right), \text{ as soon as } x \geq 7232121212. \quad (4)$$

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \leq e^\gamma \log x \left(1 + \frac{0.2}{\log^3 x}\right), \text{ when } x \geq 2278382. \quad (5)$$

$$\theta(p_k) \geq k \left(\log k + \log_2 k - 1 + \frac{\log_2 k - 2.050735}{\log k}\right), \text{ when } p_k \geq 10^{11}. \quad (6)$$

$$p_k \leq k \left(\log k + \log_2 k - 1 + \frac{\log_2 k - 1.95}{\log k}\right), \text{ when } k \geq 178974. \quad (7)$$

3 Proof of Theorem 1.1

To begin with, for $n$ such that $K := K(n) \geq K_1 = 164607$ we have $p_K \geq 2228382$. This implies by Lemmas [2.4, 2.7] that

$$\frac{n}{\varphi(n)} \leq \prod_{p \leq p_K} \frac{p}{p - 1} \leq e^\gamma \log p_K \left(1 + \frac{0.2}{\log^3 p_K}\right). \quad (8)$$

On the other hand, according to inequality (4), once $K \geq K_2 = 7232121212$, it follows that

$$\log_2 N_K = \log \theta(p_K) \geq \log p_K - \frac{0.01}{\log^3 p_K}. \quad (9)$$

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Now, with some care, one can write for $K \geq K_2$ the following

$$e^\gamma \log p_K \left(1 + \frac{0.2}{\log^3 p_K}\right) = e^\gamma \log p_K + \frac{0.2 e^\gamma}{\log^2 p_K}$$

$$= e^\gamma \log p_K \left(1 - \frac{0.01}{\log^2 p_K}\right) + \frac{(0.2 + 0.01)e^\gamma}{\log^2 p_K}$$

$$= e^\gamma \log p_K \left(1 - \frac{0.01}{\log^3 p_K}\right) + 0.3741$$

Hence, taking into account that the function $e^\gamma t + \frac{0.3741}{t^2}$ is increasing for $t \geq 1$, we easily deduce from inequality (9) that

$$e^\gamma \left(\log p_K - \frac{0.01}{\log^2 p_K}\right) + \frac{0.3741}{\log^2 p_K} < e^\gamma \log_2 N_K + \frac{0.3741}{\log_2 N_K},$$

and then

$$\frac{n}{\varphi(n)} \leq e^\gamma \log_2 N_K + \frac{0.3741}{\log_2 N_K}, \forall K \geq K_2.$$  

By computer, the last inequality is shown to be also valid when $2 \leq K < K_2$. Consequently, invoking again the increase of the function $e^\gamma t + \frac{0.3741}{t^2}$, one gets for $n \geq N_2$, and then for $n \geq 3$ that

$$\frac{n}{\varphi(n)} \leq e^\gamma \log_2 n + \frac{0.3741}{\log_2 n}.$$

Finally, as the inequality $\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)}$ holds (see [13, page 193]) for $n \geq 1$, the theorem follows.  

The following direct consequence joins the upper bounds of $\frac{\sigma(n)}{n}$ in the form $(1 + \epsilon)e^\gamma \log_2 n$ given in [2] for different values of $\epsilon$. The value $\epsilon = 0.0000123$ obtained below, once $n \geq 5041$, remains stable until the best value $\epsilon = 0.005558981\ldots$ obtained in [2], as soon as $n \geq 2521$.

**Corollary 3.1.** For every integer $n \geq 5041$, we have

$$\frac{\sigma(n)}{n} \leq (1.0000123)e^\gamma \log_2 n.$$

**Proof.** The idea is to take the term $\frac{0.3741}{\log_2 n}$ from Theorem 1.1, divide it by $e^\gamma \log_2 n$, then calculate the image of $10^{10^{10}}$. The remainder is guaranteed by Lemma 2.6.  

4 **Proof of Theorem 1.2**

By inequality (5) we infer that for every $k \geq K_1 = 164607$:

$$\frac{N_k}{\varphi(N_k)} = \prod_{p \leq p_k} \frac{p}{p - 1} \leq e^\gamma \log p_k \left(1 + \frac{0.2}{\log^3 p_k}\right).$$

However; see [12], we have

$$k \log k \leq p_k \leq k(\log k + \log_2 k),$$
once $k \geq 6$. So, we obtain the following inequalities:

$$2 \log_2 k \leq \log p_k \leq \log k + \log_2 k + \frac{\log_2 k}{\log k}, \forall k \geq 6,$$

which implies successively for $k \geq K_1$:

$$\frac{N_k}{\varphi(N_k)} \leq e^\gamma \left( \log p_k + \frac{0.2}{\log^2 p_k} \right) \leq e^\gamma \left( \log k + \log_2 k + \frac{\log_2 k}{\log k} + \frac{0.2}{4 \log^2 k} \right).$$

Then, it comes by computer that the last upper bound also holds for $k \geq 10$. Hence, one gets for all $n \geq N_{10}$, according to Lemma 2.4, that

$$\frac{n}{\varphi(n)} \leq e^\gamma \left( \log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} + \frac{0.2}{4 \log^2 K(n)} \right). \quad (10)$$

Now, let us go back to the ratio $\frac{\sigma(n)}{n}$. According to [13], this quantity takes maximal values on so called colossally abundant (CA) numbers, and if Robin’s inequality is true on consecutive CA numbers $CA_i$ and $CA_{i+1}$, then it is also true for all integer $n \in [CA_i, CA_{i+1}]$. We say that $n$ is colossally abundant if there exists a positive $\epsilon$ for which:

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(k)}{k^{1+\epsilon}}, \forall k > 1.$$

Thus, to complete our proof, it suffices to check inequality (10) for $\frac{\sigma(n)}{n}$ only on the CA numbers less than $N_{10}$, namely: 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, 21621600, 367567200 and 6983776800. □

Next, this leads us to discuss a possible reformulation of RH in terms of arithmetic functions. First, we observe that the following proposition

**Proposition 1.** We have, when $205 \leq n \leq CA_{160}$, the inequality

$$\frac{\sigma(n)}{n} \leq e^\gamma \left( \log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right),$$

where $CA_{160} > 10^{326}$.

**Proof.** It suffices to check the list of terms of the sequence registered as A004490 of CA numbers in OEIS [15]. This extends the inequality to all integers between 205 and $CA_{160}$.

The following table shows part of the calculations, where $e^\gamma A(n)$ is the upper bound of Proposition 1.
| $n$                  | $\sigma(n)/n$ | $K(n)$ | $e^\gamma A(n) - \sigma(n)/n$ |
|---------------------|---------------|--------|-------------------------------|
| $CA_{150}$ = $N_{121}N_{11}N_5N_3^2N_1^4$ | 11.570817 | 127    | 0.44727552                   |
| $CA_{151}$ = $N_{122}N_{11}N_5N_3^2N_1^4$ | 11.588010 | 128    | 0.44658941                   |
| $CA_{152}$ = $N_{123}N_{11}N_5N_3^2N_1^4$ | 11.605127 | 129    | 0.44584657                   |
| $CA_{153}$ = $N_{124}N_{11}N_5N_3^2N_1^4$ | 11.622118 | 130    | 0.44509823                   |
| $CA_{154}$ = $N_{125}N_{11}N_5N_3^2N_1^4$ | 11.638937 | 131    | 0.44439327                   |
| $CA_{155}$ = $N_{126}N_{11}N_5N_3^2N_1^4$ | 11.655541 | 132    | 0.44377752                   |
| $CA_{156}$ = $N_{127}N_{11}N_5N_3^2N_1^4$ | 11.671980 | 133    | 0.44320089                   |
| $CA_{157}$ = $N_{128}N_{11}N_5N_3^2N_1^4$ | 11.688214 | 134    | 0.44270719                   |
| $CA_{158}$ = $N_{129}N_{11}N_5N_3^2N_1^4$ | 11.704291 | 135    | 0.44224879                   |
| $CA_{159}$ = $N_{130}N_{11}N_5N_3^2N_1^4$ | 11.720259 | 136    | 0.44178089                   |
| $CA_{160}$ = $N_{131}N_{11}N_5N_3^2N_1^4$ | 11.736118 | 137    | 0.44130365                   |

This completes the proof. \(\Box\)

In view of this numerical experiments the natural question is:

**Question 1.** Is it true that

\[
\frac{\sigma(n)}{n} \leq e^\gamma \left( \log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right),
\]

for all $n \geq 205$?

An answer to this question is linked to RH by the following proposition:

**Proposition 2.** If the Riemann Hypothesis hold, we have for every integer $n \geq 205$:

\[
\frac{\sigma(n)}{n} \leq e^\gamma \left( \log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right).
\]

**Proof.** This is deduced from Robin’s criterion and essentially from the fact that $A(n) \geq \log_2 n$, for every $n \geq 10^{322}$. Indeed, one gets from Lemma 2.3 that

\[
\log K(x) \geq \log_2 x - \log_3 x - \log \delta, \ \forall x \geq 3,
\]

(11)

\[
\log_2 K(x) \geq \log_3 x + \log \left( 1 - \frac{\log_3 x + \log \delta}{\log_2 x} \right), \ \forall x \geq 3,
\]

(12)

and from Lemma 2.2 the following

\[
\frac{\log_2 K(x)}{\log K(x)} \geq \frac{\log_4 x}{\log_3 x}, \ \forall x \geq 15.
\]

(13)

Thus, inequalities (11), (12) and (13) yield us for $x \geq 15$:

\[
A(x) \geq \log_2 x + \frac{\log_4 x}{\log_3 x} + \log \left( 1 - \frac{\log_3 x + \log \delta}{\log_2 x} \right) - \log \delta.
\]

By setting $\log_2 x = t$, the study of the following function:

\[
\frac{\log_4 x}{\log_3 x} + \log \left( 1 - \frac{\log_3 x + \log \delta}{\log_2 x} \right) - \log \delta
\]

becomes less complicated, and reveals that it is increasing and positive as soon as $x \geq 10^{322}$. 

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This implies that
\[ A(x) \geq \log_2 x, \ \forall x \geq 10^{322}. \]
Finally, if the Riemann Hypothesis holds, first we have from Robin’s criterion that \( \frac{\sigma(n)}{n} \leq e^\gamma A(n) \) for all \( n \geq 10^{322} \), and thanks to the computations of Proposition 1 for the remaining values. \( \square \)

At this level, part of Conjecture 1 is proven and the persistent question is:

**Question 2.** Is it true that if RH is false, the inequality
\[ \frac{\sigma(n)}{n} \leq e^\gamma \left( \log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right) \]
is violated for infinitely many \( n \geq N_3 \)?

A heuristic motivation runs as follows:

\[
K(n) \approx \log n / \log_2 n \implies \log K(n) \approx \log_2 n - \log_3 n \approx \log_2 n \\
\implies \log K(n) + \log_2 K(n) \approx \log_2 n \\
\implies A(n) \approx \log_2 n.
\]

Hence, according to Robin’s criterion, since \( \frac{\sigma(n)}{n} > e^\gamma \log_2 n \) infinitely often, if the Riemann Hypothesis is false, as \( A(n) \approx \log_2 n \), there may exist infinitely many \( n \) such that
\[ \frac{\sigma(n)}{n} > e^\gamma A(n). \]

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