ERGODICITY OF STOCHASTIC DAMPED OSTROVSKY EQUATION DRIVEN BY WHITE NOISE

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Abstract. The current paper is devoted to the stochastic damped Ostrovsky equation driven by white noise. By establishing the uniform estimates for the solution in $H^1$ norm, we prove the global well-posedness and the existence of invariant measure for stochastic damped Ostrovsky equation with random initial value. Moreover, we obtain the ergodicity of stochastic damped Ostrovsky equation with deterministic initial conditions.

1. Introduction. In this paper, we consider the following stochastic damped Ostrovsky equation driven by white noise with positive dispersion

$$
\begin{cases}
    du(t) = \left[ \beta \partial_x^3 u(t) - \frac{1}{2} \partial_x (u^2) + \lambda u + \gamma \partial_x^{-1} u \right] dt + \Phi dW(t), \\
    u(0) = u_0,
\end{cases}
$$

where $\lambda$ is a positive number, $\beta > 0$, $\gamma > 0$, $W(t) = \sum_{j=1}^{\infty} \beta_j(t)e_j(x)$, $e_j$ is an orthonormal basis of $L^2(R)$ and $(\beta_j)_{j \in N}$ is a sequence of mutually independent real Brownian motions in a fixed probability space and is a cylindrical Wiener process on $L^2(R)$. Equation (1) with $\lambda = 0$ and $\Phi = 0$ was proposed by Ostrovsky[13] to model the weakly nonlinear long waves in a rotating liquid. By introducing the Coriolis force into equation (1) with $\lambda = 0$ and $\Phi = 0$, it can describe the propagation of surface waves in the ocean in a rotating frame of reference. In fact, $\beta$ determines the type of dispersion, $\beta < 0$ (negative dispersion) for surface and internal waves in the ocean or surface waves in a shallow channel with an uneven bottom, and $\beta > 0$ (positive dispersion) for capillary waves on the surface of a liquid or for oblique magneto-acoustic waves in plasma, we refer it to [7].

For determined case of (1) with $\lambda = 0$, Isaza et al [8] established the well-posedness in spaces of low regularity. Isaza et al [9] studied the global Cauchy problem. In [10], the authors proved that the Cauchy problem for (1) is locally well-posed in $H^s(R)$ with $s > -\frac{3}{4}$ and ill-posed in $H^s(R)$ with $s < -\frac{3}{4}$. Yan et
al [17] proved that the Cauchy problem for (1) is locally well-posed in $H^{-\frac{2}{3}}(R)$. If $\gamma = 0$, then (1) reduces to the stochastic KdV equation. There are a lot of papers on stochastic KdV equation, we refer the readers to [2, 3, 5]. Bouard et al [3] proved that for almost surely $\omega \in \Omega$, there exist $T_\omega > 0$ and a unique solution $u(t) \in C([0,T]; H^s(\Omega)) \cap X_T^{s,\gamma}$ of stochastic equations (1) on $[0, T_\omega]$ provided with the assumption that $u_0(x, \omega) \in H^s(\Omega)$, $s > -\frac{5}{8}$ for almost surely $\omega \in \Omega$ and $\mathcal{F}_0$-measurable initial data $u_0$ and $\Phi \in L_2^0, 0 \cap L_2^0(L^2(\Omega); \tilde{H}^{-\frac{3}{2}}(\Omega))$. Moreover, Bouard et al.[3] proved the global well-posedness in $L^2(\Omega; C([0, T]; L^2(\Omega)))$ for stochastic equation (1) if $\Phi \in L^2(\Omega; L^2(R); \tilde{H}^{-\frac{3}{2}}(\Omega))$ and $u_0 \in L^2(\Omega; L^2(R))$ and is $\mathcal{F}_0$-measurable. For $s > -\frac{3}{8}$, Li [11] proved that there exist a $T_\omega > 0$, $\tilde{s} = \tilde{s}(s), b = b(s, \tilde{s})$ which satisfies $\tilde{s} < 0$ and $\tilde{s}$ can sufficiently approach zero and $0 < b < \frac{1}{2}$, stochastic equations (1) possesses a unique solution on $[0, T_\omega]$ which satisfies $u \in C([0, T_\omega]; H^s(\Omega)) \cap X_T^{s,\gamma}$. Recently, Yan et al. [18] prove that the Cauchy problem for stochastic Ostrovsky equation with positive dispersion is locally well-posed for the initial data $u_0(\cdot, \omega) \in H^s(\Omega)(a.e. \omega \in \Omega)$ which is $\mathcal{F}_0$-measurable with $s > -\frac{3}{8}$ and $\Phi \in L^2(\Omega, 0 \cap L^2(L^2(\Omega); \tilde{H}^{-\frac{3}{2}}(\Omega)), \tilde{H}^{-\frac{3}{2} + \epsilon}(\Omega))$, and it is globally well-posed of stochastic Ostrovsky equation with initial data $u_0(x, \omega) \in L^2(\Omega)(a.e. \omega \in \Omega)$ and $\Phi \in L^2(\Omega, 0 \cap L^2(L^2(\Omega), \tilde{H}^{-\frac{3}{2} + \epsilon}(\Omega))$.

There are a lot of papers to investigate the ergodicity for stochastic partial equation with Gaussian noise and non-Gaussian noise, for example, [2, 3] established the local well-posedness for stochastic KdV equation with additive noise and multiplicative noise respectively. For the general framework, we refer it to [16] and [14]. But there is few papers on the ergodicity of stochastic dispersive differential equations such as stochastic damped KdV equation and stochastic damped Schrödinger equation due to the weak dispersion effect. Dankel[4] established the existence of an invariant measure, Ekren et al [6, 5] showed the existence of invariant measures for the stochastic damped KdV equation and stochastic damped Schrödinger equation respectively, but they could not prove the ergodicity for stochastic KdV equation and stochastic damped Schrödinger. Recently, [1] proved the existence of invariant measure for stochastic Schrödinger equation with jump process, but they also did not provide with the uniqueness of the invariant measure. To the best of our knowledge, there is no result on the ergodicity on stochastic damped Ostrovsky equation.

In this paper, we will investigated the ergodicity of stochastic damped Ostrovsky equation with Gaussian noise. We prove the global well-posedness and the existence of invariant measure for stochastic damped Ostrovsky equation with random initial value. Moreover, we obtain the ergodicity of stochastic damped Ostrovsky equation with deterministic initial conditions. Comparing with stochastic damped KdV equation, the nonlocal term $\gamma \partial_x^{-1}u$ of stochastic equation (1) leads to more complexity and difficulty to deal with. To overtake the difficulty caused by dispersive term $u_{xxx}$, not dissipative term $u_{xx}$, motivated by the idea from [6], it required to establish the uniform estimates for $L^2$ norm and $H^1$ norm respectively. To overtake the difficulty caused by nonlocal term $\gamma \partial_x^{-1}u$, we need to modify the conserved quantity $I_2(u(t)) = \int_{-\infty}^{\infty} (\beta (\partial_x u)^2 + \frac{\gamma}{2} (\partial_x^{-1} u)^2 + \frac{1}{2} u^2) dx$ in the space $X$, where the $\partial_x^{-1} f \in L^2(\Omega)$ instead of the one in KdV equations, we refer it to [12] for details.

The rest of paper is organized as follows. In Section 2, some function setting and useful lemmas or technique theorem are provided. In section 3, the uniformly bounded of solutions in $H^1$ space are established, then the global well-posedness is
proved. Finally, the existence of invariant measure for stochastic damped Ostrovery equation is shown by proving the strong Feller property and tightness in Section 4, furthermore, the invariant measure is unique for determined initial value.

2. preliminaries. In this section, we will recall some basic concepts and some inequalities, which plays the crucial role in establishing the main theorems, which are taken from [3, 12].

Let $H$ be a Hilbert space, $L^2_0(L^2(R), H^s)$ the space of Hilbert-Schmidt operators from $L^2(R)$ into $H^s$ with the norm

$$\|\Phi\|^2_{L^2_0} = \sum\limits_{j \in \mathbb{N}} |\Phi e_j|^2_{H^s}.$$ 

$H^s(R)$ is the Sobolev space with norm

$$\|f\|_{H^s(R)} = \|\langle \xi \rangle^s \mathcal{F} f\|_{L^2_0(R)},$$

where $\langle \xi \rangle^s = (1 + \xi^2)^{s/2}$ for any $\xi \in R$, $\mathcal{F} u$ and $\mathcal{F}^{-1} u$ denotes the Fourier transformation and the Fourier inverse transformation of $u$ with respect to its space variable respectively. $H^{s_1+s_2}(R)$ is the Sobolev space with norm

$$\|f\|_{H^{s_1+s_2}(R)} = \|\langle \xi \rangle^{s_1} |\xi|^{s_2} \mathcal{F} f\|_{L^2_0(R)}.$$ 

and $\dot{H}^s = \dot{H}^{0,s}$. With this choice of the antiderivative we have, $\partial_x^{-1} f = \left(\frac{\mathcal{F}(\xi)}{\xi}\right)^\vee$, so it is natural to define the function space $X_s$ as one in [12],

$$X_s = \{ f \in H^s(R) : \partial_x^{-1} f \in L^2(R) \}, \quad s \in R.$$ 

Space $\mathcal{S}(R^2)$ is the Schwartz space and $\mathcal{S}'(R^2)$ is its dual space. $\mathcal{F} u$ and $\mathcal{F}^{-1} u$ denotes the Fourier transformation and the Fourier inverse transformation of $u$ with respect to its all variables respectively.

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{E} f = \int_\Omega f d\mathbb{P}$.

Assume that $W(t) = \sum\limits_{j=1}^{\infty} \beta_j(t) e_j(x)$ is a cylindrical Wiener process on $L^2(R)$ associated with the filtration $(\mathcal{F}_t)_{t \geq 0}$, where $(e_k)_{k \in \mathbb{N}}$ is an orthogonal basis of $L^2(R)$, and the sequence $(\beta_k)_{k \in \mathbb{N}}$ is real, mutually independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$. Denote

$$\phi(\xi) = \beta \xi^3 + \frac{\gamma}{\xi},$$

$$U(t) u_0 = e^{-\lambda t} \int_R e^{i(\xi - t\phi(\xi))} \mathcal{F} u_0(\xi) d\xi,$$

$$\|f\|_{L^p_{\alpha}} = \left(\int_R \left(\int_R |f(x, t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}, \quad \|f\|_{L^p_{\alpha \beta \sigma}} = \|f\|_{L^p_{\alpha \beta \sigma}}.$$ 

Then the solution to (1) is equivalent to the following integral equation

$$u(t) = U(t) u_0 + \frac{1}{2} \int_0^t U(t - s) \partial_x (u^2) ds + \int_0^t U(t - s) \phi dW.$$ (2)

The following Theorem is the key tool to prove the ergodicity for stochastic equation (1), which is from [16].
**Theorem 2.1** ([16], Proposition 3.2.7). An invariant probability measure for the semigroup $P_t, t \geq 0$, is ergodicity if and only if it is an extremal point of the set of all the invariant probability measures for the semigroup $P_t, t \geq 0$.

3. Uniform estimate and global well-posedness for equation (1). In this section, we will show the uniform estimate of the solution $u$ in $L^2$ norm and $H^1$ norm respectively, and then prove the global existence of the solution.

**Lemma 3.1.** Let $s \in \mathbb{R}$ and $b < \frac{1}{2}$, $\Phi \in L^0_2(L^2(R); \dot{H}^s(R))$, for $t \in [0, T]$ and

$$\bar{\pi} = \int_0^t U(t-s)\Phi dW(s).$$

Then, we have

$$\psi \bar{\pi} \in L^2(\Omega; X_s), \mathbb{E}\left(\|\psi \bar{\pi}\|^2_{X_s}\right) \leq CM(b, \psi)\|\Phi\|^2_{L^0_2(L^2(R); \dot{H}^s)},$$

where $M(b, \Psi)$ is a constant depending only on $b$, $\|\psi\|_{H^1}, \|\psi\|_{L^2}, \|\psi\|_{L^\infty}$.

**Proof.** Lemma 3.1 can be proved similarly to Proposition 2.1 of [3], we omit the proof here.

**Theorem 3.2.** Let $u_0(x, \omega) \in L^2(\Omega; H^s(R))$ with $s > -\frac{3}{2}$ and $\Phi \in L^0_2 \cap L^0_2(L^2(R), H^s, -\frac{1}{2} + \epsilon(R))$ and $u_0$ be $\mathcal{F}_t$-measurable and $b = \frac{1}{2} - 2\epsilon$. Then, for a.e. $\omega \in \Omega$, there exists a $T_\omega > 0$ and a unique solution of the Cauchy problem for (1) on $[0, T_\omega]$ satisfying

$$u \in C([0, T_\omega]; H^s(R)) \cap X^T_\omega.$$ 

**Proof.** It can be proved by slightly modifying the proof of Theorem 3.2 in [17], we omit the proof of Lemma 3.2 here.

**Lemma 3.3.** Let $u_0 \in L^2(R)$, $\Phi \in L^0_2(L^2(R); H^0, \dot{H}^0, -\frac{1}{2} + \epsilon(R))$. For any $m > 0$ and $T > 0$, the unique solution $u^m$ P-a.s to (1) satisfies

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \|u\|^2_{L^2} \leq C(\|u_0\|_{L^2}, T, \|\Phi\|_{L^0_2}),ight.$$

for any integer $p \geq 1$.

**Proof.** It can be proved by slightly modifying the proof of Lemma 5.1 in [18], we omit the proof of Lemma 3.3 here.

**Lemma 3.4.** Let $u_0 \in H^1(R)$, $\Phi \in L^0_2(H^1(R); H^0, \dot{H}^0, -\frac{1}{2} + \epsilon(R))$. For any $T > 0$, the unique solution $u$ P-a.s to (1.1) satisfies

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \left\|\partial_x u(t)\right\|^2_{H^1} \leq C(\|u_0\|_{H^1}, T, \|\Phi\|_{L^0_2}),ight.$$ 

**Proof.** Let

$$I(u) = \int_R \beta (\partial_x u)^2 + \frac{\gamma}{2} (\partial_x^{-1} u)^2 + \frac{1}{3} u^4 dx.$$ 

Applying the Itô formula to $I(u)$, we obtain

$$dI(u) \leq (u^2, \Phi dW(t)) + 2\beta (\partial_x u, \partial_x \Phi dW(t)) + 2 \gamma (\partial_x^{-1} u, \partial_x^{-1} \Phi dW(t)) + \alpha(t) dt,$$

where

$$\alpha(t) = \|\partial_x \Phi\|^2_{HS(L^2, L^2)} + \|\partial_x^{-1} \Phi\|^2_{HS(L^2, L^2)} + \sum_i \int_R u |(\Phi e_i)(x)|^2 dx.$$
Direct computation for the three stochastic terms one by one in (7), we have
\[
\mathbb{E}((\partial_x u, \partial_x \Phi dW(t))) = \sum_i (\partial_x u, \partial_x \Phi e_i)^2 dt \leq \|\partial_x u\|^2_{L^2} \|\Phi\|_{HS(L^2, H^1)} < \infty,
\]
\[
\mathbb{E}((u^2, \Phi dW(t))) = \sum_i (u^2, \Phi e_i)^2 dt \leq \|u\|^4_{L^4} \|\Phi\|_{HS(L^2, L^2)},
\]
\[
\mathbb{E}((\partial_x^{-1} u, \partial_x^{-1} \Phi dW(t))) = \sum_i (\partial_x^{-1} u, \partial_x^{-1} \Phi e_i)^2 dt \leq \|\partial_x^{-1} u\|^2_{L^2} \|\Phi\|_{HS(L^2, H^{-1})} < \infty.
\]
Then for \(\alpha(t)\), it holds that
\[
\left\|\partial_x \Phi\right\|_{HS(L^2, L^2)}^2 \leq \|\Phi\|^2_{HS(L^2, H^1)} < C,
\]
\[
\left\|\partial_x^{-1} \Phi\right\|_{HS(L^2, L^2)}^2 \leq \|\Phi\|^2_{HS(L^2, H^{-1})} < C,
\]
and
\[
\sum_i \int_R u |(\Phi e_i)(x)|^2 dx \leq \|u\|_{L^\infty} \sum_i \int_R |(\Phi e_i)(x)|^2 dx = \|\Phi\|_{HS(L^2, L^2)} \|u\|_{L^\infty} \leq C \|u\|_{L^\infty} \leq C \|u\|_{L^2}^3 \|\partial_x u\|_{L^2}^7 \leq C \left(\|\partial_x u\|_{L^2} + \|u\|_{L^2}^4\right).
\]
Therefore, we have
\[
|\alpha(t)| \leq C(1 + \|u\|^4_{L^2} + \|\partial_x u\|^2_{L^2}).
\]
and
\[
d\mathbb{E}[I(u)] \leq C(1 + \mathbb{E}[\|u\|^4_{L^2} + \mathbb{E}[\|\partial_x u\|^2_{L^2}]).
\]
The Gronwall inequality yields that
\[
\mathbb{E}[I(u)] \leq C \left(\mathbb{E}[I(u_0)] + \mathbb{E}\left[\|u_0\|^4_{L^2}\right] + 1\right).
\]
It follows from Young’s inequality that
\[
\int_{-\infty}^{\infty} u^3(x, t) dx \leq \|u(t)\|_{L^\infty} \|u_0\|^2_{L^2} \leq \sqrt{2} \|u_0\|^{5/2}_{L^2} \|\partial_x u(t)\|_{L^2}^{1/2} \leq \frac{\beta}{2} \|\partial_x u(t)\|^2_{L^2} + C. \tag{9}
\]
which implies that
\[
\beta \|\partial_x u(t)\|^2_{L^2} + \frac{\gamma}{2} \|\partial_x^{-1} u(t)\|^2_{L^2}
= I_2(u_0) - \frac{1}{2} \|u(t)\|^3_{L^3} \leq I_2(u_0) + \frac{\beta}{2} \|\partial_x u(t)\|^2_{L^2} + C. \tag{10}
\]
Thus, for any positive real number \(\beta\) and \(\gamma\), it holds
\[
\beta \mathbb{E}[\|\partial_x u(t)\|^2_{L^2}] + \gamma \mathbb{E}[\|\partial_x^{-1} u(t)\|^2_{L^2}] \leq C. \tag{11}
\]
The proof of Lemma 3.4 is complete. \(\Box\)
Lemma 3.5. Assume that $u_0$ is determined initial value, there exist a constant $C$ which depends on $u_0$ such that

$$E\left[ \sup_{t \in [0,T]} \|u(t)\|_{H^1}^2 \right] \leq C. \quad (12)$$

Proof. Direct calculation shows that

$$E\left[ \sup_{t \in [0,T]} \|u(t)\|_{H^1}^2 \right] \leq C E\left[ \sup_{t \in [0,T]} |I(u(t))| \right]$$

$$\leq |I(u_0)| + C \int_0^T E[|\alpha(s)|^2] \, ds + C \int_0^T \sum_i (\partial_x u, \partial_x \Phi_i)^2 + (u^2, \Phi_i)^2 \, ds$$

$$+ C \int_0^T \sum_i (\partial_x^{-1} u, \partial_x^{-1} \Phi_i)^2 \, ds$$

where

$$\alpha(t) = \|\partial_x \phi\|_{H^S}^2 + \|\partial_x^{-1} \phi\|_{H^S}^2 + \sum_i \int_R u \|\Phi_i(x)\|^2 \, dx.$$ 

We deduce from Lemma 3.4 that

$$E\left[ \sup_{[0,T]} \|\partial_x u\|_{L^2}^2 \right] \leq C(\|u_0\|_{H^1}, T).$$

Hence, it holds that

$$E\left[ \sup_{t \in [0,T]} \|u(t)\|_{H^1}^2 \right]$$

$$\leq |I(u_0)| + \int_0^T C(\|u_0\|_{L^2}^2 + \|u_0\|_{H^1}^2 + 1) \, ds + C \int_0^T M(t)^2 \, dt$$

$$\leq C(\|u_0\|_{H^1}, T).$$

Thus, the proof of Lemma 3.5 is complete.

Lemma 3.6. Let $u_0 \in \dot{H}^{-1}(R)$, $\Phi \in L^2_2(H^1(R); \dot{H}_{-1}^0(R))$. For any $T > 0$, the unique solution $u$ $P$-a.s. to (1.1) satisfies

$$E\left( \sup_{t \geq 0} \|\partial_x^{-1} u(t)\|_{L^2}^2 \right) \leq C(\|\partial_x^{-1} u_0\|_{L^2}, \|u_0\|_{H^1}, T, \|\Phi\|_{L^2_2}, \|\partial_x^{-1} \Phi\|_{L^2_2}), \quad (13)$$

Proof. Lemma 3.6 can be proved similarly to Lemma 3.4.

After get the uniform bound in $H^1(R)$, we can extend Theorem 3.2 to the whole interval $[0, T]$.

Theorem 3.7. Let $u_0(x, \omega) \in L^2(\Omega; H^1(R))$, $u_0(x, \omega) \in L^2(\Omega; \dot{H}^{-1}(R))$ and $\Phi \in L^2_2 \cap L^2(\Omega; \dot{H}^{-1}(R))$ and $u_0$ is $\mathcal{F}_0$-measurable. Then the solution to the Cauchy problem for (1) is global and belongs to

$$L^2(\Omega; C([0, T]; H^1(R)) \cap L^2(\Omega; C([0, T]; \dot{H}^1(R))$$

for any $T > 0$. 

4. Ergodicity of stochastic equation (1). In this section, we first prove the
Strong Feller property of semigroup $P(t), t \geq 0$, then prove the ergodicity of stochastic equation (1) by Theorem 2.1.

Theorem 4.1. Assume that $u_0 \in H^1(R) \cap \dot{H}^{-1}(R)$, $\Phi \in L^1_0(L^2(R); H^1(R)) \cap L^1_2(L^2(R); \dot{H}^{-1}(R))$. Then $P_t$ is a strong Feller semigroup on $H^1(R) \cap \dot{H}^{-1}(R)$.

Proof. It is suffices to prove that
\[
\mathbb{E}[\|u(t)\| - \|v(t)\|] \to 0 \quad \text{as} \quad v_0 \to u_0,
\]
for any $u_0, v_0 \in H^1(R) \cap \dot{H}^{-1}(R), t > 0$ and $\xi \in B_b(H^1 \cap \dot{H}^{-1}, R)$. To the end, let $R_0 = \|v_0\|_{H^1} + 1, M = \sup_{v \in H^1 \cap \dot{H}^{-1}} \|\xi(v)\|$, and $\|v_0 - u_0\| \leq 1$. By using Chebychev’s inequality, we derive that there exists some constant $C(R_0) > 0$ such that
\[
\mathbb{P}\{\max\{\sup_{s \in [0, t]} \|u(s)\|_{H^1}, \sup_{s \in [0, t]} \|v(s)\|_{\dot{H}^{-1}}\} \geq R\} \leq \frac{C(R_0)}{R}.
\]
Choosing a sufficient large positive number $R$ such that $\frac{C(R_0)}{R} \leq \frac{\epsilon}{6M}$, where $\epsilon$ is a small enough positive number. Then
\[
\mathbb{P}\{\max\{\sup_{s \in [0, t]} \|u(s)\|_{H^1}, \sup_{s \in [0, t]} \|v(s)\|_{\dot{H}^{-1}}\} \geq R\} \leq \frac{\epsilon}{6M}. \quad (14)
\]

Let $\pi = \int_0^t U(t - s)\Phi dW(s)$, and
\[
T_h(g)(s) = U(s)h - \int_0^s U(s - r)g(r)\partial x g(r)dr + \bar{u}(s).
\]
Then $u(s) = T_{u_0}(u)(s)$ and $v(s) = T_{v_0}(v)(s)$. Define the stopping time
\[
\tau = \inf\{s \geq 0 : 4\tilde{C}(t)s^\alpha(\tilde{C}(t)R + \|\bar{u}\|_{X(s)} + \|\int_0^s U(s - r)\partial x u^2 dr\|_{X(s)}) > 1\},
\]
where $X(s) = \{C([0, s], H^1 \cap \dot{H}^{-1})\}$. Then we have
\[
\mathbb{P}(\tau < s) \leq \mathbb{E}[4\tilde{C}(t)s^\alpha(\tilde{C}(t)R + \|\bar{u}\|_{X(s)} + \|\int_0^s U(s - r)\partial x u^2 dr\|_{X(s)}) > 1]
\leq \mathbb{E}[4\tilde{C}(t)s^\alpha(\tilde{C}(t)R + \|\bar{u}\|_{X(s)} + \|\int_0^s U(s - r)\partial x u^2 dr\|_{X(s)})]
\leq 4\tilde{C}^2(t)s^\alpha(R + 1) \leq C(R)s^\alpha.
\]
It follows that
\[
\mathbb{E}[\tau] = \int_0^\infty \mathbb{P}(\tau > s)ds = \int_0^\infty (1 - \mathbb{P}(\tau < s))ds
\geq \int_0^t (1 - \mathbb{P}(\tau < s))ds \geq \int_0^t (1 - 1 \wedge (C(R)s^\alpha))ds \geq \frac{1}{C_0(R)}.
\]
Let $\tau_0 = \tau$, define a sequence of stopping times $\tau_{k+1}$ by

$$\tau_{k+1} = \inf \left\{ s \geq \tau_k : 4\tilde{C}(t)(s - \tau_k)\alpha(\tilde{C}(t)R) + \|u_{\tau_k}\|_{X(\tau_k, s)} + \left| \int_{\tau_k}^{s} U(s-r)\partial_x u^2 dr \right|_{X(\tau_k, s)} > 1 \right\},$$

where $\tilde{u}_{\tau_k}(s) = \int_{\tau_k}^{s} U(t-s)\Phi dW(s)$, and $X(\tau_k, s)$ is defined on $[\tau_k, s]$. Then $\tau_{k+1} - \tau_k$ and $\tau$ are independent with the same distribution.

We can deduce from the law of large number that

$$\frac{\tau_n}{n} = \frac{1}{n} \sum_{0 \leq i \leq n} (\tau_i - \tau_{i-1}) \to E[\tau] \geq \frac{1}{C_0(R)},$$

which implies that

$$P(\tau_n \leq t) \to 0, \quad \text{as } n \to \infty.$$

Hence, there exists $n \in \mathbb{N}$ such that

$$P(\tau_n \leq t) \leq \frac{\varepsilon}{6M}.$$

We obtain that for any $v_0$ with $\|v_0 - u_0\|_{H^1} \leq 1$

$$E[\|\xi(u(t) - \xi(v(t))\|_{H^1}] \leq E[\|\xi(u(t) - \xi(v(t))\|_{H^1}] + E[\|\xi(u(t) - \xi(v(t))\|_{H^1}] + E[\|\xi(u(t) - \xi(v(t))\|_{H^1}] \leq R\|\xi(v(t))\|_{H^1} \leq t$$

$$\leq \frac{2}{3} \varepsilon + E[\|\xi(u(t) - \xi(v(t))\|_{H^1}] \leq t$$

With simply modify of Lemma 4.5 in [5], we have

$$\|u(t) - v(t)\|_{H^1} \leq (2\tilde{C}(t))^{k+1}\|u(0) - v(0)\|_{H^1} \to 0, \text{ as } v_0 \to u_0.$$!

Thus, $E[\|\xi(u(t)) - \xi(v(t))\|_0] \to 0$. The proof of Theorem 4.1 is complete.

**Definition 4.2.** Let $(X, T)$ be a topological space, and let $\{X_n, n \in \mathbb{N}\}$, $X_0$ be a $(X, T)$-valued random variable, $X_n$ is said to converge to $X_0$ in distribution, if for any bounded continuous function $F : X \to \mathbb{R}$,

$$\lim_{n \to \infty} E[F(X_n)] = E[F(X_0)].$$

**Theorem 4.3.** Assume that $u_0 \in H^1(R)$, $\Phi \in L_0^2(L^2(R); H^1(R))$, for any sequence of deterministic initial conditions $\{u_n(0)\}$ with $R = \sup_{n \in \mathbb{N}}\|u_0\|_{H^1} < \infty$,

$\{P_{t_n}(u_n(0), \cdot) : n \in \mathbb{N}\}$ is tight on $H^1$ for any $t_n > 0$ with $t_n \to \infty$ as $n \to \infty$.

**Proof.** Without loss of generality, we assume that $t_n$ is an increasing sequence. Denote by $u_n(t)$ the solution with the initial condition $u_n(0)$. We claim that $\{u_n(t)\}_{n=1}^\infty$ converges in $L^2_{t_n}(R)$. In fact, since $E[\sup_{0 \leq t \leq t_n} \|u_n\|_{H^1}^2] \leq C(1 + E[\|u_0\|_{H^1}^2])$, then we have:

$$\sup_{t \geq 0} E[\|u_n(t)\|_{H^1}^2] \leq C(\sup_n\|u_n(0)\|_{H^1}^2 + \sup_n\|u_n(0)\|_{L^2}^2) + 1) := C(R).$$
Recall that any bounded sets in $H^1(R)$ is relatively compact in $L^2_{loc}(R)$. Hence, there exists $L^2_{loc}(R)$-valued variable $\xi$ and a subsequence $u_{n_i}(t_{n_i})$ of $u_n(t_n)$ such that $u_{n_i}(t_{n_i})$ converges to $\xi$ in $L^2_{loc}(R)$ in the distribution sense. For convenience, we also denote $u_{n_i}(t_{n_i})$ by $u_n(t_n)$. Let $\{f_i\}$ be the family of the smooth orthonormal basis of $H^1(R)$ with compactly supported set, then we have

$$
\mathbb{E}[\sum_i (u_n(t_n), f_i)_{H^1}^2 \land M^2] \to \mathbb{E}[\sum_i (\xi, f_i)_{H^1}^2 \land M^2].
$$

Therefore,

$$
\mathbb{E}[\sum_i (\xi, f_i)_{H^1}^2 \land M^2] \leq \mathbb{E}[||\xi||^2_{H^1} \land M^2] \leq C(R)
$$

and $\mathbb{E}[\sum_i (\xi, f_i)_{H^1}^2] \leq C(R)$ as $M$ increases, which means that $\xi$ takes value in $H^1$.

Next we shall show the convergence in $L^2(R)$. Let $\{g_i\}$ be the family of the smooth and orthonormal basis of $H^1(R)$ with compactly supported set, then we have

$$
\mathbb{E}[\sum_i (u_n(t_n), g_i)_{L^2}^2 \land M^2] \to \mathbb{E}[\sum_i (\xi, g_i)_{L^2}^2 \land M^2].
$$

Since $\mathbb{E}[||u_n(t_n)||^2_{L^2}] \leq C(\mathbb{E}[||u_n(0)||^2_{L^2}] + 1)$, then we have for increasing $M$,

$$
\mathbb{E}[\sum_i (u_n(t_n), g_i)_{L^2}^2] \to \mathbb{E}[\sum_i (\xi, g_i)_{L^2}^2].
$$

Therefore,

$$
\mathbb{E}\left[\sum_{i=N+1}^{\infty} (u_n(t_n), g_i)_{L^2}^2\right] \to \mathbb{E}\left[\sum_{i=N+1}^{\infty} (\xi, g_i)_{L^2}^2\right].
$$

Since

$$
\mathbb{E}\left[\sum_{i=N+1}^{\infty} (\xi, g_i)_{L^2}^2\right] \to 0, \quad N \to \infty.
$$

Then we have

$$
\lim_{N \to \infty} \sup_n \mathbb{E}\left[\sum_{i=N+1}^{\infty} (u_n(t_n), g_i)_{L^2}^2\right] = 0.
$$

Thus, $u_n(t_n)$ converges to $\xi \in L^2(R)$ in the sense of distribution, and we have

$$
\mathbb{E}[||\partial_x \xi ||_{L^2}^2] = \lim_{n \to \infty} \mathbb{E}[||\partial_x u_n(t_n) ||_{L^2}^2].
$$

Hence, $u_n(t_n)$ converges to $\xi \in H^1(R)$ in the sense of distribution, and $\{P_{t_n}(u_n(0), \cdot) : n \in \mathbb{N}\}$ is tight on $\{H^1\}$. Thus the proof of Theorem 4.3 is complete.

**Theorem 4.4.** Assume that $u_0 \in H^1(R)$, $\Phi \in L^2_{loc}(L^2(R); H^1(R))$. If $K$ is a compact set of $\{H^1\}$, then the sequence of measures $\{P_s(v, \cdot) : s \in [0, 1], v \in K\}$ is tight on $H^1$.

**Proof.** It suffices to prove that $\{P_s(v, \cdot) : s \in [0, 1], v \in K\}$ possesses a convergent subsequence. Assume that $(s_n, v_n) \in [0, 1] \times K$, then $(s_n, v_n) \in [0, 1] \times K$ possesses a convergent subsequence. Without loss of generality, we assume that $(s_n, v_n) \in [0, 1] \times K$ converges to $(s, v) \in [0, 1] \times K$. Let $u_n(t)$ be the solution with the initial value $v_n(0)$, and $u(t)$ the solution with the initial value $v$. Since $u \in C([0, 1]; H^1)$, then we have

$$
\lim_{n \to \infty} ||u(s_n) - u(s)||_{H^1} = 0, \text{P.a.s.}
$$
Next, we are going to prove that there exists a subsequence \((s_{n_k}, v_{n_k})\) of sequence \((s_n, v_n)\) such that
\[
\lim_{k \to \infty} (\|u(s_{n_k}) - u(s)\|_{H^1} + \|u_{n_k}(s) - u(s)\|_{H^1}) = 0.
\]
To the end, let \(R_0 = \sup_{v \in K} \|v\|_{H^1} + 1\), and choose \(R > 0\) such that \(\frac{C(R_0)}{R} \leq \frac{\varepsilon}{2}\), then we have
\[
\mathbb{P}\{\max\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|u_n(s)\|_{H^1}^2\} \geq R\} \\
\leq \mathbb{P}\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2 + \sup_{s \in [0,t]} \|u_n(s)\|_{H^1}^2 \geq R\} \\
\leq \frac{1}{R} \mathbb{E}[\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2 + \sup_{s \in [0,t]} \|u_n(s)\|_{H^1}^2] \leq \frac{C(R_0)}{R} \leq \frac{\varepsilon}{2}.
\]
Define a stopping time
\[
\tau = \inf\{s \geq 0 : 4\tilde{C}(t)s^\alpha(\tilde{C}(t)R + \|\tilde{u}\|_{X(s)} + \|\int_0^s U(s - t)\partial_x u^2 dt\|_{X(s)}) > 1\}.
\]
Let \(\tau_0 = \tau\), and
\[
\tau_{k+1} = \inf\{s \geq \tau_k : 4\tilde{C}(t)(s - \tau_k)^\alpha(\tilde{C}(t)R + \|\tilde{u}_{\tau_k}\|_{X(\tau_{k},s)} + \|\int_{\tau_k}^s U(s - t)\partial_x u^2 dt\|_{X(\tau_{k},s)}) > 1\},
\]
\[
A_n = \{\max\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|u_n(s)\|_{H^1}^2\} \leq R\},
\]
and choose some proper \(N\) such that \(P(\tau_N \leq 1) \leq \frac{\varepsilon}{2}\), we obtain
\[
\sup_{s \in [0,1]} \|u(s) - u_n(s)\|_{H^1} \leq (2\tilde{C}(t))^{N+1}\|v - v_n\|_{H^1} \to 0
\]
on the interval \(A_n \cap \{\tau_N \geq 1\}\).

Let \(n\) be large enough such that \((2\tilde{C}(t))^{N+1}\|v - v_n\|_{H^1} < \delta\). Then we have
\[
\mathbb{P}(\sup_{s \in [0,1]} \|u(s) - u_n(s)\|_{H^1} \leq \delta) \geq P(A_n \cap \{\tau_N \geq 1\}) \geq 1 - \varepsilon,
\]
which implies that
\[
\lim_{n \to \infty} \sup_{s \in [0,1]} \|u(s) - u_n(s)\|_{H^1} \to 0, \text{P.a.s.}
\]
Therefore, there exists a sequence \(n_k\) such that
\[
\lim_{k \to \infty} \|u(s) - u_{n_k}(s)\|_{H^1} \to 0, \text{P.a.s.}
\]
We can deduce that for any real valued uniformly continuous function \(\xi \in H^1(R)\),
\[
|\mathbb{P}_{s_{n_k}} \xi(v_{n_k}) - P_s \xi(v)| \leq E[|\xi(u_{n_k}(s_{n_k}) - \xi(u(s_{n_k}))| + E[|\xi(u(s_{n_k}) - \xi(u(s))| \to 0.
\]
Therefore, \(\{P_{s}(v, \cdot) : s \in [0,1], v \in K\}\) has a convergent subsequence. Thus, the proof of Theorem 4.4 is complete. \(\square\)

**Theorem 4.5.** Assume that \(u_0 \in H^1(R)\), \(\Phi \in L^0_2(L^2(R); H^1(R))\). Then \(\mu_n(\cdot) = \frac{1}{n} \int_0^n P_t(0, \cdot) dt\), \(n = 1, 2, \ldots\) is tight on \(H^1(R)\).
Proof. For any $\varepsilon > 0$, $\{P_n(0, \cdot) : n \geq 0\}$ is tight, and we can choose a compact set $K_\varepsilon \subset H^1(R)$, such that $\sup_n \{P_n(0, K_\varepsilon)\} \leq \frac{\varepsilon}{2}$.

Since $\{P_n(s, \cdot) : s \in [0, 1], v \in K\}$ is tight on $H^1$, we can choose a compact set $A_\varepsilon \subset H^n(R)$, such that $\sup_{s \in [0, 1], v \in K_\varepsilon} \{P_n(0, A_\varepsilon)\} \leq \frac{\varepsilon}{2}$. Then we have

$$
\mu_n(A_\varepsilon) = \frac{1}{n} \int_0^n P_t(0, A_\varepsilon) dt \\
= \frac{1}{n} \sum_{i=0}^{n-1} \int_{t_{i+1}}^{t_i} \int_{H^1} P_t(0, dy) P_{t-i}(y, A_\varepsilon) dt \\
= \frac{1}{n} \sum_{i=0}^{n-1} \int_{t_{i+1}}^{t_i} \left\{ \int_{K_\varepsilon} P_t(0, dy) P_{t-i}(y, A_\varepsilon) + \int_{K_\varepsilon^c} P_t(0, dy) P_{t-i}(y, A_\varepsilon^c) \right\} dt \\
\leq \frac{1}{n} \sum_{i=0}^{n-1} \int_{t_{i+1}}^{t_i} \left( \frac{\varepsilon}{2} \int_{K_\varepsilon} P_t(0, dy) + \int_{K_\varepsilon^c} P_t(0, dy) \right) dt \\
\leq \frac{1}{n} \sum_{i=0}^{n-1} \int_{t_{i+1}}^{t_i} \varepsilon dt = \varepsilon.
$$

It follows from the definition of tightness that $\mu_n(\cdot) = \frac{1}{n} \int_0^n P_t(0, \cdot) dt$, $n = 1, 2, \ldots$ is tight on $H^1(R) \cap H^{-1}(R)$.

**Theorem 4.6.** Assume that $u_0 \in H^1(R)$, $\Phi \in L^0_{\theta}(L^2(R); H^1(R)) \cap L^0_{\theta}(L^2(R); \dot{H}^{-1}(R))$. Then equation (1) admits an invariant measures. Moreover, equation (1) with the deterministic initial condition posses an ergodic invariant measure.

**Proof.** By using Krylov-Bogoliubov, Theorem 4.1 and Theorem 4.3, we can prove the existence of invariant measures for equation (1). Denote $K$ be the set of all the invariant measure, then it is easy to check that $K$ is convex. Let $\{\mu_n\}_{n \in \mathbb{N}^+}$ be a sequence of invariant measures in $K$. Then there exists some constant $C$ such that

$$
\sup_n \int \|u\|^2_{H^1} \mu_n(du) \leq C \quad \text{and} \quad \sup_n \int \|u\|^4_{H^1} \mu_n(du) \leq C.
$$

For any deterministic initial condition, $\{(\mu_n P_t)(\cdot) : n \in \mathbb{N}^+\} = \{\mu_n(\cdot) : n \in \mathbb{N}^+\}$ is tight. Since $K$ is a close set, then $K$ is compact. It follows from Krein-Milman theorem, a convex compact set posses extremal point. Theorem 2.1 yields that this extremal point is ergodic. Therefore, stochastic equation (1) admits an ergodic invariant measure. Thus, the proof of Theorem 4.6 is complete.

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