A METHOD FOR OBTAINING QUANTUM DOUBLES FROM THE YANG-BAXTER $R$-MATRICES

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ABSTRACT

We develop the approach of refs. [1] and [2] that enables one to associate a quasi-triangular Hopf algebra to every regular invertible constant solution of the quantum Yang-Baxter equations. We show that such a Hopf algebra is actually a quantum double.

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1. It is well known [1] that any invertible constant matrix solution $R$ of the quantum
Yang-Baxter equation (QYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad (1)$$
naturally generates a bialgebra $A_R = \{1, t_{ij}\}$ defined by

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}, \quad \Delta(T_1) = T_1 \otimes T_1, \quad \varepsilon(T) = 1,$$

(generators $t_{ij}$ form a matrix $T$, $\Delta$ is a coproduct and $\varepsilon$ a counit) and also another
bialgebra $U_R = \{1, l^+_{ij}, l^-_{ij}\}$ with

$$R_{12} L^+_2 L^+_1 = L^+_1 L^+_2 R_{12}, \quad (2)$$
$$R_{12} L^-_2 L^-_1 = L^-_1 L^-_2 R_{12}, \quad (3)$$
$$\Delta(L^+_1) = L^+_1 \otimes L^+_1, \quad \varepsilon(L^\pm) = 1, \quad (4)$$

which is paired to $A_R$. This pairing [1, 2] is established by the relations

$$< T_1, L^+_2 >= R_{12}, \quad < T_1, L^-_2 >= R^{-1}_{21},$$

obeys the duality conditions

$$< \alpha \beta, a >= < \alpha \otimes \beta, \Delta(a) >, \quad < \Delta(\alpha), a \otimes b >= < \alpha, ab >,$$

and appears to be degenerate. With some additional effort (quotienting by appropriate
null bi-ideals) these bialgebras can be made Hopf algebras $\hat{A}_R$ and $\hat{U}_R$, dual to each
other. Their antipodes are defined by

$$< T_1, S(L^+_2) > = < S(T_1), L^+_2 > = R^{-1}_{12},$$
$$< T_1, S(L^-_2) > = < S(T_1), L^-_2 > = R_{21}.$$  

With essential use of this duality Majid [3] showed that in fact, with a certain
reservation, $\hat{U}_R$ proves to be a quasitriangular Hopf algebra with the universal $R$-
matrix given by implicit formulas originated from $< T_1 \otimes T_2, R > = R_{12}$. By the way, Majid claims [2] that $\hat{U}_R$ is ‘more or less’ of the form of a quantum double. In the
present note we argue that, modulo the same reservation, $\hat{U}_R$ is actually a quantum
double.

2. Recall that a quantum double $D(A)$ is the Hopf algebra of the following type
([3], see also [1, 2, 4]). Let $A \otimes A^{\circ}$ be the tensor product of the Hopf algebra $A$
and its antidual $A^{\circ}$. Antiduality (i.e. the duality with opposite coproduct and inverse
antipode) means $< e^i, e_j > = \delta^i_j$ and

$$< \alpha \beta, a >= < \alpha \otimes \beta, \Delta(a) >, \quad < \Delta(\alpha), a \otimes b >= < \alpha, ab >,$$
$$\varepsilon(a) = < 1, a >, \quad \varepsilon(\alpha) = < \alpha, 1 >, \quad < S(\alpha), a >= < \alpha, S^{-1}(a) >, \quad (5)$$
where \( a, b \in A, \) \( \alpha, \beta \in A^\circ \), and \( \{ e_j \}, \{ e^i \} \) are the corresponding bases. To equip \( A \otimes A^\circ \) with the Hopf algebra structure of the quantum double, one must define a very specific cross-multiplication recipe. If
\[
e_i e_j = c^k_{ij} e_k, \quad \Delta(e_i) = f_{ij}^k (e_j \otimes e_k), \quad S(e^i) = \sigma_j^i e^j,
\]

it reads
\[
e^i e_j = O_{jq}^{ip} e_p e^q, \quad \text{where} \quad O_{jq}^{ip} = e_{p}^{q} c_{i}^{q} \sigma_{r}^{s} f_{s}^{r} f_{j}^{l} f_{j}^{m}.
\]

In invariant form this looks like
\[
aa = \sum \sum < S(\alpha_{(1)}), a_{(1)} > < \alpha_{(2)}, a_{(3)} > a_{(2)} \alpha_{(2)},
\]

where
\[
\Delta^2(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}, \quad \Delta^2(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.
\]

Here the usual notation for coproducts (cf. \( \Delta(a) = \sum a_{(1)} \otimes a_{(2)} \)) is used. The resulting Hopf algebra proves to be quasitriangular with the universal \( \mathcal{R} \)-matrix
\[
\mathcal{R} = \sum_i (e_i \otimes 1) \otimes (1 \otimes e^i).
\]

One easily finds that \( gl_q(2) \) and other simple examples of quantum universal enveloping algebras are both the \( \hat{U}_R \)-type algebras and quantum doubles. Can it happen that \( \hat{U}_R \) would be a quantum double for any \( R \)? Majid’s approach based on the \( \hat{U}_R \leftrightarrow A_R \) duality does not readily answer this question. That is why we choose another way: not to use \( A_R \) at all. The key observation is that there exists an inherent antiduality between \( U^+_R \) and \( U^-_R \) which is precisely of the form required for the quantum-double construction.

3. Let us define bialgebras \( U^+_R = \{ 1, \lambda^+_l \} \) and \( U^-_R = \{ 1, \lambda^-_l \} \) by eqs. (2),(4). Note that the cross-multiplication relation (3) is not yet imposed, so \( U^+_R \) and \( U^-_R \) are considered to be independent so far. However, the very natural pairing between them can be introduced. It is generated by
\[
< L^-_l, \lambda^+_l > = R^{-1}_{l_2}, \quad < L^-, 1 > = < 1, L^+ > = < 1, 1 > = 1
\]

and in the general case looks like
\[
\left< L^{-}_{l_1} \ldots L^{-}_{l_m}, \lambda^+_j \ldots \lambda^+_p \right> = R^{-1}_{i_1 j_2} \ldots R^{-1}_{i_q j_p} \ldots R^{-1}_{i_{m, j_1}},
\]

where the r.h.s. is a product of \( mn \) \( R^{-1} \)-matrices corresponding to all pairs of indices \( i_q j_p \) with \( j \)-indices ordered from right to left. The consistency of (8) and (5) with (4) is evident, while the proof of the consistency with (2) reduces to manipulations like
\[
< L_0^-, R_{12} L_1^+ - L_1^+ R_{12} > = \left< L_0^-, L_0^- \right> R_{12} (L_1^+ \otimes L_2^+) - (L_2^+ \otimes L_1^+) R_{12} = R_{12} R_{01}^{-1} R_{02}^{-1} - R_{02}^{-1} R_{01}^{-1} R_{12} = 0
\]

and repeated use of QYBE (1).
For general $R$, this pairing is degenerate. To remove the degeneracy, i.e., to transform pairing into antiduality, one should factor out appropriate bi-ideals [2]. In simple cases this procedure is explicitly carried out and works well. For general $R$ it is of course not under our control.\textsuperscript{4} The situation is quite similar to [2]: we are to rely on algebra structure. Let us introduce an antipode $S$.

For general $R$, admitting the Hopf algebra structure. Let us introduce an antipode $S$ in $U_R$ and an inverse antipode $S^{-1}$ in $U_R^\pm$ by the relations

\[
\begin{align*}
&< S(L^-), 1 > = 1, \quad S^{-1}(L^+) = L^- S^{-1}(L^+), R_{12}, \\
&\text{extending them on the whole of } U_R^\pm. \text{ or } U_R^- \text{ as antimorphisms of algebras and coalgebras. The definition is correct due to} \\
&< m \circ (S \otimes id) \circ \Delta(L^-), L^+ > = < m \circ (S(L^-) \otimes L^-), L^+ > \\
&= < S(L^-) \otimes L^+, L^- \otimes L^+ > = R_{12}^{-1} = 1 = < \varepsilon(L^-), L^+ >,
\end{align*}
\]

\[
< S(L^-), R_{12} L_2^+ L_1^+ - L_1^+ L_2^+ R_{12} > = \Delta \circ S(L^-), R_{12}(L_1^- \otimes L_1^-) - (L_1^- \otimes L_1^-)R_{12} >
\]

\[
\begin{align*}
&= < S(L^-) \otimes S(L^-), R_{12}(L_2^+ \otimes L_1^+) - (L_1^+ \otimes L_2^+)R_{12} > = R_{12}R_{02}R_{01} - R_{01}R_{02}R_{12} = 0, \\
&\text{extending them on the whole of } U_R^\pm. \text{ or } U_R^- \text{ as antimorphisms of algebras and coalgebras. The definition is correct due to} \\
&< m \circ (S \otimes id) \circ \Delta(L^-), L^+ > = < m \circ (S(L^-) \otimes L^-), L^+ > \\
&= < S(L^-) \otimes L^+, L^- \otimes L^+ > = R_{12}^{-1} = 1 = < \varepsilon(L^-), L^+ >,
\end{align*}
\]

\[
< S(L^-), R_{12} L_2^+ L_1^+ - L_1^+ L_2^+ R_{12} > = \Delta \circ S(L^-), R_{12}(L_1^- \otimes L_1^-) - (L_1^- \otimes L_1^-)R_{12} >
\]

\[
\begin{align*}
&= < S(L^-) \otimes S(L^-), R_{12}(L_2^+ \otimes L_1^+) - (L_1^+ \otimes L_2^+)R_{12} > = R_{12}R_{02}R_{01} - R_{01}R_{02}R_{12} = 0, \\
&< S(L^-) \otimes S(L^-), R_{12}(L_2^+ \otimes L_1^+) - (L_1^+ \otimes L_2^+)R_{12} > = R_{12}R_{02}R_{01} - R_{01}R_{02}R_{12} = 0,
\end{align*}
\]

\[
\begin{align*}
&< S(L^-) \otimes S(L^-), R_{12}(L_2^+ \otimes L_1^+) - (L_1^+ \otimes L_2^+)R_{12} > = R_{12}R_{02}R_{01} - R_{01}R_{02}R_{12} = 0,
\end{align*}
\]

\[
\begin{align*}
&< S(L^-) \otimes S(L^-), R_{12}(L_2^+ \otimes L_1^+) - (L_1^+ \otimes L_2^+)R_{12} > = R_{12}R_{02}R_{01} - R_{01}R_{02}R_{12} = 0,
\end{align*}
\]

\[
\begin{align*}
&\text{Thus we regain eq.(3) as the quantum-double cross-multiplication condition!}
\end{align*}
\]

\[
\begin{align*}
&\text{Our conclusion is that } R\text{-matrices obeying QYBE generate the algebraic structures} \\
&\text{of quantum double in quite a natural way.}
\end{align*}
\]

\[
\begin{align*}
&\text{5. To illustrate the proposed scheme, consider the } sl_q(2) \text{ } R\text{-matrix}
\end{align*}
\]

\[
R_q = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad R_q^{-1} = R_{q^{-1}}.
\]

\[
\begin{align*}
&\text{Here the bialgebras } U_{R_q}^\pm \text{ have 8 generators } l_{ij}^\pm. \text{ The bi-ideals to be factored out are generated by the relations} \\
&l_{21}^- = 0, \quad l_{12}^+ = 0, \quad l_{11}^+ l_{22}^- - l_{22}^- l_{11}^- = 1, \quad l_{11}^+ l_{22}^+ = l_{22}^+ l_{11}^+ = 1.
\end{align*}
\]
After factorization the number of independent generators is reduced to 4. We denote them \(X^\pm, H, H'\) (note that \(H' \neq H\) so far):

\[
L^+ = \begin{pmatrix}
q^{H/2} & 0 \\
(q^{1/2} - q^{-3/2})X^+ & q^{-H/2}
\end{pmatrix}, \quad L^- = \begin{pmatrix}
q^{-H/2} & (q^{-1/2} - q^{3/2})X^- \\
0 & q^{H/2}
\end{pmatrix}.
\]

The multiplication rules (inside each algebra), coproducts and antipodes are:

\[
[H, X^+] = 2X^+, \quad [H', X^-] = -2X^-,
\]

\[
\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(H') = H' \otimes 1 + 1 \otimes H',
\]

\[
\Delta(X^+) = X^+ \otimes q^{H/2} + q^{-H/2} \otimes X^+, \quad \Delta(X^-) = X^- \otimes q^{H/2} + q^{-H/2} \otimes X^-,
\]

\[
S(X^\pm) = -q^{\pm1}X^\pm, \quad S(H) = -H, \quad S(H') = -H'.
\]

The quantum-double cross-multiplication rules (6) take the form

\[
[H', X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [H, H'] = 0,
\]

\[
[X^+, X^-] = (q^{(H+H')/2} - q^{-(H+H')/2})/(q - q^{-1}).
\]

The identification \(H' \equiv H\) leads to the ordinary \(sl_2(2)\).

6. To give one more illustration, let us consider a bialgebra introduced in [3]. In a slightly simplified form it has generators \(\{1, t^j, u^i, E_j, F^i\}\) which obey the following relations (here we prefer to display all the indices):

\[
R^{ij}_{mn} t^m_p t^n_q = R^{mn}_{pq} t^i_p t^j_q, \quad E_p t^j_q = R^{mn}_{pq} t^i_p E_m,
\]

\[
\Delta(t^i_j) = t^i_k \otimes t^k_j, \quad \varepsilon(t^i_j) = \delta^i_j, \quad \Delta(E_j) = E_i \otimes t^j_i + 1 \otimes E_j, \quad \varepsilon(E_j) = 0,
\]

\[
R^{ij}_{mn} u^m_p u^n_q = R^{mn}_{pq} u^j_m u^i_p, \quad F^i u^j_p = R^{ij}_{mn} u^m_p F^n,
\]

\[
\Delta(u^j_i) = u^k_i \otimes u^j_k, \quad \varepsilon(u^i_j) = \delta^i_j, \quad \Delta(F^i) = F^i \otimes 1 + u^j_i \otimes F^j, \quad \varepsilon(F^i) = 0,
\]

\[
R^{ij}_{mn} u^m_p t^n_q = R^{mn}_{pq} t^i_m u^j_p, \quad E_j F^i - F^i E_j = t^i_j - u^i_j,
\]

\[
u^i_p E_q = R^{mn}_{pq} E_n u^i_m, \quad t^i_j F^j = R^{ij}_{mn} F^m t^n_p,
\]

with \(R\) obeying QYBE (1). This is not a bialgebra of the form (2)-(4). Rather it is of the ‘inhomogeneous quantum group’ type [3]. Let us make sure that it is a quantum double as well.

Consider \(T, E\)-bialgebra (11),(12) and \(U, F\)-bialgebra (13),(14) firstly as being independent and fix nonzero pairings on the generators by

\[
\langle u^i_j, t^p_q \rangle = R^{ip}_{jq}, \quad \langle u^i_j, 1 \rangle = \langle 1, t^i_j \rangle = \langle F^i, E_j \rangle = \delta^i_j,
\]

extending them to the whole bialgebras with the help of (5). The definition is correct due to

\[
\langle F^i, E_p t^j_q - R^{mn}_{pq} t^i_n E_m \rangle = \langle F^i \otimes 1 + u^i_k \otimes F^k, t^j_q \otimes E_p - R^{mn}_{pq} (E_m \otimes t^i_n) \rangle = R^{kj}_{pq} \delta^i_p - \delta^i_m R^{mn}_{pq} \delta^j_k = 0,
\]

\[
\langle F^i u^j_p - R^{ij}_{mn} u^m_p F^n, E_q \rangle = \langle F^i \otimes u^j_p - R^{ij}_{mn} (u^m_p \otimes F^n), E_k \otimes t^k_q + 1 \otimes E_q \rangle = \delta^i_k R^{kj}_{pq} - R^{ij}_{mn} \delta^m_p \delta^q_n = 0.
\]
After factoring out the corresponding null bi-ideals, we may define antipodes on the generators as follows:

\[ < S(u^i_j), t^m_q >= u^i_j, S^{-1}(t^p_q) >= (R^{-1})^{ip}_{jq}, \]

\[ < S(u^i_j), 1 >= 1, S^{-1}(t^i_j) >= \delta^i_j, \]

\[ S(F^i) = -S(u^i_j)F^j, \]

\[ S(E^j) = -E^i_j t^i_j. \]

The proof of correctness is in complete analogy with the \( \tilde{U}_R \)-case.

Now a direct application of the recipe (6) exactly reproduces the cross-multiplication relations (15),(16). For example,

\[ F^i E^j = < S(F^i), E_m > < 1, t^m_n > t^m_n + < S(u^i_n), 1 > < 1, t^m_n > E_m F^m \]

\[ + < S(u^i_n), 1 > < F^m, E_j > t^m_n = -t^m_n + E_j F^i + u^i_j, \]

because of

\[ < S(F^i), E_m >= -< S(u^i_k) \otimes F^k, E_n \otimes t^m_n + 1 \otimes E_m >= -\delta^i_k. \]

Therefore, bialgebras of the type (11)-(14) are also transformed into the quantum double using our method.

7. Consider at last a bialgebra that is known to be related to bicovariant differential calculus on quantum groups. Its coalgebra structure is given by (12), whereas the multiplication relations (11) are to be supplemented by

\[ t^i_j E_q + f^{i}_{nm} t^m_p t^q_p = R^{nm}_{pq} E_m t^i_n + f^{n}_{pq} t^i_n, \]

\[ E_i E_j - R^{mn}_{ij} E_n E_m = f^{m}_{ij} E_m. \]

\( f_{ij} \) being new structure constants. This bialgebra, unlike its ancestor (11),(12), exhibits the \( R \)-matrix-type representation

\[ R_{12} T_1 T_2 = T_2 T_1 R_{12}, \quad \Delta(T) = T \otimes T, \]

where, in terms of multi-indices like \( I = \{0, i\} \),

\[ T^I_j = \begin{pmatrix} 1 & E^i_j \\ 0 & t^i_j \end{pmatrix}, \quad R^{I,J}_{MN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta^i_j & 0 & 0 \\ 0 & 0 & \delta^i_m & f^{m}_{mn} \\ 0 & 0 & 0 & R^{ij}_{mn} \end{pmatrix}. \]

Of course, \( R \) must satisfy the QYBE (1) which now involves the structure constants \( f_{mn}^i \) as well as \( R^{ij}_{mn} \). Note that, due to (18),(19), the bialgebra (11),(12) is not restored from (20) by mere setting \( f_{mn}^i \equiv 0 \).

Now let us try to develop a quantum double from the bialgebra (20). However, it seems to be quite uneasy task. A natural Ansatz for the candidate antidual bialgebra is

\[ U^I_j = \begin{pmatrix} 1 & 0 \\ F^i & u^i_j \end{pmatrix}. \]
which causes the corresponding $R$-matrix to be

$$R^{ij}_{MN} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \delta^i_n & 0 & 0 \\
0 & 0 & \delta^m_i & 0 \\
0 & \bar{f}^{ij}_{mn} & 0 & R^{ij}_{mn}
\end{pmatrix}$$

with different structure constants $\bar{f}$ and another QYBE system involving $R$ and $\bar{f}$.

Now, attempting to fix a pairing in the form

$$<U_1, T_2> = Q_{12}$$

with a certain numerical matrix $Q$, we immediately arrive at the following general statement:

Let $R$ and $\bar{R}$ be invertible solutions of QYBE. If there exists an invertible solution $Q$ of the equations

$$Q_{12}Q_{13}R_{23} = R_{23}Q_{13}Q_{12},$$
$$\bar{R}_{12}Q_{13}Q_{23} = Q_{23}Q_{13}\bar{R}_{12},$$

then (21) is a correct pairing between the $T$- and $U$-bialgebras generated by $R$ and $\bar{R}$, respectively, and, assuming a proper quotienting procedure to be performed, the antipodes can be defined by the relations

$$<S(U_1), T_2> = <U_1, S^{-1}(T_2)> = Q^{-1}_{12}$$

and the quantum-double structure can be established on the tensor product of these bialgebras by the cross-multiplication formula

$$Q_{12}U_1T_2 = T_2U_1Q_{12}.$$  

Whether such a program can really be carried through in interesting cases (e.g. for $R$ and $\bar{R}$ given above) is the subject of further investigation.

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