HIGHER CHORDALITY II: TORIC CHORDALITY VIA THE MCMULLEN–WEIL LEFSCHETZ MAP

KARIM A. ADIPRASITO

Abstract. We put the fundamental graph-theoretic notion of chordality into a proper context within the cohomology rings of toric varieties and framework rigidity.

Our main focus is the relation of higher chordality to the Hard–Lefschetz Theorem for simplicial polytopes of Saito, McMullen and others. Toric chordality allows us to state a powerful Quantitative Lower Bound Theorem which relates the “defect” to a chordal complex to the $g$-numbers of the same polytope. More refinedly, the primitive $(i-1)$-st Betti numbers can be related to the geometric change between a class in dimension $(i+1)$ and its image under the (McMullen–Weil) Lefschetz map.

The perspective of chordality within stress spaces also enables us to generalize some of the classical results of graph chordality to higher dimension context, including the fundamental relation to the Leray property and the propagation principle. In particular, we obtain a new and simple proof of the Generalized Lower Bound Theorem for simplicial polytopes, and can draw stronger conclusions including the regularity and shellability of $k$-stacked triangulations.

Introduction. A notion of at the very foundation of graph theory, chordality is being established as an important notion in commutative algebra and toric algebraic geometry.

A central contribution to a proper understanding of chordality in this context was provided in the work of Kalai [Kal87] which connects chordality to framework rigidity of simplicial polytopes as studied by Cauchy (and many others following him): Chordality, in presence of framework rigidity, detects minimal rigidity of a simplicial polytope (compare also Gromov’s treatment [Gro86, Section 2.4]).

The second ingredient, developed in parallel to the work of Gromov and Kalai, is the relation between framework rigidity, Stanley–Reisner Theory [Sta96], toric algebraic geometry, moment-angle complexes and the weight and polytope algebras [McM93, McM96, Lee96].

Combined, these developments give a wealth of interpretations and aspects of the simple, graph-theoretic notion that have only been touched upon superficially. One of the main purposes of this paper is to give a proper explanation of this connection, and provide further tools to its understanding.

A second purpose of this paper is to introduce “toric chordality” a notion of higher chordality that lives, naturally, in the toric variety and more generally within McMullen–Lee stress spaces and skeletal rigidity (as defined by Tay–White–Whiteley [TWW95]). Additionally, this perspective allows us to quantify the relation between primitive Betti numbers and chordality. This has a variety of applications which we will explore here and some following projects, cf. [ANS15b].

We contrast the notion of toric chordality with the notion of homological chordality developed in [ANS15a], focussing on an important property of chordal graphs: graph chordality implies chordality in all higher dimensions. This important propagation principle fails in homological chordality, but

Date: March 24, 2015.

K. A. Adiprasito acknowledges support by an IPDE/EPDI postdoctoral fellowship, a Minerva postdoctoral fellowship of the Max Planck Society, and NSF Grant DMS 1128155.
it holds beautifully in the toric context. Finally, we conclude with some applications to polytope theory.

**Toric chordality and the Hard Lefschetz Theorem.** We start this paper by recalling some simple homological notions of chordality that are based on examining the homology of induced subcomplexes of a simplicial complex.

The main part of this paper is devoted to a notion first seen in the study of simplicial polytopes, specifically the Hard Lefschetz Theorem. Historically, the perspective we adopt to see chordality in the Lefschetz theorem goes back to Weil, who provided a fundamental duality of the Hard Lefschetz theorem [Wei58].

Rather than working directly in this setting, we will work more generally in the stress/weight algebra introduced by McMullen and Tay–Whiteley, referring to this setting as the McMullen–Weil framework. Once we establish stress spaces as the natural notion and setting for higher-dimensional chordality, most of the results we intend to prove fall off naturally and without problems.

**Quantitative Lower Bound Lemma for Stress Groups I.** Let $P$ denote any simplicial $d$-polytope. Then the map

$$S_{k+1}^{\text{lin}}(P) \xrightarrow{\delta = \sum \frac{d}{\pi_i}} S_k^{\text{lin}}(P),$$

has cokernel bounded above in dimension by $-g_{k+1}^-(\Delta)$. Moreover, there exists a set $\mathcal{E}(P) \subset P^{(0)}$ of at most $((k+1)g_{k+1} + (d+1-k)g_k)(P)$ vertices of $P$ such that for every $\gamma \in S_{k+1}^{\text{lin}}(P)$, we have

$$\gamma^{(0)} \setminus \mathcal{E}(P) \subset (\delta \gamma)^{(0)} \subset \gamma^{(0)}.$$

Here, $\mathcal{E}(P)$ is a set of “exceptional” vertices, and $S_r^{\text{lin}}(P)$ is the $r$-th stress group as defined by Lee [Lee96] and Tay–Whiteley [TW00] (and which is naturally isomorphic to the $(d-r)$-th Minkowski weight group of the polar dual $P^*$, cf. [McM96]).

Clearly, Lemma I is a quite elementary corollary of the Hard Lefschetz Theorem, but it is the central pointer for us on the way to a notion of higher chordality.

A particular consequence of Lemma I is that, with respect to the stress groups, a simplicial polytope $P$ with $g_k = 0$ (where $2k \leq \dim P$) is “toric $k$-chordal”, namely one has $\mathcal{E}(P) = \emptyset$ and a surjection $S_{k+1}^{\text{lin}}(P) \twoheadrightarrow S_k^{\text{lin}}(P)$ in the conclusion of Lemma I. In terms of implications, we shall see that toric chordality is the “proper” interpretation of graph chordality.

**Propagation of Toric Chordality II.** Let $\Delta$ denote any toric $k$-chordal geometric, proper simplicial $d$-complex without missing faces of dimension $> k$. Then $\Delta$ is toric $\ell$-chordal for all $\ell \geq k$.

Here, “proper” means that the vertices of the geometric simplicial complex $\Delta$ are in sufficiently general position. For proper embeddings of sufficiently regular complexes, we shall see that toric chordality implies homological chordality.

To appreciate the propagation property, one has to contrast it with a folklore theorem on chordal graphs going back at least to work of Dirac [Dir61], compare also [LB63], that asserts a propagation principle for graph chordality, cf. [ANS15a] for a detailed discussion.

**Applications to combinatorial geometry and geometric topology.** We summarize some of the applications to combinatorics we derive here.

The Generalized Lower Bound Theorem. An important part of this paper then lies in relating toric chordality to resolution chordality. After this translation is done, we in particular obtain a new proof of the Generalized Lower Bound Theorem of Kalai [Kal87] and Murai–Nevo [MN13].
Generalized Lower Bound Theorem III. Let $P$ denote any simplicial $d$-polytope, and let $2k \leq d$. Then the following are equivalent:

1. $P$ satisfies $g_k(P) = 0$.
2. $\partial P$ is decomposition $k$-chordal and has no missing faces of dimension $l$ for $k \leq l \leq d - k$.
3. $P$ admits a $k$-stacked triangulation, i.e. a triangulation without interior faces of dimension $\leq d - k$.

Using our new, elementary proof based on the propagation property, we immediately obtain a strengthening of the Generalized Lower Bound Theorem, conjectured by McMullen [McM04] and Bagchi–Datta.

**Theorem IV.** A $k$-stacked triangulation of a simplicial $d$-polytope, $k \leq \frac{d}{2}$, is regular, and in particular shellable.

The Quantitative (Generalized) Lower Bound Theorem. A key problem in the study of simplicial polytopes is to study polytopes away from the extremal primitive Betti numbers. Such a theorem is provided here.

**Quantitative Lower Bound Theorem V.** Let $P$ denote any simplicial $d$-polytope and $\bar{P} = \mathrm{Cl}_k \partial P$. Then, for every subset $W \subset P^{(0)}$ and $\mathcal{E}(P)$ as in Theorem I, we have

$$\beta_{k-1}(\bar{P}|_{W \cup \mathcal{E}(P)}) \leq \max\{-g_{k+1}(P), 0\}.$$ 

In particular, $\beta_{k-1}(\mathrm{Cl}_k \partial P|_W) = 0$ for all $W \subset P^{(0)}$ if $g_k = 0$ for $2k \leq d$.

This is not the end of the consequences of toric chordality, however: For $g_k > 0$, we obtain a powerful geometric quantification of the Generalized Lower Bound Theorem, cf. [ANS15b], resolving a conjecture of Kalai.

**Basic notation.** Throughout, we allow any simplicial complex to be a relative simplicial complex, i.e. a pair of abstract simplicial complexes $\Psi = (\Delta, \Gamma)$, $\Gamma \subset \Delta$ (where an abstract simplicial complex is a downclosed subset of the powerset $2^S$ for some finite set $S$). A geometric (relative) simplicial complex is a simplicial complex $\Psi$ with a map of the vertices to some vector space $k^n$ over any field $k$. It is proper if the image of every $k$-face, $(k < n)$, linearly spans a subspace of dimension $k + 1$.

A $k$-dimensional simplicial complex is complete if it coincides with the $k$-skeleton of some simplex. The deletion $\Delta - \sigma$ of a face $\sigma$ of $\Delta$ is the maximal subcomplex of $\Delta$ that does not contain $\sigma$; this is naturally extended to deletions of subcomplexes from $\Delta$.

Now, let $I \subset \{-1, 0, 1, 2, \ldots\}$ denote any subset, and let $\Delta$ denote any simplicial complex. We then denote by $\Delta^{(I)}$ the collection of faces $\sigma \in \Delta$ with $\dim \sigma \in I$. As a special case, we obtain the $k$-skeleton $\mathrm{Sk}_k \Delta := \Delta^{(\leq k)}$ of $\Delta$, and the collection of $k$-faces $\mathcal{F}_k := \Delta^{(k)}$.

A $k$-clique is a pure simplicial complex of dimension $k$ that contains all possible faces of dimension $\leq k$ on its vertex set. With this, one can associate to any simplicial complex its complex of $k$-cliques, defined as $\mathrm{Cl}_k \Delta := \{\sigma \subset \mathcal{F}_0(\Delta) : \mathrm{Sk}_k \sigma \subseteq \Delta\}$.

Recall that the star and link of a face $\sigma$ in $\Delta$ are the subcomplexes $\mathrm{st}_\sigma \Delta := \cup_{\tau \in K} 2^\tau$ and $\mathrm{lk}_\sigma := \Delta \setminus \{\tau : \sigma \subset \tau \in K\}$.

A nonface of $\Delta$ is, naturally, a simplex on $\Delta^{(0)}$ that is not a face of $\Delta$. A minimal nonface, or missing face, of $\Delta$ is an inclusion minimal nonface of $\Delta$. Equivalently, a simplex $\sigma$ is a missing face of $\Delta$ iff $\partial \sigma \subset \Delta$, but $\sigma \not\subset \Delta$. 


If $c$ is any simplicial $k$-chain in $\Delta$ (a formal sum of $k$-simplices enriched by some coefficients), and $\Gamma$ is any subcomplex of $\Delta$, then $c|_{\Gamma}$ denotes the restriction of $c$ to the summands supported in $\Gamma$.

Finally, $f_k(\Delta) = \# \Delta^{(k)}$ denotes the number of $k$-faces of $\Delta$. The entries of the $h$-vector resp. $g$-vector of $\Delta$ are computed as

$$h_i(\Delta) = \sum_{k=0}^{i} (-1)^{i-k} \binom{d-k}{i-k} f_{k-1}(\Delta) \quad \text{and} \quad g_0 = 1, \quad g_i = h_i(\Delta) - h_{i-1}(\Delta).$$

**Part A. Homological Chordality**

Homological chordality [ANS15a] is a natural generalization of graph chordality to higher dimensions; it is considerably more natural and easier to work with, but also much weaker in terms of implications. One natural test of strength for a generalization of chordality is its propagation principle, which homological chordality does not enjoy in a satisfying form. We recall the situation, for purposes of contrast, in the remainder of this part:

1. **Resolution of cycles.**

Let $G$ denote a chordal graph, and let $\overline{G} = \text{Cl}_1 G$ denote the complex induced by its 1-cliques. Then, if $z$ is any 1-cycle, there exists a 2-chain $c$ with $\partial c = z$ and $c(0) = z(0)$. Equivalently, $z$ can be written as a sum of 1-cycles of length 3 that contain no vertices that are not already vertices of $z$. This leads us to basic notions of homological chordality.

Consider any simplicial homology theory $H_*$ that arises from a facewise constant ring of coefficients with chain complex $C_*$. We say that a $(k+1)$-chain $c \in C_{k+1}(\Delta)$ is a resolution of a $k$-cycle $z \in Z_k$ if $c(0) = z(0)$ and $\partial c = z$. We say that a (relative) complex $\Psi$ in which every $k$-cycle admits a resolution is (resolution) $k$-chordal.

2. **The propagation property of homological chordality**

The Leray property encodes the fact that a simplicial complex is $k$-chordal for all $k \leq \ell$ (for some $\ell$). A fundamental property of graph chordality, going back at least to Dirac [Dir61] is that implies the Leray property immediately once the trivial obstruction vanishes.

Homological chordality falls short of this aspect in higher dimensions. We sum this up in the following result:

**Theorem 2.1** (cf. [ANS15a]). Let $\Delta$ denote any (abstract) simplicial complex without missing faces of dimension $> k$. The following are equivalent:

1. $\Delta$ is resolution $i$-chordal for $i \in [k, 2k - 1]$.
2. $\Delta$ is $k$-Leray, i.e., it is resolution $i$-chordal for $i \geq k$.

However, for every $k \geq 2$, there is a simplicial complex $\Delta_k$ that is

1. resolution $i$-chordal for $i \in [k, 2k - 2]$,
2. has no missing face of dimension $> k$, but
3. is not resolution $(2k - 1)$-chordal.

Therefore, a better notion of higher chordality is needed.
Part B. Toric chordality

In this section, we will define and study toric chordality, a form of higher chordality inspired by a close study of the Hard Lefschetz Theorem. Instead of directly working with the cohomology ring of the toric variety, we work with two models for it: the Stanley–Reisner ring (corresponding to the intersection ring generated by divisors) and, more importantly, Minkowski weights and McMullen–Lee stress spaces, corresponding to the Chow cohomology.

In this latter setting, we will define toric chordality, which we will show to satisfy a stronger form of the propagation theorem; see Theorem 6.1. We start by demonstrating a quantitative analogue of the GLBT for stress spaces of polytopes.

3. Stress groups, Stanley–Reisner Theory and the Hard Lefschetz Theorem

If $k$ denotes (here and in the following) any field of characteristic 0, and $\Delta$ is a simplicial complex on $n$ vertices, let $k[\Delta]$ denote its Stanley–Reisner $k[x]$-module [Sta96], where $k[x] = k[x_1, \ldots, x_n]$. A collection of linear forms $\Theta = (\theta_1, \ldots, \theta_\ell)$ in the polynomial ring $k[x]$ supporting $k[\Delta]$ is a (partial) linear system of parameters if $\dim k[\Delta]/\Theta k[\Delta] = \dim k[\Delta] - \ell$ for dim the Krull dimension.

Specifically, we consider the $i$-th stress space (cf. [Lee96, TW00]), as a $k$-subspace of $k[x]$, the latter equipped with the standard inner product on monomials $<x^a, x^b> = \delta_{a,b}$: Identify $k[\Delta]$ with the $k[x]$-submodule $\text{span}(x^a : \text{supp}(a) \in \Delta)$ of $k[x]$ and let

$$S_i^{\text{lin}}(\Delta, \Theta) := (\Theta k[x] + I_\Delta) / \cap k[x]_i \cong (k[\Delta]/\Theta k[\Delta])_i.$$  

Alternatively, notice that one can construct an action of the Stanley–Reisner ring on the stress space by observing that the adjoint operator to multiplication with a linear form $\theta = \theta(x)$ is the differential $\theta = \theta(\partial/\partial x)$ (cf. [Fra13]); with this we have

$$S_i^{\text{lin}}(\Delta, \Theta_j) = \ker \left[ \theta_j^* : S_i^{\text{lin}}(\Delta, \Theta_{j-1}) \longrightarrow S_{i-1}^{\text{lin}}(\Delta, \Theta_{j-1}) \right].$$  

If $\ell = \dim k[\Delta] = \dim \Delta + 1$, then $\Theta$ is full, and $S_i^{\text{lin}}(\Delta)$ is determined by its squarefree terms, i.e. the restriction map of $S_i^{\text{lin}}(\Delta)$ to its squarefree terms is an injection, cf. [Lee96]. The space of squarefree coefficients of $S^{\text{lin}}$ is also called the space of Minkowski weights [KP08].

Note that $\Theta$ induces a map $\Delta^{(0)} \to k^\ell$ by associating to the vertices of $\Delta$ the coordinates $(v_1, \ldots, v_n) = \Theta^t \in k^{\ell \times n}$. Hence, for a geometric simplicial complex $\Delta$ in $k^\ell$, we denote by $S_i^{\text{lin}}(\Delta, (v_1, \ldots, v_n)^t)$, and $R[\Delta] = k[\Delta]/(v_1, \ldots, v_n)^t k[\Delta]$ the canonical associated stress groups and reduced Stanley–Reisner rings, respectively, that are associated to the geometric realization.

Conversely, any proper geometric realization of any simplicial complex in $k^\ell$, namely one realization such that for every face $F$ of cardinality at most $\ell$, $(v)_{v \in F}$ is linearly independent, gives rise to a (partial) l.s.o.p. $\Theta$.

We say $\Theta$ is regular if (up to degree $k$), for every truncation $\Theta_s = (\theta_1, \ldots, \theta_s)$, we have a surjection

$$S_i^{\text{lin}}(\Delta, \Theta_{j-1}) \stackrel{\theta_j^*}{\longrightarrow} S_{i-1}^{\text{lin}}(\Delta, \Theta_{j-1}),$$  

for every $j \leq \ell$ and $i$ (at most $k$). The depth of a simplicial complex is defined as the length of the longest regular sequence that its Stanley–Reisner module admits. The following results are central to our investigation.

1. The (McMullen–Weil) Hard Lefschetz Theorem for stress spaces, [McM93, Lee96]. If $P$ is a simplicial $d$-polytope, $\Delta = \partial P$ and $\delta = \sum \frac{d}{\partial x_i}$ denotes the canonical differential on $k[x]$ associated
to the homogenizing embedding $P \hookrightarrow \mathbb{R}^d \times \{1\} \subset \mathbb{R}^{d+1}$, then
\[ S_{d-k}^{\text{lin}}(\Delta) \xrightarrow{\delta^{d-2k}} S_k^{\text{lin}}(\Delta) \]
is an isomorphism for every $k \leq \frac{d}{2}$.

(2) 
*The Hochster–Reisner–Hibi Theorem*, cf. [Sta96]. A simplicial complex $\Delta$ of dimension $\geq d - 1$ is of depth $\geq d$ if and only if its $(d - 1)$-skeleton $\Delta^{(<d)}$ is Cohen–Macaulay, i.e., for every face $\sigma$ of $\Delta$, the reduced homology of $\text{lk}_\sigma \Delta^{(<d)}$ is concentrated in dimension $d - \dim \sigma - 2$.

We shall generally prefer to work with stress spaces rather than the dual (and often more standard) Stanley–Reisner theory and the version of the Lefschetz Theorem stated above; this has the advantage that we can talk about the “support” of a $k$-stress in a straightforward way. For the results that follow, we always think of every simplicial complex as coming with an explicit coordinatization, or realization in a vectorspace over $k$. Recall that for us, it is in general not required that this realization be an embedding of the simplicial complex; rather, we usually require some properness of the coordinates of vertices.

## 4. The Quantitative Lower Bound Theorem (for stress groups)

We can now provide the first and more refined version of the Quantitative Lower Bound Theorem for polytopes, and more generally to all hereditary weak Lefschetz spheres, that is, simplicial complexes that satisfy the a weak version of the Lefschetz theorem for both the total complex (a homology sphere) and the links of faces.

An algebraic form of the well–known $g$-conjecture asserts that all homology spheres, and equivalently all Gorenstein* complexes, are Hard Lefschetz (or at least weak Lefschetz, see below).

**Definition 4.1.** Assume that there exists a $j$ and a linear differential $\delta$ such that the map
\[ S_{i+1}^{\text{lin}}(\Delta) \xrightarrow{\delta} S_i^{\text{lin}}(\Delta) \]
is an injection for all $i \geq j$ and a surjection for all $i < j$. In this case, $(\Delta, \delta)$ is called weak Lefschetz. If the same applies to every link, we call $(\Delta, \delta)$ hereditary weak Lefschetz.

Recall that if $\Delta$ is any Gorenstein* $(d - 1)$-complex, then

1. **Dehn Sommerville.** $g_k = -g_{d-k+1}$ for all $k$.
2. **Unimodality.** If $\Delta$ is any weak Lefschetz Gorenstein* $(d - 1)$-complex, map (2) is an injection for $i \geq \frac{d}{2}$, and a surjection otherwise.

**Theorem 4.2.** Consider any hereditary weak Lefschetz Gorenstein* $(d - 1)$-complex $(\Delta, \delta)$. Then, there exists a set $\mathcal{E}(\Delta) \subset \Delta^{(0)}$ of at most $(k+1)g_{k+1}(\Delta) + (d+1-k)g_k(\Delta)$ vertices such that, for every $\gamma \in S_k^{\text{lin}}(\Delta)$, we have
$$\gamma^{(0)} \setminus \mathcal{E}(\Delta) \subset (\delta \gamma)^{(0)} \subset \gamma^{(0)}.$$

**Lemma 4.3.** In the situation of Theorem 4.2, the map
\[ S_{k+1}^{\text{lin}}(\text{st}_v \Delta, \partial \text{st}_v \Delta) \xrightarrow{\delta} S_k^{\text{lin}}(\text{st}_v \Delta, \partial \text{st}_v \Delta), \quad v \in \Delta^{(0)}, \]
is an injection for all but at most $((k+1)g_{k+1} + (d+1-k)g_k)(\Delta)$ vertices of $\Delta$.

Let us recall three simple facts:

1. **McMullen’s integral formula for the g-vector ([Swa06, Prop.4.10]).** For any pure $d$-complex $\Delta$, we have
$$\sum_{v \in \Delta^{(0)}} g_k(\text{lk}_v \Delta) = ((k+1)g_{k+1} + (d+1-k)g_k)(\Delta)$$
(2) Cone Lemma ([TWW95, Cor.1.5] & [Lee96, Thm.7]). For any vertex \( v \in \Delta \), we have natural isomorphisms
\[
S^\text{lin}_{k+1}(st_v \Delta, \partial st_v \Delta) \cong S^\text{lin}_k(lk_v \Delta) \cong S^\text{lin}_k(st_v \Delta).
\]

Proof of Lemma 4.3. It suffices to show that \( g_k(lk_v \Delta) \leq 0 \) for all but at most \((k+1)g_{k+1} + (d+1-k)g_k\) vertices. But \( g_k(lk_v \Delta) \geq 0 \) for all \( v \) by the weak Lefschetz property, so the claim follows by McMullen’s integral formula. \( \square \)

Proof of Theorem 4.2. By Lemma 4.3, the map (3) is an isomorphism except for a small set \( \mathcal{E} \) of vertices. Examining the effect for any vertex \( v \in \Delta^{(0)} \setminus \mathcal{E} \) relative the class \((\gamma|_{st_v \Delta}, \delta|_{st_v \Delta}) \in S^\text{lin}_{k+1}(st_v \Delta, \partial st_v \Delta)\) gives the desired. \( \square \)

5. Toric chordality and propagation

The notion of stress spaces allows us to state a refined, more powerful version of higher chordality, that allows us to see the results of the previous section in a new light. The arguably most important corollary of the notion of toric chordality is the propagation principle it satisfies.

Consider a geometric simplicial complex \( \Delta \), and a linear differential \( \delta \). We say that \((\Delta, \delta)\) is toric \( k \)-chordal if \( \delta \) induces an surjection
\[
S^\text{lin}_{k+1}(\Delta) \xrightarrow{\delta} S^\text{lin}_k(\Delta)
\]
that extends to a surjection \( S^\text{lin}_{k+1}(st_v \Delta, \partial st_v \Delta) \xrightarrow{\delta} S^\text{lin}_k(st_v \Delta, \partial st_v \Delta), v \in \Delta^{(0)} \), and an injection
\[
S^\text{lin}_k(st_v \Delta) \xleftarrow{\delta} S^\text{lin}_{k-1}(st_v \Delta)
\]
for all vertices \( v \) of \( \Delta \). We weaker call the pair \((\Delta, \delta)\) is weakly toric \( k \)-chordal if the injection (5) applies. We observe as above:

Lemma 5.1. If \((\Delta, \delta)\) is toric \( k \)-chordal, then for every \( \gamma \) in \( S^\text{lin}_{k+1}(\Delta) \), we have
\[
(\delta \gamma)^{(0)} = (\gamma)^{(0)}.
\]

Proof. Use Lemma 4.3 and conclude as in Theorem 4.2. \( \square \)

We conclude a well-known fact from commutative algebra, cf. [MMRN11].

Corollary 5.2. Weak toric chordality propagates, i.e. a weakly toric \( k \)-chordal complex is also weakly toric \( \ell \)-chordal for every \( \ell \geq k \).

Let us justify the notion of toric chordality by concluding from Theorem 4.2:

Corollary 5.3. Consider any hereditary Lefschetz \((d-1)\)-sphere \( \Delta \). If \( g_k(\Delta) = g_{k+1}(\Delta) = 0 \), then \( \Delta \) is toric \( k \)-chordal.

The main motivation for toric chordality, however, is that it satisfies the propagation principle; clearly in the form of Theorem 2.1, but even after it is satisfied only in a single dimension.

Theorem 5.4. Assume that a geometric, properly embedded simplicial complex \((\Delta, \delta)\)

\( \circ \) has no missing faces of dimension \( \geq k \), and

\( \circ \) \( \Delta \) is toric \( k \)-chordal.

Then \( \Delta \) is toric \( i \)-chordal for all \( i \geq k \).
Proof of Theorem 5.4. Let $\delta_v := \frac{d_v}{d_{v+1}}$, and consider the stress $\delta_v \gamma_v$, were $\gamma_v$ is a relative $(i+1)$-stress in $(s_t \Delta, \partial s_t \Delta)$; by induction, said stress is in the image of $\delta : S_{i+1}^{\text{lin}}(\Delta) \to S_i^{\text{lin}}(\Delta)$.

We proceed to show that there exists a relative stress $\tilde{\gamma}_v$ in $(s_t \Delta, \partial s_t \Delta)$ that maps to $(\delta^{-1} \circ \delta_v)(\gamma_v)$ under $\delta_v$. To construct $\tilde{\gamma}_v$, it suffices to show that $(\delta^{-1} \circ \delta_v)(\gamma)$ is supported in $s_t \Delta$ for this, it suffices to show that for every $w \neq v \in \Delta^{(0)}$, $(\delta_w \circ \delta^{-1} \circ \delta_v)(\gamma)$ is supported in the star of $v$ (by assumption on the missing faces and the cone lemma).

Lemma 5.5. In the situation of Theorem 5.4, and assume that for some $k$-stress $\gamma$ of $\Delta$, $\delta(\gamma) \in \text{Im}(\delta_v)$. Then $\gamma$ is in the image of $\delta_v$.

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
S_{k}^{\text{lin}}(s_t \Delta) & \xrightarrow{\delta} & S_{k-1}^{\text{lin}}(s_t \Delta) \\
S_{k+1}^{\text{lin}}(s_t \Delta, \partial s_t \Delta) & \xrightarrow{\delta} & S_{k}^{\text{lin}}(s_t \Delta, \partial s_t \Delta)
\end{array}
\]

Since the horizontal maps $\delta$ are injections (for the top map) and surjections (for the bottom map), respectively, we can consider the element $(\delta^{-1} \circ \delta_v)(\gamma) \in S_{k+1}^{\text{lin}}(\Delta)$. The image of this element under $\delta_v$ is the desired stress.

Now,

\[
(\delta \circ \delta_w \circ \delta^{-1} \circ \delta_v)(\gamma) = \delta_w(\delta_v(\gamma))
\]

is supported in the star of $v$, so that the claim follows by the previous lemma. This proves the surjectivity

\[
S_{i+2}^{\text{lin}}(s_t \Delta, \partial s_t \Delta) \xrightarrow{\delta} S_{i+1}^{\text{lin}}(s_t \Delta, \partial s_t \Delta).
\]

Finally, let $\gamma$ denote a $(i+1)$-stress in $\Delta$, and let $\gamma_v$ denote is restriction to the relative stress in $(s_t \Delta, \partial s_t \Delta)$. By the previous argument, we can construct relative stresses $\tilde{\gamma}_v$ that map to $\gamma_v$ under $\delta$. Now, by the cone lemma, $\delta$ is injective of $\delta$ on $S_{i+1}^{\text{lin}}(s_t \Delta, \partial s_t \Delta)$ for every edge $e$ of $\Delta$, so that we can sum the Minkowski weights corresponding to $\tilde{\gamma}_v$ over the vertices of $\Delta$ to obtain the preimage of $\gamma$ itself, proving the surjectivity

\[
S_{i+2}^{\text{lin}}(\Delta) \xrightarrow{\delta} S_{i+1}^{\text{lin}}(\Delta).
\]

Corollary 5.6. Let $\Delta$ denote a proper geometric simplicial complex. Assume that

1. $\Delta$ is toric $k$-chordal, and
2. $\Delta$ has no missing faces of dimension $\geq k$, and
3. that $(\Theta = (v_1, \ldots, v_n)^t, \delta)$ is regular up to degree $k$.

Then $\Theta$ is a regular system of parameters, and $\Delta$ is Cohen–Macaulay.

Proof. Let $(\theta_1, \ldots, \theta_d)$ denote the differentials given by $\Theta = (v_1, \ldots, v_n)^t$. We argue by induction: By Theorem 5.4,

\[
\delta : S^{\text{lin}}(\Delta, (\theta_1, \ldots, \theta_i)) \longrightarrow S^{\text{lin}}(\Delta, (\theta_1, \ldots, \theta_i))
\]

is surjective for all $i$. Now, let us decompose $S^{\text{lin}}(\Delta, (\theta_1, \ldots, \theta_{d-1})) = \bigoplus_{i \geq 0} S_i$, where $S_i = \ker \theta_{d+1}^{i+1} \cap (\ker \theta_d^i)$. Then by toric propagation, we have isomorphisms $S_i \xrightarrow{\theta_d^i} S_{i-1}$, so that in particular $\theta_d$ is a surjection on $S^{\text{lin}}(\Delta, (\theta_1, \ldots, \theta_{d-1}))$. Repeating the same argument proves that the differentials $\theta_i$ are surjections on $S^{\text{lin}}(\Delta, (\theta_1, \ldots, \theta_{i-1}))$. \qed
Then we finish this section with the following link lemma for toric chordality. This can be shown easily by an induction on \( \delta \) would interest us. This is remedied by the following results:

**Corollary 5.7.** Let \( \Theta : \Delta^{(0)} \to k^d \), \( \Delta \) of dimension \( d - 1 \), and assume \( \Delta \) has no missing faces in dimension \( \geq k \). If \( (\Delta, \delta) \) is toric \( k \)-chordal then \( S^{\text{lin}}_1(\Delta; (\Theta, \delta)) = 0 \) for all \( i \geq k \).

If further \( \Delta \) is Cohen-Macaulay of dimension \( d - 1 \) then \( h_i(\Delta) = 0 \) for all \( i \geq k \).

We finish this section with the following link lemma for toric chordality.

6. Relation to resolution chordality

So far, toric chordality is only an interesting concept, without any real connection to results that would interest us. This is remedied by the following results:

**Theorem 6.1 (Toric \( k \)-chordality and resolution chordality, I).** Let \( \Delta \) be any geometric simplicial complex. Assume that

(a) \( \Delta \) is \( d \)-regular, and

(b) \( \dim \ker[\delta : S_{k+1}^{\text{lin}}(\Delta) \to S_k^{\text{lin}}(\Delta)] = \alpha \) for some linear \((d - 1)\)-proper embedding of \( \Delta \) into \( \mathbb{R}^d \) and

(c) that \( \Delta \) is weakly toric \( k \)-chordal.

Then \( \beta_{k-1}(\Delta_W, \mathbb{R}) \leq \alpha \) for every \( W \subset \Delta^{(0)} \). In particular, if \((\Delta, \delta)\) is toric \( k \)-chordal then \( \Delta \) is resolution \((k - 1)\)-chordal.

**Corollary 6.2.** Let \( P \) be any simplicial \( d \)-polytope, and let \( k \geq \frac{d}{2} \). Then \( \beta_{k-1}(\Delta_W, \mathbb{R}) \leq -g_{k+1}(P) \) for \( W \subset P^{(0)} \).

**Proof.** Let \( p \) denote a generic orthogonal projection to a \( k \)-dimensional subspace \( H \) of \( \mathbb{R}^d \), and let \( B \) denote the projection of \( \Delta \) to \( H \), so that \( B \) is in particular \((k - 1)\)-proper. We may assume that \( H \) is the coordinate subspace corresponding to the first \( k \) coordinates of \( \mathbb{R}^d \). Let \( (\theta_1, \cdots, \theta_d) \) denote the linear differentials corresponding to the embedding of \( \Delta \).

Following the homological interpretation of Tay–Whiteley [TW00], we have a natural isomorphism

\[
\left[ \text{coker} : S_{k+1}^{\text{lin}}(B|_W) \xrightarrow{\delta} S_k^{\text{lin}}(B|_W) \right] \cong \bar{H}_{k-1}(B|_W; \mathbb{R}),
\]

Now, by regularity we obtain an surjection

\[
\left[ \text{coker} : S_{k+1}^{\text{lin}}(\Delta) \xrightarrow{\delta} S_k^{\text{lin}}(\Delta) \right] \twoheadrightarrow \left[ \text{coker} : S_{k+1}^{\text{lin}}(B|_W) \xrightarrow{\delta} S_k^{\text{lin}}(B|_W) \right].
\]

This can be shown easily by an induction on \( d - k \): Let us restrict, for simplicity of notation, the case \( d - 1 = k \). Let \( \bar{S}_d^{\text{lin}}(B, B|_W) := \left[ \text{Im} : S_d^{\text{lin}}(B) \to S_d^{\text{lin}}(B, B|_W) \right] \). By regularity of \( \theta_d \), we have a surjection

\[
\bar{S}_d^{\text{lin}}(B, B|_W) \xrightarrow{\theta_d} \left[ \text{coker} : S_d^{\text{lin}}(B|_W) \xrightarrow{\delta} S_{d-1}^{\text{lin}}(B|_W) \right],
\]

and therefore a surjection

\[
\bar{S}_d^{\text{lin}}(\Delta, \Delta|_W) := \left[ \text{Im} : S_d^{\text{lin}}(B) \to S_d^{\text{lin}}(\Delta, \Delta|_W) \right] \xrightarrow{\theta_d} \left[ \text{coker} : S_d^{\text{lin}}(B|_W) \xrightarrow{\delta} S_{d-1}^{\text{lin}}(B|_W) \right]
\]

by the isomorphism of Equation (1). Consider now \( \bar{W} = \Delta^{(0)} \setminus W \), and \( \delta_{\bar{W}} := \sum_{v \in \bar{W}} \frac{d}{d_{x_v}} \). Then

\[
\delta_{\bar{W}} \bar{S}_d^{\text{lin}}(\Delta, \Delta|_W) \subset S_{d-1}^{\text{lin}}(\Delta)
\]

and for every element \( \gamma \) in \( \delta_{\bar{W}} \bar{S}_d^{\text{lin}}(\Delta, \Delta|_W) \) that is in the image of \( \delta \), the stress

\[
\delta^{-1}(\gamma) \mod \left[ \text{Im} : S_{d+1}^{\text{lin}}(B|_W) \xrightarrow{\delta} S_d^{\text{lin}}(B|_W) \right]
\]
We also obtain a new proof of the Generalized Lower Bound Theorem, for simplicial polytopes and simplicial \(\Delta\).

Theorem 7.1.

Theorem 6.3

This is no accident, as we will see now. The following result can be seen (as a very simple special case) within the philosophy of Moishezon and Kodaira who provided methods to construct projective structures on certain Kähler manifolds [Moishezon66], [Kodaira54].

\[ \text{Proof.} \]

Part C. Applications to polytope theory

7. The Generalized Lower Bound Theorem

We also obtain a new proof of the Generalized Lower Bound Theorem, for simplicial polytopes and for Lefschetz spheres, which includes a new characterization in terms of toric chordality (similar to the one of Kalai for the special case \(k = 2\), cf. [Kalai87]).

Theorem 7.1. Let \(P\) denote any simplicial \(d\)-polytope. Then the following are equivalent:

(a) \(g_k(P) = 0\) for some \(2k \leq d\).
(b) \(\partial P\) is toric \(k\)-chordal and has no missing faces of dimension \([k, d-k]\).
(c) \(P\) admits a \(k\)-stacked triangulation.

\[ \text{Proof.} \ (c) \implies (a) \text{ is easy [MW71],} \ (a) \implies (b) \text{ follows from Theorem 4.2 (where } \mathcal{E} = \emptyset). \text{ Finally, for} \ (b) \implies (c), \text{ we use the fact that } Cl_k(P) \text{ is Cohen–Macaulay by Corollary 5.6. The rest of the proof is as in [MN13]: by the work of McMullen [McM04], } Cl_k(P) \text{ is a geometric subcomplex of } \mathbb{R}^d. \text{ That } Cl_k(P) \text{ is acyclic by the Leray property, and thus triangulates } P. \]

The proof of the Generalized Lower Bound Theorem presented here points us to a generalization of Theorem 7.1 when we keep the relation between the Maxwell–Cremona principle and Minkowski weights in mind:

Consider, for a moment, the (non-simplicial) polytope \(\tilde{P} = \text{conv}(P \times \{1\} \cup \{0, \cdots, 0, 1\})\). If the facet corresponding to \(P\) is refined to \(Q\), we obtain a simplicial complex \(\tilde{Q}\). Even though it is not expected to hold in general, it can be shown using toric propagation that the Hodge–Riemann–Minkowski bilinear relations for \(Q\) (with respect to the standard differential) hold.

This is no accident, as we will see now. The following result can be seen (as a very simple special case) within the philosophy of Moishezon and Kodaira who provided methods to construct projective structures on certain Kähler manifolds [Moishezon66], [Kodaira54].

Theorem 7.2. A \(k\)-stacked triangulation of a simplicial \(d\)-polytope, \(k \leq \frac{d}{2}\), is regular. In particular, it is shellable.
Here, **regular** means that $Q$ lifts to a convex, piecewise linear functions whose domains of linearity project to $Q$. To prove this property, we use a connection between ample classes in the toric variety, positivity of stresses and regularity of triangulations which goes back to Maxwell and Cremona.

**Proof of Theorem 7.2.** For simplicity, we again consider $Q$, and call, for $\delta$ the standard differential $\sum \frac{d}{d\tau_i}$, the elements of

$$S^\text{aff}_k(\Delta) := \ker \delta : S^\text{lin}_k(\Delta) \to S^\text{lin}_{k+1}(\Delta)$$

the **affine stresses**. To make clear that we focus on squarefree coefficients, we adopt the notation of (Minkowski) weights.

By Theorem 5.4, the McMullen–Weil Lefschetz map $\delta$ pulls back the linear stresses $S^\text{lin}_{k+1}(Q)$ in the $k$-skeleton of $Q$ (and therefore stresses of $P$) to $d$-stresses $S^\text{lin}_d(Q)$ of $Q$. Now, recall the preimage of a positive $i$-weight $S^\text{lin}_i(Q)$, $i \geq k + 1$, is positive because it is positive in each link; hence, we obtain a positive $d$-weight on $Q$ from a positive $(k + 1)$-weight on $Q$; the latter exist because the $k$-skeleton of $Q$ lies in $P$, which is convex. Finally, a positive $d$-weight in $S^\text{aff}_d(Q, \partial Q)$ gives a convex lifting following [McM73, Ryb99]. □

**Remark 7.3.** The proof above relies on the inductive procedure to pull back weights given by the proof of Theorem 5.4 fact that two consecutive posi lie entirely in the boundary, and that $\delta$ acts as an isomorphism on the Minkowski weights to pull these stresses back from a stress in the boundary.

An more instructive phrasing of the above argument (“obtain a positive weight on $(Q, \partial Q)$ as a pullback from a weight on $\partial Q$ using the Lefschetz map”) can be provided as follows: Recall that the notion of tangent spaces gives a canonical geometric definition of geometric links, cf. [AZ15]. For stresses in spherical links, additionally refer to [McM73]. We adopt this notation for the remainder of the remark. We also repeatedly use the Maxwell–Cremona principle relating positive weights to liftings.

Let $Q$ be our triangulation, and let $\sigma$ denote a $(d-k)$-face of $Q$.

**Lemma 7.4.** $(\text{st}_\sigma Q, \partial \text{st}_\sigma Q)$ admits a canonical positive $d$-weight.

**Proof.** This follows at once as $\text{st}_\sigma Q - \sigma$ is in the boundary of $P$, and therefore in convex position. The positive weight on $(\text{st}_\sigma Q, \partial \text{st}_\sigma Q)$ is therefore the pullback of the corresponding weight on $(\text{lk}_\sigma Q, \partial \text{lk}_\sigma Q)$. □

Now, notice that the canonical positive weight glue together along stars of $(d - k + 1)$-faces. It remains to show that no new cycles are introduced that can prevent the positive stress to glue together globally to a stress on $(Q, \partial Q)$. But such cycles would only appear around faces of dimension $\leq d - k - 2$, which all lie in $\partial Q$. The claim follows. □

**Remark 7.5** (A “flexible” argument). The argument above uses the Lefschetz map to prescribe a canonical $d$-stress, and therefore a “canonical” lifting of $Q$ from a $k$-weight of $\partial Q$.

On the other hand, the case $k = 2$ allows for a more flexible explanation, as we can construct the positive weight, one $(d - 1)$-face at a time, cf. [McM04].

For $k > 3$, such a flexible argument, to our best knowledge, does not work. For $k = 3$, however, it is still possible, at least if we assume that $Q$ is $(d - 2)$-decomposable in the sense of Billera and Provan [BP79]. The key to this fact is the observation that every on every 2-fan, a conewise linear convex function given on a connected subfan can be extended to a convex conewise linear function on the fan. Hence, we can construct the desired positive weight iteratively along stars of $(d - 2)$-faces. As all faces of dimension $d - 3$ and lower lie in the boundary, all other obstructions to extend this stress globally vanish. This gives the desired positive $d$-stress on $(Q, \partial Q)$, and therefore a lifting.
Acknowledgements. We express our gratitude to Satoshi Murai, Eran Nevo, Ed Swartz, Robert MacPherson and June Huh for useful comments and inspiring conversations.

References

[AZ15] K. A. Adiprasito and G. M. Ziegler, Many projectively unique polytopes, Invent. Math. 199 (2015), no. 3, 581–652.

[ANS15a] K.A. Adiprasito, E. Nevo, and J. A. Samper, Higher chordality I: from graphs to complexes, 2015, preprint, arXiv:1503.05620.

[ANS15b] , Higher chordality III: The geometric lower bound theorem, 2015, preprint.

[BP79] L. J. Billera and J. S. Provan, A decomposition property for simplicial complexes and its relation to diameters and shellings, Second International Conference on Combinatorial Mathematics (New York, 1978), Ann. New York Acad. Sci., vol. 319, New York Acad. Sci., New York, 1979, pp. 82–85.

[Dir61] G. A. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg 25 (1961), 71–76.

[Fra13] G. Francois, Cocycles on tropical varieties via piecewise polynomials, Proc. Amer. Math. Soc. 141 (2013), no. 2, 481–497.

[Gro86] M. Gromov, Partial differential relations, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 9, Springer-Verlag, Berlin, 1986.

[Ka87] G. Kalai, Rigidity and the lower bound theorem. I, Invent. Math. 88 (1987), no. 1, 125–151.

[KP08] E. Katz and S. Payne, Piecewise polynomials, Minkowski weights, and localization on toric varieties, Algebra Number Theory 2 (2008), no. 2, 135–155.

[Kod54] K. Kodaira, On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties), Ann. of Math. (2) 60 (1954), 28–48.

[Lee96] C. W. Lee, P.L.-spheres, convex polytopes, and stress, Discrete Comput. Geom. 15 (1996), no. 4, 389–421.

[LB63] C. G. Lekkerkerker and J. Ch. Boland, Representation of a finite graph by a set of intervals on the real line, Fund. Math. 51 (1962/1963), 45–64.

[McM73] P. McMullen, Representations of polytopes and polyhedral sets, Geometriae Dedicata 2 (1973), 83–99.

[McM93] , On simple polytopes., Invent. Math. 113 (1993), no. 2, 419–444.

[McM96] , Weights on polytopes, Discrete Comput. Geom. 15 (1996), no. 4, 363–388.

[McM04] , Triangulations of simplicial polytopes, Beiträge Algebra Geom. 45 (2004), no. 1, 37–46.

[MW71] P. McMullen and D. W. Walkup, A generalized lower-bound conjecture for simplicial polytopes., Mathematika, Lond. 18 (1971), 264–273.

[MMRN11] J. C. Migliore, R. M. Miró-Roig, and U. Nagel, Monomial ideals, almost complete intersections and the weak Lefschetz property, Trans. Amer. Math. Soc. 363 (2011), no. 1, 229–257.

[Mol66] B. G. Moishezon, On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions. I, Izv. Akad. Nauk SSSR, Ser. Mat. 30 (1966), 133–174 (Russian).

[MN13] S. Murai and E. Nevo, On the generalized lower bound conjecture for polytopes and spheres., Acta Math. 210 (2013), no. 1, 185–202.

[Ryb99] K. Rybnikov, Stresses and liftings of cell-complexes, Discrete Comput. Geom. 21 (1999), no. 4, 481–517.

[Sta96] R. P. Stanley, Combinatorics and commutative algebra, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston Inc., Boston, MA, 1996.

[Swa06] E. Swartz, g-elements, finite buildings and higher Cohen-Macaulay connectivity, J. Combin. Theory Ser. A 113 (2006), no. 7, 1305–1320.

[TWW95] T.-S. Tay, N. White, and W. Whiteley, Skeletal rigidity of simplicial complexes. I, European J. Combin. 16 (1995), no. 4, 381–403, 503–523.

[TW00] T.-S. Tay and W. Whiteley, A homological interpretation of skeletal rigidity, Adv. in Appl. Math. 25 (2000), no. 1, 102–151.

[Wei58] A. Weil, Introduction à l’étude des variétés kählériennes, Publications de l’Institut de Mathématique de l’Université de Nancago, VI. Actualités Sci. Ind. no. 1267, Hermann, Paris, 1958.