Null energy in plane Kaluza-Klein worlds

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November 1, 2021

Abstract
We study the energy problem in the $R\!0\!X$ spacetimes in 5D. We found that for the Einstein, Landau-Lifschitz and the Moller complexes are null.
1 Introduction

Energy is a standard definition that comes from the birth of the analytical mechanics that makes easier the study of the dynamics of a physical system. We can always (with few exceptions) define certain conserved quantity—conjugated to coordinate time—that allow us to perform predictions and establish relations with other dynamical measurable observables. Although, in the very foundations of the General Relativity the definition of energy is rather than difficult sometimes uncertain. As it is well-known, the general relativist abandons the concept of force by using the principle of free-fall observers. In a mathematical language, we can write a 4D version of the Newton law of motion in curved spacetime where the “acceleration” is null

\[
\frac{D^2a^\mu}{D\tau^2} = 0. 
\]  

Therefore, we change our dynamical vision of the interaction as force by curvature:

\[
\frac{D^2n^\mu}{D\tau^2} + R^\mu_{\nu\rho\tau} n^\nu \frac{d\xi^\rho}{d\tau} \frac{d\xi^\tau}{d\tau} = 0, 
\]  

where \(n^\mu\) is relative position of two near particle trajectories (geodesics), \(\xi^\nu\) the coordinate in the manifold and \(R^\mu_{\nu\rho\tau}\) is the Riemann curvature tensor. Since no longer the concept of force is compatible with the General Relativity, we cannot perform adequately a definition of energy. For instance, this problem made difficult the journey for those who attempt to construct a consistent and complete quantum theory of gravitation \[1\], mainly because every quantum theory needs a well-defined Hamiltonian operator and the Hamiltonian constraint for GR is null for every dynamical solution.

This problem occupied, first, Einstein himself \[2\], who proposed a complex of gravitational Energy-Momentum which main technical disadvantage is that it is not tensor by definition. The Einstein complex do not satisfy a fully covariant local conservation property (a null divergence), so as calculation tool is efficient only in Cartesian coordinates, as observed by Xulu \[3\], who has exhaustive study this topic. In order to solve some Einstein complex, Landau and Lifshitz \[4\] propose a complex with tensor properties only on geodesic coordinate systems (where all \(\partial g_{\mu\nu}/\partial \xi^\rho = 0\)). Finally, and not less, Moller \[5\] derived a complex that generates a energy-momentum 4-vector that transform explicitly as that. Others complexes of gravitational energy has been created in the literature in order to give a more complete analysis of the interaction, like Papapetrou-Gupta \[6\]—who uses the spin operator of the Einstein-Hilbert Lagrangian but mixes the flat Minkowski geometry—and Weinberg \[7\]—specially for weak field approximation—.

In \[8\], we call \(R0X\) spacetimes in \(N\) dimensions to those vacuum solutions to Einstein equations that 1.- posses plane symmetry, 2.- own a particular structure that depends only on an arbitrary function we named generating function and a set of parameters that define a manifold with \(N - 3\) dimension, and 3.-
have exact solutions to the geodesic equations. The “X” can be “T” when the generating function is well-tempered, “B” when it is bounded, “U” when it is unbounded and “L” when it has a logarithmic structure. In this paper, we show some interesting results about the behavior of energy-momentum in R0X spacetimes (with \(N = 5\)) by making use of the Einstein, Landau and Moller complexes. The R0X metric we use has the general form

\[
ds^2 = e^{A_0} dt^2 - (e^{A_1} dx^2 + e^{A_2} dy^2 + e^{A_3} dz^2) - \left( \frac{dx}{d\eta} \right)^2 e^{\chi} d\eta^2
\]  

with the usual convention \(\xi^0 = t, \xi^1 = x, \xi^2 = y, \xi^3 = z, \xi^4 = \eta\).

### 2 Results and conclusions

For both Einstein and Moller energy complexes, we need to define the non-tensorial quantity:

\[
h_B^{\ CD} = \frac{g_{BM}}{\sqrt{g}} \frac{\partial}{\partial \xi^N} \left[ g \left( g_{CM} g_{DN} - g_{DM} g_{CN} \right) \right]
\]  

The Einstein energy momentum complex is then given by (in natural units):

\[
\Theta_B^C = \frac{1}{16\pi} \frac{\partial}{\partial \xi^D} h_B^{\ CD}
\]  

and the Moller complex is

\[
S_B^C = \frac{1}{10\pi} \frac{\partial}{\partial \xi^D} \left( \delta_B^C h_M^{\ MD} - \delta_B^D h_M^{\ MC} - 2 h_B^{\ CD} \right)
\]  

For the metric in (3), the non-null Einstein pseudo tensor components are (for \(C = 0, 1, 2, 3\))

\[
\Theta_B^C = \frac{1}{32\pi} d\chi \left( 1 - \sum_{B=0}^4 A_B \right) \left( \sum_{B=0, B \neq C}^4 A_B \right)
\]  

and for the Moller one:

\[
S_B^C = \frac{1}{16\pi} d\chi \left( 1 - \sum_{B=0}^4 A_B \right) (1 + 2A_C)
\]  

From [8] and [9], the \(A_B\) parameters satisfy the relation \(\sum_{B=0}^4 A_B = 1\), a direct consequence of the Einstein field equations. Therefore, for every R0X the Einstein and Moller complexes are null.

Now, for the Landau-Lifshitz energy momentum complex given in [4] and [3], we have after some simplifications
\[ L_C^L = \frac{1}{32\pi} e^\chi \left( \frac{d\chi}{d\eta} \right)^2 (A_C - 1) \left[ 2\frac{d^2\chi}{d\eta^2} \left( \frac{d\chi}{d\eta} \right)^{-1} - A_C - \frac{1}{2} \right] \] (9)

Landau-Lifshitz energy-momentum definition is valid only for geodesic coordinate system \((\partial g_{BC}/\partial \xi^D = 0)\). Therefore, a geodesic coordinate system is that satisfies the following two conditions from the derivatives of the metric tensor at certain \(\eta_0\)

\[ \left. \frac{d\chi}{d\eta} \right|_{\eta_0} = 0 \] (10)

\[ 2 \left( \left. \frac{d^2\chi}{d\eta^2} \right|_{\eta_0} \right) + \left( \left. \frac{d\chi}{d\eta} \right|_{\eta_0} \right)^2 = 0 \] (11)

Notice that these conditions ensure \(d\chi/d\eta|_{\eta_0} = d^2\chi/d\eta^2|_{\eta_0} = 0\). It is easy to picture spacetimes that could contain one or more geodesic coordinate system from the freedom to choose the generating function. For instance, a periodic generating function \(\chi\) that produces a R0B spacetime would have a infinite number of geodesic coordinate systems. Thus, \(L_C^L = 0\), in any geodesic coordinate system.

If we extend our analysis to R0L spaces where the generating function has a the generic form \(\chi = \ln\theta\), where \(\theta\) is a function of \(\eta\), some new subtleties come out from the functional structure of the metric. The conditions for geodesic systems becomes then

\[ \left[ \theta (\eta_0) \right]^{A_0 - 1} \left. \frac{d\theta}{d\eta} \right|_{\eta_0} = 0 \] (12)

\[ \left. \frac{d\theta}{d\eta} \right|_{\eta_0} \left[ 2\theta (\eta_0) \left( \left. \frac{d^2\theta}{d\eta^2} \right|_{\eta_0} \right) - \left( \left. \frac{d\theta}{d\eta} \right|_{\eta_0} \right)^2 \right] = 0 \] (13)

To avoid problems with infinities at the Christoffel connections, and thus with the definition of energy, let be \(\theta (\eta_0) \neq 0\) for any suitable geodesic system. Therefore, \(d\theta/d\eta|_{\eta_0} = 0\) is a sufficient condition that a geodesic coordinate system satisfy and that \(d^2\theta/d\eta^2|_{\eta_0}\) is not necessary to be 0. For R0L spaces the conditions for constructing geodesic coordinate systems differ from the others by a sign. Then, the Landau-Lifshitz complex for R0L spacetimes is

\[ L_C^L = \frac{1}{16\pi\theta} (A_C - 1) \left[ \theta \frac{d^2\theta}{d\eta^2} - A_C \left( \frac{d\theta}{d\eta} \right)^2 \right] \] (14)

\[ = \frac{1}{16\pi} (A_C - 1) \frac{d^2\theta}{d\eta^2} \] (15)
So, in R0L spacetimes the Landau-Lifshitz complex is not necessary null unless we find a \( \theta \)-function that certain \( \eta_0 \) is both an extreme and an inflexion point. In other hand, the gravitational energy density in the \( (L_i^j) \) is null always if the spacetime structure constant \( C_0 = 1 \), a possible value for the specific parametrization of \( N = 5 \) R0X as we can see in [9].

Let summarize our results. In principle, the gravitational complexes are artificial structures that emulate a energy-momentum tensor \( T_{BC} \) for the gravitational interaction. Thus, in Kaluza-Klein theories we must find the most appropriate interpretation to each its components over the extra-dimensions. By following [10] and [11], a time-like signature for the extra-dimension \( \eta \) would imply several undesirable features, like the wrong sign in the 4D Maxwell action relative to the Einstein one, the prediction of tachyons and closed timelike curves that allow causality violations. Hence, the choice of space-like signature for \( \eta \) ensures no inconsistencies with our standard ideas about spacetime.

Thus, by analogy with the standard definition of the energy-momentum tensor in the standard four-dimensional spacetime, \( T_{\eta i} \) \( (i = 1, 2, 3) \) are the projection on the \( \eta \)-direction of forces that act on matter across a unit surface with normal vector \( \mathbf{e}_\eta \). In other words, \( T_{\eta i} \) is the \( \eta \)-components of the momentum flux in the direction \( i \). Thus, our result \( (T^\eta_i = 0) \) is consistent with the intuitive idea that there is no matter momentum flux over the extra-dimension. In the order side, \( T_{\eta} \) is energy flux on the \( \eta \)-direction and as we should expect, there is no energy interchange into the “hidden” dimension.

Since all energy-momentum components in all the three complexes we have studied are null, we can conjecture that when we unify the theory of gravitation and the electromagnetism the “effective” energy of hole system is null for any R0X spacetime in 5D. Hence, for a self-gravitating electromagnetic system described by the Kaluza-Klein 5D vacuum gravity, there could exist a balance between the two interactions that prevents the formation of gravitational effective masses, in a similar matter that do Bianchi I spacetimes [12]. Therefore, quantities like linear and angular momentum are always conserved with trivial values. We must say that in the context of non-compactified Kaluza-Klein theories (where the extra-dimension is not compact, but intuitively must small) other R0X solutions can be admitted, for instance, all those which generating function \( \chi = \chi (\xi^\nu) (\nu = 0, 1, 2, 3) \). These R0X solutions that depends on conventional four-dimensional parameters still conserve the property of null gravitational energy, as one can easily verify by the extreme symmetry of the plane coordinates we choose for the metric.

Then, R0X worlds are energetically equivalent to vacuum universes.

Acknowledgment. (DS) would like to thank Jorge A. Diaz and Ms. Eda Maria Arce for their kindness and hospitality at LANOTECH.

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