Ampleness of the CM line bundle on the moduli space of canonically polarized varieties

Zsolt Patakfalvi and Chenyang Xu

March 31, 2015

Abstract
We prove that the CM line bundle is ample on the proper moduli space which parametrizes KSBA stable varieties.

Contents
1 Introduction 1
2 Preliminary 3
  2.1 KSBA stable family 3
  2.2 CM line bundle 4
  2.3 Dlt blow up 5
  2.4 Bigness and nefness of relative canonical bundle 6
3 Proof of Main Theorems 8
  3.1 Log canonical case 8
  3.2 Semi-log canonical case 8

1 Introduction
Through the note, we work over a ground field $k$ which is of characteristic zero. The moduli space $M^{\text{can}}$ of canonically polarized manifolds as well as its natural geometric compactification have attracted considerable interest over the past few decades. While it has been shown geometric invariant theory (GIT) can be used to construct a moduli space of $M^{\text{can}}$ (cf. [Vie95, Don01]), to find a natural compactification by directly applying GIT methods, which was successful in dimension 1, fails in higher dimension (see [WX14]). However, using the recent progress in minimal model theory, the framework which was first proposed in [KSB88] turns out to give a satisfying compactification $M^{\text{ksba}}$ (see also [Aie90]). Although the coarse moduli space $M^{\text{ksba}}$ first only exists as an algebraic space, [Kol90] has developed a strategy to verify its projectivity,
which was later completed in [Fuj12] for the case of varieties and in [KP15] for pairs.

We can apply Knudsen-Mumford determinant construction to the sequence of ample line bundles on $M^\text{ksba}$ constructed by Kollár. The coefficient of the leading term is the CM line bundle (see Section 2.2), which was first introduced in [Tia97] and later formulated in this way in [PT10]. The curvature calculation on the Weil-Petersson metric of CM line bundle suggests that it is ample. However, due to the presence of possibly singular fibers, this is only completely worked out for $M^\text{can}$ (see [Sch12]).

In this note we give a purely algebraic proof of the fact that CM line bundle is ample on $M^\text{ksba}$. Our approach is inspired by the recent interplay of studying $M^\text{ksba}$ from both algebraic and differential geometry viewpoint, especially the equivalence of KSBA stability and K-stability for canonically polarized varieties (see [Oda12]).

**Theorem 1.1.** The CM line bundle is ample on the KSBA moduli space.

Using Nakai-Moishezon criterion and the formula of CM line bundle for a family of KSBA stable varieties, we immediately see that this is implied by the following theorem.

**Theorem 1.2.** Let $T$ be a normal variety and $(Z, \Delta) \to T$ a family of $n$-dimensional KSBA stable pairs with finite fiber isomorphism equivalence classes, then $f_*( (K_{Z/T} + \Delta)^{n+1})$ is ample on $T$.

At the end, we want to remark that the positivity of CM line bundle is expected for spaces parametrizing Kähler-Einstein varieties or even polarized varieties with constant scalar curvature(see e.g. [PT10]). But in general, not much is known. In the case of moduli space of Kähler-Einstein Fano varieties, we are still lack of understanding of the positivity of CM line bundle with algebro-geometric tools. Using the fact that the curvature of CM line is the Weil-Petersson metric for a smooth family and some deep results in analysis, one can verify that it induces an embedding when restricting to the locus which parametrizes Kähler-Einstein Fano manifolds (see [LWX15]). It is an interesting and challenging question to prove similar results using only algebraic geometry.

**Acknowledgement:** We thank Chi Li and Xiaowei Wang for comments, discussions and references.

Partial financial support to CX was provided by The National Science Fund for Distinguished Young Scholars. A large part of this work was done while CX enjoyed the inspiring environment at the Institute for Advanced Studies, supported by Ky Fan and Yu-Fen Fan Membership Funds, S.S. Chern Fundation and NSF: DMS-1128155, 1252158.
2 Preliminary

2.1 KSBA stable family

In this section, we briefly introduce the concept of KSBA family. See [Kol15] for more background.

Definition 2.1. A pair \((X, \Delta)\) over \(k\) is **KSBA stable**, if it is proper over \(k\), it has slc singularities and \(K_X + \Delta\) is ample.

We define the notion of KSBA family in full generality only for the boundary free case, since that is what we need in Theorem 1.1. In the case of the presence of a boundary we define the notion of KSBA family only over normal bases, as that is sufficient for the purposes of Theorem 1.2.

Definition 2.2. For any scheme \(T\) over \(k\), a family \(f : X \to T\) is a KSBA stable family if \(f\) is flat, \(X_t\) is KSBA stable for each \(t \in T\), and it satisfies the following Kollár condition: \(\omega_{X/T}^{[m]}\) is compatible with base-change for each integer \(m\), that is, if \(S \to T\) is a morphism, then \(\omega_{X/S}^{[m]} \cong (\omega_{X/T}^{[m]})_S\).

A proper flat morphism \((X, D) \to T\) onto a normal variety is a KSBA stable family, if \(D\) avoids the generic and the singular codimension one points of each fiber, \((X_t, D_t)\) is KSBA stable for each \(t \in T\) and \(K_{X/T} + D\) is \(\mathbb{Q}\)-Cartier.

Remark 2.3. The above two definition are compatible. That is, if \(X \to T\) is a KSBA stable family in the second sense then it is automatically a KSBA stable family in the first sense, i.e. satisfies the Kollár condition, according to [Kol11, 4.4] and [Kol08, Cor 25].

Definition 2.4 (Maximal variation family). Let \(f : (X, D) \to T\) be a KSBA stable family over a normal base of dimension \(d\). We define the variation \(\text{var} f\) of the family to be \(d - c\), where \(c\) is the dimension of a general isomorphism equivalence class of the fibers. If \(\text{var} f = \dim d\), that is a general fiber is isomorphic to only finitely many others, we say the family has maximal variation.

Definition 2.5 (Finite fiber isomorphism equivalence classes). Let \(f : (X, D) \to T\) be a KSBA stable family over a normal base of dimension \(d\). We say that \(f\) has finite fiber isomorphism equivalence classes, if for each \(t \in T\), the set

\[
\{ u \in T \mid (X'_u, D'_u) \cong (X''_u, D''_u) \}
\]

is finite.

Remark 2.6. By the existence of Isom schemes of KSBA stable stable families [KP15, Prop 5.8] there is an open set \(U \subseteq T\), such that for every \(u \in U\), the locus \(\{ t \in T \mid (X_t, D_t) \cong (X_u, D_u) \}\) is a locally closed subset of the same (general) dimension.

Proposition 2.7. [KP15, Cor 5.20] Given \(f : (X, D) \to T\) a family of stable log-varieties over a normal variety \(T\), there is a generically finite proper map
$T' \to T$ from a normal variety, another proper map $T' \to T''$ to a normal variety and a family of stable log varieties $f'' : (X'', D'') \to T''$ with maximal variation and finite fiber isomorphism equivalence classes such that the pullbacks of the above two families over $T'$ are isomorphic.

**Lemma 2.8.** Given $f : (X, D) \to T$ a maximal variation KSBA stable family over a normal variety with $n$-dimensional fibers, and $H$ is an ample divisor on $X$, then $f_* H^{n+1}$ is $\mathbb{Q}$-linearly equivalent to an effective cycle with support $S$, such that $f_S : (X, D) \times_T S \to S$ is of maximal variation.

**Proof.** Let us search for the above required effective cycle $E$ in the form

$$f_* \left( \prod_{i=1}^{n+1} H_i \right),$$

where $H_i \in [mH]$ for some integer $m \gg 0$. We are ready as soon as we make sure that each component of $E$ intersects $U$ of Remark 2.6. Let $Z_l (l = 1, \ldots, t)$ be the components of $T \setminus U$. Then it is enough to guarantee that no component of $E$ contains any of the $Z_l$. For that, just choose $H_i$ inductively to be general members of $[mH]$, which therefore does not contain any irreducible component of $f^{-1}Z_l \cap \left( \bigcap_{j=1}^{n+1} H_j \right)$. This way, $f^{-1}Z_l \cap \left( \bigcap_{j=1}^{n+1} H_j \right)$ will have dimension $\dim Z_l - 1$, which shows that no component of $E$ contains $Z_l$. \hfill $\square$

### 2.2 CM line bundle

For the reader’s convenience, in this section we recall some basic background on CM line bundles. For more details see [Tia97, PT10, FR06, PRS08, Wan12, WX14]. CM line bundle was first introduced in [Tia97]. Unlike Chow line bundle, it is not positive on the entire Hilbert scheme (see [FR06]). However, it is expected to be positive on the locus where the fibers are K-polystable.

Let $f : X \to B$ be a proper flat morphism of schemes of constant relative dimension $n \geq 1$ and let $A$ be a relatively ample line bundle on $X$. We will assume throughout that $B$ is normal, $X$ is $S_2$ and has pure dimension. We also assume that $f$ has $S_2, G_1$ fibers and $K_{X/B}$ is $\mathbb{Q}$-Cartier.

Then Mumford-Knudsen’s determinant bundle construction shows that there are line bundles $\lambda_0, \ldots, \lambda_{n+1}$ such that the following formula

$$\det f_! (A^\otimes k) = \det R^nf_! (A^\otimes k) = \lambda_{n+1}^k \otimes \lambda_n^k \otimes \cdots \lambda_0$$

holds.

Let $\mu = - \left( K_X, A_{|x_t}^{n-1} \right) / A_{|x_t}^n$, then

$$\lambda_{CM} = \lambda_{CM}(X/B, A) = \lambda_{n+1}^{\mu+n(n+1)} \otimes \lambda_n^{-2(n+1)}.$$

A straightforward calculation using Grothendieck-Riemann-Roch formula (see e.g. [FR06]) shows that

$$c_1(\lambda_{n+1}) = f_*(c_1(A)^{n+1}) \quad \text{and} \quad nc_1(\lambda_{n+1}) - 2c_1(\lambda_n) = f_*(c_1(A)^nc_1(K_{X/B})).$$
Hence
\[ c_1(\lambda_{CM}) = f_*(n\mu c_1(A)^{n+1} + (n + 1)c_1(K_{X/B})c_1(A)^n) . \]
In particular, if \( A = K_{X/B} \), we simply have
\[ c_1(\lambda_{CM}) = f_*(K_{X/B})^{n+1} . \]
Similarly, a log extension as in [WX14] shows that if we consider the log setting and \( A = K_{X/B} + D \) (and in particular, we assume \( K_{X/B} + D \) to be \( \mathbb{Q} \)-Cartier instead of \( K_{X/B} \)), then
\[ c_1(\lambda_{CM}((X, D)/B)) = f_*((K_{X/B} + D)^{n+1}) . \]
(See [WX14 2.8, 2.9]).

2.3 Dlt blow up

**Proposition 2.9.** Let \( g : (Z, \Delta) \to T \) be a KSBA stable family over a smooth variety \( T \). We further assume that the generic fiber \((Z_t, \Delta_t)\) is log canonical. Then for each \( 0 < \epsilon \ll 1 \) there is a pair \( (X, D_\epsilon) \) and a divisor \( 0 \leq D \) on \( X \) with a morphism \( p : X \to Z \), such that

(a) \( K_X + D = p^*(K_Z + \Delta) \),

(b) \( (X, D_\epsilon) \) is klt,

(c) \( f : (X, D_\epsilon) \to T \) is a KSBA stable family,

(d) \( D - D_\epsilon \) is effective and its support is contained in \( \mathrm{Ex}(p) \cap \mathrm{Supp}(p_*^{-1}\Delta = 1) \), and furthermore,

(e) if the variation of \((Z, \Delta) \to T \) is maximal then so is the variation of \((X, D_\epsilon) \).

**Proof.** Let \( \tilde{p} : \tilde{X} \to Z \) be a \( \mathbb{Q} \)-factorial dlt modification of \( Z \) [KK10 Thm 3.1] and write
\[ \tilde{p}^*(K_Z + \Delta) = K_{\tilde{X}} + \tilde{D} . \]
Denote by \( \tilde{D}^{=1} = [\tilde{D}] \) and \( \tilde{D}^{<1} = \tilde{D} - \tilde{D}^{=1} \). By [BCHM10], we may take \( p : X \to Z \), the relative log canonical model of \((\tilde{X}, (1 - \epsilon)\tilde{D}^{=1} + \tilde{D}^{<1}) \) over \( Z \). Let \( q : \tilde{X} \to X \) be the induced morphism and \( D_\epsilon \), \( D^{=1} \) and \( D^{<1} \) the corresponding pushforwards. Define then \( D_\epsilon := (1 - \epsilon)D^{=1} + D^{<1} \), whence \( (b) \) and \( (d) \) follows. Note that by [BCHM10 Thm E, p 414], \( X \) is the same for all \( 0 < \epsilon \ll 1 \).

Since \( -\epsilon \tilde{D}^{=1} \equiv_Z K_{\tilde{X}} + (1 - \epsilon)\tilde{D}^{=1} + \tilde{D}^{<1} \), we have that \( -\epsilon D^{=1} \equiv_Z K_X + D_\epsilon \) is ample over \( Z \). Furthermore,
\[
K_X + D_\epsilon = q_* \left( K_{\tilde{X}} + (1 - \epsilon)\tilde{D}^{=1} + \tilde{D}^{<1} \right)
= q_* \left( q^* p^*(K_Z + \Delta) - \epsilon \tilde{D}^{=1} \right)
= p^*(K_Z + \Delta) - \epsilon D^{=1}. \quad (1)
\]
So, since $K_Z + \Delta$ is ample over $T$, $K_X + D_t$ is ample over $T$ as well for $0 < \epsilon \ll 1$. A computation similar to \([1]\) but with $D$ instead of $D_t$ shows \([a]\)

Let $n$ be the relative dimension of $Z$ over $Y$. To obtain (c), we need to show that $X$ is flat over $T$, $D_t$ does not contain any component of $X_t$ or any divisor in the singular locus of $X_t$ for any $t \in T$ and $(X_t, D_{t,t})$ is slc of dimension $n$ for all $t \in T$. We work on a neighborhood of an arbitrary point $t \in T$.

To see the above statements, note first that if $W \subset X_t$ is a component, then $\dim W \geq n$. Let $H_1, ..., H_d$ be $d$ general hypersurfaces passing through $t$ where $d = \dim(T)$. Then as $(Z, \Delta + g^*(\sum_{i=1}^d H_i))$ is crepant birational to $(X, D + f^*(\sum_{i=1}^d H_i))$, the latter is log canonical. As $(X, D + f^*(\sum_{i=1}^d H_i))$ is log canonical at $W$, $f^*H_i$ are Cartier divisors and $W \subset \bigcap_{i=1}^d f^*H_i$, by \([dFKX12, 34]\), we know that $(X, f^*(\sum_{i=1}^d H_i))$ is snc at the generic point of $W$ and $W$ is not contained in $D_t$, which has the same support as $D_{t,t}$. Therefore $\dim W = n$ and $X_t$ is smooth at the generic point of $W$. In particular, $X_t$ is reduced and equidimensional.

Since $(X, D_t)$ is klt, then $X$ is Cohen-Macaulay (see \([KM98, 5.22]\)). Since $f : X \to T$ is an equidimensional morphism, $X_t$ is CM and $f$ is flat. Let $C = \bigcap_{i=1}^{d-1} H_i$, which is a smooth curve passing through $t$. Then $X_C : = X \times_T C$ is normal and

$$(X_C, D_{t,C} : = D_t \times_tC)$$

satisfies that $(X_C, D_{t,C} + X_t)$ is log canonical by adjunction. This implies that $(X_C, D_{t,C}) \to C$ is a KSBA family. Therefore, $D_t$ has to avoid the general point $\eta$ of any codimensional one component of the singular locus of $X_t$. Thus $(X_t, D_{t,t})$ is slc by adjunction and we conclude $(X, D_t) \to T$ is a KSBA family over $T$.

To prove (e), just note that for a general $t \in T$, $(Z_t, \Delta_t)$ is the log canonical model of $(X_t, D_t)$. So, for general $t, u \in T$ and for $0 < \epsilon \ll 1$, $(X_t, D_{t,u}) \cong (X_u, D_{t,u})$ if and only if $(X_t, D_t) \cong (X_u, D_u)$, from which it follows that $(Z_t, \Delta_t) \cong (Z_u, \Delta_u)$. This shows \([e]\).

\[\square\]

### 2.4 Bigness and nefness of relative canonical bundle

We first collect some results about the push forwards of the powers of the relative canonical bundle.

**Definition 2.10.** A torsion-free coherent sheaf $F$ on a normal variety $X$ is big, if for an ample line bundle $H$ on $X$ there is an integer $a > 0$ and a generically surjective homomorphism $\bigoplus H \to S^a(F) := (S^a(F))^{**}$.

**Theorem 2.11.** If $f : (X, D) \to T$ is a maximal variation KSBA stable family over a normal projective variety $T$ with klt general fibers, then $f_*O_X(r(K_X/T + D))$ is big for every divisible enough integer $r > 0$.

**Proof.** This follows from \([KPI15, Theorem 7.1]\). \[\square\]
Remark 2.12. Note that the klt assumption in the above theorem cannot be weakened to log canonical according to [KP15, Example 7.5-7.7].

Theorem 2.13. If \( f : (X, D) \to T \) is a KSBA stable family over a normal projective variety \( T \), then \( f_*\mathcal{O}_X(r(K_{X/T} + D)) \) is nef for every divisible enough integer \( r > 0 \).

Proof. This follows from [Fuj12, Theorem 1.13].

Corollary 2.14. If \( f : (X, D) \to T \) is a KSBA stable family over a normal projective variety \( T \), then

\[
f_*((K_{X/T} + D)^{n+1})
\]

is nef.

Proof. Since \( f_*((K_{X/T} + D)^{n+1}) \) is compatible with base-change, and nefness is decided on curves, we may assume that \( T \) is a curve. However, then we are supposed to only prove that \( 0 \leq \deg f_*((K_{X/T} + D)^{n+1}) \), which follows if we show that \( (K_{X/T} + D)^{n+1} \) is the limit of effective cycles. However, the latter is a consequence of nefness.

Proposition 2.15. If \( f : (X, D) \to T \) is a maximal variation KSBA stable family over a smooth projective variety \( T \) such that the generic fiber \((X_t, D_t)\) is log canonical, then \( K_{X/T} + D \) is big and nef for every divisible enough integer \( r > 0 \).

Proof. Since there is an embedding

\[
X \subset \mathbb{P}_T(f_*\mathcal{O}_X(r(K_{X/T} + D)))
\]

for \( r \) sufficiently large and

\[
\mathcal{O}(1)|_X \cong r(K_{X/T} + D),
\]

then the nefness of \( K_{X/T} + D \) is straightforward consequence of Theorem 2.13.

For the bigness, when the general fiber is klt, By Theorem 2.11, there is a generically surjective

\[
f^*(\oplus H) \to f^*S^n f_*\mathcal{O}_X(r(K_{X/T} + D)) \to \mathcal{O}_X(ar(K_{X/T} + D)),
\]

for some \( a \) and sufficiently divisible \( r \). After replacing \( a \) by its multiple, and tensoring \( A := r(K_{X/T} + D) \), we know that the there is a nontrivial morphism

\[
f^*(\oplus mH) \otimes \mathcal{O}_X(r(K_{X/T} + D)) \to \mathcal{O}_X((ma + 1)r(K_{X/T} + D)),
\]

which implies that \( K_{X/T} + D \) is big.
In general, applying Proposition 2.9, we know that we can find a birational model \( h : Y \to X \), such that if we define \( D_Y \) so that

\[
h^*(K_X + D_Y) = K_Y + D_Y
\]

holds, then there exists a divisor \( D_Y' \leq D_Y \) for which \((Y, D_Y') \to T\) is a maximal variation of KSBA stable family with generic klt fibers. Thus \( K_{Y/T} + D_Y' \) is big which implies \( K_{X/T} + D \) is big.

### 3 Proof of Main Theorems

In this section, we prove Theorem 1.2 and hence Theorem 1.1. By induction, we may assume the base is of dimension \( d \) and Theorem 1.2 already holds when the base is of dimension at most \( d - 1 \). Note that the \( d = 0 \) case is tautologically true, hence the starting point of the induction is fine.

#### 3.1 Log canonical case

**Lemma 3.1.** Theorem 1.2 is true if a general fiber \((X_t, D_t)\) is log canonical.

**Proof.** Set \( A := K_{X/T} + D \) and \( d := \dim(T) \). According to Corollary 2.14 by Nakai-Moishezon criterion and resolution of singularity, it is enough to show that when \( T \) is a \( d \)-dimensional smooth projective variety and \((X, D)/T\) is a KSBA family of maximal variation, then \((f_*(A^{n+1}))^d > 0\).

Write \( A = H + F \) for some ample \( \mathbb{Q} \)-divisor \( H \) and effective \( \mathbb{Q} \)-divisor \( F \). According to Lemma 2.8, \( f_*(H^{n+1}) \) is linearly equivalent to a \( d-1 \) dimensional effective \( \mathbb{Q} \)-cycle over whose support \((X, D)\) has a maximal variation. Hence by induction we have

\[
(f_*(A^{n+1}))^{d-1} \cdot f_*(H^{n+1}) > 0.
\]

So we only need to show that for any \( 0 \leq k \leq n \),

\[
(f_*(A^{n+1}))^{d-1} \cdot f_*(H^k \cdot F \cdot A^{n-k}) \geq 0.
\]

This again follows from the induction since \( f_*(H^k \cdot F \cdot A^{n-k}) \) is a limit of effective \((d-1)\)-dimensional \( \mathbb{Q} \)-cycles. For any \( d-1 \) smooth projective variety \( P \to T \), by the projection formula

\[
\left(f_*(A^{n+1})_{|P}\right)^{d-1} = (f_*(A^{n+1}))^{d-1} \cdot P.
\]

Then by induction we have

\[
(f_*(A^{n+1}))^{d-1} \cdot P \geq 0,
\]

and this implies what we need by resolution of singularity. 

\[\square\]
3.2 Semi-log canonical case

Let $f : (Z, \Delta) \to S$ be a KSBA stable family over a normal variety $T$. Taking a normalization $f : X \to Z$, we get

$$(X, D) = \bigsqcup_{i=1}^{m} (X_i, D_i) \to T$$

with a conductor divisor $E$ and $D_i$ is the sum of the conductor divisor and the pull back $\Delta_{X_i}$ of $\Delta$. Furthermore, there is an involution $\tau : E^n \to E^n$ on the normalization $E^n$ of $E$ which preserves the difference divisor $\text{Diff}_{E^n}(\Delta_X)$. In fact, we know there is one to one correspondence between $(Z, \Delta)/S$ and $(X, D, E, \tau)/S$ as above (see [Kol13, Theorem 5.13]).

**Lemma 3.2.** Let $f : (Z, \Delta) \to S$ be a KSBA stable family over a normal variety $T$, Taking a normalization $f : X \to Z$, we get

$$(X, D) = \bigsqcup_{i=1}^{m} (X_i, D_i) \to T.$$  

Then $f_i : (X_i, D_i) \to T$ is a KSBA stable family over $T$.

**Proof.** Let $0 \in T$ be a closed point, $(\bar{X}_{i,0}, \bar{D}_{i,0}) \to (X_{i,0}, D_{i,0})$ the normalization, and $\bar{D}_{i,0}$ the sum of the pull back of $D_{i,0}$ and the conductor divisor. Then $(\bar{X}_{i,0}, \bar{D}_{i,0})$ is a KSBA stable pair.

We first want to check $X \to T$ is flat at the codimension one point of $X_0$. In fact, to see this, by cutting the fiber using general hypersurfaces, we can assume that $Z \to T$ has relative dimension one, i.e., the fibers are nodal curves. Let $E$ be the conductor divisor of the normalization $g : X \to Z$. By definition, $g$ is isomorphic outside $E$. Furthermore, if a codimension one point $P$ of the fiber is contained in $E$, then $g(P)$ is a nodal point of the fiber, and analytically locally around $g(P)$, we can write it as $\mathcal{O}_T[[x, y]]/(xy - a)$ for some element $a \in \mathcal{O}_T$.

However, since $P \in E$, we conclude $a = 0$, which implies that $X \to T$ is smooth along $E$.

Thus we can apply the numerical stability in [Kol15, Section 13] and conclude that for a general point $s \in T$,

$$(K_{\bar{X}_{i,0}} + \bar{D}_{i,0})^n \geq (K_{X_{i,s}} + \bar{D}_{i,s})^n.$$  

Furthermore, the equality holds if and only if $(X_i, D_i) \to T$ is a KSBA stable family over an open neighborhood of $0 \in T$.

On the other hand, we know that

$$\sum_{i=1}^{m} (K_{\bar{X}_{i,s}} + \bar{D}_{i,s})^n = (K_{Z_0} + \Delta_0)^n$$  

$$= (K_{Z_0} + \Delta_0)^n$$  

$$= \sum_{i=1}^{m} (K_{\bar{X}_{i,0}} + \bar{D}_{i,0})^n,$$
where the second equality follows from the fact that \((Z, \Delta)\) is a stable family over \(T\). Thus we conclude for each \(i\),
\[(K_{X_{i,0}} + D_{i,0})^n = (K_{X_{i,s}} + D_{i,s})^n.\]

\[\square\]

Remark 3.3. We give a sketch of a more straightforward argument for a weaker statement than Lemma 3.2, which says that there exists a proper dominant generically finite morphism \(T' \to T\), such that the normalization \((X'_i, D'_i)\) of \((Z, \Delta) \times_T T'\) is a KSBA stable family over \(T'\). This is enough for our calculation in the proof of Theorem 1.2 for the general case.

By generic flatness, there is an open set \(T^0\), such that \((X^0_i, D^0_i) := (X_i, D_i) \times_T T^0 \to T^0\) is a KSBA family over \(T^0\). Applying [AK00], we can assume there is a proper dominant generically finite base change \(g : T' \to T\) such that
\[(X_i \times_T T', D_i \times_T T')\]

admits a weak semistable reduction. It follows from [HX13 Theorem 1] that by running a relative MMP of \((X_i \times_T T', D_i \times_T T')\) over \(T'\), we obtain a relative good minimal model \((X^m_i, \Delta^m_i)\). A similar argument as in the proof of Proposition of 2.2 shows that \((X^m_i, \Delta^m_i)\) is flat over \(T'\), for any \(t \in T'\), \(\Delta^m_{i,t}\) does not contain any component or codimension one singular point of \(X^m_{i,t}\) and \((X^m_{i,t}, \Delta^m_{i,t})\) is slc.

Then the injectivity theorem (see [Fuj13 Theorem 6.4]) implies that the relative log canonical models \((X^c_i, \Delta^c_i)\) of \((X^m_i, \Delta^m_i)\) over \(T'\) fiberwisely gives the log canonical model of \((X^c_{i,t}, \Delta^c_{i,t})\). Furthermore, \((X^c_i, \Delta^c_i)\) is flat over \(T'\) by Grauert’s criterion. Thus \((X^c_i, \Delta^c_i)\) is a KSBA stable family over \(T'\).

Let
\[(X', D') = \bigsqcup_{i=1}^{m} (X^c_i, D^c_i) \to T'.\]

Over \(g^{-1}(T_0)\), \((X', D')\) extends the family of \((X, D) \times_T g^{-1}(T^0)\). Also we easily see both \(E \times_T g^{-1}(T^0)\) and the involution condition
\[\tau \times_T g^{-1}(T^0) : (E \times_{T^0} g^{-1}(T^0))^{\text{norm}} \to (E \times_{T^0} g^{-1}(T^0))^{\text{norm}}\]

extend to corresponding data \(E'\) and \(\tau'\) over \(T'\).

Thus \((X', D', E', \tau')/T'\) induces a KSBA stable family \((Z', \Delta')/T'\) over \(T'\) by [Kol13 Theorem 5.13]. It satisfies that
\[(Z', \Delta') \times_{T'} g^{-1}(T^0) = (Z, \Delta) \times_T g^{-1}(T^0).\]

By the separateness of the functor of KSBA stable family, we conclude that
\[(Z', \Delta') = (Z, \Delta) \times_T T',\]
as both of them give KSBA stable families over \(T\) which are isomorphic over the generic point.
Proof of Theorem 1.2. We use the notations of Lemma 3.2. Applying Proposition 2.7, there is a smooth projective variety $T'$ with a generically finite morphism $h_i : T' \to T_i$ to smooth projective varieties such that if we denote by $X'_i := X_i \times_T T'$ and $D'_i := D_i \times_T T'$, then

$$(X'_i, D'_i) \cong (Y_i, E_i) \times_{T_i} T'$$

where $g_i : (Y_i, E_i) \to T_i$ are KSBA stable families over $T_i$ with finite fiber isomorphism equivalence classes and log canonical generic fibers. Furthermore, we have that

$$h : T' \to T_1 \times \cdots \times T_m$$

is a generically finite morphism.

According to Lemma 3.1 and induction, we know that $(g_i)_* \left( (K_{Y_i/T_i} + E_i)^{n+1} \right)$ is ample on $T_i$. Denote by $f'_i : X'_i \to T'$ and $p_i : T_1 \times \cdots \times T_m \to T_i$ the induced morphisms. Since $h$ is generically finite, and

$$(f_* \left( (K_{X/T} + \Delta)^{n+1} \right))_{T'} = \sum_i (f'_i)_* \left( (K_{X'_i/T'} + D'_i)^{n+1} \right)$$

$$= \sum_i h^*_i (g_i)_* \left( (K_{Y_i/T_i} + E_i)^{n+1} \right) = h^* \left( \sum_i p^*_i (g_i)_* \left( (K_{Y_i/T_i} + E_i)^{n+1} \right) \right)$$

is big and nef on $T'$. Thus the above computation concludes the proof.

Proof of Theorem 1.1. We apply Nakai-Moishezon criterion (cf. [Kol90]). So it suffices to check for any $d$-dimensional irreducible subspace $B \subset M_{\text{ksba}}$, the top intersection $\lambda_{\text{CM}}^d : B > 0$.

By [Kol90] 2.7, we can replace $B$ by a finite surjective base change $\pi : B' \to B$ such that $B'$ is normal and $B' \to M_{\text{ksba}}$ lifts to $B' \to M_{\text{ksba}}$ where $M_{\text{ksba}}$ is the fine moduli DM stack which parametrizes KSBA families. Thus there is a KSBA family of finite fiber isomorphism equivalence classes $X/B'$. In particular,

$$\lambda_{\text{CM}}^d \cdot B = \frac{1}{\deg(\pi)} \lambda_{\text{CM}}^d \cdot B' > 0$$

by Theorem 1.2.

Remark 3.4. A large part of our argument works in the log setting, i.e., KSBA family of log pairs. However, due to the subtlety of the definition of KSBA functor itself, we will not discuss it here.

References

[Ale96] Valery Alexeev, *Moduli spaces $M_{g,n}(W)$ for surfaces*, Higher-dimensional complex varieties (Trento, 1994), 1996, pp. 1–22.
[AK00] Dan Abramovich and Kalle Karu, Weak semistable reduction in characteristic 0, Invent. Math. 139 (2000), no. 2, 241–273.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher Hacon, and James McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[dFKX12] Tommaso de Fernex, János Kollár, and Chenyang Xu, The dual complex of singularities, to appear in Adv. Stud. Pure Math., Professor Kawamata’s 60th birthday volume, arXiv:1212.1675 (2012).

[Don01] Simon K. Donaldson, Scalar curvature and projective embeddings, I, J. Differential Geom. 59 (2001), 479-522.

[FR06] J. Fine and J. Ross, A note on positivity of the CM line bundle, Int. Math. Res. Not. (2006), 14 pages. Article ID95875.

[Fuj12] Osamu Fujino, Semipositivity theorems for moduli problems, arXiv:1210.5784 (2012).

[Fuj13] , Injectivity theorems, arXiv:1303.2404 (2013).

[HX13] Christopher Hacon and Chenyang Xu, Existence of log canonical closures, Invent. Math. 192 (2013), no. 1, 161-195.

[KM98] János Kollár and Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original.

[Kol90] János Kollár, Projectivity of complete moduli, J. Differ. Geom. 32 (1990), 235-268.

[Kol95] , Flatness criteria, J. Algebra 175 (1995), no. 2, 715-727.

[Kol08] , Hulls and Husks, arXiv:0805.0576 (2008).

[Kol11] , A local version of the Kawamata-Viehweg vanishing theorem, Pure Appl. Math. Q. 7 (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1477–1494.

[Kol13] , Singularities of the minimal model program, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[Kol15] , Moduli of varieties of general type (2015). Book in preparation.

[KK10] János Kollár and Sándor J Kovács, Log canonical singularities are Du Bois, J. Amer. Math. Soc. 23 (2010), no. 3, 791–813, DOI 10.1090/S0894-0347-10-00663-6.

[KP15] Sándor Kovács and Zsolt Patakfalvi, Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension, arXiv:1503.02952 (2015).

[KSBB88] János Kollár and N. I. Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. 91 (1988), no. 2, 299-338.

[LWX15] Chi Li, Xiaowei Wang, and Chenyang Xu, Quasi-projectivity of the moduli space of smooth Kähler-Einstein Fano manifolds, arXiv:1502.06532 (2015).

[Oda06] Yuji Odaka, The Calabi Conjecture and K-stability, Int. Math. Res. Not. 10 (2012), 2272-2288.

[PRS08] D.H. Phong, Julius Ross, and Jacob Sturm, Deligne pairing and the Knudsen-Mumford expansion, J. Differential Geom. 78 (2008), no. 3, 475-496.

[PT10] Sean Paul and Gang Tian, CM stability and the generalized Futaki invariant II, Astérisque 328 (2010), 339-354.

[Sch12] Georg Schumacher, Positivity of relative canonical bundles and applications, Invent. Math. 190 (2012), 1-56.

[Tia97] Gang Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), no. 1, 1–37.
[Wan12] Xiaowei Wang, *Height and GIT weight*, Math. Res. Lett. 19 (2012), no. 4, 906-926.

[WX14] Xiaowei Wang and Chenyang Xu, *Nonexistence of asymptotic GIT compactification*, Duke Math. J. 163 (2014), 2217-2241.

[Vie95] Eckart Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 30, Springer-Verlag, Berlin, 1995.

Department of Mathematics, Princeton University, Fine Hall, Washington Road, NJ 08544-1000, USA
E-mail address: pzs@math.princeton.edu

Beijing International Center of Mathematics Research, 5 Yiheyuan Road, Beijing 100871, China
E-mail address: cyxu@math.pku.edu.cn