Transport equation and hard thermal loops in noncommutative Yang-Mills theory

F. T. Brandt\textsuperscript{a}, Ashok Das\textsuperscript{b}, J. Frenkel\textsuperscript{a}, D. G. C. McKeon\textsuperscript{c} and J. C. Taylor\textsuperscript{d}

\textsuperscript{a} Instituto de Física, Universidade de São Paulo, São Paulo, SP 05315-970, BRAZIL
\textsuperscript{b}Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171, USA
\textsuperscript{c} Department of Applied Mathematics, The University of Western Ontario, London, ON N6A5B7, CANADA and
\textsuperscript{d} Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, UK

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We show that the high temperature limit of the noncommutative thermal Yang-Mills theory can be directly obtained from the Boltzmann transport equation of classical particles. As an illustration of the simplicity of the Boltzmann method, we evaluate the two and the three-point gluon functions in the noncommutative $U(N)$ theory at high temperatures $T$. These amplitudes are gauge invariant and satisfy simple Ward identities. Using the constraint satisfied at order $T^2$ by the covariantly conserved current, we construct the hard thermal loop effective action of the noncommutative theory.

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I. INTRODUCTION

In recent years, there has been a lot of interest in the study of the dynamics of hot plasma in non-Abelian gauge theories. It is known now that the $n$-point gluon functions in such a theory, when evaluated at temperatures which are large compared with all momenta $p$ of the external gauge fields, are gauge invariant and have a leading behavior proportional to $T^2$. Here, $T$ represents the temperature of the plasma. In order to account for all the leading order contributions consistently as well as to obtain a meaningful gauge invariant result for physical quantities, it is necessary to perform a resummation of hard thermal loop contributions \([1]\). The hard thermal loops are defined by the relation

\[ p \ll k \sim T \]  \hspace{1cm} (1.1)

where $p$ represents a characteristic external momentum while $k$ denotes the internal loop momentum. In conventional QCD, such a procedure, although very insightful, is technically quite involved. However, it has also been noted that it is possible to use a classical transport equation in a more direct and transparent way, in order to derive the effective action which includes all the contributions associated with the hard thermal loops \([2, 3, 4, 5]\). In this approach, which has been successfully applied to a variety of gauge theories, one pictures the constituents of the plasma as classical charged particles interacting in a self-consistent manner. The main reason why such a classical description works in yielding a quantum effective action is that, for soft gauge fields, the occupation number per unit mode, in a hot plasma, is quite high due to the Bose-Einstein enhancement.

In more recent years, developments in string theory have renewed interest in noncommutative field theories \([6, 7, 8, 9, 10, 11]\). These are theories defined on a noncommutative manifold – noncommutative field theories – exhibit some very interesting features and there are continued attempts at understanding better the structures of such theories, in particular, those of noncommutative gauge theories.

In the paper \([12]\), the structure of the two- and three-gluon amplitudes, incorporating the hard thermal loop contributions, were calculated in the pure noncommutative $U(N)$ Yang-Mills theory at high temperature and the results were quite interesting. We note that while \([13]\) defines the hard thermal loops, in the presence of independent dimensional parameters, such as $\theta^{\mu \nu}$, we can also have regimes satisfying $\theta < \frac{1}{T}$ or $\theta > \frac{1}{T}$, in addition to Eq. \((1.1)\). Here, $\theta$ can be thought of as the magnitude of $\theta^{\mu \nu}$ (or, in more physical terms as the largest or the smallest eigenvalues of $\theta^{\mu \nu}$ for the first and the second inequalities respectively). In the presence of a characteristic momentum scale, $p$, one can also have regimes satisfying $\theta p T < 1$ or $\theta p T > 1$. In fact, it was shown in \([14]\) that the calculation of hard thermal loops simplifies enormously in the two limits $\theta p T \ll 1$ and $\theta p T \gg 1$. We note that, in the presence of a dimensional parameter, such as $\theta$, the hard thermal loop condition, Eq. \((1.1)\), can be written as

\[ \theta p^2 \ll \theta p T \ll \theta T^2 \]  \hspace{1cm} (1.3)

\[ [x^{\mu}, x^{\nu}] = i \theta^{\mu \nu} \]  \hspace{1cm} (1.2)

where $\theta^{\mu \nu}$ is assumed to be a constant anti-symmetric tensor. Furthermore, to avoid problems with unitarity, one also assumes that $\theta^{\mu \mu} = 0$, namely, only the spatial coordinates do not commute. Note that, by definition, $\theta^{\mu \nu}$ has the dimensions of the square of a length. Quantum field theories defined on such a manifold – noncommutative field theories – exhibit some very interesting features and there are continued attempts at understanding better the structures of such theories, in particular, those of noncommutative gauge theories.
The actual hard thermal loop calculations [12] show that, the leading $T^2$ contributions from the $U(1)$ sector of the $U(N)$ theory become suppressed by powers of $\theta p T$ when $\theta p T \ll 1$, while, in the limit $\theta p T \gg 1$, the amplitudes are all proportional to $N$ as would be the case for a large $N$ theory.

It is worth noting that a similar behavior (at zero temperature) is present in the case of a noncommutative supersymmetric $U(N)$ gauge theory [13]. Namely, it was shown there that the $U(1)$ sector of the theory behaves differently for $p < \theta^{-\frac{1}{2}}$ and $p > \theta^{-\frac{1}{2}}$ such that in the low energy limit, the noncommutative $U(N)$ theory behaves like an ordinary $SU(N)$ theory (We prefer “ordinary” to “commutative” for obvious reasons.). As is the case in [12], here, too, we will like to emphasize that the behavior of the theory in the hard thermal loop approximation should not be taken to imply that temperature somehow breaks the $U(N)$ symmetry. In fact, in connection with the work of [13], gauge invariant completions of the action involving open Wilson lines have already been proposed in [14] [15]. Physically, the results of [12], in both the limits, may be understood as follows. The limit, $\theta p T \ll 1$, can clearly be thought of as corresponding to weak non-commutativity, in which case, the self couplings of the $U(1)$ sector is negligible compared to that of the $SU(N)$ sector. Therefore, the theory, to the leading order, behaves like an ordinary $SU(N)$ theory with the effects of non-commutativity leading to small corrections. In the limit $\theta p T \gg 1$, on the other hand, we can think of the non-commutativity as strong and it is well known [16] that in the limit of maximal non-commutativity, a noncommutative theory has a stringy behavior. (Intuitively, this is seen as follows. If the noncommutative theories are supposed to describe dipoles [17, 18], then, for distances much much smaller than the dipole length, $\frac{1}{p} \ll \theta p$, the string nature would become manifest.) In particular, in the case of a noncommutative gauge theory, this would correspond to having dominant planar diagrams. (The non-planar diagrams oscillate rapidly at high temperature and yield only subleading contributions.)

It is an interesting question to ask whether a classical transport equation can be written for such theories which can determine the effective action in the hard thermal loop approximation in a simple manner, as is the case for “ordinary” theories. Of course, as we have mentioned earlier, there are two limits, $\theta p T \ll 1$ and $\theta p T \gg 1$, in which one can analyze the effective action. Since the region $\theta p T \ll 1$ corresponds to a weaker non-commutativity, in this paper, we restrict ourselves to the region where $\theta p T \gg 1$ and which genuinely exhibits noncommutative hard thermal loop effects. We will present a classical transport equation for the noncommutative $U(N)$ gauge theory which leads, in a simple way, to the leading order terms of the two and three gluon amplitudes in the hard thermal loop approximation, calculated in [12]. The color current, defined as

$$J^C_{\mu}(x, A) = \frac{\delta \Gamma[A]}{\delta A^C_{\mu}(x)}$$

where $\Gamma[A]$ represents the effective action in the hard thermal loop approximation, can be derived from the classical transport equation. Here, $A^\mu$ is the gluon field and $J^C_{\mu}$ consists of an infinite set of graphs. The above equation can be integrated and leads to the gauge invariant effective action which incorporates all the hard thermal loop contributions.

The paper is organized as follows. In section II, we derive the transport equation for the noncommutative $U(1)$ theory in two alternate ways. The first is through the conventional use of the dynamical equations for a relativistic particle in a noncommutative manifold. The second is through the use of the conservation equations for the current as well as the stress tensor. Both these methods yield the same transport equation which reduces, in the $\theta \rightarrow 0$ limit, to the conventional transport equation for QED. In the limit $\theta p T \gg 1$, we show how this transport equation can be used to derive, in a simple manner, the two and the three point functions for the $U(1)$ gauge boson, which agrees with the diagrammatic calculations in [12]. The derivation of the transport equation for the $U(N)$ case is rather involved for various technical reasons. We do not fully understand a first principle derivation in this case. In section III, we discuss some of the challenges that one faces in the $U(N)$ case and with guidance from the $U(1)$ case, propose a transport equation for the noncommutative $U(N)$ theory. This equation reduces to that for the conventional $SU(N)$ theory in the limit $\theta \rightarrow 0$. However, in the limit $\theta p T \gg 1$, it leads to genuine noncommutative hard thermal loop effects (which have a stringy character). We evaluate the two and three point gluon functions from the transport equation and these agree with the explicit diagrammatic calculations in [12]. In section IV, we derive from the transport equation an all order expression for the color current [Eq. (4.4)], which is manifestly covariantly conserved. With the help of this result, we obtain a gauge invariant form for the effective action [Eq. (4.6)] which includes all the hard thermal loop effects in noncommutative Yang-Mills theory.

II. TRANSPORT EQUATION FOR NONCOMMUTATIVE $U(1)$ THEORY

In this section, we consider the noncommutative $U(1)$ gauge theory without any matter fields. As is well known, such a theory is self-interacting much like the conventional Yang-Mills theories. In this case, the field strength is
defined to be
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu]_M \quad (2.1) \]

where
\[ [A_\mu, A_\nu]_M = A_\mu * A_\nu - A_\nu * A_\mu \quad (2.2) \]
defines the Moyal commutator and the star product that introduces self-interactions into the system is given by
\[ A_\mu(x) * A_\nu(x) = \left[ e^{\frac{i}{2} \theta^{\alpha\beta} \partial_\alpha (\partial_\beta)^{(n)}} A_\mu(x + \xi) A_\nu(x + \eta) \right]_{\xi=\eta=0} \quad (2.3) \]

The Euler-Lagrange equation, in this case, is obtained to be
\[ \partial^\mu F_{\mu\nu} - i g [A^\mu, F_{\mu\nu}]_M \equiv \mathcal{D}^\mu F_{\mu\nu} = J_\nu. \quad (2.4) \]

where \( J_\mu(x) \) represents the source for the gauge fields and \( \mathcal{D}_\mu \) denotes the Moyal covariant derivative. It follows from Eq. (2.4) that the current \( J_\mu \) is covariantly conserved. This property ensures that the effective action, which leads to the current through a functional derivation, namely,
\[ J^\mu(x, A) = \frac{\delta \Gamma[A]}{\delta A_\mu(x)}, \quad (2.5) \]
is gauge invariant. This follows since, under an infinitesimal noncommutative \( U(1) \) gauge transformation,
\[ \frac{\delta \Gamma[A]}{\delta \omega(x)} = \int dy \frac{\delta A_\mu(y)}{\delta \omega(x)} \frac{\delta \Gamma[A]}{\delta A_\mu(y)} \equiv \mathcal{D}_\mu J^\mu(x) = 0. \quad (2.6) \]

Here, \( \omega(x) \) represents the infinitesimal parameter of gauge transformation. Let us note that the covariant conservation of the current is a consequence (as well as a reflection) of the fact that, in the noncommutative \( U(1) \) theory, the current transforms in the adjoint representation under a star gauge transformation,
\[ J_\mu(x) \rightarrow U(x) * J_\mu(x) * U^{-1}(x) \quad (2.7) \]

This has to be contrasted with the behavior of the current in an “ordinary” \( U(1) \) theory.

There are several ways one can derive the transport equation for the noncommutative \( U(1) \) theory. Following the conventional method, we define the current, \( J^\mu(x) \), in terms of the statistical distribution functions, \( f(x, k) \), as
\[ J_\mu(x) = g \sum_k \int dK k^\mu f(x, k), \quad (2.8) \]

where the sum is over contributions from all particle species and helicities. Let us note that, unlike in conventional \( U(1) \) theory, in the noncommutative case, charge neutral particles belong to the adjoint representation of the group and contribute to the current as well. In fact, in the present case, where we are investigating a pure \( U(1) \) gauge theory (noncommutative), there is no charged matter present and the gauge field is charge neutral so that the classical particle that we are considering also is charge neutral (nonetheless has a nontrivial current). Furthermore, the integration measure for momentum is defined to be
\[ dK \equiv \frac{d^4 k}{(2\pi)^3} 2 \theta(k_0) \delta(k^2 - m^2) \quad (2.9) \]

which guarantees positivity of the energy as well as the on-shell evolution of the thermal excitations. Let us note next from Eqs. (2.7) and the definition of the current in (2.8) that, in this case, the distribution function \( f(x, k) \) transforms covariantly under a star gauge transformation much like the current, namely
\[ f(x, k) \rightarrow U(x) * f(x, k) * U^{-1}(x). \quad (2.10) \]

Once again, this behavior has to be contrasted with the conventional case and we note that when \( \theta^{\mu\nu} \rightarrow 0 \) and the star product reduces to ordinary products, we recover the conventional behavior for the distribution function.

In the case of a collisionless plasma, the distribution function \( f(x, k) \) may be determined from a classical collisionless transport equation, which can be derived in a simple manner as follows. Since \( f(x, k) \) transforms in the adjoint
representation, Eq. (2.10), much like the current, it follows that the distribution function must also be covariantly constant along the trajectory of the particle, namely,

\[ \mathcal{D}_\tau f(x, k) = \frac{df}{d\tau} - ig \frac{dx}{d\tau} [A_\mu, f]_M = 0, \]  

(2.11)

Here \( \tau \) is the proper time which parameterizes the trajectory and \( \mathcal{D}_\tau \) is the Moyal covariant derivative along the trajectory of the particle. Furthermore, using the equations of motion

\[ m \frac{dx^\mu}{d\tau} = k^\mu; \quad m \frac{dk^\mu}{d\tau} = g F^{\mu\nu} k^\nu \]  

(2.12)

which hold unchanged in the noncommutative theory, Eq. (2.11) leads to the transport equation,

\[ m \frac{df}{d\tau} = (k \cdot \partial) f - g k^\mu F^{\mu\nu} \star \frac{\partial f}{\partial k^\nu} = i g [k \cdot A, f]_M, \]  

(2.13)

This may also be written in the simpler form,

\[ (k \cdot \mathcal{D}) f(x, k) = g k^\mu F^{\mu\nu} \star \frac{\partial f}{\partial k^\nu} \]  

(2.14)

This equation, which generalizes the Boltzmann equation for the “ordinary” U(1) gauge theory, has the correct covariance properties under star gauge transformations, which characterize the noncommutative theory.

The transport equation can also be derived alternatively without using directly the equations of motion of the particle. For example, we have already noted that the current can be written in terms of the statistical distribution function. Similarly, generalizing the conventional ideas, we also observe that the stress tensor for the theory can also be described in terms of the function \( f(x, k) \) as

\[ J_\mu(x) = g \sum \int dK k_\mu f(x, k) \]

\[ T_{\mu\nu}(x) = \sum \int dK k_\mu k_\nu f(x, k) \]  

(2.15)

It can be easily checked that requiring the conservation equations for the current and the stress tensor to hold, namely,

\[ \mathcal{D}_\mu J^\mu = 0 \]  

(2.16)

\[ \mathcal{D}_\mu T^{\mu\nu} = -F^{\mu\nu} \star J_\mu \]  

(2.17)

also leads to the transport equation (2.14). We note that, unlike in the conventional U(1) theory, here the stress tensor transforms in the adjoint representation which is the reason for the (Moyal) covariant derivative in Eq. (2.17).

It is worth pointing out that, in the limit \( \theta^{\mu\nu} \to 0 \), we recognize Eq. (2.14) to correspond to the transport equation for conventional QED. In the present case of a pure gauge theory, the self-interactions of the gauge bosons go to zero in this limit so that we do not expect hard thermal loop corrections in the absence of charged particles in Eq. (2.15). That this follows from the transport equation as well, is seen easily from the fact that since we are considering a charge neutral particle (which belongs to the adjoint representation of the star gauge group), in the vanishing \( \theta \) limit, the current associated with it also vanishes and the transport equation, (2.14) becomes trivial.

As an example of the convenience of this method in the evaluation of the hard thermal loop contributions, we will derive next the leading order contributions to the two- and three-point functions in the hard thermal loop approximation, using the transport equation. Let us expand the distribution function in a power series in \( g \)

\[ f = f^{(0)} + g f^{(1)} + g^2 f^{(2)} + \cdots, \]  

(2.18)

where \( f^{(0)} \) is the equilibrium Bose-Einstein distribution function

\[ f^{(0)}(k_0) = \frac{1}{e^{\frac{k_0}{T}} - 1}. \]  

(2.19)

On the other hand, \( f^{(1)} \) and \( f^{(2)} \) represent the leading order corrections to this function, which can be determined order by order from the transport equation, Eq. (2.14), to be

\[ f^{(1)}(x, k) = \frac{1}{k \cdot \partial} \left( k \cdot \partial A^\nu - \partial^\nu k \cdot A \right) \frac{\partial f^{(0)}}{\partial k^\nu}. \]  

(2.20)
and
\[
\begin{align*}
f^{(2)}(x, k) &= \frac{i}{k \cdot \partial} \left( [k \cdot A, f^{(1)}]_M - [k \cdot A, A^\nu]_M \frac{\partial f^{(0)}}{\partial k^\nu} - i(k \cdot \partial A^\nu - \partial^\nu k \cdot A) \ast \frac{\partial f^{(1)}}{\partial k^\nu} \right) \\
&= \frac{i}{k \cdot \partial} \left( [k \cdot A, f^{(1)}]_M - [k \cdot A, A^\nu]_M \frac{\partial f^{(0)}}{\partial k^\nu} + \cdots \right),
\end{align*}
\]
where \( \cdots \) represent the third term which we have neglected since it leads to a subleading contribution at high temperature.

Substituting these into the definition of the current in (2.18), we obtain the current as a series in powers of the gauge field. Transforming to the momentum space and using the definition in Eq. (2.5), we obtain the two point function as
\[
\Pi_{\mu \nu}(p) = \frac{\delta^2 \Gamma[A]}{\delta A^\mu(p) \delta A^\nu(-p)} = \frac{\delta J_\mu(p)}{\delta A^\nu(-p)} = \frac{2g^2}{(2\pi)^3} \int \frac{d^3k}{|k|} \left( \frac{k^\mu k^\nu}{k^2} - \frac{p_\mu k_\nu + p_\nu k_\mu}{k \cdot p} + \eta_{\mu \nu} \right),
\]
where \( k_\mu = |\vec{k}|(1, \hat{k}) \equiv |\vec{k}| \hat{k}_\mu \) denotes the on-shell 4-momenta of the thermal particle in the plasma. Here, the restriction on the functional derivatives stands for setting all the fields to zero after differentiation. We note that the expression inside the parenthesis in (2.22) is a homogeneous function of \( k_\mu \) of degree zero, so that it is actually independent of the magnitude \( |\vec{k}| \). This fact allows us to do the integration over \( |\vec{k}| \) in (2.22) leaving us with an angular integral, which at high temperatures has the form
\[
\Pi_{\mu \nu}(p) = -\frac{g^2 T^2}{24\pi} \int d\Omega \left( \frac{k^\mu k^\nu}{(k \cdot p)^2} - \frac{p_\mu \hat{k}_\nu + p_\nu \hat{k}_\mu}{k \cdot p} + \eta_{\mu \nu} \right) \equiv -\frac{g^2 T^2}{24\pi} \int d\Omega G_{\mu \nu}(p, \hat{k}),
\]
where \( d\Omega \) denotes integration over all angular directions of the unit vector \( \hat{k} \) and we have, for simplicity, defined
\[
G_{\mu \nu}(p, \hat{k}) = \left( \frac{k^\mu k^\nu}{(k \cdot p)^2} - \frac{p_\mu \hat{k}_\nu + p_\nu \hat{k}_\mu}{k \cdot p} + \eta_{\mu \nu} \right).
\]
As expected, the two point function is independent of the parameter of non-commutativity, \( \theta^{\mu \nu} \), and is manifestly transverse. Furthermore, this result agrees completely with the results from the explicit perturbative calculations in [12] for the case \( \theta pT \gg 1 \).

Proceeding along similar lines, we can also obtain the three-point function which, to leading order, has the form
\[
\Gamma^0_{\mu \nu \lambda} = \frac{\delta^2 J_\mu(p_1)}{\delta A^\nu(p_2) \delta A^\lambda(p_3 = -(p_1 + p_2))} = \sin \left( \frac{p_{1 \alpha} \theta^{\alpha \beta} p_{2 \beta}}{2} \right) \Gamma_{\mu \nu \lambda}(p_1, p_2, p_3)
\]
where \( \theta^{\alpha \beta} \) is the parameter of non-commutativity and
\[
\Gamma_{\mu \nu \lambda}(p_1, p_2, p_3) = -\frac{i g^3 T^2}{24\pi} \int d\Omega \frac{1}{k \cdot p_3} \left[ G_{\mu \nu}(p_2, \hat{k}) \hat{k}_\lambda + \hat{k}_\mu \hat{k}_\nu H(\nu, p_2, \hat{k}) - (p_1 \leftrightarrow p_2) \right].
\]
\( G_{\mu \nu} \) was already given in Eq. (2.24) and we have defined the vector
\[
H(\nu, p_2, \hat{k}) = \frac{1}{k \cdot p_2} \left( \frac{p_2 \cdot p_3}{k \cdot p_3} \hat{k}_\nu \hat{k}_\lambda - p_{2 \lambda} \right).
\]
As expected, the parameters \( \theta^{\mu \nu} \) are explicitly present in the three-point function reflecting the star product in the interaction terms. With the help of some simple algebraic identities, the expression (2.26) can also be written in a manifestly Bose symmetric form as follows
\[
\Gamma_{\mu \nu \lambda}(p_1, p_2, p_3) = \frac{i g^3 T^2}{24\pi} \int d\Omega \left[ \frac{1}{2} \left( \frac{p_1^2}{(k \cdot p_1)^2} - \frac{1}{k \cdot p_2} \right) \hat{k}_\mu \hat{k}_\nu \hat{k}_\lambda + \left( \frac{p_{1 \lambda}}{k \cdot p_1} - \frac{p_{2 \lambda}}{k \cdot p_2} \right) \hat{k}_\mu \hat{k}_\nu + \text{two cyclic permutations} \right]
\]
The above form for the three-point function is in complete agreement with the hard thermal loop results [12] obtained by standard Feynman diagrammatic calculations in the region \( \theta pT \gg 1 \). The expression (2.28) is the same as the one which occurs in the momentum dependent part of the corresponding quantity in commutative QCD (where the “sin” in (2.25) is replaced by the structure constants). Throughout this paper, we assume that none of the denominators like \( k \cdot p_i \) vanish for any value of the light like vector \( \hat{k} \) (so that the inverses \( 1/k \cdot p_i \) are well defined).
III. TRANSPORT EQUATION FOR THE NONCOMMUTATIVE $U(N)$ THEORY

In contrast to the $U(1)$ case, the derivation of the transport equation, in the case of noncommutative $U(N)$ theory, presents a few challenges some of which we discuss below. To appreciate the difficulties, let us recapitulate very briefly how the derivation of the transport equation in a conventional $SU(N)$ theory is carried out. In this case, the standard equations for the motion of a relativistic particle carrying a color charge and interacting with an external Yang-Mills field have the forms

$$m \frac{dx^\mu}{d\tau} = k^\mu; \quad m \frac{dk^\mu}{d\tau} = g Q^a F^{\mu\nu} a k_\nu; \quad a = 1, 2, \cdots, N^2 - 1,$$  

(3.1)

where $\tau$ is the proper time of the particle and $Q^a$'s are the non-Abelian color charges. To derive the transport equation, one needs to supplement these with the evolution equation for the color charges,

$$m \frac{dQ^a}{d\tau} = -gf^{abc}(k \cdot A^b)Q^c$$  

(3.2)

which can be obtained from a quantum field theory in a mean-field theoretic manner \cite{12} and is consistent with the constraint on the charges

$$Q^a Q^a = q_2 = \text{constant}$$  

(3.3)

where $q_2$ represents the second Casimir of the group $SU(N)$.

Together, Eqs. (3.1) and (3.2) are known as Wong’s equations \cite{20} and lead, in the commutative $SU(N)$ case, to the transport equation in a straightforward manner. In going to a noncommutative Yang-Mills theory, first of all, we have to generalize the symmetry group to $U(N)$ which is straightforward. Generalizing equations (3.1) also poses no particular problem. However, since in a noncommutative Yang-Mills theory only a few representations of the gauge group are allowed \cite{21}, a classical color charge is not conceptually meaningful (There are, of course, also difficulties associated with finding a sufficiently localized state to carry out a mean field calculation, but we will not go into these.). Even if one formally assumes the existence of such a classical charge, there are still difficulties associated with the measure in the color space in the following way. It is known that it is impossible to find local, gauge invariant observables in a noncommutative gauge theory. Then, a constraint such as (3.3) is no longer gauge invariant and, therefore, is not meaningful. As a result, the definition of the integration measure in the color space, in such a classical theory, is unclear at the present. Furthermore, in the presence of a constraint involving the third Casimir (which involves the symmetric structure constants of the group, as would be the case in the $U(N)$ theory), the surface terms arising from integration by parts are non-trivial and technically much harder to calculate.

Thus, we do not fully understand a first principle derivation of the transport equation, in the case of the noncommutative $U(N)$ gauge theory, starting from the analog of Wong’s equations and this question is presently under study. However, motivated by the discussions in the case of the $U(1)$ theory and taking guidance from the actual perturbative calculations in \cite{12}, we propose a transport equation for the noncommutative $U(N)$ theory which has the correct limiting behavior and reproduces the perturbative results of \cite{12}. To discuss the transport equation, let us generalize the conventional definition of the current to this case

$$J^C_\mu(x) = g \sum_a \int dK dQ \ Q^C k_\mu f(x,k,Q),$$  

(3.4)

where $f(x,k,Q)$ is the appropriate distribution function and we are assuming that the measure for the integration over the color charges is understood. Here, $C = 0, 1, 2, \cdots, N^2 - 1$ represents the $U(N)$ indices. We use the notations and conventions in \cite{22}. Similarly, the stress tensor for the particles can be written as

$$T^{\mu\nu}(x) = \sum_a \int dK dQ \ k^\mu k'^\nu f(x,k,Q).$$  

(3.5)

It is clear now that, for these to reduce to the current and the stress tensor studied in the last section for $N = 1$, both $f$ and $T^{\mu\nu}$ must transform under the adjoint representation of the $U(1)$ subgroup of the $U(N)$ group. Thus, we have

$$D_\mu f = \partial_\mu f - \frac{ig}{2} BCC (A^B_\mu f) = \partial_\mu f - \frac{ig}{2} BCC [A^B_\mu , f]_M$$  

(3.6)

so that the conservation equation for the stress tensor can be written as

$$D_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} - \frac{ig}{2} BCC [A^B_\mu , T^{\mu\nu}]_M = -F^{\mu\nu} B \ast J^B_\mu,$$  

(3.7)
Here, $d^{ABC}$ represent the completely symmetric structure constants of the gauge group $U(N)$ (see [24] for details). Note that the above form of (2.7) reduces to Eq. (2.17) when $N = 1$ and to the conventional equation in the commutative limit, $\theta \to 0$, when the Moyal bracket vanishes, in which case $T^{\mu \nu}$ is manifestly gauge invariant. The current, on the other hand, has to satisfy the standard covariant conservation law

$$D_\mu j^{\mu} = \partial_\mu j^{\mu} + \frac{ig}{2} d^{ABC} [A_\mu^B, j^{\mu} C]_M + \frac{g}{2} f^{ABC} \{ A_\mu^B, j^{\mu} C \}_M = 0 \quad (3.8)$$

where $f^{ABC}$ represent the anti-symmetric structure constants of $U(N)$ and we have defined $\{ A, B \}_M = A \star B + B \star A$.

Using the above properties of the current and the stress tensor, we propose the following collisionless transport equation for the noncommutative $U(N)$ theory, which generalizes Eq. (2.14).

$$(k \cdot D) f(x, k, Q) = g Q^A k^\mu f^{\mu \nu}_{A} + \frac{\partial f(x, k, Q)}{\partial k^\nu}, \quad (3.9)$$

where $D_\mu$ is given by

$$D_\mu f = \partial_\mu f + \frac{ig}{2} d^{ABC} Q^A \left[ A_\mu^B, \frac{\partial f}{\partial Q^C} \right]_M + \frac{g}{2} f^{ABC} Q^A \left\{ A_\mu^B, \frac{\partial f}{\partial Q^C} \right\}_M. \quad \quad (3.10)$$

We observe that the above expression reduces, in the limit $\theta \to 0$, to the expected classical transport equation for the conventional non-Abelian $SU(N)$ Yang-Mills theory. However, as we have mentioned, our main interest here is concerned with the hard thermal effects in the region $\theta p T \gg 1$.

An iterative solution of the transport equation (3.9) allows us to evaluate in an efficient way the hard thermal loop contributions. For example, in order to derive the high temperature behavior of the two and three-gluon functions, we find recursively the leading order corrections to the Bose-Einstein distribution $f^{(0)}$ which are given by

$$f^{(1)}(x, k, Q) = \frac{1}{k \cdot \partial} Q^B \left( k \cdot \partial A^B - \partial_\mu k \cdot A^B \right) \frac{\partial f^{(0)}}{\partial k^\mu} \quad (3.11)$$

$$f^{(2)}(x, k, Q) = \frac{i Q^A}{2 k \cdot \partial} \left\{ -d^{ABC} \left[ k \cdot A^B, \frac{\partial f^{(1)}}{\partial Q^C} \right]_M + i f^{ABC} \left\{ k \cdot A^B, \frac{\partial f^{(1)}}{\partial Q^C} \right\}_M \right. \right.$$

$$\left. - \left( d^{ABC} \left[ k \cdot A^B, A^C \right]_M + i f^{ABC} \left\{ k \cdot A^B, A^C \right\}_M \right) \frac{\partial f^{(0)}}{\partial k^\nu} - 2i(k \cdot \partial A^A - \partial^\nu k \cdot A^A) \star \frac{\partial f^{(1)}}{\partial k^\nu} \right\} \quad (3.12)$$

Once again, it can be checked that the last term in (3.13) leads to a subleading contribution at high temperature. So, we neglect this term. Proceeding as in the previous case, and using the normalization

$$\int dQ Q^A Q^B = N \delta^{AB}, \quad (3.13)$$

one arrives in a straightforward manner, at the following hard thermal amplitudes

$$\Pi^{AB}_{\mu \nu}(p) = N \delta^{AB} \Pi^{AB}_{\mu \nu}(p), \quad (3.14)$$

where the gluon self-energy $\Pi_{\mu \nu}$ is given in Eq. (2.23), and

$$\Gamma^{ABC}_{\mu \nu \lambda}(p_1, p_2, p_3) = N \left[ f^{ABC} \cos \left( \frac{p^\alpha_1 \theta^\alpha \beta_2 p^\beta_2}{2} \right) + d^{ABC} \sin \left( \frac{p^\alpha_1 \theta^\alpha \beta_2 p^\beta_2}{2} \right) \right] \Gamma^{\mu \nu \lambda}_{\mu \nu \lambda}(p_1, p_2, p_3) \quad (3.15)$$

where the three point function $\Gamma^{ABC}_{\mu \nu \lambda}(p_1, p_2, p_3)$ is given in Eq. (2.28). Note that the above results reduce to the ones in the $U(1)$ case, when we set $f^{ABC} = 0$ and $d^{ABC}$ equal to 1.

The above hard thermal amplitudes, which are proportional to $T^2$, are gauge invariant and satisfy simple Ward identities. For example, one can easily verify the transversality property

$$p^\mu \Pi^{AB}_{\mu \nu}(p) = 0, \quad (3.16)$$

as well as the Ward identity which relates the two and three point gluon functions

$$p^\lambda \Gamma^{ABC}_{\mu \nu \lambda}(p_1, p_2, p_3) = i g \left[ f^{ABE} \cos \left( \frac{p^\alpha_1 \theta^\alpha \beta_2 p^\beta_2}{2} \right) + d^{ABE} \sin \left( \frac{p^\alpha_1 \theta^\alpha \beta_2 p^\beta_2}{2} \right) \right] \left[ \Pi^{EC}_{\mu \nu}(p_1) - \Pi^{EC}_{\mu \nu}(p_2) \right] \quad (3.17)$$
IV. THE EFFECTIVE ACTION FOR HARD THERMAL LOOPS

In order to determine the effective action which generates the hard thermal amplitudes, we will first derive a manifestly covariantly conserved expression for the current. To this end, it is convenient to write the current in Eq. (4.4) as that the current (4.4) which satisfies the equation

where the only remaining integration to be performed is that over the angular directions of the unit vector \( \hat{k} \), namely

The quantity \( j^C_\mu(x, \hat{k}) \) may be evaluated from Eq. (3.4) as

where we have considered massless particles with two helicities. Using the transport equation (3.9), it may be verified that, as far as the leading thermal contributions are concerned, \( j^C_\mu(x, \hat{k}) \) effectively satisfies the constraint

where \( D \) is the Moyal covariant derivative given by Eq. (3.10). This relation may be solved for \( j^C_\mu(x, \hat{k}) \) and the result substituted into Eq. (4.2). Then, after performing an integration by parts and doing the \( |\hat{k}| \) and \( k_0 \) integrations, one arrives at the following form for the current

As we have seen, this property is crucial to ensure the gauge invariance of the effective action \( \Gamma [A] \) which generates the hard thermal amplitudes. From an examination of these amplitudes [see, for example, Eqs. (3.14-3.17)], one notices the following properties of the angular integrands:

(a) The non-localities, in configuration space, have the form of products of operators \((k \cdot \partial)^{-1}\).

(b) They have a Lorentz covariant structure, which involves homogeneous functions of \( k \) of degree zero.

(c) They are gauge invariant and satisfy simple Ward identities, similar to those of the tree amplitudes.

Using analogous arguments to those given in references [23][24][25], one can show that these properties, together with the results for the lowest order amplitudes, are sufficient to fix uniquely the effective action, which is expected to be

Here, \( \text{Tr} \) stands for the trace over color matrices as well as for the appropriate space-time integrations and the operator \( 1/k \cdot D \) may be represented perturbatively as a series of nested Moyal commutators. The above form is manifestly gauge invariant. A straightforward calculation shows that it generates the same two- and three-point functions as the one obtained from the current. Given the uniqueness of the action, this is sufficient to ensure that Eq. (4.6) should represent the correct generating functional of hard thermal loops in the noncommutative \( U(N) \) theory.

The result (4.6) for the effective action can also be obtained in a more direct way. To this end, we need to show that the current (4.4) which satisfies the equation

\[ J_\mu = t^C \frac{\delta \Gamma [A]}{\delta A^C_\mu} \]
is derivable from the action
\[
\Gamma[A] = \int \frac{d\Omega}{4\pi} \gamma(\hat{k}) \equiv \frac{g^2 T^2}{6} N \int \frac{d\Omega}{4\pi} Tr \left[ \left( \frac{k^\alpha}{k \cdot D} F_{\mu \alpha} \right) \ast \left( \frac{k^\beta}{k \cdot D} F_{\mu \beta} \right) \right]. \tag{4.8}
\]

In order to verify the above property, we note that under a variation \( \delta A_\mu \) of the gauge field, one must have the relation
\[
\delta \gamma(\hat{k}) = \text{Tr} \left[ j^\mu(x, \hat{k}) \ast \delta A_\mu(x) \right]. \tag{4.9}
\]

To check this equation, we take advantage of the fact that it is gauge invariant, and compute each side in a gauge in which
\[
k^\mu A_\mu = 0. \tag{4.10}
\]

Evaluating the left and right hand side of Eq. (4.9), respectively from the relations (4.8) and (4.4), we find that in this gauge both sides are equal to
\[
-\frac{g^2 T^2}{3} N \text{Tr} \left[ A_\mu - k^\mu \int_0^\theta du \partial^\nu A_\nu(x + k u) \right] \ast \delta A_\mu \tag{4.11}
\]

Since Eq. (4.9) is gauge invariant and holds in the gauge (4.10), it must be true in any gauge.

Thus, Eq. (4.6) gives the correct expression which describes the effective action for hard thermal loops in the strong noncommutative regime \( \theta p T \gg 1 \). On the other hand, in the limit \( \theta \to 0 \), the form (4.6) also reduces to the expected effective action in the commutative theory [1]. This result may be considered as a first step towards the construction of an effective action for the noncommutative Yang-Mills theory at finite temperature, which may be of interest for a further understanding of noncommutative QCD.

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