Group Quantization of Quadratic Hamiltonians in Finance

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Abstract
The Group Quantization formalism is a scheme for constructing a functional space that is an irreducible infinite dimensional representation of the Lie algebra belonging to a dynamical symmetry group. We apply this formalism to the construction of functional space and operators for quadratic potentials - gaussian pricing kernels in finance. We describe the Black-Scholes theory, the Ho-Lee interest rate model and the Euclidean repulsive and attractive oscillators.

The symmetry group used in this work has the structure of a principal bundle with base (dynamical) group a semi-direct extension of the Heisenberg-Weyl group by SL(2, R), and structure group (fiber) R⁺.

By using a R⁺ central extension, we obtain the appropriate commutator between the momentum and coordinate operators [\\hat{p}, \hat{x}] = 1 from the beginning, rather than the quantum-mechanical [\\hat{p}, \hat{x}] = -\i \hbar. The integral transformation between momentum and coordinate representations is the bilateral Laplace transform, an integral transform associated to the symmetry group.

Keywords
Black-Scholes – Lie Groups – Quadratic Hamiltonians – Central Extensions – Oscillator – Linear Canonical Transformations – Mellin Transform – Laplace Transform

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Introduction
The Group Quantization formalism ([1], [2], [3], [6]) is a scheme for constructing a functional space that is an irreducible infinite dimensional representation of the Lie algebra belonging to a dynamical symmetry group.

This formalism utilizes Cartan geometries in a framework similar to the Hamiltonian framework, where, in finance, the coordinates represent prices, rates ... and the conjugate momenta operators are the corresponding deltas.

We apply the Group Quantization formalism to a modified WSp(2, R) group. WSp(2, R) is the semi direct product of the two-dimensional real symplectic group 
\[ Sp(2, \mathbb{R}) \simeq SL(2, \mathbb{R}) \]
by the Heisenberg-Weyl group. Our modification consists in that the embedded 2-dimensional translation subgroup in the Heisenberg-Weyl group has been extended by \[ \mathbb{R}^+ \], rather than \[ U(1) \], which is the usual extension group in the physics and mathematical literature.

Our interest in the WSp(2, R) group (called Group of Inhomogeneous Linear Transformations in [32]) is that this group is the symmetry group of the second-order parabolic differential equations. Using the WSp(2, R) symmetry, one can find coordinate systems and operators that map the equations of motion and the corresponding solutions ([13], [32]).

The use in finance of mathematical methods previously de-

\[ ^1 \text{The infinite-dimensional representation theory of SL}(2, \mathbb{R}) \text{ was used by Bargmann ([40], [41]) for the description of the free non relativistic quantum mechanical particle.} \]
veloped in in physics is often based in the formal similarities between the Black-Scholes equation and the quantum mechanical Schrodinger equation. These formal similarities have been explored both from the point of view of Lie algebra invariance ([12], [53],[36], [17] ) and global symmetries of the Black-Scholes Hamiltonian operator ([20], [23],[22],[21]).

However, the mathematical properties of a wave equation solution such as the Schrodinger equation are totally different than the properties exhibited by a parabolic differential equation. Moreover, hermiticity and unitarity do not play a prominent role in finance: unlike the case of quantum mechanics, in finance there is no probabilistic interpretation of solutions of the pricing equation, and time evolution is irreversible and non-unitary.

The Group Quantization formalism makes the most of the principal fiber bundle structure linked to the central extension of a Lie group. Although this formalism shares some features with the Geometric Quantization scheme ([43], [44], [45]), contrary to Geometric Quantization, the Group Quantization formalism does not require the previous existence of a Poisson algebra. In both formalisms, the word quantization signifies irreducible representation.

As an example of the applicability of this formalism to finance, we will obtain the Black-Scholes theory, the Ho-Lee model and the Euclidean attractive and repulsive oscillators.

0.1 Outline

Section 1 shows that a Galilean transformation on the space of Black-Scholes solutions constitutes a numeraire change. Strict Galilean invariance plays the role of phase invariance in quantum mechanics.

We describe the main features of Group Quantization in section 2. Particularly important are the definition of the connection form, the polarization algebra and the concept of higher polarization. Group Quantization considers the action of the group on itself, as opposed to the group acting on an external manifold. This guarantees the existence of two sets of commuting generators, the right invariant fields and the left invariant fields. Left invariant fields provide naturally a set of polarization constraints that result in pricing equations, while the right invariant fields provide operators compatible with these constraints.

Section 3 gives a brief survey of the \(WSp(2, \mathbb{R})\) group, the \(SL(2, \mathbb{R})\) group and the Linear Canonical transformations. By using a \(\mathbb{R}^+\) central extension to create the embedded Heisenberg-Weyl group, Group Quantization provides the appropriate commutator between the momentum and coordinate operators \([\hat{p}, \hat{x}] = 1\), compatible with \(\hat{p}\) representing a delta, rather than the quantum-mechanical \([\hat{p}, \hat{x}] = -i\hbar\), without resorting to analogies with any quantum theory or rotating to a fictitious Euclidean time.

Section 4 applies the Group Quantization formalism to a parabolic \(SL(2, \mathbb{R})\) subgroup in order to construct the Black-Scholes theory. We obtain polarization constraints, operators and pricing Kernel both in the momentum space and in coordinate space. This Heisenberg-Weyl commutator makes the bilateral Laplace transform the integral transform mapping momentum and coordinate spaces. We give examples of pricing in momentum space in appendix B.

Sections 5 to 7 discuss the application of Group Quantization to theories with deformed Black-Scholes equations, i.e., Black-Scholes with a (at most quadratic) potential.

The Group Quantization of the linear potential, that in finance represents the Hee-Lo interest rate theory, is discussed in section 5. We obtain the polarized functions and pricing equations both in momentum space and in coordinate space. We compare some of our results with the similarity methods developed in [13] and [32].

The harmonic and repulsive oscillators are studied in sections 6 and 7. The Heisenberg-Weyl commutator \([\hat{p}, \hat{q}] = 1\) implies that the harmonic oscillator is generated by a hyperbolic \(SL(2, \mathbb{R})\) subgroup, a change of scale operator, not a rotation. We discuss in detail the polarization constraints in phase space and in a real analog of the Fock space. In spite of our non-standard \([\hat{p}, \hat{q}] = 1\), we recover the usual ladder operators when we construct higher order polarized functions. The Mehler kernel is obtained first by building the Kernel from the (Hermite) polarized functions, and later by applying the Linear Canonical Transformation technique described in [32].

From the standpoint of the Group Quantization formalism, the repulsive oscillator can be obtained from the harmonic and oscillator by making the frequency parameter pure imaginary. Both oscillators, which as an interest rate theory represent quadratic interest rates, have also been recently proposed as models for stock returns ([61], [33]).

Appendix A contains some general definitions in order to make this paper more self-contained and to establish notation. Appendix C shows examples of the Lagrangian formalism and the relationship of the connection form in the Group Quantization formalism and the Poincaré-Cartan form in classical mechanics.

We have favored clarity over mathematical rigor, and important topics such as group cohomology are just glossed over. For simplicity, we use coordinates as much as possible instead of a more compact notation. We refer the reader to the publications in the bibliography for a detailed and rigorous treatment of the mathematical concepts relevant to this article.
1. Galilean Transformations as Numeraire Change

As a motivation for this work, we show that the action of the Galilei group on the space of Black-Scholes solutions constitutes a numeraire change.

The Black-Scholes equation for a stock that pays no dividends is

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV$$

(1)

where $\sigma$ is the stock volatility, $r$ is the risk-free rate, $S$ is the stock price, $t$ is time, and $V = V(S,t)$ is the price of a financial instrument. For simplicity, let’s assume that $r$ and $\sigma$ are constant. Under the change of variables

$$S \to S' = S e^{v't}, \quad t \to t'$$

(2)

where $v'$ is constant, the Black-Scholes equation becomes

$$\frac{\partial V'}{\partial t'} = -\frac{1}{2}\sigma^2 S'^2 \frac{\partial^2 V'}{\partial S'^2} - (r + v')S' \frac{\partial V'}{\partial S'} + rV'$$

(3)

where $V' = V(S',t')$.

The transformations (5) can be expressed in log-stock coordinates as

$$x' = x + v't, \quad t' = t$$

(4)

were $x = \ln(S)$. We recognize the familiar Galilean transformations, with the position $x$ representing the logarithm of the stock price, and the velocity $v$ the stock’s growth rate. Given that the action of the Galilei group is linear in the log-stock coordinates, it is convenient to express the Black-Scholes equation (1) in terms of the logarithm of the stock price

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - \mu \frac{\partial V}{\partial x} + rV$$

with $\mu = r - \frac{1}{2}\sigma^2$. Equation (3) becomes

$$\frac{\partial V'}{\partial t'} = -\frac{1}{2}\sigma^2 x'^2 \frac{\partial^2 V'}{\partial x'^2} - (\mu + v')\frac{\partial V'}{\partial x'} + rV'$$

(5)

where $V' = V(x',t')$.

The Black-Scholes equation is only covariant under the action of the Galilei group. Strict Galilean invariance can be restored by multiplying the solution $V'$ by an exponential factor

$$\bar{V}(x',t') = e^{\varepsilon(x',t')} V'(x',t')$$

$$\varepsilon(x',t') = \frac{v'}{\sigma^2} \left( \frac{1}{2} v't' - (x' - \mu t') \right)$$

(6)

One has

$$\frac{\partial \bar{V}}{\partial t'} = -\frac{1}{2}\sigma^2 x'^2 \frac{\partial^2 \bar{V}}{\partial x'^2} - \mu \frac{\partial \bar{V}}{\partial x'} + r\bar{V}$$

Numeraire invariance is analogous to Phase Invariance in quantum mechanics. However, the relevant Black-Scholes gauge group is not the quantum mechanical $U(1)$, but rather the multiplicative positive real line $\mathbb{R}^+$. Exponential gauge factors (cocycles) analogous to one in equation (6) play a pivotal role in the Group Quantization formalism.

2. The Group Quantization Formalism

We describe the main features of the Group Quantization formalism (1), (2), (3), (6).

2.1 The Quantization Group

A Quantization Group $\tilde{G}$ is a connected Lie group $\tilde{G}$ which is a central extension of a Lie group $G$ (called the dynamical group) by another Lie group $U$ called the structural group.

The central extension defines a principal bundle $(G, \tilde{G}, \pi, U)$. The structure group acts on the fiber by left multiplication. $G$ is called base manifold of the bundle and $U$ is called the fiber. The projection $\pi$ is a continuous and surjective map $\pi : \tilde{G} \to G$.

A Quantization Group $\tilde{G}$ with structural group $U$ is a Cartan geometry. The Cartan geometry is the geometry of spaces that are locally (infinitesimally) like quotient spaces $G = G/U$. Although $G$ is not a subgroup of $\tilde{G}$, $\tilde{G}/U \simeq G$ as topological spaces any $\tilde{g} \in \tilde{G}$ can be decomposed in two parts, $\tilde{g} = (g, u), g \in G, u \in U$. This means that, locally, $\tilde{G}$ looks like the Cartesian product of $G$ and $U$.

In this paper, we will only consider Quantization Groups with structural group $U = \mathbb{R}^+$.  

Figure 1. One can visualize $\tilde{G}$ by imagining that through each point in the base manifold $G$ there is a line (fiber) of points with different values of the fiber coordinate $\zeta \in U$.

\footnote{A Lie group is a differential manifold $G$ endowed with a group composition law $F : G \times G \to G$ such as $F$ and its inverse are smooth, differentiable applications.}
2.2 Invariant Vector Fields

The Group Quantization method considers the action of the group on itself, as opposed to considering the group acting on an external manifold. This guarantees the existence of two sets of commuting group operators, the right invariant fields and the left invariant fields.

The left and right translations, $L_g$ and $R_g$ are defined as

$$
L_g : G \rightarrow G \quad / \quad L_g(g') = gg' \\
R_g : G \rightarrow G \quad / \quad R_g(g') = g'g
$$

$L_g$ and $R_g$ are diffeomorphisms of $G$. Note that $L_g$ and $R_g$ commute.

Let $g, g', g'' \in G$. In an abuse of notation, we write $g'' = g'g$ for the base group $G$ composition law. We denote an element $\tilde{g} \in \tilde{G}$ by $\tilde{g} = (g, \zeta)$, with $\zeta$ in the structural group $\mathbb{R}^+$. The composition law for $\tilde{g}$ is

$$
\tilde{g}'' = (g'', \zeta') = (g' \cdot g, \zeta' \cdot \exp(\epsilon(g', g)) )
$$

(7)

where $G \in G, \zeta \in \mathbb{R}^+$ and $\epsilon(g', g) \in \mathbb{R}$ is the extension cocycle.

A vector field $X$ is called a left invariant vector field (LIVF) if

$$
L_g X = X_{gg'}
$$

and a right invariant vector field (RIVF) if

$$
R_g X = X_{g'g}
$$

In local coordinates, the LIVF are given by

$$
X^L_i = \sum_k \eta_{ik} \frac{\partial}{\partial g_k} + \lambda_i \Xi
$$

with

$$
\Xi = \zeta \frac{\partial}{\partial \xi} \quad \lambda_i = \frac{\partial \epsilon(g', g)}{\partial g_i} \bigg|_{g'=g'=e}
$$

and the RIVFs

$$
X^R_i = \sum_k \eta_{ik} \frac{\partial}{\partial g_k} + \gamma_i \Xi
$$

where

$$
\Xi = \zeta \frac{\partial}{\partial \xi} \quad \gamma_i = \frac{\partial \epsilon(g', g)}{\partial g_i} \bigg|_{g'=g,e}
$$

The $i$-th component of a LIVF $X^L_i$ is calculated as the derivative respect to unprimed coordinates evaluated at the identity: first set $g' = g$ in the derivative, then set $g = e$. The $i$-th component of a LIVF $X^L_i$ is calculated as the derivative respect to primed coordinates evaluated at the identity: first set $g = g'$ in the derivative, then set $g' = e$.

Note that, by construction, RIVFs and RIVFs commute.

2.3 Cocycles

The function $\epsilon$ in equation (7) is called a cocycle. Cocycles are restricted by the group law properties.

Associativity of the group law implies that $\epsilon$ must satisfy the following functional equation

$$
\epsilon(g'', g') + \epsilon(g'' g, g) = \epsilon(g'', g' g) + \epsilon(g', g)
$$

and existence of an inverse element

$$
\epsilon(g, e) = \epsilon(e, g) = \epsilon(e, e) = 0
$$

where $e$ is the identity element of $G$.

2.4 Coboundaries

A coordinate change in the fiber $U$ generates a mathematically trivial cocycle. If the fiber elements $\zeta$ undergo the coordinate change $\zeta \rightarrow \zeta e^{(\xi)}$ in the group law (7), the extension cocycle $\epsilon$ gets an extra additive factor $\delta_\xi(f)$

$$
\epsilon(g, g') \rightarrow \epsilon(g, g') + \delta_\xi(f)(g, g')
$$

with

$$
\delta_\xi(f)(g, g') = f(gg') - f(g) - f(g')
$$

Trivial cocycles are called coboundaries. The coboundary $\delta_\xi(f)(g, g')$ can be undone by a change of coordinates.

Although group extensions whose cocycles differ in a coboundary mathematically define the same group representation, the dynamical effects generated by coboundaries are not necessarily trivial.

One useful analogy is a change of variables in a differential equation: differential equations have canonical forms to which they can be reduced, however it is often convenient to work in non-canonical coordinates. For instance, even if the Black-Scholes equation can be reduced to the simpler heat equation, in practice its solution is calculated using variables which allow to implement naturally the instrument’s pricing boundary conditions.

2.5 Connection. Vertical Form

A connection on $\tilde{G}$ is a smooth choice of horizontal subspace $H$ and vertical subspace $V$ such that $\tilde{G}$ can be decomposed as a direct sum, $\tilde{G} = H \oplus V$.

The Group Quantization formalism uses a Cartan connection, where the geodesics coincide with the flows of an integrable LIVF algebra. Concretely, the connection form in the Group...
Quantization formalism is the vertical part $\Theta$ of the Maurer-Cartan form\(^6\)

In this work, the vertical space is the structural group $U = \mathbb{R}^+$ with a unique generator $\Xi = \zeta \frac{\partial}{\partial \zeta}$ (see equation (7)).

$\Theta$ provides a natural definition of horizontality by requiring that the horizontal fields belong to the kernel of $\Theta$.

$$i_X \Theta = \Theta(X^L) = 0 \quad i_\Xi \Theta = \Theta(\Xi) = 1 \quad (8)$$

where $X^L$ stands for all LIVFs (Left Invariant Vector Fields). In coordinates, equations (8) read (see section 2.2)

$$\Theta = \sum_i \theta_i dg_i + d\Xi \equiv \frac{1}{\zeta} d\zeta$$

where the $\theta_i$ are solutions of a linear algebraic system

$$\sum_i \theta_i X_{ik}^L + \lambda_i = 0$$

2.6 Characteristic Module

In the Group Quantization formalism, the elements of the characteristic module $\mathcal{C}_\Theta$ are interpreted as evolution operators. $\mathcal{C}_\Theta$ is an integrable system analogous to Hamiltonian

$$d\varphi^i = -\frac{1}{2} \sum_{jk} c^i_{jk} \theta^j \wedge \theta^k.$$  

express the differential of the form in terms of the $c^i_{jk}$, the structure constants of the Lie algebra $\mathfrak{g}$.

fields in classical mechanics, and the integral flows of the $\mathcal{C}_\Theta$ elements are the geodesics of the Cartan connection. $\mathcal{C}_\Theta$ is defined as the set of LIVFs that leave the connection form (vertical form) $\Theta$ strictly invariant \(^7\)

$$X \in \mathcal{C}_\Theta \quad \Rightarrow \quad d(i_X \Theta) = 0 \quad i_X (d\Theta) = 0 \quad (9)$$

The curvature form $\omega = d\Theta$ can be written in local coordinates as

$$\omega = \sum_{ij} \frac{\partial \theta_i}{\partial g_j} dg_j \wedge dg_i = \sum_{ij} \left( \frac{\partial \theta_i}{\partial g_j} - \frac{\partial \theta_j}{\partial g_i} \right) dg_j \otimes dg_i$$

then, the second of equations (9) reads, using the results in section 2.5

$$\sum_j (\frac{\partial \theta_i}{\partial g_j} - \frac{\partial \theta_j}{\partial g_i}) X_j = 0$$

Note that from the Maurer-Cartan equations, $\omega(X,Y) = 0$ implies $\Theta([X,Y]) = 0$.

2.7 Polarization Algebra

The Group Quantization formalism constructs an irreducible representation of $\mathcal{G}$ starting from the ring $\mathcal{F}$ of functions with domain $\mathcal{G}$ and range $\mathbb{R}$.

The reduction is achieved by restricting the arguments of the functional space $\mathcal{F}$ using a set of horizontal operators and building a polarized function space $\mathcal{F}_\mathcal{P}$ that provides a representation of the group.

The action of vertical fields (generators of the structural group $U$) $X^V$ on $\mathcal{F}_\mathcal{P}$ is

$$X^V \Psi = f(X^V) \Psi \quad \forall \Psi \in \mathcal{F}_\mathcal{P}$$

where $f(X^V)$ is a function associated to the generator $X^V$. We say that $\mathcal{F}_\mathcal{P}$ is a $U$-invariant functional space. In this document, $U = \mathbb{R}^+$ and $f(X^V) = 1$.

We will see later in this document that, for the Black-Scholes and the Ho-Lee groups, polarized functions have the meaning of financial instrument prices.

2.7.1 First Order Polarization

The Group Quantization formalism provides a natural choice for polarization by requiring invariance respect to a horizontal algebra $\mathcal{P}$ (polarization algebra) that imposes constraints on $\mathcal{F}$.

\(^6\)The Maurer-Cartan form, $\Gamma : G \to \mathfrak{g}$ takes values in the Lie algebra of $\mathcal{G}$, $\mathfrak{g}$ and it is the unique left-invariant 1-form such that $\Gamma_{ie}$ is the identity map.

\(^7\)A vector field $X$ is a symmetry of a 1-form $\Gamma$ if $X$ leaves $\Gamma$ semi-invariant, that is, the Lie derivative of $\Gamma$ with respect to $X$ is a total differential

$$I_X \Gamma = df^X \quad \Rightarrow \quad d(I_X \Gamma) + i_X (d\Gamma) = df^X$$

where $f^X$ is some function associated with the vector field $X$. Then, $\mathcal{C}_\Theta = \ker \Theta \cap \ker d\Theta$. 

Figure 2. Transformation along the fiber $U$ are vertical, quantities defined on the base manifold $G \simeq \mathcal{G}$ are horizontal.
A first-order polarization (or just polarization) $\mathcal{P}$ is defined as a maximal horizontal commutative left invariant algebra containing the characteristic module $C_\Theta$, so that the constraints are compatible with the evolution operators. The polarized functions $\mathcal{F}_\mathcal{P}$ will be characterized by conditions of the form

$$\mathcal{F}_\mathcal{P} = \{ \Psi \in \mathcal{F} \mid X^P \Psi = 0 \ \forall X^P \in \mathcal{P} \}$$

Since the elements of the algebra $\mathcal{P}$ are integrable vector fields, a first order polarization defines a foliation of $\hat{G}$. This means that it is possible to select functional subspaces on on $\hat{G}$ by requiring them to be constant along integral leaves of the foliation.

### 2.7.2 Higher Order Polarization

When a first order polarization is not able to provide the required functional constraints, a higher order polarization can be used.

As opposed to first-order polarizations described in (2.7.1), higher-order polarizations [8] contain higher-order differential operators belonging to the left enveloping algebra. Higher order polarizations do not define a foliation of $\hat{G}$.

A higher-order polarization $\mathcal{P}_{\mathcal{F}}$ is a maximal subalgebra of the left-invariant enveloping algebra that has no intersection with the generators of the structural group $U$ and commutes with the first order polarization $\mathcal{P}$.

Commuting with the first order polarization ensures compatibility with the action of the dynamical operators (RIVFs). As we shall see in the next sections, higher order polarizations can be constructed by using a Casimir operator.

### 2.8 Operators

Commutativity of the right and left generators makes the RIVFs good candidates for quantum operators, that is, operators that can reduce the representation from phase space, with coordinates and momenta, to an irreducible group representation with only coordinates or momenta. Note that, since $\mathcal{P}$ is spanned by LIVFs, if $X^R$ is a right generator and $\Phi$ is polarized

$$X^P(X^R \Phi) = X^R(X^P \Phi) = 0 \ \forall X^P \in \mathcal{P}, \forall \Phi \in \mathcal{F}_\mathcal{P}$$

so the action of the RIVFs on the space of polarized functions is well defined. Hence the quantum operators are the restriction of the RIVFs to the polarized $U$-invariant functional space.

### 2.9 Noether’s theorem

One can easily verify that the connection $\Theta$ is right invariant, $\Theta(X^R) = 0$ for all RIVFs. The inner product of the vertical form $\Theta$ and the RIVFs gives the classical Hamiltonian constants of motion.

### 2.10 Lagrangian

The classical Lagrangian $\mathcal{L}(x, \dot{x})$ is obtained by the projection onto the base manifold $G$ of the connection form $\Theta$ along the $C_\Theta$ flows (trajectories). See appendix C for a discussion of the Poincaré-Cartan form of classical mechanics and the Lagrangian function.

## 3. The $\text{WSp}(2, \mathbb{R})$ Group

### 3.1 Heisenberg-Weyl Group

The Heisenberg-Weyl group $W$ is a central extension of the two dimensional Euclidean translation group. Its algebra is generated by three elements, $P$, $Q$ and $I$, where $I$ is the identity operator. The only non trivial commutator is

$$[P, Q] = \gamma I \quad \gamma \in \mathbb{C} \ (10)$$

It can be proven that the choices for $\gamma$ are equivalent to select $\gamma$ real or $\gamma$ pure imaginary.

Let $p, q \in \mathbb{R}$, $\theta \in \mathbb{C}$ and define

$$W(\theta, x, q) \equiv \exp(\theta I + xP + pQ)$$

The operators $W$ specify a group composition law under multiplication.

$$W''(p'', x'', \theta'') = W'(p', x', \theta') W(p, x, \theta)$$

Using the the Baker-Campbell-Hausdorff formula, we obtain

$$p'' = p + p'$$

$$x'' = x + x'$$

$$\theta'' = \theta + \theta' + \frac{1}{2} \gamma (px' - xp')$$

The group law (11) provides a coordinate representation of the abstract operators $P$, $Q$ and $I$ acting on the functional space.

### 8 The action of $W$ on $sl(2, \mathbb{R})$ is

$$W(aQ + bP + \eta I)W^{-1} = aQ + bP + (\eta + w(a p - b x)) I$$

therefore, $W$ acts on $\gamma$ as a three-dimensional space with Cartesian coordinates $(p, x, \theta)$.

### 9 Exponentiation and composition of operators are to be considered in the context of formal operator series.

### 10 The Baker-Campbell-Hausdorff formula reads

$$\ln(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - [Y, [X, Y]] + \ldots$$

when $[X, Y]$ is a number

$$e^{X+Y} = e^X e^Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \ldots$$

One can prove that

$$e^X e^{-X} = e^X ye^{-X} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \ldots \ (11)$$

These are right invariant generators, corresponding to the action of $W'$. 
We get
\[ \Psi(p, q, \theta) = \exp(\theta) \psi(p, q) \]
that provides the required Heisenberg-Weyl commutation relations
\[
P \mapsto X_p = \frac{\partial}{\partial p} - \frac{1}{2} \gamma x \frac{\partial}{\partial \theta}
Q \mapsto X_q = \frac{\partial}{\partial q} + \frac{1}{2} \gamma p \frac{\partial}{\partial \theta}
1 \mapsto X_\theta = \frac{\partial}{\partial \theta}
\]
We get
\[ [X_p, X_q] = \gamma X_\theta \]

### 3.2 The \( SL(2, \mathbb{R}) \) group

\( SL(2, \mathbb{R}) \), the special (unimodular) linear group in two real dimensions, has a natural representation by \( 2 \times 2 \) real matrices with determinant one
\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad ad - bc = 1
\]

The \( SL(2, \mathbb{R}) \) group is an automorphism of the \( W \) group, preserving the algebra commutators. Define new operators \( P' \) and \( Q' \) as linear combinations of \( P \) and \( Q \), using the matrix (13)
\[
P' = aP + bQ \\
Q' = cP + dQ
\]
then, by direct computation,
\[
[P', Q'] = [P', Q]
\]

\( SL(2, \mathbb{R}) \) belongs to a class of Lie groups called the symplectic groups \( Sp(2n, \mathbb{R}) \) which leave invariant a skew-symmetric form and play an important role in the geometry of phase space and Hamiltonian systems. One can verify the isomorphism \( SL(2, \mathbb{R}) \approx Sp(2, \mathbb{R}) \) for the two-dimensional symplectic matrix \( \Omega \)
\[
M^\top \Omega M = \Omega
\]

### 3.3 The \( WSp(2, \mathbb{R}) \) group

\( WSp(2, \mathbb{R}) \) is a real, non-compact, connected, simple Lie group that is the semi direct product of the two-dimensional real symplectic group \( Sp(2, \mathbb{R}) \approx SL(2, \mathbb{R}) \) by the Heisenberg-Weyl group \( W \).

The \( WSp(2, \mathbb{R}) \) group is a subgroup of the Schrodinger group in one dimension, and is called\(^{12} \) **Group of Inhomogeneous Linear Transformations** in reference [32].

Let an element \( g \in WSp(2, \mathbb{R}) \) be parametrized by \( g(M, u, \zeta) \) where \( M \in SL(2, \mathbb{R}) \), \( \zeta \in \mathbb{R}^+ \) and \( u \equiv (p, x) \in \mathbb{R}^2 \). The \( WSp(2, \mathbb{R}) \) composition law is
\[
g''(M'', u'', \zeta'') = g'(M', u', \zeta') g(M, u, \zeta)
\]
with
\[
M'' = M'M \\
u'' = u'M + u \\
\zeta'' = \zeta' \zeta \exp \left( - \frac{1}{2} \gamma u'M\Omega u^T \right)
\]
where \( \Omega \) is the two-dimensional symplectic matrix in (15) and \( \gamma \) is given by the \([P', Q] \) Heisenberg-Weyl commutator in (10).

#### 3.3.1 Weyl Commutation Relations

In the mathematics literature, such as [32] and [12], the usual convention is that the momentum and position operator commutator is \([P', Q] = -i\), which corresponds to an extension of the 2-dimensional translations by \( U(1) \). Then, for diffusive equations such as the heat equation, one makes the theory Euclidean at the last step by setting the time variable \( t \) to \( it \).

In this work we set \( \gamma = 1 \), corresponding to a central extension of the Euclidean translations by \( \mathbb{R}^+ \). The reason is that in finance we want \( P \) to represent a change (a delta) in the quantity \( Q \), which usually represents a price or a rate. This choice has the added advantage that there is no need to consider a passage to a fictitious Euclidean time, since the time variable in the group represents the actual calculation time.

When we quote results from to the mathematics literature (for instance in section 3.4) we will assume, unless otherwise indicated, that the results are obtained with the usual commutator \([P', Q] = -i\).

#### 3.3.2 Orbits and Subgroups

The \( WSp(2, \mathbb{R}) \) generates the dynamics for quadratic Hamiltonians (13), [12], [2], [32]) by the action of higher order operators on \( W \). The adjoint action of the group consist of six distinct orbits [32]. Representative of these orbits are given by the following operators, that include quadratic elements from the enveloping algebra of \( W \)
\[
P^2, P^2 + Q, P^2 + Q^2, P^2 - Q^2, P, 1
\]
We have the following isomorphisms: \( g(M, 0, 1) \approx SL(2, \mathbb{R}) \) and \( g(1, u, \zeta) \approx W \). The unit element is \( g(1, 0, 1) \) and the inverse element is \( g(M^{-1}, -uM^{-1}, 1/\zeta) \)

The different quantization groups considered in this paper will be distinguished subgroups of \( WSp(2, \mathbb{R}) \).

In physics, the \( WSp(2, \mathbb{R}) \) includes as subgroups the symmetry group of the free particle, the gravitational free-fall, as well as the symmetry group of the ordinary harmonic oscillator\(^ {13} \).

\(^{12}\) Our composition law differs from [32] in that the extension group in (16) is \( \mathbb{R}^+ \), not \( \mathbb{R} \).

\(^{13}\) In quantum mechanics the cocycle in (16) is multiplied by a factor \( 1/\hbar \) for dimensionality reasons, and \([P, Q] = -ih\).
and the repulsive harmonic oscillator (with imaginary frequency). In finance, we will obtain the Black-Scholes theory, the Ho-Lee model and the Euclidean attractive and repulsive oscillators.

3.4 Linear Canonical Transformations

\( WS(2, \mathbb{R}) \) acts on functional spaces as integral transforms\(^{14}\) called Linear Canonical Transformations, whose kernel is a \( SL(2, \mathbb{R}) \) matrix [32]

\[
g(M, u)(f(x)) = \int_{\mathbb{R}} W(M, x, x') f(x') dx' \tag{17}
\]

with kernel (using the notation of equation (13))

\[
W(M, x, x') = e^{-ix/4} \sqrt{2\pi b} \exp(i(ax^2 - 2xx' + dx^2)/(2b)) \tag{18}
\]

Linear Canonical Transformations (LICs), have the important property that composition of the transforms is equivalent to multiplication of their \( SL(2, \mathbb{R}) \) kernels ([2], [32]). LICs kernels can be analytically continued to \( SL(2, \mathbb{C}) \), subject to some restrictions [2]. In this work, we will use, when indicated, equation (17) with the mapping \( b \rightarrow -ib \)

\[
W(M, x, x') = \frac{1}{\sqrt{2\pi b}} \exp(-(ax^2 - 2xx' + dx^2)/(2b)) \tag{19}
\]

which has been modified from (18) because our choice for the Heisenberg-Weyl commutator \([P, Q] = 1\).

The Fourier transform, the Laplace transform, the Bargmann transform and the Mellin transform are examples of LICs.

3.4.1 Fourier Transform

The Fourier transform

\[
\mathcal{F}(f)(z) = \int_{-\infty}^{\infty} f(p) e^{ipz} dp
\]

has the \( SL(2, \mathbb{R}) \) kernel

\[
F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

3.4.2 Double Sided Laplace Transform

The double sided Laplace transform

\[
\mathcal{L}(f)(z) = \int_{-\infty}^{\infty} f(p) e^{pz} dp
\]

can be obtained from \( \mathcal{F} \) by the formal map \( p \rightarrow ip \). Its kernel is a \( SL(2, \mathbb{C}) \) matrix

\[
L = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]

In spite of formal similarities, existence of a Fourier transform exists does not automatically imply the existence of a Laplace transform [32].

The inverse transform is

\[
\mathcal{L}^{-1}(f)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{zp} f(p)
\]

The integration contour is a vertical line on the complex plane, such that all singularities of the integrand lie on the left of it.

3.4.3 Bargmann Transform

The Bargmann transform has the complex kernel

\[
B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}
\]

3.4.4 Mellin Transform

The Mellin transform \( \mathcal{M} \)

\[
\mathcal{M}(f)(z) = \int_{0}^{\infty} y^{-z-1} f(y) dy \tag{20}
\]

is obtained from the Laplace transform \( \mathcal{L} \) by the change of coordinates \( p \rightarrow -\ln(y) \).

The resolution of identity

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z} y^{s} ds = y\delta(x-y)
\]

leads to the Mellin inversion formula

\[
\mathcal{M}^{-1}(f)(y) = f(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{s} F(z) dz
\]

The Mellin transform has been more widely used in finance than the double sided Laplace transform ([55], [57], [59], [60]). The sign of the transform variable \( z \) in (20) differs from the usual definition in the mathematics literature in order to calculate transforms of positive powers of the stock price.
4. Black-Scholes

4.1 Quantization Group

We apply the Group Quantization formalism to the $WSp(2, \mathbb{R})$ subgroup generated by the parabolic $SL(2, \mathbb{R})$ matrix

$$M_{BS} = \begin{bmatrix} 1 & \sigma^2 t \\ 0 & 1 \end{bmatrix} \quad \sigma^2 \in \mathbb{R} \quad t \in \mathbb{R}$$

(21)

$M_{BS}$ has only one eigenvalue $\lambda = +1$ and it cannot be diagonalized. $M_{BS}$ is a shear transformation on $\mathbb{R}^2$: it leaves the upper plane invariant, while it displaces the lower plane by $\sigma^2 t$. This will be significant when finding a first order polarization in section 4.5.

Composition Law

The $WSp(2, \mathbb{R})$ subgroup generated by the matrix (21) constitutes the Black-Scholes quantization group $\tilde{B}$. Its composition law is obtained from the $WSp(2, \mathbb{R})$ composition law (16) and the matrix $M_{BS}$, with some cocycle modifications that we will explain below.\(^{15}\)

$$t'' = t + t'$$
$$p'' = p' + p$$
$$x'' = x' + x + \sigma^2 p' t$$
$$\zeta'' = \zeta' \zeta e^{\sigma_2 x'}, \ \ (22)$$

where $t, p, x \in \mathbb{R}$ and $\zeta \in \mathbb{R}^\uparrow$. When deriving the group law, we have used that

$$M'_{BS} = M'_{BS} M_{BS} \Rightarrow t'' = t + t'$$

The composition law for $(t, p, x)$ gives the familiar Galilean transformations: the Galilei group $G$ is the base group of $\tilde{B}$, $G \simeq \tilde{B}/\mathbb{R}^\uparrow$, with $\sigma^2 p$ corresponding to the Galilean boost.

Our interpretation of the group coordinates is that $x$ represents the (dimensionless) logarithm of the stock price, $\dot{S} \equiv S_0 e^x$, $p$ is the conjugate momentum of $x$, $t$ is the calendar time, and $\sigma$ is the stock volatility. Notice that interest rates are not explicitly present in the group law. Interest rates in the Black-Scholes theory are not dynamical quantities and they will be incorporated in section (4.4.2) as a coordinate change in the fiber $\zeta$.

The extension cocycle $\varepsilon_{BS}$ is the sum of the true cocycle\(^{16}\) $\varepsilon_G$ and the coboundary $\varepsilon_N$.

$$\varepsilon_{BS}(g, g') = \varepsilon_G(g, g') + \varepsilon_N(g, g')$$

For later convenience when defining the Laplace transform, we use as Galilean cocycle\(^{17}\)

$$\varepsilon_G(g, g') = p' x + \sigma^2 p' t + \frac{1}{2} \sigma^2 p'^2 t$$

(23)

$\varepsilon_N$ represents a numeraire choice\(^{18}\)

$$\varepsilon_N(g, g') \equiv \delta_t(f_N) = \mu p' t$$

(24)

and has the generating function (see section 2.4)

$$f_N(x) = \frac{\mu}{\sigma^2 x}$$

4.2 Stock Volatility

Bargmann ([40]. [41]) used a central extension of the Galileo group with $U(1)$ for the description of the free non relativistic quantum mechanical particle. The cocycle used by Bargmann was

$$\varepsilon_B(g, g') = -\frac{i}{2\hbar} m(x'v' - xv' + vv' t)$$

(25)

which is labeled by the particle mass $m$. Bargmann proved that two cocycles of the form (25) with different values of $m$ are not equivalent, they cannot be transformed into each other by a coboundary.

The cocycle (25) equals the $WSp(2, \mathbb{R})$ cocycle

$$\frac{1}{2} u'M_{BS} \Omega u^T = \frac{1}{2} (xp' - px' + \sigma^2 p'^2 t)$$

with the substitutions

$$\sigma^2 \rightarrow \frac{\hbar}{m} \quad p \rightarrow \frac{v}{\sigma^2}$$

The cohomological importance of mass in quantum mechanics has been extensively studied in the literature. The stock volatility plays a similar role in finance. Formally, the Black-Scholes theory is the quantum mechanics of a free particle with an imaginary mass. The classical limit $\hbar \rightarrow 0$ corresponds to the zero volatility limit.

\(^{15}\)With no modifications, the Group Quantization formalism leads to the heat equation.

\(^{16}\)All true Galilean cocycles with the same $\sigma$ differ in a coboundary from the Galilean cocycle (23), i.e., they can be undone by a change of coordinates. The exception occurs only for the one dimensional Galilean group. In one dimension there is another true cocycle which will be used in section 5.

\(^{17}\)\varepsilon_G equals the $WSp(2, \mathbb{R})$ cocycle in (16) plus the coboundary generated by $(px)/2$ (see section 2.4).

The $WSp(2, \mathbb{R})$ cocycle is

$$\varepsilon_B(g, g') = -\frac{1}{2} u'M_{BS} \Omega u^T = \frac{1}{2} (xp' - px' + \sigma^2 p'^2 t)$$

The coboundary generated by $(px)/2$ is

$$\delta_t(f_N) = \left( \frac{1}{2} (xp' + x + \sigma^2 p'^2 t) 

\right)$$

\(^{18}\)This coboundary will provide a constant first derivative coefficient in the Black-Scholes equation. It is possible to add a new group coordinate, much like a quantum gauge potential (see [7]), which allows the introduction of numeraires that depend on stock price and time. This approach will be explored in a future work.
4.3 Lie Algebra

Left Invariant Vector Fields

\[ X_p^L = \frac{\partial}{\partial p} + x \Xi \quad X_t^L = \frac{\partial}{\partial t} + \sigma^2 p \frac{\partial}{\partial x} + E(p) \Xi \]

\[ X_t^L \Xi = \Xi = \zeta \frac{\partial}{\partial \zeta} \]

Commutators

The non-zero commutators are

\[ [X_t^L, X_p^L] = -\sigma^2 X_t^L - \mu \Xi \quad [X_p^L, X_t^L] = -\Xi \] (26)

4.4 Connection

The expression for the vertical form \( \Theta \) and the curvature form \( d\Theta \) are (see section 2.5)

\[ \Theta = -xdp - E(p) dt + d\Xi \]

\[ d\Theta = dp \wedge dx - (\sigma^2 p + \mu) dp \wedge dt \] (27)

4.4.2 Interest Rates

One has

\[ f_t^L \frac{\partial}{\partial t} \rightarrow f_t^L \frac{\partial}{\partial t} - r \Xi \]

which induces the following change in the left time generator

\[ X_t^L \rightarrow X_t^L - r \Xi \]

Note that one obtains the same Lie algebra commutators (26) in the new coordinates. The RIVFs are not modified by this coordinate change.

In the rest of this section we use the expression of the vectors fields and connection in these new coordinates and we omit the notation \( \tilde{\zeta}, \tilde{\Theta}, \) etc. for brevity.

4.5 First Order Polarization

4.5.1 Black-Scholes Equation in Momentum Space

In section 4 we noted that \( M_{BS} \) is a shear mapping where only the \( p \) space (upper plane) is invariant, and that \( M_{BS} \) cannot be diagonalized. These features indicate that the only possible first order polarization is a polarization in \( p \)-space, as we will verify below.

The first order polarization algebra \( \mathcal{P} \) is spanned \(^{20}\) by the \( x \)-translations and the \( \mathcal{C}_{\Theta} \) generator \( X_C \)

\[ \mathcal{P} = \langle X_t^L, X_C \rangle \]

The polarized functions \( \Psi \in \mathcal{F}_{\mathcal{P}} \) are found by imposing the polarization constraints on the functional space \( \mathcal{F}_{\mathcal{P}} \)

\[ \mathcal{F}_{\mathcal{P}} = \{ \Psi : \tilde{\Theta} \rightarrow \mathbb{R} / \Psi(x,p,t,\zeta) = \zeta \Psi(x,p,t) \} \]

One has

\[ X_t^L \Psi = 0 \rightarrow \Psi = \zeta \psi(p,t) \]

\[ X_C \Psi = 0 \rightarrow \frac{\partial \psi}{\partial t} + \frac{1}{2} \sigma^2 p^2 \psi + \mu p \psi - r \psi = 0 \] (30a)

Equation (30a) is the Black-Scholes equation in momentum space.

\(^{20}\)\( X_p^L \) cannot belong to \( \mathcal{P} \) because

\[ [X_t^L, X_p^L] = -\sigma^2 X_t^L - \mu \Xi \]
4.5.2 Polarized Functions

A separable solution of the Black-Scholes equation (30a) is
\[ \psi(p,t) = e^{-E_r(p)t} \Phi(p) \]
with \( \Phi(p) \) an arbitrary function of \( p \) and
\[ E_r(p) = \frac{1}{2} \sigma^2 p^2 + \mu p - r \]
We can write a general polarized function as an inverse Laplace
transform (a Linear Canonical Integral transform)
\[ \Psi(\zeta,x,t) = \zeta \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(p,t) \exp(px) \Phi(p) dp \]  
with
\[ K(p,t) = \exp(-E_r(p)t) \]  
The integration measure \( dp \) is the dual form of \( X^L \), the vector
field absent from the polarization algebra. We will justify the
use of the inverse Laplace transform in section 4.8.

In the Black-Scholes theory, polarized functions represent
prices of financial derivatives, whereas the \( \Phi(p) \) correspond
to terminal (payoff) conditions. Appendix B shows how to
price financial instruments in momentum space.

4.6 High Order Polarization

4.6.1 Black-Scholes Equation in Coordinate Space

The Black Scholes equation can be obtained directly in the
coordinate representation by using a high-order polarization
(section 2.7.2).

The second order operator
\[ C_p = X^L_p + \mu X^L_p + \frac{1}{2} \sigma^2 X^L_p X^L_p \]
is a Casimir operator commuting with all Black-Scholes LIVFs.
Since \( X^L_p \) and \( X^L_p \) do not commute, \( C_p \) defines two higher
order polarizations. The \( x \)-space polarization is generated by
\( \{ C_p, X^L_p \} \)
\[ X^R_p \Psi = 0 \rightarrow \Psi = \zeta e^{-p x} \psi(x,t) \]
\[ X^L_p \Psi = 0 \rightarrow \frac{\partial \psi}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 \psi}{\partial x^2} + \mu \frac{\partial \psi}{\partial x} - r \psi = 0 \] (33a)
Equation (33a) is the Black-Scholes equation in coordinate
(price) space.

4.7 Operators in Momentum Space

4.7.1 Right Invariant Vector Fields

we have
\[ [X^R_p, X^R_p] = \sigma^2 X^R_p + \mu Z \quad [X^R_p, X^R_p] = Z \]
All other brackets are zero.\(^{21}\)
Irreducibility is achieved by considering the action of the right
invariant fields on the constants of motion \( \Phi(p) \) in (31). After
straightforward algebra, one finds
\[ X^R_p : \Phi(p) \rightarrow \frac{\partial}{\partial p} \Phi(p) \quad X^R_p : \Phi(p) \rightarrow p \Phi(p) \] (34)
and
\[ -X^R_p : \Phi(p) \rightarrow \left( \frac{1}{2} \sigma^2 p^2 + \mu p - r \right) \Phi(p) \]
From (34) the price operator, \( \hat{x} \), the momentum operator \( \hat{p} \) and
the time evolution operator (Hamiltonian) \( \hat{H} \) are given by
\[ \hat{x} = \frac{\partial}{\partial p} \quad \hat{p} = p \quad \hat{H} \equiv \frac{1}{2} \sigma^2 \hat{p}^2 + \mu \hat{p} - r \] (35)

4.8 Operators in Coordinate Space

The expressions (35) for the operators in momentum space
suggest the following definition for the coordinate operator, \( \hat{x} \)
and the (non hermitian) momentum, \( \hat{p} \) operator in \( x \)-space
\[ \hat{x} = x \quad \hat{p} = \frac{\partial}{\partial x} \quad [\hat{p}, \hat{x}] = 1 \] (36)
Equation (36) is a direct consequence of our extension of the
Heisenberg-Weyl subgroup by \( \mathbb{R}^+ \) (equation (11)). The 2-
dimensional translation subgroup of the \( \text{WSp}(2, \mathbb{R}) \) group acts
on real exponentials, thus justifying the definitions (36) and
the use for the inverse bilateral Laplace transform in (31). This
is an opposition to quantum mechanics, where the Heisenberg-
Weyl subgroup acts on the unit circle instead, making the
Fourier transform the mapping between the coordinate and
momentum spaces.

Using equations (36), the Black-Scholes Hamiltonian\(^{22}\) in
coordinate space is
\[ \hat{H}_{BS} = \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} - r \] (37)

\(^{21}\)The inner product of the vertical form \( \Theta \) and the RIVFs gives the
(Galilean) Hamiltonian constants of motion
\[ \Theta(X^p) = 0 \rightarrow p \equiv p_0 \]
\[ \Theta(X^R_p) = 0 \rightarrow \frac{1}{2} \sigma^2 p^2 + \mu p - r \equiv E_0 \]
\[ \Theta(X^R_p) = 0 \rightarrow x - (\sigma^2 p + \mu) r \equiv x_0 \]

\(^{22}\)The non hermiticity of the Black-Scholes Hamiltonian has been exten-
sively studied in the literature ([47], [48], [54]). Non hermiticity is relatively
mild, with eigenvalues either real or appearing in complex conjugate pairs.
4.8.1 Numeraire Coboundary Value

The value of the numeraire parameter \( \mu \) can be determined by requiring the stock price \( S \equiv e^\theta \) to be a zero eigenvalue\(^{23}\) of the Hamiltonian operator\(^{24}\).

Using the Black-Scholes Hamiltonian (37), we find

\[
\hat{H}_{BS} e^\theta = 0 \quad \Rightarrow \quad \mu = r - \frac{1}{2} \sigma^2
\]

4.9 Pricing Kernel

The kernel \( K_{BS}(x', \tau) \) for the Black-Scholes equation (33a)

\[
\Psi(t,x) = \int K_{BS}(x', \tau) \Psi(0, x') dx
\]

is obtained by an in inverse Laplace transform from the momentum representation in (32)

\[
K_{BS}(x', \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{-\epsilon p} e^{\rho (x-x')}
\]

The integral (38) does not exist for \( t > 0 \), in accordance to the fact that pricing problems in finance are time irreversible final value problems.

Let \( t < 0 \) and define \( \tau = -t \). Then the integral (38) exists and we recover the well known Gaussian pricing kernel

\[
K_{BS}(x', \tau) = e^{-\epsilon \tau} \frac{1}{\sqrt{2\pi \sigma^2 \tau}} e^{-\frac{1}{2\sigma^2 \tau} (x'-x-\mu \tau)^2}
\]

4.9.1 Derivation using LCTs

The Black-Scholes pricing kernel can also be obtained with the methods developed in references \([13]\) and \([32]\), using the properties of the \( WSp(2, \mathbb{R}) \) group and its representation as Linear Canonical Transformations.

From equation (19), the \( SL(2, \mathbb{R}) \) matrix (21) generates the LCT

\[
W(M_{BS}, x, x') = \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp(- (x'-x)^2 / (2 \sigma^2 t))
\]

which is the heat equation kernel (or Weierstrass transform \( W[] \)).

\( K_{BS}(x', \tau) \) is obtained by multiplying the heat kernel \( W(M_{BS}) \) by the discount factor \( \exp(-\tau \mu) \) and setting \( \tau = T - t \), \( x \rightarrow x + \mu \tau \). We note that \( x \rightarrow x + \mu \tau \) is the Galilean transformation (4) representing a numeraire change.

\( K_{BS} \) can also be written in terms of the pseudo-differential operators\(^{25}\) associated with \( W(M_{BS}) \)\(^{32}\)

\[
e^{\epsilon \tau \hat{H}_{BS}} f(x) = e^{-\tau \epsilon} e^{-\frac{1}{2} \sigma^2 \tau \frac{d^2}{dx^2} + \mu \tau \frac{d}{dx}} f(x) = \int_{-\infty}^{\infty} dx' K_{BS}(x, x', \tau) f(x')
\]

5. Linear Potential. Ho-Lee Model

5.1 Quantization Group

The \( SL(2, \mathbb{R}) \) parabolic shear mapping (21) that has been used in the Black-Scholes theory will be used in this section for constructing the linear potential model.

The composition law for the new quantization group \( \tilde{I} \) is

\[
t'' = t + t'
\]

\[
p'' = p + p'
\]

\[
x'' = x + x' + \sigma^2 p' t
\]

\[
\zeta'' = \zeta' \zeta e^{i \epsilon (x', 0)}
\]

where \( t, p, x \in \mathbb{R} \) and \( \zeta \in \mathbb{R}^+ \) with the cocycle \( \epsilon_{IR} \)

\[
\epsilon_{IR}(g, g') = \epsilon_{G}(g, g') + \epsilon_{V}(g, g') + \epsilon_{I}(g, g')
\]

\( \epsilon_{G}(g, g') \) is the Galilean cocycle in equation (23) and \( \epsilon_{V}(g, g') \) is the coboundary term given in (24). We have added a new true cocycle (not a coboundary) \( \epsilon_{\beta}(g, g') \)

\[
\epsilon_{\beta}(g, g') = \beta t (x' + \frac{1}{2} \sigma^2 p' t)
\]

\( \beta \in \mathbb{R} \)

We interpret the group coordinates\(^{26}\) as \( x \) representing a short rate, \( p \) its conjugate momentum, and \( t \) the calendar time. The parameter \( \sigma \) is the short rate volatility.

We will verify in the next sections that this group describes the Ho–Lee interest rate model. In quantum physics, the \( U(1) \) extension of this group describes the free fall (linear potential). The Group Quantization formalism applied to the linear potential in physics can be found in \([2]\).

\(^{23}\)This is actually a martingale condition \([24]\).

\(^{24}\)In momentum space, the zero eigenvalue condition reads

\[
\left( \frac{1}{2} \sigma^2 p' + \mu p - r \right) \Psi_0(p) = 0 \rightarrow \Psi_0(p) = \delta \left( \frac{1}{2} \sigma^2 p' + \mu p - r \right)
\]

It is straightforward to prove, using the properties of the Dirac delta and the inverse Laplace transform, that if we identify \( \Psi_0(x) \) with the stock price \( S \), \( \Psi_0(x) \sim \exp(x) \), this requires \( \mu = r - \frac{1}{2} \sigma^2 \)

\(^{25}\)For smooth functions

\[
e^{\epsilon \tau} f(x) = f(x+\alpha)
\]

Combining the result above with the Gaussian integral

\[
e^{\epsilon \tau} = \int_{-\infty}^{\infty} e^{\epsilon y} e^{-y^2/2} dy
\]

we get

\[
e^{\epsilon \tau} f(x) = \int_{-\infty}^{\infty} f(x-y) e^{-y^2} dy
\]

\(^{26}\)Identifying group parameters is usually done after analyzing the polarization, however in this case the identification is simple enough.
5.2 Lie Algebra

5.2.1 Left Invariant Vector Fields

From the composition law (41) and (42), we obtain

\[ X_\xi^L \equiv \Xi = \zeta \frac{\partial}{\partial \zeta} \]
\[ X_x^L = \frac{\partial}{\partial x} \]
\[ X_p^L = \frac{\partial}{\partial p} + x \Xi \]
\[ X_t^L = \frac{\partial}{\partial t} + \sigma^2 p \frac{\partial}{\partial x} + (E(p) + \beta x) \Xi \]

where

\[ E(p) \equiv \frac{1}{2} \sigma^2 p^2 + \mu p \]

with the non-zero Lie Brackets

\[ [X_\xi^L, X_x^L] = -\beta \Xi \]
\[ [X_\xi^L, X_p^L] = -\sigma^2 X_x^L - \mu \Xi \]
\[ [X_p^L, X_t^L] = -\Xi \]

Note that, as opposed to the Black-Scholes case (section 4.3), the time generator and the \( x \)-generator do not commute.

5.3 Connection

The vertical form \( \Theta \) and the curvature form \( d\Theta \) are given by

\[ \Theta = -x dp - E(p) dt - \beta x dt + d\Xi \]
\[ d\Theta = dp \wedge dx - (\sigma^2 p + \mu) dp \wedge dt - \beta dx \wedge dt \]

with

\[ d\Xi = \frac{1}{\zeta} d \zeta \]

5.3.1 Characteristic Module

\( \mathcal{G}_\Theta \) is spanned by a unique field \( X_C \)

\[ X_C = X_\xi^L + \mu X_x^L - \beta X_p^L = \]
\[ \frac{\partial}{\partial \zeta} - \beta \frac{\partial}{\partial p} + (p \sigma^2 + \mu) \frac{\partial}{\partial x} + \left( \frac{1}{2} \sigma^2 p^2 + \mu p \right) \Xi \]

5.4 First Order Polarization

5.4.1 Ho-Lee Equation in Momentum Space

As in the Black-Scholes case, there is only one first order \( p \)-space polarization \( \mathcal{P} \) spanned by the \( x \)-translations and the \( \mathcal{G}_\Theta \) generator \( X_C \).

The polarized functions \( \Psi \in \mathcal{F}_\mathcal{P} \) are found by imposing the polarization constraints on the functional space \( \mathcal{F}_\mathcal{P} \)

\[ \mathcal{F}_\mathcal{P} = \{ \Psi : \tilde{I} \rightarrow \mathbb{R} / \Psi(x,p,t,\zeta) = \zeta \Psi(p,q,t) \} \]

One has

\[ X_x^L \Psi = 0 \Rightarrow \Psi = \zeta \Psi(p,t) \]
\[ X_p^L \Psi = 0 \Rightarrow \]
\[ \frac{\partial \Psi}{\partial t} - \beta \frac{\partial \Psi}{\partial p} + \frac{1}{2} \sigma^2 p^2 \Psi + \mu p \Psi = 0 \] (44a)

Equation (44a) is the Ho-Lee Equation in Momentum Space. The general solution\(^{27} \) of (44a) is

\[ \Psi(p,t) = \Phi(p + \beta t) e^{\frac{1}{2} \mu p^2 + \beta \sigma^2 p^3} \]

where \( \Phi(p) \) is an arbitrary function of the momentum \( p \).

The bilaterial Laplace transform relates the \( p \)-space and the \( x \)-space. From equation (45), the general expression of a polarized function in \( x \)-space can be written as

\[ \Psi(\zeta,x,t) = \zeta e^{-\beta t} \int_{c-i\infty}^{c+i\infty} dp G(p,t) e^{p\zeta} \Phi(p) \]

where

\[ G(p,.t) \equiv e^{\frac{1}{2} \beta (p-\beta t)^2 + \mu \sigma^2 (p-\beta t)^3} \]

The integration contour is a vertical line on the complex plane, such that all singularities of the integrand lie on the left of it.

5.5 High Order Polarization.

5.5.1 Ho-Lee Equation in Coordinate Space

The pricing (evolution) equation can be found directly in \( x \)-space by using a high-order polarization (see section 2.7.2). Let \( X_C \) be the \( \mathcal{G}_\Theta \) generator. The second order operator

\[ X_p^L = X_C + \frac{1}{2} \sigma^2 X_x^L X_t^L \]

is a Casimir operator commuting with all LIVFs (equation 43). Since \( X_x^L \) and \( X_p^L \) do not commute, \( X_t^L \) defines two higher order polarizations, the \( p \)-space polarization \( \mathcal{P}_P = \{ X_p^L, X_x^L \} \), and the \( x \)-space polarization \( \mathcal{P}_x = \{ X_x^L, X_t^L \} \).

The \( x \)-space polarized functions are obtained by imposing the \( \mathcal{P}_x \) polarization constraints on functions (sections) of the form \( \Psi(\zeta,p,x,t) = \zeta \Psi(p,x,t) \)

\[ X_p^L \Psi = 0 \Rightarrow \Psi = \zeta V(x,t) \]
\[ X_t^L \Psi = 0 \Rightarrow \]
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} + \beta x V = 0 \] (46a)

Equation (46a) is the Ho-Lee equation in coordinate space. If we write

\[ V(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp \Phi(p,t) e^{px} \]

and substitute in (46a), we recover the first order polarization equations (44a).

\(^{27}\text{We will show in section 5.5 that the polarized functions can be expressed in the } x \text{-space using Airy functions. This can be anticipated by the identity} \]
\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{p(x+ip^3)} = \frac{1}{3!} A_i \left( \frac{x}{3!} \right)^3 \]
\[ c > 0 \]

The solutions that are not proportional to \( A_i \) grow very rapidly at infinity and they need to be constructed with different contours in the complex plane.
Using the Feynman-Kac\textsuperscript{28} theorem, the solution of (46a) can be written as
\[ V(x,t) = E(e^{-\beta \int_{0}^{t} X(s) ds} \phi(X_{T}) | X(t) = x) \]
where the expectation is taken with respect to a normal process with volatility \( \sigma \) and drift \( \mu \) (\( W \) is a Brownian motion)
\[ dX = \mu dt + \sigma dW \]

### 5.6 Airy Expansion

After performing the following change of function and independent variable
\[ V(x,t) \equiv e^{-\frac{\mu}{2\sigma^2} x} e^{\lambda t} U(y) \]
where
\[ y \equiv \left( \frac{2}{\sigma^2} \right)^{\frac{1}{2}} \left( \frac{\mu}{2\sigma^2} - \lambda - \beta x \right) \]
the polarization equation (46a) becomes the Airy equation
\[ \frac{\partial^2 U}{\partial y^2} - y U(y) = 0 \quad (47) \]
Therefore, the polarized functions can then be written as a superposition of Airy functions\textsuperscript{29}
\[ V(x,t) = e^{-\frac{\mu}{2\sigma^2} x} \sum_{i=0}^{\infty} e^{\lambda i} (a_i A_i(y_i) + b_i B_i(y_i)) \]
where \( a_i, b_i, \lambda_i \in \mathbb{R} \)
\[ y_i \equiv \left( \frac{2}{\sigma^2} \right)^{\frac{1}{2}} \left( \frac{\mu}{2\sigma^2} - \lambda - \beta x \right) \]
This form is convenient for the analysis of complex boundary conditions in bond pricing problems, as shown in [38].

\textsuperscript{28}Consider the differential operator \( L \)
\[ L \equiv \frac{1}{2} \sigma^2 (x,t) \frac{\partial^2}{\partial x^2} + \mu(x,t) \frac{\partial}{\partial x} \]
where \( \mu(x,t), \sigma(x,t) \) and \( r(x,t) \) are functions of \( (x,t) \), with \( x \) defined in a real domain \( D \subset \mathbb{R} \) and \( t \) a positive real number, \( t \in [0, T] \) Then, subject to technical conditions, the unique solution of the PDE
\[ \left( \frac{\partial}{\partial t} + L + r(x,t)) f(x,t) = 0 \quad x \in D, 0 \leq t \leq T \]
with terminal value \( f(x,T) = g(x) \), is given by the Feynman-Kac Formula
\[ f(x,t) = \mathbb{E}[e^{-\int_{0}^{t} r(X(s)) ds} g(X_{T}) | X(t) = x] \]
where the expectation is taken with respect to the transition density induced by the SDE
\[ dX = \mu(X,t) dt + \sigma(X,t) dW \]
with \( W \) the Brownian motion.

\textsuperscript{29}There is a distinguished solution of equation (47), called \( A_i \), that decays rapidly as \( y \to \pm \infty \), while a second linearly independent solution \( B_i \) grows rapidly in this limit. Also, like Bessel functions, both \( A_i \) and \( B_i \) are oscillatory with a slow decay for large values of their arguments. See reference [16].

boundary conditions constrain the values for the expansion coefficients \( a_i, b_i \) and \( \lambda_i \).

Note that the group contraction \( \beta \to 0 \) is smooth and the group law (41) reduces to the Black-Scholes group law, however \( \beta = 0 \) is a singular point in the integral for the momentum polarized functions, (45) and the polarization equation (44a).

### 5.7 Pricing Kernel

#### 5.7.1 Similarity Methods

One can use the methods developed in [13] and [32] in order to find coordinate changes mapping Black-Scholes solutions into solutions of the equation (46a)

For instance, it can be shown that if \( V_{BS}(x,t) \) a solution of the Black-Scholes equation (33a) with \( r = 0 \), then
\[ V(x,t) = \Omega_{\beta}(x,t) V_{BS}(x - \frac{1}{2} \beta \sigma^2 t^2, t) \]
with
\[ \Omega_{\beta}(x,t) = e^{\beta x + \frac{1}{2} \beta^2 \sigma^2 t^2 + \frac{1}{2} \mu \beta t^2} \quad (48) \]
is a solution of (46a). Hence, the Ho-Lee pricing Kernel can be obtained from the Black-Scholes pricing Kernel (39) (with \( r = 0 \)) in an analogous manner
\[ K_{\lambda}(x,x', \tau) = \Omega_{\beta}(x, \tau) K_{BS}(x - \frac{1}{2} \beta \sigma^2 t^2, x', \tau) \quad (49) \]
where we have switched to the variable \( \tau = T - t \).

Notice that \( \Omega_{\beta}(x, \tau) \) is a solution of (46a). In fact, \( \Omega_{1}(x, \tau) \) is the expression for the Ho-Lee bond maturing at \( T \)
\[ \Omega_{1}(x, \tau) = e^{-\frac{\beta}{2}((T-t) - \frac{1}{2} \beta^2 (T-t)^2 + \frac{1}{2} \beta (T-t)^2} \quad t \in [0, T] \]

#### 5.7.2 Pseudo-Differential Operators

The pseudo-differential operator methods in section 4.9 can also be used to compute the evolution operator \( U \).

From equation (46a).
\[ U(\tau) = \exp(\tau H_{I}) \]
\[ H_{1} \equiv \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} + \beta x \]
where \( \tau = -t \). We split \( H_{1} \) into two operators
\[ A \equiv \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \]
\[ B = \beta x \]

Given that the only non zero commutators are
\[ [A, B] = \beta \sigma^2 \frac{\partial}{\partial x} + \mu \beta \]
\[ [B, [A, B]] = -\beta^2 \sigma^2 \]
it is possible to apply the left oriented extended version of the Baker-Campbell-Hausdorff formula [29]
\[ e^{\tau A + \tau B} = e^{\frac{1}{2} \tau^2 (2[B, [A, B]] + [A, [B, A]])} e^{\frac{1}{2} \tau^2 [A, B]} e^{\tau A} \]
We obtain
\[ e^{\mathcal{H}t} = e^{-\frac{1}{2} \beta^2 \sigma^2 \tau^2} e^{\frac{1}{2} \beta \sigma \tau^2} \]
\[ \times e^{\frac{1}{2} \beta \sigma \tau^2} e^{\frac{1}{2} \beta \sigma \tau} e^{\frac{1}{2} \sigma^2 \tau^2} e^{\frac{1}{2} \sigma^2 \tau} e^{\frac{1}{2} \sigma^2 \tau^2} e^{\frac{1}{2} \sigma^2 \tau} e^{\frac{1}{2} \sigma^2 \tau^2} e^{\frac{1}{2} \sigma^2 \tau} \]
\[ = \Omega_\beta(x, \tau) e^{\frac{1}{2} \beta \sigma \tau^2} e^{\frac{1}{2} \sigma^2 \tau^2} e^{\frac{1}{2} \sigma^2 \tau} e^{\frac{1}{2} \sigma^2 \tau^2} e^{\frac{1}{2} \sigma^2 \tau} e^{\frac{1}{2} \sigma^2 \tau^2} e^{\frac{1}{2} \sigma^2 \tau} e^{\frac{1}{2} \sigma^2 \tau^2} e^{\frac{1}{2} \sigma^2 \tau} \]
(50)
where \( \Omega_\beta(x, \tau) \) has been defined in (48). We apply the operators in (50) from right to left, and use the Weierstrass theorem to recover equation (49)
\[ K_f(x', x) = \Omega(x, \tau) K_{BS}(x - \frac{1}{2} \beta \sigma^2 \tau^2, x', \tau) \]

### 6. Harmonic Oscillator

As described in section 3.1, our version of the Heisenberg-Weyl group consists in the 2-dimensional translations centrally extended by \( \mathbb{R}^+ \), such that the commutator between the momentum and coordinate operators is \([\hat{p}, \hat{x}] = 1\).

We will verify in this section that in our modified version of the Heisenberg-Euclidean plane, whereas (52) corresponds to a rotation. Both SL is pure imaginary, \( \mathcal{G} \) and commutator between the momentum and coordinate operators is \( [\hat{p}, \hat{x}] = 1, \) and the harmonic oscillator quantization group \( \mathcal{H} \)
\[ M_\mathcal{H} = \begin{pmatrix} \cosh \omega t & \lambda^{-2} \sinh \omega t \\ \lambda^2 \sinh \omega t & \cosh \omega t \end{pmatrix} \]
\[ \lambda = \frac{\sqrt{\omega}}{\sigma} \]
(51)
where \( \omega, \sigma, t \in \mathbb{R} \)

By contrast, in quantum mechanics ([3], [5]), where the 2-dimensional translations are centrally extended by \( U(1) \), the commutator between the momentum and coordinate operators is pure imaginary, \([\hat{p}, \hat{x}] = -i \hbar \), and the harmonic oscillator quantization group is generated by an elliptic \( SL(2, \mathbb{R}) \) subgroup
\[ \begin{pmatrix} \cos \omega t & \lambda^{-2} \sin \omega t \\ -\lambda^2 \sin \omega t & \cos \omega t \end{pmatrix} \]
(52)
The \( SL(2, \mathbb{R}) \) matrix (51) acts as a squeeze mapping of the Euclidean plane, whereas (52) corresponds to a rotation. Both subgroups can be transformed into each other by the taphi mapping \( \omega \rightarrow i \omega \). Notice that the mapping between subgroups is not the Euclidean time rotation \( t \rightarrow it \).

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6.0.1 Group Composition Law

Using equation (16), \( M_\mathcal{H} \) generates the following composition law:
\[ t'' = t' + t \]
\[ p'' = p + p' \cosh \omega t + \lambda^2 x' \sinh \omega t \]
\[ x'' = x + x' \cosh \omega t + \lambda^{-2} p' \sinh \omega t \]
\[ \zeta'' = \zeta' \zeta \exp(\epsilon_\mathcal{H}(g', g)) \]
(53)
with the cocycle \( \epsilon_\mathcal{H} \)
\[ \epsilon_\mathcal{H}(g, g') = \frac{1}{2} (p g' - x g') \cosh \omega t + \frac{1}{2} (\lambda^{-2} p g' - x g') \sinh \omega t \]
The matrix \( M_\mathcal{H} \) reduces to the Black-Scholes generating matrix \( M_{BS} \) in the limit \( w \rightarrow 0 \), and correspondingly, the quantization group \( \mathcal{H} \) contracts to the Black-Scholes quantization group \( \mathcal{G} \) (modulo coboundaries) in this limit. For brevity, we have omitted the numeraire coboundary \( (31) (24) \) in the composition law (53).

6.1 Polarization using Orthogonal Coordinates

The \( SL(2, \mathbb{R}) \) matrix \( R \)
\[ R = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda & -1/\lambda \\ \lambda & 1/\lambda \end{pmatrix} \]
transforms \( M_\mathcal{H} \) into a change of scale (squeeze) operator
\[ D_\mathcal{H} = R^{-1} M_\mathcal{H} R = \begin{pmatrix} \epsilon_\omega & 0 \\ 0 & e^{-\omega t} \end{pmatrix} \]
The change of coordinates
\[ A = \frac{1}{\sqrt{2}} \left( \lambda p - \lambda x \right) \]
\[ B = \frac{1}{\sqrt{2}} \left( \lambda p + \lambda x \right) \]
(55)
splits the phase space \((p, x)\) into orthogonal subspaces. The group law (53) is written as
\[ t'' = t' + t \]
\[ A'' = A + A' e^{-\omega t} \]
\[ B'' = B + B' e^{-\omega t} \]
\[ \zeta'' = \zeta' \zeta e^{\epsilon_\mathcal{H}(g', g)} \]
with
\[ \epsilon_\mathcal{H}(g, g') = \frac{1}{2} (B' A e^{\omega t} - A B' e^{-\omega t}) \]

---

\[ ^{31} \text{The generating function for this coboundary is } \mu x, \text{ and the final expression is } \mu (\lambda^{-2} p' \sinh \omega t + x' \cosh \omega t - x'), \text{ with } \mu \text{ real.} \]
We choose the first order polarization algebra spanned by

\[ X^L_t = \frac{\partial}{\partial t} - \omega A \frac{\partial}{\partial A} + \omega B \frac{\partial}{\partial B} \]
\[ X^L_A = \frac{\partial}{\partial A} + \frac{1}{2} B \Xi \quad X^L_B = \frac{\partial}{\partial B} - \frac{1}{2} A \Xi \]

with Lie Brackets

\[ [X^L_t, X^L_A] = w X^L_t \quad [X^L_t, X^L_B] = -w X^L_B \]
\[ [X^L_A, X^L_B] = -\Xi \]

Connection

The vertical form \( \Theta \) and the curvature form \( d\Theta \) are given by

\[ \Theta = \frac{1}{2} (A dB - B dA) - w A B dt + d \Xi \]
\[ d\Theta = dA \wedge dB - w A dB \wedge dt - w B dA \wedge dt \]

\( \mathcal{C}_\Theta \) is spanned by the time generator \( X^L_t \).

First Order Polarization

We choose the first order polarization algebra spanned by

\[ \mathcal{P} = \{ X^L_B, X^L_t \} \]

The polarized functions \( \Psi \equiv \zeta \Psi(A, B, t) \) are found by imposing

\[ X^L_B \Psi = 0 \rightarrow \Psi = \zeta e^{\frac{d\zeta}{\lambda \Xi}} \Psi(A, t) \]
\[ X^L_t \Psi = 0 \rightarrow \frac{\partial \Psi}{\partial t} - \omega A \frac{\partial \Psi}{\partial A} = 0 \]

(56a)

The readers familiar with quantum mechanics will find equation (56a) formally similar to quantum harmonic oscillator coherent state equations in Fock space ([3], [5]), except that here the coordinate \( A \) is real, not complex.

We give the solution of these polarization constraints in equation (62). Equation (56a) coincides with equation (61) obtained in the next section 6.2.

6.2 First Polarization in Phase Space

We use now the original phase space coordinates \((p, q)\) in the composition law (53).

Since the generating matrix \( M \) does not leave invariant either the \( p \) or the \( x \) space, a first order polarization cannot map the phase space \((p, x)\) into either the momentum or the coordinate space. However, the results in this section will be illustrative of a phase space formulation for the harmonic oscillator and will also be useful for finding a higher order polarization.
6.3 First Order Polarized Functions

Using the separable solution of (61)

\[ e^{\alpha t} A^\dagger e^{-\alpha t} \quad n \in \mathbb{N}^+ \]

the polarized functions are expressed as an infinite series

\[ \Psi(\zeta, p, x, t) = \zeta \sum_{n=0}^{\infty} e^{\alpha n t} a_n A^n e^{\frac{1}{2} A B} \tag{62} \]

where \( a_n \in \mathbb{R} \) are the expansion coefficients.

First order polarization in non orthogonal \((p, q)\) coordinates leads naturally to a financial theory in phase space, with

\[ F(p, q) = e^{\frac{1}{2} A B} = e^{\frac{1}{4} (\lambda^{-2} p^2 - \lambda^2 x^2)} \]

the analog of a Husimi quasi-probability. The expression for the quantum harmonic coherent states in the Fock basis and the Bargmann-Segal transform can be obtained\(^{32} \) from (62) by mapping the phase space \((p, q)\) into \( \mathbb{C} \) by the simple correspondence \( p \rightarrow i p \).

6.4 High Order Polarization

6.4.1 Harmonic Oscillator Equation in Coordinate Space

The pricing equation can be obtained directly in \( x \)-space by using a high-order polarization.

The second order operator

\[ X_p = X_t + \frac{1}{2} \omega \lambda^{-2} X_p^L X_t + \frac{1}{2} \omega \lambda^2 X_t X_p^L \]

is a Casimir operator commuting with all \( \lambda \)-LIVFs. Since \( X_t^L \) and \( X_p^L \) do not commute, \( X_p \) defines two higher order polarizations, \( X_p, X_t^L \) and \( X_t, X_p^L \). The coordinate space representation is generated by \( \{X_p, X_t^L\} \). The coordinate space

\[ X_p^L \psi = 0 \rightarrow \Psi(\zeta, p, q, t) = \zeta e^{-\zeta t^2/2} \psi(x, t) \]

\[ X_p \Psi(\zeta, p, q, t) \rightarrow \frac{\partial \Psi}{\partial t} + \frac{1}{2} \omega \left( \lambda^{-2} \frac{\partial^2 \Psi}{\partial x^2} - \lambda^2 \chi^2 \Psi \right) = 0 \tag{63a} \]

Since in finance one is usually interested in final value problems, we make the change \( \tau = T + t \), where \( T \) is a maturity time and \( t \leq T \). Equation (63a) becomes

\[ \frac{\partial \Psi}{\partial \tau} = H_I \Psi \tag{64} \]

wit \( H_I \) the harmonic oscillator Hamiltonian in coordinate space

\[ H_I = \frac{1}{2} \omega \left( \lambda^{-2} \frac{\partial^2 }{\partial x^2} - \lambda^2 \chi^2 \right) \tag{65} \]

Note that \( a, a^\dagger \) are the regular ladder operators that are used for the construction of the harmonic oscillator Fock space in quantum mechanics. The Heisenberg-Weyl commutation relations \( [\hat{p}, \hat{x}] = 1 \) imply the commutator \([a, a^\dagger] = 1 \) since \( SL(2, \mathbb{R}) \) is an automorphism of the Heisenberg-Weyl subgroup (equation(14)).

6.4.2 Polarized Functions in Coordinate Space

It is well known that the solution of equation (64) can be represented as a series in Hermite functions \( \varphi_n \)

\[ \psi(\tau, x) = \sum_{n=0}^{\infty} e^{-\sigma(n + \frac{1}{2})} \sigma_n \varphi_n(\lambda x) \tag{66} \]

with \( \alpha_n \in \mathbb{R} \) the expansion coefficients. The Hermite functions \( \varphi_n \) have the following expression(16)

\[ \varphi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{1}{2} x^2} \tag{67} \]

where \( H_n \) are the Hermite polynomials. \( \varphi_n \) form an orthonormal basis in \( L^2(\mathbb{R}) \) and fulfill the eigenvalue condition

\[ \left( \frac{\partial^2 }{\partial x^2} - \chi^2 \right) \varphi_n(x) = -(2n + 1) \varphi_n(x) \]

The relationship between the (first) polarized functions (62) and the polarized functions (66) is given by a modified Bargmann transform

\[ \varphi_n(\lambda x) = \frac{1}{\sqrt{n! \sqrt{\pi}}} \times \frac{1}{2 \pi i} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx A^\dagger e^{-B^2/2 + \sqrt{2} \lambda \chi x - \lambda^2 x^2 / 2} e^{-A B} \]

where \( A = A(p, q) \) and \( B = B(p, q) \) are the variables defined in (55). The integral over the momentum \( p \) is an inverse double sided Laplace transform.
6.5 Pricing Kernel

The orthogonality of the expansion (66) makes the pricing kernel $K_f(x, x', \tau)$, defined by

$$\Psi(\tau, x) = \int K_f(x, x', \tau) \Psi(0, x')dx$$

as a sum of products of Hermite polynomials in $x$ and $x'$

$$K_f(x, x', \tau) = \lambda e^{-\frac{1}{2}a^2 \tau} \sum_{n=0}^{\infty} \left(\frac{e^{-\omega^2 \tau}}{2^n n!}\right)^n \times H_n(\lambda x) H_n(\lambda x') e^{\frac{1}{2} \lambda^2 (x^2 + x'^2)}$$

(68)

By applying the Mehler formula ([39])

$$\sum_{n=0}^{\infty} \frac{\rho^n}{2^n n!} H_n(x)H_n(y) \exp\left(\frac{1}{2}(x^2 + y^2)\right) = \frac{1}{\sqrt{1 - \rho^2}} \exp\left(\frac{4\rho xy - (1 + \rho^2)(x^2 + y^2)}{2(1 - \rho^2)}\right)$$

to the equation (68), with $\rho = \exp(-\omega \tau)$, one obtains the Mehler kernel, a generalized bivariate Gaussian probability density

$$K_f(x, x', \tau) = \frac{\lambda}{\sqrt{2\pi \sinh \omega \tau}} \exp\left(\frac{1}{2} \lambda^2 \lambda x x'\right)$$

(69)

where we have made the change $\tau = T - t$, $\tau \geq 0$, and $x = (x, x')$. $A$ is the $SL(2, \mathbb{R})$ matrix

$$A = \begin{pmatrix} -\coth \omega \tau & \text{csch} \omega \tau \\ \text{csch} \omega \tau & -\coth \omega \tau \end{pmatrix}$$

with coth and csch the hyperbolic cotangent and cosecant, respectively.

$A$ is singular when $\omega \to 0$. However

$$\lim_{\omega \to 0} A \equiv \frac{1}{\tau} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

As expected from the group law (53), the Mehler kernel maps into the heat kernel (hence the Black-Scholes theory) when $\omega \to 0$

$$\lim_{\omega \to 0} K_f(x, x', \tau) = \frac{1}{\sqrt{2\pi a^2 \tau}} e^{\frac{1}{2} a(x^2 + x'^2)}$$

6.5.1 Derivation using LCTs

The Mehler kernel (69) can also be obtained using the LCT associated ([13], [32]) to the harmonic oscillator generating matrix (51). From (19)

$$W(M_H, x, x') = \frac{1}{\sqrt{2\pi b}} \exp\left(-\frac{(ax^2 - 2xx' + dx^2)}{(2b)}\right)$$

where $a, b, c, d$ refer to the elements of (51). By direct substitution, we get

$$W(M_H, x, x') = K_f(x, x', \tau)$$

A discussion of heat kernels and Mehler-type formulas based on group-invariant solutions can be found in reference [14]. Chapter 9 of [32] gives Baker-Campbell-Hausdorff relations between pseudo-differential operators for the harmonic oscillator based on composition of $SL(2, \mathbb{R})$ matrices.

6.6 Financial Interpretation

Using the Feynman-Kac formula, we can write the solution of (63a) as

$$\Psi(x, t) = \mathbb{E}(e^{-\frac{1}{2} T \int_0^T x(s)^2 ds} \Psi(x_T, T) | X(t) = x)$$

to the equation with $\gamma \equiv \omega^2 / \sigma^2$ and where the expectation is taken with respect to a normal process with volatility $\sigma$. This equation describes a derivative with a payoff that is discounted quadratically with the oscillator level $x$. A drift term can be easily added with a coboundary generated by $x$.

A more interesting approach is to use the harmonic oscillator as a process describing stocks with non-normal returns with a correlation given by the Mehler Kernel (69). The reference [61] proposes a quantum harmonic oscillator as a model for the market force which draws a stock return from short-run fluctuations to the long-run equilibrium.

7. Repulsive Oscillator

Using the arguments in section 6 for the harmonic oscillator, because of the $\mathbb{R}^+$ central extension in the Heisenberg-Weyl subgroup of $WSp(2, \mathbb{R})$, the quantization group for the repulsive oscillator $H_R$ is generated by the elliptic $SL(2, \mathbb{R})$ matrix

$$M_R = \begin{pmatrix} \cos \omega t & \lambda^{-2} \sin \omega t \\ -\lambda^2 \sin \omega t & \cos \omega t \end{pmatrix}$$

where $\omega, \sigma, t \in \mathbb{R}, \lambda \equiv \sqrt{\omega} / \sigma$

$M_R$ is obtained from the harmonic oscillator generating matrix (51) by the mapping

$$\omega \mapsto i \omega$$

(70)

For brevity, in this article we will not apply the full Group Quantization scheme to the repulsive oscillator. Most results for the repulsive oscillator can be readily obtained from the harmonic oscillator results and the correspondence (70).

The repulsive oscillator group composition law is

$$t'' = t + t'$$

$$p'' = p + p' \cos \omega t + \lambda^2 x' \sin \omega t$$

$$x'' = x + x' \cos \omega t + \lambda x \sin \omega t$$

$$\xi'' = \xi \zeta \exp\left(\varepsilon_R(g', g)\right)$$

with the cocycle $\varepsilon_R$

$$\varepsilon_R(g, g') = \frac{1}{2}(px' - xp') \cos \omega t + \frac{1}{2}(\lambda^{-2} pp' - \lambda^2 xx') \sin \omega t$$
One can easily verify that the high-polarization equation for the repulsive oscillator is (compare with (63a))

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \psi}{\partial x^2} + \gamma x^2 \psi = 0 \quad \gamma \equiv \frac{\omega^2}{\sigma^2} \quad (71)$$

whose separable solutions are time exponentials and Parabolic cylinder functions in $x$ ([16], [32], [35], [31]). Note that upon the analytic continuation (70) the Hermite functions (67) turn into Parabolic cylinder functions.

The quantum repulsive oscillator [34] is a well known model in the literature. A complete survey of the repulsive oscillator in the context of $\mathbb{W} Sp(2, \mathbb{R})$ can be found in [32]. Chapter 9 of [32] gives special Baker-Campbell-Hausdorff relations between pseudo-differential operators for the repulsive oscillator.

### 7.1 Financial Interpretation

Using the Feynman-Kac theorem, the solution of (71) can be written as

$$\psi(x, t) = \mathbb{E}(e^{\frac{1}{2} \int_t^T X(s)^2 \, ds} \psi(X_T, T) | X(t) = x)$$

where the expectation is taken with respect to a normal process with volatility $\sigma$. This equation describes a derivative with a payoff that grows quadratically with the oscillator level $x$.

Reference [33] uses a repulsive anharmonic oscillator model to explain the distribution of financial returns in a stock market when the market exhibits an upward trend.

### 8. Conclusions

We have presented a methodology, the Group Quantization formalism, for constructing a financial theory from symmetry arguments. The $\mathbb{W} Sp(2, \mathbb{R})$ group has been used to describe the Black-Scholes model, the Ho-Lee model and the harmonic and repulsive oscillators.

The choice of $\mathbb{R}^+$ as the structural group in the Heisenberg-Weyl group ensures the appropriate commutator between the momentum and coordinate operators $[\hat{p}, \hat{x}] = 1$. This choice is compatible with the financial interpretation of $\hat{p}$ as a delta.

In addition, extending translations by $\mathbb{R}^+$ formulates the theory directly in real (calculation) time, without the need to switch to an Euclidean time in order to derive pricing quantities.

The case of Black-Scholes has been studied in detail. We have constructed the polarized functions and the position, momentum and Hamiltonian operators both in coordinate and in momentum spaces. The role of variance as cohomological invariant has been identified.

First polarizations for the harmonic oscillator have been derived in phase space and in orthogonal coordinates. A high polarization was necessary to obtain polarized functions in coordinate (price) space.

Group Quantization generates representations of a financial theory in different functional spaces, allowing for alternative pricing frameworks such as the use of Laplace and Mellin transforms. In the case of Black-Scholes and Ho-Lee, the polarized functions have the meaning of prices of financial instruments, while their meaning for the oscillators is object of active research.

As we have shown in this article, among other interesting features, the Group Quantization formalism provides naturally the functional constraints (polarization algebra or higher polarization) for deriving the pricing equations. This makes Group Quantization a versatile methodology for constructing a financial theory from symmetry arguments alone, using solely the principal bundle structure of a centrally extended Lie group.
Appendices

1. Definitions

Vector Fields

Let $M$ be a $N$-dimensional manifold with local coordinates $x_i, i = 1, 2, \ldots N$. A vector field $X$ is an application that associates a first order differential operator $X(x)$ to a point $x \in M$. $X(x)$ can be expressed in local coordinates as a linear combination of the base fields

$$ e_i \equiv \frac{\partial}{\partial x_i} \quad i = 1, 2, \ldots N $$

that is

$$ X(x) = X_i(x) \frac{\partial}{\partial x_i} $$

with $X_i(x), i = 1, 2, \ldots N$ differentiable functions on $M$. The space of the vector fields is called the tangent space of $M$, $T(M)$. For brevity we will write $X$ instead of $X(x)$.

The integral curves of $X$ are the solution to the set of ordinary differential equations

$$ \frac{dx_i}{ds} = X_i(x) $$

where $s$ is an integration parameter. Note that the invariance condition $Xf = 0$ implies that $f$ is constant along the integral curves of $X$.

Forms

A 1-form $\Gamma$ is an application that associates to every point $x \in M$ an element of the dual space of $T^*(M)$.

$$ \Gamma : x \rightarrow \Gamma(x) \quad / \quad \Gamma(x)(X(x)) = f(X) $$

with $f(X)$ a differentiable function.

As with vector fields, we write $\Gamma$ for $\Gamma(x)$. The space of 1-forms is called the cotangent space of $M$ and is denoted by $T^*(M)$. A convenient representation for the basis of $T^*(M)$ is

$$ u_i \equiv dx_i \quad i = 1, 2, \ldots N $$

and its action on the basis of $T(M)$ is

$$ u_i(e_j) \equiv dx_i \left( \frac{\partial}{\partial x_j} \right) = \delta_{ij} \quad i, j = 1, 2, \ldots N $$

2-forms, 3-forms, etc. are linear combinations of tensor products of 1-forms. A function $f$ is considered a zero-form. An important operation on forms is the differential $d$. The differential of a $n$-form is either a $(n+1)$-form or zero. For a function $f$, $d$ is the ordinary differential

$$ df(x) = \frac{\partial f}{\partial x_i} dx_i $$

For a 1-form $\Gamma$

$$ d\Gamma = \frac{\partial \Gamma_i}{\partial x_j} dx_i \wedge dx_j $$

where the wedge operator $\wedge$ is the antisymmetric combination

$$ dx_i \wedge dx_j = dx_i \otimes dx_j - dx_j \otimes dx_i $$

One can define in an analogous way differentials of higher order forms. Note that the antisymmetry of the wedge operator implies that $d^2 f = 0$ and $d^2 \Gamma = 0$. Also, if a $n$-form acts on $n - k$ vector fields one obtains a $k$-form. For instance the 1-form $df$ acting on a field $X$ gives a zero-form

$$ df(X) = \frac{\partial f}{\partial x_i} dx_i \left( \frac{\partial}{\partial x_j} \right) = X_i(x) \frac{\partial f}{\partial x_i} = X(f) $$

Analogously, one can check that $d\Gamma$ acting on $X$ alone gives a 1-form

$$ d\Gamma(X, \cdot) = \frac{\partial \Gamma_i}{\partial x_j} (X dx_j - X_j dx_i) $$

We use the inner product notation to denote the action of a n-form $\Omega$ on a vector field $X$

$$ i_X (\Omega) = \Omega(X) $$

Lie Derivative

The Lie derivative evaluates the change of vector fields and forms along the flow defined by another vector field.

The Lie derivative of a function $f$ with respect to a vector field $X$ is

$$ L_X f = X(f) = df(X) $$

For two vector fields $X, Y$, the Lie derivative of $Y$ with respect to $X$, $L_X Y$, is a vector field defined by

$$ L_X Y = \left[ X, Y \right] = \left( X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} $$

The Lie derivative of a 1-form $\Gamma$ with respect to a vector field $X, L_X \Gamma$ is also a 1-form, and has the meaning of the rate of change of $\Gamma$ along the integral lines (flow lines) of $X$. One finds, in local coordinates

$$ L_X \Gamma = (L_X \Gamma)_i dx_i $$

where

$$ (L_X \Gamma)_i = X_j \frac{\partial \Gamma_i}{\partial x_j} + \Gamma_j \frac{\partial X_i}{\partial x_j} $$

or, in a more concise notation

$$ L_X \Gamma = d(\Gamma(X)) + d\Gamma(X, \cdot) \equiv i_X d\Gamma + d(i_X \Gamma) $$
Lie Algebra

A Lie algebra \( \mathfrak{g} \) is a vector space over a field \( \mathbb{F} \) equipped with a bilinear map \([ , ] : (\mathfrak{g}, \mathfrak{g}) \to \mathfrak{g}\) such that \([X, Y] = -[Y, X]\) and \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\) (Jacobi identity). This map is called Lie bracket. The Lie algebra for a Lie group \( \mathfrak{g} \) is defined using the tangent vectors at the identity \( e \) as vector space of and the Lie derivative as bracket operator. One has

\[
[X_i, X_j] = c^k_{i,j} X^k \quad i, j = 1, 2 \ldots \dim G
\]

where the coefficients \( c^k_{i,j} \in \mathbb{F}, i, j, k = 1, 2 \ldots \dim G \) are called structure constants.

Since \( X_k \) is a LIVF if and only if \( X_k = L_{\tau}^2 X_e \) and an RIVF if and only if \( X_k = R_{\tau}^2 X_e \), there is a biunivocal correspondence between the LIVFs (RIVFs) and the set of the tangent vectors at the identity \( e \) of \( G \), with the following brackets

\[
\begin{align*}
[X^I, X^J] &= c^k_{I,J} X_k^L \quad [X^L, X^J] = -c^k_{I,J} X_k^R \\
[X^L, X^J] &= 0 \quad \forall i, j, k = 1, 2 \ldots \dim G
\end{align*}
\]

2. Instrument Prices in Momentum Space

In this section we present examples of derivatives pricing in the less familiar momentum (Laplace) space.

The price of a financial instrument \( \Psi(x, t) \) maturing at \( T \) is found by performing an inverse Laplace transform

\[
\Psi(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{sp} K(p, t) \Psi(p, T) \tag{72}
\]

where \( \tau = t - T, t \leq T \) and \( \Psi(p, T) = \exp(E(p)T) \) is the payoff at maturity in the Laplace space

\[
\Psi(x, T) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{sp} \Psi(p, T) \tag{73}
\]

B.1 Instrument Prices using the Mellin Transform

The inverse Mellin transform \(^{33}\) gives the instrument’s price in terms of the stock value instead than the log-stock. The pricing equations (72) and (73) read now

\[
\begin{align*}
V(S, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{sp} \exp(-E(p)\tau)V(p, T)S^p \\
V(S, T) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp S^p V(p, T) \tag{74}
\end{align*}
\]

with

\[
E_r(p) = \frac{1}{2} \sigma^2 p^2 + \mu p - r \quad \mu = 1 - \frac{1}{2} \sigma^2
\]

The final condition in Mellin space is given by

\[
C(p, 0) = \int_0^{+\infty} V(S, T) S^{-p-1} dS
\]

B.2 Call Option Price

We price a call option with strike \( X \) and maturity \( T \) using the Mellin transform. The payoff is, from (73)

\[
C(p, 0) = \int_0^{+\infty} (S - X)^+ S^{-p-1} dS = \frac{X^{1-p}}{p(p-1)}
\]

with \( \Re(p) > 1 \). The expression of the call option price in the Mellin space is, using (74)

\[
C(p, \tau) = e^{-r\tau} \frac{X^{1-p}}{p(p-1)} \exp(-\frac{1}{2} \sigma^2 p^2 + \mu p) \tau
\]

where \( \tau = T - t, t \leq T \).

The Black-Scholes call option formula in the price (coordinate) representation is obtained by the inverse Mellin transform

\[
V(S, t) = e^{-r\tau} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} C(p, \tau) S^p dp
\]

where \( c \in (1, +\infty) \) and \( \Re(p) > 1 \).

Details of the calculation of this integral can be found in reference [55].

3. Lagrangian Formulation

In the Group Quantization formalism, \( \Theta \mid_\varphi \), the projection onto the base manifold of the vertical form \( \Theta \) along the integral trajectories of the characteristic module \( C_\Theta \) provides the classical action \( \mathcal{S} \) \(^{34}\)

\[
\mathcal{S} = \int \Theta \mid_\varphi = \int \mathcal{L}(x, \dot{x}) dt
\]

The classical Lagrangian \( \mathcal{L}(x, \dot{x}) \) is obtained by the projection onto the base manifold. Note that \( \Theta \) and \( \Theta + df \) generate equivalent Lagrangians that differ in a total derivative and do not change the action \( \mathcal{S} \).

The vertical (connection) form \( d\Theta \) restricted to the quotient space \( \hat{G}/U \) is analogous to the Poincaré-Cartan form of classical mechanics \( \Theta_{PC} \), which can be written in local coordinates \( q, p \) as

\[
\Theta_{PC} = \sum_i p_i dq_i - H(p, q) dt \tag{75}
\]

where \( H(p, q) \) is the Hamiltonian function, and \( q, p \) are conjugate pairs of coordinates and momenta. In classical mechanics,

\(^{33}\) if \( \mathcal{M}^{-1} \) exists only for complex values of \( y \) so that \( c = \Re(y) > 0 \) is within certain (possibly multiple) convergence strips. Each strip leads to different results for \( \mathcal{M}^{-1}(f) \) \(^{[57]}\), \(^{[58]}\), \(^{[59]}\), \(^{[60]}\).
the equations of motion are just the flows associated to the Hamiltonian field
\[ X_H = \frac{\partial}{\partial t} + \sum_i \frac{\partial H}{\partial p_i} \frac{d}{dt} q_i - \frac{\partial H}{\partial q_i} \frac{d}{dt} p_i, \]
so along Hamiltonian trajectories
\[ \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \]
Note that \( \Theta_{PC} \) has the structure of a Legendre transformation if one identifies \( p_i = \partial \mathcal{L} / \partial \dot{q}_i \), where \( \mathcal{L} \) is the Lagrangian function. Along the Hamiltonian flows
\[ \Theta_{PC}|_H = \mathcal{L}(q, \dot{q}) dt \]
The differential of \( \Theta_{PC} \) along the Hamiltonian trajectories is a symplectic form.
\[ d\Theta_{PC}|_H = \sum_i d\dot{p}_i \wedge dq_i, \]
The Lagrangian function is used for the application of path integral methods in computational finance (\[24\], \[26\], \[27\], \[15\]).

C.1 Black-Scholes Lagrangian
We find the Black Scholes Lagrangian by restricting the Black-Scholes connection \( \Theta \) (57) to the characteristic module trajectories on the base manifold.

\( \mathcal{C}_g \) is generated by the time translations \( X^L \) whose integral flows on the base manifold are (see (28))
\[ \frac{dt}{ds} = 1, \quad \frac{dp}{ds} = 0, \quad \frac{dx}{ds} = \sigma^2 p + \mu, \]
\[ \rightarrow t = s, \quad p = p_0, \quad x = x_0 + (\sigma^2 p_0 + \mu) t \] (76)
where \( x_0, p_0, \xi, \zeta \) are integration constants and \( s \) is the integration parameter.

We first add the total differential \( d(px) \) to \( \Theta \) so the equations adopt the Poincaré-Cartan expression (75)
\[ \Theta \rightarrow d(px) + \Theta = p dx - E(p) dt \]
then
\[ \Theta|_\mathcal{C} = p(\sigma^2 p + \mu) dt - \left( \frac{1}{2} \sigma^2 p^2 + \mu p - r \right) dt = \]
\[ \frac{1}{2\sigma^2} (\dot{x} - \mu)^2 dt - r dt \]
where we have used equation (76)
\[ \frac{dx}{dt} = \dot{x} = \sigma^2 p + \mu \rightarrow p = \frac{\dot{x} - \mu}{\sigma^2} \]
with the result
\[ \mathcal{L}(x, \dot{x}) = \frac{1}{2\sigma^2} (\dot{x} - \mu)^2 - r \] (77)
The solution \( x(t) \) of the Lagrangian (77) coincides with the expected value of the coordinate (log-price) using the \( K_{BS} \) kernel in (39)
\[ \langle \dot{x} \rangle = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2\sigma^2} \left( \frac{x^2}{\sigma^2} - \mu x - \tau \right)^2} = x' + \mu \tau \]

C.2 Euclidean Oscillator Lagrangian
Following the same steps than in section C.1, and using the expressions (57) and (58), we find that the \( \mathcal{C}_g \) trajectories
\[ \frac{dt}{ds} = 1, \quad \frac{dp}{ds} = \omega \lambda^2 x, \quad \frac{dx}{ds} = \omega \lambda^2 p \]
generates the following Lagrangian for the Euclidean harmonic oscillator
\[ \mathcal{L}_H(x, \dot{x}) = \frac{1}{2\sigma^2} \left( \dot{x}^2 + \frac{1}{2} \omega^2 x^2 \right) \]
As expected, the mapping \( \omega \rightarrow i\omega \) provides the Lagrangian for the Euclidean repulsive oscillator
\[ \mathcal{L}_R(x, \dot{x}) = \frac{1}{2\sigma^2} \left( \dot{x}^2 - \frac{1}{2} \omega^2 x^2 \right) \]

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Disclaimer
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References

[1] V. Aldaya and J. A. de Azcarraga, Quantization as a consequence of the symmetry group: An approach to geometric quantization, J. Math. Phys., 23, 1982.

[2] V. Aldaya and J. A. de Azcárraga and K.B. Wolf Quantization, symmetry, and natural polarization, J. Math. Phys. 25 (3), March 1984.

[3] V. Aldaya, J. A. de Azcárraga and S. García, Group manifold approach to field quantisation, Journal of Physics A: Mathematical and General, 21, 23, 1988.

[4] Santiago García, Hidden invariance of the free classical particle, arXiv:hep-th/9306040 American Journal of Physics, 62, 6, 1994.

[5] V. Aldaya, J. Bisquert, J. Guerrero, J. Navarro-Salas Group-theoretical construction of the quantum relativistic harmonic oscillator, Reports on Mathematical Physics Volume 37, Issue 3, June 1996, Pages 387-418, 1997.

[6] Aldaya V., Guerrero J., Marmo G. Quantization on a Lie Group: Higher-Order Polarizations, In: Gruber B., Ramek M. (eds) Symmetries in Science X. Springer, Boston, MA, arXiv:physics/9710002v1, 1998.

[7] M. Calixto, V. Aldaya; and M. Navarro, Quantum Field Theory in Curved Space from a Second Quantization of a Group, Int. J. Mod. Phys. A15, https://arxiv.org/abs/hep-th/9701180, Mar 2000.

[8] V. Aldaya, J. Guerrero, Lie Group Representations and Quantization, Reports on Mathematical Physics, 47, 2001.

[9] J. Wei and E. Norman, Lie algebraic solution of linear differential equations, J. Math. Phys., 4(4):575–581, 1963.

[10] Szymon Charzyński and Marek Kus, Wei-Norman equations for a unitary evolution, Journal of Physics A: Mathematical and Theoretical, Volume 46, Number 26, 2013.

[11] C. F. Lo, Lie-Algebraic Approach for Pricing Zero-Coupon Bonds in Single-Factor Interest Rate Models, Journal of Applied Mathematics, Volume 2013, Article ID 276238, 2013.

[12] E.G. Kalnins, W. Miller, Jr., G.S. Pogosyan Complete sets of invariants for dynamical systems that admit a separation of variables, Journal of Mathematical Physics 43(7), pp.–3592–3609, July 2002.

[13] W. Miller, Jr., The Scrodinger and Heat Equations, Encyclopedia of Mathematics and its Applications, Vol. 4, Cambridge University Press, 2010.

[14] F. Gungor, Equivalence and Symmetries for Linear Parabolic Equations and Applications Revisited, arXiv:1501.01481 [math-ph], 2017.

[15] Vadim Linetsky, The Path Integral Approach to Financial Modeling and Options Pricing, Computational Economics, 11,1, 1997.

[16] Abramowitz, Milton, and Irene A. Stegun. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, Ninth printing, 1970.

[17] Roman Kozlov On Lie Group Classification of a Scalar Stochastic Differential Equation, Journal of Nonlinear Mathematical Physics, 18,1, 2010.

[18] A. Pazy, Semigroups of Linear Operator and Applications to Partial Differential Equations, Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[19] F. Soto-Eguibar and H. M. Moya-Cessa, Solution of the Schrodinger Equation for a Linear Potential using the Extended Baker-Campbell-Hausdorff Formula, Appl. Math. Inf. Sci. 9, No. 1, 175-181.

[20] Juan M Romero, Elio Martínez Miranda and Ulises Lavana, Conformal symmetry in quantum finance, J. Phys. Conf. Ser. 512, 2014.

[21] Juan M Romero, Elio Martínez Miranda and Ulises Lavana, Schrodinger group and quantum finance https://arxiv.org/abs/1304.4995

[22] U. Niederer, The maximal kinematical invariance group of the free Schrodinger equation, Helv. Phys. Acta, 45, 1972.

[23] C.R. Hagen, Scale and conformal transformations in galilean-covariant field theory, Phys. Rev. 5.2, 1972.

[24] Belal Baaquie, Quantum Finance: Path Integrals and Hamiltonians for Options and Interest Rates, Cambridge University Press, 2004.

[25] Belal Baaquie, Interest Rates and Coupon Bonds in Quantum Finance, Cambridge University Press, 2009.

[26] Belal Baaquie, Financial modeling and quantum mathematics, Computers and Mathematics with Applications 65 (2013) 1665–1673,

[27] Belal Baaquie, Claudio Coriano, Marakani Srikant, Quantum Mechanics, Path Integrals and Option Pricing: Reducing the Complexity of Finance, arXiv:cond-mat/0208191 [cond-mat.soft], 2002.

[28] R. K. Gazizov, N. H. Ibragimov, Lie Symmetry Analysis of Differential Equations in Finance, Nonlinear Dynamics, 17, 4, 1998

[29] F. Casas, A. Murua M. Nadini, Efficient computation of the Zassenhaus formula, Computer Physics Communications, Volume 183, Issue 11, pp 2386–2391, November 2012

[30] K.B. Wolf, On Time-Dependent Quadratic Quantum Hamiltonians, SIAM Journal on Applied Mathematics, Vol. 40, No. 3, Jun., 1981, pp. 419-431

[31] C. Munoz, J. Rueda-Paz and K.B. Wolf, Discrete repulsive oscillator wavefunctions, J. Phys. A: Math. Theor. 42 (2009) 485210 (12pp)
