A Construction of Linear Codes over $\mathbb{F}_{2^t}$ from Boolean Functions

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Abstract

In this paper, we present a construction of linear codes over $\mathbb{F}_{2^t}$ from Boolean functions, which is a generalization of Ding’s method [1, Theorem 9]. Based on this construction, we give two classes of linear codes $\tilde{C}_f$ and $\hat{C}_f$ (see Theorem 1 and Theorem 6) over $\mathbb{F}_{2^t}$ from a Boolean function $f : \mathbb{F}_q \to \mathbb{F}_2$, where $q = 2^n$ and $\mathbb{F}_{2^t}$ is some subfield of $\mathbb{F}_q$. The complete weight enumerator of $\tilde{C}_f$ can be easily determined from the Walsh spectrum of $f$, while the weight distribution of the code $\hat{C}_f$ can also be easily settled. Particularly, the number of nonzero weights of $\tilde{C}_f$ and $\hat{C}_f$ is the same as the number of distinct Walsh values of $f$. As applications of this construction, we show several series of linear codes over $\mathbb{F}_{2^t}$ with two or three weights by using bent, semibent, monomial and quadratic Boolean function $f$.

Index Terms

Linear codes, Walsh spectrum, Boolean function, weight enumerator, complete weight enumerator, bent and semibent functions.

I. INTRODUCTION

Throughout this paper, let $q = 2^n$ for any positive integer $n \geq 2$, $\mathbb{F}_q$ be the finite field with $q$ elements and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. By viewing $\mathbb{F}_q$ as a $n$-dimensional vector space $\mathbb{F}_2^n$ with respect to a fixed $\mathbb{F}_2$-basis of $\mathbb{F}_q$, a function $f$ from $\mathbb{F}_2^n$ (or $\mathbb{F}_2^m$) to $\mathbb{F}_2$ is called a Boolean function with $n$ variables, which can be expressed by

$$f = f(x) : \mathbb{F}_q \to \mathbb{F}_2.$$ 

The Walsh transform of $f$ is defined by

$$W_f(y) = \sum_{x \in \mathbb{F}_q} (-1)^{f(x) + \text{Tr}(xy)} \in \mathbb{Z},$$

where $y \in \mathbb{F}_2^n$ and Tr is the trace function from $\mathbb{F}_2^n$ to $\mathbb{F}_2$. The Walsh spectrum of $f$ is the following multiset

$$\{W_f(y) : y \in \mathbb{F}_2^n\}.$$  

For convenience, we denote the Walsh spectrum of $f$ by

$$[w_1]^{m_1}[w_2]^{m_2} \cdots [w_r]^{m_r}$$

if the multiset has $r$ distinct values $w_1, w_2, \ldots, w_r$, where $m_i = |\{y \in \mathbb{F}_q : W_f(y) = w_i\}|$ for each positive integer $i$ satisfying $1 \leq i \leq r$. It is obvious that $m_1 + m_2 + \cdots + m_r = q$.

Recently, C. Ding [1] presented a construction of binary linear code $C_f$ from a Boolean function $f$ such that the weight distribution of $C_f$ can be derived directly from the Walsh spectrum of $f$ [1, Theorem 9].

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The number of distinct nonzero weights of $C_f$ is the same as the number $r$ of distinct Walsh values of $f$. Particularly, from bent and semibent functions he gets a series of binary linear codes with two and three weights, respectively. In this paper, we show a generalization (see Theorem 1) of Ding’s construction [1] by the following way. Firstly, from any Boolean function $f: \mathbb{F}_q \rightarrow \mathbb{F}_2$ we construct two classes of linear codes $\tilde{C}_f$ and $C_f$ over some subfield $\mathbb{F}_{2^t}$ of $\mathbb{F}_q$ for some $t$ satisfying a certain condition. Secondly, the complete weight enumerator of $\tilde{C}_f$ and the weight enumerator of $C_f$ can be determined in terms of the Walsh spectrum of $f$. Particularly, we show that the code $\tilde{C}_f$ is ‘nearly’ a composition constant code which means that for each codeword $c = (c_1, c_2, ..., c_N) \in \tilde{C}_f$, $N_c(\alpha)$ are the same for all $\alpha \in \mathbb{F}_{2^t}^*$, where

$$N_c(\alpha) = |\{1 \leq i \leq N : c_i = \alpha\}|$$

is the number of $\alpha$-components of the codeword $c$.

Now we recall some terminology on linear codes. An $[N, k]$ linear code $C$ over $\mathbb{F}_2^N$ with dimension $\dim_{\mathbb{F}_2} C = k$. For each codeword $c = (c_1, c_2, ..., c_N) \in C$ and $\alpha \in \mathbb{F}_2^*$, let $N_c(\alpha)$ be the number of $\alpha$-components of the codeword $c$. Namely, $N_c(\alpha)$ is defined by (3). Then $\sum_{\alpha \in \mathbb{F}_2^*} N_c(\alpha) = N$. The complete weight enumerator of $C$ is a polynomial on $x_\alpha (\alpha \in \mathbb{F}_2^*)$ defined by

$$K_C(X) = K_C(x_\alpha : \alpha \in \mathbb{F}_2^*) = \sum_{c \in C} \prod_{\alpha \in \mathbb{F}_2^*} x_\alpha^{N_c(\alpha)} \in \mathbb{Z}[x_\alpha : \alpha \in \mathbb{F}_2^*].$$

This is a homogeneous polynomial with degree $N$. Let $x_0 = 1$ and $x_\alpha = x$ for all $\alpha \in \mathbb{F}_2^*$ in (4), we get the weight enumerator of $\tilde{C}$:

$$E_\tilde{C}(x) = 1 + \sum_{i=1}^{N} A_i x^i \in \mathbb{Z}[x],$$

where

$$A_i = |\{c \in C : \text{wt}_H(c) = \sum_{\alpha \in \mathbb{F}_2^*} N_c(\alpha) = i\}|$$

is the number of codewords $c$ in $C$ with the Hamming weight $\text{wt}_H(c) = i$. $(1, A_1, ..., A_N)$ is called the weight distribution of $C$. The code $C$ is said to be $v$ weights if the number of nonzero $A_i$ in the sequence $(A_1, A_2, ..., A_N)$ is equal to $v$. For binary case $t = 1$, it is obvious that the complete weight enumerator of $\tilde{C}$ is essentially the same as the weight enumerator. When $t \geq 2$, the complete weight enumerator $K_C(X)$ of $C$ gives more information on the code $\tilde{C}$ than the weight enumerator of $\tilde{C}$.

This paper is organized as follows. In Section III we present a generalization (see Theorem 1) of Ding’s construction. From a Boolean function $f: \mathbb{F}_q \rightarrow \mathbb{F}_2$ we construct two classes of linear codes $\tilde{C}_f$ and $C_f$ over $\mathbb{F}_{2^t}$, while the complete weight enumerator of $\tilde{C}_f$ and the weight enumerator of $C_f$ can be determined in terms of the Walsh spectrum of $f$, where $t$ is a divisor of $n$ satisfying a certain condition. If $f$ has $r$ distinct Walsh values, then both of $\tilde{C}_f$ and $C_f$ have $r$ distinct nonzero weights. As applications of Theorem 1, Section III gives several series of linear codes $C_f$ over $\mathbb{F}_{2^t}$ with two or three weights constructed by using bent, semibent, monomial and quadratic Boolean function $f: \mathbb{F}_q \rightarrow \mathbb{F}_2$. Moreover, we also present a method (see Theorem 6) by concatenation based on Theorem 1 and get more linear codes over $\mathbb{F}_{2^t}$ with three weights and more flexible length. The last Section IV is conclusion.

II. A CONSTRUCTION OF LINEAR CODES OVER $\mathbb{F}_{2^t}$

The objective of this section is to present a generalization of Ding’s construction for linear codes and construct two classes of linear codes $\tilde{C}_f$ and $C_f$ over $\mathbb{F}_{2^t}$ from a Boolean function $f$, where $t$ is a divisor of $n$ satisfying a certain condition.

Let $f = f(x)$ be a Boolean function from $\mathbb{F}_{2^t}$ to $\mathbb{F}_2$. In this section, we consider a subfield $\mathbb{F}_{2^t}$ of $\mathbb{F}_{2^n}$, where $t$ satisfies the following condition:

$$f(\alpha x) = f(x) \text{ for all } \alpha \in \mathbb{F}_{2^t}^* \text{ and } x \in \mathbb{F}_q.$$
Namely, \( f \) is a constant on each coset of subgroup \( \mathbb{F}_{2^n}^* \) in \( \mathbb{F}_q^* \).

Define

\[
\tilde{D} = \tilde{D}(f) = \{ x \in \mathbb{F}_q^* : f(x) = 0 \}. 
\]

(6)

The condition (5) implies that the set \( \tilde{D} \) is a union of \( N \) distinct cosets \( x_1 \mathbb{F}_{2^n}^*, \ldots, x_N \mathbb{F}_{2^n}^* \) in \( \mathbb{F}_q^* \). Therefore, \( \tilde{N} = |\tilde{D}| = (2^t - 1)N \). Let \( D = \{ x_1, x_2, \ldots, x_N \} \) and \( \tilde{D} = \{ x_1, x_2, \ldots, x_{N+1}, \ldots, x_{\tilde{N}} \} \). Now we define two classes of linear codes over \( \mathbb{F}_{2^n} \) as follows:

\[
\tilde{C} = \tilde{C}_f = \{ \tilde{c}_b = (\text{Tr}_t^i(bx_1), \text{Tr}_t^i(bx_2), \ldots, \text{Tr}_t^i(bx_N)) : b \in \mathbb{F}_q \}
\]

(7)

and

\[
C = C_f = \{ c_b = (\text{Tr}_t^i(bx_1), \text{Tr}_t^i(bx_2), \ldots, \text{Tr}_t^i(bx_N)) : b \in \mathbb{F}_q \},
\]

(8)

where \( \text{Tr}_t^i \) is the trace function from \( \mathbb{F}_{2^n} \) to \( \mathbb{F}_2 \).

The following theorem is the main result of this paper.

**Theorem 1.** Let \( q = 2^n \) and \( f(x) : \mathbb{F}_q \to \mathbb{F}_2 \) be a Boolean function with the Walsh spectrum \([w_1]|m_1|w_2|m_2| \ldots |w_r|m_r\).

Assume that

- \( W_f(0) = w_i \) for some \( i \) with \( 1 \leq i \leq r \);
- \( w_i - w_j \neq -q \) for all positive integer \( j, 1 \leq j \leq r \); and
- \( t \) is an positive integer satisfying \( t|n \) and the condition (5).

Then

1) the linear code \( \tilde{C} = \tilde{C}_f \) of (7) has parameters \([\tilde{N}, n/t] \) and the complete weight enumerator

\[
K_{\tilde{C}}(X) = x_0^N + (m_i - 1)x_0 \prod_{y \in \mathbb{F}_q^*} x_y^{2^{t+1} - 1} + \sum_{1 \leq j \leq r(j \neq i)} m_j x_0^{N_j}(\prod_{y \in \mathbb{F}_2^*} x_y^{M_j}),
\]

where

\[
\tilde{N} = 2^{n-1} + \frac{1}{2}(w_i - 1 - (-1)^{f(0)})
\]

\[
N_j = 2^{t-1}(q + w_j + (2^t - 1)w_j) - 2^{-1}(1 + (-1)^{f(0)})
\]

\[
M_j = 2^{t-1}(q + w_j - w_j)
\]

for each positive integer \( j \) within \( 1 \leq j \leq r \) and \( i \) is determined by \( W_f(0) = w_i \). The weight enumerator of \( \tilde{C} \) is

\[
E_{\tilde{C}}(x) = 1 + (m_i - 1)x^{(2^t - 1)q/2^{t+1}} + \sum_{j=1, j \neq i}^{r} m_j x^{(2^t - 1)M_j}.
\]

(10)

2) the linear code \( C = C_f \) of (8) has parameters \([N, n/t] \) and the weight enumerator of \( C \) is

\[
E_{C}(x) = 1 + (m_i - 1)x^{q/2^{t+1}} + \sum_{j=1, j \neq i}^{r} m_j x^{M_j},
\]

(11)

where \( N = \tilde{N}/(2^t - 1) \).

Particularly, both of \( \tilde{C} \) and \( C \) have \( r \) distinct (nonzero) weights.
Proof: 1) By definition and assumption, the length of $\tilde{C}$ is

$$\tilde{N} = \left| \{ x \in \mathbb{F}_q^* : f(x) = 0 \} \right|$$

$$= \frac{1}{2} \sum_{y \in \mathbb{F}_2} \sum_{x \in \mathbb{F}_q} (-1)^{f(x)} - \frac{1}{2}(1 + (-1)^{f(0)})$$

$$= \frac{1}{2} \sum_{x \in \mathbb{F}_q} (1 + (-1)^{f(x)}) - \frac{1}{2}(1 + (-1)^{f(0)})$$

$$= \frac{1}{2}(q + W_f(0) - 1 - (-1)^{f(0)})$$

$$= 2^{n-1} + \frac{1}{2}(w_i - 1 - (-1)^{f(0)}).$$

Next we calculate the number

$$N_{\tilde{C}(b)}(\gamma) = \left| \{ x \in \mathbb{F}_q^* : f(x) = 0 \text{ and } \text{Tr}_{p}(bx) = \gamma \} \right|$$

for any $b \in \mathbb{F}_q^*$ and any $\gamma \in \mathbb{F}_{2^n}$.

By definition and the basic facts of additive characters, we have

$$N_{\tilde{C}(b)}(\gamma) = \frac{1}{2^{n+1}} \sum_{x \in \mathbb{F}_q^*} (1 + (-1)^{f(x)}) \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha(\text{Tr}_{p}(bx) + \gamma))}$$

$$= \frac{1}{2^{n+1}} \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha \gamma)} \sum_{x \in \mathbb{F}_q^*} (1 + (-1)^{f(x)})(-1)^{\text{Tr}(bx \alpha)}$$

$$= \frac{1}{2^{n+1}} \left( \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha \gamma)} \sum_{x \in \mathbb{F}_q} (1 + (-1)^{f(x)})(-1)^{\text{Tr}(bx \alpha)} - \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha \gamma)} (1 + (-1)^{f(0)}) \right)$$

$$= \begin{cases} \frac{1}{2^{n+1}} \left( \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha \gamma)} \sum_{x \in \mathbb{F}_q} (1 + (-1)^{f(x)})(-1)^{\text{Tr}(bx \alpha)} - \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha \gamma)} (1 + (-1)^{f(0)}) \right) & \text{if } \gamma \neq 0 \\ \frac{1}{2^{n+1}} \left( \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha \gamma)} \sum_{x \in \mathbb{F}_q} (1 + (-1)^{f(x)})(-1)^{\text{Tr}(bx \alpha)} - \sum_{\alpha \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}_{p}(\alpha \gamma)} (1 + (-1)^{f(0)}) \right) & \text{if } \gamma = 0 \end{cases}$$

But for any $b \in \mathbb{F}_q^*$ and any $\alpha \in \mathbb{F}_{2^n}$, we have

$$W_f(\alpha b) = \sum_{x' \in \mathbb{F}_q} (-1)^{f(x') + \text{Tr}(\alpha bx')}$$

$$= \sum_{x' \in \mathbb{F}_q} (-1)^{f(x' + \alpha x')}$$

(Let $x' = \alpha x$)

$$= \sum_{x' \in \mathbb{F}_q} (-1)^{f(x')} + \text{Tr}(bx')$$

(By the condition (5))

$$= W_f(b).$$

(13)

In view of (13), formula (12) becomes that

$$N_{\tilde{C}(b)}(\gamma) = \begin{cases} \frac{1}{2^{n+1}} (q + W_f(0) - W_f(b)) & \text{if } \gamma \neq 0 \\ \frac{1}{2^{n+1}} (q + W_f(0) + (2^t-1)W_f(b)) - \frac{1}{2}(1 + (-1)^{f(0)}) & \text{if } \gamma = 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2^{n+1}} (q + w_i - w_j) & \text{if } \gamma \neq 0 \\ \frac{1}{2^{n+1}} (q + w_i + (2^t-1)w_j) - \frac{1}{2}(1 + (-1)^{f(0)}) & \text{if } \gamma = 0 \end{cases}$$

(14)
By assumption that $W_f(0) = w_i$, $W_f(b) = w_j$ and the walsh spectrum of $f$ is $[w_1][w_2][w_3] \cdots [w_r]$, we have

$$W_f(b) = \begin{cases} 
  w_1 & \text{occurs } m_1 \text{ times,} \\
  \vdots & \\
  w_i & \text{occurs } m_i - 1 \text{ times,} \\
  w_{i+1} & \text{occurs } m_{i+1} \text{ times,} \\
  \vdots & \\
  w_r & \text{occurs } m_r \text{ times,} 
\end{cases} \quad (15)$$

when $b$ runs through $\mathbb{F}_q^n$. Thus we get the formula (9) on $K_c(X)$. The formula (10) can be derived directly by taking $x_0 = 1$ and $x_j = x$ for all $\gamma \in \mathbb{F}_q^n$ in $K_c(X)$. Meanwhile, by assumption that $w_i - w_j \neq -q$ for all positive integer $j$ satisfying $1 \leq j \leq r$, we know that the Hamming weight of $c(b)$ is

$$\text{wt}_H(c(b)) = (2^t - 1)M_j = \frac{2^t - 1}{2^t}(q + w_i - w_j) \neq 0$$

for each $b \in \mathbb{F}_q^n$. This means that the code $\tilde{C}_f$ has $q$ distinct codewords. Hence, the dimension of the code $\tilde{C}_f$ is $\dim_{\mathbb{F}_q} \tilde{C} = \log_2 q = \frac{n}{t}$.

2) For each $b \in \mathbb{F}_q^n$, we have the codeword $\tilde{c}(b) = (\text{Tr}_{i}^q(bx))_{x \in D}$ in the code $\tilde{C}$ and $c(b) = (\text{Tr}_{i}^q(bx))_{x \in D}$ in the code $C$, where $\tilde{D} = \bigcup_{x \in D} xF_{2^t}$ is a disjoint union of cosets $xF_{2^t}$ ($x \in D$). It is obvious that $\tilde{N} = |\tilde{D}| = (2^t - 1)|D| = (2^t - 1)N$. For each element $\alpha x$ ($x \in D$, $\alpha \in \mathbb{F}_{2^t}$), $\text{Tr}_{i}^q(bx\alpha) = \alpha \text{Tr}_{i}^q(bx)$. Therefore, $\text{Tr}_{i}^q(bx\alpha) \neq 0$ if and only if $\text{Tr}_{i}^q(bx) \neq 0$. This implies that $\text{wt}_H(\tilde{c}(b)) = (2^t - 1)\text{wt}_H(c(b))$ for all $b \in \mathbb{F}_q^n$. Then the formula (11) on weight enumerator $E_C(x)$ can be derived from the formula (10). This completes the proof of Theorem 1.

Let $t = 1$. It is clear that the condition (5) is true. Form Theorem 1 we get the following corollary on binary linear codes which essentially is [1, Theorem 9].

**Corollary 2.** Let $f = f(x) : \mathbb{F}_q \to \mathbb{F}_2$ be a Boolean function with Walsh spectrum $[w_1][w_2][w_3] \cdots [w_r]$. Let Tr be the trace mapping from $\mathbb{F}_q$ to $\mathbb{F}_2$. Assume that $W_f(0) = w_i$ and $w_i - w_j \neq -q$ for all $j$ ($1 \leq j \leq r$). Let $D = D_f = \{x_1, x_2, \cdots, x_N\}$ be the set of zeros of $f(x)$ in $\mathbb{F}_q^n$. Then the subset $C = C_f$ of $\mathbb{F}_2^N$ defined by

$$C = C_f = \{c(b) = (\text{Tr}(bx_1), \text{Tr}(bx_2), \cdots, \text{Tr}(bx_N)) : b \in \mathbb{F}_q\}$$

is a binary linear code with the parameters $\{N, n\}$ and the weight enumerator

$$1 + (m_i - 1) x^{q/4} + \sum_{1 \leq j \leq r, j \neq i} m_j x^{(q^i - w_i - w_j)/4},$$

where $N = 2^{n-1} + \frac{1}{2}(w_i - 1 - (-1)^{f(0)})$.

We remark that Theorem 1 open a way to construct linear codes $\tilde{C}_f$ and $C_f$ over $\mathbb{F}_{2^t}$ with $r$ distinct nonzero weights from any Boolean function $f$ with $r$ distinct Walsh values.

**III. Linear Codes over $\mathbb{F}_{2^t}$ with Two or Three Weights**

In this section, we show several examples on linear codes with two or three weights as applications of Theorem 1. Moreover, we also give a method by concatenation based on Theorem 1 and get more linear codes with three weights.

For $t=1$, Ding has presented such examples on binary linear codes from bent, semibent, almost bent and quadratic Boolean functions [1, Corollary 10, 11, 13 and Theorem 14]. Thus we focus on construction of linear codes over $\mathbb{F}_{2^t}$ for the case $t \geq 2$. Since the parameters of the code $C_f$ is better of the code $\tilde{C}_f$ in Theorem 1, from now on we focus on the linear code $C_f$. 
A. Two weights linear codes over $\mathbb{F}_{2^t}$

In this subsection, we construct several classes of linear codes over $\mathbb{F}_{2^t}$ with two weights by using bent functions.

Let $f$ be a Boolean function from $\mathbb{F}_q$ to $\mathbb{F}_2$ throughout this subsection. If $f$ is bent and $n = 2m$, then the Walsh spectrum of $f$ is

$$[2^m]2^{n-1} + 2^{m-1}(-1)^{f(0)} [-2^m]2^{n-1} - 2^{m-1}(-1)^{f(0)}.$$

The conclusion of the following theorem is straightforward from Theorem 1 and its proof is omitted.

**Theorem 3.** Let $f$ be a bent function from $\mathbb{F}_q$ to $\mathbb{F}_2$, $q = 2^t$, $n = 2m$ and $W_f(0) = \varepsilon \cdot 2^m$ with $\varepsilon \in \{1, -1\}$. If $\mathbb{F}_{2^t}$ is a subfield of $\mathbb{F}_q$ where $t | n$ and the condition (5) is satisfied, then the code $C = C_f$ defined by (5) is a two weights linear code over $\mathbb{F}_{2^t}$ with the parameters $[\frac{1}{2^t}(2^{n-1} + \frac{1}{2}(\varepsilon 2^m - 1 - (-1)^{f(0)})], n/t]$ and the weight enumerator

$$E_C(x) = 1 + (2^{n-1} + \varepsilon \cdot 2^{m-1}(-1)^{f(0)} - 1) x^{2^{n-1}} + (2^{n-1} - \varepsilon \cdot 2^{m-1}(-1)^{f(0)}) x^{2^{n-1} + \varepsilon 2^{m-1}}.$$

It is obvious that Theorem 5 essentially is [11] Corollary 10 if $t = 1$.

Many bent functions have been found since Rothaus published his original paper in 1976 [10]. We refer the reader to [2], [3], [4], [5], [6], [7], [8], [9] and the references therein for detailed information. Here we only consider two cases which fit with Theorem 5 for the case $t \geq 2$.

**Case 1: Monomial Polynomials.**

The following corollary is a direct consequence of Theorem 3.

**Corollary 4.** Let $f(x) = \text{Tr}(\alpha x^d)$ be a bent function from $\mathbb{F}_q$ to $\mathbb{F}_2$, where $q = 2^n$, $n = 2m$, $2 \leq d \leq q - 2$ and $\alpha \in \mathbb{F}_q^*$. Suppose that $t \mid n$ and $(2^t - 1) \mid d$. Then the code $C_f$ defined by (5) is a two weights linear code over $\mathbb{F}_{2^t}$ with the parameters $[\frac{1}{2^t}(2^{n-1} + \frac{1}{2}(\varepsilon 2^m - 2)], n/t]$ and the weight enumerator

$$E_C(x) = 1 + (2^{n-1} + \varepsilon \cdot 2^{m-1} - 1) x^{2^{n-1}} + (2^{n-1} - \varepsilon \cdot 2^{m-1}) x^{2^{n-1} + \varepsilon 2^{m-1}},$$

where $\varepsilon \cdot 2^m = W_f(0) = \sum_{x \in \mathbb{F}_q} (-1)^{\text{Tr}(\alpha x^d)}$.

Let $\theta$ be a generator of $\mathbb{F}_q^*$. The known monomial bent functions $f(x) = \text{Tr}(\alpha x^d)$ can be summarized in Table I for $\alpha \in \mathbb{F}_q^*$ and even $n = 2m$.

| Name of index $d$ | $d$ | Conditions | Reference |
|------------------|-----|-------------|-----------|
| Gold             | $2^h + 1$ | 2 \mid \frac{n}{\gcd(n, 2^h)}; $\alpha \neq 0$ and $\alpha \notin \theta^{2^h + 1}$ | [11] |
| Dillon           | $2^m - 1$ | $\alpha \in \mathbb{F}_{2^m}^*$ and $\sum_{x \in \mathbb{F}_{2^m}} (-1)^{\text{Tr}(x^{d} + \alpha x)} = -1$ | [11] |
| Kasami           | $2^{2h} - 2^h + 1$ | $3 \mid m$, $\gcd(h, n) = 1$, $\alpha \neq 0$ and $\alpha \notin \theta^h$ | [11] |
| Leander          | $(2^{h} + 1)^2$ | $n = 4h$ and $2 \nmid h$ | [11] |
| CCK              | $2^{2h} + 2^h + 1$ | $n = 6h$, $\alpha \in \text{GF}(2^m)^*$ and $\text{Tr}_n^m(\alpha) = 0$ | [12] |

In order to construct linear codes over $\mathbb{F}_{2^t}$ from the monomial bent functions by using Corollary 4, we need $t$ satisfying the following extra condition (except the conditions listed in Table I)

$t \mid n$ and $(2^t - 1) \mid d$ (16)

since $(2^t - 1) \mid d$ implies the condition (5).

**Example 1.** In order to construct linear codes over $\mathbb{F}_q$ by using bent functions of Table I, we need the extra condition which is reduced to be $3 \mid d$ for $t = 2$. Namely, the extra condition is $2 \nmid h$ for Gold and Kasami bent functions, $2 \mid h$ for CCK bent functions and $2 \mid m$ for Dillon bent functions in the Table I.
Extra condition is needed for Leander bent functions. Put this extra condition in Table 4, we get two-weight linear codes over \( \mathbb{F}_4 \) with the parameter and weight enumerator given by Corollary 4.

**Example 2.** For each divisor \( t \geq 2 \) of \( m \), we have \((2^t - 1)(2^m - 1)\). Therefore from Dillon bent function \( f(x) \) listed in Table 4, we can construct a two-weight linear code over \( \mathbb{F}_{2t} \) with the parameter and weight enumerator given by Corollary 4 for any divisor \( t \) of \( m \).

**Case 2: Quadratic Functions.**

Let the quadratic Boolean function \( f : \mathbb{F}_q \rightarrow \mathbb{F}_2 \) be defined by

\[
f(x) = \sum_{i=1}^{m-1} \text{Tr}(c_i x^{1+2^i}) + \text{Tr}_1(c_m x^{1+2^m}), \quad c_m \in \mathbb{F}_{2^m}, \ c_i \in \mathbb{F}_q, \ 1 \leq i \leq m-1,
\]

where \( n = 2m \) and \( q = 2^n \). Many quadratic bent functions have been obtained by using quadratic form theory. We refer the reader to [9], [15] and the references therein for detailed information. Here we consider some cases fitting in Theorem 3 for the case \( t = 2 \).

Let \( n = 2m \). For \( t = 2 \), if \( f(x) \) is a quadratic bent function with the following form

\[
f(x) = \sum_{\lambda=0}^{[m/2]-1} \text{Tr}(c_\lambda x^{1+2^{2\lambda+1}}), \quad c_\lambda \in \mathbb{F}_q, \ 0 \leq \lambda \leq [m/2] - 1
\]

or

\[
f(x) = \sum_{\lambda=0}^{s-1} \text{Tr}(c_\lambda x^{1+2^{2\lambda+1}}) + \text{Tr}_1(c_m x^{1+2^m}), \quad m = 2s + 1, \ c_\lambda \in \mathbb{F}_q, \ 0 \leq \lambda \leq s-1, \ c_m \in \mathbb{F}_{2^m}, \ (18)
\]

then all index \( 1 + 2^{2\lambda+1} \) and \( 1 + 2^m \) for odd \( m \) can be divided by \( 3 = 2^2 - 1 \). From such quadratic bent functions \( f(x) \) in (17) and (18), we can construct linear codes over \( \mathbb{F}_4 \) by Theorem 3.

**Example 3.** As an special case of (18), we consider the following form:

\[
f(x) = \sum_{\lambda=0}^{s-1} c_\lambda \text{Tr}(x^{1+2^{2\lambda+1}}) + \text{Tr}_1(x^{1+2^m}), \quad n = 2m, \ m = 2s + 1, \ c_\lambda \in \mathbb{F}_2, \ 0 \leq \lambda \leq s-1.
\]

Furthermore, the simple special case of (19) is

\[
f(x) = \text{Tr}(x^{1+2^{2\lambda+1}}) + \text{Tr}_1(x^{1+2^m}), \quad n = 2m, \ m = 2s + 1, \ 0 \leq \lambda \leq s-1.
\]

Let \( c(x) = \sum_{\lambda=0}^{s-1} c_\lambda (x^{2^{2\lambda+1}} + x^{m-(2\lambda+1)}) + x^m \in \mathbb{F}_2[x] \). By [9 Corollary 1], we have

- the function \( f(x) \) of (19) is a bent function from \( \mathbb{F}_q \) to \( \mathbb{F}_2 \) if and only if \( \gcd(c(x), x^m + 1) = 1 \);
- the function \( f(x) \) of (20) is a bent function from \( \mathbb{F}_q \) to \( \mathbb{F}_2 \) if and only if \( \gcd(3(2\lambda+1), m) = 1 \).

When \( f(x) \) is a bent function with the form (19) or (20), then the linear code constructed by Theorem 3 is a two weights linear code over \( \mathbb{F}_4 \).

**B. Three weights linear codes over \( \mathbb{F}_2 \)**

For most of known Boolean functions \( f(x) : \mathbb{F}_q \rightarrow \mathbb{F}_2 \) with three Walsh values, the values of \( W_f(w) \) are \( w_1 = 0, w_2 = A \) and \( w_3 = -A \) and \( A = 2^l \) for some \( l \) satisfying \( l \geq [n/2] + 1 \). If \( l = [n/2] + 1 \) and \( n = 2m \) or \( n = 2m + 1 \), the function \( f \) is called semibent.

Let the Walsh spectrum of \( f \) be \([w_1]^{m_1}[w_2]^{m_2}[w_3]^{m_3}\), where \((w_1, w_2, w_3) = (0, A, -A)\) and

\[
m_i = |\{w \in \mathbb{F}_q : W_f(w) = w_i\}|
\]
for each \( i \in \{1, 2, 3\} \). Then we have the following system of equations on \( m_i(i = 1, 2, 3) \):

\[
\begin{align*}
  m_1 + m_2 + m_3 &= q_s, \\
  m_1 w_1 + m_2 w_2 + m_3 w_3 &= \sum_{x \in \mathbb{F}_q, w \in \mathbb{F}_q} (-1)^{f(x) + \text{Tr}(wx)} = q \cdot (-1)^{f(0)}, \\
  m_1 w_1^2 + m_2 w_2^2 + m_3 w_3^2 &= \sum_{w \in \mathbb{F}_q} (\sum_{x \in \mathbb{F}_q} (-1)^{f(x) + \text{Tr}(wx)})^2 = q^2,
\end{align*}
\]

(Solving this system of equations gets)

\[
\begin{align*}
  m_1 &= q - \left(\frac{q}{4}\right)^2, \\
  m_2 &= \frac{1}{2} \left(\frac{q}{4}\right)^2 + \frac{q}{4}(-1)^{f(0)}, \\
  m_3 &= \frac{1}{2} \left(\frac{q}{4}\right)^2 - \frac{q}{4}(-1)^{f(0)}.
\end{align*}
\]

Thus, the Walsh spectrum of \( f(x) \) is

\[
[0]q - \left(\frac{q}{4}\right)^2 [A] \left(\frac{q}{4}\right)^2 + \frac{q}{4}(-1)^{f(0)} [-A] \left(\frac{q}{4}\right)^2 - \frac{q}{4}(-1)^{f(0)}.
\]

By Theorem 1, we get the following three-weight linear codes \( C_f \) over \( \mathbb{F}_{2^q} \).

**Theorem 5.** Suppose that \( q = 2^n \) and \( f(x) \) is a Boolean function from \( \mathbb{F}_q \) to \( \mathbb{F}_2 \) with the Walsh spectrum defined by (22). Let \( (w_1, w_2, w_3) = (0, A, -A), \) \( W_f(0) = w_i \) for some \( i \in \{1, 2, 3\} \) and \( (m_1, m_2, m_3) \) be given by (22). Assume that \( w_i - w_j \neq -q \) for each \( j \in \{1, 2, 3\} \) and \( j \neq i \). If \( t \mid n \) and the condition (5) is satisfied, then the code \( C_f \) of (5) is a three weights linear code over \( \mathbb{F}_{2^q} \) with the parameters \( \left[\frac{1}{2^q - 1}(2^{n-1} + \frac{1}{2}(w_i - 1 - (1)^{f(0)})), n/t \right] \) and the weight enumerator

\[
E_{C_f}(x) = 1 + (m_i - 1) x^{2^{n-1} - 1} + \sum_{1 \leq j \leq 3, j \neq i} m_j x^{2^{n-1} + \frac{1}{2}(w_i - w_j)}.
\]

Many Boolean functions with three Walsh values have been found. Here we only consider two cases. **Case 1: Monomial Polynomials.**

For any nonzero integer \( b \), let \( v_2(b) \geq 1 \) be the 2-adic exponential valuation of \( b \) defined by \( 2^{v_2(b) + 1} \nmid b \) and \( 2^{b_2(b) + 1} \nmid b \). The known monomial Boolean functions \( f(x) = \text{Tr}(x^d) \) with three Walsh values \( 0, A, -A \) can be summarized in Table II which are listed in [13], where \( q = 2^n \) and \( 1 \leq d \leq q - 1 \). We refer the reader to [13] and the references therein for detailed information.

| Series | \( d \) | Conditions | \( A \) |
|---|---|---|---|
| I | \( 2^h + 1 \) | \( v_2(h) \geq v_2(n) \) | \( \sqrt{2^n d(h, n)} q \) |
| II | \( 2^h - 2^h + 1 \) | \( v_2(h) \geq v_2(n) \) | \( \sqrt{2^n d(h, n)} q \) |
| III | \( 2^m + 2^m - 1 \) | \( n = m, 2 \mid m \) | \( \sqrt{2^q} \) |
| IV | \( 2^m + 3 \) | \( n = m, 2 \mid m \) | \( \sqrt{2^q} \) |
| V | \( 2^m + 3 \) | \( n = m + 1 \) | \( \sqrt{2^q} \) |
| VI | \( 2 \cdot 3^h + 1 \) | \( n \nmid 4h + 1 \) | \( \sqrt{2^q} \) |

In Table III, it is clear that the series (III-VI) are semibent functions, and the series (I-II) are semibent functions if and only if \( gcd(d, n) = 1 \) or \( gcd(d, n) = 2 \).

For \( f(x) = \text{Tr}(x^d) \), we have \( f(0) = 0 \). Thus, the length of the code \( C_f \) constructed in Theorem 5 becomes \( \frac{1}{2^{2^q - 1}}(2^n + w_i - 2) \). In order to construct three weights linear codes \( C_f \) over \( \mathbb{F}_{2^q} \), by using \( f(x) = \text{Tr}(x^d) \) listed in Table III, we need extra conditions \( t \mid n \) and \( 3^t - 1 \mid d \) which imply the condition (5).

**Example 4.** In order to construct the code \( C_f \) over \( \mathbb{F}_{2^q} \) by using \( f(x) = \text{Tr}(x^{2 \cdot 3^h + 1}) \) of the series (VI) in Table III, the extra conditions are \( 5 \mid n \) and \( 31 \mid 2 \cdot 3^h + 1 \) plus the conditions \( n \nmid 4h + 1 \) and \( 2 \cdot 3^h + 1 \leq 2^n - 1 \). By a computation of elementary number theory, these conditions are reduced to be \( h = 30l + 21, 5 \mid n, n \nmid 120l + 85 \) and \( 2 \cdot 3^h + 1 \leq 2^n - 1 \), where \( l \) is any non-negative integer.
To get linear codes over $\mathbb{F}_{2^t}$ with more flexible length, now we give a method as a consequence of Theorem 1. Based on this method, we can get more linear codes over $\mathbb{F}_{2^t}$ with three weights.

Let $q_1 = 2^{n_1}$ and $q_2 = 2^{n_2}$ for any positive integers $n_1 \geq 2$ and $n_2 \geq 2$. Define a Boolean function

$$f(x_1, x_2) = f_1(x_1) + f_2(x_2) : \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \rightarrow \mathbb{F}_2,$$

where $f_1(x_1) : \mathbb{F}_{q_1} \rightarrow \mathbb{F}_2$ and $f_2(x_2) : \mathbb{F}_{q_2} \rightarrow \mathbb{F}_2$ are Boolean functions.

By definition, we know that the Walsh values of $f$

$$W_f(y_1, y_2) = \sum_{(x_1, x_2) \in \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}} (-1)^{f(x_1, x_2)}(-1)^{Tr_{n_1}^1(y_1) + Tr_{n_2}^2(y_2)} = W_{f_1}(y_1)W_{f_2}(y_2)$$

for all $(y_1, y_2) \in \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$. From this we know that if the Walsh spectrums of $f_1$ and $f_2$ are $[w_1]^{m_1}[w_2]^{m_2} \cdots [w_{r_1}]^{m_{r_1}}$ and $[\omega_1]^{\mu_1}[\omega_2]^{\mu_2} \cdots [\omega_{r_2}]^{\mu_{r_2}}$ respectively, then the Walsh spectrum of $f$ is $\prod_{1 \leq i \leq r_1, 1 \leq j \leq r_2}[w_i\omega_j]^{m_{ij}}$, which can be reduced to

$$[\Omega_1]^{M_1}[\Omega_2]^{M_2} \cdots [\Omega_r]^{M_r} \quad (\Omega_1 < \Omega_2 < \cdots < \Omega_r, M_i \geq 1, 1 \leq i \leq r)$$

by using $[w_1^{m_1}] [w_1]^{m_2}$ is $[w_1]^{m_1+m_2}$. Thus the number of the Walsh values of $f$ may be less than $r_1r_2$.

We define

$$\bar{D} = \bar{D}_f = \{(x_1, x_2) \in \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \setminus \{(0,0)\} : f(x_1, x_2) = 0\}$$

and $\bar{N} = |\bar{D}|$. Let $\mathbb{F}_{2^t}$ be a subfield of $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$, namely, $t | \gcd(n_1, n_2)$. Then $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$ is a $\mathbb{F}_{2^t}$-vector space by $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$ for all $\alpha \in \mathbb{F}_{2^t}$, $x_1 \in \mathbb{F}_{q_1}$ and $x_2 \in \mathbb{F}_{q_2}$. The dimension of the vector space $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$ over $\mathbb{F}_{2^t}$ is $\frac{1}{t}(n_1 + n_2)$. Moreover, if $t$ satisfies the following condition:

$$f_1(\alpha x_1) = f_1(x_1) \text{ and } f_2(\alpha x_2) = f_2(x_2) \text{ for all } \alpha \in \mathbb{F}_{2^t}, x_1 \in \mathbb{F}_{q_1} \text{ and } x_2 \in \mathbb{F}_{q_2} ,$$

then

$$f(\alpha(x_1, x_2)) = f(\alpha x_1, \alpha x_2) = f(\alpha x_1) + f(\alpha x_2) = f_1(x_1) + f_2(x_2) = f(x_1, x_2).$$

Therefore the set $\bar{D}$ of (25) is a disjoint union of $N$ sets $P^{(i)}\mathbb{F}_{2^t}$ if the condition (26) is satisfied, where $P^{(i)} = (x_1^{(i)}, x_2^{(i)}) \in \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}, 1 \leq i \leq N$ and $N = \frac{\bar{N}}{t^{n_1}}$. We define

$$D = D_f = \{P^{(1)}, P^{(2)}, \cdots , P^{(N)}\}$$

and the following linear code over $\mathbb{F}_{2^t}$ by

$$C = C_f = \{c_b : b = (b_1, b_2) \in \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}\},$$

where

$$c_b = (Tr_{n_1}^1(b_1 x_1^{(1)}) + Tr_{n_2}^2(b_2 x_2^{(1)}), Tr_{n_1}^1(b_1 x_1^{(2)}) + Tr_{n_2}^2(b_2 x_2^{(2)}), \cdots , Tr_{n_1}^1(b_1 x_1^{(N)}) + Tr_{n_2}^2(b_2 x_2^{(N)})) \in \mathbb{F}_{2^t}^N.$$

We have the following theorem. Its proof is similar to that of Theorem 1 and is omitted.

**Theorem 6.** Let $f$ be a Boolean function given by (24), $t | \gcd(n_1, n_2)$, $t$ satisfies the condition (26), and let the Walsh spectrums of $f_1$ and $f_2$ be $[w_1]^{m_1}[w_2]^{m_2} \cdots [w_{r_1}]^{m_{r_1}}$ and $[\omega_1]^{\mu_1}[\omega_2]^{\mu_2} \cdots [\omega_{r_2}]^{\mu_{r_2}}$, respectively. Assume that the Walsh spectrums of $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ is reduced to $[\Omega_1]^{M_1}[\Omega_2]^{M_2} \cdots [\Omega_l]^{M_l}$, where $M_\lambda \geq 1$ for each $\lambda \in \{1, 2, \cdots , l\}$ and all $\Omega_\lambda (1 \leq \lambda \leq l)$ are distinct integers. Then the code $C_f$ of (29) is a $[\frac{1}{2(t^2-1)}(q_1 q_2 + \Omega_i - 1 - (-1)^{f_1(0)+f_2(0)}) \sum_{1 \leq j \leq l, j \neq i} M_j x^j] \cdot \mathbb{F}_{2^t}$ linear code over $\mathbb{F}_{2^t}$ with the weight enumerator

$$1 + (M_i - 1) x^{\frac{M_i q_2}{2}} + \sum_{1 \leq j \leq l, j \neq i} M_j x^j.$$
where $P_j = \frac{1}{2^{n/2}}(q_1 q_2 + \Omega_i - \Omega_j)$ for each $j$ within $1 \leq j \leq l$ and $i$ is determined by $\Omega_i = W_f(0,0) = W_{f_1}(0)W_{f_2}(0)$ for some $i$ within $1 \leq i \leq l$.

As special cases of Theorem 6 we give some examples as follows.

**Example 5.** Let $n_1$ be even, $f_1(x_1)$ be a bent function, and $f_2(x_2) = Tr_{1}^{q_2}(c x_2)$ with $c \in \mathbb{F}_{q_2}^*$. Then the Walsh spectrums of $f_1$ and $f_2$ are $\left[\sqrt{q_1}\right]^{1/2}q_1 + \frac{1}{q_1}\sqrt{q_1}$ and $\left[0\right]^{q_2} - 1/q_2$ respectively. Thus the Walsh spectrum of $f$ is $\left[0\right]^{q_1(q_2 - 1)}q_2\sqrt{q_1}\left[\frac{1}{q_1} + \sqrt{q_1}\right] - q_2\sqrt{q_1}\left[\frac{1}{q_1} - \sqrt{q_1}\right]$. Let $t = 1$. Then the linear code $C_f$ of (29) constructed by Theorem 6 is a binary linear code with the following three weights:

$$\frac{1}{2}(q_1 q_2 + \Omega_i - \Omega_1), \quad \frac{1}{2}(q_1 q_2 + \Omega_i - \Omega_2), \quad \frac{1}{2}(q_1 q_2 + \Omega_i - \Omega_3),$$

and the weight enumerator can be given by (31), where $(\Omega_1, \Omega_2, \Omega_3) = (0, q_2\sqrt{q_1}, -q_2\sqrt{q_1})$ and $i$ is determined by $\Omega_i = W_{f_1}(0)W_{f_2}(0)$.

Note that this kind function $f$ in Example 5 is called partially-bent function in [14] (see [14, Theorem 2.1]).

**Example 6.** Let $t = 5$, $n_1 = 2m$, $m = 5\mid m$, $l$ be a non-negative integer, $h = 30l + 21, 5|n_2, n_2|20l + 85$ and $2^3h + 1 \leq 2^n - 1$. We have a bent function $f_1(x_1) = Tr_{1}^{q_1}(\alpha x_1^m - 1)$ in Example 2 and a Boolean function $f_2(x_2) = Tr_{1}^{q_2}(x_2^2 \cdot 3^h + 1)$ in Example 4. It is clear that $\mathbb{F}_{25}$ is a subfield of $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$. The condition (26) is satisfied since $2^5 - 1 = 2^m - 1$ and $2^5 - 1 = 2 \cdot 3^h + 1$. By definition, Examples 2 and 4 and Table 7 we know that the Walsh spectrums of $f_1$ and $f_2$ are $\left[\sqrt{q_1}\right]^{m+ - \sqrt{q_1}m-}$ and $\left[0\right]^{q_0}\left[\sqrt{q_2}\right]^{\mu+ - \sqrt{q_2}\mu-}$, respectively. Thus the Walsh spectrum of $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ is

$$\left[0\right]^{(m_+ + m_-)q_0}\left[\sqrt{q_1}\right]^{m_+ - \sqrt{q_1}m_-} - \sqrt{q_1}\left[\sqrt{q_2}\right]^{\mu_+ - \sqrt{q_2}\mu_-},$$

where $(m_+, m_-) = \left(\frac{1}{2}(q_1 + \sqrt{q_1}), \frac{1}{2}(q_1 - \sqrt{q_1})\right)$ and $\{\mu_0, \mu_+ , \mu_-\}$ is determined by (22). Therefore the code $C_f$ constructed by Theorem 6 is a three weights linear code over $\mathbb{F}_{25}$ and the weight enumerator of the code $C_f$ is given by (37).

**Example 7.** Let $A_1 \geq 1$ and $A_2 \geq 1$, and let $f_1(x_1) : \mathbb{F}_{q_1} \to \mathbb{F}_2$ and $f_2(x_2) : \mathbb{F}_{q_2} \to \mathbb{F}_2$ be Boolean functions with the Walsh spectrums of $\left[0\right]^{m_0}\left[A_1\right]^{m_+ - A_1}\left[A_1\right]^{m_-}$ and $\left[0\right]^{q_0}\left[A_2\right]^{\mu_+ - A_2}\left[A_2\right]^{\mu_-}$, respectively. Then the Walsh spectrum of $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ is $\left[0\right]^{M_0}\left[A_1 A_2\right]^{M_+ - A_1 A_2}\left[A_1 A_2\right]^{M_-}$, where

$$\{M_0, M_+, M_-\} = \{m_0(\mu_0 + \mu_+ + \mu_-) + (m_+ + m_-)\mu_0, m_+ \mu_+ + m_- \mu_- + m_\mu_+, \mu_0 + \mu_+ + \mu_- + m_\mu_+\}.$$

If $\mathbb{F}_2^i$ is a subfield of $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$, the code $C_f$ constructed by Theorem 6 is a three weights linear code over $\mathbb{F}_2^i$ and the weight enumerator of the code $C_f$ is given by (37).

**Case 2: Quadratic Functions.**

For all quadratic functions with the following form:

$$f(x) = \sum_{i=1}^{n/2} Tr(c_i x^1 x^{2^i}), \quad (c_i \in \mathbb{F}_q, q = 2^n),$$

the Walsh spectrum of $f(x)$ is $\left[0\right]^{2n-2h}\left[2^n-h\right]^{2h-1} + 2^{h-1} - 2^n-h\left[2^{h-1}-2^{h-1}\right]$, where $h$ can be determined by quadratic form theory (see [15]). If $2h = n$, then $f$ is a bent function. Otherwise, $2h < n$, $f$ has three Walsh values and the code $C_f$ is a binary linear code with three weights. The weight enumerators of the code $C_f$ is given by Theorem 5. Remark that $f$ is semibent if and only if $n - h = \frac{n-1}{2}$.

Recently, some functions $f$ with three Walsh values $(w_1, w_2, w_3)$ and $w_i \neq 0$ for each $i \in \{1, 2, 3\}$ have been found (see [16, Theorem 3.2]). For such functions $f$, the binary linear code $C_f$ has three weights and its weight enumerator can be calculated by Theorem 11.
IV. CONCLUDING REMARKS

In this paper, we generalized the construction of linear codes of Ding’s method. Base on this generalization, we constructed two classes of linear codes \( \tilde{C}_f \) and \( C_f \) (see Theorem 1) over \( \mathbb{F}_{2^t} \) from a Boolean function \( f : \mathbb{F}_q \to \mathbb{F}_2 \) and their weight distributions determined by the Walsh spectrum of this Boolean function \( f \), where \( q = 2^n \) and \( \mathbb{F}_{2^t} \) is some subfield of \( \mathbb{F}_q \). Moreover, the parameters of more linear codes over \( \mathbb{F}_{2^t} \) with a few weights can be derived directly. Particularly, a number of classes of two-weight and three-weight codes are derived from some known classes of bent, semibent, monomial and quadratic Boolean functions. Instead of writing down all these codes, we documented a few classes of them as examples in this paper. In addition, the complete weight enumerator of the linear code \( \tilde{C}_f \) and the weight enumerator of the linear code \( C_f \) presented in this paper were settled in simple way in terms of the Walsh spectrum of \( f \).

For further generalization and research in \( \mathbb{F}_{p^n} \) with prime \( p \geq 3 \) case, we refer the reader to [17], [18] and [19].

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