Vassiliev-Kontsevich invariants and Parseval’s theorem

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Abstract

We use an example to provide evidence for the statement: the Vassiliev-Kontsevich invariants \( k_n \) of a knot (or braid) \( k \) can be redefined so that \( k = \sum_0^\infty k_n \). This constructs a knot from its Vassiliev-Kontsevich invariants, like a power series expansion. The example is pure braids on two strands \( P_2 \cong \mathbb{Z} \), which leads to solving \( e^r = q \) for \( r \) a Laurent series in \( q \). We set \( r = \sum_1^\infty (-1)^{n+1}(q^n - q^{-n})/n \) and use Parseval’s theorem for Fourier series to prove \( e^r = q \). Finally we describe some problems, particularly a Plancherel theorem for braid groups, whose solution would take us towards a proof of \( k = \sum_0^\infty k_n \).

1 Introduction

Throughout we think of knots as being in \( \mathbb{R}^3 \) and braids as being elements of a group. Sometimes we will say knot when we mean an isotopy class of knots, and a braid when we mean a realisation of a braid in \( \mathbb{R}^3 \). Often we will need finite, and sometimes convergent infinite, formal sum of knots or braids. The context will usually make clear which is meant. For example, in \( k = \sum_0^\infty k_n \) the quantity \( k \) is the isotopy class of a knot, and each \( k_n \) is a convergent formal sums of isotopy classes of knots. Usually, \( b \) will refer to a realisation of a braid.

The Vassiliev-Kontsevich invariant \[ b_n = b_n(b) \] of a braid \( b \) can be calculated by using the height \( h \) to slice the \( b \) into slices and then performing an iterated \( n \)-slice integral over the simplex \( 1 \geq h_1 > \ldots > h_n \geq 0 \). The integrand measures the ‘twistyness’ of the slice, and composition of braids is used to glue the slices together. Each invariant \( b_n \) lies in a finite-dimensional vector space, which is usually taken to be a quotient \( V_n/V_{n+1} \) in the Vassiliev filtration (see [5] and Section 3 below).

To calculate \( k_n \) of a knot \( k \) the same method can be used, except that the height function \( h \) will have critical points, each of which makes a contribution that is glued into answer. In this paper we will use \( * \) to denote, as appropriate, either the group law for braids or the connected sum operator for knots. We can also define \( * \) on the \( k_n \). In particular, if \( k \) and \( k' \) are two knots (or braids) it then follows that \( (k * k')_n = \sum_{i+j=n} k_i * k'_j \).

It is not known if a knot \( k \) is determined by its invariants \( k_n \). We approach this problem by finding a space \( K \) which contains \( k \), and then lifting \( k_n \) from \( V_n/V_{n+1} \) and into \( K \). One can then ask if \( k = \sum k_n \). We show that this approach works for braids on two strands and suggest how it might be extended to more strands and to knots.

Throughout let \( q \) be a generator for the group \( P_2 \cong \mathbb{Z} \) of pure braids on two strands. We think of \( q \) as two strands rotating around each other in \( \mathbb{R}^3 \). Because each slice is simply a rotation of any other, the integrand is constant. It follows that the integrand for \( b_n = b_n(q) \) is the \( n \)-fold \(*\)-product \( t^n \) of the integrand \( t \) for \( b_1(q) \).

The region of integration is the unit \( n \)-simplex, with volume \( 1/n! \), and so \( b_n(q) = t^n/n! \) and thus at least formally \( \sum_0^\infty b_n = e^t \).

For \( k = \sum k_n \) to hold, the integrand must be special. In particular, it must be a sum of knots (or braids). Some simple calculations, which we omit, show that the sum must be infinite and so questions of convergence arises. Throughout we will use \( \mathcal{K} \) to denote formal infinite sums of (isotopy classes of) knots, whose coefficients are \( L^2 \)-convergent and similarly \( \mathcal{P}_m \) for \( P_m \). We use \( \mathcal{S} \cong \mathcal{S} \ast \mathcal{P}_m \) to denote ‘pure braid changes’ to a slice \( S \) on \( m \) strands.

Recall that we wish to solve \( q = \sum b_n(q) \), as a special case of \( k = \sum k_n \). Let us now write \( \tau \) for \( b_1(q) \). The problem now amount to solving \( q = e^r := \sum_1^\infty \tau^n/n! \) for \( \tau \) in a vector space that also contains \( \tau^n \), for \( n > 1 \). To obtain a candidate for the solution \( \tau \in \mathcal{P}_2 \) we use a trick. Write \( p = q^{-1} \). We can write \( q = (1 + q)/(1 + p) \) and so at least formally our candidate is \( \ln(1 + q) - \ln(1 + p) \).
Definition 1.1.
\[ \tau = \sum_{1}^{\infty} (-1)^{n+1} (q^n - p^n)/n \in P_2 \]

Because \( \sum_{1}^{\infty} 1/n^2 \) is absolutely convergent, \( \tau \) is in \( P_2 \). Note that \( f(z) = \sum_{1}^{\infty} (-1)^{n+1} (z^n - z^{-n})/n \) is nowhere absolutely convergent.

2 Proof of \( e^{\tau} = q \)

Earlier we saw that this is a special case of \( k = \sum k_n \). In this section we write \( P_2 \) as \( L^2(Z) \). We will prove

Theorem 2.1. \( \tau \in L^2(Z) \), as defined in Definition 1.1, satisfies \( \exp(\tau) = q \).

This is a shorthand for saying first that the convolutions \( \tau, \tau^2, \tau^3, \ldots \) all lie in \( L^2(Z) \) and second that the sum \( 1 + \tau + \tau^2/2! + \ldots \) converges to \( q \in L^2(Z) \). To prove this result we use Fourier series and Parseval’s theorem.

For any integrable function \( f \) defined on \([-\pi, \pi]\) we as usual let
\[ c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta) \, d\theta \]
denote the \( n \)-th complex Fourier coefficient of \( f \). We now state

Theorem 2.2 (Parseval’s theorem). Let \( A(x) \) and \( B(x) \) be integrable functions on \([-\pi, \pi]\) with complex Fourier coefficients \( a_n \) and \( b_n \). Then
\[ \sum_{-\infty}^{\infty} a_n b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(x) B(x) \, dx .\]

For the function \( f(\theta) = \theta \) we have
\[ c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \theta \, d\theta = \frac{i}{2n\pi} e^{-in\theta} \bigg|_{-\pi}^{\pi} - \frac{i}{2n\pi} \int_{-\pi}^{\pi} e^{-in\theta} \, d\theta = \frac{i(-1)^n}{n} \]
for \( n \neq 0 \), while \( c_0(f) = \int_{-\pi}^{\pi} \theta \, d\theta = 0 \). Thus, as a series \( \tau \) is the Fourier transform of \( i\theta \).

We can extend this result as follows (the proof will come later). For \( \psi \in L^2(Z) \) we use \( c_n(\psi) \) to denote \( \psi_n \), which we also interpret as the coefficient of \( q^n \).

Theorem 2.3.
\[ c_n(\tau^n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} (i\theta)^n \, d\theta \]

Proof of Theorem 2.1. The algebraic part of the proof, which relies on Theorem 2.3 is
\[ c_n(\exp(\tau)) = \sum \frac{c_n(\tau_m)}{m!} \]
\[ = \frac{1}{2\pi} \sum \int_{-\pi}^{\pi} e^{-in\theta} \frac{(i\theta)^m}{m!} \, d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \sum \frac{(i\theta)^m}{m!} \, d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} e^{i\theta} \, d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(1-n)\theta} \, d\theta \]
and hence \( c_1 = 1 \) and \( c_n = 0 \) otherwise. The analytic part is that the sum-integral is absolutely convergent and so, by Fubini’s theorem, we can perform the integration first (which then allows us to simplify the sum).
Proof of Theorem 2.3. We rewrite the result to be proved as

\[ c_n(\tau^m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (i\theta)^{m-1} \times (i\theta) e^{-in\theta} d\theta \]

and apply Parseval’s theorem with \( A = (i\theta)^{m-1} \) and \( B = i\theta e^{-in\theta} \) (and an induction hypothesis). This tells us that the right hand side is equal to \( \sum c_k(\tau^{m-1})c_k(B) \) and as

\[ c_k(i\theta e^{-in\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} i\theta e^{in\theta} e^{-ik\theta} d\theta = c_{n-k}(\tau) \]

the result follows. \( \square \)

3 Taking values in \( \mathcal{P}_m \), not \( V_n/V_{n+1} \)

Here we discuss how to extend the main result to \( \mathcal{P}_3 \). This will also help us understand better the result for \( \mathcal{P}_2 \). Prior knowledge of Bar-Natan’s paper \[1\] would help the reader. Here we outline the standard construction, but draw attention to differences. Recall that the Vassiliev-Kontsevich invariants can be evaluated by gluing together slice contributions. In \( \mathcal{P}_2 \) each slice is effectively the same as any other, and \( \mathcal{P}_2 \) is commutative. This makes the definition of \( \tau \) quite simple.

For \( \mathcal{P}_3 \) the rôle of the slice is not so clear. We presented \( e^\tau = q \) as a calculation of \( q \) from its Vassiliev-Kontsevich invariants. (This is the inverse problem to computing \( b_n \) from \( b \).) To compute \( b_1(b) \) of a braid \( b \) one divides \([0,1]\) into slices and sum the contributions made by each slice. This contribution uses \( \int dt/(z_1 - z_2) \) to measure the twist in the slice. But we want, for example, \( b_1(q) \) to be \( \tau \). This can be obtained by adding a factor of \( \tau \) to the integrand. However, this factor must be introduced geometrically, as a slice contribution (see Figure 1).

![Figure 1: e^\tau = q.](image)

In the usual Kontsevich definition each \( b_n \) lies in a finite dimensional vector space, which can be taken to be the quotient \( V_n/V_{n+1} \) in the Vassiliev ‘braids with \( n \) double points’ filtration of the vector space of finite formal sums of braids (also known as the group ring).
Now suppose we have a slice $S$ with $m \geq 3$ strands in it. Each relative motion between a pair of strands contributes to the slice. In the usual definition the integrand value space for this contribution is $V_1/V_2$. To achieve $k = \sum k_n$ we require $S \cong S \star \mathcal{P}_m$ as the value space (see Figure 2). This is an important difference.

When the value space is $V_1/V_2$ we can ignore the other strands when we compute the contribution made by a pair. But some simple examples (not given here) show that when $S$ is the value space we have to link in the other strands, and it seems likely that every element of $S \star \mathcal{P}_m$ will so appear.

Note that in $\mathcal{P}_2$ the difference $q^b - 1/n \sum q^i$ lies in the Vassiliev subspace $V_1$ and in $\mathcal{P}_2$ the corresponding sequence converges to $q^0$. The same argument also shows that in $\mathcal{K}$ and $\mathcal{P}_m$ the Vassiliev subspaces are dense. The Kontsevich invariants are an analogue of differentiation, which is well known not to be a continuous operator on $L^2$ spaces.

4 Problems

Here we state some problems related to proving $k = \sum k_n$.

**Problem 4.1.** Suppose we have a slice $S$ with $m$ strands. What is the contribution, which lies in $S \cong S \star \mathcal{P}_m$, of that slice? In particular, for each $b \in \mathcal{P}_m$ what is coefficient of $b$ in the slice contribution?

Note that $b$ is a member of a braid group, while $S$ is (part of) the realisation of a braid. Here the difference is important. We have already solved this problem, in the case of two strands. Let $\theta$ be the twisting or ‘fractional winding number’ of the two strands and let $b$ be an element of $\mathcal{P}_2$. We know that if $b = q^n$ then the contribution is $\theta \times (-1)^{n+1}/n \times b$. To extend the main result to $\mathcal{P}_3$ we need a similar formula for each $b \in \mathcal{P}_3$.

**Problem 4.2.** Suppose slice $S$ has three strands and $b$ is in $\mathcal{P}_3$. Produce a formula that depends on the pairwise twisting in $S$ and also say $b_1(b) \in V_1/V_2$ and $b_2(b) \in V_2/V_3$ that generalises the two-strand case. (See Figure 2)

Here is a hint. One might expect the twistyness $\theta$ in the slice to be divided into parts, with each part going to some $b$ in $\mathcal{P}_3$. In particular, might be looking to solve $\sum c(b) = \theta$, where the sum is over the elements $b$ of $\mathcal{P}_3$. However, in the case of $\mathcal{P}_2$ the twistyness of $q_n$ is (of course) $n$, and so the corresponding sum is $\sum_{n=1}^{\infty} (-1)^{n+1}(n - (-n))/n = (1 - 1) - (1 - 1) + \ldots$ which may be thought of as $-2\zeta(0)$, where $\zeta$ is the analytic continuation of $\sum n^{-s}$, and $\zeta(0) = -1/2$. (Similarly, naively applying $b_m(q^n) = n^m/m!(b_1(q)^m$ to $\tau$ leads to the divergent sum $\sum n^{m-1}$.)

There is in addition a constraint. A realisation in $\mathbb{R}^3$ of a braid $b$ on $n$ strands can be deformed into another realisation. This should not change the value of say $b_2(b) \in S$. When $b_2$ takes values in $V_2/V_3$ this is a consequence of the integrand satisfying the Arnold identity [11 §4.2]. When we use $S$ this makes this constraint considerably more exacting. It seems to require every element of $\mathcal{P}_m$ to appear. We can add critical points to the representation.
of a knot by adding an \(N\)-shaped kink in to a vertical line. This does not, of course, change knot invariants. Thus, in addition to the slices, the critical points also contribute. Bar-Natan et al. [2] have found an explicit formula for this contribution, when values are taken in \(V_n/V_{n+1}\). They call this ‘wheeling’, from the shape of some diagrams used.

**Problem 4.3.** Extend wheeling so that it work for \(K\).

Here are two braid group questions.

**Problem 4.4.** Let \(a\) and \(b\) be any two elements in \(\mathcal{P}_3\), the space of \(L^2\) formal sums of braids in \(P_3\). Is the product \(a \ast b\) absolutely convergent?

**Problem 4.5.** Is there a Plancherel theorem for \(P_3\)?

Here are two more general questions.

**Problem 4.6.** Drinfeld’s associator [3] is an alternative to the Kontsevich’s integral approach. Is there a way of refining it to produce values in \(\mathcal{P}_n\)?

**Problem 4.7.** Is \(k = \sum k_n\) a new connection between the mathematics of knots and quantum field theory?

5 Summary

We saw that the problem \(k = \sum k_n\) for knots leads to solving \(e^\tau = q\), whose solution relies on Parseval’s theorem. There is a local description of this, in terms of the contribution made by the slices in the Kontsevich integral. If each slice \(S\) made a suitable contribution lying in the braid modifications \(S \cong S \ast P_m\) of \(S\) we would have \(b = \sum b_n\) for braids. Further, if the wheeling at critical points can be similarly extended, then we have \(k = \sum k_n\) for knots. In the previous section we described some problems that would need to be solved, for this program to be carried out.

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