COMBINATORIAL MODEL FOR THE CLUSTER CATEGORIES OF TYPE E

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Abstract. In this paper we give a geometric-combinatorial description of the cluster categories of type $E$. In particular, we give an explicit geometric description of all cluster tilting objects in the cluster category of type $E_6$. The model we propose here arises from combining two polygons, and it generalizes the description of the cluster category of type $A$ given by Caldero-Chapoton-Schiffler in [CCS06].

1. Introduction

Caldero-Chapoton-Schiffler defined in [CCS06] categories arising from homotopy classes of paths between two vertices of a regular $(n + 3)$-polygon. Independently, Buan-Marsh-Reiten-Reineke-Todorov defined in [BMR+06] cluster categories as certain orbit categories of the bounded derived category of hereditary algebras. The latter are algebras arising from oriented graphs $Q$ with no oriented cycles. When $Q$ is an orientation of a Dynkin graph of type $A_n$, the category constructed in [CCS06] coincides with the one of [BMR+06].

The model for cluster categories of type $E$ we propose here, is based on the idea of doubling the set of oriented diagonals in a given regular polygon and combining the dynamics of these two sets in an appropriate way.

More precisely, in the case of $E_6$ we start with a set of oriented diagonals of a regular heptagon $\Pi$. Then we reinterpret a work of the author, [Lam12], to see that the oriented diagonals in $\Pi$ give rise to an additive category equivalent to the 2-repetitive cluster categories of type $A_4$. Doubling the set of oriented diagonals of $\Pi$ and identifying certain pairs of diagonals, we obtain an additive category $\mathcal{C}$. The objects of $\mathcal{C}$ are single and paired colored oriented diagonals of $\Pi$, and the morphisms are defined by a class of rotations inside $\Pi$ between them, modulo relations. Changing slightly the morphism spaces of $\mathcal{C}$ we define a second additive category $\mathcal{C}$ associated to $\Pi$. In this way the Auslander-Reiten quiver of $\mathcal{C}$ sits on a cylinder, and the Auslander-Reiten quiver of $\mathcal{C}$ sits on a Möbius band.

Theorem 4.1. There is an equivalence between the cluster category of type $E_6$ and the category $\mathcal{C}$ associated to $\Pi$.

Under this equivalence, we are able to describe all cluster tilting objects of $\mathcal{C}_{E_6}$ in our model. The strategy for this will be to first determine two fundamental families of cluster configurations $\mathcal{F}_1$ and $\mathcal{F}_2$ corresponding to cluster tilting sets of $\mathcal{C}_{E_6}$. From these two families we deduce the remaining cluster tilting sets of $\mathcal{C}_{E_6}$ using the rotation inside $\Pi$, induced from the Auslander-Reiten translation $\tau$ in $\mathcal{C}_{E_6}$, as well as a symmetry $\sigma$ of our model.
**Theorem (6.7).** In $\Pi$

- 350 different cluster configurations have one long paired diagonal, and they arise from $F_1$ through $\tau$.
- 483 other cluster configurations arise from $F_2$ through $\sigma$ and $\tau$.

This characterization of cluster tilting sets of $C_{E_6}$ allows us to deduce geometrical moves describing the mutation process between cluster tilting objects in $C_{E_6}$. The geometrical moves we find extend the mutation process of cluster categories of type $A_n$, as described by Caldero-Chapoton-Schiffler in [CCS06], to the setting of colored oriented diagonals. In fact, most of the mutations inside $C_{E_6}$ will correspond to flips of colored oriented diagonals of $\Pi$.

Geometrical models for cluster categories of other types have been investigated also in Sch08 [BZ11, Tor]. Moreover, in Fomin-Pylyavskyy [FP12], Fomin-Pylyavskyy also model the cluster algebra structure of the homogeneous coordinate ring $\mathbb{C}[Gr_{3,7}]$ of the affine cone over the Grassmannian $Gr_{3,7}$ of three dimensional subspaces in a seven dimensional complex vector space. By a result of Scott, [Sco06], it is known that the ring $\mathbb{C}[Gr_{3,7}]$ is a cluster algebra type of $E_6$. The approach of [FP12] however is different then the one we propose here, as it relies on relations satisfied by tensor diagrams, called skein relations of tensor diagrams.

The paper is organized as follows. In Section 2, some preliminary results and definitions are given. In particular we are going to define and state the fundamental properties of orbit categories. In this section we also remind the reader on the action of the shift functor on the Auslander-Reiten quiver of $D^b(\text{mod} \, kE)$ will be given.

In Section 3, the additive categories $C^o$ and $C$ associated to $\Pi$ are constructed. Both categories originate from the geometric model of 2-repetitive cluster categories of type $A_4$. The class of objects of both categories $C^o$ and $C$ are given by single and paired colored oriented diagonals of $\Pi$. The morphism spaces of the two categories are slightly different, but in both cases they are generated by rotations of single and paired colored oriented diagonals modulo certain equivalence relations. The difference between $C^o$ and is that the Auslander-Reiten quiver of $C^o$ sits on a cylinder while the Auslander-Reiten quiver of $C$ sits on a Möbius band.

Section 4 is devoted to the proof of the main result Theorem 4.11 stating the equivalence between $C$ and the cluster category of type $E_6$, $C_{E_6}$. In this section we also describe the essentially surjective functor from $C_{E_6}$ to the cluster category of type $A_4$, $C_{A_4}$.

In Section 5, we investigate in $\Pi$ the extension spaces of the cluster category of type $E_6$, using the projection of $C_{E_6}$ to $C_{A_4}$. To describe the lift of the Ext-spaces of $C_{A_4}$ to $C_{E_6}$ we introduce the concept of curves of colored oriented diagonals of $\Pi$. In Proposition 5.3 we use these curves to determine the dimensions of the extension space for each pair of indecomposable objects in $C_{E_6}$.
In Section 6, we define cluster configurations of colored oriented diagonals of $\Pi$ and determine two fundamental families of cluster configurations in $\Pi$. These configurations are in bijection with cluster tilting set, or cluster tilting objects, of $\mathcal{C}_{E_6}$. Then in Theorem 6.7 we classify all cluster configurations in $\Pi$.

In Section 7 results concerning the mutation process of cluster configurations will be stated, see Proposition 7.3. For this we adapt the concept of flips of (unoriented) diagonals to the context of colored oriented diagonals of $\Pi$.

Finally, in Section 8 we indicate how our geometric construction can be extended to cluster categories of type $E_7$, and $E_8$.

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## 2. Preliminaries

Let $k$ be an algebraically closed field and let $Q$ be an orientation of a simply laced Dynkin graph of type $A_n$, $E_6$, $E_7$ or $E_8$. Let $\text{mod}_k Q$ be the abelian category of $k$-finite dimensional right-modules over the path algebra $kQ$. Let $D := D_Q := D^b(\text{mod}_k Q)$ be the bounded derived category of $\text{mod}_k Q$ endowed with the shift functor $\Sigma : D \to D$ and the Auslander-Reiten translation $\tau : D \to D$ characterized by $\text{Hom}_D(X, -)^* \cong \text{Hom}_D(-, \Sigma \circ \tau X)$, for all $X \in D$.

### 2.1. Orbit categories of $D$.

We are interested in the orbit categories $\mathcal{C}_i^Q$ of $D$, $i \in \{1, 2\}$, generated by the action of cyclic group generated by the auto-equivalences $F_1 := \tau^{-1} \circ \Sigma$, resp. $F_2 := (\tau^{-1} \circ \Sigma)^2 = \tau^{-2} \circ \Sigma^2$. The objects of $\mathcal{C}_i^Q$ are the same as the objects of $D$ and

$$\text{Hom}_{\mathcal{C}_i^Q}(X, Y) := \bigoplus_{p \in \mathbb{Z}} \text{Hom}_D(X, (F^i)^p Y).$$

Morphisms are composed in a natural way.

When $i = 1$, $\mathcal{C}_Q^1$ is the cluster category of type $Q$ defined in [BMR+06], and independently in [CCS06] in geometric terms for $Q$ of type $A_n$. When $i = 2$, $\mathcal{C}_Q^2$ is the 2-repetitive cluster category studied by the author in [Lam12] for $Q$ of type $A_n$, and introduced by Zhu in [Zhu11] for general acyclic quivers.

### 2.2. Fundamental properties of orbit categories.

Like $D$, the categories $\mathcal{C}_Q^i$ are Krull-Schmidt and have finite dimensional Hom-spaces. The categories $\mathcal{C}_Q^i$ are triangulated categories, and the projection functor $\pi_i : D \to \mathcal{C}_i^Q$, $i = 1, 2$ is a triangle functor, see [Kel05] Theorem 1]. The induced shift functor is again denoted by $\Sigma$. Moreover, the categories $\mathcal{C}_Q^i$ have AR-triangles and the AR-translation $\tau$ is induced from $D$. The categories $\mathcal{C}_Q^i$ also have the Calabi-Yau property, i.e. $(\tau \circ \Sigma)^m \sim \Sigma^n$ as triangle functors, hence we identify $\tau \circ \Sigma$ with the Serre functor of $\mathcal{C}_Q^i$. In particular, in $\mathcal{C}_Q$ we have that $n = 2$ and $m = 1$, hence $\mathcal{C}_Q$ is Calabi-Yau of dimension 2.
In $C_Q^2$ we have that $m = 2$ and $n = 4$ in the above isomorphism of triangle functors, thus $C_Q^2$ is said to be a Calabi-Yau category of fractional dimension $\frac{4}{2}$. Notice that a Calabi-Yau category of fractional dimension $\frac{4}{2}$ is in general not a Calabi-Yau category of dimension 2. In the categories considered in the sequel we adopt the convention $\text{Ext}_{C_Q^2}(X, Y) := \text{Hom}_{C_Q^2}(X, \Sigma Y)$.

2.3. **Auslander-Reiten quiver of a Krull-Schmidt category.** A stable translation quiver $(\Gamma, \tau)$ in the sense of Riedtmann, [Rie80], is a quiver $\Gamma$ without loops nor multiple edges, together with a bijective map $\tau : \Gamma \rightarrow \Gamma$ called translation such that for all vertices $x$ in $\Gamma$ the set of starting points of arrows which end in $x$ is equal to the set of end points of arrows which start at $\tau(x)$.

For $(\Gamma, \tau)$ one defines mesh category as the quotient category of the additive path category of $\Gamma$ by the mesh ideal, see for example [Kel10]. In particular, the mesh category of $(\Gamma, \tau)$ is an additive category.

In the next result, let $ZQ$ be the repetitive quiver of $Q$, see [Hap88, I,5.6] for a reminder on this construction. Let $\tau : ZQ \rightarrow ZQ$ be the automorphism defined on the vertices $(n, i)$ of $ZQ$ by $\tau(n, i) = (n - 1, i)$, for $n \in \mathbb{Z}$, $i$ a vertex of $Q$.

**Theorem 2.1.** [Hap87, I.5.5] If $Q$ is an orientation of a simply laced Dynkin graph then the AR-quiver of $D$ is isomorphic (as stable translation quiver) to $(ZQ, \tau)$.

Let $\text{ind} D$ be the full subcategory of $D$ of indecomposable objects.

**Theorem 2.2.** [Hap88, I.5.6] Let $Q$ be an orientation of a simply laced Dynkin graph. The mesh category of $(ZQ, \tau)$ is equivalent to $\text{ind} D$.

A first important consequence of this result is that the AR-quiver of $D$ is independent of the orientation of $Q$.

2.4. **Induced action of $\Sigma$ on $ZQ$.** Next we point out some known facts about the group of automorphisms of $ZQ$ commuting with $\tau$. These considerations, together with Theorem 2.1 will enable us to determine the precise shape of the AR-quiver of the orbit categories $C_Q^2$. From [MY01, Chap. 4] one deduces that the induced action of $\Sigma$ on $ZE_6$ coincides with the action of $\rho \circ \tau^{-6}$, where $\rho$ is an order two automorphism defined on $ZE_6$. The cases $E_7$ and $E_8$ will be treated in Section 3.

3. **Categories of arcs in heptagons**

3.1. **Paired diagonals of $\Pi$.** Let $\Pi$ be a regular heptagon, with vertices numbered in the clockwise order by the group $\mathbb{Z}/7\mathbb{Z}$. Denote by $(i, j)$ the unoriented diagonal of $\Pi$ joining the vertices $i$ and $j$, and we denote by $[i, j]$ the oriented diagonal of $\Pi$ starting at $i$ and ending at $j$. We do not consider boundary segments as oriented diagonals. Double this set of oriented diagonals of $\Pi$ and distinguish each set with colors using subscripts $R, B$, e.g. $[1, 3]_R$ is the red diagonal linking the vertex 1 to 3 of $\Pi$.

Next we group some of the colored oriented diagonals into pairs. For every $i \in \mathbb{Z}/7\mathbb{Z}$, $[i, i + 2]_P = [i + 2, i]_P := \{|i, i + 2|_R, [i + 2, i]_B\}$ is a short paired diagonal, and $[i, i + 3]_P = [i + 3, i]_P := \{|i, i + 3|_R, [i + 3, i]_B\}$ is a long paired
diagonal of $\Pi$. For all paired diagonals we replace the subscripts $R$ and $B$ with $P$.

In this way we obtain 14 paired diagonals, 7 short and 7 long, together with 28 single colored oriented diagonals. Notice that once colored oriented diagonals are paired, they stop existing in our model as single colored oriented diagonals.

We refer to all these as colored oriented diagonals of $\Pi$ and we write $P$ for the set of all such diagonals. Thus, we have

$$P := \{ [i, i + 2]_P, [i, i + 3]_P, [i, i + 4]_R, [i, i + 5]_R, [i + 4, i]_B, [i + 5, i]_B, \ i \in \mathbb{Z}/7\mathbb{Z}\}.$$  

Notice that each unoriented diagonal $(i, j)$ gives rise to precisely three diagonals in $P$, namely $[i, j]_R, [j, i]_B$ and $[i, j]_P = \{ [j, i]_R, [i, j]_B \}$.

3.2. Minimal clockwise rotations. Below we define the minimal rotations between diagonals of $P$. Let $i \in \mathbb{Z}/7\mathbb{Z}$, in the next diagram we indicate with an arrow all the possible minimal rotations between diagonals of $P$:

For unoriented diagonals, minimal clockwise rotations have been introduced in [CCS06, §2] with the aim of modeling irreducible morphisms in $\mathcal{C}_n$.

3.3. Translation on $\Gamma^\circ$. Let $\tau^\circ : \Gamma^\circ \to \Gamma^\circ$ be the map defined on the vertices of $\Gamma^\circ$ by $\tau^\circ(D) := D'$ if $D'$ is obtained from $D$ by an anticlockwise rotation through $\frac{2\pi}{7}$ around the center of $\Pi$. Notice that by knowing how $\tau$ acts on single red or blue oriented diagonals, allows to deduce the action of $\tau$ on paired diagonals: $\tau([i, j]_P) = [i - 1, j - 1]_P$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Minimal rotations between paired diagonals}
\end{figure}
It is straightforward to check that \((\Gamma^\circ, \tau^\circ)\) is a stable translation quiver. Denote the additive category generated by the mesh category of \((\Gamma^\circ, \tau^\circ)\) by \(C^\circ\).

3.4. The quiver \(\Gamma\). The quiver \(\Gamma^\circ\) has all the properties needed to model the AR-quiver of \(C_{E_6}\), except that it lies on a cylinder instead of a Möbius strip. To solve this issue we construct a new quiver \(\Gamma\), lying on a Möbius strip, simply by replacing the arrows: \([7, 5]_R \to [1, 5]_R\) by \([7, 5]_R \to [5, 1]_B\), and \([5, 7]_B \to [5, 1]_B\) by \([5, 7]_B \to [1, 5]_R\). All other arrows stay unchanged.

In Figure 3 an illustration of \(\Gamma\) is provided, in the last slice we indicate the identifications occurring.

In the sequel, we will often need the automorphism \(\rho: \Gamma \to \Gamma\) of order two defined by

\[
\rho([i, j]_c) = \begin{cases} 
[j, i]_B & \text{if } c = R, \\
[j, i]_R & \text{if } c = B, \\
[j, i]_P = [i, j]_P & \text{otherwise.}
\end{cases}
\]

Since paired diagonals are invariant under change of colors and orientations, \(\rho \equiv \text{id}\) on \([i, i+2]_P, [i, i+3]_P, i \in \mathbb{Z}/7\mathbb{Z}\). Geometrically, the action of \(\rho\) can be seen as follows. On \(\Pi\) the map \(\rho\) is the symmetry sending red oriented diagonals to blue diagonals of opposite orientation, see Figure 2.

![Figure 2. Symmetry \(\rho\).](image)

3.5. Translation on \(\Gamma\). Next we equip \(\Gamma\) with the translation \(\tau: \Gamma \to \Gamma\) defined by:

\[
\tau([i, j]_R) = \begin{cases} 
[i - 1, j - 1]_R & \text{when } [i, j]_R \neq [1, 5]_R, [1, 6]_R \\
[4, 7]_B & i = 1, j = 5 \\
[5, 7]_B & i = 1, j = 6,
\end{cases}
\]

and

\[
\tau([i, j]_B) = \begin{cases} 
[i - 1, j - 1]_B & \text{when } [i, j]_B \neq [5, 1]_B, [6, 1]_B \\
[7, 4]_R & i = 5, j = 1 \\
[7, 5]_R & i = 6, j = 1.
\end{cases}
\]

By knowing how \(\tau\) acts on single red or blue oriented diagonals, we deduce that \(\tau([i, j]_P) = [i - 1, j - 1]_P\).

Inside \(\Pi\) the action of \(\tau\) is given by an anticlockwise rotation of \(2\frac{2\pi}{7}\) around the center of \(\Pi\), composed with change of color and orientation at the vertex 1 of \(\Pi\). In particular, \(\tau\) on \(\Gamma\) coincides with \(\tau^\circ\) on \(\Gamma^\circ\) on all diagonals different then \([1, 5]_R, [1, 6]_R\) and \([5, 1]_B, [6, 1]_B\).
Lemma 3.1. \((\Gamma, \tau)\) is a connected stable translation quiver.

Proof. We have to check three things. First, that \(\Gamma\) is connected, has no loops, and is locally finite. Second, for every vertex of the quiver we have to check that the number of arrows going to the given vertex equals the number of arrows leaving. Third, we have to check that the map \(\tau\) is bijective. It is straightforward to see that the quiver \((\Gamma^\circ, \tau^\circ)\) has the mentioned properties. We conclude observing that these properties are preserved under the adjustments defining \((\Gamma, \tau)\). \(\square\)

4. Equivalence of categories

Let \(\Pi\) be a regular heptagon and let \(C\) be the additive category generated by the mesh category associated \((\Gamma, \tau)\), see Section 3.3.

Theorem 4.1. There is an equivalence of additive categories between the cluster category \(C_{\text{E}_6}\) and the category \(C\) associated to \(\Pi\).

Proof. Since \(\text{ind}\ C_{\text{E}_6}\) is equivalent to the mesh category of its AR-quiver, we only have to check that there is an isomorphism of stable translation quivers between the AR-quiver of \(C_{\text{E}_6}\) and the quiver of diagonals \((\Gamma, \tau)\). Then the equivalence between \(\text{ind}\ C_{\text{E}_6}\) and the mesh category of \((\Gamma, \tau)\) induces an equivalence between \(C_{\text{E}_6}\) and the category \(C\) associated to \(\Pi\).

From the discussion in Section 2.4 it follows that the AR-quiver of \(C_{\text{E}_6}\) is isomorphic to the quotient \(\mathbb{Z}Q/ (\tau^{-1}\Sigma)\), with \(Q\) an orientation of the Dynkin graph of type \(E_6\). The translation \(\tau\) on \(\mathbb{Z}Q/ (\tau^{-1}\Sigma)\) is induced from \((\mathbb{Z}Q, \tau)\). Next, from the construction of \((\Gamma, \tau)\) given in Section 3.4 we deduce that \(\Gamma\) is as in Figure 3 with translation \(\tau\) given by a horizontal shift to the left. From these observations it is clear that \(\mathbb{Z}Q/(\tau^{-1}\Sigma) \cong \Gamma\), as stable translation quivers.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure3.png}
\caption{The quiver of diagonals \(\Gamma\).}
\end{figure}

The equivalence of the previous result allows us to define an auto-equivalence on \(C\), still denoted by \(\Sigma\), induced by the shift functor \(\Sigma\) of \(C_{\text{E}_6}\). Proceeding as in [Lam13, Lemma 3.10-Thm. 3.11] one can deduce that \(C\) is a triangulated category. Here we will not need the triangulated structure on \(C\).
4.1. The repetitive cluster category $C_{A_4}^2$. Let $(\tau^{-1} \circ \Sigma)^2$ be the automorphism of $ZA_4$ induced by the auto-equivalence $(\tau^{-1} \circ \Sigma)^2$ of $DA_4$. Let $\Gamma^o|_P$ be the quiver having as vertices the oriented red diagonals of $\Pi$ and where arrows between two vertices are drawn whenever there is a minimal clockwise rotation between the corresponding oriented red diagonals. Thus, $\Gamma^o|_P$ is the restriction of $\Gamma^o$ in Section 3.2 to the case where only one set of oriented diagonals is associated to $\Pi$.

In Figure 4 an illustration of $\Gamma^o|_R$ is provided.

![Figure 4](image)

**Figure 4.** The quiver $\Gamma^o|_R$ lying on a cylinder, and the sub-quivers $Q_1$ and $Q_2$.

The action of $\tau^o|_R$ on $\Gamma^o|_R$ is simply an horizontal shift to the left. Moreover, $(\Gamma^o|_R, \tau^o|_R) \cong (ZA_4/(\tau^{-1} \circ \Sigma)^2; \tau)$, as stable translation quivers, and it follows that: $(\Gamma^o|_R, \tau^o|_R) \cong (AR(C_{A_4}^2), \tau)$, see Lam[2] Prop. 3.7 of more details.

The construction of $\Gamma$ and $\Gamma^o$ was motivated by the following idea: glue two copies of the AR-quiver of $C_{A_4}^2$ to obtain the AR-quiver of $C_{E_6}$. To glue two copies of the AR-quiver of $C_{A_4}^2$ along two disjoint $\tau$-orbits resulted in pairing diagonals (as in Section 3.1).

More general AR-quivers can be glued together in this way, and in this context one can ask.

**Problem 4.2.** How do categorical properties of the original category behave under this gluing operation? How do cluster tilting sets behave under this operation?

4.2. Projections. Let $\Gamma_P$ be the quiver with vertices given by pairs of colored oriented diagonals of different color and opposite orientations: $\{[i, i+j]_R, [i, i+j]_B\}$, for $1 \leq i \leq 7$, $2 \leq j \leq 5$ (operations modulo 7). Two vertices of $\Gamma_P$ are linked with an arrow whenever there is a minimal clockwise rotation around a common vertex of $\Pi$ linking the two pairs of colored oriented diagonals. The translation on the vertices of $\Gamma_P$ is given by the anti clockwise rotation of $\frac{2\pi}{7}$ around the center of $\Pi$. Notice that $\Gamma_P$ lies on a cylinder.

Consider the projection $\pi_1 : \Gamma \rightarrow \Gamma_P$ defined in the following fashion. On single colored oriented diagonals $\pi_1$ is given by $\pi_1([i, i+j)_c] := \{[i, i+j]_c, \rho([i, i+j]_c)\}$, where $c \in \{R, B\}$, $j = 3, 4$ and $i \in \mathbb{Z}/7\mathbb{Z}$. On paired diagonals $\pi_1$ is the identity. On arrows $\pi_1$ is defined as follows. Let $k = 1, 2$ and $i \in \mathbb{Z}/7\mathbb{Z}$, and recall that $\rho \circ \rho = id$. Then on the arrows between paired diagonals $\pi_1$ is the identity, while arrows $[i, i+3]_P \rightarrow \rho^k([i, i+4]_R)$ are mapped to $[i, i+3]_P \rightarrow \{[i, i+4]_R, \rho([i, i+4]_R)\}$ and arrows $\rho^k([i, i+4]_R) \rightarrow \rho^{k+1}([i, i+5]_R)$ are mapped to $\{[i, i+4]_R, \rho([i, i+4]_R)\} \rightarrow \{[i, i+5]_R, \rho([i, i+5]_R)\}$. Thus, $\pi_1$ folds $\Gamma$ along its horizontal central line.
Next, let $\Gamma_\Pi$ be the quiver of (unoriented) diagonals associated to $\Pi$, as introduced first by [CCS06]. The vertices of $\Gamma_\Pi$ are the unoriented single (red) diagonals of $\Pi$, and the arrows correspond to minimal clockwise rotations around the vertices of $\Pi$ between unoriented single (red) diagonals of $\Pi$. Let $\tau_\Pi$ be the anticlockwise rotation in $\Pi$ through $2\pi/7$ restricted to unoriented red single diagonals. It is well known that $(\Gamma_\Pi, \tau_\Pi)$ is a stable translation quiver, and that $(\Gamma_\Pi, \tau_\Pi) \cong AR(C_{A_4})$, see [CCS06, Thm. 5.1].

Taking the natural projection $\pi_2 : \Gamma_P \to \Gamma_\Pi$, which maps $\{[i, i+j]_R, [i+j, i]_B\}$ to the unoriented diagonal $(i, i+j)$ of $\Pi$, $1 \leq i \leq 7$, $2 \leq j \leq 5$, and each arrow in $\Gamma_P$ to one arrow in $\Gamma_\Pi$, we get a surjective map of translation quivers $\tilde{\pi} := \pi_2 \circ \pi_1 : \Gamma \to \Gamma_\Pi$. This map then induces an essentially surjective functor $\pi : \mathcal{C}_{E_6} \to \mathcal{C}_{A_4}$, after identifying $\mathcal{C}_{E_6}$, resp. $\mathcal{C}_{A_4}$, with the additive category generated by the mesh category of the translation quiver $\Gamma$, resp. the quiver $\Gamma_\Pi$.

**Remark 4.3.** Notice that $\Gamma_P$ is isomorphic, as a stable translation quiver, to the quiver of red oriented diagonals $\Gamma^o |_R$.

### 4.3. Almost positive roots.

From Theorem 4.1 together with [BMR+06, Prop. 4.1] it follows that there is a bijection between the vertices of $\Gamma$ and the set of almost positive roots $\Phi_{\geq -1}$ consisting of the positive roots together with the negative simple roots, of the root system of type $E_6$.

Fixing an initial choice of a triangulation of a regular $(n+3)$-gon, an explicit bijection between all diagonals of the polygon and the set $\Phi_{\geq -1}$ associated to the root system of type $A_n$ was given in [CCS06].

**Problem 4.4.** Describe geometrically the bijection between the colored oriented diagonals of $\Pi$ and the almost positive roots $\Phi_{\geq -1}$ of the root system of type $E_6$.

### 5. Extension spaces

#### 5.1. Hammocks.

The support of $\text{Hom}(\tau^{-1}X, -)$ in $\mathcal{C}_Q$ is called the front Ext-hammock of $X$. The support of $\text{Hom}(-, \tau X)$ in $\mathcal{C}_Q$ is called the back Ext-hammock of $X$. These hammocks can be deduced from the AR-quiver using mesh relations, or using starting and ending functions, see [BMR+06, Chap. 8]. For AR-quivers isomorphic to $\mathbb{Z}Q$ with $Q$ an orientation of a Dynkin graph, the support of $\text{Hom}(-, -)$ has been described in detail in [Bon84].

Identifying again $\mathcal{C}_{E_6}$, resp. $\mathcal{C}_{A_4}$ with the additive categories generated by the mesh categories of $\Gamma$, resp. $\Gamma_\Pi$, in the sequel we view the hammocks inside $\Gamma$ or $\Gamma_\Pi$. Moreover, let $\text{Ext}^1_{\Pi}(D_X, D_Y) := \text{Ext}^1_{\mathcal{C}_{E_6}}(X, Y)$, for colored oriented diagonals $D_X$, $D_Y$ in $\Pi$ and indecomposable objects $X$ and $Y$ in $\mathcal{C}_{E_6}$ corresponding to $D_X$ and $D_Y$ by the equivalence of Theorem 4.1.

**Remark 5.1.** As $\mathcal{C}_{E_6}$ is 2 Calabi-Yau: $D\text{Ext}^1_{\mathcal{C}_{E_6}}(X, Y) \cong \text{Ext}^1_{\mathcal{C}_{E_6}}(X, Y)$, for all objects $X, Y$ in $\mathcal{C}_{E_6}$. Therefore, the back and front Ext-hammocks in $\mathcal{C}_{E_6}$ coincide for all objects. On the other hand, in $\mathcal{C}_{A_4}$ the back and front hammocks are disjoint, as the category is not 2-Calabi-Yau.
5.2. Lift of hammocks. Consider again the projection $\tilde{\pi} : \Gamma \to \Gamma_{\Pi}$. Let $D_{(i,j)} := \tilde{\pi}(D_X)$. For each $D_X$ we define two connected sub-quivers of $\Gamma$, $I_1(D_X)$ and $I_2(D_X)$, as follows. The vertices of both $I_1(D_X)$ and $I_2(D_X)$, are the vertices of $\Gamma$ in the Ext-hammock of $D_X$ in $\Gamma$ and in the preimage under $\tilde{\pi}$ of the Ext-hammock of $D_{(i,j)}$ in $\Gamma_{\Pi}$. The arrows of $I_1(D_X)$ and $I_2(D_X)$ coincide with the arrows of $\Gamma$. Then $I_1(D_X)$ contains the vertex $\tau^{-1}(D_X)$ and will be called the front crossing of $D_X$, $I_2(D_X)$ contains $\tau(D_X)$ and will be called the back crossing of $D_X$.

Note that for all $D_X$, the subquivers $I_1(D_X)$ and $I_2(D_X)$ are disjoint. In addition, all colored oriented diagonals in $\Pi$ crossing $D_X$ in an interior point of $D_X$ are vertices of $I_1(D_X) \cup \rho(I_1(D_X))$ and $I_2(D_X) \cup \rho(I_2(D_X))$.

See Figure 5, were the vertices of inside the front and back crossings of $D_X$ are in heptagons with bold boundary, for $D_X$ a colored oriented diagonal in the first slice of $\Gamma$.

![Figure 5. The Ext-hammocks of the diagonals with vertices of $I_1$ and $I_2$ represented in heptagons with bold boundary.](image_url)

In the next result, we assume that $D_X$ is a colored oriented diagonal of $\Pi$ in the first slice of $\Gamma$. This assumption can be dropped using $\tau$-shifts, or renumbering the vertices of $\Pi$. Moreover, we write $\partial \Pi$ to indicate the boundary of $\Pi$, and for two colored oriented diagonals $D_X$ and $D_Y$ we say that $D_Y$ enters the smaller region bounded by $D_X$ and $\partial \Pi$ if the arrow head of $D_Y$ goes to a vertex of $\partial \Pi$ inside the region and different from the vertices joined by $D_X$.

**Proposition 5.2.** Let $D_X, D_Y$ be colored oriented diagonals of $\Pi$. Assume $D_X$ is in the first slice of $\Gamma$, and that $D_X$ crosses $D_Y$.

- If $D_X$ is a paired diagonal, then $\dim_k(\operatorname{Ext}^1_{\Pi}(D_X, D_Y)) = 1$.
- If $D_X$ is a single diagonal, and $D_Y$ enters the smaller region bounded by $D_X$ and $\partial \Pi$, then $\dim_k(\operatorname{Ext}^1_{\Pi}(D_X, D_Y)) = 1$.

**Proof.** If $D_X$ is paired, $I_1(D_X)$ coincides with $I_1(D_X) \cup \rho(I_1(D_X))$ and $I_2(D_X)$ coincides with $I_2(D_X) \cup \rho(I_2(D_X))$, thus the vertices of $I_1(D_X)$ and $I_2(D_X)$ are all the oriented colored diagonals of $\Pi$ crossing $D_X$. 


If \( D_X \) is a single colored diagonal of \( \Pi \), we need to distinguish between the diagonals inside \( I_i(D_X) \) and \( \rho(I_i(D_X)) \), \( i = 1, 2 \). Then we observe that the colored oriented diagonals in \( I_1(D_X) \) and \( I_2(D_X) \) are precisely the ones satisfying the assumptions of the proposition.

5.3. **Curves of oriented colored diagonals.** The aim of this section is to divide the Ext-hammocks in \( C_{E_6} \) into curves. The reason why we do this is because for each colored oriented diagonal \( D_X \) we want to find a uniform geometric description of the elements inside the Ext-hammock of \( D_X \). Since the hammocks in \( C_{E_6} \) are very big, this goal seems hopeless. However, dividing the Ext-hammock of \( D_X \) into smaller sets, allows us to describe the elements of each such set in geometric terms. We will call these sets curves.

Let \( X \) be an indecomposable object of \( C_{E_6} \) and let \( D_X \) be the corresponding colored oriented diagonal viewed as a vertex of \( \Gamma \). For \( r \in \{2, 4\} \), the curves \( C_1(D_X), \ldots, C_r(D_X) \) of \( D_X \) in \( \Gamma \) are \( r \) collections of oriented colored diagonals having non-vanishing extensions with \( D_X \). Each collection \( C_i(D_X) \) has the shape of a curve in \( \Gamma \).

We begin defining the curves of \( D_X \), for \( D_X \) in the first slice of \( \Gamma \). For all other vertices \( D_X \) of \( \Gamma \), curves can be defined from the previous ones by \( \tau \)-shifts. The first curve of \([1,6]_R\), denoted by \( C_1([1,6]_R) \), is defined as follows:

\[
C_1([1,6]_R) := \{[7, 2 + i]_c, 0 \leq i \leq 3, c \in \{R, P\}\} \\
= \{[5, 7 + i]_c, 0 \leq i \leq 3, c \in \{R, P\}\} \\
= \{[6, 3]_R\}.
\]

The second curve of \([1,6]_R\), denoted by \( C_2([1,6]_R) \), is defined as follows:

\[
C_2([1,6]_R) := \{[5 - i, 7]_c, 0 \leq i \leq 3, c \in \{B, P\}\} \\
= \{[2, 7 + i]_c, 0 \leq i \leq 3, c \in \{B, P\}\} \\
= \{[1,4]_B\}.
\]

By definition \( C_1([1,6]_R) \) is obtained by the sequence of minimal clockwise rotations around the vertices 7, 5, 3 of \( \Pi \) starting in \( \tau^{-1}([1,6]_R) = [2, 7]_R \) and ending with \([6,3]_R\). Dually, \( C_2([1,6]_R) \) is obtained by a sequence of minimal anticlockwise rotations around the vertices 7, 2, 4 starting in \( \tau([1,6]_R) = [5,7]_B \) and ending in \([1,4]_B) \).

By construction the Ext-hammock of \([1,6]_R \) is \( C_1([1,6]_R) \cup C_2([1,6]_R) \). In Figure 5(a) the elements of \( C_1([1,6]_R) \) are drawn in the upper half of \( \Gamma \), while the elements of \( C_2([1,6]_R) \) are in the lower half.

Next, we are going to associate four curves to \([1,5]_R \). The first curve of \([1,5]_R \), \( C_1([1,5]_R) \), is the set containing colored oriented diagonals of \( \Pi \) obtained by a sequence of minimal clockwise rotations around the vertices 6, 4, 2 starting in \( \tau^{-1}([1,5]_R) = [5,2]_R \) and ending in \([5,2]_R \). The third curve \( C_3([1,5]_R) \) is obtained by a sequence of minimal anticlockwise rotations around the vertices 7, 2, 4 starting with \( \tau([1,5]_R) \) and ending in \([4,1]_B \). Moreover, \( C_4([1,5]_R) \) coincides with \( C_1([1,6]_R) \), and \( C_4([1,5]_R) \) coincides with \( C_2([1,6]_R) \).
For $D_X \in \{[1,6]_R, [1,5]_R\}$, the curves of $C_1(\rho(D_X)), \ldots, C_r(\rho(D_X))$ of $\rho(D_X)$ are defined by $\rho(C_1(D_X)), \ldots, \rho(C_r(D_X))$, $r \in \{2,4\}$.

We are left with defining the curves of the paired diagonals $[1,3]_P$ and $[1,4]_P$. For $[1,3]_P$ we have $C_1([1,3]_P)$ given by the set containing both single and paired colored oriented diagonals obtained by a sequence of minimal rotations in the clockwise order around the vertices 2, 7, 5 starting in $\tau^{-1}([1,3]_P)$ and ending in $[5,1]_P$. Similarly $C_2([1,3]_P)$ is obtained by a sequence of minimal anticlockwise rotations around the vertices 2, 4, 5 starting in $\tau([1,3]_P)$ and ending in $[6,2]_P$.

Next, $C_1([1,4]_P)$ is obtained by a sequence of of minimal clockwise rotations starting in $\tau^{-1}([1,4]_P)$ and ending in $[5,1]_P$. $C_3([1,4]_P)$ is obtained rotating in the anticlockwise order $\tau([1,4]_P)$ to $[4,7]_P$. Finally, $C_2([1,4]_P) = \tau^{-1}(C_1([1,3]_P))$ and $C_4([1,4]_P) = C_2([1,3]_P)$.

Let $D_X$ be in the first (and last) slice of $\Gamma$. In Figure 6(a)-(f) we represent the various curves of $D_X$. The numbers 1, 2, 3, 4 indicate the starting term of the curves $C_1(D_X), \ldots, C_4(D_X)$ and the colors of the heptagons indicate the curves they intersect.

5.4. Intersections of curves.

**Proposition 5.3.** Let $D_X, D_Y$ be vertices of $\Gamma$. Let $C_1(D_X), \ldots, C_r(D_X)$ be the curves of $D_X$, $r \in \{2,4\}$. Then $\dim_k(\Ext^1_H(D_X,D_Y))$ is equal to the number of curves of $D_X$ intersecting with $D_Y$ in $\Gamma$ (0 up to 3).

**Proof.** First, the Ext-hammocks in the AR-quiver of $C_{E_6}$ are invariant under $\tau$-shifts. After changing the image of the projective objects of mod$kE_6$ in the equivalence of Theorem 4.1 we can assume that $X$ or $Y$ corresponds to a diagonal in the first slice of $\Gamma$. By remark 5.1 we can treat the cases where $D_X$, or $D_Y$ belongs to the first slice of $\Gamma$ in the same way. Second, the curves of $D_X$ are by construction such that their intersection points coincide with the vertices $D_Y$ in $\Gamma$ for which $\dim_k(\Ext^1_H(D_X,D_Y)) \geq 1$. From the definition of morphisms in the mesh category of a stable translation quiver,
it follows that the number of curves intersecting in $D_Y$ is the dimension of $\Ext^1_{C_{E_6}}(D_X, D_Y)$. □

Let $D_X$ be in the first slice of $\Gamma$. In Figure 3(a)-(f) the dimension of the space $\dim_k(\Ext^1_{C_{E_6}}(D_X, D_Y))$ is expressed by the numbers of colors filling the heptagon containing $D_Y$. If $D_Y$ is in a white heptagon of $\Gamma$ then $\dim_k(\Ext^1_{C_{E_6}}(D_X, D_Y)) = 0$.

More precisely, the two curves represented in Figure 3(a) and (f) never intersect, and $\dim_k(\Ext^1_{C_{E_6}}(D_X, D_Y)) = 1$ for $D_Y$ in a colored heptagon, and $D_X$ in the first slice of $\Gamma$.

In Figure 3(b) and (e) the curve $C_1(D_X)$ intersects $C_2(D_X)$ in two vertices. Similarly, for $C_3(D_X)$ and $C_4(D_X)$. The curve $C_2(D_X)$ intersects $C_4(D_X)$ only once. The five heptagons where two curves meet have two colors (boundary and interior of the heptagon). Then $\dim_k(\Ext^1_{C_{E_6}}(D_X, D_Y)) = 2$ for $D_Y$ corresponding to one of these heptagons.

In Figure 3(c) there are two heptagons where three curves meet. They are drawn with three colors, and hence $\dim_k(\Ext^1_{C_{E_6}}(D_X, D_Y)) = 3$ for $D_Y$ corresponding to one of these two. Moreover, in nine heptagons two curves meet, and they are drawn in two colors.

Finally, in Figure 3(d) two curves are drawn, and they intersect only in one vertex of $\Gamma$.

6. Cluster tilting objects of $C_{E_6}$

Let $Q$ be an orientation of a simply-laced Dynkin graph with $n$ vertices. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ be a set of pairwise non isomorphic indecomposable objects of $C_Q$. If $\Ext^1_{C_Q}(T_i, T_j) = 0$ for all $T_i, T_j \in \mathcal{T}$, then one says that $\mathcal{T}$ is a cluster tilting set of $C_Q$. A cluster tilting object in $C_Q$ is the direct sum of all objects of a cluster tilting set in $C_Q$. Observe that knowing a cluster tilting objects allows to determines a cluster tilting set and viceversa. Moreover, given a cluster tilting set $\mathcal{T}$, one says that $\overline{\mathcal{T}} = \bigoplus_{j \neq i} T_j$, $T_j \in \mathcal{T}$ is an almost complete cluster tilting object if there is an indecomposable object $T_i^*$ in $C_Q$ such that $\overline{\mathcal{T}} \oplus T_i^*$ is a cluster tilting object of $C_Q$. The object $T_i^*$ is called the complement of $T_i$.

The mutation at $i$ of a cluster tilting object $\mathcal{T}$ in $C_Q$, for $1 \leq i \leq n$, is the operation which replaces the indecomposable summand $T_i$ in $\bigoplus_{j=1}^n T_j$ with the complement $T_i^*$ of $T_i$ in $\overline{\mathcal{T}} = \bigoplus_{j \neq i} T_j$.

The statements in the next Theorem are shown in [BMR+06].

**Theorem 6.1.** Let $\mathcal{T}$ be a cluster tilting object in $C_Q$.

- each almost complete cluster tilting object $\overline{\mathcal{T}}$ in $C_Q$ has exactly two complements, $\mathcal{T}$ and $\mathcal{T}^*$.

- If $\mathcal{T}$ and $\mathcal{T}^*$ are complements of $\overline{\mathcal{T}}$ then $\dim_k(\Ext^1_{C_Q}(T, T^*)) = 1$. On the other side, if $\dim_k(\Ext^1_{C_Q}(T, T^*)) = 1$, then there is an almost complete cluster tilting object $\overline{\mathcal{T}}$ such that $\mathcal{T}$ and $\mathcal{T}^*$ are complements of $\overline{\mathcal{T}}$.

After Proposition 3.8 in [FZ03] and [BMR+06] Thm. 4.5 we know that there are 833 cluster tilting sets, hence cluster tilting objects, in $C_{E_6}$. 
6.1. Cluster tilting objects in $\mathcal{C}_{A_n}$. After work of Caldero-Chapoton-Schiffler, see [CCS06], we know that cluster tilting objects of $\mathcal{C}_{A_n}$ are in bijection with the formal direct sums of diagonals belonging to a maximal collection of non-crossing diagonals in a regular $(n+3)$-gon. In this context mutations corresponds to flips of diagonals. More precisely, a flip replaces a diagonal $D_i$ in a given triangulation $\Delta$ with the unique other diagonal $D^*_i$ crossing $D_i$ and completing $\Delta \setminus D_i$ to a new triangulation of the regular $(n+3)$-gon.

6.2. First fundamental family of cluster configurations of $\Pi$.

**Definition 6.2.** A cluster configuration is a family of pairwise different colored oriented diagonals of $\Pi$, $\mathcal{T} = \{D_1, D_2, \ldots, D_6\}$, with the property that $\text{Ext}^1_{\Pi}(D_i, D_j) = 0$ for all $D_i, D_j \in \mathcal{T}$. A colored oriented diagonal $D^*_i$ is called complement of $D_i$ in $\mathcal{T}$ if $D^*_i \neq D_i$ and $\mathcal{T}'$ obtained from $\mathcal{T}$ after replacing $D_i$ by $D^*_i$ is a cluster configuration of $\Pi$.

Consider two heptagons, and a long paired diagonal $L_P := [i, i+3]$ for $i \in \mathbb{Z}/7\mathbb{Z}$ of $\Pi$. Our next goal is to complete $L_P$ to a set of colored oriented diagonals inside the two heptagons, giving rise to a cluster configuration of $\Pi$. For this we remark that $L_P$ divides each $\Pi$ into the quadrilateral $\Pi_4$ with boundary vertices $\{i, i+1, i+2, i+3\}$, and the pentagon $\Pi_5$ with boundary vertices $\{i, i+3, i+4, i+5, i+6\}$, $i \in \mathbb{Z}/7\mathbb{Z}$.

In Lemma 6.3 below we will see that triangulating $\Pi_4$ with a short paired diagonal, and each $\Pi_5$ with single diagonals of the appropriate color gives rise to cluster configurations. Notice that the two triangulations of $\Pi_5$ can be different, and the color is uniquely determined by the position of $L_P$ in $\Pi$, resp. in $\Gamma$.

**Lemma 6.3.** Let $L_P := [i, i+3]$ for $i \in \mathbb{Z}/7\mathbb{Z}$.

- For $i \neq 1$, triangulating each $\Pi_5$ with single diagonals of the same color, and $\Pi_4$ with a short paired diagonal gives a cluster configuration $T_{L_P}$ of $\Pi$.
- All cluster configurations of $\Pi$ containing $[j, j+3]$ for $1 \leq j, k \leq 7$ arise as $\tau^k(T_{L_P})$.

**Proof.** Since $i \neq 1$ we can assume that the region in $\Gamma$ outside the Ext-hammock of $L_P$ has only blue diagonals below $L_P$ and only red diagonals above $L_P$. Observes that the diagonals outside the Ext-hammock are precisely the diagonals involved in triangulations of the two copies of $\Pi_5$ and $\Pi_4$.

Then chose a short paired diagonal $S_P$ triangulating $\Pi_4$ with a short paired diagonal. Then triangulating a copy of $\Pi_5$ with only single red diagonals, and triangulating the second copy of $\Pi_5$ with only single blue diagonals yields a cluster configuration. With Proposition 5.2 we deduce that the arcs obtained in this way have no extension in each region above and below $L$ in $\Gamma$. One can then check that the Ext-hammocks in one region do not pass through the other region, nor though $S_P$. Thus, to each red triangulation one can choose a blue triangulations of $\Pi_5$, and all choices are possible. Similarly, one can complete $\{L_P, S^*_P\}$ to a cluster configuration, where $S^*_P$ is the other short paired diagonal triangulating $\Pi_4$. Notice that there are no other
possibilities to complete $L_P$ to a cluster configuration of $\Pi$. Next, there are 7 choices for $L_P$ in $\Gamma$. For each choice of $L_P$ the associated cluster configurations are obtained from the previous by rotation through $\tau$. Adjustment of the colors-orientations of the single diagonals triangulating $\Pi_5$ are needed if $L_P$ is the first slice of $\Gamma$.

We call the cluster configurations arising from a long paired diagonal triangulating two copies of $\Pi_5$ and $\Pi_4$ as describe in the first part of Lemma 6.3 the first fundamental family of cluster configurations, and we denote it by $\mathcal{F}_1$.

6.3. Second fundamental family of cluster configurations of $\Pi$. We saw in Lemma 6.3 that many cluster configurations correspond to two triangulations of $\Pi$. Our next goal is to describe a second family of cluster configurations describing the remaining cluster tilting set of $C_{E_6}$. For this the following general observation is needed.

For $i \in \mathbb{Z}/7\mathbb{Z}$, consider the long single red diagonal $L = [i, i+4]_R$ of $\Pi$. Then $L$ divides $\Pi$ into the quadrilateral $\Pi_4 := \{i + 4, i + 5, i + 6, i\}$, and the pentagon $\Pi_5 := \{i, i+1, i+2, i+3, i+4\}$. Let $T_L$ be a cluster configuration of $\Pi$ containing $L$. Then $T_L$ necessarily also contains one of the two short single diagonals triangulating $\Pi_4$, neighboring $L$ in $\Gamma$. Similarly for $\rho(L) = [i+4, i]_B$. More precisely,

Lemma 6.4. Let $i \in \mathbb{Z}/7\mathbb{Z}$, $L = [i, i+4]_R$ in $\Pi$, and $T_L$ be a cluster configuration containing $L$.

- If $i \neq 1$, exactly one of $[i, i+5]_R, [i+6, i+4]_R$ is in $T_L$.
- If $i = 1$, exactly one of $[i, i+5]_R, \rho([i+6, i+4]_R)$ is in $T_L$.

Similarly for $\rho(L)$. Moreover, in each case the two diagonals are complements to each other.

Proof. Let $i \neq 1$ and consider the Ext-hammock of $L$ in $\Gamma$. Since $L$ is not in the first slice of $\Gamma$ the diagonals triangulating $\Pi_4$ have the same color as $L$. Then one can check that all Ext-hammocks of objects outside the Ext-hammock of $L$, which are different from $[i, i-2]_R$ and $[i-1, i-3]_R$, never contain single diagonals inside the quadrilateral $\Pi_4$ in $\Pi$. Thus, by maximality we deduce that all cluster tilting sets containing $L$ necessarily also contain one of the diagonals inside $\Pi_4$. Taking one diagonal triangulating $\Pi_4$ rules out the other, thus the two single diagonals triangulating $\Pi_4$ are complements to each other. For $i = 1$, $L$ is in the first slice of $\Gamma$. Then one can proceed as before adjusting the color of the diagonal triangulating $\Pi_4$.

In view of the next result, we point out that the short single diagonals of Lemma 6.4 triangulating $\Pi_4$ and neighboring $L$ in $\Gamma$, are displayed in filled light gray heptagons in Figure 7(a)-(n).

Lemma 6.5. Every six-tuple of diagonals of Figure 7(a)-(n) determines a cluster configuration of $\Pi$.

Proof. For each choice of a short single diagonal of Lemma 6.4 triangulating $\Pi_4$ and neighboring $L$ in $\Gamma$, the claim can be verified by checking that the
diagonals in the highlighted heptagons have no extension among each other.

In the following, we refer to the collection of cluster configurations of Figure 7(a)-(n) as the second fundamental family of cluster configurations of Π, and we denote this family by $F_2$.

6.4. Symmetries in Π leading to cluster configurations. We determine two symmetries in Π leading to cluster configurations. One symmetry simply switches colors and orientations of the colored oriented diagonals of a given cluster configuration. The second one arises from a left-right symmetry of Γ, and corresponds to a reflection in Π.

There are two reasons why these symmetries are important. First, using these symmetries we can deduce all cluster configurations starting from the sets in $F_2$. Second, knowing how a cluster configuration behaves under mutation, allows to understand how the symmetric ones behave.

Let $i \in \mathbb{Z}/7\mathbb{Z}$, and let $h_i$ be the line in Π passing through $i$ and the middle point of $i + 3, i + 4$. Let $\sigma_i : \Pi \to \Pi$ the symmetry of Π defined as follows.

On all colored oriented diagonals not perpendicular to $h_i$, the map $\sigma_i$ is the reflection along $h_i$ followed by a switch of orientation. E.g. $\sigma_6([5, 2]_R) = [4, 7]_R, \sigma_6([2, 5]_B) = [7, 4]_B$. Let $c \in \{R, B, P\}$, then on $[i \pm 1, i \mp 1]_c$ the map $\sigma_i$ simply switches color and orientation, while $\sigma([i \pm 2, i \mp 2]_c) = [i \pm 2, i \mp 2]_c$.

Lemma 6.6. Let $T$ be a cluster configuration belonging to $F_2$. Then $\rho(T)$, and $\sigma_5(T)$ are also cluster configurations.
Proof. Apply the map $\rho$, resp. $\sigma_5$ to the cluster configuration of Lemma 6.6. Because of the shape of the Ext-hammocks one indeed produces cluster configurations. □

Notice that the set $\rho(\mathcal{T})$ of a cluster configuration $\mathcal{T}$ is an elements of the $\tau$-orbit of $\mathcal{T}$, while $\sigma_i(\mathcal{T})$ does not belong to any $\tau$-orbit of a cluster configuration of $\mathcal{F}_2$ (nor of $\mathcal{F}_1$).

6.5. Classification of cluster tilting sets of $\mathcal{C}_{E_6}$. From [FZ03, Prop. 3.8] and [BMR+06, Thm. 4.5] we know that there are 833 cluster tilting sets in $\mathcal{C}_{E_6}$. In the next result we give a complete geometric classification of all cluster tilting sets in $\mathcal{C}_{E_6}$ in terms of cluster configurations of $\Pi$.

Theorem 6.7. In $\Pi$

- 350 different cluster configurations have one long paired diagonal, and they arise from $\mathcal{F}_1$ through $\tau$.
- 483 other cluster configurations arise from $\mathcal{F}_2$ through $\sigma$ and $\tau$.

Corollary 6.8. In $\Pi$:

- 224 cluster configurations have precisely one short paired diagonal,
- 175 cluster configurations have precisely two short paired diagonals,
- 84 cluster configurations have no paired diagonals of $\Pi$.

All these cluster configurations are different.

Proofs of Theorem 6.7 and Corollary 6.8. The first part of the claim follows from Lemma 6.3. In fact, we saw that for each long paired diagonal $L_p$, there are 25 ways to triangulate one of the two pentagons $\Pi_5$ with single colored diagonals. Moreover, there are two ways to triangulate $\Pi_4$ with short paired diagonals. Thus, each $L_p$ gives rise to 50 different cluster configurations. Since there are 7 choices for $L_p$ in $\Pi$, the first claim follows.

For the second part of the claim the idea is to consider different cases, depending on the number of short paired diagonals leading to cluster configurations. First case: the only paired diagonal of $\mathcal{T}$ is a short one. Then $\mathcal{T}$ arises from the collection highlighted in (a),(b) or (c) in Figure 1, up to $\tau$-shifts and the $\sigma$-symmetry of Lemma 6.6. Moreover, after Lemma 6.7 for each colored oriented diagonal $[i, i-3]_c$, $c \in \{R, B\}$ in $\mathcal{T}$ there are two possible choices of neighboring short single diagonals in $\mathcal{T}$. Consequently, up to $\tau$-shifts, the there are 4 different cluster configurations arising from a collection of diagonals as in (a). Similarly for (b). The collection in (c) gives rise to 8 different cluster configurations up to $\tau$-shifts, as there are 4 choices for short single diagonals, and further 4 arise by taking the $\sigma$-symmetric case.

Summing up, the cluster configurations in (a), (b), (c) give rise to 224 different cluster configurations.

Second case: $\mathcal{T}$ has exactly two short paired diagonals. Then one distinguishes further into (d),(e) and (f) which have single colored oriented diagonals of the form $[i, i+3]_c$, $c \in \{R, B\}$. While the cluster configurations in (g),(i),(b) and (l) only contain short single diagonals of the form $[i, i+2]_c$, $c \in \{R, B\}$. Proceeding as before, taking into account the symmetry $\sigma$ of Lemma 6.6, one obtains the claimed number.

In the third case we count the cluster configurations arising from $\tau$-shifts of the cluster configurations in (m) and (n) having no paired diagonals. As
before we deduce that there are 28 the cluster configurations arising from \( \tau \)-shifts of (m) and 56 arising from \( \tau \)-shifts of the 4 cluster configurations in (n). Together this gives 84 cluster configurations without paired diagonals. □

6.6. The geometry of cluster configurations. In Section 6.2 we saw that cluster configurations in \( \mathcal{F}_1 \) correspond to triangulations with colored oriented diagonals of two copies of \( \Pi \). Cluster configurations of \( \mathcal{F}_2 \) are not as simple to describe. In Theorem 6.9 below we can give a general statement concerning the geometry of cluster configurations in \( \Pi \).

**Theorem 6.9.** All cluster configurations can be expressed as configurations of non-crossing colored oriented diagonals inside two heptagons.

**Proof.** First, given a cluster configurations in \( \mathcal{F}_1 \cup \mathcal{F}_2 \) we divide the colored oriented diagonals inside two heptagons by color. Paired diagonals appear in both heptagons.

Then we observe that all cluster configurations in \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are crossing free. Moreover, the symmetry \( \sigma \) produces new configurations of diagonals which are again crossing free. Taking \( \tau \)-shifts only rotates the entire configurations inside the two heptagons, occasionally switching colors and orientation according to the action of \( \tau \) inside \( \Pi \). Hence the crossing free property is preserved under \( \tau \)-shifts and the claim follows. □

The cluster configuration \( \mathcal{T} = \{ [5,3]_R, [5,2]_R, [5,1]_P, [5,6]_P, [3,5]_B, [2,5]_B \} \) is expressed in two heptagons in the center of Figure 10, the numbering of the vertices of one heptagon is highlighted in the figure. Paired diagonals appear in both heptagons and have labels.

The converse statement of Theorem 6.9 is not true, as configurations of non-crossing colored diagonals different then cluster configurations of \( \Pi \) are not cluster tilting sets of \( C_{E_6} \).

6.7. Symmetric cluster configurations. Let \( \mathcal{T}_{A_4} \) be a triangulation of \( \Pi \) consisting of unoriented arcs. We saw that \( \mathcal{T}_{A_4} \) is in bijection with a cluster tilting set of \( C_{A_4} \). Our next goal is to understand cluster configurations projecting to \( \mathcal{T}_{A_4} \) through \( \pi : C_{E_6} \to C_{A_4} \). We call cluster configurations of this type symmetric cluster configurations.

Examples of symmetric cluster configurations can be found in \( \mathcal{F}_1 \) when one takes the same triangulation of \( \Pi_5 \) consisting of single diagonals differing only in color and orientation. In \( \mathcal{F}_2 \) the only symmetric cluster configurations arises from (7d).

Next, let \( \pi_1 : \Gamma^o|_R \to \Gamma_\Pi, \rho : \Gamma \to \Gamma \) be as before, and divide \( \Gamma^o|_R \) into the isomorphic sub-quivers \( \Gamma_1 \) and \( \Gamma_2 \), as in Figure 4. For \( k = 1, 2 \), let \( \mathcal{T}^o_k := \pi_1^{-1}(\mathcal{T}_{A_4}) \cap \Gamma_k \) be in \( \Gamma^o|_R \) and let \( \mathcal{T}_k := \mathcal{T}^o_k \cup \rho(\mathcal{T}^o_k) \) be in \( \Gamma \).

Below we give a condition ensuring that \( \mathcal{T}_k \) is indeed a cluster configuration, \( k = 1, 2 \).

**Proposition 6.10.** Let \( k = 1, 2 \) and let \( \mathcal{T}_k \) be as before. Assume \( |\mathcal{T}_k| = 6 \) and \( \text{Ext}_1^1(D_i, D_j) = 0 \) for all \( D_i, D_j \) in \( \mathcal{T}_k \), \( 1 \leq i, j \leq 4 \). Then \( \mathcal{T}_k \) is a symmetric cluster configuration.

**Proof.** We have to show that \( \text{Ext}_1^1(D_i, D_j) = 0 \) for all \( D_i, D_j \) in \( \mathcal{T}_k \). It is clear that \( \text{Ext}_1^1(D_i, D_j) = 0 \) for all \( D_i, D_j \in \mathcal{T}_k \), and for all \( D_i, D_j \in \rho(\mathcal{T}_k) \).
In addition, since the Ext-hammocks of paired diagonals are \( \rho \)-symmetric, the Ext-spaces of paired diagonals vanishes on all objects in \( T_k \).

Next, we consider the case \( T_1 \) and the case \( T_2 \) separately. For the first case, let \( D_i \) be a single colored diagonal in \( T_1 \), \( 1 \leq i \leq 4 \). From the shape of the Ext-hammocks it is clear that \( \rho(D_i) \) is never in the Ext-hammock of \( D_i \). Second, one can check that if the Ext-hammock of \( D_i \) would contain a single diagonal \( D_j \) in \( \rho(T_1) \) then \( \dim_k(\text{Ext}_1^\Pi(D_i, \rho(D_j))) \neq 0 \), contradicting the assumption. Similarly, for the \( \rho \)-symmetric case.

Next, assume \( D_i \) is a single colored diagonal in \( T_2 \), \( 1 \leq i \leq 4 \). Then we will see that if the Ext-hammock of \( D_i \) contains a single diagonal \( D_j \) in \( \rho(T_2) \) different then \( \rho(D_i) \), then one gets a contradiction. In fact, if it was so then either the set consisting of \( D_i \) and \( D_j \) cannot be completed to two symmetric triangulations of \( \Pi \) with diagonals belonging to \( \Gamma_2 \), contradiction. Or \( \rho(D_j) \) is in the Ext-hammock of \( D_i \), contradiction.

\[ \Box \]

7. Mutations of cluster tilting objects in \( \mathcal{C}_{E_6} \)

In this section we will see that in many cases it is possible to deduce the mutation process in \( \mathcal{C}_{E_6} \) from the mutations process inside \( \mathcal{C}_{A_4} \). Moreover, we can deduce the mutation process for the families in \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) and taking \( \tau \)-shifts deduce it for the remaining cluster configurations.

In the next definitions we indicate by \( \overline{D} \) the unoriented single diagonal corresponding to a colored oriented diagonal \( D \) of \( \Pi \).

Definition 7.1. Let \( D_P \) be a paired oriented diagonal and let \( \mathcal{T} \) be a cluster configuration containing \( D_P \). The paired diagonal \( D_P^* \) is the flip-complement of \( D_P \), if \( D_P \) and \( D_P^* \) are related by a flip.

Definition 7.2. Let \( D_S \) be a single colored oriented diagonal and let \( \mathcal{T} \) be a cluster configuration containing \( D_S \). The single colored oriented diagonal \( D_S^* \) is the flip-complement of \( D_S \) in \( \mathcal{T} \), if \( \overline{D_S} \) and \( \overline{D_S^*} \) are related by a flip and \( \mathcal{T} \setminus D_S \cup D_S^* \) is a cluster configuration.

Notice that if a flip-complement exists then the color and orientation is uniquely determined by Lemma 6.3 and Lemma 6.4.

Proposition 7.3. Let \( D_L \) be a long colored oriented diagonal of \( \Pi \) giving rise to a cluster configuration.

- If \( D_L \) is paired: single colored diagonals triangulating \( \Pi_5 \), and paired diagonals triangulating \( \Pi_4 \) have a flip-complement.
- If \( D_L \) is single: single diagonals triangulating \( \Pi_4 \) have a flip-complement.

Proof. Let \( D_L \) be a long colored oriented diagonal dividing \( \Pi \) into the quadrilateral \( \Pi_4 \) and the pentagon \( \Pi_5 \). In Lemma 6.3 we saw that if \( D_L \) is paired, triangulating \( \Pi_4 \) with a paired diagonal, and two copies of \( \Pi_5 \) with single oriented diagonals, always gives a cluster configuration. Hence removing a single diagonal of a copy of \( \Pi_5 \) or a paired diagonal triangulating \( \Pi_4 \) can only be completed to a cluster configuration in two ways, namely with diagonals being flip-complement of each other. If \( D_L \) is single, the claim follows from Lemma 6.3. \( \Box \)
Further instances of the mutation process in $C_{E_6}$ can be described by flips of colored oriented diagonals in II, but not all mutations allow a description of this type. This is unsurprising, as for example not all mutations in the cluster algebra $\mathbb{C}[Gr_{3,7}]$ can be described through Plücker relations, see [Sco06].

With Figure 7 it is not hard to deduce all the remaining mutations occurring. For example, the cluster configuration in (a) can be mutated to (b), (c), (e), (l), (m) and to the flip of a diagonal inside one light gray colored heptagon. Similarly, we have a list for the other families. Some instances of the more complicated geometric exchanges can be found on the upper pentagon of Figure 10.

7.1. Exchange graph. In Figure 11 we display a part of the exchange graph of $C_{E_6}$. For each heptagon appearing in the figure the numbering of its vertices is as shown on the central heptagon. The vertices of the graph correspond to cluster configurations, hence to cluster tilting sets of $C_{E_6}$, edges are drawn when two cluster configurations are related by a single mutation. In the two central heptagons of Figure 10 the configuration of $T = \{(5,3)_R, (5,2)_R, (5,1)_P, (5,6)_P, (3,5)_B, (2,5)_B\}$ is displayed. The 8 neighboring configurations are placed on the vertices of the two central pentagons sharing the vertex corresponding to $T$. These 8 sets are obtained from $T$ through repeated flips of single diagonals, as described in Proposition 7.3. The vertices of the left pentagon are obtained after mutating $[6,2]_P$ in $T$.

7.2. Cluster tilted algebras. Let $T = T_1 \oplus \cdots \oplus T_n$ be a cluster tilting object of $C_Q$, then $\text{End}_{C_Q}(T)$ is the cluster tilted algebra of type $Q$. The quiver $Q_T$ of $\text{End}_{C_Q}(T)$ has no loops nor 2-cycles and it encodes precisely the exchange matrix of the cluster associated to $T$, see [BMR08] and [CK08]. In $C_{A_n}$, the quiver $Q_T$ can be read off from the triangulation $T$, see [CCS06]. The vertices of $Q_T$ are the diagonals of the triangulation and an arrow between $D_i$ and $D_j$ is drawn, whenever $D_i$ and $D_j$ bound a common triangle. The orientation of the arrow is $D_i \rightarrow D_j$, if $D_j$ is linked to $D_i$ by an anticlockwise rotation around the common vertex.

Extending the definition of $Q_T$ to the case at hand, the quivers corresponding to the cluster tilting sets of Figure 11 can be deduced. One could also read off the quivers, and relations, directly from $\Gamma$ by determining the spaces $\text{Hom}_{C_{E_6}}(T,T)$.

8. Cluster categories of type $E_7$ and $E_8$

When $Q$ is an orientation of a Dynkin diagram of type $E_7$, it was shown in [MY01, Chap. 4] that $\Sigma$ acts as $\tau^{-9}$ on $\mathbb{Z}E_7$. For $Q$ of type $E_8$, the action of $\Sigma$ coincides with $\tau^{-15}$ on $\mathbb{Z}E_8$. It follows that $\tau^{-1}\Sigma \cong \tau^{-10}$ in $\mathbb{Z}E_7$, and $\tau^{-1}\Sigma \cong \tau^{-16}$ in $\mathbb{Z}E_8$.

From these observations we deduce that the appropriate polygons to be considered in these cases are the regular 10-gon, resp. the regular 16-gon. Proceeding with a similar construction as in Section 3 we are able to show that a category of oriented colored diagonals in a 10-gon, resp. 12-gon, is
equivalent to $\mathcal{C}_{E_7}$, resp. to $\mathcal{C}_{E_8}$. Notice that since the Auslander-Reiten quivers of $\mathcal{C}_{E_7}$, resp. $\mathcal{C}_{E_8}$ lie on cylinders the identifications of Section 3.4 are omitted.

Moreover, also for the cluster category of type $E_7$, resp. $E_8$, cluster tilting sets can be described using geometric configurations of colored oriented diagonals inside a heptagon and octagon, resp. a heptagon and a nonagon. For example the cluster tilting set arising from the projection of the projective modules in $\text{mod}kE_7$, resp. $\text{mod}kE_8$ is represented in Figure 8-9.

Figure 8. A cluster tilting set of $\mathcal{C}_{E_7}$.

Figure 9. A cluster tilting set of $\mathcal{C}_{E_8}$.
Figure 10. Part of the exchange graph for a cluster category of type $E_6$. At each vertex of the graph, the diagonals with the same labels are identified.
References

[BMR+06] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov. Tilting theory and cluster combinatorics. Adv. Math., 204(2):572–618, 2006.

[BMR08] Aslak Bakke Buan, Robert J. Marsh, and Idun Reiten. Cluster mutation via quiver representations. Comment. Math. Helv., 83(1):143–177, 2008.

[Bon84] Klaus Bongartz. Critical simply connected algebras. Manuscripta Math., 46(1-3):117–136, 1984.

[BZ11] Thomas Brüstle and Jie Zhang. On the cluster category of a marked surface without punctures. Algebra Number Theory, 5(4):529–566, 2011.

[CCS06] P. Caldero, F. Chapoton, and R. Schiffler. Quivers with relations arising from clusters (An case). Trans. Amer. Math. Soc., 358(3):1347–1364, 2006.

[CK08] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. Invent. Math., 172(1):169–211, 2008.

[FP12] Sergey Fomin and Pavlo Pylyavskyy. Tensor diagrams and cluster algebras. ArXiv e-prints. October 2012.

[FZ03] Sergey Fomin and Andrei Zelevinsky. Y-systems and generalized associahedra. Ann. of Math. (2), 158(3):977–1018, 2003.

[Hap87] Dieter Happel. On the derived category of a finite-dimensional algebra. Comment. Math. Helv., 62(3):339–389, 1987.

[Hap88] Dieter Happel. Triangulated categories in the representation theory of finite-dimensional algebras, volume 119 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1988.

[Kel05] Bernhard Keller. On triangulated orbit categories. Doc. Math., 10:551–581, 2005.

[Kel10] Bernhard Keller. Cluster algebras, quiver representations and triangulated categories. In Triangulated categories, volume 375 of London Math. Soc. Lecture Note Ser., pages 76–160. Cambridge Univ. Press, Cambridge, 2010.

[Lam12] Lisa Lamberti. Repetitive higher cluster categories of type An. Accepted for publication in the Journal of Algebra and its Applications, 2012.

[Lam13] Lisa Lamberti. A geometric interpretation of the triangulated structure of m-cluster categories. Accepted for publication in Communications in Algebra, 2013.

[MY01] Jun-ichi Miyachi and Amnon Yekutieli. Derived Picard groups of finite-dimensional hereditary algebras. Compositio Math., 129(3):341–368, 2001.

[Rie80] C. Riedtmann. Algebren, Darstellungsköcher, Überlagerungen und zurück. Comment. Math. Helv., 55(2):199–224, 1980.

[Sch08] Ralf Schiffler. A geometric model for cluster categories of type Dn. J. Algebraic Combin., 27(1):1–21, 2008.

[Sco06] Joshua S. Scott. Grassmannians and cluster algebras. Proc. London Math. Soc. (3), 92(2):345–380, 2006.

[Tor] H. A. Torkildsen. A geometric realization of the m-cluster category of type tilde-a.

[Zhu11] Bin Zhu. Cluster-tilted algebras and their intermediate coverings. Comm. Algebra, 39(7):2437–2448, 2011.

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