Local vs. Global Variables for Spin Glasses

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Abstract

We discuss a framework for understanding why spin glasses differ so remarkably from homogeneous systems like ferromagnets, in the context of the sharply divergent low temperature behavior of short- and infinite-range versions of the same model. Our analysis is grounded in understanding the distinction between two broad classes of thermodynamic variables – those that describe the global features of a macroscopic system, and those that describe, or are sensitive to, its local features. In homogeneous systems both variables generally behave similarly, but this is not at all so in spin glasses. In much of the literature these two different classes of variables were commingled and confused. By analyzing their quite different behaviors in finite- and infinite-range spin glass models, we see the fundamental reason why the two systems possess very different types of low-temperature phases. In so doing, we also reconcile apparent discrepancies between the infinite-volume limit and the behavior of large, finite volumes, and provide tools for understanding inhomogeneous systems in a wide array of contexts. We further propose a set of ‘global variables’ that are definable and sensible for both short-range and infinite-range spin glasses, and allow a meaningful basis for comparison of their low-temperature properties.

KEY WORDS: spin glass; Edwards-Anderson model; Sherrington-Kirkpatrick model; replica symmetry breaking; mean-field theory; pure states; metastates; domain walls; interfaces

∗Partially supported by the National Science Foundation under grant DMS-01-02587.
†Partially supported by the National Science Foundation under grant DMS-01-02541.
1 Local variables and thermodynamic limits

In recent years, attempts have been made \[1, 2\] to draw a distinction between the thermodynamic limit as a ‘mathematical tool’ of limited physical relevance, and the physical behavior of large, finite systems, the real-world objects of study. In this note we discuss why this distinction is spurious, and show through several examples that this ‘tool’ is useful precisely for determining the behavior of large, finite systems. We will also discuss why this attempted distinction has caused confusion in the case of spin glasses, and how resolving it introduces some important new physics. For ease of discussion, we confine ourselves throughout to Ising spin systems.

One of the incongruities of the ‘thermodynamic limit vs. finite volume’ debate is that true thermodynamic states are in fact measures of the \textit{local} properties of a macroscopic system, while discussions of finite-volume properties — at least in the context of infinite-range spin glasses — focus entirely on \textit{global} quantities.

Of course, traditional thermodynamics (as opposed to statistical mechanics) is entirely a study of global properties of macroscopic systems. Quantities like energy or magnetization are collective properties of \textit{all} of the individual spins in a given finite- or infinite-volume configuration. Other global measures cannot be discussed in terms of individual spin configurations but rather are meaningful only in the context of a Gibbs distribution (also known as Gibbs state or thermodynamic state — we will use these terms interchangeably throughout). Entropy is the obvious example of such a variable. All of these together — energy, entropy, magnetization, and the various free energies associated with them — convey only coarse information about a system (though still extremely valuable).

What they convey little information about is the actual spatial structure of a state, or of the relationships among different states. Even a quantity like the staggered magnetization, which gives some information about spatial structure, does not shed significant light on local properties.

But for real systems one often does want information about local properties. In order to analyze local spatial and temporal structures one generally needs to employ \textit{local thermodynamic variables} — for example, $1-$, $2-$, \ldots, $n-$point correlation functions. In fact, these functions taken altogether convey \textit{all} of the information that can be known about that state in equilibrium — a Gibbs state is a specification of all possible \textit{n}-point (spatial) correlation functions, for every positive integer $n$.

Alternatively, one can define a thermodynamic state either as a convergent sequence (or subsequence) of finite-volume states as volume tends to infinity, or else intrinsically through the DLR equations \[3\]. But we will avoid a technical discussion here in the interest of keeping the discussion
focused on physical objects. We henceforth assume familiarity with concepts such as thermodynamic mixed state and thermodynamic pure state, which have been used extensively throughout much of the spin glass literature, and refer the reader who wishes to learn more to Section 4 of [4].

Are there any global quantities that say something about the spatial structure of a state? The answer is yes, and one in particular has proven to be very useful in comparing different pure state structures in the infinite-range Sherrington-Kirkpatrick (SK) model [5] of a spin glass. The SK Hamiltonian (in volume $N$) is

$$\mathcal{H}_N = -(1/\sqrt{N}) \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j$$

(1)

where the couplings $J_{ij}$ are independent, identically distributed random variables chosen, e.g., from a Gaussian distribution with zero mean and variance one; the $1/\sqrt{N}$ scaling ensures a sensible thermodynamic limit for free energy per spin and other thermodynamic quantities.

In a series of papers, Parisi and collaborators [6, 7, 8, 9] proposed, and worked out the consequences of, an extraordinary ansatz for the nature of the low-temperature phase of the SK model. Following the mathematical procedures underlying the solution, it came to be known as replica symmetry breaking (RSB). The starting point of the Parisi solution was that the low-temperature spin glass phase comprised not just a single spin-reversed pair of states, but rather “infinitely many pure thermodynamic states” [7], not related by any simple symmetry transformations.

But how were they related? To answer this question, Parisi introduced a global quantity — exactly of the sort we were just asking about — to quantify such relationships. The actual notion of pure ‘state’ in the SK model is problematic, as discussed, e.g., in [4, 10, 11, 12]. We’ll ignore that problem for now, though, and assume that somehow two SK pure states $\alpha$ and $\beta$ have been defined. Then their overlap $q_{\alpha\beta}$ is defined as

$$q_{\alpha\beta} = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta$$

(2)

where $\langle \cdot \rangle_\alpha$ is a thermal average in pure state $\alpha$, and dependence on $J$ and $T$ has been suppressed. So $q_{\alpha\beta}$ is a quantity measuring the similarity between states $\alpha$ and $\beta$.

We noted above that quantities referring to individual pure states are problematic in the SK model, since there is no known procedure for constructing such states in a well-defined way. However, what is really of interest is the distribution of overlaps, which can be sensibly defined by using the finite-$N$ Gibbs state. The overlap distribution is constructed by choosing, at fixed $N$ and $T$, two of the many pure states present in the Gibbs state. The probability that their overlap lies
between \( q \) and \( q + dq \) is then given by the quantity \( P_J(q) dq \), where

\[
P_J(q) = \sum_{\alpha} \sum_{\beta} W_\alpha^{\alpha} W_\beta^{\beta} \delta(q - q_{\alpha\beta}).
\]  

(3)

As before, we suppress the dependence on \( T \) and \( N \) for ease of notation. The average \( P(q) \) of \( P_J(q) \) over the disorder distribution is commonly referred to as the \textit{Parisi overlap distribution}, and serves as an order parameter for the SK model.

Because there is no spatial structure in the infinite-range model, the overlap function does seem to capture the essential relations among the different states. However, it might already be noticed that such a global quantity would miss important information in short-range models — assuming that such models also have many pure states. There is no information in \( P_J(q) \) about local correlations. This is acceptable, even desirable, in an infinite-range model such as SK which has no geometric structure, measure of distance, or notion of locality or neighbor. But all of these are well-defined objects in short-range models, and carry a great deal of information about any state, pair of states, or collection of many states. This is one of the sources of the difficulties one encounters (see, e.g., [11]) when attempting to apply conclusions to short-range spin glasses that were derived for the SK model.

## 2 Nearest-Neighbor Ising Ferromagnets

To illustrate some of these ideas in a simple context, consider the uniform nearest-neighbor Ising ferromagnet on \( \mathbb{Z}^d \), with Hamiltonian

\[
\mathcal{H} = - \sum_{x \neq y} \sigma_x \sigma_y.
\]  

(4)

It is natural in models such as this to take periodic or free boundary conditions when considering the finite-volume Gibbs state. In any fixed \( d \geq 2 \), consider for \( T < T_c \) a sequence of volumes \( \Lambda_L \) (for specificity, \( L^d \) cubes centered at the origin) tending to infinity with, for example, periodic b.c.’s. The Gibbs state — which, as emphasized in the preceding section, describes behavior of the local spin variables — converges in the infinite-volume limit to the symmetric mixture of a pure plus state with \( \langle \sigma_x \rangle > 0 \) (which, because of translation symmetry, is independent of \( x \)) and a pure minus state with \( \langle \sigma_x \rangle < 0 \). In a similar way, one could choose to study instead a global variable, say the magnetization per spin \( M_L = |\Lambda_L|^{-1} \sum_{x \in \Lambda_L} \sigma_x \). It is easy to show that this variable has a distribution that converges in the limit to a symmetric mixture of \( \delta \)-functions at \( \pm \langle \sigma_x \rangle \). In this case
descriptions of the system in terms of both local and global variables are interesting, and more to the point, they agree.

But even in the simple case of the uniform, nearest-neighbor Ising ferromagnet this need not always be true. Consider ‘Dobrushin boundary conditions’ [14]. These are b.c.’s in which the boundary spins above the ‘equator’ (a plane or hyperplane parallel to two opposing faces of $\Lambda_L$ and cutting it essentially in half by passing just above the origin) are chosen to be plus and the boundary spins at and below the equator are minus. Boundary conditions such as these are useful for studying interface structure in spin models.

Now (below the roughening temperature) translation-invariance is lost (in one direction), and the local and global variables disagree. The Gibbs state is one where $\langle \sigma_z \rangle > 0$ for $z > 0$ (taking $z = 0$ to be the equator) and $\langle \sigma_z \rangle < 0$ when $z \leq 0$. The magnitude of $\langle \sigma_z \rangle$ will depend on the value of $z$ (though it remains independent of the coordinates in all transverse directions). Moreover for $0 < T < T_c$ there is additional dimension-dependence of the behavior of the local variables (related to the roughening transition) which we will not discuss here.

The magnetization global variable is no longer even interesting. Its distribution converges to a $\delta$-function at 0 at all temperatures in all dimensions. It therefore conveys very little information about the nature of the state. One could instead choose a more appropriate global variable that better matches the boundary conditions, e.g., an order parameter such as

$$\tilde{M} = \lim_{L \to \infty} |\Lambda_L|^{-1} \sum_{x \in \Lambda_L} g(x) \langle \sigma_x \rangle$$

where $g(x) = +1$ if $x$ is above the equator and $-1$ if below.

One might also consider a sort of ‘quasi-global’ variable, that looks at block magnetizations in blocks that are large compared to the unit lattice spacing but small compared to entire system size $L$. One could then examine the ‘spatial’ distribution of the block magnetization as the location of the block varies through the system. Above $T_c$ this is simply a $\delta$-function at zero, but below $T_c$ one gets a symmetric mixture of $\delta$-functions at $\pm \lim_{z \to \infty} \langle \sigma_z \rangle$ for all $d$. This is still not as sensitive as the actual Gibbs state, which can distinguish between the rough and nonrough interfaces (e.g., below $T_c$ in $d = 2$ compared to $T < T_R$ in $d = 3$, where $T_R$ is the 3D roughening temperature) by having a different expression for the limiting Gibbs state in the two cases.

3 Finite- and Infinite-Range Spin Glasses

In the case of spin glass models, we have found [10, 11, 15, 16, 17, 18, 19] a much sharper disparity between finite- and infinite-range models than is the case for any homogeneous statistical
mechanical model of which we are aware. Consider first the Edwards-Anderson (EA) nearest-
neighbor model [20] on $\mathbb{Z}^d$. Its Hamiltonian in zero external field is given by

$$\mathcal{H} = -\sum_{x,y \mid |x-y|=1} J_{xy} \sigma_x \sigma_y,$$

(6)

where the nearest-neighbor couplings $J_{xy}$ are defined in exactly the same way as the $J_{ij}$ in the
SK Hamiltonian (1). In this model thermodynamic pure, mixed, and ground states are standard,
well-defined (see, e.g., [4, 19]) objects, constructed according to well-established prescriptions of
statistical mechanics [3, 21, 22, 23, 24, 25, 26]. Local thermodynamic variables therefore convey
in principle all of the essential information about any state.

Global variables such as Parisi overlap functions can be defined for the EA model as well, but
are now very prescription-dependent: for the same system, very different overlap functions can
be obtained through use of different boundary conditions, or by changing the order of taking the
thermodynamic limit and breaking the replica symmetry. Because of this, they may not convey
reliable information about the number of states or the relationships among them. For a detailed
review of these issues and problems, see [4].

Turning to the SK model, we find that a unique situation arises. Pure states are in principle
defined for a fixed realization of all of the couplings; but in the SK model the physical couplings
$J_{ij}/\sqrt{N}$ scale to zero as $N \to \infty$. As a result, there does not now exist any known way of
constructing thermodynamic pure states in an SK spin glass. It has been proposed [27, 28] that one
way of defining such objects is through the use of a modified ‘clustering’ property: if $\alpha$ denotes a
putative pure state in the SK model, then one can demand it satisfy:

$$\langle \sigma_i \sigma_j \rangle_\alpha - \langle \sigma_i \rangle_\alpha \langle \sigma_j \rangle_\alpha \to 0 \quad \text{as} \quad N \to \infty,$$

(7)

for any fixed pair $i, j$, in analogy with the clustering property obeyed by ‘ordinary’ pure states in
conventional statistical mechanics. At this time though, there exists no known operational way to
construct such an $\alpha$ as appears in (7). But even if such a construction were available, the definition
of pure states through (7) leads to bizarre conclusions in the SK model (see [11, 13]).

Are these problems simply a consequence of infinite-range interactions? No, because they are
absent in the Curie-Weiss model of the uniform ferromagnet. Though physical couplings scale to
zero there also, they “reinforce” each other, being nonrandom, so one may still talk about positive
and negative magnetization states — in analogy with what one sees in finite-range models — in
the $N \to \infty$ limit (of course, one can no longer talk of interface states). So the unique behavior
of the SK model arises (at the least) from the combination of two properties: coupling magnitudes
scaling to zero as $N \to \infty$, and quenched disorder in their signs. (Some success has been achieved in defining states in mean-field Hopfield models — see, e.g., \[29, 30, 31\] and references therein — where the correct order parameters are known \textit{a priori}.)

Although individual pure states have not so far been (and perhaps cannot even in principle be) constructed for the SK model, we have nevertheless proposed methods, based on chaotic size dependence \[32\], that can detect the \textit{presence} of multiple pure states. One can then examine the nature of objects that are analogous to states and that are defined through local variables, using the usual prescriptions of statistical mechanics. However, an analysis \[11\] of the properties of these state-like objects shows that they behave in completely unsatisfactory — in fact, absurd — ways. For example, using the traditional definition of a ground state — or equivalently, the modified clustering property of (7) — one can prove that (for almost every fixed coupling realization $J$) \textit{every} infinite-volume spin configuration is a ground state. That is, as $N$ increases, any fixed finite set of correlation functions cycles through all of its possible sign configurations infinitely many times. Of course, this cannot happen in short-range spin glasses in any dimension, nor in any other statistical mechanical model based on any sort of physical system.

The upshot is that global variables (like overlap distributions for the whole system) capture interesting phenomena in the SK model, while local variables are not so interesting there; in fact, their use can even be dangerous in drawing conclusions about realistic spin glass models. This may be because the SK model itself is \textit{a priori} a global (or at least nonlocal) model, and does not lend itself even in principle to any sort of local analysis.

A large body of evidence compiled by the authors \[10, 11, 15, 16, 17, 18, 19\] shows that local variables and states in the EA model do not behave anything like those in the SK model. The same conclusion applies to global variables in the EA model constructed in close analogy with those from the SK model — that is, in a way intended to convey information about states. This will be further discussed in \[13\]. Consequently, we expect that attempts to derive conclusions about the EA model in terms of local properties (i.e., pure state behavior) from the global behavior of the SK model will not work.

4 \ A New Global Order Parameter for Spin Glasses?

In this section we consider an interesting speculative question motivated by a comment of A. Bovier \[33\] that the usual description of Gibbs measures for short-range models is inadequate for mean field spin glasses. We have presented a large body of evidence that any quantity, describing spin glass properties and that is derived from or based on properties of pure states, cannot connect the be-
behavior of short-range and infinite-range spin glasses. But can one construct a new type of global variable that is meaningful for both short-range and infinite-range spin glass models, and allows a direct comparison of their properties? One obvious candidate is a ‘global’ overlap function not related to pure states; that is, rather than computing $P_L(q)$ in a ‘window’ [4, 18, 19] far from $\partial \Lambda_L$, one would compute it in the entire volume $\Lambda_L$. As discussed in [4, 18, 19], the resulting quantity may be unrelated to pure state structure. An analogous situation is the ferromagnet above the roughening temperature (but below $T_c$). Even though there are no domain wall states, employing Dobrushin boundary conditions (see Sec. 2) will generate spin configurations that on very large scales (say, of order $L$) look almost indistinguishable from those belonging to domain wall states. But it’s unclear whether doing this generates any useful or nonobvious information.

Similarly (see [18] for a more detailed discussion) there is reason to doubt whether a global overlap distribution would be any more useful. In the SK model boundary conditions are not an issue; but in short-range models, overlaps are potentially very sensitive to them. One consequence [4, 34] of this sensitivity is that spin overlap functions are unreliable indicators of pure state multiplicity: they can possess a trivial structure (cf. Fig. 1(d) in the companion paper in this volume [13]) in systems with infinitely many pure states, and a more complicated structure [34] in systems with only a single pure state. In particular, spin overlap functions computed in systems with quenched disorder can easily display complicated and nonphysical behavior that simply reflects the ‘mismatch’ between the boundary condition choice and the local coupling variables. Consequently, nontrivial spin overlaps invariably generated by a change in boundary conditions (e.g., by switching from periodic to antiperiodic in spin glasses) may carry no more significance than that, say, in the 2D Ising ferromagnet (for $0 < T < T_c$) generated by Dobrushin boundary conditions. These considerations make it difficult to believe that a spin overlap variable — even when confined to a window — is likely to uncover generally useful information in short-range spin glasses.

Nevertheless, one can conceive and construct different, and possibly more useful, global quantities that can provide useful statistical mechanical information. Of greater interest, they can be used in principle to compare and contrast physically meaningful behaviors of short-range and infinite-range spin glasses. We propose here that one such set of quantities are those related to interfaces between spin configurations drawn at random from finite-volume Gibbs distributions at fixed $L$ and $T$.

The interface between two such configurations $\sigma^L$ and $\sigma'^L$ is simply the set of all couplings that are satisfied in one of the two configurations but not the other. An interface separates regions where the spins in the two configurations agree from those where they disagree. We propose that
the global variables of interest are those — in particular, density and energy — characterizing the physical properties of these interfaces.

The use of interfaces as a probe of spin glass structure is not new. Our purpose here is to argue, based on the overall approach described in this note, that their properties provide a significantly more natural and useful set of global spin glass variables than the spin overlap function.

The interface density in particular has of course been studied in earlier papers [2, 19, 35, 36, 37, 38, 39], and its potential significance has been particularly emphasized in [40, 41, 42, 43, 44]. The quantity studied is usually the edge overlap $q^{(L)}_{e}(\sigma, \sigma')$ between $\sigma^L$ and $\sigma'^L$:

$$q^{(L)}_{e}(\sigma, \sigma') = N_{b}^{-1} \sum_{x,y \in \Lambda_L, |x-y|=1} \sigma_x \sigma_y \sigma'_x \sigma'_y$$  \hspace{1cm} (8)

where $N_b$ denotes the number of bonds inside $\Lambda_L$. In the SK model, the edge overlap is defined similarly, except that the sum runs over every pair of spins.

In the SK model, there is a trivial relationship between the edge and spin overlaps. Consider two spin configurations $\sigma$ and $\sigma'$ in an $N$-spin system; their spin overlap is defined in the usual way as

$$q^{(N)}_{s}(\sigma, \sigma') = N^{-1} \sum_{i=1}^{N} \sigma_i \sigma'_i.$$  \hspace{1cm} (9)

It is easy to see that

$$q^{(SK)}_{e}(\sigma, \sigma') = \left(q^{(SK)}_{s}(\sigma, \sigma')\right)^2 + O(1/N).$$  \hspace{1cm} (10)

In short-range models, including spin glasses, however, there is no simple relationship between the two. For example, the uniform spin configuration and that with a single domain wall running along the equator (generated, e.g., using Dobrushin boundary conditions on the ferromagnet at zero temperature in dimensions greater than or equal to two) have an edge overlap of one and a spin overlap of zero.

We emphasize, however, that the edge overlap is only one interesting quantity providing information (in this case, density) about the interface. We will argue that by itself it does not provide sufficient information to distinguish among various interesting pictures of the spin glass phase. Other quantities, in particular the energy scaling of the interface, are also required for a useful description to emerge.

We can now list several reasons why the density, energy, and possibly other variables associated with interfaces between states constitute a useful set of global spin glass variables.

0) (Preliminary technical point.) The quantities being studied can be clearly defined. This is accomplished through the metastate (see Sec. 5) and its natural extensions. As discussed elsewhere,
a probability measure on low-energy interfaces can be generated and studied through the periodic boundary condition uniform perturbation metastate [39], while one on higher-energy interfaces can be constructed via a modification of that used in constructing the excitation metastate in [36].

1) The quantities proposed are truly global; i.e., there is no need to use a ‘window’. The reason for this is presented in the next point.

2) The edge overlap, computed in the entire volume, can provide unambiguous information about pure state multiplicity. This is in contrast to the spin overlap function. A rigorous formulation and proof of this statement (in the case of zero temperature) was provided in [19]. (The results can be extended to nonzero temperatures by “pruning” from the interfaces small thermally induced droplets [19, 39].) Informally stated, the theorem presented in that paper stated that if edge overlaps were space-filling (that is, their density did not tend to zero with \( L \)), then there must be multiple pure state pairs (e.g., in the appropriate periodic b.c. metastate). Otherwise, there is only a single pair.

It needs to be noted here that this result is restricted to boundary conditions chosen independently of the couplings (which is always the case in numerical simulations and theoretical computations). It is conceivable that appropriately chosen coupling-dependent boundary conditions can generate ‘interface states’, separated by domain walls of vanishing density, as occur below the roughening temperature in Ising ferromagnets; but no procedure for constructing such boundary conditions has yet been found.

3) Interface structure also provides information on thermodynamically relevant non-pure state structure, in particular, the distribution and energies of excitations. This is a potentially important use of the procedures of Krzakala-Martin [45] and Palassini-Young [46], and is described in more detail in [39].

4) The scaling of the edge overlap with energy allows one to distinguish between different scenarios of the low-temperature phase at zero temperature. At zero temperature, the spin overlap function in any volume is identical in the scaling-droplet [34, 47, 48, 49, 50, 51, 52], chaotic pairs [41, 10, 16, 17, 18, 19, 32], and RSB scenarios [11, 2, 6, 7, 8, 9, 55] (see the companion paper [13] for a detailed description of these three pictures), but the interface density and energy, when used together, can distinguish among all of these pictures (as well as the ‘KM/PY’ scenario of Krzakala-Martin [45] and Palassini-Young [46]). This is summarized in Fig. 1.

5) The interface properties discussed here provide a means of comparing behavior of SK- and short-range spin glasses. That is, use of interface properties allows comparison without requiring recourse to pure state notions, which as discussed above are poorly defined in the SK model. So, while they provide much (although not all – see below) useful information about pure state
Figure 1: (Adapted from Ref. [53].) Table illustrating the correspondence between interface properties and different scenarios for the structure of the low-temperature phase in short-range spin glasses.

structure in short-range models, they are also well-defined in SK models and allow for direct comparison of the two.

It should always be kept in mind, however, that ultimately the interface variables we have been discussing discard a significant amount of important information on local correlations; this is to be expected from any global variable. These variables should therefore be viewed as providing additional useful information to the usual local (i.e., thermodynamic) variables; it is dangerous to view them as a replacement for those variables. Consequently, even while providing the arguments in this section, we emphasize that interface — or any other global — variables can play at best a secondary role in describing the statistical mechanics of finite-range models, where well-defined state-based quantities already exist. It is only in the SK model, where such quantities are mostly absent, that global quantities play a more primary — perhaps the only — role.

Throughout much of this section we referred to useful properties of a thermodynamic object we have called the metastate. We conclude by defining this object, which in turn enables us to return to the observation made at the beginning of this note.

5 What Are Metastates, Anyway?

We are interested in the thermodynamic behavior of large, finite systems, containing on the order of $10^{23}$ interacting degrees of freedom. Corresponding infinite volume limits serve two purposes.
Mathematically, they enable precise definitions of quantities corresponding to physical variables; physically, they allow one to approximate large finite systems (usually when surface effects can be ignored compared to bulk phenomena). They allow a deep conceptual understanding of important physical phenomena; probably the best-known example is understanding phase transitions in terms of singularities or discontinuities of thermodynamic functions. It is of course a fact that such phenomena correspond to true mathematical singularities only in strictly infinite systems, while common sense dictates that phase transitions in physical systems are quite real, and involve behavior as singular or discontinuous as can be found anywhere in the physical world. That is precisely why infinite volume limits are properly regarded as convenient and useful mathematical descriptions of large finite systems.

When dealing with the usual sort of global variable such as energy, we see no serious issues arising in disordered systems, nor do we expect that there is any conceptual divergence from homogeneous systems. Thus, for example, the calculation of the SK free energy per spin in the Parisi solution relies completely on taking the $N \to \infty$ limit (see, e.g., [28]). Any difficulties involved are really of a technical, rather than a conceptual, nature, and they have finally been rigorously resolved [54, 55].

It is when dealing with local variables that serious problems arise, and here the behavior of models with quenched disorder seems to diverge dramatically from that of homogeneous ones. In almost all well-known homogeneous models, such as uniform ferromagnets, there is simple convergence, as volume goes to infinity, to a single Gibbs state at high temperature; or to several, related via well-understood symmetry transformations, at low temperature. There the nature of the finite-volume approximation to the infinite-volume limit is conceptually clear and no difficulties arise.

But what if — as has often been conjectured for finite-range spin glasses — there are many pure states and they are not simply related to each other by symmetry transformations? Mathematically, this means that if one looks at a local variable, such as a single-spin or two-spin correlation function, there exist many possible (subsequence) limits. When this happens, it is not even immediately clear how to obtain a well-defined infinite-volume limit. This is largely a consequence of chaotic size dependence, first demonstrated in [32] as an unavoidable signature of many states. Briefly put, local variables, such as correlations, will vary chaotically and unpredictably as volume size changes (with, say, periodic boundary conditions on each volume). In fact, this chaotic size dependence was proposed by us as a test of the presence of many pure state pairs. Consequently, convergence of states to a thermodynamic limit is no longer as simple as just taking a sequence of volumes of arithmetically, say, or geometrically increasing lengths — a process that works fine for
homogeneous systems.

It turns out that such limits can be defined, but it takes a little work [10] [15]. However, the existence of limiting thermodynamic states turns out not to be the essential problem. As noted, such states do exist and are well-defined; but they turn out not to contain the information needed to fully understand the system in large finite volumes. So now there does, at first glance, seem to be a conflict between the thermodynamic limit and behavior in large finite volumes. But such a conclusion is premature — with more work, not only can the two be reconciled again, but an entirely new set of concepts and insights arises.

So we turn to the question: is it even possible in such systems to describe the nature of large finite volume systems via a single infinite volume object, and if so, how? The answer to the first question is yes [16], and to the second is: by an infinite volume object that captures the nature of the behavior of the finite systems as volume increases — i.e., by the metastate [4, 10, 16, 17, 18, 56] describing the empirical distribution of local variables as volume increases without bound.

Such behavior in $L$ is analogous to chaotic behavior in time $t$ along the orbit of a chaotic dynamical system. In each case the behavior is deterministic but effectively unpredictable. Consequently, it can be modelled via random sampling from some distribution $κ$ on the space of states. In the case of dynamical systems, one can in principle reconstruct $κ$ by keeping a record of the proportion of time the particle spends in each coarse-grained region of state space. Similarly, one can prove [10, 16] that for inhomogeneous systems like spin glasses, a similar distribution exists: even in the presence of chaotic size dependence, the fraction of volumes in which a given thermodynamic state $Γ$ appears, converges (at least along a sparse sequence of volumes). By saying that a thermodynamic state $Γ$ (which is an infinite-volume quantity) ‘appears’ within a finite volume $Λ_L$, we mean the following: within a window deep inside the volume, correlation functions computed using the finite-volume Gibbs state $ρ_L$ are the same as those computed using $Γ$ (with negligibly small deviations).

Hence, the metastate allows one to reconstruct the behavior of large finite volumes from an infinite-volume object, which contains far more information than any mixed thermodynamic state, such as those often used as a starting point in RSB analyses. Because technical details have been provided in several other places [10, 16], we do not repeat them here. We do however refer the reader to [13] where we examine more closely the nature of the low-temperature spin glass phase, from the point of view of metastates.

Acknowledgments. This research was partially supported by NSF Grants DMS-01-02587 (CMN) and DMS-01-02541 (DLS).
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