Entanglement between two superconducting qubits via interaction with non-classical radiation

Mauro Paternostro†, Giuseppe Falci‡, Myungshik Kim‡, and G. Massimo Palma‡

†School of Mathematics and Physics, The Queen’s University, Belfast BT7 1NN, United Kingdom
‡Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), Università’ di Catania,
viale A. Doria 6, 95125 Catania, Italy & MATIS-INFM, Unità’ di Catania
§Dipartimento di Tecnologie dell’Informazione, Università’ di Milano,
Via Bramante 65, 26013 Crema, Italy & NEST-INFM

(Dated: April 1, 2022)

We propose a scheme to physically interface superconducting nano-circuits and quantum optics. We address the transfer of quantum information between systems having different physical natures and defined in Hilbert spaces of different dimensions. In particular, we investigate the transfer of the entanglement initially in a non-classical state of a continuous-variable system to a pair of superconducting charge qubits. This set-up is able to drive an initially separable state of the qubits into an almost pure, highly entangled state suitable for quantum information processing.

PACS numbers: 03.67.-a, 03.67.Mn, 03.67.Hk, 85.25.Dq, 42.50.Dv

Control of the dynamics of a complex quantum system requires a trade-off between tunability and protection against noise. To this end one can be interested in processes where some physical properties of a subsystem are reliably transferred onto the state of a second one (of perhaps different nature) where information can be manipulated. The connection between the two subsystems is effectively realized via a physical interface. An interface is a communication channel used to connect the elements of a quantum register to perform quantum information processing or a physical mechanism that gives full access to the system under investigation and allows to manipulate it.

To investigate this problem, in this paper we describe the coupling between a nano-electronic circuit implementing a pair of quantum bits and a two-mode electromagnetic field. We discuss a mechanism for the transfer of entanglement from a two-mode squeezed state to the pair of qubits. Here, the information sheltered in the electromagnetic medium may be manipulated, using just single-qubit operations, when transferred to the solid-state subsystem. This may offer advantages with respect to integrability and scalability. In particular we consider a pair of (initially independent) Superconducting-Quantum-Interference-Devices (SQUIDs) implementing two charge qubits \( Q_a \) and \( Q_b \), whereas each field mode is modeled as an LC circuit (Fig. 1(a)). The SQUIDs can be individually addressed by gate voltages \( V_g \) whereas an external magnetic flux \( \phi_{ext} \) allows to change the Josephson coupling \( E_J(\phi_{ext}) \) and to modulate the interaction among the subsystems \( a \) and \( b \). Direct experimental evidence of the use of these systems as controllable coherent two-level systems has already been provided \( 1, 2, 3 \).

We first analyze a single SQUID plus LC oscillator (Fig. 1(b)). We introduce the phase drop across the SQUID \( \phi_a \) and across the LC circuit \( \phi_c \). The conjugated variables are the excess charge on the SQUID is-

![Diagram](image-url)
representing $m$ Cooper pairs in excess in the island, and implements a charge Josephson qubit [1]. Typical values of $C_{ij} \approx 10^{-15} F$ and $C_{\text{s}} \approx 10^{-17} F$ guarantee $\epsilon^2 / 2C \sim 1 K \gg E_i^0 \sim 100 \text{ mK}$. In this system, the main sources of decoherence are noise of electrostatic origin, voltage fluctuations of the circuit [1, 3] or stray polarization due to charged impurities located close to the device [2]. If we set $C_j V_j = e |0\rangle_s \langle 1|$, they have the same electrostatic energy and the SQUID is not affected, at first order, by this charge noise [2, 3]. At this working point, $\hat{H}_{\text{squid}} = \frac{e}{2} E_f \hat{\phi}_{\text{ext}} \hat{\sigma}_z$ with a computational basis $\{|+\rangle, \{-\rangle\}_s$, where $|\pm\rangle_s = (1/\sqrt{2}) (|1\rangle \pm |0\rangle)_s$ are eigenstates of $\hat{\sigma}_z$ split by $E_f / \hbar \sim 10 \text{ GHz}$. We introduce the operators $\hat{a}$ and $\hat{a}^\dagger$ ([$\hat{a}, \hat{a}^\dagger$] = 1) via $\hat{P}_b = (\hbar \omega C_2 / 2)^{1/2} (\hat{a} + \hat{a}^\dagger)$, $\hat{\varphi}_b = i (\hbar / 2 \omega C_2)^{1/2} (\hat{a} - \hat{a}^\dagger)$ and we get $H_{\text{em}} = \hbar \omega (\hat{a} \hat{a}^\dagger + 1/2)$. Taking $C_2 \approx 1 \text{ pF}$ and $L_o \approx 10 \text{ nH}$, achievable by present day technology, we have $\omega \approx 10 \text{ GHz}$. The coupling between the SQUID and the field mode can be tuned on and off resonance by modulating the energy splitting of the qubit via $\phi_{\text{ext}}$. If $E_f / \hbar \approx 10 \text{ GHz}$ is set to be much different from $\omega$, the coupling is effectively turned off and the qubit evolves independently from the field mode. On the other hand, for the quasi-resonant condition $E_f / \hbar \approx \omega$, we use $Q_{\hat{a}} = 2 e \hat{\sigma}_z$, so that

$$\hat{H}_c = \hbar \Omega \left[ (\hat{a} \hat{\sigma}_z + \hat{a}^\dagger \hat{\sigma}_z) + (\hat{a}^\dagger \hat{\sigma}_z + \hat{a} \hat{\sigma}_z) \right],$$

where $\Omega = e \sqrt{2 \omega C_2 / \hbar C_F^2}$ is the Rabi frequency of the interaction and $\hat{\sigma}_z = \hat{\sigma}_z |+\rangle \langle -|$. The Hamiltonian Eq. (2) is frequently found in quantum optics problems. The first and second term preserve the total number of excitations in the system and allow for the restriction of the computational basis in $H_{\text{squid}} \otimes H_{\text{em}}$ to $\{|\pm, n\rangle, |n, n-1\rangle\}_s$. The other (counter-rotating) terms induce leakage from this subspace. They can be neglected in the Rotating Wave Approximation (RWA) that is applicable when $\Omega \ll \omega, E_f / \hbar \approx 10 \text{ GHz}$, so that $C_1 \approx 10^{-11} F$ and $\Omega \approx 0.1 \text{ GHz}$. In this regime the eigenstates of SQUID+field mode are entangled states forming a series of doublets split by $\hbar \sqrt{\Omega}$. It is worth stressing that at the working point $V_g = e / C_g$, intra-doublet transitions are forbidden [2]. The system is thus protected, to a certain extent, against decoherence.

We now describe the interaction of a pair of SQUIDs with non-classical radiation. We consider the two-mode squeezed state $|S(r)\rangle_{ab} = \sum_{n=0}^{\infty} \eta_n |n, n\rangle_{ab}$, where $r$ is the squeezing parameter and $\eta_n = (\tanh r^n / \cosh r)$. The entanglement between modes $a, b$ is a function of $r$. Squeezed microwaves can be generated off-line with Josephson parametric oscillators [8] and then used for our protocol. The SQUIDs can be integrated in the waveguides used for the transmission of the signal [8], with the gate-plates orthogonal to the direction of propagation of the fields. Quality factors $\sim 10^4$ for a superconducting transmission-line are within the state of the art. For $\omega \sim 10 \text{ GHz}$, this gives photon-lifetimes $\sim 1 \mu \text{s}$, allowing for a coherent dynamics. The SQUIDs are prepared in a pure separable state $\rho_{12}(0) = \rho_1(0) \otimes \rho_2(0)$. The interaction between each SQUID and a field mode is driven by the co-rotating part of Eq. (2), $H_{\text{int}}^{\text{iwa}}$, with $i = 1, 2$ and $j = a, b$. The time evolution operator is $\hat{U}(t) = \otimes_{i,j} e^{-i H_{\text{int}}^{ij} \omega t}$. However, the reduced state of the SQUIDs $\rho_{12}(t)$ is inseparable, in general, because the evolution could have transferred quantum correlations from the fields to the qubits. To see it, we derive the operator-sum representation of the SQUIDs evolution $\rho_{12}(t) = T r_{ab} \left\{ \hat{U}(t) \rho_{12}(0) \otimes \rho_{ab}(0) \hat{U}^\dagger(t) \right\} = \sum_{\mu} \sum_{p=0}^{\infty} \hat{K}_{\mu}^p \rho_{12}(0) \hat{K}_{\mu}^p$, where we the Kraus operators $\hat{K}_{\mu}^p = \sum_{n=0}^{\infty} \eta_n(r) \langle m, p | \hat{U}(t) | n, n \rangle$ have been introduced. Calculating the matrix elements of $\hat{U}(t)$ over the number states of the field modes, a set of five Kraus operators is found. If the initial state of the two SQUIDs is specified, a simplification is possible and the number of Kraus operators is reduced. We assume $\rho_{12}(0) = |\cdots, -\rangle_2 |\cdots, -\rangle_1$, that can be prepared using standard techniques [1]. We get the effective representation $\rho_{12}(t) = \sum_{\mu=1}^{3} \sum_{n=0}^{\infty} \hat{K}_{\mu}^{|-\rangle_n \langle -| \hat{K}_{\mu}^\dagger$, where

$$\hat{K}_{1}^m = \eta_m \cos^2(\Omega \sqrt{m} t) |-, -\rangle_1 \langle -, -|$$

$$\hat{K}_{2}^m = \eta_m \sin^2(\Omega \sqrt{m} t) |+, +\rangle_1 \langle +, +|$$

$$\hat{K}_{3}^m = \eta_m \cos(\Omega \sqrt{m} t) \sin(\Omega \sqrt{m} t) |-, +\rangle_1 \langle -, +|.$$

$\hat{K}_{1}^m$ is responsible for zero and two-photon processes that leave the two field modes with the same number of photons. $\hat{K}_{2}^m$ and $\hat{K}_{3}^m$ describe single-photon processes in which one of the SQUIDs absorbs an incoming photon. Using Eqs. (3), the density matrix of the SQUIDs, in the ordered basis $\{|ij\rangle_1, |j, i = +, -\rangle$, takes the form

$$\rho_{12}(t) = \begin{pmatrix}
A(t, r) & 0 & 0 & -D(t, r) \\
0 & B(t, r) & 0 & 0 \\
0 & 0 & B(t, r) & 0 \\
-D(t, r) & 0 & 0 & C(t, r)
\end{pmatrix}.$$ (4)

Here $A(t, r) = \sum_{\mu=0, \mu=0}^{\infty} \chi_{\mu} \cos^4(\Omega \sqrt{\mu} t)$, $B(t, r) = \sum_{\mu=0, \mu=0}^{\infty} \chi_{\mu} \sin^2(\Omega \sqrt{\mu} t) \cos^2(\Omega \sqrt{\mu} t)$, $D(t, r) = \sum_{\mu=0, \mu=0}^{\infty} \chi_{\mu+1} \sin^2(\Omega \sqrt{\mu+1} t) \cos^2(\Omega \sqrt{\mu+1} t)$ and $C(t, r) = 1 - 2 B(t, r) - A(t, r)$, with $\chi_{\mu} = \eta_{\mu} \eta_m$. To quantify the entanglement between the qubits, we choose the negativity of partial transposition (NPT). NPT is a necessary and sufficient condition for entanglement of any bipartite qubit state $|11\rangle$. The corresponding entanglement measure is defined as $\text{E}_{\text{NPT}} = -2 \lambda^2(r, t)$, where $\lambda(r, t)$ is the unique negative eigenvalue of the two-qubit partially transposed density matrix $\rho_{12}^{PT}$ [11]. In our case, just $\lambda(r, t) = B(t, r) - D(t, r)$ can be negative for some value of $r$ and $t$ and it is used to compute
the entanglement. $\mathcal{E}_{NPT}$ is shown in Fig. 2 as a function of the degree of squeezing $r$ and the rescaled interaction time $\tau = \Omega t$. It turns out that $\mathcal{E}_{NPT}$ never becomes negative and only asymptotically goes to zero as $r$ is increased. Once the interaction starts, the entanglement is transferred to the qubits, collapsing and reviving as the interaction time increases. The maximum of the transferred entanglement is $\mathcal{E}_{NPT}^{max} = 0.87$, obtained for $\tau \simeq 3\pi/2$ and $\tilde{r} = 0.86$. $\mathcal{E}_{NPT}$ is not a monotone function of $r$ as can be seen in Fig. 3(a). It is known that the correlations in $|S(r)\rangle_{ab}$ approach those of the maximally entangled EPR state when $r \to \infty$. Increasing $r$, the contribution by higher photon-number terms in the squeezed state becomes more relevant. Each qubit is exposed to a distribution of different Rabi frequencies (each equal to $\Omega \sqrt{n}$). These induce Rabi floppings at different times that interfere spoiling the degree of entanglement between the SQUIDs. This shows that a perfectly correlated continuous variable state can not be mapped onto a maximally entangled state of two qubits. On the other hand, the discreteness of this distribution induces the entanglement to collapse and revive as time goes by. This analysis is confirmed by considering the entanglement of formation (EoF) $\mathcal{E}_{NPT}^{max}$. In Fig. 3 the two entanglement measures are compared, as functions of both $r$ and $\tau$. From the behavior of EoF, we argue that almost one EPR singlet is required to prepare $\rho_{12}(\tilde{r}, \tilde{\tau})$.

For both the measures, the SQUIDs are separable just for a short initial amount of time ($t \simeq 1\ nsec$) after which entanglement is set, persisting in time, even if fluctuating. The maximum amount of transferred entanglement is reached for $t \sim 50\ nsec$. This is within the coherence time for the capacitive coupling considered here and within the lifetime of the radiation, as already stated. A further analysis of $\rho_{12}(\tilde{r}, \tilde{\tau})$ shows that $|B(\tilde{r}, \tilde{\tau})| \ll |A(\tilde{r}, \tilde{\tau})|, |C(\tilde{r}, \tilde{\tau})|, |D(\tilde{r}, \tilde{\tau})|$. If, in zero-order approximation, we neglect $B(\tilde{r}, \tilde{\tau})$ in $\rho_{12}(\tilde{r}, \tilde{\tau})$, we get a density matrix close to that of the pure, non maximally entangled state $\langle \sqrt{A(\tilde{r}, \tilde{\tau})} \rangle - \sqrt{C(\tilde{r}, \tilde{\tau})} |++\rangle_{12}$. In general, $D(\tilde{r}, \tilde{\tau}) \neq \sqrt{A(\tilde{r}, \tilde{\tau})} C(\tilde{r}, \tilde{\tau})$, so that the state is mixed. The degree of mixedness in this purified version of $\rho_{12}(\tilde{r}, \tilde{\tau})$ is quantified using the linearized entropy $S_{l}(\rho_{s}(\tilde{r}, \tilde{\tau})) = 4/3 \left[1 - Tr(\rho_{s}^{2}(\tilde{r}, \tilde{\tau}))\right]$, that ranges from 0 (pure states) to 1 (maximally mixed ones). We get $S_{l}(\rho_{12}(\tilde{r}, \tilde{\tau})) \simeq 0.01$ that shows that, for these parameters, the two SQUIDs are in a nearly pure state. This result is interesting: it has been proved, for example, that a bipartite mixed state becomes useless for quantum teleportation whenever its linearized entropy exceeds $1 - (2/[N(N+1)])$, with $N$ the dimension of each subsystem. For qubits, the threshold is $2/3 \gg S_{l}(\rho_{12}(\tilde{r}, \tilde{\tau}))$ and the state of the entangled SQUIDs could be used as a quantum channel in protocols for distributed quantum computation. We have calculated the purity of the state when $B(\tilde{r}, \tilde{\tau})$ is included, finding the same order of magnitude of the previous result.

We now consider the average value of the transferred entanglement as the preparation of the initial state of the SQUIDs is varied. This allows to investigate about the dynamics of the superconducting qubits once different separable states as $\{|\alpha\rangle + e^{i\phi} |\beta\rangle e^{i\psi} \rangle \rangle_{2}$ are considered. Here $|\alpha, \beta\rangle \in [0, 2\pi]$ and $\phi, \psi \in [0, \pi]$, so that the entire Bloch sphere is explored. For simplicity, we take $\varphi = \psi = 0$ and we use $\mathcal{E}_{NPT}$ to calculate the average value of the entanglement. The evolution of the SQUIDs involves the complete set of Kraus operators. However, the density matrix $\rho_{12}(\tilde{r}, \tilde{\tau})$, averaged over an uniform distribution for $|\alpha, \beta\rangle \in [0, 2\pi]$ and $|\phi, \psi\rangle \in [0, \pi]$, so that the entire Bloch sphere is explored. For simplicity, we take $\varphi = \psi = 0$ and we use $\mathcal{E}_{NPT}$ to calculate the average value of the entanglement. The evolution of the SQUIDs involves the complete set of Kraus operators. However, the density matrix $\rho_{12}(\tilde{r}, \tilde{\tau})$, averaged over an uniform distribution for $|\alpha, \beta\rangle \in [0, 2\pi]$ and $|\phi, \psi\rangle \in [0, \pi]$, so that the entire Bloch sphere is explored. For simplicity, we take $\varphi = \psi = 0$ and we use $\mathcal{E}_{NPT}$ to calculate the average value of the entanglement.
the first peak in Fig. [2]. The previous result suggests that $|-, -\rangle_{12}$ plays a privileged role in the process of entanglement transfer. To support this idea, we look for the optimal preparation of the SQUIDs. Assuming small

![Diagram](image)

FIG. 4: (a): Transferred entanglement averaged over the possible preparations of the SQUIDs. The maximum entanglement is reduced with respect to the case of $|-, -\rangle_{12}$. This is due to the contribution from $|+, +\rangle_{12}$, that is separable for wide ranges of $\tau$ and $r$ and spoils the average entanglement. (b): The amount of entanglement between the qubits as a function of the preparation of the initial states when $r = \tilde{r}$ and $\tau = \tilde{\tau}$. The state $|-, -\rangle_{12}$, obtained for $\alpha = \beta = 0$, corresponds to the maximum of the transferred entanglement.

values of $\alpha$, $\beta$, up to the their second power, we have

$$\rho_{12}(r, t) \simeq N \begin{pmatrix} a & \beta b + \alpha d & \beta d + \alpha b & -c \\ \beta b + \alpha d & a' & 0 & \alpha f - \beta d \\ \alpha b + \beta d & 0 & a' & \beta f - \alpha d \\ -c & \alpha f - \beta d & \beta f - \alpha d & a'' \end{pmatrix}$$

(5)

with $N = (1 - \alpha^2 - \beta^2)$ and $a, a', a'', b, c, d, f$ combinations of trigonometric functions involving both $r$ and $\tau$. An analytical expression for the eigenvalues of the partial transpose is, this time, hard to be obtained. More insight is given by specifying $r$ and $\tau$. In Fig. [4b] we plot $\mathcal{E}_{NPT}$ versus $\alpha$ and $\beta$ for $r = \tilde{r}$ and $\tau = \tilde{\tau}$. The transferred entanglement has a maximum equal to 0.87 for $\alpha = \beta = 0$ and slowly decays. This could be important, experimentally, because small errors in the preparation of the initial state do not dramatically spoil the amount of entanglement transferred to the qubits. The same qualitative behavior is found for other values of $r$ and $t$. Thus, the initial preparation $|-, -\rangle_{12}$ provides the maximum achievable entanglement transfer. The entanglement of the SQUIDs, after the interaction, can be revealed by detecting the population $B(r, \tau)$ and the coherence $D(r, \tau)$ of the density matrix using local resonant pulses on the SQUIDs, along the same lines depicted in [17].

![Diagram](image)

We have proposed a physical interface between quantum optics and a system of two charge qubits. When a quantum-correlated state of light is considered, an entanglement-transfer from the field to the qubits can be efficiently tailored. This work has been supported in part by the UK Engineering and Physical Science Research Council grant GR/S14023/01. MP acknowledge IRCEP for financial support.

[1] Y. Makhlin, G. Schön, A. Shnirman, Rev. Mod. Phys. 73, 357 (2001); M. Tinkham, Introduction to Superconductivity (McGraw-Hill International Editions, 1996).
[2] F. Plastina and G. Falci, Phys. Rev. B 67, 224514 (2003).
[3] O. Buisson and F.W.J. Hekking, in Macroscopic Quantum Coherence and Quantum Computing, D.V. Averin, B. Ruggiero and P. Silvestrini Eds., Kluwer (New York, 2001); F. Marquardt and C. Bruder, Phys. Rev. B 63, 054514 (2001).
[4] Y. Nakamura et al., Nature 398, 786 (1999); J.M. Martinis et al., Phys. Rev. Lett. 89, 117901 (2002); A. Yu. et al., Nature 421, 823 (2003).
[5] D. Vion et al., Science 296, 886 (2002).
[6] G. Falci, E. Paladino and R. Fazio, Proceedings of the International School Enrico Fermi on "Quantum Phenomena of Mesoscopic Systems", B. Altshuler and V. Tognetti Eds., IOS Bologna (2003).
[7] E. Paladino, L. Faoro, G. Falci and R. Fazio, Phys. Rev. Lett. 88, 228304 (2002).
[8] M. O. Scully and M. S. Zubairy, Quantum optics (Cambridge University Press, 1997).
[9] R. Movshovich, et al., Phys. Rev. Lett. 65, 1419 (1990); B. Yurke et al., Phys. Rev. Lett. 60, 764 (1988); B. Yurke et al., Phys. Rev A 39, 2519 (1989).
[10] W. Son, M. S. Kim, J. Lee, D. Ahn, J. Mod. Opt. 49, 1739 (2002).
[11] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A 223, 1 (1996); A. Peres, Phys. Rev. Lett. 77, 1413 (1996); J. Lee et al., J. Mod. Opt. 47, 2151 (2000).
[12] As long as correlations are present between the two modes, entanglement is set in the state of the qubits. Furthermore, we have checked that mixedness of the state of radiation due to imperfections in the generation process (squeezing thermofields instead of vacuum, for example) does not affect the entanglement transfer.
[13] M. S. Kim and J. Lee, Phys. Rev. A 64, 012309 (2001).
[14] For two qubits, EoF is a monotonous function of NPT since $\mathcal{E}_{NPT}$ is equivalent to the concurrence. See C. H. Bennett et al., Phys. Rev. A 54, 3824 (1996); W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[15] We stress that the interaction between qubits and field modes is tuned on/off resonance via the external flux $\phi_{ext}$. Properly lasting magnetic pulses allow for the control of the interaction times.
[16] This means that the usage of this state does not give a fidelity of teleportation larger than the classical one, as pointed out by S. Bose and V. Vedral, Phys. Rev. A 61, 040101(R) (2000).
[17] C. A. Sackett et al., Nature 404, 256 (2000); F. Plastina, R. Fazio and G. M. Palma, Phys. Rev. B 64, 113306 (2001); O. Astafiev et al. quant-ph/0402619 (2003).