Comparisons of Hyvärinen and pairwise estimators in two simple linear time series models

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Abstract

The aim of this paper is to compare numerically the performance of two estimators based on Hyvärinen’s local homogeneous scoring rule with that of the full and the pairwise maximum likelihood estimators. In particular, two different model settings, for which both full and pairwise maximum likelihood estimators can be obtained, have been considered: the first order autoregressive model (AR(1)) and the moving average model (MA(1)). Simulation studies highlight very different behaviours for the Hyvärinen scoring rule estimators relative to the pairwise likelihood estimators in these two settings.

Keywords: Full likelihood, homogeneous scoring rules, Hyvärinen score, pairwise likelihood, first order autoregressive model, first order moving average model.

1 Introduction

Recent years have seen growing interest in composite likelihood methods, due to their computational advantages in estimating parameters of very complex statistical models: see Varin et al. (2011) for an overview. A key feature of these methods is their ability to avoid the calculation of the normalizing constant of the model, which will typically depend on the parameter. Determination of this constant, essential for full likelihood-based inference, can be a very challenging task, entailing multidimensional integration of the full joint density. Composite likelihood approaches can avoid this, by maximizing the product of low-dimensional marginal or conditional likelihoods. The most used composite likelihood in applications is the pairwise likelihood (Le Cessie & Van Houwelingen 1994), defined as the product of bivariate marginal densities.

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Composite likelihood estimation methods form a subset of a more general class of methods based on proper scoring rules, estimation being conducted by minimizing the empirical score over distributions in the model (Dawid & Musio, 2014; Dawid et al., 2014). Some important proper scoring rules are the log-score, \( S(x,q) = -\log q(x) \) (Good, 1952), which recovers the full (negative log) likelihood, and the Brier score (Brier, 1950). A particularly interesting special case, which entirely avoids the need to compute the normalizing constant, is the score matching method of Hyvärinen (2005), which is based on minimizing the following objective function:

\[
S(x, Q) = \Delta \ln q(x) + \frac{1}{2} || \nabla \ln q(x) ||^2,
\]

where \( q(\cdot) \) is the density function of a distribution \( Q \) proposed for a random variable \( X \), and \( x \) is the realized value of \( x \). In (1), \( \nabla \) denotes the gradient operator, and \( \Delta \) the Laplacian operator, with respect to \( x \). This assumes that the random variable \( X \) is continuous-valued and defined over the entire \( \mathbb{R}^k \) supplied with the standard norm \( || \cdot || \), and that \( q \) is differentiable over \( \mathbb{R}^k \).

The score matching technique was subsequently generalized to the case of a Riemannian manifold (Dawid & Lauritzen, 2005), to the case of a non-negative real domain \( \mathbb{R}_+^k \) or \( \{ \mathbb{R}_+ \cup 0 \}^k \), and for binary variables (Hyvärinen, 2007).

The objective function in (1), the “Hyvärinen scoring rule”, is a 2-local homogeneous proper scoring rule: see Parry et al. (2012). Inference performed using any homogeneous scoring rule does not require the knowledge of the normalizing constant of the distribution, since the value of the score is unaffected by applying a positive scale factor to the density \( q \). Works considering estimation based on the Hyvärinen score include Musio & Dawid (2013); Dawid & Musio (2014); Forbes & Lauritzen (2013). In a full natural exponential family, score matching delivers a linear estimating equation, which could be used as a starting point of iterative methods as in the \( \text{R} \) package \texttt{gRc} for Gaussian graphical model with symmetries (Forbes & Lauritzen, 2013; Højsgaard & Lauritzen, 2007).

The principal concern of this work is to investigate and compare the behaviours of the estimators obtained from the Hyvärinen score, the pairwise likelihood, and the full likelihood. We confine our attention to two different model settings: autoregressive and moving average processes. The loss in efficiency in using pairwise likelihood methods may be slight in the former case, or very large, in the latter (Davis & Yau, 2011; Jin, 2010).

The paper unfolds as follows. Section 2 introduces basic notions on scoring rules. Section 3 describes estimation procedures for first order autoregressive and moving average processes. Section 4 summarizes the results of the simulation studies conducted. We conclude in Section 5.
2 Scoring rules

A scoring rule is a loss function designed to measure the quality of a proposed probability distribution \( Q \), for a random variable \( X \) taking values in \( \chi \), in view of the outcome \( x \) of \( X \). Specifically, if a forecaster quotes a predictive distribution \( Q \) for \( X \) and the event \( X = x \) realizes, then the loss will be \( S(x, Q) \). The expected value of \( S(X, Q) \) when \( X \) has distribution \( P \) is denoted by \( S(P, Q) \). The scoring rule \( S \) is proper (relative to the class of distributions \( \mathcal{P} \)) if

\[
S(P, Q) \geq S(P, P), \quad \text{for all } P, Q \in \mathcal{P}.
\]

It is strictly proper if equality in (2) obtains only when \( Q = P \).

2.1 Estimation

Let \((x_1, \ldots, x_\nu)\) be independent realizations of a random variable \( X \), having distribution \( P_\theta \) depending on an unknown parameter \( \theta \in \Theta \), where \( \Theta \) is an open subset of \( \mathbb{R}^k \). Given a proper scoring rule \( S \), let \( S(x, \theta) \) denote \( S(x, P_\theta) \). Inference for the parameter \( \theta \) may be performed by minimising the total empirical score,

\[
S(\theta) = \sum_{i=1}^{\nu} S(x_i, \theta),
\]

resulting in the minimum score estimator,

\[
\hat{\theta}_S = \arg \min_{\theta} S(\theta).
\]

Under broad regularity conditions on the model (see e.g. Barndorff-Nielsen & Cox (1994)), \( \hat{\theta}_S \) is a root of the score equation:

\[
s(\theta) := \sum_{i=1}^{\nu} s(x_i, \theta) = 0,
\]

where \( s(x, \theta) \) denotes the gradient vector of \( S(x, \theta) \) with respect to \( \theta \): \( s(x, \theta) = \nabla S(x, \theta) \). The score equation is an unbiased estimating equation (Dawid & Lauritzen, 2005). When \( S \) is the log-score, the minimum score estimator coincides with the maximum likelihood estimator.

From the general theory of unbiased estimating functions, under broad regularity conditions on the model the minimum score estimate \( \hat{\theta}_S \) is asymptotically consistent and normally distributed:

\[
\hat{\theta}_S \sim N_k(\theta, \{\nu G(\theta)\}^{-1}),
\]

where \( G(\theta) \) denotes the Godambe information matrix (see Dawid et al. (2014); Dawid & Musio (2014)):

\[
G(\theta) := K(\theta) J(\theta)^{-1} K(\theta),
\]
where \( J(\theta) = E \left[ s(X, \theta) s(X, \theta)^T \right] \) is the variability matrix, and \( K(\theta) = E \left[ \nabla s(X, \theta) \right] \) is the sensitivity matrix; in contrast to the case for full likelihood, \( J \) and \( K \) are different in general. It is possible to define test statistics, analogous to those based on the full likelihood, starting from an arbitrary proper scoring rules: e.g. scoring rule Wald-type, scoring rule score-type and scoring rule ratio statistics [Dawid et al. 2014].

### 2.2 Standard errors

Estimation of the matrix \( J(\theta) \), and (to a somewhat lesser extent) of the matrix \( K(\theta) \), is not an easy task. Here, we review the methods we use to estimate these two matrices in the simulation studies.

Let \((y_1, \ldots, y_\nu)\) be independent observations from a \( T \)-dimensional distribution \( P_\theta \). If \( \nu \) is quite large, empirical estimation of the two matrices could be done as

\[
\hat{J} = \frac{1}{\nu} \sum_{i=1}^{\nu} s(y_i, \hat{\theta}_S) s(y_i, \hat{\theta}_S)^T, \quad \hat{K} = \frac{1}{\nu} \sum_{i=1}^{\nu} \left. \frac{\partial s(y_i, \theta)}{\partial \theta} \right|_{\theta = \hat{\theta}_S},
\]

with \( y_i = (y_{1i}, \ldots, y_{Ti}) \).

When it is possible to simulate directly from the complete model, the two matrices could be estimated by recovering to their Monte Carlo estimates, i.e.

\[
\hat{J} = \frac{1}{B} \sum_{b=1}^{B} s(y^{(b)}, \hat{\theta}_S) s(y^{(b)}, \hat{\theta}_S)^T, \quad \hat{K} = \frac{1}{B} \sum_{b=1}^{B} \left. \frac{\partial s(y^{(b)}, \theta)}{\partial \theta} \right|_{\theta = \hat{\theta}_S},
\]

where \( y^{(1)}, \ldots, y^{(B)} \) are \( B \) independent realizations from the model obtained by assuming \( \theta_S \) as the true parameter value.

We refer to Varin (2008) and Varin et al. (2011) for a detailed account on the estimation of the two matrices under the composite likelihood setting. Cattelan & Sartori (2014) compare the performances of the composite likelihood based statistics (Wald-type, score-type, and some adjustments of the composite likelihood ratio) obtained by estimating \( K \) and \( J \) empirically with the ones produced by using Monte Carlo simulation of the two matrices.

### 3 The models

This section will be devoted to two examples both dealing with multivariate normal distributions: the first order autoregressive and moving average models which are two simple examples of linear time series models. They are chosen so that we can calculate both the full and pairwise likelihood estimators.
3.1 First order autoregressive models

The stationary univariate autoregressive process of order 1, denoted by AR(1), is defined by

\[ y_t - \mu = \phi(y_{t-1} - \mu) + z_t, \quad \text{with} \quad t = 2, \ldots, T. \]

where \((z_t)\) is Gaussian white noise process with mean 0 and variance \(\sigma^2\), independent of the initial random variable \(y_1\) which is a Gaussian random variable with mean \(\mu\) and variance \(\sigma^2/(1-\phi^2)\). Here \(\phi\), with \(|\phi| < 1\), is the autoregressive parameter. Then \(y_1, \ldots, y_T\) are jointly normal with mean vector \(\mu_1\) (where \(1_T\) is the \(T\)-dimensional unit vector), and covariance matrix \(\Psi\) having components \(\psi_{lm} = \sigma^2 \phi^{|l-m|}/(1-\phi^2)\) \((l, m = 1, \ldots, T)\).

The full log-likelihood function for the unknown parameter \(\theta = (\mu, \sigma^2, \phi)\), based on data \(y = (y_1, \ldots, y_T)\), is (see for example Pace et al. (2011)):

\[ l(\theta) = -\frac{1}{2\sigma^2} \left\{ \sum_{t=1}^{T} (y_t - \mu)^2 + \phi^2 \sum_{t=2}^{T-1} (y_t - \mu)^2 - 2\phi \sum_{t=2}^{T} (y_t - \mu)(y_{t-1} - \mu) \right\} \]

\[ - \frac{T}{2} \log \sigma^2 + \frac{1}{2} \log(1-\phi^2). \]

As in Davis & Yau (2011), we shall consider the consecutive pairwise likelihood, rather than the complete pairwise likelihood, since in the time series considered dependence decreases in time, so that adjacent observations are more closely related than the others. Since, for \(t = 2, \ldots, T\), \((y_t, y_{t-1})\) has a bivariate Gaussian distribution, with common mean \(\mu\) and variance \(\sigma^2/(1-\phi^2)\), and covariance \(\sigma^2 \phi/(1-\phi^2)\), the consecutive pairwise log-likelihood for \(\theta = (\mu, \sigma^2, \phi)\) is (see Pace et al. (2011)):

\[ pl(\theta) = -\frac{1}{2\sigma^2} \left\{ \sum_{t=2}^{T} (y_t - \mu)^2 + \sum_{t=2}^{T} (y_{t-1} - \mu)^2 - 2\phi \sum_{t=2}^{T} (y_t - \mu)(y_{t-1} - \mu) \right\} \]

\[ - (T-1) \log \sigma^2 + \frac{(T-1)}{2} \log(1-\phi^2). \]

When it is known that \(\mu = 0\), the pairwise likelihood estimator, when both \(\phi\) and \(\sigma^2\) are of interest, has components

\[ \hat{\phi}_p = 2 \left( \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_t^2 + y_{t-1}^2} \right) \]

\[ \hat{\sigma}^2_p = \left( \frac{\sum_{t=2}^{T} y_t^2 + y_{t-1}^2}{2(T-1)} \right)^2 (1-\hat{\phi}_p^2). \]

Note that \(\hat{\phi}_p\) is the Yule-Walker estimator (Davis & Yau 2011).
By using basic differentiation rules, it is easy to find the Hyvärinen score for the model:

\[
H(y, \theta) = \frac{1}{2\sigma^4} \sum_{t=2}^{T-1} \left[ (1 + \phi^2)(y_t - \mu) - \phi(y_{t-1} + y_{t+1} - 2\mu) \right]^2 - \frac{2 + (T-2)(1 + \phi^2)}{\sigma^2} \\
+ \frac{(y_d - \mu - \phi(y_{T-1} - \mu))^2}{2\sigma^4} + \frac{(y_1 - \mu - \phi(y_2 - \mu))^2}{2\sigma^4}.
\]

The minimum score estimate of \(\theta\), \(\hat{\theta}_H\), can be found by minimising the Hyvärinen score in the above equation.

### 3.2 First order moving average models

The univariate moving average process of order 1, denoted by MA(1), is defined by the equation

\[
y_t - \mu = \alpha z_{t-1} + z_t, \quad (t = 1, \ldots, T),
\]

where \(|\alpha| < 1\) and \(z_0, \ldots, z_T\) are independent Gaussian random variables with 0 mean and variance \(\sigma^2\). Then the random variables \(y_1, \ldots, y_T\) are jointly normal, each having mean \(\mu\) and variance \(\sigma^2(1 + \alpha^2)\). The variables \(y_t\) and \(y_{t+k}\) are independent for \(|k| > 1\), while \(y_t\) and \(y_{t+1}\) have covariance \(\sigma^2\alpha\) \((t = 1, \ldots, T-1)\). Hence, the covariance matrix \(\Omega\) of \(y = (y_1, y_2, \ldots, y_T)\) has components \(\omega_{ss} = \sigma^2(1 + \alpha^2), \omega_{st} = \sigma^2\alpha\) if \(|s-t| = 1\), \(\omega_{st} = 0\) otherwise.

Let \(\theta = (\mu, \sigma^2, \alpha)\) be the vector of model parameters, dropping all constant terms, the full log-likelihood function of a single series is (see for instance [Hamilton (1994, pag.128)])

\[
l(\theta) = -\frac{1}{2} \log |\Omega| - \frac{1}{2} (y - \mu)\Omega^{-1}(y - \mu)^T.
\]

The maximum likelihood estimator \(\hat{\theta}\) can be found by maximizing numerically the above objective function.

As before we consider the consecutive pairwise likelihood. For \(t = 2, \ldots, T\), the pair \((y_t, y_{t-1})\) has a bivariate Gaussian density, in which the two components have both mean \(\mu\) and variance \(\sigma^2(1 + \alpha^2)\), and have covariance \(\sigma^2\alpha\). The pairwise likelihood for contiguous pairs of observations of a single series is thus

\[
pl(\theta) = -\frac{1}{2\sigma^2} \sum_{t=2}^{T-1} \frac{(y_t - \mu)^2 + (y_{t-1} - \mu)^2}{1 + \alpha^2 + \alpha^4} - \frac{2(y_t - \mu)(y_{t-1} - \mu)\alpha}{1 + \alpha^2 + \alpha^4} \\
- \frac{(T-1)}{2} \log (1 + \alpha^2 + \alpha^4) - (T-1) \log \sigma^2.
\]

The pairwise likelihood estimator \(\hat{\theta}_p\) can be found by maximizing numerically the pairwise log-likelihood function.
By using basic differentiation rules, it is easy to find the Hyvärinen score based on variables \((y_1, y_2, \ldots, y_T)\):

\[
H(y, \theta) = -\sum_{i=1}^{T} \omega_{ii} + \frac{1}{2} \sum_{i=1}^{T} \left\{ \sum_{t=1}^{T} \omega_{it}(y_t - \mu) \right\}^2,
\]

where \(\omega_{ij}\) denotes the \((i, j)\) element of the inverse of the matrix \(\Omega\).

### 3.3 \(\nu\) independent series

In the remainder of this paper we consider \(\nu\) independent series of length \(T\). We assume that \(T\) is fixed while \(\nu\) increases to infinity. We also specialise to the case that the common mean \(\mu\) and variance \(\sigma^2\) are known; without loss of generality we shall assume \(\mu = 0, \sigma^2 = 1\).

So consider now \(\nu\) independent and identically distributed first order autoregressive processes \(Y_1, \ldots, Y_\nu\), having autoregressive parameter \(\phi\). Let the \((\nu \times T)\) random matrix \(Y\) have the vector \(Y_i\) as its \(i\)th row: thus each row of \(Y\) is independent of the others, and has the \(T\)-variate normal distribution with mean-vector 0 and variance covariance matrix \(\Psi\) say. An estimating function for the parameter \(\phi\) can be obtained by summing the \(\nu\) individual Hyvärinen scores, or \(\nu\) score equations, or \(\nu\) pairwise score equations. But we can also take into consideration the fact that the sum-of-squares-and-products matrix \(S = Y^T Y\) is a sufficient statistic for the multivariate normal model, having the Wishart distribution with \(\nu\) degrees of freedom and scale matrix \(\Psi\). Then inference for the parameter \(\phi\) can be performed by resorting to the Hyvärinen score based directly on the Wishart model. The same approach can be taken if we have \(\nu\) independent first order moving average processes with the same moving average parameter \(\alpha\): Dawid and Musio (2014) apply this method to a similar, but non-stationary, model having a tridiagonal covariance matrix.

Assuming \(\nu \geq T\) so that the joint distribution of the upper triangle \((s_{ij} : 1 \leq i \leq j \leq T)\) of the sum-of-squares-and-products random matrix \(S\) (which has a Wishart distribution with parameters \(\nu\) and \(\Lambda\)) has a density, and taking into consideration all of the properties of the derivatives of traces and determinants, it can be shown that the Hyvärinen score based on this joint density is

\[
HW(S, \Lambda) = -\frac{(\nu - T - 1)}{2} \sum_{i=1}^{T} (s_{ii})^2 + \frac{1}{2} \sum_{i,j=1}^{T} \left\{ \frac{(\nu - T - 1)}{2} s_{ij} - \frac{1}{2} \lambda_{ij} \right\}^2,
\]

where \(s_{ij}, \lambda_{ij}\) are the elements of the inverse matrices \(S^{-1}\) and \(\Lambda^{-1}\), respectively. If the scale matrix \(\Lambda\) is modelled in terms of a scalar parameter \(\lambda\) (where \(\lambda = \phi\) or \(\alpha\) in our models), the associated estimate \(\hat{\lambda}_{HW}\) is now found by minimising \(HW(S, \Lambda)\) with respect to \(\lambda\).

However, for both our models, the Godambe Information needed to estimate the standard error of \(\hat{\lambda}_{HW}\) is not easy to derive analytically. The derivative of
$HW(S, \lambda)$ with respect to $\lambda$ is

$$HW_\lambda(S, \Lambda) = -\frac{1}{2} \sum_{i,j=1}^{T} \left\{ \frac{(\nu - T - 1)}{2} s^{ji} - \frac{1}{2} \lambda^{ji} \right\} \frac{\partial \lambda^{ji}}{\partial \lambda}, \quad (5)$$

and $E \{HW_\lambda(S, \Lambda)\} = 0$ since $E (s^{ij}) = \lambda^{ij} / (\nu - T - 1)$ (see Kollo & von Rosen (2005, p. 257)). Moreover, $K(\lambda) = E \{HW_{\lambda\lambda}(S, \Lambda)\} = \frac{1}{4} \sum_{i,j=1}^{T} (\partial \lambda^{ji} / \partial \lambda)^2$.

Given the simple form of the inverse of the matrix $\Psi$ in the AR(1) model, a tridiagonal matrix with elements above and below the main diagonal equal to $-\phi$, and all diagonal elements equals to $(1 + \phi^2)$ except for the elements $\psi_{11}$ and $\psi_{TT}$ which are equal to 1 (see for instance Davison (2003)), the function $K$ reduces to

$$K(\phi) = \frac{T - 1 + 2\phi^2(T - 2)}{2}. \quad (6)$$

The function $K$ for the MA(1) model entails more lengthy calculations since the elements of the inverse of the matrix $\Omega$ are (see for example Shaman (1969))

$$\omega^{ij} = (-\alpha)^{j-i} \frac{1 + \alpha^2 + \ldots + \alpha^{2(i-j)}}{(1 + \alpha^2 + \ldots + \alpha^{2T})}, \quad j \geq i. \quad (7)$$

The derivation of the function $J(\lambda)$, which after taking account of the square of $[5]$ reduces to

$$J(\lambda) = \frac{(\nu - T - 1)^2}{16} \sum_{i,j,k,l=1}^{T} \left( \frac{\partial \lambda^{ji}}{\partial \lambda} \right)^2 \text{cov} (s^{ji}, s^{kl}), \quad (8)$$

involves calculations requiring the covariance matrix of the random matrix $S^{-1}$, which has an Inverse Wishart distribution with scale matrix $\Lambda^{-1}$: see von Rosen (1988) for details on the components of the covariance matrix.

It should be pointed out that this approach cannot be used if we have a single time series of length $T$ with $T$ increasing to $\infty$, since for non-singularity of the Wishart distribution we need to assume $\nu \geq T$.

### 4 Simulation studies

We designed two simulation studies to assess and compare the behaviours of the estimators found by using the Hyvärinen scoring rule and the full and pairwise maximum likelihood estimators. In Simulation 1 we assume a first order autoregressive model, while in Simulation 2 we consider a first order moving average process. Various parameter settings are considered in both simulation studies. All calculations have been done in the statistical computing environment R (R Core Team 2013). In both simulations, 1000 replicates are generated of $\nu = 200$ processes of length $T = 50$. (Similar results, not reported here, were obtained with $\nu$ increased to 300.)
In Simulation 1, the values of the model parameters are $\mu = 0$ and $\sigma = 1$, with the autoregressive parameter $\phi \in \{-0.9, -0.8, \ldots, 0.8, 0.9\}$. Results are summarized in Table 1, which shows average estimates of the autoregressive parameter $\phi$ using the full likelihood ($\hat{\phi}$), the pairwise likelihood ($\hat{\phi}_p$), the sum of $\nu$ Hyvärinen scores ($\hat{\phi}_H$), and the Hyvärinen score based on the Wishart model ($\hat{\phi}_{HW}$). Moreover, it provides the asymptotic standard deviations ($sd$) and the relative asymptotic efficiency ($ARE$) with respect to the maximum likelihood estimator $\hat{\phi}$, i.e. the ratio between the Fisher information and the Godambe function.

Table 1: Estimated mean ($Est.$), asymptotic standard deviation ($sd$), and asymptotic relative efficiency ($ARE$) of estimators of the parameter $\phi$ in the AR(1) model, for $\nu = 200$, $T = 50$, and varying values of $\phi$.

| $\phi$ | $\hat{\phi}$       | $\hat{\phi}_p$ | $\hat{\phi}_H$ | $\hat{\phi}_{HW}$ |
|-------|---------------------|-----------------|-----------------|---------------------|
|       | Est. $sd$           | Est. $sd$       | Est. $sd$       | Est. $sd$           |
| $-0.9$| -0.8997 0.0041      | -0.9008 0.0050  | -0.9004 0.0044  | -0.9007 0.0048  |
| $-0.8$| -0.9000 0.0059      | -0.9038 0.0061  | -0.9035 0.0055  | -0.9037 0.0057  |
| $-0.7$| -0.9002 0.0071      | -0.9044 0.0077  | -0.9040 0.0072  | -0.9042 0.0074  |
| $-0.6$| -0.9002 0.0090      | -0.9049 0.0100  | -0.9044 0.0096  | -0.9048 0.0098  |
| $-0.5$| -0.9001 0.0087      | -0.9048 0.0097  | -0.9043 0.0093  | -0.9046 0.0095  |
| $-0.4$| -0.9002 0.0092      | -0.9049 0.0100  | -0.9044 0.0096  | -0.9048 0.0098  |
| $-0.3$| -0.9000 0.0098      | -0.9048 0.0100  | -0.9044 0.0096  | -0.9048 0.0098  |
| $-0.2$| -0.9003 0.0099      | -0.9048 0.0100  | -0.9044 0.0096  | -0.9048 0.0098  |
| $-0.1$| -0.9097 0.0101      | -0.9097 0.0101  | -0.9097 0.0101  | -0.9097 0.0101  |
| $0$   | 0.0002 0.0101       | 0.0002 0.0101   | 0.0002 0.0101   | 0.0002 0.0101   |
| $0.1$ | 0.1015 0.0101       | 0.1015 0.0101   | 0.1015 0.0101   | 0.1015 0.0101   |
| $0.2$ | 0.1997 0.0099       | 0.1997 0.0102   | 0.1997 0.0104   | 0.1997 0.0106   |
| $0.3$ | 0.2997 0.0096       | 0.2997 0.0102   | 0.2997 0.0104   | 0.2997 0.0106   |
| $0.4$ | 0.3993 0.0092       | 0.3993 0.0101   | 0.3997 0.0115   | 0.3997 0.0125   |
| $0.5$ | 0.5002 0.0087       | 0.5003 0.0097   | 0.5006 0.0122   | 0.5006 0.0130   |
| $0.6$ | 0.5997 0.0080       | 0.5997 0.0097   | 0.5998 0.0130   | 0.5998 0.0137   |
| $0.7$ | 0.6992 0.0071       | 0.6992 0.0097   | 0.6997 0.0138   | 0.6997 0.0146   |
| $0.8$ | 0.8001 0.0058       | 0.8001 0.0064   | 0.8006 0.0149   | 0.8006 0.0165   |
| $0.9$ | 0.8998 0.0041       | 0.8998 0.0044   | 0.8999 0.0150   | 0.8997 0.0244   |

In Simulation 2, the values of the model parameters are $\mu = 0$ and $\sigma = 1$, with the moving average parameter $\alpha \in \{-0.9, -0.8, \ldots, 0.8, 0.9\}$. Results are summarized in Table 2, which shows the estimates of the moving average parameter $\alpha$ using the full likelihood ($\hat{\alpha}$), the pairwise likelihood ($\hat{\alpha}_p$), the sum of $\nu$ Hyvärinen scores ($\hat{\alpha}_H$), and the Hyvärinen score based on the Wishart
model ($\hat{\alpha}_{HW}$) with the average of the associated standard errors (sd) and the asymptotic relative efficiency with respect to the maximum likelihood estimator $\hat{\alpha}$ (ARE).

Table 2: Estimated mean ($\text{Est.}$), asymptotic standard deviation (sd), and asymptotic relative efficiency (ARE) of estimators of the parameter $\alpha$ in the $MA(1)$ model, for $\nu = 200$, $T = 50$, and varying values of $\alpha$.

| $\alpha$ | $\hat{\alpha}$ | sd | $\hat{\alpha}_p$ | sd | ARE | $\hat{\alpha}_H$ | sd | ARE | $\hat{\alpha}_{HW}$ | sd | ARE |
|---------|----------------|----|----------------|----|-----|----------------|----|-----|----------------|----|-----|
| 0.9     | -0.8998 0.0055 |    | -0.8996 0.0167 | 0.1064 |    | -0.8999 0.0064 | 0.7208 |    | -0.8993 0.0074 | 0.5471 |    |
| 0.8     | -0.7997 0.0066 |    | -0.7996 0.0176 | 0.1390 |    | -0.7998 0.0075 | 0.7566 |    | -0.7992 0.0091 | 0.5177 |    |
| 0.7     | -0.6997 0.0075 |    | -0.6996 0.0183 | 0.1692 |    | -0.6997 0.0086 | 0.7583 |    | -0.6993 0.0106 | 0.5020 |    |
| 0.6     | -0.6004 0.0083 |    | -0.6005 0.0182 | 0.2080 |    | -0.6007 0.0095 | 0.7553 |    | -0.6003 0.0119 | 0.4878 |    |
| 0.5     | -0.5004 0.0089 |    | -0.4999 0.0169 | 0.2757 |    | -0.5007 0.0101 | 0.7646 |    | -0.5002 0.0129 | 0.4743 |    |
| 0.4     | -0.4000 0.0093 |    | -0.3997 0.0148 | 0.3984 |    | -0.4003 0.0104 | 0.8038 |    | -0.4001 0.0136 | 0.4713 |    |
| 0.3     | -0.3003 0.0097 |    | -0.3000 0.0126 | 0.5905 |    | -0.3006 0.0105 | 0.8527 |    | -0.3006 0.0139 | 0.4838 |    |
| 0.2     | -0.2000 0.0099 |    | -0.2002 0.0111 | 0.7926 |    | -0.2001 0.0104 | 0.9119 |    | -0.1999 0.0135 | 0.5408 |    |
| 0.1     | -0.1003 0.0101 |    | -0.1004 0.0103 | 0.9456 |    | -0.1004 0.0101 | 0.9882 |    | -0.1006 0.0124 | 0.6557 |    |
| 0       | 0.0001 0.0101  |    | 0.0001 0.0101  | 1.0082 |    | 0.0001 0.0101  | 1.0101 |    | 0.0005 0.0117  | 0.7429 |    |
| 0.1     | 0.1000 0.0101  |    | 0.1000 0.0103  | 0.9526 |    | 0.1001 0.0101  | 0.9933 |    | 0.0997 0.0124  | 0.6554 |    |
| 0.2     | 0.2000 0.0099  |    | 0.2000 0.0111  | 0.7932 |    | 0.2000 0.0104  | 0.9171 |    | 0.1994 0.0135  | 0.5402 |    |
| 0.3     | 0.2994 0.0097  |    | 0.2996 0.0126  | 0.5853 |    | 0.2994 0.0105  | 0.8475 |    | 0.2992 0.0139  | 0.4835 |    |
| 0.4     | 0.4000 0.0093  |    | 0.4006 0.0148  | 0.3979 |    | 0.4000 0.0105  | 0.7938 |    | 0.3994 0.0137  | 0.6369 |    |
| 0.5     | 0.5002 0.0089  |    | 0.5000 0.0169  | 0.2760 |    | 0.5004 0.0101  | 0.7672 |    | 0.5000 0.0129  | 0.4721 |    |
| 0.6     | 0.6001 0.0083  |    | 0.6000 0.0182  | 0.2075 |    | 0.6001 0.0095  | 0.7643 |    | 0.5993 0.0119  | 0.4850 |    |
| 0.7     | 0.6999 0.0075  |    | 0.6997 0.0182  | 0.1707 |    | 0.6999 0.0086  | 0.7682 |    | 0.6996 0.0106  | 0.5047 |    |
| 0.8     | 0.7999 0.0066  |    | 0.7997 0.0175  | 0.1402 |    | 0.8000 0.0075  | 0.7639 |    | 0.7995 0.0091  | 0.5209 |    |
| 0.9     | 0.8999 0.0055  |    | 0.8997 0.0167  | 0.1072 |    | 0.9000 0.0064  | 0.7300 |    | 0.8995 0.0074  | 0.5504 |    |

It should be noted that for the $MA(1)$ model no analytic expressions for the derivatives of $\phi$ are available. Numerical evaluation of scoring rule derivatives has been carried out using the R package numDeriv.

The standard deviations of $\hat{\phi}_H$ and $\hat{\alpha}_H$ are empirical estimates of the square root of the Godambe information function, which is obtained by compounding the empirical estimates of $J$ and $K$. The standard deviations of the pairwise maximum likelihood estimator and the maximum likelihood estimator are obtained by using the analytic expressions (see Pace et al. [2011]) for the AR(1) model and the empirical counterparts for the $MA(1)$ model. The Godambe information function of $\hat{\phi}_{HW}$ and $\hat{\alpha}_{HW}$ are estimated by Monte Carlo simula-
tions: specifically, in the AR(1) model we resort to analytic derivatives of (4) for the implementation of J and to the analytical form of K in equation (6); while in the MA(1) model we use numerical derivatives of (4) for calculating both K and J.

The left and right-hand panels of Figure 1 depict the asymptotic relative efficiency as a function of φ for the AR(1) model and as a function of α for the MA(1) model for ν = 200 and T = 50, respectively. The left and right-hand panels of Figure 2 show the standard errors as a function of φ for the AR(1) model and as a function of α for the MA(1) model, for ν = 200 and T = 50.

Figure 1: Asymptotic relative efficiency of estimators for the AR(1) model (left panel) and for the MA(1) model (right panel). Based on 1000 replications of ν = 200 series of length T = 50.

4.1 Discussion

Results from Simulations 1 and 2 reveal that the estimators considered produce estimates very close to the true values of the parameters. However, results not shown here suggest that when the length T of the series is small the pairwise likelihood estimator performs worse in terms of bias than the other estimators in both the models. The numerical results in Table 1 and in the left-hand panel of Figure 1 suggest that \( \hat{\phi}_H \) and \( \hat{\phi}_{HW} \) do not have high efficiency as \( \phi \) approaches 1: in particular, the asymptotic efficiency of \( \hat{\phi}_{HW} \) tends to 0 for large values of |\( \phi \)|. In contrast, under the same model setting, there is only a modest loss of efficiency for the pairwise likelihood-based estimator. Simulation 2 shows that the univariate Hyvärinen estimator \( \hat{\alpha}_H \) achieves the same efficiency as the MLE in the MA(1) model for values of the moving average parameter near 0; see Table 2 and the right-hand panel of Figure 1. However, the loss
in efficiency of the univariate Hyvärinen estimator $\hat{\alpha}_H$ is modest even when the absolute value of the moving average parameter reaches 1. The standard errors of the univariate and the multivariate Hyvärinen estimators increase as $|\alpha|$ increases from 0 to 0.3 and decrease as $|\alpha|$ increases from 0.3 to 0.9; see the right-hand panel of Figure 2. In contrast, the pairwise method shows very poor performances in terms of asymptotic relative efficiency: the $ARE$ ranges from 1 to 0.1 as $|\alpha|$ increases. These results are in agreement with the findings of Davis & Yau (2011) who focus on pairwise likelihood-based methods for linear time series.

5 Conclusions

We have investigated the performance of two estimators based on the Hyvärinen scoring rule, which can be regarded as a surrogate for a complex full likelihood. The properties of the estimators found using this scoring rule are compared with the full and pairwise maximum likelihood estimators. Two examples are discussed: the first a stationary first order autoregressive model, and the second a first order moving average model. In the first example the pairwise method produces good estimators; in contrast, in the second example this method leads to poor estimators. The opposite behaviour is observed for the univariate and multivariate Hyvärinen estimators. For the moving average process, there can be a large gain in efficiency, as compared to the pairwise likelihood method, by using the univariate or multivariate Hyvärinen score. For the autoregressive model, in contrast, the Hyvärinen score methods suffer a loss of efficiency as $|\phi|$ approaches 1. In both examples, a great improvement in the performances of
the minimum Hyvärinen score based on the Wishart model is observed as the ratio $T/\nu$ becomes negligible. It is known that the algorithm used to generate a Wishart random matrix as the sum-of-squares-and-products matrix of independent multivariate normals is not efficient (see for example [Kroese et al. (2011) pag. 150]). A question which arises is whether the inefficiency of this algorithm might be affecting the observed behaviour of the multivariate Hyvärinen score. However, results not shown here reveal that no big improvement arises if we generate directly from the Wishart distribution, using for example the `rwish` function of the `MCMCpack` package, which use the Bartlett’s decomposition (see Kollo & von Rosen [2005] p. 240)). It is clear that the loss of efficiency incurred in using the Hyvärinen scoring rule or pairwise likelihood can be quite substantial, but this depends on the underlying model. The multivariate Hyvärinen estimator has the apparent advantage over the other estimators (apart from full maximum likelihood) of being based on the sufficient statistic of the model; nevertheless the univariate Hyvärinen methods shows good performance in terms both of standard errors and efficiency. The Hyvärinen scoring rule methods may represent viable alternatives to the pairwise log-likelihood approach for inference in high-dimensional models where the computation of the normalizing constant is not feasible and the pairwise likelihood leads to poor estimators. In particular, the multivariate Hyvärinen scoring rule may be convenient for studies in which a large number of models with the same parameter should be estimated. It would be of interest to analyse the performance of the univariate and the multivariate Hyvärinen scoring rule estimators both when nuisance parameters are present and when interest focus on the complete vector of parameters.

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