Regular Blaschke Para-Umbilical Hypersurfaces in the Conformal Space \( \mathbb{Q}^n_s \)

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Abstract

In [2] we have classified the Blaschke quasi-umbilical submanifolds in the conformal space \( \mathbb{Q}^n_s \). In this paper we shall classify the Blaschke para-umbilical hypersurfaces in the conformal space \( \mathbb{Q}^n_s \). That may be also considered as the extension of the classification of the conformal isotropic submanifolds in the conformal space \( \mathbb{Q}^n_s \).

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§ 1. Introduction.

Let \( \mathbb{R}^N_s \) denote pseudo-Euclidean space, which is the real vector space \( \mathbb{R}^N \) with the non-degenerate inner product \( \langle \cdot, \cdot \rangle \) given by

\[
\langle \xi, \eta \rangle = - \sum_{i=1}^s x_i y_i + \sum_{i=s+1}^N x_i y_i,
\]

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where $\xi = (x_1, \cdots, x_N), \eta = (y_1, \cdots, y_N) \in \mathbb{R}^N$.

Let

$$C^{n+1} := \{\xi \in \mathbb{R}^{n+2}_{n+1} | \langle \xi, \xi \rangle = 0, \xi \neq 0\},$$

$$Q^n_s := \{[\xi] \in \mathbb{R}^{P^{n+1}} | \langle \xi, \xi \rangle = 0\} = C^{n+1}/(\mathbb{R}\{0\}).$$

We call $C^{n+1}$ the light cone in $\mathbb{R}^{n+2}_{n+1}$ and $Q^n_s$ the conformal space (or projective light cone) in $\mathbb{R}^{P^{n+1}}$.

The standard metric $h$ of the conformal space $Q^n_s$ can be obtained through the pseudo-Riemannian submersion $\pi: C^{n+1} \rightarrow Q^n_s, \xi \mapsto [\xi]$.

We can check $(Q^n_s, h)$ is a pseudo-Riemannian manifold.

We define the pseudo-Riemannian sphere space $S^n_s(r)$ and pseudo-Riemannian hyperbolic space $H^n_s(r)$ with radius $r$ by

$$S^n_s(r) = \{u \in \mathbb{R}^{n+1}_s | \langle u, u \rangle = r^2\}, \quad H^n_s(r) = \{u \in \mathbb{R}^{n+1}_s | \langle u, u \rangle = -r^2\}.$$

When $r = 1$ we usually omit the radius $r$. When $s = 1$ and $r = 1$ we call them de Sitter space $S^n_1$ and anti-de Sitter space $H^n_1$.

We may assume $Q^n_s$ to be the common compactification of $\mathbb{R}^n_s$, $S^n_s$ and $H^n_s$, and $\mathbb{R}^n_s$, $S^n_s$ and $H^n_s$ to be the subsets of $Q^n_s$ when referring only to the conformal geometry.

When $s = 0$, our analysis in this text can be reduced to the Moebius submanifold geometry in the sphere space (see [4]).

This paper is organized as follows. In Section 2 we recall the submanifold theory in the conformal space $Q^n_s$ and give the relations between conformal invariants and isometric ones for submanifolds in several particular surroundings. In Section 3 we classify the conformal surfaces in $Q^3_s$. In Section 4 we classify the Blaschke para-umbilical hypersurfaces in $Q^n_s$.

§ 2. Fundamental equations.

We recall the scheme of submanifold theory in the conformal space $Q^n_s$ first. A classical theorem tells us that

**Theorem 2.1.** (see [3]) The conformal group of the conformal space $Q^n_s$ is $O(n - s + 1, s + 1)/\{\pm 1\}$. If $\varphi$ is a conformal transformation on $Q^n_s$, then there is $A \in O(n - s + 1, s + 1)$, such that $\varphi = \Phi_A$ and $\Phi_A([X]) = [XA]$.

Suppose that $x: M^n_m \rightarrow Q^n_s(s \geq 1)$ is an $m$-dimensional Riemannian or pseudo-Riemannian submanifold with index $t(0 \leq t \leq s)$. That is, $x_*(TM)$ is non-degenerate subbundle of $(TQ^n_s, h)$ with index $t(0 \leq t \leq s)$. When $t = 0$ we
call $M$ space-like submanifold. When $t > 0$ we call $M$ pseudo-Riemmanian submanifold. Especially when $t = 1$, $M$ is called Lorentzian submanifold or time-like submanifold. From now on, we always assume that the submanifold $x$ has index $t(0 \leq t \leq s)$.

Let $y : U \rightarrow C^{n+1}$ be a lift of $x : M \rightarrow Q^n_s$ defined on an open subset $U$ of $M$. We denote by $\Delta$ and $\rho$ the Laplacian operator and the scalar curvature of the local non-degenerate metric $\langle dy, dy \rangle$. Then we have

**Theorem 2.2.** (cf. [3]) Suppose that $x : M \rightarrow Q^n_s$ is an $m$-dimensional Riemannian or pseudo-Riemannian submanifold with index $t(0 \leq t \leq s)$. On $M$ the 2-form

$$g := \pm (\langle \Delta y, \Delta y \rangle - \frac{m}{m-1} \rho) \langle dy, dy \rangle$$

is a globally defined conformal invariant of $x$.

**Definition 2.1.** We call an $m$-dimensional submanifold $x : M \rightarrow Q^n_s$ a regular submanifold if the 2-form $g := \pm (\langle \Delta y, \Delta y \rangle - \frac{m}{m-1} \rho) \langle dy, dy \rangle$ is non-degenerate. $g$ is called the conformal metric of the regular submanifold $x : M \rightarrow Q^n_s$.

In this paper we assume that $x : M \rightarrow Q^n_s$ is a regular submanifold. Since the metric $g$ is non-degenerate (we call it the conformal metric), there exists a unique lift $Y : M \rightarrow C^{n+1}$ such that $g = \langle dY, dY \rangle$ up to sign. We call $Y$ the canonical lift of $x$.

**Definition 2.2.** The two submanifolds $x, \tilde{x}$ are conformally equivalent, if there exists a conformal transform $\sigma : Q^n_s \rightarrow Q^n_s$, such that $\tilde{x} = \sigma \circ x$.

It follows from Theorem 2.1 that

**Theorem 2.3.** Two submanifolds $x, \tilde{x} : M \rightarrow Q^n_s$ are conformally equivalent if and only if there exists $T \in O(n-s+1, s+1)$ such that $\tilde{Y} = TY$, where $Y, \tilde{Y}$ are canonical lifts of $x, \tilde{x}$, respectively.

Let $\{e_1, \ldots, e_m\}$ be a local basis of $M$ with dual basis $\{\omega^1, \ldots, \omega^m\}$. Denote $Y_i = e_i(Y)$. We define

$$N := -\frac{1}{m} \Delta Y - \frac{1}{2m^2} \langle \Delta Y, \Delta Y \rangle Y.$$

Analogous to the corresponding calculation of [13], we have

$$\langle N, Y \rangle = 1, \langle N, N \rangle = 0, \langle N, Y_k \rangle = 0, \quad 1 \leq k \leq m.$$ 

We may decompose $\mathbb{R}^{n+2}_s$ such that

$$\mathbb{R}^{n+2}_s = \text{span}\{Y, N\} \oplus \text{span}\{Y_1, \ldots, Y_m\} \oplus \mathcal{V}$$

where $\mathcal{V} \perp \text{span}\{Y, N, Y_1, \ldots, Y_m\}$. We call $\mathcal{V}$ the conformal normal bundle for $x : M \rightarrow Q^n_s$. Let $\{\xi_{m+1}, \ldots, \xi_n\}$ be a local basis of the bundle $\mathcal{V}$ over $M$. 

3
Then \( \{Y, N, Y_1, \cdots, Y_m, \xi_{m+1}, \cdots, \xi_n\} \) forms a moving frame in \( \mathbb{R}^{n+2} \) along \( M \).

We adopt the conventions on the ranges of indices in this paper without special claim:

\[ 1 \leq i, j, k, l, p, q \leq m; \quad m + 1 \leq \alpha, \beta, \gamma, \nu \leq n. \]

We may write the structural equations as follows

\[ dY = \sum_i \omega^i Y_i; \quad dN = \sum_i \psi^i Y_i + \sum_\alpha \phi^\alpha \xi_\alpha; \quad (2.1) \]

\[ dY_i = -\psi_i Y - \omega_i N + \sum_j \omega^j Y_j + \sum_\alpha \omega^\alpha \xi_\alpha; \quad (2.2) \]

\[ d\xi_\alpha = -\phi_\alpha Y + \sum_i \omega^i_\alpha Y_i + \sum_\beta \omega^\beta_\alpha \xi_\alpha, \quad (2.3) \]

where the coefficients of \( \{Y, N, Y_i, \xi_\alpha\} \) are 1-forms on \( M \).

It is clear that \( A := \sum_i \psi_i \otimes \omega^i, B := \sum_{i, \alpha} \omega^\alpha_i \otimes \omega^\alpha \xi, \Phi := \sum_\alpha \phi^\alpha \xi_\alpha \) are globally defined conformal invariants. Let

\[ \psi_i = \sum_j A_{ij} \omega^j, \quad \omega^\alpha_i = \sum_j B^\alpha_{ij} \omega^j, \quad \phi^\alpha = \sum_i C^\alpha_i \omega^i. \]

Denote the covariant derivatives of these tensors with respect to conformal metric \( g \) as follows:

\[ \sum_j C^\alpha_{ij} \omega^j = dC^\alpha_i - \sum_j C^\alpha_j \omega^j + \sum_\beta C^\beta_i \omega^\beta; \]

\[ \sum_k A_{ij,k} \omega^k = dA_{ij} - \sum_k A_{ik} \omega^k - \sum_k A_{kj} \omega^i; \]

\[ \sum_k B^\alpha_{ij,k} \omega^k = dB^\alpha_{ij} - \sum_k B^\alpha_{ik} \omega^k - \sum_k B^\alpha_{kj} \omega^i + \sum_\beta B^\beta_{ij} \omega^\alpha. \]

The curvature forms \( \{\Omega^i_j\} \) and the normal curvature forms \( \{\Omega^\alpha_\beta\} \) of the submanifold \( x : M \to \mathbb{Q}^n_\alpha \) can be written by

\[ \Omega^i_j = \frac{1}{2} \sum_{k,l} R^i_{jkl} \omega^k \wedge \omega^l = \omega^i \wedge \psi_j + \psi^i \wedge \omega_j - \sum_\alpha \omega^\alpha_i \wedge \omega^\alpha_j; \]

\[ \Omega^\alpha_\beta = \frac{1}{2} \sum_{k,l} R^\alpha_{\beta kl} \omega^k \wedge \omega^l = -\sum_i \omega^\alpha_i \wedge \omega^\beta_j. \]

Denote

\[ g_{ij} = \langle Y_i, Y_j \rangle, \quad g_{\beta\gamma} = \langle \xi_\beta, \xi_\gamma \rangle, \quad (g^{ij}) = (g_{ij})^{-1}, \quad (g^{\beta\gamma}) = (g_{\beta\gamma})^{-1}, \]
\[ R_{ijkl} = \sum_p g_{ip} R^p_{jkl}, \quad R_{\alpha\beta kl} = \sum_\nu g_{\alpha\nu} R^{\nu}_{\beta kl}. \]

Then the integrable conditions of the structure equations are

\[ A_{ij,k} - A_{ik,j} = -\sum_{\alpha\beta} g_{\alpha\beta}(B^\alpha_{ij} C^\beta_k - B^\alpha_{ik} C^\beta_j); \quad B^\alpha_{ij,k} - B^\alpha_{ik,j} = g_{ij} C^\alpha_k - g_{ik} C^\alpha_j; \]

\[ C^\alpha_{i,j} - C^\alpha_{j,i} = \sum_{kl} g^{kl}(B^\alpha_{ik} A_{lj} - B^\alpha_{jk} A_{li}); \quad R_{\alpha\beta ij} = \sum_{kl\gamma\nu} g_{\alpha\gamma} g_{\beta\nu} g^{kl}(B^\gamma_{ik} B^\nu_{lj} - B^\nu_{ik} B^\gamma_{lj}); \]

\[ R_{ijkl} = \sum_{\alpha\beta} g_{\alpha\beta}(B^\alpha_{ik} B^\beta_{jl} - B^\beta_{ik} B^\alpha_{jl}) + (g_{ik} A_{jl} - g_{il} A_{jk}) + (A_{ik} g_{jl} - A_{il} g_{jk}). \]

Furthermore, we have

\[ \text{tr}(A) = \frac{1}{2m}(\frac{m}{m-1}\rho \pm 1); \quad R_{ij} = \text{tr}(A) g_{ij} + (m-2) A_{ij} - \sum_{k\alpha\beta} g^{kl} g_{\alpha\beta} B^\alpha_{ik} B^\beta_{lj}; \]

\[ (1-m) C^\alpha_i = \sum_{jk} g^{jk} B^\alpha_{ij,k}; \quad \sum_{ijkl\alpha\beta} g^{ij} g^{kl} g_{\alpha\beta} B^\alpha_{ik} B^\beta_{lj} = \pm \frac{m-1}{m}; \quad \sum_{ij} g^{ij} B^\alpha_{ij} = 0. \]

From above we know that in the case \( m \geq 3 \) all coefficients in the PDE system (2.1)-(2.3) are determined by the conformal metric \( g \), the conformal second fundamental form \( B \) and the normal connection \( \{\omega^\alpha\} \) in the conformal normal bundle. Then we have

**Theorem 2.4.** Two hypersurfaces \( x : M^m_t \to \mathbb{Q}^{m+1}_s \) and \( \tilde{x} : \tilde{M}^m_t \to \mathbb{Q}^{m+1}_s (m \geq 3) \) are conformal equivalent if and only if there exists a diffeomorphism \( f : M \to \tilde{M} \) which preserves the conformal metric and the conformal second fundamental form. In another word, \( \{g, B\} \) is a complete invariants system of the hypersurface \( x : M^m \to \mathbb{Q}^{m+1}_s (m \geq 3) \).

When \( \epsilon = 1, 0, -1 \), let the pseudo-Riemannian space form \( R^n_s(\epsilon) \) denote \( \mathbb{S}^n_s, \mathbb{R}^n_s, \mathbb{H}^n_s \), respectively. Let \( \sigma_\epsilon : R^n_s(\epsilon) \to \mathbb{Q}^n_s \) be the standard conformal embedding (see [3]).

Next we give the relations between the conformal invariants induced above and isometric invariants of \( u : M^m_t \to R^n_s(\epsilon) \). Let \( \{e_1, \ldots, e_m\} \) be an local basis for \( u \) with dual basis \( \{\omega^1, \ldots, \omega^m\} \). Let \( \{e_{m+1}, \ldots, e_n\} \) be a local basis of the normal bundle of \( u \). Then we have the first and second fundamental forms \( I, II \) and the mean curvature vector \( \tilde{H} \). We may write

\[ I = \sum_{ij} I_{ij} \omega^i \otimes \omega^j, \quad II = \sum_{i\alpha} h^i_{\alpha} \omega^i \otimes \omega^\alpha. \]
\[(I^{ij}) = (I_{ij})^{-1}, \quad \vec{H} = \frac{1}{m} \sum_{ij\alpha} I^{ij} h^\alpha_{ij} e_\alpha = \sum_\alpha H^\alpha e_\alpha.\]

From the structure equations
\[
du = \sum_i \omega^i u_i, \quad du_i = \sum_j \theta^j_i u_j + \sum_\alpha \theta^\alpha_i u_j - \epsilon \omega_i, \quad dc_\alpha = \sum_j \theta^j_\alpha u_j + \sum_\beta \theta^\beta_\alpha e_\beta,
\]
we have
\[
\Delta_I u = m(\vec{H} - \epsilon u), \quad \rho_I = m(m-1)\epsilon + (m^2|\vec{H}|^2 - |II|^2),
\]
where
\[
|\vec{H}|^2 = \sum_{\alpha\beta} I_{\alpha\beta} H^\alpha H^\beta, \quad |II|^2 = \sum_{ijk\alpha\beta} I_{\alpha\beta} I^{ij} h^\alpha_{ij} h^\beta_{kl}.
\]

For the global lift \(y : M \to C^{m+1}\), the conformal factor of \(y\) is
\[
e^{2\tau} = \pm \frac{m}{m-1} (|II|^2 - m|\vec{H}|^2).
\]

Furthermore, we have
\[
\Delta_I u = m(\vec{H} - \epsilon u), \quad \rho_I = m^2|\vec{H}|^2 - |II|^2;
\]
\[
A_{ij} = \tau_i \tau_j + \sum_\alpha h^\alpha_{ij} H_\alpha - \tau_{i,j} - \frac{1}{2}(\sum_{ij} I^{ij} \tau_i \tau_j + |\vec{H}|^2 - \epsilon) I_{ij},
\]
\[
B^\alpha_{ij} = e^\tau (h^\alpha_{ij} - H^\alpha I_{ij}), \quad e^\tau C^\alpha_i = H^\alpha \tau_i - \sum_j h^\alpha_{ij} \tau^j - H^\alpha_{ij},
\]
where \(\tau_{i,j}\) is the Hessian of \(\tau\) respect to \(I\) and \(H^\alpha_{ij}\) is the covariant derivative of the mean curvature vector field of \(u\) in the normal bundle \(N(M)\) respect to \(I\).

\section{3. Conformal surfaces in \(Q^3_s\).}

In this section let \(x : M^{m}_{t} \to Q^{m+1}_{s}\) be an \(m\)-dimensional regular hypersurface with index \(t(0 \leq t \leq s)\). We use the notations in Section 2 and omit all normal scripts in the formulas because the codimension now is one. Let
\[
A^i_j = \sum_k g^{ik} A_{kj}, \quad A = (A^i_j),
\]
\[
B^i_j = \sum_k g^{ik} B_{kj}, \quad B = (B^i_j).
\]
We rewrite some equations occurred previously in the new form as follows

\[ \sum_{ij} B^i_j B^j_i = \frac{m-1}{m}, \quad \sum_i B^i_i = 0, \quad (3.1) \]

\[ B_{ij,k} - B_{ik,j} = g_{ij}C_k - g_{ik}C_j, \quad A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \quad (3.2) \]

\[ C_{i,j} - C_{j,i} = \sum_k (B_{ik}A^k_j - B_{jk}A^k_i), \quad (3.3) \]

\[ \sum_i A^i_i = \frac{1}{2m}(\frac{m}{m-1}\rho \pm 1). \quad (3.4) \]

**Definition 3.1.** We call an m-dimensional regular submanifold \( x: M \to \mathbb{Q}^n_s \) conformal if the conformal form \( \Phi \equiv 0 \).

Let \( x: M \to \mathbb{Q}^n_s \) be a regular space-like surface. We can write the structural equations as

\[ e_i(N) = \sum_j A^i_j Y_j + C_i \xi, \quad e_i(\xi) = C_i Y + \sum_j B^i_j Y_j, \quad (3.5) \]

\[ e_j(Y_i) = -A_{ij} Y - g_{ij} N + \sum_k \Gamma^k_{ij} Y_k + B_{ij} \xi. \]

Since \( m = 2 \), we can find an orthonormal basis \( e_1, e_2 \) of \( x \) from (3.1) such that

\[ B = \text{diag}(\frac{1}{2}, -\frac{1}{2}). \]

If \( x \) is a conformal surface, we have \( C_i = 0, i = 1, 2 \). It implies from (3.2) that \( B_{ij,k}, A_{ij,k} \) are all symmetric with respect to the subscripts. For the same reason that \( x \) has vanishing conformal form, by (3.3), we can modify the orthonormal basis \( e_1, e_2 \) such that

\[ A = \text{diag}(a, b). \]

Taking \( i, j \) various values in

\[ \sum_k B_{ij,k} \omega^k = dB_{ij} - \sum_k B_{ik} \omega^k_j - \sum_k B_{kj} \omega^i_k, \quad (3.6) \]

we have

\[ B_{11,i} = B_{22,i} = 0, i = 1, 2. \]

Therefore \( B_{12,i} = 0, i = 1, 2 \). Letting \( i = 1, j = 2 \) in (3.6), we get the connection of \( x \) is flat, i.e., \( \omega^1_2 = 0 \). It follows from (3.4) that

\[ a + b = -\frac{1}{4}. \quad (3.7) \]
In addition, we may assume that there exist local co-ordinates $u, v$ such that

$$e_1 = \frac{\partial}{\partial u}, \quad e_2 = \frac{\partial}{\partial v}.$$ 

Taking $i = 1, j = 2$ in

$$\sum_k A_{ij,k} \omega^k = dA_{ij} - \sum_k A_{ik} \omega^j - \sum_k A_{kj} \omega^i,$$  

(3.8)

and noting $\omega^1_2 = 0$, we have

$$A_{12,i} = 0, \quad i = 1, 2.$$ 

So, when taking $i = j = 1$ and $i = j = 2$ in (3.8) respectively, we know that $a_v = b_u = 0$. Adding (3.7) we shall see that $a, b$ are both constant.

Next, we have the structural equations as the following new form

$$N_u = aY_u, \quad N_v = bY_v, \quad \xi_u = \frac{1}{2}Y_u, \quad \xi_v = -\frac{1}{2}Y_v,$$  

(3.9)

$$Y_{uv} = 0, \quad Y_{uu} = -aY - N + \frac{1}{2}\xi, \quad Y_{vv} = -bY - N - \frac{1}{2}\xi.$$  

(3.10)

So, we know from $Y_{uv} = 0$ that $Y$ can be split as

$$Y = F(u) + G(v).$$

Substituting it into the structural equations, we have

$$F''' + (2a - \frac{1}{4})F' = 0, \quad G''' + (2b - \frac{1}{4})G' = 0.$$  

(3.11)

By (3.7) we have

$$(2a - \frac{1}{4}) + (2b - \frac{1}{4}) = -1.$$  

(3.12)

In the following we discuss the resolve into three essential cases by noting the character of the coefficients of the above PDEs (3.11).

Case I: $2a - \frac{1}{4} < 0, 2b - \frac{1}{4} < 0$.

Let $2a - \frac{1}{4} = -r^2$. Then $2b - \frac{1}{4} = r^2 - 1$ and $0 < r < 1$. We have a particular resolve

$$F = (r \cosh(ru), 0, r \sinh(ru), 0, 1),$$

$$G = (0, \sqrt{1 - r^2} \cosh(\sqrt{1 - r^2}v), 0, \sqrt{1 - r^2} \sinh(\sqrt{1 - r^2}v), 0).$$
And we know that any resolve \((Y, N, Y_u, Y_v, \xi)\) of PDEs (3.9) and (3.10) is different from the initial resolve \((Y, N, Y_u, Y_v, \xi)_0\) up to an isometric transformation \(T\) in \(\mathbb{R}^5_2\), i.e., \((Y, N, Y_u, Y_v, \xi) = T(Y, N, Y_u, Y_v, \xi)_0\) so

\[ Y = F + G := (x, 1) \]

\[ = (r \cosh(ru), \sqrt{1-r^2} \cosh(\sqrt{1-r^2}v), r \sinh(ru), \sqrt{1-r^2} \sinh(\sqrt{1-r^2}v), 1) \]

locally determines a surface \(x : \mathbb{H}^1(r) \times \mathbb{H}^1(\sqrt{1-r^2}) \rightarrow \mathbb{H}^3_1\) whose canonical lift is \(Y\).

Case II: \(2a - \frac{1}{4} < 0, 2b - \frac{1}{4} > 0\).

Let \(2a - \frac{1}{4} = -r^2 - 1\). Then \(2b - \frac{1}{4} = r^2\) and \(r > 0\). We have a particular resolve

\[ F = (1, r \cosh(\sqrt{r^2 + 1}u), r \sinh(\sqrt{r^2 + 1}u), 0, 0), \]

\[ G = (0, 0, 0, \sqrt{r^2 + 1} \cos(rv), \sqrt{r^2 + 1} \sin(rv)). \]

And we know that any resolve \((Y, N, Y_u, Y_v, \xi)\) of PDEs (3.9) and (3.10) is different from the initial resolve \((Y, N, Y_u, Y_v, \xi)_0\) up to an isometric transformation \(T\) in \(\mathbb{R}^5_2\), i.e., \((Y, N, Y_u, Y_v, \xi) = T(Y, N, Y_u, Y_v, \xi)_0\) so

\[ Y = F + G := (1, x) \]

\[ = (1, r \cosh(\sqrt{r^2 + 1}u), r \sinh(\sqrt{r^2 + 1}u), \sqrt{r^2 + 1} \cos(rv), \sqrt{r^2 + 1} \sin(rv)) \]

locally determines a surface \(x : \mathbb{H}^1(r) \times S^1(\sqrt{r^2 + 1}) \rightarrow S^3_1\) whose canonical lift is \(Y\).

Case III: \(2a - \frac{1}{4} = -1, 2b - \frac{1}{4} = 0\).

We have a particular resolve

\[ F = (0, \cosh u, \sinh u, 0, 0), \]

\[ G = (\frac{v^2}{2}, 0, 0, v, \frac{v^2}{2} - 1). \]

And we know that any resolve \((Y, N, Y_u, Y_v, \xi)\) of PDEs (3.9) and (3.10) is different from the initial resolve \((Y, N, Y_u, Y_v, \xi)_0\) up to an isometric transformation \(T\) in \(\mathbb{R}^5_2\), i.e., \((Y, N, Y_u, Y_v, \xi) = T(Y, N, Y_u, Y_v, \xi)_0\) so

\[ Y = F + G := (\frac{(x, x)}{2}, x, \frac{(x, x)}{2} - 1) \]

\[ = (\frac{v^2}{2}, \cosh u, \sinh u, v, \frac{v^2}{2} - 1) \]

locally determines a surface \(x : \mathbb{H}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^3_1\) whose canonical lift is \(Y\).
Summing up, we obtain

**Theorem 3.1** If $x: \mathbb{M}^2 \to \mathbb{Q}^3_1$ is a space-like conformal surface, then it must be locally conformally equivalent to one of the three standard embedding surfaces: $\mathbb{H}^1(r) \times \mathbb{H}^1(\sqrt{1-r^2}) \subset \mathbb{H}^3_1$, $\mathbb{H}^1(r) \times \mathbb{S}^1(\sqrt{r^2+1}) \subset \mathbb{S}^3_1$, and $\mathbb{H}^1 \times \mathbb{R}^1 \subset \mathbb{R}^3_1$, where all radii of sphere or hyperbolic forms should be positive.

Similarly, we shall get

**Theorem 3.2** If $x: \mathbb{M}^2_1 \to \mathbb{Q}^3_1$ is a time-like conformal surface, then it must be locally conformally equivalent to one of the five standard embedding surfaces: $\mathbb{H}^1(r) \times \mathbb{H}^1(\sqrt{1-r^2}) \subset \mathbb{H}^3_1$, $\mathbb{S}^1(r) \times \mathbb{H}^1(\sqrt{1+r^2}) \subset \mathbb{H}^3_1$, $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \subset \mathbb{S}^3_1$, $\mathbb{R}^1 \times \mathbb{S}^1 \subset \mathbb{R}^3_1$, and $\mathbb{S}^1 \times \mathbb{R}^1 \subset \mathbb{R}^3_1$, where all the radii of (pseudo-Riemannian) sphere or hyperbolic forms should be positive.

§ 4. Blaschke para-umbilical hypersurfaces in $\mathbb{Q}^n_0$.

We remind readers that we shall retain the assumption on the head of Section 2. First, we give the

**Definition 4.1.** We call an $m$-dimensional regular hypersurface $x: \mathbb{M}^m \to \mathbb{Q}^m_1$ Blaschke para-umbilical if there exist a smooth function $\lambda, \mu$ on $\mathbb{M}$ such that

$$A = \lambda I_m + \mu B, \quad \text{and} \quad \Phi \equiv 0, \quad (4.1)$$

where $I_m$ means $m$ order unit matrix.

**Remark 4.1.** This definition is well-defined and it has no matter with the choose of local basis of $\mathbb{M}$. When $n = m + 1$, a Blaschke quasi-umbilical submanifold reduces to a Blaschke para-umbilical hypersurface (c.f. [2]).

We have

**Proposition 4.1.** If $u: \mathbb{M}^m_t \to \mathbb{R}^m_1(\epsilon)$ is a regular hypersurface with constant scalar curvature $\rho_t$ and mean curvature $H$, then $x = \sigma_\epsilon \circ u$ is a Blaschke para-umbilical hypersurface in $\mathbb{Q}^m_1$.

**Proof** Because of (2.4) and (2.5), we know immediately that $|\vec{H}|^2$ and $|II|^2$ are both constant. And one can easily see that the conformal factor $e^{2\tau} = \pm \frac{m}{m-1}(|II|^2 - m|\vec{H}|^2)$ =constant. If the unit normal vector of hypersurface $x$ is space-like (or time-like), then we denote $\epsilon = 1$ (or $-1$). By use of (2.6) and (2.7), it follows from above that

$$e^{2\tau} A = \epsilon H h + \frac{1}{2}(\epsilon - \epsilon H^2) I_m;$$

$$e^{\tau} B = h - H I_m, \quad C_i = 0, \forall i.$$ 

If we choose $\lambda = \frac{1}{2}e^{-2\tau}(\epsilon + \epsilon H^2)$, and $\mu = \epsilon e^{\tau} H$, we can verify that all the conditions of a Blaschke quasi-umbilical submanifold are satisfied. □
Proposition 4.2. Suppose that \( x : M^m \to Q_{m+1} \) is a Blaschke para-umbilical hypersurface in \( Q_{m+1} \). Then the smooth function \( \lambda \) in (4.1) must be constant.

**Proof** Suppose that \( \xi \) is the unit normal vector of hypersurface \( x \). Then from (3.1) and (3.5) we get
\[
e_i(N) = \lambda e_i(Y) + \mu e_i(\xi).
\]
That means,
\[
dN + \lambda dY + \mu d\xi = 0, \tag{4.2}
\]
which implies that
\[
d\lambda \wedge dY + d\mu \wedge d\xi = 0.
\]
Letting \( \lambda_i = e_i(\lambda), \mu_i = e_i(\mu) \), combining with (3.5) and the vanishing conformal form, we have
\[
\sum_{ijk} \lambda_i \omega^i \wedge \omega^j \delta^k Y_k + \sum_{ijk} \mu_i \omega^i \wedge \omega^j B^k_j Y_k = 0.
\]
Because of the linear independence of \( \{Y_1, \ldots, Y_m\} \) and the Cartan’s lemma, we have
\[
\lambda_i \delta^k_j + \mu_i B^k_j = \lambda_j \delta^k_i + \mu_j B^k_i. \tag{4.3}
\]
Because \( x \) has vanishing conformal form, by (3.3), we can choose an appropriate orthonormal basis \( \{e_1, \ldots, e_m\} \) such that
\[
A = \text{diag}(a_i), \quad B = \text{diag}(b_i).
\]
For (4.2), fixing \( i \), letting \( j = k \), and taking summation over \( j \), it follows from (3.4) that
\[
\lambda_i - \frac{1}{m-1} \mu_i b_i = 0. \tag{4.4}
\]
Taking \( i \neq j = k \) in (4.2), we get
\[
\lambda_i + \mu_i b_j = 0, \quad i \neq j. \tag{4.5}
\]
From (4.4) and (4.5) we have
\[
\mu_i (b_j + \frac{1}{m-1} b_i) = 0, \quad i \neq j. \tag{4.6}
\]
If \( \mu_i \)'s are all zero, it follows from (4.4) that \( \lambda_i \)'s are all zero. Then \( \lambda, \mu \) are both constant over \( M \).

On the contrary, if \( \mu_i \)'s are not all zero, without the loss of generality, we may assume that \( \mu_1 \neq 0 \), then combining (3.1) and
\[
b_i = -\frac{1}{m-1} b_1,
\]
11
we can adjust the orient of the unit normal vector \( \xi \) such that
\[
b_1 = \frac{m - 1}{m}, \quad b_2 = \cdots = b_m = -\frac{1}{m}. \tag{4.7}
\]
In the following we adopt the conventions on the ranges of indices
\[
2 \leq \alpha, \beta \leq m.
\]
Taking \( i, j \) various values in (3.6), we have
\[
B_{11,i} = B_{\alpha\beta,i} = 0, \forall i.
\]
Therefore \( B_{1\alpha,i} = 0, \forall i \). Taking \( i = 1, j = \alpha \) in (3.6), we have \( \omega_{1\alpha} = 0 \). Similarly as precious induction in Section 3, we have
\[
R_{1\alpha1\alpha} = 0 = \epsilon b_1 b_2 + a_1 + a_\alpha, \tag{4.8}
\]
where \( \epsilon = \langle \xi, \xi \rangle \). So we know that \( A = (a_1) \oplus (a_2 I_{m-1}) \). By (4.1) we get
\[
a_1 + a_2 = 2\lambda + (b_1 + b_2)\mu. \tag{4.9}
\]
Combining (4.7)-(4.9), we get
\[
2\lambda + \frac{m - 2}{m} \mu = \epsilon \frac{m - 1}{m^2},
\]
Therefore
\[
2\lambda_1 + \frac{m - 2}{m} \mu_1 = 0. \tag{4.10}
\]
Taking \( i = 1, j = 2 \) in (4.5), we have
\[
\lambda_1 = \frac{1}{m} \mu_1. \tag{4.11}
\]
Substituting (4.11) into (4.10), we get
\[
\mu_1 = 0.
\]
This is a contraction to the assumption \( \mu_1 \neq 0 \). So, if \( \mathbf{M} \) is connected, then \( \lambda = \text{constant}, \mu = \text{constant}. \Box

If we take trace of the first equation of (4.1), we will find by (3.4) that
\[
m\lambda = \text{tr}(A) = \frac{1}{2m} \left( \frac{m}{m - 1} \rho \pm 1 \right) = \frac{1}{2(m - 1)} \rho \pm \frac{1}{2m}. \tag{4.12}
\]
which implies that the conformal scalar curvature

$$\rho = \text{constant}. $$

Using the structural equations in Section 2, we have

$$-mN = \Delta Y + \text{tr}(A)Y. $$

From (4.12), we get

$$-mN = \Delta Y + m\lambda Y. \quad (4.13)$$

Therefore by Proposition 4.1 and (4.2) we can find a constant vector $\vec{c} \in \mathbb{R}^{n+2}$ such that

$$N = \lambda Y + \mu \xi + \vec{c}. \quad (4.14)$$

It follows from (4.13) and (4.14) that

$$\langle \vec{c}, Y \rangle = 1, \quad \langle \vec{c}, \xi \rangle = -\varepsilon \mu^2, \quad \langle \vec{c}, \vec{c} \rangle = -2\lambda + \varepsilon \mu^2, \quad (4.15)$$

where $\varepsilon = \langle \xi, \xi \rangle$.

Then we discuss into the following three cases.

**Case 1**: $\langle \vec{c}, \vec{c} \rangle = -2\lambda + \varepsilon \mu^2 = 0$.

By use of an isometric transform of $\mathbb{R}^{n+2}$ if necessary, assume that

$$\vec{c} = (1, 0, 1).$$

Letting

$$Y = (x_1, u, x_{n+2}),$$

it follows from the first equation of (4.15) and $\langle Y, Y \rangle = 0$ that

$$Y = \left(\frac{\langle u, u \rangle - 1}{2}, u, \frac{\langle u, u \rangle + 1}{2}\right).$$

Then $x$ determines a hypersurface $u : M^m_t \to \mathbb{R}^{m+1}$ with

$$I = \langle du, du \rangle = \langle dY, dY \rangle = g,$$

which implies that

$$\Delta_I = \Delta, \quad \rho_I = \rho = \text{constant}.$$

We know from [2] that

$$\xi = \varepsilon HY + (0, \zeta, 0), \quad (4.16)$$

where $\zeta$ is the unit normal vector of $u$. It follows from the first and the second equations of (4.15) and (4.16) that

$$H = -\mu^2 = \text{constant.}$$
Then, \( u \) is a regular hypersurface with constant scalar curvature and mean curvature in \( \mathbb{R}^{m+1}_s \). In this case \( x \) is locally conformally equivalent to a regular hypersurface with constant scalar curvature and mean curvature in \( \mathbb{R}^{m+1}_s \).

**Case 2:** \( \langle \vec{c}, \vec{c} \rangle = -2\lambda + \varepsilon \mu^2 := -r^2, r = \text{constant} > 0 \).

By use of an isometric transform of \( \mathbb{R}^{n+2}_s \) if necessary, assume that
\[
\vec{c} = (r, 0).
\]

Letting
\[
Y = (x_1, u/r),
\]
by similar method as above we have
\[
x_1 = 1/r.
\]
So
\[
Y = (1, u)/r, \quad \langle u, u \rangle = 1.
\]
Then \( x \) determines a hypersurface \( u : \mathcal{M}_t^m \rightarrow S^{m+1}_s \) with
\[
I/r^2 = \langle du, du \rangle / r^2 = \langle dY, dY \rangle = g,
\]
which implies that
\[
r^2 \Delta_I = \Delta, \quad \rho_I = \rho / r^2 = \text{constant}.
\]
We know from [2] that
\[
\xi = \varepsilon H Y + (0, \zeta), \quad (4.17)
\]
where \( \zeta \) is the unit normal vector of \( u \). It follows from the first and the second equations of (4.15) and (4.17) that
\[
H = -\mu^2 = \text{constant}.
\]
Then, \( u \) is a regular hypersurface with constant scalar curvature and mean curvature in \( S^{m+1}_s \). In this case \( x \) is locally conformally equivalent to a regular hypersurface with constant scalar curvature and mean curvature in \( S^{m+1}_s \).

**Case 3:** \( \langle \vec{c}, \vec{c} \rangle = -2\lambda + \varepsilon \mu^2 := r^2, r = \text{constant} > 0 \).

By use of an isometric transform of \( \mathbb{R}^{n+2}_s \) if necessary, assume that
\[
\vec{c} = (0, r).
\]
Letting
\[
Y = (u/r, x_{n+2}),
\]
\[
\]
similarly we have
\[ x_{n+2} = 1/r. \]
So
\[ Y = (u, 1)/r, \quad \langle u, u \rangle = -1. \]
Then \( x \) determines a hypersurface \( u : M^m_t \to \mathbb{H}^{m+1}_s \) with
\[ I/r^2 = \langle du, du \rangle/r^2 = \langle dY, dY \rangle = g, \]
which implies that
\[ r^2 \Delta I = \Delta, \quad \rho I = \rho/r^2 = \text{constant}. \]
We know from [2] that
\[ \xi = \varepsilon HY + (\zeta, 0), \quad (4.18) \]
where \( \zeta \) is the unit normal vector of \( u \). It follows from the first and the second equations of (4.15) and (4.18) that
\[ H = -\mu^2 = \text{constant}. \]
Then, \( u \) is a regular hypersurface with constant scalar curvature and mean curvature in \( \mathbb{H}^{m+1}_s \). In this case \( x \) is locally conformally equivalent to a regular hypersurface with constant scalar curvature and mean curvature in \( \mathbb{H}^{m+1}_s \).

So combining Proposition 4.1 we get

**Theorem 4.1.** Any Blaschke para-umbilic hypersurface in \( Q^n_s \) is locally conformally equivalent to a regular hypersurface with constant scalar curvature and mean curvature in \( \mathbb{R}^n_s, S^n_s \), or \( \mathbb{H}^n_s \).

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