Gravitational drag on a point mass in hypersonic motion through a gaseous medium

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ABSTRACT
We explore a ballistic orbit model to infer the gravitational drag force on an accreting point mass $M$, such as a black hole, moving at a hypersonic velocity $v_0$ through a gaseous environment of density $\rho_0$. The streamlines blend in the flow past the body and transfer momentum to it. The total drag force acting on the body, including the non-linear contribution of those streamlines with small impact parameter that bend significantly and pass through a shock, can be calculated by imposing conservation of momentum. In this fully analytic approach, the ambiguity in the definition of the lower cut-off distance $r_{\text{min}}$ in calculations of the effect of dynamical friction is removed. It turns out that $r_{\text{min}} = \sqrt{\pi G M / 2 v_0^2}$. Using spherical surfaces of control of different sizes, we carry out a successful comparison between the predicted drag force and the one obtained from a high-resolution, axisymmetric, isothermal flow simulation. We demonstrate that ballistic models are reasonably successful in accounting for both the accretion rate and the gravitational drag.

Key words: black hole physics – hydrodynamics – stars: formation – ISM: clouds – ISM: kinematics and dynamics.

1 INTRODUCTION
A body moving in a background medium loses momentum due to its gravitational interaction with its own gravitationally induced wake. This process is often referred to as dynamical friction. Chandrasekhar (1943) estimated the dynamical friction on a massive particle passing through a homogeneous and isotropic background of light stars. In the case where the perturber moves in a gaseous medium, the gravitational drag is traditionally inferred as the gravitational attraction between the perturber and its own wake.

In this approach, the density structure of the wake is derived in linear perturbation theory by assuming that the body produces a small perturbation in the ambient medium (Dokuchaev 1964; Ruderman & Spiegel 1971; Just & Kegel 1990; Ostriker 1999; Kim & Kim 2007; Sánchez-Salcedo 2009; Namouni 2010). For a perturber moving on a rectilinear orbit at constant velocity, the steady-state linear theory predicts that the drag force vanishes for subsonic perturbers, while it becomes similar to the collisionless drag force for supersonic bodies. Ostriker (1999) considered the linear-theory drag as a time-dependent rather than steady state problem and arrived at the following formula for the gravitational drag force,

$$F_g = \frac{4 \pi \rho_0 G^2 M^2}{v_0^2} \left\{ \frac{1}{2} \ln \left( \frac{1 + M}{1 - M} \right) - M \right\} \quad \text{if} \ M < 1;$$
$$\frac{1}{2} \ln \left( 1 - M^2 \right) + \ln \left( \frac{v_{\text{inj}}}{r_{\text{min}}} \right) \quad \text{if} \ M > 1. \quad (1)$$

The perturber of mass $M$, which moves at velocity $v_0$ and Mach number $M$ in a rectilinear orbit through a homogeneous medium with density $\rho_0$ and sound speed $c_0$, is assumed to be formed at $t = 0$. The minimum radius $r_{\text{min}}$ is the typical size of the perturber. This formula has enjoyed widespread theoretical application (Narayan 2000; Escala et al. 2004; Kim 2007; Conroy & Ostriker 2008; Tanaka & Haiman 2009; Villaver & Livio 2009; Chavarría et al. 2010; Nejad-Ashgar 2010). Because of the linear-theory assumption, the above equation is properly valid only at $r > R_{\text{BH}}$ where $R_{\text{BH}}$ is the Bondi–Hoyle radius ($R_{\text{BH}} = GM / [c_0^2 (1 + M^2)]$). Therefore, equation (1) is strictly valid for extended perturbers with a softening radius much larger than the Bondi–Hoyle radius. In fact, for extended perturbers, Sánchez-Salcedo & Brandenburg (1999) found good agreement between the gravitational drag in full hydrodynamical simulations and Ostriker’s formula. In particular, for Plummer perturbers with softening radius $r_s$ much larger than $R_{\text{BH}}$, they found that $r_{\text{min}} = 2.25 r_s$. An extension of Ostriker’s formula for extended bodies orbiting in a stratified gaseous sphere was given in Sánchez-Salcedo & Brandenburg (2001).

In the case of point-like perturbers, like massive black holes, it is reasonable to assume that $r_{\text{min}}$ should be of the order of a few $R_{\text{BH}}$, but a non-linear analysis is required to fix the uncertainty in the definition of $r_{\text{min}}$. In adiabatic simulations of axisymmetric accretion flows past a gravitating absorbing object, Shima et al. (1985) computed the drag by considering two contributions: the aerodynamic force, which is due to the accretion of momentum over the body surface, and the gravitational force on the perturber by its
own wake. They found that the numerical results were consistent with the estimates in linear theory.

The problem of the gravitational drag on a point-mass particle has revived new interest to estimate the time-scale of the orbital decay of massive black hole binary in the centre of galaxies. Escala et al. (2004) simulated the orbital decay of a single black hole moving initially on a circular orbit in an isothermal gaseous sphere. They found that the gravitational drag is less peaked at $1 \leq M < 2$ than predicted by Ostriker’s formula with $v_0/r_{\text{ms}} = 3.1$. Tanaka & Haiman (2009) combined the prescriptions of Ostriker (1999) and Escala et al. (2004) into a formula that is used as a prescription of the gaseous drag on black holes in numerical simulations. In order to isolate the physical reason of the failure of Ostriker’s formula, the gaseous drag on black holes in numerical simulations. In order to isolate the physical reason of the failure of Ostriker’s formula, Kim & Kim (2009) and Kim (2010) carried out axisymmetrical simulations of a massive body in rectilinear orbit with different values of the strength of the gravitational perturbation due to the body as measured by

$$A = \frac{GM}{c^2 r_s},$$

(2)

where $r_s$ is the softening radius of the Plummer perturber. They find that the functional form of the gravitational drag is not so peaked as the linear theory predicts and conclude that the discrepancy between the numerical and Ostriker results are most likely due to the nonlinear effect. It is important to note that in the simulations of Escala et al. (2004), Kim & Kim (2009) and Kim (2010), the perturber simply provides a smooth gravitational potential and does not hold any absorbing surface. Without any absorbing inner boundary condition, a hydrostatic envelope with front-back symmetry is formed near the perturber. Because of the front-back symmetry, this large envelope provides a negligible contribution to the gravitational drag force.

However, it is well known that accretion is a crucial ingredient in point-like objects, such as black holes or stars. Different boundary conditions in the high-density region of the wake are expected to change the gas dynamics near the perturber and the strength of the gravitational drag (e.g. Fryxell, Taam & McMillan 1987; Naiman, Ramírez-Ruiz & Lin 2011). For example, Ruffert (1996) simulated a 3D quasi-isothermal flow with an absorbing boundary surrounding the point-like object. Moekel & Throop (2009) carried out similar simulations, but then also included an accretion disc orbiting the point source.

The aim of this paper is to describe the contribution of the non-linear inner wake to the gravitational drag on hypersonic perturbers by using the ballistic orbit theory (Bondi & Hoyle 1944; Lyttleton 1972; Bisnovatyi-Kogan et al. 1979). Whereas this theory (the so-called model of line-accretion) has been extensively used as a powerful framework to describe the gravitational interaction between a moving massive body and the surrounding gaseous medium in the context of supermassive Bondi–Hoyle–Lyttleton accretion (e.g. Koide, Matsuda & Shima 1991; Edgar 2004), it has been traditionally ignored as a tool to quantify the gravitational drag. In fact, all analytical studies about the gravitational drag in gaseous media have been based on the linear perturbation theory, following on the analysis of Dokuchaev (1964), Ruderman & Spiegel (1971) and Rephaeli & Salpeter (1980). In this paper we develop the ballistic orbit theory to provide analytical expressions of, not only the mass accretion rate, but also the non-linear drag force on a hypersonic compact body. These estimates will be compared with numerical results of an axisymmetric isothermal hydrodynamical simulation.

### 2 THE FREE-STREAMING FLOW SOLUTION

Let us consider the axisymmetric flow generated by a point mass $M$ which moves hypersonically at a constant velocity $v_0$ inside a homogeneous gaseous environment (see also Bisnovatyi-Kogan et al. 1979). Fig. 1 shows a schematic diagram illustrating the trajectory of a fluid parcel in a frame of reference at rest with respect to the point mass.

Because $v_0$ (the upstream environmental velocity, see Fig. 1) is hypersonic, we neglect the pressure force and consider the ballistic trajectory of the fluid parcels in the gravitational potential of the point mass. As they have a positive $E = v_0^2/2$ energy (per unit mass), the trajectories of the fluid parcels are hyperbolic of the form

$$r = \frac{\xi^2}{\xi_0 (1 + \cos \theta) + \xi \sin \theta},$$

(3)

where $\xi$ is the impact parameter of the fluid parcel (see Fig. 1) and $\xi_0 = GM/v_0^2$.

With $G$ being the gravitational constant. For deriving equation (3) one has to consider a generic hyperbolic trajectory of the form

$$r = \frac{p[(1 + \epsilon \cos (\theta - \theta_0))]}{\epsilon},$$

and then impose the upstream boundary condition and the conserved angular momentum $\xi v_0$ to determine the constants $p$, $\epsilon$ and $\theta_0$.

From equation (3), we see that the streamline intercepts the symmetry axis (i.e. $\theta = 0$, see Fig. 1) at a position

$$x_0 = \frac{\xi^2}{2 \xi_0},$$

(5)

downstream from the perturber. The material will therefore pile up in a narrow, downstream wake surrounding the symmetry axis, forming a dense column of gas.

From the equation for the streamlines (equation 3) one can calculate the velocity components of the free-streaming flow along the x- and y-axes:

$$v_x = \frac{v_0}{\xi} (\xi + \xi_0 \sin \theta); \quad v_y = -\frac{v_0 \xi_0}{\xi} (1 + \cos \theta).$$

(6)

Assuming that the environment has a homogeneous density $\rho_0$ far upstream from the source, it is possible to obtain the density $\rho(x, y)$ of the free-streaming flow as a function of position:

$$\frac{\rho}{\rho_0} = \frac{\xi^3}{y [2 \xi_0 (r + x) + \xi (y)]} = \frac{\xi^2}{y (2 \xi - y)},$$

(7)

with

$$r = \sqrt{x^2 + y^2}; \quad \xi = \frac{1}{2} \left[ y + \sqrt{y^2 + 4 \xi_0 (r + x)} \right].$$

(8)

![Figure 1. Schematic diagram showing the trajectory of an environmental fluid parcel in hypersonic motion with respect to a point mass $M$. The initial velocity $v_0$ of a parcel with impact parameter $\xi$ is parallel to the $x$-axis, and its trajectory is the $P(\theta)$ curve. The problem has cylindrical symmetry, with $x$ being the symmetry axis and $y$ the cylindrical radius.](https://example.com/figure1.png)
Note that \( x \) is the distance along the symmetry axis and \( y \) the cylindrical radius. It is simple to see that \( \rho \geq \rho_0 \). Equation (7) is not valid in the shocked column of gas near the positive \( x \)-axis. The density enhancement in this approach is different from that derived in linear theory which predicts zero-enhancement outside the Mach cone (e.g. Ostriker 1999). Koide et al. (1991) found a good accordance between the analytical solutions (6)–(8) and the numerical solution even for Mach numbers as low as 1.4.

From equation (6) we see that at the point in which the streamlines intercept the symmetry axis (i.e. for \( \theta = 0 \), the flow velocity has an \( x \)-component \( v_x(0) = v_0 \) identical to the far upstream flow velocity and independent of the impact parameter \( \xi \) of the flow parcel) and a \( y \)-component \( v_y(0) = -2v_0\xi/\xi^2 \). This latter component of the velocity will be thermalized in a shock surrounding the downstream wake. We will assume that the post-shock thermal energy is radiated away instantaneously.

Now, the kinetic+potential energy per unit mass of the flow at \( x \rightarrow -\infty \) is \( E_0 = v_0^2/2 \). When the flow hits the symmetry axis (at \( \theta = 0 \), the energy associated with the \( y \)-velocity is thermalized, so that the kinetic+potential energy is reduced to a value

\[
E_i = E_0 - \frac{v_i^2(0)}{2} = \frac{v_0^2}{2} \left[ 1 - \left( \frac{2\xi_0}{\xi} \right)^2 \right].
\] (9)

From equation (9) it is clear that \( E_i \leq 0 \) (i.e. the post-shock material is gravitationally bound) if the condition

\[
\xi \leq 2\xi_0 \tag{10}
\]

is met. Therefore all of the material arriving with impact parameters \( \leq 2\xi_0 \) will eventually be accreted on to the body. A streamline with impact parameter \( \xi_1 = 2\xi_0 \) (with \( \xi_0 \) given by equation 4) crosses the symmetry axis at a distance

\[
x_1 = 2\xi_0, \tag{11}
\]
downstream from the body (see equation 5 and Fig. 1).

The material within the downstream wake will have a complex flow pattern. From equation (6) it is clear that the material enters the tail with a positive \( x \)-velocity. The material with impact parameter \( \xi \leq 2\xi_0 \) (which is gravitationally bound, see above) will therefore enter the wake flowing in the \( +x \)-direction, so that it will first flow away from the body, and eventually reverse and fall back on to the body. The distance \( x_m \) from the body (along the \( x \)-axis) at which the flow reverses can be obtained from the condition of zero velocity for a radial motion in the gravitational potential. This condition gives

\[
x_m = \frac{2\xi_0}{(2\xi_0/\xi)^2 - 1}. \tag{12}
\]

We see that when \( \xi \rightarrow \xi_1, x_m \rightarrow \infty \). Consequently, streamlines with impact parameter close to and smaller than \( \xi_1 \), will take a long time to be accreted.

The material in the wake that remains gravitationally unbound when entering the wake (i.e. the material with impact parameters \( \xi > 2\xi_0 \), see above), will flow away from the body along the \( x \)-axis, reaching infinity with a velocity

\[
v_\infty = v_0 \left[ 1 - \left( \frac{2\xi_0}{\xi} \right)^2 \right]^{1/2}. \tag{13}
\]

3 THE MASS ACCRETION RATE AND THE DRAG FORCE

3.1 The accretion rate

From the solution of Section 2, the accretion rate on to the point mass can be obtained (Hoyle & Lyttleton 1939; Bondi & Hoyle 1944). All of the material with impact parameters \( \xi \leq 2\xi_0 \) (see equation 10) will fall on to the body. Therefore, the mass accretion rate is

\[
M_{\text{acc}} = \pi(2\xi_0)^2 \rho_0 v_0 = \frac{4\pi(GM)^2 \rho_0}{v_0^3}, \tag{14}
\]

where we have used equation (4) for the second equality.

3.2 The gravitational drag

We calculate the gravitational drag on the perturber by computing the net \( x \)-momentum per unit of time, \( \Pi_x \), going through a spherical control volume of radius \( R > 2\xi_0 \) centred on the body. It is clear that the contribution to the drag by the gas within the sphere is equal to \( \Pi_x \).

We consider the three streamlines shown in the schematic diagram of Fig. 2:

(i) a streamline with impact parameter \( \xi_1 = 2\xi_0 \), which crosses the axis at \( x_1 = 2\xi_0 \) (see equation 11) (all of the material with \( \xi \leq \xi_1 \) is accreted on to the body);
(ii) a streamline with impact parameter \( \xi_2 = \sqrt{2\xi_0 R} \), which crosses the axis at a distance \( R \) downstream from the perturber (where \( R \) is the radius of the control sphere, see above);
(iii) a streamline with impact parameter \( \xi_3 \), which tangentially touches the control sphere (at a point with polar angle \( \theta_3 \), see Fig. 2).

In order to obtain \( \xi_1 \) and \( \theta_3 \) we first set \( r = R \) (i.e. a radius equal to the radius of the control sphere) in equation (3), and invert this equation to find

\[
\sin \theta_3 = \frac{\xi_3^2 - \xi_0 R \pm \xi_0 \sqrt{R(2\xi_0 + R) - \xi_3^2}}{R(\xi_3^2 + \xi_0^2)}, \tag{15}
\]

which gives the two values of \( \theta \) at which the streamline with impact parameter \( \xi \) cuts the control sphere. The angle \( \theta_3 \) (obtained with the + sign of the right-hand term of equation 15) corresponds to the point in which the streamline enters the control sphere, and \( \theta_3 \).
Gravitational drag

Figure 3. Exact (solid line, see equation 30) and approximate (dashed line, see equation 31) forms of the \( f \) function, which gives the dependence of the drag force as a function of the radius \( R \) of the control volume.

where \( x_0 \) is the distance along the \( x \)-axis at which the streamline intercepts the axis as given by equation (5). Here we have used that the \( x \)-component of the velocity is \( v_0 \), as derived in equation (6). Using equations (25 and 26), the integral in equation (24) can be performed analytically to obtain

\[
\dot{\Pi}_{x,a} = 4\pi \xi_0^2 \rho_0 v_0^2 f_a,
\]

where

\[
f_a = 1 - \frac{(2w_0)^{3/2}}{2w_0} + \frac{1}{\sqrt{1 + 2w_0}} \ln \left[ \frac{2w_0 + \sqrt{2w_0(1 + 2w_0)}}{1 + \sqrt{1 + 2w_0}} \right].
\]

Finally, the net drag force of the gas on the body is obtained as

\[
F_d = \dot{\Pi}_{x,in} - \dot{\Pi}_{x,b} - \dot{\Pi}_{x,a} = \frac{4\pi (GM)^2 \rho_0}{v_0^2} f(\xi_0, R),
\]

where \( \xi_0 \) is given by equation (4) and

\[
f(\xi_0, R) \approx \ln \left( \frac{2R}{\xi_0} \right) - \frac{1}{2} \left( 1 - \frac{\xi_0}{R} \right).
\]

A comparison between the full (equation 30) and approximate (equation 31) forms of \( f \) is shown in Fig. 3. It is clear that for radii larger than \( \sim 5\xi_0 \) the two forms of \( f \) agree to better than \( \sim 10 \) per cent.

From equation (31) we see that the drag force diverges logarithmically for large values of \( R \), as occurs in linear perturbation theory. Therefore, it is possible to match the solution found in linear theory (equation 1), which is valid at far enough distances from the body, with that found in the non-linear analysis. This can be accomplished by replacing \( R \) for \( v_0 t \), where \( t = 0 \) is the time at which the body is formed.\(^1\) Moreover, we see that the ambiguity in the definition of the minimum radius \( r_{\text{min}} \) that appears in linear theory is removed in our framework. In fact, we find that \( r_{\text{min}} = \sqrt{\xi_0/2} \), where the

\[\xi_0 = \sqrt{R(2\xi_0 + R)},\]

and using equation (15) we then obtain

\[
\sin \theta = \frac{R(2\xi_0 + R)}{R + \xi_0}.
\]

Now, the \( x \)-momentum entering the control sphere from the upstream region can be calculated as

\[
\dot{\Pi}_{x,in}(R) = 2\pi \rho_0 v_0 \int_0^{\xi_1} v_x(\xi, \theta) \xi \, d\xi,
\]

where \( \xi_1 \) is given by equation (16), \( v_x(\xi, \theta) \) by equation (6) and \( \theta \) is obtained with the \( + \) sign of equation (15). This integral can be solved analytically to obtain

\[
\dot{\Pi}_{x,in}(R) = 4\pi \xi_1^2 \rho_0 v_0^2 f_{in},
\]

where

\[
f_{in} = \frac{1 + 3w_0}{4w_0^2} - \frac{1}{2} \left[ \sqrt{1 + 2w_0} - 1 + (1 + w_0) \ln \left( \frac{1 + w_0}{1 + w_0 + \sqrt{1 + 2w_0}} \right) \right].
\]

with \( \omega_0 = \xi_0/\xi_0 < 1/2 \).

The \( x \)-momentum rate leaving the control domain has two terms:

(i) the rate \( \dot{\Pi}_{x,b} \) of \( x \)-momentum leaving through the boundary of the spherical domain,

(ii) the rate \( \dot{\Pi}_{x,a} \) of \( x \)-momentum hitting the symmetry axis and exiting the domain through a narrow wake along the \( x \)-axis.

The momentum rate leaving the sphere through the boundary of the control volume is given by

\[
\dot{\Pi}_{x,b} = 2\pi \rho_0 v_0 \int_0^{\xi_1} v_x(\xi, \theta) \xi \, d\xi.
\]

This integral can be performed analytically to obtain

\[
\dot{\Pi}_{x,b} = 4\pi \xi_1^2 \rho_0 v_0^2 f_b,
\]

where

\[
f_b = \frac{1 + w_0}{4w_0^2} + \frac{1}{2} \left[ 1 + (1 + w_0) \ln \left( \frac{w_0}{1 + w_0} \right) \right].
\]

The momentum rate exiting the control region through the wake is given by

\[
\dot{\Pi}_{x,a} = \int_{\xi_1}^{\xi_2} v_R \, dm,
\]

where

\[
dm = 2\pi \xi_0^2 \rho_0 \xi \, d\xi,
\]

and the velocity \( v_R \) along the axis with which the material leaves the control domain is given by the kinetic-potential energy conservation condition

\[
\frac{v_R^2}{2} = \frac{GM}{x_0} = \frac{v_0^2}{2} - \frac{GM}{R}.
\]

\(^1\) In a realistic situation, \( R \) will increase with time until it reaches the boundary of the cloud.
factor $\sqrt{e}$ comes from inserting the term $-1/2$ that appears in the right-hand side of equation (31) in the argument of the log.

4 AN AXISYMMETRIC NUMERICAL SIMULATION

We have computed an axisymmetric numerical simulation, solving the Euler equations for an isothermal flow in a uniform, cylindrical computational grid. We have used the ‘flux vector splitting’ algorithm of van Leer (1982), with the second order (time and space) implementation described by Raga, Navarro-González & Villagrá-Muniz (2000).

The simulation that we have presented can be compared with the work of Ruffert (1996) and Moeckel & Throop (2009), who computed 3D simulations of basically the same physical situation. The main difference with this previous work is that our simulation is 2D (axisymmetric), and has $\sim$ two orders of magnitude higher resolution.

The computational domain has an axial extent of $15\xi_0$ (with $\xi_0$ being the gravitational radius given by equation 4) and a radial extent of $7.5\xi_0$, resolved with $9000 \times 4500$ (axial $\times$ radial) grid points. A point mass (influencing the flow only through its gravitational attraction) is placed in the middle of the axial extent of the domain. A spherical volume of radius $0.05\xi_0$ (30 pixel) around the body is artificially kept at a low density at all times, so that the material entering this volume from the rest of the computational domain is effectively removed. We simulate in this way the accretion of gas on to the object.

In the left boundary of the domain we impose an inflow of density $\rho_0$ and velocity $v_0$, parallel to the symmetry axis. A reflection condition is applied on the symmetry axis, and a zero gradient condition is applied in the remaining two boundaries of the computational domain. In the initial condition, the domain is filled with a uniform flow (of velocity $v_0$ and density $\rho_0$) parallel to the symmetry axis. The isothermal sound speed is chosen to be $c_0/v_0 = 5$ (i.e. the flow entering the domain has a Mach number of 5).

The results obtained after time-integrations of 10, 40 and 70$\xi_0/v_0$ are shown in Fig. 4. This figure is a zoom of an inner region of the computational domain, showing the highly time-dependent wake formed downstream of the body.

We take the density and flow velocity time frames obtained from the simulation, and compute the net mass $M_{acc}$, and momentum fluxes through a control sphere of arbitrary radius $R$ centred on the point mass. Thereby, only $M_{acc}$ computed with $R = 2\xi_0$ corresponds exactly to the mass accretion rate on to the body. We also compute the gravitational force exerted on the body by the material within the control volume. In Fig. 5, we show the mass flux $M_{acc}$, the drag force $F_d$ and the gravitational force $F_g$ computed with a control volume of radius $R = 5\xi_0$. $F_g$ is inferred as the gravitational attraction between the body and the perturbed medium. $M_{acc}$ and $F_d$ show a peak at $t \approx 5\xi_0/v_0$, and have fluctuating values for $t \geq 10\xi_0/v_0$. The gravitational force $F_g$ initially grows as more material enters the wake behind the object, and also shows fluctuating values as a function of time. The fact that $M_{acc}$, $F_d$ and $F_g$ have strong fluctuations is not surprising given the strongly time-dependent structure of the flow (see Fig. 4).

In order to carry out a comparison with the analytic model (see Section 3.2), we have calculated the average values and the dispersions of $M_{acc}$, $F_d$ and $F_g$ in the interval $10\xi_0/v_0 \leq t \leq 66\xi_0/v_0$, in which the fluctuations of these quantities appear to be statistically stationary (see Fig. 5). We then plot these time-averaged values as a function of the radius $R$ of the control volume in Fig. 6.

We have considered control volumes with radii $2\xi_0 \leq R \leq 7\xi_0$, the lower boundary being fixed by the derivation of the analytic model (in which it was assumed that $R \geq 2\xi_0$, see Section 3) and the upper boundary given by the approach to the outer edge of the computational grid. It is clear from Fig. 6 that the dispersions of the $M_{acc}$ and $F_d$ values grow as a function of $R$ (due to the fact that larger, more massive eddies are seen at larger distances downstream from the body, see Fig. 4), while the dispersion of $F_g$ (which is a quantity integrated over the volume of the control sphere) remains approximately constant.

The analytic model predicts that $M_{acc} = 4\pi\xi_0^2\rho_0v_0$ (see equation 14) for all control spheres with $R \geq 2\xi_0$. The top panel of Fig. 6 shows that the time-averaged values obtained from the numerical simulation closely reproduce this result. The central panel of Fig. 6 shows the drag force $F_d$ calculated using equation (29), which has values that differ from the results from the numerical simulations by less than $\sim 15$ per cent (though the slope of the $F_d$ versus $R$ dependence appears to be higher in the numerical results than in the analytic model).

In the bottom panel of Fig. 6 we show the gravitational force on the body due to the density structure of the numerical model. The gravitational force is larger than the net drag because momentum accretion on to the body produces an accelerating force. We also plot the gravitational force from the numerical simulation but excluding the contribution from the dense wake behind the shock (see the lower sequence of points in the bottom frame of Fig. 6). This force is comparable to the one obtained from the analytic density stratification given by equation (7), which only refers to the material in the free-streaming region before entry into the axial wake. A comparison between the gravitational force with (upper sequence of points) and without (lower sequence) the material within the wake indicates that $\sim 90$ per cent of the gravitational force on the body comes from the material within the wake.

5 SUMMARY

We present an analytic model for the flow generated by a point mass moving hypersonically within a homogeneous environment. This model is based on the ballistic orbit theory (see e.g. Bisnovatyi-Kogan et al. 1979), and is developed so as to obtain analytic expressions for the mass accretion rate and the non-linear gravitational drag. Since we include the contribution of the non-linear inner wake, there is no ambiguity in the definition of the minimum cut-off distance of the interaction, which turns out to be $\approx 0.82\xi_0$.

We find that the predicted mass accretion rate and gravitational drag agree satisfactorily with the results from an axisymmetric, isothermal simulation.

(i) For the mass accretion rate we essentially find full agreement between the analytic and numerical results (see equation 14 and Fig. 6). This result in principle differs from the one of Moeckel & Throop (2009), who obtain a significantly lower value for the accretion rate from their numerical simulation. The fact that we obtain a better agreement could be due to the considerably higher resolution of our simulation, or to the fact that we carry out a much longer time integration (extending to $\sim 70\xi_0/v_0$, compared to $\sim 1.5\xi_0/v_0$ for the simulation of Moeckel & Throop 2009).

(ii) For the net drag force $F_d$ we obtain an agreement within $\sim 20$ per cent between the prediction from the analytic model and the numerical simulations (see Fig. 6). Though the analytic and numerical drag forces have a reasonable quantitative agreement, it appears that the analytic model predicts a shallower $F_d$ versus $R$
Figure 4. Density (in units of $\rho_0$, with the colour scale given by the top right bar) and velocity field (white arrows) from the axisymmetric simulation described in Section 4, obtained for integration times $t = 10\xi_0/v_0$ (panel a), $40\xi_0/v_0$ (panel b) and $70\xi_0/v_0$ (panel c). The axes are in units of $\xi_0$. The perturber is on the abscissa at position $x = 0$, and the flow enters the domain from the left. Only a limited region of the computational domain is shown (see Section 4). The $x$ (symmetry axis) and $y$ (cylindrical radius) axes are labelled in units of $\xi_0$. 
Mass flux (in units of $\xi^2_0\rho_0 v_0$, top), drag force (in units of $\xi^2_0\rho_0 v_0^2$, centre) and gravitational force on the body (in units of $\xi_0\rho_0 v_0^2$, bottom) as a function of time (in units of $\xi_0 v_0$). These parameters were computed from the results of the axisymmetric simulation (described in the text) using a spherical control volume of radius $R = 5\xi_0$.

There are several possible sources for this discrepancy between the analytic and numerical drag forces. It appears that the limited numerical resolution of the simulation is not responsible for this effect, because we have repeated the simulation at 1/2 and 1/4 of the resolution (of the simulation presented in Section 4) and obtain basically the same drag force. A possible source of the differences between the analytic and numerical $F_d$ is the fact that the simulation has a finite Mach number ($M = 5$ for the upstream flow, see Section 4), while the analytic model essentially has an infinite Mach number (i.e. zero gas pressure). Another difference is that the numerical simulation has a rather broad wake region, while in the analytic solution it is assumed that the tail occupies a very narrow region surrounding the symmetry axis. A third difference is that while in the analytic model the perturbed environmental gas effectively extends to infinity, the numerical simulation of course is carried out in a finite domain (see Section 4). Given these clear differences between the numerical and analytic models, the agreement that we find between the two can be regarded as quite successful.

In this paper we have therefore derived an analytic recipe for the drag force $F_d$ (from a ballistic flow model), which is successfully reproduced by an axisymmetric numerical simulation. This recipe for $F_d$ will be useful for carrying out simulations of compact bodies in motions influenced by gravitational drag. Possible examples are the motions of young stars within molecular clouds (see e.g. Throop & Bally 2008; Chavarría et al. 2010), or the orbital decay of black holes in the centre of merging galaxies (Narayan 2000; Escala et al. 2004, 2005; Dotti, Colpi & Haardt 2006). Kim & Kim (2009) computed the non-linear gravitational drag on a massive Plummer perturber in adiabatic axisymmetric simulations and found that it is smaller than the linear theory predicts for supersonic bodies. This reduction of the drag force is accounted for correctly in our drag formula.

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REFERENCES

Bisnovatyi-Kogan G. S., Kazhdan Ya. M., Klypin A. A., Lutskii A. E., Shakura N. I., 1979, SvA, 23, 201
Bondi H., Hoyle F., 1944, MNRAS, 104, 273
Chandrasekhar S., 1943, ApJ, 97, 255
Chavarría L., Mardones D., Garay G., Escala A., Bronfman L., Lizano S., 2010, ApJ, 710, 583
Conroy C., Ostrikov J. P., 2008, ApJ, 681, 151
Dokuchaev V. P., 1964, SvA, 8, 23
Dotti M., Colpi M., Haardt F., 2006, MNRAS, 367, 103
Edgar R., 2004, New Astron. Rev., 48, 843
Escala A., Larson R. B., Coppi P. S., Mardones D., 2004, ApJ, 607, 765
Escala A., Larson R. B., Coppi P. S., Mardones D., 2005, ApJ, 630, 152
Fryxell B. A., Taam R. E., McMillan S. L. W., 1987, ApJ, 315, 556
Hoyle F., Lyttleton R. A., 1939, Proc. Cambridge Philos. Soc., 35, 405
Just A., Kegel W. H., 1990, A&A, 232, 447
Kim W.-T., 2007, ApJ, 667, L5
Kim W.-T., 2010, ApJ, 725, 1069
Kim H., Kim W.-T., 2007, ApJ, 665, 432
Kim H., Kim W.-T., 2009, ApJ, 703, 1278
Koide H., Matsuda T., Shima E., 1991, MNRAS, 252, 473
Lyttleton R. A., 1972, MNRAS, 160, 255
Moekel N., Throop H. B., 2009, ApJ, 707, 258
Naiman J. P., Ramírez-Ruiz E., Lin D. N. C., 2011, ApJ, 735, 25
Namouni F., 2010, MNRAS, 401, 319
Narayan R., 2000, ApJ, 536, 663
Nejad-Asghar M., 2010, MNRAS, 406, 1253
Ostriker E. C., 1999, ApJ, 513, 252
Raga A. C., Navarro-González R., Villagrán-Muniz M., 2000, Revista Mexicana Astron. Astrofisica, 36, 67
Rephaeli Y., Salpeter E. E., 1980, ApJ, 240, 20
Ruderman M. A., Spiegel E. A., 1971, ApJ, 165, 1
Ruffert M., 1996, A&A, 311, 817
Sánchez-Salcedo F. J., 2009, MNRAS, 392, 1573
Sánchez-Salcedo F. J., Brandenburg A., 1999, ApJ, 522, L35
Sánchez-Salcedo F. J., Brandenburg A., 2001, MNRAS, 322, 67
Shima E., Matsuda Y., Takeda H., Sawada K., 1985, MNRAS, 217, 367
Tanaka T., Haiman Z., 2009, ApJ, 696, 1798
Throop H. B., Bally J., 2008, AJ, 135, 2380
van Leer B., 1982, ICASE Report No. 82–30
Villaver E., Livio M., 2009, ApJ, 705, L81

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