ON CENTERS OF BLOCKS
WITH NON-CYCLIC DEFECT GROUPS

YOSHIHIRO OTOKITA

Abstract. In this short note we study the center \( Z_B \) of a block \( B \) of a finite group over an algebraically closed field of prime characteristic through its Loewy length \( \ll Z_B \). A result of Okuyama in 1981 gave an upper bound for \( \ll Z_B \) in terms of defect group of \( B \). The purpose of this note is to improve this bound for non-cyclic defect groups.

1. Introduction

In this short note we study the center of a block of a finite group over an algebraically closed field of prime characteristic through its Loewy length \( \ll Z_B \). Let \( G \) be a finite group and \( F \) an algebraically closed field of characteristic \( p > 0 \). For a block \( B \) of the group algebra \( F G \) we denote by \( Z_B \) its center. In order to examine the structure of \( Z_B \) we use its Loewy length \( \ll Z_B \), that is, the nilpotency index of the Jacobson radical \( J_ZB \). A result of Okuyama [4] states that \( \ll Z_B \leq p^d \) where \( d \) is the defect of \( B \). In addition Koshitani-Külshammer-Sambale [2] determines \( \ll Z_B \) for cyclic defect groups. By this, we consider the other cases in this note. More precisely, we prove the following theorem:

Theorem 1. Let \( B \) be a block of \( FG \) with non-cyclic defect group of order \( p^d \). Then

\[ \ll Z_B \leq p^{d-1} + p - 1. \]

Now let us take a block \( B \) with defect group \( D \) of order \( 3^5 \) as an example. Then \( \ll Z_B \leq 243 \) by Okuyama’s formula. Theorem [1] implies that \( D \) is cyclic provided \( 84 \leq \ll Z_B \leq 243 \). In this case \( \ll Z_B = 243 \) or 122 by [2]. In all other cases we have \( \ll Z_B \leq 83 \).

We remark that the converse of this theorem is not true in general. For instance a block \( B \) with cyclic defect group \( C_{p^2} \) and inertial quotient group \( C_{p-1} \) satisfies \( \ll Z_B = p + 2 \leq 2p - 1 \) whenever \( p \geq 3 \).

2. Preliminaries

We prepare some notations. For a conjugacy class \( C \in \text{Cl}(G) \) its defect group \( \delta(C) \) is defined as a Sylow \( p \)-subgroup of \( CG(x) \) where \( x \in C \). For a \( p \)-subgroup \( P \) of \( G \) we set

\[ I_G(P) = \sum_{C \in \text{Cl}(G), \delta(C) \leq G} FC^+, \quad \tilde{I}_G(P) = \sum_{C \in \text{Cl}(G), \delta(C) \leq gP} FC^+ \]

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where $C^+$ is the class sum of $C$. These are ideals of the center $ZFG$ of $FG$. Furthermore we denote by $t(P)$ the Loewy length of $FP$ following Wallace [9]. Here we refine a lemma in Passman [7].

**Lemma 2.** Let $P$ be a $p$-subgroup of $G$. Then the following hold:

1. $I_G(P) \cdot JZFG^t(Z(P)) \subseteq \bar{I}_G(P)$,
2. $I_G(P) \cdot JZFG^{t-1/p-1} = 0$ where $|P| = p^a$.

**Proof.** It remains only to prove (1) by [7, Lemma 3 (ii)]. Let $Br_P : ZFG \to ZFC_G(P)$ be the Brauer homomorphism associated to $P$. Since $Br_P$ maps nilpotent elements to nilpotent elements we have $Br_P(I_G(P)) \subseteq ZFC_G(P)$. On the other hand $Br_P(I_G(P)) \subseteq I_{C_G(P)}(Z(P))$ holds (see the proof of [7, Lemma 3 (i)]). Thus it follows from [7, Lemma 2 (i)] that

$$Br_P(I_G(P) \cdot JZFG^t(Z(P))) \subseteq I_{C_G(P)}(Z(P)) \cdot JZFC_G(P)^t(Z(P))$$

$$= JFZ(P)^t(Z(P)) \cdot I_{C_G(P)}(Z(P)) = 0$$

since $Z(P)$ is central in $C_G(P)$. Therefore we deduce

$$I_G(P) \cdot JZFG^t(Z(P)) \subseteq \text{Ker} Br_P \cap I_G(P) = \bar{I}_G(P)$$

as claimed. $\square$

3. **Proof of main theorem**

We first improve Külshammer-Sambale [3, Theorem 12 and Proposition 15] by using Lemma [2]

**Proposition 3.** Let $B$ be a block of $FG$ with non-abelian defect group of order $p^d$. Then $|ZB| < p^{d-1}$.

**Proof.** We may assume $p \neq 2$ by [3, Proposition 15]. Let $D$ be a defect group of $B$. If $Z(D)$ is cyclic of order $p^{d-2}$ then $D$ is one of the following types:

$$M'_d := \langle x, y \mid x^{p^{d-1}} = y^p = 1, y^{-1}xy = x^{p^{d-2}+1} \rangle,$$

$$W'_d := \langle x, y, z \mid x^{p^{d-2}} = y^p = z^p = [x, y] = [x, z] = 1, [y, z] = x^{p^{d-3}} \rangle$$

where $d \geq 3$. In both cases $|ZB| < p^{d-1}$ (see [3, Proposition 10 and Lemma 11]). If $D$ has a cyclic subgroup of index $p$ then $D \cong M'_d$ (e.g. see [1, Chapter 5, Theorem 4.4] ). Thereby we need only consider the other cases. Since $Z(D)$ is non-cyclic or has order at most $p^{d-3}$, we have $\lambda_0 := p^{d-3} + p - 1 \geq t(Z(D))$. Thus we first obtain from Lemma [2] (1) that

$$I_G(D) \cdot JZFG^{\lambda_0} \subseteq I_G(D) \cdot JZFG^{t(Z(D))} \subseteq \bar{I}_G(D) = \sum_{D_1 < D} I_G(D_1).$$

By our assumptions above, $D_1$ is non-cyclic or has order at most $p^{d-2}$. In both cases we have $\lambda_1 := p^{d-2} + p - 1 \geq t(Z(D_1))$. Thus

$$I_G(D) \cdot JZFG^{\lambda_0+\lambda_1} \subseteq \sum_{D_1 < D} I_G(D_1) \cdot JZFG^{\lambda_1} \subseteq \sum_{D_1 < D} I_G(D_1) \cdot JZFG^{t(Z(D_1))}$$

$$\subseteq \sum_{D_1 < D} \bar{I}_G(D_1) = \sum_{D_2 < D_1 < D} I_G(D_2).$$
Finally, it follows from Lemma 2 (2) that
\[ I_G(D) \cdot JZF G_{\lambda_0 + \lambda_1 + \lambda_2} \subseteq \sum_{D_2 < D_1 < D} I_G(D_2) \cdot JZF G_{\lambda_2} = 0 \]
where \( \lambda_2 := p^d - 1/p - 1 \) since \( |D_2| \leq p^{d-2} \). Now let \( e \) be the block idempotent of \( B \). Then
\[ JZF B_{\lambda_0 + \lambda_1 + \lambda_2} = eJZF G_{\lambda_0 + \lambda_1 + \lambda_2} \subseteq I_G(D) \cdot JZF G_{\lambda_0 + \lambda_1 + \lambda_2} = 0 \]
and this means \( \ll ZB \leq \lambda_0 + \lambda_1 + \lambda_2 \). Accordingly, \( \ll ZB < p^{-1} \) except for one case that \( p = 3 \) and \( d = 4 \). Hence we consider this case in the following. From [3] (see [5] proof of Theorem 1.3), there exists a non-trivial \( B \)-subsection \( (u, b) \) such that
\[ \ll ZB \leq (|u| - 1)\ll Zb + 1 \]
where \( b \) is the unique block of \( F[C_G(u)/\langle u \rangle] \) dominated by \( b \). We may assume that \( b \) has defect group \( C_D(u)/\langle u \rangle \) by Sambale [8] Lemma 1.34. We put \( |u| = 3^r \) and \( |C_D(u)| = 3^s \). If \( r \leq d - 2 \) then
\[ \ll ZB \leq (3^r - 1)3^{-r} + 1 \leq (3^s - 1)3^{d-s-2} + 1 < 3^{d-1}. \]
In case of \( r = d - 1 \), we may \( r > s \) by our assumptions and thus
\[ \ll ZB \leq (3^s - 1)3^{-r} + 1 = 3^r - 3^{-r} + 1 < 3^r = 3^{d-1} \]
as required. Therefore we may assume \( d = r \), so that \( u \in Z(D) \). Hence \( |u| = 3 \) and \( D/\langle u \rangle \) is isomorphic to \( C_3 \times C_3 \times C_3, C_9 \times C_3, M_{27} \) or \( W_{27} \) by our assumptions. In all cases \( \ll Zb \leq 11 \) by [3] Theorem 1, Proposition 10 and Lemma 11. Consequently, \( \ll ZB \leq 23 < 27 = p^{d-1} \). Our claim is completely proved.

Theorem 1 is an immediate corollary to Proposition 3.

Proof of Theorem 4. We may assume \( p^{d-1} < \ll ZB \) and thus \( D \) is abelian by Proposition 3. In this case Külshammer-Sambale [5] has proved that \( \ll ZB \leq t(D) \). Hence our claim follows.

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YOSHIHIO OTOKITA:
DEPARTMENT OF MATHEMATICS AND INFORMATICS
GRADUATE SCHOOL OF SCIENCE
CHIBA UNIVERSITY
E-mail address: otokita@chiba-u.jp