Non-Hermitian quantum rings

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(Dated: May 21, 2014)

We investigate the spectral and dynamical properties of a quantum particle constrained on a ring threaded by a magnetic flux in presence of a complex (non-Hermitian) potential. For a static magnetic flux, the quantum states of the particle on the ring can be mapped into the Bloch states of a complex crystal, and magnetic flux tuning enables to probe the spectral features of the complex crystal, including the appearance of exceptional points. For a time-varying (linearly-ramped) magnetic flux, Zener tunneling among energy states is realized owing to the induced electromotive force. As compared to the Hermitian case, striking effects are observed in the non-Hermitian case, such as a highly asymmetric behavior of particle motion when reversing the direction of the magnetic flux and field-induced delayed transparency.

PACS numbers: 03.65.-w, 11.30.Er, 73.23.-b

I. INTRODUCTION

The coherent motion of charge carriers in doubly connected (ring) topologies plays a fundamental role in quantum and mesoscopic physics. Quantum mechanical experiments in ring geometries have long fascinated physicists. For example, the quantum orbital motion of electrons in mesoscopic normal-metal rings threaded by a magnetic flux produces striking interference phenomena such as the Aharonov-Bohm effect [1] and persistent currents [2,3]. Experimental evidence for Aharonov-Bohm oscillations has been detected in the mesoscopic regime in metallic [5,6] and semiconducting [7,8] rings. Because of the periodic boundary conditions enforced by the single valuedness of the wave function, the eigenstates of the electron in a ring look like Bloch waves in a crystal, where the circumference of the ring corresponds to the lattice constant and the Bloch wave number (quasimomentum) in the crystal is taken up by the flux parameter [4,9]. The effect of a superimposed nonuniform potential $V(\phi)$ in the ring simulates the structure of a crystal with a crystal and forbidden energy bands and gaps, among which Zener transitions can be induced by the electromotive force created by a linear variation in time of the magnetic flux [9]. Such previous studies on quantum rings have been mostly limited to consider an underlying Hermitian Hamiltonian. A noticeable exception is provided by the works of Hatano and Nelson [10], who investigated non-Hermitian localization in a random Schrödinger equation subjected to a constant imaginary vector potential.

In this work we study the spectral and dynamical properties of a quantum particle on a ring in the non-Hermitian case by allowing the external potential $V(\phi)$ (rather than the vector potential) to be complex-valued. Non-Hermitian quantum mechanics has received an increasing interest in recent years [11, especially in the context of Hamiltonians showing space-time reflection ($PT$) symmetry [12–14]. $PT$ Hamiltonians admit of an entirely real-valued energy spectrum below a phase transition (symmetry-breaking) point, above which pairs of complex-conjugate energies appear [12]. An important class of non-Hermitian systems is provided by complex periodic potentials [15–20], which realize a kind of synthetic complex crystals. As compared to ordinary crystals, complex crystals show unusual scattering and transport properties, which have been investigated in several recent works [16,22]. Complex crystals have been experimentally realized in different physical systems, including open two-level atomic systems interacting with near resonant light [16] and optical structures with gain and loss regions [21,22]. Here we show that a quantum particle on a ring threaded by a magnetic flux in presence of a complex (non-Hermitian) potential $V$ behaves like a Bloch particle in a complex crystal, where the magnetic flux determines the Bloch wave number of the particle. In particular, by tuning the magnetic flux the full spectral band structure of the equivalent complex crystal can be probed, including the onset of spectral singularities [19,23] and the transition from a real to a complex energy spectrum. Striking effects are found for non-stationary (linearly-increasing) magnetic fields, where multilevel Landau-Zener (LZ) transitions arise owing to the induced electromotive force. Analytical and numerical results are presented for a particle in the complex potential $V(\phi) = V_0 \cos(\phi) + V_0 \alpha \sin(\phi)$ threaded by a linearly-increasing magnetic flux $f = \beta t$, where LZ transitions occur among quantum states with different winding (angular momentum) numbers. As compared to LZ transition in the Hermitian case ($\alpha = 0$), striking effects are found in the non-Hermitian case ($\alpha \neq 0$), including strong asymmetric behavior for reversal the direction of the magnetic flux and field-induced delayed transparency at the $PT$ symmetry-breaking transition.

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II. QUANTUM PARTICLE ON A RING THREAD BY A MAGNETIC FLUX

A. The model

The time-dependent Schrödinger equation for an electron of mass \(m\) and charge \(e\) moving on a ring of radius \(R\) in presence of the external potential \(V(\varphi)\) and threaded by a magnetic flux \(\phi = \phi(t)\) [see Fig.1(a)] reads

\[
i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2mR^2} \left( -i \frac{\partial}{\partial \varphi} - f \right)^2 \psi + V(\varphi)\psi \equiv \hat{H}(t)\psi \tag{1}\]

where \(\varphi\) is the azimuthal angle that measures the position of the electron on the ring, \(f = \phi/\phi_0\), and \(\phi_0 = \hbar c/e\) is the flux quantum. The ring boundary condition

\[
\psi(\varphi + 2\pi, t) = \psi(\varphi, t) \tag{2}\]

applies to the single-valued wave function \(\psi\). For a time-independent magnetic flux, Eq.(1) can be reduced to a standard one-dimensional Schrödinger equation after the gauge transformation \(\psi(\varphi, t) = F(\varphi, t) \exp(\text{i}f\varphi)\), which simplifies Eq.(1) into the following one

\[
i\hbar \frac{\partial F}{\partial t} = -\frac{\hbar^2}{2mR^2} \frac{\partial^2 F}{\partial \varphi^2} + V(\varphi)F. \tag{3}\]

The magnetic flux controls the boundary condition for the function \(F\), namely one has

\[
F(\varphi + 2\pi, t) = F(\varphi, t) \exp(-2\pi f) \tag{4}\]

where the additional phase term on the right hand side of Eq.(4) is the Aharonov-Bohm phase.

The Schrödinger equation (1) is usually introduced to describe the coherent electronic motion in mesoscopic metal rings, however it can be found in other physical contexts as well, where the introduction of a complex-valued external potentials \(V(\varphi)\) might be feasible. For example, Eq.(1) can describe the temporal dynamics of a dilute and rotating Bose-Einstein condensate in an annular trap [24], or spatial propagation of monochromatic light waves in an annular fiber with a twisted axis [25]. In the latter optical system, a complex potential \(V(\varphi)\) describes the effects of an azimuthal index (real part of \(V\)) and loss/gain (imaginary part of \(V\)) guiding.

B. Energy spectrum in a static magnetic flux

In the absence of the external potential, \(V(\varphi) = 0\), and for a stationary magnetic flux the eigenstates and corresponding energies of the Hamiltonian \(\hat{H}\) are given by

\[
\psi_n(\varphi) = \frac{1}{\sqrt{2\pi}} \exp(\text{i}n\varphi) \tag{5}\]

\[
E_n = \frac{\hbar^2(n-f)^2}{2mR^2} \tag{6}\]

where \(n = 0, \pm 1, \pm 2, \pm 3, \ldots\) is the winding number that determines the angular momentum \(nh\) of the rotating electron. In the presence of the external potential \(V(\varphi)\), the energy spectrum and corresponding eigenfunctions of \(\hat{H}\) can be mapped into the spectrum and Bloch-Floquet states of the associated periodic potential \(V(\varphi + 2\pi) = V(\varphi)\), where the normalized magnetic flux \(f\) plays the role of the wave number \(k\) of the electron in the crystal [9]. To show such an equivalence, let us look for a solution to the eigenvalue problem \(\hat{H}\psi(\varphi) = E\psi(\varphi)\) of the form \(\psi(\varphi) = F(\varphi) \exp(\text{i}f\varphi)\). The function \(F(\varphi)\) then satisfies the stationary Schrödinger equation

\[
-\frac{\hbar^2}{2mR^2} \frac{d^2 F}{d\varphi^2} + V(\varphi)F = EF \tag{7}\]

with the boundary condition

\[
F(\varphi + 2\pi) = F(\varphi) \exp(-2\pi f) \tag{8}\]

Since \(V(\varphi + 2\pi) = V(\varphi)\), Eq.(7) can be viewed as the stationary Schrödinger equation of an electron in the periodic potential \(V(\varphi)\) with lattice constant \(\alpha = 2\pi\). According to the Bloch-Floquet theorem, the most general solution to Eq.(7) is a Bloch state, \(F(\varphi) = u_n(\varphi, k) \exp(\text{i}k\varphi)\) and \(E = E_n(k)\), where \(k\) is the crystal wave number (quasi-momentum), that varies in the interval \(-1/2 \leq k < 1/2\), \(E_n(k)\) is the energy dispersion curve of the \(n\)-th band of the crystal, and \(u_n(\varphi, k)\) is the periodic part of the Bloch eigenfunction. The boundary condition (8) requires \(k = -f\), i.e.
the normalized magnetic flux $f$ fixes the quasi momentum $k$ of the electron in the lattice. The eigenfunctions and corresponding energies of the quantum-ring Hamiltonian $\hat{H}$ are thus given by $\psi_n(\varphi) = \varphi_n(\varphi, -f)$ and $E_n = E_n(-f)$. Hence by tuning the normalized magnetic flux from $f = -1/2$ to $f = 1/2$ one can probe the entire band structure of the periodic potential $V(\varphi)$.

For a complex potential, we can generally write

$$V(\varphi) = V_R(\varphi) + i\alpha V_I(\varphi), \quad (9)$$

where $V_R(\varphi)$, $\alpha V_I(\varphi)$ are the real and imaginary parts of $V(\varphi)$, respectively, $\alpha \geq 0$ is a real parameter that determines the strength of non-Hermiticity of the potential, and $V_R(\varphi)$, $V_I(\varphi)$ are the profiles of the real and imaginary potential terms. Note that $\alpha = 0$ corresponds to the ordinary Hermitian problem. Of particular interest is the case of a $\mathcal{PT}$ symmetric complex crystal, which requires $V(-\varphi) = V^*(\varphi)$. In this case, a critical value $\alpha_c$ of $\alpha$ does exist such that the energy spectrum of the crystal is entirely real-valued for $\alpha \leq \alpha_c$ (unbroken $\mathcal{PT}$ phase), whereas complex-conjugate energies appear for $\alpha > \alpha_c$ (broken $\mathcal{PT}$ phase). The complex-conjugate energies above the symmetry breaking point emanate from the wave numbers $k = 0$ or $k = \pm \pi$, i.e. at the center or at the edge of the Brillouin zone $[19]$. For example, for the potential

$$V(\varphi) = V_0 \cos(\varphi) + i\alpha V_0 \sin(\varphi) \quad (10)$$

one has $\alpha_c = 1$ $[18]$. Interestingly, at the $\mathcal{PT}$ symmetry breaking point spectral singularities, corresponding to poles in the resolvent of the Hamiltonian in the continuous part of the spectrum, are found at either the center ($k = 0$) or at the edge ($k = \pm 1/2$) of the Brillouin zone $[19]$. The physical implications of spectral singularities in non-Hermitian systems have been highlighted in recent works, especially in connection to resonance-like behavior in scattering problems and instability thresholds in optical systems (see, for instance, $[23]$). In particular, in complex crystals spectral singularities at the $\mathcal{PT}$ symmetry breaking point are responsible for a secular growth in time of the wave function $\psi(\varphi, t)$ in spite of the real energy spectrum of the Hamiltonian $[19]$. Mathematical implications of spectral singularities have been investigated as well $[23]$ $[26]$ $[28]$, and the question whether resolution of the identity operator is possible for a Hamiltonian possessing spectral singularities has been debated. To this regard, contrary to earlier indications it was rigorously proven in Ref. $[24]$ that the contribution of the spectral singularity to the resolution of identity operator depends on the class of functions employed for physical states, and that there is no obstruction to completeness originating from a spectral singularity (see also Ref. $[28]$). In case of a non-Hermitian quantum ring considered in this paper, it should be nevertheless noticed that the spectrum of the Hamiltonian is point like, and spectral singularities of the associated complex crystal are mapped into exceptional points $[11]$ $[29]$ $[31]$. Thus, for the quantum particle

on the ring threaded by a magnetic flux with the external potential $V(\varphi)$ at $\alpha = \alpha_c$, the eigenfunctions of $\hat{H}$ form a complete basis for a magnetic flux $f$ different than either (or both) $f = n$, $f = \pm 1/2 + n$ ($n = 0, \pm 1, \pm 2, \ldots$), whereas in the opposite case coalescence of pairs of eigenfunctions and eigenvectors (corresponding to exceptional points) are found.

As an example, let us consider the periodic potential $[10]$ at $\alpha = \alpha_c = 1$, i.e.

$$V(\varphi) = V_0 \exp(i\varphi). \quad (11)$$

The energy spectrum and corresponding eigenfunctions of $\hat{H}$ can be calculated in a closed form (see, for instance, $[15]$ $[19]$). In particular, the energy spectrum turns out to be the same as that of a free particle, i.e. $E_n$ is given by Eq.(6). For $f \neq (2l + 1)/2$ (with $l = 0, \pm 1, \pm 2, \pm 3, \ldots$) the eigenvalues of $\hat{H}$ are simple (non-degenerate) and the corresponding eigenfunctions form a complete set. As $f$ approaches a value close to half an integer, i.e. as $f \to (2l + 1)/2$, two energies coalesce in pairs, namely $E_{l+1} - E_l \to 0$, and the corresponding set of eigenfunctions ceases to be complete because of the exceptional point at $E = E_l$. The appearance of the exceptional point leads to a secular growth in time of an initial wave function with a defined winding number. In fact, let us expand the wave function $\psi(\varphi, t)$ on the basis of functions with defined winding number, defined by Eq.(5), i.e. let us set

$$\psi(\varphi, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=\infty}^{n=-\infty} c_n(t) \exp(i\varphi - iE_n t) \quad (12)$$

where $E_n$ are given by Eq.(6). After substitution of Eq.(12) into Eq.(1) and assuming the potential (11), the following evolution equations for the amplitude probabilities $c_n(t)$ are readily found

$$i\hbar \frac{dc_n}{dt} = V_0 c_{n-1} \exp[i(E_n - E_{n-1})t]. \quad (13)$$

Let us assume that the particle is initially prepared in a state with a definite angular momentum, corresponding to the winding number $n = n_0$, i.e. that $c_n(0) = \delta_{n,n_0}$. The solution to the coupled equations (13) can be derived from the following recursive relations

$$c_n(t) = 0 \quad (n < n_0) \quad (14)$$

$$c_{n_0}(t) = 1$$

$$c_n(t) = - \frac{iV_0}{\hbar} \int_0^t d\xi c_{n-1}(\xi) \exp[i(2n - 2f - 1)\xi] \quad (n > n_0)$$

From Eqs.(14) it follows that the solution $c_{l+1}(t)$ for $l \geq n_0$ secularly grows in time provided that $2l - 2f + 1 = 0$, which is satisfied for a normalized magnetic flux $f$ given by $f = (2l + 1)/2$. Hence tuning the magnetic flux at an exceptional point leads to a secular growth in time of the wave function.

As a final comment, it should be noted that, while there is a close connection between the problem of a
FIG. 2. (Color online). Evolution of a quantum particle on a ring threaded by a linearly-varying magnetic flux in momentum space [left panels, snapshot of $|c_n(\tau)|^2$] and in real space [right panels, snapshot of $|\psi(\varphi, \tau)|^2$] in the Hermitian case ($\alpha = 0$) for (a) $\sigma = -0.003$, and (b) $\sigma = 0.003$. The particle is initially at rest and fully delocalized in the ring, corresponding to $\psi(\varphi, 0) = 1/\sqrt{2\pi}$. The amplitude of the external potential is $V_0 m R^2/\hbar^2 = 0.08$.

quantum particle on a ring and the related complex crystal problem, the evolution of a quantum wave packet in the two cases can show rather distinctive features as a result of the restricted spectrum of $\hat{H}$ in the quantum ring problem. For example, let us consider the complex potential $V(\varphi) = V_0 \exp(i\varphi)$ at the $\mathcal{PT}$-symmetry breaking point. In a complex crystal, the spectral singularities of $\hat{H}$ are responsible for an initial growth a normalizable state (a wave packet), however the wave function growth is limited because of the zero measure of the spectral singularities, as discussed in Ref.[19]. Contrary, in the quantum ring problem the growth is not clamped. The reason thereof is that the spectral singularities are transformed into exceptional points in the quantum ring problem, which belong to the point spectrum of $\hat{H}$.

C. Time-varying magnetic flux: Zener transitions

A particularly interesting case is the one of a magnetic flux which is varied in time [9, 32]. When the flux $\phi(t)$ threading the ring is linearly increased in time, i.e.

$$f(t) = \beta t$$

(15)
a constant electric field (electromotive force) which accelerates the particle is induced in the ring according to Faraday’s law [35]. Since $k = -f$, a linearly-varying magnetic flux $f = \beta t$ corresponds to a Bloch electron moving at a constant speed $\beta$ in the momentum ($k$) space. An electron in a pure Bloch state will follow the flux change adiabatically if the induced electric field is infinitesimal, i.e., it will be backscattered to the same energy band each time it reaches a zone boundary. If the field strength [i.e. the $\beta$ parameter in Eq.(15)] is increased, Zener tunneling between different bands can occur. For a real-valued potential $V(\varphi)$, i.e. in the Hermitian case, the general problem of Zener tunneling of a quantum particle on a ring threaded by a ramped magnetic flux can be investigated rather generally by decomposing the wave function $\psi(\varphi, t)$ as a superposition of the adiabatic Bloch states, namely

$$\psi(\varphi, t) = \sum_n a_n(t) u_n(\varphi; k = -\beta t),$$

(16)

and looking for the evolution of the amplitudes $a_n(t)$ (see, for instance, [9]). Zener tunneling between adjacent bands is ruled by a cascade of two-level LZ tunneling events, which occur as the magnetic flux $f$ crosses the edge of the Brillouin zone. While such a method can be extended to the non-Hermitian case [20], it has a limited validity since it fails in the presence of exceptional points. The reason thereof is that, as magnetic flux $f(t)$ crosses an exceptional point, the adiabatic eigenstates of $\hat{H}$ lack of completeness. Therefore, in the non-Hermitian case it is more convenient to study the quantum evolution by expanding the wave function $\psi(\varphi, t)$ on the basis of the angular momentum eigenfunctions (5). After setting

$$\psi(\varphi, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=\pm\infty} c_n(t) \exp(in\varphi)$$

(16)

and

$$V(\varphi) = \sum_{n=\pm\infty} V_n \exp(in\varphi),$$

(17)
substitution of Eqs.(16) and (17) into Eq.(1) yields the following evolution equations for the amplitudes $c_n(t)$

$$i\hbar \frac{dc_n}{dt} = \frac{\hbar^2}{2mR^2} (n - \beta t)^2 c_n + \sum_m V_{n-m} c_m.$$  \hfill (18)

In their present form, Eqs.(18) can be regraded as a multi-level LZ problem \[34\], where successive crossings of the bare (diabatic) energy levels between states $n$ and $m$ occurs at the times $t_{n,m} = (n + m)/(2\beta)$.

### III. MULTILEVEL NON-HERMITIAN LANDAU-ZENER TRANSITIONS AND FIELD-INDUCED DELAYED TRANSPARENCY

In this section we focus our attention to the multi-level LZ problem [Eqs.(18)] for the specific potential given by Eq.(10) with $\alpha \leq \alpha_c = 1$, and highlight distinct features of non-HERMITIAN versus HERMITIAN case. In particular, striking effects are predicted in the non-Hermitian case at the $\mathcal{PT}$ symmetry breaking point, as discussed below.

For the potential (10), after introduction of the normalized time $\tau = \hbar t/(2mR^2)$, Eqs.(18) read

$$i\frac{dc_n}{d\tau} = (n - \sigma \tau)^2 c_n + S_1 c_{n-1} + S_2 c_{n+1}$$ \hfill (19)

where we have set

$$S_1 = \frac{V_0 m R^2}{\hbar^2} (1 + \alpha), \quad S_2 = \frac{V_0 m R^2}{\hbar^2} (1 - \alpha) \quad (20)$$

and $\sigma = 2mR^2 \beta/\hbar$. Note that $S_1 = S_2$ in the Hermitian case ($\alpha = 0$), whereas $S_1 \neq 0, S_2 = 0$ at the $\mathcal{PT}$ symmetry breaking point ($\alpha = \alpha_c = 1$). The coupled equations (19) can be regarded as a generalization, to the non-Hermitian case, of a multi-level LZ problem \[35\, 36\]. A particularly interesting case is that of a shallow potential, corresponding to $V_0 \ll \hbar^2/(2mR^2)$ (i.e. $S_1, S_2 \ll 1$), and a slow increase of the magnetic flux, $\beta \ll \hbar/(2mR^2)$ (i.e. $|\sigma| \ll 1$), with $S_{1,2}/\sqrt{|\sigma|}$ of the order (or larger than) $\sim 1$. In this case, the multilevel LZ problem reduces to the cascade of Zener transitions between two levels $n$ and $(n+1)$, as one can see from the energy level diagram corresponding to Eqs.(19) and shown in Fig.1(b). Since $S_{1,2} \ll 1$, the diabatic energies of the levels are far apart each other that transitions are not allowed, except in the neighborhood of the times

$$\tau_n = \frac{2n + 1}{2\sigma} \quad (21)$$

$(n = 0, 1, 2, 3, ...)$, where crossing of the energies between level $n$ and level $(n+1)$ occurs [see Fig.1(b)]. This means that, apart from the dynamical phase term, the amplitude $c_n(t)$ does not change in time, except for sudden changes at the two crossings times $\tau_{n-1}$ and $\tau_n$. For example, at $\tau \sim \tau_n$ the change of amplitudes $c_n$ and $c_{n+1}$ can be obtained by solving the two-level LZ problem

$$\frac{dc_n}{d\tau} = (n - \sigma \tau)^2 c_n + S_2 c_{n+1} \quad (22)$$

$$\frac{dc_{n+1}}{d\tau} = (n + 1 - \sigma \tau)^2 c_{n+1} + S_1 c_n \quad (23)$$

which shows a linear level crossing at $\tau = \tau_n$. The scattering matrix, that relates the values of $c_n, c_{n+1}$ at times $\tau \ll \tau_n$ and $\tau \gg \tau_n$, does not depend on the winding number $n$ and its form can be rather generally expressed in terms of parabolic cylinder functions \[37\]. It should be noted that, below the $\mathcal{PT}$ symmetry breaking point, i.e. for $\alpha < 1$, the non-Hermitian multilevel LZ problem (19) can be readily mapped into the corresponding Hermitian one. In fact, after introduction of the amplitudes $a_n(t)$ by the relation

$$c_n(t) = a_n(t) \left( \frac{1 + \alpha}{1 - \alpha} \right)^{n/2}$$ \hfill (24)

Eqs.(19) take the form

$$i\frac{da_n}{d\tau} = (n - \sigma \tau)^2 a_n + S(a_{n-1} + a_{n+1})$$ \hfill (25)

where we have set $S \equiv S_1[(1 - \alpha)/(1 + \alpha)]^{1/2} = S_2[(1 + \alpha)/(1 - \alpha)]^{1/2}$. In their present form, Eqs.(25) describe the multilevel LZ problem (19) in the Hermitian case, with $S_1 = S_2 = S$. Hence the evolution for the amplitudes $c_n(t)$ in the non-Hermitian case, below the $\mathcal{PT}$ phase transition, can be readily obtained from the behavior $a_n(t)$ of the associated Hermitian problem after the substitution defined by Eq.(24). The major effect of non-Hermiticity in the LZ problem is the breakdown of the time reversal symmetry. This implies an asymmetric behavior of the particle motion when the sign of the magnetic flux, i.e. the direction of the magnetic field threading the ring, is reversed. In fact, let us first observe that, if $a_n(t, \sigma)$ is a solution to Eq.(25) with a magnetic flux $f = \sigma \tau$, then one has $a_n(t, -\sigma) = a_{-n}(t, \sigma)$, i.e. reversal of the direction of the magnetic field (and hence of the electromotive force) in the Hermitian case merely corresponds to reverse the direction of motion on the ring $(n \rightarrow -n)$. On the other hand, for the non-Hermitian case, from Eq.(24) it follows that

$$c_n(t, \sigma) = a_n(t, \sigma) \left( \frac{1 + \alpha}{1 - \alpha} \right)^{n/2}$$ \hfill (26)

$$c_n(t, -\sigma) = a_{-n}(t, \sigma) \left( \frac{1 + \alpha}{1 - \alpha} \right)^{n/2}$$ \hfill (27)

and hence:

$$c_n(t, -\sigma) = c_{-n}(t, \sigma) \left( \frac{1 + \alpha}{1 - \alpha} \right)^{n}$$ \hfill (28)

i.e. the invariance of the dynamics to the transformations $\sigma \rightarrow -\sigma, n \rightarrow -n$ is broken. As an example, let us consider the dynamics of a quantum particle that is
initially at rest and fully delocalized on the ring, i.e. let us assume \( c_n(0) = \delta_{n,0} \). In Figs.2 and 3 we show the numerically-computed quantum evolution of the particle state, both in the physical and momentum space, in the Hermitian (Fig.2) and non-Hermitian (Fig.3) case and for two opposite values of the magnetic flux rate \( \sigma \). Parameter values have been chosen such that, at each LZ crossing, the probability of Zener tunneling from one level to the coupled one, given by \( P_2 = 1 - \exp(-\pi S^2/\sigma) \), is close to one. Hence, in the Hermitian case the effect of the linearly-increasing magnetic flux is to induce a drift of the particle motion in momentum space; the direction of the drift is reversed as the sign of the magnetic flux is flipped, as shown in Fig.2. In the non-Hermitian case, a similar behavior is observed, however the norm of the wave function is not conserved; according to Eq.(28), amplification or damping of the wave function is observed, depending on the sign of \( \sigma \) (see Fig.3).

More striking features can be observed in the non-Hermitian case at the \( \mathcal{PT} \) symmetry breaking point \( \alpha = \alpha_c = 1 \). In this case, the non-Hermitian problem can not be mapped into the Hermitian one by the transformation (24) owing to a divergence at \( \alpha = 1 \), and a direct analysis of Eqs.(19) with \( S_2 = 0 \) should be considered. In the limit \( |\sigma| \ll 1 \), \( |S_1| \ll 1 \) and \( S_1/\sqrt{|\sigma|} \) larger than (or of the order of) \( \sim 1 \), a simple analytical expression to the solution to Eqs.(19) can be derived by an asymptotic analysis. After setting

\[
c_n(\tau) = a_n(\tau) \exp \left[-i \int_0^\tau dt(n-at)^2 \right]
\]

for \( \tau \geq 0 \) one obtains

\[
a_n(\tau) \simeq \begin{cases} a_n(0) & n \leq 0 \\ a_n(0) - iS_1 \sqrt{\tau/a} \exp(i\sigma\tau_{n-1})a_{n-1}(\tau_{n-1})H(\tau - \tau_{n-1}) & n \geq 1 \end{cases}
\]

(30)

for \( \sigma > 0 \), whereas

\[
a_n(\tau) \simeq \begin{cases} a_n(0) & n \geq 1 \\ a_n(0) - iS_1 \sqrt{\tau/a} \exp(i\sigma\tau_{n-1})a_{n-1}(0)H(\tau - \tau_{n-1}) & n \leq 0 \end{cases}
\]

(31)

for \( \sigma < 0 \). In Eqs.(30) and (31), \( \tau_n \) is defined by Eq.(21), whereas \( H(\tau) \) is the step (Heaviside) function, i.e. \( H(\tau) = 0 \) for \( \tau < 0 \) and \( H(\tau) = 1 \) for \( \tau > 0 \). Like in the previous case, i.e. below the \( \mathcal{PT} \) symmetry breaking, the wave function evolution is strongly asymmetric for reversal of the magnetic flux. As an example, in Fig.4 we show the numerically-computed evolution, both in real and momentum space, of the wave function corresponding to the initial particle at rest and fully delocalized on the ring, i.e. \( c_n(0) = \delta_{n,0} \), for parameter values \( V_0 m R^2/\hbar^2 = 0.02 \), \( \alpha = 1 \), and for \( \sigma = -0.003 \) [Fig.4(a)] and \( \sigma = 0.003 \) [Fig.4(b)]. The behavior of the (exact) numerically-computed wave function evolution in momentum space clearly reproduces the predictions based on the asymptotic (approximate) solutions as given by Eqs.(30) and (31). In particular, in the \( \sigma < 0 \) case [Fig.4(a)] the dynamics is frozen \( |a_n(\tau) \approx a_n(0) = \delta_{n,0} | \) i.e. the potential \( V(\varphi) \) appears to be invisible (like in Refs. [18, 20]) and the electromotive force does not increase anymore the angular momentum of the particle. Conversely, for \( \sigma > 0 \) higher winding number states are generated owing to a sequence of LZ transitions, see Fig.4(b). Hence at the \( \mathcal{PT} \)-symmetry breaking point the LZ transitions are unidirectional.

A striking effect, that we refer to as field-induced delayed transparency, is the possibility for \( \sigma < 0 \) to make the external potential \( V(\varphi) \) "invisible" after some time delay \( T \) from the initial time \( \tau = 0 \) by application of a linearly growing magnetic flux. In other words, the particle motion is affected by the external potential \( V(\varphi) \) up to the time \( \tau = T \), whereas for times \( \tau > T \) the particle motion occurs as if the external potential \( V \) were switched off (in spite it is still there). Such a counter-intuitive effect can be explained on the basis of unidirectional Zener tunneling between adjacent levels that enables to freeze the particle motion in momentum space after some target time \( T \geq 0 \). In fact, let us assume that at time \( \tau = 0 \) the particle is prepared in a rather arbitrary state with amplitude probabilities \( c_n(0) \) in momentum space. Owing to the convergence of the series (16), one has \( c_n(0) \to 0 \) as \( n \to \pm \infty \). In practice, we may assume that \( c_n(0) \approx 0 \) for \( n \leq M \), where \( M \) is some integer number (possibly negative and larger in absolute value). At time \( \tau = 0 \), let us apply a linearly varying magnetic flux \( f(\tau) = \sigma(\tau - \tau_0) \), where the parameter \( \tau_0 \) -to be determined- is the time at which the magnetic flux vanishes. Assuming \( \sigma < 0 \), from Eqs.(19) -with \( S_2 = 0 \) and with \( \tau \) replaced by \( \tau - \tau_0 \) on the right hand side of the equations- it follows that, apart from the dynamical phase, the amplitude \( c_n(\tau) \) is not affected by the external potential \( V(\varphi) \) at times \( \tau > \tau_0 + \tau_{n-1} \) because Zener tunneling is prevented. Moreover, if \( c_n(0) = 0 \) for \( n \leq M \), for the unidirectionality of Zener tunneling it readily follows that \( c_n(\tau) \approx 0 \) at any time \( \tau > 0 \) for \( n \leq M \). Hence the dynamics of the system is expected to freeze at times \( \tau > \tau_M + \tau_0 \). Physically, this means that the potential \( V \)
FIG. 4. (Color online). Same as Fig.3, but at the $PT$-symmetry breaking point ($\alpha = 1$). In (a) $\sigma = -0.003$, in (b) $\sigma = 0.003$. The amplitude of the external potential is $V_0mR^2/\hbar^2 = 0.02$.

becomes ”invisible” after a time delay

$$T = \tau_M + \tau_0 = \frac{2M + 1}{2\sigma} + \tau_0. \quad (32)$$

i.e. at time $\tau > T$ the particle dynamics is not affected anymore by the external potential $V$. From Eq.(32) it follows that the delay $T$ can be made arbitrary by an appropriate choice of $\tau_0$. The occurrence of such a delayed invisibility, predicted by the asymptotic analysis, has been checked by direct numerical integration of Eqs.(19). An example of delayed invisibility is shown in Fig.5 for parameter values $\sigma = -0.003, \alpha = 1$ and $mR^2V_0/\hbar^2 = 0.02$.

As an initial condition, we choose a Gaussian distribution in momentum space, namely $c_n(0) = N \exp\left[-(n+4)^2/9\right]$, where $N$ is a normalization constant. Such a distribution has a negligible occupation amplitudes for $n<M \simeq -7$.

Assuming $\sigma = -0.003$, to obtain a target delay of e.g. $T = 1200$ according to Eq.(32) we take $\tau_0 \simeq -967$.

The numerically-computed evolution of the wave function, both in momentum and real space, is shown in Figs.5(a) and (b), respectively. From Fig.5(a) it can be seen that the winding number occupation probabilities are frozen after a time $\tau \sim T$, where LZ transitions are forbidden. To check that the evolution of the wave function at times $\tau > T$ is not influenced anymore by the external potential $V$, i.e. that the external potential is effectively invisible at times $\tau > T$, we compared the wave function evolution in real space at times $\tau > T$ with that obtained by switching off the external potential at $\tau > T$, i.e. by letting $V(\varphi) = V_0 \exp(i\varphi)$ for $\tau < T$ and $V(\varphi) = 0$ for $\tau > T$. As an example, in Fig.5(c) we show the behavior of the real part of the wave function at the azimuthal angle $\varphi = 0$ over a time interval after $T$, as obtained in the two cases. Note the good overlap of the two curves, which indicates that the particle motion is effectively insensitive to the external potential.

FIG. 5. (Color online). An example of delayed invisibility. Wave function evolution, in momentum (a) and real space (b), for an initial Gaussian distribution in momentum space. Parameter values are given in the text. Note that at times $\tau > T$ the dynamics of occupation probabilities in momentum space is frozen. In (c) the evolution of the real part of the wave function $\psi(\varphi,t)$, at the angular position $\varphi = 0$, is shown by the solid curve. The dotted curve, almost overlapped with the solid one, is the corresponding wave function evolution that one would observe by switching off the external potential $V$ at times $\tau > T$.

IV. CONCLUSIONS

Quantum mechanics in doubly-connected (ring) topologies in presence of a magnetic field, i.e. in the so-called quantum ring systems, has long fascinated physicists, mainly because of the manifestation of important physical effects such as the Aharonov-Bohm effect and persistent currents. In this work we extended the theory of quantum rings by allowing for an external non-Hermitian potential. For a static magnetic flux, the quantum states of the particle on the ring can be mapped onto the Bloch states of a complex crystal, and magnetic flux tuning enables to probe the spectral features of the complex crystal, including the appearance of exceptional points. For a time-varying (linearly-ramped) magnetic flux, Zener tunneling among energy states is realized owing to the induced electromotive force on the ring. As compared to the Hermitian case, striking effects have been predicted to occur in the non-Hermitian case as a result of an asymmetric Zener tunneling. In particular, we discussed the possibility to observe delayed invisibility of an external potential at the $PT$ symmetry breaking point. Hence non-Hermitian quantum rings could provide a means to probe the spectral properties of com-
plex crystals and to observe unusual phenomena, like delayed transparency. The possibility to physically implement a non-Hermitian quantum ring Hamiltonian (1) remains an open question, which however goes beyond the scope of the present study. As briefly mentioned in Sec.IIA, possible physical systems where non-Hermitian quantum rings might be in principle realized include light propagation in twisted fibers [25] or cold atoms in rotating annular traps [24]. However, for such systems the experimental and technological feasibility to implement a non-Hermitian potential, like the one considered in the present work [Eq.(10)], remains a rather challenging task. Different and experimentally more feasible physical implementations, for example based on propagation of optical pulses in recirculating fiber loops with gain and loss [21], should be investigated. Finally, our analysis could be extended to study the dynamics of non-Hermitian quantum rings including a possible imaginary vector potential in the Schrödinger equation [10], in addition to the external potential.

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