A thermodynamical approach to dissipation range turbulence

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Abstract

A model to explain the statistics of the velocity gradients in the dissipation range of a turbulent flow is presented. The experimentally observed non-gaussian statistics is theoretically predicted by means of a thermodynamical analogy using the maximum entropy principle of ordinary statistical mechanics.

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The dynamics of fluids at high Reynolds number is one of the most interesting subject of statistical physics. In the limit of infinite Reynolds number, the celebrated theory of Kolmogorov (K41) suggests that the small scales statistics is characterized only by the mean rate of energy dissipation per unit mass $\epsilon$ and the scale $l$. The K41 theory is based upon the concept of self-similarity of the inertial range $[l_d,L]$ and implies that the velocity gradients $\delta v_\ell(x) = |v(x+\ell) - v(x)|$ scale as

$$<\delta v_\ell^q> \sim (\epsilon l)^{q/3}. \quad (1)$$

The experimental evidence of the breakdown of eq.(1), see [2,3,4], due to the presence of intermittency, induced Kolmogorov and Obukhov [5,6] to modify the K41 theory introducing the fluctuation of the mean flux of energy $\epsilon$. Indeed, as pointed out by Siggia [7], the turbulent flow can be described as the superposition of two fluid states, the coherent structures and the random fluctuations. Hence, in the inertial range, the flow preserves organized structures as, for instance, vortex sheets and filaments. The coherent structures are “rare” events that contribute to the non-gaussian high-amplitude fluctuations. Many models for such superposition have been proposed, either based
on a mapping-closure theory or on the fractal approach \cite{9, 10, 11, 12, 13}. However, also at very small scales \(l \sim l_d\), when the coherent structures are annihilated by dissipation, the tails of the velocity gradients statistics are deeply non gaussian. Indeed, experimental results show that the tails of the probability density function (PDF) of the velocity gradients are close to exponential \cite{14} for small \(l\). In this paper, we conjecture that the PDF of the velocity gradients in the dissipation range, \(l \sim l_d\), can be described by means of ordinary statistical mechanics. We propose to consider the velocity gradients \(\delta v_l\) of the random incoherent flow like the momentum \(p\) of a particle in an ideal gas with a “temperature” \(T\) that changes slowly. Specifically, we consider this temperature not to be fixed but \(\chi^2\)-distributed with degree \(n = 3\); the resulting PDF agrees both with the experimental results and with the theoretical structure function.

In the ordinary description of an ideal gas, the momentum \(p\) of a particle with mass \(m\) may be considered as a random variable. From the maximum entropy principle one obtains the density distribution

\[
\psi(p) = \frac{1}{Z} e^{-\frac{\beta p^2}{2m}},
\]

where \(Z\) is the partition function and \(\beta = \frac{1}{kT}\). Here \(T\) is the temperature of the thermal bath and \(k\) is Boltzmann’s constant.

Now, we consider the temperature of the thermal bath not to be constant in time, but fluctuating with a given distribution \(f(T)\). In practice, we suppose that the time scale on which \(T\) fluctuates is much larger than the typical time to reach equilibrium. Let \(\psi(p|T)\) be the conditional probability, and let \(\psi(p)\) be the probability to observe a certain value of \(p\) no matter what \(T\) is. Then, the following relation holds

\[
\psi(p) = \int_0^{+\infty} \psi(p|T)f(T)dT.
\]

Due to the maximum entropy principle of ordinary statistical mechanics, the conditional probability \(\psi(p|T)\) is gaussian but the probability to observe a certain value of \(p\) is not.

The main idea of the paper is to consider the velocity gradients \(\delta v_l(x)\) in the dissipation range and in the limit of infinite Reynolds number, as the momentum \(p\) of a particle fulfilling eq. (2) with \(m = k = 1\). In our approach, the “temperature” corresponds to the averaged kinetic energy fluctuation \(<\delta v_l^2>\) and will be referred with the symbol \(T_l\).

As shown by Kraichnan \cite{15}, if we observe the statistics of the velocity gradients in the inertial range, considering both the coherent structures and the
random fluctuations, the equipartition of energy does not apply. However, in
the dissipation range, if we suppose that the fluctuations of the velocity gra-
dients are given only by the incoherent fluctuations, we argue (and assume)
that the equipartition of energy holds. The physical implication of this model
is that the velocity fluctuations, in the dissipation range, may be described as
a cascade of thermal baths. At every scale the ordinary statistical mechanics
holds, i.e. the conditional probability for a fixed temperature $T_t$ is gaussian,
but, due to the changes of such temperature, the final distribution will be
different from the gaussian one.

Due to the three dimensional nature of the turbulent flow, we suggest that
the temperature $T_t$ fluctuates with a $\chi^2$- distribution with degree $n = 3$. Act-
ually, such physical model for the distribution of the averaged kinetic energy
fluctuations $f(T_t)$ has been proposed recently by Beck [16] in the formalism
of nonextensive statistical mechanics.

The temperature $T_t$ has to be considered as a random variable given by the
expression

$$T_t = \sum_{i=1}^{3} X_i^2,$$

(4)

where $X_i$ are independent gaussian variables with zero average. The mean
value of the averaged kinetic energy fluctuations $T_t$ depends on the scale $l$. In
particular, at scale $l$, the mean value $\langle T_t \rangle$ is $3g(l)$, where $g(l) = < X^2 >$.
This means that the probability density $f(T_t)$ is given by

$$f(T_t) = \frac{1}{\Gamma(3/2)} \left( \frac{1}{2g(l)} \right)^{3/2} T_t^{1/2} e^{-\frac{T_t}{2g(l)}}.$$

(5)

If we introduce eq.(5) into eq.(3) we obtain the following integral

$$\psi(\delta v_l) = C \frac{g(l)^{3/2}}{\Gamma(3/2)} \int_{0}^{+\infty} e^{-\frac{\delta v_l^2}{2T_t}} e^{-\frac{T_t}{2g(l)}} dT_t,$$

(6)

where $C = (4\sqrt{\pi}\Gamma(3/2))^{-1}$ is the normalization constant. The latter integral
can be evaluated exactly. Indeed, eq.(6) can be rewritten as

$$\psi(\delta v_l) = \frac{2C}{g(l)^{1/2}} K_1 \left( \frac{\delta v_l}{g(l)^{1/2}} \right),$$

(7)

where $K_1(\cdot)$ is the modified Bessel function of the second kind.

Eq.(7) provides a theoretical prediction of the statistics of the velocity gra-
dients of the incoherent turbulence in the limit of infinite Reynolds number.
Obviously, the statistics of eq.(7) depends on the scaling law $g(l)$ of the mean
temperature $\langle T_t \rangle$. 

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Given the relation between the mean temperature \(< T_t >\) and the scale \(l\), the moments of the velocity gradients can be evaluated easily by means of eq.(7). Indeed, we obtain

\[
< \delta v_l^q > = f(q) g(l)^{q/2},
\]

where \(f(q)\) is a function of \(q\) depending on the moments of the Bessel function \(K_1(\cdot)\)

\[
f(q) = 2C \int y^{q+1} K_1(y) dy.
\]

Equation (8) shows that the moments of the velocity gradients depend only on the scaling law of the mean temperature \(< T_t >\).

At the very small scales, \(l \sim l_d\), the velocity field is smooth due to the lack of singularities. Therefore, the structure function scales as \(< \delta v_l^q > \sim l^q\). Then, due to eq.(8), we propose the following scaling law

\[
g(l) = l^2.
\]

With this assumption, the theoretical prediction of the velocity gradients statistics is given by the following equation

\[
\psi(\delta v_l) = \frac{2C}{l^2} \delta v_l K_1 \left( \frac{\delta v_l}{l} \right).
\]

We will show that such theoretical prediction is in complete agreement with the experimental results.

Actually, experimental data in the dissipation range are difficult to acquire. We show the comparison between the theoretical prediction and the experimental data of Van de Water [14]. These experimental data correspond to the transverse PDF of a turbulent flow behind a grid in a closed wind tunnel at a nearly-dissipative scale \((l_d \sim 10)\). Figure 1 shows the agreement between the experimental results and the theoretical prediction.

Although the moments of the velocity gradients distribution of the incoherent turbulence depend on the scaling law of the mean temperature \(< T_t >\), the probability \(\psi(\delta v_l)\) has an asymptotic decay close to the exponential function whatever is the relation between the scale \(l\) and the mean temperature \(< T_t >\). Indeed, the asymptotic shape of the velocity gradients is given by

\[
\psi(\delta v_l) \sim e^{-\frac{\delta v_l}{c(l)}},
\]

where \(c(l)\) is a function of the scale \(l\) that depends only on \(g(l)\).

As shown by Benzi et al. [17], the PDF of the velocity gradients observed
Figure 1: Log-linear plot of the probability distributions $\psi(\delta v_l)$ of the velocity gradients. The theoretical prediction is obtained with $l=0.58$ (dashed-dotted line). The experimental data of Ref.[14] are indicated by symbols.

experimentally in the inertial range can be obtained by a non-trivial superposition of stretched exponentials, corresponding to the various exponents of the multifractal approach. Moreover, such distribution is explicitly dependent on the Reynolds number. On the contrary, at very small scales $l \sim l_d$, the tails of the PDF are exponential-like and independent of the Reynolds number. In practice, at very small scales, the exponential like decay of the PDF is a universal feature that, using the present approach, may be described by means of ordinary statistical mechanics.

In conclusion, we have presented a description of turbulence at very small scales $l \sim l_d$, where the coherent structures may be assumed to have been annihilated by dissipation. A thermodynamical model for incoherent turbulence, providing an exact expression for the PDF (in terms of the modified Bessel function of the second kind), has been presented. We have shown that the velocity gradients fluctuations at very small scales may be described as the momentum ${\bf p}$ of a particle in a gas with a temperature $T$ that fluctuates slowly. Hence, the present model explains how even the random incoherent fluctuations in the dissipative range may give rise to non-gaussian statistics. We hope that this effort will results in a deeper understanding of the physics underlying the phenomenon of fully developed turbulence and, more in gen-
eral, of complex processes characterized by the coexistence of equilibrium and non-equilibrium conditions.

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References

[1] A.N. Kolmogorov, Dokl. Akad. SSSR 30, 299 (1941)
[2] G.K. Batchelor, A.A. Townsend, Proc. Roy. Soc. A 199, 238 (1949)
[3] F. Anselmet, Y. Gagne, E.J. Hopfinger, R.A. Antonia, J. Fluid Mech. 140, 63 (1984)
[4] A. Vincent, M. Meneguzzi, J. Fluid Mech 225, 1 (1991)
[5] A.N. Kolmogorov, J. Fluid Mech. 13, 82 (1962)
[6] A.M. Obukhov, J. Fluid Mech. 13, 77 (1962)
[7] E.D. Siggia, J. Fluid Mech. 107, 375 (1981)
[8] R.H. Kraichnan, Phys. Rev. Lett. 65, 575 (1990)
[9] U. Frisch, P.L. Sulem, M. Nelkin, J. Fluid Mech. 87, 719 (1978)
[10] R. Benzi, G. Paladin, G. Parisi, A. Vulpiani, J. Phys.A 17, 3521 (1984)
[11] C. Meneveau, K.R. Sreenivasan, Phys. Rev. Lett. 59, 1424 (1987)
[12] Z.-S. She, Phys. Rev. Lett. 66, 600 (1991)
[13] Z.-S. She, E. Leveque, Phys. Rev. Lett. 72, 336 (1994)
[14] W. van de Water, Physica B 228, 185 (1996)
[15] R.H. Kraichnan, Advanc. Math. 16, 305 (1975)
[16] C. Beck, Phys. Rev. Lett. 87, 180601 (2001)
[17] R. Benzi, L. Biferale, G. Paladin, A. Vulpiani, M. Vergassola, Phys. Rev. Lett 67, 2299 (1991)