Perpetual motion and backscattering oscillations of a mobile impurity in a quantum fluid

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We study the dynamics of a mobile impurity in a quantum fluid at zero temperature. Two related settings are considered. In the first setting an impurity is injected in a fluid with some initial velocity $v_0$, and we are interested in its velocity at infinite time, $v_\infty$. We derive a rigorous upper bound on $|v_0 - v_\infty|$ for initial velocities smaller than some critical velocity. In the limit of vanishing impurity-fluid coupling this bound can be regarded as a rigorous proof of the Landau criterion of superfluidity. In the case of a finite coupling the velocity of the impurity can drop, but not to zero; the bound quantifies the maximal possible drop. In the second setting a small constant force acts on the impurity. We identify two possible regimes of the impurity dynamics – backscattering oscillations of the impurity velocity and saturation of the velocity without oscillations. Which regime is realized depends on the spectrum of the fluid and the mass of the impurity. In particular, backscattering oscillations are possible for sufficiently heavy impurities in fluids with a roton feature.

I. INTRODUCTION

What happens to an impurity particle injected in a quantum fluid at zero temperature? According to the Landau criterion of superfluidity \cite{1}, generalized to the case of a finite impurity mass \cite{2}, if the initial velocity of the impurity is less than the (mass-dependent) critical velocity $v_c$, the impurity keeps moving forever without dissipation. However, the kinematical arguments underlying the (generalized) Landau criterion \cite{1, 2} are non-rigorous: they are based on the energy conservation law in which the impurity-host interaction (which is assumed to be small) is disregarded. Generally speaking, these arguments do not exclude a possibility that the velocity of the impurity does relax to a zero or nonzero value with a small rate controlled by the interaction strength. A rigorous proof of the possibility of a perpetual motion of an impurity in a quantum fluid has been presented in \cite{3}. It has been proven in \cite{3} that

$$v_{GS}(p) \equiv \langle GS(p) | \hat{V}_i | GS(p) \rangle = \frac{\partial E(p)}{\partial p}, \quad (1)$$

where $|GS(p)\rangle$ is a ground state of the impurity-host system in a subspace with a total momentum $p$, $E(p)$ is the dispersion of the impurity-host system and $\hat{V}_i$ is the impurity velocity operator.

Eq. (1) implies that there exist eigenstates in which the expectation value of the impurity velocity is nonzero. However this result does not relate initial velocity of the injected impurity, $v_0$, to its velocity $v_\infty$ at infinite time. The latter relation has been established in \cite{3}: An upper bound on $|v_\infty - v_0|$ has been rigorously derived for an impurity injected in the one-dimensional Tonks-Girardeau gas (see \cite{3, 5–8} for a comprehensive investigation of dynamics of an impurity in the Tonks-Girardeau gas).

The result of \cite{4} applies to a specific model of a host fluid, while the underlying kinematical arguments apply to an arbitrary host fluid. Thus one can expect that the bound \cite{4} can be generalized to all such fluids. Such generalization is the first objective of the present paper: We provide an exact upper bound on $|v_\infty - v_0|$ which depends on the dispersion of the fluid, strength of interaction between the impurity and the fluid, mass of the impurity and its initial velocity $v_0$. In the limit of vanishing interaction our result reduces to the generalized Landau criterion of superfluidity \cite{1, 2} and thus can be considered as the rigorous proof of the latter. In the case of finite interaction the velocity of the impurity does drop, but not to zero, and the derived bound quantifies the maximal possible drop.

The second question we address is as follows: What happens to an impurity immersed in a quantum fluid at zero temperature and pulled by a small constant force? This question has been extensively studied recently in the one-dimensional context \cite{10, 12}. We study this question in arbitrary dimensionality and in the limit of a weak coupling between the impurity and the fluid. The physical picture we put forward is as follows. The force accelerates the impurity freely (without dissipation due to scattering off the fluid) as long as its velocity is below the critical velocity. The scattering channel which opens first when the impurity is accelerated slightly above $v_c$ is back scattering with a critical momentum transfer $q_c$.

\footnote{A one-dimensional version of eq. (1) has been reported in \cite{3}. The generalization to higher dimensions is straightforward.}

\footnote{This has been recently verified by numerical and semi-numerical calculations in specific one-dimensional \cite{5, 6} and three-dimensional \cite{9} systems.}
Two distinct situations are possible, $q_c > 0$ and $q_c = 0$. If $q_c > 0$, the "sawtooth" oscillations of the velocity of the impurity develop, see Fig. 1(c). Oscillations of this type have been discussed from one-dimensional perspective in Ref. [12]. We refer to this regime as "backscattering oscillations". If, in contrast, $q_c = 0$, then the velocity of the impurity saturates at $v_c$ without oscillations, see Fig. 1(d). In this case the impurity dissipates the pumped energy by producing infrared excitations of the fluid. If the impurity mass is regarded as a parameter, the nonequilibrium quantum phase transition between the two regimes can occur at some critical mass $m_c$. This general picture is supported by the results of Ref. [7], in which a specific impurity mass is regarded as a parameter, the nonequilibrium quantum phase transition between the two regimes, respectively. Here and in what follows subindices include that Bloch-like oscillations of the impurity velocity is periodic in one dimension, the authors of [10–12] conclude that Bloch-like oscillations of the impurity velocity develop, provided $E(p)$ has no singular points. We critically discuss the method of [10–12] and argue that adiabaticity is broken in the thermodynamic limit even for perfectly smooth $E(p)$, and the Bloch-like oscillations do not occur.

The paper is organized as follows. We introduce our setup and notations in the next section. The impurity dynamics in the absence and presence of an external force is discussed in Sec. III and IV respectively. Sec. V is devoted to the summary and concluding remarks. The Appendix contains the proof of equivalence of two definitions of critical velocity discussed in Sec III.

II. SETUP AND NOTATIONS

We consider a single impurity particle immersed in a host fluid. The Hamiltonian of the combined impurity-host system reads

$$\hat{H} = \hat{H}_h + \hat{H}_i + \hat{U},$$

(2)

$\hat{H}_h$, $\hat{H}_i$ and $\hat{U}$ being the Hamiltonian of the host fluid, the Hamiltonian of the impurity and the impurity-host interaction, respectively. Here and in what follows subindices "h" and "i" stand for a host and an impurity, respectively. $\hat{H}_h$, $\hat{H}_i$ and $\hat{U}$ are translationally invariant and isotropic. The dimensionality of the whole system can be arbitrary. Below we discuss the three constitute of $\hat{H}$ separately.

$\hat{H}_h$ describes $N$ particles of the host fluid with arbitrary interactions between each other. Particle density is $\rho \equiv N/V$ where $V$ is the volume of the fluid. An eigenstates of $\hat{H}_h$ with an energy $E_h$ is denoted by $|E_h\rangle$. Each $|E_h\rangle$ is also an eigenstate of the momentum of the fluid, $\hat{P}_h$. The host dispersion $\varepsilon(q)$ is defined as a minimal eigenenergy which corresponds to a given momentum $q$. Equivalently,

$$\varepsilon(q) \equiv \inf_{\Phi} \langle \Phi | \hat{H} | \Phi \rangle.$$  \hspace{1cm} (3)

Due to the isotropy of $\hat{H}_h$ the dispersion depends on $q = |q|$. We use a special notation, $|GS\rangle$, for the ground state of the host fluid. We set the ground state energy to zero and assume that the momentum of the ground state is zero:

$$\hat{H}_h|GS\rangle = 0, \quad \hat{P}_h|GS\rangle = 0.$$  \hspace{1cm} (4)

This implies $\varepsilon(0) = 0$ and $\varepsilon(q) \geq 0$. We further assume that $\varepsilon(q)$ in not identically zero. We define the speed of sound as

$$v_s \equiv \varepsilon'(0).$$  \hspace{1cm} (5)

$\hat{H}_i$ is a kinetic term of the impurity. For clarity of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(color online) (a) and (b): Geometrical illustration of definitions of critical velocity $v_c$, eq. (14), and critical momentum transfer $q_c$, eq. (22), for the regimes of heavy ($m > m_c$) and light ($m < m_c$) impurities, respectively. The host gas dispersion $\varepsilon(q)$ is identical in both cases. The dot (orange online) marks critical momentum transfer, which is finite in the heavy impurity regime but vanishes in the light impurity regime. The critical velocity $v_c$ is smaller than the sound velocity $v_s$ in the heavy impurity regime and coincides with $v_s$ in the light impurity regime. (c) and (d): average velocity of impurity as a function of time in the heavy and light impurity regimes, respectively. In the heavy impurity regime the backscattering oscillations take place, with the amplitude determined by the critical momentum transfer. In the light impurity regime the velocity saturates without oscillations.}
\end{figure}
presentation we take
\[ \hat{H}_i = \frac{\hat{P}_i^2}{2m}, \]  

(6)

where \( m \) and \( \hat{P}_i \) are the mass and the momentum of the impurity, correspondingly. One can straightforwardly generalize our reasoning to an arbitrary translationally invariant isotropic self-Hamiltonian of the impurity.

Interaction \( \hat{U} \) is pairwise:
\[ \hat{U} = \sum_{n=1}^{N} U(|\mathbf{r}_n - \mathbf{r}_i|) \]

(7)

Here \( \mathbf{r}_i \) and \( \mathbf{r}_n \) are coordinates of the impurity and the \( n \)th host particle, respectively, and \( U(|\mathbf{r}|) \) is a pairwise interaction potential. We call the interaction everywhere repulsive if the potential satisfies
\[ U(x) \geq 0 \quad \forall x. \]

(8)

We denote product eigenstates of \( \hat{H}_h + \hat{H}_i \) by
\[ |E_h, \mathbf{v}\rangle \equiv |E_h\rangle \otimes |\mathbf{v}\rangle, \]

(9)

where \( |\mathbf{v}\rangle \) is the plane wave of the impurity with the momentum \( m\mathbf{v} \).

Initially the impurity is injected with some velocity \( \mathbf{v}_0 \) into the host fluid at zero temperature. The initial wave function thus reads
\[ |\text{in}\rangle = |\text{GS}, \mathbf{v}_0\rangle. \]

(10)

The quantity we are interested in is the velocity of the impurity at infinite time. It is defined as
\[ v_\infty \equiv \frac{1}{m} \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' \langle \text{GS}, \mathbf{v}_0 | e^{i\hat{H}'t'} \hat{\mathbf{P}}_i e^{-i\hat{H}'t'} | \text{GS}, \mathbf{v}_0 \rangle. \]

(11)

In what follows we assume that the eigenvalues of the total Hamiltonian \( \hat{H} \) with a given total momentum are nondegenerate. This implies
\[ v_\infty = \frac{1}{m} \sum_{|E\rangle} \left| \langle \text{GS}, \mathbf{v}_0 | E \rangle \right|^2 \langle E | \hat{\mathbf{P}}_i | E \rangle, \]

(12)

where \( |E\rangle \) is an eigenstate of the total Hamiltonian \( \hat{H} \) with energy \( E \).

III. PERPETUAL MOTION

We start from reviewing kinematical arguments which lead to the notion of critical velocity [1] [2]. Consider an impurity with a velocity \( \mathbf{v}_0 \) which scatters off the host fluid which is initially in its ground state. Let \( \mathbf{q} \) and \( E_h \geq \varepsilon(q) \) be respectively final momentum and energy of the host, where \( q \equiv |\mathbf{q}| \). If one disregards the interaction between the impurity and the host, the momentum and energy conservation leads to
\[ \varepsilon(q) + \frac{q^2}{2m} \geq \varepsilon(q) + \frac{q^2}{2m}, \]

(13)

where \( \varepsilon(q) \equiv |\mathbf{v}_0| \). If \( \varepsilon(q) \) is sufficiently small, \( \varepsilon(q) < v_c \), then for all \( \mathbf{q} \neq 0 \) the inequality (13) can not be fulfilled. The critical velocity \( v_c \) is defined as
\[ v_c \equiv \inf_{q} \frac{\varepsilon(q) + \frac{q^2}{2m}}{q}. \]

(14)

Physically, the critical velocity is the smallest velocity at which the impurity can create real excitations of the host fluid. The geometrical sense of the critical velocity can be seen from Fig. 1: the line \( v \mathbf{q} \) is a tangent to the curve \( \varepsilon(q) + \frac{q^2}{2m} \). Originally Landau defined the critical velocity in the limit \( m \to \infty \) [11]:
\[ v_{c \infty} \equiv \inf_{q} \frac{\varepsilon(q)}{q}. \]

(15)

A straightforward generalization to a finite mass of the impurity was done in [2].

A definition of critical velocity different from the definition (14) was suggested in [11] [12]. We compare the two definitions in the Appendix and prove their equivalence in the limit of vanishing impurity-host coupling, provided the excitations of the host fluid are local.

The arguments by Landau reviewed above suggest that the impurity moves without dissipation when \( \varepsilon(q) < v_c \). However this reasoning relies on an approximation: the interaction between the impurity and the host is disregarded in the energy conservation law. Our aim is to rigorously prove that the impurity can move perpetually in a host fluid at zero temperature even with account of interaction, and to quantify the effect of interaction on the value \( v_\infty \) of the impurity velocity at infinite time. To this end we prove the following theorem.
Theorem. Consider an impurity particle immersed in a host fluid. The combined impurity-host system is described by the Hamiltonian \( \hat{H} \) and prepared in the initial state \( |\psi_0\rangle \) with

\[
0 < v_0 < v_c.
\]

Assume that the eigenvalues of the total Hamiltonian \( \hat{H} \) corresponding to a total momentum \( mv_0 \) are nondegenerate. Then the difference between the initial and infinite-time velocities of the impurity is bounded from above according to

\[
|v_0 - v_\infty| \leq \frac{1}{m(v_c - v_0)} \left( \langle \text{GS}, v_0 | \hat{U} | \text{GS}, v_0 \rangle - \sum_i \langle \text{GS}, v_0 | E_i \rangle^2 \langle E_i | \hat{U} | E_i \rangle \right),
\]

(16)

where the sum is over all eigenstates \( |E_i\rangle \) of the total Hamiltonian \( \hat{H} \) with the total momentum \( mv_0 \). If the interaction between the impurity and the fermions is everywhere repulsive, i.e. the condition \( |E_i| \leq \alpha \) is fulfilled, then a more transparent bound holds:

\[
|v_0 - v_\infty| \leq \frac{\rho \mathbb{U}}{m(v_c - v_0)},
\]

(17)

where \( \mathbb{U} \equiv \int d\mathbf{r} U(|\mathbf{r}|) \).

This theorem generalises an analogous result obtained in [4] for a specific one-dimensional host fluid. The proof also follows the lines of [4].

Proof. According to (12)

\[
|v_0 - v_\infty| = \frac{1}{m} \left| \sum_i |E_i\rangle \langle v_0 - v | \langle \text{GS}, v_0 | E_i \rangle^2 \langle E_i | \hat{U} | E_i \rangle \right| \leq \sum_i \sum_j |v_0 - v| \langle \text{GS}, v_0 | E_i \rangle^2 \langle E_i | \hat{U} | E_j \rangle^2.
\]

(18)

Here the summations are performed over the eigenstates \( |E_i\rangle \) of the total Hamiltonian \( \hat{H} \) and over the eigenstates \( |E_h, v\rangle \) of the noninteracting Hamiltonian \( \hat{H}_h + \hat{H}_i \).

The key step is to notice that according to (14)

\[
|v_0 - v| \leq \frac{1}{m(v_c - v_0)} (E_h + \frac{v^2}{2m} - \frac{v_0^2}{2m})
\]

(19)

for any \( |E_h, v\rangle \). This inequality is of pure kinematical origin. It leads to

\[
\sum |v_0 - v| \langle E | \hat{E}_h + \hat{E}_i \rangle \leq \sum \langle E | \hat{H}_h + \hat{H}_i - \frac{v_0^2}{2m} | E_h, v \rangle \langle E_h, v | E \rangle \leq E - \frac{v_0^2}{2m} - \langle E | \hat{U} | E \rangle.
\]

(20)

Substituting eq. (20) into eq. (18) one obtains the desired bound (16).

If the impurity-host interaction is everywhere repulsive, one can obtain the bound (17) by omitting the second term in the brackets in the r.h.s. of (16) and rewriting the first term according to

\[
\langle \text{GS}, v_0 | \hat{U} | \text{GS}, v_0 \rangle = \rho \mathbb{U}.
\]

(21)

The fundamental importance of the above theorem is that it provides a rigorous proof of the Landau criterion of superfluidity [11] (generalised for impurities of finite mass [2]) in the limit of vanishing impurity-host coupling and quantifies the maximal possible velocity drop in the case of a finite coupling.

Let us discuss and compare the bounds (16) and (17). The former has the advantage that it is valid for an arbitrary potential. Its drawback is that one has to know exact eigenstates of the total Hamiltonian in order to calculate the the r.h.s. of eq. (16) exactly. This is generally not feasible, except the integrable cases. It can be instructive to evaluate the r.h.s. of eq. (16) perturbatively. One can immediately see that the bound (17) itself is derived without resorting to the perturbation theory, however its practical application in nonintegrable cases does require using the perturbation theory.

The assumption (5) allows us to get rid of the eigenstate-dependent term in the r.h.s. of (16). Thus obtained bound (17) can be easily applied without resorting to any approximations. The price is twofold. First, the bound (17) is valid for a narrower class of potentials (which includes, however, an important pointlike repulsive potential studied in one dimension in Refs. [3, 5–8]). Second, it is not as tight as the bound (16), since the difference \( |v_0 - v_\infty| \) is bounded by a term linear in the coupling strength.

An additional benefit of the bound (17) is that it obviates two important points. First, the bound holds equally well for a finite number of host particles and in the thermodynamic limit. Second, the inequality (17) represents a nontrivial bound even for long range interactions, provided the interaction potential decreases with distance faster than \( 1/r^D \), \( D \) being the dimensionality of the system. We expect that both observation generically hold for the bound (16) as well.

We complete the analysis of the above theorem by discussing the assumption of nondegeneracy of energy spectrum for a given total momentum. It should be emphasized that this assumption is extremely unrestrictive. It is fulfilled in a generic system. Moreover, it can be even fulfilled in an integrable system [4]. To understand this note that the momentum is quantized and we require absence of level crossings in a discrete set of points only. It is interesting to note the same assumption is essential for obtaining a related result [1]. It ensures that the lower
edge of the spectrum can be treated (at least locally, in the vicinity of each point of the above-mentioned discrete set) as a one-parametric family of states. This justifies the application of the Hellmann-Feynman theorem which has been used in [3] to obtain eq. (1).

IV. BACKSCATTERING OSCILLATIONS

In the present section we consider the motion of an impurity under the action of a small constant force. We assume that the impurity-host coupling is weak and extend the kinematical reasoning presented in the beginning of the previous section to the case with a force.

Assume that the impurity is initially at rest. The force accelerates it freely (without dissipation) until its velocity reaches \( v_c \). After that the impurity acquires a chance to scatter off the host fluid. It is clear from eq. (14) that the velocity again reaches \( v_c \) after the first scattering of the impurity with the host fluid. The critical momentum transfer \( q_c \) delivers minimum in eq. (14):

\[
v_c q_c = \varepsilon(q_c) + \frac{q^2}{2m},
\]

(22)

The geometrical meaning of \( q_c \) is illustrated in Fig. 1 (a,b): the line \( v_q \) touches the curve \( \varepsilon(q) + \frac{q^2}{2m} \) in the point \( (q_c, v_c q_c) \). Note that \( q_c \) has nothing to do with \( m v_c \).

The behavior of the impurity after the first scattering depends crucially on whether or not \( q_c \) is non-zero. Consider first the case \( q_c > 0 \), see Fig. 1 (a). After the first scattering the velocity of the impurity drops by

\[
\Delta v = q_c / m, \quad (23)
\]

and the impurity starts to freely accelerate until its velocity again reaches \( v_c \), after which the whole cycle is repeated. This is how backscattering oscillations emerge. The period of the oscillations is equal to \( q_c / F \). Note that in the limit of \( m \to \infty \) the amplitude of oscillations vanishes.

Consider now the case when \( q_c = 0 \), see Fig. 1 (b). In this case, after the velocity of the impurity reaches \( v_c \), the impurity starts to dissipate the pumped energy by producing infrared excitations of the host fluid. In the weak force limit this leads to the saturation of its velocity at \( v_c \) without oscillations.

In certain fluids one can switch between these two regimes by varying the mass of the impurity, while in other fluids only the velocity saturation regime is possible. In order to identify a condition which discriminates between these two options we consider the limit of \( m \to \infty \). If \( q_c = 0 \) in this limit, then the same is true for any mass \( m \), and the second option is realized. An example of such a fluid is the Bogolyubov gas of weakly coupled bosons. The critical velocity does not depend on the impurity mass and is equal to the sound velocity in this case.

In contrast, if \( q_c > 0 \) in the limit of \( m \to \infty \), the first option – the nonequilibrium quantum phase transition with respect to the impurity mass – is realized. The prime example of a fluid of this type is liquid helium. In this case the backscattering oscillations take place for masses greater than some critical mass \( m_c \), while for \( m < m_c \) the velocity saturates without oscillations. The critical mass is determined from the following equation:

\[
v_c(m_c) = v_c.
\]

(24)

Here we explicitly indicate the dependence of the critical velocity on the mass of the impurity: \( v_c = v_c(m) \), see eq. (14). This function is plotted in Fig. 2. One can see that its derivative is (generically) discontinuous at \( m = m_c \). The amplitude of the backscattering oscillations also (generically) experiences a jump from a finite value to zero at \( m = m_c \). This proves, in particular, that the transition over \( m_c \) is indeed a nonequilibrium quantum phase transition and not a crossover.

The general physical picture of the impurity dynamics under the action of a small force presented above is confirmed by the results of the detailed study of a specific one-dimensional model – the Tonks-Girardeau gas [7]. In this model a critical impurity mass has been found to be equal to the mass of a host particle.

The backscattering oscillations get damped at finite forces since the direction (in more-than-one-dimensional systems) and the value (in any dimensions) of the momentum transfer vary from one scattering event to another. In Ref. [2] a kinetic theory for an impurity in the Tonks-Girardeau gas valid for arbitrary forces has been developed and the damping rate has been calculated. This theory can be generalized to arbitrary host fluids. We do not study this question in the present paper.

The physical picture we put forward differs significantly from the picture developed in [10,12] for one-dimensional systems. The method of [10,12] is based on adiabatically following the dispersion of the impurity-fluid system \( E(p) \). Since \( E(p) \) is periodic in one dimension, the authors of [10,12] conclude that Bloch-like oscillations of the velocity of the impurity develop, provided \( E(p) \) is a smooth function. However many-particle nature of the problem under consideration invalidates adiabatic treatment in the thermodynamic limit. In order to see this one should estimate the probability of the Landau-Zener transition at the avoided level crossing which occurs when the total momentum reaches \( m v_c \). In the case \( m < m_c \) one can consider the crossing between the (momentum-dependent) ground state and e.g. a neighboring state in which the momentum of the impurity is lowered by the momentum quantum \( \delta k \equiv 2\pi / L \), where \( L \) is the length of the quantization box, and the momentum of the fluid is increased by the same amount. In one dimension this estimate leads to the adiabaticity requirement \( F = o(g^2 / L) \), where \( g \) is an effective impurity-fluid coupling. In the case \( m > m_c \) one should take a neighboring state in which the momentum of the impu-
urity is lowered by $q_c$ and the momentum of the fluid is increased by $q_c$. This leads to the estimate $F = o(q^2/L^2)$. One can see that in both cases the Bloch-like oscillations predicted in [10–12] do not occur in the thermodynamic limit. However the treatment of [10–12] can be relevant for finite systems realizable with cold gases.

The breakdown of adiabaticity in the case when $E(p)$ has a cusp has been discussed in [12]. In this case Ref. [12] predicts the sawtooth oscillations in the limit of strong force. They qualitatively resemble the backscattering oscillations we discuss in the opposite weak force limit. However the maximal velocity and the amplitude differ for these two types of oscillations. Moreover, the damping of oscillations which is a major effect in the strong force limit is not taken into account in [12]. A possibility of oscillations in the weak force limit is mentioned in [12] but not elaborated in any detail. A quantum phase transition due to the appearance of a cusp in $E(p)$ with increasing the impurity mass is predicted in [12]. The status of this prediction remains unclear. It is not identical to the quantum phase transition discussed in the present paper since the latter is not related to the singularities of $E(p)$. In particular, our QPT does occur in the Tonks-Girardeau gas [7] while the QPT of Ref. [12] does not.

V. SUMMARY AND CONCLUDING REMARKS

To summarize, we have studied two related problems. The first one in the dynamics of a mobile impurity injected with some initial velocity $v_0$ in a quantum fluid at zero temperature. We have rigorously derived an upper bound on the difference between the initial and the asymptotic velocities of the impurity, $|v_0 - c_\infty|$, valid for initial velocities less than the mass-dependent critical velocity $v_c$. The presented derivation is completely general and does not rely on any approximations. In the limit of vanishing impurity-fluid coupling this derivation constitutes a general proof of the generalized Landau criterion of dissipationless motion of the impurity [11–12]. If in addition the limit of $m \to \infty$ is taken, the critical velocity $v_c(m)$ approaches the celebrated Landau critical velocity $v_{cL}$. In the case of finite coupling our results show that although the velocity of the impurity can drop by a finite amount, it can not drop to zero, if the coupling is not too strong.

The second problem we have studied is the dynamics of an impurity under the action of a small constant force. We have put forward a general physical picture of the impurity dynamics. In particular, we have identified two possible dynamical regimes – backscattering oscillations of the impurity velocity and velocity saturation without oscillations. The oscillatory regime is possible for fluids in which the Landau critical velocity is less than the velocity of sound, e.g. for fluids with roton feature of the spectrum. Another precondition for backscattering oscillations is that the impurity should be sufficiently heavy. If the mass of the impurity is regarded as a parameter, a nonequilibrium quantum phase transition between the two regimes occurs at some critical mass. If the impurity is light enough or the Landau critical velocity is equal to the speed of sound, the "saturation without oscillations" regime occur. This general picture is in accordance with the results obtained for a specific model of a host fluid [7], but significantly disagree with the results of Refs. [10–12]. In the latter works the Bloch-like oscillations of the impurity velocity has been predicted to take place in one dimension. We argue that this type of oscillations is absent in the thermodynamic limit due to the breakdown of adiabaticity.

Our treatment of the first problem is valid for any strength of impurity-fluid interaction, however the weaker interaction is, the more tight the obtained bounds are. Our treatment of the second problem is valid in the weak coupling limit only. At the same time in some interesting systems, e.g. in superfluid helium, the bare coupling is not weak. In this case it can be a better choice to use a suitably renormalized Hamiltonian instead a bare one. If this is done, an effective mass of the impurity should be used instead of a bare one.

We conclude by briefly discussing the observability of the two dynamical regimes, especially the backscattering oscillations, in superfluid helium and in cold atomic and molecular systems. The lightest possible impurity in superfluid $^4$He is a neutral atom of $^3$He (apart from a neutron). The backscattering oscillations with $v_c \approx 140 m/s$ (compare this to $v_{cL} \approx 58 m/s$) and $\Delta v \approx 169 m/s$ take place for this impurity (we use experimentally determined $\varepsilon(p)$ [13] and effective mass $m_{eff} \approx 2.4 m^{3He}$ [14]). For heavier impurities the backscattering oscillations with smaller critical velocities and amplitudes occur. Velocity saturation regime in $^4$He is impossible since sufficiently light impurities are lacking. Cold atomic and molecular systems provide another possibility to observe impurity dynamics under action of a force. A wide range of impurity masses and impurity-host couplings can be accessible in this setting. As for a spectrum with a roton-like feature which could allow the observation of backscattering oscillations, it can be realized in dipolar gases [11] and in one-dimensional [16], quasi-one-dimensional and elongated traps [17].

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**APPENDIX: EQUIVALENCE BETWEEN TWO DIFFERENT DEFINITIONS OF CRITICAL VELOCITY**

The dispersion of the combined impurity-host system is defined as

\[ \mathcal{E}(p) \equiv \inf_{\Psi: \hat{H}\Psi = \varepsilon \Psi} \langle \Psi|\hat{H}|\Psi \rangle. \quad (A.25) \]

Here we consider the limit of vanishing coupling between the impurity and the fluid. In this limit the dispersion of the combined impurity-fluid system reads

\[ \mathcal{E}(p) = \inf_q \left( \varepsilon(q) + \frac{(p-q)^2}{2m} \right). \quad (A.26) \]

The velocity tagged in [11][12] as a "thermodynamic critical velocity" is defined as

\[ v_{GS}^{\text{max}} = \sup_p v_{GS}(p), \quad (A.27) \]

where

\[ v_{GS}(p) = \frac{\partial \mathcal{E}}{\partial p}, \quad (A.28) \]

cf. eq. [1]. The purpose of the present Appendix is to demonstrate that if the host fluid supports only local excitations, then

\[ v_{GS}^{\text{max}} = v_c \quad (A.29) \]

in the limit of vanishing impurity-fluid coupling.

We introduce \( q_\ast(p) \) which delivers minimum in eq. (A.30):

\[ \mathcal{E}(p) = \varepsilon(q_\ast) + \frac{(p-q_\ast)^2}{2m}. \quad (A.30) \]

We also will use an inverse function \( p(q_\ast) \).

The geometrical representation of eq. (A.30) proves to be extremely insightful. Indeed, \( \left( p^2/(2m) - \mathcal{E}(p) \right) \) equals to the minimal "vertical distance" between the line \( \frac{p}{m}q \) and the curve \( \varepsilon(q) + \frac{q^2}{2m} \). For \( 0 < p < v_c m \) the distance is zero, \( q_\ast = 0 \) and

\[ v_{GS}(p) = \frac{p}{m}. \quad (A.31) \]

In particular,

\[ v_{GS}(v_c m - 0) = v_c, \quad (A.32) \]

therefore \( v_{GS}^{\text{max}} \geq v_c \). Now we need to prove the inverse inequality.

Consider \( p \geq v_c m \). Function \( q_\ast(p) \) is globally increasing. If \( \varepsilon(q) \) is piecewise-smooth, \( q_\ast(p) \) is also piecewise-smooth, although it can have positive jumps at discrete set of points. In what follows \( \partial q_\ast / \partial p \) is understood as a one-sided derivative when necessary.

A remarkable simple relation between the Hellmann-Feynman velocity and \( q_\ast(p) \) exists:

\[ v_{GS}(p) = \frac{p - q_\ast(p)}{m}. \quad (A.33) \]

It follows from

\[ \frac{\partial}{\partial q} \left( \varepsilon(q) + \frac{q^2}{2m} \right) \bigg|_{q=q_\ast} = \frac{p}{m}, \quad (A.34) \]

which is evident from the geometrical representation.

Let us introduce

**Definition.** Dispersion of the host fluid \( \varepsilon(q) \) is called local if for any integer nonnegative \( n_1, n_2 \) and for any positive momenta \( q_1, q_2 \)

\[ n_1 \varepsilon(q_1) + n_2 \varepsilon(q_2) \geq \varepsilon(n_1 q_1 + n_2 q_2). \quad (A.35) \]

The rational behind this definition is that if one can create an excitation \( (q_1, \varepsilon(q_1)) \) and another excitation \( (q_2, \varepsilon(q_2)) \), then one can also create arbitrary number of spatially separated excitations of these two types without paying any energy penalty due to interactions between the excitations. Note that local dispersion can be (and often is) neither convex nor concave function.

Let us focus on the case \( q_\ast \equiv q_\ast (nv_c + 0) > 0 \) first. Consider \( q_\ast \in [nq_c, (n+1)q_c) \) with \( n \in \mathbb{N} \). Then

\[ \frac{p(q_\ast)}{m} \leq \varepsilon((n+1)q_c) + \frac{(n+1)q_c^2}{2m} - \varepsilon(nq_c) - \frac{(nq_c)^2}{2m} \leq v_c + \frac{nq_\ast}{m}. \quad (A.36) \]

The first inequality is clear from the geometrical representation, see Fig. 3, while the second one – from eq. (A.35). From eqs. (A.33) and (A.36) one obtains \( v_{GS}^{\text{max}} \leq v_c \) which completes the proof.

In the case \( q_\ast = 0 \) one can repeat all the reasoning with some small \( \tilde{q}_c > 0 \) instead of \( q_\ast \) and then take the limit \( \tilde{q}_c \to +0 \).

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