Spinorial $R$ operator and Algebraic Bethe Ansatz

D. Karakhanyan$^a$, R. Kirschner$^b$

$^a$ Yerevan Physics Institute, 2 Alikhanyan br., 0036 Yerevan, Armenia

$^b$ Institut für Theoretische Physik, Universität Leipzig, PF 100 920, D-04009 Leipzig, Germany

Abstract

We propose a new approach to the spinor-spinor R-matrix with orthogonal and symplectic symmetry. Based on this approach and the fusion method we relate the spinor-vector and vector-vector monodromy matrices for quantum spin chains. We consider the explicit spinor $R$ matrices of low rank orthogonal algebras and the corresponding $RTT$ algebras. Coincidences with fundamental $R$ matrices allow to relate the Algebraic Bethe Ansatz for spinor and vector monodromy matrices.

1 e-mail: karakh@yerphi.am
2 e-mail: Roland.Kirschner@itp.uni-leipzig.de
1 Introduction

The Inverse Scattering Method for quantum integrable systems \[1, 2, 3, 4\] applies basically to symmetries of all Lie algebras types. The case of the \(A\) series has been studied in much detail, because of its applications to well known models and because of its simplicity compared to other cases. In last years the Yangian algebras of the \(B, C, D\) types and the corresponding Algebraic Bethe Ansatz (ABA) attracted increasing interest, e.g. \[5, 6, 7, 8, 9, 10, 11\].

Yangians of the orthogonal and symplectic types allow for the linear evaluation only by additional constraints on the universal enveloping of the corresponding Lie algebra. The spinor representations, where the Yangian generators are constructed from the underlying algebra \(C\) (see eq. (2.7) below), obey these constraints, i.e. the Yang Baxter \(RLL\) relation involving the fundamental \(R\) matrix with orthogonal or symplectic symmetry and the spinor \(L\) operators holds \[12\].

Reshetikhin \[13\] proposed to approach the ABA in the orthogonal case by replacing in the auxiliary space the fundamental (vector) representation by the spinor representation and to apply the fusion procedure. The ABA for the \(so(3)\) case has been shown to be related to the well known \(s\ell(2)\) ABA. Our investigation is oriented along this concept.

We consider the Yang-Baxter \(RLL\) relation in the tensor product of the fundamental and two copies of the spinor representations. It is used as the defining relation for the involved \(R\) operator intertwining the spinor representations. The spinorial \(R\) operator has been obtained in the orthogonal case in \[16\] and considered in \[12, 6\]. The uniform formulation of the orthogonal and symplectic cases and of the superalgebra case have been given in \[17\]. The result of the spinorial \(R\) operator has been given in the expansion in invariants on the tensor product of two copies of the algebra \(C\) built as contraction of anti-symmetrized (orthogonal case) or symmetrized (symplectic case) products of its generators.

In this paper we propose an alternative approach. We use instead the invariant built from the product of one generator out of each of the spinor algebras and its powers. We obtain the spinorial \(R\) operator in terms of the Euler Beta function. We consider low rank orthogonal cases. Here the spinor representations are finite dimensional. This results in characteristic polynomials obeyed by the invariant from which the explicit form of projection operators can be read off. The general result of the spinorial \(R\) operator is reduced to a finite sum, which is derived from the spectral decomposition of the invariant.

We observe coincidences of the spinorial and known fundamental (vector) \(R\) matrices. The structure of the spinorial \(R\) in even dimensions \((so(2m), D\) series) appears much simpler than the odd dimensional case \((so(2m + 1), B\) series).

In the even-dimensional cases the separation of the tensor product of spinor representation spaces, where \(R\) acts, into chiral parts results in the corresponding separation of \(R\). Both parts of \(R\) obey the Yang-Baxter relations separately. The explicit matrix forms of \(R\) have a lot of zero entries. This also appears in the corresponding monodromy matrices.

The observed coincidences of the spinorial \(R\) matrices imply coincidences of the corresponding \(RTT\) (spinorial Yangian) algebras and fundamental \(RTT\) (ordinary Yangian) algebras and as a consequence relations of the corresponding ABA.

In particular we observe the coincidences of the spinorial \(so(4)\) with fundamental \(s\ell(2) \oplus s\ell(2)\) and the spinorial \(so(6)\) with fundamental \(s\ell(4) \oplus s\ell(4)\) \(R\) matrices. Besides of the well known relations between spinorial \(so(3)\), fundamental \(s\ell(2)\) and fundamental \(sp(2)\) Yangian algebras we observe the equivalence of the spinorial \(so(5)\) and the fundamental
The paper is organized as follows:

In sect. 2 we recall the Yang-Baxter relations with orthogonal and symplectic symmetry, in particular the ones involving the fundamental and the spinor representations. In sect. 3 we describe our approach to the spinor-spinor $R$-matrix and compare with the Shankar-Witten approach. In sect. 4 we consider the relation between spinorial and vector monodromy matrices using the fusion procedure. In sect. 5 we analyze the spinorial $R$ matrices for $so(4)$ and $so(6)$ and the generated RTT algebras. The spinorial $R$ matrices for $so(3)$ and $so(5)$ are analyzed in sect. 6.

2 The spinorial $R$-matrix

The Yang-Baxter relation for the fundamental $R$ matrix reads

$$ R^{a_1 a_2}_{b_1 b_2} (u - v) R^{b_1 b_2}_{c_1 c_2} (u) R^{b_2 a_3}_{c_2 c_3} (v) = R^{a_2 a_3}_{b_2 b_3} (v) R^{a_1 b_3}_{b_1 c_3} (u) R^{b_1 b_2}_{c_1 c_2} (u).$$  \(2.1\)

$R$ looks similar for the cases of orthogonal and symplectic symmetry [14], [15].

$$ R^{a_1 a_2}_{b_1 b_2} (u) = u(u + \frac{n}{2} - \epsilon) I^{a_1 a_2}_{b_1 b_2} + (u + \frac{n}{2} - \epsilon) P^{a_1 a_2}_{b_1 b_2} - \epsilon \delta^{a_1 a_2}_{b_1 b_2},$$  \(2.2\)

where

$$ I^{a_1 a_2}_{b_1 b_2} = \delta^{a_1 b_1} \delta^{a_2 b_2}, \quad P^{a_1 a_2}_{b_1 b_2} = \delta^{a_1 b_2} \delta^{a_2 b_1}, \quad \delta^{a_1 a_2}_{b_1 b_2} = \epsilon^{a_1 a_2} \epsilon^{b_1 b_2}.$$  \(2.3\)

The choices $\epsilon = +1$ and $\epsilon = -1$ correspond to the $so(n)$ and $sp(2m)$ cases respectively. The index range is $a_1, b_1 = 1, \ldots, n$ or $1, \ldots, 2m$. $\epsilon^{ab}$ denotes the metric tensor which is symmetric in the orthogonal and anti-symmetric in the symplectic case.

The Yang-Baxter $RLL$ relation with the above $R$

$$ R^{a_1 a_2}_{b_1 b_2} (u) L^{b_1 b_2}_{c_1 c_2} (u + v) L^{b_2 a_1}_{c_2 c_1} (v) = L^{a_1 a_2}_{b_1 b_2} (v) L^{a_2 b_1}_{b_1 c_1} (u + v) R^{b_1 b_2}_{c_1 c_2} (u),$$  \(2.4\)

is fulfilled by the linear form of the $L$-operator

$$ L^a_b (u) = u \delta^a_b + G^a_b,$$  \(2.5\)

in the spinor representation case, where the matrix elements $G^a_b$ are built as

$$ G^a_b = \frac{1}{2}(c^a c_b - \epsilon c_b c^a)$$  \(2.6\)

from the underlying algebra $\mathcal{C}$ generated by $c_a$ obeying the commutation relations

$$ c^a c^b + \epsilon c^b c^a = \epsilon^{ab}, \quad \text{or} \quad (c^a)^\gamma_\gamma (c^b)^\gamma_\beta + \epsilon (c^b)^\gamma_\gamma (c^a)^\gamma_\beta = \epsilon^{ab} \delta^\gamma_\beta.$$  \(2.7\)

The linear ansatz for $L$-operator \(2.5\) implies the $so(n)$ or $sp(2m)$ Lie algebra relations,

$$ [G^{a_1 b_1}, G^{a_2 b_2}] = \delta^{a_1 a_2} G^{a_2 b_1} - \delta^{a_2 a_2} G^{a_1 b_2} + \epsilon \delta^{a_1 a_2} \epsilon_{b_1 c_2} G^{a_2 b_2} - \epsilon G^{a_2 a_2} \epsilon_{b_1 c_2} \epsilon_{b_1 b_2},$$  \(2.8\)

and the symmetry condition,

$$ \epsilon^{a_1 a_2} \epsilon_{c_1 c_2} (G^{a_1 b_1} - \beta \delta^{a_1}_{b_1}) G^{b_2 c_2} - \delta^{a_1 a_2} \epsilon_{c_1 c_2} \epsilon_{b_1 b_2},$$  \(2.9\)

as well as the additional constraint

$$ \epsilon^{a_1 a_2} \epsilon_{c_1 c_2} (G^{a_1 b_1} - \beta \delta^{a_1}_{b_1}) G^{b_2 c_2} = G^{a_1 a_2} \epsilon_{c_1 c_2} (G^{a_1 c_1} - \beta \delta^{a_1}_{c_1}) \epsilon^{c_1 c_2} \epsilon_{b_1 b_2},$$  \(2.10\)
which specifies the Yangian linear evaluation.

The Yang-Baxter relation is formulated in the tensor product of three representation spaces. We are concerned with the fundamental (vector) space \( V \) (2\( m \)- or 2\( m + 1 \)-dimensional) and the spinor space \( S \) (of dimension 2\( m \) in the orthogonal and infinite dimensional, oscillator Fock space, in the symplectic case).

In order to specify the representation involved we shall show all indices explicitly, \( a, b \) for \( V \) and \( \alpha, \beta \) for \( S \). Then (2.11) appears as

\[
R_{b_1b_2}^{a_1a_2}(u)L_{c_1\gamma}^{a}\left(u + v\right)L_{c_2\beta}^{b}\left(v\right) = L_{b_2\gamma}^{a}\left(v\right)L_{b_1\beta}^{b}\left(u + v\right)R_{c_1c_2}^{a_1a_2}(u). \tag{2.11}
\]

Along with this Yang-Baxter relation we consider

\[
\mathcal{R}_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}(u)L_{c_1\gamma}^{\beta_1}\left(u + v\right)L_{c_2\beta}^{\beta_2}\left(v\right) = L_{c_2\beta_1}^{\alpha_1}\left(v\right)L_{c_1\beta_2}^{\beta_2}\left(u + v\right)\mathcal{R}_{\gamma_1\gamma_2}^{\beta_1, \beta_2}(u). \tag{2.12}
\]

Here \( \mathcal{R}(u) \) stands for the spinorial \( R \)-operator.

We shall use also the modifications of the above \( RLL \) relations usually referred to as the check form. The \( R \) matrix is multiplied by the permutation operator interchanging the auxiliary space factors, \( R_{12} = \mathcal{P}_{12}R_{12} \). For example, the check form corresponding to (2.12) is

\[
\mathcal{R}_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}(u)L_{c_1\gamma}^{\beta_1}\left(u + v\right)L_{c_2\beta}^{\beta_2}\left(v\right) = L_{c_2\beta_1}^{\alpha_1}\left(v\right)L_{c_1\beta_2}^{\beta_2}\left(u + v\right)\mathcal{R}_{\gamma_1\gamma_2}^{\beta_1, \beta_2}(u). \tag{2.13}
\]

In [17], following the approach of [16], the spinorial \( R \)-operator was obtained in the form

\[
\mathcal{R}_{12}(u) = \sum_k \frac{r_k(u)}{k!} \sum_{\vec{a}, \vec{b}} \varepsilon_{a_1b_1} \cdots \varepsilon_{a_kb_k} c_1^{a_1} \cdots c_1^{a_k} \cdot c_2^{b_1} \cdots c_2^{b_k} \tag{2.14}
\]

Here \( c_1^{a_1} \cdots c_1^{a_k} \) symbolizes the anti-symmetrization in the orthogonal and the symmetrizer in the symplectic case. The coefficients are derived from an iterative relation as

\[
r_{2m} = \frac{2^{2m} \Gamma\left(m + \frac{n}{2}\right)}{\Gamma\left(m + 1 - \frac{n}{2}\right)} A_0(u),
\]

\[
r_{2m+1} = \frac{2^{2m} \Gamma\left(m + \frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(m + 1 - \frac{n}{2}\right)} A_1(u),
\]

where \( A_0 \) and \( A_1 \) are arbitrary functions of the spectral parameter.

Both the even (sum restricted to even \( k \)) and the odd parts obey (2.13).

Note that in the orthogonal case \( c^a \), the generators of \( C \), are fermionic and can be expressed in terms of the Dirac gamma matrices, \( c^a = \frac{1}{\sqrt{2}} \gamma^a \).

3 The alternative approach to \( \mathcal{R} \)

The spinorial \( \mathcal{R} \) operator can be regarded as an element of \( C_1 \otimes C_2 \) (2.7) or as an operator acting in the product of two copies of the spinor space, \( S_1 \otimes S_2 \). \( c_1^a \) and \( c_2^a \) (2.7) are the corresponding basic elements.

The equation (2.13) takes the form

\[
\mathcal{R}_{12}(u) \left( (u + v + \frac{e}{2}) \delta^a_c - c_1^a c_{1e} \right) \left( (v + \frac{e}{2}) \delta^e_b - c_2^e c_{2b} \right) = \tag{3.1}
\]

\[
= \left( (v + \frac{e}{2}) \delta^e_b - c_1^e c_{1e} \right) \left( (u + v + \frac{e}{2}) \delta^a_c - c_2^a c_{2b} \right) \mathcal{R}_{12}(u).
\]
We consider it as the defining relation for the wanted spinorial $R$ matrix and expand in powers of $v$. The terms proportional to $v^2$ are canceled, linear terms lead to the symmetry condition
\[
\mathcal{R}_{12}(u)(c_1^a c_{1b} + c_2^a c_{2b}) = (c_1^a c_{1b} + c_2^a c_{2b})\mathcal{R}_{12}(u),
\] (3.2)
from which one deduces that $\mathcal{R}_{12}(u)$ depends on $c_1$ and $c_2$ through the invariant combination
\[
z = -iz_{12} = -ic\epsilon_c^a \varepsilon_{ab} c^b_2 = -ic\epsilon c_2^a.
\] (3.3)
The last relation from $v^0$ has the form
\[
\mathcal{R}_{12}(u)(-uc_2^a c_{2b} + c_1^a z_{12} c_{2b}) = (-uc_1^a c_{1b} + c_2^a z_{12} c_{2b})\mathcal{R}_{12}(u).
\] (3.4)
Introducing $c^a_\pm = c_1 \pm ic_2$, we have the important relation for the following calculation
\[
ze^{a}_\pm = c^{a}_\pm (z \pm 1).
\] (3.5)
Consider first the symmetry condition (3.2). Multiplying it by $c^a_\pm \varepsilon_{da}$ from the left and by $c^b_\pm$ from the right one obtains
\[
\mathcal{R}_{12}(u|z \mp 1)A = A\mathcal{R}_{12}(u|z \pm 1),
\]
where
\[
A = c^a_\pm \varepsilon_{da}(c_1^a c_{1b} + c_2^a c_{2b})\varepsilon_{eb} c^b_\pm = (\epsilon \frac{d}{2} \pm z)(\frac{d}{2} \mp \varepsilon z) + (\pm i\epsilon \frac{d}{2} + iz)(\pm i \frac{d}{2} - i\varepsilon z) = 0,
\]
here $d \equiv \varepsilon_{ba} e^{ab} = \delta^a_b$.

For the analogous projection with the opposite choice of signs one has
\[
\mathcal{R}_{12}(u|z \mp 1)B = B\mathcal{R}_{12}(u|z \mp 1),
\]
where
\[
B = c^a_\pm \varepsilon_{da}(c_1^a c_{1b} + c_2^a c_{2b})\varepsilon_{eb} c^b_\mp = (\epsilon \frac{d}{2} \pm z)(\frac{d}{2} \pm \varepsilon z) + (\pm i\epsilon \frac{d}{2} + iz)(\pm i \frac{d}{2} - i\varepsilon z) = 2\epsilon \frac{d}{2} \pm \varepsilon z)^2.
\]
The analogous consideration of the defining relation (3.4) leads to the following equation for the opposite signs,
\[
\mathcal{R}_{12}(u|z \mp 1)C_L = C_R\mathcal{R}_{12}(u|z \mp 1),
\]
where
\[
C_L = c^a_\pm \varepsilon_{da}(-uc_2^b c_{2b} + c_1^a i\epsilon z c_{2b})\varepsilon_{eb} c^b_\pm =
\]
\[
= -u(\pm i \frac{d}{2} + iz)(\mp i \frac{d}{2} - i\varepsilon z) + (\epsilon \frac{d}{2} \pm z)i\epsilon z(\mp i \frac{d}{2} - i\varepsilon z) = (-u\epsilon \pm z)(\frac{d}{2} \pm \varepsilon z)^2,
\]
\[
C_R = c^a_\pm \varepsilon_{da}(-uc_2^b c_{2b} + c_2^a z_{12} c_{2b})\varepsilon_{eb} c^b_\pm =
\]
\[
= -u(\epsilon \frac{d}{2} \pm z)(\frac{d}{2} \pm \varepsilon z) + (\epsilon \frac{d}{2} \pm z)i\epsilon z(\mp i \frac{d}{2} - i\varepsilon z) = (-u\epsilon \pm z)(\frac{d}{2} \pm \varepsilon z)^2.
\]
The projection of (3.4) with the same signs leads to
\[
\mathcal{R}_{12}(u|z \mp 1)D_L = D_R\mathcal{R}_{12}(u|z \mp 1),
\]
where
\[
D_k = c^d_\pm \epsilon_{da} (-uc^2_1 \partial^b_2 + c^i_1 i\epsilon z b^b_2) \epsilon_{eb} c^e_\pm = \\
= -u(\pm i\epsilon \frac{d}{2} + iz)(\pm i\epsilon \frac{d}{2} - i\epsilon z) + (\pm i \frac{d}{2} \pm z)i\epsilon z(\pm i \frac{d}{2} - i\epsilon z) = (ue \mp z)(\frac{d^2}{4} - z^2),
\]
\[
D_R = c^d_\pm \epsilon_{da} (-uc^1_0 b^b_2 + c^z_1 z_1 b^2_2) \epsilon_{eb} c^e_\pm = \\
= -u(\epsilon \frac{d}{2} \pm z)(\frac{d}{2} \mp \epsilon z) + (\epsilon \frac{d}{2} \pm z)i\epsilon z(\pm \frac{d}{2} - i\epsilon z) = (-ue \pm z)(\frac{d^2}{4} - z^2).
\]
Canceling the common factor $\frac{d^2}{4} - z^2$ in both sides one obtains
\[
\mathcal{R}_{12}(u|z \mp 1)(ue \mp z) = (-ue \mp z)\mathcal{R}_{12}(u|z \pm 1),
\]
from which one deduces the wanted spinorial $R$ operator
\[
\mathcal{R}_{12}(u|z) = r(u) \Gamma(\frac{1}{2} (z + 1 - ue)) \Gamma(\frac{1}{2} (z + 1 + ue)).
\]

\section{3.1 Comparison with the Shankar-Witten form}

Comparing to the previous treatments \[16\] \[9\] \[17\] we have a simpler line of arguments and a compact general form of the spinorial $R$ operator.

The direct comparison of the two expressions for spinorial $R$-operator seems difficult due to the complicated connection between the invariants $I_k = \frac{1}{k!} \epsilon_{a_1 b_1} \cdots \epsilon_{a_k b_k} c_{1}^{a_1} \cdots c_{1}^{a_k}$. $c_{2}^{b_1} \cdots c_{2}^{b_k}$ and $z$.

\[
I_{k+1} = z I_k - \frac{k}{4} (d - \epsilon (k - 1)) I_{k-1}, \quad I_0(z,d) = 1, \quad I_1(z,d) = z.
\] (3.8)

Again the difference between orthogonal and symplectic cases consists in the presence of the sign factor $\epsilon$. In the orthogonal case $so(d)$ ($\epsilon = +1$) the series in $I_k$ terminates at $k = d + 1$, while for $sp(d)$ ($\epsilon = -1$) the series does not terminate.

For a proof of the recurrence relation (3.8) we refer to [17], where the generating function method is used for the calculation of (anti-) symmetrized products of $c^a$. It follows the same line as for eq. (5.6) of that paper:

\[
z I_m = c_{1}^{a_1} \cdots c_{1}^{a_m} c_{2}^{a_1} \cdots c_{2}^{a_m} a^2 c^a = \partial_{\kappa_1}^{a_1} \cdots \partial_{\kappa_m}^{a_m} \partial_{\kappa_{2}}^{a_1} \cdots \partial_{\kappa_{2}}^{a_m} (\partial^a_{\kappa_1} + \frac{1}{2} \kappa^a_{1}) (\partial^a_{\kappa_2} + \frac{1}{2} \kappa^a_{2}) e^{\kappa_1 \cdot c_1 + \kappa_2 \cdot c_2} |_{\kappa_i = 0} =
\] \[= I_{m+1} + \frac{m}{4} (d - (m - 1)\epsilon) I_{m-1}.
\]

\section{3.2 Symplectic cases of low rank}

It is instructive to specify the general solution for spinorial $R$-matrix for symplectic algebras of low rank. For $sp(2)$ one can realize the algebra $\mathcal{C}$ in terms of a pair of operators of multiplication and differentiation as

\[
c_1 = \partial = c^{-1}, \quad c_{-1} = x = -c, \quad z = -i(x_1 \partial_2 - x_2 \partial_1).
\]

Thus we have
\[
\mathcal{R}_{12}^{sp(2)}(u) = r(u) \Gamma(\frac{u+1}{2} - \frac{i}{2} (x_1 \partial_2 - x_2 \partial_1)) \Gamma\left(\frac{1+u}{2} - \frac{i}{2} (x_1 \partial_2 - x_2 \partial_1)\right).
\]
The \( sp(4) \) case corresponds to two such pairs.
\[
c^1 = c^\dagger = x, \quad c^2 = d^\dagger = y, \quad c^{-1} = \partial_x, \quad c^{-2} = \partial_y
\]
\[
z = -i(x_1 \partial_{x_2} - x_2 \partial_{x_1} + y_1 \partial_{y_2} - y_2 \partial_{y_1}).
\]
Thus we have
\[
\hat{R}^{sp(4)}_{12}(u) = r(u) \frac{\Gamma\left(\frac{n+1}{2} - \frac{i}{2}(x_1 \partial_{x_2} - x_2 \partial_{x_1} - \frac{i}{2}(y_1 \partial_{y_2} - y_2 \partial_{y_1}))\right)}{\Gamma\left(\frac{1-u}{2} - \frac{i}{2}(x_1 \partial_{x_2} - x_2 \partial_{x_1} - \frac{i}{2}(y_1 \partial_{y_2} - y_2 \partial_{y_1}))\right)}.
\]

### 3.3 The orthogonal case

In the orthogonal case, due to the Clifford relation for \( c^a \) (Dirac gamma matrices), the general expression for the spinor-spinor \( R \)-matrix in form of the Euler Beta-function can be transformed to a polynomial in \( z \) as well as to a finite expansion in the invariants
\[
I_k = \gamma^{a_1 \ldots a_k} \gamma^{a_1 \ldots a_k}, \quad k = 0, 1, \ldots n. \tag{3.9}
\]
In this case it is convenient to modify slightly the definition of the invariant \( z \):
\[
z = \frac{1}{2} \gamma_1^a \gamma_2^a. \tag{3.10}
\]
Besides of the absence of the imaginary unit in this definition, we suppose here that the Dirac gamma matrices \( \gamma_1^a \) and \( \gamma_2^a \) related to different spaces commute, in contrast to the previous convention where we have supposed their anticommutation in order to have the unified description with the symplectic case.

Having modified the definition of \( z \) we go through the steps of the above derivation in order to check that the the result is not changed.

Consider the RLL-relation
\[
\hat{R}_{12}(u|z)\left((v + \frac{1}{2})\gamma_1^a \gamma_1 b + (u + v + \frac{1}{2})\gamma_2^a \gamma_2 b - \gamma_1^0 z \gamma_2 b\right) = \left((v + \frac{1}{2})\gamma_1^a \gamma_1 b + (u + v + \frac{1}{2})\gamma_2^a \gamma_2 b - \gamma_1^0 z \gamma_2 b\right) \hat{R}_{12}(u|z).
\]
We multiply by \( \gamma_1^a = \gamma_1^a \pm \gamma_2^a \) from the left and by \( \gamma_1^b \) or \( \gamma_2^b \) from the right and use
\[
z \gamma_1^a = \gamma_1^a (-z \pm 1). \tag{3.11}
\]
Note that here an additional minus sign appears due the change in the commutativity convention. We obtain
\[
\hat{R}_{12}(u|z) = (u + 2v + 1 \mp z)(d^2 \pm 2z)^2 = (u + 2v + 1 \mp z)(d^2 \pm 2z)^2 \hat{R}_{12}(u|z) = (u + 2v + 1 \mp z)(d^2 \pm 2z)^2 \hat{R}_{12}(u|z) - z \mp 1),
\]
and
\[
\hat{R}_{12}(u|z) = (u + 2v + 1 \mp z)(d^2 - 4z^2) = (u + 2v + 1 \mp z)(d^2 - 4z^2) \hat{R}_{12}(u|z) - z \mp 1),
\]
from which we deduce the general solution for the \( so(n) \) case:
\[
\frac{\hat{R}_{12}(u|z)}{\hat{R}_0(u)} = \frac{\Gamma\left(\frac{n+1}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{n+1}{2} + \frac{u}{2}\right)} = B\left(\frac{u}{2} + 1, u\right). \tag{3.12}
\]
4 Spinor and vector monodromy matrices

We show how the monodromy matrices with spinor auxiliary space \(S_0\) and vector auxiliary space \(V_0\) are related by fusion. The gamma matrices are known to intertwine the vector and the product of spinor representations. Formally, the following analysis can be extended to the symplectic case. However, that case is connected with the problem of defining infinite sums over the spinor indices. We restrict ourselves to the orthogonal case, which will be considered in the remaining part of the paper.

The general vector-vector monodromy matrix is defined by the fundamental \(R\) matrix \((2.2)\):

\[
\Pi_{a_0,a_1...a_N}^{b_0,b_1...b_N}(u) = R_{a'_1 b_1}^{a_0 a_1}(u) \cdots R_{b_{bN}}^{a_{N-1} a_N}(u).
\]

We are interested in the diagonalization of the trace of this matrix,

\[
t^{a_1...a_N}_{b_1...b_N}(u) = \Pi_{a_0,a_1...a_N}^{b_0,b_1...b_N}(u),
\]

or \(t(u) = tr_0 \Pi(u)\), because it is the generating function of the integrals of motion of a periodic quantum spin chain.

Using the \(L\)-operator \((2.5)\) one can construct another monodromy matrix

\[
T^{a_1...a_N}_{\beta,b_1...b_N}(u) = (L_{a_1}^{a_0}(u) \cdots L_{a_N}^{a_0}(u))^{\alpha}_{\beta},
\]

acting in the tensor product of the same quantum space \(V_1 \otimes \cdots \otimes V_N\) and the spinor auxiliary space \(S_0\).

We consider also the related monodromy matrix

\[
\tilde{T}^{\gamma,c_1...c_N}_{\delta,b_1...b_N}(-u - \frac{1}{2}) = \left( L^{c_N}_{b_N}(-u - \frac{1}{2})L^{c_{N-1}}_{b_{N-1}}(-u - \frac{1}{2}) \cdots L^{c_1}_{b_1}(-u - \frac{1}{2}) \right)^{\gamma}_{\delta}.
\]

Because of the inversion relation,

\[
L_{b_\beta}^{a_\alpha}(u + \beta + \frac{1}{2})L_{c_\beta}^{b_\alpha}(-u - \frac{1}{2}) = -u(u + \beta + 1)\delta_{c_\beta}^{a_\alpha},
\]

it can be expressed by the inverse of the previous monodromy matrix

\[
\tilde{T}^{\gamma,c_1...c_N}_{\delta,b_1...b_N}(-u - \frac{1}{2}) = \left( -u(u + \beta + 1) \right)^N \left( T^{-1}(u + \beta + \frac{1}{2}) \right)^{\gamma,c_1...c_N}_{\delta,b_1...b_N}.
\]

It is not hard to check that the fusion of two conjugated \(L\)-operators \((2.5)\) gives the vector \(R\)-matrix \((2.2)\). Indeed,

\[
L_{c_1\beta}^{a_\alpha}(u + \beta + \frac{1}{2})(\gamma_{b_0})^{\beta}_{\gamma} L_{b_\delta}^{c_1}(\frac{1}{2} - u) = - \left( u(u + \beta)\delta_{b_1}^{a_1} \delta_{b_0}^{\alpha_0} + (u + \beta)\delta_{b_0}^{a_1} \delta_{b_1}^{\alpha_0} - u\delta_{b_1}^{a_1} \delta_{b_0} \right)(\gamma_{a_0})^{\alpha}_{\beta} =
\]

\[
= -P_{b_0b_1}^{a_0a_1}(u)(\gamma_{a_0})^{\alpha}_{\beta}.
\]

We have used that in the orthogonal case \((2.5)\) takes the form

\[
L^{ab}(u) = u_{ab}^{\gamma} - \frac{1}{2} \gamma^{ab}.
\]

Consider the product

\[
T^{a_1...a_N}_{\beta,c_1...c_N}(u + \beta - \frac{1}{2})(\gamma_{b_0})^{\beta}_{\gamma} \tilde{T}^{\gamma,c_1...c_N}_{\delta,b_1...b_N}(-u - \frac{1}{2}) =
\]
\[ L_{a_1,a_1}^\alpha(u + \beta - \frac{1}{2})L_{a_2,a_2}^\beta(u + \beta - \frac{1}{2}) \ldots L_{a_N,a_N}^\gamma(u + \beta - \frac{1}{2}) \times \]

\[ \times (\gamma_0)_{\gamma}^\delta L_{\gamma,N-1,b_N}^{\gamma,N-1}(u - \frac{1}{2} - u)L_{\gamma,N-2,b_{N-1}}^{\gamma,N-2}(u - \frac{1}{2} - u) \ldots L_{\gamma,1,c_1}^\gamma(1 - u) \]

We pick the last \( L \) factor in \( T \) and the first in \( \bar{T} \) and use (4.5) to continue the calculation

\[ = L_{a_1,a_1}^\alpha(u + \beta - \frac{1}{2})L_{a_2,a_2}^\beta(u + \beta - \frac{1}{2}) \ldots L_{a_N-2,a_N-1}^{\alpha-2,a_N-1}(u + \beta - \frac{1}{2})(\gamma_0)^\delta \times \] (4.7)

\[ \times (L_{a_1,a_1}^\alpha(u + \beta - \frac{1}{2})(\gamma_0)^\delta L_{\gamma,N-1,b_N}^{\gamma,N-1}(u - \frac{1}{2} - u)) \times \]

\[ \times L_{\gamma,N-2,b_{N-1}}^{\gamma,N-2}(u - \frac{1}{2} - u) \ldots L_{\gamma,1,c_1}^\gamma(1 - u) = \]

\[ = L_{a_1,a_1}^\alpha(u + \beta - \frac{1}{2})L_{a_2,a_2}^\beta(u + \beta - \frac{1}{2}) \ldots L_{a_N-2,a_N-1}^{\alpha-2,a_N-1}(u + \beta - \frac{1}{2}) \times \]

\[ \times (-R_{b_0b_N}^{\alpha_0}(u)(\gamma_0)^\delta) \times \]

\[ \times L_{\gamma,N-2,b_{N-1}}^{\gamma,N-2}(u - \frac{1}{2} - u) \ldots L_{\gamma,1,c_1}^\gamma(1 - u) = \]

Proceeding with the next factors in \( T \) and \( \bar{T} \) we obtain finally

\[ = (-1)^N \prod_{b_0,b_1 \ldots b_N}^\alpha (u)(\gamma_0)^\delta. \]

In this way we obtain the fusion relation between spinorial and vector monodromy matrices. The fusion of two spinorial monodromy matrices by trace over the auxiliary spinor space \( S_0 \) results in the vector monodromy matrix with \( V_0 \) as auxiliary space.

\[ T_{\beta \gamma}^{\alpha_1 \ldots \alpha_N}(u + \beta - \frac{1}{2})(\gamma_0)^\delta(T^{-1})^{\gamma_1 \ldots \gamma_N}_{\delta \beta \gamma}(u + \beta + \frac{1}{2}) = \prod_{b_0,b_1 \ldots b_N}^\alpha (u)(\gamma_0)^\delta. \] (4.8)

In general by this relation the spinorial \( RTT \) algebra, the generators of which are contained in the matrix \( T \), is mapped to the ordinary \( RTT \) algebra, the generators of which are matrix elements of \( T \). Moreover, it provides a way to solve the spectral problem for the trace of the ordinary monodromy matrix by solving the spectral problem for a trace involving the spinorial monodromy matrix.

The entries of the inverse-transpose \( \bar{T}^{-1}_{ij}(u) = \bar{T}_{N+1-j,N+1-i}^{-1}(u) \) to the monodromy matrix \( \bar{T} \) defined over the fundamental (vector) auxiliary space used in [10] is given by the quantum minors divided by the quantum determinant. In contrast, due to the inversion relation for \( L \) (1.3) the inverse of spinor monodromy matrix \( T(u) \) is given by the same matrix with the shifted spectral parameter.

5 Even-dimensional orthogonal algebras

In sect. 3 we have obtained the spinorial \( \tilde{R} \)-matrix in form of the Euler Beta-function for an arbitrary orthogonal algebra. Now we shall consider low rank examples corresponding to the \( D \) series. The universal expression (3.12) results in explicit forms using the corresponding characteristic polynomial in the invariant \( z \) and the resulting spectral decomposition. Details about the invariant \( z \) and relations following from its characteristic polynomial are considered in the Appendix.
5.1 The $so(4)$ case

In this case we have the characteristic polynomial
\[ W_4 = z(z^2 - 1)(z^2 - 4) = 0. \] (5.1)

In other words, in the case $d = 4$ any function of $z$ is represented by a polynomial of fourth degree. The roots of $W_4$ are the eigenvalues of $z$ and we are lead to the spectral decomposition
\[ R^{so(4)}(z|u) = B\left(\frac{z + 1 - u}{2}, u\right) = \sum_{k=-2}^{2} B\left(\frac{k + 1 - u}{2}, u\right) P_k, \] (5.2)

Here the sum goes over the roots of $W_4$: $z_k = 0, \pm 1, \pm 2$ and $P_k$ are projection operators on the corresponding eigenspace,
\[ P_0 = \frac{1}{4}(z^2 - 1)(z^2 - 4), \quad P_{\pm 1} = -\frac{1}{3!}(z \pm 1)(z^2 - 4), \quad P_{\pm 2} = \frac{1}{4!}(z^2 - 1)(z \pm 2). \] (5.3)

Using (5.1) one can check the properties $P_k P_k = P_k \delta_{kt}$, $\sum_{k=-2}^{2} P_k = 1$. Further, using the functional equation $B(x + 1, y) = \frac{x}{x+y} B(x, y)$, one deduces the explicit form of the spinorial $R$ matrix
\[ R^{so(4)}(z|u) = B\left(\frac{1}{2} - \frac{u}{2}, u\right) \left(1 - u\right) (P_2 + P_{-2}) + B\left(\frac{1}{2} - \frac{u}{2}, u\right) (P_1 - P_{-1}) = R^{(1)}_{so(4)}(u|z) + R^{(2)}_{so(4)}(u|z). \]

It separates into two parts given by even and odd functions of $z$, respectively. Both parts satisfy the $RRR$ Yang-Baxter relation separately.

$z$ is acting on $S_1 \otimes S_2$ and the permutation operator is given by
\[ P_{12} = P_0 + P_1 - P_{-1} - P_2 + P_{-2} = \frac{1}{6}(z^4 - 2z^3 - 7z^2 + 8z + 6) = \] (5.4)
\[ = \frac{1}{4} \left( I_{12} + \frac{1}{(1!)^2} \gamma_1^a \gamma_2^a - \frac{1}{(2!)^2} \gamma_1^{ab} \gamma_2^{ab} - \frac{1}{(3!)^2} \gamma_1^{abc} \gamma_2^{abc} + \frac{1}{(4!)^2} \gamma_1^{abcd} \gamma_2^{abcd} \right). \]

Consider now both parts of the spinorial $R$ matrix in more detail. The simpler part
\[ R^{(1)}_{so(4)}(u|z) = -P_1 + P_{-1} = \frac{z}{3}(z^2 - 4) = -\frac{1}{144} \gamma_1^{abc} c_2^{abc} + \frac{3}{4} \gamma_1^{a} \gamma_2^{a}, \]
is given by an odd function of $z$. The corresponding $R$-matrix without check, $R^{(1)} = P_{12} R^{(1)}$, is even in $z$,
\[ R^{(1)}_{so(4)}(u|z) = -P_1 - P_{-1} = \frac{z^2(z^2 - 4)}{3} = \frac{1}{2} (1 + \gamma_1^5 \gamma_2^5), \quad \gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4, \]
and is trivial, i.e. diagonal and independent of the spectral parameter, with the following non-vanishing entries:
\[ R^{12}_{12} = R^{13}_{13} = R^{21}_{21} = R^{23}_{23} = R^{31}_{31} = R^{32}_{32} = R^{42}_{42} = R^{43}_{43} = 1. \] (5.5)
The RLL Yang-Baxter relation
\[ R_{\text{so}(4)}^{(1)} L_1^{ab}(u) L_1^{bc}(v) = L_2^{ab}(v) L_2^{bc}(u) R_{\text{so}(4)}^{(1)} \]
reduces here to the identity
\[ (1 + \gamma_1^5 \gamma_2^5) \gamma_1^{ab} \gamma_2^{bc} = \gamma_2^{ab} \gamma_1^{bc} (1 + \gamma_1^5 \gamma_2^5). \]
We have also the Yang-Baxter RRR relation in the trivial form
\[ (1 + \gamma_1^5 \gamma_2^5)(1 + \gamma_1^5 \gamma_3^5)(1 + \gamma_2^5 \gamma_3^5) = (1 + \gamma_2^5 \gamma_3^5)(1 + \gamma_1^5 \gamma_3^5)(1 + \gamma_1^5 \gamma_2^5). \]
The part of the spinorial spinorial R matrix even in \( z \)
\[ R_{\text{so}(4)}^{(2)}(u|z) = \frac{u + 2}{4}(1 + \gamma_1^5 \gamma_2^5) - \frac{u}{16} \gamma_1^{ab} \gamma_2^{ab}, \]
corresponds to the \( R \)-matrix without check also even in \( z \),
\[ R_{\text{so}(4)}^{(2)}(u|z) = (u - 1)(P_2 + P_{-2}) + (1 + u)P_0 = \frac{u - 1}{12} z^2(z^2 - 1) + \frac{u + 1}{4}(z^2 - 1)(z^2 - 4) =
\[ = (u + \frac{1}{2}) \frac{1}{3}(z^4 - 4z^2 + 3) - \frac{1}{2}(z^2 - 1) = \frac{2u + 1}{4}(1 + \gamma_1^5 \gamma_2^5) - \frac{1}{16} \gamma_1^{ab} \gamma_2^{ab}, \]
The latter is represented by the matrix with the following non-vanishing entries:
\[ R_{11}^{11}(u) = R_{22}^{22}(u) = R_{33}^{33}(u) = R_{44}^{44}(u) = u + 1, \]
\[ R_{14}^{14}(u) = R_{23}^{23}(u) = R_{32}^{32}(u) = R_{41}^{41}(u) = u, \]
\[ R_{12}^{12}(u) = R_{21}^{21}(u) = 1 = R_{34}^{34}(u) = R_{43}^{43}(u). \]
Thus this matrix has the block form: its entries \( R_{\alpha\beta}^{\gamma\delta}(u) \) differ from zero only if \( \alpha, \beta, \gamma \) and \( \delta \) belong to sets \( 1, 4 \) or \( 2, 3 \).

It is important to notice that the two different parts \( R_{\text{so}(4)}^{(1)}(u) \) and \( R_{\text{so}(4)}^{(2)}(u) \), which are odd and even functions of \( z \) correspondingly, are distinguished by chirality. Indeed these solutions are proportional to the chiral projectors \( \Pi_+ \) and \( \Pi_- \), respectively, where
\[ \Pi_{\pm} = \frac{1}{2}(1 \pm \gamma_1^5 \gamma_2^5). \]
One deduces from (3.8) the useful formulae
\[ \Pi_+ = -\frac{1}{3} z^2(z^2 - 4), \quad \Pi_- = \frac{1}{3}(z^2 - 1)(z^2 - 3). \]
This chiral property ensures the consistency of these solutions: both of them intertwine a pair of \( L \)-operators i.e. obey the RLL-relation linear in \( R \), but also satisfy the trilinear RRR-relation not only separately, but also in arbitrary combination, due to their orthogonality.

The chiral projectors \( \Pi_{\pm} \) separate the 16-dimensional representation space \( S_1 \otimes S_2 \) into two eight-dimensional chiral subspaces. In particular, the subspace corresponding to \( \Pi_+ \) is spanned by eight eigenvectors of \( z \) corresponding to the eigenvalues \( \pm 1 \), while the six eigenvectors, corresponding to the zero eigenvalue of \( z \) as well as vectors corresponding
to the eigenvalues ±2 span the other chiral subspace. In terms of the projectors of the eigenspaces this reads as

\[ \Pi_+ P_{±1} = P_{±1} \Pi_+ = P_{±1}, \quad \Pi_− P_{±1} = 0 = P_{±1} \Pi_−, \]
\[ \Pi_− P_0 = P_0 \Pi_− = P_0, \quad \Pi_− P_{±2} = P_{±2} \Pi_− = P_{±2}, \]
\[ \Pi_+ P_0 = 0 = P_0 \Pi_+, \quad \Pi_+ P_{±2} = 0 = P_{±2} \Pi_+ . \]

Note that due to the definite chirality of \( \mathcal{R}^{(1)}_{so(4)}(u) \) and \( \mathcal{R}^{(2)}_{so(4)}(u) \) the expressions like \( tr_2(\mathcal{R}^{(1)}_{so(4)}(u)\gamma_2^a\mathcal{R}^{(1)}_{so(4)}(v)\gamma_2^b) \) and \( tr_2(\mathcal{R}^{(2)}_{so(4)}(u)\gamma_2^a\mathcal{R}^{(2)}_{so(4)}(v)\gamma_2^b) \) vanish, but the non-diagonal expression \( tr_2(\mathcal{R}^{(1)}_{so(4)}(u)\gamma_2^a\mathcal{R}^{(2)}_{so(4)}(v)\gamma_2^b) \) does not vanish and at \( 2v + 1 = u \) results in the fusion to the \( L \) matrix (2.5),

\[ tr_2(\mathcal{R}^{(1)}_{so(4)}(2v + 1)\gamma_2^a\mathcal{R}^{(2)}_{so(4)}(v)\gamma_2^b) = (2v + 1)\delta^{ab} - \frac{1}{2}e^{ab}_1 = L^{ab}_1(2v + 1). \]

5.2 **ABA for the** \( so(4) \) **case**

The observed simple structure of the spinorial \( \mathcal{R}^{so(4)} \) matrix is helpful for the diagonalization of the trace of the spinorial monodromy matrix

\[ T^{(2)}(u) = \mathcal{R}^{(2)}_{01}(u)\mathcal{R}^{(2)}_{02}(u) \ldots \mathcal{R}^{(2)}_{0N}(u), \quad (5.7) \]

or in components

\[ T_{\beta_0,\beta_1 \ldots \beta_N}^{\alpha_0,\alpha_1 \ldots \alpha_N}(u) = \mathcal{R}_{\gamma_0,\gamma_1}^{\alpha_0,\alpha_1}(u) \ldots \mathcal{R}_{\beta_0,\beta_N}^{\gamma_0,\gamma_N}(u). \]

The explicit form of the matrix \( \mathcal{R}^{(2)}_{0k}(u) \) (5.6) results in the the following representation as a 4 × 4 matrix in the auxiliary space \( \mathcal{S}_0 \) with operator valued elements acting in the quantum space \( \mathcal{S}_k \)

\[ \mathcal{R}^{(2)}_{0k}(u) = \begin{pmatrix} \mathcal{R}_1^1(u) & 0 & 0 & \mathcal{R}_1^2(u) \\ 0 & \mathcal{R}_2^1(u) & \mathcal{R}_2^2(u) & 0 \\ 0 & \mathcal{R}_3^1(u) & \mathcal{R}_3^2(u) & 0 \\ \mathcal{R}_4^1(u) & 0 & 0 & \mathcal{R}_4^2(u) \\ \end{pmatrix}. \quad (5.8) \]

Its diagonal 4 × 4 matrix components are diagonal

\[ \mathcal{R}_1^1(u) = diag(u + 1, 0, 0, u), \quad \mathcal{R}_2^2(u) = diag(0, u + 1, u, 0), \]
\[ \mathcal{R}_3^3(u) = diag(0, u, u + 1, 0), \quad \mathcal{R}_4^4(u) = diag(u, 0, 0, u + 1), \]

and have to be regarded as the elements of the Cartan subalgebra. The off-diagonal elements have to be regarded as lowering

\[ \mathcal{R}_4^1(u) = e_{41}, \quad \mathcal{R}_5^2(u) = e_{32}, \]

and rising

\[ \mathcal{R}_2^3(u) = e_{23}, \quad \mathcal{R}_1^4(u) = e_{14}, \]

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generators correspondingly. Consequently, the monodromy matrix \( T(2)(u) \) defined as an ordered matrix product of the factors \( (5.2) \) preserves the block form,

\[
T(2)(u) = \begin{pmatrix}
T_1^1(u) & 0 & 0 & T_4^1(u) \\
0 & T_2^2(u) & T_2^3(u) & 0 \\
0 & T_2^3(u) & T_4^3(u) & 0 \\
T_4^4(u) & 0 & 0 & T_4^4(u)
\end{pmatrix},
\]

(5.9)

with Cartan elements \( T^\alpha(u) \) and rising \( C_1(u) = T_1^1(u), C_2(u) = T_3^3(u) \) and lowering \( B_2(u) = T_2^2(u), B_1(u) = T_4^4(u) \) generators. We have obtained a representation of the spinorial RTT algebra and its decomposition into a representation of two subalgebras of the \( \mathfrak{sl}(2) \) type Yangian.

This becomes more evident by a similarity transformation with the \( 4 \times 4 \) matrix \( V \)

\[
V = e_{11} + e_{24} + e_{33} + e_{42}, \quad V^{-1} = V.
\]

(5.10)

One calculates easily

\[
VT(2)(u)V = \begin{pmatrix}
T_1^1(u) & T_4^1(u) & 0 & 0 \\
T_4^4(u) & T_4^1(u) & 0 & 0 \\
0 & 0 & T_3^3(u) & T_2^3(u) \\
0 & 0 & T_3^3(u) & T_2^3(u)
\end{pmatrix},
\]

(5.11)

and

\[
(V \otimes V)R(2)(u)(V \otimes V) = \begin{pmatrix}
R^I & 0 \\
0 & R^{II}
\end{pmatrix},
\]

(5.12)

with the \( 8 \times 8 \) block-matrices

\[
R^I(u) = \begin{pmatrix}
R_{11}^I(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_{11}^I(u) & 0 & 0 & R_{41}^I(u) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_{14}^I(u) & 0 & 0 & R_{41}^I(u) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

(5.13)

\[
R^{II}(u) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R_{33}^{II}(u) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & R_{32}^{II}(u) & 0 & 0 & R_{33}^{II}(u) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{22}^{II}(u)
\end{pmatrix}.
\]

(5.14)

Disregarding in these blocks the lines and columns with zero entries, we recognize in each of them the fundamental \( R \) matrix of \( \mathfrak{sl}(2) \).

Consider now the general RTT relation substituting for \( T \) instead of the particular monodromy matrix \( (5.7) \) a generic algebra valued \( 4 \times 4 \) matrix. Whereas in the \( \mathfrak{sl}(4) \) case the corresponding relation results in the first step in 256 non-trivial relations, here the
relation with the $so(4)$ spinorial $R$ matrix [5.6] leads to 64 non-trivial relations from the matrix elements with the indices belonging to sets $(1,4)$ and $(2,3)$, to further relations of the form $0 = 0$ and , most interesting, to such with non-trivial r.h.s. and zero l.h.s or vice versa. For example,

$$R_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}(u - v) T_{\gamma_1}^{b_1} (u) T_{\gamma_2}^{b_2} (v) = T_{\beta_2}^{a_2} (v) T_{\beta_1}^{a_1} (u) R_{\gamma_1 \gamma_2}^{\beta_1 \beta_2}(u - v),$$

(5.13)

for $(\alpha_1, \alpha_2) = (1,1)$, $(\gamma_1, \gamma_2) = (1,2)$ gives

$$T_1^1(u) T_1^1(v) = 0.$$

Combining all these relations one deduces, that the general $so(4)$ spinor monodromy matrix has the form [5.9], i.e.

$$T_2^1 (u) = T_3^1 (u) = T_4^1 (u) = T_1^2 (u) = T_4^3 (u) = 0,$$

and

$$T_2^1 (u) = T_3^1 (u) = T_1^4 (u) = T_2^4 (u) = T_3^4 (u) = 0.$$

Thus also in the case of arbitrary representations in the quantum space the $so(4)$ monodromy matrix generates an algebra equivalent to two independent $s\ell(2)$ Yangian algebras and the spectral problem for its trace leads to the ordinary $s\ell(2)$ ABA.

In particular, this applies to the spinor-vector monodromy matrix [4.3]. Recall, that it produces the $so(4)$ vector-vector monodromy matrix by fusion [4.8],

$$T_{a_0, a_1, \ldots, a_N}^{a_1, \ldots, a_N}(u) = \frac{1}{4} \text{tr}\left(T_{a_1, \ldots, a_N}^{a_1, \ldots, a_N}(u + \frac{1}{2}) \gamma_0^{a_0} T_{b_1, \ldots, b_N}^{-1}(u + \frac{3}{2}) \gamma_{a_0}\right).$$

(5.14)

We have to substitute $\beta = 1$ for $so(4)$.

Taking into account that $T$ and $\bar{T} = T^{-1}$ have the form [5.9], one can calculate explicitly the matrix elements of the vector monodromy in terms of products of the matrix elements of the spinor monodromy. In particular for the trace we obtain

$$T_{a_0}^{a_0}(u) = \frac{1}{2} \left((T_1^1 + T_4^1) (T_2^4 + T_3^4) + (T_2^2 + T_3^2) (T_1^1 + T_4^4)\right),$$

(5.15)

where we have abbreviated $T_{\alpha}^{\alpha}(u + \frac{1}{2}) = \left(L_1(u + \frac{1}{2}) \ldots L_N(u + \frac{1}{2})\right)$ and $\bar{T} = T^{-1}(u + \frac{3}{2}).$

The vector-vector $so(4)$ monodromy matrix is given by sums of products of the spinor-vector transfer matrix elements which obey $s\ell(2)$ type Yangian relations.

At the end of this subsection we consider the $RTT$ algebra generated by $\mathcal{R}^{(1)}_{so(4)}(u)$. Due to its chirality and the diagonal character it leads to a trivial solution for $T(u)$ of the RTT-relation. For instance, $(\alpha_1, \alpha_2) = (1,1)$, $(\gamma_1, \gamma_2) = (1,2)$ and $(\gamma_1, \gamma_2) = (2,1)$ by [5.11] leads to

$$T_1^1(v) T_1^1(u) = 0 = T_2^1(v) T_2^1(u).$$

The remaining RTT-relations imply that the most general monodromy matrix $T(u)$ intertwined by $\mathcal{R}^{(1)}_{so(4)}(u)$ is given by the diagonal matrix:

$$T(u) = \text{diag}(T_1^1(u), T_2^2(u), T_3^3(u), T_4^4(u))$$

which has vanishing rising and lowering generators.

We summarize the above results:
Proposition 1. The spinorial R matrix with so(4) symmetry \( \mathfrak{R}(2) \) (acting in the chiral subspace of the \( \Pi_{-} \) projection) generates the spinorial RTT algebra decomposing into two subalgebras of the \( \mathfrak{so}(2) \) Yangian type. The spinorial RTT algebra generated by \( \mathfrak{R}(1) \) (acting in the chiral subspace of the \( \Pi_{+} \) projection) is a trivial commuting algebra. In this way, the ordinary \( \mathfrak{so}(2) \) ABA allows to construct solutions of the ABA of for the spinorial Yangian of so(4) type.

5.3 The so(6) case

In this case \( z \) obeys

\[ W_6 = z(z^2 - 1)(z^2 - 4)(z^2 - 9) = 0, \]  

has seven different eigenvalues: \( 0, \pm 1, \pm 2, \pm 3 \) and the projectors on the corresponding eigenspaces are given by

\[
P_{\pm 3} = \frac{z(z^2 - 1)(z^2 - 4)(z^2 - 9)(z^2 \pm 3)}{6!}, \quad P_{\pm 2} = \frac{z(z^2 - 1)(z^2 - 9)(z^2 \pm 2)}{5!},
\]

\[
P_{\pm 1} = \frac{z(z^2 - 4)(z^2 - 9)(z^2 \pm 1)}{2 \cdot 4!}, \quad P_0 = \frac{(z^2 - 1)(z^2 - 4)(z^2 - 9)}{3!}.
\]

Further, the permutation operator has the spectral expansion

\[ \mathcal{P}_{12} = P_0 + P_1 - P_{-1} - P_2 - P_{-2} - P_3 + P_{-3}. \]  

According to (5.18) the \( R \)-operator decomposes into even and odd parts in \( z \)

\[
\mathfrak{R}_{so(6)}(z|u) = B\left(\frac{z + 1 - u}{2}, u\right) = \frac{B\left(- \frac{1}{2} - \frac{u}{2}, u\right)}{u - 1} \mathfrak{R}_{so(6)}^{(1)} + \frac{B\left(- 1 - \frac{u}{2}, u\right)}{u - 2} \mathfrak{R}_{so(6)}^{(2)} =
\]

\[
B\left(- \frac{1}{2} - \frac{u}{2}, u\right) \left((u - 1)(P_2 + P_{-2}) - (u + 1)P_0\right) +
\]

\[
B\left(- 1 - \frac{u}{2}, u\right) \left((u + 2)(P_1 + P_{-1}) + (u - 2)(P_{-3} - P_3)\right).
\]

According to (5.18) the \( R \)-matrices without check are

\[
\mathfrak{R}_{so(6)}^{(1)}(z|u) = -(u - 1)(P_2 + P_{-2}) - (u + 1)P_0,
\]

and

\[
\mathfrak{R}_{so(6)}^{(2)}(z|u) = (u + 2)(P_1 + P_{-1}) + (u - 2)(P_{-3} + P_3).
\]

We introduce in analogy to the \( so(4) \) case \( \gamma_1 \gamma_2 = \frac{4}{45}(z^6 - \frac{25}{2}z^4 + 34z^2 - \frac{45}{4}) \) and observe that the two parts of the spinorial \( R \) matrix are proportional to the chiral projectors \( \Pi_{\pm} \), respectively, where

\[
\Pi_{\pm} = \frac{1}{2}(1 \pm \gamma_1 \gamma_2) = \frac{1}{2} \pm \frac{2}{45} \left(z^6 - \frac{25}{2}z^4 + 34z^2 - \frac{45}{4}\right).
\]

Using the characteristic polynomial \( W_6 \) (5.16) one can prove easily the relations

\[
\Pi_{\pm} P_{\pm(2k+1)} = P_{\pm(2k+1)} P_{\pm}, \quad \Pi_{-} P_{\pm(2k+1)} = 0 = P_{\pm(2k+1)} \Pi_{-},
\]

\[
\Pi_{-} P_{\pm 2k} = P_{\pm 2k} = P_{\pm 2k} \Pi_{-}, \quad \Pi_{+} P_{\pm 2k} = 0 = P_{\pm 2k} \Pi_{+}, \quad k = 0, 1.
\]
Thus the general monodromy matrix has the block-form:

In this case we again deal with two spinor-spinor \( R \)-matrices \((5.20)\) and \((5.21)\) acting in the two chiral subspaces.

### 5.4.1 \( R^{(1)} \), the \( II_- \) part

According to \((5.20)\) this \( R \)-matrix has the following non-vanishing entries:

\[
\begin{align*}
R_{11}^{(1)}(u) &= R_{22}^{(1)}(u) = R_{33}^{(1)}(u) = R_{44}^{(1)}(u) = R_{55}^{(1)}(u) = R_{66}^{(1)}(u) = R_{77}^{(1)}(u) = R_{88}^{(1)}(u) = -6(u+1), \\
R_{14}^{(1)}(u) &= R_{16}^{(1)}(u) = R_{17}^{(1)}(u) = R_{23}^{(1)}(u) = R_{25}^{(1)}(u) = R_{28}^{(1)}(u) = R_{35}^{(1)}(u) = R_{38}^{(1)}(u) = R_{47}^{(1)}(u) = R_{45}^{(1)}(u) = R_{52}^{(1)}(u) = R_{53}^{(1)}(u) = R_{56}^{(1)}(u) = R_{61}^{(1)}(u) = R_{62}^{(1)}(u) = R_{63}^{(1)}(u) = R_{65}^{(1)}(u) = R_{66}^{(1)}(u) = R_{67}^{(1)}(u) = R_{71}^{(1)}(u) = R_{74}^{(1)}(u) = R_{76}^{(1)}(u) = R_{82}^{(1)}(u) = R_{83}^{(1)}(u) = R_{85}^{(1)}(u) = -\frac{1}{2}(5u+7), \\
R_{41}^{(1)}(u) &= R_{46}^{(1)}(u) = R_{47}^{(1)}(u) = R_{52}^{(1)}(u) = R_{58}^{(1)}(u) = R_{78}^{(1)}(u) = R_{38}^{(1)}(u) = R_{55}^{(1)}(u) = R_{62}^{(1)}(u) = R_{63}^{(1)}(u) = R_{65}^{(1)}(u) = R_{66}^{(1)}(u) = R_{67}^{(1)}(u) = R_{71}^{(1)}(u) = R_{74}^{(1)}(u) = R_{76}^{(1)}(u) = R_{82}^{(1)}(u) = R_{83}^{(1)}(u) = R_{85}^{(1)}(u) = -\frac{1}{2}(7u+5).
\end{align*}
\]

In analogy to the \( so(4) \) case, considering \((\alpha_1, \alpha_2) = (1, 1)\) and \((\gamma_1, \gamma_2) = (1, 2), (1, 3), (1, 5), (1, 8)\), in \((5.13)\), one obtains for the arbitrary \( 8 \times 8 \) monodromy matrix \( T(u) \):

\[
\begin{align*}
T_{11}^1(u)T_{22}^1(v) &= 0, \\
T_{11}^1(u)T_{33}^1(v) &= 0, \\
T_{11}^1(u)T_{55}^1(v) &= 0, \\
T_{11}^1(u)T_{88}^1(v) &= 0,
\end{align*}
\]

which have the solution

\[
T_{33}^1(v) = T_{55}^1(v) = T_{88}^1(v) = 0.
\]

Similarly, considering \((\alpha_1, \alpha_2) = (2, 2)\) and \((\gamma_1, \gamma_2) = (2, 1), (2, 4), (2, 6), (2, 7)\) leads to

\[
T_{11}^2(v) = T_{44}^2(v) = T_{66}^2(v) = T_{77}^2(v) = 0,
\]

and further to

\[
\begin{align*}
T_{11}^5(v) &= 0T_{22}^3(v) = T_{33}^3(v) = 0T_{44}^3(v) = 0, \\
T_{11}^5(v) &= T_{22}^5(v) = T_{33}^5(v) = 0T_{44}^5(v) = 0, \\
T_{11}^7(v) &= T_{22}^7(v) = T_{33}^7(v) = 0T_{44}^7(v) = 0, \\
T_{11}^8(v) &= T_{22}^8(v) = T_{33}^8(v) = 0T_{44}^8(v) = 0,
\end{align*}
\]

Thus the general monodromy matrix has the block-form:

\[
T(u) = \begin{pmatrix}
T_{11}^1(u) & 0 & T_{22}^1(u) & 0 & T_{33}^1(u) & 0 & T_{44}^1(u) & 0 & T_{55}^1(u) & 0 & T_{66}^1(u) & 0 & T_{77}^1(u) & 0 & T_{88}^1(u) \\
0 & T_{22}^2(u) & T_{33}^2(u) & 0 & T_{44}^2(u) & 0 & 0 & T_{55}^2(u) & 0 & 0 & T_{66}^2(u) & 0 & 0 & T_{77}^2(u) & 0 & T_{88}^2(u) \\
T_{11}^3(u) & 0 & 0 & T_{22}^3(u) & 0 & T_{33}^3(u) & 0 & T_{44}^3(u) & 0 & T_{55}^3(u) & 0 & T_{66}^3(u) & 0 & T_{77}^3(u) & 0 & T_{88}^3(u) \\
0 & T_{22}^5(u) & T_{33}^5(u) & 0 & T_{44}^5(u) & 0 & 0 & T_{55}^5(u) & 0 & 0 & T_{66}^5(u) & 0 & 0 & T_{77}^5(u) & 0 & T_{88}^5(u) \\
T_{11}^7(u) & 0 & 0 & T_{22}^7(u) & 0 & T_{33}^7(u) & 0 & T_{44}^7(u) & 0 & T_{55}^7(u) & 0 & T_{66}^7(u) & 0 & T_{77}^7(u) & 0 & T_{88}^7(u) \\
0 & T_{22}^8(u) & T_{33}^8(u) & 0 & T_{44}^8(u) & 0 & 0 & T_{55}^8(u) & 0 & 0 & T_{66}^8(u) & 0 & 0 & T_{77}^8(u) & 0 & T_{88}^8(u)
\end{pmatrix}. \tag{5.25}
\]
By renaming indices or interchanging rows and columns this matrix can be turned to the block-diagonal form, which means that the $8 \times 8$ matrix elements of spinor RTT algebra generators reduce to two independent subsets each generating a $s\ell(4)$ type RTT algebra. This becomes evident by the similarity transformation with the matrix $V$:

$$V = e_{11} + e_{27} + e_{36} + e_{44} + e_{55} + e_{63} + e_{72} + e_{88}. \quad (5.26)$$

One obtains for the monodromy matrix $[5.25]$:

$$VT(u)V = \begin{pmatrix}
T_1^1(u) & T_1^2(u) & T_1^3(u) & T_1^4(u) & 0 & 0 & 0 & 0 \\
T_2^1(u) & T_2^2(u) & T_2^3(u) & T_2^4(u) & 0 & 0 & 0 & 0 \\
T_3^1(u) & T_3^2(u) & T_3^3(u) & T_3^4(u) & 0 & 0 & 0 & 0 \\
T_4^1(u) & T_4^2(u) & T_4^3(u) & T_4^4(u) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & T_3^5(u) & T_3^6(u) & T_3^7(u) & T_3^8(u) \\
0 & 0 & 0 & 0 & T_2^5(u) & T_2^6(u) & T_2^7(u) & T_2^8(u) \\
0 & 0 & 0 & 0 & T_1^5(u) & T_1^6(u) & T_1^7(u) & T_1^8(u) \\
0 & 0 & 0 & 0 & T_1^5(u) & T_1^6(u) & T_1^7(u) & T_1^8(u)
\end{pmatrix}. \quad (5.27)$$

The $R$-matrix $\mathcal{R}_{s\ell(6)}^{(1)}(u)$ is also transformed by this similarity transformation to the block-diagonal form similar to $[5.12]$. We obtain two blocks of the $s\ell(4)$ $R$ matrix form.

**5.4.2 $\mathcal{R}^{(2)}$, the $\Pi_+$ part**

Consider now the second part acting in the chiral sector $\Pi_+$,

$$\mathcal{R}_{s\ell(6)}^{(2)}(u) = \frac{u + \frac{2}{24}}{z^2(z^2 - 4)(z^2 - 9) + \frac{u - \frac{2}{360}}{z^2(z^2 - 1)(z^2 - 4)}.$$

It appears as a matrix with the following non-vanishing entries:

$$\mathcal{R}_{15}^{15}(u) = \mathcal{R}_{27}^{27}(u) = \mathcal{R}_{36}^{36}(u) = \mathcal{R}_{45}^{45}(u) = \mathcal{R}_{54}^{54}(u) = \mathcal{R}_{63}^{63}(u) = \mathcal{R}_{72}^{72}(u) = \mathcal{R}_{81}^{81}(u) = u - 1,$$

$$\mathcal{R}_{12}^{12}(u) = \mathcal{R}_{13}^{13}(u) = \mathcal{R}_{15}^{15}(u) = \mathcal{R}_{21}^{21}(u) = \mathcal{R}_{24}^{24}(u) = \mathcal{R}_{26}^{26}(u) = \mathcal{R}_{31}^{31}(u) = \mathcal{R}_{34}^{34}(u) = \mathcal{R}_{37}^{37}(u) = \mathcal{R}_{42}^{42}(u) = \mathcal{R}_{43}^{43}(u) = \mathcal{R}_{48}^{48}(u) = \mathcal{R}_{51}^{51}(u) = \mathcal{R}_{56}^{56}(u) = \mathcal{R}_{57}^{57}(u) = \mathcal{R}_{62}^{62}(u) = \mathcal{R}_{63}^{63}(u) = \mathcal{R}_{64}^{64}(u) = \mathcal{R}_{65}^{65}(u) = \mathcal{R}_{73}^{73}(u) = \mathcal{R}_{75}^{75}(u) = \mathcal{R}_{84}^{84}(u) = \mathcal{R}_{86}^{86}(u) = \mathcal{R}_{87}^{87}(u) = u + 2,$$

$$\mathcal{R}_{45}^{45}(u) = \mathcal{R}_{36}^{36}(u) = \mathcal{R}_{45}^{45}(u) = \mathcal{R}_{51}^{51}(u) = \mathcal{R}_{57}^{57}(u) = \mathcal{R}_{84}^{84}(u) = \mathcal{R}_{86}^{86}(u) = \mathcal{R}_{87}^{87}(u) = \mathcal{R}_{81}^{81}(u) = -1.$$

Again, the RTT relation $[5.13]$ at index values $(\alpha_1, \alpha_2) = (1, 2), (1, 3), (1, 5)$, and $(\gamma_1, \gamma_2) = (1, 1), (1, 4), (1, 6), (1, 7)$ is solved by

$$T_1^2(v) = T_1^2(v) = T_6^2(v) = T_7^2(v) = 0, \quad T_2^3(v) = T_4^3(v) = T_6^3(v) = T_7^3(v) = 0,$$

$$T_1^5(v) = T_5^5(v) = T_6^5(v) = T_7^5(v) = 0.$$ 

Choosing $(\alpha_1, \alpha_2) = (1, 1), (1, 4), (1, 6), (1, 7)$, and $(\gamma_1, \gamma_2) = (1, 2), (1, 3), (1, 5)$, one deduces

$$T_2^2(v) = T_2^2(v) = T_6^7(v) = T_2^7(v) = 0,$$

$$T_3^1(v) = T_3^4(v) = T_3^6(v) = T_3^7(v) = 0.$$
Finally, considering say \((\alpha_1, \alpha_2) = (2, 1), (2, 4), (2, 6), (\gamma_1, \gamma_2) = (2, 8)\) and \((\alpha_1, \alpha_2) = (3, 7), (\gamma_1, \gamma_2) = (3, 8)\) one obtains:

\[T_1^4(v) = T_4^4(v) = T_5^g(v) = T_7^7(v) = 0,\]

and similarly

\[T_1^8(v) = T_4^8(v) = T_5^g(v) = T_7^7(v) = 0.\]

In this way one deduces that the second chiral part \(\mathfrak{R}_{so(6)}^{(2)}(u)\) also implies the general form \((5.25)\) for the general monodromy matrix.

We summarize the results on the \(so(6)\) case:

**Proposition 2.** The spinorial \(R\) matrices with \(so(6)\) symmetry of both chiral sectors can be separated into blocks of the fundamental \(s\ell(4)\) \(R\) matrix form. In each case the resulting \(RTT\) algebras are equivalent to two copies of the Yangian algebra of \(s\ell(4)\) type. The spectral problem of the trace of the \(so(6)\) monodromy matrices can be treated on the basis of the known nested ABA for the \(s\ell(4)\) Yangian.

### 5.5 The \(so(8)\) case

The characteristic polynomial in this case is given by

\[W_8 = (z^2 - 1)(z^2 - 4)(z^2 - 9)(z^2 - 16) = 0, \quad (5.29)\]

and the universal expression \((3.12)\) for the spinorial \(R\) matrix is reduced to

\[\mathfrak{R}_{so(8)}(z|u) = B(-1 - \frac{u}{2}, u)(P_{-3} - P_3 + \frac{u + 2}{u - 2}(P_1 - P_{-1}))+ \quad (5.30)\]

\[+ B(-\frac{3}{2} - \frac{u}{2}, u)(P_{-4} + P_4 + \frac{u + 3}{u - 3}(P_2 + P_{-2}) + \frac{u + 3}{u - 3}u + 1)P_0).\]

It decomposes into two parts of opposite chirality, both obeying the Yang-Baxter relations.

We observe that one of the two chiral parts is simpler: \(\mathfrak{R}_{so(8)}^{(1)}\) is linear in the spectral parameter and contains two invariant tensors, while \(\mathfrak{R}_{so(8)}^{(2)}\) is quadratic in \(u\) and contains three tensors.

### 6 Odd-dimensional orthogonal algebras

In the odd-dimensional cases the characteristic polynomial reduces to \(\widetilde{W}_d = \prod_{k}(z - z_k)\) \((8.9)\), where the product runs over \(m + 1\) of the \(2m + 1\) eigenvalues of \(z\). This leads to the decomposition of the Euler Beta-function in \((3.12)\):

\[\mathfrak{R}_{12}(z|u) = B(\frac{z + 1 - u}{2}, u) = \sum_k B(\frac{z_k + 1 - u}{2}, u)P_k,\quad (6.1)\]

where the summation goes over the \(m + 1\) roots \(z_k = (-1)^k\frac{2k + 1}{2}\) of the characteristic polynomial \(\widetilde{W}_d\).
6.1 The $so(3)$ case

The characteristic polynomial is

\[ \widetilde{W}_3 = (z + \frac{1}{2})(z - \frac{3}{2}). \]  

(6.2)

We have the decomposition

\[ \mathfrak{R}^{so(3)}(z|u) = B(-\frac{1}{4} - \frac{u}{2}, u) \cdot P_{-\frac{1}{2}} + B(\frac{3}{4} - \frac{u}{2}, u) \cdot P_{\frac{1}{2}} = \frac{B(-\frac{1}{4} - \frac{u}{2}, u) \cdot (2u(z + \frac{1}{2}) + 1)}{1 - 2u}, \]

where \( P_{-\frac{1}{2}} = -\frac{1}{2}(z - \frac{1}{2}), \ P_{\frac{1}{2}} = \frac{1}{2}(z + \frac{3}{2}) \). Unity and the permutation are given by

\[ 1 = P_{\frac{1}{2}} + P_{-\frac{1}{2}}, \ P_{12} = P_{\frac{1}{2}} - P_{-\frac{1}{2}}. \]

Thus

\[ \mathfrak{R}^{so(3)}(z|u) = 2u + z + \frac{1}{2} = 2u12 + P_{12}. \]  

(6.3)

Up to a redefinition of the spectral parameter the spinorial $so(3)$ $R$ matrix coincides with the fundamental $R$ matrix with $s\ell(2)$ symmetry.

Let us compare this expression also with the fundamental $R$ matrix (2.2) at \( \epsilon = -1 \) and \( n = 2 \) corresponding to the case $sp(2)$. This $4 \times 4$ matrix has the following non-vanishing entries:

\[ R^{-1,-1}_{1,1} = (u + 1)(u + 2) = R_{1,1}^{1,1}, \quad R^{-1,1}_{1,1} = u(u + 1) = R_{1,-1}^{1,-1}, \]

\[ R_{1,-1}^{1,-1} = 2(u + 1) = R_{-1,1}^{1,1}, \]

and can be rewritten as \( R(u) = 2(u + 1)(\frac{1}{2}I_{12} + P_{12}) \), i.e. it coincides with the $so(3)$ spinor-spinor $R$-matrix after rescaling \( 2u \to \frac{u}{2} \).

The spinorial $RTT$ relation for the $so(3)$ monodromy matrix (4.3) coincides with the one for the vector $s\ell(2)$ monodromy matrix. Indeed, denoting

\[ T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \]

(6.4)

one obtains that the commutation relations between $A$, $B$, $C$ and $D$ are the same as in $s\ell(2)$ case, because the spinor-spinor $R$-matrix intertwining them, coincide up to $u \to 2u$.

The coincidence of the spinorial $R$ matrix of $so(3)$ symmetry with the well known Yang formula for $s\ell(2)$ was a central point in the classical paper by Reshetikhin [13]. The fusion relation (4.8) results in expressions of the 9 elements of the vector monodromy matrix $T$ in terms of the 4 spinorial $RTT$ generators in $T(u)$ for all representations admitting this relation. The study of [10] resulted in relations among the matrix elements of $T$, in particular the three elements in the upper triangle are expressed in terms of one of them. This confirms that (4.8) establishes the equivalence of the spinorial $so(3)$ type $RTT$ algebra and the ordinary Yangian of $s\ell(2)$ type.

6.2 The $so(5)$ case

The characteristic polynomial is

\[ \widetilde{W}_5 = (z - \frac{1}{2})(z + \frac{3}{2})(z - \frac{5}{2}) = 0. \]  

(6.5)
The spinorial $R$ matrix acting in $S_1 \otimes S_2$ can be written as
\[
\mathcal{R}^{so(5)}(z|u) = (u + \frac{1}{2})(u - \frac{3}{2})(z - \frac{1}{2})(z + \frac{3}{2}) + 2(u + \frac{1}{2})(u + \frac{3}{2})(z + \frac{3}{2})(z - \frac{5}{2}) +
+(u - \frac{1}{2})(u + \frac{3}{2})(z - \frac{1}{2})(z - \frac{5}{2}) = (u + 1)(u + 3)I_{12} + u(u + 2)\gamma^a_1\gamma^a_2 - \frac{u(u + 1)}{2}\gamma^a_1\gamma^a_2,
\]
The permutation is given by
\[
\mathcal{P}_{12} = P_{\frac{1}{2}} - P_{-\frac{1}{2}} - P_{\frac{3}{2}}
\]
with the eigenspace projectors
\[
P_{\frac{1}{2}} = -\frac{1}{4}(z + \frac{3}{2})(z - \frac{5}{2}), \quad P_{-\frac{1}{2}} = \frac{1}{8}(z - \frac{1}{2})(z - \frac{5}{2}),
\]
\[
P_{\frac{3}{2}} = \frac{1}{8}(z - \frac{1}{2})(z + \frac{3}{2}). \quad (6.6)
\]
Their properties $P_{2}^{\frac{1}{2}} = P_{\frac{1}{2}}$, $P_{-\frac{1}{2}}^{2} = P_{-\frac{1}{2}}$, as well as $P_{\frac{1}{2}} + P_{-\frac{1}{2}} + P_{\frac{3}{2}} = 1$, $P_{\frac{1}{2}} \cdot P_{-\frac{1}{2}} = P_{2} \cdot P_{\frac{1}{2}} = P_{-\frac{1}{2}} \cdot P_{-\frac{1}{2}} = 0$, are checked by the characteristic polynomial $\mathcal{W}_5$.

The $16 \times 16$ $R$ matrix $\mathcal{R} = \mathcal{P}_{12}\mathcal{R}$ has the following non-vanishing matrix elements:
\[
\mathcal{R}_{11}^{11}(u) = \mathcal{R}_{22}^{33}(u) = \mathcal{R}_{33}^{33}(u) = \mathcal{R}_{44}^{34}(u) = (2u + 1)(2u + 3),
\]
\[
\mathcal{R}_{21}^{12}(u) = \mathcal{R}_{13}^{13}(u) = \mathcal{R}_{21}^{21}(u) = \mathcal{R}_{31}^{31}(u) = \mathcal{R}_{34}^{34}(u) = \mathcal{R}_{42}^{32}(u) = \mathcal{R}_{43}^{33}(u) = 2u(2u + 3),
\]
\[
\mathcal{R}_{14}^{14}(u) = \mathcal{R}_{23}^{23}(u) = \mathcal{R}_{32}^{32}(u) = \mathcal{R}_{41}^{41}(u) = 4u(u + 1),
\]
\[
\mathcal{R}_{24}^{12}(u) = \mathcal{R}_{31}^{32}(u) = \mathcal{R}_{42}^{33}(u) = \mathcal{R}_{43}^{34}(u) = \mathcal{R}_{24}^{34}(u) = \mathcal{R}_{21}^{42}(u) = \mathcal{R}_{24}^{42}(u) = 2u + 3,
\]
\[
\mathcal{R}_{23}^{14}(u) = \mathcal{R}_{32}^{34}(u) = \mathcal{R}_{33}^{43}(u) = \mathcal{R}_{34}^{44}(u) = 2u,
\]
\[
\mathcal{R}_{32}^{14}(u) = \mathcal{R}_{41}^{23}(u) = \mathcal{R}_{41}^{34}(u) = \mathcal{R}_{42}^{34}(u) = -2u.
\]
It can be rewritten in the following form
\[
\mathcal{R}_{\alpha_1\beta_2}^{\alpha_2\beta_1}(u) = 2u(2u + 3)\delta_{\alpha_2\beta_1} \delta_{\alpha_1\beta_2} + (2u + 3)\delta_{\alpha_1\beta_1} \delta_{\alpha_2\beta_2} - 2uK_{\alpha_1\alpha_2}^{\beta_1\beta_2}, \quad (6.7)
\]
where the elements of the matrix $K = 4P_{\frac{3}{2}}$ can be written as
\[
K_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \varepsilon_{\alpha_1\alpha_2} \varepsilon_{\beta_1\beta_2}, \quad \varepsilon_{\alpha_1\alpha_2} = (-1)^{\alpha_1+\alpha_2+5}, \quad \varepsilon_{\beta_1\beta_2} = (-1)^{\beta_1+\beta_2+5}. \quad (6.8)
\]

In this matrix after rescaling $2u \rightarrow u$ one can recognize the $sp(4)$ vector-vector $R$-matrix with the parameter $\beta = \frac{3}{2} - (-1) = 3$.

We summarize the results on $so(5)$:

**Proposition 3.** The spinorial $R$ matrix with $so(5)$ symmetry coincides with the ordinary fundamental $R$ matrix with $sp(4)$ symmetry. The diagonalization of the traces of the corresponding spinorial $so(5)$ monodromy and of the ordinary $sp(4)$ monodromy is done by the same nested ABA relations. Moreover, the fusion relation (4.8) allows to treat the diagonalization of the trace of the ordinary $so(5)$ monodromy matrix on the basis of the $sp(4)$ nested ABA.

The nested ABA for $sp(4)$ has been formulated in [9].
7 Conclusions

We have presented a new approach to the spinor $R$ matrix of orthogonal or symplectic symmetry. Comparing with the conventional approach, the derivation is simpler and the result has the compact form of the Euler Beta function of the invariant $z$.

In the orthogonal case, relying on the characteristic polynomial of $z$, we obtain explicit expressions of the spinorial $R$ matrices of low rank cases.

By the fusion argument we relate the spinor and vector monodromy matrices.

Studying the low rank examples, we observe coincidences of spinor $R$ matrices with some fundamental $R$ matrices. This implies relations for the monodromy matrices, the corresponding RTT algebras and of the Algebraic Bethe Ansatz for the spectral problem of traces of spinor and vector monodromy matrices.

For the $so(2m)$ cases ($D$ series) the spinor $R$ matrices have a particular simple structure. We have the decomposition into two chiral parts, where both obey the Yang-Baxter relations. Moreover, the $R$ matrix of each chiral part is sparse with many zeros and transforms to two independent blocks.

Coincidence relations between spinor and fundamental $R$ matrices are to be expected also at higher ranks. The simplicity of the $so(2m)$ case compared to $so(2m+1)$ will persist.

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8 Appendix: Characteristic polynomials

Recall the relation of the invariant $z = \frac{1}{2} \gamma_1 \gamma_2$ to the conventional spinor invariants $I_m$

$$I_m = \frac{1}{m!2^m} \gamma_1^{a_1 \ldots a_m} \gamma_2^{a_1 \ldots a_m},$$  \hspace{1cm} (8.1)

given by the iteration

$$I_{m+1} = z I_m - \frac{m}{4} (d - m + 1) I_{m-1}, \hspace{1cm} I_0(z, d) = 1, \ I_1(z, d) = z.$$  \hspace{1cm} (8.2)

It allows to calculate the explicit form the few first invariants:

$$I_2(z, d) = z^2 - \frac{d}{4},$$

$$I_3(z, d) = z^3 + \left(\frac{1}{2} - \frac{3}{4}d\right)z,$$

$$I_4(z, d) = z^4 + \left(2 - \frac{3}{2}d\right)z^2 + \frac{3}{16}d^2 - \frac{3}{8}d,$$

$$I_5(z, d) = z^5 + \left(5 - \frac{5}{2}d\right)z^3 + \frac{3}{2} - \frac{25}{8}d + \frac{15}{16}d^2z, \hspace{1cm} (8.3)$$
I_6(z, d) = z^6 + (10 - \frac{15}{4} d) z^4 + \left(\frac{23}{2} - \frac{105}{8} d + \frac{45}{16} d^2\right) z^2 - \frac{15}{8} d + \frac{45}{32} d^2 - \frac{15}{64} d^3,

I_7(z, d) = z^7 + \left(\frac{35}{2} - \frac{21}{2} d\right) z^5 + \left(49 - \frac{315}{2} d + \frac{105}{16} d^2\right) z^3 + \left(\frac{45}{4} - \frac{441}{16} d + \frac{105}{8} d^2 - \frac{105}{64} d^3\right) z.

The algebra \mathcal{C} implies in the orthogonal case the vanishing of \( I_m(z, d) \) starting from \( m = d + 1 \). Thus we obtain the characteristic polynomial in \( z \),

\[ W_d(z) = I_{d+1}(z, d) = 0. \tag{8.4} \]

### 8.1 Even-dimensional space

Consider rotations in an even-dimensional \( d = 2m \) Euclidean space. \( \text{so}(d) \) has the \( 2^m \)-dimensional spinor representation \( \mathcal{S} \). The invariant \( z = \frac{1}{2} \sum_{a=1}^{2m} \gamma_a^a \gamma_2^a = \sum_{a=1}^{2m} z^{(a)} \) acts in \( 2^{2m} (= 2^m \times 2^m) \)-dimensional space \( \mathcal{S}_1 \otimes \mathcal{S}_2 \). It is a sum of \( 2m \) mutually commuting operators \( z^{(a)} \) which have \( 2^m \) common eigenvectors \( \xi_i, i = 1, \ldots, 4^m \). The eigenvectors \( \xi_i \) span the \( 2^m \)-dimensional space \( \Sigma = \mathcal{S} \otimes \mathcal{S} \) which decomposes into the direct sum of two \( 2^m \) dimensional subspaces annihilated by projectors \( \Pi_+ \) and \( \Pi_- \), correspondingly. Here \( \Pi_{\pm} = \frac{1}{2} (1 \pm \gamma_1^{2m+1} \gamma_2^{2m+1}) \), which means that all these \( 2^m \) vectors are the eigenvectors of the operator \( z^{(2m+1)} = \frac{1}{2} \gamma_1^{2m+1} \gamma_2^{2m+1} \) with eigenvalue \( \frac{1}{2} \) or \( -\frac{1}{2} \). Note that due to the rotational symmetry, each of the operators \( z^{(a)}, a = 1, \ldots, 2m \) also has the eigenvalue \( \frac{1}{2} \) on half of all vectors of \( \Sigma \) and the eigenvalue \( -\frac{1}{2} \) on the other half of vectors of \( \Sigma \).

The eigenvalue of \( z \) on the vector \( \xi_i \), which is a root of the characteristic polynomial, is given by the sum of the eigenvalues of \( z^{(a)} \), \( a = 1, \ldots, 2m \) on \( \xi_i \). The largest is obtained in the configuration of "all spins up" and is equal to \( m = 2m \cdot \frac{1}{2} \). This configuration is encountered once, i.e. here is no degeneracy. The largest negative eigenvalue arises from "all spins down", is equal to \(-m = 2m(-\frac{1}{2})\) and is not degenerate either. The next configuration with "one spin down and all others up" is encountered \( 2m = \binom{2m}{1} \) times, and the corresponding eigenvalue is \( m - 1 \). Continuing this counting, we have in particular the eigenvalue \( 0 \) of highest degeneracy \( \binom{2m}{m} \). Thus we obtain the eigenvalues of \( z \) equal to \( k = -m, -m+1, \ldots, +m \) and the dimension of the eigenspace corresponding to \( k \) equal to \( \binom{2m}{m+k} \).

In this way we obtain the characteristic polynomial of the form

\[ W_d = z \prod_{k=1}^{m} (z - k)(z + k) = z \prod_{k=1}^{m} (z^2 - k^2) = 0, \]

and we check (by the binomial formular for \( (1+1)^{2m} \)) that the number of eigenvectors and eigenvalues accounting for their degeneracy equals \( 2^{2m} \), i.e. the dimension of \( \Sigma = \mathcal{S}_1 \otimes \mathcal{S}_2 \).

### 8.2 The permutation operator

The operator \( \mathcal{P}_{12} \) permutes the factors in the product \( \mathcal{S}_1 \otimes \mathcal{S}_2 \). It commutes with each \( z^{(a)} \), thus has common eigenvectors with them, i.e. the whole representation space \( \Sigma = \mathcal{S}_1 \otimes \mathcal{S}_2 \) according to action of \( \mathcal{P}_{12} \) is divided into the sum of the symmetric and antisymmetric subspaces \( \Sigma = \Sigma_+ \oplus \Sigma_- \). The eigenspaces \( \Sigma_+ \) and \( \Sigma_- \) of \( \mathcal{P}_{12} \) have the dimensions \( (2^m + 1)2^{m-1} \) and \( (2^m - 1)2^{m-1} \), respectively.
Like any invariant operator acting on $\Sigma$ the operator $\mathcal{P}_{12}$ for $so(2m)$ has the spectral expansion over the projection operators on the eigenspaces of $z$

$$\mathcal{P}_{12} = \sum_{k=-m}^{m} (-1)^{k(k-1)/2} P_k.$$  \hspace{1cm} (8.5)

Note that $P_0$ and $P_1$ always contribute with plus sign.

This formula is closely related to the well known formula

$$\mathcal{P}_{12} = \frac{1}{2m} \sum_{k=0}^{2m} (\frac{1}{2})^{k} \frac{1}{(k!)^2} \gamma_1^{a_1} \gamma_2^{a_2} \cdots \gamma_{2m}^{a_{2m}} =$$

$$= \frac{1}{2m} \left( \mathcal{P}_{12} + \frac{1}{(1!)^2} \gamma_1^2 - \frac{1}{(2!)^2} \gamma_1 \gamma_2 \gamma_2 - \frac{1}{(3!)^2} \gamma_1 \gamma_2 \gamma_2 + \cdots \right),$$

as well as with the dimension formula

$$\text{dim}(\Sigma^\pm) = \text{rank}(\frac{1}{2}(1 \pm \mathcal{P}_{12})) = \sum_{k=0}^{2m} \frac{1}{2} \left( 1 \pm (-1)^{(m+k)(m+k-1)/2} \right) \left( \begin{array}{c} 2m \\ k \end{array} \right) =$$

$$= (2^m \pm 1)2^{m-1}. \hspace{1cm} (8.7)$$

In order to emphasize the pairwise appearance of the projection operators in the even-dimensional case we rewrite the expression $\mathcal{P}_{12}$ for the permutation as

$$\mathcal{P}_{12} = \sum_{k=0}^{m+1} (-1)^k (P_{2k} + P_{2k+1}). \hspace{1cm} (8.8)$$

### 8.3 Odd-dimensional space

If the dimension of the space is odd $d = 2m+1$, the dimension of $S_1 \otimes S_2$ and the number of $z$ eigenvectors remains the same $2^{2m}$, but the operator $z$ contains an additional term which shifts its eigenvalues by one half and breaks the symmetry between positive and negative eigenvalues. With the shifted eigenvalues the relation $W_d = 0 \hspace{1cm} (8.4)$ holds. Further, $\varepsilon_{a_1,\ldots,a_{2m+1}}$ causes relations between even and odd invariants, $I_k(z) = \text{const} \; I_{2m+1-k}(z)$. As a consequence we obtain the reduction of $W_d$ to

$$\tilde{W}_d = \prod_{k=0}^{m} \left( z - (-1)^k \frac{2k+1}{2} \right) = 0. \hspace{1cm} (8.9)$$

The product runs over only $m+1$ of the $2m+1$ roots of $z$.

Consider, how the eigenspaces of $z$ change when increasing the dimension from $d = 2m$ to $d = 2m+1$. The adjacent $z$ multiplets for $d = 2m$ symmetric under the permutation $\mathcal{P}_{12}$ (e.g. $P_0$ and $P_1$) are unified into a multiplet for $d = 2m+1$ with the eigenvalue given by arithmetical mean ($P_{\frac{3}{2}}$) by addition of $\frac{1}{2}$ or $-\frac{1}{2}$. Similarly, the adjacent antisymmetric multiplets (say $P_{-2}$ and $P_{-1}$) are unified (into $P_{-\frac{3}{2}}$). Then the dimension formula $\hspace{1cm} (8.7)$ holds due to the Pascal triangle recurrence relation

$$\left( \begin{array}{c} 2m+1 \\ k \end{array} \right) = \left( \begin{array}{c} 2m \\ k \end{array} \right) + \left( \begin{array}{c} 2m \\ k-1 \end{array} \right),$$

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and accordingly the characteristic polynomial is given by $\widetilde{W}_d$ \(8.9\).

The dimensions of the subspaces corresponding to the roots of $\widetilde{W}_d$ are given by
\[
\binom{2m + 1}{2k},
\]
and their sum is the dimension of $\Sigma$,
\[
\sum_{k=0}^{m} \binom{2m + 1}{2k} = 2^{2m}.
\]

Here we took into account the binomial formula for \((1 + 1)^{2m + 1}\) and for \((1 - 1)^{2m + 1}\).

Consider, how the eigenspaces of $z$ change if we step up in dimension further from $d = 2m + 1$ to $d = 2m + 2$. The dimension of the spinor space $S$ is doubled. $\Sigma_{2m+1}$ is separated into subspaces even or odd under the permutation $P_{12}$. The $P_{12}$ even subspaces of $\Sigma_{2m+2}$ corresponding to the $z$ eigenvalue $\frac{1}{2}$ is translated to the two subspaces of $z$ in $\Sigma_{2m+2}$ with $z$ eigenvalues $0 = \frac{1}{2} - \frac{1}{2}$ and $1 = \frac{1}{2} + \frac{1}{2}$. Similarly, the $P_{12}$ odd subspaces with the $z$ eigenvalues $-\frac{3}{2}$ or $z = \frac{5}{2}$ are each translated to the two subspaces of $z$ eigenvalues $-1 = -3 + \frac{1}{2}$, $-2 = -3 + \frac{1}{2}$ or $2 = 3 - \frac{1}{2}$, $3 = \frac{5}{2} + \frac{1}{2}$.

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