Quantum Black Holes: Entropy and Entanglement on the Horizon

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ABSTRACT

We are interested in black holes in Loop Quantum Gravity (LQG). We study the simple model of static black holes: the horizon is made of a given number of identical elementary surfaces and these small surfaces all behaves as a spin-$s$ system accordingly to LQG. The chosen spin-$s$ defines the area unit or area resolution, which the observer uses to probe the space(time) geometry. For $s = 1/2$, we are actually dealing with the qubit model, where the horizon is made of a certain number of qubits. In this context, we compute the black hole entropy and show that the factor in front of the logarithmic correction to the entropy formula is independent of the unit $s$. We also compute the entanglement between parts of the horizon. We show that these correlations between parts of the horizon are directly responsible for the asymptotic logarithmic corrections. This leads us to speculate on a relation between the evaporation process and the entanglement between a pair of qubits and the rest of the horizon. Finally, we introduce a concept of renormalisation of areas in LQG.

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I. INTRODUCTION

Loop Quantum Gravity (LQG) is a canonical quantization of General Relativity, which relies on a 3+1 decomposition of space-time (for reviews, check [1]). It describes the states of 3d geometry and their evolution in time (through the implementation of a Hamiltonian constraint). The states of the canonical hypersurface are the well-known spin networks, which represent polymeric excitations of the gravitational field. The main and first achievement of the LQG framework is the implementation of quantum area operators and the derivation of their discrete spectrum. A surface is now made of elementary patches of finite quantized area. Based on this discrete structure of the 3d space, one can study in details the entropy associated to a surface and the entanglement between the patches making the surface. The main goal of the present work is to study analytically the relation between entropy and entanglement of the horizon state and discuss their physical/geometrical interpretations.

Let us discuss more precisely the concepts that were mentioned above. Entanglement can be loosely defined as an exhibition of stronger-than-classical correlations between the subsystems. For a long time, it stood up among the apparent “paradoxes” of quantum mechanics for its nonlocal connotations [2]. Recently it became one of the main resources of quantum information theory [2, 3]. We discuss its relevant properties in Sec. IIB. In our work we are not concerned with its role in quantum information processing. Instead we are interested in its relations to the entropy of black holes and the role of entanglement as a pointer to their physical properties, similarly to the recent discussion of quantum phase transitions [4].

A spin network is a graph, a network of points with links representing the relations between points. Each link or edge is labeled by a half-integer, which defines the area of a surface intersecting the link. More generally, a surface is defined through its intersections with the edges of the spin network described the underlying quantum 3d space: the surface can be thought as made from elementary patches, each corresponding to a single intersection with a link and whose area is given by the label of that link. These labels actually stand for SU(2) representations i.e spins, and are conventionally noted \( j \). Due to regularization ambiguities, there is not a definitive consensus on the precise area associated to a given spin \( j \). The original and conventional prescription is of an elementary area \( a(j) = \sqrt{j(j+1)} \), but other reasonable choices are \( a(j) = j \) or \( a(j) = j + 1/2 \) (although one would like to keep \( a(j = 0) = 0 \) in the end for both physical intuitive reasons and also mathematical consistency [5]). Now, points, or vertices, of a spin network represents chunks of volume. Mathematically, they are attached a SU(2) intertwiner - a SU(2) invariant tensor between the representations attached to all the edges linked to the considered vertex.

A generic surface on a spin network background is thus described as a set of patches, each punctured by a unique link of the spin network. A SU(2) representation \( j \) is attached to each patch. We denote its Hilbert space \( \mathbb{C}^V_j \). Intuitively, a (quantum) vector \( |jm\rangle \) in the Hilbert space corresponds to the geometrical normal vector to the surface defined by the elementary patch. Now the spin network defines how the patches, and therefore the whole surface, is embedded in the surrounding 3d space and describes how the surface folds. For a closed surface, the region of the spin network which is inside the surface exactly defines an intertwiner, invariant under SU(2), between the patches of the surface. This intertwiner contains the rotation-invariant information on the way the patches organize themselves to form the surface e.g the angle between two patches.

An important remark is that any spin \( j \) representation \( \mathbb{C}^V_j \) can be decomposed as a symmetrized tensor product of 2 spin-\( \frac{1}{2} \) representations \( \mathbb{C}^{V1/2} \). Therefore, one can interpret that a fundamental patch or elementary surface is a spin-\( \frac{1}{2} \) representation. All higher spin patches can be constructed from such elementary patches. For example, considering two spin-\( \frac{1}{2} \) patches, they can form a spin 0 representation or a spin 1 representation: in one case, the two patches are folded on one another and cancel each other, while in the later case they add coherently to form a bigger patch of spin 1. Considering an arbitrary surface, one can then look at it at the elementary (fundamental) level decomposing it into spin-\( \frac{1}{2} \) patches, or one can look at it at a coarse-grained level decomposing the same surface into bigger patches of spin \( s > 1/2 \). From this point of view, the size of the patches used to study a surface is like the choice of a ruler of fixed size used by the observer to analyze the properties of the object. When studying the entropy and entanglement on a given surface, we will therefore consider it made of a certain number of patches 1/2 at the elementary level (and not allow the size of the patches to vary when performing entropy computations). Then one can study the coarse-graining or renormalisation of these quantities when one observes the surface at a bigger scale, using bigger patches to characterize the surface.

In the present work, we would like to study some basic properties of Black Holes in the framework of Loop Quantum Gravity. Black holes are actually the main object studied in quantum gravity. Indeed, on one hand, they are especially simple objects from the point of view of theoretical general relativity, and on the other hand we expect a consistent
theory of quantum gravity to shed light on the origin of the entropy of a black hole and on the so-called “information-loss paradox.” Here we propose to analyze the basic features attached to the horizon of a black hole through a simple model inspired from Loop Quantum Gravity.

We deal with the simplest case of a Schwarzschild (non-rotating) black hole at the kinematical level. Actually, studying black holes at the dynamical level in quantum gravity should be very interesting, and should allow to understand the evaporation process at the level of space-time. We will later comment on the dynamics as induced by LQG, but this will not be the main point of our study; instead we have decided to focus on the black hole kinematics. Considering a static black hole, our main assumption is that the only information accessible to the observer outside the black hole can be seen on its (event) horizon. Considering the horizon as a closed surface, LQG describes it as made of patches of quantized area and describes the interior of the black hole in terms of (a superposition of) spin networks whose boundary puncturing the horizon defines the patches of the horizon surface. From the point of the external observer, it does not matter what is inside the black hole, but only the information which could be read off the horizon is relevant. More mathematically, the horizon surface is defined as a set of elementary patches to each of which is attached a given SU(2) representation. Let us label the patches with an index \( i = 1, \ldots, n \) and note the corresponding spins \( j_i \). The space geometry within the horizon, i.e. the black hole, is fully described by a spin network, which can possibly have support on a complicated graph within the horizon. From the point of view of the horizon, the details of the spin network inside the black hole do not matter: what actually matters is which intertwiner between the SU(2) representations on the horizon does the spin network induce. Indeed the observer outside the black hole is only interested by the state of the patches on the boundary, which lives in the tensor product \( V^{j_1} \otimes \cdots \otimes V^{j_n} \).

The only constraint on the state is that it should be globally gauge invariant, i.e. SU(2) invariant, such that the horizon state is exactly described by an intertwiner between the representations \( V^{j_i} \). From this point of view, horizon states are described by a spin network having support on a graph with a single vertex inside the horizon, i.e. a fully coarse-grained spin network. Actually, we would like to stress that our analysis applies to generic closed surface states as soon as we decide to ignore the details of the geometry of the enclosed spatial region.

The basic model for a quantum black hole that we propose to study is not new. We assume that the model describes the black hole horizon at the fundamental level. In other words, we assume that the observer describing the black hole have the maximal possible resolution and probes space(time) with a ruler of minimal area i.e. corresponding to a spin 1/2 surface. The resulting model for the horizon is to consider it made of a number \( n \) of spin 1/2 patches: a horizon state will live in the tensor product \( (V^{1/2})^\otimes n \). More exactly, we want a SU(2) invariant state, i.e. an intertwiner between these \( n \) representations. At this point, let us point out that a spin 1/2 system is usually called a qubit (in the language of quantum information), so that one can say that the horizon is made of \( n \) qubits and that a horizon state is a singlet state for these \( n \) qubits. Finally, the area of the horizon surface will by definition be \( n \times a_{1/2} \) where \( a_{1/2} \) is the area associated to an elementary spin 1/2 patch.

In this model, we will present the calculation of the entropy of the black hole and recover the entropy law, in an asymptotic limit, with a first term proportional to the horizon area and a logarithmic correction with a factor \(-3/2\). For these purpose, we develop a method to compute the number of intertwiners as a random walk modified with a mirror at the origin. Then the point that we would like to particularly stress is that the \( n \) qubits on the horizon are correlated, and more precisely entangled: the horizon is not made of \( n \) uncorrelated qubits and the \(-3/2\) factor is actually a reflection of the fact that these qubits are entangled.

Then one can renormalize, or equivalently coarse-grain, this fundamental model, now assuming that the observer probes the space(time) with a spin \( s \geq 1 \) elementary surface. A coarse-grained model of the black hole is then to consider horizon states to be given by intertwiners between \( n \) representations of a fixed spin \( s \). One can repeat the entropy calculation and check how the entropy law gets renormalized. The first term proportional to the area should simply get scaled by the ratio of the area \( a_s \) corresponding to a spin \( s \) surface to the \( a_{1/2} \). Then we will prove using the random walk analogy that the logarithmic correction factor is universally \(-3/2\) and does not get renormalized.

Our approach is different from the standard loop quantum gravity approach, which studies the classically induced boundary theory on the black hole horizon (or generically any isolated horizon). It is shown that one gets a U(1) Chern-Simons theory, which one can quantize and count the quantum degrees of freedom. In particular, some ways of computing the black hole entropy in such a context also involve parallels with random walk calculations. Nevertheless, one would ultimately like to start with the full quantum gravity theory, identify horizons within the quantum states of geometry and study the quantum induced boundary theory on the identified quantum horizon. However, searching for horizons in a spin network states requires either introducing a well-defined notion of observer at the quantum level and/or being able to extract the (semi-classical) metric from the quantum state of geometry. We do not tackle these issues in the present work. We decide to model the black hole as a region of the quantum space...
with a well-defined boundary (the horizon) such that the only information about the geometry of the internal region accessible to the external observer is information which can be measured on the boundary. Our results can therefore be applied to any closed surface in loop quantum gravity. Issues about actually identifying this closed surface as a horizon at the semi-classical level is left for future investigation.

The paper is organized as follows. The next section will detail the fundamental spin-1/2 black hole model. We will compute the entropy of the black hole and entanglement between qubits on the horizon. This will lead us to relate the entanglement of a pair of qubits (or more generally a small part of the horizon) with the rest of the horizon with its probability to evaporate from the black hole. We remark that the evaporation process can not work on a single qubit so that minimally only pairs of spins 1/2 can evaporate. Also, introducing a time scale canonically associated to a black hole, like the (minimal) time of flight of a particle on a circular orbit around the black hole, we derive the rate of evaporation from the probability computed previously. Section III deals with the generalized spin models. Using the random walk analogy, we derive the entropy law and show the universality of the 3/2 factor for the logarithmic correction. Section IV tackles the issue of the renormalisation of surface areas in the framework of LQG. Consider a surface defined microscopically on a spin network. The spin network describes how the surface is folded and embedded in space. A natural issue is what will be the coarse-grained area that an observer will assign macroscopically to that surface. We use the calculations of intertwiners to deduce the most probable area seen macroscopically assuming no knowledge of the underlying spin network. We then propose to refine this by defining a (background independent) state of a surface in (loop) quantum gravity through the probability amplitude assigned to intertwiners between the elementary patches making the surface. This information can be equally thought as data on the surrounding spin network or as data on how the surface geometry gets coarse-grained (how its geometry looks at different scales).

II. THE SPIN-1/2 BLACK HOLE MODEL

We describe the horizon of the black hole as a two-sphere punctured by the underlying spin network by 2n, n ∈ N, edges labeled by spins 1/2. The number of punctures defines the area of the horizon: \( A \equiv 2n a_{1/2} \), where \( a_{1/2} \) is the area corresponding to a spin-1/2 elementary patch. The geometry of the interior of the black hole is described by the potentially complex graph inside the two-sphere. However, this information is not available to an observer living outside the horizon. This external observer has only access to the information on the horizon, i.e., to the horizon state. It is given by the intertwiner between the 2n punctures on the horizon, which corresponds to completely coarse-graining the internal spin network into a single vertex. The state lives in \( (V^{1/2})^{\otimes 2n} \) and is SU(2)-gauge invariant. The tensor product of 2n spins 1/2 simply reads:

\[
\bigotimes_{j=0}^{2n} V^{1/2} = \bigoplus_{j=0}^{n} V^j \otimes \sigma_{n,j},
\]

where \( \sigma_{n,j} \) is the irreducible representation of the permutation group of 2n elements corresponding to the partition \([n+j, n-j]\). More details of the representation theory of the permutation group can be found in Appendix D. The dimensions \( d_{j}^{(n)} \equiv \dim(\sigma_{n,j}) \) count the degeneracy of each spin \( j \) in the tensor product \( (V^{1/2})^{\otimes 2n} \). The intertwiner space \( \mathcal{H} \) is defined as the SU(2)-invariant subspace of the tensor product i.e. the spin \( j = 0 \) subspace. It is obviously isomorphic to \( \sigma_{n,0} \). We note its dimension \( N \equiv d_{0}^{(n)} \). Let us remind that we are considering an even number of spins since there does not exist any intertwiner between an odd number of 1/2-spins.

It is known that the degeneracy factors are given in terms of the binomial coefficients:

\[
d_{j}^{(n)} = C_{2n}^{n+j} - C_{2n}^{n+j+1} = \frac{2j+1}{n+j+1} C_{2n}^{n+j}.
\]

In particular the dimension of the intertwiner space is

\[
N = \frac{1}{n+1} C_{2n}^{n}.
\]

Below we give a derivation of these formulas in terms of random walks, which will be straightforwardly generalizable
to higher spin models when we replace the fundamental 1/2-spin by an arbitrary s-spin. Using the Stirling formula\(^1\) to compute the asymptotics of the Catalan number \(C_{2n}^n/(n+1)\), we obtain the asymptotic behavior of the entropy of the quantum black hole:

\[
S \equiv \ln N \sim (2n) \ln 2 - \frac{3}{2} \ln(n) - \frac{1}{2} \ln \pi. \tag{II.4}
\]

The first term is linear in \(n\) and thus in the area, and is the usual term that one expects to appear in any black hole entropy calculation. The precise factor in front of the area \(\mathcal{A}\) actually depends on the unit used to define areas. The next term is the logarithmic correction in \(\ln(\mathcal{A})\). It is independent of the unit used to measure areas and thus the \(-3/2\) factor should be universal (invariant under coarse-graining). We will not discuss the other terms, which can be considered as negligible in the large area case. In the next section, we will show that we get the same factor in any s-spin black hole model.

Under the assumption that the external observer has no knowledge of the internal geometry of the black hole, the density matrix ascribed to the black hole horizon is the totally mixed state on the intertwiner space:

\[
\rho = \frac{1}{N} \sum_{i=1}^{N} |I_{i}\rangle\langle I_{i}|, \tag{II.5}
\]

where \(|I_{i}\rangle\) form an arbitrary orthonormal basis of the intertwiner space. It is obvious that the von Neumann entropy of this state reproduces the (micro-canonical) entropy: \(S(\rho) = -\operatorname{tr} \rho \ln \rho = \ln N\). Below we study the entanglement properties of this state and show how the logarithmic correction to the entropy formula reflects the correlations between the \(2n\) patches making the black hole horizon.

Let us end this introduction by discussing the generality of this 1/2-spin model. Thinking about rotating black holes, some might be tempted to only require invariance of the state under \(U(1)\) rotations around a particular axis -say \(J_{z}\)- and thus count all states with vanishing angular momentum \(m = 0\) instead of the stronger constraint of vanishing spin \(j = 0\). Then the dimension of the horizon state would be \(C_{2n}^n\) and we would have only a \(-1/2\) factor in front of the logarithmic correction. However the SU(2) invariance which we require is the gauge invariance present in loop quantum gravity and has nothing to do the physical \(\text{SO}(3)\) isometry group of the two-sphere. A more serious criticism would be that in loop quantum gravity one must allow spin network labeled by arbitrary spins and thus we should more generally allow punctures of arbitrary spin and not restricting them to 1/2-spins in order to have the true picture of a quantum black hole. Our counter-argument is that any spin \(k\) can be decomposed into 1/2-spins and thus allowing higher spin punctures would lead to an overcounting of the intertwiners and horizon states\(^2\). More precisely, for example, an intertwiner between one 1-spin and \((2n-2)\) 1/2-spins automatically defines a unique intertwiner between \((2n)\) 1/2-spins (through the unique intertwiner \(V^{1/2} \otimes V^{1/2} \rightarrow V^{1}\)). From this perspective, one can consider the 1/2-spin black hole model as the universal black hole description in loop quantum gravity. We will discuss this issue in more details in the later Sec. \ref{sec:area} on area renormalisation in loop quantum gravity. Let us nevertheless point out that the interested reader can find details on the structure of the intertwiner space when allowing arbitrary spins labeling the punctures in \cite{12}. Another argument in favor of the 1/2-spin black hole model is given in \cite{11} as a symmetry argument in the semi-classical limit. This is related to the fact that the intertwiner space provides a representation of the permutation group, which can be interpreted as the discrete diffeomorphisms of the black hole horizon as made as \(2n\) elementary patches. We will tackle this issue later in Sec. \ref{sec:inter}. 

\(^1\) We remind that Stirling formula gives the asymptotical behavior of \(n!\):

\[n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.\]

\(^2\) One can argue that replacing two 1/2-spins by one 1-spin actually modifies the area. This depends on the precise microscopic area spectrum. If one chooses the obvious linear spectrum \(a_j \propto j\), then one avoids this issue. Nevertheless, the standard prescription in loop quantum gravity is \(a_j \propto \sqrt{j(j+1)}\) which gives

\[
a_1 \frac{a_{1/2}}{a_{1/2}} = \sqrt{\frac{2}{3}}.
\]

If this case, the coarse-graining actually modifies the area. One should then keep in mind that a rigorous entropy count should involve
II.1. Entropy from random walk

In this section, we show how one can compute the degeneracies of the tensor product decomposition and thus the entropy as a random walk calculation. Indeed, describing the decomposition of \((V^{1/2})^{\otimes 2n}\) iteratively, the analogy with the random walk is clear. If we are at the \(k\)-spin representation \(V^k\), tensoring it with \(V^{1/2}\), we obtain the representations of spin \(k - 1/2\) and \(k + 1/2\). Only if we are at the trivial representation \(k = 0\), then \(V^0 \otimes V^{1/2}\) is simply \(V^{1/2}\). This way, it appears that the degeneracy of the \(k\)-spin in the tensor product \((V^{1/2})^{\otimes 2n}\) can be computed as the number of returns to the spot \(k\) after \(2n\) iterations of the random walk with a mirror (or a wall) at the origin 0. More precisely:

**Proposition 1** The degeneracy coefficients \(d^{(n)}_j\) in the decomposition of the tensor product \((V^{1/2})^{\otimes 2n}\) into the irreducible spin-\(j\) representations equal the number of returns \(\text{RWM}(j)\) to the spot \(j\) after \(2n\) iterations of the random walk of step 1/2 with a mirror in \(j = 0\) and starting at the origin. Moreover the random walk with a mirror is given in terms of the standard walk by the simple relation:

\[
\text{RWM}_n(j) = \text{RW}_n(j) - \text{RW}_n(j + 1), \tag{II.6}
\]

where \(\text{RW}(j)\) is the number of returns to the spot \(j\) after \(2n\) iterations of the random walk.

To prove this result, it is more convenient to deal with integer steps instead of switching between half-integers and integers after each step. Then we actually consider the tensor product as \((V^{1/2})^{\otimes n}\); at each step, we are tensoring with \((V^{1/2})^{\otimes 2} = V^0 \otimes V^1\) i.e. moving either one step up or one step down or staying at the same spot. Then it is straightforward to check that \(\text{RWM}_n(\geq j) \equiv \sum_{k \geq j} \text{RWM}_n(k)\) satisfies the same iteration law as the number of returns \(\text{RWM}_n(j)\) of the standard walk without mirror at the origin.

Then as \(\text{RW}_n(j) = C^{n+j}_{2n}\), we recover the previous formula \(d^{(n)}_j = C^{n+j}_{2n} - C^{n+j+1}_{2n}\). In particular, this shows the simple identity:

\[
\sum_j d^{(n)}_j = \text{RW}_n(0) = C^n_{2n}. \tag{II.7}
\]

Having these exact formulas, it is straightforward to derive the asymptotics of \(\text{RW}_n(j = 0)\) and \(\text{RWM}_n(j = 0)\) when \(n\) grows large, and the result is \(2^{2n}/\sqrt{n}\) and \(2^{2n}/n\sqrt{n}\), respectively.

On the other hand, it is possible to directly extract these asymptotics without computing the exact expression. Indeed, working on the unit circle \(U(1)\) and using the orthogonality of the modes \(e^{in\theta}\), one has:

\[
\text{RW}_n(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \ e^{-i(2j)\theta} (e^{i\theta} + e^{-i\theta})^{2n} = 2^{2n} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \ e^{-i(2j)\theta} (\cos \theta)^{2n}. \tag{II.8}
\]

For \(j = 0\), we evaluate this integral in the asymptotic limit \(n \to \infty\) using the saddle point approximation. First, we write

\[
\text{RW}_n(0) = 2^{2n} \times \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \ (\cos \theta)^{2n} = 2^{2n} \times \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \ e^{2n \log(\cos \theta)}. \tag{II.9}
\]

Then the exponent \(\phi(\theta) = \log(\cos \theta)\) is always negative and has a unique fixed point (which is actually a maximum) at \(\theta = 0\) with the second derivative \(\phi''(0) = -1\). Therefore we can approximate the integral by:

\[
\text{RW}_n(0) \xrightarrow{n \to \infty} 2^{2n} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \ e^{-\theta^2} \sim \frac{22^n}{n} \times \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \ e^{-x^2}, \tag{II.10}
\]

counting all the states of a given classical area \(\mathcal{A}\) up to an uncertainty \(\delta \mathcal{A}\). The dependence of the chosen scaling of \(\delta \mathcal{A}\) with \(\mathcal{A}\) would then certainly change the details of the entropy law.
and derive the right asymptotic behavior. Following the same line of thought, starting from

$$\text{RWM}_n(j = 0) = \text{RWM}_n(0) - \text{RWM}_n(1) = 2^{2n} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( 1 - e^{-2i\theta} \right) (\cos \theta)^{2n}, \quad (\text{II.11})$$

and using that this expression is real, we get:

$$\text{RWM}_n(0) = 2^{2n} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( 1 - \cos 2\theta \right) (\cos \theta)^{2n} = 2^{2n} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( 1 - \cos 2\theta \right) e^{2n \log(\cos \theta)}. \quad (\text{II.12})$$

Now we have an integral of the type $\int d\theta f(\theta)e^{-n\phi(\theta)}$. The exponent $\phi(\theta)$ has only one fixed point at $\theta = 0$. Then since $f(\theta)$ vanishes at $\theta = 0$, the leading order in the asymptotics is given by the first non-vanishing derivative of $f$:

$$\text{RWM}_n(0) \propto 2^{2n} \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \theta^2 e^{-n\theta^2} \sim \frac{2^{2n}}{n^{3/2}} \frac{2}{\pi} \int_{-\infty}^{+\infty} dx \ x^2 e^{-x^2}. \quad (\text{II.12})$$

This shows the $2^{2n}/n^{3/2}$ asymptotic behavior of the dimension of the intertwiner space and thus of the horizon state space. The same analysis will be generalized in the next section to any $s$-spin horizon model. The factor $-3/2$ of the logarithmic correction to the area-entropy law will appear to be universal and, as shown above, has a simple explanation in terms of random walks.

Although the asymptotics in $1/\sqrt{n}$ and $1/n\sqrt{n}$ have a natural interpretation from the random walk perspective, a simpler way to obtain integral formulas for the degeneracies is to use the orthogonality of the SU(2) characters. Indeed, we have:

$$d_j^{(n)} = \int dg \chi_j(g) \left( \chi_{\frac{1}{2}}(g) \right)^{2n} = \frac{2}{\pi} \int_0^\pi \sin^2 \theta d\theta \chi_j(\theta) \left( \chi_{\frac{1}{2}}(\theta) \right)^{2n}, \quad (\text{II.13})$$

where $dg$ is the normalized Haar measure on SU(2). The character $\chi_j(g)$ is the trace of the group element $g$ in the $j$-spin representation and is a function of only the rotation angle $\theta$:

$$\chi_j(\theta) = \frac{\sin(2j + 1)\theta}{\sin \theta}, \quad \chi_{\frac{1}{2}}(\theta) = 2 \cos \theta.$$

### II.2. Entanglement

The black hole model we analyze allows to deal with entanglement in a finite-dimensional setting. In addition, we consider only a bipartite entanglement, i.e., entanglement between two distinguishable parts of this system. This is the best understood part of the entanglement theory [3, 15]. In this section and in the appendices we present only the most essential elements of the analysis. The missing proofs and the generalizations to other families of states can be found in [10].

Pure state entanglement is easily identified. Pure states that cannot be written as a direct product of two states are entangled. For example, the singlet state of two qubits,

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle), \quad (\text{II.14})$$

is an entangled state. The definition is more involved for mixed states. It can be expressed as a part of the threefold hierarchy. At the lowest level there are direct product states $\rho = \rho_1 \otimes \rho_2$. Then, there are classically correlated (or separable) states,

$$\rho = \sum_i w_i \rho_i \otimes \rho'_i, \quad \sum_i w_i = 1, \quad \forall w_i > 0. \quad (\text{II.15})$$

which are mixtures (convex combinations) of the direct product states. Finally, entangled states are those that cannot be represented as separable. They have their own hierarchy which does not concern us here. Apart from two-qubit systems, no single universal criterion that allows to tell whether a mixed state is entangled or not is known.
For pure states there is a natural way to quantify entanglement. The degree of entanglement of a pure state $|\Psi\rangle$ is the von Neumann entropy of either of its reduced density matrices,

$$E(|\Psi\rangle\langle\Psi|) = S(\rho_{A,B}) = -\text{tr} \rho_{A,B} \log \rho_{A,B}, \quad \rho_A = \text{tr}_B |\Psi\rangle\langle\Psi|.$$  \hspace{1cm} (II.16)

For example, reduced density matrices of $|\Psi^-\rangle$ are the maximally mixed spin-1/2 states $\rho_{A,B} = 1_{2\times2}/2$, so its degree of entanglement is $\log 2 = 1$ bit. In general, a pure state whose degree of entanglement is $\log d$, where $d$ is the dimension of the Hilbert space of one of the subsystems, is called maximally entangled.

There are various measures of entanglement of mixed states that reflect different aspects of their preparation. We adapt here the entanglement of formation, which is defined as follows. For all possible decompositions of the state $\rho_{AB} \equiv \rho$ into mixtures of pure states,

$$\rho = \sum_i w_i |\Psi_i\rangle\langle\Psi_i|, \quad \sum_i w_i = 1, \quad \forall w_i > 0,$$  \hspace{1cm} (II.17)

the weighted average of degrees of entanglement of the constituents is calculated, and the minimum is taken over all decompositions

$$E_F(\rho) = \min_{\{\Psi_i\}} \sum_i w_i S(\rho_i).$$  \hspace{1cm} (II.18)

For pure states this expression reduces to the degree of entanglement. Entanglement of formation is zero for unentangled states. For generic states, an analytic expression for $E_F(\rho)$ exists only for two-qubit systems.

Nevertheless, we were able to calculate entanglement of $\rho$ of Eq. (II.15) for all bipartite splittings (Proposition 2 below). Moreover, it is possible to show that all standard entanglement measures coincide on $\rho$ \cite{16}, so there is no arbitrariness the choice of the entanglement measure. Henceforth we drop the subscript and simply write $E(\rho)$.

Let us split the $2n$ qubits into a group of $2k$ and a group of $2n-2k$ qubits and adapt the angular momentum basis. Hence the Hilbert space is $(V^{1/2})^{\otimes2k} \otimes (V^{1/2})^{\otimes(2n-2k)}$, for which a shorthand $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ will be used. The basis states in either of the subspaces are labeled as $|j, m, a_j\rangle$ and $|j, m, b_j\rangle$, respectively. Here $0 \leq j \leq k$ and $-j \leq m \leq j$ have their usual meaning and $a_j$ enumerate different irreducible subspaces $V^j$. Their multiplicities $d^A_j, B_j$ are given in Appendix B and $a_j = 1, \ldots, d^A_j$ and $b_j = 1, \ldots, d^B_j$. The constraint $J^2 = 0$ ensures the states $|I_i\rangle$ are the singlets on $V^j_A \otimes V^j_B$ subspaces for $j = 0, \ldots, k$,

$$|j, a_j, b_j\rangle \equiv \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j (-1)^{j-m} |j, -m, a_j\rangle \otimes |j, m, b_j\rangle.$$  \hspace{1cm} (II.19)

A more transparent notation is based on representing each subspace with a fixed $j$ as $V^j \otimes D^j$, where $V^j$ is a $2j + 1$ dimensional space that carries a spin-$j$ irreducible representation (irrep) of SU(2) and $D^j$ is the degeneracy subspace. In the case of qubits the Shur’s duality \cite{13} identifies these spaces as irreps of the permutation group $S_{2n}$ that correspond to the partition $[n + j, n - j]$ of $2n$ objects \cite{13,14}. Hence each basis state can be represented as a tensor product state on three subspaces,

$$|j, a_j, b_j\rangle \equiv |j\rangle_{AB} \otimes |a_j\rangle_{D^A} \otimes |b_j\rangle_{D^B}.$$  \hspace{1cm} (II.20)

In this notation Eq. (II.15) becomes

$$\rho = \frac{1}{N} \sum_j \sum_{a_j, b_j} |j, a_j, b_j\rangle\langle j, a_j, b_j| = \frac{1}{N} \sum_j \sum_{a_j, b_j} |j\rangle_{AB} \otimes |a_j\rangle_{D^A} \otimes |b_j\rangle_{D^B} |j\rangle_{AB} \langle j\rangle_{AB} \otimes \langle a_j\rangle_{D^A} \otimes \langle b_j\rangle_{D^B}$$  \hspace{1cm} (II.21)

Reduced density matrices of these states,

$$\rho_{j,a_j} = \text{tr}_B |j, a_j, b_j\rangle\langle j, a_j, b_j|,$$  \hspace{1cm} (II.22)

where here and in the following the tracing out is over $\mathcal{H}_B$, are independent of $b_j$. There are exactly $d^B_j$ of the matrices $\rho_{j,a_j}$, so the reduced density matrix $\rho_A = \text{tr}_B \rho$ is given by

$$\rho_A = \frac{1}{N} \sum_j \sum_{a_j} d^B_j \rho_{j,a_j}.$$  \hspace{1cm} (II.23)

With this notation, it is possible to prove the following result:
Proposition 2 The entanglement of the black hole state $\rho$ is given by

$$E(\rho) = \frac{1}{N} \sum_j \sum_{a_j} d_j^B S(\rho_{j,a_j}) = \frac{1}{N} \sum_j d_j^B d_j^A \log(2j + 1).$$  \hspace{1cm} (II.24)

The proof can be found in Appendix A. Following [10], this result can actually be generalized to more general zero-spin states:

Proposition 3 Consider an arbitrary convex combination of the zero spin states,

$$\rho = \sum_j \sum_{a_j,b_j} w_{a_j,b_j}^{(j)} \langle j |_{AB} \otimes | a_j \rangle_{D^A} \otimes | b_j \rangle_{D^B},$$

$$\sum_{a_j,b_j} w_{a_j,b_j}^{(j)} = 1$$  \hspace{1cm} (II.25)

all measures of entanglement for the state $\rho$ are equal to

$$E(\rho) = \sum_{j,a_j,b_j} w_{a_j,b_j}^{(j)} \log(2j + 1).$$  \hspace{1cm} (II.26)

As an example we consider the entanglement between the block of 2 qubits and the rest. Dividing the system into 2 and $2n-2$ qubits leads to the following degeneracies:

$$d_{0,1}^A = 1, \quad d_0^B = \left(\frac{2n-2}{n-1}\right) \frac{1}{n}, \quad d_1^B = \left(\frac{2n}{n}\right) \frac{3(n-1)}{2(2n-1)(n+1)}.$$  \hspace{1cm} (II.27)

that asymptotically satisfy $d_0^B/N \sim 1/4, d_1^B/N \sim 3/4$. Hence

$$E(\rho|2) = \frac{d_0^B}{N} \log 3 \sim \frac{3}{4} \log 3.$$  \hspace{1cm} (II.28)

Entanglement of 4 and $2n-4$ qubits is hardly more challenging, with asymptotic values of the degeneracies being

$$s_{0}^{(4)} = d_0^A d_0^B / N \sim 1/8, \quad s_{1}^{(4)} = d_1^A d_1^B / N \sim 9/16, \quad s_{2}^{(4)} = d_2^A d_2^B / N \sim 5/16.$$  \hspace{1cm} (II.29)

As a result

$$E(\rho|4) \sim \frac{9}{16} \log 3 + \frac{5}{16} \log 5.$$  \hspace{1cm} (II.30)

Fig. 1 illustrates the convergence of the entanglement to its asymptotic value.

One finds that if the 2n qubits of the horizon are separated into two sets of n qubits each, then the entanglement of the state $\rho$ (asymptotically) equals to $E(\rho|n) \sim \frac{1}{2} \log n$, while the entropy $S(\rho)$ the reduced density matrices $\rho_n$ of its halves is $S(\rho_n) \sim n \log 2$. More technically, this translates to the quantum mutual information [17] between the black hole and its halves being three times the entanglement between the two halves:

$$I_\rho(n : n) \equiv S(\rho_n) + S(\rho_n) - S(\rho) \sim 3E(\rho|n).$$  \hspace{1cm} (II.31)

Let us recall that for any pure entangled state $|\Psi\rangle$, like the singlet state of Eq. [II.14], we have:

$$I_\Psi(A : B) \equiv 2E(\Psi).$$  \hspace{1cm} (II.32)

In a mixed entangled state the classical correlations are present on top of the entanglement, so it is natural to expect that $I_\rho(A : B) > 2E(\rho|A : B)$. From the quantum information-theoretical point of view, it can be argued that the quantum mutual information represents the total amount of classical and quantum correlations [2]. Hence our result shows that the logarithmic correction to the black hole entropy represents the total amount of correlations for the symmetric splitting of the horizon state.

More generally, while the exact entanglement between the two parts of the spin network depends on k, an explicit calculation shows that Eq. [II.31] can be extended to

$$I_\rho(A : B) \approx 3E(\rho|A : B)$$  \hspace{1cm} (II.33)

for any nearly arbitrary partitions into two parts, A and B. For example, for $k = 10$ the coefficient is approximately 2.956.
Let us now focus on the possible physical meaning of the entanglement calculations computed above. Let us consider a pair of qubits. It is in a mixture of having an intertwiner $j = 0$ or $j = 1$ with the rest of the horizon patches. In the case that it has an intertwiner $j = 0$, standard loop quantum gravity tells us that the pair of qubits is actually not linked -*detached* - from the rest of the horizon qubits i.e. it is *not* on the horizon anymore. From the information point of view, a $j = 0$ intertwiner means that the pair of qubits is uncorrelated with the rest of the horizon, so intuitively it can not be part of the black hole horizon. The entanglement, or more precisely the entangled fraction, is a precise quantification of how much the pair of qubits is in the intertwiner state $j = 1$ compared to being in in the intertwiner state $j = 0$. It is therefore natural to assume a relation between the unentangled fraction and the evaporation probability of the pair of qubits (more precisely that there exists a monotonous function relating the two).

Let us consider the unentangled fraction of a pair of qubits $s_0^{(2)}$. It depends of course on the total number of patches $n$, i.e. on the area of the black hole horizon.

The probability of evaporation of the pair of qubit is given by the *Born rule*:

$$\text{Prob}_{\text{evap}} = \text{Tr}(\rho P_{j=0}),$$

where $\rho$ is the black hole density matrix and $P_{j=0}$ the projector on the subspace $\mathcal{H}_{j=0}$ of intertwiners having a link of spin $j = 0$ between the chosen pair of qubits and the rest of the horizon. Then $\rho$ is actually the identity on the intertwiner space $\mathcal{H}$ normalized by the dimension $N = \dim \mathcal{H}$, so that the probability is exactly computed as the unentangled fraction:

$$\text{Prob}_{\text{evap}} = \frac{\dim \mathcal{H}_{j=0}}{\dim \mathcal{H}} = \frac{d_0^{A=2\text{qubits}} d_0^{B=\text{rest}}}{N} = s_0^{(2)}. \quad (\text{II.35})$$

Then to go from probabilities to evaporation rate we need a time scale. In our simple model, we do not have any dynamics. The only way to introduce a time scale is to go to the semi-classical limit: for large black holes, the natural time scale is proportional to its mass $\tau \propto M \propto \sqrt{A} \propto \sqrt{n}$. For example, we can consider the black hole as a clock with its period being the time of flight of the shortest stable circular trajectory around the black hole. Finally, we can conclude that:

$$\frac{dA}{dt} \propto \frac{dn}{dt} \propto -\frac{s_0^{(2)}}{\tau} \propto \frac{s_0^{(2)}}{M} \propto \frac{s_0^{(2)}}{\sqrt{n}}, \quad (\text{II.36})$$

where the proportionality coefficients actually depend on the precise choice of time scale and the exact relation between $n$, the area $A$ and the mass $M$. Since $s_0^{(2)}$ converges to a fixed non-zero value $1/4$ when $n$ grows to infinity, we get a (approximately) fixed probability of evaporation (of the pair of qubits) for large black holes. To make the analysis
more exact, we should consider the exact dependence of the unentangled fraction $s_0^{(2)}$ on the black hole size $n$. Actually, Eq. (II.27) gives us:

$$s_0^{(2)} \sim \frac{1}{4} + \frac{3}{8n}.$$  \hspace{1cm} (II.37)

which leads to a correction to the usual evaporation formula:

$$\frac{dA}{dt} \propto \frac{dn}{dt} \propto -\frac{1}{4\sqrt{n}} - \frac{3}{8n\sqrt{n}}.$$ 

This might allow us to derive precise corrections to the evaporation formula of black holes. Nevertheless, we should then also consider the corrections to the semi-classical mass formula $M \propto \sqrt{n}$. Moreover all relies on the use of a semi-classical time scale, which would need to be derived from the exact dynamics of the underlying quantum gravity theory. At the end of the day, we recover in our simple model the standard Hawking evaporation formula:

$$\frac{dM}{dt} \propto -\frac{1}{M^2}.$$ 

Our analysis is actually very similar to the one performed by Bekenstein and Mukhanov [19].

We would like to point out that we have made the implicit assumption that the evaporation process is completely dominated by the evaporation of a pair of qubits. Indeed, to be thorough, we should consider as well the possible evaporation of bigger blocks. Nevertheless, we have computed the unentangled fraction of a block of $2^k$ qubits (we only consider blocks of even size because of the SU(2) invariance requirement):

$$s_0^{(2k)} \sim \frac{1}{\sqrt{\pi}} \left( \frac{n}{k(n-k)} \right)^{\frac{3}{2}}.$$ 

And it is clear that the unentangled fraction drops rapidly with the block size $k$. Actually, only one out of $2^{2k}$ basis states of the $2k$-qubit block does not contain smaller unentangled pieces, hence the “proper” unentangled fraction (probability of a block of size $k$ to get detached without any of its smaller piece getting detached itself) is $2^{-2k}s_0^{(2k)}$, which now drops exponentially with the block size $k$.

We would like to insist on the fact that our analysis is not dynamical: we rely on the Born rule and treat the black hole evaporation as a traditional decay (like for radioactive emission). To go further, one should consider the geometry state of the interior and exterior of the quantum black hole, and not only the horizon state, and analyze their quantum dynamics. Our main point here is that it does make sense to relate the notion of entanglement between parts of the horizon to probabilities of evaporation from the horizon.

A last point is that the pair of qubits which evaporates will likely not be the Hawking radiation: it forms a part of spacetime which is in the exterior of the black hole. The Hawking radiation will be produced by the difference of energy between the system of a black hole with horizon size $2n$ and the system of a black hole with horizon size $2(n-1)$ plus the detached pair of qubits.

Finally, we have related the entanglement calculations between parts of the horizon to the Hawking evaporation of black holes. As entanglement is a purely quantum notion, this fits with the fact that the evaporation is a purely quantum effect. Moreover, it points toward a link between (quantum) information concepts such as correlation and entanglement and the physics of (quantum) geometry.

### II.4. Black holes at the interplay between quantum gravity and quantum information

As we have illustrated above, quantum information concepts, such as entanglement, are useful for the study of black holes in quantum gravity. This relation goes beyond conceptual points and borrowing the technical tools of the entanglement theory [16]. In particular, the use of SU(2) invariant subspaces in quantum black hole has a precise analogy in the establishing of quantum communication without a shared reference frame [20].

Communication without a shared reference frame (SRF) can be formulated as follows. Assuming that there are no sources of noise, different orientations of the reference frames of the communicating parties induce a unitary map
between their states and operators that describe their experimental procedures. This unitary representation depends on the physical nature of the qubits, usually massive spin-1/2 particles. When the parties have no knowledge about the respective orientation of their frames, the state of the transmitted qubits, the initial state $\rho$ looks averaged over SU(2) as far as the receiver is concerned:

$$\Upsilon(\rho) = \int_{\text{SU}(2)} dU \rho U^\dagger,$$

(II.38)

where $dU$ is the Haar measure of SU(2). The task is to transfer information in the most efficient way despite this decoherence.

Indeed, the transmittable information is to be encoded in a SU(2)-invariant way. For this purpose, to start with, we can use the $j = 0$ subspaces of the space of the $n$ transmitted qubits. For example, in the decomposition of $(V^{1/2})^\otimes 4$, there are two distinct $j = 0$ subspaces, which can be be used to encode the states $|+\rangle$ and $|−\rangle$ of a single qubit. In the notation of Eq. (II.19) it is given, e.g., by

$$|+\rangle \rightarrow |j = 0, 1, 1\rangle, \quad |−\rangle \rightarrow |j = 1, 1, 1\rangle.$$  

(II.39)

As we have seen earlier, the Schur’s duality [13] gives in general:

$$\mathcal{H}_n \equiv \bigotimes_{j=0}^{n} \mathbb{C}^2 = \bigoplus_{j=(n^{1/2})}^{n} V^j \otimes \sigma_{n,j}.$$  

(II.40)

A SU(2) transformation on $\mathcal{H}_n$ will act on the spaces $V^j$ while leaving the degeneracy spaces $\sigma_{n,j}$ invariant. It is then straightforward to check that the map $\Upsilon$ acting on density matrices on $\mathcal{H}_n$ will randomizes the $V^j$ sector (mapping to the maximally mixed state) while preserving the information stored in $\sigma_{n,j}$. More precisely, it is possible to show the following:

**Proposition 4** Consider the space $\mathcal{H}_n$ and the map $\Upsilon$ acting on the space of density matrices over $\mathcal{H}_n$. Let us consider the space $\mathcal{H}_{2n} \equiv \mathcal{H}_n^{(A)} \otimes \mathcal{H}_n^{(B)}$ and its SU(2)-invariant subspace $\mathcal{H}_{2n}^0 \equiv \sigma_{2n,j=0}$. Then $\text{Im}(\Upsilon)$ is given exactly by the reduced density matrices defined as the partial traces over $B$ of density matrices on $\mathcal{H}_{2n}^0$:

$$\text{Tr}_B(\text{Den}(\mathcal{H}_{2n}^0)) = \text{Im}(\Upsilon_n).$$

In particular, for $\rho \in \text{Im}(\Upsilon_n)$, there exists a $\tilde{\rho}$ density matrix on $\mathcal{H}_{2n}$ such that $\rho = \text{Tr}_B(\tilde{\rho})$. Then by definition, the entropy of $\rho$ is $S(\rho) = E(\tilde{\rho}|A:B)$ the entanglement of the state $\tilde{\rho}$.

The most efficient transmission method is a block encoding, when $L$ logical qubits are encoded into $n$ physical ones. To this end the qubits are encoded into the degeneracy labels of the highest multiplicity spin-$j_{\text{max}}$ representation of SU(2). As shown in Sec. 4, the highest multiplicity spin is asymptotically $j_{\text{max}} = \sqrt{n}/2$. The number of logical qubits is the logarithm $\log_2$ of the dimension of the degeneracy space corresponding to $j_{\text{max}}$, hence the efficiency $L/n$ asymptotically approaches 1 as $1 - n^{-1} \log_2 n$.

The same setting can be applied to cryptographic purposes [21]. Imagine that Soraya tries to communicate to Marc and that they share a common reference frame to measure and send qubits. On the other hand, we consider an eavesdropper Tamara who does not have any information about that common reference frame. In this context, for all state $\rho$ that Soraya sends, Tamara will see $\Upsilon(\rho)$. One can define the quantum capacity of the secured quantum channel following [21]. We are looking for the biggest possible Hilbert space $\mathcal{H} \subseteq \mathcal{H}_n$ such that for all density matrix $\rho$ with support in $\mathcal{H}$, $\Upsilon(\rho)$ is the maximally mixed state $\rho_0$ on $\mathcal{H}$ and Tamara will not be able to extract any information about the message which Soraya wants to send to Marc. The quantum capacity is then, as usual, the logarithm of the dimension of the maximal such Hilbert space $\mathcal{H}$.

This is easily done using the tools introduced previously. If $\rho_0$ is the maximally mixed state on $\mathcal{H}$ then $\log(\text{dim}\mathcal{H}) = S(\rho_0)$. Moreover, there exists a state $\tilde{\rho}_0$ such that $\rho_0 = \text{Tr}_B(\tilde{\rho}_0)$, and then the entropy is given by the entanglement $S(\rho_0) = E(\tilde{\rho}_0|A:B))$. Finally, proposition [3] gives us:

$$E(\tilde{\rho}_0) = \sum_{j=0}^{n} \sum_{a_j, b_j} w_{a_j, b_j}^{(j)} \log(2j + 1), \quad \text{with} \quad \sum_{j=0}^{n} \sum_{a_j, b_j} w_{a_j, b_j}^{(j)} = 1.$$
Obviously, the maximal entanglement will be when $w^{(n)} = 1$ and $w^{(j)} = 0$ for all $j < n$: Soraya must use the highest possible spin $j = n$ to encode her messages. Then the quantum capacity is given by the entanglement, so that $\text{QCap} = \log(2^n + 1)$, as obtained in [21].

Finally, [21] notices that the classical capacity of this secured quantum channel is three times its quantum capacity. This factor 3 is exactly the one between the classical correlations and quantum correlations which we obtained earlier in equation \[11.33\].

To conclude this section, we have shown that this particular protocol of quantum communication without shared reference frame allows to translate directly physical properties of the quantum black hole, like its entropy or entanglement, to quantum information notions such as classical and quantum channel capacities. We hope that such a dictionary could be further developed.

III. THE SPIN-1 BLACK HOLE MODEL AND GENERALIZATION

We now describe higher spin models, where we assume that the horizon is made of $n$ elementary patches of a fixed spin $s$. We start by analyzing the 1-spin case and then we generalize the result about the entropy of the general case. These models can be interpreted as coarse-grained models of the fundamental 1/2-spin model when the observer counting the degrees of freedom does not have access to a resolution finer than the unit area $a_s$ set by the elementary patch of spin $s$.

In the spin-1 case, we are interested into the decomposition of multiple tensor products of $V^1$:

$$\bigotimes^n V^1 = \bigoplus_{j=0}^n 1d_j^{(n)} V^j. \tag{III.1}$$

We show below how to compute the degeneracies $1d_j^{(n)}$ in terms of random walk. Then we explain how the random walk calculation extends to the general case of the decomposition of multiple tensor products of $V^s$: 

$$\bigotimes^n V^s = \bigoplus_{j=0}^n s d_j^{(n)} V^j. $$

This allows us to prove that the black hole entropy $S^{(s)}(n) \equiv \log(s d_j^{(n)})$ always have a first term proportional to the horizon area and then a logarithmic correction with the same $-3/2$ factor. Similarly, Eq. \[11.33\] that compares quantum mutual information $I_\rho(A : B)$ for an arbitrary bipartite splitting of the horizon spin network with the entanglement between its halves holds also in this case.

This claim was already made in [10]. However, there, the authors start with the number of conformal blocks for the q-deformed SU(2). We believe that our approach is simpler and more transparent, and that the random walk analogy allows a straightforward generalization to a large class of gauge groups (other than SU(2)). For example, the same results with the universal $-3/2$ factor directly applies to the supersymmetric extension $osp(1|2)$, which is the relevant group for the loop quantization of $N = 1$ 4d supergravity.

III.1. Random walk analogy, entropy renormalisation and universal log correction

Following the same logic as for the 1/2-spin model, we describe the decomposition of the tensor product $\otimes^n V^s$ iteratively. Since for an arbitrary spin $j$ representation we have

$$V^s \otimes V^{j_0} = \bigoplus_{j=|j_0-s|}^{j_0+s} V^j,$$

the iterative process can be thought as a random walk. From the spot $j_0$, one can go in one step with equal probability to any spot from $j_0 - s$ to $j_0 + s$, as long as $j_0$ is larger than the fixed spin $s$. When $j_0$ is smaller than $s$, then we run into the wall at the origin $j = 0$. We formalize this in the following statement:
Proposition 5 The degeneracy of the irreducible spin representation $V_j$ in the decomposition of the tensor product $\otimes^n V^z$ is the number of returns to the spot $j$ after $n$ iterations of a random walk allowing jumps of length up to $s$ with a wall/mirror at $j = 0$. This truncated random walk is such that the number of returns can be simply computed from a standard random walk allowing jumps of length up to $s$ with no obstacle at $j = 0$:

$$s d_j^{(n)} = \text{RWM}_n^{(s)}(j) = \text{RW}_n^{(s)}(j) - \text{RW}_n^{(s)}(j + 1).$$

(III.2)

Then the number of returns of the standard random walk can be computed as an integral:

$$\text{RW}_n^{(s)}(j) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{-ij\theta} \left(e^{-is\theta} + \ldots + 1 + \ldots + e^{+is\theta}\right)^n,$$

(III.3)

for $s$ is an integer and

$$\text{RW}_n^{(s)}(j) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{-i(2j-1)\theta} \left(e^{-i(2s)\theta} + e^{-i(2k-2)\theta} + \ldots + e^{-i\theta} + e^{i\theta} + \ldots + e^{i(2s)\theta}\right)^n,$$

(III.4)

when $s$ is a half-integer.

As in the 1/2-spin case, it is actually possible to derive these integral formulas directly using the characters of $SU(2)$:

$$s d_j^{(n)} = \int dg \, \chi_j(g) \left(\chi_s(g)\right)^n = \frac{2}{\pi} \int_0^{\pi} d\theta \, \sin \theta \sin (2j + 1) \theta \left(\frac{\sin(2s + 1)\theta}{\sin \theta}\right)^n.$$

(III.5)

For $s = 1$, we get explicit formulas of $\text{RW}_n^1(0)$ and $\text{RWM}_n^1(0)$ as sums:

$$\text{RW}_n^1(0) = \frac{n!}{\pi(2\pi)^n} \sum_{l=0}^{\lceil \frac{n}{2} \rceil} \frac{n!}{l!(n-2l)!},$$

(III.6)

$$\text{RWM}_n^1(0) = \frac{n!}{\pi(2\pi)^n} \sum_{l=0}^{\lceil \frac{n}{2} \rceil} \frac{n!}{l!(n-2l)!} - \frac{n!}{\pi(2\pi)^n} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(l+1)!(n-2l-1)!},$$

(III.7)

where $\lfloor y \rfloor$ is the integer part of $y$. Distinguishing the cases with $n$ is odd or even, one can further simply the expression of the truncated random walk$^3$. We can directly evaluate the asymptotics of these numbers of return using Stirling formula. Indeed for large $n$, letting $x = 2l/n \in [0, 1]$, the sums can be approximated by the integrals:

$$\text{RW}_n^1(0) \sim \frac{1}{2\pi} \int_0^1 dx \frac{1}{x\sqrt{1 - x}} e^{\phi(x)}, \quad \text{RWM}_n^1(0) \sim \frac{1}{2\pi} \int_0^1 dx \frac{3x^2 - 2}{x^3 \sqrt{1 - x}} e^{\phi(x)},$$

where the exponent is given by

$$\phi(x) = x \ln 2 - x \ln x - (1 - x) \ln(1 - x).$$

It is straightforward to check that $\phi$ admits a unique maximum at $x = 2/3$ for which $\phi(2/3) = \ln 3$ and $\phi''(2/3) = -9/2 < 0$. We then directly extract the asymptotical behavior of the integrals using the saddle point approximation. Since $1/x \sqrt{1 - x}$ doesn’t vanish at $x = 2/3$ we get:

$$\text{RW}_n^1(0) \propto \frac{3^n}{\sqrt{n}}.$$

(III.8)

$^3$ For example when $n = 2m + 1$, we have:

$$\text{RWM}_n^1(0) = \sum_{l=0}^{m} \frac{n!}{l!(n-2l)!} \frac{3l - n + 1}{l + 1}.$$
while since \((3x - 2)/x^2 \sqrt{1 - x}\) vanishes at \(x = 2/3\) but has a non-vanishing second derivative\(^4\), we get:

\[
\text{RW}_n^1(0) \propto \frac{3^n}{n^{\sqrt{n}}}. \tag{III.9}
\]

A faster and easier calculation is using the saddle point approximation directly on the integral expression given in the previous proposition\(^{\text{[1]}}\). Indeed:

\[
\text{RW}_n^1(0) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, (1 + 2\cos \theta)^n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{n \ln(2\cos \theta + 1)} + \frac{(-1)^n}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{n \ln(2\cos \theta - 1)}. \tag{III.10}
\]

Both exponents \((2\cos \theta + 1)\) and \((2\cos \theta - 1)\) have a unique maximum at \(\theta = 0\). Thus the first term will contribute \(3^n/\sqrt{n}\) to the asymptotic behavior while the second term will only give a term of the order of \(1/\sqrt{n}\). In the end, we simply recover \(\text{RW}_n(0) \sim 3^n/\sqrt{n}\).

As \((1 - \cos \theta) \sim \theta^2/2\), the saddle point approximation directly yields \(\text{RW}_n(0) \sim 3^n/\sqrt{n}\).

This proof can be straightforwardly adapted to the generic spin-\(s\) case, so that we get:

**Proposition 6** We have the following asymptotical behavior of the degeneracy of the trivial representation \(V^0\) in the decomposition of the tensor product \(\otimes^n V^s\):

\[
\text{RW}_n^{(s)}(0) \propto \frac{(2s + 1)^n}{\sqrt{n}}, \quad s_d^{(n)} = \text{RW}_n^{(s)}(0) \propto \frac{(2s + 1)^n}{n^{\sqrt{n}}}. \tag{III.12}
\]

Since the degeneracy of the trivial representation is simply the dimension of the intertwiner space i.e. of the Hilbert space of horizon states, we derive the entropy law:

\[
S_n^{(s)} \sim n \ln(2k + 1) - \frac{3}{2} \ln n + \ldots, \tag{III.13}
\]

where we recognize the first term proportional to the horizon area and the now universal \(-3/2\) factor in front of the logarithmic correction.

Here, we showed that the \(3/2\) correction factor comes directly from the analogy with the random walk and does not depend on the particular model (choice of spin for the black hole elementary surfaces) that we chose. This proves that the \(3/2\) factor is actually invariant under coarse-graining i.e. under the choice of area unit that we use to probe the horizon of the black hole: this is therefore a universal factor and rigid prediction from these particular models of quantum black holes.

Moreover, we can refine our evaluation of the degeneracies \(s_d^{(n)}\) pushing further the approximations of the integrals giving \(\text{RWM}\):

\[
s_d^{(n)} \sim \frac{3}{8} \sqrt{\frac{3}{2\pi}} \frac{(1 + 2s)^n}{[s(1 + s)]^{5/2}} \frac{4s(1 + s)n - 3 - 6j(1 + j)}{n^{5/2}} \frac{(1 + j)}{n^{15/2}} \sim \frac{(1 + 2s)^n(1 + j)}{s(1 + s)n^{15/2}}. \tag{III.14}
\]

This allows us to show the following statement.

---

\(^4\) Let us consider the integral \(\int_0^1 dx \, f(x)e^{\alpha \phi(x)}\). Assuming that \(\phi\) has a unique maximum at \(x_0 \in [0, 1]\), we write \(\phi(x) = a - b(x - x_0)^2\) with \(b > 0\). Then we expand \(f\) around \(x_0\) and doing the change of variable to \(y = (x - x_0)/\sqrt{b}\), which gives the asymptotic behavior of the integral as:

\[
\frac{e^{\alpha b}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dy \left( f(x_0) + f'(x_0)\frac{y}{\sqrt{b}} + \frac{1}{2} f''(x_0) \frac{y^2}{b} + \ldots \right) e^{-by^2}.
\]

It is clear that when \(f(x_0)\) vanishes, the first derivative \(f'(x_0)\) doesn’t play into role in the approximation since \(\int dy \, ye^{-by^2} = 0\), so that the first relevant term is the second derivative \(f''(x_0)\).
Proposition 7 We have the following asymptotical behavior of the maximal dimensionality \( s d_j^{(n)}_{\text{max}} \) of the representation spaces \( V^j \) in the decomposition of the tensor product \( \otimes^n V^j \):
\[
j_{\text{max}} \propto n^{1/2}, \quad s d_j^{(n)}_{\text{max}} \propto \frac{(2s+1)^n}{n}. \tag{III.14}
\]
The proportionality coefficients depend on \( s \).

The calculations leading to these results can be found in Appendix B.

III.2. Elementary scale and irreducibility

The structure of spacetimes as given by LQG is fundamentally discrete. Hence the diffeomorphism symmetry that is defined on a smooth manifold is replaced by a discrete symmetry on a spin network [11]. The Hilbert space of LQG is spanned by the spin networks, and it carries a representation of the permutation group. Any representation of the permutation group of \( n \) objects \( S_n \) can be either faithful or reducible. Moreover, for large \( n \) the squared dimension of \( (\mathbb{C}^k)^{\otimes n} \) is smaller than \( n! \), so the faithful representation of the permutation group on the spin network any size is impossible. There is no a priori reason [11] to require that all the edges of a spin network are labeled by the same fundamental representation \( \mathbf{j} \), since it is not necessary for building representations of the permutation group. However, we can see that the requirement of the irreducibility of the representation is equivalent to describing spin networks only in terms of the fundamental representation \( s = \frac{1}{2} \). Namely [12, 22], the Shur’s duality that allows to decompose \( (\mathbb{C}^N)^n \) as a direct sum of tensor products of the irreps of \( SU(N) \) and \( S_k \) for the direct product of \( U(N) \) holds only when the dimension of the elementary space is \( N \). For \( SU(2) \) it takes the form
\[
\bigotimes \mathbb{C}^2 \cong \bigoplus_{j=0}^{2n} V^j \otimes \sigma_{n,j}, \tag{III.15}
\]
where \( V^j \) is the irreducible spin-\( j \) representation of \( SU(2) \), and the irreps \( \sigma_{n,j} \) correspond to the partition \( [n+j, n-j] \) of \( 2n \) objects [14]. There is no similar structure for the direct product of different vector spaces on the left hand side, and the isomorphism in the case of the direct product of \( \mathbb{C}^N \) involves the irreducible representations of \( SU(N) \).

IV. ON AREA RENORMALISATION IN LQG

Let us look at a generic surface, made of \( 2n \) elementary patches of spin-1/2. Unlike the previous case we allow for open surfaces. Then the Hilbert space of surface states is simply the tensor product \( \otimes^{2n} V^{1/2} \). The precise state describes how the geometry of the surface, the way it is folded. It is the surrounding spin network, in which the surface is embedded, which provides the information on the surface state and the way that the surface is precisely embedded in the 3d space.

Now the surface can be folded any way at the microscopic scale. Then depending on how the surface is folded, an observer at a larger will see a smooth surface which is only a approximation of the “real” surface and will measure an area smaller than the full microscopic area \( \mathcal{A}_{\text{micro}} = 2n a_{1/2} \).

Let us start by the case when we completely coarse-grain the surface state. That is we decompose \( \otimes^{2n} V^{1/2} = \bigoplus_j d_j^{(n)} V^j \), and the total spin \( j \) will describe the macroscopic area of the completely coarse-grained surface. For example, for \( 2n = 2 \), \( V^{1/2} \otimes V^{1/2} = V_0 \oplus V^1 \) and we have two coarse-grained area states: \( j = 0 \) which corresponds to the surface folded on itself and \( j = 1 \) which corresponds to the open surface with the two elementary patches side by side.

For large \( n \), one can describe the completely coarse-grained state for the trivial state of geometry i.e. when we do have no information at all the state of the surface. Then the most probable macroscopic area is given by the spin \( j \) with the largest degeneracy \( d_j^{(n)} \). Given the explicit expression of these degeneracy factors as \( d_j = C_{2n}^{n+j} - C_{2n}^{n+j+1} \), we easily write that
\[
\frac{d_j^{(n+1)}}{d_j^{(n)}} = \frac{(n-j)(2j+3)}{(n+j+2)(2j+1)},
\]
and identify the maximal entropy for:

\[ j_{\text{max}}(\frac{\phi}{2}) = \sqrt{\frac{n}{2}} \quad (\text{IV.1}) \]

One can reproduce the same result by using the Stirling formula approximation\(^5\) to \(d_j\) for large \(n\). This square-root law is natural from the point of view of the random walk analogy and it is actually true in any \(s\)-spin model, when looking at the maximal degeneracy in the decomposition of the tensor product \(\otimes^{2n} V^s\) (see Appendix B for details).

However, as soon as we have a non-trivial geometry state and have more information of the tensor product state corresponding to the surface at the microscopic level, we do not expect the macroscopic area to follow this rule anymore. Nevertheless, we would like to insist on the fact that the maximal area given by the microscopic area \(2n a_j/2\) is totally improbable at the macroscopic level: its degeneracy is simply 1 while typical degeneracies grow as \(2^{2n}/n\sqrt{n}\) and the maximal degeneracy as \(2^{2n}/n\). Therefore we believe that the study of area reorganization in LQG will not be as straightforward as assuming a simple linear rescaling of the microscopic area.

More generally, we do not need to go directly at the completely coarse-grained level. Physically, we are interested by the whole coarse-graining process: for example going from the fundamental scale set the \(1\)-\(2\)-spin to a larger semi-classical scale given a fixed large \(s\)-spin. A simple calculation one can do considering \(2n\) qubits is to take a subset of \(m\) qubits and compute the reduced density matrix of the subset from the state (possibly mixed) of the full surface. Then one will know the most likely area of this subsurface formed by the considered \(2k\) qubits. Pushing this set-up further, we can partition the full surface into \(p\) bigger patches formed by \(n/p\) \(1/2\)-spins and compute the reduced density matrices of each bigger patch and thus obtain a coarse-grained description of the surface. A first step would be to do this calculation in the framework of the totally mixed state corresponding to the black hole state.

Indeed, a reduced density matrix for a \(2k\) qubit patch is

\[ \rho(2k) = \frac{1}{N} \sum_{j=0}^{k} \sum_{a_j=1} d_{j,a}^A d_{j,a}^B \rho_{j,a}, \quad (\text{IV.3}) \]

At this stage of the analysis it is not clear which prescription to calculate a coarse-grained value of \(j\) should be used. There are three reasonable options that all lead to \(j_{\text{coarse}} \sim \sqrt{E}\), while the numerical factors are of the order of unity.

- The expectation value of the patch’s spin is

\[ \langle J \rangle = \text{tr} J \rho(2k), \quad (\text{IV.4}) \]

where \(J = \sqrt{J^2}\) or any other suitable function of \(J\). Since all the density matrices are diagonal in the angular momentum basis, this expectation value can be calculated as

\[ \langle J \rangle = \sum_{j} J(j) p_j^{(n,k)}, \quad p_j^{(n,k)} = d_j^{(2k)} d_j^{(2n-2k)}/N, \quad (\text{IV.5}) \]

and, e. g., \(J = \sqrt{j(j+1)}\). We calculate the expectation value of \(J\) by replacing the sum in Eq. \(\text{[IV.5]}\) by the integral. Using the asymptotic expansion of Eq. \(\text{[IV.2]}\) the normalization of the probability distribution \(p_j^{(n,k)}\)

---

\(^5\) Letting \(x = j/n \in [0,1]\), we have

\[ d_j^{(n)} \approx d_j^{(n)}(x) = \frac{2^{2n}}{\sqrt{n^2}} f(x) e^{-n \phi(x)}, \quad (\text{IV.2}) \]

with

\[ f(x) = \frac{1}{\sqrt{1-x^2}} \frac{2x + \frac{n}{2}}{1 + x + \frac{n}{4}}, \quad \phi(x) = (1-x) \ln(1-x) + (1+x) \ln(1+x). \]

The maximum is given by the equation \(n \phi'(x) = (\ln f)'(x)\). For \(x_{\text{max}} \approx 0\), we get \(x_{\text{max}} \sim 1/\sqrt{2n}\). Moreover if we prefer to look at the maximum of \((2j+1)d_j^{(n)}\), then one simply multiply the function \(f\) by \(2x + 1/n\) and then the maximum is located at \(x_{\text{max}} \sim 1/\sqrt{n}\).
is fixed by
\[
\int_0^\infty x^2 e^{-(\mu+1)kx^2} = N',
\]
where \(\mu \equiv k/(n-k) \approx k/n\), all the constants where absorbed in \(N'\) and \(x_{\text{max}} \approx 1/\sqrt{2k} \to 0\) was used. Taking \(J \approx j\) the expectation value becomes
\[
\langle J \rangle \approx \langle j \rangle = \frac{k}{N'} \int_0^\infty x^3 e^{-(\mu+1)kx^2} = \sqrt{\frac{2}{\sqrt{\pi}}},
\]
Since for large spins the elementary area is approximately proportional to the spin,
\[
\frac{A_{\text{macro}}}{A_{\text{micro}}} \approx \frac{2}{a_{1/2}\sqrt{\pi}} \sqrt{\frac{p}{n}},
\]
as illustrated on Fig. 2 below.

\[\begin{array}{c|c}
A_{\text{macro}} & 1750 \\
1500 & 1500 \\
1250 & 1250 \\
1000 & 1000 \\
750 & 750 \\
500 & 500 \\
250 & 250 \\
500 & 500 \\
1000 & 1000 \\
1500 & 1500 \\
2000 & 2000 \\
\end{array}\]

\[\begin{array}{c|c}
p & 500 \\
1000 & 1000 \\
1500 & 1500 \\
2000 & 2000 \\
\end{array}\]

\[\begin{array}{c|c}
0 & 0 \\
20 & 20 \\
40 & 40 \\
60 & 60 \\
80 & 80 \\
100 & 100 \\
120 & 120 \\
\end{array}\]

**FIG. 2:** The macroscopic area of the \(2n = 2000\) qubit horizon as a function of the number of patches, \(p = 2, 4, 5, \ldots , 2000\). The square-root area spectrum is assumed, \(a_j \equiv \sqrt{j(j+1)}\), and \(\langle J \rangle\) is taken to represent the patch.

- Considering \(\langle J \rangle\) as an expectation values of a spin measurement, the most likely spin value \(j\) that occurs in such a measurement is \(j_{\text{max}}^{(n,k)}\) that maximizes the probability distribution \(p_j^{(n,k)}\). When \(1 \ll k\) the spin of the highest degeneracy representation is given by Eq. (C.7), and for \(k \ll n\) it reduces to
\[
j_{\text{max}} = \sqrt{k}.
\]

- The most likely state \(|j, j_z\rangle\) (again summing over all the degeneracy labels) corresponds to \(j\) that maximizes
\[
p_{j,z}^{(n,k)} = \frac{p_j^{(n,k)}}{2j + 1}.
\]

The value of \(j\) can be determined similarly to \(j_{\text{max}}\), and for \(1 \ll k \ll n\) it is
\[
j = \sqrt{k/2}.
\]

For all the three choices of the coarse-grained \(j\) the area of the patch is
\[
A_{(2k)} = a_{1/2}/\sqrt{\alpha}
\]
where \(\alpha \sim 1\) depends on the coarse-graining procedure. As a result, the coarse-grained area of the surface is
\[
A_{\text{macro}} = p A_{(2k)} = a_{1/2}/\sqrt{\sqrt{np}}.
\]

In the end, it would be interesting to generalize these area renormalisation calculations to more generic situations in loop quantum gravity, and thereby underline the distinction between the microscopic geometry and the semi-classical measurable and observable geometry.
V. CONCLUSIONS AND OUTLOOK

We have studied simple models for black holes in the framework of loop quantum gravity. We consider the black hole horizon as a quantum surface made of a certain number \( n \) of identical elementary surfaces. These elementary quantum surfaces are mathematically modeled as spin-\( s \) systems, while the whole horizon must be a singlet state (or intertwiner) in the tensor product of these \( n \) spins \( s \). Actually our framework applies to generic quantum closed surfaces as soon as we coarse-grain the interior geometry.

We have computed the entropy of black holes in this context and recovered the usual entropy with a logarithmic correction. The factor in front of that correction is actually \(-3/2\) whatever the spin \( s \) which we choose. We also analyzed the entanglement between parts of the horizon. More technically, we computed the entanglement for splitting of the horizon into two parts of arbitrary sizes. We have shown that this entanglement is responsible for the logarithmic correction to the entropy law. More precisely, the logarithmic correction is the total correlation between the two parts of the horizon and is equal to three times the entanglement. Moreover, we also point out a potential relation with the evaporation process, which allows us to recover at first order the Hawking evaporation formula. From our perspective, quantum black holes appear at the interplay between quantum gravity and quantum information, and we show how all the objects appearing in our analysis of the black hole can be translated to the framework of quantum communication without shared reference frame.

Finally, we have used all the developed technology to illustrate a concept of renormalisation of areas in loop quantum gravity: given a surface at the microscopic level, it might be folded in ways not observable to a macroscopic observer, so that the macroscopically observed area is different to the microscopic area postulated by loop quantum gravity. And we formulate a square-root law for the renormalisation flow of the area. We see a few possible development of the present work:

- One can try to introduce the dynamics into these simple black hole models, using the quantization of the ADM energy in loop quantum gravity. This would allow to study the evolution of the entanglement, potentially look at the entanglement decay and the evaporation process.

- One can try to take into account the rotation of black holes in our models. However, this means to introduce more objects, relative to which we would define the rotation of the black hole. This would need to model the exterior of the black holes and think about spin network geometries whose semi-classical regimes are approximately the Schwarzschild metric.

- It would be interesting to push further our analysis of area renormalisation, and study this concept in more complicated situation that the black hole state which turned out to be rather simple. We could also look at the renormalisation of other geometrical quantities such as the volume.

- Finally, we can push further the use of entanglement and correlations as tools to probe the geometry of spin networks in LQG. One can naturally speculate on a link between the notion of distance and the actual correlations between subsystems of a given spin networks, similarly to what happens in spin lattices. Work on this topic will be reported elsewhere \[24\].

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APPENDIX A: ENTANGLEMENT OF THE BLACK HOLE STATE

Here we prove the Proposition \[2\] giving the exact formula for the entanglement of the black hole state. In the basis we adopted the states \( |j, a_j, b_j\rangle \) are given be the bi-orthogonal (Schmidt) decompositions \[2\], so their reduced density matrices are diagonal. Moreover, the only nonzero part of \( \rho_{j, a_j} \) is a sequence of \( 2j + 1 \) terms \( 1/(2j + 1) \). Using the
decomposition of $A$ into $V^j$ and $D^j$ subspaces,

$$
\rho_{j,a_j} = \frac{1}{2j+1} \mathbb{1}_{V^j} \otimes |a_j\rangle\langle a_j|_{D^j}.
$$

(A.1)

For the future convenience we introduce $\rho_j = \mathbb{1}_{V^j}/(2j+1)$. Two matrices $\rho_{j,a_j}$ and $\rho_{l,a_l}$ have orthogonal supports if one or both of their indices are different. The reduced density matrix $\rho_A = \text{tr}_B \rho$ is

$$
\rho_A = \frac{1}{N} \sum_j \sum_{a_j} \sum_{b_j} d_j^B \rho_j \otimes |a_j\rangle\langle a_j|_{D^j}.
$$

(A.2)

Since the von Neumann entropy of the reduced density matrices $\rho_j$ satisfies

$$
S(\rho_{j,a_j}) = S(\rho_j) = \log(2j+1),
$$

(A.3)

the weighted average of the entanglement in this decomposition is

$$
S_E(\rho) = \frac{1}{N} \sum_{j=0}^m \sum_{a_j} \sum_{b_j} \sum_{a_j'} \sum_{b_j'} c_{\alpha,j,a_j,b_j}^* c_{\alpha,j,a_j',b_j'} \rho_j \otimes |a_j\rangle\langle a_j|_{D^j} \otimes |b_j\rangle\langle b_j|_{D^j}.
$$

(A.4)

All pure states $|\Psi_\alpha\rangle$ that appear in alternative decompositions of $\rho$ ought to be some linear combinations of the states $|j, a_j, b_j\rangle$,

$$
|\Psi_\alpha\rangle = \sum_{j,a_j,b_j} c_{\alpha,j,a_j,b_j} |j\rangle_{AB} \otimes |a_j\rangle_{D^j_A} \otimes |b_j\rangle_{D^j_B}.
$$

(A.5)

The diagonal form of $\rho$ forces the coefficients $c_{\alpha,j,a_j,b_j}$ to satisfy the normalization condition

$$
\sum_{\alpha} w_\alpha c_{\alpha,j,a_j,b_j}^* c_{\alpha,j,a_j',b_j'} = \frac{1}{N} \delta_{a_j a_j'} \delta_{b_j b_j'}.
$$

(A.6)

The reduced density matrices $\rho_A(\alpha) = \text{tr}_B |\Psi_\alpha\rangle\langle \Psi_\alpha|$ are

$$
\rho_A(\alpha) = \sum_j \sum_{a_j} \sum_{a_j'} c_{\alpha,j,a_j,b_j}^* c_{\alpha,j,a_j',b_j} \rho_j \otimes |a_j\rangle\langle a_j|_{D^j_A}.
$$

(A.7)

Introducing

$$
\lambda_{\alpha,j,a_j,a_j'} = \sum_{b_j} c_{\alpha,j,a_j,b_j}^* c_{\alpha,j,a_j',b_j}, \quad \pi_j(\alpha) = \sum_{a_j} \lambda_{\alpha,j,a_j,a_j},
$$

(A.8)

we rewrite the reduced density matrix of $|\Psi_\alpha\rangle\langle \Psi_\alpha|$ as

$$
\rho_A(\alpha) = \sum_j \pi_j(\alpha) \rho_j \otimes \Lambda_j(\alpha),
$$

(A.9)

where the fictitious density matrix $\Lambda$ was introduced on the degeneracy subspace $D^j_A$,

$$
\Lambda_j(\alpha) = \frac{1}{\pi_j(\alpha)} \sum_{a_j,a_j'} \lambda_{\alpha,j,a_j,a_j'} |a_j\rangle\langle a_j|_{D^j_A}.
$$

(A.10)

From the orthogonality of the matrices $\rho_{j,a_j}$ and Eq. (A.6) it is easy to see that

$$
\sum_{\alpha} w_\alpha \lambda_{\alpha,j,a_j,a_j'} = \frac{1}{N} d_j^{B} \delta_{a_j a_j'}.
$$

(A.11)

The weighted average of the entanglement of the decomposition $\{\Phi_\alpha\}$ is $\langle S(\{\Phi_\alpha\}) \rangle \equiv \sum_{\alpha} w_\alpha S(\rho_A(\alpha))$. From Eq. (A.10) and the concavity property of entropy [2, 23] it follows that
\[
\langle S(\{\Phi_\alpha\}) \rangle \geq \sum_{\alpha,j} w_\alpha \pi_j(\alpha) [S(\rho_j) + S(A_j(\alpha))] \geq \sum_{\alpha,j} w_\alpha \pi_j(\alpha) S(\rho_j) = \frac{1}{N} \sum_j d_j^{(a)} S(\rho_j). \quad (A.12)
\]

Hence \(S_E(\rho)\) is indeed the entanglement of formation. The equality of all measures of entanglement for the black hole states is proven in [16].

APPENDIX B: EVALUATION OF THE DEGENERACIES \(s_d^{(n)}\) FOR ARBITRARY SPIN

We consider the integer spin. The half-integer spin is treated in exactly the same way. From Eq. (III.3) it follows that

\[
RWM^n_s(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta f(j,\theta)e^{n \log F(s,\theta)},
\]

where \(f(j,\theta) = \cos(j\theta) - \cos((j+1)\theta)\) and

\[
F(s,\theta) = \frac{2\cos(\frac{a\theta}{2}) \sin \left(\frac{(1+s)\theta}{2}\right)}{\sin \left(\frac{\theta}{2}\right)} - 1. \quad (B.2)
\]

For a fixed \(j\) and \(n \to \infty\) the saddle point method gives the following estimate of the degeneracy:

\[
s_d^{(n)} \approx 3\sqrt{\frac{3}{2\pi}} \frac{(1+2s)^n}{s(1+s)^{5/2}} \frac{4s(1+s)n - 3 - 6j(1+j)}{n^{5/2}} (1+j). \quad (B.3)
\]

Hence

\[
s_d^{(n)} \propto n \to \infty \frac{(1+2s)^n(1+j)}{(s(1+s)n)^{3/2}} \quad (B.4)
\]

Having in mind the search for \(j_{\text{max}}\), \(s_d^{(n)} \propto n \to \infty\), in deriving (B.3) we expanded \(f(j,\theta)\) as \(f(j,\theta) \approx a\theta^2 + b\theta^4\). Indeed, solving \(s_d^{(n)} / dj = 0\) we get

\[
j_{\text{max}} \approx \frac{\sqrt{8k(1+s)n} - 3 - 1 / 2}{6}, \quad s_d^{(n)} \propto n \to \infty \frac{(1+2k)^n}{n}. \quad (B.5)
\]

A comparison with the exact results for the degeneracies for \(s = \frac{1}{2}\) and \(s = 1\) shows an excellent agreement with the asymptotics for \(j \lesssim \sqrt{n}\). In both cases numerics show that the asymptotic estimate of \(j_{\text{max}}\) given above differs from its exact value by a factor of \(\sqrt{2/3}\) and that the maximal degeneracies are of the same order of magnitude.

APPENDIX C: ENTANGLEMENT CALCULATIONS

Since Eq. (A.4) gives the entanglement of formation of \(\rho\), it is necessary to find the multiplicities of the spin-\(j\) subspaces. For \(s = \frac{1}{2}\) the result is

\[
d_j^{(2k)} = \binom{2k}{k} \frac{2j + 1}{k + j + 1}. \quad (C.1)
\]

where instead of the subspace labels \(A, B\) we write the number of qubits. The multiplicities satisfy the following normalization conditions:

\[
\sum_{j=0}^{k} d_j^{(2k)} d_j^{(2k-2k)} = N, \quad \sum_{j=0}^{k} d_j^{(2k)} = \binom{2k}{k}, \quad \sum_{j=0}^{k} d_j^{(2k)} (2j + 1) = 2^{2k}. \quad (C.2)
\]
We start the entanglement evaluation by considering two equal subsystems of \( n \) qubits each. In this case \( d_{j}^{A,B} = d_{j} \), so

\[
E(\rho|n:n) = \frac{1}{N} \sum_{j=0}^{n/2} d_{j}^{2} \log(2j + 1). \tag{C.3}
\]

In the leading order

\[
E(\rho|n:n) \sim \log(2j_{\text{max}}^{(n/2)} + 1) \cdot 1, \tag{C.4}
\]

where the coefficient \( d_{j}^{(2k)} \) reaches its maximal value at

\[
j = j_{\text{max}}^{(k)} \approx \frac{1}{2} \sqrt{2k + 2} - 1, \tag{C.5}
\]

so

\[
E(\rho|n:n) \sim \frac{1}{2} \log n. \tag{C.6}
\]

Numerical simulations show that \( E(\rho|n:n) - \frac{1}{2} \log n \approx 0.0183 \).

In a generic case of \( 2k : 2n - 2k \) splitting for \( 1 \ll k \leq n/2 \) the sum in Eq. (A.4) can be evaluated similarly to Eq. (C.4). In this case

\[
j_{\text{max}}^{(n,k)} \approx \frac{1}{2} \sqrt{2 + 3n + 4k(n-k)} \cdot n + 1 - 1, \tag{C.7}
\]

which reduces to the result of Eq. (C.5) for the \( n : n \) splitting. It is also interesting to note the fraction of the unentangled states in the decomposition of \( \rho \)

\[
s_{0}^{(2k)} = \frac{d_{0}^{(2k)} d_{0}^{(2n-2k)}}{N}, \tag{C.8}
\]

go down with \( k, 1 \ll k \leq n/2 \),

\[
s_{0}^{(2k)} \sim \frac{1}{\sqrt{n}} \left( \frac{n}{k(n-k)} \right)^{3/2}. \tag{C.9}
\]

The quantum mutual information was introduced in Sec. 2.2. The reduced density matrix of a smaller block for \( 2k : 2n - 2k \) splitting is

\[
\rho(2k) = \frac{1}{N} \sum_{j=0}^{k} \sum_{a_{j}} d_{j}^{A} d_{j}^{B} \rho_{j,a_{j}}, \tag{C.10}
\]

and it entropy is

\[
S(\rho(2k)) = - \sum_{j=0}^{k} (2j + 1) d_{j}^{A} \left( \frac{d_{j}^{B}}{N} \frac{1}{2j + 1} \right) \log \left( \frac{d_{j}^{B}}{N} \frac{1}{2j + 1} \right). \tag{C.11}
\]

In particular, for the \( n : n \) decomposition

\[
S(\rho(n)) = E(\rho) + \sum_{j=0}^{n/2} d_{j}^{2} \frac{N}{d_{j}} \log \frac{N}{d_{j}} = E(\rho) + \log N - \sum_{j=0}^{n/2} d_{j}^{2} \log d_{j}. \tag{C.12}
\]
Again we approximate
\[
\sum_{j=0}^{n/2} \frac{d_j^2}{N} \log d_j \sim \sum_{j=0}^{n/2} \frac{d_j^2}{N} \log d_{j+1} = \log d_{j+1},
\]
where \( \log d_{j+1} \sim n \log 2 - \log n \), so
\[
S(\rho_{(n)}) \sim E(\rho|n : n) + n \log 2 - \frac{1}{2} \log n + \ldots \sim n \log 2.
\]
As a result,
\[
I_\rho(n : n) \equiv 2S(\rho_{(n)}) - S(\rho) \sim 3E(\rho|n : n).
\]

Using the results of Appendix B it is possible to show that the above relation is true for any fundamental spin \( s \).

Indeed, from Eq. (B.4) it follows that \( N_s(\rho) \equiv \dim H_s \) satisfies
\[
\log N_s(\rho) = \log^* d_j(\rho)(0) \sim n \log(1 + 2k) - \frac{3}{2} \log n,
\]
from (B.5) and (C.13) the entanglement between the halves is
\[
E(\rho) \sim \frac{1}{2} \log n,
\]
and the entropy of one-half of the horizon state is
\[
S(\rho_{n/2}) \sim \frac{n}{2} \log(1 + 2k) - \log n.
\]
As a result,
\[
I_\rho^s(n : n) \sim \frac{3}{2} \log n.
\]

**APPENDIX D: OVERVIEW OF THE REPRESENTATIONS OF THE PERMUTATION GROUP**

We list some of the basic properties of the representations of \( S_n \). An exhaustive discussion can be found, e.g., in \cite{13, 14, 22}. Irreducible representations of a permutation group are labeled by the Young tableaux, and those that correspond to the irreps \( \sigma_{n,j} \) consist of just two rows.

\[
\begin{bmatrix}
1 & 2 & \ldots & n + j \\
n + j + 1 & \ldots & 2n
\end{bmatrix}
\]

The Young tableau of Eq. (D.1) is a *normal table*, where the numbers 1, 2, \ldots, 2n appear in order from left to right and from the upper row to the lower row. The general formulas that define the dimensionality of the irreducible representations of the permutation group reduce in the case of \( \sigma_{n,j} \) to \( d_j(2n) \) whose properties are described in Appendix C. The dimension of a representation equals to the number of distinct *standard tableaux*, where number in each row appear increasing (not necessarily in strict order) to the right and in each column appear increasing to the bottom. The rule for their ordering will be given below. For each partition \( [\nu] \) we enumerate the standard tableaux as \( ([\nu], m) \), where \( m = 1, \ldots, \dim \sigma_{[\nu]} \).

The standard orthonormal basis for the irreducible representation of \( S_n \) that corresponds to the partition \( [\nu] \) (Young-Yamanouchi basis) is labeled by the irreducible representations of the groups in the subgroup chain \( S_n \supset S_{n-1} \supset \ldots \supset S_2 \) to which it belongs. A simple rule describes this construction: from a given Young tableau \( ([\nu], m) \) of \( n \) objects one obtains another tableau \( ([\nu'], m') \) of \( n - 1 \) objects by removing the box that contains the number \( n \), etc. For example, a particular basis vector of the [31] irrep of \( S_4 \) is labeled by the following chain of irreps:

\[
\begin{bmatrix}
1 & 3 & 4 \\
2 & 1 & 3 \\
2 & 1
\end{bmatrix}
\]
These two quantities define the irreducible (or Young) symmetrizer. Then the symmetrizer $S$ i.e. this irreducible basis vector belongs the irrep $^{[31]}$ of $S_4$, $^{[21]}$ of $S_3$, and $^{[11]}$ of $S_2$. This labeling uniquely specifies the basis vectors. They are ordered by the Yamanouchi symbols $(r_nr_{n-1}\ldots r_1)$, where $r_i$ is the row number of $i$. For example, the above vector corresponds to the Yamanouchi symbol $(1121)$. Once all the standard tableaux of $^{[\nu]}$ are labelled by their Yamanouchi symbols, they can be linearly ordered, with the vector with the largest Yamanouchi symbol being labelled $^{[\nu]}$, the vector with the second largest symbol being $^{[\nu]}$, etc. In practice, the irreducible vectors are found as the simultaneous eigenvectors of the complete commuting set of operators. In the case of a permutation group, this set consists of the 2-cycle class operators for all subgroups of the chain $S_n \supset S_{n-1} \supset \ldots \supset S_2$.

The matrix elements of all operators that act on the irreducible representation space $^{[\nu]}$ of $S_n$ can be reconstructed from the representations of $n-1$ generators of $S_n$, which are the 2-cycles permutations $(12),(23),\ldots,(n-1,n)$. Their matrix elements in the $^{[\nu]}m$ basis are obtained from the following set of rules:

1. If $i-1$ and $i$ are in the same row or column of a standard tableau $(^{[\nu]}r_1 \ldots ^{[\nu]}r_n)$, then

$$<i-1,i>|^{[\nu]}m, \rangle = \pm|^{[\nu]}m, \rangle, \quad (D.3)$$

with the plus sign taken when both numbers are in the same row.

2. If $i-1$ and $i$ are not in the same row or column, then the matrix element $D^{[\nu]}_{m'm}$ is defined with the help of the axial distance $l$. It is calculated as follows: starting from the box that contains $i-1$ one proceeds by a rectangular route, stepping one box each time, until the box with $i$ is reached. Each step right or upward contributes $+1$ to the axial distance, while a step downwards or left decreases $s$ by 1. The matrix elements

$$D^{[\nu]}_{m'm} = \begin{cases} 
\frac{1}{l}, & m' = m \\
\sqrt{l^2-1}/l, & ^{[\nu]}(i-1,1)\equiv (i-1,1)|^{[\nu]}m, \rangle \\
0, & \text{otherwise}
\end{cases} \quad (D.4)$$

are given by

For a given standard tableau $(^{[r_1]}r_2 \ldots r_n)$ and a permutation $(i-1,i)$ the axial distance is

$$l = c_i - c_{i-1} - (r_i - r_{i-1}), \quad (D.6)$$

where $c_i$ is a column number of the object $i$. For the two-row Young tables the column numbers are easily obtained from the corresponding Yamanouchi symbol as

$$c_i = \begin{cases} 
2i - \sum_{j=1}^{i} r_j, & r_i = 1 \\
\sum_{j=1}^{i} r_j - i, & r_i = 2
\end{cases} \quad (D.7)$$

For example

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 3 \end{pmatrix} (34) = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 3 \end{pmatrix} (34) = \begin{pmatrix} 1 & 2 & 3 \\ 4 \end{pmatrix} = \frac{\sqrt{8}}{3} \quad (D.8)$$

An alternative method of constructing irreps of $S_n$ is to work on its group algebra $\mathbb{C}S_n$. For a given Young tableau $\lambda \equiv (^{[\nu]}r_1 \ldots r_n)$ one defines a subgroup $P \subset S_n$ that preserves the rows of $\lambda$, and the subgroup $Q_\lambda$ that preserves its columns. Then the symmetrizer $a_\lambda$ and the antisymmetrizer $b_\lambda$ are defined as

$$a_\lambda = \sum_{g \in P} g, \quad b_\lambda = \sum_{g \in Q} \text{sign}(g)g. \quad (D.9)$$

These two quantities define the irreducible (or Young) symmetrizer

$$s_\lambda = a_\lambda b_\lambda, \quad (D.10)$$
which generates the irreducible representation $\lambda$ that corresponds to the partition $[\nu]$ by right multiplication on $CS_n$. The symmetrizers that are obtained from different standard tableaux of the same partition lead to the distinct equivalent irreps.

[1] C. Rovelli, *Loop Quantum Gravity*, (Cambridge University Press, Cambridge, 2004);
T. Thiemann, *Lectures on Loop Quantum Gravity*, Lect. Notes Phys. **631**, 41-135 (2003), gr-qc/0210094.
A. Ashtekar and J. Lewandowski, *Background Independent Quantum Gravity: A Status Report*, Class. Quant. Grav. **21** (2004) R53, gr-qc/0404018.

[2] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).

[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, New York, 2000);
M. Keyl, *Fundamentals of Quantum Information Theory*, Phys. Rep. **369**, 431 (2002), quant-ph/0202122.

[4] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, 2001);
G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, *Entanglement in Quantum Critical Phenomena*, Phys. Rev. Lett. **90**, 227902 (2003), quant-ph/0211074.

[5] A. Corichi, *Comments on area spectra in Loop Quantum Gravity*, gr-qc/0402061.

[6] R. M. Wald *The Thermodynamics of Black Holes*, in Living Rev. Relativity 4, 6 (2001) [Online article: http://www.livingreviews.org/Articles/Volume4/2001-6wald].

[7] A. Peres and D. R. Terno, *Quantum Information and Relativity Theory*, Rev. Mod. Phys. **76**, 93 (2004), quant-ph/0212023.

[8] A. Ashtekar, J. Baez, K. Krasnov, *Quantum Geometry of Isolated Horizons and Black Hole Entropy*, Adv.Theor.Math.Phys. 4 (2000) 1-94, gr-qc/0005126.

[9] M. Domagala, J. Lewandowski, *Black Hole Entropy from Quantum Geometry*, Class.Quant.Grav. 21 (2004) 5233-5244, gr-qc/0407051.

[10] R. K. Kaul, S. K. Rama, *Black Hole Entropy from Spin One Punctures*, Phys. Rev. **D68**, 024001 (2003), gr-qc/0301128.

[11] O. Dreyer, F. Markopoulou, and L. Smolin, *Symmetry and Entropy of Black Hole Horizons*, hep-th/0409056.

[12] F. Girelli, E. R. Livine, *Reconstructing Quantum Geometry from Quantum Information: Spin Networks as Harmonic Oscillators*, Class.Quant.Grav. **22** (2005) 3295-3314, gr-qc/0501075.

[13] R. Goodman and N. R. Wallach, *Representations and Invariants of the Classical Groups* (Cambridge University Press, 1998).

[14] W.-K. Tung, *Group Theory in Physics* (World Scientific, Singapore 1985).

[15] Quant. Info. Comp. **1** (1) (2001); F. Mintert, A. R. R. Carvalho, M. Kus, and A. Buchleitner, *Measures and Dynamics of Entangled States*, Phys. Rep. **415**, 207 (2005).

[16] E. R. Livine and D. R. Terno, *Entanglement of Zero Angular Momentum Mixtures and Black Hole Entropy*, Phys. Rev. A **72**, 022307 (2005), quant-ph/0502043.

[17] N. J. Cerf and C. Adami, *Von Neumann Capacity of Noisy Quantum Channels*, Phys. Rev. A **56**, 3470 (1997), quant-ph/9609024.

[18] B. Grossman, S. Popescu, and A. Winter, *On the Quantum, Classical and Total Amount of Correlations in a Quantum State*, quant-ph/0410091.

[19] J.D. Bekenstein and V. F. Mukhanov, *Spectroscopy of the Quantum Black Hole*, Phys.Lett. **B360**, 7 (1995), gr-qc/9505012.

[20] S. D. Bartlett and D. R. Terno, *Relativistically Invariant Quantum Information*, Phys. Rev. A **71**, 012302 (2005), quant-ph/0403014.

[21] S. D. Bartlett, T. Rudolph, R. W. Spekkens, *Decoherence-Full Subsystems and the Cryptographic Power of a Private Shared Reference Frame*, Phys. Rev. Lett. **91**, 027901 (2003), quant-ph/0302111.

[22] F. Girelli, E. R. Livine, D. R. Terno, *Reconstructing Quantum Geometry from Quantum Information: Entanglement as a Measure of Distance*, in preparation.