ON THE POSSIBILITY OF QUANTUM GRAVITY EFFECTS AT ASTROPHYSICAL SCALES*

MARTIN REUTER
HOLGER WEYER
Institute of Physics, University of Mainz
Staudingerweg 7
D-55099 Mainz, Germany

Received Day Month Year
Revised Day Month Year
Communicated by Managing Editor

The nonperturbative renormalization group flow of Quantum Einstein Gravity (QEG) is reviewed. It is argued that at large distances there could be strong renormalization effects, including a scale dependence of Newton’s constant, which mimic the presence of dark matter at galactic and cosmological scales.

Keywords: Quantum gravity; dark matter; galaxies.

1. Introduction

By now it appears increasingly likely that Quantum Einstein Gravity (QEG), the quantum field theory of gravity whose underlying degrees of freedom are those of the spacetime metric, can be defined nonperturbatively as a fundamental, “asymptotically safe” theory [1,17]. By construction, its bare action is given by a non–Gaussian renormalization group (RG) fixed point. In the framework of the “effective average action” a suitable fixed point is known to exist within certain approximations. They suggest that the fixed point should also exist in the exact theory, implying its nonperturbative renormalizability.

The general picture regarding the RG behavior of QEG as it has emerged so far points towards a certain analogy between QEG and non–Abelian Yang–Mills theories, Quantum Chromo Dynamics (QCD) say. For example, like the Yang–Mills coupling constant, the running Newton constant \( G = G(k) \) is an asymptotically free coupling, it vanishes in the ultraviolet (UV), i.e. when the typical momentum scale \( k \) becomes large. In QCD the realm of asymptotic freedom is realized for momenta \( k \) larger than the mass scale \( \Lambda_{\text{QCD}} \) which is induced dynamically. In QEG the analogous role is played by the Planck mass \( m_{\text{Pl}} \). It delimits the asymptotic

*Invited contribution to the Int. J. Mod. Phys. D special issue on dark matter and dark energy.
scaling region towards the infrared (IR). For \( k \gg m_{Pl} \) the RG flow is well described by its linearization about the non–Gaussian fixed point. Both in QCD and QEG simple local approximations (truncations) of the running Wilsonian action (effective average action) are sufficient above \( \Lambda_{QCD} \) and \( m_{Pl} \), respectively. However, as the scale \( k \) approaches \( \Lambda_{QCD} \) or \( m_{Pl} \) from above, many complicated, typically nonlocal terms are generated in the effective action \( 18 \). In fact, in the IR, strong renormalization effects are to be expected because gauge (diffeomorphism) invariance leads to a massless excitation, the gluon (graviton), implying potential IR divergences which the RG flow must cure in a dynamical way. Because of the complexity of the corresponding flow equations it is extremely difficult to explore the RG flow of QCD or QEG in the IR, far below the UV scaling regime, by analytical methods. In QCD, lattice results and phenomenology suggest that the nonperturbative IR effects modify the classical Coulomb term by adding a confinement potential to it which increases (linearly) with distance: \( V(r) = -a/r + \kappa r \) \( 19 \).

The problem of the missing mass or “dark matter” is one of the most puzzling mysteries of modern astrophysics \( 20 \). It is an intriguing idea that the apparent mass discrepancy is not due to an unknown form of matter but rather indicates that we are using the wrong theory of gravity, Newton’s law in the non–relativistic and General Relativity in the relativistic case. If one tries to explain the observed non–Keplerian rotation curves of galaxies or clusters \( 21 \) in terms of a modified Newton law, a nonclassical term needs to be added to the \( 1/r \)-potential whose relative importance grows with distance \( 22 \). In “MOND” \( 23 \) for instance, a point mass \( M \) produces the potential \( \phi(r) = -GM/r + \sqrt{a_0 GM} \ln(r) \) and it is tempting to compare the \( \ln(r) \)-term to the qualitatively similar confinement potential in (quenched) QCD. It seems not unreasonable to speculate that the “confinement” potential in gravity is a quantum effect which results from the antiscreening character of quantum gravity \( 2 \) in very much the same way as this happens in Yang–Mills theory. If so, the missing mass problem could get resolved in a very elegant manner without the need of introducing dark matter on an ad hoc basis. In Refs. \( 24, 25 \) this idea has been explored within a semi–phenomenological analysis of the effective average action of quantum gravity \( 2 \). (See Refs. \( 26-34 \) for similar work on gravitational “RG improvement”. Earlier investigations of IR quantum gravity effects include Refs. \( 35-38 \).)

2. RG running of the gravitational parameters

The effective average action \( \Gamma_k[g_{\mu\nu}] \) is a “coarse grained” Wilson type action functional which defines an effective field theory of gravity at the variable mass scale \( k \). Roughly speaking, the solution to the associated effective Einstein equations \( \delta \Gamma_k/\delta g_{\mu\nu} = 0 \) yields the metric averaged over a spacetime volume of linear extension \( k^{-1} \). In a physical situation with a typical scale \( k \), the equation \( \delta \Gamma_k/\delta g_{\mu\nu} = 0 \) “knows” about all quantum effects relevant at this particular scale. For \( k \) fixed, the functional \( \Gamma_k \) should be visualized as a point in “theory space”, the space of all
action functionals. When the RG effects are “switched on”, one obtains a curve in this space, the RG trajectory, which starts at the bare action \( S \equiv \Gamma_{k \to \infty} \) and ends at the ordinary effective action \( \Gamma \equiv \Gamma_{k \to 0} \).

The average action is defined in terms of a modified functional integral over all metrics, \( \int \mathcal{D}g_{\mu\nu} \exp(-S[g]) \), the difference with respect to the conventional setting being that this integral has a built–in IR cutoff. It extends only over metric fluctuations with covariant momenta \( p^2 > k^2 \). The modes with \( p^2 < k^2 \) are given a momentum dependent (mass) \( 2 \propto R_k(p^2) \) and are suppressed therefore. As a result, \( \Gamma_k \) describes the dynamics of metrics averaged over spacetime volumes of the size \( k^{-1} \), i.e. \( \Gamma_k[g_{\mu\nu}] \) gives rise to an effective field theory valid near \( k \): when evaluated at tree level, \( \Gamma_k \) correctly describes all quantum gravitational phenomena, including all loop effects, provided the typical momentum scales involved are all of order \( k \).

(See Ref. 2 and the references therein for a precise definition of these notions.)

The RG trajectory \( k \mapsto \Gamma_k[\cdot] \) can be obtained by solving an exact functional RG equation. In practice one has to resort to approximations. Nonperturbative approximate solutions can be obtained by truncating the space of action functionals, i.e. by projecting the RG flow onto a (typically finite–dimensional) subspace which encapsulates the essential physics.

The “Einstein–Hilbert truncation”, for instance, approximates \( \Gamma_k \) by a linear combination of the monomials \( \int \sqrt{g} R \) and \( \int \sqrt{g} \). Their prefactors contain the running Newton constant \( G(k) \) and the running cosmological constant \( \Lambda(k) \). Their \( k \)-dependence is governed by a system of two coupled ordinary differential equations.

The flow equations resulting from the Einstein–Hilbert truncation are most conveniently written down in terms of the dimensionless “couplings” \( g(k) \equiv k^{d-2} G(k) \) and \( \lambda(k) \equiv \Lambda(k)/k^2 \) where \( d \) is the dimensionality of spacetime. Parameterizing the RG trajectories by the “RG time” \( t \equiv \ln k \) the coupled system of differential equations for \( g \) and \( \lambda \) reads \( \partial_t \lambda = \beta_\lambda, \partial_t g = \beta_g \), where the \( \beta \)-functions are given by

\[
\beta_\lambda(\lambda, g) = -(2 - \eta_N) \lambda + \frac{1}{2} (4\pi)^{1-d/2} g \\
\times \left[ 2d(d+1) \Phi_{d/2}^1(-2\lambda) - 8d \Phi_{d/2}^1(0) - d(d+1) \eta_N \Phi_{d/2}^1(-2\lambda) \right] \\
\beta_g(\lambda, g) = (d - 2 + \eta_N) g.
\]

(1)

Here \( \eta_N \), the anomalous dimension of the operator \( \int \sqrt{g} R \), has the representation

\[
\eta_N(g, \lambda) = \frac{g B_1(\lambda)}{1 - g B_2(\lambda)}.
\]

(2)
The functions $B_1(\lambda)$ and $B_2(\lambda)$ are defined by
\begin{align}
B_1(\lambda) & \equiv \frac{1}{4} (4\pi)^{1-d/2} \left[ d(d+1) \Phi_{d/2-1}^1(\lambda) - 6d(d-1) \Phi_{d/2}^2(\lambda) \\
-4d \Phi_{d/2-1}(0) - 24 \Phi_{d/2}^2(0) \right] \\
B_2(\lambda) & \equiv -\frac{1}{6} (4\pi)^{1-d/2} \left[ d(d+1) \Phi_{d/2-1}^1(-\lambda) - 6d(d-1) \Phi_{d/2}^2(-\lambda) \right].
\end{align}

The above expressions contain the “threshold functions” $\Phi^p_n$ and $\Phi^\tilde{p}_n$. They are given by
\begin{equation}
\Phi^p_n(w) = \frac{1}{\Gamma(n)} \int_0^\infty dz \ z^{n-1} \frac{R^{(0)}(z) - z R^{(0)'}(z)}{[z + R^{(0)}(z) + w]^p}
\end{equation}
and a similar formula for $\Phi^\tilde{p}_n$ without the $R^{(0)'}$-term. In fact, $R^{(0)}$ is a dimensionless version of the cutoff function $R_k$, i.e. $R_k(p^2) \propto k^2 R^{(0)}(p^2 / k^2)$. Eq. (4) shows that $\Phi^p_n(w)$ becomes singular for $w \to -1$. (For all admissible cutoffs, $z + R^{(0)}(z)$ assumes its minimum value 1 at $z = 0$ and increases monotonically for $z > 0$.) If $\lambda > 0$, the $\Phi$’s in the $\beta$–functions are evaluated at negative arguments $w \equiv -2\lambda$. As a result, the $\beta$–functions diverge for $\lambda > 1/2$ and the RG equations define a flow on a half-plane only: $-\infty < g < \infty$, $-\infty < \lambda < 1/2$.

This point becomes particularly clear if one uses a sharp cutoff. Then the $\Phi$’s either display a pole at $w = -1$,
\begin{equation}
\Phi^p_n(w) = \frac{1}{\Gamma(n)} \frac{1}{p-1} \frac{1}{(1+w)^{p-1}} \quad \text{for } p > 1,
\end{equation}
or, in the special case $p = 1$, they have a logarithmic singularity at $w = -1$:
\begin{equation}
\Phi^1_n(w) = -\Gamma(n)^{-1} \ln(1+w) + \varphi_n.
\end{equation}
The constants $\varphi_n \equiv \Phi^1_n(0)$ parameterize the residual cutoff scheme dependence which is still present after having opted for a sharp cutoff. We shall take them equal to the corresponding $\Phi^1_n(0)$–value of a smooth exponential cutoff, but their precise value has no influence on the qualitative features of the RG flow. The corresponding $\Phi$’s are constant for the sharp cutoff: $\Phi^1_n(w) = \delta_{p1} / \Gamma(n+1)$.

From now on we continue the discussion in $d = 4$ dimensions. Then, with the sharp cutoff, the coupled RG equations assume the following form:
\begin{align}
\partial_t \lambda &= -(2 - \eta_N) \lambda - \frac{g}{\pi} \left[ 5 \ln(1 - 2\lambda) - \varphi_2 + \frac{5}{4} \eta_N \right] \quad \text{(7a)} \\
\partial_t g &= (2 + \eta_N) g \quad \text{(7b)} \\
\eta_N &= -\frac{2g}{6\pi + 5g} \left[ \frac{18}{1 - 2\lambda} + 5 \ln(1 - 2\lambda) - \varphi_1 + 6 \right]. \quad \text{(7c)}
\end{align}

In Fig. 1 the exponential cutoff with “shape parameter” $s = 1$ is used. In $d = 4$, the only $\varphi$’s we need are $\varphi_1 = \zeta(2)$ and $\varphi_2 = 2\zeta(3)$. See Ref. [4] for a detailed discussion.
Solving the system numerically we obtain the phase portrait shown in Fig. 1. The RG flow is dominated by two fixed points \((g_*, \lambda_*)\): a Gaussian fixed point (GFP) at \(g_* = \lambda_* = 0\), and a non–Gaussian fixed point (NGFP) with \(g_* > 0\) and \(\lambda_* > 0\). There are three classes of trajectories emanating from the NGFP: trajectories of Type Ia and IIIa run towards negative and positive cosmological constants, respectively, and the single trajectory of Type IIa (“separatrix”) hits the GFP for \(k \to 0\). The short–distance properties of QEG are governed by the NGFP; for \(k \to \infty\), in Fig. 1 all RG trajectories on the half–plane \(g > 0\) run into this point.

The conjectured nonperturbative renormalizability of QEG is due to the NGFP: if it is present in the full RG equations, it can be used to construct a microscopic quantum theory of gravity by taking the limit of infinite UV cutoff along one of the trajectories running into the NGFP, thus being sure that the theory does not develop uncontrolled singularities at high energies. By definition, “QEG” is the theory whose bare action \(S\) equals the fixed point action \(\lim_{k \to \infty} \Gamma_k\left[g_{\mu\nu}\right]\).

Let us pause here for a moment and comment on the physics encoded in the beta functions and the flow they imply. They express the key property of QEG, namely the antiscreening character of the gravitational interaction. In fact, the scale dependence of Newton’s constant is governed directly by the anomalous dimension \(\eta_N\). In \(d = 4\) dimensions, say, its flow equation is \(\partial_k g = (2 + \eta_N) g\) which translates to

\[ k \frac{\partial}{\partial k} G(k) = \eta_N G(k) \] (8)
for the dimensionful \( G = g/k^2 \). The form of the expression (2) for the anomalous dimension illustrates the nonperturbative character of the beta functions. For \( g \beta(\lambda) < 1 \), Eq. (2) can be expanded as

\[
\eta_N = g B_1(\lambda) \sum_{n \geq 0} g^n B_2(\lambda)^n
\]

which shows that even a simple truncation can sum up arbitrarily high powers of the couplings. It is instructive to consider the approximation where only the lowest order is retained in (9). In \( d = 4 \), and for \( \lambda(k) \approx 0 \), one obtains \( \eta_N = B_1(0) G_0 k^2 + \mathcal{O}(G_0^2 k^4) \) with \( G_0 \equiv G(k = 0) \), and integrating (8) yields

\[
G(k) = G_0 \left[ 1 + \frac{1}{2} B_1(0) G_0 k^2 + \mathcal{O}(G_0^2 k^4) \right].
\]

Here \( B_1(0) \) is a \( R_k \)-dependent constant which, however, can be shown to be negative for all admissible cutoff functions \( R_k \). One sees that at least in the regime where (10) is valid, \( G(k) \) is a decreasing function of \( k \): Newton’s constant is large (small) on low (high) momentum scales. Interpreting \( k \) as an inverse distance, \( G \) is an increasing function of the distance scale. This amounts to the antiscreening behavior mentioned above.

The validity of the approximation (10) requires \( k \ll m_{Pl} \) with the Planck mass defined by the IR value of Newton’s constant, \( m_{Pl} \equiv G_0^{−1/2} \), as well as \( \lambda(k) \approx 0 \). As a result, it applies to the lower part of the separatrix since there both \( k/m_{Pl} \) and \( \lambda(k \to 0) \) are small. In the other regimes numerical methods must be used. In Fig. 2 we plot both the dimensionful and dimensionless Newton and cosmological constants along the separatrix as a function of \( k \). One finds that \( G(k) \) decreases monotonically with the momentum scale all the way from \( k = 0 \) up to \( k^* = \infty \). For \( k \to \infty \) the scaling governed by the NGFP sets in, and \( G(k) \approx g^* / k^2 \) vanishes \( \propto 1/k^2 \) for \( k \to \infty \). The cosmological constant, on the other hand, increases monotonically with \( k \) and diverges \( \propto k^2 \) in the NGFP regime. The logarithmic plots in Fig. 2 illustrate that for most \( k \)-values the trajectory is either close to the NGFP or the GFP and follows the corresponding power law scalings. At \( k \approx m_{Pl} \) it “crosses over” very rapidly from the NGFP to the GFP. The trajectories of Type Ia have similar properties, the main difference being that \( \Lambda(k) \) becomes negative below a certain scale.

The trajectories of Type IIIa have an important property which is not resolved in Fig. 1. Within the Einstein–Hilbert approximation they cannot be continued all the way down to the infrared \( (k = 0) \) but rather terminate at a finite scale \( k_{\text{term}} > 0 \). At this scale they hit the singular boundary \( \lambda = 1/2 \) where the \( \beta \)–functions diverge. As a result, the flow equations cannot be integrated beyond this point. The value of \( k_{\text{term}} \) depends on the trajectory considered.

In Ref. 4 the behavior of \( g \) and \( \lambda \) close to the boundary was studied in detail. The aspect which is most interesting for the present discussion is the following. As the trajectory gets close to the boundary, \( \lambda \) approaches 1/2 from below. In this
domain the anomalous dimension \( \eta_N \) is dominated by its pole term:

\[
\eta_N \approx -\frac{36}{6\pi + 5g} \frac{1}{1 + 2\lambda}
\]  

(11)

Obviously \( \eta_N \rightarrow -\infty \) for \( \lambda \rightarrow 1/2 \), and eventually \( \eta_N = -\infty \) at the boundary. This behavior has a dramatic consequence for the (dimensionful) Newton constant. Since the running of \( G(k) \) is given by \( \partial_k G = \eta_N G \), the large and negative anomalous dimension causes \( G \) to grow very strongly when \( k \) approaches \( k_{\text{term}} \) from above. This behavior is sketched schematically in Fig. 3.
At moderately large scales \( k \), well below the NGFP regime, \( G \) is approximately constant. As \( k \) is lowered towards \( k_{\text{term}} \), \( G(k) \) starts growing because of the pole in \( \eta_N \propto 1/(1-2\lambda) \), and finally, at \( k = k_{\text{term}} \), it develops a vertical tangent, \( (dG/dk)(k_{\text{term}}) = -\infty \). The cosmological constant is finite at the termination point: \( \Lambda(k_{\text{term}}) = k_{\text{term}}^2/2 \).

By fine-tuning the parameters of the trajectory the scale \( k_{\text{term}} \) can be made as small as we like. Since it happens only very close to \( \lambda = 1/2 \), the divergence at \( k_{\text{term}} \) is not visible on the scale of Fig. 1 (Note also that \( g \) and \( G \) are related by a decreasing factor of \( k^2 \)).

The phenomenon of trajectories which terminate at a finite scale is not special to gravity, it occurs also in truncated flow equations of theories which are understood much better. Typically it is a symptom which indicates that the truncation used becomes insufficient at small \( k \). In QCD, for instance, thanks to asymptotic freedom, simple local truncations are sufficient in the UV, but a reliable description in the IR requires many complicated (nonlocal) terms in the truncation ansatz. Thus the conclusion is that for trajectories of Type IIIa the Einstein–Hilbert truncation is reliable only well above \( k_{\text{term}} \). It is to be expected, though, that in an improved truncation those trajectories can be continued to \( k = 0 \). The IR growth of \( G(k) \) can be understood in very general terms as being due to an “instability driven renormalization” \cite{25,42,43}.

We believe that while the Type IIIa trajectories of the Einstein–Hilbert truncation become unreliable very close to \( k_{\text{term}} \), their prediction of a growing \( G(k) \) for decreasing \( k \) in the IR is actually correct. The function \( G(k) \) obtained from the differential equations (7) should be reliable, at least at a qualitative level, as long as \( \lambda \ll 1 \). For special trajectories the IR growth of \( G(k) \) sets in at extremely small scales.
scales $k$ only. Later on we shall argue on the basis of a gravitational “RG improvement” that this IR growth might perhaps be responsible for the non–Keplerian rotation curves observed in galaxies.

The other trajectories with $g > 0$, the Types Ia and IIa, do not terminate at a finite scale. The analysis of Ref. 4 suggests that they are reliably described by the Einstein–Hilbert truncation all the way down to $k = 0$.

3. The RG trajectory “realized in Nature”

In Ref. 25 we hypothesized that the matter fields present in the real world do not change the qualitative features of the Einstein–Hilbert flow and then, on the basis of this hypothesis, tried to pin down the specific RG trajectory of QEG which is realized in Nature. Conceptually the procedure is the same as in QED, for instance, where one fixes the corresponding trajectory by measuring the electron mass and the fine structure constant. Likewise, in QEG, the input data are the observed values of Newton’s constant and the cosmological constant. They point towards the highly “non–generic” trajectory of Type IIIa sketched in Fig. 4.

For $k \to \infty$ it starts infinitesimally close to the NGFP. Then, lowering $k$, the trajectory spirals about the NGFP and approaches the “separatrix”, the distinguished trajectory which ends at the GFP. It runs almost parallel to the separatrix for a very long “RG time”; only in the “very last moment” before reaching the GFP, at the turning point $T$, it gets driven away towards larger values of $\lambda$. In Fig. 4 the points $P_1$ and $P_2$ symbolize the beginning and the end of the regime in which classical general relativity is valid (“GR regime”). The classical regime starts soon after the turning point $T$ which is passed at the scale $k_T \approx 10^{-30} m_{Pl}$. In this regime $G(k)$ and $\Lambda(k)$ have almost no $k$-dependence.

In Ref. 25 we speculated that to the right of the point $P_2$ there starts a regime
of strong IR renormalization effects which might become visible at astrophysical and cosmological length scales. As we mentioned already, trajectories of Type IIIa terminate at some \( k \neq 0 \) near \( \lambda = 1/2 \) (close to the question mark in Fig. 4). Before it starts becoming invalid, the Einstein–Hilbert approximation suggests that \( G \) will increase, while \( \Lambda \) decreases, as \( \lambda \nearrow 1/2 \).

The Type IIIa trajectory of QEG which Nature has selected is highly special in the following sense. It is fine–tuned in such a way that it gets extremely close to the GFP before “turning left”. The coordinates \( g_T \) and \( \lambda_T \) of the turning point are both very small: \( g_T \approx \lambda_T \approx 10^{-60} \). The coupling \( g \) decreases from \( g(k) = 10^{-70} \) at a typical terrestrial length scale of \( k^{-1} = 1 \) m to \( g(k) = 10^{-92} \) at the solar system scale of \( k^{-1} = 1 \) AU, and finally reaches \( g(k) = 10^{-120} \) when \( k \) equals the present Hubble constant \( H_0 \).

In fact, the Hubble parameter \( k = H_0 \) is approximately the scale where the Einstein-Hilbert trajectory becomes unreliable. The observations indicate that today the cosmological constant is of the order \( H_0^2 \). Interpreting this value as the running \( \Lambda(k) \) at the scale \( k = H_0 \) we have \( \Lambda(H_0) \approx H_0^2 \); as a result, the dimensionless \( \lambda(k) \), at this scale, is of order unity: \( \lambda(H_0) = \Lambda(H_0)/H_0^2 = \mathcal{O}(1) \). Thus one arrives at a conclusion which is quite remarkable and intriguing: the scale at which the IR renormalization effects set in, if they exist, is predicted to be the present Hubble scale.

The “unnaturalness” of Nature’s gravitational RG trajectory has an important consequence. Because it gets so extremely close to the GFP it spends a very long RG time in its vicinity because the \( \beta \)-functions are small there. As a result, the termination of the trajectory at \( \lambda = 1/2 \) is extremely delayed, by 60 orders of magnitude, compared to a generic trajectory where this happens for \( k \) near the Planck mass. This non–generic feature of the trajectory is a necessary condition for a long classical regime with \( G, \Lambda \approx \text{const} \) to emerge, and any form of classical physics to be applicable.

It was shown\(^{25}\) that for any trajectory which actually does admit a long classical regime the cosmological constant in the classical regime is automatically small. In fact, the fine–tuning behind the “unnatural” trajectory Nature has selected is of a much more general kind than the traditional cosmological constant problem\(^{44}\), the primary issue is the emergence of a classical spacetime; once this is achieved, the extreme smallness of the observed \( \Lambda \) (compared to \( m^2_{\text{Pl}} \)) comes for free.

Stated differently, if the picture based upon the Einstein–Hilbert truncation is qualitatively correct all quantum theories (i.e. QEG based upon any of its trajectories) have the property that if it makes any sense at all to use \( S = (16\pi G)^{-1} \int d^4x \sqrt{-g} \ (R - 2\Lambda) \) as a classical action, then \( \Lambda/m^2_{\text{Pl}} \equiv \Lambda G \) is guaranteed to be a very small number. QEG seems to resolve the cosmological constant problem in its original form by restricting the form of possible classical limits.

In principle it should be possible to work out the predictions of the theory for cosmological scales by an ab initio calculation within QEG. Unfortunately, because of the enormous technical complexity of the RG equations, this has not been pos-
sible in practice yet. In this situation one can adopt a phenomenological strategy, however. One makes an ansatz for the RG trajectory which has the general features discussed above, derives its consequences, and confronts them with the observations. In this manner the observational data can be used in order to learn something about the RG trajectory in the nonperturbative regime which is inaccessible to an analytic treatment for the time being. Using this strategy, the cosmological consequences of a very simple scenario for the $k \to 0$ behavior has been worked out; the assumption proposed in Refs. [29, 30] is that the IR effects lead to the formation of a second NGFP into which the RG trajectory gets attracted for $k \to 0$. This hypothesis leads to a phenomenologically viable late–time cosmology with a variety of rather attractive features. It predicts an accelerated expansion of the universe and explains, without any fine–tuning, why the corresponding matter and vacuum energy densities are approximately equal.

4. Galaxy rotation curves

Given the encouraging results indicating that the IR effects are possibly “at work” in cosmology, by continuity, it seems plausible to suspect that somewhere between solar system and cosmological scales they should first become visible. In Refs. [24, 25] we therefore investigated the idea that they are responsible for the observed non–Keplerian galaxy rotation curves. The calculational scheme used there was a kind of “RG improvement”, the basic idea being that upon identifying the scale $k$ with an appropriate geometric quantity comparatively simple (local) truncations effectively mimic much more complicated (nonlocal) terms in the effective action [34]. Considering spherically symmetric, static model galaxies, the scale $k$ was taken to be the inverse of the radial proper distance which boils down to $1/r$ in leading order. Since the regime of galactic scales turned out to lie outside the domain of validity of the Einstein–Hilbert approximation (see below) the only practical option was to make an ansatz for the RG trajectory $\{G(k), \Lambda(k), \ldots\}$ and to explore its observable consequences. In particular a relationship between the $k$-dependence of $G$ and the rotation curve $v(r)$ of the model galaxy has been derived [24].

The idea of the approach proposed in Refs. [34] and [24] is to start from the classical Einstein–Hilbert action $S_{EH} = \int d^4 x \sqrt{-g} L_{EH}$ with the Lagrangian $L_{EH} = (R - 2 \Lambda)/(16\pi G)$ and to promote $G$ and $\Lambda$ to scalar fields. This leads to the modified Einstein–Hilbert (mEH) action

$$S_{mEH}[g, G, \Lambda] = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left\{ \frac{R}{G(x)} - 2 \frac{\Lambda(x)}{G(x)} \right\}. \quad (12)$$

The resulting theory has certain features in common with Brans–Dicke theory; the main difference is that $G(x)$ (and $\Lambda(x)$) is a prescribed “background field” rather than a Klein–Gordon scalar as usual. Upon adding a matter contribution the action (12) implies the modified Einstein equation

$$G_{\mu\nu} = -\Lambda(x) g_{\mu\nu} + 8\pi G(x) \left(T_{\mu\nu} + \Delta T_{\mu\nu}\right). \quad (13)$$
Here $\Delta T_{\mu\nu}$ is a new contribution to the energy–momentum tensor due to the $x$-dependence of $G$:

$$\Delta T_{\mu\nu} \equiv \frac{1}{8\pi} \left( D_\mu D_\nu - g_{\mu\nu} D^2 \right) \frac{1}{G(x)}.$$  

(14)

(In Refs. [34] and [24] a further contribution, $\theta_{\mu\nu}$, was added to the energy–momentum tensor in order to describe the 4-momentum of the field $G(x)$. Its form is not completely fixed by general principles. As it does not affect the Newtonian limit [24] we set $\theta_{\mu\nu} \equiv 0$ here.) The field equation (13) is mathematically consistent provided $\Lambda(x)$ and $G(x)$ satisfy a “consistency condition” which insures that the RHS of (13) has a vanishing covariant divergence.

In Ref. [24] we analyzed the weak field, slow–motion approximation of this theory for a time–independent Newton constant $G = G(x)$ and $\Lambda \equiv 0$. In this (modified) Newtonian limit the equation of motion for massive test particles has the usual form, $\ddot{x}(t) = -\nabla \phi$, but the potential $\phi$ obeys a modified Poisson equation,

$$\nabla^2 \phi = 4\pi G \rho_{\text{eff}}$$  

(15a)

with the effective energy density

$$\rho_{\text{eff}} = \rho + (8\pi G)^{-1} \nabla^2 N.$$  

(15b)

In deriving (15) it was assumed that $T_{\mu\nu}$ describes pressureless dust of density $\rho$ and that $G(x)$ does not differ much from the constant $G$. We use the parameterization

$$G(x) = \bar{G} \left[ 1 + N(x) \right]$$  

(16)

and assume that $N(x) \ll 1$. More precisely, the assumptions leading to the modified Newtonian limit are that the potential $\phi$, the function $N$, and typical (squared) velocities $v^2$ are much smaller than unity; all terms linear in these quantities are retained, but higher powers ($\phi^2$, etc.) and products of them ($\phi N$, etc.) are neglected. (In the application to galaxies this is an excellent approximation.) Apart from the rest energy density $\rho$ of the ordinary (“baryonic”) matter, the effective energy density $\rho_{\text{eff}}$ contains the “vacuum” contribution

$$(8\pi \bar{G})^{-1} \nabla^2 N(x) = (8\pi \bar{G}^{2/3})^{-1} \nabla^2 G(x)$$  

(17)

which is entirely due to the position dependence of Newton’s constant. Since it acts as a source for $\phi$ on exactly the same footing as $\rho$ it mimics the presence of “dark matter”.

As the density (17) itself contains a Laplacian $\nabla^2$, all solutions of the Newtonian field equation (15) have a very simple structure:

$$\phi(x) = \hat{\phi}(x) + \frac{1}{2} N(x).$$  

(18)

Here $\hat{\phi}$ is the solution to the standard Poisson equation $\nabla^2 \hat{\phi} = 4\pi \bar{G} \rho$ containing only the ordinary matter density $\rho$. The simplicity and generality of this result is quite striking.
On the Possibility of Quantum Gravity Effects at Astrophysical Scales

Up to this point the discussion applies to an arbitrary prescribed position dependence of Newton’s constant, not necessarily related to a RG trajectory. At least in the case of spherically symmetric systems the identification of the relevant geometric cutoff is fairly straightforward, \( k \propto 1/r \), so that we may consider the function \( G(k) \) as the primary input, implying \( G(r) \equiv G(k = \xi/r) \). Writing again \( G \equiv \overline{G} [1 + N] \) we assume that \( G(k) \) is such that \( N \ll 1 \). Then, to leading order, the potential for a point mass reads, according to (18):

\[
\phi(r) = -\frac{\overline{G} \, M}{r} + \frac{1}{2} N(r). \tag{19}
\]

Several comments are in order here.

(a) The reader might have expected to find a term \(-\overline{G} \, M \, N(r)/r\) on the RHS of (19) resulting from Newton’s potential \( \phi_N \equiv -\overline{G} M/r \) by the “improvement” \( \overline{G} \to G(r) \). However, this term \( \phi_N \, N \) is of second order with respect to the small quantities we are expanding in. In the envisaged application to galaxies, for example, \( \phi_N \, N \) is completely negligible compared to the \( 1/2 N \)-term in (19).

(b) According to (19), the renormalization effects generate a nonclassical force (per unit test mass) given by \(-N'(r)/2\) which adds to the classical \( 1/r^2 \)-term. This force is attractive if \( G(r) \) is an increasing function of \( r \) and \( G(k) \) a decreasing function of \( k \). This is in accord with the intuitive picture of the antiscreening character of quantum gravity \(^{(2)}\) “Bare” masses get “dressed” by virtual gravitons whose gravitating energy and momentum cannot be shielded and lead to an additional gravitational pull on test masses therefore.

(c) The solution (19) is not an approximation artifact. In Ref. 24 we constructed exact solutions of the full nonlinear modified Einstein equations (with \( N \) not necessarily small) which imply (19) in their respective Newtonian regime. Those exact solutions can be interpreted as a “deformation” of the Schwarzschild metric (\( M \neq 0 \)) or the Minkowski metric (\( M = 0 \)) caused by the position dependence of \( G \). The solutions related to the Minkowski metric are particularly noteworthy. They contain no ordinary matter (no point mass), but describe a curved spacetime, a kind of gravitational “soliton” which owes its existence entirely to the \( x \)-dependence of \( G \). At the level of Eq. (19) they correspond to the \( M = 0 \)-potential \( \phi = \frac{1}{2} N \) which solves the modified Poisson equation if the contribution \( \propto \nabla^2 N \) is the only source term. In the picture where dark matter is replaced with a running of \( G \) this solution corresponds to a pure dark matter halo containing no baryonic matter. The fully relativistic \( M = 0 \)-solutions might be important in the early stages of structure formation\(^{(24)}\).

Let us make a simple model of a spherically symmetric “galaxy”. For an arbitrary density profile \( \rho = \rho(r) \) the solution of Eq. (15) reads

\[
\phi(r) = \int_0^r dr' \frac{\overline{G} \, M(r')}{r'^2} + \frac{1}{2} N(r). \tag{20}
\]
M. Reuter and H. Weyer

where $\mathcal{M}(r) \equiv 4\pi \int_0^r dr' r'^2 \rho(r')$ is the mass of the ordinary matter contained in a ball of radius $r$. We are interested in periodic, circular orbits of test particles in the potential (20). Their velocity is given by $v^2(r) = r \phi'(r)$ so that we obtain the rotation curve

$$v^2(r) = \frac{\overline{G} \mathcal{M}(r)}{r} + \frac{1}{2} r \frac{d}{dr} \mathcal{N}(r). \quad (21)$$

We identify $\rho$ with the density of the ordinary luminous matter and model the luminous core of the galaxy by a ball of radius $r_0$. The mass of the ordinary matter contained in the core is $M(r_0) \equiv M_0$, the “bare” total mass of the galaxy. Since, by assumption, $\rho = 0$ and hence $\mathcal{M}(r) = M_0$ for $r > r_0$, the potential outside the core is $\phi(r) = -\frac{G M_0}{r} + \frac{\mathcal{N}(r)}{2}$. We refer to the region $r > r_0$ as the “halo” of the model galaxy.

As an example, let us make the scale free power law ansatz $G(k) \propto k^{-q}$. For $q > 0$ Newton’s constant increases in the IR. We assume that this $k$-dependence starts inside the core of the galaxy (at $r < r_0$) so that $G(r) \propto r^q$ everywhere in the halo. For the modified Newtonian limit to be realized, the position dependence of $G$ must be weak. Therefore we shall tentatively assume that the exponent $q$ is very small ($0 < q \ll 1$); applying the model to real galaxies this will turn out to be the case actually. Thus, expanding to first order in $q$, $r^q = 1 + q \ln(r) + \cdots$, we obtain

$$G(r) = \overline{G} \left[ 1 + \mathcal{N}(r) \right]$$

with $\mathcal{N}(r) = q \ln(\kappa r)$ (22)

where $\kappa$ is a constant. In principle the point $\overline{G}$ about which we linearize is arbitrary, but in the present context the usual laboratory value $G_{\text{lab}}$ is the natural choice. In the halo, Eq. (22) leads to a logarithmic modification of Newton’s potential

$$\phi(r) = -\frac{\overline{G} M_0}{r} + \frac{q}{2} \ln(\kappa r). \quad (23)$$

The corresponding rotation curve is

$$v^2(r) = \frac{\overline{G} M_0}{r} + \frac{q}{2}. \quad (24)$$

Remarkably, at large distances $r \to \infty$ the velocity approaches a constant $v_\infty = \sqrt{q/2}$. Obviously the rotation curve implied by the $k^{-q}$-trajectory does indeed become flat at large distances — very much like those we observe in Nature.

Typical measured values of $v_\infty$ range from 100 to 300 km/sec so that, in units of the speed of light, $v_\infty \approx 10^{-3}$. Thus, ignoring factors of order unity for a first estimate, we find that the data require an exponent of the order

$$q \approx 10^{-6}. \quad (25)$$

The smallness of this number justifies the linearization with respect to $\mathcal{N}$. It also implies that the variation of $G$ inside a galaxy is extremely small. The relative variation of Newton’s constant from some $r_1$ to $r_2 > r_1$ is $\Delta G/G = q \ln(r_2/r_1)$. As
the radial extension of a halo comprises only 2 or 3 orders of magnitude the variation between the inner and the outer boundary of the halo is of the order \( \Delta G/G \approx q \), i.e. Newton’s constant changes by one part in a million only.

Including the core region, the complete rotation curve reads

\[
v^2(r) = \frac{G M(r)}{r} + \frac{q}{2}.
\]

The \( r \)-dependence of this velocity is in qualitative agreement with the observations. For realistic density profiles, \( M(r)/r \) is an increasing function for \( r < r_0 \), and it decays as \( M_0/r \) for \( r > r_0 \). As a result, \( v^2(r) \) rises steeply at small \( r \), then levels off, goes through a maximum at the boundary of the core, and finally approaches the plateau from above. Some galaxies indeed show a maximum after the steep rise, but typically it is not very pronounced, or is not visible at all. The prediction of (24) for the characteristic \( r \)-scale where the plateau starts is \( 2G M_0/q \); at this radius the classical term \( G M_0/r \) and the nonclassical one, \( q/2 \), are exactly equal. With \( q = 10^{-6} \) and \( M_0 = 10^{11} M_\odot \) one obtains 9 kpc, which is just the right order of magnitude.

The above \( v^2(r) \) is identical to the one obtained from standard Newtonian gravity if one postulates dark matter with a density \( \rho_{DM} \propto 1/r^2 \). We see that if \( G(k) \propto k^{-q} \) with \( q \approx 10^{-6} \) no dark matter is needed. The resulting position dependence of \( G \) leads to an effective density \( \rho_{eff} = \rho + q/(8\pi G r^2) \) where the \( 1/r^2 \)-term, well known to be the source of a logarithmic potential, is present as an automatic consequence of the RG improved gravitational dynamics.

We consider these results a very encouraging indication pointing in the direction that quantum gravitational renormalization effects could perhaps explain the observed non–Keplerian galaxy rotation curves. If so, the underlying RG trajectory of QEG is characterized by an almost constant anomalous dimension \( \eta_N = -q \approx -10^{-6} \) for \( k \) in the range of galactic scales.

Is the Einstein–Hilbert truncation sufficient to search for this trajectory? Unfortunately the answer is no. According to Eq. (15), \( \eta_N \) is proportional to \( g \) which is extremely tiny in the regime of interest, smaller than its solar system value \( 10^{-92} \). In order to achieve a \( |\eta_N| \) as large as \( 10^{-6} \), the smallness of \( g \) must be compensated by large IR enhancement factors. As a result, \( \lambda \) should be extremely close to 1/2, in which case the RHS of (15) is dominated by the pole term: \( \eta_N \approx -(6g/\pi) (1 - 2\lambda)^{-1} \). Assuming \( g \approx 10^{-92} \) as a rough estimate, a \( q \)-value of \( 10^{-6} \) would require \( 1 - 2\lambda \approx 10^{-86} \). It is clear that when \( 1 - 2\lambda \) is so small the Einstein–Hilbert trajectory is by far too close to its termination point to be a reliable approximation of the true one. Moreover, \( \eta_N(g(k), \lambda(k)) \) is not approximately \( k \)-independent in this regime. Thus we must conclude that an improved truncation will be needed for an investigation of the conjectured RG behavior at galactic scales.

It is clear that the above model of a galaxy is still quite simplistic and does not yet reproduce all phenomenological aspects of the mass, size, and angular momentum dependence of the rotation curves for different galactic systems. In particular
$v_\infty$ is a universal constant here and does not obey the empirical Tully–Fisher relation. As we explained in Ref. [24] to which the reader is referred for further details these limitations are due to the calculational scheme used here (“cutoff identification”, etc.). Usually this scheme can provide a first qualitative or semi–quantitative understanding, but if one wants to go beyond this first approximation, a full fledged calculation of $\Gamma[g_{\mu\nu}]$ would be necessary which is well beyond our present technical possibilities.

5. Conclusion

The above analysis indicates that if the observed non–Keplerian rotation curves are due to a renormalization effect, the scale dependence of Newton’s constant should be roughly similar to $G(k) \propto k^{-q}$. Knowing this, it will be the main challenge for future work to see whether a corresponding RG trajectory is actually predicted by the flow equations of QEG. For the time being an ab initio calculation of this kind, while well–defined conceptually, is still considerably beyond the state of the art as far as the technology of practical RG calculations is concerned. In contrast to phenomenological theories such as MOND it is nevertheless possible to predict at least the scale on which the IR effects are to be expected. Given the measured values of $G$ and $\Lambda$, the RG trajectory is fixed. The “new physic” is expected to become visible at scales $k$ for which $\lambda(k)$ gets close to $1/2$. For the trajectory “realized in Nature” this is the case for $k$ slightly above the present Hubble parameter $H_0$.

For reliable calculation of the RG trajectory in the IR it might help to rewrite the nonlocal terms generated during the flow in terms of local field monomials by introducing extra fields besides the metric. This is a standard procedure in the Wilsonian approach which often allows for a simple local description of the effective IR dynamics. It is tempting to speculate that the resulting local effective field theory might be related to the generalized gravity theory in Ref. [45] which includes a Kalb–Ramond field; it is fully relativistic and explains the galaxy and cluster data with remarkable precision.

References

1. S. Weinberg in *General Relativity, an Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel (Cambridge University Press, 1979);
   S. Weinberg, hep-th/9702027
2. M. Reuter, *Phys. Rev.* D57, 971 (1998) and hep-th/9605030
3. O. Lauscher and M. Reuter, *Phys. Rev.* D65, 025013 (2002) and hep-th/0108040
4. M. Reuter and F. Saueressig, *Phys. Rev.* D65, 065016 (2002) and hep-th/0110054
5. O. Lauscher and M. Reuter, *Class. Quant. Grav.* 19, 483 (2002) and hep-th/0110021
   *Phys. Rev.* D66, 025026 (2002) and hep-th/0205062
6. W. Souma, *Prog. Theor. Phys.* 102, 181 (1999).
7. M. Reuter and F. Saueressig, *Phys. Rev.* D66, 125001 (2002) and hep-th/0206145
   *Fortschr. Phys.* 52, 650 (2004) and hep-th/0311056
8. D. Dou and R. Percacci, *Class. Quant. Grav.* 15, 3449 (1998).
9. S. Falkenberg and S. Odintsov, *Int. J. Mod. Phys.* **A13**, 607 (1998).
10. R. Percacci and D. Perini, *Phys. Rev.* **D67**, 081503 (2003);
    *Phys. Rev.* **D68**, 044018 (2003);
    D. Perini, *Nucl. Phys. Proc. Suppl.* **127C**, 185 (2004);
    R. Percacci and D. Perini, *Class. Quant. Grav.* **21**, 5035 (2004);
    A. Codellino and R. Percacci, *Phys. Rev. Lett.* **97**, 221301 (2006).
11. A. Bonanno and M. Reuter, *JHEP* **0502**, 035 (2005) and hep-th/0410191.
12. O. Lauscher and M. Reuter, *JHEP* **0510**, 050 (2005) and hep-th/0508202.
13. M. Reuter and J.-M. Schwindt, *JHEP* **0601**, 070 (2006) and hep-th/0511021.
14. F. Girelli, S. Liberati, R. Percacci and C. Rahmede, gr-qc/0607030.
15. D. Litim, *Phys. Rev. Lett.* **92**, 201301 (2004); F. Fischer and D. Litim, *Phys. Lett.* **B638**, 492 (2006).
16. For reviews see O. Lauscher and M. Reuter, hep-th/0511260;
    M. Niedermaier and M. Reuter, *Liv. Rev. Rel.* to appear.
17. P. Forgács and M. Niedermaier, hep-th/0207028;
    M. Niedermaier, *JHEP* **12**, 066 (2002); *Nucl. Phys. B673*, 131 (2003).
18. M. Reuter and C. Wetterich, *Phys. Rev.* **D56**, 7893 (1997);
    H. Gies, *Phys. Rev.* **D66**, 025006 (2002).
19. See, for instance, M. Böhm, A. Denner and H. Joos,*Gauge Theories of the Strong and Electroweak Interaction* (Teubner, Stuttgart, 2001).
20. See, for instance, T. Padmanabhan, *Theoretical Astrophysics*, Vol. 3
    (Cambridge University Press, 2002).
21. See, for instance, F. Combes, P. Boissé, A. Mazure and A. Blanchard,
    *Galaxies and Cosmology* (Springer, New York, 2002).
22. For a review see A. Aguirre, C. P. Burgess, A. Friedland and D. Nolte,
    *Class. Quant. Grav.* **18**, R 223 (2001).
23. M. Milgrom, *Astrophys. J.* **270**, 371 (1983); **270**, 384 (1983); **270**, 365 (1983);
    M. Milgrom in *Dark matter in astrophysics and particle physics*, eds. H. V. Klapdor–Kleingrothaus and L. Baudis (Heidelberg, 1998);
    *Acta Phys. Polon.* **B32**, 3613 (2001).
24. M. Reuter and H. Weyer, *Phys. Rev.* **D70**, 124028 (2004) and hep-th/0410117.
25. M. Reuter and H. Weyer, *JCAP* **0412**, 001 (2004) and hep-th/0410119.
26. A. Bonanno and M. Reuter, *Phys. Rev.* **D62**, 043008 (2000) and hep-th/0002196.
27. A. Bonanno and M. Reuter, *Phys. Rev. D65*, 043508 (2002) and hep-th/0106133.
28. M. Reuter and F. Saueressig, *JCAP* **0509**, 012 (2005) and hep-th/0507167.
29. A. Bonanno and M. Reuter, *Phys. Lett.* **B527**, 9 (2002) and astro-ph/0106468.
30. E. Bentivegna, A. Bonanno and M. Reuter, *JCAP* **0401**, 001 (2004) and astro-ph/0303150.
31. A. Bonanno, G. Esposito and C. Rubano, *Gen. Rel. Grav.* **35**, 1899 (2003);
    *Class. Quant. Grav.* **21**, 5005 (2004);
    A. Bonanno, G. Esposito, C. Rubano and P. Scudellaro, *Class. Quant. Grav.* **23**, 3103 (2006).
32. T. Tsuneyama, hep-th/0401110.
33. I. L. Shapiro, J. Sola and H. Stefancic, *JCAP* **0501**, 012 (2005).
34. M. Reuter and H. Weyer, *Phys. Rev.* **D69**, 104022 (2004) and hep-th/0311196.
35. N. C. Tsamis and R. P. Woodard, *Phys. Lett.* **B301**, 351 (1993);
    *Ann. Phys.* **238**, 1 (1995); *Nucl. Phys.* **B474**, 235 (1996).
36. I. Antoniadis and E. Mottola, *Phys. Rev.* **D45**, 2013 (1992).
37. O. Bertolami, J. M. Mourão and J. Pérez–Mercader, *Phys. Lett.* **B311**, 27 (1993); O. Bertolami and J. Garcia–Bellido, *Int. J. Mod. Phys.* **D5**, 363 (1996); T. Goldman, J. Pérez–Mercader, F. Cooper and M. Martin Nieto, *Phys. Lett.* **B281**, 219 (1992).
38. O. Bertolami and J. García–Bellido, *Nucl. Phys. Proc. Suppl.* **48**, 122 (1996).
39. C. Wetterich, *Phys. Lett.* **B301**, 90 (1993).
40. For a review see: J. Berges, N. Tetradis and C. Wetterich, *Phys. Rep.* **363**, 223 (2002); C. Wetterich, *Int. J. Mod. Phys.* **A16**, 1951 (2001).
41. M. Reuter and C. Wetterich, *Nucl. Phys.* **B417**, 181 (1994), *Nucl. Phys.* **B427**, 291 (1994), *Nucl. Phys.* **B391**, 147 (1993), *Nucl. Phys.* **B408**, 91 (1993); M. Reuter, *Phys. Rev.* **D55**, 4430 (1996), *Mod. Phys. Lett.* **A12**, 2777 (1997).
42. J. Alexandre, V. Branchina and J. Polonyi, *Phys. Lett.* **B445**, 351 (1999); J. Polonyi, [hep-lat/9610030](https://arxiv.org/abs/hep-lat/9610030); V. Branchina, H. Mohrbach and J. Polonyi, *Phys. Rev.* **D60**, 045006, 045007 (1999).
43. O. Lauscher, M. Reuter and C. Wetterich, *Phys. Rev.* **D62**, 125021 (2000) and [hep-th/0006099](https://arxiv.org/abs/hep-th/0006099).
44. S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989); T. Padmanabhan, *Curr. Sci.* **88**, 1057 (2005); M. Reuter and C. Wetterich, *Phys. Lett.* **B188**, 38 (1987).
45. J. Moffat, *JCAP* **0505**, 2003 (2005); J. R. Brownstein and J. Moffat, *Astrophys. J.* **636**, 721 (2006); *Mon. Not. Roy. Astron. Soc.* **367**, 527 (2006).