SIDON-RAMSEY AND $B_h$-RAMSEY NUMBERS

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Abstract. For a given positive integer $k$, the Sidon-Ramsey number $SR(k)$ is defined as the minimum value of $n$ such that, in every partition of the set $[1, n]$ into $k$ parts, there exists a part that contains two distinct pairs of numbers with the same sum. In other words, there is a part that is not a Sidon set. In this paper, we investigate the asymptotic behavior of this parameter and two generalizations of it. The first generalization involves replacing pairs of numbers with $h$-tuples, such that in every partition of $[1, n]$ into $k$ parts, there exists a part that contains two distinct $h$-tuples with the same sum. Alternatively, there is a part that is not a $B_h$ set. The second generalization considers the scenario where the interval $[1, n]$ is substituted with a non-necessarily symmetric $d$-dimensional box of the form $\prod_{i=1}^{d}[1, n_i]$. For the general case of $h \geq 3$ and non-symmetric boxes, before applying our method to obtain the Ramsey-type result, we needed to establish an upper bound for the corresponding density parameter.

1. Introduction

A subset $S$ of an additive group $G$ is called a Sidon set if the sums of any two elements (possibly equal) of $S$ are distinct. In other words, if $x, y, z, w \in S$ satisfy

$$x + y = z + w,$$

then $\{x, y\} = \{z, w\}$, which means that the equation above has only trivial solutions in $S$. For a given $X \subset G$, an important problem is to determine the maximum size of a Sidon set contained in $X$. This problem has been mainly studied when $G = \mathbb{Z}$ and $X = [1, n]$. We use $F_2(n)$ to denote the size of the largest Sidon set contained in $[1, n]$. It is known that

$$n^{1/2}(1 - o(1)) \leq F_2(n) \leq n^{1/2} + 0.998 n^{1/4}. \quad (1)$$

The upper bound has been progressively improved [ET41, Lin69, Cil10], the best being recently established by Balogh, Füredi and Roy [BFR23]. The lower bound may be inferred from several known constructions; in particular the one provided by Singer concerning maximal Sidon sets in $X = G = \mathbb{Z}_n$, where $n = q^2 + q + 1$ and $q$ is a prime power [Sin38]. For more information about problems related with Sidon sets the reader may consult the survey paper of O’Bryant [O’B04].

As with many density theorems, there is a Ramsey version of the problem of maximizing the size of a Sidon set. For a given positive integer $k$, a Sidon $k$-partition of $X \subset G$ is a partition of $X$ into $k$ parts, all of which are Sidon sets. Let $SR(k)$ be the Sidon-Ramsey number, defined as the minimum $n$ such that there is no Sidon $k$-partition of $[1, n]$. This parameter can be found in different contexts under different names. For instance, the existence of the Sidon-Ramsey numbers

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is a consequence of a theorem of Rado \cite{Rad43} with the matrix
\[
\begin{pmatrix}
1 & 1 & -1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 & 0 & 1
\end{pmatrix},
\]
where two new variables and two new equations are introduced to assure that no trivial solutions to the equation \(x + y = z + w\) are considered. Liang et al. \cite{LLXX13} and Xu et al. \cite{XLL18}, using computer assistance, found the exact values of \(SR(k)\) for \(k \leq 5\) and gave specific bounds for \(k \leq 19\). They use the fact that \(SR(k)\) can be bounded from above using the pigeonhole principle, that is,
\[
(2) \quad SR(k) \leq (t - 1)k + 1 \text{ for any } k, t \geq 2 \text{ satisfying } (F_2(t) - 1)/(t - 1) > k.
\]
By computing the upper bound derived from (2) and (1), along with the well-known Singer construction of Sidon sets, it can be easily deduced that the Sidon-Ramsey number \(SR(k)\) behave asymptotically as \(k^2\) (Theorem 1). Due to the interesting connection between Sidon sets and \(C_4\)-free graphs, the proof of the lower bound of Theorem 1 closely resembles the proof presented in \cite{CG75} for Theorem 3. However, we have included this proof in Section 2 to ensure completeness.

**Theorem 1.** Let \(k\) be a positive integer, then
\[
k^2 - O(k^c) \leq SR(k) \leq k^2 + Ck^{3/2} + O(k),
\]
where \(c \leq 1.525\) depends on the distribution of the prime numbers and \(C\) can be taken close to 1.996 and depends on the best upper bound for Sidon numbers.

In this paper, we investigate analogous results to Theorem 1 corresponding to some generalizations of \(F_2(n)\). Specifically, we study the Ramsey-type parameter for \(B_h\) sets in the interval \(X = [1, n] \subset \mathbb{Z}\) (precise definitions will be provided in Section 2). Additionally, we explore the problem in higher dimensions by studying the size of the largest \(B_h\)-set within a non-necessarily symmetric \(d\)-dimensional box \(X = \prod_{j=1}^{d}[1, n_j] \subset \mathbb{Z}^d\).

The paper is organized as follows. In Section 2 we deal with everything related to the one dimensional case. First, we define a \(B_h\) set in \(\mathbb{Z}\), its density parameter \(F_h(n)\), and its corresponding Ramsey-type parameter \(BR_h(k)\). If \(h = 2\) this coincides with the definition of a Sidon set, \(F_2(n)\), and \(SR(k)\). Then, we present the proof of Theorem 1 which states the asymptotic behavior of \(SR(k)\) (corresponding to the case \(h = 2\)) separated from the proof of Theorem 2, which states the asymptotic behavior of \(BR_h(k)\) for \(h \geq 3\).

Section 3 is devoted to the work in higher dimensions. Given positive integers \(n_1 \leq \cdots \leq n_d\), we use \(F_h(n_1, \ldots, n_d)\) to denote the size of the largest \(B_h\)-set contained in a \(d\)-dimensional box \([1, n_1] \times \cdots \times [1, n_d]\). If \(n_1 = \cdots = n_d = n\), we said that the box is symmetric. Previous research by Lindström \cite{Lin72}, Cilleruelo \cite{Cil10}, and Rackham and Šarka \cite{RS10} has studied this parameter in specific scenarios: symmetric boxes with \(h = 2\), asymmetrical boxes with \(h = 2\), and symmetric boxes for \(h \geq 2\), respectively. To complete this investigation, we employ techniques developed by Cilleruelo \cite{Cil10} and Rackham and Šarka \cite{RS10} to establish an upper bound for \(F_h(n_1, \ldots, n_d)\) in the most general case, where the box is not necessarily symmetric and \(h \geq 3\) (refer to Theorem 5). For the Ramsey-type version, it is convenient to define \(br_h(n_1, \ldots, n_d)\) to be the largest positive integer \(k\) such that there is no \(k\)-partition of \(\prod_{i \leq d}[1, n_i]\) in which all its parts are \(B_h\) sets. Note that, in dimension \(d = 1\), \(br_h(n)\) is the counterpart of \(BR_h(k)\) in the sense that \(br_h(BR_h(k)) = k\).

Tables 1 and 2 serve to identify the notation that we will use and locate the bounds of each parameter in the manuscript.
Table 1. We use $F_h(n)$ to denote the size of the largest $B_h$-set contained in $[1, n]$, and so $F_2(n)$ is the classical Sidon number. The corresponding Ramsey-type parameter is denoted by $BR_h(k)$, and so $SR(k) = BR_2(k)$.

| $F_h(n)$ | $h = 2$ | $h \geq 3$ |
|-----------|---------|-------------|
| $n_1 = \cdots = n_d = n$ | (1) | (3) |
| $N = n_1 n_2 \ldots n_d$ | Theorem 1 | Theorem 2 |

Table 2. We use $F_h(n_1, \ldots, n_d)$ to denote the size of the largest $B_h$-set contained in a $d$-dimensional box $[1, n_1] \times [1, n_2] \times \cdots \times [1, n_d]$. The corresponding Ramsey-type parameter is denoted by $br_h(n_1, \ldots, n_d)$. When $n_1 = n_2 = \cdots = n_d = n$ we said that the box is symmetric. In other case, the results are expressed in terms of the product $N = n_1 n_2 \ldots n_d$.

| $F_h(n_1, \ldots, n_d)$ | $br_h(n_1, \ldots, n_d)$ |
|------------------------|-------------------------|
| $n_1 = \cdots = n_d = n$ | $N = n_1 n_2 \ldots n_d$ |
| $N = n_1 n_2 \ldots n_d$ | (13) | (14) |

2. $B_h$ and $B_h$-Ramsey numbers in intervals

A subset $S$ of an additive group $G$ is called a $B_h$-set if all sums of the form $a_1 + \cdots + a_h$, where $a_1, \ldots, a_h \in A$, are distinct; note that a $B_2$-set is a Sidon set. We use $F_h(n)$ to denote the size of the largest $B_h$-set contained in $[1, n]$. It is known that

\[(1 + o(1))n^{1/h} \leq F_h(n) \leq c_h n^{1/h}.\]

The lower bound was proved by Bose and Chowla [BC63], while the constant $c_h$ in the upper bound has been successively improved [DR84, Jia93, Che94, Cil01]. Currently, the best bounds are due to Green [Gre01], who proved that $c_3 < 1.519$, $c_4 < 1.627$ and $c_h \leq \frac{1}{2} (h + (\frac{1}{2} + o(1)) / n(h))$.

As with Sidon sets, we can define the Ramsey version of this problem. For a given positive integer $k$ and a subset $X$ of an additive group $G$, a $B_h$ $k$-partition of $X$ is a partition of $X$ into $k$ parts, all of which are $B_h$-sets. Let $BR_h(k)$ be the $B_h$-Ramsey number, defined as the minimum $n$ such that there is no $B_h$-$k$-partition of $[1, n]$. We shall note that $BR_2(k) = SR(k)$.

Proof of Theorem 1. From the upper bound in (1) and using the pigeonhole principle we deduce that the interval $[1, n]$ cannot be partitioned into less than

\[
k \geq \frac{n}{F_2(n)} \geq \frac{n}{n^{1/2} + 0.998 n^{1/4}}
\]

Sidon sets. This can be simplified, using the geometric series, to

\[
k \geq \frac{n^{1/2}}{1 + 0.998 n^{-1/4}}
\]

\[
= n^{1/2} \left(1 - 0.998 n^{-1/4} + O \left(n^{-1/2} \right) \right)
\]

\[
= \left(n^{1/4} - 0.499 \right)^2 + O(1).
\]
Solving this for $n$ gives
\[
  n \leq \left( (k + O(1))^{1/2} + 0.499 \right)^4 \\
  = (k + O(1))^2 + 1.996(k + O(1))^{3/2} + O(k) \\
  = k^2 + 1.996k^{3/2} + O(k),
\]
which provides the upper bound.

To prove the lower bound, notice that a Sidon set in $\mathbb{Z}_n$ is also a Sidon set in $[1, n] \subset \mathbb{Z}$. Therefore, any Sidon-Ramsey $k$-partition of $\mathbb{Z}_n$ induces a Sidon-Ramsey $k$-partition of $[1, n]$. Singer proved that, if $q$ is a prime power, there exists a Sidon set $S$ in $\mathbb{Z}_{q^2+q+1}$ of size $q+1$ [Sin38]. Assume that $S = \{s_1, s_2, \ldots, s_{q+1}\}$ and consider the Sidon sets $S_i = S - s_i$. Since $|S_i| = |S| = q+1$ and $S_i \cap S_j = \{0\}$, with $i \neq j$, we have that
\[
  \big| \bigcup_{i=1}^{q+1} S_i \big| = (q+1)(q+1) - (q+1) + 1 = q^2 + q + 1,
\]
then $\bigcup_{i=1}^{q+1} S_i = \mathbb{Z}_{q^2+q+1}$ and so $\bigcup_{i=1}^{q+1} S_i$ covers $\mathbb{Z}_{q^2+q+1}$. Therefore, we may construct a Sidon-Ramsey $(q+1)$-partition by taking subsets of the elements of this cover. Let $k \in \mathbb{N}$ and let $p$ be the largest prime less than $k$. It is known that $p = k - O(k^{0.525})$ (see e.g. [BHP01]). As there exists a Sidon-Ramsey $(p+1)$-partition of $[1, p^2 + p + 1]$, we conclude that $SR(k) \geq SR(p+1) \geq p^2 + p + 1 = k^2 - O(k^{1.525})$, which completes the proof. \hfill $\square$

For $h \geq 2$ we have a result analogous to the Theorem 1. We can derive an upper bound of $BR_h(k)$ from Green’s bound [Gre01] and the pigeonhole principle. The lower bound comes from considering translates of the $B_h$-set constructed by Ruzsa (for $h = 2$) [Ruz93] and by Gómez-Trujillo (for $h > 2$) [GT11]. Note that the construction used for the $h = 2$ case here, is not the same as the one used in the proof of Theorem 1, although it gives the same bound.

**Theorem 2.** Let $k, h$ be positive integers, $h \geq 2$, then
\[
  k^{\frac{h}{k-h}} - O\left( k^{1+\frac{h}{k-h}} \right) \leq BR_h(k) \leq C_h k^{\frac{h}{k-h}},
\]
where $c \leq 0.525$ depends on the distribution of the prime numbers and $C_h$ depends on the best upper bound for $F_h$ numbers.

**Proof.** First, we work the upper bound for the $B_h$–Ramsey numbers. The case $h = 2$ follows from Theorem 1. If $h > 2$, the best known upper bound for $B_h$ sets, established by Green in 2001 [Gre01], is
\[
  F_h(n) \leq c_h n^{1/h},
\]
where $c_h \leq \frac{1}{2^h} (h + (3/2 + o(1)) \log(h))$. From this and using the pigeonhole principle we deduce that the interval $[1, n]$ cannot be partitioned into less than
\[
  k \geq \frac{n}{F_h(n)} \geq \frac{n}{c_h n^{1/h}}
\]
$B_h$ sets. Solving this for $n$ gives
\[
  n \leq C_h k^{\frac{h}{k-h}},
\]
where $C_h = c_h ^{\frac{h}{k-h}}$. Which provides the upper bound in Theorem 2 for $h > 2$.

To prove the lower bound in Theorem 2, notice that a $B_h$ set in $\mathbb{Z}_n$ is also a $B_h$ set in $[1, n] \subset \mathbb{Z}$. We conclude that any $B_h$-Ramsey $k$-partition of $\mathbb{Z}_n$ induces a $B_h$-Ramsey $k$-partition of $[1, n]$. The
main idea to get the lower bounds. In general, for any positive integers \( a \leq b \) and any function \( f : \mathbb{Z}_a \to \mathbb{Z}_b \), the set
\[
C = \{ (t, f(t)) : t \in \mathbb{Z}_a \}
\]
and their translations \( C + (0, 1), C + (0, 2), \ldots, C + (0, b-1) \) give a partition of \( \mathbb{Z}_a \times \mathbb{Z}_b \). Ruzsa proved that for any \( p \) prime number, there is a function \( f : \mathbb{Z}_{p^{-1}} \to \mathbb{Z}_p \), such that the set
\[
\{(a, f(a)) : a \in \mathbb{Z}_{p^{-1}} \} \subseteq \mathbb{Z}_{p^{-1}} \times \mathbb{Z}_p
\]
is a \( B_2 \)-set (see [Ruz93]). For \( h > 2 \), Gómez and Trujillo proved that for any \( p \) prime number there exist a function \( f : \mathbb{Z}_p \to \mathbb{Z}_{p^{h-1}} \) such that the set
\[
\{(a, f(a)) : a \in \mathbb{Z}_p \} \subseteq \mathbb{Z}_p \times \mathbb{Z}_{p^{h-1}}
\]
is a \( B_h \)-set (see [GT11]). Then, for any prime number \( p \), there is a \( B_2 \) Ramsey partition of \( [1, p^2 - p] \) in \( p \) parts, and for \( h > 2 \) there is a \( B_h \)-Ramsey partition of \( [1, p^h - p] \) in \( p^{h-1} - 1 \) parts.

Let \( k \in \mathbb{N} \) and, for \( h > 2 \), let \( p \) be the largest prime number such that \( p^{h-1} - 1 \leq k \) (for \( h = 2 \) we consider \( p \) to be the largest prime number less than \( k \)). It is known that \( p = (k + 1)^{1/(h-1)} - O((k + 1)^{0.525/(h-1)}) \) (for \( h = 2 \), we have \( p = k - O(k^{0.525}) \)) (see [Dus99]). In the case that \( h > 2 \) we have that
\[
BR_h(k) \geq BR_h(p^{h-1} - 1) \geq p^h - p = (k + 1)^{1/(h-1)} - O((k + 1)^{0.525/(h-1)})^h - (k + 1)^{1/(h-1)} + O((k + 1)^{0.525/(h-1)}) = (k + 1)^{h/(h-1)} - O((k + 1)^{1+0.525/(h-1)}).
\]
In the case \( h = 2 \) we have that
\[
BR_2(k) \geq BR_2(p) \geq p^2 - p = (k - O(k^{0.525}))^2 - (k - O(k^{0.525})) = k^2 - O(k^{1.525}) \quad \square
\]

3. \( B_h \) and \( B_h \)-Ramsey numbers in \( d \)-dimensional boxes

It is an interesting problem to study \( B_h \)-sets in higher dimensions. For a fixed positive integer \( d \) and positive integers \( n_1 \leq \cdots \leq n_d \), we seek to bound the largest cardinality \( F_h(n_1, \ldots, n_d) \) of a \( B_h \)-set in the \( d \)-dimensional (not necessarily symmetric) box
\[
X = [1, n_1] \times [1, n_2] \times \cdots \times [1, n_d] = \prod_{j=1}^{d} [1, n_j] \subset \mathbb{Z}^d.
\]

Regarding the lower bounds, there is a natural way to map one-dimensional \( B_h \)-sets to \( d \)-dimensional \( B_h \)-sets. Any integer \( 0 \leq a \leq n_1 \cdots n_d - 1 \) has exactly one representation of the form \( a_1 + a_2 n_1 + a_3 n_1 n_2 + \cdots + a_d - 1 n_1 \cdots n_d - 1 \), where \( 0 \leq a_i < n_i \). Let \( \varphi : \mathbb{Z} \to \mathbb{Z}^d \) be such that \( \varphi(a) = (a_1, a_2, \ldots, a_d) \). Note that any \( B_h \)-set in \( [0, n_1 \cdots n_d - 1] \) gets sent into a \( B_h \)-set in \([0, n_1 - 1] \times \cdots \times [0, n_d - 1] \), since \( \varphi(x_1) + \cdots + \varphi(x_h) = \varphi(y_1) + \cdots + \varphi(y_h) \) implies that \( x_1 + \cdots + x_h = y_1 + \cdots + y_h \). Using this property of \( \varphi \), we immediately have that
\[
F_h(n_1 \cdots n_d) \leq F_1(n_1, \ldots, n_d).
\]
So lower bounds in the \( d \)-dimensional case can be obtained from lower bounds in the one-dimensional case. This was observed for Sidon sets by Cilleruelo [Cil10].
As for the upper bounds, the first result was given by Lindström for the case \( n_1 = \cdots = n_d = n \) and \( h = 2 \) [Lin72], i.e. Sidon sets in symmetrical boxes in high dimensions. The bound obtained was
\[
F_2(n, \ldots, n) \leq n^{d/2} + O \left( n^{d^2 + 2} \right),
\]
which together with (4) and (3) gives
\[
n^{d/2}(1 - o(1)) \leq F_2(n, \ldots, n) \leq n^{d/2} + O \left( n^{d^2 + 2} \right).
\]

The best result obtained for asymmetrical boxes with \( h = 2 \) is due to Cilleruelo [Cil10], he proved that
\[
F_2(n_1, \ldots, n_d) \leq N^{1/2} \left( 1 + O \left( \frac{N_s - 1}{N^{1/2}} \right) \right),
\]
where \( N_0 = 1, N_i = \prod_{j=1}^{i} n_j \) for \( 1 \leq i \leq d \), \( N = N_d \) and \( s \) is the least integer such that \( N^{1/2} \leq n_s^{d-s+2} N_{s-1} \). This match the lower bound obtained by (4) and (3),
\[
N^{1/2}(1 - o(1)) \leq F_2(n_1, \ldots, n_d) \leq N^{1/2} \left( 1 + O \left( \frac{N_s - 1}{N^{1/2}} \right) \right).
\]

For \( h \geq 2 \), the best upper bound so far for symmetrical boxes is due to Rackham and Šarka [RS10] who showed that
\[
F_h(n_1, \ldots, n) \leq \begin{cases} 
\frac{t^h(t!)}{h} n^{\frac{h}{d}} + O \left( n^{\frac{d^2}{d+s+1}} \right) & \text{if } h = 2t, \\
\frac{t^{h-1}(t!)}{h} n^{\frac{h}{d}} + O \left( n^{\frac{d^2}{d+s+1}} \right) & \text{if } h = 2t - 1,
\end{cases}
\]
which again match the lower bound obtained from (4) and (3),
\[
(1 + o(1)) n^{d/h} \leq F_h(n_1, \ldots, n) \leq \begin{cases} 
\frac{t^h(t!)}{h} n^{\frac{h}{d}} + O \left( n^{\frac{d^2}{d+s+1}} \right) & \text{if } h = 2t, \\
\frac{t^{h-1}(t!)}{h} n^{\frac{h}{d}} + O \left( n^{\frac{d^2}{d+s+1}} \right) & \text{if } h = 2t - 1.
\end{cases}
\]

For large enough \( h \), Rackham and Šarka gave the improvement
\[
F_h(n_1, \ldots, n) \leq \begin{cases} 
(\pi d)^{\frac{h}{d}} (1 + \epsilon(h)) t^{\frac{h}{d}}(t!) \frac{n^{\frac{h}{d}}}{h} + O \left( n^{\frac{d^2}{d+s+1}} \right) & \text{if } h = 2t, \\
(\pi d)^{\frac{h-1}{d}} (1 + \epsilon(h)) t^{\frac{h-1}{d}}(t!) \frac{n^{\frac{h}{d}}}{h} + O \left( n^{\frac{d^2}{d+s+1}} \right) & \text{if } h = 2t - 1,
\end{cases}
\]
where \( \epsilon(h) \) is a function that approach 0 as \( h \to \infty \).

Note that the bound (7) by Cilleruelo in the symmetrical case and the bound (9) by Rackham and Šarka for \( h = 2 \) coincide with the bound (5) by Lindström. In Theorem 5 we provide an upper bound for \( F_h(n_1, \ldots, n_d) \) in the most general case, where the box is not necessarily symmetric and \( h \geq 2 \). We use a mix of techniques used by Cilleruelo and Rackham and Šarka. Before continuing we need a couple of definitions and lemmas.

Given subsets \( A \) and \( B \) of an additive group \( G \), and \( g \in G \), we define the sumset of \( A \) and \( B \) as
\[
A + B = \{ a + b : a \in A, b \in B \},
\]
and the additive energy between $A$ and $B$ as
\[
\sum_{g \in G} d_A(g)d_B(g),
\]
where
\[
d_X(g) = |\{(x, x') : x, x' \in X, x - x' = g\}|.
\]

**Lemma 3 ([Cil10]).** Let $G$ be an additive group and let $A, B \subset G$. Then
\[
|A|^2 \leq \frac{|A + B|}{|B|^2} \sum_{g \in G} d_A(g)d_B(g).
\]

The following lemma is a generalization of Lemmas 4.2.1 and 4.3.1 from [RS10]. The two lemmas from [RS10] require that $B$ is a symmetrical box, but this fact is not actually used in the proof. We use $tA$ to denote the set of sums of $t$ (not necessarily different) elements of $A$, and $t \ast A$ to denote the set of sums of $t$ distinct elements of $A$.

**Lemma 4.** Let $h \geq 2$, $A$ be a $B_h$ set in $\mathbb{N}^d$ and $B = [0, i_1 - 1] \times \ldots \times [0, i_d - 1]$.

1. If $h = 2t$ then
   \[
   \sum_{z \in \mathbb{Z}^d} d_{tA}(z)d_B(z) \leq |B|^2 + O(|B||A|^{h-1}).
   \]

2. If $h = 2t - 1$ then
   \[
   \sum_{z \in \mathbb{Z}^d} d_{t \ast A}(z)d_B(z) \leq \frac{|A|}{t}|B|^2 + O(|B||A|^h).
   \]

Using Lemma 3 and Lemma 4 together with techniques previously used by Cilleruelo [Cil10] and Rackham and Sarka [RS10] we obtain the following theorem.

**Theorem.** Let $n_1 \leq n_2 \leq \ldots \leq n_d$ be positive integers, set $N_0 = 1$, $N_i = \prod_{j=1}^i n_j$ for $1 \leq i \leq d$, $N = N_d$, and let $s$ be the least integer such that $N^{1/h} \leq n^d - s^2 N^{s-1}$. Then, for $h \geq 2$,
\[
F_h(n_1, \ldots, n_d) \leq \begin{cases} 
\binom{t!}{\frac{h}{2}} \frac{1}{N^\frac{h}{2}} \left(1 + O\left(\frac{N_{s-1}}{N^{\frac{h}{2}}}\right)^{\frac{1}{d+s^2}}\right) & \text{if } h = 2t, \\
\binom{t!}{\frac{h-1}{2}} \frac{1}{N^\frac{h}{2}} \left(1 + O\left(\frac{N_{s-1}}{N^{\frac{h}{2}}}\right)^{\frac{1}{d+s^2}}\right) & \text{if } h = 2t - 1.
\end{cases}
\]

**Proof.** First, we will work the case $h = 2t$. Let $A$ be a $B_h$ set in $X = \prod_{j=1}^d [1, n_j] \subset \mathbb{Z}^d$, $1 \leq s \leq d$ and let $r_j = 0$ for $j < s$ and $r_j = \lceil tn_j M \rceil$ for $j \geq s$ and some $0 < M < 1$ fixed. Applying Lemma 3 with $tA$ and $B = [0, r_1] \times \cdots \times [0, r_d]$, and using Lemma 4 we have that
\[
\binom{|tA|^2}{|t|} \frac{|B|^2}{|tA + B|} \leq \sum_{z \in \mathbb{Z}^d} d_{tA}(z)d_B(z) \leq |B|^2 + O(|B||A|^{2t-1}).
\]

The number of elements in $tA$ is $\binom{|A|}{t}$. A basic lower estimation to this binomial coefficient is
\[
\binom{\frac{|A||t|}{|A|}}{t} \leq \binom{|A|}{t},
\]
then it follows that $\binom{|A|}{t} \leq |tA|$. Similarly, we get that $\binom{|A|^h}{h} \leq |hA|$ and $hA \subset hX$, which implies $\binom{|A|^h}{h} \leq |hX| = h^d N$, i.e. $|A| = O(N^{\frac{h}{d}})$. Then we can estimate (12) as
\[
\binom{|A|^2}{(t!)^2} \frac{|B|^2}{|tA + B|} \leq |B|^2 + O(|B||A|^{2t-1}) \leq |B|^2 + O(|B||N|^\frac{h}{d}).
\]
or equivalently
\[ |A|^{2t} \leq (t!)^2 |tA + B| \left( 1 + O \left( \frac{N^{\frac{h}{2} - 1}}{|B|} \right) \right). \]

Notice that \( tA + B \subseteq [1, tn_1 + r_1] \times \cdots \times [1, tn_d + r_d] \). As \( r_j \leq tn_j M < r_j + 1 \) then
\[
|tA + B| \leq \prod_{j=1}^{d} (tn_j + r_j) = t^d N \prod_{j=1}^{d} \left( 1 + \frac{r_j}{tn_j} \right) = t^d N \prod_{j=s}^{d} \left( 1 + \frac{r_j}{tn_j} \right)
\]
\[
\leq t^d N (1 + M)^{d-s+1} = t^d N (1 + O(M)),
\]
and
\[ |B| = \prod_{j=s}^{d} (1 + r_j) \geq \prod_{j=s}^{d} tn_j M = t^{d-s+1} \frac{N}{N_{s-1}} M^{d-s+1}. \]

We conclude that
\[ |A|^{2t} \leq (t!)^2 t^d N (1 + O(M)) \left( 1 + O \left( \frac{N^{\frac{h}{2} - 1}}{N^{d-s+1}} \right) \right). \]

To minimize the order of the last expression, take \( M = (N_{s-1}/N^{1/h})^{1/(d-s+2)} \) and \( s \) as the least integer such that \( N^{1/h} \leq n_s^{d-s+2} N_{s-1} \). Then
\[ |A|^{2t} \leq (t!)^2 t^d N \left( 1 + O \left( \frac{N_{s-1}}{N^{d+2}} \right) \right). \]

We conclude the proof of this case by taking the \( 2t \)-th root.

The proof in the case \( h = 2t - 1 \) is pretty similar to the proof in the case \( h = 2t \). Let \( A \) be a \( B_h \)-set in \( X \), \( 1 \leq s \leq d \) and \( 0 < M < 1 \). We take \( M, r_j \) and \( B \) as the even case proof. Applying Lemma 3 with \( t \) and \( B \) and using Lemma 4 we have that
\[
\frac{|tA| |B|^2}{|tA + B|} = \sum_{z \in \mathbb{Z}^d} d_{tA}(z) d_B(z) \leq \frac{|A|}{t} |B|^2 + O(|B||A|^{2t-1}).
\]

The number of elements in \( t \) is \( \left( \begin{array}{c} d \\ t \end{array} \right) \). A basic lower estimation of the binomial coefficient is
\[
\frac{|A|^t}{t^t} \left( 1 - \frac{c_t}{|A|} \right) \leq |tA| \quad \text{where} \quad c_t \text{ depends only on } t.
\]
Using that, and the facts that \( |B| = O(N) \) and \( |A| = \Omega(N^*) \) (this last follows from (3) and (4)), and from \( \frac{|A|^t}{t^t} \leq h^d N \), we have that
\[
\frac{|A|^{2t} \left( 1 - \frac{c_t}{|A|} \right)^2 |B|^2}{(t!)^2 |tA + B|} \leq \frac{|A|}{t} \left( |B|^2 + O(|B||A|^{2t-2}) \right),
\]
or equivalently
\[
|A|^{2t} \leq \frac{(t!)^2 |A|}{t} |tA + B| \left( 1 + O \left( \frac{|A|^{2t-2}}{|B|} \right) \right) \left( 1 - \frac{c_t}{|A|} \right) \left( 1 + \frac{N^{\frac{h}{2} - 1}}{|B|} \right).
\]
\[
\leq \frac{(t!)^2 |A|}{t} |tA + B| \left( 1 + O \left( \frac{|A|^{2t-2}}{|B|} \right) \right) \left( 1 + \frac{1}{|A|} \right) \left( 1 - \frac{c_t}{|A|} \right) \left( 1 + \frac{N^{\frac{h}{2} - 1}}{|B|} \right).
\]
Similarly as in the even case, we get that $|B| \geq t^{d-s+1} \frac{N}{N_{s-1}} M^{d-s+1}$ and $|sA+B| \leq t^d N(1 + O(M))$. Then

$$|A|^{2t-1} \leq \frac{(t!)^2}{t} t^d N(1 + O(M)) \left( 1 + O \left( \frac{N_{s-1}^{\frac{1}{t}}} {N M^{d-s+1}} \right) \right).$$

In order to minimize the last expression we take $M = (N_{s-1}/N^{1/h})^{1/(d-s+2)}$ and $s$ as the least integer such that $N^{1/h} \leq n^{d-s+2} N_{s-1}$. Then

$$|A|^{2t-1} \leq (t!)^2 t^{d-1} N \left( 1 + O \left( \frac{N_{s-1}}{N^{1/h}} \right) \right)^{1/2}.$$

We conclude the proof taking the $(2t-1)^{th}$ root. \hfill \Box

When $h$ is large enough, it is sometimes possible to improve Theorem 5. If the box $X = \prod_{j=1}^{d-1} [1, n_j]$ is sufficiently symmetric, we may consider a slightly larger symmetrical box $X’$ that contains $X$, then it is possible to use (11) instead of (9) to obtain the improvement.

For the Ramsey version it is convenient to bound the size of the partition in terms of the dimensions of the box. Let $b_{Br}(n_1, \ldots, n_d)$ be the largest positive integer $k$ such that there is no $B_h k$-partition of $\prod_{i \leq d} [1, n_i]$. In dimension $d = 1$, $b_{Br}(n)$ is the counterpart of $BR_{br}(k)$ in the sense that $b_{Br}(BR_{br}(k)) = k$.

The second result of this section is a lower and upper bound for $b_{Br}(n_1, \ldots, n_d)$.

**Theorem 6.** Given positive integers $n_1 \leq n_2 \leq \cdots \leq n_d$, let $N_0 = 1$, $N_i = \prod_{j \leq i} n_j$ for $1 \leq i \leq d$, $N = N_d$, and let $s$ be the least index such that $N^{1/h} \leq n^{d-s+2} N_{s-1}$. Define

$$\gamma_h(n_1, \ldots, n_d) = \begin{cases} \frac{N_{s-1}}{(t!)^{d-s+1}} \left( 1 - O \left( \frac{N_{s-1}}{N^{1/h}} \right) \right) & \text{if } h = 2t, \\ \frac{N_{s-1}}{(t!)^{d-s+1}} \left( 1 - O \left( \frac{N_{s-1}}{N^{1/h}} \right) \right) & \text{if } h = 2t - 1, \end{cases}$$

then

$$\gamma_h(n_1, \ldots, n_d) \leq b_{Br}(n_1, \ldots, n_d) \leq N^{\frac{h-1}{h}} \left( 1 + O \left( N^{\frac{1}{h}} \right) \right),$$

where $c \leq 0.525$ depends on the distribution of the prime numbers.

**Proof.** The lower bound is obtained by using the pigeonhole principle with the bounds given in Theorem 5. The upper bound follows from the lower bound in Theorem 2 and the fact that any $B_h$ $k$-partition in $[1, N]$ can be mapped to a $B_h k$-partition in $[1, n_1] \times \cdots \times [1, n_d]$. \hfill \Box

Therefore, the $B_h$-Ramsey number $b_{Br}(n_1, \ldots, n_d)$ behaves asymptotically as $N^{(h-1)/h}$.

Note that the lower bound in Theorem 6 depends on the best upper bound for $F_h$. In particular, for large enough $h$, it is sometimes possible to use (11) instead of (9) to obtain a better lower bound for $b_{Br}$. To finish the paper we evaluate some particular cases of Theorem 6. When $h = 2$ and the box is symmetric,

$$n^{d/2} - O \left( n^{d/2 + \alpha} \right) \leq b_{br}(n, \ldots, n) \leq n^{d/2} + O \left( n^{\alpha/2} \right).$$

When $h = 2$ and the box is not necessarily symmetric,

$$N^{1/2} \left( 1 - O \left( \frac{N_{s-1}}{N^{1/2}} \right) \right) \leq b_{br}(n_1, \ldots, n_d) \leq N^{1/2} + O \left( N^c/2 \right).$$
Finally, when \( h \geq 3 \) and the box is symmetric

\[
\gamma_h(n, \ldots, n) = \begin{cases} 
  n^d(h-1) - O\left( n^\frac{d(h-1)}{h} \right) & \text{if } h = 2t, \\
  n^d(h-1) - O\left( n^\frac{d(h-1)}{h} \right) & \text{if } h = 2t-1,
\end{cases}
\]

and then,

\[
(15) \quad \gamma_h(n, \ldots, n) \leq br_h(n, \ldots, n) \leq n^d(h-1) + O\left( n^\frac{d(c+h-2)}{h} \right).
\]

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