GENERIC PROPERTIES OF CLOSED ORBITS FOR LAGRANGIAN FLOWS ON SURFACES

JOSÉ ANTÔNIO G. MIRANDA

Abstract. We prove a local perturbation theorem for the $k$-jets of the Poincaré map over a closed orbit of the flow of a Lagrangian system $L : TM \to \mathbb{R}$ on a closed surface $M$. The perturbations consist of adding to the Lagrangian $L$ a $C^\infty$-potential $u : M \to \mathbb{R}$. Therefore we obtain generic properties of closed orbits in the sense of Mañé.

1. Introduction.

Let $L : TM \to \mathbb{R}$ be a smooth Tonelli Lagrangian defined in a closed smooth manifold $M$, i.e., $L$ satisfy the two conditions: convexity: for each fiber $T_xM$, the restriction $L(x,v)$ has positive defined Hessian, and superlinearity: $\lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = \infty$ uniformly in $x \in M$. In this paper, we assume also that $M$ has dimension two.

The action of $L$ over an absolutely continuous curve $\gamma : [a,b] \to M$ is defined by:

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt.$$ 

The extremal curves of the action are given by solutions of the Euler-Lagrange equations that in local coordinates can be written as:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} = 0.$$ 

Since $L$ is convex and $M$ is compact, the Euler-Lagrange equations define a complete flow $\phi^L_t : TM \to TM$, that is called the Lagrangian flow of $L$ and is defined by

$$\phi^L_t(x_0, v_0) = (\gamma(t), \dot{\gamma}(t)),$$

where $\gamma : \mathbb{R} \to M$ is the solution of (1) with initial conditions $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v_0$.

Let $\omega_0$ be the canonical symplectic structure in the cotangent bundle $T^*M$. Then the Lagrangian flow of $L$ is conjugated to a Hamiltonian flow in $(T^*M, \omega_0)$ by the Legendre transformation $L : TM \to T^*M$, defined by:

$$L(x,v) = \left( x, \frac{\partial L}{\partial v}(x,v) \right).$$

The corresponding Hamiltonian $H : T^*M \to \mathbb{R}$, is given by:

$$H(x,p) = \max_{v \in T_xM} \{ p(v) - L(x,v) \} = E_L(L^{-1}(x,p)),$$

where $E_L : TM \to \mathbb{R}$ is the Energy function, that is defined as

$$E_L(x,v) = \frac{\partial L}{\partial v}(x,v) \cdot v - L(x,v).$$

2000 Mathematics Subject Classification. 37J99, 37C27, 37C20, 37B40.
Given \( c \in \mathbb{R} \), the energy level \( E_{L}^{-1}(c) \subset TM \) is compact and invariant by the Lagrangian flow. Suppose that \( \theta_{t} = \phi_{t}^{L}(\theta) \) is a closed (or periodic) orbit with period \( T_{0} > 0 \) in \( E_{L}^{-1}(c) \) and let \( \Sigma \subset E_{L}^{-1}(c) \) be a local transversal section in the energy level \( E_{L}^{-1}(c) \) at the point \( \theta \). We say that \( \theta_{t} \) is nondegenerate if the linearized Poincaré map \( d_{\theta}P = d_{\theta}P(\theta, \Sigma, L) : T_{\theta}\Sigma \to T_{\theta}\Sigma \) does not admit a root of unity as eigenvalue. We say that \( \theta_{t} \) is hyperbolic if \( d_{\theta}P \) does not have eigenvalues with norm equal to 1, and that \( \theta_{t} \) is elliptic if all the eigenvalues of \( d_{\theta}P \) have norm one but they are not roots of unity. For surfaces, a nondegenerate closed orbit is either elliptic or hyperbolic. Given two hyperbolic periodic orbits \( \theta_{t} \) and \( \eta_{t} \) of the Lagrangian flow \( \phi_{t}^{L} \), a heteroclinic orbit from \( \theta_{t} \) to \( \eta_{t} \) is an orbit whose \( \alpha \)-limit is \( \theta_{t} \) and its \( \omega \)-limit is \( \eta_{t} \). The strong stable and strong unstable manifolds of the hyperbolic periodic orbit \( \theta_{t} \) at the point \( \theta_{t} \) are defined as

\[
W^{ss}(\theta_{t}) = \{ v \in E_{L}^{-1}(c); \lim_{t \to \infty} d(\theta_{t}, \phi_{t}^{L}(v)) = 0 \}, \quad \text{and} \quad W^{su}(\theta_{t}) = \{ v \in E_{L}^{-1}(c); \lim_{t \to -\infty} d(\theta_{t}, \phi_{t}^{L}(v)) = 0 \}
\]

respectively. The (weak) stable and (weak) unstable manifolds of the hyperbolic periodic orbit \( \theta_{t} \) are defined as

\[
W^{s}(\theta_{t}) = \bigcup_{t \in \mathbb{R}} \phi_{t}^{L}(W^{ss}(\theta_{t})) \quad \text{and} \quad W^{u}(\theta_{t}) = \bigcup_{t \in \mathbb{R}} \phi_{t}^{L}(W^{su}(\theta_{t}))
\]

respectively. The sets \( W^{s}(\theta_{t}) \) and \( W^{u}(\theta_{t}) \) are \( \phi_{t}^{L} \)-invariant and they are immersed submanifolds of \( E_{L}^{-1}(c) \). A heteroclinic orbit from \( \theta_{t} \) to \( \eta_{t} \) is in the intersection \( W^{s}(\theta_{t}) \cap W^{u}(\eta_{t}) \). When this intersection is transversal in \( E_{L}^{-1}(c) \) we say that the heteroclinic orbit is transversal.

Let us recall some facts about the jet space for symplectic maps in \((\mathbb{R}^{2n}, \omega_{0} = dx \wedge dy)\). Let \( \text{Diff}_{\omega_{0}}(\mathbb{R}^{2n}, 0) \) be the space of smooth symplectic diffeomorphisms \( f : (\mathbb{R}^{2n}, \omega_{0}) \to (\mathbb{R}^{2n}, \omega_{0}) \), such that \( f(0) = 0 \). Given \( k \in \mathbb{N} \), consider the equivalence relation \( \sim_{k} \) in \( \text{Diff}_{\omega_{0}}(\mathbb{R}^{2n}, 0) \), defined as:

\[
f \sim_{k} g \Leftrightarrow \text{the Taylor polynomials of degree } k \text{ in zero are equal}.
\]

We define the \( k \)-jet of \( f \in \text{Diff}_{\omega_{0}}(\mathbb{R}^{2n}, 0) \) that we denote by \( j^{k}(f) = j^{k}(f)(0) \), as the equivalence class of \( f \) with respect to the relation \( \sim_{k} \). The space of symplectic \( k \)-jets \( J_{k}^{s}(n) \) is the set of all equivalence classes with respect to the relation \( \sim_{k} \) of elements of \( \text{Diff}_{\omega_{0}}(\mathbb{R}^{2n}, 0) \). Observe that \( J_{k}^{s}(n) \) is a vector space that is also a Lie group, if we consider the product defined by

\[
j^{k}(f) \cdot j^{k}(g) = j^{k}(f \circ g), \quad \forall f, g \in \text{Diff}_{\omega_{0}}(\mathbb{R}^{2n}, 0).
\]

When \( k = 1 \), we can identify \( J_{1}^{s}(n) \) with the classic Lie group \( \text{Sp}(n) \). We say that a subset \( Q \subset J_{k}^{s}(n) \) is invariant if

\[
\sigma \cdot Q \cdot \sigma^{-1} = Q, \quad \forall \sigma \in J_{k}^{s}(n).
\]

Note that if \( \theta_{t} = \phi_{t}^{L}(\theta) \) is a periodic orbit in some energy level \( E_{L}^{-1}(c) \) and \( \Sigma \subset E_{L}^{-1}(c) \) is a local transversal section at the point \( \theta_{t} \), then the symplectic form \( \mathcal{L}^{\ast} \omega_{0} \) on \( TM \) induces a symplectic form on \( \Sigma \) and the Poincaré map \( P(\theta, \Sigma, L) : \Sigma \to \Sigma \) becomes a symplectic diffeomorphism. Therefore, using Darboux coordinates, we can assume that \( j^{k}(P(\theta, \Sigma, L)) \in J_{k}^{s}(1) \). Given an invariant subset \( Q \subset J_{k}^{s}(1) \) and a closed orbit \( \theta_{t} \), it follows from \( \text{(3)} \) that the property :the \( k \)-jet the Poincaré map over \( \theta_{t} \) belongs to \( Q \), is well defined, because it is independent of the section \( \Sigma \) and the coordinate system.

In this paper we will consider generic properties in the sense of to R. Mañé. In [Man90], he shows “how the theory of minimizing measures becomes much stronger and more accurate if we restrict it to generic Lagrangians”. Let \( C^{\infty}(M) \) be the space of smooth functions \( u : M \to \mathbb{R} \) endowed with the \( C^{\infty} \) topology. Recall that a subset \( \mathcal{O} \subset C^{\infty}(M) \) is called residual if it contains a countable intersection of open and dense subsets. We say that a property is generic (in the sense of Mañé), if
for each Lagrangian $L$, there exists a residual subset $\mathcal{O} \subset C^\infty(M)$, such that the property holds for all modified Lagrangians $L - u$, $u \in \mathcal{O}$.

In [Oli08], E. Oliveira proves a conservative version of the Kupka-Smale Theorem for generic Lagrangians on surfaces. More precisely, he proves that, if in the configuration space $M$ has dimension two, for each $c \in \mathbb{R}$, there exists a residual set $\mathcal{O} = \mathcal{O}(c) \subset C^\infty(M)$, such that, every Lagrangian $L - u$, $u \in \mathcal{O}$ satisfies:

$$\begin{align*}
P_{K-S} : \quad & (i) \quad E_{L}^{-1}(c) \text{ is a regular energy level,} \\
                   & (ii) \quad \text{all closed orbits in } E_{L}^{-1}(c) \text{ are either hyperbolic or elliptic} \\
                   & (iii) \quad \text{all heteroclinic intersections in } E_{L}^{-1}(c) \text{ are transversal.}
\end{align*}$$

The main goal of the present paper is to extend this result to include conditions on the higher order derivatives of the Poincaré maps of closed orbits. It is motivated by the fact that some important properties of the dynamical behavior near elliptic closed orbits depend on the higher order derivatives of the corresponding Poincaré map.

The extension that we prove here is analogous to what has been done for other classes of conservative systems. Let us consider, for instance, generic properties of closed geodesics, and recall that a bumpy metric is a metric such that all closed geodesics are non-degenerate. In this case the subset of bumpy metrics in the space of smooth Riemannian metrics on $M$ is residual. This theorem is attributed to R. Abraham [Abr70]; also see D.V. Anosov [Ano82], where a complete proof is given. In [KT72], Klingenberg and Takens extend the bumpy metric theorem to include conditions on the $k$-jets of the Poincaré map over closed orbits for geodesic flows. For the class of magnetic flows on surfaces, a complete study of generic properties of closed orbits can be seen in [Mir06].

Using the notation above, we can state our local perturbation theorem.

**Theorem 1.** Let $Q \subset J_k^s(1)$ open and invariant, such that $j^k(P(\theta, \Sigma, L)) \in \overline{Q}$. Then there exists a smooth potential $u : M \rightarrow \mathbb{R}$, arbitrarily $C^r$-close to zero, with $r > k$, such that

- $\theta_t$ is also a closed orbit of the Lagrangian flow for $L - u$ and
- $j^k(P(\theta, \Sigma, L - u)) \in Q$.

Combining the Kupka-Smale Theorem and the above theorem, we obtain:

**Theorem 2.** Let $M$ be a closed two dimensional manifold. Given an open and dense invariant subset $Q \subset J_k^s(1)$ and $c \in \mathbb{R}$, the property

$$P_Q : \text{ the } k\text{-jet of the Poincaré map of every closed orbit in } E_{L}^{-1}(c) \text{ belongs to } Q$$

is generic, in the sense of Mañé, for Lagrangian systems on $M$.

We would like to point out that we don’t know if the above properties are generic for Lagrangians on manifolds of arbitrary dimensions. In the proof of Theorem 1 we shall use the local perturbation result in [Oli08, Theorem 4.5] which was proved only in dimension two.

As an application of Theorem 1 we obtain a result about the topological entropy of Lagrangian flows. The topological entropy is a dynamical invariant that, roughly speaking, measures its orbit structure complexity. Its precise definition can be found in [Bow75]. The relevant question about the topological entropy is whether it is positive or vanishes. It is well know that if a flow contains a transversal homoclinic orbit, then it has positive topological entropy.

**Proposition 3.** Let $M$ be a closed two dimensional manifold. Suppose that the Lagrangian flow $\phi^t_L : TM \rightarrow TM$ has a non-hyperbolic closed orbit in an energy level $E_{L}^{-1}(c)$. Then, there is a potential function $u : M \rightarrow \mathbb{R}$ of norm arbitrarily small in the $C^\infty$-topology, such that the perturbed Lagrangian flow $\phi^t_{L-u}$ restricted to $E_{L-u}^{-1}(c)$ has positive topological entropy.
In order to prove this proposition, we apply the Theorem [4] to approximate \( L \) by \( L - u \) such that the Poincaré map of the non-hyperbolic orbit becomes a generic exact twist map in a small neighborhood of the elliptic fixed point. Then a result of Le Calvez [LC91] implies that this twist map has homoclinic orbits, it implies positive topological entropy. In particular, the flow has infinite many closed orbits. The details of this arguments will be given in section [4].

2. The local perturbation of the \( k \)-jet.

In order to prove the Theorem [1] we take the Hamiltonian point of view. Then, given the closed orbit \( \theta_t = (\gamma(t), \dot{\gamma}(t)) \) of the Lagrangian flow, let \( \Gamma(t) = (\gamma(t), p(t)) \) be the corresponding closed orbit of the Hamiltonian flow, defined by (2). Recall that, by the Kupka-Smale Theorem we can suppose that \( \Gamma \) is nondegenerate and the energy level \( H^{-1}(c) \) that contain \( \Gamma \) is regular. Observe that perturbations \( L - u \) of \( L \) are equivalent to perturbations of the kind \( H + u \), where \( u : M \to \mathbb{R} \) is a smooth function.

2.1. \( k \)-General family of linear symplectic maps. First, we produce a perturbation of the 1-jet of the Poincaré map along the periodic orbit to put it in a particular position.

Let \( \mathbb{R}[x, y]^k \) be the space of real homogeneous polynomials of degree \( k \) in the variables \( x, y \). We fix the polynomial \( F(x, y) := x^k \). For each \( k \in \mathbb{N} \), let \( G_k \) be the set defined as:

\[
G_k = \{ (\sigma_1, ..., \sigma_k) \in Sp(1)^k; \{ F(x, y), F(\sigma_1(x, y)), ..., F(\sigma_k(x, y)) \} \text{ is a basis for } \mathbb{R}[x, y]^k \}.
\]

**Definition 4.** A one parameter family \( \sigma : [a, b] \to Sp(1) \), with \( \sigma(a) = \sigma_0 = I \), is \( k \)-general, if there exist times \( t_1, ..., t_k \in (a, b) \) such that \( (\sigma_{t_1}, ..., \sigma_{t_k}) \) is a element of the set \( G_k \).

First we observe that:

**Proposition 5.** For each \( k \in \mathbb{N} \), the subset \( G_k \) is open and dense in \( Sp(1)^k \).

**Proof.** Consider the one parameter family of symplectic matrix \( \sigma : [0, 1] \to Sp(1) \), such that:

\[
\sigma_t(x, y) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ty \\ y \end{pmatrix}.
\]

It is easy to see that there exist values \( t_1, ..., t_k \in (0, 1] \) for what the polynomials \( F(x, y) \) and \( F(\sigma_{t_n}(x, y)) = (x + t_n y)^k \), with \( n = 1, ..., k \), form a basis of \( \mathbb{R}[x, y]^k \). By fixing a basis in \( \mathbb{R}[x, y]^k \), we have that \( G_k \) is the complement in \( Sp(1)^k \) of the set:

\[
G^c_k = \{ (\sigma_1, ..., \sigma_k) \in Sp(1)^k; \det [F(\sigma_i)]_{0 \leq i \leq k} = 0 \}.
\]

Therefore \( G_k \) is a non empty complementary set of a algebraic subset. This implies that \( G_k \) is open and dense in \( Sp(1)^k \).

Now, let \( T > 0 \) be such that the arc \( \gamma((0, T]) \subset M \) has not self-intersection points of \( \gamma \), and take \( W \subset M \) a tubular neighborhood of \( \gamma((0, T]) \), with \( W \cap \gamma \subset \gamma((0, T]) \). We consider

\[
\mathcal{F}^1 = \mathcal{F}^1(\gamma, T, W) = \{ u \in C^\infty(M); j^1(u)(\gamma([0, T])) \equiv 0 \text{ and Supp}(u) \subset W \}.
\]

Note that, for each \( u \in \mathcal{F}^1 \), the curve \( \Gamma(t) \) is also a closed orbit of the Hamiltonian flow \( \psi^{H+u}_t : T^*M \to T^*M \) and that the energy levels \( H^{-1}(c) \) and \( (H + u)^{-1}(c) \) are tangent along of the curve \( \Gamma(t) \) (see also remark [3]). Choose \( t_0 \in (0, T) \), and let \( \mathcal{N}(t_0) \) be the linear subspace of \( T_{\Gamma(t_0)}(T^*M) \) that is the symplectic orthogonal of the Hamiltonian vector field \( X^{H}(\Gamma(t_0)) \). Then, we can consider the map

\[
S_{t_0} : \mathcal{F}^1 \to Sp(1)
\]
defined by \( S^1_{t_0}(u) = d_{\Gamma(t_0)}P_{t_0}(u) \), were \( d_{\Gamma(t_0)}P_{t_0}(u) : \mathcal{N}(0) \to \mathcal{N}(t_0) \) denotes the derivative of the Poincaré map \( P_{t_0} = P(\Gamma(0), \Sigma(t_0), H + u) \), for the transversal sections \( \Sigma_t \subset H^{-1}(c) \) at \( \Gamma(t) \), such that \( T_{\Gamma(t)}\Sigma_t = \mathcal{N}(t) := d_{\Gamma(t_0)}\psi^{H+u}_{t-t_0} \cdot \mathcal{N}(0) \), for \( t \in [0, t_0] \).

**Remark 6.** Its was proved in [Ohl08], that the derivative at zero of the map \( S^1_{t_0} : F^1 \to Sp(1) \) is onto in the Lie algebra \( sp(1) \) of the classical Lie group \( Sp(1) \). This implies that \( S^1_{t_0} \) is an open map in a neighborhood of zero. Therefore we can found a potential \( u : M \to \mathbb{R} \), arbitrarily \( C^\infty \)-close to zero, that is adapted to each symplectic matrix in a small enough open ball centered at \( S^1_{t_0}(0) = d_{\Gamma(0)}P_{t_0}(H) \). Moreover the support of \( u \) can be choose arbitrarily small. See the proof of Theorem 4.5 in [Ohl08] for the details.

Combining its remark and Proposition 5 we obtain:

**Lemma 7.** For each integer \( k > 2 \), there exists a smooth potential \( u_0 : M \to \mathbb{R} \), with \( C^\infty \)-norm arbitrarily small such that \( \Gamma(t) \) is also a closed orbit of the perturbed Hamiltonian flow \( \psi^{H+u_0} \) and such that the one parameter family \( t \mapsto d_{\Gamma(0)}P_t(H + u_0) \), for \( t \in [0, T] \), is \( k \)-general.

**Proof.** Given \( k \in \mathbb{N} \), we set \( t_0 = t_0(k, T) \in (0, T] \) such that \( T = (k + 1)t_0 \). We divide \( \gamma \) in \( k + 1 \) segments \( \gamma_i : [0, t_0] \to M \) given by \( \gamma_i(t) := \gamma(t + it_0) \), with \( 0 \leq i \leq k \). Consider the map

\[ S : F^1 \to Sp(1)^k \]

defined by

\[ S(u) = (d_{\Gamma(0)}P_{t_0}(u), \ldots, d_{\Gamma(0)}P_{kt_0}(u)) \]

Then, by Remark 5 each component of the map \( S \) is a local submersion near \( 0 \in F^1 \). Since the \( G^k \subset Sp(1)^k \) is dense (Proposition 5), we can choose a potential \( u = u_1 + \cdots + u_k \in F^1 \), with \( \text{Supp}(u_i) \cap \gamma_j = \emptyset \) for \( i \neq j \) and the \( C^\infty \)-norm arbitrarily small such that the one parameter family of linear symplectic maps associated to the linearized Poincaré map of the perturbation \( H + u \) is \( k \)-general.

### 2.2. Perturbation of the k-jet.

Let us now fix a local coordinate system in a neighborhood of a segment of the closed orbit \( \Gamma(t) \) that we will use to give a local description of the Hamiltonian flow and to define an appropriated perturbation space. Let \( T_0 \) the minimal period of \( \Gamma(t) \). We choose \( T \in (0, T_0) \) such that the segment \( \gamma([0, T]) \subset M \) has not self-intersection and take coordinates \( x_1, x_2 \) in a neighborhood \( W \subset M \) of \( \gamma([0, T]) \) such that:

- \( \gamma(t) = (t, 0) \) and
- \( \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}_{[t, 0]} \) is a orthogonal basis for \( T_{\gamma(t)}M \), for all \( t \in [0, T] \).

For each \( x \in W \), let \( \{dx_1, dx_2\} \subset T^*_xM \) be the dual basis for \( \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\} \subset T_xM \). Then \( (x_1, x_2, dx_1, dx_2) \) is a local coordinate system in a neighborhood of \( \Gamma([0, T]) \). In these coordinates we have \( \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \), and the Hamiltonian vector field \( X_H \) is

\[
X_H = \sum_{i=1}^{2} \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^{2} \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i}.
\]

Now we are going to define our perturbation space. Let \( \delta, \beta : \mathbb{R} \to \mathbb{R} \) be smooth functions such that \( \delta \in \{h \in C^\infty (\mathbb{R}) ; \text{ Supp}(h) \subset (0, T)\} \) (later we will take \( \delta(t) \) as a smooth approximation of a Dirac delta function) and \( \beta : \mathbb{R} \to \mathbb{R} \) satisfies:

- \( \text{Supp}(\beta) \subset (-\epsilon, \epsilon) \), with \( \epsilon \) sufficiently small and
- \( j^{k+1}(\beta)(0) = x_2^{k+1} \),

where \( j \) is the embedding and \( k \) is the dimension of the manifold.
for $k > 1$. Let $\mathcal{F}^k = \mathcal{F}(W, \gamma, H, T, k)$ be the subset of functions $u : M \to \mathbb{R}$ such that, in the local coordinates $(x_1, x_2)$, have the form

$$u(x_1, x_2) = \delta(x_1)\beta(x_2).$$

We will consider perturbations of the kind $(H + u)$ for $u \in \mathcal{F}^k$. Then $X_{H+u} = X_H + u$, and locally by (4) we have

$$\partial H = \delta'(x_1)\beta(x_2) \frac{\partial}{\partial y_1} - \delta(x_1)\beta'(x_2) \frac{\partial}{\partial y_2} = -\delta(x_1)\beta'(x_2) \frac{\partial}{\partial y_2} + \mathcal{O}(x^{k+1}).$$

First, we choose a family of local hypersurfaces $[0, T] \to \Lambda(t)$ in $T^*M$, given by $\Lambda(t) = [x_1 = t]$. Then $\Lambda(t)$ is transversal to $\Gamma(t)$, for all $t \in [0, T]$. For each $t \in [0, T]$, we consider the map

$$S_t(u) : (\Lambda(0), l(0)) \to (\Lambda(0), l(0))$$

defined as

$$S_t(u) = \hat{P}_t - 1 \circ \hat{P}_t,$$

where $\hat{P}_t : \Lambda(0) \to \Lambda(t)$ and $\hat{P}_t' : \Lambda(0) \to \Lambda(t)$ denote the Poincaré maps in an open neighborhood of $\Gamma(0) \in \Lambda(0)$ to $\Lambda(t)$ with respect to $X_H$ and $(X_H + u)$ respectively. Note that the vector field $X_u$ satisfy:

- $j^{(k-1)}(X_u)(\Gamma(t)) = 0$, $\forall t \in [0, T]$,
- $\Gamma(0), \Gamma(T) \notin \text{Supp}(X_u)$ and
- $X_u|_{\Lambda(t)}$ is $k$-tangent to $\Lambda(t)$, for all $t \in [0, T]$.

The following proposition holds for abstract vector fields satisfying the three conditions above. A proof can be seen in [KT72, section 2].

**Proposition 8.** The $k$-jet of $S_T(u)$ at the point $\Gamma(0)$ is equal to the $k$-jet of the flow at time $T$ associated to the non-autonomous vector field $\hat{P}_t^*\left(X_u|_{\Lambda(t)}\right)$ at the point $\Gamma(0)$.

Let $\Sigma(t) \subset T^*M$ be the submanifold given by

$$\Sigma(t) = \Lambda(t) \cap H^{-1}(c), \quad t \in [0, T].$$

Then $\omega$ induces a symplectic structure on $\Sigma(t)$ and the restriction $\hat{P}_t|_{\Sigma(t)} : \Sigma(0) \to \Sigma(t)$ is a symplectic map for all $t \in [0, T]$. Since $\Gamma(0), \Gamma(T) \notin \text{Supp}(u)$, $\hat{P}_t|_{\Sigma(0)} : \Sigma(0) \to \Sigma(T)$ is a symplectic map too.

Observe that $\hat{P}_t(\Gamma(0)) \equiv 1$. Then we can parameterize $\Sigma(t)$ in terms of the coordinates $x_2, y_2$, this is, for each $t \in [0, T]$ there is an open set $V_t \subset \mathbb{R}^2$ and a function $\alpha_t : V_t \to \mathbb{R}$, such that

$$\Sigma(t) = \left\{(x_2, y_2) : (x_2, y_2) \in V_t, \quad (x_2, y_2) \in V_t \right\}.$$

Since $T\Sigma(t) \subset \text{Ker}(dx_1)$, the symplectic structure induced by $\omega$ in $\Sigma(t)$ is given by $\omega|_{\Sigma(t)} = dx_2 \wedge dy_2$. For each $u \in \mathcal{F}^k$ and $t \in [0, T]$, we consider the Hamiltonian function $K_{u,t} : \Sigma(t) \to \mathbb{R}$ given by

$$K_{u,t} = u|_{\Sigma(t)} = \delta(t)\beta(x_2)$$

and we denote by $Y_{u,t}$ its Hamiltonian vector field. Then

$$j^{k+1}(K_{u,t})(\Gamma(t)) = \delta(t)x_2^{k+1},$$

and this defines a family, parameterized by $t$, of multiples of the polynomial $F(x_2, y_2) := x_2^{k+1}$.

**Remark 9.** The submanifold $\Sigma(0)$ is not invariant by the map $S_t$, if $(t, x_2) \in \text{Supp}(u)$. But, by (7), the unique component of the field $X_u$ that has the $k$-jet vanishing at $\Gamma(t)$ is the component in the direction $\frac{\partial}{\partial y_2}$ and this direction is tangent to $\Sigma(t)$ along $\Gamma(t)$ (because $dH|_{\Gamma(t)}(\frac{\partial}{\partial y_2}) = \omega|_{\Gamma(t)}(X_H, \frac{\partial}{\partial y_2}) \equiv 0$). Then the vector fields $Y_{u,t}$ and $X_u|_{\Sigma(t)}$ in $\Sigma(t)$ have the same $k$-jet along $\Gamma(t)$.

Moreover, since $\hat{P}_t(\Sigma(0)) = \Sigma(t)$, we have that the non-autonomous vector field $\hat{P}_t^*(X_u|_{\Sigma(t)})$ is $k$-jet
tangent to $\Sigma(0) \subset \Lambda(0)$. Therefore the submanifolds $\Sigma(0)$ and $S_t|\Sigma(0)$ have a tangency of order $k$ at $\Gamma(0)$. Then, to study the k-jet of $S_t$ at the point $\Gamma(0)$ we can assume that $S_t$ leaves $\Sigma(0)$ invariant for all $t \in [0, T]$.

**Lemma 10.** The k-jet of $S_t|\Sigma(0)$ in $\Gamma(0)$ is equal to the k-jet at $\Gamma(0)$ of the Hamiltonian flow at time $t$ that corresponds to the non-autonomous Hamiltonian $[\hat{\delta}(t) F \circ (\hat{P}_t|\Sigma(0))]$ in $\Sigma(0)$.

**Proof:** Combining the remarks 8 and 9 we conclude that the k-jet of $S_t|\Sigma(t)$ is equal to the k-jet of the flow at time $t$ associated to the field $\hat{P}_t^*(Y_{u,t})$. On the other hand, if $X$ denotes the Hamiltonian field for the non autonomous Hamiltonian $[K_{u,t} \circ (\hat{P}_t|\Sigma(0))]$, then using that $\hat{P}_t|\Sigma(0) : \Sigma(0) \to \Sigma(t)$ is a symplectic map, we have:

$$\omega(X, \cdot)|_{\Sigma(0)} = d \left( K_{u,t} \circ (\hat{P}_t|\Sigma(0)) \right) = \hat{P}_t^* \omega(Y_{u,t} \cdot)|_{\Sigma(1)} = \omega(\hat{P}_t^*(Y_{u,t}) \cdot)|_{\Sigma(0)}.$$ 

And, since $\omega|_{\Sigma(0)}$ is no degenerate, we have that $X = \hat{P}_t^*(Y_{u,t})$. Hence the k-jet of $\hat{P}_t^*(Y_{u,t})$ in $\Gamma(0)$ is determined by the $(k+1)$-jet of the Hamiltonian $[K_{u,t} \circ (\hat{P}_t|\Sigma(0))]$ in $\Gamma(0)$, that, by 10, is equal to the k-jet of the Hamiltonian $[\hat{\delta}(t) F \circ (\hat{P}_t|\Sigma(0))]$. This completes the proof. \hfill $\Box$

**Remark 11.** Recall that $J^k_s(1)$ is a Lie Group with the group structure defined by $j^k(f) \cdot j^k(g) = j^k(f \circ g)$. Let $3^k_s(1)$ be the space of the k-jets in $0 \in \mathbb{R}^2$ of the symplectic vector fields in $(\mathbb{R}^2, dx_1 \wedge dx_2)$ that are zero in the origin. We define the bracket $[\cdot, \cdot]^k : 3^k_s(1) \times 3^k_s(1) \to 3^k_s(1)$ by $[j^k(X), j^k(Y)]^k = -j^k([X, Y])$. Since $X, Y$ are zero in the origin, $[\cdot, \cdot]^k$ depends only on the k-jets of $X$ and $Y$. Then $[\cdot, \cdot]^k$ defines a Lie algebra structure in $3^k_s(1)$. Moreover, $3^k_s(1)$ is the Lie algebra of $J^k_s(1)$ and the exponential map $\exp : 3^k_s(1) \to J^k_s(1)$ is given by $\exp(t \cdot j^k(X)) = j^k(\psi_t)$, where $\psi_t$ is the local flow associated to $X$. For more details and proofs, see [KMS93] [IV].

**Lemma 12.** Let $\pi_k : j^k_s(1) \to j^{k-1}_s(1)$ be the canonical projection. Given an integer $k \geq 2$, consider the map

$$S^k_T : F^k \to \text{Ker}(\pi_k) \subset J^k_s(1)$$

$$u \mapsto j^k(S^k_T(u)|\Sigma(0))(\Gamma(0)).$$

If the one parameter family $[0, T] \to d\Gamma(0)\hat{P}_t|\Sigma(0) \subset j^1_s(1) = S^1_p(1)$, is $(k+1)$-general for some $k > 1$, then the map $S^k_T$ is a local submersion in neighborhood of $0 \in F^k$.

**Proof.** Since $d\Gamma(0)\hat{P}_t|\Sigma(0)$ is $(k+1)$-general, for $0 \leq t \leq T$, there are $t_1, ..., t_{k+1} \in (0, T)$, such that

$$\left\{ F(x_2, y_2), F(d\hat{P}_{t_1}(x_2, y_2)), ..., F(d\hat{P}_{t_{k+1}}(x_2, y_2)) \right\}$$

is a basis for $\mathbb{R}[x_2, y_2]^{k+1}$, where $F(x_2, y_2) = x_2^{k+1} \in \mathbb{R}[x_2, y_2]^{k+1}$. For each $0 \leq i \leq k+1$ and $\lambda > 0$ sufficiently small, let $\delta_{\lambda}(t_i) : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ approximation of the Dirac delta function at the point $t_i$ with support in the interval $[t_i - \lambda, t_i + \lambda]$. We consider $u_i = u_i(\lambda) = \delta_{\lambda}(t_i)\beta(x_2) \in F^k$, for $i \in \{0, ..., k+1\}$. By Lemma 10 and the properties of the exponential map, as defined in the Remark 11 we have:

$$D_0S^k_T(u_i) = \left. \frac{\partial}{\partial s} \right|_{s=0} S^k_T(s, u_i) = \left. \frac{\partial}{\partial s} \right|_{s=0} \exp(t \cdot j^k(s \cdot X_i)),$$

where $X_i$ denotes the Hamiltonian field in $\Sigma(0)$ corresponding to the non autonomous Hamiltonian $[\delta_{\lambda}(t_i) F \circ (\hat{P}_t|\Sigma(0))]$. Computing the derivative with respect to $t$ in the above equality, we obtain:

$$\frac{d}{dt}(D_0S^k_T(u_i)) = \left. \frac{\partial}{\partial s} \right|_{s=0} \left( \frac{\partial}{\partial t} \exp(t \cdot j^k(s \cdot X_i)) \right) = \left. \frac{\partial}{\partial s} \right|_{s=0} \left( \frac{\partial}{\partial t} \exp(t \cdot j^k(s \cdot X_i)) \right) =$$
\frac{\partial}{\partial s} \bigg|_{s=0} (d_{(t, j^k(s, X_i))} \exp \cdot j^k(s, X_i)) = \frac{\partial}{\partial s} \bigg|_{s=0} j^k(s, X_i) = j^k(X_i).

Then

\[ D_0 S^k_T \cdot (u_i) = \int_0^T j^k(X_i) \, dt. \]

By definition of \( X_i \) and (7), we have that \( \lambda \) converges to 0 then \( D_0 S^k_T \cdot (u_i) \) converges to the k-jet in \( \Gamma(0) \) of the Hamiltonian field in \( \Sigma(0) \) correspondent to the autonomous Hamiltonian 

\[ H = \left[ F \circ (\hat{P}_t|_\Sigma(0)) \right]. \]

Computing the \((k+1)\)-jet of \( H \) in \( \Gamma(0) \), we obtain:

\[ j^{k+1}(H_i) = \left[ F \circ (d_{\Gamma(0)} \hat{P}_t|_\Sigma(0)) \right]. \]

Since \( \{F \circ (d_{\Gamma(0)} \hat{P}_t|_\Sigma(0))\}_{0 \leq i \leq k+1} \) is a basis for \( \mathcal{R}[x_2, y_2]_{k+1} \), we have that for \( \lambda \) sufficiently small \( \{D_0 S^k_T \cdot (\xi_i)\}_{0 \leq i \leq k+1} \) is a basis for the Lie Algebra of the Lie subgroup \( \text{Ker}(\pi_k) \). Hence the map \( S^k_T \) is a local submersion. \( \square \)

2.3. Proof of the Theorem 1. Let \( \Gamma(t) = (\gamma(t), p(t)) \) be a closed orbit of the Hamiltonian flow \( H^\iota_t \) of minimal period \( T_0 > 0 \). Since the number of self-intersection points is finite, we can choose \( T \in (0, T_0] \), such that the segment \( \gamma([0, T]) \) does not contain self-intersection points of the curve \( \gamma \) and a tubular neighborhood \( W \subset M \) of \( \gamma([0, T]) \), sufficiently small, such that \( W \cap \gamma = \gamma([0, T]) \).

By this, we can choose a local coordinates system \( x_1, x_2 \) in \( W \) and a family of local transversal sections \( \Sigma(t) = \Lambda(t) \cap H^{-1}(c) \subset T^*M \), and maps \( \hat{P}_t : \Lambda(0) \to \Lambda(t) \), as in subsection 2.2. Then \( P_t := P(\Gamma(0), \Sigma(0), \Sigma(t), H) = \hat{P}_t|_\Sigma(0) \). By Lemma 4 there exists a smooth potential \( u \) with \( C^\infty \)-norm arbitrarily small, such that the correspondent one parameter family \( [0, T] \to d_{\Gamma(0)} P_t \) is s-general, for any \( s = 2, 3, \ldots, k+1 \).

We set \( \mathcal{F}^i = \mathcal{F}^i(W, \gamma, H, T, i) \subset C^\infty(M), 2 \leq i \leq k \), as in section 2 and

\[ \mathcal{F} = \mathcal{F}(W, \gamma, T) = \left\{ u \in C^\infty(M); \ u|_{\gamma([0, T])} \equiv 0 \text{ and } \text{Supp} \ (u) \subset W \right\}. \]

It is easy to see that \( \mathcal{F}^i \subset \mathcal{F} \) for all \( i \in \{1, \ldots, k\} \). We define the map:

\[ S : \mathcal{F} \to J^k_s(1) \]

\[ u \mapsto j^k(S(u))(\Gamma(0)) \]

where \( S(u) = P(\Sigma(0), \Sigma(0), H)^{-1} \circ P(\Sigma(0), \Sigma(0), H+u) \).

**Remark 13.** Since \( \text{Supp}(u) \subset W \) for all \( u \in \mathcal{F} \) and

\[ P(\Sigma(0), \Sigma(0), q) = P(\Sigma(0), \Sigma(T), q) \circ P(\Sigma(T), \Sigma(T_0), q), \]

with \( q = \{H, H+u\} \), its follows that \( S(u) = S_T(u) = P_T(H)^{-1} \circ P_T(H+u) \).

By Remark 6 and Lemma 12 we have that each \( r = 2, \ldots, k \), the map \( S^k_T \) is an open map in a neighborhood of \( 0 \in \mathcal{F} \). Since \( j^k(P(\theta, \Sigma, H)) \in Q \), the openness of the maps \( S^k_T \) ( for \( r = 2, \ldots, k \) ) in a neighborhood of zero implies that there exists \( u \in \mathcal{F} \) arbitrarily \( C^\infty \)-close to zero, such that the k-jet of \( S_T(u) \) is an element of the set \( Q \). By Remark 13 its proves the theorem.

3. Proof of Theorem 2

Let \( Q \) be an open, dense and invariant subset of \( J^k_s(1) \) and \( c \in \mathcal{R} \) ( fixed ). As in the Kupka-Smale Theorem, for each \( n \in \mathcal{N} \), let \( \mathcal{O}(c, n) \subset C^\infty(M) \) be such that \( E^{1-}_{L-u}(c) \) are regular and every closed orbit of the flow \( \phi^L_{t-u} \) in \( E^{1-}_{L-u}(c) \) with \( \text{period} \leq n \) are nondegenerate, for all \( u \in \mathcal{O}(c, n) \). Then \( \mathcal{O}(c, n) \) is open and dense subset of \( C^\infty(M) \) with the \( C^\infty \)-topology ( see Lemma 3.3 in [Oli08] ).
\( \mathcal{G}(n) \subset \mathcal{O}(c, n) \) be the set of \( C^\infty \)-potentials \( u : M \to \mathbb{R} \) such that the k-jet of the Poincaré map of every closed orbit of \( \phi_t^{L - u} \) contained in \( E^{-1}_{L - u}(c) \) and with period \( \leq n \) belongs to \( Q \). Since the set of periodic orbits of \( \phi_t^{L - u} \) in \( E^{-1}_{L - u}(c) \) with period \( \leq n \) is finite, for all \( u \in \mathcal{G}(n) \), and by continuity of the Poincaré map, we have that \( \mathcal{G}(n) \) is open. By Theorem 11, \( \mathcal{G}(n) \) is also \( C^\infty \)-dense subset of \( C^\infty(M) \). Therefore

\[
\mathcal{G} = \bigcap_{n \in \mathbb{N}} \mathcal{G}(n)
\]

is the residual subset that we are looking for. \( \square \)

4. Lagrangians flows with a non-hyperbolic closed orbit.

In this section we prove the Proposition 3. For this, we follow the strategy in the proof of an analogous result for geodesic flows on \( S^2 \) [CBP02 Prop. 3.3]. In [Mir07 Th. 1.1], we used similar arguments for magnetic flows on surfaces with a non-hyperbolic closed orbit.

Let us recall the Birkhoff’s normal form, for a proof see [SM95 p 222]

**Theorem 14.** Let \( f \) be a \( C^4 \) diffeomorphism defined in a neighborhood of \( 0 \in \mathbb{R}^2 \) such that \( f \) preserves the area form \( dx \wedge dy \) and \( f(0) = 0 \). Suppose that the eigenvalues of \( d_0 f \) satisfy: \( |\lambda| = 1 \) and \( \lambda^n \neq 1 \), for all \( n \in \{1, ..., 4\} \). Then there exists a \( C^4 \) diffeomorphism \( h \), defined in a neighborhood of \( 0 \) such that \( h(0) = 0 \), \( h \) preserves the form \( dx \wedge dy \) and, in polar coordinates \((r, \theta)\), we have:

\[
h^{-1} \circ f \circ h(r, \theta) = (r, \theta + \alpha + \beta r^2) + \mathcal{O}(r^4).
\]

Moreover, the property of \( \beta \neq 0 \) uniquely depends of \( f \).

We say that a homeomorphism \( f : [a, b] \times S^1 \to [a, b] \times S^1 \) is a twist map if for all \( \theta \in S^1 \) the function \( [a, b] \to \pi_2 \circ f(\cdot, \theta) \in S^1 \) is strictly monotonous. Observe that if the coefficient \( \beta = \beta(f) \) in the normal form is not zero, then for \( |r| \leq \epsilon \), with \( \epsilon \) small enough, \( f \) is conjugated to a twist map in \([0, \epsilon] \times S^1\).

We shall use the following result:

**Proposition 15** (Le Calvez [LC91 Remarques pg.26]). Let \( f \) be a diffeomorphism of the annulus \( \mathbb{R} \times S^1 \) such that it is a twist map, it is area preserving, the form \( f^*(R \, d\theta) - R \, d\theta \) is exact and

(i) If \( x \) is a periodic point for \( f \) and \( q \) is its period, the eigenvalues of \( d_x f^q \) are not roots of unity.

(ii) The stable and unstable manifolds of hyperbolic periodic orbits of \( f \) intersect transversally (i.e. whenever they meet, they meet transversally).

Then \( f \) has periodic orbits with homoclinic points.

We are now ready to show the Proposition 3.

4.1. **Proof of Proposition 3.** Let \( \theta_t = \phi_t^L(\theta) \) a non-hyperbolic closed orbit of minimal period \( T > 0 \), contained in \( E^{-1}_L(c) \). Let \( P(\theta, \Sigma, L) \) the Poincaré map for a local transversal section \( \Sigma \subset E^{-1}_L(c) \) in \( \theta \). Since \( \theta_t \) is non-hyperbolic, we have that the eigenvalues of \( d_0 P \) are of the form \( e^{\pm 2\pi i \alpha} \), with \( \alpha \in [0, 1) \).

We consider the subset \( Q \subset \mathcal{J}_h^3(1) \), defined as:

\[
Q = \left\{ \sigma \circ f_{\alpha, \beta} \circ \sigma^{-1} : \sigma \in \mathcal{J}_h^3(1), \, \beta > 0, \, \text{ and } \alpha \notin \left\{ \frac{0}{3}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\} \right\},
\]

where \( f_{\alpha, \beta} : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by \( f_{\alpha, \beta}(r, \theta) = (r, \theta + \alpha + \beta r^2) + \mathcal{O}(r^4) \), in polar coordinates. By the Birkhoff’s normal form, the subset \( Q \subset \mathcal{J}_h^3(1) \) is open and invariant. Since the orbit \( \theta_t \) is non-hyperbolic, we have that \( \mathcal{J}_h^3(P(\theta, \Sigma, L)) \subset \mathcal{Q} \). So, the Theorem 11 implies the existence of \( u : M \to \mathbb{R} \)
arbitrarily $C^\infty$-close to $0 \in C^\infty(M)$ such that: $\theta_t$ is a closed orbit of same period for the flow $\phi_{t-L}^u$ and $j^3(P(\theta, \Sigma, L-u)) \in Q$.

Observe that $\theta_t$ is elliptic for the perturbed flow $\phi_{t-L}^u$. Therefore, there is a neighborhood $\mathcal{U} \subset C^\infty(M)$ of $u$, such that for all $\mathcal{U} \in \mathcal{U}$, the flow $\phi_{t-L}^u|_{E_2^{-1}(c)}$ has an elliptic closed orbit $\bar{\theta}_t$ close to $\theta_t$ that we call analytic continuation of $\theta_t$. Since $Q$ is open, if the neighborhood $\mathcal{U}$ is small enough, we can assume that $j^3(P(\bar{\theta}, \Sigma, L-\mathcal{U})) \in Q$, for all $\mathcal{U} \in \mathcal{U}$.

By Kupka-Smale Theorem, we can approximate $u$ for $\mathcal{U} \in \mathcal{U}$ such that $\mathcal{U}$ is the analytic continuation of $\theta_t$, $f = P(\bar{\theta}, \Sigma, L-M)$ satisfies the conditions (i) and (ii) of the Proposition 15, $j^3(f) \in Q$ and, via Darboux coordinates, $f$ is a diffeomorphism in a neighborhood of $0 \in \mathcal{R}^2$ that preserves the area form $dx \wedge dy$.

By definition of $Q$, the map $f$ is conjugated to a twist map $f_0 = hfh^{-1}$, in polar coordinates. In order to apply the Proposition 15, we need to do a change of coordinates which transforms $f_0$ in a twist map $T : \mathcal{H}^+ \times S^1 \to \mathcal{H}^+ \times S^1$, such that the 1-form $T^*(Rd\theta) - Rd\theta$ is exact. Then the existence of a homoclinic orbit implies the existence of a non-trivial hyperbolic set.

In fact, we consider the following maps:

\[
\begin{array}{ccc}
(x, y) & \rightarrow & (r, \theta) \\
\mathbb{D} & \stackrel{P}{\rightarrow} & \mathbb{R}^+ \times S^1 \\
& \downarrow f & \downarrow f_0 \\
\mathbb{D} & \rightarrow & \mathbb{R}^+ \times S^1 \\
& \downarrow T & \\
& \mathbb{R}^+ \times S^1 \\
\end{array}
\]

where $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1 \}$, $P^{-1}(r, \theta) = (r \cos \theta, r \sin \theta)$. Let $G(x, y) = (\frac{1}{2}r^2, \theta) = (R, \theta)$. Then $\lambda := G^*(R d\theta) = \frac{1}{2}(x dy - y dx)$. Observe that $d\lambda = dx \wedge dy$ is the area form $\mathbb{D}$. Since that $\mathbb{D}$ is contractible, we have that $f_0^*(\lambda) - \lambda$ is exact. Therefore $T^*(R d\theta) - Rd\theta$ is exact. Since $R(r) = \frac{1}{2} r^2$ strictly increasing on $r > 0$, $T$ is a twist map if and only if $f_0$ is a twist map.

\[\square\]

Acknowledgments

I would like to thank Carlos Carballo for carefully checking the draft and for the many comments and suggestions that contributed to the improvement of this exposition. I am grateful to G. Contreras and M. J. Dias Carneiro, for the helpful conversations.

References

[Abr70] R. Abraham, *Bumpy metrics*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 1–3.

[Ano82] D. V. Anosov, *Generic properties of closed geodesics*, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 4, 675–709, 896.

[Bow75] Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, Vol. 470, Springer-Verlag, Berlin, 1975.

[CBP02] Gonzalo Contreras-Barandiarán and Gabriel P. Paternain, *Genericity of geodesic flows with positive topological entropy on $S^2$*, J. Differential Geom. 61 (2002), no. 1, 1–49.

[KMS93] Ivan Koliář, Peter W. Michor, and Jan Slováč, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993.

[KT72] Wilhelm Klingenberg and Floris Takens, *Generic properties of geodesic flows*, Math. Ann. 197 (1972), 323–334.

[LC91] Patrice Le Calvez, *Propriétés dynamiques des difféomorphismes de l’anneau et du tore*, Astérisque (1991), no. 204, 131.

[Mañ96] Ricardo Mañé, *Generic properties and problems of minimizing measures of Lagrangian systems*, Nonlinearity 9 (1996), no. 2, 273–310.
[Mir06] José Antônio Gonçalves Miranda, *Generic properties for magnetic flows on surfaces*, Nonlinearity 19 (2006), no. 8, 1849–1874.

[Mir07] ———, *Positive topological entropy for magnetic flows on surfaces*, Nonlinearity 20 (2007), no. 8, 2007–2031.

[Oli08] Elismar R. Oliveira, *Generic properties of Lagrangians on surfaces: the Kupka-Smale theorem*, Discrete Contin. Dyn. Syst. 21 (2008), no. 2, 551–569.

[SM95] C. L. Siegel and J. K. Moser, *Lectures on celestial mechanics*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Translated from the German by C. I. Kalme, Reprint of the 1971 translation.

Universidade Federal de Minas Gerais, DMat / ICEX, Av. Antônio Carlos, 6627/ C. P. 702, 30123-970. Belo Horizonte, MG, Brasil.

E-mail address: jan@mat.ufmg.br