MEASURED FOLIATIONS AND HILBERT 12TH PROBLEM

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Abstract. Yu. I. Manin conjectured that the maximal abelian extensions of the real quadratic number fields are generated by the pseudo-lattices with real multiplication. We prove this conjecture using theory of measured foliations on the modular curves.

1. Introduction

The Hilbert 12th problem consists in a generalization of the Kronecker-Weber theorem for abelian extensions of the rational numbers to any base number field. It therefore asks for analogues of the roots of unity, and of their appearance as particular values of a special function, in this case the exponential function; the requirement is that such numbers should generate a whole family of further number fields that are analogues of the cyclotomic fields and their subfields. The classical theory of complex multiplication solves this problem for the case of any imaginary quadratic field, by using modular functions and elliptic functions with respect to a particular period lattice related to the field in question. Shimura extended this to general CM fields.

A description of abelian extensions of real quadratic number fields in terms of coordinates of points of finite order on abelian varieties associated with certain modular curves was obtained by [Shimura 1972] [9]. Around that time, [Stark 1976] [11] formulated a number of conjectures on abelian extension of arbitrary number fields, which in the real quadratic case amount to specifying generators of these extensions using special values of Artin L-functions. In the case of an arbitrary number field, a description of the abelian extensions is given by class field theory, but an explicit description of the generators of these abelian extensions, in the sense sought by Kronecker and Hilbert, is still open.

The classical theory of complex multiplication on elliptic curves shows that the maximal abelian extension of $\mathbb{Q}(\tau)$, where $\tau$ is an imaginary quadratic irrationality, can be obtained by adjoining the special values $\wp(\tau, z)$ and $j(\tau)$ of modular functions $j$ and elliptic functions $\wp$, and roots of unity, where $\tau$ generates the imaginary quadratic field and $z$ represents a torsion point on the corresponding elliptic curve. In particular, when $\tau$ is imaginary quadratic, so that the lattice $\mathbb{Z} + \mathbb{Z}\tau$ has complex multiplication, the field $\mathbb{Q}(\tau, j(\tau))$ is the Hilbert class field of $\mathbb{Q}(\tau)$, that is the maximal abelian unramified extension of $\mathbb{Q}(\tau)$. In particular $\mathbb{Q}(\tau, j(\tau))$ is a finite Galois extension of $\mathbb{Q}(\tau)$ and $[\mathbb{Q}(\tau, j(\tau)) : \mathbb{Q}(\tau)]$ equals the class number of $\mathbb{Q}(\tau)$. Moreover, the ideal class group of $\mathbb{Q}(\tau, j(\tau))$ is isomorphic to the Galois group of $\mathbb{Q}(\tau, j(\tau))$ over $\mathbb{Q}(\tau)$, whose action on $j(\tau)$ can be explicitly described, see e.g. [Silverman 1994] [10, Ch. II].
In his theory of real multiplication [Manin 2004] [4] proposed finding the analogue of this very specific type of result for real quadratic fields, using so-called “pseudo-lattices” \( \mathbb{Z} + \mathbb{Z} \theta \), where \( \theta \) is real, with non-trivial real multiplications. In the classical theory, a lattice is a \( \mathbb{Z} \)-module \( \mathbb{Z} + \mathbb{Z} \tau \) of rank 2 with \( \tau \) not real, isomorphic lattices differing by a non-zero scaling in \( \mathbb{C} \). The lattice \( \mathbb{Z} + \mathbb{Z} \tau \) has non-trivial endomorphisms if and only if \( \tau \) is imaginary quadratic, and the endomorphism ring is then an order in the field \( \mathbb{Q}(\tau) \). In Manin’s theory, a pseudo-lattice is a \( \mathbb{Z} \)-module \( \mathbb{Z} + \mathbb{Z} \theta \) of rank 2 with \( \theta \) real. Pseudo-lattices are again considered up to scaling, but have non-trivial endomorphism rings if and only if \( \theta \) is real quadratic. The endomorphism ring \( R = \mathbb{Z} + fO \) is then an order in the field \( k = \mathbb{Q}(\theta) \), where \( O \) is the ring of integers of \( k \), and \( f \) is the conductor of \( R \). Manin calls these pseudo-lattices with real multiplication; we shall call it here Manin pseudo-lattices.

Let \( \Lambda_M \) be Manin’s pseudo-lattice and \( R = \text{End} (\Lambda_M) \) its endomorphism ring; denote by \( k = R \otimes \mathbb{Q} \) the real quadratic field associated to the ring \( R \); notice that \( R = \mathbb{Z} + fO_k \) is an order in \( k \), where \( f \geq 1 \) is the conductor of the order. The class field theory says that for each abelian extension \( K \) of the field \( k \) it holds \( \text{Gal} (K/k) \cong \text{Cl} (k) \), where \( \text{Cl} (k) \) is the (abelian) group of ideal classes of the ring \( R \); such an extension is called a \textit{ring class field} of \( k \) modulo \( f \) (conductor) \( f \). Notice that the ring class field modulo \( f = 1 \) coincides with the Hilbert class field, i.e. the maximal unramified extension of \( k \) [Silverman 1994] [10]. We shall focus on the following real multiplication problem: \textit{To construct explicit generators of the ring class field of \( k \) modulo \( f \geq 1 \).}

Let \( N > 0 \) be an integer and \( \Gamma_0(N) := \{ (a, b, c, d) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N \} \) a finite index subgroup of the modular group. By \( S_2(\Gamma_0(N)) \) one understands a collection of the cusp forms (of weight two) on the extended upper half-plane \( \mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \). Recall that the Riemann surface \( X_0(N) := \mathbb{H}^*/\Gamma_0(N) \) is called a \textit{modular curve}. We identify \( S_2(\Gamma_0(N)) \) with the linear space \( \Omega_{hol}(X_0(N)) \) of the holomorphic differentials on \( X_0(N) \) via the formula \( f(z) \mapsto \omega = f(z)dz \).

By \( T_2 := \mathbb{Z}[T_1, T_2, \ldots] \) we denote a commutative algebra of the Hecke operators \( T_n \) acting on the space \( S_2(\Gamma_0(N)) \) by the formula \( T_nf = \sum_{m \in \mathbb{Z}} \gamma(m)q^m \), where \( \gamma(m) = \sum_{(a,b) \in \text{gcd}(m,n)} \text{gcd}(m,n)^m \) and \( f(z) = \sum_{m \in \mathbb{Z}} c(m)q^m \) is the Fourier series of the cusp form \( f \) at \( q = e^{2\pi iz} \). The common eigenvector \( f \in S_2(\Gamma_0(N)) \) of all \( T_n \in \mathbb{T}_2 \) is referred to as a \textit{Hecke eigenform}. By \( K_f \) we understand an algebraic number field generated by the Fourier coefficients of the Hecke eigenform \( f \); it is known that \( K_f \) is totally real and \( \deg (K_f/\mathbb{Q}) \leq g \), where \( g \) is the genus of surface \( X_0(N) \) [Diamond & Shurman 2005] [1], pp. 234-235. When \( \deg (K_f/\mathbb{Q}) = g \) the Hecke eigenform \( f \) will be called \textit{maximal}; in what follows we work exclusively with this class of the Hecke eigenforms. For a Hecke eigenform \( f \) one considers a \( \mathbb{Z} \)-module \( \int_{H_1(X_0(N), Sing, \omega_2)} \mathbb{R} (\omega) = \mathbb{Z} \lambda_1 + \cdots + \mathbb{Z} \lambda_g \), where \( \omega = f(z)dz \) is a holomorphic differential on \( X_0(N) \); such a module belongs to the field \( K_f \) (lemma 2.1).

The vector \( v_A = (\lambda_1, \ldots, \lambda_g) \) coincides with the Perron-Frobenius eigenvector of a positive integer matrix \( A \) corresponding to the eigenvalue \( \lambda_A \in K_f \); the algebraic number \( \lambda_A \) we shall call a \textit{Hecke unit} of the field \( K_f \).

On the other hand, every measured foliation \( \mathcal{F} \) on a surface \( X \) ([Hubbard & Masur 1979] [2], [Thurston 1988] [12]) defines a pseudo-lattice \( \Lambda = \text{Jac} (h(\mathcal{F})) \), where \( h(\mathcal{F}) \) is an induced measured foliation on the torus and \( \text{Jac} (h(\mathcal{F})) \) its jacobian (lemma 3.1); in particular if \( \mathcal{F} = F_{\lambda} \) is a measured foliation of the surface \( X_0(N) \) by the vertical trajectories \( \mathbb{R} (fdz) = 0 \) of a Hecke eigenform \( f \) then \( \text{Jac} (h(F_{\lambda})) \)
is a Manin pseudo-lattice \( \Lambda_M \) (lemma 3.5). By a \( j \)-invariant of \( \Lambda_M \) we understand the algebraic number \( j(\Lambda_M) := \lambda_A \), where \( \lambda_A \) is the Hecke unit of the field \( K_f \); it is indeed an invariant independent of basis in the \( \mathbb{Z} \)-modules \( \mathbb{Z} \lambda_1 + \cdots + \mathbb{Z} \lambda_g \) and \( \text{Jac} \ (h(F_N)) \), and as such generalizes the classical \( j \)-invariant defined on the upper-half plane \( \mathbb{H} \) to the (quadratic irrational) points at the boundary of \( \mathbb{H} \). We shall consider the ring \( R = \mathbb{Z} + fO_k := \text{End} \ (\Lambda_M) \) and the real quadratic field \( k = R \otimes \mathbb{Q} \). Our main result can be formulated as follows.

**Theorem 1.1.** The extension \( K = k(j(\Lambda_M)) \) is a ring class field of \( k \) modulo \( f \geq 1 \).

The structure of the article is as follows. The notation and preliminary facts are introduced in Section 2. Theorem 1.1 is proved in Section 3. In Section 4 we compare theorem 1.1 with the results of [Shimura 1972] [9].

2. Preliminaries

2.1. Measured foliations and their Jacobians. By a \( p \)-dimensional \( C^r \) foliation of an \( m \)-dimensional manifold \( M \) one understands a decomposition of \( M \) into a union of disjoint connected subsets \( \{ \mathcal{L}_\alpha \}_{\alpha \in A} \) called leaves of the foliation. The leaves must satisfy the following property: Every point in \( M \) has a neighborhood \( U \) and a system of local class \( C^r \) coordinates \( x = (x^1, \ldots, x^m) : U \to \mathbb{R}^m \) such that for each leaf \( \mathcal{L}_\alpha \), the components of \( U \cap \mathcal{L}_\alpha \) are described by the equations \( x^{p+1} = \text{Const}, \ldots, x^m = \text{Const} \). Such a foliation is denoted by \( \mathcal{F} = \{ \mathcal{L}_\alpha \}_{\alpha \in A} \). The number \( q = m - p \) is called a codimension of the foliation \( \mathcal{F} \) [Lawson 1974] [3] p.370. The codimension \( q \) class \( C^r \) foliations \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are said to be \( C^r \)-conjugate \((0 \leq s \leq r)\) if there exists a diffeomorphism of \( M \) of class \( C^s \), which maps the leaves of \( \mathcal{F}_0 \) onto the leaves of \( \mathcal{F}_1 \). If \( s = 0 \) \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are topologically conjugate, ibid., p.388.

The foliation \( \mathcal{F} \) is called singular if the codimension \( q \) of the foliation depends on the leaf. We further assume that \( q \) is constant for all but a finite number of leaves. Such a set of the exceptional leaves will be denoted by \( \text{Sing} \ \mathcal{F} := \{ \mathcal{L}_\alpha \}_{\alpha \in E} \), where \( |E| < \infty \); note that if \( \text{Sing} \ \mathcal{F} = \emptyset \) one gets the usual definition of a (non-singular) foliation. A quick example of the singular foliations is given by trajectories of a non-trivial differential form on the manifold \( M \) vanishing in a finite number of points of \( M \); the set of zeroes of such a form corresponds to the exceptional leaves of foliation.

Roughly speaking, measured foliation ([Hubbard & Masur 1979] [2], [5, Section 0.3.2]) is a singular codimension one \( C^r \)-foliation, induced by trajectories of a closed differential \( \phi \) on a two-dimensional manifold (surface) \( X \); formally speaking, a measured foliation \( \mathcal{F} \) on a surface \( X \) is a partition of \( X \) into the singular points \( x_1, \ldots, x_n \) of order \( k_1, \ldots, k_n \) and regular leaves (1-dimensional submanifolds). On each open cover \( U_i \) of \( X - \{x_1, \ldots, x_n\} \) there exists a non-vanishing real-valued closed 1-form \( \phi_i \) such that: (i) \( \phi_i = \pm \phi_j \) on \( U_i \cap U_j \); (ii) at each \( x_i \) there exists a local chart \( (u, v) : V \to \mathbb{R}^2 \) such that for \( z = u + iv \), it holds \( \phi_i = Im (z^{k_i} dz) \) on \( V \cap U_i \) for some branch of \( z^{k_i} \). The pair \( (U_i, \phi_i) \) is called an atlas for the measured foliation \( \mathcal{F} \). Finally, a measure \( \mu \) is assigned to each segment \( (t_0, t) \in U_i \), which is transverse to the leaves of \( \mathcal{F} \), via the integral \( \mu(t_0, t) = \int_{t_0}^t \phi_i \). The measure is invariant along the leaves of \( \mathcal{F} \), hence the name. Note that measured foliation can have singular points of the half-integer index; those cannot be given by the trajectories of a closed form. Yet when all \( k_i \) are even integers, the leaves of foliation can
be continuously oriented and $\mathcal{F}$ is called oriented in this case; such foliations are given by trajectories of a closed differential form on the surface $X$. In what follows we work with the oriented measured foliations.

Let $\mathcal{F}$ be measured foliation on a compact surface $X$ and \{\(\gamma_1, \ldots, \gamma_n\)\} a basis in the relative homology group $H_1(X, \text{Sing} \; \mathcal{F}; \mathbb{Z})$; it is known that $n = 2g + |\text{Sing} \; (\mathcal{F})| - 1$, where $g$ is the genus of $X$ [Hubbard & Masur 1979] [2]. By $\lambda_i \in \mathbb{R}$ we denote the periods $\int_{\gamma_i} \phi$ of $\phi$ in the above basis. By a jacobian of the measured foliation $\mathcal{F}$ one understand a $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ regarded as a subset of the real line $\mathbb{R}$; we shall denote the jacobian by $\text{Jac} \; (\mathcal{F})$. The jacobian is independent of the choice of basis in the homology group $H_1(X, \text{Sing} \; \mathcal{F}; \mathbb{Z})$ and depends solely on the foliation $\mathcal{F}$ [7]; moreover the measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are topologically conjugate if and only if $\text{Jac} \; (\mathcal{F}') = \mu \text{Jac} \; (\mathcal{F})$, where $\mu > 0$ is a real number $\text{ibid}$.

Let $\varphi : X \to X$ be an orientation-preserving automorphism of a compact surface $X$; it is known that such an automorphism is either (i) of finite order or (ii) an infinite order automorphism preserving certain simple closed curves or else (iii) a pseudo-Anosov, i.e., an infinite order automorphism which does not preserve any simple closed curve on $X$ [Thurston 1988] [12]. In case (iii) there exist a stable $\mathcal{F}_u$ and unstable $\mathcal{F}_u$ mutually orthogonal measured foliations on $X$ such that $\varphi(\mathcal{F}_u) = \frac{1}{\lambda_u} \mathcal{F}_u$ and $\varphi(\mathcal{F}_u) = \lambda_u \mathcal{F}_u$, where $\lambda_u > 1$ is called the dilatation of $\varphi$. The invariant measured foliation $\mathcal{F} = \mathcal{F}_u$ is called a pseudo-Anosov foliation; it is known that its jacobian belongs to the number field $\mathbb{Q}(\lambda_u)$ [Thurston 1988] [12], p.427-428.

### 2.2. Foliations on modular curves.

Let $N > 1$ be a natural number and consider a finite index subgroup of the modular group given by the formula $\Gamma_0(N) = \{(a, b, c, d) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N\}$. Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and let $\Gamma_0(N)$ act on $\mathbb{H}$ by the linear fractional transformations; consider an orbifold $\mathbb{H}/\Gamma_0(N)$. To compactify the orbifold at the cusps, one adds a boundary to $\mathbb{H}$, so that $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and the compact Riemann surface $X_0(N) = \mathbb{H}^*/\Gamma_0(N)$ is called a modular curve. The meromorphic functions $f(z)$ on $\mathbb{H}$ that vanish at the cusps and such that $f\left(\frac{az + b}{cz + d}\right) = (cz + d)^2 f(z)$, $\forall \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_0(N)$, are called cusp forms of weight two; the (complex linear) space of such forms will be denoted by $S_2(\Gamma_0(N))$. The formula $f(z) \mapsto \omega = f(z)dz$ defines an isomorphism $S_2(\Gamma_0(N)) \cong \Omega_{hol}(X_0(N))$, where $\Omega_{hol}(X_0(N))$ is the space of holomorphic differentials on the Riemann surface $X_0(N)$. Note that $\dim_{\mathbb{C}}(S_2(\Gamma_0(N))) = \dim_{\mathbb{C}}(\Omega_{hol}(X_0(N))) = g$, where $g = g(N)$ is the genus of the surface $X_0(N)$. Recall that there exists a natural involution $i$ on the space $S_2(\Gamma_0(N))$ defined by the formula $f(z) \mapsto f^*(z)$, where $f(z) = \sum c_n q^n$ and $f^*(z) = \sum c_n q^n$. A subspace, $S_2^0(\Gamma_0(N))$, fixed by the involution, consists of the cusp forms, whose Fourier coefficients are the real numbers. Clearly, $\dim_{\mathbb{R}}(S_2^0(\Gamma_0(N))) = g$. The map $i$ induces an involution $i_\phi$ on the space $\Phi_{X_0(N)}$ of all measured foliations on $X_0(N)$; in a proper coordinate system the involution $i_\phi$ acts by the formula $(\lambda_1, \ldots, \lambda_g, \lambda'_1, \ldots, \lambda'_g) \mapsto (\lambda'_1, \ldots, \lambda'_g, \lambda_1, \ldots, \lambda_g)$. A subspace of $\Phi_{X_0(N)}$ fixed by involution $i_\phi$ we shall denote by $\Phi_{X_0(N)}^\phi$; it consists of measured foliations of the form $(\lambda_1, \ldots, \lambda_g, \lambda_1, \ldots, \lambda_g)$. Thus $\text{Jac} \; (\mathcal{F}) = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_g$ for $\forall \mathcal{F} \in \Phi_{X_0(N)}^\phi$.

Let $f \in S_2(\Gamma_0(N))$ be a (normalized) Hecke eigenform, such that $f(z) = \sum_{n=1}^{\infty} c_n(f)q^n$ its Fourier series. We shall denote by $K_f = \mathbb{Q}(\{c_n(f)\})$
the algebraic number field generated by the Fourier coefficients of \( f \). Let \( g \) be the genus of the modular curve \( X_0(N) \). It is well known that \( 1 \leq \text{deg} \ (K_f | \mathbb{Q}) \leq g \) and \( K_f \) is a totally real field, see e.g. [Diamond & Shurman 2005] [1, pp. 234-235]. Let \( \mathcal{F}_N \) be a measured foliation given by the lines \( Re \ (f dz) = 0 \), where \( f \) is the maximal Hecke eigenform. We shall use the following fact.

**Lemma 2.1.** ([7]) \( \mathcal{F}_N \) is pseudo-Anosov and \( \text{Jac} \ (\mathcal{F}_N) \) is a \( \mathbb{Z} \)-module in the field \( K_f \).

### 3. Proof of theorem 1

For the sake of clarity, we outline main ideas of the proof. Given a measured foliation \( \mathcal{F} \) on a surface \( X \), one can associate to \( \mathcal{F} \) the canonical measured foliation, \( F \), on torus \( T^2 \) and *vice versa* provided the set \( \text{Sing} \ \mathcal{F} \) is specified. This is the standard fact following from the Riemann-Hurwitz construction of a ramified covering of one Riemann surface (torus \( T^2 \)) by another (surface \( X \)). Such a map between foliations is well defined and unique up to the topological conjugacy.

Since \( \text{Jac} \ (\mathcal{F}) \) is invariant of the topological conjugacy classes, \( \text{Jac} \ (\mathcal{F}) \) must be related to \( \text{Jac} \ (F) \); it is indeed so and such a relation is the inclusion \( \text{Jac} \ (F) \subseteq \text{Jac} \ (\mathcal{F}) \).

In case \( \mathcal{F} \) is a pseudo-Anosov measured foliation, the inclusion \( \text{Jac} \ (F) \subseteq \text{Jac} \ (\mathcal{F}) \) can be exploited to construct (finite) extensions of the real quadratic number fields. Namely, whenever \( \mathcal{F} \) is such a foliation, it is known that \( \text{Jac} \ (\mathcal{F}) \) belongs (as a \( \mathbb{Z} \)-module) to a real algebraic field \( K \) generated by dilatation \( \lambda_\varphi \) of the pseudo-Anosov automorphism \( \varphi \) of surface \( X \). The automorphism descends canonically to such of torus \( T^2 \) and, therefore, \( \text{Jac} \ (F) \) belongs to a real quadratic field \( k \); in such a way one gets an extension \( k \subseteq K \) of the field \( k \).

The above construction is general and (in most cases) the arithmetic of the field \( k \) and such of the field \( K \) are independent of each other. However, it is not so for the modular curve \( X = X_0(N) \) and the foliation \( \mathcal{F}_N \) given by the vertical trajectories of the corresponding Hecke eigenform. It will develop, that in this special case the ideal class group \( \text{Cl} \ (k) \) of the field \( k \) controls the Galois group \( \text{Gal} \ (K|k) \) of extension \( k \subseteq K \), so that \( \text{Gal} \ (K|k) \cong \text{Cl} \ (k) \).

Why \( \text{Gal} \ (K|k) \cong \text{Cl} \ (k) \) anyway? Recall that there are \( \{f_1, \ldots, f_g\} \) linearly independent Hecke eigenforms, where \( g \) is the genus of \( X_0(N) \). For each \( 1 \leq i \leq g \) the Fourier coefficients of eigenform \( f_i \) generate an algebraic number field \( K = K_f \) such that \( \text{deg} \ (K_f | \mathbb{Q}) = g \) and \( \text{Jac} \ (F_i) \subseteq K \), where \( F_i \) is a measured foliation by vertical trajectories of the form \( f_i dz \). On the other hand, it was shown that on the torus \( T^2 \) one gets \( \{F_1, \ldots, F_g\} \) measured foliations corresponding to \( F_i \). Because \( f_i \) are conjugate Hecke eigenforms, the jacobians \( \text{Jac} \ (F_i) \) will have the same endomorphism ring, yet being itself pairwise distinct; it means that \( |\text{Cl} \ (k)| = g \). With a little extra work (see below), one constructs the required isomorphism \( \text{Gal} \ (K|k) \cong \text{Cl} \ (k) \).

In view of the above construction, \( \text{Jac} \ (F_i) \) are Manin pseudo-lattices; therefore one gets a solution to the real multiplication problem of Yu. I. Manin. (The reader can verify that all maps above are equivariant and canonical with respect to the natural morphisms between the objects in use.) These remarks complete the sketch of proof of our main result.
We shall pass to a detailed argument by splitting the proof in a series of lemmas starting with the following Riemann-Hurwitz construction for measured foliations.

**Lemma 3.1.** Every measured foliation $\mathcal{F}$ on a surface $X$ (with fixed set $\text{Sing} \mathcal{F}$) covers a measured foliation $h(\mathcal{F})$ on torus $T^2$ and vice versa; the map $h$ is a bijection between the classes of topologically conjugate measured foliations on $X$ and such on $T^2$.

**Proof.** Let $\mathcal{F}$ be measured foliation on a surface $X$; let $\text{Jac} (\mathcal{F}) = Z\lambda_1 + \cdots + Z\lambda_n$ be its jacobian. We shall consider an $n$-dimensional torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$ endowed with a codimension one foliation $\mathcal{F}'$ defined by the formula $\int_{H_1(T^n;\mathbb{Z})} \phi_{\mathcal{F}'} = \text{Jac} (\mathcal{F})$, where $\phi_{\mathcal{F}'}$ is a closed differential tangent to foliation $\mathcal{F}'$; such a foliation always exists [Plante 1975] [8]. Notice that $\text{Jac} \mathcal{F}$ coincides with the so-called group of periods $P(\mathcal{F}') := \int_{H_1(T^n;\mathbb{Z})} \phi_{\mathcal{F}'}$ of foliation $\mathcal{F}'$; such a group is defined in [Plante 1975] [8], p. 346.

Let $T^2$ be a two-dimensional torus; let $E = (T^n, T^2, p)$ be a fiber bundle $T^n$ over $T^2$ with the fiber $p^{-1}(x) \cong T^{n-2}$ over the point $x \in T^2$. Recall that every foliation $F$ on $T^2$ induces a foliation $p^{-1}(F)$ on $T^n$ defined by the fiber map $p : T^n \to T^2$ [Lawson 1974] [3]. p. 373. Denote by $F$ a foliation on the torus $T^2$ such that $p^{-1}(F) = F'$. Since $\mathcal{F}'$ is given by a closed form, so will be the foliation $F$; in other words, $F$ is a measured foliation. (Notice that the rank of $\text{Jac} (\mathcal{F})$ is less or equal to such of $\text{Jac} (\mathcal{F}')$ in general; however, $\text{rank} \ (\text{Jac} (\mathcal{F})) = 2$ and $\text{rank} \ (\text{Jac} (\mathcal{F}')) = n$ for generic foliations $F$ and $\mathcal{F}'$. Thus we have a correctly defined map

$$h : F \to \mathcal{F}' \to F.$$  \hfill (3.1)

If $E' = (T^n, T^2, p')$ is another fiber bundle with the base $T^2$ then the corresponding foliation $F'$ will be topologically conjugate to the foliation $F$; we leave this claim as an exercise to the reader.

Conversely, let $F$ be measured foliation on the two-torus $T^2$; if $E = (T^n, T^2, p)$ is a fiber bundle, then $\mathcal{F}' = p^{-1}(F)$ is an induced codimension one foliation on $T^n$ [Lawson 1974] [3], p. 373. The latter is given by a closed form $\phi_{\mathcal{F}'}$ and has the group of periods $P(\mathcal{F}') = \int_{H_1(T^n;\mathbb{Z})} \phi_{\mathcal{F}'}$ [Plante 1975] [8], p. 346. Fixing a set $\text{Sing} \mathcal{F}$ one constructs measured foliation $\mathcal{F}$ on a surface $X$, such that $\text{Jac} (\mathcal{F}) = P(\mathcal{F}')$. Clearly, every measured foliation $\mathcal{F}$ on $X$ can be constructed in this way. Thus the map $h : F \to F$ is a bijection between the classes of topological conjugacy of measured foliations on $X$ (with fixed set $\text{Sing} \mathcal{F}$) and such on the torus $T^2$. Lemma 3.1 follows. \hfill \Box

**Lemma 3.2.** If $\mathcal{F}$ is measured foliation on a surface $X$ then $\text{Jac} (h(\mathcal{F})) \subseteq \text{Jac} (\mathcal{F})$.

**Proof.** Recall that the integral $\int \phi$ defines a pairing $H_1(M; \mathbb{R}) \times H^1(M; \mathbb{R}) \to \mathbb{R}$ between the first homology and cohomology groups of a manifold $M$. The pairing is a scalar product on the vector space $H_1(M; \mathbb{R}) \cong H^1(M; \mathbb{R})$; we shall fix an orthonormal basis $(e_i, e_j) = \delta_{ij}$ in $H_1(M; \mathbb{R})$. Let $X$ be a surface with measured foliation $\mathcal{F}$. It was shown that $\text{Jac} (\mathcal{F}) = \text{Jac} (F')$, where $\mathcal{F}'$ is a codimension one foliation on the torus $T^n$ and $F = h(\mathcal{F})$ is a foliation on $T^2$ induced by the fiber map $p : T^n \to T^2$. Since $p$ is a continuous map it defines a homomorphism $p_* : H_1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n \to H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$. In a proper basis the map $p_*$ coincides with the projection map on the first coordinates, i.e. a map $p_* : \mathbb{Z}^n \to \mathbb{Z}^2$ which acts by the formula $(z_1, z_2, \ldots, z_n) \mapsto (z_1, z_2, 0, \ldots, 0)$. One can compare the following
system of equations
\[
\begin{align*}
\text{Jac} (\mathcal{F}) &= \int_{H_1(T^n;\mathbb{Z})} \phi = \lambda_1 \mathbb{Z} + \cdots + \lambda_n \mathbb{Z} \\
\text{Jac} (h(\mathcal{F})) &= \int_{H_1(T^2;\mathbb{Z})} p^n(\phi) = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z},
\end{align*}
\]  
(3.2)

where \(\lambda_i\) are coordinates of the form \(\phi\) in \(H^1(T^n;\mathbb{R})\) and \(p^n(\phi)\) its projection in \(H^1(T^2;\mathbb{R})\). Thus \(\text{Jac} (h(\mathcal{F})) \subseteq \text{Jac} (\mathcal{F})\) and the equality holds if and only if \(n = 2\). Lemma 3.2 is proved.

**Lemma 3.3.** If \(\mathcal{F}\) is a pseudo-Anosov measured foliation on a surface \(X\) then \(K = \text{End} (\text{Jac} (\mathcal{F})) \otimes \mathbb{Q}\) is a number field; the field \(K = k(\lambda_\varphi)\), where \(k = \text{End} (\text{Jac} (h(\mathcal{F}))) \otimes \mathbb{Q}\) is either a real quadratic or rational field.

**Proof.** Let \(\mathcal{F}\) be measured foliation which is invariant of an infinite-order (pseudo-Anosov) automorphism \(\varphi\) of a surface \(X\); let \(\lambda_\varphi > 1\) be the dilatation of automorphism \(\varphi\). It is known that in this case the generators \((\lambda_1, \ldots, \lambda_n)\) of \(\text{Jac} (\mathcal{F})\) are coordinates of the Perron-Frobenius eigenvector corresponding to the eigenvalue \(\lambda_\varphi\) of a positive integer matrix \(A_\varphi \in GL_n(\mathbb{Z})\). Therefore \(\lambda_i\) can be scaled to belong to the algebraic number field \(K = \mathbb{Q}(\lambda_\varphi)\); clearly the endomorphism ring \(\text{End} (\text{Jac} (\mathcal{F}))\) is an order in the field \(K\) and \(K\) itself can be written as \(K = \text{End} (\text{Jac} (h(\mathcal{F}))) \otimes \mathbb{Q}\).

The matrix \(A_\varphi\) defines an automorphism of the torus \(T^n\); the automorphism commutes with the projection map \(p_\varphi : H_1(T^n;\mathbb{Z}) \to H_1(T^2;\mathbb{Z})\) defined earlier. We shall write \(A \in GL_2(\mathbb{Z})\) to denote an automorphism of \(T^2\) induced by \(A_\varphi\). Since \(A_\varphi\) has an infinite order so will be the automorphism \(A\); therefore the eigenvalues \(\lambda\) and \(\lambda\) of \(A\) are either quadratic irrational numbers or \(\lambda = \overline{\lambda} = \pm 1\). When \(\lambda > 1\) is a quadratic irrationality the \(\text{Jac} (h(\mathcal{F}))\) is generated by coordinates of the eigenvector corresponding to \(\lambda\); the latter can be scaled to belong to the field \(K = \mathbb{Q}(\lambda)\). Thus \(\text{End} (\text{Jac} (h(\mathcal{F}))) \subseteq \mathbb{Q}\) is a real quadratic number field which coincides with the field \(K\). When \(\lambda = \overline{\lambda} = \pm 1\) then \(\text{Jac} (h(\mathcal{F}))\) belongs to the field \(\mathbb{Q}\) so that \(\text{End} (\text{Jac} (h(\mathcal{F}))) \otimes \mathbb{Q} \cong \mathbb{Q}\). In view of lemma 3.2 one arrives at the following inclusion of the number fields
\[
\text{End} (\text{Jac} (h(\mathcal{F}))) \otimes \mathbb{Q} \subseteq \text{End} (\text{Jac} (\mathcal{F})) \otimes \mathbb{Q}.
\]  
(3.3)

In other words, the field \(K = k(\lambda_\varphi)\), where \(k\) is either a real quadratic or rational number field. Lemma 3.3 follows.

**Remark 3.4.** Notice that in general the extension \(K|k\) is not a Galois extension; if it is such an extension, the group \(\text{Gal} (K|k)\) is not necessarily defined by the ideal class group \(\text{Cl} (k)\). Yet \(\text{Gal} (K|k) \cong \text{Cl} (k)\) whenever \(\mathcal{F} = \mathcal{F}_N\) is measured foliation by the vertical trajectories of a Hecke eigenform \(f \in S_2(\Gamma_0(N))\) (see lemma 3.6).

Recall that \(f \in S_2(\Gamma_0(N))\) is called a maximal Hecke eigenform if \(\text{deg} (K_f) = g\), where \(g\) is the genus of surface \(X_0(N)\); consider a measured foliation \(\mathcal{F}_N = \Re (f dz)\) generated by \(f\).

**Lemma 3.5.** \(\text{Jac} (h(\mathcal{F}_N))\) is a Manin pseudo-lattice.

**Proof.** The foliation \(\mathcal{F}_N\) is the invariant measured foliation of a pseudo-Anosov automorphism of \(X_0(N)\) (lemma 2.1); thus to prove our claim it remains to show that \(\text{End} (\text{Jac} (h(\mathcal{F}_N))) \otimes \mathbb{Q} \neq \mathbb{Q}\) (lemma 3.3). Consider a Hecke operator \(T_n \in T_2\); it acts on \(\text{Jac} (\mathcal{F}_N) = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_g\) by a symmetric matrix \((t_{ij}) \in M_g(\mathbb{Z})\), i.e. \(\lambda'_j = \sum t_{ij}\lambda_i\) [Diamond & Shurman 2005] [1]. Let us calculate an induced action of \(T_n\) on \(\text{Jac} (\mathcal{F}_N) = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2\). In notation of lemma 3.5 we have \(p_\varphi(\lambda_1, \ldots, \lambda_g) = \sum t_{ij}\lambda_i\)
(\lambda_1, \lambda_2, 0, \ldots, 0)$, where $p_*$ is a projection map $H_1(X_0(N), \text{Sing}(\mathcal{F}_\mathbb{N}; \mathbb{Z}) \to H_1(T^2; \mathbb{Z})$; thus the action of $T_n$ on $\text{Jac}(h(\mathcal{F}_\mathbb{N}))$ is given by a symmetric matrix $\tau_n = (t_{11}, t_{12}, t_{12}, t_{22}) \in M_2(\mathbb{Z})$. Since $\tau_n$ is an integer matrix it defines a non-trivial endomorphism of $\text{Jac}(h(\mathcal{F}_\mathbb{N}))$; such an endomorphism can be given as multiplication of points of $\mathbb{R}$ by a real number $\alpha$, i.e. $\alpha \lambda_1 = t_{11} \lambda_1 + t_{12} \lambda_2$, $\alpha \lambda_2 = t_{12} \lambda_1 + t_{22} \lambda_2$. Since $\theta = \lambda_2/\lambda_1$, it satisfies the equation $\theta = \frac{t_{12} - t_{11}}{t_{11} + t_{12}} \lambda_1$; the latter is equivalent to the quadratic equation $t_{12} \theta^2 + (t_{11} - t_{22}) \theta - t_{12} = 0$ whose determinant $D = (t_{11} - t_{22})^2 + 4t_{12}^2 \geq 0$. Such a quadratic equation has two real roots; these roots cannot be rational since $\tau_n$ is a non-trivial endomorphism. Therefore $\theta$ is a quadratic irrationality. Lemma 3.5 follows. \hfill \Box

**Lemma 3.6.** The number field $K = \text{End}(\text{Jac}(\mathcal{F}_\mathbb{N})) \otimes \mathbb{Q}$ is a ring class field of the real quadratic field $k = \text{End}(\text{Jac}(h(\mathcal{F}_\mathbb{N})) \otimes \mathbb{Q}$.

**Proof.** To outline the proof, let $R = \text{End}(\text{Jac}(h(\mathcal{F}_\mathbb{N})))$ be an order in the real quadratic field $k = R \otimes \mathbb{Q}$ and $\text{Cl}(R)$ the ideal class group of the order $R$. To prove lemma 3.6 one needs to verify the isomorphism $\text{Cl}(R) \cong \text{Gal}(K|k)$, where $\text{Gal}(K|k)$ is the Galois group of the extension $K|k$; in turn, such an isomorphism can be given by the action of group $\text{Cl}(R)$ on generators of the extension $K|k$. To give a rough idea, look at a basis of the Hecke eigenforms $\{f_1, \ldots, f_g\}$ in the space $S_2(\Gamma_0(N))$; such a basis is known to exist, see e.g. [Diamond & Shurman 2005] [1, Theorem 5.8.2]. Each $f_i$ defines a Manin pseudo-lattice and one gets $\{\Lambda_M^{(1)}, \ldots, \Lambda_M^{(g)}\}$ pairwise non-isomorphic pseudo-lattices; notice that $\text{End}(\Lambda_M^{(i)}) \cong R$ for all $1 \leq i \leq g$, i.e. $h_R = g$, where $h_R$ is the class number of the order $R$. But $f_i$ define the Hecke units $\lambda_i$ of the field $K$ which are algebraically conjugate numbers; the formula $j(\Lambda_M^{(i)}) = \lambda_i$ gives the required action of the group $\text{Cl}(R)$ on the generators of the field $K$. We shall pass to a detailed argument.

(i) For a Hecke eigenform $f$ consider its Hecke unit $\lambda \in K_f$; denote by $p \in \mathbb{Z}[x]$ a minimal polynomial of the algebraic number $\lambda = \lambda_1$. Since $p(x)$ splits in the totally real field $K_f$ one can write

$$p(x) = (x - \lambda_1) \ldots (x - \lambda_g),$$

where $\lambda_i \in K_f$ are the Hecke units of some Hecke eigenforms $f_i$; the eigenforms make a basis $\{f_1, \ldots, f_g\}$ of the space $S_2(\Gamma_0(N))$. By lemma 3.5 there exist Manin’s pseudo-lattices $\Lambda_M^{(i)}$ corresponding to $f_i$ and we shall define the j-invariant of the latter as $j(\Lambda_M^{(i)}) := \lambda_i$.

(ii) Let $R = \text{End}(\Lambda_M^{(1)})$ and let $h_R$ be the class number of the order $R$. Then $h_R = g$, where $g$ is the genus of the surface $X_0(N)$.

(a) Let us show that $g \leq h_R$. Indeed, a basis $\{f_1, \ldots, f_g\}$ of the Hecke eigenforms $f_i$ in the space $S_2(\Gamma_0(N))$ defines Manin’s pseudo-lattices $\Lambda_M^{(1)}, \ldots, \Lambda_M^{(g)}$, such that $\text{End}(\Lambda_M^{(i)}) = R$. Thus, the order $R$ has at least $g$ ideal classes, i.e. $g \leq h_R$.

(b) Let us show that $g \geq h_R$. Indeed, let $\Lambda_M^{(1)}, \ldots, \Lambda_M^{(h_R)}$ be a full list of Manin’s pseudo-lattices in the order $R$. Since $R \cong \mathbb{Z}/I$, where $I$ is a fixed ideal of the ring of Hecke operators $\mathbb{Z}$, we conclude that there exists at least $h_R$ Hecke eigenforms in the space $S_2(\Gamma_0(N))$. Thus $g \geq h_R$.

It follows from (a) and (b) that $g = h_R$. 

(iii) Let us establish an explicit formula for the isomorphism $Cl(R) \to Gal(K|k)$; since $Gal(K|k)$ is an automorphism group of the field $K$, it will suffice to define the action of an element $a \in Cl(R)$ on generators of $K$. Let $\Lambda_M^{(i)} \subseteq R$ be a Manin pseudo-lattice and let $[\Lambda_M^{(i)}]$ be its ideal class in $R$. Since $[\Lambda_M^{(i)}] \in Cl(R)$, the element $a \ast [\Lambda_M^{(i)}] \in Cl(R)$ for all $a \in Cl(R)$. We let $\Lambda_M^{(i)}$ be a Manin pseudo-lattice, such that $[\Lambda_M^{(i)}] = a \ast [\Lambda_M^{(i)}]$. For the sake of brevity, we write $\Lambda_M^{(i)} = a \ast [\Lambda_M^{(i)}]$.

The action of an element $a \in Cl(R)$ on the generators $j(\Lambda_M^{(i)})$ of the field $K$ is given by the following formula:

$$a \ast j(\Lambda_M^{(i)}) := j(a \ast [\Lambda_M^{(i)}]), \quad \forall a \in Cl(R).$$

We leave it to the reader to verify that the last formula gives an isomorphism $Cl(R) \to Gal(K|k)$; this argument completes the proof of lemma 3.6. □

Theorem 1.1 follows from lemma 3.6. □

4. Shimura’s method revisited

Theorem 1.1 says that there exists an integer $N_{\psi}(d)$, such that the Fourier coefficients of a Hecke eigenform $f \in S_2(\Gamma_0(N_{\psi}(d)))$ generate the ring class field of a real quadratic field $k = \mathbb{Q}(\sqrt{d})$ modulo the conductor $\psi \geq 1$. In 1972 Goro Shimura calculated the integer $N_{\psi}(d)$ for certain prime values of $d$ [Shimura 1972] [9]. The general formula can be found in [6]. In this section we use Shimura’s results to illustrate theorem 1.1 by proving the following proposition.

Proposition 4.1. If $p$ is a prime number such that $p \equiv 1 \mod 4$, then $N_{\psi}(p) = p$ for some $\psi \geq 1$. The corresponding values of the conductor $\psi$ and the $j$-invariant $j(\Lambda_M)$ for $p \leq 197$ are compiled in Table 1.

Proof. Recall that $\Gamma_1(N)$ is a subgroup of $\Gamma_0(N)$ defined by the congruence relations $a \equiv d \equiv 1 \mod N$, see Section 2.2 for the notation; consider a character $\psi$ of $(\mathbb{Z}/N\mathbb{Z})^\times$ such that $\psi(-1) = 1$. Let $S_2(\Gamma_1(N), \psi)$ be a subspace of the space of cusp forms $S_2(\Gamma_1(N))$ consisting of function $f$ such that $f \left( \frac{az+b}{cz+d} \right) = \psi(d)(cz+d)^2f(z), \quad \forall \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N)$. For each positive integer $n$ one can define the Hecke operator $T(n)$ on $S_2(\Gamma_1(N))$ [Diamond & Shurman 2005] [1, p. 168]; a restriction of $T(n)$ to $S_2(\Gamma_1(N), \psi)$ will be denoted by $T(n, \psi)$. The corresponding Hecke eigenform we shall write as $f(z, \psi)$ and the algebraic number field generated by the Fourier coefficients of $f(z, \psi)$ we denote by $K_{f, \psi}$. Unless $\psi$ is a trivial character, $K_{f, \psi}$ is a CM-field, i.e. a totally imaginary quadratic extension of a totally real algebraic number field [Shimura 1972] [9, Proposition 1.3]. The maximal real subfield of the field $K_{f, \psi}$ will be denoted by $F$.

We further assume that $\psi$ is a non-trivial real quadratic character $\psi^2 = 1$. It is not hard to see, that such a character generates an automorphism of order 2 of the field $F$ and, therefore, $Gal(K|\mathbb{Q})$ contains a subgroup of order 2. Let $k$ be the corresponding real quadratic subfield of $F$. Shimura noticed that $F$ can be interpreted as a ray class field of $k$ connected to an integral ideal $\mathfrak{c}$ of $F$. In particular, if $N$ is a prime number $p \equiv 1 \mod 4$ and $\psi(a) = \left( \frac{a}{\mathfrak{c}} \right)$, then $k = \mathbb{Q}(\sqrt{N})$ [Shimura 1972] [9, p.147]. In this case Shimura gives numerical examples for levels $29 \leq N \leq 233$ with conductors $N(\mathfrak{c})$, where $N(\mathfrak{c})$ is the norm of ideal $\mathfrak{c}$ [Shimura 1972] [9, pp.149-150].
Shimura’s method is linked to the measured foliations as follows. Let $\mathcal{F}_\psi := \mathbb{R} \langle f(z, \psi)dz \rangle$ be measured foliation on the surface $X_1(N)$. Consider the jacobian $Jac(\mathcal{F}_\psi) = \mathbb{Z} \lambda_1 + \cdots + \mathbb{Z} \lambda_n$ of foliation $\mathcal{F}_\psi$; it is clear that $\lambda_i \in F$. We claim that if $\psi$ is a quadratic character and companions [Shimura 1972] [9, p.134] of $f(z, \psi)$ span $S_2(\Gamma_1(N), \psi)$, then

$$Jac(\mathcal{F}_\psi) = Jac(\mathcal{F}),$$

(4.1)

where $\mathcal{F} = \mathbb{R} \langle g(z)dz \rangle$ is measured foliation on surface $X_0(N)$ corresponding to a maximal Hecke eigenform $g \in S_2(\Gamma_0(N))$. Indeed, we have $Jac(\mathcal{F}) \subseteq Jac(\mathcal{F}_\psi)$, since $S_2(\Gamma_0(N)) \subset S_2(\Gamma_1(N))$ and the Hecke operator $T(n, \psi)$ acts on both spaces [Diamond & Shurman 2005] [1, p.168]. Notice that for the trivial character $\psi = 1$ the following three fields coincide: $K_{f,1} = F = K_g$. Thus, for quadratic character $\psi$ field $K_{f,\psi}$ is a quadratic extension of $F = K_g$. But elements of the field $K_{f,\psi}$ are periods of the holomorphic differential $f(z, \psi)dz$ on surface $X_1(N)$; the real parts of such periods generate the maximal real subfield of $K_{f,\psi}$, i.e. the field $F$. Thus the ranks of $Jac(\mathcal{F}_\psi)$ and $Jac(\mathcal{F})$ must coincide, hence equation (4.1).

In view of formula (4.1), one concludes that Shimura’s real quadratic field $k$ (modulo integral ideal $c$) attached to quadratic character $\psi$ coincides with such obtained from Manin’s pseudo-lattice $\Lambda_M$ modulo the conductor $f = N(c)$. Table 1 comprises all Shimura’s examples except for the non-companionate cusp forms $N = 109, 157$ and 229. In case $N = 233$ the value of conductor $N(c)$ is unknown, hence must be excluded. Proposition 4.1 follows.

| $p$ | $k$ | $f$ | $j(\Lambda_M)$ |
|-----|-----|-----|----------------|
| 29  | $\mathbb{Q}(\sqrt{29})$ | 5   | 1              |
| 37  | $\mathbb{Q}(\sqrt{37})$ | 1   | 1              |
| 41  | $\mathbb{Q}(\sqrt{41})$ | 1   | 1              |
| 53  | $\mathbb{Q}(\sqrt{53})$ | 7   | $1 + \sqrt{2}$ |
| 61  | $\mathbb{Q}(\sqrt{61})$ | 13  | $2 + \sqrt{3}$ |
| 73  | $\mathbb{Q}(\sqrt{79})$ | 89  | $(1 + \sqrt{5})$ |
| 89  | $\mathbb{Q}(\sqrt{89})$ | 5   | a fundamental unit of the splitting field of polynomial $x^3 + 17x^2 + 83x + 125 = 0$ |
| 97  | $\mathbb{Q}(\sqrt{97})$ | 467 | $x^3 + 27x^2 + 204x + 467 = 0$ |
| 101 | $\mathbb{Q}(\sqrt{101})$ | 5   | $x^4 + 13x^3 + 51x^2 + 67x + 20 = 0$ |
| 113 | $\mathbb{Q}(\sqrt{113})$ | 97  | $x^4 + 19x^3 + 122x^2 + 297x + 194 = 0$ |
| 137 | $\mathbb{Q}(\sqrt{137})$ | 109 | $x^5 + 23x^4 + 188x^3 + 670x^2 + 989x + 436 = 0$ |
| 149 | $\mathbb{Q}(\sqrt{149})$ | 61  | $x^6 + 20x^5 + 148x^4 + 499x^3 + 766x^2 + 465x + 61 = 0$ |
| 173 | $\mathbb{Q}(\sqrt{173})$ | 13  | $x^7 + 20x^6 + 151x^5 + 542x^4 + 972x^3 + 833x^2 + 276x + 13 = 0$ |
| 181 | $\mathbb{Q}(\sqrt{181})$ | 435 | $x^8 + 23x^7 + 210x^6 + 974x^5 + 2441x^4 + 3234x^3 + 2030x + 435 = 0$ |
| 197 | $\mathbb{Q}(\sqrt{197})$ | 7   | $x^8 + 24x^7 + 228x^6 + 1095x^5 + 2834x^4 + 3942x^3 + 2795x^2 + 925x + 112 = 0$ |

Table 1.
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