Characteristic polynomials for random band matrices near the threshold

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Abstract

The paper continues [9], [8] which study the behaviour of second correlation function of characteristic polynomials of the special case of \( n \times n \) one-dimensional Gaussian Hermitian random band matrices, when the covariance of the elements is determined by the matrix

\[
J = (-W^2 \Delta + 1)^{-1}.
\]

Applying the transfer matrix approach, we study the case when the bandwidth \( W \) is proportional to the threshold \( \sqrt{n} \).

1 Introduction

As in [9], [8], we consider Hermitian \( n \times n \) matrices \( H \) whose entries \( H_{ij} \) are random complex Gaussian variables with mean zero such that

\[
E\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}J_{ij}, \tag{1.1}
\]

where

\[
J_{ij} = (-W^2 \Delta + 1)_{ij}^{-1}, \tag{1.2}
\]

and \( \Delta \) is the discrete Laplacian on \( \mathcal{L} = [1,n] \cap \mathbb{Z} \) with Neumann boundary conditions. It is easy to see that the variance of matrix elements \( J_{ij} \) is exponentially small when \( |i-j| \gg W \), and so \( W \) can be considered as the width of the band.

The density of states \( \rho \) of the ensemble is given by the well-known Wigner semicircle law (see [1], [6]):

\[
\rho(E) = (2\pi)^{-1} \sqrt{4 - E^2}, \quad E \in [-2, 2]. \tag{1.3}
\]

Random band matrices (RBM) provide a natural and important model to study eigenvalue statistic and quantum transport in disordered systems as they interpolate between classical Wigner matrices, i.e. Hermitian random matrices with all independent identically distributed elements, and random Schrödinger operators, where only a random on-site potential is present in addition to the deterministic Laplacian on a regular box in \( d \)-dimension lattice. Such matrices have various application in physics: the eigenvalue statistics of RBM is in relevance in quantum chaos, the quantum dynamics associated with RBM can be used to model conductance in thick wires, etc.

One of the main long standing problem in the field is to prove a fundamental physical conjecture formulated in late 80th (see [3], [5]). The conjecture states that the eigenvectors of \( n \times n \) RBM are completely delocalized and the local spectral statistics governed by random

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matrix (Wigner-Dyson) statistics for large bandwidth \( W \), and by Poisson statistics for a small \( W \) (with exponentially localized eigenvectors). The transition is conjectured to be sharp and for RBM in one spatial dimension occurs around the critical value \( W = \sqrt{n} \). This is the analogue of the celebrated Anderson metal-insulator transition for random Schrödinger operators.

The conjecture on the crossover in RBM with \( W \sim \sqrt{n} \) is supported by physical derivation due to Fyodorov and Mirlin (see [5]) based on supersymmetric formalism, and also by the so-called Thouless scaling. However, there are only partial results on the mathematical level of rigour (see reviews [2], [7] and references therein for the details).

The only result that rigorously demonstrate the threshold around \( W \sim \sqrt{n} \) for a certain eigenvalue statistics was obtain in [9] (regime \( W \gg \sqrt{n} \)), [8] (regime \( W \ll \sqrt{n} \)). Instead of eigenvalue correlation functions these papers deal with more simple object which is the second correlation functions of characteristic polynomials:

\[
F_2(x_1, x_2) = \mathbb{E}\left\{ \det(x_1 - H) \det(x_2 - H) \right\}.
\]

(1.4)

The main results of [9], [8] concern the asymptotic behaviour of this function for

\[
x_{1,2} = E + \frac{\xi_{1,2}}{n \rho(E)}, \quad E \in (-2, 2), \quad \xi_{1,2} \in [-C, C].
\]

Namely, let

\[
D_2 = F_2(E, E), \quad \tilde{F}_2(x_1, x_2) = D_2^{-1} \cdot F_2(x_1, x_2).
\]

Then we have the following theorem

**Theorem 1.1** ([9], [8]) *For the 1d RBM of (1.1) – (1.2) we have*

\[
\lim_{n \to \infty} \tilde{F}_2 \left( E + \frac{\xi}{2n \rho(E)}, E - \frac{\xi}{2n \rho(E)} \right) = \begin{cases} 
\frac{\sin \pi \xi}{\pi \xi}, & W \geq n^{1/2+\theta}; \\
1, & 1 \ll W \leq \sqrt{\frac{n}{C_* \log n}},
\end{cases}
\]

*where the limit is uniform in \( \xi \) varying in any compact set \( C \subset \mathbb{R} \). Here \( E \in (-2, 2) \), and \( \rho(x) \) is defined in (1.3).*

The purpose of the present paper is to complete Theorem 1.1 by the study of correlation functions of characteristic polynomials (1.4) near the threshold \( W \sim \sqrt{n} \). The main result is

**Theorem 1.2** *For the 1d RBM of (1.1) – (1.2) with \( n = C_* W^2 \) we have*

\[
\lim_{n \to \infty} \tilde{F}_2 \left( E + \frac{\xi}{2n \rho(E)}, E - \frac{\xi}{2n \rho(E)} \right) = (e^{-C^* \Delta_U - i \xi \hat{\nu} \cdot 1}, 1),
\]

*where \( C^* = C_*/(2\pi \rho(E))^2 \). In this formula \((\cdot, \cdot)\) is an inner product on a 2-dimensional sphere \( \mathbb{S}^2 \), \( \Delta_U \) is a Laplace operator on \( \mathbb{S}^2 \)

\[
\Delta_U = -\frac{d}{dx} x(1-x) \frac{d}{dx}, \quad x = |U_{12}|^2,
\]

\( U \) is a \( 2 \times 2 \) unitary matrix, and \( \hat{\nu} \) is an operator of multiplication on

\[
\nu(U) = 1 - 2|U_{12}|^2 \quad (1.5)
\]

*on \( \mathbb{S}^2 \).*
Remark 1.1 It is easy to see that if \( W \gg \sqrt{n} \) (and so \( C \to 0 \)), then we have
\[
(e^{-C^*\Delta_U - \pi i \hat{\xi} \cdot 1}, 1) \sim (e^{-\pi i \hat{\xi} \cdot 1}, 1) = \frac{\sin \pi \xi}{\pi \xi}.
\]
Similarly if \( W \ll \sqrt{n} \) (and so \( C \to \infty \)), then we get
\[
(e^{-C^*\Delta_U - \pi i \hat{\xi} \cdot 1}, 1) \sim (e^{-C^* \Delta_U \cdot 1}, 1) = 1.
\]
Thus the result of Theorem 1.2 "glue" together two parts of Theorem 1.1.

Remark 1.2 The study of eigenfunctions and spectral statistics in the critical regime (near the threshold) is of independent interest. Critical wave-functions at the point of the Anderson localization transition are expected to be multifractal. Moreover, multifractal structure occurs in a critical regime of power-law banded random matrices (see the review [4] and reference therein for the details). Although the correlation functions of characteristic polynomials (1.4) are not reach enough to feel this phenomena, the techniques developed in the paper can be useful in studying the usual correlation functions of 1d RBM near the threshold.

The proof of Theorem 1.2 is based on the techniques of [8]. Namely, we apply the version of transfer matrix approach introduced in [8] to the integral representation obtained in [9] by the supersymmetry techniques (note that the integral representation does not contain Grassmann integrals, see Proposition 2.1).

The paper is organized as follows. In Section 2 we rewrite \( F_2 \) as an action of the \( n \)-th degree of some transfer operator \( K_\xi \) (see (2.5) below) and outline the proof of Theorem 1.2. In Section 3 we collect all preliminaries results obtained in [8]. Section 4 deals with the proof of Theorem 1.2.

We denote by \( C \), \( C_1 \), etc. various \( W \) and \( n \)-independent quantities below, which can be different in different formulas. To reduce the number of notations, we also use the same letters for the integral operators and their kernels.

2 Outline of the proof of Theorem 1.2

First, we rewrite \( F_2 \) as an action of the \( n - 1 \)-th degree of some transfer operator, as it was done in [8].

For \( X \in \text{Herm}(2) \) define
\[
f := F(X) = \exp \left\{ -\frac{1}{4} \text{Tr} \left( X + \frac{i \Lambda_0}{2} \right)^2 + \frac{1}{2} \text{Tr} \log \left( X - \frac{i \Lambda_0}{2} \right) - C_+ \right\},
\]
\[
f_\xi := F_\xi(X) = F(X) \cdot \exp \left\{ -\frac{i}{2 n \rho(E)} \text{Tr} X \hat{\xi} \right\}
\]
with \( \hat{\xi} = \text{diag} \{ \xi, -\xi \} \), \( \Lambda_0 = E \cdot I_2 \),
\[
a_\pm = \pm \sqrt{1 - E^2 / 4}
\]
\[
C_+ = \frac{1}{4} \text{Tr} \left( a_+ I + \frac{i \Lambda_0}{2} \right)^2 - \frac{1}{2} \text{Tr} \log \left( a_+ I - \frac{i \Lambda_0}{2} \right).
\]
Set also $\mathcal{H} = L_2[\text{Herm}(2)]$, and let $K, K_\xi : \mathcal{H} \to \mathcal{H}$ be operators with the kernels
\begin{align*}
K(X,Y) &= \frac{W^4}{2\pi^2} F(X) \exp \left\{ - \frac{W^2}{2} \text{Tr} (X - Y)^2 \right\} F(Y); \\
K_\xi(X,Y) &= \frac{W^4}{2\pi^2} F_\xi(X) \exp \left\{ - \frac{W^2}{2} \text{Tr} (X - Y)^2 \right\} F_\xi(Y).
\end{align*}

As it was proved in [8], Section 2, we have

**Proposition 2.1** ([8]) The second correlation function of characteristic polynomials of (1.4) for 1D Hermitian Gaussian band matrices (1.1) – (1.2) can be represented as follows:

\begin{equation}
F_2 \left( E + \frac{\xi}{n\rho(E)}, E - \frac{\xi}{n\rho(E)} \right) = -C_n(\xi) \cdot W^{-4n} \det^{-2} J \cdot (K_\xi^{n-1}f, \bar{\xi}), \tag{2.6}
\end{equation}

where $(\cdot, \cdot)$ is a standard inner product in $\mathcal{H}$, $\rho$ is defined in (1.3), and

\begin{equation*}
C_n(\xi) = \exp \left\{ 2nC_+ + \xi^2/n\rho(E)^2 \right\}
\end{equation*}

with $C_+$ of (2.3).

For arbitrary compact operator $M$ denote by $\lambda_j(M)$ the $j$th (by its modulo) eigenvalue of $M$, so that $|\lambda_0(M)| \geq |\lambda_1(M)| \geq \ldots$.

The idea of the transfer operator approach is very simple and natural. Let $\mathcal{K}(X,Y)$ be the matrix kernel of the compact integral operator in $\bigoplus_{i=1}^p L_2[X, d\mu(X)]$. Then

\begin{equation*}
\int g(X_1)K(X_1,X_2)\ldots K(X_{n-1},X_n)f(X_n)\prod d\mu(X_i) = (K^{n-1}f, \bar{g})
\end{equation*}

\begin{equation*}
= \sum_{j=0}^\infty \lambda_j^{n-1}(\mathcal{K})c_j, \quad \text{with} \quad c_j = (f, \psi_j)(g, \bar{\psi}_j),
\end{equation*}

where $\psi_j$ are eigenvectors corresponding to $\lambda_j(\mathcal{K})$, and $\bar{\psi}_j$ are the eigenvectors of $\mathcal{K}^\ast$. Hence, to study the correlation function, it suffices to study the eigenvalues and eigenfunctions of the integral operator with the kernel $\mathcal{K}(X,Y)$.

The main difficulties in application of this approach to (2.6) are the complicated structure and non self-adjointness of the corresponding transfer operator $K_\xi$ of (2.5).

In fact, since the analysis of eigenvectors of non self-adjoint operators is rather involved, it is simpler to work with the resolvent analog of (2.6)

\begin{equation}
(K_\xi^{n-1}f, \bar{\xi}) = -\frac{1}{2\pi i} \oint_{\mathcal{L}} z^{n-1}(G_\xi(z)f, \bar{f}_\xi)dz, \quad G_\xi(z) = (K_\xi - z)^{-1}, \tag{2.7}
\end{equation}

where $\mathcal{L}$ is any closed contour which enclosed all eigenvalues of $K_\xi$.

To explain the idea of the proof, we start from the definition

**Definition 2.1** We shall say that the operator $A_{n,W}$ is equivalent to $B_{n,W}$ ($A_{n,W} \sim B_{n,W}$) on some contour $\mathcal{L}$ if

\begin{equation*}
\int_\mathcal{L} z^{n-1}((A_{n,W} - z)^{-1}f, \bar{g})dz = \int_\mathcal{L} z^{n-1}((B_{n,W} - z)^{-1}f, \bar{g})dz (1 + o(1)), \quad n,W \to \infty,
\end{equation*}

with some $f, g$ depending of the problem.
The idea is to find some operator equivalent to $K_\xi$ whose spectral analysis we are ready to perform.

It is easy to see that the stationary points of the function $F$ of (2.1) are

$$X_+ = a_+ \cdot I_2, \quad X_- = a_- \cdot I_2; \quad (2.8)$$

$$X_+(U) = a_+ U L U^*, \quad U \in \tilde{U}(2),$$

where $a_\pm$ is defined in (2.2), $\tilde{U}(2) := U(2)/U(1) \times U(1)$, $L = \text{diag} \{1, -1\}$. Notice also that the value of $|F|$ at points (2.8) is 1.

Roughly speaking, the first step in the proof of Theorem 1.2 is to show that if we introduce the projection $P_s$ onto the $W^{-1/2} \log W$-neighbourhoods of the saddle points $X_+$, $X_-$ and the saddle "surface" $X_{\pm}$, then in the sense of Definition 2.1

$$K_\xi \sim P_s K_\xi P_s =: K_{m,\xi}. $$

To study the operator $K_{m,\xi}$ near the saddle "surface" $X_{\pm}$ we use the "polar coordinates". Namely, introduce

$$t = (a_1 - b_1)(a_2 - b_2), \quad p(a, b) = \frac{\pi}{2}(a - b)^2, \quad (2.9)$$

and denote by $dU$ the integration with respect to the Haar measure on the group $\tilde{U}(2)$: in the standard parametrization

$$U = \begin{pmatrix} \cos \varphi & \sin \varphi \cdot e^{i\theta} \\ -\sin \varphi \cdot e^{-i\theta} & \cos \varphi \end{pmatrix}, \quad (2.10)$$

we have

$$dU = \frac{1}{\pi} du \, d\varphi \, d\theta, \quad u = |\sin \varphi| \in [0, 1], \quad \theta \in [0, 2\pi).$$

Consider the space $L_2[\mathbb{R}^2, p] \times L_2[\tilde{U}(2), dU]$. The inner product and the action of an integral operator in this space are

$$(f, g)_p = \int f(a, b) \bar{g}(a, b)p(a, b) \, da \, db;$$

$$(Mf)(a_1, b_1, U_1) = \int M(a_1, b_1, U_1; a_2, b_2, U_2) f(a_2, b_2, U_2) p(a_2, b_2) \, da \, db \, dU_2.$$

Changing the variables

$$X = U^* \Lambda U, \quad \Lambda = \text{diag} \{a, b\}, \quad a > b, \quad U \in \tilde{U}(2),$$

we obtain that $K_\xi = K + \tilde{K}_\xi$ can be represented as an integral operator in $L_2[\mathbb{R}^2, p] \times L_2[\tilde{U}(2), dU]$ defined by the kernel

$$K_\xi(X, Y) = K(a_1, a_2, b_1, b_2, U_1, U_2) + \tilde{K}_\xi(a_1, a_2, b_1, b_2, U_1, U_2), \quad (2.11)$$

where

$$K(a_1, a_2, b_1, b_2, U_1, U_2) = t^{-1} A(a_1, a_2) A(b_1, b_2) K_s(t, U_1, U_2);$$

$$K_s(t, U_1, U_2) := W^2 t \cdot e^{i W^2 t U_1 U_2 L(U_1^* U_2^*)^* L/4 - t W^2/2}; \quad (2.12)$$

$$\tilde{K}_\xi(a_1, a_2, b_1, b_2, U_1, U_2) = K(a_1, a_2, b_1, b_2, U_1, U_2) \left( e^{\nu(a_1 - b_1, U_1) + \nu(a_2 - b_2, U_2)}/n - 1 \right);$$

$$\nu(x, U) = -\frac{i \xi x}{4 \rho(E)} \text{Tr} U U^* L.$$
Here is a contribution of the unitary group \( \hat{U}(2) \), and \( \nu(x, U) \) is a perturbation of \( F \) appearing in \( F_\xi \) (see (2.1)). Operator \( A \) is a contribution of eigenvalues \( a, b \) that has the form

\[
A(x, y) = (2\pi)^{-1/2}W e^{-g(x)/2}e^{-W^2(y-x)^2/2}e^{-g(y)/2},
\]

\[
g(x) = (x + iE/2)^2/2 - \log(x - iE/2) - C_+;
\]

Note also that

\[
\|\widetilde{K}_\xi\| \leq C/n
\]

with some absolute \( C > 0 \).

The main properties of \( K_\ast \) are given in the following proposition:

**Proposition 2.2** If we consider \( K_\ast(t, U_1, U_2) \) of (2.12) as a kernel of the self-adjoint integral operator in \( L_2[\hat{U}(2), dU] \), then its eigenvectors \( \{\phi_{j}(U)\} \) \( (j = (j, s), j = 0, 1, \ldots, s = -j, \ldots, j) \) do not depend on \( t \) and are the standard spherical harmonics:

\[
\phi_{j,s}(U) = l_{j,s} P_j^s(\cos 2\varphi) e^{is\theta} = l_{j,s} \left( \frac{d}{dx} \right)^s P_j(x) \bigg|_{x=1-2|U_1|^2} (2\bar{U}_{11}U_{12})^s,
\]

where \( U \) has the form (2.10), and \( P_j^s \) is an associated Legendre polynomial

\[
P_j^s(\cos x) = (\sin x)^s \left( \frac{d}{d\cos x} \right)^s P_j(\cos x), \quad P_j(x) = \frac{1}{2j!} \frac{d^j}{dx^j}(x^2 - 1)^j,
\]

where \( l_{j,s} = \sqrt{\frac{(2j + 1)(j-s)!}{(j+s)!}} \).

Moreover, the subspace \( L_2[u,dU] \subset L_2[\hat{U}(2), dU] \) of the functions depending on \( \varphi \) only is invariant under \( K_\ast \), and the restriction of \( K_\ast \) to \( L_2[u, dU] \) has eigenvectors

\[
\phi_j(U) := \phi_{j,0}(U).
\]

The corresponding eigenvalues \( \{\lambda_j(t)\}_{j=0}^\infty \), if \( t > d > 0 \), where \( d \) is some absolute positive constant, have the form

\[
\lambda_0(t) = 1 - e^{-W^2t},
\]

\[
\lambda_j(t) = (1 - e^{-W^2t}) \left[ 1 - \frac{j(j+1)}{W^2t} \right] (1 + O(j^2/W^2t)).
\]

Notice that since

\[
\text{Tr } U^* LUL = 2(1 - 2u^2),
\]

functions \( F, F_\xi \) do not depend on \( \theta \) of (2.10), and hence according to Proposition 2.2 in what follows we can consider the restriction of \( K, K_\ast \) and \( \widetilde{K}_\xi \) of (2.12) to \( L_2[u, dU] \) (to simplify notations we will denote these restriction by the same letters).

In addition, it follows from Proposition 2.2 that if we introduce the following basis in \( L_2[\mathbb{R}^2, p] \times L_2[u, dU] \)

\[
\Psi_{k,j}(a, b, U) = \Psi_k(a, b) \phi_j(U),
\]

\[
\Psi_k(a, b) = \sqrt{\frac{2}{\pi}} (a - b)^{-1} \psi_k(a) \psi_k(b),
\]

\[6\]
where \( \bar{\kappa} = (k_1, k_2) \), and \( \{\psi_k(x)\}_{k=0}^{\infty} \) is a certain basis in \( L_2[\mathbb{R}] \), then the matrix of \( K \) of (2.12) in this basis has a “block diagonal structure”, which means that

\[
(K\Psi_{k',j}, \Psi_{k,j})_p = 0, \quad j \neq j_1
\]

\[
(K\Psi_{k',j}, \Psi_{k,j})_p = (K_j \Psi_{k'}, \Psi_{k})_p
\]

\[
= \int \lambda_j(t)A(a_1, a_2)A(b_1, b_2)\psi_k(a_1)\psi_{k_2}(b_1)\psi_{k_2}(a_2)\psi_{k_2}(b_2)da_1db_1da_2db_2.
\]

The next step in the proof of Theorem 1.2 is to show that only the neighbourhood of the saddle “surface” \( X_\pm \) gives the main contribution to the integral, and moreover we can restrict the number of \( \phi_j \) to \( l = \lfloor \log W \rfloor \). More precisely, we are going to show that in the sense of Definition 2.1

\[
K_{m,\xi} \sim P_l K_{m,\xi} P_l =: K_{m,l,\xi},
\]

where \( P_l \) is the projection on the linear span of \( \{\Psi_{\bar{k},j}(a, b, U)\}_{j \leq l, |\bar{k}| \leq m} \).

For the further resolvent analysis we want to put \( t \) in the definition of \( K_\ast \) and \( a_1 - b_1, a_2 - b_2 \) in the definition of \( K_{\xi} \) (see (2.9), (2.11) – (2.12)) equal to their saddle-point value \( t^* = (a_+ - a_-)^2 = 4\pi^2 \rho(E)^2 \) and \( a_+ - a_- = 2\pi \rho(E) \) correspondingly. More precisely we want to show that in the sense of Definition 2.1

\[
K_{m,l,\xi} \sim A_m \otimes K_{s,\xi,l}
\]

where

\[
K_{s,\xi,l} = Q_l K_{s,\xi} Q_l,
\]

\[
K_{s,\xi}(U_1, U_2) = W^2 t^* \cdot e^{-t^* W^2 \nu U_1 U_2^* L U_1^* L / 2 - W^2 / 2} \cdot e^{n-1 \nu (2\pi \rho(E), U_1) + n-1 \nu (2\pi \rho(E), U_2)}
\]

and \( Q_l \) is the projection on \( \{\phi_j(U)\}_{j \leq l} \). The operator \( A_m \) in (2.19) is defined as

\[
A_m = P_m A(a_1, a_2)A(b_1, b_2)P_m,
\]

where \( P_m \) is the projection on \( \{\Psi_{k}(a, b)\}_{|k| \leq m} \).

Now (2.19), (2.7) and Definition 2.1 give

\[
F_2 \left( E + \frac{\xi}{2n \rho(E)}, E - \frac{\xi}{2n \rho(E)} \right) = C_n \left( \left( K_{s,\xi,l}^{-1} \otimes A_m^{-1} \right) f_\xi, \bar{f}_\xi \right)(1 + o(1))
\]

\[
= (A_m^{-1} f_1, \bar{f}_1) (K_{s,\xi,l}^{-1} 1, 1)(1 + o(1)),
\]

where we used that \( f_\xi \) asymptotically can be replaced by \( f_1 \otimes 1 \), where \( f_1 \) does not depend on \( \xi \) and \( U_j \). Similarly

\[
D_2 = C_n (K_{s,0}^{-1} \otimes A_m^{-1} f, \bar{f})(1 + o(1)) = (A_m^{-1} f_1, \bar{f}_1) (K_{s,0}^{-1} 1, 1)(1 + o(1)),
\]

and so

\[
F_2 \left( E + \frac{\xi}{2n \rho(E)}, E - \frac{\xi}{2n \rho(E)} \right) = (K_{s,\xi,l}^{-1} 1, 1)(1 + o(1)),
\]

since according to Proposition 2.2 \( \phi_0(U) = 1 \) is eigenvector of \( K_\ast \) with an eigenvalue 1, thus

\[
(K_{s,0}^{-1} 1, 1) = 1.
\]

Observe that the Laplace operator \( \Delta U \) on \( U(2) \) is also reduced by \( E_0 \) and has the same eigenfunctions as \( K_{s,0} \) with eigenvalues \( \lambda_j^* = j(j + 1) \). Hence, in the regime \( W^{-2} = C_\ast n^{-1} \) we can write \( K_{s,\xi,l} \) as

\[
K_{s,\xi,l} \sim 1 - n^{-1}(C_\ast \Delta U + i\xi \pi \nu) \Rightarrow (K_{s,\xi,l}^{-1} 1, 1) \rightarrow (e^{-C_\ast \Delta U - i\xi \pi \nu} 1, 1),
\]

where \( C_\ast = C_\ast / t^* \), which gives Theorem 1.2.
3 Preliminary results

Recall that stationary points $X_+, X_-$, and $X_\pm(U)$ of the function $\mathcal{F}$ of (2.1) are defined in (2.8).

Put

$$X = \begin{pmatrix} a_1 & (x_1 + iy_1)/\sqrt{2} \\ (x_1 - iy_1)/\sqrt{2} & b_1 \end{pmatrix}, \quad Y = \begin{pmatrix} a_2 & (x_2 + iy_2)/\sqrt{2} \\ (x_2 - iy_2)/\sqrt{2} & b_2 \end{pmatrix}.$$  

Considering the operators $K, K_\xi$ near the points $X_+$ and $X_-$, we are going to extract the contribution from the diagonal elements of $X, Y$. To this end, rewrite $K(X, Y), K_\xi(X, Y)$ of (2.4) – (2.5) as

$$K_\xi(X, Y) = K(X, Y) + \tilde{K}_\xi(X, Y),$$

$$K(X, Y) = A(a_1, a_2) A(b_1, b_2) A_1(X, Y),$$

where the kernels $A$ (the contribution of the diagonal elements) is defined in (2.13), and $A_1$ (the contribution of the off-diagonal elements, which however depends on diagonal elements as well) has the form

$$A_1(X, Y) = (2\pi)^{-1} W^2 F_1(X) \cdot \exp\{W^2(x_1 - x_2)^2/2 - W^2(y_1 - y_2)^2/2\} \cdot F_1(Y);$$

$$F_1(X) = \exp \left\{ -\frac{1}{4}(x_1^2 + y_1^2) + \frac{1}{2} \log \left( 1 - \frac{x_1^2 + y_1^2}{2(a_1 - iE/2)(b_1 - iE/2)} \right) \right\}.$$

The perturbation kernel $\tilde{K}_\xi$ in this coordinates is

$$\tilde{K}_\xi(X, Y) = A(a_1, a_2) A(b_1, b_2) A_1(X, Y) \left( e^{-\frac{i}{2\pi(E/2)}(\xi(a_1 - b_1) + \xi(a_2 - b_2))} - 1 \right).$$

It is easy to check that for $g$ defined in (2.13)

$$g(a_\pm + x) - g(a_\pm) = c_\pm x^2 + c_3 x^3 + \ldots$$

with

$$c_\pm = a_\pm(\sqrt{4 - E^2} \pm iE)/2, \quad \Re c_+ = \Re c_- > 0,$$

and some constants $c_3, c_4, \ldots$

Representation of $K, K_\xi$ near $X_\pm(U)$ was described in (2.11) – (2.12).

Following [8], define the orthonormal in $L_2[\mathbb{R}]$ system of functions

$$\psi^0_k(x) = e^{-\alpha W x^2} \sqrt{\alpha W/\pi},$$

$$\psi^0_k(x) = h_k^{-1/2} e^{-\alpha W x^2} e^{2\Re \alpha W x^2} \left( \frac{d}{dx} \right)^k e^{-2\Re \alpha W x^2} p_k(x),$$

$$h_k^\pm = k!(4\Re \alpha \cdot W)^{k-1/2} \sqrt{2\pi}, \quad k = 1, 2, \ldots,$$

with some $\alpha$ such that $\Re \alpha > 0$, and set

$$\psi_k \pm(x) = \psi_k \pm(x - a_\pm)$$

(3.6)
with
\[ \alpha_{\pm} = \sqrt{\frac{c_+}{2}} \left(1 + \frac{c_+}{2W^2}\right)^{1/2} \]

Now choose \(W, n\)-independent \(\delta > 0\), which is small enough to provide that the domain
\[ \Omega_\delta = \{X : |F(X)| > 1 - \delta\} \]
contains three non-intersecting subdomains \(\Omega_\delta^+, \Omega_\delta^-, \Omega_\delta^\pm\), such that each of \(\Omega_\delta^+, \Omega_\delta^-\) contains one of the points \(X_+, X_-\), and \(\Omega_\delta^\pm\) contains the surface \(X_\pm(U)\) of (2.8).

Set
\[ m = [\log^2 W], \tag{3.7} \]
and consider the system of functions
\[ \{\Psi_{\tilde{k},j,\delta}\}_{|\tilde{k}| \leq m, j \leq (mW)^{1/2}}, \tag{3.8} \]
\[ \tilde{k} = (k_1, k_2), \ |\tilde{k}| = \max\{k_1, k_2\}, \]
obtained by the Gram-Schmidt procedure from
\[ \{1_{\Omega_\delta^\pm} \Psi_{\tilde{k},j}\}_{|\tilde{k}| \leq m, j \leq (mW)^{1/2}}, \]
where
\[ \Psi_{\tilde{k},j}(a, b, U) = \Psi_{\tilde{k}}(a, b)\phi_j(U), \tag{3.9} \]
\[ \Psi_{\tilde{k}}(a, b) = \sqrt{\frac{\pi}{2}} (a - b)^{-1} \psi_{k_1}^+(a) \psi_{k_2}^-(b). \]

Similarly, consider the system of functions \(\{\Psi_{\tilde{k},\delta}^+\}_{|\tilde{k}| \leq m}\) (with \(\tilde{k} = (k_1, k_2, k_3, k_4)\), \(|\tilde{k}| = \max\{k_i\}\)) obtained by the Gram-Schmidt procedure from
\[ \{1_{\Omega_\delta^\pm} \psi_{k_1}^+(a) \psi_{k_2}^+(b) \psi_{k_3}^+(x + a_+) \psi_{k_4}^+(y + a_+))\}_{|\tilde{k}| \leq m}, \]
and define \(\{\Psi_{\tilde{k},\delta}^-\}_{|\tilde{k}| \leq m}\) by the same way. Denote \(P_\pm, P_+, P_-\) the projections on the subspaces spanned on these three systems. Evidently these three projection operators are orthogonal to each other. Set
\[ P = P_\pm + P_+ + P_-, \quad \mathcal{H}_1 = P\mathcal{H}, \quad \mathcal{H}_2 = (1 - P)\mathcal{H}, \quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{3.10} \]
where \(\mathcal{H} = L_2[\text{Herm}(2)]\). Besides, note that for any \(\varphi\) supported in some domain \(\Omega\) and any \(C > 0\)
\[ (K\varphi)(X) = O(e^{-cW^2}) \text{ for } X : \text{dist}\{X, \Omega\} \geq C > 0. \tag{3.11} \]

Now consider the operator \(K\) as a block operator with respect to the decomposition (3.10). It has the form
\[ K^{(11)} = K_\pm + K_+ + K_- + O(e^{-cW}), \tag{3.12} \]
\[ K_\pm := P_\pm KP_\pm, \quad K_+ := P_+ KP_+, \quad K_- := P_- KP_-, \]
\[ K^{(12)} = P_\pm K(I_\pm - P_\pm) + P_+ K(I_+ - P_+) + P_- K(I_- - P_-) + O(e^{-cW}), \]
\[ K^{(21)} = (I_\pm - P_\pm) KP_\pm + (I_+ - P_+) KP_+ + (I_- - P_-) KP_- + O(e^{-cW}), \]
where $I_+, I_-$, and $I_+^-$ are operators of multiplication by $1_{\Omega^\pm}, 1_{\Omega^+_\delta}$, and $1_{\Omega^-_{\delta}}$ respectively. Indeed, it is easy to see from (3.11) and from the relation
\[ \psi_k(x) = O(e^{-cW}) \text{ for } |x| \geq C, \quad k \leq m \]
that, e.g., $P_+K P_- f = O(e^{-cW})$, $P_\pm K(I_+ - P_+) f = O(e^{-cW})$, etc.

Note that by (2.17) $K_\pm$ also has a block diagonal structure:
\[ K_\pm = \left( mW \right)^{1/2} \sum_{j=0}^{(mW)^{1/2}} K_\pm^{(j)}, \quad K_\pm^{(j)} = \mathcal{P}_j P_\pm K P_\pm \mathcal{P}_j. \]  \hfill (3.13)

Here and below we denote by $P_j$ the projection on \{ $\Psi(a,b)\phi_j(U)$ \}.

Let us denote by $p$ and $q$ some absolute exponents which could be different in different formulas.

Chose the contour $\mathcal{L}$ as follows:
\[ \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2, \]  \hfill (3.14)
where
\[ \mathcal{L}_2 = \left\{ z : |z| = |\lambda_0(K)| \left( 1 - \frac{\log^2 W}{(a_+ - a_-)^2 W^2} \right) \right\}, \]  \hfill (3.15)
and
\[ \mathcal{L}_1 = L^0 \cup L^1, \]  \hfill (3.16)
\[ L^0 = \{ z : |z - \lambda_0(K)| = \frac{D^2}{(a_+ - a_-)^2 W^2} \}; \]
\[ L^1 = \bigcup_{j=D}^{J} L_j; \quad L_j = \{ z : |z - \lambda_{j,*} \cdot \lambda_0(K)| = \frac{\gamma}{W^2} \} \]
with
\[ l = \log W. \]  \hfill (3.17)

Here
\[ \lambda_{j,*} = 1 - \frac{j(j+1)}{W^2(a_+ - a_-)^2}, \]  \hfill (3.18)
\[ \gamma > 0 \text{ and } D > 0 \text{ are sufficiently large (but } \gamma < D/2(a_+ - a_-)^2). \] Notice that
\[ \text{dist}\{L^0, L^1\} \geq \frac{D}{3(a_+ - a_-)^2 W^2}, \]  \hfill (3.19)
\[ \text{dist}\{\mathcal{L}_1, \mathcal{L}_2\} \geq \frac{C \log W}{W^2}. \]  \hfill (3.20)

Denote also
\[ G_\xi^0(z) = (\mathcal{A}_m \otimes K_{\xi,l_\xi} - z)^{-1}, \]  \hfill (3.21)
where $\mathcal{A}_m$, $K_{\xi,l_\xi}$ are defined in (2.21) and (2.20).

We start with the following theorem
Theorem 3.1 For the operators $K$ defined in (2.4) we have

(i) For $z$ outside of the contour $L$ of (3.14) we have $\| (K - z)^{-1} \| \leq CW^2$;

(ii) Given $z$ such that

$$|z - \lambda_{j,*} \cdot |\lambda_0(K)|| \geq \frac{\gamma}{W^2}, \quad |z| \geq |\lambda_0(K)| \left(1 - \frac{\log^2 W}{(a_+ - a_-)^2 W^2}\right)$$

with sufficiently big $\gamma > 0$, consider $G^{(j)}(z) = (K^{(j)}_\pm - z)^{-1}$. Then

$$\| G^{(j)} \| \leq C_1 W^2 / \gamma$$

(3.22)

with some absolute constant $C_1$ which does not depend on $\gamma$.

In addition, for any $z$ such that $|\lambda_0(K)| \left(1 - \frac{\log^2 W}{(a_+ - a_-)^2 W^2}\right) \leq |z| \leq 1 + C_2 / n$

$$\| (K_+ - z)^{-1} \| \leq CW, \quad \| (K_- - z)^{-1} \| \leq CW, \quad \| (K^{(22)} - z)^{-1} \| \leq CW/m^{1/3}. \quad (3.23)$$

(iii) We have

$$\| K^{(21)} \| \leq C m^{3/2} / W^{3/2}, \quad \| K^{(12)} \| \leq C m / W, \quad (3.24)$$

and for $z$ outside of $L$ we also have

$$\| (K^{(11)} - z)^{-1} K^{(12)} \| \leq C m^p, \quad \| K^{(21)} (K^{(11)} - z)^{-1} \| \leq C m^p. \quad (3.25)$$

Same statements are valid for $K_\xi$ of (2.5). In addition, given (3.21),

$$|G^0(z)| \leq CW^2$$

(3.26)

for $z$ outside of the contour $L$.

Proof of Theorem 3.1. The proof of the theorem for $K$ and (3.26) follows from Lemmas 4.1 – 4.3 and Proposition 4.1 of [8].

To obtain the result for $K_\xi$ set

$$G_{1,\xi} = (K^{(11)}_\xi - z)^{-1} = (K^{(11)}_\xi + \bar{K}^{(11)}_\xi - z)^{-1}, \quad (3.27)$$

$$G_{2,\xi} = (K^{(22)}_\xi - z)^{-1} = (K^{(22)}_\xi + \bar{K}^{(22)}_\xi - z)^{-1}.$$ 

Now using Schur’s formula we get

$$(K_\xi - z)^{-1} = \begin{pmatrix} G^{(11)}_\xi & -G^{(11)}_\xi K^{(12)}_\xi G_{2,\xi} \\ -G_{2,\xi} K^{(21)}_\xi G^{(11)}_\xi & G_{2,\xi} + G_{2,\xi} K^{(21)}_\xi G^{(11)}_\xi K^{(12)}_\xi G_{2,\xi} \end{pmatrix},$$

(3.28)

where

$$G^{(11)}_\xi = (K^{(11)}_\xi - z - K^{(12)}_\xi G_{2,\xi} K^{(21)}_\xi)^{-1} = (1 - G_{1,\xi} K^{(12)}_\xi G_{2,\xi} K^{(21)}_\xi)^{-1} G_{1,\xi}.$$
Denoting
\[ R = (1 - G_{1,\xi} K_{\xi}^{(12)} G_{2,\xi} K_{\xi}^{(21)})^{-1}, \]  
we get
\[ G_{\xi}^{(11)} = R G_{1,\xi}. \]  
Notice that
\[ \| G_{1,\xi} K_{\xi}^{(12)} \| = \|(K^{(11)} - z + \tilde{K}_{\xi}^{(11)})^{-1} (K^{(12)} + \tilde{K}_{\xi}^{(12)})\| \]  
\[ = \| (1 + (K^{(11)} - z)^{-1} \tilde{K}_{\xi}^{(11)})^{-1} (K^{(11)} - z)^{-1} (K^{(12)} + \tilde{K}_{\xi}^{(12)}) \|. \]  
Moreover, part (ii) of the Theorem for operator $K$ yield
\[ \| (K^{(11)} - z)^{-1} \| \leq \frac{C_1 n}{\gamma}, \]  
where $\gamma$ is sufficiently big and $C_1$ does not depend on $\gamma$. Hence
\[ \| (K^{(11)} - z)^{-1} \tilde{K}_{\xi}^{(11)} \| \leq C < 1. \]  
Thus according to (3.31), (3.25) for $K$, and (2.14)
\[ \| G_{1,\xi} K_{\xi}^{(12)} \| \leq C \| (K^{(11)} - z)^{-1} (K^{(12)} + \tilde{K}_{\xi}^{(12)}) \| \]  
\[ \leq C \| (K^{(11)} - z)^{-1} K^{(12)} \| + \| (K^{(11)} - z)^{-1} \tilde{K}_{\xi}^{(12)} \| \]  
\[ \leq C (\log^p W + C_1) \leq C \log^p W. \]  
Similarly
\[ \| K_{\xi}^{(21)} G_{1,\xi} \| \leq C \| (K^{(21)} + \tilde{K}_{\xi}^{(21)}) (K^{(11)} - z)^{-1} \| \leq C \log^p W. \]  
The bound (3.24) for $K_{\xi}$ trivially follow from (3.24) for operator $K$ and (2.14), which finishes the proof of (iii) for $K_{\xi}$.  
In addition, due to the last bound of (3.23) for operator $K$ and (2.14) we have
\[ \| G_{2,\xi} \| = \| (K^{(22)} + \tilde{K}_{\xi}^{(22)} - z)^{-1} \| \]  
\[ = \| (1 + (K^{(22)} - z)^{-1} \tilde{K}_{\xi}^{(22)})^{-1} (K^{(22)} - z)^{-1} \| \leq C W / m^{1/3} \]  
which gives the last bound of (3.23) for operator $K_{\xi}$. This implies
\[ \| G_{2,\xi} K_{\xi}^{(21)} \| \leq \frac{\log^p W}{W^{1/2}}. \]  
Thus
\[ \| G_{1,\xi} K_{\xi}^{(12)} G_{2,\xi} K_{\xi}^{(21)} \| \leq \| G_{1,\xi} K_{\xi}^{(12)} \| \cdot \| G_{2,\xi} K_{\xi}^{(21)} \| \leq \frac{C \log^p W}{W^{1/2}}, \]  
and so
\[ \| R \| \leq C. \]  
This, (3.29) – (3.30), and (3.32) yield
\[ \| G_{\xi}^{(11)} \| \leq C n. \]
Similarly (3.30) gives
\[ \|G^{(11)}_{\xi}K^{(12)}_{\xi}\| = \|RG_{1,\xi}K^{(12)}_{\xi}\| \leq \|R\| \cdot \|G_{1,\xi}K^{(12)}_{\xi}\| \leq C \log^p W, \]
which implies
\[ \|G^{(11)}_{\xi}K^{(12)}_{\xi}G_{2,\xi}\| \leq C \log^p W \cdot W. \quad (3.37) \]
It is easy to see that
\[ D^{-1}C(A - BD^{-1}C)^{-1} = (D - CA^{-1}B)^{-1}CA^{-1}, \]
thus
\[ G_{2,\xi}K^{(21)}_{\xi}G^{(11)}_{\xi} = (K^{(22)}_{\xi} - z - K^{(21)}_{\xi}G_{1,\xi}K^{(12)}_{\xi})^{-1}K^{(21)}_{\xi}G_{1,\xi}, \]
But
\[ \|G_{2,\xi}K^{(21)}_{\xi}G_{1,\xi}K^{(12)}_{\xi}\| \leq \|G_{2,\xi}K^{(21)}_{\xi}\| \cdot \|G_{1,\xi}K^{(12)}_{\xi}\| \leq \frac{C \log^p W}{W^{1/2}}, \]
hence using (3.25) for \( K_{\xi} \) we obtain
\[ \|G_{2,\xi}K^{(21)}_{\xi}G^{(11)}_{\xi}\| \leq C \|G_{2,\xi}\| \cdot \|K^{(21)}_{\xi}\| \cdot \|G^{(11)}_{1,\xi}\| \leq C \log^p W \cdot W. \quad (3.38) \]
We also can write
\[ \|G_{2,\xi}K^{(21)}_{\xi}G^{(11)}_{\xi}G_{2,\xi}\| \leq \|G_{2,\xi}\|^2 \cdot \|K^{(21)}_{\xi}\| \cdot \|G^{(11)}_{\xi}\| \leq C \log^p W \cdot W^{1/2} \quad (3.39) \]
which finishes the proof of (i) for \( K_{\xi} \).
Bounds (3.22) – (3.23) for \( K_{\xi} \) can be obtained easily from those for \( K \) and from (2.14).
\( \square \)

4 Proof of Theorem 1.2

The key step in the proof of Theorem 1.2 is the following theorem.

Theorem 4.1 Given \( G_{\xi}(z) = (K_{\xi} - z)^{-1} \) with \( K_{\xi} \) of (2.5), \( f_{\xi} \) of (2.1), and the contour \( \mathcal{L} \)
defined in (3.14) – (3.17), we can write for the integral in (2.7)
\[ \int_{\mathcal{L}} z^{n-1}(G_{\xi}(z)f_{\xi}, \tilde{f}_{\xi})dz \]
\[ = \int_{\mathcal{L}} z^{n-1}(G_{\xi}^{0}(z)(f_{1,\pm} \otimes 1), (\tilde{f}_{1,\pm} \otimes 1))dz + |\lambda_{0}(K)|^{n-1} \cdot \|f_{1}\|^2 \cdot O\left(\frac{1}{\log W}\right), \quad (4.1) \]
where
\[ f_{1} = Pf, \quad (4.2) \]
where \( P \) is the orthogonal projector to the the space \( \mathcal{H}_{1} \) (see (3.11)), and \( G_{\xi}^{0} \) is defined in (3.21). Here \( f_{1,\pm} \) is a projection of \( f \) on the linear span of \( \{\Psi_{k,0}(a,b), |k| \leq m\} \) of (3.9).

The contour \( \mathcal{L} \) encircles all eigenvalues of \( \mathcal{A}_{m} \otimes K_{*,\xi,l} \) defined in (2.21) and (2.20), and
\[ (\mathcal{A}_{m}^{-1}f_{1,\pm}, f_{1,\pm}) = |\lambda_{0}(K)|^{n-1} \cdot \|f_{1}\|^2 \cdot (1 + o(1)). \quad (4.3) \]
Let us assume that Theorem 4.1 is proved and derive the assertion of Theorem 1.2.

Indeed, since $L$ encircles all eigenvalues of $A_m \otimes K_{s\xi,l}$, according to the Cauchy theorem we get

$$-\frac{1}{2\pi i} \int_L z^{-n-1}(G^0_{\xi}(z)(f_1, \pm \otimes 1), (\tilde{f}_1, \pm \otimes 1))dz = ((A_m \otimes K_{s\xi,l})^{n-1}(f_1, \pm \otimes 1), (\tilde{f}_1, \pm \otimes 1))$$

$$= (A_m^{n-1}f_1, \pm, \tilde{f}_1, \pm) \cdot (K_{s\xi,l}^{n-1}1,1).$$

Now

$$K_{s\xi,l} = K_{s0,l} - \frac{i\pi}{n} \nu + O(n^{-2}),$$

where $K_{s0,l}$ is a diagonal (in basis $\{\phi_j\}_{j \leq l}$ of (2.15)) operator with eigenvalues $\{\lambda_{j,*}\}_{j \leq l}$ of (3.18). Since the Laplace operator $\Delta_U$ on $U(2)$ has the same eigenfunctions as $K_{s0}$ with eigenvalues

$$\lambda_j^* = j(j+1),$$

we get for $n = C_\ast W^2$

$$K_{s\xi,l} \sim 1 - n^{-1}(C^* \Delta_U + i\xi \pi \nu) + O(n^{-2}) \Rightarrow (K_{s\xi,l}^{n-1}1,1) \rightarrow (e^{-C^* \Delta_U - i\xi \hat{\nu}}1,1), \quad (4.4)$$

where $C^* = C_s/\ell^*$ as in Theorem 1.2.

This and (4.3) imply that

$$-\frac{1}{2\pi i} \int_L z^{-n-1}(G^0_{\xi}(z)(f_1, \pm \otimes 1), (\tilde{f}_1, \pm \otimes 1))dz$$

is of order

$$|\lambda_0(K)|^{n-1} \cdot \|f_1, \pm\|^2,$$

and so (4.1) can be rewritten as

$$-\frac{1}{2\pi i} \int_L z^{-n-1}(G_{\xi}(z)f_{\xi}, \tilde{f}_{\xi})dz = (A_m^{n-1}f_{1, \pm}, \tilde{f}_{1, \pm}) \cdot (K_{s\xi,l}^{n-1}1,1)(1 + o(1)), \quad n \rightarrow \infty.$$ 

This, a similar relation with $\xi = 0$, (2.6), and (2.7), yield

$$D_2^{-1}F_2\left(E + \frac{\xi}{2n\rho(E)}, \frac{\xi}{2n\rho(E)} \right)$$

$$= (A_m^{n-1}f_{1, \pm}, \tilde{f}_{1, \pm}) \cdot (K_{s\xi,l}^{n-1}1,1)$$

$$= (A_m^{n-1}f_{1, \pm}, \tilde{f}_{1, \pm}) \cdot (K_{s0,l}^{n-1}1,1)(1 + o(1)) = (K_{s\xi,l}^{n-1}1,1)(1 + o(1)).$$

Here we used (2.22). This relation and (4.4) complete the proof of Theorem 1.2.
4.1 Proof of Theorem 4.1

We are left to prove Theorem 4.1.

First we decompose \( f = (f_1, f_2) \) with respect to decomposition (3.10). Observe that since

\[ |F(X)| \leq 1, \]

and \( F(X) \) exponentially decreases at \( \infty \) (in eigenvalues \( a, b \)), we have \( \|f\| = \text{const} \leq 1. \) Moreover it is easy to see that

\[ \|f_1\|_2^2 \geq \|f_1, \pm\|_2^2 \geq \|F(X)\bar{\Psi}_{0,0}\|_2^2 = O\left(\frac{2}{W}\right), \]

with \( \bar{\Psi}_{0,0} \) of (3.9). Therefore

\[ \|f_1\|_2 \geq C/W. \]

We start with the following simple lemma

**Lemma 4.1** The main contribution to the integral in (2.7) is given by the integral over the contour \( \mathcal{L}_1 \) of (3.16), i.e.

\[
\int_{\mathcal{L}_2} z^{n-1}(G_\xi(z)f_\xi, \bar{f}_\xi)dz = \int_{\mathcal{L}_2} z^{n-1}(G_\xi(z)f_1, \bar{f}_1)dz + |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\left(\frac{\log W}{W}\right),
\]

where \( f \) is defined in (2.1). In addition,

\[
\int_{\mathcal{L}_2} z^{n-1}(G_\xi^0(z)(f_1, \pm 1), (\bar{f}_1, \pm 1))dz = |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot o\left(e^{-C\log^2 W}\right),
\]

where \( \mathcal{L}_2 \) is defined in (3.15), and \( G_\xi^0(z) \) is defined in (3.21).

**Proof of Lemma 4.1** Since for \( z \in \mathcal{L}_2 \) we have

\[ |z|^{n-1} \leq |\lambda_0(K)|^{n-1} \cdot e^{-C\log^2 W}, \]

we get using \( \|G_\xi(z)\| \leq CW^2 \) (see part (i) of Theorem 3.1 for \( K_\xi \)) that

\[
\int_{\mathcal{L}_2} z^{n-1}(G_\xi(z)f_\xi, \bar{f}_\xi)dz \leq C_1|\lambda_0(K)|^{n-1} \cdot e^{-C_2\log^2 W} \cdot W^2
\]

\[ = |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot o\left(e^{-C\log^2 W}\right). \]

Here we used (4.5). Similarly one can obtain (4.6) from (3.26).

Besides,

\[ |\mathcal{L}_1| \leq C\log W/W^2, \]

and for \( z \in \mathcal{L}_1 \)

\[ |z|^{n-1} \leq C|\lambda_0(K)|^{n-1}. \]

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Thus, since $\|f - f_\xi\| \leq C/n$, we get according to (4.5)

$$\left| \int_{L_1} z^{n-1}(G_\xi(z)(f_\xi - f), \bar{f}_\xi)dz \right| \leq C|\lambda_0(K)|^{n-1} \cdot W^2 \cdot \|f - f_\xi\| \cdot |L_1|$$

$$\leq |\lambda_0(K)|^{n-1} \cdot \frac{\log W}{W^2} \leq |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\left(\frac{\log W}{W}\right),$$

which gives the lemma.

\[\Box\]

Lemma 4.1 yields that we can prove (4.1) for $L_1$ instead of $L_1$.

The next step is to prove that we can consider only the upper-left block $K^{(11)}_\xi$ of $K_\xi$ (see (3.12)). More precisely, we are going to prove

**Lemma 4.2** Given (3.27) and (4.2), we have

$$\int_{L_1} z^{n-1}(G_\xi(z)f, \bar{f})dz = \int_{L_1} z^{n-1}(G_\xi(z)f_1, \bar{f}_1)dz + |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\left(\frac{\log^p W}{W^{1/2}}\right),$$

**Proof of Lemma 4.2** According to (3.28) we have

$$\int_{L_1} z^{n-1}((K_\xi - z)^{-1}f, \bar{f})dz = \int_{L_1} z^{n-1}(G^{(11)}_\xi f_1, \bar{f}_1)dz - \int_{L_1} z^{n-1}(G^{(11)}_\xi K^{(12)}_\xi G_{2,\xi}f_2, \bar{f}_1)dz$$

$$- \int_{L_1} z^{n-1}(G_{2,\xi}K^{(21)}_\xi G^{(11)}_\xi f_1, \bar{f}_2)dz + \int_{L_1} z^{n-1}((G_{2,\xi} + G_{2,\xi}K^{(21)}_\xi G^{(11)}_\xi K^{(12)}_\xi G_{2,\xi})f_2, \bar{f}_2)dz$$

Thus, we get using (3.37) – (3.38), (4.7) – (4.8), $\|f_2\| \leq C$, and (4.5)

$$\left| \int_{L_1} z^{n-1}(G^{(11)}_\xi K^{(12)}_\xi G_{2,\xi}f_2, \bar{f}_1)dz \right| \leq \|G^{(11)}_\xi K^{(12)}_\xi G_{2,\xi}\| \cdot \|f_1\| \cdot \|f_2\| \cdot \int_{L_1} |z|^{n-1}|dz|$$

$$\leq \frac{C \log^p W \cdot W}{W^2} \cdot |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\left(\frac{\log^p W}{W^{1/2}}\right) \cdot |\lambda_0(K)|^{n-1},$$

$$\left| \int_{L_1} z^{n-1}(G_{2,\xi}K^{(21)}_\xi G^{(11)}_\xi f_1, \bar{f}_2)dz \right| \leq \|G_{2,\xi}K^{(21)}_\xi G^{(11)}_\xi\| \cdot \|f_1\| \cdot \|f_2\| \cdot \int_{L_1} |z|^{n-1}|dz|$$

$$\leq \frac{C \log^p W \cdot W}{W^2} \cdot |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\left(\frac{\log^p W}{W^{1/2}}\right) \cdot |\lambda_0(K)|^{n-1}.$$
Thus (3.39) and (4.5) yield
\[
\left| \int_{\mathcal{L}_1} z^{n-1}(G_{2,\xi}K^{(1)}_{\xi}G^{(1)}_{\xi}K^{(12)}_{\xi} G_{2,\xi} f_2; \bar{f}_2)dz \right|
\leq \|G_{2,\xi}K^{(1)}_{\xi}G^{(1)}_{\xi}K^{(12)}_{\xi} G_{2,\xi}\| \cdot \|f_2\|^2 \cdot \int_{\mathcal{L}_1} |z|^{n-1}|dz|
\leq \frac{C \log W}{W^{1/2}} \cdot |\lambda_0(K)|^{n-1} \leq O\left(\frac{C \log W}{W^{1/2}}\right) \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1}.
\]

Besides, according to (3.30) and (3.35)
\[
\left| \int_{\mathcal{L}_1} z^{n-1}((G^{(1)}_{\xi} - G_{1,\xi})f_1, \bar{f}_1)dz \right| \leq \|R\| \cdot \|G_{1,\xi}\| \cdot \|f_1\|^2 \cdot \int_{\mathcal{L}_1} |z|^{n-1}|dz|
\leq \frac{C \log^p W}{W^{1/2}} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1} = O\left(\frac{C \log W}{W^{1/2}}\right) \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1}.
\]
These bounds imply Lemma 4.2 \(\square\)

Now write \(K^{(1)}_{\xi} - z, K^{(1)}_{\xi} - z\) in the block form
\[
K^{(1)}_{\xi} - z = \begin{pmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{pmatrix}, \quad K^{(1)}_{\xi} - z = \begin{pmatrix} M_{1,\xi} & M_{12,\xi} \\ M_{21,\xi} & M_{2,\xi} \end{pmatrix}
\]
according to decomposition
\[
\mathcal{H}_1 = \mathcal{M}_1 \oplus \mathcal{M}_2,
\]
where \(\mathcal{M}_1\) is a linear span of \(\{\Psi_{j,k,\delta}, j \leq \log W, |k| \leq m\}\) (see (3.8)). Then (see (3.12), (3.13))
\[
M_1 = \sum_{j=0}^{\log W} K^{(j)}_\pm, \quad K^{(j)}_\pm = \mathcal{P}_j P_\pm K P_\pm \mathcal{P}_j,
\]
\[
M_2 = K_+ + K_- + \sum_{j=0}^{(mW)^{1/2}} K^{(j)}_\pm,
\]
\[
M_{12} = O(e^{-cW}), \quad M_{21} = O(e^{-cW}),
\]
where \(\mathcal{P}_j\) is the projection on \(\{\Psi_{\xi}(a,b)\mathcal{P}_j(U)\}\).

Set
\[
G_{1,t,\xi}(z) = (K_{m,t,\xi} - z)^{-1} = (M_{1,\xi})^{-1},
\]
where \(K_{m,t,\xi}\) is defined in (2.18). Notice also that, since \(f_1\) does not depend on \(\{U_j\}\), the part of \(f_1\) corresponding to \(\mathcal{M}_1\) is \(f_{1,\pm} \otimes 1\).

The next step is to show

Lemma 4.3 The operator \(K^{(1)}_{\xi}\) of (3.12) can be replaced by \(K_{m,t,\xi}\) of (2.18), i.e. we can write
\[
\int_{\mathcal{L}_1} z^{n-1}(G_{1,\xi}(z) f_1, \bar{f}_1)dz
= \int_{\mathcal{L}_1} z^{n-1}(G_{1,\xi}(z) (f_{1,\pm} \otimes 1), (\bar{f}_{1,\pm} \otimes 1))dz + |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\left(\frac{1}{\log W}\right).
\]
Proof of Lemma 4.4. Denote
\[ D_\xi = M_{1,\xi} - M_{12,\xi} M_{2,\xi}^{-1} M_{21,\xi}, \quad D_{0,\xi} = 1 - M_{12,\xi} M_{2,\xi}^{-1} M_{21,\xi} M_{1,\xi}^{-1} \]
and write \( f_1 = (f_\pm \otimes 1, f_{12}) \) according to the decomposition (4.9).
Using Schur’s formula we get
\[ G_{1,\xi} = \left( \begin{array}{cc} D_\xi^{-1} & -D_\xi^{-1} M_{12,\xi} M_{1,\xi}^{-1} \\ -M_{2,\xi}^{-1} M_{21,\xi} D_\xi^{-1} & M_{2,\xi}^{-1} + M_{2,\xi}^{-1} M_{21,\xi} D_\xi^{-1} M_{12,\xi} M_{2,\xi}^{-1} \end{array} \right) \] (4.12)
Notice that according to (ii) of Theorem 3.1 \( M_{2,\xi}^{-1} \) is analytic inside of \( L_1 \), and so
\[ \int_{L_1} z^{n-1} (M_{2,\xi}^{-1} f_{12}, \bar{f}_{12}) dz = 0, \]
thus
\[ \int_{L_1} z^{n-1} (G_{1,\xi}(z)f_1, \bar{f}_1) dz = \int_{L_1} z^{n-1} (D_\xi^{-1} f_1, (\bar{f}_1 \pm \otimes 1)) dz \]
\[ - \int_{L_1} z^{n-1} (D_\xi^{-1} M_{12,\xi} M_{2,\xi}^{-1} f_{12}, (\bar{f}_{1,\pm} \otimes 1)) dz - \int_{L_1} z^{n-1} (M_{2,\xi}^{-1} M_{21,\xi} D_\xi^{-1} (f_{1,\pm} \otimes 1), \bar{f}_{12}) dz \]
\[ + \int_{L_1} z^{n-1} (M_{2,\xi}^{-1} M_{21,\xi} D_\xi^{-1} M_{2,\xi}^{-1} f_{12}, \bar{f}_{12}) dz \] (4.13)
Let \( z \in L_1 \). Then using (3.13) and (3.20) we can write (recall that \( \log W \sim \log n \))
\[ \|M_2^{-1}\| \leq C n / \log n. \]
In addition,
\[ \|K^{(11)}_\xi - K^{(11)}\| \leq C / n, \]
\[ \|M_2^{-1}_\xi\| = \|M_2^{-1} (1 + (M_2 \xi - M_2) M_2^{-1})^{-1}\| \leq \frac{C_1 n}{\log n} \cdot (1 - \frac{C_2}{\log n})^{-1} \leq C n / \log n, \]
\[ \|M_{12,\xi}\| \leq C / n, \quad \|M_{21,\xi}\| \leq C / n. \] (4.14)
Here we used (2.14). Part (ii) of Theorem 3.1 also gives (recall \( n = C_* W^2 \))
\[ \|M_{1,\xi}^{-1}\| \leq C n. \] (4.15)
In addition, using the resolvent identity we obtain
\[ D_\xi^{-1} - M_{1,\xi}^{-1} = M_{1,\xi}^{-1} M_{12,\xi} M_{2,\xi}^{-1} M_{21,\xi} M_{1,\xi}^{-1} D_{0,\xi}^{-1}. \] (4.16)
According to (4.14) – (4.15) we get
\[ \|M_{12,\xi} M_{2,\xi}^{-1} M_{21,\xi} M_{1,\xi}^{-1}\| \leq C / \log n, \]
thus
\[ \|D_{0,\xi}^{-1}\| \leq C. \] (4.17)
In view of (4.16)

\[ \|D_{\xi}^{-1} - M_{1,\xi}^{-1}\| \leq \frac{Cn}{\log n}. \]

Therefore, since according to (3.18), we have for \( z \in L_j \) of (3.16)

\[ |z|^{n-1} \leq C_1|\lambda_0(K)|^{n-1} \cdot e^{-C_2j(j+1)}, \]

and \( |L_j| = 2\pi\gamma/W^2 \), we get

\[
\left| \int_{\mathcal{L}_1} z^{-n+1} \left((D_{\xi}^{-1} - M_{1,\xi}^{-1})(f_1,\pm 1), (f_1,\pm 1)dz \right) \right| \\
\leq \frac{Cn}{\log n} \cdot \|f_1,\pm\| \cdot |\lambda_0(K)|^{n-1} \cdot \sum_{j=1}^{l} |L_j| \cdot e^{-C_2j(j+1)} \\
\leq \frac{C}{\log n} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1} \cdot e^{-C_2(j+1)} \leq \frac{C}{\log n} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1}.
\]

Now consider another integrals in (4.13). Using \( D_{\xi} = D_{0,\xi}^{-1} M_{1,\xi}^{-1} \), we obtain similarly

\[
\left| \int_{\mathcal{L}_1} z^{-n+1}(D_{\xi}^{-1} M_{12,\xi}^{-1} M_{2,\xi}^{-1} f_{12}, (f_1,\pm 1)dz) \right| \\
\leq \frac{Cn}{\log n} \cdot \|f_1,\pm\| \cdot \|f_{12}\| \cdot |\lambda_0(K)|^{n-1} \cdot \sum_{j=1}^{l} |L_j| \cdot e^{-C_2j(j+1)} \\
\leq \frac{C}{\log n} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1},
\]

and by the same argument

\[
\left| \int_{\mathcal{L}_1} z^{-n+1}(M_{2,\xi}^{-1} M_{21,\xi}^{-1} D_{\xi}^{-1} f_1, \bar{f}_{12})dz \right| \leq \frac{C}{\log n} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1},
\]

\[
\left| \int_{\mathcal{L}_1} z^{-n+1}(M_{2,\xi}^{-1} M_{21,\xi}^{-1} D_{\xi}^{-1} M_{12,\xi}^{-1} M_{2,\xi}^{-1} f_{12})dz \right| \leq \frac{C}{\log n} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1}.
\]

This implies the lemma.

\[ \square \]

Now we have the integral

\[ \int_{\mathcal{L}_1} z^{-n+1}(G_{1,t}(z)(f_1,\pm 1), (f_1,\pm 1))dz. \]

The last step is to show

**Lemma 4.4** The operator \( K_{m,t,\xi} \) of (2.18) can be replaced by \( A_{m} \otimes K_{t,\xi} \) (see (2.20) – (2.21)), i.e. we have

\[
\int_{\mathcal{L}_1} z^{-n+1}(G_{1,t}(z)(f_1,\pm 1), (f_1,\pm 1))dz \\
= \int_{\mathcal{L}_1} z^{-n+1}(G_{t,\xi}^0(z)(f_1,\pm 1), (f_1,\pm 1))dz + |\lambda_0(K)|^{n-1} \cdot \|f_1\|^2 \cdot O\left(\frac{\log^p W}{W^{1/2}}\right),
\]

where \( G_{t,\xi}^0 \) is defined in (3.21).
Proof of Lemma 4.4. Using the resolvent identity we can write
\[ G_{1,l}(z) - G_{\xi}^0(z) = -G_{\xi}^0(z)(M_{1,\xi} - A_m \otimes K_{s,\xi,l})G_{1,l}(z) \]
Since for (3.5)
\[ \psi_k^0(x) = O(e^{-c \log^2 W}), \quad |x| \geq 2W^{-1/2} \log W, k \leq m, \]
we get that both \( K_{m,l,\xi}, A_m \otimes K_{s,\xi,l} \) are concentrated in the \( \log W/2 \)-neighbourhoods of \( a_\pm \) (see [8], for details). In this neighbourhood
\[ a_1 - b_1 = a_+ - a_- + O\left(\frac{\log W}{W^{1/2}}\right), \quad a_2 - b_2 = a_+ - a_- + O\left(\frac{\log W}{W^{1/2}}\right), \]
\[ t = (a_+ - a_-)^2 + O\left(\frac{\log W}{W^{1/2}}\right) = t_0 + O\left(\frac{\log W}{W^{1/2}}\right). \]
Thus according to (2.16)
\[ \|K_{m,l,0} - A_m \otimes K_{s,0,l}\| \leq \frac{C \log W}{W^{5/2}}, \]
where \( K_{m,l,0}, A_m \otimes K_{s,0,l} \) are \( K_{m,l,\xi}, A_m \otimes K_{s,\xi,l} \) with \( \xi = 0 \). In addition, in this neighbourhood
\[ \|\tilde{K}_\xi(X,Y) - \tilde{K}_\xi(X,Y)\|_{x = y = x_\pm} \leq \frac{C \log W}{n \sqrt{W}}. \]
Hence, since \( n \sim W^2 \), we get
\[ \|K_{m,l,\xi} - A_m \otimes K_{s,\xi,l}\| \leq \frac{C \log W}{W^{5/2}}, \]
and so
\[ \left| \int_{L_1} z^{n-1} \left( (G_{1,l}(z)(f_1, \pm \otimes 1), (\bar{f}_1, \pm \otimes 1)) - (G_{\xi}^0(z)(f_1, \pm \otimes 1), (\bar{f}_1, \pm \otimes 1)) \right) dz \right| \]
\[ \leq C|L_1| \cdot \frac{C W^4 \cdot \log^p W}{W^{5/2}} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1} \leq \frac{C \log^p W}{W^{1/2}} \cdot \|f_1\|^2 \cdot |\lambda_0(K)|^{n-1} \]
\[ \square \]
We are left to prove (4.3).
According to (2.21) and the choice of \( \Psi_k \) in (3.3) we have
\[ A_m = A_m^{(\pm)} \otimes A_m^{(-)} + O(e^{-c \log^2 W}), \]
where
\[ A_m^{(\pm)} = P_{\pm} A P_{\pm}, \]
where \( P_+ \) and \( P_- \) are the projections on the subspaces spanned on the systems \( \{\psi^+_k\}_{k=0}^m \) and \( \{\psi^-_{k,\xi}\}_{k=0}^m \) respectively (see (3.6)). The behaviour of \( A_m^{(\pm)} \) was studied in [8]. In particular, it was proved in Lemma 3.3, [8] that \( |\lambda_1(A_m^{(\pm)})| \leq |\lambda_0(A_m^{(\pm)})| \cdot (1 - c/W) \), and so for any \( g \)
\[ (A_m^{n-1} g, \bar{g}) = \lambda_0(A_m^{(\pm)})^{n-1} \cdot \lambda_0(A_m^{(\pm)})^{n-1} |(g, \Psi_{0,0})|^2 (1 + o(1)). \]
Since also \( \lambda_0(K) = \lambda_0(A_m^{(+)} + \lambda_0(A_m^{(-)}) + O(e^{-c\log^2 W}) \) (see [8], eq. (4.22)), we get
\[
(A_m^{-1} f_1, \pm \bar{f}_1, \pm) = \lambda_0(K)^{-1} \cdot |(f_1, \Psi_{0,0})|^2 (1 + o(1)),
\]
where we used that \((f_1, \pm \Psi_{0,0}) = (f_1, \Psi_{0,0})\).

According to the definition of \(\{\Psi_{k,\bar{k}}\}_{|\bar{k}| \leq m}\) it is also easy to see that
\[
\|f_1\|^2 = |(f_1, \Psi_{0,0})|^2 (1 + O(1/W)).
\]
Thus
\[
(A_m^{-1} f_1, \pm f_1) = \lambda_0(K)^{-1} \cdot \|f_1\|^2 (1 + o(1)),
\]
which completes the proof of Theorem 4.1.

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