SIMPLE AMPLITUDES FOR $\phi^3$ FEYNMAN LADDER GRAPHS

by

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Abstract

Recently, we proposed a new approach for calculating Feynman graphs amplitude using the Gaussian representation for propagators which was proven to be exact in the limit of graphs having an infinite number of loops. Regge behaviour was also found in a completely new way and the leading Regge trajectory calculated. Here we present symmetry arguments justifying the simple form used for the polynomials in the Feynman parameters $\bar{\alpha}_\ell$, where $\bar{\alpha}_\ell$ is the mean-value for these parameters, appearing in the amplitude for the ladder graphs. (Taking mean-values is equivalent to the Gaussian representation for propagators).

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Quite a long time ago, ladder graphs and the Bethe-Salpeter equations were used to derive Regge behaviour and Regge trajectories in the context of scalar $\phi^3$ field theory. Some years ago, a renewal of interest for these graphs was to be noted. In the same period of time, we derived a rigorous derivation of the Gaussian representation for propagators in superrenormalizable scalar field theories which is valid in the infinite number of loops limit. This Gaussian representation is equivalent to fixing the Feynman $\alpha$-parameter $\alpha_\ell$ of some propagator $\ell$ to a constant value $\bar{\alpha}_\ell$ which will be, in fact, the mean-value for this parameter.

Then, once an overall scale for all $\alpha_\ell$'s has been separated and integrated over one gets a particularly simple expression for any Feynman graph amplitude $G$, when $I$ the number of propagators and $L$ the number of loops tend to infinity, which reads (the coupling $\gamma$ is taken equal to -1) in the Euclidean region

$$F_G = (4\pi)^{-dL/2}h_0[P_G\{\bar{\alpha}\}]^{-d/2} \left[ Q_G(P,\{\bar{\alpha}\}) + m^2h_0 \right]^{-(I-dL/2)} \Gamma(I - dL/2)h_0^{I-1}/(I - 1)!$$

where $h_0 = \sum \bar{\alpha}_\ell$ is an arbitrary constant and $h_0^{I-1}/(I - 1)!$ is the phase-space volume of the $\alpha_\ell$'s. (This expression has been obtained by applying the mean-value theorem for the $I$ variables $\alpha_\ell$). $P_G(\{\alpha\})$ and $Q_G(P,\{\alpha\})$ are defined by

$$P_G(\{\alpha\}) = \sum_\mathcal{T} \prod_{\ell \notin \mathcal{T}} \alpha_\ell$$

$$Q_G(P,\{\alpha\}) = P_G^{-1}(\{\alpha\}) \sum_C s_C \prod_{\ell \in C} \alpha_\ell$$

where the sum $\sum_\mathcal{T}$ is the sum over all spanning trees of $G$ (a spanning tree of $G$ is incident with every vertex of $G$) and $\sum_C$ is the sum over all cuts $C$ of $G$ (a cut $C$ is the complement of a spanning tree $\mathcal{T}$ plus one propagator which cuts $\mathcal{T}$ into two disjoint parts). External momenta $P_v$ are used to define

$$s_C = \left( \sum_{v \in G_1} P_v \right)^2 = \left( \sum_{v \in G_2} P_v \right)^2$$
where $G_1$ and $G_2$ are two disjoint parts of $G$ (which are bound together by the propagator cutting $T$ used above to define $C$). In a recent letter we derived the Regge behaviour for the sum of infinite ladders and the corresponding leading Regge trajectory. But we needed the expressions of $P_G(\{\bar{\alpha}\})$ and $Q_G(P, \{\bar{\alpha}\})$ for ladder graphs. In fact, we displayed the very simple expressions

$$P_G(\{\bar{\alpha}\}) = (\bar{\alpha}_-)^L \exp(yL) f(y) \quad (4a)$$

$$Q_G(P, \{\bar{\alpha}\}) = (t/2) L \bar{\alpha}_+ + s \bar{\alpha}_- \exp(-yL)[f(y)]^{-1} \quad (4b)$$

with

$$f(g) = \frac{1}{2} y (1 + y^{-1})^2 \quad (5a)$$

$$y = (2\bar{\alpha}_+ / \bar{\alpha}_-)^{1/2} \quad (5b)$$

$\bar{\alpha}_+$ being the mean-value of the $\alpha_\ell$’s for the propagators parallel to the ladder and $\bar{\alpha}_-$ being the mean-value of the $\alpha_\ell$’s for the central propagators.

It will be the purpose of the present letter to derive the expressions (4a) and (4b) for $P_G(\{\bar{\alpha}\})$ and $Q_G(P, \{\bar{\alpha}\}$ as well as expressions for some useful parameters. To understand how $P_G(\{\bar{\alpha}\})$ is obtained let us look first at fig. 1a) where a ladder with $L$ loops is displayed. In order to obtain a tree $T$ from such a ladder we have to delete $L$ propagators whose $\alpha_\ell$’s will be making one product in (2a). There are two different ways to delete propagators, either in the center of the ladder bringing up a factor $\bar{\alpha}_-$ or on the sides bringing up a factor $\bar{\alpha}_+$. If we consider the removing of $p$ propagators in the center, then $(L - p)$ will be removed on the sides. Each time we remove a propagator in the center we are fusing two neighbour loops into what we call a “cell” which is built by propagators surrounding it, making it a “closed cell”. An open cell is built when one propagator on the border of the closed cell is removed.

The three different topologies shown in fig. 1 are a) the case where only closed cells are made removing center propagators, b) the case where one open cell is made removing
one end propagator c) the case where both end center propagators are removed. Then, each remaining closed cell has to be opened by removing one propagator on the sides. If the number of side-propagators of a closed cell is $2\ell_i$ (for a “length” $\ell_i$ for this closed cell), then, there are $2^{\ell_i}$ ways of removing a side-propagator from a closed cell to make an open cell. Once only open cells remain there is no way to remove again propagators because we have already a tree $T$. The three topologies give rise to contributions to $P_G(\{\bar{\alpha}\})$ that are respectively

$$P_G^{(a)}(\{\bar{\alpha}\}) = \sum_{p=0}^{L-1} (\bar{\alpha}_-)p(\bar{\alpha}_+)^{L-p} \sum_{\ell_1+\cdots+\ell_{L-p}=L} 2^{L-p} \ell_1 \cdots \ell_{L-p}$$

$$P_G^{(b)}(\{\bar{\alpha}\}) = 2 \sum_{p=1}^{L} (\bar{\alpha}_-)p(\alpha_+)^{L-p} \sum_{\ell_1+\cdots+\ell_{L-p+1}=L} 2^{L-p} \ell_2 \cdots \ell_{L-p+1}$$

$$P_G^{(c)}(\{\bar{\alpha}\}) = \sum_{p=2}^{L} (\bar{\alpha}_-)p(\alpha_+)^{L-p} \sum_{\ell_1+\cdots+\ell_{L-p+2}=L} 2^{L-p} \ell_2 \cdots \ell_{L-p+1}$$

where $p$ center-propagators have been removed and $L-p$ side-propagators have also been removed. Remark that for symmetry reasons given below we have taken the same value $\bar{\alpha}_-$ for all mean-values of $\alpha_\ell$’s belonging to center propagators and also the same value $\bar{\alpha}_+$ for all mean-values of $\alpha_\ell$’s belonging to side propagators. We will see next that, indeed, $P_G(\{\bar{\alpha}\})$ is a symmetric function of all $\bar{\alpha}_-$’s and of all $\bar{\alpha}_+$’s. The same will also be true for the numerator of $Q_G(P,\{\bar{\alpha}\}) \sum_{C} s_C \prod_{\ell \in C} \alpha_\ell$, (see (2b)), where for the ladder, $s_C$ is either $t$, the momentum-transfer square invariant or $s$, the large invariant when looking for Regge behaviour (its logarithm is known in phenomenology, as the total “rapidity”).

Now, let us look at the sums in (6), not making any assumption about the mean-values

$$\sum_p 2^{L-p} \sum_{\{\ell_+\},\{\ell_-\}} \prod_{i_+ \in \{\ell_+\}} \bar{\alpha}_{i_+} \prod_{i_- \in \{\ell_-\}} \bar{\alpha}_{i_-} \prod_{\ell_k \in \{\ell_k\}} \ell_k$$

with the constraints $\sum_i \bar{\alpha}_i = h_0$, $\sum_k \ell_k = L$, $\{i_+\}$ containing $L-p$ elements and $\{i_-\}$ containing $p$ elements (or indices). It is obvious that any two $\bar{\alpha}_{i_+}$’s with $i_+ \in \{i_+\}$ can be
interchanged leaving (7) unchanged. The same is true for any two $\bar{\alpha}_{i_-}$’s with $i_- \in \{i_-\}$. The same is again true when interchanging two $\bar{\alpha}_{i_+}$’s with $i_+ \notin \{i_+\}$ or two $\bar{\alpha}_{i_-}$’s with $i_- \notin \{i_-\}$. The more delicate case arises interchanging two $\bar{\alpha}_+$’s with one $i_+ \in \{i_+\}$ and the other $i_+ \notin \{i_+\}$ or two $\bar{\alpha}_-$’s with one $i_- \in \{i_-\}$ and the other $i_- \notin \{i_-\}$.

Let us take first the (−) case. We can decompose $\{i_-\}$ in $\{i'_-\} \cup j_-$ with

$$\sum \prod_{\{i_-\}} \bar{\alpha}_{i_-} = \sum \prod_{j_- \notin \{i'_-\}} \bar{\alpha}_{j_-} \prod_{i'_- \in \{i'_-\}} \bar{\alpha}_{i'_-}. \tag{8}$$

So $\bar{\alpha}_{j_{1-}}$ with $j_{1-} \in \{i_-\}$, $\notin \{i'_-\}$ will be exchanged with $\bar{\alpha}_{j_{2-}}$ with $j_{2-} \notin \{i_-\}$, this for a specific $\{i_-\}$. For another $\{i_-\}$ the roles of $\bar{\alpha}_{j_{1-}}$ and $\bar{\alpha}_{j_{2-}}$ might be reversed. Then, in the sum over all ensembles $\{i_-\}$, we have an invariance under the interchange $\bar{\alpha}_{j_{1-}} \leftrightarrow \bar{\alpha}_{j_{2-}}$ (we remark that no $\{i'_-\}$ contains $j_{1-}$ nor $j_{2-}$ and that $\{i_-\}$ can be either $\{i'_-\} \cup j_{1-}$, or $\{i'_-\} \cup j_{2-}$, or $\{i'_-\} \cup j_{k-}$ with $j_{k-} \neq j_{1-}, j_{2-}$). The reasoning concerning the exchange $j_{1-} \leftrightarrow j_{2-}$ would be over if the factor $\prod_{i_+ \in \{i_+\}} \bar{\alpha}_{i_+} \prod_{\ell_k \in \{\ell_k\}} \ell_k$ in (7), multiplying (8), was invariant under this exchange. This is what we examine next. We displayed in fig. 2 the two configurations discussed above for the exchange $\bar{\alpha}_{j_{1-}} \leftrightarrow \bar{\alpha}_{j_{2-}}$.

The total length of the cells enclosing $j_{1-}$ is $L_{k_1}$ and the total length of the cells enclosing $j_{2-}$ is $L_{k_2}$. Because there is only one $\bar{\alpha}_{i_+}$ factor for each cell, we see that going from a) to b) (i.e. making the exchange $j_{1-} \leftrightarrow j_{2-}$) we have to suppress one $\bar{\alpha}_{i_+}$ in one of the cells enclosing $j_{1-}$ in a) and put an additional one in one of the cells enclosing $j_{2-}$ in b). This amounts to make an exchange $i_{1+} \leftrightarrow i_{2+}$ or $\bar{\alpha}_{i_{1+}} \leftrightarrow \bar{\alpha}_{i_{2+}}$. So, if we have symmetry under this exchange, we have symmetry under the exchange $\bar{\alpha}_{j_{1-}} \leftrightarrow \bar{\alpha}_{j_{2-}}$ provided the last factor $\prod_{\ell_k \in \{\ell_k\}} \ell_k$ is also symmetric. And, in fact, going from a) to b) we have to make an interchange in the order of $\ell_k$ factors under which $\prod_{\ell_k \in \{\ell_k\}} \ell_k$ is invariant and then, an exchange of the lengths $L_{k_1} \leftrightarrow L_{k_2}$ under which $\sum_{\{\ell_k\}} \prod_{k \in \{\ell_k\}} \ell_k$ is invariant because $L_{k_1}$ and $L_{k_2}$ are summed over (they are “dummy variables”). So, indeed, symmetry under interchange of the $\bar{\alpha}_{i_+}$’s entails symmetry under interchange of the $\bar{\alpha}_{i_-}$’s. Now, we will argue that all $\bar{\alpha}_{i_+}$’s should be given the same value.
We proved some years ago\(^7\) that all \(\bar{\alpha}_\ell\)'s can be written \(\bar{\alpha}_\ell = C_\ell(h_0/I)\) where \(C_\ell\) is some constant not depending on \(I\). The basic reason for this is given by the simple factoring of the total phase-space \(h_0^{I-1}/(I-1)!\) into \(I\) factors which gives for each \(\alpha_\ell\) a phase-space \(\simeq eh_0/I\).

The equations\(^7,8\) which determine \(C_\ell\) make it depend on \(t\) and \(s\) (or in general any invariant \(s_C\)) through \(Q_G(P,\{\bar{\alpha}\})\) which is a ratio of two polynomials of degree \(L+1\) (for \(\sum_C s_C \prod_{\ell \subset C} \alpha_\ell\)) and degree \(L\) (for \(P_G(\{\bar{\alpha}\})\)). Therefore \(Q_G(P,\{\bar{\alpha}\})\) is homogeneous to only one power of \(\bar{\alpha}_\ell\) and behaves a priori like \(O(h_0/I)\). However, in the special case of ladders the number of ways we can cut across any \(T\) in order to get \(s_C = t\) is in fact \(L\) (each open cell of length \(\ell_i\) can be cut \(\ell_i\) times) which is proportional to \(I\). (And, we have to multiply by \(1/2\) in order to avoid the double-counting of cuts). So we have \(Q_G(P,\{\bar{\alpha}\})\) proportional to \(t C_\ell(L/2)(h_0/I) \sim C_\ell t\). (The second term is negligible because the factor multiplying \(s\) is proportional to \((h_0/I)/\exp(C^{st}I)\), the exponential coming from the exponential number of spanning trees \(T\) in \(G\) as \(I \to \infty\). This is visible in the final expression (4b) for \(Q_G(P,\{\bar{\alpha}\})\). In the Regge limit where \(s \to \infty\), the saddle point one finds when summing over all \(L\) gives \(s \exp(-yL) = C^{st}\) or \(L \simeq \ell ns\) so that even in this case the \(\bar{\alpha}_-\) factor \(\sim (h_0/I)\) kills the term proportional to \(s\). So, we conclude that the \(C_\ell\)'s, in the case of ladders only depend on the invariant \(t\). Now, assuming that \(\bar{\alpha}_i\) varies along the ladder means that the corresponding constant, \(C_+\), should also depend on \(x_i = L_i/L\) if \(L_i\) is the number of propagators along one side of the ladder separating the propagator \(i_+\) from, for instance, an external line on the left of the ladder.

Now, let us isolate a sub-ladder \(G_1\) (of the ladder \(G\)) with length \(L_1\) (with \(L_1 \to \infty\)). We note that for \(G_1\), the invariant \(t\) is the same as for \(G\), because \(t\) is conserved along the ladder \(G\). So, \(C_+\), along \(G_1\) only depends on \(x_i = L_i/L_1\), if \(L_i/L_1\) is the distance of an \(i_+\) propagator to the left end of \(G_1\). And the function \(C_+(x_i,t)\) should be the same for
\( G \) and \( G_1 \), the equations determining it being the same. That is, we should have

\[
C_+(x_i, t) = C_+(x_{i_1}, t)
\]  

(9)

for any sub-ladder \( G_1 \), which entails that \( C_+ \) should not depend on \( x_i \), i.e. \( \bar{\alpha}_{i_+} \) should be the same along \( G \), and therefore \( \bar{\alpha}_{i_-} \) too according to the first part of our argument.

Of course, writing (9), we implied that the amplitude for \( G_1 \) factorized from the rest of \( G \). However, we proved\(^9\) that, precisely, the weighted sum of all spanning trees \( \mathcal{T} \) of a graph \( G \) containing an infinite number of loops could be factorized into independent factors for a connected sub-graph containing also an infinite number of loops and its complement graph on \( G \). \( G \) and \( G_1 \) having an infinite number of loops, and \( G_1 \) being connected, the factorization property holds in the present case. (In the more restraint domain of multiperipheral dynamics\(^3\) this factorization is also implied by the well-known short range correlation property along ladders).

A straightforward way of seeing the factorization appear in the case of ladders is to consider the amplitude for \( G_1 \), \( A_{G_1}(t, L) \), and to couple it through integration over the momenta of its external lines to its neighbour ladders \( G_0 \) and \( G_2 \). Because \( A_{G_1}(t, L) \) only depends on \( t \) and \( L \), integrating over the momenta in the loops at the ends of \( G_1 \), we see an exact factorization appear (\( g(t) \) is the coupling between two adjacent ladders)

\[
A_G(t, L) = A_{G_0}(t, L_0) g(t) A_{G_1}(t, L_1) g(t) A_{G_2}(t, L_2)
\]  

(10)

which justify its use in deriving (9). This happens because the dependence over the other invariant (\( s_1 \) for \( G_1 \)) only appears through the evaluation of the sum over all \( L_1 \)'s, giving a saddle-point\(^8\) of \( L_1 \simeq \ell n s_1 \) when \( s_1 \) is allowed to go to infinity.

For \( Q_G(P, \{\bar{\alpha}\}) \), the factor \( \sum_C s_C \prod_{i \in C} s_C \) in (2b) can be decomposed like \( P_G(\{\bar{\alpha}\}) \) into three distinct contributions corresponding to the three topologies of fig. 1. These are

\[
\tilde{Q}_G^{(a)}(P, \{\bar{\alpha}\}) = s \bar{\alpha}_{L+1} + (-2\bar{\alpha}_- m^2 + (t/2)L\bar{\alpha}_+) P_G^{(a)}(\{\bar{\alpha}\})
\]  

(11a)
\( \widetilde{Q}^{(b)}_G(P, \{\bar{\alpha}\}) = -\bar{\alpha}_- m^2 P^{(b)}_G(\{\bar{\alpha}\}) + \sum_{\ell_1=1}^{L-1} \left[-2\ell_1 m^2 + (t/2)(L - \ell_1)\right] \bar{\alpha}_+ P^{(b)}_{G,\ell_1}(\{\bar{\alpha}\}) \) \hspace{1cm} (11b)

\[
\widetilde{Q}^{(c)}_G(P, \{\bar{\alpha}\}) = \sum_{\ell_1=1, \ell_{L-p+2}=1}^{L-2} \left[-2(\ell_1 + \ell_{L-p+2}) m^2 + (t/2)(L - \ell_1 - \ell_{L-p+2})\right].
\]

\[
\bar{\alpha}_+ P_{G,\ell_1,\ell_{L-p+2}}(\{\bar{\alpha}\}) \hspace{1cm} (11c)
\]

where \( P^{(b)}_{G,\ell_1}(\{\bar{\alpha}\}) \) refers to \( P^{(b)}_G(\{\bar{\alpha}\}) \) with a fixed \( \ell_1 \) and \( P^{(c)}_{G,\ell_1,\ell_{L-p+2}}(\{\bar{\alpha}\}) \) refers to \( P^{(c)}_G(\{\bar{\alpha}\}) \) with fixed \( \ell_1 \) and \( \ell_{L-p+2} \). Summing over \( \ell_1 \), or over \( \ell_1 \) and \( \ell_{L-p+2} \), these quantities have finite mean-values that we can neglect relatively to \( L \). We then get

\[ Q_G(P, \{\bar{\alpha}\}) = (t/2) L \bar{\alpha}_+ + s \bar{\alpha}_+^{L+1} [P_G(\{\bar{\alpha}\})]^{-1}. \] \hspace{1cm} (12)

It is clear that all the reasoning done specifically for \( P_G(\{\bar{\alpha}\}) \) deriving the symmetry properties \( \bar{\alpha}_{j_1} \leftrightarrow \bar{\alpha}_{j_2} \) if there is a \( \bar{\alpha}_{i_1} \leftrightarrow \bar{\alpha}_{i_2} \) symmetry can be redone for \( Q_G(\ell, \{\bar{\alpha}\}) \), \( \prod_{j_+ \in G} \bar{\alpha}_{j_+} \) replacing \( (\bar{\alpha}_-)^{L+1} \) and \( \sum_{j_+ \notin \{i_+\}} \bar{\alpha}_{i_+} \) replacing \( L\bar{\alpha}_+ \). Our next step will be to perform the sums in (6) in order to obtain a compact expression for \( P_G(\{\bar{\alpha}\}) \).

We use the formulae

\[
\int_{x_i \geq 0} \prod_{i=1}^n dx_i \ x_i^{p_i-1} \delta \left(S - \sum_{i=1}^n x_i\right) = \frac{\partial}{\partial S} I(S) \hspace{1cm} (13a)
\]

and\(^{10}\)

\[
I(S) = \int_{x_i \geq 0} \prod_{i=1}^n dx_i \ x_i^{p_i-1} \theta \left(S - \sum_{i=1}^n x_i\right) = S^{\sum_{i=1}^n p_i} \prod_{i=1}^n \Gamma(p_i)/\Gamma \left(1 + \sum_{i=1}^n p_i\right). \hspace{1cm} (13b)
\]

Replacing \( S \) by \( L, p_i \) by 2 and \( n \) by \( L - p \) we get

\[
P_G^{(a)}(\{\bar{\alpha}\}) = \sum_{p=0}^{L-1} (\bar{\alpha}_-)^p (\bar{\alpha}_+)^{L-p} 2^{L-p} L^{2(L-p)-1}/[(L-p) - 1] !
\]

\[
\simeq (\bar{\alpha}_-)^L (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2} \text{sh} \ [(2\bar{\alpha}_+/\bar{\alpha}_-) L]\] \hspace{1cm} (14a)

\[
P_G^{(b)}(\{\bar{\alpha}\}) = 2(\bar{\alpha}_-)^L \left\{ \text{ch} \left( (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2} L \right) - 1 \right\} \hspace{1cm} (14b)
\]
\[ P_G^{(c)}(\{\bar{\alpha}\}) = (\bar{\alpha}_-)^L (2\bar{\alpha}_+ / \bar{\alpha}_-)^{-1/2} \text{sh} \left[ (2\bar{\alpha}_+ / \bar{\alpha}_-)^{1/2} L \right] \]  

and therefore obtain the expressions (4) for \( P_G(\{\bar{\alpha}\}) \) and \( Q_G(\{\bar{\alpha}\}) \). In the calculation of \( \bar{\alpha}_i^- \) or \( \bar{\alpha}_{i+} \) we decompose \( P_G(\{\bar{\alpha}\}) \) into:

\[ P_G(\{\bar{\alpha}\}) = a_i + b_i \bar{\alpha}_i = b_i (a_i/b_i + \bar{\alpha}_i) \]  

where \( i \) is either an \( i_+ \) or an \( i_- \) propagator.

The formulae (13) then allow us to calculate \( \mu_+ = a_+ / b_+ \) and \( \mu_- = a_- / b_- \). We have for \( b_+ \bar{\alpha}_{i+} \) a cell which is cut at \( i_+ \) and for \( P_G(\{\bar{\alpha}\}) \) the same cell can be cut \( 2\ell \) times if the cell has length \( \ell \). Then

\[ <1/(2\ell)> = b_+ \bar{\alpha}_{i+} / P_G(\bar{\alpha}) = \bar{\alpha}_+ / (\mu_+ + \alpha_+) \]  

where \( <1/(2\ell)> \) is the mean-value of \( 1/(2\ell) \) in the sum defining \( P_G(\{\bar{\alpha}\}) \). Then, taking the mean-value, one has to replace \( \sum_{i=1}^{n} p_i \) by \( \left( \sum_{i=1}^{n} p_i \right) - 1 \), and multiply by \( 1/2 \), which gives

\[ <1/(2\ell)> = 1/2 (2\bar{\alpha}_+ / \alpha_-)^{1/2} = y/2 \]  

\[ \mu_+ / \bar{\alpha}_+ = 2/y - 1 \]  

\[ (17a) \]

\[ (17b) \]

For the (−) case, for \( a_- \) we have the end propagator \( i_- \) of a cell of length \( \ell_- \) which is not removed, while for \( b_- \bar{\alpha}_{i-} (\ell_- - 1) \) propagators \( i_- \) can be removed from the cell. This gives

\[ <1/\ell> = \bar{\alpha}_- / P_G(\{\bar{\alpha}\}) = \mu_- / (\mu_- + \bar{\alpha}_-) \]  

This amount again to replace \( \sum_{i=1}^{n} p_i \) by \( \left( \sum_{i=1}^{n} p_i \right) - 1 \) (without dividing by two), so we have

\[ <1/\ell> = \mu_- / (\mu_- + \bar{\alpha}_-) = y \]  

\[ (19a) \]

or

\[ \mu_- / \bar{\alpha}_- = 1/(y^{-1} - 1) \]  

\[ (19b) \]

These expressions are crucial to define all parameters appearing in the equations\(^{7,8}\) defining \( \bar{\alpha}_+ \) and \( \bar{\alpha}_- \).
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Figure Captions

**Fig. 1** We display the three topologies for spanning trees $\mathcal{T}$ in the ladder graphs. $L - p$ counts the number of open cells of length $\ell_i$. Thick dashed lines indicate removed $-$ propagators. Gaps along the ladder indicate removed $+$ propagators.

**Fig. 2** A propagator $j_1 -$ is exchanged with a propagator $j_2 -$ (one being removed and the other not) going from a) to b). A propagator $i_1 +$ is also exchanged with a propagator $i_2 +$ to keep only one $+$ removed propagator per cell. The total length of the cells enclosing $j_1 -$ in a) and $j_2 -$ in b) are respectively $L_{k_1}$ and $L_{k_2}$. 
Fig. 1
Fig. 2