A Survey of Some of the Recent Developments in
Leavitt Path Algebras

Kulumani M. Rangaswamy
Department of Mathematics, University of Colorado,
Colorado Springs, CO. 80918

1 Introduction

Leavitt path algebras are algebraic analogues of graph C*-algebras and, ever since they were introduced in 2004, have become an active area of research. Many of the initial developments during the 2004 - 2014 period have been nicely described in the recent book [2] and in the excellent survey article [1]. Our goal in this article is to report on some of the recent developments in the investigation of the algebraic aspects of Leavitt path algebras not included in [2], [1]. Because the Leavitt path algebras grew as algebraic analogues of graph C*-algebras, their initial investigation involved mostly the ideas and techniques used in the study of graph C*-algebras such as the graph properties of Conditions (K) and (L), and the ring properties of being simple, purely infinite simple, prime/primitive etc. An important starting goal in this initial study was to work out the algebraic analogue of the deep and powerful Kirchberg Phillips theorem to classify purely infinite simple Leavitt path algebras $L := L_K(E)$ up to isomorphism or up to Morita equivalence by means the Grothendieck groups $K_0(L)$ and the sign of the determinant $\det(I - A_E)$ where $A_E$ is the adjacent matrix of the graph $E$.

After such initial progress, there has been an explosion of articles dealing with not only the various different aspects of Leavitt path algebras, but also many natural generalizations such as Leavitt path algebras over commutative rings, of separated graphs, of high rank graphs, Steinberg algebras and groupoids etc. In the background of many of these investigations is the special feature that every Leavitt path algebra $L$ is endowed with three mutually compatible structures: $L$ is a $K$-algebra, $L$ is a $\mathbb{Z}$-graded ring and $L$ is a ring with involution $\ast$. Our focus in this survey is to describe a selection of recent graded and non-graded ring-theoretic and module-theoretic investigations of Leavitt path algebras. My apologies to authors whose work has not been included due to the constrained focus, limitation of time and length of the paper.

In the first part of this survey, we describe graphical conditions on $E$ under which the corresponding Leavitt path algebra $L_K(E)$ belongs to well-known classes of rings. The interesting fact is that often a single graph property of $E$ seems to imply multiple ring properties of $L_K(E)$ and these properties for
general rings are usually independent of each other. The poster child of such a phenomenon is the graph property for a finite graph $E$ that no cycle in $E$ has an exit. In this case $L_K(E)$ possesses at least nine completely different ring properties! (see Theorem 147). Because of such connections between $E$ and $L_K(E)$, Leavitt path algebras can be effective tools in the construction of examples of rings with various desired properties. If we do not impose any graphical conditions on $E$ and just look at $L_K(E)$ as a $\mathbb{Z}$-graded ring, a really interesting result by Hazrat [23] states that $L_K(E)$ is a graded von Neumann regular ring. Because of this, the graded one-sided and two-sided ideals of $L_K(E)$ possess many desirable properties.

The module theory over Leavitt path algebras is still at an infant stage. The second part of this survey gives an account of some of the recent advances in this theory. Naturally, the initial investigations focussed on the simplest of the modules, namely, the simple modules over $L_K(E)$. We begin with outlining a few methods of constructing graded and non-graded simple left/right $L_K(E)$-modules. A special type of simple modules, called Chen simple modules introduced by Chen [19], play an important role. This is followed by characterizing Leavitt path algebras over which all the simple modules possess some special properties, such as, when all the simple modules are flat, or injective, or finitely presented or graded etc. For example, very recently, A.A. Ambily, R. Hazrat and H. Li ([10]) have proved that every simple left/right $L_K(E)$-modules is flat if and only if $L_K(E)$ is von Neumann regular, thus showing, in the case of Leavitt path algebras, an open question in ring theory has an affirmative answer. Likewise, it was shown in [37] that $\text{Ext}^1_{L_K(E)}(S, S) \neq 0$ for a Chen simple module $S$ induced by a cycle. It can then be shown that if all the simple left $L_K(E)$-modules are injective, then $L_K(E)$ is von Neumann regular. The converse easily holds if $E$ is a finite graph, since it that case $L_K(E)$ is semi-simpleartinian. In contrast, if $R$ is an arbitrary non-commutative ring, the injectivity of all simple left $R$-modules need not imply von Neumann regularity of $R$ (see [20]). Our next result in this section describes Leavitt path algebras of finite graphs having finite irreducible representation type, that is, when there are only finitely many isomorphism classes of simple modules. Interestingly, this class of algebras turns out to be precisely the class of Leavitt path algebras of finite graphs having finite Gelfand-Kirillov dimension.

The last section deals with one-sided ideals of a Leavitt path algebra $L$. Four years ago it was shown in [37] that finitely generated two-sided ideals of $L$ are principal ideals. Recently, G. Abrams, F. Mantese and A.Tonolo ([17]) generalized this by showing that every finitely generated one-sided ideal of $L$ is a principal ideal. Such rings are called Bézout rings. Using a deep theorem of G. Bergman, Ara and Goodearl ([12]) showed that one-sided ideals of $L$ are projective. From these two results, it follows that the sum and the intersection of principal one-sided ideals of $L$ are again principal. Thus the principal one-sided ideals of $L$ form a sublattice of the lattice of all one-sided ideals of $L$. A well-known theorem, proved originally for graph C*-algebras and later for Leavitt path algebras $L_K(E)$, states that every two-sided ideal of $L_K(E)$ is a graded ideal if and only if $E$ satisfies Condition (K), equivalently $L_K(E)$ is a
weakly regular ring. What happens when every one-sided ideal of \( L_K(E) \) is graded? The last theorem of this section answers this question, namely, every one-sided ideal of \( L_K(E) \) is graded if and only if every simple \( L_K(E) \)-module is graded if and only if \( L_K(E) \) is a von Neumann regular ring (see \ref{24}).

In summary, this survey is intended to showcase a small sample of some of the recent research on the algebraic aspects of Leavitt path algebras. Hopefully, this provides the reader with some insights into this theory and generates further interest in this exciting and growing field of algebra.

\section{Preliminaries}

For the general notation, terminology and results in Leavitt path algebras, we refer to \cite{2} and \cite{1}. We give below an outline of some of the needed basic concepts and results.\n
A (directed) graph \( E = (E^0, E^1, r, s) \) consists of two sets \( E^0 \) and \( E^1 \) together with maps \( r, s : E^1 \to E^0 \). The elements of \( E^0 \) are called vertices and the elements of \( E^1 \) edges. A vertex \( v \) is called a sink if it emits no edges and a vertex \( v \) is called a regular vertex if it emits a non-empty finite set of edges. An infinite emitter is a vertex which emits infinitely many edges. For each infinite emitter \( v \), we call \( e^* \) a ghost edge. We let \( r(e^*) \) denote \( s(e) \), and we let \( s(e^*) \) denote \( r(e) \). A path \( \mu \) of length \( n > 0 \) is a finite sequence of edges \( \mu = e_1 e_2 \cdots e_n \) with \( r(e_i) = s(e_{i+1}) \) for all \( i = 1, \ldots, n - 1 \). In this case \( \mu^* = e_n^* \cdots e_2^* e_1^* \) is the corresponding ghost path. A vertex is considered a path of length \( 0 \).

A path \( \mu = e_1 \cdots e_n \) in \( E \) is closed if \( r(e_n) = s(e_1) \), in which case \( \mu \) is said to be based at the vertex \( s(e_1) \). A closed path \( \mu \) as above is called simple provided it does not pass through its base more than once, i.e., \( s(e_i) \neq s(e_1) \) for all \( i = 2, \ldots, n \). The closed path \( \mu \) is called a cycle if it does not pass through any of its vertices twice, that is, if \( s(e_i) \neq s(e_j) \) for every \( i \neq j \). An exit for a path \( \mu = e_1 \cdots e_n \) is an edge \( e \) such that \( s(e) = s(e_i) \) for some \( i \) and \( e \neq e_i \).

For any vertex \( v \), the tree of \( v \) is \( T_E(v) = \{ w \in E^0 : v \geq w \} \). We say there is a bifurcation at a vertex \( v \) or \( v \) is a bifurcation vertex, if \( v \) emits more than one edge. In a graph \( E \), a vertex \( v \) is called a line point if there is no bifurcation or a cycle based at any vertex in \( T_E(v) \). Thus, if \( v \) is a line point, the vertices in \( T_E(v) \) arrange themselves on a straight line path \( \mu \) starting at \( v \) (\( \mu \) could just be \( v \)) such as \( \bullet \to \bullet \cdots \bullet \to \bullet \cdots \) which could be finite or infinite.

If \( p \) is an infinite path in \( E \), say, \( p = e_1 \cdots e_n e_{n+1} \cdots \), we follow Chen \cite{19} to define, for each \( n \geq 1 \), \( \tau^{\leq n} (p) = e_1 \cdots e_n \) and \( \tau^{> n} (p) = e_{n+1} e_{n+2} \cdots \). Two infinite paths \( p, q \) are said to be tail-equivalent if there are positive integers \( m, n \) such that \( \tau^{> m} (p) = \tau^{> n} (q) \). This defines an equivalence relation among the infinite paths in \( E \) and the equivalence class containing the path \( p \) is denoted by \([p]\). An infinite path \( p \) is said to be a rational path if it is tail-equivalent to an infinite path \( q = c c c \cdots \), where \( c \) is a closed path.

Given an arbitrary graph \( E \) and a field \( K \), the Leavitt path algebra \( L_K(E) \) is defined to be the \( K \)-algebra generated by a set \( \{ v : v \in E^0 \} \) of pair-wise
orthogonal idempotents together with a set of variables \( \{e, e^* : e \in E^1\} \) which satisfy the following conditions:

1. \( s(e)e = e = er(e) \) for all \( e \in E^1 \).
2. \( r(e)e^* = e^* = e^*s(e) \) for all \( e \in E^1 \).
3. (The “CK-1 relations”) For all \( e, f \in E^1 \), \( e^*e = r(e) \) and \( e^*f = 0 \) if \( e \neq f \).
4. (The “CK-2 relations”) For every regular vertex \( v \in E^0 \),
   \[
   v = \sum_{e \in E^1, s(e) = v} ee^*.
   \]

An arbitrary element \( a \in L := L_K(E) \) can be written as
\[
a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^*
\]
where \( \alpha_i, \beta_i \) are paths and \( k_i \in K \). Here \( r(\alpha_i) = s(\beta_i^*) = r(\beta_i) \).

Every Leavitt path algebra \( L_K(E) \) is a \( \mathbb{Z} \)-graded algebra, namely, \( L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n \) induced by defining, for all \( v \in E^0 \) and \( e \in E^1 \), \( \deg(v) = 0 \), \( \deg(e) = 1 \), \( \deg(e^*) = -1 \). Here the \( L_n \) are abelian subgroups satisfying \( L_m L_n \subseteq L_{m+n} \) for all \( m, n \in \mathbb{Z} \). Further, for each \( n \in \mathbb{Z} \), the homogeneous component \( L_n \) is given by \( L_n = \{ \sum k_i \alpha_i \beta_i^* : \alpha, \beta \in \text{Path}(E), |\alpha_i| - |\beta_i| = n \} \). (For details, see Section 2.1 in [2]). An ideal \( I \) of \( L_K(E) \) is said to be a graded ideal if \( I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n) \).

Throughout this paper, \( E \) will denote an arbitrary graph (with no restriction on the number of vertices or on the number of edges emitted by each vertex) and \( K \) will denote an arbitrary field. For convenience in notation, we will denote, most of the times, the Leavitt path algebra \( L_K(E) \) by \( L \).

We shall first recall the definition of the Gelfand-Kirillov dimension of associative algebras over a field.

Let \( A \) be a finitely generated \( K \)-algebra, generated by a finite dimensional subspace \( V = K_{a_1} \oplus \cdots \oplus K_{a_m} \). Let \( V^0 = K \) and, for each \( n \geq 1 \), let \( V^n \) denote the \( K \)-subspace of \( A \) spanned by all the monomials of length \( n \) in \( a_1, \cdots, a_m \).

Set \( V_n = \sum_{i=0}^{n} V^i \). Then the Gelfand-Kirillov dimension of \( A \) (for short, the \( \text{GK-dimension} \) of \( A \)) is defined by
\[
\text{GK-dim}(A) := \limsup_{n \to \infty} \log_n (\dim V_n).
\]

It is known that the \( \text{GK-dim}(A) \) is independent of the choice of the generating subspace \( V \).

If \( A \) is an infinitely generated \( K \)-algebra, then the \( \text{GK-dimension} \) of \( A \) is defined as
\[
\text{GK-dim}(A) := \sup_B \text{GK-dim}(B)
\]
where \( B \) runs over all the finitely generated \( K \)-subalgebras of \( A \).
The GK-dimension of an algebra $A$ measures the growth of the algebra $A$ and might be considered the non-commutative analogue of the classical Krull dimension for commutative algebras. Some useful examples (see [29]) are: The GK-dimension the matrix ring $M_A(K)$ is 0 and the GK-dimension of the matrix ring $M_{A'}(K[x,x^{-1}])$ is 1, where $A, A'$ are arbitrary possibly infinite index sets.

**Grading of a matrix ring over a $\mathbb{Z}$-graded ring:** We wish to recall the grading of matrices of finite order and then indicate how to extend this to the case of infinite matrices in which at most finitely many entries are non-zero (see [22] and [32]). This will be used in Section 4. In the following, the length of a path $p$ will be denoted by $|p|$.

Let $\Gamma$ be an additive abelian group, $A$ be a $\Gamma$-graded ring and $(\delta_1, \cdots, \delta_n)$ an $n$-tuple where $\delta_i \in \Gamma$. Then $M_n(A)$ is a $\Gamma$-graded ring where, for each $\lambda \in \Gamma$, its $\lambda$-homogeneous component consists of $n \times n$ matrices

$$M_n(A)(\delta_1, \cdots, \delta_n)_\lambda = \begin{pmatrix}
A_{\lambda+\delta_1-\delta_1} & A_{\lambda+\delta_2-\delta_1} & \cdots & A_{\lambda+\delta_n-\delta_1} \\
A_{\lambda+\delta_1-\delta_2} & A_{\lambda+\delta_2-\delta_2} & \cdots & A_{\lambda+\delta_n-\delta_2} \\
& & & \\
A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & \cdots & A_{\lambda+\delta_n-\delta_n}
\end{pmatrix}. \quad (1)$$

This shows that for each homogeneous element $x \in A,$

$$\deg(e_{ij}(x)) = \deg(x) + \delta_i - \delta_j, \quad (2)$$

where $e_{ij}(x)$ is a matrix with $x$ in the $ij$-position and with every other entry 0.

Now let $A$ be a $\Gamma$-graded ring and let $I$ be an arbitrary infinite index set. Denote by $M_I(A)$ the matrix with entries indexed by $I \times I$ having all except finitely many entries non-zero and for each $(i, j) \in I \times I$, the $ij$-position is denoted by $e_{ij}(a)$ where $a \in A$. Considering a "vector" $\bar{\delta} := (\delta_i)_{i \in I}$ where $\delta_i \in \Gamma$ and following the usual grading on the matrix ring (see (1), (2)), define, for each homogeneous element $a,$

$$\deg(e_{ij}(a)) = \deg(a) + \delta_i - \delta_j. \quad (3)$$

This makes $M_I(A)$ a $\Gamma$-graded ring, which we denote by $M_I(A)(\bar{\delta})$. Clearly, if $I$ is finite with $|I| = n$, then the graded ring coincides (after a suitable permutation) with $M_n(A)(\delta_1, \cdots, \delta_n)$.

Suppose $E$ is a finite acyclic graph consisting of exactly one sink $v$. Let $\{p_i : 1 \leq i \leq n\}$ be the set of all paths ending at $v$. Then it was shown in (Lemma 3.4, [4])

$$L_K(E) \cong M_n(K) \quad (4)$$

under the map $p_i p_i^* \mapsto e_{ij}$. Now taking into account the grading of $M_n(K)$, it was further shown in (Theorem 4.14, [22]) that the same map induces a graded isomorphism

$$L_K(E) \rightarrow M_n(K)(|p_1|, \cdots, |p_n|) \quad (5)$$
In the case of a comet graph $E$ (that is, a finite graph $E$ with a cycle $c$ without exits and a vertex $v$ on $c$ such that every path in $E$ which does not include all the edges in $c$ ends at $v$), it was shown in (Lemma 2.7.1, [2]) that the map

$$L_K(E) \to M_n(K[x, x^{-1}])$$

given by

$$p_i c_k p_j^* \mapsto e_{ij}(x^k)$$

where the $e_{ij}$ are matrix units, induces an isomorphism. Again taking into account the grading, it was shown in (Theorem 4.20, [22]) that the map

$$L_K(E) \to M_n(K[x|c|, x^{-|c|}])$$

given by

$$p_i c_k p_j^* \mapsto e_{ij}(x^{|c|})$$

induces a graded isomorphism. Later in the paper [3], the isomorphisms (4) and (6) were extended to infinite acyclic and infinite comet graphs respectively (see Proposition 3.6 [3]). The same isomorphisms with the grading adjustments will induce graded isomorphisms for Leavitt path algebras of such graphs. We now describe this extension below.

Let $E$ be a graph such that no cycle in $E$ has an exit and such that every infinite path contains a line point or is tail-equivalent to a rational path $c c c \cdots$. where $c$ is a cycle (without exits). Define an equivalence relation in the set of all line points in $E$ by setting $u \sim v$ if $T_E(u) \cap T_E(v) \neq \emptyset$. Let $X$ be the set of representatives of distinct equivalence classes of line points in $E$, so that for any two line points $u, v \in X$ with $u \neq v$, $T_E(u) \cap T_E(v) = \emptyset$. For each vertex $v_i \in X$, let $\tilde{p} v_i := \{p_s^v_i : s \in \Lambda_i\}$ be the set of all paths that end at $v_i$, where $\Lambda_i$ is an index set which could possibly be infinite. Denote by $|\tilde{p} v_i| = \{|p_s^v_i| : s \in \Lambda_i\}$.

Let $Y$ be the set of all distinct cycles in $E$. As before, for each cycle $c_j \in Y$ based at a vertex $w_j$, let $\tilde{q} w_j := \{q_r^{w_j} : r \in \Upsilon_j\}$ be the set of all paths that end at $w_j$ but do not include all the edges of $c_j$, where $\Upsilon_j$ is an index set which could possibly be infinite. Let $|\tilde{q} w_j| := \{|q_r^{w_j}| : r \in \Upsilon_j\}$. Then the isomorphisms (5) and (7) extend to a $\mathbb{Z}$-graded isomorphism

$$L_K(E) \cong_{gr} \bigoplus_{v_i \in X} M_{\Lambda_i}(K)(|\tilde{p} v_i|) \oplus \bigoplus_{w_j \in Y} M_{\Upsilon_j}(K[x|c_j|, x^{-|c_j|}](|\tilde{q} w_j|))$$

where the grading is as in (3).
3 Leavitt path algebras satisfying a polynomial identity

Observe that Leavitt path algebras in general are highly non-commutative. For instance, if the graph $E$ contains an edge $e$ with $u = s(e) \neq r(e) = v$, then $ue = e$, but $eu = 0$. Indeed, it is an easy exercise to conclude that if $E$ is a connected graph, then $L_K(E)$ is a commutative ring if and only if the graph $E$ consists of just a single vertex $v$ or is a loop $e$, that is a single edge $e$ with $s(e) = r(e) = v$. In this case, $L_K(E)$ is isomorphic to $K$ or $K[x, x^{-1}]$.

Note that to say a ring $R$ is commutative is equivalent to saying that $R$ satisfies the polynomial identity $xy - yx = 0$. An algebra $A$ over a field $K$ is said to satisfy a polynomial identity (or simply, a PI-algebra), if there is a polynomial $p(x_1, \cdots, x_n)$ in finitely many non-commuting variable $x_1, \cdots, x_n$ with coefficients in $K$ such that $p(a_1, \cdots, a_n) = 0$ for all choices of elements $a_1, \cdots, a_n \in A$. For example, the Amitsur-Levitzky theorem (see [34]) states that the ring $M_n(R)$ of $n \times n$ matrices over a commutative ring $R$ satisfies the so called standard polynomial identity $P_n(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} \epsilon_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}$ where $S_n$ is the symmetric group of $n!$ permutations of the set $\{1, \cdots, n\}$ and $\epsilon_{\sigma} = 1$ or $-1$ according as $\sigma$ is even or odd. A natural question is to characterize the Leavitt path algebras which satisfy a polynomial identity. This is completely answered in the next theorem.

Theorem 3.1 ([17]) Let $E$ be an arbitrary graph. Then the following properties are equivalent for $L_K(E)$:

(a) $L_K(E)$ satisfies a polynomial identity;

(b) No cycle in $E$ has an exit, there is a fixed positive integer $d$ such that the number of distinct paths that end at any vertex $v$ is $\leq d$ and the only infinite paths in $E$ are paths that are eventually of the form $ggg \cdots$, for some cycle $g$;

(c) There is a fixed positive integer $d$ such that $L_K(E)$ is a subdirect product of matrix rings over $K$ or $K[x, x^{-1}]$ of order at most $d$.

If the graph $E$ is row-finite, then the Leavitt path algebra $L_K(E)$ in Theorem 3.1 actually decomposes as a ring direct sum of matrix rings over $K$ or $K[x, x^{-1}]$ of order at most a fixed positive integer $d$. This shows that satisfying a polynomial identity imposes a serious restriction on the structure of Leavitt path algebras.

4 Four Important Graphical Conditions

In this section, we shall illustrate how specific graphical conditions on the graph $E$ give rise to various algebraic properties of $L_K(E)$. We illustrate this by choosing four different graph properties of $E$. Interestingly, a single graph theoretical property of $E$ often implies several different ring properties for $L_K(E)$. It is amazing that a single property that no cycle in a finite graph $E$ has an
exit implies that the corresponding Leavitt path algebra $L_K(E)$ possesses several different ring properties such as being directly finite, self-injective, having bounded index of nilpotence, a Baer ring, satisfying a polynomial identity, having GK-dimension $\leq 1$, etc. (see Theorem 4.7 below). Consequently, Leavitt path algebras turn out to be useful tools in the construction of various examples of rings. We will also describe the interesting history behind the terms Condition (K) and Condition (L) which play an important role in the investigation of both the graph $C^*$-algebras and Leavitt path algebras (see [2], [40], [42]).

Recall, a ring $R$ is said to be von Neumann regular if to each element $a \in R$ there is an element $b \in R$ such that $a = aba$. The ring $R$ is said to be $\pi$-regular (strongly left or right $\pi$-regular) if to each element $a \in R$, there is a $b \in R$ and an integer $n \geq 1$ such that $a^n = a^nba^n$ ($a^n = a^{n+1}b$ or $a^n = ba^{n+1}$). In general, these ring properties are not equivalent. But as the next theorem shows, they all coincide for Leavitt path algebras.

A graph $E$ is said to be acyclic if $E$ contains no cycles. The next theorem characterizes the von Neumann regular Leavitt path algebras.

**Theorem 4.1** [7] For an arbitrary graph $E$, the following conditions are equivalent for $L := L_K(E)$:

(a) The graph $E$ is acyclic;
(b) $L$ is von Neumann regular;
(c) $L$ is $\pi$-regular;
(d) $L$ is strongly left/right $\pi$-regular.

Another important graph property is Condition (K). In some sense this property is diametrically opposite of being acyclic.

**Definition 4.2** A graph $E$ satisfies Condition (K) if whenever a vertex $v$ lies on a simple closed path $\alpha$, $v$ also lies on another simple closed path $\beta$ distinct from $\alpha$.

The Condition (K) implies a number of ring properties;

**Definition 4.3** (i) A ring $R$ is said to be left/right weakly regular if for every left/right ideal $I$ of $R$, $I^2 = I$;

(ii) A ring $R$ is said to be an exchange ring if given any left/right $R$-module $M$ and two direct decompositions of $M$ as $M = M' \oplus A$ and $M = \bigoplus_{i=1}^n A_i$, where $M' \cong R$, there exist submodules $B_i \subseteq A_i$ such that $M = M' \oplus \bigoplus_{i=1}^n B_i$.

**Theorem 4.4** ([13], [10], [40]) Let $E$ be an arbitrary graph. Then the following conditions are equivalent for $L := L_K(E)$:

(i) The graph $E$ satisfies Condition (K);
(ii) $L$ is an exchange ring;
(iii) $L$ is left/right weakly regular;
(iv) Every two-sided ideal of $L$ is a graded ideal.
Definition 4.5 A graph $E$ is said to satisfy Condition (L), if every cycle in $E$ has an exit.

Theorem 4.6 [36] Let $E$ be an arbitrary graph. Then the following are equivalent for $L_K(E)$:

(i) $E$ satisfies Condition (L);

(ii) $L$ is a Zorn ring, that is, every (non-nil) right/left ideal $I$ contains a non-zero idempotent;

(iii) Every element $a \in L$ is the von Neumann inverse of another element $b \in L$; that is, to each $a \in L$, there is an element $b \in L$ such that $bab = b$.

An interesting history of Conditions (K) and (L): One may wonder about the choice of the letters K and L in the terms Condition (K) and Condition (L). There is an interesting narrative about the origins of these terms. I am grateful to Mark Tomforde for outlining this history to me which he will also be including in his forth-coming book on Graph Algebras ([42]). Both these two graph conditions were originally introduced by graph C*-algebraists. It all started when Cuntz and Krieger (whom some consider the founders of graph C*-algebras), introduced in their original paper ([21]) a condition on matrices with entries in $\{0, 1\}$ and called it Condition (I). Assuming that the "I" is the English letter I and not the Roman numeral one, Pask and Raeburn introduced in 1996 a Condition (J) in their paper ([33]), as J is the letter that follows I in the English alphabet. (They apparently did not recognize that Cuntz and Krieger also introduced a follow-up Condition (II), thus indicating, in their view, I and II stand for Roman numerals). Conforming to this pattern, when Kumjian, Pask, Raeburn and Renault introduced a new condition in their 1997 paper ([30]), they chose the letter K to denote this new condition and called it Condition (K). Continuing this pattern yet again, Kumjian, Pask and Raeburn introduced Condition (L) in 1998 ([31]). Actually, Astrid and Huef later showed that Condition (L) coincides with Condition (I) for graphs of finite matrices. Moreover, Condition (K) is considered analogous to Condition (II) for Cuntz-Krieger algebras. In all the investigations that followed in graph C*-algebras and also in Leavitt path algebras, Conditions (K) and (L) emerged as important graph conditions. Poor Condition (J) remains neglected!

Recall, Condition (L) requires every cycle to have an exit. We next consider a graph property that is diametrically opposite to Condition (L), namely, no cycle in the graph has an exit. This implies several interesting ring/module properties.

First, consider a finite graph $E$ in which no cycle has an exit. In this case, $L_K(E)$ is a ring with identity. We begin recalling a number of ring properties.

A ring $R$ with identity 1 is said to be directly finite if for any two elements $x, y$, $xy = 1$ implies $yx = 1$. This is equivalent to $R$ being not isomorphic to any proper direct summand of $R$ as a left or a right $R$-module. A ring $R$ with identity is called a Baer ring if the left/right annihilator of every subset $X$ of $R$ is generated by an idempotent. A $\Gamma$-graded ring $R$ is said to be a graded Baer ring, if the left/right annihilator of every subset $X$ of homogeneous elements is
generated by a homogeneous idempotent. A ring \( R \) is said to have **bounded index of nilpotence** if there is a positive integer \( n \) which is such that \( a^n = 0 \) for every nilpotent element \( a \in R \).

**Theorem 4.7** (\[9\], \[17\], \[25\], \[27\], \[28\], \[43\]) For a finite graph \( E \), the following conditions are equivalent for \( L := L_K(E) \):

(i) No cycle in \( E \) has an exit;
(ii) \( L \) is directly finite;
(iii) \( L \) is a Baer ring;
(iv) \( L \) is a graded Baer ring;
(v) \( L \) is a graded left/right self-injective ring;
(vi) \( L \) satisfies a polynomial identity;
(vii) \( L \) has bounded index of nilpotence;
(viii) \( L \) is graded semi-simple;
(ix) \( L \) has GK-dimension \( \leq 1 \);
(x) \( L \) is finite over its center.

Thus if \( E \) is the following graph,

```
• −→ • → • ←− •
```

then \( L_K(E) \) will possess all the stated nine ring properties.

For a finite graph \( E \), if \( L_K(E) \) satisfies any of the equivalent conditions in the preceding theorem, \( L_K(E) \) decomposes as a graded direct sum of finitely many matrix rings of finite order over \( K \) and/or \( K[x, x^{-1}] \) which are given the matrix gradings indicated in equations (5) and (7) in the Preliminaries section.

For a ring \( R \) without identity, but with local units, \( R \) is said to be directly finite if for every \( x, y \in R \) and an idempotent \( u \in R \) satisfying \( ux = x = xu, uy = y = yu \), we have \( xy = u \) implies \( yx = u \). Every commutative ring is trivially directly finite.

If \( R \) is a ring without identity, \( R \) is called a **locally Baer ring** (locally graded Baer ring) if for every idempotent (homogeneous idempotent) \( e \), the corner \( eRe \) is a Baer (graded Baer) ring.

**Theorem 4.8** (\[27\], \[28\]) Let \( E \) be an arbitrary graph. Then the following conditions are equivalent for \( L := L_K(E) \):

(i) No cycle in \( E \) has an exit, \( E \) is row-finite and every infinite path ends at a sink or a cycle.
(ii) \( L \) is a locally Baer ring;
(iii) \( L \) is a graded locally Baer ring;
(iv) \( L \) is a graded left/right self-injective ring;
(v) \( L \) is graded isomorphic to a ring direct sum of matrix rings.
\[ L_K(E) \cong_{gr} \bigoplus_{v_i \in X} M_{\Lambda_i}(K)((p^{v_i} - v_i^{\infty})) \oplus \bigoplus_{w_j \in Y} M_{\Upsilon_j}(K[x_{t_j}, x^{-t_j}])((q^{w_j} - w_j^{\infty})) \]

where \(\Lambda_i, \Upsilon_j\) are suitable index sets, the \(t_j\) are positive integers, \(X\) is the set of representatives of distinct equivalence classes of line points in \(E\) and \(Y\) is the set of all distinct cycles (without exits) in \(E\).

5 Simple modules over Leavitt path algebras

In this section, we shall indicate the methods of constructing simple modules over Leavitt path algebras by graphical methods.

As noted in [2], every element \(a\) of a Leavitt path algebra \(L_K(E)\) over a graph \(E\) can be written in the form \(a = \sum_{i=1}^{n} \alpha_i \beta_i^*\) and that the map \(\sum_{i=1}^{n} \alpha_i \beta_i^* \mapsto \sum_{i=1}^{n} \beta_i \alpha_i^*\) induces an isomorphism \(L_K(E) \rightarrow (L_K(E))^\text{op}\). Consequently, \(L_K(E)\) is left-right symmetric. So in this and the next section, we shall only be stating results on left ideals and left modules over \(L_K(E)\). The corresponding results on right ideals and right modules hold by symmetry.

**Definition 5.1** A vertex \(v\) is called a Laurent vertex if \(T_E(v)\) consists of the set of all vertices on a single path \(\gamma = \mu c\) where \(\mu\) is a path without bifurcations starting at \(v\) and \(c\) is a cycle without exits based on a vertex \(u = r(\mu)\).

An easy example of a Laurent vertex is the vertex \(v\) in the following graph:

\[
\begin{array}{c}
  \bullet \\
  \rightarrow \\
  \leftarrow \\
  v \\
  \rightarrow \\
  \leftarrow \\
  \bullet
\end{array}
\]

The next theorem gives conditions under which a vertex in the graph \(E\) generates a simple left graded simple left ideal of \(L_K(E)\).

**Theorem 5.2** ([2], [22]) Let \(E\) be an arbitrary graph and let \(v\) be a vertex. Then

(a) The left ideal \(L_K(E)v\) is a simple/minimal left ideal of \(L_K(E)\) if and only if \(v\) is a line point;

(b) The left ideal \(L_K(E)v\) is a graded simple/minimal left ideal of \(L_K(E)\) if and only if \(v\) is either a line point or a Laurent vertex.

Next we shall describe the general methodology used by Chen ([19]) and extended in ([25], [38]) to construct left simple and graded simple modules over \(L_K(E)\) by using special vertices or cycles in the graph \(E\).
I) Definition of the module $A_u$: Let $u$ be a vertex in a graph $E$ which is either a sink or an infinite emitter. Let $A_u$ be the $K$-vector space having as a basis the set $B = \{p : p$ is a path in $E$ with $r(p) = u\}$. We make $A$ a left $L_K(E)$-module as follows: Define, for each vertex $v$ and each edge $e$ in $E$, linear transformations $P_v, S_e$ and $S_{e^*}$ on $A$ by defining their actions on the basis $B$ as follows:

For all $p \in B$,

(I) $P_v(p) = \begin{cases} p, & \text{if } v = s(p) \\ 0, & \text{otherwise} \end{cases}$

(II) $S_e(p) = \begin{cases} ep, & \text{if } r(e) = s(p) \\ 0, & \text{otherwise} \end{cases}$

(III) $S_{e^*}(p) = \begin{cases} p', & \text{if } p = ep' \\ 0, & \text{otherwise} \end{cases}$

(IV) $S_{e^*}(u) = 0$.

The endomorphisms $\{P_v, S_e, S_{e^*} : v \in E^0, e \in E^1\}$ satisfy the defining relations (1) - (4) of the Leavitt path algebra $L_K(E)$. This induces an algebra homomorphism $\phi$ from $L_K(E)$ to $\text{End}_K(A_u)$ mapping $v$ to $P_v$, $e$ to $S_e$ and $e^*$ to $S_{e^*}$. Then $A_u$ can be made a left module over $L_K(E)$ via the homomorphism $\phi$. We denote this $L_K(E)$-module operation on $A_u$ by $\cdot$.

**Theorem 5.3** ([19], [38]) If the vertex $u$ is either a sink or an infinite emitter, then $A_u$ is a simple left $L_K(E)$-module.

If the vertex $u$ lies on a cycle without exits, then in the Definition of $A_u$, define the basis $B = \{pq^* : p, q \text{ path in } E \text{ with } r(q^*) = s(q) = u\}$. We then get the following result.

**Theorem 5.4** ([25]) If a vertex $u \in E$ lies on a cycle without exits, then $A_u$ is a graded simple left $L_K(E)$-module graded isomorphic to the graded minimal left ideal $L_K(E)u$ and $A_u$ is not a simple left $L_K(E)$-module.

**Remark 5.5** In [25], the module $A_u$ is defined by using an algebraic branching system and is denoted as $N_{vc}$. Here we have defined the module $A_u$ differently, but the proof of the above theorem is just the proof of Theorem 3.5(1) of [25].

With a slight modification of the definition of $A_u$, Chen [19] shows one more way of constructing simple modules by using the infinite paths that are tail-equivalent to a fixed infinite path in $E$. Recall, two infinite paths $p = e_1 \cdots e_r \cdots$ and $q = f_1 \cdots f_s \cdots$ are said to be **tail-equivalent** if there exist fixed positive integers $m, n$ such that $e_{n+k} = f_{m+k}$ for all $k \geq 1$. Let $[p]$ denote the tail-equivalence class of all infinite paths equivalent to $p$. Let $A_{[p]}$ denote the $K$-vector space having $[p]$ as a basis. As in the definition of $A_u$, for each vertex $v$ and each edge $e$ in $E$, define the linear transformations $P_v, S_e$ and $S_{e^*}$ on $A$ by defining their actions on the basis $[p]$ satisfying the conditions (I),(II),(III), but not (IV) above. As before, they satisfy the defining relations of a Leavitt path algebra and thus induce a homomorphism $\varphi : L_K(E) \to A_{[p]}$. The vector space $A_{[p]}$ then becomes a left $L_K(E)$-module via the map $\varphi$. 

12
Theorem 5.6 ([19]) The module $A_{[p]}$ is a simple left $L_K(E)$-module and for two infinite paths $p, q$, $A_{[p]} \cong A_{[q]}$ if and only if $[p] = [q]$.

It can be shown (see [38]) that the simple modules $A_u, A_v$ and $A_{[p]}$ corresponding respectively to a sink $u$, an infinite emitter $v$ and an infinite path $p$, are all non-isomorphic.

A special infinite path is the so called a rational infinite path induced by a simple closed path (and in particular, a cycle) $c$. This is the infinite path $ccc\ldots$. We denote this path by $c^\infty$. We shall be using the corresponding simple $L_K(E)$-module $A_{c^\infty}$ subsequently.

6 Leavitt path algebras with simple modules having special properties

We shall describe when all the simple modules over a Leavitt path algebra are flat or injective or finitely presented or graded etc.

An open problem, raised by Ramamuthi ([35]) some forty years ago, asks whether a non-commutative ring $R$ with 1 is von Neumann regular if all the simple left $R$-modules are flat. Using a more general approach of Steinberg algebras, Ambily, Hazrat and Li ([10]) obtain the following theorem which shows that Ramamurthi’s question has an affirmative answer in the case of Leavitt path algebras.

Theorem 6.1 ([10]) Let $E$ be an arbitrary graph. Then every simple left $L_K(E)$-module is flat if and only if $L_K(E)$ is von Neumann regular.

Next we consider the case when $L_K(E)$ is a left V-ring, that is, when every simple left $L_K(E)$-module is injective. Kaplansky showed that if $R$ is a commutative ring, then every simple $R$-module is injective if and only if $R$ is von Neumann regular. A natural question is under what conditions every simple left $L_K(E)$-module is injective. It was shown in [26] that, in this case, $L_K(E)$ becomes a weakly regular ring. However, recently Abrams, Mantese and Tonolo ([5]) showed that, if $c$ is a cycle in a graph $E$, then the corresponding simple left $L_K(E)$-module $A_{c^\infty}$ satisfies $\text{Ext}_1^{L_K(E)}(A_{c^\infty}, A_{c^\infty}) \neq 0$. This implies that the module $A_{c^\infty}$ cannot be an injective module. So, if every simple left $L_K(E)$-module is injective, then necessarily $E$ contains no cycles. Then by ([17]) $L_K(E)$ must be von Neumann regular. Thus we obtain the following new result and its corollary.

Theorem 6.2 Let $E$ be an arbitrary graph. If every simple left $L_K(E)$-module is injective, then $L_K(E)$ is a von Neumann regular ring.

Conversely, if $L_K(E)$ is a von Neumann regular ring then is the graph $E$ contains no cycles ([17]) and, if $E$ is further a finite graph, then $L_K(E)$ is a direct sum of finitely many matrix rings of finite order over $K$ (Theorem 2.6.17, [2]). In this case, $L_K(E)$ is a direct sum of left/right simple modules and hence
Corollary 6.3 Let \( E \) be a finite graph. Then every simple left/right \( L_K(E) \)-module is injective if and only if \( L_K(E) \) is a von Neumann regular ring.

When \( E \) is an arbitrary graph, it is an open question whether the von Neumann regularity of \( L_K(E) \) implies that every simple left/right \( L_K(E) \)-module is injective.

Next we consider Leavitt path algebras \( L_K(E) \) whose simple modules are all finitely presented. When \( E \) is a finite graph, \( L_K(E) \) possesses a number of interesting properties as noted in the following theorem

Theorem 6.4 (13) For any finite graph \( E \), the following properties of the Leavitt path algebra \( L := L_K(E) \) are equivalent:

(i) Every simple left \( L \)-module is finitely presented;
(ii) No two cycles in \( E \) have a common vertex;
(iii) There is a one-to-one correspondence between isomorphism classes of simple \( L \)-modules and primitive ideals of \( L \);
(iv) The Gelfand-Kirillov dimension of \( L \) is finite.

The preceding theorem has been generalized in (39) to the case when \( E \) is an arbitrary graph with several similar equivalent conditions.

It is easy to observe that every simple left module over a Leavitt path algebra \( L := L_K(E) \) is of the form \( L_v/N \) for some vertex \( v \) and a maximal left submodule \( N \) of \( L_v \). A natural question is, given a vertex \( u \), can we estimate the number \( \kappa_u \) of distinct maximal left \( L \)-submodules \( M \) of \( L_u \) such that \( L_u/M \) is isomorphic to \( L_v/N \)? In [39] it is shown that \( \kappa_u \leq |uL_v\backslash N| \) and consequently the cardinality of the set of all such simple modules corresponding to various vertices is \( \leq \sum_{u \in E} |uL_v\backslash N| \) and thus is \( \leq |L| \).

More generally, one may try to estimate the number of non-isomorphic simple modules over a Leavitt path algebra. In this connection, observe the following: Suppose a graph \( E \) contains two cycles \( g, h \) which share a common vertex \( v \), such as the two cycles in the following graph.

![Graph Image]

We then wish to show that there are uncountably many non-isomorphic simple modules over the corresponding \( L_K(E) \). By Theorem 5.6, we need only to produce uncountably many non-equivalent infinite paths in \( E \). With that in mind, consider the infinite rational path \( p = ggg \cdots \) which, for convenience, we
write as \( p = g_1g_2g_3 \cdots \) indexed by the set \( \mathbb{P} \) of positive integers, where \( g_i = g \) for all \( i \). Now, for every subset \( S \) of \( \mathbb{P} \), define an infinite path \( p_S \) by replacing \( g_i \) by \( h \) if and only if \( i \in S \). Observe that this gives rise to uncountably many distinct infinite paths. From the definition of equivalence of paths, it can be derived that, given an infinite path \( q \) there are at most countably many infinite paths that are equivalent to \( q \). From this one can then establish that there are uncountably many non-isomorphic simple left \( L_K(E) \)-modules.

Leavitt path algebras having only finitely many non-isomorphic simple modules are characterized in the next theorem. Recall, a ring \( R \) is called left/right semi-artinian if every non-zero left/right \( R \)-module contains a simple submodule.

**Theorem 6.5** ([14]) Let \( E \) be an arbitrary graph and \( K \) be any field. Then the following are equivalent for the Leavitt path algebra \( L = L_K(E) \):

(i) \( L \) has at most finitely many non-isomorphic simple left/right \( L \)-modules;

(ii) \( L \) is a left and right semi-artinian von Neumann regular ring with finitely many two-sided ideals which form a chain under set inclusion;

(iii) The graph \( E \) is acyclic and there is a finite ascending chain of hereditary saturated subsets \( \{0\} \subsetneq H_1 \subsetneq \cdots \subsetneq H_n = E_0 \) such that for each \( i < n \), \( H_{i+1} \setminus H_i \) is the hereditary saturated closure of the set of all the line points in \( E \setminus H_i \).

## 7 One-sided Ideals in a Leavitt Path Algebra

In this section, we shall describe some of the interesting properties of one-sided ideals of a Leavitt path algebra.

Using a deep theorem of G. Bergman [18], Ara and Goodearl proved the following result that Leavitt path algebras are hereditary.

**Theorem 7.1** ([12]) If \( E \) is an arbitrary graph, then every left/right ideal of \( L_K(E) \) is a projective left/right \( L_K(E) \)-module.

A von Neumann regular ring \( R \) has the characterizing property that every finitely generated one-sided ideal of \( R \) is a principal ideal generated by an idempotent. Four years ago, it was shown in [37] that every finitely generated two-sided ideal of a Leavitt path algebra is a principal ideal. Recently, Abrams, Mantese and Tonolo have proved that this interesting property holds for one sided ideals too, as indicated in the next theorem.

**Theorem 7.2** ([6]) Let \( E \) be an arbitrary graph. Then \( L := L_K(E) \) is a Bézout ring, that is, every finitely generated one-sided ideal is a principal ideal.

Thus if \( La \) and \( Lb \) are two principal left ideals of \( L \), then, being finitely generated, \( La + Lb = Lc \), a principal left ideal. So the sum of any two principal left ideals of \( L \) is again a principal left ideal. What about their intersection? Should the intersection of two principal left ideals of \( L \) be again a principal left ideal? This is answered in the next theorem. Since this result is new, we outline its proof which is straightforward.
Theorem 7.3 Let $E$ be an arbitrary graph. Then both the sum and the intersection of two principal left ideal of $L := L_K(E)$ are again principal left ideals. Thus the principal left ideals of $L$ form a sublattice of the lattice of all left ideals of $L$.

Proof. Suppose $A, B$ are two principal left ideals of $L$. Consider the following exact sequence where the map $\theta$ is the additive map $(a, b) \rightarrow a + b$

$$0 \rightarrow K \rightarrow A \oplus B \xrightarrow{\theta} A + B \rightarrow 0$$

where $K = \{(x, -x) : x \in A \cap B\} \cong A \cap B$. Now $A + B$ is a (finitely generated) left ideal of $L_K(E)$. By Theorem 7.1 it is a projective module and hence the above exact sequence splits. Consequently, $A \cap B$ is isomorphic a direct summand of $A \oplus B$. Since $A \oplus B$ is finitely generated, so is $A \cap B$. By Theorem 7.2 $A \cap B$ is then a principal left ideal. □

As we noted in Theorem 4.4, every two-sided ideal of $L_K(E)$ is graded if and only if the graph $E$ satisfies Condition (K). What happens if every one-sided ideal of $L_K(E)$ is graded? This is answered in the next and the last theorem of this section.

Theorem 7.4 (25) Let $E$ be an arbitrary graph. Then the following properties are equivalent for $L = L_K(E)$:

(i) Every left ideal of $L$ is a graded left ideal;
(ii) Every simple left $L$-module is a graded module;
(iii) The graph $E$ contains no cycles;
(iv) $L$ is a von Neumann regular ring.

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