Waring-Goldbach Problem: Two Squares and Three Biquadrates

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Abstract. Assume that $\psi$ is a function of positive variable $t$, monotonically increasing to infinity and $0 < \psi(t) \ll \log t/\log \log t$. Let $R_3(n)$ denote the number of representations of the integer $n$ as sums of two squares and three biquadrates of primes and we write $E_3(N)$ for the number of integers $n$ satisfying $n \leq N$, $n \equiv 5, 53, 101 \pmod{120}$ and

$$|R_3(n) - \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \mathcal{S}_3(n)n^{3/4}} \log^5 n| \geq \frac{n^{3/4}}{\psi(n) \log^5 n},$$

where $0 < \mathcal{S}_3(n) \ll 1$ is the singular series. In this paper, we prove $E_3(N) \ll N^{23/48+\epsilon} \psi^2(N)$ for any $\epsilon > 0$. This result constitutes a refinement upon that of Friedlander and Wooley [2].

1. Introduction

The celebrated Waring problem involving two squares still remains one of the most elegant problems in additive number theory. Here we outline several pieces of research about it.

Let $v(n)$ denote the number of representations of $n$ as sums of two squares and three nonnegative cubes. In 1972, Linnik [10] proved that $v(n) \gg n^{2/3-\epsilon}$ for all large integers $n$ and any $\epsilon > 0$. In 1981, Hooley [3] improved upon the work of Linnik by obtaining the expected asymptotic formula for $v(n)$. In 2000, he [6] also obtained the asymptotic formula for the number of representations of $n$ as sums of three squares and a $k$-th power.

Let $R_s(n)$ denote the number of representations of natural number $n$ as sums of two squares and $s$ biquadrates. The expected asymptotic formula for $R_s(n)$ can be established for $s \geq 5$, see Hooley [4]. But for $s \leq 4$, all techniques fail to obtain the expected asymptotic formula for $R_s(n)$. Let $E_s(N)$ be the number of integers $n \leq N$ such that the expected asymptotic formula for $R_s(n)$ fails to be valid. In 2014, Friedlander and Wooley [2] showed

$$E_3(N) \ll N^{1/2+\epsilon} \quad \text{and} \quad E_4(N) \ll N^{1/4+\epsilon}.$$
Later on, Zhao [13] strengthened these results by showing
\[ E_3(N) \ll N^{3/8+\varepsilon} \quad \text{and} \quad E_4(N) \ll N^{1/8+\varepsilon}. \]

Let \[ \Omega = \{ n \in \mathbb{N} : n \equiv 5, 53, 101 \pmod{120} \}. \]

The purpose of this paper is to investigate the cognate problem concerning the representation of integers \( n \) in \( \Omega \) such that
\[
(1.1) \quad n = p_1^2 + p_2^2 + p_3^4 + p_4^4, 
\]
where \( p_i \) are prime numbers. The congruence condition is necessary here, since we have \( p^2 \equiv 1 \) or \( 49 \pmod{120} \) and \( p^4 \equiv 1 \pmod{120} \) for primes \( p > 5 \). Denote by \( R_3(n) \) the number of representations of natural number \( n \in \Omega \) as the form (1.1). By applying a pruning process into the Hardy-Littlewood method, we obtain the following result, which constitutes an improvement upon that of Friedlander and Wooley [2].

**Theorem 1.1.** For a function \( \psi \) of a positive variable \( t \), monotonically increasing to infinity and \( 0 < \psi(t) \ll \log t / (\log \log t) \), let \( E_3(N) \) be the number of integers \( n \in \Omega \) and \( n \leq N \) such that
\[
\left| R_3(n) - \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \mathcal{S}_3(n)n^{3/4} \log^5 n \right| \geq n^{3/4} \psi(n) \log^5 n, \]
where
\[
\mathcal{S}_3(n) = \sum_{q=1}^{\infty} \frac{1}{\varphi^5(q)} \sum_{a(q)\ast} S_2(q,a)^2S_4(q,a)^3e_q(-an) \quad \text{and} \quad S_k(q,a) = \sum_{r(q)\ast} e\left(\frac{ar^k}{q}\right). 
\]

Then for any \( \varepsilon > 0 \), we have
\[ E_3(N) \ll N^{23/48+\varepsilon} \psi^2(N). \]

2. Notations and some preliminary lemmas

In this paper, \( \varepsilon \in (0, 10^{-100}) \) and the value of \( \varepsilon \) may change from line to line. Let \( N \) denote a sufficiently large positive integer in terms of \( \varepsilon \). The constants in \( O \)-term and \( \ll \) symbol depend at most on \( \varepsilon \). The letter \( p \), with or without subscript, is reserved for a prime number. As usual, \( \varphi(n) \) denotes Euler’s function. We use \( e(\alpha) \) to denote \( e^{2\pi i\alpha} \) and \( e_q(\alpha) = e(\alpha/q) \). We denote by \( \sum_{x(q)\ast} \) a sum with \( x \) running over a reduced system of residues modulo \( q \). For a set \( \mathcal{F} \), \( |\mathcal{F}| \) denotes the cardinality of \( \mathcal{F} \).
Lemma 2.1. Let
\[ g_k(\alpha) = \sum_{2 \leq p \leq N^{1/k}} e(\alpha p^k). \]
Then for \( \alpha = a/q + \lambda, (a,q) = 1, q \leq Q \) and \( |\lambda| \leq Q/(qN) \), we have
\[ g_2(\alpha) \ll Q^{1/2}N^{11/40+\epsilon} + V_2(\alpha), \]
where
\[ (2.1) \quad V_2(\alpha) = \frac{N^{1/2}\log c N}{q^{1/2-\epsilon}(1 + N|\lambda|)^{1/2}} \]
and \( c > 0 \) denotes some absolute constant.

Proof. It follows from [9, Theorem 2].

Lemma 2.2. Let
\[ S_k(q,a) = \sum_{r(q)^*} e \left( \frac{ar^k}{q} \right). \]
Then for \((q,a) = 1\), we have
(i) \( |S_k(q,a)| \ll q^{1/2+\epsilon}; \)
(ii) \( |S_k(p,a)| \leq ((k,p - 1) - 1)p^{1/2} + 1; \)
(iii) \( S_k(p^l,a) = 0 \) for \( l \geq \gamma(p) \), where
\[ \gamma(p) = \begin{cases} \theta + 2 & \text{if } p^\theta \parallel k, p \neq 2 \text{ or } p = 2, \theta = 0, \\
\theta + 3 & \text{if } p^\theta \parallel k, p = 2, \theta > 0. \end{cases} \]

Proof. For (i), see [7, Lemma 8.5]. For (ii), see [11, Lemma 4.3]. For (iii), see [7, Lemma 8.3].

Lemma 2.3. Let \( 2 \leq k_1 \leq k_2 \leq \cdots \leq k_s \) be natural numbers such that
\[ \sum_{i=j+1}^{s} \frac{1}{k_i} \leq \frac{1}{k_j}, \quad 1 \leq j \leq s - 1. \]
Then we have
\[ \int_0^1 \left| \prod_{i=1}^{s} g_{k_i}(\alpha) \right|^2 d\alpha \leq N^{1/k_1+\cdots+1/k_s+\epsilon}. \]

Proof. By considering the number of solutions of the underlying equation, Lemma 2.3 follows from [1, Lemma 1].
Lemma 2.4. Let $\mathcal{F}(N)$ denote a subset of integers in the interval $(N/2, N]$ and $Z = |\mathcal{F}(N)|$. Let $\xi : \mathbb{Z} \to \mathbb{C}$ be a function with $|\xi(n)| \leq 1$ for all $n \in \mathbb{Z}$, and set
$$K(\alpha) = \sum_{n \in \mathcal{F}(N)} \xi(n)e(-n\alpha).$$

Then we have
\begin{enumerate}[(i)]  
  
  \item \[\int_0^1 |g_2(\alpha)^2 K(\alpha)^2| \, d\alpha \ll Z^2 N^\varepsilon + ZN^{1/2};\]
  
  \item \[\int_0^1 |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \ll N^{2/3 + \varepsilon} Z + N^{11/12 + \varepsilon} Z^{1/2}.\]
\end{enumerate}

Proof. By [12, (2.4)] and the bound $|\xi(n)| \leq 1$, we have
$$\int_0^1 |g_2(\alpha)^2 K(\alpha)^2| \, d\alpha = \sum_{p_1, p_2 \leq N^{1/2}} \sum_{m, n \in \mathcal{F}(N)} \xi(m)\overline{\xi(n)}$$
$$\ll \sum_{p_1, p_2 \leq N^{1/2}} \sum_{m, n \in \mathcal{F}(N)} 1$$
$$\ll Z^2 N^\varepsilon + ZN^{1/2}.$$ 

By Hölder’s inequality and (i), we have
$$\int_0^1 |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha$$
$$\ll \left(\int_0^1 |g_2(\alpha)^2 g_4(\alpha)^4| \, d\alpha\right)^{5/12} \left(\int_0^1 |g_4(\alpha)|^{16} \, d\alpha\right)^{1/12} \left(\int_0^1 |g_2(\alpha)^2 K(\alpha)^2| \, d\alpha\right)^{1/2}$$
$$\ll N^{5/12 + \varepsilon} N^{1/4 + \varepsilon} (Z^2 N^\varepsilon + ZN^{1/2})^{1/2}$$
$$\ll N^{2/3 + \varepsilon} Z + N^{11/12 + \varepsilon} Z^{1/2},$$

where Hua’s inequality and Lemma 2.3 are used. This completes the proof. \qed

In order to apply the Hardy-Littlewood method, we first define the Farey dissection. For this purpose, we set
$$A = 10^{100} (1 + c), \quad Q_0 = \log^A N, \quad Q_1 = N^{1/4} \quad \text{and} \quad Q_2 = N^{3/4},$$

where $c$ is defined by (2.1). For $(a, q) = 1$, $0 \leq a < q$, we put
$$M_0(q, a) = \left(\frac{a}{q} - \frac{Q_0^A}{N}, \frac{a}{q} + \frac{Q_0^A}{N}\right), \quad M(q, a) = \left(\frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2}\right),$$
$$\mathcal{M}_0 = \bigcup_{q \leq Q_0^A} \bigcup_{a=1}^q M_0(q, a), \quad \mathcal{M} = \bigcup_{q \leq Q_1} \bigcup_{a=1}^q M(q, a),$$
$$\mathcal{J} = \left(-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\right), \quad m_1 = \mathcal{J} \setminus \mathcal{M}, \quad m_2 = \mathcal{M} \setminus \mathcal{M}_0, \quad m = m_1 \cup m_2.$
Then we have the Farey dissection

\[ (2.2) \quad \mathcal{I} = \mathcal{M}_0 \cup \mathcal{m}. \]

**Lemma 2.5.** For \( \alpha \in \mathcal{m}_1 \), we have

\[ |g_2(\alpha)| \ll N^{7/16+\varepsilon}. \]

**Proof.** It follows from [3, Theorem 1]. \( \Box \)

**Lemma 2.6.** For \((a, q) = 1\), let \( \mathcal{R}_0(q, a) = \left( \frac{a}{q} - \frac{1}{qN^{7/8}}, \frac{a}{q} + \frac{1}{qN^{7/8}} \right) \). Then we have

(i) \[
\sum_{q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} \int_{\mathcal{R}_0(q, a)} |V_2(\alpha)|^{1/6} \, d\alpha \ll N^{-77/96+\varepsilon};
\]

(ii) \[
\sum_{q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} \int_{\mathcal{R}_0(q, a)} |V_2(\alpha)|^2 \, d\alpha \ll Q_0^3,
\]

where \( V_2(\alpha) \) is defined by \((2.1)\).

**Proof.** By \((2.1)\), we have

\[
\sum_{q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} \int_{\mathcal{R}_0(q, a)} |V_2(\alpha)|^{1/6} \, d\alpha \ll \sum_{q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} q^\varepsilon \int_{|\lambda| \leq 1/(qN^{7/8})} \frac{N^{1/12} \log^{c/6} N}{(q + qN|\lambda|)^{1/12}} \, d\lambda
\]

\[
\ll N^{-11/12+\varepsilon} \sum_{q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} q^{-1/12+\varepsilon} \int_{|u| \leq N^{1/8}/q} \frac{1}{(1 + u)^{1/12}} \, du
\]

\[
\ll N^{-11/12+\varepsilon} Q_0^{23/12+\varepsilon} \int_0^{N^{1/8}} \frac{1}{(1 + u)^{1/12}} \, du \ll N^{-77/96+\varepsilon}.
\]

Now, (i) is proved, and (ii) can be proved by similar arguments. \( \Box \)

**Lemma 2.7.** Let

\[ v_k(\lambda) = \sum_{2 < n \leq N} \frac{e(n\lambda)}{n^{1-1/k} \log n}. \]

Then for \( \alpha = a/q + \lambda \in \mathcal{M}_0 \), we have

\[ g_k(\alpha) = \frac{S_k(q, a)}{\varphi(q)} v_k(\lambda) + O(N^{1/k} \exp(-\log^{1/3} N)). \]
Proof. See [7] Lemma 7.15.

Lemma 2.8. Let $\Omega = \{n \in \mathbb{N} : n \equiv 5, 53, 101 \pmod{120}\}$, and let

$$A_3(q, n) = \frac{1}{\varphi^5(q)} \sum_{a(q) *} S_2(q, a)^2 S_4(q, a)^3 e_q(-an) \quad \text{and} \quad \mathcal{S}_3(n) = \sum_{q=1}^{\infty} A_3(q, n).$$

Then the series $\mathcal{S}_3(n)$ is convergent and $\mathcal{S}_3(n) > 0$ for $n \in \Omega$.

Proof. The convergence of $\mathcal{S}_3(n)$ follows from Lemma 2.2(i). By Lemma 2.2(iii) and the fact that $A_3(q, n)$ is multiplicative in $q$, we get

$$\mathcal{S}_3(n) = (1 + A_3(2, n) + A_3(4, n) + A_3(8, n) \prod_{p>2} (1 + A_3(p, n))).$$

When $p > 22$, we conclude from Lemma 2.2(ii) that

$$|A_3(p, n)| \leq \frac{(p^{1/2} + 1)^2(3p^{1/2} + 1)^3}{(p-1)^4} \leq \frac{100}{p^{3/2}}.$$

So we get

$$\prod_{p>22} (1 + A_3(p, n)) \geq \prod_{p>22} \left(1 - \frac{100}{p^{3/2}}\right) > c > 0.$$

Let $L(q, n)$ denote the number of solutions to the congruence

$$x_1^2 + x_2^3 + x_3^4 + x_4^4 + x_5^4 \equiv n \pmod{q}, \quad 1 \leq x_i \leq q, \ (x_i, q) = 1.$$

Then by [7] Lemma 8.6, we have

$$1 + A_3(2, n) + A_3(4, n) + A_3(8, n) = \frac{L(8, n)}{2^7},$$

$$1 + A_3(p, n) = \frac{pL(p, n)}{(p-1)^5}.$$

For $n \equiv 5, 53, 101 \pmod{120}$, it is easy to verify that

$$L(8, n) > 0 \quad \text{and} \quad L(p, n) > 0 \quad \text{for} \ 2 < p \leq 19.$$

Now the conclusion $\mathcal{S}_3(n) > 0$ follows from (2.3)–(2.7).

3. Mean value estimates

Let

$$I_j = \int_{m_j} |g_2(\alpha)^2 g_4(\alpha)^3 K(\alpha)| \, d\alpha, \quad j = 1, 2,$$

where $K(\alpha)$ is defined as in Lemma 2.4.
Proposition 3.1. We have
\[ I_1 \ll N^{71/96+\varepsilon}Z + N^{95/96+\varepsilon}Z^{1/2}, \]
where \( Z \) is defined as in Lemma 2.4.

Proof. By Lemma 2.4(ii) and Lemma 2.5, we have
\[ I_1 \ll \sup_{\alpha \in \mathfrak{m}_1} |g_2(\alpha)|^{1/6} \int_0^1 |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \]
\[ \ll N^{7/96+\varepsilon} (N^{2/3+\varepsilon} Z + N^{11/12+\varepsilon} Z^{1/2}) \]
\[ \ll N^{71/96+\varepsilon} Z + N^{95/96+\varepsilon} Z^{1/2}. \]

Proposition 3.2. We have
\[ I_2 \ll N^{3/4} Q_0^{-A/4} Z + N^{95/96+\varepsilon} Z^{1/2}. \]

Proof. For \( \alpha \in \mathfrak{m}_2 \), it follows from Lemma 2.1 with \( Q = N^{1/4} \) that
\[ |g_2(\alpha)| \ll V_2(\alpha) + N^{2/5+\varepsilon}, \]
where \( V_2(\alpha) \) is defined by (2.1). By (3.1) and Lemma 2.4(ii), we get
\[ I_2 \ll \int_{\mathfrak{m}_2} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \]
\[ \ll N^{1/5+\varepsilon} \int_0^1 |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \]
\[ \ll N^{11/15+\varepsilon} Z + N^{59/60+\varepsilon} Z^{1/2}. \]
Let
\[ \mathfrak{N}_0(q,a) = \left( \frac{a}{q} - \frac{1}{q N^{7/8}}, \frac{a}{q}, 1 + \frac{1}{q N^{7/8}} \right), \quad \mathfrak{N}(q,a) = \left( \frac{a}{q} - \frac{1}{q Q_0}, \frac{a}{q}, 1 + \frac{1}{q Q_0} \right), \]
and
\[ \mathfrak{N}_1(q,a) = \mathfrak{N}(q,a) \setminus \mathfrak{N}_0(q,a). \]
From Dirichlet’s approximation theorem, we have
\[ \int_{\mathfrak{m}_2} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \]
\[ \leq \sum_{q \leq Q_0} \sum_{a=-q}^{2q} \int_{\mathfrak{m}_2 \cap \mathfrak{N}_1(q,a)} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \]
\[ + \sum_{q \leq Q_0} \sum_{a=-q}^{2q} \int_{\mathfrak{m}_2 \cap \mathfrak{N}_0(q,a)} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha. \]
For $\alpha = a/q + \lambda \in \mathcal{N}_1(q, a)$, it is easy to see that $q(1 + N|\lambda|) \gg N^{1/8}$, hence
\begin{equation}
(3.4) \quad \sup_{\alpha \in \mathcal{N}_1(q, a)} |V_2(\alpha)| \ll N^{7/16+\varepsilon}.
\end{equation}

By (3.4), we obtain
\begin{equation}
(3.5) \quad \sum_{q \leq Q_0} \sum_{a = -q}^{2q} \int_{m_2 \cap \mathcal{N}_1(q, a)} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \\
\ll \sup_{\alpha \in \mathcal{N}_1(q, a)} |V_2(\alpha)|^{1/6} \int_{0}^{1} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \\
\ll N^{7/96+\varepsilon} (N^{2/3+\varepsilon} Z + N^{11/12+\varepsilon} Z^{1/2}) \\
\ll N^{71/96+\varepsilon} Z + N^{95/96+\varepsilon} Z^{1/2},
\end{equation}

where Lemma 2.4(ii) is used. For $\alpha \in m_2$, we have $q + qN|\lambda| \gg Q_0^4$. Then it follows from [8, Lemma 3.3] that
\begin{equation}
(3.6) \quad \sup_{\alpha \in \mathcal{m}_2} |g_4(\alpha)| \ll N^{31/128+\varepsilon} + \frac{N^{1/4} \log^4 N}{q^{1/8-\varepsilon}(1 + N|\lambda|)^{1/8}} \ll \frac{N^{1/4}}{Q_0^{4/9}}.
\end{equation}

Moreover for $\alpha \in \mathcal{N}_0(q, a)$, by Lemma 2.1 with $Q = N^{1/8}$, we have
\begin{equation}
(3.7) \quad |g_2(\alpha)| \ll V_2(\alpha) + N^{27/80+\varepsilon}.
\end{equation}

From (3.6), (3.7) and Lemma 2.6(i)(ii), we get
\begin{align}
&\sum_{q \leq Q_0} \sum_{a = -q}^{2q} \int_{m_2 \cap \mathcal{N}_0(q, a)} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \\
&\ll \sum_{q \leq Q_0} \sum_{a = -q}^{2q} \int_{m_2 \cap \mathcal{N}_0(q, a)} |V_2(\alpha)|^{1/6} |g_2(\alpha)|^{11/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \\
&\quad + N^{99/160+\varepsilon} \sum_{q \leq Q_0} \sum_{a = -q}^{2q} \int_{m_2 \cap \mathcal{N}_0(q, a)} |V_2(\alpha)|^{1/6} |g_4(\alpha)^3 K(\alpha)| \, d\alpha \\
&\ll Z \sup_{\alpha \in \mathcal{m}_2} |g_4(\alpha)|^{3} \sum_{q \leq Q_0} \sum_{a = -q}^{2q} \int_{\mathcal{N}_0(q, a)} |V_2(\alpha)|^{2} \, d\alpha \\
&\quad + ZN^{99/160+\varepsilon} \sup_{\alpha \in \mathcal{m}_2} |g_4(\alpha)|^{3} \sum_{q \leq Q_0} \sum_{a = -q}^{2q} \int_{\mathcal{N}_0(q, a)} |V_2(\alpha)|^{1/6} \, d\alpha \\
&\ll ZN^{3/4} Q_0^{-3A/8} Q_0^3 + ZN^{99/160+\varepsilon} N^{3/4} Q_0^{-A/2} N^{77/96+\varepsilon} \\
&\ll N^{3/4} Q_0^{-A/4} Z,
\end{align}
where the trivial bound $|K(α)| ≤ Z$ is used. Now from (3.2), (3.3), (3.5) and (3.8), we get

$$I_2 \ll N^{3/4}Q_0^{-A/4}Z + N^{65/96+ε}Z^{1/2}.$$ 

**Proposition 3.3.** For $N/2 \leq n \leq N$, we have

$$\int_{\mathfrak{m}_0} g_2(α)^2g_4(α)^3e(-nα) dα = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \mathfrak{S}_3(n)n^{3/4} + O \left( \frac{n^{3/4}\log\log n}{\log^6 n} \right).$$

**Proof.** For $α = a/q + λ$, let $f_k(α) = \frac{S_k(q,α)}{φ(q)}v_k(λ)$. Then it follows from Lemma 2.7 that

$$\int_{\mathfrak{m}_0} g_2(α)^2g_4(α)^3e(-nα) dα = \int_{\mathfrak{m}_0} f_k(α)^2g_4(α)^3e(-nα) dα + O(n^{3/4}\exp(-\log^{1/4} n)).$$

It is easy to see that

$$\int_{\mathfrak{m}_0} f_k(α)^2g_4(α)^3e(-nα) dα = \sum_{q ≤ Q_0^A} A_3(q, n) \int_{|λ| ≤ Q_0^{A}/N} v_2(λ)^2v_4(λ)^3e(-nλ) dλ.$$

It follows from [7] Lemma 7.16] that

$$\int_{|λ| ≤ Q_0^{A}/N} v_2(λ)^2v_4(λ)^3e(-nλ) dλ = \int_0^1 v_2(λ)^2v_4(λ)^3e(-nλ) dλ + O \left( \int_0^1 \frac{1}{λ^{7/4}\log^2 N} dλ \right)$$

$$= \int_0^1 v_2(λ)^2v_4(λ)^3e(-nλ) dλ + O(N^{3/4}Q_0^{-3A/4}).$$

Similar to [7] Lemma 7.19], we have

$$\int_0^1 v_2(λ)^2v_4(λ)^3e(-nλ) dλ = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} n^{3/4} + O \left( \frac{n^{3/4}\log\log n}{\log^6 n} \right).$$

By (3.11) and (3.12), we have

$$\int_{|λ| ≤ Q_0^{A}/N} v_2(λ)^2v_4(λ)^3e(-nλ) dλ = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} n^{3/4} + O \left( \frac{n^{3/4}\log\log n}{\log^6 n} \right).$$

From Lemma 2.2(i) and the inequality $φ(q) ≥ q/\log q$, we get

$$\sum_{q ≤ Q_0^A} A_3(q, n) = \mathfrak{S}_3(n) + O \left( \sum_{q > Q_0^A} q^{-3/2+ε} \right) = \mathfrak{S}_3(n) + O(Q_0^{-A/2+ε}).$$

Now combining (3.9), (3.10), (3.13) and (3.14), we have

$$\int_{\mathfrak{m}_0} g_2(α)^2g_4(α)^3e(-nα) dα = \frac{\Gamma^2(1/2)\Gamma^3(1/4)}{\Gamma(7/4)} \mathfrak{S}_3(n)n^{3/4} + O \left( \frac{n^{3/4}\log\log n}{\log^6 n} \right).$$
4. Proof of Theorem 1.1

By the Farey dissection (2.2), we have

\[(4.1) \quad R_3(n) = \int_{m_0} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha) \, d\alpha + \int_{m} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha) \, d\alpha.\]

Let \(\psi\) be a function of positive variable \(t\), monotonically increasing to infinity and \(0 < \psi(t) \ll \log t / (\log \log t)\). By Proposition 3.3 and Lemma 2.8, we may define \(\mathcal{F}(N)\) to be the set of integers \(n \in \Omega, \, N/2 \leq n \leq N\) such that

\[(4.2) \quad \left| R_3(n) - \Gamma^2(1/2) \Gamma^3(1/4) \, \frac{S_3(n) n^{3/4}}{\log^5 n} \right| \geq \frac{n^{3/4}}{\psi(n) \log^5 n}.\]

For \(n \in \mathcal{F}(N)\), by (4.1), (4.2) and Proposition 3.3, we get

\[(4.3) \quad \left| \int_{m} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha) \, d\alpha \right| \geq \frac{n^{3/4}}{\psi(n) \log^5 n}.\]

For \(n \in \mathcal{F}(N)\), let \(\xi(n)\) be defined by the following equation

\[(4.4) \quad \left| \int_{m} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha) \, d\alpha \right| = \xi(n) \int_{m} g_2(\alpha)^2 g_4(\alpha)^3 e(-n\alpha) \, d\alpha.\]

Then it is easy to see that \(|\xi(n)| \leq 1\). Write \(Z(N) = |\mathcal{F}(N)|\). From (4.3), (4.4), we have

\[(4.5) \quad \frac{Z(N) N^{3/4}}{\psi(N) \log^5 N} \ll \sum_{n \in \mathcal{F}(N)} \frac{n^{3/4}}{\psi(n) \log^5 n} \ll \int_{m} |g_2(\alpha)^2 g_4(\alpha)^3 K(\alpha)| \, d\alpha \ll \left( \int_{m_1} + \int_{m_2} \right) |g_2(\alpha)^2 g_4(\alpha)^3 K(\alpha)| \, d\alpha,
\]

where

\[K(\alpha) = \sum_{n \in \mathcal{F}(N)} \xi(n) e(-n\alpha).\]

From (4.5), Propositions 3.1 and 3.2, we obtain

\[(4.6) \quad \frac{Z(N) N^{3/4}}{\psi(N) \log^5 N} \ll N^{3/4} Q_0^{-A/4} Z(N) + N^{95/96+\varepsilon} Z(N)^{1/2}.\]

It follows from (4.6) that

\[(4.7) \quad Z(N) \ll N^{23/48+\varepsilon} \psi^2(N).\]

Now by (4.7), we have

\[E_3(N) \ll N^{1/3} + \sum_{1 \leq 2^j \leq N^{2/3}} Z \left( \frac{N}{2^j} \right) \ll N^{23/48+\varepsilon} \psi^2(N),\]

and the proof of the Theorem 1.1 is completed.
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