Period differential equations for families of $K3$ surfaces derived from 3 dimensional reflexive polytopes with 5 vertices

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Abstract

In this article we study the families of $K3$ surfaces derived from 3 dimensional 5 verticed reflexive polytopes with at most terminal singularity. We determine the lattice structures, the period differential equations and the projective monodromy groups for these families.

Introduction

A $K3$ surface $S$ is characterized by the condition $K_S = 0$ and simply connectedness. It means that $S$ is a 2-dimensional Calabi-Yau manifold. V. V. Batyrev [Ba] introduced the notion of the reflexive polytope for the study of Calabi-Yau manifolds.

In this article we use the 3-dimensional reflexive polytopes with at most terminal singularities. Such a polytope $P$ is defined by the intersection of several half spaces

$$a_jx + b_jy + c_jz \leq 1, \quad (a_j, b_j, c_j) \in \mathbb{Z}^3 \quad (j = 1, \cdots, s)$$

in $\mathbb{R}^3$ with the conditions

(i) every vertex is a lattice point,
(ii) the origin is the unique inner lattice point,
(iii) only the vertices are the lattice points on the boundary.

Moreover, if a reflexive polytope satisfies the condition

(iv) every face is triangle and its 3 vertices generate the lattice,

it is called a Fano polytope.

All 3-dimensional 5-vertexed reflexive polytopes with at most terminal singularity are listed up (see [KS] or [O]):

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}.$$

Among them, $P_2, P_3, P_4$ and $P_5$ are Fano polytopes.

We can find a family of $K3$ surfaces for each polytope by a natural method. In this article we study the polytopes $P_1, P_2$ and $P_3$. Namely, we determine the lattice structure, the period differential equation and the projective monodromy group for each of them.

Keywords: $K3$ surfaces ; period differential equations ; toric varieties

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T. Ishige \[I\] has made a detailed research on the family of $K3$ surfaces coming from the polytope $P_4$. Especially he noticed characterization of the corresponding monodromy group by a numerical approach.

Inspired by Ishige’s discovery, we have studied families of $K3$ surfaces derived from the other polytopes $P_1, P_2, P_3$ and $P_5$. We have made an intensive study on the polytope $P_5$ in our previous article \[Na\]. There, we have studied the period map for a family, saying $F = \{S(\lambda, \mu)\}$, of $K3$ surfaces, where

$$S(\lambda, \mu) : xyz^2(x + y + z + 1) + \lambda xyz + \mu.$$  

Namely, we have determined the lattice structure of a generic member of the family $F$, the period differential equation and the projective monodromy group using the Torelli type theorem for polarized $K3$ surfaces and the lattice theory. Furthermore, we have shown that our differential equation coincides with the uniformizing differential equation of the Hilbert modular orbifold for $Q(\sqrt{5})$ studied by Sasaki-Yoshida \[SY\] and T. Sato \[Sa\].

Here, we study the remaining cases $P_1, P_2$ and $P_3$. Namely we investigate corresponding families $F_j$ ($j = 1, 2, 3$) of $K3$ surfaces.

In Section 1, we show explicit defining equations for the families $F_j = \{S_j(\lambda, \mu)\}$ ($j = 1, 2, 3$) and $F_5$ and we introduce elliptic fibrations for these families. The singular fibres of each elliptic fibration are described in Table 1.

In Section 2, we determine the lattice structure for a generic member of each family $F_j$ ($j = 1, 2, 3$). Namely, we obtain the Néron-Severi lattice $NS(S_j(\lambda, \mu))$ ($j = 1, 2, 3$) as in Table 2. Note that in the case $P_5$ we could determine $NS(S(\lambda, \mu))$ for $S(\lambda, \mu) \in F$ by a naive method (see \[Na\]). In this article we need more advanced theory of the Mordell-Weil lattice due to T. Shiota \[Sh1\].

For 95 weighted projective $K3$ surfaces, there is a result of S. M. Belcastro \[Be\]. And for $K3$ surfaces with non-symplectic involution, there is a result of V. V. Nikulin \[Ni\]. Our case is not contained in these results. Furthermore, we note that the result of K. Koike \[Koi\] and our result in this article support the mirror symmetry conjecture (see Remark \[2.1\]).

In Section 3, we determine the period differential equations (Theorem \[3.2\]). Furthermore, we obtain their monodromy groups (Theorem \[3.3\]).

1 Families of $K3$ surfaces and elliptic fibrations

We obtain a family of algebraic surfaces by the following canonical procedure from $P_j$ ($j = 1, 2, 3$):

(i) Make a toric 3-fold $X_j$ from the reflexive polytope $P_j$. This is a rational variety.

(ii) Take a divisor $D$ on $X_j$ that is linearly equivalent to $-K_{X_j}$.

(iii) Generically $D$ is represented by a $K3$ surface.

We obtain the corresponding families of $K3$ surfaces $F_j = \{S_j(\lambda, \mu)\}$ for $P_j$ ($j = 1, 2, 3$) given by

$$S_1(\lambda, \mu) : xyz(x + y + z + 1) + \lambda x + \mu y = 0,$$

$$S_2(\lambda, \mu) : xyz(x + y + z + 1) + \lambda x + \mu = 0,$$

$$S_3(\lambda, \mu) : xyz(x + y + z + 1) + \lambda z + \mu xy = 0.$$  

We can find an elliptic fibration for every surface of our family $F_j$ ($j = 1, 2, 3$). Moreover we can describe these surfaces in the form

$$y^2 = x^3 - g_2(z)x - g_3(z),$$

where $g_2$ ($g_3$, resp.) is a polynomial of $z$ with $5 \leq \deg(g_2) \leq 8$ ($7 \leq \deg(g_3) \leq 12$, resp.). In this paper we call it the Kodaira normal form. From the Kodaira normal form we can obtain singular fibres of elliptic fibration. Corresponding singular fibres of our elliptic fibration of $F_j$ ($j = 1, 2, 3$) are shown in Table 1.
Family & $\mathcal{F}_1$ & $\mathcal{F}_2$ & $\mathcal{F}_3$
--- & --- & --- & ---
Singular Fibres & $I_9 + I_3 + 6I_1$ & $I_1^1 + I_1 + 6I_1$ & $I_9 + I_9 + 6I_1$

Table 1.

1.1 $\mathcal{F}_1$

**Proposition 1.1.** (1) The surface $S_1(\lambda, \mu)$ is birationally equivalent to the surface defined by the equation

$$z_1^2 = y_1^3 + (\mu^2 + 2\mu x_1 + x_1^2 - 4x_1^3)y_1^2 + (-8\lambda\mu x_1^3 - 8\lambda x_1^4)y_1 + 16\lambda^2 x_1^6.$$  \hfill (1.4)

This equation gives an elliptic fibration of $S_1(\lambda, \mu)$.

(2) The elliptic surface given by (1.4) has the holomorphic section

$$P : x_1 \mapsto (x_1, y_1, z_1) = (x_1, 0, 4\lambda x_1^3).$$  \hfill (1.5)

**Proof.** (1) By the birational transformation

$$x = -\frac{2x_1^2 y_1}{-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1}, \quad y = \frac{y_1^2}{2x_1(-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1)}, \quad z = -\frac{4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1}{2x_1 y_1},$$

(1.6) is transformed to (1.4).

(2) This is apparent.

(1.4) gives an elliptic fibration for the surface $S_1(\lambda, \mu)$. Set

$$\Lambda_1 = \{(\lambda, \mu) \in \mathbb{C}^2 | \lambda \mu(729\lambda^2 - 54\lambda(27\mu - 1) + (1 + 27\mu)^2 \neq 0)\}.$$  \hfill (1.6)

**Proposition 1.2.** Suppose $(\lambda, \mu) \in \Lambda_1$. The elliptic surface given by (1.4) has the singular fibres of type $I_9$ over $x_1 = 0$, of type $I_3$ over $x_1 = \infty$, and other six fibres of type $I_1$.

**Proof.** (1.4) is described in the Kodaira normal form

$$z_1^2 = y_2^3 - g_2(x_1)y_2 - g_3(x_1), \quad x_1 \neq \infty,$

with

$$\begin{align*}
g_2(x_1) &= -\left(-\frac{\mu}{3} - \frac{4\mu^3 x_1^3}{3} - 2\mu x_1^2 + \frac{4\mu x_1^3}{3} - 8\lambda\mu x_1^3 + x_1^4 - 8\lambda x_1^4 + \frac{16\mu x_1^3}{3} + \frac{8x_1^5}{3} - \frac{16x_1^6}{3}\right), \\
g_3(x_1) &= -\left(\frac{2\mu^6}{27} + \frac{4\mu^5 x_1}{9} + \frac{10\mu^4 x_1^2}{9} + \frac{4\mu^3 x_1^3}{3} + \frac{8\mu^2 x_1^4}{3} + \frac{8\mu x_1^5}{3} + \frac{10\mu x_1^6}{3}\right. \\
&\quad \left.+ 32\lambda^2 x_1^6 + \frac{32\mu^2 x_1^6}{9} + \frac{8\lambda x_1^7}{3} + \frac{16\mu x_1^7}{3} + \frac{2x_1^6}{27} + \frac{8x_1^5}{9} + 16\lambda^2 x_1^6 - \frac{32\mu x_1^6}{9}\right) \\
&\quad - \frac{32\lambda x_1^7}{3} + \frac{32\mu x_1^7}{9} - \frac{8x_1^7}{9} - \frac{32x_1^7}{3} + \frac{64\mu x_1^7}{9} + \frac{32\mu x_1^7}{9} - \frac{128x_1^9}{27},
\end{align*}$$

and

$$z_1^2 = y_3^3 - h_2(x_2)y_3 - h_3(x_2), \quad x_2 \neq \infty,$

with

$$\begin{align*}
h_2(x_2) &= -\left(-\frac{16x_2^2}{3} + \frac{8x_2^3}{3} - \frac{x_2^4}{3} - 8\lambda x_2^4 + \frac{16\mu x_2^3}{3} - \frac{4\mu x_2^5}{3} - 8\lambda x_2^5 + \frac{8\mu^2 x_2^5}{3} - 2\mu^2 x_2^6 - \frac{4\mu^3 x_2^7}{3} - \frac{\mu^4}{3}\right), \\
h_3(x_2) &= -\left(-\frac{128x_2^3}{27} + \frac{32x_2^3}{9} - \frac{8x_2^3}{9} - \frac{32\lambda x_2^5}{9} + \frac{64\mu x_2^3}{9} + \frac{2x_2^6}{27} + \frac{8x_2^5}{3} + 16\lambda^2 x_2^6 - \frac{32\mu x_2^5}{9}\right) \\
&\quad - \frac{32\lambda x_2^7}{3} + \frac{32\mu x_2^7}{9} + \frac{4\mu x_2^7}{9} + 8\lambda x_2^7 - \frac{16\mu x_2^7}{3} + \frac{10\mu x_2^8}{9} + 8\lambda^2 x_2^8 - \frac{32\mu x_2^8}{9}\right) \\
&\quad + \frac{40\mu^6 x_2^{11}}{27} + \frac{8\lambda x_2^{10}}{9} + \frac{8\mu^4 x_2^{10}}{9} + \frac{10\mu^2 x_2^{10}}{9} + \frac{4\mu^5 x_2^{11}}{9} + \frac{2\mu^6 x_2^{12}}{27},
\end{align*}$$

\(3\)
where \( x_1 = 1/x_2 \). We have the discriminant of the right hand side for \( y_1(y_2, \text{resp.}) \):
\[
\begin{align*}
D_0 &= 256\lambda^2x_1^2(\lambda y^3 - \mu + 3\lambda y^2x_1 - 4\mu^3x_1 + 3\lambda x_1^2 - 6\mu^2x_1^2 + \lambda x_1^3 + 27\lambda^2x_1^3 \\
&\quad - 4\mu x_1^3 - 36\lambda x_1^2 + 8\mu^2x_1^2 - x_1^4 - 36x_1^4 + 16\lambda x_1^4 + 8x_1^4 - 16x_1^4), \\
D_\infty &= 256\lambda^2x_2^2(-16 + 8x_2 - x_2^2 - 36\lambda x_2^2 + 16\mu x_2^2 + \lambda x_2^3 + 27\lambda^2x_2^3 - 4\mu x_2^3 - 36\lambda x_2^3 \\
&\quad 8\mu^2x_2^3 + 3\lambda x_2^4 - 6\mu^2x_2^4 + 3\lambda^2x_2^4 - 4\mu^3x_2^4 + \lambda^3x_2^4 - \mu^4x_2^4).
\end{align*}
\]
From these data, we obtain the required statement (see [Kod]).

\[1.2\]

\[\mathcal{F}_2\]

**Proposition 1.3.** (1) The surface \( S_2(\lambda, \mu) \) is birationally equivalent to the surface defined by the equation
\[
z_1^2 = x_1^3 + (-4\lambda y + y^2 + 2y^3 + y^4)x_1^2 + (-8\mu y^3 - 8\mu y^4)x_1 + 16\mu^2 y^4. \tag{1.7}
\]
This equation gives an elliptic fibration of \( S_2(\lambda, \mu) \).

(2) The elliptic surface given by (1.7) has the holomorphic section
\[
P : y \mapsto (x_1, y, z_1) = (0, y, 4\mu y^2) \tag{1.8}
\]

**Proof.** (1) By the birational transformation
\[
x = \frac{x_1^2}{2y(x_1 y - 4\mu y^2 + x_1 y + z_1)}, \quad z = -\frac{x_1 y - 4\mu y^2 + x_1 y + z_1}{2x_1 y},
\]
(1.2) is transformed to (1.7).

(2) This is apparent.

(1.7) gives an elliptic fibration for \( S_2(\lambda, \mu) \). Set
\[
\Lambda_2 = \{ (\lambda, \mu) \in C^2 | \lambda \mu(\lambda^2 + 2\lambda^2 - 2\lambda(1 + 189\lambda) + (1 + 576\lambda)\mu^2 - 256\mu^3) \neq 0 \}. \tag{1.9}
\]

**Proposition 1.4.** Suppose \( (\lambda, \mu) \in \Lambda_2 \). The elliptic surface given by (1.7) has the singular fibres of type \( I_1^* \) over \( y = 0 \), of type \( I_{11} \) over \( y = \infty \), and other six fibres of type \( I_1 \).

**Proof.** (1.7) is described in the Kodaira normal form
\[
z_1^2 = x_1^3 - g_2(y)x_2 - g_3(y), \quad y \neq \infty,
\]
with
\[
\begin{align*}
g_2(y) &= -(\frac{-16\lambda y^3}{3} + \frac{8\lambda y^3}{3} - 8\mu y^3 - \frac{y^4}{3} + \frac{16\lambda y^4}{3} - 8\mu y^4 - \frac{4y^5}{3} + \frac{8\lambda y^5}{3} - 2y^6 - \frac{4y^7}{3} - \frac{y^8}{3}), \\
g_3(y) &= -\biggl(\frac{-128\lambda^3 y^3}{27} + \frac{32\lambda^2 y^4}{9} - \frac{32\lambda^2 y^4}{3} + 16\mu^2 y^4 - 8\lambda y^5 + \frac{64\lambda^2 y^5}{9} + 8\mu y^5 - \frac{32\lambda y^5}{3} + \frac{2y^6}{27} - \frac{32\lambda^2 y^6}{9} \\
&\quad + 8\mu y^6 + \frac{4y^7}{9} - \frac{16\lambda y^7}{3} + 8\mu y^7 + \frac{10y^8}{9} - \frac{32\lambda y^8}{9} + \frac{8y^9}{3} + \frac{40y^9}{27} - \frac{8y^9}{9} + \frac{10y^{10}}{9} + \frac{4y^{11}}{9} + \frac{2y^{12}}{27}\biggr),
\end{align*}
\]
and
\[
z_2^2 = x_2^3 - h_2(y_1)x_2 - h_3(y_1), \quad y_1 \neq \infty,
\]
with
\[
\begin{align*}
h_2(y_1) &= -\biggl(\frac{-1}{3} - \frac{4y_1}{3} - \frac{2y_1^2}{3} + \frac{8\lambda y_1^3}{3} - \frac{y_1^4}{3} + \frac{116\lambda y_1^4}{3} - 8\mu y_1^4 + \frac{8\lambda y_1^5}{3} - 16\lambda y_1^5 \biggr), \\
h_3(y_1) &= -\biggl(\frac{2}{27} + \frac{4y_1}{9} + \frac{10y_1^2}{9} + \frac{40y_1^3}{27} - \frac{8\lambda y_1^4}{9} + \frac{10y_1^4}{9} + \frac{32\lambda y_1^5}{3} + \frac{8y_1^5}{3} - 16\lambda y_1^5 \biggr) \\
&\quad + \frac{2y_1^6}{27} - \frac{32\lambda y_1^6}{9} + \frac{32\lambda^2 y_1^6}{9} + 8\mu y_1^6 - \frac{8\lambda y_1^7}{9} + \frac{64\lambda^2 y_1^7}{9} + \frac{8\mu y_1^7}{3} \\
&\quad - \frac{32\lambda y_1^8}{3} + \frac{32\lambda^2 y_1^8}{3} - \frac{32\mu y_1^8}{3} + 16\mu^2 y_1^8 - \frac{128\lambda^3 y_1^8}{27}\biggr),
\end{align*}
\]
where \( y = 1/y_1 \). We have the discriminant of the right hand side for \( x_2(x_3, \text{ resp.}): \)

\[
\begin{align*}
D_0 &= -256\mu^2y^7(16\lambda^3 - 8\lambda^2y + 36\mu\lambda y - 27\mu^2y^2 + \lambda y^2 - 16\lambda^2y^2 - \mu y^2 + 36\mu\lambda y^2 + 4\lambda y^3 \\
-8\lambda^2y^3 - 3\mu y^3 + 6\lambda y^4 - 3\mu y^4 + 4\lambda y^5 - \mu y^5 + \lambda y^6), \\
D_\infty &= -256\mu^2y_1^7(\lambda + 4\lambda y_1 - \mu y_1 + 6\lambda y_1^2 - 3\mu y_1^2 + 4\lambda y_1^3 - \lambda y_1^3 - \mu y_1^3 + \lambda^3 y_1^3 \\
-\mu y_1^3 + 36\lambda^2 y_1^4 - 8\lambda^2y_1^4 + 36\lambda y_1^5 - 27\mu^2y_1^5 + 16\lambda^3y_1^6).
\end{align*}
\]

From these data, we obtain the required statement.

\[\square\]

### 1.3 \( \mathcal{F}_3 \)

**Proposition 1.5.** (1) The surface \( S_3(\lambda, \mu) \) is birationally equivalent to the surface defined by the equation

\[
y_1^2 = z_1^3 + (\lambda^2 + 2\lambda x_1 + x_1^2 - 4\mu x_1^2 - 4x_1^3)z_1^2 + 16\mu x_1^5.
\]

This equation gives an elliptic fibration of \( S_3(\lambda, \mu) \).

(2) The elliptic surface given by (1.10) has the holomorphic sections

\[
\begin{align*}
P : z_1 \mapsto (x_1, y_1, z_1) &= (x_1, 4\mu x_1^2(x_1 + \lambda), 4x_1^2\mu), \\
O' : z_1 \mapsto (x_1, y_1, z_1) &= (x_1, 0, 0).
\end{align*}
\]

The section \( O' \) satisfies \( 2O' = O \).

**Proof.** (1) By the birational transformation

\[
x = \frac{2x_1^4(4\mu x_1^2 - z_1)}{y_1 + \lambda z_1 + x_1 z_2}, \quad y = \frac{y_1 + \lambda z_1 + x_1 z_1}{2x_1(4\mu x_1^2 - z_1)}, \quad z = -\frac{z_1(4\mu x_1^2 - z_1)}{2x_1(y_1 + \lambda z_1 + x_1 z_1)},
\]

(1.3) is transformed to (1.10).

(2) This is apparent.

(1.10) gives an elliptic fibration for \( S_3(\lambda, \mu) \). Set

\[
\Lambda_3 = \{ (\lambda, \mu) \in \mathbb{C}^2 | \lambda\mu(729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1 + 12\mu)) \neq 0 \}.
\]

**Proposition 1.6.** Suppose \((\lambda, \mu) \in \Lambda_3 \). The elliptic surface given by (1.10) has the singular fibres of type \( I_{10} \) over \( z = 0 \), of type \( I_2^* \) over \( z = \infty \), and other six fibres of type \( I_1 \).

**Proof.** (1.10) is described in the Kodaira normal form

\[
y_1^2 = z_3^3 - g_2(x_1)z_2 - g_3(x_1), \quad x_1 \neq \infty,
\]

with

\[
\begin{align*}
g_2(x_1) &= -\left( -\frac{\lambda^4}{3} - \frac{4\lambda^3 x_1}{3} - 2\lambda^2 x_1^2 + \frac{8\lambda^2 \mu x_1^3}{3} - \frac{4\lambda x_1^4}{3} + \frac{8\lambda^2 x_1^3}{3} + \frac{16\lambda \mu x_1^4}{3} \\
&- \frac{x_1^4}{3} + \frac{16\lambda x_1^4}{3} + \frac{8\mu x_1^4}{3} - \frac{16\mu x_1^4}{3} + \frac{8x_1^5}{3} + \frac{16\mu x_1^5}{3} - \frac{16x_1^6}{3}\right), \\
g_3(x_1) &= \left( \frac{2\lambda^6}{27} + \frac{4\lambda^5 x_1}{9} + \frac{10\lambda^4 x_1^2}{9} - \frac{8\lambda^2 \mu x_1^3}{9} + \frac{40\lambda^3 x_1^3}{27} - \frac{8\lambda^4 x_1^3}{9} - \frac{32\lambda^3 \mu x_1^3}{9} + \frac{10\lambda^2 x_1^4}{9} - \frac{32\lambda^3 x_1^4}{9} \\
&- \frac{16\lambda^2 x_1^5}{9} + \frac{32\lambda^2 \mu x_1^5}{9} + \frac{4\lambda x_1^6}{9} - \frac{16\lambda^2 x_1^6}{9} + \frac{32\mu x_1^6}{9} + \frac{16\mu \mu x_1^6}{9} + \frac{2x_1^6}{27} - \frac{32\mu x_1^6}{9} + \frac{32\lambda^2 x_1^6}{9} \\
&- \frac{8\mu x_1^6}{9} + \frac{32\mu x_1^6}{9} + \frac{32 \mu x_1^6}{9} - \frac{128\mu x_1^6}{27} - \frac{8\lambda x_1^6}{9} + \frac{16\lambda x_1^6}{9} + \frac{64\mu x_1^6}{9} + \frac{32\mu x_1^6}{9} + \frac{64\mu x_1^6}{9} - \frac{128\mu x_1^6}{27}\right),
\end{align*}
\]

and

\[
y_2^2 = z_2^3 - h_2(x_2)z_2 - h_3(x_2), \quad x_2 \neq \infty,
\]

5
with

\[
\begin{align*}
 h_2(x_2) &= -\left( -16x_2^3 + \frac{8x_2^3}{3} + \frac{16\mu x_2^3 - x_1^4}{3} + \frac{16\lambda x_2^3}{3} + \frac{8\mu x_2^4}{3} - \frac{16\mu^2 x_2^4}{3} - \frac{4\lambda x_2^5}{3} \\
 & \quad + \frac{8\lambda x_2^5}{3} + \frac{\lambda x_2^6 - 2\lambda^2 x_2^6 + \frac{8\lambda^2\mu x_2^6}{3} + \frac{4\lambda^3 x_2^7}{3} - \frac{\lambda^4 x_2^8}{3} }{3} \right), \\
 h_3(x_2) &= -\left( -\frac{128x_2^3}{27} + \frac{32x_2^4}{9} + \frac{64\mu x_2^4}{9} - \frac{8x_2^5}{9} + \frac{64\lambda x_2^5}{9} + \frac{2x_2^6}{27} - \frac{32\mu x_2^6}{9} - \frac{32\lambda x_2^6}{9} - \frac{8\mu x_2^6}{3} \\
 & \quad + \frac{32\mu x_2^6}{9} + \frac{32\mu x_2^6}{9} - \frac{128\mu^2 x_2^6}{9} + \frac{4\lambda x_2^7}{9} - \frac{16\lambda x_2^7}{9} - \frac{32\mu x_2^9}{9} + \frac{32\lambda x_2^9}{9} - \frac{16\lambda^2 x_2^9}{9} - \frac{32\lambda^2 x_2^9}{9} \\
 & \quad + \frac{40\lambda x_2^{10}}{27} - \frac{8\lambda x_2^{10}}{9} - \frac{32\lambda^2 x_2^{10}}{9} + \frac{10\lambda^2 x_2^{10}}{9} - \frac{8\lambda^4 x_2^{10}}{9} + \frac{4\lambda^5 x_2^{11}}{9} + \frac{2\lambda^6 x_2^{12}}{9} \right),
\end{align*}
\]

where \(x_1 = 1/x_2\). We have the discriminant of the right hand side for \(z_2\) (resp): \(z_3\)

\[
\begin{align*}
 D_0 &= -256\mu^3 x_1^9(\lambda^4 + 4\lambda^3 x_1 + 6\lambda^2 x_1^2 - 8\lambda^2\mu x_1^3 + 4\lambda x_1^3 - 8\lambda^3 x_1^3 - 16\lambda x_1^3) \\
 & \quad + x_1^3 - 16\lambda x_1^4 - 8\mu x_1^4 + 16\mu^2 x_1^4 - 8\mu x_1^5 - 32\mu x_1^5 + 16\mu^5, \\
 D_\infty &= -256\mu^2 x_2^9(16 - 8x_2 - 32\mu x_2 + x_2^2 - 16\lambda x_2^2 - 8\mu x_2^3 + 16\mu^2 x_2^3 + 4\lambda x_2^4 - 8\lambda^2 x_2^4 + 6\lambda^2 x_2^4 + 4\lambda^3 x_2^4 + 4\lambda^4 x_2^5).
\end{align*}
\]

From these data, we obtain the required statement.

We need another elliptic fibration.

**Proposition 1.7.** (1) The surface \(S_3(\lambda, \mu)\) is birationally equivalent to the surface defined by the equation
\[
y_1^2 = x_1^3 + (\mu^2 + 2\mu z + z^2 + 2\mu z^2 + 2z^3 + 4z^4) x_1^3 + (-8\lambda \mu z^3 - 8\lambda^2 z - 8\lambda^2 z^5)x_1 + 16\lambda^2 z^6. \tag{1.13}
\]
This equation gives an elliptic fibration of \(S_3(\lambda, \mu)\).

(2) The elliptic surface given by \(1.13\) has the holomorphic sections
\[
\begin{align*}
 P : z \mapsto (x_1', y_1', z) &= (0, 4\lambda z^3, z), \\
 Q : z \mapsto (x_1', y_1', z) &= (0, -4\lambda z^3, z). \tag{1.14}
\end{align*}
\]

*Proof.* (1) By the birational transformation
\[
x = -\frac{4\lambda z^2}{x_1}, \quad y = -\frac{\mu x_1' - y_1' - x_1' z - x_1 z^2 + 4\lambda z^3}{2x_1' z},
\]
\(1.13\) is transformed to \(1.13\).

(2) This is apparent.

**Proposition 1.8.** Suppose \((\lambda, \mu) \in \Lambda_3\). The elliptic surface given by \(1.13\) has the singular fibres of type \(I_0\) over \(z = 0\), of type \(I_9\) over \(z = \infty\), and other six fibres of type \(I_1\).

*Proof.* \(1.13\) is described in the Kodaira normal form
\[
y_1^2 = x_1^3 - g_2(z)x_1^2 - g_3(z), \quad z \neq \infty,
\]
with

\[
\begin{align*}
 g_2(z) &= -\left( -\frac{\mu^4}{3} - \frac{4\mu^3 z}{3} - 2\mu^2 z^2 - \frac{4\mu^3 z^2}{3} - \frac{4\mu^3 z^3}{3} - 8\lambda \mu z^3 - 4\mu^2 z^3 - \frac{\mu^4}{3} - \frac{8\lambda z^4}{3} \\
 & \quad - 4\mu^4 - 2\mu^2 z^4 - \frac{4\mu^3 z^4}{3} - \frac{8\lambda z^5 - 4\mu z^5 - 2z^6 - \frac{4\mu z^6}{3} - \frac{4z^7}{3}}{3} - \frac{z^8}{3} \right), \\
 g_3(z) &= -\left( \frac{2\mu^6}{27} + \frac{4\mu^5 z}{9} + \frac{10\mu^4 z^2}{9} + \frac{4\mu^5 z^2}{9} + \frac{4\mu^3 z^3}{3} + \frac{40\mu^3 z^3}{3} + \frac{8\lambda \mu z^3}{3} + \frac{2\mu^4 z^3}{9} + \frac{10\mu^4 z^3}{9} + 8\lambda \mu z^4 + \frac{40\mu^3 z^4}{9} \\
 & \quad + \frac{10\mu^4 z^4}{9} + \frac{3\mu^5 z^4}{9} + 8\mu z^5 + \frac{40\mu^2 z^5}{9} + \frac{8\mu z^5}{9} + \frac{4\mu^3 z^5}{27} + \frac{8\lambda z^6}{3} + \frac{16\lambda z^6}{3} + \frac{20\mu z^6}{9} + 16\lambda \mu z^6 + \frac{20\mu z^6}{9} + \frac{20\mu z^6}{9} \\
 & \quad + \frac{20\mu^2 z^6}{9} + \frac{40\mu^3 z^6}{27} + \frac{4z^7}{9} + 8\lambda z^7 + \frac{40\mu z^8}{9} + \frac{10\mu z^8}{27} + \frac{40\mu z^8}{9} + \frac{8\lambda z^9}{3} + \frac{20\mu z^9}{9} + \frac{10\mu z^9}{9} + \frac{4z^{11}}{9} + \frac{2z^{12}}{27} \right),
\end{align*}
\]
and
\[ y_2^2 = x_3^3 - h_2(z_1)x_3 - h_3(z_1), \quad z_1 \neq \infty, \]
with
\[
\begin{align*}
h_2(z_1) &= - \left( -\frac{\mu^4}{3} - \frac{4\mu^3 z}{3} - 2\mu^2 z^2 - \frac{4\mu^3 z^2}{3} - \frac{\lambda^4}{3} - 8\lambda\mu z^3 - 4\mu^2 z^3 - \frac{\mu^4}{3} - 8\lambda z^4 \\
&\quad - 4\mu z_1^4 - 2\mu^2 z_1^2 - \frac{4\mu z_1^5}{3} - 8\lambda\mu z_1^5 - 4\mu^2 z_1^5 - 2\mu^2 z_1^6 - \frac{4\mu^3 z_1^6}{3} - \frac{4\mu^3 z_1^7}{3} - \frac{\mu^4 z_1^8}{3} \right), \\
h_3(z_1) &= - \left( \frac{2}{27} + \frac{10 z_1}{9} + \frac{4\mu z_1^2}{27} + \frac{40 z_1^3}{3} + 20\mu z_1^3 + \frac{10 z_1^4}{9} + 8\lambda^2 z_1^4 + \frac{40\mu z_1^4}{9} + \frac{10\mu^2 z_1^4}{9} \\
&\quad + \frac{4 z_1^5}{9} + 8\lambda^2 z_1^5 + \frac{40\mu z_1^5}{9} + 8\lambda^2 z_1^5 + \frac{40\mu z_1^5}{9} + \frac{2 z_1^6}{27} + \frac{4\mu z_1^6}{9} + 8\lambda^2 z_1^6 + \frac{40\mu z_1^6}{9} + \frac{10\mu^2 z_1^6}{9} \\
&\quad + 8\lambda^2 z_1^8 + \frac{40\mu z_1^8}{9} + \frac{40\mu z_1^8}{27} + \frac{40\mu z_1^8}{3} + \frac{20\mu^2 z_1^8}{9} + \frac{10\mu z_1^8}{9} + \frac{4\mu^3 z_1^8}{9} + \frac{4\mu^3 z_1^8}{9} + \frac{2\mu^6 z_1^{12}}{27} \right),
\end{align*}
\]
where \( z = 1/z_1 \) We have the discriminant of the right hand side for \( x_2' (x_3', \text{resp.}) \):
\[
\begin{align*}
D_0 &= 256\lambda^3 z^9 (\mu^3 + 3\mu^2 z + 3\mu z^2 + 3\mu z^2 + z^3 + 27\lambda z^3 + 6\mu z^3 + 3z^4 + 3\mu z^4 + 3z^5 + z^6), \\
D_\infty &= 256\lambda^3 z_1^9 (1 + 3z_1 + 3z_1^2 + 3z_1^2 + z_1^3 + 27\lambda z_1^3 + 6\mu z_1^3 + 3z_1^4 + 3\mu z_1^4 + 3z_1^5 + \mu z_1^6). 
\end{align*}
\]
From these data, we obtain the required statement.

## 2 Lattices for \( \mathcal{F}_j \)

In this section, we determine the lattice structure of a generic member of \( \mathcal{F}_j \) \((j = 1, 2, 3)\).

For a general \( K3 \) surface \( S \), \( H_2(S, \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module of rank 22. The intersection form of \( H_2(S, \mathbb{Z}) \) is given by
\[
E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U,
\]
where
\[
E_8(-1) = \left( \begin{array}{ccccc}
-2 & 1 & 1 & 1 & 1 \\
1 & -2 & 1 & 1 & 1 \\
1 & 1 & -2 & 1 & 1 \\
1 & 1 & 1 & -2 & 0 \\
1 & 0 & -2 & 1 & -2 \\
\end{array} \right),
\]
\[
U = \left( \begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array} \right).
\]
Let \( \text{NS}(S) \) denote the sublattice in \( H_2(S, \mathbb{Z}) \) generated by the divisors on \( S \). It is called the Néron-Severi lattice. The rank of \( H_2(S, \mathbb{Z}) \) is called the Picard number. We call the orthogonal complement of \( \text{NS}(S) \) in \( H_2(S, \mathbb{Z}) \) the transcendental lattice. We note that the Picard number is equal to \( \dim_{\mathbb{Q}}(\text{NS}(S) \otimes_{\mathbb{Z}} \mathbb{Q}) \).

**Theorem 2.1.** Let \( j \in \{1, 2, 3\} \). The Picard number of a generic member of the family \( \mathcal{F}_j \) is equal to 18.

As a principle, we obtain the above theorem by the method exposed in the section 2 of the article [Na].

Because we shall have the lattice \( L_1 \) \((L_2, L_3, \text{resp.}) \) in [2, 23] \((2, 10, 2, 15, \text{resp.}) \) for \( \mathcal{F}_1 \) \((\mathcal{F}_2, \mathcal{F}_3, \text{resp.}) \), we have rank\( (\text{NS}(S_j(\lambda, \mu))) \geq 18 \). Let \( j \in \{1, 2, 3\} \). Take \((\lambda_0, \mu_0) \in \Lambda_j \). Take a small neighborhood \( \delta \) of \((\lambda_0, \mu_0) \) in \( \Lambda_j \) so that we have a local trivialization
\[
\tau : \{S_j(\lambda, \mu) \mid (\lambda, \mu) \in \delta \} \to \mathbb{S};(\lambda_0, \mu_0) \times \delta.
\]
We note that $\tau$ may preserve the lattice $L_j$. Let $\omega_j(\lambda, \mu)$ be the unique holomorphic 2-form on the K3 surface $S_j(\lambda, \mu)$ up to a constant factor. By using the pairing
\[
\langle \cdot, \cdot \rangle : H^2(S_j(\lambda_0, \mu_0), \mathbb{C}) \times H_2(S_j(\lambda_0, \mu_0)) \to \mathbb{C},
\]
we define a period $\Phi(\lambda_0, \mu_0) \in \mathbb{P}^{21}(\mathbb{C})$ of $S_j(\lambda_0, \mu_0)$ given by $\langle \omega_j(\lambda_0, \mu_0), \gamma_k \rangle$ ($k = 1, \cdots, 22$) for a fixed basis $\{\gamma_1, \cdots, \gamma_{22}\}$ of $H_2(S_j(\lambda_0, \mu_0), \mathbb{Z})$. We have a natural extension $\Phi(\lambda, \mu)$ for $(\lambda, \mu) \in \delta$ by using $\langle \omega(\lambda, \mu), \tau^{-1}(\gamma_k) \rangle$ ($k = 1, \cdots, 22$). Then we can define a local period map
\[
\Phi_3 : \delta \to \mathbb{P}^{21}(\mathbb{C}).
\]
It is sufficient to have that $\Phi_3$ is injective on $\delta$ to prove $\text{rank}(\text{NS}(S_j(\lambda, \mu))) = 18$ for generic $(\lambda, \mu) \in \Lambda_j$. In this situation, we have $\dim(\Phi_3(\delta)) = 2$. It implies $\text{rank}(\text{NS}(S_j(\lambda, \mu)^\perp)) = 4$ for generic $(\lambda, \mu) \in \Lambda_j$. But, to assure this assertion, we need a delicate observation exposed in the argument to obtain Theorem 2.2 in \cite{Na}.

We have the following fact for the elliptic fibration of $S_j(\lambda, \mu)$ stated in Section 1 by the same argument to prove Lemma 1.1 in \cite{Na}.

**Fact 2.1.** Let $j \in \{1, 2, 3\}$ and $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Lambda_j$. Two elliptic surfaces $(S_j(\lambda_1, \pi_1, \mathbb{P}^1(\mathbb{C})))$ and $(S_j(\lambda_2, \pi_2, \mathbb{P}^1(\mathbb{C})))$ are isomorphic as elliptic surfaces if and only if $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.

Also, we have the following fact.

**Fact 2.2.** (\cite{Na} Lemma 2.1) Let $S$ be a K3 surface with elliptic fibration $\pi : S \to \mathbb{P}^1(\mathbb{C})$, and let $F$ be a fixed general fibre. Then $\pi$ is the unique elliptic fibration up to $\text{Aut}(\mathbb{P}^1(\mathbb{C}))$ which has $F$ as a general fibre.

Because we established Fact 2.1 and Fact 2.2 by the same argument to obtain Proposition 2.1 in \cite{Na}, we have the same marked K3 surfaces $S_j(\lambda_1, \mu_1)$ and $S_j(\lambda_2, \mu_2)$ if and only if $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$. According to the Torelli type theorem for K3 surfaces, we obtain that $\Phi_3$ is injective.

We need explicit lattice structures of the Néron-Severi lattices and the transcendental lattices for further study.

In the article \cite{Na} we could determine the explicit Néron-Severi lattice for the polytope $P_3$ in a naive way, for we have $\det(\text{NS}(S(\lambda, \mu))) = -5$ which does not contain any square factor.

However, it is much more difficult to determine the explicit Néron-Severi lattice for the polytopes $P_j$ ($j = 1, 2, 3$), for $\det(\text{NS}(S_j(\lambda, \mu))) = -9 = -3^2$. In this section, we prove the following theorem.

**Theorem 2.2.** For a generic point $(\lambda, \mu) \in \Lambda_j$ ($j = 1, 2, 3$), we have the intersection matrices of Néron-Severi lattices $\text{NS}$ and the transcendental lattices $\text{Tr}$ as in Table 2.

| Family | $\mathcal{F}_1$ | $\mathcal{F}_2$ | $\mathcal{F}_3$ |
|--------|----------------|----------------|----------------|
| $\text{NS}$ | $E_{8(-1)} \oplus E_{8(-1)} \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ | $E_{8(-1)} \oplus E_{8(-1)} \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$ | $E_{8(-1)} \oplus E_{8(-1)} \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}$ |
| $\text{Tr}$ | $A_1 := U \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ | $A_2 := U \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}$ | $A_3 := U \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$ |

Table 2.

**Remark 2.1.** K. Koike \cite{Koi} has made a research on the families of K3 surfaces derived from the dual polytopes of 3 dimensional Fano polytopes. The polytopes $P_2$ and $P_3$ in our notation are the Fano polytopes. Due to Koike we have Néron-Severi lattices for the dual polytopes $P_2^*$ and $P_3^*$ (given by Table 3).
By comparing Table 3 and Table 2, we can assure the mirror symmetry conjecture for the reflexive polytopes $P_2$ and $P_3$.

### 2.1 The Mordell-Weil lattices

Let us recall the theory of Mordell-Weil lattices due to T. Shioda. For detail, see [Sh1] and [Sh2].

Let $S$ be a compact complex surface and $C$ be an algebraic curve. Let $\pi : S \to C$ be an elliptic fibration with sections. For generic $v \in C$, the fibre $\pi^{-1}(v)$ is an elliptic curve. In the following we assume that the elliptic fibre $\pi : S \to C$ has singular fibres. $\mathbb{C}(C)$ denotes the algebraic function field on $C$. If $C = \mathbb{P}^1(\mathbb{C})$, the field $\mathbb{C}(C)$ is isomorphic to the rational function field $\mathbb{C}(t)$.

In this article, $(\cdot)$ denotes the intersection number and $E(\mathbb{C}(C))$ denotes the Mordell-Weil group of sections of $\pi : S \to C$. For all $P \in E(\mathbb{C}(C))$ and $v \in C$, we have $(P \cdot \pi^{-1}(v)) = 1$. Note that the section $P$ intersects an irreducible component with multiplicity 1 of every fibre $\pi^{-1}(v)$. Let $O$ be the zero of the group $E(\mathbb{C}(C))$. The section $O$ is given by the set of the points at infinity on every generic fibre.

Set

$$R = \{ v \in C|\pi^{-1}(C) \text{ is a singular fibre of } \pi \}.$$

For all $v \in R$ we have

$$\pi^{-1}(v) = \Theta_{v,0} + \sum_{j=1}^{m_v-1} \mu_{v,j} \Theta_{v,j}, \quad (2.1)$$

where $m_v$ is the number of irreducible components of $\pi^{-1}(v)$, $\Theta_{v,j}$ ($j = 0, \cdots, m_v - 1$) are irreducible components with multiplicity $\mu_{v,j}$ of $\pi^{-1}(v)$, and $\Theta_{v,0}$ is the component with $\Theta_{v,0} \cap O \neq \emptyset$.

Let $F$ be a generic fibre of $\pi$. Set

$$T = \langle F, O, \Theta_{v,j}|v \in R, 1 \leq j \leq m_v - 1 \rangle_\mathbb{Z} \subset \text{NS}(S).$$

We call $T$ the trivial lattice for $\pi$. For $P \in E(\mathbb{C}(C))$, $(P) \in \text{NS}(S)$ denotes the corresponding element.

**Theorem 2.3.** (T. Shioda [Sh1] (see also [Sh2] Theorem (3·10)))

1. The Mordell-Weil group $E(\mathbb{C}(C))$ is a finitely generated Abelian group.
2. The Néron-Severi group $\text{NS}(S)$ is a finitely generated Abelian group and torsion free.
3. We have the isomorphism of groups $E(\mathbb{C}(C)) \simeq \text{NS}(S)/T$ given by

$$P \mapsto (P) \mod T.$$

We set $\hat{T} = (T \otimes_\mathbb{Z} \mathbb{Q}) \cap \text{NS}(S)$ for the trivial lattice $T$.

**Corollary 2.1.** ([Sh1], see also [Sh2] Proposition (3·11))

1. We have

$$\text{rank}(E(\mathbb{C}(C))) = \text{rank}(\text{NS}(S)) - 2 - \sum_{v \in R} (m_v - 1).$$

2. We have

$$E(\mathbb{C}(C))_{\text{tor}} \simeq \hat{T}/T,$$

where $E(\mathbb{C}(C))_{\text{tor}}$ is the torsion part of $E(\mathbb{C}(C))$. 

| Dual Polytope | $P_2$ | $P_3$ |
|--------------|-------|-------|
| Néron-Severi lattice | [0 3] | [0 3] |
|                | [3 -2] | [3 2] |

Table 3.
Set
\[ E(\mathbb{C}(C))^0 = \{ P \in E(\mathbb{C}(C)) | P \cap \Theta_{v,0} \neq \phi \text{ for all } v \in R \} . \]
We have
\[ E(\mathbb{C}(C))^0 \subset E(\mathbb{C}(C))/E(\mathbb{C}(C))_{tor} \]
(see [Sh1], see also [Sh2] Section 5).
Let \( v \in R \). Under the notation (2.1), we set
\[ (\pi^{-1}(v))^\sharp = \bigcup_{0 \leq j \leq m_v-1, \mu_{v,j}=1} \Theta_{v,j}^\sharp, \]
where \( \Theta_{v,j}^\sharp = \Theta_{v,j} - \{ \text{singular points of } \pi^{-1}(v) \} \). Set
\[ m_v^{(1)} = \sharp \{ j|0 \leq j \leq m_v-1, \mu_{v,j} = 1 \}. \]

**Theorem 2.4.** ([Ne], [Kod], see also [Sh2] Section 7) Let \( v \in R \). The set \( (\pi^{-1}(v))^\sharp \) has a canonical group structure.

**Remark 2.2.** Especially, for the singular fibre \( \pi^{-1}(v) \) of type \( I_b \) (\( b \geq 1 \)),
\[ (\pi^{-1}(v))^\sharp \simeq \mathbb{C}^\times \times (\mathbb{Z}/b\mathbb{Z}). \]
For the singular fibre \( \pi^{-1}(v) \) of type \( I_b^\ast \) (\( b \geq 0 \)),
\[ (\pi^{-1}(v))^\sharp \simeq \begin{cases} \mathbb{C} \times (\mathbb{Z}/4\mathbb{Z}) & (b \in 2\mathbb{Z} + 1), \\ \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2 & (b \in 2\mathbb{Z}). \end{cases} \]
For each \( v \in C \), we introduce the map
\[ sp_v : E(\mathbb{C}(C)) \to (\pi^{-1}(v))^\sharp : P \mapsto P \cap \pi^{-1}(v). \]
Note that
\[ P \cap \pi^{-1}(v) = (x,a) \in \left( \mathbb{C}^\times \right) \times \{ \text{finite group} \} \]
(see [Sh2] Section 7). We call \( sp_v \) the specialization map.

**Theorem 2.5.** ([Sh2] Section 7) For all \( v \in C \), the specialization map
\[ sp_v : P \mapsto (x,a) \in \left( \mathbb{C}^\times \right) \times \{ \text{finite group} \} \]
is a homomorphism of groups.

**Remark 2.3.** Especially for the singular fibre \( \pi^{-1}(v) \) of type \( I_b \) (\( I_b^\ast \), resp.), the projection of \( sp_v \)
\[ E(\mathbb{C}(C)) \to (\mathbb{Z}/b\mathbb{Z}) \quad ((\mathbb{Z}/4\mathbb{Z}) \text{ or } (\mathbb{Z}/2\mathbb{Z})^2, \text{ resp.}) \]
is a homomorphism of groups.

### 2.2 \( \mathcal{F}_1 \)
The elliptic fibration given by (1.3) is described in Figure 1.
The trivial lattice for this fibration is
\[ T_1 = \langle a_1, a_2, a_3, a_4, a'_4, a'_3, a'_2, a'_1, c_1, b_1, b_2, b_3, c_2, c_3, O, F \rangle_{\mathbb{Z}}. \]

Let \( P \) be the section in (1.5). \( P \cap a_3 \neq \phi \) at \( x_1 = 0 \) and \( P \cap c_2 \neq \phi \) at \( x_1 = \infty \). Set
\[ L_1 = \langle P, T_1 \rangle_{\mathbb{Z}}. \]  

This is a subgroup of \( \text{NS}(S_1(\lambda, \mu)) \). We have \( \text{det}(L_1) = -9 \). According to Theorem 2.1 and Theorem 2.3 (3), we obtain \( \text{NS}(S_1(\lambda, \mu)) \otimes \mathbb{Q} = L_1 \otimes \mathbb{Q} \), and we obtain also
\[ \text{NS}(S_1(\lambda, \mu)) = ((P)_{\mathbb{Q}} \cap \text{NS}(S_1(\lambda, \mu))) + \hat{T}_1 \]  

for generic \((\lambda, \mu) \in \Lambda_1\). We have
\[ [\text{NS}(S_1(\lambda, \mu)) : L_1] = 1 \text{ or } [\text{NS}(S_1(\lambda, \mu)) : L_1] = 3. \]  

In the following, we prove
\[ [\text{NS}(S_1(\lambda, \mu)) : L_1] = 1. \]

**Lemma 2.1.** For generic \((\lambda, \mu) \in \Lambda_1\), we have \( \hat{T}_1 = T_1 \).

**Proof.** From (2.4) and (2.5) it is necessary that \( \hat{T}_1 = T_1 \) or \( [\hat{T}_1 : T_1] = 3 \). We assume \( [\hat{T}_1 : T_1] = 3 \). Then, according to Corollary 2.1 (2),
\[ E(\mathbb{C}(x_1))_{\text{tor}} \simeq \hat{T}_1/T_1 \simeq \mathbb{Z}/3\mathbb{Z}. \]  

Therefore there exists \( S_0 \in E(\mathbb{C}(x_1))_{\text{tor}} \) such that \( 3S_0 = O \). By Remark 2.3 and (2.2), we can assume that \( S_0 \cap a_3 \neq \phi \) at \( x_1 = 0 \) and \( S_0 \cap c_0 \neq \phi \) at \( x_1 = \infty \). Put \((S_0 \cdot O) = k \in \mathbb{Z}\). Set \( \hat{T}_1 = \langle T_1, S_0 \rangle_{\mathbb{Z}} \). By calculating the intersection matrix, we have
\[ \text{det}(\hat{T}_1) = -72(1 + k + k^2) \neq 0. \]  

On the other hand, due to (2.6), we have \( \text{rank}(\hat{T}_1) = 17 \). So it follows \( \text{det}(\hat{T}_1) = 0 \). This contradicts (2.7).
By the above lemma, we have

$$\text{NS}(S_1(\lambda, \mu)) = ((P)_{\mathbb{Q}} \cap \text{NS}(S_1(\lambda, \mu))) + T_1. \quad (2.8)$$

**Lemma 2.2.** For generic $(\lambda, \mu) \in \Lambda_1$, we have $\text{NS}(S_1(\lambda, \mu)) = L_1$.

**Proof.** It is sufficient to prove $[\text{NS}(S_1(\lambda, \mu)) : L_1] = 1$. We assume $[\text{NS}(S_1(\lambda, \mu)) : L_1] = 3$. By (2.8) there exists $R \in E(C(x_1))$ such that $3R = P$. According to Remark 2.3,

$$(R \cdot c_3) = 1, \quad \text{at } x_1 = \infty$$

and

$$
\begin{align*}
(R \cdot a_1) &= 1, \\
\text{or} \\
(R \cdot a_4) &= 1, \quad \text{at } x_1 = 0. \\
\text{or} \\
(R \cdot a_7) &= 1,
\end{align*}
$$

We can assume $(R \cdot O) = 0$, for $P$ in (1.5) does not intersect $O$. By the addition theorem for elliptic curves, we have $2P$ and we can check $2P$ does not intersect $O$. So, we assume $(R \cdot P) = 0$ also. Set $\tilde{L}_1 = (L_1, R)_{\mathbb{Z}}$. By calculating the intersection matrix, we have

$$\det(\tilde{L}_1) = \begin{cases}
12 & \text{if } (R \cdot a_1) = 1, \\
-30 & \text{if } (R \cdot a_4) = 1, \\
6 & \text{if } (R \cdot a_7) = 1.
\end{cases} \quad (2.9)$$

On the other hand, we have $\text{rank}(\tilde{L}_1) = 18$ from Theorem 2.1. Hence, we obtain $\det(\tilde{L}_1) = 0$. This contradicts (2.9). Therefore, we have $[\text{NS}(S_1(\lambda, \mu)) : L_1] = 1$. \qed

**Lemma 2.3.** The lattice $L_1$ is isomorphic to the lattice given by the intersection matrix

$$
\begin{pmatrix}
E_8(-1) & 0 \\
0 & E_8(-1) \\
1 & 3
\end{pmatrix},
$$

and its orthogonal complement is given by the intersection matrix

$$A_1 = 
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 3 \\
0 & 3 & 0
\end{pmatrix}.\]
Proof. We obtain the corresponding intersection matrix $M_1$ for the lattice $L_1$:

\[
\begin{pmatrix}
-2 & 1 & 1 & \cdots & 1 \\
1 & -2 & 1 & \cdots & 1 \\
1 & -2 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & -2 & 1 & \cdots & 1 \\
1 & -2 & 1 & \cdots & 1
\end{pmatrix}
\]

Let $U_1$ be the unimodular matrix

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
11 & 8 & 91 & \cdots & 1 \\
22 & 16 & 182 & \cdots & 1 \\
33 & 24 & 273 & \cdots & 1 \\
26 & 19 & 214 & \cdots & 1 \\
19 & 14 & 155 & \cdots & 1 \\
12 & 9 & 96 & \cdots & 1 \\
5 & 4 & 37 & \cdots & 1 \\
-2 & -1 & -2 & \cdots & 1 \\
2 & 2 & 18 & \cdots & 1 \\
4 & 4 & 36 & \cdots & 1 \\
6 & 6 & 54 & \cdots & 1 \\
8 & 8 & 72 & \cdots & 1 \\
10 & 10 & 90 & \cdots & 1 \\
7 & 7 & 63 & \cdots & 1 \\
5 & 5 & 45 & \cdots & 1 \\
-1 & -1 & 0 & \cdots & 1 \\
18 & 13 & 150 & \cdots & 1 \\
-1 & -4 & -5 & -36 & \cdots & 1 \\
\end{pmatrix}
\]

We have

\[
^tU_1 M_1 U_1 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.
\]

Therefore we obtain Theorem 2.2 for $\mathcal{F}_1$.

2.3 $\mathcal{F}_2$

The elliptic fibration given by (1.7) is described in Figure 2.
The trivial lattice for this fibration is
\[ T_2 = \langle a_1, a_2, a_3, a_4, c_1, b_0, b_1, c_2, c_3, O, F \rangle_{\mathbb{Z}}. \]

Let \( P \) be the section in (1.8). Note \( P \cap a_4 \neq \emptyset \) and \( P \cap c_2 \neq \emptyset \). Set
\[ L_2 = \langle P, T_2 \rangle_{\mathbb{Z}}. \] (2.10)

This is a subgroup of \( NS(S_2(\lambda, \mu)) \). We have \( \det(L_2) = -9 \). As in the case \( F_1 \), so we obtain
\[ NS(S_2(\lambda, \mu)) = (\langle P \rangle_{\mathbb{Q}} \cap NS(S_2(\lambda, \mu))) + \hat{T}_2 \]
for generic \((\lambda, \mu) \in \Lambda_2\). We have
\[ [NS(S_2(\lambda, \mu)) : L_2] = 1 \text{ or } [NS(S_2(\lambda, \mu)) : L_2] = 3. \] (2.11)

In the following, we prove \([NS(S_2(\lambda, \mu)) : L_2] = 1\).

**Lemma 2.4.** For generic \((\lambda, \mu) \in \Lambda_2\), we have \( \hat{T}_2 = T_2 \).

**Proof.** By a direct calculation, we have \( \det(T_2) = -44 \). From (2.11), we have \( \hat{T}_2 = T_2 \).

Therefore we obtain
\[ NS(S_2(\lambda, \mu)) = (\langle P \rangle_{\mathbb{Q}} \cap NS(S_2(\lambda, \mu))) + T_2. \] (2.12)

**Lemma 2.5.** For generic \((\lambda, \mu) \in \Lambda_2\), we have \( NS(S_2(\lambda, \mu)) = L_2 \).

**Proof.** We assume \([NS(S_2(\lambda, \mu)) : L_2] = 3\). From (2.12), there exists \( R \in E(C(y)) \) such that \( 3R = P \). According to Remark 2.3, we obtain \( (R \cdot a_4) = 1 \) and \( (R \cdot c_4) = 1 \). Because the section \( P \) in (1.8) and the section \( 2P \) do not intersect \( O \), we have \( (R \cdot O) = 0 \) and \( (R \cdot P) = 0 \). Set \( \hat{L}_2 = \langle L_2, R \rangle_{\mathbb{Z}} \). Calculating its intersection matrix, we have \( \det(\hat{L}_2) = -38 \). As in the proof of Lemma 2.2, this contradicts to Theorem 2.1.

**Lemma 2.6.** The lattice \( L_2 \) is isomorphic to the lattice given by the following intersection matrix
\[
\begin{pmatrix}
E_8(-1) & E_8(-1) \\
0 & 3 \\
3 & 2
\end{pmatrix},
\]
and its orthogonal complement is given by the intersection matrix

\[
A_2 = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 3 \\
3 & -2
\end{pmatrix}.
\]

**Proof.** We obtain the corresponding intersection matrix \(M_2\) for \(L_2\):

\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1
\end{pmatrix}
\]

Let \(U_2\) be the unimodular matrix

\[
\begin{pmatrix}
5 & 1 & 0 & 56 & 27 \\
4 & -2 & 0 & 13 & 6 \\
15 & 3 & 0 & 162 & 80 \\
26 & 8 & 0 & 311 & 154 \\
13 & 1 & 0 & 120 & 60 \\
1 & 10 & 0 & 0 & 100 & 50 \\
1 & 8 & 0 & 0 & 80 & 40 \\
1 & 6 & 0 & 0 & 60 & 30 \\
1 & 4 & 0 & 0 & 40 & 20 \\
1 & 2 & 0 & 0 & 20 & 10 \\
1 & 12 & 6 & 0 & 170 & 84 \\
24 & 12 & 0 & 340 & 168 \\
36 & 18 & 0 & 510 & 252 \\
30 & 15 & 0 & 425 & 210 \\
18 & 9 & 0 & 255 & 126 \\
-4 & 0 & 1 & -28 & -14 \\
24 & 12 & 0 & 340 & 168 \\
-8 & 1 & 0 & -56 & -28
\end{pmatrix}
\]

We have

\[
t^t U_2 M_2 U_2 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}.
\]

Therefore, we obtain Theorem 2.2 for \(F_2\).

### 2.4 \(F_3\)

The elliptic fibration given by (1.10) is described in Figure 3.
Figure 3.

The trivial lattice for this fibration is

\[ T_3 = \langle a_1, a_2, a_3, a_4, a'_0, a'_4, a'_3, a'_2, a'_1, c_1, b_0, b_1, b_2, c_2, c_3, O, F \rangle \mathbb{Z}. \]

Let \( P \) be the section in (1.11). Set

\[ L'_3 = \langle P, T_3 \rangle \mathbb{Z}. \]

This is a subgroup of \( \text{NS}(S_3(\lambda, \mu)) \) and we have \( \det(L'_3) = -36 \). Moreover, the section \( O' \) in (1.11) is a 2-torsion section for this elliptic fibration. Due to Corollary 2.1, \([\hat{T}_3 : T_3]\) is divided by 2. Hence, we have

\[ \text{rank}(\hat{L}'_3) = 18 \text{ or } \text{rank}(\hat{L}'_3) = 6. \tag{2.13} \]

**Lemma 2.7.** For generic \((\lambda, \mu) \in \Lambda_3\), we have \([\hat{T}_3 : T_3]\) = 2.

*Proof.* We have \( \det(T_3) = -40 \). From (2.13), we obtain \([\hat{T}_3 : T_3]\) = 2. \qed

**Lemma 2.8.** For generic \((\lambda, \mu) \in \Lambda_3\), we have \([\text{NS}(S_3(\lambda, \mu)) : L'_3]\) = 2.

*Proof.* We shall show that \([\text{NS}(S_3(\lambda, \mu)) : L'_3]\) = 2. We assume \([\text{NS}(S_3(\lambda, \mu)) : L'_3]\) = 6. From Lemma 2.7, there exists \( R \in E(\mathbb{C}(x_1)) \) such that \( 3R = P \). According to Remark 2.3, it is necessary that \( R \cdot c_2 = 1 \) and \( R \cdot a_4 = 1 \). Also we have \( (R \cdot O) = 0 \), for \( P \) in (1.11) does not intersect \( O \). Moreover we can assume that \( (R \cdot P) = 0 \) or 1, for the section \( 2P \) does not intersect \( O \) at \( x_1 \neq \infty \). Set \( \hat{L}'_3 = \langle L'_3, R \rangle \mathbb{Z} \). Calculating the intersection matrix, we have

\[
\det(\hat{L}'_3) = \begin{cases} 
-16 & \text{(if } (R \cdot P) = 0) \\
-112 & \text{(if } (R \cdot P) = 1) 
\end{cases}.
\tag{2.14}
\]

On the other hand, Theorem 2.1 implies that \( \text{rank}(\hat{L}_3) = 18 \) and \( \det(\hat{L}_3) = 0 \). This is a contradiction to (2.14). \qed

Due to the above lemma, we have

\[ |\det(\text{NS}(S_3(\lambda, \mu)))| = 9 \]

for generic \((\lambda, \mu) \in \Lambda_3\).

To determine the explicit lattice structure for \( F_3 \) we use another elliptic fibration defined by (1.13). This fibration is described in Figure 4.
Figure 4.

Let \( P_0 \) and \( Q_0 \) be the sections in (1.14) for this elliptic fibration. Set

\[
L_3 = \langle d_1, d_2, d_3, d_4, d'_3, d'_2, d'_1, e_1, e_2, e_3, e_4, e'_3, e'_2, P_0, Q_0, O, F \rangle_{\mathbb{Z}}.
\]

(2.15)

We have \( L_3 \otimes \mathbb{Q} = \operatorname{NS}(S_3(\lambda, \mu)) \otimes \mathbb{Q} \) for generic \((\lambda, \mu) \in \Lambda_3\) and \( \det(L'_3) = -9 \). Therefore we have

\[
L_3 = \operatorname{NS}(S_3(\lambda, \mu))
\]

for generic \((\lambda, \mu) \in \Lambda_3\).

**Lemma 2.9.** The lattice \( L_3 \) is isomorphic to the lattice given by the intersection matrix

\[
\begin{pmatrix}
E_8(-1) \\
E_8(-1) \\
0 & 3 \\
3 & -2
\end{pmatrix},
\]

and its orthogonal complement is given by the intersection matrix

\[
A_3 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 3 & \\
0 & 3 & 2
\end{pmatrix}.
\]
Proof. We obtain the corresponding intersection matrix $M_3$ for the lattice $L_3$:

$$
\begin{pmatrix}
-2 & 1 & 1 & 1 \\
1 & -2 & 1 & 1 \\
1 & -2 & 1 & 1 \\
1 & -2 & 1 & 1 \\
1 & -2 & 1 & 1 \\
1 & -2 & 1 & 1 \\
-2 & 1 & 1 & 1 \\
1 & -2 & 1 & 1 \\
1 & -2 & 1 & 1 \\
1 & -2 & 1 & 1 \\
0 & 0 & -2 & 1 \\
0 & 0 & -2 & 1 \\
0 & 0 & -2 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}
$$

Let $U_3$ be the unimodular matrix

$$
\begin{pmatrix}
1 & 28 & 5 & 468 \\
1 & 56 & 10 & 936 \\
1 & 84 & 15 & 1404 \\
27 & 5 & 432 \\
21 & 1 & 4 & 378 \\
15 & 1 & 3 & 324 \\
10 & 1 & 2 & 216 \\
5 & 1 & 1 & 108 \\
1 & 34 & 6 & 576 \\
1 & 68 & 12 & 1152 \\
1 & 102 & 18 & 1728 \\
1 & 51 & 9 & 864 \\
-1 & 1 & 36 \\
-1 & 1 & 0 & 18 \\
1 & 0 & 35 \\
1 & 85 & 15 & 1440 \\
1 & -1 & 0 & 54 \\
1 & -16 & -3 & -252 \\
\end{pmatrix}
$$

We have

$$\trans{U_3}M_3U_3 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}.$$

Therefore, we obtain Theorem 2.2 for $\mathcal{F}_3$.

3 Period differential equations

In this section, we determine the system of period differential equations and its projective monodromy group for the family $\mathcal{F}_j$ ($j = 1, 2, 3$).

Set

$$
\begin{cases}
F_1(x, y, z) = xyz(x + y + z + 1) + \lambda x + \mu y, & (\lambda, \mu) \in \Lambda_1, \\
F_2(x, y, z) = xyz(x + y + z + 1) + \lambda x + \mu, & (\lambda, \mu) \in \Lambda_2, \\
F_3(x, y, z) = xyz(x + y + z + 1) + \lambda z + \mu xy, & (\lambda, \mu) \in \Lambda_3.
\end{cases}
$$
The unique holomorphic 2-form on the K3 surface $S_j(\lambda, \mu) \in \Lambda_j$ \((j = 1, 2, 3)\) is given by

$$\omega_j = \frac{dz \wedge dx}{\partial F_j/\partial y}$$

up to a constant factor.

First, we consider a period of $S_j(\lambda, \mu)$ \((j = 1, 2, 3)\).

**Theorem 3.1.** We can find a 2-cycle $\Gamma_j$ \((j = 1, 2, 3)\) so that we have the following power series expansion of the period $\mathcal{J}_{\Gamma_j} \omega_j$ which is valid in a sufficiently small neighborhood of $(\lambda, \mu) = (0, 0)$.

1. (A period for $F_1$) We have a period of $S_1(\lambda, \mu)$:

   $$\eta_1(\lambda, \mu) = \mathcal{J}_{\Gamma_1} \omega_1 = (2\pi i)^2 \sum_{n,m=0}^{\infty} \frac{(3m + 3n)!}{(n!)^2(m!)^2(m + n)!} \lambda^n \mu^m. \quad (3.1)$$

2. (A period for $F_2$) We have a period of $S_2(\lambda, \mu)$:

   $$\eta_2(\lambda, \mu) = \mathcal{J}_{\Gamma_2} \omega_2 = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^n \frac{(4m + 3n)!}{(m!)^2(2n!)^2((m + n)!)^2} \lambda^n \mu^m. \quad (3.2)$$

3. (A period for $F_3$) We have a period of $S_3(\lambda, \mu)$:

   $$\eta_3(\lambda, \mu) = \mathcal{J}_{\Gamma_3} \omega_3 = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^n \frac{(3m + 2n)!}{(m!)^2(3n!)^2} \lambda^n \mu^m. \quad (3.3)$$

**Proof.** Let $j \in \{1, 2, 3\}$. By the same argument in the proof of Theorem 3.1 of the article [Na], we can choose a certain 2-cycle $\Gamma_j$ on $S_j(\lambda, \mu)$ so that the period integral $\mathcal{J}_{\Gamma_j} \omega_j$ is given by a power series of $(\lambda, \mu)$. \(\square\)

**Remark 3.1.** In the case $P_1$, our period is reduced to the Appell $F_1$ (see [Koi]):

$$\eta_1(\lambda, \mu) = F_1\left(\frac{1}{3}, \frac{2}{3}, 1, 1; 27\lambda, 27\mu \right) = F\left(\frac{1}{3}; \frac{2}{3}, 1; x \right) F\left(\frac{1}{3}; \frac{2}{3}, 1; y \right),$$

where $F$ is the Gauss hypergeometric function and $x(1 - y) = 27\lambda, y(1 - x) = 27\mu$.

Secondary, we apply the theory of the GKZ hypergeometric functions to obtain the system of differential equations whose solution is the period integral in Theorem 3.1. In the following, set

$$\theta_{\lambda} = \frac{\lambda}{\partial \lambda}, \quad \theta_{\mu} = \frac{\mu}{\partial \mu}.$$

**Proposition 3.1.** (1) (The GKZ system of equations for $F_1$) Set

\[
\begin{aligned}
L_1^{(1)} &= \lambda \theta_{\mu}^2 - \mu \theta_{\lambda}^2, \\
L_2^{(1)} &= \lambda(3\theta_{\lambda} + 3\theta_{\mu})(3\theta_{\lambda} + 3\theta_{\mu} - 1)(3\theta_{\lambda} + 3\theta_{\mu} - 2).
\end{aligned}
\quad (3.4)
\]

It holds

$$L_1^{(1)} \eta_1(\lambda, \mu) = L_2^{(1)} \eta_1(\lambda, \mu) = 0.$$

(2) (The GKZ system of equations for $F_2$) Set

\[
\begin{aligned}
L_1^{(2)} &= \lambda \theta_{\mu}^2 + \mu \theta_{\lambda}(3\theta_{\lambda} + 4\theta_{\mu} + 1), \\
L_2^{(2)} &= \theta_{\lambda}(\theta_{\lambda} + \theta_{\mu})^2 + \lambda(3\theta_{\lambda} + 4\theta_{\mu} + 1)(3\theta_{\lambda} + 4\theta_{\mu} + 2)(3\theta_{\lambda} + 4\theta_{\mu} + 3).
\end{aligned}
\quad (3.5)
\]

It holds

$$L_1^{(2)} \eta_2(\lambda, \mu) = L_2^{(2)} \eta_2(\lambda, \mu) = 0.$$
Theorem 3.2. (1) (The period differential equation for $\mathcal{F}_3$) Set

\[
\begin{align*}
L_1^{(3)} &= \theta^2_\lambda - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\
L_2^{(3)} &= \theta^3_\lambda + \lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2)(3\theta_\lambda + 2\theta_\mu + 3).
\end{align*}
\] (3.6)

It holds

\[L_1^{(3)}\eta_3(\lambda, \mu) = L_2^{(3)}\eta_3(\lambda, \mu) = 0.\]

Proof. Set

\[
\begin{align*}
A_1 &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix},
\end{align*}
\]

and

\[
\beta = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.
\]

Let $j \in \{1, 2, 3\}$. From the matrix $A_j$ and the vector $\beta$, we have the system of the GKZ system of equations concerned with the period $\eta_j(\lambda, \mu)$ in Theorem 3.1. For detail, see the proof of Proposition 3.1 in $\text{[Na]}$. \qed

Each system in the above proposition has the 6-dimensional space of solutions. On the other hand, Theorem 2.1 says that the rank of transcendental lattice for $\mathcal{F}_3$ is 4. It implies that there are the system of period differential equations for the family $\mathcal{F}_j$ ($j = 1, 2, 3$) with the 4-dimensional space of solutions.

**Theorem 3.2.** (1) (The period differential equation for $\mathcal{F}_1$) Set

\[
\begin{align*}
L_1^{(1)} &= \lambda\theta^2_\mu + \mu\theta_\lambda(3\theta_\lambda + 4\theta_\mu + 1), \\
L_3^{(1)} &= \lambda\theta_\lambda(3\theta_\lambda + 2\theta_\mu) + \mu\theta_\lambda(1 - \theta_\lambda) + 9\lambda^2(3\theta_\lambda + 4\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2).
\end{align*}
\] (3.7)

It holds

\[L_1^{(1)}\eta_1(\lambda, \mu) = L_3^{(1)}\eta_1(\lambda, \mu) = 0.\]

The space of solutions of the system $L_1^{(1)}u = L_3^{(1)}u = 0$ is 4-dimensional.

(2) (The period differential equation for $\mathcal{F}_2$) Set

\[
\begin{align*}
L_1^{(2)} &= \lambda\theta^2_\mu + \mu\theta_\lambda(3\theta_\lambda + 4\theta_\mu + 1), \\
L_3^{(2)} &= \lambda\theta_\lambda(3\theta_\lambda + 2\theta_\mu) + \mu\theta_\lambda(1 - \theta_\lambda) + 9\lambda^2(3\theta_\lambda + 4\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2).
\end{align*}
\] (3.8)

It holds

\[L_1^{(2)}\eta_2(\lambda, \mu) = L_3^{(2)}\eta_2(\lambda, \mu) = 0.\]

The space of solutions of the system $L_1^{(2)}u = L_3^{(2)}u = 0$ is 4-dimensional.

(3) (The period differential equation for $\mathcal{F}_3$) Set

\[
\begin{align*}
L_1^{(3)} &= \theta^2_\lambda - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\
L_3^{(3)} &= \theta_\lambda(3\theta_\lambda - 2\theta_\mu) + 9\lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2) + 4\mu\theta_\lambda(3\theta_\lambda + 2\theta_\mu + 1).
\end{align*}
\] (3.9)

It holds

\[L_1^{(3)}\eta_3(\lambda, \mu) = L_3^{(3)}\eta_3(\lambda, \mu) = 0.\]

The space of solutions of the system $L_1^{(3)}u = L_3^{(3)}u = 0$ is 4-dimensional.
Proof. We determine these systems by the method of indeterminate coefficients. For detail, see the proof of Theorem 3.2 in [Na].

In the following we prove that those spaces of solutions is 4-dimensional.

(1) Set $\varphi = t^1(1, \theta_\lambda, \theta_\mu, \theta_\kappa^2)$. We obtain the corresponding Pfaffian system $\Omega_1 = A_1 d\lambda + B_1 d\mu$ with $d\varphi = \Omega_1 \varphi$ by the following way. Setting

$$t_1 = 729\lambda^2 - 54\lambda(27\mu - 1) + (1 + 27\mu)^2,$$

we have

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1/9 & -1/2 & -1/2 & -(1 + 27\lambda + 27\mu)/(54\lambda) \\ a_{11}/t_1 & a_{12}/(2t_1) & a_{23}/(2t_1) & a_{24}/(2t_1) \end{pmatrix},$$

with

$$\begin{cases} a_{11} = 3\lambda(1 - 27\lambda + 27\mu), \\ a_{12} = 3\lambda(5 - 35\lambda + 135\mu), \\ a_{13} = 27\lambda(1 - 3\lambda + 27\mu), \\ a_{14} = 3(-729\lambda^2 + (1 + 27\mu)^2), \end{cases}$$

and

$$B_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1/9 & -1/2 & -1/2 & -(1 + 27\lambda + 27\mu)/(54\lambda) \\ 0 & 0 & 0 & \mu/\lambda \\ b_{11}/t_1 & b_{12}/(2t_1) & b_{13}/(2t_1) & b_{14}/(2t_1) \end{pmatrix},$$

with

$$\begin{cases} b_{11} = 3\lambda(1 + 27\lambda - 27\mu), \\ b_{12} = 27\lambda(1 + 27\lambda - 3\mu), \\ b_{13} = 3\lambda(5 + 135\lambda - 351\mu), \\ b_{14} = (1 + 27\lambda)^2 + 108(27\lambda - 1)\mu - 3645\mu^2. \end{cases}$$

We have $d\Omega_1 = \Omega_1 \wedge \Omega_1$. Therefore the system $L^{(1)}_1 u = L^{(1)}_3 u = 0$ has the 4-dimensional space of solutions.

(2) Set $\varphi = t^1(1, \theta_\lambda, \theta_\mu, \theta_\kappa^2)$. We obtain the corresponding Pfaffian system $\Omega_2 = A_2 d\lambda + B_2 d\mu$ with $d\varphi = \Omega_2 \varphi$ as the following way. Setting

$$\begin{cases} t_2 = \lambda^2(1 + 27\lambda)^2 - 2\lambda(1 + 189\lambda) + (1 + 576\lambda)\mu^2 - 256\mu^3, \\ s_2 = 1 + 108\lambda - 288\mu, \end{cases}$$

we have

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{11}/s_2 & a_{12}/(2s_2) & a_{13}/(s_2) & a_{14}/(2s_2) \\ a_{21}/(t_2s_2) & a_{22}/(t_2s_2) & a_{23}/(t_2s_2) & a_{24}/(t_2s_2) \end{pmatrix},$$

with

$$\begin{cases} a_{11} = -9\lambda, \\ a_{12} = -(81\lambda^2 + \mu - 144\lambda\mu), \\ a_{13} = -54\lambda, \\ a_{14} = -3\lambda(1 + 27\lambda - 144\mu) + \mu, \\ a_{21} = -6\lambda^3(1 + 1458\lambda^2 - 2592\lambda\mu + 6\mu(-55 + 4608\mu)), \\ a_{22} = -3\lambda^2(1 + 54\lambda(5 + 351\lambda)) + \lambda(1 + 4\lambda(61 + 810\lambda(5 + 72\lambda)))\mu + 64(17 + 2808\lambda)\mu^3 - 147456\mu^4 - 2(1 + 9\lambda(53 + 32\lambda(131 + 864\lambda)))\mu^2, \\ a_{23} = -8\lambda^3(2 - 27\lambda)^2 + 9(-133 + 2160\lambda)\mu + 28944\mu^2, \\ a_{24} = 3r_2s_2 + 162\lambda r_2 - 3\lambda s_2(\lambda + 81\lambda^2 + 1458\lambda^3 - 378\lambda\mu + \mu(-1 + 288\mu)), \end{cases}$$

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and
\[
B_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
\frac{b_{11}}{s_2} & \frac{b_{12}}{(2\lambda s_2)} & \frac{b_{13}}{s_2} & \frac{b_{14}}{(2\lambda s_2)} \\
\frac{b_{21}}{(s_2)} & \frac{b_{22}}{(\lambda^2 s_2)} & \frac{b_{23}}{s_2} & \frac{b_{24}}{(\lambda^2 s_2)} \\
\frac{b_{31}}{(t_2 s_2)} & \frac{b_{32}}{(2\lambda t_2 s_2)} & \frac{b_{33}}{(t_2 s_2)} & \frac{b_{34}}{(2\lambda t_2 s_2)}
\end{pmatrix},
\]
with
\[
\begin{aligned}
b_{11} &= -9\lambda, \\
b_{12} &= -(81\lambda^2 + \mu - 144\lambda\mu), \\
b_{13} &= -54\lambda, \\
b_{14} &= -3\lambda(1 + 27\lambda - 144\mu) + \mu, \\
b_{21} &= 36\mu, \\
b_{22} &= \mu(\lambda(-1 + 54\lambda) + 2\mu), \\
b_{23} &= 216\mu, \\
b_{24} &= (3(1 - 54\lambda)\lambda - 2\mu)\mu, \\
b_{31} &= 3\lambda(81\lambda^3(1 + 27\lambda) + \lambda(-1 + 36\lambda)(-5 + 108\lambda)\mu + 3(-1 + 32\lambda)(1 + 432\lambda)\mu^2 + 768\mu^3, \\
b_{32} &= 2187\lambda^5(1 + 27\lambda) - (1 + 192\lambda(11 + 1164\lambda))\lambda^3 + 256(1 + 864\lambda)\mu^4 - \lambda^3(2 + 27\lambda(4 + 9\lambda(77 + 864\lambda)))\mu + \lambda(5 + \lambda(1279 + 864\lambda(85 + 864\lambda)))\mu^2, \\
b_{33} &= 2\lambda(3\lambda^2(1 + 27\lambda)(-1 + 135\lambda) + 2\lambda(23 + 54\lambda(-11 + 972\lambda))\mu + 9(-3 + 64\lambda)(1 + 432\lambda)\mu^2 + 6912\mu^3, \\
b_{34} &= -(81\lambda^4(1 + 27\lambda)^2 + \lambda^2(-7 + 9\lambda(-58 + 27\lambda(-125 + 3456\lambda)))\mu + \lambda(8 + 9\lambda(425 + 24192\lambda))\mu^2 - (1 + 3456\lambda(1 + 162\lambda))\mu^3 + 256(1 + 1440\lambda)\mu^4.
\end{aligned}
\]

We see \(d\Omega_2 = \Omega_2 \wedge \Omega_2\). Therefore the system \(L_1 u = L_3 u = 0\) has the 4-dimensional solution space.

(3) Set \(\varphi = t_4(1, \theta_\lambda, \theta_\mu, \theta_\chi^2)\). We obtain the corresponding Pfaffian system \(\Omega_3 = A_3 d\lambda + B_3 d\mu\) with \(d\varphi = \Omega_3 \varphi\) as the following way. Setting
\[
\begin{aligned}
t_3 &= 729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1 + 12\mu), \\
s_3 &= -54\lambda + (1 - 4\mu)^2,
\end{aligned}
\]
we have
\[
A_3 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\frac{a_{11}}{s_3} & \frac{a_{12}}{(2s_3)} & \frac{a_{13}}{s_3} & \frac{a_{14}}{(2s_3)} \\
\frac{a_{21}}{(t_3 s_3)} & \frac{a_{22}}{(t_3 s_3)} & \frac{a_{23}}{(t_3 s_3)} & \frac{a_{24}}{(t_3 s_3)}
\end{pmatrix},
\]
with
\[
\begin{aligned}
a_{11} &= 9\lambda, \\
a_{12} &= 81\lambda + 4(1 - 4\mu)\mu, \\
a_{13} &= 27\lambda, \\
a_{14} &= 3 + 81\lambda - 48\mu^2, \\
a_{21} &= -2\lambda(-2187\lambda^2 + 27\lambda(4\mu - 9)(4\mu - 1) - (-1 + 4\mu)^3(3 + 8\mu)), \\
a_{22} &= 3\lambda(9477\lambda^2 + (1 - 4\mu)^2(-11 + 4\mu(-9 + 16\mu)) - 27\lambda(25 + 4\mu(-31 + 40\mu))), \\
a_{23} &= 2\lambda(729\lambda^2 + (-1 + 4\mu)^3(11 + 16\mu) + 27\lambda(-1 + 4\mu)(19 + 20\mu)), \\
a_{24} &= 81\lambda(-2 + 27\lambda + 8\mu)(1 + 27\lambda - 16\mu^2),
\end{aligned}
\]
and
\[
B_3 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
\frac{b_{11}}{s_3} & \frac{b_{12}}{(2s_3)} & \frac{b_{13}}{s_3} & \frac{b_{14}}{(2s_3)} \\
\frac{b_{21}}{s_3} & \frac{b_{22}}{(s_3)} & \frac{b_{23}}{s_3} & \frac{b_{24}}{s_3} \\
\frac{b_{31}}{(t_3 s_3)} & \frac{b_{32}}{(2t_3 s_3)} & \frac{b_{33}}{(t_3 s_3)} & \frac{b_{34}}{(2t_3 s_3)}
\end{pmatrix},
\]
Remark 3.2. From the Puffian systems in the above proof, we obtain the singular locus of the system (3.2):

\[
\begin{align*}
\lambda &= 0, \quad \mu = 0, \quad 729\lambda^2 - 54\lambda(27\mu - 1) + (1 + 27\mu)^2 = 0, \\
\end{align*}
\]

the singular locus of the system (3.3):

\[
\begin{align*}
\lambda &= 0, \quad \mu = 0, \quad \lambda^2(1 + 27\lambda)^2 - 2\lambda\mu(1 + 189\lambda) + (1 + 576\lambda)^2 - 256\mu^3 = 0,
\end{align*}
\]

and the singular locus of the system (3.9):

\[
\begin{align*}
\lambda &= 0, \quad \mu = 0, \quad 729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1 + 12\mu) = 0.
\end{align*}
\]

Omitting these locus from \(\mathbb{C}^2\) we have the domain \(\Lambda_j\) (\(j = 1, 2, 3\)) in (1.6), (1.9) and (1.12).

Finally, we determine the projective monodromy groups.

Let \(j \in \{1, 2, 3\}\). For generic \((\lambda, \mu) \in \Lambda_j\), we can take a basis \(\\{\gamma_5, \ldots, \gamma_{22}\}\) of \(\text{NS}(S_j(\lambda, \mu))\) such that the intersection matrix \((\gamma_i \cdot \gamma_j)_{5 \leq k, l \leq 22}\) is equal to the matrix in Theorem 2.2. This basis is extended to a basis \((\gamma_1, \ldots, \gamma_4, \gamma_5, \ldots, \gamma_{22})\) of \(H_2(S_j(\lambda, \mu))\). Let \(\\{\gamma_1', \ldots, \gamma_{22}'\}\) be its dual basis (namely \((\gamma_k \cdot \gamma_j') = \delta_{k,l}\)). By Theorem 2.2 we have \((\gamma_k' \cdot \gamma_j') = A_j\).

Using this basis \((\gamma_1, \ldots, \gamma_{22})\), we define the local period map as in the beginning of Section 2. Moreover, we define the multivalued period map

\[
\Phi_j : \Lambda_j \to \mathbb{P}^3(\mathbb{C})
\]

by the analytic continuation of the local period map along any arc in \(\Lambda_j\).

Set

\[
\mathcal{D}_j = \{ \xi \in \mathbb{P}^3(\mathbb{C}) | \xi A_j^t \xi = 0, \xi A_j \xi > 0 \}.
\]

By the Riemann-Hodge relation, we have \(\Phi_j(\Lambda_j) \subset \mathcal{D}_j\).

The fundamental group \(\pi_1(\Lambda_j, \ast)\) acts on \(\Phi_j(\Lambda_j)\) by the analytic continuation of the local period map. This action induces a group of projective linear transformations which is a subgroup of \(PGL(4, \mathbb{Z})\). We call it the projective monodromy group of the multivalued period map \(\Phi_j\).

Note that \(\mathcal{D}_j\) is composed of two connected components: \(\mathcal{D}_j = \mathcal{D}_j^+ \cup \mathcal{D}_j^-\). Set \(PO(\Lambda_j, \mathbb{Z}) = \{ g \in GL(4, \mathbb{Z}) | g A_j g = A_j \}\). It acts on \(\mathcal{D}_j\) by \(g \xi \mapsto g' \xi\). \((\xi \in \mathcal{D}_j, g \in PO(\Lambda_j, \mathbb{Z}))\). Let \(PO^+(\Lambda_j, \mathbb{Z})\) be the subgroup of \(PO(\Lambda_j, \mathbb{Z})\) given by \(\{ g \in PO(\Lambda_j, \mathbb{Z}) | g(\mathcal{D}_j^+) = \mathcal{D}_j^+ \}\).
Theorem 3.3. Let \( j \in \{1, 2, 3\} \). The projective monodromy group of the period differential equation for the family \( F_j \) is equal to \( PO^+(A_j, \mathbb{Z}) \).

Proof. Because the projective monodromy group \( G_j \) of the multivalued period map \( \Phi_j \) is equal to that of the period differential equation for \( F_j \), we determine \( G_j \). It is obvious \( G_j \subset PO^+(A_j, \mathbb{Z}) \). However, we need a delicate observation to prove the converse \( PO^+(A_j, \mathbb{Z}) \subset G_j \). For precise argument, see Section 4 in \cite{Na}.

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References

[Ba] V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Algebraic Geometry 3, 1994, 493-535.

[Be] S. M. Belcastro, Picard lattices of families of K3 surfaces, Comm. Algebra 30 (1), 2002, 61-82.

[I] T. Ishige, A Family of K3 Surfaces connected with the Hilbert Modular Group for \( \sqrt{2} \) and the GKZ Hypergeometric Differential Equation, preprint, 2010.

[Kod] K. Kodaira, On compact analytic surfaces. II, III, Ann. Math., 77, 1963, 563-626; Ann. Math., 78, 1963, 1-40.

[Koi] K. Koike, K3 surfaces induced from polytopes, Master Thesis, Chiba Univ., 1998.

[KS] M. Kreuzer and H. Sharke, Classification of Reflexive Polyhedra in Three Dimensions, Adv. Theor. Math. Phys., 2, 847-864.

[Na] A. Nagano, A period differential equations for a family of K3 surfaces and the Hilbert modular orbifold for the field \( \mathbb{Q}(\sqrt{5}) \), preprint, arXiv:1009.5725, 2010.

[Ne] A. Néron, Modèles minimaux des variétés abéliennes sur les corps locaux et globaux, Publ. Math. I. H. E. S., 21, 1964.

[Ni] V. V. Nikulin, Integral symmetric bilinear forms and some of their applications, Math. USSR Izv., 14, 1980, No.1, 103-167.

[O] M. Otsuka, 3-dimensional reflexive polytopes, Master Thesis, Chiba Univ., 1998.

[SY] T. Sasaki and M. Yoshida, Linear differential equations in two variables of rank four I-II, Math. Ann. 282, 1988, 69-111.

[Sa] T. Sato, Uniformizing differential equations of several Hilbert modular orbifolds, Math. Ann. 291, 1991, 179-189.

[Sh1] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan, 1972, 24, 20-59.

[Sh2] T. Shioda, Mordell-Weil lattice no riron to sono onyou (in Japanese), Todai seminar note, Univ. of Tokyo.
