Some Classes of Minimally Almost Periodic Topological groups *

W.W. Comfort† Franklin R. Gould‡

October 14, 2014

Abstract

A topological group $G = (G, T)$ has the small subgroup generating property (briefly: has the SSGP property, or is an SSGP group) if for each neighborhood $U$ of 1, there is a family $\mathcal{H} \subseteq \mathcal{P}(U)$ of subgroups of $G$ such that $\langle \bigcup \mathcal{H} \rangle$ is dense in $G$. The class of SSGP groups is defined and investigated with respect to the properties usually studied by topologists (products, quotients, passage to dense subgroups, and the like), and with respect to the familiar class of minimally almost periodic groups (the m.a.p. groups). Additional classes SSGP($n$) for $n < \omega$ (with SSGP(1) = SSGP) are defined and investigated, and the class-theoretic inclusions SSGP($n$) $\subseteq$ SSGP($n + 1$) $\subseteq$ m.a.p. are established and shown proper.

In passing the authors also establish the presence of SSGP(1) or SSGP(2) in many of the early examples in the literature of abelian m.a.p. groups.

2010 Mathematics Subject Classification: Primary 54H11; Secondary 22A05

Key words and phrases: SSGP group, m.a.p. group, f.p.c. group

1 Introduction

Conventions 1.1. (a) For $X$ a set we write $\mathcal{P}(X) := \{A : A \subseteq X\}$. This is the power set of $X$.

(b) The topological spaces we hypothesize, in particular our hypothesized topological groups, are assumed to be completely regular and Hausdorff (i.e., to be Tychonoff spaces). When a topology is defined or constructed on a set or a group, the Tychonoff property will be verified explicitly (if it is not obvious). In

---

*This paper derives from and extends selected portions of the Doctoral Dissertation [18], written at Wesleyan University (Middletown, Connecticut, USA) by the second-listed co-author under the guidance of the first-listed co-author.

†Department of Mathematics and Computer Science, Wesleyan University, Wesleyan Station, Middletown, CT 06459; phone: 860-685-2632; FAX 860-685-2571 Email: wcomfort@wesleyan.edu

‡Department of Mathematical Sciences, Central Connecticut State University, New Britain, 06050; Email: gouldfrr@ccsu.edu
this context we recall ([23](8.4)) that in order that a topology with continuous algebraic operators on a group be a Tychonoff topology, it suffices that it satisfy the Hausdorff separation property.

(c) For \( X \) a space and \( x \in X \) we write
\[ \mathcal{N}_X(x) := \{ U \subseteq X : U \text{ is a neighborhood of } x \} . \]
When ambiguity is unlikely we write \( \mathcal{N}(x) \) in place of \( \mathcal{N}_X(x) \).

(d) The identity of a group \( G \) is denoted 0 or \( 1_G \); if \( G \) is known or assumed to be abelian and additive notation is in play, the identity may be denoted 0 or \( 0_G \).

(e) When \( G \) is a group and \( \kappa \geq \omega \), we use the notations \( \bigoplus_\kappa \ G \) and \( G^{(\kappa)} \) interchangeably:
\[
\bigoplus_\kappa \ G = G^{(\kappa)} := \{ x \in G^\omega : |\{ \eta < \kappa : x_\eta \neq 1_G \}| < \omega \} .
\]
When \( G \) is a topological group, \( \bigoplus_\kappa \ G \) has the topology inherited from \( G^\omega \).

The minimally almost periodic groups (briefly: the m.a.p. groups) to which our title refers are by definition those topological groups \( G \) for which every continuous homomorphism \( \phi : G \to K \) with \( K \) a compact group satisfies \( \phi[G] = \{ 1_K \} \). It follows from the Gel’fand-Raikov Theorem [16] (see [23](§22) for a detailed development and proof) that every compact group \( K \) is algebraically and topologically a subgroup of a group of the form \( \Pi_{i \in I} U_i \) with each \( U_i \) a (finite-dimensional) unitary group [23](22.14). Therefore, to check that a topological group \( G \) is m.a.p. it suffices to show that each continuous homomorphism \( \phi : G \to U(n) \) with \( U(n) \) the \( n \)-dimensional unitary group satisfies \( \phi[G] = \{ 1_{U(n)} \} \). Similarly, since every compact abelian group \( K \) is algebraically and topologically isomorphic to a subgroup of a group of the form \( T^I \) [23](22.17), to check that an abelian topological group \( G \) is m.a.p., it suffices to show that each continuous homomorphism \( \phi : G \to T \) satisfies \( \phi[G] = \{ 1_T \} \).

Sometimes for convenience we denote by m.a.p. the (proper) class of m.a.p. groups, and if \( G \) is a m.a.p. group we write \( G \in \text{m.a.p.} \). Similar conventions apply to the classes \( \text{SSGP}(n) \) \( (0 \leq n < \omega) \) defined in Definition 2.3.

Algebraic characterizations of those abelian groups which admit an m.a.p. group topology has been achieved only recently [8]. For a brief historical account of the literature touching this issue, see Discussion 4.1 below.

Acknowledgement 1.2. We gratefully acknowledge helpful comments received from Dieter Remus, Dikran Dikranjan, and Saak Gabriyelyan. Each of them improved the exposition in a pre-publication version of this manuscript, and enhanced our historical commentary with additional bibliographic references.

2 SSGP Groups: Some Generalities

Definition 2.1. Let \( G = (G, T) \) be a topological group and let \( A \subseteq G \). Then \( A \) topologically generates \( G \) if \( \langle A \rangle \) is dense in \( G \).

Definition 2.2. Let \( G = (G, T) \) be a topological group. Then \( G \) has the small subgroup generating property if for every \( U \in \mathcal{N}(1_G) \) there is a family \( \mathcal{H} \) of subgroups of \( G \) such that \( \mathcal{H} \subseteq P(U) \) and \( \bigcup \mathcal{H} \) topologically generates \( G \).

A topological group with the small subgroup generating property is said to have the SSGP property, or to be an SSGP group, or simply to be SSGP.

Now for \( 0 \leq n < \omega \) the classes \( \text{SSGP}(n) \) are defined as follows.

Definition 2.3. Let \( G = (G, T) \) be a Hausdorff topological group. Then

(a) \( G \in \text{SSGP}(0) \) if \( G \) is the trivial group.

(b) \( G \in \text{SSGP}(n + 1) \) for \( n \geq 0 \) if for every \( U \in \mathcal{N}(1_G) \) there is a family \( \mathcal{H} \) of subgroups of \( G \) such that
(1) $H \subseteq \mathcal{P}(U)$.
(2) $H := \bigcup \mathcal{H}$ is normal in $G$, and
(3) $G/H \in \text{SSGP}(n)$.

Remarks 2.4. (a) For $0 \leq n < \omega$ the class-theoretic inclusion $\text{SSGP}(n) \subseteq \text{SSGP}(n + 1)$ holds, hence $\text{SSGP}(n) \subseteq \text{SSGP}(m)$ when $n < m < \omega$. To see this, note that when $G \in \text{SSGP}(n)$ and $U \in \mathcal{N}(1_G)$ then we have, taking $\mathcal{H} := \{\{1_G\}\}$, that $H := \bigcup \mathcal{H} = \{1_G\}$ and $G/H \simeq G \in \text{SSGP}(n)$, so indeed $G \in \text{SSGP}(n + 1)$.

(b) Clearly the class $\text{SSGP}$ of Definition 2.2 coincides with the class $\text{SSGP}(1)$ of Definition 2.3.

A topological group $G$ is said to be precompact if $G$ is a (dense) topological subgroup of a compact group. It is a theorem of Weil [38] that a topological group $G$ is precompact if and only if $G$ is totally bounded in the sense that for each $U \in \mathcal{N}(1_G)$ there is finite $F \subseteq G$ such that $G = FU$.

It is obvious that a precompact group $G$ with $|G| > 1$ is not m.a.p. Indeed if $G$ is dense in the compact group $\overline{G}$ then the continuous function $\text{id} : G \rightarrow \overline{G}$ does not satisfy $\text{id}[G] = \{1_{\overline{G}}\}$.

Theorem 2.5. The class-theoretic inclusion $\text{SSGP}(n) \subseteq \text{m.a.p.}$ holds for each $n < \omega$.

Proof. The proof is by induction on $n$. Clearly if $G \in \text{SSGP}(0)$ and $\phi \in \text{Hom}(G, U(m))$ then $\phi[G] = \{1_{U(m)}\}$, so $G \in \text{m.a.p.}$. Suppose now that $\text{SSGP}(n) \subseteq \text{m.a.p.}$, let $G$ be a topological group such that $G \in \text{SSGP}(n + 1)$, and let $\phi : G \rightarrow U(m)$ be a continuous homomorphism. Choose $V \in \mathcal{N}(1_{U(m)})$ so that $V$ contains no subgroups of $U(m)$ other than $\{1_{U(m)}\}$. Then $U := \phi^{-1}[V] \in \mathcal{N}(1_G)$, and $\phi$ maps every subgroup of $U$ to $1_{U(m)}$. Let $\mathcal{H} \subseteq \mathcal{P}(U)$ be a family of subgroups of $G$ such that $H := \langle \mathcal{H} \rangle$ is normal in $G$ and such that $G/H \in \text{SSGP}(n)$. Since a homomorphism maps subgroups to subgroups we have $\phi[H] = \{1_{U(m)}\}$.

It follows that $\phi$ defines a continuous homomorphism $\tilde{\phi} : G/H \rightarrow U(m)$ (given by $\tilde{\phi}(xH) := \phi(x)$). By the induction hypothesis, $\tilde{\phi}$ is the trivial homomorphism, so $\phi$ is trivial as well; the relation $G \in \text{m.a.p.}$ follows. □

Now in 2.6, 2.10 we clarify what is and is not known about the classes of groups mentioned in Theorem 2.5. We begin with a simple lemma and a familiar definition.

Lemma 2.6. Let $G$ be a nontrivial (Hausdorff) topological group for which some $U \in \mathcal{N}(1_G)$ contains no subgroup other than $\{1_G\}$. Then there is no $n < \omega$ such that $G \in \text{SSGP}(n)$.

Proof. Clearly $G \notin \text{SSGP}(0)$. Suppose there is a minimal $n > 0$ such that $G \in \text{SSGP}(n)$, and let $U \in \mathcal{N}(1_G)$ be as hypothesized. Then the only choice for $\mathcal{H} \subseteq \mathcal{P}(U)$ is $\mathcal{H} := \{\{1_G\}\}$, yielding $H = \langle \cup \mathcal{H} \rangle = \{1_G\}$. Thus, $G/H = G \in \text{SSGP}(n - 1)$, which contradicts the assumption that $n$ is minimal. □

Definition 2.7. Let $G$ be a group and let $1_G \notin C \subseteq G$. Then $C$ cogenerates $G$ if every subgroup $H$ of $G$ such that $|H| > 1$ satisfies $H \cap C \neq \emptyset$.

Theorem 2.8. Let $G$ be a nontrivial finitely cogenerated topological group. Then there is no $n < \omega$ such that $G \in \text{SSGP}(n)$.

Proof. Let $C$ be a finite set of cogenerators for $G$, and choose $U \in \mathcal{N}(1_G)$ such that $U \cap C = \emptyset$. Then $U$ contains no subgroup other than $\{1_G\}$, and the statement follows from Lemma 2.6. □
We have noted for every \( n < \omega \) the class-theoretic inclusion \( \text{SSGP}(n) \subseteq \text{m.a.p.} \). On the other hand, there are many examples of \( G \in \text{m.a.p.} \) such that \( G \in \text{SSGP}(n) \) for no \( n < \omega \). But more is true: There are groups which admit an m.a.p. topology which admit an \( \text{SSGP}(n) \) topology for no \( n < \omega \). Indeed from Corollary 2.13 and Theorem 2.8 respectively we see that the groups \( G = \mathbb{Z} \) and \( G = \mathbb{Z}(p^\infty) \) (cogenerated by suitable \( C \subseteq G \) with \( |C| = p - 1 < \omega \)) admit no \( \text{SSGP}(n) \) topology; while Ajtai, Havas, and Komlós [4], and later Zelenyuk and Protasov [39], have shown the existence of m.a.p. topologies for \( \mathbb{Z} \) and for \( \mathbb{Z}(p^\infty) \).

In Theorem 2.12 we show that in the context of abelian groups, Theorem 2.8 can be strengthened. We use the following basic facts from the theory of abelian groups.

**Lemma 2.9.** A finitely cogenerated group is the direct sum of finite cyclic \( p \)-groups and groups of the form \( \mathbb{Z}(p^\infty) \), hence is torsion ([11](3.1 and 25.1)).

**Lemma 2.10.** A finitely generated abelian group is the direct sum of cyclic free groups and cyclic torsion groups ([11](15.5)).

**Lemma 2.11.** If \( G \) is a finitely generated abelian group and \( H \) is a torsionfree subgroup then there is a decomposition \( G = K \oplus T \) where \( T \) is the torsion subgroup, \( K \) is torsionfree and \( H \subseteq K \) ([11], Chapter III).

**Theorem 2.12.** A nontrivial abelian group which is the direct sum of a finitely generated group and a finitely cogenerated group does not admit an \( \text{SSGP}(n) \) topology for any \( n < \omega \).

**Proof.** We proceed by induction on the torsionfree rank, \( r_0(G) \). Suppose first that \( r_0(G) = 0 \). Then \( G \) is finitely co-generated and does not admit an \( \text{SSGP}(n) \) topology by Theorem 2.8. Now suppose that the theorem has been proved up to rank \( r - 1 \) and we have \( r_0(G) = r \geq 1 \) and \( G = F \oplus T \), with \( F \) finitely generated and \( T \) finitely co-generated. Using Lemmas 2.9 and 2.10 we rewrite \( G \) in the form \( G = F' \oplus T' \) where \( T' \) is the (finitely co-generated) torsion subgroup and \( F' \) is free. Then \( r_0(F') = r_0(G) = r \). Let \( a \in F' \) be an element of infinite order and choose \( U \in \mathcal{N}(0) \) so that \( a \notin U \) and so that \( U \cap C = \emptyset \) where \( C \) is a finite set of cogenerators of \( T' \) (with \( 0 \notin C \)). If all subgroups contained in \( U \) are torsion, then each such subgroup is a subgroup of \( T' \) and is therefore the zero subgroup, since it misses \( C \). In that case, \( G \) does not have SSGP. Alternatively, if \( U \) contains a cyclic subgroup \( H \) of infinite order, we have \( r_0(H) > 0 \). Furthermore, since \( H \subseteq U \), we have \( H \cap T' = \{0\} \). It follows from Lemma 2.11 that there is a decomposition \( G = F'' \oplus T'' \) which is isomorphic to the original decomposition and is such that \( H \subseteq F'' \). Since a quotient of a finitely generated group is also finitely generated, it follows that \( F''/\overline{T} \) is finitely generated. Then we have \( G/\overline{H} = (F''/\overline{T}) \oplus T'' \). We also have that \( r_0(G/\overline{H}) < r \) because \( r_0(G) = r_0(\overline{H}) + r_0(G/\overline{H}) \), ([11](§16, Ex. 3(d))]. Also, \( G/\overline{H} \) is nontrivial since \( \overline{T} \subseteq \overline{T} \) and \( a \notin \overline{T} \). It follows by our induction assumption that \( G/\overline{H} \) does not admit \( \text{SSGP}(n) \), and so by Theorem 2.13(b) (below), neither does \( G \). \( \square \)

**Corollary 2.13.** The group \( \mathbb{Z} \) does not admit an \( \text{SSGP}(n) \) topology for any \( n < \omega \).

The following theorem lists several inheritance properties for groups in the classes \( \text{SSGP}(n) \).

**Theorem 2.14.** (a) If \( K \) is a closed normal subgroup of \( G \), with \( K \in \text{SSGP}(n) \) and \( G/K \in \text{SSGP}(m) \) then \( G \in \text{SSGP}(m + n) \).
(b) If $G \in \text{SSGP}(n)$ and $\pi : G \to B$ is a continuous homomorphism from $G$ onto $B$, then $B \in \text{SSGP}(n)$. In particular, if $K$ is a closed normal subgroup of $G \in \text{SSGP}(n)$ then $G/K \in \text{SSGP}(n)$.

(c) If $K$ is a dense subgroup of $G$ and $K \in \text{SSGP}(n)$ then $G \in \text{SSGP}(n)$.

(d) If $G_i \in \text{SSGP}(n)$ for each $i \in I$ then $\bigoplus_{i \in I} G_i \in \text{SSGP}(n)$ and $\prod_{i \in I} G_i \in \text{SSGP}(n)$.

Proof. We proceed in each case by induction on $n$. Each statement is trivial when $n = 0$. We address (a), (b), (c) and (d) in order, assuming in each case for $1 \leq n < \omega$ that the statement holds for $n - 1$.

(a) Let $U \in \mathcal{N}(1_G)$, so that $U \cap K \in \mathcal{N}(1_K)$. Then there is a family $\mathcal{H} \subseteq \mathcal{P}(U \cap K)$ of subgroups of $K$ such that $K/H \in \text{SSGP}(n - 1)$ where $H := \langle \cup \mathcal{H} \rangle^K = \langle \cup \mathcal{H} \rangle^G$. Since $G/K$ is topologically isomorphic with $(G/H)/(K/H)$, we have $(G/H)/(K/H) \in \text{SSGP}(m)$ along with $K/H \in \text{SSGP}(n - 1)$. Then by the induction hypothesis, $G/H \in \text{SSGP}(m + n - 1)$. Since $\mathcal{H} \subseteq \mathcal{P}(U)$ with $U$ arbitrary, we have $G \in \text{SSGP}(m + n)$, as required.

(b) Given $G \in \text{SSGP}(n)$ and continuous $\pi : G \to B$, let $U \in \mathcal{N}(1_B)$. Then $\pi^{-1}[U] \in \mathcal{N}(1_G)$ and there is a family $\mathcal{H} \subseteq \mathcal{P}(\pi^{-1}[U])$ of subgroups such that $G/\langle \cup \mathcal{H} \rangle \in \text{SSGP}(n - 1)$. Let $\tilde{\mathcal{H}}$ be the family of subgroups of $B$ given by $\tilde{\mathcal{H}} := \{ \pi[L] : L \in \mathcal{H} \}$. Then $\tilde{\mathcal{H}} \subseteq \mathcal{P}(U)$. Set $H := \langle \cup \tilde{\mathcal{H}} \rangle$ and set $\tilde{H} := \langle \cup \tilde{\mathcal{H}} \rangle^G$. Then $\tilde{H}$ is normal in $B$ since by assumption $H$ is normal in $G$. By invoking the induction hypothesis we will show that $B/\tilde{H} \in \text{SSGP}(n - 1)$ and thus that $B \in \text{SSGP}(n)$. Note that $H \subseteq \pi^{-1}[\tilde{H}]$ since $\langle \cup \mathcal{H} \rangle \subseteq \pi^{-1}[(\cup \tilde{\mathcal{H}})]$ and $\pi^{-1}[\tilde{H}]$ is closed. We have that $G/H \in \text{SSGP}(n - 1)$ so by induction, $(G/H)/(\pi^{-1}[\tilde{H}]/\tilde{H}) \in \text{SSGP}(n - 1)$ and this is topologically isomorphic with $G/\pi^{-1}[\tilde{H}]$ by the second topological isomorphism theorem. Now, we claim that the algebraic isomorphism $\pi : G/\pi^{-1}[\tilde{H}] \to B/\tilde{H}$ induced by $\pi$ is continuous (though it may not be open). Clearly, $\pi$ maps cosets of $\pi^{-1}[\tilde{H}]$ to cosets of $\tilde{H}$. If $\tilde{V}$ is an open union of cosets of $\tilde{H}$, then $\pi^{-1}[\tilde{V}]$ is an open union of cosets of $\pi^{-1}[\tilde{H}]$ and the claim follows. Composing maps gives a continuous homomorphism from $G/H$ onto $B/\tilde{H}$; we conclude that $B/\tilde{H} \in \text{SSGP}(n - 1)$ and thus $B \in \text{SSGP}(n)$, as required.

(c) Given $G$ and $K$ as hypothesized, let $U \in \mathcal{N}(1_G)$. Since $U \cap K \in \mathcal{N}(1_K)$, there is a family $\mathcal{H} \subseteq \mathcal{P}(U \cap K)$ of subgroups of $K$ such that $K/H \in \text{SSGP}(n - 1)$, where $H := \langle \cup \mathcal{H} \rangle^K$. Note that $H = \overline{H} \cap K$. Let $\phi : K\overline{H}/\overline{H} \to K/(K \cap \overline{H})$ be the natural isomorphism from the first (algebraic) isomorphism theorem for groups. The corresponding theorem for topological groups says that $\phi$ is an open map, i.e., $\phi^{-1}$ is a continuous map. Then from part (a) of this theorem, $K\overline{H}/\overline{H} \in \text{SSGP}(n - 1)$. Now $K\overline{H}/\overline{H}$ is dense in $G/\overline{H}$, because the subset of $G$ that projects onto the closure of $K\overline{H}/\overline{H}$ must be closed and must contain $K\overline{H}$. Then $G/\overline{H} \in \text{SSGP}(n - 1)$ by the induction hypothesis. Since $\overline{H} = \langle \cup \mathcal{H} \rangle^G$, we have $G \in \text{SSGP}(n)$, as required.

(d) Since $\bigoplus_{i \in I} G_i$ is dense in $\prod_{i \in I} G_i$, it suffices by part (c) to treat the case $G := \bigoplus_{i \in I} G_i$. Let $U \in \mathcal{N}_G(1_G)$, without loss of generality with $U = \prod_{i \in I} U_i$ where $U_i \in \mathcal{N}(1_G)$ and $U_i = G_i$ for $i > N_U$. For each $i$ there is a family $\mathcal{H}_i \subseteq \mathcal{P}(U_i)$ of subgroups of $G_i$ such that $G_i/H_i \in \text{SSGP}(n - 1)$ where $H_i := \langle \cup \mathcal{H}_i \rangle$. Now consider the family of subgroups of $G$ given by $\mathcal{H} := \{ \bigoplus_{i \in I} L_i : L_i \in \mathcal{H}_i \}$. Then $\mathcal{H} \subseteq \mathcal{P}(U)$, $\langle \cup \mathcal{H} \rangle$ is identical to $\bigoplus_{i \in I} \langle \cup \mathcal{H}_i \rangle$, and $H := \langle \cup \mathcal{H} \rangle$ is identical to $\bigoplus_{i \in I} H_i$. We also have that $G/H$ is topologically isomorphic with $\bigoplus_{i \in I} G_i/H_i$ (cf. Part 6.9). From the induction hypothesis we have $G/H \in \text{SSGP}(n - 1)$, so $G \in \text{SSGP}(n)$, as required. \hfill \Box
Remark 2.15. Certain other tempting statements of inheritance or permanence type, parallel in spirit to those considered in Theorem 2.14, do not hold in general. We give some examples.

(a) We show below, using a construction of Hartman and Mycielski [21] and of Dierolf and Warken [6], that a closed subgroup of an SSGP group may lack the SSGP($n$) property for every $n < \omega$. Indeed, every topological group can be embedded as a closed subgroup of an SSGP group (Theorem 2.18).

(b) The conclusion of part (a) of Theorem 2.14 can fail when $s < m + n$ replaces $m + n$ in its statement. For example, the construction used in Lemma 3.4 shows that a topological group $G / \in SSGP(n)$ may have a closed normal subgroup $K \in SSGP(1)$ with also $G/K \in SSGP(n)$, so $s = m + n$ is minimal when $m = 1$. We did not pursue the issue of minimality of $m + n$ in Theorem 2.14(a) for arbitrary $m, n > 1$.

(c) The converse to Theorem 2.14(c) can fail. In [18] a certain monothetic m.a.p. group constructed by Glasner [17] is shown to have SSGP, but we noted above in Corollary 2.13 that $\mathbb{Z}$ admits an SSGP($n$) topology for no $n < \omega$.

In contrast to that phenomenon, it should be mentioned that (as has been noted by many authors) in the context of m.a.p. groups, a dense subgroup $H$ of a topological group $G$ satisfies $H \in$ m.a.p. if and only if $G \in$ m.a.p. Thus in particular in the case of Glasner’s monothetic group, necessarily the dense subgroup $\mathbb{Z}$ inherits an m.a.p. topology.

We now restrict our discussion to abelian groups and to the class SSGP=SSGP(1), and examine which specific abelian groups do and do not admit an SSGP topology. We have already noted (Theorem 2.12) that the product of a finitely co-generated abelian group with a finitely generated abelian group does not admit an SSGP topology even though it may admit an m.a.p. topology. We now give additional examples of abelian groups which admit not only an m.a.p. topology but also an SSGP topology.

Theorem 2.16. The following abelian groups admit an SSGP topology.

(a) $\mathbb{Q}$, and those subgroups of $\mathbb{Q}$ in which some primes are excluded from denominators, as long as an infinite number of primes and their powers are allowed;

(b) $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}'/\mathbb{Z}$ where $\mathbb{Q}'$ is a subgroup of $\mathbb{Q}$ as in described in (a);

(c) direct sums of the form $\bigoplus_{i<\omega} \mathbb{Z}_{p_i}$ where the primes $p_i$ all coincide or all differ;

(d) $\mathbb{Z}^{(\omega)}$ (the direct sum);

(e) $\mathbb{Z}^{(\lambda)}$ (the full product);

(f) $G^{(\lambda)}$ for $|G| > 1$ and $\lambda \geq \omega$;

(g) $F^\lambda$ for $1 < |F| < \omega$ and $\lambda \geq \omega$;

(h) arbitrary sums and products of groups which admit an SSGP topology.

Item (h) is a special case of Theorem 2.14(d), and item (e) is demonstrated in the second author’s paper [19]. The “coincide” case of item (c) follows from item (f), the “differ” case is established below in Theorem 2.20 (c) and (d)) below demonstrates the validity of item (f) for $\omega \leq \lambda \leq \mathfrak{c}$. This together with (h) gives (d) and (f) in full generality. Item (g) then follows from the relation $F^{\lambda} \simeq \bigoplus_{2^\lambda} F$ ([11](8.4, 8.5)). The remaining items are demonstrated in [18].

There are many examples of nontrivial SSGP(1) groups (that is, of SSGP groups). It has been shown by Hartman and Mycielski [21] that every topological group $G$ embeds as a closed subgroup into a connected, arcwise connected group $G^*$; two decades later Dierolf and Warken [6], working independently and without reference to [21], found essentially the same embedding $G \subseteq G^*$.
and showed that $G^* \in \text{m.a.p.}$. Indeed the arguments of \cite{6} show in effect that $G^* \in \text{SSGP}$ (of course with property SSGP not yet having been named). We now describe the construction and we give briefly the relevant argument.

**Definition 2.17.** Let $G$ be a Hausdorff topological group. Then algebraically $G^*$ is the group of step functions $f : [0, 1) \to G$ with finitely many steps, each of the form $[a, b)$ with $0 \leq a < b \leq 1$. The group operation is pointwise multiplication in $G$. The topology $T$ on $G^*$ is the topology generated by (basic) neighborhoods of the identity function $1_{G^*} \in G^*$ of the form

$$N(U, \epsilon) := \{ f \in G^* : \lambda(\{ x \in [0, 1) : f(x) \notin U \}) < \epsilon \},$$

where $\epsilon > 0$, $U \in \mathcal{N}_G(1_G)$, and $\lambda$ denotes the usual Lebesgue measure on $[0, 1)$.

**Theorem 2.18.** Let $G$ be a topological group. Then

(a) $G$ is closed in $G^* = (G^*, T)$;

(b) $G^*$ is arcwise connected; and

(c) $G^* \in \text{SSGP}$.

**Proof.** Note first that the association of each $x \in G$ with the function $x^* \in G^*$ (the function given by $x^*(r) := x$ for all $r \in [0, 1)$) realizes $G$ algebraically as a subgroup of $G^*$. Furthermore the map $x \mapsto x^*$ is a homeomorphism onto its range, since for $\epsilon < 1$, $U \in \mathcal{N}(1_G)$ and $x \in G$ one has

$x \in U \iff x^* \in N(U, \epsilon)$.

(a) Let $f_0 \in G^*$ and $f_0 \notin G$. There are distinct (disjoint) subintervals of $[0, 1)$ on which $f_0$ assumes distinct values $g_0, g_1 \in G$ respectively. By the Hausdorff property there is $U \in \mathcal{N}(1_G)$ such that $g_1 U \cap g_2 U = \emptyset$. Choose $\epsilon$ smaller than the measure of either of the two indicated intervals. Then $f_0 N(U, \epsilon)$ is a neighborhood of $f_0$ such that $f_0 N(U, \epsilon) \cap G = \emptyset$. Therefore, $G$ is closed in $G^*$.

(b) Let $f \in G^*$ and for each $t \in [0, 1)$ define $f_t : [0, 1) \to G$ by $f_t(x) = f(x)$ for $0 \leq x < t$ and $f_t(x) = 1_G$ for $t \leq x < 1$; and define $f_1 := f$. Then $t \mapsto f_t$ is a continuous map from $[0, 1)$ to $G^*$ such that $f_0 = 1_G$, and $f_1 = f$. To show that the map is continuous, let $f_t N(U, \epsilon)$ be a basic neighborhood of $f_t$ and let $s \in (t - \epsilon/4, t + \epsilon/4) \cap [0, 1)$. Then $f_s \in f_t N(U, \epsilon)$, since

$$\lambda(\{ x \in [0, 1) : f_s(x) - f_t(x) \notin U \}) < \epsilon.$$

We conclude that $G^*$ is arcwise connected.

(c) Let $N(U, \epsilon) \in \mathcal{N}(1_G^*)$, and for each interval $I = [t_0, t_1) \subseteq [0, 1)$ with $t_1 - t_0 < \epsilon$ let

$$F(I) := \{ f \in G^* : \text{f is constant on I}, \ f \equiv 1_G \text{ on } [0, 1) \setminus I \}.$$

Then $F(I)$ is a subgroup of $G^*$ and $F(I) \subseteq N(U, \epsilon)$, and with $\mathcal{H}_e := \{ F(I) \}$ we have that each $f \in G^*$ is the product of finitely many elements from $\bigcup \mathcal{H}_e$—i.e., $f \in (\bigcup \mathcal{H}_e) \subseteq (\bigcup \mathcal{H}_e)$. It follows that $G^* \in \text{SSGP} \subseteq \text{m.a.p.}$.

There are also countable subgroups of $G^*$ which retain properties (a) and (c) (but not (b)) of Theorem 2.18. We make the following definition.

**Definition 2.19.** Let $G$ be a topological group and let $A \subseteq [0, 1)$ where $A$ is dense in $[0, 1)$ and $0 \in A$. Then $G_A^* = (G_A^*, T)$ is the subgroup of $(G^*, T)$ obtained by restriction of step functions on $[0, 1)$ to those steps $[a, b)$ such that $a, b \in A \cup \{1\}$, $a < b$.

**Theorem 2.20.** Let $G$ be a topological group. Then

(a) $G$ is closed in $G_A^* = (G_A^*, T)$;

(b) $G_A^*$ is dense in $G^*$;

(c) $G_A^* \in \text{SSGP}$; and

(d) if $G$ is abelian, then the groups $G_A^*, G(\lambda)$ (with $\lambda = |A|$) are isomorphic as groups.
Proof. With the obvious required change, the proofs of (a) and (c) coincide with the corresponding proofs in Theorem 2.18.

(b) Let \( f \in G^* \) have \( n \) steps \((n < \omega)\) and let \( f \cdot N(U, \epsilon) \in N_G^\alpha(f) \). Then there is \( \tilde{f} \in f \cdot N(U, \epsilon) \cap G^*_A \) such that \( \tilde{f} \) has step end-points in \( A \cup \{1\} \), each within \( \epsilon/n \) of the corresponding end-point for \( f \).

(d) We give an explicit isomorphism. \( G^{(\alpha)} \) can be expressed as the set of functions \( \phi : A \to G \) with finite support and pointwise addition. Each such function is the sum of finitely many elements of the form \( \phi_{a, g} \) where \( a \in A \), \( g \in G \), \( \phi_{a, g}(a) = g \) and \( \phi_{a, g}(x) = 0 \) for \( x \neq a \). Now we define corresponding functions \( f_{a, g} \in G^*_A \). Let \( f_{a, g}(x) = g \) for all \( x \in [0, 1) \) and for \( a > 0 \), let \( f_{a, g} \) be the two-step function defined by \( f_{a, g}(x) = g \) for \( 0 \leq x < a \) and \( f_{a, g}(x) = 0 \) for \( a \leq x < 1 \). Then the map \( \phi_{a, g} \mapsto f_{a, g} \) extends linearly to an isomorphism from \( G^{(\lambda)} \) onto \( G^*_A \).

Remark 2.21. Note that Theorem 2.14(c) cannot be used to prove (c) from (b) in Theorem 2.20. Note also that the isomorphism given in the proof of (d) provides a way of imposing an SSGP topology on \( G^{(\lambda)} \) for \( \omega \leq \lambda \leq c \) and \( G \) an abelian group. A corresponding mapping can be given when \( G \) is nonabelian but it need not be an isomorphism. In that case, it is still possible to write each element of \( G^*_A \) as a product of two-step functions, but then it is necessary to specify the order in which they are to be multiplied.

Some other SSGP groups arise as a consequence of the following fact.

Theorem 2.22. Let \( G = (G, T) \) be a (possibly nonabelian) torsion group of bounded order such that \( (G, T) \) has no proper open subgroup. Then \( G \in \text{SSGP} \).

Proof. There is an integer \( M \) which bounds the order of each \( x \in G \), and then \( N := M! \) satisfies \( x^N = 1_G \) for each \( x \in G \).

We must show: Each \( U \in N(1_G) \) contains a family \( H \) of subgroups such that \( \langle \bigcup H \rangle \) is dense in \( G \). Given such \( U \), let \( V \in N(1_G) \) satisfy \( V^N \subseteq U \). For each \( x \in V \) we have \( x^k \in U \) for \( 0 < k < N \), hence \( x \in V \Rightarrow \langle x \rangle \subseteq U \). Thus with \( H := \{ \langle x \rangle : x \in V \} \) we have: \( H \) is a family of subgroups of \( U \) (that is, of subsets of \( U \) which are subgroups of \( G \)). Then \( V \subseteq \bigcup H \), so \( G = \langle V \rangle \subseteq \langle \bigcup H \rangle \) — the first equality because \( \langle V \rangle \) is an open subgroup of \( G \).

In Corollaries 2.23 and 2.26 we record two consequences of Theorem 2.22.

Corollary 2.23. If \( (G, T) \) is a (possibly nonabelian) connected torsion group of bounded order, then \( (G, T) \in \text{SSGP} \).

Proof. A connected group has no proper open subgroup, so Theorem 2.22 applies.

Lemma 2.24. Let \( G \in \text{m.a.p.} \) and \( G \) abelian. Then \( G \) does not contain a proper open subgroup.

Proof. Suppose that \( H \) is a proper open subgroup of \( G \). Since \( G/H \) is a nontrivial abelian discrete (and therefore locally compact) group, there is a nontrivial (continuous) homomorphism \( \phi : G/H \to T \). Then the composition of \( \phi \) with the projection map from \( G \) to \( G/H \) is a nontrivial continuous homomorphism from \( G \) to a compact group, contradicting the m.a.p. property of \( G \).

Remark 2.25. We are grateful to Dikran Dikranjan for the helpful reminder that Lemma 2.24 fails when the “abelian” hypothesis is omitted. Examples to this effect abound, samples including: (a) the infinite algebraically simple groups whose only group topology is the discrete topology, as concocted by
Shelah [37] under [CH], and by Hesse [22] and Ol’shanskii [28] (and later by several others) in [ZFC]; and (b) such matrix groups as $SL(2, \mathbb{C})$, shown by von Neumann [26] to be m.a.p. even in the discrete topology (the later treatments [27], [23] (22.22(h)) and [2] (9.11) of this specific group follow closely those of [26]).

**Corollary 2.26.** For an abelian torsion group $G$ of bounded order, these conditions are equivalent for each group topology $T$ on $G$.

(a) $(G, T) \in \text{SSGP}$;
(b) $(G, T) \in \text{m.a.p.} $; and
(c) $(G, T)$ has no proper open normal subgroup.

**Proof.** The implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a) are given respectively by Theorem 2.6, Lemma 2.24, and Theorem 2.22. □

**Remark 2.27.** It is worthwhile to note that connected torsion groups of bounded order, as hypothesized in Theorem 2.23 do exist. For the reader’s convenience, drawing freely on the expositions [36] and [2] (2.3–2.4), we outline the essentials of $\cite{36}$, Prodanov [31] (of course, the classes SSGP($\omega$) were not formally defined in 1980); he provides on the direct sum $G := \mathbb{Z}^{(\omega)} = \bigoplus_{n=1}^{\omega} \mathbb{Z}$ a topological group topology $T$ which satisfies (as we show below) $(G, T) \in \text{SSGP}(2)$ and $(G, T) \notin \text{SSGP}(1)$. Given $G$, Prodanov [31] constructs a basis at $0$ for a group topology $T$ as follows: Let $\epsilon_n (n = 1, 2, \ldots)$ be the canonical basis for $G$. Then, use induction to define a sequence of finite subsets of $\mathbb{Z}^{(\omega)}$.

## 3 SSGP Groups: Some Specifics

The question naturally arises whether for $n < \omega$ the class-theoretic inclusion SSGP($n \subset \mathbb{Z}^{(\omega)}$) is proper. The issue is addressed in part by Prodanov [31] (of course, the classes SSGP($n$) had not been formally defined in 1980); he provided on the direct sum $G := \mathbb{Z}^{(\omega)} = \bigoplus_{n=1}^{\omega} \mathbb{Z}$ a topological group topology $T$ which satisfies (as we show below) $(G, T) \in \text{SSGP}(2)$ and $(G, T) \notin \text{SSGP}(1)$. Given $G$, Prodanov [31] constructs a basis at $0$ for a group topology $T$ as follows: Let $\epsilon_n (n = 1, 2, \ldots)$ be the canonical basis for $G$. Then, use induction to define a sequence of finite subsets of $\mathbb{Z}^{(\omega)}$:...
"Let $A_1 = \{e_1 - e_2, e_2\}$, and suppose that the sets $A_1, A_2, \ldots, A_{m-1}$ $(m = 2, 3, \ldots)$ are already defined. By $\alpha_m$ we denote an integer so large that the $s$-th co-ordinates of all elements of $A_1 \cup A_2 \cup \ldots \cup A_{m-1}$ are zero for $s \geq \alpha_m$. Now we define $A_m$ to consist of all differences

$$(1) \ e_{i+k\alpha_m} - e_{i+(k+1)\alpha_m} \quad (1 \leq i \leq m, \ 0 \leq k \leq 2^{m-1} - 1)$$

and of the elements

$$(2) \ e_{i+2^{m-1}\alpha_m} \quad (1 \leq i \leq m).$$

Thus the sequence $\{A_n\}_{n=1}^{\infty}$ is defined.

Now for arbitrary $n \geq 1$ we define

$$(3) \ U_n := (n+1)!Z(\omega) \pm A_n \pm 2A_{n+1} \pm \ldots \pm 2^l A_{n+l} \pm \ldots$$

(By the notation of (3) Prodanov means that $U_n$ consists of those elements of $\mathbb{Z}(\omega)$ which can be represented as a finite sum consisting of an element divisible by $(n+1)!$ plus at most one element of $A_n$ with arbitrary sign, plus at most two elements of $A_{n+1}$ with arbitrary signs, plus at most four elements of $A_{n+2}$ with arbitrary signs, and so on.)

"It follows directly from that definition that the sets $U_n$ are symmetric with respect to 0, and that $U_{n+1} \cup U_{n+1} \subset U_n$ ($n = 1, 2, \ldots$). Therefore they form a fundamental system of neighborhoods of 0 for a group topology $\mathcal{T}$ on $\mathbb{Z}(\omega)$.

Since we need it later, we give a careful proof of an additional fact outlined only briefly by Prodanov [31].

**Theorem 3.1.** The group $\mathbb{Z}(\omega)$ with the group topology $\mathcal{T}$ defined above is Hausdorff.

**Proof.** It suffices to show $\bigcap_{n<\omega} U_n = \{0\}$.

Let $0 \neq g = \sum_i a_i e_i$ where the $e_i$ form the canonical basis. Then there is a least integer $r$ such that $a_i = 0$ for $i \geq r + 1$, and there is a least integer $s$ such that $(s+1)!$ does not divide $g$. Let $n := \max(r, s)$. We claim that if $n > 0$ then $g \notin U_n$. Suppose otherwise. Since $(n+1)!$ does not divide $g$, there is some $p \leq n$ such that $(n+1)!$ does not divide $a_p$. Thus any representation of $g$ in the form (3) must include, for at least one $m > n$, one or more terms of the form $\pm(e_p - e_{p+\alpha_m})$, all with the same sign. This means that the components $\pm e_{p+\alpha_m}$ must be cancelled by the same components from additional terms of the form $\pm(e_{p+\alpha_m} - e_{p+2\alpha_m})$. This chain of implications continues until $k$ reaches its maximum value, $2^{m-1} - 1$, with the inclusion of terms $\pm(e_{p+(2^{m-1}-1)\alpha_m} - e_{p+2^{m-1}\alpha_m})$. Finally, the components $\pm e_{p+2^{m-1}\alpha_m}$ must be cancelled by terms of type (2) with $i = p$ and the same value of $m$. This means that we have necessarily included at least $2^{m-1} + 1$ elements from $A_m$ in our expansion of $g$, contradicting the requirement that no more than $2^{m-n}$ elements of $A_m$ be included as summands for such representations of $g \in U_n$. □

**Theorem 3.2.** Prodanov’s group $(G, \mathcal{T})$ satisfies $(G, \mathcal{T}) \in \text{SSGP}(2)$, $(G, \mathcal{T}) \notin \text{SSGP}(1)$.

**Proof.** First we show that $(\mathbb{Z}(\omega), \mathcal{T}) \in \text{SSGP}(2)$. Since the $U_n$ form a basic set of neighborhoods of 0, every neighborhood of 0 contains a subgroup of the form $(n+1)!\mathbb{Z}(\omega)$ for some $n$. Thus each $U_n$ generates $\mathbb{Z}(\omega)$. This means that $\mathbb{Z}(\omega)$ has no proper open subgroups, hence for fixed $n$ the group $G_n := \mathbb{Z}(\omega)/(n+1)!\mathbb{Z}(\omega)$ contains no proper open subgroup. Further, $G_n$ is of bounded order. Thus $G_n \in \text{SSGP}(1)$ by Theorem [7.22] and therefore $(\mathbb{Z}(\omega), \mathcal{T}) \in \text{SSGP}(2)$.
We show that \((Z^{(\omega)} , T) \notin SSGP(1)\): From the definition of a basic neighborhood \(U_n\) it is clear that any subgroup of \(Z^{(\omega)}\) which \(U_n\) contains must also be a subgroup of \((n+1)! Z^{(\omega)}\). Thus the condition \((Z^{(\omega)} , T) \in SSGP\) would imply that each subgroup \((n+1)! Z^{(\omega)}\) is dense in \(Z^{(\omega)}\). Thus every nonempty open set is dense in \(Z^{(\omega)}\), contradicting Theorem 3.1 (\((Z^{(\omega)} , T)\) is a Hausdorff topological group.

Another way to show that the class-theoretic inclusion \(SSGP(1) \subseteq SSGP(2)\) is proper is to find a topological group \(G \notin SSGP\) with a closed normal subgroup \(H\) such that \(H \subseteq SSGP\) and \(G/H \in SSGP\) (for the case \(n = m = 1\) of Theorem 2.14(a) then shows \(G \in SSGP(2)\)). Such examples were given in the second author’s dissertation [18]. We generalize that construction to show by induction for arbitrary \(n > 1\) the existence of topological groups which have \(SSGP(n)\) but not \(SSGP(n-1)\).

The case \(n = 1\) is demonstrated by any of our nontrivial SSGP examples. However, our construction by induction for the case \(n > 1\) will require additional properties, namely, that our example groups be abelian, countable, torsionfree, and have a group topology defined by a metric. Let \(H\) be the topological group \(Z_A^n\) as in Definition 2.19 where \(Z\) has the discrete topology and \(A \subseteq [0, 1)\) consists of points of the form \(\frac{1}{m}n\) for \(m, t \in \mathbb{N}\) and \(0 \leq t \leq 2^m\). It is clear that \(H\) is abelian, countable and torsionfree and is not the trivial group. \(H\) also has \(SSGP(1)\) (Theorem 2.20). The topology on \(H\) is the metric topology given by the norm \(\|h\| := \lambda(Supp(h))\) for \(h \in H\), where \(Supp(h)\) is the support of \(h\) as a function on \([0, 1)\). (Here the “norm” designation follows historical precedent; we use it both out of respect and for convenience, but we do not require that \(|Ng| = |N| \cdot \|g\|\).)

Fix \(n > 1\) and suppose there is a countable, torsionfree abelian group \(G_{n-1}\) with a metric \(\rho\) that defines a group topology on \(G_{n-1}\) such that \(G_{n-1} \subseteq SSGP(n-1)\), and \(G_{n-1} \notin SSGP(n-2)\). Now, define (algebraically) \(G_n := H \oplus G_{n-1}\); we give \(G_n\) a topology which is different from the product topology, using a technique borrowed from M. Ajtai, I. Havas, and J. Komlós [1]. We create a metric group topology on \(G_n\) starting with a function \(\nu : S \rightarrow \mathbb{R}^+\), where \(S\) is a specified generating set for \(G_n\) which does not contain \(1_{G_n}\). We refer to \(\nu\) together with the generating set \(S\) as a “provisional norm” (in terms of which a norm on the group will be defined). For \(g \in G_n\), we write \(g = (h,g')\) with \(h \in H\) and \(g' \in G_{n-1}\).

Let \(e_{m,t}\) be the element of \(H\) which has value 1 on the interval \([\frac{m+1}{2^m}, \frac{m+2}{2^m})\) and has value 0 elsewhere. Let \(U_m := \{g' \in G_{n-1} : \|g'\| \leq \frac{1}{2^m}\}\) for \(m \in \mathbb{Z}\) and let \(g'_{m,t}\) for \(t < \omega\) be a list of the elements in \(U_m\). In addition, let \(r(m,t)\) be an enumeration of the pairs \((m,t)\). We define the set \(S \subseteq G_n\) to be the set of elements \(s\) in the following provisional norm assignments, \(\nu(s)\):

1. \(\nu((p \cdot e_{m,t} , 0)) = \|p \cdot e_{m,t}\| = \frac{1}{2^m}\) for \(m \in \mathbb{N}_0\), \(0 < |p| < \omega\), \(1 \leq t \leq 2^m\)
2. \(\nu((f_r , g_{m,t})) = \|g'_{m,t}\| \leq \frac{1}{2^m}\) for \(m \in \mathbb{Z}\), \(t < \omega\)

where \(f_r = \sum_{i=1}^{2r} e_{r,2i-1}\) and \(r = r(m,t)\).

Notice that (1) gives the same provisional norm to every non-zero element in a subgroup of \(G_n\), whereas (2) is for a linearly independent set of elements of \(G_n\).

Now we define a seminorm \(\|\cdot\|\) on \(G_n\) in terms of the provisional norm \(\nu\).

**Definition 3.3.** For \(g \in G_n\),
\|g\| := \inf \left\{ \sum_{i=1}^{N} (\alpha_i \nu(s_i)) : g = \sum_{i=1}^{N} \alpha_i s_i, \; s_i \in S, \; \alpha_i \in \mathbb{Z}, \; N < \omega \right\}.

This defines a seminorm because \( S \) generates \( G_n \) and because the use of the infimum in the definition guarantees that the triangle inequality will be satisfied. Therefore, the neighborhoods of 0 defined by this seminorm will generate a (possibly non-Hausdorff) group topology on \( G_n \). Again, we do not require that a seminorm (or a norm) satisfy \( \|mg\| = |m| \cdot \|g\| \) because this property is not needed in order to generate a group topology.

Now in Lemma 3.4 we use the notation and definition just introduced.

**Lemma 3.4.** (a) \( G_n \) is a torsionfree, countable abelian group;
(b) the seminorm on \( G_n \) is a norm (resulting in a Hausdorff metric);
(c) \( G_n \in \text{SSGP}(n) \); and
(d) \( G_n \notin \text{SSGP}(n-1) \).

**Proof.** (a) is clear.

(b) To show that \( \| \cdot \| \) is a norm on \( G_n \), we need to show that for \( 0 \neq g \in G_n \) we have \( \|g\| > 0 \). Let \( g = (h, g') \). If \( g' \neq 0 \) then an expansion of \( (h, g') \) by elements of \( S \) must include elements as in (2). For those elements, we have \( \nu((f_r, g')) = \|g'\| \) so from the triangle inequality in \( G_{n-1} \) we can conclude that \( \|(h, g')\|_{G_n} \geq \|g'\|_{G_{n-1}} \). On the other hand, if \( g' = 0 \) then there is an expression for \((h, 0)\) in terms of elements of \( S \) of type \((p \cdot e_{m, t}, 0)\) such that \( \sum_{i=1}^{N} |\alpha_i| \nu(s_i) = \|h\| = \lambda(\text{supp}(h)) \) and this value is minimal. If, instead, the expansion includes elements of type \( s = (f_r, g_{m, t}) \) then there is a minimal \( \nu \)-value such a term can have. This is because there is a minimal size, \( \frac{1}{|r|} \), for an interval on which \( h \) is constant. An expansion of \((h, 0)\) by elements of \( S \) that includes an element \( s = (f_r, g_{m, t}) \) such that \( r(m, t) > M \) would also have to include \( 2^r \) elements of \( S \) of the form \((e_{r, i}, 0)\), each with coefficient \(-1\). The contribution of these terms to the sum \( \sum_{i=1}^{N} |\alpha_i| \nu(s_i) \) is greater than or equal to \( \frac{1}{2} \).

Such an expansion could have no effect on the infimum. So if \( \|(h, 0)\| \neq \|h\|_{H} \) then \( \|(h, 0)\| \geq \min(\{\|g_{m, t}\|_{G_n} : r(m, t) < M\}) \). We conclude that \( \|(h, g')\| \) is bounded away from 0 except when \((h, g') = (0, 0)\).

(c) We show next that \( G_n \in \text{SSGP}(n) \). We use the fact that \( H \) has SSGP in its subgroup topology. (This is clear because the provisional norm was defined for each element of the form \((h, 0)\), assigning to each the norm \( \|h\| \) inherited from \( \mathbb{Z} \).) It follows that \( \|(h, 0)\| \leq \|h\| \) for each \( h \in H \), so any \( \varepsilon \)-neighborhood of \((0, 0)\) contains a family \( \mathcal{H} \) of subgroups such that \( \langle \mathcal{H} \rangle = H \). We will show that the quotient topology for \( G_n/H \) coincides with the original topology for \( G_{n-1} \) (which also implies that \( \overline{\mathcal{H}}^{G_n} = H \)). We use the fact that the quotient map is open, (**23(5.26)**). As we just pointed out, for each \( g' \in G_{n-1} \) there is an \( h \in H \) such that \( \|(h, g')\| = \|g'\|_{G_{n-1}} \). We conclude that \( g' \) is in the \( \varepsilon \)-neighborhood of \( 0 \in G_{n-1} \) if and only if there is \( h \in H \) such that \((h, g')\) is in the \( \varepsilon \)-neighborhood of \((0, 0) \in G_{n} \). In other words, the neighborhoods of 0 in \( G_{n-1} \) coincide with the projections onto \( G_n/H \) of the neighborhoods of \((0, 0)\) in \( G_n \). Thus the topologies of \( G_n/H \) and \( G_{n-1} \) coincide. (Note, however, that the subgroup topology on \( G_{n-1} \) does not coincide with its original topology.) Since by assumption \( G_{n-1} \in \text{SSGP}(n-1) \), it follows from the definition that \( G_n \in \text{SSGP}(n) \).

(d) It remains to show \( G_n \notin \text{SSGP}(n-1) \). Suppose the contrary. Then every \( \varepsilon \)-neighborhood \( U_\varepsilon \) of \((0, 0) \in G_n \) contains a family \( \mathcal{K}_t \) of subgroups such that \( G_n/\bigcup \mathcal{K}_t \in \text{SSGP}(n-2) \). Let \( G \in \mathcal{K}_t \) and \((h, g') \in G \) with \( g' \neq 0 \). We
claim then \( \epsilon \geq \frac{1}{4} \). For \( N < \omega \) we must have \( \|(Nh, Ng')\| < \epsilon \). This means that each \((Nh, Ng')\) has an expansion

\[
(Nh, Ng') = \sum_{i=1}^{M} \alpha_i^{(N)}(h_i, 0) + \sum_{j=1}^{L} \beta_j^{(N)}(h'_j, g'_j) \quad \text{such that}
\]

\[
\sum_{i=1}^{M} \eta(\alpha_i^{(N)}) \nu(h_i, 0)) + \sum_{j=1}^{L} |\beta_j^{(N)}| \nu(h'_j, g'_j)) < \epsilon
\]

where each \((h_i, 0), (h'_j, g'_j) \in S\) and where \( \eta(\alpha_i^{(N)}) := 1 \) when \( \alpha_i^{(N)} \neq 0 \) and \( \eta(\alpha_i^{(N)}) := 0 \) when \( \alpha_i^{(N)} = 0 \).

We consider two cases.

Case 1. In the expansion above, the coefficients \( \beta_i^{(N)} \) are of the form \( N\beta_i^{(1)} \) for all \( N < \omega \). Then clearly for sufficiently large \( N \) we have \( \|(Nh, Ng')\| > \frac{1}{4} \) (or, for that matter, \( \|(Nh, Ng')\| > \epsilon \)).

Case 2. There is some \( N \) where the expansion for \((Nh, Ng')\) is such that \( \beta_i^{(N)} \neq N\beta_i^{(1)} \) for some value of \( i \). For the \( H \)-component of the given expansion of \((Nh, Ng')\), we have

\[
Nh = \sum_{i=1}^{M} \alpha_i^{(N)} h_i + \sum_{j=1}^{L} \beta_j^{(N)} h'_j
\]

where \( M = \max(M_1, M_N) \) and \( L = \max(L_1, L_N) \). Multiplying the specified expansion of \((h, g')\) by the number \( N \), we also have

\[
Nh = \sum_{i=1}^{M} N\alpha_i^{(1)} h_i + \sum_{j=1}^{L} N\beta_j^{(1)} h'_j
\]

Equating the two expansions and re-arranging, we can write

\[
\sum_{j=1}^{L} (\beta_j^{(N)} - N\beta_j^{(1)}) h'_j = \sum_{i=1}^{M} (N\alpha_i^{(1)} - \alpha_i^{(N)}) h_i
\]

where, for some index \( j \), we have \( (\beta_j^{(N)} - N\beta_j^{(1)}) h'_j \neq 0 \). Since the \( h'_j \) are linearly independent, each \( h'_j \) that has a nonzero coefficient in the expression above must be balanced by terms on the right. This implies that \( \sum_{i=1}^{M} \eta(\alpha_i^{(1)}) \geq \frac{1}{2} \), which in turn means that either \( \sum_{i=1}^{M} \eta(\alpha_i^{(1)}) \geq \frac{1}{4} \) or \( \sum_{i=1}^{M} \eta(\alpha_i^{(N)}) \geq \frac{1}{4} \). We conclude that \( \|(n, g')\| \geq \frac{1}{4} \) or \( \|(Nh, Ng')\| \geq \frac{1}{4} \), as claimed. Returning to the family \( \mathcal{K}_e \) of subgroups, we see that if \( \epsilon < \frac{1}{4} \), then \( \mathcal{K}_e \subseteq \mathcal{P}(H) \) so that \( \langle \cup \mathcal{K}_e \rangle \) is a closed subgroup of \( H \). In such cases we have \( G_n/\langle \cup \mathcal{K}_e \rangle \notin \text{SSGP}(n-2) \) because, by Theorem \ref{212} part (b), \( G_n/\langle \cup \mathcal{K}_e \rangle \in \text{SSGP}(n-2) \) would imply that \( \langle G_n/\langle \cup \mathcal{K}_e \rangle \rangle/\langle H/\langle \cup \mathcal{K}_e \rangle \rangle \in \text{SSGP}(n-2) \) or, equivalently, that \( G_n/H \simeq G_{n-1} \in \text{SSGP}(n-2) \), contrary to assumption.

We emphasize the essential content of Lemma \ref{84}.

**Theorem 3.5.** For \( 1 \leq n < \omega \) there is an abelian topological group \( G \) such that \( G \in \text{SSGP}(n) \) and \( G \notin \text{SSGP}(n-1) \).

While Theorem \ref{212} furnishes a vast supply of well-behaved abelian groups which admit no \( \text{SSGP}(n) \) topology, we have found that an \( \text{SSGP} \) topology can be constructed for many of the standard building blocks of infinite abelian groups.

We give now verify Theorem \ref{211}c, that is, we give a construction of an \( \text{SSGP} \) topology for groups of the form \( G := \oplus p_i \mathbb{Z}_{p_i} \) (with \( p_i \) a sequence of distinct primes); this illustrates the method used throughout the second-listed co-author’s thesis \cite{18}.

Using additive notation, write \( 0 = 0_G \) and let \( \{e_i : i = 1, 2, \ldots\} \) be the canonical basis for \( G \), so that \( p_1 e_1 = p_2 e_2 = \ldots = p_i e_i = \ldots = 0 \). We define a provisional norm \( \nu \), much as in the description preceding Lemma \ref{84}. This will generate a norm \( \|\cdot\| \) via Definition \ref{33} in such a way that in the generated topology every neighborhood of 0 contains sufficiently many subgroups to generate a dense subgroup of \( G \). Suppose we can show that \( G \) is Hausdorff and that each
$U \in \mathcal{N}(0)$ contains a family of subgroups $\mathcal{H}$ such that $G/\mathcal{H}$ is torsion of bounded order, where $H := (\cup \mathcal{H})$. Then if also $G/\mathcal{H}$ has no proper open subgroup, we have from Theorem 2.22 that $G/\mathcal{H} \in \text{SSGP}$, so that $G \in \text{SSGP}(2)$. Our plan is to choose a norm so that $G/H$, and thus $G/\mathcal{H}$ is actually finite. Then if $G$ contains no proper open subgroup, it is necessarily the case that $\mathcal{H} = G$. Thus we attempt to define a norm $\| \cdot \|$ so that

1. Every neighborhood of 0 contains a set of subgroups of $G$ whose union generates a subgroup $H$ such that $G/H$ is finite.

2. $G$ has no proper open subgroups, or equivalently, every neighborhood of 0 generates $G$.

3. $G$ is Hausdorff.

First, define $\nu(me_n) = \frac{1}{n}$ for every $m < \omega$ such that $m \neq 0 \mod p_n$. The neighborhood of 0 defined by $\|g\| < \frac{1}{n}$ will then contain subgroups which generate $H := p_1p_2...p_{n-1}G$. Then $G/H$ is finite, as desired.

To satisfy (2) define $e_n := \sum_{i=1}^{n} e_i$ for $n < \omega$, and define $\nu(e_n) := \frac{1}{n}$ for each $n < \omega$. All that then remains (the most difficult piece) is to show that $G$ with this topology is Hausdorff. We will then have the following result.

**Theorem 3.6.** Let $G = \bigoplus_{i<\omega} \mathbb{Z}_{p_i}$, where $p_1 < p_2 < p_3 < ...$ are primes. Let $S = \{me_n : n < \omega, 0 < m < p_n\} \cup \{e_n : n \in \mathbb{N}\}$, with $e_n, e_n$ defined as above. Let $\nu(me_n) = \frac{1}{n}$ for $0 < m < p_n$, and let $\nu(e_n) = \frac{1}{n}$. Then the norm defined by

$$\|g\| = \inf \left\{ \sum_{i=1}^{n} |\alpha_i|\nu(s_i) : g = \alpha_1s_1 + ... + \alpha_ns_n, s_i \in S, \alpha_i \in \mathbb{Z}, n < \omega \right\}$$

generates an SSGP topology on $G$.

**Proof.** As noted above, our construction for the norm $\| \cdot \|$ guarantees that every $\epsilon$-neighborhood $U$ of 0 generates $G$ and also contains subgroups whose union generates an $H$ such that $G/H$ is finite. Then, as also noted, if $G$ is Hausdorff we are done.

Suppose $0 \neq g \in G$ and $n$ is the highest nonzero coordinate index for $g$. We need to show that $\|g\|$ is bounded away from 0. In fact, we show that $\|g\| \geq \frac{1}{n}$.

For convenience we extend the domain of $\nu$ to all formal finite sums of elements from $S$ with coefficients from $\mathbb{Z}$:

$$\varphi = \sum_{i=M}^{N} (a_ie_i + b_i\hat{e}_i), \quad \text{let } \nu(\varphi) = \sum_{i=M}^{N} (\eta_i + |b_i|)\frac{1}{i}$$

where each $\eta_i$ is either 0 or 1, according as to whether or not $a_i \equiv 0 \mod p_i$.

In addition, we will assume

1. $g = \text{val}(\varphi)$, which means that the formal sum $\varphi$ evaluates to $g \in G$;

2. each $e_i$ and each $\hat{e}_i$ appears at most once in any formal sum; and

3. $0 \leq a_i < p_i$ for each $i$,

item (2) being justified by the fact that we are ultimately interested in the norm, which minimizes $\nu(\varphi)$.

Let $\mathcal{F}(M, N)$ be the set of such formal sums where $M$ is the smallest coordinate index for a nonzero coefficient $a_M$ or $b_M$ and where $N$ is the largest such index. (Here for $b_i$, “non-zero” indicates that $b_i$ is not a multiple of $p_1p_2...p_i$.)
We want to show that $\nu(\varphi) \geq \frac{1}{n}$ where $g = \text{val}(\varphi)$, where $\varphi = \sum_{i=M}^{N} (a_i e_i + b_i \bar{e}_i)$ and where either $a_M$ or $b_M$ is nonzero and either $a_N$ or $b_N$ is nonzero. In other words, $\varphi \in F(M,N)$. This is clear if $M \leq n$.

Suppose first that $N = M = n + 1$. Then, since the $n + 1$ component of $g$ is 0 we have that $a_{n+1} + b_{n+1} \equiv 0 \mod p_{n+1}$. Both coefficients are 0 only if $g = 0$, so either both are nonzero or else $a_{n+1} = 0$ and $b_{n+1} = mp_{n+1}$ for some $m \neq 0$. In the first case we have $\nu(\varphi) \geq \frac{2}{n+1} > \frac{1}{n}$ and in the second case we have $\nu(\varphi) \geq \frac{mp_{n+1}}{n+1} > 1 \geq \frac{1}{n}$.

Suppose instead that $M = N > n + 1$. In this case, we know that the $(N-1)$ component of $g$ is 0. In order for that to be true when $g \neq 0$ can be written as $\varphi = a_N e_N + b_N \bar{e}_N$, it must be the case that $b_N = mp_{N-1}$ for some $m \neq 0$. But then we have $\nu(\varphi) \geq \frac{p_{N-1}}{N} \geq 1 \geq \frac{1}{n}$.

Finally, we fix $M$ and use induction on $N$. Assume that we have already shown that $\nu(\varphi) \geq \frac{1}{n}$ when $\varphi \in F(M,Q)$ for $M \leq Q \leq N - 1$, and suppose that $\varphi \in F(M,N)$. We treat three cases separately.

(a) Case 1. $|b_{N-1} + b_N| \geq p_{N-1}$. Then

$$\nu(\varphi) \geq \frac{|b_{N-1}|}{N-1} + \frac{|b_N|}{N} \geq \frac{p_{N-1}}{N} \geq 1 \geq \frac{1}{n}.$$ 

(b) Case 2. $b_{N-1} + b_N = 0$. Then we can delete the terms

$$b_{N-1} \bar{e}_{N-1} + b_N \bar{e}_N + a_N e_N$$

without affecting the value of $\varphi$, and our induction assumption applies.

(c) Case 3. $|b_{N-1} + b_N| < p_{N-1}$ and $b_{N-1} + b_N \neq 0$. Then if we let $\varphi'$ be the formal sum obtained from $\varphi$ by deleting $a_N e_N + b_N \bar{e}_N$ and replacing $b_{N-1} \bar{e}_{N-1}$ with $(b_{N-1} + b_N) \bar{e}_{N-1}$, we have $\text{val}(\varphi') = \text{val}(\varphi) = g$, and

$$\nu(\varphi') - \nu(\varphi) = \frac{|b_{N-1} + b_N|}{N-1} \left( \frac{|b_{N-1}|}{N-1} + \frac{|b_N|}{N} + 1 \right) \leq \frac{|b_N|}{N(N-1)} - \frac{1}{N}.$$ 

We see that this difference is negative or zero as long as $|b_N| \leq N - 1$. Then, since $\varphi' \in F(M, N - 1)$, our induction assumption applies. If, on the contrary, $|b_N| \geq N$, we already have $\nu(\varphi) \geq 1 \geq \frac{1}{n}$, and we can conclude that $G$ is Hausdorff, as desired. \hfill \Box

\section{Conclusion Remarks}

\textbf{Discussion 4.1.} With no pretense to completeness, we here discuss briefly some of the literature relating to the development of the class of m.a.p. groups.

(a) As we indicated earlier, in effect the class m.a.p. was introduced in 1930 by von Neumann \cite{29}, who then together with Wigner \cite{27} proved that (even in its discrete topology) the matrix group $SL(2, \mathbb{C})$ is an m.a.p. group.

(b) In the period 1940–1952, several workers showed that certain real topological linear spaces are m.a.p. groups; several examples, with detailed verification, are given by Hewitt and Ross \cite{23, 32}.

(c) We have quoted above at length from the 1980 paper of Prodanov \cite{31}, which showed by \textquotedblleft elementary means\textquotedblright{} that the group $\bigoplus_{\omega} \mathbb{Z}$ admits an m.a.p. topology.

(d) Ajtai, Havas and Komlós \cite{1} proved that each group $G$ of the form $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$, or $\bigoplus_{p} \mathbb{Z}(p_{n})$ (with all $p_n \in \mathbb{P}$ either identical or distinct) admits a
m.a.p. group topology.

(c) Protasov [32] and Remus [34] asked whether every infinite abelian group admits an m.a.p. group topology; the question was deftly settled in the negative by Remus [35] with the straightforward observation that for distinct $p, q \in \mathbb{P}$, every group topology on the infinite group $G := \mathbb{Z}(p) \times (\mathbb{Z}(q))^\kappa$ (with $\kappa \geq \omega$) has the property that the homomorphism $x \to qx$ maps $G$ continuously onto the compact group $\mathbb{Z}(p)$. (See [3](3.J), [5](4.6) for additional discussion.)

(f) Remus [34] showed that every free abelian group, also every infinite divisible abelian group, admits an m.a.p. topology.

(g) In view of the cited examples $\mathbb{Z}(p) \times (\mathbb{Z}(q))^\kappa$ of Remus [35], it was natural for Comfort [8](3.J.1) to raise the question: Does every abelian group which is not torsion of bounded order admit an m.a.p. topology? What about the countable case?

(h) Motivated by question (g), Gabriyelyan [13], [14] showed that every infinite finitely generated abelian group admits an m.a.p. topology, indeed the witnessing topology may be chosen complete in the sense that every Cauchy net converges. Gabriyelyan [15] showed further that an abelian torsion group of bounded order admits an m.a.p. topology if and only if each of its leading Ulm-Kaplansky invariants is infinite. (The reader unfamiliar with the Ulm-Kaplansky invariants might consult [12]§(77); those cardinals also play a significant role in [4] in a setting closely related to the present paper.)

(i) Complete and definitive characterizations of those (not necessarily torsion) abelian groups $G$ which admit an m.a.p. topology were given recently by Dikranjan and Shakhmatov [8]. Among them are these: (1) $G$ is connected in its Zariski topology; (2) $m \in \mathbb{Z} \Rightarrow mG = \{0\}$ or $|mG| \geq \omega$; (3) the group $\text{fin}(G)$ is trivial, i.e., $\text{fin}(G) = \{0\}$. (The group $\text{fin}(G)$, whose study was initiated in [7](4.4) and continued in [4](§2), may be defined by the relation $\text{fin}(G) = \langle \bigcup\{mG : m \in \mathbb{Z}, |mG| < \omega\} \rangle$.

Detailed subsequent analysis of the theorems and techniques of [8] have allowed those authors to answer the following two questions in the negative; these questions were posed in [18] and in a privately circulated pre-publication copy of the present manuscript.

(1) Let $G$ be a group with a normal subgroup $K$ for which $K$ and $G/K$ admit topologies $U$ and $V$ respectively such that $(K, U) \in \text{m.a.p.}$ and $(G/K, V) \in \text{m.a.p.}$ Is there then necessarily a group topology $T$ on $G$ such that $(K$ is closed in $(G, T)$ and $(K, U) = (K, T|K)$ and $(G/K, U) = (G/K, T_q)$ with $T_q$ the quotient topology?

(2) Let $G$ be a group with a normal subgroup $K$ such that both $K$ and $G/K$ admit m.a.p. topologies. Must $G$ admit a m.a.p. topology?

**Remark 4.2.** In the dissertation [18], the second-listed co-author found it convenient to introduce the class of weak SSGP groups (briefly, the WSSGP groups), that is, those topological groups $G = (G, T)$ which contain no proper open subgroup and have the property that for every $U \in \mathcal{N}(1_G)$ there is a family of subgroups $\mathcal{H} \subseteq \mathcal{P}(U)$ such that $H = \bigcup \mathcal{H}$ is normal in $G$ and $G/H$ is torsion of bounded order. Subsequent analysis (as in Theorem 2.22 above) along with the definitions of the classes SSGP($n$) has revealed the class-theoretic inclusions $\text{SSGP}(1) \subseteq \text{WSSGP} \subseteq \text{SSGP}(2)$. A consequence of Theorem 2.22 is that the Markov-Graev-Remus examples (as in Remark 2.27) are not just WSSGP but are, in fact, SSGP. And Prodonov’s group, discussed above, belongs to the class of SSGP(2) groups (implying the m.a.p. property). From these facts we conclude that the class of WSSGP groups contributes little additional useful information to the present inquiry, and we have chosen to suppress its systematic discussion in this paper.
In Theorems 2.8 and 2.12 we have identified several classes of groups which do not admit an SSGP topology. That suggests the following natural question.

**Question 4.3.** What are the (abelian) groups which admit an SSGP topology?

Our work also leaves open this intriguing question:

**Question 4.4.** Does every abelian group which for some \( n > 1 \) admits an SSGP\((n)\) topology also admit an SSGP topology?

There is another important and much-studied class of m.a.p. groups: those whose every continuous action on a compact space has a fixed point, the so-called f.p.c. groups. (See, for example, [17], [30] and [10].) The study of f.p.c. groups, also known as extremely amenable groups, led to the formulation of a difficult long-standing open question in abelian topological group theory:

**Question 4.5.** Do the f.p.c. abelian groups constitute a proper subclass of the m.a.p. abelian groups?

This question was raised by E. Glasner in 1998 [17]. Even the characterization of abelian m.a.p. groups and abelian f.p.c. groups by different big set conditions (with the one characterizing f.p.c. groups being the stronger) did not settle the question. (See [29] and [9] for details.) Unfortunately, the SSGP property has so far not shed light on this question, either. It is known [17] that there are f.p.c. topologies for \( \mathbb{Z} \), so f.p.c. groups need not have SSGP.

**References**

[1] M. Ajtai, I. Havas, and J. Komlós, *Every group admits a bad topology*, Studies in Pure Mathematics. To the Memory of P. Turán (Paul Erdős, ed.), Birkhäuser Verlag, Basel and Akadémiai Kiado, Budapest, 1983, pp. 21–34.

[2] W. W. Comfort, *Topological groups*, In: Handbook of Set-theoretic Topology (Kenneth Kunen and Jerry E. Vaughan, eds.), pp. 1143–1263. North-Holland, Amsterdam, 1984.

[3] W. W. Comfort, *Problems on Topological Groups and Other Homogeneous Spaces*, In: Open Problems in Topology, (Jan van Mill and George M. Reed, eds.) pp. 314–347. North-Holland, 1990.

[4] W. W. Comfort and Dikran Dikranjan, *The density nucleus of a topological group*, Topology Proc. 44 (2014), 325–356.

[5] W. W. Comfort and Dieter Remus, *Long chains of topological group topologies—a continuation*, Topology and Its Applications 75 (1997), 51–79. Correction: 96 (1999), 277–278.

[6] S. Dierolf and S. Warken, *Some examples in connection with Pontryagin’s duality theorem*, Arch. Math. 30 (1978), 599-605.

[7] Dikran Dikranjan and Dmitri Shakhmatov, *The Markov-Zariski topology of an abelian group*, J. Algebra 324 (2010), 1125–1158.

[8] Dikran Dikranjan and Dmitri Shakhmatov, *Final solution of Protasov-Comfort’s problem on minimally almost periodic group topologies*. Manuscript in preparation, 2014.

[9] R. Ellis and H. B. Keynes, *Bohr compactifications and a result of Følner*, Israel J. Math. 12 (1972), 314–330.
[10] Ilijas Farah and Slawomir Solecki, *Extreme amenability of $L_0$, a Ramsey theorem, and Lévy groups*, www.math.uiuc.edu/~ssolecki (unpublished), March 20, 2007.

[11] László Fuchs, *Infinite Abelian Groups, vol. I*, Academic Press, New York–San Francisco–London, 1970.

[12] László Fuchs, *Infinite Abelian Groups, vol. II*, Academic Press, New York and London, 1973.

[13] S. S. Gabriyelyan, *Finitely generated subgroups as von Neumann radicals of an Abelian group*, Mat. Stud. 38 (2012), 124–138.

[14] S. S. Gabriyelyan, *Minimally almost periodic group topologies on countably infinite abelian groups*, Proc. Amer. Math. Soc. To appear.

[15] S. S. Gabriyelyan, *Bounded subgroups as a von Neumann radical of an Abelian group*, Topology and Its Applications 178 (2014), 185–199.

[16] I. M. Gel’fand and D. Raĭkov, *Irreducible unitary representations of locally bicom pact groups*, Matem. Sbornik N.S. 13 (1943), 301–316. In Russian.

[17] E. Glasner, *On minimal actions of Polish groups*, Topology and Its Applications, 85 (1998), 119-125.

[18] F. R. Gould, *On certain classes of minimally almost periodic groups*, Doctoral Dissertation, Wesleyan University (Connecticut, USA), 2009.

[19] F. R. Gould, *An SSGP topology for $Z^n$*, Topology Proceedings 44 (2014), pp. 389–392

[20] M. I. Graev, *Free topological groups*, In: Topology and Topological Algebra, Translations Series 1, vol. 8, American Mathematical Society, 1962, pp. 305–364. Russian original in: Известия Акад. Наук СССР Сер. Мат. 12 (1948), 279-323.

[21] S. Hartman and Jan Mycielski, *On the embedding of topological groups into connected topological groups*, Colloq. Math. 5 (1958), 167–169.

[22] G. Hesse, *Zur Topologisierbarkeit von Gruppen*, Ph.D. thesis, Universität Hannover, Hannover (Germany), 1979.

[23] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis, volume I*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, volume 115, Springer Verlag, Berlin-Göttingen-Heidelberg, 1963.

[24] A. A. Markov, *On free topological groups*, Doklady Akad. Nauk SSSR 31 (1941), 299–301.

[25] A. A. Markov, *On free topological groups*, In: Topology and Topological Algebra, Translations Series 1, vol. 8, American Mathematical Society, 1962, pp. 195–272. Russian original in: Известия Акад. Наук СССР Сер. Мат. 9 (1945), 3–64.

[26] J. von Neumann, *Almost periodic functions in a group I*, Trans. Amer. Math. Soc. 36 (1934), 445–492.

[27] J. von Neumann and E. P. Wigner, *Minimally almost periodic groups*, Annals of Math. (Series 2) 41 (1940), 746–750.
[28] A. Yu. Ol’shanskiĭ, *A remark on a countable non-topologizable group*, Vestnik Mosk. Gos. Univ, Ser. I, Mat. Mekh. 3 (1980), 103. In Russian.

[29] V. G. Pestov, *Some universal constructions in abstract topological dynamics*, In: Topological Dynamics and its Applications. A Volume in Honor of Robert Ellis, Contemp. Math. 215 (1998), pp. 83–99.

[30] V. Pestov, *Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups.*, Israel J. Math, 127 (2002), 317-357.

[31] I. Prodanov, *Elementary example of a group without characaters*, In: Mathematics and Mathematical Education (Sophia, Bulgaria), Bulgarian Acad. Science, 1980, pp. 79–81. Proc. 9th Spring Conference (April, 1980), Bulgarian Math. Soc.

[32] I. V. Protasov, *Review of [1]*, Zentralblatt für Matematik 535 (1984), 93.

[33] Dieter Remus, *Die Anzahl von $T_2$-präkompakten Gruppentopologien auf unendlichen abelschen Gruppen*, Rev. Roumaine Math. Pures Appl. 31 (1986), 803–806.

[34] Dieter Remus, *Topological groups without non-trivial characters*, In: General Topology and Its Relations to Modern Analysis and Algebra VI (Z. Frolík, ed.), pp. 477–484. Proc. Sixth 1986 Prague Topological Symposium, Heldermann Verlag, Berlin, 1988.

[35] Dieter Remus, *Letter to W. W. Comfort*, September, 1989.

[36] Barbara V. Smith-Thomas, *Free topological groups*, Topology Appl. 4 (1974), 51–72.

[37] Saharon Shelah, *On a problem of Kurosh, Jónsson groups and applications*, In: Word Problems II (S. I. Adian, W. W. Boone, and G. Higman, eds.), pp. 373–394. North-Holland Publishing Company, Amsterdam, 1980.

[38] André Weil, *Sur les Espaces à Structure Uniforme et sur la Topologie Générale*, Publ. Math. Univ. Strasbourg, vol. 551, Hermann & Cie, Paris, 1938.

[39] E. G. Zelenyuk and I. V. Protasov, *Topologies on Abelian groups*, Math. USSR Izvestiya 37, No. 2 (1991), 445-460.