Quantum Symmetric Pairs and Their Zonal Spherical Functions

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Abstract
We study the space of biinvariants and zonal spherical functions associated to quantum symmetric pairs in the maximally split case. Under the obvious restriction map, the space of biinvariants is proved isomorphic to the Weyl group invariants of the character group ring associated to the restricted roots. As a consequence, there is either a unique set, or an (almost) unique two-parameter set of Weyl group invariant quantum zonal spherical functions associated to an irreducible symmetric pair. Included is a complete and explicit list of the generators and relations for the left coideal subalgebras of the quantized enveloping algebra used to form quantum symmetric pairs.

INTRODUCTION
The representation theory of semisimple Lie algebras and Lie groups has been closely intertwined with the theory of symmetric spaces since E. Cartan's pioneering work in the 1920's. A beautiful classical result shows that the zonal spherical functions of these spaces can be identified with a family of orthogonal polynomials. With the introduction of quantum groups in the 1980's, it was natural to look for and study quantum versions of symmetric spaces. This search became especially compelling as q orthogonal polynomials, which are obvious candidates for quantum zonal spherical functions, appeared in the literature (see for example [Ma] and [K2]). However, the theory of quantum symmetric spaces was initially slow to develop because it was not obvious how to form them.

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Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\theta$ be an involution of $\mathfrak{g}$. A classical (infinitesimal) symmetric pair consists of the Lie algebra $\mathfrak{g}$ and the fixed Lie subalgebra $\mathfrak{g}^\theta$. In his fundamental paper [K1], Koornwinder was the first to use an infinitesimal approach to construct quantum symmetric spaces. In particular, he found quantum analogs of a symmetric pair of Lie algebras for $\mathfrak{g} = \mathfrak{sl}_2$ using twisted primitive elements. As this method was generalized to other examples, it became clear that coideal analogs of $U(\mathfrak{g}^\theta)$ inside the quantized enveloping algebra $U_q(\mathfrak{g})$ had the potential to produce a “good” theory of quantum symmetric spaces (see for example [N],[NS], and [DN].) In his 1996 survey paper [Di], Dijkhuizen describes the philosophy of this approach in three major steps which we briefly rephrase here.

1. Find coideals inside the quantized enveloping algebra which can be used to form quantum symmetric pairs. Show that the finite-dimensional spherical modules can be parametrized using the dominant integral restricted roots.

2. Determine the image of the space of biivariant functions associated to a quantum symmetric pair inside the character ring of the root system of $\mathfrak{g}$.

3. Realize zonal spherical functions as $q$-orthogonal polynomials by computing the radial components of appropriate central elements of the quantized enveloping algebra.

This paper is one of a series by the author intended to answer these questions when the triangular decomposition of $\mathfrak{g}$ has been chosen so that the Cartan subalgebra is maximally split with respect to $\theta$. Quantum analogs of the enveloping algebra $U(\mathfrak{g}^\theta)$ are constructed in [L2] and [L4] by describing their generators. These subalgebras of $U_q(\mathfrak{g})$ are further characterized as the unique maximal left coideal subalgebras which specialize to $U(\mathfrak{g}^\theta)$ as $q$ goes to 1. The finite-dimensional spherical modules associated to quantum symmetric pairs are classified in [L3], thus completing problem (1).

The main result of this paper is a comprehensive answer to problem (2). Let $\mathcal{B}_q$ denote the set of coideal subalgebras associated to $\mathfrak{g}, \mathfrak{g}^\theta$. For each pair of subalgebras $B$ and $B'$ in $\mathcal{B}_q$, the space of biiinvariants $B'B(\mathcal{H}_B)$ consists of the left $B'$ and right $B$ invariants inside the quantized function algebra $R_q[G]$ corresponding to $U_q(\mathfrak{g})$. The vector space $B'B(\mathcal{H}_B)$ is an algebra and can
be written as a direct sum of eigenspaces with respect to the action of the center of $U_q(g)$. The eigenvectors with respect to this action are called zonal spherical functions.

Elements of the quantized function algebra can be thought of as functions on $U_q(g)$. Recall that $U_q(g)$ contains a multiplicative subgroup $T$ corresponding to the root lattice of the root system of $g$. Let $\Sigma$ denote the restricted root system associated to $g, \theta$ and let $C[2\Sigma]$ be the character group ring of the weight lattice corresponding to $2\Sigma$. We prove that restriction to $T$ of the space of biinvariants (considered as functions on $U_q(g)$) induces an injection $\Upsilon$ from $B'$ into $C[2\Sigma]$.

Let $W$ denote the restricted Weyl group associated to $\Sigma$. Let $H$ denote the group of Hopf algebra automorphisms of $U_q(g)$ which fix elements of $T$. Note that $H$ acts on $B_\theta$ by sending an algebra to its image under the automorphism. Given $B \in B_\theta$ and any $H$ orbit $O$ of $B_\theta$ we find $B' \in O$ such that $\Upsilon(B'H_B)$ is $W$ invariant. We then show, in this case, that $\Upsilon(B'H_B)$ is equal to $C[2\Sigma]^W$.

The action of $H$ on $B_\theta$ can be extended componentwise to an action of the group $H \times H$ on $B_\theta \times B_\theta$. We show that the images under $\Upsilon$ of two different spaces of biinvariants associated to pairs in the same $H \times H$ orbit are related by an automorphism of $C[2\Sigma]$. Thus the image of any space of biinvariants under $\Upsilon$ is just a translation of $C[2\Sigma]^W$ via an automorphism of $C[2\Sigma]$. Furthermore, this automorphism restricts to an eigenvalue preserving function on the corresponding sets of zonal spherical functions.

A zonal spherical family associated to a $H \times H$ orbit is the image under $\Upsilon$ of a specially chosen basis of zonal spherical functions in $B'H_B$ for some pair $(B, B')$ in this orbit. We show that there is a virtually unique $W$ invariant zonal spherical family associated to each $H \times H$ orbit of $B_\theta \times B_\theta$. The obstruction to uniqueness, a small finite subgroup of $\text{Aut}C[2\Sigma]^W$, is just the trivial group for most choices of $g$ and $\theta$.

The pair $g, \theta$, or more precisely, $g, \theta$, is called irreducible if $g$ cannot be written as the direct sum of two semisimple Lie subalgebras which both admit $\theta$ as an involution. In [A] (see also [He, Chapter X, Section F]), Araki gave a complete list of all the irreducible pairs $g, \theta$. In the final section of the paper, using this list and the construction ([L4, Section 7]) of the algebras in $B_\theta$, we explicitly describe the generators and relations for every coideal subalgebra in $B_\theta$ associated to each irreducible pair $g, \theta$. It follows from this classification that if $g, \theta$ is irreducible, either $B_\theta \times B_\theta$ consists of a single $H \times H$ orbit or a
two-parameter set of orbits. In the first case, there is a unique $W$ invariant zonal spherical family associated to $\mathcal{B}_\theta$. For the latter case, there is either a unique two-parameter set of $W$ invariant zonal spherical families associated to $\mathcal{B}_\theta$ or exactly two two-parameter sets.

In a future paper, we will address problem (3) and realize these $W$ invariant zonal spherical families as sets of $q$ hypergeometric polynomials. Apparently, when $\Sigma$ is reduced and $\mathcal{B}_\theta$ is a single $H \times H$ orbit, the zonal spherical functions are Macdonald polynomials. Here, one of the parameters is a power of the other. This would generalize the results in [N] as well as those announced in [NS] concerning quantum symmetric pairs which can be constructed using solutions to reflection equations. Other $q$ hypergeometric functions such as Askey-Wilson polynomials with two indeterminates arise in the remaining cases. This should remind the reader of the zonal spherical functions computed on the quantum 2-sphere in [K1] and on the family of quantum projective spaces studied in [DN].

The remainder of the paper is organized as follows. Section 1 sets notation and reviews basic facts about involutions, restricted root systems, and quantized enveloping algebras. In Section 2, we review the construction of the quantum analogs of $U(g^\theta)$ inside $U_q(g)$. Section 3 is a study of spherical modules and their spherical vectors. Fine information about the possible weights of weight vectors which show up as summands of spherical vectors is obtained. The next three sections of the paper are devoted to the space of biinvariants and their zonal spherical functions. In Section 4, the map $\Upsilon$ restricted to a space of biinvariants is shown to be injective with image contained in $C[2\Sigma]$. Section 5 establishes a criterion for choosing $B$ and $B'$ so that $B' \mathcal{H}_B$ is $W$ invariant. Section 6 is a study of zonal spherical families. We show how to relate them using elements of the group $H \times H$ and determine necessary and sufficient conditions for a zonal spherical family to be $W$ invariant. In Section 7, we give a complete and explicit list of generators and relations for the algebras in $\mathcal{B}_\theta$ associated to each irreducible pair $g, \theta$.

1 Background and Notation

Let $C$ denote the complex numbers, $R$ denote the real numbers, $Q$ denote the rational numbers, $Z$ denote the integers, and $N$ denote the nonnegative integers. Let $g = n^- \oplus h \oplus n^+$ be a semisimple Lie algebra of rank $n$ over
C with Cartan matrix \((a_{ij})\). Write \(\Delta\) for the set of roots of \(g\) and let \(\pi = \{\alpha_1, \ldots, \alpha_n\}\) be a fixed set of positive simple roots. Set \(Q(\pi) = \sum_{1 \leq i \leq n} \mathbb{Z}\alpha_i\) and \(Q^+(\pi) = \sum_{1 \leq i \leq n} N\alpha_i\). Let \(P^+(\pi)\) denote the set of dominant integral weights associated to the root system \(\Delta\) and \(P(\pi)\) denote the weight lattice. Write \(\langle \ , \ \rangle\) for the Cartan inner product. Let \(\leq\) denote the standard partial order on \(Q(\pi)\). In particular, given two elements \(\lambda\) and \(\beta\) in \(Q(\pi)\), we say that \(\lambda \leq \beta\) if and only if \(\lambda - \beta \in Q^+(\pi)\).

Let \(\theta\) be a maximally split Lie algebra involution of \(g\) with respect to the fixed Cartan subalgebra \(h\) and triangular decomposition of \(g\) in the sense of [L4, Section 7]. Write \(g^\theta\) for the corresponding fixed Lie subalgebra. The involution \(\theta\) induces an involution \(\Theta\) of the root system of \(g\) and thus an automorphism of \(h^*\). Let \(\pi_{\Theta}\) be the set of simple roots \(\{\alpha_i | \Theta(\alpha_i) = \alpha_i\}\) and write \(\Delta_{\Theta}\) for the corresponding root system generated by \(\pi_{\Theta}\). Let \(p\) be the permutation on \(\{1, \ldots, n\}\) corresponding to a diagram automorphism of \(\pi\) such that \(\Theta(\alpha_i) + \alpha_{p(i)} \in \sum_{\alpha_i \in \pi_{\Theta}} \mathbb{Z}\alpha_i\) for each \(\alpha_i \in \pi - \pi_{\Theta}\). (For more information about involutions, see for example [D, 1.13] and [L4, Section 7]).

Given \(\beta \in \mathfrak{h}^*\), set \(\tilde{\beta} = 1/2(\beta - \Theta(\beta))\). Let \(\Sigma\) denote the restricted root system associated to \(g, \theta\) ([Kn, Chapter VI, Section 4], [He, Chapter X, Section F], or [L4, Section 7]). Note that \(\Sigma\) can be identified with the set \(\{\tilde{\alpha} | \alpha \in \Delta - \Delta_{\Theta}\}\) using the Cartan inner product as the inner product for \(\Sigma\). Moreover, \(\{\tilde{\alpha} | \alpha \in \pi - \pi_{\Theta}\}\) is a set of positive simple roots for the root system \(\Sigma\). Set \(Q(\Sigma)\) equal to the root lattice \(\sum_i \mathbb{Z}\tilde{\alpha}_i\) of \(\Sigma\) and set \(P(\Sigma)\) equal to the weight lattice associated to the root system \(\Sigma\). Let \(P^+(\Sigma)\) denote the subset of \(P(\Sigma)\) consisting of dominant integral weights. Set \(P(2\Sigma) = \{2\lambda | \lambda \in P(\Sigma)\}\).

Let \(q\) be an indeterminate and set \(q_i = q^{(\alpha_i, \alpha_i)/2}\) for each \(1 \leq i \leq n\). Write \(U = U_q(g)\) for the quantized enveloping algebra generated by \(x_i, y_i, t_i^{\pm 1}, 1 \leq i \leq n\), over the algebraic closure \(\mathbb{C}\) of \(\mathbb{C}(q)\) (See [L4, Section 1, (1.4)-(1.10)] or [Jo, 3.2.9] for relations.) Let \(U^-\) be the subalgebra of \(U\) generated by \(y_i, 1 \leq i \leq n\) and let \(U^+\) be the subalgebra of \(U\) generated by \(x_i, 1 \leq i \leq n\). Given an integral domain \(D\) containing \(\mathbb{C}\), we set \(U_D = U \otimes_{\mathbb{C}} D\).

The algebra \(U\) is a Hopf algebra with comultiplication map \(\Delta\), antipode \(\sigma\), and counit \(\epsilon\). Let \(U_+\) denote the augmentation ideal of \(U\), which is the kernel of the counit map \(\epsilon\). Given a subalgebra \(A\) of \(U\), we write \(A_+\) for the intersection of \(A\) with \(U_+\).
We use Sweedler notation for the coproduct. In particular, we write
\[ \Delta(a) = \sum a(1) \otimes a(2) \]
for each \( a \in U \). Recall that a subalgebra \( S \) of \( U \) is called a left coideal subalgebra if \( \Delta(a) \in U \otimes S \) for all \( a \in S \).

Let \( \text{ad}_r \) denote the right adjoint action defined by
\[ (1.1) \quad (\text{ad}_r a)b = \sum \sigma(a(1))ba(2) \]
for all \( a \in U \) and \( b \in U \). For example,
\[ (1.2) \quad (\text{ad}_r x_j)b = -t_j^{-1}x_jb + t_j^{-1}bx_j \quad \text{and} \quad (\text{ad}_r y_j)b = by_j - y_j t_j b t_j^{-1} \]
for \( b \in U \) and \( 1 \leq j \leq n \).

Let \( T \) be the group generated by the \( t_i \), \( 1 \leq i \leq n \) and let \( \tau \) be the isomorphism from \( Q(\pi) \) to \( T \) which sends \( \alpha_i \) to \( t_i \). Write \( U^\circ \) for the group algebra generated by \( T \). Define the subgroup \( T_\Theta \) of \( T \) by
\[ T_\Theta = \{ \tau(\lambda) | \lambda \in Q(\pi) \text{ and } \Theta(\lambda) = \lambda \}. \]

Let \( M \) be a \( U \) module. We say that a nonzero vector \( v \in U \) has weight \( \lambda \in \mathfrak{h}^* \) provided that \( \tau(\lambda) \cdot v = q^{(\gamma, \lambda)}v \) for all \( \tau(\lambda) \in T \). For any vector subspace \( V \subset M \), write \( V_\gamma \) for the subspace of \( V \) spanned by the \( \gamma \) weight vectors. If \( V \) is a vector subspace of \( U \), then \( V_\gamma \) is the subspace of \( V \) consisting of \( \gamma \) weight vectors with respect to the adjoint action. Now suppose that \( F \) is both a subalgebra and \( \text{ad} T \) submodule of \( U \). Then for any subgroup \( S \) of \( T \), we write \( FS \) for the subalgebra of \( U \) generated by \( F \) and \( S \). Note that \( FS \) is spanned as a vector space over \( C \) by elements as with \( a \in F \) and \( s \in S \).

2 Quantum Symmetric Pairs

Quantum analogs of the pair \( U(\mathfrak{g}), U(\mathfrak{g}^\theta) \) associated to a maximally split involution are constructed and characterized up to Hopf algebra automorphism in [L2] and [L4, Section 7]. These analogs consist of \( U \) and a maximal left coideal subalgebra \( B \) of \( U \) which specializes to \( U(\mathfrak{g}^\theta) \) as \( q \) goes to 1 (The reader is referred to [L4, Section 1, following (1.10)] for the precise definition
of specialization used here, [L4, (7.25)] for the notion of maximal, and [L4, Theorem 7.5].) We review the construction of these subalgebras here.

From now on, Θ will denote an involution of the root system Δ induced by a maximally split involution of g. Let \( \mathcal{M} \) be the subalgebra of \( U \) generated by \( x_i, y_i, t_i^{\pm 1} \) for \( \alpha_i \in \pi_\Theta \). Set \( \mathcal{M}^+ \) equal to the subalgebra of \( \mathcal{M} \) generated by the \( x_i, \alpha_i \in \pi_\Theta \).

Let \( [\Theta] \) be the set of all maximally split involutions of \( g \) which induce the involution \( \Theta \) on \( \Delta \). Note that two involutions in \( [\Theta] \) are conjugate to each other via a Lie algebra automorphism of \( g \) which fixes the Cartan subalgebra.

Let \( \theta \) be the particular choice of maximally split involution in \( [\Theta] \) as described in [L4, Section 7, the discussion following (7.5)] and \( \tilde{\theta} \) be the automorphism of \( U \) defined in [L4, Theorem 7.1] which specializes to \( \theta \). We describe the action of \( \tilde{\theta} \) on \( y_i \) for \( \alpha_i \notin \pi_\Theta \). (Note that replacing each \( x_j \) by \( e_j \), each \( t_j \) with 1, and setting \( q = 1 \) will recover the action of \( \theta \) on \( f_i \). Since \( \theta \) is the identity on the positive and negative root vectors corresponding to roots in \( \pi_\Theta \) and the action of \( \theta \) on \( h \) corresponds to the action of \( \Theta \) on \( h^* \), this information is enough to determine \( \theta \).

Let \( \pi^* \) be the subset of \( \pi - \pi_\Theta \) consisting of all \( \alpha_i \notin \pi_\Theta \) such that either \( i = p(i) \) or \( i < p(i) \). Write \( x_i^{(m)} \) and \( y_i^{(m)} \) where \( m \in \mathbb{N} \) for the divided powers of \( x_i \) and \( y_i \) respectively (see [Jo, 4.3.14 and 1.2.12]). Given \( i \) such that \( \alpha_i \in \pi^* \), there exists a sequence \( \alpha_i, \ldots, \alpha_r \) consisting of elements in \( \pi_\Theta \) and positive integers \( m_1, \ldots, m_r \) subject to

(2.1) \[ \tilde{\theta}(y_i) = \left( \text{ad}_r x_{i_1}^{(m_1)} \right) \cdots \left( \text{ad}_r x_{i_r}^{(m_r)} \right) t_{p(i)}^{i-1} x_{p(i)} \]

and

(2.2) \[ \tilde{\theta}(y_{p(i)}) = (-1)^{m_1+\cdots+m_r} \left( \text{ad}_r x_{i_r}^{(m_r)} \right) \cdots \left( \text{ad}_r x_{i_1}^{(m_1)} \right) t_{i}^{-1} x_i. \]

Using (1.1), it is straightforward to check that \( \left( \text{ad}_r y_j \right) t_{i}^{-1} x_i = 0 \) whenever \( i \neq j \). Thus \( t_{i}^{-1} x_i \) is a lowest weight vector with respect to the action of \( \text{ad}_r \mathcal{M} \) for all \( \alpha_i \notin \pi_\Theta \). It follows as in the classical case ([L4, discussion following (7.6)]) that these sequences satisfy the following property. For each \( r \geq s > 0 \),

(2.3) \[ X_s = \left( \text{ad}_r x_{i_s}^{(m_s)} \right) \left( \text{ad}_r x_{i_{s+1}}^{(m_{s+1})} \right) \cdots \left( \text{ad}_r x_{i_r}^{(m_r)} \right) t_{p(i)}^{i-1} x_{p(i)} \]

is a highest weight vector for the action of \( \text{ad}_r x_{i_s} \) and \( X_{s+1} \) is a lowest weight vector for the action of \( \text{ad}_r y_{i_s} \). Moreover, \( \tilde{\theta}(y_i) \) is a highest weight vector for the action of \( \text{ad}_r \mathcal{M} \).
Set $S$ equal to the subset of $\pi^*$ consisting of $\alpha_i$ such that $\Theta(\alpha_i) = -\alpha_i$ and $2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ is even for all $\alpha_j$ such that $\Theta(\alpha_j) = -\alpha_j$. Let $D$ denote the subset of $\pi^*$ consisting of those $\alpha_i$ such that $i \neq p(i)$ and $(\alpha_i, \Theta(\alpha_i)) \neq 0$. Let $Z$ be an integral domain containing the complex numbers $C$ and contained in some field extension $C(Z)$ of $C$. Write $Z^*$ for the nonzero elements of $Z$. Define

$$S(Z) = \{ s \in Z^n | s_i = 0 \text{ for } \alpha_i \not\in S \}$$

and

$$D(Z) = \{ d \in (Z^*)^n | d_i = 1 \text{ for } \alpha_i \not\in S \}.$$ 

Given $s \in S(Z)$ and $d \in S(Z)$, the subalgebra $B_{\theta,s,d}$ of $U_{C(Z)}$ is generated by $T_\Theta$, $M$, and the elements $B^\theta_{i,s_i,d_i}$ for $\alpha_i \in \pi - \pi_\Theta$ where

$$B^\theta_{i,s_i,d_i} = y_i t_i + d_i \tilde{\theta}(y_i) t_i + s_i t_i.$$ 

We abbreviate $B^\theta_{i,s_i,d_i}$ by $B_i$ when $s_i, d_i,$ and $\tilde{\theta}$ can be understood from the context. Note that if $i = p(i)$, then the definitions of $B_i$ and $B_{p(i)}$ using (2.4) agree.

The next theorem follows as in the proof of [L4, Theorem 7.2]. (See also [L4, Variations 1 and 2].)

**Theorem 2.1** Let $Z$ be an extension field $C$. For each $s \in S(Z)$ and $d \in D(Z)$, the algebra $B_{\theta,s,d}$ is a left coideal subalgebra of $U_Z$.

Recall that a pair $g, \theta$ is called irreducible if $g$ cannot be written as the direct sum of two semisimple Lie subalgebras which both admit $\theta$ as an involution. In Section 7, we give a complete list of coideal subalgebras $B_{\theta,s,d}$ associated to irreducible pairs $g, \theta$. From this list, it can be seen that if $g, \theta$ is irreducible, then either both $S$ and $D$ are the empty set, or one is the empty set and the other consists of exactly one root.

Set $B_\Theta$ equal to the union of the orbits of all $B_{\theta,s,d}$ under the action of $H$ where $s \in S(C)$ and $d \in D(C)$. More generally, given an integral domain $Z$ containing $C$, let $B_\Theta(Z)$ be the set of algebras $B$ which are isomorphic via a Hopf algebra automorphism fixing $MT_\Theta$ of $U_Z$ to $B_{\theta,s,d}$, for some $s \in S(Z)$ and $d \in D(Z)$. Note that if both $S$ and $D$ are empty, then $B_\Theta$ consists of exactly one orbit under this action. It follows from the description of the group of Hopf algebra automorphisms of $U$ in [Jo, Section 10.4] that the set
\{B_{s,d}| s \in S(C) \text{ and } d \in D(C)\} \text{ is a complete set of distinct representatives for the orbits of } B \text{ under } H.

Let \( \hat{U} \) denote the \( C[q,q^{-1}] \) subalgebra of \( U \) generated by \( x_i, y_i, t_i^{\pm 1}, \) and \( (t_i - 1)/(q - 1) \) for \( 1 \leq i \leq n. \) Let \( B_{s,d} \) be the set of maximal left coideal subalgebras of \( U \) such that \( B \cap \hat{U} \) specializes to \( U(g^\theta) \) as \( q \) goes to 1. In particular, \( B_{s,d} \) is the set of quantum analogs of \( U(g^\theta) \) in the sense of [L2] and [L4]. By [L4, Theorem 7.5], every algebra in \( B_{s,d} \) is isomorphic to an appropriate \( B_{s,d} \) for some \( s \in S(C[q,q^{-1}]) \) and \( d \in D(C[q,q^{-1}]) \) via some Hopf algebra automorphism in \( H. \)

It should be noted that for certain \( \theta \), some of the algebras in \( B_{s,d} \) have appeared in the literature in papers not written by the author. The most prominent of these algebras is the nonstandard quantum analog \( U'_q(\mathfrak{so}_n) \) of \( U(\mathfrak{so}_n) \) introduced by Gavrilik and Klimyk in [GK]. This algebra is equal to \( B_{s,d} \) when \( g, \theta \) is of type \( \text{A}_1 \) as described in Section 7. The representation theory of \( U'_q(\mathfrak{so}_n) \) has been extensively investigated in a series of papers by Gavrilik, Klimyk, and others (see for example [GI], [GIK], and [HKP]). In [NS], Noumi and Sugitani construct subalgebras \( U_{tw}(g^\theta) \) of \( U \) which are quantum analogs of \( U(g^\theta) \) for many of the pairs \( g, \theta \), when \( g \) is of classical type. Their approach is completely different and involves solutions to reflection equations. Nevertheless, it is shown in [L2, Section 6], that for a given pair \( g, \theta \), each subalgebra \( U_{tw}(g^\theta) \) constructed in [NS] belongs to \( B_{s,d}. \)

3 Spherical modules

Let \( L(\lambda) \) denote the finite-dimensional simple \( U \) module with highest weight \( \lambda \) where \( \lambda \in P^+(\pi). \) Given a left coideal subalgebra \( B \) in \( B_{s,d} \) of \( U, \) we say that \( L(\lambda) \) is spherical with respect to \( B \) if \( L(\lambda)^B = \{ v \in L(\lambda) | bv = c(b)v \} \) is a one-dimensional space. Necessary and sufficient conditions for \( L(\lambda) \) to be spherical with respect to elements \( B \) in \( B_{s,d} \) are established in [L3, Section 4]. This result is extended in this paper to all algebras in \( B_{s,d}. \)

Let \( \{\zeta_i|\alpha_i \in S\} \cup \{\eta_j|\alpha_j \in D\} \) be a set of \( |S| + |D| \) indeterminates. Note that \(-1\) cannot be written as a sum of squares in the field \( \mathbb{R}(q)(\zeta, \eta) \) generated by these indeterminates over \( \mathbb{R}(q). \) It follows that \( \mathbb{R}(q)(\zeta, \eta) \) is a formally real field. Thus we can choose a set of positive elements for the field \( \mathbb{R}(q)(\zeta, \eta) \) containing \( \eta_j \) for each \( \alpha_j \in D. \) Let \( \mathcal{K} \) denote the algebraic closure of \( \mathbb{R}(q)(\zeta, \eta). \) Set \( \mathcal{R} \) equal to the real algebraic closure of \( \mathbb{R}(q)(\zeta, \eta). \)
inside of $\mathcal{K}$ and let $\mathcal{R}^+$ denote the set of positive elements of $\mathcal{R}$. (For more information on formally real fields and real algebraic closures, the reader is referred to [J, Chapter 11].)

Note that $\mathcal{K} = \mathcal{R} + i\mathcal{R}$ and thus complex conjugation extends to $\mathcal{K}$. Let $\kappa$ be the conjugate linear antiautomorphism of $U_K$ defined by $\kappa(x_i) = y_it_i$, $\kappa(y_i) = t_i^{-1}x_i$ and $\kappa(t) = t$ for all $t \in T$. In particular, $\kappa$ restricts to an antiautomorphism of $U_R$ and acts as conjugation on elements of $C$. Note that $\kappa^2$ is just the identity. Let $\mathbb{H}_\mathcal{R}$ denote the subgroup of $\mathbb{H}$ whose elements restrict to Hopf algebra automorphisms of $U_\mathcal{R}$.

Let $s(\zeta) = (s(\zeta_1),\ldots,s(\zeta_n))$ be the element of $S(\mathcal{K})$ such that $s(\zeta_i) = \zeta_i$ for all $\alpha_i \in S$. Similarly, let $d(\eta) = (d(\eta_1),\ldots,d(\eta_n))$ be the element of $D(\mathcal{K})$ such that $d(\eta_j) = \eta_j$ for all $\alpha_j \in D$. Since $d(\eta_j) = 1$ for $\alpha_j \notin D$, it follows that $d(\eta_j) \in \mathcal{R}^+$ for all $1 \le j \le n$. By the definition of real algebraic closures, every positive element in $\mathcal{R}$ has a square root in $\mathcal{R}$. Let $d(\eta_i)^{1/2}$ denote the positive square root of $d(\eta_i)$ for each $i$. We have the following generalizations of [L3, Lemma 3.2] and [L4, Theorem 7.6].

**Theorem 3.1** There exists a Hopf algebra automorphism $\varphi \in \mathbb{H}_\mathcal{R}$ such that $\varphi \kappa \varphi^{-1}(B_{\theta,s(\zeta),d(\eta)}) = B_{\theta,s(\zeta),d(\eta)}$. Moreover $B_{\theta,s(\zeta),d(\eta)}$ acts semisimply on all finite-dimensional $U_\mathcal{K}$-modules.

**Proof:** The first assertion implies the second statement by [L4, Section 2]. The proof presented here for the first assertion does not follow the original argument given in [L3] which uses specialization at $q = 1$. Instead, it is based closely on the discussion leading up to [L4, Theorem 7.6].

Set $B = B_{\theta,s(\zeta),d(\eta)}$ and $B_i = B_{\theta,i,s(\zeta),d(\eta)}$ for each $\alpha_i \notin \pi_\Theta$. Note that

$$(\varphi \kappa \varphi^{-1})(\mathcal{M}T_\Theta) = \mathcal{M}T_\Theta$$

and $(\varphi \kappa \varphi^{-1})^2 = \kappa^2$ is the identity on $U_\mathcal{K}$ for all $\varphi \in \mathbb{H}_\mathcal{R}$. Hence it is sufficient to find $\varphi \in \mathbb{H}_\mathcal{R}$ such that $\varphi \kappa \varphi^{-1}(B)$ contains $B_i$ for each $\alpha_i \notin \pi_\Theta$.

A straightforward computation using (1.2) shows that

$$q(\lambda,\alpha_i)cT(\lambda)y_i - y_it_i cT(\lambda)t_i^{-1} = [(\text{ad}_r y_i) c] \tau(\lambda)$$

for all $c \in U$, $1 \le i \le m$, and $\tau(\lambda) \in T$. Thus $[(\text{ad}_r y_i) b\tau(\lambda)^{-1}] \tau(\lambda)t \in B$ for any $\tau(\lambda) \in T$, $t \in T_\Theta$, $\alpha_i \in \pi_\Theta$, and $b \in B$. As in [L4, following the proof of Theorem 7.5], we have

$$(\text{ad}_r y_i^{m_1}) \cdots (\text{ad}_r y_i^{m_r}) \tilde{\theta}(y_{p(i)}) = (-1)^{m_1+\cdots+m_r} t_i^{-1} x_i$$
and \( \kappa((\text{ad}_r x_j)b) = -(\text{ad}_r y_j)\kappa(b) \) for all \( b \in U \) and \( 1 \leq j \leq n \). Thus (2.1) implies that
\[
\kappa(\tilde{\theta}(y_i)) = (-1)^{(m_1 + \cdots + m_r)}(\text{ad}_r y^{(m_1)}_{i_1}) \cdots (\text{ad}_r y^{(m_r)}_{i_r})y_{p(i)}.
\]
Moreover, it follows that
\[
(3.1) \quad \kappa(\tilde{\theta}(y_i))t_i + d(\eta_{p(i)})t_i^{-1}x_it_i + s(\zeta_i)t_i
\]
\[
= (-1)^{(m_1 + \cdots + m_r)}[(\text{ad}_r y^{(m_1)}_{i_1}) \cdots (\text{ad}_r y^{(m_r)}_{i_r})B_{p(i)}t_i^{-1}]t_i(t_i^{-1})t_i
\]
is an element of \( B \).

Let \( \varphi \) be the Hopf algebra automorphism in \( H_R \) defined by
\[
\varphi(x_j) = q^{(\alpha_j, -\alpha_j + \Theta(\alpha_j))/4}d(\eta_j)^{1/2}x_j \text{ for } \alpha_j \notin \pi_{\Theta}.
\]
Temporarily write \( \kappa_1 = \varphi\kappa\varphi^{-1} \). By (2.1) and (2.2), we see that \( \Theta(\alpha_i) - \alpha_i = \Theta(\alpha_{p(i)}) - \alpha_{p(i)} \). Hence
\[
(\alpha_i, \Theta(\alpha_i) - \alpha_i) = (\alpha_i, \Theta(\alpha_{p(i)}) - \alpha_{p(i)}) = (\alpha_i, \Theta(\alpha_{p(i)})) - (\alpha_i, \alpha_{p(i)})
\]
\[
= (\Theta(\alpha_i) - \alpha_i, \alpha_{p(i)}) = (\alpha_{p(i)}, \Theta(\alpha_{p(i)}) - \alpha_{p(i)}).
\]
Thus, a straightforward computation using the fact that \( s(\zeta_i) = 0 \) for \( \alpha_i \notin \mathcal{S} \) shows that
\[
q^{(\alpha_i, \Theta(\alpha_i))/2}\kappa_1(\tilde{\theta}(y_i))t_i + d(\eta_{p(i)})t_i^{-1}x_it_i + \tilde{s}_it_i
\]
\[
= d(\eta_{p(i)})d(\eta_i)^{-1}(y_it_i + d(\eta_{p(i)})\tilde{\theta}(y_i)t_i + s(\zeta_i)t_i) = d(\eta_{p(i)})d(\eta_i)^{-1}B_i
\]
is an element of \( \kappa_1(B) \) for all \( \alpha_i \in \pi_{\Theta} \). Therefore \( B = \kappa_1(B) \). \( \square \)

Recall that \( P^+(\Sigma) \) is the subset of \( \mathfrak{h}^* \) consisting of dominant integral weights associated to the root system \( \Sigma \). Let \( P^+_\Theta \) be the set of all \( \lambda \in P^+(\pi) \) such that

(i) \( (\lambda, \beta) = 0 \) for all \( \beta \in \mathfrak{h}^*_\Theta \).

(ii) \( (\lambda, \tilde{\beta})/(\tilde{\beta}, \tilde{\beta}) \) is an integer for every restricted root \( \tilde{\beta} \in \Sigma \).

By [He2, Chapter II, Theorem 4.8], \( P^+_\Theta \) is just the intersection of \( P(2\Sigma) \) with \( P^+(\Sigma) \).

Given \( \lambda \in P^+(\pi) \) and an integral domain \( \mathcal{Z} \) containing \( \mathcal{C} \), we set \( L(\lambda)_{\mathcal{Z}} = L(\lambda) \otimes_{\mathcal{C}} \mathcal{Z} \). We have the following criterion for \( L(\lambda)_{\mathcal{K}} \) to be a spherical module with respect to the the algebra \( B_{\theta, s(\zeta_i), d(\eta_i)} \).
Theorem 3.2 Let \( B = B_{\theta, s(\zeta), d(\eta)} \) Then

\[
\dim L(\lambda)_K^B \leq 1
\]

for all \( \lambda \in P^+(\pi) \). Moreover equality holds in \( (3.2) \) and thus \( L(\lambda)_K^B \) is spherical with respect to \( B_{\theta, s(\zeta), d(\eta)} \) if and only if \( \lambda \in P^+_{\Theta} \).

**Proof:** The first assertion follows as in the proof of [L4, Theorem 7.7(i)] and the second assertion follows as in the proof of [L3, Theorem 4.3] using Theorem 3.1. \( \square \)

Fix \( \lambda \in P^+_{\Theta} \). Given a weight \( \mu \) of \( L(\lambda) \), let \( \{v^i_\mu\}_i \) be a basis of weight vectors for the \( \mu \) weight space of \( L(\lambda) \) where \( i \) varies over a finite set. Note that the \( \lambda \) weight space of \( L(\lambda) \) has dimension 1 and so \( \{v^1_\lambda\}_i \) has only one element. Set \( v_\lambda = v^1_\lambda \). Given a vector \( v = \sum_{i, \mu} g^i_\mu v^i_\mu \) in \( L(\lambda)_K^B \), set

\[
\text{Supp}(v) = \{ \mu | g^i_\mu \neq 0 \text{ for some } i \}.
\]

Write \( C[\zeta, \eta] \) for the polynomial ring \( C[\zeta_i, \eta_j | \alpha_i \in S \text{ and } \alpha_j \in D] \). Set

\[
C_k = x_k + q_k^2 d(\eta_{\mu(k)})^{-1} \kappa(\bar{\theta}(y_k)) t_k + q_k^2 s(\zeta_k)(t_k - 1)
\]

for \( \alpha_k \notin \pi_{\Theta} \) and set \( C_k = x_k \) for \( \alpha_k \in \pi_{\Theta} \). Note that \( \bar{\theta}(y_k) \in U_C[\zeta, \eta] \) for each \( \alpha_k \notin \pi_{\Theta} \). Hence, by (3.1) and the definition of the counit of \( U \), each \( C_k \) is an element of \( B_+ \cap U_{C[\zeta, \eta]} \).

The next result provides finer information about the invariant vectors of spherical modules.

**Lemma 3.3** Suppose that \( \tilde{\xi}_\lambda \) is a nonzero \( B_{\theta, s(\zeta), d(\eta)} \) invariant vector of \( L(\lambda)_K^B \). Then

\[
\tilde{\xi}_\lambda \in v_\lambda + \sum_{\mu < \lambda, k} C[\zeta, \eta] v^k_\mu
\]

up to multiplication by a nonzero scalar.

**Proof:** Let \( a_\lambda \) denote the coefficient of \( v_\lambda \) in \( \tilde{\xi}_\lambda \) written as a linear combination of the basis vectors \( \{v^i_\mu\} \) of \( L(\lambda) \). Rescaling if necessary, we may assume that \( a_\lambda \) equals 0 or 1.

We can write \( \xi_\lambda \) as a linear combination of weight vectors: \( \xi_\lambda = \sum_{\mu} w_\mu \). For each weight \( \gamma \) of \( L(\lambda) \), let \( y^{\gamma}_{-\lambda+\gamma} \) be weight vectors in \( U_{\lambda-\gamma} \) such that
\[ y_{\lambda-\gamma}^i v_\lambda = v_i^j. \]

It is well known that there exists a set of weight vectors \( \{x_i^\lambda-\gamma\} \) in \( U_{\lambda-\gamma}^+ \) such that \( x_i^\lambda-\gamma v_j^i = \delta_{ij} v_\lambda \). It follows that \( \sum_i y_{\lambda-\gamma}^i x_i^\lambda-\gamma w_\gamma = w_\gamma \). In particular, \( w_\gamma \in \sum_i Ux_i w_\gamma \) for each \( \gamma \in \text{Supp}(\tilde{\xi}_\lambda) - \{\lambda\} \).

Partition \( \text{Supp}(\tilde{\xi}_\lambda) \) into two sets \( I_1 \) and \( I_2 \) as follows. The weight \( \gamma \) is in \( I_1 \) provided \( w_\gamma \) is a nonzero element of \( a_\lambda Uc_{[\kappa,\eta]} v_\lambda \). The second set \( I_2 \) is simply the complement of \( I_1 \) in \( \text{Supp}(\tilde{\xi}_\lambda) \). Note that \( I_1 \cup I_2 \) is nonempty since \( \tilde{\xi}_\lambda \) is nonzero. We show that \( I_2 \) is empty. This in turn implies that \( I_1 = \text{Supp}(\tilde{\xi}_\lambda) \) and \( a_\lambda = 1 \), which proves the theorem.

Assume \( I_2 \) is nonempty and let \( \gamma \) be a maximal element of \( I_2 \). Fix \( i \). The maximality of the choice of \( \gamma \) and the definition of \( C_k \) ensures that

\[
C_k \sum_{\mu \in I_2} w_\mu \in x_k w_\gamma + \sum_{\mu \notin \gamma + \alpha_k} (L(\lambda)K)_\mu.
\]

Recall that \( C_k \tilde{\xi}_\lambda = 0 \). Thus

\[
C_k \sum_{\mu \in I_1} w_\mu \in -x_k w_\gamma + \sum_{\mu \notin \gamma + \alpha_k} (L(\lambda)K)_\mu.
\]

However, since \( C_k \in Uc_{[\kappa,\eta]} \), it follows that

\[
C_k \sum_{\mu \in I_1} w_\mu \in a_\lambda Uc_{[\kappa,\eta]} v_\lambda.
\]

Thus \( x_k w_\gamma \in a_\lambda Uc_{[\kappa,\eta]} v_\lambda \). for all \( 1 \leq k \leq n \). It follows that

\[
w_\gamma \in \sum_k Ux_k w_\gamma \in a_\lambda Uc_{[\kappa,\eta]} v_\lambda,
\]

a contradiction. Therefore \( I_2 \) is empty. \( \square \)

Note that Theorem 3.2 was originally proved in [L3] only for subalgebras in \( B_{\theta,1} \). Now consider spherical vectors corresponding to a subalgebra \( B \) of \( U \) in the larger set \( B_\theta \). The argument used to sketch the proof of [L4, Theorem 7.7(i)] shows that

\[
\dim L(\lambda)^B \leq 1
\]

for all \( \lambda \in P^+(\pi) \). Moreover the proof of [L3, Theorem 4.3] can be applied in this more general setting to show that if \( L(\lambda) \) admits a spherical vector then \( \lambda \in P_0^+ \). However, the proof of the existence of a spherical vector associated to \( B \) in \( B_{\theta,1} \) relies on the fact that the \( B \in B_{\theta,1} \) act semisimply on
finite-dimensional $U$ modules. Unfortunately, we have not proved a version of Theorem 3.1 for all subalgebras $B$ of $U$ in $\mathcal{B}_\theta$. Nevertheless, we obtain a version of Theorem 3.2, which applies to all the algebras in $\mathcal{B}_\theta$, using the preceding lemma.

**Theorem 3.4** Let $Z$ be an integral domain containing $C$ and let $B$ be an algebra in $\mathcal{B}_\theta(Z)$. Then

\[(3.4) \quad \dim L(\lambda)_Z^B \leq 1.\]

Moreover equality holds in (3.4) (and thus $L(\lambda)_Z$ is spherical with respect to $B$) if and only if $\lambda \in P_\ominus^+$. 

Proof: Applying a Hopf algebra automorphism of $U_Z$ to $B$ if necessary, we reduce to the case when $B = B_{\theta,s,d}$ for some $s = (s_1, \ldots, s_n) \in S(Z)$ and $d = (d_1, \ldots, d_n) \in D(Z)$. Without loss of generality, we may assume that $Z$ is the $C$ algebra generated by the $s_i$ for $\alpha_i \in S$ and the $d_j$ for $\alpha_j \in D$. By the discussion preceding the theorem, we need only show that there exists a $B$ invariant vector in $L(\lambda)_Z$ for all $\lambda \in P_\ominus^+$.  

Fix $\lambda \in P_\ominus^+$. By Lemma 3.3, there exist polynomials $f_k^\mu$ in $C[\zeta, \eta]$ such that

$$\tilde{\xi}_\lambda = v_\lambda + \sum_{\mu \prec \lambda, k} f_k^\mu v_\mu^k$$

is a $B_{\theta,s(\zeta),d(\eta)}$ invariant vector. Let $J$ be the ideal in $C[\zeta, \eta]$ generated by $\zeta_i - s_i$ as $\alpha_i$ runs over $S$ and $\eta_j - d_j$ as $\alpha_j$ runs over $D$. Let $\tilde{f}_k^\mu$ denote the image of $f_k^\mu$ in $Z$ obtained by modding out by the ideal $J$. Set

$$\xi_\lambda = v_\lambda + \sum_{\mu \prec \lambda, k} \tilde{f}_k^\mu v_\mu^k.$$ 

Note that $\xi_\lambda$ is nonzero since the coefficient of $v_\lambda$ is 1. Also, the image of $B_{\theta,s(\zeta),d(\eta)} \cap (U_C[\zeta, \eta])_+$ in $U_Z$ obtained by modding out by $J$ is just $B_+$. Furthermore, the action of $U_Z$ on $L(\lambda)_Z$ induced by taking the action of $U_C[\zeta, \eta]$ on $L(\lambda)_C[\zeta, \eta]$ and modding out by $J$ is the same as the standard action of $U_Z$ on $L(\lambda)_Z$. Thus $\xi_\lambda$ is a nonzero $B_+$ invariant vector in $L(\lambda)_Z$. $\Box$

Recall that $P(\Sigma)$ is the weight lattice associated to the root system $\Sigma$ and hence contains the root lattice associated to $\Sigma$. Since $\tilde{\alpha}_i \in \Sigma$ it follows that $\tilde{\alpha}_i \in P(\Sigma)$ for each $\alpha_i \in \pi - \pi_{\Theta}$. Sometimes, a smaller scalar multiple of $\tilde{\alpha}_i$ is also in $\Sigma$. This happens when $S$ is nonempty as explained in the following lemma.
Lemma 3.5  If $\alpha_i \in S$ then $\tilde{\alpha}_i/2 \in P(\Sigma)$.

Proof: It is sufficient to prove the lemma when $\mathfrak{g}, \theta$ is an irreducible pair. Given $\alpha_i \in S$, we must check that

\[(3.5) \quad \frac{2(\tilde{\alpha}_i/2, \tilde{\alpha}_j)}{(\tilde{\alpha}_j, \tilde{\alpha}_j)} \in \mathbb{Z}\]

for all $\tilde{\alpha}_j \in \Sigma$. This is clearly true when $i = j$. Hence it is sufficient to check the case when $i \neq j$. By the classification in Section 7, there are only four possibilities with $S$ nonempty: AIII, Case 2; CI; DIII Case 1; EVII. The tables in [A, Section 5] and [He, Chapter X, Section F] include the Dynkin diagram associated to the restricted root systems in the column labelled “$\Delta$”. In each of these four cases, the restricted root system is of type C and the root $\tilde{\alpha}_i$, for $\alpha_i \in S$, corresponds to the unique long root. Hence (3.5) follows. $\blacksquare$

Define an ordering on the restricted weights $\{\tilde{\lambda} | \lambda \in Q(\pi)\}$ by $\tilde{\beta} \geq_r \tilde{\beta}$ if and only if $\tilde{\beta}' - \tilde{\beta} \in \sum_i N\tilde{\alpha}_i$. Now suppose that $\lambda$ and $\mu$ are two elements of $P(\pi)$ such that $\lambda \geq \mu$. Then there exist nonnegative integers $n_i$ such that $\lambda - \mu = \sum_i n_i \alpha_i$. It follows that $\tilde{\lambda} - \tilde{\mu} = \sum_i n_i \tilde{\alpha}_i$. Hence $\tilde{\lambda} \geq_r \tilde{\mu}$ with strict inequality if and only if $n_i$ is nonzero for some $\alpha_i \in \pi - \pi_{\Theta}$.

The next result gives additional information about the spherical vectors which will be used in later sections.

Theorem 3.6  Let $\mathcal{Z}$ be an integral domain containing $\mathcal{C}$, let $B \in B_{\theta}(\mathcal{Z})$ and let $\lambda \in P^+_{\Theta}$. Write $\xi_\lambda$ for a nonzero $B$ invariant vector of $L(\lambda)_{\mathcal{Z}}$. Then $\text{Supp}(\xi_\lambda) \subset P(2\Sigma)$, $\lambda \in \text{Supp}(\xi_\lambda)$, and if $\beta \in \text{Supp}(\xi_\lambda) - \{\lambda\}$ then $\tilde{\beta} <_r \tilde{\lambda}$.

Proof: Without loss of generality, we may assume that $B = B_{\theta,s,d}$ for some $s \in S(\mathcal{Z})$ and $d \in D(\mathcal{Z})$. By Theorem 3.4, rescaling if necessary, we can write

$$\xi_\lambda = v_\lambda + \sum_{\mu < \lambda} w_\mu$$

where each $w_\mu$ is a weight vector of weight $\mu$ in $L(\lambda)_{\mathcal{Z}}$. In particular, $\lambda \in \text{Supp}(\xi_\lambda)$.

Recall that $\{\tilde{\alpha}_i | \alpha_i \in \pi - \pi_{\Theta}\}$ is a set of simple roots for $\Sigma$. It follows that that $2\tilde{\alpha}_i \in P(2\Sigma)$ for all $\alpha_i \in \pi - \pi_{\Theta}$. By Lemma 3.5, $\tilde{\alpha}_i \in P(2\Sigma)$ for all $i$.
such that $\alpha_i \in S$. Set

\[(3.6) \quad N = \sum_{\alpha_i \in \pi - \pi_0 - S} N(2\tilde{\alpha}_i) + \sum_{\alpha_i \in S} N\tilde{\alpha}_i\]

and

\[\lambda - N = \{\lambda - \gamma | \gamma \in N\} \subset \lambda - N.\]

Now $\lambda \in P_0^+$ and $P_0^+$ is a subset of $P(2\Sigma)$. Hence $\lambda - N \subset P(2\Sigma)$. Furthermore, every element $\beta \in \lambda - N$ such that $\beta \neq \lambda$ satisfies $\beta <_r \lambda$. We complete the proof of the theorem by showing that

\[\text{Supp}(\xi) \subset \lambda - N.\]

Consider $\alpha_i \in \pi_0$ and recall that $x_i\xi = 0$. Note that if $\beta \neq \gamma$, then $x_iw_{\beta}$ and $x_iw_\gamma$ have the same weight if and only if they are both zero. Thus $x_iw_{\beta} = 0$ for each $\beta \in \text{Supp}(\xi)$ and $\alpha_i \in \pi_0$.

Let $I_1$ be the subset of $\text{Supp}(\xi)$ consisting of those weights $\beta$ such that $\beta \in \lambda - N$. Since $\lambda \in \text{Supp}(\xi)$ and $\lambda \subseteq \lambda - N$, it follows that $\lambda \subseteq I_1$. Let $I_2$ be the complement of $I_1$ in $\text{Supp}(\xi)$. We argue that $I_2$ is empty. Choose $\beta \in I_2$ such that $\beta$ is a maximal element in $I_2$ with respect to the partial ordering $>_r$. Since $\lambda \neq \beta$, $w_\beta$ is not a highest weight vector in $L(\lambda)_Z$. Hence there exists $i$ such that $x_iw_\beta \neq 0$. By the previous paragraph, we must have that $\alpha_i \in \pi - \pi_0$. Note that $x_iw_\beta$ is a weight vector of weight $\alpha_i + \beta$. Assume first that $\alpha_i \notin S$. It follows from the definition of $C_i$ (see (3.3)) that $C_i = x_i + Y_i$ where $Y_i$ is a weight vector in $U$ of weight $\Theta(\alpha_i)$. Now $C_i\xi = 0$ since $C_i \in B_+$. Hence $x_iw_\beta = -Y_iw_\gamma$ for some $\gamma \in \text{Supp}(\xi)$. Moreover $\gamma = \alpha_i - \Theta(\alpha_i) + \beta = 2\tilde{\alpha}_i + \beta$. It follows that $\gamma - \beta = 2\tilde{\alpha}_i$ and thus $\gamma >_r \beta$. Moreover, $\beta \notin \lambda - N$ implies that $\gamma \notin \lambda - N$ which contradicts the choice of $\beta$.

Now assume $\alpha_i \in S$. Hence $\Theta(\alpha_i) = -\alpha_i$ and $\tilde{\alpha}_i = \alpha_i$. Note that $B_+$ contains the element $C_i = x_i + q_i^2y_it_i + q_i^2s_i(t_i - 1)$. Hence we can write $x_iw_\beta = -q_i^2y_it_iw_{\gamma_1} - q_i^2s_i(t_i - 1)w_{\gamma_2}$ where $\gamma_1 = \beta + 2\tilde{\alpha}_i$ and $\gamma_2 = \beta + \tilde{\alpha}_i$. Since $x_iw_\beta$ is nonzero, at least one of $\gamma_1, \gamma_2$ is contained in $\text{Supp}(\xi)$. But neither $\gamma_1$ nor $\gamma_2$ can be in the subset $I_1$ of $\lambda - N$ since $\beta \notin \lambda - N$. Once again this contradicts the choice of $\beta$ and the lemma follows. \(\square\)
4 Quantum Zonal Spherical Functions

In this section, we review some basic facts about the quantized function algebra and define quantum zonal spherical functions associated to pairs $B, B'$ of algebras in $B_\theta$. We then prove an important injectivity result relating the space of biinvariants to the character group ring associated to $\Sigma$.

Let $R_q[G]$ denote the quantized function algebra as defined in [Jo, Chapter 9.1]. Recall that $R_q[G]$ is a right and left $U$ module. Let $L(\lambda)^*$ denote the $U$ module dual to $L(\lambda)$. Viewing $L(\lambda)$ as a left $U$ module, $L(\lambda)^*$ is given its natural right $U$ module structure.

According to the quantum Peter-Weyl theorem ([Jo, 9.1.1 and 1.4.13]), there is an isomorphism as right and left $U$ modules:

\[(4.1) \quad R_q[G] \cong \oplus_{\lambda \in P^+(\pi)} L(\lambda) \otimes L(\lambda)^*.
\]

Given an element $w \otimes w^* \in L(\lambda) \otimes L(\lambda)^*$, we write the corresponding element of $R_q[G]$ as $c^\lambda_{w^*,w}$. As a vector space, $R_q[G]$ is spanned by vectors $c^\lambda_{w^*,w}$ where $\lambda \in P^+(\pi)$, $w \in L(\lambda)$, and $w^* \in L(\lambda)^*$. Note that elements of $R_q[G]$ can be thought of as functions on the quantized enveloping algebra $U$. In particular,

\[c^\lambda_{w^*,w}(u) = w^*(uw)\]

for all $u \in U$.

Let $B$ and $B'$ be two subalgebras of $U$ in $B_\theta$. Define the subspace of $B' \mathcal{H}_B$ of left $B'$ and right $B$ invariants inside of $R_q[G]$ by

\[B' \mathcal{H}_B = \{ \varphi \in R_q[G] | b' \cdot \varphi = \epsilon(b') \varphi \text{ and } \varphi \cdot b = \epsilon(b) \varphi \text{ for all } b \in B \text{ and } b' \in B' \}.
\]

By [KS, Chapter 11, Proposition 68], $B' \mathcal{H}_B$, which we refer to as the space of biinvariants associated to the pair $(B, B')$, is a subalgebra of $R_q[G]$.

Set

\[B' \mathcal{H}_B(\lambda) = B' \mathcal{H}_B \cap (L(\lambda) \otimes L(\lambda)^*)\]

where we identify $L(\lambda) \otimes L(\lambda)^*$ with a subspace of $R_q[G]$ using (4.1). By Theorem 3.4, $B' \mathcal{H}_B(\lambda)$ is 0 if $\lambda \notin P^+_\Theta$ and is a one-dimensional trivial left $B'$ and right $B$ module otherwise. Thus (4.1) implies the following direct sum decomposition into trivial one-dimensional left $B'$ and right $B$ modules:

\[(4.2) \quad B' \mathcal{H}_B \cong \oplus_{\lambda \in P^+_\Theta} B' \mathcal{H}_B(\lambda).
\]
For each \( \lambda \in P^+_\Theta \), nonzero elements of \( B' \mathcal{H}_B(\lambda) \) are called zonal spherical functions associated to the pair \( B, B' \). Fix a nonzero vector \( c_{B,B}' \in B' \mathcal{H}_B(\lambda) \). It follows from (4.2) that \( \{ c_{B,B}' \lambda \in P^+_\Theta \} \) is a basis for \( B' \mathcal{H}_B \).

Note that \( L(\lambda) \otimes L(\lambda)^* \) is the joint eigenspace for the action (either right or left) of the center of \( U \) on \( R_q[G] \) with eigenvalue given by the central character of \( L(\lambda) \). Thus the zonal spherical functions are just the eigenvectors of \( B' \mathcal{H}_B \) for the action of the center of \( U_q(\mathfrak{g}) \). Moreover, the eigenvalue of \( c_{B,B}' \lambda \) with respect to the central element \( a \) is given by the action of \( a \) on \( L(\lambda) \).

Given \( \lambda \in P(\pi) \), define the function \( z^\lambda \) on \( T \) by
\[
\tau(\beta) = q^{(\lambda,\beta)}.
\]
Write \( C[P] \) for the group algebra over \( C \) generated by the multiplicative group \( \{ z^\lambda | \lambda \in P(\pi) \} \). Let \( \Upsilon \) be the algebra homomorphism from \( R_q[G] \) to \( C[P] \) obtained by restricting elements of \( R_q[G] \), considered as functions on \( U \), to the subalgebra \( U^o \) generated by \( T \).

Suppose that \( Z \) is an integral domain containing \( C \). Set \( Z[P] = C[P] \otimes_C Z \). The map \( \Upsilon \) extends to a homomorphism from \( R_q[G] \otimes_C Z \) to \( Z[P] \) which we also denote by \( \Upsilon \). The definitions of \( B' \mathcal{H}_B \) and \( c_{B,B}' \lambda \) also extend in the obvious way to pairs of algebras \( B, B' \) in \( B_\Theta(Z) \).

Suppose that \( w \) and \( w^* \) are weight vectors in \( L(\lambda), L(\lambda)^* \) respectively. Assume that the weight of \( w \) is \( \beta \). For each \( \tau(\mu) \) in \( T \), we have
\[
c^\lambda_{w*,w}(\tau(\mu)) = w^*(\tau(\mu)w) = q^{(\beta,\mu)} w^*(w) = w^*(w) z^\beta \cdot \tau(\mu).
\]
In particular, \( \Upsilon(c^\lambda_{w*,w}) \) is a scalar multiple of \( z^\beta \) where \( \beta \) is the weight of \( w \). Note that if the weight of \( w \) and \( w^* \) differ, then \( w^*(w) = 0 \) and hence \( \Upsilon(c^\lambda_{w*,w}) = 0 \).

The next lemma transfers information about spherical vectors from the last section to the zonal spherical functions.

**Lemma 4.1** Let \( s \in S(Z) \) and \( d \in D(Z) \) where \( Z \) is an integral domain containing \( C \). Suppose that \( B = \chi(B_\Theta,s,d) \) and \( B' = \chi'(B_\Theta,s',d) \) where \( \chi \) and \( \chi' \) are Hopf algebra automorphisms of \( U_Z \) fixing \( T \) and \( M \). Then for each \( \lambda \in P^+_\Theta \),
\[
\Upsilon(c^\lambda_{B,B'}) = z^\lambda + \sum_{\beta < \lambda} f_\beta z^\beta
\]
up to a nonzero scalar where each \( f_\beta \in Z \) and \( \beta \) runs over weights of \( L(\lambda) \) which are contained in \( P(2\Sigma) - \{ \lambda \} \).

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Proof: Let \( \{ v_\mu | \mu \leq \lambda \} \) be a basis consisting of weight vectors for \( L(\lambda) \) as in Section 3. Let \( \{ v_\mu^\ast | \mu \leq \lambda \} \) be a basis of weight vectors for \( L(\lambda)^\ast \) such that \( v_\mu^\ast (v_\mu^j) = \delta_{ij} \).

By Theorem 3.4, \( L(\lambda)_Z \) contains a \( B' \)-invariant vector \( \xi \) and \( L(\lambda)^\ast_Z \) contains a \( B \)-invariant vector \( \xi^\ast \). Rescaling if necessary, we may write

\[ \xi = v_\lambda + \sum_{\beta < r} f_\beta^i v_\beta^i \]

and

\[ \xi^\ast = v_\lambda^\ast + \sum_{\beta < r} f_\beta^{i\ast} v_\beta^{i\ast} \]

where \( f_\beta^i \) and \( f_\beta^{i\ast} \) are elements of \( Z \) for each \( \beta \) and \( i \). Furthermore, by Theorem 3.6, if either \( f_\beta^i \) or \( f_\beta^{i\ast} \) is nonzero for some \( i \), then \( \beta \in \Sigma(2\Sigma) \). Set \( f_\beta = \sum_i f_\beta^i f_\beta^{i\ast} \). It follows that up to a nonzero scalar, \( \Upsilon(c^\lambda_{B,B'}) \) equals \( z^\lambda + \sum_\beta f_\beta z^\beta \) where \( \beta \) runs over the weights of \( L(\lambda) \) contained in \( \Sigma(2\Sigma) - \{ \lambda \} \).

Consider \( \lambda \in P^+_\Theta \) and \( B, B' \in B_\Theta \). For the remainder of the paper, we assume that \( c^\lambda_{B,B'} \) has been chosen so that when \( \Upsilon(c^\lambda_{B,B'}) \) is written as a linear combination of the \( z^\gamma \), the coefficient of \( z^\lambda \) is equal to 1. For each \( \lambda \in P^+_\Theta \), we set \( \varphi^\lambda_{B,B'} = \Upsilon(c^\lambda_{B,B'}) \).

Set \( C[2\Sigma] \) equal to the group subalgebra of \( C[P] \) corresponding to the group \( \{ z^{2\lambda} | \lambda \in P(\Sigma) \} \). The next result allows us to identify the space spanned by the zonal spherical functions with a subalgebra of \( C[2\Sigma] \).

**Theorem 4.2** The restriction map \( \Upsilon \) from \( B' \mathcal{H}_B \) to \( U^\circ \) defines an injection of \( B' \mathcal{H}_B \) into \( C[2\Sigma] \).

**Proof:** Let \( \lambda \in P^+_\Theta \) and recall that \( P^+_\Theta \subset P(2\Sigma) \). By Lemma 4.1, there exists \( a_\beta \in C \) such that

\[ \varphi^\lambda_{B,B'} = z^\lambda + \sum_{\beta < r} a_\beta z^\beta \]

where the \( \beta \) run over a finite number of elements in \( P(2\Sigma) \). Thus each \( \varphi^\lambda_{B,B'} \in C[2\Sigma] \). It follows that \( \Upsilon(B' \mathcal{H}_B) \) is a subalgebra of \( C[2\Sigma] \).
Recall that the set \( \{ c_{B,B'}^\lambda \mid \lambda \in P_\Theta^+ \} \) is a basis for \( B' \mathcal{H}_B \). Consider a typical element \( X = \sum_i b_i c_{B,B'}^\lambda_i \) in \( B' \mathcal{H}_B \) and note that \( \Upsilon(X) = \sum_i b_i \varphi_{B,B'}^\lambda_i \). Choose a maximal weight \( \lambda_1 \) in the set \( \{ \lambda \mid b_i \neq 0 \} \). Then \( \Upsilon(X) \) is in

\[
b_1 z^{\lambda_1} + \sum_{\mu \notin \lambda_1} C z^\mu
\]

and so is nonzero. Hence \( \Upsilon \) is injective. \( \square \)

5 A Criterion for Invariance

Let \( W \) be the Weyl group associated to the restricted root system \( \Sigma \). Note that \( W \) acts on \( C[2\Sigma] \) via \( w \cdot z^\beta = z^{w \beta} \) for all \( \beta \in P(2\Sigma) \) and \( w \in W \). A classical result shows that the zonal spherical functions correspond to the \( W \) invariant functions of the character ring of the restricted root system. In this section, we obtain a similar result in the quantum case. In particular, given \( B \in B_\theta \) we determine how to choose \( B' \) so that the image of \( B' \mathcal{H}_B \) under \( \Upsilon \) is the entire invariant ring \( C[2\Sigma]^W \).

Let \( \rho \) denote the half sum of the positive roots in \( \Delta \), so \( (\rho, \alpha_i) = (\alpha_i, \alpha_i)/2 \) for each \( 1 \leq i \leq n \). Note that if \( \Theta(\alpha_i) = -\alpha_p(i) \) then \( \tilde{\theta}(y_i) = t^{-1}_p x_p(i) \) and \( \tilde{\theta}(y_p(i)) = t^{-1}_i x_i \). Hence

\[
\tilde{\theta}(y_p(i)) t^{-1}_p x_p(i) = t^{-1}_i x_i \tilde{\theta}(y_i)
\]

in this case. We show that a similar result is true in general.

Recall that \( \mathcal{M}^+ \) is the subalgebra of \( \mathcal{M} \) generated by the \( x_i, \alpha_i \in \pi_\Theta \).

**Lemma 5.1** For each \( \alpha_i \notin \pi_\Theta \),

\[
q^{(\rho,\Theta(\alpha_i)+\alpha_i)} \tilde{\theta}(y_p(i)) t^{-1}_p x_p(i) \in t^{-1}_i x_i \tilde{\theta}(y_i) + \mathcal{M}^+ U + U \mathcal{M}^+.
\]

**Proof:** Recall the sequences used in Section 2 (see (2.1) and (2.2)) to define \( \tilde{\theta}(y_i) \) for \( \alpha_i \notin \pi_\Theta \). Using the form of \( (\text{ad}_r x_i) \) given in (1.2), we have that

\[
t^{-1}_i x_i \tilde{\theta}(y_i) = t^{-1}_i x_i [(\text{ad}_r x_i)^{m_1} \cdots (\text{ad}_r x_i)^{m_r} t^{-1}_p x_p(i)]
\]

\[
\in (-1)^{m_1 + \cdots + m_r} t^{-1}_i x_i [(t^{-1}_1 x_1)^{m_1} \cdots (t^{-1}_r x_r)^{m_r} t^{-1}_p x_p(i)] + \mathcal{M}^+ U + U \mathcal{M}^+.
\]

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A straightforward computation shows that for each positive integer $k$ and each $j$,

$$
(t_j^{-1}x_j)^k = q^{-k(k+1)(\alpha_j,\alpha_j)/2}x_j^{k-k}.
$$

(5.2)

Note that $x_{i_{s+1}}^{m_{s+1}} \ldots x_{i_r}^{m_r} x_{p(i)}$ and $X_{s+1}$ (defined in (2.3)) have the same weight. Recall that $X_s$ is a highest weight vector for the action of $ad_r x_{i_s}$ and that $X_{s+1}$ is a lowest weight vector for the action of $ad_r y_{i_s}$. Set $\lambda_s$ equal to the weight of $X_s$. It follows that $(\lambda_s, \alpha_i) = \frac{m_s}{2}(\alpha_i, \alpha_i)$ and $(\lambda_{s+1}, \alpha_i) = -\frac{m_s}{2}(\alpha_i, \alpha_i)$ for each $s$. Hence (5.2) implies

$$
(t_{i_s}^{-1}x_{i_s})^{m_s} x_{i_{s+1}}^{m_{s+1}} \ldots x_{i_r}^{m_r} x_{p(i)}
$$

$$
= q^{-m_s(m_s+1)(\alpha_{i_s},\alpha_{i_s})/2} x_{i_s}^{m_s} x_{i_{s+1}}^{m_{s+1}} \ldots x_{i_r}^{m_r} x_{p(i)} x_{i_s}^{-m_s}.
$$

(5.3)

Note that

$$
-(\sum_{j=1}^r m_j \alpha_i) = \Theta(\alpha_i) - \alpha_{p(i)}.
$$

Hence repeated applications of (5.3) yields

$$
(t_{i_1}^{-1}x_{i_1})^{(m_1)} \ldots (t_{i_r}^{-1}x_{i_r})^{(m_r)} t_{p(i)}^{-1} x_{p(i)}
$$

$$
= q^{(\rho, \Theta(\alpha_i) - \alpha_{p(i)})} x_{i_1}^{(m_1)} \ldots x_{i_r}^{(m_r)} t_{i_1}^{-m_1} \ldots t_{i_r}^{-m_r} t_{p(i)}^{-1} x_{p(i)}
$$

(5.4)

Now

$$
(\Theta(\alpha_i) + \alpha_{p(i)}, \Theta(\alpha_i) - \alpha_i) = (\Theta(\alpha_i), \Theta(\alpha_i) - \alpha_i + \alpha_{p(i)}) - (\alpha_{p(i)}, \alpha_i)
$$

$$
= (\Theta(\alpha_i), \Theta(\alpha_{p(i)})) - (\alpha_{p(i)}, \alpha_i) = 0.
$$

Furthermore,

$$
(\rho, \Theta(\alpha_i) - \alpha_{p(i)}) + (\alpha_i, \alpha_i) = (\rho, \Theta(\alpha_i) + \alpha_i).
$$

Set $m = m_1 + \ldots + m_r$. Expressions (5.1) and (5.4) imply that $t_{i_1}^{-1}x_i \tilde{\theta}(y_i)$ is an element of

$$
(-1)^m q^{(\rho, \Theta(\alpha_i)+\alpha_i)} t_{i_1}^{-m_r} \ldots t_{i_r}^{-m_1} x_{i_1}^{(m_r)} \ldots x_{i_1}^{(m_1)} t_{p(i)}^{-1} x_{p(i)} + UM^+_i.
$$

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Using (1.2) we obtain \( t_i^{-1}t_i^{-mr} \cdots t_i^{-m_1}x_i^{(m_r)} \cdots x_i^{(m_1)}t_i^{-1} \) is an element of

\[
[(\text{ad}_i x_i^{(m_r)}) \cdots (\text{ad}_i x_i^{(m_1)})]t_i^{-1}x_i \in M_i^+ U + M_i^- U
\]

Thus

\[
t_i^{-1}x_i\hat{\theta}(y_i) \in q^{(\rho, \Theta(\alpha_i) + \alpha_i)}\hat{\theta}(y_i) + M_i^+ U + U M_i^+.
\]

\( \square \)

The next result provides a criterion for determining when \( B \mathcal{H}_B \) is \( W \) invariant. In order to prove the result, we need to extend \( U \) to a larger algebra. First, enlarge \( T \) to the group \( \tilde{T} \) generated by \( t_i^{1/m} \) for each \( 1 \leq i \leq n \) and each positive integer \( m \). We can extend the isomorphism \( \tau \) to an isomorphism between \( \sum \mathbb{Q}\alpha_i \) and \( \tilde{T} \). Let \( \tilde{U} \) be the \( C \) algebra generated by \( U \) and \( \tilde{T} \) such that

\[
\tau(\lambda)v_\mu = q^{(\mu, \lambda)}v_\mu \tau(\lambda)
\]

for each \( \tau(\lambda) \in \tilde{T} \) and weight vector \( v_\mu \in U \) of weight \( \mu \).

Note that each finite-dimensional simple \( U \)-module \( L(\lambda) \) easily becomes a \( \tilde{U} \)-module as follows. Recall that \( L(\lambda) \) can be written as a direct sum of weight spaces. Let \( v \) be a weight vector in \( L(\lambda) \) of weight \( \beta \). Set

\[
\tau(\mu) \cdot v = q^{(\beta, \mu)}v
\]

for all \( \tau(\mu) \in \tilde{T} \). We can similarly extend the action of \( U \) on \( L(\lambda)^* \) to \( \tilde{U} \). It follows that elements of \( R_q[G] \) extend to functions on \( \tilde{U} \). Given an algebra \( B \) in \( \mathcal{B}_\theta \), let \( \tilde{B} \) be the \( C \) subalgebra of \( \tilde{U} \) generated by \( B \) and the group \( \tilde{T}_\Theta = \{ \tau(\lambda) \in \tilde{T} | \Theta(\lambda) = \lambda \} \). Note that \( B \mathcal{H}_B \) is invariant as a left \( \tilde{B} \) and right \( \tilde{B} \) module.

Note that the subalgebra \( \tilde{U}^\circ \) of \( \tilde{U} \) generated by elements of \( \tilde{T} \) is a \( C[P] \) module with the following action:

\[
\sum c_\gamma z^\gamma \cdot \tau(\lambda) = \sum c_\gamma q^{(\gamma, \lambda)}.
\]

**Lemma 5.2** Fix \( j \) such that \( \alpha_j \notin \pi_\Theta \) and let \( \tilde{s}_j \) denote the reflection in \( W \) corresponding to \( \tilde{\alpha}_j \). Assume that for all \( \lambda \in \sum \mathbb{Q}\tilde{\alpha}_i \) such that \( (\lambda, \alpha_j) = 0 \) that

\[
(5.5) \quad \tau(\lambda)(\tau(k\tilde{\alpha}_j) - \tau(-k\tilde{\alpha}_j)) \in \tilde{B}_+ \tilde{U} + \tilde{U} \tilde{B}_+^t
\]

for some nonzero rational number \( k \). Then \( \Upsilon(B \mathcal{H}_B) \) is \( \tilde{s}_j \) invariant.
Proof: Recall that $Q(\Sigma)$ is the root lattice associated to $\Sigma$. Suppose that $\beta \in Q(\Sigma)$ and $(\beta, \alpha_j) \neq 0$. Set $r = (\beta, \alpha_j)/(\alpha_j, \alpha_j)$ and $\lambda = k\beta/r - k\alpha_j$. Then $\beta = (r/k)(\lambda + k\alpha_j)$, $\lambda \in \sum_i Q\alpha_i$, and $(\lambda, \alpha_j) = 0$. Hence $\tilde{s}_j(\beta) = (r/k)(\lambda - k\alpha_j)$.

By (5.5), $\tau(k\beta/r) - \tau(k\tilde{s}_j\beta)/r \in \tilde{B}_+ \tilde{U} + \tilde{U}\tilde{B}_+$. Let $X$ be an element of $B\mathcal{H}_B$ and set $\Upsilon(X) = \sum_\gamma b_\gamma z^\gamma$. Note that $X(a) = \Upsilon(X) \cdot (a)$ for all $a \in \tilde{T}$ and $X(b) = 0$ for all $b \in \tilde{B}_+ \tilde{U} + \tilde{U}\tilde{B}_+$. It follows that

$$\sum_\gamma b_\gamma (\beta, \gamma ) = 0.$$ 

Hence

$$\sum_\gamma b_\gamma z^\gamma (\beta, \gamma ) = \sum_\gamma b_\gamma z^\gamma (\tilde{s}_j, \gamma )/r.$$ 

This forces

$$(5.6) \quad \sum_{\gamma |(\gamma, k\beta/r) = s} b_\gamma = \sum_{\gamma |(\gamma, k\tilde{s}_j\beta)/r = s} b_\gamma$$

for each integer $s$. We can rewrite (5.6) as

$$\sum_{\gamma |(\gamma, \beta) = m} b_\gamma = \sum_{\gamma |(\gamma, \tilde{s}_j\beta) = m} b_\gamma$$

for each integer $m$. Thus we may conclude that

$$(5.7) \quad \sum_\gamma b_\gamma z^\gamma (\beta, \gamma ) = \sum_\gamma b_\gamma z^\gamma (\tilde{s}_j, \gamma )/r = \sum_\gamma b_\gamma z^\gamma (\tilde{s}_j, \gamma )/r$$

for all $\beta$ such that $\beta \in Q(\Sigma)$ and $(\beta, \alpha_j) \neq 0$. Now suppose that $\beta \in Q(\Sigma)$ and $(\beta, \alpha_j) = 0$. Then $\tilde{s}_j \beta = \beta$ and so (5.7) also holds in this case. Since $\sum_\gamma b_\gamma z^\gamma$ is in $C[2\Sigma]$, it follows that $\sum_\gamma b_\gamma z^\gamma$ is $\tilde{s}_j$ invariant. □

Given $B$ in $\mathcal{B}_\theta$, we show how to choose another algebra $B'$ in $\mathcal{B}_\theta$ such that $B'\mathcal{H}_B$ is $W$ invariant. For each $c \in \mathbf{D}(C)$, define the Hopf algebra automorphism $\chi_c$ in $H$ of $U$ as follows. For all $i$ such that $\alpha_i \in \pi - \pi_\theta$, set

$$\chi_c(x_i) = q^{-1/2(\rho, \theta(\alpha_i) - \alpha_i)} c_i^{-1} x_i$$

and

$$\chi_c(y_i) = q^{1/2(\rho, \theta(\alpha_i) - \alpha_i)} c_i y_i.$$

Given $a$ and $b$ in $\mathbf{D}(C)$ and integers $m, r$, set $a^m b^r$ equal to the $n$-tuple in $\mathbf{D}(C)$ with entries $a_i^m b_i^r$. 

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**Theorem 5.3** Let $B = B_{q,s,d}$ be in $B_\theta$. If $B' = \chi_c(B_{q,s',c^2d})$ for some $c \in D(C)$ and $s' \in S(C)$, then $\Upsilon(B' H_B)$ is $W$ invariant.

**Proof:** Recall the polynomial ring $\mathcal{C}[\zeta, \eta]$, its algebraic closure $\mathcal{K}$, and the elements $s(\zeta) \in S(\mathcal{K})$ and $d(\eta) \in D(\mathcal{K})$ defined in Section 3. Given $c \in D(\mathcal{C})$, note that the $\mathcal{C}$ Hopf algebra automorphism $\chi_c$ of $U$ extends to a $\mathcal{K}$ Hopf algebra automorphism of $U_K$ which we also refer to as $\chi_c$. Choose $c \in D(\mathcal{C})$ and $s' \in S(\mathcal{C})$. Set $B = B_{\theta,s(\zeta),d(\eta)}$ and $B' = \chi_c(B_{\theta,s',c^2d(\eta)})$. By Lemma 4.1 and the proof of Theorem 3.4, it is enough to show that $\Upsilon(B' H_B)$ is $W$ invariant.

Recall the definition of $\pi^*$ from Section 2. Fix $j$ such that $\alpha_j \in \pi^*$. Let $\lambda$ be an element in $\sum_i Q \tilde{a}_i$ such that $(\lambda, \alpha_j) = 0$. Since $\lambda = \tilde{\lambda}$, it follows that $\Theta(\lambda) = -\lambda$ and thus $(\lambda, \Theta(\alpha_j)) = 0$. Thus $\tau(\lambda)$ commutes with both $y_j$ and $\tilde{\theta}(y_j)$.

Assume first that $S$ is empty. Note that

$$B_k = y_k t_k + d(\eta_k) \tilde{\theta}(y_k) t_k$$

is in $B$ for $k = j$ and $k = p(j)$. Recall (see proof of Theorem 3.1) that $\Theta(\alpha_j) - \alpha_j = \Theta(\alpha_j) - \alpha_j$. Since $j \leq p(j)$, both $c_p(j)$ and $d(\eta_j)$ are equal to 1. Hence it follows from the definition of $\chi_c$ and (2.1) that $\chi_c(\tilde{\theta}(y_j)) = q^{-1/2(q(\Theta(\alpha_j) - \alpha_j))} \tilde{\theta}(y_j)$. Moreover,

$$B'_j = (c_j y_j t_j + q^{-(p(\Theta(\alpha_j) - \alpha_j))} c_j^2 d(\eta_j) \tilde{\theta}(y_j) t_j)$$

and

$$B'_{p(j)} = (y_{p(j)} t_{p(j)} + q^{-(p(\Theta(\alpha_j) - \alpha_j))} c_j^{-1} \tilde{\theta}(y_{p(j)}) t_{p(j)})$$

are elements of $B'$.

By Lemma 5.1, we have

$$B_k t_k^{-1} x_k = (y_k t_k + d(\eta_k) \tilde{\theta}(y_k) t_k) t_k^{-1} x_k$$

(5.8)

$$\in \frac{-(t_k - t_k^{-1})}{(q_k - q_k^{-1})} + t_k^{-1} x_k y_k t_k + q^{-(p(\Theta(\alpha_k) - \alpha_k))} d(\eta_k) t_{p(k)}^{-1} x_{p(k)} \tilde{\theta}(y_{p(k)}) t_k + \mathcal{M}_+^U + U \mathcal{M}_+$$

for $k = j$ and $k = p(j)$.
Consider first the case when \( j = p(j) \). In particular, \( \alpha_j \notin \mathcal{D} \) and thus \( c_j = d(\eta_j) = 1 \). It follows that
\[
B'_j = y_jt_j + q^{-\rho(\theta(\alpha_j) - \alpha_j)} \tilde{\theta}(y_j)t_j.
\]
Furthermore, by (5.8),
\[
B_j t_j^{-1} x_j \in \frac{-\left(t_j - t_j^{-1}\right)}{(q_j - q_j^{-1})} + t_j^{-1} x_j B'_j + \mathcal{M}_+^tU + U\mathcal{M}_+^t.
\]
Since \( \tau(\lambda) \) commutes with \( y_j \) and \( \theta(y_j) \), we obtain
\[
(B_j t_j^{-1} x_j) \tau(\lambda) - \tau(\lambda)(t_j^{-1} x_j B'_j)
\]
\[
\in \left(\frac{-2\left(t_j - t_j^{-1}\right)}{(q_j - q_j^{-1})}\right) \tau(\lambda) + \mathcal{M}_+^tU + U\mathcal{M}_+^t.
\]
Note that \( t_j^{-1/2} \tau(\Theta(\alpha_j))^{-1/2} - 1 \) is \( \tilde{B}_+ \) since \( t_j^{-1/2} \tau(\Theta(\alpha_j))^{-1/2} \in \tilde{T}_\Theta \). Therefore
\[
\tau(\lambda)(t_j^{1/2} \tau(\Theta(\alpha_j))^{1/2} - t_j^{-1/2} \tau(\Theta(\alpha_j))^{1/2})
\]
\[
\in \tilde{B}_+ \tilde{U} + \tilde{U} \tilde{B}_+.
\]
Thus by Lemma 5.2, \( B'_j \mathcal{H}_B \) is \( \tilde{s}_j \) invariant.

Now consider the case when \( j < p(j) \). Note that \( t_k t_{p(k)}^{1} - 1 \) and \( t_k^{-1} t_{p(k)} - 1 \) are both elements of \( (\mathcal{M}T_\Theta)_+ \) for each \( 1 \leq k \leq n \). Hence \( (t_{p(j)} - t_{p(j)}^{-1}) t_{p(j)}^{-1} t_j = (t_j - t_j^{-1}) + (\mathcal{M}T_\Theta)_+ \). Set \( a = c_j^{-1} d(\eta_j) - 1 \) and note that \( a + 1 \) is nonzero since \( d(\eta_j) \) is an indeterminate. Using (5.8), a straightforward computation yields
\[
a B_j t_j^{-1} x_j + B_{p(j)} t_{p(j)}^{-1} x_{p(j)} t_{p(j)} t_j + (a + 1) \frac{(t_j - t_j^{-1})}{(q_j - q_j^{-1})}
\]
\[
\in c_j^{-1} a t_j^{-1} x_j B'_j + t_{p(j)}^{-1} x_{p(j)} B'_{p(j)} t_{p(j)}^{-1} t_j + (\mathcal{M}T_\Theta)_+ U + U(\mathcal{M}T_\Theta)_+.
\]
In particular, (5.9) holds in this case as well and the image of \( B'_j \mathcal{H}_B \) is \( s_{\alpha_j} \) invariant when \( j \) is strictly less than \( p(j) \).

Now assume that \( \Theta(\alpha_i) = -\alpha_i \) and \( \alpha_i \in \mathcal{S} \). Then \( B_+ \) contains
\[
C_i = y_i t_i + q_i^{-2} x_i + s(\xi_i)(t_i - 1)
\]
and $B'_+$ contains
\[ C'_i = y_i t_i + x_i + s'_i (t_i - 1). \]

Furthermore
\[ (s(\zeta_i) - q_i^{-1} s'_i) (t_i^{1/2} - t_i^{-1/2}) \tau(\lambda) = C'_i t_i^{-1/2} \tau(\lambda) - q_i^{-1} \tau(\lambda) t_i^{-1/2} C'_i. \]

Note that $(s(\zeta_i) - q_i^{-1} s'_i)$ is nonzero since $s(\zeta_i)$ is an indeterminate while $s'_i \in \mathcal{C}$. It follows that \( \tau(\lambda) (t_i^{1/2} - t_i^{-1/2}) \in \tilde{B}_+ \tilde{U} + \tilde{U} \tilde{B}_+ \). Another application of Lemma 5.2 yields that the image of $B' \mathcal{H}_B$ under $\Upsilon$ is $\tilde{s}_j$ invariant in this case.

We have shown that the image of $B' \mathcal{H}_B$ under $\Upsilon$ is $\tilde{s}_j$ invariant for all reflections $\tilde{s}_j$ in the set $\{ \tilde{s}_j | \alpha_j \in \pi^* \}$. Note that the set $\{ \tilde{\alpha}_j | \alpha_j \in \pi^* \}$ is equal to the set of simple roots of $\Sigma$. The theorem now follows from the fact $W$ is generated by the reflections in $\{ \tilde{s}_j | \alpha_j \in \pi^* \}$. \( \square \)

For each $\lambda \in P^+_\Theta$, set
\[ m_\lambda = \sum_{w \in W} z^{2w_\lambda}. \]

It is well known that the set
\[ \{ m_\lambda | \lambda \in P^+_\Theta \} \]
forms a basis for $\mathcal{C}[2\Sigma]^W$. Now consider a subset $\{ \varphi_\lambda | \lambda \in P^+_\Theta \}$ of $\mathcal{C}[2\Sigma]$ which satisfies the following conditions. For each $\lambda \in P^+_\Theta$

\begin{equation}
\varphi_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda, \mu} m_\mu \quad \text{for some } a_{\lambda, \mu} \in \mathcal{C}.
\end{equation}

\begin{equation}
\varphi_\lambda \in \mathcal{C}[2\Sigma]^W
\end{equation}

Then the set $\{ \varphi_\lambda | \lambda \in P^+_\Theta \}$ forms a basis for $\mathcal{C}[2\Sigma]^W$.

Suppose that $B, B' \in \mathcal{B}_\Theta$ are chosen as in Theorem 5.3. The next result shows that the image of the zonal spherical functions associated to the pair $B, B'$ in $\mathcal{C}[2\Sigma]$ satisfies (5.10) and (5.11).

**Corollary 5.4** Let $B = B_{\theta,s,d}$ be in $\mathcal{B}_\Theta$ and suppose that $B' = \chi_c(B_{\theta,s',c^2 d})$ for some $c \in \mathcal{D}$ and $s' \in \mathcal{D}$. Then the set $\{ \varphi^\lambda_{B,B'} | \lambda \in P^+_\Theta \}$ satisfies condition (5.10) and (5.11). Moreover, $\Upsilon$ defines an isomorphism of $B' \mathcal{H}_B$ onto $\mathcal{C}[2\Sigma]^W$. 

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Proof: By Theorem 4.2, $\varphi_{B,B'}^\lambda$ is an element of $C[2\Sigma]$ for each $\lambda \in P_\Theta^\dagger$. On the other hand, Theorem 5.3 implies that each $\varphi_{B,B'}^\lambda$ is $W$ invariant and hence is an element of $C[2\Sigma]^W$. It follows that the set $\{\varphi_{B,B'}^\lambda | \lambda \in P_\Theta^\dagger\}$ satisfies condition (5.11).

Fix $\lambda \in P_\Theta^\dagger$. Since $\varphi_{B,B'}^\lambda$ is in $C[2\Sigma]^W$, we can write $\varphi_{B,B'}^\lambda$ as a linear combination of the $m_\gamma, \gamma \in P_\Theta^\dagger$:

$$\varphi_{B,B'}^\lambda = \sum_{\gamma \in P_\Theta^\dagger} r_\gamma m_\gamma.$$  

By Lemma 4.1, we can also write

$$(5.12) \quad \varphi_{B,B'}^\lambda = z^\lambda + \sum_{\beta < Level} c_\beta z^\beta$$

for some choice of scalars $c_\beta$ where $\beta$ runs over weights of $L(\lambda)$ contained in $P(2\Sigma) - \{\lambda\}$. Hence, if $r_\gamma \neq 0$, then $\gamma$ is a weight of $L(\lambda)$ contained in $P_\Theta^\dagger$. Thus $r_\gamma \neq 0$ implies that $\gamma = \lambda$ or $\gamma < _r \lambda$. Since $z^\lambda$ appears with coefficient 1 in (5.12), it follows that $r_\lambda = 1$. Therefore, $\varphi_{B,B'}^\lambda$ satisfies condition (5.10). The last assertion now follows from the discussion preceding the corollary and Theorem 4.2. $\Box$

6 Zonal Spherical Families

We say that a function $\lambda \rightarrow \varphi_\lambda$ from $P_\Theta^\dagger$ to $C[2\Sigma]$ is a zonal spherical family associated to $B_\Theta$ if there exists $B$ and $B'$ in $B_\Theta$ such that $\varphi_\lambda = \varphi_{B,B'}^\lambda$ for all $\lambda \in P_\Theta^\dagger$. For many practical purposes, we may think of a zonal spherical family as a set $\{\varphi_\lambda | \lambda \in P_\Theta^\dagger\}$ indexed by $P_\Theta^\dagger$. Let $F_\Theta$ denote the set of zonal spherical families associated to $B_\Theta$. The group $H \times H$ acts on $B_\Theta \times B_\Theta$ in the obvious way:

$$(\chi_1, \chi_2) \cdot (B, B') = (\chi_1(B), \chi_2(B')).$$

In this section, we study the relationship between orbits of $B_\Theta \times B_\Theta$ under the action of $H \times H$ and orbits of $F_\Theta$ under a subgroup of $\text{Hom}(Q(\pi), C^\times)$. This allows us to attach a unique $W$ invariant zonal spherical family (up to possible sign) to each orbit of $B_\Theta \times B_\Theta$.

Recall the definition of the lattice $N$ generated by $2\Sigma$ and the set $\{\tilde{\alpha}_i | \alpha_i \in S\}$ given in (3.6). Consider a Hopf algebra automorphism $\chi$ of $U$ in $H$. Note
that $\chi$ induces a group homomorphism $\bar{\chi}$ from $Q(\pi)$ to the nonzero elements of $C$ such that
\[ \chi(u) = \bar{\chi}(\text{wt } u)u \]
for each weight vector $u \in U$. Since $N$ is a sublattice of $Q(\pi)$, $\bar{\chi}$ restricts to an element of $\text{Hom}(N, C^\times)$.

**Lemma 6.1** The map $\Psi : H \times H \to \text{Hom}(N, C^\times)$ defined by $(\chi_1, \chi_2) \mapsto \bar{\chi}_1^{-1} \bar{\chi}_2$ is a surjective homomorphism.

**Proof:** Note first that $H$ is an abelian group. Hence the map $(\chi_1, \chi_2) \mapsto \bar{\chi}_1^{-1} \bar{\chi}_2$ defines a group homomorphism of $H \times H$ onto $H$. Thus it is sufficient to show that the map $\chi \mapsto \bar{\chi}$ defines a surjective homomorphism from $H$ onto $\text{Hom}(N, C^\times)$.

Note that $N$ is just the free abelian group with generating set $(\pi - \pi_\Theta - S) \cup \{\tilde{\alpha}_i/2|\alpha_i \in S\}$. Set $\frac{1}{2}N = \{\frac{1}{2}\beta|\beta \in N\}$. Let $\psi$ be an element of $\text{Hom}(N, C^\times)$. Taking square roots allows us to extend $\psi$ to an element of $\text{Hom}(\frac{1}{2}N, C^\times)$. We may further extend $\psi$ to an element of $\text{Hom}(\frac{1}{2}N + Q(\frac{1}{2}\pi)^\Theta, C^\times)$ by insisting that $\hat{\psi}(\gamma) = 0$ for all $\gamma \in Q(\frac{1}{2}\pi)^\Theta$. Since $Q(\pi) \subset \frac{1}{2}N + Q(\frac{1}{2}\pi)^\Theta$, it follows that $\hat{\psi}$ restricts to an element of $\text{Hom}(Q(\pi), C^\times)$ which is zero on $Q(\pi)^\Theta$. In particular, $\hat{\psi} = \bar{\chi}$ where $\chi \in H$ is chosen so that $\chi(u) = \hat{\psi}(\text{wt } u)u$ for all weight vectors $u \in U$. \(\square\)

Note that the action of $H \times H$ on $B_\theta \times B_\theta$ induces an action of $H \times H$ on $F_\theta$. In particular, given $B$ and $B'$ in $B_\theta$ and $(\chi_1, \chi_2) \in H \times H$, set
\[ (\chi_1, \chi_2) \cdot \varphi_{B,B'}^\lambda = \varphi_{\chi_1(B), \chi_2(B')}^\lambda \]
for all $\lambda \in P_\Theta^+$.

The group $\text{Hom}(P(\Sigma), C^\times)$ acts on $C[2\Sigma]$ via $g \cdot z^\beta = g(\beta)z^\beta$ for all $g \in \text{Hom}(P(\Sigma), C^\times)$ and $\beta \in P(2\Sigma)$. This restricts to an action of the group $\text{Hom}(N, C^\times)$ on the group ring $C[N]$ generated by $z^\beta$, $\beta \in N$.

Let $\{\varphi_\lambda|\lambda \in P_\Theta^+\}$ be a zonal spherical family in $F_\theta$. By Theorem 3.6 and its proof, $\varphi_\lambda$ is the unique element of the vector space $C\varphi_\lambda$ such that
\[ \varphi_\lambda \in z^\lambda + \sum_{\beta <_\lambda} Cz^\beta \text{ and } \varphi_\lambda \in z^\lambda C[N] \]
for each $\lambda \in P_\Theta^+$. 

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Lemma 6.2 For all $B, B' \in \mathcal{B}_\theta$, $(\chi_1, \chi_2) \in H \times H$, and $\lambda \in P^+_\Theta$

$$\varphi_{B,B'}^\lambda = z^\lambda \hat{\chi}_1^{-1} \chi_2 \cdot (z^{-\lambda} \varphi_{B,B'}^\lambda).$$

Proof: Let $(\chi_1, \chi_2) \in H \times H$. Note that there exists a positive integer $m$ such that $P(\pi)$ is a subset of $\sum_i Z(\alpha_i/m)$. Hence for each $i$, the map $\hat{\chi}_i$ can be extended to a (not necessarily unique) homomorphism, which we also call $\chi_i$, from $\sum_j Z(\alpha_j/m)$ to $C^\times$.

Consider $\lambda \in P^+(\pi)$ and fix $i \in \{1, 2\}$. The map $\chi_i$ induces linear transformations on $L(\lambda)$ and $L(\lambda)^*$, both denoted by $\hat{\chi}_i$, such that $\hat{\chi}_i(v) = \hat{\chi}_i(wt v)v$ for each weight vector $v$ in $L(\lambda)$ or $L(\lambda)^*$. Suppose that $v \in L(\lambda)$ and $v' \in L(\lambda)^*$ are weight vectors. Note that

$$(6.2) \quad \chi_i(u) \hat{\chi}_i(v) = \hat{\chi}_i(uv) \quad \text{and} \quad \hat{\chi}_i^{-1}(v') \chi_i(u) = \hat{\chi}_i^{-1}(v'u)$$

for all $u \in U$. It follows that

$$c^\lambda_{\chi_i^{-1}(v'), v} = \chi_i(wt v')(c^\lambda_{v', v}) \quad \text{and} \quad c^\lambda_{v', \chi_i(v)} = \chi_i(wt v)(c^\lambda_{v', v}).$$

Thus

$$(6.3) \quad \Upsilon(c^\lambda_{\chi_i^{-1}(v'), \chi_2(v)}) = \chi_1^{-1}(wt v') \chi_2(wt v) \Upsilon(c^\lambda_{v', v}) = (\chi_1^{-1} \chi_2) \cdot (\Upsilon(c^\lambda_{v', v})).$$

Let $\xi$ denote a $B'$ invariant vector of $L(\lambda)$ and let $\xi^*$ denote a $B$ invariant vector of $L(\lambda)^*$. Assertion (6.2) ensures that $\hat{\chi}_2(\xi)$ is a nonzero $\chi_2(B')$ invariant element of $L(\lambda)$ and $\hat{\chi}_1^{-1}(\xi^*)$ is a $\chi_1(B)$ invariant vector of $L(\lambda)^*$. Thus $c^\lambda_{\hat{\chi}_1(B), \chi_2(B')}(\varphi_{\chi_1(B), \chi_2(B')}(\varphi_{B,B'}^\lambda) = c^\lambda_{\chi_1^{-1}(\xi^*), \hat{\chi}_2(\xi)}$ up to a nonzero scalar. It follows from assertion (6.3) that $\varphi_{\chi_1(B), \chi_2(B')}(\varphi_{B,B'}^\lambda) = (\chi_1^{-1} \chi_2) \cdot (\varphi_{B,B'}^\lambda)$ up to a nonzero scalar.

Now $\hat{\chi}_1^{-1} \chi_2(\varphi_{B,B'}^\lambda)$ is just a nonzero scalar multiple of $z^\lambda(\hat{\chi}_1^{-1} \chi_2) \cdot (z^{-\lambda} \varphi_{B,B'}^\lambda)$. The lemma follows from the fact that $z^\lambda(\hat{\chi}_1^{-1} \chi_2) \cdot (z^{-\lambda} \varphi_{B,B'}^\lambda)$ satisfies the conditions of (6.1). 

It follows from Lemma 6.2, that the kernel of the action of $H \times H$ on $\mathcal{F}_\theta$ is contained in $\text{Ker}(\Psi)$ where $\Psi$ is the homomorphism of Lemma 6.1. Thus the action of $H \times H$ on $\mathcal{F}_\theta$ induces an action of $\text{Hom}(N, C^\times)$ on $\mathcal{F}_\theta$. Furthermore, this action is given by

$$g \cdot \varphi_\lambda = z^\lambda g(z^{-\lambda} \varphi_\lambda)$$

for all $\lambda \in P^+_\Theta$, $g \in \text{Hom}(N, C^\times)$, and $\{\varphi_\lambda | \lambda \in P^+_\Theta\} \in \mathcal{F}_\theta$. The following result is an immediate consequence of Lemma 6.1 and Lemma 6.2.
Theorem 6.3 The map

\[(B, B') \rightarrow \{\varphi_{B, B'}^\lambda \mid \lambda \in P_\theta^+\}\]

from \(B \times B \) to \(F_\theta\) is a \(H \times H\) equivariant map which induces a surjection from the set of orbits of \(B \times B \) under the action of \(H \times H\) onto the set of orbits of zonal spherical families under the action of \(\text{Hom}(N, C^\times)\).

A family \(\{\varphi_\lambda \mid \lambda \in P_\theta^+\} \in F_\theta\) is called \(W\) invariant provided each \(\varphi_\lambda \in C[2\Sigma]_W\). We next determine which elements in \(\text{Hom}(N, C^\times)\) preserve \(W\) invariance. Recall that \(Q(2\Sigma)\) is the root lattice associated to \(2\Sigma\).

Lemma 6.4 Let \(g \in \text{Hom}(N, C^\times)\) and let \(\{\varphi_\lambda \mid \lambda \in P_\theta^+\}\) be a \(W\) invariant zonal spherical family. Then \(g \cdot \varphi_\lambda\) is \(W\) invariant for all \(\lambda \in P_\theta^+\) if and only if \(g\) acts trivially on \(Q(2\Sigma)\).

Proof: Suppose that \(\Sigma\) is of type \(BC_r\). Then the weight lattice of \(\Sigma\) is equal to the weight lattice of the subroot system \(\Sigma_1\) in \(\Sigma\) of type \(B_r\). In particular, if the lemma holds when \(\Sigma\) is of type \(B_r\), then it follows for \(\Sigma\) of type \(BC_r\). Hence we can reduce to the case where \(\Sigma\) is a reduced root system.

Consider \(g \in \text{Hom}(N, C^\times)\). Note that \(N\) is a free \(\mathbb{Z}\) module. Furthermore, there exists a positive integer \(m\) such that \(P(\Sigma)\) is a sublattice of \(\{\beta/m \mid \beta \in N\}\). Thus we may extend \(g\) to a (not necessarily unique) element of \(\text{Hom}(P(\Sigma), C^\times)\). Hence, it is sufficient to prove the assertions of the theorem with \(\text{Hom}(N, C^\times)\) replaced by \(\text{Hom}(P(\Sigma), C^\times)\).

Recall that \(\{\varphi_\lambda \mid \lambda \in P_\theta^+\}\) is a basis for \(C[2\Sigma]_W\). Thus, the first assertion of the lemma is equivalent to \(g\) restricts to an action on \(C[2\Sigma]_W\) if and only if \(g\) acts trivially on \(Q(2\Sigma)\).

Assume first that \(g\) sends \(C[2\Sigma]_W\) to itself. Given \(X = \sum_\beta a_\beta z^\beta\), set \(\text{Supp}(X) = \{z^\beta \mid a_\beta \neq 0\}\). Let \(\delta\) denote the half sum of the positive restricted roots in \(\Sigma\). By [H, Corollary 10.2], \(2(\delta, \alpha_i) = (\tilde{\alpha}_i, \alpha_i)\) and \(\tilde{s}_i(\delta) = \delta - \tilde{\alpha}_i\) for each \(\tilde{\alpha}_i \in \Sigma\). It follows that \(2\delta \in P(2\Sigma)\). By say [H, Theorem 10.3(e)], \(w\delta = \delta\) if and only if \(w = 1\) and thus \(w\delta = w'\delta\) if and only if \(w = w'\). Thus for each \(\tilde{\alpha}_i\) such that \(\alpha_i \in \Sigma\), we can write

\[
m_\delta = z^{2\delta} + z^{2\delta - 2\tilde{\alpha}_i} + \sum_{\beta \neq 2\delta, \beta \neq 2\delta - 2\tilde{\alpha}_i} a_\beta z^\beta
\]

for some scalars \(a_\beta\).
By (6.4), we have
\begin{equation}
(6.5) \quad g \cdot m_\delta = g(2\delta)z^\delta + g(2\delta - 2\tilde{\alpha}_i)z^{\delta - \tilde{\alpha}_i} + \sum_{\beta \neq \delta, \beta \neq \delta - \tilde{\alpha}_i} g(\beta)a_\beta z^\beta.
\end{equation}

Note that \(\text{Supp}(g \cdot m_\delta) = \text{Supp}(m_\delta)\) and \(g \cdot m_\delta \in \mathcal{C}[2\Sigma]^W\). Recall that the \(m_\beta\), for \(\beta \in P_\Theta^+\), form a basis for \(\mathcal{C}[2\Sigma]^W\). Furthermore, by [H, Section 13.2, Lemma A], the sets \(\text{Supp}(m_\beta)\) are pairwise disjoint. Hence (6.5) implies that \(g \cdot m_\delta\) is equal to \(g(2\delta)m_\delta\). It further follows from (6.5) that \(g(2\delta) = g(2\delta - 2\tilde{\alpha}_i) = g(2\delta)g(2\tilde{\alpha}_i)^{-1}\). Therefore \(g(2\tilde{\alpha}_i) = 1\) for all \(i\). Hence \(g\) acts trivially on \(Q(2\Sigma)\).

Now assume that \(g \in \text{Hom}(P(\Sigma), \mathcal{C}^\times)\) such that \(g(2\tilde{\alpha}_i) = 1\) for all \(\alpha_i \in \pi - \pi_\Theta\). Note that the set \(2\Sigma\) is a root system with respect to the Cartan inner product (of the same type as \(\Sigma\)) with set of positive simple roots equal to \(\{2\tilde{\alpha}_i|\alpha_i \in \pi - \pi_\Theta\}\). Since we are assuming that \(\Sigma\) is reduced, we can find a semisimple Lie algebra \(L\) with root system equal to \(2\Sigma\). It is well known that the characters of the finite dimensional simple modules associated to \(L\) are parametrized by the dominant integral weights \(P_\Theta^+\) and form a basis for \(\mathcal{C}[2\Sigma]^W\). Let \(X_\lambda\) be the character associated to the finite dimensional simple \(L\) module \(V(\lambda)\) of highest weight \(\lambda\) for \(\lambda \in P_\Theta^+\). Now the root vectors of \(L\) have weight \(\beta\), for \(\beta \in 2\Sigma\). Set \(\lambda + Q(2\Sigma) = \{\lambda + \beta|\beta \in Q(2\Sigma)\}\). It follows that the weights of \(V(\lambda)\) are contained in the set \(\lambda + Q(2\Sigma)\). In particular, \(\text{Supp}(X_\lambda) \subset \lambda + Q(2\Sigma)\). Since \(g\) acts as the identity on \(Q(2\Sigma)\), it follows that \(g \cdot z^{\lambda - \beta} = g(\lambda)z^{\lambda - \beta}\) for all \(\beta \in Q(2\Sigma)\). Therefore \(g \cdot X_\lambda\) is equal to \(g(\lambda)X_\lambda\) for each \(\lambda \in P_\Theta^+\). Thus \(g\) sends \(\mathcal{C}[2\Sigma]^W\) to itself. \(\Box\)

Set \(G_\theta\) equal to the subgroup of \(\text{Hom}(N, \mathcal{C}^\times)\) which acts trivially on \(Q(2\Sigma)\). In particular, \(G_\theta\) is isomorphic to \(\text{Hom}(N/Q(2\Sigma), \mathcal{C}^\times)\). Note that \(N/Q(2\Sigma)\) is isomorphic to the direct product of \(|S|\) copies of \(\mathbb{Z}_2\). Thus \(G_\theta \cong \text{Hom}(N/Q(2\Sigma), \{1, -1\}) \cong \mathbb{Z}_2^{[S]}\).

Recall that \(\{B_{\theta,s,d}|s \in \mathcal{S}(C)\} and d \in \mathcal{D}(C)\}\) is a complete set of distinct representatives for the orbits of \(B_\theta\) under the action of \(H\). Set \(O(B_{\theta,s,d}, B_{\theta,s',d'})\) equal to the orbit containing \((B_{\theta,s,d}, B_{\theta,s',d'})\) under the action of \(H \times H\). Note that \(\{O(B_{\theta,s,d}, B_{\theta,s',d'})|s, s' \in \mathcal{S}(C)\) and \(d, d' \in \mathcal{D}(C)\}\) is a complete list of the distinct orbits of \(B_\theta \times B_\theta\) under this action. Given \(s, s' \in \mathcal{S}(C)\) and \(d, d' \in \mathcal{D}(C)\), set
\[\varphi_{s,s',d,d'}^\lambda = \varphi_{B,B'}^\lambda\]
where \( B = B_{\theta,s,d} \), \( B' = \chi_c(B_{\theta,s',d'}) \), and \( c \in D(C) \) is chosen to satisfy \( c^2 = d^{-1}d' \).

**Theorem 6.5** The set \( \{ \varphi_\lambda | \lambda \in P_\Theta^+ \} \) is a \( W \) invariant zonal spherical family associated to an element of \( O(B_{\theta,s,d}, B_{\theta,s',d'}) \) if and only if there exists \( g \in G_\theta \) such that

\[
\varphi_\lambda = g \cdot \varphi_{s,s',d,d'}^\lambda
\]

for all \( \lambda \in P_\Theta^+ \). In particular, if \( S \) is empty, then \( g \) in (6.6) is just the identity map.

**Proof:** Let \( B = B_{\theta,s,d} \) and \( B' = \chi_c(B_{\theta,s',d'}) \), where \( c \) is the element of \( D \) which satisfies \( c^2 = d^{-1}d' \) chosen above. By Corollary 5.4, \( \{ \varphi_{s,s',d,d'}^\lambda | \lambda \in P_\Theta^+ \} \) is a \( W \) invariant zonal spherical family associated to \( (B, B') \). Let \( \{ \varphi_\lambda | \lambda \in P_\Theta^+ \} \) be another zonal spherical family associated to an element in \( O_1 \). By Theorem 6.3, there exists \( g \in \text{Hom}(N, C^\times) \) such that

\[
\{ \varphi_\lambda | \lambda \in P_\Theta^+ \} = g \cdot \{ \varphi_{s,s',d,d'}^\lambda | \lambda \in P_\Theta^+ \}.
\]

The result now follows from Lemma 6.4. \( \square \)

Consider the case when \( g, \theta \) is an irreducible pair. There are three possibilities:

(i) \( S \) and \( D \) are both empty.

(ii) \( S \) is empty and \( D \) has exactly one element.

(iii) \( S \) has exactly one element and \( D \) is empty.

Note that when \( S \) is empty, \( G_\theta \) is the trivial group. In the first case, \( B_\theta \) is a single orbit under the action of \( H \). Thus Theorem 6.5 implies that we can associate a unique \( W \) invariant zonal spherical family to \( B_\theta \) when both \( S \) and \( D \) are empty. Suppose \( D = \{ \alpha_i \} \) as in case (ii). Then the set \( D(C) \) is a set of \( n \)-tuples which vary only in the entry \( d_i \). Thus, there is a unique two-parameter set of \( W \) invariant zonal spherical families associated to \( B_\theta \). If \( S \) and \( D \) satisfy condition (iii) then \( G_\theta \) is a cyclic group of order 2. In this case, we can associate exactly two two-parameter sets of \( W \) invariant zonal spherical families to \( B_\theta \).
It should be noted that the existence of “two-parameter” families of $W$ zonal spherical functions associated to a quantum symmetric pair has appeared in the literature in a few special cases. Indeed, two-parameter families of zonal spherical functions are studied in [K1] when $g = \mathfrak{sl}_2$ and $g, \theta$ is of type $A_1$ and in [DK] when $g, \theta$ is of type $A_{IV}$. (Types $A_1$ and $A_{IV}$ are explained in the next section.)

7 Irreducible Symmetric Pairs

By [A, 2.5 and 5.1], the pair $g, \theta$ is irreducible if and only if

(7.1) $g$ is simple or

(7.2) $g = g_1 \oplus g_2$ where both $g_1$ and $g_2$ are simple. Here $g_1$ is generated by $\{e_i, f_i, h_i|1 \leq i \leq m\}$ and $g_1$ is generated by $\{e_{i+m}, f_{i+m}, h_{i+m}|1 \leq i \leq m\}$. Moreover,

$$e_i \rightarrow e_{i+m}, f_i \rightarrow f_{i+m}, h_i \rightarrow h_{i+m}$$

defines an isomorphism from $g_1$ to $g_2$, and $\theta$ is defined by

$$\theta(e_i) = f_{i+m}, \quad \theta(f_i) = e_{i+m}, \quad \theta(h_i) = h_{i+m}.$$ 

Assume for the moment that $g$ and $\theta$ are as described in (7.2). In this case, both $\mathcal{D}$ and $\mathcal{S}$ are empty sets. The group $T_\theta$ is generated by $t_i t_i^{-1}$, $1 \leq i \leq m$ and the algebra $B_{\theta,s,d}$ is generated by $T_\theta$ together with the elements $B_i = y_i t_i + t_i^{-1} x_{i+m} t_i$ and $B_{i+m} = y_{i+m} t_{i+m} + t_{i+m}^{-1} x_{i} t_i$ for $1 \leq i \leq m$.

Now consider the case when $g$ is simple. We use the classification in [A] (see also [He, Chapter X, Section F]) to give a complete list of the subalgebras $B_{\theta,s,d}$ associated to irreducible pairs $g, \theta$ for $g$ simple. In each case, we describe $\pi_\theta$, $\mathcal{S}$, $\mathcal{D}$, the permutation $p$ when $p$ is not the identity, and $B_i, \alpha_i \notin \pi_\theta$. This information is enough to determine all the generators of $B_{\theta,s,d}$ since $\mathcal{M}$ is just the quantized enveloping algebra associated to the root system $\pi_\theta$ considered as a subset of $\pi$ and $T_\theta$ is generated by the sets $\{t_i|\alpha_i \in \pi_\theta\}$ and $\{t_i t_{p(i)}^{-1}|\alpha_i \notin \pi_\theta\}$.

We use below the numbering of the vertices of the Dynkin diagram given in [H, Section 11.4]. It should be noted that when $g$ is of type $D_n$, the roots $\alpha_n$ and $\alpha_{n-1}$ can be reordered with $n$ and $n - 1$ interchanged.
**Type Al** $g$ is of type $A_n$, $\pi_\Theta$ is the empty set, $S$ and $D$ are both empty, $B_i = y_i t_i + q_i^{-2} x_i$ for each $1 \leq i \leq n$.

**Type AII** $g$ is of type $A_n$, where $n = 2m + 1$ is odd and $n \geq 3$, $\pi_\Theta = \{\alpha_{2j+1}|0 \leq j \leq m\}$, $S$ and $D$ are both empty, $B_i = y_i t_i + [(ad_r x_{i-1} x_{i+1})t_i^{-1} x_i] t_i$, for $i = 2j, 1 \leq j \leq m$.

**Type AIII** Case 1: $g$ is of type $A_n$, $r$ is an integer such that $2 \leq r \leq n/2$, $\pi_\Theta = \{\alpha_j| r + 1 \leq j \leq n - r\}$, $S$ is empty, $D = \{\alpha_r\}$, $p$ sends $i$ to $n - i + 1$ for each $1 \leq i \leq n$,

$$B_i = y_i t_i + q_i^{-2} x_p(i) t_i^{-1} x_i \quad \text{for} \quad 1 \leq i \leq r - 1 \quad \text{and} \quad n - r + 2 \leq i \leq n,$$

$$B_r = y_r t_r + d_r [(ad_r x_{r+1})(ad_r x_{r+2}) \cdots (ad_r x_{n-r})(ad_r x_{n-r+1})t_r^{-1} x_p(r)] t_r,$$

and

$$B_p(r) = y_p(r) t_p(r) + (-1)^{n-2r} [(ad_r x_{n-r})(ad_r x_{n-r+1}) \cdots (ad_r x_{r+1}) t_r^{-1} x_p(r)].$$

Case 2: $g$ is of type $A_n$ where $n = 2m + 1$, $\pi_\Theta$ is empty, $D$ is empty, $S = \{\alpha_{m+1}\}$, $p$ sends $i$ to $n - i + 1$ for $1 \leq i \leq n$, $B_i = y_i t_i + q_i^{-2} x_p(i) t_i^{-1} x_i$ for $1 \leq i \leq m$ and $m + 2 \leq i \leq n$, and $B_{m+1} = y_{m+1} t_{m+1} + q_{m+1} x_{m+1} + s_{m+1} t_{m+1}$.

**Type AIV** Same as type AIII, case 1, with $r = 1$.

**Type BI** $g$ is of type $B_n$, $r$ is an integer such that $2 \leq r \leq n$, $\pi_\Theta = \{\alpha_i|r+1 \leq i \leq n\}$, $S$ and $D$ are empty, $B_i = y_i t_i + q_i^{-2} x_i$ for $1 \leq i \leq r - 1$ and

$$B_r = y_r t_r + [(ad_r x_{r+1} \cdots x_{n-1} x_n^{(2)} x_{n-1} \cdots x_{r+1}) t_r^{-1} x_r] t_r^{-1}.$$

**Type BII** Same as type BI, Case 1, only $r = 1$.

**Type CI** $g$ is of type $C_n$, $\pi_\Theta$ is empty, $D$ is empty, $S = \{\alpha_n\}$, $B_i = y_i t_i + q_i^{-2} x_i$ for $1 \leq i \leq n - 1$, and $B_n = y_n t_n + q_n^{-2} x_n + s_n t_n$.

**Type CII** Case 1: $g$ is of type $C_n$, $r$ is an even integer such that $1 \leq r \leq (n - 1)$, $\pi_\Theta = \{\alpha_{2j-1}|1 \leq j \leq r/2\} \cup \{\alpha_j| r + 1 \leq j \leq n\}$, $S$ and $D$ are both empty,

$$B_i = y_i t_i + [(ad_r x_{i-1} x_{i+1}) t_i^{-1} x_i] t_i$$

for $i = 2j, 1 \leq j \leq (r - 2)/2$ and

$$B_r = y_r t_r + [(ad_r x_{r-1} x_{r+1} \cdots x_{n-3} x_{n-1} x_{n-3} \cdots x_{r+1}) t_r^{-1} x_r] t_r.$$
Case 2: \( g \) is of type \( C_n \) where \( n \) is even, \( \pi_\Theta = \{\alpha_{2j-1} | 1 \leq j \leq n/2\} \), \( S \) and \( D \) are both empty, \( B_i = y_i t_i + (\text{ad}_r x_{i-1}x_{i+1})t_i^{-1}x_i \) for \( i = 2j, 1 \leq j \leq (n-2)/2 \) and \( B_n = y_n t_n + [(\text{ad}_r x_{n-1}^{(2)})t_i^{-1}x_n]t_n \).

Type DI Case 1: \( g \) is of type \( D_n \), \( r \) is an integer such that \( 2 \leq r \leq n-2 \), \( \pi_\Theta = \{\alpha_i | r + 1 \leq i \leq n\} \), \( S \) and \( D \) are both empty, \( B_i = y_i t_i + q_i^{-2}x_i \) for \( 1 \leq i \leq r - 1 \), and

\[ B_r = y_r t_r + [(\text{ad}_r x_{r+1} \cdots x_{n-2}x_{n-1}x_{n-2} \cdots x_{r+1})t_r^{-1}x_r]t_r. \]

Case 2: We assume \( n \geq 4 \) (the case when \( n = 3 \) is the same as type AI.) \( g \) is of type \( D_n \), \( \pi_\Theta \) is empty, \( S \) and \( D \) are both empty, \( p(i) = i \) for \( 1 \leq i \leq n-2 \), \( p(n-1) = n \), and \( p(n) = n-1 \) and \( B_i = y_i t_i + t_i^{-1}x_{p(i)} t_i \) for \( 1 \leq i \leq n \).

Case 3: \( g \) is of type \( D_n \), \( \pi_\Theta \) is empty, \( S \) and \( D \) are both empty, and \( B_i = y_i t_i + q_i^{-2}x_i \) for \( 1 \leq i \leq n \).

Type DII This is the same as DI, Case 1, with \( r = 1 \).

Type DIII Case 1: \( g \) is of type \( D_n \) where \( n \) is even, \( \pi_\Theta = \{\alpha_{2j-1} | 1 \leq i \leq n/2\} \), \( D \) is empty, \( S = \{\alpha_n\} \), \( B_i = y_i t_i + (\text{ad}_r x_{i-1}x_{i+1})t_i^{-1}x_i \) for \( i = 2j, 1 \leq j \leq (n-2)/2 \), and \( B_n = y_n t_n + q_n^{-2}x_n + s_n t_n \).

Case 2: \( g \) is of type \( D_n \), \( n \) is odd, \( \pi_\Theta = \{\alpha_{2j-1} | 1 \leq i \leq (n-1)/2\} \), \( S \) is empty, \( D = \{\alpha_{n-1}\} \), \( B_i = y_i t_i + (\text{ad}_r x_{i-1}x_{i+1})t_i^{-1}x_i \) for \( i = 2j, 1 \leq j \leq (n-3)/2 \), \( p(j) = j \) for \( 1 \leq j \leq n-2 \), \( p(n) = n - 1 \), and \( p(n-1) = p(n) \), \( B_{n-1} = y_{n-1} t_{n-1} + d_{n-1}[(\text{ad}_r x_{n-2})t_{n-1}^{-1}x_{n-1}]t_n \) and \( B_n = y_n t_n - [(\text{ad}_r x_{n-2})t_{n-1}^{-1}x_{n-1}]t_n \).

Type E1, EV, EVIII \( g \) is of type E6,E7, E8 respectively, \( \pi_\Theta \) is empty, \( S \) and \( D \) are both empty, and \( B_i = y_i t_i + q_i^{-2}x_i \) for all \( \alpha_i \in \pi \).

Type EII \( g \) is of type E6, \( \pi_\Theta \) is empty, both \( S \) and \( D \) are empty, \( p(1) = 6, p(3) = 5, p(4) = 4, p(2) = 2, p(5) = 3 \), and \( p(6) = 1 \), and \( B_i = y_i t_i + q_i^{-2}x_{p(i)} t_{p(i)}^{-1} t_i \) for \( 1 \leq i \leq 6 \).

Type EIII \( g \) is of type E6, \( \pi_\Theta = \{\alpha_3, \alpha_4, \alpha_5\} \), \( S \) is empty, \( D = \{\alpha_1\} \), \( p(1) = 6, p(3) = 5, p(4) = 4, p(2) = 2, p(5) = 3, p(6) = 1 \),

\[ B_1 = y_1 t_1 + d_1[(\text{ad}_r x_3)(\text{ad}_r x_4)(\text{ad}_r x_5)t_0^{-1}x_6]t_1, \]

\[ B_6 = y_6 t_6 + [(\text{ad}_r x_5)(\text{ad}_r x_4)(\text{ad}_r x_3)t_1^{-1}x_1]t_6, \]

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and

\[ B_2 = y_2 t_2 + [(\text{ad}_r x_4 x_3 x_5 x_4) t_2^{-1} x_2] t_2. \]

**Type EIV** \( \mathfrak{g} \) is of type E6, \( \pi_\theta = \{\alpha_3, \alpha_4, \alpha_5, \alpha_2\} \), \( \mathcal{S} \) and \( \mathcal{D} \) are both empty,

\[ B_1 = y_1 t_1 + [(\text{ad}_r x_3 x_4 x_5 x_2 x_4 x_3) t_1^{-1} x_1] t_1 \]

and

\[ B_6 = y_6 t_6 + [(\text{ad}_r x_5 x_4 x_3 x_2 x_4 x_5) t_6^{-1} x_6] t_6. \]

**Type EVI** \( \mathfrak{g} \) is of type E7, \( \pi_\theta = \{\alpha_7, \alpha_5, \alpha_2\} \), both \( \mathcal{S} \) and \( \mathcal{D} \) are empty,

\[ B_6 = y_6 t_6 + [(\text{ad}_r x_7 x_5) t_6^{-1} x_6] t_6, \quad B_4 = y_4 t_4 + [(\text{ad}_r x_2 x_5) t_4^{-1} x_4] t_4, \]

and \( B_i = y_i t_i + q_i^{-2} x_i \) for \( i = 1, 3 \).

**Type EVII** \( \mathfrak{g} \) is of type E7, \( \pi_\theta = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\} \), \( \mathcal{D} \) is empty, \( \mathcal{S} = \{\alpha_7\} \),

\[ B_1 = y_1 t_1 + [(\text{ad}_r x_3 x_4 x_2 x_5 x_4 x_3) t_1^{-1} x_1] t_1, \]

\[ B_6 = y_6 t_6 + [(\text{ad}_r x_5 x_4 x_2 x_3 x_4 x_5) t_6^{-1} x_6] t_6, \]

and \( B_7 = y_7 t_7 + q_7^{-2} x_7 + s_7 t_7 \).

**Type EIX** \( \mathfrak{g} \) is of type E8, \( \pi_\theta = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\} \), \( \mathcal{D} \) and \( \mathcal{S} \) are both empty,

\[ B_1 = y_1 t_1 + [(\text{ad}_r x_3 x_4 x_2 x_5 x_4 x_3) t_1^{-1} x_1] t_1, \]

\[ B_6 = y_6 t_6 + [(\text{ad}_r x_5 x_4 x_2 x_3 x_4 x_5) t_6^{-1} x_6] t_6, \]

and \( B_i = y_i t_i + q_i^{-2} x_i \) for \( i = 7, 8 \).

**Type FI** \( \mathfrak{g} \) is of type F4, \( \pi_\theta \) is empty, \( \mathcal{S} \) and \( \mathcal{D} \) are empty, \( B_i = y_i t_i + q_i^{-2} x_i \) for \( i = 1, 2, 3, 4 \).

**Type FII** \( \mathfrak{g} \) is of type F4, \( \pi_\theta = \{\alpha_1, \alpha_2, \alpha_3\} \), both \( \mathcal{D} \) and \( \mathcal{S} \) are empty, and

\[ B_1 = y_4 t_4 + [(\text{ad}_r x_3 x_2 x_1 x_3 x_2 x_3) t_4^{-1} x_4] t_4 \]

**Type G** \( \mathfrak{g} \) is of type G2, \( \pi_\theta \) is empty, \( \mathcal{S} \) and \( \mathcal{D} \) are both empty, and \( B_i = y_i t_i + q_i^{-2} x_i \) for \( i = 1, 2 \).

Set \( B_i = y_i t_i \) for \( \alpha_i \in \pi_\theta \). Recall that \( \mathcal{M}^+ = \mathcal{M} \cap U^+ \). Given \( \alpha_i \notin \pi_\theta \), let \( Z_i \) be the element in \( \mathcal{M}^+ \) such that

\[ \tilde{\theta}(y_i) = (\text{ad}_r Z_i) t_{p(i)}^{-1} x_{p(i)}. \]
Theorem 7.1 The left coideal subalgebra $B_{\theta,s,d}$ of $U$ is the algebra generated over $\mathcal{M}^+ T_{\Theta}$ by the elements $B_i$, $1 \leq i \leq n$ subject to the following relations:

(i) $\tau(\lambda) B_i \tau(-\lambda) = q^{-(\lambda, \alpha_i)} B_i$ for all $\tau(\lambda) \in T_{\Theta}$ and $1 \leq i \leq n$.

(ii) $t^{-1}_j x_i B_j - B_i t^{-1}_j x_j = \delta_{ij} (t_j - t_j^{-1})/(q_j - q_j^{-1})$ for all $\alpha_j \in \pi_{\Theta}$ and $1 \leq i \leq n$.

(iii) If $a_{ij} = 0$, then
$$B_i B_j - B_j B_i = \delta_{ij} (q_j - q_j^{-1})^{-1} (d_i [(\text{ad}_r Z_i) t^{-2}_j] t_i t_p(i) - d_{ip}(i) [(\text{ad}_r Z_{p(i)}) t^{-2}_j] t_i t_p(i)).$$

(iv) If $a_{ij} = -1$, then
$$B_i^2 B_j - (q_i + q_i^{-1}) B_i B_i B_j + B_j B_i^2 = \delta_{i,p(i)} q_i^{-1} [(\text{ad}_r Z_i) t^{-2}_j] t_i^2 B_j - \delta_{i,p'(i)} (q_i + q_i^{-1}) B_i (d_i q_i^{-1} t_j^{-1} t_i + d_j q_j^2 t_i^{-1} t_j).$$

(v) If $a_{ij} = -2$, then
$$B_i^3 B_j + (q_i^2 + 1 + q_i^{-2})(-B_j B_j B_j + B_j B_j B_j^2) - B_j B_i^3 = \delta_{i,p'(i)} q_i^{-1} (q_i + q_i^{-1})^2 (B_i B_j - B_j B_i).$$

(vi) If $a_{ij} = -3$, then
$$B_i^4 B_j + B_j B_i^4 - (q_i^3 + q_i + q_i^{-1} + q_i^{-3}) B_i^3 B_j B_i + B_i B_j B_i^3 B_j = (q_i^5 + 2q_i^{-3} + 4q_i^{-1} + 2q_i + q_i^3)(B_j B_j B_j + B_j B_j B_j)$$
$$- (q_i^4 + 4q_i^2 + 5 + 5q_i^{-2} + 4q_i^{-4} + q_i^{-6}) B_i B_j B_i - (q_i^2 + 1 + q_i^{-2}) B_j.$$
The relations in Theorem 7.1 can be checked directly in a case-by-case fashion using the list of algebras \( B_{\theta,s,d} \) above. Indeed, this is the approach used in [L1] which handles most of the cases when \( \Theta(\alpha_i) = -\alpha_{p(i)} \) for \( 1 \leq i \leq n \). Similar computations can be made in the other cases. However, in general, these computations are rather brutal. In this paper, we use the alternate, quicker method derived from the proof of [L2, Theorem 7.4]. First, we review some notation from [L4] and prove a small technical lemma.

Let \( G^+ \) be the subalgebra of \( U \) generated by \( x_i t_i^{-1}, 1 \leq i \leq n \). As in [L4, (4.2)], we have the following two direct sum decompositions

\[
U = \sum_{\lambda,\mu} U_{-\lambda} G^+_\mu U^0 \quad \text{and} \quad U^0 = \sum_{\lambda} C\tau(\lambda).
\]

Set \( \pi_{\lambda,\mu} \) equal to the projection of \( U \) onto \( U_{-\lambda} G^+_\mu U^0 \) using the above decomposition of \( U \) and set \( P_\gamma \) equal to the projection of \( U^0 \) onto \( C\tau(\gamma) \) using the decomposition of \( U^0 \).

Let \( Z \) be an element of \( M^+ \). The form of the right adjoint action (1.2) implies that \( [(ad_r Z)t^{-2}_j)x_j] \) is an element of \( M^+ t^{-2}_j \). Since \( t^{-1}_p t_i \) is in \( T_\Theta \), it follows that \( [(ad_r Z)t^{-2}_p t_i]t_i \) is an element of \( B_{\theta,s,d} \).

**Lemma 7.2** Let \( Z \in M^+ \) be a vector of weight \( \gamma \). For all \( \alpha_j \in \pi_\Theta \) and each \( \mu \in Q^+(\pi) \),

\[
\pi_{\alpha_j,\mu} \circ \Delta((ad_r Z)t^{-1}_j)x_j) = \delta_{\mu,\gamma} t^{-1}_j x_j \otimes [(ad_r Z)t^{-2}_j]t_j.
\]

**Proof:** We show how the argument works when \( \gamma \) is a simple root. The general case follows similarly using induction on the weight of \( Z \). Thus, assume that \( Z = x_i \) for some \( \alpha_i \in \pi_\Theta \). Recall ([Jo, 3.2.9]) that

\[
\Delta(x_k) = x_k \otimes 1 + t_k \otimes x_k
\]

for each \( 1 \leq k \leq n \). Hence (1.2) implies that

\[
\Delta((ad_r x_i)(t^{-1}_j x_j)) = -(x_i t^{-1}_i \otimes t^{-1}_j + 1 \otimes x_i t^{-1}_i)(t^{-1}_j x_j \otimes t^{-1}_j + 1 \otimes t^{-1}_j x_j) + (t^{-1}_i \otimes t^{-1}_i)(t^{-1}_j x_j \otimes t^{-1}_j + 1 \otimes t^{-1}_j x_j)(x_i \otimes 1 + t_i \otimes x_i).
\]

(7.1)
Inspection of (7.1) shows that $\pi_{\alpha, \mu} \circ \Delta((\text{ad}_r x_i)t^{-1}_jx_j) = 0$ unless $\mu = \alpha_i$. Thus, a straightforward computation using (7.1) yields

$$\pi_{\alpha, \nu} \circ \Delta((\text{ad}_r x_i)t^{-1}_jx_j) = t^{-1}_jx_j \otimes (-x_i t^{-1}_i t^{-1}_j + q^{-\langle \alpha, \alpha \rangle} t^{-1}_i t^{-1}_j x_i) = t^{-1}_jx_j \otimes [(\text{ad}_r x_i)t^{-2}_j t_j].$$

\[\square\]

**Proof of Theorem 7.1:** Relations (i) and (ii) are just (i) and (ii) of [L4, Theorem 7.4]. We check relations (iii) through (vi).

Given $B_i$ and $B_j$, set

$$Y = Y(B_i, B_j) = \sum_{m=0}^{1-a_{ij}} (-1)^m \left[ 1 - \frac{a_{ij}}{m} \right] q_i B_i^{-a_{ij} - m} B_j B_i^m.$$

Set $\lambda = (1 - a_{ij})\alpha_i + \alpha_j$. By the proof of [L4, Theorem 7.4] (see also [L4, Variations 1 and 2]),

$$0 = ((P_\lambda \circ \pi_{0,0}) \otimes \text{Id}) \Delta(Y).$$

Let $Y' \in B$ such that

$$\tau(\lambda) \otimes Y' = ((P_\lambda \circ \pi_{0,0}) \otimes \text{Id})(\Delta(Y) - \tau(\lambda) \otimes Y).$$

Then by (7.2), $Y + Y' = 0$. Hence, we obtain relations (iii)-(vi) by evaluating

$$((P_\lambda \circ \pi_{0,0}) \otimes \text{Id})(\Delta(Y) - \tau(\lambda) \otimes Y)$$

to find $Y'$.

By the Lemma 7.1 and [L4, (6.5) and (7.15)], we have

$$\Delta(B_i) = \Delta(y_i t_i + d_i[(\text{ad}_r Z_i)t^{-1}_{p(i)} x_{p(i)}] t_i + s_i t_i)$$

$$= y_i t_i \otimes 1 + t_i \otimes B_i + t^{-1}_i x_{p(i)} t_i \otimes d_i[(\text{ad}_r Z_i)t^{-2}_{p(i)}] t_{p(i)} t_i$$

$$+ \sum_{\gamma > \alpha_i} G^+_\gamma t_i \otimes B.$$

Note that the $s_i$ are hidden in the above form of the coproduct. This fact is the key reason that the relations are independent of the choice of $s_i$. (See also [L2, Lemma 5.7].)

The case when $\alpha_i$ is in $\pi_\Theta$ follows directly from [L4, 7.20]. Indeed, checking the possibilities, $a_{ij}$ must equal $-1$ or $-2$ in this case. Furthermore

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\[ \delta_{p(i),j} = \delta_{p(j),i} = \delta_{i,p'(i)} = 0 \] and thus (iii)-(v) look just like the quantum Serre relations between \( y_i t_i \) and \( y_j t_j \). This is exactly what is explained in the paragraph concerning [L4, 7.20]. Thus for the remainder of the proof, we assume that \( \alpha_i \notin \pi_\Theta \).

Suppose that \( \alpha_j \notin \pi_\Theta \). By (i), it follows \( B_j \) commutes with each \( t_i x_i^{-1} \), \( \alpha_i \in \pi_\Theta \). Hence \( B_j \) commutes with every element of \( \mathcal{M} \cap G^+ \). In particular, \([\text{ad}_r Z_i] t_{p(i)}^{-2} t_{p(i)}^2\), which is an element of \( \mathcal{M} \cap G^+ \) for each \( \alpha_i \notin \pi_\Theta \), commutes with \( B_j \) for all \( \alpha_j \notin \pi_\Theta \).

**Case 1:** \( a_{ij} = 0 \). In this case, \( Y = B_i B_j - B_j B_i \) and \( \lambda = \alpha_i + \alpha_j \). Note that this expression is symmetric in \( i \) and \( j \) up to a negative sign. In particular, we may assume that \( \alpha_j \) is also not in \( \pi_\Theta \). Expression (7.3) implies that and

\[
(P_\lambda \circ \pi_{0,0}) \otimes Id)(\Delta(B_i B_j) - \tau(\lambda) \otimes B_i B_j) = (P_\lambda \circ \pi_{0,0})([t^{-1}_p x_p t_i y_j t_j \otimes d_i][\text{ad}_r Z_i] t_{p(i)}^{-2} t_{p(i)} t_i).
\]

Since \( t^{-1}_p x_p t_i y_j t_j = \delta_{p(i),j}(q_i - q_i^{-1})^{-1} t_i (t_j - t_j^{-1}) \), it follows that

\[
(P_\lambda \circ \pi_{0,0})(t^{-1}_p x_p t_i y_j t_j) = \delta_{p(i),j}(q_i - q_i^{-1})^{-1}.
\]

Thus

\[
(P_\lambda \circ \pi_{0,0}) \otimes Id)(\Delta(B_i B_j) - \tau(\lambda) \otimes B_i B_j) = \delta_{p(i),j}(q_i - q_i^{-1})^{-1} t_i t_j \otimes (d_i)[\text{ad}_r Z_i] t_{p(i)}^{-2} t_{p(i)} t_i.
\]

The same argument shows that

\[
(P_\lambda \circ \pi_{0,0}) \otimes Id)(\Delta(B_i B_j) - \tau(\lambda) \otimes B_i B_j) = \delta_{p(j),i}(q_i - q_i^{-1})^{-1} t_j t_i \otimes (d_j)[\text{ad}_r Z_j] t_{p(j)}^{-2} t_{p(j)} t_j.
\]

Relation (iii) follows from the fact that \( \delta_{p(j),i} = \delta_{p(i),j} = \delta_{p'(i),j} \).

**Case 2:** \( a_{ij} = -1 \). In this case

\[ Y = B_i^2 B_j - (q_i + q_i^{-1}) B_i B_j B_i + B_j B_i^2. \]

Recall that if \( p(i) = i \), then \( d_i = 1 \). Furthermore, checking the possibilities, we have \( i = p(j) \) if and only if \( \Theta(\alpha_i) = -\alpha_{p(i)} \) and \( \Theta(\alpha_j) = -\alpha_{p(j)} \). Thus

\[
(P_\lambda \circ \pi_{0,0}) \otimes Id)(\Delta(Y) - \tau(\lambda) \otimes Y) = (P_\lambda \circ \pi_{0,0})(\delta_{i,p(i)}(G_1) + \delta_{i,p(j)}(G_2))
\]

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where
\[ G_1 = t_i^{-1}x_it_it_it_j \otimes [(\operatorname{ad}_r Z_i)t_i^{-2}]t_it_iB_j \]
\[- (q_i + q_i^{-1})t_i^{-1}x_it_it_jy_it_i \otimes [(\operatorname{ad}_r Z_i)t_i^{-2}]t_it_iB_j \]
\[ + t_it_it_i^{-1}x_it_it_jy_it_i \otimes B_j [(\operatorname{ad}_r Z_i)t_i^{-2}]t_it_i \]
and
\[ G_2 = (t_j^{-1}x_it_it_jy_it_j \otimes d_it_j^{-1}t_it_iB_i + t_it_j^{-1}x_it_it_j \otimes B_id_it_j^{-1}t_i \]
\[- (q_i + q_i^{-1})(t_i^{-1}x_it_it_jy_it_i \otimes d_it_j^{-1}t_it_iB_i + t_it_i^{-1}x_it_it_jy_it_i \otimes d_jB_it_j^{-1}t_j) \]
\[ + (t_i^{-1}x_it_it_jy_it_i \otimes d_jt_j^{-1}t_jB_i + t_i^{-1}x_it_it_jy_it_i \otimes d_it_j^{-1}t_jB_i). \]

Suppose that \( \beta_1, \beta_2, \beta_3 \) are elements of \( Q(\pi) \) such that \( \beta_1 + \beta_2 + \beta_3 = 2\alpha_i \).
Recall that \( \lambda = 2\alpha_i + \alpha_j \). Note that
\[ (P_\lambda \circ \pi_{0,0})(\tau(\beta_1)x_j \tau(\beta_1)y_j \tau(\beta_3)) = (q_i - q_i^{-1})^{-1}q_i^{-2(\beta_1, \alpha_j)}t_i^2t_j. \]
Hence
\[ (P_\lambda \circ \pi_{0,0})(G_1) = (q_i - q_i^{-1})[t_i^2t_j(q_i^{-2} - (q_i + q_i^{-1})q_i^{-1} + q_i^{-2}) \otimes [(\operatorname{ad}_r Z_i)t_i^{-2}]t_i^2B_j \]
\[ - q_i^{-1}t_i^2t_j \otimes [(\operatorname{ad}_r Z_i)t_i^{-2}]t_i^2B_j. \]
A similar computation shows that
\[ (P_\lambda \circ \pi_{0,0})(G_2) = (q_i - q_i^{-1})[t_i^2t_j(q_i^{-3}q_i^{-1} + q_i - (q_i + q_i^{-1})q_i^{-1}q_i^{-3}) \otimes d_iB_it_j^{-1}t_i \]
\[ + t_i^2t_j(-(q_i + q_i^{-1})q_i + q_i^{-1}q_i^{-3} + q_iq_i^{-2}) \otimes d_jB_it_j^{-1}t_j] \]
\[ = t_i^2t_j \otimes [q_i^{-1}(q_i + q_i^{-1})d_iB_it_j^{-1}t_i + q_i^2(q_i + q_i^{-1})d_jB_it_j^{-1}t_j]. \]

**Case 3:** \( a_{ij} = -2 \). Checking the possibilities, we see that \( \Theta(\alpha_i) = -\alpha_i \) and \( \Theta(\alpha_j) = -\alpha_j \). Furthermore, \( S \) is equal to \( \{\alpha_j\} \). Thus \( B_i = y_it_i + q_i^{-2}x_i \) and \( B_j = y_it_j + q_j^{-2}x_j + s_jt_j \). It follows that \( \Delta(B_i) = B_i \otimes 1 + t_i \otimes B_i \) and \( \Delta(B_j) = (y_it_j + q_j^{-2}x_j) \otimes 1 + t_j \otimes B_j \). So
\[ (P_\lambda \circ \pi_{0,0}) \otimes Id)(\Delta(Y) - \tau(\lambda) \otimes Y) \]
\[ = (P_\lambda \circ \pi_{0,0}) \otimes Id)(H_1 \otimes B_iB_j + H_2 \otimes B_jB_i) \]
where
\[ H_1 = B_i^2t_it_j + B_it_iB_tj + t_iB_iB_tj \]
\[ + (q_i^2 + 1 + q_i^{-2})[-t_iB_it_jB_i - B_it_it_jB_i + t_it_jB_iB_i] \]

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and
\[ H_2 = (q_i^2 + 1 + q_i^{-2})(-B_i B_j t_i t_j + B_i B_j t_i t_j + B_i B_j t_i B_i) \\
- t_j B_i B_j t_i - t_j B_i B_j t_i - t_j B_i B_i. \]

Let \( \tau_1 = \tau(\beta_1), \tau_2 = \tau(\beta_2) \) and \( \tau_3 = \tau(\beta_3) \) where \( \beta_1, \beta_2, \) and \( \beta_3 \) are three elements in \( Q(\pi) \) such that \( \beta_1 + \beta_2 + \beta_3 = \alpha_i + \alpha_j. \) Relation (iv) now follows from the fact that
\[ (P_\lambda \circ \pi_{0,0}) \tau_1 B_i \tau_2 B_i \tau_3 = (P_\lambda \circ \pi_{0,0})(\tau_1 q_i^{-2} x_i \tau_2 y_i t_i \tau_3) = q^{-(\beta_2, \alpha_i)}(q_i - q_i^{-1})^{-1} t_i^3 t_j. \]

**Case 4:** \( a_{ij} = -3. \) Again, checking the possibilities, we must have that \( \Theta(\alpha_i) = -\alpha_i, \Theta(\alpha_j) = -\alpha_j, \) and \( S \) is empty. Thus \( B_k = y_k t_k + q_k^{-2} x_k \) and \( \Delta(B_k) = B_k \otimes 1 + t_k \otimes B_k \) for \( k = 1, 2. \) The argument follows in a similar, although lengthier, manner to Case 3. \( \square \)

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