An optimization approach to parameter identification in variational inequalities of second kind

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Abstract This paper is concerned with the inverse problem of parameter identification in variational inequalities of the second kind that does not only treat the parameter linked to a bilinear form, but importantly also the parameter linked to a nonlinear non-smooth function. A new abstract framework covers frictional contact as well as other non-smooth problems from continuum mechanics. We investigate the dependency of the solution of the forward problem on these parameters and prove Lipschitz continuity results. We formulate an optimization approach to the parameter identification problem and provide solvability results. Moreover we establish a convergence result for finite dimensional approximation in the optimization approach.

Keywords Variable parameter identification · Ellipticity parameter · Friction parameter · Trilinear form · Semisublinear form · Mosco set convergence · Galerkin method

1 Introduction

Off the mainstream in the mathematical field of inverse problems dealing with parameter identification in linear problems with ordinary or with partial differential equations [8, 24, 26, 31, 32] this paper is concerned with the identification of variable parameters in nonlinear problems, namely in elliptic variational inequalities (VI) that are of the second kind following the terminology of [11]. A prominent example of
this class is the following direct problem: Find the function \( u \) in the standard Sobolev space \( H^1(\Omega) = \{ v \in L^2(\Omega) : \nabla v \in (L^2(\Omega))^d \} \) on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) such that for any \( v \in H^1(\Omega) \) there holds

\[
\int_{\Omega} e(x) [\nabla u \cdot \nabla (v - u) + u(v - u)] \, dx + \int_{\partial \Omega} f(s) |u||v| - |u| \, ds \geq \int_{\Omega} g(x)(v - u) \, dx. \tag{1}
\]

This VI is related to the Helmholtz partial differential equation \(-\Delta u + u = g\) rendering the coercive bilinear form \( \nabla u \cdot \nabla v + u v \). (1) provides a simplified scalar model of the Tresca frictional contact problem in linear elasticity, as detailed later in the text. By the classic theory of variational inequalities [25] there is a unique solution \( u \) of (1) if the datum \( g \) that enters the right-hand side is given in \( L^2(\Omega) \) and moreover, the “ellipticity” parameter \( e > 0 \) in \( L^\infty(\Omega) \) and the “friction” parameter \( f > 0 \) in \( L^\infty(\partial \Omega) \) are known. Here we study the inverse problem that asks for the distributed parameters \( e \) and \( f \), when the state \( u \) or, what is more realistic, some approximation \( \tilde{u} \) from measurement is known. In other words, we are interested in the variable parameters \( e \) and \( f \) such that \( u(e, f) = \tilde{u} \). However, due to the lack of regularity in the measured data, it is unrealistic to expect such coefficients. Consequently, the inverse problem of parameter identification will be posed as an optimization problem which aims to minimize the misfit function, namely the gap between the solution \( u = u(e, f) \) and the measured data \( \tilde{u} \). This approach has been precisely the case with simpler inverse problems. To the best of our knowledge, this is the first work on the inverse problem of parameter identification in variational inequalities that does not only treat the parameter \( e \) linked to a bilinear form, but importantly also the parameter \( f \) linked to a nonlinear nonsmooth function, like the modulus function above.

A related, but different identification problem has been studied by Le and Ang in [29]. There the friction parameter in the full vectorial system of viscoelasticity is sought. But unilateral Signorini conditions and nonsmooth friction conditions are replaced by an interface model that permits penetration. Thus the weak formulation of the direct problem does not lead to a VI, but to a nonlinear variational equality instead. Abda et al. [1] deal with another different inverse problem of identification of unknown boundaries with Signorini boundary condition.

There are some works on parameter identification in variational inequalities (see [14, 19, 20, 22, 27]). The early paper of Kluge [27] deals with the existence problem of parameter determination problems for parametric VIs and quasi-VIs among others. In contrast to the most papers on inverse problems where an optimization framework is the preferred choice, Hoffmann and Sprekels [22] develop an iterative scheme that is based on the construction of certain regularized time-dependent problems containing the original problem as asymptotic steady state. Hasanov [19] treats the identification of the unknown coefficient in a monotone nonlinear elliptic operator with mixed and Signorini-type boundary conditions; with the set of admissible coefficients compact, the existence of a quasisolution of the inverse problem (minimizing an appropriate misfit function) is obtained. Moreover, for the application of an inverse diagnostic problem for an axially symmetric elastoplastic body a numerical method and com-

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putational results are also presented. Hintermüller [20] formulates inverse coefficient problems for variational inequalities of first kind as optimal control problems. By the application of a primal-dual penalization technique he establishes first order optimality conditions which utilizing a complementarity function are rewritten as a set of equalities for numerical realization. González [14] considers an identification problem associated to an elliptic variational inequality of first kind subject to a bilateral restriction, where all the parameters involved in the elliptic operator as well as the parameters defining the constraints are identified. Based on compactness of the feasible set of parameters and by the continuous dependence of the solution of the direct problem on these parameters, the well-posedness of the identification problem is shown.

Gockenbach and Khan developed in [13] a convenient abstract framework for identification in partial differential equations using trilinear forms which has been recently extended to saddle point problems by Jadamba et al. in [23]. The present paper aims to study the inverse problem in VIs of second kind by exploring the advantages and by extending this trilinear form based framework. Inspiration for the present paper comes also from the paper of Adly et al. [3] on the related optimal control problem of a quasi-variational obstacle problem.

This paper is organized as follows. The next Sect. 2 collects some more variational inequalities of the second kind which model frictional contact and are drawn from other non-smooth problems in continuum mechanics. Here we also propose an abstract framework for parameter identification of the ellipticity parameter linked to the bilinear form and of a friction parameter linked to a non-smooth function. Then in Sect. 3 we investigate the dependence of the solution of the forward problem on these parameters and prove basic Lipschitz continuity results. In Sect. 4 we formulate an optimization approach to the parameter identification problem and provide a solvability result. In Sect. 5 we combine the optimization approach with Galerkin discretization and establish a convergence result for finite dimensional approximation based on Mosco set convergence. The paper ends with a short outlook to related parameter identification problems.

2 Some more variational inequalities and an abstract framework for parameter identification

Let $\Omega$ be a bounded domain $\subset \mathbb{R}^d$ ($d = 2, 3$) with Lipschitz boundary $\Gamma$ and nonempty boundary part $\Gamma_C \subset \subset \Gamma$. Further, let $0 < e \in L^\infty(\Omega)$, $0 < f \in L^\infty(\Gamma_C)$, $g \in L^2(\Omega)$, $V = \{ v \in H^1(\Omega) : v|\Gamma \Gamma_C = 0 \}$. Then one may consider the VI: Find $u \in V$ such that for all $v \in V$,

$$\int_\Omega e \nabla u \cdot \nabla (v - u) + \int_{\Gamma_C} f (|v| - |u|) \geq \int_\Omega g(v - u).$$

(2)

This scalar VI is more related than (1) to the Tresca frictional contact problem which reads as follows: Find $u \in V := \{ v \in H^1(\Omega, \mathbb{R}^d) : v|\Gamma \Gamma_C = 0 \}$ ($d = 2, 3$) such that for all $v \in V$, 

$\n$
\[ \int_{\Omega} [E \sigma(u) : \sigma(v - u)] + \int_{\Gamma_C} f (|v \cdot n| - |u \cdot n|) \geq \int_{\Omega} g \cdot (v - u), \]  

where now \( E \in L^\infty(\Omega, \mathbb{R}^{d \times d}) \), \( E > 0 \) (that is, \( E \) is positive definite) is the anisotropic elasticity tensor, \( \sigma = \sigma(u), \sigma = 1/2 (\nabla u + (\nabla u)^T) \) denotes the strain field associated to the displacement field \( u \), \( n \) stands for the outward unit normal, and now \( g \in L^2(\Omega, \mathbb{R}^d) \).

A vectorial VI of second kind similar to (3) appears in Stokes flow with leaky boundary condition or with Tresca friction boundary condition; see [5, 6, 10].

When replacing the functional \( \int_{\Gamma_C} f |w| \) by \( \int_{\Omega} f |
abla w| \) in (1) or in (2) one obtains a VI of second kind that models laminar flow of a Bingham fluid, see e.g. [11]. More general than Bingham fluid is the vectorial viscoplastic fluid flow problem studied in [12].

All these VIs can be covered by the following abstract framework. Let (as above) \( V \) be a Hilbert space; moreover \( E_+, F_+ \subset F \). Let as with [13], \( t : E \times V \times V \rightarrow \mathbb{R}, (e, u, v) \mapsto t(e, u, v) \) a trilinear form and \( l : V \rightarrow \mathbb{R}, v \mapsto l(v) \) a linear form. Assume that \( t \) is continuous such that \( t(e, \cdot, \cdot) \) is \( V \)-elliptic for any fixed \( e \in \text{int } E_+ \). Now in addition we have a “semisublinear form” \( s : F \times V \rightarrow \mathbb{R}, (f, u) \mapsto s(f, u) \), that is, for any \( u \in V \), \( s(f, u) \) is linear in its first argument \( f \) on \( F \) and for any \( f \in F_+ \), \( s(f, \cdot) \) is sublinear, continuous, and nonnegative on \( V \). Moreover assume that \( s(f, 0_V) = 0 \) for any \( f \in F \).

Then the forward problem is the following VI: Given \( e \in \text{int } E_+ \) and \( f \in F_+ \), find \( u \in V \) such that

\[ t(e; u, v - u) + s(f; v) - s(f; u) \geq l(v - u), \forall v \in V. \]  

Now with given convex closed subsets \( E^{ad} \subset \text{int } E_+ \) and \( F^{ad} \subset F_+ \), we seek to identify two parameters, namely the “ellipticity” parameter \( e \in E^{ad} \) and the “friction” parameter \( f \) in \( F^{ad} \).

In the model problem we have some convex closed cone \( E_+ \subset \{ e \in L^\infty(\Omega) : e \geq 0 \text{ a.e. on } \Omega \} \) containing the convex closed “feasible” set \( E^{ad} = \{ e \in E_+ : e \leq \bar{e} \text{ a.e. on } \Omega \} \), where the bounds \( e < \bar{e} \) are given in \( \mathbb{R}^+_+ = \{ r \in \mathbb{R} : r > 0 \} \). Likewise we have some convex closed cone \( F_+ \subset \{ f \in L^\infty(\Gamma_C) : f \geq 0 \text{ a.e. on } \Gamma_C \} \) containing the convex closed “feasible” set \( F^{ad} = \{ f \in F_+ : f \leq \bar{f} \text{ a.e. on } \Gamma_C \} \), where the bounds \( 0 \leq \bar{f} < \bar{f} \) are given.

As with Gockenbach and Khan in [13] we can assume

\[ t(e; u, v) \leq \bar{t} ||e||_E ||u||_V ||v||_V, \forall e \in E, u \in V, v \in V \]  

\[ t(e; u, u) \geq \underline{t} ||u||^2_V, \forall e \in E^{ad} \subset E, u \in V. \]  

In fact, \( \bar{t} < \infty \) directly follows from the assumed continuity of \( t \), where in the model problem \( \bar{t} = 1 \), and \( \underline{t} > 0 \) comes from Poincaré inequality; respectively in the elastic friction contact problem from Korn’s inequality. Moreover our assumption that \( s(f, \cdot) : V \rightarrow \mathbb{R}^+_+ \) is sublinear and l.s.c. for all \( f \in F_+ \) is satisfied in the model problem with

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\[ s(f, v) = \int_{\Gamma_c} f |v| \, ds, \]

also in the elastic friction contact problem with \(|v|\) replaced by \(|v \cdot n|\), respectively with

\[ s(f, v) = \int_{\Omega} f |\nabla v| \, dx \]

in the Bingham flow or (simplified) plasticity problem.

3 Some basic Lipschitz estimates

In this section we study how the solution of the VI (4) depends on the ellipticity and friction parameters and derive some Lipschitz estimates.

3.1 How does solution depend on ellipticity parameter?

For simplicity fix \( f \in F_+ \) and define \( \varphi(u):= \varphi_f(u):= s(f, u) \). By assumptions, the solution map \( e \in E^{ad} \mapsto u = S_f(e) \) is uniquely defined. In the following we show that \( S_f : E^{ad} \subset E \to V \) is Lipschitz. This extends [13, Theorem 2.2] to variational inequalities of second kind.

**Proposition 3.1** Suppose that the trilinear form \( t \) satisfies (5) and (6). Then there holds

\[ ||S_f(e_1) - S_f(e_2)||_V \leq \tilde{t} \tilde{l}^{-1} \min \{ \tilde{l}^{-1} ||l||_{V^*}, ||S_f(e_1)||_V, ||S_f(e_2)||_V \} ||e_1 - e_2||_E, \tag{7} \]

hence in particular,

\[ ||S_f(e_1) - S_f(e_2)||_V \leq \tilde{t} \tilde{l}^{-2} ||l||_{V^*} ||e_1 - e_2||_E. \]

Thus \( S_f : E^{ad} \subset E \to V \) is Lipschitz, independent of \( f \in F_+ \).

**Proof** Let \( e_i \in E^{ad}, u_i \in S_f(e_i) \). Then

\[ t(e_1; u_1, v - u_1) + \varphi(v) - \varphi(u_1) \geq l(v - u_1), \forall v \in V ; \]

\[ t(e_2; u_2, v - u_2) + \varphi(v) - \varphi(u_2) \geq l(v - u_2), \forall v \in V. \]

Choose \( v = u_2 \), respectively \( v = u_1 \) and add. This gives

\[ t(e_1; u_1, u_2 - u_1) + t(e_2; u_2, u_1 - u_2) \]

\[ = t(e_1; u_1 - u_2, u_2 - u_1) + t(e_2 - e_1; u_2, u_1 - u_2) \]

\[ \geq 0. \]
Rearranging terms leads to
\[ t(e_1; u_1 - u_2, u_1 - u_2) \leq t(e_2 - e_1; u_2, u_1 - u_2). \]
This gives the estimate
\[ L||u_1 - u_2||_V^2 \leq \bar{t} ||e_2 - e_1||_E ||u_2||_V ||u_1 - u_2||_V. \]
Hence
\[ ||u_1 - u_2||_V \leq \bar{t} L^{-1} ||u_2||_V ||e_1 - e_2||_E. \]
Exchange the role of solutions \( u_1 = S_f(e_1), \ u_2 = S_f(e_2) \) and obtain
\[ ||u_1 - u_2||_V \leq \bar{t} L^{-1} ||u_1||_V ||e_1 - e_2||_E. \]
Moreover set \( \nu = 0 \) in the first VI above and obtain
\[ t(e_1; u_1, u_1) + \varphi(u_1) \leq t(u_1). \]
By \( \varphi(u_1) \geq 0 \) this results in
\[ L||u_1||_V^2 \leq ||\nu||_V^{\ast} ||u_1||_V. \]
and thus in the a priori bound
\[ ||u_1||_V \leq L^{-1} ||\nu||_V^{\ast}. \]
Altogether arrive at (7).

3.2 How does solution depend on friction coefficient?

Fix \( e \in E^\text{ad} \subset \text{int} E_+ \). By assumptions, the solution map \( f \in F_+ \mapsto u = S_e(f) \) is uniquely defined. We can show that \( S_e : F^\text{ad} \subset F_+ \to V \) is Lipschitz provided that there holds with some constant \( \bar{s} \)

\[ |s(f; u_2) - s(f; u_1)| \leq \bar{s} \| f \|_F \| u_2 - u_1 \|_V, \ \forall f \in F; u_1 \in V, u_2 \in V. \quad (8) \]

Note that (8) is satisfied in the model problem, since

\[
\int_{\Gamma_C} f \ (|u_2| - |u_1|) \leq \int_{\Gamma_C} |f| \ |u_2 - u_1| \\
\leq ||i||_{F \to L^\infty} ||f||_F \|\gamma_{TC} \|_{V \to L^1} \|u_2 - u_1\|_V.
\]
where we use that the imbedding \( i : F \to L^\infty(\Gamma_C) \) is continuous with norm \( \| i \|_{F \to L^\infty} \) and the trace map \( \gamma_{\Gamma_C} : V \to L^1(\Gamma_C) \) is continuous with norm \( \| \gamma_{\Gamma_C} \|_{V \to L^1} \).

**Proposition 3.2** Suppose that the trilinear form \( t \) satisfies (6) and that the semisublinear form \( s \) satisfies (8). Then there holds

\[
\| S(e_1, f_1) - S(e_2, f_2) \|_V \leq \ell^{-1} \| f_1 - f_2 \|_E ,
\]
what is independent of \( e \in E^{ad} \). Hence the solution map is Lipschitz in the friction parameter.

**Proof** Let \( u_i \in S(e_i, f_i) \), \( f_i \in F^{ad} \). Then

\[
t(e; u_1, v - u_1) + s(f_1; v) - s(f_1; u_1) \geq l(v - u_1), \forall v \in V, \]
\[
t(e; u_2, v - u_2) + s(f_2; v) - s(f_2; u_2) \geq l(v - u_2), \forall v \in V.
\]

Choose \( v = u_2 \), resp. \( v = u_1 \) and add. Thus

\[
t(e; u_1 - u_2, u_2 - u_1) + s(f_1 - f_2; u_2) - s(f_1 - f_2; u_1) \geq 0
\]

Hence conclude by (6) and (8) that there holds (9).

\[\Box\]

### 3.3 Solution map is Lipschitz both in ellipticity and in friction parameter

Here we consider the uniquely defined solution map \( (e, f) \in E^{ad} \times F^{ad} \mapsto u = S(e, f) \).

**Theorem 3.3** Suppose that the trilinear form \( t \) satisfies (5) and (6) and that the semisublinear form \( s \) satisfies (8). Let \( e_i \in E^{ad}, f_i \in F^{ad} \) \((i = 1, 2)\). Then there holds for some constant \( c > 0 \)

\[
\| S(e_2, f_2) - S(e_1, f_1) \|_V \leq c \{ \| e_1 - e_2 \|_E + \| f_1 - f_2 \|_F \}.
\]

**Proof** From the previous estimates (7) and (9) conclude by the triangle inequality

\[
\| S(e_2, f_2) - S(e_1, f_1) \|_V \\
\leq \| S(e_2, f_2) - S(e_2, f_1) \|_V + \| S(e_2, f_1) - S(e_1, f_1) \|_V \\
\leq \ell^{-1} \| f_1 - f_2 \|_F + \ell^{-2} \| l \|_{V^*} \| e_1 - e_2 \|_E \\
\leq \ell^{-1} \max \{ \ell^{-1} \| l \|_{V^*}, \| e_1 - e_2 \|_E \} \{ \| f_1 - f_2 \|_F + \| e_1 - e_2 \|_E \}.
\]

\[\Box\]

To conclude this section let us point out that Lipschitz continuity results for parametric VIs are of interest in its own right. For such results for parametric VIs of first kind we can refer to [30,33].
4 The optimization approach

Let observation $\tilde{u} \in V$ be given. Then the parameter identification problem studied in this paper reads: Find parameters $e \in E^{ad}$, $f \in F^{ad}$ such that $u = S(e, f)$ “fits best” $\tilde{u}$, where $u \in V$ satisfies the VI (4), namely

$$t(e; u, v - u) + s(f; v) - s(f; u) \geq l(v - u), \forall v \in V.$$  

Similar to [13,14,20] and similar to parameter estimation in linear elliptic equations [7] we follow an optimization approach and introduce the “misfit function”

$$j(e, f) := \frac{1}{2} \|S(e, f) - \tilde{u}\|^2$$

to be minimized.

Here we assume similar to [20] that the sought ellipticity and friction parameters are smooth enough to satisfy with compact imbeddings

$$E^{ad} \subset \hat{E} \subset \subset E; F^{ad} \subset \hat{F} \subset \subset F.$$  

Some examples are in order. By the Rellich–Kondrachev Theorem [2, Theorem 6.3], $H^1(\Omega) \subset \subset C^0_B(\Omega)$, the space of bounded, continuous functions on $\Omega$, provided $\Omega$ satisfies the cone condition; clearly $C^0_B(\Omega) \subset L^\infty(\Omega)$. Thus $\hat{E} = H^1(\Omega) \subset \subset L^\infty(\Omega)$. Further, $\hat{\Gamma} = H^{\frac{1}{2}}(\Gamma_R) \subset \subset L^2(\Gamma_R)$, see e.g. [28]. More general Sobolev spaces of fractional order can also be used, see [4] for the identification of the ellipticity parameter in linear elliptic Dirichlet problems.

For simplicity let $\hat{E}, \hat{F}$ be Hilbert spaces (or more generally reflexive Banach spaces). Thus with given weights $\beta > 0, \gamma > 0$ we pose the stabilized optimization problem

$$(OP) \quad \text{minimize } j(e, f) + \frac{\beta}{2} \|e\|^2_{\hat{E}} + \frac{\gamma}{2} \|f\|^2_{\hat{F}}$$

subject to $e \in E^{ad}$, $f \in F^{ad}$

Under these assumptions we have the following solvability theorem.

**Theorem 4.1** Suppose the above compact imbeddings. Suppose that the trilinear form $t$ satisfies (5) and (6) and that the semisublinear form $s$ satisfies (8). Then (OP) admits an optimal (not necessarily unique!) solution $(e^*, f^*, u) \in E^{ad} \times F^{ad} \times V$, where $u = S(e^*, f^*)$. i.e. $u \in V$ solves the VI (4).

**Proof** The reasoning follows rather standard lines. Let $\delta := \inf (OP)$, then $\delta \geq 0$. Take minimizing sequences $(e_n), (f_n)(n \in \mathbb{N})$. Thanks to the stabilizing terms $\|e\|^2_{\hat{E}}, \|f\|^2_{\hat{F}}$, the sequences $(e_n)$, $(f_n)$ are bounded; up to some subsequences weakly convergent to some $e^* \in E^{ad}, f^* \in F^{ad}$. Use compact imbedding and Lipschitz continuity of the solution map $S$ by Theorem 3.3 to derive that $S(e_n, f_n) \rightarrow S(e^*, f^*)$
in norm. Conclude by the norm convergence in the misfit function, respectively by the weak lower semicontinuity of the stabilizing functions $\|e\|_{\hat E}^2, \|f\|_{\hat F}^2$ that

$$\delta \leq \|S(e^*, f^*) - \tilde u\|^2 + \frac{\beta}{2} \|e^*\|_{\hat E}^2 + \frac{\gamma}{2} \|f^*\|_{\hat F}^2$$

$$\leq \liminf_{n \in \mathbb{N}} \|S(e_n, f_n) - \tilde u\|^2 + \frac{\beta}{2} \|e_n\|_{\hat E}^2 + \frac{\gamma}{2} \|f_n\|_{\hat F}^2 \leq \delta$$

\[\square\]

5 Finite dimensional approximation

5.1 Galerkin method and Mosco set convergence

As a first step towards numerical realization of the optimization approach we study finite dimensional approximation. To this end we employ the Galerkin method, consider a positive parameter $h \to 0$, and introduce finite dimensional ansatz spaces $V_h \subset V, E_h \subset \hat E, F_h \subset \hat F$ with

$$V = \text{cl} \bigcup_{h > 0} V_h, \ \hat E = \text{cl} \bigcup_{h > 0} E_h, \ \hat F = \text{cl} \bigcup_{h > 0} F_h.$$ 

Further for the approximation of $E_{ad}$ and of $F_{ad}$ we have closed convex sets $E_{h}^{ad}$ and $F_{h}^{ad}$. We do not require that simply $E_{h}^{ad} = E_h \cap E_{ad}$ and $F_{h}^{ad} = F_h \cap F_{ad}$. Instead as explained in Remark 5.1, we admit nonconforming approximation $E_{h}^{ad} \not\subset E_{ad}$, $F_{h}^{ad} \not\subset F_{ad}$ and employ the concept of Mosco set convergence.

Let us recall that the set sequence $(M_h)_{h > 0}$ (in a Hilbert space or reflexive Banach space) converges to the set $M$ in the Mosco sense, simply written $M_h \to M$ for $h \to 0$, if the following two conditions hold (with $\rightharpoonup$ denoting weak convergence in contrast to strong convergence denoted by $\to$):

(i) If $z_h \in M_h$ for $h > 0$ and $z_h \rightharpoonup z$, then $z \in M$.

(ii) For any $z \in M$ there exist $z_h \in M_h$ such that $z_h \to z$.

Remark 5.1 For numerical realization of the Galerkin ansatz by the finite element method the concept of Mosco convergence has to be refined somewhat, see [11,21] for details. Very often in applications (see e.g. the model problem) the feasible sets $E_{ad}$, $F_{ad}$ are defined by inequality constraints in a function space that are not tractable by numerical computations. Therefore to obtain approximations $E_{h}^{ad}, F_{h}^{ad}$ that are amenable to numerical computation one imposes the inequality constraints only pointwise at the finite set of the quadrature points of the appropriate quadrature rule, what leads to nonconforming approximation with piecewise polynomial approximation that is higher than piecewise linear. Also the nonlinear functional $s : F \times V \to \mathbb{R}$, as given by an integral (see e.g. the model problem) has to be approximated by a finite sum $s_h : F_h \times V_h \to \mathbb{R}$, using an appropriate quadrature rule. For further details of finite element approximation in the forward problem we can refer to [15,17].
5.2 The finite dimensional optimization problem and convergence

In the setting given we are led to replace \((OP)\) by its finite dimensional counterpart

\[
(\text{OP})_h \quad \text{minimize } j_h(e_h, f_h) + \frac{\beta}{2} \|e_h\|_E^2 + \frac{\gamma}{2} \|f_h\|_F^2
\]

subject to \(e_h \in E^\text{ad}_h, f_h \in F^\text{ad}_h,\)

where

\[
j_h(e_h, f_h) := \frac{1}{2} \|S_h(e_h, f_h) - \tilde{u}\|^2
\]

and \(u_h = S_h(e_h, f_h) \in V_h\) solves the finite dimensional VI

\[
t(e_h; u_h, v_h - u_h) + s(f_h; v_h) - s(f_h; u_h) \geq l(v_h - u_h), \forall v_h \in V_h.
\] (11)

Clearly the datum \(\tilde{u}\), which appears in the finite dimensional misfit function \(j_h\), can be replaced by some approximations \(\tilde{u}_h\) such that \(\tilde{u}_h \to \tilde{u}\).

Since we generally have \(E^\text{ad}_h \not\subset E^\text{ad}\), \(F^\text{ad}_h \not\subset F^\text{ad}\) we have to assume that (6) holds also for all \(e \in E^\text{ad}_h\) and moreover that

\[
E^\text{ad} \cap \bigcap_{h>0} E^\text{ad}_h \neq \emptyset, \quad F^\text{ad} \cap \bigcap_{h>0} F^\text{ad}_h \neq \emptyset.
\]

Next we provide the following lemma as an essential tool for our final convergence result.

**Lemma 5.2** Let \(e_h \in E^\text{ad}_h, f_h \in F^\text{ad}_h\), and \(u_h = S_h(e_h, f_h) \in V_h\) solve the VI (11). Suppose that \((e_h, f_h) \to (e, f)\) in \(E \times F\), and moreover \(u_h \to u\) for \(h \to 0\). Then \(u_h \to u\) for \(h \to 0\).

**Proof** By Mosco convergence (i), clearly \(e \in E^\text{ad}\) and \(f \in F^\text{ad}\).

To show the claimed norm convergence, use (6) and estimate

\[
t \limsup_{h \to 0} \|u_h - u\|^2 \leq \limsup_{h \to 0} t(e_h; u_h - u, u_h - u) = \limsup_{h \to 0} t(e; u_h, u_h - u).
\] (12)

By density there exist \(\tilde{u}_h \in V_h\) such that \(\tilde{u}_h \to u\). From (11),

\[
\limsup_{h \to 0} t(e_h; u_h, u_h - \tilde{u}_h) \leq \limsup_{h \to 0} \left[ s(f_h; \tilde{u}_h) - s(f_h; u_h) + l(u_h - \tilde{u}_h) \right] \]

\[
= \limsup_{h \to 0} \left[ s(f_h; \tilde{u}) - s(f_h; u_h) \right].
\]
On the other hand, since \( e_h \to e \), by (5),
\[
\lim_{h \to 0} [t(e; u_h, u_h - u) - t(e_h; u_h, u_h - \bar{u}_h)] = 0
\]
and since \( f_h \to f \), by (f, 0) = 0 and (8),
\[
\lim_{h \to 0} [s(f_h, u_h) - s(f, u_h)] = 0, \quad \lim_{h \to 0} [s(f_h, \bar{u}_h) - s(f, \bar{u}_h)] = 0.
\]
Hence by (12),
\[
0 \leq \frac{1}{2} \lim \sup_{h \to 0} \|u_h - u\|^2 \leq \lim_{h \to 0} s(f; \bar{u}_h) - \lim \inf_{h \to 0} s(f; u_h) \leq 0.
\]
\(\square\)

Now we can present the following result that ensures convergence of the finite dimensional approximation.

**Theorem 5.3** For any \( h > 0 \), \((OP)_h\) has a minimizer \((e^*_h, f^*_h) \in E^{ad}_h \times F^{ad}_h\) with \( u^*_h = S_h(e^*_h, f^*_h) \in V_h\) solving (11).

Let \( E^{ad}_h \to E^{ad} \) and \( F^{ad}_h \to F^{ad} \) in \( \hat{E} \times \hat{F} \). Then there exists a subsequence of \((e^*_h, f^*_h)_{h>0}\) that converges weakly in \( \hat{E} \times \hat{F} \) and strongly in \( E \times F \) to \((e^*, f^*) \in E^{ad} \times F^{ad}\) and moreover \( u^*_h \) converges strongly in \( V \) to \( u^* \), where \( u^* = S(e^*, f^*) \) and \((e^*, f^*)\) is a solution to \((OP)\).

**Proof** The existence of a minimizer \((e^*_h, f^*_h) \in E^{ad}_h \times F^{ad}_h\) follows from the coercivity of \( t(e^*_h, \cdot, \cdot)\) due to (6).

The proof of the convergence statement runs in several steps.

Step 1. A priori bounds

Insert \( v_h = 0 \in V_h \) in (11) and obtain from (6)
\[
\frac{1}{2} \|u^*_h\|^2 \leq t(e^*_h, u^*_h, u^*_h) \leq l(u^*_h) \leq \|l\| \|u^*_h\|.
\]
Hence \( u^*_h \) is uniformly bounded.

Let \( e_0 \in E^{ad} \cap \bigcap_{h>0} E^{ad}_h \), \( f_0 \in F^{ad}_h \cap \bigcap_{h>0} F^{ad}_h \). Then from optimality,
\[
j_h(e^*_h, f^*_h) + \frac{\beta}{2} \|e^*_h\|^2_E + \frac{\gamma}{2} \|f^*_h\|^2_F \leq C + \frac{\beta}{2} \|e_0\|^2_E + \frac{\gamma}{2} \|f_0\|^2_F.
\]
where \( j_h(e_0, f_0) = \frac{1}{2} \|S_h(e_0, f_0) - \bar{u}\|^2 \leq C \), since \( S_h(e_0, f_0) \) is bounded, what follows from (6) as above. Hence \((e^*_h, f^*_h)\) is uniformly bounded in \( \hat{E} \times \hat{F} \).
Step 2. Convergences

Up to some subsequence, \((u_h^*, e_h^*, f_h^*)\) converges weakly in the reflexive space 
\(V \times \hat{E} \times \hat{F}\) to some \((u^*, e^*, f^*)\) for \(h \to 0\). By Mosco convergence (i), \((e^*, f^*)\) lies 
in \(E^{ad} \times F^{ad}\). By compact imbedding, there exists a subsequence, still denoted by 
\((e_h^*, f_h^*)_{h>0}\), such that \((e_h^*, f_h^*)\) converges strongly to \((e^*, f^*)\) in 
\(E \times F\) for \(h \to 0\). Therefore in virtue of Lemma 5.2, \(u_h^*\) converges strongly to \(u^*\).

Step 3. Claim \(u^* = S(e^*, f^*)\)

By density, there exist \(v_h \in V_h\) for any fixed \(v \in V\), such that \(v_h \to v\). Then insert \(v_h\) in (11) with 
e_h := e_h^*, f_h := f_h^*, u_h := u_h^*. Passing to the limit \(h \to 0\) yields (4) with 
e := e^*, f := f^*, u := u^*. Hence \(u^* = S(e^*, f^*)\) and thus \(j_h(e_h^*, f_h^*) \to j(e^*, f^*)\) 
as \(h \to 0\).

Step 4. Claim \((e^*, f^*)\) solves \((OP)\)

Let \((e', f')\) be arbitrary in \(E^{ad} \times F^{ad}\). By Mosco convergence (ii) there exist 
\((e_h', f_h') \in E^{ad}_h \times F^{ad}_h\) such that \((e_h', f_h') \to (e', f')\) in \(\hat{E} \times \hat{F}\), by imbedding also in 
\(E \times F\). Let \(u_h' = S_h(e_h', f_h')\). Then \(u_h'\) is bounded, what follows from (6) as above 
in Step 1. Thus there exists a subsequence, still denoted by \((u_h')_{h>0}\), such that \(u_h'\) 
converges weakly to some \(u'\), in fact by Lemma 5.2, \(u_h'\) converges strongly to \(u'\). 
Moreover, similar as seen in Step 3, \(u' = S(e', f')\) and thus \(j_h(e_h', f_h') \to j(e', f')\) 
as \(h \to 0\). Therefore we arrive at

\[
j(e^*, f^*) + \frac{\beta}{2} \| e^* \|_{\hat{E}}^2 + \frac{\gamma}{2} \| f^* \|_{\hat{F}}^2 \leq \liminf_{h \to 0} \left\{ j_h(e_h^*, f_h^*) + \frac{\beta}{2} \| e_h^* \|_{\hat{E}}^2 + \frac{\gamma}{2} \| f_h^* \|_{\hat{F}}^2 \right\} \\
\leq \lim_{h \to 0} \left\{ j_h(e_h', f_h') + \frac{\beta}{2} \| e_h' \|_{\hat{E}}^2 + \frac{\gamma}{2} \| f_h' \|_{\hat{F}}^2 \right\} \\
= j(e', f') + \frac{\beta}{2} \| e' \|_{\hat{E}}^2 + \frac{\gamma}{2} \| f' \|_{\hat{F}}^2.
\]

This shows the claim and completes the proof. \(\square\)

6 Concluding remarks: an outlook

In this paper we have studied a parameter identification problem for nonlinear non-
smooth problems in the setting of variational inequalities of the second kind. Thus 
our results are confined to the class of inverse problem where the associated direct 
problem is a convex variational problem.

On the other hand, there are interesting nonconvex contact problems that describe 
adhesion and delamination phenomena; see e.g. [18] for the forward problem. Here 
the identification of the nonmonotone contact laws is a challenging task.

Another avenue of research are time-dependent variational inequalities and evolu-
tionary variational inequalities, where parameter identification is a widely open field; 
see e.g. [9,16] for the forward problem.

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