EXPONENTIAL ERGODICITY AND REGULARITY FOR EQUATIONS WITH LÉVY NOISE

ENRICO PRIOLA, ARMEN SHIRIKYAN, LIHU XU, AND JERZY ZABCZYK

Abstract. We prove exponential convergence to the invariant measure, in the total variation norm, for solutions of SDEs driven by \( \alpha \)-stable noises in finite and in infinite dimensions. Two approaches are used. The first one is based on Harris theorem, and the second on Doeblin’s coupling argument \cite{11}. Irreducibility, Lyapunov function techniques, and uniform strong Feller property play an essential role in both approaches. We concentrate on two classes of Markov processes: solutions of finite-dimensional equations, introduced in \cite{29}, with Hölder continuous drift and a general, non-degenerate, symmetric \( \alpha \)-stable noise, and infinite-dimensional parabolic systems, introduced in \cite{32}, with Lipschitz drift and cylindrical \( \alpha \)-stable noise. We show that if the nonlinearity is bounded, then the processes are exponential mixing. This improves, in particular, an earlier result established in \cite{30} using the weak convergence induced by the Kantorovich–Wasserstein metric.

Keywords: stochastic PDEs, \( \alpha \)-stable noise, Hölder continuous drift, Harris’ theorem, coupling, total variation, exponential mixing, Ornstein–Uhlenbeck processes.

Mathematics Subject Classification (2000): 60H15, 47D07, 60J75, 35R60.

Contents

1. Introduction
2. Main results
   2.1. Notations and assumptions
   2.2. Structural properties of solutions
   2.3. Ergodic results for finite-dimensional equations
   2.4. Ergodic results in the infinite-dimensional case
   2.5. Two approaches to exponential ergodicity
3. Proof of structural properties, \( \dim H < \infty \)

The first author was supported by the M.I.U.R. research project Prin 2008 “Deterministic and stochastic methods in the study of evolution problems”. The third author gratefully acknowledges the support by Junior program Stochastics of Hausdorff Research Institute for Mathematics. His research is partially supported by the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement nr. 258237. The fourth author gratefully acknowledges the support by the Polish Ministry of Science and Higher Education grant “Stochastic equations in infinite dimensional spaces” N N201 419039.
1. Introduction

This paper is concerned with ergodic properties of the stochastic equation
\begin{equation}
    dX_t = [AX_t + F(X_t)]dt + dZ_t, \quad X_0 = x,
\end{equation}
both in finite- and infinite-dimensional real Hilbert spaces $H$. Here $A$ is a linear operator, $F$ is a bounded mapping, and $Z$ is a symmetric $\alpha$-stable process. Under suitable conditions, we establish exponential convergence of the solutions to the invariant measure in the variation norm. Note that many nonlinear stochastic PDEs, including semilinear heat equations perturbed by Lévy noise, can be written in the form (1.1) with an infinite-dimensional phase space $H$.

Irreducibility and uniform strong Feller properties play an essential role in our approach. They are established in the paper when the space $H$ is finite-dimensional, $Z$ is a non-degenerate symmetric $\alpha$-stable process, and $F$ is an $\eta$-Hölder continuous function with $1 - \frac{\alpha}{2} < \eta \leq 1$ and $1 < \alpha < 2$. Under stronger assumptions on the drift $F$ and on the noise process $Z$, those properties were derived in [32] in infinite dimensions. The finite-dimensional result established in this paper is an important contribution of independent interest.

Stochastic PDEs driven by Lévy noises have been intensively studied since some time; e.g., see the papers [4, 2, 28, 26, 20, 32, 41], the book [27], and the references therein. Invariant measures and long-time asymptotics for stochastic systems driven by Lévy noises were studied in a number of papers. In particular, the linear case ($F \equiv 0$) was investigated in [36, 44] for finite-dimensional spaces and in [6, 33, 11] for the infinite dimension. The case of nonlinear equations was studied in [34, 27, 22, 11, 42]. However, there are no many results on ergodicity and exponential mixing (cf. [42, 15, 30]). The paper [15] studied the exponential mixing of finite-dimensional stochastic systems with jump noises, which include one-dimensional SDEs driven by $\alpha$-stable noise.

Some ergodic properties for SPDEs like (1.1) were also studied in [30]. It was proved that if the supremum norm of $F$ is small, then there exists a unique invariant measure, which is exponential mixing under the weak convergence induced by the Kantorovich–Wasserstein metric. Here we improve substantially this result, showing that the convergence to the invariant measure holds exponentially fast in...
the total variation norm without any smallness assumption on $F$. To prove this result, we have to impose a slightly stronger regularity condition on the noise compared to that of [30]; this is, however, a really mild assumption (see Remark 2.3 and Example 2.9).

As mentioned before, we also establish exponential mixing in the total variation norm for finite-dimensional stochastic equations of the form (1.1) with a less regular drift term $F$ and a more general noise $Z$. It seems that, even in one dimension (when $Z$ reduces to a standard symmetric rotationally invariant $\alpha$-stable noise), our result on exponential mixing is new (cf. [11,15]).

We have two proofs for the exponential mixing results. Even though they give the same result, we included both proofs in the paper since they are based on some additional properties of independent interest, such as exponential estimates for hitting times of balls. The first proof is based on Harris’ theorem, while the other uses the classical coupling argument, see Section 2.5 and also [19]. In both approaches, irreducibility and uniform strong Feller property play a crucial role. The Harris approach only needs to check some conditions involving Lyapunov functions, but it is not intuitive. The coupling proof is more involved, but gives an intuition for understanding the way in which the dynamics converges to the ergodic measure.

Let us sketch our methods for proving the well-posedness and the structural properties of finite-dimensional stochastic systems, since it has independent interest. To prove the existence and pathwise uniqueness of solutions, we only need to modify slightly the argument in [29]. We stress that the condition $1 - \frac{\alpha}{2} < \eta \leq 1$ is needed to have existence and uniqueness of solutions (cf. [29]). The irreducibility and uniform strong Feller property will be established in the following two steps. First, we prove irreducibility and (uniform) gradient estimates for finite-dimensional Ornstein–Uhlenbeck processes driven by non-degenerate symmetric $\alpha$-stable processes (related gradient estimates under different assumptions from ours are given in the recent paper [43]). Then we proceed as in [32] and deduce irreducibility and uniform gradient estimates for solutions to (1.1). Note that if $\eta < 1$ then the deterministic equation may have many solutions as classical examples show. Currently, there is a great interest in understanding pathwise uniqueness for SDEs when $F$ is not Lipschitz, see the references given in [7,29].

The paper is organized as follows. In Section 2 we formulate basic structural properties of the solutions of (1.1) and our main ergodic results—Theorems 2.7 and 2.8. In Section 3 we concentrate on proving the new structural properties of finite-dimensional systems. Section 4 contains decay $L_p$-estimates for solutions of (1.1), which are needed to prove exponential ergodicity; here we concentrate on the infinite-dimensional case since in finite dimensions these estimates are straightforward. The two proofs for the exponential mixing of infinite dynamics are established in Sections 5 and 6 respectively, the former applying Harris’ theorem...
and the latter using coupling argument. Section 6 is quite involved, in particular, exponential estimates for the first hitting time of balls are of independent interest. In Section 7, we show the exponential ergodicity for finite-dimensional systems (Theorem 2.7) in a sketchy way. We have only shown the full details for the proof of Theorem 2.8 concerning SPDEs, since the finite-dimensional result can be proved by similar and easier methods.

Acknowledgements. We would like to thank C. Odasso for patiently discussing with us his paper [24] and writing a note for us on the proof of inequality (6.6). We also would like to thank M. Hairer for pointing out to us the proof of Theorem 2.8 by the Harris approach.

2. Main results

2.1. Notations and assumptions. Let $H$ be a real separable Hilbert space with an inner product $\langle \cdot , \cdot \rangle$ and the corresponding norm $| \cdot |$. We denote by $\{e_k\}_{k \geq 1}$ an orthonormal basis, so that any vector $x \in H$ can be written as $x = \sum_{k \geq 1} x_k e_k$, where $\sum_k |x_k|^2 < \infty$. Denote by $B_b(H)$ the Banach space of bounded Borel-measurable functions $f : H \to \mathbb{R}$ with the supremum norm $\|f\|_0 := \sup_{x \in H} |f(x)|$.

Let $B(H)$ be the Borel $\sigma$-algebra on $H$ and let $\mathcal{P}(H)$ be the set of probabilities on $(H, B(H))$. Recall that the total variation distance between two measures $\mu_1, \mu_2 \in \mathcal{P}(H)$ is defined by

$$\|\mu_1 - \mu_2\|_{TV} = \frac{1}{2} \sup_{f \in B_b(H), \|f\|_0 = 1} |\mu_1(f) - \mu_2(f)| = \sup_{\Gamma \in B(H)} |\mu_1(\Gamma) - \mu_2(\Gamma)|.$$ 

Let $z(t)$ be a one-dimensional symmetric $\alpha$-stable process with $0 < \alpha < 2$. Its infinitesimal generator $A$ is given by

$$(2.1) \quad Af(x) := \frac{1}{C_\alpha} \int_{\mathbb{R}} \frac{f(y + x) - f(x)}{|y|^{\alpha+1}} dy, \quad x \in \mathbb{R},$$

where $C_\alpha = -\int_{\mathbb{R}} (\cos y - 1) \frac{dy}{|y|^{\alpha+1}}$; see [85] and [3]. It is well known that $z(t)$ has the following characteristic function:

$$\mathbb{E}[e^{i\lambda z(t)}] = e^{-t|\lambda|^\alpha},$$

t $\geq 0$, $\lambda \in \mathbb{R}$. A multidimensional generalization of $z(t)$ is obtained by considering an $n$-dimensional non-degenerate symmetric $\alpha$-stable process $Z = (Z_t)$. This is a Lévy process with the additional property that

$$(2.2) \quad \mathbb{E}[e^{i\langle Z_t, u \rangle}] = e^{-t\psi(u)}, \quad \psi(u) = -\int_{\mathbb{R}^d} \left( e^{i\langle u, y \rangle} - 1 - i\langle u, y \rangle 1_{\{|y| \leq 1\}}(y) \right) \nu(dy),$$
\( u \in \mathbb{R}^n, \ t \geq 0 \), where the Lévy (intensity) measure \( \nu \) is of the form

\[
(2.3) \quad \nu(D) = \int_S \mu(d\xi) \int_0^\infty 1_D(r\xi) \frac{dr}{r^{1+\alpha}} \quad D \in \mathcal{B}(\mathbb{R}^n),
\]

for some symmetric, non-zero finite measure \( \mu \) concentrated on the unit sphere \( S = \{ y \in \mathbb{R}^d : |y| = 1 \} \) (see [35, Theorem 14.3]). Note that formula \((2.3)\) implies that \( \psi(u) = c_\alpha \int_S |\langle u, \xi \rangle|^\alpha \mu(d\xi), \ u \in \mathbb{R}^n \) (see also [35, Theorem 14.13]).

The non-degeneracy hypothesis on \( Z \) is the assumption that there exists a positive constant \( C_\alpha \) such that, for any \( u \in \mathbb{R}^n \),

\[
(2.4) \quad \psi(u) \geq C_\alpha |u|^\alpha.
\]

This is equivalent to the fact that the support of \( \mu \) is not contained in a proper linear subspace of \( \mathbb{R}^n \) (see [29] for more details). Recall that the infinitesimal generator \( \mathcal{A} \) of the process \( Z \) is given on the space of infinitely differentiable functions with compact support \( C^\infty_c(\mathbb{R}^n) \) by the formula

\[
\mathcal{A} f(x) = \int_{\mathbb{R}^d} \left( f(x + y) - f(x) - 1_{\{|y| \leq 1\}} \langle y, Df(x) \rangle \right) \nu(dy), \ f \in C^\infty_c(\mathbb{R}^n),
\]

see [35, Section 31]. Note that \( Z_t = \sum_{1 \leq j \leq n} \beta_j z_j(t) e_j \) (where \( \{z_j(t)\}_{1 \leq j \leq n} \) are i.i.d. one-dimensional symmetric \( \alpha \)-stable processes) is in particular a non-degenerate symmetric \( \alpha \)-stable process if each \( \beta_j \neq 0 \).

We will make two sets of assumptions on \((1.1)\) depending on the dimension of the Hilbert space \( H \). They are similar but more restrictive if \( \dim H = \infty \).

**Assumption 2.1.** \([\dim H = n < \infty]\)

(A1) \( A \) is an \( n \times n \) matrix and \( \max_{1 \leq i \leq n} \Re(\gamma_i) < 0 \), where \( \gamma_1, \ldots, \gamma_n \) are the eigenvalues of \( A \) counted according to their multiplicity.

(A2) \( Z = (Z_t) \) is a symmetric non-degenerate \( n \)-dimensional \( \alpha \)-stable process with \( 1 < \alpha < 2 \).

(A3) \( F : H \to H \) is bounded and \( \eta \)-Hölder continuous with \( 1 - \frac{\alpha}{2} < \eta \leq 1 \).

**Assumption 2.2.** \([\dim H = \infty]\)

(A1) \( A \) is a dissipative operator defined by

\[
A = \sum_{k \geq 1} (-\gamma_k) e_k \otimes e_k
\]

with \( 0 < \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_k \leq \ldots \) and \( \gamma_k \to \infty \) as \( k \to \infty \).

(A2) \( Z_t \) is a cylindrical \( \alpha \)-stable process with \( Z_t = \sum_{k \geq 1} \beta_k z_k(t) e_k \), where \( \{z_k(t)\}_{k \geq 1} \) are i.i.d. symmetric \( \alpha \)-stable processes with \( 0 < \alpha < 2 \) and \( \beta_k \) are positive constants such that \( \sum_{k \geq 1} \frac{\beta_k^\alpha}{\gamma_k^{\alpha+1}} < \infty \) for some \( \varepsilon \in (0, 1) \).

(A3) \( F : H \to H \) is Lipschitz and bounded.

(A4) There exist some \( \theta \in (0, 1) \) and \( C > 0 \) so that \( \beta_k \geq C \gamma_k^{-\theta+1/\alpha} \).
Remark 2.3. Let us comment on Assumption 2.2. The Lipschitz property guarantees that Eq. (1.1) has a unique solution, and (A4) ensures that the solution is strong Feller. The condition \( \sum_{k \geq 1} \frac{\beta_k}{\gamma_k} \epsilon_k < \infty \) in (A2) implies that the solution to (1.1) evolves in linear subspace with compact embedding into \( H \), see Section 4. Note that in [30] it is only required that (A2) holds for \( \epsilon = 0 \) (i.e., that \( X^x_t \in H \), a.s.). However, our present assumption with \( \epsilon > 0 \) is really a mild assumption (compare also with Example 2.9).

2.2. Structural properties of solutions. In this subsection we formulate the structural properties of solutions in both finite and infinite dimensions; see Theorems 2.4 and 2.5. These structural properties will play an important role in proving the exponential ergodicity. The proof of the next theorem is quite involved and is postponed to Section 3.

**Theorem 2.4.** Let \( H = \mathbb{R}^n \). Under Assumption 2.1, there exists a unique strong solution \( X^x_t \) for (1.1). The solutions \( (X^x_t)_{x \in H} \) form a Markov process with transition semigroup \( P_t \),

\[
P_t f(x) = \mathbb{E}[f(X^x_t)], \quad f \in B_b(H),
\]

which is irreducible and such that there exists \( C > 0 \) with

\[
|P_t f(x) - P_t f(y)| \leq C \| f \|_0 |x - y|^{\frac{1}{\alpha} \wedge 1}, \quad x, y \in H, \quad t > 0, \quad f \in B_b(H).
\]

(2.5)

The following infinite-dimensional result is analogous to the previous one and is proved in [32]. Note that the noise \( Z \) considered here reduces in finite dimension to a particular case of the noise in Theorem 2.4.

**Theorem 2.5.** Under Assumption 2.2 there exists a unique mild solution \( X^x_t \) for (1.1),

\[
X^x_t = e^{At} x + \int_0^t e^{A(t-s)} F(X^x_s) ds + \int_0^t e^{A(t-s)} dZ_s.
\]

(2.6)

The solutions \( (X^x_t)_{x \in H} \) form a Markov process with the transition semigroup \( P_t \). The process is irreducible and there exists \( C > 0 \) such that

\[
|P_t f(x) - P_t f(y)| \leq C \| f \|_0 |x - y|^{\frac{1}{\theta} \wedge 1}, \quad x, y \in H, \quad t > 0,
\]

(2.7)

where \( \theta \) is given in (A4) of Assumption 2.2.

**Remark 2.6.** Note if \( \dim H = \infty \) then, in general, trajectories of \( (X^x_t) \) do not have a càdlàg modifications (see [5]).
2.3. Ergodic results for finite-dimensional equations. Let us denote by \((P_t)_{t \geq 0}\) the Markov semigroup associated with (1.1) and by \((P^*_t)_{t \geq 0}\) the dual semigroup acting on \(\mathcal{P}(H)\).

The main result for the finite-dimensional case is as follows:

**Theorem 2.7.** Under Assumption 2.1, the system (1.1) is ergodic and exponentially mixing. More precisely, there exists \(\mu \in \mathcal{P}(H)\) such that, for any \(p \in (0, \alpha)\) and any measure \(\nu \in \mathcal{P}(H)\) with finite \(p\)th moment, we have

\[
\|P^*_t \nu - \mu\|_{TV} \leq Ce^{-ct} \left(1 + \int_H |x|^p \nu(dx)\right),
\]

where \(C = C(p, \alpha, A, \|F\|_0)\) and \(c = c(p, \alpha, A, \|F\|_0)\).

One can easily adapt our proof to show that the previous theorem is also true when \((Z_t)\) is Gaussian.

2.4. Ergodic results in the infinite-dimensional case. The following theorem describing the long-time behaviour of \((X^*_t)\) is the main result of the infinite-dimensional case.

**Theorem 2.8.** Under Assumption 2.2, the system (1.1) is ergodic and exponentially mixing. More precisely, there exists \(\mu \in \mathcal{P}(H)\) so that for any \(p \in (0, \alpha)\) and any measure \(\nu \in \mathcal{P}(H)\) with finite \(p\)th moment, we have

\[
\|P^*_t \nu - \mu\|_{TV} \leq Ce^{-ct} \left(1 + \int_H |x|^p \nu(dx)\right),
\]

where \(C = C(p, \alpha, \theta, \beta, \gamma, \epsilon, \|F\|_0)\) and \(c = c(p, \alpha, \theta, \beta, \gamma, \epsilon, \|F\|_0)\) with \(\beta = (\beta_k)\), \(\gamma = (\gamma_k)\).

We will apply the above theorem in the following example which was considered in [30].

**Example 2.9.** Consider the following semilinear parabolic SPDE in a bounded domain \(D \subset \mathbb{R}^d\) with smooth boundary \(\partial D\):

\[
\begin{align*}
&dX(t, \xi) = [\Delta X(t, \xi) + F(X(t, \xi))]dt + dZ_t(\xi), \\
&X(0, \xi) = x(\xi), \\
&X(t, \xi) = 0, \quad \xi \in \partial D,
\end{align*}
\]

where \(Z_t\) and \(F\) are specified below. The Laplace operator \(-\Delta\) with the Dirichlet boundary condition has a discrete spectrum. We denote by \(\{\epsilon_k\}\) the set of its normalised eigenfunctions and by \(\{\gamma_k\}\) the corresponding eigenvalues written in increasing order and repeated according to multiplicity. It is well known that \(\gamma_k = C_d k^{2/d}(1 + \epsilon_k)\), where \(C_d\) is a constant depending on \(d\) and \(D\), and \(\{\epsilon_k\}\) is a sequence going to zero as \(k \to \infty\); see [1].
We study the dynamics defined by (2.10) in the Hilbert space \( H = L^2(D) \) with the orthonormal basis \( \{e_k\} \). Let us assume that \( Z = (Z_t) \) is a cylindrical \( \alpha \)-stable noise written in the form
\[
Z_t = \sum_{k=1}^{\infty} \beta_k z_k(t)e_k,
\]
where \( \{z_k(t)\}_k \) are i.i.d. symmetric \( \alpha \)-stable processes with \( \alpha \in (0, 2) \). A straightforward calculation using the above-mentioned asymptotics of \( \gamma_k \) shows that (A2) and (A4) are satisfied simultaneously if and only if \( 2\alpha(\theta - \varepsilon) > d \). Thus, if \( d \leq 3 \), one can choose \( \alpha, \theta, \) and \( \varepsilon \) for which Assumption 2.2 holds, and we get the property of exponential mixing in the total variation norm for the dynamics of (2.10). This improves earlier results established in Theorems 2.5 and 2.6 of [30] according to which strong mixing holds under essentially the same hypotheses and exponential mixing is true in the Kantorovich–Wasserstein metric if, in addition, the norm \( \|F\|_0 \) is sufficiently small.

2.5. Two approaches to exponential ergodicity. We shall prove the exponential ergodicity results by two approaches. The first one is by applying classical Harris’ theorem and the other is by coupling argument.

We shall use the following Harris’ theorem. For a surprisingly short and nice proof, we refer to Hairer’s lecture notes [13].

**Theorem 2.10 (Harris).** Let \( P_t \) be a Markov semigroup in the Polish space \( X \) such that there exists \( T_0 > 0 \) and \( V : X \to \mathbb{R}_+ \) which satisfies the following properties:

(i) there exists \( \gamma < 1 \) and \( K > 0 \) such that \( P_{T_0}V(x) \leq \gamma V(x) + K, x \in X \).

(ii) for every \( R > 0 \) there exists \( \delta > 0 \) such that
\[
\|P_{T_0}^*\delta_x - P_{T_0}^*\delta_y\|_{TV} \leq 1 - \delta,
\]
for all \( x, y \in X \) such that \( V(x) + V(y) \leq R \).

Then there exist some \( T > 0 \) and \( \beta < 1 \) such that
\[
\int_X (1 + V(x))|P_T^*\mu - P_T^*\nu|(dx) \leq \beta \int_X (1 + V(x))|\mu - \nu|(dx).
\]

The key point for Harris’ theorem approach is to find a Lyapunov function \( V \) and to check conditions (i) and (ii).

To sketch the coupling approach, let us fix a large constant \( T > 0 \) and consider the restriction of the Markov process \( (X^x_t), x \in H, \) to the times proportional to \( T \). We denote by \( (Y_k) \) the resulting discrete-time Markov process, by \( P_x \) the corresponding family of probability measures, and by \( P_k(x, \Gamma) \) the transition function. The dissipativity of \( A \), the boundedness of \( F \), and the non-degeneracy of \( Z \) imply that \( (Y_k) \) is irreducible, and the first hitting time of any ball has a finite exponential moment. Furthermore, as will follow from Theorems 2.4 and 2.5 if the initial
points \(x_1, x_2 \in H\) are such that \(|x_1 - x_2| \leq r\), with a sufficiently small \(r > 0\), then
\[
\|P_1(x_1, \cdot) - P_1(x_2, \cdot)\|_{TV} \leq \frac{1}{2}.
\]

Now let \((Y_k^1, Y_k^2)\) be a homogeneous discrete-time Markov process in the extended phase space \(H \times H\) such that the following properties hold for the pair \((Y_k^1, Y_k^2)\) under the law \(\mathbb{P}_{(x_1,x_2)}\) corresponding to the initial point \((x_1, x_2)\):

(a) The laws of \(Y_k^1\) and \(Y_k^2\) coincide with \(P_1(x_1, \cdot)\) and \(P_1(x_2, \cdot)\), respectively.
(b) If \(\max(|x_1|, |x_2|) > r\) and \(x_1 \neq x_2\), then the random variables \(Y_k^1\) and \(Y_k^2\) are independent.
(c) If \(\max(|x_1|, |x_2|) \leq r\) and \(x_1 \neq x_2\), then
\[
\mathbb{P}_{(x_1,x_2)}\{Y_k^1 \neq Y_k^2\} = \|P_1(x_1, \cdot) - P_1(x_2, \cdot)\|_{TV}.
\]
(d) If \(x_1 = x_2\), then \(Y_k^1 = Y_k^2\) with probability 1.

Such a chain can be constructed with the help of maximal coupling of measures; see Section 6. Combining properties (a)–(d) with irreducibility of \((Y_k)\) and inequality (2.11), it is possible to prove that the stopping time \(\rho = \min\{k \geq 0 : Y_k^1 = Y_k^2\}\) is \(\mathbb{P}_{(x_1,x_2)}\)-almost surely finite and has a finite exponential moment. Moreover, it follows from (d) that \(Y_k^1 = Y_k^2\) for \(k \geq \rho\). We can thus write
\[
|P_k(x_1, \Gamma) - P_k(x_2, \Gamma)| = |\mathbb{E}_{(x_1,x_2)}(I_{\Gamma}(Y_k^1) - I_{\Gamma}(Y_k^2))| \leq \mathbb{P}_{(x_1,x_2)}\{\rho > k\},
\]
where \(\Gamma \subset H\) is an arbitrary Borel subset and \(I_{\Gamma}\) stands for its indicator function. Since \(\rho\) has a finite exponential moment, the right-hand side of (2.12) can be estimated by \(\text{const} \, e^{-\gamma k}\). Taking the supremum over all Borel subsets \(\Gamma\), we conclude that the total variation distance between \(P_k(x_1, \Gamma)\) and \(P_k(x_2, \Gamma)\) goes to zero exponentially fast for any initial points \(x_1, x_2 \in H\). This implies the required uniqueness and exponential mixing.

In conclusion, let us note that, in the context of randomly forced PDE’s, the coupling argument can be modified to cover the case of degenerate noises. We refer the reader to [16, 21, 37] for discrete-time random perturbations, to [14, 12, 17, 38, 25] for a white noise, to [23] for a compound Poisson process, and to the book [18] for further references on this subject. We believe that a similar approach can be developed in the case of dissipative PDE’s driven by Lévy noises.

3. Proof of structural properties, \(\dim H < \infty\)

In this section, we concentrate on proving Theorem 2.4 which can be done in the following steps.

**Step 1. Existence and uniqueness.** Since (with \(X_t = X_t^x\))
\[
X_t = x + \int_0^t AX_s ds + \int_0^t F(X_s) ds + Z_t,
\]
defining \( v(t) = X_t - Z_t \), one can construct a càdlàg adapted solution, by working \( \omega \) by \( \omega \) and using a compacteness argument.

Uniqueness holds even in the limiting case \( \alpha = 1 \). When \( A = 0 \) it follows directly from [29]. In the present case of \( A \neq 0 \), since the drift in [29] was supposed to be bounded and \( x \mapsto Ax \) is an unbounded mapping, to prove pathwise uniqueness one can proceed into two different ways. First one can adapt the computations in [29] using a standard stopping time argument. To this purpose, we only note that if \( X_t \) is one solution starting from \( x \in \mathbb{R}^n \) then formula in [29, Lemma 4.2] continue to hold if \( t \) is replaced by \( t \wedge \tau_R \), \( R > 0 \), where

\[
\tau_R = \inf\{t \geq 0; |X_t| \leq R\}.
\]

Another method consists in introducing the process \( Y_t = e^{-At}X_t \). Clearly \( Y_t \) satisfies the following equation

\[
(3.2) \quad dY_t = e^{-At}F(e^{At}Y_t) + e^{-At}dZ_t.
\]

According to [29] with small modifications (due to the fact that now the drift is bounded but also time-dependent), (3.2) has a unique strong solution such that

\[
Y_t = x + \int_0^t e^{-As}F(e^{As}Y_s)ds + \int_0^t e^{-As}Z_s,
\]

and this is equivalent to (3.1).

**Step 2. Markov property.** This follows from the uniqueness by standard considerations.

**Step 3. Uniform strong Feller estimate (2.7).**

In order to adapt the method used in the proof of [32, Theorem 5.7], we need gradient estimates like

\[
(3.3) \quad \|DR_tf\|_0 \leq \frac{c}{t^{1/\alpha}}\|f\|_0, \quad t \in (0, 1], \quad f \in B_b(H),
\]

for the OU semigroup \( R_t \) corresponding to \( F = 0 \) in (3.1).

**Remark 3.1.** Some related estimates were obtained in a recent paper [43] which however does not cover the present situation. We also mention [39] which contains a Bismut–Elworthy–Li formula for jump diffusion semigroups (even without a Gaussian part). We cannot apply [39] since our Lévy measure \( \nu \) in general does not have a \( C^1 \)-density with respect to the Lebesgue measure in \( \mathbb{R}^n \setminus \{0\} \).

The next result seems to be of independent interest.

**Theorem 3.2.** Let \( H = \mathbb{R}^n \). Assume that \( Z = (Z_t) \) is an \( n \)-dimensional symmetric non-degenerate \( \alpha \)-stable process, \( \alpha \in (0, 2) \). Consider any real \( n \times n \) matrix \( A \).

Then gradient estimates (3.3) holds for the OU semigroup \( R_t \) associated with

\[
dX_t = AX_t dt + dZ_t, \quad X_0 = x.
\]
Proof. Let us fix \( f \in B_b(H) \) and \( t \in (0,T] \). It is known (see, for instance, [31]) that
\[
R_t f(x) = \int_H f(e^{tA}x + y)p_t(y)(dy),
\]
where \( p_t(y) = \frac{1}{(2\pi)^n} \int_H e^{-i(y,h)} \exp \left( - \int_0^t \psi(e^{sA}h)ds \right)dh, \)
where \( \psi \) is the exponent (or symbol) of the Lévy process \( Z \) (see (2.2)). We write
\[
R_t f(x) = \frac{1}{(2\pi)^n} \int_H f(z) \left( \int_H e^{-i(z,h)} e^{i(e^{tA}h,x)} e^{-\int_0^t \psi(e^{sA}h)ds} dh \right)dz.
\]
(1) Recall the rescaling property
\[
\psi(us) = s^\alpha \psi(u), \quad s \geq 0,
\]
and \( u \in H \). The non-degeneracy assumption (2.4) implies that there exists the directional derivative along any fixed direction \( l \in H, \|l\| = 1 \) (cf. Section 3 in [29]),
\[
D_l R_t f(x) = \frac{i}{(2\pi)^n} \int_H f(z) \left( \int_H e^{-i(z,h)} e^{i(e^{tA}h,x)} \langle e^{tA}h, l \rangle e^{-\int_0^t \psi(e^{sA}h)ds} dh \right)dz.
\]
Let \( e^{tA}h = k \). We have
\[
D_l R_t f(x) = \frac{i e^{-t tr(A)}}{(2\pi)^n} \int_H f(z) \left( \int_H e^{-i(z,e^{-tA}k)} e^{i(k,x)} \langle k, l \rangle e^{-\int_0^t \psi(e^{sA}k)ds} dk \right)dz
\]
\[
= \frac{i}{(2\pi)^n} \int_H f(e^{tA}x) \left( \int_H e^{i(k,x)} \langle k, l \rangle e^{-\int_0^t \psi(e^{-rA}k)dr} dk \right)dx
\]
\[
= \frac{i}{(2\pi)^n} \int_H f(e^{tA}x) \left( \int_H e^{i(k,x-\xi)} \langle k, l \rangle e^{-\int_0^t \psi(e^{-rA}k)dr} dk \right)dx.
\]
Let us introduce
\[
\phi_t(v) = \frac{1}{(2\pi)^n} \int_H e^{i(k,v)} \langle k, l \rangle e^{-\int_0^t \psi(e^{-rA}k)dr} dk.
\]
It is clear that we get
\[
\|D_l R_t f\|_0 \leq \frac{C_1}{t^{1/\alpha}} \|f\|_0, \quad t \in (0,1].
\]
(and so [33]) if we are able to prove that
\[
\|\phi_t\|_{L^1(H)} \leq \frac{C_1}{t^{1/\alpha}}, \quad t \in (0,1],
\]
where \( L^1(H) = L^1(\mathbb{R}^n) \) with respect to the Lebesgue measure.

(2) Let us check (3.4). Using the rescaling property, we have
\[
\phi_t(v) = \frac{1}{(2\pi)^n} \int_H e^{i(k,v)} \langle k, l \rangle \exp \left\{ - \frac{1}{t} \int_0^t \psi(e^{-rA}t^{1/\alpha}k)dr \right\} dk.
\]
\[= \frac{1}{(2\pi)^n t^{n/\alpha}} \int_H \exp \left\{ i \left( \frac{h}{t^{1/\alpha}}, v \right) \langle h, l \rangle \right\} \exp \left\{ -\frac{1}{t} \int_0^t \psi(e^{-rA^*}h) dr \right\} dh \]

\[= \frac{1}{t^{1/\alpha}} \frac{1}{(2\pi)^n t^{n/\alpha}} \int_H \exp \left\{ i \left( \frac{v}{t^{1/\alpha}}, h \right) \langle h, l \rangle \right\} \exp \left\{ -\frac{1}{t} \int_0^t \psi(e^{-rA^*}h) dr \right\} dh.\]

Since (with the change of variable: \( v/t^{1/\alpha} = w \))

\[\int_H |\phi_t(v)| dv = \frac{1}{(2\pi)^n t^{1/\alpha}} \int_H \left| e^{i(w,h)} \right| \langle h, l \rangle \exp \left\{ -\frac{1}{t} \int_0^t \psi(e^{-rA^*}h) dr \right\} dh |dw|,\]

in order to prove (3.5) we need to show that

\[\|\varphi_t\|_{L^1(H)} \leq C_1, \quad t \in (0, 1],\]

where

\[\varphi_t(w) = \frac{1}{(2\pi)^n} \int_H e^{-i(w,h)} \langle h, l \rangle \exp \left\{ -\frac{1}{t} \int_0^t \psi(e^{-rA^*}h) dr \right\} dh.\]

(3) Let us now show (3.5). Write \( \psi = \psi_1 + \psi_2 \),

\[\psi_1(u) = \int_{|y| \leq 1} \left( 1 - \cos(u, y) \right) \nu(dy), \quad \psi_2 = \psi - \psi_1,\]

so that

\[\varphi_t(w) = \frac{1}{(2\pi)^n} \int_H e^{-i(w,h)} \langle h, l \rangle \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) dr \right\} \exp \left\{ -\frac{1}{t} \int_0^t \psi_2(e^{-rA^*}h) dr \right\} dh.\]

Now consider the random variable

\[Y_t = \frac{1}{t^{1/\alpha}} \int_0^t e^{-(t-s)A} dZ_s^2, \quad t \in (0, 1],\]

where \( Z^2 = (Z_t^2) \) is a Lévy process having exponent \( \psi_2 \). It is easy to check that its law \( \mu_t \) has characteristic function \( e^{-\frac{1}{2} \int_0^t \psi_1(e^{-rA^*}h) dr} \), i.e.,

\[\hat{\mu}_t(h) = \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) dr \right\}, \quad h \in H.\]

Now suppose that there exists \( g_t \in L^1(H), t \in (0, 1] \), such that

\[\hat{g}_t(h) = \langle h, l \rangle \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) dr \right\}.\]

Then, by well known properties of the Fourier transform (see Proposition 2.5 in [35]) we would get

\[\hat{g}_t \cdot \hat{\mu}_t = \hat{g}_t \ast \hat{\mu}_t\]

and, using the Fourier inversion formula,

\[\varphi_t(w) = (g_t \ast \mu_t)(w),\]
so that \(\|\varphi_t\|_{L^1} \leq \|g_t\|_{L^1}, \ t \in (0, 1]\). Thus to prove (3.5) and get the assertion, it remains to show that (3.6) holds and moreover that

\[
(3.7) \quad \|g_t\|_{L^1(H)} \leq C_1, \ t \in (0, 1].
\]

(4) Now we show (3.6) and (3.7). Note that

\[
\exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) \, dr \right\} = \exp \left\{ -\frac{1}{t} \int_0^t \, dr \int_{|y| \leq 1} \left( 1 - \cos((e^{-rA^*}h, y)) \right) \nu(dy) \right\}
\]

\[
= \exp \left\{ -\frac{1}{t} \int_0^t \psi(e^{-rA^*}h) \, dr \right\} \exp \left\{ \frac{1}{t} \int_0^t \, dr \int_{|y| > 1} \left( 1 - \cos((e^{-rA^*}h, y)) \right) \nu(dy) \right\}
\]

\[
\leq \exp \{ 2\nu(|y| > 1) \} \exp \left\{ \frac{-C_0}{t} \int_0^t |e^{-rA^*}h|^\alpha \, dr \right\}.
\]

Since \(|h| \leq c_2|e^{-rA^*}h|, h \in H, r \in [0, T]\), it follows that

\[
(3.8) \quad \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) \, dr \right\} \leq c_1 e^{-c_3|\alpha|}, \ h \in H, \ t \in (0, 1].
\]

We find easily that \(\psi_1 \in C^\infty(H)\) and so, using also (3.8) we deduce that the mapping \(h \mapsto (h, l) e^{-\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) \, dr} \) is in the Schwartz space \(S(H)\), for any \(t \in (0, 1]\). It follows that there exists \(g_t \in S(H)\) such that (3.6) holds. By the inversion formula,

\[
g_t(w) = \frac{1}{(2\pi)^n} \int_H e^{-i\langle w, h \rangle} \langle h, l \rangle \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) \, dr \right\} \, dh, \ w \in H.
\]

Now we show (3.7), by proving that for any multiindex \(\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n\), there exists \(C_T\) such that (with \(w^\beta := w_1^{\beta_1} \cdots w_n^{\beta_n}\))

\[
(3.9) \quad \sup_{w \in H} |w^\beta g_t(w)| = c_1 < \infty, \ t \in [0, 1]
\]

(note that the constant \(c_1\) is independent of \(t\)). Indeed once (3.5) is proved then

\[
\|g_t\|_{L^1} \leq c_1' \int_H \frac{1}{1 + |w|^{2n}} \, dw = c_1' < \infty.
\]

We will check (3.7) only for \(w^\beta = w_j\), i.e. \(\beta = (0, \ldots, 1, \ldots, 0)\) with 1 in the \(j\)-th position. The proof in the general case is similar.

We have, integrating by parts and using estimate (3.8),

\[
w_j g_t(w) = \frac{1}{(2\pi)^n} \int_H w_j e^{-i\langle w, h \rangle} \langle h, l \rangle \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) \, dr \right\} \, dh
\]

\[
= \frac{i}{(2\pi)^n} \int_H \partial_{h_j} (e^{-i\langle w, h \rangle}) \langle h, l \rangle \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) \, dr \right\} \, dh
\]

\[
= -\frac{i}{(2\pi)^n} \int_H e^{-i\langle w, h \rangle} l_j \exp \left\{ -\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) \, dr \right\} \, dh
\]
\[-\frac{i}{(2\pi)^n} \int_H e^{-i\langle w, h \rangle} \langle h, l \rangle e^{-\frac{1}{t} \int_0^t \psi_1(e^{-rA^*}h) dr} \left(-\frac{1}{t} \int_0^t \langle D\psi_1(e^{-rA^*}h), e^{-rA^*}e_j \rangle dr \right) dh.\]

Using (3.8) and the fact that $|D\psi_1(u)| \leq c_5 |u|$, we get easily that
\[
\sup_{w \in H} \left| w_j g_t(w) \right| = c_1 < \infty, \quad t \in [0, 1].
\]
The proof is complete. \qed

**Step 4. Irreducibility.** We cannot argue as in the proof of [32, Theorem 5.3] since the drift $F$ is only H"older continuous. Note, however, that if we prove that the Ornstein–Uhlenbeck process $Z_A = (Z_A(t))$,
\[
(3.10) 
Z_A(t) = \int_0^t e^{A(t-s)} dZ_s
\]
(starting at $x = 0$), is irreducible then we can obtain irreducibility for the solution $X^x$ using the following quite general result of independent interest.

**Proposition 3.3.** Assume that for each $t > 0$ the support of $Z_A(t)$ is the whole space. Then the process $(X^x)$ is irreducible, for any $x \in H$.

**Proof.** Fix $t > 0$, $a > 0$ and let $r > 0$ be any positive number. Then
\[
X_{t+a} = e^{Aa}X_t + \int_t^{t+a} e^{A(t+a-s)} F(X_s) ds + \int_t^{t+a} e^{A(t+a-s)} dZ_s.
\]
Let $z$ be any element in the support of the distribution of the random variable $e^{Aa}X_t$. Then, by the very definition, the event
\[
B = \{|e^{Aa}X_t - z| < r/3\}
\]
is of positive probability. Since $\|F\|_0 < \infty$, there exists $c > 0$ such that for each $t \geq 0$ and for each positive $b$ with probability 1
\[
\left| \int_t^{t+b} e^{A(t+b-s)} F(X_s) ds \right| \leq cb,
\]
In particular, the above inequality holds for $b = a$. Let us fix $x$ and $y$ in $H$. Then
\[
X_{t+a} - y = (e^{Aa}X_t - z) + \int_t^{t+a} e^{A(t+a-s)} F(X_s) ds + \left( \int_t^{t+a} e^{A(t+a-s)} dZ_s - y + z \right).
\]
Define the event
\[
C = \left\{ \left| y - z - \int_t^{t+a} e^{A(t+a-s)} dZ_s \right| < r/3 \right\},
\]
which, by assumption, is of positive probability. The events $B$ and $C$ are independent and therefore the probability of $B \cap C$ is positive. On this event, and thus with positive probability, we have the estimate:
\[
|X_{t+a} - y| \leq \frac{r}{3} + ca + \frac{r}{3}.
\]
Starting from number $a$ such that $ca < r/3$ we have with positive probability

$$|X_{t+a} - y| \leq r.$$ 

To finish the proof we should replace $t + a$ and $t$ with $t$ and $t - a$. □

By the previous result, we know that the proof of Step 4 is complete once the following theorem has been proved.

**Theorem 3.4.** Let $H = \mathbb{R}^n$. Assume that $Z = (Z_t)$ is an $n$-dimensional symmetric non-degenerate $\alpha$-stable process, $\alpha \in (0, 2)$. Consider any real $n \times n$ matrix $A$. Then the Ornstein–Uhlenbeck process $X(t) = Z_A(t)$ (given in (3.10) and starting at $x = 0$) is irreducible i.e., for any $t > 0$ the support of the distribution of $X(t)$ is $H$.

**Proof.** By the non-degenerate assumption (2.3) there exists $n$ points $a_1, \ldots, a_n \in S$ such that $a_k \in \text{supp}(\mu)$ for $1 \leq k \leq n$ and $\text{span}\{a_1, \ldots, a_n\} = \mathbb{R}^n$. Since $\mu$ is symmetric, $-a_1, \ldots, -a_n \in \text{supp}(\mu)$. It is clear that for any $\varepsilon > 0$, $\mu(B_s(\pm a_k, \varepsilon)) > 0$ where $B_s(a_k, \varepsilon) = \{y \in S; |y - a_k| < \varepsilon\}$.

For each $k$, let us now consider the affines $F_{k,+} := \{ra_k, r > 1\}$ and $F_{k,-} := \{-ra_k, r > 1\}$. For any point $y_k \in \{ra_k, -\infty < r < \infty\}$, there exist $y_{k,+} \in F_{k,+}$ and $y_{k,-} \in F_{k,-}$ such that $y_k = y_{k,+} + y_{k,-}$. Define $F_{k,+} := \{(x, r) : x \in B_s(a_k, \varepsilon), r > 1\}$, $F_{k,-} = \{(x, r) : x \in B_s(-a_k, \varepsilon), r > 1\}$. Take $\varepsilon > 0$ small enough to make $F_{i,+} \cap F_{j,-} = \emptyset$ for $i \neq j$ and $F_{i,+} \cap F_{i,-} = \emptyset$ for each $i$.

Decompose $\nu$ as the sum of two measures $\nu_1, \nu_2$ such that

$$\nu = \nu_1 + \nu_2,$$

and one of the measures, say $\nu_1 = \nu_1(\cup_{k=1}^{n} F_{k,+} \cup \cup_{k=1}^{n} F_{k,-} \nu)$, is finite. We can assume that the process $Z$ is the sum of two independent Lévy processes $Z^1$ and $Z^2$, with the Lévy measures $\nu_1$ and $\nu_2$ respectively. Note that

$$X^1(t) := \int_0^t e^{A(t-s)}dZ^1_s, \quad t \geq 0,$$

is a compound Poisson process. Since $\text{supp}(\mu_1) \subset \text{supp}(\mu_1 * \mu_2)$ for any two measures $\mu_1$ and $\mu_2$, it is enough to prove the irreducibility of $X^1$.

Let us fix $t > 0$, $y \in H$ and $r > 0$. It is enough to show that

$$\mathbb{P}(|X^1(t) - y| < r) > 0.$$ 

Let $M$ be a number such that for all $s \in (0, 1)$:

$$|e^{As}z| \leq M|z|, \quad |(e^{As} - I)z| \leq Ms|z|, \quad z \in H.$$ 

Write $y = \sum_{k=1}^{n} y_k a_k$ where $y_1, \ldots, y_n \in \mathbb{R}$, for each $k$ we have two points $y_{k,+} \in F_{k,+}$ and $y_{k,-} \in F_{k,-}$ and positive number $\delta < 1$ such that:

$$y_{k,+} + y_{k,-} = y_k a_k, \quad \delta M (|y_{k,+}| + |y_{k,-}|) < \frac{r}{2n},$$
Choose \( \varepsilon > 0 \) sufficiently small, the probability that the process \( Z^1 \) will perform exactly \( 2n \) jumps \( \xi_{1,-} \in F_{1,\varepsilon}, \xi_{1,+} \in F_{1,\varepsilon}^+ \), \( \ldots, \xi_{n,-} \in F_{n,\varepsilon}, \xi_{n,+} \in F_{n,\varepsilon}^+ \) before \( t \) at moments \( \tau_{1,-} < \tau_{1,+} < \tau_{2,-} < \tau_{2,+} < \ldots < \tau_{n,-} < \tau_{n,+} < t \) such that

\[
\tau_{1,-} > t - \delta, \quad |\xi_{k,-} - y_{k,-}| < \frac{r}{4nM}, \ldots, \quad |\xi_{k,+} - y_{k,+}| < \frac{r}{4nM}, \quad k = 1, \ldots, n,
\]
is positive. Therefore, at least with the same probability, the following relations hold:

\[
\begin{align*}
\left| \int_0^t e^{(t-s)A}dZ^1_s - y \right| &= \left| \sum_{j=1}^n e^{A(t-\tau_{j,-})} \xi_{j,-} + e^{A(t-\tau_{j,+})} \xi_{j,+} - y \right| \\
&= \left| \sum_{j=1}^n e^{A(t-\tau_{j,-})}(\xi_{j,-} - y_{j,-}) + e^{A(t-\tau_{j,+})}(\xi_{j,+} - y_{j,+}) \right| \\
&\quad + \left| \sum_{j=1}^n (e^{A(t-\tau_{j,-})} - I)y_{j,-} + (e^{A(t-\tau_{j,+})} - I)y_{j,+} \right| \\
&\leq \sum_{j=1}^n M (|\xi_{j,-} - y_{j,-}| + |\xi_{j,+} - y_{j,+}|) + \sum_{j=1}^n \delta M (|y_{j,-}| + |y_{j,+}|) < r.
\end{align*}
\]

This finishes the proof. \( \square \)

The proof of Theorem 2.4 is now complete.

4. ESTIMATES OF THE SOLUTION, \( \dim H = \infty \)

This section contains some preparation for the proof of Theorem 2.8, giving some estimates for the solution (2.6). Recall that the Ornstein–Uhlenbeck process is defined by

\[
Z_A(t) = \int_0^t e^{A(t-s)}dZ_s = \sum_{k \geq 1} Z_{A,k}(t)e_k
\]

where

\[
Z_{A,k}(t) = \int_0^t e^{-\gamma t_{k,s}}\beta_k dz_k(s).
\]

For any \( \varepsilon \geq 0 \), define

\[
H^\varepsilon = \left\{ x = \sum_{k \geq 1} x_k e_k \in H : \sum_{k \geq 1} \gamma_k^{2\varepsilon} |x_k|^2 < \infty \right\}
\]

Note that \( H^\varepsilon \) coincides with the domain of \( (-A)^\varepsilon \) and that \( H^0 = H \). Denote further by \( |\cdot|_\varepsilon \) the norm of \( H^\varepsilon \). For \( x \in H^\varepsilon \) and \( R > 0 \), we denote by \( B_\varepsilon(x,R) \)
the closed ball in $H^\varepsilon$ of radius $R$ centered at $x$. We shall write $B_\varepsilon(R) := B_\varepsilon(0, R)$ and $B(x, R) := B_0(x, R)$.

**Lemma 4.1.** The following assertions hold:

(i) $Z_A(t) \in H^\varepsilon$ a.s. for all $t > 0$.

(ii) For any $p \in (0, \alpha)$, we have

$$E |Z_A(t)|_p^p \leq C \left( \sum_{k \geq 1} |\beta_k|^\alpha \frac{1 - e^{-\alpha \gamma_k t}}{\alpha \gamma_k^{1-\alpha \varepsilon}} \right)^{\frac{p}{\alpha}},$$

where $C = C(\alpha, p) > 0$.

**Proof.** (i). By (4.7) in [32] we have

$$E[e^{iA(t)}] = e^{-|\lambda_c|q\varepsilon(t)},$$

where $c_k(t) = \beta_k \left( \frac{1 - e^{-\alpha \gamma_k t}}{\alpha \gamma_k} \right)^{1/\alpha}$. Hence, $Z_A(t)$ has the same distribution as $c_k(t)\xi_k$ for all $k \geq 1$ where $\{\xi_k\}_{k \geq 1}$ are i.i.d. with $E[e^{i\lambda t}] = e^{-|\lambda|^\alpha}$. We shall use Proposition 3.3 in [32], which claims that

$$(q_k\xi_k)_{k \geq 1} \in l^2 \text{ a.s. } \iff \sum_{k \geq 1} |q_k|^\alpha < \infty,$$

where $q_k \in \mathbb{R}$ for all $k$. From this it is easy to check that

$$\sum_{k \geq 1} (\gamma_k)^{2\varepsilon} [c_k(t)\xi_k]^2 < \infty \text{ a.s. } \iff \sum_{k \geq 1} \frac{\beta_k^2 \varepsilon}{\gamma_k^{1-\alpha \varepsilon}} < \infty.$$

Since $Z_A(t)$ has the same distribution as $(c_k(t)\xi_k)_{k \geq 1}$, (i) is clearly true.

(ii). We follow the argument in the proof of [32] Theorem 4.4. Take a Rademacher sequence $\{r_k\}_{k \geq 1}$ in a new probability space $(\Omega', \mathcal{F}', P')$, i.e. $\{r_k\}_{k \geq 1}$ are i.i.d. with $P\{r_k = 1\} = P\{r_k = -1\} = \frac{1}{2}$. Recall the following Khintchine inequality: for any $p > 0$, there exists some $C(p) > 0$ such that for arbitrary real sequence $\{h_k\}_{k \geq 1}$,

$$\left( \sum_{k \geq 1} h_k^2 \right)^{1/2} \leq C(p) \left( E^' \left[ \sum_{k \geq 1} r_k h_k \right]^{p/2} \right)^{1/p}.$$

By this inequality, one has

$$E |Z_A(t)|_p^p = E \left( \sum_{k \geq 1} \gamma_k^{2\varepsilon} |Z_{A,k}(t)|^2 \right)^{p/2} \leq C'E' \sum_{k \geq 1} r_k \gamma_k^{\varepsilon} Z_{A,k}(t)^p$$

$$= C'E' \left| \sum_{k \geq 1} r_k \gamma_k^{\varepsilon} Z_{A,k}(t) \right|^p,$$

where $C = C(\alpha, p) > 0$. 

**Proof.**
where $C = C^p(p)$. In view of the equality $|r_k| = 1$ and formula (4.7) of [32], for any $\lambda \in \mathbb{R}$ one has

$$
\mathbb{E} \exp \left\{ i\lambda \sum_{k \geq 1} r_k \gamma_k Z_{A,k}(t) \right\} = \exp \left\{ -|\lambda|^\alpha \sum_{k \geq 1} |\beta_k|^\alpha \gamma_k \int_0^t e^{-\alpha \gamma_k(t-s)} ds \right\} = \exp \left\{ -|\lambda|^\alpha \sum_{k \geq 1} \gamma_k \alpha c_k(t) \right\}.
$$

Now we use (3.2) in [32]: if $X$ is a symmetric random variable satisfying $\mathbb{E} [e^{i\lambda X}] = e^{-\sigma^\alpha |\lambda|^\alpha}$ for some $\alpha \in (0,2)$ and any $\lambda \in \mathbb{R}$, then $\mathbb{E}|X|^p = C(\alpha, p) \sigma^\alpha$ for all $p \in (0, \alpha)$. Since $\sum_{k \geq 1} \gamma_k \alpha c_k(t) < \infty$, it is clear to see

$$
\mathbb{E} \left| \sum_{k \geq 1} r_k \gamma_k Z_{A,k}(t) \right|^p = C(\alpha, p) \left( \sum_{k \geq 1} |\beta_k|^\alpha \frac{1 - e^{-\alpha \gamma_k t}}{\alpha \gamma_k 1 - \alpha} \right)^{\frac{p}{\alpha}},
$$

from which and (1.3) we get (1.2). \hfill \square

**Lemma 4.2.** Let $(X_t^x)$ be the solution to Eq. (1.1) with $x \in H^\varepsilon$. For any $p \in (0, \alpha)$, there exist some constants $C_1 = C_1(p) > 0$ and $C_2 = C_2(p, \varepsilon, \gamma, \beta, \|F\|_0) > 1$ such that

(4.4) \hspace{1cm} $\mathbb{E}|X_t^x|^p \leq C_1 e^{-p\gamma t}|x|^p + C_2, \quad \forall \ t > 0,$

where $C_1(p) \leq 1$ for $p \in (0, 1]$ and $C_1(p) = 3^{p-1}$ otherwise.

**Proof.** By (2.6), we have

$$
X_t = e^{At} x + \int_0^t e^{A(t-s)} F(X_s) ds + Z_A(t).
$$

It is easy to see

$$
|e^{At} x|_x \leq e^{-\gamma t}|x|_x.
$$

By the easy inequality $|(\sigma)^\varepsilon e^{At}|_{L(H)} \leq C(\sigma) t^{-\sigma}$, $t \geq 0$, $\sigma > 0$, one has

$$
\left| \int_0^t e^{A(t-s)} F(X_s) ds \right|_x \leq \int_0^t \left| (\sigma)^\varepsilon e^{A(t-s)/2} F(X_s) \right|_x ds \leq C(\varepsilon) \int_0^t (t-s)^{-\varepsilon} e^{-\gamma_1(t-s)/2} ds \|F\|_0 \leq C(\varepsilon, \gamma_1) \|F\|_0.
$$

for all $t > 0, x \in H$ and $\omega \in \Omega$. Furthermore, from (1.2),

$$
\mathbb{E}|Z_A(t)|^p \leq C(p, \alpha, \beta, \gamma, \varepsilon), \quad \forall \ p \in (0, \alpha).
$$
Now we use the following trivial inequality: for any $a, b, c \geq 0$,
\[
(a + b + c)^p \leq (a^p + b^p + c^p), \quad p \leq 1;
\]
\[
(a + b + c)^p \leq 3^{p-1} (a^p + b^p + c^p), \quad p > 1.
\]
Combining the above three estimates and the inequality, we can easily see that (4.4) is true. □

**Lemma 4.3.** Let $(X_t^x)$ be the solution to Eq. (1.1). For any $p \in (0, \alpha)$, we have
\[
(4.5) \quad \mathbb{E}|X_t^x|^p \leq C \left( (t^{-\varepsilon p}|x|^p + t^{p-\varepsilon p}\|F\|_0^p + 1 \right)
\]
for all $t > 0$, where $C = C(p, \alpha, \beta, \gamma, \varepsilon)$.

**Proof.** By (2.6) and (4.2), we have
\[
\mathbb{E}|X_t^x|^p \leq C_1 \left[ A^x e^{At} |x|^p + \mathbb{E} \left( \int_0^t |A^x e^{A(t-s)}|L(H) |F(X_s^x)|ds \right)^p + \mathbb{E}|Z_A(t)|^p \right]
\]
\[
\leq C_2 \left[ t^{-\varepsilon p}|x|^p + \left( \int_0^t (t-s)^{-\varepsilon} ds \right)^p \|F\|_0^p + 1 \right]
\]
\[
\leq C_3 \left[ t^{-\varepsilon p}|x|^p + t^{p-\varepsilon p}\|F\|_0^p + 1 \right],
\]
where $C_1 = C_1(p)$ and $C_i = C_i(p, \alpha, \beta, \gamma, \varepsilon)$ ($i = 2, 3$). □

5. **Proof of Theorem 2.8 by Harris’ approach, dim $H = \infty$**

Let us split the proof into the following three steps.

**Step 1.** The existence of an invariant measure was established in [30]. Let us prove that any invariant measure $\mu$ has finite $p^{th}$ moment ($p < \alpha)$:
\[
(5.1) \quad m_p(\mu) := \int_H |x|^p \mu(dx) < \infty \quad \text{for any } p \in (0, \alpha).
\]
Indeed, by (2.6) and the trivial inequality
\[
(a + b) \wedge c \leq a \wedge c + b \wedge c, \quad a, b, c \in \mathbb{R}^+,
\]
for all $t > 0$ and $n \in \mathbb{N}$, we have
\[
|X_t^x|^p \wedge n \leq \left[ (C_p e^{-\rho^*t} |x|^p) \wedge n + C_p \left( \int_0^t e^{A(t-s)} F(X_s^x) ds \right)^p + C_p |Z_A(t)|^p \right].
\]
Using a similar calculation as in Lemma 4.2 we obtain
\[
\mathbb{E}(\|X_t^x\|^p) \leq \left( C_p e^{-\rho^*t} |x|^p \right) \wedge n + C,
\]
where $C = C(\alpha, \beta, \gamma, p, \|F\|_0)$. Integrating this inequality against $\mu(dx)$, we get
\[
\mu(|x|^p \wedge n) \leq \mu \left[ (C_p e^{-\rho^*t} |x|^p) \wedge n \right] + C.
\]
Passing to the limit first as $t \to \infty$ and then as $n \uparrow \infty$, we complete the proof of (5.1).

**Step 2.** To prove the uniqueness of an invariant measure and inequality (2.9), it suffices to show that

\[
\|P_{kT}(x_1, \cdot) - P_{kT}(x_2, \cdot)\|_{TV} \leq C \left(1 + |x_1|^p + |x_2|^p\right)e^{-ckT}, \quad x_1, x_2 \in H,
\]

where $C$ and $c$ are positive constants not depending on $x_1$, $x_2$, and $k$. Indeed, if (5.2) is established, then for any measures $\nu_1, \nu_2 \in \mathcal{P}(H)$ with finite $p$th moment we derive

\[
\|P_{kT}^* \nu_1 - P_{kT}^* \nu_2\|_{TV} \leq C \left(1 + m_p(\nu_1) + m_p(\nu_2)\right)e^{-ckT}, \quad k \in \mathbb{N}.
\]

This implies, in particular, that an invariant measure is unique. Moreover, writing any $t \geq 0$ in the form $t = kT + s$ with $0 \leq s < T$ and using inequalities (5.3) and (4.4), we obtain

\[
\|P_{kT}^* \nu_1 - P_{kT}^* \nu_2\|_{TV} = \|P_{kT}^* (P_s^* \nu_1) - P_{kT}^* (P_s^* \nu_2)\|_{TV}
\leq C \left(1 + m_p(P_s^* \nu_1) + m_p(P_s^* \nu_2)\right)e^{-ckT}
\leq C_1 \left(1 + m_p(\nu_1) + m_p(\nu_2)\right)e^{-ct}.
\]

This estimate readily implies the required inequality (2.9).

Note that (5.2) holds if we are able to apply Theorem 2.10 to equation (1.1) with $V(x) = |x|^p$ and $p \in (0, \alpha)$. Indeed, once this is done, we obtain that there exists $T > 0$ such that

\[
\|P_{kT}(x_1, \cdot) - P_{kT}(x_2, \cdot)\|_{TV} \leq \int_H \left(1 + V(x)\right)|P_{kT}^* \delta_{x_1} - P_{kT}^* \delta_{x_2}|(dx)
\leq \beta^k \int_H \left(1 + V(x)\right)|\delta_{x_1} - \delta_{x_2}|(dx)
\leq 2\beta^k \left(1 + |x_1|^p + |x_2|^p\right), \quad k \geq 1.
\]

This immediately implies (5.2).

**Step 3.** It remains to check the conditions (i) and (ii) in Theorem 2.10. Choosing $V(x) = |x|^p$ with $p \in (0, \alpha)$ and applying Lemma 4.2 with $\varepsilon = 0$ and $T_0 > \frac{\log(1+C_1)}{p\gamma}$, one immediately get (i).

To prove (ii), we shall use the following lemma proved in [32].

**Lemma 5.1 (Theorem 5.4, [32]).** Let $(X^x_t)$ be the solution to Eq. (1.1). Then $(X^x_t)$ is irreducible on $H$, i.e., for any $t > 0$ and $B(y, r)$ with arbitrary $y \in H$ and $r > 0$, we have

\[
\mathbb{P}(X^x_t \in B(y, r)) > 0.
\]
Let $x$ and $y$ satisfy $|x|^p + |y|^p \leq R$. By Lemma 4.3 we know that, for any fixed $T_0 > 0$,
\[ \mathbb{E}[|X_{T_0}^x|] + \mathbb{E}[|X_{T_0}^y|] \leq C(|x|^p + |y|^p + 1) \leq C_1. \]
It follows that there exists some $R_1 > 0$ such that
\[ \mathbb{P}\left(|X_{T_0}^x| \leq R_1 \right) > 1/2, \quad \mathbb{P}\left(|X_{T_0}^y| \leq R_1 \right) > 1/2. \]

Since $\gamma_k \to \infty$, $B_\delta(M)$ is compact in $H$. By Lemma 5.1 for any $r > 0$ we have some $\delta(r) > 0$ such that
\[ (5.5) \quad \inf_{x \in B_\delta(R_1)} \mathbb{P}\left(X_{T_0}^x \in B(r)\right) \geq 2\delta. \]

By Markov property and the above three inequalities,
\[ \mathbb{P}\left(X_{T_0}^x \in B(r)\right) > \delta, \quad \mathbb{P}\left(X_{T_0}^y \in B(r)\right) > \delta. \]
Without loss of generality, in the next computations we assume that $X_t^x$ and $X_t^y$ are independent (this is true if the driving noises of $X_t^x$ and $X_t^y$ are independent).

By Markov property and Theorem 2.5,
\[
\|P_{T_0}^x \delta - P_{T_0}^y \delta\|_{TV} = \frac{1}{2} \sup_{\|\phi\|_0 \leq 1} \left| \mathbb{E}[P_{T_0}^x \phi(X_{2T_0}^x) - P_{T_0}^y \phi(X_{2T_0}^y)] \right|
\leq \left[ 1 - \mathbb{P}\{X_{2T_0}^x \in B(\delta), X_{2T_0}^y \in B(\delta)\} \right]
+ \frac{1}{2} \mathbb{E} \left\{ \sup_{\|\phi\|_0 \leq 1} \left| P_{T_0}^x \phi(X_{2T_0}^x) - P_{T_0}^y \phi(X_{2T_0}^y) \right| X_{2T_0}^x \in B(\delta), X_{2T_0}^y \in B(\delta) \right\}
\leq 1 - \mathbb{P}\{X_{2T_0}^x \in B(\delta), X_{2T_0}^y \in B(\delta)\} + C r \mathbb{P}\{X_{2T_0}^x \in B(\delta), X_{2T_0}^y \in B(\delta)\}
\leq 1 - (1 - C r)\delta^2.
\]
Taking $r > 0$ sufficiently small, we complete the proof.

6. Proof of Theorem 2.8 by coupling, $\dim H = \infty$

In this section, we shall prove Theorem 2.8 by the Doeblin coupling argument, which gives much more intuitions for understanding the way that the dynamics converges to the ergodic measure.

6.1. Construction of the coupling chain. Let us first give some preliminary about maximal coupling.

**Definition 6.1.** Let $\mu_1$, $\mu_2 \in \mathcal{P}(H)$. A pair of random variables $(\xi_1, \xi_2)$ defined on the same probability space is called a coupling for $(\mu_1, \mu_2)$ if $\mathcal{D}(\xi_i) = \mu_i$ for $i = 1, 2$, where $\mathcal{D}(\cdot)$ denotes the distribution of random variable. A coupling $(\xi_1, \xi_2)$ is said to be maximal if
\[ (6.1) \quad \mathbb{P}\{\xi_1 \neq \xi_2\} = \|\mu_1 - \mu_2\|_{TV}, \]
Lemma 6.2. For any two measures \( \mu_1, \mu_2 \in \mathcal{P}(H) \), there exists a maximal coupling. Moreover, if \( (\xi_1, \xi_2) \) is a maximal coupling, then we have

\[
\mathbb{P}(\xi_1 \in A, \xi_2 \in A) \geq \mathbb{P}(\xi_1 \in A) \mathbb{P}(\xi_2 \in A), \quad \forall \ A \in \mathcal{B}(H).
\]

In what follows, we shall the need the following lemma whose proof can be found in [10, 19, 18].

Now let us construct an auxiliary Markov chain in the extended phase space \( H \times H \). Let \( T > 0 \) be some fixed real number to be chosen later. For any \( x := (x_1, x_2) \in H \times H \), denote by \( M(x) = (M_1(x), M_2(x)) \) the maximal coupling of \( (P_T)^* \delta_{x_1} \) and \( (P_T)^* \delta_{x_2} \). Let us define a transition function \( \tilde{P}_T(x, \cdot) \) on the space \( H \times H \) such that

\[
\tilde{P}_T(x; A_1 \times A_2) = \begin{cases} 
P_T(x_1, A_1 \cap A_2) & \text{if } x_1 = x_2, \\
D(M_1(x), M_2(x))(A_1 \times A_2) & \text{if } x_1, x_2 \in B(r) \text{ with } x_1 \neq x_2, \\
P_T(x_1, A_1)P_T(x_2, A_2) & \text{otherwise},
\end{cases}
\]

where \( A_1, A_2 \in \mathcal{B}(H) \) are arbitrary sets, \( P_T(x_1, \cdot) \) is the transition probability of \( X_T^{x_i} \) for \( i = 1, 2 \), and \( D(\cdot) \) denotes the distribution of a random variable. For any \( A \in \mathcal{B}(H \times H) \), \( \tilde{P}_T(x, A) \) is uniquely defined by a classical approximation procedure. Now the transition function \( \tilde{P}_T(x, \cdot) \) is well defined.

6.2. Hitting times \( \tau^\varepsilon \) and \( \tau \). We denote by \( (X_1(kT), X_2(kT))_{k \in \mathbb{Z}^+} \) the Markov chain whose transition function is equal to \( \tilde{P}_T(x, \cdot) \); here \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \). Clearly, for each \( i = 1, 2 \), \( (X_i(kT)) \) is also a Markov chain and has the same distribution as \( (X_T^{x_i}) \). We shall write \( X(kT) = (X_1(kT), X_2(kT)) \) for \( k \in \mathbb{Z}^+ \).

For any \( r, M > 0 \), define the hitting times

\[
\tau^\varepsilon = \inf\{kT; |X_1(kT)|_\varepsilon + |X_2(kT)|_\varepsilon \leq M\},
\]

\[
\tau = \inf\{kT; |X_1(kT)| + |X_2(kT)| \leq r\},
\]

where \( \varepsilon \in (0, 1) \) is the constant in Assumption 2.2. Recall that the infimum over an empty set is equal to \( +\infty \).

\footnote{Inequality (6.3) is true for any pair of random variables that are independent conditioned on the event \( \{\xi_1 \neq \xi_2\} \).}
6.2.1. Estimates of the hitting time $\tau^\varepsilon$. The main result of this subsection is the following theorem, which is in fact a step for estimating $\tau$.

**Theorem 6.3.** For any $p \in (0, \alpha)$ and sufficiently large $T > 0$ there is a constant $M = M(p, T, \alpha, \beta, \gamma, \varepsilon)$ such that, for any $x = (x_1, x_2) \in H \times H$,

$$
E_x [e^{\eta \tau^\varepsilon}] \leq C \left(1 + |x_1|^p + |x_2|^p\right)
$$

where $\eta > 0$ is sufficiently small, and $C = C(p, T, \alpha, \beta, \gamma, \varepsilon, \|F\|_0, \eta)$

To prove Theorem 6.3, we first establish two auxiliary lemmas.

**Lemma 6.4.** For any $p \in (0, \alpha)$, the Markov chain $(X(T))$ satisfies the inequality

$$
E_x (|X_1(T)|^p + |X_2(T)|^p) \leq C_1 e^{-p \gamma T} \left( |x_1|^p + |x_2|^p \right) + 2C_2,
$$

where $C_1$ and $C_2$ are the same as in Lemma 4.2.

**Proof.** By definition of coupling and Lemma 4.2, we have

$$
E_x X_1(T)^p = E X_1^p \leq C_1 e^{-p \gamma T} |x_1|^p + C_2.
$$

for $i = 1, 2$. From the above inequality, we complete the proof. □

**Lemma 6.5.** For any $p \in (0, \alpha)$ and sufficiently large $T > 0$, there exist positive constants $q = q(p, \gamma) \in (0, 1)$ and $M = M(p, T, \alpha, \beta, \gamma, \varepsilon, \|F\|_0, \varepsilon)$ such that

$$
P_x (\tau^\varepsilon > kT) \leq q^k \left(1 + |x_1|^p + |x_2|^p\right)
$$

for any $x = (x_1, x_2) \in H \times H^\varepsilon$.

**Proof.** The proof follows the idea in [9]. Let us take $T > 0$ so large that the coefficient in front of $|x|_\varepsilon^p$ in inequality (4.4) is smaller than 1. In this case, setting $P = P_x$, $E = E_x$, and

$$
|x|_\varepsilon^p = |x_1|_\varepsilon^p + |x_2|_\varepsilon^p,
$$

we can write

$$
E \left[|X(kT + T)|^p \right| F_{kT} \right] \leq q^2 |X(kT)|^p + 2C_2
$$

where $q > 0$ is defined by the relation $q^2 = C_1 e^{-p \gamma T} < 1$. By Chebyshev inequality,

$$
P \left(|X(kT + T)|_\varepsilon > M |F_{kT}\right) \leq \frac{q^4}{M^p} |X(kT)|^p + \frac{2C_2}{M^p}.
$$

Denote

$$
B_k = \{|X(jT)|_\varepsilon > M; j = 0, \ldots, k\}
$$

and

$$
p_k = P(B_k), \ e_k = E \left(|X(kT)|^p 1_{B_k}\right),
$$

where
integrating (6.8) over $B_k$, one has
\begin{equation}
 p_{k+1} \leq \frac{q^2}{M^p} e_k + \frac{2C_2}{M^p} p_k.
\end{equation}
Moreover, by integrating (6.7) over $B_k$,
\begin{equation}
 e_{k+1} \leq \mathbb{E}(|X(kT + T)|_\varepsilon^p 1_{B_k}) \leq q^2 e_k + 2C_2 p_k.
\end{equation}
From (6.9) and (6.10), one has
\begin{equation}
 e_{k+1} p_{k+1} \leq \left( \frac{q^2}{M^p} 2C_2 \right) \left( e_k p_k \right),
\end{equation}
which clearly implies
\begin{equation}
 q^2 e_{k+1} + 2C_2 p_{k+1} \leq \left( q^2 + \frac{2C_2}{M^p} \right) (q^2 e_k + 2C_2 p_k)
\end{equation}
We can choose $M = M(p, T, \alpha, \beta, \gamma, \varepsilon, \|F\|_0)$ so that
\begin{equation}
 q^2 + \frac{2C_2}{M^p} \leq q.
\end{equation}
Thus we clearly have from (6.12)
\begin{equation}
 q^2 e_k + 2C_2 p_k \leq q^k (q^2 e_0 + 2C_2 p_0),
\end{equation}
This inequality, together with the easy fact $p_k = \mathbb{P}_x(\tau^\varepsilon > kT)$, immediately implies the required estimate (6.6) since $C_2 > 1$ in inequality (4.4).

\begin{proof}[Proof of Theorem 6.3]
By the definition of coupling and (4.5), for any $p \in (0, \alpha)$ we have
\begin{equation}
 \mathbb{E}_x \left( |X_1(T)|_\varepsilon^p + |X_2(T)|_\varepsilon^p \right) = \mathbb{E}_x X_{T_1}^\varepsilon|_{e}^p + \mathbb{E}_x X_{T_2}^\varepsilon|_{e}^p \leq C_4 (1 + |x_1|^p + |x_2|^p)
\end{equation}
where $C_4 = C_4(p, T, \alpha, \beta, \gamma, \varepsilon, \|F\|_0)$.
For any $x = (x_1, x_2) \in H \times H$, by Markov property, (6.6) and the above inequality, we easily have
\begin{equation}
 \mathbb{E}_x \left[ e^{\eta \tau^\varepsilon} \right] = \mathbb{E}_x \left( e^{\eta \tau^\varepsilon} 1_{\{\tau^\varepsilon \leq T\}} \right) + \mathbb{E}_x \left( e^{\eta \tau^\varepsilon} 1_{\{\tau^\varepsilon > T\}} \right)
\end{equation}
\begin{equation}
 \leq e^{\eta T} + \mathbb{E}_x \left( 1_{\{\tau^\varepsilon > T\}} \mathbb{E}_X(\tau^\varepsilon|_{e}^p) \right)
\end{equation}
\begin{equation}
 \leq e^{\eta T} + C_5 \mathbb{E}_x \left( 1 + |X_1(T)|_\varepsilon^p + |X_2(T)|_\varepsilon^p \right)
\end{equation}
\begin{equation}
 \leq C_6 \left( 1 + |x_1|^p + |x_2|^p \right)
\end{equation}
where $C_i = C_i(p, \alpha, \eta, \beta, \varepsilon, \|F\|_0, T)$ ($i = 5, 6$).
\end{proof}
6.2.2. Estimates of the hitting time $\tau$.

**Theorem 6.6.** For any $p \in (0, \alpha)$ and sufficiently large $T > 0$, there exist positive constants $\lambda = \lambda(T, p, \alpha, \beta, \gamma, \|F\|_0, r)$ and $C = C(p, \alpha, \beta, \gamma, \|F\|_0, r, T)$ such that

\[
E_x[e^{\lambda \tau}] \leq C(1 + |x_1|^p + |x_2|^p).
\]

The key point of the proof is to use Theorem 6.3 and Lemma 6.7 below. The argument is quite general, for simplicity, let us give its heuristic idea by using $(X_{kT})$, (note the difference between $X_{kT}$ and $X(kT)$), as follows:

(i) Since $B_\varepsilon(M)$ is compact in $H$, by irreducibility and uniform strong Feller property we have that $\inf_{z \in B_\varepsilon(0, M)} P_T(z, B(R)) = p > 0$. Therefore, as long as $X_{kT}$ is in $B_\varepsilon(M)$, it has the probability at least $p$ to jump into $B(R)$ at $(k+1)T$.

(ii) Suppose that $(X_{kT})$ enters $B_\varepsilon(M)$ for $j$ times before it jumps into $B(R)$, by strong Markov property and (i) this event happens with some probability less than $(1 - p)^j$.

(iii) If $\tau = kT$ for some large $kT$ (i.e. the process first enters $B(R)$ at $kT$), $j$ is also large. Thus $\mathbb{P}(\tau = kT) \leq (1 - p)^j$ is small.

Let us now make the above heuristic argument rigorous for $(X(kT))$. We first need to establish the following lemma.

**Lemma 6.7.** For any compact set $K \subset H \times H$ and any $R > 0$, there exists some constant $\delta = \delta(K, R) > 0$ such that

\[
\inf_{x \in K} \mathbb{P}_x\{X(T) \in B(R) \times B(R)\} > 0.
\]

**Proof.** To show (6.16), we split the argument into the following three cases.

(i) As $x \notin B(R) \times B(R)$ with $x_1 \neq x_2$, $X_1(T)$ and $X_2(T)$ are independent. Therefore, by Lemma 5.1 one has

\[
\mathbb{P}_x(X(T) \in B(R) \times B(R)) = \mathbb{P}_x(X_1(T) \in B(R)) \mathbb{P}_x(X_2(T) \in B(R)) = \mathbb{P}(X_1^{x_1} \in B(R)) \mathbb{P}(X_2^{x_2} \in B(R)) > 0.
\]

(ii) As $x = (x_1, x_2)$ with $x_1 = x_2$, we have $X_1(T) = X_2(T)$. Hence,

\[
\mathbb{P}_x(X(T) \in B(R) \times B(R)) = \mathbb{P}(X_1^{x_1} \in B(R)) > 0.
\]

(iii) As $x \in B(R) \times B(R)$ with $x_1 \neq x_2$, by the maximal coupling property (6.2), one has
\[
\mathbb{P}_x(X(T) \in B(R) \times B(R)) = \mathbb{P}_x(M(x) \in B(R) \times B(R)) \\
\geq \mathbb{P}_x(M_1(x) \in B(R))\mathbb{P}_x(M_2(x) \in B(R)) \\
= \mathbb{P}(X^{x_1}_T \in B(R))\mathbb{P}(X^{x_2}_T \in B(R)) > 0,
\]
where \(M(x) = (M_1(x), M_2(x))\) is the maximal coupling of \((P^*_\epsilon x_1, P^*_\epsilon x_2)\).

\[
\text{From (i)-(iii) it is clear that}
\]
\[
\mathbb{P}_x(X(T) \in B(R) \times B(R)) \geq \mathbb{P}(X^{x_1}_T \in B(R))\mathbb{P}(X^{x_2}_T \in B(R)).
\]

By Feller property of \(P_T\) and Lemma 6.1, for any open subset \(O \subset H\) the function \(x \mapsto P_T(x, O)\) is positive and lower semi-continuous. Hence, it is separated from zero on any compact subset. Therefore, there is a constant \(\delta = \delta(x, R, T) > 0\) so that

\[
\inf_{x \in K} \mathbb{P}(X^{x_1}_T \in B(R))\mathbb{P}(X^{x_2}_T \in B(R)) > 0.
\]

\text{From the above two inequality, we complete the proof.} \qed

\textbf{Proof of Theorem 6.6.} Take \(M = M(p, T, \alpha, \beta, \gamma, \|F\|_0)\) defined in Theorem 6.3 and simply write

\[
|x|^p = |x_1|^p + |x_2|^p, \quad x = (x_1, x_2) \in H \times H.
\]

Let us prove the theorem in the following four steps:

\textit{Step 1.} Write \(\tau_0^\varepsilon = 0, \tau_1^\varepsilon = \tau^\varepsilon\) and define

\[
\tau_{k+1}^\varepsilon = \inf\{jT > \tau_k^\varepsilon; |X_1(jT)|\leq M, |X_2(jT)|\leq M\}
\]

for all integer \(k \geq 1\). Since \((X(kT))\) is a discrete time Markov chain, it is strong Markovian. By Theorem 6.3 and Poincare inequality \(|z| \leq \frac{1}{\tau_1^\varepsilon} |z|_e\) for any \(z \in H^e\), we have

\[
\mathbb{E}_x\left[ e^{\eta(\tau_{k+1}^\varepsilon - \tau_k^\varepsilon)} \right] \leq C(1 + |X(\tau_k^\varepsilon)|^p) \leq c(1 + M^p),
\]

where \(c = C(1 + 2^p/\gamma_1^p)\) and \(C = C(p, \alpha, \beta, \gamma, \|F\|_0, r, T)\) is the same as in Theorem 6.3. The above inequality, together with strong Markov property, implies

\[
\mathbb{E}_x\left[ e^{\eta \tau_k^\varepsilon} \right] \leq c^k(1 + M^p)^{k-1}(1 + |x|^p).
\]

\textit{Step 2.} Since \(B_\varepsilon(M) \subset H\), by Lemma 6.7 we have
\[
\inf_{y \in B(M) \times B(M)} \mathbb{P}_y \left( X(T) \in B(r) \times B(r) \right) = \sigma,
\]
for all \( r > 0 \), where \( \sigma = \sigma(\varepsilon, M, r, T) > 0 \). Therefore, for some \( \sigma \in (0, 1) \),
\[
(6.20) \quad \inf_{|y| \leq M} \mathbb{P}_y \left( X(T) \in B(r) \times B(r) \right) \geq \sigma,
\]
where \( |y| = |y_1| + |y_2| \).

**Step 3.** Given any \( k \in \mathbb{N} \), define
\[
\rho_k = \sup \{ j ; \tau_\varepsilon^j \leq kT \}.
\]
Clearly, \( \tau_{\rho_k+1} > kT \). For any \( k \in \mathbb{N} \), one has
\[
\mathbb{P}_x(\tau = kT) = \sum_{j=0}^{l} \mathbb{P}_x(\tau = kT, \rho_k = j)
\]
\[
(6.21) \quad = \sum_{j=0}^{l} \mathbb{P}_x(\tau = kT, \rho_k = j) + \sum_{j=l+1}^{k} \mathbb{P}_x(\tau = kT, \rho_k = j)
\]
where \( l < k \) is some integer number to be chosen later.

**Step 4.** Let us estimate the above \( I_1 \) and \( I_2 \). By the definition of \( \rho_k \), Chebyshev inequality and strong Markov property, we have
\[
\mathbb{P}_x(\tau = kT, \rho_k = j) \leq \mathbb{P}_x(\tau_\varepsilon^j > kT/2) + \mathbb{P}_x(\tau_\varepsilon^j \leq kT/2, \rho_k = j)
\]
\[
\leq \mathbb{P}_x(\tau_\varepsilon^j > kT/2) + \mathbb{P}_x(\tau_\varepsilon^j \leq kT/2, \tau_{\rho_k+1} > kT)
\]
\[
\leq e^{-\eta kT/2} \mathbb{E}_x \left[ e^{\eta \tau_\varepsilon^j} \right] + \mathbb{E}_x \left[ \mathbb{P}_X(\tau_\varepsilon^{j+1} > \tau_\varepsilon^j > kT/2) \right]
\]
By \( 6.19 \) and \( 6.18 \), the above inequality implies
\[
\mathbb{P}_x(\tau = kT, \rho_k = j) \leq c^j (1 + M^p)^{-j-1} (1 + |x|^p) e^{-\eta kT/2} + c(1 + M^p) e^{-\eta kT/2}.
\]
Hence,
\[
(6.22) \quad I_1 \leq \left[ c^{j+1} (1 + M^p)^{j+1} (1 + |x|^p) + l c(1 + M^p) \right] e^{-\eta kT/2}
\]
\[
\leq c^{j+2} (1 + M^p)^{j+2} (1 + |x|^p) e^{-\eta kT/2}.
\]
Now we estimate \( I_2 \). For \( j > l \), by the definitions of \( \tau \) and \( \rho_k \), strong Markov property and \( 6.20 \), we have
\[
\mathbb{P}_x(\tau = kT, \rho_k = j) \leq \mathbb{P}_x \left( |X(\tau_1^j)| > r, \ldots, |X(\tau_j^j)| > r \right) \leq (1 - \sigma)^j.
\]
Hence,

\[(6.23)\quad I_2 \leq \frac{1}{\sigma}(1 - \sigma)^{l+1}.\]

Taking \(\bar{\eta} = \frac{n}{4\log(c+\sqrt{TP})}\) and \(l = [\eta kT]\), we have

\[I_1 \leq e^{-knT/4} (1 + |x|^p), \quad I_2 \leq \frac{1}{\sigma} \exp \{-kT\bar{\eta} \log \frac{1}{1-\sigma}\}.\]

Combining the above estimates of \(I_1\) and \(I_2\), and taking \(2\lambda = \frac{\sigma}{4} \wedge \bar{\eta} \log \frac{1}{1-\sigma}\), we have

\[\mathbb{P}_x(\tau = kT) \leq \left(c^2 + \frac{1}{\sigma}\right) e^{-2\lambda kT} (1 + |x|^p)\]

From the above inequality, we immediately obtain the desired estimate. \(\square\)

6.3. Final part of the coupling proof. It is divided into two steps.

Step 1. By the same reason as in Steps 1 and 2 in Section 5, to prove the uniqueness of an invariant measure and inequality \(\text{[2.9]}\), it suffices to show that

\[(6.24)\quad \|P_{kT}(x_1, \cdot) - P_{kT}(x_2, \cdot)\|_{TV} \leq C (1 + |x_1|^p + |x_2|^p)e^{-ckT}, \quad x_1, x_2 \in H,\]

where \(C\) and \(c\) are positive constants not depending on \(x_1, x_2,\) and \(k\). Let \((X_1(t), X_2(t)), t \in T\mathbb{Z}\), be the chain constructed in Section 6.1. Define the stopping time

\[\rho = \min\{kT : k \in \mathbb{N}, X_1(kT) = X_2(kT)\},\]

where the minimum over an empty set is equal to \(+\infty\). Suppose we have proved that

\[(6.25)\quad \mathbb{P}_x\{\rho > kT\} \leq Ce^{-\eta kT} (1 + |x_1|^p + |x_2|^p),\]

where \(x = (x_1, x_2) \in H \times H\) is arbitrary, and the positive constants \(\eta\) and \(C\) do not depend on \(x\). In this case, using the fact that \(X_1(kT) = X_2(kT)\) for \(k \geq l\) as soon as \(X_1(lT) = X_2(lT)\), we can write

\[|P_{kT}(x_1, \Gamma) - P_{kT}(x_2, \Gamma)| = |\mathbb{E}_x 1_{\Gamma}(X_1(kT)) - \mathbb{E}_x 1_{\Gamma}(X_2(kT))|\]

\[= \mathbb{E}_x \left(1_{\{\rho > kT\}} 1_{\Gamma}(X_1(kT)) - 1_{\Gamma}(X_2(kT))\right)\]

\[\leq \mathbb{P}_x\{\rho > kT\}.\]

Using \(\text{[6.25]}\), we obtain

\[|P_{kT}(x_1, \Gamma) - P_{kT}(x_2, \Gamma)| \leq Ce^{-\eta kT} (1 + |x_1|^p + |x_2|^p).\]

Taking the supremum over all \(\Gamma \in \mathcal{B}(H)\), we arrive at the required inequality \(\text{[5.2]}\).

Step 2. Thus, it remains to establish \(\text{[6.25]}\). To this end, we first note that if \(r > 0\) is sufficiently small, then

\[(6.26)\quad \mathbb{P}_x\{X_1(T) \neq X_2(T)\} \leq 1/2 \quad \text{for any} \ x \in B(r) \times B(r).\]
Indeed, by Theorem 2.4 for any function \( f \in B_b(H) \) with \( \|f\|_0 \leq 1 \) we have
\[
\|(P_T(x_1, \cdot), f) - (P_T(x_2, \cdot), f)\| = \|P_T f(x_1) - P_T f(x_2)\| \leq C_1 |x_1 - x_2| \quad \text{for } x_1, x_2 \in H.
\]

Recalling the definition of the total variation distance, we see that
\[
\|P_T(x_1, \cdot) - P_T(x_2, \cdot)\|_{TV} \leq 1/2, \quad x_1, x_2 \in B(r),
\]
where \( r > 0 \) is sufficiently small. Since \( (X_1(T), X_2(T)) \) is a maximal coupling for the pair \( (P_T(x_1, \cdot), P_T(x_2, \cdot)) \), by (6.25) we arrive at (6.26).

We now introduce the iterations \( \{\tau_n\} \) of the stopping time \( \tau \) defined by (6.4):
\[
\tau_1 = \tau, \quad \tau_{n+1} = \inf\{jT > \tau_n : |X_1(jT)| + |X_2(jT)| \leq r\}.
\]

An argument similar to that used in Step 1 of the proof of Theorem 6.6 shows that
\[
\mathbb{E}_x e^{\lambda \tau_n} \leq K^n (1 + |x_1|^p + |x_2|^p),
\]
where \( K > 1 \) and \( \lambda > 0 \) do not depend on \( x_1, x_2 \in H \) and \( n \geq 1 \). By the Chebyshev inequality, it follows that
\[
(6.27) \quad \mathbb{P}_x \{\tau_n > kT\} \leq e^{-\lambda k T} K^n (1 + |x_1|^p + |x_2|^p).
\]

Let us define the events
\[
\Gamma_n = \{X_1(\tau_m + T) \neq X_2(\tau_m + T) \text{ for } 1 \leq m \leq n\}
\]
and set \( P_n(x) = \mathbb{P}_x(\Gamma_n) \). By (6.26) and the strong Markov property, we have
\[
\mathbb{P}_x \{X_1(\tau_n + T) \neq X_2(\tau_n + T) \mid \mathcal{F}_{\tau_n}\} \leq \mathbb{P}_{X(\tau_n)} \{X_1(T) \neq X_2(T)\} \leq 1/2
\]

It follows that
\[
P_n(x) = \mathbb{P}_x (\Gamma_{n-1} \cap \{X_1(\tau_n + T) \neq X_2(\tau_n + T)\})
\]
\[
= \mathbb{E}_x (1_{\Gamma_{n-1}} \mathbb{P}_x \{X_1(\tau_n + T) \neq X_2(\tau_n + T) \mid \mathcal{F}_{\tau_n}\}) \leq \frac{1}{2} P_{n-1}(x),
\]
whence, by iteration, we get \( P_n(x) \leq 2^{-n} \) for any \( n \geq 1 \). Combining this with (6.27), for any integers \( n, k \geq 1 \) we obtain
\[
\mathbb{P}_x \{\rho > kT\} = \mathbb{P}_x \{\rho > kT, \tau_n < kT\} + \mathbb{P}_x \{\rho > kT, \tau_n \geq kT\}
\]
\[
\leq \mathbb{P}_x (\Gamma_n) + \mathbb{P}_x \{\tau_n \geq kT\}
\]
\[
\leq 2^{-n} + e^{-\lambda k T} K^n (1 + |x_1|^p + |x_2|^p).
\]

Taking \( n = \varepsilon k \) with a sufficiently small \( \varepsilon > 0 \), we arrive at the required inequality (6.25). The proof of Theorem 2.8 is complete.
7. PROOFS OF EXPONENTIAL MIXING WHEN $\dim H < \infty$

First of all, by Theorem 2.5 of [30], the system in (3.1) has at least one invariant measure. To prove Theorem 2.7 we can use the Harris method or the coupling argument.

In both approaches we need also the decay estimates for solutions given in Lemmas 4.2 and 4.3. These can be easily adapted to the strong solution $X_t$ in (3.1) (indeed, by the Gronwall lemma, starting from (3.1), we get $\mathbb{E}|Z_A(t)|^p < \infty$ for any $p \in (0, \alpha)$).

For the Harris approach, in order to verify the two conditions in Theorem 2.10 we can repeat the same argument as in Section 5.

For the coupling approach, the key point is irreducibility and gradient estimates of Theorem 2.4. Using a similar (but easier) argument as in Section 6 we can prove Theorem 2.7 in the following three steps:

1. constructing the coupling and defining the stopping time $\tau$ exactly as in Section 6.1;
2. proving the exponential estimate (6.15);
3. using the same argument as in Section 6.3 which involves the coupling time.

Finally, let us emphasize that unlike the infinite-dimensional setting, we do not need to introduce $H^\varepsilon$ and $\tau^\varepsilon$ to get some compactness, since any finite-dimensional closed ball is automatically compact.

REFERENCES

1. S. Agmon, Lectures on elliptic boundary value problems, Van Nostrand, Toronto–New York–London, 1965.
2. S. Albeverio, V. Mandrekar, and B. Rüdiger, Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise, Stochastic Process. Appl. 119 (2009), no. 3, 835–863.
3. S. Albeverio, B. Rüdiger, and J. L. Wu, Invariant measures and symmetry property of Lévy type operators, Potential Anal. 13 (2000), no. 2, 147–168.
4. S. Albeverio, J. L. Wu, and T. S. Zhang, Parabolic SPDEs driven by Poisson white noise, Stochastic Process. Appl. 74 (1998), no. 1, 21–36.
5. Z. Brzezniak, B. Goldys, P. Imkeller, S. Peszat, E. Priola and J. Zabczyk, Time irregularity of generalized Ornstein-Uhlenbeck processes, C. R. Acad. Sci. Paris Ser. Math. 348 (2010), 273–276.
6. A. Chojnowska-Michalik, On processes of Ornstein–Uhlenbeck in Hilbert spaces, Stochastics 21 (1987), 251-286.
7. G. Da Prato and F. Flandoli, Pathwise uniqueness for a class of SDE in Hilbert spaces and applications, J. Funct. Anal. 259 (2010), no. 1, 243–267.
8. G. Da Prato and J. Zabczyk, Ergodicity for infinite-dimensional systems, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996.
9. A. Debussche, Stochastic Navier-Stokes equations: well posedness and ergodic properties, preprint (available on http://php.math.unifi.it/users/cime/).
10. W. Doeblin, Éléments d’une théorie générale des chaînes simples constantes de Markoff, Ann. Sci. École Norm. Sup. (3) 57 (1940, 61–111.
11. T. Funaki and B. Xie, A stochastic heat equation with the distributions of Lévy processes as its invariant measures, Stochastic Process. Appl. 119 (2009), no. 2, 307–326.
12. M. Hairer, Exponential mixing properties of stochastic PDEs through asymptotic coupling, Probab. Theory Related Fields 124 (2002), no. 3, 345–380.
13. J. C. Mattingly, Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics, Comm. Math. Phys. 230 (2002), no. 3, 421–462.
14. A. M. Kulik, Exponential ergodicity of the solutions to SDE’s with a jump noise, Stochastic Process. Appl. 119 (2009), no. 2, 602–632.
15. S. Kuksin and A. Shirikyan, A coupling approach to randomly forced nonlinear PDEs. I, Comm. Math. Phys. 221 (2001), no. 2, 351–366.
16. J. C. Mattingly, Exponential mixing for the 3D stochastic Navier-Stokes equations, Comm. Math. Phys. 270 (2007), no. 1, 109–139.
17. E. Priola, Exponential mixing for stochastic PDEs: the non-additive case, Probab. Theory Related Fields 140 (2008), no. 1-2, 41–82.
18. B. Øksendal, Stochastic partial differential equations driven by multi-parameter white noise of Lévy processes, Quart. Appl. Math. 66 (2008), no. 3, 521–537.
19. S. Peszat and J. Zabczyk, Stochastic partial differential equations with Lévy noise, Encyclopedia of Mathematics and its Applications, vol. 113, Cambridge University Press, Cambridge, 2007.
20. E. Priola, Pathwise uniqueness for singular SDEs driven by stable processes, Osaka Journal of Mathematics, to appear (available on http://www.newton.ac.uk/preprints/NI10062.pdf).
21. E. Priola, L. Xu and J. Zabczyk, Exponential mixing for some SPDEs with Lévy noise, Stochastic and Dynamics, to appear (arXiv:1010.4530).
22. E. Priola and J. Zabczyk, Densities for Ornstein-Uhlenbeck processes with jumps, Bulletin of the London Mathematical Society 41 (2009), 41–50.
23. E. Priola and J. Zabczyk, Structural properties of semilinear SPDEs driven by cylindrical stable processes, Probab. Theory Related Fields, published online (arXiv:0810.5063v1).
24. E. Priola and J. Zabczyk, On linear evolution equations with cylindrical Lévy noise, to appear in Proceedings “SPDE’s and Applications - VIII”, Quaderni di Matematica, Seconda Università di Napoli (arXiv:0908.0356v1).
25. A. Rusinek, Mean reversion for HJMM forward rate models, Adv. in Appl. Probab. 42 (2010), no. 2, 371–391.
35. K. I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.

36. K. I. Sato and M. Yamazato: *Stationary processes of Ornstein–Uhlenbeck type*, Lect. Notes in Math. **1021** (1983), 541–551.

37. A. Shirikyan, *Exponential mixing for 2D Navier-Stokes equations perturbed by an unbounded noise*, J. Math. Fluid Mech. **6** (2004), no. 2, 169–193.

38. ______, *Exponential mixing for randomly forced partial differential equations: method of coupling*, Instability in models connected with fluid flows. II, Int. Math. Ser. (N. Y.), vol. 7, Springer, New York, 2008, pp. 155–188.

39. A. Takeuchi, *The Bismut–Elworthy–Li type formulae for stochastic differential equations with jumps*, Journal of Theoretical Probability, to appear [arXiv:1002.1384].

40. H. Thorisson, *Coupling, stationarity, and regeneration*, Springer-Verlag, New York, 2000.

41. L. Xu and B. Zegarlinski, *Ergodicity of the finite and infinite dimensional α-stable systems*, Stoch. Anal. Appl. **27** (2009), no. 4, 797–824.

42. ______, *Existence and exponential mixing of infinite white alpha-stable systems with unbounded interactions*, Electron. J. Probab. **15** (2010), 1994–2018.

43. F.Y. Wang, *Gradient estimate for Ornstein–Uhlenbeck jump processes*, Stoch. Proc. Appl., to appear [arXiv:1005.5023].

44. J. Zabczyk, *Stationary distributions for linear equations driven by general noise*, Bull. Pol. Acad. Sci. **31** (1983), 197–209.

---

**Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy**

*E-mail address*: enrico.priola@unito.it

**Department of Mathematics, University of Cergy–Pontoise, CNRS UMR 8088, 2 avenue Adolphe Chauvin, 95302 Cergy–Pontoise, France**

*E-mail address*: Armen.Shirikyan@u-cergy.fr

**TU Berlin, Fakultät II, Institut für Mathematik, Straße des 17. Juni 136, D-10623 Berlin, Germany**

*E-mail address*: xu@math.tu-berlin.de

**Institute of Mathematics, Polish Academy of Sciences, P-00-950 Warszawa, Poland**

*E-mail address*: zabczyk@impan.pl