OPTIMIZED ESTIMATES OF THE REGULARITY OF THE CONDITIONAL DISTRIBUTION OF THE SAMPLE MEAN

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Abstract. We give an improved estimate for the regularity of the conditional distribution of the empirical mean of a finite sample of IID random variables, conditional on the sample "fluctuations", extending the well-known property of Gaussian IID samples. Specifically, we replace the bounds in probability, established in our earlier works, by those in distribution, and this results in the optimal regularity exponent in the final estimate.

1. Introduction

Consider a sample of $N$ IID (independent and identically distributed) random variables with Gaussian distribution $\mathcal{N}(0, 1)$, and introduce the sample mean $\xi = \xi_N$ and the "fluctuations" $\eta_i$ around the mean:

$$\xi_N = \frac{1}{N} \sum_{i=1}^{N} X_i, \quad \eta_i = X_i - \xi_N, \quad i = 1, \ldots, N.$$ 

It is well-known from elementary courses of the probability theory that $\xi_N$ is independent from the sigma-algebra $\mathcal{F}_\eta$ generated by $\{\eta_1, \ldots, \eta_N\}$ (the latter are linearly dependent, and have rank $N - 1$). To see this, it suffices to note that $\eta_i$ are all orthogonal to $\xi_N$ with respect to the standard scalar product in the linear space formed by $X_1, \ldots, X_N$ given by

$$\langle Y, Z \rangle := \mathbb{E}[Y Z],$$

where $Y$ and $Z$ are real linear combinations of $X_1, \ldots, X_N$ (recall: $\mathbb{E}[X_i] = 0$).

Therefore, the conditional probability distribution of $\xi_N$ given $\mathcal{F}_\eta$ coincides with the unconditional one, so $\xi_N \sim \mathcal{N}(0, N^{-1})$, thus $\xi_N$ has bounded density

$$p_\xi(t) = \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi N^{-1}}} \leq \frac{N^{1/2}}{\sqrt{2\pi}}.$$ 

Moreover, for any interval $I \subset \mathbb{R}$ of length $|I|$, we have

$$\text{ess sup} \mathbb{P} \left\{ \xi_N(\omega) \in I \mid \mathcal{F}_\eta \right\} = \mathbb{P} \left\{ \xi_N(\omega) \in I \right\} \leq \frac{N^{1/2}}{\sqrt{2\pi}} |I|. \quad (1.1)$$

The essential supremum in the above LHS is a bureaucratic tribute to the formal rule saying that $\mathbb{P} \left\{ \cdot \mid \mathcal{F}_\eta \right\}$ is a random variable (which is $\mathcal{F}_\eta$-measurable), and as such is defined, generally speaking, only up to subsets of measure zero.

In some applications to the eigenvalue concentration estimates in the theory of multi-particle random, Anderson-type Hamiltonians, one has to estimate the
Figure 1. In this example, \( N = 2, \xi = \frac{1}{2}(X_1 + X_2) \) and \( \eta = \frac{1}{2}(X_1 - X_2) \). One has to assess the probability of the pink curvilinear strip \( \{ (X_1, X_2) : \xi \in [a(\eta), a(\eta) + s] \} \).

The probability of the form
\[
P\{ \xi_N(\omega) \in I(\eta) \},
\]
where the interval \( I(\eta) = [f(\eta), f(\eta) + \epsilon] \) is determined only by the fluctuations \( \eta \), and \( f \) is some measurable (in fact, Lipschitz continuous\(^1\)) function. For example, with \( N = 2 \),
\[
\xi = \xi_2 = \frac{X_1 + X_2}{2}, \quad \eta = \eta_1 = \frac{X_1 - X_2}{2},
\]
one may consider the probability
\[
P\{ \xi \in [\eta^2, \eta^2 + s] \} = (2\pi)^{-1} \int_{\mathbb{R}^2} dX_1 dX_2 e^{-\frac{1}{2}(x_1^2 + x_2^2)} 1_A(x_1, x_2)
\]
where, e.g.,
\[
A := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{(x_1 - x_2)^2}{4} \leq \frac{x_1 + x_2}{2} \leq \frac{(x_1 - x_2)^2}{4} + s \right\}, \quad s > 0.
\]

In this particular case – for Gaussian samples – the conditional regularity of the sample mean \( \xi_N \) (given the fluctuations) is granted, but is not always so, as shows the following elementary example where the common probability distribution of the sample \( X_1, X_2 \) is just excellent: \( X_i \sim \text{Unif}([0, 1]) \), so \( X_i \) admit a compactly supported probability density bounded by 1. In this simple example the random vector \( (X_1, X_2) \) is uniformly distributed in the unit square \([0, 1]^2\), and the condition \( \eta = c \) selects a straight line in the two-dimensional plane with coordinates \( (X_1, X_2) \), parallel to the main diagonal \( \{X_1 = X_2\} \). The conditional distribution of \( \xi \) given \( \{\eta = c\} \) is the uniform distribution on the segment
\[
J_c := \{ (x_1, x_2) : x_1 - x_2 = 2c, \; 0 \leq x_1, x_2 \leq 1 \}
\]
of length vanishing at \( 2c = \pm 1 \). For \( |2c| = 1 \), the conditional distribution of \( \xi \) on \( J_c \) is concentrated on a single point, which is the ultimate form of singularity.

\(^1\)We refer to the applications where \( f \) is an eigenvalue of some self-adjoint operator, and by the min-max principle, such EVs are Lipschitz continuous functions of the parameters upon which the operator depends.
2. An application to the Wegner-type bounds

Let $\Lambda$ be a finite graph, with $|\Lambda| = N \geq 1$, and $H(\omega) = H_\Lambda(\omega)$ be a random DSO acting in the finite-dimensional Hilbert space $\mathcal{H} = \ell^2(\Lambda)$, with IID random potential $V : \Lambda \times \Omega \to \mathbb{R}$, relative to a probability space $(\Omega, \mathcal{F}, P)$. Decomposing the random field $V$ on $\Lambda$,

$$V(x; \omega) = \xi_N(\omega) + \eta_x(\omega),$$

we can represent $H(\omega)$ as follows:

$$H(\omega) = \xi_N(\omega) \mathbf{1} + A(\omega),$$

where the self-adjoint operator $A(\omega)$ is $\mathcal{F}_\eta$-measurable, and so are its eigenvalues $\tilde{\mu}_j(\omega)$, $j = 1, \ldots, N$. It is readily seen that $A(\omega)$ is a DSO with potential having zero sample mean. Since $A(\omega)$ commutes with the scalar operator $\xi_N(\omega) \mathbf{1}$, the eigenvalues $\lambda_j(\omega)$ of $H(\omega)$ have the form

$$\lambda_j(\omega) = \xi_N(\omega) + \mu_j(\omega).$$

The numeration of the eigenvalues $\lambda_j(\omega)$, $\mu_j(\omega)$ is, of course, not canonical, but they can be consistently defined as random variables on $\Omega$.

The representation (2.1) implies immediately the following EVC bound: for any interval $I_s = [t, t + s]$,

$$\mathbb{P} \{ \text{tr} P_{I_s} (H(\omega)) \geq 1 \} \leq \sum_{j=1}^{N} \mathbb{P} \{ \lambda_j(\omega) \in I_s \} = \sum_{j=1}^{N} \mathbb{P} \{ \xi_N(\omega) + \mu_j(\omega) \in I_s \}
= \sum_{j=1}^{N} \mathbb{E} \left[ \mathbb{P} \{ \xi_N(\omega) + \mu_j(\omega) \in I_s | \mathcal{F}_\eta \} \right]
= \sum_{j=1}^{N} \mathbb{E} \left[ \mathbb{P} \{ \xi_N(\omega) \in [-\mu_j(\omega) + t, -\mu_j(\omega) + t + s] | \mathcal{F}_\eta \} \right]$$

(2.2)

Further, omitting the argument $\omega$ for notational brevity, we have

$$\mathbb{P} \{ \xi_N + \tilde{\mu}_j \in I_s | \mathcal{F}_\eta \} = \mathbb{P} \{ \xi_N \in [\tilde{\mu}_j + t, \tilde{\mu}_j + t + s] | \mathcal{F}_\eta \}
= \mathbb{P} \{ \xi_N \in [\tilde{\mu}_j, \tilde{\mu}_j + s] | \mathcal{F}_\eta \}$$

where $\tilde{\mu}_j(\omega) := -\mu_j(\omega) + t$ are $\mathcal{F}_\eta$-measurable, i.e., fixed under the conditioning. Now introduce the conditional continuity modulus of $\xi_N$, given $\mathcal{F}_\eta$:

$$\nu_N(s) := \sup_{t \in \mathbb{R}} \text{ess sup} \mathbb{P} \{ \xi_N \in [t, t + s] | \mathcal{F}_\eta \}, \quad s > 0.$$ 

Obviously,

$$\mathbb{P} \{ \lambda_j \in I_s | \mathcal{F}_\eta \} \leq \nu_N(s),$$

thus the unconditional probability $\mathbb{P} \{ \lambda_j \in I_s \}$ can be assessed by analyzing the probability distribution of the random conditional continuity modulus $\nu_N(s; \omega)$.

In this section, we discuss by way of example the Wegner-type bounds for a conventional, single-particle DSO, but in applications to the multi-particle EVC bounds, similar objects turn out to be of interest:

$$s \mapsto \mathbb{P} \{ \xi_N(\omega) \in [\tilde{\mu}(\omega), \tilde{\mu}(\omega) + s] \},$$

(2.3)
where an $\tilde{\mathcal{F}}_\mathcal{K}$-measurable random variable $\tilde{\mu}$ is given by an eigenvalue of yet another operator $\tilde{H}(\omega)$ which is not necessarily independent of $H(\omega)$. The most difficult case is where $H(\omega)$ and $\tilde{H}(\omega)$ are stochastically correlated in a very strong way: every "local" random variable, representing the disorder in a multi-particle Anderson model, which affects $H(\omega)$ also affects $\tilde{H}(\omega)$, and vice versa. As a result, there is little one can say about $\tilde{\mu}(\omega)$, except that it is a measurable function.

3. Reduction to the local analysis in the sample space

Assume that the support $\mathcal{S} \subset \mathbb{R}$ of the common continuous marginal probability measure $\mathbb{P}_\mathcal{V}$ of the IID random variables $X_j$, $1 \leq j \leq N$, is covered by a finite or countable union of intervals:

$$\mathcal{S} \subset \bigcup_{k \in \mathcal{K}} J_k,$$

where $\mathcal{K} \subset \mathbb{Z}$, $J_k = [a_k, b_k]$, $a_{k+1} \geq b_k$.

Let $\mathbf{K} = \mathcal{K}^N$, and for each $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbf{K}$, denote

$$J_\mathbf{k} = \times_{i=1}^N J_{k_i}.$$

Owing to the continuity of the marginal measure, $J_k$ are "essentially" disjoint: for all $k \neq l$, $\mathbb{P}_\mathcal{V}(J_k \cap J_l) = 0$. Respectively, the family of the parallelepipeds $\{J_\mathbf{k}, \mathbf{k} \in \mathbf{K}\}$ forms a partition $\mathcal{K}$ of the sample space, which we will often identify with the probability space $\Omega$. Further, let $\tilde{\mathcal{F}}_\mathcal{K}$ be the sub-sigma-algebra of $\tilde{\mathcal{F}}$ generated by the partition $\mathcal{K}$. Now the quantities of the general form (2.3) can be assessed as follows:

$$\mathbb{P} \{ \xi_N \in [\bar{\mu}, \bar{\mu} + s] \} = \mathbb{E} \left[ \mathbb{P} \{ \xi_N \in [\bar{\mu}, \bar{\mu} + s] \mid \tilde{\mathcal{F}}_\mathcal{K} \} \right]$$

$$= \sum_{\mathbf{k} \in \mathbf{K}} \mathbb{P} \{ J_\mathbf{k} \} \mathbb{P} \{ \xi_N \in [\bar{\mu}, \bar{\mu} + s] \mid J_\mathbf{k} \}.$$

Let $\mathbb{P}_\mathbf{k} \{ \cdot \}$ be the conditional probability measure, given $\{X \in J_\mathbf{k}\}$, $\mathbb{E}_\mathbf{k} [\cdot]$ the respective expectation, and $p_\mathbf{k} = \mathbb{P} \{ J_\mathbf{k} \}$. Then we have

$$\mathbb{P} \{ \xi_N \in [\bar{\mu}, \bar{\mu} + s] \} = \sum_{\mathbf{k} \in \mathbf{K}} p_\mathbf{k} \mathbb{E}_\mathbf{k} \left[ \mathbb{P}_\mathbf{k} \{ \xi_N \in [\bar{\mu}, \bar{\mu} + s] \mid \tilde{\mathcal{F}}_\mathcal{K} \} \right]$$

$$\leq \sup_{\mathbf{k} \in \mathbf{K}} \mathbb{E}_\mathbf{k} \left[ \mathbb{P}_\mathbf{k} \{ \xi_N \in [\bar{\mu}, \bar{\mu} + s] \mid \tilde{\mathcal{F}}_\mathcal{K} \} \right]. \quad (3.1)$$

This simple formula shows that one may seek a satisfactory upper bound on the LHS of (3.1) by assessing the "local" conditional probabilities $\mathbb{P}_\mathbf{k} \{ \xi_N \in [\bar{\mu}, \bar{\mu} + s] \mid \tilde{\mathcal{F}}_\mathcal{K} \}$, where each random variable $X_j$ is restricted to a subinterval $J_{k_j}$ of its global support, so the entire sample $X = (X_1, \ldots, X_N)$ is restricted to a parallelepiped $\mathbf{J} \subset \mathbb{R}^N$.

In the next section, we perform such analysis first in the case of a uniform marginal distribution of the IID variables $X_i$.

4. Uniform marginal distributions

Let be given a real number $\ell > 0$ and an integer $N \geq 2$. Consider a sample of $N$ IID random variables with uniform distribution $\text{Unif}([0, \ell])$, and introduce again the sample mean $\bar{\xi} = \xi_N$ and the "fluctuations" $\eta_i$ around the mean:

$$\xi_N = \frac{1}{N} \sum_{i=1}^N X_i, \quad \eta_i = X_i - \xi_N.$$
Further, consider the Euclidean space \( \sim \eta \) variables \( c \) along an element \( X \). Here, \( \tilde{\eta} \) is a natural length parameter on the elements \( X(Y) \). For later use, note that, owing to (4.4), each of the re-scaled variables \( N^{1/2}X_i \) can serve as the (normalized) length parameter on the elements \( X(Y) \).

Along an element \( X(Y) \), one can simultaneously parameterize \( \tilde{\xi} \) and the variables \( X_i \), by setting \( \tilde{\xi}(t) = c_0 + t \), \( X_j(t) = c_j + N^{-1/2}t \), with arbitrarily chosen constants \( c_j \). Here, \( \tilde{\xi}_N \) is a natural length parameter on \( X(Y) \), since the transformation \( X \mapsto (\tilde{\xi}_N, \tilde{\eta}_1, \ldots, \tilde{\eta}_{N-1}) \) is orthogonal.

Remark 4.1. For later use, note that, owing to (4.4), each of the re-scaled variables \( N^{1/2}X_i \) can serve as the (normalized) length parameter on the elements \( X(Y) \).
It follows from (4.4) that for any given \(a \in \mathbb{R}, s > 0\), and some \(a' \in \mathbb{R}\),

\[
\xi_N \in [a, a + s] \iff \xi_N \in [a', a' + N^{1/2}s]
\]  \hspace{1cm} (4.5)

Next, denote \(J^{(\ell)} = [0, \ell]^N\) and introduce the random variable

\[
\nu_N(s; J^{(\ell)}) = \nu_N(s; J^{(\ell)}; X) := \text{ess sup sup} \{ \xi_N \in [t, t + s] \mid \mathcal{F}_0 \}. \hspace{1cm} (4.6)
\]

Here the presence of ess sup is the tribute to the fact that the conditional probabilities are random variables, usually defined up to subsets of zero measure; \(\ell > 0\) is the width of the common uniform distribution of \(X_j\). Equivalently, one may write \(\nu_N(s; J^{(\ell)}; \omega)\) instead of \(\nu_N(s; J^{(\ell)}; X)\), since the sample space \(\mathbb{R}^N\) is identified with the underlying probability space \(\Omega\).

Since \(\{X_i\}\) are IID with uniform distribution on \([0, \ell]\), the distribution of the random vector \(X(\omega)\) is uniform in the cube \(J^{(\ell)} = [0, \ell]^N\), inducing a uniform conditional distribution on each element \(\mathcal{X}(Y)\). Therefore, by (4.5) and (4.6),

\[
\nu_N(s; J^{(\ell)}) = \frac{N^{1/2}s}{|\mathcal{X}(Y)|}. \hspace{1cm} (4.7)
\]

It is to be stressed that both sides of the above equality are random variables:

\(\nu_N(s; \ell) = \nu_N(s; \ell; \omega)\) by its definition in (4.6), and \(\mathcal{X}(Y) = \mathcal{X}(Y(X(\omega)))\).

5. SHORT INTERVALS ARE UNLIKELY

**Lemma 1.** Assume that the IID random variables \(X_1, \ldots, X_N\), \(N \geq 2\), admit (common) probability density \(p_Y\) with \(\|p_Y\|_\infty \leq \mathcal{F} < \infty\). Then

\[
\mathbb{P}\{ |\mathcal{X}(Y)| < r \} \leq \frac{1}{4\mathcal{F}} r^2 N. \hspace{1cm} (5.1)
\]

In particular, for \(X_i \sim \text{Unif}([0, \ell])\), one has

\[
\mathbb{P}\{ |\mathcal{X}(Y)| < r \} \leq \frac{r^2 N}{4\ell^2}. \hspace{1cm} (5.2)
\]

**Proof.** Let

\[
\overline{X} = \overline{X}(X) = \min_i X_i, \hspace{0.5cm} \underline{X} = \underline{X}(X) = \max_i X_i. \hspace{1cm} (5.3)
\]

While \(\overline{X}(X)\) and \(\underline{X}(X)\) vary along the elements \(\mathcal{X}(Y)\), their difference \(\overline{X}(X) - \underline{X}(X)\) does not; it is uniquely determined by \(\mathcal{X}(Y)\).

According to Remark 4.1, each \(N^{1/2}X_i\), \(i = 1, \ldots, N\), restricted to \(\mathcal{X}(Y)\), provides a normalized length parameter on \(\mathcal{X}(Y)\); thus the range of each \(N^{1/2}X_i|\mathcal{X}(Y)\) is an interval of length \(|\mathcal{X}(Y)|\). One can increase (resp., decrease), e.g., the value of \(X_1\), as long as all \(\{X_i, 1 \leq i \leq N\}\) are strictly smaller than \(\ell\) (resp., strictly positive). Therefore, the maximum increment of \(X_1\) (indeed, of any \(X_i\) along \(\mathcal{X}(Y)\)) is given by \(\ell - \overline{X}(X)\), and its maximum decrement equals \(\underline{X}(X)\), so the range of the normalized length parameter \(N^{1/2}X_1\) along \(\mathcal{X}(Y(X))\) is an interval of length \(N^{1/2}(\ell - \overline{X}(X) + \underline{X}(X))\):

\[
|\mathcal{X}(Y(X))| = N^{1/2}(\ell - \overline{X}(X) + \underline{X}(X)), \hspace{1cm} (5.4)
\]

Since both \(\overline{X}(X)\) and \(\ell - \overline{X}(X)\) are non-negative,

\[
\overline{X} + (\ell - \overline{X}) < t \implies \max\{\underline{X}, \ell - \overline{X}\} < t/2. \hspace{1cm} (5.5)
\]
With $0 \leq t \leq \ell$, $(\ell - X_i < t/2)$ implies $(X_i > t/2)$, thus denoting
\[ A_{i,j}(t) := \{X_i < t/2\} \cap \{\ell - X_j < t/2\}, \quad (5.6) \]
we have, for any $i$,
\[ A_{ii}(t) = \{X_i < t/2\} \cap \{\ell - X_i < t/2\} = \emptyset. \quad (5.7) \]
Therefore,
\[
\left\{ \max \left\{ X(X), \ell - X(X) \right\} < \frac{t}{2} \right\} \subset \bigcup_{i \neq j} \left\{ X_i < \frac{t}{2}, \ell - X_j < \frac{t}{2} \right\}. \quad (5.8)
\]
Thus the union $\bigcup_{i \neq j} A_{i,j}(t)$ contains all samples $X$ with $|X(Y)| < t/2$.

The sample $\{X_k\}$ is IID, with common probability density uniformly bounded by $\mathfrak{r} < \infty$, so for any $i \neq j$
\[
P \{ A_{i,j}(t) \} = P \left\{ X_i < \frac{t}{2} \right\} \cdot P \left\{ \ell - X_j < \frac{t}{2} \right\} = \frac{1}{4} \mathfrak{r}^2 t^2.
\]
Therefore,
\[
P \{ |X(Y)| < r \} = P \left\{ N^{1/2} ((\ell - X(X)) + X(X)) < r \right\} = P \left\{ ((\ell - X(X)) + X(X)) < rN^{1/2} \right\} 
\leq \sum_{i \neq j} P \left\{ A_{i,j}(rN^{-1/2}) \right\} \leq N(N - 1) \frac{(\mathfrak{r}N^{-1/2})^2}{4} \quad (5.9)
\]
\[
\leq \frac{1}{4} \mathfrak{r}^2 r^2 N.
\]

6. Regularity bound for the uniform distributions

**Theorem 1.** Let be given IID random variables $X_1, \ldots, X_N$ with $X_i \sim \text{Unif}([0, \ell])$ and a measurable function $\lambda : Y \mapsto \lambda(Y)$. In each interval $\mathcal{X}(Y) \subset \tilde{X}(Y) \equiv \mathbb{R}$, introduce the sub-interval $I_s(Y) = [\lambda(Y), \lambda(Y) + s] \cap \tilde{X}(Y)$. For any $s \in (0, 1]$,
\[
P \{ \xi(\omega) \in I_s(Y) \} \leq \frac{N^2 (8\ell + 1)}{4\ell^2} s \quad (6.1)
\]

**Proof.** $I(\omega) := |X(Y)|$

\[
P \{ \xi \in I_s(\eta) \} = E \left[ P \left\{ \xi \in I_s(\eta) \mid \Xi_\eta \right\} \right] = E \left[ I_{I(\omega) < s} P \left\{ \xi \in I_s(\eta) \mid \Xi_\eta \right\} \right] + E \left[ I_{I(\omega) \geq s} P \left\{ \xi \in I_s(\eta) \mid \Xi_\eta \right\} \right] 
\leq P \{ I(\omega) < s \} + E \left[ I_{I(\omega) \geq s} P \left\{ \xi \in I_s(\eta) \mid \Xi_\eta \right\} \right] \quad (6.2)
\]

where, by virtue of $(5.9)$,
\[
P \{ I(\omega) < s \} \leq \frac{N}{4\ell^2} s^2, \quad (6.3)
\]
yielding
\[
\sup_{s > 0} \frac{P \{ I(\omega) < s \}}{s^2} \leq \frac{N}{4\ell^2}. \quad (6.4)
\]
The second summand in the RHS of (6.2) can be assessed as follows:

\[
\mathbb{E} \left[ 1_{t \geq s} \mathbb{P} \{ \xi \in I_s(\eta) \mid \mathcal{F}_s \} \right] \leq \mathbb{E} \left[ 1_{t \geq s} \frac{1}{t} \right] = s \mathbb{E} \left[ 1_{t \geq s} t^{-1} \right] = s \int_s^\ell r^{-1} dF_t(r)
\]  

Using integration by parts for the Stiltjes integral and (6.4), we obtain

\[
\int_s^\ell r^{-1} dF_t(r) = \left. \frac{F_t(r)}{r} \right|_s^\ell + \int_s^\ell r^{-2} F_t(r) dr \\
\leq \frac{1}{\ell} + (\ell - s) \sup_{r > 0} \frac{F_t(r)}{r^2} \leq \frac{1}{\ell} + \frac{\ell N}{4\ell^2}
\]

\[
= \frac{N}{\ell} \left( \frac{1}{4} + \frac{4}{N} \right)
\]

\[
\leq \left\{ \begin{array}{ll}
\frac{9N}{4}, & N = 2; \\
\frac{2N}{\ell}, & N \geq 3.
\end{array} \right.
\]

Collecting (6.3), (6.5) and (6.6), the assertion follows:

\[
\mathbb{P} \{ \xi \in I_s(\eta) \} \leq \frac{N^2}{4\ell^2} s^2 + \frac{2N}{\ell} N s \leq \frac{2N}{\ell} s \left( \frac{Ns}{8\ell} + 1 \right) \leq \frac{N^2 (8\ell + 1)}{4\ell^2} s.
\]

\[\square\]

An elementary analysis of the RHS of the inequality (6.7) leads to the following

**Corollary 2.** Under the assumptions of Theorem 1, for any \( s \in (0, 8\ell/N) \),

\[
\mathbb{P} \{ \xi \in I_s(\eta) \} \leq 4\ell^{-1} N s;
\]

in particular, for \( \ell \geq 1 \) and \( s \in (0, 8\ell/N) \),

\[
\mathbb{P} \{ \xi \in I_s(\eta) \} \leq 4Ns.
\]

Furthermore, for any \( \ell \leq 1 \) and any \( s \in (0, 1] \),

\[
\mathbb{P} \{ \xi \in I_s(\eta) \} \leq \frac{9}{4} \ell^{-2} N^2 s < 3\ell^{-2} N^2 s,
\]

while for \( \ell \geq 1 \) and any \( s \in (0, 1] \),

\[
\mathbb{P} \{ \xi \in I_s(\eta) \} \leq \frac{9}{4} \ell^{-1} N^2 s < 3\ell^{-1} N^2 s.
\]
7. Smooth positive densities

Now we consider a richer class of probability distributions. While the conditions which we will assume are certainly very restrictive, they are quite sufficient for applications to physically realistic Anderson models.

**Theorem 3.** Assume that the common probability distribution of the IID random variables $V_j$, $j = 1, \ldots, N$, with PDF $F_V$, satisfies the following conditions:

(i) the probability distribution is absolutely continuous:

\[
F_V(v) = \rho(v) \, dv, \quad \text{supp } \rho = [a, a + \ell];
\]

(ii) the probability density $\rho(\cdot)$ has bounded logarithmic derivative on $(a, a + \ell)$:

\[
\left\| (\ln \rho)' 1_{(a, a + \ell)} \right\|_{\infty} \leq C_p' < +\infty.
\]

Then there exists a constant $C = C(F_V, \ell) < \infty$ such that for any $s \in (0, \ell N^{-2})$ and any $\mathcal{F}_N$-measurable random variable $\lambda$, setting $I_s(\omega) := [\lambda(\omega), \lambda(\omega) + s]$, one has the following bound:

\[
P\{ \xi_N(\omega) \in I_s(\omega) \} \leq CNs.
\]

**Proof.** Without loss of generality, it suffices to prove the claim for supp $\rho = [0, \ell]$, which we assume below.

\[
\rho(x) \in [e^{-\alpha_N}, e^{+\alpha_N}].
\]

Now introduce in $J_k$:

- the uniform probability distribution $\tilde{P}_k$, i.e., the normalized measure with constant density $\tilde{p}_k$ w.r.t. the Lebesgue measure;
- the probability distribution induced by $P$, conditional on $\{X \in J_k\}$, i.e., the normalized measure with density $p_k(x) = Z_k^{-1} p(x) = \frac{p(x)}{\int_{J_k} P(y) \, dy}$. 

\[
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\]
By continuity of the density $p$, \( \int_{J_k} P(y) \, dy = c|J_k| \), for some $c \in [e^{-\alpha N}, e^{+\alpha N}]$, so

\[
\frac{P_k(x)}{p(x)} = \frac{p(x)}{c} \in [e^{-2\alpha N}, e^{+2\alpha N}]
\]

Hence for any event $\mathcal{A}$, we have

\[
e^{-2\alpha N} P\{\mathcal{A}\} \leq P_k\{\mathcal{A}\} \leq e^{+2\alpha N} P\{\mathcal{A}\}
\] (7.4)

It follows from (7.4) and (3.1) that

\[
P\{\xi \in I_s(\eta)\} \leq \sup_k P_k\{\xi \in I_s(\eta)\} \leq C(F_V, \ell) N s.
\] (7.5)

Recall that this bound was proved only for $s \leq \ell/M(N) = o(\ell N^{-1})$.

\[\square\]

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