Vector-tensor gravity from a broken gauge symmetry

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Abstract. In this paper we present a Yang-Mills type gauge theory of vector-tensor gravity, where the tetrad, the spin connection and vector field are identified with components of the gauge field. This setup leads to a theory that in flat spacetime is contained in Generalized Proca theories, while in curved spacetime is closely related to beyond Generalized Proca. We solve for static and spherically symmetric spacetime and show that there are two branches of solutions, one where the metric is asymptotically Schwarzschild even though there is a cosmological constant in the action, and another one where the metric is asymptotically (anti-)de Sitter. Also, we study the effect of the vector field on homogeneous and isotropic spacetimes, finding that it contributes to the accelerated expansion of the spacetime.

Keywords: Gauge theories of gravity, Modified gravity.
1. Introduction

The greatest example of the geometrization of the fundamental interactions is Einstein’s General Relativity (GR). Since then, gravity goes hand in hand with geometry to the point that we identify the gravitational phenomena as a manifestation of a curved space-time. Moreover, Yang-Mills (YM) theory, which is the basis for the standard model of particle physics, is another geometrical theory. Although there are many differences between GR and YM there have been attempts to unify them in the framework of some classical field theory [1, 2].

To construct a unified model for the fundamental interactions, different approaches have been used. One can consider higher dimensional models of gravity where the metric is the main field, and it components describe the fundamental interactions [3, 4]. On the other hand, elementary interactions are described is by the connection associated to an internal symmetry group where the space-time is non-dynamical (but let us remember that in GR the metric is a dynamical entity). Various attempts have been made to construct Yang-Mills type gauge theories of gravity. There are interesting formulations in the literature, known as pure connection actions for gravity [5, 6, 7, 8]. The fundamental field is gauge field (for a corresponding symmetry group). Therefore, the metric is no longer the main field for describing gravity, but a derived object. Consequently, GR arises from the proposed gauge theory.

The description of the gravitational field without explicit reference to a metric, but rather to gauge fields or $p$-forms has been largely developed in the past years, with some motivation coming from the attempts to quantize gravity because of the relative simplicity that this constructions entail. In these theories the metric is reconstructed from the dynamical fields under consideration. This kind of descriptions are sometimes referred to as form theories of gravity. One of the first and best known examples of this type of theories is MacDowell-Mansouri (MM) gravity [9] (see [10] for a review). It is constructed purely from the field strength of the gauge potential on either the de-Sitter group $SO(4, 1)$ for a positive cosmological constant, or the anti de Sitter group $SO(3, 2)$ for a negative cosmological constant. The gauge potential acts as an internal connection that unifies into a single object the tetrad and spin connection used in the Palatini formulation. This is done by associating the translational part of the gauge connection to the tetrad and the Lorentz part to the spin connection. By explicitly breaking the original gauge group to its Lorentz subgroup, one obtains the action for MM gravity. This action turns out to be equivalent to Einstein-Hilbert gravity with cosmological constant, supplemented with the Euler topological term. The MM action is thus an elegant mathematical construct with deep connections to the infrared and ultraviolet physics of space-time. It naturally includes a cosmological constant and signals the way to the inclusion of topological terms that modify the quantum predictions of the theory with respect to those of pure GR [11].

Einstein-Hilbert gravity with a cosmological constant is the basis of the standard cosmological model, ΛCDM, which is in good agreement with many different
observations. Despite the observational success of the ΛCDM model, there are motivations to search for alternatives. First of all, in order to conclude that ΛCDM is our best cosmological model we need to compare it against other models. Second, the ingredients of ΛCDM – cold dark matter and a dark energy sector modelled by a cosmological constant – are difficult to reconcile with the theoretical pillars of contemporary physics. In view of this, there have been several proposals for alternative theories of gravity. A large class of these proposals modifies gravity by adding new degrees of freedom, for instance from scalar or vector fields. The development of these proposals led to theories such as Horndeski, beyond Horndeski, and degenerate scalar-tensor theories of gravity in the case of scalar fields [12, 13, 14], or generalised Proca in the case of vector fields [15, 16, 17], which admit a rich phenomenology with solutions relevant for astrophysics and cosmology. This rich phenomenology comes with a caveat: the Lagrangians for these theories include several self-interactions of the scalar or vector fields and their derivatives. However, it has been shown that subsets of these theories can be recovered from simpler Lagrangians in higher dimensional setups, such as brane-world scenarios or compactifications [18, 19, 20], or considering a Higgs mechanism [21].

Following the discussion in the previous paragraphs, it seems natural to explore how these scalar and vector-tensor theories might emerge from a construction inspired in Yang-Mills gauge theory. Here we focus on the vector-tensor theories, and we show that a modification of the MacDowell-Mansouri framework allows one to construct a model of generalised Proca. Therefore, the well established gauge symmetry, is a physical principle to derive a class of vector-tensor theories.

This work is organised as follows. In Sec. 2 we present the standard procedure to obtain GR with a cosmological constant from the MM action. In Sec. 3 we review the vector-tensor theory known as generalised Proca. In Sec. 4 we obtain a vector-tensor theory from a construction similar to the one used to obtain GR from the MM action, and we show the relation between this theory and generalised Proca. In sections 5 and 6 we explore static and cosmological solutions to this model. Finally, in section 7 we discuss our results and provide some concluding remarks.

2. GR as a broken gauge symmetry

Some years ago MacDowell and Mansouri [22] proposed a unified theory of gravitation and supergravity that is independent of the metric, and instead takes as fundamental field the gauge fields of a certain group. In this formulation the metric is not present in the action, but it can be recovered from the gauge fields of the theory. To develop this theory we first consider a gauge potential, or connection, associated to a fiber bundle $SO(3, 2)$ (or $SO(4, 1)$) on a 4-dimensional base space-time. We denote this connection as $\omega_\mu^{AB}$, where greek indices run from 0 to 3 and correspond to the base space-time, while capital Latin indices (internal indices) run from 0 to 4 and correspond to the fiber anti-de Sitter group. The connection is antisymmetric in its internal indices, $\omega_\mu^{AB} = -\omega_\mu^{BA}$. 
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The curvature associated to this connection is

\[ R_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{\, AB} - \partial_\nu \omega_\mu^{\, AB} + \frac{1}{2} f_{[CD][EF]}^{[AB]} \omega_\mu^{\, CD} \omega_\nu^{\, EF}, \]

where the notation \([AB]\) means antisymmetrization in the indices \(A\) and \(B\) and

\[ f_{[CD][EF]}^{[AB]} = \frac{1}{2} \left[ \eta_{CE} \delta_D^A \delta_F^B - \eta_{CF} \delta_D^A \delta_E^B + \eta_{DE} \delta_C^A \delta_F^B - \eta_{DF} \delta_C^A \delta_E^B \right] \]

are the structure constants of \(SO(3, 2)\) with \((\eta_{AB}) = \text{diag}(1, -1, -1, -\lambda^2)\) \(^{23}\). With the use of \(^2\) we rewrite the curvature as

\[ R_{\mu\nu}^{a4} = \partial_\mu \omega_\nu^{\, a4} - \partial_\nu \omega_\mu^{\, a4} + \omega_\mu^{\, 4b} \omega_\nu^{\, b} - \omega_\mu^{\, a} \omega_\nu^{\, 4b} \]

and

\[ R_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \lambda^2 \left( \omega_\mu^{\, a4} \omega_\nu^{\, 4b} - \omega_\mu^{\, 4a} \omega_\nu^{\, b} \right), \]

where lowercase Latin indices run from 0 to 3 and

\[ R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{\, ab} - \partial_\nu \omega_\mu^{\, ab} + \omega_\mu^{\, ca} \omega_\nu^{\, cb} - \omega_\mu^{\, ca} \omega_\nu^{\, cb} \]

is the usual four-dimensional Riemann curvature tensor. The proposal by MM was to identify the components of \(\omega_\mu^{\, AB}\) as a four-dimensional part \(\omega_\mu^{\, ab}\) and a vierbein

\[ \omega_\mu^{\, 4a} = e_\mu^a. \]

Thus, the mixed and 4-dimensional parts of the curvature become

\[ R_{\mu\nu}^{a4} = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + e_\mu^b \omega_\nu^{\, b} - \omega_\mu^{\, a} e_\nu^b; \]

and

\[ R_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \lambda^2 \left( e_\mu^a e_\nu^b - e_\nu^a e_\mu^b \right). \]

Then the MM action is written in terms of the 4-dimensional part of the curvature \(^1\)

\[ S = \int d^4x e^{\mu\nu\alpha\beta} \epsilon_{abcd} R_{\mu\nu}^{ab} R_{\alpha\beta}^{cd}, \]

where \(\epsilon^{\mu\nu\alpha\beta}\) is the Levi-Civita symbol. Because of the Levi-Civita symbol and the symmetry properties of \(R_{\mu\nu}^{AB}\), the action does not include \(R_{\mu\nu}^{a4}\). This term is equivalent to the torsion, therefore setting \(R_{\mu\nu}^{a4} = 0\), implies vanishing torsion. This is sometimes imposed in MacDowell-Mansouri. We can write this action explicitly in terms of the connection and the vierbein by using Eqs. \([7]\) and \([5]\),

\[ S = \int d^4x e^{\mu\nu\alpha\beta} \epsilon_{abcd} \left[ R_{\mu\nu}^{ab} R_{\alpha\beta}^{cd} - 2\lambda^2 \left( e_\mu^a e_\nu^b - e_\nu^a e_\mu^b \right) R_{\alpha\beta}^{cd} + 4\lambda^4 e_\mu^a e_\nu^b e_\alpha^c e_\beta^d \right]. \]

To see how this action is related to the standard Einstein-Hilbert action we consider a 4-dimensional space-time \(M\) with metric \(g_{\mu\nu}\) and an orthonormal frame field \(e\) (or tetrad) such that

\[ g_{\mu\nu} e_\mu^a e_\nu^b = \eta_{ab}, \]

where \(\eta_{ab}\) is the Minkowski metric.
where the lower case Latin indices correspond to the inner space basis which is endowed with an internal metric $\eta_{ab}$. The inverse frame field (or co-tetrad) defined by $e^a \epsilon^b = \delta^a_b$ is related to the space-time metric by

$$g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}. \quad (12)$$

Moreover, the Levi-Civita symbols in the internal and space-time indices are related by

$$\epsilon_{abcd} e^a_{\mu} e^b_{\nu} e^c_{\alpha} e^d_{\beta} = \epsilon_{\mu\nu\alpha\beta}, \quad (13)$$

where $e = \sqrt{-\det(g)}$. For their contraction, we use

$$e^{\mu_1 \cdots \mu_k \nu_1 \cdots \nu_n} \epsilon^{\nu_1 \cdots \nu_k} = -k! (n-k)! \delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}, \quad (14)$$

and similarly for the Levi-Civita symbols in the internal space (which also has 1 timelike dimension). The use of the relations given above enable us to rewrite (10) as

$$S = \int d^4 x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} F^a_{\mu\nu} F^b_{\alpha\beta} + 8 \lambda^2 \int d^4 x \left[ R - 12 \lambda^2 \right]. \quad (15)$$

The second term contains the Einstein-Hilbert action with a cosmological constant, while the first term is the so called Euler topological term. In four dimensions, the Euler topological term is proportional to the Gauss-Bonnet term and, since it is a total derivative, has no contribution to the field equations in the classical regime [25, 24]. Notice that a proper identification of the second term in (15) with General Relativity and a cosmological constant requires us to put by hand the appropriate gravitational constant in front of the action, otherwise we get a theory where the gravitational and cosmological constants are not independent. Another alternative is to consider a different construction, also inspired in MM, but where the resulting 4d theory contains enough free parameters for describing the gravitational and cosmological constants. This alternative construction makes use of the self-dual and anti-self-dual parts of the curvature (e.g. [26, 27]).

3. Generalised Proca theory

A few years ago, a galileon-type generalization of the Proca action containing derivative self-interactions of a vector field was proposed [15, 16]. The original Proca theory describes a massive vector field whose temporal component does not have a kinetic term, the only three propagating degrees of freedom correspond to two transverse modes plus one longitudinal mode. In the generalization of the Proca theory, the key idea is to include all the possible derivative self-interactions of the vector field preserving three propagating degrees of freedom whilst the temporal component remains non-dynamical. The theory in flat space-time is described by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{n=2}^{6} \alpha_n \mathcal{L}_n, \quad (16)$$
with

\[ L_2 = f_2(X, F, Y), \]
\[ L_3 = f_3(X) \partial_\mu A^\mu, \]
\[ L_4 = f_4(X) \left[ (\partial_\mu A^\mu)^2 - \partial_\alpha A_\beta \partial^\beta A^\alpha \right] + c_2 \tilde{f}_4(X) F_{\mu\nu} F^{\mu\nu}, \]
\[ L_5 = f_5(X) \left[ (\partial_\mu A^\mu)^3 - 3(\partial_\mu A^\mu) \partial_\alpha A_\beta \partial^\beta A^\alpha + 2 \partial_\alpha A_\beta \partial^\gamma A^\alpha \partial^\beta A^\gamma \right] + d_2 \tilde{f}_5(X) F^{\alpha\beta} \partial_\alpha A_\beta, \]
\[ L_6 = -\epsilon^{\mu\nu\rho} \epsilon_{\alpha\beta\gamma} \left[ f_6(X) \partial_\mu A^\alpha \partial_\nu A^\beta \partial_\rho A^\gamma \partial_\sigma A^\kappa + e_2 \tilde{f}_6(X) \partial_\mu A_\nu \partial_\alpha A^\beta \partial_\beta A^\gamma \partial_\sigma A^\kappa \right]. \quad (17) \]

Here \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \) \( f_{3,4,5,6} \) and \( \tilde{f}_{4,5,6} \) are arbitrary functions depending on \( X = -A_\mu A^\mu/2, \) \( F = -F_{\mu\nu} F^{\mu\nu}/4, \) \( Y = -A^\mu A^\nu F_{\mu\nu} F_{\alpha\beta}, \) and \( c_2, d_2, e_2 \) are constant coefficients. The function \( f_2 \) depends on all possible terms with \( U(1) \) symmetry and terms with no time derivatives acting on the time component of the vector field. The Lagrangian densities \( \mathcal{L}_n \) shown above were constructed exploring order by order all the possible Lorentz invariant terms that can be built and determining the suitable coefficients in order to remove the emerging ghost-instabilities. The set of parameters generated for all the possible self-interactions is then fixed with the help of the constraint equation provided by the vanishing determinant of the Hessian matrix \( \mathcal{H}_{\mathcal{L}}^{\mu\nu} = \partial^2 \mathcal{L}/\partial \dot{A}_\mu \partial \dot{A}_\nu. \) This ensures the propagation of only three degrees of freedom.

The extension of this theory to a curved space-time could be realized by promoting the partial derivatives appearing in the Lagrangian to covariant derivatives. Nevertheless, this naive covariantization propagates additional degrees of freedom. To avoid this, non-minimal couplings to the curvature are included, playing the role of counter-terms to keep only the three physical degrees of freedom and maintain second order equations of motion. On curved space-time the generalised Proca theory [17] is represented by the Lagrangian

\[ \mathcal{L}_{\text{gen.Proca}}^\text{curved} = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_{n=2}^{6} \beta_n \mathcal{L}_n, \quad (18) \]

where the \( \mathcal{L}_n \) Lagrangians are given by

\[ L_2 = G_2(X, F, Y), \]
\[ L_3 = G_3(X) \nabla_\mu A^\mu, \]
\[ L_4 = G_4(X) R + G_{4,X} \left[ (\nabla_\mu A^\mu)^2 - \nabla_\alpha A_\beta \nabla^\beta A^\alpha \right], \]
\[ L_5 = G_5(X) G_{5,\mu} \nabla^\mu A^\nu - \frac{1}{6} G_{5,X} \left[ (\nabla_\mu A^\mu)^3 - 3(\nabla_\mu A^\mu) \nabla_\alpha A_\beta \nabla^\beta A^\alpha + 2 \nabla_\alpha A_\beta \nabla^\gamma A^\alpha \partial^\beta A^\gamma \right] - g_5(X) \tilde{F}^{\alpha\mu} \tilde{F}_{\mu} \nabla_\alpha A_\beta, \]
\[ L_6 = G_6(X) P^{\mu\nu\rho} \nabla_\mu A_\nu \nabla_\alpha A_\beta - \epsilon^{\mu\nu\rho} \epsilon_{\alpha\beta\gamma} \left[ g_6(X) \nabla_\mu A^\alpha \nabla_\nu A^\beta \partial^\rho A^\gamma \nabla_\sigma A^\kappa + e_2 g_{6,X} (X) \nabla_\mu A_\nu \nabla^\alpha A^\beta \partial^\rho A^\gamma \nabla_\sigma A^\kappa \right], \quad (19) \]
where \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \), \( X = -\frac{1}{2} A_\mu A^\mu \) and \( \nabla \) represents the covariant derivative operator. \( G_2 \) is an arbitrary function of \( X, F \) and \( Y \) whereas \( G_{3,4,5,6} \) and \( g_5, g_6, e_2 \) are arbitrary functions of \( X \), with \( G_{i,X} \equiv \partial G_i/\partial X \). \( P^{\mu\nu\alpha\beta} \) and \( \tilde{F}^{\mu\nu} \) are the components of the double dual Riemann tensor and the dual strength tensor, defined, respectively, by

\[
P^{\mu\nu\alpha\beta} = \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} \epsilon_{\alpha\beta\gamma\sigma} R_{\rho\lambda\gamma\sigma} , \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} .
\]

The term multiplying \( g_6(X) \) in \( L_6 \) can be omitted since, as argued in [32, 35], this term in flat space-time is a total derivative and on curved backgrounds it can be included as a combination of the self-interactions \( L_2,3,4,5 \). It is worth to mention that for the terms \( G_3(X) \nabla_\mu A^\mu \) and \( g_5(X) \tilde{F}^{\alpha\mu} \nabla_\mu A_\beta \) appearing in \( L_3 \) and \( L_5 \), respectively, the addition of a counter-term is not required since the coupling with the connection is linear and so they do not give rise to higher order equations of motion. As commented in [33], in the construction of this theory all the possible contractions of the vector field derivative self-interactions with Riemann constructed tensors could be included, however, in order to keep second order equations the couplings must be to a divergence-free tensor constructed out of the Riemann tensor. In four dimensional space these includes the metric tensor, the Einstein tensor and the Riemann dual tensor. Hence, the term \( G_6(X) P^{\mu\nu\alpha\beta} \nabla_\mu A_\alpha \nabla_\nu A_\beta \) in \( L_6 \) yields second order equations of motion.

Furthermore, if \( A_\mu \to \nabla_\mu \pi \), the generalised covariant Galileon theory [39] is recovered.

4. Vector-tensor gravity as a broken gauge symmetry

Here we show that a simple modification of the ansatz (6) leads to a vector-tensor theory that is a linear combination of the vector Galileon actions. The ansatz we consider is

\[
\omega^a_\mu = e^a_\mu + \Phi^a_\mu ,
\]

where \( \Phi \) is a 1-form constructed out of some vector field \( A_\mu \). We assume that the tetrad is torsionless. As a consequence, \( \Phi^a_\mu \) cannot be removed by a redefinition of the tetrad. Furthermore, \( \mathcal{R}^{ab}_{\mu\nu} \) does not automatically vanish, however, this term is not present in the action. Then the 4-dimensional curvature is

\[
\mathcal{R}^{ab}_{\mu\nu} = \partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + \omega^a_\mu \omega^b_\nu - \omega^b_\mu \omega^a_\nu - \lambda^2 (e^a_\mu + \Phi^a_\mu) (e^b_\nu + \Phi^b_\nu) - (e^a_\nu + \Phi^a_\nu) (e^b_\mu + \Phi^b_\mu)
\]

\[
\equiv \mathcal{R}^{ab}_{\mu\nu} - \lambda^2 \Sigma^{ab}_{\mu\nu} .
\]

Substituting this in (9) gives

\[
S = \int d^4 x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \left[ \mathcal{R}^{ab}_{\mu\nu} \mathcal{R}^{cd}_{\alpha\beta} - 2 \lambda^2 \Sigma^{ab}_{\mu\nu} \Sigma^{cd}_{\alpha\beta} + 4 \lambda^4 (e^a_\mu + \Phi^a_\mu) (e^b_\nu + \Phi^b_\nu) (e^c_\alpha + \Phi^c_\alpha) (e^d_\beta + \Phi^d_\beta) \right]
\]

\[
\equiv S_{Euler} + S_{AR} + S_A .
\]
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The first term is the Euler topological term, while the last two terms describe interactions of the vector field non-minimally coupled to gravity.

For the rest of this work we will assume \( \Phi_{\mu}^a \) is defined as \( \Phi_{\mu}^a = \epsilon^a_v \nabla_{\mu} A^p dx^\mu \). We can verify \( \epsilon^b_v \Phi_{\mu}^a = \nabla_{\mu} A^p \) and \( \epsilon^b_v \Phi_{\mu}^a = \Phi_{\mu}^b \), which will be used thoroughly in the following. In this way, the action in (9) now is written as

\[
S = \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \left[ R_{\mu\nu}^a R_{\alpha\beta}^{cd} - 2 \lambda^2 \epsilon^a_{\mu\nu} R_{\alpha\beta}^{cd} \right. \\
+ 4 \lambda^4 (\epsilon^a_{\mu} + \epsilon^a_{\gamma} \nabla_{\mu} A^\gamma)(\epsilon^b_{\nu} + \epsilon^b_{\rho} \nabla_{\nu} A^\rho)(\epsilon^c_{\alpha} + \epsilon^c_{\sigma} \nabla_{\alpha} A^\sigma)(\epsilon^d_{\beta} + \epsilon^d_{\tau} \nabla_{\beta} A^\tau) \left. \right]. (24)
\]

The explicit calculation of \( S_{AR} \) leads to the following result:

\[
S_{AR} = \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \left[ -2 \lambda^2 R_{\mu\nu}^{ab} \epsilon_{\alpha\beta} \right. \\
+ \epsilon^{\mu\nu\alpha\beta} (\epsilon_{\rho\lambda\alpha} R_{\rho\lambda\mu\nu} A^\gamma + \epsilon_{\rho\lambda\tau} R_{\rho\lambda\mu\nu} \nabla_{\alpha} A^\tau) \\
\left. \right] = \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \left[ 8R - 16 G_{\mu\nu} \nabla_{\nu} A_{\mu} - 8 P_{\mu\nu\alpha\beta} \nabla_{\alpha} A_{\mu} \nabla_{\nu} A_{\beta} \right], (25)
\]

where \( G_{\mu\nu} \) is the Einstein tensor. The first term is the Ricci scalar. Notice that the second term can be ignored after integration by parts since the Einstein tensor satisfies \( \nabla_{\mu} G^{\mu\nu} = 0 \). Expanding the \( S_A \) part of the action we get

\[
S_A = 4 \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \left[ \epsilon_{\mu\nu\alpha\beta} + 4 \epsilon_{\mu\nu\alpha} \nabla_{\beta} A^\rho + 6 \epsilon_{\mu\nu\rho\lambda} \nabla_{\alpha} A^\rho \nabla_{\beta} A^\lambda \right. \\
\left. + 4 \epsilon_{\mu\gamma\rho} \nabla_{\nu} A^\gamma \nabla_{\alpha} A^\rho \nabla_{\beta} A^\lambda + \epsilon_{\gamma\delta\rho\lambda} \nabla_{\mu} A^\gamma \nabla_{\nu} A^\delta \nabla_{\alpha} A^\rho \nabla_{\beta} A^\lambda \right] = \int d^4x \epsilon^{\mu\nu\alpha\beta} \left[ -96 - 96 \nabla_{\alpha} A^\alpha - 48 ( (\nabla_{\alpha} A^\alpha)^2 - \nabla_{\alpha} A^\beta \nabla_{\beta} A^\alpha ) \\
- 16 \left( (\nabla_{\alpha} A^\alpha)^3 - 3 \nabla_{\rho} A^\rho \nabla_{\alpha} A^\beta \nabla_{\beta} A^\alpha + 2 \nabla_{\alpha} A^\rho \nabla_{\rho} A^\beta \nabla_{\beta} A^\alpha \right) \\
\right. \\
\left. + 4 \epsilon^{\mu\nu\alpha\beta} (\epsilon_{\gamma\delta\rho\lambda} \nabla_{\mu} A^\gamma \nabla_{\nu} A^\delta \nabla_{\alpha} A^\rho \nabla_{\beta} A^\lambda \right] . (26)
\]

Adding up the contributions of \( S_A \) and \( S_{AR} \) the full action after integration by parts is

\[
S = \int d^4x \sqrt{-g} \lambda^4 \left\{ 8R - 8 P^{\mu\nu\alpha\beta} \nabla_{\mu} A_{\nu} \nabla_{\alpha} A_{\beta} - \lambda^2 \left[ 96 + 48 ( (\nabla_{\alpha} A^\alpha)^2 - \nabla_{\alpha} A^\beta \nabla_{\beta} A^\alpha ) \\
+ 16 \left( (\nabla_{\alpha} A^\alpha)^3 - 3 \nabla_{\rho} A^\rho \nabla_{\alpha} A^\beta \nabla_{\beta} A^\alpha + 2 \nabla_{\alpha} A^\rho \nabla_{\rho} A^\beta \nabla_{\beta} A^\alpha \right) \\
- 4 \epsilon^{\mu\nu\alpha\beta} (\epsilon_{\gamma\delta\rho\lambda} \nabla_{\mu} A^\gamma \nabla_{\nu} A^\delta \nabla_{\alpha} A^\rho \nabla_{\beta} A^\lambda \right] \right\} . (27)
\]

The resulting theory is closely related to the vector Galileon theory proposed in [16]. In particular, in flat space-time we find that \( S \) is given by the following combination of vector Galileons:

\[
S = \int d^4x \sqrt{-g} \lambda^4 \left[ 96 \mathcal{L}_2 + 96 \mathcal{L}_3 + 48 \mathcal{L}_4 + 16 \mathcal{L}_5 + 4 \mathcal{L}_6 \right], (28)
\]

where \( \mathcal{L}_{2,3,4,5,6} \) are the vector Galileons given in [17] for the case with \( f_2(X, F, Y) = f_3(X) = f_4(X) = f_5(X) = f_6(X) = 1 \) and \( c_2 = d_2 = e_2 = 0 \).
On the other hand, in curved space-time we notice that the derivative self-interactions appearing in \( \mathcal{L}_4 \) are closely related to beyond-generalized Proca theories \[17\]. In fact, all the terms except \( P^{\mu\nu\alpha\beta} \nabla_{\mu} A_{\alpha} \nabla_{\nu} A_{\beta} \) are known in beyond-generalised Proca. The new term does not contribute to the Hessian matrix \( \mathcal{H}_{\mu}^{\nu} = \partial^2 \mathcal{L} / \partial \dot{A}_{\mu} \partial \dot{A}_{\nu} \) due to the symmetries of the double dual Riemann tensor, therefore it does not spoil the constraints of the full theory. Furthermore, as shown in the next section and in Appendix \[8\] the field equations remain second order both for the metric and the vector field. Nevertheless, vanishing of the Hessian determinant is not the only requirement in constructing beyond generalised Proca theories, it is also imposed that the dynamics of the longitudinal mode of the vector field is described by Horndeski and the vector field. Nevertheless, vanishing of the Hessian determinant is not the only requirement in constructing beyond generalised Proca theories, it is also imposed that the dynamics of the longitudinal mode of the vector field is described by Horndeski and the vector field. Nevertheless, vanishing of the Hessian determinant is not the only requirement in constructing beyond generalised Proca theories, it is also imposed that the dynamics of the longitudinal mode of the vector field is described by Horndeski and the vector field. Nevertheless, vanishing of the Hessian determinant is not the only requirement in constructing beyond generalised Proca theories, it is also imposed that the dynamics of the longitudinal mode of the vector field is described by Horndeski and the vector field. Nevertheless, vanishing of the Hessian determinant is not the only requirement in constructing beyond generalised Proca theories, it is also imposed that the dynamics of the longitudinal mode of the vector field is described by Horndeski and the vector field. Nevertheless, vanishing of the Hessian determinant is not the only requirement in constructing beyond generalised Proca theories, it is also imposed that the dynamics of the longitudinal mode of the vector field is described by Horndeski and the vector field. Nevertheless, vanishing of the Hessian determinant is not the only requirement in constructing beyond generalised Proca theories, it is also imposed that the dynamics of the longitudinal mode of the vector field is described by Horndeski and the vector field.

Another observation is that, if we select the unitary gauge – \( \partial \phi = 0 \) – then the term \( P^{\mu\nu\alpha\beta} \nabla_{\mu} \nabla_{\alpha} \phi \nabla_{\nu} \nabla_{\beta} \phi \) vanishes due to its symmetries. These observations point towards the possibility that the theory \[27\] describes a vector field with a healthy transverse mode – no ghost or gradient instabilities, but a longitudinal mode subject to gradient instabilities. A detailed analysis of the longitudinal sector would be intrinsically relevant since it would lead to a scalar-tensor theory constructed by a different choice of \( \Phi^A \) in \( \omega^A \). This is left for future work.

Following \[10\] we can also add counter terms to keep up to three propagated degrees of freedom and equations of motion of second order. The proper counter terms corresponding to \( \mathcal{L}_4 \) and \( \mathcal{L}_5 \) are, respectively, \( G_4 = -48X \) and \( G_5 = 96X \), so the new action is

\[
S_A = \int d^4 x e \lambda^4 \left[ -96 - 96 \nabla_{\alpha} A^{\alpha} - 48X R - 48 \left( (\nabla_{\alpha} A^{\alpha})^2 - \nabla_{\alpha} A^{\beta} \nabla_{\beta} A^{\alpha} \right) \\
+ 96X G_{\mu \nu} \nabla^{\mu} A^{\nu} - 16 \left( (\nabla_{\alpha} A^{\alpha})^3 - 3 \nabla_{\rho} A^{\rho} \nabla_{\alpha} A^{\beta} \nabla_{\beta} A^{\alpha} + 2 \nabla_{\alpha} A^{\rho} \nabla_{\rho} A^{\beta} \nabla_{\beta} A^{\alpha} \right) \\
+ \epsilon^{\mu \nu \alpha \beta} \epsilon_{\gamma \delta \rho \lambda} \nabla_{\mu} A^{\gamma} \nabla_{\nu} A^{\delta} \nabla_{\rho} A^{\alpha} \nabla_{\lambda} A^{\beta} \right].
\]

(29)

We see that setting \( A_{\mu} = \nabla_{\mu} \pi \) the generalised covariant Galileon theory is restored for the special case of \( G_2 = G_3 = -\lambda^4 96, \ G_4 = -\lambda^4 96, \ G_4 = -48\lambda^4 X \) and \( G_5 = 96\lambda^4 X \).
Vector-tensor gravity from a broken gauge symmetry

5. Static Solutions

Let us now explore static, spherically symmetric solutions for subsets of the action \([27]\). For this purpose we consider a metric of the form
\[
ds^2 = -f(r)dt^2 + h(r)^{-1}dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]
and a vector field with a temporal and a radial component
\[
(A_{\mu}) = (A_0(r), \pi(r), 0, 0),
\]
where \(A_0(r)\) and \(\pi(r)\) are functions of the radial coordinate only. For simplicity, we restrict ourselves to the terms that are invariant under \(A_{\mu} \to -A_{\mu}\) in \([27]\), i.e., we consider the following Lagrangian:
\[
\mathcal{L} = \sqrt{-g} \left[ 8R - 8\mu^{\alpha\beta} \nabla_{\alpha} A_{\mu} \nabla_{\beta} A_{\mu} - \lambda^2 \left( 96 + 48 \left( (\nabla_{\alpha} A_{\mu})^2 - \nabla_{\alpha} A_{\beta} \nabla_{\beta} A_{\alpha} \right) \right) - 4\epsilon^{\mu\nu\alpha\beta} \epsilon_{\gamma \delta \rho \lambda} \nabla_{\mu} A_{\gamma} \nabla_{\nu} A_{\alpha} \nabla_{\beta} A_{\lambda} \right].
\]
Variation w.r.t. the metric yields the following field equations,
\[
0 = A^\alpha R_{\alpha \beta} R_{\mu}^\beta - \frac{1}{2} A^\alpha R_{\mu \alpha} R + A^\alpha R^\beta R_{\mu \beta \alpha \gamma} - R_{\mu \alpha \beta \gamma} \nabla^\gamma \nabla^\beta A^\alpha + \lambda^2 (6A^\alpha R_{\mu \alpha} \\
+ 3A^\alpha R_{\mu \alpha} \nabla_{\beta} A^\beta \nabla_{\gamma} A^\gamma - 3A^\alpha R_{\mu \alpha} \nabla_{\beta} A_{\gamma} \nabla_{\beta} A^\gamma + 6A^\alpha R_{\mu \beta \alpha \gamma} \nabla^\gamma A^\beta \nabla^\delta A_{\gamma} \\
- 6A^\alpha R_{\mu \gamma \alpha \delta} \nabla_{\beta} A^\beta \nabla^\delta A_{\gamma} + 6A^\alpha R_{\mu \gamma \alpha \delta} \nabla_{\beta} A_{\gamma} \nabla_{\beta} A^\gamma - 6A^\alpha R_{\mu \beta \gamma} \nabla_{\beta} A^\gamma \nabla_{\mu} A^\beta \\
+ 6A^\alpha R_{\alpha \beta \gamma} \nabla^\gamma A^\beta \nabla_{\mu} A^\beta),
\]
while variation w.r.t. the vector field gives the following field equations,
\[
0 = 8G_{\mu \nu} - 2 (H^1_{\mu \nu}(A, R) + H^2_{\mu \nu}(A, \nabla A) + H^3_{\mu \nu}(R, \nabla A) + H^4_{\mu \nu} (\nabla A)) \\
+ \lambda^2 \left[ 48g_{\mu \nu} + 24(C^1_{\mu \nu}(A, R) + C^2_{\mu \nu}(A, \nabla A) + C^3_{\mu \nu}(\nabla A)) \\
+ 4 \left( E^1_{\mu \nu}(A, R, \nabla A) + E^2_{\mu \nu}(\nabla A) + E^3_{\mu \nu}(\nabla A, \nabla A) \right) \right].
\]
where the terms \(H^1_{\mu \nu}(A, R), H^2_{\mu \nu}(A, \nabla A), H^3_{\mu \nu}(R, \nabla A), H^4_{\mu \nu} (\nabla A), C^1_{\mu \nu}(A, R), C^2_{\mu \nu}(A, \nabla A), C^3_{\mu \nu}(\nabla A), E^1_{\mu \nu}(A, R, \nabla A), E^2_{\mu \nu}(\nabla A)\) and \(E^3_{\mu \nu}(\nabla A, \nabla A)\) are given explicitly in Appendix 8. In the next subsections we present solutions first for a flat space-time metric, then for the Schwarzschild metric, and finally for a general Schwarzschild-like spacetime.

5.1. Flat space-time metric

Starting with a Minkowski background, the field equations in \(33\) are satisfied straightforwardly, and for the field equations in \(34\) in terms of the vector field \(31\), we get
\[
\xi_{11} = \pi(r) \left[ (4r - 2A_0 A'_0) \pi' + r^2 \pi'' \right] - \pi^2 \left( A'_0^2 + A_0 A''_0 - 3\pi'^2 - 1 \right) \\
- r \left[ r \left( A'_0^2 - \pi'^2 + 1 \right) + A_0 \left( 2A'_0 + r A''_0 \right) \right] + \pi^3 \pi''
\]
\[
\xi_{22} = r^2 - \pi \left( 2r \pi' + \pi \right)
\]
\[
\xi_{33} = r \left( \pi'^2 - 1 \right) + \pi \left( r \pi'' + 2\pi' \right).
\]
The solution for $\pi(r)$ is obtained from $\xi_{22} = 0$, and after substitution of this solution into $\xi_{11} = 0$ we find a differential equation for $A_0(r)$. The solutions for $A_0(r)$ and $\pi(r)$ are,

$$
\pi(r) = \pm \sqrt{\frac{r^2}{3} + \frac{\pi_0}{r}},
$$

$$
A_0 = \pm \frac{1}{6} \left\{ 12r^2 + 72a_1 + 36\pi_0^2 + 6^{2/3} \left( \frac{a_0 - \pi_0}{\pi_0^{1/3}} \right) \left( 2\sqrt{3} \tan^{-1} \left( \frac{1}{\sqrt{3}} - \frac{2^{5/3}r}{3^{5/6}a_1^{1/3}} \right) \right) + \ln \left( \frac{6^{2/3}r + 3\pi_0^{1/3}}{r} \right)^2 - \ln \left( 6^{1/3}2r^2 - 6^{2/3}3\pi_0^{1/3}r + 3\pi_0^{2/3} \right) \right\}^{1/2},
$$

where $\pi_0$, $a_0$ and $a_1$ are integration constants. In general these solutions are not well defined across the entire range of the radial coordinate. Performing an asymptotic expansion reveals that such solutions at infinity behave as,

$$
\pi(r) = \pm \frac{r}{\sqrt{3}} \pm \frac{3\sqrt{3}\pi_0^2}{2r^2} + \frac{3\sqrt{3}\pi_0^2}{8r^5} + O \left( \frac{1}{r} \right),
$$

$$
A_0(r) = \pm \left( - \frac{r}{\sqrt{3}} - \frac{c_1}{\sqrt{3}r} - \frac{\sqrt{3}(a_0 + \pi_0)}{4r^2} + \frac{c_1^2}{2\sqrt{3}r^3} + \frac{\sqrt{3}c_1(a_0 + \pi_0)}{4r^4} \right) + O \left( \frac{1}{r} \right),
$$

where

$$
c_1 = 3a_1 + \frac{(a_0 - \pi_0) \left( \sqrt{3} \ln(3) - 3\pi \right)}{42^{1/3}3^{5/6}a_1^{1/3}},
$$

and $O(r^n)$ represents terms of order equal or higher than $r^n$. Note that $c_1$ diverges as $\pi_0$ approaches zero, but this is avoided if we take $a_0 = \pi_0$, so the solution for the $A_0$ function, according to (39), now takes the simpler form

$$
A_0(r) = \pm \sqrt{\frac{r^2}{3} + 2a_1 + \frac{\pi_0}{r}}.
$$

The divergence at $r = 0$ in $\pi_0$ and $A_0$ is avoided if $\pi_0 = 0$. Even if $\pi_0 \neq 0$, the scalar $A_\mu A^\mu$ is regular, and actually a constant, $A_\mu A^\mu = -2a_1$.

### 5.2. Schwarzschild metric

In addition, we may ask if the Schwarzschild metric is solution for this lagrangian. To show this we impose $f(r) = h(r) = 1 - 2M/r$ on the line element. In this case (33) leads to two differential equations,

$$
0 = A_0 \left[ -2\lambda^2 \left( 8M^2 - 6Mr + r^2 \right) \pi^2 + 2\lambda^2 r(r - 2M)^2 \pi \pi' + Mr \right],
$$

$$
0 = \pi \left\{ r \left[ 2\lambda^2 r^2 A_0(r - 2M)A_0' - 2\lambda^2 Mr A_0^2 + M(r - 2M) \right] + 2\lambda^2 (3M - r)(r - 2M)^2 \pi^2 \right\}.
$$
Solving this system for \( \pi(r) \) and \( A_0(r) \) we find that

\[
\pi_0(r) = \left(1 - \frac{2M}{r}\right)^{-1} \sqrt{\pi_0 r^2 + \frac{M}{3\lambda^2 r}},
\]

(46)

\[
A_0(r) = \sqrt{-\frac{2a_0 M}{r} + a_0 + \frac{8\pi_0 M^3}{r} - 4\pi_0 M^2 + \pi_0 r^2 + \frac{M}{3r\lambda^2}}.
\]

(47)

Substituting this solutions into the metric field equations we see that we require that \( \pi_0 = 1/3 \) so that (46) and (47) to be solution. Then, the solutions are

\[
\pi_0(r) = \left(1 - \frac{2M}{r}\right)^{-1} \sqrt{\frac{r^2}{3} + \frac{M}{3\lambda^2 r}},
\]

(48)

\[
A_0(r) = \sqrt{-\frac{2a_0 M}{r} + a_0 + \frac{8M^3}{3r} - \frac{4M^2}{3} + \frac{r^2}{3} + \frac{M}{3r\lambda^2}}.
\]

(49)

Now the divergences of \( \pi(r) \) and \( A_0(r) \) are protected by the horizon of the Schwarzschild metric, and the divergence of \( \pi(r) \) at \( r = 2M \) can be removed by a coordinate transformation. The scalar \( A_\mu A^\mu \) continues to be regular, taking the constant value \( A_\mu A^\mu = 4M^2/3 - a_0 \). Since there is a cosmological constant-like term in the Lagrangian, this Schwarzschild spacetime is a self-tuning solution.

### 5.3. Schwarzschild-like solution

A more general solution is obtained by only assuming \( h(r) = f(r) \) for the line element (30). In this case, the vector field equations are

\[
0 = (6\lambda^2 \pi^2 f^2 + f + 6\lambda^2 r^2 - 1) f'' + 12\lambda^2 f' \left(r + f^2 \pi f'\right) + (12\lambda^2 \pi^2 f + 1) f'^2
\]

(50)

\[
0 = \left\{f' \left[f - 6\lambda^2 \left(A_0^2 - \pi^2 f^2\right)\right] + 12\lambda^2 f \left(A_0 A'_0 + r\right)\right\} f'
\quad + \left(6\lambda^2 \pi^2 f^2 + f + 6\lambda^2 r^2 - 1\right) f f''.
\]

(51)

The solution for \( \pi_0(r) \) and \( A_0(r) \) is obtained in terms of \( f(r) \) and its derivatives with respect to \( r \),

\[
\pi(r) = f(r)^{-1} \sqrt{\pi_0 f'(r)^{-1} + (1 - f(r))(6\lambda^2)^{-1} - r^2}
\]

(52)

\[
A_0(r) = \sqrt{\pi_0 f'(r)^{-1} - a_0 f(r) - r^2 + (6\lambda^2)^{-1}}
\]

(53)

Notice that, as in the flat spacetime case, the scalar \( A_\mu A^\mu \) is constant, \( A_\mu A^\mu = -(6\lambda^2)^{-1} - a_0 \). The substitution of this solutions into the field equations for the metric leads to a second order non-lineal differential equation for \( f(r) \),

\[
0 = 3r\lambda^2 \pi_0 f'' - f' \left[3\lambda^2 \pi_0 - f' \left(r f' + f + 12r^2 \lambda^2 - 1\right)\right].
\]

(54)

When \( \pi_0 = 8M/3 \) the Schwarzschild solution, \( f(r) = 1 - 2M/r \), is obtained. The same is true when \( \lambda = 0 \). On the other hand, if \( \pi_0 = 0 \) the solution is the Schwarzschild de-Sitter metric,

\[
f(r) = 1 - \frac{2M}{r} - 4\lambda^2 r^2.
\]

(55)
Figure 1. A particular solution for a static configuration of the theory described by (32). This branch of solutions is obtained after solving (54) for \( \pi_0 = 8M/3 \) and using (56) as boundary conditions. For this solution we have used \( M = 1 \), \( \lambda = 1/10 \) and \( m_4 = 0.3 \). This solution resembles the Schwarzschild black hole solution (dashed line) at large distances.

These two solutions, Schwarzschild and Schwarzschild-de Sitter, are actually special cases of two branches of solutions. To find these branches we study approximated solutions in the limit \( r \to \infty \) assuming that, for large \( r \), \( f(r) \) behaves as \( f(r) = m_2 r^2 + m_1 r + \sum_i m_i/r^i \). Using this assumption in (54), we get two branches of solutions. The first one is selected by choosing \( \pi_0 = 8M/3 \), which leads to

\[
\begin{align*}
    f_{SB}(r) &= 1 - \frac{2M}{r} + \frac{m_4}{r^4} + \left( \frac{m_4 M - 8m_4^2}{14\lambda^2} \right) \frac{1}{r^7} + \left( \frac{M^2}{280\lambda^4} - \frac{9m_4}{35\lambda^2} + \frac{8m_4^2}{5M^2} \right) \frac{m_4}{r^{10}} \\
    &\quad + \mathcal{O}\left( \frac{1}{r^{13}} \right).
\end{align*}
\]

We label this branch \( f_{SB} \) to indicate that its asymptotic form reduces to the Schwarzschild solution when \( m_4 = 0 \). This is also a self-tuning solution.

The second branch is found with \( m_2 = -4\lambda^2 \) and \( \pi_0 = -32m_4\lambda^2/3M \), giving for the metric

\[
\begin{align*}
    f_{dSB}(r) &= 1 - \frac{2M}{r} - 4\lambda r^2 + \frac{m_4}{r^4} + \left( \frac{m_4 M - 2m_4^2}{4\lambda^2} \right) \frac{1}{r^7} + \left( \frac{M^2}{16\lambda^4} - \frac{15m_4}{8\lambda^2} + \frac{7m_4^2}{M^2} \right) \frac{m_4}{r^{10}} \\
    &\quad + \mathcal{O}\left( \frac{1}{r^{13}} \right).
\end{align*}
\]

This is labelled \( f_{dSB} \) to indicate that, asymptotically, it reduces to the Schwarzschild-de Sitter solution when \( m_4 = 0 \).

By numerically solving (54), using the expansions shown above as boundary conditions, we confirm that both \( f_{SB} \) and \( f_{dSB} \) lead to complete solutions when \( m_4 \geq 0 \). For \( f_{SB} \), an example of a solution is shown in Figure 1. Although the metric is regular, by plugging its numerical profile into curvature invariants, such as the Ricci and Kretschmann scalars, one can verify that there is a curvature singularity at \( r = 0 \), thus, the solution shown in Fig 1 is a black hole, Schwarzschild-like, solution.

In addition, by exploring the parameter space we find that the existence of the
Figure 2. Solutions in the branch with asymptotic profiles given by $f_{SB}$, with $\lambda = 0.06$. The gray dotted line corresponds to the exact Schwarzschild solution, which is obtained when $m_4 = 0$. The dashed green line corresponds to a solution with horizon ($m_4 = 0.0006$), while the solid blue line is a solution without horizon ($m_4 = 1$). In the shaded region there are other solutions with $0.0006 < m_4 < 1$. Above the blue line there are solutions with $m_4 > 1$. Below the dashed green line we could not find other solutions, except for Schwarzschild with $m_4$ exactly equal to zero.

horizon depends on the values of $\lambda$ and $m_4$. Figure 2 shows solutions for $\lambda = 0.06$ and different values of $m_4$, we see that some solutions are naked singularities, i.e. they do not have a horizon and, by the discussion above, have a curvature singularity. By setting and upper bound on $m_4$, solutions without horizon can be avoided, so that the curvature singularity remains protected. The same figure illustrates that the limit $m_4 \to 0$ does not have a smooth transition to the Schwarzschild solution. For instance, for $\lambda = 0.06$ we could not find numerical solutions with $0 < m_4 < 0.0006$, with $m_4 = 0.0006$ corresponding to the dashed green line in Figure 2. This, together with Fig. 1 hints that there is a class of solutions to equation (54) where $f(r)$ remains finite at $r = 0$. The existence of these solutions is confirmed by Taylor expanding (54) near $r = 0$, and our numerical results above show that this class of solutions matches the asymptotic Schwarzschild profile. However, we highlight that even though $f(r)$ remains finite at $r = 0$, the curvature invariants diverge, therefore these solutions do not represent regular spacetimes. In the solutions near $r = 0$, one can verify that the divergence of the curvature invariants arises from the fact that $f'(0) = 0$.

Let us now discuss the branch corresponding to $f_{dSB}$. Figure 3 shows two examples of solutions in this branch, one for de Sitter asymptotics (left) and another one for anti-de Sitter (right). For de Sitter asymptotics, a relevant difference with respect to the exact de Sitter solution appears when analysing the horizons. While for de Sitter there is a critical value ($\lambda_c$) determining whether there are two ($\lambda < \lambda_c$), one ($\lambda = \lambda_c$) or no horizons ($\lambda > \lambda_c$), for the solutions with asymptotics $f_{dSB}$ we find either one or no horizon, depending on the values of $\lambda$ and $m_4$: if $\lambda > \lambda_c$ but $m_4$ is large enough the solution does have a horizon. This is actually the case in the left panel of Figure 3. Similarly to the case of $f_{SB}$, here we also identify that the limit $m_4 \to 0$ does not have a smooth transition to the exact de Sitter solution. Examples of solutions for $\lambda = 0.06$ and
Figure 3. Solutions for a static configuration of the theory given in (32). In this case the branch of solutions were obtained by solving (54) with $\pi_0 = -\frac{32m_4\lambda^2}{3M}$ and using (56) as asymptotic boundary condition. For the free parameters we used $M = 1$, $\lambda = 1/10$ and $m_4 = 30$ for the left plot and $M = 1$, $\lambda = i/10$ and $m_4 = 30$ for the right one. At large distances the behavior of the solution is asymptotically de Sitter (left) or anti-de Sitter (right). For small distances the solutions deviate from (anti-)de Sitter. In the de Sitter case, this have an important effect on the horizons, as explained in the main text.

Figure 4. Solutions in the branch with asymptotic profiles given by $f_{dSB}$, with $\lambda = 0.06$. The gray dotted line corresponds to the exact Schwarzschild-de Sitter solution, which is obtained when $m_4 = 0$. The dashed green line corresponds to a solution with $m_4 = 0.000001$, while the solid blue line is a solution with $m_4 = 100$. In the shaded region, and above the blue line, there are solutions for other values of $m_4$. Below the dashed green line we could not find other solutions, except for Schwarzschild-de Sitter with $m_4$ exactly equal to zero.

different values of $m_4$ are shown in Figure 4. The green dashed line again corresponds to the closer solution to de Sitter that we could find, in this case for $m_4 = 10^{-7}$. The solution with anti de-Sitter asymptotic behaviour, e.g. the right panel of Figure 3, is obtained by allowing $\lambda$ to be imaginary, which amounts to a change in the signature of the internal 5d metric. In this case, the limit $m_4 \to 0$ does smoothly recover the exact anti-de Sitter solution.

Summarizing, we have several static solutions, including flat spacetime, as well as
exact and asymptotic Schwarzschild and (A)dS solutions. All of these solutions have a constant $A_\mu A^\mu$, and the trace of the full energy-momentum tensor – including the non-minimal couplings between the vector field and gravity – inherits its properties from the Ricci scalar. The BH solutions (i.e. solutions with a singularity at $r = 0$) describe singular spacetimes, but the divergence at $r = 0$ is protected by the horizon, unlike for naked singularities. However, naked singularities can be avoided by an appropriate choice of a parameter of the solution, $m_4$ (Fig. 2), since for a given $\lambda$ there is a range for the values of $m_4$ where the curvature singularity is protected by a horizon. This is similar to what happens in a Reissner-Nördstrom geometry, where the would-be naked singularity is avoided by choosing the mass and electric charge in such a way that they stay below the extremal limit.

6. Cosmological Solutions

In this section we investigate the cosmological solutions for this theory. For this purpose we assume a spatially flat, homogeneous and isotropic universe, described by the line element

$$ds^2 = -dt^2 + a(t) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),$$

with $a(t)$ being the scale factor which measures the physical distances over time. Additionally, we consider a vector field whose only non-vanishing component is $A_0(t)$, so the $A_\mu$ field takes the form

$$(A_\mu) = (A_0(t), 0, 0, 0).$$

For simplicity, we consider again the Lagrangian (32), invariant under $A_\mu \rightarrow -A_\mu$. Starting with the variation with respect to the vector field (33) and using the assumptions (58) and (59) we get the following differential equation, in terms of the Hubble parameter:

$$0 = [2H' + H^2] \left[ H^2 \left( 6\lambda^2 A_0^2 - 1 \right) + 24\lambda^2 \right],$$

whilst the 00 component of the corresponding Einstein equations (34) and the use of the same assumptions gives,

$$H^2 = -H^2 A_0 \left( \frac{1}{2} A_0 H' + \frac{1}{4} H^2 A_0 - \frac{1}{2} H A_0' \right) + \lambda^2 \left( 8 + \frac{3}{2} H^4 A_0^4 - 12HA_0 A_0' \right. \left. - 3H^3 A_0^3 A_0' + 12H^2 A_0^2 + 3A_0^4 H^2 H' \right).$$

According to (60), there are two possible branches of solutions. Let us study first the branch with

$$2H'(t) + H(t)^2 = 0,$$

leading to

$$H(t) = 2H_0 (2 + H_0 t)^{-1},$$
where the integration constant has been chosen in such way that $H(0) = H_0$, the Hubble constant at present time. Substituting this solution into (61) gives

$$A_0(t) = \pm \sqrt{\frac{d(t)}{6H_0^3\lambda^2} - \frac{4t}{H_0} - \frac{4}{6\lambda^2} + t^2}, \quad (64)$$

with

$$d(t) = H_0^6 + H_0^2\lambda^4 \left(48H_0^4t^4 + 384H_0^3t^3 + 1152H_0^2t^2 + 1536H_0t + 576\right) + H_0^2\lambda^2 \left(24a_0H_0 - 24H_0^4t^2 - 96H_0^3t - 48H_0^2\right). \quad (65)$$

The limit $\lambda \to 0$ is well defined only if the minus sign is taken for $\sqrt{d(t)}$. The solution for the scale factor, from (63), reads:

$$a(t) = \frac{1}{4} \left(H_0t + 2\right)^2, \quad (66)$$

here the integration constant has been set asking that at present time $a(0) = 1$. We can write $a(t)$ in terms of a dimensionless quantity by performing the transformation $t \to (\tau - 2)/H_0$, so now we have:

$$a(\tau) = \frac{1}{4} \tau^2, \quad (67)$$

Now it is possible to relate this scale factor with the corresponding scale factor for a perfect fluid with equation of state $P = \omega \rho$. As we know, the scale factor for a perfect fluid is, in general, of the form

$$a(t) \propto t^{\frac{2}{3(1+\omega)}}. \quad (68)$$

Hence, in the cosmological setup we are studying, the vector field has the same effect on the scale factor as a perfect fluid with $\omega = -2/3$, which is associated to a universe in accelerated expansion. The deceleration parameter, defined as $q = -(1 + H'(t)/H(t)^2)$, for (63) is $q = -1/2$, implying a cosmic acceleration.

The second branch emerging from (60) is

$$H^2 \left(6\lambda^2A_0^2 - 1\right) + 24\lambda^2 = 0. \quad (69)$$

Solving for $A_0(t)$ gives

$$A_0(t) = \pm \sqrt{(6\lambda^2)^{-1} - 4H^{-2}}. \quad (70)$$

Substituting into equation (61) and solving for $H(t)$ gives

$$H(t) = \pm 4\lambda, \quad (71)$$

which correspond to the scale factor

$$a(t) = a_0e^{\pm 4\lambda}. \quad (72)$$

Turning back to the expression in (70) and substituting the solution for $H(t)$ we find that

$$A_0(t) = \pm \frac{\lambda^{-1}}{2\sqrt{3}} i. \quad (73)$$
A complex solution for the vector field is unexpected given the initial assumptions for the present formulation, nevertheless, we can get an alternate version of this solution with a real solution for $A_0$ if we consider a complex spin connection. For instance, we can set $\omega^{a\mu} = e^a_\mu + i\Phi^a_\mu$. As consequence of this choice the action for the theory now reads

$$S = \int d^4x \sqrt{-g} \lambda^4 \left[ 8R + 8P^{\mu \nu \alpha \beta} \nabla_\mu A_\alpha \nabla_\nu A_\beta - \lambda^2 \left( 96 - 48 \left( \nabla_\alpha A^\alpha \right)^2 - \nabla_\alpha A^\beta \nabla_\beta A^\alpha \right) 
- 16i \left( \nabla_\alpha A^\alpha \right)^3 - 3\nabla_\rho A^\rho \nabla_\alpha A^\beta \nabla_\beta A^\alpha + 2\nabla_\alpha A^\rho \nabla_\rho A^\beta \nabla_\beta A^\alpha 
- 4\epsilon^{\mu \nu \alpha \beta} \epsilon_{\gamma \delta \rho \lambda} \nabla_\mu A^\gamma \nabla_\nu A^\delta \nabla_\alpha A^\rho \nabla_\beta A^\lambda \right] + h.c, \quad (74)$$

where we have added the conjugate to have a real action‡. The resulting action is given by

$$S = \int d^4x \sqrt{-g} \lambda^4 \left[ 8R + 8P^{\mu \nu \alpha \beta} \nabla_\mu A_\alpha \nabla_\nu A_\beta - \lambda^2 \left( 96 - 48 \left( \nabla_\alpha A^\alpha \right)^2 - \nabla_\alpha A^\beta \nabla_\beta A^\alpha \right) 
- 4\epsilon^{\mu \nu \alpha \beta} \epsilon_{\gamma \delta \rho \lambda} \nabla_\mu A^\gamma \nabla_\nu A^\delta \nabla_\alpha A^\rho \nabla_\beta A^\lambda \right]. \quad (75)$$

For this action we can obtain the equations of motion performing the variation with respect to the metric and the vector field. Taking into account the same assumptions given above for the line element and the vector field components, we have, for the vector field equation of motion in terms of the Hubble parameter,

$$0 = (2H' + H^2) \left[ H^2 \left( 6\lambda^2 A_0^2 + 1 \right) - 24\lambda^2 \right], \quad (76)$$

and for the 00 component of the equation of motion for the metric,

$$H^2 = H^2 A_0 \left( \frac{1}{4} A_0 H^2 - \frac{1}{2} HA_0' + \frac{1}{2} A_0 H' \right) + \lambda^2 \left( 8 + \frac{3}{2} A_0^4 H^4 - 3A_0^3 H^2 A_0' - 12A_0^2 H' 
+ 3A_0^4 H^2 H' + 12A_0 A_0' H \right). \quad (77)$$

Solving $H^2 \left( 6\lambda^2 A_0^2 + 1 \right) - 24\lambda^2 = 0$ in (76) for $A_0 = A_0(t)$, we obtain

$$A_0(t) = \sqrt{4H^{-2} - (6\lambda^2)^{-1}}. \quad (78)$$

Then, using this equation in (77), gives

$$H(t) = \pm 4\lambda. \quad (79)$$

Hence, (78) reduces to

$$A_0(t) = \pm \frac{\lambda^{-1}}{2\sqrt{3}}. \quad (80)$$

‡ Adding the hermitic conjugate to have a real action is standard in QFT.
On the other hand, taking the solution for $0 = 2H'(t) + H(t)^2$ in (76) (the same as in (63)), and substituting in (77), we get, after solving for $A_0(t)$:

$$A_0(t) = \pm \sqrt{\frac{d(t)^{1/2}}{6H_0^2\lambda^2} + \frac{4t}{H_0} + \frac{4}{H_0^2} - \frac{1}{6\lambda^2} + t^2}$$  \hspace{2cm} (81)

with

$$d(t) = H_0^4 + \lambda^2 \left( 24a_0H_0 - 24H_0^4t^2 - 96H_0^3t - 48H_0^2 \right) + \lambda^4 \left( 48H_0^4t^4 + 384H_0^3t^3 + 1152H_0^2t^2 + 1536H_0t + 576 \right).$$  \hspace{2cm} (82)

The previous results show that, for the new action, it is possible to have an exponential scale factor with a real vector.

7. Discussion

In this work we presented a novel approach for obtaining vector-tensor theories of gravity. This approach is based on the original construction of MacDowell and Mansouri to obtain general relativity with a cosmological constant from explicit symmetry breaking of a 5-dimensional gauge group. In the original construction, the fifth component of the internal metric is identified with the cosmological constant, and a set of components of the 5d gauge connection are identified with the 4d spacetime tetrad. In our modification, the internal metric remains the same, but the components of the gauge connection are now identified with the tetrad plus a contribution from an additional field. In the language of reductive Cartan geometry, where MM proposal can be given a geometric interpretation, our modification amounts to changing only the vertical projection of the 5d connection [41, 42]. A detailed study of this geometric setting is outside the scope of this work. Physically, the resulting 4d theory in flat spacetime turns out to be a linear combination of the generalised Proca Lagrangian, while in curved spacetime we get a linear combination of beyond generalised Proca Lagrangians and an additional term whose longitudinal mode does not have Horndeski dynamics. Moreover, the theory gives place to second order equations of motion.

Considering a reduced model with symmetry under $A_\mu \to -A_\mu$, we find static vacuum solutions whose asymptotic behaviour agrees with Schwarzschild or Schwarzschild-de Sitter spacetimes, depending on the value of a constant that appears in the solution for the vector field, but independent of the value of $\lambda$. We also found solutions that where the metric components are regular at $r = 0$. However, a deeper analysis shows that the curvature invariants do diverge there. Nonetheless, it would be interesting to analyse whether further modifications in the construction of the model could lead to regular black holes, like the ones known to appear in nonlinear electrodynamics [40].

We also analysed cosmological solutions with the vector field as the only source of matter, finding that it can contribute to the accelerated expansion of the universe. In particular, we find a solution that can be interpreted as GR with a barotropic perfect
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A fluid with equation of state parameter $\omega = -2/3$, indicating that the presence of the vector field $\vec{A}$ gives an accelerating scale factor. A second branch of solutions in the cosmological scenario turns out to give an imaginary vector field. This is addressed by considering a complex connection and adding the conjugate of the action and consequently obtaining a real vector field. A full cosmological analysis is left for future work.

It is noteworthy that the MacDowell-Mansouri construction that we used as a starting point in this work is not the only possibility to obtain gravity from gauge symmetry breaking. Several alternatives have been studied in the literature, such as considering (anti-)self-dual curvatures, exploring different symmetry breaking patterns, rewriting the action as a $BF$-theory, etc. Also, some connections between MM and $(2+1)$-dimensional gravity and topological M-theory are known. A combination of these proposals with our approach to introduce additional degrees of freedom would probably lead to models with a richer phenomenology, for instance, linear combinations of vector Galileons with non-fixed coefficients. Also, it is relevant to study the case where, instead of a vector field, a scalar field is considered in the construction of the theory. Preliminary results indicate that a combination of beyond Horndeski Lagrangians is obtained, together with some terms that do not seem to fall under that category. An analysis of the properties and degrees of freedom of such model, as well as of different constructions of gravity from gauge symmetry breaking, is left for future work.

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Appendix A

8. Equations of Motion

The equations of motion resulting after the variation of (27) with respect to the metric field are

$$0 = 8G_{\mu\nu} - 2 \left( H^1_{\mu\nu}(A, R) + H^2_{\mu\nu}(A, \nabla\nabla A) + H^3_{\mu\nu}(R, \nabla A) + H^4_{\mu\nu}(\nabla\nabla A) \right)$$

$$+ \lambda^2 \left[ 48g_{\mu\nu} + 24(C^1_{\mu\nu}(A, R) + C^2_{\mu\nu}(A, \nabla\nabla A) + C^3_{\mu\nu}(\nabla A)) \right. \right.$$  

$$+ 16 \left( D^1_{\mu\nu}(A, R, \nabla\nabla A) + D^2_{\mu\nu}(A, \nabla A, \nabla\nabla A) + D^3_{\mu\nu}(\nabla A) \right)$$

$$+ \left. 4 \left( E^1_{\mu\nu}(A, R, \nabla A) + E^2_{\mu\nu}(\nabla A) + E^3_{\mu\nu}(\nabla A, \nabla\nabla A) \right) \right] ,$$  

(83)

where

$$C^1_{\mu\nu} = 2A^\alpha A^\beta g_{\mu\nu}R_{\alpha\beta} - 2A^\alpha A^\beta R_{\mu\alpha} - 2A^\alpha A^\mu R_{\nu\alpha}$$  

$$C^2_{\mu\nu} = -A^\mu \nabla_\alpha \nabla^\alpha A^\nu - A^\nu \nabla_\alpha \nabla^\alpha A^\mu + A^\mu \nabla_\alpha \nabla_\mu A^\alpha + A^\alpha \nabla_\alpha \nabla_\mu A^\nu + A^\mu \nabla_\alpha \nabla_\nu A^\alpha$$

$$+ A^\alpha \nabla_\alpha \nabla_\nu A^\mu - 2A^\alpha g_{\mu\nu} \nabla_\beta \nabla_\alpha A^\beta$$  

(84)
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\[ C_{\mu\nu}^3 = -2\nabla_\alpha A_\mu \nabla^\alpha A_\nu - g_{\mu\nu} \nabla_\alpha A^\alpha \nabla_\beta A^\beta - g_{\mu\nu} \nabla_\alpha A_\beta \nabla^\beta A^\alpha + \nabla^\alpha A_\nu \nabla_\mu A_\alpha + \nabla_\alpha A^\alpha \nabla_\mu A_\nu + \nabla^\alpha A_\mu \nabla_\nu A_\alpha + \nabla_\alpha A^\alpha \nabla_\nu A_\mu \]

(86)

\[ D_{\mu\nu}^1 = -3A^\alpha A_\mu R_{\rho\alpha} \nabla_\beta A^\beta - 3A^\alpha A_\mu R_{\rho\beta} \nabla_\alpha A^\beta + \frac{3}{2} A^\alpha A_\mu R_{\alpha\beta} \nabla^\beta A_\mu + 3A^\alpha A^\beta g_{\mu\nu} R_{\alpha\beta} \nabla_\gamma A^\gamma + 3A^\alpha A^\beta g_{\mu\nu} R_{\beta\alpha} \nabla_\gamma A^\gamma + 3A^\alpha A_\mu R_{\beta\alpha} \nabla_\gamma A^\gamma
\]

(87)

\[ D_{\mu\nu}^2 = \frac{3}{2} A^\alpha \nabla_\alpha A_\mu \nabla_\beta A^\beta + \frac{3}{2} A^\alpha \nabla_\alpha A_\mu \nabla_\beta A^\beta - \frac{3}{2} A^\mu \nabla_\alpha A_\mu \nabla_\beta A^\beta
\]

(88)

\[ D_{\mu\nu}^3 = -3\nabla_\alpha A_\mu \nabla^\alpha A_\mu \nabla_\beta A^\beta - \frac{3}{2} \nabla_\alpha A_\mu \nabla^\alpha A_\mu \nabla_\beta A^\beta + \frac{3}{2} \nabla^\alpha A_\mu \nabla_\beta A_\alpha \nabla^\gamma A_\gamma + \frac{3}{2} \nabla^\alpha A_\mu \nabla_\beta A_\alpha \nabla^\gamma A_\gamma
\]

(89)

\[ E_{\mu\nu}^1 = -6A^\alpha A_\mu R_{\rho\alpha} \nabla_\beta A^\beta \nabla_\gamma A^\gamma - 6A^\alpha A_\mu R_{\rho\alpha} \nabla_\beta A^\beta \nabla_\gamma A^\gamma + 6A^\alpha A_\mu R_{\rho\beta} \nabla_\alpha A^\beta \nabla_\gamma A^\gamma - 6A^\alpha A_\mu R_{\rho\beta} \nabla_\alpha A^\beta \nabla_\gamma A^\gamma + 6A^\alpha A^\beta g_{\mu\nu} R_{\alpha\beta} \nabla_\gamma A^\gamma + 6A^\alpha A^\beta g_{\mu\nu} R_{\beta\alpha} \nabla_\gamma A^\gamma + 6A^\alpha A_\mu R_{\beta\alpha} \nabla_\gamma A^\gamma
\]

(90)

\[ E_{\mu\nu}^2 = -6\nabla_\alpha A_\mu \nabla^\alpha A_\mu \nabla_\beta A^\beta \nabla_\gamma A^\gamma + 6\nabla_\alpha A_\beta \nabla^\beta A_\mu \nabla_\gamma A^\gamma - 6\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla_\nu A_\alpha - 6\nabla_\alpha A_\gamma \nabla_\nu A_\mu \nabla_\beta A_\nu
\]
+ 6\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\alpha \nabla_\mu A^\beta + 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A^\beta \nabla_\alpha A_\nu \\
- 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\alpha - 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\alpha A_\nu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
- 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A^\beta \nabla_\nu A_\alpha \\
- 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A^\beta \nabla_\nu A_\mu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
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+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
+ 3\nabla^\alpha A_\mu \nabla_\beta A_\gamma \nabla^\gamma A_\nu A_\mu \\
+ \text{for } \alpha, \beta, \gamma, \delta = 1, 2, 3.
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\[ \begin{align*}
- A_\nu R_\alpha \nabla^\alpha A_\mu - A_\mu R_\alpha \nabla^\alpha A_\nu + A_\nu R_\alpha \nabla_\mu A_\alpha - 2A_\alpha R_\beta \nabla^\alpha \nabla_\mu A_\beta \\
+ A^\alpha R_\nu \nabla_\mu \nabla_\alpha A_\nu + A_\mu R_\nu \nabla_\alpha A_\alpha - 2A_\nu R_\mu A_\alpha A_\beta + A^\alpha R_\nu \nabla_\alpha A_\mu \\
+ 2A_\nu R_{\rho\sigma\nu} \nabla_\gamma \nabla^\gamma A_\beta - 2A_\nu R_{\rho\sigma\nu} \nabla_\gamma \nabla^\gamma A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla^\gamma A_\beta \\
- 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha \\
+ 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha \\
- 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha - 2A_\nu R_{\rho\gamma\nu} \nabla_\gamma \nabla_\beta A_\alpha \\
\end{align*} \]

\[ H^3_{\mu\nu} = R_\alpha A_\nu \nabla_\mu A_\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A^\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A^\alpha + 4R_{\nu\rho\alpha\gamma} \nabla^\gamma A^\beta \nabla_\mu A^\alpha \\
- 2R_{\nu\gamma\alpha\beta} \nabla^\gamma A^\beta \nabla_\mu A^\alpha - 2R_{\nu\gamma\alpha\beta} \nabla^\gamma A^\beta \nabla_\mu A^\alpha + R_\nu A^\alpha \nabla_\mu A_\alpha - R_{\nu\rho} A^\alpha \nabla_\mu A_\alpha \\
+ R_\nu A^\alpha \nabla_\mu A_\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A^\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A^\alpha + 4R_{\nu\rho\alpha\gamma} \nabla^\gamma A^\beta \nabla_\mu A^\alpha \\
- 2R_{\nu\gamma\alpha\beta} \nabla^\gamma A^\beta \nabla_\mu A^\alpha - 2R_{\nu\gamma\alpha\beta} \nabla^\gamma A^\beta \nabla_\mu A^\alpha + 4R_{\nu\rho\alpha\gamma} \nabla^\gamma A^\beta \nabla_\mu A^\alpha \\
+ 4R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A_\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A_\alpha + 4R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A_\alpha + 2R_{\nu\rho} \nabla_\alpha A^\beta \nabla_\mu A_\alpha \\
- g_{\mu\alpha} \nabla_\nu A^\beta A^\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta A^\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta A^\alpha - 2R_{\nu\rho} \nabla_\alpha A^\beta A^\alpha \\
- 2R_{\nu\rho} \nabla_\alpha A^\beta A^\alpha + 2R_{\nu\rho} \nabla_\alpha A^\beta A^\alpha - g_{\mu\nu} \nabla_\alpha A^\beta A^\alpha \\
+ 4g_{\mu\nu} R_{\gamma\alpha\beta} \nabla^\gamma A^\beta - 4g_{\mu\nu} R_{\gamma\alpha\beta} \nabla^\gamma A^\beta - 4g_{\mu\nu} R_{\gamma\alpha\beta} \nabla^\gamma A^\beta \\
- 2R_{\nu\rho\alpha\gamma} \nabla_\alpha A^\beta A^\alpha + 4R_{\nu\rho\alpha\gamma} \nabla_\alpha A^\beta A^\alpha + 2g_{\mu\nu} R_{\gamma\alpha\beta} \nabla^\gamma A^\beta \\
+ 4R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha + 4R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha + 3R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha \\
- R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha + 3R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha - R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha \\
- 3R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha - R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha \\
- 2R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha + 4R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha + 4g_{\mu\nu} R_{\gamma\alpha\beta} A^\beta A^\alpha \\
+ 4g_{\mu\nu} R_{\gamma\alpha\beta} A^\beta A^\alpha \\
\end{align*} \]

\[ H_1^{\alpha\beta} = 2\nabla_\alpha \nabla_\beta A_\mu \nabla^\beta A_\mu + 2\nabla_\beta \nabla_\nu A_\alpha A^\beta \nabla^\beta A_\mu + 2\nabla_\alpha \nabla_\mu A_\beta \nabla^\beta A_\nu + 2\nabla_\beta \nabla_\nu A_\alpha \nabla^\beta A_\mu + 2\nabla_\beta \nabla_\nu A_\alpha \nabla^\beta A_\mu \]

\[ \begin{align*}
R_{\nu\rho\alpha\gamma} \nabla_\alpha A^\beta A^\alpha + 4R_{\nu\rho\alpha\gamma} \nabla_\alpha A^\beta A^\alpha + 2R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha \\
- R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha - R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha \\
- 2R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha + 4R_{\nu\rho\gamma\delta} \nabla_\alpha A^\beta A^\alpha + 4g_{\mu\nu} R_{\gamma\alpha\beta} A^\beta A^\alpha \\
+ 4g_{\mu\nu} R_{\gamma\alpha\beta} A^\beta A^\alpha
\end{align*} \]

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