ON THE COMPLETE INTEGRABILITY OF THE
OSTROVSKY-VAKHNENKO EQUATION

YAREMA A. PRYKARPATSKYY

ABSTRACT. The complete integrability of the Ostrovsky-Vakhnenko equation is studied by means of symplectic gradient-holonomic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, Lax type representation and related infinite hierarchies of conservation laws are constructed.

1. INTRODUCTION

In 1998 V.O. Vakhnenko investigated high-frequency perturbations in a relaxing barotropic medium. He discovered that this phenomenon is described by a new nonlinear evolution equation. Later it was proved that this equation is equivalent to the reduced Ostrovsky equation [1], which describes long internal waves in a rotating ocean. The nonlinear integro-differential Ostrovsky-Vakhnenko equation

\[ u_t = -u u_x - D^{-1}_x u \]

on the real axis \( \mathbb{R} \) for a smooth function \( u \in C^{(\infty)}(\mathbb{R}; \mathbb{R}) \), where \( D^{-1}_x \) is the inverse-differential operator to \( D_x := \partial/\partial x \), can be derived [2] as a special case of the Whitham type equation

\[ u_t = -u u_x + \int_{\mathbb{R}} K(x, y) u_y dy. \]

Here the generalized kernel \( K(x, y) := \frac{1}{2} |x - y|, x, y \in \mathbb{R} \) and \( t \in \mathbb{R} \) is an evolution parameter. Different analytical properties of equation (1.1) were analyzed in articles [1, 2], the corresponding Lax type integrability was stated in [2].

Recently by J.C. Brunelli and S. Sakovich in [4] there was demonstrated that Ostrovsky-Vakhnenko equation is a suitable reduction of the well known Camassa-Holm equation that made it possible to construct the corresponding compatible Poisson structures for (1.1), but in a complicated enough non-polynomial form.

In the present work we will reanalyze the integrability of equation (1.1) both from the gradient-holonomic [7, 12, 13], symplectic and formal differential-algebraic points of view. As a result, we will re-derive the Lax type representation for the Ostrovsky-Vakhnenko equation (1.1), construct the related simple enough compatible polynomial Poisson structures and an infinite hierarchy of conservation laws.

2. GRADIENT-HOLONOMIC INTEGRABILITY ANALYSIS

Consider the nonlinear Ostrovsky-Vakhnenko equation (1.1) as a suitable nonlinear dynamical system

\[ \frac{du}{dt} = -u u_x - D^{-1}_x u := K[u] \]

on the smooth 2\( \pi \)-periodic functional manifold

\[ M := \{ u \in C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}) : \int_0^{2\pi} u dx = 0 \}, \]

where \( K : M \to T(M) \) is the corresponding well-defined smooth vector field on \( M \).

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We, first, will state that the dynamical system (2.4) on manifold $M$ possesses an infinite hierarchy of conservation laws, that can signify as a necessary condition for its integrability. For this we need to construct a solution to the Lax gradient equation

\begin{equation}
\varphi_t + K^\cdot \varphi = 0,
\end{equation}

in the special asymptotic form

\begin{equation}
\varphi = \exp[-\lambda t + D_x^{-1}\sigma(x;\lambda)],
\end{equation}

where, by definition, a linear operator $K^\cdot : T^*(M) \to T^*(M)$ is, adjoint with respect to the standard convolution $(\cdot,\cdot)$ on $T^*(M) \times T(M)$, the Frechet-derivative of a nonlinear mapping $K : M \to T(M)$:

\begin{equation}
K^\cdot = uD_x + D_x^{-1}
\end{equation}

and, respectively,

\begin{equation}
\sigma(x;\lambda) \simeq \sum_{j \in \mathbb{Z}_+} \sigma_j[u] \lambda^{-j},
\end{equation}

as $|\lambda| \to \infty$ with some ”local” functionals $\sigma_j : M \to C^\infty(\mathbb{R} / 2\pi \mathbb{Z};\mathbb{R})$ on $M$ for all $j \in \mathbb{Z}_+$.

By substituting (2.4) into (2.3) one easily obtains the following recurrent sequence of functional relationships:

\begin{equation}
\sigma_{j,t} + \sum_{k \leq j} \sigma_{j-k}(u\sigma_k + D_x^{-1}\sigma_{k,t}) - \sigma_{j+1} + (u\sigma_j)_x + \delta_{j,0} = 0
\end{equation}

for all $j + 1 \in \mathbb{Z}_+$ modulo the equation (2.1). By means of standard calculations one obtains that this recurrent sequence is solvable and

\begin{align*}
\sigma_0[u] &= 0, \quad \sigma_1[u] = 1, \quad \sigma_2[u] = u_x, \\
\sigma_3[u] &= 0, \quad \sigma_4[u] = u_t + 2uu_x, \\
\sigma_5[u] &= \frac{3}{2}(u^2)_{xt} + u_{tt} + 2/3(u^3)_{xx} - u_xD_x^{-1}u
\end{align*}

and so on. It is easy check that all of functionals

\begin{equation}
\gamma_j := \int_0^{2\pi} \sigma_j[u] dx
\end{equation}

are on the manifold $M$ conservation laws, that is $d\gamma_j/dt = 0$ for $j \in \mathbb{Z}_+$ with respect to the dynamical system (2.4). For instance, at $j = 5$ one obtains:

\begin{align*}
\gamma_5 &:= \int_0^{2\pi} \sigma_5[u] dx = \int_0^{2\pi} \left[3/2(u^2)_{xt} + u_{tt} + 2/3(u^3)_{xx} - u_xD_x^{-1}u\right] dx = \\
&= \frac{d^2}{dt^2} \int_0^{2\pi} u_{tt} dx - \int_0^{2\pi} u_xD_x^{-1}u dx = \frac{d^2}{dt^2} \int_0^{2\pi} u dx - uD_x^{-1}u|_0^{2\pi} + \int_0^{2\pi} u^2 dx = \\
&= \int_0^{2\pi} u^2 dx,
\end{align*}

and

\begin{align*}
d\gamma_5/dt &= 2 \int_0^{2\pi} uu_t dx - 2 \int_0^{2\pi} u(uu_x + D_x^{-1}u) dx = \\
&= -2 \int_0^{2\pi} uD_x^{-1}u dx = -\int_0^{2\pi} [(D_x^{-1}u)_x] dx = (D_x^{-1}u)_0^{2\pi} = 0,
\end{align*}

since owing to the constraint (2.2) the integrals $(D_x^{-1}u)_0^{2\pi} = 0$.

The result stated above allows us to suggest that the dynamical system (2.4) on the functional manifold $M$ is an integrable Hamiltonian system.

First, we will show that this dynamical system is a Hamiltonian flow

\begin{equation}
du/dt = -\vartheta \text{ grad } H[u]
\end{equation}
with respect to some Poisson structure $\vartheta : T^*(M) \rightarrow T(M)$ and a Hamiltonian function $H \in \mathcal{D}(M)$. Based on the standard symplectic techniques \[10, 7, 6, 12\] consider the conservation law (2.10) and present it in the scalar "momentum" form:

\[
-1/2\gamma_5 = \frac{1}{2} \int_0^{2\pi} u_x D_x^{-1} u dx = (1/2D_x^{-1}u, u_x) := (\psi, u_x)
\]

with the co-vector $\psi := 1/2D_x^{-1}u \in T^*(M)$ and calculate the corresponding co-Poissonian structure (2.14)

\[
\vartheta^{-1} := \psi' - \psi'^* = D_x^{-1},
\]

or the Poissonian structure (2.15)

\[
\vartheta = D_x.
\]

The obtained operator $\vartheta = D_x : T^*(M) \rightarrow T(M)$ is really Poissonian for (2.1) since the following determining symplectic condition (2.16)

\[
\psi_t + K^{'*} \psi = \text{grad } L
\]

holds for the Lagrangian function (2.17)

\[
L = \frac{1}{12} \int_0^{2\pi} u^3 dx.
\]

As a result of (2.16) one obtains easily that (2.18)

\[
du/dt = -\vartheta \text{grad } H[u],
\]

where the Hamiltonian function (2.19)

\[
H = (\psi, K) - L = \frac{1}{2} \int_0^{2\pi} \left[ u^3/3 - (D_x^{-1}u)^2/2 \right] dx
\]

is an additional conservation law of the dynamical system (2.1). Thus, one can formulate the following proposition.

**Proposition 2.1.** The Ostrovsky-Vakhnenko dynamical system (2.1) possesses an infinite hierarchy of nonlocal, in general, conservation laws (2.9) and is a Hamiltonian flow (2.18) on the manifold $M$ with respect to the Poissonian structure (2.15).

**Remark 2.2.** It is useful to remark here that the existence of an infinite ordered by $\lambda$-powers hierarchy of conservations laws (2.9) is a typical property \[10, 6, 7, 12\] of the Lax type integrable Hamiltonian systems, which are simultaneously bi-Hamiltonian flows with respect to corresponding two compatible Poissonian structures.

As is well known \[10, 6, 7, 12\], the second Poissonian structure $\eta : T^*(M) \rightarrow T(M)$ on the manifold $M$ for (2.1), if it exists, can be calculated as (2.20)

\[
\eta^{-1} := \tilde{\psi}' - \tilde{\psi}'^*,
\]

where $a$-covector $\tilde{\psi} \in T^*(M)$ is a second solution to the determining equation (2.16):

\[
\tilde{\psi}_t + K^{'*} \tilde{\psi} = \text{grad } \tilde{L}
\]

for some Lagrangian functional $\tilde{L} \in \mathcal{D}(M)$. It can be certainly done by means of simple enough but cumbersome analytical calculations based, for example, on the asymptotical small parameter method \[7, 12, 13\] and on which we will not stop here.

Instead of this we will shall apply the direct differential-algebraic approach to dynamical system (2.1) and reveal its Lax type representation both in the differential scalar and in canonical matrix Zakharov-Shabat forms. Moreover, we will construct the naturally related compatible polynomial Poissonian structures for Ostrovsky-Vakhnenko dynamical system (2.1) and generate an infinite hierarchy of commuting to each other nonlocal conservation laws.
3. Lax type representation and compatible Poissonian structures - the differential-algebraic approach

We will start with construction of the polynomial differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{x, t\}$ generated by a fixed functional variable $u \in \mathbb{R}\{x, t\}$ and invariant with respect to two differentiations $D_x := \partial/\partial x$ and $D_t := \partial/\partial t + u\partial/\partial x$, satisfying the Lie-algebraic commutator relationship

$$[D_x, D_t] = u_x D_x. \tag{3.1}$$

Since the Lax type representation for the dynamical system (2.1) can be interpreted as the existence of a finite-dimensional invariant differential ideal $\mathcal{I}_N\{u\} \subset \mathcal{K}\{u\}$, realizing the corresponding finite-dimensional representation of the the Lie-algebraic commutator relationship (3.1), this ideal can be presented as

$$\mathcal{I}_N\{u\} := \{ \sum_{j=0, N} g_j D_x^j f[u] \in \mathcal{K}\{u\} : g_j \in \mathcal{K}, j = 0, N \}, \tag{3.2}$$

where an element $f[u] \in \mathcal{K}\{u\}$ and $N \in \mathbb{Z}_+$ are fixed. The $D_x$-invariance of ideal (3.2) will be a priori evident, if the function $f[u] \in \mathcal{K}\{u\}$ satisfies the linear differential relationship

$$D_x^{N+1} f = \sum_{k=0}^N a_j[u] D_x^j f \tag{3.3}$$

for some coefficients $a_j[u] \in \mathcal{K}\{u\}$, $j = 0, N$, but its $D_t$-invariance strongly depends on the element $f[u] \in \mathcal{K}\{u\}$, which can be found from the functional relationship (2.3) on the element $\varphi[u; \lambda] := \text{grad} \gamma(\lambda) \in \mathcal{K}\{u\}$, $\gamma(\lambda) := \int_0^{2\pi} \sigma(x; \lambda) dx$, rewritten in the following form:

$$D_x D_t \varphi = -\varphi. \tag{3.4}$$

From the right hand side one follows that there exists an element $\eta := \eta[u] = -D_t \varphi[u] \in \mathcal{K}\{u\}$, such that

$$\varphi = D_x \eta. \tag{3.5}$$

Having substituted (3.3) into the left hand side of (3.4) one finds easily that

$$D_x D_t \eta - u_x \eta_x = D_x D_t \eta - u_x \varphi =$$

$$= D_x D_t \eta - u_x \text{grad} \gamma(\lambda) =$$

$$= D_x (D_t \eta - \gamma[u, \lambda]) = \eta, \tag{3.6}$$

where we have put, by definition, $\gamma(\lambda) := \int_0^{2\pi} \gamma(u; \lambda) dx$ for a suitably chosen density element $\gamma[u; \lambda] \in \mathcal{K}\{u\}$. As an evident result of (3.6) one derives that there exists an element $\rho := \rho[u] \in \mathcal{K}\{u\}$, such that

$$\eta = D_x \rho. \tag{3.7}$$

Turning back to the relationships (3.5) and (3.7) one obtains that the following differential representation

$$\varphi = D_x^2 \rho \tag{3.8}$$

holds.

As a further step, we can try to realize the differential ideal (3.2) by means of the generating element $f[u] \implies \rho[u] \in \mathcal{K}\{u\}$, defined by the relationship (3.8). But, as it is easy to check, the obtained this way differential ideal is not finite-dimensional. So, for a future calculating convenience, we will represent the element $\rho[u] \in \mathcal{K}\{u\}$ in the following natural factorized form:

$$\rho := \tilde{f} f, \tag{3.9}$$

where elements $f, \tilde{f} \in \mathcal{K}\{u\}$ satisfy, by definition, the adjoint pairs of the following differential relationships:

$$D_x^{N+1} f = \sum_{k=0}^N a_j[u] D_x^j f, \tag{3.10}$$

$$(-1)^{N+1} D_x^{N+1} \tilde{f} = \sum_{k=0}^N (-1)^j (D_x^j a_j[u]) \tilde{f},$$
and
\begin{equation}
(3.11) \quad D_{t}f = \sum_{j=0}^{N-1} b_{j} D_{x}^{j}f, \quad D_{t} \bar{f} = -u_{x} \bar{f} + \sum_{j=0}^{N-1} (-1)^{j+1} (D_{x}^{j} b_{j})f,
\end{equation}

for some elements \( b_{j} \in K\{u\}, j = 0, N - 1 \), and check the finite-dimensional \( D_{x} \)- and \( D_{t} \)-invariance of the corresponding ideal \( \mathcal{I}_{2} \), generated by the element \( f \in K\{u\} \).

Now it is easy to check by means of simple enough calculations, based on the relationship \( (3.12) \) and \( (3.13) \), that the following differential equalities
\begin{equation}
(3.12) \quad D_{x}(D_{t} \varphi) = -\varphi, \quad D_{x}(D_{t} \varphi) = u_{x} \varphi - D_{t} \varphi, \\
D_{x}(D_{t} \varphi) = u_{x} \varphi - 2u_{x} D_{t} \varphi - D_{t} \varphi, \\
D_{x}(D_{t} \varphi) = -(u_{x} + D_{x}^{-1}) \varphi + (4u_{x}^{2} + 3u) D_{t} \varphi - 2u_{x} D_{t}^{2} \varphi - D_{t}^{2} \varphi,\ldots,
\end{equation}

and their consequences
\begin{equation}
(3.13) \quad D_{t} D_{x}^{2} \rho = -\rho_{x}, \quad D_{x}(D_{t} \rho_{x}) = u_{x} \rho_{xx} - D_{x} \rho, \\
D_{x}^{2}(D_{t} \rho) = D_{x}(u_{x} D_{x} \rho - \rho) + u_{xx} D_{x}^{2} \rho,\ldots,
\end{equation}

hold. Taking into account the independence of the sets of functional elements \( \{ f, D_{x} f, D_{x}^{2} f, \ldots, D_{x}^{N-1} f \} \subset K\{u\} \) and \( \{ f, D_{x} f, D_{x}^{2} f, \ldots, D_{x}^{N-1} f \} \subset K\{u\} \), the relationships \( (3.13) \) jointly with \( (3.9) \), \( (3.10) \) and \( (3.11) \) make it possible to state the following lemma.

**Lemma 3.1.** The set \( \mathcal{I}_{2} \) represents a \( D_{x} \)- and \( D_{t} \)-invariant differential ideal in the ring \( K \) for all \( N \geq 2 \).

**Proof.** This result easily follows from the fact that for number \( N \geq 2 \) all of the relationships \( (3.13) \) persist to be compatible upon taking into account the differential expressions \( (3.9) \) and \( (3.11) \). Contrary to that, at \( N = 1 \) they become not compatible. \( \square \)

As a corollary of Lemma 3.1, having put in \( (3.2) \) and \( (3.11) \) the number \( N = 2 \), one finds easily by means of elementary enough calculations that the related differential ideal \( \mathcal{I}_{2}\{u\} \) lasts to be invariant, if the differential Lax type relationships
\begin{equation}
(3.14) \quad D_{x}^{3} f = -\mu \bar{u} f, \quad D_{x}^{3} \bar{f} = \mu \bar{u} \bar{f},
\end{equation}

and
\begin{equation}
(3.15) \quad D_{t} f = \mu^{-1} D_{x}^{2} f + u_{x} f, \quad D_{t} \bar{f} = -\mu^{-1} D_{x}^{2} \bar{f} - 2u_{x} \bar{f},
\end{equation}

where \( \bar{u} := u_{xx} + 1/3, \mu \in \mathbb{C} \setminus \{0\} \) is an arbitrary complex parameter, hold. Moreover, they exactly coincide with those found before in \( [5] \). The obtained above differential relationships \( (3.14) \) and \( (3.15) \) can be equivalently rewritten in the following matrix Zakharov-Shabat type form:
\begin{equation}
D_{x} h = \hat{q}[u; \mu] h, \quad D_{x} h = \hat{t}[u; \mu] h,
\end{equation}

where matrices
\begin{equation}
(3.17) \quad \hat{q}[u; \mu] := \begin{pmatrix} u_{x} & 0 & 1/\mu \\ -1/3 & 0 & 0 \\ 0 & -1/3 & -u_{x} \end{pmatrix}, \quad \hat{t}[u; \mu] := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu \bar{u} & 0 & 0 \end{pmatrix}
\end{equation}

and \( h := (f, D_{x} f, D_{x}^{2} f)^{T} \in K\{u\}^{3} \).

Based further on the obtained differential relationships \( (3.14) \) and \( (3.15) \), one obtains that the compatibility condition \( (3.1) \) gives rise to the following important relationship
\begin{equation}
(3.18) \quad \hat{\varphi} = D_{x}^{2} D_{t} \varphi = 3 \mu^{2} \eta \varphi,
\end{equation}

where the polynomial integro-differential operator
\begin{equation}
(3.19) \quad \eta := \partial^{-1} \bar{u} \bar{\partial}^{-3} \bar{u} \bar{\partial}^{-1} + 4 \partial^{-2} \bar{u} \partial^{-2} \bar{u} \partial^{-2} + 2(\partial^{-2} \bar{u} \partial^{-2} \bar{u} \partial^{-1} + \partial^{-1} \bar{u} \partial^{-2} \bar{u} \partial^{-2})
\end{equation}
is skewsymmetric on the functional manifold \( M \) and presents the second compatible Poisson structure for the Ostrovsky-Vakhnenko dynamical system \( (2.1) \).

Based now on the recurrent relationships following from substitution of the asymptotic expansion
\begin{equation}
(3.20) \quad \varphi \simeq \sum_{j \in \mathbb{Z}_{+}} \varphi_{j} \xi^{-j}, \quad \xi := -1/(3\mu^{2}),
\end{equation}
into (3.18), one can determine a new infinite hierarchy of conservations laws for dynamical system (2.1):

\[ \tilde{\gamma}_j := \int_0^1 ds(\varphi_j[u_s], u), \]

for \( j \in \mathbb{Z}_+ \), where

\[ \varphi_j = \lambda^j \varphi_0, \quad \partial \varphi_0 = 0, \]

and the recursion operator \( \Lambda := \partial^{-1} : T^*(M) \to T^*(M) \) satisfies the standard Lax type representation:

\[ \Lambda_t = [\Lambda, K^*]. \]

The obtained above results can be formulated as follows.

**Proposition 3.2.** The Ostrovsky-Vakhnenko dynamical system (2.1) allows the standard differential Lax type representation (3.14), (3.15) and defines on the functional manifold \( \mathcal{M} \) an integrable bi-Hamiltonian flow with compatible Poisson structures (2.15) and (3.19). In particular, this dynamical system possesses an infinite hierarchy of nonlocal conservation laws (3.21), defined by the gradient elements (3.22).

**Remark 3.3.** It is useful to remark here that the existence of an infinite \( \lambda \)-powers ordered hierarchy of conservations laws (2.9) is a typical property \([10, 6, 7, 12]\) of the Lax type integrable Hamiltonian systems, which are simultaneously bi-Hamiltonian flows with respect to corresponding compatible Poissonian structures.

**Remark 3.4.** It is interesting to observe that our second polynomial Poisson structure (3.19) differs from that obtained recently in \([4]\), which contains the rational power factors.

It is easy to construct making use of the differential expressions (3.14) and (3.15) a slightly different from (3.16) matrix Lax type representation of the Zakharov-Shabat form for the dynamical system (1.1). Really, if to define the "spectral" parameter \( \mu := 1/(9\lambda) \in \mathbb{C}\setminus\{0\} \) and new basis elements of the invariant differential ideal (3.2):

\[ g_1 := -3D_xf, \quad g_2 := f, \quad g_3 := 9\lambda D_x^2f + u_xf, \]

then relationships (3.14) and (3.15) can be rewritten as follows:

\[ D_t g = q[u; \lambda]g, \quad D_x g = l[u; \lambda]g, \]

where matrices

\[ q[u; \lambda] := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -u & 0 \end{pmatrix}, \quad l[u; \lambda] := \begin{pmatrix} 0 & u_x/(3\lambda) & -1/(3\lambda) \\ -1/3 & 0 & 0 \\ -u_x/3 & -1/3 & 0 \end{pmatrix} \]

coincide with those of \([5, 4]\) and satisfy the following Zakharov-Shabat type compatibility condition:

\[ D_t l = [q, l] + D_x q - l D_x u. \]

**Remark 3.5.** As it was already mentioned above, the Lax type representation (3.26) of the Ostrovsky-Vakhnenko dynamical system (1.1) was obtained in \([5]\) by means of a suitable limiting reduction of the Degasperis-Processi equation

\[ U_t - u_{xxx} + 4uu_x - 3u_xu_{xx} - uu_{xxx} = 0. \]

For convenience, let us rewrite the latter in the following form:

\[ D_z z = -3zD_xu, \quad z = u - D_x^2u, \]

where differentiations \( D_z := \partial/\partial z + u\partial/\partial x \) and \( D_x := \partial/\partial x \) satisfy the Lie- algebraic relationship (3.1). It appears to be very impressive that equation (3.28) is itself a special reduction of a new Lax type integrable Riemann type hydrodynamic system, proposed and studied (at \( s = 2 \)) recently in \([11]\):

\[ D_{t}^{-1}u = z_x^s, \quad D_{t} z = 0, \]
where $s, N \in \mathbb{N}$ are arbitrary natural numbers. Really, having put, by definition, $z := \bar{z}^s_x$ and $s = 3$, from (3.30) one easily obtains the following dynamical system:

\begin{align}
D_{t}^N - 1 u &= z, \\
D_{t} z &= -3 z D_{x} u,
\end{align}

(3.31)

coinciding with the Degasperis-Processi equation (3.29) if to make the identification $z = u - D_{x}^2 u$. As a result, we have stated that a function $u \in C^\infty(\mathbb{R}^2; \mathbb{R})$, satisfying for an arbitrary $N \in \mathbb{N}$ the generalized Riemann type hydrodynamical equation

\begin{align}
D_{t}^N - 1 u &= u - D_{x}^2 u,
\end{align}

(3.32)

simultaneously solves the Degasperis-Processi equation (3.28). In particular, having put $N = 2$, we obtain that solutions to the Burgers type equation

\begin{align}
D_{t} u &= u - D_{x}^2 u
\end{align}

(3.33)

are solving also the Degasperis-Processi equation (3.28). It means, in particular, that the reduction procedure of the work [5] can be also applied to the Lax type integrable Riemann type hydrodynamic system (3.30), giving rise to a related Lax type representation for the Ostrovsky-Vakhnenko dynamical system (1.1).

4. Conclusion

We have showed that the Ostrovsky-Vakhnenko dynamical system is naturally embedded into the general Lax type integrability scheme [10, 6, 7, 12], whose main ingredients such as the corresponding compatible Poissonian structures and Lax type representation can be effectively enough retrieved by means of direct modern integrability tools, such as the differential-geometric, differential-algebraic and symplectic gradient holonomic approaches. We have also demonstrated the relationship of the Ostrovsky-Vakhnenko equation (1.1) with a generalize Riemann type hydrodynamic system, studied recently in [11] and its reduction.

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Department of Applied Mathematics, University of Agriculture, ul. Balicka 253c, 30-198 Krakow, Poland, and the Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska str., Kyiv, Ukraine

E-mail address: yarpry@gmail.com