Damping and reaction rates and wave function renormalization of fermions in hot gauge theories

Alejandro Ayala, Juan Carlos D’Olivo
Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
Aptartado Postal 70-543, México Distrito Federal 04510, México.

Axel Weber
Instituto de Física y Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Apartado Postal 2-82 Morelia Michoacán 58040, México.

March 28, 2022

Abstract

We examine the relation between the damping rate of a chiral fermion mode propagating in a hot plasma and the rate at which the mode approaches equilibrium. We show how these two quantities, obtained from the imaginary part of the fermion self-energy, are equal when the reaction rate is defined using the appropriate wave function of the mode in the medium. As an application, we compute the production rate of hard axions by Compton-like scattering processes in a hot QED plasma starting from both, the axion self-energy and the electron self-energy. We show that the latter rate coincides with the former only when this is computed using the corresponding medium spinor modes.

PACS numbers: 11.10.Wx, 11.15.-q
1 Introduction

As is well known, the properties of a chiral fermion propagating in a medium are modified from those in vacuum. Interactions give rise to the appearance of collective modes characterized by dispersion relations and damping rates. To be precise, if $P^\mu = (\omega, \vec{p})$ is the four-momentum of a fermion mode in the rest frame of the medium, the corresponding pole of the propagator is located, in general, at a complex value

$$\omega = \omega_p - i \frac{\gamma}{2}, \quad (1)$$

where $\omega_p$ and $\gamma$ are real functions of $p$ and correspond to the energy and damping rate of the mode, respectively. In particular, $m_f \equiv \omega_{p=0}$ can be interpreted as the mass of the fermionic excitation and $\gamma^{-1}$, in accordance with linear response theory [4], gives the characteristic time scale for the decay of the mode. In a field theoretical description, $\omega_p$ and $\gamma$ are determined (for weak damping) from the real and imaginary part of the fermion self-energy, respectively.

We recall, however, that there exists a distinct concept also related to the imaginary part of the fermion self-energy. This quantity, which we will call the total reaction rate $\Gamma$, is interpreted in such a way that its inverse gives the time scale for the fermion distribution to approach equilibrium [2, 3]. It would then appear that one has two different physical quantities, the damping rate $\gamma$ and the total reaction rate $\Gamma$, both obtained from the imaginary part of the fermion self-energy. This has indeed been a source of some confusion in the literature, for instance, in a paper on the subject, two different expressions are used to compute the damping and total reaction rate in terms of the self-energy [4].

The equivalence between the above two quantities was established by D’Olivo and Nieves [5] for a chiral fermion interacting with a scalar and a massive fermion through a Yukawa coupling. They showed that in this case

$$\Gamma = \gamma, \quad (2)$$

provided $\Gamma$ is defined using the appropriate wave functions of the chiral fermion mode in the medium. In this paper we show explicitly how this equivalence is realized in the realm of hot gauge theories where the resummation method of Braaten and Pisarski [6] has provided a consistent framework for the computation of the leading-order contributions to damping rates. We also stress the role played by wave function renormalization of the medium spinor modes for the above equivalence. We illustrate this role by computing the production rate of hard axions in a hot QED plasma through Compton-like scattering processes, showing that the rate obtained by considering the axion self-energy coincides with the rate obtained from the expression for the electron self-energy only when the appropriate medium spinors are used.

This paper is organized as follows: Sections II and III are devoted to a brief recollection of some well established concepts regarding the self-energy of a chiral fermion in a medium...
and its relation to the fermion damping rate. In section II we review how the damping rate can be written in terms of the imaginary part of the self-energy and the spinors satisfying the effective Dirac equation in the medium. In section III we briefly recall how a complete leading order calculation yields gauge independent poles for the fermion propagator in the medium in a covariant gauge. In section IV we express the imaginary part of the self-energy as a total reaction rate. We identify the physical processes involved and establish the equivalence between the damping rate and the total reaction rate when the latter is expressed in terms of the spinors in the medium. In section V we compute the production rate of hard axions in a hot QED plasma. We finally discuss our results in section VI. A short appendix collects notations and definitions of some quantities appearing in section IV.

2 The damping rate

In this section we express the damping rate in terms of the self-energy, following D’Olivo and Nieves [5], thereby establishing our notation and preparing for the discussion in the next section. The inverse propagator of a chiral fermion in a medium can be written in general as

\[ iS^{-1}(P) = P - \Sigma, \]  

where \( \Sigma \) is the effective self-energy induced by the medium. Hereafter, capital letters are used to refer to momentum four-vectors. At \( T = 0 \) the theory is chirally invariant, due to the vanishing of the fermion mass. Chiral invariance is unaffected by the presence of the medium, hence the inverse propagator in the medium rest frame can be expressed as

\[ iS^{-1}(P) = A_0 \gamma_0 - A_s \vec{\gamma} \cdot \hat{p} \]

\[ = \frac{1}{2} \Delta_+^{-1}(P) (\gamma_0 + \vec{\gamma} \cdot \hat{p}) + \frac{1}{2} \Delta_-^{-1}(P) (\gamma_0 - \vec{\gamma} \cdot \hat{p}) \]  

in terms of the functions \( \Delta_{\pm}(P) = (A_0 \mp A_s)^{-1} \), where we have denoted the unit vector \( \hat{p}/p \) by \( \hat{p} \). The propagator poles are given by \( \Delta_{\pm}^{-1}(P) = 0 \). In the positive-energy sector, Eq. (4) yields four propagating modes, two with positive and two with negative helicity over chirality ratio [6]. For definitiveness, let us work with a negative-helicity solution and look for the positive-energy poles in \( \Delta_+(P) \). The same arguments apply to negative-energy and positive-helicity solutions and the poles of \( \Delta_-(P) \).

\( \Delta_+^{-1}(\omega, p) \) has in general a real and an imaginary part. Writing \( \omega \) as in Eq. (1), the equation determining the poles is

\[ \text{Re} \, \Delta_+^{-1}(\omega_p - i\gamma/2, p) + i \, \text{Im} \, \Delta_+^{-1}(\omega_p - i\gamma/2, p) = 0. \]  

In the following, we consider the case of weak damping, \( \gamma \ll \omega_p \), where the physical concept of a propagating mode is still meaningful. Eq. (5) can then be solved by expanding
it in powers of $\gamma$ and retaining terms at most linear in $\gamma$ and $\text{Im} \Delta^{-1}$. At zeroth order, $\omega_p$ is obtained from
\[ \text{Re} \Delta^{-1}_+ (\omega_p, p) = 0, \tag{6} \]
whereas $\gamma$ is given by the first-order equation, which yields
\[ \frac{\gamma}{2} = Z_p \text{Im} \Delta^{-1}_+ (\omega_p, p), \tag{7} \]
where
\[ Z_p^{-1} = \left[ \frac{\partial}{\partial \omega} \text{Re} \Delta^{-1}_+ \right]_{\omega=\omega_p}. \tag{8} \]

The factor $Z_p$ coincides with the residue at the pole of the one-particle contribution to the propagator and it is thus the normalization factor that has to be taken into account to construct the spinors representing one-particle states [8]. In the context of many-body physics, $Z_p$ represents the probability factor that needs to be considered for the various processes involving the fermion mode [9].

In order to relate $\gamma$ to $\text{Im} \Sigma$, recall that the one-particle states are represented by spinors satisfying the effective Dirac equation
\[ (\slashed{P} - \text{Re} \Sigma) u(P) = 0 \tag{9} \]
obtained by neglecting the absorptive part of the effective self-energy. Real and imaginary part of $\Sigma$ are defined, as usual, by
\[ \text{Re} \Sigma = \frac{1}{2} \left( \Sigma + \gamma^0 \Sigma^\dagger \gamma^0 \right), \]
\[ \text{Im} \Sigma = \frac{1}{2i} \left( \Sigma - \gamma^0 \Sigma^\dagger \gamma^0 \right). \tag{10} \]

In the following, we will not need the explicit solutions of Eq. (9), but rather work with the corresponding projection operators. Choosing a normalization corresponding to that of the one particle states [8], the projector for the solution under consideration can be written as
\[ u(P) \bar{u}(P) = Z_p \omega_p, L \not{n}, \tag{11} \]
where $L = (1 - \gamma_5)/2$ and $n^\mu$ is a four-vector with components $n^\mu = (1, \hat{p})$ in the plasma rest frame. From Eqs. (3), (4) and (7) we get, by a straightforward computation,
\[ \gamma = -\frac{Z_p}{2} \text{Tr} \left[ \not{n} \text{Im} \Sigma(\omega_p, p) \right] = -\frac{1}{\omega_p} \bar{u}(P) \text{Im} \Sigma(\omega_p, p) u(P), \tag{12} \]
where in the last line we have used Eq. (11). Note that in Eq. (12) the self-energy is evaluated at $\omega = \omega_p$. Eq. (12) is valid independently of any approximation scheme, perturbative or not, and it only requires that the damping rate be small compared to the energy of the mode. This equation was already used in Ref. [5] to establish the equality of damping and reaction rates in a model with Yukawa coupling. In the next section, we review how the damping rate is obtained in hot gauge theories.

3 The HTL self-energy

In a gauge theory, a plasma of massless particles is characterized by two quantities, the temperature $T$ and the dimensionless coupling constant $g$. If $g \ll 1$ there exist two distinct energy scales, $T$ and $gT$. The first is characteristic of individual particles, whereas the second is typical of collective phenomena. In the Hard Thermal Loop (HTL) approximation, bare perturbation theory is used to compute the leading contributions to the self-energies and vertices. For soft external momenta (i.e. of order $\sim gT$), these are of the same order of magnitude as the bare quantities that they modify, thus the necessity of resummation. HTLs provide effective propagators and vertices necessary for a consistent perturbative expansion. In the following, we will consider exclusively soft external momenta. The lowest order HTL contribution to the fermion propagator can be shown to be real (for time-like momenta) and gauge-independent [6]. A complete leading order calculation of the imaginary part of the self-energy requires, however, the sum of the two diagrams depicted in Fig. 1. The heavy dots indicate effective propagators and vertices. Fig. 1a is the usual diagram for the fermion self-energy modified by the use of effective propagators and the effective two-fermion one-gauge boson vertex. Fig. 1b is an additional contribution coming from the effective two-fermion two-gauge boson vertex, upon contracting the gauge boson lines.

Let us denote by $\Sigma(P)$ the effective self-energy calculated to leading order in its real

![Figure 1: The self-energy graphs for fermions at one-loop level in the effective expansion. Heavy dots denote the effective propagators and vertices arising in the HTL approximation](image)
and imaginary parts. In the imaginary time formalism of Thermal Field Theory, the contributions from Figs. 1a and 1b are, respectively

\begin{align*}
\Sigma_a(P) &= -g^2 C_F T \sum_n \int \frac{d^3k}{(2\pi)^3} \Gamma_\mu(P, P - K) D_{\mu\nu}(K) S(P - K) \Gamma_\nu(P - K, P) \\
\Sigma_b(P) &= -g^2 C_F T \sum_n \int \frac{d^3k}{(2\pi)^3} \Gamma_{\mu\nu}(P, P - K) D_{\mu\nu}(K)
\end{align*}

(13)

where \( C_F \) is the Casimir invariant corresponding to the fundamental representation of the gauge group. The effective two-fermion one-gauge boson and two-fermion two-gauge boson vertices are denoted by \( \Gamma_\mu \) and \( \Gamma_{\mu\nu} \), respectively. \( D_{\mu\nu} \) and \( S \) represent the gauge boson and fermion effective propagators, respectively.

In a covariant gauge, \( D_{\mu\nu} \) has a piece proportional to the gauge parameter \( \xi \). Let us separate its contribution to the effective self-energy writing

\[ \Sigma(P) = \Sigma_0(P) + \Sigma_\xi(P) \]  

(14)

where \( \Sigma_0(P) \) is gauge-parameter independent and \( \Sigma_\xi(P) \) is the contribution to Eq. (13) from the term proportional to the gauge parameter in \( D_{\mu\nu} \), namely, from

\[ \frac{\xi K_\mu K_\nu}{K^2 K^2} \].

By using the Ward identities satisfied by the HTLs we can express the gauge-parameter dependent term in Eq. (14) as

\[ \Sigma_\xi(P) = a(P) S^{-1}(P) \],

(15)

where we have defined

\[ a(P) \equiv \xi g^2 C_F T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(K^2)^2} - S(P - K) S^{-1}(P) \].

(16)

The function \( a(P) \) is infrared singular. In what follows we assume that a proper infrared regulator has been introduced and that this will be taken to zero only until the very end of any calculation [10].

The modification that \( \Sigma(P) \) introduces in the fermion propagator can be written, after analytical continuation to Minkowski space, as in Eq. (3). Using Eqs. (14) and (15), we have

\[ i S^{-1}(P) = P - \Sigma_0(P) - i a(P) S^{-1}(P) \].

(17)

A simple power counting shows that \( a(P) \) is of order \( g \). Therefore, to the given order, it is allowed to replace \( S^{-1}(P) \) by \( * S^{-1}(P) \) in Eq. (17) which now becomes

\[ i * S^{-1}(P) = P - * \Sigma(P) \]

\[ \approx [1 - a(P)][P - * \Sigma_0(P)], \]

(18)
from which we have

\[ \ast \Sigma(P) \equiv a(P) P + [1 - a(P)] \ast \Sigma_0(P). \quad (19) \]

Notice that in principle, there are other non-leading contributions to the real part of \( \ast \Sigma(P) \) which are not taken into account in Eq. (19). In fact, for the calculation of the damping rate to leading order, it is only necessary to consider the leading contributions to the real part of \( \ast \Sigma(P) \). Nevertheless, in the following we keep a non-leading term containing \( a(P) \) to illustrate that its presence does not affect the gauge independence of the damping rate.

Eq. (18) implies that the poles of the propagator are gauge-parameter independent, since a pole of \( [P - \ast \Sigma_0(P)]^{-1} \) is also a pole of \( \ast S \). Within our approximation, the effective Dirac equation for the one-particle fermion mode is

\[ [P - \text{Re} \ast \Sigma(P)] \ast u(P) = 0. \quad (20) \]

By neglecting the absorptive with respect to the dispersive part of \( \ast \Sigma_0(P) \), Eq. (20) becomes

\[ [P - \text{Re} \ast \Sigma_0(P)] \ast u(P) = 0. \quad (21) \]

According to Eq. (11), a factor \( [1 - \text{Re} a(P)]^{-1/2} \) has to be absorbed into the normalization of the spinor \( \ast u(P) \) since this factor is part of the residue of the fermion propagator at the pole.

From Eqs. (12) and (19), the damping rate \( \gamma \) can be written as

\[
\gamma = -\frac{1}{\omega_p} \ast \bar{u}(P) \text{Im} \ast \Sigma(\omega_p, p) \ast u(P) \\
= -\frac{1}{\omega_p} [1 - \text{Re} a(P)] \ast \bar{u}(P) \text{Im} \ast \Sigma_0(\omega_p, p) \ast u(P),
\]

where in the last line, we have used the effective Dirac equation, Eq. (21), satisfied by \( \ast u(P) \). \( \omega_p \) is the dispersion relation of the corresponding mode and is given, for example in Ref. [11]. Let us now define the spinor \( u(P) \) related to \( \ast u(P) \) by

\[ u(P) \equiv [1 - \text{Re} a(P)]^{1/2} \ast u(P), \quad (23) \]

thereby cancelling the gauge-parameter dependent factor in the normalization of \( \ast u(P) \).

The spinor \( u(P) \) satisfies Eq. (21) and is normalized according to Eq. (11). The explicit expression for the residues of the one particle contributions to the propagator is [11]

\[ Z_p = \frac{\omega_p - p^2}{2m_f^2}. \quad (24) \]
Eq. (22) is now written in terms of $u(P)$ as
\begin{equation}
\gamma = -\frac{1}{\omega_p} \bar{u}(P) \text{Im}^* \Sigma_0(\omega_p, p) u(P),
\end{equation}
which is manifestly gauge-parameter independent. Arguments to show that the expression for the damping rate in the Coulomb gauge coincides with that in a covariant gauge have been given elsewhere [6], therefore all of the above translates as well to the case of the Coulomb gauge. In the next section, we will write the explicit expression for the imaginary part of the gauge independent piece of the self-energy, $^*\Sigma_0$.

4 The reaction rate

We now proceed to establish the relation between $^*\Sigma_0$ and the total reaction rate $\Gamma$ in the HTL approximation. In order to do this, we need an explicit expression for the imaginary part of the gauge-independent part of the self-energy is obtained from Eq. (13). To evaluate the sum, we use the identity (see for example Ref. [12])
\begin{equation}
\text{Im} T \sum_n g(i\omega_n) \bar{g}(i(\omega - \omega_n)) = \pi(e^{p_0/T} + 1)
\end{equation}
where the analytical continuation $i\omega \rightarrow p_0 + i\epsilon$ has been performed. Here $f$ and $\bar{f}$ represent the statistical distributions, whereas $\rho$ and $\bar{\rho}$ are the spectral densities corresponding to the functions $g$ and $\bar{g}$, respectively. The spectral densities contain the discontinuities across the real energy axis. Their support depends on whether the momentum four-vector is inside or outside the light-cone. For time-like momenta, these spectral densities have support on the quasi-particle poles, whereas for space-like momenta they have support on the whole interval corresponding to the branch cut of the propagator.

Adding up the two contributions in Eq. (13), $\text{Im}^* \Sigma_0(\omega_p, p)$ can be expressed, by means of Eq. (26), as an integral involving products of spectral densities [8],
\begin{equation}
\text{Im}^* \Sigma_0(\omega_p, p) = \frac{\alpha_s^2 C_F(e^{\omega_p/T} + 1)}{2} \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} dk_0 dp_0' \delta(\omega_p - k_0 - p_0') f(k_0) \bar{f}(p_0') \times \left\{ \gamma^\mu \rho_{\mu\nu}(k_0, k) \bar{\rho}(p_0', \vec{p} - \vec{k}) \gamma^\nu \\
+ \gamma^\mu \rho_{\mu\nu}(k_0, k) \bar{\rho}(p_0', \vec{p} - \vec{k}) F_1^\nu(P, K) \\
+ \gamma^\mu \rho_{\mu\nu}(k_0, k) \bar{\rho}(p_0', \vec{p} - \vec{k}) F_2^\nu(P, K) \\
+ F_1^\nu(P, K) \rho_{\mu\nu}(k_0, k) \bar{\rho}(p_0', \vec{p} - \vec{k}) \gamma^\mu \\
+ F_2^\nu(P, K) \rho_{\mu\nu}(k_0, k) \bar{\rho}(p_0', \vec{p} - \vec{k}) \gamma^\mu \right\}.
\end{equation}
\[ + F_1^\mu(P, K) \rho_{\mu\nu}(k_0, k) \tilde{\rho}(p'_0, \tilde{p} - \tilde{k}) F_1^\nu(P, K) \\
+ F_2^\mu(P, K) \rho_{\mu\nu}(k_0, k) \tilde{\alpha}(p'_0, \tilde{p} - \tilde{k}) F_2^\nu(P, K) \\
+ F_2^\mu(P, K) \rho_{\mu\nu}(k_0, k) \tilde{\beta}(p'_0, \tilde{p} - \tilde{k}) F_2^\nu(P, K) \\
+ G^{\mu\nu}(P, K) \rho_{\mu\nu}(k_0, k) \} \right|_{p_0 = \omega_p}, \tag{27} \]

where \( \alpha_g = g^2/(4\pi) \) and the functions \( \rho_{\mu\nu}, \tilde{\rho}, \tilde{\alpha}, \tilde{\beta}, F_1^\mu, F_2^\mu \) and \( G^{\mu\nu} \) are defined in the appendix. Eq. (27) develops infrared divergences associated with the exchange of unscreened long wavelength magnetic gauge bosons. Nevertheless, it has been shown that the leading divergences can be resummed by a generalization of the Bloch-Nordseick model at finite temperature \[14\]. Eq. (27) is valid for any \( SU(N) \) gauge group, provided that the thermal masses include the corresponding group factors, and the generalization to the case with finite chemical potentials is also straightforward \[11\].

When computing the on-shell matrix element \( \bar{u}(P) \text{Im}^* \Sigma_0(\omega_p, p) u(P) \) using the medium spinors satisfying the normalization condition in Eq. (11), the right-hand side of Eq. (27) can be identified as a sum of rates for processes involving thermalized fermions and gauge bosons. To this end, we note that the transverse and longitudinal projection operators correspond to the products of transverse (summed over components) and of longitudinal photon polarizations, respectively, and that

\[
\left( \gamma^0 \mp \vec{\gamma} \cdot \vec{p} \right) = \frac{1}{\omega_p Z_p^\pm} \sum_s u_s^\pm(\omega_p^\pm, \vec{p}) \bar{u}_s(\omega_p^\pm, \vec{p}), \tag{28} \]

where the sum is over helicities of the (medium) fermion modes and the \( \pm \) on the r.h.s. of Eq. (28) refer to modes with either positive or negative helicity over chirality ratio. The functions \( F_1^\mu, F_2^\mu \) and \( G^{\mu\nu} \) are effective vertices and the terms containing \( \tilde{\alpha} \) or \( \tilde{\beta} \) represent products of amplitudes with an intermediate fermion propagator. Therefore, we write

\[
\bar{u}(P) \text{Im}^* \Sigma_0(\omega_p, p) u(P) = -\omega_p \Gamma, \tag{29} \]

where \( \Gamma \) is the total reaction rate, which is the sum of decay and creation rates. Detailed balance ensures that the rate for the decay processes \( \Gamma_D \), and the rate for the creation processes \( \Gamma_I \), are related by \( \Gamma_D = e^{\omega_p/T} \Gamma_I \). Eq. (29) was first computed, for the case \( p = 0 \), in Refs. \[12, 15\]. Comparing Eqs. (25) and (29) we finally obtain the relation

\[
\gamma = \Gamma. \tag{30} \]

Three different kinds of products make up the total rate: the pole-pole, pole-cut and cut-cut contributions \[19\]. The pole-pole terms represent processes involving two quasiparticles, a fermionic excitation with either positive or negative helicity over chirality ratio and a gauge boson excitation, either transverse or longitudinal, in addition to the original fermion mode. The pole-cut terms represent either scattering or annihilation processes...
involving the exchange of a fermion or a gauge boson mode. Finally the cut-cut terms represent the same scattering or annihilation processes as the pole-cut terms together with the radiation of a soft gauge boson. The physical origin of the two latter kinds of processes is Landau damping [17]. For specific values of the energy and momentum of the original fermion mode, some of these processes will be kinematically forbidden. It is easy to check that by restricting our attention to the pole-pole contributions in the first term of Eq. (27), one obtains the analogue to Eq. (35) in Ref. [5]. We emphasize that, although our expression for $\Gamma$ in Eq. (29) is formally the same as the one used in Ref. [2], the original fermion mode is represented by the spinor $u(P)$ corresponding to a quasiparticle propagating in the medium, and not by a free-particle spinor, from which it differs by the normalization.

5 Hard axion production in a hot QED plasma

As an application, consider the computation of the production rate of hard axions in a hot QED plasma through the Compton-like scattering diagram depicted in Fig. 2. This process is free from infrared divergences and thus it better illustrates the necessity of including the correct fermion wave function normalization in the calculation.

The tree level calculation involves an integral over the three momentum transferred $p'$ of the virtual electron. Since in the soft $p'$ region HTL corrections to the electron propagator are not suppressed by powers of $e$, we need to use the effective electron propagator. Also, notice that since the axion momentum $k$ is hard, then, for soft $p'$, the momentum of the incoming electron $p$ must also be hard. The contribution to the hard-axion production rate, $\Gamma(k)$, from the soft $p'$ region is most conveniently found from the imaginary part of the axion self energy depicted in Fig. 3. The heavy dot in the loop electron line represents the effective propagator.

![Figure 2: Axion photoproduction by Compton-like scattering.](image-url)
In Euclidean space, the expression for the axion self-energy is written as

\[ \Pi(k) = \lambda^2 T \sum_n \int \frac{d^3p'}{(2\pi)^3} \text{Tr}[K\gamma_5 S(P') S_H(P' + K) K\gamma_5], \quad (31) \]

where \( \lambda \) is the effective axion-electron coupling with units of inverse energy and \( S_H \) is the hard electron propagator. Given that at least one of the momenta flowing into the vertices is hard, there is no need to consider HTL corrections to the vertices. \( \Gamma(k) \) is related to \( \Pi(k) \) by

\[ \Gamma(k) = \frac{1}{k_0} \frac{\text{Im}\Pi(k)}{1 - e^{k_0/T}}, \quad (32) \]

where the factor \( (1 - e^{k_0/T})^{-1} \) selects the processes in which axions are produced. We now use the analog of Eq. (26) for the fermion-antifermion case to compute the imaginary part of the axion self-energy. The terms involved are

\[ \text{Im} T \sum_n S(P') S_H(P' + K) = \pi (1 - e^{k_0/T}) \int_0^\infty dp'_0 \int_0^\infty dp_0 \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \tilde{f}(p'_0) \tilde{f}(-p_0) \delta(k_0 + p'_0 - p_0) \tilde{\rho}(p'_0) \tilde{\rho}(p_0), \quad (33) \]

where in the right-hand side we have performed the analytical continuation to Minkowski space. Notice that since \( p_0 \) are hard and \( p^2_0 - p^2 > 0 \)

\[ \rho_\pm(p_0, p) = 2\pi Z_\pm(p) \delta(p_0 \mp \omega_\pm(p)), \quad (34) \]

since for hard momentum, the modes with negative helicity over chirality ratio decouple. Keeping only the contribution from positive energy electrons, the hard axion production rate in the soft \( p' \) region is written as

\[ \Gamma_{soft}(k) = \frac{\lambda^2 \pi}{k} \int \frac{d^3p'}{(2\pi)^3} \int_0^\infty dp'_0 \int_0^\infty dp_0 \tilde{f}(p'_0) \tilde{f}(-p_0) \delta(k_0 + p'_0 - p_0) \]

\[ \times 2\pi Z_+(p) \delta(p_0 - \omega_+(p)) \text{Tr}[K \tilde{\rho}(p'_0) K (\gamma_0 - \vec{\gamma} \cdot \hat{p})]. \quad (35) \]

Figure 3: Lowest order hard axion self-energy. The heavy dot in one of the internal electron lines represents the HTL effective electron propagator.
After integration over the direction of $\bm{p}'$ in Eq. (35), we obtain two kinematical restrictions for $p'_0$ and $p'$, namely $p'^2 - p'^2_0 > 0$ and $2k > p' - p'_0$, therefore, after some simplifications

$$\Gamma_{\text{soft}}(k) = \frac{\lambda^2}{8k\pi^4} \int_{q^*} \int_0^p dp' dp'_0 \bar{f}(p'_0) \bar{f}(-p'_0 - k) \frac{(p'_0 + k)}{2p'k} \theta(2k + p'_0 - p') Z_+(p)$$

$$\times \left\{ \beta_+(p'_0, p') \text{Tr} [ K (\gamma_0 - \vec{\gamma} \cdot \hat{p}) K (\gamma_0 - \vec{\gamma} \cdot \hat{p})] \right\} + \beta_-(p'_0, p') \text{Tr} [ K (\gamma_0 + \vec{\gamma} \cdot \hat{p}) K (\gamma_0 + \vec{\gamma} \cdot \hat{p})] ,$$

(36)

where we have introduced the intermediate scale $q^*$ with $eT \ll q^* \ll T^{[18]}$. Carrying out the calculation of the traces and using that for hard $k$

$$\bar{f}(-p'_0 - k) = \frac{1}{e^{-(k_0 + p'_0)/T} + 1} \approx \frac{1}{e^{-k/T} + 1} \approx 1,$$

(37)

we get

$$\Gamma_{\text{soft}}(k) = \frac{\lambda^2}{8k\pi^4} \int_{q^*} dp' \int_{-p'}^{p'} dp'_0 \bar{f}(p'_0) Z_+(p'_0 + k) \theta(2k + p'_0 - p')(p'^2 - p'^2_0)$$

$$\times \left\{ (p' - p'_0)(2k + p'_0 + p')\beta_+(p'_0, p') + (p'_0 + p')(2k + p'_0 - p')\beta_-(p'_0, p') \right\} \quad (38)$$

We notice that Eq. (38) involves a factor $Z_+(p'_0 + k)$ which arises from cutting the electron line with hard momentum in the loop diagram of Fig. 3. This factor signals that the electron at hand corresponds to an excitation in the medium.

We now ask whether the same expression for the hard axion production rate can be recovered from considering the self-energy of this hard electron in the medium. As we will show, the expression thus obtained will coincide with Eq. (38) provided that we include the proper normalization of the medium electron.

Consider the diagram in Fig. 4 representing the self-energy of a hard electron in the medium. The dotted line represents the axion. As before, in the soft $p'$ region, HTL corrections to the electron propagator are not suppressed by powers of $e$ and have to be

![Figure 4: Lowest order hard electron self-energy. The heavy dot in the internal electron line represents the HTL effective electron propagator.](image)
resummed into the effective electron propagator represented by the line with the heavy dot. In Euclidean space, the expression for the electron self-energy is

$$\Sigma(P) = -\lambda^2 T \sum_n \int \frac{d^3p'}{(2\pi)^3} (P+P') \gamma_5 G(P+P') S(P')(P+P') \gamma_5,$$

where $G(P)$ is the bare axion propagator. Production of axions is equivalent to destruction of the original hard electrons. The rate of destruction of these electrons is given, according to Eqs. (32) and (33), by

$$\Gamma(p) = -Z_+(p) \frac{\text{Tr}[(\gamma_0 - \vec{\gamma} \cdot \hat{p}) \text{Im} \Sigma(P)]}{(1 + e^{-p_0/T})},$$

where the factor $(1 + e^{-p_0/T})^{-1}$ selects the processes where the hard electron disappears from the medium and we have also summed over helicity states of the incoming electron. Notice that in Eq. (40) we have included explicitly the factor $Z_+(p)$, as corresponds to a rate that involves medium electrons. Using Eq. (24), we have

$$\text{Im} T \sum_n G(P + P') S(P') = \pi (e^{p_0/T} + 1) \int_\infty^\infty \frac{dk_0}{2\pi} \int_\infty^\infty \frac{dp_0'}{2\pi} f(k_0) f(-p_0')$$

$$\times \delta(p_0 - k_0 + p_0') \tilde{\rho}(p_0') 2\pi \delta[k_0^2 - (\vec{p} + \vec{p}')^2],$$

where we used that $2\text{Im} G(P + P') = 2\pi \delta[(P + P')^2]$ and have also performed the analytical continuation to Minkowsky space. From Eqs. (40) and (41) we get after carrying out the angular integration

$$\Gamma_{\text{soft}}(p) = \frac{\lambda^2}{16\pi^4} \int_0^{q^*} dp' p' \int_{-p'}^{p'} dp_0' \theta(2p + p_0' - p') \frac{Z_+(p)}{p} e^{p_0/T} f(p_0 + p_0') \tilde{f}(-p_0')$$

$$\times \text{Tr}[(\gamma_0 - \vec{\gamma} \cdot \hat{p})(P+P') \rho(p_0')(P+P')],$$

where the integration over $p'$ is up to the intermediate scale $q^*$. Notice that since $p_0$ is hard and $p_0'$ is soft,

$$e^{p_0/T} f(p_0 + p_0') = \frac{e^{p_0/T}}{e^{(p_0+p_0')/T} - 1} \approx \frac{e^{p_0/T}}{e^{p_0/T}} = 1.$$

Thus, after computing the traces involved, we get

$$\Gamma_{\text{soft}}(p) = \frac{\lambda^2}{16\pi^4} \int_0^{q^*} dp' p' \int_{-p'}^{p'} dp_0' \theta(2p + p_0' - p') \tilde{f}(-p_0') \frac{Z_+(p)}{p} (p^2 - p_0'^2)$$

$$\times \{(\beta_+(p_0', p')(p' - p_0')(2p - p' + p_0') + \beta_-(p_0', p')(p' + p_0')(2p + p' + p_0')\}.$$
In order to transform the rate of electron loss into the rate of axion production, we use that \( p = k - p'_0 \), as required by kinematics, and that \( k \) is hard. Therefore,

\[
\frac{1}{p^2} = \frac{1}{(k - p'_0)^2} \approx \frac{1}{k^2}
\]

\[
Z_+(p) = Z_+(k - p'_0)
\]

\[
(2p + p'_0 \mp p') = (2k - p'_0 \mp p').
\]

We now introduce unity in the form

\[
1 = \int d^4k \delta(K - P - P') ,
\]

to write

\[
\Gamma_{\text{soft}}(p) = \int d^4k \delta(K - P - P') \Gamma_{\text{soft}}(k)
\]

where \( \Gamma_{\text{soft}}(k) \) is given by

\[
\Gamma_{\text{soft}}(k) = \frac{\lambda^2}{8k^2 \pi^4} \int_0^{q^*} \int_{-p'}^{p'} dp'_0 dp' \bar{f}(-p'_0)Z_+(k - p'_0)\theta(2k - p'_0 - p')(p'^2 - p'^2_0)

\times \{(p'_0 - p'_0)(2k - p'_0 - p')\beta_+(p'_0, p') + (p'_0 + p'_0)(2k - p'_0 + p')\beta_-(p'_0, p')\}.
\]

Eq. (48) coincides with Eq. (38) as can be checked by means of the substitution \( p'_0 \rightarrow -p'_0 \) for which \( \beta_+ \leftrightarrow \beta_- \).

It is now straightforward to compute the explicit expression for \( \Gamma_{\text{soft}}(k) \). We use that in the soft \( p' , p'_0 \) region \( \bar{f}(p'_0) \approx 1/2 \) and that \( k \gg p_0, p'_0 \), therefore

\[
\Gamma_{\text{soft}}(k) = \frac{\lambda^2 Z_+(k)}{8k^2 \pi^4} \int_0^{q^*} dp' \int_{-p'}^{p'} dp'_0 (p'^2 - p'^2_0)

\times \{(p'_0 - p'_0)\beta_+(p'_0, p') + (p'_0 + p'_0)\beta_-(p'_0, p')\}.
\]

The integral is computed by using the sum rules satisfied by \( \beta_{\pm} \) [11]. The result is

\[
\Gamma_{\text{soft}}(k) = \frac{\lambda^2 Z_+(k)}{2k^2 \pi} m_f \left\{ \frac{g^2}{m^2_f} - \int_0^\infty dx \left[ \left(x^3 - y_+(x)x^2 - y^2_+(x)x + y^3_+(x)\right) Z_+(x)\right]

+ \left(x^3 + y_-(x)x^2 - y^2_-(x)x - y^3_-(x)\right) Z_-(x) \right\},
\]

where we have defined the dimensionless function \( y_{\pm}(x) = \omega_{\pm}(x)/m_f \) and have extended the limit of integration in the second term to \( \infty \). The remaining integral in Eq. (50) is also a dimensionless quantity that can be computed numerically and its value is approximately 1.3.

We now proceed to compute the contribution to the hard axion production rate from the hard momentum transfer region. For this case, the tree level electron propagator
must be used. Notice that the tree level propagator can be recovered from the effective propagator by simply neglecting \( m_f^2 \) in the denominators of Eqs. (61). In this manner, the approximate spectral densities become

\[
\beta_{\pm}(p'_0, p') \approx \frac{\pi^3 m_f^2/p'^2}{(p' + p'_0)}.
\]  

(51)

Thus, the contribution to the axion production rate from the hard \( p' \) region is

\[
\Gamma_{\text{hard}}(k) = \frac{\lambda^2 Z_+(k)}{2k\pi} m_{f}^2 \int_{q^*}^{\infty} dp' \int_{-p'}^p dp'_0 \tilde{f}(p'_0) \frac{(p'^2 - p'_0^2)}{p'^2} \theta(2k + p'_0 - p').
\]  

(52)

It is convenient to further decompose the hard region \( p' > q^* \) into a (I) low \( (p'_0 < q^*) \) and a (II) high \( (p'_0 > q^*) \) frequency regions. In the low frequency region, we can still use the approximation \( \tilde{f}(p'_0) \approx 1/2 \). The remaining integrals are readily performed and the result is

\[
\Gamma_{\text{hard}}^{(I)}(k) = \frac{\lambda^2 Z_+(k)}{2k\pi} m_{f}^2 (-q^*)^2.
\]  

(53)

In the high frequency region, \( \tilde{f} \) provides the cutoff for the integrals. Recalling that \( q^* \ll T \) we get

\[
\Gamma_{\text{hard}}^{(II)}(k) = \frac{\lambda^2 Z_+(k)}{2k\pi} m_{f}^2 \left\{ 2k^2 + 2kT \ln(1 + e^{-k/T}) \right\}.
\]  

(54)

Adding Eqs. (50), (53) and (54), the dependence on the intermediate scale \( q^* \) cancels, as it should, and the final expression for the hard axion production rate is written as

\[
\Gamma(k) = \frac{\lambda^2 Z_+(k)}{2k\pi} m_{f}^2 \left\{ k^2 m_f^2 + \frac{kT}{m_f^2} \ln(1 + e^{-k/T}) - 0.65 \right\},
\]  

(55)

which is valid for \( k \sim T \). Inclusion of a possible electron chemical potential is straightforward. Equation (55) represents the leading contribution to the photoproduction rate of axions in relativistic plasmas \[14\] at large \( k \). Though this process is known to be sub-dominant as compared to nucleon bremsstrahlung of axions in this kind of plasmas \[19\], a complete leading order treatment such as that of Eq. (55) could have significant consequences for astrophysical processes in strongly magnetized relativistic plasmas \[21\].

We should emphasize that in order for the rate obtained from the axion self-energy to formally coincide with the rate obtained from the electron self-energy, the latter has to be weighed with the factor \( Z_+ \) as corresponds to a medium electron mode.

Finally, we note that for hard \( k \)[11]

\[
Z_+(k) \approx 1 + \frac{m_f^2}{2k^2} \left( 1 - \ln \left( \frac{2k^2}{m_f^2} \right) \right),
\]  

(56)
thus, the error involved when ignoring the factor $Z_+$ is small. This is in contrast to the situation when $k$ is soft for which

$$Z_+(k) \approx \frac{1}{2} \pm \frac{k}{3m_f}.$$  

(57)

6 Discussion

In this paper, we have argued, in the realm of hot gauge theories, the equality of the damping rate of a fermion mode propagating in a medium and its total reaction rate, to leading order, provided the latter is computed using the wave function of the mode in the medium. This result is based on the explicit gauge independence of the on-shell matrix element of the absorptive part of the fermion self-energy, according to the effective perturbative expansion of Braaten and Pisarski. It also unifies two apparently distinct concepts, both related to $\text{Im} \, \Sigma_0$. In addition, Eq. (30) also sheds light on the distinct interpretation of a quantity like $\Gamma$ in the work of Weldon [2] and in ours. In effect, according to Eq. (11), the spinors $u(P)$ are normalized to $Z_p$, related to the probability of finding a fermion mode with energy $\omega_p$ and momentum $\vec{p}$ pertaining to the medium. On the other hand, if the spinor $u(P)$ was taken as free, it would represent a mode in vacuum. Thus, if we take spinors from the medium, $\Gamma$ can be interpreted as the rate to approach equilibrium for an excitation in the medium — slightly out of equilibrium — which, according to Eq. (30), is equivalent to the rate at which the excitation is damped. On the other hand, if the spinors are from vacuum, the corresponding reaction rate can be interpreted as the rate to approach equilibrium for a test particle, taken from the vacuum and dropped into the medium.

We have used the equivalence between damping and reaction rates of fermions in a medium to compute the hard axion production rate in a hot QED plasma by Compton-like scattering processes. We have shown that the rate obtained from the axion self-energy coincides with the rate obtained from the electron self-energy only when the proper medium wave function is used in this latter computation.

Another interesting point, that we do not touch upon here, has to do with whether Eq. (30) is a general result, independent of perturbation theory, which is related to the way the thermalization of gauge degrees of freedom is treated [22]. The fact that at least to lowest, but complete order, the expression for the fermion damping rate in hot gauge theories is identical in the Coulomb or in covariant gauges, hints towards the validity of the result to all orders.

Acknowledgments

We would like to thank J. Nieves for useful and stimulating discussions. Support for this work has been received in part by CONACyT-México under grants Nos. 127212 and
Appendix

Here, we list all of the functions introduced in section IV. Capital letters are used throughout to denote momentum four-vectors, all of them in Minkowski space and \( \hat{Q}^\mu = (-1, \hat{q}) \). We start with the functions \( F_1^\mu \), \( F_2^\mu \) and \( G^{\mu\nu} \), related to the explicit expressions for the HTL effective vertices \( \Gamma^\mu_{\mu} \) and \( \Gamma_{\mu\nu} \) \[1\].

\[
F_1^\mu(P, K) = -m_f^2 \int \frac{d\Omega}{4\pi} \, \frac{\hat{Q}^\mu \hat{Q}^\nu}{[P \cdot \hat{Q}]} \mathcal{P} \left( \frac{1}{(P - K) \cdot \hat{Q}} \right),
\]
\[
F_2^\mu(P, K) = -m_f^2 \int \frac{d\Omega}{4\pi} \, \frac{\hat{Q}^\mu \hat{Q}^\nu}{[P \cdot \hat{Q}]} \delta((P - K) \cdot \hat{Q}),
\]
\[
G^{\mu\nu}(P, K) = -m_f^2 \int d\Omega \frac{\hat{Q}^\mu \hat{Q}^\nu}{[P \cdot \hat{Q}]} \delta((P - K) \cdot \hat{Q}). \tag{58}
\]

\( \mathcal{P} \) denotes the Cauchy principal value \[2\]. These functions depend on the quantity \( \Delta = P^2(P - K)^2 - (P \cdot (P - K))^2 \) and vanish for \( \Delta > 0 \).

Next, let us define the spectral densities, \( \rho_{\mu\nu} \), \( \bar{\rho} \), \( \bar{\alpha} \) and \( \bar{\beta} \),

\[
\rho_{\mu\nu}(k_0, k) = \frac{k^2}{k_0^2 - k^2} \rho_L(k_0, k) P_L^{\mu\nu} + \rho_T(k_0, k) P_T^{\mu\nu},
\]
\[
\bar{\rho}(p_0, \vec{p}) = \rho_+(p_0, p)(\gamma_0 - \gamma \cdot \hat{p}) + \rho_-(p_0, p)(\gamma_0 + \gamma \cdot \hat{p}),
\]
\[
\bar{\alpha}(p_0, \vec{p}) = \alpha_+(p_0, p)(\gamma_0 - \gamma \cdot \hat{p}) + \alpha_-(p_0, p)(\gamma_0 + \gamma \cdot \hat{p}),
\]
\[
\bar{\beta}(p_0, \vec{p}) = \beta_+(p_0, p)(\gamma_0 - \gamma \cdot \hat{p}) + \beta_-(p_0, p)(\gamma_0 + \gamma \cdot \hat{p}), \tag{59}
\]

where \( P_L \) and \( P_T \) are the longitudinal and transverse projection operators in four dimensions. The quantities \( \rho_{L,T} \), \( \rho_{\pm} \) are the standard HTL spectral densities corresponding to the gauge boson and fermion propagator, respectively \[1\]. We have also defined

\[
\alpha_\pm(p_0, p) = \frac{(2\pi) \left[ p_0 \mp p - \frac{m_f^2}{p} \left( \frac{(p + p_0)}{2p} \ln \left| \frac{p_0 + p}{p_0 - p} \right| \pm 1 \right) \right]}{\left[ p_0 \mp p - \frac{m_f^2}{p} \left( \frac{(p + p_0)}{2p} \ln \left| \frac{p_0 + p}{p_0 - p} \right| \pm 1 \right) \right]^2 + \left[ \frac{\pi m_f^2}{2p} \left( 1 \mp \frac{p_0}{p} \right) \right]^2}, \tag{60}
\]
\[
\beta_\pm(p_0, p) = -\frac{\pi^3 m_f^2}{p} \left( \frac{(p + p_0)}{p} \right) \theta(p^2 - p_0^2) \left[ p_0 \mp p - \frac{m_f^2}{p} \left( \frac{(p + p_0)}{2p} \ln \left| \frac{p_0 + p}{p_0 - p} \right| \pm 1 \right) \right]^2 + \left[ \frac{\pi m_f^2}{2p} \left( 1 \mp \frac{p_0}{p} \right) \right]^2. \tag{61}
\]
References

[1] M. Le Bellac, Thermal Field Theory, Cambridge University Press (1996) p. 115.

[2] H.A. Weldon, Phys. Rev. D 28, 2007 (1983).

[3] L.P. Kadanoff and G. Baym, Quantum Statistical Mechanics, Benjamin Cummings, Reading, MA. (1962) p. 39.

[4] T. Altherr, E. Petitgirard and T. del Rio Gaztelurrutia, Phys. Rev. D 47, 703 (1993).

[5] J.C. D’Olivo and J.F. Nieves, Phys. Rev. D 52, 2987 (1995).

[6] R.D. Pisarski, Phys. Rev. Lett. 63, 1129 (1989); E. Braaten and R.D. Pisarski, Nucl. Phys. B 337, 569 (1990); ibid. B 339, 310 (1990).

[7] H.A. Weldon, Phys. Rev. D 40, 2410 (1989).

[8] J.F. Nieves, Phys. Rev. D 40, 866 (1989).

[9] D.A. Kirzhnits, Field Theoretical Methods in Many-body Systems, Pergamon Press, London (1967) p. 260.

[10] A. Rebhan, Phys. Rev. D 46, 4779 (1992).

[11] M. Le Bellac, ibid, chapters 6 and 7.

[12] E. Braaten and R.D. Pisarski, Phys. Rev. D 46, 1829 (1992).

[13] It is possible to use Eq. (26) since in Eq. (13), the vertex $\Gamma_{\mu\nu}$ can be shown to depend on the four vectors $P$ and $P - K$.

[14] J.P. Blaizot and E. Iancu, Phys. Rev. Lett. 76, 3080 (1996).

[15] R. Kobes, G. Kunstatter and K. Mak, Phys. Rev. D 45, 4632 (1992).

[16] A similar analysis has been carried out for the case of the photon damping rate in the quark-gluon plasma by E. Braaten, R.D. Pisarski and T.C. Yuan, Phys. Rev. Lett. 64, 2242 (1990).

[17] E.M. Lifshitz and L.P. Pitaevskii, Physical Kinetics, Course of Theoretical Physics Vol. 10, Pergamon Press (1981).

[18] E. Braaten and T.C. Yuan, Phys. Rev. Lett. 66, 2183 (1991).

[19] See also T. Hatsuda and M. Yoshimura, Phys. Lett. B 203, 469 (1988).

[20] G.G. Raffelt, Phys. Rept. 198, 1 (1990).
[21] A.V. Borisov and V. Yu. Grishina, Zh. Eksp. Teor. Fiz. 110, 1575-1588 (1996) [JETP 83, 868 (1996)].

[22] J.C. D’Olivo and J.F. Nieves, Phys. Lett. B 359, 148 (1995).

[23] Similar expressions to the functions $F_{1,2}^\mu$ and $G^{\mu\nu}$ have been introduced and evaluated by M.H. Wong, Z. Phys. C 53, 465 (1992) and by J. Frenkel and J.C. Taylor, Nucl. Phys. B 334, 199 (1990).