On Combination Networks with Cache-aided Relays and Users

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Abstract—Caching is an efficient way to reduce peak hour network traffic congestion by storing some contents at the user’s cache without knowledge of later demands. Coded caching strategy was originally proposed by Maddah-Ali and Niesen to give an additional coded caching gain compared the conventional uncoded scheme. Under practical consideration, the caching model was recently considered in relay network, in particular the combination network, where the central server communicates with \( K = \binom{N}{r} \) users (each is with a cache of \( M \) files) through \( H \) immediate relays, and each user is connected to a different \( r \)-subsets of relays. Several inner bounds and outer bounds were proposed for combination networks with end-user-caches. This paper extends the recent work by the authors on centralized combination networks with end-user caches to a more general setting, where both relays and users have caches. In contrast to the existing schemes in which the packets transmitted from the server are independent of the cached contents of relays, we propose a novel caching scheme by creating an additional coded caching gain to the transmitted load from the server with the help of the cached contents in relays. We also show that the proposed scheme outperforms the state-of-the-art approaches.

I. INTRODUCTION

A. Shared Link Networks

Caching content at the end-user’s memories mitigates peak-hour network traffic congestion. The seminal paper [1] by Maddah-Ali and Niesen (MAN) proposed an information theoretic model for cache-aided shared link networks. Such a model comprises a server with \( N \) files of \( B \) bits each, \( K \) users with local memory of size \( M \) files, and a single error-free broadcast “bottleneck” link. A caching scheme comprises two phases. (i) Placement phase: during off-peak hours, the server places parts of its library into the users’ caches without knowledge of what the users will later demand. Centralized caching systems allow for coordination among users during the placement phase, while decentralized ones do not. When pieces of files are simply copied into the cache, the cache placement phase is said to be uncoded; otherwise it is coded. (ii) Delivery phase: each user requests one file during peak-hour time. According to the user demands and cache contents, the server transmits \( RB \) bits in order to satisfy all user demands. The goal is to determine \( R^* \), the minimum load that satisfies arbitrary/worst-case user demands.

The coded caching strategy (with coded delivery) originally proposed in [1] gives an additional multiplicative global caching gain compared to uncoded caching schemes. For centralized systems, each file is split into a number of non-overlapping equal size and uncoded pieces that are strategically placed into the user caches. During the deliver phase, coded multicast messages are sent through the shared link so that a single transmission simultaneously serves several users. In [2], we showed that the MAN scheme is optimal under the constraint of uncoded cache placement when \( K \leq N \). In [3], the MAN scheme was shown to have redundant multicast transmissions when \( K > N \). The achieved load in [3] was proved to be optimal under the constraint of uncoded placement, and that it is optimal to within a factor of 2 [4].

B. Combination Networks

Since users may communicate with the central server through intermediate relays, recently caching was considered in relay networks. The caching problem with general relay networks was originally considered in [5], where a caching scheme including uncoded cache placement and linear network coding was proposed. In [6], it was proved that under the constraint of uncoded cache placement and of the separation between caching and multicast message generation on one hand, and message delivery on the other hand (i.e., the generation of the multicast messages is independent of the communication network topology), the proposed scheme is order optimal within a factor of 24 among the separation schemes.

Since it is difficult to analyze general relay networks, a symmetric network, known as combination network, has received a significant attention [7]. In this network as illustrated...
in Fig. [1] a server equipped with $N$ files is connected to $H$ relays. Each of the ($^H\binom{N}{r}$) users, with a memory of $M$ files each, is connected to a unique $r$-subset of $H$ relays. The goal is to minimize the maximum load among all links, which are assumed to be error-free and orthogonal.

The existing achievable schemes for centralized combination networks could be divided into two classes, based on uncoded cache placement [8], [9], [10], [11] and cache placement [12], [13], [14], respectively. The authors in [8], [9] proposed delivery schemes to deliver MAN multicast messages. With MAN placement, we proposed a delivery by generating multicast messages based on the network topology in [10]. The caching scheme proposed in [11] used the Placement Delivery Array (PDA) to reduce the sub-packetization of the schemes in [8], [12] for the case $r$ divides $H$. The combination network was treated as H uncoordinated shared-link models in [12] by using an $(H,r)$ MDS precoding. By leveraging the connectivity of $K$ users and $H$ relays respectively, two asymmetric coded placements were proposed in [13], [14] which can lead a symmetric delivery phase. Outer bounds (based on cut-set or acyclic directed graph for the corresponding indexing coding problem) were proposed in [7], [9]. Some existing schemes are known to be optimal under the constraint of uncoded placement for some system parameters [9], [10], [13], [11].

**C. Beyond Combination Networks**

The existing inner and outer bounds for combination networks with end-user caches to more general settings:

1) Combination networks with cache-aided relays and users was considered in [12], [10], where the main idea of [12], [10] is to divide each file into two parts, the part only cached in relays and the remaining part. The first parts of the demanded files are directly transmitted from relay to user and the delivery of the second parts is equivalent to the combination network with end-user-caches.

2) The proposed scheme for centralized systems in [10] was extended to decentralized systems with dMAN cache placement.

3) In [10], we extended the proposed inner bound to more general relay networks than combination networks, where each user is connected to an arbitrary subset of relays.

**D. Contributions**

In this paper, we consider combination networks with cache-aided relays and users, based on the asymmetric coded placement in [14], we propose a caching placement strategy where the cached contents in relays are treated as the additional side informations of the connected users which can also help users to decode the coded messages transmitted from the server and thus can further reduce the transmitted load from the server to relays. We also show that the proposed scheme outperforms the state-of-the-art schemes.

**II. SYSTEM MODEL AND RELATED RESULTS**

In Section II-A we introduce the notation convention used in this paper. In Section II-B we introduce the system model for combination network with cache-aided relays and users. Finally, in Section II-C we revise the asymmetric coded placement proposed by us in [14], which will be used in our proposed scheme for combination networks with cache-aided relays and users.

**A. Notation Convention**

A collection is a set of sets, e.g., $\{\{1,2\},\{1,3\}\}$. Calligraphic symbols denote sets or collections, bold symbols denote vectors, and sans-serif symbols denote system parameters. We use $|\cdot|$ to represent the cardinality of a set or the absolute value of a real number; $[a:b] := \{a,a+1,\ldots,b\}$ and $[n] := [1:n]$; $\oplus$ represents bit-wise XOR. We define the set

$$\text{arg max}_{x \in \mathcal{X}} f(x) := \{x \in \mathcal{X} : f(x) = \max_{y \in \mathcal{X}} f(y)\}. \quad (1)$$

We define that

$$K_i := \begin{pmatrix} H & i \\ H+1 & r-i \end{pmatrix}, \quad i \in [0:r], \quad (2)$$

where $K_0 = K$ is the number of users in the system, $K_1$ is the number of users connected to each relay, and $K_i$ represents the number of users that are simultaneously connected to $i$ relays. Our convention is that $(\frac{a}{b}) = 0$ if $x < 0$ or $y < 0$ or $x < y$.

**B. System Model for Combination Networks with Cache-aided Relays and Users**

In a $(H,r,M_{\text{relay}},M_{\text{user}},N)$ combination network, a server has $N$ files, denoted by $F_1,\ldots,F_N$, each composed of $B$ i.i.d uniformly distributed bits. The server is connected to $H$ relays through $H$ error-free orthogonal links. The relays are connected to $K := K_0$ users through $r$ error-free orthogonal links. Each user is connected to a distinct subset of $r$ relays. Each relay can store $M_{\text{relay}}B$ bits and each user can store $M_{\text{user}}B$ bits, for $(M_{\text{relay}},M_{\text{user}}) \in [0,N]^2$. The subset of users connected to relay $i$ is denoted by $U_i, i \in [K]$. The subset of relays connected to user $k$ is denoted by $H_k, k \in [K]$. For each subset of users $W \subseteq [K]$, the set of relays each of which is connected to all the users in $W$ is denoted by

$$R_W := \{h \in [H] : W \subseteq U_h\}. \quad (3)$$

For the network in Fig. [1] for example, $U_1 = \{1,2,3\}, H_1 = \{1,2\}$, and $R_{\{1,2\}} = \{1\}$.

In the placement phase, relay $h \in [H]$ and user $k \in [K]$ store information about the $N$ files in its cache of size $M_{\text{relay}}B$ and $M_{\text{user}}B$ bits, respectively. The cache content of relay $h \in [H]$ is denoted by $Z_{h}^{\text{relay}}$ and the one of user $k \in [K]$ is denoted by $Z_{k}^{\text{user}}$; let $Z := (Z_{1}^{\text{relay}},\ldots,Z_{H}^{\text{relay}},Z_{1}^{\text{user}},\ldots,Z_{K}^{\text{user}})$. During the delivery phase, user $k \in [K]$ requests file $d_{k} \in [N]$; the demand vector $d := (d_{1},\ldots,d_{K})$ is revealed to all nodes. Given $(d,Z)$, the server sends a message $X_h$ of $BR_h(d,Z)$ bits to relay $h \in [H]$. Then, relay $h \in [H]$ transmits a message $X_{h \rightarrow k}$ of $BR_{h \rightarrow k}(d,Z)$ bits to user $k \in U_h$. User $k \in [K]$
must recover its desired file $F_{d_k}$ from $Z_k$ and $(X_{h \rightarrow k} : h \in \mathcal{H}_k)$ with high probability when $B \rightarrow \infty$. The objective is to determine the load (number of transmitted bits in the delivery phase) pairs

$$(R^{\rightarrow_r}, R^{\rightarrow_u}) = \left( \max_{h \in [H]} R_h(d, Z), \max_{h \in [H], k \in U_h} R_{h \rightarrow k}(d, Z) \right)$$

for the worst case demands $d$ for a given placement $Z$.

In practice, the throughput of transmission from the server to relays may be much lower than the throughput from the relays to their local connected users. For example, in wireless networks where the throughput from small cell base stations to users is much higher than that from the macro base stations to small base stations if all use sub-6GHz wireless communications. In this paper, for combination networks with cache-aided relays and users, we mainly want to minimize the max-link load from the server to relays, i.e., $R^{\rightarrow_r} = \max_{h \in [H]} R_h(d, Z)$.

For a caching scheme with max-link load among all the links from the server to relays $R^{\rightarrow_r}$, we say it attains a coded caching gain of $g$ if

$$R^{\rightarrow_r} = \frac{R^{\rightarrow_r}_{\text{routing}}}{g}, \text{ for } g \geq K_1$$

$$R^{\rightarrow_r}_{\text{routing}} := K_1 \max \left\{ \frac{1 - (rM_{\text{relay}} + M_{\text{user}})/N, 0)}{H}, \max \frac{1 - (rM_{\text{relay}} + M_{\text{user}})/N, 0)}{r} \right\},$$

where $R^{\rightarrow_r}_{\text{routing}}$ is achieved by routing. By the cut-set bound [8] we have $g \leq K_1$ (recall that $K_1$ is the number of users connected to each relay).

### C. Asymmetric Coded Placement in [14]

In this part, we introduce the caching scheme based on an asymmetric coded placement for the case $M_{\text{relay}} = 0$ proposed in [14], which treats the combination network as $H$ coordinated shared-link models and leverages the connectivity among the divided models.

We aim to achieve coded caching gain $g \in [2 : K_1]$, that is, every coded multicast message is simultaneously useful for $g$ users. So each subfile should be cached by at least $g - 1$ users. We consider each subset of users $W$ with cardinality $g - 1$ for which there exists at least one relay connected to all the users in $W$, that is, we define the collection

$$Z_g := \{ W \subseteq [K] : |W| = g - 1, R_W \neq \emptyset \},$$

where $R_W$ defined in [3]. For example, consider the combination network in Fig. 1, we have

$$Z_3 = \{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\} \}.$$ 

Each subfile corresponds to one set in $Z_g$.

**Placement phase:** We define

$$S_1(g) := \sum_{a=1}^{r} \frac{\binom{K_a}{a}(g - 1)}{\left( \begin{array}{c} g - 2 \end{array} \right)}.$$  

$$S_2(g) := \sum_{a=1}^{r} \frac{(K_a - 1)}{\left( \begin{array}{c} g - 2 \end{array} \right)}.$$  

We divide each file $i \in [N]$ into $S_1(g)$ non-overlapping and equal-length pieces, which are then encoded by a $([Z_i], S_1(g))$ MDS code; denote the MDS coded symbols/subfiles as $(f_{i,W} : W \in Z_g)$. For each $W \in Z_g$, $f_{i,W}$ is cached by the users in $W$. Each MDS coded symbol includes $\frac{B}{S_1(g)}$ bits, and thus by the inclusion-exclusion principle [15 Theorem 10.1], we can compute that the needed memory size is

$$M_{\text{user}}^{(g)} = \frac{NS_2(g)}{S_1(g)}.$$  

**Delivery phase:** Each user $k \in [K]$ needs to recover all the MDS coded symbols $f_{d_k, W}$ where $W \in Z_g$, $k \notin W$ and $R(k)_{\cup W} \neq \emptyset$ (but not those for which $R(k)_{\cup W} = \emptyset$). For those MDS coded symbols needed by user $k$, we divide $f_{d_k, W}$ into $|R(k)_{\cup W}|$ non-overlapping and equal-length pieces, $f_{d_k, W} = \{ f_{d_k, W, h} : h \in R(k)_{\cup W} \}$. After considering all the MDS coded symbols demanded by all the users, for each relay $h \in [H]$ and each set $J \subseteq U_h$ where $|J| = g$, we create the multicast message

$$W_h^J := \oplus_{k \in J} f_{d_k, J\setminus\{k\}, h}.$$  

which will be sent to relay $h$ who will then forward it to the users in $J$.

Hence from the placement and delivery phase, each user $k \in [K]$ obtains the MDS coded symbol $f_{d_k, W}$ where $W \in Z_g$ and $R(k)_{\cup W} \neq \emptyset$. By the inclusion-exclusion principle [15 Theorem 10.1], user $k$ totally obtains $S_1(g)$ MDS coded symbols of $F_{d_k}$ such that it can recover its desired file $F_{d_k}$.

**Max-link load:** It can be proved that each demanded MDS coded symbol is multicasted with other $g - 1$ demanded MDS coded symbols with the same length and thus the coded caching gain is $g$ and thus the max link-load is $K(1 - \frac{M_{\text{user}}^{(g)}}{N})/\langle H_g \rangle$.

It was also shown in [14] that when $g - 1 \leq K_2$, the achieved max link-load by the proposed approach is strictly lower than the one by [12]. However, when $g - 1 > K_2$, we have $|R_W| = 1$ for each set $W \in Z_g$, and thus we do not leverage the coordination among relays. In this case it is equivalent to the scheme in [12].

### III. Combination Networks with Cache-aided Relays and Users

In Section III-A we will revise the caching scheme in [12], which divides each file into two parts and the packets transmitted from the server are independent of the cached contents of relays. In Section III-B we propose a novel caching scheme, in which the users can leverage the cached contents of the connected relays to decode the coded messages transmitted from the server.
A. Caching in [12] for Combination Networks with Cache-aided Relays and Users

The memories-loads tradeoff of the scheme in [12] is the lower convex envelope of the two groups of points.

1) \((M_{\text{relay}}, M_{\text{user}}) = (0, Nt_2/K_1)\) where \(t_2 \in [0 : K_1]\).

For each point in this group, we can see that relays do not have memory and the scheme is equivalent to the one for combination networks with end-user-caches. The combination network is treated as H uncoordinated shared-link models.

**Placement Phase:** Each file \(F_i\), where \(i \in [N]\), is divided into \(r\) non-overlapping and equal-length pieces, which are then encoded by using a \((H, r)\) MDS code; the \(h\)-th MDS coded symbol is denoted by \(s_{ih}\), of size \(|s_{ih}| = B/r\) for \(h \in [H]\). For each \(h \in [H]\), \(s_{ih}\) is divided into \(K_h\) non-overlapping and equal-length pieces, i.e., \(s_{ih} = \{s_{ih,W} : W \subseteq U_h, |W| = t_2\}\). Each user \(k \in U_h\) caches \(s_{ih,W}\) if \(k \in W\).

**Delivery Phase:** For each relay \(h \in [H]\), the MAN-like multicast messages

\[
\begin{align*}
   w_{i,j}^{h} &= \sum_{k \in J \setminus \{k\}} s_{ih,W} \\
   \forall J \subseteq U_h : |J| = t_2 + 1, h \in [H],
\end{align*}
\]

are delivered from the server to relay \(h\), and then relay \(h\) then forwards \(w_{i,j}^{h}\) to the users in \(J\). It can be seen that user \(k\) connected to relay \(h\) can recover \(s_{ih,k}\), and eventually \(F_{ik}\).

**Max-link load:** Each demanded subfile is transmitted with other \(t_2\) subfiles from the server. So \(R^{\text{relay}} = \frac{K(1 - M_{\text{user}}/N)}{Nt_2 + 1}\). Each user receives the uncoded part of its demanded file with totally \((1 - M_{\text{user}}/N)B\) bits from its \(r\) connected relays. So \(R^{\text{user}} = \frac{(1 - M_{\text{user}}/N)/r}{r}\).

2) \((M_{\text{relay}}, M_{\text{user}}) = (N/r, Nt_1/K_1)\) where \(t_1 \in [0, K_1]\).

In this case, each relay directly caches \(s_{ih}\) such that the server needs not to transmit any packets to relays.

If \(t_1 = 0\), each user does not cache any bits. In the delivery phase, each relay \(h \in [H]\) transmits \(s_{ih}\) to each user \(k \in U_h\). So we have \((R^{\text{relay}}, R^{\text{user}}) = (0, 1/r)\).

If \(t_1 = K_1\), in the placement phase, each user \(k\) cached \(s_{ih}\) for \(h \in [H]\) and \(i \in [N]\). So it caches all the \(N\) files such that \((R^{\text{relay}}, R^{\text{user}}) = (0, 0)\).

Notice that in the above two points, \(R^{\text{relay}}\) is always equal to \((1 - M_{\text{user}}/N)/r\).

**B. Proposed Scheme for Combination Networks with Cache-aided Relays and Users**

We start by an example of [13] for combination networks with end-user-caches, which will be used later to derive our proposed method for combination network with cache-aided relays and users.

**Example 1** \((H = 5, r = 3, N = 10, M_{\text{relay}} = 0\) and \(g = 3)\).

In this example, we have \(U_1 = \{6\}\), \(U_2 = \{1, 2, 3, 7, 8, 9\}\), \(U_3 = \{1, 4, 5, 7, 8, 10\}\), \(U_4 = \{2, 4, 6, 7, 9, 10\}\), \(U_5 = \{3, 5, 6, 8, 9, 10\}\).

**Example 2** \((H = 5, r = 3, N = 10, M_{\text{relay}} = 25/12\) and \(M_{\text{user}} = 5/12, g = 3)\). The network topology is the same as Example 1. In this example, we also impose that each demanded subfile of each user which is neither stored in its memory nor the memories of its connected relays, is transmitted from the server in one linear combination including other \(g - 1 = 2\) subfiles. We aim to let each user benefit from the cached content in its connected relays as its own cache contents.

**Placement phase:** As in Example 1, we also divide each \(F_i\) into \(36\) non-overlapping and equal-length pieces, which are then encoded by \((45, 36)\) MDS code. The length of each MDS symbol is \(B/36\). For each \(W \in \mathcal{Z}_3\), there is one MDS coded symbol/subfile denoted by \(f_{i,W}(\text{composed of bits})\) which is cached by all the users in \(W\). Thus the memory size \(M_{\text{user}} = Ns_2(g)/S_1(g) = 2.5\).

**Delivery phase:** We let each user \(k\) recover \(f_{d_k,W}\) where \(k \notin W\). \(W \in \mathcal{Z}_g\) and \(R_{\{k\} \cup W} \neq \emptyset\).

For each such \(f_{d_k,W}\), we divide it into \(\mathcal{R}_{\{k\} \cup W}\) non-overlapping and equal-length pieces, \(f_{d_k,W} = \{f_{d_k,W,h} : h \in \mathcal{R}_{\{k\} \cup W}\}\). After considering all the MDS coded symbol demanded by all the users, for each relay \(h \in [H]\) and each set \(J \subseteq U_h\) where \(|J| = g\), we create the multicast message in (10) to be sent to relay \(h\) and then forwarded to the users in \(J\). Hence each demanded MDS coded symbol is transmit in one linear combination which also includes other \(g - 1 = 2\) demanded MDS coded symbols with identical length and thus the coded caching gain is \(g = 3\).

In conclusion, the minimum needed memory size to achieve \(g = 3\) of the proposed scheme is \(M_{\text{user}} = 2.5\) while the ones of [12] is \(10/3 \approx 3.333\).

Our proposed scheme for combination networks with cache-aided relays and users illustrated in the next example is based on the caching scheme in Example 1.
For example, consider $f_{i,1}\{2\}$ which is divided into $|R_{i,2}|+1 = 3$ non-overlapping pieces. Each relay $h \in R_{i,2} = \{1,2\}$ caches $f_{i,1}\{2\}, h$ with $\frac{M_{\text{relay}}B_{i}}{|N_{i}|g_{i}} = B/72$ bits. For the last piece, we have $|f'_{i,1}\{2\}| = B/36 - 2B/72 = 0$ and thus no user caches any bits of $f'_{i,1}\{2\}$.

Consider now $f_{i,1}\{6\}$ which is divided into $|R_{i,1}|+1 = 2$ non-overlapping pieces. Each relay $h \in R_{i,1} = \{1\}$ caches $f_{i,1}\{6\}, h$ with $\frac{M_{\text{relay}}B_{i}}{N_{i}g_{i}} = B/72$ bits. So $|f'_{i,1}\{6\}| = B/36 - B/72 = B/72$. Thus each user in $\{1\}$ caches $f'_{i,1}\{6\}$ with B/72 bits.

We then focus on user $1$. For each set \( W \subseteq U \), \( W \in \{\{1\},\{1,3\},\{1,4\},\{1,5\},\{1,7\},\{1,8\}\} \), we have $|R_W| = 2$ and $|f'_{i,W}| = 0$. For each set \( W \subseteq \{\{1\},\{1,9\},\{1,10\}\} \), we have $|R_W| = 1$ and $|f'_{i,W}| = B/72$. So user 1 caches $3NB/72 = M_{\text{relay}}B$ bits.

For each set \( W \subseteq Z_3 \), $R_{k,W} \neq \emptyset$. There are three steps in delivery phase:

1) In the first step, for each relay $h \in H$ and each user $k \in U_h$, relay $h$ delivers all the cached bits of $f_{d_h}$ to user $k$. More precisely, for each set $W \subseteq U_h$ where $|W| = g - 1$, relay $h$ delivers $f_{d_h,W}$ to user $k$. So by this step and the placement phase, each user $k \notin K$ can recover $f_{d_h,W}$ where $W \in Z_3$ and $k \notin W$. User $k$ can also recover $f_{d_h,W,s}$ where $W \in Z_3$, $k \notin W$, $R_{k,W} \neq \emptyset$ and $h \in (R_W \cap H_k)$.

2) In the second step, we also focus on each relay $h \in H$ and each user $k \in U_h$. For each set $W' \subseteq U_h$ and each $k' \subseteq [K] \setminus U_h$, where $|W'| = g - 1$, $k \notin W'$ and $R_{k',W'} \neq \emptyset$, relay $h$ delivers $f_{d_h,W'}$ to user $k$. These additional side information of user $k$ will help him decode the multicast messages transmitted from the server in the second step.

3) In the last step, as Example [7] we let each user $k$ recover $f_{d_h,W} \setminus \{f_{d_h,W,s} : h \in (R_W \cap H_k)\}$ where $W \in Z_3$, $k \notin W$ and $R_{k,W} \neq \emptyset$. More precisely, we let $G_{k,W} = f_{d_h,W} \setminus \{f_{d_h,W,s} : h \in (R_W \cap H_k)\}$ representing the unknown bits in $f_{d_h,W}$ of user $k$. We divide $G_{k,W}$ into $|R_{k,W}|$ non-overlapping and equal-length pieces, $G_{k,W} = \{G_{k,W,s} : h \in R_{k,W}\}$. After considering all the MDS coded symbols demanded by all the users, for each relay $h \in H$ and each set $J \subseteq U_h$ where $|J| = g - 3$, we create the multicast message

$$V_{J,H} = \bigoplus_{k \in J} G_{k,J \setminus \{k\},h}$$

12 to be sent to relay $h$ and then forwarded to the users in $J$, where we use the same convention as that in the literature when it comes to ‘summing’ sets.

For example, consider relay 1 and set $J_1 = \{1,2,3\}$. It can be seen that $G_{1,1,2,3} = f_{d_1,1,2,3} \setminus \{f_{d_1,1,2,3,1} \cup f_{d_2,1,2,3,2}\} = \emptyset$. Similarly $G_{2,1,2,3} = G_{3,1,2,3} = \emptyset$. So $V_{J_1} = \emptyset$.

Consider now relay 1 and set $J_2 = \{1,2,4\}$. It can be seen that $G_{1,2,4} = f_{d_1,2,4} \setminus f_{d_1,2,4,1} \cup f_{d_2,2,4,2}$. Since $R_{1,2,4} = \{1\}$, we don’t further partition $G_{1,2,4}$ which includes $B/72$ bits. Similarly, each of $G_{2,1,4}$ and $G_{4,1,2}$ has $B/72$ bits. Hence, $V_{J_2} = G_{1,2,4} \cap G_{2,1,4} \cap G_{4,1,2}$ including $B/72$ is transmitted from the server to relay 1, which then forwards it to users in $\{1,2,4\}$.

Hence, we achieve $g = 3$ and $(R^{\text{relay}}, R^{\text{user}}) = \left(\frac{2}{3}, \frac{41}{72}\right) \approx (0.29, 0.57)$ while the scheme in [12] described in Section III-A gives $(R^{\text{relay}}, R^{\text{user}}) = \left(\frac{11}{12}, \frac{22}{72}\right) \approx (0.46, 0.32)$. It can be seen the link load from the server to relays achieved by the proposed method is less than the other one achieved by the scheme in [12].

Comparing the proposed scheme and the scheme in [12], there are two main advantages. On one hand, we can see that the cached contents of relays help users to decode the packets transmitted from the server which can lead an additional coded caching gain. For example, $f_{d_1,1,2,1}$ is cached by relay 1 and delivered to user 1 through coding gain. In the first step of delivery, $f_{d_1,1,2,1}$ is transmitted from relay 1 to user 1 and $f_{d_2,1,7,3}$ is transferred to user 3 such that user 1 knows them. In the second step of delivery, the server transmits $f_{d_1,2,7,4} \oplus f_{d_2,1,7,3} \oplus f_{d_2,1,1,2}$ to relay 2 and user 1 can use $f_{d_1,2,7,1} \oplus f_{d_2,1,7,3}$ to decode $f_{d_1,2,7,1}$. On the other hand, our proposed scheme is based on the asymmetric coded placement in [13] which is proved to be better than the scheme in [12].

We now generalize the proposed scheme in Example [2]. Notice that in this example, $f_{i,1,2}$ with $B/36$ bits is divided into $|R_{i,2}|+1 = 3$ non-overlapping pieces where $f_{i,1,2,1} = f_{i,1,2,2}$ and $f_{i,1,2,3} = \frac{M_{\text{relay}}B_{i}}{N_{i}} = B/72$ bits and $|f'_{i,1,2}| = 0$. It can be seen that if we increase the relay cache size, we still desire to achieve $g = 3$, we have $|f_{i,1,2,1}| + |f_{i,1,2,2}| > f_{i,1,2}$ and thus these two pieces are overlapped which leads to redundancy. In other words, not all the bits of $f_{d_i}$ cached in relays 1 and 2 are useful to user 1. So in this paper, we only consider the case

$$M_{\text{relay}} \leq \max_{W \subseteq \mathcal{O}|\mathcal{S}(g)} \left\{ \frac{N_{g_{i}}}{K_{i+1}} \right\},$$

where $S(i,g)$ is the length of each MDS symbol generated by the scheme in [13] (described in Section II-C).

The memories-loads tradeoff of the proposed scheme is the lower convex envelope of the three groups of points.

1) $(M_{\text{relay}}, M_{\text{user}}) = \left(0, \text{NS}(g)\right)$ for each $g \in \{1 : K_{i}\}$. For each point in this group, we can see that relays do not have memory and the scheme is equivalent to the one for combination networks with end-user-caches. We use the scheme in Section II-C which leads to $(R^{\text{relay}}, R^{\text{user}}) = (K(1 - M_{\text{user}}/N)/(H_S), (1 - M_{\text{user}}/N)/r)$.

2) $(M_{\text{relay}}, M_{\text{user}}) \leq \left\{ \begin{array}{ll} \frac{N_{g_{i}}}{K_{i+1}} \max_{y \in \mathcal{O}|\mathcal{S}(g)} & \\
\frac{N_{g_{i}}}{S_{i}(g)} \max_{y \in \mathcal{O}|\mathcal{S}(g)} & \end{array} \right.$
gain $g \in [2 : K_2 + 1]$. In this case, for each user $k$, each relay $h \in H_k$ caches $f_{i,W,h}$ where $k \in W$, $|W| = g - 1$ and $W \subseteq U_h$, we have

$$
\sum_{i \in [N]} \sum_{W \subseteq Z_g} \sum_{h \in H_k} \sum_{W \subseteq U_h} |f_{i,W,h}|
= \sum_{i \in [N]} \sum_{h \in H_k} \sum_{W \subseteq U_h} |f_{i,W,h}|
= \frac{N(\binom{K_1}{g-1}r)}{\max\{g \in [r] : K_y \geq g - 1\}S_1(g)}.
$$

(13)

In addition, since each user $k$ caches $f'_{i,W}$ where $k \in W$, we have

$$
M_{user} = \sum_{i \in [N]} \sum_{W \subseteq Z_g} |f'_{i,W}|.
$$

(14)

Hence, from (13) and (14) we have

$$
\sum_{i \in [N]} \sum_{W \subseteq Z_g} |f'_{i,W}| + \sum_{i \in [N]} \sum_{W \subseteq Z_g} \sum_{h \in H_k} |f_{i,W,h}|
= \frac{N S_2(g)}{S_1(g)} = \sum_{i \in [N]} \sum_{W \subseteq Z_g} |f_{i,W}|.
$$

Hence, we can use the proposed scheme in Example 2 Placement phase: We also divide each $F_i$ into $S_1(g)$ non-overlapping and equal-length pieces, which are then encoded by $(\{Z_g\}, S_1(g))$ MDS code. The length of each MDS symbol is $B/S_1(g)$. For every $W \subseteq Z_g$, there is one MDS symbol denoted by $f_{i,W}$ and we divide $f_{i,W}$ into $|R_W|$ + 1 non-overlapping parts, $f_{i,W} = \{f_{i,W,h} : h \in R_W\}$. For each $h \in R_W$, $f_{i,W,h}$ is cached by relay $h$ where

$$
|f_{i,W,h}| = \frac{M^{relay} B}{N(\binom{K_1}{g-1})} = \frac{1}{\max\{g \in [r] : K_y \geq g - 1\}S_1(g)}.
$$

In addition, $f'_{i,W}$ is cached by each user in $W$ where

$$
|f'_{i,W}| = \frac{B/S_1(g) - |R_W|M^{relay} B}{N(\binom{K_1}{g-1})} = \frac{|R_W|B}{S_1(g) \max_{W \subseteq Z_g} |R_W|}.
$$

Hence, for each $W \subseteq Z_g$, if $W \in \arg\max_{W \subseteq Z_g} |R_W|$, we have $|f'_{i,W}| = 0$; otherwise, $|f'_{i,W}| > 0$.

Delivery phase: We let each user $k$ recover $f_{d_k,W}$ where $W \in Z_g$ and $R_{\{k\} \cup W} \neq \emptyset$. There are two steps in delivery phase:

a) For each relay $h \in [H]$ and each user $k \in U_h$, relay $h$ delivers all the cached bits of $F_{d_k}$ to user $k$. More precisely, for each set $W \subseteq U_h$ where $|W| = g - 1$, relay $h$ delivers $f_{d_k,W,h}$ to user $k$. In addition, for each set $W' \subseteq U_h$ where $|W'| = g - 1$, $k \in W'$ and $|R_{W'}| > 1$, relay $h$ delivers $f_{d_k,W',h}$ to user $k'$ where $k' \subseteq [K] \setminus U_h$ and $R_{\{k'\} \cup W'} \neq \emptyset$.

b) We let each user $k$ recover $f_{d_k,W} \setminus \{f_{d_k,W,h} : h \in (R_{W} \cap H_k)\}$ where $W \in Z_g$, $k \notin W$ and $R_{\{k\} \cup W} \neq \emptyset$. More precisely, we let $G_{e,W} = f_{d_k,W} \setminus \{f_{d_k,W,h} : h \in (R_{W} \cap H_k)\}$ representing the unknown bits in $f_{d_k,W}$ of user $k$. We divide $G_{e,W}$ into $|R_{\{k\} \cup W}|$ non-overlapping and equal-length pieces, $G_{e,k,W} = \{G_{e,k,W,h} : h \in R_{\{k\} \cup W}\}$. After considering all the MDS coded symbols demanded by all the users, for each relay $h \in [H]$ and each set $J \subseteq U_h$ where $|J| = g$, we create the multicast message in [12], which is to be sent to relay $h$ and then forwarded to the users in $J$. It is also easy to check that each subfile in the multicast message in [12] has the same length.

We can compute that

$$
R^{s\rightarrow r} = \frac{K \max \{1 - (rM^{relay} + M_{user})/N, 0\}}{H g},
$$

$$
R^{r\rightarrow u} = \frac{1 - (M_{user}/N)}{r} + \frac{\sum_{h=2}^{r} (-\binom{1}{h-1}) \binom{K_1 - K_2}{h-2} (K_y - 1)^h}{\max\{g \in [r] : K_y \geq g - 1\}S_1(g)}.
$$

3) $(M^{relay}, M_{user}) = (N/r, Nt_1/K_1)$ where $t_1 \in [0, K_1]$. In this case, each relay directly caches $s_i^h$ such that the server needs not to transmit any packets to relays. If $t_1 = 0$, each user does not cache any bits. In the delivery phase, each relay $h \in [H]$ transmits $s_i^h$ to each user $k \in U_h$. So we have $(R^{s\rightarrow r}, R^{r\rightarrow u}) = (0, 1/r)$.

If $t_1 = K_1$, in the placement phase, each user $k$ caches $s_i^h$ for $h \in [H]$ and $i \in [N]$. So it caches all the $N$ files and $(R^{s\rightarrow r}, R^{r\rightarrow u}) = (0, 0)$.

Remark 1. In [14], an improved scheme was proposed for combination network with end-user-caches when $g \in [K_3 + 2, K_2 + (H-2)]$. We can also use the above method to extend this improved scheme to combination networks with cache-aided relays and users. In this paper, for simplicity, we do not give the details.

It is also straightforward to derive the following theorem.

Theorem 1. In a $(H, r, M^{relay}, M_{user}, N)$ combination network, the achieved max link-load from the server to relays by the proposed scheme is not larger than the one achieved by the caching scheme in [12].

IV. NUMERICAL COMPARISONS AND CONCLUSIONS

In Fig. 2 we compare the performance of the proposed scheme with the scheme in [12] for the centralized combination network with $H = 5, r = 3$, and $N = K = 10$ and $M^{relay} = 1$. We can see our proposed scheme provides a lower max link-load from the server to relays with the tradeoff of an increase of the max link-load of the max link-load from the relays to users. This is because that in the second step of delivery phase, each relay transmit some bits which are not from the demanded file of each of its connected users and
these bits can lead an increased coded caching gain on the number of bits sent from the server.

To conclude the paper, we proposed a novel caching scheme for combination networks with cache-aided relays and users, which aimed to create multicasting opportunities from the caches of relays and users. The proposed scheme was shown to be achieve a max-link load from the server to relays not larger than the best scheme known in the literature.

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