First Order Hardy Inequalities Revisited

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Abstract. In this paper, we consider the first order Hardy inequalities using simple equalities. This basic setting not only permits to derive quickly many well-known Hardy inequalities with optimal constants, but also supplies improved or new estimates in miscellaneous situations, such as multipolar potential, the exponential weight, hyperbolic space, Heisenberg group, the edge Laplacian, or the Grushin type operator.

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1 Introduction

The Hardy inequalities go back to G.H. Hardy, who showed in [23] a very famous estimate: Let $p > 1$, then

$$\int_{0}^{\infty} |u'(x)|^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{0}^{\infty} \frac{|u(x)|^p}{x^p} dx, \quad \forall u \in C^1(\mathbb{R}_+), \quad u(0) = 0. \quad (1.1)$$

Since one century, the Hardy type inequalities have been enriched extensively and broadly, they play important roles in many branches of analysis and geometry. More generally, we call first order Hardy inequalities, the estimates like

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\[ \int_{\mathcal{M}} V |\nabla u|^p d\mu \geq \int_{\mathcal{M}} W |u|^p d\mu \]

with positive weights \(V, W\) and \(u\) in suitable function spaces. They are also called weighted Poincaré or Hardy-Poincaré inequalities.

A huge literature exists on the Hardy inequalities, it is just impossible nor our intension to mention all the progress even for the first order case, we refer to the classical and recent books [2, 22, 28, 31, 32] for interested readers. The modest objective here is to show that many first order Hardy inequalities can be derived naturally and quickly from simple equalities, which on one hand yield classical Hardy inequalities with optimal constants, and on the other hand provide new Hardy inequalities or improve some well-known results.

The most well-known first order Hardy inequality is the following: Let \(\alpha \in \mathbb{R}\),
\[
\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^\alpha} dx \geq \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^\alpha+2} dx, \quad \forall u \in C^1_c(\mathbb{R}^n \setminus \{0\}). \tag{1.2}
\]

Here and after, \(|\cdot|\) denotes the Euclidean norm. The optimal constant could be firstly shown in [24, p. 259] with \(n = 1\). For simplicity, we consider only real valued functions, and without special remark, the functions \(u\) are \(C^1\), compactly supported away from the singularities of involved weights. In general, applying density argument, many estimates hold true in larger functional spaces.

The inequality (1.2) can be seen as a direct consequence of the following equality (see for instance [16] with \(\alpha = 0\) and [33, Lemma 2.3(i)] with \(\alpha = 2\)). For any \(u \in C^1_c(\mathbb{R}^n \setminus \{0\})\) and \(\alpha \in \mathbb{R}\), if \(v = |x|^{\frac{n-2-\alpha}{2}} u\), there holds
\[
\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^\alpha} dx = \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^\alpha+2} dx + \int_{\mathbb{R}^n} \frac{|\nabla v|^2}{|x|^{n-2}} dx. \tag{1.3}
\]

There is also a radial derivative version of the above equality: \(\forall u \in C^1_c(\mathbb{R}^n \setminus \{0\})\),
\[
\int_{\mathbb{R}^n} \frac{|\partial_r u|^2}{|x|^\alpha} dx = \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^\alpha+2} dx + \int_{\mathbb{R}^n} \frac{|T_\alpha(u)|^2}{|x|^\alpha} dx, \tag{1.4}
\]
where \(\partial_r\) is the radial derivative \(\partial_r = \frac{x \nabla}{r}\), and \(T_\alpha = \partial_r + \frac{2-2-\alpha}{2} r\). Indeed, the equalities (1.3) and (1.4) are equivalent, since
\[
|\nabla u|^2 = |\partial_r u|^2 + \sum_{j=1}^n |L_j u|^2,
\]
where
\[
L_j u = \partial_j u - \frac{x_j}{r} \partial_r u, \quad \forall 1 \leq j \leq n.
\]
Therefore,
\[
\left| \nabla \left( |x|^{\frac{n-2}{2}} \cdot u \right) \right|^2 = |x|^{n-2-\alpha} \left( |T_\alpha(u)|^2 + \sum_{j=1}^{n} |L_j u|^2 \right).
\]

From (1.3) and (1.4), as the remainder terms are nonnegative, we deduce easily the optimal Hardy inequalities (1.2), and the non-attainability of equality in (1.2) over the corresponding Banach space, for example, \( D^{1,2}(\mathbb{R}^n) \) if \( \alpha = 0 \) and \( n \geq 3 \).

The formula (1.4) was found firstly by Brezis-Vázquez in 1997 for \( \alpha = 0 \). They called it the “magical” computation in [9, p. 454], and applied it to get a famous improved Hardy inequality: For any bounded domain \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \),
\[
\int_{\Omega} |\nabla u|^2 dx - \frac{(n-2)^2}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} dx \geq H_2 \left( \frac{\omega_n}{|\Omega|} \right)^{\frac{n}{2}} \int_{\Omega} |u|^2 dx, \quad \forall u \in H^1_0(\Omega). \tag{1.5}
\]

Here, \( H_2 \) stands for the first eigenvalue of \(-\Delta \) in \( H^1_0(\mathbb{B}^2) \). In this paper, \( \mathbb{B}^n \) denotes always the Euclidean unit ball in \( \mathbb{R}^n \). The above inequality is the departure point for a rich literature on the improvement of classical Hardy inequalities, over bounded or not domains in \( \mathbb{R}^n \).

More recently, in the important works of Ghoussoub-Moradifam [21, 22], they introduced a notion of Bessel pair to study Hardy type inequalities. Let \( V, W \) be positive \( C^1 \) radial functions on \( \mathbb{B}^n \setminus \{0\} \) such that
\[
\int_0^1 \frac{1}{r^{n-1} V(r)} dr + \int_0^1 r^{n-1} V(r) dr < \infty, \tag{1.6}
\]
they proved that the following assertions are equivalent:

(i) \((V, W)\) is a \( n \)-dimensional Bessel pair on \((0, 1)\) with constant \( c > 0 \), that is, the ODE
\[
y''(r) + \left( \frac{n-1}{r} + \frac{V'}{V} \right) y'(r) + \frac{cW}{V} y(r) = 0 \tag{1.7}
\]
has a positive solution on the interval \((0, 1)\).

(ii) For all \( u \in C^1_c(\mathbb{B}^n) \), there holds
\[
\int_{\mathbb{B}^n} V |\nabla u|^2 dx \geq \beta_{V, W} \int_{\mathbb{B}^n} W u^2 dx. \tag{1.8}
\]

Here, \( \beta_{V, W} \) is the best constant, given as the supremum of \( c \) satisfying (i).
Ghoussoub-Moradifam used the above Bessel pair notion to improve, extend and unify many results on weighted Hardy inequalities, and studied the corresponding best constants. Their ideas have inspired many researches in the last decade and turn out to be very successful.

Here we attempt a more general approach using another point of view, where the departure point is an easy equality as follows. Let $(\mathcal{M}, g)$ be a Riemannian manifold and $\Omega \subset \mathcal{M}$ be open. Consider $V \in C^1(\Omega)$ and $\vec{F} \in C^1(\Omega, T_g \mathcal{M})$, then for any $u \in C^1_c(\Omega)$, any family of inner product $\langle \cdot, \cdot \rangle \in C^1(\Omega, \Lambda^2 T_g \mathcal{M})$ (not necessarily that given by $g$), there holds

$$\int_{\Omega} V \| \nabla_g u \|^2 dg = \int_{\Omega} \left[ \text{div}_g (V \vec{F}) - V \| \vec{F} \|^2 \right] u^2 dg + \int_{\Omega} V \| \nabla_g u + u \vec{F} \|^2 dg. \tag{1.9}$$

The formula (1.9) is proved just by developing $\| \nabla_g u + u \vec{F} \|^2$ and integration by parts. Here, $\nabla_g$ and $dg$ are respectively the gradient and volume form with respect to the metric $g$; $\| \cdot \|$ denotes the norm associated to $\langle \cdot, \cdot \rangle$; and $\text{div}_g$ is the adjoint operator of $\nabla_g$ with respect to $\langle \cdot, \cdot \rangle$ and $g$.

In particular, let $\vec{F} = -\nabla_g f$ with $f$ positive in $C^2(\Omega)$, we see that

$$\int_{\Omega} V \| \nabla_g u \|^2 dg = -\int_{\Omega} \frac{\text{div}_g (V \nabla_g f)}{f} u^2 dg + \int_{\Omega} V \left\| \nabla_g u - \frac{u}{f} \nabla_g f \right\|^2 dg. \tag{1.10}$$

Moreover, if $V = 1$, and the inner product $\langle \cdot, \cdot \rangle$ is that given by $g$, we arrive at

$$\int_{\Omega} \| \nabla_g u \|^2 dg = -\int_{\Omega} \frac{\Delta_g f}{f} u^2 dg + \int_{\Omega} f^2 \left\| \nabla_g \left( \frac{u}{f} \right) \right\|^2 dg. \tag{1.11}$$

The idea to use (1.9), (1.10) or (1.11) to get Hardy inequalities has been used in various situations, for example [19, Lemma 2.1] presents (1.9) in Euclidean spaces. The following are some classical or more recent Hardy inequalities which can be derived by the equalities (1.9)-(1.11).

- Taking $f = |x|^{\frac{n-2}{2} - \alpha}$ and $V = |x|^{-\alpha}$ in (1.10), we get

$$W = -\frac{\text{div}(V \nabla f)}{f} = \frac{(n-2-\alpha)^2}{4|x|^{\alpha+2}}$$

in $\mathbb{R}^n \setminus \{0\}$ which gives (1.3). In other words, equality (1.3) is a special example of (1.10).
In 1934, J. Leray [25] proved a famous inequality in dimension two
\[
\int_{B^2} |\nabla u|^2 dx \geq \frac{1}{4} \int_{B^2} \frac{u^2}{|x|^2 (\ln |x|)^2} dx, \quad \forall u \in H^1_0(B^2). \tag{1.12}
\]
Wang-Willem showed in [36] a generalization of Leray’s inequality: Let \( n \geq 1, \alpha \leq n - 2, \Omega = \mathbb{R}^n \setminus \overline{B^n} \) or \( B^n \). Then for any \( u \in C^1_c(\Omega) \),
\[
\int_{\Omega} \frac{|\nabla u|^2}{|x|^\alpha} dx \geq \frac{(n-2-\alpha)^2}{4} \int_{\Omega} \frac{u^2}{|x|^\alpha+2} dx + \frac{1}{4} \int_{\Omega} \frac{u^2}{|x|^\alpha+2 (\ln |x|)^2} dx. \tag{1.13}
\]
The estimate (1.13) can be proved using \( f = |x|^{\frac{2-2\alpha}{\alpha}} |\ln |x||^{\frac{1}{2}}, V = |x|^{-\alpha} \) in (1.10).

Carron [11] proved the following well-known Hardy inequality for non compact manifold: Let \( (\mathcal{M}, g) \) be a non parabolic complete Riemannian manifold such that there is a nonnegative function \( \rho \) satisfying \( \| \nabla \rho \| = 1; \Delta_g \rho \geq -C/\rho \) in \( \mathcal{D}'(\mathcal{M}) \) with \( C > 1 \); and the zero set of \( \rho \) is compact with null capacity. Then we have
\[
\int_{\mathcal{M}} \| \nabla u \|^2 dx \geq \frac{(C-1)^2}{4} \int_{\mathcal{M}} \frac{u^2}{\rho^2} dx, \quad \forall u \in C^1_c(\mathcal{M}).
\]
His proof is equivalent to use \( f(\rho) = \rho^{\frac{1+C}{2}} \) in (1.11), hence \( W = -\Delta_g f \geq \frac{(C-1)^2}{4\rho^2} \).

Brock-Chiacchio-Mercaldo [10] used the following inequality to get some weighted isoperimetric inequalities in cones: Let \( n \geq 3, \Omega = \{ x_1 > 0, x_n > 0 \} \), \( d\mu = x^k_1 |x|^m dx \), \( k \geq 1 \) and \( m \in \mathbb{R} \). Denote by \( E \) the closure of \( C^1(\overline{\Omega}) \) functions vanishing on \( \{ x_1 = 0 \} \) in \( H^1(\Omega, d\mu) \), then
\[
\int_{\Omega} |\nabla u|^2 d\mu \geq \frac{(n+m+k)^2}{4} - m \int_{\Omega} \frac{u^2}{|x|^2} d\mu, \quad \forall u \in E.
\]
This can be derived from (1.10) by taking \( V = x^k_1 |x|^m \) and \( f(x) = x_1 |x|^{-\frac{n+m+k}{2}} \).

Let \( f(x) = y(r) \), we can check that Eq. (1.7) is equivalent to
\[
-\text{div}(V \nabla f) = cWf \quad \text{in} \quad B^n \setminus \{0\}.
\]
By (1.10), for any \( u \in C^1_c(\mathbb{B}^n \setminus \{0\}) \), the Hardy inequality (1.8) holds true with \( c \) tending to \( \beta_{V,W} \). The technical assumption (1.6) permits to extend (1.8) for \( u \in H^1_0(\mathbb{B}^n) \). Conversely, as indicated by [21, Lemma 2.4], the validity of (1.8) yields the existence of a positive function \( f \) such that \( -\text{div}(V \nabla f) \geq cWf \) in \( \mathbb{B}^n \setminus \{0\} \).
As illustrated by above examples, the equalities (1.9)-(1.11) permit to find very quickly many first order Hardy inequalities. More examples will be displayed later. The best constants can be obtained with suitable choice of $\vec{F}, f, V$ and optimization on the sequel weight $W$. For example, to obtain (1.13), we test $f(x) = |x|^\beta |\ln |x||^\gamma$ with $\beta, \gamma \in \mathbb{R}$ and $V = |x|^{-\alpha}$. There holds, for any $|x| \neq 1$,

$$-\frac{\text{div}(V \nabla f)}{f} = -\frac{\beta(n-2-\alpha+\beta)}{|x|^{2+\alpha}} - \frac{\gamma(n-2-\alpha+2\beta)}{|x|^{2+\alpha} |\ln |x||} + \frac{\gamma(1-\gamma)}{|x|^{2+\alpha} (|\ln |x||)^2}.$$  

To maximize the coefficient for the first term on right hand side, we choose $\beta = \frac{2-n+\alpha}{2}$, which cancels the second term, and we optimize the last one with $\gamma = \frac{1}{2}$. Hence, (1.13) is valid for $u \in C^1_c(\Omega \setminus \{0\})$, we conclude eventually with the density argument if $\Omega = \mathbb{B}^n$.

Furthermore, in (1.10) or (1.11), we remark that the equality holds true without the last term, if and only if $u/f$ is locally constant, this resolves quickly the attainability issue, and suggests us the idea to construct appropriate approximation sequence showing the eventual optimality of involved constants.

The use of vector fields $\vec{F}$ in (1.9) can be helpful to handle anisotropic situation (even the final choice could be a gradient field). For example, Tidblom [34] and Filippas-Tertikas-Tidblom [19] used (1.9) to get various Hardy-inequalities on orthogonal cones in $\mathbb{R}^n$. See also the consideration for multipolar Hardy inequalities in [8, 13].

In the sequel, we will handle many other situations just using the identity (1.9) (hence (1.10) or (1.11)), and we show miscellaneous improved or new Hardy inequalities.

- For multipolar potential case, we improve and generalize several estimates given by Felli-Marchini-Terracini [17], Bosi-Dolbeault-Esteban [8], Cazacu-Zuazua [13].

- We answer an open problem of Blanchet-Bonforte-Dolbeault-Grillo-Vázquez [5] (see also [21]), by showing best constants for Hardy inequality with weight $(1+|x|^2)^\beta$.

- For the exponential weight in $\mathbb{R}^n$, we improve an inequality of Escobar-Kavian [17].

- We show an improved Hardy inequality on the unit disc of hyperbolic space.

- We consider also Heisenberg group, the Grushin type operators or the edge Laplacian, and supply various examples of new Hardy inequalities.

At last, we give some discussion for more general $L^p$ setting and Bessel pair.
2 Multipolar Hardy inequalities

Consider $n \geq 3$ and $m$ distinct points $(a_i)_{1 \leq i \leq m}$ in $\mathbb{R}^n$, if we denote $r_i = |x-a_i|$, Felli-Marchini-Terracini studied the multipolar Hardy inequality with $\lambda_i > 0$ as follows:

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \sum_{i=1}^m \lambda_i \int_{\mathbb{R}^n} \frac{u^2}{r_i^2} dx, \quad \forall u \in H^1(\mathbb{R}^n). \quad (2.1)$$

They proved that (2.1) holds true if and only if $\lambda_1 + \cdots + \lambda_m \leq \left(\frac{n-2}{4}\right)^2$, see [18, Theorem 1.1]. In the special borderline case when all $\lambda_i$ are equal to $\left(\frac{n-2}{4}\right)^2$, Bosi-Dolbeault-Esteban improves (2.1) in [8, Lemma 8]: For any $u \in H^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{4}\right)^2 \sum_{1 \leq i \leq m} \lambda_i \int_{\mathbb{R}^n} \frac{u^2}{r_i^2} dx + \sum_{1 \leq i \neq j \leq m} \frac{(|a_i-a_j|^2)}{r_i^2 r_j^2} u^2 dx. \quad (2.2)$$

Here we present a multipolar Hardy inequality which improves and generalizes completely (2.1) and (2.2).

**Theorem 2.1.** Let $n \geq 3$, and $(\lambda_i) \in (0, \infty)^m$ satisfy

$$\sum_{i=1}^m \lambda_i = \left(\frac{n-2}{4}\right)^2. \quad (2.3)$$

Then for any $u \in H^1(\mathbb{R}^n)$, and any family of $m$ distinct points $(a_i)_{1 \leq i \leq m}$ in $\mathbb{R}^n$, there holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \sum_{i=1}^m \lambda_i \int_{\mathbb{R}^n} \frac{u^2}{r_i^2} dx + \frac{4}{(n-2)^2} \sum_{1 \leq i \neq j \leq m} \lambda_i \lambda_j \int_{\mathbb{R}^n} \frac{|a_i-a_j|^2}{r_i^2 r_j^2} u^2 dx, \quad (2.4)$$

where $r_i = |x-a_i|$. 

**Proof.** Let $\alpha = (a_i) \in \mathbb{R}^m$ and

$$\bar{F}_\alpha(x) = \sum_{1 \leq i \leq m} \alpha_i \frac{x-a_i}{r_i^2}.$$

Then direct calculation gives

$$\text{div} \left( \bar{F}_\alpha \right) - |\bar{F}_\alpha|^2 = \sum_{1 \leq i \leq m} \frac{\alpha_i(n-2)}{r_i^2} - \sum_{1 \leq i \leq m} \frac{\alpha_i^2}{r_i^2} - \sum_{1 \leq i \neq j \leq m} \frac{\alpha_i \alpha_j}{r_i^2 r_j^2} \frac{(x-a_i,x-a_j)}{r_i^2 r_j^2}.$$
\[
\sum_{1 \leq i \leq m} \frac{a_i(n - 2) - a_i^2}{r_i^2} + \frac{1}{2} \sum_{1 \leq i \neq j \leq m} \left( \frac{|a_i - a_j|^2}{r_i^2 r_j^2} - \frac{1}{r_i^2} - \frac{1}{r_j^2} \right) a_i a_j \\
= \sum_{1 \leq i \leq m} \frac{a_i(n - 2) - S_{\alpha}}{r_i^2} + \frac{1}{2} \sum_{1 \leq i \neq j \leq m} \frac{|a_i - a_j|^2}{r_i^2 r_j^2} a_i a_j.
\]

Here \(S_{\alpha} = \sum_{1 \leq i \leq m} a_i\), and we used the equality
\[
\frac{|a_i - a_j|^2}{r_i^2 r_j^2} = |(x - a_i) - (x - a_j)|^2 \frac{1}{r_i^2 r_j^2} - 2 \frac{\langle x - a_i, x - a_j \rangle}{r_i^2 r_j^2}.
\]

Now we want to choose \(\lambda_i = (n - 2 - S_{\alpha}) a_i\) with \(a_i > 0\) and \(S_{\alpha} < n - 2\). By (2.3),
\[
\frac{(n - 2)^2}{4} = (n - 2 - S_{\alpha}) S_{\alpha}, \quad \text{hence} \quad S_{\alpha} = \frac{n - 2}{2}, \quad a_i = \frac{2\lambda_i}{n - 2}.
\]

Finally, with the above values of \((a_i)\), we obtain
\[
\text{div}(\vec{F}_n) - |\vec{F}_n|^2 = \sum_{1 \leq i \leq m} \frac{\lambda_i}{r_i^2} + \frac{4}{(n - 2)^2} \sum_{1 \leq i < j \leq m} \lambda_i \lambda_j \frac{|a_i - a_j|^2}{r_i^2 r_j^2}.
\]

By (1.9), the estimate (2.4) holds true for \(u \in C^1_c(\mathbb{R}^n \setminus \{a_i, 1 \leq i \leq m\})\), and remains valid in \(H^1(\mathbb{R}^n)\) by capacity or approximation argument as \(n \geq 3\).  

We should mention that the method in [8] was somehow the vector field approach as (1.9), they used the special \(\vec{F}_n\) where all \(a_i = \frac{n - 2}{2m}\). Recently, using \(\vec{F}_n\) with all \(a_i = \frac{n - 2}{2m}\), that is
\[
\vec{F}(x) = \frac{n - 2}{m} \sum_{1 \leq i \leq m} \frac{x - a_i}{r_i^2},
\]

Cazacu-Zuazua obtained in [13] another optimal multipolar Hardy inequality
\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n - 2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \frac{|a_i - a_j|^2}{r_i^2 r_j^2} u^2 dx, \quad \forall u \in H^1(\mathbb{R}^n). \quad (2.5)
\]

The following is a family of Hardy inequalities which generalizes (2.5).

**Theorem 2.2.** Let \(n \geq 3\), and \((\mu_i) \in (0, \infty)^m\) satisfy \(\sum_{1 \leq i \leq m} \mu_i = n - 2\). Then for any \(u \in H^1(\mathbb{R}^n)\) and any family of \(m\) distinct points \((a_i)_{1 \leq i \leq m}\) in \(\mathbb{R}^n\), there holds
\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \sum_{1 \leq i < j \leq m} \mu_i \mu_j \int_{\mathbb{R}^n} \frac{|a_i - a_j|^2}{r_i^2 r_j^2} u^2 dx, \quad (2.6)
\]

where \(r_i = |x - a_i|\).
The proof uses exactly the same computation as for (2.4), with now \( \alpha = (\mu_i) \) and \( S_\alpha = n - 2 \), so we omit it.

## 3 Best constant with weight \( (1 + |x|^2)\beta \)

To study the asymptotic behavior of solutions to fast diffusion equation via entropy estimate \([5, 6]\), Blanchet-Bonforte-Dolbeaul-Grillo-Vázquez showed some spectral gap estimates, using Hardy inequalities with weights \( (1 + |x|^2)^\beta \). More precisely, they studied the following Hardy type estimate: Let \( n \geq 3, \alpha \in \mathbb{R}, \)

\[
\int_{\mathbb{R}^n} (1 + |x|^2)^\alpha |\nabla u|^2 \, dx \geq C \int_{\mathbb{R}^n} (1 + |x|^2)^{\alpha - 1} u^2 \, dx, \quad \forall u \in C_c^1(\mathbb{R}^n). \tag{3.1}
\]

As indicated in \([6]\), under suitable initial conditions for the fast diffusion equation \( \partial_t v = \Delta (v^{1+\frac{1}{2}}) \) in \( \mathbb{R}^n \), the best constant in (3.1) provides the sharp decay rate of the entropy. They supplied in \([7, \text{Theorem 2}]\) the best constant in (3.1) for \( \alpha < 0 \), see also \([21]\). The best constant with \( \alpha > 0 \) was left open, except for \( \alpha = n \), see \([5, \text{Table 1}]\) and \([22, \text{Open problem (3)}]\). Here we give a complete answer.

**Theorem 3.1.** Let \( n > 2 \), let \( C_{\alpha,n} \) denote the best constant for (3.1). We have

\[
\begin{cases}
C_{\alpha,n} = \frac{(n+2\alpha-2)^2}{4}, & \text{if } \frac{2-n}{2} \leq \alpha \leq \frac{n+2}{2}, \\
C_{\alpha,n} = 2(\alpha - 1)n, & \text{if } \alpha > \frac{n+2}{2}.
\end{cases}
\]

**Proof.** Let \( V = (1 + r^2)^\gamma, \ f = (1 + r^2)^\gamma \) with \( \gamma \in \mathbb{R} \) and \( r = |x| \). Direct calculation yields

\[
W := -\frac{\text{div}(V \nabla f)}{f} = -\frac{V'f' - Vf}{f} = \left[ Q(\gamma) + \frac{4\gamma(\gamma + \alpha - 1)}{1 + r^2} \right] (1 + r^2)^{\alpha - 1},
\]

where

\[
Q(\gamma) = -2\gamma(2\gamma + n + 2\alpha - 2) = \frac{(n + 2\alpha - 2)^2}{4} - \left( 2\gamma + \frac{n + 2\alpha - 2}{2} \right)^2.
\]

Hence, the maximum

\[
\max_{\mathbb{R}} Q(\gamma) = \frac{(n + 2\alpha - 2)^2}{4}
\]

and it is attained at

\[
\gamma = \gamma^* = -\frac{n + 2\alpha - 2}{4}.
\]
Fixing $\gamma = \gamma^*$, there holds

$$4\gamma^* (\gamma^* + \alpha - 1) = -(n+2\alpha-2)(2\alpha-2-n) = \frac{n^2 -(2\alpha-2)^2}{4(1+r^2)} =: T(r).$$

So the discussion depends on the sign of $\frac{n^2 -(2\alpha-2)^2}{4}$.

- If $\frac{2-n}{2} \leq \alpha \leq \frac{n+2}{2}$, taking $\gamma = \gamma^*$, as $\min_{\mathbb{R}^+} T(r) = 0$, we claim that $C_{\alpha,n} \geq \frac{(n+2\alpha-2)^2}{4}$, since it is known that $C_{\alpha,n} \leq \frac{(n+2\alpha-2)^2}{4}$, see [20].

- Assume now $\alpha > \frac{n+2}{2}$. Let $f(x) = (1+r^2)^{1-\alpha}$, there holds $(1+r^2)^{1-\alpha}W \equiv 2(\alpha-1)n$. In the spirit of (1.10), it means that $C_{\alpha,n} \geq 2(\alpha-1)n$. Moreover, as $(1+r^2)^{\alpha-1}f^2+(1+r^2)^{\alpha} |\nabla f|^2 \in L^1(\mathbb{R}^n)$, applying standard cut-off argument, we conclude readily that $C_{\alpha,n} \leq 2(\alpha-1)n$.

The proof is complete.

Moreover, when $\alpha > \frac{n+2}{2}$, (1.10) means that $\lambda(1+r^2)^{1-\alpha}$ are the unique minimizers which realize the best constant in the corresponding weighted Sobolev space used by [4, 5], obtained as the completion of $C^0_\infty(\mathbb{R}^n)$ with respect to the norm

$$\|v\|_\alpha = \|v\|_{L^2(\mathbb{R}^n,d\mu_{\alpha-1})} + \|\nabla v\|_{L^2(\mathbb{R}^n,d\mu_{\alpha})},$$

where

$$d\mu_\beta = (1+|x|^2)^\beta dx.$$

**Remark 3.1.** It is worthy to mention that when $\alpha < \frac{2-n}{2}$, as $(1+r^2)^{\alpha-1} \in L^1(\mathbb{R}^n)$, the inequality (3.1) is meaningful only under additional condition such as

$$\int_{\mathbb{R}^n} ud\mu_{\alpha-1} = \int_{\mathbb{R}^n} (1+r^2)^{\alpha-1}udx = 0.$$

The best constant to the Hardy inequality (3.1) under the above constrain was given for all $\alpha < \frac{2-n}{2}$ in [7, Theorem 2].

## 4 On an inequality of Maz’ya

Maz’ya established a weighted Hardy inequality as follows, see [27] or [28, Theorem 1, p. 214]: For $n \geq 2$, there holds

$$\int_{\mathbb{R}^n} x_n |\nabla u|^2 dx \geq \frac{1}{16} \int_{\mathbb{R}^n} \frac{u^2}{(x_{n-1}^2 + x_n^2)^{2\alpha}} dx, \quad u \in C^1_c(\mathbb{R}^n_+).$$

(4.1)
Consequently, consider $u(x) = |x_n|^{-\frac{1}{\alpha}} v(x)$ with $v \in C^1_c(\mathbb{R}^n_+)$, we have

$$
\int_{\mathbb{R}^n_+} |\nabla v|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{v^2}{x_n^2} \, dx + \frac{1}{16} \int_{\mathbb{R}^n_+} x_n \frac{v^2}{(x_n^2 + x_n^2)^{\frac{1}{2}}} \, dx. \quad (4.2)
$$

Later on, Tidblom raised the coefficient $\Lambda = \frac{1}{16}$ to $\frac{1}{8}$ in (4.2), see [34, Corollary 3.1]. However, this does not mean that the corresponding improvement holds for (4.1). Because $v$ vanishes on $\partial \mathbb{R}^n_+$ will force $u$ to be zero on $\partial \mathbb{R}^n_+$, and the use of weight like $x_n^{\frac{1}{2}}$ in [34, Lemma 2.1] seems to prevent the approximation or cut-off argument with respect to $x_n$. Finally, Maz'ya-Shaposhnikova [29] showed that the optimal constant in (4.1) is equal to

$$
\lambda : = \inf_{g \in C^1([0,\pi])} \int_0^\pi \left[ g'(t)^2 + \frac{1}{4} g(t)^2 \right] \sin t \, dt
$$

and $\lambda = 0.1564..$ by numerical approximation. Here, with (1.9), we will not reach the optimal constant, but we can improve easily (4.1) with $\frac{1}{8}$ and an explicit remainder term.

**Theorem 4.1.** Let $n \geq 2$, then for any $u \in C^1_c(\mathbb{R}^n_+)$,

$$
\int_{\mathbb{R}^n_+} x_n |\nabla u|^2 \, dx \\
\geq \frac{1}{8} \int_{\mathbb{R}^n_+} \frac{u^2}{(x_n^2 + x_n^2)^{\frac{1}{2}}} \, dx + \frac{7}{32} \int_{\mathbb{R}^n_+} \left[ \frac{x_n}{x_n^2 + x_n^2} - \frac{x_n^2}{(x_n^2 + x_n^2)^{\frac{3}{2}}} \right] u^2 \, dx. \quad (4.3)
$$

**Proof.** Consider $a, b \in \mathbb{R}$,

$$
\vec{F} = (0, \ldots, 0, \frac{bx_{n-1}}{\rho^2}, a \frac{x_n}{\rho^2} + b \frac{x_n}{\rho^2})
$$

with $\rho = |(x_{n-1},x_n)| \geq 0$ and $V = x_n$. Then for $\rho > 0$, there holds

$$
W : = \text{div} (V \vec{F}) - V |\vec{F}|^2 \\
= a \frac{a^2 + b^2 - b}{\rho^2} x_n - \frac{a(1+2b)x_n^2}{\rho^3} \\
= \frac{a}{\rho} \left[ a^2 + b^2 - b + a(1+2b) \right] \frac{x_n}{\rho^2} + a(1+2b) \frac{x_n(\rho - x_n)}{\rho^3}.
$$
We hope that \( a(1+2b) \geq 0 \) and
\[
a^2 + b^2 - b + a(1+2b) = \left( b + \frac{2a-1}{2} \right)^2 + \frac{8a-1}{4} \leq 0.
\]
In this sense, the best choice could be \( a = \frac{1}{8} \) and \( b = \frac{3}{8} \), which gives
\[
W = \frac{1}{8\rho} + \frac{7 x_n (\rho - x_n)}{32 \rho^3}.
\] (4.4)

Furthermore, let \( u \in C^1_c(\mathbb{R}^n_+ \setminus \{\rho = 0\}) \), even \( u \) does not vanish always on \( \partial \mathbb{R}^n_+ \), the equality (1.9) remains true with \( \Omega = \mathbb{R}^n_+ \setminus \{\rho = 0\} \), thanks to \( V = 0 \) on \( \partial \mathbb{R}^n_+ \). Combining with (4.4), we see that (4.3) is valid for any \( u \in C^1_c(\mathbb{R}^n_+) \).

Finally, let \( \varphi \) be a cut-off function in \( \mathbb{R} \), using \( u_\varepsilon(x) = u(x) \left[ 1 - \varphi(\rho / \varepsilon) \right] \), we can prove readily that the result holds true for all \( u \in C^1_c(\mathbb{R}^n) \), we omit the detail.

**Remark 4.1.** It is interesting to notice that the vector field \( \vec{F} \) used here is not a gradient vector field.

## 5 Exponentially weighted Hardy inequalities

The Hardy inequalities with weight \( K(x) = e^{\frac{|x|^2}{4}} \) in \( \mathbb{R}^n \) play important roles in the study of heat equation, see [17, 35]. Let \( V = K(x) \) and \( f_\alpha(x) = |x|^{\frac{2-n}{2}} e^{\alpha |x|^2} \), we have
\[
- \frac{\text{div}(V \nabla f_\alpha)}{f_\alpha} = K(x) \left[ \frac{(n-2)^2}{4|x|^2} + \frac{n-2-16\alpha}{4} - \alpha(4\alpha+1)|x|^2 \right].
\]
Hence, for \( n \geq 3 \), \( u \in H^1_K(\mathbb{R}^n) \) the weighted Sobolev space with respect to the measure \( K(x) dx \) as in [17, 35], there holds
\[
\int_{\mathbb{R}^n} |\nabla u|^2 Kdx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} Kdx + \frac{n-2-16\alpha}{4} \int_{\mathbb{R}^n} u^2 Kdx - \alpha(4\alpha+1) \int_{\mathbb{R}^n} u^2 Kdx.
\] (5.1)
Taking \( \alpha = -\frac{1}{8} \), we get the following result.

**Theorem 5.1.** Let \( n \geq 3 \) and \( u \in H^1_K(\mathbb{R}^n) \), then
\[
\int_{\mathbb{R}^n} |\nabla u|^2 Kdx \geq \frac{1}{16} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} Kdx + \frac{n}{4} \int_{\mathbb{R}^n} u^2 Kdx + \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} Kdx.
\] (5.2)
The estimate (5.2) improves the consideration of Escobedo-Kavian [17] by adding the last positive term with classical potential \( \frac{(n-2)^2}{4|x|^2} \).

With \( \alpha = -\frac{1}{4} \) in (5.1), we obtain another Hardy inequality showed by Vázquez-Zuazua [35, Theorem 9.1], that is, for \( u \in H^1_{K}(\mathbb{R}^n) \) with \( n \geq 3 \),

\[
\int_{\mathbb{R}^n} |\nabla u|^2 Kdx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} Kdx + \frac{n+2}{4} \int_{\mathbb{R}^n} u^2 Kdx. \tag{5.3}
\]

6 Hyperbolic disc

Consider the Poincaré hyperbolic space \( \mathbb{H}^n \) over \( \mathbb{B}^n \) with the metric

\[ ds^2 = \frac{4dx^2}{(1-r^2)^2}, \quad r = |x|. \]

It is well known that the volume form of \( \mathbb{H}^n \) is \( dv_{\mathbb{H}} = \frac{2^n}{(1-r^2)^n} |dx|^2 \), the associated gradient operator and Laplacian are respectively

\[
\nabla_{\mathbb{H}} = \frac{1-r^2}{2} \nabla, \quad \Delta_{\mathbb{H}} = \frac{1-r^2}{4} \left[ (1-r^2) \Delta + 2(n-2)x \cdot \nabla \right].
\]

Recall that \( \nabla, \Delta \) are the usual gradient and Laplacian in \( \mathbb{R}^n \). Denote \( \rho = \frac{1}{2} \ln \frac{1+r}{1-r} \) the hyperbolic distance from the origin to \( x \in \mathbb{H}^n \). For hyperbolic radial function, i.e. function \( \varphi(\rho) \), there holds

\[
\Delta_{\mathbb{H}} \varphi = \frac{\partial^2 \varphi}{\partial \rho^2} + (n-1) \coth \rho \frac{\partial \varphi}{\partial \rho}.
\]

For example, let \( f = (\ln \tanh \frac{\rho}{2})^a (\sinh \rho)^b \) with \( a, b \in \mathbb{R} \), we have

\[
\frac{\Delta_{\mathbb{H}} f}{f} = b(n-1+b) + \frac{b(n-2+b)}{(\sinh \rho)^2} + \frac{a(a-1)}{(\sinh \rho)^2 (\ln \tanh \frac{\rho}{2})^2} + \frac{a(n-2+2b) \coth \rho}{\sinh \rho \ln \tanh \frac{\rho}{2}}.
\]

Taking \( a = \frac{1}{2} \) and \( b = \frac{2-n}{2} \), the equality (1.11) means that for any \( u \in C^1_{c}(\mathbb{H}^n \setminus \{0\}) \),

\[
\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} u|^2 dv_{\mathbb{H}} \geq \frac{n(n-2)}{4} \int_{\mathbb{H}^n} \frac{u^2}{(\sinh \rho)^2} dv_{\mathbb{H}} + \frac{(n-2)^2}{4} \int_{\mathbb{H}^n} \frac{u^2}{(\sinh \rho)^2 (\ln \tanh \frac{\rho}{2})^2} dv_{\mathbb{H}} + \frac{1}{4} \int_{\mathbb{H}^n} \frac{u^2}{(\sinh \rho)^2 (\ln \tanh \frac{\rho}{2})^2} dv_{\mathbb{H}}.
\]
This is the optimal Hardy-type inequality showed by [4, Theorem 2.10].

Consider \( f = \rho^a (\sinh \rho)^b (\cosh \rho)^\alpha \) with \( a, b, \alpha \in \mathbb{R} \), then

\[
\frac{\Delta_H f}{f} = \frac{a(a-1)}{\rho^2} + \frac{b(n-2+b)}{\sinh^2 \rho} + \frac{a(n-1+2b)}{\rho} \coth \rho \\
+ \alpha(n+2b) + b(n-1+b) + \alpha(n-1) \tanh^2 \rho + 2a \alpha \tanh \rho \frac{\tanh \rho}{\rho}.
\]  

(6.1)

Set first \( \alpha = b = 0 \) and \( a = 2 - \frac{n}{2} \), applying (1.11), we have

\[
\int_{\mathbb{H}^n} |\nabla_H u|^2 dv_H \geq \frac{(n-2)^2}{4} \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dv_H + \frac{(n-1)(n-2)}{2} \int_{\mathbb{H}^n} \frac{\rho \coth \rho - 1}{\rho^2} u^2 dv_H,
\]

which was given in [20, Theorem 1.3]. Let \( a = \frac{1}{2} \) and \( b = 1 - \frac{n}{2} \), (6.1) yields

\[
-\frac{\Delta_H f}{f} = \frac{1}{4\rho^2} + \frac{(n-1)(n-3)}{4\sinh^2 \rho} + \frac{(n-1)^2}{4} + I_{\alpha}(\rho),
\]

where

\[
I_{\alpha}(\rho) := -\alpha \left[ (\alpha-1) \tanh^2 \rho + \frac{\tanh \rho}{\rho} + 1 \right].
\]

Taking still \( \alpha = 0 \) and using again (1.11), there holds

\[
\int_{\mathbb{H}^n} |\nabla_H u|^2 dv_H \geq \frac{1}{4} \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dv_H + \frac{(n-1)(n-3)}{4} \int_{\mathbb{H}^n} \frac{u^2}{\sinh^2 \rho} dv_H \\
+ \frac{(n-1)^2}{4} \int_{\mathbb{H}^n} u^2 dv_H,
\]

(6.2)

which was given by [3, Theorem 2.1] and [20, Theorem 1.4].

Consider now the unit hyperbolic ball \( B_{\mathbb{H}} := \{ x \in \mathbb{B}^n, \rho < 1 \} = \{ r < \tanh 1 \} \). We want to obtain an improvement of (6.2) over \( B_{\mathbb{H}} \) with a better choice of \( \alpha \). Clearly \( I_{\alpha}(\rho) < 0 \) for \( \alpha \geq 1 \); for \( 0 < \alpha < 1 \), we have still \( I_{\alpha}(\rho) < 0 \) since \( \tanh^2 \rho < 1 \) for any \( \rho > 0 \). Hence, \( \alpha > 0 \) are not appropriate. Let \( \alpha < 0 \) and rewrite

\[
I_{\alpha}(\rho) = -\alpha \tanh^2 \rho [\alpha - 1 + J(\rho)] \quad \text{with} \quad J(\rho) := \frac{\coth \rho}{\rho} + \coth^2 \rho.
\]

Obviously \( J \) is decreasing in \((0, \infty)\), so that for \( \rho \in [0, 1], I_{\alpha}(\rho) \geq G(\alpha) \tanh^2 \rho \) with

\[
G(\alpha) := -\alpha(\alpha-1) - J(1) \alpha = -\alpha^2 - \left( \coth 1 + \csch^2 1 \right) \alpha.
\]

Choosing \( \alpha = -\frac{\coth 1 - \csch^2 1}{2} < 0 \), which reaches the maximum of \( G \), we arrive at the following theorem.
Theorem 6.1. Let $n \geq 3$. For any $u \in H^1_0(B_H)$, there holds,

$$\int_{B_H} |\nabla_H u|^2 dv_H \geq \frac{1}{4} \int_{B_H} \frac{u^2}{\rho^2} dv_H + \frac{(n-1)(n-3)}{4} \int_{B_H} \frac{u^2}{\sinh^2 \rho} dv_H$$

$$+ \frac{(n-1)^2}{4} \int_{B_H} u^2 dv_H + \frac{(\coth 1 + \text{csch}^2 1)^2}{4} \int_{B_H} (\tanh \rho)^2 u^2 dv_H.$$

7 Hardy inequality for edge Laplacian

Let $X$ be an open set of $\mathbb{R}^n$ and $Y$ be an open set in $\mathbb{R}^q$ containing the origin, $n, q \geq 1$. Consider $E = (0,1) \times X \times Y$, which can be regarded as a local model of stretched edge-manifolds (i.e. manifolds with edge singularities, see [12, 30]) with dimension $N = n + q + 1$, and the Riemannian metric

$$t^{-2} dt^2 + dx^2 + t^{-2} dy^2 \quad \text{for} \quad (t,x,y) \in E.$$

Hence, the associated gradient and volume form are

$$\nabla_E = (t \partial_t, \partial_{x_1}, \ldots, \partial_{x_n}, t \partial_{y_1}, \ldots, t \partial_{y_q}), \quad d\sigma = t^{-1-q} dt dx dy.$$

The corresponding Laplace-Beltrami operator (called edge Laplacian) is the following degenerate elliptic operator:

$$\Delta_E = (t \partial_t)^2 - q t \partial_t + \Delta_x + t^2 \Delta_y = t^2 \partial_{tt} + (1-q) t \partial_t + \Delta_x + t^2 \Delta_y.$$

Theorem 7.1. Let $n \geq 2$ and $u \in C^1_c(E)$, we have

$$\int_E |\nabla_E u|^2 d\sigma \geq \int_E \left[ \frac{(n-2)^2}{4\psi} + \frac{q^2}{4} + \frac{(n-2)e^2}{8} q W_0 \right] u^2 d\sigma,$$

where

$$\psi = e^{-\frac{t}{2}} + |x|^2 + |y|^2, \quad W_0 = \frac{t^{-2} e^{-\frac{t}{2}}}{\psi}.$$

The above result is motivated by [12, Proposition 3.5], where the following Hardy inequality was used to handle the existence of solution to a Dirichlet problem with edge Laplacian and singular potential

$$\int_E |\nabla_E u|^2 d\sigma \geq \frac{n^2}{4} \int_E W_0 u^2 d\sigma, \quad \forall u \in C^1_c(E).$$
For \( n \geq 2 \), as \( t^{-2} e^{-\frac{1}{t^2}} \leq e^{-1} \) if \( t \in (0, 1) \),

\[
\frac{W}{W_0} \geq \frac{(n-2)^2 e}{4} + \frac{(n-2)e^2}{8} q,
\]

where

\[
W = \frac{(n-2)^2}{4\psi} + \frac{q^2}{4} + \frac{(n-2)e^2}{8} q W_0.
\]

Hence, (7.1) improves greatly (7.2), especially for large \( n \) and \( q \).

**Proof.** Set \( f = g(t)\psi^a \) with \( a \in \mathbb{R} \), direct computation yields

\[
\frac{\Delta_E f}{f} = t^2 \partial_{tt} f + (1-q) t \partial_t f + \Delta_x f + t^2 \Delta_y f
\]

\[
= t^2 \left[ \frac{g''}{g} + 4a \frac{g'}{g} t^3 e^{-\frac{1}{t^2}} - \frac{6a}{\psi t^4} e^{-\frac{1}{t^2}} + \frac{4a}{\psi^6 t^6} e^{-\frac{1}{t^2}} + \frac{4a(a-1)}{\psi^2 t^6} e^{-\frac{2}{t^2}} \right]
\]

\[
+ (1-q) t \left[ \frac{g'}{g} + \frac{2a}{\psi t^3} e^{-\frac{1}{t^2}} \right] + \frac{4a(a-1)}{\psi^2} |x|^2 + 2n \frac{a}{\psi} + t^2 \left[ \frac{4a(a-1)}{\psi^2} |y|^2 + \frac{2qa}{\psi} \right]
\]

\[
= \frac{4a(a-1)}{\psi^2} \left[ t^{-4} e^{-\frac{2}{t^2}} + |x|^2 + t^2 |y|^2 \right] + \frac{(2n+2qt^2)a}{\psi}
\]

\[
+ \frac{ae^{-\frac{1}{t^2}}}{t^4 \psi} \left[ \frac{4}{g} + \frac{4t^3 g'}{g} - 2(q+2) t^2 \right] + \frac{(1-q) t g'}{g} + t^2 \frac{g''}{g}.
\]

Taking \( g(t) = t^b \) with \( b \in \mathbb{R} \), there holds

\[
\frac{\Delta_E f}{f} = \frac{4a(a-1)}{\psi^2} \left[ t^{-4} e^{-\frac{2}{t^2}} + |x|^2 + t^2 |y|^2 \right] + \frac{(2n+2qt^2)a}{\psi}
\]

\[
+ \frac{ae^{-\frac{1}{t^2}}}{t^2 \psi} \left( \frac{4}{t^2} + 4b - 2q - 4 \right) + b(b-q).
\]

As \( t^{-4} e^{-\frac{1}{t^2}} \leq 4e^{-2} < 1 \) for \( t \in (0, 1) \),

\[
t^{-4} e^{-\frac{2}{t^2}} + |x|^2 + t^2 |y|^2 \leq \psi.
\]

Let now \( a \leq 0 \) and \( b = \frac{q}{2} \), we conclude that

\[
-\frac{\Delta_E f}{f} \geq \frac{4(1-a)}{\psi} - \frac{(2n+2qt^2)a}{\psi} - a W_0 \left( \frac{4}{t^2} - 4 \right) + \frac{q^2}{4}
\]
\[
\begin{align*}
\geq \frac{4(1-a)a-2na}{\psi} - 2qaW_0 t e^\frac{t^2}{4} + \frac{q^2}{4} \\
\geq \frac{4(1-a)a-2na}{\psi} - \frac{e^2aq}{2}W_0 + \frac{q^2}{4}.
\end{align*}
\]

The proof is completed by choosing \(a = \frac{2-n}{4}\). \(\square\)

Our approach can also improve (7.2) in dimension one. Indeed, for \(a = 0\), \(b = \frac{q}{2}\) and \(n = 1\), the above proof still works and implies

\[
\int_E |\nabla_E u|^2 d\sigma \geq \frac{q^2}{4} \int_E u^2 d\sigma.
\]

**8 On Heisenberg group**

Let \(n \geq 1\) and \(H_n\) be the standard Heisenberg group, i.e. \(\mathbb{R}^{2n+1}\) endowed with the following group law: For any \(\zeta = (x,y,t)\), \(\tilde{\zeta} = (\tilde{x},\tilde{y},\tilde{t}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\),

\[
\zeta \circ \tilde{\zeta} = \left( x + \tilde{x}, y + \tilde{y}, t + \tilde{t} + 2 \sum_{i=1}^{n} (x_i \tilde{y}_i - y_i \tilde{x}_i) \right).
\]

We equip \(H_n\) with the norm

\[
\|
\zeta
\|_{H_n} := \rho = \left( r^4 + t^2 \right)^{\frac{1}{4}},
\]

where

\[
\rho^2 = |(x,y)|^2 = |x|^2 + |y|^2.
\]

Define the vector fields

\[
X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad 1 \leq i \leq n.
\]

The associated horizontal gradient and Kohn hypoelliptic Laplacian on \(H_n\) are respectively denoted by \(\nabla_{H_n}\) and \(\Delta_{H_n}\), that is

\[
\nabla_{H_n} = (X_1, \ldots, X_n, Y_1, \ldots, Y_n),
\]

\[
\Delta_{H_n} = \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} Y_i^2 = \Delta_{(x,y)} + 4r^2 \frac{\partial^2}{\partial t^2} + 4 \sum_{i=1}^{n} \left( y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right) \frac{\partial}{\partial t}.
\]

We denote by \(Q := 2n + 2\) the homogeneous dimension of \(H_n\). Here is a series of sharp and new Hardy inequalities.
Theorem 8.1. The following estimates hold true:

\[ \int_{\mathcal{H}_n} |\nabla H_n u|^2 d\xi \geq \frac{Q^2}{4} \int_{\mathcal{H}_n} \frac{r^2}{\rho^4} u^2 d\xi + \int_{\mathcal{H}_n} \frac{r^2}{l^2} u^2 d\xi, \quad \forall u \in C^1_c(\mathcal{H}_n \{t=0\}), \tag{8.1} \]

\[ \int_{\mathcal{H}_n} |\nabla H_n u|^2 d\xi \geq \left( \frac{(Q+2)^2}{4} \right) \int_{\mathcal{H}_n} \frac{r^2}{\rho^4} u^2 d\xi, \quad \forall u \in C^1_c(\mathcal{H}_n \{0\}), \tag{8.2} \]

\[ \int_{\mathcal{H}_n} |\nabla H_n u|^2 d\xi \geq \left( \frac{(Q-4)^2}{4} \right) \int_{\mathcal{H}_n} \frac{r^2}{\rho^4} u^2 d\xi + 9 \int_{\mathcal{H}_n} \frac{r^2}{l^2} u^2 d\xi, \quad \forall u \in C^1_c(\mathcal{H}_n \{r=0\}). \tag{8.3} \]

Here, \(d\xi\) denotes the Lebesgue measure in \(\mathbb{R}^{2n+1}\). The inequality (8.1) improves [26, Theorem 1.1] where Luan-Yang proved similar estimate with \(\frac{Q^2-4}{4}\) instead of the sharp coefficient \(\frac{Q^2}{4}\).

Proof. To prove (8.1), clearly we need only to consider \(u\) supported in \(\mathcal{H}_n^+\), the half space where \(t > 0\). With integration by parts, we can check easily that (1.11) becomes

\[ \int_{\mathcal{H}_n^+} |\nabla H_n u|^2 d\xi = \int_{\mathcal{H}_n^+} \frac{-\Delta H_n f}{f} u^2 d\xi + \int_{\mathcal{H}_n^+} |\nabla H_n u - \frac{u}{f} \nabla H_n f|^2 d\xi. \]

Let \(f = \rho^a r^b t^c\) with \(a, b, c \in \mathbb{R}\), there holds

\[ -\frac{\Delta H_n f}{f} = -\frac{b(Q-4+b)}{r^2} - a(Q-2+2b+4c+a) \frac{r^2}{\rho^4} - 4c(c-1) \frac{r^2}{l^2}. \]

Set \(a = -\frac{Q}{2}, b = 0\) and \(c = \frac{1}{2}\), we get (8.1).

The estimates (8.2) and (8.3) are obtained similarly, with different choice of parameters in \(f\), working on the corresponding domain.

- (8.2) is obtained with \(a = -\frac{Q+2}{2}, b = 0\) and \(c = 1\).
- Taking \(a = -3, b = -\frac{Q-4}{2}\) and \(c = 1, (8.3)\) is proved.

\[ \square \]

9 Gurshin type operators

Let \(\xi = (x,y) \in \mathbb{R}^{d+k}\) with \(d,k \geq 1\). Let \(\gamma > 0\), consider

\[ X_i := \frac{\partial}{\partial x_i}, \quad Y_j := |x|^{\gamma} \frac{\partial}{\partial y_j}, \quad 1 \leq i \leq d, \quad 1 \leq j \leq k. \]
Here we denote by $\rho$ the Grushin distance from the origin to $\xi = (x, y)$
\[ \rho = \| \xi \|_C = \left[ |x|^{2+2\gamma} + (1+\gamma)^2 |y|^2 \right]^{\frac{1}{2+2\gamma}}. \]

D’Ambrosio in [15, Theorem 3.3] proved a well-known Hardy inequality (among many others) for Grushin operators: For any $u \in C^1_c((\mathbb{R}^d\setminus\{0\}) \times \mathbb{R}^k)$,
\begin{align*}
\int_{\mathbb{R}^{d+k}} |\nabla_\gamma u|^2 \, d\xi &\geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{u^2}{|x|^2} \, d\xi \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{u^2}{|y|^2} \, d\xi. \quad (9.1) \\
\end{align*}

When $d \geq 3$, applying density argument, the above estimate holds true for $u \in D^{1,2}_\gamma(\mathbb{R}^{d+k})$, the closure of $C^1_c(\mathbb{R}^{d+k})$ endowed with the norm $\|\nabla_\gamma u\|_{L^2}$. In the following, we show some examples of improved Hardy inequalities.

**Theorem 9.1.** For any $u \in C^1_c((\mathbb{R}^d\setminus\{0\}) \times (\mathbb{R}^k\setminus\{0\}))$, we have
\begin{align*}
\int_{\mathbb{R}^{d+k}} |\nabla_\gamma u|^2 \, d\xi &\geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{u^2}{|x|^2} \, d\xi + \frac{(k-2)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{|x|^{2\gamma} u^2}{|y|^2} \, d\xi \\
&\quad + (1+\gamma)^2 \int_{\mathbb{R}^{d+k}} \frac{|x|^{2\gamma} u^2}{\rho^{2+2\gamma}} \, d\xi, \quad (9.2) \\
\int_{\mathbb{R}^{d+k}} |\nabla_\gamma u|^2 \, d\xi &\geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{u^2}{|x|^2} \, d\xi + \frac{k^2(1+\gamma)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{|x|^{2\gamma} u^2}{\rho^{2+2\gamma}} \, d\xi, \quad (9.3) \\
\int_{\mathbb{R}^{d+k}} |\nabla_\gamma u|^2 \, d\xi &\geq \frac{(k-2)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{|x|^{2\gamma} u^2}{|y|^2} \, d\xi + \frac{(d+2\gamma)^2}{4} \int_{\mathbb{R}^{d+k}} \frac{|x|^{2\gamma} u^2}{\rho^{2+2\gamma}} \, d\xi. \quad (9.4)
\end{align*}

Clearly, (9.3) is valid for $u \in C^1_c((\mathbb{R}^d\setminus\{0\}) \times \mathbb{R}^k)$, hence improves (9.1). When $d,k \geq 3$, (9.2)-(9.4) hold true for $u \in D^{1,2}_\gamma(\mathbb{R}^{d+k})$, in particular (9.2) improves also (9.1) if $k \geq 3$.

**Proof.** Let $f = |x|^a |y|^b \rho^c$ with $a,b,c \in \mathbb{R}$, there holds
\begin{align*}
\frac{\Delta_\gamma f}{f} &\geq \frac{\Delta_x f + |x|^{2\gamma} \Delta_y f}{f^2} = \frac{a(d-2+a) |x|^{2\gamma}}{|x|^2} + \frac{b(k-2+b) |x|^{2\gamma}}{|y|^2} \\
&\quad + \frac{c [c-2+d+2a+(1+\gamma)(k+2b)] |x|^{2\gamma}}{\rho^{2+2\gamma}}.
\end{align*}

Using integration by parts, we can apply (1.11) with $\nabla_\gamma, \Delta_\gamma$ and $d\xi$. 
Choosing \( a = \frac{2-d}{2} \), \( b = \frac{2-k}{2} \) and \( c = -(1+\gamma) \), we get (9.2).

(9.3) is given by \( a = \frac{2-k}{2}, b = 0 \) and \( c = -\frac{k(1+\gamma)}{2} \).

We obtain (9.4) with \( a = 0, b = \frac{2-k}{2} \) and \( c = -\frac{d+2\gamma}{2} \).

The proof is complete.

\[ \square \]

10 Further remarks

10.1 Bessel pair and capacity

By (1.10), we can generalize Ghoussoub-Moradifam’s definition of Bessel pair. \((V,W)\) is called a Bessel pair over an open set \( \Omega \subset (M,g) \) if there exists a positive function \( f \in C_1(\Omega) \) such that

\[ -\text{div}_g(V \nabla_g f) \geq Wf \quad \text{in} \quad \Omega. \]

Applying the equality (1.10), we see that

\[ \int_{\Omega} V \| \nabla_g u \|^2 dg \geq \int_{\Omega} W u^2 dg, \quad \forall u \in C_c^1(\Omega). \]

When \( V, W \in C^1(\Omega \setminus \Sigma) \cap L^1_{\text{loc}}(\Omega) \) forms a Bessel pair over \( \Omega \setminus \Sigma \) with a negligible set \( \Sigma \subset \subset \Omega \), a natural question is to know when we can extend the couple \((V,W)\) as a Bessel pair over \( \Omega \). Here is a sufficient condition in the spirit of [28].

**Definition 10.1.** Let \( \Omega \subset (M,g) \) be open and \( \Sigma \subset \subset \Omega \). We say that \( \text{cap}_V(\Sigma, \Omega) = 0 \) if \( \text{vol}_g(\Sigma) = 0 \), and for any open set \( \Omega' \) satisfying \( \Sigma \subset \subset \Omega' \subset \Omega \), there holds \( \text{cap}_{V,\Omega'}(\Sigma) = 0 \) where

\[ \text{cap}_{V,\Omega'}(\Sigma) = \inf \left\{ \int_{\Omega'} V \| \nabla_g \eta \|^2 dg, \eta \in C^1_c(\Omega'), \Sigma \subset \subset \{ \eta = 1 \} \right\}. \]

The following result could be well-known, we show a proof here for the convenience of readers.

**Proposition 10.1.** Let \( V, W \in C^1(\Omega \setminus \Sigma) \cap L^1_{\text{loc}}(\Omega) \) be a Bessel pair over \( \Omega \setminus \Sigma \) where \( \Sigma \subset \subset \Omega \). If \( \text{cap}_V(\Sigma, \Omega) = 0 \), then \((V,W)\) is a Bessel pair in \( \Omega \).
Proof. Given $\delta > 0$, let $\Omega_\delta = \{ x \in \Omega, d(x, \Sigma) < \delta \}$, then $\Omega_\delta \subset \Omega$. As $\text{cap}_{V, \Omega_\delta} (\Sigma) = 0$, there exists a sequence $(\eta_k) \in C^1_c (\Omega_\delta)$ such that

$$\Sigma \subset \subset \{ \eta_k = 1 \}, \quad \lim_{k \to \infty} \int_{\Omega_\delta} V \| \nabla_g \eta_k \|^2 dg = 0.$$ 

For $u \in C^1_c (\Omega)$, $u \zeta_k \in C^1_c (\Omega \setminus \Sigma)$ with $\zeta_k = 1 - \eta_k$. Therefore

$$\int_{\Omega \setminus \Sigma} V \| \nabla_g (u \zeta_k) \|^2 dg \geq \int_{\Omega \setminus \Sigma} W (u \zeta_k)^2 dg. \quad (10.1)$$

Moreover, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\int_{\Omega \setminus \Sigma} V \| \nabla_g (u \zeta_k) \|^2 dg \leq (1 + \epsilon) \int_\Omega V \| \nabla_g u \|^2 dg + C_\epsilon \int_\Omega V u^2 \| \nabla_g \zeta_k \|^2 dg$$

$$\leq (1 + \epsilon) \int_\Omega V \| \nabla_g u \|^2 dg + C_\epsilon \| u \|^2 \int_\Omega V \| \nabla_g \zeta_k \|^2 dg.$$

Taking $k \to \infty$, there holds

$$(1 + \epsilon) \int_\Omega V \| \nabla_g u \|^2 dg \geq \int_{\Omega \setminus \Sigma} W (u \zeta_k)^2 dg \geq \int_{\Omega \setminus \Omega_\delta} W u^2 dg.$$ 

On the other hand,

$$\lim_{\delta \to 0^+} \int_{\Omega \setminus \Omega_\delta} W u^2 dg = \int_{\Omega \setminus \Sigma} W u^2 dg = \int_\Omega W u^2 dg.$$ 

Combining the above estimates and let $\delta \to 0^+$,

$$(1 + \epsilon) \int_\Omega V \| \nabla_g u \|^2 dg \geq \int_\Omega W u^2 dg.$$ 

Taking $\epsilon \to 0^+$, we are done. \qed

10.2 $L^p$ Hardy inequalities

Until now, we concentrate our discussion for Hardy inequalities in $L^2$ setting, here we point out that the idea to use equalities works also for general $L^p$ setting with $p > 1$. 

Let \( u \in C^1_c(\Omega) \), \( p > 1 \), \( V \in C^1(\Omega) \) be a nonnegative weight. Consider \( \vec{F} \in C^1(\Omega,TgM) \) and a family of inner product \( \langle \cdot , \cdot \rangle \in C^1(\Omega,\Lambda^2TgM) \),

\[
\int_{\Omega} V \| \nabla_g u \|^p dg = -\int_{\Omega} V \left[ (p-1)\|\vec{F}\|^p|u|^p + \|\vec{F}\|^{p-2}\langle \vec{F}, \nabla_g (|u|^p) \rangle \right] dg + \int_{\Omega} VR(u,\vec{F})dg
\]

\[
= \int_{\Omega} \left[ \text{div}_g (V\|\vec{F}\|^{p-2}\vec{F}) - (p-1)V\|\vec{F}\|^{p} \right] |u|^p dg + \int_{\Omega} VR(u,\vec{F})dg,
\]

(10.2)

where

\[
\mathcal{R}(u,\vec{F}) = (p-1)\|\vec{F}\|^p|u|^p + \|\vec{F}\|^{p-2}\langle \vec{F}, \nabla_g (|u|^p) \rangle + \|\nabla_g u\|^p
\]

\[
= (p-1)\|\vec{F}\|^p|u|^p + \|\nabla_g u\|^p + p\|\vec{F}\|^{p-2}|u|^p\|\vec{F}\|^{p-2}\langle \vec{F}, \nabla_g u \rangle \geq 0.
\]

The last estimate is given by the Cauchy-Schwarz inequality for inner product and Young’s inequality. Therefore, we obtain a \( L^p \)-Hardy inequality

\[
\int_{\Omega} V \| \nabla_g u \|^p dg \geq \int_{\Omega} W |u|^p dg
\]

(10.3)

with

\[
W = \text{div}_g (V\|\vec{F}\|^{p-2}\vec{F}) - (p-1)V\|\vec{F}\|^{p}.
\]

In particular, if \( \vec{F} = -\nabla_g f \) with \( f > 0 \), there holds

\[
W = -\frac{\text{div}_g (V\|\nabla_g f\|^{p-2}\nabla_g f)}{f^{p-1}} = -\frac{L_{V,p}f}{f^{p-1}}.
\]

(10.4)

The following examples are direct consequence of the estimate (10.3), hence somehow of the equality (10.2).

- Let \( V = 1 \), \( p > 1 \) and \( f = |x|^{1-\frac{p}{n}} \) in \( \mathbb{R}^n \), we have

\[
-\frac{\text{div}(|\nabla f|^{p-2}\nabla f)}{f^{p-1}} = \left| \frac{n-p}{p} \right| |x|^{-p} \quad \text{in} \; \mathbb{R}^n \backslash \{0\},
\]

which means that

\[
\int_{\mathbb{R}^n} |\nabla u|^p dx \geq \left| \frac{n-p}{p} \right| \int_{\mathbb{R}^n} \frac{u^p}{|x|^p} dx, \; \forall u \in C^1_c(\mathbb{R}^n \backslash \{0\}).
\]

This is the well-known \( L^p \)-Hardy inequality, which generalizes (1.1) to any dimension.
Consider \( \langle \vec{v}, \vec{w} \rangle_A = \langle A(x)\vec{v}, \vec{w} \rangle \) in \( \mathbb{R}^n \) with a smooth family of positive definite symmetric matrix \( A(x) \) and \( V = 1 \). Now

\[
L_{V,p}f = \text{div} \left( A(x)|\nabla f|^p \nabla f \right)^{-1} =: \mathcal{L}_{A,p}f.
\]

Take \( f = E^{1-\frac{1}{p}} \) with a positive function \( E \) and \( -\mathcal{L}_{A,p}E = \mu \), we derive from (10.3)-(10.4) that for any \( u \in W_{0}^{1,p}(\Omega), p > 1, \)

\[
\int_{\Omega} |\nabla u|_A^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{\nabla E|E_A^p|u|^p dx}{E^p} + \left( \frac{p-1}{p} \right)^{p-1} \int_{\Omega} \frac{|u|^p dx}{E^{p-1}} d\mu,
\]

which is the inequality (9.34) in [22]. Moreover, if \( \mu \) is a nonnegative finite measure on \( \Omega \), we get

\[
\int_{\Omega} |\nabla u|_A^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{\nabla E|E_A^p|u|^p dx}{E^p}, \quad \forall u \in C_c^1(\Omega),
\]

which is given in [1].

Let \( \mathcal{H}_n \) be the Heisenberg group. Here all the notations are that in Section 8. Let \( V = r^{\beta-p}\rho^{2p-\alpha} \) with \( \alpha, \beta \in \mathbb{R}, p > 1, n \geq 1 \) and \( f = \rho^b \). As \( |\nabla_{\mathcal{H}_n}\rho| = \frac{r}{\rho} \) for \( \rho > 0 \), there holds, in \( \mathcal{H}_n \setminus \{0\}, \)

\[
-\text{div}_{\mathcal{H}_n}(V|\nabla_{\mathcal{H}_n}f|^{p-2}\nabla_{\mathcal{H}_n}f) = -|b|^{p-2}b [2n+2+b(p-1)+\beta-\alpha] \frac{r^b}{\rho^\alpha}.
\]

Choosing \( b = -\frac{2n+2+\beta-\alpha}{p} \), we arrive at

\[
\int_{\mathcal{H}_n} r^{\beta-p}\rho^{2p-\alpha} |\nabla_{\mathcal{H}_n}u|^p dx \geq \left( \frac{2n+2+\beta-\alpha}{p} \right)^p \int_{\mathcal{H}_n} r^b \rho^a u^p dx
\]

for any \( u \in C_c^1(\mathcal{H}_n \setminus \{0\}) \). This enables us [14, Theorem 3.2].

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References

[1] Adimurthi and A. Sekar, *Role of the fundamental solution in Hardy-Sobolev-type inequalities*, Proc. Roy. Soc. Edinburgh Sect. A 136(6) (2006), 1111–1130.

[2] A. A. Balinsky, W. D. Evans and R. T. Lewis, *The Analysis and Geometry of Hardy’s Inequality*, Springer, 2015.

[3] E. Berchio, D. Ganguly and G. Grillo, *Sharp Poincaré-Hardy and Poincaré-Rellich inequalities on the hyperbolic space*, J. Funct. Anal. 272 (2017), 1661–1703.

[4] E. Berchio, D. Ganguly, G. Grillo and Y. Pinchover, *An optimal improvement for the Hardy inequality on the hyperbolic space and related manifolds*, Proc. Roy. Soc. Edinburgh Sect. A 150 (2020), 1699–1736.

[5] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, *Hardy-Poincaré inequalities and applications to nonlinear diffusions*, C. R. Math. Acad. Sci. Paris 344 (2007), 431–436.

[6] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, *Asymptotics of the fast diffusion equation via entropy estimates*, Arch. Rational Mech. Anal. 191 (2009), 347–385.

[7] M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proc. Natl. Acad. Sci. USA 107 (2010), 16459–16464.

[8] R. Bosi, J. Dolbeault and M. J. Esteban, *Estimates for the optimal constants in multipolar Hardy inequalities for Schrödinger and Dirac operators*, Commun. Pure Appl. Anal. 7 (2008), 533–562.

[9] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid. 10 (1997), 443–469.

[10] F. Brock, F. Chiacchio and A. Mercaldo, *Weighted isoperimetric inequalities in cones and applications*, Nonlinear Anal. 75 (2012), 5737–5755.

[11] G. Carron, *Inégalités de Hardy sur les variétés riemanniennes non-compactes*, J. Math. Pures Appl. 76 (1997), 883–891.

[12] H. Chen, X. C. Liu and Y. W. Wei, *Dirichlet problem for semilinear edge-degenerate elliptic equations with singular potential term*, J. Differential Equations 252 (2012), 4289–4314.

[13] C. Cazacu and E. Zuazua, *Improved multipolar Hardy inequalities, in: Studies in phase space analysis with applications to PDEs*, PNDLDE, Vol. 84, Birkhäuser/Springer, (2013), 35–52.

[14] L. D’Ambrosio, *Some Hardy inequalities on the Heisenberg group*, Differential Equations 40(4) (2004), 552–564.

[15] L. D’Ambrosio, *Hardy inequalities related to Grushin type operators*, Proc. Amer. Math. Soc. 132 (2004), 725–734.

[16] J. Dolbeault and B. Volzone, *Improved Poincaré inequalities*, Nonlinear Anal. 75 (2012), 5985–6001.
[17] M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation*, Nonlinear Anal. 11 (1987), 1103–1133.

[18] V. Felli, E. Marchini and S. Terracini, *On Schrödinger operators with multipolar inverse-square potentials*, J. Funct. Anal. 250 (2007), 265–316.

[19] S. Filippas, A. Tertikas and J. Tidblom, *On the structure of Hardy-Sobolev-Maz’ya inequalities*, J. Eur. Math. Soc. 11 (2009), 1165–1185.

[20] J. Flynn, N. Lam, G. Z. Lu and S. Mazumdar, *Hardy’s identities and inequalities on Cartan Hadamard manifolds*, arXiv:2103.12788v1.

[21] N. Ghoussoub and A. Moradifam, *Bessel pairs and optimal Hardy and Hardy-Rellich inequalities*, Math. Ann. 349 (2011), 1–57.

[22] N. Ghoussoub and A. Moradifam, *Functional inequalities: new perspectives and new applications*, Mathematical Surveys and Monographs 187, AMS, 2013.

[23] G. H. Hardy, *Note on a theorem of Hilbert*, Math Z. 6 (1920), 314–317.

[24] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1952.

[25] J. Leray, *Sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Math. 63 (1934), 193–248.

[26] J. W. Luan and Q. H. Yang, *A Hardy type inequality in the half-space on \( \mathbb{R}^n \) and Heisenberg group*, J. Math. Anal. Appl. 347 (2008), 645–651.

[27] V. Maz’ya, *On a degenerating problem with directional derivative*, Math. USSR Sb. 16(3) (1972), 429–469.

[28] V. Maz‘ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften 342, Springer, 2011.

[29] V. Maz‘ya and V. Shaposhnikova, *A collection of sharp dilation invariant integral inequalities for differentiable functions*, in: Sobolev spaces in mathematics. I, International Mathematical Series Vol. 8, Springer, (2009), 223–247.

[30] V. E. Nazaikinskii, A. Y. Savin, B. W. Schulze and B. Y. Sternin, *Elliptic Theory on Singular Manifolds*, in: *Differential and Integral Equations and Their Applications*, CRC Press, 2006.

[31] B. Opic and A. Kufner, *Hardy-type Inequalities*, in: *Pitman Research Notes in Mathematics Series* 219, Longman Scientific & Technical, 1990.

[32] M. Ruzhansky and D. Suragan, *Hardy Inequalities on Homogeneous Groups*, in: *Progress in Mathematics* 327, Birkhäuser/Springer, 2019.

[33] A. Tertikas and N. B. Zographopoulos, *Best constants in the Hardy-Rellich inequalities and related improvements*, Adv. Math. 209 (2007), 407–459.

[34] J. Tidblom, *A Hardy inequality in the half-space*, J. Funct. Anal. 221 (2005), 482–495.

[35] J. L. Vázquez and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. 173 (2000), 103–153.

[36] Z. Q. Wang and M. Willem, *Caffarelli-Kohn-Nirenberg inequalities with remainder terms*, J. Funct. Anal. 203 (2003), 550–568.