Approximation Schemes for Capacitated Clustering in Doubling Metrics

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Abstract

We consider the classic uniform capacitated $k$-median and uniform capacitated $k$-means problems in bounded doubling metrics.

We provide the first QPTAS for both problems and the first PTAS for the $k$-median version for points in $\mathbb{R}^2$.

This is the first improvement over the bicriteria QPTAS for capacitated $k$-median in low-dimensional Euclidean space of Arora, Raghavan, Rao [STOC 1998] (1 + $\varepsilon$-approximation, 1 + $\varepsilon$-capacity violation) and arguably the first polynomial-time approximation algorithm for a non-trivial metric. Our result relies on a new structural proposition that applies to any metric space and that may be of interest for developing approximation algorithms for the problem in other metric spaces, such as for example planar or minor-free metrics.

1 Introduction

The capacitated $k$-median and $k$-means problems are infamous problems: no constant factor approximation is known for any non-trivial metric, even when the capacities are uniform. Given a set of points $C$ in a metric space together with an integer $\eta$, the capacitated clustering problem asks for a set $C$ of $k$ points, called centers, together an assignment $\mu : C \mapsto C$ that assigns at most $\eta$ clients to any cluster and such that the sum of the $p$th power of the distance from each point to the center it is assigned to is minimized (see a more formal definition in Section 1.1). When $p = 1$, this is known as the capacitated $k$-median problem with uniform capacities, while the case $p = 2$ is the capacitated $k$-means problem with uniform capacities.

The best known algorithm is folklore and is an $O(\log k)$-approximation arising from Bartal’s embedding into trees and a simple dynamic program for solving the problem exactly in time $n^{O(t)}$ in graphs of treewidth at most $t$ (in this case $t = 1$). From a theory perspective, finding a constant factor approximation for the problem in general metric spaces or showing that none exists unless P=NP is an important challenge that has received a lot of attention (see for example the large amount of work on bicriteria approximations or on the facility location version of the problem [7, 28, 29, 50, 17, 6, 9, 8], and the recent work on approximation algorithm with running exp$(k)$ poly$(n)$) [11, 54, 15]). The only known hardness of approximations bounds are constant and are the ones obtained for the uncapacitated $k$-median, see [21] for general metrics and [12] for $\ell_p$-metrics. In fact, the capacitated $k$-median problem has been presented as one of the most fundamental problems for which determining whether $O(1)$-approximation is possible is still an open problem [1]. This hardness seems to extend to any non-trivial metric (bounded treewidth graphs excepted) since no constant factor approximation when the input consists for example of point in $\mathbb{R}^2$ is known. This stands in sharp contrast with the uncapacitated variant of the problem for which approximation schemes are known.
Thus, since the breakthrough of Arora et al. on clustering problems in low-dimensional Euclidean space, it has remained an important open problem to obtain at least a constant factor approximation for capacitated clustering problems even in \( \mathbb{R}^2 \). Since their work, the community has developed two main techniques for obtaining approximation schemes for clustering problems in metrics of fixed doubling dimension or low-dimensional Euclidean space: the approach of Kolliopoulos and Rao and the local search algorithm. Unfortunately, the approach of Kolliopoulos and Rao requires to reassign clients among the optimal set of centers and so, cannot be adapted to the case where centers have capacities (as also pointed out by Sariel Har-Peled in the comments of a StackExchange discussion). Furthermore, it is easy to come up with a set of points in \( \mathbb{R}^2 \) where local search approach may have an arbitrarily bad approximation ratio. Thus, the best algorithm for the problem in \( \mathbb{R}^2 \) is the 20-year old bicriteria QPTAS of Arora et al. (see again the discussion at [23]).

Applications in \( \mathbb{R}^2 \) and \( \mathbb{R}^{O(1)} \). The capacitated clustering problem has received a lot of attention through the years. The study of the problem in low-dimensional metric stem from prepositioning resources and redistricting. For example, consider the problem of positioning emergency supplies to support disaster relief by dispatching a certain number of emergency items, such as medicine, food, centers on a map. There has been a large body of work in this direction and it has been argued that the capacity constraint of the centers is often a hard constraint, in particular when it comes to medical centers where the cost of service increases very quickly when the capacity is violated [18] see also [24, 32, 25, 4] for more discussions on hard capacities for prepositioning emergency resources and hard capacity constraints in plant locations.

A slightly less dramatic motivation also comes from bike-sharing systems: the goal is place bike-stations (i.e.: facilities) so as to cover a certain demand of people (users) so that, at the end of its trip, a user can find a spot at a station to leave its bike: the number of spots at a given station is then a hard constraint.

The redistricting problem is the problem of dividing a region into a number of electoral districts under some hard constraints, often coming from the constitution of the country. The first of the hard constraints is the number of districts, which is our number of clusters \( k \), and which is, in many country like France or the US, fixed by law for a given region. Thus, computing a redistricting into \((1+\varepsilon)k\) districts is not an option. The second of the hard constraints is the size of the districts. In the US for example, even though the Supreme Court has declined to name a specific percentage limit on how much populations of districts can differ, we observe from [20] p. 499 that “a 2002 Pennsylvania redistricting plan was struck down because one district had... 19 more people than another”. It follows that since \( \eta \) is a few thousands for these instances, having a capacity violation
of \((1 + \varepsilon)\eta\) would not be satisfactory, unless \(\varepsilon\) could be made very tiny. For example, some states accept a 1.5\% difference in size, but to obtain such a small imbalance in sizes, current techniques would require a running time of \(n^{60}\) at best, while producing a poor approximate solution.

Finally, as shown experimentally by [20], the \(k\)-means or \(k\)-median objectives are suitable objective functions for evaluating the quality of a solution (or what is referred to as its compactness, see also [27]). The idea behind this being that, for the Euclidean plane, for a given set of centers, the best assignment of clients to centers under the capacity constraint can be phrased as a Voronoi diagram in \(\mathbb{R}^3\) [14]. Hence, the districts are convex, an appealing property. In most of these works, assuming that the input points are points in \(\mathbb{R}^2\) is fairly standard assumption (see [20, 14] and references therein).

Therefore, designing good approximation algorithms for the capacitated \(k\)-median and \(k\)-means problems in \(\mathbb{R}^2\) and more generally in metric spaces of fixed doubling dimension has become an important challenge.

1.1 Our Results

We give the first PTAS for \(k\)-median with uniform capacities in \(\mathbb{R}^2\) and the first QPTAS for \(k\)-median, \(k\)-means with uniform capacities in metrics of doubling dimension. The problem at hand is the following

**Definition 1.** Let \(\mathcal{X} = (X, \text{dist})\) be a metric space. Given a set of clients \(\mathcal{C} \subseteq X\) in a metric space, a set of candidate centers \(\mathcal{A} \subseteq X\), a capacity \(\eta\), and \(p \geq 1\), the capacitated \(k\)-clustering problem asks for set \(C \subseteq \mathcal{A}\) of size at most \(k\) and an assignment \(\mu: \mathcal{C} \rightarrow C\) such that

- for any \(f \in C\), \(\{|c | c \in \mathcal{C} \text{ and } \mu(c) = f\}| \leq \eta\), and
- \(\sum_{c \in \mathcal{C}} \text{dist}(c, \mu(c))^p\) is minimized.

Given a solution \((C, \mu)\), we refer to the candidate centers in \(C\) as centers or facilities. We say that \(f \in C\) serves a client \(c\) if \(\mu(c) = f\).

Our result in \(\mathbb{R}^2\) is as follows. It holds for any \(\ell_p\)-metric in \(\mathbb{R}^2\), where \(p = O(1)\).

**Theorem 1.1.** There exists an algorithm that given an instance of size \(n\) of the capacitated \(k\)-median problem in \(\mathbb{R}^2\) (capacitated clustering problem with \(p = 1\)) outputs a \((1 + \varepsilon)\)-approximate solution in time \(n^{1/\varepsilon O(1)}\).

Our result extends to metric of bounded doubling dimension. The doubling dimension of a metric is the smallest integer \(d\) such that any ball of radius \(2r\) can be covered by \(2^d\) balls of radius \(r\). The result is as follows.

**Theorem 1.2.** There exists an algorithm that given an instance of size \(n\) of the capacitated \(k\)-clustering problem with parameter \(p\) in a metric space of doubling dimension \(d\) outputs a \((1 + \varepsilon)\)-approximate solution in time \(n^{((\frac{p}{d})^p \log n)^{O(d)}}\).

1.2 Techniques

Our main technical contribution and the meat of the paper is Proposition 4.1 which, interestingly, holds in any metric space and so could perhaps be of use for solving the open problem of getting an \(O(1)\)-approximation for the problem in general metric spaces, or maybe more likely to obtain similar in other contexts like for planar graph inputs. We first provide some intuition on how we use Proposition 4.1 before describing what the proposition gives us.
To understand better our contribution, we quickly review the classic quad-tree dissection and split-tree decomposition techniques of Arora [3] and Talwar [33]. The general ideas behind the decomposition consists in recursively partitioning the input into regions and forcing the optimal solution to connect points in different regions through a set of portals of size say $\rho^d$. By doing so, one obtains small-size interfaces between regions that enables dynamic programming techniques.

Concretely, the technique ensures that for a client in a given region $R$ that is assigned to a center outside the region, the detour paid to connect the client to its center through the set of portals is $1/\rho$ times the diameter of $R$. The crux of the analysis is to show that the probability that a client $u$ and a facility $v$ at distance $\delta$ are in different clusters of diameter $D$ is roughly $\delta/D$. It follows that the expected detour becomes $(D/\rho) \cdot \delta/D$ and so at most $1/\rho$ times the distance between $u$ and $v$ in the original metric. Since this is the original cost of serving $u$ by $v$, the cost of the optimal solution that is forced to go through the portal is at most $(1 + 1/\rho)$ higher than the optimal solution that does not have to satisfy this constraint. This works fine when the distance is equal to the cost (namely when $p = 1$).

However, for $p = 2$ or larger the expected cost of the detour now becomes $(D/\rho)^2 \cdot \delta/D = \delta D/\rho^2$. On the other hand, the original cost of serving $u$ by $v$ is $d^2$ and so, when $D = \omega(d)$, the detour incurred by going to the closest portal may be too expensive. This is one of the reasons why no PTAS was known for the uncapacitated $k$-means problem until the work of [13, 19]. Unfortunately, the algorithm of [13, 19] is local search and it is easy to come-up with an instance where local search can have arbitrarily bad approximation ratio for the capacitated version of the problem.

Our technique circumvents the above problem as follows. Observe that in the above discussion, for any client $u$ and the facility $f$ that serves $u$ in the optimal solution, if the regions that contain $u$ and do not contain $f$ have diameter at most $(\log n) \text{dist}(u, f)/\varepsilon$, then one can use a portal set of size $\rho = (\log n/\varepsilon)^{O(d)}$ and guarantee that the detour paid is in total at most $\varepsilon \text{dist}(u, f)^2$ which is $\varepsilon$ times the cost paid by $u$ in the solution.

Thus, we only have to worry about pairs of clients and facilities for which the decomposition does not provide such a nice structure. This leads us to say that a facility or a client $p$ is “badly cut” (see formal definition in the next section) which, at the intuitive level goes as follows. A point is “badly cut” if, at some point in the decomposition, there exists a region of diameter $D$ that contains $p$ but that does not contain some point that is at distance $D/\text{poly log } n$ from $p$. In other words: $p$ is very close (relatively to $D$) to the boundary of the region of diameter $D$.

As we argued before, for any point $p$ that is not badly cut, we are in good shape, we can afford to connect $p$ to the facility that serves it OPT through the portal. The question is what to do with badly cut points. First, we will show that a point is badly cut only with a tiny probability, say $\varepsilon$ for a constant $\varepsilon$. For example, if we were interested in a solution opening up to $(1 + O(\varepsilon))k$ centers, we would almost be done: each facility of OPT has probability at most $\varepsilon$ of being badly cut so we would have at most $\varepsilon k$ badly cut facilities in expectation. Therefore, we could simply decide to consider a solution opening $2^{O(d)}$ facilities instead of one for each badly cut facility: for the region where the facility is badly cut, open one facility on each child region. Then we would be guaranteed that no client is separated from its facility at a too high level. We could then use dynamic programming to find such a solution.

However, our goal is to satisfy all constraints: at most $k$ facilities open and at most $\eta$ points assigned to each facility. To handle this, we use our main result, Proposition [11] which helps us deal with the badly cut clients and facilities and prove that there exists a near-optimal solution that can be found through a dynamic program.

The approach is as follows, we compute a $\gamma$-approximate solution $L$ to the problem. For any

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1One has to guarantee that this holds recursively, so the argument would be slightly more involved
point $p$ that is badly cut, we move $p$ to the location of the center serving $p$ in solution $L$. This yields a new instance where any solution to the new instance can be lifted back to the original instance by paying an extra $O(\varepsilon \text{cost}(L))$ in expectation. We then need to argue that one can find a $(1 + \varepsilon)$-approximation in the new instance. To do so, we show that there exists a near-optimal solution that is well-behaved in the new instance: we show show that there exists a solution of cost at most $\text{cost}(\text{OPT}) + O(\varepsilon \text{cost}(L))$ that contains each badly cut facility of $L$. At first, this may seem unrealistic since we want to end up with a $(1 + \varepsilon)$-approximation while still opening at most $k$ centers and preserving the capacity constraints. This is where Proposition 4.1 comes into place.

**Main Result: Proposition 4.1** As we have described the major ingredient is Proposition 4.1. Loosely speaking, Proposition 4.1 states that given a solution $(C, \mu)$ of cost $X$ and a random process which picks each center of $C$ with probability $\varepsilon^2$, then with probability at least $1 - \varepsilon$, there exists a solution which contains the selected centers and that: (1) meets the capacity constraints (2) has at most $k$ centers, and (3) that is of cost at most $\text{cost}(\text{OPT}) + O(\varepsilon X)$.

The result is obtained by designing a careful rerouting scheme of the clients, involving min-cost max-flow techniques. While the result for uncapacitated version of the problem can be obtained through a simple lemma; obtaining the same bound for the capacitated case is much more challenging.

We then make use of the proposition as follows. This provides us with a very good instance where (1) clients that are badly cut are moved to the facility that serves them in $L$ and (2) badly cut facilities of $L$ are now part of the solution we are trying to compute. This is enough to conclude: Consider a client $c$. If it is not badly cut, then we don’t have to worry about paying the detour through portals. If it is badly cut, then it is now located at the center that serves it in $L$. Moreover, if this center is badly cut then it is open and so the service cost for this client is 0. If this center is not badly cut, then one can afford to make the detour to connect the client to its closest facility through the portals. Making this reasoning rigorous is a bit challenging and shown in the next sections.

A few more details still have to be addressed. Another problem we have to solve for making the entire approach work is the following. Note that the solution we obtain has cost at most $(1 + \varepsilon)\text{cost}(\text{OPT}) + \varepsilon(\text{cost}(L))$ with probability at least $1 - \varepsilon$. This probability can be boosted and we indeed boost it to $1 - \varepsilon/\log \log n$ by repeating $\log n$ times. This is critical since as discussed in the intro there is no $O(1)$-approximation algorithm and so, the solution computed has cost at most $(1 + \varepsilon)\text{OPT} + \varepsilon \text{cost}(L)$ which is $\varepsilon \log n \cdot \text{cost}(\text{OPT})$. However, this is enough to allow us to bootstrap: we use the solution obtained to get a solution of cost at most $\varepsilon^2 \log n \cdot \text{cost}(\text{OPT})$. By repeating this process $\log \log n$ times, we finally obtain a near-optimal solution.

**Organization of the Paper**

We provide definitions and preliminaries in the remainder of this section. Our structural result, Proposition 4.1 is presented in Section 4. To motivate the proposition, we first show how it is used in Section 3 (Lemma 3.2). From there a simple QPTAS follows, see Section 5 and a more involved PTAS is presented in Section 6.
2 Preliminaries

Throughout the following sections, let $\varepsilon > 0$. Moreover, we will assume that $k\eta \geq n$ since otherwise, the problem has no solution. The following observation and preprocessing step are folklore. Given a set of centers $C$, the assignment $\mu$ minimizing the cost can be computed using a min cost flow algorithm by defining a sink with capacity $\eta$ for each center of $C$, placing a demand of 1 on each client, and for each client $c$ and center $f \in C$, defining an edge $(c, f)$ with capacity one and cost $\dist(c, f)^p$. Thus, given a set of centers the best assignment can always be computed in polynomial time.

The following lemma will be useful to derive our results when $p > 1$.

Lemma 2.1 (E.g.: [16]). Given a metric space $(X, \dist)$, and $p \geq 0$ and $1/2 > \varepsilon > 0$, we have for any $a, b, c \in X$, we have $\dist(a, b)^p \leq (1 + \varepsilon)^p \dist(a, c)^p + \dist(c, b)^p (1 + 1/\varepsilon)^p$.

2.1 Doubling Metric Spaces and Decompositions

Without loss of generality, we can assume that the aspect-ratio of the input, namely the ratio of the maximum distance between pairs of points in $A \cup C$ to the minimum distance between pairs of points in $A \cup C$, is at most $O(n^4)$. Indeed, consider the following preprocessing step: compute an $O(\log n)$-approximation and let $v$ be the cost of the solution computed. Then, while there is a pair of point $x, y$ that are at distance less than $\varepsilon v/n^3$, remove $x$ and add a client at $y$ (note that the two clients at $y$ may not necessarily be assigned to the same facility in a solution). In the instance obtained at the end of this process, a point is at distance at most $\varepsilon v/n^3$ from its original location, and so any solution for this instance can be converted back to the original instance with a cost increase of at most $\varepsilon v/n^2 \leq \varepsilon \text{cost}(\text{OPT})$. Finally, also note that, up to dividing all distance by the minimum distance between any pair of points of the input, we can assume that the minimum distance is 1 and so the diameter of the pointset, namely the maximum distance between any pair of points, is equal to the aspect-ratio.

A $\delta$-net of $X$ is a set of point $Y$ such that $\forall v \in X, \exists x \in Y \mid d(v, x) \leq \delta$ and $\forall x, y \in Y, d(x, y) > \delta$. The cardinality of a net in metrics of doubling dimension $d$ is bounded by the following lemma.

Lemma 2.2 ([22]). Let $(X, \dist)$ by a metric space with doubling dimension $d$ and diameter $\Delta$, and let $Y$ be a $\delta$-net of $X$. Then $|Y| \leq 2^{d[\log(\Delta/\delta)]}$

We define the rings centered at a point $c$ as follows: the $i$th ring centered at $c \in A \cup C$ is the set of all points at distance $[2^i, 2^{i+1}]$ from $c$. The rings of $c$ is the collection of all the rings centered at $c$ which contains at least one point of the input. The following fact follows from the definition and having aspect-ratio bounded by $O(n^4/\varepsilon)$.

Fact 1. The number of rings for any point $c$ is at most $O(\log(n/\varepsilon))$.

We use the randomized split-trees of Talwar [33] for doubling metrics and the randomized dissection of Arora [2] for the plane. Since random split-trees are standard tools, we point to [33] for a more detailed introduction. We use the exact same definition as [33], with a slight change in notation; We avoid talking about clusters but use the name boxes instead. A decomposition of the metric $X$ is a partitioning of $X$ into subsets, which we call boxes. A hierarchical decomposition is a sequence of decompositions $P_0, P_1, \ldots, P_\ell$ such that every $P_i$ is a refinement of $P_{i+1}$, namely each box of $P_i$ is a subset of a box of $P_{i+1}$. The boxes of $P_i$ are the level-$i$ boxes. A split-tree decomposition will be one where $P_{\ell} = \{X\}$ and $P_0 = \{\{x\} \mid x \in X\}$.

For any point $p$ and $x > 0$, we say that the ball $B(p, x)$ is cut at level $i$, if there are $P_1, P_2 \in P_i$, and $P_3 \in P_{i+1}$ such that $P_1 \cap B(p, x) \neq \emptyset$ and $P_2 \cap B(p, x) \neq \emptyset$ and $B(p, x) \subseteq P_3 \in P_{i+1}$.
We obtain, a decomposition achieving the following properties (see [33]):

1. The total number of levels \( \ell \) is \( O(\log n) \) (since the aspect ratio of the input metric is \( O(n^4/\varepsilon) \), and \( \varepsilon \) is constant).

2. Each box of level \( i \) has diameter at most \( 2^{i+1} \), namely the maximum pairwise distances between points in a box of level \( i \) is at most \( 2^{i+1} \).

3. Each box of level \( i \) is the union of at most \( 2^{O(d)} \) level \( i - 1 \) boxes.

4. For any point \( u, x > 0 \) and level \( i \), the probability that the ball \( B(u, x) \) is cut at level \( i \) is \( O(d \cdot x/2^i) \).

Condition 4 is a direct corollary of the definition of the decomposition and not stated precisely like this in [33] but is fairly standard, see e.g., [11] for a proof.

For any point \( c \), for any ring \( j \) of \( c \), we say that it suffers a bad cut if the ball \( B(c, 2^j) \) is cut at a level \( i \) higher than \( \log(d(\log n/(\varepsilon/(p+1))^{6p})) + j \), namely \( 2^i > d(\log n/(\varepsilon/(p+1))^{6p})2^j \). We have:

**Lemma 2.3.** For any point \( p \), the probability that a ring \( j \) centered at \( p \) suffers a bad cut is at most \( O((\varepsilon/(p+1))^{6p}/\log n) \).

**Proof.** By Condition 4, the probability to be cut at level \( \log(d(\log n/(\varepsilon/(p+1))^{6p})) + j + i \) is \( O((\varepsilon/(p+1))^{6p}/(2^i \log n)) \). Then, taking a union bound over all levels higher than \( \log(d(\log n/(\varepsilon/(p+1))^{6p})) + j \): we have that the total probability of suffering a bad cut is at most \( O((\varepsilon/(p+1))^{6p}/\log n) \sum_{i=1}^{O(\log n)} 1/2^i \) and so at most \( O((\varepsilon/(p+1))^{6p}/\log n) \). \( \square \)

## 3 Decompositions of Clustering Instances

The goal of this section is to show how to use split-tree decompositions and our structural result (Proposition 4.1) so as to use a dynamic program to solve capacitated clustering instances.

Consider a metric space \((X, \text{dist})\) together with an instance \( C, A, \eta, p \) of the capacitated \( k \)-clustering problem on \((X, \text{dist})\). Assume that \( X \) has been preprocessed so as to have an aspect-ratio of at most \( O(n^4/\varepsilon) \) as described in the previous section and by removing points of \( X \) that are not in \( C \cup A \). Let \( D \) be a split-tree decomposition of \((X, \text{dist})\). Let \( L \) be a solution to the problem and \( OPT \) denote an optimal solution. For each point \( c \in L \cup OPT \cup C \), we say that \( c \) is badly cut if at least one of its rings suffers a bad cut.

**Lemma 3.1.** For a given point \( c \in L \cup OPT \cup C \), the probability that \( c \) is badly cut is \( O((\varepsilon/(p+1))^{6p}) \).

**Proof.** By Lemma 2.3 each ring suffers a bad cut with probability \( O((\varepsilon/(p+1))^{6p}/\log n) \). Thus, since the number of rings is \( O(\log(n/\varepsilon)) \), taking a union bound on the probability of each ring suffering a bad cut implies the lemma. \( \square \)

Given a solution \((L, \mu_L)\) to \( C, A, \eta, p \) and a random decomposition \( D \) of \((X, \text{dist})\), we define a new instance of the \( k \)-clustering problem in metric \((X, \text{dist})\) as follows. Any client \( c \) that is badly cut is “moved” to the facility \( L(c) \) that serves it in solution \( L \). Namely, given the set of badly cut clients \( B_P \), the new instance is defined by the following tuple \( C - B_D \cup \{L(c) \mid c \in B_P\}, A, \eta, p \), where \( C - B_D \cup \{L(c) \mid c \in B_P\} \) is a multiset. We identify clients of the new instance with their counterpart in the original instance so that an assignment \( \mu_0 \) of clients in the original instance can
be translated naturally into an assignment $\mu'_0$ of clients in the new instance: each client $c$ of the new instance is assigned to the same center than its counterpart in the original instance.

We say that a solution for the new instance, namely a set of centers $S$ and a mapping $\mu$, is valid if it contains all the badly cut facilities of $L$ and that all the clients served by a badly cut facility $\ell$ in $L$ are mapped to $\ell$ in $\mu$.

For a given instance of the $k$-clustering problem, the instance that is defined as described above is a function of $D$, we thus refer to this instance by $I_{D,L}$. Note that the set of candidate centers in $I_{D,L}$ is identical to the set of candidate centers in the original instance.

Therefore, for a given set of $k$ centers $S\subset A$ and an assignment of clients to centers $\mu$, we denote by $\text{cost}(S,\mu)$ the cost of solution $(S,\mu)$ in the original instance and by $\text{cost}_{I_{D,L}}(S,\mu)$ the cost of solution $S$ in instance $I_{D,L}$. For any set of centers $S$, we let the optimal assignment of clients to $S$ be $\mu_S$.

We let

$$\nu_{I_{D,L}} = \max_{|S|\leq k} \max_{S\subseteq A} \left( \text{cost}(S,\mu_S) - (1+3\varepsilon)\text{cost}_{I_{D,L}}(S,\mu_S), \right)$$

$$(1-3\varepsilon)\text{cost}_{I_{D,L}}(S,\mu_S) - \text{cost}(S,\mu_S)).$$

This can be seen as how much a solution is “distorted” in the instance $I_{D,L}$. We say that an instance $I_{D,L}$ is good if the following conditions hold:

- $\nu_{I_{D,L}} = O(\varepsilon \cdot \text{cost}(L,\mu_L))$
- There exists a valid solution $\hat{G},\mu$ such that $\text{cost}(\hat{G},\mu) \leq (1+O(\varepsilon))\text{cost(OPT,}\mu_{\text{OPT}}) + O(\varepsilon \text{cost}(L,\mu_L))$.

The next lemma shows that an instance is good with probability at least $1-\varepsilon$. We first step back and provide an informal explanation on how to conclude from there; Assume that the instance is good, then for the clients two things can happen: if the client is not badly cut then it is at its location in the original instance and none of its ring suffers a bad cut. In that case, its assignment in the best valid solution is not distorted by the use of portals. Otherwise, the client is badly cut, and then either the facility $\ell$ it is assigned to in the local solution is also badly cut or not. If $\ell$ is also badly cut then the client is assigned to $\ell$ and so its service cost is 0 (no need of portals). Otherwise its distance to the facility it is assigned to in the best valid solution is not distorted by portals. Then, since the cost of the best valid solution is small, we can make use of dynamic programming and solve instance $I_{D,L}$.

Here we conclude the section by showing that an instance is good with probability $1-\varepsilon$.

**Lemma 3.2.** Given a randomized split-tree decomposition $D$, the probability that $I_{D,L}$ is good is at least $1-\varepsilon$.

**Proof.** We first bound the probability that $\nu_{I_{D,L}} \leq O(\varepsilon \cdot \text{cost}(L,\mu_L))$. By definition, we have that for any solution $S,\mu$, $\text{cost}(S,\mu) - \text{cost}_{I_{D,L}}(S,\mu) \leq \varepsilon$.
\[
\sum_{\text{badly cut client } c} \text{dist}(c, \mu(c))^p - \text{dist}(\mu(c), \mu_L(c))^p \\
\leq \sum_{\text{badly cut client } c} (1 + 3\varepsilon)\text{dist}(\mu(c), \mu_L(c))^p + \\
\frac{\text{dist}(c, \mu_L(c))^p}{(\varepsilon/(p + 1))^p} - \text{dist}(\mu(c), \mu_L(c))^p \\
\leq \sum_{\text{badly cut client } c} 3\varepsilon \cdot \text{dist}(\mu(c), \mu_L(c))^p + \frac{\text{dist}(c, \mu_L(c))^p}{(\varepsilon/(p + 1))^p}
\]

Where we have used Lemma 2.1 to go from the first to second line. Thus,

\[
\text{cost}(S, \mu) - (1 + 3\varepsilon)\text{cost}_{I_D, L}(S, \mu) \leq \sum_{\text{badly cut client } c} \frac{\text{dist}(c, \mu_L(c))^p}{(\varepsilon/(p + 1))^p} \tag{1}
\]

Similarly, we have that \(\text{cost}_{I_D, L}(S, \mu) - \text{cost}(S, \mu) \leq\)

\[
\sum_{\text{badly cut client } c} \text{dist}(\mu(c), \mu_L(c))^p - \text{dist}(c, \mu(c))^p \\
\leq \sum_{\text{badly cut client } c} (1 + 3\varepsilon)\text{dist}(c, \mu(c))^p + \\
\frac{\text{dist}(c, \mu_L(c))^p}{(\varepsilon/(p + 1))^p} - \text{dist}(c, \mu(c))^p \\
\leq \sum_{\text{badly cut client } c} 3\varepsilon \cdot \text{dist}(c, \mu(c))^p + \frac{\text{dist}(c, \mu_L(c))^p}{(\varepsilon/(p + 1))^p}
\]

and so,

\[
(1 - 3\varepsilon)\text{cost}_{I_D, L}(S, \mu) - \text{cost}(S, \mu) \leq \sum_{\text{badly cut client } c} \frac{\text{dist}(c, \mu_L(c))^p}{(\varepsilon/(p + 1))^p} \tag{2}
\]

Now, observe that the right hand side of both Equations 1 and 2 does not depend on \(S\). Therefore, the expected value of \(\nu_{I_{D,L}}\) is

\[
E[\nu_{I_{D,L}}] \leq \sum_c Pr[c \text{ badly cut}] \cdot \frac{\text{dist}(c, \mu_L(c))^p}{(\varepsilon/(p + 1))^p} \leq
\]

\[
O(\varepsilon^5 \text{cost}(L)),
\]

where we have used Lemma 3.1. We then apply Markov’s inequality and obtain that \(I_{D,L}\) satisfies the first condition with probability at least \(1 - \varepsilon/3\). Let \(E_\nu\) be the event that \(I_{D,L}\) satisfies the first condition.

We now show that there exists a valid solution \(\hat{G}, \hat{\mu}\) such that \(\text{cost}(\hat{G}, \hat{\mu}) \leq (1 + O(\varepsilon))\text{cost}(\text{OPT}, \mu_{\text{OPT}}) + O(\varepsilon \cdot \text{cost}(L, \mu_L))\). Consider an optimal solution \(\text{OPT}\) and apply Proposition 4.1 to \(\text{OPT}\) and \(L\). We let \(\hat{L}, \hat{L}, \hat{F}, \hat{F}\) as defined by the proposition.
We let $\mathcal{E}_1$ be the event that there are at most $\varepsilon^3 |\hat{L}|$ facilities of $\hat{L}$ that are badly cut. We have that by Lemma 3.1, the expected number of badly cut facilities in $\hat{L}$ is at most $\varepsilon^3 |\hat{L}|$. Applying Markov’s inequality, we have that $\mathcal{E}_1$ holds with probability at least $1 - \varepsilon/3$. Condition on event $\mathcal{E}_1$ happening. Consider $G^*$ as defined per Property 2 of the proposition. This solution contains $k - \Omega(\varepsilon \cdot |\hat{L}|)$. Thus, let $T$ be the temporary solution defined as $G^*$ plus de badly cut facilities of $\hat{L}$. Since we condition on event $\mathcal{E}_1$ happening, we have that $T$ has at most $k$ centers. Hence, $T$ has at most $k$ centers with probability at least $1 - \varepsilon/3$.

We finally make use Property 3 of the proposition to incorporate the remaining badly cut facilities of $\hat{L}$, i.e., the badly cut facilities of $\hat{L}$. We apply Property 3 to our random procedure for defining badly cut facilities in $\hat{L}$, with probability at least 1 - $\varepsilon/3$. Taking a union bound over the probability that $\mathcal{E}_2$ and $\mathcal{E}_1$ do not happen yields the lemma.

4 Structural Result

We start by providing some intuition on what our main structural aims to achieve. Our goal is to show that there exists a near-optimal solution which contains the badly cut facilities. To achieve this, consider the bipartite graph obtained by having one vertex for each facility of our current solution on one side, one vertex for each facility of the optimal solution on the other side, and an edge from the vertex corresponding to a facility of OPT to the vertex corresponding to the closest facility of our current solution.

Now, what we would like to argue is that if we pick a random facility $f$ of our current solution whose corresponding vertex in the bipartite graph has at least one incoming edge from some facility of our current solution, then we can replace the facility corresponding to vertex $v$ in OPT with $f$ and leave the cost unchanged, up to a factor $(1 + 1/k)$. In the uncapacitated setting, one can formalize this intuition. It then remains to address the issue of facilities that have no incoming edge. Since the numbers of facilities of OPT and our current solution are the same, for each facility that has no incoming edge, there is another facility with more than one incoming edge. This allows to delete some facilities of OPT to make room for facilities of our current solution. See [11] for a complete proof of this argument in the uncapacitated setting.

For capacitated versions of the problem the picture changes drastically since a replacement may incur a large reassignment of clients. In some cases, reassigning the clients of the facility that was deleted to the badly cut facility that replaces it may result in an arbitrarily bad cost. Thus, finding a careful reassignment of the clients is crucial.

We now turn to the formal proof. Let OPT be an optimal solution and $L$ be any solution. Define the charge of a facility $f$ in a solution $S$ to be the total number of client assigned to $f$ in $S$.

**Proposition 4.1.** Let $1/2 > \xi > 0$ be a fixed constant. Let OPT be an optimal solution and $L$ be any solution. Let $P$ be any random process such that each facility of OPT is selected with probability at most $\xi^2$. Then, there exists a partition of OPT into two sets $\hat{F}, \tilde{F}$ and a partition of $L$ into two sets $\hat{L}, \tilde{L}$ such that

1. $|\hat{F}| = |\hat{L}|$, and so $|\tilde{L}| = |\tilde{F}|$.
2. There exists $\hat{F}^* \subseteq \hat{F}$ of size at least $\xi |\hat{F}|/3$ such that the set $G^* = OPT - \hat{F}^*$ is a solution of cost at most $\text{cost(OPT)} + O(\xi (\text{cost(OPT)} + \text{cost(L)}))$.
3. Let $\hat{F}^*$ be the set of facilities selected by the random process $P$. Then, there exists a 1-to-1 mapping $\phi : \hat{F} \rightarrow \tilde{L}$ that satisfies the following. Let $\tilde{L}^* = \bigcup_{f \in \hat{F}^*} \phi(f)$. With probability at
least \(1 - \xi\), the solution \(\mathcal{G} = \mathcal{G}^* - \tilde{F}^* \cup \tilde{L}^*\) and where each client served by a facility \(\ell \in \tilde{L}^*\) in solution \(L\) is served by \(\ell\) in solution \(\mathcal{G}\), has cost at most

\[
\text{cost}(\text{OPT}) + O(\xi(\text{cost}(\text{OPT}) + \text{cost}(L)))
\]

We consider the following bipartite graph \(\Phi = (A, B, E)\) with both capacities and costs (or weights) on the edges defined as follows. The set \(A\) contains one vertex for each facility of \(\text{OPT}\) plus a special vertex \(t\). The set \(B\) contains one vertex for each facility of \(L\) plus a special vertex \(s\). We slightly abuse notation and call the vertex representing facility \(f\) by \(f\) as well. The set of edges is as follows: for each facility \(f \in \text{OPT}\) and \(\ell \in L\), for each client \(c\) that is served by \(f\) in \(\text{OPT}\) and \(\ell\) in \(L\), add a directed edge \(e\) from \(f\) to \(\ell\) in \(\Phi\). We refer to \(e\) as the edge corresponding to client \(c\). The capacity of the edge is 2 and the cost of the edge is \(\text{dist}(c, f) + \text{dist}(c, \ell) = g_c + \ell_c\). Note that this may create parallel edges – parallel edges are kept in \(\Phi\).

Furthermore, for each vertex of \(f \in A - \{t\}\), we add \(\lfloor \eta/2 \rfloor\) directed edges from \(s\) to \(f\) each with capacity 2 and cost 0. Finally, from each vertex of \(\ell \in B - \{s\}\), we add \(\lfloor \eta/2 \rfloor\) edges directed from \(\ell\) to \(t\) each with capacity 2 and cost 0.

**Preprocessing step when \(\eta\) is not a multiple of 2** We now apply a preprocessing step for the case when \(\eta\) is not a multiple of 2. We assign a fractional weight of \(1/\eta\) to each edge that connect a vertex of \(A - \{t\}\) to a vertex of \(B - \{s\}\) of \(\Phi\). This defines a fractional matching over the vertices of \(A - \{t\} \cup B - \{s\}\) where each vertex of \(A - \{t\} \cup B - \{s\}\) that serves \(\eta\) clients is such that the total weight of the edges adjacent to it is 1. Therefore, there exists an integral matching where each vertex of \(A - \{t\} \cup B - \{s\}\) that serves \(\eta\) clients is matched (see e.g., [31]). Consider such a matching and delete the edge of the matching. We refer to the clients corresponding to the deleted edge by the *deleted* clients. The degree of each vertex after the preprocessing step differs by at most 1 from the original degree.

In the remaining, we let \(\eta' = \eta - 1\) if \(\eta\) is not a multiple of 2, and \(\eta' = \eta\) otherwise. Hence \(\eta'\) is a multiple of 2. For each facility \(f \in A - \{t\}\), we denote by \(\eta(f)\) the outgoing degree of \(f\) after the preprocessing step and for each \(f \in B - \{s\}\), we denote by \(\eta(f)\) the incoming degree of \(f\) after the preprocessing step. We put a demand of \(2\lfloor \eta(f)/2 \rfloor\) on each vertex of \(f \in A - \{t\}\) and a demand of \(2\lfloor \eta(f)/2 \rfloor\) on each vertex of \(f \in B - \{s\}\).

**Lemma 4.2.** There exists a flow \(\mathcal{F}_0\) in \(\Phi\) from \(s\) to \(t\) of cost at most \(\text{cost}(L) + \text{cost}(\text{OPT})\) and that satisfies:

- **Integrality:** each edge between \(A - \{t\}\) and \(B - \{s\}\) receives a flow of either 0 or 2.
- **Demand:** each vertex \(f \in A - \{t\}\) receives a flow of at least \(2\lfloor \eta(f)/2 \rfloor\); each vertex \(f \in B - \{s\}\) receives a flow of at least \(2\lfloor \eta(f)/2 \rfloor\).

**Proof.** We will show the following claim:

**Claim 1.** There exists a flow that satisfies the demand constraint and the capacities of the edges, but that does not necessarily satisfy the integrality constraint.

Then, assuming Claim 1, the lemma follows: Classic results (e.g., [31]) on the integrality of flows show that if the edges all have capacities 2, the demands are multiple of 2, and there exists a fractional flow satisfying the demands and capacities, then there is a flow that sends either a flow of 0 or a flow of 2 in each edge and that satisfies the demand constraints. Thus, we turn to the proof of Claim 1.
Consider sending a flow from $s$ to each vertex $f$ of $A - \{t\}$ of a value $2[\eta(f)/2]$. Since the total capacity from $s$ to $f$ is $2[\eta(f)/2]$ this is possible and the current cost of the flow is 0. Now, consider for each non-deleted client $c$ served by $f$ in OPT, to send a flow of 1 from $f$ to the facility of $L$ that serves it in solution $L$. This corresponds to sending a flow of 1 through the edge corresponding to client $c$. By the definition of the graph, for each such client $c$ there exists an edge with capacity 2 between the two facilities. This ensures that the demand at each facility $f \in A - \{t\}$ is met.

Finally, observe that each vertex $f$ of $B - \{s\}$ receives a flow that corresponds to the number of non-deleted clients served by the center in solution $L$. Thus, it is possible to complete the assignment by sending the flow arriving in each vertex $f$ of $B - \{s\}$ to $t$ using the edge of capacity $2[\eta(f)/2]$ and cost 0 and the demand at $f$ is met.

Let $F$ denote a maximal integral flow satisfying the demand and integrality constraints, as per Lemma 4.2. We say that a facility is saturated if the total flow it receives is $\eta'$. We say that an edge is $F$-saturated if the total flow in the flow $F$ that goes through the edge is 2.

**Lemma 4.3.** The cost of $F$ is at most $2(cost(L) + cost(OPT))$.

**Proof.** This follows from the fact that when all the edges of the graph are saturated the total cost is $2(cost(L) + cost(OPT))$. □

We now define $U$ to be the set of facilities of $A$ such that $\eta(f) \geq \eta'/2$, i.e., $U = \{f \mid f \in A, \eta(f) \geq \eta'/2\}$. We will refer to the facilities of $U$ as heavy facilities. Let $\Lambda$ be the set of facilities of $U$ whose corresponding vertices in $\Phi$ that are saturated by flow $F$. Define $\Lambda = A - \{t\} - \Lambda$. Let $\zeta$ be the set of facilities of $B - \{s\}$ that are saturated by flow $F$. Define $\bar{\zeta} = B - \{s\} - \zeta$.

We now aim at matching vertices of $\Lambda$ and $\zeta$ to vertices of $B - \{s\}$ and $A - \{t\}$ respectively. We will make use of the following classic theorem (see e.g., [31]).

**Theorem 4.4 ([31]).** Let $G = (A, B, E)$ be a bipartite graph with edge weights $w : E \to \mathbb{R}_+$. Let $M_0 : E \to [0, 1]$ be a fractional matching of weight $W = \sum_e M_0(e) \cdot w(e)$. There exists an integral matching $M_1 : E \to \{0, 1\}$ that satisfies:

- Each vertex $u \in A \cup B$ such that $\sum_{(u,v) \in E} M_0((u,v)) = 1$ is matched in $M_1$, i.e., $\exists (u,v) \in E$ s.t. $M_1((u,v)) = 1$, and;
- The weight of $M_1$ is at most $W$, i.e., $\sum_{e \in E} M_1(e) \cdot w(e) \leq W$, and;
- $M_1((u,v)) = 1 \implies M_0((u,v)) \neq 0$.

Consider rescaling the amount of flow $F$ sent through each edge by a factor $1/\eta'$ and denote by $M_0$ the underlying flow. Seeing $M_0$ as a matching of weight at most $2(OPT + cost(L))/\eta'$ we have the following application of Theorem 4.4.

**Corollary 1.** There exists an integral matching $M$ in $\Phi$ that satisfies:

1. Each facility of $\Lambda$ is matched to a facility of $B - \{s\}$, and;
2. Each facility of $\zeta$ is matched to a facility of $A - \{t\}$, and;
3. The weight of the matching is at most $2(cost(OPT) + cost(L))/\eta'$, and;
4. If a facility $f$ is matched to a facility $\ell$, then it must be that the flow $F$ going from $f$ to $\ell$ is positive.
We define $\mathcal{M}_A$ to be the set of vertices of $A - \{\ell\}$ that are matched and $\mathcal{M}_B$ the set of vertices of $B - \{s\}$ that are matched. Note that $\Lambda \subseteq \mathcal{M}_A$ and $\zeta \subseteq \mathcal{M}_B$.

Consider a facility $\ell \in \mathcal{M}_B$, we define the following mapping. Let $f(\ell)$ be the facility of $A$ that is matched to $\ell$ in $\mathcal{M}$.

We now consider each pair of matched vertices $\ell, f(\ell)$ and define a function $p$ that maps each edge incoming to $\ell$ to either an edge outgoing of $f(\ell)$ or to $f(\ell)$ directly. For each vertex $\ell \in \mathcal{M}_B$, define $t(\ell)$ and $\bar{t}(\ell)$ to be respectively the numbers of non-$\mathcal{F}$-saturated and $\mathcal{F}$-saturated edges ingoing to $\ell$ and not originating from $f(\ell)$. For each vertex $f \in \mathcal{M}_A$ define $s(f)$ and $\bar{s}(f)$ to be respectively the numbers of non-$\mathcal{F}$-saturated and $\mathcal{F}$-saturated edges outgoing from $f$ and not going to $\ell$.

The mapping $p$ is defined as follows. Consider a pair of matched vertices $\ell, f(\ell)$.

1. If $t(\ell) > s(f(\ell))$, choose an arbitrary subset of size $s(f(\ell))$ among the non-$\mathcal{F}$-saturated edges ingoing to $\ell$ and define a one-to-one mapping from these edges to the edges in $s(f(\ell))$. For the $t(\ell) - s(f(\ell))$ remaining edges, map them to $f(\ell)$. This defines the mapping $p$ for the non-$\mathcal{F}$-saturated edges ingoing to $\ell$ for the case ($t(\ell) > s(f(\ell))$).

Otherwise, when $t(\ell) \leq s(f(\ell))$, simply define an arbitrary injective function from the non-$\mathcal{F}$-saturated edges ingoing to $\ell$ to the non-$\mathcal{F}$-saturated edges outgoing from $f(\ell)$.

2. Proceed similarly with the $\mathcal{F}$-saturated edges that are incoming to $\ell$ and outgoing from $f(\ell)$.

We now consider vertices of $A - \mathcal{M}_A - \{t\}$. For each such vertex $f$, for each edge $e$ outgoing from $f$, we define $S(e)$ to be the sequence obtained by recursively applying the mapping $p$ (i.e., $S(e) = e, p(e), p(p(e)), \ldots$) until we can’t apply $p$ again, namely either we reach an edge $e’$ such that $p(e’) = f’$ where $f’ \in \mathcal{M}_A$, or we reach an edge $e’ = (f’, \ell’)$ where $\ell’ \in B - \mathcal{M}_B - \{s\}$. Let $S$ be the set of all the sequences defined above. We have the following lemma.

**Lemma 4.5.** For each sequence $S(e) \in S$, we have that each edge of the graph appears at most once in $S(e)$ and so, $S(e)$ is finite. Moreover, for each edge $e’$ of the graph, there is at most one sequence in $S$ containing $e’$.

*Proof.* We first argue that $S(e)$ is finite. Let $e = (f, \ell)$ with $f \in A - \mathcal{M}_A - \{t\}$. Recall that $p$ maps the edges adjacent to a facility $\ell \in \mathcal{M}_B$ and not coming from $f(\ell)$ either to a facility $f(\ell)$ in which case, the sequence stops, or is an injective mapping to the edges outgoing from $f(\ell)$ and not going to $\ell$.

Furthermore, observe that for any edge $(f^j, \ell^j)$ of the sequence, $p((f^j, \ell^j))$ is an edge which starts at a matched vertex. Therefore, except for the first edge $e$, no edge of the sequence is adjacent to $f$ since $f$ is unmatched, i.e., no edge in the sequence $p(e), p(p(e)), \ldots$ is adjacent to $f$.

Assume towards contradiction that there is an edge that appears twice in the sequence and consider the first one in the order of the sequence. Let $(v_i, u_i)$ be this edge. By the above argument, we have that $v_i \neq f$ since otherwise there would be an edge in the subsequence $p(e), p(p(e)), \ldots$ that is adjacent to $f$.

Thus, we have $v_i \neq f$, and so $p^{-1}((v_i, u_i))$ is also twice in the sequence since $p$ is injective on the edges. This is a contradiction since $(v_i, u_i)$ is the first one of the sequence, it follows that $S(e)$ is finite.

Finally, since for any $S((f, \ell)) \in \mathcal{F}$, we have that $f$ is an unmatched vertex, the edge $(f, \ell)$ cannot appear in another sequence $S(e) \in \mathcal{F}$. Thus applying the same reasoning as above, an edge cannot appear in two different sequences. 

\[\square\]
We now distinguish two types of sequences. We say that an edge \( e \) is a route-to-matched if the sequence \( S(e) \) stops at a vertex \( f \in \mathcal{M}_A \), and a route-to-unmatched if the sequence \( S(e) \) stops at a vertex \( \ell \in B - \mathcal{M}_B - \{s\} \). We have the following lemma.

**Lemma 4.6.** Consider a facility \( f \in A - \mathcal{M}_A - \{t\} \) and such that \( \eta(f) \geq \eta'/2 \). The number of edges \( e \) adjacent to \( f \) and such that \( S(e) \) is a route-to-unmatched sequence is at most \( \eta'/2 - 1 \).

**Proof.** Since \( f \) is unmatched, we have that the total flow going through \( f \) in the flow \( F \) is at most \( \eta' \). Thus, there are at most \( \eta'/2 - 1 \) edges that adjacent to \( f \) and that are \( F \)-saturated. We will show that for each edge \( e \) adjacent to \( f \) such that \( S(e) \) is a route-to-unmatched sequence, we have that \( e \) is \( F \)-saturated.

Now, suppose towards contradiction that there exists an edge \( e \) such that \( S(e) \) is a route-to-unmatched sequence and consider the path induced by the sequence \( S(e) \). By Lemma 4.5, this sequence is finite and so let \( (f_j, \ell_j) \) be the last edge of the sequence. Since \( S(e) \) is a route-to-unmatched, \( \ell_j \) is unmatched.

Since we have that \( e \) is not-\( F \)-saturated and by definition of \( p \), a direct induction shows that all the edges in the sequence \( S(e) \) are not-\( F \)-saturated, and so these are all edges with positive capacities in the residual graph \( \Phi^F \). Moreover, observe that for each matched pair \( \ell_x, f(\ell_x) \), Corollary 4.8 implies that there are at least 2 units of flow going from \( f(\ell_x) \) to \( \ell_x \) in flow \( F \). This induces an edge with positive capacity from \( \ell_x \) to \( f(\ell_x) \) in \( \Phi^F \).

Thus, consider the subgraph of \( \Phi^F \) induced by the edges of \( S(e) \) and the edges between matched pairs \( \ell_x, f(\ell_x) \) and consider a simple path from \( f \) to \( \ell_j \) in this graph. This path uses each edge at most once and so it is possible to route at least one unit of flow through this path without violating the capacities of the edges of the path.

Furthermore, \( \ell_j \) and \( f \) are not matched and so there is at least one edge with positive capacity from \( \ell_j \) to \( t \) and an edge with positive capacity from \( s \) to \( f \) in \( \Phi^F \). Therefore there is a path with positive capacity from \( s \) to \( t \) in \( \Phi^F \). Furthermore, observe that routing a unit of flow through this path can only increase the flow going through any of the vertices of the graph. Thus, there is a flow with higher value which satisfies the demand constraints, a contradiction to the maximality of \( F \) that concludes the proof.

**Assignment \( \mu \)** For each facility \( f \in U - \mathcal{M}_A \), namely an unmatched heavy facility, consider the edges \( e = (f, \ell) \) such that \( S(e) \) is a route-to-matched sequence. For the client \( c \) associated with edge \( e \), we let \( \mu(c) \) map to the matched vertex of \( \mathcal{M}_A \) that terminates the sequence \( S(e) \). Let \( \mathcal{C}_1 \) be the set of these clients. For clients in \( \mathcal{C} - \mathcal{C}_1 \) (including deleted clients), we let \( \mu(c) \) be the facility that serves it in OPT.

**Sequences to paths** For each facility \( f \in U - \mathcal{M}_A \), namely an unmatched heavy facility, consider the edges \( e = (f, \ell) \). For each such edge \( e \), we define the path associated to sequence \( S(e) \) as follows. The first edge of the path is \( (f, \ell) \), the second edge of the path is \( (\ell, f(\ell)) \), the third edge of the path is \( p(e) = (f(\ell), \ell_1) \). For \( i > 1 \), the \( 2i \)-th and \( 2i + 1 \)-st edges of the path are edges \( (f(\ell_{i-1}), \ell_i) \) and \( (\ell_i, f(\ell_i)) \). If there are multiple edges \( (\ell_i, f(\ell_i)) \), the one with smallest weight is chosen. We let \( P(e) \) denote the path associated to edge \( S(e) \). By the triangle inequality, the length of the path is simply the sum of the weights of the edges.

We show the following lemma, which will be used in two different ways:

1. to bound the cost of reassigning a client whose corresponding edge is a route-to-matched client to the facility of \( \mathcal{M}_A \) at the end of the sequence;
2. to bound the cost of reassigning a client whose corresponding edge is a route-to-unmatched client to the facility of $B - \mathcal{M}_B$ at the end of the sequence.

**Lemma 4.7.** The sum over all facility $f \in U - \mathcal{M}_A$, of the sum over all edges $e = (f, \ell)$ of the length of the paths associated to $S(e)$ is at most $4(\text{cost}(\text{OPT}) + \text{cost}(L))$. In other words,

$$\sum_{f \in U - \mathcal{M}_A} \sum_{e=(f, \ell)} \text{length}(\mathcal{P}(e)) \leq 4(\text{cost}(\text{OPT}) + \text{cost}(L)).$$

**Proof.** Observe that the by Lemma 4.5, the paths are edge disjoint, except for the edges of the path that are connecting two vertices that are matched together (namely, the even edges of the path). More concretely, in the path $e = (f, \ell)$, $\text{dist}(\ell, f(\ell)), p(e) = (f(\ell), \ell_1), \text{dist}(\ell_1, f(\ell_1)), \ldots$ associated to sequence $S(e)$, the edges $(f, \ell), p((f, \ell)), p(p((f, \ell))), \ldots$ appear in at most one sequence. Thus, the sum over all sequences of the edges that are not connecting two matched vertices is bounded by the total sum of edge weights of the graph and so at most $(\text{cost}(\text{OPT}) + \text{cost}(L))$.

We now bound the number of times $\text{dist}(\ell, f(\ell))$ is going to appear in the sum of the lengths of the paths of all the sequences. We first observe that the number of paths in which this edge appears is bounded by the incoming degree of $\ell$ which corresponds to the number of clients served by $\ell$ in $L$ and so at most $\eta'$. Thus, we have that $\text{dist}(\ell, f(\ell))$ appears at most $\eta'$ times in the sum. Finally, Corollary [1] Property [4] combined with the triangle inequality shows that $\sum_{e \in \mathcal{M}_B} \text{dist}(\ell, f(\ell)) \leq \sum_{e \in \mathcal{M}_B} w(e((\ell, f(\ell)))) \leq 2(\text{cost}(\text{OPT}) + \text{cost}(L))/\eta'$, where $e((\ell, f(\ell))$ is the edge matching $\ell$ to $f(\ell)$ and $w$ its weight. Thus, since $\text{dist}(\ell, f(\ell))$ appears at most $\eta'$ times for each matched pair $\ell, f(\ell)$, we have that the total cost induced by these edges is at most $2(\text{cost}(\text{OPT}) + \text{cost}(L))$. \hfill $\square$

**Remark on the case** $p > 1$. For any objective where the cost of assigning client $c$ to facility $f$ is $\text{dist}(c, f)^p$, for $p > 1$, Lemma 4.7 does not allow to relate the cost of assigning a client $c$ to the facility that is at the end of the path of the sequence $S(e)$ where $e$ is the edge corresponding to client $c$. Indeed, for example for $p = 2$, the cost for a client in $C_1$ is going to be the square of the sum of the weights of the edges in the path and this cannot be related to the cost of an optimal solution and the cost of the local solution directly since the solutions pays the sum of the lengths squared (instead of the sum of the lengths).

The way to handle this is to modify the definition of the length of a path associated to $S(e)$. Consider first a route-to-matched sequence $S(e) = e, p(e), p(p(e)), \ldots$ and let $e = (f, \ell), p(e) = (f(\ell), \ell_1), p'(e) = (f(\ell_{i-1}), \ell_i)$, for $i > 1$. Let $f(\ell_{i-1})$ be the matched vertex that terminates the sequence. We define the length of the path associated to $S(e)$ as $\text{length}(\mathcal{P}(e)) = (\text{dist}(f, \ell) + \text{dist}(\ell, f(\ell))^p + \sum_i (\text{dist}(f(\ell_{i-1}), \ell_i) + \text{dist}(\ell_i, f(\ell_i)))^p$.

Observe that $(a + b)^p \leq 2^p(a^p + b^p)$ and so we have that

$$\text{length}(\mathcal{P}(e)) \leq 2^p(\text{dist}(f, \ell)^p + \text{dist}(\ell, f(\ell))^p + \sum_i (\text{dist}(f(\ell_{i-1}), \ell_i)^p + \text{dist}(\ell_i, f(\ell_i))^p).$$

Now recall that for each edge of the graph $(f', \ell')$, the weight is given by $\text{dist}(c, f')^p + \text{dist}(c, \ell')^p$ where $c$ is the client associated to the edge. Thus we have that $\text{dist}(f', \ell')^p \leq 2^p(\text{dist}(c, f')^p + \text{dist}(c, \ell')^p)$. Therefore, mimicking the proof of Lemma 4.7 we have that $\sum_{f \in U - \mathcal{M}_A} \sum_{e=(f, \ell)} \text{length}(\mathcal{P}(e))$ is at most $2^{O(p)}(\text{cost}(\text{OPT}) + \text{cost}(L))$. 

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Now, the length of a path $P(e)$ does not correspond to the cost of assigning of the client $c$ associated to edge $e$ to the facility at the end of the path. Instead, it corresponds to the cost of assigning $c$ to $f(\ell)$ plus the cost of assigning the client $c_1$ of $f(\ell)$ to $f(\ell_1)$ whose edge is in the sequence, plus the cost of assigning the client $c_2$ of $f(\ell_1)$ to $f(\ell_2)$ whose edge is in the sequence, and more generally the cost of assigning the client of $f(\ell_t)$ whose edge is in the sequence to $f(\ell_{t+1})$, for all $i < s$. Indeed, the total cost of such a reassignment is given by $(\text{dist}(f, \ell) + \text{dist}(\ell, f(\ell))) + \sum_i (\text{dist}(f(\ell_i), \ell_i) + \text{dist}(\ell_i, f(\ell_i))) = \text{length}(P(e))$ and so bounded by $2^{O(p)}(\text{cost}(\text{OPT}) + \text{cost}(L))$.

In the case of $p > 1$, we let $\mu^p$ be the reassignment defined above. It is easy to see that if $\mu$ meets the capacities then $\mu^p$ also meets the capacities.

We now turn to show that the capacities are met for assignment $\mu$.

**Lemma 4.8.** Consider the solution defined by the set of centers of OPT together with the assignment $\mu$. This solution satisfies the following properties:

1. Each facility $f$ of OPT whose corresponding vertex is unmatched, i.e., $f \notin \mathcal{M}_A$, is assigned at most $\lfloor \eta/2 \rfloor$ clients. In other words, $|\{c \mid \mu(c) = f\}| \leq \lfloor \eta/2 \rfloor$.

2. Each facility $f$ of OPT whose corresponding vertex is matched, i.e., $f \in \mathcal{M}_A$, is assigned at most $\eta$ clients. In other words, $|\{c \mid \mu(c) = f\}| \leq \eta$.

3. For each client $c \in \mathcal{C} - \mathcal{C}_1$, its cost is identical to its cost in OPT. Moreover, $\sum_{c \in \mathcal{C}_1} \text{dist}(c, \mu(c)) \leq 2^{O(p)}(\text{cost}(\text{OPT}) + \text{cost}(L))$.

**Proof.** We first prove Property 1. From Lemma 4.6, the only clients that are assigned to an unmatched facility in $\mu$ are the one for which sequence $S(e)$ of the corresponding edge $e$ is a route-to-unmatched plus possibly one deleted client. It follows that the total number of clients assigned is $\eta'/2 = \lfloor \eta/2 \rfloor$.

To prove Property 2 we start with the following observation. Consider a pair of matched vertices, $\ell, f(\ell)$. The total number of new elements that can be assigned to $f(\ell)$ in mapping $\mu$ is, by definition of $p$, the number of edges that are incoming to $\ell$ and not originating from $f(\ell)$ minus the number of edges outgoing from $f(\ell)$ and not going to $\ell$, or in other words $\max(\ell(\ell) - \bar{s}(f(\ell)), 0) + \max(t(\ell) - s(f(\ell)), 0)$. Let $m_\ell$ and $\bar{m}_\ell$ respectively denote the number of non-$\mathcal{F}$-saturated and $\mathcal{F}$-saturated edges between $\ell$ and $f(\ell)$.

It follows that the total number of non-deleted clients served by $f(\ell)$ in assignment $\mu$ is at most

$$\nu = \bar{s}(f(\ell)) + s(f(\ell)) + m_\ell + \bar{m}_\ell + \max(\ell(\ell) - \bar{s}(f(\ell)), 0) + \max(t(\ell) - s(f(\ell)), 0).$$

We aim at showing that $\nu$ is at most $\eta'$. We have the following equations:

- $\bar{s}(f(\ell)) + s(f(\ell)) + m_\ell + \bar{m}_\ell \leq \eta'$, since the degree of $f(\ell)$ in $\Phi$ (after preprocessing) is at most $\eta'$;
- $\ell(\ell) + t(\ell) + m_\ell + \bar{m}_\ell \leq \eta'$, since the degree of $\ell$ in $\Phi$ (after preprocessing) is at most $\eta'$;
- $2\left[\frac{\ell(\ell) + t(\ell) + m_\ell + \bar{m}_\ell}{2}\right] \leq 2\bar{s}(f(\ell)) + 2m_\ell \leq \eta'$ since the flow $\mathcal{F}$ going through $\ell$ is at least $\left[\frac{\ell(\ell) + t(\ell) + m_\ell + \bar{m}_\ell}{2}\right]$ by the definition of the demand at $\ell$ and at most $\eta'$ since the outgoing capacity from $\ell$ is $\eta'$.
- Moreover, each edge of $\ell(\ell)$ and $m_\ell$ carries 2 units of flow.
- $2\left[\frac{s(f(\ell)) + m_\ell + \bar{m}_\ell}{2}\right] \leq 2\bar{s}(f(\ell)) + 2\bar{m}_\ell \leq \eta'$, for the same reason than the above case.
In the case where either both $\bar{\ell}(\ell) \geq \bar{s}(f(\ell))$ and $t(\ell) \geq s(f(\ell))$ or both $\bar{\ell}(\ell) \leq \bar{s}(f(\ell))$ and $t(\ell) \leq s(f(\ell))$, then we have that $\nu$ is at most $\eta'$ by combining directly with the first two equations of the above list.

We thus turn to the case where $\bar{\ell}(\ell) \geq \bar{s}(f(\ell))$ and $t(\ell) \leq s(f(\ell))$. First, if $\bar{\ell}(\ell) = \bar{s}(f(\ell))$, then both max are 0 and so $\nu \leq \bar{s}(f(\ell)) + s(f(\ell)) + m_\ell + \bar{m}_\ell \leq \eta'$ by the first of the above equations.

So we assume $\bar{\ell}(\ell) > \bar{s}(f(\ell))$ and $t(\ell) \leq s(f(\ell))$. Thus, we have $\nu \leq s(f(\ell)) + m_\ell + \bar{m}_\ell + \bar{\ell}(\ell)$.

From the fourth of the above equations we have that $\bar{s}(f(\ell)) + s(f(\ell)) + m_\ell + \bar{m}_\ell - 1 \leq 2s(f(\ell)) + 2\bar{m}_\ell$ and so $s(f(\ell)) + m_\ell + \bar{m}_\ell - 1 \leq \bar{s}(f(\ell)) + 2\bar{m}_\ell$. We then combine with the upper bound on $\nu$ to obtain

$$\nu \leq s(f(\ell)) + m_\ell + \bar{m}_\ell + \bar{\ell}(\ell) \leq \bar{s}(f(\ell)) + \bar{\ell}(\ell) + 2\bar{m}_\ell + 1.$$ 

Therefore, since $\bar{s}(f(\ell)) < \bar{\ell}(\ell)$, we conclude that $\nu \leq 2\bar{\ell}(\ell) + 2\bar{m}_\ell \leq \eta'$, using the third equation. The case where $\bar{\ell}(\ell) \leq \bar{s}(f(\ell))$ and $t(\ell) \geq s(f(\ell))$ is symmetric.

We now need to incorporate possibly one deleted client of $f(\ell)$. Namely a client served by $f(\ell)$ in OPT whose edge has been deleted during the preprocessing step and so that is still assigned to $f(\ell)$ (recall that at most 1 client served by a facility is deleted during the preprocessing step). Observe that there is a deleted client only if $\eta$ is not a multiple of 2. In which case we have that $\eta' = \eta - 1$ and so, the total number of clients assigned to $f(\ell)$ is $\eta' + 1 \leq \eta$ as claimed.

Finally, to bound the sum of the length of the edges in all the paths associated to the route-to-matched sequences we simply invoke Lemma 4.17. It follows, the total cost of the assignment $\mu$ for the vertices of $C_1$ is at most $4(cost(OPT) + cost(L))$, or using the above remark, at most $2O(p)(cost(OPT) + cost(L))$ for the case $p > 1$. \[\square\]

We can now prove the main proposition.

Proof of Proposition 4.1. For each unmatched facility $f \in OPT$, we let $\xi(f)$ denote the unmatched facility of $L$ that is the closest to $\xi(f)$. We then divide the unmatched facilities of OPT into two groups, $U_1, U_2$ as follows. Let $U_1 = \{f \mid f \in OPT \text{ and there is no facility } f' \neq f \text{ s.t. } \xi(f') = \xi(f)\}$. Let $U_2$ be the rest of the unmatched facilities of OPT.

We let $\bar{F} = U_1 \cup M_A$ and let $\bar{\ell} = \{\ell \mid \exists f \in U_1, s.t. \xi(f) = \ell \} \cup M_B$, and $\phi$ be the 1-to-1 mapping of the facilities of $\bar{F}$ to $\bar{\ell}$ defined by the matching $M$ and the function $\xi$ on $U_1$.

We define $\bar{F} = F - \bar{F} = U_2$ and $\bar{\ell} = L - \bar{\ell}$. By the pigeonhole principle we immediately have that $|\bar{\ell}| = |\bar{\ell}|$ and $|\bar{F}| = |\bar{F}|$. To finish the proof of the proposition, we need to prove Properties 2 and 3.

We first aim at proving Property 2. Consider the mapping $\xi$ of the facilities of $\bar{F}$. We let $\chi(\ell) = \{f \mid \xi(f) = \ell\}$. We now proceed as follows: for each facility $\ell$ such that $|\chi(\ell)| > 1$, we pair up the facilities of OPT such that $\xi(f) = \ell$. Let $(f_1, g_1), (f_2, g_2), \ldots$ be the list of $|\chi(\ell)|/2$ pairs. For each pair, we will consider closing one facility. We need to guarantee two things: first that the capacities are met and second that the total service cost is bounded.

To ensure that the capacities are met, we make use of Lemma 4.18. For each pair $(f_i, g_i)$, we follow the assignment $\mu$ for the set of clients that they serve in OPT. Without loss of generality, assume that $f_i$ is farther away to $\ell = \xi(f_i) = \xi(g_i)$ than $g_i$. This guarantees that both facilities
serve at most \( \lfloor \eta/2 \rfloor \) clients. We consider the cost of closing down \( f_i \) and serving its clients by \( g_i \). Moreover, \( \mu \) reassigns clients served by the unmatched facilities to matched facilities and Lemma 4.8 shows that the total number of clients assigned to a matched facilities is at most \( \eta \). It follows that the total number of clients assigned to \( f_i \) and \( g_i \) is at most \( \eta \) and so removing one of the two facilities still yield a feasible solution.

We now turn to bounding the cost of closing one facility per pair \((f_i, g_i)\). Consider first for simplicity the case \( p = 1 \). The reassignment we have designed is as follows:

1. For the clients \( c \) whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-matched sequence and such that the facility serving \( c \) belongs to a pair \((f_i, g_i)\), the assignment is the same as in \( \mu \).

2. For the clients \( c \) whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-unmatched sequence, and s.t. \( c \) is served by a facility \( f_i \) in a pair \((f_i, g_i)\), the assignment is now \( g_i \). Let \( \ell \) be the facility such that \( \xi(f_i) = \ell \) and \( \xi(g_i) = \ell \). The cost of the assignment is \( \text{dist}(c, \ell) + \text{dist}(\ell, g_i) \leq \text{dist}(c, \ell) + \text{dist}(\ell, f_i) = 2\text{dist}(c, \ell) + \text{OPT}(e) \), by the triangle inequality and since \( \text{dist}(\ell, f_i) \geq \text{dist}(\ell, g_i) \). We redefine \( \mu(c) = g_i \).

3. For the remaining clients, the assignment is the same than in OPT.

Consider the clients \( c \) whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-matched sequence and such that the facility serving \( c \) belongs to a pair \((f_i, g_i)\). Lemma 4.8 shows that the sum over all pairs \((f_i, p_i)\) of the cost of the reassignment of their clients whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-matched sequence is bounded by \( O(\text{cost}(\text{OPT}) + \text{cost}(L)) \).

For the clients \( c \) whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-unmatched sequence, and s.t. \( c \) is served by a facility \( f_i \) in a pair \((f_i, g_i)\). Let \( \ell \) be the facility such that \( \xi(f_i) = \ell \) and \( \xi(g_i) = \ell \). We have that the cost is \( \text{dist}(c, \mu(c)) \leq 2\text{dist}(c, \ell) + \text{OPT}(e) \). Now observe that \( \text{dist}(c, \ell) \leq \text{length}(P(e)) \) since \( \ell \) is the closest unmatched facility to \( f_i \). Thus applying Lemma 4.7, the sum over all the facilities \( f_i \) that are closed of the reassignment cost of their clients \( c \) whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-unmatched sequence is at most \( O(\text{cost}(L) + \text{cost}(\text{OPT})) \).

For the remaining clients, their cost is the same than in OPT.

We thus have that: \( \sum_{(f_i, g_i)} \sum_{c \text{ served by } f_i \text{ or } g_i} \text{dist}(c, \mu(c)) \) is \( O(\text{cost}(\text{OPT}) + \text{cost}(L)) \). Moreover, \( \mu(c) \) does not assign more than \( \eta \) clients to any facility.

Now consider selecting each pair \((f_i, g_i)\) with probability \( \varepsilon \) and closing down \( f_i \). For each selected pair, we follow the assignment prescribed above and for the remaining pairs, we follow the optimal assignment. The assignment is feasible no matter what are the selected pairs since we only consider reassigning clients served by the selected pairs in OPT to matched facilities or to one of the facility of the pair. By Lemma 4.8, we know that we can reassign all clients of the pairs and still get a feasible solution, therefore the solution obtained is definitely feasible.

By the above discussion, the expected cost of the assignment for the clients that are served by a facility of a pair \((f_i, g_i)\), is at most

\[
\sum_{(f_i, g_i)} \left( \text{pr}[(f_i, g_i) \text{ is selected}] \cdot \sum_{c \text{ served by } f_i \text{ or } g_i} \text{dist}(c, \mu(c)) + \sum_{c \text{ served by } f_i \text{ or } g_i} \text{OPT}(c) \right) + \sum_{(f_i, g_i)} \left( 1 - \text{pr}[(f_i, g_i) \text{ is selected}] \right) \sum_{c \text{ served by } f_i \text{ or } g_i} \text{OPT}(c)
\]
which is at most

\[ \sum_{(f_i, g_i) \text{ served by } f_i \text{ or } g_i} \sum_{c \text{ served by } f_i \text{ or } g_i} \text{OPT}(c) + O(\varepsilon(\text{cost(OPT)} + \text{cost}(L))). \]

Therefore, since for the rest of the clients, the cost is optimal, there exists a solution \( G^* \) of cost at most \( \text{cost(OPT)} + O(\varepsilon(\text{cost(OPT)} + \text{cost}(L))). \)

We finally prove Property \( \mathbf{K} \) of the proposition. Consider a facility \( f \in \tilde{F} \) and a facility \( \ell \in \tilde{L} \) such that \( \phi(f) = \ell \). Let \( c(f) \) be the cost of replacing \( f \) by \( \ell \) in solution \( G^* \) and serving the set \( N(\ell) \) of all the clients served by \( \ell \) in \( L \) by \( \ell \) in the solution \( G^* \).

Our bound of \( c(f) \) is in 2 steps. We first bound the cost of serving by \( \ell \) all the clients assigned to \( f \) in assignment \( \mu \). This is an intermediate solution that does not satisfy that the clients served by \( \ell \) in \( L \) are also served by \( \ell \) in solution \( G^* \). We will then modify the intermediate solution to ensure this last property.

Consider first the case where \( f \) is an unmatched facility. We will reassign the clients of \( f \) in two ways. First, for the clients \( c \) whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-matched sequence. In that case, we use \( \mu(c) \) as a reassignment and so these clients are served by a different facility than \( f \). Again, this is compatible with the previous reassignment since the mapping \( \mu \) that reassigns all clients of unmatched facility ensures that no matched facility receives more than \( \eta \) clients by Lemma \([1, 8] \).

Second, for the set \( N(f) \) of clients whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-unmatched sequence, we temporarily assign them to \( \ell \) and we can bound the cost for such clients by \( \text{length}(P(e)) \), since \( \ell \) is the closest unmatched facility.

Now consider the case where \( f \) is a matched facility. We proceed identically but we cannot use the bound on the length of the path since this bound only applies to unmatched facilities. In that case, we use the bound given by the matching. We have that the cost paid by each client \( c \) of \( f \) is \( \text{dist}(c, f) + \text{dist}(f, \phi(f)) \). By the triangle inequality, \( \text{dist}(f, \phi(f)) \) is at most the weight of the edge in the matching. Serving all the clients assigned to \( f \) in mapping \( \mu \) incurs an additional cost (in addition to what they are paying due in mapping \( \mu \)) \( \eta \left( \sum_{e \in M_{\ell \ell}} \text{dist}(\ell, f(\ell)) \right) \) which is by Corollary \( \mathbf{I} \) at most \( \eta \left( 2(\text{cost(OPT)} + \text{cost}(L))/\eta \right) \).

Therefore, the reassignment performed for the intermediate solution has cost at most \( O(\text{cost}(L) + \text{cost(OPT)}) \). Note that this indeed takes into account the reassignment of the client \( c \) whose corresponding edge \( e \) is s.t. \( S(e) \) is a route-to-matched sequence.

We now move from the intermediate solution to a solution where for each selected pair \((\ell, f)\), the clients served by \( \ell \) in \( L \) are also served by \( \ell \) in \( \text{OPT} \). Let’s assign the clients of \( N(\ell) \) by \( \ell \). By doing so, we may have exceeded the capacity of \( \ell \): we have \(|N(\ell)|, |N(\ell)| \leq \eta \text{ but } |N(f)| + |N(\ell)| > \eta \). To fix this, we use the room left out on the other facilities by the \(|N(\ell)| \) clients served by \( \ell \) in solution \( L \). Indeed, since these clients are now served by \( \ell \), they leave some free room to the other clients. Thus, consider an arbitrary set \( B \) of \(|N(\ell)| + |N(f)| - \eta \) clients of \(|N(f)| \) (note that \(|N(f)| \geq |N(\ell)| + |N(f)| - \eta \) and define a 1-to-1 mapping from this set to an arbitrary subset \( B' \) of size \(|N(\ell)| + |N(f)| - \eta \) of \( N(\ell) \) (again note that \(|N(\ell)| \geq |N(\ell)| + |N(f)| - \eta \) so this is possible). Now, each client of \( B \) is assigned to the facility that serves the client it is mapped to in \( B' \) in the solution \( G^* \). Since \( G^* \) is feasible, this solution is also feasible. By the triangle inequality, the increase in cost for doing so is at most \( L(c) + \text{OPT}(c) \) where \( c \) is the client in \( B' \) and so, in total for the clients in \( N(\ell) \) at most \( \sum_{c \in N(\ell)} L(c) + \text{OPT}(c) \).
Summing up over all such facilities and using Lemma 4.7, we have that $\sum_{f \in F} c(f) = O(\text{cost}(L) + \text{cost}(\text{OPT}))$. Thus, if each facility $f$ is replaced by $\phi(\ell)$ with probability $\xi^2$, we have that the expected cost of the solution is $\sum_{f \in F} pr[f \text{ selected by the random process}] \cdot c(f) = \sum_{f \in F} \xi^2 c(f) = O(\xi^2 \text{cost}(L) + \text{cost}(\text{OPT}))$. By Markov inequality, we have that the resulting solution has cost at most $(1 + O(\xi))\text{cost}(\text{OPT}) + O(\xi(\text{cost}(L) + \text{cost}(\text{OPT})))$ as claimed. This concludes the proof of the proposition.

To handle the case $p > 1$, one needs to proceed as prescribed in the previous remark.

5 A Simple QPTAS for Doubling Metrics – Proof of Theorem 1.2

In this section, we give a simple approach for obtaining an algorithm running in time $\exp(((d\rho\varepsilon^{-1})^{1/\rho} \log n)^{O(d)})$, which is a quasi-polynomial bound for any fixed $d$. In this section and the next, it will be convenient to see an assignment of a client $c$ to a center $\ell$ as a path from $c$ to $\ell$ that may intersect portals and whose length is simply the $p$th power of the sum of the length of the segments of the path.

Our algorithm is very simple. Let $\varepsilon > 0$ be a sufficiently small constant. Assume we know how to compute a $\gamma$-approximate solution $L$. We show how to compute a solution of cost at most $(1 + \varepsilon)\text{cost}(\text{OPT}) + \varepsilon \cdot \text{cost}(L)$. At start, the algorithm computes a randomized split-tree $\mathcal{D}$. Let’s condition on the event that $\mathcal{I}_{\mathcal{D},L}$ is a good instance (w.r.t. $L$). This happens with probability at least $1 - \varepsilon$ by Lemma 5.2. The algorithm then computes $\mathcal{I}_{\mathcal{D},L}$ and works in $\mathcal{I}_{\mathcal{D},L}$, its goal being to find the best valid solution in $\mathcal{I}_{\mathcal{D},L}$. We design a dynamic program that given $\mathcal{I}_{\mathcal{D},L}$ and $L$ computes a $(1 + \varepsilon)$-approximation to the best valid solution. We then preprocess this new instance as follows. For each facility of $\ell$ that is badly cut, we force it open: our dynamic program is forced to pick $\ell$ in its solution. Since the dynamic program aims at finding the best valid solution, we have that the best solution in the preprocessed instance $\mathcal{I}'_{\mathcal{D},L}$ can be transformed into a valid solution in $\mathcal{I}_{\mathcal{D},L}$ with the same cost.

The dynamic program proceeds on $\mathcal{D}$ from the leaves to the the root. The algorithm then computes a hierarchy of nets: namely for each box $P$ of level $i$, it computes a $\rho 2^{i+1}$-net which is a superset of the nets computed for the descendant boxes of $P$. The net of a box is then used as a set of portals. It follows that the number of portals at a given box is $\rho^{-O(d)}$.

The definition a cell $C$ of the dynamic program is a tuple

$$ (B, \langle n_{p_1,d_1}^{\text{in}}, n_{p_1,d_2}^{\text{in}}, \ldots, n_{p_1,d_{\text{max}}}^{\text{out}} \rangle, \langle n_{p_2,d_1}^{\text{in}}, \ldots, n_{p_2,d_{\text{max}}}^{\text{out}} \rangle, \ldots, \langle n_{p_{\rho-O(d)},d_1}^{\text{in}}, \ldots, n_{p_{\rho-O(d)},d_{\text{max}}}^{\text{out}} \rangle, k_B) $$

where $p_1, \ldots, p_{\rho-O(d)}$ are the portals of box $B$, and $d_i$ are power of $(1 + \varepsilon^2/\log n)$ in the range [minimum distance; maximum distance]. Given such a cell $C$ we say that $n_{p_i,d_j}^{\text{out}}$ is a parameter of $C$.

Then, the value of such a table cell is the the value of the best valid solution for the clients in box $B$ under the constraints that:

1. For each portal $p$ of $B$, $n_{p_i,d_j}^{\text{in}}$ clients from the inside of $B$ at distance in $[d_j; (1 + \varepsilon)d_j]$ that are served outside $B$ and crossing at $p$, $n_{p_i,d_j}^{\text{out}}$ clients from the outside of $B$ at distance in $[d_j; (1 + \varepsilon)d_j]$ that are served inside $B$ and crossing at $p$. Note that only one of the two can be positive;
2. The solution opens \( k_B \) of centers open inside \( B \), including the badly cut centers of \( L \) inside \( B \);

Eventually, we consider the solutions at the root \( R \), with the following set of parameters, each portal \( p \) of \( R \) is such that \( n_p = 0 \), and \( k_R = k \). Among all these solutions, the algorithm outputs the one with minimum cost.

The base cases of the dynamic program consist of smallest-size cells where at most one facility can be opened at a given location. The value of each base-case cell can be obtained by computing the one with minimum cost.

The number of clients within the cells that are assigned outside should be at most \( n_{\text{out}} \) for each portal \( p_i \), and the total number of clients in the cell minus \( n_{\text{in}} \) plus \( n_{\text{out}} \) should be at most \( \eta \).

Computing the value of a non-base-case cell \( C \)

\[
(B, \langle (n_{p_1,d_1}^{\text{in}}, n_{p_1,d_1}^{\text{out}}, \ldots, n_{p_1,d_{\max}}^{\text{out}}), \ldots, (n_{p_k,d_1}^{\text{in}}, n_{p_k,d_1}^{\text{out}}, \ldots, n_{p_k,d_{\max}}^{\text{out}}), k_B \rangle),
\]

of the DP can be done by iterating over all tuples of cells of the DP such that each child box \( B' \) of \( B \) appears in exactly one DP-cell of the tuple, and all DP cells of the tuples are associated with child boxes of \( B \), and taking the tuple that is compatible and whose sum of values of entry cells is minimized. The tuple is compatible with cell \( C \) if for each \( n_{p_i,d_j}^{\text{in}} \) of the definition of \( C \), one can assign \( n_{p_i,d_j}^{\text{in}} \) clients to each portal \( p_i \) under the constraint that one can assign at most \( n_{p_i,d_j}^{\text{in}} \) from a each portal \( p_u \) of a DP-cell \( C' \) of the tuple such that \( n_{p_u,d_j}^{\text{in}} \) is part of the definition of \( C' \) and \( d_v + \text{dist}(p_u,p_i) = (1 \pm \varepsilon^2/\log n) d_j \). Moreover the sum of the \( k_{B'} \) for each \( B' \) of the tuple has to be at most \( k_B \). The verification of the compatibility of an assignment is done through enumeration of all possibilities.

We show that our dynamic program outputs a solution of cost at most \((1 + \varepsilon)\operatorname{cost}_{I_{D,L}} \operatorname{cost}(\hat{\mathcal{G}})\), where \( \hat{\mathcal{G}} \) is as defined per the definition of valid solution.

**Lemma 5.1.** The above dynamic program produces a solution of cost at most \((1 + \varepsilon)\operatorname{cost}_{I_{D,L}} \operatorname{cost}(\hat{\mathcal{G}})\), where \( \hat{\mathcal{G}} \) is as per the definition of valid solution. Moreover, the running time is at most \( \exp((\rho/\varepsilon)^{-O(d)}) \).

**Proof.** To prove the approximation guarantee, we need to argue that:

1. Forcing the assignment path of each client to make a detour through the closest portal whenever it leaves a region in the new instance does not increase the cost by a factor of more than \((1 + \varepsilon)\); and

2. Rounding the number of clients coming through each portals to a power of \((1 + \varepsilon^2/\log n)\) yields a solution of cost at most \((1 + \varepsilon)\operatorname{cost}_{I_{D,L}} \operatorname{cost}(\hat{\mathcal{G}})\).

We first prove (1). Consider instance \( I_{D,L} \). Consider a client \( c \) that is not badly cut and \( I'_{D,L} \) together with the facility \( \hat{\mathcal{G}}(c) \) that serves it in solution \( \hat{\mathcal{G}} \). We have that \( c \) and \( \hat{\mathcal{G}}(c) \) are separated at a level \( u(c) = \log(d \cdot \log n \cdot (c/(p+1)^{d})) + \text{dist}(c, \hat{\mathcal{G}}(c)) + 1 \), by the definition of (not) badly cut. Thus, consider the solution where, in each level where \( c \) and \( \hat{\mathcal{G}}(c) \) are in different boxes, the path makes
a detour to the closest portal in the box containing \( c \). This incurs a detour of \( \rho^{2^{i+1}} \) for each such box of level \( i \). We have that the total detour for the path is at most

\[
\sum_{i=0}^{u(c)} \rho^{2^{i+1}} \leq \sum_{i=0}^{u(c)} \rho^{2^{u(c)+1-i}} = \sum_{i=0}^{u(c)} \rho \frac{2^{u(c)+1}}{2^i} \\
\leq \rho 2^{u(c)+2} \\
\leq 16 \rho (d - \log n \cdot \log (\varepsilon/(p+1))^{\rho q}) \cdot \text{dist}(c, \hat{G}(c))
\]

Setting \( \rho = \varepsilon (16d - \log n \cdot \log (\varepsilon/(p+1))^{\rho q})^{-1} \) shows that the overall detour is at most \( \varepsilon \text{dist}(c, \hat{G}(c)) \).

Now consider a client \( c \) that is badly cut. In instance \( I_{D,L} \), we have that \( c \) is relocated to the facility \( f \) of \( L \) that serves it in solution \( L \). Then two things can happen: if \( f \) is not badly cut, then in instance \( I_{D,L} \) we have that \( c \) is \( c \) and the facility \( \hat{G}(c) \) that serves it in \( \hat{G} \) are separated at a level \( u(c) \leq \log(d - \log n \cdot \log (\varepsilon/(p+1))^{\rho q}) + \log \text{dist}(c, \hat{G}(c)) + 1 \). Hence, the cost increase for \( c \) due to enforcing the detours to the closest portals in instance \( I_{D,L} \) is at most \( (1 + \varepsilon) \) times the assignment cost of \( c \) in solution \( \hat{G}(c) \). Otherwise, \( f \) is badly cut and so, by definition of \( \hat{G} \), \( f \) serves \( c \) in \( \hat{G} \) and so the portals have no effect.

An immediate induction shows that our dynamic program computes a valid solution \( S' \) such that \( \text{cost}_{I_{D,L}}(S') \leq (1 + O(\varepsilon)) \text{cost}_{I_{D,L}}(\hat{G}) \). Namely, the cost incurred by rounding distances to power of \((1 + \varepsilon^2/\log n)\) over the \( O(\log n) \) levels of the dynamic program only incurs an overall cost increase of \((1 + \varepsilon^2/\log n \log n = (1 + O(\varepsilon^2)) \). The rest of the dynamic program is exact. Thus, Lemma 3.2 implies that cost\((S') \leq (1 + \varepsilon)\text{cost}(\text{OPT}) + \varepsilon\text{cost}(L) \).

The running time follows from the definition of the dynamic program and the choice of the number of portals and the fact that \( \frac{\max \text{ distance}}{\min \text{ distance}} = n^{\Omega(1)} \). The running time is \( \exp((\rho/\varepsilon)^{-O(d)}) \).

The proof of Theorem 1.2 almost follows from combining Lemma 5.1 and Lemma 3.2. The only thing that remains to be proven is that we can find a solution \( L \) of cost \( O(\text{cost}(\text{OPT})) \). Unfortunately, nothing better than an \( O(\log n) \)-approximation is known. We thus repeat the above algorithm until we find a solution of cost at most \( (1 + \varepsilon)\text{cost}(\text{OPT}) \). Namely, we start with \( L \) being an \( O(\log n) \)-approximation. We then apply the algorithm to obtain a solution \( L_1 \) of cost at most \( (1 + \varepsilon)\text{cost}(\text{OPT}) + \varepsilon\text{cost}(L) \) with probability at least \( 1 - \varepsilon/\log n \); boosting the probability is always possible by repeating the random step (i.e.: computing a new randomized split-tree decomposition) and outputting the best solution among the different solution computed. We then apply the algorithm again to \( L_1 \) and find a solution \( L_2 \) of cost at most \( (1 + \varepsilon)\text{cost}(\text{OPT}) + \varepsilon\text{cost}(L_1) \) with probability at least \( 1 - \varepsilon/\log n \). Repeating this process \( s = O(\log \log n) \) times yields a solution \( L_s \) of cost at most \( (1 + \varepsilon)\text{cost}(\text{OPT}) \) with probability at least \( 1 - \varepsilon \). We note that similar techniques have been used in [5].

6 A PTAS for Capacitated \( k \)-Median in \( \mathbb{R}^2 \) – Proof of Theorem 1.1

We aim at providing a faster algorithm for the Euclidean plane with running time \( 2^{\rho^{-1}} \text{poly}(n) \), which as we will see is \( n^{\varepsilon^{-O(1)}} \).

Our algorithm differs from the the algorithm described in Section 5 in two ways:

1. the size of the net in each box is \( \rho^{-1} = f(\varepsilon) \log n \) for some computable function \( f \).
2. the table entries of the dynamic program are smaller: the number \( n_p \in [n] \) that is kept for each portal \( p \) is a power of \( (1 + \varepsilon) \).
In the case of the plane, the algorithm uses the randomized quad-tree dissection of Arora. This also satisfies the properties 1-4 described in Section 3. Any box of the dissection of level \( i \) is a \( 2^{i+1} \times 2^{i+1} \) square. We condition on the event that the instance \( \mathcal{I}_{D,L} \) is good which by Lemma 5.2 happens with probability at least \( 1 - \varepsilon \).

Reducing the size of the net simply follows the standard analysis of Arora et al. 8. We only place portals on the boundaries of boxes and consider solutions that are such that each path connecting a client \( c \) to the facility \( f \) that serves it in OPT is forced to make a detour for each boundary of a box \( B \) it crosses. When using a portal set of size \( \rho \), the length of each such detour is then \( \rho \epsilon B \), where \( \partial B \) denotes the perimeter of box \( B \) and so less than twice is diameter.

Therefore, the discussion of Section 5 still holds: from Lemma 5.2 and the definition of a valid instance, the badly cut clients are not a problem anymore: their assignment cost is either 0 (in case their center in \( L \) is also badly cut) or well approximated by portals (c.f. Eq 3). Similarly for the clients that are not badly cut, their assignment cost is well approximated by portal (c.f. Eq 3 again). More formally, let \( B(c) \) denotes the set of box that contains \( c \) and that do not contain the facility \( f \) that serves \( c \) in OPT. Thus, we write that the total detour for using portals is at most \( \sum_{B \in B(c)} \rho^\ell(B(c)) \epsilon B \), where \( \ell(B(c)) \) is the level of box \( B(c) \). Therefore, the overall detour is at most \( 16 \epsilon \text{dist}(c, f) \) since we condition on the event that \( \mathcal{I}_{D,L} \) is a good instance.

This implies an algorithm running in time \( \exp(O(f(\varepsilon) \log^2 n)) \) through the following naive implementation. The dynamic program defines an entry by a tuple \((B, \langle n_{p_1}^{in}, n_{p_1}^{out}, \ldots, n_{p_{\rho-1}}^{in}, n_{p_{\rho-1}}^{out} \rangle, k_B)\), where \( p_1, \ldots, p_{\rho-1} \) are the portals of box \( B \). Then, the value of such a table entry is the value of the best valid solution for the clients in box \( B \) under the constraints that:

1. for each portal \( p \) of box \( B \), \( n_{p}^{in} \) clients coming from the inside of \( B \) and that are assigned to a center outside \( B \) and crossing at \( p \), and \( n_{p}^{out} \) clients outside \( B \) are assigned to a center inside \( B \) and are crossing at \( p \).
2. the solution opens at most \( k_B \) centers inside \( B \).

The running time of this algorithm is \( \exp O(\rho^{-1} \log n) \). However, we aim at getting a truly polynomial-time algorithm. We thus makes the following optimization.

We now show how to speed this up. For each box \( B \), we pick an arbitrary portal of \( p, p_B^* \) – when \( B \) is clear from the context we simply refer to it by \( p^* \). Since portals are placed on the perimeter of a square, we can impose an ordering from \( p^* \) in clockwise manner: \( p^* = p_0, p_1 \) is the next portal on the boundary after \( p^* \) in the clockwise order, and so on.

Our algorithm only considers table entries \((B, \langle n_{p_1}^{in}, n_{p_1}^{out}, \ldots, n_{p_{\rho-1}}^{in}, n_{p_{\rho-1}}^{out} \rangle, k_B)\) that are such that \( \varepsilon^2 n_{p_1} \leq n_{p_{\rho-1}} \leq n_{p_1}/\varepsilon^2 \). We say that such a table entry satisfies constraint 1. A solution satisfies constraint 1 if it can be described by a table entry that satisfies constraint 1 for each box \( B \).

Moreover, our algorithm only considers table entries \((B, \langle n_{p_1}^{in}, n_{p_1}^{out}, \ldots, n_{p_{\rho-1}}^{in}, n_{p_{\rho-1}}^{out} \rangle, k_B)\) that satisfy constraint 1 and where each \( n_{p}^{in} \) and \( n_{p}^{out} \) is a multiple of \((1 + \varepsilon_0) \), for some \( \varepsilon_0 \) that will be chosen later, except for \( p_B^* \) for which the value is still in \([n]\). We refer to this constraint imposed on solutions as constraint 2: namely, a solution that can be described through by such table entries is a solution that satisfies constraint 2. Similarly, a table entry is said to satisfy constraint 2 if each of the \( (n_{p}^{in}, n_{p}^{out}) \) are power of \((1 + \varepsilon_0) \), except for \( p^* \).

**Claim 2.** The running time of the dynamic programming algorithm that only considers entries that satisfy constraint 2 (and so also constraint 1) is \( n^{O(1)} \exp(\rho^{-1} \log((\varepsilon_0 \rho)^{-1})) \).

\(^2\)Note that in an optimal assignment at most one of \( n_{p}^{in}, n_{p}^{out} \) is non-negative but this does not impact the asymptotic running time.
Proof. Fix a given box $B$. There are $n^2$ possible values of $(n^\text{in}_p, n^\text{out}_p)$ and $k$ possible values for $k_B$. Then, for the $(n^\text{in}_p, n^\text{out}_p)$, each value is in the range $\varepsilon^2 n_{p,i} \leq n_{p,i} \leq n_{p,i-1}/\varepsilon^2$ and a power of $(1 + \varepsilon_0)$ and so there are at most $O(\log(1/\varepsilon/\varepsilon_0))$ possibilities. This shows that the total number of entries $(B, (n^\text{in}_{p_1}, n^\text{out}_{p_1}), (n^\text{in}_{p_2}, n^\text{out}_{p_2}), \ldots, (n^\text{in}_{p_{n-1}}, n^\text{out}_{p_{n-1}}), k_B)$ that satisfy constraint 2 is at most $(\log(1/\varepsilon)/\varepsilon_0)^{\rho-1} n^3$. It follows that the running time of the algorithm is at most $\exp \rho^{-1} \log((\varepsilon/\varepsilon_0)^{-1}) n^{O(1)}$. \hfill $\square$

To conclude, we now need to prove that there exists a near-optimal solution that satisfies constraint 2, this is done in the following claim. The proof of the theorem then follows immediately.

### Claim 3.
If $\varepsilon_0 \leq \varepsilon^5$, there exists a $(1 + O(\varepsilon))$-approximate solution that satisfies constraint 2 and forces the assignment path of each client to leave a box only at its portals.

Proof. We first observe that our analysis of the detours paid by each client to reach the facility it is assigned to in an optimal solution is as follows. Each client $c$ going through a portal of a box $B$ of level $i$ pays a detour of up to $\alpha \rho 2^i$ in the worst-case, where $\alpha$ is a large enough constant. In the case where each client pays a worst-case detour, the cost of the best solution that forces the assignment path of each client to leave a box only at its portals remains at most $(1 + \varepsilon)OPT$ (for the right choice of $\rho$, as a function of $\alpha$).

Thus, we will give each client $c$ an additional budget of $\alpha \rho 2^i$ for level $i$. The best solution that forces the assignment path of each client to leave a box only at its portals plus pays the budget is at most $(1 + 2\varepsilon)OPT$. Namely, any $(1 + \varepsilon)$-approximate solution such that each client pays its budget remains a $(1 + O(\varepsilon))$-approximation to an optimal solution.

We first consider the cost of making a solution satisfy constraint 1. We first claim that for two portals $p, p'$ that are consecutive on the boundary of a box $B$ of level $x$, there exists a near-optimal solution such that $\varepsilon^2 n_{p'} \leq n_p \leq n_{p'}/\varepsilon^2$ (and each client its assignment path leaves a box only at its portals). Indeed, consider the optimal solution that forces the assignment path of each client to leave a box only at its portals, and assume that $\varepsilon^2 n_{p'} > n_p$. Then, consider forcing $\varepsilon^2 n_{p'}$ clients that go to $n_{p'}$ to make an extra-detour to $n_p$. The length of this detour is at most $\rho 2^x + 1$. Hence, the extra-cost is $n_{p'} \rho 2^x + 1$. Now, observe that this is at most $\varepsilon^2$ times the budget of all the clients going to $n_{p'}$. A similar argument holds in the case where $n_p > n_{p'}/\varepsilon^2$. Thus, consider the boundary of $B$ and an arbitrary portal $p_0$ of $B$. Let $p_0, p_1, \ldots$ be the portals in the order given by a clockwise walk on the boundary of $B$ starting at $p_0$. Visit the portals in that order and ensure that $\varepsilon^2 p_i \leq p_{i+1}$ for all $i$ using the above transformation iteratively. What is the overall cost? observe that the clients $n(p_i)$ that are initially going through portal $p_i$ may now be assigned to a portal $p_j$ where $j$ is much larger than $i$. What is the total cost for this? We have that at each portal at most an $\varepsilon$ fraction of the clients can be moved again. Thus, the total extra cost for the clients of $n(p_i)$ is at most $n_{p_i}/2^x \leq p_{i+1}$. This choices that there exists a near-optimal solution that forces the assignment path of each client to leave a box only at its portals and that satisfies constraint 1.

We now turn to prove that given such a solution, there exists a solution that forces the assignment path of each client to leave a box only at its portals and that satisfies constraint 2. We show that that except for one portal denoted by $p^*$, the numbers $n_p$ could be approximated to power of $(1 + \varepsilon^5)$ in the following way. We again consider the portals in clockwise order, starting from $p_0 = p^*$. The initial number of clients $n_{p_i}^0$ assigned to portal $p_i$ is the one prescribed after the
above transformation. For the \( i \)th portal \( p_i \), \( i > 0 \), let \( n_{p_i} \) number of clients assigned to \( p_i \) when the procedure visits \( p_i \). Let \( n_{p_i} \) be the power of \((1 + \epsilon^5)\) that is the closest to \( n_{p_i} \) and smaller than \( n_{p_i} \). We reassign \( n_{p_i} \) to \( n_{p_i} \leq \epsilon^3 n_{p_i} \) clients of \( n(p_i) \) to \( p_{i+1} \).

By doing so iteratively, we end up with an assignment where, except for \( p^* \) which may receive from \( p_{\rho - 1} \) and not give to any other portal, \( n_{p_i} \) is a power of \((1 + \epsilon^5)\). We now bound the cost of the reassignment. We first show that \( n_{p_i} \leq (1 + \epsilon^2)n_{p_i}^0 \). This is true for \( i \in \{1, 2\} \) since \( n_{p_1}^0 \leq n_{p_2}^0 / \epsilon \) and the total number of clients moved from \( p_1 \) to \( p_2 \) is at most \( \epsilon^5 n_{p_1}^0 \) and so \( n_{p_2} \leq (1 + \epsilon^5) n_{p_2}^0 \). We assume that this is true up to \( p_{i-1} \) and show that it holds for \( p_i \). The number of clients received by \( p_i \) is thus at most \( \epsilon^5 (1 + \epsilon^2)n_{p_i-1}^0 / \epsilon^2 \) by the inductive hypothesis. This is at most \( \epsilon^5 (1 + \epsilon^2)n_{p_i}^0 / \epsilon^2 \leq \epsilon^3 (1 + \epsilon^2)n_{p_i}^0 \leq \epsilon^2 n_{p_i}^0 \) for any \( \epsilon \leq 1/2 \).

It follows that the clients of \( n(p_i) \) that are reassigned to \( p_{i+1} \) can be chosen from the clients that are initially assigned to \( p_i \) and so each client that is assigned to portal \( p_i \) (in the solution satisfying constraint 1) is now assigned either to portal \( p_i \) or to portal \( p_{i+1} \). It follows that the extra cost is at most the total budget of level \( i \) for the clients going to the portals and the claim follows. \(\square\)

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