Some curiosities of the algebra of bounded Dirichlet series
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Abstract. It is shown that the algebra $\mathcal{H}^\infty$ of bounded Dirichlet series is not a coherent ring, and has infinite Bass stable rank. As corollaries of the latter result, it is derived that $\mathcal{H}^\infty$ has infinite topological stable rank and infinite Krull dimension.

1. Introduction

The aim of this short note is to make explicit two observations about algebraic properties of the ring $\mathcal{H}^\infty$ of bounded Dirichlet series. In particular we will show that

1. $\mathcal{H}^\infty$ is not a coherent ring. (This is essentially an immediate consequence of Eric Amar’s proof of the noncoherence of the Hardy algebra $H^\infty(\mathbb{D}^n)$ of the polydisk $\mathbb{D}^n$ for $n \geq 3$ [1].)

2. $\mathcal{H}^\infty$ has infinite Bass stable rank. (This is a straightforward adaptation of the first author’s proof of the fact that the stable rank of the infinite polydisk algebra is infinite [12]). As corollaries, we obtain that $\mathcal{H}^\infty$ has infinite topological stable rank, and infinite Krull dimension.

Before giving the relevant definitions, we briefly mention that $\mathcal{H}^\infty$ is a closed Banach subalgebra of the classical Hardy algebra $H^\infty(\mathbb{C}_{>0})$ consisting of all bounded and holomorphic functions in the open right half plane

$$\mathbb{C}_{>0} := \{ s \in \mathbb{C} : \text{Re}(s) > 0 \},$$

and it is striking to compare our findings with the corresponding results for $H^\infty(\mathbb{C}_{>0})$:

|             | $H^\infty(\mathbb{C}_{>0})$ | $\mathcal{H}^\infty$ |
|-------------|--------------------------|-----------------------|
| Coherent?   | Yes (See [11])           | No                    |
| Bass stable rank | 1 (See [17])         | $\infty$             |
| Topological stable rank | 2 (See [16])       | $\infty$             |
| Krull dimension | $\infty$ (See [13])  | $\infty$             |

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Nevertheless the above results for $\mathcal{H}^\infty$ lend support to Harald Bohr’s idea of interpreting Dirichlet series as functions of infinitely many complex variables, a key theme used in the proofs of the main results in this note. We recall the pertinent definitions below.

1.1. The algebra $\mathcal{H}^\infty$ of bounded Dirichlet series. $\mathcal{H}^\infty$ denotes the set of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

(1.1)

where $(a_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers, such that $f$ is holomorphic and bounded in $\mathbb{C}_{>0}$. Equipped with pointwise operations and the supremum norm,

$$\|f\|_\infty := \sup_{s \in \mathbb{C}_{>0}} |f(s)|, \quad f \in \mathcal{H}^\infty,$$

$\mathcal{H}^\infty$ is a unital commutative Banach algebra. In [8, Theorem 3.1], it was shown that the Banach algebra $\mathcal{H}^\infty$ is precisely the multiplier space of the Hilbert space $\mathcal{H}$ of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for which

$$\|f\|^2_\mathcal{H} := \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$ 

The importance of the Hilbert space $\mathcal{H}$ stems from the fact that its kernel function $K_\mathcal{H}(z, w)$ is related to the Riemann zeta function $\zeta$:

$$K_\mathcal{H}(z, w) = \zeta(z + \overline{w}).$$

For $m \in \mathbb{N}$, let $\mathcal{H}_m^\infty$ be the closed subalgebra of $\mathcal{H}^\infty$ consisting of Dirichlet series of the form (1.1) involving only integers $n$ generated by the first $m$ primes $2, 3, \ldots, p_m$.

1.2. $\mathcal{H}^\infty = H^\infty(\mathbb{D}^\infty)$. In [8, Lemma 2.3 and the proof of Theorem 3.1], it was established that $\mathcal{H}^\infty$ is isometrically (Banach algebra) isomorphic to $H^\infty(\mathbb{D}^\infty)$, a certain algebra of functions analytic in the infinite dimensional polydisk, defined below. As this plays a central role in what follows, we give an outline of this based on [8], [15] and [10].

A seminal observation made by H. Bohr [3], is that if we put

$$z_1 = \frac{1}{2^s}, \quad z_2 = \frac{1}{3^s}, \quad z_3 = \frac{1}{5^s}, \ldots, \quad z_n = \frac{1}{p_n^s}, \ldots,$$

where $p_n$ denotes the $n$th prime, then, in view of the Fundamental Theorem of Arithmetic, formally a Dirichlet series in $\mathcal{H}_n^\infty$ or $\mathcal{H}^\infty$ can be considered as a power series of infinitely many variables. Indeed, each $n$ has a unique expansion

$$n = p_1^{\alpha_1(n)} \cdots p_r^{\alpha_r(n)}(n),$$
with nonnegative $\alpha_j(n)$s, and so, from (1.1), we obtain the formal power series

$$F(z) = \sum_{n=1}^{\infty} a_n z_{\alpha_1(n)} \cdots z_{\alpha_r(n)}^{(n)},$$

where $z = (z_1, \cdots, z_m)$ or $z = (z_1, z_2, z_3, \cdots)$ depending on whether $f$ is a function in $\mathcal{H}_m^\infty$ or in $\mathcal{H}^\infty$. Let us recall Kronecker’s Theorem on diophantine approximation [7, Chapter XXIII]:

**Proposition 1.1.** For each $m \in \mathbb{N}$, the map
t \mapsto (2^{-it}, 3^{-it}, \cdots, p_m^{-it}) : (0, \infty) \to \mathbb{T}^m
has dense range in $\mathbb{T}^m$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Using the above and the Maximum Principle, it can be shown that for $f \in \mathcal{H}_m^\infty$,

$$\|f\|_\infty = \|F\|_\infty,$$

where the norm on the right hand side is the $H_\infty$ norm of these functions $F_m$ denotes the usual Hardy algebra of bounded holomorphic functions on the polydisk $\mathbb{D}^m$, endowed with the supremum norm:

$$\|F\|_\infty := \sup_{z \in \mathbb{D}^m} |F(z)|, \quad F \in H_\infty(\mathbb{D}^m).$$

In [8], it was shown that this result also holds in the infinite dimensional case. In order to describe this result, we introduce some notation. Let $c_0$ be the Banach space of complex sequences tending to 0 at infinity, with the induced norm from $\ell_\infty$, and let $B$ be the open unit ball of that Banach space. Thus with $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$,

$$B = c_0 \cap \mathbb{D}^\mathbb{N}.$$ 

For a point $z = (z_1, \cdots, z_m, \cdots) \in B$, we set $z^{(m)} := (z_1, \cdots, z_m, 0, \cdots)$, that is, $z_k = 0$ for $k > m$. Substituting $z^{(m)}$ in the argument of $F$ given formally by (1.2), we obtain a function

$$(z_1, \cdots, z_m) \mapsto F(z^{(m)}),$$

which we call the $m$th-section $F_m$ (after Bohr’s terminology “$m$te abschnitt”). $F$ is said to be in $H_\infty(\mathbb{D}^\infty)$ if the $H_\infty$ norm of these functions $F_m$ are uniformly bounded, and denote the supremum of these norms to be $\|F\|_\infty$. Using Schwarz’s Lemma for the polydisk, it can be seen that for $m < \ell$,

$$|F(z^{(m)}) - F(z^{(\ell)})| \leq 2\|f\|_\infty \cdot \max\{|z_j| : m < j \leq \ell\},$$

and so we may define

$$F(z) = \lim_{m \to \infty} F(z^{(m)}).$$

It was shown in [8] that (1.3) remains true in the infinite dimensional case, and so we may associate $\mathcal{H}_m^\infty$ with $H_\infty(\mathbb{D}^\infty)$.

**Proposition 1.2 ([8]).** There exists a Banach algebra isometric isomorphism $\iota : \mathcal{H}_m^\infty \to H_\infty(\mathbb{D}^\infty)$. 
1.3. Coherence.

Definition 1.3. Let $R$ be a unital commutative ring, and for $n \in \mathbb{N}$, let $R^n = R \times \cdots \times R$ ($n$ times).

For $f = (f_1, \cdots, f_n) \in R^n$, a relation $g$ on $f$ is an $n$-tuple $g = (g_1, \cdots, g_n)$ in $R^n$ such that

$$g_1 f_1 + \cdots + g_n f_n = 0.$$ 

The set of all relations on $f$ is denoted by $f^\perp$.

The ring $R$ is said to be coherent if for each $n$ and each $f \in R^n$, the $R$-module $f^\perp$ is finitely generated.

A property which is equivalent to coherence is that the intersection of any two finitely generated ideals in $R$ is finitely generated, and the annihilator of any element is finitely generated [4]. We refer the reader to the article [5] and the monograph [6] for the relevance of the property of coherence in commutative algebra. All Noetherian rings are coherent, but not all coherent rings are Noetherian. (For example, the polynomial ring $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is not Noetherian because the sequence of ideals $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, x_3 \rangle \subset \cdots$ is ascending and not stationary, but $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is coherent [6, Corollary 2.3.4].)

In the context of algebras of holomorphic functions in the unit disk $D$, we mention [11], where it was shown that the Hardy algebra $H^\infty(D)$ is coherent, while the disk algebra $A(D)$ isn’t. For $n \geq 3$, Amar [1] showed that the Hardy algebra $H^\infty(D^n)$ is not coherent. (It is worth mentioning that whether the Hardy algebra $H^\infty(D^2)$ of the bidisk is coherent or not seems to be an open problem.) Using Amar’s result, we will prove the following result:

Theorem 1.4. $H^\infty$ is not coherent.

1.4. Stable rank. In algebraic $K$-theory, the notion of (Bass) stable rank of a ring was introduced in order to facilitate $K$-theoretic computations [2].

Definition 1.5. Let $R$ be a commutative ring with an identity element (denoted by 1).

An element $(a_1, \cdots, a_n) \in R^n$ is called unimodular if there exist elements $b_1, \cdots, b_n$ in $R$ such that

$$b_1 a_1 + \cdots + b_n a_n = 1.$$ 

The set of all unimodular elements of $R^n$ is denoted by $U_n(R)$.

We say that $a = (a_1, \cdots, a_{n+1}) \in U_{n+1}(R)$ is reducible if there exists an element $(x_1, \cdots, x_n) \in R^n$ such that

$$(a_1 + x_1 a_{n+1}, \cdots, a_n + x_n a_{n+1}) \in U_n(R).$$

The Bass stable rank of $R$ is the least integer $n \in \mathbb{N}$ for which every $a \in U_{n+1}(R)$ is reducible. If there is no such integer $n$, we say that $R$ has infinite stable rank.
Using the same idea as in [12, Proposition 1] (that the infinite polydisk algebra \( A(\mathbb{D}^\infty) \) has infinite Bass stable rank), we show the following.

**Theorem 1.6.** The Bass stable rank of \( \mathcal{H}^\infty \) is infinite.

For Banach algebras, an analogue of the Bass stable rank, called the topological stable rank, was introduced by Marc Rieffel in [14].

**Definition 1.7.** Let \( R \) be a commutative complex Banach algebra with unit element 1. The least integer \( n \) for which \( U_n(R) \) is dense in \( R^n \) is called the topological stable rank of \( R \). We say \( R \) has infinite topological stable rank if no such integer \( n \) exists.

**Corollary 1.8.** The topological stable rank of \( \mathcal{H}^\infty \) is infinite.

**Proof.** This follows from the inequality that the Bass stable rank of a commutative unital semisimple complex Banach algebra is at most equal to its topological stable rank; see [14, Corollary 2.4]. \( \square \)

**Definition 1.9.** The Krull dimension of a commutative ring \( R \) is the supremum of the lengths of chains of distinct proper prime ideals of \( R \).

**Corollary 1.10.** The Krull dimension of \( \mathcal{H}^\infty \) is infinite.

**Proof.** This follows from the fact that if a ring has Krull dimension \( d \), then its Bass stable rank is at most \( d + 2 \); see [9]. \( \square \)

2. **Noncoherence of \( \mathcal{H}^\infty \)**

We will use the following fact due to Amar [1, Proof of Theorem 1.(ii)].

**Proposition 2.1.** \((z_1 - z_2, z_2 - z_3)^\perp\) is not a finitely generated \( H^\infty(\mathbb{D}^3)\)-module.

**Proof of Theorem 1.4.** The main idea of the proof is that, using the isomorphism \( \iota \), essentially we boil the problem down to working with \( H^\infty(\mathbb{D}^\infty) \). Let

\[
\begin{align*}
f_1 &:= \frac{1}{2^s} - \frac{1}{3^s}, \\
f_2 &:= \frac{1}{3^s} - \frac{1}{5^s}.
\end{align*}
\]

Then \( \iota(f_1) = z_1 - z_2 \) and \( \iota(f_2) = z_2 - z_3 \). Suppose that \((f_1, f_2)^\perp\) is a finitely generated \( \mathcal{H}^\infty\)-module, say by

\[
\begin{bmatrix}
g_1^{(1)} \\
g_1^{(2)}
\end{bmatrix}, \ldots, \begin{bmatrix}
g_r^{(1)} \\
g_r^{(2)}
\end{bmatrix} \in (\mathcal{H}^\infty)^2.
\]

We will show that the 3rd section of the image under \( \iota \) of the above elements generate \((z_1 - z_2, z_2 - z_3)^\perp\) in \( H^\infty(\mathbb{D}^3) \), contradicting Proposition 2.1. If

\[
\begin{bmatrix}
G^{(1)} \\
G^{(2)}
\end{bmatrix} \in \left(H^\infty(\mathbb{D}^3)\right)^2 \cap (F_1, F_2)^\perp,
\]

\]
then $F_1G^{(1)} + F_2G^{(2)} = 0$, and by applying $\iota^{-1}$, we see that 
\[
\begin{bmatrix}
\iota^{-1}G^{(1)} \\
\iota^{-1}G^{(2)}
\end{bmatrix} \in (f_1, f_2)^\perp.
\]
So there exist $\alpha^{(1)}, \ldots, \alpha^{(r)} \in \mathcal{H}^\infty$ such that 
\[
\begin{bmatrix}
\iota^{-1}G^{(1)} \\
\iota^{-1}G^{(2)}
\end{bmatrix} = \alpha^{(1)} \begin{bmatrix} g_1^{(1)} \\ g_1^{(2)} \end{bmatrix} + \cdots + \alpha^{(r)} \begin{bmatrix} g_r^{(1)} \\ g_r^{(2)} \end{bmatrix}.
\]
Applying $\iota$, we obtain 
\[
\begin{bmatrix} G^{(1)} \\ G^{(2)} \end{bmatrix} = \iota(\alpha^{(1)}) \begin{bmatrix} \iota(g_1^{(1)}) \\ \iota(g_1^{(2)}) \end{bmatrix} + \cdots + \iota(\alpha^{(r)}) \begin{bmatrix} \iota(g_r^{(1)}) \\ \iota(g_r^{(2)}) \end{bmatrix}.
\]
Finally taking the 3rd section, we obtain 
\[
\begin{bmatrix} G^{(1)}(z_1, z_2, z_3) \\ G^{(2)}(z_1, z_2, z_3) \end{bmatrix} = \sum_{j=1}^r \begin{bmatrix} (\iota(g_1^{(j)}))(z^{(3)}) \\ (\iota(g_2^{(j)}))(z^{(3)}) \end{bmatrix}.
\]
So it follows that 
\[
\begin{bmatrix} (\iota(g_1^{(1)}))(z^{(3)}) \\ (\iota(g_1^{(2)}))(z^{(3)}) \\ \cdots \\ (\iota(g_r^{(1)}))(z^{(3)}) \\ (\iota(g_r^{(2)}))(z^{(3)}) \end{bmatrix}
\]
generate $(z_1 - z_2, z_2 - z_3)^\perp$, a contradiction to Amar’s result, Proposition 2.1. 
\[
\square
\]

3. Stable rank of $\mathcal{H}^\infty$

The proof of Theorem 1.6 is a straightforward adaptation of the first author’s proof of the fact that the Bass stable rank of the infinite polydisk algebra is infinite [12, Proposition 1]. In [12], the infinite polydisk algebra $A(\mathbb{D}^\infty)$ is the uniform closure of the algebra generated by the coordinate functions $z_1, z_2, z_3, \ldots$ on the countably infinite polydisk $\mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \cdots$.

**Proof of Theorem 1.6:** Fix $n \in \mathbb{N}$. Let $g \in \mathcal{H}^\infty$ be given by 
\[
g(s) := \prod_{j=1}^n \left( 1 - \frac{1}{(p_j p_{n+j})^s} \right) \in \mathcal{H}^\infty. \tag{3.1}
\]
Set 
\[
f := \left( \frac{1}{2^n}, \ldots, \frac{1}{p_n^n}, g \right) \in (\mathcal{H}^\infty)^{n+1}.
\]
We will show that $f \in U_{n+1}(\mathcal{H}^\infty)$ is not reducible. First let us note that $f$ is unimodular. Indeed, by expanding the product on the right hand side of (3.1), we obtain 
\[
g = 1 + \frac{1}{2^n} \cdot g_1 + \cdots + \frac{1}{p_n^n} \cdot g_n,
\]
for some appropriate \( g_1, \ldots, g_n \in \mathcal{H}^\infty \). Now suppose that \( f \) is reducible, and that there exist \( h_1, \ldots, h_n \in \mathcal{H}^\infty \) such that
\[
\left( \frac{1}{2^s} + gh_1, \ldots, \frac{1}{p_n^s} + gh_n \right) \in U_n(\mathcal{H}^\infty).
\]
Let \( y_1, \ldots, y_n \in \mathcal{H}^\infty \) be such that
\[
\left( \frac{1}{2^s} + gh_1 \right) y_1 + \cdots + \left( \frac{1}{p_n^s} + gh_n \right) y_n = 1.
\]
Applying \( \iota \), we obtain
\[
(z_1 + \iota(g)(\iota(h_1)))\iota(y_1) + \cdots + (z_n + \iota(g)(\iota(h_n)))\iota(y_n) = 1. \tag{3.2}
\]
Let \( h := (\iota(h_1), \ldots, \iota(h_n)) \). For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), we define
\[
\Phi(z) = \begin{cases} 
-h(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, 0, \ldots) \prod_{j=1}^n (1 - |z_j|^2) 
& \text{for } |z_j| < 1, j = 1, \ldots, n, \\
0 & \text{otherwise.}
\end{cases}
\]
Then \( \Phi \) is a continuous map from \( \mathbb{C}^n \) into \( \mathbb{C}^n \). But \( \Phi \) vanishes outside \( \mathbb{D}^n \), and so
\[
\max_{z \in \mathbb{D}^n} \| \Phi(z) \|_2 = \sup_{z \in \mathbb{C}^n} \| \Phi(z) \|_2.
\]
This implies that there must exist an \( r \geq 1 \) such that \( \Phi \) maps \( K := r\mathbb{D}^n \) into \( K \). As \( K \) is compact and convex, by Brouwer’s Fixed Point Theorem it follows that there exists a \( z_* \in K \) such that
\[
\Phi(z_*) = z_*.
\]
Since \( \Phi \) is zero outside \( \mathbb{D}^n \), we see that \( z_* \in \mathbb{D}^n \). Let \( z_* = (\zeta_1, \ldots, \zeta_n) \). Then for each \( j \in \{1, \ldots, n\} \), we obtain
\[
0 = \zeta_j + (\iota(h_j)(\iota(g)))(\zeta_1, \ldots, \zeta_n, \overline{\zeta}_1, \ldots, \overline{\zeta}_n, 0, \ldots) \prod_{k=1}^n (1 - |\zeta_k|^2)
= \zeta_j + (\iota(h_j)\iota(g))(\zeta_1, \ldots, \zeta_n, \overline{\zeta}_1, \ldots, \overline{\zeta}_n, 0, \ldots). \tag{3.3}
\]
But from (3.2), we know that
\[
\sum_{j=1}^n (z_j + \iota(h_j)(\iota(g)))\iota(y_j) = 1,
\]
and this contradicts (3.3). As the choice of \( n \in \mathbb{N} \) was arbitrary, it follows that the Bass stable rank of \( \mathcal{H}^\infty \) is infinite. \( \square \)

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