Shell Model Symmetries

Levering Wolfe and Larry Zamick
Department of Physics and Astronomy,
Rutgers University, Piscataway, New Jersey 08854

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Abstract

We discuss $L=0$ vs $L=2$ couplings, symmetries of the pairing interaction for neutrons and protons, and the $J_{max}$ interaction. We show that certain schematic interactions yield exponentially decreasing transition strengths. We compare shell model calculations of $B(E2)$'s and quadrupole moments in the p-f shell with collective model results.

It is a pleasure to be able to contribute to proceedings honoring this milestone of Franco Iachello’s career. We will start with what may be regarded as early elementary precursors to his seminal papers with Akito Arima and others [1,2] on the interaction boson approximation -IBA and IBM. Then, as a change of pace we will discuss most recent work where we focus on matrix models of strength distributions for which there are many problems that are not yet understood.

MBZE WAVE FUNCTIONS

We first discuss single $j$ shell calculations in the $f_{7/2}$ shell which were carried out around 1964 by McCullen, Bayman and Zamick [3], Ginocchio and French [4], and with improved input (MBZE) in 2006 [5]. As our input Hamiltonian we take matrix elements from experiment—from the spectrum of $^{42}$Sc. The $T=1$ matrix elements for $J=0,2,4,6$ (in MeV) are 0.000, 1.5803, 2.8153, and 3.2420. The $T=0$ ones for $J=1,3,5,7$ are 0.6111, 1.4903, 1.5161 and 0.6163.

In Table 1 we show wave functions of $^{44}$Ti in the single j shell model. The columns are amplitudes for the protons to couple to $J_p$ and neutrons to $J_n$ for a state of total angular momentum $J=0$.

| Interaction | $J_p$ | $J_n$ |
|-------------|-------|-------|
| MBZE        | 0.6111| 1.4903|
| MBZE        | 0.6163| 1.5161|

Table 1: Wave functions of the lowest $J=0^+T=0$ state in $^{44}$Ti for various interactions—also for unique $J=0^+T=2$ state.
Note that the even J 2-body matrix elements above have isospin $T=1$ and the odd ones have isospin $T=0$. The even states are antisymmetric and can occur for 2 neutrons, 2 protons and for a neutron and a proton. The odd J states are symmetric and can only exist for the proton-neutron system $^{42}$Sc. Although $J=0$ is the lowest there are also low-lying $J=1$ and $J=J_{\text{max}}=7$ two-body matrix elements. This results in the fact that in the ground state of $^{44}$Ti there is a high probability that the 2 protons couple to $J_p=2$ and neutrons to $J_n=2$. For MBZE$^5$ the amplitude is 0.5616 or 31.5% probability. This may be regarded as a precursor to works which show the importance of $L=2$ couplings and of d bosons including IBA $^{1,2}$.

We also show the wave functions for a $J=0$ pairing interaction and likewise $J=J_{\text{max}}=7$. In the former case there is less (2,2) coupling than MBZE. One might wonder why one does not get 100% (0,0). This is shown in the last column where the unique $J=0$ $T=2$ state is shown. This is a double analog of a unique $J=0$ state of 4 neutrons i.e. $^{44}$Ca and so the wave function is unique and it necessary has 25% of the (0,0) strength. For $J=J_{\text{max}}$ one gets more (2,2) coupling than for MBZE.

### SENIORITY FOR PROTON NEUTRON SYSTEMS ($J=0$ PAIRING): ALSO $J_{\text{max}}$ INTERACTION

For the most part seniority considerations are generally applied to systems of identical particles, and there are well documented works including $^{6,7}$. However there are examples where seniority is relevant to systems of mixed neutrons and protons.

One example is the set of single j wave functions of $^{48}$Cr. In the $f_{7/2}$ model space the valence nucleons are at mid-shell. The quantity $s=(-1)^x$ with $x=(v_p+v_n)/2$ is a good quantum number. Some states have $s=+1$ and others $s=-1$. This was shown by Escudeors, Zamick and Bayman $^5$ and further work was done by Neergaard $^8$ and Kingan et al. $^9$. Note that neither $v_p,v_n$, or $v$ itself are good quantum numbers in the case-only $s$.

Another example is the pairing Hamiltonian of Edmond and Flowers$^{10}$. The states of mixed neutron-proton systems have quantum numbers $(v,T,t)$ where $v$ is the seniority, $T$ the total isospin and $t$ the reduced isospin. The latter is the isospin of nucleons not coupled to $J=0$. The energies of the states for $n$ nucleons is given by:

| $J_p,J_n$ | $J=0$ pairing | $J=7$ pairing | MBZE | $J=0$ $T=2$ |
|-----------|----------------|----------------|------|-------------|
| 0 0       | 0.8660         | 0.6486         | 0.7878 | -0.5000     |
| 2 2       | 0.2152         | 0.7143         | 0.5616 | 0.3737      |
| 4 4       | 0.2887         | 0.1452         | 0.2208 | 0.5000      |
| 6 6       | 0.3469         | 0.0058         | 0.1234 | 0.6009      |
\[ C \left\{ \frac{(n-v)(4j+8-n-v)}{4} - T(T+1) + t(t+1) \right\}. \]

There are several selection rules for M1 transitions in this model, as noted by Harper and Zamick[11]. The most interesting one is that although seniority can change by 2 units one cannot change seniority and reduced isospin at the same time. This is in contrast to the case for identical particles where the M1 operator cannot change the seniority. To explain this we note that the M1 operator must act on an np pair. If the M1 operator acts on a \( J=0 \) \( (T=1) \) pair it creates a \( J=1 \) \( (T=0) \) pair. Since the new pair has \( T=0 \) it will not affect the reduced isospin. Alternately if we act on a \( J=1 \). \( T=0 \) pair to create a \( J=0 \) \( T=1 \) pair, we note that because the initial pair has \( T=0 \) it does not affect the reduced isospin. On the other hand the seniority has been change by 2 units.

We look at the Edmond Flowers energy formula above[10] in some detail using \( ^{96}\text{Cd} \) as an example–2 proton holes and 2 neutron holes in the \( g_{9/2} \) shell. Note that there is no \( J \) in the formula. This leads to high degeneracy. With appropriate \( C \) and scaling the ground state energy is \( E=0 \) with seniority \( v=0 \). For \( v=2 \) we get states at \( E=1 \) MeV with even \( J \) values 2,4,6 and 8. Then we get \( v=4 \) states with \( J=10,12,14 \) and 16 at \( E=2.24 \) MeV. Note the big break in the spectrum from \( J=8 \) and 10. Zamick[12] claims that with realistic interactions there is also a mid-J break, although not so pronounced[10]. Using the realistic interaction of Qi [13] the differences \( E(J)-E(J+2) \) for \( J=0,2,4—16 \) are respectively 0.892,1.113, 1.041, 0.348, 1.681, 0.528, 0.128, and 0.216. We see that there is indeed a gap \( E(8)-E(10) =1.681 \) MeV. This is much larger than the neighboring differences. The concept of seniority splitting is behind this break. It deserves further investigation.

With regards to identical particles Escuderos and Zamick [14] noted that although seniority should not be good beyond \( j=7/2 \) there were 2 unique \( v=4 \) states in the \( (g_{9/2})^4 \) configuration, one with \( J=4^+ \) and the other with \( J=6^+ \) which remained eigenstates even when seniority violating interactions were used[14]. See further work by Y. Quian and C. Qi 15] and references therein.

Consider 2 holes in the \( g_{9/2} \) shell. We define the \( J_{\text{max}} \) interactions as being zero except when \( J=J_{\text{max}} \) in which case it is a negative constant. Zamick and Escuderos [16] compared the spectrum of 2 proton holes and 2 neutron holes (i.e. \( ^{96}\text{Cd} \)) for such an interaction with realistic CCGI [17] as well as one with half \( E(J_{\text{max}}) \) and half \( E(0) \). Results are shown in Table 2.

| \( J \) | 0  | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 |
|------|----|----|----|----|----|----|----|----|----|
| CCGI | 0  | 1.081 | 2.112 | 2.888 | 3.230 | 4.882 | 5.339 | 5.403 | 5.224 |
| \( E(9) \) | 1.059 | 1.059 | 1.059 | 1.051 | 1.046 | 0.967 | 0.657 | 0.128 | 0.000 |
| \( E(9,9) \) | 0  | 1.274 | 1.858 | 2.393 | 2.512 | 3.214 | 3.135 | 2.847 | 2.168 |

We see that the CCGI interaction [17] gives a steady increase in excitation energy with angular momentum except for \( J=16 \). The \( 16^+ \) state is correctly predicted to lie below the \( 14^+ \)state. This explains why the \( 16^+ \) state is isomeric.
With $E(9)$ we see a terrible spectrum with for the most part the excitation energy decreasing with angular momentum and having $J_{\text{max}}=16$ as the ground state. At first this might sound surprising, but the $J=16^+$ wave function can be written as $(pn)J=9$ $(pn)J=9$ (antisymmetrized). We can see how this feels the attraction of 2 nucleons with $J=9$. Things are more reasonable if one takes half and half mixtures of $E(0)$ and $E(9)$.

Despite the bad spectrum with the $E(9)$ interaction it turns out the wave functions are quite good. Furthermore they are to an excellent approximation proportional to unitary $9j$ coefficients. In some cases the proportionality is exact [18,19,16].

Just to give an example we compare the lowest 2 $J=2^+$ states in $^{96}\text{Cd–}E(9)$ interaction results vs U9j. The U9j columns are suitably normalized sets $(jj)^9(jj)^{-J_x}|(jj)^{-J_x}(jj)^{J_x})_{J=2}$. In the first case $J_x=9$ and in the second case $J_x=7$. We see that $J_x$ is an approximate quantum number.

As seen in Table 3 there is stunning agreement between the wave function amplitudes obtained by a diagonalization with the $E(9)$ interaction and those from U9j. This leads to the question as to why there is no more mixing of the 2 $J_x$ states. Part of the reason lies in the value of the U9j $(jj)^9(jj)^9| (jj)^9(jj)^7)_{J=2}$. We note it is fairly large for $j=3/2$ but becomes increasingly small with $j$. This lead us[19] to study the asymptotic value of this U9j. Whereas the expression for this U9j is very complicated the asymptotic value is simple. For large $j$ the expression for this U9j is $A_j m e^{-\alpha j}$ with $m=3/2$ and $\alpha=4 \ln(2)$. [18]We see that it deceases almost exponentially with $j$. To obtain this result we used the Stirling approximation: $\ln (n!)= n \ln(n)-n + \ln(\sqrt{2\pi n})$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$J_{p,n}$ & $E(9)$ & U9j ($J_x=9$) & $E(9)$ & U9j ($J_x=7$) \\
\hline
E* & 1.069 & . & 3.059 & . \\
0,2 & 0.5334 & 0.5338 & 0.1349 & 0.1351 \\
2,2 & -0.4707 & -0.4708 & 0.5569 & 0.5567 \\
2,4 & 0.3025 & 0.3035 & 0.3188 & 0.3189 \\
4,4 & -0.1388 & -0.1390 & 0.6300 & 0.6299 \\
4,6 & 0.0531 & 0.0531 & 0.1320 & 0.1320 \\
6,6 & -0.0137 & -0.0138 & 0.1350 & 0.1350 \\
6,8 & 0.0025 & 0.0025 & 0.0114 & 0.0014 \\
8,8 & -0.0003 & -0.0003 & 0.0052 & 0.0052 \\
\hline
\end{tabular}
\caption{Comparison of the wave functions of the 2 lowest $2^+$ states: $E(9)$ vs. U9j}
\end{table}

**CALCULATED INTER-BAND B(E2)’s IN THE P-F SHELL**

Previous studies of even-even Ti isotopes showed reasonably strong $B(E2)$’s in the yrast band–$J=0_1$ to $2_1$, $2_1$ to $4_1$, e.t.c.[20]. In this work we study transitions
from states in the yrast band to a second group of states: $1_1, 2_2, 3_1, 4_2, 5_1$ i.e. second excited states of even $J$ and lowest states of odd $J$. We use the shell model code NushellX [21,22]. We make comparisons with the rotational model as described by Bohr and Mottelson[23].

In Table 4 we show the yrast transitions. The largest ones are from $J=0$ to $J=2$. The values for the Ti isotopes with $A=44, 46, 48$ and 50 in $e^2 fm^4$ are respectively 526, 624, 521 ad 502 $e^2 fm^4$. There is a large increase in $^{48}$Cr, 1254 $e^2 fm^4$. It was noted in ref[20] that in both the rotational model and vibrational model [23], the $J\rightarrow J+2$ B(E2)'s increase with $J$, but as seen in table 1 in the shell mode they decrease with $J$.

We next make a comparison of the behavior in the Ti isotopes with what occurs in more deformed nuclei. It is convenient to choose the work of Clement et al. [24] on $^{98}$Sr because they show several measured B(E2)'s between states in the yrast band and those in the next band. The comparison is somewhat hybrid because we are listing experimental results for Sr and theoretical results for Ti. The B(E2's) in Weisskopf units (WU) are 19.4 in $^{46}$Ti and 95.5 in $^{98}$Sr. This shows that the latter nucleus is indeed more strongly deformed than any of the Ti isotopes.

In their Table 4 Clement et al.[24] show reduced matrix elements. In our Table 5 we show rather the ratio of a given B(E2) to the intraband 0(1)$\rightarrow$ 2(1) B(E2). The ratio of this transition to 2(1)$\rightarrow$ 2(2) in $^{98}$Sr is quite small whereas for $^{44}$Ti and $^{46}$Ti the values are 0.2909 and 0.1694 respectively. A ratio close to 0.2 is also found for 0(1)$\rightarrow$2(2) in $^{48}$Ti.

In Table 6 we show B(E2)'s within the second group for $^{46}$Ti. They are in general much larger than the interband transitions between the 2 groups.
Table 4: Calculated yrast B(E2) $J \rightarrow (J+2)$ e$^2$fm$^4$

| J  | $^{44}$Ti | $^{46}$Ti | $^{48}$Ti | $^{50}$Ti | $^{48}$Cr |
|----|-----------|-----------|-----------|-----------|-----------|
| 0  | 526       | 624       | 521       | 502       | 1254      |
| 2  | 246       | 286       | 269       | 176       | 609       |
| 4  | 155       | 228       | 87.8      | 67.9      | 487       |
| 6  | 94.7      | 190       | 102       | .426      | 403       |
| 8  | 114       | 134       | 68.2      | 57.5      | 261       |
| 10 | 64.1      | 49.8      | 28.8      | 56.5      | 194       |
| 12 | 45.0      | 5.26      | 13.6      | .426      | 148       |
| 14 | 0.062     | 7.25      | 22        | 71.3      |           |

Table 5: Ratio B(E2)/B(E2) $0(1) \rightarrow 2(1)$

| Ji→Jf | $^{88}$Sr | $^{44}$Ti | $^{46}$Ti | $^{48}$Ti | $^{50}$Ti | $^{48}$Cr |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0(1)→2(2) | 0.00799  | 0.05380   | 0.006458  | 0.1902    | 0.000142  | 0.00246   |
| 2(1)→0(2) | 0.02556  | 0.0113    | 0.0208    | 0.0195    | 0.00219   | 0.0221    |
| 2(1)→2(2) | 0.000767*| 0.2909    | 0.1694    | 0.0845    | 0.00703   | 0.0451    |
| 4(1)→2(2) | 0.004603 | 0.02567   | 0.01651   | 0.00263   | 0.00365   | 0.00606   |
| 2(1)→3(1) | 0.0123   | 0.009295  | 0.0595    | 0.000448  | 0.0151    |
| 4(1)→3(1) | 0.04297  | 0.006490  | 0.04626   | 0.0307    | 0.00177   |

Table 6: $^{46}$Ti B(E2) e$^2$fm$^4$ in the second group

| Ji→Jf | $^{46}$Ti B(E2) e$^2$fm$^4$ in the second group |
|-------|------------------------------------------------|
| 0(2)  | x                                               |
| 2(2)  | x                                               |
| 4(2)  | 3.21E+01 1.26E+01 1.76E+01 1.29E+01 5.06E+01 |
| 6(2)  | 2.81E+01 2.05E+01 4.14E+00 7.19E+00 1.54E+00 |
| 8(2)  | 1.07E+00 3.49E+01 3.79E+00 4.98E+00 1.07E+02 |
| 10(2) | 8.16E+01 1.69E+01 2.00E+00 3.81E+00 7.93E+01 |
| 12(2) | 6.42E+01 2.16E+01 1.22E+01 3.41E+01 1.39E+01 |
| 14(2) | 1.17E+01 2.22E+00 2.99E+00 2.05E+01 1.13E-01 |
| 16(2) | 9.73E-02 9.85E-01 1.16E+02 4.19E-01 8.59E-01 |

Table 7: Electric quadrupole moments (e fm$^2$)

| Ji→Jf | $^{44}$Ti | $^{46}$Ti | $^{48}$Ti | $^{50}$Ti | $^{48}$Cr |
|-------|-----------|-----------|-----------|-----------|-----------|
| 2(1)  | 6.01      | -13.6     | -14.5     | 6.53      | -30.8     |
| 2(2)  | -0.89     | 7.1       | 5.02      | 13.3      | 21.9      |

Note that for $^{46}$Ti,$^{48}$Ti and $^{48}$Cr the quadrupole moments of the 2(1) states are negative and those of the 2(2) states are positive. In the rotational model the first group would be prolate and the second group would be oblate. Indeed, the quadrupole moments of $J=2^+$ for a K=2 band are equal and opposite of those of a K=0 band.

In closing we note that the main point of this work is that inter-band B(E2)’s
show variations as one goes from one isotope to another and as one goes from less deformed to more strongly deformed nuclei. The behaviors are not as well understood as those for yrast transitions. They deserve to be studied more both experimentally and theoretically. We note that the calculated intra-band transitions in the first group are large and likewise for the second group. The inter-band transitions between the 2 groups are very small and it is hard to see a trend with increasing neutron number.

**MATRIX STUDIES**

In almost anything we do, be it shell model, IBM, or anything else, somewhere we are diagonalizing a matrix. So we thought it would be a good idea to study the properties of matrices in general, somewhat divorced from specific experiments. To this end with Arun Kingan [25,26] we started with the matrix shown in Figure 1 (a) (although ours were 11 by 11). The diagonal is $\eta E$ and there are constant interaction matrix elements are $v$ only next to the diagonal. We defined 2 possible transition operators $\langle n | T_A(n+1) | n+1 \rangle = 1$ and $\langle n | T_B(n+1) | n+1 \rangle = \sqrt{n+1}$, with all other transition elements being zero. With the B choice we found a near exponential decrease of transition strength with excitation energy for all $v$. We found a similar behavior with the A choice for small $v$ but for large $v$ we find 2 exponential behaviors- one for even $n$ and one for odd $n$ [26].

Here for simplicity we take the B choice and consider only the case $v=1$. As seen in Fig.2 we do indeed get a simple exponential behavior for the tridiagonal case-on a log plot an almost straight line with a negative slope. We then consider a new pentadiagonal matrix (Figure 1 (b)).

**Figure 1:** Study of Tridiagonal (a) and Pentadiagonal Matrices (b).

\[
\begin{pmatrix}
0 & v & 0 & 0 & 0 \\
v & E & v & 0 & 0 \\
0 & v & 2E & v & 0 \\
0 & 0 & v & 3E & v \\
0 & 0 & 0 & v & 4E
\end{pmatrix}
\begin{pmatrix}
0 & v & v & 0 & 0 \\
v & E & v & v & 0 \\
v & v & 2E & v & v \\
0 & v & v & 3E & v \\
0 & 0 & v & v & 4E
\end{pmatrix}
\]

(a) Tridiagonal  
(b) Pentadiagonal

**Figure 2:** Transition strength versus excitation energy
Figure 2 (a) (Tridiagonal) does indeed display a simple near exponential fall off of the transition strength. Although we just said we are divorcing ourselves from experiment we note that exponential fall offs have been frequently observed or calculated e.g. in gamma cascades following neutron capture [27,28,29]; also in calculated magnetic dipole strength. The ground state wavefunction has an interesting structure in the weak coupling limit: $a_0=1$, $a_1 = -\frac{v}{E}$, $a_2 = \frac{v^2}{E^2}$, $a_3 = -\frac{v^3}{E^3}$, ..., $a_n = \frac{(-v)^n}{E^n}$. This coincides with the Taylor series for the exponential function $e^{-\frac{v}{E}}$. We can understand the factorials in the denominator the fact that in n’th order perturbation theory the energy denominators are $(E_0-E_1)(E_0-E_2) \ldots \ldots \ (E_0-E_n)$, which, with our choice of a matrix is $(-E)^n n!$.

In Fig 2 (b) (Pentadiagonal) we see more structure, one can indeed draw a straight line through many of the points but at low energies there is a dip. These studies are in their infancy and there is still much to learn.

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