G-Structures and Wrapped NS5-Branes

Jerome P. Gauntlett*,†  Dario Martelli*  Stathis Pakis* and Daniel Waldram*

*Department of Physics
Queen Mary, University of London
Mile End Rd, London E1 4NS, U.K.

†Isaac Newton Institute for Mathematical Sciences
University of Cambridge
20 Clarkson Road, Cambridge, CB3 0EH, U.K.

ABSTRACT

We analyse the geometrical structure of supersymmetric solutions of type II supergravity of the form $\mathbb{R}^{1,9-n} \times M_n$ with non-trivial NS flux and dilaton. Solutions of this type arise naturally as the near-horizon limits of wrapped NS fivebrane geometries. We concentrate on the case $d = 7$, preserving two or four supersymmetries, corresponding to branes wrapped on associative or SLAG three-cycles. Given the existence of Killing spinors, we show that $M_7$ admits a $G_2$-structure or an $SU(3)$-structure, respectively, of specific type. We also prove the converse result. We use the existence of these geometric structures as a new technique to derive some known and new explicit solutions, as well as a simple theorem implying that we have vanishing NS three-form and constant dilaton whenever $M_7$ is compact with no boundary. The analysis extends simply to other type II examples and also to type I supergravity.

1 E-mail: j.p.gauntlett@qmul.ac.uk
2 E-mail: d.martelli@qmul.ac.uk
3 E-mail: s.pakis@qmul.ac.uk
4 E-mail: d.j.waldram@qmul.ac.uk
1 Introduction

Solutions of type II supergravity corresponding to NS fivebranes wrapping supersymmetric cycles provide an interesting arena for studying the holographic duals of supersymmetric Yang-Mills (SYM) theory [1]. Solutions, in the “near-horizon limit”, have now been found for a number of different cases [2]–[10]. In each case the final geometry is of the form $\mathbb{R}^{1,5-d} \times M_{4+d}$, where $d$ is the dimension of the cycle on which the fivebrane is wrapped, and the NS three-form $H$ and the dilaton are non-trivial.

In [7] various aspects of the geometry of such supersymmetric solutions was elucidated. The key input is the existence of Killing spinors describing the preserved supersymmetries. In type II supergravity, the vanishing of the supersymmetry variation of the gravitino implies that any Killing spinor is parallel with respect to one of two connections $\nabla^\pm$ with totally anti-symmetric torsion $\pm \frac{1}{2} H$. This implies that $M_{4+d}$ admits certain geometric structures and the vanishing of the variation of the dilatino imposes additional conditions on the structures. It was shown in [7], following [11], that the resulting structures give rise to generalised calibrations [12].

Here we would like to continue the investigations of [7], again assuming only the existence of particular sets of Killing spinors. Thus while motivated by considering solutions for wrapped NS five-branes, the results apply generally to supersymmetric backgrounds with non-trivial NS-flux $H$ and dilaton. For definiteness we will focus on seven-dimensional geometries $M_7$, though it is clear that the analysis generalises to all cases discussed in [7]. Two distinct types of geometry arise. The first is when the Killing spinors are all parallel with respect to the same connection, say $\nabla^+$. Geometrical structures of this type have been discussed in [13, 14, 15, 16, 17, 18] as well as [7]. The second, new, type [8] is when there are some Killing spinors parallel with respect to $\nabla^+$ and some with respect to $\nabla^-$. For our particular example, seven-dimensional geometries arise when fivebranes wrap associative three-cycles or SLAG three-cycles. The geometries are distinguished by the fact that the former is of the first type with a single Killing spinor parallel with respect to $\nabla^+$. The latter case, on the other hand, is of the second type, preserving twice as much supersymmetry, with two Killing spinors $\epsilon^\pm$, parallel with respect to $\nabla^\pm$ respectively. For the generic case with a single Killing spinor, the seven-dimensional geometry admits a $G_2$-structure of a specific type, to be reviewed below. On the other hand, the seven-dimensional geometries with two Killing spinors $\epsilon^\pm$ admit two $G_2$-structures, or more precisely an $SU(3)$ structure, again of a specific type to be discussed. Note that because of the non-vanishing intrinsic torsion the $SU(3)$-
structure does not imply that the manifold is a direct product of a six-dimensional geometry with a one-dimensional geometry. However, there is a product structure which does allow the metric to be put into a canonical form with a six-one split as we shall discuss.

The geometrical structures defined by the preserved supersymmetries can equivalently be specified by tensor fields satisfying first-order differential equations. These give a set of necessary conditions imposed by preservation of supersymmetry and the equations of motion. We will show that for the particular cases of $G_2$- and $SU(3)$-structures mentioned above, these conditions are also sufficient. From the derivation, it is clear that should always be possible to find such a set of conditions. Note that, for the cases of single $G_2$- and $Spin(7)$-structures, this issue was also analysed in [16, 17] and [18] respectively, although these works did not consider the relationship between Killing spinors and the equations of motion, as we shall here.

One of the motivations for this work was to see if these sufficient conditions just mentioned could provide a new method for constructing supersymmetric solutions describing wrapped fivebranes. This would provide an alternative to the “standard” two-step construction [19, 2] of first finding a solution of $D = 7$ gauged supergravity and then uplifting to $D = 10$. We shall show that some known solutions can be recovered in this way. In addition to providing a direct $D = 10$ check of the solutions, this makes their underlying geometry manifest. We will also use the method to construct a new solution that describes a fivebrane wrapping a non-compact associative three-cycle. It is a co-homogeneity one solution with principle orbits given by $SU(3)/U(1) \times U(1)$.

Recall that the solution of [6, 7] describing fivebranes wrapping SLAG three-cycles was argued to be dual to pure $\mathcal{N} = 2$, $D = 3$ SYM. By analogy with [2, 3] it seems likely that there are more general solutions that would be dual to $\mathcal{N} = 2$, $D = 3$ SYM with a Chern–Simons term. These seem difficult to find using gauged supergravity. Unfortunately they also seem to be difficult to obtain using the methods to be described here. In particular, to recover the known solution of [6, 7] one is first naturally led to partial differential equations, and it is somewhat miraculous that there is change of coordinates that leads one to the relatively simple solution obtained in [6, 7] using the gauged supergravity approach.

The type II supergravity solutions for wrapped branes that are dual to quantum field theories are non-compact. As somewhat of an aside we also prove a simple vanishing theorem for compact manifolds. In particular, we show that the expression for the three-form $H$ in terms of the $G_2$-structure allows one to prove that on a
compact manifold without boundary given \( dH = 0 \), the three-form must necessarily vanish and the dilaton \( \Phi \) is constant. There are analogous expressions for \( H \) in terms of generalised calibrations for other fivebrane geometries \([\ddagger]\), and hence this result generalises easily.

The plan of the rest of the paper is as follows. We begin in section 2 with some general discussion of \( G \)-structures and \( G \)-invariant tensors using \( G_2 \) as our example. In section 3 we review and extend what is known about the geometry with \( G_2 \) structure that arises when type II fivebranes wrap associative three-cycles. We also prove the vanishing theorem for compact manifolds. Section 4 discusses the geometry that arises when fivebranes wrap SLAG three-cycles. In this case there are two \( G_2 \) structures or equivalently an \( SU(3) \) structure. Section 5 uses the necessary and sufficient conditions for \( G_2 \)-structures admitting Killing spinors as a technique to rederive some known solutions as well as deriving a new solution that describes a fivebrane wrapping a non-compact associative three-cycle. In section 6 we use the analogous conditions for \( SU(3) \)-structures to derive BPS equations for solutions corresponding to fivebranes wrapped on SLAG three-cycles. We conclude in section 7 by discussing how the results would extend to fivebranes wrapping other supersymmetric cycles as discussed in \([\ddagger]\) and we also briefly comment on the extension to type I supergravity.

2 \( G \)-structures

In this section we review the notion of \( G \)-structure and \( G \)-invariant tensors on a Riemannian manifold \( M \) and the relation to intrinsic torsion and holonomy. Though the discussion is general, our examples will concentrate on the case relevant here of \( G_2 \)-structure on a seven-manifold. Further details can be found, for example, in \([20]\) and \([21]\).

In general the existence of a \( G \)-structure on an \( n \)-dimensional Riemannian manifold means that the structure group of the frame bundle is not completely general but can be reduced to \( G \subset O(n) \). Thus, for \( G_2 \)-structures on a seven-manifold, the structure group reduces to \( G_2 \subset SO(7) \subset O(7) \).

An alternative and sometimes more convenient way to define \( G \)-structures is via \( G \)-invariant tensors. A non-vanishing, globally defined tensor \( \eta \) is \( G \)-invariant if it is invariant under \( G \subset O(n) \) rotations of the orthonormal frame. Since \( \eta \) is globally defined, by considering the set of frames for which \( \eta \) takes the same fixed form, one can see that the structure group of the frame bundle must then reduce to \( G \) (or a subgroup of \( G \)). Thus the existence of \( \eta \) implies we have a \( G \)-structure. Typically,
the converse is also true. Recall that, relative to an orthonormal frame, tensors of a
given type form the vector space, or module, for a given representation of $O(n)$. If
the structure group of the frame bundle is reduced to $G \subset O(n)$, this module can
be decomposed into irreducible modules of $G$. Typically there will be some type of
tensor that will have a component in this decomposition which is invariant under $G$.
The corresponding vector bundle of this component must be trivial, and thus will
admit a globally defined non-vanishing section $\eta$.

To see how this works in the case of $G_2$-structures, consider the three-form on $\mathbb{R}^7$
given by

$$\phi_0 = dx^{136} + dx^{235} + dx^{145} - dx^{246} - dx^{127} - dx^{347} - dx^{567}$$  (2.1)

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$ and let $g_0 = dx_1^2 + ... + dx_7^2$ denote the standard Euclidean
metric. The group $G_2$ can be defined as the subgroup of the $O(n)$ symmetries of $g_0$ which leaves $\phi_0$ invariant. A seven-dimensional manifold $M_7$ then admits a
$G_2$-structure if and only if there is a globally defined three-form $\phi$ on $M_7$ which is $G_2$-
invariant. That is, at each point on $M_7$ we can consistently identify the three-form $\phi$ with the standard
$G_2$-invariant three-form $\phi_0$. Note that given $\phi_0$ we also have the metric $g_0$, an orientation $dx^{1...7}$ and the Hodge-dual four-form $\ast \phi_0$. Thus given a
$G_2$-invariant $\phi$ on $M_7$ we actually also get an associated metric $g$ and four-form $\ast \phi$ on $M_7$ such that $(\phi, \ast \phi, g)$ are identified under the map to $\mathbb{R}^7$ with $\phi_0, \ast \phi_0, g_0$.

It will be useful to give explicitly some of the tensor decompositions in the $G_2$
case. For instance, for two-forms one finds

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$$  (2.2)

where

$$\Lambda^2_7 = \{ \alpha \in \Lambda^2 : \ast(\phi \wedge \alpha) = -2\alpha \} = \{ \beta \phi : \beta \in TM \},$$

$$\Lambda^2_{14} = \{ \alpha \in \Lambda^2 : \ast(\phi \wedge \alpha) = \alpha \}.  \tag{2.3}$$

Recall that the space of two-forms $\Lambda^2$ is isomorphic to the Lie algebra or adjoint
representation $so(7)$. Thus this decomposition is just the decomposition of $so(7)$
under $G_2$, namely $21 \rightarrow 14 + 7$, where $14$ is the Lie algebra $g_2 \cong \Lambda^2_{14}$ of $G_2$. There
is, similarly, a decomposition of three-forms

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$  (2.4)

following $35 \rightarrow 1 + 7 + 27$ under $G_2 \subset SO(7)$. Note that elements of the singlet $\Lambda^3_1$
module are simply multiples of the $G_2$-invariant three-form $\phi$. 

4
Riemannian manifolds with $G_2$-structures have been classified some time ago by Fernandez and Gray [22]. The idea is the same for any $G$-structure on a Riemannian manifold, as discussed for example in [21]. Given some $G$-invariant form $\eta$ defining a $G$-structure, the derivative of $\eta$ with respect to the Levi–Civita connection, $\nabla \eta$, can be decomposed into $G$-modules. The different types of $G$-structures are then specified by which of these modules are present, if any.

In more detail, one first uses the fact that there is no obstruction to finding some connection $\nabla'$ such that $\nabla' \eta = 0$. Choosing one, then $\nabla - \nabla'$ is a tensor which has values in $\Lambda^1 \otimes \Lambda^2$. Since $\Lambda^2 \cong so(n) = g \oplus g^\perp$ where $g^\perp$ is the orthogonal complement of the Lie algebra $g$ in $so(n)$, we conclude that $\nabla \eta = (\nabla - \nabla') \eta$ can be identified with an element $K$ of $\Lambda^1 \otimes g^\perp$. Furthermore, this element is a function only of the particular $G$-structure, independent of the choice of $\nabla'$. It is in one-to-one correspondence with what is known as the intrinsic torsion $T_0$. Explicitly, we have in components, acting on a $q$-form

$$\nabla_m \eta_{n_1...n_q} = -K_{m n_1}^p \eta_{p n_2...n_q} - K_{m n_2}^p \eta_{n_1 p...n_q} - \cdots - K_{m n_q}^p \eta_{n_1...n_{q-1} p}, \quad (2.5)$$

where for $K_{m n}^p \in \Lambda^1 \otimes g^\perp$, the $m$ and antisymmetric $n, p$ indices label the one-form $\Lambda^1$ and two-form $g^\perp \subset \Lambda^2$ modules respectively.

In the $G_2$ case, from the decomposition (2.2), we see that $g^\perp \cong \Lambda_7^2$, while $\Lambda^1$ is simply the $7$ representation of $G_2$. Thus specifying $\nabla \phi$ is equivalent to giving elements in the four $G_2$-modules in the decomposition of $K$

$$7 \times 7 \rightarrow 1 + 7 + 14 + 27 \quad (2.6)$$

Given the general relation (2.3) with $\eta = \phi$, we see that $d \phi$ and $d^\dagger \phi \equiv -\star d \star \phi$ pick out different parts in this decomposition. For example, following (2.2), the two-form $d^\dagger \phi$ contains the $7 + 14$ pieces as follows

$$d^\dagger \phi = i_\theta \phi + \alpha_{14}. \quad (2.7)$$

Here $\alpha_{14} \in \Lambda_{14}^2$ and $\theta$ corresponds to the $7$, is called the Lee form and is given by

$$3 \theta \equiv (d^\dagger \phi \wedge \star \phi), \quad (2.8)$$

or in components $\theta_a = -\frac{1}{6} \phi_{abc} \nabla_c \phi^{abc}$. Similarly, the four-form $d \phi$ can be decomposed into $1 + 7 + 27$ pieces, and so contains all but the $14$ in (2.6). Note, in particular, since it is derived from the same general expression (2.5), the $7$ in this decomposition must be proportional to the Lee form defined in the decomposition of $d^\dagger \phi$. 

5
It is clear from the above discussion that $\nabla^0 \equiv \nabla + K$ canonically defines a connection for which $\nabla^0 \eta = 0$. It is the unique connection with torsion given by the intrinsic torsion $T_0$. Since the holonomy of this connection, and any connection $\nabla'$ for which $\nabla' \eta = 0$, must stabilise $\eta$ we conclude that its holonomy, $\text{Hol}(\nabla^0)$, must be contained within $G$. On the other hand demanding that specific types of connection have holonomy in $G$, in general restricts the $G$-structure. For example, for the Levi–Civita connection to have $\text{Hol}(\nabla) \subseteq G$ we require that all the elements in the decomposition of $K$ vanish so that $\nabla \eta = 0$. The $G$-structure is then said to be “torsion-free”.

This is probably the most familiar case of $G_2$-structure. With torsion-free structure, so $\nabla \phi = 0$, $(M_7, \phi, g)$ is said to be a “$G_2$ manifold”. It means that the Levi–Civita connection $\nabla$ has holonomy contained in $G_2$ and $g$ is a Ricci-flat metric. Given the preceding discussion it is clear that the condition $\nabla \phi = 0$ is equivalent to requiring

$$d\phi = d^\ast \phi = 0$$

since all the relevant $G_2$-modules in $\nabla \phi = 0$ appear either in $d\phi$ or $d^\ast \phi$. This equivalence has been exploited in [23] to provide a method for finding $G_2$ holonomy metrics for manifolds of co-homogeneity one. The strategy is the following. Write down an ansatz for the associative three form $\phi$ in terms of several arbitrary functions of one radial variable. Find the associated metric and impose the conditions (2.9) to obtain a system of first-order differential equations for the arbitrary functions. Solving these one obtains a $G_2$ holonomy metric.

For type II supergravity solutions describing NS fivebranes wrapping supersymmetric three-cycles one finds seven manifolds with $G_2$ structures of a different type [7] since the connection with holonomy in $G_2$ is not the Levi–Civita connection $\nabla$. This is reviewed in the next section. We can derive an analogous pair of necessary and sufficient conditions to (2.9). We will then exploit these, generalising [23], to find new solutions in a later section.

3 $G_2$-structure and NS fivebranes on associative three-cycles

The action for the bosonic NS-NS fields of type IIA or type IIB supergravity is given by

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left( R + 4(\nabla \Phi)^2 - \frac{1}{12} H^2 \right),$$

where $\kappa^2$ is the gravitational coupling constant, $g$ is the determinant of the metric, $\Phi$ is the scalar field, $\nabla$ is the covariant derivative, $R$ is the scalar curvature, and $H$ is the four-form field strength.
with \( H = dB \). The corresponding equations of motion read

\[
R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma} + 2 \nabla_\mu \nabla_\nu \Phi = 0,
\]

\[
\nabla^2 \Phi - 2 (\nabla \Phi)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0,
\]

\[
\nabla^\mu (e^{-2\Phi} H_{\mu\nu}) = 0. 
\]

As shown in [2], the IIB supergravity solution describing fivebranes wrapped on an associative three-cycle in a manifold of \( G_2 \)-holonomy is of the form \( \mathbb{R}^{1,2} \times M_7 \) where \( M_7 \) admits a single \( \text{Spin}(7) \) spinor satisfying

\[
\nabla^+ \epsilon \equiv \left( \nabla_m + \frac{1}{8} H_{mnp} \gamma^{np} \right) \epsilon = 0,
\]

\[
H_{mn} \gamma^{mn} \epsilon = -12 \partial_n \Phi \gamma^n \epsilon,
\]

where \( \nabla^+ \) is a connection with totally antisymmetric torsion \( \frac{1}{2} H \). Here the \( \text{Spin}(7) \) Dirac matrices \( \gamma^m \) are imaginary, antisymmetric and satisfy

\[
\{ \gamma_m, \gamma_n \} = 2 \delta_{mn}
\]

with \( \gamma^1 \cdots \gamma^7 = -i \).

From the existence of a single solution to the first equation in (3.3) it immediately follows that we have a \( G_2 \)-structure on \( M_7 \) and the \( \text{Hol}(\nabla^+) = G_2 \). To see this, we first define a three form \( \phi \) in terms of the Killing spinor by:

\[
\phi_{mn} = i \epsilon^T \gamma_{mn} \epsilon
\]

where we have normalized the spinors to satisfy \( \epsilon^T \epsilon = 1 \). This \( G_2 \)-structure then satisfies

\[
\nabla^+ \phi = 0
\]

and hence \( \text{Hol}(\nabla^+) \subseteq G_2 \). Using the second equation in (3.3) we find that the \( G_2 \)-structure also satisfies

\[
- * e^{2\Phi} d(e^{-2\Phi} \phi) = H
\]

\[
d(e^{-2\Phi} * \phi) = 0
\]

\[
d\phi \wedge \phi = 0
\]

That is, for a solution to the equations of motion to preserve supersymmetry, \( M_7 \) must admit a \( G_2 \) structure which satisfies the conditions (3.7) with \( H \) closed. This form
of the conditions [7] naturally displays the connections with generalised calibrations [12].

A converse result has also been proved in [16, 17]. Let us summarise the idea behind it before extending it. One assumes the existence of a $G_2$-structure satisfying the last two equations in (3.7). Recalling the definition of the Lee-form introduced in (2.7) it is easy to see that $d(e^{-2\Phi} \ast \phi) = 0$ is equivalent to the statement that (i) the Lee form is given by $\theta = -2d\Phi$ and (ii) that $\alpha_{14}$ vanishes. It was shown in [16] that the second condition is the necessary and sufficient condition for the existence of a unique connection $\nabla^+ = \nabla + \frac{1}{2}H$, with totally anti-symmetric torsion $\frac{1}{2}H$, that preserves the three form $\phi$ and admits parallel spinors. The idea behind this is rather simple. First recall from (2.4) that $H$, which is in the 35 of $SO(7)$, decomposes under $G_2$ as $35 \rightarrow 1 + 7 + 27$. On the other hand as we discussed in section 2, the different types of $G_2$ structure correspond to the modules given in the decomposition (2.6) of $K$. It is thus clear that totally anti-symmetric torsion is associated with vanishing $\alpha_{14}$ in $K$. Moreover, it was shown in [17] that the $G_2$ singlet piece in $H$ is proportional to $*(d\phi \wedge \phi)$ and vanishes if and only if the supersymmetry variation of the dilatino vanishes. The point is that the Clifford action of the 27 piece of $H$ on $\epsilon$ vanishes.

In other words it was proved in [17] that a manifold with a $G_2$ structure $(M_7, g, \phi)$ admits solutions to (3.3) with varying dilaton and non vanishing NS three form $H$ providing that the $G_2$ structure satisfies:

\[
d\phi \wedge \phi = 0
\]
\[
* (\phi \wedge d^i \phi) = -2d^i \phi
\]
\[
\theta = -2d\Phi
\]

or equivalently

\[
d\phi \wedge \phi = 0
\]
\[
d(e^{-2\Phi} \ast \phi) = 0
\]

The torsion of the unique connection with totally anti-symmetric torsion preserving the $G_2$ structure is then given by $H = - \ast e^{2\Phi} d(e^{-2\Phi} \phi)$.

Note that supersymmetry alone is not sufficient to ensure that we have a solution to the type II field equations. We also need to impose at least the closure of $H$. In fact this is all we need as we now show using the integrability conditions of the Killing
spinor equations (3.3). As shown in the Appendix these imply that

\[
\left( R_{mn} - \frac{1}{4} H_{mpq} H^{pq}_n + 2 \nabla_m \nabla_n \Phi \right) \gamma^n \epsilon = \frac{1}{12} dH_{npq} \gamma^{npq} \epsilon + \frac{1}{2} e^{2\Phi} \nabla^p (e^{-2\Phi} H_{pmn}) \gamma^n \epsilon
\]

(3.10)

and

\[
\left( \nabla^2 \Phi - 2 [\nabla \Phi]^2 + \frac{1}{12} H_{mpn} H^{mpn} \right) \epsilon = -\frac{1}{48} dH_{npq} \gamma^{npq} \epsilon - \frac{1}{4} e^{2\Phi} \nabla^m (e^{-2\Phi} H_{mpn}) \gamma^m \epsilon
\]

(3.11)

The assumptions on the $G_2$ structure (3.9) mean that the $H$ equation of motion (3.2) is automatically satisfied. We thus immediately conclude from (3.11) that, if we also impose $dH = 0$, then the dilaton equation of motion is satisfied. The other equation (3.10) is then of the form $A_{mn} \gamma^n \epsilon = 0$ which implies $A_{mn} A^{mn} = 0$. On a Riemannian manifold we then deduce $A_{mn} = 0$ which is precisely the Einstein equations.

In summary, we have shown that a solution of the equations of motion of the form $\mathbb{R}^{1,2} \times M_7$ admits a single Killing spinor if and only if

\[
d\phi \wedge \phi = 0,
\]

\[
d(e^{-2\Phi} \ast \phi) = 0,
\]

\[
dH = 0,
\]

(3.12)

where $H = - \ast e^{2\Phi} d(e^{-2\Phi} \phi)$. This result is the analog of (2.9) for $G_2$-manifolds and in principle provides a method for finding new supersymmetric solutions with non-zero $H$. One starts with an ansatz for $\phi$, finds the associated metric and imposes these equations to obtain, in the case of a metric of co-homogeneity one, ordinary differential equations for the arbitrary metric functions. We give examples of this technique in section 5.

It is interesting to note that the expression for $H$ implies a simple vanishing theorem\(^1\) on a compact manifold without boundary the only solutions to (3.12) have $H = 0$ and $\Phi$ constant, that is, $M_7$ is a $G_2$-manifold. To see this, we first note, given the expression for $H$

\[
\int_{M_7} e^{-2\Phi} H \wedge \ast H = - \int_{M_7} H \wedge d(e^{-2\Phi} \phi) = 0,
\]

(3.13)

\(^1\)A different vanishing theorem was proved in [24], which assumed vanishing dilaton.
where in the final equation we integrate by parts and use $dH = 0$. Since the first integrand is positive definite, we conclude that $H = 0$. Integrating the dilaton equation of motion then implies by a similar argument that $d\Phi = 0$ so $\Phi$ is constant. The conditions (3.12) then reduce to $d\phi = d*\phi = 0$ which imply that $M_7$ is a $G_2$-manifold. In [4] we derived analogous expressions for $H$ in terms of the generalised calibrations for other geometries in dimensions six and eight arising when fivebranes wrap calibrated cycles. Since only this expression and the $\Phi$ equation of motion entered the above argument, clearly this theorem easily generalises. For all compact supersymmetric manifolds $M$ without boundary, the flux vanishes $H = 0$ and $\Phi$ is constant.

4 $SU(3)$-structure and NS fivebranes on SLAG three-cycles

It was shown in [5] that the type II supergravity solutions describing fivebranes wrapping SLAG three-cycles are also of the form $\mathbb{R}^{1,2} \times M_7$ where now $M_7$ admits a pair of $Spin(7)$ spinors $\epsilon^\pm$ satisfying

$$\nabla^\pm_m \epsilon^\pm \equiv \left( \nabla_m \pm \frac{1}{8} H_{mnp} \gamma^{np} \right) \epsilon^\pm = 0,$$

$$H_{mnr} \gamma^{mnr} \epsilon^\pm = \mp 12 \partial_n \Phi \gamma^n \epsilon^\pm,$$

(4.1)

where $\nabla^\pm$ are two connections with anti-symmetric torsion $\pm \frac{1}{2} H$. From the discussion of the previous section, it follows that the two spinors $\epsilon^\pm$ define two distinct $G_2$-structures of the type characterised by the conditions (3.12). The two connections $\nabla^\pm$ have holonomy contained in two different $G_2$ subgroups of $SO(7)$. The $G_2$-invariant three forms are constructed from the Killing spinors

$$\phi^\pm_{mnr} = i \epsilon^{\pm T} \gamma_{mnr} \epsilon^\pm,$$

(4.2)

where we have again chosen the normalization $\epsilon^{\pm T} \epsilon^\pm = 1$, as we can always do.

The appearance of two $G_2$-structures is again quite general, depending only on the requirement that there are two distinct solutions $\epsilon^+$ and $\epsilon^-$. We can analyse this structure further, making only one further assumption, as in [5], that the the Killing spinors are orthogonal to each other $i.e.$ $\epsilon^{+ T} \epsilon^- = 0$. We expect that this should cover the general class of supersymmetric solutions describing fivebranes wrapped on SLAG three-cycles. This is because we can deduce what projections we expect to be imposed on the preserved supersymmetries by considering the supersymmetries
preserved by a fivebrane probe wrapped on a SLAG three-cycle, as described in [7]. Note in particular, that this condition was satisfied for the specific supersymmetric supergravity solutions of [6, 7]. It is equivalent to the statement that the two $G_2$ structures satisfy

$$\phi^+_m r^1 r^2 \phi^-_{[m} r^1 r^2 = 0,$$

as can be shown by Fierz rearrangement.

Apart from the two $G_2$ three-forms we can also construct various other forms using the Killing spinors

$$K_n = i \epsilon^+ T \gamma_n \epsilon^-, \quad \omega_{mn} = \epsilon^+ T \gamma_{mn} \epsilon^-, \quad \chi_{mnp} = i \epsilon^+ T \gamma_{mnp} \epsilon^-.$$

These are the basic objects in the sense that the two $G_2$ structures can be constructed from $K$, $\omega$ and $\chi$ as follows

$$\phi^\pm = \pm i K \ast \chi - K \wedge \omega.$$

$K, \omega, \chi$ satisfy a series of algebraic relations that follow from Fierz rearrangements. First, given the normalization of the Killing spinors, we find that:

$$K_m K^m = 1, \quad \omega_{mn} \omega^{mn} = 6, \quad \chi_{mnp} \chi^{mnp} = 24,$$

and also

$$\omega^m r^r \omega^r_n = -\delta^m_n + K_m K^n,$$

$$i_K \chi = 0,$$

$$i_K \omega = 0,$$

$$(i_K \ast \chi)_{mnr} = \omega_m \chi_{nrl} = \omega_{[m} \chi_{nrl]}.$$

In addition, we can calculate the covariant derivatives of these forms using the first Killing spinor equation to get

$$\nabla_m K_n = \frac{1}{4} H_{m t_1 t_2} \chi_n t_{t_1 t_2},$$

$$\nabla_m \omega_{n_1 n_2} = \frac{1}{4} H_{m t_1 t_2} \ast \chi_{n_1 n_2} t_{t_1 t_2},$$

$$\nabla_m \chi_{n_1 n_2 n_3} = \frac{3}{2} H_{m [n_1 n_2} K_{n_3]} - \frac{1}{4} H_{m t_1 t_2} \ast \omega_{n_1 n_2 n_3} t_{t_1 t_2}. $$
From the dilatino equations we then deduce the following relations for the exterior derivatives of the forms
\[
\begin{align*}
    d(e^{-\Phi}K) &= 0, \\
    d(e^{-\Phi}\omega) &= 0, \\
    e^{\Phi}d(e^{-\Phi}\chi) &= -H \wedge K,
\end{align*}
\] (4.9)
as well as for the $G_2$-structures as in (3.7),
\[
e^{2\Phi}d(e^{-2\Phi}\phi^\pm) = \mp \ast H.
\] (4.10)

It is also not difficult to show in addition that
\[
\begin{align*}
    d(i_K \ast \chi) \wedge i_K \ast \chi &= 0, \\
    d(i_K \ast \chi) \wedge K \wedge \omega &= 0.
\end{align*}
\] (4.11)

### 4.1 $SU(3)$ structure

Let us now discuss what the presence of these three invariant forms $K$, $\omega$ and $\chi$ implies about the type of a $G$-structure we have on $M_7$. Recall that the existence of $\phi^\pm$, or equivalently $\epsilon^\pm$, implied that there were two distinct $G_2$-structures. Of course there can only be one actual structure group $G$ of the frame bundle, so the implication is that $G$ must be a common subgroup of these two distinct embeddings of $G_2$ in $SO(7)$. The largest possible such group is $SU(3)$. This is consistent with the existence of two Killing spinors since the $8$ spinor of $Spin(7)$ includes two singlets in its decomposition $1 + 1 + 3 + \bar{3}$ under $SU(3)$. Thus we might expect that in fact there is actually an $SU(3)$-structure on $M_7$. By considering each of the invariant forms $K$, $\omega$ and $\chi$ in turn we will see that this is indeed the case. Each will further restrict the $G$-structure until we are left with $SU(3)$.

We start with $K$. Clearly, a fixed vector is left invariant by $SO(6) \subset SO(7)$ rotations of the orthonormal frame. Thus we see that $K$ defines an $SO(6)$-structure. Equivalently we can introduce
\[
\Pi_m{}^n = 2K_mK^n - \delta_m{}^n
\] (4.12)
satisfying $\Pi^2 = \delta$, since $K^2 = 1$, and hence defining an almost product structure. It is metric compatible in the sense that $\Pi g \Pi^T = g$, or equivalently $\Pi_{mn} = \Pi_{mr}\delta_{rn}$ is symmetric. It is also integrable in that its Nijenhuis tensor defined by
\[
N_{mn}{}^r = \Pi_m{}^k \partial_n \Pi_k{}^r - \Pi_n{}^k \partial_m \Pi_k{}^r
\] (4.13)
vanishes using $d(e^{-\Phi}K) = 0$. This implies that we in fact have a product structure. It follows that we can find coordinates such that $\Pi$ is diagonal, or equivalently $K = e^{\Phi} dx^7$. In these coordinates the seven-dimensional metric takes the form

$$ds_7^2 = g_{ab} dx^a dx^b + e^{2\Phi} dx_7^2.$$  (4.14)

Note that the geometry is not a direct product since $g_{ab}$ and $\Phi$ are allowed to depend on all the coordinates. The metrics of the solutions presented in [6, 7] indeed have this form, as we shall show in section 6.

Now consider $\omega$. The pair $(K, \omega)$ define what is known as an almost contact metric structure (see for example [25]). This means, in general, that the structure group of the frame bundle on a manifold $M_{2k+1}$ reduces from $SO(2k + 1)$ to $U(k)$, implying here that we have an $U(3)$-structure. It is the analog of an almost hermitian structure for odd-dimensional manifolds. A manifold $M_{2k+1}$ is said to have an almost contact metric structure if it admits a $(1, 1)$ tensor $\omega_{mn}$ and a one-form $K$ satisfying the first equation of (4.7), and furthermore $\omega$ is metric compatible so that $\omega g \omega^T = g$, or equivalently $\omega_{mn} = \omega_{mr} g_{rn}$ is a two form. Note that this implies $i_K \omega = 0$. Essentially, the existence of $K$ allows one to consistently decompose the tangent space into $2k$-dimensional piece and a one-dimensional piece. The two-form $\omega$ then defines an almost hermitian structure on the $2k$-dimensional piece, so that the corresponding complexified tangent space splits into the sum of a $k$-dimensional complex space and its complex conjugate. Thus, in general, we have the decomposition $TM_{2k+1} \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} \oplus (T_K \otimes \mathbb{C})$. There is an integrability condition similar to that of an almost hermitian structure and if it is integrable the almost contact metric structure is called normal. In the geometries we are interested in the structure is not integrable in general.

It is interesting to note that restricting to the six-dimensional part of the metric (4.14) by setting $x_7$ constant we obtain a conventional almost hermitian structure. However this is again not integrable and the six-dimensional manifold is not a complex manifold. This is perhaps surprising since such solutions describe fivebranes wrapped on SLAG three cycles in Calabi-Yau manifolds and hence one might naively have expected that the six-dimensional part of the geometry is complex.

Finally we come to $\chi$. We first note that we can define a complex three-form

$$\vartheta = \chi - i(i_K \ast \chi).$$  (4.15)

We see that $\vartheta$ is normal to $K$, and also, from the third equation of (4.7), that it is a $(3, 0)$-form with respect to the almost contact structure $(K, \omega)$. In other words it is a
section of $\Lambda^3 T^{1,0}$. In this sense it is the analog of the holomorphic three-form on the original Calabi–Yau manifold. In fact, in an exactly analogous way, it is easy to see that the subgroup of $U(3)$ which preserves $\vartheta$ is $SU(3)$. Thus we conclude that we do indeed have an $SU(3)$-structure on $M_7$.

Since $d(e^{-\Phi} K) = d(e^{-\Phi} \omega) = 0$ it follows that

$$*H = e^{2\Phi} d(e^{-2\Phi} \text{Im} \vartheta)$$

which shows that $\vartheta$ is a generalized calibration. This mirrors the fact that for a Calabi-Yau 3-fold the imaginary part of the holomorphic three-form calibrates SLAG three-cycles.

### 4.2 The necessary and sufficient conditions

We have shown that given two spinors $\epsilon^\pm$ satisfying (4.1), with $\epsilon^T \epsilon^- = 0$, $M_7$ necessarily has an $SU(3)$-structure given by $(K, \omega, \chi)$. The structure is not general, but as in the case of $G_2$-structure above is restricted. We have seen already that, for instance, the almost product structure defined by $K$ is integrable, though the almost contact structure $(K, \omega)$ is not. In general, we showed that we have the conditions

$$dK = d\Phi \wedge K$$
$$d\omega = d\Phi \wedge \omega$$
$$d\chi \wedge K = d\Phi \wedge \chi \wedge K$$
$$d(i_K * \chi) \wedge i_K * \chi = 0$$
$$d(i_K * \chi) \wedge K \wedge \omega = 0$$

(4.17)

These are also sufficient conditions for the existence of a solution to (4.1). To see this one needs to show that $\phi^\pm$ defined by (4.3) satisfy (4.3), which follow from the algebraic properties of $(K, \omega, \chi)$ and also the each satisfy the conditions (3.12). The latter is straightforward to show using the fact that $*\phi^\pm = \pm \chi \wedge K - \frac{1}{2} \omega \wedge \omega$.

In other words, a spacetime of the form $\mathbb{R}^{1,2} \times M_7$ admits two orthogonal Killing spinors $\epsilon^\pm$ satisfying (4.1), if and only if we have an $SU(3)$-structure on $M_7$ satisfying the above conditions (4.17). In addition, the antisymmetric torsion $\pm \frac{1}{2} H$ is given by (4.16). Furthermore, this gives a supersymmetric solution of type II supergravity if and only if we impose in addition the closure of $H$ as before.

Once again we can in principle use this result as a method for finding solutions. In section 6 we shall recover the solutions that were found in [6, 7] using gauged supergravity techniques.
Constructing solutions with a single $G_2$ structure

In this section we will use the results of section 3 to construct examples of the geometries with a single $G_2$-structure that were described there. These correspond to fivebranes wrapping associative three-cycles. We focus on co-homogeneity one manifolds.

We generalise the method presented in [23]. One first makes an ansatz for the $G_2$ structure $\phi$ satisfying appropriate symmetries and then finds the associated metric from the expression:

$$g_{ij} = (\det s_{ij})^{-1/9} s_{ij}$$

$$s_{ij} = \frac{1}{144} \phi_{imn2n3n6n7} \epsilon^{n1n2n3n4n5n6n7}, \quad \epsilon^{1234567} = 1$$

(5.1)

The three form $\phi$ must be stable in the sense of [31] to ensure that it is generic enough to make the metric non-degenerate. We then impose equations (3.9). If these are satisfied we have a solution to (3.3). One must then impose the closure of $H$ to obtain a solution to the full supergravity theory.

5.1 The example of [2]

Let us first demonstrate this method by recovering the example presented in [2]. This is a co-homogeneity one example with principle orbits $S^3 \times S^3$. Our starting point is an ansatz for the three form that has appeared in constructions of $G_2$ holonomy metrics in [26],[27]:

$$\phi = ab \, dt \wedge \sum_{a=1}^3 (\Sigma_a - \frac{1}{2} \sigma_a) \wedge \sigma_a + \frac{a^3}{3!} \epsilon_{abc} \, \sigma_a \wedge \sigma_b \wedge \sigma_c - \frac{ab^2}{2!} \epsilon_{abc} \sigma_a \wedge (\Sigma_b - \frac{1}{2} \sigma_b) \wedge (\Sigma_c - \frac{1}{2} \sigma_c)$$

(5.2)

where $(\Sigma_a, \sigma_a)$ are left invariant one-forms on $SU(2) \times SU(2)$, satisfying $d\sigma_1 = -\sigma_2 \wedge \sigma_3$, $d\Sigma_1 = -\Sigma_2 \wedge \Sigma_3$ plus cyclic permutations, and $t$ is a radial variable. The two arbitrary functions $a$ and $b$ depend on the radial variable only. The associated metric is given by

$$ds^2_7 = dt^2 + b^2 \sum_{a=1}^3 (\Sigma_a - \frac{1}{2} \sigma_a)^2 + a^2 \sum_{a=1}^3 (\sigma_a)^2$$

(5.3)
Introduce a frame

\[ e^t = dt \]
\[ e^a = b(\Sigma_a - \frac{1}{2} \sigma_a) \]
\[ \tilde{e}^a = a \sigma_a \]  \hspace{1cm} (5.4)

In this frame the three form \( \phi \) and its dual are given by (with \( e^{123123} = -1 \))

\[ \phi = e^t \wedge e^a \wedge \tilde{e}^a + \frac{1}{3!} \epsilon_{abc} e^a \wedge e^b \wedge \tilde{e}^c - \frac{1}{2!} \epsilon_{abc} \tilde{e}^a \wedge e^b \wedge e^c \]
\[ *\phi = \frac{1}{3!} \epsilon_{abc} e^t \wedge e^a \wedge e^b \wedge e^c + \frac{1}{2!} \epsilon_{abc} e^t \wedge \tilde{e}^a \wedge e^b \wedge e^c + \frac{1}{2!} \tilde{e}^a \wedge e^a \wedge e^b \wedge e^b \]  \hspace{1cm} (5.5)

and it is straightforward to calculate:

\[ d\phi = \frac{1}{2} \{(\log a)' - \frac{b}{4a^2}\} \epsilon_{abc} e^t \wedge \tilde{e}^a \wedge e^b \wedge \tilde{e}^c - \frac{1}{2} \{(\log b^2 a)' - \frac{1}{b}\} \epsilon_{abc} e^t \wedge \tilde{e}^a \wedge e^b \wedge e^c \]
\[ d*\phi = \{(\log ab)' - \frac{1}{2b} - \frac{b}{8a^2}\} e^t \wedge \tilde{e}^a \wedge e^a \wedge \tilde{e}^b \wedge e^b \]  \hspace{1cm} (5.6)

First note that \( d\phi \wedge \phi = 0 \) is automatically satisfied. Also we have \( d*\phi = d(2\Phi) \wedge *\phi \) with:

\[ d(2\Phi) = \left[ (\log a^2b^2)' - \frac{b}{2} - \frac{b}{4a^2} \right] dt \]  \hspace{1cm} (5.7)

So all the conditions (3.8) are satisfied and we have a solution to the Killing spinor equations (3.3). Note that the two functions \( a, b \) are still arbitrary. This is because we started with a very special ansatz which guaranteed from the beginning that all the conditions were satisfied. However we still need to impose the closure of \( H \). This will give us second order equations in principle but as we shall see, in this case they are trivially integrated once. The torsion \( H \) is constructed from (3.7) and we find that:

\[ H = \frac{1}{3!} \epsilon_{abc} F e^a \wedge e^b \wedge e^c + \frac{1}{2!} \epsilon_{abc} G e^a \wedge e^b \wedge \tilde{e}^c \]  \hspace{1cm} (5.8)

where

\[ F = \{(\log b^2 a^{-1})' - \frac{1}{b} + \frac{b}{2a^2}\} \]
\[ G = \{- (\log a)' + \frac{b}{4a^2}\} \]  \hspace{1cm} (5.9)

Imposing \( dH = 0 \) we get the equations

\[ F' + F(\log b^3)' = 0 \]  \hspace{1cm} (5.10)
\[ G' + G(\log ba^2)' = 0 \]  \hspace{1cm} (5.11)
\[ G \frac{1}{2b} - F \frac{b}{8a^2} = 0 \]  \hspace{1cm} (5.12)
The first two are trivially integrated to give $Fb^3 = C_1$ and $Gba^2 = C_2$ while the third implies that $C_1 = 4C_2 \equiv -\mu$. Using the definitions of $F$ and $G$ we thus arrive at a system of first order equations for the metric functions $a, b$:

\begin{align*}
    b' &= \frac{1}{2} (1 - \frac{\mu}{b^2}) (1 - \frac{b^2}{4a^2}) \quad (5.13) \\
    a' &= \frac{b}{4a} (1 + \frac{\mu}{b^2}) \quad (5.14)
\end{align*}

These equations are precisely those derived at the end of section 3.1.1 of [2]. For the special case $\mu = 0$ the torsion and the dilaton vanish and the equations can be integrated to recover the $G_2$ holonomy metric on the spin bundle of $S^3$ [28]. The solution with $b^2 = \mu$, $a^2 = \sqrt{\mu r}$ was found in [2] using gauged supergravity methods. It corresponds to fivebranes wrapped on the associative three-sphere of the $G_2$-holonomy manifolds of [28], in the near horizon limit. The general solution of these equations remains an outstanding problem.

One can extend this analysis in a relatively straightforward way to recover the solution first presented in [3], but the formulae are rather lengthy so we shall not present the details here.

### 5.2 New Solution

Another example is to start with a cohomogeneity one manifold with principal orbits $SU(3)/U(1) \times U(1)$. Such $G_2$ structures have appeared in [29], [30], and solutions have been found for a $G_2$ metric on the $\mathbb{R}^3$ bundle over $\mathbb{CP}^2$ [28]. Here we use the results of [29] to find and solve the BPS equations for solutions that describe fivebranes wrapped on the $\mathbb{R}^3$ fibres, which are non-compact associative three cycles, in such $G_2$ manifolds.

Let $\{e_a\}$ be the left invariant one forms on $SU(3)$. We define

\begin{align*}
    \omega_1 &= e_{12}, \quad \omega_2 = e_{34}, \quad \omega_3 = e_{56} \quad (5.15)
\end{align*}

and also a basis for the $SU(3)$ invariant three forms:

\begin{align*}
    \alpha &= e_{246} - e_{235} - e_{145} - e_{136} \quad \beta = e_{135} - e_{146} - e_{236} - e_{245} \quad (5.16)
\end{align*}

where $e_{12} \equiv e_1 e_2$ etc, and the exterior product of forms is understood. It then follows
that these satisfy

\[ d\omega_1 = d\omega_2 = d\omega_3 = \frac{1}{2} \alpha \]
\[ d\alpha = 0 \]
\[ d\beta = -2(\omega_1 \land \omega_2 + \omega_2 \land \omega_3 + \omega_3 \land \omega_1) \]
\[ d(\omega_i \land \omega_j) = 0, \quad i \neq j \quad (5.17) \]

The \( G_2 \) structure and it's associated metric are given by:

\[ \phi = (f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3) \land dt + f_1 f_2 f_3 (\cos \theta \alpha + \sin \theta \beta) \quad (5.18) \]

\[ ds^2_7 = dt^2 + f_1^2 g_1 + f_2^2 g_2 + f_3^2 g_3 \quad (5.19) \]

where \( f_i, \theta \) are arbitrary functions of \( t \) and

\[ g_1 = e_1^2 + e_2^2, \quad g_2 = e_3^2 + e_4^2, \quad g_3 = e_5^2 + e_6^2 \quad (5.20) \]

We find that

\[ * \phi = f_2^2 f_3^2 \omega_2 \land \omega_3 + f_3^2 f_1^2 \omega_3 \land \omega_1 + f_1^2 f_2^2 \omega_1 \land \omega_2 + f_1 f_2 f_3 (\cos \theta \beta - \sin \theta \alpha) \land d(e_1^2 + e_2^2) \quad (5.21) \]

and hence

\[ d*\phi = \left( \frac{1}{2}(f_1^2 + f_2^2 + f_3^2) - (f_1 f_2 f_3 \cos \theta') \alpha \right) \land dt - (f_1 f_2 f_3 \sin \theta') \beta \land dt - 2 f_1 f_2 f_3 \sin \theta (\omega_1 \land \omega_2 + \omega_2 \land \omega_3 + \omega_3 \land \omega_1) \quad (5.22) \]

\[ d*\phi = \left( (f_2^2 f_3^2)' - 2 f_1 f_2 f_3 \cos \theta \right) \omega_2 \land \omega_3 \land dt + \left( (f_3^2 f_1^2)' - 2 f_1 f_2 f_3 \cos \theta \right) \omega_3 \land \omega_1 \land dt + \left( (f_1^2 f_2^2)' - 2 f_1 f_2 f_3 \cos \theta \right) \omega_1 \land \omega_2 \land dt \quad (5.23) \]

Let us first consider the equation \( d*\phi = d(2\Phi) \land *\phi \). This gives:

\[ \Phi' = \log(f_i f_j)' - \frac{f_k}{f_i f_j} \cos \theta, \quad i \neq j \neq k \quad (5.24) \]

which defines the dilaton and also imposes:

\[ f_k \log f_i' + \cos \theta \frac{f_i}{f_j} = f_k \log f_j' + \cos \theta \frac{f_j}{f_i}, \quad i \neq j \neq k \quad (5.25) \]
which is two independent equations. Next we impose \((\star \phi, d\phi) = 0\) and we conclude that:

\[
\theta' = -\sin \theta \frac{f_1^2 + f_2^2 + f_3^2}{f_1 f_2 f_3}
\]  

Having satisfied these conditions we need to impose the closure of \(H\). We find that the torsion is given by:

\[
H = 2f_1 f_2 f_3 \sin \theta \left( \frac{f_3^2}{f_1^2 f_2} \omega_3 + \frac{f_1^2}{f_2^2 f_3} \omega_1 + \frac{f_2^2}{f_1^2 f_3} \omega_2 \right) \wedge dt
\]

\[
- \left[ 2\Phi' f_1 f_2 f_3 \sin \theta - (f_1 f_2 f_3 \sin \theta)' \right] \alpha
\]

\[
- \left[ (f_1 f_2 f_3 \cos \theta)' - 2\Phi' f_1 f_2 f_3 \cos \theta - \frac{1}{2} (f_1^2 + f_2^2 + f_3^2) \right] \beta
\]

For \(H\) to be closed we thus need to impose the following:

\[
(f_1 f_2 f_3 \cos \theta)' - 2\Phi' f_1 f_2 f_3 \cos \theta - \frac{1}{2} (f_1^2 + f_2^2 + f_3^2) = 0
\]

\[
[2\Phi' f_1 f_2 f_3 \sin \theta - (f_1 f_2 f_3 \sin \theta)']' + f_1 f_2 f_3 \sin \theta(\frac{f_3^2}{f_1^2 f_2} + \frac{f_1^2}{f_2^2 f_3} + \frac{f_2^2}{f_1^2 f_3}) = 0
\]

Now (5.29) is a second order equation and also we have five equations for four unknown functions. However the second order equation follows from the four first order equations so there is no inconsistency. To see this we first rearrange equations (5.25),(5.26),(5.28), and write them as:

\[
(\log f_i)' = \frac{1}{2 \cos \theta} \frac{f_1^2 + f_2^2 + f_3^2}{f_1 f_2 f_3} - \cos \theta \frac{f_i^2}{f_1 f_2 f_3}, \quad i = 1, 2, 3
\]

\[
\theta' = -\sin \theta \frac{f_1^2 + f_2^2 + f_3^2}{f_1 f_2 f_3}
\]

Then we note that we can write (5.29) as:

\[
\frac{1}{2} (\tan \theta(f_1^2 + f_2^2 + f_3^2))' + f_1 f_2 f_3 \sin \theta(\frac{f_3^2}{f_1^2 f_2} + \frac{f_1^2}{f_2^2 f_3} + \frac{f_2^2}{f_1^2 f_3}) = 0
\]

This is satisfied given (5.30). So indeed we have arrived at a system of BPS equations for the four unknown functions.

To solve the BPS equations we first define a new radial variable by \(dt = f_1 f_2 f_3 d\lambda\). In terms of this the equations become:

\[
\frac{d}{d\lambda} (\log f_i) = \frac{1}{2 \cos \theta} \sum_i f_i^2 - \cos \theta f_i^2
\]

\[
\frac{d\theta}{d\lambda} = -\sin \theta \sum_i f_i^2
\]
Define $u_i = f_i^2 \tan \theta$ then using the above we find:

\[ \frac{d(u_i^{-1})}{d\lambda} = 2 \frac{\cos^2 \theta}{\sin \theta} \]
\[ \frac{d\theta}{d\lambda} = -\cos \theta \sum_i u_i \]

(5.33)

Now define another radial variable by $d\rho = (2 \cos^2 \theta / \sin \theta) d\lambda$. Now in terms of this we can solve for $u_i$ and then for $\sin \theta$. We find that:

\[ u_i = \frac{1}{\rho - \alpha_i} \]
\[ \sin \theta = (Mq(\rho, \alpha_i))^{-1/2} \]

(5.34)

where $q(\rho, \alpha_i) \equiv \prod_i (\rho - \alpha_i)$. The $\alpha_i$ and $M$ are four arbitrary integration constants. By rescaling the radial coordinate we find that the solution takes the form:

\[ ds^2 = \frac{d\rho^2}{4 \sqrt{q - M^2}} + \sum_i \sqrt{q - M^2} \frac{1}{\rho - \alpha_i} g_i \]
\[ e^{2\Phi} = e^{2\Phi_0} (1 - \frac{M^2}{q}) \]
\[ H = M \left( \sum_i \frac{\omega_i}{(\rho - \alpha_i)^2} \wedge d\rho - \frac{1}{2} \sum_i \frac{1}{\rho - \alpha_i} \right) \]

(5.35)

In the limit $M \to 0$ the torsion vanishes, the dilaton tends to a constant and we recover the metric of [30]. This is a $G_2$ holonomy metric with a conical singularity for generic values of the $\alpha_i$ but is regular when two of these constants are equal. In this case one obtains the $G_2$ holonomy metric on the $\mathbb{R}^3$ bundle over $\mathbb{C}P^2$ [28]. For non-zero $M$ the torsion is non-vanishing and in the large $\rho$ limit the solution approaches the one in [30]. In the interior we see that the radial variable is constrained by $\rho \geq \rho_0$ where $\rho_0$ is the solution of $q - M^2 = 0$. Note that we always have $\rho_0 \geq \alpha_i$. At $\rho = \rho_0$ the metric is singular for all values of $\alpha_i$.

When $M = 0$ the $G_2$ holonomy manifolds do not have any compact associative three-cycles on which to wrap a fivebrane, but they do have non-compact associative three-cycles. In the example of the $G_2$-holonomy metric on the $\mathbb{R}^3$ bundle over $\mathbb{C}P^2$ there is a co-associative $\mathbb{C}P^2$ bolt, and the $\mathbb{R}^3$ fibres are non-compact and associative. It is thus natural to interpret the solutions with $M \neq 0$ as describing fivebranes wrapping such a non-compact associative three-cycle, in the near horizon limit.

Finally, we point out that it should be very straightforward to generalise the solutions in this section to co-homogeneity one solutions where the principle orbits are $\mathbb{C}P^3$. These include the $G_2$ holonomy metric on the $\mathbb{R}^3$ bundle over $S^4$ [28].
6 Recovering the solution of \([6, 7]\)

In this section we will use a similar procedure to recover the solutions of \([6, 7]\). In so doing we will explicitly demonstrate the \(SU(3)\) structure of these solutions.

We first introduce frame one forms as in \([7]\):

\[
\begin{align*}
\nu^1 &= d\theta \\
\nu^2 &= \sin \theta d\phi \\
S^1 &= \cos \phi \frac{\sigma^1}{2} - \sin \phi \frac{\sigma^2}{2} \\
S^2 &= \sin \theta \frac{\sigma^3}{2} - \cos \theta \left( \sin \phi \frac{\sigma^1}{2} + \cos \phi \frac{\sigma^2}{2} \right) \\
S^3 &= -\cos \theta \frac{\sigma^3}{2} - \sin \theta \left( \sin \phi \frac{\sigma^1}{2} + \cos \phi \frac{\sigma^2}{2} \right)
\end{align*}
\]

(6.1)

where \(\theta, \phi\) are angles on a two-sphere and \(\sigma^a\) are the usual left invariant one forms on \(SU(2)\) satisfying \(d\sigma^a = \frac{1}{2} \epsilon^{abc} \sigma^b \wedge \sigma^c\). These satisfy the exterior algebra:

\[
\begin{align*}
dS^1 &= 2S^2 \wedge S^3 + \nu^2 \wedge S^3 + A \wedge S^2 \\
dS^2 &= 2S^3 \wedge S^1 - \nu^1 \wedge S^3 - A \wedge S^1 \\
dS^3 &= 2S^1 \wedge S^2 + \nu^1 \wedge S^2 - \nu^2 \wedge S^1 \\
d\nu^1 &= 0 \\
d\nu^2 &= -A \wedge \nu^1
\end{align*}
\]

(6.2)

where \(A = \cos \theta d\phi\). We introduce a frame:

\[
\begin{align*}
e^r &= a(r, x_7)dr \\
e^a &= b(r, x_7)S^a \\
\tilde{e}^a &= c(r, x_7)(\nu^a + S^a) \\
e^3 &= b(r, x_7)S^3 \\
e^7 &= e^\Phi dx_7
\end{align*}
\]

(6.3)

where \(a = 1, 2\), and make an ansatz for the \(SU(3)\) invariant forms:

\[
\begin{align*}
K &= e^7 \\
\omega &= e^r \wedge e^3 + e^1 \wedge e^2 - e^2 \wedge \tilde{e}^1 \\
\chi &= e^r \wedge (-e^1 \wedge e^2 + \tilde{e}^1 \wedge \tilde{e}^2) - e^3 \wedge (e^1 \wedge \tilde{e}^1 + e^2 \wedge \tilde{e}^2) \\
i_K \chi &= e^r \wedge (e^1 \wedge \tilde{e}^1 + e^2 \wedge \tilde{e}^2) - e^3 \wedge (e^1 \wedge e^2 - \tilde{e}^1 \wedge \tilde{e}^2)
\end{align*}
\]

(6.4)
corresponding to the metric:

\[ ds^2 = a^2 dr^2 + b^2 d\Omega_3^2 + c^2 \sum_{\alpha=1,2} (\nu^\alpha + S^\alpha)^2 + e^{2\Phi} dx_7^2 \]  

(6.5)

where the orientation is taken to be \( \epsilon_31\hat{1}\hat{2}\hat{7} = -1 \).

The following identities are useful:

\[ d\alpha = -\frac{1}{2} d\beta = S^3 \wedge \gamma \]
\[ d\gamma = 2S^3 \wedge \beta \]  

(6.6)

where

\[ \alpha = S^1 \wedge S^2 \]
\[ \beta = S^1 \wedge \nu^2 - S^2 \wedge \nu^1 + \nu^1 \wedge \nu^2 \]
\[ \gamma = S^1 \wedge \nu^1 + S^2 \wedge \nu^2 \]  

(6.7)

Using these we can write

\[ \omega = ab \, dr \wedge S^3 + bc \, dS^3 \]  

(6.8)

\[ \chi = a(c^2 - b^2) dr \wedge \alpha + ac^2 \, dr \wedge \beta + \frac{cb^2}{2} d\beta \]  

(6.9)

\[ i_K * \chi = abc \, dr \wedge \gamma + b(c^2 - b^2) S^3 \wedge \alpha + \frac{bc^2}{2} d\gamma \]  

(6.10)

Imposing the necessary and sufficient conditions discussed in section 4.2 we find the following equations must be imposed:

\[ \partial_{x^7} (e^{-\Phi} ab) = 0 \]
\[ \partial_{x^7} (e^{-\Phi} bc) = 0 \]
\[ \partial_r (e^{-\Phi} bc) = e^{-\Phi} ab \]
\[ \partial_r b = \frac{ac}{b} \]  

(6.11)

and that the torsion is given by

\[ H = F_1 \, d\alpha + F_2 \, dr \wedge \alpha + F_3 \, dr \wedge \beta + F_4 \, dx_7 \wedge \alpha + F_5 \, dx_7 \wedge \beta \]  

(6.12)

where

\[ F_1 = -b^2 c e^{-\Phi} \partial_{x^7} \log(abc \, e^{-2\Phi}) \]
\[ F_2 = -a e^{-\Phi} (b^2 \partial_{x^7} \log(bc^2 e^{-2\Phi}) - c^2 \partial_{x^7} \log(b^3 e^{-2\Phi})) \]
\[ F_3 = ac^2 e^{-\Phi} \partial_{x^7} \log(b^3 e^{-2\Phi}) \]
\[ F_4 = e^{\Phi} \left( \frac{b^2}{a} \partial_r \log(bc^2 e^{-2\Phi}) - \frac{c^2}{a} \partial_r \log(b^3 e^{-2\Phi}) - \frac{2b^2}{c} + \frac{2c^3}{b^2} \right) \]
\[ F_5 = e^{\Phi} \left( -\frac{c^2}{a} \partial_r \log(b^3 e^{-2\Phi}) + \frac{2c^3}{b^2} \right) \]  

(6.13)
Now imposing closure of $H$ we find that
\[
\begin{align*}
\partial_r F_1 + 2F_3 - F_2 &= 0 \\
\partial_{x_7} F_1 + 2F_5 - F_4 &= 0 \\
\partial_{x_7} F_2 - \partial_r F_4 &= 0 \\
\partial_{x_7} F_3 - \partial_r F_5 &= 0
\end{align*}
\] (6.14)

Using the first order equations (6.11) we find that the above equations reduce to the single second order equation
\[
abe^{-\Phi} \partial_{x_7}^2 b + \partial_r (ce^\Phi) = 0
\] (6.15)

Now (6.11) imply that $abe^{-\Phi} = h(r)$ and by choosing the radial coordinate appropriately we can set $h \equiv 1$. Then the rest of the equations determine the dilaton and $a, c$ in terms of $b$ via:
\[
\begin{align*}
a^2 &= \frac{b}{r} \partial_r b \\
c^2 &= a^2 r^2 \\
e^\Phi &= ab
\end{align*}
\] (6.16)

where $b$ satisfies the second order non-linear pde:
\[
\frac{\partial^2}{\partial r^2} b^3 + 3 \frac{\partial^2}{\partial x_7^2} b = 0
\] (6.17)

We recover the solution of [7] by making a change of variables to $(z, \psi)$ such that:
\[
\begin{align*}
r &= \sqrt{z} B(z) \sin \psi \\
x_7 &= A(z) \cos \psi
\end{align*}
\] (6.18)

where $A(z)$ and $B(z)$ satisfy
\[
\begin{align*}
A'(z) &= B(z) \\
B'(z) &= A(z) - \frac{B(z)}{2z}
\end{align*}
\] (6.19)

so that
\[
\begin{align*}
A(z) &= \frac{z^{1/4}}{\sqrt{2}} (I_{-1/4}(z) + \mu \ K_{1/4}(z)) \\
B(z) &= \frac{z^{1/4}}{\sqrt{2}} (I_{3/4}(z) - \mu \ K_{3/4}(z))
\end{align*}
\] (6.20)
Here $\mu$ is an integration constant. The solution is then just

$$b^2 = z$$  \hspace{1cm} (6.21)

Thus we have explicitly demonstrated the $SU(3)$ structure of the solution found in [6, 7].

It seems a formidable challenge to find the general solution of (6.17). Let us just note that it easy to construct solutions which do not depend on $x_7$. We then have

$$b = (\lambda_1 r + \lambda_2)^{1/3}$$  \hspace{1cm} (6.22)

These solutions might be interpreted as solutions corresponding to wrapped NS five-branes that are smeared over the $x^7$ direction. Note in particular, that the torsion is non vanishing for any choice of the constants $\lambda_1, \lambda_2$ so that we do not recover the pure geometry $CY_3 \times S^1$, where $CY_3$ is the deformed conifold, as one might have expected. The reason for this is simply that a more general ansatz for the $SU(3)$ structure is required. Enlarging our ansatz would also allow one in principle to obtain more general wrapped NS fivebrane solutions as well, but we expect that the pdes will be intractable without further insight.

7 Discussion

We have analysed supersymmetric type II geometries of the form $\mathbb{R}^{1,5-d} \times M_{d+4}$ with non-trivial NS three-form flux and dilaton, motivated by the fact that the near-horizon limits of wrapped NS fivebranes geometries are of this type. In particular, we considered the examples of seven-dimensional manifolds arising from branes wrapped on associative or SLAG three-cycles. These geometries admit a $G_2$ or an $SU(3)$ structure, respectively, of a specific type that we determined. We also proved a converse result, namely that given such a geometric structure then one obtains a supersymmetric solution to the equations of motion. We used the converse result as a method to construct solutions. Note that for both cases the group $G$ in the $G$-structure is exactly the same as that of the underlying special holonomy group of the manifold containing the supersymmetric cycle on which fivebrane is wrapped.

It is straightforward to extend the results for these specific examples to the geometries arising when type II NS5-branes wrap other supersymmetric cycles. In [7] we analysed the holonomy of the connections $\nabla^\pm$ that would arise in each case, and a summary appeared in table 1 of that reference. In $n$ dimensions, in the cases where just one of the connections $\nabla^\pm$ has special holonomy $G \subset SO(n)$, the geometry is
specified by a $G$-structure of a type that can be easily specified, by following the discussion of section 2 (for related work see [13, 14, 15, 18]). In the cases where both $\nabla^\pm$ have special holonomy contained in $G' \subset SO(d)$ say, we find that the manifolds admit a $G \subset G'$-structure. For both cases, one finds that the group $G$ of the $G$-structure that appears in the final geometry is exactly the same as the special holonomy group of the manifold, just as for the examples explicitly discussed in this paper.

For example, $D = 6$ geometries can arise when NS fivebranes wrap Kähler two-cycles in Calabi-Yau three-folds or two-folds. In the former case one of the connections $\nabla^\pm$ has special holonomy $SU(3)$ while the other has general holonomy $SO(6)$. The resulting geometry has an $SU(3)$-structure which was discussed in [13, 14, 15]. Examples of this geometry were presented in [1]. On the other hand when NS-fivebranes wrap two-cycles in a Calabi-Yau two-fold we find that both connections $\nabla^\pm$ have $SU(3)$-holonomy. In this case the structure group of the six-manifold is in fact $SU(2)$. This structure includes a product structure which allows one to choose co-ordinates with a four-two split to the metric, but it is not a direct product. Examples of this kind of geometry were presented in [4, 5].

Finally, it is also worth mentioning that much of the discussion applies to type I supergravity. The action and supersymmetry transformations are recorded in the Appendix. For this case there is only a single connection with totally anti-symmetric torsion $\nabla^+$ and so supersymmetry will just give rise to a single $G$-structure. Consider for example the $D = 7$ case. Since the variation of the dilatino and gravitino for the type I theory are the same as for $\epsilon^+$ we deduce that the $G_2$ structure is exactly the same as that discussed in section 3. In addition we need to ensure the vanishing of the supersymmetry variation of the gaugino

$$F_{mn} \gamma^{mn} \epsilon = 0 \quad (7.1)$$

This implies, following [32] that $F$ must satisfy the $G_2$ instanton equation $F_{mn} = \frac{1}{2} \phi_{mn}^{pq} F_{pq}$ i.e. the two-form $F$ is in the 14 in the decomposition (2.2). This is the type of geometry dictated by supersymmetry that would arise when type I fivebranes wrap associative three-cycles and also SLAG three-cycles. To obtain a solution to the equations of motion for type I supergravity we have to solve $dH = 2 \alpha' \text{Tr} F \wedge F$. Using the integrability conditions given in the appendix it is clear that these conditions are also sufficient to obtain a supersymmetric solution to the equations of motion, by generalising the argument in section 2. As yet there are no known solutions of this kind with non-vanishing $F$. Such solutions would have the geometry naturally expected for type I or heterotic “gauge” fivebranes [33] wrapping associative three-
cycles. For the case of type I fivebranes wrapping SLAG three-cycles the interesting possibility arises that there will in fact be an $SU(3)$ structure, despite the fact that it is not dictated by supersymmetry alone.

It is interesting to note that the type II solutions give rise to type I solutions with $F = 0$. These correspond to type I or heterotic “neutral” fivebranes wrapping associative three-cycles. The type II solutions corresponding to fivebranes wrapping SLAG three-cycles thus give rise to type I solutions describing type I fivebranes wrapping SLAG three-cycles that have an $SU(3)$ structure. This is some evidence that this will also occur for wrapped gauge-fivebranes.

In type I or heterotic string theory anomaly cancellation implies that the Bianchi identity is modified by higher order corrections in $\alpha'$. To leading order this is most informatively written as $dH = 2\alpha' \text{Tr}[F \wedge F - R(\Omega^-) \wedge R(\Omega^-)]$ where $\Omega^- = \omega - H/2$ [34]. This should be viewed as implicitly defining $H$. One can ask whether one can solve this for wrapped branes by identifying $A$ with $\Omega^-$ as this would be the analogue of “symmetric fivebranes” [33]. For supersymmetric fivebranes wrapping associative three cycles, only $\Omega^+ = \omega + H/2$ has holonomy contained in $G_2$ and hence identifying $A$ with $\Omega^-$ would not be supersymmetric. Interestingly, for supersymmetric fivebranes wrapping SLAG three-cycles both $\Omega^\pm$ have holonomy contained in $G_2$ and hence one can obtain supersymmetric solutions for these cases. More explicitly, the solution constructed in [7] automatically gives a solution of the heterotic or type I string theory with non-vanishing gauge-fields if we simply identify $A = \Omega^-$. This argument equally applies to the solutions found in [4, 5] corresponding to fivebranes wrapping two-cycles in Calabi-Yau two-folds. For the type II theory these are holographically dual to a slice of the Coulomb branch of pure $\mathcal{N} = 2$ super-Yang-Mills theory [4]. The corresponding type I solution has half the supersymmetry and so should holographically encode information about the $\mathcal{N} = 1$ gauge theories arising on type I or heterotic fivebranes wrapping two-cycles in Calabi-Yau two-folds. It would be interesting to study this further.

Acknowledgements

We would like to thank Gary Gibbons and Chris Hull for helpful discussions. All authors are supported in part by PPARC through SPG #613. DW also thanks the Royal Society for support.
8 Appendix

We will derive the integrability conditions in the context of type I SUGRA. The bosonic fields are the same as the NS sector of the type II supergravity supplemented by a gauge field in the adjoint of some gauge group, with field strength $F$. For gauge group $SO(32)$ or $E_8 \times E_8$ this is part of the low-energy effective action of type I or heterotic string theory. The action is given by:

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left( R + 4(\nabla \Phi)^2 - \frac{1}{12} H^2 - \alpha' \text{Tr} F^2 \right)$$  \hspace{1cm} (8.1)

with

$$dH = 2\alpha' \text{Tr} F \wedge F \hspace{1cm} \text{(8.2)}$$

The equations of motion are given by

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}_{\nu} + 2\nabla_\mu \nabla_\nu \Phi - 2\alpha' \text{Tr} F_\mu F_{\nu} = 0$$

$$\nabla^2 \Phi - 2(\nabla \Phi)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{\alpha'}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} = 0$$

$$\nabla_\mu (e^{-2\Phi} H^{\mu\nu}) = 0$$

$$2e^{2\Phi} D_\mu (e^{-2\Phi} F_{\mu\nu}) - F^{\mu\rho} H_{\mu\nu\rho} = 0$$  \hspace{1cm} (8.3)

Supersymmetric configurations have vanishing variation of the gravitino, dilatino and gaugino:

$$\delta \psi_\mu \sim \nabla_\mu \epsilon + \frac{1}{8} H_{\mu\nu\rho} \Gamma^{\nu\rho} \epsilon = 0$$

$$\delta \lambda \sim \Gamma^\mu \partial_\mu \Phi \epsilon + \frac{1}{12} H_{\mu\nu\rho} \Gamma_{\mu\nu\rho} \epsilon = 0$$

$$\delta \chi \sim F_{\mu\nu} \Gamma_{\mu\nu} \epsilon = 0$$  \hspace{1cm} (8.4)

where $\epsilon$ is a Majorana-Weyl spinor of $Spin(1,9)$. Note that the first two conditions are half of the conditions arising in the type II theories. We now deduce some consequences of the integrability conditions of these equations.

First, take the covariant derivative of the variation of the gravitino and antisymmetrise, to get

$$R_{\mu\nu\sigma_1\sigma_2} \Gamma^{\sigma_1\sigma_2}_{\epsilon} = -\nabla_{[\mu} H_{\nu]\sigma_1\sigma_2} \Gamma^{\sigma_1\sigma_2}_{\epsilon} - \frac{1}{2} H_{[\mu \sigma_1} |_{\rho|} H_{\nu]\rho\sigma_2} \epsilon$$  \hspace{1cm} (8.5)

Next multiply this expression by $\Gamma^\nu$ and use a Bianchi identity to obtain an expression for $R_{\mu\nu} \Gamma^\nu \epsilon$. Then use $H_{\mu\nu\rho} \Gamma^{\nu\rho}$ times the dilatino variation, the covariant derivative
of the dilatino as well as $F_{\mu\nu}\Gamma^\nu$ times the variation of the gaugino to get

$$(R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}^{\rho\sigma} + 2\nabla_\mu \nabla_\nu \Phi - 2\alpha' \text{Tr} F^\rho_{\mu} F^\rho_{\nu}) \Gamma^\nu \epsilon =$$

$$\frac{1}{12} (dH - 2\alpha' \text{Tr} F \wedge F)_{\mu\rho\sigma} \Gamma^{\mu\rho\sigma} \epsilon + \frac{1}{2} e^{2\phi} \nabla^\rho (e^{-2\phi} H_{\rho\mu\nu}) \Gamma^\nu \epsilon$$

(8.6)

Similar manipulations on $\Gamma^\mu \nabla_\mu$ acting on the variation of the dilatino implies

$$(\nabla^2 \Phi - 2(\nabla \Phi)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{\alpha'}{2} F_{\mu\nu} F^{\mu\nu}) \epsilon =$$

$$- \frac{1}{48} (dH - 2\alpha' F \wedge F)_{\mu\rho\sigma} \Gamma^{\mu\rho\sigma} \epsilon - \frac{1}{4} e^{2\phi} \nabla^\mu (e^{-2\phi} H_{\nu\mu\rho}) \Gamma^{\nu\rho} \epsilon$$

(8.7)

while $\Gamma^\mu \nabla_\mu$ acting on the variation of the gaugino yields

$$(2 e^{2\phi} \nabla^\mu (e^{-2\phi} F_{\mu\nu}) - F^{\rho\sigma} H_{\rho\sigma\nu}) \Gamma^\nu \epsilon = 3D F_{\mu\nu} \Gamma^{\mu\nu} \epsilon$$

(8.8)

We next note that if the Bianchi identities for $H$ and $F$ satisfied as well as the $H$ equation of motion then we deduce the dilaton equation of motion. The equation (8.8) is of the form $A_{\mu} \Gamma^\mu \epsilon = 0$ which implies $A^\mu A_\mu = 0$. Similarly (8.6) is of the form $B_{\mu\nu} \Gamma^\nu = 0$ which implies $B_{\mu\nu} B^{\mu\nu} = 0$. If we assume that we have a solution of the form $\mathbb{R}^{1,9-n} \times M_n$ then we can deduce that $A_m = B_{mn} = 0$ which give the gauge and Einstein equations of motion. In other words the Killing spinor equations, combined with the Bianchi identities for $H$ and $F$, plus the $H$ equations of motion imply all equations of motion are satisfied.

References

[1] J. M. Maldacena and C. Nunez, “Towards the large N limit of pure N = 1 super Yang Mills,” Phys. Rev. Lett. 86 (2001) 588 [arXiv:hep-th/0008001].

[2] B. S. Acharya, J. P. Gauntlett and N. Kim, “Fivebranes wrapped on associative three-cycles,” Phys. Rev. D 63 (2001) 106003 [arXiv:hep-th/0011119].

[3] J. Maldacena and H. Nastase, “The supergravity dual of a theory with dynamical supersymmetry breaking,” JHEP 0109, 024 (2001) [arXiv:hep-th/0105049].

[4] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, “Wrapped fivebranes and $\mathcal{N} = 2$ super Yang-Mills theory,” Phys. Rev. D 64 (2001) 106008 [arXiv:hep-th/0106117].
[5] F. Bigazzi, A. L. Cotrone and A. Zaffaroni, “$\mathcal{N} = 2$ gauge theories from wrapped five-branes,” Phys. Lett. B 519 (2001) 269 [arXiv:hep-th/0106160].

[6] J. Gomis and J. G. Russo, “$D = 2 + 1\mathcal{N} = 2$ Yang-Mills theory from wrapped branes,” JHEP 0110 (2001) 028 [arXiv:hep-th/0109177].

[7] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, “Fivebranes wrapped on SLAG three-cycles and related geometry,” JHEP 0111 (2001) 018 [arXiv:hep-th/0110034].

[8] J. Gomis, “On SUSY breaking and $\chi$SB from string duals,” Nucl. Phys. B 624 (2002) 181 [arXiv:hep-th/0111060].

[9] R. Apreda, F. Bigazzi, A. L. Cotrone, M. Petrini and A. Zaffaroni, “Some Comments on N=1 Gauge Theories from Wrapped Branes,” arXiv:hep-th/0112236.

[10] K. Hori and A. Kapustin, “Worldsheet descriptions of wrapped NS five-branes,” arXiv:hep-th/0203147.

[11] J. P. Gauntlett, N. Kim, S. Pakis and D. Waldram, “Membranes wrapped on holomorphic curves,” Phys. Rev. D 65 (2002) 026003 [arXiv:hep-th/0105250].

[12] J. Gutowski, G. Papadopoulos and P. K. Townsend, “Supersymmetry and generalized calibrations,” Phys. Rev. D 60 (1999) 106006 [arXiv:hep-th/9905156].

[13] A. Strominger, “Superstrings With Torsion,” Nucl. Phys. B 274 (1986) 253.

[14] C. M. Hull, “Superstring Compactifications With Torsion And Space-Time Supersymmetry,” in Turin 1985, Proceedings, Superunification and Extra Dimensions, 347-375.

[15] S. Ivanov and G. Papadopoulos, “A no-go theorem for string warped compactifications,” Phys. Lett. B 497 (2001) 309 [arXiv:hep-th/0008232].

[16] T. Friedrich and S. Ivanov, “Parallel spinors and connections with skew-symmetric torsion in string theory,” arXiv:math.dg/0102142.

[17] T. Friedrich and S. Ivanov, “Killing spinor equations in dimension 7 and geometry of integrable $G_2$-manifolds,” arXiv:math.dg/0112201.

[18] S. Ivanov, “Connection with torsion, parallel spinors and geometry of Spin(7) manifolds,” arXiv:math.dg/0111210.
[19] J. Maldacena and C. Nunez, “Supergravity description of field theories on curved manifolds and a no go theorem,” Int. J. Mod. Phys. A 16 (2001) 822 [arXiv:hep-th/0007018].

[20] D.D. Joyce, Compact Manifolds with Special Holonomy, Oxford Mathematical Monographs, Oxford University Press, 2000.

[21] S. Salamon, Riemannian Geometry and Holonomy Groups, Vol. 201 of Pitman Research Notes in Mathematics, Longman, Harlow, 1989.

[22] M. Fernandez, A. Gray, Riemannian Manifolds with Structure Group $G_2$, Ann. Mat. Pura Appl. 32 (1982), 19-45.

[23] A. Brandhuber, $G_2$ Holonomy Spaces from Invariant Three-Forms, [hep-th/0112113].

[24] I. Agricola, Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory, [math.DG/0202094].

[25] D.E. Blair, “Conact manifolds in Riemannian geometry”, Lect. Notes No. 509, Springer-Verlag, (1976).

[26] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, A $G_2$ Unification of the Deformed and Resolved Conifold, [hep-th/0112138].

[27] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, Gauge Theory at Large $N$ and New $G_2$ Holonomy Metrics, [hep-th/0106034].

[28] R.L. Bryant and S. Salamon, “On the construction of some complete metrics with exceptional holonomy”, Duke Math. J. 58 (1989) 829.

[29] R. Cleyton and A. Swann, Cohomogeneity-one $G_2$-structures, [math.DG/0111056].

[30] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, Cohomogeneity One Manifolds of Spin(7) and $G_2$ Holonomy, [hep-th/0108245].

[31] N. Hitchin, Stable Forms and special metrics, [math.DG/0107101]

[32] M. Gunaydin and H. Nicolai, “Seven-dimensional octonionic Yang-Mills instanton and its extension to an heterotic string soliton,” Phys. Lett. B 351, 169 (1995) [Addendum-ibid. B 376, 329 (1996)] [arXiv:hep-th/9502009].
[33] A. Strominger, “Heterotic Solitons,” Nucl. Phys. B 343, 167 (1990) [Erratum-ibid. B 353, 565 (1991)],
C. G. Callan, J. A. Harvey and A. Strominger, “World Sheet Approach To Heterotic Instantons And Solitons,” Nucl. Phys. B 359, 611 (1991),

[34] E.A. Bergshoeff and M. de Roo, Nucl. Phys. B328 (1989) 439.