Identifying Influential Links for Event Propagation on Twitter: A Network of Networks Approach

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Abstract—Patterns of event propagation in online social networks provide novel insights on the modeling and analysis of information dissemination over networks and physical systems. This paper studies the importance of follower links for event propagation on Twitter. Three recent event propagation traces are collected with the user languages being used to identify the Network of Networks (NoN) structure embedded in the Twitter follower networks. We first formulate event propagation on Twitter as an iterative state equation, and then propose an effective score function on follower links accounting for the containment of event propagation via link removals. Furthermore, we find that utilizing the NoN model can successfully identify influential follower links such that their removals lead to remarkable reduction in event propagation on Twitter follower networks. Experimental results find that the between-network follower links, though only account for a small portion of the total follower links, are crucial to event propagation on Twitter.

Index Terms—event propagation model, information dissemination, spectral graph theory, social network

I. INTRODUCTION

Patterns of event propagation in online social networks are closely related to the modeling and analysis of information dissemination over certain networks and physical systems. Examples include epidemic processes in contact networks [1], [2], information diffusion in social networks and social media [3]–[8], and malware propagation in technological networks [9]–[12], among others.

This paper exploits the network structure embedded in online social networks for identifying influential links for event propagation. Specifically, we use Twitter follower networks to study and develop an effective link score function that reflects the importance of a follower link in event propagation. An event on a Twitter follower network can be a uniform resource locator (URL) of a web address or a hashtag in a tweet. A follower who has seen a tweet and decided (not) to retweet the event is called a retweeter (non-retweeter). A typical example of event propagation on Twitter is the announcement of the discovery of a Higgs boson-like particle in July 2012 [13]. Given a Twitter follower network, our proposed method effectively identifies important follower links affecting event propagation based on the network connectivity structure without requiring prior knowledge such as where the event is originally posted and how the event is retweeted.

We model event propagation using an iterative state equation, and then propose a Left Eigenvector Score (LES) for each follower link. We show that LES is able to identify influential follower links for event propagation in the sense that the removal of those links is effective in reducing the event propagation. Although our method requires only the information of the network’s connectivity structure, it can be easily extended to incorporate certain additional user information to further improve the effectiveness of the proposed method. Specifically, we utilize the Network of Networks (NoN) structure in Twitter follower networks as additional user information. The NoN model is a general approach for characterizing a network at different scales. A large-scale network is often composed of several sub-networks, and the interconnectivity and interdependency between these sub-networks are known to be crucial to information dissemination and network robustness [14]–[19].

To validate the effectiveness of LES and the NoN structure of event propagation on Twitter, we collect three recent event propagation traces on Twitter using the Application Programming Interface (API) provided by Twitter for public data retrieval, which in turn offers new platforms for tracking and collecting real-world event propagation traces on Twitter at large scales. The user language is used to identify the sub-networks within the Twitter follower network under consideration. We find that the between-network links play an important role in event propagation, as they account for information dissemination from one user language to another. Experimental results demonstrate that link removals based on LES can successfully reduces event propagation in real-world traces, especially when the between-network follower links are used for LES calculation.

The rest of this paper is organized as follows. Sec. II provides an overview of event propagation in networks. Sec. III illustrates the NoN structure from the collected Twitter traces. Sec. IV provides a theoretical framework for identifying influential follower links for event propagation, including modeling event propagation via an iterative state equation, specifying a surrogate function for event propagation, and proposing a novel link score function (LES) for evaluating the importance of follower links in event propagation. Sec. V uses the collected Twitter traces to compare the performance of different score functions for identifying influential follower links. Finally, Sec. VI concludes this paper.

1Twitter REST APIs. Available at https://dev.twitter.com/rest/public
The main notations used in this paper are given in Table I.

### Table I: List of main notations.

| Notation | Description |
|----------|-------------|
| $n$      | number of users |
| $m$      | number of follower links |
| $A$      | adjacency matrix of follower links |
| $A_{\text{bet}}$ | (within-network) follower links |
| $\lambda_{\text{max}}(A)$ | largest eigenvalue of $A$ |
| $r_i$    | binary event propagation status vector |
| $T$      | entry-wise threshold function |
| $s$      | upper bound on the total number of retweeters |
| $y$      | left leading eigenvector of $A$ |
| $E_R$    | link removal set |
| $q$      | number of removed links |

**II. RELATED WORK**

Event propagation in networks has been actively studied in different fields. In [1], [2], [20], [21], event propagation is studied in the context of epidemic processes in physical and engineering networks. Each node in the network is categorized into a few states (e.g., the susceptible-infected-recovered (SIR) model) for analyzing and predicting collective behaviors, such as the emergence of epidemic spreads, or the monitoring of malware propagation. In online social networks, event propagation is studied in the context of information diffusion [3], [13], [22], influence maximization [23], influential user identification [4]–[6], and locating rumor sources [24]. In signal processing, event propagation is studied in the context of diffusion estimation among agents in a network [25], [26], and extracting patterns based on the diffusion of graph signals [27]–[29].

Many existing event propagation models, such as the SIR model [1] for epidemic processes, and the independent cascade model (ICM) [30] for influence maximization, assume pairwise event propagation parameters. For example, for the SIR model, an infected node can infect a susceptible node within the contact range with some probability, or can transition to the recovered state with some probability, where these infection and recover probabilities are governed by certain parameters.

Similarly, for the ICM, each node can be activated for further information propagation with some probability provided that one of the neighboring nodes has been activated.

Different from these parametric event propagation models, we formulate event propagation as an iterative state equation that is only associated with the network structure. Our event propagation model then leads to the surrogate function for event propagation, which is explained in Sec. IV-B. It allows us to evaluate the importance of every link in the network without assuming any pairwise event propagation parameters, which is discussed in Sec. IV-C. Such an event propagation model is particularly appealing for studying event propagation in online social networks such as Twitter, since the network structure (i.e., the follower connectivity pattern) can be easily obtained from the Twitter API.

**III. ILLUSTRATION OF THE NO-N STRUCTURE OF EVENT PROPAGATION ON TWITTER**

To illustrate event propagation, we collected the traces of three recent events on Twitter during a period of two weeks through the Twitter API. These events include URLs and hashtags summarized as follows. The details of the collected event traces from Twitter, including the description and collection duration, are given in Appendix A.

- **Obama FB**: A URL that links to U.S. President Obama’s personal Facebook page created in 2015.
- **Premier 12**: A hashtag of an international baseball tournament in 2015.
- **AlphaGo**: A hashtag about a board game algorithm defeating a European Go champion in 2016 [31].

We also collected each user’s language setting on Twitter, which is used as the network identity. The source of an event need not to be unique. For example, the same URL can be independently posted by some users and then be retweeted by their followers. Fig. I displays the Network of Networks (NoN) structure in the retweeter network of the aforementioned events. It is observed that the propagation patterns of these events share some common features. (i) For each event, there are some hub users such that their posts are retweeted by many other users. For the Obama FB event, one hub user is President Obama’s personal Twitter account, and another hub user is White House’s Twitter account. For the Premier 12 event, one hub user is the tournament organizer’s official Twitter account. For the AlphaGo event, one hub user is Google’s Twitter account. (ii) The events are originally posted by some “seed users” of different languages, and other users tend to retweet the event from a user of the same language. Take Premier 12 as an example, the tweets regarding Premier 12 are first tweeted by some seed users of different languages, including Dutch, English, Spanish, Korean, zh-TW and Italian. Then most of the tweets are retweeted by users of the same language.

**IV. METHODOLOGY**

**A. Event propagation model**

Consider a Twitter follower network consisting of $n$ users and $m$ follower links. Let $A$ be an $n \times n$ binary adjacency matrix representing the follower relationship in the network, where its entry of the $i$-th row and the $j$-th column $[A]_{ij} = 1$ if user $i$ follows user $j$, and $[A]_{ij} = 0$ otherwise. We divide the time period of event propagation into $F$ non-overlapping frames, and let $A_t$ be an $n \times n$ binary adjacency matrix indicating the follower links that have been activated for event propagation during the $t$-th time frame, $t = 1, 2, \ldots, F$. In other words, $[A_t]_{ij} = 1$ indicates that user $i$ follows user $j$, while $[A_t]_{ij} = 1$ indicates that user $i$ retweets user $j$ during the $t$-th time frame. Let $r_i$ be an $n$-dimensional binary vector indicating the event propagation status of every user, where $r_i$’s $i$-th entry $[r_i]_i = 1$ if the event has ever been posted or retweeted by the $i$-th user since the beginning to the $t$-th time
frame, and \(|r_t|_i = 0\) otherwise. In addition, let \(r_0\) be a binary vector such that its nonzero entries indicate the set of users who first post the event. Then the event propagation model can be written as an iterative state equation

\[
r_{t+1} = \mathbb{T} \left( r_t + \mathbb{T} \left( A^T_{t+1} r_t \right) \right), \ orall \ t = 0, 1, 2, \ldots, F - 1, \tag{1}
\]

where \(A^T_{t+1}\) is the matrix transpose of \(A_{t+1}\), and \(\mathbb{T}(\cdot)\) is an entry-wise threshold function defined as \(|\mathbb{T}(x)|_i = 1\) if \(|x|_i > 1\) and \(|\mathbb{T}(x)|_i = |x|_i\) if \(0 \leq |x|_i \leq 1\), for any nonnegative vector \(x\). The term \(\mathbb{T} \left( A^T_{t+1} r_t \right)\) can be viewed as the increment vector for event propagation in the \(t+1\)-th time frame. The derivation of the event propagation model in (1) is given in Appendix B.

The event propagation model in (1) can be easily adapted to incorporate the NoN structure of a Twitter follower network. Let \(A_{\text{bet}}\) and \(A_{\text{wit}}\) denote the adjacency matrix of the between-network and within-network follower links, respectively. The event propagation model can be rewritten as

\[
r_{t+1} = \mathbb{T} \left( r_t + \mathbb{T} \left( A^\text{bet}_{t+1} T^T r_t \right) + \mathbb{T} \left( A^\text{wit}_{t+1} T^T r_t \right) \right) \tag{2}
\]

for all \(t = 0, 1, 2, \ldots, F - 1\). The matrices \(A^\text{bet}_{t+1}\) and \(A^\text{wit}_{t+1}\) are defined similarly as \(A_{t+1}\) such that \(A_t = A^\text{bet}_t + A^\text{wit}_t\). The terms \(\mathbb{T} \left( A^\text{bet}_{t+1} T^T r_t \right)\) and \(\mathbb{T} \left( A^\text{wit}_{t+1} T^T r_t \right)\) in (2) account for the event propagation increment caused by between-network and within-network follower links, respectively.

B. Surrogate function for event propagation

Since we are interested in investigating the effect of link removals on a Twitter follower network prior to actual event propagation, in practice only the adjacency matrix \(A\) of the Twitter follower network is known, whereas the event propagation status vector \(r_t\) and the adjacency matrix \(A_t\) affecting actual event propagation are unknown. Nonetheless, we will show that the largest eigenvalue of \(A\), denoted by \(\lambda_{\max}(A)\),
can be used as a surrogate function for the containment of event propagation, as it is associated with an upper bound on the increment of event propagation. In addition, \( \lambda_{\text{max}}(A) \) is known to be related to the information dissemination threshold of some parametric epidemic models \[\text{[1], [22]}.\]

Specifically, let \( \|x\|_0 \) denote the number of nonzero entries of an \( n \)-dimensional vector \( x \), which is also known as the \( \ell_0 \) norm or the sparsity level of \( x \). Under the sparsity assumption that \( \|r_F\|_0 \leq s \), where \( s \leq n \) is a trivial upper bound on \( s \), we can obtain a surrogate function of the increment \( \|T(A_{t+1}^T r_t)\|_0 \) in terms of \( \lambda_{\text{max}}(A) \), \( s \) and \( n \), which is:

\[
\|T(A_{t+1}^T r_t)\|_0 \leq s \cdot \lambda_{\text{max}}(A) + \sqrt{ns} \tag{3}
\]

for all \( t = 0, 1, 2, \ldots, F-1 \). The derivation is given in Appendix C. It is clear from (3) that minimizing the largest eigenvalue \( \lambda_{\text{max}}(A) \) of the adjacency matrix \( A \) can be effective in containing event propagation, since \( \lambda_{\text{max}}(A) \) is associated with an upper bound on the event propagation increment \( \|T(A_{t+1}^T r_t)\|_0 \) for each iteration in \( t \).

Applying the results in (3) to the event propagation model with NoN structure in (2), we can obtain upper bounds on the increments \( T(A^\text{bet}_{t+1}^T r_t) \) and \( T(A^\text{wit}_{t+1}^T r_t) \) associated with between-network and within-network follower links in terms of \( \lambda_{\text{max}}(A^\text{bet}) \) and \( \lambda_{\text{max}}(A^\text{wit}) \), which are:

\[
\|T(A_{t+1}^\text{bet}^T r_t)\|_0 \leq s \cdot \lambda_{\text{max}}(A^\text{bet}) + \sqrt{ns}; \tag{4}
\]

\[
\|T(A_{t+1}^\text{wit}^T r_t)\|_0 \leq s \cdot \lambda_{\text{max}}(A^\text{wit}) + \sqrt{ns}. \tag{5}
\]

C. LES: left eigenvector score

Since in Sec. IV-B the largest eigenvalue of the adjacency matrix of a Twitter follower network, \( \lambda_{\text{max}}(A) \), is shown to be an important factor affecting event propagation, we propose a score function on follower links such that link removals based on the score function of decreasing order become an effective reducer in the largest eigenvalue. Specifically, we use the left eigenvector \( y \) of the adjacency matrix \( A \) to define a score for each follower link for evaluating every follower link’s importance in event propagation. By the Perron-Frobenius theorem \[\text{[33]}, \] the largest eigenvalue of an adjacency matrix is always real and nonnegative, and its associated left eigenvector \( y \) has nonnegative entries and unit Euclidean norm, i.e., \( y_i \geq 0 \) for all \( i \) and \( \sum_{i}y_i^2 = 1 \). Since \( y \) satisfies the eigenfunction \( A^T y = \lambda_{\text{max}}(A) y \), it can be viewed as the vector of eigenvector centrality of each user based on every user’s follower connectivity pattern in the Twitter follower network, for which the eigenvector centrality is a measure of importance among influential nodes in a network \[\text{[34]} \].

Let \( (i, j) \) denote a follower link in the Twitter follower network representing the relation that user \( i \) follows user \( j \). The follower link score we propose for assessing the influence in event propagation, which we call the Left Eigenvector Score (LES), is defined as:

\[
\text{LES}(i, j) = [y]_i \cdot [y]_j. \tag{6}
\]

Since \( y \) is the vector of eigenvector centrality based on each user’s follower connectivity pattern, high LES for a follower link \( (i, j) \) means that the followers of both user \( i \) and user \( j \) play an important role in the Twitter follower network, and hence the follower link \( (i, j) \) is crucial to event propagation.

Moreover, we show that removing the follower links of top LES can be effective in reducing the largest eigenvalue \( \lambda_{\text{max}}(A) \), and hence is able to contain event propagation increment according to (3). Let \( E_R \) denote a subset of follower links in a Twitter follower network such that \( (i, j) \in E_R \) if the follower link \( (i, j) \) will be removed from the Twitter follower network. For any follower link removal set \( E_R \) with cardinality \( |E_R| = q \geq 1 \), let \( \tilde{A}(E_R) \) be the adjacency matrix after removing the follower links in \( E_R \) from the Twitter follower network. If \( \sum_{(i,j)\in E_R} [y]_i [y]_j > 0 \), then:

\[
\lambda_{\text{max}}(A) - \sum_{(i,j)\in E_R} [y]_i [y]_j \leq \lambda_{\text{max}}(\tilde{A}(E_R)); \tag{7}
\]

\[
\lambda_{\text{max}}(A) - c \cdot \sum_{(i,j)\in E_R} [y]_i [y]_j \geq \lambda_{\text{max}}(\tilde{A}(E_R)), \tag{8}
\]

where \( c = \frac{\epsilon}{2}, \epsilon = \sum_{(i,j)\in E_R} [\bar{y}]_i [\bar{y}]_j \), and \( \bar{y} \) is the left leading eigenvector of \( \tilde{A}(E_R) \). The proof is given in Appendix D.

Since the number of nonzero entries in \( A \) is the total number of follower links \( m \), computing the left eigenvector \( y \) takes \( O(m) \) time by power iteration methods, and reporting the top \( q \) follower links of LES takes \( O(nmq) \) time. Therefore, the overall computational complexity for finding the removal set \( E_R \) of cardinality \( q \) is \( O(nmq) \).

Similar analysis to (7) and (8) can be directly applied to the largest eigenvalues \( \lambda_{\text{max}}(A^\text{bet}) \) and \( \lambda_{\text{max}}(A^\text{wit}) \) in (4) and (5) by using their corresponding left leading eigenvectors. As a result, the proposed LES can be easily adapted to the NoN structure in the Twitter follower network.

V. EXPERIMENTS ON TWITTER TRACES

A. Experiment setup and dataset description

To study the effect of follower link removals on event propagation, we collected three real-world event propagation traces and user languages from Twitter as described in Sec. III. We also collected the users who have seen but have not retweeted the event (i.e., non-retweeters) and their user

### TABLE II: Statistics of the collected events and Twitter follower networks

| Dataset       | Event                    | Users   | Follower Links | Networks (Languages) | Between-Network Follower Links | Within-Network Follower Links |
|---------------|--------------------------|---------|----------------|----------------------|-------------------------------|-------------------------------|
| Obama FB      | [http://Facebook.com/POTUS](http://Facebook.com/POTUS) | 5,169,477 | 7,272,858      | 117                  | 19.74%                        | 80.26%                        |
| Premier 12    | #premier12               | 7,557,534 | 9,702,942      | 90                   | 22.11%                        | 77.89%                        |
| AlphaGo       | #AlphaGo                 | 9,259,187 | 9,794,702      | 141                  | 29.35%                        | 70.65%                        |
languages to form a Twitter follower network for testing the effect of link removals on event propagation. In other words, the collected Twitter follower networks include the follower connectivity structure of retweeters and non-retweeters of an event, and their user languages are used to identify the NoN structure. The statistics of the collected datasets are summarized in Table II. One notable NoN feature of these Twitter follower networks is that the between-network follower links only account for a portion of from 20% to 30% of the total number of follower links.

**Evaluation Metric.** A link score function for assessing link influence on event propagation is a function of the adjacency matrix and the NoN identities of a Twitter follower network. The actual event propagation traces are only used to compare the performance of different link scores. We use the event reachability as the performance metric, which is defined as the fraction of users who can still post or retweet the events after some follower links are removed from the original Twitter follower network. The event fails to propagate further to a user’s follower if the corresponding follower link has been removed. As a result, the set of link removals that lead to lower event reachability are the links that have more influence on event propagation.

**Follower Link Scores.** We compare the effect of removing top \( q \) follower links on event reachability based on different link score functions, for which the score function of a follower link \((i, j)\) takes the form

\[
\text{score}(i, j) = |x_i| \cdot |\bar{x}_j|,
\]

where \( x \) and \( \bar{x} \) are nonnegative \( n \)-dimensional vectors. The score function can be easily incorporated with centrality measures on users based on the Twitter follower network topology. However, since the Twitter follower network is often not a connected graph, i.e., there is not a path connecting any two users in the network, centrality measures defined on connected graphs, such as the closeness and betweenness centrality measures \([34]\), cannot be used as a score function.

The following summarizes different score functions for performance comparison, including the scenario where the network identity of every user is known and the NoN model is applied such that the between-network and within-network follower links are used separately for link score computation. The computational complexity of returning top \( q \) follower links for different follower link score functions is summarized in Table III.

The implementation details and computational complexity analysis are given in Appendix E.

| Score function       | Complexity |
|----------------------|------------|
| LES                  | \( O(mq) \) |
| InDeg                | \( O(mq) \) |
| NetMelt              | \( O(mq) \) |
| NoN-LES-Bet (NoN-LES-Wit) | \( O(mq + n) \) |
| NoN-InDeg-Bet (NoN-InDeg-Wit) | \( O(mq) \) |
| NoN-NetMelt-Bet (NoN-NetMelt-Wit) | \( O(mq + n) \) |

**B. Performance evaluation**

Fig. 2 displays the event reachability with respect to different link removal methods as described in Sec. IV-A. Comparing to the link removal methods without using the NoN structure (LES, InDeg and NetMelt), it can be observed that incorporating the NoN structure (user languages) can further reduce event reachability. In particular, the NoN-LES-Bet method outperforms other methods in the Premier 12 and AlphaGo datasets. For the Obama FB dataset, LES and NoN-LES-Wit can be more effective than other methods for the first few follower link removals. However, as the number of removals increases these two methods soon lose their appeals, and NoN-LES-Bet significantly outperforms other methods. For example, if we are able to remove 0.25% of follower links from the Obama FB dataset, NoN-LES-Bet can reduce the event reachability to roughly 20%, whereas the second best method (NoN-InDeg-Bet) only reduces the event reachability to roughly 35%, which means that NoN-LES-Bet is 15% more effective in finding important links as compared to other methods. These results suggest that LES can better reflect the level of importance of a follower link for event propagation. More interestingly, the success of NoN-LES-Bet in reducing event propagation on Twitter implies that although between-network follower links correspond to under 30% of the total number of follower links in these datasets, they are crucial to event propagation.

The effectiveness of LES in reducing event propagation can be explained by the fact it is a minimizer of an upper bound on the increment of event propagation as established in Sec. IV. Furthermore, link score functions based on in-degrees or NetMelt do not result in as much reduction as compared with the LES-based methods. The finding that the LES-based
methods are superior to the InDeg-based methods suggests that event propagation not only depends on the number of followers, but also on the role of each user’s followers in event propagation. This is also consistent with the importance of social ties for event propagation in online social networks [36], [37].

Fig. 3 displays the fraction of between-network follower links of different link removal methods. We find that although NoN-LES-Bet and NoN-NetMelt-Bet lead to similar fraction of between-network follower link removals, NoN-LES-Bet achieves lower event reachability than NoN-NetMelt-Bet as shown in Fig. 2. This implies that the proposed LES is indeed more effective in identifying most important follower links that influence event propagation.

VI. CONCLUSION

The contributions of this paper are twofold. First, we have discovered and investigated the Network of Networks (NoN) structure embedded in real-world event propagation patterns. Second, by minimizing an upper bound on event propagation increment, a left eigenvector score (LES) is proposed to identify the level of importance in event propagation. Experiments on Twitter data show that the proposed method is able to exploit the NoN structure over the different languages used by Twitter users. In particular, we show that the LES successfully identifies most important links influencing event propagation.

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of removed between-network follower links of NoN-LES-Bet and NoN-NetMelt-Bet are similar, the follower links identified by NoN-LES-Bet are more influential in event propagation as their removals result in lower event reachability.

![Fig. 3: Fraction of between-network follower links in different link removal methods. Comparing to Fig. 2, although the fraction of removed between-network follower links of NoN-LES-Bet and NoN-NetMelt-Bet are similar, the follower links identified by NoN-LES-Bet are more influential in event propagation as their removals result in lower event reachability.](image)

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A. Details of the collected real-world event propagation traces on Twitter

To illustrate event propagation, we collected the traces of three recent events on Twitter during a period of two weeks through the Twitter API. These events include URLs and hashtags specified as follows.

- **Obama FB:** we tracked the tweets including the URL [http://Facebook.com/POTUS](http://Facebook.com/POTUS) from November 9th to November 23rd in 2015. The URL links to U.S. President Obama’s personal Facebook page, and was firstly being posted by his personal Twitter account on November 9th 2015.

- **Premier 12:** we tracked the tweets including the hashtag “#premier12” from November 19th to December 3rd in 2015. Premier 12 is a flagship international baseball tournament organized by the World Baseball Softball Confederation (WBSC), featuring the twelve best-ranked national baseball teams in the world.

- **AlphaGo:** we tracked the tweets including the hashtag “#AlphaGo” from January 27th to February 10th in 2016. AlphaGo is a computer program developed by Google DeepMind in London to play the board game Go. On January 27th 2016, the news of AlphaGo defeating a European Go champion was announced along with the algorithm published in Nature [31].

B. Derivation of the iterative state equation in (7)

Since \( A_t \) accounts for the adjacency matrix of activated follower links for event propagation during the \( t \)-th time frame, the \( i \)-th entry of the vector \( A_{t+1}^T r_t \) can be expressed as

\[
[A_{t+1}^T r_t]_i = \sum_{j=1}^{n} [A_{t+1}]_{ij} [r_t]_j,
\]

which is the number of tweets regarding the event that user \( i \) decides to share on Twitter during the \( t + 1 \)-th time frame. Therefore, the entry-wise thresholded binary vector \( T(A_{t+1}^T r_t) \) indicates the status of new users participating in event propagation during the \( t + 1 \)-th time frame. Lastly, since \( T(A_{t+1}^T r_t) \) represents the vector of event propagation increment, \( r_{t+1} = T(r_t + T(A_{t+1}^T r_t)) \) accounts for the event propagation status of all users since the beginning to the \( t + 1 \)-th time frame.

C. Proof of the upper bound in (9)

First, observe from (1) that the sparsity level \( \| r_t \|_0 \) of \( r_t \) is a non-decreasing function in \( t \). Therefore, the condition that \( \| r_F \|_0 \leq s \) implies \( \| r_t \|_0 \leq s \) for all \( t \leq F \). Let \( I_n \) denote the \( n \)-dimensional column vector of all ones. Then the sparsity level \( \| T(A_{t+1}^T r_t) \|_0 \) of the binary vector \( T(A_{t+1}^T r_t) \) can be expressed as

\[
\| T(A_{t+1}^T r_t) \|_0 = 1_n^T T(A_{t+1}^T r_t).
\]

Decomposing the term \( 1_n^T T(A_{t+1}^T r_t) \) in (10), we have

\[
1_n^T T(A_{t+1}^T r_t) = r_t^T T(A_{t+1}^T r_t) + (1_n - r_t)^T T(A_{t+1}^T r_t),
\]

Let \( x = (\sum_{i=1}^{n} |x_i|^2)^{-1/2} \) denote the Euclidean norm of a vector \( x \). We can derive an upper bound on the term \( r_t^T T(A_{t+1}^T r_t) \) in (11), which is

\[
r_t^T T(A_{t+1}^T r_t) \leq (a) r_t^T A_{t+1} r_t \leq (b) r_t^T A_{t+1} r_t \leq (c) \| r_t \|_2^2 \cdot \frac{r_t^T A_{t+1} r_t}{\| r_t \|_2^2} \leq (d) \| r_t \|_2^2 \cdot \max_{x : \| x \|_2 = 1} x^T A_{t+1} x \leq (e) \| r_t \|_2^2 \cdot \lambda_{\max}(A_{t+1}) \leq (f) \lambda_{\max}(A_{t+1}) \leq (g) \lambda_{\max}(A),
\]

where \( (a) \) is due to the fact that \( T(\cdot) \) is a threshold function and \( A_{t+1}^T r_t \) is a nonnegative vector, \( (b) \) is true since \( r_t^T A_{t+1} r_t \) is a real value, \( (c) \) is a simple arithmetic operation, \( (d) \) is due to the fact that \( \| r_t \|_2^2 \) is a vector of unit Euclidean norm, \( (e) \) is from the Courant-Fischer theorem [33]. \( (f) \) uses the fact that \( r_t \) is a binary vector such that \( \| r_t \|_2^2 = \sum_{i=1}^{n} [r_t]_i = \sum_{i=1}^{n} |[r_t]_i| = \| r_t \|_0 \leq s \), and \( (g) \) is due to the fact that all the nonzero entries in \( A_{t+1} \) also appear in \( A \), and hence \( \lambda_{\max}(A_{t+1}) \leq \lambda_{\max}(A) \), which can be verified by using the matrix perturbation theorem [33].

Next, using the Cauchy-Schwarz inequality, the term \( (1_n - r_t)^T T(A_{t+1}^T r_t) \) in (11) is upper bounded by

\[
(1_n - r_t)^T T(A_{t+1}^T r_t) \leq \| 1_n - r_t \|_2 \cdot \| T(A_{t+1}^T r_t) \|_2 \leq \sqrt{n} \cdot s.
\]

\[
\| 1_n - r_t \|_2 \leq \| 1_n \|_2 = \sqrt{n} \text{ is a trivial upper bound since } r_t \text{ is a binary vector. } \| T(A_{t+1}^T r_t) \|_2 \leq \sqrt{s} \text{ since } \| T(A_{t+1}^T r_t) \|_2^2 = \| T(A_{t+1}^T r_t) \|_0 \leq \| r_{t+1} \|_0 \leq s \text{ by the iterative state equation.}
\]

\[
(11)
\]

\[
(12)
\]

\[
(13)
\]
perturbation equation gives $\lambda$.

Combining the two established upper bounds (equations (12) and (13)) on $r_i^T T(A_{i+1} r_i)$ and $(1_n - r_i)^T T(A_{i+1} r_i)$, we obtain the upper bound on $\|HT(A_{i+1} r_i)\|$ as in (8).

**D. Proof of the bounds in (7) and (8)**

Given a follower link removal set $E_R$ with cardinality $|E_R| = q \geq 1$, the adjacency matrix $A(E_R)$ after removing the follower links in $E_R$ from the original network can be written as a matrix perturbation to the adjacency matrix $A$ of the original Twitter follower network, which takes the form

$$\tilde{A}(E_R) = A - \sum_{(i,j) \in E_R} e_i e_j^T,$$

(14)

where $e_i$ denotes the $n$-dimensional column vector of zeros except that its $i$-th entry is 1.

Left and right multiplying the left leading eigenvector $y$ of $A$ to the matrix perturbation equation, we obtain

$$y^T (A - \sum_{(i,j) \in E_R} e_i e_j^T) y = \lambda_{\text{max}}(A) - \sum_{(i,j) \in E_R} [y_i, y_j]$$

$$= y^T \tilde{A}(E_R) y$$

$$\leq \lambda_{\text{max}}(\tilde{A}(E_R)),$$

(15)

where the inequality is from the Courant-Fischer theorem [33] that $\lambda_{\text{max}}(\tilde{A}(E_R)) = \max_{\|x\| = 1} x^T \tilde{A}(E_R) x$. Therefore, we obtain the lower bound on $\lambda_{\text{max}}(\tilde{A}(E_R))$ as in (7).

To obtain an upper bound on $\lambda_{\text{max}}(\tilde{A}(E_R))$ in terms of $\lambda_{\text{max}}(A)$ and $\sum_{(i,j) \in E_R} [y_i, y_j]$, we first note that $(|y_i| - |y_j|)^2 = |y_i|^2 - 2|y_i||y_j| + |y_j|^2 \geq 0$ for any $i$ and $j$. Summing this inequality over the set $E_R$ gives

$$0 \leq \sum_{(i,j) \in E_R} [y_i, y_j]$$

$$\leq 2q - 2 \sum_{(i,j) \in E_R} |y_i||y_j|,$$

(16)

where $2q$ is due to the fact that $y$ has unit Euclidean norm such that $\sum_{(i,j) \in E_R} |y_i|^2 \leq |E_R| \cdot \max_i |y_i|^2 \leq |E_R| \cdot 1 = q$. Therefore, with (16) we obtain the inequality

$$\sum_{(i,j) \in E_R} [y_i, y_j] \leq q.$$

(17)

Lastly, assume $\sum_{(i,j) \in E_R} [y_i, y_j] > 0$ and let $\tilde{y}$ denote the left leading eigenvector of $\tilde{A}(E_R)$ such that $\sum_{(i,j) \in E_R} [\tilde{y}_i, \tilde{y}_j] = \epsilon$. Left and right multiplying $\tilde{y}$ of $\tilde{A}(E_R)$ to the matrix perturbation equation gives

$$\lambda_{\text{max}}(\tilde{A}(E_R)) \leq \lambda_{\text{max}}(A) - \sum_{(i,j) \in E_R} [\tilde{y}_i, \tilde{y}_j]$$

$$= \lambda_{\text{max}}(A) - \sum_{(i,j) \in E_R} [\tilde{y}_i, \tilde{y}_j] \sum_{(i,j) \in E_R} [y_i, y_j]$$

$$\leq \lambda_{\text{max}}(A) - \frac{\epsilon}{q} \sum_{(i,j) \in E_R} [y_i, y_j],$$

(18)

which leads to the upper bound on $\lambda_{\text{max}}(\tilde{A}(E_R))$ as in (8).

**E. Implementation of follower link score functions and computational complexity analysis**

We consider the score function of a follower link $(i, j)$ that takes the form

$$\text{score}(i, j) = |x_i| \cdot |x_j|,$$

where $x$ and $\bar{x}$ are nonnegative $n$-dimensional vectors.

The following reports on the implementation and computational complexity of returning $q$ follower links of the highest score for different follower link score functions.

- **LES**: $x = \bar{x} = y$, where $y$ is the left leading eigenvector of the adjacency matrix $A$. The computational complexity is $O(mq)$, which is analyzed in Sec. [IV-C].
- **InDeg**: $x = \bar{x} = d_{in}$, where $d_{in}$ is the vector of in-degree of each user, and its $j$-th element $d_{in}[j] = \sum_{i=1}^{n} |A|_{ij}$ is the number of followers of user $j$. The computational complexity is $O(mq)$.
- **NetMelt**: NetMelt [35] is an edge removal algorithm proposed to decrease the largest eigenvalue $\lambda_{\text{max}}(A)$ of the adjacency matrix $A$. The computational complexity is $O(mq + n)$.
- **NoN-LES-Bet (NoN-LES-Wit)**: NoN-LES-Bet (NoN-LES-Wit) exploits the NoN structure and evaluates the score function using $x = \bar{x} = y_{\text{bet}}$ ($x = \bar{x} = y_{\text{wit}}$), where $y_{\text{bet}}$ ($y_{\text{wit}}$) denotes the left leading eigenvector of the between-network (within-network) adjacency matrix $A_{\text{bet}}$ ($A_{\text{wit}}$). The computational complexity is $O(mq)$.
- **NoN-InDeg-Bet (NoN-InDeg-Wit)**: NoN-InDeg-Bet and NoN-InDeg-Wit are extensions of the InDeg score tailored to the NoN structure. Specifically, for NoN-InDeg-Bet (NoN-InDeg-Wit) we set $x = \bar{x} = d_{in}^{\text{bet}}$ ($x = \bar{x} = d_{in}^{\text{wit}}$), where $d_{in}^{\text{bet}}$ ($d_{in}^{\text{wit}}$) is the in-degree vector that only accounts for the between-network (within-network) follower links in the Twitter follower network. The computational complexity is $O(mq)$.
- **NoN-NetMelt-Bet (NoN-NetMelt-Wit)**: Non-NetMelt-Bet and Non-NetMelt-Wit are NetMelt algorithms that incorporate the NoN structure. For Non-NetMelt-Bet (NoN-NetMelt-Wit), we set $x = y_{\text{bet}}$ and $\bar{x} = z_{\text{bet}}$ ($x = y_{\text{wit}}$ and $\bar{x} = z_{\text{wit}}$), where $y_{\text{bet}}$ and $z_{\text{bet}}$ ($y_{\text{wit}}$ and $z_{\text{wit}}$) denote the left and right leading eigenvectors of $A_{\text{bet}}$ ($A_{\text{wit}}$). The computational complexity is $O(mq + n)$.

We also implemented score functions based on the right leading eigenvector of the adjacency matrix. However, its effect on reducing event propagation is not prominent, so we omit the results in the paper.