Zero-energy corner states protected by generalized chiral symmetry in $C_4$ symmetric crystals

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Abstract

We investigated the corner-states characteristics of $C_4$-symmetric higher-order topological crystalline insulators. By evaluating the charge polarization and fractional corner charge, we obtained the abundant topological properties for different intercell hopping amplitudes and identified that the non-trivial fractional corner charges are accompanied by second-order topological corner states. Interestingly, there are two different types of corner states topologically protected by generalized chiral symmetry, which guarantees that both corner states are pinned to zero energy. One of the corner states is localized at the corners like the one protected by generalized chiral symmetry of three-atom sublattice. However, the other corner states are localized at the two sublattices nearest to the corners, which has no counterpart in a lattice with generalized chiral symmetry. Moreover, both of them have robustness against defects and disorder.

1. Introduction

Conventional topological insulators (TIs) support edge states that the wave propagation with unique robustness to the defects and disorder, which are topologically protected due to quantized bulk dipole moments [1–4]. Recently, the theory of dipole moments in TIs has been extended to higher multipole moments, which exhibit boundary-localized moments of lower dimension [5, 6]. Such as, a quadrupole insulator in two-dimension (2D) has edge dipole moments and fractional corner charges [7–11], and an octupole insulator in three dimensions has surface quadrupole moments, hinge dipole moments and fractional corner charges [12, 13]. More generally, $n$-dimensional TIs support $n - k$-dimensional topological boundary states ($2 \leq k \leq n$) [14]. This characteristic is extended to wider range of TIs and broadly referred to as higher-order topological insulators (HOTIs) [15–20].

For 2D HOTIs, the most fascinating characteristic is its second-order corner states. Interestingly, some other 2D HOTIs have the properties of zero-dimensional corner states, although they have no bulk quadrupole moments. Especially in $C_n$-symmetric TIs, fractional corner charge caused by filling anomalies often means the appearance of corner states, which has been widely studied in various fields [21]. For example, electric circuits [22, 23], acoustics [24, 25], photonics [26–30], mechanics [31, 32], spintronics [14, 33, 34], and recently, photonic nanocavities [35, 36]. One of the most important models is the 2D Su–Schrieffer–Heeger (SSH) model [37, 38] (or primitive generator $h_4^{(4)}$ in [21]). At first, its edge states were widely studied [39, 40], but its corner states were ignored. This is because the energy of the corner states fall into the energy spectrum of the bulk states, which mean the corner states are embedded in the continuum [25]. Thus, the 2D SSH model can support a kind of corner-localized bound states in the...
continuum [41]. However, the corner states can be observed and applied by surrounding a nontrivial sample by a trivial one [25, 27, 29, 35, 36, 42].

In this paper, we study the primitive generators $h^{(4)}_{1b}$ and $h^{(4)}_{1b}$ (2D SSH) of $C_4$-symmetry and generalized chiral symmetry in Kagome lattice. Our model is composed of the primitive generators $h^{(4)}_{1b}$ and $h^{(4)}_{1b}$, which still maintains $C_4$ symmetry and time reversal symmetry (TRS). By changing the parameters of Hamiltonian, we get abundant topological phases. Then, we calculate the $C_4$-symmetric topological index $\chi^{(4)}$ for different phase. In addition to the edge-induced filling anomaly, the topological phase has a corner-induced filling anomaly with secondary topological indices protected by $C_4$ symmetry and TRS for the each band. With open boundaries, corner-induced filling anomaly means that there will be fractional corner charge in the system [21, 41]. In general, the fractional corner charge is accompanied by zero-energy corner-localized states when chiral symmetry is present [21, 29, 41]. However, our model does not hold the chiral symmetry. But we find that the model has generalized chiral symmetry like that in the Kagome lattice [24, 30, 43]. Interestingly, we find two different types of corner states, one localized at one of sublattices and the other localized at two of sublattices. They are all protected by generalized chiral symmetry, which ensures that the energy of corner states is pinned to zero energy. And, they all have robustness against defects and disorder. We believe that the corner states localized at two of sublattices can enrich the theory of generalized chiral symmetry. And our finding can provide a simple way for designing topological materials with second-order corner states.

2. Model and topological phase

The lattice configuration we consider is shown in figure 1(a). It consists of four sites per unit cell with the intra-cell hopping $\gamma$ and next nearest neighbor (NNN) hoppings $\lambda$ between sites in different unit cells. $\vec{a}_1 = (1, 0)$ and $\vec{a}_2 = (0, 1)$ are the lattice vectors. If there is no NNN hopping, the model becomes the 2D SSH [37, 38] or the primitive generator $h^{(4)}_{1b}$ of $C_4$-symmetric topological crystalline insulators (TCIs) [21]. When the inter-cell hopping $\gamma = 0$, the model becomes the primitive generator $h^{(4)}_{1b}$ of $C_4$-symmetric TCIs [21]. The primitive generators $h^{(4)}_{1b}$ and $h^{(4)}_{1b}$ are $C_4$-symmetric, have 2 and 1 filled bands respectively, and have Wannier centers at the maximal Wyckoff position $b$ [21]. So, the Bloch Hamiltonian of this model can be expressed as the sum of Hamiltonian of primitive generators $h^{(4)}_{1b}$ and $h^{(4)}_{1b}$. $H^{(4)}_1$ and $H^{(4)}_2$ are Hamiltonian of $h^{(4)}_{1b}$ and $h^{(4)}_{1b}$, respectively

$$H = H^{(4)}_1 + H^{(4)}_2$$

$$H^{(4)}_1 = \begin{pmatrix}
0 & t + \gamma e^{i\vec{k}_x} & 0 & t + \gamma e^{i\vec{k}_y} \\
t + \gamma e^{-i\vec{k}_x} & 0 & t + \gamma e^{i\vec{k}_y} & 0 \\
0 & t + \gamma e^{-i\vec{k}_y} & 0 & t + \gamma e^{i\vec{k}_x} \\
0 & 0 & t + \gamma e^{i\vec{k}_x} & 0
\end{pmatrix}$$

$$H^{(4)}_2 = \begin{pmatrix}
0 & t & \lambda e^{(k_x+k_y)} & t \\
t & 0 & t & \lambda e^{(b_y-k_y)} \\
\lambda e^{-(k_x+k_y)} & t & 0 & t \\
t & \lambda e^{(k_x-k_y)} & t & 0
\end{pmatrix}.$$

Considering the actual model, $2t$ of Hamiltonian $H$ is modified to $t$. The Hamiltonian preserves TRS and satisfies $H\left(\vec{k}\right) = H^\dagger\left(-\vec{k}\right)$. Thus, Chern number of the Hamiltonian is zero [21]. And the Hamiltonian with $C_4$ symmetry obeys $\hat{r}_4 H^{(4)}_2 = H\left(R_4 \vec{k}\right)$, where

$$\hat{r}_4 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

and $R_4$ is the rotation operation that rotates the crystal momenta counter clockwise by $\pi$, i.e. $R_4 \left(k_x, k_y\right) = \left(-k_y, k_x\right)$. By solving the Hamiltonian of model, its bulk energy spectrum is shown in figures 1(c)–(f). There is always bands degeneracy at $\Gamma$ and $M$ in the energy spectrum, which is protected by TRS and $C_4$ symmetry. The four energy bands can be named the 1st, 2nd, 3rd and 4th bands from the highest to the lowest.

Figures 1(c)–(f) show the energy spectrum corresponding to different $\gamma$ values in the case of fixed $t$ value and $\lambda$ value. The primitive generator $h^{(4)}_{1b}$ has topological phase in the case of $\lambda > t$ [21]. In order to
Figure 1. (a) Lattice configurations of H. Black dotted line ($t$) and solid line ($\gamma$) represent the intra- and intercell hoppings, respectively. The NNN hoppings between the different cells is indicated by red dotted line ($\lambda$). (b) Brillouin zone of the model with $C_4$ symmetry. M is fourfold high symmetry points (HSPs), X and Y are twofold HSPs. $\Gamma$ is an $n$-fold HSPs in $C_n$ symmetric crystals. (c)–(f) are bulk energy bands along high symmetry lines of the first Brillouin zone. In the all plots, $\lambda/t = 3$, and the $\gamma/t$ of (c)–(f) are 0.6, 1.6, 3.1, 4.6, respectively.

clearly observe the effect of $\gamma$ on the topological phase of the primitive generator $h^{(4)}_{2b}$, so the change of bulk energy spectrum is under the condition of $\lambda \gg t$ in figures 1(c)–(f). From figure 1(c), one can find that when $0 < \gamma < t$, the 1st and 2nd bands are cross. Since the band gap of the 1st and 2nd bands are not opened, the topological phase is the same as primitive generator $h^{(4)}_{2b}$ [21]. But with the increase of $\gamma$, the 1st and 2nd bands move up and down respectively. If the intensity of $\gamma$ is further increased, once $\gamma > t$, the 1st and 2nd bands will be separated and a band gap will appear. However, as long as $\gamma < \lambda - t$, that is, $\gamma$ satisfies $t < \gamma < \lambda - t$, the 3rd and the 4th bands is basically unchanged (figure 1(d)). In the case of $\lambda - t < \gamma < \lambda + t$, the lower three bands are mixed together (figure 1(e)). When $\lambda + t < \gamma$, the 2nd band and 3rd bands are combined, and a gap appears between the 3rd and 4th bands [figure 1(f)]. These different band structures endow the system rich topological characteristics.

We can distinguish nontrivial topological phase arising from the $C_4$ symmetry through the symmetry representations that the occupied energy bands take at the HSPs of the Brillouin zone in figure 1(b) [21, 41]. However, it will not be able to distinguish the topological phase of $0 < \gamma < t$ and $t < \gamma < \lambda - t$ if we only take into account the two lowest occupied bands in figures 1(c)–(d). So, we focus on the topological phase of each band rather than the topology below Fermi level, so the definition of corner-induced filling anomaly need not the vanishing of polarization in insulators [21, 41]. The literature [21] gives the topological classes of $C_4$ symmetry by the indices

$$\chi^{(4)} = \left( [X_1^{(2)}], [M_1^{(4)}], [M_2^{(4)}] \right)$$

(2)
Zero-energy corner states protected by generalized chiral symmetry

3. Two types of zero-energy corner states

Here, we present the energy spectrum of a finite lattice (1600 sites) for \( \lambda/t = 7 \), and with varying ratio \( \gamma/t \) in figure 2(a). According to the different topological phases in table 2, we divide the figure 2(a) into three parts by red dotted lines. The left part corresponds to the phase of \( 0 < \gamma < t \), the middle part corresponds to the phase of \( t < \gamma < \lambda - t \), and the right part corresponds to the phase of \( \lambda - t < \gamma < \lambda + t \). In the left part of figure 2(a), there are three types of boundary modes in the gap between bulk states: gapped edge states, topologically protected zero-energy corner states 1 (ZEOCS1) in figure 2(b) and gapped corner states in the limit of \( \gamma = 0 \). Although the edge states are in the gap, they are not topologically protected because of

Table 1. The symmetry representations at the HSPs of each band.

| Phase       | Bands | \( C_4 \Gamma M \) | \( C_2 \Gamma X Y \) |
|-------------|-------|-------------------|-------------------|
| \( 0 < \gamma < t \) | 1,2   | \{+1,+1\} | \{+1,+1\} |
|             | 3,4   | \{+1,-1\} | \{-1,+1\} |
|             | 1     | \{+1\} | \{+1+1\} |
|             | 2     | \{-1\} | \{+1\} |
| \( t < \gamma < \lambda - t \) | 3,4   | \{+1\} | \{+1\} |
|             | 1     | \{+1\} | \{+1\} |
| \( \lambda - t < \gamma < \lambda + t \) | 2,3   | \{+1\} | \{+1\} |
|             | 1     | \{+1\} | \{+1\} |
| \( \lambda + t < \gamma \) | 4     | \{+1\} | \{+1\} |

where \( \Pi_p^{(n)} = \#\Pi_p^{(n)} - \#\Pi_p^{(n)} \), and \( \Pi_p^{(n)} \) is the number of states in a particular band at HSPs with rotation eigenvalues \( \Pi_p^{(n)} = e^{2ni(p-1)/n} \), for \( p = 1, 2, \ldots, n \). II represents the HSPs \( X, Y, M \). Table 1 gives the representation for \( C_4 \) and \( C_2 \) symmetries. The value of polarizations expressed by topological index \( \chi^{(4)} \) are (detailed in reference [21])

\[
\bar{P} = \frac{e}{2} \left[ X^{(4)}_1 \right] (\bar{a}_1 + \bar{a}_2) \mod e. \tag{3}
\]

In the \( \chi^{(4)} = (2, 0, 0) \) and \( \chi^{(4)} = (-2, 0, 0) \) phase, their polarization is trivial and two Wannier centers are located at Wyckoff position \( b \) [21]. There is only one Wannier center located at Wyckoff position \( b \) in the \( \chi^{(4)} = (-1, -1, 0) \) and \( \chi^{(4)} = (-1, 1, 0) \) phase. And in the case of \( \chi^{(4)} = (1, 1, 0) \), there are three Wannier centers located at Wyckoff position \( b \). In non-trivial phase, \( \bar{P} = (\frac{\gamma}{2}, \frac{\gamma}{2}, \frac{\gamma}{2}) \), which leading to the fractional charge on the edges [21]. In addition to polarization, the topological phase has a secondary topological indice \( Q_{\text{corner}}^{(4)} \), which is the fractional corner charge caused by the corner-induced filling anomaly [21]. It can be calculated by

\[
Q_{\text{corner}}^{(4)} = \frac{e}{4} \left( \left[ X^{(4)}_1 \right] + 2 \left[ M^{(4)}_1 \right] + 3 \left[ M^{(4)}_2 \right] \right) \mod e. \tag{4}
\]

In the \( \chi^{(4)} = (1, 1, 0) \) phase, the fractional corner charge \( Q_{\text{corner}}^{(4)} = \frac{e}{2} \) is due to the fact that three Wannier center located at Wyckoff position \( b \). The values of polarization and corner charges for different phase are indicated in table 2. Fractional corner charge is protected by TRS and \( C_4 \) symmetry [21]. In order to calculate the corner charge numerically, the system must keep polarization \( \bar{P} = (0, 0) \) in the filled energy band. Therefore, we only numerically calculate the corner charge of \( \frac{e}{2} \) in the case of \( 0 < \gamma < t \) to prove the robustness of corner charge (see appendix).
the polarization $\vec{P} = (0, 0)$ in the left part of figure 2(a). And, since the gapped corner states only exist in the limit of $\gamma = 0$, they are not topologically protected. Therefore, we are only interested in topology protected ZECS1, rather than the gapped edge states and corner states in the left part of figure 2(a). And in the case of $0 < \gamma < t < \lambda$, the dimers connected by $\lambda$ make up the bulk of the finite lattices, and they occupy energy $\pm \lambda$.

In the middle and right parts of figure 2(a), there is another corner states pinned to zero energy, so we call them zero-energy corner states 2 (ZECS2) in figure 2(d). In the case of $\gamma = t$, an entire structure splits into the dimers connected by $\lambda$ and a chain connected by $\gamma$ (or $t$) at the edges. This chain is connected end to end and naturally includes the sites of ZECS1 and ZECS2 in figure 2(c). When $t < \gamma$, there are four isolated sites at corners, dimers at the edges, and tetramers with NNN hoppings $\lambda$ in the finite lattices. The sites that make up the dimers and tetramers are only connected by $\gamma$ (black solid line) in figure 1(a). The dimers occupied energy $\pm \gamma$. The tetramer occupies energy $\pm 2\gamma + \lambda$, and two degenerate energies $-\lambda$. So the four isolated sites at corners form the corner states, and the dimers at the edges form the edge states. Therefore, although there are different phases in the middle and right parts of figure 2(a), the ZECS2 are localized at one of sublattices in the corners.

### 3.2. Zero-energy corner states 1

Given the polarization $\vec{P}$ and corner-induced filling anomaly with secondary topological indices $Q^{(4)}_{\text{corner}}$ for our system in table 2. Trivial polarization indicates that there is no edge states, but the system has a non-trivial fractional corner charges in the case of $0 < \gamma < t$, which means that the corner-localized states may appear [27, 29]. From figures 3(a) and (b), one can see that both gapped corner states and edge states disappear, only ZECS1 exists in the gap. This result proves that gapped corner states and edge states are not topologically protected. Obviously, the energy of this corner states is pinned to zero energy in the case of $0 < \gamma < t$ in figure 3(a) and the probability densities of the corner states reveals their zero-dimensional nature in figure 2(b). And the ZECS1 appears in the middle of the gap under the limit of $\gamma = 0$ as shown in figure 2(a), which is similar to the quantized microwave quadrupole insulator [10], photonic mid-gap cavity modes with hexagonal structure [26], and 1D SSH model [44]. However, ZECS1 are not protected by chiral symmetry because the Hamiltonian (equation (1)) of the system breaks the chiral symmetry. In the Kagome lattice and 2D SSH model, the generalized chiral symmetry also ensures that the corner states are pinned to zero energy [24, 33, 43]. Thus, ZECS1 seem to be protected by generalized chiral symmetry, we can define the generalized chiral symmetry for a unit cell containing four sites

$$\Gamma^{-1}_4 H_0 \Gamma_4 = H_1$$
$$\Gamma^{-1}_4 H_1 \Gamma_4 = H_2$$
$$\Gamma^{-1}_4 H_2 \Gamma_4 = H_3$$
$$H_0 + H_1 + H_2 + H_3 = 0.$$  \(5\)
Figure 3. (a) Energy spectrum for the system in finite lattice with $\gamma/t = 0.1, \lambda/t = 3$. (b) The corresponding density of states of (a). (c) Energy spectrum in finite lattice with $\gamma/t = 3, \lambda/t = 3$. (d) The corresponding density of states of (c). In all plots, there are 20 cells in both $x$ and $y$ directions.

From equation (5), we combine the last equation with the previous three, thus we can get that

$$\Gamma^{-1} \mathbf{H}_3 \Gamma_4 = \mathbf{H}_0$$

and

$$[\mathbf{H}_0, \Gamma_4^4] = 0.$$  

It implies $\Gamma_4^4 = I$ and the eigenvalues are given by $1$, $\exp(\pi i/2)$, $\exp(\pi i)$, and $\exp(-\pi i/2)$. So, we can write

$$\Gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi/2} & 0 & 0 \\ 0 & 0 & e^{i\pi} & 0 \\ 0 & 0 & 0 & e^{-i\pi/2} \end{pmatrix}. \tag{6}$$

The $\mathbf{H}_0$, $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$ have the same eigenvalue because of the transformations in equation (5) are unitary. By taking the trace of the last equation of equation (5), we find

$$\text{Tr} [\mathbf{H}_0 + \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3] = 4 \text{Tr} [\mathbf{H}_0] = 0,$$

which means that the sum of eigenvalues of Hamiltonian $\mathbf{H}_0$ with generalized chiral symmetry is zero. If given an eigenstate $|\psi_1\rangle = \frac{1}{\sqrt{2}} (0, 1, 0, 1)^T$ or $|\psi_2\rangle = \frac{1}{\sqrt{2}} (0, 1, 0, -1)^T$ that has support in sublattice 2 and 4 in figure 1(a), it will obey

$$\mathbf{H}_0 |\psi_1\rangle = E_1 |\psi_1\rangle$$

$$\mathbf{H}_0 |\psi_2\rangle = E_2 |\psi_2\rangle$$

$$\Gamma_4 |\psi_1\rangle = i |\psi_2\rangle. \tag{7}$$

If $E_1$ and $E_2$ are non-zero numbers, then $E_1$ and $E_2$ will not be opposite to each other. Thus,

$$(\mathbf{H}_0 + \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3) |\psi_1\rangle = (\mathbf{H}_0 + \Gamma_4^{-1} \mathbf{H}_3 \Gamma_4 + \Gamma_4^{-2} \mathbf{H}_2 \Gamma_4^2 + \Gamma_4^{-3} \mathbf{H}_1 \Gamma_4^3) |\psi_1\rangle = (2E_1 + 2E_2) |\psi_1\rangle = 0. \tag{8}$$

Thus, $E_1$ and $E_2$ can only be equal to zero. If the fourfold degenerate zero-energy states exist, these states could localize at two of the sublattices. This result perfectly confirms the correctness of the numerical analysis in figure 2(b).

### 3.3. Zero-energy corner states 2

In case of $\lambda - t < \gamma < \lambda + t$, the polarization $\vec{P} = \left( \frac{\xi}{4}, \frac{\zeta}{4} \right)$ and secondary topological indices $Q^{(4)}_{\text{corner}} = \frac{\xi}{2}, \frac{\zeta}{4}$ for the first and lower band, respectively. Non-trivial polarization and fractional corner charges indicate that the appearance of topologically protected edge states and corner states. Thus, we can see that the edge states
and ZECS2 exist in the band gap from figures 3(c) and (d). And, just like ZECS1, the corner states are also pinned to zero energy and fourfold degeneracy in figure 3(c), which is defined as ZECS2. ZECS2 only localize at one of the sublattices, which is similar to the zero energy corner states protected by generalized chiral symmetry in Kagome lattice [24, 43].

Given an eigenstate \( |\psi_j\rangle\) that has support in only sublattice \( j\), it will obey

\[
H_0 |\psi_j\rangle = E |\psi_j\rangle
\]

\[
\Gamma_4 |\psi_j\rangle = e^{i\theta_j} |\psi_j\rangle
\]

where \( \theta_j = \frac{\pi}{2} (j - 1), j = 1, 2, 3, 4 \). Thus, we find

\[
(H_0 + H_1 + H_2 + H_3) |\psi_j\rangle = (H_0 + \Gamma_4^{-1}H_0\Gamma_4 + \Gamma_4^{-2}H_0\Gamma_4^2 + \Gamma_4^{-3}H_0\Gamma_4^3) |\psi_j\rangle = 4E |\psi_j\rangle = 0.
\]

The result shows that the mode with \( E = 0 \) is localized at one sublattice. We have verified numerically the above result in figure 2(d). The bulk states with energy \(-2\gamma + \lambda\) will be zero in the case of \( \gamma = \frac{\lambda}{2} \) in figure 2(a). At this time, the bulk states and the ZECS2 are mixed together. The similar phenomenon is discussed in the 2D SSH model [25, 41] and the Kagome lattices [24].

4. Robustness of zero-energy corner states

In order to examine the robustness of zero-energy corner states, we introduce defects and disorder into the system. Firstly, we investigate the behavior of corner states under artificial lattice defects [14, 43]. From figures 4(a)–(c), we remove one unit cell from the corner, edge and bulk of the lattice. We can see that the defects of edge and bulk have no effect on ZECS1 and ZECS2. But, the defect in the corner makes one of the zero modes no longer present. Other zero modes are not affected by the defect, as shown in figures 4(d) and (h). Interestingly, the defect introduce new zero-energy corner states in the figure 4(h). The introduction of the above three type of lattice defects breaks the \( C_4 \) symmetry of the lattice. This proves that the generalized chiral symmetry, as a local symmetry, provides a stronger protection mechanism than the crystal symmetry: even without the \( C_4 \) symmetry, the zero energy state still present [43].

Then, we introduce the on-site potential \( V_{\text{dis}} = \delta V_{4 \times 4} \) to all lattices as disorder, where \( V \) is a random number from \(-1\) to \(1\), \( I_{4 \times 4} \) is an identity matrix and \( \delta \) represents the strength of disorder [8, 33, 34]. From figures 5(a) and (b), the energy of bulk states are significantly modified with the increasing of disorder strength. Because the introduction of on-site potential \( V_{\text{dis}} \) breaks the generalized chiral symmetry, the
energy of zero-energy corner states is not pinned at zero energy. And, the corner states does not hold fourfold degenerate.

5. Conclusions

In this paper, we have studied that the system is composed of primitive generators $h_{1b}^{(4)}$ and $h_{2b}^{(4)}$. Since the system still maintains $C_4$ symmetry and TRS, polarization $\vec{P}$ and fractional corner charges can be obtained by calculating the symmetry representations at all high-symmetry points. Therefore, we obtain the topological indices for different phases. By studying the energy spectrum of finite lattices, we find two different types of zero-energy corner states (ZECS1 and ZECS2) which are protected by generalized chiral symmetry in quadripartite lattices. ZECS1 are localized at two of sublattices, which extends the theory of generalized chiral symmetry. ZECS2 are localized at one of sublattices, which is similar to the Kagome lattice with generalized chiral symmetry for three-atom sublattice. Moreover, these two types of corner states are robust against defects and disorder due to the topological protection from the generalized chiral symmetry. And the evolution of edge states is also an interesting issue for future study. Because of its simple structure, it can be directly used to design the new materials with second-order corner states.

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Appendix. Robustness of fractional corner charge

In the case of $0 < \gamma < t$, there is two electron per unit cell and its Wannier center is located at the maximal Wyckoff position $b$ at $\frac{1}{4}$-filling. Consequently, the $\vec{P} = (0,0)$ and $Q_{\text{Corner}}^{(4)} = \frac{2}{9}$. In this section, we add an infinitesimal $\delta \Gamma_0$ to the Hamiltonian $H$ in equation (1), where $\Gamma_0 = \sigma_3 \sigma_0$, $\sigma_3$ and $\sigma_0$ are Pauli matrices for the degrees of freedom within a unit cell. The sign of $\delta$ determines whether the corner modes at one diagonal or the other are filled [6]. The formula of corner charge can be found in references [5, 6, 8]. Fractional corner charge is protected by $C_4$ symmetry and TRS [21]. Next, we break the $C_4$ symmetry to observe if the fractional corner charge can still be quantized.

Then, we add the intracellular next-nearest-neighbor hopping to break the $C_4$ symmetry. The perturbation Hamiltonian has the form

$$h = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I_{\text{NN}} & 0 \\ 0 & 0 \end{pmatrix}$$ (A.1)
thus, the Hamiltonian of the system become

$$H_{NN} = H + h + \delta \Gamma_0. \quad \text{(A.2)}$$

The perturbation Hamiltonian $h$ is only the coupling of sublattice 1 and 3 in unit cell, which breaks $C_4$ symmetry down to only $C_2$ symmetry. Figure 6(a) shows the non-quantized corner charge when the coupling terms $h_{NN}$ break the $C_4$ symmetry down to only $C_2$ symmetry. Then all the coupling parameters are multiplied by the imaginary number $i$ to break the TRS. From figure 6(b), we can see that there is no corner charge in the system. Because the corner charge is protected by $C_4$ symmetry and TRS, when the TRS is broken, the corner charge also cannot be quantized [21].

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References

[1] King-Smith R D and Vanderbilt D 1993 Phys. Rev. B 47 1651
[2] Vanderbilt D and King-Smith R 1993 Phys. Rev. B 48 4442
[3] Thouless D J 1983 Phys. Rev. B 27 6083
[4] Thouless D J, Kohmoto M, Nightingale M P and den Nijs M 1982 Phys. Rev. Lett. 49 405
[5] Benalcazar W A, Bernevig B A and Hughes T L 2017 Phys. Rev. B 96 245115
[6] Benalcazar W A, Bernevig B A and Hughes T L 2017 Science 357 61
[7] Mittal S, Orre V V, Zhu G, Gorlach M A, Poddubny A and Hafezi M 2019 Nat. Photon. 13 692
[8] Li C-A and Wu S-S 2020 Phys. Rev. B 101 195309
[9] Imhof S et al 2018 Nat. Phys. 14 925
[10] Peterson C W, Benalcazar W A, Hughes T L and Bahl G 2018 Nature 555 346
[11] Serra-Garcia M, Peri V, Süssstrunk R, Bilal O R, Larsen T, Villanueva L G and Huber S D 2018 Nature 555 342
[12] Xi N, Li M, Weiner M, Al A and Khanikaev A B 2020 Nat. Commun. 11 1
[13] Xue H, Ge Y, Sun H-X, Wang Q, Jia D, Guan Y-J, Yuan S-Q, Chong Y and Zhang B 2020 Nat. Commun. 11 1
[14] Li Z, Cao Y, Yan P and Wang X 2019 npj Comput. Mater. 5 1
[15] Schindler F, Cook A M, Vergniory M G, Wang Z, Parkin S S P, Bernevig B A and Neupert T 2018 Sci. Adv. 4 eaao3046
[16] Song Z, Fang Z and Fang C 2017 Phys. Rev. Lett. 119 246402
[17] Khalaf F 2018 Phys. Rev. B 97 205136
[18] Van Miert G and Ortiz C 2018 Phys. Rev. B 98 081110
[19] Hsu C-H, Stano P, Klinovaja J and Loss D 2018 Phys. Rev. Lett. 121 196801
[20] Franca S, van den Brink J and Fulga I 2018 Phys. Rev. B 98 201114
[21] Benalcazar W A, Li T and Hughes T L 2019 Phys. Rev. B 99 245151
[22] Erazo M 2018 Phys. Rev. B 98 201402
[23] Yang H, Li Z-X, Liu Y, Cao Y and Yan P 2020 Phys. Rev. Res. 2 022028
[24] Ni X, Weiner M, Alu A and Khanikaev A B 2019 Nat. Mater. 18 113
[25] Chen Z-G, Xu C, Al Jadhali R, Mei J and Wu Y 2019 Phys. Rev. B 100 075120
[26] Noh J, Benalcazar W A, Huang S, Collins M J, Chen K P, Hughes T L and Rechtsman M C 2018 Nat. Photon. 12 408–15
[27] Xie B-Y, Wang H-F, Wang H-X, Zhu X-Y, Jiang J-H, Lu M-H and Chen Y-F 2018 Phys. Rev. B 98 205147
[28] Kim M and Rho J 2020 Nanophotonics 9 3227
[29] Xie B-Y, Su G-X, Wang H-F, Su H, Shen X-P, Zhan P, Lu M-H, Wang Z-L and Chen Y-F 2019 Phys. Rev. Lett. 122 233903
[30] Li M, Zhirinhi D, Gorlach M, Ni X, Filonov D, Slobozhanyuk A, Alu A and Khanikaev A B 2020 Nat. Photon. 14 89
[31] Wakao H, Yoshida T, Araki H, Mizoguchi T and Hatsuagari Y 2020 Phys. Rev. B 101 094107
[32] Fan H, Xia B, Tong L, Zheng S and Yu D 2019 Phys. Rev. Lett. 122 204301
[33] Li Z-X, Cao Y, Wang X and Yan P 2020 Phys. Rev. B 101 184404
[34] Li Z-X, Cao Y, Wang X and Yan P 2020 Phys. Rev. Appl. 13 064058
[35] Ota Y, Liu F, Katsumi R, Watanabe K, Wakabayashi K, Arakawa Y and Iwamoto S 2019 Optica 6 786
[36] Xie X et al 2020 Laser Photon. Rev. 14 1900425
[37] Obana D, Liu F and Wakabayashi K 2019 Phys. Rev. B 100 075437
[38] Liu F and Wakabayashi K 2017 Phys. Rev. Lett. 118 076803
[39] Liu S et al 2019 Research 2019 8609875

Figure 6. (a) The charge density under $C_4$ symmetry is broken. (b) The charge density under TRS symmetry is broken. In all plots, there are 20 cells in both x and y directions and $\lambda/\tau = 5$, $\gamma/\tau = 0.4$ and $\delta/\tau = 10^{-7}$. $t_{NN}/\tau = 0.4$ for (a).
[40] Zheng L-Y, Achilleos V, Richoux O, Theocharis G and Pagneux V 2019 *Phys. Rev. Appl.* 12 034014
[41] Benalcazar W A and Cerjan A 2020 *Phys. Rev. B* 101 161116
[42] Chen X-D, Deng W-M, Shi F-L, Zhao F-L, Chen M and Dong J-W 2019 *Phys. Rev. Lett.* 122 233902
[43] Kempkes S N, Slot M R, van den Broeke J J, Capiod P, Benalcazar W A, Vanmaekelbergh D, Bercioux D, Swart I and Morais Smith C 2019 *Nat. Mater.* 18 1292
[44] Asbth J K, Oroszlny L and Plyi A 2016 *A Short Course on Topological Insulators* (Berlin: Springer) p 1C22