RING WORMHOLES IN D-DIMENSIONAL EINSTEIN AND DILATON GRAVITY

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Abstract

On the basis of exact solutions to the Einstein—Abelian gauge—dilaton equations in $D$-dimensional gravity, the properties of static axially symmetric configurations are discussed. Solutions free of curvature singularities are selected; they can be attributed to traversible wormholes with cosmic string-like singularities at their necks. In the presence of an electromagnetic field some of these wormholes are globally regular, the string-like singularity being replaced by a set of twofold branching points. Consequences of wormhole regularity and symmetry conditions are discussed. In particular, it is shown that (i) regular, symmetric wormholes have necessarily positive masses as viewed from both asymptotics and (ii) their characteristic length scale in the big charge limit ($GM^2 \ll Q^2$) is of the order of the “classical radius” $Q^2/M$.

1 Introduction

Wormholes as regular 3-geometries with a neck connecting two flat asymptotics have apparently first attracted the researchers’ attention as particular spatial sections of the extended Schwarzschild (Kruskal) space-time. The possible existence of such geometries invoked such attractive ideas as “charge without charge” or “mass without mass” \[1\]. However, the Schwarzschild and Reissner-Nordström wormholes are not traversible for a non-tachyonic particle due to the dynamic nature of the relevant space-time. Later on appeared static, spherically symmetric wormhole metrics describing traversible wormholes \[2, 3\], but at the expense of introducing unusual types of matter (scalar) fields, since in a certain domain near a wormhole neck one must have $\varepsilon + p_r < 0$ where $\varepsilon$ and $p_r$ are the energy density and pressure in the radial direction, respectively. Other spherically symmetric wormholes were found in Refs. \[4\] (in particular, a model with electric field and a domain of neutral dust with negative density), \[5\] (as particle-like solutions of a class of models of nonlinear electrodynamics), \[6\] (in higher-dimensional models with sigma-model-type behavior of effective scalar fields in a 4-dimensional reformulation of the theory), \[7\] (in multidimensional dilaton gravity) and some others. Clement \[6\] also obtained an axially symmetric generalization of his solutions, describing a chain of wormholes situated along a symmetry axis.

In the recent years wormholes have been discussed as possible time machines \[8\] et al.; the discussion concerned the consequences of wormhole existence rather than models where they can appear. This interesting set of problems is beyond the scope of

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this paper. We also do not touch upon the vast recent work concerning Euclidean and quantum wormholes.

Instead, we would like to discuss a class of Lorentzian, axially symmetric wormholes having no spherically symmetric analog — we suggest to call them ring wormholes — whose distinctive feature is the existence of a ring of branching points like those in Riemannian surfaces of analytic functions. Unlike spherically symmetric counterparts, these ones do not require unusual matter for their existence: in the field model under consideration a necessary condition for their regularity is just a kind of equilibrium condition between the electric and scalar fields.

The field model, used here in a phenomenological manner, is $D$-dimensional dilaton gravity. The latter (more precisely, Einstein-gauge-dilaton-axion gravity) is known to form the bosonic part of effective low energy string theory [9], one of the theories pretending to become “a theory of everything”. This is the main motivation for a large number of studies of its solutions and predictions (see, for instance, [10] and references therein). Another problem of interest (in string theory among others) is that of possible effects and manifestations of extra space-time dimensions. One of ways to take them into account is to treat extra-dimension scale factors as separate dynamic variables, as is done, e.g., in [12, 13, 14, 17, 16]. This is the approach adopted here.

We start from the action

$$S = \int d^D x \sqrt{|g|} \left[ R + g^{MN} \phi_M \phi_N - e^{2\lambda \phi} F^2 \right]$$

(1)

where $g_{MN}$ is the $D$-dimensional metric, $D g = |\det g_{MN}|$, $\phi$ is the dilaton scalar field and $F^2 = F^{MN} F_{MN}$, $F = dU$ is an Abelian gauge field, to be interpreted as the electromagnetic field.

The field-theoretic limit of string theory corresponds to the specific value of the coupling constant $\lambda = \lambda_{\text{string}} = \pm (D - 2)^{-1/2}$ [3, 7]. However, we retain an arbitrary value of $\lambda$ in order to cover a wider spectrum of possible field theories, such as Kaluza-Klein type ones, considered, e.g., in Ref. [13]. The value $\lambda = 0$ evidently corresponds to $D$-dimensional (in particular, 4-dimensional) general relativity (GR) with a minimally coupled scalar field.

It should be noted that the action (1) is written in the so-called Einstein conformal gauge, convenient for solving the field equations. However, if the underlying theory is string theory, then a more fundamental role is played by the “string metric”, or “$\sigma$ model metric”

$$\hat{g}_{AB} = e^{-2\lambda \phi} g_{AB}$$

(2)

rather than $g_{AB}$ from [1] (see, e.g., [17, 13] and references therein). Therefore it makes sense to discuss such conformal gauge-dependent issues as the nature of singularities (if any) in terms of $\hat{g}_{AB}$. Strictly speaking, this argument applies only to $\lambda = \lambda_{\text{string}}$, but, for convenience, we deal with $\hat{g}_{AB}$ for any $\lambda$, keeping in mind that this automatically leads to correct results for the case of GR ($\lambda = 0$) as well.

Throughout the paper capital Latin indices range from 0 to $D - 1$, Greek ones from 0 to 3.
2 Static, axially symmetric solutions to the field equations

2.1 Equations

Consider a $D$-dimensional pseudo-Riemannian manifold $V^D$ with the structure

$$V^D = M^4 \times V_1 \times \ldots \times V_n; \quad \text{dim } V_i = N_i; \quad D = 4 + \sum_{i=1}^{n} N_i,$$

where $M^4$ plays the role of the conventional space-time and $V_i$ are Ricci-flat manifolds of arbitrary dimensions and signatures, with the line elements $ds_i^2$, $i = 1, \ldots, n$. The $D$-metric is

$$ds_D^2 = g_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + \sum_{i=1}^{n} e^{2\beta_i(x^\mu)} ds_i^2.$$

For studying static, axially symmetric systems let us use the 4-dimensional formulation of the theory. If the gauge field $F$ is purely 4-dimensional (only $F_{\mu\nu} \neq 0$), the action (1) after integrating out the extra-dimension coordinates reads (up to a constant factor and a divergence):

$$S = \int d^4x \sqrt{4g} e^\sigma \left( 4R - \sigma^\mu \sigma_\mu + \sum_{i=1}^{n} N_i \beta_i;\mu \beta_i;\mu + \varphi^\mu \varphi_\mu - e^{2\lambda \varphi} F^2 \right)$$

where $4R$ is the curvature derived from $g_{\mu\nu}$, the 4-dimensional part of $g_{MN}$, and, as before, $\sigma = \sum_{i=1}^{n} N_i \beta_i$. Eq. (3) corresponds to the original $D$-Einstein conformal gauge. The 4-dimensional Einstein gauge with the metric $\bar{g}_{\mu\nu}$ is obtained after the conformal mapping

$$\bar{g}_{\mu\nu} = e^\sigma g_{\mu\nu}$$

leading the action to the form

$$S = \int d^4x \sqrt{\bar{g}} \left( 4\bar{R} + \frac{1}{2} \sigma^\mu \sigma_\mu + \sum_{i=1}^{n} N_i \beta_i;\mu \beta_i;\mu + \varphi^\mu \varphi_\mu - e^{\sigma+2\lambda \varphi} F^2 \right)$$

where $\bar{g}_{\mu\nu}$ is used to form the curvature $4\bar{R}$ and to raise and lower the indices. Let us now adopt the following assumptions:

(i) the static, axially symmetric metric $\bar{g}_{\mu\nu}$ takes the Weyl canonical form

$$d\bar{s}^2 = e^{2\gamma} dt^2 - e^{-2\gamma} \left[ e^{2\beta}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \right];$$

(ii) the gauge field $U$ has the only component $U_0 = U(\rho, z)$ (that is, only the electric field is present);

(iii) there is only one “extra” space $V_1$ (we will denote $\text{dim } V_1 = d$, $\beta_1 = \xi$).
Assumption (ii) is not very restrictive since a magnetic component of \( F \), compatible with axial symmetry and regularity on the axis, may be easily introduced by a duality rotation applied to the purely electric field \( F \). Assumption (iii) is adopted mostly to save space, since, as seen from (7), a generalization to \( n > 1 \) is straightforward.

The vacuum field equations (for \( \varphi \equiv U \equiv 0 \)) can be written in the form

\[
\Delta \chi = \Delta \nu = \Delta W = 0; \\
\beta_\rho = \rho \left[ \frac{d}{2a_1} \Delta_1 \chi + \frac{1}{2a_2} \Delta_1 \nu + K_1 \Delta_1 W \right], \\
\beta_z = \rho \left[ \frac{d}{a_1} \chi_\rho \chi_z + \frac{2}{a_2} \nu_\rho \nu_z + 2K_1 W_\rho W_z \right]
\]

where the indices \( \rho \) and \( z \) denote partial derivatives, \( a_0 = \frac{1}{2} d^2 \) and \( \Delta \) is the “flat” Laplace operator in the cylindrical coordinates:

\[
\Delta = \rho^{-1} \partial_\rho (\rho \partial_\rho) + \partial_z \partial_z.
\]

Eqs. (9) coincide with the scalar-vacuum equations in conventional GR; the “scalar field” \( \xi \) enters into the equations in the same way as the metric function \( \gamma \), and, just as in GR, their solution by quadratures reduces to that of the flat-space Laplace equation [20].

It can be shown that in the general case (7), under a certain additional assumption, the field equations essentially reduce to (9). Indeed, introduce the combinations of the unknowns \( \gamma, \xi \) and \( \varphi \)

\[
\chi = -(d + 2) \lambda \xi + \varphi, \\
\nu = \gamma + \frac{d + 2}{2a_1} (d \xi + 2 \lambda \varphi), \\
\omega = \gamma - \frac{1}{2} d \xi - \lambda \varphi,
\]

and assume that (iv) the potential \( U(\rho, z) \) is functionally related to \( \omega \).

Then the field equation for \( U \) takes the form of a linear first-order equation with respect to \( f(\omega) = (dU/d\omega)^2 \), which gives

\[
e^{2\omega(\rho,z)} = a_3 (c_0 + 2c_1 U + U^2)
\]

where \( c_0 \) and \( c_1 \) are integration constants. Defining the function \( W(U) \) by the relation

\[
dW/dU = e^{-2\omega},
\]

it is an easy matter to bring the remaining field equations to a form like (3):

\[
\Delta \chi = \Delta \nu = \Delta W = 0; \\
\beta_\rho = \rho \left[ \frac{d}{2a_1} \Delta_1 \chi + \frac{1}{2a_2} \Delta_1 \nu + K_1 \Delta_1 W \right], \\
\beta_z = \rho \left[ \frac{d}{a_1} \chi_\rho \chi_z + \frac{2}{a_2} \nu_\rho \nu_z + 2K_1 W_\rho W_z \right]
\]

As will be seen further, \( e^{2\omega(\rho,z)} = \hat{g}_{00} \) is the static gravitational potential in string metric. Therefore Assumption (iv) means that the gravitational and electromagnetic field strength vectors are everywhere parallel. In particular, these two fields must have common sources, if any.
where we have introduced the constants
\[
\begin{align*}
a_1 &= d + 2\lambda^2(d + 2); \\
a_2 &= 1 + \frac{d + 2}{a_1}; \\
a_3 &= 1 + \frac{a_1}{d + 2} = \frac{a_1a_2}{d + 2}; \\
K_1 &= a_3(c_1^2 - c_0).
\end{align*}
\]

A similar procedure was applied to this field system in 4-dimensions in Ref. [22]. As in (9), the integrability conditions for \(\beta\) in (13) are fulfilled automatically. Thus the solutions are obtained by quadratures provided the harmonic functions \(\chi, \nu, W\) are known. The original functions \(\gamma, \xi, \varphi\) are restored using the reverse transformation
\[
\begin{align*}
\gamma &= \nu / a_2 + \omega / a_3, \\
\xi &= -2\lambda\chi / a_1 + 2(\nu - \omega) / (a_1a_2), \\
\varphi &= d \cdot \chi / a_1 + 2\lambda(\nu - \omega) / a_3
\end{align*}
\]
(note that the determinant of (10) is \(a_1a_2 = a_1 + d + 2 > 0\)).

It should be noted that in the special case of our field model \(d = \lambda = 0\) (GR, \(D = 4\), scalar electrovacuum), when the above formulas are not directly applicable due to \(a_1 = 0\), Eqs. (13) still hold with \(d/a_1 = 1, \omega \equiv \gamma\) and \(\chi \equiv \varphi\), while the coefficient \(1/a_2\) vanishes and the function \(\nu\) does not appear at all. So all the subsequent formulas are also valid with these substitutions; in particular, in the monopole solutions of Subsec. 2.3 one should put \(q_2 = 0\) since this is the “charge” corresponding to \(\nu\).

### 2.2 Multipole solutions

Following [21, 23], let us seek solutions in the coordinates \((x, y)\), connected with \(\rho\) and \(z\) by
\[
\rho^2 = L^2(x^2 + \varepsilon)(1 - y^2), \quad z = Lxy
\]
where \(L\) is a fixed positive constant and \(\varepsilon = 0, \pm 1\), so that \(x\) and \(y\) are spherical \((\varepsilon = 0)\), prolate spheroidal \((\varepsilon = -1)\), or oblate spheroidal \((\varepsilon = +1)\) coordinates, respectively; \(-1 < y < 1; x > 1\) for \(\varepsilon = -1\) and \(x > 0\) for \(\varepsilon = 0, +1\).

The Laplace equation \(\Delta f = 0\) acquires the form
\[
\partial_x(x^2 + \varepsilon)\partial_x f + \partial_y(1 - y^2)\partial_y f = 0.
\]

Separating the variables in Eq. (13) (the first line), i.e., putting, for example, \(\chi(x, y) = \chi_1(x)\chi_2(y)\), one obtains
\[
\begin{align*}
[(x^2 + \varepsilon)\chi_{1x}]_x - \lambda_0 \chi_1 &= 0, \\
[(1 - y^2)\chi_{2y}]_y + \lambda_0 \chi_2 &= 0
\end{align*}
\]
where \(\lambda_0\) is the separation constant. Solutions to (13), finite on the symmetry axis \(y = \pm 1\), are the Legendre polynomials \(P_l(y)\), while \(\lambda_0 = l(l + 1)\) with \(l = 0, 1, 2, \ldots\). The corresponding solutions to (18) are combinations of Legendre functions of the first and second kinds.
The equations for \( \nu \) and \( W \) are solved in a similar way. This is the way to obtain a general class of solutions containing arbitrary multipolarities \( l \): after writing out the solutions to the three Laplace equations (each has, in general, the form of an infinite series — a superposition of different multipolarities), \( \beta(\rho, z) \) is found by quadratures from the equations (obtained from (13))

\[
\begin{align*}
\beta_x &= \frac{d}{2a_1} \Delta_x \chi + \frac{1}{a_2} \Delta_x \nu + K_1 \Delta_x W \\
\beta_y &= \frac{d}{2a_1} \Delta_y \chi + \frac{1}{a_2} \Delta_y \nu + K_1 \Delta_y W
\end{align*}
\]  

(20)

where, for any function \( f \),

\[
\begin{align*}
\Delta_x f &= \frac{1}{L^2(x^2 + y^2)} \left[ x \rho^2 f_x^2 - 2 y \rho^2 f_x f_y - L^2 x (1 - y^2)^2 f_y^2 \right], \\
\Delta_y f &= \frac{1}{L^2(x^2 + y^2)} \left[ L^2 y (x^2 + 1)^2 f_x^2 + 2 x \rho^2 f_x f_y - y \rho^2 f_y^2 \right].
\end{align*}
\]  

(21)

In what follows, however, we restrict ourselves to \( l = 0 \) (monopole solutions), under the asymptotic flatness conditions: \( \gamma = \xi = \varphi = U = 0 \) at spatial infinity.

### 2.3 Monopole solutions

A monopole solution to Eq. (19), regular at \( y = \pm 1 \), is just a constant, so that \( \chi = \chi(x) \). Eq. (18) takes the form \((x^2 + \varepsilon) d\chi/dx = \text{const.} \). Its integration leads to the following expressions for \( \chi(x) \) satisfying the asymptotic flatness condition:

\[
\chi = \begin{cases} 
-\frac{1}{2} q_1 \ln \frac{x + 1}{x - 1}, & \varepsilon = -1, \\
-q_1/x, & \varepsilon = 0, \\
-q_1 \arccot x, & \varepsilon = +1.
\end{cases}
\]  

(22)

The monopole solutions for \( \nu \) and \( W \) are found in a similar way and are described by (22) with the replacements

\[
q_1 \mapsto q_2 \quad \text{for } \nu; \quad q_1 \mapsto q_3 \quad \text{for } W
\]  

(23)

The expressions for \( \beta(x, y) \) satisfying the asymptotic flatness condition \( \beta(\infty, y) = 0 \) are

\[
e^{2\beta} = \begin{cases} 
(x^2 - 1)^K (x^2 - y^2)^{-K}, & \varepsilon = -1, \\
\exp[-K(1 - y^2)/x^2], & \varepsilon = 0, \\
(x^2 + y^2)^K (x^2 + 1)^{-K}, & \varepsilon = +1
\end{cases}
\]  

(24)

\[
K = \frac{d}{2a_1} q_1^2 + \frac{1}{a_2} q_2^2 + K_1 q_3^2,
\]  

(25)

where \( K_1 \) is defined in (14).
At spatial infinity our monopole solutions are asymptotically spherically symmetric. Indeed, assuming \( y = \cos \theta \), where \( \theta \) is the conventional polar angle, the line element (8) transformed by (16) is spherically symmetric if

\[ e^{2\beta} = (x^2 + \varepsilon)/(x^2 + \varepsilon y^2). \]

The condition (26) holds for all the solutions in the limit \( x \to \infty \) where they have Schwarzschild asymptotics. As for the whole space, the condition (26) is fulfilled under the additional requirement \( K = -\varepsilon \). Unlike the vacuum case, when \( K \) is positive-definite and the above condition can hold for \( \varepsilon = 0, +1 \) only in the trivial case when the space-time is flat, in the presence of an electric field \( K \) can have any sign, hence the sphericity condition can be fulfilled with any \( \varepsilon \). One naturally obtains the known solutions described in [13, 7].

The solutions with \( \varepsilon = 0, -1 \) turn out to possess naked singularities in all nontrivial (nonspherical) cases. So let us pay more attention to the solution with \( \varepsilon = +1 \), which can have no curvature singularity. Although a preferred conformal gauge does exist (the string one), it is remarkable that the most important features of these configuration are insensitive to conformal factors of the forms \( \exp(\text{const} \times \sigma) \) and \( \exp(\text{const} \times \varphi) \), due to global regularity of both \( \varphi \) and \( \sigma \).

3 Ring wormholes

3.1 The wormhole geometry

For \( \varepsilon = +1 \), the 4-dimensional scalars \( \chi, \nu \) and \( W \) are finite functions of \( x \):

\[ (\chi, \nu, W) = -(q_1, q_2, q_3) \arccot x. \]

Thus the metric \( g_{\mu\nu} \) and the field \( F \) are regular if and only if \(|\omega| < \infty \), i.e., if \( U \) nowhere tends to infinity or a root of the trinomial (11). If we require that at spatial infinity \( U = \omega = 0 \), then \( c_0 = 1/a_3, \ K_1 = a_3c_1^2 - 1 \) and the trinomial’s discriminant is \( 4a_3^2(c_1^2 - c_0) = 4a_3K_1 \), so that (11) gives:

\[
W - W_0 = \begin{cases} 
-1/2a_3K_1 \ln \frac{U + c_1 - \sqrt{K_1/a_3}}{U + c_1 + \sqrt{K_1/a_3}}, & K_1 > 0, \\
1/a_3(U + c_1), & c_1^2 = a_3, \\
-1/\sqrt{-a_3K_1} \arccot \frac{U + c_1}{\sqrt{-K_1/a_3}}, & K_1 < 0,
\end{cases}
\]

where \( W_0 = \text{const} \) must be chosen to satisfy \( W = 0 \) for \( U = 0 \), according to (27).

Evidently \( U \) cannot tend to a root of (11), since otherwise we would get \( W \to \infty \), whereas actually \( W(x) \) ranges over a finite interval. To prevent \( U \to \pm \infty \) it is sufficient to require that the corresponding value of \( W \) do not belong to this interval. We will now assume that this condition holds and postpone its further discussion till Subsec. 3.2.
The absence of a curvature singularity does not necessarily mean, however, that the space-time is globally regular. Let us study the limit $x \to 0$ in some detail.

The functions $\varphi$, $\xi$, $\gamma$ and $e^\beta$ are finite at $x = 0$. The curve $x = 0$, $y = 0$ lies in the plane $z = 0$ and forms a ring $\rho = L$ of finite length (Fig. 1).

The surface $x = 0$, $y > 0$ is a disk bounded by the above ring and parametrized by the coordinates $y$ and $\phi$. This metric is flat if and only if $K = 0$. Otherwise the disk is curved but has a regular center at $y = 1$ (the upper small black circle in Fig. 1). The limit $x \to 0$ corresponds to approaching the disk from the half-space $z > 0$.

Another similar disk, the lower half-space one, corresponds to $y < 0$. The two disks are naturally identified when our oblate spheroidal coordinates are used in flat space. In our case, a possible identification of points $(x = 0, y = y_0$, $\phi = \phi_0)$ and $(x = 0, y = -y_0, \phi = \phi_0)$, where $\phi_0$ is arbitrary and $0 < y_0 \leq 1$, leads to a finite discontinuity of the extrinsic curvature of the surfaces identified, or, physically, to a finite discontinuity of forces acting on test particles. This may be interpreted as a membrane-like matter distribution, bounded by the ring $x = y = 0$.

The field discontinuity across the surface $x = 0$ is avoided if one continues the coordinate map to negative $x$, as is done for the vacuum case in Ref. [15]. As a result, appears another “copy” of the 3-space, so that a particle crossing the regular disk $x = 0$ along a trajectory with fixed $y$, threads a path through the ring and can ultimately get to another flat spatial infinity, $x \to -\infty$ with new asymptotic values of $\gamma$, $\xi$ and $\varphi$. The function $\beta$ is even with respect to $x$ and hence coincides at both asymptotics. We obtain a wormhole configuration, nonsymmetric with respect to its “neck” $x = 0$, having no curvature discontinuity, except maybe the ring $x = y = 0$.

To study the geometry near the ring, let us consider a 2-surface of fixed $\phi$ at small $x$ and $y$. In the polar coordinates $(r, \psi)$ $(x = r \cos \psi$, $y = r \sin \psi$), and still new ones $\mu$ and

![Figure 1: Axial section of the neighborhood of the ring $x = y = 0$. The points $A$ and $B$, marked by big black circles, belong to the ring, the thick lines connecting them show the upper and lower disks $x = 0$, $y > 0$.](image-url)
\( \eta \) defined by
\[
r = [(K + 2)\mu]^{1/(K+2)}, \quad \psi = \eta/(K + 2),
\]
the corresponding 2-dimensional metric is (up to a constant factor)
\[
dl_{(x,y)}^2 = r^{2K+2}(dr^2 + r^2d\psi^2) = d\mu^2 + \mu^2d\eta^2
\]
Thus the metric near the ring is locally flat. However, it is locally flat on the ring itself only if \( \eta \) ranges over a segment of length 2\( \pi \). Is this the case?

Given \( x > 0 \), the polar angle \( \psi \) is defined on the segment \([-\pi/2, \pi/2]\), hence \( \eta \in [-\pi - K\pi/2, \pi + K\pi/2] \). Consequently, the ring enjoys local flatness only in the simplest case \( K = 0 \), when \( \beta \) = const. If the electric field were absent, that would mean that \( q_1 = q_2 = 0 \) and the space-time is flat [13]; however, with \( q_3 \neq 0 \), \( K = 0 \) no longer implies the global flatness (see [24]).

In the wormhole case \( x \) can have either sign, hence
\[
\psi \in [-\pi, \pi] \Rightarrow \eta \in [-2 + K\pi, (2 + K)\pi]
\]
Thus the axially symmetric wormhole solution contains in general a cosmic string-like ring singularity with a polar angle excess greater or smaller than 2\( \pi \) for \( K > 0 \) and \( K < 0 \), respectively. The case \( K = 0 \) actually means that there is no singularity and the ring geometry exactly corresponds to what should occur near the neck of such a wormhole. Indeed, making one revolution through the ring at fixed \( \phi \) (completing an angle \( \eta_2 - \eta_1 = 2\pi \)), an observer finds himself at a similar position, but in the “second world”, with (in general) other values of \( \varphi, \xi \) and \( W \), and returns to his original position only completing an angle of 4\( \pi \).

For \( K = 0 \) the \((x, y)\) surface behaves near \( x = y = 0 \) like the two-sheeted Riemann surface of the analytic function \( \sqrt{x + iy} \); and, indeed, the above transition \((x, y) \equiv (r, \psi) \rightarrow (\mu, \eta)\) may be described as a conformal mapping in the complex plane with this analytic function: \( r e^{i\psi} = \sqrt{\mu e^{i\eta}} \). Other examples of branching points in the space-time are known in some Einstein-Maxwell fields in 4 dimensions [24].

### 3.2 Wormhole parameters

Let us use for further interpretation the string metric (2), where the 4-dimensional part \( g_{\mu \nu} \) of \( g_{AB} \) is connected with our \( \overline{g}_{\mu \nu} \) by
\[
\overline{g}_{\mu \nu} = e^{d\xi}g_{\mu \nu}.
\]
(Note that conformal mappings do not affect angles, so that the ring regularity condition does not suffer). Consequently, the string metric is
\[
d\hat{s}_D^2 = e^{2\hat{\xi}}ds_1^2 + e^{2\omega}dt^2 - e^{2\omega - 4\gamma}[e^{2\beta}(d\rho^2 + dz^2) + \rho^2d\phi^2]
\]
where
\[
\hat{\xi} = \xi - \lambda \varphi = -\frac{d + 2}{a_1}\lambda \chi + \frac{2}{a_1a_2}[1 - (d + 2)\lambda^2](\nu - \omega)
\]

and the other functions are defined as before. Note that for \( \lambda = \lambda_{\text{string}} \) one has just \( \hat{\xi} = -\chi \).

Our solution depends on four integration constants (the “charges” \( q_i \) and \( c_1 \)), two input parameters \( d \) and \( \lambda \) and the length scale \( L \) which appeared in the transformation (10) and, like the integration constants, is a parameter of the family of solutions.

Let us connect the wormhole electric charge and mass and their other properties with the input and integration constants.

First, let us restrict ourselves to regular wormholes, for which \( K = 0 \), hence \( K_1 \leq 0 \) (\( K_1 = 0 \) is suitable only in the case \( q_1 = q_2 = 0, \ q_3 \neq 0 \)). The case of all \( q_i = 0 \) is trivial and reduces to flat space without fields.

Second, we will try to select symmetric wormholes, i.e., those with equal asymptotic values of the time rate, i.e., \( e^{\omega(\pm \infty)} = 1 \); other symmetry relations will then follow. An equal time rate means, in particular, a possibility to “immerse” the two asymptotics (“mouths”) of a wormhole to the same external Minkowski space-time without clock rescaling.

Let us fix

\[
- K_1 = 1 - a_3 c_1^2 = \cos^2 \mu > 0, \quad \sin \mu = \sqrt{a_3 c_1}
\]

where \( \mu \) is a new parameter introduced for convenience. By (11), \( e^{2\omega} \) is positive for all \( U \) and, as \( W(x) \) is a monotone function and \( dW/dU = e^{-2\omega} U(x) \) is also a monotone function. This immediately implies that the electric charges of the wormhole as seen from the two asymptotics (determined, up to a positive factor, by \( dU/d|x| \)) have different signs.

The third line of (28) can be rewritten as

\[
\sqrt{a_3} \frac{U + c_1}{\cos \mu} = - \tan [\sqrt{a_3} \cos \mu (q_3 \arccot x + W_0)]
\]

where \( W_0 \) is a constant determined from the condition that \( U \) and \( W \) both vanish as \( x \to \infty \):

\[
\tan \mu = - \tan [\sqrt{a_3} \cos \mu \cdot W_0] \Rightarrow W_0 = - \mu / (\sqrt{a_3} \cos \mu).
\]

The expressions for masses and charges can be obtained from the field behavior at the asymptotics. Thus, for \( x \to \infty \) the mass \( M_+ \) is determined by the Schwarzschild asymptotic of \( \hat{g}_{\mu\nu} \):

\[
e^{2\omega} = 1 - 2GM/r_+ + o(1/r_+)
\]

where \( r_+ \) is the asymptotic expression for the spherical radius: \( r_+ = Lx \). Comparing (38) with the actual asymptotic of \( e^{2\omega} \), one obtains

\[
GM_+ = Lc_1 a_3 q_3.
\]

Thus \( M_+ \) is positive if and only if the constants \( c_1 \) and \( q_3 \) (and hence also \( \mu \)) have the same sign.

The charge \( Q_+ \) at the same asymptotic is determined by the electric field behavior: the asymptotic field magnitude must be

\[
E^i = \sqrt{G} Q_+ r_+^i / r_+^3
\]
where the index $i$ enumerates 3-vector components in the asymptotic Minkowski space and the appearance of $\sqrt{G}$ conforms to the way the electromagnetic field was introduced in the Lagrangian. A comparison to the actual field shows that
\[ \sqrt{G}Q_+ = L q_3. \] (41)

Comparing (39) and (41), one obtains a necessary condition for wormhole regularity in terms of masses and charges:
\[ GM_+/Q_+^2 = a_3 \sin^2 \mu < a_3 = 1 + \frac{d}{d+2} + 2\lambda^2. \] (42)

The same condition could be obtained on equal grounds for the asymptotic $x \to -\infty$ but with slight complications connected with the other appearance of the boundary conditions.

Another restriction on the solution parameters is connected with the dependence $U(x)$ (36),(37). By construction, the argument of the tangent in the r.h.s. of (36) is within the interval $(-\pi/2, \pi/2)$ at $x \to \infty$ (it just equals $\mu$). For $U(x)$ to remain regular at all $x$, the above argument should not ever leave this interval. Taking into account that, for $M_+ > 0$, the quantities $\mu$, $c_1$ and $q_3$ have the same sign, it is easy to obtain this second regularity condition in the form
\[ \pi \sqrt{a_3} \cos \mu \cdot |q_3| - |\mu| < \pi/2, \] (43)

or
\[ |q_3| < (\pi/2 + |\mu|)/(\pi \sqrt{a_3} \cos \mu), \] (44)

a restriction upon $q_3$ provided the parameter $\mu$ (connected with the charge-to-mass ratio) is specified.

The combined conditions $K = 0$ (whence follows, in particular, (12)), and (13) or (14) are not only necessary, but also sufficient for our solution’s regularity.

We did not so far apply the symmetry condition. Putting in (11) $e^{2\omega} = 1$ and recalling that $a_3 c_0 = 1$, we find two possible values for $U$: 0 and $-2c_1$. As $U = 0$ corresponds to $x \to +\infty$, for a nontrivial solution $U(-\infty) = -2c_1$. Let us substitute this into (13), then, with (37),
\[ \pi \sqrt{a_3 q_3} \cos \mu = 2\mu. \] (45)

Eq. (13) has a number of consequences:

(i) (13) implies that $\mu$ and $q_3$ have the same sign. That is, if a wormhole is regular and symmetric, it can have only a positive mass (and it is positive as seen from either of the asymptotics).

(ii) Given (13), (14) is fulfilled automatically. Thus $K = 0$ plus symmetry is sufficient for regularity.

(iii) (13) connects $q_3 = \sqrt{G}Q_+/L$ with $\mu$, or with the charge-to-mass ratio. In particular, if we know the mass and the charge, (13) yields the relevant length scale $L$ of the solution:
\[ L = \pi \sqrt{a_3 G Q_+} \cos \mu/(2\mu). \] (46)
For big charge-to-mass ratios, such as those known for particles (about $10^{18}$ for a proton), the gravitational constant in (46) is cancelled and we obtain $L$ of the order of the “classical radius” $Q^2/M$:

$$L \approx \pi a_3 Q^2/M_+,$$  \hspace{1cm} (47)

which may be treated as a hint that regular soliton-like particle models can be obtained on this trend.

With (45), the quantity $U + c_1$ becomes an odd function of $x$,

$$U + c_1 = \frac{\cos \mu}{\sqrt{a_3}} \tan \left( \frac{2\mu}{\pi} \arctan x \right).$$ \hspace{1cm} (48)

The function $\omega(x)$ is even:

$$e^{2\omega} = a_3(U + c_1)^2 + \cos^2 \mu.$$ \hspace{1cm} (49)

However, the whole solution is not entirely symmetric with respect to $x = 0$. Indeed, since at least one of the charges $q_1$ and $q_2$ is nonzero, there is a nontrivial monotone $x$-dependence of $\chi$ or $\nu$ (or both). Consequently, either the $\phi$ field, or the internal space scale factor $\xi$, or both have different finite limits at $x = +\infty$ and $x = -\infty$. By the conventional terminology, the solution behaves as a kink with respect to these fields. (An attempt to find a relation between $d$ and $\lambda$ such that the $x$-dependence of $\phi$ and $\xi$, other than via $\omega(x)$, would disappear, fails since the emerging set of algebraic equations has no solution.) As is easily verified, this nonsymmetry conclusion applies to all conformal gauges connected by functions of $\phi$.

Finally, let us address to the “purely electric wormhole” solution $q_1 = q_2 = 0$, $q_3 \neq 0$, $K = K_1 = 0$, also satisfying the regularity condition connected with the geometry near the ring $x = y = 0$. In this case all fields are expressed in terms of $\omega(x)$:

$$\varphi = -2\lambda \omega/a_3, \quad \xi = -2\omega/(a_1 a_2), \quad \gamma = \omega/a_3.$$ \hspace{1cm} (50)

The condition $K_1 = 0$ implies $c_1 = \pm 1/\sqrt{a_3}$. Eq. (28) subject to the condition that $U$ and $W$ both vanish as $x \to \infty$ yields

$$U = c_1 W/(c_1 - W),$$ \hspace{1cm} (51)

where $W$ is, as before, expressed by (27). The regularity condition for $U$ (that $W \neq c_1$ for any real $x$) is valid if

$$-c_1/q_3 < 0 \quad \text{(for sign } c_1 = \text{ sign } q_3),$$

$$-c_1/q_3 > \pi \quad \text{(for sign } c_1 \neq \text{ sign } q_3).$$ \hspace{1cm} (52)

Thus regular solutions are obtained. However, the explicit form of $\omega(x)$

$$e^{2\omega} = \left( 1 + \frac{U}{c_1} \right)^2 = \left( 1 + \frac{q_3}{c_1} \arccot x \right)^{-2}$$ \hspace{1cm} (53)

shows that in the nontrivial case $q_3 \neq 0$ the function $\omega(x)$ is monotone. That means that not only the wormhole cannot be symmetric, but the mass has different signs at the two asymptotics. Hence this solution can hardly be treated as a satisfactory one, despite its regularity.
4 Concluding remarks

4.1. The ring wormholes described here are a type of nonsingular Lorentzian wormholes able to exist in spaces of any dimension. A somewhat similar solution — stationary, axially symmetric, 5-dimensional, was found by G. Clement in Ref. [26], but there the function $e^\beta$ (in our notation) grew to infinity as $x \to 0$, $y \to 0$ and it was concluded that “the singular ring is at infinity”. The author mentioned three problems with the physical interpretation of this solution. First, the solution parameters, including the mass, as measured at $x \to -\infty$, are the opposite of those measured at $x \to \infty$ (though, the isometry exchanging the “end points” might be interpreted as classical mass and charge conjugation). Second, there was no definite sign of the mass. And third, the existence of unobserved long-range scalar forces.

For the present symmetric wormholes the first two problems do not exist: the mass is manifestly positive “from both sides”. The third problem is common for all models with long-range scalar fields, be they of dilatonic or extra-dimensional origin, or Goldstone, or Brans-Dicke ones, etc. We would like just to recall that the existence of at least one such field ($\varphi$ or $\xi$) in our model is necessary in order to provide the wormhole regularity, in other words, to maintain a kind of equilibrium with the electromagnetic field.

4.2. Unlike the spherically symmetric case, where wormholes can appear only with matter violating the usual energy conditions, e.g., unconventional scalar fields ([2, 3, 6] and others) or special kinds of nonlinear electrodynamics [5], under axial symmetry they appear rather naturally in vacuum, scalar-vacuum and electrovacuum systems both in GR and dilaton gravity. Other types of nonspherical wormholes, connected with cosmic strings, are considered in Ref. [25].

4.3. A tentative study shows that the present class of regular wormhole models can be generalized to include the harmonic functions (“scalar fields”) $\nu$, $\chi$, $\omega$ with higher multipole moments ($l > 0$), which may be treated as perturbations to purely monopole models. The higher-$l$ parts of the fields are strongly constrained by the asymptotic flatness conditions and those of regularity at the ring $x = y = 0$. We hope to give a more detailed treatment of such models elsewhere.

4.4. For the present class of static, axially symmetric configurations, addition of new scalar fields to electrically neutral models (for instance, to pure vacuum in GR) affects the properties of the solutions only quantitatively; this applies, in particular, to extra-dimension scale factors which, in the 4-dimensional setting of the problem, actually behave as massless, minimally coupled scalar fields. Unlike that, the electric field (all the same in GR or dilaton gravity) creates qualitatively new features: there appear new types of singularities and globally regular axially symmetric wormholes. See, however, the last sentence of item 4.1.

4.5. The 4-dimensional version of our models with $\lambda = 0$ are readily reformulated in terms of a broad (Bergmann-Wagoner) class of scalar-tensor theories (STT) of gravity in 4 dimensions, in particular, the Brans-Dicke theory and GR with a conformally coupled scalar field. Models of different STT differ from each other by conformal mappings with
regular ($\varphi$-dependent) conformal factors which change the explicit form of wormhole symmetry conditions but do not affect the neck regularity. Note that the scalar field of the STT can be interpreted in terms of an extra-dimension scale factor, while different theories correspond to different conformal gauges of the 4-dimension section. In the case $\lambda \neq 0$ one can speak of generalized STT with a coupling between the scalar and electromagnetic fields.

To conclude, both globally regular solutions and those with “strings” may be of interest for describing late stages of gravitational collapse and/or cosmological dark matter. Their monopole nature probably means that they cannot decay by gravitational wave emission. So it can be of significant interest to study their generalizations and stability.

Acknowledgement

This work was supported in part by the Russian Ministry of Science and by CNPq Grant No. . One of us (K.B.) is sincerely grateful to the Department of Physics of UFES for kind hospitality.

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