FABER-KRAHN INEQUALITY FOR ANISOTROPIC EIGENVALUE PROBLEMS WITH ROBIN BOUNDARY CONDITIONS

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Abstract. In this paper we study the main properties of the first eigenvalue \( \lambda_1(\Omega) \) and its eigenfunctions of a class of highly nonlinear elliptic operators in a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \), assuming a Robin boundary condition. Moreover, we prove a Faber-Krahn inequality for \( \lambda_1(\Omega) \).

1. Introduction

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), \( n \geq 2 \). This paper is devoted to the study of the following problem:

\[
\lambda_1(\Omega) = \min_{u \in W^{1,p}(\Omega), u \neq 0} J(u),
\]

where

\[
J(u) = \frac{\int_{\Omega} [H(\nabla u)]^p dx + \beta \int_{\partial \Omega} |u|^p H(\nu) d\sigma}{\int_{\Omega} |u|^p dx}.
\]

1 < \( p < +\infty \), \( \nu \) is the outer normal to \( \partial \Omega \), and \( \beta \) is a fixed positive number. Moreover, we suppose that \( H \) is a sufficiently smooth norm of \( \mathbb{R}^n \) (see Sections 2 and 3 for the precise assumptions). The minimizers of (1.1) satisfy the equation

\[
-\text{div} \left( [H(\nabla u)]^{p-1} H_\xi(\nabla u) \right) = \lambda_1(\Omega) |u|^{p-2} u \text{ in } \Omega,
\]

with Robin conditions on the boundary:

\[
[H(\nabla u)]^{p-1} H_\xi(\nabla u) \cdot \nu + \beta H(\nu)|u|^{p-2} u = 0 \text{ on } \partial \Omega.
\]

The operator in (1.3) reduces to the \( p \)-Laplacian when \( H \) is the Euclidean norm of \( \mathbb{R}^n \). For a general norm \( H \), it is an anisotropic, highly nonlinear operator, and it has attracted an increasing interest in last years. We refer, for example, to \([1,20,21]\) (\( p = 2 \)) and \([1,19,22]\) (\( 1 < p < +\infty \)) where Dirichlet boundary conditions are considered. Moreover, for Neumann boundary values see, for instance, \([15,35]\) (\( p = 2 \)), while overdetermined problems are studied in \([12,35]\) (\( p = 2 \)). In this paper we are interested in considering the eigenvalue problem (1.3) with the Robin boundary conditions (1.4). In particular, our main objective is to obtain a Faber-Krahn inequality by studying the shape optimization problem

\[
\min_{|\Omega|=m} \lambda_1(\Omega)
\]

among all the Lipschitz domains with given measure \( m > 0 \). To study problem (1.5), we first have to investigate the basic properties of the first eigenvalue and of the relative eigenfunctions of (1.3), (1.4), as existence, sign, simplicity and regularity.

Date: November 15, 2013.

2010 Mathematics Subject Classification. 35P15, 35P30, 35J60.

Key words and phrases. Eigenvalue problems, nonlinear elliptic equations, Faber-Krahn inequality, Wulff shape.
In the Euclidean case, problem (1.1) reduces to

\[ \lambda_1,\varepsilon(\Omega) = \min_{\substack{u \in W^{1,p}(\Omega) \\
u \neq 0}} \frac{\int_{\Omega} |Du|^p dx + \beta \int_{\partial \Omega} |u|^p d\sigma}{\int_{\Omega} |u|^p dx}, \]

and the minimizers satisfy the problem

\[
\begin{cases}
- \text{div} (|Du|^{p-2} Du) = \lambda_1,\varepsilon(\Omega) |u|^{p-2} u & \text{in } \Omega, \\
|Du|^{p-2} \frac{\partial u}{\partial \nu} + \beta |u|^{p-2} u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

In such a case, problem (1.5) has been first investigated by Bossel for \( p = 2 \), when \( \Omega \) varies among smooth domains of \( \mathbb{R}^2 \) with fixed measure. More precisely, in [7] she proved that

\[ (1.6) \lambda_1,\varepsilon(\Omega) \geq \lambda_1,\varepsilon(B), \]

where \( B \) is a disk such that \( |B| = |\Omega| \). This result has been generalized to any dimension \( n \geq 2 \) for Lipschitz domains in [17]. As regards the case \( 1 < p < +\infty \), the inequality (1.6) has been proved by [16] for smooth domains, and by [9] in the case of Lipschitz domains. The equality cases are also addressed in [9, 16]. As regards the case \( \beta < 0 \), we refer the reader to [25] and the references therein.

In the anisotropic case, our result reads as follows. Let \( H^o \) be the polar function of \( H \), and denote by \( W_R \) the Wulff shape, that is the \( R \)-sublevel set of \( H^o \), such that \( |W_R| = |\Omega| \) (see Section 2 for the definitions). If \( \Omega \neq W_R \) is a Lipschitz set of \( \mathbb{R}^n \), then

\[ \lambda_1(\Omega) > \lambda_1(W_R). \]

Hence, the unique minimizer of (1.5) is the Wulff shape. Such result relies in the so-called anisotropic isoperimetric inequality (see for example [1]), and it is in agreement with the Faber-Krahn inequality for the first eigenvalue of (1.3) in the homogeneous Dirichlet case (see [4]).

As a matter of fact, we may ask if the first eigenvalue \( \lambda_1(\Omega) \) is bounded from above in terms of the Lebesgue measure of \( \Omega \). Indeed, in the Euclidean setting, this is the case for the first nonvanishing Neumann Laplacian eigenvalue (see [37], and also [8, 11] for related results), but this does not happen for the first Dirichlet Laplacian eigenvalue. In this order of ideas, by a result given in [29] it follows that the first Robin Laplacian eigenvalue among the sets of fixed measure is unbounded from above. Here we prove a lower bound for the first eigenvalue \( \lambda_1(\Omega) \) of our anisotropic Robin problem in a convex set \( \lambda_1(\Omega) \) in terms of the anisotropic inradius of \( \Omega \). This will imply that, among all Lipschitz sets with fixed measure \( m > 0 \),

\[ \sup_{|\Omega| = m} \lambda_1(\Omega) = +\infty. \]

The paper is organized as follows. In Section 2, we recall some basic definitions and properties of \( H \) and of its polar function \( H^o \). In Section 3, we state and prove some properties of the first eigenvalue of (1.3), (1.4). More precisely, under suitable assumptions on \( H \), we show that there exists a first eigenvalue \( \lambda_1(\Omega) \) which is simple. Moreover, we prove that the first eigenfunctions are in \( C^{1,\alpha}(\Omega) \cap C(\bar{\Omega}) \), for some \( 0 < \alpha < 1 \). Furthermore, a solution of the eigenvalue problem is a first eigenfunction if and only if it has a fixed sign. In Section 4 we investigate the eigenvalue problem when \( \Omega \) is a Wulff shape, while in Section 5 we give a representation formula for \( \lambda_1(\Omega) \) by means of the level sets of the first eigenfunctions. Using such results, in Section 6 we state precisely the main result and give a proof.
2. Notation and preliminaries

Let \( H : \mathbb{R}^n \to [0, +\infty[ , \ n \geq 2 \), be a \( C^2(\mathbb{R}^n \setminus \{0\}) \) function such that
\[
H(t\xi) = |t|H(\xi), \quad \forall \xi \in \mathbb{R}^n , \ \forall t \in \mathbb{R},
\]
and such that any level set \( \{ \xi \in \mathbb{R}^n : H(\xi) \leq t \} \), with \( t > 0 \) is strictly convex.

Moreover, suppose that there exist two positive constants \( a \leq b \) such that
\[
a|\xi| \leq H(\xi) \leq b|\xi|, \quad \forall \xi \in \mathbb{R}^n.
\]

**Remark 2.1.** We stress that the homogeneity of \( H \) and the convexity of its level sets imply the convexity of \( H \). Indeed, by (2.1), it is sufficient to show that, for any \( \xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\} \),
\[
H(\xi_1 + \xi_2) \leq H(\xi_1) + H(\xi_2).
\]

By the convexity of the level sets, we have
\[
H\left(\xi_1 \over H(\xi_1) + H(\xi_2) + \xi_2 \over H(\xi_1) + H(\xi_2)\right) =
\[
H\left(\frac{H(\xi_1)}{H(\xi_1) + H(\xi_2)} \xi_1 \over H(\xi_1) + H(\xi_2) \right) + \frac{H(\xi_2)}{H(\xi_1) + H(\xi_2)} \xi_2 \over H(\xi_1) + H(\xi_2) \right) \leq 1,
\]
and by (2.1) we get (2.3).

We define the polar function \( H^o : \mathbb{R}^n \to [0, +\infty[ \) of \( H \) as
\[
H^o(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{H(\xi)}.\]

It is easy to verify that also \( H^o \) is a convex function which satisfies properties (2.1) and (2.2). Furthermore,
\[
H(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{H^o(\xi)}.
\]

The set
\[
\mathcal{W} = \{ \xi \in \mathbb{R}^n : H^o(\xi) < 1 \}
\]
is the so-called Wulff shape centered at the origin. We put \( \kappa_n = |\mathcal{W}| \), where \( |\mathcal{W}| \) denotes the Lebesgue measure of \( \mathcal{W} \). More generally, we denote with \( \mathcal{W}_r(x_0) \) the set \( r\mathcal{W} + x_0 \), that is the Wulff shape centered at \( x_0 \) with measure \( \kappa_n r^n \), and \( \mathcal{W}_r(0) = \mathcal{W}_r \).

The following properties of \( H \) and \( H^o \) hold true (see for example [3]):
\[
H^o(\xi) \cdot \xi = H(\xi), \quad H^o_\xi(\xi) \cdot \xi = H^o(\xi),
\]
\[
H\left( H^o_\xi(\xi) \right) = H^o(\xi), \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},
\]
\[
H^o(\xi) H^o_\xi(\xi) = H(\xi) H^o_\xi(\xi) = \xi, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.
\]

**Definition 2.1.** (Anisotropic area functional and perimeter (2.10)). Let \( M \) be an oriented \((n-1)\)-dimensional hypersurface in \( \mathbb{R}^n \). The anisotropic area functional of \( M \) is
\[
\sigma_H(M) := \int_M H(\nu) \, d\sigma,
\]
where \( \nu \) denotes the outer normal to \( M \) and \( \sigma \) is the \((n-1)\)-dimensional Hausdorff measure.

The anisotropic area of a set \( M \) is finite if and only if the usual Euclidean hypersurface area \( \sigma(M) \) is finite. Indeed, by property (2.2) we have that
\[
\alpha \sigma(M) \leq \sigma_H(M) \leq \gamma \sigma(M),
\]
An isoperimetric inequality for the anisotropic area holds, namely for $K \subset M$ open set of $\mathbb{R}^n$ with Lipschitz boundary,
\begin{equation}
\sigma_H(\partial K) \geq \kappa \frac{1}{n} |K|^{1-\frac{1}{n}},
\end{equation}
and the equality holds if and only if $K$ is homothetic to a Wulff shape (see for example [10], [13], [26], [1]). We stress that in [21] an isoperimetric inequality for the anisotropic relative perimeter in the plane is studied.

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, and $d_H(x)$ the anisotropic distance of a point $x \in \Omega$ to the boundary $\partial \Omega$, that is
\begin{equation}
d_H(x) = \inf_{y \in \partial \Omega} H^\nu(x - y).
\end{equation}
By the property [26], the distance function $d_H(x)$ satisfies
\begin{equation}
H(Dd_H(x)) = 1.
\end{equation}
Finally, we recall that when $\Omega$ is convex $d_H(x)$ is concave. In a natural way, the anisotropic inradius of a convex, bounded open set $\Omega$ is the value
\begin{equation}
R_{H,\Omega} = \sup \{d_H(x), x \in \Omega\}
\end{equation}
For further properties of the anisotropic distance function we refer the reader to [13].

3. The first eigenvalue problem

In this section we prove some properties of the minimizers of (1.1), which are the weak solutions of the following Robin boundary value problem:
\begin{equation}
\begin{cases}
-\text{div} (F_p(Du)) = \lambda_1(\Omega)|u|^{p-2}u & \text{in } \Omega, \\
F_p(Du) \cdot \nu + \beta H(\nu)|u|^{p-2}u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}
where
\begin{equation}
F_p(Du) := [H(Du)]^{p-1}H_x(Du).
\end{equation}
For weak solution of problem (3.1) we mean a function $u \in W^{1,p}(\Omega)$ such that
\begin{equation}
\int_{\Omega} F_p(Du) \cdot D\psi \, dx + \beta \int_{\partial \Omega} u^{p-1}\psi H(\nu) \, d\sigma = \lambda_1(\Omega) \int_{\Omega} |u|^{p-2}u \psi \, dx, \quad \psi \in W^{1,p}(\Omega).
\end{equation}
Obviously, $\lambda_1(\Omega)$ in (1.1) (and then in (3.1)) depends also on $\beta$. In general, we will consider $\beta > 0$ fixed. Anyway, when it will be necessary, to emphasize the dependence on $\beta$ we will denote the first eigenvalue of (3.1) with $\lambda_1(\Omega, \beta)$.

For the Euclidean case we refer to [31], where the eigenvalue problem for the $p$-Laplacian under several boundary conditions is considered.

From now on, we assume that $H$ is a convex function as in Section 2 assuming also that it verifies the following hypothesis:
\begin{equation}
H \in C^2(\mathbb{R}^n \setminus \{0\}), \quad \text{with} \quad \sum_{i,j=1}^n \frac{\partial}{\partial \xi_j} ((H(\eta)|\xi|^p-1 H_{\xi_i}(\eta)) \xi_i \xi_j \geq \gamma |\eta|^{p-2}|\xi|^2,
\end{equation}
for some positive constant $\gamma$, for any $\eta \in \mathbb{R}^n \setminus \{0\}$ and for any $\xi \in \mathbb{R}^n$.

**Theorem 3.1.** There exists a function $u_p \in C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$ which realizes the minimum in (1.1), and satisfies the problem (3.1). Moreover, $\lambda_1(\Omega)$ is the first eigenvalue of (3.1), and the first eigenfunctions are positive (or negative) in $\Omega$. 
Proof. The proof makes use of standard arguments. We briefly recall the main steps. The direct method of the Calculus of Variations guarantees that the infimum in (1.1) is attained at a function $u_p \in W^{1,p}(\Omega)$. We may assume that $u_p \geq 0$, being also $|u_p|$ a minimizer in (1.1). Moreover, the function $u_p$ is a weak solution of (3.1). In order to obtain that $u_p \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$, we first claim that a $L^\infty$-estimate for $u_p$ holds. To get the claim, we take $\varphi = [T_M(u_p)]^{k+1}$ as test function, with $k, M$ positive numbers, and $T_M(s) = \min\{s, M\}$, $s \geq 0$. Using (2.4) and (2.2), we easily get
\[
\alpha (kp + 1)\int_{u_p \leq M} |D u_p|^p u_p^k \, dx \leq \int_\Omega F_p(D u_p) \cdot D \varphi \, dx + \beta \int_{\partial \Omega} u_p^{p-1} \varphi H(\nu) \, d\sigma \leq \lambda_1(\Omega) \int_\Omega u_p^{p(k+1)} \, dx,
\]
and then
\[
\int_\Omega |D T_M(u_p)|^{k+1} \, dx + \int_\Omega |T_M(u_p)|^{p(k+1)} \, dx \leq \left( \frac{(k+1)^p}{\alpha} \lambda_1(\Omega) + 1 \right) \int_\Omega u_p^{p(k+1)} \, dx.
\]
Applying the Sobolev inequality and the Fatou lemma, we get that
\[
\|u_p\|_{(k+1)p^*} \leq S^{\frac{1}{k+1}} \left( \frac{(k+1)^p}{kp + 1} \frac{\lambda_1(\Omega)}{\alpha} + 1 \right)^{\frac{1}{k+1}} \|u_p\|_{(k+1)p^*},
\]
where $S$ is the Sobolev constant. Using the standard Moser iteration technique for the $L^p$-norms, we get the claim. For sake of completeness, we give the complete proof (see also [27]).

First of all, we have that there exists a constant $c$ independent of $k$ such that
\[
\left( \frac{(k+1)^p}{kp + 1} \frac{\lambda_1(\Omega)}{\alpha} + 1 \right)^{\frac{1}{k+1}} \leq c.
\]
Then,
\[
\|u_p\|_{(k+1)p^*} \leq S^{\frac{1}{k+1}} c^{\frac{1}{k+1}} \|u_p\|_{(k+1)p^*}.
\]
Choosing $k_n$ in (3.4) such that $(k_1 + 1)p = p^*$, and $k_n$, $n \geq 2$, such that $(k_n + 1)p = (k_{n-1} + 1)p^*$, by induction we obtain
\[
\|u_p\|_{(k_n+1)p^*} \leq S^{\frac{1}{k_n+1}} c^{\frac{1}{k_n+1}} \|u_p\|_{(k_{n-1}+1)p^*}.
\]
Hence, using iteratively the above inequality, we get
\[
\|u_p\|_{(k_n+1)p^*} \leq S^{\sum_{i=1}^{n} \frac{1}{k_i+1}} c^{\sum_{i=1}^{n} \frac{1}{k_i+1}} \|u_p\|_{p^*}.
\]
Being $k_n + 1 = (p^*/p)^n$, and $p^*/p > 1$, it follows that for any $n \geq 1$
\[
(3.5) \quad \|u_p\|_{(k_n+1)p^*} \leq C \|u\|_{p^*},
\]
as $r_n = (k_n + 1)p^* \to +\infty$ as $n \to +\infty$. The estimates in (3.5) imply that $u \in L^\infty(\Omega)$. Indeed, if by contradiction the exist $\varepsilon > 0$ and $A \subset \Omega$ with positive measure such that $|u| > C \|u\|_{p^*} + \varepsilon = K$ in $A$, we have
\[
\liminf_n \|u_n\|_{p^*} \geq \liminf_n \left( \int_A K^{r_n} \right)^{\frac{1}{r_n}} = K > C \|u\|_{p^*},
\]
which is in contrast with (3.5).

Now the $L^\infty$-estimate, the hypothesis (3.3) and the properties of $H$ allow to apply standard regularity results (see [23], [33]), in order to obtain that $u \in C^{1,\alpha}(\Omega)$. As matter of fact, as observed in [9] it is possible to follow the argument in [30] pages 466-467 to get the
continuity of $u_p$ up to the boundary. Finally, $u_p$ is strictly positive in $\Omega$ by the Harnack inequality (see [34]).

**Theorem 3.2.** The first eigenvalue $\lambda_1(\Omega)$ of (3.1) is simple, that is the relative eigenfunctions are unique up to a multiplicative constant.

**Proof.** We follow the idea of [4, 5]. Let $v, w$ two positive minimizers of (1.1) in $\Omega$ such that $\|v\|_p = \|w\|_p = 1$, and consider $\eta_t = (tv^p + (1-t)w^p)^{1/p}$, with $t \in [0, 1]$. Obviously, $\|\eta_t\|_p = 1$. Moreover, using the homogeneity and the convexity of Theorem 3.2.

The first eigenvalue inequality (see [34]).

(3.6)

$$
[H(D\eta_t)]^p = \eta_t^p \left[ H \left( t \left( \frac{v}{\eta_t} \right)^p \frac{Dv}{v} + (1-t) \left( \frac{w}{\eta_t} \right)^p \frac{Dw}{w} \right) \right]^p
$$

$$
\leq \eta_t^p \left[ s(x) H \left( \frac{Dv}{v} \right) + (1-s(x)) H \left( \frac{Dw}{w} \right) \right]^p
$$

$$
\leq t v^p \left[ H \left( \frac{Dv}{v} \right) \right]^p + (1-t) w^p \left[ H \left( \frac{Dw}{w} \right) \right]^p
$$

$$
= t[H(Dv)]^p + (1-t)[H(Dw)]^p.
$$

Hence, recalling (1.2), the inequalities in (3.6) and the definition of $\eta_t$ give that

$$
J(\eta_t) \leq tJ(v) + (1-t)J(w) = \lambda_1(\Omega),
$$

and then $\eta_t$ is a minimizer for $J$. This implies that the inequalities in (3.6) become equalities. The equality between the third and the fourth row of (3.6) holds if and only if $H(Dv/v) = H(Dw/w)$. Hence, the strict convexity of the level sets of $H$ guarantees from the equalities in (3.6) that $Dv/v = Dw/w$ in $\Omega$, that is $v/w$ is constant. The norm constraint on $v$ and $w$ implies the uniqueness, and this concludes the proof.

**Remark 3.1.** We stress that the nonnegative solution $u_p \in C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$ of (3.1) cannot be identically zero on $\partial\Omega$. Indeed, in such a case, taking $v = 1$ as test function in (3.2), we obtain

$$
\int_{\Omega} u_p^{p-1} dx = 0,
$$

contradicting the positivity of $u_p$ in $\Omega$. As a matter of fact, if we suppose $\partial\Omega$ to be a connected $C^2$ manifold, then the Hopf boundary point Lemma holds (see [14]), which implies that $u$ cannot vanish on $\partial\Omega$.

**Theorem 3.3.** Any nonnegative function $v \in W^{1,p}(\Omega)$, $v \neq 0$, which satisfies, in the sense of (3.2),

(3.7)

$$
\begin{cases}
-\text{div}(F_p(Dv)) = \lambda v^{p-1} & \text{in } \Omega, \\
F_p(Dv) \cdot \nu + \beta H(\nu)v^{p-1} = 0 & \text{on } \partial\Omega.
\end{cases}
$$

is a first eigenfunction of (3.1), that is $\lambda = \lambda_1(\Omega)$ and $v = u_p$, where $u_p$ is given in Theorem 3.1. up to multiplicative constant.

For analogous results in the Dirichlet case, see for example [28] and the references therein.

**Proof of Theorem 3.3.** The same arguments of Theorem 3.1 allow to prove that the given nonnegative solution $v$ of (3.7) is in $C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$ and it is positive in $\Omega$. Moreover, the function $u_p \in C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$ satisfies

(3.8)

$$
\int_{\Omega} [H(Du_p)]^p dx + \beta \int_{\partial\Omega} u_p^p H(\nu) d\sigma = \lambda_1(\Omega) \int_{\Omega} u_p^p dx,
$$

is a first eigenfunction of (3.7), that is $\lambda = \lambda_1(\Omega)$ and $v = u_p$, where $u_p$ is given in Theorem 3.1. up to multiplicative constant.
while, choosing $u_p^p/(v + \varepsilon)^{p-1}$, with $\varepsilon > 0$, as test function for $v$, we get

\begin{equation}
\int_\Omega \left[H\left(\frac{u_p}{v + \varepsilon}Dv\right)\right]^{p-1}H_\varepsilon(Dv) \cdot Du_p\,dx - (p-1)\int_\Omega \left[H\left(\frac{u_p}{v + \varepsilon}Dv\right)\right]^p\,dx + 
+ \beta \int_{\partial\Omega} \frac{v^{p-1}}{(v + \varepsilon)^{p-1}} u_p^p H(\nu)\,d\sigma = \lambda \int_\Omega \frac{v^{p-1}}{(v + \varepsilon)^{p-1}} u_p^p\,dx.
\end{equation}

Subtracting (3.9) by (3.8), being $H_\varepsilon$ zero homogeneous, and observing that $v/(v + \varepsilon) \leq 1$, we get

\[
\int_\Omega \left\{[H(Du_p)]^p - F_p\left(\frac{u_p}{v + \varepsilon}Dv\right) \cdot Du_p + (p-1)\left[H\left(\frac{u_p}{v + \varepsilon}Dv\right)\right]^p\right\}\,dx \leq \int_\Omega \left[\lambda_1(\Omega) - \frac{v^{p-1}}{(v + \varepsilon)^{p-1}}\lambda\right] u_p^p\,dx.
\]

The convexity of $H^p$ guarantees that the left-hand side in the above inequality is nonnegative. Hence, as $\varepsilon \to 0$, the monotone convergence gives that

\[(\lambda_1(\Omega) - \lambda) \int_\Omega u_p^p\,dx \geq 0,
\]

and this can hold if and only if $\lambda \leq \lambda_1(\Omega)$. Being $\lambda_1(\Omega)$ the smallest possible eigenvalue, necessarily we have that $\lambda = \lambda_1(\Omega)$. The uniqueness of the first eigenfunction implies that, up to some positive multiplicative constant, $v = u_p$. \qed

In order to show a lower bound for $\lambda_1(\Omega)$ when $\Omega$ is a convex set of $\mathbb{R}^n$ in terms of the anisotropic inradius of $\Omega$, we need an Hardy-type inequality for functions which, in general, do not vanish on the boundary. To this aim, we impose further regularity on $H$. More precisely, we assume also that

\begin{equation}
\partial W = \{x: H^0(x) = 1\}
\end{equation}

has positive Gaussian curvature in any point.

If $\Omega$ is $C^2$, this assumption ensures that the anisotropic distance from the boundary of $\Omega$ is $C^2$ in a tubular neighborhood of $\partial \Omega$ (see for instance [13]).

**Lemma 3.1.** Let $\Omega$ be a bounded convex open set of $\mathbb{R}^n$ with $C^2$ boundary and suppose that $H^0$ satisfies also (3.10). Then, for any $\alpha > 0$ and $\vartheta > 0$, the following Hardy-type inequality holds:

\begin{equation}
\int_\Omega [H(Du)]^p\,dx + \vartheta^{p-1} \int_{\partial\Omega} |u|^p H(\nu)\,d\sigma \geq (p-1)(\alpha \vartheta)^{p-1}(1 - \alpha \vartheta) \int_\Omega \frac{|u|^p}{(d_H + \alpha)^p}\,dx,
\end{equation}

where $u \in W^{1,p}(\Omega)$ and $d_H$ is the anisotropic distance from the boundary of $\Omega$, defined in (2.3).

**Proof.** It is sufficient to prove the thesis for $u \geq 0$. Moreover, using an approximation argument, we can suppose that $u \in C^1(\Omega)$. For $\delta$ positive, let us define $H_\delta(\xi) = H^\delta(\xi) + \delta$, where $H^\delta$ is the $\delta$-mollification of $H$. By the convexity of $H(\xi)$, the function $H^\delta$ is convex and we have, for any $\xi_1, \xi_2 \in \mathbb{R}^n$,

\[
[H_\delta(\xi_1)]^p \geq [H_\delta(\xi_2)]^p + p[H_\delta(\xi_2)]^{p-1}(H_\delta)_{\xi_2}(\xi_2) \cdot (\xi_1 - \xi_2).
\]

We apply the above inequality to $\xi_1 = Du$ and $\xi_2 = \frac{\alpha \vartheta u}{d_H + \alpha} \cdot d^e$, where $\alpha > 0$, $\vartheta > 0$, and $d^e$ is the $\varepsilon$-mollification of $d_H$. The convexity of $\Omega$ gives that the function $d_H$, and then $d^e$, are
concave functions. We have:

\begin{equation}
(3.12) \quad \int_{\Omega} [H_{\delta}(Du)]^p \, dx \geq (\alpha \vartheta)^p \int_{\Omega} \frac{u^p}{(d^c + \alpha)^p} [H_{\delta}(Dd^c)]^p \, dx + p(\alpha \vartheta)^{p-1} \int_{\Omega} \frac{u^{p-1}}{(d^c + \alpha)^{p-1}} [H_{\delta}(Dd^c)]^{p-1}(H_{\delta})\xi(Dd^c) \cdot Du \, dx + \left(\frac{p}{p-1}\right)(\alpha \vartheta)^p \int_{\Omega} \frac{u^p}{(d^c + \alpha)^p} [H_{\delta}(Dd^c)]^{p-1}(H_{\delta})\xi(Dd^c) \cdot Dd^c \, dx
\end{equation}

Passing to the limit as \( \delta \to 0 \) and using (2.5), the sum of the first and the third terms in the right-hand side of (3.12) converge to

\[-(p-1)(\alpha \vartheta)^p \int_{\Omega} \frac{u^p}{(d^c + \alpha)^p} [H(Dd^c)]^p \, dx.\]

Moreover, by the divergence theorem we have that

\begin{equation}
(3.13) \quad p \int_{\Omega} \frac{u^{p-1}}{(d^c + \alpha)^{p-1}} [H_{\delta}(Dd^c)]^{p-1}(H_{\delta})\xi(Dd^c) \cdot Du \, dx = \frac{1}{p} \int_{\Omega} \frac{1}{(d^c + \alpha)^{p-1}} (H_{\delta})^p \xi(Dd^c) \cdot D(u^p) \, dx = \frac{1}{p} \int_{\partial \Omega} (d^c + \alpha)^{p-1} (H_{\delta})^p \xi(Dd^c) \cdot Dd^c \, d\sigma + \frac{1}{p} \int_{\Omega} \frac{1}{(d^c + \alpha)^{p-1}} \text{div}((H_{\delta})^p \xi(Dd^c)) \, dx + \frac{p-1}{p} \int_{\partial \Omega} (d^c + \alpha)^{p-1} (H_{\delta})^p \xi(Dd^c) \cdot Dd^c \, d\sigma \geq \frac{1}{p} \int_{\partial \Omega} (d^c + \alpha)^{p-1} (H_{\delta})^p \xi(Dd^c) \cdot Dd^c \, d\sigma + \frac{p-1}{p} \int_{\Omega} (d^c + \alpha)^p (H_{\delta})^p \xi(Dd^c) \cdot Dd^c \, dx.
\end{equation}

Last inequality follows from the fact that \( \text{div}((H_{\delta})^p \xi(Dd^c)) \) is nonnegative. Indeed, it is the trace of the product of the matrices \( [H_{\delta}^p \xi(Dd^c)] \) and \( [-D^2d^c] \), which are both positive semidefinite, being \( H_{\delta}^p \) convex and \( d^c \) concave.

Passing to the limit as \( \delta \to 0 \) in (3.13), and using (2.5), we get

\begin{equation}
p \int_{\Omega} \frac{u^{p-1}}{(d^c + \alpha)^{p-1}} [H(Dd^c)]^{p-1}(H)\xi(Dd^c) \cdot Du \, dx \geq \frac{1}{p} \int_{\partial \Omega} (d^c + \alpha)^{p-1} (H)^p \xi(Dd^c) \cdot Dd^c \, d\sigma + (p-1) \int_{\Omega} \frac{u^p}{(d^c + \alpha)^p} [H(Dd^c)]^p \, dx
\end{equation}

Then, as \( \delta \to 0 \) in (3.12), the above computations gives that

\begin{equation}
(3.14) \quad \int_{\Omega} [H(Du)]^p \, dx - \frac{(\alpha \vartheta)^{p-1}}{p} \int_{\partial \Omega} u^p (H)^p \xi \left( \frac{Dd^c}{d^c + \alpha} \right) \cdot Dd^c \, d\sigma \geq (p-1)(\alpha \vartheta)^{p-1}(1 - \alpha \vartheta) \int_{\Omega} \frac{u^p}{(d^c + \alpha)^p} [H(Dd^c)]^p \, dx.
\end{equation}

Now we pass to the limit for \( \epsilon \to 0 \). Recalling that under our assumptions \( d_H \) is \( C^2 \) in a tubular neighborhood of \( \partial \Omega \), by uniform convergence we get

\begin{equation}
(3.15) \quad \int_{\Omega} [H(Du)]^p \, dx - (\alpha \vartheta)^{p-1} \int_{\partial \Omega} \frac{u^p}{(d_H + \alpha)^p} [H(Dd_H)]^{p-1} H(\xi(Dd_H) \cdot Dd_H \, d\sigma \geq (p-1)(\alpha \vartheta)^{p-1}(1 - \alpha \vartheta) \int_{\Omega} \frac{u^p}{(d_H + \alpha)^p} [H(Dd_H)]^p \, dx.
\end{equation}
Being $H(DdH) = 1$ a.e. in $\Omega$, and $dH = 0$ on $\partial \Omega$, choosing $\vartheta^{p-1} = \beta$, and recalling that $\nu = -DdH/|DdH|$ on $\partial \Omega$, by (2.1) and (2.5) we get the thesis. \hfill \Box

An immediate application of the previous Lemma is the following result.

**Proposition 3.1.** If $\Omega$ is a convex set of $\mathbb{R}^n$ with $C^2$ boundary and if $H$ satisfies also (3.11), then

$$\lambda_1(\Omega) \geq \left( \frac{p-1}{p} \right)^p \frac{\beta}{R_{H,\Omega} \left( 1 + \beta \frac{1}{p-1} R_{H,\Omega} \right)^{p-1}},$$

where $R_{H,\Omega}$ is the anisotropic inradius of $\Omega$, as defined in (2.11).

**Proof.** Let $\beta = \vartheta^{p-1}$. Then, by (3.11) and the definitions of $\lambda_1(\Omega)$ and of the anisotropic inradius $R_{H,\Omega}$ we get that

$$\lambda_1(\Omega) \geq \frac{(p-1)\beta}{(R_{H,\Omega} + \alpha)^p (1 - \beta \frac{1}{p-1} \alpha)^{p-1}}.$$

Then, maximizing the right-hand side of the above inequality we obtain that

$$\lambda_1(\Omega) \geq \left( \frac{p-1}{p} \right)^p \frac{\beta}{R_{H,\Omega} \left( 1 + \beta \frac{1}{p-1} R_{H,\Omega} \right)^{p-1}}.$$

\hfill \Box

**Remark 3.2.** As a consequence of the previous Proposition, we have that

$$\sup_{|\Omega|=m} \lambda_1(\Omega) = +\infty,$$

among all the Lipschitz domains with given measure $m > 0$.

Finally, we have the following scaling property.

**Proposition 3.2.** For any $t > 0$, we have that $\lambda_1(t\Omega, \beta) = t^{-p} \lambda_1(\Omega, t^{p-1} \beta)$.

**Proof.** By the homogeneity of $H$, we have:

$$\lambda_1(t\Omega, \beta) = \min_{v \in W^{1,p}(t\Omega) \setminus \{0\}} \frac{\int_{t\Omega} [H(Dv(x))]^p dx + \beta \int_{\partial(t\Omega)} |v(x)|^p H(\nu(x))d\sigma(x)}{\int_{t\Omega} |v(x)|^p dx} = \min_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{t^{n-p} \int_{\Omega} [H(Du(y))]^p dy + t^{n-1} \beta \int_{\partial\Omega} |u(y)|^p H(\nu(y))d\sigma(y)}{t^n \int_{\Omega} |u(y)|^p dy} = t^{-p} \lambda_1(\Omega, t^{p-1} \beta).$$

\hfill \Box

### 4. The eigenvalue problem in the anisotropic radial case

In this section we study the properties of the minimizers of (1.1) when $\Omega$ is homothetic to the Wulff shape, that is, for $R > 0$, the functions $v_p$ such that

$$J(v_p) = \min_{u \in W^{1,p}(W_R) \setminus \{0\}} \frac{\int_{W_R} [H(Du)]^p dx + \beta \int_{\partial W_R} |u|^p H(\nu)d\sigma}{\int_{W_R} |u|^p dx},$$

(4.1)
where $W_R = RW = \{ x : H^o(x) < R \}$, with $R > 0$, and $W$ is the Wulff shape centered at the origin. By Theorem 3.1 such functions solve the following problem:

\begin{equation}
\begin{aligned}
- \text{div}(F_p(Dv)) &= \lambda_1(W_R)|v|^{p-2}v \quad \text{in } W_R, \\
F_p(Dv) \cdot \nu + \beta H(\nu)|v|^{p-2}v &= 0 \quad \text{on } \partial W_R.
\end{aligned}
\end{equation}

**Theorem 4.1.** Let $v_p \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ be a positive solution of problem (4.2). Then, there exists a decreasing function $\varrho_p = \varrho_p(r)$, $r \in [0, R]$, such that $\varrho_p \in C^\infty(0, R) \cap C^1([0, R])$, and

\begin{equation}
\begin{aligned}
v_p(x) &= \varrho_p(H^o(x)), \quad x \in W_R, \\
\varrho'_p(0) &= 0, \\
-(-\varrho'_p(R))^{p-1} + \beta(\varrho_p(R))^{p-1} &= 0.
\end{aligned}
\end{equation}

**Proof.** Let $B_R$ be the Euclidean ball centered at the origin, $B_R = \{ x \in \mathbb{R}^n : |x| < R \}$, and consider the $p$-Laplace eigenvalue problem in $B_R$, that is (4.2) with $H(\xi) = |\xi|$

\begin{equation}
\begin{aligned}
-\Delta_p w &= \lambda_{1,\xi}(B_R)|w|^{p-2}w \quad \text{in } B_R, \\
|Dw|^{p-2}\frac{\partial w}{\partial \nu} + \beta|w|^{p-2}w &= 0 \quad \text{on } \partial B_R,
\end{aligned}
\end{equation}

where $\lambda_{1,\xi}(B_R)$ denotes the first eigenvalue. It is known (see, for example, [16]) that problem (4.3) admits a positive radially decreasing solution $w_p(x) = \varrho_p(|x|)$, $0 \leq |x| \leq R$, such that $\varrho_p \in C^\infty(0, R) \cap C^1([0, R])$ and verifies

\begin{equation}
\begin{aligned}
-\varrho''_p(r)(p-1)(-\varrho'_p(r))^{p-2}\varrho'_p(r) + \frac{n-1}{r}(-\varrho'_p(r))^{p-1} &= \lambda_{1,\xi}(B_R)\varrho_p(r)^{p-1}, \quad r \in [0, R], \\
\varrho'_p(0) &= 0, \\
-(-\varrho'_p(R))^{p-1} + \beta\varrho_p(R)^{p-1} &= 0.
\end{aligned}
\end{equation}

Let $v_p(x) = \varrho_p(H^o(x))$, $x \in W_R$. Using properties (2.5) - (2.7), for $x \in W_R \setminus \{0\}$ we have that

\begin{equation}
H(Dv_p(x)) = -\varrho'_p(H^o(x))H(DH^o(x)) = -\varrho'_p(H^o(x)),
\end{equation}

and

\begin{equation}
DH(Dv_p(x)) = -DH(DH^o(x)) = -\frac{x}{H^o(x)},
\end{equation}

which imply that

\begin{equation}
F_p(Dv_p) = -(\varrho'(H^o(x)))^{p-1}\frac{x}{H^o(x)},
\end{equation}

and then, by (4.3),

\begin{equation}
-\text{div}(F_p(Dv_p)) = -(p-1)(-\varrho'_p(H^o(x)))^{p-2}\varrho''_p(H^o(x)) + \frac{n-1}{H^o(x)}(-\varrho'_p(H^o(x)))^{p-1} = \lambda_{1,\xi}(B_R)v_p(x)^{p-1} \quad \text{for } x \in W_R \setminus \{0\}.
\end{equation}

As regards the boundary condition, observing that $\nu(x) = DH^o(x)/|DH^o(x)|$, by (4.5), the properties (2.5), (2.6), and (4.3) we have that

\begin{equation}
F_p(v_p(x)) \cdot \nu(x) + \beta H(\nu(x))v_p(x)^{p-1} = \frac{1}{|DH^o(x)|} \left( -(-\varrho'(R))^{p-1} + \beta\varrho_p(R)^{p-1} \right) = 0 \quad \text{for } x \in \partial W_R.
\end{equation}

Hence, integrating (4.6) on $W_R \setminus W_{\epsilon}$, we can use the divergence theorem and the boundary condition (4.7), and let $\epsilon$ going to 0, obtaining that $v_p$ verifies

\begin{equation}
\begin{aligned}
- \text{div}(F_p(Dv_p)) &= \lambda_{1,\xi}(B_R)v_p^{p-1} \quad \text{in } W_R, \\
F_p(Dv) \cdot \nu + \beta H(\nu)v_p^{p-1} &= 0 \quad \text{on } \partial W_R.
\end{aligned}
\end{equation}
But Theorem 3.3 guarantees that a positive solution of (4.8) has to be a first eigenfunction, and
\[ \lambda_1(W_R) = \lambda_{1,E}(B_R). \]
This concludes the proof. \(\square\)

Remark 4.1. We observe that the proof of the above theorem shows that, for any convex function \(H\) we can consider, the first eigenvalue in the ball \(W_R = \{H^\alpha(x) < R\}\) is the same, and coincides with the first eigenvalue for the \(p\)-Laplacian problem (4.3) in the Euclidean ball \(B_R\) (with the same \(R\)).

Next two lemmata will be useful in the proof of the main result. Their proofs are analogous to the ones obtained in [9]. For the sake of completeness, we write them in details.

Lemma 4.1. If \(0 < r < s\), then \(\lambda_1(W_r) > \lambda_1(W_s)\).

Proof. Let \(v_p\) a minimizer of (4.1), with \(R = r\), and take \(w(x) = v_p(\frac{r}{s}x)\), \(x \in W_s\). Then, by the homogeneity of \(H\) we get
\[
\lambda_1(W_s) \leq \frac{\int_{W_s} [H(Dw)]^p dx + \beta \int_{\partial W_s} |w|^p H(\nu) d\sigma}{\int_{W_s} |w|^p dx} = \frac{\left(\frac{r}{s}\right)^p \int_{W_r} [H(Dv_p)]^p dx + \beta \frac{r}{s} \int_{\partial W_r} |v_p|^p H(\nu) d\sigma}{\int_{W_r} |v_p|^p dx} < \frac{\int_{W_r} [H(Dv_p)]^p dx + \beta \int_{\partial W_r} |v_p|^p H(\nu) d\sigma}{\int_{W_r} |v_p|^p dx} = \lambda_1(W_r)
\]
We stress that by (4.4), if \(v_p(x) = \rho_p(H^\alpha(x))\) is the positive solution in \(W_R\) we found in Theorem 4.1 we have that, for \(x \in \partial W_R\),
\[
\beta = \frac{[H(Dv_p(x))]^{p-1}}{v_p(x)^{p-1}}.
\]
Then, for every \(0 \leq r \leq R\), we define
\[
\beta_r = \frac{[H(Dv_p(x))]^{p-1}}{v_p(x)^{p-1}}, \quad \text{for } H^\alpha(x) = r.
\]
Let us observe that \(\beta_0 = 0\) and \(\beta_R = \beta\).

Lemma 4.2. If \(0 \leq r < s \leq R\), then \(\beta_r < \beta_s\).

Proof. We first observe that, similarly as in the proof of Theorem 4.1 for \(0 < r < R\), the function \(v_p\) is such that
\[
\begin{cases}
- \text{div}(F_p(Dv_p)) = \lambda_1(W_R)v_p^{p-1} & \text{in } W_r, \\
F_p(Dv_p) \cdot \nu + \beta_r H(\nu) v_p^{p-1} = 0 & \text{on } \partial W_r.
\end{cases}
\]
Then, denoted by $\lambda_1(W_r, \beta_r)$ the first eigenvalue in $W_r$ with $\beta = \beta_r$, by Theorem 3.3 we have necessarily $\lambda_1(W_R) = \lambda_1(W_r, \beta_r)$ for all $r \in [0, R]$. Hence, by Lemma 4.1 we obtain, for $0 < r < s \leq R$, that

$$
\int_{W_r} [H(Dv)]^p dx + \beta_r \int_{\partial W_r} v_p H(\nu) d\sigma = \lambda_1(W_r, \beta_r) = \lambda_1(W_s, \beta_s) \leq \int_{W_r} [H(Dv)]^p dx + \beta_s \int_{\partial W_r} v_p H(\nu) d\sigma,
$$

and then $\beta_r < \beta_s$. $\square$

5. A REPRESENTATION FORMULA FOR $\lambda_1(\Omega)$

Now we prove a level set representation formula for the first eigenvalue $\lambda_1(\Omega)$. To this aim, we will use the following notation. Let $\tilde{u}_p$ be the first positive eigenfunction such that $\max \tilde{u}_p = 1$. Then, for $t \in [0, 1]$,

$$
U_t = \{x \in \Omega: \tilde{u}_p > t\}, \\
S_t = \{x \in \Omega: \tilde{u}_p = t\}, \\
\Gamma_t = \{x \in \partial \Omega: \tilde{u}_p > t\}.
$$

First of all, it is worth to observe that the anisotropic areas of the sets $\partial U_t$, $S_t$ and $\Gamma_t$, defined in 2.1, are related in the following way.

**Lemma 5.1.** There exists a countable set $Q \subset [0, 1]$ such that

$$
(5.1) \quad \sigma_H(\partial U_t) \leq \sigma_H(\Gamma_t) + \sigma_H(S_t), \quad \forall t \in [0, 1] \setminus Q.
$$

**Proof.** The proof follows similarly as in [9]. The continuity up to the boundary of the eigenfunction $\tilde{u}_p$, given in Theorem 3.1 guarantees that

$$
\partial U_t \cap \Omega \subseteq S_t, \quad \partial U_t \cap \partial \Omega \subseteq \tilde{\Gamma}_t
$$

for any $t \in [0, 1]$, where $\tilde{\Gamma}_t = \{x \in \partial \Omega: \tilde{u}_p \geq t\}$. Moreover, by [32] Section 1.2.3 we have that

$$
\int_0^\infty \sigma_H(\Gamma_t) dt = \int_0^\infty \sigma_H(\tilde{\Gamma}_t) dt = \int_{\partial \Omega} \tilde{u}_p d\sigma_H \leq \sigma_H(\partial \Omega) < +\infty.
$$

Hence $\sigma_H(\Gamma_t) \leq \sigma_H(\tilde{\Gamma}_t) < +\infty$ and then $\sigma_H(\Gamma_t) = \sigma_H(\tilde{\Gamma}_t)$ for a.e. $t \in [0, 1]$. Moreover, being $\sigma_H(\Gamma_t)$ and $\sigma_H(\tilde{\Gamma}_t)$ monotone decreasing in $t$, they are continuous in $[0, 1]$ up to a countable set $Q$. Hence,

$$
\sigma_H(\partial U_t) = \sigma_H(\partial U_t \cap \Omega) + \sigma_H(\partial U_t \cap \partial \Omega) \leq \sigma_H(S_t) + \sigma_H(\Gamma_t),
$$

for all $t \in [0, 1] \setminus Q$. $\square$
If we formally divide both terms in the equation in (3.1) by \( \tilde{u}_p^{p-1} \), and integrate in \( U_t \), by (2.10) and the boundary condition we get

\[
(5.2) \quad \lambda_1(\Omega)|U_t| = \int_{U_t} - \frac{\text{div} \left( F_p(D\tilde{u}_p) \right)}{\tilde{u}_p^{p-1}} \, dx =
\]
\[
= -(p - 1) \int_{U_t} \frac{|H(D\tilde{u}_p)|^{p-1} H(p) \cdot D\tilde{u}_p}{\tilde{u}_p^{p-1}} \, dx - \int_{\partial U_t} \frac{|H(D\tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}} H(p) \cdot \nu \, d\sigma =
\]
\[
= -(p - 1) \int_{U_t} \frac{|H(D\tilde{u}_p)|^{p}}{\tilde{u}_p^{p-1}} \, dx + \int_{S_t} \frac{|H(D\tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}} H(\nu) \, d\sigma + \beta \int_{\Gamma_t} H(\nu) \, d\sigma =
\]
\[
= |U_t| \mathcal{F}_\Omega \left( U_t, \frac{|H(D\tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}} \right),
\]
where

\[
(5.3) \quad \mathcal{F}_\Omega(U_t, \varphi) = \frac{1}{|U_t|} \left( -(p - 1) \int_{U_t} \varphi^p \, dx + \int_{S_t} \varphi H(\nu) \, d\sigma + \beta \int_{\Gamma_t} H(\nu) \, d\sigma \right),
\]
with \( \varphi \) nonnegative measurable function in \( \Omega \). The formal computations in (5.2) give a representation formula of \( \lambda_1(\Omega) \) which will be rigorously proved in the result below.

**Theorem 5.1.** Let \( \tilde{u}_p \in C^{1,0}(\Omega) \cap C(\overline{\Omega}) \) be the positive minimizer of (1.1) such that \( \max \tilde{u}_p = 1 \). Then, for a.e. \( t \in [0, 1] \),

\[
(5.4) \quad \lambda_1(\Omega) = \mathcal{F}_\Omega \left( U_t, \frac{|H(D\tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}} \right).
\]

**Proof.** Let \( 0 < \varepsilon < t < 1 \), and

\[
\psi_\varepsilon = \begin{cases} 
0 & \text{if } \tilde{u}_p \leq t \\
\frac{u - t}{\varepsilon} \frac{1}{\tilde{u}_p^{p-1}} & \text{if } t < \tilde{u}_p < t + \varepsilon \\
\frac{1}{\tilde{u}_p^{p-1}} & \text{if } \tilde{u}_p \geq t + \varepsilon.
\end{cases}
\]

The functions \( \psi_\varepsilon \) are in \( W^{1,p}(\Omega) \) and increasingly converge to \( \tilde{u}_p^{-(p-1)} \chi_{U_t} \) as \( \varepsilon \searrow 0 \). Moreover,

\[
D\psi_\varepsilon = \begin{cases} 
0 & \text{if } \tilde{u}_p < t \\
\frac{1}{\varepsilon} \left( (p - 1) \frac{t}{\tilde{u}_p} + 2 - p \right) \frac{D\tilde{u}_p}{\tilde{u}_p^{p-1}} & \text{if } t < \tilde{u}_p < t + \varepsilon \\
-(p - 1) \frac{D\tilde{u}_p}{\tilde{u}_p^{p-1}} & \text{if } \tilde{u}_p > t + \varepsilon.
\end{cases}
\]

Then, choosing \( \psi_\varepsilon \) as test function in (3.2), we get that the first integral is

\[
- (p - 1) \int_{U_t + \varepsilon} \frac{|H(D\tilde{u}_p)|^p}{\tilde{u}_p^{p-1}} \, dx + \frac{1}{\varepsilon} \int_{U_t \setminus U_{t+\varepsilon}} \frac{|H(D\tilde{u}_p)|^p}{\tilde{u}_p^{p-1}} \left( (p - 1) \frac{t}{\tilde{u}_p} + 2 - p \right) \, dx =
\]
\[
= -(p - 1) \int_{U_t + \varepsilon} \frac{|H(D\tilde{u}_p)|^p}{\tilde{u}_p^{p-1}} \, dx + \frac{1}{\varepsilon} \int_{t+\varepsilon}^{t} \left( (p - 1) \frac{t}{\tau} + 2 - p \right) \int_{S_t} \frac{|H(D\tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}} H(\nu) \, d\sigma,
\]
where last equality follows by the coarea formula. Then, reasoning similarly as in [9], we get that

\[
\int_{\Omega} |H(D\tilde{u}_p)|^{p-1} H(p, D\tilde{u}_p) \cdot D\psi_\varepsilon \, dx \xrightarrow{\varepsilon \to 0} - (p - 1) \int_{U_t} \frac{|H(D\tilde{u}_p)|^p}{\tilde{u}_p^{p-1}} \, dx + \int_{S_t} \frac{|H(D\tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}} H(\nu) \, d\sigma.
\]
As regards the other two integrals in $(5.2)$, we have that
\[ \beta \int_{\partial \Omega} \tilde{u}_p^{-1} \psi \kappa H(\nu) \, d\sigma = \beta \int_{\Gamma_{t+\varepsilon}} H(\nu) \, d\sigma + \beta \int_{\Gamma_t \setminus \Gamma_{t+\varepsilon}} \frac{u - t}{\varepsilon} H(\nu) \, d\sigma \xrightarrow{\varepsilon \to 0} \beta \int_{\Gamma_t} H(\nu) \, d\sigma, \]
and, by monotone convergence theorem and the definition of $\psi_{\varepsilon}$,
\[ \lambda_1(\Omega) \int_{\Omega} \tilde{u}_p^{-1} \psi_{\varepsilon} \, dx \xrightarrow{\varepsilon \to 0} \lambda_1(\Omega) |U_t|. \]
Summing the three limits, we get $(5.5)$. \qed

**Theorem 5.2.** Let $\varphi$ be a nonnegative function in $\Omega$ such that $\varphi \in L^p(\Omega)$. If $\varphi \neq \frac{[H(D\tilde{u}_p)]^{p-1}}{\tilde{u}_p^{-1}}$, where $\tilde{u}_p$ is the eigenfunction given in Theorem 5.1 and $F_{\Omega}$ is the functional defined in $(5.3)$, then there exists a set $S \subset ]0,1[$ with positive measure such that for every $t \in S$ it holds that
\[ (5.5) \quad \lambda_1(\Omega) > F_{\Omega}(U_t, \varphi). \]

**Proof.** The proof is similar to the one obtained in [9], and we only sketch it here. It can be divided in two main steps. First, we claim that, if
\[ w(x) := \varphi - \frac{[H(D\tilde{u}_p)]^{p-1}}{\tilde{u}_p^{-1}}, \quad I(t) := \int_{U_t} w H(D\tilde{u}_p) \, dx, \]
then $I: ]0,1[ \to \mathbb{R}$ is locally absolutely continuous and
\[ (5.6) \quad F_{\Omega}(U_t, \varphi) \leq \lambda_1(\Omega) - \frac{1}{|U_t|^{p-1}} \left( \frac{d}{dt} t^p I(t) \right), \]
for almost every $t \in ]0,1[$. Second, we show that the derivative in $(5.6)$ is strictly positive in a subset of $]0,1[$ of positive measure.

In order to prove $(5.6)$, writing the representation formula $(5.3)$ in terms of $w$, it follows that, for a.e. $t \in ]0,1[$,
\[ F_{\Omega}(U_t, \varphi) = \lambda_1(\Omega) \left( \int_{S_t} w H(\nu) \, d\sigma - (p - 1) \int_{U_t} \left( \varphi^p - \frac{[H(D\tilde{u}_p)]^{p-1}}{\tilde{u}_p} \varphi \right) \, dx \right) \]
\[ \leq \lambda_1(\Omega) \left( \int_{S_t} w H(\nu) \, d\sigma - p \int_{U_t} \frac{H(D\tilde{u}_p)}{\tilde{u}_p} \, dx \right) \]
\[ = \lambda_1(\Omega) \left( \int_{S_t} w H(\nu) \, d\sigma - p I(t) \right), \]
where the inequality in $(5.7)$ follows from the inequality $\varphi^p \geq \varphi^p + p' \varphi^{p-1} (\varphi - v)$, with $\varphi, v \geq 0$. Applying the coarea formula, it is possible to rewrite $I(t)$ as
\[ I(t) = \int_{U_t} \frac{H(D\tilde{u}_p)}{\tilde{u}_p} \, dx = \int_t^1 \frac{1}{\tau} \int_{S_\tau} w H(\nu) \, d\sigma. \]
This assures that $I(t)$ is locally absolutely continuous in $]0,1[$ and, for almost every $t \in ]0,1[$ we have
\[ -\frac{d}{dt} (t^p I(t)) = t^p \int_{S_t} w H(\nu) \, d\sigma - p I(t). \]
Substituting in $(5.7)$, the inequality $(5.6)$ follows. In order to conclude the proof, arguing by contradiction exactly as in [9] Theorem 3.2, it is possible to show that $G(t) := t^p I(t)$ has positive derivative in a set of positive measure. Together with $(5.6)$, this implies $(5.5)$. \qed
6. Main result

Now we are in position to state and prove the desired Faber-Krahn inequality.

**Theorem 6.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, and $H: \mathbb{R}^n \rightarrow [0, +\infty]$ a function with strictly convex sublevel sets which satisfies (2.1), (2.2), and (3.3). Then,

$$\lambda_1(\Omega) \geq \lambda_1(\mathcal{W}_R),$$

where $\mathcal{W}_R$ is the Wulff shape centered at the origin such that $|\mathcal{W}_R| = |\Omega|$. The equality holds if and only if $\Omega$ is a Wulff shape.

**Proof.** The first step in order to prove the result is to construct a suitable test function in $\Omega$ for (5.3). Let $v_p$ be a positive eigenfunction of the anisotropic radial problem (4.2) in $\mathcal{W}_R$.

By Theorem 4.1, $v_p$ is a function depending only by $H^\sigma(x)$, and then we are able to define, as in (4.9), the function

$$\beta_r = \varphi_\ast(x) = \frac{[H(Dv_p(x))]^{p-1}}{v_p(x)^{p-1}}, \text{ with } x \in \overline{\mathcal{W}_R}, \text{ i.e. } H^\sigma(x) = r \in [0, R].$$

As before, let $\tilde{u}_p$ be the first eigenfunction of (6.1) in $\Omega$ such that $\|\tilde{u}_p\|_{\infty} = 1$. Using the same notation of Section 3 for any $t \in [0, 1]$ we consider $\mathcal{W}_{r(t)}$, the Wulff shape centered at the origin, where $r(t)$ is the positive number such that $|U_t| = |\mathcal{W}_{r(t)}|$. Then, for $x \in \Omega$ and $\tilde{u}_p(x) = t$, we define

$$\varphi(x) := \beta_{r(t)}.$$

Similarly as in [9], $\varphi$ is a measurable function. Thanks to this test function, we can compare $\mathcal{F}_\Omega(U_t, \varphi)$ with $\mathcal{F}_{\mathcal{W}_R}(B_{r(t)}, \varphi_\ast)$. Indeed, we claim that

$$\mathcal{F}_\Omega(U_t, \varphi) \geq \frac{1}{|\mathcal{W}_{r(t)}|} \left( \frac{(p-1)}{2} \int_{\mathcal{W}_{r(t)}} \varphi_{\ast}' dx + \int_{\partial \mathcal{W}_{r(t)}} \varphi_{\ast} H(\nu) d\sigma \right)$$

for all $t \in [0, 1] \setminus Q$, where $Q$ is the set of Lemma 5.1. In order to show (6.2), we first observe that by (3.2) Section 1.2.3, being $|U_t| = |\mathcal{W}_{r(t)}|$ for all $t \in [0, 1]$

$$\int_{U_t} \varphi_{\ast}' dx = \int_{\mathcal{W}_{r(t)}} \varphi_{\ast}' dx.$$

Moreover, the anisotropic isoperimetric inequality (2.8), Lemma 5.1 and being, by Lemma 4.2 $\beta_{r(t)} \leq \beta$ for any $t$, we have that

$$\int_{\partial \mathcal{W}_{r(t)}} \varphi_{\ast} H(\nu) d\sigma = \beta_{r(t)} \sigma_H(\partial \mathcal{W}_{r(t)}) \leq$$

$$\leq \beta_{r(t)} \sigma_H(\partial U_t) \leq \beta_{r(t)} \sigma_H(S_t) + \beta_{r(t)} \sigma_H(\Gamma_t) \leq \int_{S_t} \varphi H(\nu) d\sigma + \beta \int_{\Gamma_t} H(\nu) d\sigma.$$

Hence, joining (5.4) and (5.3) we get (6.2). Then, applying the level set representation formula (5.4) in the anisotropic radial case, and (5.5), by (6.2) we get

$$\lambda_1(\mathcal{W}_R) = \mathcal{F}_{\mathcal{W}_R}(\mathcal{W}_{r(t)}, \varphi_\ast) \leq \mathcal{F}_\Omega(U_t, \varphi) \leq \lambda_1(\Omega)$$

for some $t \in [0, 1]$, which gives (5.1).

In order to conclude the proof, we study the equality case. Let us suppose that $\lambda_1(\Omega) = \lambda_1(\mathcal{W}_R)$.

We first claim that, for a.e. $t \in [0, 1]$, $U_t$ is homothetic to a Wulff shape. Indeed, by (5.5) and (6.2)

$$\lambda_1(\mathcal{W}_R) = \lambda_1(\Omega) \geq \mathcal{F}_\Omega(U_t, \varphi) \geq \mathcal{F}_{\mathcal{W}_R}(\mathcal{W}_{r(t)}, \varphi_\ast) = \lambda_1(\mathcal{W}_R)$$
for $t$ in a set of positive measure $S \subset [0,1]$. Then by Theorems 5.1 and 5.2 it follows necessarily that $\varphi = \frac{H(Du_\tilde{u})^{-1}}{p-1}$. This implies that for almost every $t \in [0,1]$, the equality in (6.2) and in (6.3) holds. In particular, $\sigma_H(\partial W_{\epsilon(t)}) = \sigma_H(\partial U_{t})$ for a.e. $t$. By the equality case in the anisotropic isoperimetric inequality, we get the claim. Since $U_{t}$, $t \in [0,1]$ are nested sets all homothetic to Wulff shapes, it follows that also $\Omega = \bigcup_{t \in [0,1]} U_{t}$ is homothetic to a Wulff shape, up to a measure zero set. The Lipschitz assumption on the boundary of $\Omega$ guarantees that $\Omega = W_{R}$, up to translations.

\[ \square \]

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