NONLINEAR DIFFERENTIAL EQUATIONS ARISING FROM
BOOLE NUMBERS AND THEIR APPLICATIONS

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Abstract. In this paper, we study nonlinear differential equations satisfied by
the generating function of Boole numbers. In addition, we derive some explicit
and new interesting identities involving Boole numbers and higher-order Boole
numbers arising from our nonlinear differential equations.

1. Introduction

The Boole polynomials, $Bl_n (x \mid \lambda), \ (n \geq 0)$, are given by the generating function

\[ \frac{1}{1 + (1 + t)^{\lambda}} (1 + t)^x = \sum_{n=0}^{\infty} Bl_n (x \mid \lambda) \frac{t^n}{n!}, \quad (\text{see } [5-8, 10, 11, 18]), \]

where we assume that $\lambda \neq 0$.

When $x = 0$, $Bl_n (x) = Bl_n (0 \mid \lambda)$, $(n \geq 0)$, are called the Boole numbers. The
higher-order Boole polynomials (or Peters polynomials) are also defined by the
generating function

\[ \left( \frac{1}{1 + (1 + t)^{\lambda}} \right)^r (1 + t)^x = \sum_{n=0}^{\infty} Bl^{(r)}_n (x \mid \lambda) \frac{t^n}{n!}, \quad (r \in \mathbb{N}), \quad (\text{see } [18]). \]

The first few Boole and higher-order Boole polynomials are as follows:

$Bl_0 (x \mid \lambda) = \frac{1}{2}$, \quad $Bl_1 (x \mid \lambda) = \frac{1}{4} (2x - \lambda)$, \quad $Bl_2 (x \mid \lambda) = \frac{1}{4} (2x (x - \lambda - 1) + \lambda)$,
and

$Bl^{(r)}_0 (x \mid \lambda) = 2^{-r}$, \quad $Bl^{(r)}_1 (x \mid \lambda) = 2^{-(r+1)} (2x - \lambda)$,
$Bl^{(r)}_2 (x \mid \lambda) = 2^{-(r+2)} (4x (x - 1) + (2 - 4x) \lambda r + r (r - 1) \lambda^2)$, \ldots

With the viewpoint of umbral calculus, Boole numbers and polynomials have
been studied by several authors (see [1-20]).

Recently, Kim-Kim has studied the following nonlinear differential equations(see [6, 8]):

\[ \left( \frac{d}{dt} \right)^N F (t) = \frac{(-1)^N}{(1 + t)^N} \sum_{j=2}^{N+1} (j-1)! (N-1)! H_{N-1,j-2} F (t)^j, \quad (N \in \mathbb{N}), \]

where

$H_{N,0} = 1$, \quad for all $N$,

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2. Nonlinear differential equations arising from the generating function of Boole numbers

Let

\[ F(t; \lambda) = \frac{1}{(1 + t)^{\lambda} + 1}. \]

Then, by (2.1), we get

\[ F^{(1)} = \frac{d}{dt} F(t) \]

\[ = \left( \frac{1}{(1 + t)^{\lambda} + 1} \right)^2 \frac{(-1)^{\lambda}}{(1 + t)^{\lambda}} (1 + t)^\lambda \]

\[ = \frac{(-1)^{\lambda}}{1 + t} \left( \frac{1}{(1 + t)^{\lambda} + 1} \right)^2 (1 + t)^\lambda - 1 \]

\[ = \frac{(-1)^{\lambda}}{1 + t} \left( F - F^2 \right), \]

and

\[ F^{(2)} = \frac{dF^{(1)}}{dt} \]

\[ = \frac{(-1)^{\lambda}}{(1 + t)^2} \left( F - F^2 \right) - \frac{\lambda}{1 + t} \left( F^{(1)} - 2FF^{(1)} \right) \]

\[ = \frac{(-1)^{\lambda}}{(1 + t)^2} \left( F - F^2 \right) + \frac{(-1)^{\lambda}}{(1 + t)^2} (1 - 2F) (F - F^2) \]

\[ = \frac{(-1)^{\lambda}}{(1 + t)^2} \left( (1 + \lambda) F - (1 + 3\lambda) F^2 + 2\lambda F^3 \right). \]

Continuing this process, we set

\[ F^{(N)} = \left( \frac{d}{dt} \right)^N F(t) = \frac{(-1)^{N\lambda}}{(1 + t)^N} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) F^i, \]

where \( N = 0, 1, 2, \ldots. \)

From (2.4), we have

\[ F^{(N+1)} \]
\[ d\frac{F^{(N)}}{dt} = \frac{-1}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) F^i + \frac{-1}{1+t} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) i F^{i-1} F^{(1)} \]

\[ = \frac{-1}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) F^i + \frac{-1}{1+t} \sum_{i=1}^{N+1} i a_{i-1} (N; \lambda) F^{i-1} (F - F^2) \]

\[ = \frac{-1}{(1+t)^{N+1}} \sum_{i=1}^{N+1} (N+i\lambda) a_{i-1} (N; \lambda) F^i - \sum_{i=2}^{N+2} (i-1) \lambda a_{i-2} (N; \lambda) F^i \]

\[ = \frac{-1}{(1+t)^{N+1}} \left\{ (N+\lambda) a_0 (N; \lambda) F - (N+1) \lambda a_N (N; \lambda) F^{N+2} \right\} \]

On the other hand, replacing \( N \) by \( N + 1 \) in (2.4), we get

\[ F^{(N+1)} = \frac{-1}{(1+t)^{N+1}} \sum_{i=1}^{N+2} a_{i-1} (N+1; \lambda) F^i. \]

From (2.5) and (2.6), we can derive the following relations:

\[ a_0 (N+1; \lambda) = (N+\lambda) a_0 (N; \lambda), \]

\[ a_{N+1} (N+1; \lambda) = - (N+1) \lambda a_N (N; \lambda) \]

and

\[ a_{i-1} (N+1; \lambda) = - (i-1) \lambda a_{i-2} (N; \lambda) + (N+i\lambda) a_{i-1} (N; \lambda), \]

where \( 2 \leq i \leq N+1 \).

By (2.1) and (2.4), it is easy to show that

\[ F = F^{(0)} = \lambda a_0 (0; \lambda) F. \]

By comparing the coefficients on both sides of (2.10), we have

\[ a_0 (0; \lambda) = \frac{1}{\lambda}. \]

From (2.2) and (2.4), we note that

\[ \frac{-1}{1+t} \frac{\lambda}{1+t} (F - F^2) = F^{(1)} \]

\[ = \frac{-1}{1+t} \frac{\lambda}{1+t} (a_0 (1; \lambda) F + a_1 (1; \lambda) F^2) \]

Thus, by (2.12), we get

\[ a_0 (1; \lambda) = 1, \text{ and } a_1 (1; \lambda) = -1. \]

(2.13) \[ a_0 (N+1; \lambda) = (N+\lambda) a_0 (N; \lambda) = (N+\lambda) (N+\lambda-1) a_0 (N-1; \lambda) \]

\[ ; \]

\[ \vdots \]
From (2.9), we can derive the following equations:

\( a_1 (N + 1; \lambda) \)
\( = -\lambda a_0 (N; \lambda) + (N + 2\lambda) a_1 (N; \lambda) \)
\( = -\lambda a_0 (N; \lambda) + (N + 2\lambda) \{ -\lambda a_0 (N - 1; \lambda) + (N - 1 + 2\lambda) a_1 (N - 1; \lambda) \} \)
\( = -\lambda (a_0 (N; \lambda) + (N + 2\lambda) a_0 (N - 1; \lambda)) + (N + 2\lambda) (N + 2\lambda - 1) a_1 (N - 1; \lambda) \)
\( = -\lambda \{ a_0 (N; \lambda) + (N + 2\lambda) a_0 (N - 1; \lambda) + (N + 2\lambda) (N + 2\lambda - 1) a_0 (N - 2; \lambda) \} \)
\( + (N + 2\lambda) (N + 2\lambda - 1) (N + 2\lambda - 2) a_1 (N - 2; \lambda) \)
\( = -\lambda \sum_{i=0}^{N-1} (N + 2\lambda) a_0 (N - i; \lambda) + (N + 2\lambda) a_1 (N; \lambda) \)
\( = -\lambda \sum_{i=0}^{N} (N + 2\lambda) a_0 (N - i; \lambda) , \)

\( a_2 (N + 1; \lambda) \)
\( = -2\lambda a_1 (N; \lambda) + (N + 3\lambda) a_2 (N; \lambda) \)
\( = -2\lambda a_1 (N; \lambda) + (N + 3\lambda) \{ -2\lambda a_1 (N - 1; \lambda) + (N + 3\lambda - 1) a_2 (N - 1; \lambda) \} \)
\( = -2\lambda \{ a_1 (N; \lambda) + (N + 3\lambda) a_1 (N - 1; \lambda) \} \)
\( + (N + 3\lambda) (N + 3\lambda - 1) \{ -2\lambda a_1 (N - 2; \lambda) + (N + 3\lambda - 2) a_2 (N - 2; \lambda) \} \)
\( = -2\lambda \{ a_1 (N; \lambda) + (N + 3\lambda) a_1 (N - 1; \lambda) + (N + 3\lambda) (N + 3\lambda - 1) a_1 (N - 2; \lambda) \} \)
\( + (N + 3\lambda) (N + 3\lambda - 1) (N + 3\lambda - 2) a_2 (N - 2; \lambda) \)
\( = \cdots \)

where

\( (x)_n = x (x - 1) (x - 2) \cdots (x - n + 1) , \quad (n \geq 0) . \)
We have the following recurrence relations:

**Theorem 1.** We have the following recurrence relations:

(i) \( a_0 (0; \lambda) = \frac{1}{\lambda}; \) \( a_0 (1; \lambda) = 1, \) \( a_1 (1; \lambda) = -1, \)

(ii) \( a_0 (N + 1; \lambda) = (N + \lambda) a_{N+1} (N + 1; \lambda) = (-1)^{N+1} \lambda^N (N + 1)!, \)

(iii) \( a_k (N + 1; \lambda) = -k \lambda \sum_{i_1 = 0}^{N-k+1} (N + (k + 1) \lambda)_{i_1} a_{k-1} (N - i_1; \lambda), \)

for \( 1 \leq k \leq N. \)

Now, we observe that

\[
(2.21) \quad a_1 (N + 1; \lambda) = -\lambda \sum_{i_1 = 0}^{N} (N + 2 \lambda)_{i_1} a_0 (N - i_1; \lambda)
\]

\[
= -\lambda \sum_{i_1 = 0}^{N} (N + 2 \lambda)_{i_1} (N + \lambda - i_1 - 1)_{N-i_1-1},
\]

(2.22) \( a_2 (N + 1; \lambda) \)
(2.23) \[ = -2\lambda \sum_{i_2=0}^{N-1} (N + 3\lambda)_{i_2} a_1 (N - i_2; \lambda) \]
\[ = (-1)^2 2! \lambda^2 \sum_{i_2=0}^{N-1} \sum_{i_1=0}^{N-i_2-1} (N + 3\lambda)_{i_2} (N + 2\lambda - i_2 - 1)_{i_1} \]
\[ \times (N + \lambda - i_2 - i_1 - 2)_{N-i_2-i_1-2}, \]
and

(2.24) \[ a_3 (N + 1; \lambda) \]
\[ = -3\lambda \sum_{i_3=0}^{N-2} (N + 4\lambda)_{i_3} a_2 (N - i_3; \lambda) \]
\[ = (-1)^3 3! \lambda^3 \sum_{i_3=0}^{N-2} \sum_{i_2=0}^{N-i_3-2} \sum_{i_1=0}^{N-i_3-i_2-2} (N + 4\lambda)_{i_3} (N + 3\lambda - i_3 - 1)_{i_2} \]
\[ \times (N + 2\lambda - i_3 - i_2 - 2)_{i_1} \]
\[ \times (N + \lambda - i_3 - i_2 - i_1 - 3)_{N-i_3-i_2-i_1-3}. \]

Continuing this process, we have

(2.25) \[ a_j (N + 1; \lambda) \]
\[ = (-1)^j j! \lambda^j \]
\[ \times \sum_{i_j=0}^{N-j+1} \sum_{i_{j-1}=0}^{N-j+1-i_j} \cdots \sum_{i_1=0}^{N-j+1-i_2-i_3-\cdots-i_j} (N + (j + 1)\lambda)_{i_j} (N + j\lambda - i_j - 1)_{j-1} \]
\[ \times \cdots \times (N + 2\lambda - i_j - \cdots - i_2 - (j - 1))_{i_1} \]
\[ \times (N + \lambda - i_j - \cdots - i_1 - j)_{N-i_j-\cdots-i_1-j}, \]
where \(1 \leq j \leq N.\)

From (2.25), we note that the matrix \((a_i (j; \lambda))_{0 \leq i,j \leq N}\) is given by

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & \cdots & (N-1)\lambda^{N-1} \\
0 & 1 & (1+\lambda) & (2+\lambda) & \cdots & (N+\lambda-1)\\n1 & -1 & & & & \\
2 & & & & & (-1)^2 \lambda^2! \\
3 & & & & & (-1)^3 \lambda^3! \\
\vdots & & & & & \\
N & & & & & (-1)^N \lambda^{N-1} N! \\
\end{pmatrix}
\]

Therefore, by Theorem 1, (2.4), and (2.25), we obtain the following theorem.
Theorem 2. The nonlinear differential equations

\[ F^{(N)} = \frac{(-1)^N \lambda}{(1 + t)^N} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) F^i, \quad (N \in \mathbb{N}), \]

have a solution \( F = F(t, \lambda) = \frac{1}{(1 + t)^N + 1} \),

where \( a_0 (N; \lambda) = (N + \lambda - 1)_{N-1}, \ a_N (N; \lambda) = (-1)^N \lambda^{N-1} N! \),

\[ a_j (N; \lambda) = (-1)^j j! \lambda^j \prod_{i=0}^{N-j-1-i_j} \frac{(N + (j+1) \lambda - 1)_{i_j}}{i_j!} \times (N + j \lambda - \lambda - 2)_{i_{j-1}} \cdots (N + 2 \lambda j - i_{j-2} - j)_{i_1} \times (N + \lambda - i_{j-1} - \cdots - i_1 - j - 1)_{N-i_{j-1} - \cdots - i_1 - 1}, \quad (1 \leq j \leq N - 1). \]

Recall that the Boole numbers, \( B_l (\lambda), \ (k \geq 0) \), are given by the generating function

(2.27) \[ \frac{1}{(1 + t)^\lambda + 1} = \sum_{k=0}^{\infty} B_l (\lambda) \frac{t^k}{k!}. \]

Thus, by (2.27), we get

\[ F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, \lambda) \]

\[ = \left( \frac{d}{dt} \right)^N \left( \frac{1}{(1 + t)^\lambda + 1} \right) \]

\[ = \sum_{k=N}^{\infty} B_k (\lambda) (k)_{N} \frac{t^{k-N}}{k!} \]

\[ = \sum_{k=0}^{\infty} B_{k+N} (\lambda) \frac{(k + N)_{N} t^k}{(k + N)!} \]

\[ = \sum_{k=0}^{\infty} B_{k+N} (\lambda) \frac{t^k}{k!}, \quad (N \in \mathbb{N}). \]

From (1.2), Theorem 2 and (2.27), we have

(2.28) \[ \sum_{k=0}^{\infty} B_{k+N} (\lambda) \frac{t^k}{k!} = F^{(N)} \]

\[ = \frac{(-1)^N \lambda}{(1 + t)^N} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) \left( \frac{1}{(1 + t)^\lambda + 1} \right)^i \]

\[ = (-1)^N \lambda (1 + t)^{-N} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) \left( \frac{1}{(1 + t)^\lambda + 1} \right)^i. \]
\[
\begin{align*}
&= (-1)^N \lambda \left( \sum_{l=0}^{\infty} (-1)^l (N + l - 1) t^l / l! \right) \left( \sum_{n=0}^{\infty} \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) B_{m}^{(i)} (\lambda) t^n / m! \right) \\
&= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) \left( \sum_{l=0}^{\infty} (-1)^l (N + l - 1) t^l / l! \right) \left( \sum_{m=0}^{\infty} B_{m}^{(i)} (\lambda) t^n / m! \right) \\
&= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) \left( \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} (-1)^l (N + l - 1) B_{k-l}^{(i)} (\lambda) \right) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left( (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) \sum_{l=0}^{k} \binom{k}{l} (-1)^l (N + l - 1) B_{k-l}^{(i)} (\lambda) \right) \frac{t^k}{k!},
\end{align*}
\]

where \( N \in \mathbb{N} \).

By comparing the coefficients on both sides of (2.28), we obtain the following theorem.

**Theorem 3.** For \( N \in \mathbb{N} \) and \( k \in \mathbb{N} \cup \{0\} \), we have

\[
B_{k+N} (\lambda) = (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1} (N; \lambda) \sum_{k=0}^{n} \binom{k}{l} (-1)^l (N + l - 1) B_{k-l}^{(i)} (\lambda).
\]

By replacing \( t \) by \( e^t - 1 \) in (1.1), we get

\[
(2.29) \quad \frac{1}{2} \left( \frac{2}{e^{\lambda t} + 1} \right) = \sum_{k=0}^{\infty} B_k (\lambda) \frac{1}{k!} (e^t - 1)^k \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} B_k (\lambda) S_2 (n, k) \right) \frac{t^n}{n!},
\]

where \( S_2 (n, k) \) are the Stirling numbers of the second kind.

As is well known, Euler numbers are given by the generating function

\[
(2.30) \quad \left( \frac{2}{e^t + 1} \right) = \sum_{n=0}^{\infty} E_n t^n / n!, \quad \text{(see [6])}.
\]

From (2.29) and (2.30), we have

\[
(2.31) \quad 2^{-1} \lambda^n E_n = \sum_{k=0}^{n} B_k (\lambda) S_2 (n, k), \quad (n \geq 0).
\]

It is well known that the higher-order Euler numbers are also defined by the generating function

\[
(2.32) \quad \left( \frac{2}{e^t + 1} \right)^r = \sum_{n=0}^{\infty} E_n^{(r)} t^n / n!, \quad \text{(see [19])}.
\]

Now, we observe that

\[
(2.33) \quad \left( \frac{1}{e^{\lambda t} + 1} \right) = \left( \frac{1}{(e^t - 1 + 1)^\lambda + 1} \right)^i \\
= \sum_{k=0}^{\infty} B_{k}^{(i)} (\lambda) \frac{1}{k!} (e^t - 1)^k
\]

\( \text{(see [17])} \).
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} B_{k}^{(i)} (\lambda) S_2 (n, k) \right) \frac{t^n}{n!}.
\]

Thus, by (2.32) and (2.33), we get
\[
2^{-i} \lambda^n E_n^{(i)} = \sum_{k=0}^{n} B_{k}^{(i)} (\lambda) S_2 (n, k), \quad (n \geq 0, i \in \mathbb{N}).
\]

From (1.1) and (2.30), we note that
\[
2^{-i} \lambda^n E_n^{(i)} = \sum_{k=0}^{n} E_k \frac{\lambda^k}{k!} (\log (1 + t))^k
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} E_k \lambda^k S_1 (n, k) \right) \frac{t^n}{n!},
\]
where \( S_1 (n, k) \) are the Stirling numbers of the first kind.

Thus, by (2.34), we get
\[
(2.35) \quad B_l (\lambda) = \frac{1}{2} \sum_{k=0}^{n} E_k \lambda^k S_1 (n, k), \quad (n \geq 0).
\]

By (2.32), we easily get
\[
(2.36) \quad \left( \frac{2}{(1 + t)^{\lambda} + 1} \right)^i = \left( \frac{2}{e^{\lambda \log(1+t)} + 1} \right)^i
\]
\[
= \sum_{k=0}^{\infty} E_k \frac{1}{k!} \lambda^k (\log (1 + t))^k
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} E_k \lambda^k S_1 (n, k) \right) \frac{t^n}{n!}, \quad (i \in \mathbb{N}).
\]

From (2.32) and (2.36), we have
\[
(2.37) \quad 2^i B_l (\lambda) = \sum_{k=0}^{n} E_k^{(i)} \lambda^k S_1 (n, k), \quad (n \geq 0, i \in \mathbb{N}).
\]

Therefore, by Theorem 3, (2.36), and (2.37), we obtain the following theorem.

**Theorem 4.** For \( k \in \mathbb{N} \cup \{ 0 \} \) and \( N \in \mathbb{N} \), we have
\[
\frac{1}{2} \sum_{n=0}^{k+N} E_n \lambda^n S_1 (k + N, n)
\]
\[
= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1} (N ; \lambda) \sum_{l=0}^{k} \binom{k}{l} (-1)^l (N + l - 1) \sum_{n=0}^{k-l} 2^{-i} E_n^{(i)} \lambda^n S_1 (k - l, n).
\]
References

1. H. Alzer and R. Chapman, *On Boole’s formula for factorials*, Australas. J. Combin. 59 (2014), 333–336.
2. A. Bayad and J. Chikhi, *Apostol-Euler polynomials and asymptotics for negative binomial reciprocals*, Adv. Stud. Contemp. Math. (Kyungshang) 24 (2014), no. 1, 33–37.
3. A. Bayad and T. Kim, *Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials*, Russ. J. Math. Phys. 18 (2011), no. 2, 133–143.
4. D. Kang, J. Jeong, S.-J. Lee, and S.-H. Rim, *A note on the Bernoulli polynomials arising from a non-linear differential equation*, Proc. Jangjeon Math. Soc. 16 (2013), no. 1, 37–43.
5. D. S. Kim and T. Kim, *A note on Boole polynomials*, Integral Transforms Spec. Funct. 25 (2014), no. 8, 627–633. MR 3195946
6. ———, *Some identities of Boole and Euler polynomials*, Ars Combin. 118 (2015), 349–356.
7. D. S. Kim, T. Kim, and J. J. Seo, *A note on q-analogue of Boole polynomials*, Appl. Math. Inf. Sci. 9 (2015), no. 6, 3135–3158. MR 3386346
8. T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. Number Theory 132 (2012), no. 12, 2854–2865. MR 2965196
9. ———, *Degenerate Euler zeta function*, Russ. J. Math. Phys. 22 (2015), no. 4, 469–472.
10. ———, *On the degenerate higher-order Cauchy numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) 25 (2015), no. 3, 417–421.
11. T. Kim, D. V. Dolgy, and D. S. Kim, *Symmetric identities for degenerate generalized Bernoulli polynomials*, J. Nonlinear Sci. Appl. 9 (2016), no. 2, 677–683.
12. T. Kim, D. S. Kim, D. V. Dolgy, and J.-J. Seo, *Bernoulli polynomials of the second kind and their identities arising from umbral calculus*, J. Nonlinear Sci. Appl. 9 (2016), no. 3, 860–869.
13. T. Kim, D. S. Kim, H.-I. Kwon, J.-J. Seo, and D. V. Dolgy, *Some identities of q-Euler polynomials under the symmetric group of degree n*, J. Nonlinear Sci. Appl. 9 (2016), no. 3, 1077–1082.
14. J. Kwon and J.-W. Park, *A note on (h, q)-Boole polynomials*, Adv. Difference Equ. (2015), 2015:198, 11.
15. J. K. Kwon, *A note on weighted Boole polynomials*, Global J. Pure Appl. Math. 11 (2015), no. 5, 2055–2063.
16. M. Nuzzetti, *Toward a history of the algebra of logic (from Boole to Sheffer)*, Metalogicon 27 (2014), no. 1, 45–62. MR 3289454
17. A. Osipov, *On a G. Boole’s identity for rational functions and some trace formulas*, Complex Anal. Oper. Theory 5 (2011), no. 3, 889–900. MR 2836331 (2012i:47047)
18. S. Roman, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984. MR 741185 (87c:05015)
19. E. Şen, *Theorems on Apostol-Euler polynomials of higher order arising from Euler basis*, Adv. Stud. Contemp. Math. (Kyungshang) 23 (2013), no. 2, 337–345. MR 3088764
20. S. L. Uckelman, *Computing with concepts, computing with numbers: Llull, Leibniz, and Boole*, Programs, proofs, processes, Lecture Notes in Comput. Sci., vol. 6158, Springer, Berlin, 2010, pp. 427–437. MR 2678155 (2012c:03022)

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