Birationally rigid Fano cyclic covers

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We prove birational superrigidity of Fano cyclic covers over hypersurfaces in the projective space.

1. The main result

Let $\mathbb{P} = \mathbb{P}^{M+1}$ be the complex projective space, $M \geq 5$, $Q = Q_m \subset \mathbb{P}$ a smooth hypersurface of degree $m \geq 1$,

$$\sigma: V \to Q$$

the $K:1$ cyclic cover, branched over a smooth divisor $W \cap Q$, where $W = W_{Kl} \subset \mathbb{P}$ is a hypersurface of degree $Kl$. Introducing a new coordinate $u$ of weight $l$, we can realize $V$ as a complete intersection of type $m \cdot Kl$ in the weighted projective space

$$\mathbb{P}(1, \ldots, 1, l),$$

namely, $V$ is given by the following system of equations

$$\begin{cases} f(x_0, \ldots, x_{M+1}) = 0 \\ u^K = g(x_0, \ldots, x_{M+1}), \end{cases}$$

(1)

where $f(x_s)$ and $g(x_s)$ are homogeneous polynomials of degrees $m$ and $Kl$, respectively.

If the integers $m, l$ and $K$ satisfy the relation

$$m + (K - 1)l = M + 1,$$

then $V$ is a primitive Fano variety of dimension $M$, that is, $\text{Pic} V = \mathbb{Z}K_V$ and $(-K_V)$ is ample.

The purpose of this note is to sketch a proof of the following

**Theorem 1.** A general (in the sense of Zariski topology) variety $V$ is birationally superrigid.

In particular, $V$ admits no non-trivial structures of a rationally connected fibration, any birational map $V \dasharrow V^\sharp$ onto a Fano variety with $\mathbb{Q}$-factorial terminal singularities and $\text{rk Pic} V^\sharp = 1$ is an isomorphism and the groups of birational and biregular self-maps coincide:

$$\text{Bir} V = \text{Aut} V.$$

**Remark 1.** Birational superrigidity means that for any movable linear system $\Sigma$ on $V$ its virtual and actual thresholds of canonical adjunction coincide:

$$c_{\text{virt}}(\Sigma) = c(\Sigma),$$

(2)
see [1-3] for the definitions, also for a (very simple) proof that 2 implies all the other claims of Theorem 1.

2. The regularity conditions

Let us give a precise description of the concept of the variety $V$ being general. Since $V$ is determined by two polynomials $f$ and $g$, of degrees $m$ and $Kl$, respectively, we can look at $V$ as a point in the parameter space,

$$(f, g) \in H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(m)) \times H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(Kl)) = \mathcal{V}.$$  

Now we will describe the set $U_{\text{reg}} \subset V$ of regular varieties $V$, such that $(f, g) \in U_{\text{reg}}$ implies birational superrigidity. The set $U_{\text{reg}}$ is defined by the regularity conditions which must be satisfied at every point $o \in V$. These regularity conditions are similar to those used in [2-5]. We consider the two cases, when $\sigma(o) \not\in W$ and $\sigma(o) \in W$, separately.

(R1) The regularity condition outside the ramification divisor.

Set $p = \sigma(o) \not\in W$, take a system of affine coordinates $z_1, \ldots, z_{M+1}$ with the origin at $p$ (we can assume that $z_i = x_i/x_0$), and set $y = u/x_0^l$. Now the standard affine set

$$\mathbb{A}^{M+2}_{(z_1, \ldots, z_{M+1}, y)}$$

is a chart for $\mathbb{P}(1, \ldots, 1, l)$. With respect to these coordinates we have two equations of $V$ (abusing our notations, we denote the corresponding non-homogeneous polynomial by the same symbol as the homogeneous one):

$$f = q_1 + \ldots + q_m = 0,$$
$$y^K = g = w_0 + \ldots + w_{Kl},$$

$q_j$ and $w_j$ are homogeneous polynomials of degree $j$ in $z_\ast$.

By assumption, $w_0 \neq 0$ and we can assume that $w_0 = 1$ and $y(o) = 1$. Similar to [2-5], set

$$g^{1/K} = (1 + w_1 + \ldots + w_{Kl})^{1/K} = 1 + \sum_{i=1}^{\infty} \gamma_i(w_1 + \ldots + w_{Kl})^i =$$
$$= 1 + \sum_{i=1}^{\infty} \Phi_i(w_1, \ldots, w_{Kl}),$$

where $\gamma_i \in \mathbb{Q}$ are defined by the Taylor expansion at zero

$$(1 + s)^{1/K} = 1 + \sum_{i=1}^{\infty} \gamma_i s^i$$

and $\Phi_i(w_1(z_\ast), \ldots, w_{Kl}(z_\ast))$ are homogeneous polynomials of degree $i \geq 1$ in $z_\ast$.  

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Set also
\[ [g^{1/K}]_k = 1 + \sum_{i=1}^{k} \Phi_i(w_1, \ldots, w_K), \]
and for \( k = 1, \ldots, m \)
\[ f_k = q_1 + \ldots + q_k. \]

Now we can formulate the regularity condition at the point \( o \): if \( m \leq Kl \), then the sequence
\[ q_1, \ldots, q_m, \Phi_{l+1}(w_*(z_*)), \ldots, \Phi_{Kl-1}(w_*(z_*)) \]
is regular in \( O_{o,\mathbb{C}^M+1} \).
If \( m \geq Kl + 1 \), then we require that the sequence
\[ q_1, \ldots, q_{m-1}, \Phi_{l+1}(w_*(z_*)), \ldots, \Phi_{Kl}(w_*(z_*)) \]
be regular in \( O_{o,\mathbb{C}^M+1} \).

(R2) The regularity condition on the ramification divisor.
In the notations above, \( w_0 = 0 \) in this case. We require that the sequence
\[ q_1, \ldots, q_m, w_1, \ldots, w_K \]
be regular in \( O_{o,\mathbb{C}^M+1} \).

**Proposition 1.** The set \( U_{reg} \) of regular pairs \((f, g) \in V \) (that is, the pairs satisfying (R1) or (R2) at every point \( o \in V \)) is a non-empty Zariski open subset in \( V \).

**Proof** is similar to the proof of Proposition 1.1 in [2], see [5] for more details.

3. The technique of hypertangent divisors
Here we give a sketch of the proof of Theorem 1.

By the Lefschetz theorem,
\[ \text{Pic} V = A^1 V = \mathbb{Z} H, \]
where \( H = -K_V \) is the anticanonical class, and
\[ A^2 V = \mathbb{Z} K_V^2 = \mathbb{Z} H^2 \]
(the group of codimension 2 cycles modulo numerical equivalence). By the sufficient condition of birational superrigidity (see [2], Proposition 2.1, also [5,6]), in order to prove Theorem 1 we need to show that for any irreducible subvariety \( Y \subset V \) of codimension 2 and any point \( o \in Y \) the following inequality
\[ \frac{\text{mult}_o Y}{\deg Y} = \frac{\text{mult}_o Y}{\deg Y} \leq \frac{4}{\deg V} \]
holds, where \( \deg Y = (Y \cdot H^{M-2}) \) and \( \deg V = H^M = mK \).
The inequality (3) is proved by means of the technique of hypertangent divisors [1-6]. Assume at first that the point \( o \) lies outside the ramification divisor, \( \sigma(o) \notin W \). For any \( i \leq \min(m - 1, K\ell - 1) \) set

\[
\Lambda = \left| \sum_{j=0}^{i} s_{i-j} f_j + \sum_{k=K}^{i} s_{i-k}^* (y - [g^{1/K}]_k) \right|
\]

to be the \( i \)-th hypertangent linear system at \( o \), where \( s_a, s_a^* \) are arbitrary homogeneous polynomials in \( z_0 \) of degree \( a \) and we assume that the value of the coordinate function \( y \) at \( o \) is 1 (otherwise replace \( y \) by \( y\xi \), where \( \xi \in \mathbb{C}^* \) is the appropriate root of 1) and we also assume that \( \sum_{k=K}^{i} \) is equal to zero if \( i \leq K - 1 \). It is easy to see that

\[
\Lambda_i \subset \left| iH \right|, \quad \text{mult}_o D \geq i + 1
\]

for any divisor \( D \in \Lambda_i \).

Now we proceed as in [4]: set

\[
\mathcal{M} = \{1, \ldots, m - 1\}, \quad \mathcal{L} = \{l, \ldots, K\ell - 2\}.
\]

Here we consider the case \( l \geq 3 \) and \( m \leq K\ell \). Set

\[
c_e = \sharp[3, e] \cap \mathcal{M} + \sharp[3, e] \cap \mathcal{L}.
\]

For \( e \leq 2 \) we get \( c_e = 0 \), for \( e \geq \max\{m - 1, K\ell - 2\} \) we get that \( c_e = M - 3 \). Obviously, \( c_{e+1} \geq c_e \). Define the ordering function

\[
\chi: \{1, \ldots, M - 3\} \to \mathbb{Z}_+
\]

by the formula

\[
\chi([c_{e-1} + 1, c_e] \cap \mathbb{Z}_+) = e.
\]

(4)

If \( c_{e-1} = c_e \), then the set \([c_{e-1} + 1, c_e] \) is empty and the formula (4) gives no information. Note that

\[
c_{e+1} - c_e \in \{0, 1, 2\}
\]

so that \( \chi \) can take the same value at at most two neighbor points. Let

\[
\mathbb{D} = \{D_i \in \Lambda_i, \quad i = 1, \ldots, M - 3\}
\]

be a general set of hypertangent divisors.

**Lemma 1.** For every \( i = 1, \ldots, M - 3 \) (and a sufficiently general \( \mathbb{D} \)) the closed algebraic set

\[
R_i(\mathbb{D}) = \bigcap_{j=1}^{i} D_i \cap Y
\]

is of codimension \((i + 2)\) near the point \( o \).

**Proof:** this follows from the regularity condition (R1), see [1-6].

Now we construct in the usual way a sequence of irreducible subvarieties

\[
Y_2 = Y, Y_3, \ldots, Y_{M-1},
\]

such that
• codim $Y_i = i$,

• $Y_{i+1}$ is an irreducible component of the effective cycle $(Y_i \circ D_{i-1})$ of scheme-theoretic intersection of $Y_i$ and $D_{i-1}$,

• the crucial inequality

$$\frac{\text{mult}_o Y_{i+1}}{\deg Y_{i+1}} \geq \frac{\chi(i-1) + 1}{\chi(i-1)} \cdot \frac{\text{mult}_o Y_i}{\deg Y_i}$$

holds.

Thus

$$\frac{\text{mult}_o Y \cdot 4}{3} \cdot \frac{\text{mult}_o Y \cdot m}{m-1} \cdot \frac{l+1}{l} \cdot \ldots \cdot \frac{Kl-1}{Kl-2} = \frac{\text{mult}_o Y \cdot m}{3} \cdot \frac{Kl-1}{l} \leq \frac{\text{mult}_o Y_{M-1}}{\deg Y_{M-1}} \leq 1,$$

whence we get

$$\frac{\text{mult}_o Y}{\deg Y} \leq \frac{1}{mK} \left( \frac{3Kl}{Kl-1} \right) < \frac{4}{mK},$$

as required. Thus the estimate (3) is proved outside the ramification divisor.

The remaining cases ($l = 2$ or $m \geq Kl$) are treated in a similar way.

4. The ramification case

Now let us prove the estimate (2) assuming that the point $o$ lies on the ramification divisor. The hypersurface $W$ is given by the equation

$$w_1 + \ldots + w_{Kl}$$

(with respect to a fixed system of coordinates $z_*$ with the origin at $p = \sigma(o) \in \mathbb{P}$).

Set

$$h_k = w_1 + \ldots + w_k$$

to be the truncated polynomial, $k \leq K - 1$.

**Lemma 2.** The multiplicity of the divisor $(\sigma^*(h_k | Q) = 0)$ at the point $o$ is at least $k + 1$.

**Proof.** This is obvious since

$$\sigma^*(h_k | Q) = (y^K - \sigma^*((w_{k+1} + \ldots + w_{Kl}) | Q))$$

$y(o) = 0$ and $K \geq k + 1$. Q.E.D. for the lemma.

Now arguing as in the case outside the ramification divisor, and applying the regularity condition (R2), we get:

$$\frac{\text{mult}_o Y \cdot 3}{2} \cdot \frac{\text{mult}_o Y \cdot m}{m-1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{K}{K-1} =$$
\[
\text{mult}_o Y \cdot \left( \frac{mK}{4} \right) \leq 1,
\]
whence we immediately get the required estimate (3).

This completes our proof of the Theorem 1.

5. Further results and concluding remarks

Using the techniques of [7] and making the regularity conditions (R1), (R2) stronger in the same way as in [7], one obtains the following

**Theorem 2.** A sufficiently general (in the sense of Zariski topology) variety \( V \in \mathcal{V} \) is divisorially canonical, that is, for any effective divisor \( D \in | - nK_V | \) the pair

\[(V, \frac{1}{n} D)\]

has canonical singularities.

In particular, the property of being birationally superrigid is stable with respect to the operation of taking the direct product by \( V \).

Using the technique of hypertangent divisors (Sec. 3 and 4 above) in a more delicate way, one can drop the condition of \( V \) being a cyclic cover of \( Q \). Namely, the second equation in the system (1) can be replaced by the polynomial

\[u^K + g_1(x_*)u^{K-1} + \ldots + g_K(x_*),\]

where \( g_i(x_*) \) are homogeneous of degree \( il \).

For \( K = 2 \) this generalization gives nothing new; however, for \( K \geq 3 \) it makes the class of varieties much bigger. Denote by \( \mathcal{V}^+ \) the corresponding parameter space, \( \mathcal{V} \subset \mathcal{V}^+ \).

**Theorem 3.** A sufficiently general (in the sense of Zariski topology) variety \( V \in \mathcal{V}^+ \) is divisorially canonical (in particular, birationally superrigid).

**Remarks.** (i) The estimate (3) can be sharpened at the expense of making the regularity conditions (R1), (R2) stronger. This sharpening makes it possible to prove birational rigidity of Fano fiber spaces \( X/P^1 \), the fiber of which is a Fano multiple hypersurface, see [3,4].

(ii) The standard scheme of proving birational superrigidity is to establish that for any moving linear system \( \Sigma \subset | - nK_V | \) and a general divisor \( D \in \Sigma \) the pair

\[(V, \frac{1}{n} D)\]

has canonical singularities. If this is not the case, then there exist a birational morphism

\[\varphi: \tilde{V} \to V\]

(one can assume \( \varphi \) to be a sequence of blow ups) and an exceptional divisor \( E \subset \tilde{V} \) such that the Noether-Fano inequality

\[\nu_E(\Sigma) > n \cdot a(E)\] (5)
is satisfied (such an exceptional divisor is called a maximal singularity of the system \( \Sigma \)). The idea underlying the sufficient condition of birational superrigidity, used in [1-6] and in the present paper above, is that if the estimate (3) holds, then the Noether-Fano inequality (5) is not possible.

It was noted by Cheltsov [8], that if we consider, instead of rationally connected fiber spaces, the structures of a \( K \)-trivial fibration on \( V \), then what we need is just to replace the Noether-Fano inequality (5) by its non-strict version:

\[
\nu_E(\Sigma) \geq n \cdot a(E).
\]

(6)

But the point is, if the centre

\[ B = \text{centre}(E) = \varphi(E) \subset V \]

of the discrete valuation \( E \) on \( V \) is a subvariety of codimension at least 3, then for general divisors \( D_1, D_2 \in \Sigma \) the non-strict Noether-Fano inequality (6) still implies that

\[
\text{mult}_B(D_1 \circ D_2) \geq 4n^2
\]

(no modification of the proof is necessary) and the equality takes place only in the case

\[
\text{codim } B = 3, \quad \text{mult}_B \Sigma = 2n,
\]

(7)

so that the estimate (3) still gives a contradiction in all cases but (7) and the case \( \text{codim } B = 2 \).

Thus, provided that the estimate (3) holds for any irreducible subvariety of codimension two, one has just to study these two simple cases. For instance, it is easy to see that the situation (7) is impossible for Fano cyclic covers, considered above. Thus the inequality (6) implies that \( B \sim H^2 \) and one can show easily that in fact

\[ B = \sigma^{-1}(H_1 \cap H_2), \]

where \( H_i \) are hyperplane sections of \( Q \), so that all \( K \)-trivial structures on \( V \) are given by pencils in the anticanonical linear system \( |-K_V| \).

This argument is absolutely typical: whenever one has a proof of birational (super)rigidity, it remains to do a very easy job to cover the \( K \)-trivial (or elliptic) case, either. For this reason, I believe that the \( K \)-trivial results have very little independent value. The same applies to the “Fano structures with canonical singularities”, see for instance [9] and other papers in that series.

(iii) As always, one of the crucial parameters of Fano variety is its anticanonical degree. This can be seen directly from the inequality (3); as the degree gets higher, the required estimate (2) gets sharper, and, accordingly, harder to prove. On the other hand, it means that the varieties of small degree are much easier to investigate. More precisely, primitive Fano varieties \( X \) of degree less or equal than 4 create no problems at all; in fact, they are all covered (even if they are singular, provided that the singularities are sufficiently mild) by the test class construction of V.A.Iskovskikh and Yu.I.Manin in its original form [10] (plus its higher-dimensional extension [11].
or inversion of adjunction [12]). For this reason, the papers like [13] are rather of exercise-doing level and hardly represent any real progress in the field.

(iv) The results of this paper remain true if we take $V$ to be an iterated Fano cover in the spirit of [5].

(v) One can replace, with no damage to the results, the hypersurface $Q$ by a complete intersection of type $d_1 \ldots d_k$, $d_1 + \ldots + d_k + (K - 1)l = M + 1$, where $k$ is less than $\frac{1}{2} \dim V$, see [5] and [6].

The detailed proofs of all results discussed above will be published elsewhere.

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