About stabilization in large of gyrostat programmed motion with cavity filled with viscous fluid

S P Bezglasnyi

Construction mechanics, engineering geology, basics and foundations Department, Samara State Technical University, Molodogvardeyskaya street, 242, Samara, 443100, Russian Federation

Abstract. This paper is concerned with the control problem of nonautonomous programmed attitude motions around the mass centre of a gyrostat with fluid. The goal of this research is to investigate how to solve the problem of stabilization in large the specified programmed attitude motion of gyrostat. Within the framework of this note, the model of single-rotor dynamically symmetrical gyrostat which includes a spherical cavity filled with highly viscous fluid is considered. The program control forces and the stabilizing control forces acted upon the gyrostat are received. The main research methods proposed in this article are the method of Lyapunov functions and the method of limit equations and limit systems. Numerical simulation results illustrate and verify that stabilizing in large programmed attitude motions problem of a gyrostat with viscous fluid is solved.

1. Introduction

Problems dealing with the attitude motion of satellites and aircraft in orbit can be referred to as an important applied field. They are investigated by many researchers that have extensive supporting literature. The design of attitude motions of rigid bodies or systems, for example, gyrostats, has been an area of study for several decades. It resulted in an urgent need to investigate the attitude motions of aircraft regarding the mass centre. Satellites can contain one or some spinning rotors to provide gyroscopic stability of spatial orientation. Some notes are concerned with gyrostat satellite attitude motions, for example [1-12]. The main principles and methods of rotational motions controls of bodies and systems have been investigated in papers [1-3]. Paper [4] presents the attitude motion equations of a multibody gyrostat and introduces its analytical solutions. An analytical solution to dynamical variables of asymmetric gyrostat has been presented in [5]. Also, finding Euler angles in quadratures have been got. Attempts for problems of resonance modes and bifurcations of stationary motions of aircraft have been made with considerable success in [6, 7]. Modern foreign and domestic researchers hard investigated methods of elimination of chaotic movements in notes [8, 9]. In papers [10, 11] researchers investigated problems dealing with stabilization of the program motions sets of various structure gyrostats. Also, in notes [12] it could be seen that the task on stationary rotation motion stabilization of an axisymmetric satellite in a circular orbit without velocities measurement has been solved.

In addition to solid attitude motion problems, important problems of solids dynamic motions with cavities filled with fluid have been widely investigated from the middle of the previous century. World-famous is, that the literature on solid with fluid modeling generally deals with two main research ways [13, 14]. The first method and interesting results on the theory of rigid bodies with
cavities filled with liquid are presented in [14]. The vast majority of researchers used this approach. But, the applicability of this method is restricted by the availability of partial differential equations. This point, undoubtedly, imposes a serious limitation on research and leads to serious difficulties.

Unlike the first method, another way is presented in paper [13]. The author, academician Chernous’ko, suggested a motion model of a solid body with a cavity entirely filled with highly viscous fluid. The advantage of this approach lies in the fact that the fluid influence on the body movement is described using kinematic characteristics of the body. So, the method allows us to use a mathematical model without partial differential equations. This way is widely used in papers of modern authors, for example [15-17].

The goal of this paper is to present new results in investigating into the problem dealing with attitude motion stabilization in large of gyrostat with cavity filled with viscous fluid. Firstly, this research uses a gyrostat mathematical model of academician Chernous’ko [13]. Secondly, this paper presents the gyrostat movement equations in the Lagrange equations form. Thirdly, in this notes formulation of a problem dealing with the program motions stabilization in large of gyrostat with fluid is given. And finally, as mentioned before, this paper presents the acted upon the gyrostat program control and development of stability in large control which was constructing by the feedback principle in analytical form.

2. Control problem definition and motion equations

We consider and investigate the attitude motion of a gyrostat which has a cavity filled with very viscous fluid. It can be modeled by two connected solids with a common rotation axis. The first rigid body is the carrier or the base. $Oxyz$ is conveniently chosen body-fixed reference frame. The solid carrier has a spherical cavity filled with highly viscous fluid. Let the main inertia moments of a carrier with fluid be $A_1$, $B_1$ and $C_1$. The second rigid body is the dynamically symmetric rotor. We denote main inertia moments of the rotor as $I_2 = I_3$ and $I_1$. Let here the frame $OXYZ$ be an absolutely fixed coordinate system. Also, the point $O$ is fixed. Without loss of any generality, we assume that it coincides with the system’s mass centre. In addition, point $O$ is located on the dynamic symmetry axis of the second body (figure 1).

![Figure 1. Gyrostat.](image)

Let rotor be perfectly aligned and rotates around the axis $Oz$ with the passage of time. The rotation of a rotor around the solid carrier we define as the rotation angle $\delta$.

We construct the mathematical model by the motion equations of the gyrostat. We will use the Lagrange equations of the second type:
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = \mathbf{Q}
\]

(1)

Here \( \mathbf{q} = (\psi, \theta, \varphi, \delta)^T \) is the vector of general coordinates. The vector of general velocities is given as \( \dot{\mathbf{q}} = (\dot{\psi}, \dot{\theta}, \dot{\varphi}, \dot{\delta})^T \). Symbol \((\ )^T\) means transposition. Values \( \psi, \theta, \varphi \) are Euler variables. We define that the general forces are a sum \( \mathbf{Q} = \mathbf{Q}' + \mathbf{Q}^\ast + \mathbf{Q}''. \) Here we mean that symbol \( \mathbf{Q}' \) defines the force torque acting upon the carrier from the cavity with fluid. We appoint the designations \( \mathbf{Q}'', \mathbf{Q}' \) as the program control and the stabilizing control.

For further analysis, and without loss of generality, we assume that the gyrostat with viscous fluid should make the attitude motion around its mass centre according to the specified law as \( \mathbf{r}(t) = (\psi'(t), \theta'(t), \varphi'(t), \delta'(t))^T \). Here values \( \psi'(t), \theta'(t), \varphi'(t), \delta'(t) \) should be the determined continuous and bounded functions of time. We assume them to be known initially or be functions given. Hereinafter the function \( \mathbf{r}(t) \) will be called the programmed motion of gyrostat.

According to the main idea of this paper, we can now state the problem of stabilization in large of gyrostat programmed attitude motion. Namely, we should find the acted upon the carrier control forces \( \mathbf{Q}'', \mathbf{Q}' \) which make the gyrostat programmed motion \( \mathbf{r}(t) \) asymptotically stable in large.

Firstly, we remark that the same problem on stabilization in small was solved in [18] for gyrostat with three degrees of freedom without coordinate \( \delta \).

To solve this problem about programmed motion stabilization in large we intend to use the second Lyapunov method of stability theory. In addition, we would like to attract a method of the limit equations and limit systems from [19]. We are designing active control using the feedback principle.

The kinetic energy of the gyrostat is
\[
2T = A p^2 + B q^2 + C r^2 + C_2 r \sigma + C_3 \sigma^2
\]

(2)

Here \( \mathbf{w} = (p, q, r)^T \) is the absolute angular velocity vector of the carrier. It is calculated according to the coordinate system \( Oxyz \). Values \( A = A_1 + I_1, B = B_1 + I_2, C = C_1 + I_3 \) are the main gyrostat inertia moments. They are calculated according to the coordinate system \( Oxyz \), too. The angular velocity \( \sigma = \sigma(t) = \dot{\delta} \) of the rotor rotation is general velocity. Equations (1) are motion equations of gyrostat with viscous fluid. As everyone knows, they may be closed, for example, with Euler kinematic equations

\[
\begin{aligned}
    p &= \dot{\psi} \cos \varphi + \dot{\varphi} \sin \theta \sin \varphi \\
    q &= -\dot{\theta} \sin \varphi + \dot{\varphi} \sin \theta \cos \varphi \\
    r &= \dot{\varphi} + \dot{\psi} \cos \theta
\end{aligned}
\]

(3)

Then we can format the kinetic energy formula, as follows:
\[
2T = \left( A \sin^2 \theta \sin^2 \varphi + B \sin^2 \theta \cos^2 \varphi + C \cos^2 \theta \right) \dot{\psi}^2 + \left( A \cos^2 \varphi + B \sin^2 \varphi \right) \dot{\theta}^2 + 2 \left( A - B \right) \psi \dot{\psi} \sin \theta \sin \varphi \cos \varphi + C \phi^2 + 2C \dot{\psi} \dot{\varphi} \cos \theta + 2I_1 \psi \sigma_3 \cos \theta + 2I_2 \dot{\phi} \sigma_3 + I_3 \sigma^2
\]

(4)

As you know, the kinetic energy (4) is described as the following sum of the forms \( T = T_z + T_\theta + T_\phi \). Here \( T_\phi = 0 \) is a scalar function. The term \( T_\theta = 0 \) is a linear form of the general velocities \( \dot{\mathbf{q}} \). The last term \( T_z = 0.5 \mathbf{q}^T \mathbf{A}(\mathbf{q}) \dot{\mathbf{q}} \) is a quadratic form of the velocities. In addition, we note that the matrix \( \mathbf{A}(\mathbf{q}) \) is a positive definite and bounded value. Its elements look like.
\[ a_{11} = A \sin^2 \theta \sin^2 \varphi + B \sin^2 \theta \cos^2 \varphi + C \cos^2 \theta, \quad a_{12} = a_{21} = (A - B) \sin \theta \sin \varphi \cos \varphi, \quad a_{13} = a_{31} = C \cos \theta, \]
\[ a_{22} = A \cos^2 \varphi + B \sin^2 \varphi, \quad a_{23} = a_{32} = 0, \quad a_{33} = C, \quad a_{43} = a_{34} = I_3 \cos \theta, \quad a_{24} = a_{42} = 0, \quad a_{44} = a_{43} = I_3, \quad a_{44} = I_3 \]

We use the matrix \( \mathbf{A}(\mathbf{q}) \) to rewrite the gyrostat movement equations (1) in a vector-matrix form

\[
\mathbf{A} \ddot{\mathbf{q}} + \mathbf{\Lambda} = \mathbf{Q}' + \mathbf{Q}'' + \mathbf{Q}'''
\]  

Vector \( \mathbf{\Lambda} = \mathbf{\Lambda}(\mathbf{q}, \dot{\mathbf{q}}) \) has named the vector with components

\[
\mathbf{\Lambda}_i = \sum_{j,k=1}^3 \frac{\partial a_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j - \frac{1}{2} \sum_{j,k=1}^3 \frac{\partial a_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j, \quad (i = 1, 4)
\]

Earlier we identified the general forces acted upon the gyrostat as \( \mathbf{Q} = \mathbf{Q}' + \mathbf{Q}'' + \mathbf{Q}''' \). Here \( \mathbf{Q}' = (Q'_x, Q'_y, Q'_z) \) is the force torque acting upon the solid carrier from the cavity with fluid. It has components

\[
\begin{align*}
Q'_x &= m_z \sin \theta \sin \varphi + m_y \sin \theta \cos \varphi + m_x \cos \theta \\
Q'_y &= m_z \cos \varphi - m_y \sin \varphi \\
Q'_z &= m_x \\
Q''_z &= 0
\end{align*}
\]

Vector’s components \( \mathbf{m} = (m_x, m_y, m_z)^T \) in the right parts of (7) according to the model suggested in paper [13] are calculated as

\[
\mathbf{m} = -\frac{P}{\nu} \left[ \begin{array}{c}
\dot{p} + q \dot{r} - r \dot{q} \\
\dot{q} + p \dot{r} - r \dot{p} \\
\dot{r} + p \dot{q} - q \dot{p}
\end{array} \right]
\]

Here \( \dot{\mathbf{w}} = (\dot{p}, \dot{q}, \dot{r}) \) is the angular acceleration vector of the solid carrier. \( P = \frac{8 \pi a^7}{525} \) is coefficient. It describes the cavity form (sphere with radius \( a \)). Value \( \rho \) is a density of fluid. Value \( \nu \) is a kinematic viscosity of fluid. In our paper, we assume that the cavity is filled with highly viscous fluid: \( \nu^{-1} \ll 1 \).

Papers [13, 17] allow us to use the mathematical model of academician Chernous’ko for our search. According to it, we can calculate the following components of the vector (8)

\[
\begin{align*}
m_x &= \frac{eP}{ABC_1} \left[ p [q^2 (C_1 - A - B)(B - A)B + r^2 (C_1 + A - B)(A - C)C_1] - C_1 I_3 \sigma^2 p + C_1 (B - 2C) I_3 pr \sigma \right] \\
m_y &= \frac{eP}{ABC_1} \left[ q [r^2 (A - C - B)(C - B)C_1 + p^2 (A - C_1 + B)(B - A)A] - C_1 I_3 q^2 \sigma^2 + C_1 (A - 2C) I_3 qr \sigma \right] \\
m_z &= \frac{eP}{ABC_1} \left[ p^2 (B - A - C_1) A \{(A - C)r - I_3 \sigma\} + q^2 B(B - A + C_1) \times \{(C - B)r + I_3 \sigma\} \right]
\end{align*}
\]

3. Basic results
First at all, we calculate the programmed control torque \( \mathbf{Q}' \). To obtain directly this control from the attitude motion equations we should substitute function \( r(t) \) in the (5). We will have

\[
\mathbf{Q}' = \mathbf{A}(\mathbf{r}(t)) \ddot{\mathbf{r}} + \mathbf{\Lambda}(\mathbf{r}(t), \dot{\mathbf{r}}(t)) - \mathbf{Q}'(\mathbf{r}(t), \dot{\mathbf{r}}(t))
\]
The control (9) provides the programmed motion \( r(t) \) of the gyrostat with cavity filled with viscous fluid. It means, that the function \( r(t) \) is the solution of the equations (5) under the assumption that the programmed control (9) is acting upon the gyrostat. But, as everyone knows, in the presence of initial deviations or actions of small perturbations this solution may be not stable. So, in addition to the programmed control (9), we should construct one more additional stabilizing control torque \( Q' \) which makes the gyrostat programmed motion \( r(t) \) asymptotically stable in large.

As mentioned before, to solve the problem of stabilization in large of the gyrostat programmed motion, we use the second method of Lyapunov functions. This approach is widely recognized in the field of the classical stability theory. It allows us to investigate the solution stability of differential equations without obtaining all solutions. The second Lyapunov method uses some functions with the specified properties to prove the solution stability. However, the applicability of this method is restricted to some difficulties. The main of the which faced by the vast majority of researchers is that there are no general methods for constructing the Lyapunov functions. This point, obviously, imposes a serious limitation on using the second Lyapunov method. To simplify the search of the Lyapunov function, we use a method of limit equations and limit systems [19]. This approach allows us to extend the class of the Lyapunov functions to the functions with nonpositive derivatives.

Let us introduce the new generalized coordinates (deflections) \( x \) according to the following equality \( x = (x_1, x_2, x_3, x_4) = (\psi - \psi'(t), \theta - \theta'(t), \phi - \phi'(t), \delta - \delta'(t)) \) or \( x = q - r(t) \). These values describe the behaviour of the solutions of system (5) in the neighborhood of the programmed motion \( r(t) \).

Then we can rewrite (5) with control (9) in new variables by way

\[
\Lambda(t + x) + \Lambda + \Lambda' + \Lambda'' = Q + Q' + Q''
\] (10)

Here components of the vectors \( \Lambda, \Lambda', \Lambda'' \) look like

\[
\begin{align*}
\Lambda_j &= \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_j} \dot{x}_j - \frac{1}{2} \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_i} \dot{x}_j \\
\Lambda'_j &= \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_j} \dot{x}_j - \frac{1}{2} \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_i} \dot{x}_j + \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_i} \dot{x}_j - \frac{1}{2} \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_j} \dot{x}_j \\
\Lambda'' &= \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_j} \dot{x}_j - \frac{1}{2} \sum_{j=1}^{4} \frac{\partial \Lambda_{ij}}{\partial x_i} \dot{x}_j, \quad (i = 1, 4)
\end{align*}
\]

For the analysis to follow, it is useful to introduce the Lyapunov function with bounded and positive definite matrix \( C \), defined as

\[
V(x, \dot{x}) = \frac{1}{2} x^T C x + \frac{1}{2} \dot{x}^T A \dot{x}
\] (11)

Worth noting is that the function (11) is positive definite. Consider the following calculation of the total time-derivative of the function (11)

\[
\frac{dV}{dt} = \dot{x}^T \left( C x - \Lambda'' \right) + \frac{1}{2} \dot{x} \left( \frac{\partial \Lambda}{\partial x} \right)^T \dot{x} + \dot{x} \left( \frac{\partial^2 \Lambda}{\partial x^2} \right) \dot{x} - A \dot{x} + \frac{1}{2} x^T N + Q' + Q'' + Q''
\]

Here some intermediate notation \( N = N(q, \dot{q}) \) is defined as the vector with components

\[
N_i = \sum_{j=1}^{4} \frac{\partial N_{ij}}{\partial x_j} \dot{x}_j - \frac{1}{2} \sum_{j=1}^{4} \frac{\partial N_{ij}}{\partial x_i} \dot{x}_j, \quad (i = 1, 4)
\]

The method of the limit equations and the limit systems [19] allows us to construct the stabilization control as

\[
Q' = -C x - D \dot{x} + A \dot{x} + \Lambda'' - \frac{1}{2} \left( \frac{\partial \Lambda}{\partial x} \right) \dot{x} - \frac{1}{2} N - Q' - Q''
\] (12)
Assuming that the matrix $D$ is positive definite and bounded, we find the total derivative of the function (11) depending on time. Sure, we should calculate it according to (10) along with controls (9) and (12). It can be trivially shown, that

$$\frac{dV}{dt} = -\dot{\mathbf{x}}^T D \dot{\mathbf{x}} \leq -d_0 \|\dot{\mathbf{x}}\| \leq 0 \quad (0 < d_0 = \text{const})$$  \hspace{1cm} (13)$$

The formula (13) means that the derivative of Lyapunov function (11) is negative definite determined by speeds. In addition, the derivative (13) is equal to zero on the set $\{\dot{x} = 0\}$. The limit system to the system (10), along with (9) and (12) on a set $\{\dot{x} = 0\}$ has solution $x = 0$ only. It does not have other solutions. For this reason, we can receive that the program motion $\mathbf{r}(t) = (\psi'(t), \theta'(t), \varphi'(t), \delta'(t))^T$ of the gyrostat is asymptotically stable. To prove this fact, we should use the theorem from paper [19]. Aside from that we mention, that $V(x, \dot{x}) \rightarrow \infty$ is satisfied under $x \rightarrow \infty$, $\dot{x} \rightarrow \infty$. Then the solution $x = 0$ is asymptotically stable in large. It can be concluded according to the theorem from [20].

4. Simulation and numerical solution

To illustrate and verify the analytical results, we present the numerical solution of the equations of control attitude motion of the gyrostat. We integrate it by “Wolfram Mathemtica 7.0”. Without loss of generality, we can assign the following parameters of the mechanical system. Let carrier and rotor be modeled as solids with inertia moments $A = 0.81$, $B = 0.65$, $C = 0.47$, $I_3 = 0.02$ kg/m$^2$. We accept that the radius of the sphere with fluid is $a = 0.05$ m. Kinematic viscosity of fluid is $\nu = 1000$ m$^2$/s. In addition, we choose the program motion according to $\psi'(t) = 6\cos(2t)$, $\theta'(t) = \pi/2 + \sin(2t)$, $\varphi'(t) = \sin(10t)$ rad. Let the angular velocity of rotor rotation about the carrier be $\sigma = 1$ s$^{-1}$.

We assign that $x(0) = (0.21, 0.15, 0.1, 0.3)^T$ rad and $\dot{x}(0) = (0.1, 0.15, 0.22, 0.1)^T$ rad/s are the initial deviations at $t = 0$. To obtain the numerical solution we integrate the control motion equations over the time interval $[0,30]$ s. We choose the coefficients of the matrices $C$ and $D$ as $c_i = d_i = 0.2$, $c_j = d_j = 0$, $i \neq j$, $i, j = 1,2,3,4$. 

![Figure 2. Value $x_1(t)$](image1)

![Figure 3. Value $x_2(t)$](image2)
Figures 2-5 illustrate the behavior of the components of the vector \( \mathbf{x}(t) \). They are the deviations of the general coordinates \( \psi, \theta, \varphi, \delta \) of the gyrostat with cavity filled with viscous fluid for its programmed attitude motion \( \mathbf{r}(t) = (\psi^*(t), \theta^*(t), \varphi^*(t), \delta^*(t))^T \). This motion takes place under the influence of programmed forces (9) and stabilizing forces (12). The graphs illustrate the asymptotical stable in large of the obtained solutions.

We should note that the same problem was solved in [21] for gyrostat with three degrees of freedom without coordinate \( \delta \).

5. Conclusion

We can conclude that within the framework of this research the problem of stabilization in large of nonautonomous programmed attitude motions around the mass centre of a gyrostat with viscous fluid is solved. A mathematical model of single-rotor dynamically symmetrical gyrostat with a spherical cavity filled with highly viscous fluid is constructed and investigated. The main result is that the acted upon the gyrostat the programmed control forces and the stabilizing control forces are received. They solve stabilizing in large programmed motions problem of a gyrostat with viscous fluid. The method of Lyapunov functions and the method of limit equations and limit systems are used. Apart from that to verify the main analytical result, the numerical simulation of the equations of control motion of the gyrostat is carried out. Main results can be used for gyroscopic control systems designing.

Acknowledgments

This work is supported by Russian Science Foundation grant No. 19-01-00791.

References

[1] Chernous’ko F L 1963 J. Appl. Math. And Mech. 27 708–22
[2] Letov A M 1969 Flight Dynamic and Control (Moscow: Nauka) p 369
[3] Rumiantsev V V 1961 J. Appl. Math. And Mech. 25 9–19
[4] Wittenburg 1977 Dynamics of Systems of Rigid Bodies (Teubner, Stuttgart) p 292
[5] Cochran J E, Shu P H and Rew S D 1982 J. Guidance Control Dynamics 5 37–42
[6] Krasilnikov P S 2013 Nelinejnaja Dinamika 9 671–96
[7] Orel O E and Ryabov P E 1998 Regular and Chaotic Dynamics 3 82–91
[8] Hall C and Rand R 1994 J. Guidance Control Dynamics 17 30–7
[9] El-Gohary A 2009 Chaos, Solitons and Fractals 42 2842–51
[10] Bezglasnyi S P 2014 J. Appl. Math. And Mech. 78 551-9
[11] Bezglasnyi S P 2015 Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2015, WCE 2015 pp 7-9
[12] Andreev A S and Petrovicheva Y 2020 Proceedings - 2020 2nd International Conference on Control Systems, Mathematical Modeling, Automation and Energy Efficiency, SUMMA 2020 pp 153-5 9280815
[13] Chernous'ko F L 1968 Motion of a Rigid Body with Cavities Containing a Viscous Fluid (Moscow: Nauka) p 309
[14] Moiseev N N and Rumiantsev V V 1965 Dynamics of Bodies with Fluid-Filled Cavities (Moscow: Nauka) p 439
[15] Akulenko L D, Sinkevich Ya S, Leshchenko D D and Rachinskaya A L 2011 Cosmic Research 49 440-51
[16] Bezglasnyi S P 2014 Russian aeronautics 57 333-8
[17] Bezglasnyi S P 2017 J. of Computer and System Sciences International 56 749-58
[18] Bezglasnyi S P and Krasnikov V S 2016 Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2016 pp 191-4
[19] Andreev A S 1984 J. Appl. Math. and Mech. 48 225–32
[20] Barbashin E A and Krasovskii N N 1952 Dokl. Akad. Nauk SSSR 86 453-6
[21] Bezglasny S P 2017 Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, WCE 2017 pp 152-6