CHARACTERIZATIONS OF ALGEBRAS OF RAPIDLY DECREASING GENERALIZED FUNCTIONS

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Abstract. The well-known characterizations of Schwartz space $S$ of rapidly decreasing functions is extended to the algebra $G_S$ of rapidly decreasing generalized functions and to the algebra $G_S^\infty$ of regular rapidly decreasing generalized functions.

1. Introduction

The Schwartz space $S$ of rapidly decreasing functions on $\mathbb{R}^n$ and its generalizations, in view of their importance in many domains of analysis, have been characterized differently by many authors, e.g. see [10], [15], [17], [4], [1] and [11]. To built a Fourier analysis within the generalized functions of [5], the algebra of rapidly decreasing generalized functions on $\mathbb{R}^n$, denoted $G_S$, was first constructed in [18] and recently studied in [8] and [7]. The algebra of regular rapidly decreasing generalized functions on $\mathbb{R}^n$, denoted $G_S^\infty$, is fundamental in the characterization of the local regularity of a Colombeau generalized function by its Fourier transform and also for developing a generalized microlocal analysis.

Let

$$S^* = \left\{ f \in C^\infty : \forall \alpha \in \mathbb{Z}_+^n, \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| < \infty \right\},$$

$$S_* = \left\{ f \in C^\infty : \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \mathbb{R}^n} |x^{\beta} f(x)| < \infty \right\},$$

then, inspired by the work of [15], the authors of [4] proved the following result:

$$S = S^* \cap S_*$$

The aim of this work is to characterize the algebras $G_S$ and $G_S^\infty$ in the spirit of the characterization of the Schwartz space $S$ done in [4]. In fact we do more, this characterization is given in the general context of the algebra $G_S^R (\Omega)$ of $R$–regular rapidly decreasing generalized functions on an open set $\Omega$ of $\mathbb{R}^n$, see [7] and [3]. The sixth section of this paper gives such an extension, i.e. the characterization of the algebra $G_S^R (\Omega)$ provided $\Omega$ is a box of $\mathbb{R}^n$. The seventh section gives a characterization of $G_S^R (\mathbb{R}^n)$ of $R$–regular rapidly decreasing generalized functions on the whole space $\mathbb{R}^n$ using the Fourier transform. The last section, as corollaries of the results of the paper, gives the characterizations of the classical algebras $G_S$ and $G_S^\infty$.

1991 Mathematics Subject Classification. 46F30; 46F05; 42B10.

Key words and phrases. Schwartz space $S$, Rapidly decreasing generalized functions, Colombeau generalized functions, Fourier transform.
2. Regular sets of sequences

We will adopt the notations and definitions of distributions and Colombeau generalized functions, see [14] and [13].

Definition 1. A non void subset $\mathcal{R}$ of $\mathbb{R}_+^Z$ is said to be regular, if

For all $(N_m)_{m \in \mathbb{Z}^+} \in \mathcal{R}$ and $(k, k') \in \mathbb{Z}_+^2$, there exists $(N'_m)_{m \in \mathbb{Z}^+} \in \mathcal{R}$ such that

\[(R1) \quad N_{m+k} + k' \leq N'_m, \quad \forall m \in \mathbb{Z}^+\]

For all $(N_m)_{m \in \mathbb{Z}^+}$ and $(N'_m)_{m \in \mathbb{Z}^+}$ in $\mathcal{R}$, there exists $(N''_m)_{m \in \mathbb{Z}^+} \in \mathcal{R}$ such that

\[(R2) \quad \max (N_m, N'_m) \leq N''_m, \quad \forall m \in \mathbb{Z}^+\]

For all $(N_m)_{m \in \mathbb{Z}^+}$ and $(N'_m)_{m \in \mathbb{Z}^+}$ in $\mathcal{R}$, there exists $(N''_m)_{m \in \mathbb{Z}^+} \in \mathcal{R}$ such that

\[(R3) \quad N_{l_1} + N'_{l_2} \leq N''_{l_1+l_2}, \quad \forall (l_1, l_2) \in \mathbb{Z}_+^2\]

Example 1. The set $\mathbb{R}_+^+$ of all positive sequences is regular.

Example 2. The set $\mathcal{A}$ of affine sequences defined by

$$\mathcal{A} = \{(N_m)_{m \in \mathbb{N}} \in \mathbb{R}_+^Z : \exists a \geq 0, \exists b > 0, \forall l \in \mathbb{Z}_+, N_l \leq al + b\}$$

is regular.

Example 3. The set $\mathcal{B}$ of all bounded sequences of $\mathbb{R}_+^Z$ is regular.

The notion of regular set is extended to the sets of double sequences.

Definition 2. A non void subset $\tilde{\mathcal{R}}$ of $\mathbb{R}_+^{Z^2}$ is said to be regular if

For all $(N_{q,l})_{(q,l) \in \mathbb{Z}_+^2} \in \tilde{\mathcal{R}}$ and $(k, k', k'') \in \mathbb{Z}_+^3$, there exists $(N'_{q,l})_{(q,l) \in \mathbb{Z}_+^2} \in \tilde{\mathcal{R}}$ such that

\[\tilde{(R1)} \quad N_{q+k,l+k'} + k'' \leq N'_{q,l}, \quad \forall (q,l) \in \mathbb{Z}_+^2\]

For all $(N_{q,l})_{(q,l) \in \mathbb{Z}_+^2}$ and $(N'_{q,l})_{(q,l) \in \mathbb{Z}_+^2}$ in $\tilde{\mathcal{R}}$, there exists $(N''_{q,l})_{(q,l) \in \mathbb{Z}_+^2} \in \tilde{\mathcal{R}}$ such that

\[\tilde{(R2)} \quad \max (N_{q,l}, N'_{q,l}) \leq N''_{q,l}, \quad \forall (q,l) \in \mathbb{Z}_+^2\]

For all $(N_{q,l})_{(q,l) \in \mathbb{Z}_+^2}$ and $(N'_{q,l})_{(q,l) \in \mathbb{Z}_+^2}$ in $\tilde{\mathcal{R}}$, there exists $(N''_{q,l})_{(q,l) \in \mathbb{Z}_+^2} \in \tilde{\mathcal{R}}$ such that

\[\tilde{(R3)} \quad N_{q_1,l_1} + N'_{q_2,l_2} \leq N''_{q_1+q_2,l_1+l_2}, \quad \forall (q_1, q_2, l_1, l_2) \in \mathbb{Z}_+^4\]

Example 4. The set $\mathbb{R}_+^{Z^2}$ of all positive double sequences is regular.

Example 5. The set $\mathcal{B}$ of all bounded sequences of $\mathbb{R}_+^{Z^2}$ is regular.

The following lemma, not difficult to prove, is needed in the formulation of the principal theorem of this paper.

Lemma 1. Let $\tilde{\mathcal{R}}$ be a regular subset of $\mathbb{R}_+^{Z^2}$, then

(i) The subset $\mathcal{R}^0 := \{N_{0} : N \in \tilde{\mathcal{R}}\}$ is regular in $\mathbb{R}_+^{Z^2}$.

(ii) The subset $\mathcal{R}_0 := \{N_{0} : N \in \tilde{\mathcal{R}}\}$ is regular in $\mathbb{R}_+^{Z^2}$. 
3. THE ALGEBRA OF $\mathcal{R}$–REGULAR BOUNDED GENERALIZED FUNCTIONS

Let
\[ S^* (\Omega) = \left\{ f \in C^\infty (\Omega) : \forall \alpha \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |\partial^\alpha f(x)| < \infty \right\}, \]

and $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^\mathbb{Z}$, if we define
\[
\mathcal{E}^\mathcal{R}_{S^*} (\Omega) = \left\{ (u_\epsilon)_\epsilon \in S^* (\Omega)^I : \exists N \in \mathcal{R}, \forall \alpha \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |\partial^\alpha u_\epsilon(x)| = O (\epsilon^{-N_{|\alpha|}}), \epsilon \to 0 \right\},
\]
\[
\mathcal{N}^\mathcal{R}_{S^*} (\Omega) = \left\{ (u_\epsilon)_\epsilon \in S^* (\Omega)^I : \forall N \in \mathcal{R}, \forall \alpha \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |\partial^\alpha u_\epsilon(x)| = O (\epsilon^{N_{|\alpha|}}), \epsilon \to 0 \right\},
\]
where $I = [0, 1]$, then the properties of $\mathcal{E}^\mathcal{R}_{S^*} (\Omega)$ and $\mathcal{N}^\mathcal{R}_{S^*} (\Omega)$, easy to verify, are given by the following results.

**Proposition 2.** (i) The space $\mathcal{E}^\mathcal{R}_{S^*} (\Omega)$ is a subalgebra of $S^* (\Omega)^I$.

(ii) The space $\mathcal{N}^\mathcal{R}_{S^*} (\Omega)$ is an ideal of $\mathcal{E}^\mathcal{R}_{S^*} (\Omega)$.

(iii) We have $\mathcal{N}^\mathcal{R}_{S^*} (\Omega) = \mathcal{N}_{S^*} (\Omega)$, where
\[
\mathcal{N}_{S^*} (\Omega) = \left\{ (u_\epsilon)_\epsilon \in S^* (\Omega)^I : \forall m \in \mathbb{Z}_+, \forall \alpha \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |\partial^\alpha u_\epsilon(x)| = O (\epsilon^m), \epsilon \to 0 \right\}
\]

We have also the null characterization of the ideal $\mathcal{N}_{S^*} (\Omega)$ provided $\Omega$ is a box.

**Definition 3.** An open subset $\Omega$ of $\mathbb{R}^n$ is said to be a box if
\[ \Omega = I_1 \times I_2 \times \ldots \times I_n, \]
where each $I_i$ is a finite or infinite open interval in $\mathbb{R}$.

**Proposition 3.** Let $\Omega$ be a box, then an element $(u_\epsilon)_\epsilon \in \mathcal{E}^\mathcal{R}_{S^*} (\Omega)$ belongs to $\mathcal{N}_{S^*} (\Omega)$ if and only if the following condition is satisfied
\[ \forall m \in \mathbb{Z}_+, \sup_{x \in \Omega} |u_\epsilon(x)| = O (\epsilon^m), \epsilon \to 0 \]

**Proof.** Suppose that $(u_\epsilon)_\epsilon \in \mathcal{E}^\mathcal{R}_{S^*} (\Omega)$ satisfies [1]. It suffices to show that $(\partial_i u_\epsilon)_\epsilon$ satisfies the $\mathcal{N}_{S^*} (\Omega)$ estimates for all $i = 1, \ldots, n$. Suppose that $u_\epsilon$ is real valued, in the complex case, we shall carry out the following calculus separately on its real and imaginary part. Let $m \in \mathbb{Z}_+$, we have to show
\[ \sup_{x \in \Omega} |\partial_i u_\epsilon(x)| = O (\epsilon^m), \epsilon \to 0 \]

Since $(u_\epsilon)_\epsilon \in \mathcal{E}^\mathcal{R}_{S^*} (\Omega)$, then
\[ \exists N \in \mathcal{R}, \sup_{x \in \Omega} |\partial_i^2 u_\epsilon(x)| = O (\epsilon^{-N_2}), \epsilon \to 0 \]

Since $(u_\epsilon)_\epsilon$ satisfies [1], we have
\[ \forall m \in \mathbb{Z}_+, \sup_{x \in \Omega} |u_\epsilon(x)| = O (\epsilon^{N_{2+m}}), \epsilon \to 0 \]

By Taylor’s formula, we have
\[ u_\epsilon(x + \epsilon^{N_{2+m}} e_i) = u_\epsilon(x) + \partial_i u_\epsilon(x) \epsilon^{N_2+m} + \frac{1}{2} \partial_i^2 u_\epsilon(x + \theta \epsilon^{N_2+m} e_i) \epsilon^{2(N_2+m)}, \]
where \( \theta \in ]0,1[ \) and \( \epsilon \) is sufficiently small as \( \Omega \) is a box. It follows that
\[
|\partial_i u_\epsilon(x)| \leq \left| u_\epsilon \left( x + \epsilon^{N_2+m} e_i \right) \right| \epsilon^{-N_2-m} + \left| u_\epsilon(x) \right| \epsilon^{-N_2-m} + \epsilon^{N_2+m} \partial_i^2 u_\epsilon \left( x + \theta \epsilon^{N_2+m} e_i \right) \tag{\ast}
\]

From (\ref{1}), we have (\ast) and (***) are of order \( O(\epsilon^m) \), \( \epsilon \to 0 \); and from (\ref{2}), we have (****) is of order \( O(\epsilon^m) \), \( \epsilon \to 0 \).

**Definition 4.** Let \( \mathcal{R} \) be a regular subset of \( \mathbb{R}^\mathbb{Z}_+ \), the algebra of \( \mathcal{R} \)-regular bounded generalized functions, denoted by \( \mathcal{G}^\mathcal{R}_S(\Omega) \), is the quotient algebra
\[
\mathcal{G}^\mathcal{R}_S(\Omega) = \frac{\mathcal{E}^\mathcal{R}_S(\Omega)}{\mathcal{N}_S(\Omega)}
\]

**Remark 1.** When \( \mathcal{R} \) is the set of all positive sequences the algebra \( \mathcal{G}^\mathcal{R}_S(\Omega) \) is denoted by \( \mathcal{G}_{L^\infty}(\Omega) \) in \([2]\) and \([3]\) as it is constructed on the differential algebra \( D_{L^\infty}(\Omega) \) of Schwartz \([19]\). So it is more correct to write \( \mathcal{G}^\mathcal{R}_S(\Omega) \) instead of \( \mathcal{G}_{L^\infty}(\Omega) \). The null characterization of negligible elements of \( \mathcal{G}_{L^\infty}(\Omega) \) in the case \( \Omega = \mathbb{R}^n \) is given in \([9]\).

4. **The algebra of \( \mathcal{R} \)-regular roughly decreasing generalized functions**

Let
\[
\mathcal{S}_S(\Omega) = \left\{ f \in \mathcal{C}^\infty(\Omega) : \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |x^\beta f(x)| < \infty \right\},
\]
and \( \mathcal{R} \) be a regular subset of \( \mathbb{R}^\mathbb{Z}_+ \), if we define
\[
\mathcal{E}^\mathcal{R}_S(\Omega) = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}_S(\Omega)^I : \exists N \in \mathcal{R}, \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |x^\beta u_\epsilon(x)| = O(\epsilon^{-N_{|\beta|}}), \epsilon \to 0 \right\},
\]
\[
\mathcal{N}^\mathcal{R}_S(\Omega) = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}_S(\Omega)^I : \forall N \in \mathcal{R}, \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |x^\beta u_\epsilon(x)| = O(\epsilon^{N_{|\beta|}}), \epsilon \to 0 \right\},
\]
then the following properties of \( \mathcal{E}^\mathcal{R}_S(\Omega) \) and \( \mathcal{N}^\mathcal{R}_S(\Omega) \) are easy to verify.

**Proposition 4.** (i) The space \( \mathcal{E}^\mathcal{R}_S(\Omega) \) is a subalgebra of \( \mathcal{S}_S(\Omega)^I \).
(ii) The space \( \mathcal{N}^\mathcal{R}_S(\Omega) \) is an ideal of \( \mathcal{E}^\mathcal{R}_S(\Omega) \).
(iii) We have \( \mathcal{N}^\mathcal{R}_S(\Omega) = \mathcal{N}_S(\Omega) \), where
\[
\mathcal{N}_S(\Omega) = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}_S(\Omega)^I : \forall m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |x^\beta u_\epsilon(x)| = O(\epsilon^m), \epsilon \to 0 \right\}
\]

The following proposition characterizes \( \mathcal{N}_S(\Omega) \).

**Proposition 5.** Let \( (u_\epsilon)_\epsilon \in \mathcal{E}^\mathcal{R}_S(\Omega) \), then \( (u_\epsilon)_\epsilon \in \mathcal{N}_S(\Omega) \) if and only if the following condition is satisfied
\[
\forall m \in \mathbb{Z}_+, \sup_{x \in \Omega} |u_\epsilon(x)| = O(\epsilon^m), \epsilon \to 0 \tag{\ref{5}}
\]

**Proof.** Suppose that \( (u_\epsilon)_\epsilon \in \mathcal{E}^\mathcal{R}_S(\Omega) \) satisfies (\ref{5}), since \( (u_\epsilon)_\epsilon \in \mathcal{E}^\mathcal{R}_S(\Omega) \), then \( \exists N \in \mathcal{R}, \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \Omega} |x^{2\beta} u_\epsilon(x)| = O(\epsilon^{-N_{2|\beta|}}), \epsilon \to 0 \).

From (\ref{5}), we have
\[
\forall m \in \mathbb{Z}_+, \sup_{x \in \Omega} |u_\epsilon(x)| = O(\epsilon^{2m+N_{2|\beta|}}), \epsilon \to 0
\]

Therefore $\forall x \in \Omega$,
\[ |x^\beta u_\epsilon (x)|^2 = |x^{2\beta} u_\epsilon (x)| |u_\epsilon (x)| = O (\epsilon^m), \; \epsilon \to 0, \]
hence
\[ \forall m \in \mathbb{Z}_+, |x^\beta u_\epsilon (x)| = O (\epsilon^m), \; \epsilon \to 0 \]

**Remark 2.** The $C^\infty$ regularity in the definition of elements of $G^R_{S_\epsilon} (\Omega)$ is not in fact needed in the proof of the principal results of this work.

5. **The algebra of $R$–regular rapidly decreasing generalized functions**

Let
\[ S (\Omega) = \left\{ f \in C^\infty (\Omega) : \forall (\alpha, \beta) \in \mathbb{Z}_+^{2n}, \sup_{x \in \Omega} |x^\beta \partial^\alpha f (x)| < \infty \right\}, \]
called the space of rapidly decreasing functions on $\Omega$, and let $\tilde{R}$ be a regular subset of $\mathbb{R}_+^{2n}$, if we define
\[ E_{S_\epsilon}^R (\Omega) = \left\{ (u_\epsilon)_\epsilon \in S (\Omega)^I : \exists N \in \tilde{R}, \forall (\alpha, \beta) \in \mathbb{Z}_+^{2n}, \sup_{x \in \Omega} |x^\beta \partial^\alpha u_\epsilon (x)| = O (\epsilon^{-N|\alpha|,|\beta|}), \epsilon \to 0 \right\}, \]
\[ N_{S_\epsilon}^R (\Omega) = \left\{ (u_\epsilon)_\epsilon \in S (\Omega)^I : \forall N \in \tilde{R}, \forall (\alpha, \beta) \in \mathbb{Z}_+^{2n}, \sup_{x \in \Omega} |x^\beta \partial^\alpha u_\epsilon (x)| = O (\epsilon^N|\alpha|,|\beta|), \epsilon \to 0 \right\}, \]
then we have the following results.

**Proposition 6.** We have the following assertions

(i) The space $E_{S_\epsilon}^R (\Omega)$ is a subalgebra of $S (\Omega)^I$.

(ii) The space $N_{S_\epsilon}^R (\Omega)$ is an ideal of $E_{S_\epsilon}^R (\Omega)$.

(iii) We have $N_{S_\epsilon}^R (\Omega) = N_S (\Omega)$, where
\[ N_S (\Omega) = \left\{ (u_\epsilon)_\epsilon \in S (\Omega)^I : \forall m \in \mathbb{Z}_+, \forall (\alpha, \beta) \in \mathbb{Z}_+^{2n}, \sup_{x \in \Omega} |x^\beta \partial^\alpha u_\epsilon (x)| = O (\epsilon^m), \epsilon \to 0 \right\} \]

**Proof.** The proof is not difficult, it follows from the properties of the set $\tilde{R}$. \hfill \Box

**Definition 6.** Let $\tilde{R}$ be a regular subset of $\mathbb{R}_+^{2n}$, the algebra of $\tilde{R}$–regular rapidly decreasing generalized functions on $\Omega$, denoted by $G^R_{S_\epsilon} (\Omega)$, is the quotient algebra
\[ G_{S_\epsilon}^R (\Omega) = \frac{E_{S_\epsilon}^R (\Omega)}{N_S (\Omega)} \]

**Example 6.** (i) For $\tilde{R} = \mathbb{R}_+^{2n}$, we obtain the algebra $G_S (\Omega)$ of rapidly decreasing generalized functions on $\Omega$, see [13].

(ii) Pour $\tilde{R} = \tilde{B}$, we obtain the algebra $G_{S_\epsilon}^R (\Omega)$ of regular rapidly decreasing generalized functions on $\Omega$, see [8].
6. Characterization of $\mathcal{R}$–regular rapidly decreasing generalized functions

Let us mention that the theorem of [4] can be extended to an open subset $\Omega$ of $\mathbb{R}^n$ provided $\Omega$ is a box.

**Proposition 7.** If $\Omega$ is a box of $\mathbb{R}^n$, then

$$\mathcal{S} (\Omega) = \mathcal{S}^* (\Omega) \cap \mathcal{S}_* (\Omega)$$

**Proof.** The proof is the same as in [4], noting that, in Taylor’s expansion, the hypothesis that $\Omega$ is a box assures that $(x_1 + h, x')$ stays in $\Omega$ for all $(x_1, x') \in \Omega$ and $h > 0$ sufficiently small. □

The principal result of this section is an extension of (6) to the algebra of $\mathcal{R}$–regular rapidly decreasing generalized functions. It is the first characterization of the algebra $\mathcal{G}_\mathcal{R}^\mathcal{R} (\Omega)$.

**Theorem 8.** If $\Omega$ is a box, then

$$\mathcal{G}_\mathcal{R}^\mathcal{R} (\Omega) = \mathcal{G}^{\mathcal{R}_0}_\mathcal{S}_* (\Omega) \cap \mathcal{G}^{\mathcal{R}_0}_\mathcal{S}^* (\Omega)$$

**Proof.** We have to show that $\mathcal{S} (\Omega) = \mathcal{S}^* (\Omega) \cap \mathcal{S}_* (\Omega)$.

The inclusions $\mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega) \subset \mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega) \cap \mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega)$ and $\mathcal{N}^\mathcal{R}_\mathcal{S} (\Omega) \subset \mathcal{N}^\mathcal{R}_\mathcal{S} (\Omega) \cap \mathcal{N}^\mathcal{R}_\mathcal{S} (\Omega)$ are obvious. In order to show the reverse inclusions, first let $(u_\epsilon)_\epsilon \in \mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega) \cap \mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega)$, then $(u_\epsilon)_\epsilon \in \mathcal{S}^* (\Omega) \cap \mathcal{S}_* (\Omega)' \cap \mathcal{S}_* (\Omega)' = \mathcal{S} (\Omega)'$. In order to show that $(u_\epsilon)_\epsilon$ satisfies the estimate of $\mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega)$, set $x = (x_1, x') \in I_1 \times (I_2 \times I_3 \times \ldots \times I_n) := \Omega$ and consider in first the case $x_1 > 0$. For $h > 0$ sufficiently small, a Taylor’s expansion of $u_\epsilon$ with respect to $x_1$ gives

$$u_\epsilon (x_1 + h, x') = u_\epsilon (x_1, x') + h \partial_1 u_\epsilon (x_1, x') + \frac{h^2}{2} \partial_1^2 u_\epsilon (\xi, x'),$$

for $\xi \in [x_1, x_1 + h]$. The hypothesis $(u_\epsilon)_\epsilon \in \mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega) \cap \mathcal{S}^\mathcal{R}_\mathcal{S} (\Omega)$ gives

$$\exists L \in \mathcal{R}_0, \forall k \in \mathbb{Z}_+, \sup_{x_1 > 0} (1 + |x|)^k |u_\epsilon (x)| = O (\epsilon^{-L_k}), \epsilon \longrightarrow 0$$

$$\exists M \in \mathcal{R}_0, \sup_{x_1 > 0} |\partial_1^2 u_\epsilon (x)| = O (\epsilon^{-M_2}), \epsilon \longrightarrow 0$$

We have

$$\sup_{x_1 > 0} (1 + |x|)^k |u_\epsilon (x_1 + h, x')| \leq \sup_{x_1 > 0} (1 + |(x_1 + h, x')|)^k |u_\epsilon (x_1 + h, x')| = O (\epsilon^{-L_k}), \epsilon \longrightarrow 0$$

It follows from (7)

$$|\partial_1 u_\epsilon (x_1, x')| \leq \frac{1}{h} |u_\epsilon (x_1 + h, x')| + |u_\epsilon (x_1, x')| + \frac{h}{2} |\partial_1^2 u_\epsilon (\xi, x')|$$

Therefore

$$\sup_{x_1 > 0} (1 + |x|)^k |\partial_1 u_\epsilon (x)| = O (\epsilon^{-L_k-M_2}), \epsilon \longrightarrow 0$$

From (R3) of definition [2], there exists $N' \in \mathcal{R}$ such that

$$L_k + M_2 \leq N'_{2, k}$$

consequently

$$\forall \beta \in \mathbb{Z}_+^n, \sup_{x_1 > 0} |x^\beta \partial_1 u_\epsilon (x)| \leq C \sup_{x_1 > 0} (1 + |x|)^{|\beta|} |\partial_1 u_\epsilon (x)| = O (\epsilon^{-N'_{2, |\beta|}}), \epsilon \longrightarrow 0$$
If \( x_1 < 0 \), consider \( v_\epsilon \) such that \( v_\epsilon (x) = u_\epsilon (-x_1, x') \). We see that \( (v_\epsilon)_\epsilon \in \mathcal{E}^R_{S^*} (\Omega) \cap \mathcal{E}^C_{S^*} (\Omega) \) and consequently the above arguments give the existence of \( N'' \in \mathcal{R} \) such that

\[
\sup_{x_1 > 0} |x^\beta \partial_1 v_\epsilon (x)| = \sup_{x_1 < 0} |x^\beta \partial_1 u_\epsilon (x)| = O \left( \epsilon ^{N''} \right) , \epsilon \to 0
\]

Now from \( (\mathcal{R}1) \) and \( (\mathcal{R}2) \) of definition \( \mathcal{R} \) there exists \( N \in \mathcal{R} \) such that

\[
\max \left( N'_2 |\beta|, N''_2 |\beta| \right) \leq N_1 |\beta| ,
\]

so

\[
\sup_{x \in \Omega} |x^\beta \partial_1 u_\epsilon (x)| = O \left( \epsilon ^{N_1 |\beta|} \right) , \epsilon \to 0
\]

Analogously we show

\[
\exists N \in \mathcal{R}; \forall \beta \in \mathbb{Z}^n_+, \sup_{x \in \Omega} |x^\beta \partial_1 u_\epsilon (x)| = O \left( \epsilon ^{N_2 |\beta|} \right) , \epsilon \to 0 , i = 2, ..., n
\]

Therefore, by induction, we obtain

\[
\exists N \in \mathcal{R}; \forall \alpha \in \mathbb{Z}^n_+, \forall \beta \in \mathbb{Z}^n_+, \sup_{x \in \Omega} |x^\beta \partial_1 u_\epsilon (x)| = O \left( \epsilon ^{N_{2.1} |\beta|} \right) , \epsilon \to 0 ,
\]

i.e. \( (u_\epsilon)_\epsilon \in \mathcal{E}^\mathbb{R}_{S^*} (\Omega) \).

Suppose now that \( (u_\epsilon)_\epsilon \in \mathcal{N}_{S^*} (\Omega) \cap \mathcal{N}_{S^*} (\Omega) \), then

\[
\forall m \in \mathbb{Z}^n_+, \forall k \in \mathbb{Z}^n_+, \sup_{x_1 > 0} (1 + |x|)^k |x^\beta \partial_1 u_\epsilon (x)| = O \left( \epsilon ^{m} \right) , \epsilon \to 0
\]

\[
\forall m \in \mathbb{Z}^n_+, \sup_{x_1 > 0} |\partial_1^2 u_\epsilon (x)| = O \left( \epsilon ^{m} \right) , \epsilon \to 0
\]

We have

\[
\sup_{x_1 > 0} (1 + |x|)^k |u_\epsilon (x_1 + h, x')| \leq \sup_{x_1 > 0} (1 + |(x_1 + h, x')|)^k |u_\epsilon (x_1 + h, x')| = O \left( \epsilon ^{m} \right) , \epsilon \to 0
\]

It follows from \( (\mathcal{R}7) \)

\[
\sup_{x_1 > 0} (1 + |x|)^k |\partial_1 u_\epsilon (x)| = O \left( \epsilon ^{m} \right) , \epsilon \to 0
\]

Consequently

\[
\forall m \in \mathbb{Z}^n_+, \forall \beta \in \mathbb{Z}^n_+, \sup_{x_1 > 0} \left| x^\beta \partial_1 u_\epsilon (x) \right|^2 \leq C_1 \sup_{x_1 > 0} (1 + |x|)^{|\beta|} |\partial_1 u_\epsilon (x)| = O \left( \epsilon ^{m} \right) , \epsilon \to 0
\]

If \( x_1 < 0 \), consider \( v_\epsilon \) such that \( v_\epsilon (x) = u_\epsilon (-x_1, x') \) as above, then we obtain

\[
\sup_{x_1 > 0} \left| x^\beta \partial_1 v_\epsilon (x) \right|^2 = \sup_{x_1 < 0} \left| x^\beta \partial_1 u_\epsilon (x) \right|^2 = O \left( \epsilon ^{m} \right) , \epsilon \to 0
\]

Therefore, by induction, we have

\[
\forall m \in \mathbb{Z}^n_+, \forall \alpha \in \mathbb{Z}^n_+, \forall \beta \in \mathbb{Z}^n_+, \sup_{x \in \Omega} |x^\beta \partial_1 u_\epsilon (x)| = O \left( \epsilon ^{m} \right) , \epsilon \to 0
\]

Thus \( \mathcal{N}_{S^*} (\Omega) \cap \mathcal{N}_{S^*} (\Omega) \subset \mathcal{N}_{S^*} (\Omega) \) and consequently \( \mathcal{G}_{S^*}^\mathbb{R} (\Omega) = \mathcal{G}_{S^*}^C (\Omega) \cap \mathcal{G}_{S^*}^C (\Omega) \). \( \square \)

The propositions \( \mathcal{R}3 \) and \( \mathcal{R}5 \) give the following result which characterizes the negligible elements of the algebra \( \mathcal{G}_{S^*}^\mathbb{R} (\Omega) \).
Corollary 9. If $\Omega$ is a box, then an element $(u_\epsilon)_\epsilon \in \mathcal{E}_S^\mathcal{R}(\Omega)$ is in $\mathcal{N}_S(\Omega)$ if and only if the following condition is satisfied
\[\forall m \in \mathbb{Z}_+, \sup_{x \in \Omega} |u_\epsilon(x)| = O(\epsilon^m), \epsilon \to 0\]

7. Characterization of $\mathcal{R}$—regular rapidly decreasing generalized functions via Fourier transform

The direct Fourier transform of $u \in \mathcal{S}$, denoted $\hat{u}$, is defined by
\[\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\xi} u(x) \, dx\]

Definition 7. The Fourier transform of $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_S^\mathcal{R}$, denoted by $\mathcal{F}_S(u)$, is defined by
\[\mathcal{F}_S(u) = \hat{u} = [(\hat{u}_\epsilon)_\epsilon] \text{ in } \mathcal{G}_S^\mathcal{R}\]

Remark 3. The inverse Fourier transform of $u \in \mathcal{S}$, denoted $\tilde{u}$, and the map $\mathcal{F}_S^{-1}$ are defined as usually and in the same way.

The following proposition gives one of the main results of the Fourier transform $\mathcal{F}_S$ and is easy to prove.

Proposition 10. The map
\[\mathcal{F}_S : \mathcal{G}_S^\mathcal{R} \to \mathcal{G}_S^\mathcal{R}\]
is an algebraic isomorphism.

Let
\[\mathcal{S}^\mathcal{S} = \left\{ f \in \mathcal{C}^\infty : \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \hat{f}(\xi)| < \infty \right\},\]
and let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+^n}$, if we define
\[\mathcal{E}_S^{\mathcal{R}^0} = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^\mathcal{S} : \exists N \in \mathcal{R}^0, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \hat{u}_\epsilon(\xi)| = O(\epsilon^{-N|\beta|}), \epsilon \to 0 \right\},\]
\[\mathcal{N}_S^{\mathcal{R}^0} = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^\mathcal{S} : \forall N \in \mathcal{R}^0, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \hat{u}_\epsilon(\xi)| = O(\epsilon^N|\beta|), \epsilon \to 0 \right\},\]
then the following proposition is easy to prove.

Proposition 11. (i) The space $\mathcal{E}_S^{\mathcal{R}^0}$ is a subalgebra of $\mathcal{S}^\mathcal{S}$.
(ii) The space $\mathcal{N}_S^{\mathcal{R}^0}$ is an ideal of $\mathcal{E}_S^{\mathcal{R}^0}$.
(iii) The ideal $\mathcal{N}_S^{\mathcal{R}^0} = \mathcal{N}_S$, where
\[\mathcal{N}_S := \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^\mathcal{S} : \forall m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \hat{u}_\epsilon(\xi)| = O(\epsilon^m), \epsilon \to 0 \right\}\]

The following proposition characterizes $\mathcal{N}_S$.

Proposition 12. Let $(u_\epsilon)_\epsilon \in \mathcal{E}_S^{\mathcal{R}^0}$, then $(u_\epsilon)_\epsilon \in \mathcal{N}_S^{\mathcal{R}^0}$ if and only if the following condition is satisfied
\[\forall m \in \mathbb{Z}_+, \sup_{\xi \in \mathbb{R}^n} |\hat{u}_\epsilon(\xi)| = O(\epsilon^m), \epsilon \to 0\]
Proof. The proof is similar to that of proposition 5. □

Definition 8. The algebra $G_{S_z}^{R_0}$ is defined as the quotient algebra

\[ G_{S_z}^{R_0} = \frac{E_{S_z}^{R_0}}{N_{S_z}}. \]

The next theorem is the second characterization of $G_{S}^{\bar{R}}$.

Theorem 13. We have

\[ G_{S}^{\bar{R}} = G_{S_z}^{R_0} \cap G_{S_z}^{R_0}. \]

Proof. Let $(u_\epsilon) \in E_{S_z}^{R_0}$, then $\exists C > 0$ such that

\[ \int |x^\beta \hat{u}_\epsilon(x)| \, dx \leq C \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^n |x^\beta \hat{u}_\epsilon(x)| \]

\[ = O \left( \epsilon^{-N|\beta| + 2n} \right), \quad \epsilon \to 0 \]

\[ = O \left( \epsilon^{-N|\beta|} \right), \quad \epsilon \to 0, \]

for some $N \in \mathcal{R}^0$. The continuity of $F$ from $L^1$ to $L^\infty$ gives

\[ ||\partial^\beta u_\epsilon||_{L^\infty} = O \left( \epsilon^{-N|\beta|} \right), \quad \epsilon \to 0, \]

which shows that $(u_\epsilon) \in E_{S_z}^{R_0}$ and therefore $E_{S_z}^{R_0} \subset E_{S_z}^{R_0}$. Consequently $E_{S_z}^{R_0} \cap E_{S_z}^{R_0} \subset E_{S}^{\bar{R}}$. In order to show the inverse inclusion let us mention, at first, that from [4], we have

\[ (u_\epsilon) \in S^l \iff (u_\epsilon) \in S^l \cap \hat{S}^l \]

which implies in particular that $S \subset \hat{S}$. On the other hand if $(u_\epsilon) \in E_{S}^{\bar{R}}$, then

\[ \int |\partial^\beta u_\epsilon(x)| \, dx \leq C \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^n |\partial^\beta u_\epsilon(x)| \]

\[ = O \left( \epsilon^{-N|\beta| + 2n} \right), \quad \epsilon \to 0 \]

\[ = O \left( \epsilon^{-N|\beta|,0} \right), \quad \epsilon \to 0, \]

i.e.

\[ \int |\partial^\beta u_\epsilon(x)| \, dx = O \left( \epsilon^{-N|\beta|} \right), \quad \epsilon \to 0, \]

for some $N \in \mathcal{R}^0$. The continuity of $F$ from $L^1$ to $L^\infty$ gives

\[ ||\xi^\beta \hat{u}_\epsilon||_{L^\infty} = O \left( \epsilon^{-N|\beta|} \right), \quad \epsilon \to 0, \]

which shows that $(u_\epsilon) \in E_{S_z}^{R_0}$ and consequently $(u_\epsilon) \in E_{S_z}^{R_0} \cap E_{S_z}^{R_0}$. Thus $E_{S_z}^{\bar{R}} \subset E_{S_z}^{R_0} \cap E_{S_z}^{R_0}$, it follows that $E_{S_z}^{\bar{R}} = E_{S_z}^{R_0} \cap E_{S_z}^{R_0}$. A similar proof shows that $N_S = N_{S_z} \cap N_{S_z}$. Therefore $G_{S}^{\bar{R}} = G_{S_z}^{R_0} \cap G_{S_z}^{R_0}$. □

The following corollary gives a second characterization of the space $N_S$.

Corollary 14. An element $(u_\epsilon) \in E_{S}^{\bar{R}}$ is in $N_S$ if and only if the following condition is satisfied

\[ \forall m \in \mathbb{Z}_+, \sup_{\xi \in \mathbb{R}^n} |\hat{u}_\epsilon(\xi)| = O \left( \epsilon^m \right), \quad \epsilon \to 0 \]
8. Consequences

We know that when \( \tilde{\mathcal{R}} = \mathbb{R}^2_+ \) we obtain \( \mathcal{G}_{S^2}^\mathbb{Z} \mathcal{S} = \mathcal{G}_S \). Theorem 8 gives the following corollary which is a characterization of the algebra of rapidly decreasing generalized functions.

**Corollary 15.** We have
\[
\mathcal{G}_S = \mathcal{G}_{S^*} \cap \mathcal{G}_{S_*},
\]
where
\[
\mathcal{G}_{S^*} := \left\{ (u_\epsilon)_{\epsilon} \in S^*: \forall \alpha \in \mathbb{Z}^n_+, \exists m \in \mathbb{Z}_+, \sup_{x \in \mathbb{R}^n} |\partial^\alpha u_\epsilon (x)| = O (\epsilon^{-m}), \epsilon \rightarrow 0 \right\},
\]
and
\[
\mathcal{G}_{S_*} := \left\{ (u_\epsilon)_{\epsilon} \in S_*: \forall \beta \in \mathbb{Z}^n_+, \exists m \in \mathbb{Z}_+, \sup_{x \in \mathbb{R}^n} |x^\beta u_\epsilon (x)| = O (\epsilon^m), \epsilon \rightarrow 0 \right\}
\]

We have also the following corollary which is another characterization of the algebra \( \mathcal{G}_S \).

**Corollary 16.** We have
\[
\mathcal{G}_S = \mathcal{G}_{S^*} \cap \mathcal{G}_{S_*},
\]
where
\[
\mathcal{G}_{S^*} := \left\{ (u_\epsilon)_{\epsilon} \in \hat{\mathcal{S}}^*: \forall \beta \in \mathbb{Z}^n_+, \exists m \in \mathbb{Z}_+, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \hat{u}_\epsilon (\xi)| = O (\epsilon^m), \epsilon \rightarrow 0 \right\}
\]
and
\[
\mathcal{G}_{S_*} := \left\{ (u_\epsilon)_{\epsilon} \in \hat{\mathcal{S}}_*: \forall \beta \in \mathbb{Z}^n_+, \exists m \in \mathbb{Z}_+, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \hat{u}_\epsilon (\xi)| = O (\epsilon^{-m}), \epsilon \rightarrow 0 \right\}
\]

We also know that when \( \tilde{\mathcal{R}} = \tilde{\mathcal{B}} \) we obtain \( \mathcal{G}_S^{\infty} = \mathcal{G}_S^{\infty} \). The next result, which is a corollary of theorem 8 for \( \mathcal{R} = \tilde{\mathcal{B}} \), gives a characterization of \( \mathcal{G}_S^{\infty} \).

**Corollary 17.** We have
\[
\mathcal{G}_S^{\infty} = \mathcal{G}_{S^*} \cap \mathcal{G}_{S_*},
\]
where
\[
\mathcal{G}_{S^*} := \left\{ (u_\epsilon)_{\epsilon} \in S^*: \exists m \in \mathbb{Z}_+, \forall \alpha \in \mathbb{Z}^n_+, \sup_{x \in \mathbb{R}^n} |\partial^\alpha u_\epsilon (x)| = O (\epsilon^{-m}), \epsilon \rightarrow 0 \right\},
\]
and
\[
\mathcal{G}_{S_*} := \left\{ (u_\epsilon)_{\epsilon} \in S_*: \exists m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}^n_+, \sup_{x \in \mathbb{R}^n} |x^\beta u_\epsilon (x)| = O (\epsilon^{-m}), \epsilon \rightarrow 0 \right\}
\]

We have also the following result obtained as a corollary of the theorem 13.
Corollary 18. We have
\[ G_\infty^c = G_\infty^c \cap G_\infty^c \]
where
\[
G_\infty^c := \left\{ (u_\epsilon)_\epsilon \in \widehat{S}^i : \exists m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}^n_+, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \widehat{u_\epsilon}(\xi)| = O(\epsilon^{-m}), \epsilon \to 0 \right\}
\]

\[
\left\{ (u_\epsilon)_\epsilon \in \widehat{S}^i : \forall m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}^n_+, \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \widehat{u_\epsilon}(\xi)| = O(\epsilon^m), \epsilon \to 0 \right\}
\]

REFERENCES

[1] J. Alvarez, H. Obiedat, Characterizations of the Schwartz space $S$ and the Beurling-Björck space $S_\omega$. Cubo 6, p. 167-183, (2004).
[2] H. Biagioni, M. Oberguggenberger, Generalized solutions to the Korteweg-de Vries and the generalized long-wave equations. SIAM J. Math Anal., 23(4), p. 923-940, (1992).
[3] C. Bouzar, K. Benmeriem, Ultraregular generalized functions. Oran-Essenia University, Preprint (2006).
[4] S.-Y. Chung, D. Kim et S. Lee. Characterization for Beurling-Bjorck space and Schwartz space. Proc. A.M.S. Vol. 125, No 11, p. 3229-3234, (1997).
[5] J. F. Colombeau, Elementary introduction to new generalized functions, North Holland, (1985).
[6] J. F. Colombeau, A. Heibig, M. Oberguggenberger, Generalized solutions to PDEs of evolution type. Acta Appl. Math. 45, p.115–142. (1996).
[7] A. Delcroix, Fourier Transform of rapidly decreasing generalized functions. Application to microlocal regularity. J. Math. Anal. Appl., 327, p. 564-584, (2007).
[8] C. Garetto, Pseudo-differential operators in algebras of generalized functions and global hypoellipticity. Acta Appl. Math., 80(2):1, p. 123-174, (2004).
[9] C. Garetto, Topological structures in Colombeau algebras: investigation of the duals of $G_c(\Omega), G(\Omega)$ and $G(\mathbb{R}^n)$. Monatsh. Math., 146(3), p. 203–226, (2005).
[10] S. Gindikin, L. Volevich. The Cauchy problem and related problems for convolution equations. Uspehi Mat. Nauk, vol. 27, no. 4, 65–143, (1972).
[11] K. Gröchenig, G. Zimmermann, Spaces of test functions via the STFT, J. Funct. Spaces Appl. 2, p. 25–53, (2004).
[12] I. M. Gelfand, G. E. Shilov. Generalized functions, vol. 2, Academic Press, (1967).
[13] M. Grosser, M Kunzinger, M. Oberguggenberger, R. Steinbauer. Geometric theory of generalized functions with applications to general relativity. Kluwer, (2001).
[14] L. Hörmander, Distributions theory and Fourier analysis. Springer, (1983).
[15] A. I. Kashipirovski, Equality of the spaces $S_\beta^0$ and $S_\infty^0 \cap S_\beta$, Funct. Anal. Appl., 14, p. 129, (1980).
[16] M. Oberguggenberger, Regularity theory in Colombeau algebras. Bull. T. CXXXIII. Acad. Serbe Sci. Arts, Cl. Sci. Math. Nat. Sci. Math., 31, p. 147-162 , (2006).
[17] N. Ortner, P. Wagner, Applications of weighted $D\epsilon_{\rho,l}$-spaces to the convolution of distributions. Bull. Polish Acad. Sc. Math., Vol. 37, N° 7-12, p. 579-595, (1989).
[18] Ya. V. Radyno, Sabra Ramdan, Ngo Fu Tkhan, The Fourier transform in an algebra of new generalized functions. Russian Acad. Sci. Dokl. Math. Vol. 46, N°3, p. 414-417 , (1993).
[19] L. Schwartz, Théorie des distributions. Herman, Paris, 2ème Ed., (1966).

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