Bayesian Group Learning for Shot Selection of Professional Basketball Players

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Abstract

In this paper, we develop a group learning approach to analyze the underlying heterogeneity structure of shot selection among professional basketball players in the NBA. We propose a mixture of finite mixtures (MFM) model to capture the heterogeneity of shot selection among different players based on Log Gaussian Cox process (LGCP). Our proposed method can simultaneously estimate the number of groups and group configurations. An efficient Markov Chain Monte Carlo (MCMC) algorithm is developed for our proposed model. Ultimately, our proposed learning approach is further illustrated in analyzing shot charts of several plays in the NBA’s 2017–2018 regular season.

Keywords: Basketball Shot Charts; Heterogeneity Pursuit; Log Gaussian Cox Process; Mixture of Finite Mixtures; Nonparametric Bayesian

1 Introduction

Shot charts are important summaries for basketball coaches. A shot chart is a spatial representation of a basketball player’s shot selections. Good defense strategies depend on thorough understandings of shot selections for different players. Shot selection data
have been discussed from different statistical perspectives. Reich et al. (2006) developed a spatially varying coefficients model for shot-chart data. Miller et al. (2014) analyzed the underlying spatial structure among professional basketball players based on spatial point processes. Franks et al. (2015) combined spatial and spatio-temporal processes, matrix factorization techniques and hierarchical regression models for characterizing the spatial structure of locations for shot attempts. The shot attempts can be viewed as a spatial point process, where the locations are random. Spatial point process are well discussed in many statistical literatures, such as the Poisson process (Geyer, 1998), the Gibbs process (Goulard et al., 1996), and the Log Gaussian Cox process (LGCP; Møller et al., 1998). In addition, spatial point process have been applied to different areas, such as ecological studies (Thurman et al., 2015; Jiao et al., 2020), environmental sciences (Veen and Schoenberg, 2006; Hu et al., 2019), and sports analytics (Miller et al., 2014; Jiao et al., 2019). Most existing literatures concentrate on parametric (Guan, 2008) or nonparametric (Guan, 2008; Geng et al., 2019) estimation of the underlying intensities for spatial point process and analysis of second-order properties (Diggle et al., 2007). There are very limited literatures discussing the grouping pattern of multiple point processes. Knowing the group information of different point processes will lead to discovery of the underlying heterogeneity structure of different players.

Chen et al. (2019) developed a group linked Cox process model for analyzing point of interest (POI) data in Beijing. To determine the number of groups, starting from the most complicated model where each observation has its own group, a loss function is used in a series of hierarchical merging steps to combine the groups. Such procedure will ignore uncertainty in estimation of the number of groups. In contrast, Bayesian models such as Dirichlet process (DP; Ferguson, 1973) offer a natural solution that simultaneously estimates the number of groups and group configurations. However, Miller and Harrison (2013) shows that Dirichlet process mixture model (DPMM) tends to create tiny extraneous groups. In other words, DPMM does not produce consistent estimator of the number of groups. In this paper, we employ the mixture of finite mixture (MFM; Miller and Harrison, 2018) approach for learning group structure of
multiple spatial point processes, which, on the contrary, provides consistent estimation for group numbers.

The contribution of this paper is two-fold. First, we propose a Bayesian group learning method to simultaneously estimate the number of groups and the group configurations. In particular, we use an LGCP to model the spatial pattern of the shot attempts. Based on similarity matrices of fitted intensity among different players, a MFM model is incorporated for group learning. Moreover, the MFM model has a Pólya urn scheme similar to the Chinese restaurant process, which is exploited to develop an efficient Markov chain Monte Carlo MCMC algorithm without reversible jump or even allocation samplers. In addition, our proposed Bayesian approach reveals interesting shooting patterns of professional basketball players which summaries better characterize player types than any traditional categorization.

The rest of the paper is organized as follows. In Section 2, the shot chart data of different players from the 2017–2018 NBA regular season is introduced. In Section 3, we discuss the LGCP and develop the Bayesian group learning method based on MFM. Details of the Bayesian inference are presented in Section 4, including the MCMC algorithm and post MCMC inference methods. Applications of the proposed methods to NBA players data are reported in Section 6. Section 7 concludes the paper with a discussion.

2 Motivating Data

Our data consists of made and missed field goal attempt locations from the offensive half court of games in the 2017–2018 National Basketball Association (NBA) regular season. The data is available at http://nbasavant.com/index.php. We focus on players that have made more than 400 field goal attempts (FTA). Also, players who just started their careers in the 2017–2018 season, such as Lonzo Ball and Jayson Tatum, are not considered. A total of 191 plays who meet the two criteria above are selected for our analysis.
We model a player’s shooting location choices and outcomes as a spatial point pattern on the offensive half court, a 47 ft by 50 ft rectangle, which is the standard size for NBA. We assume the spatial domain \( D \in [0, 47] \times [0, 50] \). Indexing the players with \( i \in \{1, \ldots, 191\} \), the locations of shots for player \( i \) are denoted as \( X_i = \{x_{i,1}, \ldots, x_{i,T_i}\} \), \( \forall x_{i,T_i} \in D \), where \( T_i \) is the total number of attempts made by player \( i \) on the offensive half court. The outcome of each shot attempt is denoted by \( x_{i,t} \) for \( t = 1, \ldots, T_i \), where \( x_{i,t} = 1 \) indicates that player \( n \)’s \( i \)-th shot was made, and \( x_{i,t} = 0 \) if missed. We select nine players and visualize their shot charts in Figure 1. It can be seen that Clint Capela has more shots in the painted area, as most of his field goals are slam dunks. JJ Redick, however, has more shots outside the painted area. Our goal is to find groups of similar shooting location habits among the NBA basketball players.
3 Method

3.1 Log-Gaussian Cox Process

Spatial point pattern data are modeled by spatial point processes (Diggle et al., 1976) characterized by a quantity called intensity. Within an area \( B \), the intensity on any location \( s \in B \) can be represented as \( \lambda(s) \), which is defined as:

\[
\lambda(s) = \lim_{|ds| \to 0} \left( \frac{E[N(ds) | ds]}{|ds|} \right),
\]

where \( ds \) is an infinitesimal region around \( s \), \(|ds|\) represents its area, and \( N(ds) \) shows the number of events that happened over \( ds \). The Poisson distribution is often used to model count data, and the spatial Poisson point process is widely used for point pattern data. A spatial point pattern is a dataset \( y = (s_1, s_2, ..., s_\ell) \) consisting of the locations of points, denoted as \( (s_1, s_2, ..., s_\ell) \), that are observed in a bounded region \( B \subseteq \mathbb{R}^2 \), which is a realization of the spatial point process \( Y \). We denote by \( N_Y(A) = \sum_{i=1}^\ell 1(s_i \in A) \) the counting process associated with the spatial point process \( Y \), which counts the number of points of \( Y \) for an area \( A \subseteq B \). For the Poisson process \( Y \) over \( B \), which has the intensity function \( \lambda(s) \), \( N_Y(A) \sim \text{Poisson}(\lambda(A)) \), where \( \lambda(A) = \int_A \lambda(s)ds \). In addition, if two areas \( A_1 \) and \( A_2 \) are disjoint, then \( N_Y(A_1) \) and \( N_Y(A_2) \) are independent. Based on such property of the Poisson process, it is easy to obtain \( E(N_Y(A)) = \text{Var}(N_Y(A)) = \lambda(A) \). When \( \lambda(s) = \lambda \), we have constant intensity over the space \( B \) and in this special case, \( Y \) reduces to a homogeneous Poisson process (HPP). For a more general case, \( \lambda(s) \) can be spatially varying, which leads to a nonhomogeneous Poisson process (NHPP). For the NHPP, the log-likelihood on \( B \) for the observed dataset \( y \) is given by

\[
\ell = \sum_{i=1}^k \log \lambda(s_i) - \int_B \lambda(s)ds,
\]
where $\lambda(s_i)$ is the intensity function for location $s_i$. We signify that a set of points $y = (s_1, s_2, \ldots, s_\ell)$ follows a Poisson process as

$$y \sim \mathcal{PP}(\lambda(\cdot)).$$  \hfill (2)

A log-Gaussian Cox process (LGCP) is a doubly-stochastic Poisson process with a spatially varying intensity function modeled as an exponentiated Gaussian Process (Rasmussen and Williams, 2006). The LGCP can be written hierarchically as

$$y \sim \mathcal{PP}(\lambda(\cdot)),$$

$$\lambda(\cdot) = \exp(Z(\cdot)),$$

$$Z(\cdot) \sim \mathcal{GP}(0, k(\cdot, \cdot)),$$  \hfill (3)

where $k(\cdot, \cdot)$ is the covariance function of the Gaussian Process, $Z(\cdot)$.

Next, we will introduce how to calculate the similarities of shooting pattern among different players based on the intensity surface $\lambda(\cdot)$. In our analysis, we have $n$ different players with their respective point processes, $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$. Based on LGCP, we have their underlying intensity surfaces $\lambda^{(1)}(\cdot), \lambda^{(2)}(\cdot), \ldots, \lambda^{(n)}(\cdot)$. We compute player similarity using a certain measure of distance in the space of intensity surfaces. With our main goal being to group players who share similar shooting position choices over the court, an appropriate metric is needed to quantify similarities among the intensities. First, construct a matrix $C \equiv (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)})$. We denote $C^{(i)} = \lambda^{(i)}$. Then, we compute players’ distance matrix $H$ as:

$$H_{i,j} = \left\| \frac{C^{(i)}}{\sum C^{(i)}} - \frac{C^{(j)}}{\sum C^{(j)}} \right\|^2,$$  \hfill (4)

where $i, j \in \{1, \ldots, n\}$. It can be seen that $H$ is symmetric, and $H \in \mathbb{R}^{n \times n}$. 


3.2 Group Learning via Point Process Intensity

With the matrix $H$ obtained, we employ nonparametric Bayesian methods to detect grouped patterns in the intensities. Our initial step is to transform the distance matrix $H$ so that each entry $H_{i,j}$ is within the range of a Gaussian distribution. Denote the transformed distance matrix as $S$. Its $(i,j)$th element is calculated as

$$S_{ij} = \log \left( \frac{H_{i,j} - \min(H)}{\max(H) - H_{i,j}} \right),$$  \hspace{1cm} (5)$$

where $\max(H)$ and $\min(H)$ denote the maximum and minimum values in $H$, respectively. A larger value of $S_{ij}$ indicates higher similarity of intensities. We further assume that

$$S_{ij} | \mu, \tau, k \sim N(\mu_{ij}, \tau_{ij}^{-1}), \quad \mu_{ij} = U_{zi, zj} \hspace{1cm} \tau_{ij} = T_{zi, zj}, \quad 1 \leq i < j \leq n,$$  \hspace{1cm} (6)$$

where $k$ denotes the number of groups, $N()$ denotes the normal distribution, $z_i \in \{1, \ldots, k\}$ denotes the group membership of player $i$ for $i = 1, \ldots, 191$. The matrices $U = [U_{rs}] \in (-\infty, +\infty)^{k \times k}$ and $T = [T_{rs}] \in (0, +\infty)^{k \times k}$ are both symmetric, with $U_{rs} = U_{sr}$ indicating the mean closeness between any two fitted intensity surfaces in groups $r$ and $s$, respectively, and $T_{rs} = T_{sr}$ indicating the precision.

Denote by $Z_{n,k} = \{(z_1, \ldots, z_n) : z_i \in \{1, \ldots, k\}, 1 \leq i \leq n\}$ the set of all possible partitions of $n$ players into $k$ groups. With certain $z \in Z_{n,k}$, denote by $\mathcal{S}_{[rs]}$ the $n_r \times n_s$ sub-matrix of $\mathcal{S}$ consisting of entries $\mathcal{S}_{ij}$ where $z_i = r$ and $z_j = s$. Under model (6), the joint likelihood of $\mathcal{S}$ can be written as

$$P(\mathcal{S} | z, U, T, k) = \prod_{1 \leq r \leq s \leq k} P(\mathcal{S}_{[rs]} | z, U, T),$$  \hspace{1cm} (7)$$

where

$$P(\mathcal{S}_{[rs]} | z, U, T) = \prod_{1 \leq i < j \leq n; z_i = r, z_j = s} \frac{1}{\sqrt{2\pi T_{rs}}} \exp \left\{ -\frac{T_{rs}(\mathcal{S}_{ij} - U_{rs})^2}{2} \right\}.$$
Assuming that the number of groups $k$ is given, independent prior distributions are often assigned to $z, U, \text{ and } T$. Such specification can be conveniently incorporated into a finite mixture model. When $k$ is unknown, however, the Dirichlet process mixture prior models (Antoniak, 1974) can be employed as:

$$
\mathcal{S}_i \sim F(\cdot, \theta_i), \quad \theta_i \sim G(\cdot), \quad G \sim DP(\alpha G_0),
$$

with $\mathcal{S}_i = (\mathcal{S}_{i1}, \mathcal{S}_{i2}, \ldots, \mathcal{S}_{in})$, $\theta_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{in})$ and $\theta_{ij} = (\mu_{ij}, \tau_{ij})$. The process $G$ is parameterized by a base measure $G_0$ and a concentration parameter $\alpha$. With $\theta_i$ for $i = 1, \ldots, n$ drawn from $G$, a conditional prior distribution for a newly drawn $\theta_{n+1}$ can be obtained via integration (Blackwell et al., 1973):

$$
p(\theta_{n+1} | \theta_1, \ldots, \theta_n) = \frac{1}{n + \alpha} \sum_{i=1}^{n} \delta_{\theta_i}(\theta_{n+1}) + \frac{\alpha}{n + \alpha} G_0(\theta_{n+1}),
$$

with $\delta_{\theta_i}(\theta_j) = I(\theta_j = \theta_i)$ being the point mass at $\theta_i$. The model can be equivalently obtained with the introduction of group membership $z_i$’s and having $K$, the number of groups, approach infinity (Neal, 2000):

$$
\mathcal{S}_i | z_i, \theta^{*} \sim F(\theta^{*}_{z_i}), \\
z_i | \pi \sim \text{Discrete}(\pi_1, \ldots, \pi_K), \\
\theta^{*}_{z} \sim G_0, \\
\pi \sim \text{Dirichlet}(\alpha/K, \ldots, \alpha/K),
$$

where $\pi = (\pi_1, \ldots, \pi_K)$. It can be seen that under this construction, the group-specific distribution $F(\cdot | \theta^{*}_{z_i})$ solely depends on the vector of parameters $\theta^{*}_{z}$.

In construction (10), the prior distribution of $(z_1, \ldots, z_n)$, which would allow for automatic inference on the number of groups $k$, can be obtained by integrating out $\pi$, the mixing proportions. This is also known as the Chinese restaurant process (CRP; Aldous, 1985; Pitman, 1995; Neal, 2000). The conditional distribution for $z_i$ is defined
through the metaphor of a Chinese restaurant (Blackwell et al., 1973):

\begin{equation}
P(z_i = c \mid z_1, \ldots, z_{i-1}) \propto \begin{cases} |c|, & \text{at an existing table labeled } c \\ \alpha, & \text{if } c \text{ is a new table} \end{cases}, \tag{11}
\end{equation}

where $|c|$ denotes the size of group $c$. Despite its ability to simultaneously estimate the number of groups and group configuration, the CRP has been shown by Miller and Harrison (2018) to produce redundant tail groups, causing inconsistency in estimation for the number of groups even with the sample size going to infinity. Miller and Harrison (2018) also proposed a modification of the CRP, known as the mixture of finite mixtures (MFM) model, to mitigate this problem. The MFM model can be formulated as:

\begin{equation}
k \sim p(\cdot), \quad (\pi_1, \ldots, \pi_k) \mid k \sim \text{Dirichlet}(\gamma, \ldots, \gamma), \quad z_i \mid k, \pi \sim \sum_{h=1}^{k} \pi_h \delta_h, \quad i = 1, \ldots, n, \tag{12}
\end{equation}

with $p(\cdot)$ being a proper probability mass function on the set of positive integers, and $\delta_h$ being a point mass at $h$. Define a coefficient $V_n(w)$ as

\[ V_n(w) = \sum_{k=1}^{\infty} \frac{k(w)}{(\gamma k) \cdot n} p(k), \]

where $w$ denotes the number of “existing tables”, $k(w) = k(k-1) \ldots (k-w+1)$, and $(\gamma k)^{(n)} = \gamma k(\gamma k+1) \ldots (\gamma k+n-1)$, $x^{(0)} = 1$, and $x^{(0)} = 1$. Introduction of a new table is slowed down by $V_n(w+1)/V_n(w)$, which yields the following conditional prior of $\theta$:

\begin{equation}
P(\theta_{n+1} \mid \theta_1, \ldots, \theta_n) \propto \sum_{i=1}^{w} (n_i + \gamma) \delta_{\theta_i^*} + \frac{V_n(w+1)}{V_n(w)} \gamma G_0(\theta_{n+1}), \tag{13}
\end{equation}

with $\theta_1^*, \ldots, \theta_w^*$ being the unique values taken by $\theta_1, \ldots, \theta_n$. Conditional distribution
for the group membership can be expressed analogous to (11) as:

\[ P(z_i = c | z_1, \ldots, z_{i-1}) \propto \begin{cases} |c| + \gamma, & \text{at an existing table labeled } c \\ V_n(w+1)/V_n(w)\gamma, & \text{if } c \text{ is a new table} \end{cases} \]  

(14)

Adapting MFM to our model setting for functional grouping, the model and prior can be expressed hierarchically as:

\[ k \sim p(\cdot), \text{where } p(\cdot) \text{ is a p.m.f on } \{1, 2, \ldots\} \]

\[ T_{rs} = T_{sr} \overset{\text{ind}}{\sim} \text{Gamma}(\alpha, \beta), \quad r, s = 1, \ldots, k, \]

\[ U_{rs} = U_{sr} \overset{\text{ind}}{\sim} \text{N}(\mu_0, k_0^{-1}T_{rs}^{-1}), \quad r, s = 1, \ldots, k, \]

\[ \Pr(z_i = j | \pi, k) = \pi_j, \quad j = 1, \ldots, k, \quad i = 1, \ldots, n, \]  

(15)

\[ \pi | k \sim \text{Dirichlet}(\gamma, \ldots, \gamma), \]

\[ S_{ij} | z, U, T, k \overset{\text{ind}}{\sim} \text{N}(\mu_{ij}, \tau_{ij}^{-1}), \quad \mu_{ij} = U_{zi}z_j, \quad \tau_{ij} = T_{zi}z_j, \quad 1 \leq i < j \leq n. \]

We assume \( p(\cdot) \) is a Poisson(1) distribution truncated to be positive through the rest of the paper, which has been proved by Miller and Harrison (2018); Geng et al. (2019) to guarantee consistency for the mixing distribution and the number of groups. We refer to the hierarchical model above as MFM-PPGrouping.

4 Bayesian Inference

In this section, we will discuss Bayesian estimation of LGCP via integrated nested Laplace approximation (INLA), collapsed sampler algorithm for MFM-PPGrouping approach, and posterior inference on MCMC samples.

First, we will introduce intensity estimation of LGCP via INLA. The LGCP model accounts for spatial dependence by incorporating a Gaussian random field (GRF). GRFs are spatially continuous random processes in which random variables at any location in a space are normally distributed and are correlated with random variables
at other locations according to a continuous correlation process. The GRF is approximated by the solution to a stochastic partial differential equation (SPDE; see, Lindgren et al., 2011, for a review), as SPDEs provide an efficient way of approximating the GRF in continuous space (Simpson et al., 2016). Model-based Markov chain Monte Carlo (MCMC) can be very time-consuming for LGCP. In this paper, we compute the LGCP model using the R-package inlabru (Bachl et al., 2019), which provides an easy access to Bayesian inference for spatial point processes by using integrated nested Laplace approximation (INLA; Rue et al., 2009). INLA is an alternative to Markov chain Monte Carlo (MCMC) for fitting latent Gaussian models. It provides a fast and accurate way to fit a potential model, and facilitates computationally efficient inference on point processes. For more details about INLA, we refer the reader to the R-INLA project website at http://www.r-inla.org. The benefit of inlabru is that it provides methods for fitting spatial density surfaces, as well as for prediction, while not requiring knowledge of SPDE theory.

When defining the model, the mesh and observation matrix for the piecewise linear basis functions need to be specified. In INLA, the basis functions used are typically “tent” (finite element) functions constructed over a triangulation of the domain. The mesh is composed of two regions: the interior mesh, which is where the actions happen; and the exterior mesh, which is designed to alleviate the boundary effects. We create the mesh based on Cervone et al. (2016). The mesh for our shot chart data using the tent functions is shown in Figure 2.

Then, the SPDE can be constructed on the mesh using the function inla.spde2.pcmatern(). The “pc” in “pcmatern” is short for “penalized complexity”, and it is used to refer to prior distributions over the hyperparameters that are both interpretable and have interesting theoretical properties (see, Simpson et al., 2017, for a discussion).

Based on INLA, we obtain estimated intensity surfaces $\hat{\lambda}^{(1)}(\cdot), \hat{\lambda}^{(2)}(\cdot), \ldots, \hat{\lambda}^{(n)}(\cdot)$, and then obtain $\mathcal{S}$ by (4) and (6). Next, we use MFM-PPGrouping for group learning based on $\mathcal{S}$. The sampler presented in Algorithm 1 is used to sample from the posterior distributions for unknown parameters, including $k, z = (z_1, \ldots, z_n) \in \{1, \ldots, k\}^n$ and
Figure 2: Triangulation for the shot data locations over which the “tent” functions are constructed (black line), and the observation locations are inside the blue rectangle.

\( \lambda = (\lambda_1, \ldots, \lambda_n) \) in (15). As it marginalizes over the distribution of \( k \), the sampler does not depend on reversible jump, and is more efficient than allocation samplers.

After obtaining posterior samples of \( \{z_1, z_2, \ldots, z_n\} \), posterior inference for the group configurations needs to be carried out so that the values are nominal integers denoting group belongings. This renders the posterior mean unsuitable for our purpose. We adopt Dahl’s method (Dahl, 2006). Define a membership matrix \( B^{(\ell)} \) as:

\[
B^{(\ell)} = (B^{(\ell)}(i, j))_{i, j \in \{1:n\}} = (z_i^{(\ell)} = z_j^{(\ell)})_{n \times n},
\]

(16)

where \( \ell = 1, \ldots, B \) indexes the number of MCMC iterations, \( z_i^{(\ell)} \) and \( z_j^{(\ell)} \) denote the memberships for players \( i \) and \( j \), respectively. An \( B^{(\ell)}(i, j) \) equals 1 if \( z_i^{(\ell)} = z_j^{(\ell)} \), and 0 otherwise. An element-wise mean of the membership matrices can be obtained as

\[
\overline{B} = \frac{1}{B} \sum_{t=1}^{B} B^{(t)},
\]

where the summation is also element-wise. The posterior iteration with the smallest
Algorithm 1 Collapsed Sampler for MFM-PPGrouping

1: procedure c-MFM-PPGrouping
2: Initialize: let $z = (z_1, \ldots, z_n), U = (U_{rs}), T = (T_{rs})$.
3: for each iter = 1 to M do
4: Update $T = (T_{rs})$ conditional on $z$ as
   \[
p(T_{rs} \mid \mathcal{S}, z) \sim \text{Gamma} \left( \alpha + n_{rs}/2, \beta + (n_{rs} - 1) \text{var}(A_{[rs]})/2 + \frac{k_0 n_{rs} (\bar{A}_{[rs]} - \mu_0)^2}{2(k_0 + n_{rs})} \right)
\]
5: Update $U = (U_{rs})$ conditional on $z$ as
   \[
p(U_{rs} \mid \mathcal{S}, T_{rs}, z) \sim \mathcal{N} \left( \frac{k_0 \mu_0 + n_{rs} \bar{A}_{[rs]}}{k_0 + n_{rs}}, ((k_0 + n_{rs})T_{rs})^{-1} \right)
\]
   where $A_{[rs]} = (\mathcal{S}_{ij}; z_i = r, z_j = s, i \neq j)$, $\bar{A}_{[rs]} = (\sum_{z_i = r, z_j = s, i \neq j} \mathcal{S}_{ij})/n_{rs}$ and $n_{rs} = \sum_{i \neq j} I(z_i = r, z_j = s), r = 1, \ldots, k; s = 1, \ldots, k$. Note that $k$ denotes the number of groups yielded by the current $z$.
6: Update $z = (z_1, \ldots, z_n)$ conditional on $U = (U_{rs})$ and $T = (T_{rs})$. For each $i$ in $(1, \ldots, n)$, $P(z_i = c \mid z_{-i}, \mathcal{S}, U, T)$ can be obtained in closed form as:
   \[
   \propto \left\{ \begin{array}{ll}
   \prod_{j > i} \frac{1}{\sqrt{2\pi T_{ij}}} e^{-\frac{T_{ij}(S_{ij} - U_{ij})^2}{2}} \prod_{k < i} \frac{1}{\sqrt{2\pi T_{ik}}} e^{-\frac{T_{ik}(S_{ik} - U_{ik})^2}{2}} \\ V_0\Gamma(C_{-i} + 1) \Gamma(C_{-i}) \gamma m(\mathcal{S}_i)
   \end{array} \right.
   \]
   at an existing table $c$,
   \[
   \left( \prod_{j > i} \frac{1}{\sqrt{2\pi T_{ij}}} e^{-\frac{T_{ij}(S_{ij} - U_{ij})^2}{2}} \prod_{k < i} \frac{1}{\sqrt{2\pi T_{ik}}} e^{-\frac{T_{ik}(S_{ik} - U_{ik})^2}{2}} \right) \frac{1}{V_0\Gamma(C_{-i} + 1) \Gamma(C_{-i})} \gamma m(\mathcal{S}_i)
   \]
   if $c$ is a new table
   with $C_{-i}$ being the partition obtained by removing $z_i$ and
   \[
m(\mathcal{S}_i) = \prod_{t=1}^{C_{-i}} \frac{\Gamma(\alpha_n)}{\Gamma(\beta_n^\alpha)} \beta_n^{\alpha_n} \left( \frac{k_0}{k_0 + n_t} \right)^{\frac{1}{2}} \frac{n_t}{(2\pi)^{\frac{n_t}{2} - 1}},
   \]
   where $\alpha_n = \alpha + n_t/2, n_t = \sum_{i \neq j} I(z_j = t), \beta_n = \beta + (n_t - 1) \text{var}(A_{[t]})/2 + \frac{k_0 n_t (\bar{A}_{[t]} - \mu_0)^2}{2(k_0 + n_t)}$, $A_{[t]} = (\mathcal{S}_{ij}; z_j = t, i \neq j)$ and $\bar{A}_{[t]} = (\sum_{z_j = t, i \neq j} \mathcal{S}_{ij})/n_t$.
7: end for
8: end procedure
The squared distance to $\mathcal{B}$ is obtained by

$$C_{LS} = \arg\min_{c \in (1:B)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathcal{B}(c)(i,j) - \mathcal{B}(i,j))^2.$$ \hspace{1cm} (17)

The estimated parameters, together with the group assignments $z$, are obtained from $C_{LS}$th post burn-in iteration. With the Dahl’s method, our Bayesian grouping method is summarized in Algorithm 2.

**Algorithm 2** Bayesian Group Learning Procedure for Basketball Players

1: Fit LGCPs for $n$ different players $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ via *inlabru* and get $n$ underlying intensity surface $\lambda^{(1)}(\cdot), \lambda^{(2)}(\cdot), \ldots, \lambda^{(n)}(\cdot)$,
2: Use (4) to construct matrix $\mathcal{S}$ and matrix $\mathcal{S}$ and based on $\lambda^{(1)}(\cdot), \lambda^{(2)}(\cdot), \ldots, \lambda^{(n)}(\cdot)$,
3: Get $B$ posterior samples of $z^{(1)}, z^{(2)}, \ldots, z^{(B)}$ from $\mathcal{S}$ via Algorithm 1,
4: Summary posterior samples by Dahl’s method.

## 5 Simulation

### 5.1 Simulation Setup

We simulate data using three different basis spatial structure from the player’s shot intensity. We fix the true number of groups at three, and the three base intensities are visualized in Figure 3. The first group corresponds to players most of whose shots are in the painted area; the second group corresponds to players whose shot locations are widely distributed in every location from painted area to three-point line. A player in the third group has more shot at the three-point line and inside the painted area. For each group, 25 observations are generated by adding a normal noise term of mean 0 and variance 0.5 to the basis intensity, yielding 75 observations in total.

We run a total of 50 replicates. The final grouping performance is evaluated using the estimated number of groups and compared against the true setting in terms of modulo labeling of group assignment using Rand index (RI; Rand, 1971), the computation of which is facilitated by the R-package *fossil* (Vavrek, 2011). The RI ranges
Figure 3: Visualization of three bases for the three groups in simulation design.

from 0 to 1 with a higher value indicating better agreement between a grouping scheme and the true setting. In particular, a value of 1 indicates perfect agreement.

5.2 Simulation Results

We run our algorithm with 1,000 MCMC iterations, with the first 500 iterations as burn-in for each replicate data. We examine it is sufficient for the chain to converge and stabilize. The numbers are chosen to be sufficiently large for the chain to converge and stabilize. To verify this, with a single replicate of data, 50 separate MCMC chains are run with different random seeds and hence initial values, and 50 final grouping schemes are obtained. The RI is calculated for these 50 chains at each of their iteration, giving 50 traces, which are visualized in Figure 4. It can be observed that convergence is attained after a small number of iterations, and the band of the 50 traces is rather tight after convergence.

Proceeding to 50 separate replicates of data, our proposed algorithm was run, and 50 RI values are obtained by comparing with the true setting. They average to 0.9512, which indicates rather accurate grouping ability of the proposed approach. In addition, performance comparisons of our proposed method with three competing methods are made. We compare our method to K-means algorithm, Density-based spatial grouping of applications with noise (DBSCAN) and mean shift grouping. Grouping recovery performances of all four methods are measured using the RI. The 50 final RI’s obtained
Figure 4: Rand Index trace plot for single replicate simulated data. Dark grey line indicates for each random seed. Red line is the average rand index for 50 random seeds.

for the three competitors average to 0.8853, 0.7521, and 0.7690, respectively, indicating the superior performance of our proposed approach.

6 Analysis of NBA Players

In this section, we apply the proposed method to the analysis of players’ shot data in the 2017-2018 NBA regular season. Only the locations of shots are considered regardless of the players’ positions on the court (e.g., point guard, power forward, etc.). Players’ positions (e.g., point guard, power forward, etc) are not considered, but only the shots’ positions. As a starting point, a predictive intensity matrix is obtained for each player using \texttt{inlabru}. Algorithms 1 and 2 are subsequently used to identify the groups. We run 1,000 MCMC iterations and the first 500 iterations as burn-in period. The result from the MFM model suggests that the 191 players are to be classified into nine groups. The sizes of the nine groups are 56, 7, 12, 3, 8, 19, 51, 24, and 11 respectively. Visualizations of intensity matrices for two selected players from each group are presented in Figure 5.

Several interesting observations are made from the visualization results. The fitted intensities for groups 1 and 7 have similar patterns. Shots are concentrated over the
painted area, and then beyond the three point line. However, the ratios of fitted intensities over the painted area and beyond the three point line distinguishes the two groups.

Group 2, group 3 and group 4 share some similarity. The majority of shots are around the painted area and close to the hoop. The difference, again, lies in the contrast of intensities in these concentrated areas over others, which is the largest for group 4, followed by group 2 and group 3. It means that the players in group 3, on a relative scale when compared to those in group 4 and group 2, make more shots outside the painted area. Group 4 is a special and interesting example, as the majority of shot for this group is close to the hoop or even around the hoop. Steve Adams, Clint Capela, and DeAndre Jordan, for example, are all good at making alley-oops and slam dunks. Only very few shots are made by these players outside the painted area.

Group 5 and group 6 are similar group. The fitted intensities in the painted area is around 12.5, which is smaller than those of groups 2, 3, and 4. Close scrutiny of the visualizations reveal that players in group 5 make more shots when facing the hoop, while players in group 6 appear to be comfortable making shots from a broader range of angles.

Players in group 8 make shots at almost all locations, including three-pointers, perimeter shot, and over the painted area. The fact that most shots are around the painted area indicate that they drive well. Shots are evenly distributed in areas other than within the painted area, where there is higher intensity. In group 9, however, while the players make a large proportion of shots beyond the three-point line, they also make shots within the painted area, mostly around the hoop. JJ Redick and JR Smith, the well-known shooters, both fall in this group.

As further verification, we use multidimensional scaling to lower the dimension of the fitted intensity matrices for players to 2 so that similarities in their shooting habits can be visualized. See Figure 6. Separation of the nine groups is quite clear. Group 4, for example, with its unique strong preference for alley-oops and slam dunks, stands far from others.
Finally, to make sure the group configuration presented here is not a random occurrence but reflects the true pattern demonstrated by the data, we run 50 separate MCMC chains with different random seeds and initial values, and obtained 50 final grouping schemes. The RI between each scheme and the present grouping scheme is calculated, and they average to 0.9166, indicating high concordance of conclusion regardless of random seeds.

Figure 5: Fitted intensities for two selected players in each of the nine identified groups.

7 Conclusion

In this paper, we proposed using MFM to capture heterogeneity of different NBA players based on LGCP. Our group learning method provides a quantitative summary of different players shot habits other than traditional position categorization. Our simulation results indicated that our proposed methods achieve good grouping accuracy.

The real data application give us the information about player’s shooting habit location. Players can understand their own shooting habits, and they can also strengthen
their weaker shooting locations. On the other hand, the professional coach can formulate a defensive strategy to reduce the opponent’s score with these information. Our grouping results will provide a good guidance for team managers trading the players with similar shot pattern.

A few topics beyond the scope of this paper are worth further investigation. In this paper, a two-stage group learning method is proposed. An unified approach is an interesting alternative in future work. In addition, incorporating auxiliary information such as player position or historical information could also be taken into account for grouping in our future work. Jointly modeling spatial field goal percentage and shot selection will provide more detail instructions for professional coaches.

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