In drawing on an analogy with the flavor mixing observed in the quark sector we discuss a pattern of large flavor mixing angles in the lepton sector. Simple arguments based on a democratic symmetry and its violation in the lepton sector allow us to determine the flavor mixing matrix of leptons. The mixing angle relevant for solar neutrino oscillations is maximal (close to 45°), while the angle relevant for atmospheric neutrino oscillations is given by \( \sin^2 2\theta = 8/9 \). The emerging pattern is consistent with the results of the solar and atmospheric neutrino experiments.

1. Main Chapter

In this talk I shall concentrate on phenomenological issues of the neutrino mixing phenomenon and in particular on analogies between the leptonic mixing and the mixing of the quark flavors. The mixing of quark flavors is known for almost forty years, and there is little doubt that there will be parallelisms between the mixings of leptonic and quark flavors. Nevertheless, substantial differences might exist.

Let me first discuss some general features of the quark and charged lepton mass spectra, and of flavor mixing.

I should like to emphasize, that the term “neutrino mixing” which is often used is misleading. If neutrino oscillations exist, they manifest a general leptonic mixing phenomenon and a mismatch between the neutrino mass spectrum and the mass spectrum of the charged leptons, in analogy to the quarks. Just as the flavor mixing angles in the quark sector are related intrinsically to the quark masses, the parameters of the neutrino oscillations or in general the leptonic mixing angles are related directly both to the neutrino and the charged lepton mass terms. In particular the pattern of the charged lepton masses will be significant for the leptonic flavor mixing and for the magnitude of the neutrino oscillations.

The mass spectra of the quarks and of the charged leptons are similar. Both for the quarks and for the charged leptons the spectra are largely dominated by the members of the third family.
95% of the lepton masses are provided by the $\tau$–lepton. The $b$–quark contributes about 97% to the masses in the charge $-1/3$ sector. The charge $+2/3$ sector is dominated to 98% by the $t$–quark. Both in case of the charged leptons and of the quarks the contribution of the first family to the total mass in the corresponding channel is almost negligible. At the present time it is not known whether such a conspicuous hierarchy in the mass spectra of the charged fermions is accompanied by a similar hierarchy of the neutrinos. If such a hierarchy would also exist in the neutrino sector, one would expect that the $\tau$–neutrino is the heaviest neutrino, accompanied by a relatively light $\mu$–neutrino and an almost massless $e$–neutrino.

For the discussion of flavor mixing it is often useful to treat the fermion masses as parameters, which can be changed arbitrarily and in particular set to zero or infinity. Obviously the physics of the leptons and quarks will not be changed significantly, if we set the masses of the first and second family to zero. The departure from the real world will be about 5%. Of course, due to our ignorance about the origin of the fermion masses we do not know whether such a change of these masses is indeed possible. Within the framework of the standard model it is, of course, easy to make such changes just by modifying the coupling parameters describing the interaction of the fermions and the scalar field.

If the masses of the first two families vanish, the mass matrices of the fermions become matrices of rank one:

$$ M = C \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

(1)

In the limit there appears to be a mass gap $C$, given by the mass of the $t$–quark, the $b$–quark or the $\tau$–lepton respectively. By a suitable orthogonal transformation the mass matrix $M$ can be brought into a form, in which all matrix elements are identical:

$$ \overline{M} = \frac{C}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. $$

(2)

Such a mass matrix, which is often called a democratic mass matrix, also has rank one. As far as the charged leptons and the quarks are concerned, we can speak either of a hierarchy basis ($M$) or of a democratic basis ($\overline{M}$). Both are, of course, equivalent. However, in the democratic basis one realizes a new symmetry described by the discrete group $S(3)_L \times S(3)_R$.

Before coming back to this symmetry, let me describe how such a situation could arise. If we look at the quarks or charged leptons using the democratic basis one finds that there are universal transitions between all three fermion states. One is reminded of the so–called “pairing force”, which is responsible for the appearance of a mass gap in superconductivity or for giant resonances in nuclear physics. I should also like to mention that the mass spectrum of the neutral pseudoscalar mesons in QCD ($\pi^0, \eta, \eta'$) is described by democratic–type mass matrix. Between the various $\bar{q}q$–states there are large transition elements provided by the gluonic interaction. These transitions are universal in the chiral limit due to the universality of the gluonic interaction. In the pseudoscalar channel these transitions are particularly strong and lead to large mixing effects, due to large non–perturbative effects. It is due to these gluonic transitions that in the limit of chiral $SU(3)_L \times SU(3)_R$ the $0^{-+}$–mesons segregate into a massive singlet and a massless octet.
In the case of the pseudoscalar mesons we can also denote the transformation between the hierarchy basis and the democratic basis. The eigenstates of the $S(3)$–symmetry are nothing but the states $\bar{u}u$, $\bar{d}d$ and $\bar{s}s$. The connection between the mass eigenstates and the $\bar{q}q$–states is given by:

$$
\begin{align*}
\pi^0 &= \frac{1}{\sqrt{2}} ( \bar{u}u - \bar{d}d ) , \\
\eta &= \frac{1}{\sqrt{6}} ( \bar{u}u + \bar{d}d - 2\bar{s}s ) , \\
\eta' &= \frac{1}{\sqrt{3}} ( \bar{u}u + \bar{d}d + \bar{s}s ) .
\end{align*}
\tag{3}
$$

In analogy let me write down the mass eigenstates of charged leptons in the democratic limit in terms of the eigenstates $l_1, l_2$ and $l_3$ of the democratic symmetry:

$$
\begin{align*}
e &= \frac{1}{\sqrt{2}} ( l_1 - l_2 ) , \\
\mu &= \frac{1}{\sqrt{6}} ( l_1 + l_2 - 2l_3 ) , \\
\tau &= \frac{1}{\sqrt{3}} ( l_1 + l_2 + l_3 ) .
\end{align*}
\tag{4}
$$

Similar relations can be written down for the quarks. In reality the democratic symmetry is not exact, but broken by small terms. These symmetry breaking effects have been discussed some time ago by a number of authors, and we shall refer to the literature.

It is interesting to discuss the description of the flavor mixing in the context of the democratic symmetry and its violation. In the limit of the democratic symmetry one expects for the quarks that no flavor mixing is present. In other words, the flavor mixing angles must be related to the violation of the symmetry and in particular to the masses of the first two families, or rather to the mass ratios of the masses of the first two families and the mass of the corresponding representative of the third family. The best way of describing the flavor mixing would be one in which the parameters for the flavor mixing, e. g. the flavor mixing angles, are smooth functions of the symmetry breaking parameters. In view of this we have recently studied all possible ways for describing the flavor mixing of the quarks. As discussed in Ref. there exist nine different ways in general to describe the mixing of three families. But only one description obeys the constraints discussed above. In particular the so-called "standard" parametrization advocated by Particle Data Group does not obey these constraints. Instead one is lead to the following parametrization:

$$
V = \begin{pmatrix}
c_u & s_u & 0 \\
-s_u & c_u & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
e^{-i\varphi} & 0 & 0 \\
0 & c & s \\
0 & -s & c
\end{pmatrix}
\begin{pmatrix}
c_d & -s_d & 0 \\
s_d & c_d & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
s_u & s_d & c + c_u c_d e^{-i\varphi} & c_u c_d - s_u s_d e^{-i\varphi} & s_u c_d e^{-i\varphi} & c_u s \end{pmatrix} \\
-c_s & s_d & c - s_u c_d e^{-i\varphi} & s_u c_d c + s_u s_d e^{-i\varphi} & s_u s_d & c \end{pmatrix}.
\tag{5}
$$

Here $c_u = \cos \theta_u$, $s_d = \sin \theta_d$, $c = \cos \theta$, etc.

It is interesting to note that this way of describing the flavor mixing matrix of the quarks is in the absence of the complex phase $\varphi$ identical to the rotation matrix given originally by Euler. Since
it will turn out that the corresponding mixing angles are small, one finds in a good approximation:

\[
V = \begin{pmatrix}
  e^{-i\varphi} & s_u - s_d e^{-i\varphi} & s_u s \\
  s_d - s_u e^{i\varphi} & 1 & s \\
  -s_d s & -s & 1
\end{pmatrix}.
\]  (6)

As discussed in Ref. 5, the three rotation angles \(\theta_u, \theta_d\) and \(\theta\) have a precise physical meaning. The angle \(\theta\) is a combined effect arising from the mixing between the second and third family (heavy quark mixing). The angle \(\theta_u\) primarily describes a mixing between \(u\) and \(c\) quarks, and the angle \(\theta_d\) primarily describes a mixing between \(d\) and \(s\) quarks. The angle \(\theta\) is essentially given by the magnitude of \(V_{cb}\). The angles \(\theta_u\) and \(\theta_d\) are determined as follows:

\[
\tan \theta_u = \left| \frac{V_{ub}}{V_{cb}} \right|, \\
\tan \theta_d = \left| \frac{V_{td}}{V_{ts}} \right|.  
\]  (7)

In a simple model of symmetry breaking (see Refs. 7, 8) these two angles are related in a simple way to the ratio of quark mass eigenvalues:

\[
\theta_u = \arctan \sqrt{\frac{m_u}{m_c}}, \\
\theta_d = \arctan \sqrt{\frac{m_d}{m_s}}.  
\]  (8)

Recently the angles as well as the complex phase \(\varphi\) have been determined with relatively high accuracy from a global analysis of current data 9. One finds

\[
\theta = 2.30^\circ \pm 0.09^\circ, \\
\theta_u = 4.87^\circ \pm 0.86^\circ, \\
\theta_d = 11.71^\circ \pm 1.09^\circ, \\
\varphi = 91.1^\circ \pm 11.8^\circ.  
\]  (9)

We note that the values obtained here are in very good agreement with the expectations from the quark masses (see Ref. 8). Furthermore the complex phase is in very good agreement with the expectation of 90\(^\circ\), as expected from a very simple symmetry breaking 10, 11. The fact that the phase is 90\(^\circ\) signifies that \(CP\) violation is maximal in the sense described in Ref. 10. Furthermore the experimental data support that the mass matrix of the quarks in the hierarchy basis has the following structure:

\[
M(q) = \begin{pmatrix}
  0 & a & 0 \\
  a^* & b' & b \\
  0 & b & c
\end{pmatrix}.  
\]  (10)

The complex phases of the quark mass matrices can be arranged such that they appear primarily in the (1,2) and (2,1) matrix elements.

Since the mass spectrum of the charged leptons exhibits a similar hierarchical pattern as the quarks, it is most natural to suppose that the matrix structure and the texture properties of the charged lepton mass matrix is analogous to those of the quark mass matrices.
The question arises whether the neutrinos also exhibit a hierarchical mass pattern. It may well be that the neutrino masses are not hierarchical at all. If they were, we could write the neutrino mass matrix in analogy to the charged lepton matrix in the democratic basis as follows:

\[ M(\nu) = \frac{C_\nu}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \Delta M(\nu), \]

where \( \Delta M(\nu) \) denotes the perturbative correction. The constant \( C_\nu \) describes the strength of the pairing-force term in the neutrino channel. The magnitude of this term is related to the mass of the heaviest neutrino and could be at most about 30 eV, i.e. it must be about eight orders of magnitude smaller than the charged lepton term \( (C_l = m_\tau) \). It is hard to believe that the ratio \( C_\nu / C_l \) is simply a tiny number by accident. It would be much more natural to suppose that the leading pairing-force term is completely absent in the neutrino channel. This possibility was discussed in Ref. 12. The absence of the leading pairing force term for the neutrinos would have drastic consequences for the mixing pattern of the leptons. An interesting possibility is that the eigenstates of the democratic symmetry for the neutrinos are identical to the mass eigenstates. Following Ref. 12 we write down the following mass matrices for the charged leptons and the neutrinos:

\[
M(l^-) = C_l \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \delta_l & 0 & 0 \\ 0 & \rho_l & 0 \\ 0 & 0 & \varepsilon_l \end{pmatrix},
\]

\[
M(\nu) = 0 + \begin{pmatrix} \delta_\nu & 0 & 0 \\ 0 & \rho_\nu & 0 \\ 0 & 0 & \varepsilon_\nu \end{pmatrix}.
\]

(12)

Obviously large mixing phenomena are generated due to the absence of the pairing-force term for the neutrinos. The electroweak doublets of leptons can be written as:

\[
\left( \begin{array}{c} \nu_1 \\ l_1 \\ \nu_2 \\ l_2 \\ \nu_3 \\ l_3 \end{array} \right).
\]

(13)

Here the upper components, the neutrino states, are the mass eigenstates while their electroweak partners are identical to the democratic eigenstates \( l_i \). We can also perform a unitary transformation and write down the mass eigenstates for the charged leptons, neglecting the small breaking terms for the democratic symmetry, and obtain:

\[
\left( \begin{array}{c} \frac{1}{\sqrt{2}} (\nu_1 - \nu_2) \\ \frac{1}{\sqrt{6}} (\nu_1 + \nu_2 - 2\nu_3) \\ \frac{1}{\sqrt{3}} (\nu_1 + \nu_2 + \nu_3) \end{array} \right).
\]

(14)

In analogy to the case of the quarks we describe the leptonic flavor mixing matrix as follows:

\[
V_l = \begin{pmatrix} c_\nu & s_\nu & 0 \\ -s_\nu & c_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\psi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_l & -s_l & 0 \\ s_l & c_l & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_\nu s_l c - c_\nu c_l e^{-i\psi} & s_\nu c_l c - c_\nu s_l e^{-i\psi} & s_\nu s \s_l c \\ -s_\nu s_l c - c_\nu c_l e^{-i\psi} & c_\nu c_l c + s_\nu s_l e^{-i\psi} & c_\nu s \s_l \end{pmatrix}.
\]

(15)

The leptonic mixing angles are given by \( \theta_l \), describing a mixing for the charged leptons, an angle \( \theta \), describing a mixing between the second and the third family, and an angle \( \theta_\nu \), describing the mixing
in the neutrino channel. The complex phase, causing CP violation for the leptons, is denoted by $\psi$. For simplicity we assume $CP$ symmetry to be conserved in the leptonic sector.

The electroweak doublets written above give the following leptonic flavor mixing matrix:

$$V_L = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

We can read off the following mixing angles:

$$\theta_L = 0, \quad \theta_\nu = \arcsin \frac{1}{\sqrt{2}} = 45^\circ, \quad \theta = \arcsin \frac{2}{\sqrt{6}} = 54.7^\circ. \quad (17)$$

We note that $\sin^2 2\theta_\nu = 1$ and $\sin^2 2\theta = 8/9$. Using the arguments given in Ref. [1], we can also write down the corrections to the above (lowest-order) leptonic mixing matrix, given by the violation of the democratic symmetry for the charged leptons. As an illustrative example, we obtain

$$V_L' = V_L + \frac{m_\mu}{m_\tau} \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{6}} & 0 \end{pmatrix} - \sqrt{\frac{m_e}{m_\mu}} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (18)$$

The corrections to $\theta_L$ and $\theta$ are small, i.e., $\theta_L = -4.1^\circ$ and $\theta = 52.3^\circ$. The value of $\theta_\nu$ is essentially unchanged. Correspondingly we find $\sin^2 2\theta = 0.94$. A similar result for $\sin^2 2\theta$ has also been obtained in Ref. [1].

We have obtained large mixing angles for all neutrinos. Each flavor eigenstate ($\nu_e, \nu_\mu$ or $\nu_\tau$) is a linear superposition of three mass eigenstates $\nu_1, \nu_2$ and $\nu_3$, described by $V_L^\dagger$ instead of $V_L$. The electron neutrino is in lowest order given by

$$\nu_e = \frac{1}{\sqrt{2}} (\nu_1 - \nu_2). \quad (19)$$

An electron neutrino produced in the sun would oscillate between the states $\nu_e = (\nu_1 - \nu_2)/\sqrt{2}$ and $\bar{\nu}_e = (\nu_1 + \nu_2)/\sqrt{2}$. Note that this state is neither a $\mu$–neutrino nor a $\tau$–neutrino, but rather a mixture of the two. The solar neutrino experiments are consistent with a large mixing angle in the scheme of long-wavelength vacuum oscillations:

$$P(\nu_e \rightarrow \nu_e) = 1 - \sin^2 2\theta_{\text{sun}} \sin^2 \left(1.27 \frac{\Delta m^2_{\text{sun}} L}{|P|} \right) \quad (20)$$

with $\sin^2 2\theta_{\text{sun}} \approx 0.7 \ldots 1$ and $\Delta m^2_{\text{sun}} \approx (0.6 \ldots 1.1) \times 10^{-10}$ eV$^2$. In our case we have $\sin^2 2\theta_{\text{sun}} = \sin^2 2\theta_\nu = 1$. Then $|m_2^2 - m_1^2| = \Delta m^2_{\text{sun}} \sim 10^{-10}$ eV$^2$, i.e., the two neutrinos $\nu_1$ and $\nu_2$ must be degenerate to a very high degree of accuracy.

In terms of mass eigenstates the $\mu$– and $\tau$–neutrinos are given approximately by:

$$\nu_\mu = \frac{1}{\sqrt{6}} (\nu_1 + \nu_2 - 2\nu_3),$$

$$\nu_\tau = \frac{1}{\sqrt{2}} (\nu_1 + \nu_2 + \nu_3). \quad (21)$$
A $\mu$–neutrino will in general oscillate into all three neutrinos. However, due to the high degeneracy between the $\nu_1$ and $\nu_2$–states, oscillations between $\mu$–neutrinos and electron–neutrinos will appear only at very large distances. Oscillations between $\mu$–neutrinos and $\tau$–neutrinos could show up at smaller distances, if the mass difference between the $(\nu_1, \nu_2)$–states and the $\nu_3$–state is sizable. For the sake of our discussion let us suppose that the first two neutrino states are completely degenerate, in which case we can perform a $45^\circ$–rotation among the two states without changing the physical situation.

The atmospheric neutrino experiments, in particular the recent Superkamiokande measurements [1] are consistent with a large mixing angle in the $\nu_\mu \leftrightarrow \nu_\tau$ oscillations:

$$P(\nu_\mu \rightarrow \nu_\tau) = \sin^2 2\theta_{\text{atm}} \sin^2 \left(1.27 \frac{\Delta m_{\text{atm}}^2 L}{|\mathbf{P}|}\right)$$

with $\sin^2 2\theta_{\text{atm}} \approx (0.7 \ldots 1)$ and $\Delta m_{\text{atm}}^2 \approx (0.3 \ldots 8) \times 10^{-3} \text{ eV}^2$. In our case we obtain $\sin^2 2\theta_{\text{atm}} = \sin^2 2\theta = 0.83$, while $|m_3^2 - m_2^2| = \Delta m_{\text{atm}}^2 \sim 10^{-3} \text{ eV}^2$ implies the weaker degeneracy between $\mu$– and $\tau$–neutrinos.

Thus the observational hints towards neutrino oscillations both for solar and atmospheric neutrinos indicate a mass pattern for the three neutrino states as follows. The first two neutrinos $\nu_1$ and $\nu_2$ are almost degenerate, while the mass of the third neutrino $\nu_3$ is slightly larger or smaller. If this picture is correct, it would imply that there is no way to obtain evidence for neutrino oscillations using neutrino beams from accelerators, unless one performs a long distance experiment. For example, there would be no way to accommodate the results obtained in the LSND experiment [3].

The pattern for the neutrino mass differences discussed above can be realized in two qualitatively different ways. Either there is a hierarchical pattern in the neutrino mass spectrum or the three neutrino masses are nearly degenerate. If the spectrum is hierarchical, the first two mass eigenstates $\nu_{1,2}$ must be extremely light. For example one could have $m_1 \sim 0$ and $m_2 \sim 10^{-5} \text{ eV}$. The third neutrino $\nu_3$ would have a mass in the range $(0.02 \ldots 0.1) \text{ eV}$. Thus one has $\Delta m_{\text{sun}}^2 \approx m_2^2$ and $\Delta m_{\text{atm}}^2 \approx m_3^2$. In this case the neutrinos would play only a minor role in cosmology.

In the case of a degenerate neutrino mass spectrum the three mass eigenstates would have nearly the same mass:

$$m_1 = \delta_\nu$$
$$m_2 = \delta_\nu + \kappa_\nu$$
$$m_2 = \delta_\nu + \kappa_\nu'$$

with the constraints $\Delta m_{\text{sun}}^2 \approx 2\delta_\nu \kappa_\nu \sim 10^{-10} \text{ eV}^2$ and $\Delta m_{\text{atm}}^2 \approx 2\delta_\nu \kappa_\nu' \sim 10^{-3} \text{ eV}^2$. If neutrinos are a significant part of the dark matter component in the universe, the sum of the three neutrino states is expected to be in the range between 4 eV and 8 eV. As an illustrative example we could take $\delta_\nu = 2 \text{ eV}$, i.e., $m_1 = 2 \text{ eV}$, $m_2 = (2 + 2 \times 10^{-11}) \text{ eV}$, and $m_3 = (2 + 10^{-3}) \text{ eV}$.

I should like to emphasize that we have arrived at these conclusions and especially at the high degree of degeneracy in the neutrino mass spectrum by using the constraints from the solar neutrino and atmospheric neutrino experiments. From the theoretical point of view discussed initially there would be no need to have the three neutrino masses to be highly degenerate. It may well be that
the neutrino mass degeneracy is a hint towards another symmetry (see, e.g., Ref. 18 for a SO(10) grand unification model accommodating degenerate neutrino masses).

Thus far we did not discuss whether the neutrino mass eigenstates are of Majorana type or Fermi–Dirac type. In fact, both cases are possible. In the Majorana case one needs to worry about the implications for the neutrinoless double-$\beta$–decay 19. These were discussed in Ref. 12.

Finally we should like to stress again that the case of nearly degenerate neutrinos, possibly of masses of the order of 2 eV, is an interesting possibility. Large mixing angles are readily obtained in such a model. Both the mass spectrum and the flavor mixing pattern of the leptons might differ substantially from those observed for the quarks.

2. Acknowledgements

One of us (H.F.) would like to thank Dr. G. Raffelt and Prof. L. Okun for discussions.

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