Characterizations of non-associative rings by the properties of their fuzzy ideals

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ABSTRACT

In this paper, we extend the characterizations of Kuroki [Regular fuzzy duo rings. Inform Sci. 1996;96:119–139], by initiating the concept of fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals in a class of non-associative and non-commutative rings (LA-ring). We characterize regular (intra-regular, both regular and intra-regular) LA-rings in terms of such ideals.

1. Introduction

In ternary operations, the abelian law is given by \(abc = cba\). Kazim et al. \cite{9} have generalized this notion by introducing the parenthesis on the left side of this equation \(abc = cba\) to get a new pseudo associative law, that is \((ab)c = (cb)a\). This law \((ab)c = (cb)a\) is called the left invertive law. A groupoid \(S\) is left almost semigroup (abbreviated as LA-semigroup ), if it satisfies the left invertive law. An LA-semigroup is a midway structure between a abelian semigroup and a groupoid. Ideals in LA-semigroups have been investigated by Protic et al. \cite{16}.

A groupoid \(S\) is medial (resp. paramedial ), if \((ab)(cd) = (ac)(bd)\) (resp. \((ab)(cd) = (db)(ca)\)) in \cite{4} (resp. \cite{1}). An LA-semigroup is medial, but in general an LA-semigroup need not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al. \cite{16} and also satisfies these conditions \(a(bc) = b(ac), (ab)(cd) = (dc)(ba)\).

The notion of LA-semigroup is extended to the left almost group (abbreviated as LA-group) by Kamran \cite{5}. An LA-semigroup \(S\) is left almost group, if there exists a left identity \(e \in S\) such that \(ea = a\) for all \(a \in S\) and for every \(a \in S\) there exists \(b \in S\) such that \(ba = e\).

Shah et al. \cite{19} discussed the left almost ring (abbreviated as LA-ring) of finitely nonzero functions, which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set \(R\) with at least two elements such that \((R, +)\) is an LA-group, \((R, \cdot)\) is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring \((R, +, \cdot)\), we can always obtain an LA-ring \((R, \oplus, \cdot)\) by defining for all \(a, b \in R\), \(a \oplus b = b - a\) and \(a \cdot b\) is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

A non-empty subset \(A\) of an LA-ring \(R\) is an LA-subring of \(R\) if \(a - b\) and \(ab \in A\) for all \(a, b \in A\). \(A\) is a left (resp. right) ideal of \(R\) if \((A, +)\) is an LA-group and \(RA \subseteq A\) (resp. \(AR \subseteq A\)). A is an ideal of \(R\) if it is both a left ideal and a right ideal of \(R\).

A non-empty subset \(A\) of an LA-ring \(R\) is interior ideal of \(R\), if \((A, +)\) is an LA-group and \(\text{and } RA \subseteq A\). A non-empty subset \(A\) of an LA-ring \(R\) is quasi-ideal of \(R\), if \((A, +)\) is an LA-group and \(AR \cap RA \subseteq A\). An LA-subring \(A\) of \(R\) is bi-ideal of \(R\) if \((AR)A \subseteq A\). An LA-ring \(R\) is a generalized bi-ideal of \(R\) if \((A, +)\) is an LA-group and \((AR)A \subseteq A\).

We define the concept of fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring \(R\). We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) LA-rings by the properties of fuzzy left (right, quasi-, bi-, generalized bi-) ideals.

2. Fuzzy ideals in left almost ring (LA-rings)

First time the concept of fuzzy set was introduced by Zadeh in his classical paper \cite{23}. This concept has
provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, Languages, robotics, coding theory and others.

It soon invoked a natural question concerning a possible connection between fuzzy sets and algebraic systems like (set, group, semigroup, ring, near-ring, semiring, measure) theory, groupoids, real analysis, topology, differential equations and so forth. Rosenfeld [18], was the first, who introduced the concept of fuzzy set in a group. The study of fuzzy set in semigroup was established by Kuroki [10]. He studied fuzzy ideals and fuzzy (interior, quasi-, bi-, generalized bi-, semi-prime) ideals of semigroups.

Liu [12] introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings [2, 3, 11, 13, 14, 22]. Gupta et al. [3] gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [11] characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

Kausar et al. [20] initiated the concept of intuitionistic fuzzy normal subrings over a non-associative ring and also characterized the non-associative rings by their intuitionistic fuzzy bi-ideals in [6]. Recently Kausar [8] explored the notion of direct product of finite intuitionistic anti-fuzzy normal subrings over non-associative rings.

A fuzzy subset $\mu$ of an LA-ring $R$ is a function from $R$ into the closed unit interval $[0, 1]$, that is $\mu : R \to [0, 1]$, the complement of $\mu$ is denoted by $\mu^c$, is also a fuzzy subset of $R$ defined by $\mu^c(x) = 1 - \mu(x)$ for all $x \in R$. $F(R)$ denotes the collection of all fuzzy subsets of $R$.

A fuzzy subset $\mu$ of an LA-ring $R$ is a fuzzy LA-subring of $R$ if $\mu(x + y) \geq \min[\mu(x), \mu(y)]$ and $\mu(xy) \geq \min[\mu(x), \mu(y)]$ for all $x, y \in R$. Equivalent definition: $\mu$ is a fuzzy LA-subring of $R$ if $\mu(x + y) \geq \min[\mu(x), \mu(y)]$, $\mu(-x) \geq \mu(x)$ and $\mu(xy) \geq \min[\mu(x), \mu(y)]$ for all $x, y \in R$. $\mu$ is a fuzzy left (resp. right) ideal of $R$ if $\mu(x - y) \geq \min[\mu(x), \mu(y)]$ and $\mu(xy) \geq \mu(y)$ (resp. $\mu(xy) \geq \mu(x)$) for all $x, y \in R$. $\mu$ is called a fuzzy ideal of $R$ if it is both a fuzzy left ideal and a fuzzy right ideal of $R$.

Let $A$ be a non-empty subset of an LA-ring $R$. Then the characteristic function of $A$ is denoted by $\chi_A$ and defined by

$$\chi_A(a) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A. \end{cases}$$

We note that an LA-ring $R$ can be considered a fuzzy subset of itself and we write $R = C_R$, i.e. $R(x) = 1$ for all $x \in R$.

Let $\mu$ and $\gamma$ be two fuzzy subsets of an LA-ring $R$. The inclusion $\mu \subseteq \gamma$ is defined by $\mu(x) \leq \gamma(x)$ for all $x \in R$. The symbols $\mu \cap \gamma$ and $\mu \cup \gamma$ are defined by $(\mu \cap \gamma)(x) = \min[\mu(x), \gamma(x)]$ and $(\mu \cup \gamma)(x) = \max[\mu(x), \gamma(x)]$ all $x \in R$. The product of $\mu$ and $\gamma$ is denoted by $\mu \circ \gamma$ and defined by

$$((\mu \circ \gamma)(x)) = \begin{cases} \forall x = \sum_{i=1}^{n} a_i \chi_i & \text{if } x = \sum_{i=1}^{n} a_i \chi_i, a_i, b_i \in R \\ 0 & \text{if } x \neq \sum_{i=1}^{n} a_i \chi_i. \end{cases}$$

A fuzzy subset $\mu$ of an LA-ring $R$ is a fuzzy interior ideal if $\mu(x - y) \geq \mu(x) \cap \mu(y)$ and $\mu((xy)z) \geq \mu(y)$ for all $x, y, z \in R$. A fuzzy subset $\mu$ of an LA-ring $R$ is a fuzzy quasi-ideal of $R$ if $(\mu \circ \gamma)(x) \cap (R \circ \mu) \subseteq \mu$ and $\mu(x - y) \geq \mu(x) \cap \mu(y)$ for all $x, y \in R$. A fuzzy LA-subring $\mu$ of an LA-ring $R$ is a fuzzy bi-ideal of $R$ if $\mu((xy)z) \geq \min[\mu(x), \mu(z)]$ for all $x, y, z \in R$. A fuzzy subset $\mu$ of an LA-ring $R$ is a fuzzy generalized bi-ideal of $R$ if $\mu((xy)z) \geq \mu(x) \cap \mu(y)$ and $\mu((xy)z) \geq \mu(x) \cap \mu(z)$ for all $x, y, z \in R$. A fuzzy ideal $\mu$ of $R$ is a fuzzy idempotent if $\mu \circ \mu = \mu$.

Now we give the imperative properties of such ideals of an LA-ring $R$, which will play a vital role in the later sections.

**Lemma 2.1:** Let $R$ be an LA-ring. Then the following properties hold.

1. $(\mu \circ \gamma)(x) \circ \beta = (\beta \circ \gamma)(x) \circ \mu$.
2. $(\mu \circ \gamma)(x) \circ (\beta \circ \delta) = (\mu \circ \beta)(x) \circ (\gamma \circ \delta)$ for all fuzzy subsets $\mu, \gamma, \beta$ and $\delta$ of $R$.

**Proof:** Let $\mu, \gamma$ and $\beta$ be fuzzy subsets of an LA-ring $R$. We have to show that $(\mu \circ \gamma)(x) \circ \beta = (\beta \circ \gamma)(x) \circ \mu$.

$$((\mu \circ \gamma)(x)) = \bigvee_{x = \sum_{i=1}^{n} a_i \chi_i} \left( \bigwedge_{i=1}^{n} (\mu \circ \gamma)(a_i) \cap \beta(b_i) \right)$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i \chi_i} \left( \bigwedge_{i=1}^{n} \left( \bigvee_{a_i = \sum_{i=1}^{n} c_i d_i} \left( \bigwedge_{i=1}^{n} (\mu(c_i) \land \gamma(d_i)) \cap \beta(b_i) \right) \right) \right)$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i \chi_i} \left( \bigwedge_{i=1}^{n} \left( \bigvee_{a_i = \sum_{i=1}^{n} c_i d_i} \left( \bigwedge_{i=1}^{n} (\mu(c_i) \land \gamma(d_i)) \cap \beta(b_i) \right) \right) \right)$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i \chi_i} \left( \bigwedge_{i=1}^{n} \left( \bigvee_{a_i = \sum_{i=1}^{n} c_i d_i} \left( \bigwedge_{i=1}^{n} (\mu(c_i) \land \gamma(d_i)) \cap \beta(b_i) \right) \right) \right)$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i \chi_i} \left( \bigwedge_{i=1}^{n} \left( \bigvee_{a_i = \sum_{i=1}^{n} c_i d_i} \left( \bigwedge_{i=1}^{n} (\mu(c_i) \land \gamma(d_i)) \cap \beta(b_i) \right) \right) \right)$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i \chi_i} \left( \bigwedge_{i=1}^{n} \left( \bigvee_{a_i = \sum_{i=1}^{n} c_i d_i} \left( \bigwedge_{i=1}^{n} (\mu(c_i) \land \gamma(d_i)) \cap \beta(b_i) \right) \right) \right)$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i \chi_i} \left( \bigwedge_{i=1}^{n} \left( \bigvee_{a_i = \sum_{i=1}^{n} c_i d_i} \left( \bigwedge_{i=1}^{n} (\mu(c_i) \land \gamma(d_i)) \cap \beta(b_i) \right) \right) \right)$$

Similarly, we can prove (2).

**Proposition 2.1:** Let $R$ be an LA-ring with left identity $e$. Then the following assertions hold.
Theorem 2.1: Let $A$ and $B$ be two non-empty subsets of an LA-ring $R$. Then the following assertions hold.

(1) If $A \subseteq B$ then $\chi A \subseteq \chi B$.
(2) $\chi A \cap \chi B = \chi_{A \cap B}$.
(3) $\chi A \cup \chi B = \chi_{A \cup B}$.

Proof: Suppose that $A \subseteq B$ and $a \in R$. If $a \in A$, this implies that $a \in B$. Hence $\chi_A(a) = 1 = \chi_B(a)$, i.e. $\chi A \subseteq \chi B$. If $a \notin A$, so $a \notin B$. Thus $\chi_A(a) = 0 = \chi_B(a)$, i.e. $\chi A \subseteq \chi B$.

(2) Let $x \in R$ such that $x = ab$. This means that $x = ab$ for some $a \in A$ and $b \in B$. Now

\[ (\chi A \cap \chi B)(x) = \bigwedge_{i=1}^{\infty} a_i b_i \{ \chi_A(a_i) \wedge \chi_B(b_i) \} \geq \chi_A(a) \wedge \chi_B(b) = 1 \wedge 1 = 1 = \chi_{A \cap B}(x). \]

(iii) If $x \notin A \cup B$, then

\[ (\chi A \cup \chi B)(x) = \bigvee_{i=1}^{\infty} a_i b_i \{ \chi_A(a_i) \lor \chi_B(b_i) \} = 0 \lor 0 = 0 = \chi_{A \cup B}(x). \]

(iv) If $a \notin A$ and $a \notin B$, then

\[ (\chi A \cup \chi B)(a) = \chi_A(a) \lor \chi_B(a) = 0 \lor 1 = 1 = \chi_{A \cup B}(a). \]

If $a \notin A \cup B$, this implies that $a \notin A$ and $a \notin B$. Then obvious $\chi A \cup \chi B = \chi_{A \cup B}$. Hence in all cases $\chi A \cup \chi B = \chi_{A \cup B}$.

Theorem 2.2: Let $A$ be a non-empty subset of an LA-ring $R$. Then the following properties hold.

(1) $A$ is an LA-subring of $R$ if and only if $\chi A$ is a fuzzy LA-subring of $R$.
(2) $A$ is a left (resp. right, two-sided) ideal of $R$ if and only $\chi A$ is a fuzzy left (resp. right, two-sided ) ideal of $R$.

Proof: (1) Let $A$ be an LA-subring of $R$ and $a, b \in R$. If $a, b \in A$, then by definition $\chi_A(a) = 1 = \chi_B(b)$. Thus $\chi A \subseteq \chi B$, i.e. $\chi A$ is a fuzzy LA-subring of $R$.

Conversely, suppose that $\chi A$ is a fuzzy LA-subring of $R$ and let $a, b \in A$. Since $\chi_A(a \cdot b) \geq \chi_A(a) \cdot \chi_A(b) = 1$ and $\chi_A(ab) \geq \chi_A(a) \cdot \chi_A(b) = 1$, $\chi A$ is a fuzzy LA-subring of $R$.

(2) Let $A$ be a left ideal of $R$ and $a, b \in R$. If $a, b \in A$, then by definition $\chi_A(a) = 1 = \chi_B(b)$. Thus $\chi A \subseteq \chi B$.

Conversely, assume that $\chi A$ is a fuzzy left ideal of $R$. Let $a, b \in A$ and $z \in R$. Since $\chi_A(a \cdot z) \geq \chi_A(a) \cdot \chi_A(z) = 1$ and $\chi_A(za) \geq \chi_A(a) \cdot \chi_A(z) = 1$, $\chi A$ is a fuzzy left ideal of $R$.

Remark 2.1: (i) $A$ is an additive LA-subgroup of $R$ if and only if $\chi A$ is a fuzzy additive LA-subgroup of $R$.

Theorem 2.3: Let $\mu$ be a fuzzy subset of an LA-ring $R$. Then the following assertions hold.

(1) $\mu$ is a fuzzy LA-subring of $R$ if and only if $\mu \cap \mu \subseteq \mu$ and $\mu \subseteq \mu$.
(2) $\mu$ is a fuzzy left (resp. right) ideal of $R$ if and only if $R \cap \mu \subseteq \mu$ (resp. $\mu \cap R \subseteq \mu$) and $\mu(x \cdot y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in R$.
(3) $\mu$ is a fuzzy ideal of $R$ if and only if $R \cap \mu \subseteq \mu \cap R \subseteq \mu$ and $\mu(x \cdot y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in R$.

Proof: (1) Suppose that $\mu$ is a fuzzy LA-subring of $R$ and $x \in R$. For $\mu \cap \mu \subseteq \mu$. 

Lemma 2.2: If $\mu$ and $\gamma$ are two fuzzy LA-subrings (resp. (left, right, two-sided) ideals) of an LA-ring $R$, then $\mu \cap \gamma$ is also a fuzzy LA-subring (resp. (left, right, two-sided) ideal) of $R$.

**Proof:** Let $\mu$ and $\gamma$ be two fuzzy LA-subrings of $R$. We have to show that $\mu \cap \gamma$ is also a fuzzy LA-subring of $R$. Now

$$(\mu \cap \gamma)(x - y) = (\mu(x - y) \land \gamma(x - y))$$

$$\geq \{\mu(x) \land \mu(y)\} \land \{\gamma(x) \land \gamma(y)\}$$

$$= \mu(x) \land \gamma(x) \land \gamma(y)$$

$$= \{\mu(x) \land \gamma(x)\} \land \{\gamma(y)\} \land \gamma(y)$$

$$= (\mu \cap \gamma)(x) \land (\mu \cap \gamma)(y).$$

Similarly, we have $(\mu \cap \gamma)(xy) \geq (\mu \cap \gamma)(x) \land (\mu \cap \gamma)(y).$ Hence $\mu \cap \gamma$ is a fuzzy LA-subring of $R$. Similarly, for ideals.

Lemma 2.3: If $\mu$ and $\gamma$ are two fuzzy LA-subrings of an LA-ring $R$, then $\mu \circ \gamma$ is also a fuzzy LA-subring of $R$.

**Proof:** Let $\mu$ and $\gamma$ be two fuzzy LA-subrings of $R$. We have to show that $\mu \circ \gamma$ is also a fuzzy LA-subring of $R$. Now $(\mu \circ \gamma)^2 = (\mu \circ \gamma) \circ (\mu \circ \gamma) = (\mu \circ \mu) \circ (\gamma \circ \gamma) \subseteq \mu \circ \gamma$. Since $\gamma - \gamma \subseteq \gamma$, $\gamma$ being a fuzzy LA-subring of $R$. This implies that $\mu \circ (\gamma - \gamma) \subseteq \mu \circ \gamma$, i.e. $\mu \circ \gamma - \mu \circ \gamma \subseteq \mu \circ \gamma$. Hence $\mu \circ \gamma$ is a fuzzy LA-subring of $R$.

Remark 2.2: If $\mu$ is a fuzzy LA-subring of an LA-ring $R$, then $\mu \circ \mu$ is also a fuzzy LA-subring of $R$.

Lemma 2.4: Let $R$ be an LA-ring with left identity $e$. Then $RR = R$ and $eR = R = Re$.

**Proof:** Since $RR \subseteq R$ and $x = ex \in RR$, i.e. $RR = R$. Since $e$ is the left identity of $R$, i.e. $eR = R$. Now $Re = (RR)e = (eR)R = RR = R$.

Lemma 2.5: Let $R$ be an LA-ring with left identity $e$. Then every fuzzy right ideal of $R$ is a fuzzy ideal of $R$.

**Proof:** Let $\mu$ be a fuzzy right ideal of $R$ and $x, y \in R$. Now $\mu(xy) = \mu((ex)y) = \mu((yx)e) \geq \mu(yx) \geq \mu(y)$. Thus $\mu$ is a fuzzy ideal of $R$.

Lemma 2.6: If $\mu$ and $\gamma$ are two fuzzy left (resp. right) ideals of an LA-ring $R$ with left identity $e$, then $\mu \circ \gamma$ is also a fuzzy left (resp. right) ideal of $R$.

**Proof:** Let $\mu$ and $\gamma$ be two fuzzy left ideals of $R$. We have to show that $\mu \circ \gamma$ is also a fuzzy left ideal of $R$. Since $\mu \circ \gamma = \mu \circ (\mu \circ \gamma) \subseteq \mu \circ \gamma$ by the Lemma 2.3. Now $R \circ (\mu \circ \gamma) = (R \circ R) \circ (\mu \circ \gamma) \subseteq R \circ \gamma$. Hence $\mu \circ \gamma$ is a fuzzy left ideal of $R$. 

Therefore $\mu$ is a fuzzy left ideal of $R$. Similarly, we can prove (3). ■
Remark 2.3: If $\mu$ is a fuzzy left (resp. right) ideal of an LA-ring $R$ with left identity $e$, then $\mu \circ \mu$ is a fuzzy ideal of $R$.

Lemma 2.7: If $\mu$ and $\gamma$ are two fuzzy ideals of an LA-ring $R$, then $\mu \circ \gamma \subseteq \mu \cap \gamma$.

Proof: Let $\mu$ and $\gamma$ be two fuzzy ideals of $R$ and $x \in R$. If $(\mu \circ \gamma)(x) = 0$, then $\mu \circ \gamma \subseteq \mu \cap \gamma$, otherwise we have

$$(\mu \circ \gamma)(x) = \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{m} \{ \mu(a) \wedge \gamma (b_i) \}]$$

$\leq \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{m} \{ \mu(a) \land \gamma (a_i b_i) \}]$

$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{m} (\mu \cap \gamma) (a_i b_i)]$

$= (\mu \cap \gamma)(x)$.

$\Rightarrow \mu \circ \gamma \subseteq \mu \cap \gamma$. ■

Remark 2.4: If $\mu$ is a fuzzy ideal of an LA-ring $R$, then $\mu \circ \mu \subseteq \mu$.

Lemma 2.8: Let $R$ be an LA-ring. Then $\mu \circ \gamma \subseteq \mu \cap \gamma$ for every fuzzy right ideal $\mu$ and for every fuzzy left ideal $\gamma$ of $R$.

Proof: Same as Lemma 2.7. ■

Theorem 2.4: Let $A$ be a non-empty subset of an LA-ring $R$. Then the following conditions are true.

1. $A$ is an interior ideal of $R$ if and only if $\chi_A$ is a fuzzy interior ideal of $R$.
2. $A$ is a quasi-ideal of $R$ if and only if $\chi_A$ is a fuzzy quasi-ideal of $R$.
3. $A$ is a bi-ideal of $R$ if and only if $\chi_A$ is a fuzzy bi-ideal of $R$.
4. $A$ is a generalized bi-ideal of $R$ if and only if $\chi_A$ is a fuzzy generalized bi-ideal of $R$.

Proof: (1) Let $A$ be an interior ideal of $R$. This implies that $A$ is an additive LA-subgroup of $R$. Then $\chi_A$ is a fuzzy additive LA-subgroup of $R$ by Remark 2.1. Let $x, y, a \in R$. If $a \in A$, then by definition $\chi_A(a) = 1$. Since $(x)a y \in A$, being an interior ideal of $R$, this means that $\chi_A((x)a y) = 1$. Thus $\chi_A((x)a y) \geq \chi_A(a)$. Similarly, we have $\chi_A((x)a y) \geq \chi_A(x)$, when $a \not\in A$. Hence $\chi_A$ is a fuzzy interior ideal of $R$.

Conversely, suppose that $\chi_A$ is a fuzzy interior ideal of $R$. This implies that $\chi_A$ is a fuzzy additive LA-subgroup of $R$. Then $\chi_A$ is an additive LA-subgroup of $R$ by Remark 2.1. Let $x, y, a \in R$. If $a \in A$, so $\chi_A(a) = 1$. Since $\chi_A((x)a y) \geq \chi_A(a) = 1$, $\chi_A$ being a fuzzy interior ideal of $R$, this means that $\chi_A((x)a y) = 1$, i.e. $(x)a y \in A$. Hence $A$ is an interior ideal of $R$.

(2) Let $A$ be a quasi-ideal of $R$, this implies that $A$ is an additive LA-subgroup of $R$. Then $\chi_A$ is a fuzzy additive LA-subgroup of $R$. Now

$$(\chi_A \circ R) \cap (R \circ \chi_A) = (\chi_A \circ R) \cap (R \circ \chi_A)$$

$= \chi_{AR} \cap \chi_{RA} = \chi_{AR \setminus RA} \subseteq \chi_A$, by Theorem 2.1

Therefore $\chi_A$ is a fuzzy quasi-ideal of $R$.

Conversely, assume that $\chi_A$ is a fuzzy quasi-ideal of $R$, this means that $\chi_A$ is a fuzzy additive LA-subgroup of $R$. Then $A$ is an additive LA-subgroup of $R$. Let $x$ be an element of $AR \cap RA$. Now

$$\chi_A(x) \supseteq ((\chi_A \circ R) \cap (R \circ \chi_A))(x)$$

$$= \min\{\chi_A(R(x)), (R \circ \chi_A)(x)\}$$

$$= \min\{\chi_A(Ax), (Ax \circ \chi_A)(x)\}$$

$$= \min\{\chi_{AR}, \chi_{RA}\}$$

$$= (\chi_{AR} \cap \chi_{RA})(x) = \chi_{AR \setminus RA}(x) = 1.$$}

This implies that $x \in A$, i.e. $AR \cap RA \subseteq A$. Therefore $A$ is a quasi-ideal of $R$.

(3) Let $A$ be a bi-ideal of $R$, this implies that $A$ is an LA-subring of $R$. Then $\chi_A$ is a fuzzy LA-subring of $R$ by Theorem 2.2. Let $x, y, a \in R$. If $x, y \in A$, then by definition $\chi_A(x) = 1 = \chi_A(y)$. Since $(x)a y \in A$, $A$ being a bi-ideal of $R$, this means that $\chi_A((x)a y) = 1$. Thus $\chi_A((x)a y) \geq \chi_A(x) \wedge \chi_A(y)$. Similarly, we have $\chi_A((x)a y) \geq \chi_A(x) \wedge \chi_A(y)$, when $x, y \not\in A$. Hence $\chi_A$ is a fuzzy bi-ideal of $R$.

Conversely, suppose that $\chi_A$ is a fuzzy bi-ideal of $R$, this means that $\chi_A$ is a fuzzy LA-subring of $R$. Then $A$ is an LA-subring of $R$ by Theorem 2.2. Let $x, y, a \in R$ and $x, y \in A$, so $\chi_A(x) = 1 = \chi_A(y)$. Since $\chi_A((x)a y) \geq \chi_A(x) \wedge \chi_A(y) = 1$, $\chi_A$ being a fuzzy bi-ideal of $R$. Thus $\chi_A((x)a y) = 1$, i.e. $(x)a y \in A$. Hence $A$ is a bi-ideal of $R$. Similarly, we can prove (4). ■

Theorem 2.5: Let $\mu$ be a fuzzy subset of an LA-ring $R$. Then $\mu$ is a fuzzy interior ideal of $R$ if and only if $(\circ \mu) \circ R \subseteq \mu \circ \mu$ and $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in R$.

Proof: Suppose that $\mu$ is a fuzzy interior ideal of $R$ and $x \in R$. If $(\circ \mu) \circ R(x) = 0$, then $(\circ \mu) \circ R \subseteq \mu$, otherwise there exist $a_i, b_i, c_i, d_i \in R$ such that $x = \sum_{i=1}^{n} a_i b_i$ and $a_i = \sum_{i=1}^{n} c_i d_i$. Since $\mu$ is a fuzzy interior ideal of $R$, this implies that $\mu(x) \geq \mu(x \wedge \mu(y))$. Now

$$(\circ \mu) \circ R(x)$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{(\circ \mu)(a_i) \land R(b_i)\}]$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{ \vee_{a_i = \sum_{i=1}^{n} c_i d_i} [\bigwedge_{i=1}^{n} \{(R(c_i) \land \mu(d_i)) \setminus R(b_i)\}] \wedge R(b_i) \} ]$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{ \vee_{a_i = \sum_{i=1}^{n} c_i d_i} [\bigwedge_{i=1}^{n} \{ R(c_i) \land \mu(d_i) \} \setminus 1 \} ] \wedge 1 \} ]$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{ \vee_{a_i = \sum_{i=1}^{n} c_i d_i} [\bigwedge_{i=1}^{n} \mu(d_i) \setminus 1 \} ] \wedge 1 \} ]$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{ \vee_{a_i = \sum_{i=1}^{n} c_i d_i} [\bigwedge_{i=1}^{n} \mu(d_i) \setminus 1 \} ] \wedge 1 \} ]$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{ \vee_{a_i = \sum_{i=1}^{n} c_i d_i} [\bigwedge_{i=1}^{n} \mu(d_i) \setminus 1 \} ] \wedge 1 \} ]$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{ \vee_{a_i = \sum_{i=1}^{n} c_i d_i} [\bigwedge_{i=1}^{n} \mu(d_i) \setminus 1 \} ] \wedge 1 \} ]$$

$$= \bigvee_{x = \sum_{i=1}^{n} a_i b_i} [\bigwedge_{i=1}^{n} \{ \vee_{a_i = \sum_{i=1}^{n} c_i d_i} [\bigwedge_{i=1}^{n} \mu(d_i) \setminus 1 \} ] \wedge 1 \} ]$$
Lemma 2.9: Let \( \mu \) be a fuzzy subset of an LA-ring \( R \). Then \( \mu \) is a fuzzy ideal of \( R \) if and only if \( \mu \circ \gamma \) is also a fuzzy bi-ideal of \( R \), where \( \gamma \) is a fuzzy ideal of \( R \).

Proof: Let \( \mu \) and \( \gamma \) be two fuzzy bi-ideals of \( R \). We have to show that \( \mu \circ \gamma \) is also a fuzzy bi-ideal of \( R \). Since \( \mu \) and \( \gamma \) are fuzzy LA-subrings of \( R \), then \( \mu \circ \gamma \) is also a fuzzy LA-subring of \( R \) by the Lemma 2.3. Now

\[
((\mu \circ \gamma) \circ (R \circ \gamma)) \circ (\mu \circ \gamma) = (((\mu \circ \gamma) \circ (R \circ \gamma)) \circ (\mu \circ \gamma) = ((\mu \circ R) \circ (\gamma \circ R)) \circ (\mu \circ \gamma) = ((\mu \circ R) \circ (\gamma \circ R)) \circ (\mu \circ \gamma) \subseteq (\mu \circ \gamma).
\]

Therefore \( \mu \circ \gamma \) is a fuzzy bi-ideal of \( R \).

Lemma 2.11: Every fuzzy ideal of an LA-ring \( R \) is a fuzzy interior ideal of \( R \). The converse is not true in general.

Proof: Straight forward.

Example 2.1: Let \( R = \{0, 1, 2, 3, 4, 5, 6, 7\} \) be an LA-ring under the following + and \( \cdot \) defined as below

\[
\begin{align*}
+ & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\
0 & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\
1 & \quad 2 \quad 0 \quad 3 \quad 1 \quad 6 \quad 4 \quad 7 \quad 5 \\
2 & \quad 1 \quad 3 \quad 0 \quad 2 \quad 5 \quad 7 \quad 4 \quad 6 \\
3 & \quad 3 \quad 2 \quad 1 \quad 0 \quad 7 \quad 6 \quad 5 \quad 4 \\
4 & \quad 4 \quad 5 \quad 6 \quad 7 \quad 0 \quad 1 \quad 2 \quad 3 \\
5 & \quad 6 \quad 4 \quad 7 \quad 5 \quad 2 \quad 0 \quad 3 \quad 1 \\
6 & \quad 5 \quad 7 \quad 4 \quad 6 \quad 1 \quad 3 \quad 0 \quad 2 \\
7 & \quad 7 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0 \\
\cdot & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\
0 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
1 & \quad 0 \quad 4 \quad 4 \quad 0 \quad 0 \quad 4 \quad 4 \quad 0 \\
2 & \quad 0 \quad 4 \quad 4 \quad 0 \quad 0 \quad 4 \quad 4 \quad 0 \\
3 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
4 & \quad 4 \quad 3 \quad 3 \quad 0 \quad 0 \quad 3 \quad 3 \quad 0 \\
5 & \quad 5 \quad 0 \quad 7 \quad 7 \quad 0 \quad 0 \quad 7 \quad 7 \quad 0 \\
6 & \quad 6 \quad 0 \quad 7 \quad 7 \quad 0 \quad 0 \quad 7 \quad 7 \quad 0 \\
7 & \quad 7 \quad 0 \quad 3 \quad 3 \quad 0 \quad 0 \quad 3 \quad 3 \quad 0 
\end{align*}
\]

Let \( \mu \) be a fuzzy subset of an LA-ring \( R \) and define \( \mu(0) = \mu(4) = 0.7 \), \( \mu(1) = \mu(2) = \mu(3) = \mu(5) = \mu(6) = \mu(7) = 0 \). Then \( \mu \) is a fuzzy interior ideal of \( R \), but not a fuzzy ideal of \( R \), because \( \mu \) is not a fuzzy right ideal of \( R \), as

\[
\begin{align*}
\mu(41) & = \mu(3) = 0 \\
\mu(4) & = 0.7 \\
\mu(41) & \neq \mu(4).
\end{align*}
\]

Proposition 2.2: Let \( \mu \) be a fuzzy subset of an LA-ring \( R \) with left identity \( e \). Then \( \mu \) is a fuzzy ideal of \( R \) if and only if \( \mu \) is a fuzzy interior ideal of \( R \).
Proof: Let \( \mu \) be a fuzzy interior ideal of \( R \) and \( x,y \in R \). Now \( \mu(xy) = \mu((ex)y) \geq \mu(x) \), thus \( \mu \) is a fuzzy right ideal of \( R \). Hence \( \mu \) is a fuzzy ideal of \( R \) by Lemma 2.11. Converse is true by Lemma 2.11.

Lemma 2.12: Every fuzzy left (resp. right, two-sided) ideal of an LA-ring \( R \) is a fuzzy bi-ideal of \( R \). The converse is not true in general.

Proof: Suppose that \( \mu \) is a fuzzy right ideal of \( R \) and \( x,y,z \in R \). Now \( \mu((xy)z) = \mu(xy) \geq \mu(x) \) and \( \mu((y)z)z) = \mu((z)y) \geq \mu(z) \), this implies that \( \mu((xy)z) = \mu(x) \wedge \mu(z) \). Hence \( \mu \) is a fuzzy bi-ideal of \( R \).

Lemma 2.13: Every fuzzy bi-ideal of an LA-ring \( R \) is a fuzzy bi-ideal of \( R \). The converse is not true in general.

Proof: Obvious.

Lemma 2.14: Every fuzzy left (resp. right, two-sided) ideal of an LA-ring \( R \) is a fuzzy quasi-ideal of \( R \). The converse is true by Lemma 2.15.

Proof: Let \( \mu \) be a fuzzy quasi-ideal of \( R \). Since \( \mu \circ \mu \subseteq \mu \circ R \) and \( \mu \circ \mu \subseteq R \circ \mu \), i.e. \( \mu \circ \mu \subseteq \mu \circ R \cap R \circ \mu \). So \( \mu \) is a fuzzy LA-subring of \( R \).

Proposition 2.3: Every fuzzy quasi-ideal of an LA-ring \( R \) is a fuzzy LA-subring of \( R \).

Proof: Let \( \mu \) be a fuzzy quasi-ideal of \( R \). Since \( \mu \circ \mu \subseteq \mu \circ R \cap R \circ \mu \), this implies that there exists an element \( a \in R \), such that \( \mu \circ (\mu(a)) = \mu(\mu(a)) = \mu(a) \). Hence \( \mu \) is a fuzzy quasi-ideal of \( R \).

Proposition 2.4: Let \( \mu \) be a fuzzy right ideal and \( \gamma \) be a fuzzy left ideal of an LA-ring \( R \), respectively. Then \( \mu \wedge \gamma \) is a fuzzy quasi-ideal of \( R \).

Proof: We have to show that \( \mu \wedge \gamma \) is a fuzzy quasi-ideal of \( R \). Since \( (\mu \wedge \gamma)(x - y) \geq (\mu \wedge \gamma)(x) \wedge (\mu \wedge \gamma)(y) \) by Lemma 2.9 and \( ((\mu \wedge \gamma) \circ R) \cap R \circ ((\mu \wedge \gamma) \circ R) \subseteq (\mu \circ R) \cap R \circ \gamma \subseteq \mu \cap \gamma \). Therefore \( \mu \wedge \gamma \) is a fuzzy quasi-ideal of \( R \).

Lemma 2.15: Let \( R \) be an LA-ring with left identity \( e \), such that \( (xe)x = xe \) for all \( x \in R \). Then every fuzzy quasi-ideal of \( R \) is a fuzzy bi-ideal of \( R \).

Proof: Let \( \mu \) be a fuzzy quasi-ideal of \( R \). Since \( \mu \circ \mu \subseteq \mu \) by Proposition 2.3. Now \( (\mu \circ R) \circ \mu \subseteq (R \circ R) \circ \mu \subseteq R \circ \mu \) and \( (\mu \circ R) \circ \mu \subseteq (R \circ R) \circ \mu = (\mu \circ R) \circ (e \circ R) = (\mu \circ e) \circ (e \circ R) = \mu \circ R \).

Hence \( \mu \) is a fuzzy bi-ideal of \( R \).

Proposition 2.5: If \( \mu \) and \( \gamma \) are two fuzzy quasi-ideals of an LA-ring \( R \) with left identity \( e \), such that \( (xe)x = xe \) for all \( x \in R \). Then \( \mu \circ \gamma \) is a fuzzy bi-ideal of \( R \).

Proof: Let \( \mu \) and \( \gamma \) be two fuzzy quasi-ideals of \( R \), this implies that \( \mu \) and \( \gamma \) are two fuzzy bi-ideals of \( R \), by Lemma 2.15. Then \( \mu \circ \gamma \) is also a fuzzy bi-ideal of \( R \) by Lemma 2.10.

3. Regular left almost rings

An LA-ring \( R \) is regular if for every \( x \in R \), there exists an element \( a \in R \) such that \( x = (xa)x \). In this section, we characterize regular LA-rings by the properties of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 3.1: Every fuzzy right ideal of a regular LA-ring \( R \) is a fuzzy ideal of \( R \).

Proof: Suppose that \( \mu \) is a fuzzy right ideal of \( R \). Let \( x,y \in R \), this implies that there exists an element \( a \in R \), such that \( x = (xa)x \). Thus \( \mu(xy) = \mu((xa)x)(y) = \mu((xy)(xa)) \geq \mu(xy) \geq \mu(y) \). Hence \( \mu \) is a fuzzy ideal of \( R \).

Lemma 3.2: Every fuzzy ideal of a regular LA-ring \( R \) is a fuzzy idempotent.

Proof: Assume that \( \mu \) is a fuzzy ideal of \( R \) and \( \mu \circ \mu \subseteq \mu \). We have to show that \( \mu \subseteq \mu \circ \mu \). Let \( x \in R \), this means that there exists an element \( a \in R \) such that \( x = (xa)x \).

\[
\begin{align*}
(\mu \circ \mu)(x) &= \bigvee_{x=a_{i}b_{i}}^{a_{i}b_{i} \in \{a_{i}b_{i}\}} [\mu(a_{i}) \wedge \mu(b_{i})] \leq \mu(x) \\
&= \mu \subseteq \mu \circ \mu.
\end{align*}
\]

Therefore \( \mu \subseteq \mu \circ \mu \).

Remark 3.1: Every fuzzy right ideal of a regular LA-ring \( R \) is a fuzzy idempotent.

Lemma 3.3: Let \( \mu \) be a fuzzy subset of a regular LA-ring \( R \). Then \( \mu \) is a fuzzy ideal of \( R \) if and only if \( \mu \) is a fuzzy interior ideal of \( R \).

Proof: Suppose that \( \mu \) is a fuzzy interior ideal of \( R \). Let \( x,y \in R \), then there exists an element \( a \in R \), such that \( x = (xa)x \). Thus \( \mu(xy) = \mu((xa)x)y) = \mu((xy)(xa)) \geq \mu(x) \), i.e. \( \mu \) is a fuzzy right ideal of \( R \). So \( \mu \) is a fuzzy ideal of \( R \) by Lemma 3.1. Converse is true by Lemma 2.11.

Remark 3.2: The concept of fuzzy (interior, two-sided) ideals coincides in regular LA-rings.
Proposition 3.1: Let R be a regular LA-ring. Then \((\mu \circ R) \cap (R \circ \mu) = \mu\), for every fuzzy right ideal \(\mu\) of R.

Proof: Assume that \(\mu\) is a fuzzy right ideal of R. Then 
\((\mu \circ R) \cap (R \circ \mu) \subseteq \mu\), because every fuzzy right ideal of R is a fuzzy quasi-ideal of R by Lemma 2.14. Let \(x \in R\), this implies that there exists an element \(a \in R\), such that 
\(x = (xa)x\). Thus 
\[(\mu \circ R)(x) = \bigvee_{x = \sum_{i=1}^{n} a_{i}} [\bigwedge_{i=1}^{n} (\mu(a_{i}) \wedge R(b_{i}))] \geq \mu(xa) \wedge R(x) \geq \mu(x) \wedge 1 = \mu(x)\] 
\[\Rightarrow \mu \subseteq \mu \circ R.\]

Similarly, we have \(\mu \subseteq R \circ \mu\), i.e. \(\mu \subseteq (\mu \circ R) \cap (R \circ \mu)\). Therefore \((\mu \circ R) \cap (R \circ \mu) = \mu.\)

Lemma 3.4: Let R be a regular LA-ring. Then \(\mu \circ \gamma = \mu \wedge \gamma\), for every fuzzy right ideal \(\mu\) and for every fuzzy left ideal \(\gamma\) of R.

Proof: Since \(\mu \circ \gamma \subseteq \mu \wedge \gamma\) for every fuzzy right ideal \(\mu\) and every fuzzy left ideal \(\gamma\) of R by Lemma 2.8. Let \(x \in R\), this means that there exists an element \(a \in R\) such that 
\(x = (xa)x\). Thus 
\[(\mu \circ \gamma)(x) = \bigvee_{x = \sum_{i=1}^{n} a_{i}} [\bigwedge_{i=1}^{n} (\mu(x_{i}) \wedge \gamma(b_{i}))] \geq \mu(xa) \wedge \gamma(x) \geq \mu(x) \wedge \gamma(x) = (\mu \wedge \gamma)(x)\] 
\[\Rightarrow \mu \cap \gamma \subseteq \mu \wedge \gamma.\]

Hence \(\mu \circ \gamma = \mu \wedge \gamma\).

Lemma 3.5 ([6, Lemma 8]): Let R be an LA-ring with left identity e. Then Ra is the smallest left ideal of R containing a.

Lemma 3.6 ([6, Lemma 9]): Let R be an LA-ring with left identity e. Then aRa is a left ideal of R.

Proposition 3.2 ([6, Proposition 5]): Let R be an LA-ring with left identity e. Then aRa is the smallest right ideal of R containing a.

Theorem 3.1: Let R be an LA-ring with left identity e, such that \((xe)x = xR\) for all \(x \in R\). Then the following conditions are equivalent.

1. \(R\) is a regular.
2. \(\mu \wedge \gamma = \mu \circ \gamma\) for every fuzzy right ideal \(\mu\) and for every fuzzy left ideal \(\gamma\) of R.
3. \(\beta = (\beta \circ R) \circ \beta\) for every fuzzy quasi-ideal \(\beta\) of R.

Proof: Suppose that (1) holds and \(\beta\) be a fuzzy quasi-ideal of R. Then \((\beta \circ R) \circ \beta \subseteq \beta\), because every fuzzy quasi-ideal of R is a fuzzy bi-ideal of R by Lemma 2.15. Let \(x \in R\), this implies that there exists an element \(a \in R\) such that 
\(x = (xa)x\). Thus 
\[(\beta \circ R)(x) = \bigvee_{x = \sum_{i=1}^{n} a_{i}} [\bigwedge_{i=1}^{n} (\beta(a_{i}) \wedge R(b_{i}))] \geq (\beta \circ R)(xa) \wedge R(x) \geq (\beta \circ R)(x) \wedge 1 = (\beta \circ R)(x)\] 
\[\Rightarrow \beta \subseteq (\beta \circ R) \circ \beta.\]

Therefore \(\beta = (\beta \circ R) \circ \beta\), i.e. (1) implies (3). Assume that (3) holds. Let \(\mu\) be a fuzzy right ideal and \(\gamma\) be a fuzzy left ideal of R. This means that \(\mu \) and \(\gamma\) be fuzzy quasi-ideals of R by Lemma 2.14, so \(\mu \wedge \gamma\) be also a fuzzy quasi-ideal of R. Then by our assumption, \(\mu \wedge \gamma = ((\mu \wedge \gamma) \circ R) \circ (\mu \wedge \gamma) \subseteq (\mu \circ R) \circ (\mu \wedge \gamma) \subseteq \mu \circ \gamma\), i.e. \(\mu \wedge \gamma \subseteq \mu \circ \gamma\). Since \(\mu \circ \gamma \subseteq \mu \wedge \gamma\), hence \(\mu \circ \gamma = \mu \wedge \gamma\), i.e. (3) \(\Rightarrow\) (2). Suppose that (2) is true and \(a \in R\). Then Ra is a left ideal of R containing a by Lemma 3.5 and aRa is a right ideal of R containing a by Proposition 3.2. So \(\chi_{Ra}\) is a fuzzy left ideal and \(\chi_{aRa}\) is a fuzzy right ideal of R, by Theorem 2.2. Then by our supposition \(aRa \in R \cap aRa = \chi_{Ra} \circ aRa\) i.e. \(\chi_{aRa} = aRa\) by Theorem 2.1. Thus \(aRa \cup Ra = aRa \cup Ra\). Since a \(\in (aRa) \cap (aRa) \), i.e. a \(\in (aRa)\), so a \(\in (aRa) \cup (aRa)\). This implies that a \(\in (aRa)\) or a \(\in (aRa)\). If a \(\in (aRa)\), then a = (ax)/(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a = ((exe)y)a = (a(xey))a for any \(x, y \in R\). If a \(\in (aRa)\), then \((Ra)Ra = (Re)a(Ra) = (ae)(a(Ra) = (aRa)(Ra)\), i.e. a \(\in (aRa)\).

So a is a regular, i.e. R is a regular. Hence (2) \(\Rightarrow\) (1).

Theorem 3.2: Let R be an LA-ring with left identity e, such that \((xe)x = xR\) for all \(x \in R\). Then the following conditions are equivalent.

1. \(R\) is a regular.
2. \(\mu = (\mu \circ R) \circ \mu\) for every fuzzy quasi-ideal \(\mu\) of R.
3. \(\beta = (\beta \circ R) \circ \beta\) for every fuzzy bi-ideal \(\beta\) of R.
4. \(\delta = (\delta \circ R) \circ \delta\) for every fuzzy generalized bi-ideal \(\delta\) of R.

Proof: (1) \(\Rightarrow\) (4) is obvious. Since (4) \(\Rightarrow\) (3), every fuzzy bi-ideal of R is a fuzzy generalized bi-ideal of R by Lemma 2.13. Since (3) \(\Rightarrow\) (2), every fuzzy quasi-ideal of R is a fuzzy bi-ideal of R by Lemma 2.15. (2) \(\Rightarrow\) (1), by Theorem 3.1.

Theorem 3.3: Let R be an LA-ring with left identity e, such that \((xe)x = xR\) for all \(x \in R\). Then the following conditions are equivalent.

1. \(R\) is a regular.
(2) \( \mu \cap \gamma = (\mu \circ \gamma) \circ \mu \) for every fuzzy quasi-ideal \( \mu \) and for every fuzzy ideal \( \gamma \) of \( R \).

(3) \( \beta \cap \gamma = (\beta \circ \gamma) \circ \beta \) for every fuzzy bi-ideal \( \beta \) and for every fuzzy right ideal \( \gamma \) of \( R \).

(4) \( \delta \cap \gamma = (\delta \circ \gamma) \circ \delta \) for every fuzzy generalized bi-ideal \( \delta \) and for every fuzzy ideal \( \gamma \) of \( R \).

**Proof:** Suppose that (1) holds. Let \( \delta \) be a fuzzy generalized bi-ideal and \( \gamma \) be a fuzzy ideal of \( R \). Now \( (\delta \circ \gamma) \circ \delta \subseteq (R \circ \gamma) \circ R \subseteq \gamma \circ R \subseteq \gamma \) and \( (\delta \circ \gamma) \circ \delta \subseteq (\delta \circ \gamma) \circ \delta \subseteq \delta \circ \gamma \), i.e. \( (\delta \circ \gamma) \circ \delta \subseteq \delta \circ \gamma \). Let \( x \in R \), this means that there exists an element \( a \in R \) such that \( x = (xa)x \). Now \( xa = ((xa)x)a = (ax)(xa) \).

Thus

\[
((\delta \circ \gamma) \circ \delta)(x) = \bigvee_{x=a} \left\{ (\delta \circ \gamma)(a) \land \delta(b) \right\}
\]

\[
\geq (\delta \circ \gamma)(xa) \land \delta(x)
\]

\[
= \bigvee_{x=a} \left\{ (\delta(b)) \land \gamma(q) \right\}
\]

\[
\geq (\delta(x) \land \gamma((\gamma)(x)).
\]

\[\Rightarrow \delta \cap \gamma \subseteq (\delta \circ \gamma) \circ \delta.\]

Hence \( \delta \cap \gamma = (\delta \circ \gamma) \circ \delta, \) i.e. (1) \( \Rightarrow \) (4). It is clear that (4) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (2). Assume that (2) is true. Then \( \mu \cap \gamma = (\mu \circ \gamma) \circ \mu \), where \( R \) itself is a fuzzy two-sided ideal of \( \mu \), so \( \mu = (\mu \circ \gamma) \circ \mu \). Therefore \( R \) is a regular by Theorem 3.1, i.e. (2) \( \Rightarrow \) (1).

**Theorem 3.4:** Let \( R \) be an LA-ring with left identity \( e \), such that \((xe)R = xR \) for all \( x \in R \). Then the following conditions are equivalent.

(1) \( R \) is a regular.

(2) \( \mu \cap \gamma \leq \gamma \circ \mu \) for every fuzzy quasi-ideal \( \mu \) and for every fuzzy right ideal \( \gamma \) of \( R \).

(3) \( \beta \cap \gamma \leq \gamma \circ \beta \) for every fuzzy bi-ideal \( \beta \) and for every fuzzy right ideal \( \gamma \) of \( R \).

(4) \( \delta \cap \gamma \leq \gamma \circ \delta \) for every fuzzy generalized bi-ideal \( \delta \) and for every fuzzy right ideal \( \gamma \) of \( R \).

**Proof:** (1) \( \Rightarrow \) (4), is obvious. Since (4) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (2). Suppose that (2) is true, this implies that \( \gamma \cap \mu = \mu \cap \gamma \subseteq \gamma \circ \mu \), where \( \mu \) is a fuzzy left ideal of \( R \). Since \( \gamma \circ \mu \subseteq \gamma \cap \mu \), so \( \gamma \cap \mu = \gamma \circ \mu \). Hence \( R \) is a regular by Theorem 3.1, i.e. (2) \( \Rightarrow \) (1).

**Theorem 3.5:** Let \( R \) be an LA-ring with left identity \( e \), such that \((xe)R = xR \) for all \( x \in R \). Then the following conditions are equivalent.

(1) \( R \) is a regular.

(2) \( \mu \cap \gamma \cap \lambda \leq (\mu \circ \gamma) \circ \lambda \) for every fuzzy quasi-ideal \( \mu \), for every fuzzy right ideal \( \gamma \) and for every fuzzy left ideal \( \lambda \) of \( R \).

(3) \( \beta \cap \gamma \cap \lambda \leq (\beta \circ \gamma) \circ \lambda \) for every fuzzy bi-ideal \( \beta \), for every fuzzy right ideal \( \gamma \) and for every fuzzy left ideal \( \lambda \) of \( R \).

(4) \( \delta \cap \gamma \cap \lambda \leq (\delta \circ \gamma) \circ \lambda \) for every fuzzy generalized bi-ideal \( \delta \), for every fuzzy right ideal \( \gamma \) and for every fuzzy left ideal \( \lambda \) of \( R \).

**Proof:** Suppose that (1) holds. Let \( \delta \) be a fuzzy generalized bi-ideal, \( \gamma \) be a fuzzy right ideal and \( \lambda \) be a fuzzy left ideal of \( R \). Let \( x \in R \), this implies that there exists an element \( a \in R \) such that \( x = (xa)x \). Now

\[
x = (xa)x = ((xa)x)a = (ax)(xa) = x((ax)a).
\]

Thus

\[
((\delta \circ \gamma) \circ \lambda)(x) = \bigvee_{x=a} \left\{ (\delta \circ \gamma)(a) \land \lambda(b) \right\}
\]

\[
\geq (\delta \circ \gamma)(xa) \land \lambda(x)
\]

\[
= \bigvee_{x=a} \left\{ (\delta(b)) \land \gamma(q) \right\}
\]

\[
\geq (\delta(x) \land \gamma((\gamma)(x)).
\]

\[\Rightarrow \delta \cap \gamma \subseteq (\delta \circ \gamma) \circ \lambda.\]

Hence (1) \( \Rightarrow \) (4). It is clear that (4) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (2). Assume that (2) holds. Then \( \mu \cap \gamma \cap \lambda \subseteq (\mu \circ \gamma) \circ \lambda \), where \( \mu \) is a right ideal of \( R \), i.e. \( \mu \cap \gamma \cap \lambda \subseteq \mu \circ \lambda \). Since \( \mu \circ \lambda \subseteq \mu \lambda \), so \( \mu \circ \lambda = \mu \lambda \). Therefore \( R \) is a regular by Theorem 3.1, i.e. (2) \( \Rightarrow \) (1).

4. Intra-regular left almost rings

An LA-ring \( R \) is intra-regular if for every \( x \in R \), there exist elements \( a, b \in R \) such that \( x = \sum_{i=1}^{n}(ax^2)b \). In this section, we characterize intra-regular LA-rings by the properties of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals.

**Lemma 4.1:** Every fuzzy left (resp. right) ideal of an intra-regular LA-ring \( R \) is a fuzzy ideal of \( R \).

**Proof:** Suppose that \( \mu \) is a fuzzy right ideal of \( R \). Let \( x, y \in R \), this implies that there exist elements \( a, b \in R \),
such that \( x = \sum_{i=1}^{n} (a_i, x^2) b_i \). Thus

\[
\mu(xy) = \mu(((a_i, x^2) b_i) = \mu((y b_i)(a_i, x^2)) \\
\geq \mu(y b_i) \geq \mu(y).
\]

Hence \( \mu \) is a fuzzy ideal of \( R \).

Lemma 4.2: Let \( R \) be an intra-regular LA-ring with left identity \( e \). Then every fuzzy ideal of \( R \) is a fuzzy idempotent.

Proof: Assume that \( \mu \) is a fuzzy ideal of \( R \) and \( \mu \circ \mu \leq \mu \). Let \( x \in R \), this means that there exist elements \( a_i, b_i \in R \), such that \( x = \sum_{i=1}^{n} (a_i, x^2) b_i \). Now

\[
x = (a_i, x^2) b_i = (a_i(x x)) b_i = (x(a_i x)) b_i \\
= (x(a_i x))(e b_i) = (xe)((a_i x) b_i) = (a_i x)((xe) b_i).
\]

Thus

\[
(\mu \circ \mu)(x) = \bigvee_{x=\sum_{i=1}^{n} p a_i \{i \{1 \} \{1 \} \{1 \}} (a_i (\mu (p_i) \wedge \mu (q_i))) \\
\geq \mu (a_i (x)) \wedge \mu ((a_i x) b_i) \geq \mu (x) \wedge \mu (x) = \mu (x) \\
\Rightarrow \mu \leq \mu \circ \mu.
\]

Therefore \( \mu = \mu \circ \mu \).

Proposition 4.1: Let \( \mu \) be a fuzzy subset of an intra-regular LA-ring \( R \) with left identity \( e \). Then \( \mu \) is a fuzzy ideal of \( R \) if and only if \( \mu \) is a fuzzy interior ideal of \( R \).

Proof: Suppose that \( \mu \) is a fuzzy interior ideal of \( R \). Let \( x, y \in R \), then there exist elements \( a_i, b_i \in R \), such that \( x = \sum_{i=1}^{n} (a_i, x^2) b_i \). Thus

\[
\mu(xy) = \mu(((a_i, x^2) b_i) y) = \mu((y b_i)(a_i, x^2)) \\
= \mu((y b_i)(a_i (x))) = \mu((y b_i)(x(a_i x))) \\
= \mu((y x)(b_i(a_i x))) \geq \mu(x).
\]

So \( \mu \) is a fuzzy right ideal of \( R \), hence \( \mu \) is a fuzzy ideal of \( R \) by Lemma 4.1. Converse is true by Lemma 2.11.

Remark 4.1: The concept of fuzzy (interior, two-sided) ideals coincides in intra-regular LA-rings with left identity.

Lemma 4.3: Let \( R \) be an intra-regular LA-ring with left identity \( e \). Then \( \gamma \cap \mu \leq \mu \circ \gamma \), for every fuzzy left ideal \( \mu \) and for every fuzzy right ideal \( \gamma \) of \( R \).

Proof: Let \( \mu \) be a fuzzy left ideal and \( \gamma \) be a fuzzy right ideal of \( R \). Let \( x \in R \), this implies that there exist elements \( a_i, b_i \in R \) such that \( x = \sum_{i=1}^{n} (a_i, x^2) b_i \). Now

\[
x = (a_i, x^2) b_i = (a_i(x x)) b_i = (x(a_i x)) b_i \\
= (x(a_i x))(e b_i) = (xe)((a_i x) b_i) = (a_i x)((xe) b_i).
\]

Thus

\[
(\mu \circ \gamma)(x) = \bigvee_{x=\sum_{i=1}^{n} p a_i \{i \{1 \} \{1 \} \{1 \}} (a_i (\mu (p_i) \wedge \gamma (q_i))) \\
\geq \mu (a_i (x) \wedge \gamma ((x e) b_i) \geq \mu (x) \wedge \gamma (x) \\
= \gamma (x) \wedge \mu (x) = (\gamma (x) \wedge \mu (x)) \\
\Rightarrow \gamma \cap \mu \leq \mu \circ \gamma.
\]

Theorem 4.1: Let \( R \) be an LA-ring with left identity \( e \), such that \( (xe)R = xR \) for all \( x \in R \). Then the following conditions are equivalent.

1. \( R \) is an intra-regular.
2. \( \gamma \cap \mu \leq \mu \circ \gamma \) for every fuzzy left ideal \( \mu \) and for every fuzzy right ideal \( \gamma \) of \( R \).

Proof: (1) \( \Rightarrow \) (2) is obvious by Lemma 4.3. Suppose that (2) holds and \( a \in R \). Then \( Ra \) is a left ideal of \( R \) containing \( a \) by Lemma 3.5 and \( aR \cup Ra \) is a right ideal of \( R \) containing \( a \) by Proposition 3.2. So \( \chi_{aR} \) is a fuzzy left ideal and \( \chi_{aR \cup Ra} \) is a fuzzy right ideal of \( R \), by Theorem 2.2. By our supposition \( \chi_{aR \cup Ra} \cap R = \chi_{aR} \circ \chi_{aR \cup Ra} \), i.e. \( \chi_{aR \cup Ra} \subseteq \chi_{R} \) by Theorem 2.1. Thus \( (aR \cup Ra) \cap Ra \subseteq Ra(aR \cup Ra) \). Since \( a \in (aR \cup Ra) \cap Ra \), i.e. \( a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra) \). This implies that \( a \in (Ra)(aR) \) or \( a \in (Ra)(Ra) \). If \( a \in (Ra)(aR) \), then

\[
(Ra)(aR) = (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\
= (Ra)((ae)(R)) = (Ra)((aR))R \\
= (Ra)(R)(a) = (Ra)(aR) = ((Ra)(aR)R) \\
= ((Ra)(ea))R = (R(a)(ae))R = (RaR)^2, \\
\]

so \( a \in (RaR)^2 \). If \( a \in (Ra)(Ra) \), then obvious \( a \in (Ra)^2 \). Thus means that \( a \) is an intra-regular. Hence \( R \) is an intra-regular, i.e. (2) \( \Rightarrow \) (1).

Theorem 4.2: Let \( R \) be an LA-ring with left identity \( e \), such that \( (xe)R = xR \) for all \( x \in R \). Then the following conditions are equivalent.

1. \( R \) is an intra-regular.
2. \( \mu \circ \gamma = (\mu \circ \gamma) \circ \mu \) for every fuzzy quasi-ideal \( \mu \) and for every fuzzy ideal \( \gamma \) of \( R \).
3. \( \beta \cap \gamma = (\beta \circ \gamma) \circ \beta \) for every fuzzy bi-ideal \( \beta \) and for every fuzzy ideal \( \gamma \) of \( R \).
4. \( (\delta \circ \gamma) \circ \delta \) for every fuzzy generalized bi-ideal \( \delta \) and for every fuzzy ideal \( \gamma \) of \( R \).

Proof: Suppose that (1) holds. Let \( \delta \) be a fuzzy generalized bi-ideal and \( \gamma \) be a fuzzy ideal of \( R \). Now \( (\delta \circ \gamma) \circ \delta \subseteq (R \circ \gamma) \circ \delta \subseteq \gamma \circ \delta \subseteq \gamma \) and \( (\delta \circ \gamma) \circ \delta \subseteq (\delta \circ \delta) \circ \delta \subseteq \delta \), thus \( (\delta \circ \gamma) \circ \delta \subseteq \delta \cap \gamma \). Let \( x \in R \), this implies that there exist elements \( a_i, b_i \in R \) such that
$$x = \sum_{i=1}^{n}(a_ix)^2 b_i.$$ Now

$$x = (a_i x^2) b_i = (a_i xx) b_i = (a_i x) b_i.$$ 

Thus

$$((\delta \circ \gamma) \circ \delta) (x) = \vee_{x = \sum_{i=1}^{n} p_{qi}} \{\lambda_{i=1}^{n} [(\delta \circ \gamma) (p_{i}) \land \delta (q_{i})] \} \geq (\delta \circ \gamma) (b_{i}(a_i x)) \land \delta (x) \geq (x \land \gamma) (x) \geq \delta (x) \land (\delta \land \gamma) (x). \Rightarrow \delta \land \gamma \subseteq \delta \circ \gamma.$$ 

Hence $$\delta \land \gamma \subseteq (\delta \circ \gamma) \circ \delta \circ \delta,$$ which is our desired result.

Theorem 4.3: Let $$R$$ be an LA-ring with left identity $$e,$$ such that $$(xe) R = xR$$ for all $$x \in R.$$ Then the following conditions are equivalent.

(1) $$R$$ is an intra-regular.

(2) $$\mu \land \gamma \land \lambda \subseteq (\gamma \circ \delta) \circ \lambda,$$ for every fuzzy quasi-ideal $$\mu,$$ for every fuzzy left ideal $$\gamma$$ and for every fuzzy right ideal $$\lambda$$ of $$R.$$

(3) $$\beta \land \gamma \land \lambda \subseteq (\gamma \circ \beta) \circ \lambda,$$ for every fuzzy bi-ideal $$\beta,$$ for every fuzzy left ideal $$\gamma$$ and for every fuzzy right ideal $$\lambda$$ of $$R.$$

(4) $$\delta \land \gamma \land \lambda \subseteq (\delta \circ \gamma) \circ \lambda,$$ for every fuzzy generalized bi-ideal $$\delta,$$ for every fuzzy general ideal $$\gamma$$ and for every fuzzy right ideal $$\lambda$$ of $$R.$$

Proof: Suppose that (1) holds. Let $$\delta$$ be a fuzzy generalized bi-ideal, $$\gamma$$ be a fuzzy left ideal of $$R.$$ Let $$x \in R,$$ then there exist elements $$a_i, b_i \in R$$ such that $$x = \sum_{i=1}^{n}(a_i^2 x^2) b_i.$$ Now $$x = (a_i xx) b_i = (a_i x) b_i = (a_i x) b_i = (a_i x) b_i.$$ Thus

$$(x(a_i x)) b_i = (b_i (a_i x)) x.$$ 

Hence (1) implies (4). It is clear that (4) $$\Rightarrow$$ (3) and (3) $$\Rightarrow$$ (2). Suppose that (2) holds. Let $$\mu$$ be a fuzzy right ideal and $$\gamma$$ be a fuzzy left ideal of $$R.$$ Since every fuzzy right ideal of $$R$$ is a fuzzy quasi-ideal of $$R,$$ this implies that $$\mu$$ is a fuzzy quasi-ideal of $$R.$$ Then $$\mu \subseteq \gamma \subseteq \gamma \circ \mu.$$ Thus $$R$$ is an intra-regular by Theorem 4.1, i.e. (2) $$\Rightarrow$$ (1).

Theorem 4.4: Let $$R$$ be an LA-ring with left identity $$e,$$ such that $$(xe) R = xR$$ for all $$x \in R.$$ Then the following conditions are equivalent.

(1) $$R$$ is an intra-regular.

(2) $$\mu \land \gamma \land \lambda \subseteq (\gamma \circ \delta) \circ \lambda,$$ for every fuzzy quasi-ideal $$\mu,$$ for every fuzzy left ideal $$\gamma$$ and for every fuzzy right ideal $$\lambda$$ of $$R.$$

(3) $$\beta \land \gamma \land \lambda \subseteq (\gamma \circ \beta) \circ \lambda,$$ for every fuzzy bi-ideal $$\beta,$$ for every fuzzy left ideal $$\gamma$$ and for every fuzzy right ideal $$\lambda$$ of $$R.$$

(4) $$\delta \land \gamma \land \lambda \subseteq (\delta \circ \gamma) \circ \lambda,$$ for every fuzzy generalized bi-ideal $$\delta,$$ for every fuzzy left ideal $$\gamma$$ and for every fuzzy right ideal $$\lambda$$ of $$R.$$
5. Regular and intra-regular left almost rings

In this section, we characterize both regular and intra-regular LA-rings by the properties of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals.

**Theorem 5.1:** Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a both regular and an intra-regular.
2. $\mu \circ \mu = \mu$ for every fuzzy bi-ideal $\mu$ of $R$.
3. $\mu_1 \cap \mu_2 = (\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1)$ for all fuzzy bi-ideals $\mu_1$ and $\mu_2$ of $R$.

**Proof:** Suppose that (1) holds. Let $\mu$ be a fuzzy bi-ideal of $R$ and $\mu \circ \mu \subseteq \mu$. Let $x \in R$, this implies that there exists an element $a \in R$ such that $x = (xa)x$, also there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Then

$$x = (xa)x = ((xa)x)(a) = (a_i x^2)(b_i) = (a_i x^2)b_i = c_i(a_i x^2).$$

Hence $\mu = (\mu \circ \mu)\cap (\mu \circ \mu)\subseteq \mu$.

Similarly, we have $\mu_1 \cap \mu_2 \subseteq \mu_2$, thus $(\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1) \subseteq \mu_2$. Hence $\mu_1 \cap \mu_2 = (\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1)$, i.e. (2) $\Rightarrow$ (3). Suppose that (3) holds. Let $\mu$ be a fuzzy right ideal and $\gamma$ be a fuzzy ideal of $R$, then $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$, i.e. $\mu \cap \gamma \subseteq \mu \circ \gamma$ and $\mu \cap \gamma \subseteq \gamma \circ \mu$, where $\gamma$ is also a fuzzy left ideal of $R$. Since $\mu \cap \gamma \subseteq \mu \cap \gamma$, thus $\mu \cap \gamma = \mu \circ \gamma$ and $\mu \cap \gamma \subseteq \gamma \circ \mu$. Hence $R$ is both a regular and an intra-regular, i.e. (3) $\Rightarrow$ (1).

**Theorem 5.2:** Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is both regular and intra-regular.
2. Every fuzzy quasi-ideal of $R$ is a fuzzy idempotent.

**Proof:** Suppose that $R$ is both a regular and an intra-regular. Let $\mu$ be a fuzzy quasi-ideal of $R$. Then $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$, i.e. $\mu \cap \gamma \subseteq \mu \circ \gamma$ and $\mu \cap \gamma \subseteq \gamma \circ \mu$. Hence $\mu \cap \gamma = \mu \circ \gamma$ and $\mu \cap \gamma \subseteq \gamma \circ \mu$. Conversely, assume that every fuzzy quasi-ideal of $R$ is a fuzzy idempotent. Let $a \in R$, then $Ra$ is a left ideal of $R$ containing $a$ by Lemma 3.5. This implies that $Ra$ is a quasi-ideal of $R$, so $X_{Ra}$ is a fuzzy quasi-ideal of $R$ by Theorem 2.4. By our assumption $X_{Ra} = X_{Ra} \circ X_{Ra} = X_{Ra}(Ra)$, i.e. $Ra = (Ra)(Ra)$. Since $a \in Ra$, i.e. $a \in (Ra)(Ra)$, thus $a$ is both a regular and an intra-regular by Theorems 3.1 and 4.1, respectively. Hence $R$ is both a regular and an intra-regular.

**Theorem 5.3:** Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is both regular and intra-regular.
2. $\mu \cap \gamma \subseteq \mu \circ \gamma$ for all fuzzy quasi-ideals $\mu$ and $\gamma$ of $R$. 


(3) $\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy quasi-ideal $\mu$ and for every fuzzy bi-ideal $\gamma$ of $R$.
(4) $\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy bi-ideal $\mu$ and for every fuzzy quasi-ideal $\gamma$ of $R$.
(5) $\mu \cap \gamma \subseteq \mu \circ \gamma$ for all fuzzy bi-ideals $\mu$ and $\gamma$ of $R$.
(6) $\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy bi-ideal $\mu$ and for every fuzzy generalized bi-ideal $\gamma$ of $R$.
(7) $\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy generalized bi-ideal $\mu$ and for every fuzzy quasi-ideal $\gamma$ of $R$.
(8) $\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy quasi-ideal $\mu$ and for every fuzzy generalized bi-ideal $\gamma$ of $R$.
(9) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for all fuzzy quasi-ideals $\mu$ and $\gamma$ of $R$.
(10) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy quasi-ideal $\mu$ and for every fuzzy bi-ideal $\gamma$ of $R$.
(11) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy quasi-ideal $\mu$ and for every fuzzy generalized bi-ideal $\gamma$ of $R$.
(12) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for all fuzzy bi-ideals $\mu$ and $\gamma$ of $R$.
(13) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy bi-ideal $\mu$ and for every fuzzy generalized bi-ideal $\gamma$ of $R$.
(14) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for all fuzzy generalized bi-ideals $\mu$ and $\gamma$ of $R$.

Proof: Suppose that (1) holds. Assume that $\mu$ and $\gamma$ be two fuzzy generalized bi-ideals of $R$. Let $x \in R$, this implies that there exists an element $a \in R$ such that $x = (xa)x$, and also there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n a_i b_i$. Since $x = (xa)x = ((xw)x)x$ by Theorem 5.1. Thus

$$
(\mu \circ \gamma)(x) = \bigwedge_{x=\sum_{i=1}^n a_i b_i} \left(\mu \circ \gamma \left( \bigwedge_{i=1}^n (\mu \circ \gamma (q_i)) \right) \right) \\
\geq \mu((xw)x) \wedge \gamma(x) \\
\geq (\mu \wedge \mu \wedge \gamma \wedge \gamma)(x) = (\mu \cap \gamma)(x) \\
\Rightarrow \mu \cap \gamma \subseteq \mu \circ \gamma.
$$

Hence (1) $\Rightarrow$ (9). It is clear that (9) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (4) $\Rightarrow$ (2) and (9) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (3). Assume that (2) holds. Let $\mu$ be a fuzzy right ideal and $\gamma$ be a fuzzy left ideal of $R$. Since every fuzzy right ideal and fuzzy left ideal of $R$ is a fuzzy quasi-ideal of $R$ by Lemma 2.14. By our assumption, $\mu \cap \gamma \subseteq \mu \circ \gamma$. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, so $\mu \cap \gamma = \mu \circ \gamma$, i.e. $R$ is a regular. Again by our assumption, $\mu \cap \gamma = \gamma \cap \mu \subseteq \gamma \circ \mu$, i.e. $R$ is an intra-regular. Hence $R$ is both a regular and an intra-regular, i.e. (2) $\Rightarrow$ (1). In similar way, we can prove that (3) $\Rightarrow$ (1).

Theorem 5.4: Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) $R$ is both regular and intra-regular.
(2) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy right ideal $\mu$ and for every fuzzy left ideal $\gamma$ of $R$.
(3) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy quasi-ideal $\mu$ and for every fuzzy quasi-ideal $\gamma$ of $R$.
(4) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy quasi-ideal $\mu$ and for every fuzzy bi-ideal $\gamma$ of $R$.
(5) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy right ideal $\mu$ and for every fuzzy generalized bi-ideal $\gamma$ of $R$.
(6) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy right ideal $\mu$ and for every fuzzy quasi-ideal $\gamma$ of $R$.
(7) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy left ideal $\mu$ and for every fuzzy bi-ideal $\gamma$ of $R$.
(8) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy left ideal $\mu$ and for every fuzzy generalized bi-ideal $\gamma$ of $R$.

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