EXAMPLES OF COMPLETE MANIFOLDS OF POSITIVE RICCI CURVATURE WITH NILPOTENT ISOMETRY GROUPS

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It is well known [4] that the isometry group of a complete riemannian manifold $M$ with strictly positive sectional curvature is always compact. This is no longer true in general when $M$ has Ricci curvature $\text{Ric} > 0$. The first example was given in [7] for $\dim M = 4$. In this note we shall prove

**Theorem.** Let $L$ be an $n$-dimensional simply connected nilpotent Lie group. Then for all sufficiently large $p$, the product manifold $M^{p+n} = \mathbb{R}^p \times L$ admits complete riemannian metrics with strictly positive Ricci curvature such that the isometry group of $M$ contains $L$.

Using a theorem of Malcev [8], we have as an immediate consequence:

**Corollary.** Every finitely generated torsion-free nilpotent group can be realized as the fundamental group of a complete riemannian manifold with strictly positive Ricci curvature.

On the other hand, every finitely generated subgroup of the fundamental group of any complete manifold with $\text{Ric} \geq 0 \ (K \geq 0)$ is nilpotent (abelian) up to finite index [6, 5, 4].

**Proof of the Theorem.** Our construction is inspired by [2]. We first apply an observation in [3, pp. 126–127] to obtain a family of almost flat metrics $g_r$ on $L$, $0 < r < \infty$.

Choose a triangular basis $\{X_1, \ldots, X_n\}$ for the Lie algebra $\mathfrak{l}$ of $L$, i.e., $[X, X_i] \in l_{i-1}$ whenever $X \in \mathfrak{l}$, and $l_{i-1}$ is spanned by $X_1, \ldots, X_{i-1}$. For $X = \sum_{i=1}^n a_i X_i$ set $\|X\|^2 = \sum_{i=1}^n h_i^2(r)a_i^2$, where $h_i(r) = (1 + r^2)^{-\alpha_i}$, and $\alpha_n = \alpha > 0$, $2\alpha_i - 4\alpha_{i+1} = 1$, $1 \leq i \leq n - 1$. The above norm gives rise to a corresponding almost flat left invariant metric $g_r$. Then

$$\text{(1) } |\text{Ric}_L(X_i)| \leq c(1 + r^2)^{-1},$$

where $c$ is a constant depending on $n$ and the structure constants.

Now we define a warped product metric $g$ on $M$ by

$$g = dr^2 + f^2(r) \, ds^2 + g_r,$$

where $ds^2$ is the canonical euclidean metric on the sphere $S^{p-1} \subset \mathbb{R}^p$, $f(r) = r(1 + r^2)^{-1/4}$. $g$ is a complete metric on $M$, since $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f(r) > 0$ for $r > 0$, $h_i(r) > 0$ for $r \geq 0$, $h'_i(0) = 0$ for $1 \leq i \leq n$.

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It is clear that the isometry group of $g$ contains $L$.

Let $H = \partial/\partial r$ and $U = f(r)^{-1}v$ for a unit tangent vector $v$ of $S^{p-1}$. Straightforward calculation yields:

$$\text{Ric}(H, U) = 0,$$
$$\text{Ric}(X_i, H) = \text{Ric}(X_i, U) = 0, \quad (1 \leq i \leq n),$$
$$\text{Ric}(X_i, X_j) = 0, \quad (i \neq j, 1 \leq i, j \leq n).$$

$$\text{Ric}(X_i, X_i) = -\frac{g''}{g_i} - (p - 1)\frac{f'g'_i}{fg_i} + \text{Ric}_L(X_i) - \sum_{i \neq j} \frac{g'_i g'_j}{g_ig_j}$$
$$\geq \left\{ -2\alpha_i[(2\alpha_i + 1)r^2 - 1] + (p - 1)\alpha_i(2 + r^2) - c(1 + r^2) - \sum_{i \neq j} 4\alpha_i\alpha_jr^2 \right\} / (1 + r^2)^2$$
$$\quad + \sum_{i = 1}^{n} \frac{f''}{f}$$
$$\geq \left\{ -\sum_{i = 1}^{n} 2\alpha_i[(2\alpha_i + 1)r^2 - 1] + (p - 1)\frac{r^2 + 6}{4} \right\} / (1 + r^2)^2.$$  \hfill (2)

$$\text{Ric}(H, H) = -\sum_{i = 1}^{n} \frac{g''}{g_i} - (p - 1)\frac{f''}{f}$$
$$= \left\{ -\sum_{i = 1}^{n} 2\alpha_i[(2\alpha_i + 1)r^2 - 1] + (p - 1)\frac{r^2 + 6}{4} \right\} / (1 + r^2)^2.$$  \hfill (3)

$$\text{Ric}(U, U) = -\frac{f''}{f} + \frac{p - 2}{f^2} - (p - 2)\left(\frac{f'}{f}\right)^2 - \sum_{i = 1}^{n} \frac{f'g'_i}{fg_i}.$$  \hfill (4)

Since $1 - (f')^2 \geq 0$, $f'' \leq 0$, we have $\text{Ric}(U, U) > 0$ in (4). Positivity of the Ricci curvature in the equations (2) and (3) follows for $p$ sufficiently large. Observe that every term of the right-hand side decays at a rate of order at least $r^{-2}$. This completes the proof of the theorem.

REMARK. The smallest $p$ that yields positive Ricci curvature on $M^{p+n} = \mathbb{R}^p \times L$ by means of our construction is quite large in general. For example, in the case of the three-dimensional Heisenberg group $L = H^3$, we have to choose $p > 673$. (With a slightly refined choice of functions, $p > 26$ will already work.) We don’t know whether or not $p$ can be chosen much smaller. However, it follows from [1] that necessarily $p \geq 4$ when $L = H^3$.

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