Damped Oscillator with δ-kicked frequency in probability representation of quantum mechanics

Vladimir N. Chernega, Olga V. Man’ko

1Lebedev Physical Institute, Russian Academy of Sciences
Leninskii Prospect 53, Moscow 119991, Russia
2Bauman Moscow State Technical University
The 2nd Baumanskaya Street 5, Moscow 105005, Russia
*Corresponding author e-mail: omanko@sci.lebedev.ru

Abstract

We obtain the tomogram of squeezed correlated states of a quantum parametric damped oscillator in an explicit form. We study the damping within the framework of the Caldirola–Kanai model and chose the parametric excitation in the form of a very short pulse simulated by a δ-kick of frequency; the squeezing phenomenon is reviewed for both models. The cases of strong and weak damping are investigated.

Keywords: parametric oscillator, Caldirola–Kanai model, probability representation of quantum mechanics, tomograms, squeezing coefficient.

1 Introduction

In quantum mechanics, the quantum system state is described by the wave function (pure state) [1] or the density matrix (mixed state) [2, 3]; see references in [4]. In the probability representation of quantum mechanics [5, 6, 7], the quantum states are associated with tomograms. The quantum tomogram is the probability distribution function of the position $X$ measured in an ensemble of reference frames. Due to this circumstances, the quantum tomogram depends on additional parameters $\mu$ and $\nu$ determining the reference frame in the classical phase space.
The density matrix and wave function of the quantum state can be reconstructed from the tomogram. The symplectic tomography method was discussed for measuring quantum states in [5]. The method uses the Fourier transform of the marginal distribution for a measurable squeezed and rotated quadrature (symplectic tomogram) instead of the Radon transform [8], which is used in optical tomography to reconstruct the Wigner function [9] from the optical tomogram [10, 11].

The tomograms determine completely the quantum state and measuring this distribution implies reconstructing the quantum state. In [12], the scheme of optical tomography was used for reconstruction of the Wigner function of the electromagnetic field state. Experiments with reproducible measurements of squeezed vacuum state of light generated by an optical parametric oscillator were performed in [13].

The problem of the time-dependent harmonic oscillator is interesting due to various applications of this model in different areas of physics. For example, it describes the motion of a single particle in a Paul trap [14, 15]. In [16, 17], it was shown that the solutions for systems with quadratic Hamiltonians are expressed in terms of the classical trajectory of the system. In [18], the correlated squeezed states of a quantum oscillator with $\delta$-kicked frequencies were found, and the possibility of obtaining the squeezing phenomenon due to the parametric excitation in the form of $\delta$-kicks of frequency was shown.

In [19], it was shown that any nonadiabatic change in the frequency of a harmonic oscillator leads to squeezing. The appearance of squeezing phenomenon due to different types of a sudden change in frequency was considered in [20, 21, 22]. Repeated jumps between two frequencies were shown to generate fast increasing squeezing if the time intervals between the jumps were chosen to have special values [23]. The squeezing properties of a harmonic oscillator with a series of sudden changes between two frequencies were calculated in [24], where infinite, quasiperiodic, and irregular behaviors of the squeezing were investigated.

In this paper, we consider the damping within the framework of the Caldirola–Kanai model [25, 26]. This model is a partial case of the multidimensional system described
by nonstationary Hamiltonian, which is a general quadratic form in the position and momentum operators considered in [16, 17]. The Gaussian wave packets in the case of weak damping were investigated in [27, 28]. In [29], the propagator, wave function, and uncertainty relation for a time-dependent damped harmonic oscillator were evaluated, and the classical solution of the quantum systems was analyzed. In [30], the time dependent quantum harmonic oscillator subject to a sudden change in its mass was considered. Squeezing and resonance in a generalized Caldirola–Kanai-type quantum parametric oscillator was investigated in [31]. In [32], it was shown that the ground state of well-known pseudo-stationary states for the Caldirola–Kanai Hamiltonian is a generalized minimum uncertainty state, and their dependences on the damping factor and frequency were found. In [33], it was shown that a generalization of the Caldirola–Kanai Hamiltonian, which describes dissipative systems, can be fulfilled by replacing the standard exponential function with the $q$-exponential one. The different problems of quantum oscillator with a time-dependent frequency were solved in [34–38].

The aim of this work is to consider the parametric excitation of a quantum damped oscillator in the probability representation of quantum mechanics and obtain an explicit form of the tomographic probability distribution of quadrature in squeezed correlated states of a quantum damped parametric oscillator, as well as to review the influence of damping on the squeezing phenomenon for the kicked oscillator.

This paper is organized as follows.

In Sec. 2, we review the symplectic tomography scheme on the example of a quantum parametric damped oscillator and obtain symplectic and optical tomograms of squeezed correlated states. In Sec. 3, following [39] we review the appearance of squeezing phenomenon due to the parametric excitation, which is chosen in the form of very short pulse simulated by a δ-kick of frequency. We consider the cases of weak and strong damping and the case of a free particle. The parametric excitation under consideration is interesting due to the possibility of finding the classical trajectory in an explicit form. In Sec. 4, we conclude by listing the main results obtained in this study.
2 Tomography of a Quantum Damped Parametric Oscillator

It was shown [5, 6] that for the generic linear combination of quadratures, which is a measurable observable (we use dimensionless variables and put $\hbar = 1$),

$$\hat{X} = \mu\hat{q} + \nu\hat{p},$$  \hspace{1cm} (1)

where $\hat{q}$ and $\hat{p}$ are operators of the position and momentum, respectively. The tomogram $w(X, \mu, \nu)$ (normalized with respect to the variable $X$), being dependent on the two extra real parameters $\mu$ and $\nu$, is related to the state of the quantum system expressed in terms of the density operator $\hat{\rho}$ as follows:

$$w(X, \mu, \nu) = \text{Tr} \left[ \hat{\rho} \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}) \right].$$  \hspace{1cm} (2)

The physical meaning of the parameters $\mu$ and $\nu$ is that they describe an ensemble of rotated and scaled reference frames, in which the position $X$ is measured. For $\mu = \cos \varphi$ and $\nu = \sin \varphi$, the tomogram (2) is the distribution of homodyne output variable used in optical tomography [10, 11]. Formula (2) can be inverted, and the density operator of the state can be expressed in terms of the tomogram [40],

$$\hat{\rho} = \frac{1}{2\pi} \int w(X, \mu, \nu) e^{i(X\hat{1} - \mu\hat{q} - \nu\hat{p})} dX d\mu d\nu.$$  \hspace{1cm} (3)

First, we discuss the tomogram for squeezed and correlated state of a damped parametric oscillator; it has the Gaussian form,

$$w_\alpha(X, \mu, \nu, t) = \frac{1}{\sqrt{2\pi}\sigma_X(t)} \exp \left\{ -\frac{(X - \bar{X})^2}{2\sigma_X(t)} \right\},$$  \hspace{1cm} (4)

in which, in view of (1), one can express the mean value of observable as follows:

$$\bar{X} = \mu\langle q \rangle + \nu\langle p \rangle,$$  \hspace{1cm} (5)

where $\langle p \rangle$ and $\langle q \rangle$ are the quadrature means in the state (4), and $\alpha$ is a complex number.
The dispersion of the observable $X$ is

$$\sigma_X(t) = \mu^2 \sigma_{q^2} + \nu^2 \sigma_{p^2} + 2\mu\nu\sigma_{pq}, \quad (6)$$

where the parameters $\sigma_{q^2}$, $\sigma_{p^2}$, and $\sigma_{pq}$ are dispersions and covariance of quadratures in the state (4). The tomogram (4) is the probability distribution function; it is normalized, and completely determines the squeezed correlated quantum state of a damped parametric oscillator.

Let us consider the quantum damped parametric oscillator within the framework of the Caldirola–Kanai model [25, 26]. The Hamiltonian of the system reads

$$\hat{H} = \frac{1}{2} me^{2\gamma t} \omega^2(t) \hat{q}^2 + \frac{1}{2m} e^{-2\gamma t} \hat{p}^2, \quad (7)$$

where $m$ is the mass of oscillator, $\gamma$ is damping coefficient, $\hat{q}$ is the position operator, $\hat{p}$ is the momentum operator, and $\omega(t)$ is the time-dependent frequency of oscillator. For simplicity, we assume put $m = 1$ and $\hbar = 1$.

The classical equations of motion for the classical position $q$ and momentum $p$ corresponding to the Caldirola–Kanai model are

$$\dot{q} = p e^{-2\gamma t}, \quad \dot{p} = -\omega^2(t) e^{2\gamma t} q, \quad \ddot{q} + 2\gamma \dot{q} + \omega^2(t) q = 0. \quad (8)$$

We consider the function $\varepsilon(t)$, which is the solution of the equation of motion

$$\ddot{\varepsilon}(t) + 2\gamma \dot{\varepsilon}(t) + \omega^2(t) \varepsilon(t) = 0, \quad \Omega^2(0) = \omega^2(0) - \gamma^2, \quad (9)$$

with the initial conditions $\varepsilon(0) = 1$, $\dot{\varepsilon}(0) = i\Omega(0)$, and it satisfies the constraint

$$e^{2\gamma t} (\dot{\varepsilon} \varepsilon^* - \dot{\varepsilon}^*) = 2i\Omega(0). \quad (10)$$

The quantum dispersion of the position in state (4) is expressed through the classical trajectory $\varepsilon(t)$ (9)

$$\sigma_{q^2} = |\varepsilon|^2/2. \quad (11)$$

For the quantum dispersion of the momentum in the state (4), one can obtain the following expression:

$$\sigma_{p^2} = \frac{e^{4\gamma t} |\dot{\varepsilon}|^2}{2\Omega^2(0)}. \quad (12)$$
The covariance of the position and momentum in the state (4) is

$$\sigma_{qp} = \frac{1}{2}\sqrt{\frac{e^{4\gamma t}|\dot{\varepsilon}|^2}{\Omega^2(0)}} - 1. \quad (13)$$

The mean values of quadratures in the state (4) are

$$\langle p \rangle = \frac{e^{2\gamma t}}{\sqrt{2}\Omega(0)} (\alpha\dot{\varepsilon}^* - \alpha^* \dot{\varepsilon}), \quad \langle q \rangle = \frac{1}{\sqrt{2}} (\alpha\dot{\varepsilon}^* + \alpha^* \dot{\varepsilon}). \quad (14)$$

The mean value of observable $X$ (5) in squeezed correlated states (4) is expressed through the classical trajectory as follows:

$$\bar{X} = \frac{1}{\sqrt{2}} \left[ \mu (\alpha\dot{\varepsilon}^* + \alpha^* \dot{\varepsilon}) + \nu \frac{e^{2\gamma t}}{\Omega(0)} (\alpha\dot{\varepsilon}^* - \alpha^* \dot{\varepsilon}) \right]. \quad (15)$$

For dispersion of observable $X$ (6) in squeezed correlated states (4), one has

$$\sigma_X = \frac{1}{2} \left( \mu^2|\varepsilon|^2 + \frac{\nu^2}{\Omega^2(0)} e^{4\gamma t}|\dot{\varepsilon}|^2 + \nu \mu \sqrt{\frac{e^{4\gamma t}|\dot{\varepsilon}|^2}{\Omega^2(0)}} - 1 \right). \quad (16)$$

The explicit expression for the symplectic tomogram, which determines the squeezed correlated states of a quantum damped parametric oscillator, reads

$$w_\alpha (X, \mu, \nu) = \left[ \pi \left( \mu^2|\varepsilon|^2 + \frac{\nu^2}{\Omega^2(0)} e^{4\gamma t}|\dot{\varepsilon}|^2 + \mu \nu \sqrt{\frac{e^{4\gamma t}|\dot{\varepsilon}|^2}{\Omega^2(0)}} - 1 \right) \right]^{-1/2} \times \exp \left[ -\frac{(X - \mu \left( \frac{\alpha \dot{\varepsilon}^* + \alpha^* \dot{\varepsilon}}{\sqrt{2}} \right) - \left\{ \nu e^{2\gamma t} (\alpha\dot{\varepsilon}^* - \alpha^* \dot{\varepsilon}) / \sqrt{2}\Omega(0) \} \right)^2}{\mu^2|\varepsilon|^2 + \frac{\nu^2 e^{4\gamma t}|\dot{\varepsilon}|^2}{\Omega^2(0)} + \nu \mu \sqrt{e^{4\gamma t}|\dot{\varepsilon}|^2/\Omega^2(0)} - 1} \right]. \quad (17)$$

The optical tomogram of squeezed correlated states of quantum damped parametric oscillator has the Gaussian form

$$w_{\alpha}^{\text{opt}} (X, \theta) = \left[ \pi \left( \cos^2 \theta|\varepsilon|^2 + \frac{\sin^2 \theta}{\Omega^2(0)} e^{4\gamma t}|\dot{\varepsilon}|^2 + \sin \theta \cos \theta \sqrt{\frac{e^{4\gamma t}|\dot{\varepsilon}|^2}{\Omega^2(0)}} - 1 \right) \right]^{-1/2} \times \exp \left[ -\frac{(X - \cos \theta \left( \frac{\alpha \dot{\varepsilon}^* + \alpha^* \dot{\varepsilon}}{\sqrt{2}} \right) - \sin \theta e^{2\gamma t} \left( \frac{\alpha\dot{\varepsilon}^* - \alpha^* \dot{\varepsilon}}{\sqrt{2}\Omega(0)} \right) \right)^2}{\cos^2 \theta|\varepsilon|^2 + \frac{\sin^2 \theta e^{4\gamma t}|\dot{\varepsilon}|^2}{\Omega^2(0)} + \sin \theta \cos \theta \sqrt{e^{4\gamma t}|\dot{\varepsilon}|^2/\Omega^2(0)} - 1} \right]. \quad (18)$$
The optical tomogram (18) satisfies the entropic inequality

$$- \int \left[ w_{\alpha}^{\text{opt}}(X, \theta) \ln w_{\alpha}^{\text{opt}}(X, \theta) + w_{\alpha}^{\text{opt}}(X, \theta + \pi/2) \ln w_{\alpha}^{\text{opt}}(X, \theta + \pi/2) \right] dX \geq \ln(\pi e).$$

(19)

This inequality can be checked experimentally in the optical tomography measurement scheme.

The squeezing coefficient in the states determined by tomograms (17) and (18) reads

$$k = \sigma_{q^2}(t)/\sigma_{q^2}(0) = |\varepsilon|^2.$$  

(20)

It is also expressed through the classical trajectory $\varepsilon(t)$. In the case where $|\varepsilon|^2$ is smaller than unity, which means that the dispersion of position at the some moment of time $t$ is less than that at the initial moment of time, the squeezing phenomenon appears. Due to this, the states (4) have the name of squeezed correlated states, as well as in the case without damping. So one can see that all physical characteristics of the system are expressed through the solution of the classical equation of motion $\varepsilon(t)$.

The only remaining problem is to find an explicit expression for the function $\varepsilon$. We devote the following sections to finding the explicit expressions for classical trajectories for different regimes of damping.

3 Damped Oscillator with Parametric Excitation in the Form of a $\delta$-Kick of Frequency

We consider a quantum damped oscillator with the time-dependent frequency, which is varied in a special manner of $\delta$-kick

$$\omega^2(t) = \omega_0^2 - 2\kappa \delta(t),$$

where $\omega_0$ is the constant part of frequency, and $\delta$ is the Dirac delta-function. We have the following equation for the function $\epsilon(t)$:

$$\ddot{\epsilon}(t) + 2\gamma \dot{\epsilon}(t) + \omega_0^2 \epsilon(t) - 2\kappa \delta(t) = 0.$$  

(21)
3.1 The Case of Weak Damping

In this section, we consider the case of weak damping when $\omega_0 > \gamma$. Before and after the kick of frequency, the solution for (21) is given by

$$\epsilon_k(t) = A_k e^{-\gamma t + i\Omega t} + B_k e^{-\gamma t - i\Omega t}, \quad k = 0, 1. \tag{22}$$

where $\Omega = (\omega_0^2 - \gamma^2)^{1/2}$. Due to the continuity conditions,

$$\epsilon_0(0) = \epsilon_1(0), \quad \dot{\epsilon}_1(0) - \dot{\epsilon}_0(0) = 2\kappa \epsilon_0(0). \tag{23}$$

The coefficients $A_k$ and $B_k$ must satisfy the relations, which in the matrix form read

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1 - i\kappa/\Omega & -i\kappa/\Omega \\ i\kappa/\Omega & 1 + i\kappa/\Omega \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{24}$$

If $\epsilon(-0) = 1$ and $\dot{\epsilon}(-0) = i\Omega$ at the initial time instant, then $A_0 = 1 - i\gamma/2\Omega$ and $B_0 = i\gamma/2\Omega$, and the classical trajectory after the kick can be expressed as

$$\epsilon_1(t) = \left(1 - \frac{i(\kappa + \gamma/2)}{\Omega}\right) \exp(-\gamma t + i\Omega t) - \frac{i(\kappa + \gamma/2)}{\Omega} \exp(-\gamma t - i\Omega t). \tag{25}$$

If before the first $\delta$-kick of the frequency the quantum oscillator was in the coherent state determined by formula (17) with

$$\epsilon(t) = e^{-\gamma t} \left[e^{i\Omega t} + (\gamma/\Omega) \sin \Omega t\right], \tag{26}$$

which can be considered as the coherent state of a damped oscillator due to the analogy with undamped case, then the parametric excitation will transform it into a squeezed correlated state determined by the tomogram (17) with the function $\epsilon(t)$ given by (25).

It is not difficult to calculate the quantum dispersion of position in excited correlated squeezed state; it reads

$$\sigma_{q^2}(t) = \frac{e^{-2\gamma t}}{2\Omega} \left[1 + \frac{\sin^2 \Omega t}{\Omega^2}(2\kappa + \gamma)^2 + (2\kappa + \gamma) \frac{\sin 2\Omega t}{\Omega}\right]. \tag{27}$$

From the above expressions, we see that the maximum and minimum of $\sigma_{q^2}(t)$ and of squeezing coefficient $k^2(t) = \sigma_{q^2}(t)/\sigma_{q^2}(0)$ depend on the ratios of force of delta-kick and
damping constant to the frequency of oscillations, while the lower limit of the squeezing coefficient is

\[ k^2 = \left[ 1 + \frac{2(\kappa + \gamma/2)^2}{\Omega^2} - \frac{2(\kappa + \gamma/2)}{\Omega^2} \sqrt{(\kappa + \gamma/2)^2 + \Omega^2} \right] \times \exp \left[ \frac{\gamma}{\Omega} \cos^{-1} \left( \frac{\Omega}{\sqrt{(\kappa + \gamma/2)^2 + \Omega^2}} \right) - \frac{\pi \gamma}{\Omega} (2n - 1) \right], \quad n = 0, 1, \ldots \] (28)

From the above formulas, one can see that the squeezing phenomenon can be achieved for all values of the damping coefficient. So choosing more strong kicks of frequency (increasing the force of the delta-kick), we can squeeze quantum noise in the oscillator position even in the case of large (but less than \( \omega_0 \)) damping coefficient \( \gamma \).

### 3.2 Strong Damping

Now we consider a quantum damped oscillator in the regime of strong damping, where \( \gamma > \omega_0 \). In this case, the solution to Eq. (21) before and after the kick of frequency is \( \epsilon_k = A_k e^{(\Omega - \gamma)t} + B_k e^{-(\gamma + \Omega)t} \), with frequency \( \Omega = (\gamma^2 - \omega_0^2)^{1/2} \). Using the same procedure as in the previous section, one can obtain that after the \( \delta \)-kick of frequency the coefficients \( A_1, B_1 \) are connected with the initial ones in the same way through the matrix equation,

\[
\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1 + \kappa/\Omega & \kappa/\Omega \\ -\kappa/\Omega & 1 - \kappa/\Omega \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}.
\] (29)

Taking the initial conditions in the form \( \epsilon(0) = 1 \) and \( \dot{\epsilon}(0) = i\Omega \), one arrives at \( A_0 = \frac{1}{2}(1 + i + \gamma/\Omega) \) and \( B_0 = \frac{1}{2}(1 - i - \gamma/\Omega) \), and obtain for the classical trajectory \( \epsilon(t) \) after the \( \delta \)-kick of frequency the following expression:

\[
\epsilon(t) = e^{-\gamma t} \left[ \cosh \Omega t + \sinh \Omega t \left( i + \frac{\gamma}{\Omega} + \frac{2\kappa}{\Omega} \right) \right].
\] (30)

If before the first \( \delta \)-kick of frequency the quantum oscillator was in the coherent state determined by (17) with function \( \epsilon \) determined by (26), then after the \( \delta \)-kick of frequency
it occurs in squeezed correlated state determined by tomogram (17) with function \( \epsilon(t) \) given by (30), and the dispersion of position takes the form

\[
\sigma_q^2(t) = \frac{\hbar e^{-2\gamma t}}{2\Omega} \left[ \cosh 2\Omega t + \left( \frac{2\kappa + \gamma}{\Omega} \right)^2 \cosh 2\Omega t - 1 - \frac{2\kappa + \gamma}{2} \sqrt{\cosh^2 2\Omega t - 1} \right].
\]  

(31)

Due to the property of \( \cosh \alpha \) not being less than unity, one can see that the dispersion (31) can never be less than \( \frac{\hbar e^{-2\gamma t}}{2\Omega} \), so the squeezing phenomenon cannot be achieved in the system under study by a \( \delta \)-kick of frequency in the regime of strong damping.

### 3.3 Parametric Excitation of Free particle Motion

In this section, we consider the case where the constant part of frequency is equal to zero but the parametric excitation acts on the free particle motion. The linear integrals of motion and Gaussian wave packets for such systems without parametric excitation were considered in [27]. In the case \( \omega_0 = 0 \), the equation for classical trajectory reads

\[
\ddot{\epsilon}(t) + 2\gamma \dot{\epsilon}(t) - 2\kappa \delta(t) = 0.
\]  

(32)

Before and after the delta-kick of frequency, the solution for this equation is given by the expression \( \epsilon_k = A_k + B_k e^{-2\gamma t} \). Applying the procedure used before and the continuity conditions, one can obtain the relation between the coefficients after \( \delta \)-kick and the initial ones in the form

\[
\begin{pmatrix}
A_1 \\
B_1
\end{pmatrix} = \begin{pmatrix}
1 + \kappa/\gamma & \kappa/\gamma \\
-\kappa/\gamma & 1 - \kappa/\gamma
\end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}.
\]  

(33)

Taking into account the coefficients \( A_0 = 1 + i/2, B_0 = -i/2 \), which coincide with the initial conditions considered above, the expression for classical trajectory after the kick can be obtained as follows:

\[
\epsilon(t) = 1 + \frac{\kappa}{\gamma} \left( 1 - e^{-2\gamma t} \right) + \frac{i}{2} \left( 1 - e^{-2\gamma t} \right).
\]  

(34)
The excited states are determined by tomogram (17), where the function $\varepsilon$ is given by (34), and the squeezing coefficient is equal to

$$k^2 = 1 + \left(\frac{\kappa^2}{\gamma^2}\right)(1 - e^{-2\gamma t})^2 + 2 \frac{\kappa}{\gamma(1 - e^{-2\gamma t})}. \quad (35)$$

From expression (35), we can see that the squeezing coefficient $k^2$ can never be smaller than unity because the function $e^{-2\gamma t}$ is always greater than unity since we investigate the problem in positive moments of time ($t > 0$), and the damping coefficient is a positive number ($\gamma > 0$). The squeezing phenomenon cannot be achieved for free damped particle by a one-kick of frequency. In the case of zero damping ($\gamma = 0$) and in the limit of free particle ($\omega_0 = 0$), one $\delta$-kick of frequency does not produce any squeezing [41].

4 Conclusions

In the probability representation of quantum mechanics and within the framework of the Caldirola–Kanai model, we considered the parametric excitation of a damped oscillator and obtain the tomograms of squeezed correlated states in an explicit form; formulas (17) and (18). We discussed the influence of different regimes of damping on the possibility of appearance of squeezing phenomenon in the system under study. We mention the possibility of appearance of squeezing phenomenon due to the parametric excitation chosen in a special form of a $\delta$-kick of frequency in the case of weak damping (for all $\gamma$ less than $\omega_0$) by choosing different forces of the kick. In the regime of strong damping and for free particle motion, it is impossible to reach the squeezing phenomenon by a $\delta$-kick of frequency.

It is worth noting that the results of papers [20, 23, 24, 38] performed under the supervision of Professor Janszky provided important contribution in the theory of the phenomena under discussion.

We were happy to have the opportunity to visit Hungary in connection with the 5th International Conference on Squeezed States and Uncertainty Relations (Lake Balaton,
Hungary, May 27–31, 1997) and the Wigner Centennial Conference (Pecs, Hungary, July 8–12, 2002) organized by Professor Janszky, and we dedicate this article to his memory.

References

[1] E. Schrödinger, *Ann. Phys* (Liepzig), **79**, 489 (1926).

[2] L. D. Landau, *Z. Phys.*, **45**, 430 (1927).

[3] J. von Neumann, *Nach. Ges. Wiss. Göttingen*, **11**, 245 (1927).

[4] V V. Dodonov, and V. I. Man’ko, *Invariants and the Evolution of Nonstationary Quantum Systems, Proceedings of the Lebedev Physical Institute*, Nauka, Moscow (1987), Vol. 183; [English translation: Nova Science Publishers, New York (1989)].

[5] S. Mancini, V. I. Man’ko, and P. Tombesi, *Phys. Lett. A*, **213**, 1 (1996).

[6] S. Mancini, V. I. Man’ko and P. Tombesi, *Found. Phys.*, **27**, 801 (1997).

[7] O. V. Man’ko, and V. I. Man’ko, *J. Russ. Laser Res.*, **18**, 407(1997).

[8] J. Radon, *Berichte Sächsischen Akad. Wissensch., Leipzig*, **29**, 262(1917).

[9] E. Wigner, *Phys. Rev.*, **40**, 749 (1932).

[10] J. Bertrand, and P. Bertrand, *Found. Phys.*, **17**, 397 (1989).

[11] K. Vogel and H. Risken, *Phys. Rev. A*, **40**, 2847 (1989).

[12] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, *Phys. Rev. Lett.*, **70**, 1244 (1993).

[13] S. Schiller, G. Breitenbach, S. F. Pereira, et al., *Phys. Rev. Lett.*, **77**, 2933 (1996).

[14] W. Paul, *Rev. Mod. Phys.*, **62**, 531 (1990).
[15] L. S. Brown, *Phys. Rev. Lett.*, **66**, 527 (1991).

[16] I. A. Malkin, V. I. Man’ko, and D. A. Trifonov, *J. Math. Phys.*, **14**, 576 (1973).

[17] V. V. Dodonov, I. A. Malkin, and V. I. Man’ko, *Int. J. Theor. Phys.*, **14**, 37 (1975).

[18] V. V. Dodonov, O. V. Man’ko, and V. I. Man’ko, *J. Sov. Laser Res.*, **13**, 196 (1992).

[19] C. F. Lo, *J. Phys. A: Math. Gen.*, **23**, Ii 55 (1990).

[20] J. Janszky and Y. Y. Yushin, *Opt. Commun.*, **59**, 151 (1986).

[21] M. S. AbdalIa and R. I. C. Colegrave, *Phys. Rev. A*, **48**, 1526 (1993).

[22] B. Baseia, R. Vyas, and V. S. Bagnato, *Quantum Opt.*, **5**, 155 (1993).

[23] J. Janszky and P. Adam, *Phys. Rev. A*, **46**, 6091 (1992).

[24] T. Kiss, P. Adam, and J. Janszky, *Phys. Lett. A*, **192**, 311 (1994).

[25] P. Caldirola, *Nuovo Cimento*, **18**, 393 (1941).

[26] E. Kanai, *Prog. Theor. Phys.*, **3**, 440 (1948).

[27] R. W. Hasse, *J. Math. Phys.*, **16**, 2005 (1975).

[28] P. Caldirola and L. Lugiato, *Physica A*, **116**, 248 (1982).

[29] Akpan N. Ikot, *Arab. J. Sci. Eng.*, **37**, 217 (2012).

[30] H. Moya-Cesa, *Rev. Mex. Fis.*, **53**, 42(2007).

[31] Sirin A. Büyükas, *J. Math. Phys.*, **59**, 082104 (2018).

[32] S. P. Kim, *J. Phys. A: Math. Gen.*, **36**, N48 (2003).

[33] J. R. Choi, *IJOABJ*, **5**, 1 (2014).

[34] V. V. Dodonov, O. V. Man’ko, and V. I. Man’ko, *Phys. Lett. A*, **175**, 1 (1993).
[35] G. S. Agarwal and S. A. Kumar, Phys. Rev. Lett, 67, 3665 (1995).

[36] Y. S. Kim and V. I. Man’ko, Phys. Lett. A, 157, 226 (1991).

[37] O. V. Man’ko and Leehwa Yeh, Phys. Lett. A, 189, 268 (1994).

[38] T. Kiss, J. Janszky, and P. Adam, Phys. Rev. A, 49, 4935 (1994).

[39] O. V. Man’ko, in: D. Han, K. Peng, Y. S. Kim, and V. I. Man’ko, (Eds.), Proceedings of the IV International Conference on Squeezed States and Uncertainty Relation, China, 1995, NASA Conference Publication, MD (1995), Vol. 3322, p. 235.

[40] O. V. Man’ko, V. I. Man’ko, and G. Marmo, J. Phys. A: Math. Gen., 35, 699 (2002).

[41] D. J. C. Fernandez and B. Mielnik, J. Math. Phys., 35, 2083 (1994).