Measurement-induced entanglement of two superconducting qubits

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Abstract. We study the problem of two superconducting quantum qubits coupled via a resonator. If only one quanta is present in the system and the number of photons in the resonator is measured with a null result, the qubits end up in an entangled Bell state. Here we look at one source of errors in this quantum nondemolition scheme due to the presence of more than one quanta in the resonator, previous to the measurement. By analyzing the structure of the conditional Hamiltonian with arbitrary number of quanta, we show that the scheme is remarkably robust against these type of errors.

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Superconducting qubits have emerged in the last decade as a reliable benchtop onto which quantum-information experiments can be implemented. Architectures that demonstrate entanglement at the level of two qubits have been proposed and demonstrated experimentally (for extensive references, see [1]). Here we concentrate on a scheme that has been proposed in [1], namely entanglement of the two qubits via nondestructive measurements of the number of quanta in third quantum system coupled to the two qubits.

This third quantum system can be for example a third qubit, a Josephson junction, an electromagnetic resonator, or a nanomechanical resonator. Schemes such as these have been proposed initially in quantum optics [2], and, together with interaction-free experiments [3] and partial-collapse measurements [4], they have no classical analogue as they explore the consequences of the projective measurements in quantum mechanics. If the number of excitations present in the system is 1, there is no principal difference between a two-level system and a resonator as the intermediating system. However, in the case of a resonator as a connecting system, it can well be that the number of excitations present in the system, $n$, is larger than 1. We analyze here precisely this situation.

The Hamiltonian of two qubits (biased at the optimal points) coupled via a resonator is

$$H = -\sum_{j=1,2} \frac{E_{J_j}}{2} \sigma_j^z + \omega_r a^+ a + i \sum_{j=1,2} g_j (a^+ \sigma_j^- - a \sigma_j^+)$$

In the following, we take the two qubits identical and on-resonance with the coplanar waveguide resonator, $E_{J_1} = E_{J_2} = \omega_r$, and the same values for the qubit-resonator coupling constants $g_1 = g_2 = g$.

We now consider the situation in which the resonator is out-coupled to a detector. As a detector, one can use a single large Josephson junction and perform switching-current


Table 1. Eigenvectors and eigenvalues of $-iH_c^{(n)}$ for $\kappa = \Gamma = 0$.

| eigenvalue | eigenvector |
|------------|-------------|
| 0          | $\sqrt{\frac{n}{2n-1}}(\sqrt{\frac{n-1}{n}}, 0, 0, 1)^T$ |
| 0          | $(0, -1, 1, 0)^T$ |
| $-ig\sqrt{2(2n-1)}$ | $\sqrt{\frac{n-1}{2(2n-1)}}(-\sqrt{\frac{n-1}{n-1}}, -i\sqrt{\frac{2n-1}{2(n-1)}}, -i\sqrt{\frac{2n-1}{2(n-1)}})^T$ |
| $ig\sqrt{2(2n-1)}$ | $\sqrt{\frac{n-1}{2(2n-1)}}(-\sqrt{\frac{n-1}{n-1}}, i\sqrt{\frac{2n-1}{2(n-1)}}, i\sqrt{\frac{2n-1}{2(n-1)}})^T$ |

experiments [5]. Under the condition that no event is detected by the junction, and including the dissipative terms $\Gamma_1 = \Gamma_2 = \Gamma$ for the qubits and the resonator, the conditional non-Hermitian Hamiltonian [6], in the interaction picture, reads:

$$H_c = \omega_r \left(a_1^\dagger a - \frac{\sigma_1^+}{2} - \frac{\sigma_2^+}{2}\right) + ig \left(a_1^\dagger \sigma_1^- + a_1 \sigma_2^- - a_1^\dagger \sigma_1^+ - a_1 \sigma_2^+\right) - ika^\dagger a - i\Gamma \left(\sigma_1^+ \sigma_1^- + \sigma_2^+ \sigma_2^-\right)$$

We notice that the conditional Hilbert space of the system splits naturally into 4-dimensional subspaces with $n$ excitations distributed between the two qubits and the resonator. The Hamiltonian $H_c$ projected onto the subspace spanned by the basis states $|\tilde{1}, n\rangle \equiv |n\rangle_p \otimes |\uparrow\rangle$, $|\tilde{2}, n\rangle \equiv |n-1\rangle_p \otimes |\downarrow\rangle$, $|3, n\rangle \equiv |n-1\rangle_p \otimes |\downarrow\rangle$, and $|4, n\rangle \equiv |n-2\rangle_p \otimes |\downarrow\rangle$ takes the matrix form

$$-iH_c^{(n)} = \begin{bmatrix}
nk & -g\sqrt{n} & -g\sqrt{n} & 0 \\
g\sqrt{n} & \Gamma + (n-1)k & 0 & -g\sqrt{n-1} \\
g\sqrt{n} & 0 & \Gamma + (n-1)k & -g\sqrt{n-1} \\
0 & g\sqrt{n-1} & g\sqrt{n-1} & 2\Gamma + (n-2)k
\end{bmatrix}.$$  

This non-Hermitian matrix, $-iH_c^{(n)}$, for $k = \Gamma = 0$, has two eigenvectors with eigenvalue 0, and two other with nonzero eigenvalue (see Table 1). We evolve now an initial state with $n$ excitations $|\psi(0)\rangle$ by the conditional Hamiltonian $H_c^{(n)}$ and find the state at any time $t$ under the condition that the resonator did not decay, $\exp(-iH_c^{(n)}t)|\psi(0)\rangle$. The results are shown in Fig. (1).

From Fig. (1) we see that indeed for $n = 1$ the probabilities corresponding to the states $|\tilde{2}, n = 1\rangle$ and $|3, n = 1\rangle$ are equal. It is in fact known [1, 2], that, if the system starts in the states $|\tilde{3}, n = 1\rangle$ or $|\tilde{2}, n = 1\rangle$ in the asymptotic state the qubits will be projected to a maximally entangled Bell state, $|\tilde{3}, n = 1\rangle = |\tilde{2}, n = 1\rangle_+ = (1/\sqrt{2})|0\rangle_p \otimes (|\uparrow\rangle - |\downarrow\rangle)$ (we have verified the also the minus sign numerically). This corresponds to the first eigenvalue of the nondissipative Hamiltonian from Table 1.

In the case of $n \neq 1$ photons a completely different state results, namely the asymptotic state corresponds to the first eigenvalue in Table 1. This statement is valid no matter which of the four states (or combinations of them) $|\tilde{1}, n\rangle$, $|\tilde{2}, n\rangle$, $|3, n\rangle$, $|4, n\rangle$ is taken as the initial one, although of course the intermediate-time evolution is different.

From Fig. (1) we see that the ratios of the two surviving probabilities are indeed $P_{|\tilde{1}, n\rangle}/P_{|\tilde{4}, n\rangle} = (n-1)/n$; we have also checked numerically the signs. The asymptotic state is then

$$|\psi_{asym}\rangle = \sqrt{\frac{n}{2n-1}} \left(\sqrt{\frac{2n-1}{n}} |\tilde{1}, n\rangle + |\tilde{4}, n\rangle \right)$$

(3)
Figure 1. The conditional occupation probabilities for the four states $|\tilde{1}, n\rangle$, $|\tilde{2}, n\rangle$, $|\tilde{3}, n\rangle$, and $|\tilde{1}, n\rangle$, for $n = 1, 2, 3, 4$ and initial state $|\tilde{2}, n\rangle$.

Interestingly, in the limit of large $n$ in which we can take $|n\rangle_p \approx |n - 2\rangle_p$ and factor out the resonator, this state results in another two-qubit Bell state, $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$. It is intuitively clear that due to the presence of extra excitations in the system the probability of decay will increase and there will be no asymptotic stationary state as in the case $n = 1$. To understand this quantitatively, we calculate the probability to have no photon emission until time $t$. This is given by the norm of the wavefunction evolved with the non-Hermitian conditional Hamiltonian with $n$ excitations,

$$P_0^{(n)}(t) = ||e^{-iH_c^{(n)}t}\psi(0)||^2.$$  \hfill (4)

This probability is plotted in Fig. 2 for $n = 1, 2, 3, 4$. To make the interpretation straightforward, we took $\Gamma_1 = \Gamma_2 = \Gamma = 0$, in other words the only possibility of decay is through the resonator. We see that for $n = 1$ and an initial state $|\tilde{2}, n = 1\rangle$ (or equivalently $|\tilde{3}, n = 1\rangle$) the probability $P_0^{(n=1)}$ goes asymptotically to 0.5, as it should in the case $k \gg \Gamma$, as was found in $[1, 2]$, corresponding to an eigenvalue of the form $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$. For the other possible initial state with $n = 1$, namely $|\tilde{1}, n = 1\rangle$, the probability decays fast, therefore the errors introduced by these states will be asymptotically suppressed.

For $n \neq 1$ the probability of no decay goes very fast to zero: this means that the efficiency of the scheme is much smaller than in the case $n = 1$, in the sense that there will be much less "favorable" events in which no decay of the resonator is detected. To make things more
formal, suppose we start with an initial mixture of states $|\tilde{2}, n = 1 \rangle$ (desirable, probability $P^{(0)}$) and thermal excitations or other kind of errors and fluctuations giving $n \neq 1$ excitations in the system with probabilities $P^{(n)}_{ex}$. The explicit form of this initial state does not matter, because the asymptotic result is the same: the asymptotic state will be a mixture of the Bell state we want to obtain, $(|\uparrow \downarrow \rangle + |\downarrow \uparrow \rangle)/\sqrt{2}$, which comes with probability $1/2$ and states corresponding to $|\psi_{asym} \rangle$, which come with (the much smaller) probabilities $P^{(0)}_{0}$,

$$
\rho_{asym} = \frac{0.25 P^{(0)}}{0.5 P^{(0)} + \sum_m P^{(m)}_{0} P^{(m)}_{ex} (|\downarrow \uparrow \rangle + |\downarrow \downarrow \rangle)(\langle \downarrow \downarrow | + \langle \downarrow \uparrow | \otimes |0\rangle_{pp} \langle 0|) \\
+ \sum_n \frac{P^{(n)}_{0} P^{(n)}_{ex}}{0.5 P^{(0)} + \sum_m P^{(m)}_{0} P^{(m)}_{ex}} |\psi_{asym}\rangle \langle \psi_{asym}|).
$$

In conclusion, irrespective to the initial weights $P^{(0)}$, $P^{(n)}_{ex}$, this density matrix approaches exponentially fast the desirable Bell state, due to the fast decay of the probabilities $P^{(n)}_{0}$. Of course, the initial state and $P^{(0)}$, $P^{(n)}_{ex}$ do have a role to play in the efficiency of this process (how many favorable non-counting events we get), but not in the structure of the final state under the condition of no de-excitation events detected.

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