AGRARIAN AND $L^2$-INVARIANTS

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Abstract. We develop the theory of agrarian invariants, which are algebraic counterparts to $L^2$-invariants. Specifically, we introduce the notions of agrarian Betti numbers, agrarian acyclicity, agrarian torsion and agrarian polytope.

We use the agrarian invariants to solve the torsion-free case of a conjecture of Friedl–Tillmann: we show that the marked polytopes they constructed for two-generator one-relator groups with nice presentations are independent of the presentations used. We also show that, for such groups, the agrarian polytope encodes the splitting complexity of the group. This generalises theorems of Friedl–Tillmann and Friedl–Lück–Tillmann. Finally, we prove that for agrarian groups of deficiency 1, the agrarian polytope admits a marking of its vertices which controls the Bieri–Neumann–Strebel invariant of the group, improving a result of the second author and partially answering a question of Friedl–Tillmann.

1. Introduction

In 2017, Friedl–Lück [FL17] broadened the arsenal of $L^2$-invariants: to the more classical $L^2$-Betti numbers and $L^2$-torsion, they added the universal $L^2$-torsion and the $L^2$-torsion polytope. The definitions involve some operator algebra technology, and a little $K$-theory. Practically however, it transpired, e.g., in [FK18, Kie18], that these new $L^2$-invariants can be computed using linear algebra over skew fields. The current article redevelops much of the $L^2$-theory from this (more algebraic) point of view. The resulting theory of agrarian invariants is at the same time conceptually simpler and applicable to a greater number of groups.

The invariants. Recall that a group $G$ is agrarian if its integral group ring $\mathbb{Z}G$ embeds in a skew field. This terminology was introduced in [Kie18], but the idea dates back to Malcev [Mal48], and is a central theme of the work of Cohn [Coh95].

Taking a specific agrarian embedding $\mathbb{Z}G \hookrightarrow D$ for some skew field $D$ allows us to define the notion of $D$-agrarian Betti numbers: when $G$ acts cellulary on a CW-complex $X$, we simply compute the $D$-dimension of the homology of $D \otimes_{\mathbb{Z}G} C_\ast$, where $C_\ast$ is the cellular chain complex of $X$. When $D$ is the skew field $D(G)$ introduced by Linnell in [Lin93] (assuming that $G$ is torsion-free and satisfies the Atiyah conjecture), the agrarian Betti numbers are precisely the $L^2$-Betti numbers.

When the agrarian Betti numbers vanish and $G$ acts on $X$ cocompactly, we define the agrarian torsion, in essentially the same way as Whitehead or Reidemeister torsion is defined. Again, when $D(G)$ is Linnell’s skew field, we obtain an invariant very closely related to the universal $L^2$-torsion. In fact, in this case agrarian and universal $L^2$-torsion often contain the same amount of information by a theorem of Linnell–Lück [LL18].

The vanishing of $L^2$-Betti numbers is guaranteed when $X$ fibres over the circle due to a celebrated theorem of Lück; the agrarian Betti numbers also vanish in this setting, as we show in Theorem 3.10.
The final invariant, the agrarian polytope, is a little more involved. In the context of \( L^2 \)-invariants, one can write the universal \( L^2 \)-torsion as a fraction of two elements of a (twisted) group ring of the free part of the abelianisation of \( G \). Both the numerator and the denominator can be converted into polytopes, using the Newton polytope construction, and the \( L^2 \)-torsion polytope is defined as the formal difference of these Newton polytopes. The \( L^2 \)-torsion polytope naturally lives in the polytope group of \( G \), defined in [FL17] and investigated further by Funke [Fun16].

In the agrarian setting, we need to replace the given skew field \( D \) by what we call \( D_K \) – another skew field containing \( ZG \), with the additional property that elements thereof can be expressed as fractions in the same way as above. The agrarian polytope is then constructed analogously to the \( L^2 \)-torsion polytope.

The main advantage of agrarian invariants over \( L^2 \)-invariants lies in the fact that they are defined for a group \( G \) as long as \( ZG \) embeds into any skew field – not necessarily the one known to exist if \( G \) were to satisfy the Atiyah conjecture. Agrarian groups form an a priori larger class than torsion-free groups satisfying the Atiyah conjecture. Moreover, there are explicit classes of groups of topological interest, such as surface-by-surface groups, which are known to be agrarian, but not known to satisfy the Atiyah conjecture (see [Kie18, Section 4] for a more substantial discussion).

**Applications.** We introduce the theory of agrarian invariants not only for its general appeal, but also with specific applications in mind.

The definition of the \( L^2 \)-torsion polytope generalised the polytope \( P_\pi \) introduced by Friedl–Tillmann [FT15] for two-generator one-relator groups with nice (see Definition 7.8) presentations. In fact, one can trace the definition of the Friedl–Tillmann polytope to Brown’s algorithm [Bro87, Theorem 4.3]. The polytope \( P_\pi \) is endowed with a marking of its vertices, and the resulting marked polytope \( M_\pi \) determines the Bieri–Neumann–Strebel (BNS) invariants of the group in question.

The Friedl–Tillmann polytope is constructed using the group presentation, and it was not clear whether it is actually a group invariant – in fact, this problem appears as [FT15, Conjecture 1.2]. The question was partially answered by Friedl–Tillmann themselves [FT15, Theorem 1.3] for residually \{torsion-free elementary amenable\} groups, and by Friedl–Lück [FL17, Remark 5.5] for torsion-free groups satisfying the Atiyah conjecture. We offer a complete resolution of the torsion-free case of this conjecture.

**Theorem 7.13.** If \( G \) is a torsion-free group admitting a nice \((2,1)\)-presentation \( \pi \), then \( M_\pi \subset H_1(G; \mathbb{R}) \) is an invariant of \( G \) (up to translation).

We prove the above statement by showing that \( P_\pi \) arises as an agrarian polytope of \( G \). The result implies that for groups \( G \) as above the agrarian polytope is independent of the choice of the agrarian embedding \( ZG \hookrightarrow D \). This way, we also answer [FT15] Question 9.2, which asked whether \( M_\pi \) can be defined intrinsically from \( G \).

Friedl–Tillmann also observed that when \( G \) is residually \{torsion free elementary amenable\}, the polytope \( P_\pi \) encodes information about the splitting complexity of \( G \); given an epimorphism \( \varphi: G \to \mathbb{Z} \), the corresponding splitting complexity \( c(G, \varphi) \) is the minimal rank of a finitely generated subgroups \( B \) such that \( G \) can be written as an HNN extension of a finitely generated group \( A \) (containing \( B \)) over \( B \) in such a way that the canonical map onto \( \mathbb{Z} \) coming from the HNN structure coincides with \( \varphi \). An analogous result was proven by Friedl–Lück–Tillmann [FLT16, Theorem 5.2] when \( G \) is torsion free and satisfies the Atiyah conjecture. We prove the following, more general result:
Theorem 7.19. Let $G$ be a torsion-free group admitting a nice $(2,1)$-presentation $\pi$. Then for any epimorphism $\varphi : G \to \mathbb{Z}$ we have
\[ c(G, \varphi) = c_f(G, \varphi) = \text{th}_\varphi(\mathcal{P}_\pi) + 1. \]

Here, $\text{th}_\varphi(\mathcal{P}_\pi)$ denotes the thickness of the agrarian polytope $\mathcal{P}_\pi$ measured with respect to $\varphi$ (see Definition 5.12 for details), and $c_f(G, \varphi)$ denotes the free splitting complexity, where the subgroup $B$ is additionally required to be free.

Lastly, we turn our attention to agrarian groups of deficiency 1. These groups were already investigated in Kie18 Section 5.6], where it was shown that for such a group $G$ there exists a marked polytope determining the BNS invariant $\Sigma^1$ of $G$. The marking however was of a more general character than is the case for $\mathcal{M}_\pi$, where only vertices and no higher-dimensional faces can be marked. We generalise this result and thus make progress towards answering [FT15, Question 9.4] in the following way:

Theorem 7.23. Let $G$ be a $D$-agrarian group of deficiency 1, and denote by $K$ the kernel of the projection onto the free part of its abelianisation. There exists a marking of the vertices of the agrarian polytope $D^K$ such that for every $\varphi \in H^1(G; \mathbb{R}) \setminus \{0\}$ we have $\varphi \in \Sigma^1(G)$ if and only if $\varphi$ is marked.

We refer to Definition 7.4 for details on how the marking on the polytope yields a marking of classes in the first cohomology.

Let us also mention another connection of interest: in the setting of 3-manifolds, the $L^2$-torsion polytope was shown in FL17 to coincide with the Thurston polytope, that is the dual of the unit ball of the Thurston norm. This point of view allowed Funke and the second author to define the Thurston norm in the setting of free-by-cyclic groups FK18. Since the $L^2$-torsion polytope is a particular example of an agrarian polytope, one can expect that the theory of agrarian invariants will be useful in defining further generalisations of the Thurston norm.

The Thurston polytope is also very closely related to the BNS invariants, via [Thu86, Theorem 5] and [BNS87, Corollary F]. A new proof of this relation, essentially using agrarian methods, was given by the second author in Kie18.

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2. Agrarian groups and Ore fields

In the following, all rings are associative, unital, and not necessarily commutative; ring homomorphisms preserve the unit. Modules are understood to be left modules unless specified otherwise. As a general exception, we consider a ring $R$ to be an $R$-$R$-bimodule.

Definition 2.1. Let $G$ be a group. An agrarian embedding for $G$ is an injective ring homomorphism $i : \mathbb{Z}G \to D$ into a skew field $D$. A morphism of agrarian embeddings is an inclusion of skew fields that commutes with the specified embeddings of $\mathbb{Z}G$. If $G$ admits an agrarian embedding (into $D$), it is called a $(D)$-agrarian group.
An agrarian group $G$ is necessarily torsion free; also, it satisfies the Kaplansky zero divisor conjecture, that is, $ZG$ has no non-trivial zero divisors.

At present, there are no known torsion-free examples of groups which are not agrarian. There is however a plethora of positive examples of agrarian groups: If a torsion-free group satisfies the Atiyah conjecture over $Q$, then its group ring embeds into the skew field $D(G) = D(G; Q)$ of Linnell (by the same argument as used in [Lie02 Theorem 10.39]), and hence the group is agrarian. Similarly, any biorderable group is agrarian, since its integral group ring embeds into the Malcev–Neumann completion, which is a skew field by [Mal48,Neu49] (see also [Coh95, Theorem 2.4.5]).

Agrarian groups also enjoy a number of convenient inheritance properties. For a more substantial discussion, see [Kie18, Section 4] by the second-named author. One example of agrarian groups which does not feature in the discussion in [Kie18] is that of torsion-free one-relator groups. All such groups have been shown to be agrarian by Lewin–Lewin [LL78].

2.1. Twisted group rings. We now recall the construction of twisted group rings, which will later allow us to obtain for any agrarian group an agrarian embedding into a localised twisted polynomial ring.

**Definition 2.2.** Let $R$ be a ring and let $G$ be a group. Let functions $c : G \rightarrow \text{Aut}(R)$ and $\tau : G \times G \rightarrow R^\times$ be such that

$$c(g) \circ c(g') = c_{r(g,g')} \circ c(gg'),$$

$$\tau(g,g')\tau(gg',g'') = c(g)(\tau(g',g''))\tau(g,g''),$$

where $g,g',g'' \in G$, and where $c_r \in \text{Aut}(R)$ for $r \in R^\times$ denotes the conjugation map $x \mapsto rxr^{-1}$. The functions $c$ and $\tau$ are called structure functions. We denote by $RG$ the free $R$-module with basis $G$ and write elements of $RG$ as finite $R$-linear combinations $\sum_{g \in G} \lambda_g \cdot g$ of elements of $G$. The structure functions endow $RG$ with the structure of an (associative) twisted group ring by declaring

$$g \cdot r \cdot 1 = c(g)(r) \ast g \text{ and } g \cdot g' = \tau(g,g') \ast gg',$$

and extending linearly.

The usual, untwisted group ring corresponds to the special case of trivial structure functions. In the following, group rings with $R = \mathbb{Z}$ will always be understood to be untwisted.

The fundamental example of a twisted group ring arises in the following way:

**Example 2.3.** Let $\alpha : G \rightarrow H$ be a group epimorphism with kernel $K$. We choose any set-theoretic section $s : H \rightarrow G$, i.e., a map between the underlying sets such that $\alpha \circ s = \text{id}_H$. We denote by $(\mathbb{Z}K)H$ the twisted group ring defined by the structure functions $c(h)(r) = s(h)r s(h)^{-1}$ and $\tau(h,h') = s(h)s(h')s(hh')^{-1}$. The untwisted group ring $\mathbb{Z}G$ is then isomorphic to the twisted group ring $(\mathbb{Z}K)H$ via the map

$$g \mapsto \left(g \cdot (s \circ \alpha)(g)^{-1}\right) \cdot \alpha(g).$$

Observe that the structure maps of the twisted group ring $(\mathbb{Z}K)H$ we just defined depend on the section $s : H \rightarrow G$. The global ring structure however does not, since for any section the resulting twisted group ring is isomorphic (as a ring) to $\mathbb{Z}G$.

The construction of a twisted group ring out of a group epimorphism can be extended to agrarian embeddings. This technique is formulated in the following technical lemma, which will be our main source of twisted group rings.
Lemma 2.4. Let $G$ be an agrarian group with an agrarian embedding $i: ZG \hookrightarrow D$. Let $N \leq G$ be a normal subgroup and set $Q := G/N$. Then $i$ restricts to an agrarian embedding $ZN \hookrightarrow D$ for $N$ that is equivariant with respect to the conjugation action of $G$. Moreover, for any section $s: Q \rightarrow G$ of the quotient map, $i$ extends to an embedding

$$(ZN)Q \hookrightarrow DQ,$$

where $(ZN)Q$ is as defined in Example 2.3, and $DQ$ is a twisted group ring with the same structure functions as $(ZN)Q$. The twisted group ring structure of $DQ$ is independent of the choice of the section $s$ up to isomorphism, and this isomorphism can be chosen to map $\sum_{q \in Q} u_q q$ to $\sum_{q \in Q} v_q q$ such that for every $q \in Q$, the elements $u_q$ and $v_q$ differ only by an element of $D^\times$.

Proof. By definition, $i$ restricts to an agrarian embedding of $ZN \leq ZG$ into $D$. Since $N$ is normal in $G$, the action of $G$ on $ZG$ by conjugation preserves $ZN$. The embedding $ZN \hookrightarrow D$ is equivariant with respect to the conjugation action of $G$ since it factors through $i$.

Let $s: Q \rightarrow G$ be a set-theoretic section of the group epimorphism $pr: G \rightarrow G/N = Q$, i.e., a map that satisfies $pr \circ s = \text{id}_Q$. Then every $q \in Q$ defines an automorphism $c_s(q)$ of $ZN$ given by conjugation by $s(q)$. We denote by $(ZN)Q$ the twisted group ring associated to the section $s$ as in Example 2.3. Since the automorphisms $c_s(q)$ of $ZN$ extends to an automorphism of $D \supset ZN$, which is again given by conjugation by $c_s(q)$, we can apply the twisted group ring construction also to $D$ and $Q$ such that the embedding $(ZN)Q \hookrightarrow DQ$ extends $ZN \hookrightarrow D$.

Let $s_1$ and $s_2$ be two set-theoretic sections of $pr: G \rightarrow Q$. Denote by $D_{s_1}Q$ and $D_{s_2}Q$ the associated twisted group ring structures on $DQ$. We claim that the map $\alpha: D_{s_1}Q \rightarrow D_{s_2}Q$ given by

$$\sum_{q \in Q} u_q \star q \mapsto \sum_{q \in Q} (u_q s_1(q) s_2(q)^{-1}) \star q$$

is a ring isomorphism. Since $s_1(q) s_2(q)^{-1} \in G \subset D^\times$ for all $q \in Q$, it is clear that $\alpha$ is an isomorphism between the underlying free $D$-modules and changes the coefficients by a unit in $D$ only.

It thus remains to check that $\alpha$ respects the ring multiplications, which is fully determined by the cases $q \cdot u$ and $q \cdot q'$ for $q, q' \in Q$ and $u \in D$. We denote by $\tau_i$ for $i = 1, 2$ the respective structure function of $D_{s_i}Q$ and compute that

$$\alpha(q \cdot q') = \alpha(\tau_1(q, q') \star qq')$$

$$= \alpha(s_1(q) s_1(q') s_2(qq')^{-1} \star qq')$$

$$= s_1(q) s_1(q') s_1(qq')^{-1} s_1(qq') s_2(qq')^{-1} \star qq'$$

$$= s_1(q) s_1(q') s_2(qq')^{-1} \star qq'$$
and
\[
\alpha(q \cdot u \ast 1) = \alpha(s_1(q)u s_1(q)^{-1} \ast q \cdot 1 \ast 1) \\
= \alpha(s_1(q)u s_1(q)^{-1}s_1(q)s_1(1)s_1(q)^{-1} \ast q) \\
= \alpha(s_1(q)us_1(1)s_1(q)^{-1} \ast q) \\
= s_1(q)us_1(1)s_1(q)^{-1}s_1(q)s_2(q)^{-1} \ast q \\
= s_1(q)us_1(1)s_2(q)^{-1} \ast q \\
\alpha(q) \cdot \alpha(u \ast 1) = (s_1(q)s_2(q)^{-1} \ast q) \cdot (us_1(1)s_2(1)^{-1} \ast 1) \\
= (s_1(q)s_2(q)^{-1}s_2(q)us_1(1)s_2(1)^{-1}s_2(q)^{-1} \ast q) \cdot 1 \\
= s_1(q)us_1(1)s_2(1)^{-1}s_2(q)^{-1}s_2(q)s_2(1)s_2(q)^{-1} \ast q \\
= s_1(q)us_1(1)s_2(q)^{-1} \ast q.
\]

Hence \(\alpha\) is an isomorphism of rings. \(\square\)

2.2. Associated Ore embeddings. We now consider the special case where \(N\) is the kernel \(K\) of an epimorphism from \(G\) onto a finitely generated free abelian group \(H\). Lemma 2.4 then provides us with a twisted group ring \(DH\) and an embedding \(\mathbb{Z}G \cong (\mathbb{Z}N)H \hookrightarrow DH\). Since \(H\) is free abelian, it is in particular biorderable and hence \(DH\) contains no non-trivial zero divisors (this is a standard fact following from the existence of an embedding of \(DH\) into its Malcev–Neumann completion; for details see [Kie18 Theorem 4.10]). It then follows from [Kie18 Theorem 2.11], an extension of a result of Tamari [Tam54] to twisted skew field coefficients, that \(DH\) satisfies the Ore condition, and thus its Ore localisation is a skew field.

Recall that a ring \(R\) without zero divisors satisfies the Ore condition if for every \(p, q \in R\) with \(q \neq 0\) there exists \(r, s \in R\) with \(s \neq 0\) such that
\[
ps = qr.
\]

This equality allows for the conversion of a left fraction \(q^{-1}p\) into a right fraction \(rs^{-1}\), which in turn makes it possible to multiply fractions (in the obvious way). The Ore condition also facilitates the existence of common denominators, and thus allows for addition of fractions. Thanks to these properties, the ring \(R\) embeds into its Ore localisation
\[
\text{Ore}(R) := \{q^{-1}p \mid p, q \in R, q \neq 0\},
\]
which is evidently a skew field. For details see the book of Passman [Pas85 Section 4.4].

Our construction is summarised in

**Definition 2.5.** Let \(G\) be an agrarian group with agrarian embedding \(i: \mathbb{Z}G \hookrightarrow D\). Let \(K\) be a normal subgroup of \(G\) such that \(H := G/K\) is finitely generated free abelian. The associated \(K\)-Ore embedding of \(G\) is the agrarian embedding
\[
\mathbb{Z}G \cong (\mathbb{Z}K)H \hookrightarrow DH \hookrightarrow \text{Ore}(D; K \leq G),
\]
where \(\text{Ore}(D; K \leq G)\) denotes the Ore localisation \(\text{Ore}(DH)\) of the twisted group ring \(DH = D(G/K)\).

**Lemma 2.6.** Let \(G\) be a finitely generated agrarian group with agrarian embedding \(i: \mathbb{Z}G \rightarrow D\). Denote by \(pr: G \rightarrow H\) the projection onto the free part \(H\) of the abelianisation of \(G\). Let \(\varphi: G \rightarrow H'\) be an epimorphism onto a finitely generated free abelian group, inducing the following commutative diagram of epimorphisms:

\[
\begin{array}{ccc}
G & \xrightarrow{pr} & H \\
\downarrow \varphi \quad & & \downarrow \psi \\
& H' & \\
\end{array}
\]
Denote the kernels of \( pr, \varphi \) and \( \overline{\varphi} \) by \( K_1, K_\varphi \) and \( K_{\overline{\varphi}} \), respectively. Further let \( s \) and \( t \) be sections of the epimorphisms \( pr \) and \( \overline{\varphi} \), respectively. Then

\[
\beta: (DK_{\overline{\varphi}}) H' \to DH
\]

is an isomorphism between twisted group rings constructed using the sections \( s \), \( t \) and \( s \circ t \). It extends to an isomorphism

\[
\beta: \text{Ore}(D; K \leq K_\varphi; K_\varphi \leq G) \cong \text{Ore}(D; K \leq G)
\]
of skew fields.

**Proof.** Left \( D \)-bases of \((DK_{\overline{\varphi}}) H'\) and \( DH \) are given by \( k \star h' \) and \( k t(h') \) respectively for \( k \in K_{\overline{\varphi}} \) and \( h' \in H' \). These bases are identified bijectively by \( \beta \) with inverse \( h \mapsto h t(\overline{\varphi}(h)^{-1}) \star \overline{\varphi}(h) \). It follows that \( \beta \) is an isomorphism of left \( D \)-modules. Checking that \( \beta \) respects the twisted group ring multiplication is a tedious but direct computation that we will omit.

Since \( DK_{\overline{\varphi}} \) is a subring of \( DH \), \( \beta \) extends to an injection \( \text{Ore}(DK_{\overline{\varphi}}) H' \hookrightarrow \text{Ore}(DH) \) that contains \( DH \) in its image. Ore localising again, this implies that \( \beta \) extends to an isomorphism \( \text{Ore}(\text{Ore}(DK_{\overline{\varphi}}) H') \to \text{Ore}(DH) \), the domain of which is precisely \( \text{Ore}(\text{Ore}(D; K \leq K_\varphi; K_\varphi \leq G)) \).

\( \square \)

### 3. Agrarian Betti numbers

From now on, \( G \) will denote a \( D \)-agrarian group with a fixed agrarian embedding \( \mathbb{Z}G \hookrightarrow D \) and all tensor products will be taken over \( \mathbb{Z}G \).

#### 3.1. Definition of agrarian Betti numbers.

Let \( C_* \) be a \( \mathbb{Z}G \)-chain complex and let \( n \in \mathbb{Z} \). Viewing \( D \) as a \( \mathbb{Z}G \)-bimodule via the agrarian embedding, \( \mathbb{Z}G \otimes C_* \) becomes a \( D \)-chain complex. Since \( D \) is a skew field, the \( D \)-module \( H_n(D \otimes C_*) \) is free and we can consider its dimension \( \dim_D H_n(D \otimes C_*) \in \mathbb{N} \cup \{ \infty \} \). This leads to

**Definition 3.1.** Let \( G \) be an agrarian group with agrarian embedding \( i: \mathbb{Z}G \hookrightarrow D \) and \( C_* \) a \( \mathbb{Z}G \)-chain complex. For \( n \in \mathbb{Z} \), the \( n \)-th \( D \)-Betti number of \( C_* \) with respect to \( i \) is defined as

\[
b^D_n(C_*):=\dim_D H_n(D \otimes C_*) \in \mathbb{N} \cup \{ \infty \},
\]

A \( \mathbb{Z}G \)-chain complex is called \( D \)-acyclic if all of its \( D \)-Betti numbers are equal to 0.

In the special case where \( G \) satisfies the Atiyah conjecture and the agrarian embedding is chosen to be \( \mathbb{Z}G \hookrightarrow D(G) \), the \( D \)-Betti numbers of any \( \mathbb{Z}G \)-chain complex \( C_* \) agree with its \( L^2 \)-Betti numbers \( b^{(2)}_n(C_*; G) \) by [FL16, Theorem 3.6 (2)]. Note that the assumption that \( C_* \) is projective is not used in the proof, the theorem thus holds for arbitrary \( \mathbb{Z}G \)-chain complexes.

We will mainly be concerned with the agrarian Betti numbers assigned to \( G \)-CW-complexes, which are equivariant analogues of CW-complexes and very convenient models for \( G \)-spaces. A typical example of a \( G \)-CW-complex is the universal covering of a connected CW-complex \( X \) with \( G = \pi_1(X) \).

**Definition 3.2.** Let \( G \) be a (discrete) group. A \( G \)-CW-complex is a CW-complex \( X \) together with an implicit (left) \( G \)-action mapping \( p \)-cells to \( p \)-cells and such that any cell mapped into itself is fixed pointwise by the action. An action satisfying these properties is called cellular. The \( p \)-skeleton of a \( G \)-CW-complex \( X \), denoted by
Remark 3.6 A $G$-CW-complex $X$ is called free if its $G$-action is free. It is called of finite type if for every $p \geq 0$ there are only finitely many $p$-dimensional $G$-cells in $X$. If the total number of $G$-cells of any dimension in $X$ is finite, the $G$-CW-complex $X$ is called finite.

Definition 3.3. A $G$-CW-complex $X$ is called free if its $G$-action is free. It is called of finite type if for every $p \geq 0$ there are only finitely many $p$-dimensional $G$-cells in $X$. If the total number of $G$-cells of any dimension in $X$ is finite, the $G$-CW-complex $X$ is called finite.

Definition 3.4. A $\mathbb{Z}G$-chain complex $C_*$ is called free (of finite type) if $C_n$ is a free (finitely generated) $\mathbb{Z}G$-module for every $n \in \mathbb{Z}$. It is called bounded if there is $N \in \mathbb{N}$ such that $C_i = 0$ for $i > N$ and $i < -N$.

A $G$-CW-complex is free (of finite type) if and only if its associated cellular chain complex $C_*(X)$ is free (of finite type). If it is finite, then its cellular chain complex is bounded and of finite type.

We now define agrarian Betti numbers for some $G$-CW-complexes, in particular for those associated to a cellular chain complex:

Definition 3.5. Let $X$ a $G$-CW-complex. For $p \geq 0$, the $p$-th $D$-Betti number of $X$ with respect to $i$ is defined as

$$b^D_p(X) := b^D_p(C_*(X)) \in \mathbb{N} \cup \{\infty\}.$$ 

A $G$-CW-complex $X$ is called $D$-acyclic if all of its $D$-Betti numbers are equal to 0.

If $X$ is a $G$-CW-complex of finite type, the $D$-Betti numbers of $X$ will all be non-negative integers.

3.2. Dependence on the agrarian embedding. Note that the agrarian Betti numbers may depend not only on the skew field $D$ but also on the particular choice of agrarian embedding. In the following, we will always consider a fixed such embedding and thus suppress $i$ from the notation.

Remark 3.6. Recall that a ring homomorphism $f: R \to S$ is called epic if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for any ring homomorphisms $g_1, g_2: S \to T$. Any agrarian embedding $i: \mathbb{Z}G \to D$ factors in a unique way through an epic agrarian embedding, namely the embedding into the subfield $D'$ of $D$ generated by $i(\mathbb{Z}G)$. Now any $D$-module, and in particular $D$ itself, is flat as a right module over the skew field $D'$. Hence, we get

$$\dim_D H_n(D \otimes D' C_*) = \dim_D H_n(D \otimes D' D' \otimes C_*) = \dim_D D \otimes D' H_n(D' \otimes C_*) = \dim_D H_n(D' \otimes C_*)$$

for any $\mathbb{Z}G$-chain complex $C_*$ and $n \in \mathbb{Z}$, i.e., $D$ and $D'$ yield the same agrarian Betti numbers. We can thus restrict our attention to epic agrarian embeddings.

If $H \leq G$ is a subgroup of a $D$-agrarian group $G$, then it is obviously again agrarian. If $i: \mathbb{Z}G \to D$ is a fixed agrarian embedding of $G$, we obtain a canonical agrarian embedding $i_H: \mathbb{Z}H \to \mathbb{Z}G \to D$, based on which we define the agrarian Betti numbers for $\mathbb{Z}H$-chain complexes. By the previous discussion, we could have equivalently considered the subfield of $D$ generated by $\mathbb{Z}H$. 
Lemma 3.7. Let $G$ be a finitely generated agrarian group. The agrarian Betti numbers for any connected finite free $G$-CW-complex are independent of the choice of agrarian embedding if and only if there exists an epic agrarian embedding for $G$ that is unique up to isomorphism commuting with the embeddings.

Proof. If there is a unique isomorphism type of epic agrarian embeddings for $G$, then by the preceding discussion every choice of an agrarian embedding factors through an epic agrarian embedding of this type and hence gives the same agrarian Betti numbers even for all $ZG$-complexes.

Now let $i_1: ZG \hookrightarrow D_1$ and $i_2: ZG \hookrightarrow D_2$ be non-isomorphic epic agrarian embeddings. By [Coh95, Theorem 4.3.5] there exists an $m \times n$-matrix $A$ over $ZG$ which becomes invertible when viewed as a matrix over $D_1$, but becomes singular over $D_2$ (without loss of generality). We realise $A$ topologically by constructing the skeleton of a connected finite free $G$-CW-complex $X$ step by step as follows: Choose a generating set $S = \{x_1, \ldots, x_k\}$ for $G$ and consider the Cayley graph $C(G, S)$ as a 1-dimensional $G$-CW-complex $X_1$. We now attach $m$ 2-dimensional spheres to each vertex, and extend the action of $G$ in the obvious way; this way we arrive at the 2-skeleton $X_2$ of $X$. For every $1 \leq j \leq n$, we then attach a 3-dimensional $G$-cell to $X_2$ in such a way that the resulting boundary map $C_3(X) \rightarrow C_2(X)$ in the cellular chain complex of the resulting space coincides with the matrix $A$. This concludes the construction of the connected finite free $G$-CW-complex $X$.

The cellular chain complex $C_*(X)$ is concentrated in degrees 0, 1, 2 and 3 and looks as follows:

$$
ZG^m \xrightarrow{A} ZG^n \xrightarrow{0} ZG^k \xrightarrow{(x_1^{-1}, \ldots, x_k^{-1})} ZG.
$$

As $A$ becomes invertible over $D_1$, but singular over $D_2$, we have

$$
H_2(X; D_1) = 0 \neq H_2(X; D_2),
$$

and hence the $D_1$- and $D_2$-Betti numbers of $X$ differ. \hfill \Box

Translated into our setting, in [Lew74, Section V], Lewin constructs two non-isomorphic epic agrarian embeddings of $F_6$, the free group on six generators. Taking the previous lemma into consideration, we conclude that the notion of the agrarian Betti numbers of an $F_6$-CW-complex is not well-defined. Nonetheless, we will later give examples of complexes for which the $D$-Betti numbers can be shown to not depend on $D$.

3.3. Computational properties. In order to formulate and prove agrarian analogues of the properties of $L^2$-Betti numbers, as collected by Lück in [Lüc02, Theorem 1.35], we have to introduce a few classical constructions on $G$-CW-complexes and chain complexes.

Recall that for a free $G$-CW-complex $X$ and a subgroup $H \leq G$ of finite index, the $H$-space $res_H^G(X)$ is obtained from $X$ by restricting the action to $H$. A free (finite, finite type) $H$-CW-structure for this space can be obtained from a free (finite, finite type) $G$-CW-structure of $X$ by replacing a $G$-cell with $|G:H|$ many $H$-cells.

If $H \leq G$ is any subgroup and $Y$ is a free $H$-CW-complex, then $G \times_H Y$ is the $H$-space $G \times Y/(g, y) \sim (gh^{-1}, hy)$. A free (finite, finite type) $H$-CW-structure of $Y$ determines a free (finite, finite type) $G$-CW-structure of $G \times_H Y$ by replacing an $H$-cell with a $G$-cell.

We now consider a chain complex $C_*$ with differentials $c_*$. Its suspension $\Sigma C_*$ is the chain complex with $C_{n-1}$ as the module in degree $n$ and $n$-th differential equal to $-c_{n-1}$. If $f_*: C_* \rightarrow D_*$ is a chain map between chain complexes with differentials $c_*$ and $d_*$, the mapping cone $cone(f_*)$ is the chain complex with
cone_n(f_*) = C_{n-1} \oplus D_n and n-th differential given by
\[
C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}} C_{n-2} \oplus D_{n-1}.
\]

The mapping cone of f_* fits into the following short exact sequence:
\[
0 \to D_* \to \text{cone}_n(f_*) \to \Sigma C_* \to 0.
\]

The following theorem covers all the properties of agrarian Betti number we will use in computations:

**Theorem 3.8.** The following properties of D-Betti numbers hold:

1. **(Homotopy invariance).** Let f : X \to Y be a G-map of free G-CW-complexes of finite type. If the map \( H_p(f; \mathbb{Z}) : H_p(X; \mathbb{Z}) \to H_p(Y; \mathbb{Z}) \) induced on cellular homology with integral coefficients is bijective for \( p \leq d - 1 \) and surjective for \( p = d \), then
   \[
   b^D_p(X) = b^D_p(Y) \quad \text{for} \quad p \leq d - 1;
   \]
   \[
   b^D_d(X) \geq b^D_d(Y).
   \]

   In particular, if \( f \) is a weak homotopy equivalence, we get for all \( p \geq 0 \):
   \[
   b^D_p(X) = b^D_p(Y).
   \]

2. **(Euler-Poincaré formula).** Let \( X \) be a finite free G-CW-complex. Let \( \chi(X/G) \) be the Euler characteristic of the finite CW-complex \( X/G \), i.e.,
   \[
   \chi(X/G) := \sum_{p \geq 0} (-1)^p \cdot \beta_p(X/G),
   \]
   where \( \beta_p(X/G) \) denotes the number of \( p \)-cells of \( X/G \). Then
   \[
   \chi^D(X) := \sum_{p \geq 0} (-1)^p \cdot b^D_p(X) = \chi(X/G).
   \]

3. **(Upper bound).** Let \( X \) be a free G-CW-complex. With \( \beta_p(X/G) \) as before, for all \( p \geq 0 \) we have
   \[
   b^D_p(X) \leq \beta_p(X/G).
   \]

4. **(Zeroth agrarian Betti number).** If \( G \) is non-trivial, then for any connected free G-CW-complex \( X \) of finite type we have
   \[
   b^D_0(X) = 0.
   \]

5. **(Transfer formula).** Let \( X \) be a free G-CW-complex of finite type and let \( H \leq G \) be a subgroup of finite index. For \( p \geq 0 \) we have
   \[
   b^D_p(G \times_H \text{res}_G^H(X)) = |G : H| \cdot b^D_p(X).
   \]

6. **(Amenable groups).** Let \( X \) be a free G-CW-complex of finite type and assume that \( G \) is amenable. Then
   \[
   b^D_p(X) = \dim_D(D \otimes H_p(X; \mathbb{Z}G)).
   \]

**Proof.**
(1) We replace \( f \) by a homotopic cellular map. Consider the ZG-chain map
\[
f_* : C_*(X) \to C_*(Y)
\]
induced by \( f \) on the cellular chain complexes and its mapping cone \( \text{cone}_n(f_*) \), which fits into a short exact sequence
\[
0 \to C_*(Y) \to \text{cone}_n(f_*) \to \Sigma C_* \to 0
\]
of ZG-chain complexes. Applying the assumptions on the map \( H_p(f; \mathbb{Z}) \) to the long exact sequence in homology associated to this short exact sequence, we obtain that \( H_p(\text{cone}_n(f_*)) = 0 \) for \( p \leq d \).
Claim. The homology of \( D \otimes \text{cone}_\ast(f_\ast) \) vanishes in degrees \( p \leq d \).

Assume for the moment that this indeed holds. Since \( D \otimes \text{cone}_\ast(f_\ast) = \text{cone}_\ast(id_D \otimes f_\ast) \) and \( D \otimes \Sigma C_\ast(X) = \Sigma(D \otimes C_\ast(X)) \), the short sequence
\[
0 \to D \otimes C_\ast(Y) \to D \otimes \text{cone}_\ast(f_\ast) \to D \otimes \Sigma C_\ast(X) \to 0
\]
is also exact. We now consider the associated long exact sequence in homology, in which the terms \( H_p(D \otimes \text{cone}_\ast(f_\ast)) \) for \( p \leq d \) vanish by the claim. The exactness of the sequence then implies that the differentials \( H_p(D \otimes \Sigma C_\ast(Y)) \overset{\cong}{\to} H_{p-1}(D \otimes C_\ast(X)) \) are isomorphisms for \( p \leq d \) and the differential \( H_{d+1}(D \otimes \Sigma C_\ast(Y)) \to H_d(D \otimes C_\ast(X)) \) is an epimorphism. Applying \( \dim_\mathbb{Z} \) and using the definition of the suspension then yields the desired statement.

We are left with proving the claim. Since \( \text{cone}_\ast(f_\ast) \) is bounded below and consists of free modules, we can inductively construct a \( \mathbb{Z}G \)-chain homotopy equivalent \( \mathbb{Z}G \)-chain complex \( Z_\ast \), which vanishes in degrees \( p \leq d \). Tensoring with \( D \) then yields a \( D \)-chain homotopy equivalence between \( D \otimes \text{cone}_\ast(f_\ast) \) and \( D \otimes Z_\ast \). As \( Z_p = 0 \) for \( p \leq d \), the same holds true for \( D \otimes Z_\ast \) and hence \( H_p(D \otimes \text{cone}_\ast(f_\ast)) = H_p(D \otimes Z_\ast) = 0 \) for \( p \leq d \).

(2) This is a consequence of two immediate facts: first, the Euler characteristic of a chain complex over a skew field does not change when passing to homology; second, we have the identity \( \beta_p(X/G) = \dim_\mathbb{Z} D \otimes C_p(X) \).

(3) This holds since \( H_p(X; D) \) is a subquotient of \( D \otimes C_p(X) \) and the latter has dimension \( \beta_p(X/G) \) over \( D \) (as remarked above).

(4) We will first argue that, without loss of generality, we may assume \( X/G \) to have exactly one 0-cell. If \( X \) is empty, then the claim is trivially true. Otherwise, let \( T \) be a maximal tree in the 1-skeleton of the CW-complex \( X/G \) and denote by \( q: X/G \to (X/G)/T \) the associated cellular quotient map, which is a homotopy equivalence. Note that \( (X/G)/T \) has a single 0-cell. Let \( p: (X/G)/T \to X/G \) be a cellular homotopy inverse of \( q \). We denote by \( X' \) the total space in the following pullback of the \( G \)-covering \( X \to X/G \) along \( p \):

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow & & \downarrow \\
X/G & \xrightarrow{q} & (X/G)/T
\end{array}
\]

Alternatively, we can see \( X' \) as being obtained from \( X \) by collapsing each lift of \( T \) individually to a point. Since \( (X/G)/T \) is a connected free \( G \)-CW-complex of finite type, \( X \to X/G \) is a \( G \)-covering and \( X \) is connected, the \( G \)-CW-complex \( X' \) is also connected, free and of finite type. Furthermore, \( X' \) is \( G \)-homotopy equivalent to \( X \) via any \( G \)-equivariant lift of the homotopy equivalence \( p \). By Theorem 3.8 [1] the \( D \)-Betti numbers of \( X \) and \( X' \) agree, so we may assume without loss of generality that \( X \) has a single equivariant 0-cell.

Since \( X \) is a free \( G \)-CW-complex of finite type which has a single 0-cell, the differential \( c_1: C_1(X) \to C_0(X) \) in its cellular chain complex is of the form
\[
\mathbb{Z}G \overset{\oplus_{i=1}^n (1-g_i)}{\to} \mathbb{Z}G
\]
for \( g_i \in G, i = 1, \ldots, n, n \in \mathbb{N} \) for any choice of a \( \mathbb{Z}G \)-basis of \( C_\ast(X) \) consisting of cells. The image of the differential is contained in the augmentation ideal of \( \mathbb{Z}G \), and, as \( X \) is assumed to be connected, has to coincide with it.
for $H_0(X;\mathbb{Z})$ to be isomorphic to $\mathbb{Z}$. Since $G$ is assumed to be non-trivial, there is $1 \leq i \leq n$ such that $1 - g_i \neq 0$. But this means that $1 - g_i$ becomes a unit in $D$, and hence $H_0(X;D) = 0$.

(5) Fix a set of coset representatives of $G/H$ and identify the finitely generated free $\mathbb{Z}H$-modules $\mathbb{Z}G$ and $\mathbb{Z}H^{[G:H]}$ accordingly. We obtain for every free $\mathbb{Z}G$-module $V \cong \bigoplus_j \mathbb{Z}G$ with chosen basis the following chain of isomorphisms of $D$-modules:

$$D \otimes \mathbb{Z}G \otimes_{\mathbb{Z}H} V \cong D \otimes_{\mathbb{Z}H} \left( \bigoplus_i \mathbb{Z}G \right) \cong D \otimes_{\mathbb{Z}H} \left( \bigoplus_i \mathbb{Z}H^{[G:H]} \right) \cong \bigoplus_i D^{[G:H]} \otimes V$$

On cellular chain complexes, $\text{res}_G^H$ corresponds to regarding a $\mathbb{Z}G$-chain complex as a $\mathbb{Z}H$-chain complex and $G \times H$ translates into applying the functor $\mathbb{Z}G \otimes_{\mathbb{Z}H} \cdot$. We thus get

$$D \otimes C_\ast(G \times H \text{ res}_G^H(X)) \cong D \otimes \mathbb{Z}G \otimes_{\mathbb{Z}H} C_\ast(X) \cong D^{[G:H]} \otimes C_\ast(X).$$

Since $D^{[G:H]}$ is a free and hence a flat right $D$-module, taking homology commutes with the tensor products and we conclude

$$H_p(G \times H \text{ res}_G^H(X); D) \cong H_p(X; D^{[G:H]}) \cong D^{[G:H]} \otimes_D H_p(X;D),$$

from which the statement follows by applying $\text{dim}_D$.

(6) As $G$ is agrarian, its group ring $\mathbb{Q}G$ does not admit zero divisors (this is immediate, since $\mathbb{Q}G$ embeds into the same skew field $D$ as $\mathbb{Z}G$ does). A result of Tamari [Tam54] we used already for free abelian groups now implies that since $G$ is amenable, it admits an Ore field of fractions $F$ of $\mathbb{Q}G$, and hence of $\mathbb{Z}G$. In particular, $F$ is flat over $\mathbb{Z}G$ and every embedding of $\mathbb{Z}G$ into a skew field, such as the agrarian embedding $i$: $\mathbb{Z}G \hookrightarrow D$, factors through the natural inclusion $\mathbb{Z}G \hookrightarrow F$. We thus obtain for any $p \geq 0$, using first that $F \hookrightarrow D$ is flat and then that $\mathbb{Z}G \hookrightarrow F$ is flat:

$$b^D_p(X) = \text{dim}_D H_p(X;D) = \text{dim}_D D \otimes_F H_p(X;F) = \text{dim}_D D \otimes_F F \otimes H_p(X;\mathbb{Z}G) = \text{dim}_D D \otimes H_p(X;\mathbb{Z}G).$$

\[ \Box \]

3.4. Mapping tori. In subsequent sections, we will study invariants of CW complexes with vanishing agrarian Betti numbers. In the context of $L^2$-invariants, an extremely useful way of showing the vanishing of $L^2$-Betti numbers comes from a celebrated theorem of Lück [Lüc02, Theorem 1.39] (which answered a question of Gromov). Below, we offer a straightforward generalisation of Lück’s result to the setting of agrarian Betti numbers.

Definition 3.9. Let $f: X \rightarrow X$ be a selfmap of a path-connected space. The mapping torus $T_f$ of $f$ is obtained from the cylinder $X \times [0,1]$ by identifying $(x,1)$ with $(f(x),0)$ for any $x \in X$. The canonical projection is the map $T_f \rightarrow S^1$ sending $(x,t)$ to $\exp(2\pi it)$. It induces an epimorphism $\pi_1(T_f) \rightarrow \pi_1(S^1) = \mathbb{Z}$.

If $X$ has the structure of a CW-complex with $\beta_p(X)$ cells of dimension $p$, then $T_f$ can be endowed with a CW-structure with $\beta_p(T_f) = \beta_p(T_f) + \beta_{p-1}(T_f)$ cells of dimension $p$.

Theorem 3.10. Let $f: X \rightarrow X$ be a cellular selfmap of a connected CW-complex $X$ and $\pi_1(T_f) \xrightarrow{\varphi} G \xrightarrow{\psi} \mathbb{Z}$ be any factorisation into epimorphisms of the epimorphism induced by the canonical projection. Let $\tilde{T}_f$ be the covering of the mapping torus $T_f$ associated to $\varphi$, endowed with the structure of a connected free $G$-CW-complex.
Assume that the d-skeleton of X (and thus of \(T_f\)) is finite for some \(d \geq 0\). Then for all \(p \leq d\) we have

\[ b^D_p(T_f) = 0. \]

**Proof.** The proof is an adaption of the proof for \(L^2\)-Betti numbers, see [Lück02, Theorem 1.39].

Fix \(p \geq 0\). For any \(n \geq 1\), define \(G_n \leq G\) as the preimage of the subgroup \(n \cdot \mathbb{Z} \leq \mathbb{Z}\) under \(\psi: G \to \mathbb{Z}\). Since \(G_n\) has index \(n\) in \(G\), we deduce from Theorem 3.8 (5)

\[ (1) \quad b^D_p(G \times G_n \mathbb{Z}_T(T_f)) = \frac{1}{n} b^D_p((G \times G_n \mathbb{Z}_T(T_f))) \]

Reparametrising yields a homotopy equivalence \(h: T_{f^n} \xrightarrow{\sim} T_f / G_n\) of CW-complexes, where \(f^n\) denotes the \(n\)-fold composition of \(f\). Let \(T_{f^n}\) be the \(G_n\)-space obtained as the following pullback, or equivalently, as the covering of \(T_{f^n}\) corresponding to the kernel of \(\pi_1(T_{f^n}) \cong \pi_1(T_f/G_n) \to G_n\):

\[
\begin{array}{ccc}
T_{f^n} & \xrightarrow{\bar{h}} & \text{res}_{G_n}^G(T_f) \\
\downarrow & & \downarrow \\
T_f / G_n & \xrightarrow{h} & T_f / G_n
\end{array}
\]

Since \(h\) is a homotopy equivalence between base spaces of \(G_n\)-coverings, \(\bar{h}\) is a \(G_n\)-homotopy equivalence. It induces a \(G\)-homotopy equivalence \(G \times G_n \xrightarrow{\bar{h}} G \times G_n \mathbb{Z}_T(T_f)\). By Theorem 3.8 (1) we obtain

\[ (2) \quad b^D_p(G \times G_n \mathbb{Z}_T(T_f)) = b^D_p(G \times G_n \mathbb{Z}_T(T_f)) \]

for \(p \geq 0\). Since \(T_{f^n}\) has a CW-structure with \(\beta_p(X) + \beta_{p-1}(X)\) cells of dimension \(p\) and this number is finite by assumption, using Theorem 3.8 (3) we conclude:

\[ b^D_p(T_f) \cong \frac{1}{n} b^D_p(G \times G_n \mathbb{Z}_T(T_f)) \cong \frac{1}{n} b^D_p(G \times G_n \mathbb{Z}_T(T_f)) \cong \frac{1}{n} b^D_p(G \times G_n \mathbb{Z}_T(T_f)) \leq \frac{\beta_p(X) + \beta_{p-1}(X)}{n}. \]

Letting \(n \to \infty\) finishes the proof. \(\square\)

4. **Agrarian Torsion**

Having introduced agrarian Betti numbers together with computational tools allowing us to prove their vanishing for certain spaces, we will now present a secondary invariant for such spaces. This invariant will be called **agrarian torsion** and arises as Reidemeister torsion with values in the abelianised units of the skew field \(D\). It is motivated by the construction of **universal \(L^2\)-torsion** by Friedl and Lück [FL17]. We will reference the rather general treatment of torsion by Cohen [Coh73] throughout this section.

As usual, in this section \(G\) will always be an agrarian group with a fixed agrarian embedding \(i: \mathbb{Z}G \to D\).

4.1. **Contractible \(D\)-chain complexes.** In order to define agrarian torsion, we require a contractible \(D\)-chain complex. In our case, contractibility is governed by the agrarian Betti numbers because of

**Proposition 4.1** ([Ros94, Proposition 1.7.4]). Let \(R\) be a ring and \(C_*\) an \(R\)-chain complex. If \(C_*\) is acyclic, vanishes in sufficiently small degree and consists of projective \(R\)-modules, then \(C_*\) is contractible.

**Lemma 4.2.** A finite \(\mathbb{Z}G\)-chain complex \(C_*\) is \(D\)-acyclic if and only if \(D \otimes C_*\) is contractible.
Proof. Since $C_*$ is finite, the $D$-chain complex $D \otimes C_*$ is in particular bounded below. All its modules are free because $D$ is a skew field, and hence the statement follows from Proposition 4.1.

Agrarian torsion, being constructed as non-commutative Reidemeister torsion, naturally takes values in the first $K$-group of $D$:

**Definition 4.3.** Let $R$ be a ring. Denote by $\text{GL}(R)$ the direct limit of the groups $\text{GL}_n(R)$ of invertible $n \times n$ matrices over $R$ with the embeddings given by adding an identity block in the bottom-right corner. The $K_1$-group $K_1(R)$ is defined as the abelianisation of $\text{GL}(R)$. The reduced $K_1$-group $\tilde{K}_1(R)$ is defined as the quotient of $K_1(R)$ by the subgroup $\{1\}$.

We now consider a $D$-acyclic finite free $ZG$-chain complex $(C_*, c_*)$. Such a complex will be called *based* if it comes with a choice of preferred bases for all chain modules. By the previous lemma, we can find a chain contraction $c_\gamma$ of $D \otimes C_*$. Set $C_{\text{odd}} := \bigoplus_i C_i$ and $C_{\text{even}} := \bigoplus_i C_i$. Note that $D$-acyclicity guarantees that $\dim D \otimes C_{\text{odd}} = \dim D \otimes C_{\text{even}}$.

**Lemma 4.4.** In the situation above, the map $c_* + c_\gamma : D \otimes C_{\text{odd}} \to D \otimes C_{\text{even}}$ is an isomorphism of finitely generated based free $D$-modules and the class in $K_1(D)$ defined by the matrix representing it in the preferred basis does not depend on the choice of $c_\gamma$.

Proof. That the map is an isomorphism is the content of [Col73 (15.1)], the independence is covered by [Col73 (15.3)].

### 4.2. The Dieudonné determinant.

The $K_1$ groups of skew fields can be determined using a generalisation of the classical determinant of a matrix over a field to matrices over skew fields, which is known as the Dieudonné determinant. As opposed to the situation for fields, there is no longer a polynomial expression in terms of the entries of the matrix; rather, the Dieudonné determinant is defined by an inductive procedure:

**Definition 4.5.** Let $A = (a_{ij})$ be an $n \times n$ matrix over a skew field $D$. The canonical representative of the Dieudonné determinant $\det^c A \in D$ is defined inductively as follows:

1. If $n = 1$, then $\det^c A = a_{11}$.
2. If the last row of $A$ consists of zeros only, then $\det^c A = 0$.
3. If $a_{nn} \neq 0$, then we form the $(n-1) \times (n-1)$ matrix $A' = (a'_{ij})$ by setting $a'_{ij} = a_{ij} - a_{in}a_{nj}a_{nn}$, and declare $\det^c A = \det^c A' \cdot a_{nn}$.
4. Otherwise, let $j < n$ be maximal such that $a_{nj} \neq 0$. Let $A'$ be obtained from $A$ by interchanging rows $j$ and $n$. Then set $\det^c A = -\det^c A'$.

The Dieudonné determinant $\det A$ of $A$ is defined to be the image of $\det^c A$ in $D^\times/[D^\times, D^\times] \cup \{0\}$, i.e., in the abelianised unit group of $D$ if $\det^c A \neq 0$. We also write $D_{ab}^\times$ for this group.

As a convention, we will write the group operation of the abelian group $D_{ab}^\times$ (and its quotients) additively.

If $D$ is a (commutative) field, then the Dieudonné determinant coincides with the usual determinant as the matrix $A$ is brought into upper-diagonal form during the inductive procedure defining $\det^c A$.

The Dieudonné determinant is multiplicative on all matrices and takes non-zero values on invertible matrices [Die43].
Proposition 4.6 ([Ros94, Corollary 2.2.6]). Let $D$ be a skew field. Then the Dieudonné determinant \(\det : \text{GL}(D) \to D^*_\ab\) induces group isomorphisms
\[
\det : K_1(D) \xrightarrow{\cong} D^*_\ab \quad \text{and} \quad \det : \widetilde{K}_1(D) \xrightarrow{\cong} D^*_\ab/\{\pm 1\}.
\]

4.3. Definition and properties of agrarian torsion. Relying on the explicit description of \(\widetilde{K}_1(D)\) obtained above, we can motivate

Definition 4.7. The \(D\)-agrarian torsion of a \(D\)-acyclic finite based free \(\mathbb{Z}G\)-chain complex \((C_*, e_*)\) is defined as
\[
\rho_D(C_*) := \det([e_* + \gamma_*]) \in D^*_\ab/\{\pm 1\},
\]
where \([e_* + \gamma_*] \in \widetilde{K}_1(D)\) is the class determined by the (representing matrix of the) isomorphism constructed in Lemma 4.3.

The usual additivity property for torsion invariants directly carries over to the agrarian setting in the following form:

Lemma 4.8 ([Coh73 (17.2)]). Let \(0 \to C'_* \to C_* \to C''_* \to 0\) be a short exact sequence of finite based free \(\mathbb{Z}G\)-chain complexes such that the preferred basis of \(C_*\) is composed of the preferred basis of \(C'_*\) and preimages of the preferred basis elements of \(C''_*\). Assume that any two of the complexes are \(D\)-acyclic. Then so is the third and
\[
\rho_D(C_*) = \rho_D(C'_*) + \rho_D(C''_*).
\]

The difference in agrarian torsion between \(\mathbb{Z}G\)-chain homotopy equivalent chain complexes is measured by the Whitehead torsion of the chain homotopy equivalence, analogously to the statement of [FL17, Lemma 2.10] for universal \(L^2\)-torsion:

Lemma 4.9. Let \(f : C_* \to E_*\) be a \(\mathbb{Z}G\)-chain homotopy equivalence of finite based free \(\mathbb{Z}G\)-chain complexes. Denote by \(\rho_\pm(\text{cone}(f_*)) \in \widetilde{K}_1(\mathbb{Z}G)\) the Whitehead torsion of the contractible finite based free \(\mathbb{Z}G\)-chain complex \(\text{cone}(f_*))\). If one of \(C_*\) and \(E_*\) is \(D\)-acyclic, then so is the other and we get
\[
\rho_D(E_*) - \rho_D(C_*) = \det(i_*(\rho_+(\text{cone}(f_*)))),
\]
where \(i_* : \widetilde{K}_1(\mathbb{Z}G) \to \widetilde{K}_1(D)\) is induced by \(i : \mathbb{Z}G \to D\).

Proof. Since \(f_*\) is a \(\mathbb{Z}G\)-chain homotopy equivalence, the finite free \(\mathbb{Z}G\)-chain complex \(\text{cone}(f_*))\) is contractible and hence its Whitehead torsion \(\rho_\pm(\text{cone}(f_*))\) is defined. The finite free \(D\)-chain complex \(D \otimes \text{cone}(f_*))\) is again contractible and since the matrix defining its agrarian torsion are already invertible over \(\mathbb{Z}G\), we get that \(\rho_D(\text{cone}(f_*)) = \det(i_*(\rho_\pm(\text{cone}(f_*))))\).

We now apply Lemma 4.8 to the short exact sequence \(0 \to E_* \to \text{cone}(f_*)) \to \Sigma C_* \to 0\) with \(\text{cone}(f_*))\) and one of \(\Sigma C_*\) and \(E_*\) being \(D\)-acyclic. Since \(\rho_D(\Sigma C_*) = -\rho_D(C_*)\), as is readily observed from the definition of \(\rho_D\), we obtain that \(\rho_D(E_*) = \rho_D(C_*) = \rho_D(\text{cone}(f_*)) = \det(i_*(\rho_\pm(\text{cone}(f_*))))\). \(\Box\)

Our goal is to apply the concept of \(D\)-agrarian torsion to \(G\)-CW-complexes. Since the free cellular chain complexes associated to such complexes do not admit a canonical \(\mathbb{Z}G\)-basis, but only a canonical \(\mathbb{Z}\)-basis (up to orientation), we have to account for this additional indeterminacy by passing to a further quotient of \(D^*_\ab\):

Definition 4.10. Let \(X\) be a \(D\)-acyclic finite free \(G\)-CW-complex. The \(D\)-agrarian torsion of \(X\) is defined as
\[
\rho_D(X) := \rho_D(C_*(X)) \in D^*_\ab/\{\pm g \mid g \in G\},
\]
where $C_*(X)$ is endowed with any $\mathbb{Z}G$-basis that projects to a $\mathbb{Z}$-basis of $C_*(X/G)$ consisting of (unequivariant) cells.

That $\rho_D(X)$ is indeed well-defined can be seen from [Coh73 (15.2)].

4.4. Comparison with universal $L^2$-torsion. A rich source of agrarian groups is the class of torsion-free groups that satisfy the Atiyah conjecture. For these groups, there is a canonical skew field $D(G)$ in which the group ring $\mathbb{Z}G$ embeds. In the case of $D = D(G)$, agrarian torsion coincides with the determinant of the universal $L^2$-torsion introduced by Friedl and Lück in [FL17], as we will see now.

Universal $L^2$-torsion naturally lives in a weak version of the $K_1$-group of the group ring, which is defined as follows:

**Definition 4.11** ([FL17 Definition 2.1]). Let $G$ be a group. Denote by $K_1^w(\mathbb{Z}G)$ the weak $K_1$-group defined in terms of the following generators and relation:

**Generators:** $[A]$ for square matrices $A$ over $\mathbb{Z}G$ that become invertible after the change of rings $\mathbb{Z}G \to D(G)$

**Relations:**
- $[AB] = [A] + [B]$ for matrices $A$ and $B$ of compatible sizes and such that $A$ and $B$ become invertible over $D(G)$.
- $[D] = [A] + [C]$ for a block matrix
  $$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$
  with $A$ and $C$ square and invertible over $D(G)$.

Define the weak Whitehead group $\text{Wh}^w(G)$ as the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup generated by the $1 \times 1$-matrices $(\pm g)$ for all $g \in G$.

Note that there are canonical maps $K_1(\mathbb{Z}G) \to K_1^w(\mathbb{Z}G)$ and $K_1^w(\mathbb{Z}G) \to K_1(D(G))$ given by $[A] \mapsto [A]$ and $[A] \mapsto [1 \otimes A]$ on generators, respectively.

The following result by Linnell and Lück indicates that the abelian groups in which agrarian torsion and universal $L^2$-torsion take values coincide up to isomorphism for a large class of groups:

**Theorem 4.12** ([LL18]). Let $C$ be the smallest class of groups which contains all free groups and is closed under direct union and extensions by elementary amenable groups. Let $G$ be a torsion-free group which belongs to $C$. Then $D(G)$ is a skew field and the composite map

$$K_1^w(\mathbb{Z}G) \to K_1(D(G)) \xrightarrow{\text{def}} D(G)^\times_{ab}$$

is an isomorphism.

Let $X$ be a finite free $G$-CW-complex that is $L^2$-acyclic, i.e., whose $L^2$-Betti numbers vanish. Friedl and Lück [FL17 Definition 3.1] associate to such a $G$-CW-complex an element $\rho^{(2)}_u(X) \in \text{Wh}^w(G)$ called the universal $L^2$-torsion of $X$. We can obtain from this an element

$$\det(\rho^{(2)}_u(X)) \in D(G)^\times_{ab}/\{\pm g \mid g \in G\},$$

which by Theorem 4.12 carries an equivalent amount of information as $\rho^{(2)}_u$ for many groups $G$.

The statement of the following theorem is implicit in [FLT16 Section 2.3] by Friedl, Lück and Tillmann.

**Theorem 4.13.** Let $G$ be a torsion-free group that satisfies the Atiyah conjecture. Then $G$ is $D(G)$-agrarian. Furthermore, if $X$ is any finite free $G$-CW-complex, then $X$ is $D(G)$-acyclic if and only if it is $L^2$-acyclic. If this is the case, we have

$$\rho_{D(G)}(X) = \det(\rho^{(2)}_u(X)) \in D(G)^\times_{ab}/\{\pm g \mid g \in G\}.$$
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Proof. During the proof, we will use the notion of universal $L^2$-torsion for $L^2$-acyclic finite based free $\mathbb{Z}G$-chain complexes as defined in [FL17, Definition 2.7]. The universal $L^2$-torsion of a finite free $G$-CW-complex is then obtained as the universal $L^2$-torsion of the associated cellular chain complex together with any basis consisting of $G$-cells. We will also abuse notation in that we consider classes in $\tilde{K}_1^G(\mathbb{Z}G)$ to be represented by both square matrices over $\mathbb{Z}G$ (our convention) and $\mathbb{Z}G$-endomorphisms of some $\mathbb{Z}G^n$ (the convention in [FL17]).

The first statement is proved analogously to one direction of [Luc02, Lemma 10.39], the second statement then follows from Lemma 4.6 and [FL17, Lemma 2.21].

In order to prove the last statement, we want to make use of the universal property of universal $L^2$-torsion (see [FL17, Remark 2.16]). To this end, we first consider $\mathbb{Z}G$-chain complexes of the following simple form: Let $[A] \in \tilde{K}_1^G(\mathbb{Z}G)$ be represented by an $n \times n$ matrix $A$ over $\mathbb{Z}G$, and let $C^A_*$ be the $\mathbb{Z}G$-chain complex concentrated in degrees 0 and 1 with the only non-trivial differential given by $r_A : \mathbb{Z}G^n \to \mathbb{Z}G^n, x \mapsto x \cdot A$. Since $A$ becomes an isomorphism over $D(G)$, such a complex is always $D(G)$- and thus $L^2$-acyclic.

The universal $L^2$-torsion of $C^A_*$ is computed from a weak chain contraction $(\delta_u, v_u)$ of $C^A_*$ as defined in [FL17, Definition 2.4]. In this particular case, we can take $\delta_0 = \text{id}_{\mathbb{Z}G^n}, \delta_p = 0$ for $p \neq 0$ and $v_0 = v_1 = r_A, v_p = 0$ for $p \notin \{0, 1\}$. According to [FL17, Definition 2.7], the universal $L^2$-torsion of $C^A_*$ is thus given by

$$\rho_{\mathcal{A}}(C^A_*) = [v_1 \circ r_A + 0] - [v_1] = [r^2_A] - [r_A] = [A] \in \tilde{K}_1^G(\mathbb{Z}G)$$

and hence $\det(\rho_{\mathcal{A}}(C^A_*)) = \det A \in D(G)^{\mathcal{A}}/{\{\pm 1\}}$.

The $(D(G))$-agrarian torsion of $C^A_*$ is computed from a (classical) chain contraction of $D(G) \otimes C^A_*$, let $\gamma_\ast$ be such a contraction with $\gamma_0 = (\text{id}_{D(G)} \otimes r_A)^{-1}$ and $\gamma_p = 0$ for $p \neq 0$. Since $\gamma$ vanishes in odd degrees, the construction of $(D(G))$-agrarian torsion yields

$$\rho_{D(G)}(C^A_*) = \det([\text{id}_{D(G)} \otimes r_A + 0]) = \det A \in D(G)^{\mathcal{A}}/{\{\pm 1\}},$$

and hence $\det(\rho_{D(G)}(C^A_*)) = \rho_{D(G)}(C^A_*)$.

The pair $(D(G)^{\mathcal{A}}/{\{\pm 1\}}, \rho_{D(G)})$ consists of an abelian group and an assignment that associates to a $D(G)$-acyclic (i.e., $L^2$-acyclic) finite based free $\mathbb{Z}G$-chain complex an element $\rho_{D(G)} \in D(G)^{\mathcal{A}}/{\{\pm 1\}}$. The assignment is additive by Lemma 4.8 and maps complexes of the shape $\mathbb{Z}G \xrightarrow{\pm \text{id}_{D(G)}} \mathbb{Z}G$ to $1 \in D(G)^{\mathcal{A}}/{\{\pm 1\}}$ by construction. It hence constitutes an example of an additive $L^2$-torsion invariant in the sense of [FL17, Remark 2.16]. Since by [FL17, Theorem 2.12] the pair $(\tilde{K}_1^G(\mathbb{Z}G), \rho_{\mathcal{A}}(\cdot))$ is the universal such invariant, there is a unique group homomorphism $f : \tilde{K}_1^G(\mathbb{Z}G) \to D(G)^{\mathcal{A}}/{\{\pm 1\}}$ satisfying $f \circ \rho_{\mathcal{A}}(\cdot) = \rho_{D(G)}$.

It is left to check that $f$ and $\det$ agree as maps $\tilde{K}_1^G(\mathbb{Z}G) \to D(G)^{\mathcal{A}}/{\{\pm 1\}}$. We have seen already that $\det(\rho_{\mathcal{A}}(C^A_*)) = \rho_{D(G)}(C^A_*)$. But $\rho_{\mathcal{A}}(C^A_*) = [A]$, and hence $\{\rho_{\mathcal{A}}(C^A_*) | [A] \in \tilde{K}_1^G(\mathbb{Z}G)\}$ generates $\tilde{K}_1^G(\mathbb{Z}G)$ as a group. Since $f$ agrees with $\det$ on this generating set, we conclude that $f = \det$.

As will prove useful when we define the agrarian analogue of the $L^2$-torsion polytope, passing to an associated $K$-Ore field of the Linell skew field only changes the agrarian torsion by pushing forward via an inclusion of skew fields:

**Corollary 4.14.** Let $G$ be a torsion-free group that satisfies the Atiyah conjecture and let $K \leq G$ be a normal subgroup such that $H := G/K$ is finitely generated free abelian. Then $G$ is both $D(G)$- and Ore$(D(G); K \leq G)$-agrarian and the agrarian embeddings factor as $\mathbb{Z}G \leftrightarrow D(G) \leftrightarrow \text{Ore}(D(G); K \leq G)$. Furthermore, if $X$ is any
finite free $G$-CW-complex, then $X$ is $\text{Ore}(\mathcal{D}(G); K \leq G)$-acyclic if and only if it is $L^2$-acyclic. If this is the case, we have

$$\rho_{\text{Ore}(\mathcal{D}(G); K \leq G)}(X) = j_*(\det(\rho_G^2(X))) \in \text{Ore}(\mathcal{D}(G); K \leq G)^\times / \{g \mid g \in G\},$$

where $j_*: \mathcal{D}(G)^\times / \{g \mid g \in G\} \to \text{Ore}(\mathcal{D}(G); K \leq G)^\times / \{g \mid g \in G\}$ is induced by the respective inclusion of skew fields.

Proof. We abbreviate $\text{Ore}(\mathcal{D}(G); K \leq G)$ by $\mathcal{D}_K$. By [Lüc02, Lemma 10.69], the Linnell skew field $\mathcal{D}(G)$ is isomorphic to the Ore localisation of the twisted group ring $\mathcal{D}(K)H$ via an isomorphism extending the twisted group ring decomposition $\mathcal{Z}G \cong (\mathcal{Z}K)H$. In this way $\mathcal{D}(G)$ is a subfield of $\mathcal{D}_K$, which is the Ore localisation of the larger twisted group ring $\mathcal{D}(G)H$. Hence, as discussed in Remark 3.6 the agrarian embeddings $\mathcal{Z}G \hookrightarrow \mathcal{D}(G)$ and $\mathcal{Z}G \hookrightarrow \mathcal{D}_K$ yield the same agrarian Betti numbers. The first statement thus follows from Theorem 4.13.

For the statement on agrarian torsion we first note that [Lüc02, Lemma 10.34 (1)] implies that since $\mathcal{D}(G)$ is a skew field, it is rationally closed in $\mathcal{D}_K$, i.e., every matrix over $\mathcal{D}(G)$ that becomes invertible over $\mathcal{D}_K$ was already invertible in $\mathcal{D}(G)$. In particular this applies to matrices over $\mathcal{Z}G$. Hence, the invertible matrices appearing in the construction of $\rho_{\mathcal{D}_K}$ following Definition 4.4 are already invertible over the subfield $\mathcal{D}(G)$. We conclude that viewed as an element of $K_1(\mathcal{D}_K)$, the agrarian torsion $\rho_{\mathcal{D}_K}(X)$ is the image of the agrarian torsion $\rho_{\mathcal{D}(G)}(X)$ under the map $K_1(\mathcal{D}(G)) \to K_1(\mathcal{D}_K)$ induced by the respective inclusion of skew fields. Since the Dieudonné determinant is natural with respect to inclusion and $\rho_{\mathcal{D}(G)}(X)$ calculates $L^2$-torsion by Theorem 4.13 the second statement holds. \[\square\]

5. Agrarian Polytope

Our aim for this section is to extract from the agrarian torsion of a $G$-CW-complex a finite combinatorial object, namely a polytope. The polytope arises from a Newton polytope construction applied to elements of the Ore localisation of a twisted polynomial ring over a the skew field of an agrarian embedding. Our motivation for studying the Newton polytope of a torsion invariant originates from [FLI17].

5.1. The polytope group. Before we get to define the Newton polytope of an element of a twisted group ring, we have to introduce several concepts related to polytopes.

Definition 5.1. Let $V$ be a finite-dimensional real vector space. A polytope in $V$ is the convex hull of finitely many points in $V$. For a polytope $P \subset V$ and a linear map $\varphi: V \to \mathbb{R}$ we define

$$F_\varphi(P) := \{p \in P \mid \varphi(p) = \min_{q \in P} \varphi(q)\}$$

and call this polytope the $\varphi$-face of $P$. The elements of the collection $\{F_\varphi(P) \mid \varphi: V \to \mathbb{R}\}$ are the faces of $P$. A face is called a vertex if it consists of a single point.

Note that with the above definition polytopes are always compact and convex.

In the following, we will always take $V = \mathbb{R} \otimes H$ for some finitely generated free abelian group $H$.

Definition 5.2. A polytope $P$ in $V$ is called integral if its vertices lie on the lattice $H \subset V$. 

Given two (integral) polytopes $P$ and $Q$ in $V$ we can form the pointwise sum $P + Q = \{p + q \mid p \in P, q \in Q\}$. As a subset of $V$ this will again be an (integral) polytope, the vertices of which are obtained as pointwise sums of some of the vertices of $P$ and $Q$. The polytope $P + Q$ is called the Minkowski sum of $P$ and $Q$. With the Minkowski sum the set of all (integral) polytopes in $V$ becomes an abelian monoid with neutral element $\{0\}$. It is cancellative, i.e., $P + Q = P' + Q$ for polytopes $P, P'$ and $Q$ implies $P = P'$, see [Rad52, Lemma 2]. The monoid can thus be embedded into an abelian group in a universal way and gives rise to the following algebraic object introduced in [FT15, 6.3]:

**Definition 5.3.** Let $H$ be a finitely generated free abelian group. Denote by $\mathcal{P}(H)$ the polytope group of $H$, that is the Grothendieck group of the cancellative abelian monoid given by all integral polytopes in $\mathbb{R} \otimes \mathbb{Z} H$ under Minkowski sum. In other words, let $\mathcal{P}(H)$ be the abelian group with generators the formal differences $P - Q$ of integral polytopes and relations $(P - Q) + (P' - Q') = (P + P') - (Q + Q')$ as well as $P - Q = P' - Q'$ if $P + Q = P' + Q'$. The neutral element is given by the one-point polytope $\{0\}$, which we will drop from the notation. We view $H$ as a subgroup of $\mathcal{P}(H)$ via the map $h \mapsto \{h\}$. The translation-invariant polytope group of $H$, denoted by $\mathcal{P}_T(H)$, is defined to be the quotient group $\mathcal{P}(H)/H$.

While a general element of the polytope group can have many equivalent representations as a formal difference of polytopes, an element of the type $[P]$ is uniquely presented in this form (up to translation, if viewed as an element of $\mathcal{P}(H)$). Such an element is called a **single polytope**.

### 5.2. The polytope homomorphism.

As is the case for the $L^2$-torsion polytope, the following simple construction underpins the definition of the agrarian polytope:

**Definition 5.4.** Let $D$ be a skew field and let $H$ be a finitely generated free abelian group. Let $DH$ denote some twisted group ring formed out of $D$ and $H$. The Newton polytope $P(p)$ of an element $p = \sum_{h \in H} u_h \ast h \in DH$ is the convex hull of the support $\text{supp}(p) = \{h \in H \mid u_h \neq 0\}$ in $H_1(H; \mathbb{R})$.

Since $H$ is free abelian, as in Definition 5.3 we can consider the Ore localisation $\text{Ore}(DH)$ of the twisted group ring $DH$. The previous definition can then be extended to elements of $\text{Ore}(DH)$ as follows:

**Definition 5.5.** The group homomorphism

$$P: \text{Ore}(DH)^{\times}_{ab} \to \mathcal{P}(H)$$

$$pq^{-1} \mapsto P(p) - P(q)$$

is called the **polytope homomorphism** of $\text{Ore}(DH)$. It induces a homomorphism

$$P: \text{Ore}(DH)^{\times}_{ab}/\{\pm h \mid h \in H\} \to \mathcal{P}_T(H).$$

The well-definedness of $P$ is immediate from the construction of $\mathcal{P}(H)$. The fact that $P$ is a group homomorphisms is not hard, and has been shown in [Kie18, Lemma 3.12] (see also the discussion following the lemma).

### 5.3. Definition of the agrarian polytope.

In the following, $G$ will always be a finitely generated $D$-agrarian group. We will take the free abelian group $H$ to be the free part of the abelianisation of $G$. Furthermore, we denote the canonical projection onto $H$ by $pr$ and its kernel by $K$.

In [FL17], the Newton polytope is constructed for the Linnell skew field $D(G)$, which can be expressed as an Ore localisations of the twisted group ring $D(K)H$. In the more general agrarian situation, an arbitrary agrarian embedding for $G$ is of course not always an Ore localisation of a suitable twisted group ring. But, as we
have seen in the discussion leading up to Definition 2.5, we can instead consider the associated \(K\)-Ore embedding \(\Ore(D; K \leq G)\), which we will subsequently denote by \(D_K\).

**Definition 5.6.** Let \(C_*\) be a \(D_K\)-acyclic finite based free \(\mathbb{Z}G\)-chain complex. The \(D_K\)-agrarian polytope of \(C_*\) is defined as

\[
P^{D_K}(C_*) := P(-\rho_{D_K}(C_*)) \in \mathcal{P}(H).
\]

We will mostly apply this notion to \(G\)-CW-complex, where we have to account for the indeterminacy caused by the need to choose a basis of cells:

**Definition 5.7.** Let \(X\) be a \(D_K\)-acyclic finite free \(G\)-CW-complex. The \(D_K\)-agrarian polytope of \(X\) is defined as

\[
P^{D_K}(X) := P(-\rho_{D_K}(X)) \in \mathcal{T}(H).
\]

The purpose of the sign in the definition of the \(D_K\)-agrarian polytope is to get a single polytope in many cases of interest. A priori, the polytope may depend on the choice of the section of \(pr\) involved in the construction of \(D_K\). By Lemma 2.4, the twisted group rings obtained from any two choices differ by an isomorphism preserving supports, and thus the agrarian polytope is indeed well-defined.

**Proposition 5.8.** The \(D_K\)-agrarian polytope \(P^{D_K}(X)\) is a \(G\)-homotopy invariant of \(X\).

**Proof.** Let \(X\) and \(X'\) be \(D_K\)-acyclic finite free \(G\)-CW-complexes \(G\)-homotopy equivalent via \(f: X \to X'\). We denote the induced homotopy equivalence between \(X/G\) and \(X'/G\) by \(\tilde{f}\). By Lemma 4.9, the agrarian torsions of \(X\) and \(X'\) are related via

\[
\rho_{D_K}(X') - \rho_{D_K}(X) = \det(\rho(\tilde{f})).
\]

After applying the polytope homomorphism, we obtain

\[
P^{D_K}(X') - P^{D_K}(X) = P(\det(\rho(\tilde{f}))).
\]

The latter polytope is a singleton by [Kie18, Corollary 5.16] and hence \(P^{D_K}(X') = P^{D_K}(X) \in \mathcal{T}(H)\). □

Because of the previous proposition, the agrarian polytope of the universal covering of the classifying space of a group, which is only well-defined up to \(G\)-homotopy equivalence, does not depend on the choice of a particular \(G\)-CW-model. We are thus led to

**Definition 5.9.** Assume that \(G\) is of type \(F\), i.e., let there be an unequivariantly contractible finite free \(G\)-CW-complex \(EG\). We say that \(G\) is \(D_K\)-acyclic if any such \(G\)-CW complex is \(D_K\)-acyclic. If this is the case, we define the \(D_K\)-agrarian polytope of \(G\) to be

\[
P^{D_K}(G) := P^{D_K}(EG).
\]

For future reference, we record the following direct consequence of Lemma 4.8

**Lemma 5.10.** Let \(0 \to C'_* \to C_* \to C''_* \to 0\) be a short exact sequence of finite based free \(\mathbb{Z}G\)-chain complexes such that the preferred basis of \(C_*\) is composed of the preferred basis of \(C'_*\) and preimages of the preferred basis elements of \(C''_*\). Assume that any two of the complexes are \(D_K\)-acyclic. Then so is the third and

\[
P^{D_K}(C_*) = P^{D_K}(C'_*) + P^{D_K}(C''_*).
\]
5.4. Comparison with the $L^2$-torsion polytope. In the case of $D = D(G)$, the
definition of the $D_K$-agrarian polytope differs from the construction of the $L^2$-torsion
polytope in the choice of the skew field in which the torsion is computed. Whereas
$D(G)$ naturally takes the form of an Ore localisation of a twisted group ring in
which $\mathbb{Z}K$ embeds, namely $D(K)H$, in the agrarian case we are left to use the larger
twisted group ring $D(G)H$. Fortunately, the corresponding change of skew fields does not affect the polytope:

**Theorem 5.11.** Let $G$ be a torsion-free group that satisfies the Atiyah conjecture
and denote by $H$ the free part of its abelianisation, where the abelianisation is
assumed to be finitely generated. Denote by $K$ the kernel of the projection onto $H$.
Let $X$ be an $L^2$-acyclic finite free $G$-CW complex. Then

$$P^{D(G)K}(X) = P_{L^2}(X) \in \mathcal{P}_T(H).$$

**Proof.** By Corollary 4.14, the agrarian torsion of $X$ is related to its $L^2$-torsion via

$$j_* \rho_{\text{Ore}(D(G); K \leq G)}(X) = j_*(\det(\rho^{(2)}_u(X))),$$

where $j_*$ is induced by the inclusion $j: D(G) \hookrightarrow \text{Ore}(D(G); K \leq G)$. Recall
that we defined the agrarian polytope as $P^{D(G)K}(X) = P(\rho_{\text{Ore}(D(G), K \leq G)}(X))$, where
we use that $\text{Ore}(D(G); K \leq G)$ is the Ore localisation of the twisted group ring $D(G)H$. The $L^2$-torsion polytope is defined in [FL17, Definition 4.21] as

$$P_{L^2}(X) = P(-\det(\rho^{(2)}_u(X))),$$

where for applying the polytope homomorphism Friedl and Lück rely on the fact that $D(G)$ is the Ore localisation of the twisted group ring $D(K)H$. We are thus left to check that taking the support over $D(K)H$ gives the same result as pushing forward to $D(G)H$ using $j$ and taking supports there. But $j$ restricts to an inclusion of twisted group rings and thus preserves supports.

□

5.5. Thickness of Newton polytopes. The agrarian polytope is usually rather
difficult to compute for a concrete group. Its thickness along a given line is usually
more accessible. With an approach similar to [FL16], we will see in Section 6 that it
can be computed in terms of agrarian Betti numbers of a suitably restricted chain
complex.

**Definition 5.12.** Assume that $G$ is finitely generated and denote the free part of
its abelianisation by $H$. Let $\varphi: G \to \mathbb{Z}$ be a homomorphism factoring through $H$ as $\varphi: H \to \mathbb{Z}$. Let $P \in \mathcal{P}(H)$ be a single polytope. The **thickness** of $P$ along $\varphi$ is
given by

$$\text{th}_\varphi(P) := \max \{ \varphi(x) - \varphi(y) \mid x, y \in P \} \in \mathbb{Z}_{\geq 0}. $$

Since it respects the Minkowski sum and vanishes on polytopes consisting of a
single point, the assignment $P \mapsto \text{th}_\varphi(P)$ extends to a group homomorphism

$$\text{th}_\varphi: \mathcal{P}_T(H) \to \mathbb{Z}. $$

An equivalent way to think of a twisted group ring $DH$ constructed from an
agrarian embedding $\mathbb{Z}G \to D$ in the case $H = \mathbb{Z}$ is as a twisted Laurent polynomial
ring $D[t, t^{-1}]$. In order to see the correspondence, note that since $\mathbb{Z}$ is free with
one generator, we can choose a section $s$ of the epimorphism $\varphi: G \to \mathbb{Z}$ that is
itself a homomorphism. By Lemma 2.4, the resulting twisted group ring will be
independent of the choice of the (group-theoretic or not) section. If we stipulate
that $tdt^{-1} = s(1)ds(1)^{-1}$ for $d \in D$, then the ring $D[t, t^{-1}, s, \varphi]$, with $\varphi$ added as an
index to indicate the origin of the twisting, will be canonically isomorphic to $D\mathbb{Z}$.

For elements of the Laurent polynomial ring, the Newton polytope will be a
line of length equal to the degree of the polynomial. Here, the **degree** $\deg(x)$ of a
non-trivial Laurent polynomial $x$ is the difference of the highest and lowest degree
among its monomials. In particular, the degree of a single monomial is always 0
and the degree of a polynomial with non-vanishing constant term coincides with its degree as a Laurent polynomial.

Let now $G$ be a finitely generated agrarian group with agrarian embedding $\mathbb{Z}G \to \mathcal{D}$ and denote by $K$ the kernel of the projection of $G$ onto the free part of its abelianisation, which we denote $H$. Further let $\varphi: G \to \mathbb{Z}$ be an epimorphism with kernel $K_{\varphi}$, and denote the induced map $H \to \mathbb{Z}$ by $\varphi$ with kernel $K_{\varphi}$. Recall that by Lemma 2.6, the iterated Ore field $\text{Ore}(\text{Ore}(\text{Ore}(D); K \leq K_{\varphi}); K_{\varphi} \leq G) = \text{Ore}(\text{Ore}(DK_{\varphi})\mathbb{Z})$ can be identified with the Ore field $\text{Ore}(D; K \leq G) = \text{Ore}(DH)$ via the isomorphism $\beta$. We write $\text{Ore}(DK_{\varphi})\mathbb{Z}$ as a twisted Laurent polynomial ring $\text{Ore}(DK_{\varphi})[t, t^{-1}]_{\varphi}$. The idea behind the following lemma is now based on the fact that the Newton ‘lines’ of $x$ when viewed as a single-variable Laurent polynomial with more complicated coefficients.

**Lemma 5.13.** In the situation above, for any $x \in \text{Ore}(DK_{\varphi})[t, t^{-1}]_{\varphi}$ with $x \neq 0$, we have

$$\text{th}_{\varphi}(P(\beta(x))) = \deg(x).$$

**Proof.** By extending to a common Ore denominator of all $\text{Ore}(DK_{\varphi})$-coefficients of $x$, we can restrict to the case $x \in \text{Ore}(DK_{\varphi})[t, t^{-1}]_{\varphi}$. Thus $x$ will be of the form $x = \sum_{n \in \mathbb{Z}} \sum_{k \in K_{\varphi}} u_{k,n} \ast k t^n$ with $u_{k,n} \in \mathcal{D}$. Denoting the group-theoretic section of $\varphi$ used to construct the twisted Laurent polynomial ring by $s$, we obtain:

$$\beta(x) = \sum_{n \in \mathbb{Z}} \sum_{k \in K_{\varphi}} u_{k,n} \ast k s(n).$$

The elements $ks(n)$ form a basis of the free $\mathcal{D}$-module $DH$, and thus no cancellation can occur between the individual $u_{k,n}$. By the analogous argument for the twisted group ring $DK_{\varphi}$, cancellation can also be ruled out for the sum $\sum_{k \in K_{\varphi}} u_{k,n} \ast k$ for each $n \in \mathbb{Z}$. We conclude:

$$\text{th}_{\varphi}(P(\beta(x))) = \max \{ \varphi(k_1 s(n_1)) - \varphi(k_2 s(n_2)) \mid k_1, k_2 \in K_{\varphi}, n_1, n_2 \in \mathbb{Z}, u_{k_i,n_i} \neq 0 \}$$

$$= \max \{ n_1 - n_2 \mid k_1, k_2 \in K_{\varphi}, n_1, n_2 \in \mathbb{Z}, u_{k_i,n_i} \neq 0 \}$$

$$= \max \{ n_1 - n_2 \mid \exists k_i \in K_{\varphi}: u_{k_i,n_i} \neq 0 \text{ for } i = 1, 2 \}$$

$$= \max \{ n_1 - n_2 \mid \sum_{k_i \in K_{\varphi}} u_{k_i,n_i} \ast k_i \neq 0 \text{ for } i = 1, 2 \}$$

$$= \deg(x).$$

$\square$

6. **Twisted agrarian Euler characteristic**

While the shape of the agrarian polytope introduced in the previous section is often hard to determine, there is a convenient equivalent description of its thickness along a given line. To this end, we will introduce the agrarian analogue of the twisted $L^2$-Euler characteristic introduced by Friedl and Lück in [FL16]. We assume that $G$ is a finitely generated $D$-agrarian group with a fixed agrarian embedding $i: \mathbb{Z}G \to \mathcal{D}$. We use $H$ to denote the free part of the abelianisation of $G$, and let $K$ be the kernel of the canonical projection of $G$ onto $H$.

6.1. **Definition of the twisted agrarian Euler characteristic.**

**Definition 6.1.** Let $X$ be a finite free $G$-$\mathbb{Z}$-CW-complex and let $\varphi: G \to \mathbb{Z}$ be a homomorphism. We denote by $\varphi^*\mathbb{Z}[t, t^{-1}]$ the $\mathbb{Z}$-module obtained from the $\mathbb{Z}$-module $\mathbb{Z}[t, t^{-1}]$ by letting $G$ act as $g \cdot \sum_{n \in \mathbb{Z}} \lambda_n t^n = \sum_{n \in \mathbb{Z}} \lambda_n t^{n + \varphi(g)}$, where $\lambda_n \in \mathbb{Z}$.
for \( n \in \mathbb{Z} \). Consider the \( \mathbb{Z}G \)-chain complex \( C_*(X) \otimes_{\mathbb{Z}} \varphi^* \mathbb{Z}[t, t^{-1}] \) equipped with the diagonal \( G \)-action and set

\[
\begin{align*}
&b_p^D(X; \varphi) := b_p^D(C_*(X) \otimes_{\mathbb{Z}} \varphi^* \mathbb{Z}[t, t^{-1}]) \in \mathbb{N} \cup \{\infty\}, \\
h^D(X; \varphi) := \sum_{p \geq 0} b_p^D(X; \varphi) \in \mathbb{N} \cup \{\infty\}, \\
\chi^D(X; \varphi) := \sum_{p \geq 0} (-1)^p b_p^D(X; \varphi) \in \mathbb{Z}, \text{ if } h^D(X; \varphi) < \infty.
\end{align*}
\]

We say that \( X \) is \( \varphi \)-\( D \)-finite if \( h^D(X; \varphi) < \infty \), and in this case \( \chi^D(X; \varphi) \) is called the \( \varphi \)-twisted \( D \)-agarian Euler characteristic of \( X \). More generally, we will also consider the \( \varphi \)-twisted agrarian Euler characteristic \( \chi^D(C_*; \varphi) \) for any finite free \( \mathbb{Z}G \)-chain complex \( C_* \), with \( C_* \) taking the role of the specific chain complex \( C_* \).

The aim of this section is to prove the following

**Theorem 6.2.** Write \( D_K = \text{Ore}(D; K \leq G) \). Let \( X \) be a \( D_K \)-acyclic finite free \( G \)-\( CW \)-complex and \( \varphi : G \to \mathbb{Z} \) a homomorphism. Then

\[ \text{th}_\varphi(P^{D_K}(X)) = -\chi^{D_K}(X; \varphi). \]

For universal \( L^2 \)-torsion, the analogous statement has been proved by Friedl and Lück in [FL17, Remark 3.28]. Their proof is based on the fact that universal \( L^2 \)-torsion is the universal abelian invariant of \( L^2 \)-acyclic finite based free \( \mathbb{Z}G \)-chain complexes \( C_* \) that is additive on short exact sequences and satisfies a certain normalisation condition. While large parts of the verification of this universal property are purely formal, in the proof of [FL17 Lemma 2.4] it is used that the combinatorial Laplace operator on \( C_* \) induces the \( L^2 \)-Laplace operator on \( \mathcal{N}(G) \otimes C_* \), which has no analogue over a general skew field \( D \). We instead establish Theorem 6.2 via the matrix chain approach to the computation of Reidemeister torsion, as explained in [Tur01, 2.1].

### 6.2. Reduction to ordinary Euler characteristics

Before we get to the proof, we will transfer some of the helpful lemmata in [FL16 Sections 2.2 & 3.3] to the agrarian setting.

The following lemma allows us to restrict our attention to surjective twists \( \varphi : G \to \mathbb{Z} \) in the proof of Theorem 6.2

**Lemma 6.3.** Let \( X \) be a finite free \( G \)-\( CW \)-complex and let \( \varphi : G \to \mathbb{Z} \) be a group homomorphism.

1. For any integer \( k \geq 1 \) we have that \( X \) is \( (k \cdot \varphi) \)-\( D \)-finite if and only if \( X \) is \( \varphi \)-\( D \)-finite, and if this is the case we get

\[ \chi^D(X; k \cdot \varphi) = k \cdot \chi^D(X; \varphi). \]

2. Denote the trivial homomomata \( G \to \mathbb{Z} \) by \( \epsilon_0 \). The complex \( X \) is \( \epsilon_0 \)-\( D \)-finite if and only if \( X \) is \( D \)-acyclic, and if this is the case we get

\[ \chi^D(X; \epsilon_0) = 0. \]

**Proof.**

1. This follows from the direct sum decomposition \( (k \cdot \varphi)^* \mathbb{Z}[t, t^{-1}] \cong \bigoplus_{i=0}^k \varphi^i \mathbb{Z}[t, t^{-1}] \) and additivity of Betti numbers.

2. This is a direct consequence of \( C_*(X) \otimes_{\mathbb{Z}} \epsilon_0^* \mathbb{Z}[t, t^{-1}] \cong \bigoplus_{\mathbb{Z}} C_*(X) \) and additivity of Betti numbers. \( \square \)

We will now see that twisted \( D \)-agarian Euler characteristics over \( G \) can equivalently be viewed as ordinary \( D \)-agarian Euler characteristics over the kernel of the twist homomorphism.
Lemma 6.4. Let $X$ be a finite free $G$-CW-complex and let $\varphi: G \to \mathbb{Z}$ be an epimorphism. Denote the kernel of $\varphi$ by $K_\varphi$. Then $X$ is $\varphi$-D-finite if and only if $\sum_{p \geq 0} b_p^D(\text{res}_{K_\varphi}^G X) < \infty$, and in this case we have

$$\chi^D(X; \varphi) = \chi^D(\text{res}_{K_\varphi}^G X).$$

Proof. The proof is based on the following isomorphism of $ZG$-chain complexes:

$$ZG \otimes_{ZK_\varphi} \text{res}_{K_\varphi}^G C_*(X) \cong C_*(X) \otimes_{\mathbb{Z}} \varphi^* Z[t, t^{-1}]$$

$$g \otimes x \mapsto gx \otimes t^{\varphi(g)},$$

the inverse of which is given by $y \otimes t^g \mapsto g \otimes g^{-1} y$ for any choice of $g \in \varphi^{-1}(g)$. Using the isomorphism, we obtain for any $p \geq 0$:

$$H_p(D \otimes C_*(X) \otimes_{\mathbb{Z}} \varphi^* Z[t, t^{-1}]) \cong H_p(D \otimes ZG \otimes_{ZK_\varphi} \text{res}_{K_\varphi}^G C_*(X))$$

$$= H_p(D \otimes ZK_\varphi \text{res}_{K_\varphi}^G C_*(X)).$$

We conclude that $b_p^D(X; \varphi) = b_p^D(\text{res}_{K_\varphi}^G X)$ by applying $\dim_D$, which yields the claim after taking the alternating sum over $p \geq 0$. \hfill \Box

Remark 6.5. Let $G$ be a $D$-agrarian group of type $F$. Let $\varphi: G \to \mathbb{Z}$ be an epimorphism with kernel $K_\varphi$. If $K_\varphi$ is also of type $F$, then by Lemma 6.4 and Theorem 3.8 (2)

$$\chi^D(EG; \varphi) = \chi^D(\text{res}_{K_\varphi}^G EG) = \chi^D(EK_\varphi) = \chi(K_\varphi).$$

In particular, in this case the value of $\chi^D(EG; \varphi)$ does not depend on the choice of agrarian embedding.

Lemma 6.6. Let $C_*$ be a $D$-acyclic $ZG$-chain complex of finite type. Let $\varphi: G \to \mathbb{Z}$ be an epimorphism with kernel $K_\varphi$. Consider the embedding $ZG \cong (ZK_\varphi)\mathbb{Z} \hookrightarrow D\mathbb{Z} = D[t, t^{-1}]_{K_\varphi}$ constructed in Lemma 2.4 for $N = K_\varphi$ and $Q = \mathbb{Z}$. Then

$$b_n^D(\text{res}_{K_\varphi}^G C_*) = \dim_D H_n(D[t, t^{-1}]_{K_\varphi} \otimes C_*) < \infty.$$ 

In particular, the $D[t, t^{-1}]_{K_\varphi}$-modules $H_n(D[t, t^{-1}]_{K_\varphi} \otimes C_*)$ are torsion.

Proof. The proof is analogous to that of [FL16] Theorem 3.6 (4)] with $D$ taking the role of $D(K)$. The assumption that $C_*$ be projective is in fact not used in the proof of the theorem and hence is not part of the statement of Lemma 6.6 \hfill \Box

Corollary 6.7. Let $X$ be a $D$-acyclic finite free $G$-CW-complex. Let $\varphi: G \to \mathbb{Z}$ be an epimorphism with kernel $K_\varphi$. Then $X$ is $\varphi$-D-finite and

$$\chi^D(X, \varphi) = \sum_{p \geq 0} (-1)^p \dim_D H_p(D[t, t^{-1}]_{K_\varphi} \otimes C_*(X)).$$

Proof. Apply Lemmata 6.4 and 6.6 \hfill \Box

6.3. Relation to thickness of the agrarian polytope. We now come to the proof of Theorem 6.2 which will use concepts and notation from [Tur01 Section 2.1], albeit with a shift in the grading. Assume that we are given a $D$-acyclic finite based free $ZG$-chain complex $C_*$ concentrated in degrees 0 through $m$. By fixing an ordering of the preferred basis, we identify subsets of $\{1, \ldots, \text{rk } C_p\}$ with subsets of the preferred basis elements of $C_p$. We then denote by $A_p$, for $p = 1, \ldots, m$, the matrix of the differential $c_p: C_p \to C_{p-1}$ expressed in the preferred basis. The matrix $A_p$ consists of the entries $a_{jk}^p \in ZG$, where $j = 1, \ldots, \text{rk } C_p$ and $k = 1, \ldots, \text{rk } C_{p-1}$. 
Theorem 6.9. Every $D$-acyclic finite based free $ZG$-chain complex $C_\ast$ admits a non-degenerate $\tau$-chain. For any such matrix chain $\alpha$, we have

$$\rho_D(C_\ast) = \sum_{p=1}^{m} (-1)^{p-1} \det_D(S_p(\alpha)) \in D_{ab}^\times / \{ \pm g \mid g \in G \}.$$  

Proof of Theorem 6.9. We will actually prove the more general statement that for any $D$-acyclic finite based free $ZG$-chain complex $C_\ast$ concentrated in degrees 0 through $m$

$$\theta_\varphi(P(-\rho_{D_K}(C_\ast))) = -\chi^{D_K}(C_\ast; \varphi).$$

Since $\theta_\varphi$ and $P$ are homomorphisms, we can drop the signs from both sides. Using Lemma 6.3, we can further assume that $\varphi$ is an epimorphism. By Theorem 6.9, we find a non-degenerate $\tau$-chain $\alpha$ such that

$$\theta_\varphi(P(\rho_{D_K}(C_\ast))) = \theta_\varphi \left( \left. P \left( \sum_{p=1}^{m} (-1)^{p-1} \det_D(S_p(\alpha)) \right) \right|_{\varphi} \right).$$

Crucially,

$$\text{Ore}(\text{Ore}(D; K \leq K_\varphi)[t, t^{-1}]_\varphi) \cong \text{Ore}(DH) = D_K$$

via the isomorphism $\beta$ constructed in Lemma 2.6. The subring

$$\text{Ore}(D; K \leq K_\varphi)[t, t^{-1}]_\varphi$$

of the left-hand side, which contains $\beta^{-1}(ZG)$ and thus all entries of $S_p = S_p(\alpha)$, is a (non-commutative) Euclidean domain. This means that we can diagonalise the matrices $S_p$ by multiplying them from the left and right with permutation matrices and elementary matrices over this twisted Laurent polynomial ring. This diagonalisation procedure occurs as part of an algorithm that brings a matrix into Jacobson normal form, which is a non-commutative analogue of the better-known Smith normal form for matrices over commutative PIDs. For details, we refer to the proof of [Jac13] Theorem 3.10]. Recall that a permutation matrix is a matrix obtained from an identity matrix by permuting rows and columns. An elementary matrix over a ring $R$ is a matrix differing from the identity matrix in a single off-diagonal entry. The determinant of either type of matrix is 1 or $-1$, and thus the thickness in direction of $\varphi$ of its polytope vanishes. Hence, $\theta_\varphi(P(\det(S_p))) = \theta_\varphi(P(\det(T_p)))$ for the diagonal matrix $T_p$ obtained from $S_p$ in this way. We denote the diagonal entries of $T_p$ by $d_{p,i} \in \text{Ore}(D; K \leq K_\varphi)[t, t^{-1}]_\varphi$ for $i = 1, \ldots, \alpha_p$ and note that all the $d_{p,i}$ are non-zero since the $S_p$ are invertible over $D_K$. Using that both $\theta_\varphi$ and
\[ P \text{ are homomorphisms, and applying Lemma } 5.13 \text{ once more, we compute:} \]
\[
\text{th}_{\varphi}(P(\rho D_K(C_s))) = \text{th}_{\varphi}\left( P \left( \sum_{p=1}^{m} (-1)^p \det D_K(S_p(\alpha)) \right) \right) \\
= \sum_{p=1}^{m} (-1)^{p-1} \sum_{i=1}^{\alpha_p} \text{th}_{\varphi}(P(\beta(d_{p,i}))) \\
= \sum_{p=1}^{m} (-1)^{p-1} \sum_{i=1}^{\alpha_p} \deg(d_{p,i}).
\]

Now considering the right-hand side, we make use of the fact that the agrarian embedding \( \mathbb{Z}K_\varphi \hookrightarrow D_K = \text{Ore}(D; K \leq G) \) factors through the smaller skew field \( \text{Ore}(D; K \leq K_\varphi) \), and thus \( \mathbb{Z}G \cong (\mathbb{Z}K_\varphi)\mathbb{Z} \hookrightarrow D_K[t, t^{-1}]_\varphi \) factors through \( \text{Ore}(D; K \leq K_\varphi)[t, t^{-1}]_\varphi \). Since \( D_K \) is flat over \( \text{Ore}(D; K \leq K_\varphi) \), we conclude from Corollary 6.7 that
\[
\chi(D_K(C_s; \varphi)) = \sum_{p=0}^{m} (-1)^p \dim_{D_K} H_p(D_K[t, t^{-1}]_\varphi \otimes C_s) \\
= \sum_{p=0}^{m} (-1)^p \dim_{\text{Ore}(D; K \leq K_\varphi)} H_p(\text{Ore}(D; K \leq K_\varphi)[t, t^{-1}]_\varphi \otimes C_s).
\]
Since \( C_s \) is \( D_K \)-acyclic, we have \( H_m(D_K \otimes C_s) = 0 \). But \( C_{m+1} \) is trivial and hence the differential \( c_m \) must be injective. In particular, the summand corresponding to \( p = m \) vanishes. Comparing this sum with the sum expression for the left-hand side of the equality we are trying to prove with index shifted by 1, we are thus done if we can show that for all \( p = 0, \ldots, m-1 \) we have
\[
(3) \sum_{i=1}^{\alpha_p} \deg(d_{p+1,i}) = \dim_{\text{Ore}(D; K \leq K_\varphi)} H_p(\text{Ore}(D; K \leq K_\varphi)[t, t^{-1}]_\varphi \otimes C_s).
\]

In order not to overload notation, we abbreviate \( \text{Ore}(D; K \leq K_\varphi)[t, t^{-1}]_\varphi \) as \( R \).

Recall that the homology modules \( H_p(R \otimes C_s) \) are \( R \)-torsion by Lemma 6.6. Since \( R \otimes C_{p-1} \) is a free \( R \)-module and hence any torsion element maps into it trivially, we may express the homology modules as the torsion of a cokernel:
\[
H_p(R \otimes C_s) = \ker(id_R \otimes c_p) / \im(id_R \otimes c_{p+1}) \\
\cong \ker (id_R \otimes c_p; (R \otimes C_p) / \im(id_R \otimes c_{p+1}) \rightarrow R \otimes C_{p-1}) \\
= \text{tors}((R \otimes C_p) / \im(id_R \otimes c_{p+1})) \\
= \text{tors}(\text{coker}(id_R \otimes c_{p+1})).
\]

Instead of performing elementary operations on the matrix \( S_{p+1} \) to obtain the diagonal matrix \( T_{p+1} \), we can instead apply them to the entire matrix \( A_{p+1} \) representing \( id_R \otimes c_{p+1} \). This procedure, will not change the isomorphism type of the cokernel of the map given by right multiplication with this matrix. Applying further elementary operations over \( R \), we can achieve that all rows and columns of \( A_{p+1} \) not contained in \( S_{p+1} \) consist only of zeros, and the submatrix \( S_{p+1} \) is now of the form \( T_{p+1} \). Hence \( H_p(R \otimes C_s) \cong \text{tors}(\text{coker}(id_R \otimes c_{p+1})) \cong \otimes_{i=1}^{\alpha_{p+1}} R/(d_{p+1,i}) \), which yields Eq. (3) after applying \( \text{dim}_{\text{Ore}(D; K \leq K_\varphi)} \).
\]

\textbf{Corollary 6.10.} Let \( \varphi: G \rightarrow \mathbb{Z} \) be an epimorphism with kernel \( K_\varphi \leq G \). Assume that \( K_\varphi \) is of type \( \mathbb{F} \). Then \( G \) is of type \( \mathbb{F} \), acyclic with respect to any agrarian embedding, and
\[
\text{th}_{\varphi}(P^{D_K}(G)) = \chi(K_\varphi).
\]
Proof. Every epimorphism onto \( \mathbb{Z} \) splits, hence \( G \) actually decomposes as a semidirect product \( K_\varphi \rtimes \mathbb{Z} \) with associated section \( s: \mathbb{Z} \to G \). Here, \( n \in \mathbb{Z} \) acts on \( K_\varphi \) via conjugation by \( s(n) \). Since \( K_\varphi \) is of type \( F \), we can find a finite \( G \)-CW-model for \( EK_\varphi \) on which conjugation by \( s(1) \) is realised topologically by a \( K_\varphi \)-equivariant homeomorphism \( f: EK_\varphi \to EK_\varphi \). The universal covering of the mapping torus \( T_f \) is then a finite \( G \)-CW-model for \( EG \) and acyclic with respect to any agrarian embedding by Theorem 3.10. Now apply Theorem 6.2 and Remark 6.5 to this model for \( EG \). \( \square \)

7. Applications

7.1. The Bieri–Neumann–Strebel invariants and HNN extensions. In order to discuss some application of the theory of agrarian invariants, we need to first cover the BNS invariants and the HNN extensions.

Definition 7.1. Let \( G \) be a group generated by a finite subset \( S \), and let \( X \) denote the Cayley graph of \( G \) with respect to \( S \). Recall that the vertex set of \( X \) coincides with \( G \). We define the Bieri–Neumann–Strebel (or BNS) invariant \( \Sigma^1(G) \) to be the subset of \( H^1(G; \mathbb{R}) \setminus \{0\} \) consisting of the non-trivial homomorphisms \( \varphi \) of the characters \( \varphi: G \to \mathbb{R} \) for which the full subgraph of \( X \) spanned by \( \varphi^{-1}(\{0, \infty\}) \subset G \) is connected.

The BNS invariants were introduced by Bieri, Neumann and Strebel in [BNS87] via a different, but equivalent definition. It is an easy exercise to see that \( \Sigma^1(G) \) is independent of the choice of the finite generating set \( S \).

We now aim to give an interpretation of lying in the BNS invariant for integral characters \( \varphi: G \to \mathbb{Z} \). To do so, we need to introduce the notion of HNN extensions.

Definition 7.2. Let \( A \) be a group and let \( \alpha: B \to C \) be an isomorphism between two subgroups of \( A \). Choose a presentation \( \langle S \mid R \rangle \) of \( A \) and let \( t \) be a new symbol not in \( S \). Then the group \( A*_{\alpha} \) defined by the presentation
\[
\langle S, t \mid R, tbt^{-1} = \alpha(b) \ \forall b \in B \rangle
\]
is called the HNN extension of \( A \) relative to \( \alpha \). We call \( A \) the base group and \( B \) the associated group of the HNN extension.

The HNN extension is called ascending if \( B = A \).

The homomorphism \( \varphi: A*_{\alpha} \to \mathbb{Z} \) given by \( \varphi(t) = 1 \) and \( \varphi(s) = 0 \) for every \( s \in S \) is the induced character.

Proposition 7.3 ([BNS87 Proposition 4.3]). Let \( G \) be a finitely generated group, and let \( \varphi: G \to \mathbb{Z} \) be a non-trivial character. We have \( \varphi \in \Sigma^1(G) \) if and only if \( G \) is isomorphic to an ascending HNN extension with finitely generated base group and induced character \( \varphi \).

Definition 7.4. Suppose that \( G \) is finitely generated. Let \( P \) be a polytope in the \( \mathbb{R} \)-vector space \( H_1(G; \mathbb{R}) \), and let \( F \) be a face of \( P \). A dual of \( F \) is a connected component of the subspace
\[
\{ \varphi \in H^1(G; \mathbb{R}) \mid F_\varphi(P) = F \}
\]

A marked polytope is a pair \( (P, m) \), where \( P \) is a polytope in \( H_1(G; \mathbb{R}) \), and \( m \) is a marking, that is a function \( m: H^1(G; \mathbb{R}) \to \{0, 1\} \), which is constant on duals of faces of \( F \), and such that \( m^{-1}(1) \) is open.

The pair \( (P, m) \) is a polytope with marked vertices if \( m^{-1}(1) \) is a union of some duals of vertices of \( P \).

The marking \( m \) will usually be implicit, and the characters \( \varphi \) with \( m(\varphi) = 1 \) will be called marked.
Friedl–Tillmann use a different notion of a marking of a polytope, which corresponds to a polytope with marked vertices in our terminology where the marking \( m \) is additionally required to be constant on all duals of a given vertex. Thus, our notion is more general, and the two notions differ when the polytope in question is a singleton in a 1-dimensional ambient space: with our definition of marking, such a polytope admits four distinct markings (just as every compact interval of non-zero length does), whereas with the Friedl–Tillmann definition such a polytope admits only two markings in which either every character is marked or none is.

7.2. Two-generator one-relator groups. The story of the usefulness of agrarian invariants for two-generator one-relator groups begins with the following

**Theorem 7.5** ([LL78, Theorem 1]). *Torsion-free one-relator groups are agrarian.*

In order to describe the cellular chain complex of the universal coverings of classifying spaces for two-generator one-relator groups, we will use Fox derivatives, which were originally defined in [Fox53]. Let \( F \) be a free group on generators \( x_i, i \in I \). The Fox derivative with respect to \( x_i \) is then defined to be the unique \( Z \)-linear map \( \frac{\partial}{\partial x_i} : ZF \to ZF \) satisfying the conditions

\[
\frac{\partial 1}{\partial x_i} = 0, \quad \frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad \text{and} \quad \frac{\partial uv}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}
\]

for all \( u, w \in F \).

**Proposition 7.6.** Let \( G = \langle x, y \mid r \rangle \) be a two-generator one-relator group that is not the free group on two generators. Then \( G \) is \( D \)-acyclic with respect to any agrarian embedding \( ZG \hookrightarrow D \).

**Proof.** A CW-model for the classifying space \( BG \) is given by the 2-complex corresponding to the given presentation of \( G \), see [LS01, Chapter III, Proposition 11.1]. The \( ZG \)-chain complex of the universal covering \( \tilde{E}G \) of \( BG \) then takes the following form in terms of the Fox derivatives \( \frac{\partial r}{\partial x} \) and \( \frac{\partial r}{\partial y} \), see [Fox53]:

\[
ZG \xrightarrow{\left( \frac{\partial r}{\partial x} \quad \frac{\partial r}{\partial y} \right)} ZG^2 \xrightarrow{(x-1) \quad (y-1)} ZG.
\]

If we pass to the associated \( D \)-chain complex, we note that the second differential becomes surjective since \( x - 1 \neq 0 \) and the first differential becomes injective unless \( \frac{\partial r}{\partial x} = \frac{\partial r}{\partial y} = 0 \). But, by the fundamental formula of Fox calculus [Fox53, (2.3)], the identity \( s - 1 = \frac{\partial r}{\partial x} (x - 1) + \frac{\partial r}{\partial y} (y - 1) \) holds for every \( s \in ZG \) and in particular for \( r \). Since \( G \) is not a free group by assumption and hence \( r \) is not trivial, we conclude that at least one of its two Fox derivatives is non-zero and hence both differentials have maximal rank. It follows that \( b_p^D(G) = 0 \) for \( p \neq 1 \) and thus also \( b_1^D(G) = \chi(G) = 0 \) by Theorem 3.8 (2) \( \square \)

**Lemma 7.7.** Let \( G = \langle x, y \mid r \rangle \) be a two-generator one-relator group that is not the free group on two generators and choose some \( D \)-agrarian embedding of \( G \). Denote the universal covering of the presentation 2-complex of \( G \) associated to this presentation by \( \tilde{E}G \). If \( y \) is non-trivial in \( G \), then

\[
\rho_D(\tilde{E}G) = -\left[ \frac{\partial r}{\partial x} \right] + [y-1] \in D^\times_{ab},
\]

where \([-] : D^\times \to D^\times_{ab} \) is the canonical quotient map. If \( y = 0 \), then the analogous expression holds with \( x \) and \( y \) interchanged.
Proof. We again choose the presentation 2-complex of $G$ as a model for $BG$ and obtain the following commutative diagram with exact columns:

\[
\begin{array}{cccc}
ZG & \xrightarrow{\partial x} & ZG & \\
\downarrow & & \downarrow & \\
ZG & \xrightarrow{\partial y} & ZG^2 & \xrightarrow{(x^{-1}, y^{-1})} ZG \\
\downarrow & & \downarrow & \\
ZG & \xrightarrow{y^{-1}} & ZG & \\
\end{array}
\]

The middle row is $C_*(EG)$ and the diagram represents a short exact sequence of finite based free $ZG$-chain complexes with preferred bases induced from $C_*(EG)$. The middle row is $D$-acyclic by Proposition 7.6 and the lower row is $D$-acyclic since $y$ is assumed to be non-trivial in $G$. By Lemma 4.8, the upper row is also $D$-acyclic and we conclude that $\rho_D(C_*(EG))$ is the sum of the agrarian torsion of the upper and the lower row. Taking into account that the upper row is shifted, we obtain the claimed expression for $\rho_D(EG)$. \qed

Definition 7.8. A $(2, 1)$-presentation $\langle x, y \mid r \rangle$ giving rise to a group $G$ is called somewhat nice if

(1) $r$ is a non-empty word,
(2) all cyclic permutations of $r$ are reduced.

It is called nice if in addition $b_1(G) = 2$.

The following definition is motivated by [FT15, Proposition 3.5] and in the case of a nice $(2, 1)$-presentation is an alternative construction of the polytope defined by Friedl and Tillmann.

Definition 7.9. Let $\pi = \langle x, y \mid r \rangle$ be a somewhat nice $(2, 1)$-presentation giving rise to a group $G$. Denote by $H$ the free part of the abelianisation of $G$. Then we set

$$\mathcal{P}_\pi := P\left(\frac{\partial r}{\partial x}\right) - P(y - 1) \in \mathcal{P}_T(H)$$

if $y$ represents a non-trivial element in $G$, otherwise we make the same definition with $x$ and $y$ interchanged.

It is shown in [FT15, Proposition 3.5] that the element $\mathcal{P}_\pi \in \mathcal{P}_T(H)$ defined in this way is indeed represented by a single polytope. This can alternatively be seen from [Fun17, Lemma 4.3]. In [FT15], Friedl and Tillmann also endow $\mathcal{P}_\pi$ with a marking of vertices, turning it into a marked polytope $\mathcal{M}_\pi$. A vertex of $\mathcal{P}_\pi$ is declared marked if any of its duals contains a character lying in $\Sigma^1(G)$. Friedl–Tillmann prove in [FT15, Theorem 1.1] that if $\pi$ is nice, then every character lying in any dual of a marked vertex lies in $\Sigma^1(G)$, and hence the markings of $\mathcal{P}_\pi$ and $\Sigma^1(G)$ determine one another.

Conjecture 7.10 ([FT15, Conjecture 1.2]). If $G$ is a group admitting a nice $(2, 1)$-presentation $\pi$, then $\mathcal{M}_\pi \subset H_1(G; \mathbb{R})$ is an invariant of $G$ (up to translation).

As evidence for this conjecture, Friedl and Tillmann prove

Theorem 7.11 ([FT15, Theorem 1.3]). If $G$ is a group admitting a nice $(2, 1)$-presentation $\pi$ and $G$ is residually {torsion-free elementary amenable}, then $\mathcal{M}_\pi \subset H_1(G; \mathbb{R})$ is an invariant of $G$ (up to translation).
Making use of their construction of universal $L^2$-torsion, Friedl and Lück resolved this conjecture and provided a construction of $\mathcal{M}_\pi$ intrinsic to the group $G$ under the additional assumption that $G$ is torsion-free and satisfies the Atiyah conjecture:

**Theorem 7.12** ([FL17, Remark 5.5]). If $G$ is a torsion-free group admitting a nice $(2,1)$-presentation $\pi$ and $G$ satisfies the Atiyah conjecture, then $\mathcal{M}_\pi \subset H_1(G;\mathbb{R})$ is an invariant of $G$ (up to translation). Moreover, $\mathcal{P}_\pi = P_{L^2}(G)$.

By using agrarian torsion instead of universal $L^2$-torsion, we are able to remove the additional assumptions on $G$:

**Theorem 7.13.** If $G$ is a torsion-free group admitting a nice $(2,1)$-presentation $\pi$, then $\mathcal{M}_\pi \subset H_1(G;\mathbb{R})$ is an invariant of $G$ (up to translation). Moreover, $\mathcal{P}_\pi = P^{D\kappa}(G) \in \mathcal{P}_T(\mathbb{Z}^2)$.

**Proof.** First note that $P^{D\kappa}(G)$ is defined in terms of the $G$-space $EG$, which is unique up to $G$-homotopy equivalence, and $P^{D\kappa}(-)$ is $G$-homotopy invariant by Proposition 5.8. Hence $P^{D\kappa}(G)$ is well-defined and does not depend on the particular choice of $\pi$. Now the equality $\mathcal{P}_\pi = P^{D\kappa}(G)$ follows directly from the definitions of $\mathcal{P}_\pi$ and $P^{D\kappa}(G)$ by the computation done in Lemma 7.7.

We conclude from [FT15, Theorem 1.1] that once $\mathcal{P}_\pi$ is known to be an invariant of $G$, the same is true for the marked version $\mathcal{M}_\pi$ since marked vertices are determined by the BNS invariant $\Sigma^1(G)$ of the group $G$.

Friedl and Tillmann remark in [FT15, Section 8] that they can define $\mathcal{P}_\pi$ even when $\pi$ is only assumed to be somewhat nice. They also produce a marked polytope $\mathcal{M}_\pi$ for a somewhat nice presentation $\pi$ under the additional assumption that the associated group is not isomorphic to a Baumslag-Solitar group

$$B(\pm 1, n) := \langle x, y \mid xy^\pm 1x^{-1} = y^n \rangle.$$  

The problem with the Baumslag–Solitar groups $B(\pm 1, n)$ is that the resulting polytope is a singleton lying in a 1-dimensional $\mathbb{R}$-vector space. Since $\Sigma^1(B(\pm 1, n))$ is non-trivial, there is no marking of $\mathcal{P}_\pi$ in the Friedl–Tillmann sense which would correctly control the BNS invariant. Our notion of marking of vertices of a polytope circumvents this problem, and allows for a definition of $\mathcal{M}_\pi$ also for Baumslag–Solitar groups.

It follows directly from the proof of [FT15, Proposition 8.1] that $\mathcal{P}_\pi = P^{D\kappa}(G)$ continues to hold in the more general setting of somewhat nice presentations, and that $\mathcal{M}_\pi$ is in fact an invariant of the group. Hence, Theorem 7.13 extends to all somewhat nice $(2,1)$-presentations (with our more general definition of a marking being used).

**Theorem 7.14.** If $G$ is a torsion-free group admitting a somewhat nice $(2,1)$-presentation $\pi$, then $\mathcal{M}_\pi$ is an invariant of $G$ (up to translation).

**Remark 7.15.** Since the proof of Theorem 7.13 shows that $\mathcal{P}_\pi = P^{D\kappa}(G)$ holds for any $D$-agrarian embedding of the two-generator one-relator group $G$, we conclude that $P^{D\kappa}(G)$ is actually independent of the choice of agrarian embedding.

The diameter of $\mathcal{P}_\pi$ controls the minimal complexity of certain expressions of $G$ as an HNN extension over a finitely generated group. Before we state the precise connection, we need to introduce the following concept:

**Definition 7.16** ([FLT16, Section 5.1]). Let $\Gamma$ be a finitely presented group and let $\varphi : \Gamma \to \mathbb{Z}$ be an epimorphism. A *splitting* of $(\Gamma, \varphi)$ is a presentation of $\Gamma$ as an HNN extension with induced character $\varphi$ and finitely generated base and associated groups.
It is proved in [BS78, Theorem A] that any pair \((\Gamma, \varphi)\) admits a splitting. Hence we can define the splitting complexity of \((\Gamma, \varphi)\) as
\[
c(\Gamma, \varphi) := \min \{rk(B) \mid (\Gamma, \varphi) \text{ splits with associated group } B\},
\]
where \(rk(B)\) denotes the minimal number of generators of \(B\). We also define the free splitting complexity of \((\Gamma, \varphi)\) as
\[
c_f(\Gamma, \varphi) := \min \{rk(F) \mid (\Gamma, \varphi) \text{ splits with associated free group } F\},
\]
which may be infinite. We always have \(c(\Gamma, \varphi) \leq c_f(\Gamma, \varphi)\).

Friedl and Tillmann observed the following connection between the thickness of of \(\mathcal{P}_\pi\) and the (free) splitting complexity of \(G\):

**Theorem 7.17 ([FT15, Theorem 7.2]).** Let \(G\) be a residually \{torsion-free elementary amenable\} group admitting a nice \((2,1)\)-presentation \(\pi\). Then for any epimorphism \(\varphi : G \to \mathbb{Z}\) we have
\[
c(G, \varphi) = c_f(G, \varphi) = \text{th}_\varphi(\mathcal{P}_\pi) + 1.
\]

Note that every residually \{torsion-free elementary amenable\} group must itself be torsion-free. Friedl, Lück, and Tillmann then noted in [FLT16, Theorem 5.2] that the original proof could be adapted to the setting of [FL16], thereby giving the same formula for groups satisfying the Atiyah conjecture.

We will now present a common generalisation of these results. For this, we require the following strengthened form of a proposition of Harvey, which is evident from the last sentence of its original proof:

**Proposition 7.18 ([Har05, Proposition 9.1]).** Let \(D\) be a skew field and \(D[t, t^{-1}]\) a twisted Laurent polynomial ring with coefficients in \(D\). Let \(M = A + tB\) for two matrices \(A\) and \(B\) over \(D\) of shape \(l \times m\). Then the map \(r_M : D[t, t^{-1}]^l \to D[t, t^{-1}]^m\) given by right multiplication by \(M\) satisfies
\[
\dim_D \text{tors}(\text{coker}(r_M)) \leq rk_D B.
\]

We are now in a position to improve upon both [FT15, Theorem 7.2] and [FLT16, Theorem 5.2] by recasting the proof of [FT15, Theorem 7.2] in the agrarian world.

**Theorem 7.19.** Let \(G\) be a torsion-free group admitting a nice \((2,1)\)-presentation \(\pi\). Then for any epimorphism \(\varphi : G \to \mathbb{Z}\) we have
\[
c(G, \varphi) = c_f(G, \varphi) = \text{th}_\varphi(\mathcal{P}_\pi) + 1.
\]

**Proof.** The inequality \(c_f(G, \varphi) \leq \text{th}_\varphi(\mathcal{P}_\pi) + 1\) is proved for all groups admitting a nice \((2,1)\)-presentation in [FT15, Proposition 7.3]. Let \(K\) be the kernel of the projection of \(G\) onto the free part of its abelianisation. Since \(P^{DK}(G)\) agrees with \(\mathcal{P}_\pi\) by Theorem 7.13 we are left with showing that \(c(G, \varphi) \geq \text{th}_\varphi(\mathcal{P}^{DK}(G)) + 1\), which by Theorem 6.2 is further reduced to the following statement about the \(\varphi\)-twisted \(D_K\)-agrarian Euler characteristic of \(G\):
\[
c(G, \varphi) - 1 \geq -\chi^{DK}(G; \varphi).
\]

Recall from the proof of Proposition 7.6 that the Cayley 2-complex \(X\) of \(G\) serves as a model of \(EG\). By Lemmata 6.4 and 6.6 we can thus compute \(\chi^{DK}(G; \varphi)\) from the Betti numbers of the complex \(D_K[t, t^{-1}]_\varphi \otimes C_*(X)\):
\[
D_K[t, t^{-1}]_\varphi \xrightarrow{\partial Y} D_K[t, t^{-1}]_\varphi \xrightarrow{\partial X} D_K[t, t^{-1}]_\varphi.
\]
Since \(D_K[t, t^{-1}]_\varphi\) is a (non-commutative) principal ideal domain, the kernel of the differential originating from degree 2 is free. It is also seen to be torsion by Lemma 6.6 and hence \(\dim_{D_K} H_p(D_K[t, t^{-1}]_\varphi \otimes C_*(X)) = 0\) for \(p \geq 2\).
We let \( c = c(G, \varphi) \) and choose a splitting
\[
G = (A, t \mid \mu(B) = t B t^{-1})
\]
of \((G, \varphi)\) with associated group \( B \) generated by \( x_1, \ldots, x_c \); in particular \( A \subseteq \ker(\varphi) \) is finitely generated. We pick a presentation \( A = \langle g_1, \ldots, g_k \mid r_1, r_2, \ldots \rangle \), which is possible since \( G \) and thus \( A \) are countable. Denote the number of relations in this presentation by \( \ell \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \). The splitting of \((G, \varphi)\) then gives the following alternative presentation of \( G \):
\[
G = \langle g_1, \ldots, g_k, t \mid r_1, r_2, \ldots, \mu(x_1)^{-1} t x_1 t^{-1}, \ldots, \mu(x_c)^{-1} t x_c t^{-1} \rangle.
\]
Note that the \( r_i, x_j \) and \( \mu(x_j) \) can all be expressed as words in the generators of \( G \). Denote by \( Y \) the Cayley 2-complex associated to this presentation. By construction \( \pi_1(Y/G) = \pi_1(X/G) \), and thus \( Y \) can be turned into a model for \( EG \) by attaching \( G \)-cells in dimension 3 and higher only. Hence, its homology with arbitrary coefficients agrees with that of \( X \) up to dimension 1, which in particular implies that \( \dim_D \mathcal{H}_p(D_K[t, t^{-1}]_\varphi \otimes C_*(X)) = \dim_D \mathcal{H}_p(D_K[t, t^{-1}]_\varphi \otimes C_*(Y)) \) for \( p = 0, 1 \).

In conclusion, we will know \( \chi^D_K(G; \varphi) \) if we compute the first two \( D_K[t, t^{-1}]_\varphi \)-Betti numbers of the \( G \)-CW-complex \( Y \). For this, we need to consider its shape in more detail. The complex \( Y \) is a two-dimensional free \( G \)-CW-complex with one zero-cell, \( k + 1 \) one-cells and \( l + c \) two-cells, and its cellular chain complex takes the form
\[
\cdots \to 0 \to \mathbb{Z} G^{l+c} \xrightarrow{(M_0 \quad M_1)} \mathbb{Z} G \oplus \mathbb{Z} G^k \xrightarrow{v_0 \quad v_1} \mathbb{Z} G,
\]
where the (potentially infinite) block matrix \((M_0 \quad M_1)\) representing the second differential consists of the Fox derivatives of the relations with respect to \( t \) and the \( g_i \), respectively, and \( v_0 = t - 1, v_1 = (g_1 - 1, \ldots, g_k - 1)^t \). Since the relations \( r_1, r_2, \ldots \) are words in \( Z A \), their Fox derivatives with respect to \( t \) are trivial and their derivatives with respect to the \( g_i \) again lie in \( Z A \). For the other relations, we obtain
\[
\frac{\partial}{\partial t} (\mu(x_j)^{-1} t x_j t^{-1}) = \mu(x_j)^{-1} - \mu(x_j)^{-1} t x_j t^{-1} \in Z A \quad \text{and} \quad \frac{\partial}{\partial g_i} (\mu(x_j)^{-1} t x_j t^{-1}) = \frac{\partial}{\partial g_i}(\mu(x_j)^{-1}) + \mu(x_j)^{-1} t \frac{\partial}{\partial g_i} x_j \in Z A + t \cdot Z A.
\]
Hence, the matrix \( M \) is of the shape
\[
\begin{bmatrix}
0 & \cdots & \cdots \\
0 & \in Z A \\
\ell & \in Z A \\
\ell & \in Z A + t \cdot Z A \\
c & \in Z A \\
\end{bmatrix}
\]
with the block \( M_0 \) consisting of the first column of \( M \). Now consider the following chain map of \( D_K[t, t^{-1}]_\varphi \)-chain complexes, where the vertical maps are given by projections and both complexes continue trivially to the left and right:
Example 7.20. For words $x,y \in \langle a,b \rangle$, we define $x^y := y^{-1}xy$ and $[x,y] := x^{-1}y^{-1}xy$.

Consider the two-generator one-relator group $G$ defined by

$$
\left\langle a,b \mid [a,b] = [a,b], [a,b]^3 \right\rangle,
$$

which can be presented in cyclically reduced form as

$$
\pi := \left\langle a,b \mid a^{-1}bab^{-1}a^{-1}bab^{-2}a^{-1}baba^{-1}b^{-2}ab \right\rangle.
$$

We see directly from the first presentation of $G$ that the relator becomes trivial in the abelianisation, hence $b_1(G) = 2$ and $\pi$ is a nice $(2,1)$-presentation. By [LS01 Proposition II.5.18], the group $G$ is also torsion-free since the single relator is not a proper power.

We claim that $G$ is not residually solvable, i.e., not every element maps non-trivially into a solvable quotient of $G$. Since the element $[a,b]$ can be written as an arbitrarily deeply nested iterated commutator (using the relation of the first presentation above), it is contained in all derived subgroups of $G$ and hence of every quotient. But if a quotient is solvable, some derived subgroup and hence the image of $[a,b]$ will be trivial. It is thus left to show that $[a,b]$ is non-trivial in $G$. Assume that $[a,b] = 1$ in $G$. Then $G$ is abelian and hence also $[b,a] = b^{-1}a^{-1}ba = 1$ in $G$. But $[b,a]$ appears as a proper subword of the relator in $\pi$ and thus represents a non-trivial element by [LS01 Proposition II.5.29].

We conclude that a method such as the one employed in [FT13 Lemma 6.1] can not be used to deduce that $G$ is residually {torsion-free elementary amenable} and hence satisfies the assumptions of Theorem [7.11]. We deem it plausible that $G$ is even not residually {torsion-free elementary amenable} and is thus not covered by Theorem [7.11] but to the best of the authors’ knowledge no two-generator one-relator group has been shown to have this property.
If we denote the single relator of \( \pi \) by \( r \), an easy but tedious computation shows that

\[
\frac{\partial r}{\partial a} = -b^{-1}a^{-1} + b^{-1}a^{-1}b - b^{-1}a^{-1}bab^{-1}a^{-1} + b^{-1}a^{-1}bab^{-1}a^{-1}b \\
- (b^{-1}a^{-1}ba)^2b^{-2}a^{-1} + (b^{-1}a^{-1}ba)^2b^{-2}a^{-1}b - (b^{-1}a^{-1}ba)^2b^{-2}a^{-1}baba^{-1} \\
+ (b^{-1}a^{-1}ba)^2b^{-2}a^{-1}baba^{-1}b^{-2},
\]

with the image in the abelianisation of each summand noted in brackets. The convex hull of these points in \( \mathbb{R}^2 \) corresponds to an interval of length 2 in the \( b \)-direction, hence \( \mathcal{P}_\pi = P^{D_2}(G) \) is an interval of length 1 in the \( b \)-direction. The marked polytope \( \mathcal{M}_\pi \) has no markings since all abelianised monomials appear multiple times.

Let \( \varphi_b: G \to \mathbb{Z} \) be the homomorphism sending \( a \) to 0 and \( b \) to 1. Since \( \text{th}_{\varphi_b}(P^{D_2}(G)) = 1 \), we conclude from Theorem 7.19 that \( c_f(G, \varphi_b) = c(G, \varphi_b) = 2 \). A (free) splitting of \( G \) along \( \varphi_b \) of minimal rank is thus given by

\[
G = \langle a, b, x, y \mid x = [x, y], y = x^b, x = [a, b] \rangle \\
= \langle a, x, y, b \mid x = [x, y], y = x^b, ax = a^b \rangle.
\]

Note that our example is a nice version of the original example of a two-generator one-relator group which is not residually finite produced by Baumslag in [Bau69]. He considers the \((2,1)\)-presentation \( \langle a, b \mid a = [a, a^b] \rangle \), which is only somewhat nice.

7.3. **Deficiency 1 groups.** In this section we will see how agrarian polytopes connect with the theory of BNS invariants of general deficiency 1 groups.

**Definition 7.21.** Let \( G \) be a finitely presented group. The **deficiency** of \( G \) is the minimum taken over all finite presentations of \( G \) of the difference between the number of generators and the number of relators.

**Theorem 7.22 (BNS87, Theorem 7.2).** If \( G \) is a finitely presented group of deficiency at least 2, then \( \Sigma^1(G) = \emptyset \).

Because of the above result, we will focus on groups of deficiency 1. This is a very rich family of groups, containing all ascending HNN extensions of finitely generated free groups, and almost all two-generator one-relator groups.

The structure of \( \Sigma^1(G) \) for agrarian groups of deficiency 1 was studied by the second-named author in [Kie18, Section 5.6]. There, stronger results were obtained for deficiency 1 groups when they were assumed to satisfy the Atiyah conjecture. The reason for this was that there was no theory of agrarian polytopes available, in contrast to the theory of \( L^2 \)-torsion polytopes. Since we have developed the missing theory here, we can now strengthen the results and prove the following.

**Theorem 7.23.** Let \( G \) be a D-agrarian group of deficiency 1, and denote by \( K \) the kernel of the projection onto the free part of its abelianisation. There exists a marking of the vertices of the agrarian polytope \( P^{D_2}(G) \) such that for every \( \varphi \in H^1(G; \mathbb{R}) \setminus \{0\} \) we have \( \varphi \in \Sigma^1(G) \) if and only if \( \varphi \) is marked.

Before proceeding to the proof, let us first contrast the statement with [Kie18, Theorem 5.23]: here, we prove that \( \Sigma^1(G) \) coincides with the marked characters coming from a polytope with marked vertices, whereas in [Kie18] the marking came from a marked polytope, and hence was (a priori) not as rigid as we now show it.
to be. In particular, we now show that if for some character \( \varphi : G \to \mathbb{R} \), we can connect \( \varphi \) to \( -\varphi \) using a path lying in \( \Sigma^1(G) \), then in fact \( \Sigma^1(G) = H^1(G; \mathbb{R}) \setminus \{0\} \).

**Sketch proof of Theorem 7.23.** The proof follows that of [Kie18, Theorem 5.25] very closely, and therefore we will give here merely a sketch.

We take \( X \) to be a classifying space for \( G \) extending the presentation complex coming from a presentation realising the deficiency of \( G \) (which is equal to 1). We let \( C_* \) denote the cellular chain complex of the universal covering of \( X \); by choosing a cellular bases of \( C_* \) we will treat \( C_* \) as a chain complex of free finitely generated \( \mathbb{Z}G \)-modules.

If \( \Sigma^1(G) = \emptyset \) then the result follows by taking the trivial marking \( H^1(G; \mathbb{R}) \to \{0\} \). We may thus assume that \( \Sigma^1(G) \neq \emptyset \). Moreover, since \( \Sigma^1(G) \) is open (see [BNS87, Theorem A]) and stable under positive homotheties (by definition), there exists a character \( \varphi : G \to \mathbb{Z} \) inside of \( \Sigma^1(G) \). By Proposition 7.3 this means that we can write \( G \) as an ascending HNN extension with a finitely generated base group. On the level of classifying spaces, this tells us that \( G \) admits a classifying space which is a mapping torus of a selfmap of a space with finite 1-skeleton. We conclude using Theorem 5.10 that \( b_0^1(G) = 0 \).

Knowing that \( b_0^1(G) = 0 \) allows us to immediately conclude that the boundary map

\[
c_3 : C_3 \to C_2
\]

is trivial, since it has to be trivial after tensoring with \( D_K \) for reasons of dimension – recall that \( C_2 \) is a free module of rank 1 less than \( C_1 \), and \( C_0 \) has rank 1. Therefore, the 2-skeleton of \( X \) is a classifying space for \( G \), and hence we may assume that \( X \) is 2-dimensional.

Now we need to look at the chain complex \( C_* \) more closely. Since it is the cellular chain complex of the Cayley 2-complex (the universal covering of the presentation complex), we have \( C_* \) equal to

\[
\mathbb{Z}G^{n-1} \xrightarrow{A} \mathbb{Z}G^n \xrightarrow{(s-1),s} \mathbb{Z}G
\]

where \( S \) is a generating set of \( G \) of cardinality \( n \). We enumerate \( S = \{s_1, \ldots, s_n\} \) and define

\[
U_i = \{ \varphi \in H^1(G; \mathbb{R}) \mid \varphi(s_i) \neq 0 \}.
\]

We let \( A_i \) denote the matrix \( A \) with the row corresponding to \( s_i \) removed. Now we are exactly in the situation discussed in the proof of [Kie18, Theorem 5.25] (and in [Kie18, Theorem 5.23]). The argument given there shows that for every \( i \in \{1, \ldots, n\} \) there exists a marking \( m_i \) of vertices of \( P_i = P(\det D \otimes A_i) \) such that a character \( \varphi \in U_i \) lies in \( \Sigma^1(G) \) if and only if \( m_i(\varphi) = 1 \). As in the proof of Lemma 7.7 we see from Lemma 4.8 that for every \( i \) we have

\[
P^{D_K} (G) = P(\det D \otimes A_i) - P(s_i - 1)
\]

Now [Kie18, Lemma 5.12] allows us to construct a marking \( m \) of the vertices of \( P^{D_K} (G) \) which agrees with \( m_i \) on \( U_i \) for every \( i \). This finishes the proof. \( \square \)

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