Cohomology of $\mathfrak{osp}(1|2)$ acting on linear differential operators on the supercircle $S^{1|1}$

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Abstract

We compute the first cohomology spaces $H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ ($\lambda, \mu \in \mathbb{R}$) of the Lie superalgebra $\mathfrak{osp}(1|2)$ with coefficients in the superspace $\mathfrak{D}_{\lambda,\mu}$ of linear differential operators acting on weighted densities on the supercircle $S^{1|1}$. The structure of these spaces was conjectured in [4]. In fact, we prove here that the situation is a little bit more complicated. (To appear in LMP.)

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1 Introduction

The space of weighted densities with weight $\lambda$ (or $\lambda$-densities) on $S^1$, denoted by:

$$\mathcal{F}_\lambda = \left\{ f(dx)^\lambda, \, f \in C^\infty(S^1) \right\}, \quad (\lambda \in \mathbb{R}),$$

is the space of sections of the line bundle $(T^*S^1)^{\otimes \lambda}$. Let $\text{Vect}(S^1)$ be the Lie algebra of all vector fields $F \frac{d}{dx}$ on $S^1$, ($F \in C^\infty(S^1)$). With the Lie derivative, $\mathcal{F}_\lambda$ is a $\text{Vect}(S^1)$-module. Alternatively, the $\text{Vect}(S^1)$ action can be written as follows:

$$L_{F \frac{d}{dx}}^\lambda (f(dx)^\lambda) = (F f' + \lambda f F')(dx)^\lambda, \quad (1.1)$$

where $f'$, $F'$ are $\frac{df}{dx}$, $\frac{dF}{dx}$.

Let $A$ be a differential operator on $S^1$. We see $A$ as the linear mapping $f(dx)^\lambda \mapsto (Af)(dx)^\mu$ from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ ($\lambda, \mu$ in $\mathbb{R}$). Thus the space of differential operators is a $\text{Vect}(S^1)$ module, denoted $\mathfrak{D}_{\lambda,\mu}$. The $\text{Vect}(S^1)$ action is:

$$L_{L_X}^{\lambda,\mu}(A) = L_X^\lambda \circ A - A \circ L_X^\mu. \quad (1.2)$$
If we restrict ourselves to the Lie subalgebra of $\text{Vect}(S^1)$ generated by $\{ \frac{d}{dx}, x \frac{d}{dx}, x^2 \frac{d}{dx} \}$, isomorphic to $\mathfrak{sl}(2)$, we get a family of infinite dimensional $\mathfrak{sl}(2)$ modules, still denoted $D_{\lambda, \mu}$.

P. Lecomte, in [5], found the cohomology spaces $H^1(\mathfrak{sl}(2); D_{\lambda, \mu})$ and $H^2(\mathfrak{sl}(2); D_{\lambda, \mu})$. These spaces appear naturally in the problem of describing the deformations of the $\mathfrak{sl}(2)$-module $\mathcal{D}$ of the differential operators acting on $S^n = \bigoplus_{k=-n}^{n} F_{\frac{k}{n+2}}$. More precisely, the first cohomology space $H^1(\mathfrak{sl}(2); V)$ classifies the infinitesimal deformations of a $\mathfrak{sl}(2)$ module $V$ and the obstructions to integrability of a given infinitesimal deformation of $V$ are elements of $H^2(\mathfrak{sl}(2); V)$. Thus, for instance, the infinitesimal deformations of the $\mathfrak{sl}(2)$ module $\mathcal{D}$ are classified by:

$$H^1(\mathfrak{sl}(2); \mathcal{D}) = \bigoplus_{k=0}^{n} H^1(\mathfrak{sl}(2); D_{\frac{k}{n}}, \frac{1}{n}) \oplus \bigoplus_{k=-n}^{n} H^1(\mathfrak{sl}(2); D_{\frac{k}{n}}, \frac{1}{n}).$$

In this paper we are interested to the study of the corresponding super structures. More precisely, we consider here the superspace $S^{1|1}$ equipped with its standard contact structure 1-form $\alpha$, and introduce the superspace $\mathcal{F}_\alpha$ of $\alpha$-densities on the supercircle $S^{1|1}$.

Let $\mathcal{K}(1)$ be the Lie superalgebra of contact vector fields, $\mathcal{F}_\alpha$ is naturally a $\mathcal{K}(1)$-module. For each $\lambda, \mu$ in $\mathbb{R}$, any differential operator on $S^{1|1}$ becomes a linear mapping from $\mathcal{F}_\alpha$ to $\mathcal{F}$, thus the space of differential operators becomes a $\mathcal{K}(1)$-module denoted $\mathcal{D}_{\lambda, \mu}$.

To the symplectic Lie algebra $\mathfrak{sl}(2)$ corresponds the ortosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ which is naturally realized as a subalgebra of $\mathcal{K}(1)$. Restricting our $\mathcal{K}(1)$-modules to $\mathfrak{osp}(1|2)$, we get $\mathfrak{osp}(1|2)$-modules still denoted $\mathcal{F}_\alpha$, $\mathcal{D}_{\lambda, \mu}$.

We compute here the first cohomology spaces $H^1(\mathfrak{osp}(1|2); \mathcal{D}_{\lambda, \mu})$, $(\lambda, \mu$ in $\mathbb{R})$, getting a result very close to the classical spaces $H^1(\mathfrak{sl}(2); D_{\lambda, \mu})$. Especially, these spaces have the same dimension. Moreover, we give explicit formulæ for all the non trivial 1-cocycles.

These spaces arise in the classification of infinitesimal deformations of the $\mathfrak{osp}(1|2)$-module of the differential operators acting on $S^n = \bigoplus_{k=1-n}^{n} \mathcal{F}_{\frac{k}{n}}$. We hope to be able to describe in the future all the deformations of this module.

## 2 Definitions and Notations

### 2.1 The Lie superalgebra of contact vector fields on $S^{1|1}$

We define the supercircle $S^{1|1}$ through its space of functions, $C^\infty(S^{1|1})$. A $C^\infty(S^{1|1})$ has the form:

$$F(x, \theta) = f_0(x) + \theta f_1(x),$$

where $x$ is the even variable and $\theta$ the odd variable: we have $\theta^2 = 0$. Even elements in $C^\infty(S^{1|1})$ are the functions $F(x, \theta) = f_0(x)$, the functions $F(x, \theta) = \theta f_1(x)$ are odd elements. Note $p(F)$ the parity of a homogeneous function $F$.

Let $\text{Vect}(S^{1|1})$ be the superspace of vector fields on $S^{1|1}$:

$$\text{Vect}(S^{1|1}) = \{ F_0 \partial_x + F_1 \partial_\theta \; | \; F_i \in C^\infty(S^{1|1}) \},$$
where \( \partial_\theta \) and \( \partial_x \) stand for \( \frac{\partial}{\partial \theta} \) and \( \frac{\partial}{\partial x} \). The vector fields \( f(x)\partial_x \) and \( \theta f(x)\partial_\theta \) are even, the vector fields \( \theta f(x)\partial_x \) and \( f(x)\partial_\theta \) are odd. The superbracket of two vector fields is bilinear and defined for two homogeneous vector fields by:

\[
[X, Y] = X \circ Y - (-1)^{p(X)p(Y)}Y \circ X.
\]

Denote \( \mathcal{L}_X \) the Lie derivative of a vector field, acting on the space of functions, forms, vector fields, . . .

The supercircle \( S^{1|1} \) is equipped with the standard contact structure given by the following even 1-form:

\[
\alpha = dx + \theta d\theta.
\]

We consider the Lie superalgebra \( \mathcal{K}(1) \) of contact vector fields on \( S^{1|1} \). That is, \( \mathcal{K}(1) \) is the superspace of conformal vector fields on \( S^{1|1} \) with respect to the 1-form \( \alpha \):

\[
\mathcal{K}(1) = \{ X \in \text{Vect}(S^{1|1}) \mid \text{there exists } F \in C^\infty(S^{1|1}) \text{ such that } \mathcal{L}_X(\alpha) = F \alpha \}.
\]

Let us define the vector fields \( \eta \) and \( \eta' \) by

\[
\eta = \partial_\theta + \theta \partial_x, \quad \eta' = \partial_\theta - \theta \partial_x.
\]

Then any contact vector field on \( S^{1|1} \) can be written in the following explicit form:

\[
X_F = F \partial_x + \frac{1}{2}\eta(F)(\partial_\theta - \theta \partial_x) = -F\eta' + \frac{1}{2}\eta(F)\eta', \quad \text{where } F \in C^\infty(S^{1|1}).
\]

Of course, \( \mathcal{K}(1) \) is a subalgebra of \( \text{Vect}(S^{1|1}) \), and \( \mathcal{K}(1) \) acts on \( C^\infty(S^{1|1}) \) through:

\[
\mathcal{L}_{X_F}(G) = FG' + \frac{1}{2}(-1)^{p(F)+1}\eta(F) \cdot \eta(G).
\]  

(2.3)

Let us define the contact bracket on \( C^\infty(S^{1|1}) \) as the bilinear mapping such that, for a couple of homogenous functions \( F, G \),

\[
\{F, G\} = FG' - F'G + \frac{1}{2}(-1)^{p(F)+1}\eta(F) \cdot \eta(G),
\]  

(2.4)

Then the bracket of \( \mathcal{K}(1) \) can be written as:

\[
[X_F, X_G] = X_{\{F, G\}}.
\]

2.2 The superalgebra \( \mathfrak{osp}(1|2) \)

Recall the Lie algebra \( \mathfrak{sl}(2) \) is isomorphic to the Lie subalgebra of \( \text{Vect}(S^1) \) generated by

\[
\left\{ \frac{d}{dx}, x \frac{d}{dx}, x^2 \frac{d}{dx} \right\}.
\]
Similarly, we now consider the orthosymplectic Lie superalgebra as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta}, X_\theta).$$

The space of even elements is isomorphic to $\mathfrak{sl}(2)$:

$$(\mathfrak{osp}(1|2))_0 = \text{Span}(X_1, X_x, X_{x^2}) = \mathfrak{sl}(2).$$

The space of odd elements is two dimensional:

$$(\mathfrak{osp}(1|2))_1 = \text{Span}(X_{x\theta}, X_\theta).$$

The new commutation relations are

\[
\begin{align*}
[X_{x^2}, X_\theta] &= -X_{x\theta}, & [X_x, X_\theta] &= -\frac{1}{2}X_\theta, & [X_1, X_\theta] &= 0, \\
[X_{x^2}, X_{x\theta}] &= 0, & [X_x, X_{x\theta}] &= \frac{1}{2}X_\theta, & [X_1, X_{x\theta}] &= X_\theta, \\
[X_{x\theta}, X_\theta] &= \frac{1}{2}X_x.
\end{align*}
\]

### 2.3 The space of weighted densities on $S^{1|1}$

In the super setting, by replacing $dx$ by the 1-form $\alpha$, we get analogous definition for weighted densities i.e. we define the space of $\lambda$-densities as

$$\mathfrak{F}_\lambda = \{ \phi = F(x, \theta)\alpha^\lambda \mid F(x, \theta) \in C^\infty(S^{1|1}) \}. \quad (2.5)$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty(S^{1|1})$, but the Lie derivative of the density $G\alpha^\lambda$ along the vector field $X_F$ in $\mathcal{K}(1)$ is now:

$${\mathcal{L}_F}(G\alpha^\lambda) = \mathcal{L}_F^\lambda(G)\alpha^\lambda, \quad \text{with} \quad \mathcal{L}_F^\lambda(G) = \mathcal{L}_F(G) + \lambda F^\prime G. \quad (2.6)$$

Or, if we put $F = a(x) + b(x)\theta$, $G = g_0(x) + g_1(x)\theta$,

$$\mathcal{L}_F^\lambda(G) = L_{a\partial_x}^\lambda(g_0) + \frac{1}{2} bg_1 + \left(L_{a\partial_x}^{\lambda + \frac{1}{2}}(g_1) + \lambda g_0 b' + \frac{1}{2} g_0^\prime b \right) \theta. \quad (2.7)$$

Especially, we have

$$\begin{cases}
\mathcal{L}_a^\lambda(g_0) = L_{a\partial_x}^\lambda(g_0), & \quad \mathcal{L}_{X_a}^\lambda(g_1\theta) = \theta L_{a\partial_x}^{\lambda + \frac{1}{2}}(g_1), \\
\mathcal{L}_{X_{\theta}}^\lambda(g_0) = (\lambda g_0 b' + \frac{1}{2} g_0^\prime b) \theta & \text{and} \quad \mathcal{L}_{X_{\theta}}^\lambda(g_1\theta) = \frac{1}{2} bg_1.
\end{cases}$$

Of course, for all $\lambda$, $\mathfrak{F}_\lambda$ is a $\mathcal{K}(1)$-module:

$$[\mathcal{L}_F^\lambda, \mathcal{L}_G^\lambda] = \mathcal{L}_{[F, G]}^\lambda.$$
2.4 Differential Operators on Weighted Densities

A differential operator on $S^1\mid 1$ is an operator on $C^\infty(S^1\mid 1)$ of the following form:

$$A = \sum_{i=0}^{\ell} \tilde{a}_i(x, \theta) \partial_x^i + \sum_{i=0}^{\ell} \tilde{b}_i(x, \theta) \partial_x^i \partial_\theta.$$  

In [4], it is proved that any local operator $A$ on $S^1\mid 1$ is in fact a differential operator.

Of course, any differential operator defines a linear mapping from $F_\lambda$ to $F_\mu$ for any $\lambda, \mu \in \mathbb{R}$, thus the space of differential operators becomes a family of $K(1)$ and $osp(1\mid 2)$ modules denoted $\mathfrak{D}_{\lambda,\mu}$, for the natural action:

$$L_{\lambda,\mu}^X F(A) = L_\mu X F \circ A - (-1)^{p(A)p(F)} A \circ L_\lambda^X F. \quad (2.8)$$

3 The space $H^1(\mathfrak{osp}(1\mid 2); \mathfrak{D}_{\lambda,\mu})$

3.1 Lie superalgebra cohomology (see [2])

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and $A = A_0 \oplus A_1$ a $\mathfrak{g}$ module. We define the cochain complex associated to the module as an exact sequence:

$$0 \rightarrow C^0(\mathfrak{g}, A) \rightarrow \cdots \rightarrow C^{q-1}(\mathfrak{g}, A) \xrightarrow{\delta^{q-1}} C^q(\mathfrak{g}, A) \rightarrow \cdots.$$

The spaces $C^q(\mathfrak{g}, A)$ are the spaces of super skew-symmetric $q$ linear mappings:

$$C^q(\mathfrak{g}, A) = \bigoplus_{q_0+q_1 = q} \text{Hom}(\wedge^{q_0} \mathfrak{g}_0 \otimes S^{q_1} \mathfrak{g}_1, A).$$

Elements of $C^q(\mathfrak{g}, A)$ are called cochains. The spaces $C^q(\mathfrak{g}, A)$ is $\mathbb{Z}_2$ graded:

$$C^q(\mathfrak{g}, A) = C^q_0(\mathfrak{g}, A) + C^q_1(\mathfrak{g}, A), \text{ with } C^q_0(\mathfrak{g}, A) = \bigoplus_{q_0+q_1 = q, q_1+r = p \mod 2} \text{Hom}(\wedge^{q_0} \mathfrak{g}_0 \otimes S^{q_1} \mathfrak{g}_1, A_r).$$

The linear mapping $\delta^q$ (or, briefly $\delta$) is called the coboundary operator. This operator is a generalization of the usual Chevalley coboundary operator for Lie algebra to the case of Lie superalgebra. Explicitly, it is defined as follows. Take a cochain $c \in C^q(\mathfrak{g}, A)$, then for $q_0, q_1$ with $q_0 + q_1 = q + 1$, $\delta^q c$ is:
\[ \delta^q c(g_1, \ldots, g_{q_0}, h_1, \ldots, h_{q_1}) \]
\[ = \sum_{1 \leq s < t \leq q_0} (-1)^{s+t-1} c([g_s, g_t], g_1, \ldots, \hat{g}_s, \ldots, \hat{g}_t, \ldots, g_{q_0}, h_1, \ldots, h_{q_1}) \]
\[ + \sum_{s=1}^{q_0} \sum_{t=1}^{q_1} (-1)^{s-1} c(g_1, \ldots, \hat{g}_s, \ldots, g_{q_0}, \hat{g}_t, h_1, \ldots, \hat{h}_s, \ldots, \hat{h}_t, \ldots, h_{q_1}) \]
\[ + \sum_{1 \leq s < t \leq q_1} c([h_s, h_t], g_1, \ldots, \hat{g}_s, \ldots, \hat{g}_t, h_1, \ldots, \hat{h}_s, \ldots, \hat{h}_t, \ldots, h_{q_1}) \]
\[ + \sum_{s=1}^{q_0} (-1)^s g_s c(g_1, \ldots, \hat{g}_s, \ldots, g_{q_0}, h_1, \ldots, h_{q_1}) \]
\[ + (-1)^{q_0-1} \sum_{s=1}^{q_1} h_s c(g_1, \ldots, \hat{g}_s, \ldots, g_{q_0}, h_1, \ldots, \hat{h}_s, \ldots, h_{q_1}). \]

where \( g_1, \ldots, g_{q_0} \) are in \( g_0 \) and \( h_1, \ldots, h_{q_1} \) in \( g_1 \).

The relation \( \delta^q \circ \delta^{q-1} = 0 \) holds. The kernel of \( \delta^q \), denoted \( Z^q(g, A) \), is the space of \( q \) cocycles, among them, the elements in the range of \( \delta^{q-1} \) are called \( q \) coboundaries. We note \( B^q(g, A) \) the space of \( q \) coboundaries.

By definition, the \( q^{th} \) cohomolgy space is the quotient space
\[ H^q(g, A) = Z^q(g, A)/B^q(g, A). \]

One can check that \( \delta^q(C^q_p(g, A)) \subset C^{q+1}_p(g, A) \) and then we get the following sequences
\[ 0 \longrightarrow C^0_p(g, A) \longrightarrow \cdots \longrightarrow C^{q-1}_p(g, A) \xrightarrow{\delta^{q-1}} C^q_p(g, A) \cdots, \]
where \( p = 0 \) or \( 1 \). The cohomology spaces are thus graded by
\[ H^q_p(g, A) = \text{Ker} \delta^q|_{C^q_p(g, A)}/\delta^{q-1}(C^{q-1}_p(g, A)). \]

### 3.2 The main theorem

The main result in this paper is the following:

**Theorem 3.1.** The cohomology spaces \( H^1_p(g, D_{\lambda, \mu}) \) are finite dimensional. An explicit description of these spaces is the following:

1) The space \( H^1_0(\mathfrak{osp}(1|2), D_{\lambda, \mu}) \) is
\[ H^1_0(\mathfrak{osp}(1|2), D_{\lambda, \mu}) \simeq \begin{cases} \mathbb{R} & \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \]
A base for the space $H_0^1(osp(1|2), \mathfrak{D}_{\lambda,\lambda})$ is given by the cohomology class of the 1-cocycle:

$$\Upsilon_{\lambda,\lambda}(X_F) = F'. \quad (3.10)$$

2) The space $H_1^1(osp(1|2), \mathfrak{D}_{\lambda,\mu})$ is

$$H_1^1(osp(1|2), \mathfrak{D}_{\lambda,\mu}) \simeq \begin{cases} \mathbb{R}^2 & \text{if } \lambda = \frac{1-k}{2}, \mu = \frac{k}{2}, \\ 0 & \text{otherwise}. \end{cases} \quad (3.11)$$

A base for the space $H_1^1(osp(1|2), \mathfrak{D}_{\lambda,\mu})$ is given by the cohomology classes of the 1-cocycles:

$$\Upsilon_{\frac{1-k}{2}}(X_F) = \eta^2(F)\eta^{2k-1},$$

$$\tilde{\Upsilon}_{\frac{1-k}{2}}(X_F) = (k-1)\eta^4(F)\eta^{2k-3} + \eta^3(F)\eta^{2k-2}. \quad (3.12)$$

Note that the 1-cocycle $\tilde{\Upsilon}_{\frac{1-k}{2}}$ coincides with the 1-cocycle $\gamma_{2k-1}$ given by Gargoubi et al. in [4]. The proof of Theorem 3.1 will be the subject of subsection 3.4.

### 3.3 Relationship between $H^1(osp(1|2), \mathfrak{D}_{\lambda,\mu})$ and $H^1(sl(2), \mathfrak{D}_{\lambda,\mu})$

Before proving the theorem 3.1 we present here some results illustrating the analogy between the cohomology spaces in super and classical settings.

First, note that:

1) As a $sl(2)$-module, we have $\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}})$ and $osp(1|2) \simeq sl(2) \oplus \Pi(\mathfrak{h})$, where $\mathfrak{h}$ is the subspace of $\mathcal{F}_{-\frac{1}{2}}$ spanned by $\{dx^{-\frac{1}{2}}, xdx^{-\frac{1}{2}}\}$ and $\Pi$ is the change of parity.

2) As a $sl(2)$-module, we have for the homogeneous components of $\mathfrak{D}_{\lambda,\mu}$:

$$(\mathfrak{D}_{\lambda,\mu})_0 \simeq D_{\lambda,\mu} \oplus D_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \quad \text{and} \quad (\mathfrak{D}_{\lambda,\mu})_1 \simeq \Pi(D_{\lambda+\frac{1}{2},\mu} \oplus D_{\lambda,\mu+\frac{1}{2}}).$$

**Proposition 3.1.** Any 1-cocycle $\Upsilon \in Z^1(osp(1|2); \mathfrak{D}_{\lambda,\mu})$, is decomposed into $(\Upsilon', \Upsilon'')$ in $Hom(sl(2); \mathfrak{D}_{\lambda,\mu}) \oplus Hom(\mathfrak{h}; \mathfrak{D}_{\lambda,\mu})$. $\Upsilon'$ and $\Upsilon''$ are solutions of the following equations:

$$\Upsilon'([X_{g_1}, X_{g_2}]) - L_{X_{g_1}} \Upsilon'(X_{g_2}) + L_{X_{g_2}} \Upsilon'(X_{g_1}) = 0, \quad (3.13)$$

$$\Upsilon''([X_g, X_{h\theta}]) - L_{X_g} \Upsilon''(X_{h\theta}) + L_{X_{h\theta}} \Upsilon'(X_g) = 0, \quad (3.14)$$

$$\Upsilon'([X_{h_1\theta}, X_{h_2\theta}]) - L_{X_{h_1\theta}} \Upsilon''(X_{h_2\theta}) + L_{X_{h_2\theta}} \Upsilon''(X_{h_1\theta}) = 0. \quad (3.15)$$

Here, $g, g_1, g_2$ are polynomials in the variable $x$, with degree at most 2, and $h, h_1, h_2$ are affine functions in the variable $x$. 
Proof. The equations (3.13), (3.14) and (3.15) are equivalent to the fact that \( \Upsilon \) is a 1-cocycle. For any \( X_F, X_G \in \mathfrak{osp}(1|2) \),

\[
\delta \Upsilon(X_F, X_G) := \Upsilon([X_F, X_G]) - \mathfrak{L}_{X_F}^\lambda \Upsilon(X_G) + (-1)^{p(G)p(F)} \mathfrak{L}_{X_G}^\lambda \Upsilon(X_F) = 0.
\]

According to the \( \mathbb{Z}_2 \)-grading, the even component \( \Upsilon_0 \) and the odd component \( \Upsilon_1 \) of any 1-cocycle \( \Upsilon \) can be decomposed as \( \Upsilon = (\Upsilon_0, \Upsilon_1) \) and \( \Upsilon = (\Upsilon_0, \Upsilon_0, \Upsilon_1, \Upsilon_1) \), where

\[
\begin{align*}
\Upsilon_0 &: \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda,\mu}, \\
\Upsilon_{0\frac{1}{2}} &: \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}, \\
\Upsilon_{110} &: \mathfrak{h} \rightarrow \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \\
\Upsilon_{11\frac{1}{2}} &: \mathfrak{h} \rightarrow \mathcal{D}_{\lambda+\frac{1}{2},\mu}.
\end{align*}
\]

and

\[
\begin{align*}
\Upsilon_{010} &: \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \\
\Upsilon_{01\frac{1}{2}} &: \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda+\frac{1}{2},\mu}, \\
\Upsilon_{100} &: \mathfrak{h} \rightarrow \mathcal{D}_{\lambda,\mu}, \\
\Upsilon_{10\frac{1}{2}} &: \mathfrak{h} \rightarrow \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}.
\end{align*}
\]

The decomposition \( \Upsilon = (\Upsilon', \Upsilon'') \) given in proposition 3.1 corresponds to

\[
\Upsilon' = (\Upsilon_{000}, \Upsilon_{00\frac{1}{2}}, \Upsilon_{010}, \Upsilon_{01\frac{1}{2}}) \quad \text{and} \quad \Upsilon'' = (\Upsilon_{110}, \Upsilon_{11\frac{1}{2}}, \Upsilon_{100}, \Upsilon_{10\frac{1}{2}}).
\]

By considering the equation (3.13), we can see the components \( \Upsilon_{000}, \Upsilon_{00\frac{1}{2}}, \Upsilon_{010} \) and \( \Upsilon_{01\frac{1}{2}} \) as 1-cocycles on \( \mathfrak{sl}(2) \) with coefficients respectively in \( \mathcal{D}_{\lambda,\mu}, \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}, \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \) and \( \mathcal{D}_{\lambda+\frac{1}{2},\mu} \).

The first cohomology space \( H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu}) \) was computed by Gargoubi and Lecomte [3]. The result is the following:

\[
H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu}) \cong \begin{cases} 
\mathbb{R} & \text{if} \quad \lambda = \mu \\
\mathbb{R}^2 & \text{if} \quad (\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2}) \quad \text{where} \quad k \in \mathbb{N} \setminus \{0\} \\
0 & \text{otherwise.}
\end{cases}
\]

(3.16)

The space \( H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda}) \) is generated by the cohomology class of the 1-cocycle

\[
C'_\lambda(F \frac{d}{dx})(dx^\lambda) = F' f dx^\lambda.
\]

(3.17)

For \( k \in \mathbb{N} \setminus \{0\} \), the space \( H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}) \) is generated by the cohomology classes of the 1-cocycles, \( C_k \) and \( \tilde{C}_k \) defined by

\[
C_k(F \frac{d}{dx})(dx^\frac{1+k}{2}) = F' f^{(k)}(dx^\frac{1+k}{2}) \quad \text{and} \quad \tilde{C}_k(F \frac{d}{dx})(dx^\frac{1+k}{2}) = F'' f^{(k-1)}(dx^\frac{1+k}{2}).
\]

(3.18)

We shall need the following description of \( \mathfrak{sl}(2) \) invariant mappings.

**Lemma 3.2.** Let

\[
A : \mathfrak{h} \times \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu, \quad (h dx^{-\frac{1}{2}}, f dx^\lambda) \mapsto A(h, f) dx^\mu
\]
be a bilinear differential operator. If $A$ is $\mathfrak{sl}(2)$-invariant then

$$\mu = \lambda - \frac{1}{2} + k, \quad \text{where} \quad k \in \mathbb{N}$$

and the following relation holds

$$A_k(h, f) = a_k(h f(k) + k(2\lambda + k - 1)h' f(k-1)), \quad \text{where} \quad k(k-1)(2\lambda + k - 1)(2\lambda + k - 2)a_k = 0.$$

Proof. A straightforward computation.

Now, let us study the relationship between these 1-cocycles and their analogues in the super setting. We know that any element $\Upsilon \in Z^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\mu})$ is decomposed into $\Upsilon = \Upsilon' + \Upsilon''$ where $\Upsilon' \in Hom(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\mu})$ and $\Upsilon'' \in Hom(\mathfrak{h}, \mathcal{D}_{\lambda,\mu})$. The following lemma shows the close relationship between the cohomology spaces $H^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\mu})$ and $H^1(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\mu})$.

Lemma 3.3. The 1-cocycle $\Upsilon$ is a coboundary for $\mathfrak{osp}(1|2)$ if and only if $\Upsilon'$ is a coboundary for $\mathfrak{sl}(2)$.

Proof. It is easy to see that if $\Upsilon$ is a coboundary for $\mathfrak{osp}(1|2)$ then $\Upsilon'$ is a coboundary over $\mathfrak{sl}(2)$. Now, assume that $\Upsilon'$ is a coboundary for $\mathfrak{sl}(2)$, that is, there exists $\tilde{A} \in \mathcal{D}_{\lambda,\mu}$ such that for all $g$ polynomial in the variable $x$ with degree at most 2

$$\Upsilon(X_g) = \mathcal{L}^\lambda_{X_g} \tilde{A}. $$

By replacing $\Upsilon$ by $\Upsilon - \delta \tilde{A}$, we can suppose that $\Upsilon' = 0$. But, in this case, the map $\Upsilon''$ must satisfy, for all $h, h_1, h_2$ polynomial with degree 0 or 1 and $g$ polynomial with degree 0,1 or 2, the following equations

$$\mathcal{L}^\lambda_{X_g} \Upsilon''(X_{h_1}) = \mathcal{L}^\lambda_{X_g} \Upsilon''([X_g, X_{h_1}]) = 0, \quad (3.19)$$

$$\mathcal{L}^\lambda_{X_{h_1}} \Upsilon''(X_{h_2}) + \mathcal{L}^\lambda_{X_{h_2}} \Upsilon''(X_{h_1}) = 0. \quad (3.20)$$

1) If $\Upsilon$ is an even 1-cocycle then $\Upsilon''$ is decomposed into $\Upsilon''_{00} : \mathfrak{h} \otimes \mathcal{F}_{\lambda + \frac{1}{2}} \rightarrow \mathcal{F}_{\lambda}$ and $\Upsilon''_{01} : \mathfrak{h} \otimes \mathcal{F}_{\lambda} \rightarrow \mathcal{F}_{\lambda + \frac{1}{2}}$. The equation (3.19) tell us that $\Upsilon''_{00}$ and $\Upsilon''_{01}$ are $\mathfrak{sl}(2)$ invariant bilinear maps. Therefore, the expressions of $\Upsilon''_{00}$ and $\Upsilon''_{01}$ are given by Lemma 3.2. So, we must have $\mu = \lambda + k = (\lambda + \frac{1}{2}) - \frac{1}{2} + k$ (and then $\mu + \frac{1}{2} = \lambda - \frac{1}{2} + k + 1$). More precisely, using the equation (3.20), we get up to a factor:

$$\Upsilon = \begin{cases} 
0 & \text{if } k(k-1)(2\lambda + k)(2\lambda + k - 1) \neq 0 \quad \text{or} \quad k = 1 \text{ and } \lambda \notin \{0, -\frac{1}{2}\}, \\
\delta(\theta \partial_{\theta} \partial_{x}^{k}) & \text{if } (\lambda, \mu) = (\frac{-k}{2}, \frac{k}{2}), \\
\delta(\partial_{x}^{k} - \theta \partial_{\theta} \partial_{x}^{k}) & \text{if } (\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2}) \quad \text{or} \quad \lambda = \mu.
\end{cases}$$
2) If \( \Upsilon \) is an odd 1-cocycle then \( \Upsilon'' \) is decomposed into \( \Upsilon''_0 : \mathfrak{h} \otimes \mathcal{F}_\lambda \to \mathcal{F}_\mu \) and \( \Upsilon''^0 : \mathfrak{h} \otimes \mathcal{F}_{\lambda + \frac{1}{2}} \to \mathcal{F}_{\mu + \frac{1}{2}} \). As in the previous case, the expressions of \( \Upsilon''_0 \) and \( \Upsilon''^0 \) are given by Lemma 3.2. So, we must have \( \mu = \lambda - \frac{1}{2} + k \) (and then \( \mu + \frac{1}{2} = (\lambda + \frac{1}{2}) - \frac{1}{2} + k \)). More precisely, using the equation (3.22), we get:

\[
\Upsilon = \begin{cases} 
0 & \text{if } k(k - 1)(2\lambda + k - 1) \neq 0 \\
\delta(\theta) & \text{if } \mu = \lambda - \frac{1}{2}, \\
\delta(\partial_\theta) & \text{if } \mu = \lambda + \frac{1}{2}, \\
\delta(\theta \partial_x^k) & \text{if } (\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2}).
\end{cases}
\]

\( \square \)

Now, the space \( Z^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \mu}) \) of 1-cocycles is \( \mathbb{Z}_2 \)-graded:

\[
Z^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \mu}) = Z^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \mu})_0 \oplus Z^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \mu})_1.
\] (3.21)

Therefore, any element \( \Upsilon \in Z^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \mu}) \) is decomposed into an even part \( \Upsilon_0 \) and odd part \( \Upsilon_1 \). Each of \( \Upsilon_0 \) and \( \Upsilon_1 \) is decomposed into two components: \( \Upsilon_0 = (\Upsilon_{00}, \Upsilon_{11}) \) and \( \Upsilon_1 = (\Upsilon_{01}, \Upsilon_{10}) \), where

\[
\begin{cases} 
\Upsilon_{00} : \mathfrak{sl}(2) \to (\mathcal{D}_{\lambda, \mu})_0, \\
\Upsilon_{11} : \mathfrak{h} \to (\mathcal{D}_{\lambda, \mu})_1, \\
\Upsilon_{01} : \mathfrak{sl}(2) \to (\mathcal{D}_{\lambda, \mu})_1, \\
\Upsilon_{10} : \mathfrak{h} \to (\mathcal{D}_{\lambda, \mu})_0,
\end{cases}
\]

The components \( \Upsilon_{11} \) and \( \Upsilon_{10} \) of \( \Upsilon_0 \) and \( \Upsilon_1 \) are also decomposed as follows: \( \Upsilon_{11} = \Upsilon_{110} + \Upsilon_{11\frac{1}{2}} \) and \( \Upsilon_{10} = \Upsilon_{100} + \Upsilon_{10\frac{1}{2}} \), where \( \Upsilon_{110} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda, \mu} + \frac{1}{2}) \), \( \Upsilon_{11\frac{1}{2}} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda + \frac{1}{2}, \mu}) \), \( \Upsilon_{100} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda, \mu}) \), \( \Upsilon_{10\frac{1}{2}} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda + \frac{1}{2}, \mu}) \).

As in [1], the following lemma gives the general form of each of \( \Upsilon_{110} \) and \( \Upsilon_{11\frac{1}{2}} \).

**Lemma 3.4.** Up to a coboundary, the maps \( \Upsilon_{110}, \Upsilon_{11\frac{1}{2}}, \Upsilon_{100} \) and \( \Upsilon_{10\frac{1}{2}} \) are given by

\[
\begin{align*}
\Upsilon_{110}(X_{\theta}) &= a_0 h \theta \partial_x^k + a_1 h' \theta \partial_x^{k-1} \\
\Upsilon_{11\frac{1}{2}}(X_{\theta}) &= b_0 h \theta \partial_x^k + b_1 h' \theta \partial_x^{k-1},
\end{align*}
\]

\[
\begin{align*}
\Upsilon_{110}(X_{\theta}) &= c_0 h \theta \partial_x^k + c_1 h' \theta \partial_x^{k-1} \\
\Upsilon_{11\frac{1}{2}}(X_{\theta}) &= d_0 h \theta \partial_x^k + d_1 h' \theta \partial_x^{k-1},
\end{align*}
\]

where the coefficients \( a_i, b_i, c_i, \) and \( d_i \) are constants.

Proof. The coefficients \( a_i, b_i, c_i, \) and \( d_i \) a priori are some functions of \( x \), but we shall now prove \( \partial_x a_i = \partial_x b_i = 0 \) (and similarly \( \partial_x c_i = \partial_x d_i = 0 \)). To do that, we shall simply show that \( \mathcal{L}^\lambda_{\partial_x} \Upsilon_{110}(\mathcal{Y}_{11}) = 0 \).

First, for all \( h \) polynomial with degree 0 or 1, we have

\[
(\mathcal{L}^\lambda_{\partial_x} \Upsilon_{110})(X_{\theta}) = \mathcal{L}^\lambda_{\partial_x} \Upsilon_{110}(\Upsilon_{11}(X_{\theta})) - \Upsilon_{11}([\partial_x, X_{\theta}]).
\]

(3.22)

On the other hand, from Lemma 3.3 it follows that, up to a coboundary, \( \Upsilon_{00} \) is a linear combination of some 1-cocycles for \( \mathfrak{sl}(2) \) given by (3.17) and (3.18). So, we have \( \Upsilon_{00}(\partial_x) = 0 \) and then

\[
\mathcal{L}^\lambda_{X_{\theta}} \Upsilon_{00}(\partial_x) = 0.
\]
Therefore, the equation (3.22) becomes, for all \( h \),
\[
- (\mathfrak{L}^{\lambda,\mu}_{\partial_x} \Upsilon_{11})(X_{h\theta}) = \Upsilon_{11}([\partial_x, X_{h\theta}]) - \mathfrak{L}^{\lambda,\mu}_{\partial_x}(\Upsilon_{11}(X_{h\theta})) + \mathfrak{L}^{\lambda,\mu}_{X_{h\theta}}(\Upsilon_{00}(\partial_x)).
\] (3.23)

The right-hand side of (3.23) is nothing but \( \delta \Upsilon_0(\partial_x, X_{h\theta}) \). But, \( \Upsilon_0 \) is a 1-cocycle, then \( \mathfrak{L}^{\lambda,\mu}_{\partial_x}(\Upsilon_{11}) = 0 \). Lemma 3.4 is proved.

\[ \square \]

### 3.4 Proof of Theorem 3.1

The first cohomology space \( H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}) \) inherits the \( \mathbb{Z}_2 \)-grading from (3.21) and is decomposed into odd and even subspaces:
\[
H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}) = H^1_0(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}) \oplus H^1_1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}).
\]

We compute each part separately.

1) Let \( \Upsilon_0 \) be a non trivial even 1-cocycle for \( \mathfrak{osp}(1|2) \) in \( \mathfrak{D}_{\lambda,\mu} \). According to the \( \mathbb{Z}_2 \)-grading, \( \Upsilon_0 \) should retain the following general form: 
\[
\Upsilon_0 = \Upsilon_{000} + \Upsilon_{00\frac{1}{2}} + \Upsilon_{110} + \Upsilon_{11\frac{1}{2}}
\]
such that
\[
\left\{ \begin{array}{ll}
\Upsilon_{000} : & \mathfrak{sl}(2) \to \mathfrak{D}_{\lambda,\mu}, \\
\Upsilon_{00\frac{1}{2}} : & \mathfrak{sl}(2) \to \mathfrak{D}_{\lambda\frac{1}{2},\mu + \frac{1}{2}}, \\
\Upsilon_{110} : & h \to \mathfrak{D}_{\lambda,\mu + \frac{1}{2}}, \\
\Upsilon_{11\frac{1}{2}} : & h \to \mathfrak{D}_{\lambda + \frac{1}{2},\mu}.
\end{array} \right.
\] (3.24)

Then, by using Lemma 3.3, we deduce that, up to coboundary, \( \Upsilon_{000} \) and \( \Upsilon_{00\frac{1}{2}} \) can be expressed in terms of \( C'_\lambda, C_k \) and \( \tilde{C}_k \) where \( \lambda \in \mathbb{R} \) and \( k \in \mathbb{N} \setminus \{0\} \). We thus consider three cases:

i) \( \lambda = \mu, \ Upsilon_{000} = \alpha C'_\lambda \), and \( \Upsilon_{00\frac{1}{2}} = \beta C'_{\lambda\frac{1}{2}} \).

ii) \( (\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2}) \), \( \Upsilon_{000} = \alpha_1 C_k + \alpha_2 \tilde{C}_k \), and \( \Upsilon_{00\frac{1}{2}} = 0 \).

iii) \( (\lambda, \mu) = (\frac{-k}{2}, \frac{k}{2}) \), \( \Upsilon_{000} = 0 \) and \( \Upsilon_{00\frac{1}{2}} = \alpha_1 C_k + \alpha_2 \tilde{C}_k \).

Put \( \Upsilon' = \Upsilon_{000} + \Upsilon_{00\frac{1}{2}} \) and \( \Upsilon'' = \Upsilon_{110} + \Upsilon_{11\frac{1}{2}} \). In each case, the 1-cocycle \( \Upsilon_0 \) must satisfy
\[
\left\{ \begin{array}{l}
\Upsilon''[X_g, X_{\theta h}] = \mathfrak{L}^{\lambda,\mu}_{X_g} \Upsilon''(X_{\theta h}) - \mathfrak{L}^{\lambda,\mu}_{X_{\theta h}} \Upsilon''(X_g), \\
\Upsilon'[X_{\theta h_1}, X_{\theta h_2}] = \mathfrak{L}^{\lambda,\mu}_{X_{\theta h_1}} \Upsilon''(X_{\theta h_2}) + \mathfrak{L}^{\lambda,\mu}_{X_{\theta h_2}} \Upsilon''(X_{\theta h_1}),
\end{array} \right.
\] (3.25)

where \( h, h_1, \) and \( h_2 \) are polynomials of degree 0 or 1, \( g \) polynomial of degree 0, 1 or 2.

Now, thanks to Lemma 3.4, we can write
\[
\Upsilon_{110}(X_{h\theta}) = a_0 h\theta \partial_x^k + a_1 h'\partial_x^{k-1} \quad \text{and} \quad \Upsilon_{11\frac{1}{2}}(v_{h\theta}) = b_0 h\partial_v \partial_x^k + b_1 h'\partial_v \partial_x^{k-1}.
\]
Let us now solve the equations (3.25). We obtain \( \lambda = \mu \) and \( \gamma_{\lambda, \lambda}(X_F) = F' \). This completes the proof of part 1).

2) Consider a non trivial odd 1-cocycle \( \gamma_1 \) for \( \mathfrak{osp}(1|2) \) in \( \mathfrak{D}_{\lambda, \mu} \) and its decomposition

\[
\gamma_1 = \gamma_{010} + \gamma_{01} + \gamma_{100} + \gamma_{10}^{\frac{1}{2}},
\]

where

\[
\begin{align*}
\gamma_{010} &: \mathfrak{sl}(2) \to \mathfrak{D}_{\lambda, \mu + \frac{1}{2}}, \\
\gamma_{01} &: \mathfrak{sl}(2) \to \mathfrak{D}_{\lambda + \frac{1}{2}, \mu}, \\
\gamma_{100} &: \mathfrak{h} \to \mathfrak{D}_{\lambda \mu}, \\
\gamma_{10}^{\frac{1}{2}} &: \mathfrak{h} \to \mathfrak{D}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}}.
\end{align*}
\]

(3.26)

We must have \( (\lambda, \mu) = (1 - k, k - \frac{1}{2}) \) with \( k \in \mathbb{N} \setminus \{0\} \). Moreover \( \gamma_1 = \gamma_{000} + \gamma_{00} + \gamma_{110} + \gamma_{11}^{\frac{1}{2}} \) is a 1-cocycle for \( \mathfrak{K}(1) \) if and only if

\[
\begin{align*}
\gamma_{010} &= \alpha_1 C_k + \alpha_2 \tilde{C}_k, \\
\gamma_{01} &= \beta_1 C_{k-1} + \beta_2 \tilde{C}_{k-1}, \\
\gamma''[X_g, X_{\theta h}] &= \mathcal{L}^{\lambda, \mu}_{X_g} \gamma''(X_{\theta h}) - \mathcal{L}^{\lambda, \mu}_{X_{\theta h}} \gamma'(X_g), \\
\gamma'[X_{\theta h_1}, X_{\theta h_2}] &= \mathcal{L}^{\lambda, \mu}_{X_{\theta h_1}} \gamma''(X_{\theta h_2}) + \mathcal{L}^{\lambda, \mu}_{X_{\theta h_2}} \gamma''(X_{\theta h_1}),
\end{align*}
\]

(3.27)

where \( \gamma' = \gamma_{010} + \gamma_{01}^{\frac{1}{2}} \) and \( \gamma'' = \gamma_{100} + \gamma_{10}^{\frac{1}{2}} \).

As above, we then can write

\[
\gamma_{100}(X_{\theta h}) = a_0 h \theta \partial_x^k + a_1 h' \theta \partial_x^{k-1} \quad \text{and} \quad \gamma_{10}^{\frac{1}{2}}(v_{\theta h}) = b_0 h \partial_y \partial_x^k + b_1 h' \theta \partial_y \partial_x^{k-1}.
\]

According to Lemma 3.3, the map \( \gamma_1 \) is a non trivial 1-cocycle if and only if at least one of the maps \( \gamma_{010} \) and \( \gamma_{01} \) is a non trivial 1-cocycle for \( \mathfrak{sl}(2) \), that means \((\alpha_1, \alpha_2, \beta_1, \beta_2) \neq (0, 0, 0, 0)\). Let us determine the linear maps \( \gamma_{100} \) and \( \gamma_{10}^{\frac{1}{2}} \). Up to factor, we get:

\[
\gamma_1 = \alpha_1 \gamma_{10}^{\frac{1}{2}} + \alpha_2 \tilde{\gamma}_{10}^{\frac{1}{2}} + a_0 \delta(2 \theta \partial_x^k).
\]

Thus, the cohomology classes of \( \gamma_{10}^{\frac{1}{2}} \) and \( \tilde{\gamma}_{10}^{\frac{1}{2}} \) generate \( H_1^1(\mathfrak{osp}(1|2), \mathfrak{D}_{1, \frac{1}{2}}) \). The proof is now complete.

References

[1] I. Basdouri, M. Ben Ammar, N. Ben Fraj, M. Boujelbene and K. Kamoun Cohomology of the Lie Superalgebra of Contact Vector Fields on \( \mathbb{R}^{1|1} \) and Deformations of the Superspace of Symbols, [math.RT/0702645](http://arxiv.org/abs/math.RT/0702645).

[2] Fuchs D B, Cohomology of infinite-dimensional Lie algebras, Plenum Publ. New York, 1986.

[3] H. Gargoubi, Sur la géométrie de l’espace des opérateurs différentiels lineaires sur \( \mathbb{R} \), Bull. Soc. Roy. Sci. Liège. Vol. 69, 1, 2000, 2147.
[4] H. Gargoubi, N. Mellouli and V. Ovsienko, *Differential Operators on Supercircle: Conformally Equivariant Quantization and Symbol Calculus*, Letters in Mathematical Physics (2007) 79: 5165.

[5] P. B. A. Lecomte, *On the cohomology of $\mathfrak{sl}(n + 1; \mathbb{R})$ acting on differential operators and $\mathfrak{sl}(n + 1; \mathbb{R})$-equivariant symbols*, Indag. Math. NS. 11 (1), (2000), 95–114.

[6] A. Nijenhuis, R. W. Richardson Jr., *Deformations of homomorphisms of Lie groups and Lie algebras*, Bull. Amer. Math. Soc. 73 (1967), 175–179.