GLOBAL HYPOELLIPTICITY FOR A CLASS OF PSEUDO-DIFFERENTIAL OPERATORS ON THE TORUS

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Abstract. We show that an obstruction of number-theoretical nature appears as a necessary condition for the global hypoellipticity of the pseudo-differential operator $L = D_t + (a + ib)(t)P(D_x)$ on $T^1_t \times T^N_x$. This condition is also sufficient when the symbol $p(\xi)$ of $P(D_x)$ has at most logarithmic growth. If $p(\xi)$ has super-logarithmic growth, we show that the global hypoellipticity of $L$ depends on the change of sign of certain interactions of the coefficients with the symbol $p(\xi)$. Moreover, the interplay between the order of vanishing of coefficients with the order of growth of $p(\xi)$ plays a crucial role in the global hypoellipticity of $L$. We also describe completely the global hypoellipticity of $L$ in the case where $P(D_x)$ is homogeneous. Additionally, we explore the influence of irrational approximations of a real number in the global hypoellipticity.

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1. Introduction

We investigate the global hypoellipticity of pseudo-differential operators of the form

$$L = D_t + (a + ib)(t)P(D_x), \quad (t, x) \in T^1_t \times T^N_x,$$

where $a(t)$ and $b(t)$ are real smooth functions on $T^1_t$, and $P(D_x)$ is a pseudo-differential operator of order $m \in \mathbb{R}$ defined on $T^N \simeq \mathbb{R}^N/(2\pi \mathbb{Z}^N)$. The operator $P(D_x)$ is given

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by
\begin{equation}
(1.2)\quad P(D_x) \cdot u = \sum_{\xi \in \mathbb{Z}^N} e^{i\xi \cdot x} p(\xi) \hat{u}(\xi),
\end{equation}
where \( p = p(\xi) \in S^m(\mathbb{Z}^N) \) is the toroidal symbol of \( P(D_x) \) and
\[
\hat{u}(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{-i\xi \cdot x} u(x) dx, \quad \xi \in \mathbb{Z}^N,
\]
are the Fourier coefficients of \( u \).

The operator \( L \) is said to be \textit{globally hypoelliptic} on \( \mathbb{T}^1 \times \mathbb{T}^N \) if the conditions
\[
u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N) \quad \text{and} \quad Lu \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)
\]
imply that \( u \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N) \).

Even in the case of vector fields, the investigation of global hypoellipticity on the torus is a challenging problem that still have open questions. Perhaps the question without an answer that is most famous and seemingly far from a solution is the Greenfield and Wallach conjecture. It states that: if a smoothly closed manifold \( M \) admits a globally hypoelliptic vector field \( X \), then \( M \) is diffeomorphic to a torus and \( X \) is smooth conjugated to a Diophantine vector field (see [22]).

This conjecture has a geometric version stated in terms of cohomology-free dynamical systems, known as Katok conjecture, and was proved only in some few cases and in dimensions 2 and 3. For more details we refer the works of G. Forni [17], J. Hounie [27], and A. Kocsard [28].

With respect to the differential case of the operator we are interested, with \( P(D_x) = D_x \) and \( N = 1 \), J. Hounie has proved in Theorem 2.2 of [26] that \( L = D_t + (a + ib)(t)D_x \) is globally hypoelliptic on \( \mathbb{T}^2 \) if and only if \( b(t) \) does not change sign and either \( b_0 \neq 0 \) or \( a_0 \) is an irrational non-Liouville number, where
\[
a_0 = (2\pi)^{-1} \int_0^{2\pi} a(t) dt \quad \text{and} \quad b_0 = (2\pi)^{-1} \int_0^{2\pi} b(t) dt.
\]

We recall that S. Greenfield and N. Wallach have proved in [21] that the above conditions on \( a_0 \) and \( b_0 \) means that the constant coefficient operator \( D_t + (a_0 + ib_0)D_x \) is globally hypoelliptic. Therefore, the global hypoellipticity of \( D_t + (a_0 + ib_0)D_x \) is a necessary condition for the global hypoellipticity of the operator with variable coefficients \( D_t + (a + ib)(t)D_x \).

We prove that this necessity remains valid for any pseudo-differential operator \( P(D_x) \) defined on the \( N \)-dimensional torus, that is, if the operator \( L \) defined in (1.1) is globally hypoelliptic then the constant coefficient operator
\begin{equation}
(1.3)\quad L_0 = D_t + (a_0 + ib_0)P(D_x), \quad (t, x) \in \mathbb{T}^1 \times \mathbb{T}^N,
\end{equation}
is also globally hypoelliptic (see Theorem 3.5).

We also show that the global hypoellipticity of \( L_0 \) and the control of the sign of the imaginary part of the functions
\[
t \in \mathbb{T}^1 \mapsto \mathcal{M}(t, \xi) = (a + ib)(t) p(\xi), \quad \xi \in \mathbb{Z}^N,
\]
for sufficiently large \( |\xi| \), are sufficient conditions to the global hypoellipticity of \( L \) (see Theorem 3.6).

Although the global hypoellipticity of \( L_0 \) cannot be removed in the study of the global hypoellipticity of \( L \), the converse of Theorem 3.6 in general does not hold; unlike
the differential case \( P(D_x) = D_x \). In Sections 4 and 5 we exhibit examples of globally hypoelliptic operators in which the imaginary part of the functions \( t \in \mathbb{T}^1 \mapsto \mathcal{M}(t, \xi) \) changes sign for infinitely many indexes \( \xi \in \mathbb{Z}^N \) (see Examples 4.5, 4.6, 4.11, 4.13 and the first example in Subsection 5.1).

We point out that our results are not a consequence of the Hounie’s abstract results in [26], even when our operator fits in the conditions assumed in that work. Depending on \( P(D_x) \), the scales of Sobolev spaces used by Hounie are different from the usual Sobolev spaces, which implies in a different notion of global hypoellipticity. We refer the reader to [1], Section 3.3, for more details.

In Section 2 we study operators with constant coefficients giving a special attention to the case when \( P(D_x) \) is a homogeneous operator of rational degree \( m \) on \( \mathbb{T}^1 \), see Theorem 2.4. In this case, our main contribution is to shed light on the connections between hypoellipticity and certain approximations of real numbers, which are not considered in [21]. Indeed, in our approach the global hypoellipticity depends on the following approximations

\[
\left| \frac{\tau}{|\xi|^m} + p(\pm 1) \right|, \ (\tau, \xi) \in \mathbb{Z} \times \mathbb{Z},
\]

where the numbers \( \tau/|\xi|^m \) can be irrational, depending on \( m \).

As a consequence of Theorem 2.4 if \( m = \ell/q \) is irreducible, with \( \ell, q \in \mathbb{N} \), then the operator \( D_t + \alpha(D_x^2)^{m/2} \) is globally hypoelliptic if and only if \( \alpha^q \) is an irrational non-Liouville number. Notice that the global hypoellipticity of this operator does not depend on \( \ell \). For example, \( D_t + \sqrt{2}(D_x^2)^{1/2} \) is globally hypoelliptic if and only if \( q \) is odd.

Regarding the case of variable coefficients, one of the contributions of this work is to show that the global hypoellipticity of the operator \( L \) defined in (1.1) is related to the growth of the real and imaginary parts of the symbol \( p(\xi) \) when \( |\xi| \to \infty \).

In Section 4 we give a complete characterization for the global hypoellipticity of \( L \) when either \( \alpha(\xi) \) or \( \beta(\xi) \) has at most logarithmic growth, where

\[
p(\xi) = \alpha(\xi) + i\beta(\xi), \ \xi \in \mathbb{Z}^N.
\]

When both \( \alpha(\xi) \) and \( \beta(\xi) \) have at most logarithmic growth we show that the change of sign of the functions

\[
t \in \mathbb{T}^1 \mapsto \mathcal{M}(t, \xi) = a(t)\beta(\xi) + b(t)\alpha(\xi), \ \xi \in \mathbb{Z}^N,
\]
does not play any role in the global hypoellipticity of \( L \). More precisely, we prove that \( L \), defined in (1.1), is globally hypoelliptic if and only if \( L_0 \), defined in (1.3) is globally hypoelliptic. This equivalence comes from the reduction to normal form, that is a technique well explored in the works [1] [13] [14] [15] [16] [32].

If \( \beta(\xi) \) has at most logarithmic growth, but \( \alpha(\xi) \) has super-logarithmic growth, then we prove that \( L \) is globally hypoelliptic if and only if \( L_0 \) is globally hypoelliptic and \( b(t) \) does not change sign. This result remains valid if we exchange \( \beta(\xi) \) by \( \alpha(\xi) \) and \( b(t) \) by \( a(t) \), see Subsection 4.2.

When \( \alpha(\xi) \) and \( \beta(\xi) \) have super-logarithmic growth, the interactions between the functions \( a(t)\beta(\xi) \) and \( b(t)\alpha(\xi) \) play a larger role. In this case, the operator \( L \) may be non-globally hypoelliptic even if \( L_0 \) is globally hypoelliptic and both \( a(t) \) and \( b(t) \) do not change sign (see Examples 4.13, 5.3 and the second example in Subsection 5.1).
On the other hand, $L$ may be globally hypoelliptic even if both $a(t)$ and $b(t)$ changes sign provided that $\alpha(\xi)$ and $\beta(\xi)$ go to infinity with the same order of growth (see Example 5.6).

In the case where both the parts $\alpha(\xi)$ and $\beta(\xi)$ have super-logarithmic growth and $\alpha(\xi)/\beta(\xi) \to K$, as $|\xi| \to \infty$, we show that $L$ is not globally hypoelliptic if the function $t \in \mathbb{T}^1 \to a(t) + b(t)K$ changes sign (Corollary 5.2). In particular, if $p(\xi)$ has super-logarithmic growth with $\alpha(\xi) = o(\beta(\xi))$, then $L$ is not globally hypoelliptic if $a(t)$ changes sign. Analogously, $L$ is not globally hypoelliptic when $\beta(\xi) = o(\alpha(\xi))$ and $b(t)$ changes sign (see Corollary 5.3).

Another contribution we give is to present (in Subsection 5.1) a relation between the global hypoellipticity of the operator $L$ and the order of vanishing of the coefficients $a(t)$ and $b(t)$. We emphasize that this phenomenon is more common in the study of the global solvability of vector fields on the torus (see [4, 5, 12, 18]).

In Section 6 we describe completely the global hypoellipticity of $L$ in the case where $P(D_x)$ is homogeneous (see Theorem 6.1 and Corollary 6.2). For these operators the converse of Theorem 3.6 holds. Moreover, we analyze the case of sums of homogeneous operators extending Theorem 1.3 of [6], see Corollary 6.6.

For more results on the problem of global hypoellipticity and global solvability of equations and systems of equations on the torus we refer the reader to the works [2, 3, 8, 9, 10, 11, 19, 20, 24] and the references therein.

2. The constant coefficient operators

By following the approach introduced by Greenfield and Wallach in [21], we may characterize the global hypoellipticity of the operator

$$L = D_t + P(D_x), \quad (t, x) \in \mathbb{T}^1 \times \mathbb{T}^N,$$

by means of a control in its symbol

$$L(\tau, \xi) = \tau + p(\xi), \quad (\tau, \xi) \in \mathbb{Z} \times \mathbb{Z}^N.$$

**Theorem 2.1.** The operator $L$ in (2.1) is globally hypoelliptic if and only if there exist positive constants $C$, $M$ and $R$ such that

$$|\tau + p(\xi)| \geq \frac{C}{(|\tau| + |\xi|)M}, \quad \text{for all} \quad |\tau| + |\xi| \geq R.$$

The proof of this result follows the same ideas of the differential case made in [21].

Note that, if the imaginary part of $p(\xi)$ does not approach to zero rapidly, then the estimate in Theorem 2.1 is verified. More precisely, if there exists $M \geq 0$ such that

$$\liminf_{|\xi| \to \infty} |\xi|^M |\Im p(\xi)| > 0,$$

then the operator $L = D_t + P(D_x)$ is globally hypoelliptic.

This type of condition appears in Theorem 5.3 of [16], where the authors studied the relation between global hypoellipticity and simultaneous inhomogeneous Siegel conditions.
On the other hand, without this control in the imaginary part, for instance when $\Im p(\xi) \equiv 0$, Diophantine phenomena appear. When the symbol is homogeneous of rational positive degree, we present a new relation between global hypoellipticity and Liouville numbers in Theorem 2.1.

We observe that each toroidal symbol $p$ in the class $S^m(\mathbb{Z}^N)$ can be extended to an Euclidean symbol $\tilde{p} \in S^m(\mathbb{R}^N)$ such that $p = \tilde{p}|_{\mathbb{Z}^N}$ (see Theorem 4.5.3 of [33]).

**Definition 2.2.** We say that a toroidal symbol $p(\xi)$ is homogeneous of degree $m$ if it has an Euclidean extension $\tilde{p}(\xi)$ such that $p(\xi) = |\xi|^m \tilde{p}(\xi/|\xi|)$, $\xi \in \mathbb{Z}^*_N$.

In order to not overload our notation, we will use the notation $p(\xi/|\xi|)$ in place of $\tilde{p}(\xi/|\xi|)$.

When the symbol $p(\xi)$ is homogeneous of degree $m$, it follows from Theorem 2.1 that the operator $L$ given by (2.1) is globally hypoelliptic if and only if there exist positive constants $C$, $M$, and $R$, such that

$$\left| \frac{\tau}{|\xi|^m} + p \left( \frac{\xi}{|\xi|} \right) \right| \geq C \left( |\tau| + |\xi| \right)^M,$$

for all $(\tau, \xi) \in \mathbb{Z} \times \mathbb{Z}^*_N$ which satisfy $|\tau| + |\xi| \geq R$.

### 2.1. Global hypoellipticity and Liouville numbers.

Let $P(D_x)$ be an operator on $\mathbb{T}^1$ with symbol $p(\xi)$ homogeneous of degree $m$. In this case

$$p(\xi) = |\xi|^m p(\pm 1), \quad \text{for all } \xi \in \mathbb{Z}_n.$$

Thus, when $m < 0$, by using condition (2.2) we see that the operator $L$ is globally hypoelliptic if and only if $p(\pm 1) \neq 0$. Similarly, when $m = 0$, it follows that $L$ is globally hypoelliptic if and only if $p(\pm 1) \notin \mathbb{Z}$.

The case in which $m$ is a positive rational number and $\Im p(\pm 1) = 0$, is more interesting. We now move to describe it.

By using the notations $p(1) = \alpha + i\beta$ and $p(-1) = \tilde{\alpha} + i\tilde{\beta}$, we have

$$\left| \frac{\tau}{|\xi|^m} + p \left( \frac{\xi}{|\xi|} \right) \right| = \begin{cases} \left| \frac{\tau}{|\xi|^m} + (\alpha + i\beta) \right|, & \text{if } \xi > 0, \\ \left| \frac{\tau}{|\xi|^m} + (\tilde{\alpha} + i\tilde{\beta}) \right|, & \text{if } \xi < 0. \end{cases}$$

In this case, when $\beta = 0$ (respectively $\tilde{\beta} = 0$) we must control the approximations of the real number $\alpha$ (respectively $\tilde{\alpha}$) by numbers of the type $\tau/|\xi|^m$, for all $(\tau, \xi) \in \mathbb{Z} \times \mathbb{Z}_n$.

**Definition 2.3.** An irrational number $\lambda$ is said to be a Liouville number if there exists a sequence $(j_n, k_n) \in \mathbb{Z} \times \mathbb{N}$, such that $k_n \to \infty$ and

$$\left| \lambda - \frac{j_n}{k_n} \right| < (k_n)^{-n}, \quad n \in \mathbb{N}.$$

Under the previous notation we have the following result:
\textbf{Theorem 2.4.} If $p = p(\xi)$ is a homogeneous symbol of degree $m = \ell/q$ with $\ell, q \in \mathbb{N}$, and $\gcd(\ell, q) = 1$, then the operator

\[ L = D_t + P(D_x), \quad (t, x) \in \mathbb{T}^2, \]

is globally hypoelliptic if and only if $\alpha \beta = 0$ is globally hypoelliptic in this case. Therefore, in order to prove Theorem 2.4 it is enough to consider either $\beta = 0$ or $\tilde{\beta} = 0$.

\textit{Proof.} Inequality (2.2) is easily verified when $\beta \cdot \tilde{\beta} \neq 0$, consequently $L$ is globally hypoelliptic in this case. Therefore, in order to prove Theorem 2.4 it is enough to consider either $\beta = 0$ or $\tilde{\beta} = 0$.

We start by considering the case $\beta = 0$ and $\tilde{\beta} \neq 0$. In this situation, it follows from (2.2) and (2.3) that $L$ is globally hypoelliptic if and only if there exist positive constants $C, M$ and $R$ such that

\[ (2.4) \quad \left| \frac{\tau}{\xi^m} + \alpha \right| \geq C(|\tau| + \xi)^{-M}, \]

for all $(\tau, \xi) \in \mathbb{Z} \times \mathbb{N}$ such that $|\tau| + \xi \geq R$.

Since $\beta = 0$, if $\alpha = 0$ then $p(\xi) = 0$ for all $\xi > 0$ and, therefore, $L$ is not globally hypoelliptic. From now on, without loss of generality, we assume that $\alpha > 0$.

When $\alpha^q$ is a rational number, we prove that $L$ is not globally hypoelliptic by exhibiting infinitely many $(\tau, \xi) \in \mathbb{Z} \times \mathbb{N}$ such that $|\tau/\xi^m - \alpha| = 0$.

We then write $\alpha^q = \tilde{p}/\tilde{q}$, with $\tilde{p}, \tilde{q} \in \mathbb{N}$. By prime factorization we have

\[ \tilde{q} = q_1^{n_1} \cdots q_r^{n_r} \quad \text{and} \quad \tilde{p} = p_1^{n_1} \cdots p_s^{n_s}. \]

Since $\gcd(\ell, q) = 1$, there exists $(x_i, y_i) \in \mathbb{N}^2$ and $(v_j, w_j) \in \mathbb{N}^2$ such that

\[ \ell x_i - q y_i = \gamma_i, \quad i = 1, \ldots, r \quad \text{and} \quad q v_j - \ell w_j = \sigma_j, \quad j = 1, \ldots, s. \]

Define

\[ \tau_n = n^{\ell} q_1^{n_1} \cdots q_r^{n_r} p_1^{n_1} \cdots p_s^{n_s} \quad \text{and} \quad \xi_n = q q_1^{n_1} \cdots q_r^{n_r} p_1^{n_1} \cdots p_s^{n_s}. \]

It follows that $\tilde{p}\tau_n^q = \tilde{p}\xi_n^q$, for all $n \in \mathbb{N}$; hence

\[ \left( \frac{\tau_n}{\xi_n} \right)^m = \left( \frac{\tau_n}{\xi_n} \right)^{\ell/q} = \left( \frac{\tilde{p}}{\tilde{q}} \right)^{1/q} = \alpha, \quad \text{for all} \quad n \in \mathbb{N}, \]

and then $L$ is not globally hypoelliptic.

From now on, assume that $\alpha^q$ is an irrational number.

If $L$ is not globally hypoelliptic, it follows from (2.4) that there exists a sequence $(\tau_n, \xi_n) \in \mathbb{Z} \times \mathbb{N}$ such that

\[ \left| \frac{\tau_n}{\xi_n^{\ell/q}} - \alpha \right| < (|\tau_n| + \xi_n)^{-n}, \quad |\tau_n| + \xi_n \geq n. \]

By taking $j_n = -\tau_n^q$ and $k_n = \xi_n^q$ we obtain

\[ \left| \frac{j_n}{k_n} + \alpha^q \right| = \left| - \left( \frac{\tau_n}{\xi_n} \right)^q + \alpha^q \right| = \frac{\tau_n}{\xi_n^{\ell/q}} - \alpha \cdot \sum_{j=1}^{q} \left( \frac{\tau_n}{\xi_n} \right)^{q-j} \alpha^{j-1}. \]
Since \( \sum_{j=1}^{q} (\tau_n/\xi_n^{l/q})^{q-j} \alpha^{q-1} \) goes to \( q\alpha^{q-1} \), as \( n \) goes to infinity, it follows that

\[
\left| \frac{j_n}{k_n} + \alpha^{q} \right| \leq C \left| \frac{\tau_n}{\xi_n^{l/q}} - \alpha \right| \leq C (|\tau_n| + \xi_n)^{-n} \leq C k_n^{-n/\ell}, \quad \text{for all } n,
\]

where the constant \( C > 0 \) does not depend on \( j_n \) and \( k_n \).

The estimate above implies that \( \alpha^q \) is a Liouville number.

Assuming that \( \alpha^q \) is a Liouville number, let us show that \( L \) is not globally hypoelliptic. Indeed, if \( \alpha^q \) is a Liouville number, then there is a sequence \( (j_n, k_n) \in \mathbb{N}^2 \), \( j_n + k_n \geq n \), such that

\[
|j_n - \alpha^q k_n| < (j_n + k_n)^{-n}.
\]

By multiplying this inequality by

\[
j_n^{q-1 + (\ell-1)\tilde{p}\ell} k_n^q,
\]

where \( \tilde{p} \) and \( \tilde{q} \) are positive integers such that \( \tilde{p} \ell - \tilde{q} q = 1 \), we obtain

\[
|\left( j_n^{\ell+\tilde{q}(\ell-1)} k_n^q \right)^q - \alpha^q (k_n j_n^{q-1+(\ell-1)\tilde{p}})^q| < j_n^{q-1 + (\ell-1)\tilde{p}\ell} k_n^{q}(j_n + k_n)^{-n}.
\]

Suppose, by contradiction, that \( L \) is globally hypoelliptic. By (2.4), there exist positive constants \( C, M \) and \( R \) such that

\[
|\tau - \alpha^{\ell/q}| \geq C (|\tau| + \xi)^{-M},
\]

for all \( (\tau, \xi) \in \mathbb{N} \times \mathbb{N} \) such that \( \tau + \xi > R \).

Since

\[
|\tau^q - \alpha^q \xi^{\ell/q}| = |\tau - \alpha^{\ell/q}| \cdot \left| \sum_{\kappa=1}^{q} \tau^{q-\kappa} (\alpha^{\ell/q})^{q-\kappa} \right| \geq C (\tau + \xi)^{-M} \left| \sum_{\kappa=1}^{q} \tau^{q-\kappa} (\alpha^{\ell/q})^{q-\kappa} \right|,
\]

and

\[
\left| \sum_{\kappa=1}^{q} \left( j_n^{\ell+\tilde{q}(\ell-1)} k_n^q \right)^{q-\kappa} \alpha^{q-\kappa} \left( k_n^{q-1+(\ell-1)\tilde{p}} \right)^{\ell(q-1)\tilde{p}\ell} k_n^{q}(j_n + k_n)^{-n} \right| \geq \tilde{C}, \quad (j_n \geq 1, \ k_n \geq 1),
\]

for some \( \tilde{C} > 0 \), it follows that

\[
C \tilde{C} \leq \left( j_n^{\ell+\tilde{q}(\ell-1)} k_n^q + k_n^{q-1+(\ell-1)\tilde{p}} \right)^{M} j_n^{q-1 + (\ell-1)\tilde{p}\ell} k_n^{q}(j_n + k_n)^{-n},
\]

for all \( n \in \mathbb{N} \).

Now, by taking

\[
K = \max \{ \ell + \tilde{q}(\ell - 1), \tilde{q}, \tilde{p}, q - 1 + (\ell - 1)\tilde{p}, (q - 1)\ell + (\ell - 1)\tilde{p}\ell, \tilde{q} q \}
\]

we obtain

\[
0 < C \tilde{C} \leq \left( j_n^{\ell+\tilde{q}(\ell-1)} k_n^q + k_n^{q-1+(\ell-1)\tilde{p}} \right)^{M} j_n^{q-1 + (\ell-1)\tilde{p}\ell} k_n^{q}(j_n + k_n)^{-n} \leq \left( j_n^{K k_n^K} + k_n^{K k_n^K} \right)^{M} \left( j_n^{K k_n^K} \right)^{n} = 2^M (j_n + k_n)^{-n + 2K(M+1)},
\]

which contradicts the fact that \( \alpha^q \) is a Liouville number.
for all $n \in \mathbb{N}$, which is a contradiction, since the right-hand side goes to zero as $n$ goes to infinity.

Finally, in the case in which $\beta \neq 0$ and $\tilde{\beta} = 0$, a slight modification in the previous arguments give us that $L$ is globally hypoelliptic if and only if $\tilde{\alpha}^q$ is an irrational non-Liouville number. Analogously, if $\beta = 0$ and $\tilde{\beta} = 0$, then $L$ is globally hypoelliptic if and only if both $\alpha^q$ and $\tilde{\alpha}^q$ are irrational non-Liouville numbers.

\[ \Box \]

As consequence of Theorem 2.4 we obtain the following examples.

**Example 2.5.** Let $m = \ell/q$ be a positive rational number with $\gcd(\ell, q) = 1$, then $L = D_t + \alpha(D_x^2)^{m/2}$ is globally hypoelliptic if and only if $\alpha^q$ is an irrational non-Liouville number. In particular, for the non-Liouville number $\alpha = \sqrt{2}$ the operator $L = D_t + \alpha(D_x^2)^{1/2}$ is globally hypoelliptic while $L = D_t + \alpha(D_x^2)^{1/4}$ is not.

**Example 2.6.** Let $\lambda = \sum_{n=1}^{\infty} 10^{-n!}$ be the Liouville constant. For each integer $q \geq 2$ we have that $\lambda^q/2$ is a Liouville number while $\lambda^{3/2}$ is not (see [29]). Therefore, by taking $\alpha = \lambda^{3/2}$ we have that $L = D_t + \alpha(D_x^2)^{1/2q}$ is not globally hypoelliptic for each integer $q \geq 2$.

### 3. The variable coefficient operators

In this section we study the global hypoellipticity of the operator (1.1), which we recall

\[ L = D_t + (a + ib)(t)P(D_x), \quad (t, x) \in \mathbb{T}^1 \times \mathbb{T}^N, \]

where $a(t)$ and $b(t)$ are real valued smooth functions on $\mathbb{T}^1$ and $P(D_x)$ is a pseudo-differential on $\mathbb{T}^N$ with symbol $p = p(\xi), \xi \in \mathbb{Z}^N$.

Without any assumption about the behavior of $p(\xi)$, as $|\xi| \to \infty$, we will present a necessary condition and, also, sufficient conditions for the global hypoellipticity of $L$.

First, we show that the global hypoellipticity of $L_0 = D_t + (a_0 + ib_0)P(D_x)$, where

\[ a_0 = (2\pi)^{-1} \int_0^{2\pi} a(t)dt \quad \text{and} \quad b_0 = (2\pi)^{-1} \int_0^{2\pi} b(t)dt, \]

is necessary for the global hypoellipticity of $L$ (Theorem 3.5). After this, we will show that this condition is also sufficient provided that the imaginary part of the function

\[ t \in \mathbb{T}^1 \mapsto M(t, \xi \xi) \equiv (a + ib)(t)p(\xi), \quad \xi \in \mathbb{Z}^N, \]

does not change sign, for all $|\xi|$ large enough (Theorem 3.6).

By using partial Fourier series in the variable $x$, we can write a distribution $u$ in $\mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N)$ as

\[ u = \sum_{\xi \in \mathbb{Z}^N} \widehat{u}(t, \xi)e^{ix\xi}, \]

where $\widehat{u}(t, \xi) = (2\pi)^{-N}\langle u(t, \cdot), e^{-ix\xi} \rangle$. Hence, the equation $(iL)u = f$ lead us to consider the differential equations

\[ \partial_t \widehat{u}(t, \xi) + iM(t, \xi)\widehat{u}(t, \xi) = \widehat{f}(t, \xi), \quad t \in \mathbb{T}^1, \quad \text{for all } \xi \in \mathbb{Z}^N. \]

\[ (3.1) \]
With the notations
\[ M_0(\xi) = (2\pi)^{-1} \int_0^{2\pi} M(t, \xi) dt = (a_0 + ib_0)p(\xi) \]
and
\[ (3.2) \quad Z_M = \{ \xi \in \mathbb{Z}^N; \ M_0(\xi) \in \mathbb{Z} \}, \]
we have:

**Lemma 3.1.** If \( u \in D'(\mathbb{T}^1 \times \mathbb{T}^N) \) and \( iLu = f \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N) \), then equation \((3.1)\) implies that \( \hat{u}(\cdot, \xi) \) belongs to \( C^\infty(\mathbb{T}^1) \), for all \( \xi \in \mathbb{Z}^N \).

Moreover, for each \( \xi \notin Z_M \), equation \((3.1)\) has a unique solution, which can be written in the following two ways:

\[ \hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i M_0(\xi)}} \int_0^{2\pi} \exp\left(-i \int_{t-s}^t M(r, \xi) dr\right) \hat{f}(t-s, \xi) ds, \quad (3.3) \]
or
\[ \hat{u}(t, \xi) = \frac{1}{e^{2\pi i M_0(\xi)} - 1} \int_0^{2\pi} \exp\left(i \int_{t+s}^{t+s} M(r, \xi) dr\right) \hat{f}(t+s, \xi) ds. \quad (3.4) \]

Furthermore, we have the following characterization for the global hypoellipticity of \( L_0 \).

**Proposition 3.2.** The following statements are equivalent:

i) \( L_0 \) is globally hypoelliptic;

ii) There exist positive constants \( C, M, \) and \( R \) such that
\[ |\tau + M_0(\xi)| \geq C(|\tau| + |\xi|)^{-M}, \text{ for all } (|\tau| + |\xi|) \geq R; \]

iii) There exist positive constants \( \tilde{C}, \tilde{M}, \) and \( \tilde{R} \) such that
\[ |1 - e^{\pm 2\pi i M_0(\xi)}| \geq \tilde{C}|\xi|^{-\tilde{M}}, \text{ for all } |\xi| \geq \tilde{R}. \]

The equivalence \( i) \Leftrightarrow ii) \) follows from Theorem 2.1 and the equivalence \( ii) \Leftrightarrow iii) \) is a technical result that is a slight modification of the proof of Lemma 3.1 of [6].

3.1. **A necessary condition.** Our first result in this section is the following:

**Proposition 3.3.** If \( L \) is globally hypoelliptic, then the set \( Z_M \) defined in \((3.2)\) is finite.

**Proof.** If \( Z_M \) is infinite, then there exists a sequence \( \{\xi_n\} \) such that \( |\xi_n| \) is increasing and \( M_0(\xi_n) \in \mathbb{Z} \). Set
\[ c_n = \exp\left(-\int_0^{t_n} \Im M(r, \xi_n) dr\right), \]
where \( t_n \in [0, 2\pi] \) is such that
\[ \int_0^{t_n} \Im M(r, \xi_n) dr = \max_{t \in [0, 2\pi]} \int_0^t \Im M(r, \xi_n) dr. \]
For each \( \xi_n \) the function
\[ \hat{u}(t, \xi_n) = c_n \exp\left(-i \int_0^t M(r, \xi_n) dr\right) \]
is smooth on $\mathbb{T}^1$ and satisfies the equation

$$
\partial_t \tilde{u}(t, \xi_n) + i \mathcal{M}(t, \xi_n) \tilde{u}(t, \xi_n) = 0.
$$

Moreover, $|\tilde{u}(t, \xi_n)| \leq 1$, for all $t \in [0, 2\pi]$, and $|\tilde{u}(t_n, \xi_n)| = 1$. Hence,

$$
u = \sum_{n=1}^{\infty} \tilde{u}(t, \xi_n) e^{ix\xi_n} \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N) \setminus C^\infty(\mathbb{T}^1 \times \mathbb{T}^N),$$

and satisfies $Lu = 0$. Therefore, $L$ is not globally hypoelliptic. \qed

Remark 3.4. In the case of constant coefficients the previous result implies that $L_0$ is not globally hypoelliptic if $Z_M$ is infinite. Therefore, every time we assume that $L_0$ is globally hypoelliptic it is understood that $Z_M$ is finite.

Now we present our first general result on global hypoellipticity.

**Theorem 3.5.** If $L$ is globally hypoelliptic, then $L_0$ is globally hypoelliptic.

**Proof.** We assume that $L_0$ is not globally hypoelliptic and prove that $L$ is not globally hypoelliptic.

By Proposition 3.2, there is a sequence $\{\xi_n\}$ such that $|\xi_n|$ is strictly increasing, $|\xi_n| > n$, and

$$
|1 - e^{-2\pi i M_0(\xi_n)}| < |\xi_n|^{-n}, \text{ for all } n \in \mathbb{N}.
$$

By Proposition 3.3 it is enough to consider the case where $Z_M$ is finite and $\xi_n \notin Z_M$, for all $n$.

For each $n$, we may choose $t_n \in [0, 2\pi]$ so that $\int_{t_n}^{t} \Im M(r, \xi_n)dr \leq 0$, for all $t \in [0, 2\pi]$.

Indeed, for all $t \in [0, 2\pi]$ we write

$$
\int_{t_n}^{t} \Im M(r, \xi_n)dr = \int_{0}^{t} \Im M(r, \xi_n)dr - \int_{0}^{t_n} \Im M(r, \xi_n)dr,
$$

and it is enough to consider $t_n$ satisfying

$$
\int_{0}^{t_n} \Im M(r, \xi_n)dr = \max_{t \in [0, 2\pi]} \int_{0}^{t} \Im M(r, \xi_n)dr.
$$

By passing to a subsequence, we may assume that there exists $t_0 \in [0, 2\pi]$ such that $t_n \to t_0$, as $n \to \infty$.

Let $I$ be a closed interval in $(0, 2\pi)$ such that $t_0 \notin I$. Consider $\phi$ belonging to $C^\infty_c(I, \mathbb{R})$, such that $0 \leq \phi(t) \leq 1$ and $\int_{0}^{2\pi} \phi(t)dt > 0$.

For each $n$, we define $\hat{f}(\cdot, \xi_n)$ as being the $2\pi$–periodic extension of

$$
(1 - e^{-2\pi i M_0(\xi_n)}) \exp \left(-\int_{t_n}^{t} i M(r, \xi_n)dr\right) \phi(t).
$$

Since $p(\xi)$ increases slowly, $\int_{t_n}^{t} \Im M(r, \xi_n)dr \leq 0$ for all $t \in [0, 2\pi]$, and since (3.3) holds, it follows that $\hat{f}(\cdot, \xi_n)$ decays rapidly. Hence,

$$
f(t, x) = \sum_{n=1}^{\infty} \hat{f}(t, \xi_n)e^{ix\xi_n} \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N).$$
In order to exhibit a distribution \( u \in \mathcal{D}'(T^1 \times T^N) \setminus C^\infty(T^1 \times T^N) \) such that \( iLu = f \), we consider

\[
\hat{u}(t, \xi_n) = \frac{1}{1 - e^{-2\pi i M_0(\xi_n)}} \int_0^{2\pi} \exp \left( - \int_{t-s}^t i \mathcal{M}(r, \xi_n) dr \right) \hat{f}(t-s, \xi_n) ds.
\]

Note that \( 1 - e^{-2\pi i M_0(\xi_n)} \neq 0 \), since \( \xi_n \not\in \mathbb{Z} \), and \( \hat{u}(\cdot, \xi_n) \in C^\infty(T^1) \) (Lemma 3.1). Moreover, for \( t, s \in [0, 2\pi] \) such that \( t - s \geq 0 \), we have

\[
\left| \frac{1}{1 - e^{-2\pi i M_0(\xi_n)}} \exp \left( - \int_{t-s}^t i \mathcal{M}(r, \xi_n) dr \right) \hat{f}(t-s, \xi_n) \right| \leq \exp \left( \int_{t-s}^t \Im \mathcal{M}(r, \xi_n) dr + \int_{t_n}^{t-s} \Im \mathcal{M}(r, \xi_n) dr \right) = \exp \left( \int_{t_n}^t \Im \mathcal{M}(r, \xi_n) dr + 2\pi \Im M_0(\xi_n) \right).
\]

while for \( t, s \in [0, 2\pi] \) such that \( t + s < 0 \), we have

\[
\left| \frac{1}{1 - e^{-2\pi i M_0(\xi_n)}} \exp \left( - \int_{t-s}^t i \mathcal{M}(r, \xi_n) dr \right) \hat{f}(t-s, \xi_n) \right| \leq \exp \left( \int_{t-s}^t \Im \mathcal{M}(r, \xi_n) dr + \int_{t_n}^{t-s+2\pi} \Im \mathcal{M}(r, \xi_n) dr \right) = \exp \left( \int_{t_n}^t \Im \mathcal{M}(r, \xi_n) dr + 2\pi \Im M_0(\xi_n) \right).
\]

Since \( \Im M_0(\xi_n) \to 0 \), by (3.5), the estimates above imply that \( |\hat{u}(t, \xi_n)| \leq 4\pi \), for all \( t \in T^1 \) and for \( n \) sufficiently large. Hence, \( \hat{u}(\cdot, \xi_n) \) increases slowly and

\[
u(t, x) = \sum_{n=1}^{\infty} \hat{u}(t, \xi_n) e^{ix\xi_n} \in \mathcal{D}'(T^1 \times T^N).
\]

If \( t_0 \) > sup \( I \), then \( t_n \) > sup \( I \), for all \( n \) sufficiently large, and

\[
|\hat{u}(t_n, \xi_n)| = \int_{t_n-\inf I}^{t_n-\sup I} \phi(t_n - s) ds = \int_0^{2\pi} \phi(t) dt > 0,
\]

on the other hand, if \( t_0 \) < inf \( I \), then \( t_n \) < inf \( I \), for all \( n \) sufficiently large, and

\[
|\hat{u}(t_n, \xi_n)| = \left| \int_{t_n-\sup I}^{t_n-\inf I+2\pi} \exp \left( - \int_{t_n-s}^{t_n-s+2\pi} i \mathcal{M}(r, \xi_n) dr \right) \phi(t_n - s + 2\pi) ds \right| = e^{2\pi \Im M_0(\xi_n)} \int_0^{2\pi} \phi(s) ds > (1/2) \int_0^{2\pi} \phi(s) ds > 0,
\]

which implies that \( \hat{u}(\cdot, \xi_n) \) does not decay rapidly.

Hence \( u \in \mathcal{D}'(T^1 \times T^N) \setminus C^\infty(T^1 \times T^N) \), and since \( iLu = f \) (by Lemma 3.1), it follows that \( L \) is not globally hypoelliptic.
3.2. Sufficient conditions. We now present sufficient conditions to the global hypoellipticity of \( L \).

**Theorem 3.6.** If the operator \( L_0 \) given by (1.3) is globally hypoelliptic and the function \( \Im M(t, \xi) = a(t)\beta(\xi) + b(t)\alpha(\xi) \) does not change sign, for sufficiently large \(|\xi|\), then the operator \( L \) given by (1.1) is globally hypoelliptic.

**Proof.** Let \( u \in \mathcal{D}'(T^1 \times T^N) \) be a distribution such that \( iLu = f \), with \( f \in C^\infty(T^1 \times T^N) \). We will show that \( u \in C^\infty(T^1 \times T^N) \).

By using partial Fourier series in the variable \( x \), it follows that \( iLu = f \) if and only if

\[
\partial_t \hat{u}(t, \xi) + iM(t, \xi)\hat{u}(t, \xi) = \hat{f}(t, \xi),
\]

for all \( t \in T^1 \) and for all \( \xi \in \mathbb{Z}^N \).

Lemma 3.1 implies that \( \hat{u}(:, \xi) \in C^\infty(T^1) \), for each \( \xi \in \mathbb{Z}^N \). Moreover, since \( Z_M \) is finite (thanks to Remark 3.4), for \(|\xi| \) sufficiently large the equation (3.6) has a unique solution, which may be written in the form (3.3) or (3.4). Since \( t \mapsto \Im M(t, \xi) \) does not change sign for \(|\xi| \) large enough, we conveniently write

\[
\hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i M_0(\xi)}} \int_0^{2\pi} \exp \left( -i \int_{t-s}^t M(r, \xi) dr \right) \hat{f}(t-s, \xi) ds,
\]

if \( \xi \) is such that \( \Im M(t, \xi) \leq 0 \), for all \( t \in T^1 \), and

\[
\hat{u}(t, \xi) = \frac{1}{e^{2\pi i M_0(\xi)} - 1} \int_0^{2\pi} \exp \left( i \int_{t}^{t+s} M(r, \xi) dr \right) \hat{f}(t+s, \xi) ds,
\]

if \( \xi \) is such that \( \Im M(t, \xi) \geq 0 \), for all \( t \in T^1 \).

Since \( L_0 \) is globally hypoelliptic, then there exist positive constants \( C, M, \) and \( R \), so that

\[
|1 - e^{\pm 2\pi i M_0(\xi)}| \geq C|\xi|^{-M},
\]

for all \(|\xi| \geq R \) (Proposition 3.2).

Hence, for \(|\xi| \) sufficiently large, the solution \( \hat{u}(:\cdot, \xi) \) of (3.6) satisfies

\[
|\hat{u}(t, \xi)| \leq \frac{2\pi}{C} |\xi|^M \|
\hat{f}(\cdot, \xi)\|_\infty.
\]

Similar estimates holds true for the derivatives \( \partial_t^n \hat{u}(t, \xi) \).

Thus, the rapid decaying of the sequence \( \hat{f}(\cdot, \xi) \) implies that \( \hat{u}(\cdot, \xi) \) decays rapidly.

Therefore, \( u \in C^\infty(T^1 \times T^N) \); consequently, \( L \) is globally hypoelliptic.

\[\square\]

In the next sections we will see situations where \( L \) is globally hypoelliptic, but there exist infinitely many indexes \( \xi \) such that \( \Im M(t, \xi) \) changes sign. That is, the assumption that \( \Im M(t, \xi) \) does not change sign is not necessary for the global hypoellipticity of the operator \( L \).
4. Logarithmic Growth

From now on, the speed in which the symbol \( p(\xi) \) goes to infinity will play a crucial point in the study of the global hypoellipticity of

\[
L = D_t + (a + ib)(t)P(D_x), \quad (t, x) \in \mathbb{T}^1 \times \mathbb{T}^N.
\]

We recall that \( p(\xi) = \alpha(\xi) + i\beta(\xi) \), where both \( \alpha(\xi) \) and \( \beta(\xi) \) are real-valued functions in \( S^m(\mathbb{Z}^N) \). In particular

\[
|\alpha(\xi)| \leq C|\xi|^m \quad \text{and} \quad |\beta(\xi)| \leq C|\xi|^m, \quad \text{as} \quad |\xi| \to \infty.
\]

In this section our goal is to deal with the case where either \( \alpha(\xi) \) or \( \beta(\xi) \) has at most logarithmic growth.

**Definition 4.1.** A function \( r : \mathbb{Z}^N \to \mathbb{C} \) has at most logarithmic growth if

\[
r(\xi) = O(\log(|\xi|)), \quad \text{as} \quad |\xi| \to \infty,
\]

that is, there are positive constants \( \kappa \) and \( n_0 \) such that

\[
|r(\xi)| \leq \kappa \log(|\xi|), \quad \text{for all} \quad |\xi| \geq n_0.
\]

When this condition fails, we will say that \( r(\xi) \) has super-logarithmic growth.

When either \( \alpha(\xi) \) or \( \beta(\xi) \) has at most logarithmic growth, the global hypoellipticity of \( L \) is completely characterized by the following:

**Theorem 4.2.** Let \( p(\xi) = \alpha(\xi) + i\beta(\xi) \in S^m(\mathbb{Z}^N) \) be a symbol.

i) If \( \alpha(\xi) = O(\log(|\xi|)) \) and \( \beta(\xi) = O(\log(|\xi|)) \), then \( L \) is globally hypoelliptic if and only if \( L_0 \) is globally hypoelliptic.

ii) If \( \alpha(\xi) = O(\log(|\xi|)) \) and \( \beta(\xi) \) has super-logarithmic growth, then \( L \) is globally hypoelliptic if and only if \( L_0 \) is globally hypoelliptic and \( a(t) \) does not change sign.

iii) If \( \alpha(\xi) \) has super-logarithmic growth and \( \beta(\xi) = O(\log(|\xi|)) \), then \( L \) is globally hypoelliptic if and only if \( L_0 \) is globally hypoelliptic and \( b(t) \) does not change sign.

In the particular case where \( \beta \equiv 0 \) we have the following:

**Corollary 4.3.** If the symbol \( p(\xi) \) is real-valued, then the operator \( L \) is globally hypoelliptic if and only if \( L_0 \) is globally hypoelliptic and either

i) \( p(\xi) = O(\log(|\xi|)) \); or

ii) \( p(\xi) \) has super-logarithmic growth and \( b(t) \) does not change sign.

**Remark 4.4.** When \( p(\xi) \) is a real-valued symbol having at most logarithmic growth, item i) shows that the behaviour of the function \( b(t) \) plays no role in the global hypoellipticity of pseudo-differential operators of type (1.1), what means that the famous condition \( (P) \) of Nirenberg-Treves, see \([30]\) and \([31]\), is neither necessary nor sufficient to guarantee global hypoellipticity.

On the other hand, item ii) is according to the known result for vector fields \( L = D_t + (a + ib)(t)D_x \) on \( \mathbb{T}^2 \) studied by Houmie in \([25]\). We recall that in this case, the condition \( L_0 \) globally hypoelliptic means that either \( b_0 \neq 0 \) or \( a_0 \) is an irrational non-Liouville number.
We split the proof of Theorem 4.2 in two subsections. In Subsection 4.1 we prove item i) by using an argument of reduction to normal forms. The proof of items ii) and iii) are treated in Subsection 4.2 where the change of sign of the coefficients play an important role.

In Subsection 4.3 we show that the techniques developed in previous subsections can be applied to study a particular case where the symbol has super logarithmic growth.

Before proceeding with the proofs, we present two examples which illustrate that the condition $\Im M(t, \xi)$ does not change sign in Theorem 3.6 is not necessary for the global hypoellipticity of $L$.

Example 4.5. If $P(D_x) = (-\Delta_x)^{m/2}$ on $\mathbb{T}^N$, with $m < 0$, then by item i) of the Theorem 4.2 the operator $L = D_t + [1 + i \sin(t)](-\Delta_x)^{m/2}$ is globally hypoelliptic since $L_0 = D_t + (-\Delta_x)^{m/2}$, is globally hypoelliptic by Theorem 2.1. Notice that $\Im M(t, \xi) = \sin(t)|\xi|^m$ changes sign for all $|\xi| > 0$.

Example 4.6. Assume that $P(D_x) = (-\Delta_x)^{m/2} + i(-\Delta_x)^{n/2}$ on $\mathbb{T}^N$, where $m \leq 0$ and $n > 0$. Theorem 4.2 item iii) implies that the operator $L = D_t + [1 + \cos(t) - i][-\Delta_x)^{m/2} + i(-\Delta_x)^{n/2}]$ is globally hypoelliptic, since $a(t) = 1 + \cos(t) \geq 0$ and $L_0 = D_t + (1 - i)[-\Delta_x)^{m/2} + i(-\Delta_x)^{n/2}]$ is globally hypoelliptic. Indeed, the assumptions $m \leq 0$ and $n > 0$ implies that, for $(\tau, \xi) \in \mathbb{Z} \times \mathbb{Z}_N^*$ such that $|\xi| > 2^{1/n}$, we have $|\tau + (1 - i)(|\xi|^m + i|\xi|^n)| \geq |\xi|^n - |\xi|^m | \geq 1.

Hence, $L_0$ is globally hypoelliptic by Section 2.

Notice that $\Im M(t, \xi) = (1 + \cos(t))|\xi|^n - |\xi|^m$ changes sign for infinitely many indexes, since $m \leq 0$ and $n > 0$.

4.1. Reduction to normal form. In this subsection we show that, under the assumption of growth at most logarithm of the symbol, the study of the global hypoellipticity of $L$ and $L_0$ are equivalent.

In this situation we have $\alpha(\xi) = O(\log(|\xi|))$ and $\beta(\xi) = O(\log(|\xi|))$, as $|\xi| \to \infty$, and the proof of item i) of Theorem 4.2 follows from Corollary 4.8 bellow.

We introduce the following (formal) operators: for each distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N)$, we set

$$\Psi_a(u) = \sum_{\xi \in \mathbb{Z}^N} e^{-i(A(t) - a_t)p(\xi)}\hat{u}(t, \xi)e^{ix\xi},$$

and

$$\Psi_b(u) = \sum_{\xi \in \mathbb{Z}^N} e^{(B(t) - b_t)p(\xi)}\hat{u}(t, \xi)e^{ix\xi},$$
where

\[ A(t) = \int_0^t a(s)ds \quad \text{and} \quad B(t) = \int_0^t b(s)ds. \]

**Proposition 4.7.** If \( \beta(\xi) = O(\log(|\xi|)) \), then \( \Psi_a \) is an isomorphism which satisfies (4.3)

\[ \Psi_a^{-1} \circ L \circ \Psi_a = L_{a_0} \]

on both the spaces \( \mathcal{D}'(T^1 \times T^N) \) and \( C^\infty(T^1 \times T^N) \), where

\[ L_{a_0} = D_t + (a_0 + ib(t))P(D_x). \]

Analogously, if \( \alpha(\xi) = O(\log(|\xi|)) \), then \( \Psi_b \) is an isomorphism which satisfies (4.4)

\[ \Psi_b^{-1} \circ L \circ \Psi_b = L_{b_0} \]

on both the spaces \( \mathcal{D}'(T^1 \times T^N) \) and \( C^\infty(T^1 \times T^N) \), where

\[ L_{b_0} = D_t + (a(t) + ib_0)P(D_x). \]

The proof of this proposition consists in to show that \( \Psi_a \) and \( \Psi_b \) are well defined operators, in this case they are evidently linear operators with inverse

\[ \Psi_a^{-1}(v) = \sum_{\xi \in \mathbb{Z}^N} e^{i(A(t) - a_0)t}\hat{v}(t, \xi)e^{ix\xi}, \]

and

\[ \Psi_b^{-1}(v) = \sum_{\xi \in \mathbb{Z}^N} e^{-(B(t) - b_0)t}\hat{v}(t, \xi)e^{ix\xi}, \]

respectively, on both the spaces \( \mathcal{D}'(T^1 \times T^N) \) and \( C^\infty(T^1 \times T^N) \). Moreover, identities (4.3) and (4.4) are easily verified.

Before starting this proof, let us state the reduction to the normal form:

**Corollary 4.8.** If \( \beta(\xi) = O(\log(|\xi|)) \) (respectively \( \alpha(\xi) = O(\log(|\xi|)) \)), then \( L \) is globally hypoelliptic if and only if \( L_{a_0} \) (respectively \( L_{b_0} \)) is globally hypoelliptic.

**Proof.** The validity of the identity \( L_{a_0} = \Psi_a^{-1} \circ L \circ \Psi_a \) on both \( \mathcal{D}'(T^1 \times T^N) \) and \( C^\infty(T^1 \times T^N) \) imply that \( L \) is globally hypoelliptic if and only if \( L_{a_0} \) is globally hypoelliptic.

In fact, assume that \( L \) is globally hypoelliptic and let \( u \in \mathcal{D}'(T^1 \times T^N) \) such that \( L_{a_0}u = f \in C^\infty(T^1 \times T^N) \). Since \( v = \Psi_a(u) \in \mathcal{D}'(T^1 \times T^N) \) satisfy \( Lv = \Psi_a(f) \in C^\infty(T^1 \times T^N) \), it follows that \( v \in C^\infty(T^1 \times T^N) \), since \( L \) is globally hypoelliptic.

Hence, \( u = \Psi_a^{-1}(v) \in C^\infty(T^1 \times T^N) \), which implies that \( L_{a_0} \) is globally hypoelliptic. The converse is similar.

Analogously, the validity of the identity \( L_{b_0} = \Psi_b^{-1} \circ L \circ \Psi_b \) on both \( \mathcal{D}'(T^1 \times T^N) \) and \( C^\infty(T^1 \times T^N) \) will imply that \( L \) is globally hypoelliptic if and only if \( L_{b_0} \) is globally hypoelliptic.

\[ \square \]

The following estimates will be useful in the proof of Proposition 4.7.

**Lemma 4.9.** Consider \( p \in S^m(\mathbb{Z}^N) \). Given \( k \in \mathbb{N}_0 \), there are positive constants \( C \) and \( n_0 \) such that

\[ |\partial_x^k(e^{-i(A(t) - a_0)t}p(\xi))| \leq C|\xi|^{km}e^{\beta(\xi)(-A(t) + a_0t)}, \]

where

\[ A(t) = \int_0^t a(s)ds \quad \text{and} \quad B(t) = \int_0^t b(s)ds. \]
and
\[ |\partial_t^k (e^{(B(t)-b_0)t})\psi(\xi)| \leq C|\xi|^m e^{\alpha(\xi)(B(t)-b_0)t}, \]
for each $|\xi| \geq n_0$.

**Proof.** For $k = 0$ these estimates are evident. If the first estimate holds for $\ell \in \{0,1,\ldots,k\}$, then we have:

\[
|\partial_t^{k+1} (e^{-i(A(t)-a_0)t})\psi(\xi)| \leq |p(\xi)| \sum_{\ell=0}^{k} \left(\begin{array}{c} k \\ \ell \end{array}\right) |\partial_t^\ell e^{-i(A(t)-a_0)t}\psi(\xi)| \\
\times \sup_{t \in \mathbb{T}^1} |\partial_t^{k-\ell}(a(t) - a_0 t)| \\
\leq C|p(\xi)| \sum_{\ell=0}^{k} \left(\begin{array}{c} k \\ \ell \end{array}\right) |\partial_t^\ell e^{-i(A(t)-a_0)t}\psi(\xi)| \\
\leq C|p(\xi)| e^{\beta(\xi)(-A(t)+a_0 t)} \sum_{\ell=0}^{k} \left(\begin{array}{c} k \\ \ell \end{array}\right) |\xi|^{\ell m} \\
\leq C|\xi|^{(k+1)m} e^{\beta(\xi)(-A(t)+a_0 t)},
\]

where we are using $|p(\xi)| \leq C|\xi|^m$, as $|\xi| \to \infty$.

The second estimate can be obtained by using similar arguments.

\[ \square \]

**Proof of Proposition 4.7** We have to verify only that $\Psi_a$ and $\Psi_b$ are well defined linear operators on both $\mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N)$ and $C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$ whenever

\[ \beta(\xi) = O(\log(|\xi|)) \quad \text{and} \quad \alpha(\xi) = O(\log(|\xi|)), \]

respectively.

Fixed $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N)$, we must study the behavior of the Fourier coefficients

\[ \psi_a(t,\xi) = e^{-i(A(t)-a_0 t)p(\xi)} \hat{u}(t,\xi), \quad \text{for all } \xi \in \mathbb{Z}^N, \]

and

\[ \psi_b(t,\xi) = e^{(B(t)-b_0 t)p(\xi)} \hat{u}(t,\xi), \quad \text{for all } \xi \in \mathbb{Z}^N. \]

Given $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N)$, it follows by Lemma 4.9 the existence of positive constants $C$ and $M$ such that

\[ |\langle \psi_a(t,\xi), \phi \rangle| \leq C|\xi|^M \|\phi\|_M \sup_{t \in \mathbb{T}^1} |e^{\beta(\xi)(-A(t)+a_0 t)}|, \]

and

\[ |\langle \psi_b(t,\xi), \phi \rangle| \leq C|\xi|^M \|\phi\|_M \sup_{t \in \mathbb{T}^1} |e^{\alpha(\xi)(B(t)-b_0 t)}|, \]

for $|\xi|$ large enough, where

\[ \|\phi\|_M = \max \{ \|\partial^\alpha \phi(t)\|; \alpha \leq M, \ t \in \mathbb{T}^1 \}. \]

If $\beta(\xi) = O(\log(|\xi|))$, by (1.2) there exist $\kappa > 0$ and $n_0 \in \mathbb{N}$ such that

\[ |\beta(\xi)| \leq \log(|\xi|^\kappa), \quad \text{for all } |\xi| \geq n_0. \]
Now, take $\delta_1 < 0$ and $\delta_2 > 0$ satisfying
\begin{equation}
\delta_1 \leq -A(t) + a_0 t \leq \delta_2, \text{ for all } t \in T^1. 
\end{equation}

The inequalities (4.7) and (4.8) imply that, for all $|\xi| \geq n_0$ we have:
\begin{equation}
\beta(\xi)(-A(t) + a_0 t) \leq \begin{cases} 
\log(|\xi|^\kappa \delta_2), & \text{if } \beta(\xi) > 0, \\
\log(|\xi|^{-\kappa \delta_1}), & \text{if } \beta(\xi) < 0.
\end{cases}
\end{equation}

Hence,
\begin{equation}
e^{\beta(\xi)(-A(t)+a_0 t)} \leq |\xi|^{\delta_3}, \text{ as } |\xi| \to \infty,
\end{equation}
where $\delta_3 = \max\{\kappa \delta_2, -\kappa \delta_1\}$.

With similar ideas, by using the fact that $\alpha(\xi) = O(\log(|\xi|))$, we obtain $\delta_4 > 0$ such that
\begin{equation}
e^{\alpha(\xi)(B(t) - b_0 t)} \leq |\xi|^{\delta_4}, \text{ as } |\xi| \to \infty.
\end{equation}

Then, by (4.9), (4.10) and the last two inequalities
\begin{equation}
|\langle \psi_a(t, \xi), \phi \rangle| \leq C|\xi|^{M+\delta_3}||\phi||_M 
\end{equation}
and
\begin{equation}
|\langle \psi_b(t, \xi), \phi \rangle| \leq C|\xi|^{M+\delta_4}||\phi||_M,
\end{equation}
for all $|\xi|$ sufficiently large and for all $\phi \in C^\infty(T^1)$; thus $\Psi_a \cdot u \in D'(T^1 \times T^N)$ and $\Psi_b \cdot u \in D'(T^1 \times T^N)$.

Finally, if $u \in C^\infty(T^1 \times T^N)$, then Lemma (4.9) and the rapid decaying of $\hat{u}(\cdot, \xi)$ imply that for each $k \in \mathbb{N}_0$ we obtain $C_k > 0$ and $M_k \in \mathbb{R}$ such that
\begin{equation}
|\partial^k_t \psi_a(t, \xi)| \leq C_k|\xi|^{M_k}e^{\beta(\xi)(-A(t)+a_0 t)} \sum_{j=0}^{k} |\partial^j_t \hat{u}(t, \xi)|
\end{equation}
and
\begin{equation}
|\partial^k_t \psi_b(t, \xi)| \leq C_k|\xi|^{M_k}e^{\alpha(\xi)(B(t) - b_0 t)} \sum_{j=0}^{k} |\partial^j_t \hat{u}(t, \xi)|,
\end{equation}
for $|\xi|$ large enough.

By using again (4.9) and (4.10), and from the rapid decaying of $\hat{u}(\cdot, \xi)$, it follows that $\Psi_a(u)$ and $\Psi_b(u)$ are in $C^\infty(T^1 \times T^N)$, what finishes the proof of Proposition 4.7.

\[\square\]

4.2. Change of sign. Our focus now is to prove item \textit{ii)} of Theorem 4.2 in which $\alpha(\xi)$ has at most logarithmic growth and $\beta(\xi)$ has super-logarithmic growth. Notice that, in this case, the global hypoellipticity of $L$ cannot be reduced to the global hypoellipticity of a constant coefficient operator.

The proof of item \textit{iii)} of Theorem 4.2 consists in slight modifications of the techniques used in the proof of item \textit{ii)}. Since the argument is quite similar, it will be omitted.

\textbf{Proof of item \textit{ii)}} of Theorem 4.2 We recall that the hypothesis in this case are $\beta(\xi)$ has super-logarithmic growth and $\alpha(\xi) = O(\log(|\xi|))$, hence, in view of Corollary 4.8 we may assume that $b(t)$ is constant, $b \equiv b_0$. 

\[\square\]
\textbf{Sufficiency:}

Assume that $a(t)$ does not change sign and that $L_0$ is globally hypoelliptic.

Let $u \in \mathcal{D}(\mathbb{T}^1 \times \mathbb{T}^N)$ be such that $iLu = f \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$. We will show that $u \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$. By the Fourier series in the variable $x$, we are led to the equations

\begin{equation}
\hat{f}(t, \xi) = \partial_t \hat{u}(t, \xi) + i\mathcal{M}(t, \xi)\hat{u}(t, \xi)
= \partial_t \hat{u}(t, \xi) + \left[-(b_0\alpha(\xi) + a(t)\beta(\xi)) + i(a(t)\alpha(\xi) - b_0\beta(\xi))\right] \hat{u}(t, \xi),
\end{equation}

for all $t \in \mathbb{T}^1$ and for all $\xi \in \mathbb{Z}^N$.

By applying Lemma 3.1 to equation (4.11), it follows that $t \in \mathbb{T}^1 \mapsto \hat{u}(t, \xi)$ is smooth, for each $\xi \in \mathbb{Z}^N$. Since $Z_M$ is finite (Remark 3.4), for $|\xi|$ sufficiently large, equation (4.11) has a unique solution. This solution can be written by

$$\hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i M_0(\xi)}} \int_0^{2\pi} \exp \left(- \int_{t-s}^t i\mathcal{M}(r, \xi) dr \right) \hat{f}(t - s, \xi) ds,$$

if $\xi$ is such that $a(t)\beta(\xi) \leq 0$, for all $t \in \mathbb{T}^1$, and

$$\hat{u}(t, \xi) = \frac{1}{e^{2\pi i M_0(\xi)} - 1} \int_0^{2\pi} \exp \left( \int_{t-s}^{t+s} i\mathcal{M}(r, \xi) dr \right) \hat{f}(t + s, \xi) ds,$$

if $\xi$ is such that $a(t)\beta(\xi) \geq 0$, for all $t \in \mathbb{T}^1$.

Since $\alpha(\xi) = O(\log(|\xi|))$, there exists $K > 0$ such that

$$\exp(\alpha(\xi)sb_0) \leq |\xi|^{K|b_0|},$$

for $|\xi|$ sufficiently large and $s \in [0, 2\pi]$. Thus, for $|\xi|$ large enough and such that $a(t)\beta(\xi) \leq 0$, for all $t \in \mathbb{T}^1$, we have

$$\left| \exp \left(- \int_{t-s}^t i\mathcal{M}(r, \xi) dr \right) \right| = \exp \left( \alpha(\xi)sb_0 + \int_{t-s}^t a(r)\beta(\xi) dr \right) \leq e^{\alpha(\xi)sb_0} \leq |\xi|^{K|b_0|}.$$

Similarly, for $|\xi|$ large enough and such that $a(t)\beta(\xi) \geq 0$, for all $t \in \mathbb{T}^1$, we have

$$\left| \exp \left( \int_{t}^{t+s} i\mathcal{M}(r, \xi) dr \right) \right| = \exp \left( -\alpha(\xi)sb_0 - \int_{t}^{t+s} a(r)\beta(\xi) dr \right) \leq e^{-\alpha(\xi)sb_0} \leq |\xi|^{K|b_0|}.$$

Finally, as in the proof of Theorem 3.6, the global hypoellipticity of $L_0$ give us a control as in (3.7), and the rapid decaying of $\hat{f}(\cdot, \xi)$ imply that $\hat{u}(\cdot, \xi)$ decays rapidly. Hence, $u$ belongs to $C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$ and $L$ is globally hypoelliptic.

\textbf{Necessity:}

By Theorem 3.5, it is enough to prove that the changing of sign of $a(t)$ implies that $L$ is not globally hypoelliptic.

We will exhibit a smooth function

$$f(t, x) = \sum_{n=1}^{\infty} \hat{f}(t, \xi_n)e^{ix\xi_n},$$
Thus, for which $i Lu = f$ has a solution in $\mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N) \setminus C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$.

Our assumptions on $\beta(\xi)$ imply that we may choose a sequence $\{\xi_n\}$, such that $|\xi_n|\geq n$, and $|\beta(\xi_n)| \geq \log(|\xi_n|^{\alpha})$, for all $n \in \mathbb{N}$.

By passing to a subsequence we may assume that either $\beta(\xi_n) > 0$, for all $n$, or $\beta(\xi_n) < 0$, for all $n$.

Without loss of generality, we also may assume that

$$b_0 \alpha(\xi_n) + a_0 \beta(\xi_n) \leq 0,$$

for all $n \in \mathbb{N}$. Indeed, in the other case, it is enough to consider $-L$ and to change the variable $t$ by $-t$.

Suppose we are in the case $\beta(\xi_n) > 0$ for all $n$ (the other case is similar).

Set

$$M_a = \max_{0 \leq x \leq 2 \pi} \int_{t-s}^t a(r) dr = \int_{t_0-s_0}^{t_0} a(r) dr.$$

Since $a(t)$ changes sign, $M_a > 0$ and $s_0 \in (0, 2\pi)$; moreover, without loss of generality (by performing a translation in the variable $t$) we may assume that $t_0$ and $\sigma_0 = t_0 - s_0$ belong to the open interval $(0, 2\pi)$.

Let $\phi \in C^\infty((\sigma_0 - \epsilon, \sigma_0 + \epsilon))$ be a function such that $0 \leq \phi(t) \leq 1$, and $\phi(t) = 1$ in a neighborhood of $[\sigma_0 - \epsilon/2, \sigma_0 + \epsilon/2]$.

We then define $\hat{f}(\cdot, \xi_n)$ by the $2\pi$–periodic extension of the function

$$(1 - e^{-2\pi i M_0(\xi_n)}) \phi(t) \exp\left(i \int_{t_0}^{t_0} R_M(r, \xi_n) dr\right) e^{-\beta(\xi_n) M_a} e^{\alpha(\xi_n)(t - t_0) b_0}.$$

Since $b_0 \alpha(\xi_n) + a_0 \beta(\xi_n) \leq 0$, we have that $1 - e^{-2\pi i M_0(\xi_n)}$ is bounded and for $t \in [0, 2\pi]$ we have

$$e^{\alpha(\xi_n)(t - t_0) b_0} \leq e^{\alpha(\xi_n)|2\pi|b_0)},$$

which increases slowly, since $\alpha(\xi) = O(\log(|\xi|))$.

Moreover, by using estimate (3.1), the term $e^{-\beta(\xi_n) M_a}$ will imply that $\hat{f}(\cdot, \xi_n)$ decays rapidly, since $\beta(\xi_n) > \log(|\xi_n|^{\beta})$.

By Proposition 3.3 we may assume that $Z_M$ is finite and, by passing to a subsequence, that $1 - e^{-2\pi i M_0(\xi_n)} \neq 0$, then we define

$$\hat{u}(t, \xi_n) = \frac{1}{1 - e^{-2\pi i M_0(\xi_n)}} \int_0^{2\pi} \exp\left(-\int_{t-s}^t i M_M(r, \xi_n) dr\right) \hat{f}(t-s, \xi_n) ds.$$

For all $s, t \in [0, 2\pi]$, we have

$$\left|\frac{1}{1 - e^{-2\pi i M_0(\xi_n)}} \hat{f}(t-s, \xi_n)\right| \leq e^{\alpha(\xi_n)|4\pi|b_0)} e^{-\beta(\xi_n) M_a}.$$

Thus,

$$|\hat{u}(t, \xi_n)| \leq \int_0^{2\pi} \exp\left(-\beta(\xi_n)(M_a - \int_{t-s}^t a(r) dr)\right) e^{\alpha(\xi_n) b_0} e^{\alpha(\xi_n)|4\pi|b_0)} ds 
\leq 2\pi e^{\alpha(\xi_n)|6\pi|b_0)}.$$
This estimate imply that the sequence \( \hat{u}(\cdot, \xi_n) \) increases slowly, since \( \alpha(\xi) = O(\log(|\xi|)) \). Hence

\[
    u = \sum_{n=1}^{\infty} \hat{u}(t, \xi_n) e^{i2\xi_n} \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N).
\]

Note that

\[
    |\hat{u}(t_0, \xi_n)| \geq \int_{s_0 - \delta/2}^{s_0 + \delta/2} \exp \left( -\beta(\xi_n) \left( M_a - \int_{t_0}^{t_0 - s} a(r)dr \right) \right) ds.
\]

Since \( M_a - \int_{t_0}^{t_0 - s} a(r)dr \geq 0 \) and \( s_0 \) is a zero of order even, the Laplace Method for Integrals implies that

\[
    |\hat{u}(t_0, \xi_n)| \geq C \beta(\xi_n)^{-1/2} \geq C \left( 1 + |\xi_n|^2 \right)^{-m/4} \geq C 2^{-m/4} |\xi_n|^{-m/2},
\]

where \( C \) and \( m \) are positive constants and do not depend on \( n \). This estimate implies that \( \hat{u}(\cdot, \xi_n) \) does not decay rapidly.

Hence, \( u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N) \setminus C^\infty(\mathbb{T}^1 \times \mathbb{T}^N) \). Therefore, \( L \) is not globally hypoelliptic, since \( iLu = f \) by Lemma 3.1.

When \( \beta(\xi_n) < 0 \) for all \( n \), we repeat the constructions above, where now we use

\[
    M_a = \min_{0 \leq t, s \leq 2\pi} \int_{t-s}^{t} a(r)dr.
\]

The proof of Theorem 4.2 - item ii) is complete. \( \square \) \( \square \)

**Remark 4.10.** In the proof of sufficiency in Theorem 4.2 item ii), was not necessary suppose that \( \beta(\xi) \) has super-logarithmic growth. Moreover, we observe that this proof is not consequence of Theorem 3.6 since \( t \mapsto \Im M(t, \xi) = a(t) \beta(\xi) + b(t) \alpha(\xi) \) may change sign, even when \( a(t) \) does not change sign.

We finish this subsection with an additional example which also exhibit a globally hypoelliptic operator in a situation in which \( t \in \mathbb{T}^1 \mapsto \Im M(t, \xi) \) changes sign, for infinitely many indexes \( \xi \).

**Example 4.11.** Let \( b(t) \) be a \( 2\pi \)-periodic extension of a real smooth nonzero function defined on \( (0, 2\pi) \) with integral equals to zero. Let \( a(t) \) be the \( 2\pi \)-periodic extension of the function \( 1 - \psi \), where \( \psi \in C^\infty_c((0, 2\pi), \mathbb{R}), 0 \leq \psi(t) \leq 1 \) and \( \psi \equiv 1 \) in a neighborhood of support of \( b(t) \).

If \( P(D_x) \) has symbol \( p(\xi) = 1 + i(|\xi| \log(1 + |\xi|)) \), then

\[
    L = D_t + (a(t) + ib(t)) P(D_x), \quad t, x \in \mathbb{T}^1 \times \mathbb{T}^N,
\]

is globally hypoelliptic by Theorem 4.2 - item ii). Note that,

\[
    t \mapsto \Im M(t, \xi) = a(t)|\xi| \log(1 + |\xi|) + b(t),
\]

changes sign for all indexes \( \xi \in \mathbb{Z}^N \).
4.3. A particular class of operators. The aim of this subsection is to notice that there is a particular class of operators, which includes cases in which both $\alpha(\xi)$ and $\beta(\xi)$ have super-logarithmic growth, where the study of the global hypoellipticity follows from adaptations of the techniques used in the proof of Theorem 4.2.

For example, if $p(\xi) = \alpha(\xi) + i(1 + \alpha(\xi))$ and $\alpha(\xi)$ has super-logarithmic growth, we cannot apply Theorem 4.2 to study the global hypoellipticity of the operator

$$D_t + (\cos^2(t) + i \sin(t))P(D_x),$$

but notice that $\mathcal{M}(t, \xi)$ splits in the form

$$[\sin(t) + \cos^2(t)]\alpha(\xi) + \cos^2(t).$$

Hence, $\mathcal{M}(t, \xi)$ satisfies the assumptions concerning the speed of growth which was assumed in Theorem 4.2. We claim that the operator above is not globally hypoelliptic, since $|\sin(t) + \cos^2(t)|$ changes sign.

More generally, with similar arguments of those used in the proof of Theorem 4.2, we may give a complete answer about the global hypoellipticity of the operator $L$, given by (1.1), in the case where $\mathcal{M}(t, \xi)$ splits in the following way:

(4.12) $$\mathcal{M}(t, \xi) = \tilde{a}(t)\gamma(\xi) + \tilde{b}(t)\eta(\xi),$$

where $\tilde{a}(t)$ and $\tilde{b}(t)$ are real smooth functions on $\mathbb{T}^1$, and $\gamma(\xi)$ and $\eta(\xi)$ are real valued toroidal symbols, such that either $\gamma(\xi) = O(\log(|\xi|))$ or $\eta(\xi) = O(\log(|\xi|))$.

**Theorem 4.12.** Let $L$ be the operator defined in (1.1) and assume that the decomposition (4.12) is true. Then $L$ is globally hypoelliptic if and only if $L_0$ is globally hypoelliptic and $\tilde{a}(t)$ (respectively $\tilde{b}(t)$) does not change sign whenever $\gamma(\xi)$ (respectively $\eta(\xi)$) has super-logarithmic growth.

Observe that, under the assumptions in Theorem 4.12 and assuming that $\gamma(\xi)$ has super-logarithmic growth, the converse of Theorem 3.6 holds true provided that the function $\mathcal{M}(t, \xi) = \tilde{a}(t)\gamma(\xi) + \tilde{b}(t)\alpha(\xi)$ changes sign if and only if $\tilde{a}$ changes sign. However, as we saw in Example 4.11, this property does not hold in general.

Bellow we present other interesting examples in this direction.

**Example 4.13.** If $a(t)$ and $b(t)$ do not vanish identically and are $\mathbb{R}$—linearly dependent functions, we may write

$$\mathcal{M}(t, \xi) = b(t)(\alpha(\xi) + \lambda\beta(\xi)),$$

with $\lambda \in \mathbb{R} \setminus \{0\}$. In this case, Theorem 4.12 gives a complete answer about the global hypoellipticity of $L$.

When $\alpha(\xi) + \lambda\beta(\xi) = O(\log(|\xi|))$, $L$ is globally hypoelliptic if and only if $L_0$ is globally hypoelliptic.

For instance, if $a(t) = -b(t)$ and $\beta(\xi) = 1 + \alpha(\xi)$, then $\mathcal{M}(t, \xi) = -a(t)$. Hence, $L$ is globally hypoelliptic even if $b(t)$ changes sign.

When $\alpha(\xi) + \lambda\beta(\xi)$ has super-logarithmic growth, $L$ is globally hypoelliptic if and only if $L_0$ is globally hypoelliptic and $b(t)$ does not change sign.

**Example 4.14.** When $a(t)$ and $b(t)$ are $\mathbb{R}$—linearly independent functions, $L$ may be not globally hypoelliptic even if both $a(t)$ and $b(t)$ do not change sign. Indeed, we may find non-zero integers $p$ and $q$ so that $a(t)p + b(t)q$ changes sign (see Lemma 3.1 of [7]).
If, for instance, $\alpha(\xi) = q\gamma(\xi)$ and $\beta(\xi) = p\gamma(\xi)$, in which $\gamma(\xi)$ has super-logarithmic growth, then Theorem 4.12 implies that $L$ is not globally hypoelliptic.

5. Super-logarithmic growth

The purpose of this section is to present additional results about the global hypoellipticity of the operator $L$ given by (1.1), which we recall

\[ L = D_t + (a + ib)(t)P(D_x), \quad (t, x) \in \mathbb{T}^1 \times \mathbb{T}^N, \]

where $a(t)$ and $b(t)$ are real smooth functions on $\mathbb{T}^1$, and $P(D_x)$ is a pseudo-differential operator on $\mathbb{T}^N$, with symbol $p(\xi) = \alpha(\xi) + i\beta(\xi)$, $\xi \in \mathbb{Z}^N$.

We consider a more general situation where either $\alpha(\xi)$ or $\beta(\xi)$ has super-logarithmic growth and we present a necessary condition for the global hypoellipticity of $L$, which is given by a control in the sign of certain functions.

Precisely, assume that $\beta(\xi)$ has super-logarithmic growth and let

\[ E_{\alpha, \beta} = \left\{ K \in \mathbb{R}; \text{ there exists } \{\xi_n\} \subset \mathbb{Z}^N \text{ satisfying } (*) \right\}, \]

\[
(*) \quad \begin{cases} 
|\xi_n| \to \infty; \\
\alpha(\xi_n)/\beta(\xi_n) \to K, \text{ as } n \to \infty; \\
|\beta(\xi_n)| \geq n \log(|\xi_n|), \text{ for all } n \in \mathbb{N}.
\end{cases}
\]

In this case, we prove that $L$ is not globally hypoelliptic if there exists $K \in E_{\alpha, \beta}$ such that the function $t \in \mathbb{T}^1 \mapsto a(t) + b(t)K$ changes sign.

An analogous result holds when $\alpha(\xi)$ has super-logarithmic growth. In this case, $L$ is not globally hypoelliptic if there exists $C \in E_{\beta, \alpha}$ such that the function $t \in \mathbb{T}^1 \mapsto b(t) + a(t)C$ changes sign.

In particular, we obtain a necessary condition for the global hypoellipticity of $L$ when either $\alpha(\xi)$ or $\beta(\xi)$ has super-logarithmic growth and the limit $\lim_{|\xi| \to \infty} \alpha(\xi)/\beta(\xi)$ exists. When the order of growth of $\alpha(\xi)$ is faster (respectively slower) than the order of growth of $\beta(\xi)$, the operator $L$ is not globally hypoelliptic if $b(t)$ (respectively $a(t)$) changes sign (see Corollary 5.2).

**Theorem 5.1.** If $\beta(\xi)$ has super-logarithmic growth, $K \in E_{\alpha, \beta}$ and the function $t \in \mathbb{T}^1 \mapsto a(t) + b(t)K$ changes sign, then $L$ given by (1.1) is not globally hypoelliptic. Similarly, if $\alpha(\xi)$ has super-logarithmic growth, $C \in E_{\beta, \alpha}$, and $t \in \mathbb{T}^1 \mapsto b(t) + a(t)C$ changes sign, then $L$ is not globally hypoelliptic.

**Proof.** We consider the situation in which $\beta(\xi)$ has super-logarithmic growth and $K \in E_{\alpha, \beta}$. The other situation is analogous.

We assume that $t \in \mathbb{T}^1 \mapsto a(t) + b(t)K$ changes sign and prove that $L$ is not globally hypoelliptic.

The assumptions on $\beta$ and $K$ imply that there exists a sequence $\{\xi_n\}$ such that $|\xi_n|$ is strictly increasing, $|\xi_n| > n$, $|\beta(\xi_n)| \geq n \log(|\xi_n|)$, $\alpha(\xi_n)/\beta(\xi_n) \to K$, and $\xi_n \not\in \mathbb{Z}_M$, for all $n$. Note that we are assuming that $\mathbb{Z}_M$ is finite, otherwise, by Proposition 5.3, there is nothing to prove.
Without loss of generality, suppose that $b_0\alpha(\xi_n) + a_0\beta(\xi_n) \leq 0$, for all $n$. Indeed, if necessary we can consider $-L$ and perform the change of variable $t$ by $-t$.

By using a subsequence, we may assume that either $\beta(\xi_n) < 0$, for all $n$, or $\beta(\xi_n) > 0$, for all $n$.

Suppose first that $\beta(\xi_n) > 0$, for all $n$. For each $n$, set

$$M_n = \max_{0 \leq t,s \leq 2\pi} \left\{ \int_{t-s}^{t} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr \right\} = \int_{t_n-s_n}^{t_n} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr.$$

Again, by passing to a subsequence, there exist $t_0$ and $s_0$ such that $t_n \to t_0$ and $s_n \to s_0$, as $n \to \infty$. Since $\alpha(\xi_n)/\beta(\xi_n) \to K$, as $n \to \infty$, it follows that

$$\int_{t_0-s_0}^{t_0} a(r) + b(r) K dr = \max_{0 \leq t,s \leq 2\pi} \int_{t-s}^{t} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr \geq M_{ab}. $$

Since $a(t) + b(t)K$ changes sign, we have $M_{ab} > 0$ and $s_0 \in (0, 2\pi)$. Performing a translation in the variable $t$, we may assume that $t_0, s_0$ and $\sigma_0 = t_0 - s_0$ belong to $(0, 2\pi)$.

Choose $\epsilon > 0$ small enough so that $0 < \sigma_0 - \epsilon$ and $\sigma_0 + \epsilon < t_0$. Consider $\phi$ belonging to $C_c^\infty((\sigma_0 - \epsilon, \sigma_0 + \epsilon), \mathbb{R})$ such that $0 \leq \phi(t) \leq 1$ and $\phi(t) = 1$ for all $t \in [\sigma_0 - \epsilon/2, \sigma_0 + \epsilon/2]$.

Finally, we define $\hat{f}(\cdot, \xi_n)$ as being the $2\pi-$periodic extension of

$$(1 - e^{-2\pi i M_0(\xi_n)}) \phi(t) \exp \left( i \int_{t}^{t_n} \mathfrak{R} M(r, \xi_n) dr \right) e^{-\beta(\xi_n) M_n}.$$ 

Note that $1 - e^{-2\pi i M_0(\xi_n)}$ is bounded, since $b_0\alpha(\xi_n) + a_0\beta(\xi_n) \leq 0$. Thus, by estimate \ref{4.1}, the behaviour of the term $e^{-\beta(\xi_n) M_n}$ when $|\xi_n| \to \infty$ imply that $\hat{f}(\cdot, \xi_n)$ decays rapidly, since $M_n \to M_{ab} > 0$ and $\beta(\xi_n) \geq \log(|\xi_n|^\alpha)$.

It follows that

$$f(t, x) = \sum_{n=1}^{\infty} \hat{f}(t, \xi_n) e^{ix\xi_n} \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N).$$

Since $\xi_n \notin Z_M$, we may define

$$\hat{u}(t, \xi_n) = \frac{1}{1 - e^{-2\pi i M_0(\xi_n)}} \int_{0}^{2\pi} \exp \left( -i \int_{t-s}^{t} M(r, \xi_n) dr \right) \hat{f}(t-s, \xi_n) ds,$$

which belongs to $C^\infty(\mathbb{T}^1)$.

For $n$ large enough, the estimate

$$|\left(1 - e^{-2\pi i M_0(\xi_n)}\right)^{-1} \hat{f}(t-s, \xi_n)| \leq e^{-\beta(\xi_n) M_n} \leq 1$$

implies that

$$|\hat{u}(t, \xi_n)| \leq \int_{0}^{2\pi} \exp \left( -\beta(\xi_n) \left( M_n - \int_{t-s}^{t} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr \right) \right) ds \leq 2\pi.$$

Hence, $\hat{u}(\cdot, \xi_n)$ increases slowly. Then

$$u = \sum_{n=1}^{\infty} \hat{u}(t, \xi_n) e^{ix\xi_n} \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N).$$
We will show that \( u \not\in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N) \). In fact, for \( n \) sufficiently large we have \( \sigma_0 + \epsilon < t_n \), from which we can infer that
\[
|\widehat{u}(t_n, \xi_n)| = \left| \int_{t_n - \sigma_0 - \epsilon}^{t_n - \sigma_0 + \epsilon} \phi(t_n - s) \times \exp \left( -\beta(\xi_n) \left( M_n - \int_{t_n - s}^{t_n} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr \right) \right) ds \right|.
\]

Since \( t_n - s_n \to \sigma_0 \), we have
\[
t_n - s_n - \sigma_0 - \epsilon/2 < -\epsilon/4 \quad \text{and} \quad t_n - s_n - \sigma_0 + \epsilon/2 > \epsilon/4,
\]
for \( n \) large enough. Hence, for \( n \) large enough, we have
\[
(s_n - \epsilon/4, s_n + \epsilon/4) \subset (t_n - \sigma_0 - \epsilon, t_n - \sigma_0 + \epsilon)
\]
and \( \phi(t_n - s) = 1 \), for \( s \in (s_n - \epsilon/4, s_n + \epsilon/4) \). It follows that
\[
|\widehat{u}(t_n, \xi_n)| \geq \int_{s_n - \epsilon/4}^{s_n + \epsilon/4} \exp \left( -\beta(\xi_n) \left( M_n - \int_{t_n - s}^{t_n} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr \right) \right) ds.
\]

For each \( n \), the function
\[
[s_n - \epsilon/4, s_n + \epsilon/4] \ni s \mapsto \phi_n(s) = M_n - \int_{t_n - s}^{t_n} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr
\]
vanishes at \( s_n \) and \( \phi_n(s) \geq 0 \), for all \( s \). Furthermore, since \( \alpha(\xi_n)/\beta(\xi_n) \to K \) and \( t_n \to t_0 \), there exists \( C > 0 \), which does not depend on \( n \), such that
\[
|\widehat{u}(t_n, \xi_n)| \geq \int_{s_n - \epsilon/4}^{s_n + \epsilon/4} e^{-\beta(\xi_n)C(s-s_n)^2} ds.
\]

The Laplace Method for Integrals implies that
\[
|\widehat{u}(t_n, \xi_n)| \geq \tilde{C} \beta(\xi_n)^{-1/2},
\]
where \( \tilde{C} > 0 \) does not depend on \( n \).

As in the proof of necessity of item ii) in Theorem 4.2 the previous estimate implies that \( \widehat{u}(\cdot, \xi_n) \) does not decay rapidly. Hence, \( u \) belongs to \( \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N) \setminus C^\infty(\mathbb{T}^1 \times \mathbb{T}^N) \) and \( L \) is not globally hypoelliptic.

Finally, in the case \( \beta(\xi_n) < 0 \), for all \( n \), we repeat the technique above, but now we use
\[
M_n = \min_{0 \leq t, s \leq 2\pi} \left\{ \int_{t-s}^{t} a(r) + b(r) \frac{\alpha(\xi_0)}{\beta(\xi_n)} dr \right\} = \int_{t_n - s}^{t_n} a(r) + b(r) \frac{\alpha(\xi_n)}{\beta(\xi_n)} dr
\]
and
\[
M_{ab} = \min_{0 \leq t, s \leq 2\pi} \int_{t-s}^{t} a(r) + b(r) K dr = \int_{t_0 - s_0}^{t_0} a(r) + b(r) K dr < 0.
\]

The proof of Theorem 5.1 is complete. \( \square \)

**Corollary 5.2.** If \( \beta(\xi) \) has super-logarithmic growth and \( \alpha(\xi)/\beta(\xi) \to K \), as \( |\xi| \to \infty \), then \( L \) is not globally hypoelliptic if \( a(t) + b(t)K \) changes sign. Similarly, if \( \alpha(\xi) \) has super-logarithmic growth and \( \beta(\xi)/\alpha(\xi) \to C \), as \( |\xi| \to \infty \), then \( L \) is not globally hypoelliptic if \( b(t) + a(t)C \) changes sign.
We say that $\beta(\xi)$ goes to infinity faster than $\alpha(\xi)$, and use the notation $\alpha(\xi) = o(\beta(\xi))$, if for all positive constant $\kappa$ there exists a positive constant $n_0$ such that $|\alpha(\xi)| \leq \kappa |\beta(\xi)|$, for all $|\xi| \geq n_0$. Note that, in this case, $\alpha(\xi)/\beta(\xi) \to 0$, as $|\xi| \to \infty$.

**Corollary 5.3.** If $\beta(\xi)$ has super-logarithmic growth and $\alpha(\xi) = o(\beta(\xi))$, then $L$ is not globally hypoelliptic if $a(t)$ changes sign. If $\alpha(\xi)$ has super-logarithmic growth and $\beta(\xi) = o(\alpha(\xi))$, then $L$ is not globally hypoelliptic if $b(t)$ changes sign.

**Remark 5.4.** The main contribution of Theorem 5.1 and its corollaries is in the case where both $\alpha(\xi)$ and $\beta(\xi)$ have super-logarithmic growth. We invite the reader to compare this result with items ii) and iii) in Theorem 4.2.

**Example 5.5.** If $a(t) = \cos^2(t)$, $b(t) = -\sin^2(t)$, $\alpha(\xi) = \sqrt{|\xi|}$ and $\beta(\xi) = \sqrt{|\xi|} + 1$, then Theorem 5.1 implies that $L$ is not globally hypoelliptic. Note that $\alpha(\xi)/\beta(\xi) \to 1$, as $|\xi| \to \infty$, and $a(t) + b(t) = \cos^2(t) - \sin^2(t)$ changes sign.

Under the conditions in Corollary 5.2, in the case in which $\beta(\xi)$ has super-logarithmic growth with

$$\liminf_{|\xi| \to \infty} |\xi|^M|\beta(\xi)| > 0,$$

for some $M \geq 0$ and $K \equiv \lim_{|\xi| \to \infty} \alpha(\xi)/\beta(\xi)$, the operator $L$ is globally hypoelliptic provided that $a(t) + b(t)K$ never vanishes. In fact, for $|\xi|$ sufficiently large, the function

$$t \mapsto \Im M(t, \xi) = \beta(\xi)[a(t) + b(t)\alpha(\xi)/\beta(\xi)]$$

does not change sign. Moreover, $L_0$ is globally hypoelliptic, since $|a_0 + b_0K| > 0$ and $|\tau + M_0(\xi)| \geq \Im M_0(\xi) \geq |\beta(\xi)||a_0 + b_0\alpha(\xi)/\beta(\xi)| \geq |\beta(\xi)||a_0 + b_0K|/2$,

for $|\xi|$ large enough. Hence, Theorem 5.6 implies that $L$ is globally hypoelliptic.

**Example 5.6.** Assume that $a(t) = 1 + \sin(t)$ and $b(t) = 1 - \sin(t)$. If $\alpha(\xi) = \sqrt{|\xi|} + \xi$ and $\beta(\xi) = \xi$, then $\alpha(\xi)/\beta(\xi) \to 1$, as $|\xi| \to \infty$, and $a(t) + b(t) = 2$ never vanishes. Hence, $L$ is globally hypoelliptic.

On the other hand, if $a(t) + b(t)K$ does not change sign, but $a(t) + b(t)K$ vanishes, then $L$ may be non-globally hypoelliptic. In Subsection 5.1 we explore this phenomenon when $K = 0$, where we present a non-globally hypoelliptic operator in the case in which $a(t)$ does not change sign, $a(t)$ vanishes (of finite order) at a singular point, both $\alpha(\xi)$ and $\beta(\xi)$ have super-logarithmic growth, and $\alpha(\xi) = o(\beta(\xi))$.

### 5.1. Order of vanishing

The idea here is to show that certain relations between the order of vanishing of $a(t)$ and the speed in which $\alpha(\xi)$ and $\beta(\xi)$ go to infinity, play a role in the global hypoellipticity of the operators studied in this article.

We start with an example which illustrates that the operator may be globally hypoelliptic if, for $\xi$ large, the functions $\Im M(t, \xi)$ vanishes only of finite order, and the order of vanishing at each zero is appropriated to absorb the growth of $p(\xi)$. This situation is generalized in Theorem 5.7 and, in the sequence, we show that the converse of this result does not hold.

**First example:** Let $b \equiv 1$ and $a \in C^\infty(\mathbb{T}^1, \mathbb{R})$ be a function such that $a(t) = -(t-\pi)^2$ on a fixed interval $(\pi - \epsilon, \pi + \epsilon)$, $a(t)$ is increasing on $[0, \pi - \epsilon)$, and is decreasing on $(\pi + \epsilon, 2\pi]$.
Moreover, for \( t, s \in [0, 2\pi] \) and \( a(t) < 0 \) for \( t \in [0, \pi) \cup (\pi, 2\pi] \). Setting \( p(\xi) = \sqrt{|\xi| + i|\xi|\sqrt{|\xi|}, \xi \in \mathbb{Z}, \) we have
\[
\Im \mathcal{M}(t, \xi) = \sqrt{|\xi|(|\xi|a(t) + 1)}.
\]

Note that \( \Im \mathcal{M}(\cdot, \xi) \) changes sign for all but a finite number of indexes \( \xi \).

We will prove that
\[
L = D_t + (a + i)P(D_x), \quad (t, x) \in \mathbb{T}^2,
\]
is globally hypoelliptic, for this, given \( u \in \mathcal{D}'(\mathbb{T}^2) \) such that \( iLu = f \), with \( f \in C^\infty(\mathbb{T}^2) \), we must show that \( u \in C^\infty(\mathbb{T}^2) \).

Lemma 3.1 implies that \( \hat{u}(\cdot, \xi) \) belongs to \( C^\infty(\mathbb{T}^1) \) for all \( \xi \), and for \( |\xi| > -a_0^{-1} \), we may write
\[
\hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i \mathcal{M}_0(\xi)}} \int_0^{2\pi} \exp \left( -\int_{t-s}^t i\mathcal{M}(r, \xi) dr \right) \hat{f}(t-s, \xi) ds.
\]

For \( |\xi| > -a_0^{-1} \), the term \( (1 - e^{-2\pi i \mathcal{M}_0(\xi)})^{-1} \) is bounded; indeed, since \( a_0 < 0 \) we have
\[
2\pi \Im \mathcal{M}_0(\xi) = 2\pi \sqrt{|\xi|(|\xi|a_0 + 1)} \rightarrow -\infty, \quad \text{as } |\xi| \rightarrow \infty.
\]

Moreover, for \( t, s \in [0, 2\pi] \), we have
\[
-\Re \int_{t-s}^t i\mathcal{M}(r, \xi) dr = \int_{t-s}^t \mathcal{M}(r, \xi) dr
\]
\[
= \int_{t-s}^t |\xi|\sqrt{|\xi|}a(r) + \sqrt{|\xi|}dr
\]
\[
\leq \int_{\pi^{-1}/\sqrt{|\xi|}}^{\pi+1/\sqrt{|\xi|}} |\xi|\sqrt{|\xi|}a(r) + \sqrt{|\xi|}dr
\]
\[
= -\int_{\pi^{-1}/\sqrt{|\xi|}}^{\pi+1/\sqrt{|\xi|}} |\xi|\sqrt{|\xi|}(r-\pi)^2 dr + 2 = 4/3,
\]
for \( |\xi| \) sufficiently large.

Hence, the rapid decaying of \( \hat{f}(\cdot, \xi) \) and estimates (4.1) will imply that \( \hat{u}(\cdot, \xi) \) decays rapidly. Hence, \( u \in C^\infty(\mathbb{T}^2) \) and \( L \) is globally hypoelliptic.

The following result generalizes the situation presented in the previous example.

**Theorem 5.7.** Suppose that \( \beta(\xi) \) has super-logarithmic growth with
\[
\liminf_{|\xi| \to \infty} |\xi|^M |\beta(\xi)| > 0,
\]
for some \( M \geq 0 \) and \( \alpha(\xi) = o(\beta(\xi)) \). Assume that \( a(t) \) does not change sign and vanishes of finite order only. Write \( a^{-1}(0) = \{t_1 < \cdots < t_n\} \) and let \( m_j \) be the order of vanishing of \( a(t) \) at \( t_j \), \( j = 1, \ldots, n \). If for each \( j \) we have
\[
|\alpha(\xi)/\beta(\xi)|^{1/m_j} |\alpha(\xi)| = O(\log(|\xi|)),
\]
then the operator \( L \) given by (1.1) is globally hypoelliptic.
Proof. Given $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^N)$ such that $iLu = f$, with $f \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$, we must show that $u \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$.

Without loss of generality, assume that $a(t)$ is non-negative.

By Lemma 3.1, the coefficients $\hat{u}(\cdot, \xi)$ are smooth on $\mathbb{T}^1$, for all $\xi \in \mathbb{Z}^N$. Moreover, since $\Im \mathcal{M}_0(\xi) = \beta(\xi) a_0 + b_0 \alpha(\xi)$, $a_0 > 0$ and $\alpha(\xi) = o(\beta(\xi))$, for $|\xi|$ large enough we have $\Im \mathcal{M}_0(\xi) \neq 0$, and then $\mathcal{M}_0(\xi) \notin \mathbb{Z}_M$.

Hence, for $|\xi|$ sufficiently large, we may write

$$\hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i \mathcal{M}_0(\xi)}} \int_0^{2\pi} \exp \left( - \int_{t-s}^t i\mathcal{M}(r, \xi) dr \right) \hat{f}(t - s, \xi) ds,$$

if $\beta(\xi) < 0$, and

$$\hat{u}(t, \xi) = \frac{1}{e^{2\pi i \mathcal{M}_0(\xi)} - 1} \int_0^{2\pi} \exp \left( \int_{t}^{t+s} i\mathcal{M}(r, \xi) dr \right) \hat{f}(t + s, \xi) ds,$$

if $\beta(\xi) > 0$.

We must show that the sequence $\hat{u}(\cdot, \xi)$ decays rapidly. Notice that,

$$|\tau + \mathcal{M}_0(\xi)| \geq |\Im \mathcal{M}_0(\xi)| = |\beta(\xi)||(a_0 + b_0 \alpha(\xi)/\beta(\xi))| \geq C|\xi|^{-M},$$

when $|\xi| \to \infty$ and it follows by Proposition 3.2 that

$$|1 - e^{-2\pi i \mathcal{M}_0(\xi)}|^{-1} \quad \text{and} \quad |e^{2\pi i \mathcal{M}_0(\xi)} - 1|^{-1}$$

have at most polynomial growth.

Now, let $I = \bigcup_{j=1}^n I_j$ be a neighborhood of $a^{-1}(0)$ such that

$$a(t) = (t - t_j)^{m_j} a_j(t), \quad t \in I_j,$$

where $a_j(t) \geq C_j > 0$, and $m_j$ is an even number, so $a(t)$ does not change sign.

For the indexes $\xi$ such that $\beta(\xi) < 0$ and $|\xi|$ is sufficiently large, we have $\beta(\xi)(a(r) + b(r)\alpha(\xi)/\beta(\xi)) < 0$ on $\mathbb{T}^1 \setminus I$. Moreover, if

$$\beta(\xi)(a(r) + b(r)\alpha(\xi)/\beta(\xi)) = 0$$

for a certain $r$ in $I_j$, then

$$(r - t_j)^{m_j} a_j(r) = -b(r)\alpha(\xi)/\beta(\xi).$$

In particular,

$$|r - t_j| = \left| \frac{b(r)\alpha(\xi)}{a_j(r)\beta(\xi)} \right|^{1/m_j} \leq C'_j \left| \frac{\alpha(\xi)}{\beta(\xi)} \right|^{1/m_j},$$

where $C'_j = (\|b\|_\infty/C_j)^{1/m_j}$.

It follows that $\beta(\xi)(a(r) + b(r)\alpha(\xi)/\beta(\xi)) < 0$ on

$$\mathbb{T}^1 \setminus \bigcup_{j=1}^n \left[ t_j - C'_j \left| \frac{\alpha(\xi)}{\beta(\xi)} \right|^{1/m_j}, t_j + C'_j \left| \frac{\alpha(\xi)}{\beta(\xi)} \right|^{1/m_j} \right].$$
Hence, for the indexes $\xi$ such that $\beta(\xi) < 0$ and $|\xi|$ is sufficiently large, we obtain
\[
\int_{t-s}^{t} \beta(\xi)(a(r) + b(r)\alpha(\xi)/\beta(\xi))dr \leq \sum_{j=1}^{n} \int_{t_j-C_j}^{t_j+C_j} (r - t_j)^{m_j}a_j(r)\beta(\xi) + b(r)\alpha(\xi)dr.
\]
Since
\[
\int_{t_j-C_j}^{t_j+C_j} (r - t_j)^{m_j}a_j(r)\beta(\xi) + b(r)\alpha(\xi)dr \leq K_j \left| \frac{\alpha(\xi)}{\beta(\xi)} \right|^{1/m_j} |\alpha(\xi)|,
\]
for some positive constant $K_j$, and $|\alpha(\xi)/\beta(\xi)|^{1/m_j}|\alpha(\xi)| = O(\log(|\xi|))$ (by hypothesis), it follows that there exists a positive constant $M$ such that
\[
\left| \exp \left( -\int_{t-s}^{t} i\mathcal{M}(r,\xi)dr \right) \right| = \exp \left( \int_{t-s}^{t} \beta(\xi) \left( a(r) + b(r)\frac{\alpha(\xi)}{\beta(\xi)} \right)dr \right) \leq |\xi|^M,
\]
for all the indexes $\xi$ such that $\beta(\xi) < 0$ and $|\xi|$ is sufficiently large.

This same procedure may be used to verify that a similar estimate holds true for the indexes $\xi$ such that $\beta(\xi) > 0$ and $|\xi|$ is sufficiently large.

Finally, by using estimates above and (4.1), we may verify that the rapid decaying of $\tilde{f}(\cdot, \xi)$ will imply that $\tilde{u}(\cdot, \xi)$ decays rapidly. Therefore $u \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N)$ and $L$ is globally hypoelliptic.

Remark 5.8. We have a similar result when $\alpha(\xi)$ has super-logarithmic growth,
\[
\liminf_{|\xi| \to \infty} |\xi|^M |\alpha(\xi)| > 0,
\]
$\beta(\xi) = o(\alpha(\xi))$, and $b(t)$ does not change sign and vanishes only of finite order.

In the next example we show that the converse of Theorem 5.7 does not hold true.

SECOND EXAMPLE: Consider $a \in C^\infty(\mathbb{T}^1, \mathbb{R})$ as in the first example in this subsection. We will see that $L = D_t + (a(t) + i)P(D_x)$, where $(t, x) \in \mathbb{T}^2$ and $p(\xi) = \xi + i\xi^2$, is not globally hypoelliptic. Notice that $a(t)$ does not change sign, but
\[
\Im M(t, \xi) = \xi^2 a(t) + \xi
\]
changes sign for infinitely many indexes $\xi \in \mathbb{Z}$.

For $\xi > 0$ large enough, we have $\xi^2 a(t) + \xi < 0$ on
\[
[0, \pi - 1/\sqrt{\xi}] \cup (\pi + 1/\sqrt{\xi}, 2\pi]
\]
and $\xi^2 a(t) + \xi = -\xi^2 (t - \pi)^2 + \xi > 0$ on $(\pi - 1/\sqrt{\xi}, \pi + 1/\sqrt{\xi})$, so that
\[
M_\xi \equiv \max_{t, s \in [0, 2\pi]} \int_{t-s}^{t} \Im M(r, \xi)dr = \int_{\pi - 1/\sqrt{\xi}}^{\pi + 1/\sqrt{\xi}} -\xi^2 (r - \pi)^2 + \xi dr = 4\sqrt{\xi}/3.
\]
Let \( \hat{f}(\cdot, \xi) \) be the \( 2\pi \)-periodic extension of
\[
(1 - e^{-2\pi i \mathcal{M}_0(\xi)}) \exp \left( i \int_1^{\pi + 1/\sqrt{\xi}} \Re \mathcal{M}(r, \xi) dr \right) e^{-\mathcal{M}_\xi \phi_\xi(t)},
\]
in which \( \phi_\xi \in C_c^\infty((\pi - 2/\sqrt{\xi}, \pi], \mathbb{R}) \) is given by \( \phi_\xi(t) = \psi(\sqrt{\xi}(t - \pi + 1/\sqrt{\xi})) \), with \( \psi \in C_c^\infty((-1, 1], \mathbb{R}) \), \( 0 \leq \psi \leq 1 \), and \( \psi \equiv 1 \) in a neighborhood of \([-1/2, 1/2] \).

Notice that \( 1 - e^{-2\pi i \mathcal{M}_0(\xi)} \) is bounded, since \( a_0 < 0 \) implies that
\[
\Im \mathcal{M}_0(\xi) = \xi^2 a_0 + \xi < 0,
\]
for \( \xi \) large enough.

With these definitions, by using (4.1) we may see that \( \hat{f}(\cdot, \xi) \) decays rapidly. Hence
\[
f(t, x) = \sum_{\xi > -a_0^{-1}} \hat{f}(t, \xi) e^{i\pi \xi} \in C^\infty(\mathbb{T}^2).
\]

In order to exhibit \( u \in \mathcal{D}'(\mathbb{T}^2) \setminus C^\infty(\mathbb{T}^2) \) such that \( iLu = f \), consider
\[
\hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i \mathcal{M}_0(\xi)}} \int_0^{2\pi} \exp \left( - \int_{t-s}^{t} i \mathcal{M}(r, \xi) dr \right) \hat{f}(t - s, \xi) ds,
\]
for \( \xi > -a_0^{-1} \).

Note that \( 1 - e^{-2\pi i \mathcal{M}_0(\xi)} \neq 0 \); hence \( \hat{u}(\cdot, \xi) \) is well defined and belongs to \( C^\infty(\mathbb{T}^1) \).

For \( s, t \in [0, 2\pi] \), we have
\[
\left| \frac{1}{1 - e^{-2\pi i \mathcal{M}_0(\xi)}} \hat{f}(t - s, \xi) \exp \left( - \int_{t-s}^{t} i \mathcal{M}(r, \xi) dr \right) \right| \leq \| \psi \|_{\infty} \exp \left( - \left( \mathcal{M}_\xi - \int_{t-s}^{t} \Im \mathcal{M}(r, \xi) dr \right) \right) \leq 1.
\]

Thus \( |\hat{u}(t, \xi)| \leq 2\pi \), which implies that \( \hat{u}(\cdot, \xi) \) increases slowly. It follows that
\[
u = \sum_{\xi > -a_0^{-1}} \hat{u}(t, \xi) e^{i\pi \xi} \in \mathcal{D}'(\mathbb{T}^2),
\]
and Lemma 3.1 implies that \( iLu = f \).

Finally,
\[
|\hat{u}(\pi + 1/\sqrt{\xi}, \xi)| = \int_{1/\sqrt{\xi}}^{3/\sqrt{\xi}} \phi_\xi(\pi + 1/\sqrt{\xi} - s) \times \exp \left( - \left( \mathcal{M}_\xi - \int_{\pi + 1/\sqrt{\xi} - s}^{\pi + 1/\sqrt{\xi}} \Im \mathcal{M}(r, \xi) dr \right) \right) ds.
\]

Since \( 2/\sqrt{\xi} \) is a zero of order at least two of
\[
\theta_\xi(s) = \mathcal{M}_\xi - \int_{\pi + 1/\sqrt{\xi} - s}^{\pi + 1/\sqrt{\xi}} \Im \mathcal{M}(r, \xi) dr \geq 0,
\]
it follows that
\[
\theta_\xi(s) \leq (s - 2/\sqrt{\xi})^2 \| \theta''_\xi \|_{\infty} \leq (s - 2/\sqrt{\xi})^2 (\| a \|_{\infty} + 1).
\]
Hence
\[ |\hat{u}(\pi + 1/\sqrt{\xi}, \xi)| \geq \int_{1/\sqrt{\xi}}^{3/\sqrt{\xi}} \psi(2 - s \sqrt{\xi}) e^{-\xi^2(||a||_{\infty}+1)(s-2/\sqrt{\xi})^2} ds \]
\[ \geq \int_{3/(2\sqrt{\xi})}^{5/(2\sqrt{\xi})} e^{-\xi^2(||a||_{\infty}+1)(s-2/\sqrt{\xi})^2} ds \]
\[ = \int_{-1/(2\sqrt{\xi})}^{1/(2\sqrt{\xi})} e^{-\xi^2(||a||_{\infty}+1)s^2} ds. \]

As previously mentioned, the Laplace Method for Integrals implies that
\[ |\hat{u}(\pi + 1/\sqrt{\xi}, \xi)| \geq K/\xi, \]
where $K$ is a positive constant which does not depend on $\xi$. In particular, $\hat{u}(\cdot, \xi)$ does not decay rapidly and $L$ is not globally hypoelliptic.

6. Homogeneous operators

In the previous section we saw that, (in general) the converse of Theorem 3.6 does not hold, since there exist globally hypoelliptic operators of type (1.1) for which the function $t \in T^1 \rightarrow \mathcal{M}(t, \xi)$ changes sign, for infinitely many indexes $\xi$.

We present here a class of symbols where the converse holds. For instance, if $p(\xi)$ is homogeneous of order one, then this converse holds, since, in this case, condition (P) of Nirenberg-Treves is necessary for the global hypoellipticity (see [25], Corollary 26.4.8).

We will see that the converse of Theorem 3.6 holds true in the case in which $p(\xi)$ is homogeneous of any positive degree.

In the sequel, we present a class of operators composed of a sum of homogeneous pseudo-differential operators, for which the study of the global hypoellipticity follows from the techniques used in this article.

Theorem 6.1. Assume that the symbol of $P(D_x)$ is homogeneous of degree $m$.

i) If $m \leq 0$ then $L$ is globally hypoelliptic if and only if $L_0$ is globally hypoelliptic;

ii) If $m > 0$ then $L$ is globally hypoelliptic if and only if $L_0$ is globally hypoelliptic, and the function $t \mapsto \mathcal{M}(t, \xi)$ does not change sign, for all $\xi \in \mathbb{Z}^N \setminus \{0\}$.

Proof. If $m \leq 0$, the result follows from item i) of Theorem 4.2. For the case in which $m > 0$, the presented conditions are sufficient thanks to Theorem 3.6. On the other hand, if there exists $\xi_0 \in \mathbb{Z}^N \setminus \{0\}$ such that $t \mapsto \mathcal{M}(t, \xi_0)$ changes sign, then

$\quad t \mapsto (n|\xi_0|)^m \mathcal{M}(t, \xi_0/|\xi_0|)$

changes sign for all $n \in \mathbb{N}$.

Now in order to show that $L$ is not globally hypoelliptic, we may repeat the techniques in the proof of the necessity in item ii) of Theorem 4.2.

The following result is a consequence of Theorem 2.4 and Theorem 6.1.
Corollary 6.2. Let \( p = p(\xi) \) be a homogeneous symbol of degree \( m = \ell/q \), with \( \ell, q \in \mathbb{N} \), and \( \gcd(\ell, q) = 1 \). Write \( p(1) = \alpha + i\beta \) and \( p(-1) = \tilde{\alpha} + i\tilde{\beta} \). The operator
\[
L = D_t + (a + ib)(t)P(D_x), \quad (t, x) \in \mathbb{T}^2,
\]
is globally hypoelliptic if and only if the following statements occur:

1. The functions \( t \in \mathbb{T}^1 \mapsto a(t)\beta + b(t)\alpha \) and \( t \in \mathbb{T}^1 \mapsto a(t)\tilde{\beta} + b(t)\tilde{\alpha} \) do not change sign.
2. \( (a_0\alpha - b_0\beta)^q \) is an irrational non-Liouville number whenever \( a_0\beta + b_0\alpha = 0 \), and \( (a_0\tilde{\alpha} - b_0\tilde{\beta})^q \) is an irrational non-Liouville number whenever \( a_0\tilde{\beta} + b_0\tilde{\alpha} = 0 \).

6.1. Sum of homogeneous operators. The techniques used in this article allow us to study the global hypoellipticity of operators of the type
\[
(6.1) \quad L = D_t + \sum_{j=1}^{N} (a_j + ib_j)(t)P_j(D_{x_j}), \quad (t, x) \in \mathbb{T}^1 \times \mathbb{T}^N,
\]
where each \( P_j(D_{x_j}) \) is homogeneous of degree \( m_j \) (see Definition 2.2), so that its symbol \( p_j(\xi_j) \) satisfies
\[
p_j(\xi_j) = \begin{cases} 
\xi_j^{m_j} p_j(1), & \text{if } \xi_j > 0, \\
|\xi_j|^{m_j} p_j(-1), & \text{if } \xi_j < 0.
\end{cases}
\]

The results presented in this subsection generalize Theorem 1.3 of [6], see Corollary 6.6 below.

The constant coefficient operator \( L_0 \) associated to the operator \( L \) given by (6.1) is
\[
L_0 = D_t + \sum_{j=1}^{N} (a_{j0} + ib_{j0})P_j(D_{x_j}).
\]

We also set, for \( j = 1, \ldots, N \),
\[
\mathcal{M}_j(t, \xi_j) \doteq (a_j + ib_j)(t)p_j(\xi_j), \quad \mathcal{M}_{j0}(\xi_j) \doteq (a_{j0} + ib_{j0})p_j(\xi_j)
\]
\[
\mathcal{M}(t, \xi) \doteq \sum_{j=1}^{N} \mathcal{M}_j(t, \xi_j), \quad \text{and} \quad \mathcal{M}_0(\xi) \doteq \sum_{j=1}^{N} \mathcal{M}_{j0}(\xi_j).
\]

Theorem 6.3. The operator \( L \) given by (6.1) is globally hypoelliptic if the following situations occur:

1. \( L_0 \) is globally hypoelliptic.
2. For each pair \( j, k \in \{1, \ldots, N\} \) \( (j \neq k) \) such that \( m_j > 0 \) and \( m_k > 0 \), the sets of real-valued functions
\[
\mathcal{T}_{r,s} \doteq \{ \Im \mathcal{M}_j(\cdot, (-1)^r), \Im \mathcal{M}_k(\cdot, (-1)^s) \}, \quad r, s \in \{1, 2\},
\]
are \( \mathbb{R} \)-linearly dependent.
3. For each \( \xi_j \in \mathbb{Z} \setminus \{0\} \), the function \( t \in \mathbb{T}^1 \mapsto \Im \mathcal{M}_j(t, \xi_j) \) does not change sign whenever \( m_j > 0 \), \( j = 1, \ldots, N \).

On the other hand, if \( L \) is globally hypoelliptic, then conditions i) and iii) hold.
We notice that $L$ may be non-globally hypoelliptic if conditions $i)$ and $iii)$ hold, but condition $ii)$ fails. For instance, consider the operator

$$D_t + i \cos^2(t)D_{x_1} + i\sqrt{2}\sin^2(t)D_{x_2}, \ (t, x_1, x_2) \in \mathbb{T}^3.$$  

This operator satisfies $i)$ and $iii)$, but $ii)$ fails, since $\cos^2(t)$ and $\sqrt{2}\sin^2(t)$ are $\mathbb{R}$–linearly independent functions. Theorem 1.3 of [7] implies that this operator is not globally hypoelliptic.

Before presenting the proof of Theorem 6.3, we give an example which shows that condition $ii)$, in general, is not necessary for the global hypoellipticity of $L$.

**Example 6.4.** Consider

$$L = D_t + i \cos^2(t)D_{x_1}^2 + i \sin^2(t)D_{x_2}^2, \ (t, x_1, x_2) \in \mathbb{T}^3.$$  

Note that $\mathcal{M}^1(t, 1) = \cos^2(t)$ and $\mathcal{M}^2(t, 1) = \sin^2(t)$ are $\mathbb{R}$–linearly independent functions. Moreover, condition $iii)$ is satisfied and we have

$$|\tau + i\mathcal{M}^{10}(\xi_1) + i\mathcal{M}^{20}(\xi_2)| = |\tau + i(\xi_1^2 + \xi_2^2)/2| \geq 1/2,$$

for all $(\xi_1, \xi_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Hence, condition $i)$ is also satisfied.

By using partial Fourier series in the variables $(x_1, x_2)$ and proceeding as in the proof of Theorem 3.6, we see that $L$ is globally hypoelliptic.

**Sketch of the proof of Theorem 6.3**  
**Sufficient conditions:**

Given a distribution $u \in \mathcal{D}'(\mathbb{T}_1^1 \times \mathbb{T}_x^N)$ such that $iLu = f$, with $f \in C^\infty(\mathbb{T}_1^1 \times \mathbb{T}_x^N)$, we must show that $u \in C^\infty(\mathbb{T}_1^1 \times \mathbb{T}_x^N)$.

By using partial Fourier series in the variable $x = (x_1, \ldots, x_N)$, we are led to the equations

$$\partial_t \hat{u}(t, \xi) + i\hat{u}(t, \xi) \sum_{j=1}^N \mathcal{M}_j(t, \xi_j) = \hat{f}(t, \xi), \ t \in \mathbb{T}_1^1, \ \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{T}_x^N.$$  

Since $L_0$ is globally hypoelliptic, proceeding as in the proof of Proposition 3.3 we see that

$$Z_{\mathcal{M}} = \left\{ \xi \in \mathbb{Z}^N; \sum_{j=1}^N \mathcal{M}_{j0}(\xi_j) \in \mathbb{Z} \right\}$$

is finite. Hence, Lemma 3.1 implies that for all but a finite number of indexes $\xi$, $\hat{u}(t, \xi)$ is written as either

$$\hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i \mathcal{M}_0(\xi)}} \int_0^{2\pi} \exp \left( -i \int_{t-s}^t \mathcal{M}(r, \xi) \, dr \right) \hat{f}(t - s, \xi) ds,$$

or

$$\hat{u}(t, \xi) = \frac{1}{e^{2\pi i \mathcal{M}_0(\xi)} - 1} \int_0^{2\pi} \exp \left( i \int_{t+s}^t \mathcal{M}(r, \xi) \, dr \right) \hat{f}(t + s, \xi) ds,$$

where now

$$\mathcal{M}(t, \xi) = \sum_{j=1}^N \mathcal{M}_j(t, \xi_j).$$

Assume that $m_j > 0$, for $j = 1, \ldots, r$, and $m_j \leq 0$, for $j = r + 1, \ldots, N$.  

From formulas (6.2) and (6.3) we see that, in order to show that \( \hat{u}(\cdot, \xi) \) decays rapidly, it is enough to control the imaginary part of the functions

\[
t \in T^1 \to \sum_{j=1}^{r} \mathcal{M}_j(t, \xi_j).
\]

Recall that the global hypoellipticity of \( L_0 \) implies that the sequences \( (1 - e^{\pm 2\pi i M_0(\xi)})^{-1} \) increases slowly (Proposition 3.2).

For the indexes \( \xi \in \mathbb{Z}^N \) such that \( \xi_1 = \cdots = \xi_r = 0 \), we have

\[
\sum_{j=1}^{r} \Im \mathcal{M}_j(t, \xi_j) = \sum_{j=1}^{r} \Im \mathcal{M}_j(t, 0),
\]

which does not depend on \( \xi \).

Suppose now that \( \xi \in \mathbb{Z}^N \) is such that \( \xi_1 \neq 0 \). Since

\[
\Im \mathcal{M}_j(t, \xi_j) = |\xi_j|^{m_j} \Im \mathcal{M}_j(t, \pm 1), \quad \xi_j \neq 0,
\]

under assumption \( ii \) it follows that

\[
\sum_{j=1}^{r} \Im \mathcal{M}_j(t, \xi_j) = \Im \mathcal{M}_1(t, \pm 1) \sum_{j=1}^{r} \lambda_j^\pm |\xi_j|^{m_j} + \sum_{j=1}^{r} \gamma_j \Im \mathcal{M}_j(t, 0),
\]

where \( \lambda_j^\pm \) and \( \gamma_j \) are real numbers, \( j = 1, \ldots, r \). Moreover, \( \gamma_j = 0 \) when \( \xi_j \neq 0 \), and \( \lambda_j^\pm = 0 \) when \( \xi_j = 0 \).

An analogous formula holds if at least one \( \xi_j \) is non-zero, for \( j = 1, \ldots, r \).

We note that we have a finite number of such formulas which we may use to represent

\[
\sum_{j=1}^{r} \Im \mathcal{M}_j(t, \xi_j),
\]

for all indexes \( \xi \) such that at least one \( \xi_j \neq 0 \), \( j = 1, \ldots, r \).

By using these formulas and condition \( iii \) we may see that the rapid decaying of \( \hat{f}(\cdot, \xi) \) implies that \( \hat{u}(\cdot, \xi) \) decays rapidly (similar to which was done in the proof of item \( ii \) of Theorem 4.2).

Therefore, conditions \( i \) – \( iii \) imply that \( L \) is globally hypoelliptic.

**Necessary conditions:**

Proceeding as in the proof of Theorem 3.5, where now

\[
\mathcal{M}(t, \xi) = \sum_{j=1}^{N} \mathcal{M}_j(t, \xi_j),
\]

we see that condition \( i \) is necessary.

The necessity of condition \( iii \) follows from Theorem 6.1. Indeed, if

\[
L_j = D_t + (a_j + ib_j)(t)P_j(D_{x_j}), \quad (t, x_j) \in T^2,
\]

is not globally hypoelliptic, there exists \( \nu \in \mathcal{D}'(T^2_{(t,x)}) \setminus C^\infty(T^2) \) such that \( L_j \nu \in C^\infty(T^2) \).
Setting \( x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \), it follows that
\[
\mu \triangleq \nu \otimes 1_{x'} \in \mathcal{D}'(\mathbb{T}^2 \times \mathbb{T}^{N-1}) \setminus C^\infty(\mathbb{T}^2 \times \mathbb{T}^{N-1})
\]
and \( \mu \) satisfies \( L \mu = L_j \nu \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^N) \). Hence, \( L \) is not globally hypoelliptic if condition \( iii) \) fails.

\[\square\]

It follows from Theorem 1.3 of [6] that condition \( ii) \) of Theorem 6.3 is necessary if \( P_j(D_{x_j}) = D_{x_j}, \) \( j = 1, \ldots, N \). The next result gives a larger class of operators for which this necessity still holds true.

**Theorem 6.5.** Assume that the operator \( L \) defined in (6.1) is globally hypoelliptic. Then, for all \( j, k \in \{1, \ldots, N\} \) such that \( j \neq k \), \( m_j, m_k \in \mathbb{Z}_+^* \) and \( \gcd(m_j, m_k) = 1 \), the sets
\[
\Upsilon_{r,s} = \{ \mathfrak{M}_j(\cdot, (-1)^r), \mathfrak{M}_k(\cdot, (-1)^s) \}, \quad r, s \in \{1, 2\},
\]
are \( \mathbb{R} \)-linearly dependent.

**Proof.** Let \( m_1 \) and \( m_2 \) be positive integers such that \( \gcd(m_1, m_2) = 1 \) and assume that \( \mathfrak{M}_1(\cdot, 1) \) and \( \mathfrak{M}_2(\cdot, 1) \) are \( \mathbb{R} \)-linearly independent functions in \( C^\infty(\mathbb{T}^1, \mathbb{R}) \) (the other possibilities are analogous).

By Lemma 3.1 of [7] there exist non-zero integers \( p \neq q \) such that
\[
t \mapsto \mathfrak{M}_1(t, 1)p + \mathfrak{M}_2(t, 1)q
\]
changes sign and has non-zero mean.

Inspired by (2.5), we multiply this function by
\[
p^{(m_1-1)m_2+(m_2-1)\ell_2m_2}\ell_1m_1,
\]
where \( \ell_1 \) and \( \ell_2 \) are non-negative integers such that \( \ell_2m_2 - \ell_1m_1 = 1 \). Hence, the function
\[
t \mapsto \lfloor p^{\ell_1(m_2-1)+m_2q}\ell_1 \rfloor \mathfrak{M}_1(t, 1) + \lfloor p^{m_1-1+(m_2-1)\ell_2q}\ell_2 \rfloor \mathfrak{M}_2(t, 1),
\]
changes sign.

Setting \( \tilde{p} = p^{\ell_1(m_2-1)+m_2q}\ell_1 \) and \( \tilde{q} = p^{m_1-1+(m_2-1)\ell_2q}\ell_2 \), it follows that \( \tilde{p} \) and \( \tilde{q} \) are integers and
\[
n^{m_1m_2}[\tilde{p}^{m_1}\mathfrak{M}_1(t, 1) + \tilde{q}^{m_2}\mathfrak{M}_2(t, 1)] = \mathfrak{M}_1(t, \tilde{p}^{m_2}) + \mathfrak{M}_2(t, \tilde{q}^{m_1})].
\]

Notice that, changing the variable \( t \) by \( -t \) and considering \( -L \) (if necessary), we may assume that \( \mathfrak{M}_{10}(\tilde{p}) + \mathfrak{M}_{20}(\tilde{q}) < 0 \).

We then proceed as in the proof of necessity in item \( ii) \) of Theorem 4.2 in order to show that
\[
L_{12} = D_t + (a_1 + ib_1)P_1(D_{x_1}) + (a_2 + ib_2)(t)P_2(D_{x_2}), \quad (t, x_1, x_2) \in \mathbb{T}^3,
\]
is not globally hypoelliptic. As before, this implies that \( L \) is not globally hypoelliptic.

To be more precise, the technique to show that \( L_{12} \) is not globally hypoelliptic consists of using the change of sign of \( n^{m_1m_2}[\mathfrak{M}_1(t, \tilde{p}) + \mathfrak{M}_2(t, \tilde{q})] \) to construct a smooth function
\[
\hat{f}(t, x_1, x_2) = \sum_{n=1}^{\infty} \hat{f}(t, \tilde{p}^{m_2}, \tilde{q}^{m_1})e^{i(\tilde{p}^{m_2}x_1+\tilde{q}^{m_1}x_2)}
\]
such that $iL_u = f$ has a solution in $\mathcal{D}'(\mathbb{T}^3) \setminus C^\infty(\mathbb{T}^3)$. The Fourier coefficients $\hat{f}(\cdot, \tilde{\tilde{p}}n^{m_2}, \tilde{\tilde{q}}n^{m_1})$ are the $2\pi$-periodic extension of $\Theta_{1,2}(n) \phi(t) \exp \left( i \int_{t_0}^t \Re M_1(r, \tilde{\tilde{p}}n^{m_2}) + \Re M_2(r, \tilde{\tilde{q}}n^{m_1}) dr \right) e^{-n n_1 m_2 M}$, where $\Theta_{1,2}(n) = 1 - e^{-2\pi i [M_{10}(\tilde{\tilde{p}} n^{m_2}) + M_{20}(\tilde{\tilde{q}} n^{m_1})]}$ and

$M = \max_{0 \leq t, s \leq 2\pi} \int_{t-s}^t \Im M_1(r, \tilde{\tilde{p}}) + \Im M_2(r, \tilde{\tilde{q}}) dr$, which is supposed to be assumed in $t = t_0$ and $s = s_0$, and $\phi$ is a smooth cutoff function identically one in a small neighborhood of $t_0 - s_0$.

\hfill \Box

**Corollary 6.6.** Suppose that each symbol $p_j(\xi_j)$ is real-valued and homogeneous, whose degree is a positive integer $m_j$. Assume also that

$$\gcd(m_j, m_k) = 1, \quad \text{for} \quad j \neq k; \quad j, k \in \{1, \ldots, N\}.$$  

Under these assumptions, $L$ given by (6.1) is globally hypoelliptic if and only if the following occurs:

i) $L_0$ is globally hypoelliptic.

ii) $\dim \text{span} \{b_j \in C^\infty(\mathbb{T}^1, \mathbb{R}) ; \ j = 1, \ldots, N\} \leq 1$

iii) $b_j(t)$ does not change sign, for $j = 1, \ldots, N$.

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