Two-dimensional Site-Bond Percolation as an Example of Self-Averaging System.

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The Harris-Aharony criterion for a statistical model predicts, that if a specific heat exponent $\alpha \geq 0$, then this model does not exhibit self-averaging. In two-dimensional percolation model the index $\alpha = -\frac{1}{2}$. It means that, in accordance with the Harris-Aharony criterion, the model can exhibit self-averaging properties. We study numerically the relative variances $R_M$ and $R_\chi$ for the probability $M$ of a site belonging to the "infinite" (maximum) cluster and the mean finite cluster size $\chi$. It was shown, that two-dimensional site-bound percolation on the square lattice, where the bonds play the role of impurity and the sites play the role of the statistical ensemble, over which the averaging is performed, exhibits self-averaging properties.
I. INTRODUCTION

The influence of disorder on a phase transition is one of the important problems in the theory of phase transitions. In experimental measurements, the thermodynamic properties of one (or several) large-sized sample (with respect to the number of molecules) are usually studied. Therefore, it is important to know, whether a single large-sized sample with quenched realization of impurities can represent the properties of a model? The self-averaging properties of the system provide an answer to this question. If some quantity is self-averaging, the measurement for a single large sample gives a reasonable value for all samples of such size. If the quantity is not self-averaging, an increase in the system size does not make the measurement for a single sample representative.

Let us consider some statistical model with impurities on the $d$-dimensional lattice with a linear size $L$, in which we average some quantity $X(\omega)$ for a certain sample, where $\omega$ is the impurity realization from some impurity ensemble $\Omega$. Here we assume that $X(\omega)$ is the exact thermal average for impurity realization $\omega$. Let us denote by $[X]$ the average over all impurity realizations of ensemble $\Omega$: $[X] = \sum_{\omega \in \Omega} P(\omega)X(\omega)$, where $P(\omega)$ is the probability of impurity realization $\omega$. Then, variance $V_X = \sum_{\omega \in \Omega} P(\omega)X^2(\omega) - [X]^2$. Let us define the relative variance of $X$: $R_X = \frac{V_X}{[X]^2}$. This quantity characterizes the size dependent properties of the model. If $R_X(L) \sim L^x$, $x = -d$ the quantity $X$ is said to be strongly self-averaging. If $x < 0$, $X$ said to be weak self-averaging, and if $\lim_{L \to \infty} R_X(L) \to const$, than $X$ is non-self-averaging. We will now discuss self-averaging of a disordered system; therefore we examine the averaging over all impurity realizations, instead of self-averaging of a pure system, when we study the averaging over the whole statistical ensemble (for example, over all possible spin configuration for a spin models).

The Harris criterion states \cite{1} that the weak randomness does not change the critical behavior of the $d$-dimensional second-order phase-transition model if the specific heat index $\alpha < 0$, which corresponds to the pure system correlation length index $\nu > 2/d$. As was first mentioned by Brout \cite{2}, far for critiality where the system size is much greater than the correlation length $L \gg \xi$, all additive thermodynamical quantities are strongly self-averaging. The self-averaging properties of a disordered system near the critiality obeys Harris-Aharony (HA) criterion see \cite{3} and \cite{4}. This criterion states that if randomness is irrelevant the system is governed by a pure fixed point and the relative variance $R_X \sim L^{\alpha_p \nu_p}$, where $\alpha_p$ and $\nu_p$ are the critical exponents of a pure system. If however, the system is governed by a random fixed point, then $\lim_{L \to \infty} R_X = const$. This criterion explains the numerical results of many works \cite{5, 6, 7}. Wiseman and Domany in \cite{7} show that the Ashkin-Teller model with $\alpha < 0$ is weakly self-averaging.

Two-dimensional percolation, which can be treated as a $q$-state Potts model with $q \to 1$, is one of the most popular two-dimensional statistical models with the second-order phase transition. The two-dimensional percolation critical index $\alpha = -\frac{1}{2} < 0$, and, in accordance with the HA criterion we can expect that this model exhibits weak self-averaging properties. We study numerically the relative variances $R_M$ and $R_\chi$ of the probability $M$ of a site to belong the maximum cluster (the analog of magnetization) and the mean finite cluster size $\chi$ (the analog of the magnetic susceptibility). It was shown, that two-dimensional site-bound percolation, where bonds play the role of impurities and sites play the role of statistical
ensemble over which the averaging is performed, exhibits self-averaging properties. The article is arranged as follows. In Section 2 we describe the site-bond percolation model and two types of impurities distribution; in Section 3 we discuss the self-averaging criterion and its phenomenological derivation; in Section 4 we present the numerical results and Section 5 is the conclusion.

II. TWO-DIMENSIONAL SITE-BOND PERCOLATION MODEL AND CANONICAL AND GRAND-CANONICAL IMPURITY DISTRIBUTIONS

Site-bond percolation is a combination of purely site percolation and purely bond percolation. In purely site percolation, one investigates clusters of occupied sites. In purely bond percolation, one investigates clusters of sites, connected by occupied bonds. In site-bond percolation one investigates clusters of occupied sites connected by occupied bonds. Each site is occupied with probability $p_s$, and each bond is occupied with probability $p_b$. In critical point $(p^*_s, p^*_b)$ the correlation length becomes infinite. Yanuka and Engelman [8] proposed approximate formula for a critical curve in $(p_s, p_b)$ plane

$$\frac{\log(p_s)}{\log(p^*_s)} + \frac{\log(p_b)}{\log(p^*_b)} = 1$$

The two-dimensional site-bond percolation belongs the same universality class as the two-dimensional purely site percolation. Let us define by $n_C$ the mean number of clusters of size $C$ per lattice site. Then, the probability of a site to belong to the maximum cluster is

$$M = n_{C_{max}} C_{max}$$

and the mean finite cluster size

$$\chi = \frac{\sum_{C, C \neq C_{max}} n_C C^2}{\sum_{C, C \neq C_{max}} n_C C}$$

Near the critical point $(p^*_s, p^*_b)$ this quantities scales as follows

$$M(p_s, p_b = p^*_b) \sim (p_s - p^*_s)^\beta, \quad p_s > p^*_s, \quad \beta = \frac{1}{18}$$

and

$$\chi(p_s, p_b = p^*_b) \sim (p_s - p^*_s)^{-\gamma}, \quad \gamma = \frac{43}{18}$$

The correlation length $\xi(p_s, p_b = p^*_b) \sim (p_s - p^*_s)^{-\nu}$ scaling index is $\nu = \frac{4}{3}$.

As mentioned above, each bond is occupied with probability $p_b$ and is empty with probability $1 - p_b$; therefore the total number of occupied bonds on the lattice fluctuates. This method of generating impurity configurations is known as Grand Canonical (GC) by analogy with Grand Canonical statistical ensemble with a fluctuating number of particles. However we can fix the number of occupied bonds $N_b$ and then distribute them randomly on the lattice. This method of generating impurity configurations is known as Canonical (C) by analogy with the Canonical statistical ensemble with fixing number of particles.
III. SELF-AVERAGING CRITERION

Let us explain the HA criterion via simple phenomenological considerations [6]. In this section, we use the $T$ temperature as a parameter of the model but keeping in mind, that in the case of percolation, the role of this parameter is played by the site concentration $p_s$.

We characterize every sample (impurity realization) $\omega$ with size $L$ by a pseudo-critical temperature $T^*(\omega, L)$. $T^*(\omega, L)$ fluctuates about its mean value and is averaged over all impurity configurations $T^* = [T^*(\omega, L)]$. We introduce the reduced temperature for each sample

$$
\tilde{t}_\omega = \frac{T - T^*(\omega, L)}{T^*}
$$

In the vicinity of the critical point, the quantity $X$ scales as

$$
X_\omega(T, L) = L^\rho \tilde{Q}_\omega(\tilde{t}_\omega L^{y_t})
$$

Here, $\rho$ is the exponent characterizing the behavior of $[X]$ at $T^*$ and the thermal scaling index $y_t = \frac{1}{\nu}$ is assumed to be universal for all samples. The form of function $\tilde{Q}_\omega$ is assumed to be sample dependent. We assume that (this relation is numerically checked in the next section)

$$
(\delta T^*)^2 \sim (\delta \tilde{t}_\omega)^2 \sim L^{-d}
$$

So the variance of the argument of the function $\tilde{Q}_\omega$ scales as

$$
(\delta \tilde{t}_\omega L^{y_t})^2 \sim L^{-d + \frac{2}{\nu}}
$$

We keep in mind that the scaling relation [9] $\alpha = 2 - \nu d$. So, the variance $V_X$ of quantity $X$ at the critical point scales as

$$
V_X \sim L^{2\rho} (\delta \tilde{t}_\omega L^{y_t})^2 \sim L^{2\rho - d + \frac{2}{\nu}} = L^{2\rho + \frac{2 - \nu d}{\nu}} = L^{2\rho + \frac{\alpha}{\nu}}
$$

and the relative variance ($[X] \sim L^\rho$) is

$$
R_X = \frac{V_X}{[X]^2} = L^{2\rho + \frac{\alpha}{\nu} - 2\rho} = L^{\frac{\alpha}{\nu}}
$$

We thus obtain the Harris-Aharony criterion: if the scaling index $\alpha = \text{const} < 0$, then the model is self-averaging and the relative variance for quantity $X$ is $R_X \sim L^{\frac{\alpha}{\nu}}$ from simple scaling relations.

IV. NUMERICAL RESULTS

In our computation, the bond play the role of impurities; therefore, we fix the bond concentration $p_b^* = 0.875$ and consider the site concentration $p_s$ as parameter of our model.

1. we generate (by C and GC methods) the bond configuration $\omega$. 

2. For this bond configuration we generate a set of site configurations (for site we use only GC method); for each site configuration we, calculate the probability of a site belonging to the maximum cluster and the mean finite cluster size.

3. We average these quantities over site realizations and obtain mean values $M(p_s, p_b)$ and $\chi(p_s, p_b)$. We split the set of site configurations into ten series to evaluate the numerical inaccuracy $\Delta M$ and $\Delta \chi$.

4. We perform steps 1.–3. for another bond configuration.

The $\chi_\omega(p_s)$ dependence for three different bond realizations ($\omega = 1, 2, 3$) is shown in Fig.1. Let us calculate the pseudocritical site concentration $p_s^*(\omega)$ for each bond realization $\omega$. We assume, that the mean finite cluster size $\chi_\omega(p_s)$ has maximum at the pseudocritical point $p_s^*(\omega)$, and therefore, we approximate the data for $\chi_\omega$ near the maximum of $\chi_\omega$ by the parabola $\chi_\omega(p_s) \approx a - b (p_s - p_s^*(\omega))^2$ and treat $p_s^*(\omega)$ as the pseudocritical concentration for bond realization $\omega$. The locations of pseudocritical points for bond realizations $\omega = 1, 2, 3$ is shown in Fig.1 by vertical lines.

Let us find the critical point $p_s^*$. We calculate the mean critical site concentration $[p_s^*(L)]_{\text{averaged over 100 bond realizations created by GC and C methods as a function of lattice size } L}$. These numerical data and results of approximation

\[
\begin{align*}
\text{C - method} : & \quad [p_s^*(L)]_{C} \simeq 0.6519(5) - 0.34(10)L^{-0.94(9)} \\
\text{GC - method} : & \quad [p_s^*(L)]_{GC} \simeq 0.6511(4) - 0.39(15)L^{-1.02(11)}
\end{align*}
\]

is shown in Fig.2. We take the concentration $p_s^* = 0.6515$ (averaged by C and GC methods) as critical. The variance of pseudocritical concentration behaves as $(\delta p_s^*)^2 \sim L^{-2}$ (Fig.3), as we assumed in the previous section.

Let us investigate the self-averaging properties of $M$ and $\chi$ at the critical point ($p_b^* = 0.875, p_s^* = 0.6515$).

Now we describe computational procedure that we use to calculate the relative variance. Here, we follow the method described [5, 6]. First, we note that, for each impurity realization $\omega$, instead of the exact value of quantity $X_\omega$ we get some value $\bar{X}_\omega$ averaged over site configurations with a numerical error

\[
(\delta \bar{X}_\omega)^2 = \frac{\sigma_{p_b,\omega}}{N_s/\tau_\omega}
\]

Here, $N_s$ is the number of site configurations (the length of Monte-Carlo run) and $\tau_\omega$ is autocorrelation time. To calculate $(\delta \bar{X}_\omega)^2$ we split the MC sequence of site configurations of length $N_s$ into 10 subsequences and treat each subsequence as independent. We define by $[\ldots]$ the averaging over impurity – bond configurations. Thus, the error of $[\bar{X}_\omega]$ averaged over $N_b$ bond configurations is

\[
(\delta [\bar{X}_\omega])^2 = \frac{1}{N_b - 1} \sum_{\omega \in \Omega} ([\bar{X}_\omega^2] - [\bar{X}_\omega]^2)
\]
The total total error \( (\delta[\bar{X}_{\omega}])^2 \) has two contributing terms: from the sample to sample fluctuation of exact \( X_{\omega} \) about \([X_{\omega}] \) and from the fluctuations of \( \bar{X}_{\omega} \), averaged over the finite number of spin configurations, about \( X_{\omega} \) for each bond realization \( \omega \).

\[
(\delta[\bar{X}_{\omega}])^2 = \frac{V_X}{N_b} + \left[ \frac{\sigma_{p_b,\omega}}{N_b N_s/\tau_{\omega}} \right] \tag{14}
\]

Here, \( N_b \) is the number of bond configurations. We can calculate \( \left[ \frac{\sigma_{p_b,\omega}}{N_b N_s/\tau_{\omega}} \right] \) by averaging the error \( (\delta\bar{X}_{\omega})^2 \) over the bond configurations \( \omega \in \Omega \). So, we can express the relative variance via \( (\delta[\bar{X}_{\omega}])^2 \) and \( (\delta\bar{X}_{\omega})^2 \)

\[
R_X = \frac{V_X}{|X|^2} = \frac{1}{|X|^2} \left( N_b \left( \delta[\bar{X}_{\omega}] \right)^2 - \left[ (\delta\bar{X}_{\omega})^2 \right] \right) \tag{15}
\]

The numerical data for relative variance \( R_M \) and \( R_X \) computed by the C and GC methods in accordance (15), are shown in Fig. 5 and 4 respectively. We can see that points lie on the straight lines in log-log scale. Thus, we approximate the data by power function \( a L^b \) on the interval \( L \in [48, 128] \). The results of approximation are plotted in Fig. 5 and 4 and placed below. As we might expect, for the GC method the relative variance is greater, than for the C-method because of the fluctuation of bonds number.

\begin{align*}
C : R_M &\simeq 0.0041(3) L^{-0.504(19)} \\
GC : R_M &\simeq 0.0147(15) L^{-0.50(2)} \\
C : R_X &\simeq 0.0243(2) L^{-0.53(2)} \\
GC : R_X &\simeq 0.0259(5) L^{-0.51(5)}
\end{align*}

We see, that in excellent agreement with HA criterion, the relative variance of measured quantities obeys the power law dependence \( R \sim L^{-\frac{\nu}{2}} = L^\alpha \). As a result, we can state that this model is self-averaging.

\section{V. CONCLUSION}

We have shown numerically that the two-dimensional site-bond percolation exhibit self-averaging properties at critical point. In our numerical experiments bonds play the role of the quenched disorder, and the sites play role of the statistical ensemble. We have found that relative variance scales \( R_M \sim L^{-\frac{\nu}{2}}, R_X \sim L^\frac{\nu}{2} \), where \( -\frac{\nu}{2} = \frac{\alpha}{\nu} \). We assume that we can consider the sites as a quenched disorder and bonds, as a statistical ensemble, and the results will be the same.

We can also expect for the site percolation that, if we "freeze" the state of some sites and average over other sites, we get the same weakly self-averaging behavior with respect this "frozen" sites, which play the role of quenched disorder. Of course, the same is valid for bond percolation.
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Figure captions

**Figure 1.** The mean finite-cluster size $\chi(p)$ for three realization of impurity configurations. The critical concentrations $p_s^*$ for each realization $\omega$, are shown by vertical lines.

**Figure 2.** The mean pseudocritical site concentration $[p_s^*]$ as a function of lattice size $L$ for bond realizations, created by C and GC methods, the results of approximation by a power function of $L$, and the extrapolated critical concentration $p_s^* = 0.6515$.

**Figure 3.** The variance $(\delta p_s^*)^2$ of critical concentration as a function of lattice size $L$ for C (crosses) and GC (triangles) distributions, and the results of approximation by function $aL^{-b}$.

**Figure 4.** The relative variance $R_M$ of probability $M$ of a site belonging to the maximum cluster as a function of lattice size $L$ for GC (crosses) and C (triangles) distributions and the results of approximation by function $aL^{-b}$.

**Figure 5.** The relative variance $R_\chi$ of the mean finite-cluster size $\chi$ as a function of lattice size $L$ C (crosses) and GC (triangles) distributions and the results of approximation by function $aL^{-b}$. 
FIG. 1:
C-method \[ p_s^* \]

GC-method \[ p_s^* \]

\[
0.6519(5) - 0.34(10)L^{-0.94(9)}
\]

\[
0.6511(4) - 0.39(15)L^{-1.02(11)}
\]

\[ p_s^* = 0.6515 \]

FIG. 2:
C-method \((\delta p^*_s)^2\)

GC-method \((\delta p^*_s)^2\)

0.0014(7)\(L^{-2.1(1)}\)

0.0020(5)\(L^{-1.93(6)}\)

FIG. 3:
FIG. 4:
FIG. 5: