We discuss some recent work in the field of classical solitonic solutions in string theory. In particular, we construct instanton and monopole solutions and discuss the dynamics of string-like solitons. Some of the motivation behind this work is that instantons may provide a nonperturbative understanding of the vacuum structure of string theory, while monopoles may appear in string predictions for grand unification. The string-like solitons represent extended states of fundamental strings. The essential role of supersymmetry in both the saturation of the Bogomol'nyi bound and in the cancellation of higher order corrections is emphasized. Talk given at the International Workshop: “Recent Advances in the Superworld”, Houston Advanced Research Center, The Woodlands, TX, April 14-16, 1993. CTP/TAMU-21/93, April 1993.

1. Introduction

In the past few years, classical solitonic solutions in string theory with higher-membrane structure have been actively investigated. These solutions are static multi-soliton solutions obeying a zero-force condition and saturating a Bogomol’nyi bound between ADM mass and charge. In certain cases, exact solutions of bosonic and heterotic string theory may be constructed, each solution in principle corresponding to an exact conformal field theory (CFT) of the sigma-model. Although the solutions are initially conceived as perturbative expansions in the classical string parameter $\alpha'$ (the inverse string tension), the exact solutions acquire nonperturbative status. Being classical, the solitons are tree-level solutions in the (quantum) topological expansion of the string worldsheet, but are also nonperturbative in the loop parameter $e^{\phi_0}$. Therefore, full quantum string-loop extensions of these solutions await an understanding of nonperturbative string theory. Nevertheless, it is possible to use these solitons in nonperturbative calculations (such as vacuum-tunneling) since it is often the case that higher order corrections do not contribute to these effects.

In this work, we discuss the construction of two classes of solitons, instantons and monopoles\textsuperscript{1}, and examine the dynamics of string-like solitons\textsuperscript{2}. Both the monopole and instanton solutions have “fivebrane” structure in $D = 10$ (i.e. they are $5 + 1$-dimensional objects in $9 + 1$-dimensional spacetime).

The motivation for the study of these solutions includes the following: Since these solutions represent Planck scale solitons, their existence constitutes a possible test for string theory as a theory of quantum gravity. Another interest in these
solutions stems from the application of known field-theoretic techniques (and their stringy analogs) in the physics of solitons and instantons to string theory. For example, the string instanton solutions, through vacuum tunneling computations, may lead to an understanding of the structure of the vacuum in string theory, much in the same manner as instantons are used in field theory. The string monopole solutions may appear in grand unification predictions of string theory, while the macroscopic string solitons may be used to represent extended string states. An especially noteworthy feature of the monopoles shown here is the cancellation between gauge and gravitational singularities in the action, a feature, which, if it survives quantization, promises to shed light on the nature of string theory as a finite theory of quantum gravity. An important point regarding the role of supersymmetry in these solutions is that the supersymmetric solutions represent extremal limits of generalized black hole type solutions. For each class of solutions, the existence of partially unbroken spacetime supersymmetries leads to the saturation of a Bogomol’nyi bound and guarantees the stability of the solitons. In addition, for the heterotic solutions, worldsheet supersymmetry leads to the cancellation of higher order corrections. A different point of view to these solutions is the study of the resultant string-inspired low-energy field theories, which may well capture the essential behaviour of these solitons (e.g. the singularity cancellation in the monopole solutions) without requiring an expansion in the full string theory. Finally, there is the open problem of proving the string/fivebrane duality conjecture, and its implications to low energy string theory, the singularity structure of string theory and its relation to the more familiar electric/magnetic duality conjecture of Montonen and Olive.

2. Instantons and Monopoles

We first summarize the ’t Hooft ansatz for the Yang-Mills instanton, and then write down the tree-level bosonic axionic instanton solution. An exact bosonic solution, with corresponding CFT can be obtained for the special case of a linear dilaton wormhole. An exact multi-instanton solution of heterotic string theory is then obtained by combining the Yang-Mills gauge solution with the bosonic axionic instanton. An exact heterotic multi-monopole can be obtained from the same ansatz via a slight modification of the solution. The notable feature in this case is the cancellation between gauge and gravitational divergences in the effective action.

Consider the four-dimensional Euclidean action

\[
S = -\frac{1}{2g^2} \int d^4x \text{Tr} F_{\mu\nu}F^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4. \tag{1}
\]

For gauge group SU(2), the fields may be written as \(A_\mu = (g/2i)\sigma^a A^a_\mu\) and \(F_{\mu\nu} = (g/2i)\sigma^a F^{a}_{\mu\nu}\) (where \(\sigma^a, a = 1, 2, 3\) are the 2 \(\times\) 2 Pauli matrices). A self-dual solution (but not the most general one) to the equation of motion of this action is given by the ’t Hooft ansatz

\[
A_\mu = i\Sigma_{\mu\nu} \partial_\nu \ln f, \tag{2}
\]
where $\Sigma_{\mu\nu} = \eta^{i\mu}(\sigma^i/2)$ for $i = 1, 2, 3$, where

\[
\begin{align*}
\pi^{\mu\nu} &= -\pi^{\nu\mu} = \epsilon^{\mu\nu}, \\
\mu, \nu &= 1, 2, 3, \\
\pi^{\nu}\nu &= -\delta^{\nu}, \quad \nu = 4
\end{align*}
\]

and where $f^{-1} f = 0$. The ansatz for the anti-self-dual solution is similar, with the $\delta$-term in Eq. (3) changing sign. To obtain a multi-instanton solution, one solves for $f$ in the four-dimensional space to obtain

\[
f = 1 + \sum_{i=1}^{N} \frac{\rho_i^2}{|\vec{x} - \vec{a}_i|^2},
\]

where $\rho_i^2$ is the instanton scale size and $\vec{a}_i$ the location in four-space of the $i$th instanton. Note that this solution has $5N$ parameters, while the most general self-dual solution has $8N - 3$ parameters.

It turns out that there is an analog to the Yang-Mills instanton in the gravitational sector of the string, namely the axionic instanton\(^7\). In its simplest form, this instanton appears as a solution for the massless fields of the bosonic string. The bosonic sigma model action can be written as

\[
I = \frac{1}{4\pi\alpha'} \int d^2x \left( \sqrt{\gamma^{ab}} \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} + i \epsilon^{ab} \partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu\nu} + \alpha' \sqrt{\gamma} R^{(2)}(\phi) \right),
\]

where $g_{\mu\nu}$ is the sigma model metric, $\phi$ the dilaton and $B_{\mu\nu}$ the antisymmetric tensor, and where $\gamma^{ab}$ is the worldsheet metric and $R^{(2)}$ the two-dimensional curvature. The classical equations of motion of the effective action of the massless fields is equivalent to Weyl invariance of $I$. A tree-level solution is given by any dilaton function satisfying $e^{-2\phi} e^{2\phi} = 0$ with

\[
\begin{align*}
g_{\mu\nu} &= \epsilon^{2\phi} \delta_{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4, \\
g_{ab} &= \delta_{ab}, \quad a, b = 5, \ldots, 26, \\
H_{\mu\nu\lambda} &= \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi, \quad \mu, \nu, \lambda, \sigma = 1, 2, 3, 4.
\end{align*}
\]

In order to see the instanton structure of this solution, we define a generalized curvature $\hat{R}^i_{jkl}$ in terms of the standard curvature $R^i_{jkl}$ and $H_{\mu\nu\lambda}$:

\[
\hat{R}^i_{jkl} = R^i_{jkl} + \frac{1}{2} \left( \nabla_l H_{jk}^i - \nabla_k H_{jl}^i \right) + \frac{1}{4} \left( H_{jk}^m H_{lm}^i - H_{jl}^m H_{km}^i \right).
\]

One can also define $\hat{R}^i_{jkl}$ as the Riemann tensor generated by the generalized Christoffel symbols $\hat{\Gamma}^\mu_{\alpha\beta}$, where $\hat{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - (1/2) H^\mu_{\alpha\beta}$. The crucial observation for obtaining higher-loop and even exact solutions is the following. For any solution given by Eq. (6), we can express the generalized curvature in covariant form in terms of the dilaton field as\(^7\)

\[
\hat{R}^i_{jkl} = \delta_{il} \nabla_k \nabla_j \phi - \delta_{ik} \nabla_l \nabla_j \phi + \delta_{jk} \nabla_l \nabla_i \phi - \delta_{jl} \nabla_k \nabla_i \phi \pm \epsilon_{ijkm} \nabla_l \nabla_m \phi \mp \epsilon_{ijkl} \nabla_k \nabla_l \phi,
\]

where $\Sigma_{\mu\nu} = \eta^{i\mu}(\sigma^i/2)$ for $i = 1, 2, 3$, where

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\]
It easily follows that

\[ \hat{R}_{ijkl} = \frac{1}{2} \hat{C}_{kl}^{mn} \hat{R}_{jmn}. \]  

(9)

So the instanton appears in the gravitational sector of the string in the (anti) self-duality of the generalized curvature. A tree-level multi-instanton solution is therefore given by Eq. (6) with the dilaton given by

\[ e^{2\phi} = C + \sum_{i=1}^{N} \frac{Q_i}{|x - \vec{a}_i|^2}, \]  

(10)

where \( Q_i \) is the charge and \( \vec{a}_i \) the location in the four-space (1234) (the transverse space) of the \( i \)th instanton. In the spherically symmetric case \( e^{2\phi} = Q/r^2 \), we can explicitly solve the higher order equations of motion by rescaling the dilaton and fixing the metric and antisymmetric tensor in lowest order form. For example, the two-loop dilaton is given by \( e^{2\phi} = Q/r^2(1 - \alpha' Q) \). To get an exact solution in this linear dilaton case, we notice that the sigma-model action can be decomposed according to

\[ I = I_1 + I_3, \]

where \( I_1 \) is the action for a Feigin-Fuchs Coulomb gas, a one-dimensional CFT with central charge given by \( c_1 = 1 + 6\alpha' (\partial \phi)^2 \) and \( I_3 \) is the Wess–Zumino–Witten action on an \( SU(2) \) group manifold with central charge

\[ c_3 = \frac{3k}{k+2} \approx 3 - \frac{6}{k} + \frac{12}{k^2} + ... \]  

(12)

where \( k = Q/\alpha' \), the level of the WZW model, is an integer, from the quantization condition on the Wess-Zumino term. Thus \( Q \) is not arbitrary, but is quantized in units of \( \alpha' \).

We use this splitting to obtain exact expressions for the fields by fixing the metric and antisymmetric tensor field in their lowest order form and rescaling the dilaton to all orders in \( \alpha' \). The resulting expression for the dilaton is

\[ e^{2\phi} = \frac{Q}{r \sqrt{1 + \frac{2\alpha'}{r}}} \]  

(13)

for arbitrary \( \alpha' \).

We now turn to the heterotic solution. The tree-level supersymmetric vacuum equations for the heterotic string are given by

\[ \delta \psi_M = (\nabla_M - \frac{1}{4} H_{MAB} \Gamma^{AB}) \epsilon = 0, \]
\[ \delta \lambda = (\Gamma^A \partial_A \phi - \frac{1}{4} H_{AMC} \Gamma^{ABC}) \epsilon = 0, \]
\[ \delta \chi = F_{AB} \Gamma^{AB} \epsilon = 0, \]  

(14)
where $\psi_M$, $\lambda$ and $\chi$ are the gravitino, dilatino and gaugino fields. The Bianchi identity is given by

$$dH = \alpha' \left( \text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F \right).$$

(15)

The $(9 + 1)$-dimensional Majorana-Weyl fermions decompose down to chiral spinors according to $SO(9,1) \supset SO(5,1) \otimes SO(4)$ for the $M^{9,1} \rightarrow M^{5,1} \times M^4$ decomposition. Let $\mu, \nu, \lambda, \sigma = 1, 2, 3, 4$ and $a, b = 0, 5, 6, 7, 8, 9$. Then the ansatz

$$g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu},$$

$$g_{ab} = \eta_{ab},$$

$$H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi$$

(16)

with constant chiral spinors $\epsilon_{\pm}$ solves the supersymmetry equations with zero background fermi fields provided the YM gauge field satisfies the instanton (anti)self-duality condition

$$F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu}^{\lambda\sigma} F_{\lambda\sigma}.$$  

(17)

A perturbative solution representing a supersymmetric fivebrane was first derived by Strominger$^9$. In the absence of a gauge sector, the multi-fivebrane solution is identical to the tree-level type II supersymmetric fivebrane solution of Duff and Lu$^{10}$, derived in terms of the dual seven-form formulation of supergravity, with $K = e^{-\phi} H = dA$, where $A$ is the antisymmetric six-form associated with a fivebrane. An exact solution is obtained as follows. Define a generalized connection by

$$\Omega_{\pm M} = \omega_{\pm M} \pm H_{\pm M}$$

(18)

embedded in an SU(2) subgroup of the gauge group, and equate it to the gauge connection $A_{\mu}$$^{11}$ so that $dH = 0$ and the corresponding curvature $R(\Omega_{\pm})$ cancels against the Yang-Mills field strength $F$. As in the bosonic case, for $e^{-2\phi} e^{2\phi} = 0$ with the above ansatz, the curvature of the generalized connection can be written in covariant form in terms of the dilaton as in Eq. (8) from which it follows that both $F$ and $R$ are (anti)self-dual. This solution becomes exact since $A_{\mu} = \Omega_{\pm\mu}$ implies that all the higher order corrections vanish$^{7,12,13}$ The self-dual solution for the gauge connection is then given by the 't Hooft ansatz. An interesting feature of the heterotic solution is that it combines a YM instanton structure in the gauge sector with an axionic instanton structure in the gravity sector. In addition, the heterotic solution has finite action.

Note that the single instanton solution in the heterotic case carries through to higher order without correction to the dilaton. This seems to contradict the bosonic solution by suggesting that the expansion for the central charge $c_3$ terminates at one loop. This contradiction is resolved by noting that for an $N = 4$ worldsheet supersymmetric solution$^{13}$ the bosonic contribution to the central charge is given by

$$c_3 = \frac{3k'}{k' + 2},$$  

(19)
where $k' = k - 2$. This reduces to

$$c_3 = 3 - \frac{6}{k} = 3 - \frac{6\alpha'}{Q}, \quad (20)$$

which indeed terminates at one loop order. The exactness of the splitting then requires that $c_1$ not get any corrections from $(\partial \Phi)^2$ so that $c_1 + c_3 = 4$ is exact for the tree-level value of the dilaton$^{13}$.

In a similar manner, we obtain an exact heterotic multi-monopole solution$^{14}$. We do so by singling out a direction in the transverse space (say $x_4$). Since the derivation is essentially identical to that for the heterotic instanton, we simply write down the solution

$$g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu}, \quad g_{ab} = \eta_{ab},$$

$$H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \Phi,$$

$$e^{2\phi} = e^{2\phi_0} f,$$

$$A_\mu = i \sum_{i=1}^N m_i \delta_{x_i} \ln f,$$  

where in this case $f = 1 + \sum_{i=1}^N \frac{m_i}{|\vec{x}_i - \vec{a}_i|}$ and $\vec{x}$ and $\vec{a}_i$ are vectors in the space (123). If we set $\Phi = A_4$, then the gauge and scalar fields in the subspace (0123) may be simply written in terms of the dilaton as

$$\Phi^a = \mp \frac{2}{g} \delta^a_i \partial_i \phi,$$

$$A^a_k = - \frac{2}{g} \epsilon^{akj} \partial_j \phi.$$  

(22)

The above solution represents an exact multimonopole solution of heterotic string theory$^{14}$, and is stable as a result of its saturation of a Bogomol’nyi bound between ADM mass and charge$^{15}$. The saturation of the Bogomol’nyi bound in turn owes to the existence of partially unbroken spacetime supersymmetries.

In the string effective action, the term $-\alpha' F^2$ diverges near each source. However, this divergence is precisely cancelled by the term $\alpha'R^2(\Omega_{\pm})$ in the $O(\alpha')$ action. This result follows from the exactness condition $A_\mu = \Omega_{\pm\mu}$ which leads to $dH = 0$ and the vanishing of all higher order corrections in $\alpha'$. Another way of seeing this is to consider the higher order corrections to the bosonic action$^{12}$. All such terms contain the tensor $T_{MNPQ}$, a generalized curvature incorporating both $R(\Omega_{\pm})$ and $F$. The ansatz is constructed precisely so that this tensor vanishes identically$^{7,14}$. The action thus reduces to its finite lowest order form and can be calculated directly for a multi-source solution from the expressions for the massless fields in the gravity sector.

The divergences in the gravitational sector in heterotic string theory thus serve to cancel the divergences stemming from the field theory solution. This solution thus provides an interesting example of how this type of cancellation can occur in string theory, and supports the promise of string theory as a finite theory of quantum gravity. Another point of interest is that the string solution represents
a supersymmetric multimonopole solution coupled to gravity, whose zero-force condition in the gravity sector (cancellation of the attractive gravitational force and repulsive antisymmetric field force) arises as a direct result of the zero-force condition in the gauge sector (cancellation of gauge and Higgs forces of exchange) once the gauge connection and generalized connection are identified. The fulfillment of the exactness condition depends crucially on the existence of unbroken spacetime supersymmetries.

3. Dynamics of String Solitons

In earlier work\textsuperscript{16}, Dabholkar et al. presented a low-energy analysis of macroscopic superstrings and discovered several interesting analogies between macroscopic superstrings and solitons in supersymmetric field theories. The main result of this work centers on the existence of exact multi-string solutions of the low-energy supergravity super-Yang-Mills equations of motion. In addition, Dabholkar et al. find a Bogomolnyi bound for the energy per unit length which is saturated by these solutions, just as the Bogomolnyi bound is saturated by magnetic monopole solutions in ordinary Yang-Mills field theory. The solution may be outlined as follows. The action for the massless spacetime fields (graviton, axion and dilaton) in the presence of a source string can be written as

\[
S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{g} \left( R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\alpha}\phi H^2 \right) + S_{\sigma},
\]

with the source terms contained in the sigma model action \( S_{\sigma} \) given by

\[
S_{\sigma} = \frac{-\mu}{2} \int d^2\sigma (\sqrt{\gamma}^{-1} \cdot \frac{\partial X}{\partial \sigma} \cdot \frac{\partial X}{\partial \sigma} - e^{\alpha-2\phi} \gamma^{\mu\nu} B_{\mu\nu}),
\]

with \( \alpha = \sqrt{2/(D-2)} \) and \( \gamma_{mn} \) a worldsheet metric to be determined. The sigma model action \( S_{\sigma} \) describes the coupling of the string to the metric, antisymmetric tensor field and dilaton. The first part of the action \( S \) above represents the effective action for the massless fields in the spacetime frame and whose equations of motion are equivalent to conformal invariance of the underlying sigma model. The combined action thus generates the equations of motion satisfied by the massless fields in the presence of a macroscopic string source. The static solution to the equations of motion is given by

\[
ds^2 = e^A [-dt^2 + (dx^1)^2] + e^B d\vec{x} \cdot d\vec{x}
\]

\[
A = \frac{D-4}{D-2} E(r) \quad B = -\frac{2}{D-2} E(r)
\]

\[
\phi = \alpha E(r) \quad B_{01} = -e^E(r),
\]

where \( x^1 \) is the direction along the string, \( r = \sqrt{\vec{x} \cdot \vec{x}} \) and

\[
e^{-E(r)} = \begin{cases} 
1 + \frac{M}{1 - 8G\mu \ln(r)} & \text{for } D > 4 \\
1 - 8G\mu \ln(r) & \text{for } D = 4
\end{cases}
\]
for a single static string source. The solution can be generalized to an arbitrary number of static string sources by linear superposition of solutions of the \((D-2)\)dimensional Laplace’s equation. The existence of this multi-soliton solution depends on the the zero-force condition which arises from the cancellation of long-range forces of exchange of the massless fields of the string (the graviton, axion and dilaton). This is a perfect analog to the zero-force condition of Manton for magnetic monopoles, which requires that the attractive scalar exchange force precisely cancel the repulsive vector exchange force when the Bogomolnyi bound is attained. Dabholkar et al. show that a similar Bogomolnyi bound is satisfied by their string soliton solutions, further strengthening the analogy with the monopoles.

In contrast to BPS monopoles, the string solitons also obey a zero \textit{dynamical} force condition\(^2\). While the static force vanishes as a result of the cancellation of long-range forces of exchange, the force between two moving solitons is in general nonvanishing and depends on the velocities of the solitons. The most complete answer would be given by a full time-dependent solution of the equations of motion of the above action for the case of an arbitrary number of sources moving with arbitrary transverse velocities. These equations, however, are much more difficult to solve for moving sources than for a static configuration. Even a two-soliton solution is in general quite intractable for this class of actions. In what follows we briefly summarize some attempts to approximate the dynamical interaction of the solitons strings\(^3\).

We first examine the scattering of these solitons using the above test-string approach. This entails solving the constraint equation for the worldsheet metric obtained by varying the worldsheet Lagrangian \(L\). The resultant solution for the worldsheet metric along with the static solution for the spacetime metric, antisymmetric tensor field and dilaton from the static ansatz for a single source string are then substituted into the Lagrangian, whose equations yield the dynamics of the test string in the source string background. We obtain a kinetic Lagrangian of the form \((\mu/2)\dot{X}^2 + O(\dot{X}^4)\). The absence of the potential term confirms the zero static force condition while the flatness of the leading order kinetic term suggests a zero dynamical force condition, i.e. trivial scattering, in the low velocity limit.

We now address the scattering problem from a string-theoretic point of view. The winding configuration described by \(X(\sigma, \tau)\) describes a soliton string state. It is therefore a natural choice for us to compare the dynamics of these states with the above string solitons in order to determine whether we can identify these solitons with infinitely long fundamental strings. Accordingly, we study the scattering of \(n = 1\) winding states in the limit of large winding radius \(R\). We find that the Veneziano amplitude \(A_4 \to 0\) as \(R \to \infty\) except at scattering angle \(\theta = 0, \pi\). This result also indicates trivial scattering in this limit, providing evidence for the identification of the string solitons with infinitely long macroscopic fundamental strings.

Finally, we turn to soliton-soliton scattering. In the low-velocity limit, multi-soliton solutions trace out geodesics in the static solution manifold, with distance defined by the Manton metric on moduli space manifold. In the absence of a full time-dependent solution to the equations of motion, these geodesics represent a good
approximation to the low-energy dynamics of the solitons. For BPS monopoles, the Manton procedure was implemented by Atiyah and Hitchin\textsuperscript{17}. Computing the Manton metric on moduli space for the scattering of the soliton string solutions in $D = 4$ (we expect that the same result will hold for arbitrary $D \geq 4$) we find that the metric is flat to lowest nontrivial order in the string tension. This result implies trivial scattering of the string solitons and is consistent with the above two calculations, and thus provides even more compelling evidence for the identification of the string soliton with the underlying fundamental string.

For the instanton and monopole fivebranes\textsuperscript{18}, both the test-fivebrane limit and the metric on moduli space also yield a zero dynamical force condition. Since a fundamental theory of fivebranes has not yet been constructed, there is no corresponding Veneziano amplitude computation with which to compare. For the instantons, it is sufficient to demonstrate Ricci flatness of the Manton metric to obtain trivial scattering while for the monopoles a flat metric can be explicitly computed. The zero dynamical force can be seen as a direct consequence of the exactness condition of equating the gauge and spin connections and is again a consequence of worldsheet supersymmetry\textsuperscript{15}.

4. Future Directions

While all the solutions we discussed are classical, one can still conceive of situations in which quantum corrections to the instantons, for example, drop out in nonperturbative computations (such as for vacuum tunnelling). To this end, a vertex operator representation of the instantons would be highly desirable. The most interesting feature of the heterotic monopole solution is the cancellation between gauge and gravitational singularities. If this is an intrinsically “stringy” feature, then it presumably occurs in a larger context within string theory, in which case the full quantum string loop extension of this solution promises to shed light on the nature of string theory as a finite theory of quantum gravity. If this cancellation is of a more accidental nature, then it would pay to concentrate more on the corresponding low-energy field theory, whose quantization is presumably far simpler. In either case, it would be interesting to see whether the singularity cancellation occurs in a quantized solution, or in the context of blackhole type solutions. We have compelling dynamical evidence for the identification of the string solitons with macroscopic fundamental strings, but an exact heterotic solution seems most natural in the context of the conjectured dual fundamental theory of fivebranes. While the construction of such a theory remains elusive, there is so far solid evidence to support the duality conjecture.

References

1. R. R. Khuri, *Phys. Rev.* **D46** (1992) 4526.
2. R. R. Khuri, *Geodesic Scattering of Solitonic Strings*, Texas A&M preprint
3. M. J. Duff, *Class. Quan. Grav.* **5** (1988) 189; M. J. Duff and J. X. Lu, *Nucl. Phys.* **B354** (1991) 129.

4. M. J. Duff, R. R. Khuri and J. X. Lu, *Nucl. Phys.* **B377** (1992) 281.

5. C. Montonen and D. Olive, *Phys. Lett.* **B72** (1977) 117; H. Osborn, *Phys. Lett.* **B83** (1979) 321.

6. G. ’t Hooft, *Nucl. Phys.* **B79** (1974) 276; *Phys. Rev. Lett.* **37** (1976) 8; F. Wilczek, in *Quark confinement and field theory*, ed. D. Stump and D. Weingarten, (John Wiley and Sons, New York, 1977); E. Corrigan and D. B. Fairlie, *Phys. Lett.* **B67** (1977) 69; R. Jackiw, C. Nohl and C. Rebbi, *Phys. Rev.* **D15** (1977) 1642; *Particles and Fields*, ed. David Boal and A. N. Kamal, (Plenum Publishing Co., New York, 1978) p.199.

7. R. R. Khuri, *Phys. Lett.* **B259** (1991) 261; *Proceedings of XXth International Conference on Differential Geometric Methods in Theoretical Physics*, ed. S. Catto and A. Rocha (World Scientific, Jan. 1992) p.1074.

8. I. Antoniadis, C. Bachas, J. Ellis and D. V. Nanopoulos, *Phys. Lett.* **B211** (1988) 393; *Nucl. Phys.* **B328** (1989) 117.

9. A. Strominger, *Nucl. Phys.* **B343** (1990) 167.

10. M. J. Duff and J. X. Lu, *Nucl. Phys.* **B354** (1991) 141.

11. J. M. Charap and M. J. Duff, *Phys. Lett.* **B69** (1977) 445.

12. E. A. Bergshoeff and M. de Roo, *Nucl. Phys.* **B328** (1989) 439; *Phys. Lett.* **B218** (1989) 210.

13. C. G. Callan, J. A. Harvey and A. Strominger, *Nucl. Phys.* **B359** (1991) 611.

14. R. R. Khuri, *Phys. Lett.* **B294** (1992) 325; *Nucl. Phys.* **B387** (1992) 315.

15. R. R. Khuri, *A Comment on the Stability of String Monopoles*, Texas A&M preprint CTP/TAMU-81/92.

16. A. Dabholkar and J. A. Harvey, *Phys. Rev. Lett.* **63** (1989) 719; A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz, *Nucl. Phys.* **B340** (1990) 33.

17. N. S. Manton, *Phys. Lett.* **B110** (1982) 54; M. F. Atiyah and N. J. Hitchin, *Phys. Lett.* **A107** (1985) 21; *The Geometry and Dynamics of Magnetic Monopoles*, (Princeton University Press, 1988).

18. R. R. Khuri, *Nucl. Phys.* **B376** (1992) 350; C. G. Callan and R. R. Khuri, *Phys. Lett.* **B261** (1991) 363; R. R. Khuri, *Phys. Lett.* **B294** (1992) 331; A. G. Felce and T. M. Samols, *Low-Energy Dynamics of String Solitons* ITP Santa Barbara preprint NSF-ITP-92-155.