Classical information and Mermin’s non-technical proof of the theorem of Bell

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Abstract

We show that Mermin’s reasoning against our refutation of his non-technical proof for Bell-type inequalities is of limited significance or contains mathematical inconsistencies that, when taken into account, do not permit his proof to go forward. Our refutation therefore stands.

We discuss two recent notes of Mermin \cite{1,2} that deal with our refutation \cite{3} of Mermin’s non-technical proof \cite{4} of Bell’s inequalities and other publications related to this discussion \cite{5}-\cite{8}. We first show, that the latest note \cite{2} does not add any substance to our original refutation \cite{3}. Secondly, we give a more detailed explanation of our original refutation.

Mermin discusses in his latest note \cite{2} the admissibility of certain classical information that can be exchanged between the stations $S_1$ and $S_2$ in Einstein–Podolsky–Rosen (EPR) type experiments. All classical information is permitted except “...any information whatever about the setting it has randomly been given in that run.” Mermin writes: “Let us turn Hess and Philipp upside down and explore the extent to which Bell’s theorem survives, not only if, following Hess and Philipp, we take advantage of properties of the detectors correlated by the time on local synchronized clock’s, but even if we allow further correlation of the detectors through direct straightforward ongoing classical communication between them.” What Mermin does not appreciate here is, that our introduction of time as an independent variable in Bell’s functions $A, B$ adds not only time as a variable but also adds the set of all functions of time and settings. Indeed, we have stated repeatedly and clearly that our extension of Bell’s parameter space is the addition of time and setting dependent parameter random variables $\lambda_{a,t}, \lambda_{b,t}, \lambda_{c,t}$ for station $S_1$ and $\lambda_{a,t}^{**}, \lambda_{b,t}^{**}, \lambda_{c,t}^{**}$ for station $S_2$. The structure of these functions may be constituted such as to carry information about some of the history of how and when the settings were and are actually chosen and thus may contain information on the setting of the given run. Therefore these functions (parameter random variables) cannot be communicated...
during the course of each run to the other station. Mermin’s addition of classical information is then of very limited significance to our enlargement of the parameter space and we are back to the previous discussion whether our addition is sufficient to refute Mermin’s model in the first place. We add here further explanations to this discussion which should help to better understand our initial refutation of Mermin’s proof [1], [9].

To facilitate the discussion, we attempt to provide a one to one correspondence of Mermin’s notation and ours. We cannot complete our reasoning with Mermin’s all too abbreviated non-technical notation and therefore need to take this step. We also proceed now in two stages. We first consider only the parameters [4] of the original publication of Mermin which are all parameters “that in each run both particles carry to their detectors” and only later discuss the enlarged set of parameters. We therefore consider random variables $A = \pm 1$ in station $S_1$ and $B = \pm 1$ in station $S_2$ that describe the potential outcome of spin measurements and are indexed by instrument settings that are characterized by three-dimensional unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in both stations. Mermin introduces no precise counterpart for the values that these random variables $A, B$ may assume as distinguished from the actual outcomes in form of green and red detector flashes (the data). These green and red flashes correspond then also to the values that $A, B$ may assume: $+1$ which we can identify with green and $-1$ which we can identify with red.

The key assumption of Bell [10] and also of Mermin in his original non-technical proof [4] is that the random variables $A, B$ depend only on the setting in the respective station and on another random variable $\Lambda$ that carries the information with the particles that are emitted from a common source. Because of Einstein locality and the particular way EPR experiments are performed [11], the parameter random variables $\Lambda$ are independent of the settings. Mermin uses instead of $\Lambda$ instruction sets e.g. GGR meaning flash green for settings $\mathbf{a}, \mathbf{b}$ (which Mermin actually labels 1,2) and flash red for setting $\mathbf{c}$ (labelled 3 by Mermin). In our and Bell’s notation this means that for a particular value $\Lambda^*$ that the variable $\Lambda$ may assume and that corresponds to the specific instruction set GGR we have $A(\mathbf{a}, \Lambda^*) = A(\mathbf{b}, \Lambda^*) = +1$ and $A(\mathbf{c}, \Lambda^*) = -1$. The following Table 1 summarizes the eight possible instruction sets and the nine possible different $AB$ products which are used by Mermin as a model for EPR-experiments. According to Mermin’s point (i) [2], the columns

![Table 1: Possible (but exclusive) AB products](image)
of $A_aB_a$, $A_bB_b$ and $A_cB_c$ all have entries $+1$. It is clear, and this is the main point of Mermin’s argument, that each of the 8 rows of the Table 1 contains at least five entries $+1$ and at most 4 entries $-1$. Each pair of settings occurs with probability $1/9$. Denote by $p_1, ..., p_8$ the probabilities that the instruction set $\Lambda$ of rows 1, ..., 8 is carried by both particles in a given run. Then, no matter of whether or not we consider actual outcomes or potential outcomes, the average over all possibilities obeys

$$\frac{1}{9} [9p_1 + 9p_8 + (5 - 4)(p_2 + p_3 + p_4 + p_5 + p_6 + p_7)] \geq \frac{1}{9}$$

as $\sum_{i=1}^{8} p_i = 1$, thus contradicting point (ii) of references [1], [2]. This represents Mermin’s non-technical argument in a more formal way. As can easily be seen from Table 1, it is crucial that $A_a$ in Table 1 is the same in each of the three entries where it appears and that $A_a = B_a$ (and similarly for $A_b$ etc.).

We now turn to the expanded instruction set including time and setting dependent instrument parameters. We emphasize that the time dependence is crucial and all our reasoning rests on it. If it could be shown that EPR experiments contain no time dependencies and all elements of a corresponding model are also independent of time, then we have no argument. Let us therefore assume that the experiments contain time dependencies and that the theory needs to include time and setting dependent parameter random variables. It becomes then a question of which mathematical objects correspond to Mermin’s instruction sets. Following Mermin we identify now the instruction sets with instructions or operations that turn $A$ into $\pm 1$ and $B$ into $\pm 1$, i.e. in Mermin’s newest definition the instruction sets are the random variables $A$ and $B$ [12]. We can now again consider Table 1 with the same notation except that the instruction sets refer to the expanded instruction sets. However, we have lost now the clear distinction that Einstein locality necessitates for source and instrument parameters, respectively. More importantly, it can no longer be guaranteed that $A_a$ in Table 1 is the same in each of the three entries (and similar for $A_b$ etc.). Furthermore, we have $A_a = B_a$ only for the same time. This was explained by us in Eqs.(1) and (2) of reference [9]. However, this point contains some subtle reasoning and we give therefore the following elaboration. For a theory to provide valid conclusions, special care must be taken when combining, adding and/or counting different elements (actual or imagined) that are mutually exclusive. In the case that we consider there are two further subtleties. First, different elements of the theory are mutually exclusive only at the same time and second, the measurement time itself appears as a random variable. This puts restrictions on the possible measurement times which now must be different for different settings. If they are chosen to be equal, Table 1 will contain mutually exclusive alternatives that cannot simultaneously be used in mathematical operations such as counting, adding or averaging. We will return to this point below.

The instruction sets become now also time and setting dependent and the joint frequency of occurrence of the setting dependent instrument parameters may now be different in different columns of Table 1. We see that Mermin’s proof, that is based on the equality of all $A_a$, $A_b$ and $A_c$ (and similarly for the $B$’s) in Table 1, comes to a halt. His argument [1] that the expanded instruction sets for a given single experiment in one time interval must be one of his eight instruction sets is true but meaningless. It has no consequences for the statistics.
and the associated probability measure and therefore does not guarantee Mermin’s way of counting +1 or −1.

One way one still could attempt to proceed with the proof is to assume that EPR experiments are equivalent to other experiments that are made all at the same time. This does not only involve counterfactual reasoning but contradicts the assumed fact of possible time dependencies. The only other way to proceed with Mermin’s proof is to add to the possible outcomes of the Table 1 the outcomes for all the nine settings as they would have been obtained when taken at the same time and to include them into the counting of positive and negative $AB$ products. This procedure, however, again does not only contain counterfactual reasoning, but contradicts the facts because then the theory counts also impossible outcomes by adding mutually exclusive alternatives. This, in turn, has the consequence that the set of elements so counted is nine times larger than the set of experimental results. Any reasonable procedure to establish a theoretical model for an experiment will require a one to one correspondence of what is added/counted/averaged in theory to what is added/counted/averaged in the experiment. We conclude that Mermin’s proof, as well as those of other Bell type inequalities, apply only to a stationary situation that may have nothing to do with past and present EPR experiments [11].

We finish this discussion with an example of what can happen if imagined alternatives are added (or counted) that involve, simultaneously (time!), mutually exclusive alternatives. Mutually exclusive events are discussed in every probability text well before the concept of countable additivity of probability measures is introduced. When probabilities of mutually exclusive events are added, it is understood that the simultaneous occurrence of mutually exclusive events is impossible. Our example below has nothing to do with quantum mechanics. It just illustrates the rules of standard probability theory that must be obeyed. Consider a coin with a little magnet inside and a hidden bigger magnet under ground. The experimenter can set the underground magnet N or S and correspondingly the coin will show with higher probability head or tail, respectively, in any given experiment. If one performs a large number of coin tosses, the result will be biased according to the choices that are made for the underground magnet (N, S). Now in analogy to the above addition of nine terms for each single run of the experiment, construct a theory in which one simply adds or counts, simultaneously, the mutually exclusive alternatives of head or tail. Count 1 if a head shows and count 0 if a tail shows in each of the potential outcomes. Then for each toss the count of potential heads plus the count of potential tails equals 1. Because we have twice as many potential outcomes than actual tosses, the likelihood of heads equals exactly 1/2. Hence such a theory necessarily concludes that this is a fair game. This theory, however, has combined simultaneously mutually exclusive alternatives. It necessarily counts twice as many elements than any given set of experimental results contains and is therefore not admissible. This is, of course, immediately obvious to anyone.

As mentioned above, the reasoning becomes more subtle when time plays a role in the random variables and the elements become mutually exclusive at the same time, hence impossible. Then care must be exercised in the possible choices of the random variables related to measurement time or the same mistake as outlined in the above example could be made. One way to proceed safely is to label the measurement times by the actual setting that
is chosen e.g. \( t_{ab}, t_{ac} \) etc. and regard measurement times with different label as different random variables. Such labelling clearly shows the special role of measurement time as random variable.

We present now our argument in a still more technical way to bring the problems of Mermin’s reasoning into focus. Consider the random variables \( A_i = A_i(\Lambda) \) and \( B_j = B_j(\Lambda) \) with \( i, j = a, b, c \) and which, just as above, assume the values \( \pm 1 \). Let \( \mu \) be the probability measure that assigns weight \( p_l \) to the \( \Lambda \) in the \( l \)’th row. Then according to Mermin’s (i) and (ii) we have

(i) 
\[ A_j(\Lambda) = B_j(\Lambda) , \ j = a, b, c \]

and

(ii) the average over all products \( A_i B_j \) is about 0.

The average for actual as well as potential outcomes equals about:

\[
\frac{1}{9} \int (A_a(\Lambda) + A_b(\Lambda) + A_c(\Lambda))(B_a(\Lambda) + B_b(\Lambda) + B_c(\Lambda)) d\mu
\] (2)

If we now consider the original instruction sets that are independent of the setting then we have by Mermin’s condition (i)

\[ A_a(\Lambda) + A_b(\Lambda) + A_c(\Lambda) = B_a(\Lambda) + B_b(\Lambda) + B_c(\Lambda) \] (3)

for any given value of \( \Lambda \) (which could be e.g. \( \Lambda^* \)). This gives for Eq.(2)

\[
\frac{1}{9} \int (A_a(\Lambda) + A_b(\Lambda) + A_c(\Lambda))^2 d\mu \geq \frac{1}{9}
\] (4)

contradicting Mermin’s condition (ii).

However, if we use now the enlarged instruction set which includes both time and setting dependent parameters, then the \( AB \) products that involve different pairs of settings involve different times and the parameter random variables \( \Lambda \) in Eq.(3) may all be different. In fact, just as above, it can no longer be guaranteed that \( A_a \) is the same in each of the three products \( A_a B_a, A_a B_b \) and \( A_a B_c \) and that \( A_a = B_a \), independent of time. Again, Mermin’s proof comes to halt.

One final comment. It is obvious that the above argument still works for an arbitrary probability measure \( \mu \), that is for any probability distribution governing \( \Lambda \) of the original instruction set [4]. However, if the enlarged instruction set is considered, Mermin’s proof cannot be completed unless it can be guaranteed that \( A_a \) is the same in each of the three products \( A_a B_a, A_a B_b \) and \( A_a B_c \). More specifically one needs to take into account that there are nine joint distributions in operation namely \( \mu_{aa}, \mu_{ab}, \mu_{ac}, \ldots, \mu_{cc} \). Thus, unless it can be guaranteed that the first marginal distribution of the three distributions \( \mu_{aa}, \mu_{ab} \) and \( \mu_{ac} \) is the same, the proof cannot go forward.
We conclude that our refutation of Mermin's non-technical proof for the theorem of Bell stands.

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