Square Property, Equitable Partitions, and Product-like Graphs

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ABSTRACT

Equivalence relations on the edge set of a graph $G$ that satisfy restrictive conditions on chordless squares play a crucial role in the theory of Cartesian graph products and graph bundles. We show here that such relations in a natural way induce equitable partitions on the vertex set of $G$, which in turn give rise to quotient graphs that can have a rich product structure even if $G$ itself is prime.

Keywords: square property, unique square property, USP-relation, quotient graph, equitable partition, Cartesian graph product

1. INTRODUCTION

Sabidussi [19] and later Vizing [21] showed that every finite connected graph has a unique prime factorization w.r.t. the Cartesian product. This Cartesian product structure is naturally understood in terms of an equivalence relation $\sigma$ on the edge set $E(G)$ that identifies the fibers as the connected components of the subgraphs of $G$ that are induced by a single equivalence class of $\sigma$ [19]. The first polynomial time algorithm to compute the factorization of an input graph [4] explicitly constructs $\sigma$ starting from another, finer, relation $\delta$. The product relation $\sigma$ was later shown to be simply the convex hull $C(\delta)$ of the relation $\delta$ [16].

Graph bundles [17], the combinatorial analog of the topological notion of a fiber bundle [14], are a common generalization of both Cartesian products [10] and covering graphs [1]. A slight modification of the relation $\delta$ turns out to play a fundamental role for the characterization of graph bundles [23] and forms the basis of efficient algorithms to recognize Cartesian graph bundles [15, 22, 23]. Here we introduce a further generalization, termed USP-relations, that still retains the salient properties of $\delta$.

The connected components of a given equivalence class of the product relation $\sigma$, i.e., the fibers of $G$ w.r.t. to a given factor $F$, form a natural partition $\mathcal{P}_F$ of the vertex set of $G$. It is well known (see e.g. [10]) that $G$ then has a representation as $G \cong (G/\mathcal{P}_F) \boxtimes F$. It is of interest, therefore, to consider quotient graphs of Cartesian products in a more systematic way.

Equitable partitions of graphs [5, 6] were originally introduced as a means of simplifying the computation of graph spectra [20] and walks on graphs [8]. A series of recent results on so-called perfect state transfer revealed a close connection between equitable partitions of the vertex set of $G$, the corresponding quotient graphs, and the Cartesian product structure of $G$ [2, 7].

Here, we show that equitable partitions on the vertex set $V(G)$ are induced in a natural way in a more general setting, namely by equivalence relations that are coarsenings of relations with the unique square property on the edge set $E(G)$. The quotient graphs w.r.t. these equitable partitions exhibit a natural, rich product structure even when $G$ itself is prime. It can therefore be regarded as an “approximate graph product”, albeit
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in a somewhat different sense than the deviations from product structures explored e.g. in [12, 13, 11].

2. BACKGROUND AND PRELIMINARIES

2.1. Basic Definitions and Notation

In the following we assume that $G$ is a finite connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. $H$ is an induced subgraph of $G$ if $x, y \in V(H)$ and $(x, y) \in E(G)$ implies $(x, y) \in E(H)$. An induced cycle on four vertices is called chordless square.

Relations. We will consider equivalence relations $R$ on $E$, i.e., $R \subseteq E \times E$ such that (i) $(e, e) \in R$, (ii) $(e, f) \in R$ implies $(f, e) \in R$ and (iii) $(e, f) \in R$ and $(f, g) \in R$ implies $(e, g) \in R$. The equivalence classes of $R$ will be denoted by Greek letters, $\phi \subseteq E$. We will furthermore write $\phi \mathrel{\unlhd}_R \psi$ for mean that $\phi$ is an equivalence class of $R$.

A relation $Q$ is finer than a relation $R$ while the relation $R$ is coarser than $Q$ if $(e, f) \in Q$ implies $(e, f) \in R$, i.e, $Q \subseteq R$. In other words, for each class $\vartheta$ of $R$ there is a collection $\{\chi|\chi \subseteq \vartheta\}$ of $Q$-classes, whose union equals $\vartheta$. Equivalently, for all $\phi \subseteq Q$ and $\psi \subseteq R$ we have either $\phi \subseteq \psi$ or $\phi \cap \psi = \emptyset$.

To make this paper easier to read we denote in the following refinements of a given relation $R$ by $Q$ and coarse grainings of $R$ by $S$, so that $Q \subseteq R \subseteq S$.

For a given equivalence class $\phi \subseteq R$ and a vertex $u \in V(G)$ we denote the set of neighbors of $u$ that are incident to $u$ via an edge in $\phi$ by $N_\phi(u)$, i.e., $N_\phi(u) := \{v \in V(G) | [u, v] \in \phi\}$.

Equitable Partitions. A partition $\mathcal{P}$ of the vertex set $V(G)$ of a graph $G$ is equitable if, for all (not necessarily distinct) classes $A, B \in \mathcal{P}$ every vertex $x \in A$ has the same number

$$m_{AB} := |N_G(x) \cap B|$$

of neighbors in $B$. The matrix $M = \{m_{AB}\}$ is called partition degree matrix.

Quotient Graphs. Let $G$ be a graph and $\mathcal{P}$ be a partition of $V(G)$. The (undirected) quotient graph $G/\mathcal{P}$ has as its vertex set $\mathcal{P}$, i.e., the classes of the partition. There is an edge $[A, B]$ for $A, B \in \mathcal{P}$ if and only if there are vertices $a \in A$ and $b \in B$ such that $[a, b] \in E(G)$. Note that there is a loop $[A, A]$ unless the class $A$ of $\mathcal{P}$ is an independent set.

Weighted Quotient Graphs. Let $G$ be a graph and let $\mathcal{P}$ be an equitable partition of $V(G)$ with partition degree matrix $M$. The directed weighted quotient graph $\overrightarrow{G/\mathcal{P}}$ has
vertex set $V(G/P) = P$ and directed edges $(A, B)$ from $A$ to $B$ with weight $m_{AB}$ iff $m_{AB} \geq 1$. Note that $G/P$ has loops whenever $m_{AA} \geq 1$.

By construction, $m_{AB} \geq 1$ implies $m_{BA} \geq 1$. Hence $G/P$ has a well-defined underlying undirected and unweighted graph, which obviously coincides with $G/P$. The underlying simple graph, obtained by also omitting the loops, will be denoted by $\mathcal{N}(G/P) = \mathcal{N}(G/P)$.

**Cartesian Graph Product.** The Cartesian product $G \square H$ has vertex set $V(G \square H) = V(G) \times V(H)$; two vertices $(g_1, h_1)$, $(g_2, h_2)$ are adjacent in $G \square H$ if $(g_1, g_2) \in E(G)$ and $h_1 = h_2$, or $(h_1, h_2) \in E(G_2)$ and $g_1 = g_2$.

Cartesian products generalize in a natural way to directed edge-weighted graphs (with loops allowed). Their Cartesian product $G \square H$ has the edge weights

$$m((g_1, h_1), (g_2, h_2)) = \begin{cases} m_G(g_1, g_2), & \text{iff } h_1 = h_2 \text{ and } g_1 \neq g_2 \\ m_M(h_1, h_2), & \text{iff } g_1 = g_2 \text{ and } h_1 \neq h_2 \\ m_G(g_1, g_2) + m_H(h_1, h_2), & \text{iff } g_1 = g_2 \text{ and } h_1 = h_2 \\ 0, & \text{otherwise} \end{cases}$$

where $m_G(g_1, g_2)$ and $m_H(h_1, h_2)$ denotes the edge weight of the arcs $(g_1, g_2)$ in $G$ and $(h_1, h_2)$ in $H$, resp. The absence of such an arc is equivalent to $m_{X}(x_1, x_2) = 0$ for $X \in \{G, H\}$. The Cartesian product of the underlying undirected and unweighted graphs is obtained by ignoring the directions and weights in the product graph.

The Cartesian product is associative and commutative. Every finite connected graph $G$ has a decomposition $G = \square_{i=1}^n G_i$ into prime factors that is unique up to isomorphism and the order of the factors [19].

The mapping $p_i : V(\square_{i=1}^n G_i) \to V(G_i)$ defined by $p_i(v) = v_i$ for $v = (v_1, v_2, \ldots, v_n)$ is called projection on the $i$-th factor of $G$. The induced subgraph $G_{i}^w$ of $G$ with vertex set $V(G_{i}^w) = \{v \in V(G) \mid p_j(v) = w_j, \text{ for all } j \neq i\}$ is called $G_i$-layer through $w$. It is isomorphic to $G_i$.

An equivalence relation $R$ on the edge set $E(G)$ of a Cartesian product $G = \square_{i=1}^n G_i$ of (not necessarily prime) graphs $G_i$ is a product relation if $e R f$ holds if and only if there exists a $j \in \{1, \ldots, n\}$ such that $|p_j(e)| = |p_j(f)| = 2$.

**Cartesian Graph Bundles.** Given two graphs $G$ and $H$, a map $p : G \to H$ is called a graph map if $p$ maps adjacent vertices of $G$ to adjacent or identical vertices in $B$ and edges of $G$ to edges or vertices of $B$. Graph maps are also known as weak homomorphisms [10]. For instance, the projections $p_i$ of product graphs to their factors are graph maps.

A graph $G$ is a Cartesian graph bundle if there are two graphs $F$, the fiber, and $B$ the base graph, and a graph map $p : G \to B$ such that: For each vertex $v \in V(B)$, $p^{-1}(v) \cong F$ and for each edge $e \in E(B)$ we have $p^{-1}(e) \cong K_2 \square F$. The triple $(G, p, B)$ is called a presentation of $G$ as a Cartesian graph bundle. If $G = \square_{i=1}^n G_i$ is a product, then $(G, p_j, G_j)$ is a bundle presentation of $G$ with fiber $\square_{i=1, i \neq j}^n G_i$ for all $1 \leq j \leq n$. 
2.2. From Edge Partitions to Vertex Partitions

We start from an equivalence relation \( R \) on \( E(G) \). Let \( \varphi \subseteq R \). An edge \( e \) is called \( \varphi \)-edge if \( e \in \varphi \). The subgraph \( G_\varphi \) has vertex set \( V(G) \) and edge set \( \varphi \). The connected components of \( G_\varphi \) containing vertex \( x \in V(G) \) are denoted by \( G_x^\varphi \).

By construction, the set
\[
\mathcal{P}_\varphi^R := \{ V(G_x^\varphi) \mid x \in V(G) \}
\]
is a partition of \( V(G) \) for every \( \varphi \subseteq R \). The quotient graph \( G/\mathcal{P}_\varphi^R \) has as its vertex sets the connected components \( G_x^\varphi \) and edges \( (G_x^\varphi, G_y^\varphi) \) if and only if there are \( x' \in V(G_x^\varphi) \) and \( y' \in V(G_y^\varphi) \) with \( (x', y') \in E(G) \).

The projection \( p_\varphi : G \rightarrow G/\mathcal{P}_\varphi^R \) defined by \( x \mapsto G_x^\varphi \) is a graph map. If \( (x, y) \in \varphi \) then \( y \in V(G_x^\varphi) \) and hence \( G_x^\varphi = G_y^\varphi \). Thus, we have a loop in the quotient graph \( G/\mathcal{P}_\varphi^R \) for every \( V(G_x^\varphi) \neq \{x\} \). Edges that do not form a loop in \( G/\mathcal{P}_\varphi^R \) thus arise only from \( (x, y) \in E \setminus \varphi \).

In the following we will be interested in particular in the complements of \( R \)-classes, i.e., in \( \overline{\varphi} := E \setminus \varphi \). The corresponding subgraphs are denoted by \( G_{\overline{\varphi}} \), with connected components \( G_x^{\overline{\varphi}} \) for a given \( x \in V(G) \). For later reference we note following simple

**Observation 1.** It holds \( y \in V(G_x^{\overline{\varphi}}) \) if and only if there is a path \( P := (x = x_0, x_1, \ldots, x_k = y) \) from \( x \) to \( y \) such that \([x_i, x_{i+1}] \notin \varphi\) for all \( 0 \leq i \leq k - 1 \).

Just like \( \mathcal{P}_\varphi^R \), the set
\[
\mathcal{P}_{\overline{\varphi}}^R := \{ V(G_x^{\overline{\varphi}}) \mid x \in V(G) \}
\]
is a partition of \( V(G) \) for every \( \varphi \subseteq R \). To see this, we note that \( x \in V(G_x^{\overline{\varphi}}) \) holds for all \( x \in V(G) \). Thus, \( P \neq \emptyset \) for all \( P \in \mathcal{P}_{\overline{\varphi}}^R \) and \( \bigcup_{P \in \mathcal{P}_{\overline{\varphi}}^R} P = V(G) \). Furthermore, \( V(G_x^{\overline{\varphi}}) \cap V(G_y^{\overline{\varphi}}) \neq \emptyset \) if and only if \( x \) and \( y \) are in same connected component w.r.t. \( \overline{\varphi} \), i.e., if and only if \( V(G_x^{\overline{\varphi}}) = V(G_y^{\overline{\varphi}}) \). Note, Graham and Winkler showed in [9] that the particular defined equivalence relation \( R = \theta^* \) on \( E(G) \), the so-called Djoković-Winkler relation, induces a canonical isometric embedding of a graph \( G \) into a Cartesian product \( G \sqcap_{\varphi \subseteq R} G_\varphi/\mathcal{P}_\varphi^R \). Moreover, Feder showed that if we choose \( R = (\theta \cup \tau)^* \) then \( G \cong G \sqcap_{\varphi \subseteq R} G_\varphi/\mathcal{P}_\varphi^R \) and thus, \( R \) is the product relation \( \sigma \), see [3].

We furthermore will need the intersections
\[
V_R(x) := \bigcap_{\varphi \subseteq R} V(G_x^{\overline{\varphi}}).
\]

These sets form the classes of the common refinement of the partitions \( \mathcal{P}_\varphi^R, \varphi \subseteq R \), i.e.,
\[
\mathcal{P}^R := \left\{ \bigcap_{\varphi \subseteq R} V(G_x^{\overline{\varphi}}) \mid x \in V(G) \right\} = \{ V_R(x) \mid x \in V(G) \}
\]
is again a partition of $V(G)$.

**Lemma 1.** Let $Q$ and $R$ be two equivalence relations on $E(G)$ so that $Q$ is finer than $R$. Then $V_R(x) \subseteq V_Q(x)$.

**Proof.** Consider two equivalence classes $\varphi, \psi \in Q$. From $\varphi \cup \psi = \varphi \cap \psi$ we observe that $G_{\varphi \cup \psi}$ is a subgraph of both $G_\varphi$ and $G_\psi$. This remains true for the connected components containing a given vertex $x \in V$, and hence

$$V(G_{\varphi \cup \psi} \subseteq V(G_\varphi \cap V(G_\psi)).$$

Using this observation we compute

$$V_R(x) = \bigcap_{\varphi \in R} V(G_\varphi) = \bigcap_{\varphi \in R} V(G_{\varphi \cup \psi}) \subseteq \bigcap_{\varphi \in R \cap \psi} V(G_{\varphi}) = \bigcap_{\chi \in Q} V(G_\chi) = V_Q(x).$$

Thus, a coarser equivalence relation $R$ on $E(G)$ leads to smaller sets $V_R(x)$, and hence to a finer partition $\mathcal{P}^R$ of the vertex set.

### 2.3. The Square Property and the Unique Square Property

**Definition.** Two edges $e, f \in E(G)$ are in the relation $\delta$, $e \delta f$, if one of the following conditions is satisfied:

(i) $e$ and $f$ are opposite edges of a chordless square.
(ii) $e$ and $f$ are adjacent and there is no chordless square containing $e$ and $f$.
(iii) $e = f$.

**Definition.** An equivalence relation $R$ on $E(G)$ has the unique square property if it satisfies

(S1) Any two adjacent edges $e$ and $f$ from distinct equivalence classes span a unique chordless square with opposite edges in the same equivalence class $R$.

$R$ has the square property if it satisfies in addition

(S2) The opposite edges of any chordless square belong to the same equivalence class.

The unique square property was introduced by Zmazek and Žerovnik [23] as a feature of the so-called fundamental factorizations of graph bundles over simple bases. The results derived below therefore hold in particular also for this type of graph bundles.

Relations with the unique square property do not need to satisfy the square property as shown by the counterexample in Figure 1. On the other hand, from the definition it is clear, that every equivalence relation $R$ on $E(G)$ that has the square property also has the unique square property. It has been noted, e.g. in [23], that $\delta$ has the unique square property.

The following observation has been used implicitly e.g. in [15, 23].
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FIGURE 1. The square property and the unique square property are not equivalent. The line styles distinguish the two classes of the equivalence relation $R$ on the edges. It has the unique square property (S1). The edges $[1, 2]$ and $[1, 4]$ span two chordless squares $S_{Q1} = [1, 2, 5, 4]$ and $S_{Q2} = [1, 2, 3, 4]$ of which $S_{Q1}$ has opposite edges in the same equivalence class. The square $S_{Q2}$ thus violates (S2).

Proposition 1. An equivalence relation $R$ on $E(G)$ has the square property if and only if $\delta \subseteq R$.

Proof. Let $R$ be an equivalence relation on $E(G)$ and $\delta \subseteq R$. Then Condition (i) in the definition of $\delta$ directly implies Condition (S2). Let $e, f$ be two adjacent edges and suppose $(e, f) \notin R$. Then there must exist a square containing both edges, otherwise, by condition (ii), $(e, f) \in \delta \subseteq R$, a contradiction. Let this square consist of edges $e, f, e', f'$ such that $e'$ is opposite edge to $e$ and $f'$ is opposite edge to $f$. Then condition (i) implies $(e, e'), (f, f') \in \delta \subseteq R$. Assume $e, f$ are contained in another square consisting of edges $e, f, e'', f''$ such that $e''$ is opposite edge to $e$ and $f''$ is opposite edge to $f$. Then there is also a square consisting of edges $e', f', e'', f''$ such that $e''$ is opposite edge to $f'$ and $f''$ is opposite edge to $e'$. Again condition (i) implies $(e, e''), (f, f'') \in \delta \subseteq R$ as well as $(e'', f'), (f'', e') \in \delta \subseteq R$. Finally, by transitivity it follows $(e, f) \in R$, a contradiction. Thus, $R$ has the square property.

Let $R$ be an equivalence relation on $E(G)$ with the square property. We have to show that $e R f$ implies $e R f$ for all edges $e, f$. First assume $e R f$ such that $e$ and $f$ are not adjacent. Hence, either Condition (i) or (iii) is fulfilled which immediately implies $e R f$. Now, let $e$ and $f$ be adjacent and assume for contraposition that $e \not R f$. Thus, by condition (S1) there is a chordless square spanned by $e$ and $f$ and therefore $e$ and $f$ do not satisfy condition (ii). Hence, $e \not R f$ which completes the proof.

The transitive closure $\delta^*$ of $\delta$ is therefore the finest equivalence relation on $E(G)$ that has the square property. Furthermore, an equivalence relation $R$ has the square property if and only if its classes are unions of equivalence classes of $\delta^*$. Therefore, if $R$ has the square property and $R \subseteq S$, then the coarser equivalence relation $S$ also has the square property.

In contrast, there is no finest equivalence relation that has the unique square property. Moreover, if an equivalence relation $R$ satisfies the unique square property, it is still possible that there exists a coarser equivalence relation $S \supset R$ that does not have the unique square property, as shown by the example in Figure 2.

This observation motivates us to consider a slightly more general set of equivalence relations.
The equivalence relation $Q$ on the edge set $E(G)$ of the “diagonalized cube” $G$ has the four equivalence classes $\varphi_1, \varphi_2, \varphi_3$ and $\varphi_4$ depicted by solid, zigzag, dotted and dashed edges, respectively. One easily checks that $Q$ has the unique square property. The relation $R$ with classes $\psi_1 = \varphi_1 \cup \varphi_2$ and $\psi_2 = \varphi_3 \cup \varphi_4$, however, does not have the unique square property, because the edges $[1, 4]$ and $[1, 2]$ span two squares $(1, 5, 6, 4)$ and $(1, 5, 6, 4)$ with opposite edges belonging to the same class. Clearly, $R$ is a USP-relation.

**Definition.** An equivalence relation $R$ on the edge set of a connected graph $G$ is called a **USP-relation** if there exists a finer equivalence relation $Q \subseteq R$ that satisfies the unique square property.

**Observation 2.** If $R$ is a USP-relation on the edge set of a graph $G$, then any two adjacent edges of distinct $R$-classes span a (not necessarily unique) square with opposite edges in the same equivalence class.

In the remainder of this section we collect several basic properties of USP-relations. These results have originally been obtained for the relation $\delta$ in the context of graph products and later were generalized to the unique square property for applications to Cartesian graph bundles. Here we show that the statements remain true for USP-relations.

**Lemma 2.** Let $R$ be a USP-relation on the edge set of a connected graph $G$. Then each vertex of $G$ is incident to at least one edge of each $R$-class.

**Proof.** This was shown for the relation $\delta$ in [4] and later for equivalence relations $Q$ with the unique square property in [23]. Obviously, the result remains true when equivalence classes of $Q$ are united, i.e., for any USP-relation $R$ coarser than $Q$. The assertion follows immediately from the definition of USP-relations.

Hence, if $G$ is connected and $R$ is a USP-relation, then $N_\varphi(u) \neq \emptyset$ and $N_{\varphi}(u) \neq \emptyset$ for all $u \in V(G)$ and all $\varphi \in R$. Thus, neither $G_\varphi$ nor $G_{\varphi}$ has isolated vertices.

**Lemma 3.** Let $R$ be a USP-relation on $E(G)$ and let $[u, v] \in \varphi \subseteq R$. Then $R$ induces a bijection between the $\psi$-edges incident to $u$ and $\psi$-edges incident to $v$ for every $\psi \in R$. Furthermore, the vertices $u$ and $v$ have the same $\psi$-degree for every $\psi \in R$ with $\psi \neq \varphi$.

**Proof.** Again, the result was first proved for $\delta$ in [4] and then for equivalence relations with the unique square property in [23]. Now suppose $R$ is an USP-relation, i.e., there is an equivalence relation $Q \subseteq R$ such that $Q$ has the unique square property. Then each
equivalence class $\chi \subseteq R$ is the union of some $Q$-equivalence classes, $\chi = \bigcup_{\psi \subseteq \chi} \psi$. The result of [23] guarantees the existence of a bijection of the $\psi$-edges incident to $u$ and the $\psi'$-edges incident to $v$ for $[u, v] \in \varphi \neq \chi$ for all $\psi \subseteq \chi$. Since $\psi \cap \psi' = \emptyset$ for any two distinct classes $\psi, \psi' \subseteq Q$ we conclude that the disjoint union of these bijections over the $\psi \subseteq \chi$ is a bijection between the $\chi$-edges incident to $u$ and the $\chi$-edges incident to $v$ for any $[u, v] \in \varphi \neq \chi$. Clearly, this bijection is again induced by $R$. It follows immediately that the $\chi$-degrees of $u$ and $v$ are also the same for all $\chi \neq \varphi$.

The following result was proved in [16] assuming the square property. The proof uses only the existence but not the uniqueness of these squares. Thus, by Observation 2, the result remains true for USP-relations:

Lemma 4. Let $R$ be a USP-relation on $E(G)$ that contains only two equivalence classes $\varphi, \varphi'$. Then

$$|V(G^x_{\varphi}) \cap V(G^y_{\varphi'})| \geq 1$$

for all $x, y \in V(G)$.

If $R$ is a convex USP-relation, i.e., if $R$ is a product relation, then $|V(G^x_{\varphi}) \cap V(G^y_{\varphi'})| = 1$ for all $x, y \in V$ and all $\varphi \in R$ [16]. In Theorem 3 below we will show that the converse is also true.

We will need also the following technical results:

Lemma 5. Let $R$ be a USP-relation on the edge set $E(G)$ of a connected graph $G$. Let $Q \subseteq R$ be an equivalence relation on $E(G)$ with unique square property. For $[v, w] \in \chi \subseteq Q$ and $\varphi \subseteq R$ with $\varphi \cap \chi = \emptyset$ let $H$ be the subgraph of $G$ with vertex set $V(G^v_{\varphi}) \cup V(G^w_{\varphi})$ that contains only $\varphi$-edges and $\chi$-edges, respectively, that is,

$$E(H) = \{[x, y] \in \varphi \mid x, y \in V(G^v_{\varphi}) \cup V(G^w_{\varphi})\} \cup \{[x, y] \in \chi \mid x, y \in V(G^v_{\varphi}) \cup V(G^w_{\varphi})\}.$$ 

Then $Q$ restricted to $H$ has the unique square property on $H$.

Proof. It suffices to show that for any two adjacent edges $e = [x, y], f = [x, z] \in E(H)$ with $(e, f) \notin Q$ the vertex $u$ of the unique square $(x, y, u, z)$ with opposite edges in the same equivalence class spanned by $e$ and $f$ in $G$ is already contained in $V(H)$.

W.l.o.g. let $x \in V(G^v_\varphi)$ and $e \in \alpha \subseteq \varphi, \alpha \subseteq Q$. Hence, $[u, z] \in \alpha \subseteq \varphi$ and therefore $u \in V(G^x_\varphi) \subseteq V(G^v_\varphi) \cup V(G^w_\varphi) = V(H)$. 

3. RESULTS

3.1. Equitable Partitions

Lemma 6. Let $G$ be a graph and let $\varphi \neq \psi$ be two equivalence classes of a USP-relation $R$ and let $v, w \in V(G)$. Then all vertices of $G^v_{\varphi}$ have the same number of incident $\psi$-edges
connecting \(G^*_\varphi\) and \(G^w_\varphi\). More formally,

\[
|N_\psi(v) \cap V(G^w_\varphi)| = |N_\psi(x) \cap V(G^w_\varphi)|
\]

holds for all \(x \in V(G^*_\varphi)\).

**Proof.** First, we show that \(N_\psi(v) \cap V(G^w_\varphi) = \emptyset\) if and only if \(N_\psi(x) \cap V(G^w_\varphi) = \emptyset\) holds for all \(x \in V(G^*_\varphi)\).

W.l.o.g., let \([v, w] \in \psi\) and consider an arbitrary vertex \(x \in V(G^*_\varphi)\). Then there is a path \(P := (v = v_0, v_1, \ldots, v_k = x)\) from \(v\) to \(x\) in \(G^*_\varphi\). Recalling Observation 2, we can construct a walk \(Q = (w = w_0, w_1, \ldots, w_k)\) such that \([v_i, w_i] \in \varphi\) for all \(0 \leq i \leq k\) and \([v_i, v_{i+1}] \in \varphi\) for all \(0 \leq i < k - 1\). Then \(w_k \in N_\psi(x)\) and \(w_k \in V(G^*_\varphi)\) and therefore, \(N_\psi(x) \cap V(G^*_\varphi) \neq \emptyset\). Since \(x \in V(G^*_\varphi)\) was arbitrarily chosen, we can conclude that \(N_\psi(x) \cap V(G^*_\varphi) \neq \emptyset\) holds for all \(x \in V(G^*_\varphi)\). Conversely, if \(N_\psi(x) \cap V(G^*_\varphi) \neq \emptyset\) holds for all \(x \in V(G^*_\varphi)\), this is trivially fulfilled also for \(x = v\). Thus, we have \(N_\psi(v) \cap V(G^w_\varphi) = \emptyset\) if and only if \(N_\psi(x) \cap V(G^w_\varphi) = \emptyset\) holds for all \(x \in V(G^*_\varphi)\).

Now suppose that \(N_\psi(v) \cap V(G^w_\varphi) \neq \emptyset\). W.l.o.g., let \([v, w] \in \psi\). Since \(R\) is a USP-relation, there is some relation \(Q \subseteq R\) that has the unique square property, and \(\psi \subseteq R\) is the disjoint union of some equivalence classes \(\chi \subseteq Q\), \(\psi = \bigcup_{\chi \subseteq \psi} \chi\). Thus, we have

\[
|N_\psi(v) \cap V(G^w_\varphi)| = \sum_{\chi \subseteq \psi} |N_\chi(x) \cap V(G^w_\varphi)|
\]

Therefore, it suffices to show that \(|N_\chi(v) \cap V(G^w_\varphi)| = |N_\chi(x) \cap V(G^w_\varphi)|\) holds for all \(x \in V(G^*_\varphi)\) and all \(\chi \subseteq Q\) with \(\chi \subseteq \psi\). In the following we denote with \(N_{\chi|H}\) the intersection of some set \(N \subseteq V(G)\) and the vertex set of a given subgraph \(H \subseteq G\).

Suppose first, \(x \in N_\chi(v)\), i.e., \([v, x] \in \varphi \neq \psi\). By construction, \(\varphi \cap \chi = \emptyset\) holds for all \(\chi \subseteq \psi\). Using the same arguments as before, we can conclude that \(N_{\chi | H} \cap V(G^w_\varphi) = \emptyset\) if and only if \(N_\chi(v) \cap V(G^w_\varphi) = \emptyset\). Therefore, assume \(N_\chi(v) \cap V(G^w_\varphi) \neq \emptyset\). Let \(H\) be the subgraph of \(G\) with vertex set \(V(G^*_\varphi) \cup V(G^w_\varphi)\) defined as in Lemma 5. Then the restriction of \(Q\) to \(H\) satisfies the unique square property on \(H\).

If \(G^w_\varphi = G^w_\varphi\), we can conclude by Lemma 3 that

\[
|N_\chi(v) \cap V(G^w_\varphi)| = |N_\chi(v) \cap V(G^w_\varphi)| = |N_\chi(x) \cap V(G^w_\varphi)|. \tag{4}
\]

Assume now \(G^*_\varphi \neq G^w_\varphi\), and hence \(V(G^*_\varphi) \cap V(G^w_\varphi) = \emptyset\). Thus \(|N_\chi(y) | H| = |N_\chi(y) \cap V(G^*_\varphi)| + |N_\chi(y) \cap V(G^w_\varphi)|\) holds for all \(y \in V(G^*_\varphi)\) and therefore, we can conclude again from Lemma 3 and Equation (4)

\[
|N_\chi(x) \cap V(G^*_\varphi)| = |N_\chi(x) \cap V(G^w_\varphi)| - |N_\chi(v) \cap V(G^w_\varphi)| = |N_\chi(v) \cap V(G^w_\varphi)| - |N_\chi(v) \cap V(G^w_\varphi)| = |N_\chi(v) \cap V(G^w_\varphi)|,
\]

which implies \(|N_\psi(v) \cap V(G^w_\varphi)| = |N_\psi(v) \cap V(G^w_\varphi)|\).

If \(v\) and \(x\) are connected by a path in \(G^*_\varphi\), the assertion follows by induction on the length of the path. \(\blacksquare\)

**Corollary.** Let \(G\) be a connected graph and \(R\) be a USP-relation on \(E(G)\). Then \(P^R_{\psi}\) is an equitable partition of the graph \(G_{\varphi}\) for every equivalence class \(\varphi\) of \(R\).
Proof. This follows immediately from $V(G) = V(G_\varphi)$, the fact that $P_\varphi^R$ defined in Eq.(2) is a partition of $V(G)$, and Lemma 6.

If $G_\varphi$ is an induced subgraph of $G$ we have $G_\varphi / P_\varphi^R \cong N (G / P_\varphi^R)$ which follows from the fact that $[G_\varphi^x, G_\varphi^y]$ is an edge in $N (G / P_\varphi^R)$ if and only if there is an edge in $G$ connecting a vertex in $V(G_\varphi^x)$ with a vertex in $V(G_\varphi^y)$. This edge must be in $\varphi$, since otherwise $G_\varphi^x = G_\varphi^y$, and hence it is in $G_\varphi$.

Remark. The quotient graphs $B_\varphi := G_\varphi / P_\varphi^R$ provide a direct connection to the theory of graph bundles since $B_\varphi$ coincides with the base graph of the bundle presentation $(G, p, B_\varphi)$ of $G$ provided $B_\varphi$ is 2-convex [15, 18]. We recall that a subgraph $H \subseteq G$ is 2-convex w.r.t. $G$ if all shortest $G$-paths of length $\leq 2$ connecting pairs of vertices in $H$ are contained in $H$. An equivalence class $\varphi \subseteq R$ is said to be 2-convex if all connected components $G_\varphi^x$ of $G_\varphi$ are 2-convex w.r.t. $G$. Moreover, it can easily be shown that $G$ has a graph bundle presentation $(G, p, G_\varphi / P_\varphi^R)$ over a simple base if and only if $p : G_\varphi \rightarrow G_\varphi / P_\varphi^R$ is a covering projection, i.e., a locally bijective homomorphism [6, 15, 18].

**Theorem 1.** Let $R$ be a USP-relation on the edge set $E(G)$ of a connected graph $G$. Then $P_\varphi^R$ defined in Eq.(3) is an equitable partition of $G$.

To prove the Theorem, we first show the following:

**Lemma 7.** Let $G$ be a connected graph and $R$ be a USP-relation on $E(G)$. Then for an arbitrary equivalence class $\varphi$ of $R$ holds:

1. $N_\varphi(x) \cap V_R(y) \neq \emptyset$ if and only if $N_\varphi(u) \cap V_R(y) \neq \emptyset$ for all $u \in V_R(x)$.
2. $N_\varphi(x) \cap V_R(y) \neq \emptyset$ implies $N_\varphi(x) \cap V_R(y) = N_\varphi(x) \cap V(G_\varphi^y)$.

**Proof.**

1. Let $N_\varphi(x) \cap V_R(y) \neq \emptyset$ and hence, $N_\varphi(x) \cap V(G_\varphi^y) \neq \emptyset$ for all $\psi \in R$. Thus, there exists a vertex $z \in V(G)$ with $[x, z] \in \varphi$ such that $z \in V(G_\varphi^y)$ for all $\psi \in R$. Note, it holds $z \in V(G_\varphi^y)$ since for all $\varphi \neq \psi$ there is a path that is not in $\psi$ which is the particular edge $[x, z] \in \varphi$. Therefore, $G_\varphi^x = G_\varphi^y$ for all $\psi \neq \varphi$.

Now let $u \in V_R(x)$. Hence $u \in V(G_\varphi^x) = V(G_\varphi^y)$ for all $\psi \neq \varphi$. From Lemma 6 and the fact that $N_\varphi(x) \cap V(G_\varphi^x) \neq \emptyset$, we can conclude that $N_\varphi(u) \cap V(G_\varphi^y) \neq \emptyset$, i.e., there exists a vertex $w \in V(G_\varphi^y)$ such that $[u, w] \in \varphi$. This implies $w \in V(G_\varphi^x) = V(G_\varphi^y)$ for all $\psi \neq \varphi$ and therefore $w \in V_R(y)$, hence $N_\varphi(u) \cap V_R(y) \neq \emptyset$.

Conversely, if $N_\varphi(u) \cap V_R(y) \neq \emptyset$ for all $u \in V_R(x)$, this is trivially fulfilled for $u = x$.

2. Let $z \in N_\varphi(x) \cap V_R(y)$, that is, $z \in N_\varphi(x)$ and $z \in V(G_\varphi^y)$ for all $\psi \in R$, in particular, $z \in V(G_\varphi^y)$. Hence, $z \in N_\varphi(x) \cap V(G_\varphi^y)$ and therefore we have $N_\varphi(x) \cap
Now, let \( z \in N_\varphi(x) \cap V(G_{\varphi(y)}) \), which is equivalent to \([x,z] \in \varphi \) and \( z \in V(G_{\varphi(y)}) \). It follows \( z \in V(G_{\varphi(y)}) \) for all \( \psi \neq \varphi \) and thus \( z \in V(G_{\varphi(y)}) \). Hence, \( z \in N_\varphi(x) \cap V_R(y) \) and therefore \( N_\varphi(x) \cap V(G_{\varphi(y)}) \subseteq N_\varphi(x) \cap V_R(y) \), from which we can conclude equality of the sets.

Proof of Theorem 1. By construction \( \mathcal{P}^R \) is a partition of \( V(G) \). It remains to show that this partition is equitable, that is, we have to show that for arbitrary \( u, x, y \in V(G) \) with \( u \in V_R(x) \) holds

\[
|N_G(u) \cap V_R(y)| = |N_G(x) \cap V_R(y)|.
\]

Notice, that for arbitrary \( x \in V(G) \) holds \( N_G(x) = \bigcup_{\varphi \subseteq R} N_\varphi(x) \) and \( N_\varphi(x) \cap N_\psi(x) = \emptyset \) for \( \varphi \neq \psi \). Hence we have \( |N_G(x) \cap V_R(y)| = \sum_{\varphi \subseteq R} |N_\varphi(x) \cap V_R(y)| \). Therefore, it suffices to show

\[
|N_\varphi(u) \cap V_R(y)| = |N_\varphi(x) \cap V_R(y)| \quad \forall \varphi \subseteq R
\]

to prove Eq. (5). This equality, however, follows immediately from Lemma 7 together with Lemma 6.

3.2. Product Structure of Quotient Graphs

Product structures and equitable partitions are compatible in the following sense:

**Proposition 2.** [2] Let \( G = \square_{i=1}^n G_i \) and let \( \pi_i \) be an equitable partition on \( G_i \). Then there is an equitable partition \( \pi \) of \( G \) such that

\[
\square_{i=1}^n (G_i/\pi_i) = (G/\pi).
\]

Since the Cartesian product of the underlying undirected and unweighted graphs is obtained by simply omitting the weights, we also have

\[
\square_{i=1}^n (G_i/\pi_i) = (G/\pi)
\]

for the same equitable partition \( \pi \) of \( G \).

Our next result shows that equitable partitions constructed above arrange themselves as a special case of Prop. 2.

**Theorem 2.** Let \( R \) be a USP-relation on the edge set \( E(G) \) of a connected graph \( G \). Then

\[
G/\mathcal{P}^R \cong \square_{\varphi \subseteq R} G_\varphi/\mathcal{P}_{\varphi}^R.
\]
Proof. Let \( \varphi_1, \ldots, \varphi_n \) denote the equivalence classes of \( R \). Let \( x, v_1, \ldots, v_n \in V(G) \), where the \( v_i \) need not necessarily be distinct. If \( x \in V(G^u_i) \) for all \( i = 1, \ldots, n \) then \( V_R(x) = \bigcap_{i=1}^n V(G^u_i) \).

Remark, that for \( 1 \leq i \leq n \) the vertex set of \( G_{\varphi_i}/P^R_{\varphi_i} \) is given by \( V(G_{\varphi_i}/P^R_{\varphi_i}) = \{ G^u_{\varphi_i} | v_1 \in V(G) \} \). Hence, we have

\[
V(\bigtimes_{i=1}^n G_{\varphi_i}/P^R_{\varphi_i}) = \left\{ (G^u_{\varphi_1}, \ldots, G^u_{\varphi_n}) | v_1 \in V(G), i = 1, \ldots, n \right\},
\]

where \( (G^u_{\varphi_1}, \ldots, G^u_{\varphi_n}) = (G^u_{\varphi_1}, \ldots, G^u_{\varphi_n}) \) if and only if \( u_i \in V(G^u_i) \) for all \( i = 1, \ldots, n \).

We define a mapping \( V(G/P^R) \to V(\square_{i=1}^n G_{\varphi_i}/P^R_{\varphi_i}) \) as follows:

\[
V_R(x) \mapsto (G^u_{\varphi_1}, \ldots, G^u_{\varphi_n})
\]

iff \( x \in V(G^u_i) \) for all \( i = 1, \ldots, n \).

For all \( x \in V(G) \) there exist \( v_i, i = 1, \ldots, n \) such that \( x \in V(G^u_i) \), e.g. choose \( v_i = x \).

And since from \( x \in V(G^u_i) \) and \( x \in V(G^u_j) \) follows \( G^u_i = G^u_j \), this mapping is well defined.

Due to the fact that \( x \in V(G^u_i) \) and \( y \in V(G^u_j) \) implies \( G^u_i = G^u_j \), we can conclude that this mapping is injective. To prove surjectivity, it suffices to show, that \( \cap_{i=1}^n V(G^u_i) \neq \emptyset \) for arbitrary \( v_i \in V(G) \). We show by induction for all \( k \leq n \) holds \( \cap_{i=1}^k V(G^u_i) \neq \emptyset \). For \( k = 1 \) this is trivially fulfilled. Let \( k \geq 1 \) and suppose \( \cap_{i=1}^k V(G^u_i) \neq \emptyset \). We have to show, that this implies \( \cap_{i=1}^{k+1} V(G^u_i) \neq \emptyset \). From the induction hypothesis, we can conclude there must be a vertex \( x \in V(G) \) such that \( x \in V(G^u_i) \) for all \( i = 1, \ldots, k \) and hence \( \cap_{i=1}^{k+1} V(G^u_i) = \cap_{i=1}^k V(G^u_i) \) for all \( i = 1, \ldots, k \). Therefore, we have to show

\[
V(G^x_{\varphi_{k+1}}) \subseteq \bigcap_{i=1}^k V(G^u_i), \tag{6}
\]

From that and Lemma 4 we can conclude \( \emptyset \neq V(G^x_{\varphi_{k+1}}) \cap V(G^u_{\varphi_{k+1}}) \subseteq \bigcap_{i=1}^{k+1} V(G^u_i) \), from what the assumption follows.

Let \( y \in V(G^x_{\varphi_{k+1}}) \). Then there exists a path \( Q \) from \( x \) to \( y \) such that all edges of \( Q \) are in class \( \varphi_{k+1} \). Clearly, they are not in class \( \varphi_i \) for \( i = 1, \ldots, k \) and therefore \( y \in V(G^x_{\varphi_i}) \) for all \( i = 1, \ldots, k \), from what Equation (6) and finally surjectivity follows.

It remains to verify the isomorphism property, that is \( [V_R(x), V_R(y)] \) is an edge in \( G/P^R \) if and only if \( [(G^x_{\varphi_1}, \ldots, G^x_{\varphi_k})], (G^u_{\varphi_1}, \ldots, G^u_{\varphi_k})] \) is an edge in \( \square_{i=1}^k G^u_{\varphi_i}/P^R_{\varphi_i} \). Let \( [V_R(x), V_R(y)] \in E(G/P^R) \), that is, there exists a vertex \( x' \in V_R(x) \) and a vertex \( y' \in V_R(y) \) such that \( [x', y'] \) is an edge in \( G \) and therefore in \( \varphi_i \) for some \( i, 1 \leq i \leq n \). This implies \( G^x_{\varphi_j} = G^y_{\varphi_j} = G^u_{\varphi_j} \) for all \( j \neq i \) and \( [G^x_{\varphi_i}, G^u_{\varphi_i}] \in E(G_{\varphi_i}/P^R_{\varphi_i}) \).

Thus, \( [(G^x_{\varphi_1}, \ldots, G^x_{\varphi_k})], (G^u_{\varphi_1}, \ldots, G^u_{\varphi_k})] \) is an edge in \( \square_{i=1}^k G^u_{\varphi_i}/P^R_{\varphi_i} \). Conversely, let \( [(G^x_{\varphi_1}, \ldots, G^x_{\varphi_k})], (G^u_{\varphi_1}, \ldots, G^u_{\varphi_k})] \in E(\square_{i=1}^k G^u_{\varphi_i}/P^R_{\varphi_i}) \). There must be an \( i, 1 \leq i \leq n \),
such that \([G^x_{\varphi_i}, G^y_{\varphi_j}] \in E(G_{\varphi_i}/P_{\varphi_i})\) and \(G^x_{\varphi_j} = G^y_{\varphi_j}\) for all \(j \neq i\). \([G^x_{\varphi_i}, G^y_{\varphi_j}] \in E(G_{\varphi_i}/P_{\varphi_j})\) implies that there exists a vertex \(x' \in V(G^x_{\varphi_i})\) and a vertex \(y' \in V(G^y_{\varphi_j})\) such that \([x', y'] \in \varphi_i\) in \(G\). From Lemma 6, we can conclude that there exists a vertex \(z \in V(G^x_{\varphi_i})\) such that \([x, z] \in \varphi_i\). This in turn implies \(z \in V(G^x_{\varphi_i}) = V(G^y_{\varphi_j})\) for all \(j \neq i\) and thus, \(z \in V_i(y)\). Hence, \([V_R(x), V_R(y)]\) is an edge in \(G/P^R\).

**Corollary.** Suppose the conditions of Theorem 2 are satisfied. If furthermore \(G_{\varphi}\) is an induced subgraph of \(G\) for all \(\varphi \subseteq R\) then

\[
G/P^R \cong \bigoplus_{\varphi \subseteq R} N(G/P^R_{\varphi}).
\]

**Proof.** It suffices to show that \(G/P^R\) has no loops if all \(G_{\varphi}\) are induced. We will prove this by contradiction. Therefore, assume that \(G/P^R\) contains a loop \([V_R(x), V_R(x)]\) for some \(x \in V(G)\). Hence, there are vertices \(y, z \in V_R(x)\) with \([y, z] \in E(G)\). Clearly, \([y, z] \in \varphi \) for some \(\varphi \subseteq R\). But since \(y, z \in V(G^z_{\varphi})\) it follows that \(G_{\varphi}\) is not induced, a contradiction.

**Corollary.** If the conditions of Theorem 2 are satisfied, then

\[
\overline{G/P^R} \cong \bigoplus_{\varphi \subseteq R} \overline{G_{\varphi}/P^R_{\varphi}}.
\]
Proof. Since the underlying undirected and unweighted graphs of \(G/P_R\) and \(G_\varphi/P_R\) are exactly \(G/P_R\) and \(G_\varphi/P_R\), respectively, it suffices to show that the weights are transferred as in Eq. (1). This follows immediately from Lemma 7 and Lemma 6 and the fact that \(|N_G(x) \cap V_R(y)| = \sum_{\varphi \subseteq R} |N_\varphi(x) \cap V_R(y)|\).

With the help of the results obtained in this section we can strengthen a useful result of [16]:

**Theorem 3.** Let \(Q\) be a USP-relation on the edge set \(E(G)\) of a connected graph \(G\) and let \(\varphi \subseteq Q\). Then \(|V(G^e_\varphi) \cap V(G^e_\varphi)| = 1\) holds for all \(x, y \in V(G)\) if and only if \(R = \{\varphi, \varphi\}\) is a product relation.

Proof. It has been shown in [16] that for product relations, that is, convex USP-relations, holds \(|V(G^e_\varphi) \cap V(G^e_\varphi)| = 1\) for all \(x, y \in V(G)\). It remains to show, therefore, that converse is also true.

Notice that \(\varphi = \varphi\). Hence the equitable partition induced by \(R\) is \(P_R = \{V(G^e_\varphi) \cap V(G^e_\varphi) \mid x \in V(G)\}\). By assumption, \(P_R\) consists exclusively of singletons. Thus \(G = G/P_R\). Recall that \(P_R = \{V(G^e_\varphi) \mid x \in V(G)\}\) and \(P_R = \{V(G^e_\varphi) \mid x \in V(G)\}\) are the equitable partitions of the graphs \(G_\varphi\) and \(G_\varphi\) respectively. For arbitrary \(y \in V(G)\) let \(P_R^y(y)\) denote the restriction of \(P_R\) to the connected component \(G^y_\varphi\) of \(G_\varphi\), that is \(P_R^y(y) = \{V(G^e_\varphi) \cap V(G^e_\varphi) \mid x \in V(G)\}\). From Lemma 4, Lemma 6 and the definition of the quotient graphs, we can conclude that the mapping \(G_\varphi \cap G^y_\varphi \rightarrow G^y_\varphi\) defines an isomorphism \(G_\varphi/P_R^y \cong G^y_\varphi/P_R^y\) for all \(y \in V(G)\) and since \(|V(G^e_\varphi) \cap V(G^e_\varphi)| = 1\) holds for all \(x, y \in V(G)\), we even have \(G_\varphi/P_R^y \cong G^y_\varphi\). Analogously, it follows \(G_\varphi/P_R^y \cong G^y_\varphi\) for
all \( y \in V(G) \). Thus, \( G \cong G^x_\varphi \square G^y_\psi \) for all \( x, y \in V(G) \), demonstrating that \( R = \{ \varphi, \psi \} \) is a product relation.

**3.3. Refinements and Coarse Graining**

Given a graph \( G \) and a nontrivial USP-relation \( R \) on \( E(G) \) it will often be the case that \( G/P^R \) has no “real” product structure, as in the example of Figure 2. Here, \( V(G/P^R) = V(G) \) for each of the four equivalence classes, so that \( G/P^R \) is the trivial graph \( \mathcal{L}K_1 \) consisting of a single vertex with a loop. In Section 2., we have shown that a coarse graining \( S \) of a USP-relation \( R \) in general leads to a refinement \( P^S \) of the vertex partition \( P^R \). Hence we can expect to obtain larger quotient graph \( G/P^S \) with a “richer” product structure. This is indeed sometimes the case as shown by the example in Fig. 5.

However, a coarser relation \( S \supseteq R \) does not always lead to a partition \( P^S \) that is strictly finer than \( P^R \), see Fig. 6 for an example. In this section we therefore explore the conditions under which a strictly finer partition \( P^S \) of the vertex set is obtained by a coarser equivalence relation \( S \supseteq R \).

**Proposition 3.** Let \( \varphi \) and \( \psi \) be two equivalence classes of a USP-relation \( R \) on the edge set \( E(G) \) of a connected graph \( G \). Then for all \( x \in V(G) \) holds

\[
V(G^x_\varphi \cup \psi) = \bigcup_{y \in V(G^x_\psi)} V(G^y_\psi) = \bigcup_{y \in V(G^x_\varphi)} V(G^y_\varphi).
\]

**Proof.** It suffices to show the first equation. Therefore, let \( z \in \bigcup_{y \in V(G^x_\psi)} V(G^y_\psi) \), that is, there exists a vertex \( y' \in V(G^x_\psi) \) such that \( z \in V(G^y_\psi) \). Hence, there is a path \( P_{x,y'} \) from \( x \) to \( y' \) in \( \varphi \) and a path \( P_{y',z} \) from \( y' \) to \( z \) in \( \psi \). Thus, \( P_{x,y'} \cup P_{y',z} \) is a path from \( x \) to \( z \) in \( \varphi \cup \psi \) and therefore \( z \in V(G^x_\varphi \cup \psi) \) from which we can conclude \( \bigcup_{y \in V(G^x_\psi)} V(G^y_\psi) \subseteq V(G^x_\varphi \cup \psi) \).
Now, let \( z \in V(G'_{\varphi \cup \psi}) \). Clearly, the restriction of \( R \) to \( G'_{\varphi \cup \psi} \) is an equivalence relation on \( E(G'_{\varphi \cup \psi}) \) with only two equivalence classes \( \varphi \) and \( \psi \). Therefore, by Lemma 4 we can conclude that \( V(G'_{\varphi}) \cap V(G'_{\psi}) \neq \emptyset \). Let \( y \in V(G'_{\varphi}) \cap V(G'_{\psi}) \). It follows that \( G'_{\psi} = G'_{\varphi} \) and thus, \( z \in \bigcup_{y \in V(G'_{\varphi})} V(G'_{\varphi}) \) since in particular \( y \in V(G'_{\varphi}) \). From \( V(G'_{\varphi \cup \psi}) \subseteq \bigcup_{y \in V(G'_{\varphi})} V(G'_{\varphi}) \) we conclude equality of the sets.

**Proposition 4.** Let \( R \) be a USP-relation on the edge set \( E(G) \) of a connected graph \( G \) and let \( \varphi, \psi \subseteq R, \varphi \neq \psi \). Then \( V(G'_{\varphi}) \cap V(G'_{\psi}) = V(G'_{\varphi \cup \psi}) \) if and only if \( V(G'_{\varphi}) \cap V(G'_{\psi}) \subseteq V(G'_{\varphi \cup \psi}) \).

**Proof.** From Prop. 3 we can compute \( V(G'_{\varphi}) \cap V(G'_{\psi}) = V(G'_{\varphi}) \cap V(G'_{\varphi \cup \psi}) = V(G'_{\varphi}) \cap \left( \bigcup_{w \in V(G'_{\varphi})} V(G'_{w \cup \varphi \psi}) \right) = \bigcup_{w \in V(G'_{\varphi})} \left( V(G'_{\varphi}) \cap V(G'_{w \cup \varphi \psi}) \right) = V(G'_{\varphi}) \cap \left( \bigcup_{w \in V(G'_{\varphi})} V(G'_{w \cup \varphi \psi}) \right) \).

Notice that \( V(G'_{\varphi}) \subseteq V(G'_{\varphi \cup \psi}) \) if and only if \( v \in V(G'_{\varphi \cup \psi}) \), otherwise we would have \( V(G'_{\varphi}) \cap V(G'_{w \cup \varphi \psi}) = \emptyset \). Therefore,
\[
V(G'_{\varphi}) \cap V(G'_{\psi}) = \bigcup_{w \in V(G'_{\varphi}) \cap V(G'_{\psi})} V(G'_{w \cup \varphi \psi}) = V(G'_{\varphi \cup \psi}) \cup \left( \bigcup_{w \in \mathcal{X}} V(G'_{w \cup \varphi \psi}) \right)
\]

Hence, we have \( V(G'_{\varphi}) \cap V(G'_{\psi}) = V(G'_{\varphi \cup \psi}) \) if and only if \( \mathcal{X} = \emptyset \) which is equivalent to \( V(G'_{\varphi}) \cap V(G'_{\psi}) \subseteq V(G'_{\varphi \cup \psi}) \).

\[\square\]
Proposition 5. Let $R$ be a USP-relation on the edge set $E(G)$ of a connected graph $G$ with two distinct equivalence classes $\varphi, \psi \subseteq R$.

1. If $V(G^\varphi_x) \subseteq V(G^\varphi_y)$ for some $x \in V(G)$ then $V(G^\varphi_y) \subseteq V(G^\varphi_x)$ holds for all $y \in V(G^\varphi_x)$.
2. If $V(G^\psi_x) \subseteq V(G^\psi_y)$ for some $x \in V(G)$ then $V(G^\psi_y) = V(G)$.
3. If $V(G^\varphi_x) = V(G^\psi_y)$, $x \in V(G)$, then for all $y \in V(G)$ holds $V(G^\varphi_y) \cap V(G^\psi_y) \subseteq V(G^\varphi_{\varphi \cup \psi})$ if and only if $V(G^\varphi_y) \subseteq V(G^\varphi_{\varphi \cup \psi})$.

Proof.

1. Let $X := \{ v \in V(G^\varphi_x) \mid V(G^\varphi_y) \subseteq V(G^\varphi_x) \}$. If $V(G^\varphi_x) \subseteq V(G^\varphi_y)$ then $X \neq \emptyset$. Suppose $V(G^\varphi_x) \setminus X \neq \emptyset$. By connectedness of $G^\varphi_x$, there exists some vertex $y \in V(G^\varphi_x) \setminus X$ and $v \in X$ such that $[v, y] \in \psi$. Clearly, $y \notin V(G^\varphi_y)$. Since $y \notin X$, there exists a vertex $w \in V(G^\varphi_y) \setminus V(G^\varphi_y)$. Since $[v, y] \in \psi$, we can use Lemma 6 to conclude that there exist a vertex $z \in V(G^\varphi_y)$ such that $[z, w] \in \psi$. This implies $w \in V(G^\varphi_y) = V(G^\varphi_y)$, since $v \in X$, a contradiction.

2. Let $V(G^\varphi_x) \subseteq V(G^\varphi_y)$ and suppose $V(G) \setminus V(G^\varphi_y) \neq \emptyset$. By connectedness of $G$, there exist vertices $v \in V(G^\varphi_y)$ and $y \in V(G) \setminus V(G^\varphi_y)$ such that $[v, y] \in E(G)$. Obviously, $[v, y]$ must be in $\varphi$. Hence, $y \in V(G^\varphi_y)$. From the first assertion, we conclude that this implies $y \in V(G^\varphi_y)$, a contradiction.

3. Clear.

We conclude our presentation by summarizing conditions under which the joining of two equivalence classes of a USP-relation does not affect the partitioning of the vertex set.

Corollary. Let $R$ be a USP-relation on the edge set $E(G)$ of a connected graph $G$ with two distinct equivalence classes $\varphi, \psi \subseteq R$ and denote by $S$ be the USP-relation obtained from $R$ by joining $\varphi$ and $\psi$. Then:

1. $P^R = P^S$ if $\varphi$ or $\psi$ belong to a factor of $G$.
2. If there is a vertex $x \in V(G)$ with $V(G^\varphi_x) \subseteq V(G^\varphi_y)$ then $P^R = P^S$ if and only if $V(G^\varphi_y) \subseteq V(G^\varphi_{\varphi \cup \psi})$ holds for all $y \in V(G)$.
3. If $P^R = P^S$ then $\varphi \cup \psi$ belongs to a factor of $G$ if and only if both $\varphi$ and $\psi$ belong to a factor of $G$.

Proof.

1. W.l.o.g., let $\varphi$ correspond to a factor of $G$. Then it holds $V(G^\varphi_x) \cap V(G^\varphi_y) = \{ x \} \subseteq V(G^\varphi_{\varphi \cup \psi})$ for all $x \in V(G)$, which implies the assertion.
2. Follows immediately from Proposition 5.
3. If both $\varphi$ and $\psi$ correspond to factors, then clearly $\varphi \cup \psi$ also corresponds to a factor. Conversely, suppose $\varphi \cup \psi$ correspond to a factor and suppose $P^R = P^S$. 

Then $|V(G^x_{\varphi,\psi}) \cap V(G^x_{\varphi,\psi})| = 1$ and $V(G^x_{\varphi}) \cap V(G^x_{\psi}) \subseteq V(G^x_{\varphi,\psi})$ holds for all $x \in V(G)$. Note that $V(G^x_{\varphi}) \subseteq V(G^x_{\varphi,\psi})$ and hence $V(G^x_{\varphi}) \cap V(G^x_{\psi}) = (V(G^x_{\varphi}) \cap V(G^x_{\psi})) \cap V(G^x_{\varphi,\psi}) \subseteq V(G^x_{\varphi,\psi}) \cap V(G^x_{\varphi,\psi})$ holds for all $x \in V(G)$. This implies $V(G^x_{\varphi}) \cap V(G^x_{\psi}) = \{x\}$ for all $x \in V(G)$. By Theorem 3 we can now conclude that $\varphi$ belongs to a factor of $G$. Analogously, it follows that $\psi$ belongs to a factor of $G$.

ACKNOWLEDGMENTS

This work was supported in part by the Deutsche Forschungsgemeinschaft within the EUROCORES Programme EUROGIGA (project GReGAS) of the European Science Foundation.

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