An Efficient PTAS for Stochastic Load Balancing with Poisson Jobs

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Abstract

We give the first polynomial-time approximation scheme (PTAS) for the stochastic load balancing problem when the job sizes follow Poisson distributions. This improves upon the 2-approximation algorithm due to Goel and Indyk (FOCS’99). Moreover, our approximation scheme is an efficient PTAS that has a running time double exponential in $1/\epsilon$ but nearly-linear in $n$, where $n$ is the number of jobs and $\epsilon$ is the target error. Previously, a PTAS (not efficient) was only known for jobs that obey exponential distributions (Goel and Indyk, FOCS’99).

Our algorithm relies on several probabilistic ingredients including some (seemingly) new results on scaling and the so-called “focusing effect” of maximum of Poisson random variables which might be of independent interest.

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1 Introduction

We consider the following fundamental problem in scheduling theory: given \( n \) jobs with job sizes \( w_1, \ldots, w_n \geq 0 \), assign jobs to \( m \) machines such that the maximum load of any machine (i.e., the total size of jobs assigned to the machine) is minimized. In other words, we want to partition \([n]\) into sets \( S_1, \ldots, S_m \) so as to minimize \( \max_{i \in [m]} \sum_{j \in S_i} w_j \). Often referred to as load balancing or makespan minimization, this is one of the classical NP-complete problems and along with its many variants, has been extensively studied in both theoretical computer science and operations research. While the exact problem is hard, this problem admits a polynomial-time approximation scheme (PTAS) [HS87] and later work improved this to an efficient PTAS [AAWY98, Jan09, JKV16, JR19]. Several other variants of this problem have also been studied – this includes (i) the related machines case where the machines can have different speeds [Hoe88]; (ii) the unrelated machines case where the job size itself depends on the machine on which it is scheduled [LST90]; and (iii) the precedence constrained case where there are precedence constraints on schedule of jobs [CS99, CB01].

All the aforementioned variants of this problem have the common feature that the job sizes are known in advance to the algorithm designer. However, in many situations, there might be uncertainty in the job size. An obvious way to model this uncertainty is via the framework of stochastic optimization as follows – we have \( m \) machines and \( n \) jobs where the size of the \( i \)th job is given by the random variable \( W_i \). If we now assign the jobs to \( m \) machines (given by \( S_1, \ldots, S_m \)), then the load of the \( j \)th machine is given by the random variable \( \sum_{i \in S_j} W_i \). Similar to the case when the job sizes are deterministic, in stochastic load balancing, one would like to minimize the maximum load. However, since the maximum load (across machines) is itself a random variable – arguably, the most natural objective is to then minimize the expected maximum load. In other words, we seek to find a partition of \([n]\) into sets \( S_1, \ldots, S_m \) so as to minimize \( \mathbb{E} \left[ \max_{j \in [m]} \sum_{i \in S_j} W_i \right] \).

Throughout this paper, we assume that the random variables \( \{W_i\}_{i=1}^n \) are independent – that is the job sizes are independent of each other. Further, note that the algorithm designer is assumed to know the distribution of the random variables \( \{W_i\}_{i=1}^n \) (though, of course, not the actual realizations of the loads).

To our knowledge, Kleinberg, Rabani and Tardos [KRT00] were the first to consider this problem in the algorithms community. They gave an \( O(1) \)-factor approximation algorithm for this problem. Soon thereafter, Goel and Indyk [GI99] considered the problem of obtaining better approximation for special classes of random variables – in particular, (i) if each \( \{W_i\} \) is an exponential random variable, they obtain a PTAS (though not an efficient one); (ii) if each \( \{W_i\} \) is a Poisson random variable, then they obtain a 2-approximation algorithm. In fact, this 2-approximation is obtained by considering the (deterministic) instance with loads \( \{w_1, \ldots, w_n\} \) where \( w_i = \mathbb{E}[W_i] \) and then applying Graham’s heuristic [Gra66] on this instance.

Somewhat more complicated variants of this problem have also been considered – as an example, Gupta et al. [GKNS18] considered the problem of stochastic load balancing on unrelated machines. Here, the load of job \( i \) on machine \( j \) is given by a random variable \( W_{i,j} \). For this variant, [GKNS18] gave an \( O(1) \)-approximation algorithm (thus extending the guarantee of [KRT00] to the case of unrelated machines). Similarly, Molinaro [Mol19] considered the problem of minimizing the expected \( \ell_p \) norm of the loads (the version we have can be seen as minimizing...
the expected $\ell_\infty$ norm of the loads). Despite all this impressive progress, the only case where we have a PTAS for stochastic load balancing is when all the loads $\{W_i\}$ are exponential random variables. As the main result of this paper, we obtain an efficient PTAS for stochastic load balancing when all the loads are Poisson random variables.

**Theorem 1.1.** There is an algorithm PoiScheduling$(n, m, \{\lambda_i\}_{i=1}^n, \epsilon)$ that given an instance of the load balancing problem with $n$ jobs and $m$ machines where the size of the $i$th job is $W_i = \text{Poi}(\lambda_i)$ (i.e. a Poisson random variable with mean $\lambda_i$), and a parameter $0 < \epsilon < 1$, outputs a job assignment whose expected maximum load satisfies $L \leq (1 + \epsilon)L^*$, where $L^*$ is the expected maximum load of an optimal assignment. The algorithm runs in time $2^{O(1/\epsilon^2)} + O(n \log^2 n \log \log^2 n)$.

Theorem 1.1 is the first PTAS for stochastic load balancing with Poisson jobs. Prior to this result, the best known approximation algorithm for this setting was due to Goel and Indyk [GI99] (mentioned earlier) and had an approximation factor of 2. In fact, our PTAS is also an efficient PTAS – i.e., the running time remains polynomial in $n$ even for some $\epsilon = o(1)$. In contrast, the PTAS from [GI99] for exponential random variables was not an efficient PTAS. Finally, we point out that our running time is doubly exponential in the error parameter $\epsilon$. While this can be potentially improved to a singly exponential dependence in $\epsilon$, it is unlikely to be improved further – in particular, [CJZ14] showed that under the ETH, any PTAS for even the deterministic load balancing problem must have a singly exponential dependence on $\epsilon$.

1.1 Our techniques

At a high level, to design an algorithm for stochastic load balancing, we must come up with an algorithmically tractable proxy for the objective function $E[\max_{j \in [m]} \sum_{i \in S_j} W_i]$. However, the expected maxima of random variables (and more generally stochastic processes) can be notoriously difficult to reason about. Indeed, we point out that in the last fifty years, significant effort in probability theory has been devoted towards understanding the maximum of even simple families of random variables such as Gaussians [Dar83, Tal96]. Despite this challenge, the hope is that by exploiting structural properties of Poisson random variables along with appropriate algorithmic primitives, we will be able to design an efficient PTAS for stochastic load balancing for Poisson jobs.

The starting points of our algorithm are two natural heuristics which have previously been analyzed in the context of stochastic load balancing.

1. The first heuristic is to construct an instance of (deterministic) load balancing where the size of the $i$th job is $w_i = E[W_i]$. One can then apply the PTAS (say from [AAWY98]) to get an allocation of the $n$ jobs into $m$ machines. The obvious pitfall here is that the actual job size is a Poisson random variable which may typically be very far from its mean. In other words, this heuristic has a good guarantee provided

$$E[\max_{j=1}^m \text{Poi}(\mu_j)] \approx \max_{j=1}^m E[\text{Poi}(\mu_j)],$$

\footnote{One can reduce an instance of deterministic load balancing to one of Poisson load balancing by scaling up all job sizes such that they all become at least $\omega(\epsilon^{-2} \log m)$. By Chernoff bounds and union bound this reduction preserves $(1 + O(\epsilon))$-approximation.}
where $\mu_j$ is the expected load size of the $j^{th}$ machine. Of course, the above relation may be far from true and indeed, we want to point out that while the left hand side $E[\max_{j=1}^n \text{Poi}(\mu_j)]$ is just a function of $\mu_1, \ldots, \mu_j$, it is far from being a linear function of $\mu_1, \ldots, \mu_j$. It is easy to create an instance where the optimum obtained by replacing each Poisson load by its expectation is a constant factor away from the true optimum, a detailed proof is provided in Appendix A.

Despite this limitation, this heuristic is in fact of both theoretical and practical value. In particular, from a theoretical aspect, recall that $\text{Poi}(\lambda)$ concentrates around $\lambda$ (with standard deviation $\sqrt{\lambda}$). This can be leveraged to show that if the optimum allocation must necessarily have at least one machine with a (sufficiently) large load, then the allocation for the deterministic load balancing problem provides a near optimal allocation for the stochastic version as well.

2. The second heuristic is a greedy algorithm – namely, we first assign an arbitrary order to the jobs and iteratively assign each job to the machine with the least current expected load. This is the same as the Graham’s rule [Gra66], and is precisely how the authors of [GI99] obtained a 2-approximation for load balancing Poisson jobs. The underlying rationale for this rule is the following crucial fact about Poisson random variables. Suppose $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ such that $\mu_1 + \mu_4 = \mu_2 + \mu_3$. Then, $E[\max\{\text{Poi}(\mu_1), \text{Poi}(\mu_4)\}] \geq E[\max\{\text{Poi}(\mu_2), \text{Poi}(\mu_3)\}]$. This fact can in fact be extended to prove that if there is an allocation such that the expected load is the same across all machines, then that is an optimum allocation. Of course, such an allocation might not exist – however, heuristically we might hope that if all the job sizes are small, then we can approximately equalize the expected load on the machines and that such an allocation might have a near-optimal expected maximum load.

It turns out that these heuristics (even when rigorously analyzed) are not sufficient to provide a PTAS for stochastic load balancing in all regimes of job sizes and (number of) machines. To describe the other ingredients, let us define $\mu^{(0)}$ to be the total expected job size divided by $m$. The first crucial observation is that if there is a job size $\lambda$ which is more than $\mu$, then in the optimal allocation, such a job is assigned its own separate machine (Observation 4.9). This observation can be iteratively applied so that we are now left with job sizes $\{\lambda_i\}_{i=1}^{n'}$ and $m'$ machines such that

$$\max_{i=1}^{n'} \{\lambda_i\} \leq \frac{\sum_{i=1}^{n'} \lambda_i}{m'} := \mu.$$

In other words, no job is larger than the average expected load across the $m'$ machines, i.e., $\mu$. With this simplification, we discuss another familiar trick in the context of allocation problems – namely we create a rounded instance such that each job size (now call it $\{\lambda'_i\}_{i=1}^{n'}$) is now in the interval $[\epsilon \mu, 2\mu]$. The rounding procedure we apply is identical to the rounding procedure used by [JKV16] in the context of deterministic load balancing. A key property is that the number of different (expected) job sizes in this modified instance is a constant – i.e., only dependent on the target error parameter $\epsilon$.

Note that $\text{Poi}(\lambda_1) + \text{Poi}(\lambda_2) = \text{Poi}(\lambda_1 + \lambda_2)$ if $\text{Poi}(\lambda_1)$ and $\text{Poi}(\lambda_2)$ are independent, so every machine’s load is still a Poisson random variable.
This rounding step highlights a key technical challenge our algorithm faces – namely, it is possible that by “multiplicatively dilating” the job sizes, the expected maximum load of the machines can change significantly. In other words, suppose \( \mu_1, \ldots, \mu_m \geq 0 \), then is it the case that for any \( 0 < \delta < 1 \),

\[
E[\max_{j=1}^m \text{Poi}((1 + \delta)\mu_j)] \approx (1 + O(\delta))E[\max_{j=1}^m \text{Poi}(\mu_j)] \quad (1)
\]

While intuitively this looks reasonable, it is not clear if this is true in full generality. Fortunately for us, we obtain the following dichotomy:

1. When \( \mu \) is very large (this corresponds to the [Case 1 in the analysis], we are able to show that the first heuristic above provides a PTAS – in other words, just substituting each stochastic job \( W_i \) with a deterministic job \( w_i = E[W_i] \) and then applying the PTAS for the deterministic case [AAWY98, Jan09, JKV16] gives a PTAS for the stochastic case. The underlying reason is that in this case, the expected maximum is essentially the same as the heaviest expected load across the \( m \) machines.

The same algorithm also works if \( \mu \) is in a “certain intermediate range” and \( m \) is sufficiently large (this corresponds to [Case 3 in the analysis]). In fact, in this case, even the greedy heuristic described earlier provides a PTAS. The underlying reason why the deterministic PTAS works is the following: in this regime, the expected maximum remains essentially the same even if all the loads were to go up by a factor of 2.

2. Outside of the above two cases, our heuristics (greedy or deterministic scheduling) fail to provably work. However, in these cases, we are able to prove (1). In other words, we are able to show that dilating or contracting each job size by a factor of \( (1 + \delta) \) affects the expected maximum by only a factor of \( 1 \pm O(\delta) \). Thus, we can apply the rounding procedure from [AAWY98] to reduce to the case where the number of different job sizes is a constant. In fact, this is enough to obtain a PTAS for the stochastic load balancing problem though not an efficient PTAS.

Finally, to get an efficient PTAS, we leverage a third property of the “maximum of Poisson random variables” – namely, the so-called “focusing effect” [And70, ACH97, BSP09]. Roughly speaking, it says that suppose we have \( m \) independent Poisson random variables (call them \( X_1, \ldots, X_m \)), each with mean \( \mu \), then there is an integer \( I \) such that \( (\max_{i=1}^m X_i) \in [I, I + 1] \) with probability \( 1 - o(1) \) as \( m \to \infty \). We extend this to (certain instances of) independent but not identically distributed Poisson random variables. Essentially such a “focusing effect”, whenever it holds, allows us to express the expected maximum of the loads of \( m \) machines as a linear function of the allocation and then employ an integer linear program (ILP) to find the optimal allocation.

To explain how an ILP comes into the picture, first of all, we can assume that \( m \) (i.e., the number of machines) is sufficiently large in terms of the target error parameter \( \delta \). If this is not the case, then we can simply employ dynamic programming to find a good allocation (it is now an efficient PTAS because \( m \) is a constant). Once \( m \) is large, we show the following:

(a) either there is a transition point \( t = t(\mu_1, \ldots, \mu_m) \) such that \( \max\{\text{Poi}(\mu_1), \ldots, \text{Poi}(\mu_m)\} \) sharply concentrates within \( 1 \pm O(\delta) \) of the transition point. In this case, we do a binary
search to iterate over (potential) transition points and use an ILP to find the smallest $t^*$, for which there is an allocation with loads $\mu_1, \ldots, \mu_m$ such that $t^* = t(\mu_1, \ldots, \mu_m)$. Observe that in this case, the smallest such $t^*$ will minimize the expected maximum load.

(b) Otherwise, there is a transition point $t = t(\mu, m)$ such that $\max\{\text{Poi}(\mu_1), \ldots, \text{Poi}(\mu_m)\}$ sharply concentrates in the set $[t-1, t]$. Observe that $t$ only depends on $\mu$ and $m$, and hence can be easily computed. With the knowledge of $t$, we now use an ILP to find an assignment $\mu_1, \ldots, \mu_m$ which maximizes the probability that $\max\{\text{Poi}(\mu_1), \ldots, \text{Poi}(\mu_m)\} = t - 1$ and thus minimizes the expected maximum load.

### 1.2 Organization

In Section 2, we formally state the problem, establish some notations, and describe some properties of Poisson random variables that we will utilize. In Section 3 we present and prove our concentration and scaling results for the maximum of Poisson random variables. In Section 4 we present our efficient polynomial-time approximation scheme and analyze its performance.

### 2 Preliminaries

We use $\text{Poi}(\lambda)$ to denote a Poisson random variable with mean $\lambda$. Recall that $\Pr[\text{Poi}(\lambda) = k] = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k \in \mathbb{N}$. In the stochastic load balancing problem considered in this paper, we are given $n$ jobs and $m$ machines where the job sizes are independent Poisson random variables $\text{Poi} (\lambda_1), \text{Poi} (\lambda_2), \ldots, \text{Poi} (\lambda_n)$. We will call $\lambda_i$ the size of job $i$. Our goal is to assign the jobs to the machines so that the expected maximum load

$$L \overset{\text{def}}{=} \mathbb{E} \left[ \max_{j=1}^{m} \sum_{i \in S_j} \text{Poi} (\lambda_i) \right]$$

is minimized, where $S_j$ is the set of jobs assigned to machine $j$.

It is well known that the sum of two independent Poisson random variables also follows a Poisson distribution, i.e. $\text{Poi} (\lambda_1) + \text{Poi} (\lambda_2) = \text{Poi} (\lambda_1 + \lambda_2)$. Therefore if we let $\mu_j = \sum_{i \in S_j} \lambda_i$, we can write (2) as $L = \mathbb{E} \left[ \max_{j=1}^{m} \text{Poi} (\mu_j) \right]$. We will call $\mu_j$ the load of machine $j$.

Henceforth, our analysis of Poisson random variables will mainly serve the purpose of characterizing the expected maximum load, and therefore we will use $\mu$ and $\mu_1, \mu_2, \ldots, \mu_m$ to denote the means when stating useful claims about Poisson distributions.

**Definition 2.1.** We write $M(m, \mu)$ to denote the random variable whose value is the maximum of $m$ i.i.d. $\text{Poi} (\mu)$.

In [GI99] the authors proved that Poisson distributions are log-concave:

**Proposition 2.2 ([GI99]).** For any $t \geq 0$, the function

$$f_t(\mu) = \log \Pr[\text{Poi} (\mu) \leq t]$$

is decreasing and concave with respect to $\mu$. 

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For any random variables $X$ and $Y$ taking values on $\mathbb{N}$, we say $X$ stochastically dominates $Y$, denoted by $X \geq_{sd} Y$, if $\Pr[X \geq k] \geq \Pr[Y \geq k]$ holds for every $k \in \mathbb{N}$. Note that for independent $X, Y$ we have $\Pr[\max\{X, Y\} \geq k + 1] = 1 - \Pr[X \leq k] \Pr[Y \leq k]$. Now by Proposition 2.2 we have the following:

**Proposition 2.3** (Lemma 2.1 of [GI99]). Given $0 \leq \mu_1 \leq \mu_2 \leq \mu_3$, it holds that $\mu_2 = \mu_1 + \mu_3$, it holds that $\max\{\text{Poi}(\mu_1), \text{Poi}(\mu_3)\} \geq_{sd} \max\{\text{Poi}(\mu_1), \text{Poi}(\mu_2)\}$.

Poisson random variables satisfy exponential tail bounds:

**Proposition 2.4** (Theorem 4.4, Theorem 4.5 of [MU17]). Let $X$ be a Poisson random variable with mean $\mu$. For $0 < \delta < 1$ we have

\[
\Pr[ X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}, \tag{4}
\]

\[
\Pr[ X \leq (1-\delta)\mu] \leq e^{-\mu\delta^2/2}. \tag{5}
\]

In our analysis we will need to use Stirling’s approximation to deal with factorials:

**Proposition 2.5** (Stirling’s approximation [Rob55]). For any integer $n > 0$,

\[
e\left(\frac{n}{e}\right)^n \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{1/(2n+1)} \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{1/12n} \leq e n \left(\frac{n}{e}\right)^n. \tag{6}
\]

3 Concentration and Scaling Results for Maximum of Poissons

In this section we present our concentration and scaling results for the maximum of independent Poisson random variables $\text{Poi}(\mu_1), \text{Poi}(\mu_2), \ldots, \text{Poi}(\mu_m)$, which will be used to prove the correctness of our algorithm.

Throughout we assume $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0$, and define $\mu = (\sum_{j=1}^{m} \mu_j)/m$. We use $\delta$ as an error parameter, which measures how well the maximum of Poissons is concentrated. We consider five different cases based on the relationship between $\mu$, $m$, and $\mu_j$’s, and prove our results for each of them. Note that while the ranges of $\mu$ in these cases are disjoint, we prove our results below for slightly overlapping ranges of $\mu$ for ease of analyzing our algorithm in Section 4.

Fix $\delta \in (0, 1/10]$. We prove our results for the following cases respectively:

**Case 1**: $\frac{6}{\delta^2} \log m < \mu$ (Lemma 3.1).

**Case 2**: $\frac{1}{2(\delta+1)} \log m < \mu \leq \frac{6}{\delta^2} \log m$ and $m \geq 2^{2\frac{\delta}{\delta^2}}$ (Lemma 3.2).

**Case 3**: $\frac{1}{m^\delta} < \mu \leq \frac{1}{2(\delta+1)} \log m$, $m \geq 2^{\frac{H}{\delta^2}}$, and $\forall j, \mu_j \in [\mu/4, 4\mu]$ (Lemma 3.3).

**Case 4**: $\frac{4 \log m}{m} < \mu \leq \frac{1}{m^\delta}$, $m \geq 2^{100/\delta^2}$, and $\forall j, \mu_j \in [\mu/4, 4\mu]$ (Lemma 3.8).

**Case 5**: $\mu \leq \frac{4 \log m}{m}$, $m \geq 1000 (1/\delta) \log^2 (1/\delta)$, and $\forall j, \mu_j \in [\mu/4, 4\mu]$ (Lemma 3.9).
Sketch of the Proofs. We first sketch the ideas we used to prove our concentration results. Our scaling results follow from essentially the same ideas with some additional analysis. All our results rely crucially on the exponential tails of Poisson distributions.

For simplicity we assume for now that δ is a constant, and all μ_j ∈ [μ/2, 2μ]. Our proofs can basically be seen as coping with cases where μ is in the following ranges respectively:

1. When μ = ω(log m(1)), by the exponential tails of Poisson distributions and union bound we prove that μ_1 ≤ E [max_{j=1}^{m(1)} Poi (μ_j)] ≤ (1 + O(δ))μ_1. This is the idea for Case 1.

2. When μ = o(log m(1)), we define the transition point roughly as t = \frac{log m(1)}{log(1/\mu)+log(log m(1))}. The intuition is that \frac{log m(1)}{log(1/\mu)+log(log m(1))} is close to the root of (μ_j/t)^t = 1/m(1) for all j ∈ [m(1)], since all μ_j ∈ [μ/2, 2μ]. Once again by the exponential tails of Poisson distributions, when m is sufficiently large max_{j=1}^{m(1)} Poi (μ_j) is either concentrated within (1 ± O(δ))t or takes value ⌈t⌉ − 1 or ⌊t⌋ with high probability, depending on how small μ is. This is the idea for Case 3, Case 4 and Case 5.

3. μ = Θ(log m(1)) is the trickiest case since the roots of (μ_j/t)^t = 1/m(1) for different j's can vary much even though all μ_j ∈ [μ/2, 2μ]. Therefore we now define the transition point t as the largest integer satisfying \sum_{j=1}^{m(1)} Pr [Poi (μ_j) ≥ t] ≥ 1/3. Due to the exponential tails of Poisson distributions, this sum decays geometrically from t to (1 + O(δ))t and also grows geometrically from t to (1 − O(δ))t. When m(1) is sufficiently large (and thus μ, t are sufficiently large), we have that \sum_{j=1}^{m(1)} Pr [Poi (μ_j) ≥ (1 + O(δ))t] is small enough and \sum_{j=1}^{m(1)} Pr [Poi (μ_j) ≥ (1 − O(δ))t] is large enough to give us concentration. This is the idea for Case 2.

In our proofs in the following subsections we will use the following identity often, which holds for any independent random variables X, Y taking values on N:

$$E [\max \{X, Y\}] = \sum_{x=0}^{\infty} Pr [X = x] \left( x + \sum_{y=x+1}^{\infty} Pr [Y ≥ y] \right).$$ (7)

3.1 Case 1

Lemma 3.1 (Case 1). Suppose δ ∈ (0, 1/10] and μ > \frac{δ}{10} log m. Then

$$μ_1 ≤ E [\max_{j=1}^{m} Poi (μ_j)] ≤ (1 + 5δ)μ_1.$$ (8)
Proof. By union bound and Proposition 2.4 we have
\[
\Pr\left[ \max_{j=1}^m \text{Poi}(\mu_j) \geq (1 + \delta)\mu_1 \right] \leq m\Pr[\text{Poi}(\mu_1) \geq (1 + \delta)\mu_1] \leq \frac{m}{m^2} = \frac{1}{m}.
\] (9)

Then the expected maximum can bounded from above by
\[
E\left[ \max_{j=1}^m \text{Poi}(\mu_j) \right] = \sum_{k=1}^{\infty} \Pr\left[ \max_{j=1}^m \text{Poi}(\mu_j) \geq k \right]
\leq [(1 + \delta)\mu_1] + \sum_{k=1}^{\infty} m\Pr[\text{Poi}(\mu_1) \geq k]
\leq [(1 + \delta)\mu_1] + \sum_{k=1}^{\infty} m\Pr[\text{Poi}(\mu_1) \geq k]
\]

Note that
\[
\Pr\left[ \max_{j=1}^m \text{Poi}(\mu_j) \geq k \right] \leq \frac{\mu_1}{k + 1} \Pr\left[ \max_{j=1}^m \text{Poi}(\mu_j) \geq k + 1 \right] < (1 + \delta)\Pr\left[ \max_{j=1}^m \text{Poi}(\mu_j) \geq k + 1 \right].
\]

Therefore
\[
\Pr\left[ \max_{j=1}^m \text{Poi}(\mu_j) \geq k \right] \leq [(1 + \delta)\mu_1] + \sum_{k=1}^{\infty} m\Pr[\text{Poi}(\mu_1) \geq k]
\leq (1 + \delta)\mu_1 + \left\{ \sum_{k=0}^{\infty} (1 + \delta)^{-k} \right\} m\Pr[\text{Poi}(\mu_1) \geq (1 + \delta)\mu_1]
\leq (1 + \delta)\mu_1 + \frac{2}{\delta} \leq (1 + 3\delta)\mu_1
\] (10)
as desired. □

3.2 Case 2

Lemma 3.2 (Case 2). Suppose \( \delta \in (0, 1/10] \), \( \frac{1}{21\delta + \epsilon} \log m < \mu \leq \frac{12}{\delta^2} \log m \), and \( m \geq 2^{2^\frac{1}{2^3}} \). Define transition point \( t_2 = t_2(\mu_1, \mu_1, \ldots, \mu_m) \) as the largest integer satisfying
\[
\sum_{j=1}^m \Pr[\text{Poi}(\mu_j) \geq t_2] \geq \frac{1}{3}.
\] (11)

Then for any random variable \( X \) taking values on \( \mathbb{N} \), we have
\[
(1 - 6\delta)E[\max\{t_2, X\}] \leq E\left[ \max\left\{ \max_{j=1}^m \text{Poi}(\mu_j), X \right\} \right] \leq (1 + 10\delta)E[\max\{t_2, X\}],
\] (12)

and
\[
E\left[ \max\left\{ \max_{j=1}^m \text{Poi}((1 + \delta)\mu_j), X \right\} \right] \leq (1 + 16\delta)E\left[ \max\left\{ \max_{j=1}^m \text{Poi}(\mu_j), X \right\} \right].
\] (13)

\(^3\)The choice of \( \frac{1}{3} \) in the definition of \( t_2 \) is arbitrary. In principle any constant bounded away from both 1 and 0 suffices.
Before proving this lemma, we state another useful lemma, which shows that when the largest \( \mu_j \) is large enough we have that (a) the maximum load is concentrated around \( t_2 \) and (b) \( t_2 \) blows up by at most \( 1 + O(\delta) \) when we scale each \( \mu_j \) by \( 1 + \delta \).

**Lemma 3.3.** Let \( t_2 \) be the largest integer satisfying \( \sum_{j=1}^{m} \Pr[\text{Poi}(\mu_j) \geq t_2] \geq \frac{1}{3} \). Given \( \delta \in (0, \frac{1}{10}) \) where

\[
\mu_1 \geq \frac{6 \ln \left( \frac{1}{\delta} \right)}{\delta^2} \tag{14}
\]

Let \( \ell = \lfloor (1 - 4\delta)t_2 \rfloor \), \( r = \lceil (1 + 8\delta)t_2 \rceil \) where \( t_2 = t_2(\mu_1, \ldots, \mu_m) \). Then the following statements hold:

1. \( \Pr \left[ \max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j) \geq r \right] \leq \delta^2 \) and \( \sum_{k=\ell}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j) \geq k \right] \leq \delta. \)

2. \( \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq \ell \right] \geq 1 - \delta. \)

**Remark 3.4.** By Proposition 2.4 \([14]\) is a sufficient condition for \( \text{Poi}(\mu_1) \) to be concentrated within \( (1 \pm O(\delta))\mu_1 \) with probability at least \( 1 - O(\delta) \). Lemma 3.3 shows that this condition also implies the same concentration of \( \max_{j=1}^{m} \text{Poi}(\mu_j) \) around its expectation. This can be seen as a consequence of the exponential tails of Poisson distributions: the sum of tail probabilities decreases (resp. increases) geometrically from \( t_2 \) to \( (1 + O(\delta))t_2 \) (resp. \( (1 - O(\delta))t_2 \)).

Before getting into the proof of Lemma 3.3 we show how it implies Lemma 3.2.

**Proof of Lemma 3.2** By Lemma 3.3 for any integer \( x \leq r \),

\[
x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] \geq \sum_{y=1}^{x} 1 + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] \geq \sum_{y=1}^{x} 1 + \sum_{y=x+1}^{\ell} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] \geq (1 - \delta)\ell \tag{15}
\]

and with the same argument we have

\[
x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j) \geq y \right] \leq r + \delta. \tag{16}
\]

For any integer \( x > r \),

\[
x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] \geq x \tag{17}
\]

\[
x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j) \geq y \right] \leq x + \delta. \tag{18}
\]
By the definition of \( \ell \) and (15), when \( x \leq r \) we have that

\[
\begin{align*}
x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] &\geq \min \{x, (1 - 6\delta)t_2\} \geq (1 - 6\delta) \min \{x, t_2\}
\end{align*}
\]

(19)

which coupled with (7) gives us the lower bound of (12).

By the definition of \( r \) and (18), when \( x > r \) we have that

\[
\begin{align*}
x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] &\leq \min \{x, (1 + 9\delta)t_2\} + \delta \leq (1 + 10\delta) \min \{x, t_2\},
\end{align*}
\]

(20)

where the last inequality follows from that \( t_2 \geq \mu_1 > 1 \). This and (7) together give us the upper bound of (12).

By the definitions of \( \ell \) and \( r \),

\[
(1 + 16\delta)(1 - \delta)\ell \geq r + \delta.
\]

Also, \((1 + 16\delta)x \geq x + \delta\) clearly holds when \( x > r \). Therefore we have

\[
\begin{align*}
x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] &\leq (1 + 16\delta) \left( x + \sum_{y=x+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq y \right] \right)
\end{align*}
\]

(21)

for any \( x \in \mathbb{N} \), which coupled with (7) proves (13).

To prove Lemma 3.2, we first show that the transition point \( t_2 \) defined as the largest integer satisfying

\[
\sum_{j=1}^{m} \Pr[\text{Poi}(\mu_j) \geq t_2] \geq \frac{1}{3}
\]

(22)

is at least as large as \( \mu_1 \), the largest mean.

**Lemma 3.5.** If \( \mu_1 \geq 400 \), \( t_2 \geq \mu_1 \).

**Proof.** It suffices to show that \( \Pr[\text{Poi}(\mu_1) \geq \mu_1] \geq 1/3 \), which implies

\[
\sum_{j=1}^{m} \Pr[\text{Poi}(\mu_j) \geq \mu_1] \geq \frac{1}{3}
\]

(23)

and hence \( t_2 \geq \mu_1 \). Let \( \nu_1 \) be the median of \( \text{Poi}(\mu_1) \). By [Cho94] \( \nu_1 \geq \mu_1 - \ln 2 > \mu_1 - 1 \). Therefore

\[
\begin{align*}
\Pr[\text{Poi}(\mu_1) \geq \mu_1] &\geq \frac{1}{2} - \Pr[\text{Poi}(\mu_1) = \lfloor \mu_1 \rfloor] = \frac{1}{2} - e^{-\mu_1} \frac{\mu_1^{\lfloor \mu_1 \rfloor}}{\lfloor \mu_1 \rfloor}! \\
&\geq \frac{1}{2} - \frac{e^{-\mu_1 + \lfloor \mu_1 \rfloor}}{\sqrt{2\pi \lfloor \mu_1 \rfloor}} \left( \frac{\mu_1}{\lfloor \mu_1 \rfloor} \right)^{\lfloor \mu_1 \rfloor} \geq \frac{1}{2} - \frac{1}{\sqrt{2\pi \lfloor \mu_1 \rfloor}} \left( 1 + \frac{1}{\mu_1 - 1} \right)^{\mu_1} \\
&\geq \frac{1}{2} - \frac{e^2}{\sqrt{2\pi \lfloor \mu_1 \rfloor}} \geq \frac{1}{3}.
\end{align*}
\]

Here the first inequality on the second line follows from \( n! \geq \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \) by Stirling. □

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We will use the following fact in proving Lemma 3.3.

**Fact 3.6.** \( \delta < 1/10 \) and \( \mu_1 \geq \frac{6 \ln(1/\delta)}{\delta^2} \) implies that for any \( z \geq \mu_1/6, \lfloor z \rfloor \geq (1 - 0.1\delta)z \) and \( \lceil z \rceil \leq (1 + 0.1\delta)z \).

Now we prove Statement 1 of Lemma 3.3.

**Proof of Lemma 3.3 Statement 1.** Let \( r' = \lceil (1 + 2\delta)t_2 \rceil \). We first bound the tail probability \( \Pr \left[ \max_{j=1}^m \text{Poi} \left( \mu_j \right) \geq r' \right] \), and then bound the ratio of \( \Pr \left[ \max_{j=1}^m \text{Poi} \left( (1 + \delta)\mu_j \right) \geq r \right] \) to it.

By union bound, the former is at most \( \sum_{j=1}^m \Pr \left[ \text{Poi} \left( \mu_j \right) \geq r' \right] \). Since \( r' \geq (1 + 2\delta)t_2 \geq (1 + 2\delta)\mu_1 \) and \( \mu_1 \) is the largest, this sum is bounded by a geometric series of ratio \( \frac{1}{1 + 2\delta} \) and hence

\[
\sum_{j=1}^m \Pr \left[ \text{Poi} \left( \mu_j \right) \geq r' \right] \leq m \Pr \left[ \text{Poi} \left( \mu_j \right) = r' \right] \left( \sum_{i=0}^{\infty} \left( \frac{\mu_j}{r'} \right)^i \right)
\]

\[
\leq \left( \sum_{i=0}^{\infty} \left( \frac{\mu_j}{r'} \right)^i \right) \sum_{j=1}^m \Pr \left[ \text{Poi} \left( \mu_j \right) = r' \right] \leq \left( \sum_{i=0}^{\infty} \left( \frac{1}{1 + 2\delta} \right)^i \right) \sum_{j=1}^m \Pr \left[ \text{Poi} \left( \mu_j \right) = r' \right]
\]

\[
= \left( \frac{1}{1 - \frac{1}{1 + 2\delta}} \right) \sum_{j=1}^m \Pr \left[ \text{Poi} \left( \mu_j \right) = r' \right]
\]

\[
\leq \frac{1}{\delta} \sum_{j=1}^m \left( \prod_{k=t_2+2}^{r'} \frac{\mu_j}{k} \right) \Pr \left[ \text{Poi} \left( \mu_j \right) = t_2 + 1 \right]
\]

\[
= \frac{1}{\delta} \sum_{j=1}^m \left( \prod_{k=t_2+2}^{r'} \frac{\mu_j}{k} \right) \Pr \left[ \text{Poi} \left( \mu_j \right) = t_2 + 1 \right] \quad \text{(as } t_2 \delta > \mu_1 \delta > 2)\]

\[
\leq \frac{1}{\delta} \sum_{j=1}^m \left( \frac{1}{1 + \delta} \right)^{r' - \lceil (1 + \delta)t_2 \rceil} \Pr \left[ \text{Poi} \left( \mu_j \right) = t_2 + 1 \right]
\]

\[
\leq \frac{1}{\delta (1 + \delta)^{0.8\delta t_2}} \sum_{j=1}^m \Pr \left[ \text{Poi} \left( \mu_j \right) = t_2 + 1 \right]
\]

\[
\leq \frac{1}{\delta} \left( 1 - \frac{\delta}{1 + \delta} \right)^{0.8\delta t_2} \leq \frac{1}{3\delta} e^{-\delta^2 t_2/2} \leq \frac{\delta^2}{3}.
\]  

Here the penultimate line holds because \( r' - \lceil (1 + \delta)t_2 \rceil \geq (1 + 2\delta)t_2 - (1 + 0.1\delta)(1 + \delta)t_2 \geq 0.8\delta t_2 \).

The last inequality is by \( t_2 \geq \mu_1 \geq \frac{6 \ln(1/\delta)}{\delta^2} \).

We then bound how much the tail probability blows up when we scale each \( \mu_j \) by \( 1 + \delta \) and shifting \( r' \) to \( r \). Because of the similarity between the following and previous inequations, we skip some details. Similarly, by union bound this tail probability is at most \( \sum_{j=1}^m \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) \geq r \right] \), and as \( r \geq (1 + 8\delta)\mu_1 \geq (1 + 6\delta)(1 + \delta)\mu_1 \) it is bounded by a geometric series of ratio \( \frac{1}{1 + 6\delta} \).
Therefore
\[
\sum_{j=1}^{m} \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) \geq r \right] \leq \frac{1}{3 \delta} \sum_{j=1}^{m} \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) = r \right]
\]
\[
= \frac{1}{3 \delta} \sum_{j=1}^{m} \left( \prod_{k=r'+1}^{r} \frac{(1 + \delta)\mu_j}{k} \right) \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) = r' \right]
\]
\[
\leq \frac{1}{3 \delta} \sum_{j=1}^{m} \left( \frac{(1 + \delta)\mu_j}{r'} \right)^{5.8 \delta t_2} \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) = r' \right]
\]
\[
= \frac{1}{3 \delta} \sum_{j=1}^{m} \left( \frac{(1 + \delta)\mu_j}{r'} \right)^{5.8 \delta t_2} e^{-\delta \mu_j + \delta (1 + \delta) r'} \Pr \left[ \text{Poi} \left( \mu_j \right) = r' \right]
\]
\[
\leq \frac{1}{3 \delta} \sum_{j=1}^{m} \left( \frac{(1 + \delta)\mu_j}{r'} \right)^{5.8 \delta t_2} e^{(r' - \mu_j)} \Pr \left[ \text{Poi} \left( \mu_j \right) = r' \right].
\] (27)

Here the third line follows from that \( r - r' \geq (1 + 8 \delta) t_2 - (1 + 0.1 \delta)(1 + 2 \delta) t_2 \geq 5.8 \delta t_2 \).

To further analyze (27), we define \( \beta_j \geq 0 \) such that \( t_2 = (1 + \beta_j)\mu_j \). Then
\[
\left( \frac{(1 + \delta)\mu_j}{r'} \right)^{5.8 \delta t_2} \leq \left( \frac{(1 + \delta)\mu_j}{(1 + 2 \delta)(1 + \beta_j)\mu_j} \right)^{5.8 \delta (1 + \beta_j)\mu_j} \leq \left( \frac{1}{(1 + 0.9 \delta)(1 + \beta_j)} \right)^{5.8 (1 + \beta_j)\delta \mu_j}
\]
\[
\leq \left( \frac{1}{(1 + 2.5 \delta)(1 + \beta_j)} \right)^{1.9 (1 + \beta_j)\delta \mu_j} \leq \left( \frac{1}{(1 + 2.5 \delta)(1 + \beta_j)} \right)^{(1 + 2 \delta)(1 + \beta_j)\delta \mu_j}.
\] (28)

Here the second inequality follows from \( \delta < 1/10 \) and hence \( 1 + 2 \delta \geq (1 + 0.9 \delta)(1 + \delta) \). The third inequality follows from \( (1 + 0.9 \delta)^3 \geq 1 + 2.5 \delta \). The last inequality follows from \( 1.9 \geq 1 + 2.5 \delta \).

Furthermore, notice that \( x^x \geq e^{x-1} \) holds for \( x \geq 1 \), since \( x \ln x \geq x - 1 \) on \( [1, \infty) \) by comparing derivatives. Applying this to (28) gives
\[
\left( \frac{(1 + \delta)\mu_j}{r'} \right)^{5.8 \delta t_2} \leq \exp \left\{ - ((1 + 2.5 \delta)(1 + \beta_j) - 1) \mu_j \delta \right\} \leq \exp \left\{ - (r' - \mu_j) \delta \right\},
\] (29)

where the last inequality holds because \((1 + 2.5 \delta)(1 + \beta_j)\mu_j = (1 + 2.5 \delta) t_2 \geq \left[(1 + 2 \delta) t_2 \right] = r'\).

Inserting (29) into (27) gives
\[
\sum_{j=1}^{m} \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) \geq r \right] \leq \frac{1}{3 \delta} \sum_{j=1}^{m} \Pr \left[ \text{Poi} \left( \mu_j \right) = r' \right] \leq \frac{1}{3 \delta} \frac{\delta^3}{3} = \frac{\delta^2}{9},
\] (30)

where the last inequality is from (25).

Finally, by \( r \geq (1 + 8 \delta)\mu_1 \geq (1 + 6 \delta)(1 + \delta)\mu_1 \),
\[
\Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) \geq r \right] = \sum_{k=r}^{\infty} \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) = k \right] = \sum_{k=r}^{\infty} \frac{(1 + \delta)\mu_j}{k + 1} \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) = k + 1 \right]
\]
\[
\geq \sum_{k=r} \left( 1 + 6 \delta \right) \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \right) = k + 1 \right] = (1 + 6 \delta) \Pr \left[ \text{Poi} \left( (1 + \delta)\mu_j \geq r + 1 \right) \right].
\] (31)
Therefore by geometric series
\[
\sum_{k=r}^{\infty} \sum_{j=1}^{m} \Pr \left( \operatorname{Poi} \left( (1 + \delta) \mu_j \right) \geq k \right) \leq \frac{1}{\delta} \sum_{j=1}^{m} \Pr \left( \operatorname{Poi} \left( (1 + \delta) \mu_j \right) \geq r \right) \leq \frac{\delta}{2t}.
\] (32)

Thereby both claims in this lemma have been proved. \(\square\)

We then prove Statement 2 of Lemma 3.3.

**Proof of Lemma 3.3 Statement 2**. We do case analysis for \(t_2\).

1. \(t_2 \geq (1 + 2\delta)\mu_1\). It suffices to show that \( \Pr \left[ \max_{j=1}^{m} \operatorname{Poi} \left( \mu_j \right) \geq (1 - \delta) t_2 \right] \geq 1 - \epsilon \), since \( \ell \leq (1 - 4\delta)t_2 \leq (1 - \delta)t_2 \).

First we prove that the sum of tail probabilities at \((1 - \delta)t_2\) is large:
\[
\sum_{j=1}^{m} \Pr \left[ \operatorname{Poi} \left( \mu_j \right) \geq (1 - \delta) t_2 \right] = \sum_{k=\lceil (1 - \delta)t_2 \rceil}^{\infty} \sum_{j=1}^{m} \Pr \left[ \operatorname{Poi} \left( \mu_j \right) = k \right]
\geq \sum_{k=\lceil (1 - \delta)t_2 \rceil}^{\infty} \sum_{j=1}^{m} \left( \prod_{i=\lceil (1 - \delta)t_2 \rceil + 1}^{t_2} \frac{i}{\mu_j} \right) \Pr \left[ \operatorname{Poi} \left( \mu_j \right) = k + \lfloor \delta t_2 \rfloor \right]
\geq \left( \frac{(1 - \delta)t_2}{\mu_1} \right)^{0.9\delta t_2} \sum_{k=\lceil (1 - \delta)t_2 \rceil}^{\infty} \sum_{j=1}^{m} \Pr \left[ \operatorname{Poi} \left( \mu_j \right) = k + \lfloor \delta t_2 \rfloor \right]
\geq \frac{\delta^2 t_2}{2} \sum_{j=1}^{m} \Pr \left[ \operatorname{Poi} \left( \mu_j \right) \geq t_2 \right] \geq \frac{\delta^2 \mu_1}{6} \geq \ln \left( \frac{1}{\delta} \right).
\] (33)

Here the third line follows from that \((1 - \delta)t_2 \geq (1 - \delta)(1 + 2\delta)\mu_1 \geq (1 + 0.8\delta)\mu_1 \geq \mu_1\) and \([\delta t_2] \geq \delta t_2 - 1 \geq (1 - \delta)\delta t_2 \geq 0.9\delta t_2\). The first inequality on the fourth line holds because \((1 - \delta)t_2 \geq (1 + 0.8\delta)\mu_1\), and \((1 + 0.8\delta)^{0.9\delta t_2} \geq 1 + 0.72\delta^2 t_2\), by Bernoulli’s inequality. So we have \((1 + 0.8\delta)^{0.9\delta t_2} \geq 1 + 0.72\delta^2 t_2 \geq 0.72\delta^2 t_2 \geq \delta^2 t_2/2\).

With this, we bound the tail probability of the max at \((1 - \delta)t_2\) by
\[
\Pr \left[ \max_{j=1}^{m} \operatorname{Poi} \left( \mu_j \right) \geq (1 - \delta) t_2 \right] = 1 - \prod_{j=1}^{m} \left( 1 - \Pr \left[ \operatorname{Poi} \left( \mu_j \right) \geq (1 - \delta) t_2 \right] \right)
\geq 1 - \exp \left\{ - \sum_{j=1}^{m} \Pr \left[ \operatorname{Poi} \left( \mu_j \right) \geq (1 - \delta) t_2 \right] \right\} \geq 1 - \delta.
\] (34)

2. \(t_2 < (1 + 2\delta)\mu_1\). By Proposition 2.4 we have
\[
\Pr \left[ \operatorname{Poi} \left( \mu_1 \right) \geq (1 - \delta)\mu_1 \right] \geq 1 - e^{-\mu_1 \delta^2/2} \geq 1 - \delta^3.
\] (35)

Since \(\ell \leq (1 - 4\delta)t_2 \leq (1 - \delta)\mu_1\), we have
\[
\Pr \left[ \max_{j=1}^{m} \operatorname{Poi} \left( \mu_j \right) \geq \ell \right] \geq \Pr \left[ \operatorname{Poi} \left( \mu_1 \right) \geq \ell \right] \geq \Pr \left[ \operatorname{Poi} \left( \mu_1 \right) \geq (1 - \delta)\mu_1 \right] \geq 1 - \delta^3.
\] (36)

Combining these two cases finishes the proof. \(\square\)
3.3 Case 3

Lemma 3.7 (Case 3). Suppose \( \delta \in (0, 1/10) \), \( \frac{1}{2\mu} < \mu \leq \frac{1}{2^{1/3+1}} \log m \), and \( m \geq 2^{\frac{2}{3}\log \frac{2}{\delta}} \). Define transition point \( t_3 = t_3(m, \mu) = \frac{\log m}{\log \frac{1}{\mu} + \log \log m} \). Then for any random variable \( X \) on \( \mathbb{N} \) we have

\[
(1 - 4\delta) \max \{t_3, X\} \leq \max \{M(m, \mu), X\} \leq (1 + 14\delta) \max \{t_3, X\},
\]

and

\[
E[\max \{M(m, 4\mu), X\}] \leq (1 + 20\delta) E[\max \{M(m, \mu), X\}] \cdot
\]

Proof. Let \( t_3 = t_3(\mu) \) defined above and \( t'_3 = t_3(4\mu) = \frac{\log m}{\log \frac{1}{\mu} + 2\log \log m} \). Note that we have the identity

\[
\frac{\log m}{\log \frac{1}{\mu} + \log \log m} = \mu \cdot \frac{\log m^{1/\mu}}{\log \log m^{1/\mu}}.
\]

Let \( \ell = \lceil t_3 \rceil \) and \( r = \lceil (1 + 10\delta)t_3 \rceil \). Since \( \mu \leq \frac{1}{2^{1/3+1}} \log m \) we know that \( t_3 \leq \frac{\log m}{1/\delta + 1} \leq \delta \log m \).

Since \( m \geq 2^{\frac{2}{3}\log \frac{2}{\delta}} \), we have \( \log \log m \leq \delta \log m \). This coupled with \( \mu \geq \frac{1}{m^2} \) gives us \( t_3 \geq \frac{1}{2\mu} \).

Hence \( \ell \geq t_3 - 1 \geq (1 - 2\delta)t_3 \) and \( r \leq (1 + 10\delta)t_3 + 1 \leq (1 + 12\delta)t_3 \).

By \( \mu \leq \frac{1}{2^{1/3+1}} \log m \) we have \( t'_3 \leq (1 + 1.5\delta)t_3 \). By the range of \( \mu \) we can conclude that \( \ell > 8\mu \). Then we can write

\[
\Pr[\text{Poi}(\mu) = \ell] = e^{-\mu} \frac{\mu^\ell}{\ell!} \geq e^{-\mu+\ell-1} \cdot \frac{\mu^\ell}{\ell!} \geq e^{-\mu+t_3-2} \cdot \left( \frac{\mu}{t_3} \right)^{t_3} \geq \frac{1}{m} \cdot \frac{1}{\log \log m^{1/\mu}}.
\]

Here the first inequality on the first line follows from Stirling (Proposition 2.5). The first equality on the second line follows from (39). The last inequality follows from the ranges of \( \mu \) and \( m \).

Also we have

\[
\Pr[\text{Poi}(4\mu) = r] = e^{-4\mu} \frac{(4\mu)^r}{r!} \leq e^{-4\mu+r-1} \frac{(4\mu)^r}{r^r} \leq e^{-4\mu+r-1} \left( \frac{4\mu}{(1 + 10\delta)t_3} \right)^{(1 + 10\delta)t_3}
\]

\[
\leq e^{-4\mu+r-1} \left( \frac{\mu}{t_3} \right)^{(1 + 10\delta)t'_3} \geq e^{-4\mu+r-1} \frac{1}{m} \left( 1 - \frac{\log \log m^{1/4\mu}}{\log \log m^{1/4\mu}} \right)^{(1 + 8\delta)}
\]

\[
\leq e^{-4\mu+r-1} \frac{1}{m^{1+6\delta}} \leq \frac{m^{4\delta}}{m^{1+6\delta}} = \frac{1}{m^{1+2\delta}}.
\]

Here the first inequality on the first line follows from Stirling. The first inequality on the second line follows from \((1 + 10\delta)t_3 \geq (1 + 8\delta)(1 + 1.5\delta)t_3 \geq (1 + 8\delta)t'_3 \). The equality on the second line follows from (39). The first inequality on the third line follows from \( \mu \leq \frac{1}{2^{1/3+1}} \log m \). The second inequality on the third line follows from that \( r \leq 2t_3 + 2\delta \log m \).
Now we can bound the tail probability of the max at $\ell$ from below by

$$
\Pr \left[ \max_{j=1}^m \text{Poi} (\mu) \geq \ell \right] = 1 - \left( 1 - \Pr \left[ \text{Poi} (\mu) \leq \ell \right] \right)^m \geq 1 - \left( 1 - \Pr \left[ \text{Poi} (\mu) = 1 \right] \right)^m
$$

$$
\geq 1 - \left( 1 - \frac{1}{m} \right)^m \geq 1 - \exp \left\{ -m \frac{\log \log \log m^{1/\mu}}{\log \log m^{1/\mu}} \right\}
$$

$$
\geq 1 - \delta
$$

(42)

where the last inequality is by $\mu \geq \frac{1}{m^3} > \frac{\log m}{m}$ and thus $m^{\frac{\log \log \log m^{1/\mu}}{\log \log m^{1/\mu}}} > \log m > \ln(1/\delta)$.

We can also bound the tail probability at $r$ from above by union bound and geometric series:

$$
\Pr \left[ \max_{j=1}^m \text{Poi} (4\mu) \geq r \right] \leq 2m \Pr \left[ \text{Poi} (4\mu) = r \right] \leq \frac{2}{m^{2\delta}} \leq \delta.
$$

(43)

Since $r > t_3 > 8\mu$, once again by geometric series and union bound we have

$$
\sum_{k=r}^{\infty} \Pr \left[ \max_{j=1}^m \text{Poi} (4\mu) \geq k \right] \leq 4m \Pr \left[ \text{Poi} (4\mu) = r \right] \leq 2\delta.
$$

(44)

Applying these two bounds (42) and (44) to (11) finishes the proof.

3.4 Case 4

Lemma 3.8 (Case 4). Suppose $\delta \in (0, 1/10)$, $\frac{4 \log m}{m} < \mu \leq \frac{2}{m^2}$, $m \geq 2^{100/\delta^2}$, and $\mu_1, \ldots, \mu_m \in [\mu/4, 4\mu]$. Define transition point $t_4 = t_4(m, \mu) = \left\lceil \gamma_4(m, \mu) \right\rceil$ where $\gamma_4(m, \mu) = \frac{\log m}{\log m + \log \log m}$.

Let $W$ be a Bernoulli random variable taking values on $t_4 - 1$ and $t_4$ where $\Pr \left[ W = t_4 - 1 \right] = \prod_{j=1}^m \Pr \left[ \text{Poi} (\mu_j) \leq t_4 - 1 \right]$. Then for any random variable $X$ on $\mathbb{N}$ we have

$$
(1 - 5\delta)E \left[ \max \{ W, X \} \right] \leq E \left[ \max \left\{ \max_{j=1}^m \text{Poi} (\mu_j), X \right\} \right] \leq (1 + 16\delta)E \left[ \max \{ W, X \} \right],
$$

(45)

and

$$
E \left[ \max \left\{ \max_{j=1}^m \text{Poi} ((1 + \delta)\mu_j), X \right\} \right] \leq (1 + 16\delta)E \left[ \max \left\{ \max_{j=1}^m \text{Poi} (\mu_j), X \right\} \right].
$$

(46)

Proof. Since $\mu > \frac{4 \log m}{m}$, we have $\gamma_4(\mu) > 1$ and $t_4 \geq 2$. Since $\mu \leq \frac{2}{m^2}$, we have $\gamma_4(\mu) \leq 1/\delta$ and $t_4 \leq 1/\delta + 1$. We first lower bound the probability that $\text{Poi} (\mu/4)$ equals $t_4 - 1$:

$$
\Pr \left[ \text{Poi} (\mu/4) = t_4 - 1 \right] = e^{-\mu/4} \frac{(\mu/4)^{t_4-1}}{(t_4-1)!} \geq e^{-\mu/4 + t_4 - 2} \frac{1}{t_4 - 1} \geq e^{-\mu/4 + \gamma_4 - 2} \frac{(\mu/4)^{t_4-1}}{\gamma_4} \geq \frac{1}{m^{\frac{1}{\log \log m^{1/\mu}}}} \geq \frac{1}{m^{\frac{1}{2 \log \log m^{1/\mu}}}}
$$

(47)
Here the first inequality on the first line follows from Stirling (Proposition 2.5). The first equality on the second line follows from \( \frac{\log m}{\log \frac{m}{\mu} + \log \log m} = \frac{\mu}{2} \cdot \frac{\log m^{2/\mu}}{\log \log m^{2/\mu}}. \) The last inequality holds since 
\[
\mu/4 > \frac{\log m}{m}
\]
and therefore \( m^{2/\mu} > \log^{1/2} m \).

We then upper bound the probability that \( \text{Poi}(\mu/2) \) equals \( t_4 + 1 \):
\[
\Pr[\text{Poi}(\mu/4) = t_4 + 1] = e^{-\mu/4} \left( \frac{\mu/4}{t_4 + 1} \right)^{t_4 + 1} \leq e^{-\mu/4 + t_4} \left( \frac{\mu/4}{t_4 + 1} \right)^{t_4 + 1} \leq e^{-\mu/4 + t_4} \left( \frac{\mu/4}{\gamma_4} \right)^{\gamma_4 + 1} \leq e^{-\mu/4 + t_4} \frac{1}{m^{1+\delta/2}} \leq \frac{1}{m^{1+\delta/4}}.
\] (48)

Here the first inequality on the first line follows from Stirling. The first inequality on the second line follows from \( \gamma_4 \leq 1/\delta. \) The equality on the second line follows from \( \frac{\log m}{\log \frac{m}{\mu} + \log \log m} = \frac{\mu}{2} \cdot \frac{\log m^{2/\mu}}{\log \log m^{2/\mu}}. \) The first inequality on the third line follows from \( \mu \leq \frac{2}{m^{\delta}} < 1. \) The second inequality on the third line follows from that \( t_4 \leq 1/\delta + 1 \) and \( m^{\delta/4} \geq 2^{25/\delta}. \)

Now by the definition of Poisson probability function we have
\[
\Pr[\text{Poi}(4\mu) = t_4 + 1] \leq 16^{t_4 + 1} \cdot \Pr[\text{Poi}(\mu/4) = t_4 + 1] \leq \frac{1}{m^{1+\delta/8}}
\] (49)
where the last inequality again follows from \( t_4 \leq 1/\delta + 1 \) and \( m^{\delta/8} \geq 2^{12/\delta}. \)

To prove concentration, first note that
\[
\Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq t_4 - 1 \right] \geq \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu/4) \geq t_4 - 1 \right] = 1 - (1 - \Pr[\text{Poi}(\mu/4) \geq t_4 - 1])^m \geq 1 - (1 - \Pr[\text{Poi}(\mu/4) = t_4 - 1])^m \geq 1 - \exp \left\{ -m \left( \frac{\log \log m^{4/\mu}}{2 \log \log m^{4/\mu}} \right) \right\} \geq 1 - \delta
\] (50)
where the last inequality holds since \( \mu/4 > \frac{\log m}{m} \) and therefore \( m^{2/\mu} > \log^{1/2} m \).

Then also note that
\[
\Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq t_4 + 1 \right] \leq \Pr \left[ \max_{j=1}^{m} \text{Poi}(4\mu) \geq t_4 + 1 \right] \leq 2m \Pr[\text{Poi}(4\mu) = t_4 + 1] \leq \frac{2}{m^{\delta/8}} < \delta
\]
where the second inequality follows from union bound and geometric series. Then since \( t_4 \geq 2 > 10\mu, \) once again by geometric series and union bound we have
\[
\sum_{k=t_4+1}^{\infty} \Pr \left[ \max_{j=1}^{m} \text{Poi}(\mu_j) \geq k \right] \leq 4m \Pr[\text{Poi}(4\mu) = t_4 + 1] < 2\delta.
\] (51)
These two bounds (50) and (51) in conjunction with (7) proves concentration.
To prove the scaling result, consider the function $f(x) = 1 - \prod_{j=1}^{m} (1 - a_j x)$ on $[0, (\max_{j=1}^{m} a_j)^{-1}]$, where all $a_j \geq 0$. The derivative of $f$ is nonnegative and decreasing, which implies that $f(\alpha x) \leq \alpha f(x)$ for $\alpha \geq 1$. With this, we have

$$1 - \prod_{j=1}^{m} (1 - \Pr[\text{Poi}((1 + \delta)\mu_j) = t_4]) \leq (1 + \delta)^{t_4} \left(1 - \prod_{j=1}^{m} \frac{1 - \Pr[\text{Poi}((1 + \delta)\mu_j) = t_4]}{(1 + \delta)^{t_4}}\right)$$

$$\leq (1 + \delta)^{t_4} \left(1 - \prod_{j=1}^{m} (1 - \Pr[\text{Poi}(\mu_j) = t_4])\right) \leq (1 + 10t_4\delta) \left(1 - \prod_{j=1}^{m} (1 - \Pr[\text{Poi}(\mu_j) = t_4])\right)$$

where the last inequality is by that $t_4 \leq 1/\delta + 1$ and thus $(1 + \delta)^{t_4} \leq e^{1+\delta} < 10$. □

### 3.5 Case 5

**Lemma 3.9 (Case 5).** Suppose $\delta \in (0, 1/10]$, $\mu \leq \frac{8 \log m}{m}$, $m \geq 1000(1/\delta) \log^2(1/\delta)$, and $\forall j, \mu_j \in [\mu/4, 4\mu]$. Let $W$ be a 0/1 Bernoulli random variable with expectation $1 - \prod_{j=1}^{m} \Pr[\text{Poi}(\mu_j) = 0] = 1 - e^{-m\mu}$. Then for any random variable $X$ on $\mathbb{N}$ we have

$$E[\max\{W, X\}] \leq E\left[\max_{j=1}^{m} \text{Poi}(\mu_j)\right] \leq (1 + 10\delta)E[\max\{W, X\}], \quad (52)$$

and

$$E\left[\max\left\{\max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j), X\right\}\right] \leq (1 + 10\delta)E\left[\max\left\{\max_{j=1}^{m} \text{Poi}(\mu_j), X\right\}\right]. \quad (53)$$

**Proof.** First we note that

$$\Pr\left[\max_{j=1}^{m} \text{Poi}(\mu_j) \geq 1\right] = 1 - \prod_{j=1}^{m} \Pr[\text{Poi}(\mu_j) = 0] = 1 - e^{-m\mu} \geq m\mu.$$

$$\Pr\left[\max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j) \geq 1\right] = 1 - \prod_{j=1}^{m} \Pr[\text{Poi}((1 + \delta)\mu_j) = 0] = 1 - e^{-\mu(1 + \delta)^{m\mu}} \geq m\mu.$$

Since the derivative of $1 - e^{-x}$ with respect to $x$ is decreasing on $[0, \infty)$, we have

$$1 - e^{-\mu(1 + \delta)^{m\mu}} \leq (1 + \delta)(1 - e^{-m\mu}). \quad (54)$$

By union bound, geometric series, and that $\mu_j \leq 4\mu$ for all $j$ we have

$$\Pr\left[\max_{j=1}^{m} \text{Poi}(\mu_j) \geq 2\right] \leq \Pr\left[\max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j) \geq 2\right] \leq 10m(1 + \delta)^2\mu^2 \leq \frac{640 \log^2 m}{m}.$$ 

Again by geometric series, we have

$$\sum_{k=2}^{\infty} \Pr\left[\max_{j=1}^{m} \text{Poi}((1 + \delta)\mu_j) \geq k\right] \leq 15m(1 + \delta)^2\mu^2 \leq \frac{960 \log^2 m}{m} \quad (55)$$

where the last expression is at most $\delta$ by the condition on $m$. Now by (7) we have desired results. □
4 An Efficient Polynomial-time Approximation Scheme

Our PTAS for stochastic load balancing is heavily inspired by the approach of [AAWY98, JKV16] for the deterministic load balancing problem. Thus, we first begin with a recap of their approach.

4.1 Recap of the PTAS for deterministic load balancing

Consider any instance of deterministic load balancing where the job sizes are \( \{\lambda_i\}_{i=1}^n \) and we have \( m \) machines – the goal is to find an assignment with smallest maximum load. The algorithms in [AAWY98, JKV16] proceed in two phases: In phase I, we assign “big” jobs to separate machines. Here big jobs are the maximal set of jobs whose size is greater than the remaining average load. In other words, it is the maximal set \( B \subseteq [n] \) satisfying that \( \forall i \in B, \lambda_i > (\sum_{i \in B} \lambda_i)/(m - |B|) := \mu. \) Exploiting the convexity of the objective function (i.e. the function \( \max \{x_1, x_2, \ldots, x_m\} \)), [AAWY98, JKV16] show that an optimum assignment (i) assigns the jobs in \( B \) to their own separate machines; (ii) assigns the remaining jobs to the remaining machines in a way such that each of these machines have a load between \( \mu/2 \) and \( 2\mu. \)

With this, we are only left with the problem of assigning the small jobs, i.e., the jobs not in \( B. \) A second key step here is to round the sizes of the remaining jobs, such that (i) the number of different job sizes is now \( O(1/\epsilon) \) and (ii) the potential number of different assignments to any single machine is \( 2^{O(1/\epsilon)}. \) Crucially, both these numbers are just dependent on the target error parameter \( \epsilon. \) With this rounding, [AAWY98, JKV16] formulate the problem of finding an optimal assignment on the remaining (rounded) jobs as an integer linear program with \( 2^{O(1/\epsilon)} \) variables – referred to as a configuration-IP. The configuration-IP can be solved in time exponential in the number of variables and linear in the input length using algorithms in [Len83, Kan87].

**Theorem 4.1** ([Kan87]). There is an algorithm ILP that solves an integer linear program with \( p \) variables in time \( p^{O(p)}O(n) \) where \( n \) is the length of input.

This leads to an overall running time of \( 2^{O(1/\epsilon)} + O(n \log n) \) [AAWY98], where the running time \( O(n \log n) \) comes from the pre-processing step of sorting the job sizes (to find the big jobs).

**Theorem 4.2** ([AAWY98]). There is an algorithm DetScheduling that given an instance of the load balancing problem with \( n \) jobs and \( m \) machines where the jobs have deterministic sizes \( \lambda_1, \lambda_2, \ldots, \lambda_n, \) and a parameter \( 0 < \epsilon < 1, \) outputs a job assignment whose maximum load is at most \( (1 + \epsilon) \) of the maximum load of an optimum assignment. The algorithm runs in time \( 2^{O(1/\epsilon)} + O(n \log n) \).

The authors in [JKV16] then use a sparsification technique to show that the configuration-IP has an optimum solution with a small support size, which leads to an improved running time of \( 2^{O(1/\epsilon)\log^2(1/\epsilon)} + O(n \log n). \) Later by improving the runtime of solving the ILP, a running time of \( 2^{O(1/\epsilon)\log^2(1/\epsilon)} + O(n \log n) \) was achieved [JR19].

\(^4\)Recall that \( \tilde{O}(f) \) denotes \( O(f \log^c f) \) for some constant \( c. \)
4.2 Overview of our approach

We now give an overview of our approach for the stochastic load balancing problem. Recall that we have \(n\) jobs and \(m\) machines where the \(i^{th}\) job has size \(\text{Poi}(\lambda_i)\). Similar to the deterministic case [AAWY98, JKV16], we define “big” jobs as the maximal set of jobs whose (expected) size is greater than the remaining (expected) average load, i.e. the maximal set \(B \subseteq [n]\) satisfying that \(\forall i \in B\)

\[
\lambda_i > \frac{\sum_{i \in B} \lambda_i}{m - |B|}.
\]

By Proposition 2.2, the objective function is convex with respect to the machine loads (similar to [AAWY98, JKV16]). Thus, we assign the jobs in \(B\) to separate machines (see Lines 2-5 of Algorithm 2).

Assigning the small jobs (i.e., the jobs outside \(B\)) is however somewhat more complicated. As stated in Section 1.1, there are two principal difficulties in applying the approaches from [AAWY98, JKV16] to handling the remaining jobs. One is to discretize the job sizes, and the other is to formulate the problem of minimizing the expected maximum load as an integer linear program. The key to circumventing both these difficulties lies in the (technical) results on concentration and scaling of maximum of Poisson random variables presented in Section 3. To understand their role, let us begin with some notation. Let \(m^{(1)} = m - |B|\) denote the number of the remaining machines. Consider an assignment of the remaining jobs and let \(\text{Poi}(\mu_1), \text{Poi}(\mu_2), \ldots, \text{Poi}(\mu_{m^{(1)}})\) be the corresponding distributions of the machine loads. Suppose \(\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{m^{(1)}}\) and let \(\mu = (\sum_{j=1}^{m^{(1)}} \mu_j) / m^{(1)}\). By an argument similar to the deterministic case [AAWY98, JKV16] (see Observation 4.10), we can restrict ourselves to the case when \(\mu_j \in [\mu/2, 2\mu]\) for all \(j \in [m^{(1)}]\). Let \(\delta \in (0,1)\) be a target error parameter (roughly speaking, we set \(\delta \approx \Theta(\epsilon)\)).

Our results in Section 3 first imply that when \(\mu\) is sufficiently large in terms of \(1/\delta\) and \(m^{(1)}\) (the condition of Lemma 3.1), or when \(\mu\) is in a certain intermediate range but \(m^{(1)}\) is large enough in terms of \(1/\delta\) (the condition of Lemma 3.7), it suffices to find an assignment by running the deterministic load balancing algorithm in Theorem 4.2. When \(\mu\) does not satisfy the above conditions but \(m^{(1)}\) is sufficiently large in terms of \(1/\delta\), we have the following dichotomy:

1. \(\max_{j=1}^{m^{(1)}} \text{Poi}(\mu_j)\) is concentrated within \((1\pm O(\delta))\) of a certain transition point \(t = t(\mu_1, \ldots, \mu_{m^{(1)}})\) (Lemma 3.2), or
2. takes value \([t] - 1\) or \([t]\) with high probability, where the transition point \(t = t(m^{(1)}, \mu)\) only depends on \(m^{(1)}\) and \(\mu\) (Lemmas 3.8 and 3.9).

Further, in all of the cases,

\[
\mathbb{E}\left[\max_{j=1}^{m^{(1)}} \text{Poi}((1+\delta)\mu_j)\right] \leq (1 + O(\delta))\mathbb{E}\left[\max_{j=1}^{m^{(1)}} \text{Poi}(\mu_j)\right]
\]

(Lemmas 3.2, 3.8, and 3.9). Finally, when \(m\) is sufficiently small in terms of \(\delta\), we can use dynamic programming to get an efficient PTAS.

Let us now see how the above structural results are useful in algorithm design – first of all, immediately allows us to discretize the job sizes by rounding up their means to the nearest
integer power of $(1 + \delta)$, with proper handling of jobs with size below a certain threshold\footnote{The specific rounding scheme we use is identical to the one used in [JKV16].}. Once the job sizes are rounded, the existence of the transition points (rather, the precise definitions of these transition points in Section 3) can be used to construct integer linear programs which can find a near optimal solution. As an example, if we are in Case 4 as defined above, by Lemma 3.8 minimizing the expected maximum is the same as finding $\mu_1, \mu_2, \ldots, \mu_{m(1)}$ such that the probability that the maximum load is $t_4$ is minimized. This finishes our overview of the PTAS.

### 4.3 Our PTAS

We give a full description of our efficient PTAS in Algorithm 2 as PoiScheduling, which calls ROUNDED (Algorithm 1), ILPSCHEDULING (Algorithm 4), and DPSCHEDULING (Algorithm 3) as subroutines.

PoiScheduling first handles the “big” jobs in the same way as deterministic case (Lines 2-5 of Algorithm 2). Let $\mu$ still denote the remaining average machine load (i.e. the RHS of (56)) and $m(1)$ denote the number of the remaining machines. PoiScheduling then does one of the following for the remaining jobs:

1. When $\mu$ is large enough to meet the condition of Case 1 or $\mu$ and $m(1)$ meet the condition of Case 3, PoiScheduling directly uses the algorithm for the deterministic case as a blackbox (Lines 7-9 of Algorithm 2).  
2. When $\mu$ and $m(1)$ meet the condition of Case 2, Case 4, or Case 5, PoiScheduling first rounds the sizes of the remaining jobs in the same way as [JKV16] (Line 6 of Algorithm 2). Then for Case 2 it uses integer linear programming in conjunction with a binary search to find the smallest transition point $t_2$ achievable by an assignment of the remaining jobs, where the ILPs have linear objective functions and configuration-IP from [AAWY98, JKV16] as feasibility constraints (Lines 10-12 of Algorithm 2). Case 4 and Case 5 are also handled by integer linear programs (Lines 13-15 and Lines 16-17 of Algorithm 2 respectively).

3. When none of the conditions of Case 1 - Case 5 is met, namely $m(1) \leq 2^{O(1/\epsilon)}$ and $\mu \leq O(\epsilon^{-2} \log m)$, PoiScheduling finds an assignment by dynamic programming (Line 19 of Algorithm 2).

We then briefly state how the subroutines called by PoiScheduling function. Roughly speaking, ROUNDED takes a multi-set of job sizes and a parameter $\delta$ as input and outputs a multi-set of job sizes such that the number of different job sizes only depends on $\delta$. ILPSCHEDULING takes the jobs and the number of machines $m$, and a function $f$ as input. The goal of ILPSCHEDULING is to find an assignment with loads $\mu_1, \ldots, \mu_m$ such that $\sum_{j=1}^{m} f(\mu_j)$ is minimized. DPSCHEDULING takes the jobs and number of machines, and an error parameter $\epsilon$ as input and returns an assignment of jobs with at most $(1 + \epsilon)$ error with respect to an optimum assignment by a dynamic programming.

By using the algorithm in [Kan87] (Theorem 4.1) to solve the integer linear programs, PoiScheduling achieves a running time double exponential in $1/\epsilon$ and nearly-linear in $n$.  

5The specific rounding scheme we use is identical to the one used in [JKV16].
Note that although the algorithm in Theorem 4.1 needs integral coefficients, we can compute the coefficients with a high precision (inverse polynomial precision) which is sufficient to get our desired approximation and does not affect our running time, so we don’t get into the details. We also note that while it is possible to use the sparsification technique in [JKV16] to improve our running time of solving integer linear programs to single exponential in $1/\epsilon$, our dynamic program for the case when $1/\epsilon \geq \Omega(\log \log m)$ still takes time double exponential in $1/\epsilon$.

The performance of PoIScheduling is characterized in Theorem 1.1. The performances of ROUNDED and DPScheduling are characterized in Lemmas 4.3 and 4.4 respectively. We do not give a separate lemma for ILPScheduling but analyze it in our proofs directly.

Note that Lemma 4.3 below only gives guarantees for how the job sizes and individual machine loads change after rounding; the lemma itself does not make assertions about the expected maximum load. Instead, guarantees for the latter will follow from our scaling results in Section 3.

**Algorithm 1: ROUNDED($\{\lambda_i\}_{i=1}^n, \mu, \delta$)**

**Input**: Job sizes $\{\lambda_i\}_{i=1}^n$, $\mu$ s.t. all $\lambda_i \leq \mu$, and $\delta \in (0, 1)$

**Output**: Rouned job sizes $\{\lambda_i'\}_{i=1}^{n'}$

1. $n' \leftarrow 0$ and $S \leftarrow 0$
2. for $i \leftarrow 1$ to $n$
3. if $\lambda_i \geq \delta \mu$ then
4. $n' \leftarrow n' + 1$
5. $\nu \leftarrow (1 + \delta)^k \delta \mu$ where $k$ is the unique integer s.t. $(1 + \delta)^{k-1} \delta \mu < \lambda_i \leq (1 + \delta)^k \delta \mu$.
6. $\lambda_{n'} \leftarrow l \delta^2 \mu$ where $l$ is the unique integer s.t. $(l-1)\delta^2 \mu < \nu \leq l \delta^2 \mu$.
7. else
8. $S \leftarrow S + \lambda_i$.
9. $S' \leftarrow k \delta \mu$ where $k$ is the unique integer s.t. $(k-1)\delta \mu < S \leq k \delta \mu$.
10. $\lambda_i' \leftarrow \delta \mu$ for each $i = n' + 1, \ldots, n' + S'/\delta \mu$ and $n' \leftarrow n' + S'/\delta \mu$.
11. return $\{\lambda_i'\}_{i=1}^{n'}$

**Lemma 4.3 ([JKV16]).** The algorithm $\{\lambda_i'\}_{i=1}^{n'} = \text{ROUNDED}(\{\lambda_i\}_{i=1}^n, \mu, \delta)$ runs in time $O(n)$. Suppose all $\lambda_i \leq \mu$, $\delta \in (0, 1)$, and $\mu = (\sum_{i=1}^n \lambda_i)/m$ for an integer $m$. The number of different sizes in $\{\lambda_i'\}_{i=1}^{n'}$ is bounded by $O(1/\delta \log 1/\delta)$, and each $\lambda_i' = \delta \mu + k \delta^2 \mu$ for some $k \in \mathbb{Z}_{\geq 0}, k \leq 2/\delta$. $n'$ is bounded by $O(m/\delta)$. For any assignment of $\{\lambda_i\}_{i=1}^n$ to $m$ machines with loads $\mu_1, \mu_2, \ldots, \mu_m$, there is an assignment of $\{\lambda_i'\}_{i=1}^{n'}$ to $m$ machines with loads $\mu_1', \mu_2', \ldots, \mu_m'$ such that all $\mu_j' \leq (1 + 5\delta)\mu_j$. Conversely, for any assignment $\{\lambda_i'\}_{i=1}^{n'}$ to $m$ machines with loads $\mu_1', \mu_2', \ldots, \mu_m'$, there is an assignment of $\{\lambda_i\}_{i=1}^n$ to $m$ machines with loads $\mu_1, \mu_2, \ldots, \mu_m$ such that all $\mu_j \leq (1 + 5\delta)\mu_j'$, and the latter assignment can be found in $O(n)$ time given the former assignment if both $\{\lambda_i\}_{i=1}^n$ and $\{\lambda_i'\}_{i=1}^{n'}$ are sorted.
Algorithm 2: POIScheduling(n, m, \{\lambda_i\}_{i=1}^n, \epsilon)

Input: Number of jobs n, number of machines m, job sizes \{\lambda_i\}_{i=1}^n, and \epsilon \in (0, 1)

Output: A job assignment \phi: [n] \rightarrow [m]

1. \mu \leftarrow (\sum_{i=1}^n \lambda_i) / m, and sort \{\lambda_i\}_{i=1}^n such that \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.
2. n^{(1)} \leftarrow n and m^{(1)} \leftarrow m.
3. while \lambda_{n^{(1)}} > \mu do
   4. Assign the job with size \lambda_{n^{(1)}} to the empty machine m^{(1)}: \phi(n^{(1)}) \leftarrow m^{(1)}.
   5. \mu \leftarrow \frac{m^{(i)} \mu - \lambda_{n^{(i)}}}{m^{(i)} - 1}, n^{(1)} \leftarrow n^{(1)} - 1, and m^{(1)} \leftarrow m^{(1)} - 1.
   6. \delta \leftarrow \frac{\epsilon}{1000} and \{\lambda'_i\}_{i=1}^n \leftarrow \text{Rounded}(\{\lambda_i\}_{i=1}^n, \mu, \delta).
   7. if \mu > \frac{1}{\delta} \log m^{(1)} or \left(\frac{1}{(m^{(1)})^2} < \mu \leq \frac{1}{2(n^{(1)})^2} \log m^{(1)} \text{ and } m^{(1)} \geq 2^{2/\delta}\right) then
      // Case 1, Case 3
      Create a new instance with m^{(1)} machines and deterministic job sizes \{\lambda'_i\}_{i=1}^n.
      Run DETScheduling in Theorem 4.2 to find a (1 + \frac{\epsilon}{\delta})-optimum assignment 
      \phi':\{n^{(1)}\} \rightarrow [m^{(1)}].
   8. else if \frac{1}{(m^{(1)})^2} \log m^{(1)} < \mu \leq \frac{6}{\delta^2} \log m^{(1)} and m^{(1)} \geq 2^{2/\delta} then
      // Case 2
      Use binary search to find the smallest t_2 \in [\mu, 100\mu \log m^{(1)}] s.t. there is an 
      assignment of \{\lambda'_i\}_{i=1}^n with loads \{\mu_j\}_{j=1}^{m^{(1)}} s.t. \sum_{j=1}^{m^{(1)}} \Pr[\text{Poi}(\mu_j) > t_2] < \frac{1}{3}; this is by 
      letting (\text{opt}, \phi') \leftarrow \text{ILPScheduling}(\{\lambda'_i\}_{i=1}^n, m^{(1)}, x \rightarrow \Pr[\text{Poi}(x) > t_2]) for each 
      guess of t_2 and checking if \text{opt} < \frac{1}{3}.
   9. else if \frac{4\log m^{(1)}}{m^{(1)}} < \mu \leq \frac{1}{(m^{(1)})^2} and m^{(1)} \geq 2^{100/\delta^2} then
      // Case 4
      t_4 \leftarrow \left\lfloor \frac{\log m^{(1)}}{\log 2 + \log \log m^{(1)}} \right\rfloor.
      Find an assignment of \{\lambda'_i\}_{i=1}^n with loads \{\mu_j\}_{j=1}^{m^{(1)}} s.t. \prod_{j=1}^{m^{(1)}} \Pr[\text{Poi}(\mu_j) < t_4] is 
      maximized, by letting 
      (\text{opt}, \phi') \leftarrow \text{ILPScheduling}(\{\lambda'_i\}_{i=1}^n, m^{(1)}, x \rightarrow -\ln(\Pr[\text{Poi}(x) < t_4])).
   10. else if \mu \leq \frac{4\log m^{(1)}}{m^{(1)}} and m^{(1)} \geq 1000(1/\delta) \log^2(1/\delta) then
       // Case 5
       Find an assignment of \{\lambda'_i\}_{i=1}^n with loads \{\mu_j\}_{j=1}^{m^{(1)}} s.t. \prod_{j=1}^{m^{(1)}} \Pr[\text{Poi}(\mu_j) = 0] is 
       maximized, by letting 
       (\text{opt}, \phi') \leftarrow \text{ILPScheduling}(\{\lambda'_i\}_{i=1}^n, m^{(1)}, x \rightarrow -\ln(\Pr[\text{Poi}(x) = 0])).
   11. else
       \phi' \leftarrow \text{DPScheduling}(\{\lambda_i\}_{i=n^{(1)}+1}^n, \{\lambda'_i\}_{i=1}^n, m, \epsilon).
   12. If \phi' is an assignment of the rounded jobs \{\lambda'_i\}_{i=1}^n, use Lemma 4.3 to covert it to an 
    assignment of \{\lambda_i\}_{i=1}^n such every machine’s load overflows by no more than (1 + 5\delta).
   13. Return the assignment \phi \cup \phi'.

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Lemma 4.4. Given jobs \( \{\nu_i\}_{i=1}^{n(0)}, \{\lambda_i\}_{i=1}^{n(1)} \), number of machines \( m \), and \( \epsilon \in (0, 1) \). Let \( m^{(1)} = m - n^{(0)} \), \( \mu = (\sum_{i=1}^{n(1)} \lambda_i)/m^{(1)} \), and \( \delta = \max \left\{ \frac{\epsilon}{10000m^{(1)}}, 2^{-10^9/\epsilon^2} \right\} \). Suppose all \( \lambda_i \leq \mu \), all \( \nu_i > \mu \), and \( \mu \leq 6000000\epsilon^{-2} \log m^{(1)} \). Then \( \text{DPScheduling}(\{\nu_i\}_{i=1}^{n(0)}, \{\lambda_i\}_{i=1}^{n(1)}, m, \epsilon) \) finds in \( O(n^{(0)} \epsilon^{-4} \log^2 m^{(1)}) + (m^{(1)}/\delta)^{O(\frac{1}{\log \frac{1}{\delta}})} \) time an assignment with expected maximum load \( L \leq (1 + \epsilon)L^* \), where \( L^* \) is the expected maximum load of an optimum assignment.
Algorithm 4: ILPScheduling($\{\lambda_i\}_{i=1}^n, m, f$)

**Input**: Job sizes $\{\lambda_i\}_{i=1}^n$, number of machines $m$, and a function $f : \mathbb{R} \to \mathbb{R}$

**Output**: An optimum value $opt$ and a job assignment $\phi : [n] \to [m]$

1. $\mu \leftarrow (\sum_{i=1}^n \lambda_i) / m$.
2. Let $\pi_1 < \pi_2 < \ldots < \pi_d$ be all different sizes in $\{\lambda_i\}_{i=1}^n$ and $\bar{\pi} \leftarrow (\pi_1, \pi_2, \ldots, \pi_d)^T$.
3. Let $n_k$ be the number of jobs with size $\pi_k$ and $\bar{n} \leftarrow (n_1, n_2, \ldots, n_d)^T$ (so $\sum_{k=1}^d n_k = n$).
4. $Q \leftarrow \{\bar{c} \in \mathbb{Z}_{\geq 0}^d : \bar{c}^T \bar{\pi} \leq 4\mu\}$ (the set of possible assignments to a single machine).
5. Solve the following integer linear programming using Theorem 4.1

$$\begin{align*}
\min \sum_{\bar{c} \in Q} x_{\bar{c}} f(\bar{c}^T \bar{\pi}) \\
\text{s.t.} \sum_{\bar{c} \in Q} x_{\bar{c}} &= m \\
\sum_{\bar{c} \in Q} x_{\bar{c}} \bar{c} &= \bar{n} \\
x_{\bar{c}} &\in \mathbb{Z}_{\geq 0}, \forall \bar{c} \in Q.
\end{align*}$$ \hspace{1cm} (59)

6. Return the optimum value of the ILP and an assignment $\phi$ extracted from its solution.

4.4 Proof of Theorem 1.1

We first use Lemma 4.4 and the lemma below to prove Theorem 1.1, and then give detailed proofs of these two lemmas.

The following lemma shows that if Algorithm 2 does not use dynamic programming to find an assignment, then we have a good approximation.

**Lemma 4.5.** If Algorithm 2 returns an assignment without going into Line 19, then $L \leq (1 + \epsilon)L^*$, where $L, L^*$ is the expected maximum load of the assignment returned by Algorithm 2 and the optimum assignment respectively.

Roughly speaking, the proof idea behind Lemma 4.5 is as follows. For Case 1 and Case 3, where we do not use the rounded jobs, using the guarantee of the deterministic load balancing algorithm we show that the maximum expected load of any machine is bounded. Then by using the concentration results from Section 3, we can bound the expected maximum load.

For 3 other cases, where the assignment obtained is for the rounded jobs, the proof consists of two main steps. In the first step, we compare the individual machine loads of the output assignment (of unrounded jobs) and the assignment of the rounded jobs given by ILPScheduling by Lemma 4.3. Then by the concentration and scaling results from Section 3, we bound the expected maximum load of the output assignment using the transition point of rounded jobs. In the second step, we bound the expected maximum of an optimum assignment of rounded jobs by that of an optimum assignment of unrounded jobs by scaling results from Section 3. Thus, we bound the expected maximum load of an output assignment by that of an optimum assignment.

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Now we prove Theorem 1.1.

Proof of Theorem 1.1. First we analyze the running time when POISCHEDULING does not use dynamic programming. If the algorithm invokes the efficient PTAS for deterministic case in Theorem 1.2, the running time is bounded by $2^{O(1/\epsilon)} + O(n \log n)$. Otherwise the algorithm will first do the rounding. By Lemma 3.3 after the rounding the number of different job sizes in \( \{\lambda_i\}_{i=1}^{n'} \) becomes $O(1 \log 1/\epsilon)$ and all job sizes are between $\delta \mu$ and $2 \mu$. Now for each machine of 1, 2, \ldots, $m(1)$, we only need to consider the assignments to it with load at most $4 \mu$. Therefore each machine can have at most $O(1/\delta) = O(1/\epsilon)$ jobs, and the number of different job profiles that can be assigned to one machine is at most $(1/\epsilon)^{O(1/\epsilon)} = 2^{O(1/\epsilon) \log 1/\epsilon}$, which is the number of variables in the ILP. By Theorem 4.1 the ILP can be solved in $2^{O(1/\epsilon) \log 1/\epsilon} O(\log n)$ time. Since for Case 2 we need to do a binary search on interval $[\mu, 100 \mu \log m(1)]$, the total running time is bounded by $2^{O(1/\epsilon) \log 1/\epsilon} O(\log n \log \log n)$.

If POISCHEDULING uses dynamic programming, we have $\delta \geq 2^{-10^6/\epsilon^2}$ and $m(1) \leq 2^{O(1/\epsilon)}$. Therefore the total running time is bounded by $2^{O(1/\epsilon^2)} + O(n \epsilon^{-4} \log^2 n)$ by Lemma 4.4. Combining these two cases gives us the desired running time.

The approximation guarantee directly follows from Lemmas 4.5 and 4.4. □

We then give the proofs of Lemma 4.4 and Lemma 4.5.

4.4.1 Proof of Lemma 4.4

We will need the following claims for proving Lemma 4.4.

Corollary 4.6 (of Lemmas 3.7 and 3.8). Given $m \geq 2^{10000}$ and $\frac{4 \log m}{m} < \mu \leq \frac{\log m}{10000}$, we have

$$
\frac{1}{10} \cdot \frac{\log m}{\log \frac{1}{\mu} + \log \log m} \leq M(m, \mu) \leq 10 \cdot \frac{\log m}{\log \frac{1}{\mu} + \log \log m}.
$$

Proof. Setting $\delta = 1/10$ and applying Lemmas 3.7 and 3.8 gives the desired claim. □

Lemma 4.7. Given $\mu_1, \mu_2, \ldots, \mu_m \geq 0$. Let $\mu = (\sum_{j=1}^{m} \mu_j) / m$ and $u = \lceil 1000 e^{-2} \log m \max \{ \mu, 1 \} \rceil$. Suppose all $\mu_j \in [\mu/2, 2\mu]$. For any random variable $X$ taking values on $\mathbb{N}$ we have

$$
R \leq \mathbb{E} \left[ \max \left\{ \frac{m}{j=1} \text{Poi}( \mu_j ), X \right\} \right] \leq \left( 1 + \frac{\epsilon}{10} \right) R
$$

where

$$
R \overset{\text{def}}{=} \sum_{x=0}^{u-1} \mathbb{P}[X = x] \left( x + \sum_{y=x+1}^{u} \mathbb{P} \left[ \frac{m}{j=1} \text{Poi}( \mu_j ) \geq y \right] \right) + \sum_{x=u}^{\infty} \mathbb{P}[X = x] x.
$$

We will use the following theorem from [Can19] to prove it.

Theorem 4.8 ([Can19]). For any $x, \mu > 0$ we have $\mathbb{P} \left[ \text{Poi}( \mu ) \geq \mu + x \right] \leq e^{-\frac{x^2}{2(\mu+x)}}$. 25
Proof of Lemma 4.4. By Theorem 4.8 and the definition of $u$ we have

$$\Pr[\text{Poi}(2\mu) \geq u] \leq e^{-1000e^{-2}\log m} \leq \frac{e}{1000m}.$$ 

Then by union bound, geometric series, and that

$$\mathbb{E}\left[\max\left\{\max_{j=1}^{m} \text{Poi}(\mu_j), X\right\}\right] = \sum_{x=0}^{\infty} \Pr[X = x] \left(x + \sum_{y=x+1}^{u} \Pr\left[\max_{j=1}^{m} \text{Poi}(\mu_j) \geq y\right]\right)$$

we have the desired claim. □

Proof of Lemma 4.7. By Theorem 4.8 and the definition of $u$ we have

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Then by union bound, geometric series, and that

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we have the desired claim.

We first prove the running time. By the performance guarantee of ROUNDED (Lemma 4.3), we have that $d = O\left(\frac{1}{\delta} \log \frac{1}{\delta}\right)$ and $n' = O(m^{(1)})/\delta$. By our way of discretizing the loads, we have $e = O\left(\frac{1}{\delta}\right)$. Then the number of different load profiles of the $m^{(1)}$ machines is at most $m^{(1)}e = (m^{(1)})^O(1)$. The number of different job profiles for jobs $\{X_i\}_{i=1}^{n'}$ is $(n')^d = (m^{(1)}/\delta)^O(1/\delta \log 1/\delta)$. The running time for dynamic programming (Lines 7-12 of Algorithm 3) is therefore bounded by $(m^{(1)})/\delta)^O(1/\delta \log 1/\delta)$. The time needed to calculate (58) for all load profiles is $O(n^{(0)}u^2) + (m^{(1)})^O(1/\delta)m^{(1)}O(u^2)$. By $\mu \leq 6000000e^{-2}\log m^{(1)}$ and the definition of $u$ on Line 13 of Algorithm 3, the total running time of DPScheduling is $O(n^{(0)}e^{-4}\log^2 m^{(1)}) + (m^{(1)})^O(1/\delta \log 1/\delta)$.

We then prove the approximation guarantee of the returned assignment. By Lemma 4.7 it suffices to show that our dynamic programming can always find an assignment whose expected maximum load is at most $1 + \frac{\epsilon}{5}$ of the optimum.

First we consider the case when $m^{(1)} \leq 1000/2^{109}/\epsilon^2$. In this case we have $\delta = \frac{\epsilon}{10000000\mu}$. Note that our dynamic programming will achieve a load profile $\vec{m}$ where each machine overflows by at most $1 + 5\delta$ compared to the optimum solution by Lemma 4.3. Since we have

$$\mathbb{E}\left[M(m^{(1)}, \text{Poi}(5\delta \mu))\right] \leq m^{(1)} \mathbb{E}[\text{Poi}(5\delta \mu)] = 5m^{(1)} \delta \mu = \frac{\epsilon}{200}\mu,$$

(63)

the load profile $\vec{m}$’s expected maximum load is within $1 + \frac{\epsilon}{200}$ of the optimum.

We then consider the case when $m^{(1)} > 1000/2^{109}/\epsilon^2$. In this case $\delta = 2^{-109}/\epsilon^2$. By Lemmas 3.7, 3.8, and 3.9 if $\mu \leq \frac{1}{2^{100}/\epsilon+1} \log m^{(1)}$ we are guaranteed to find a $(1 + 100\delta)$-optimum solution since we have concentration and scaling results hold. Therefore we only need to consider the case when $\frac{1}{2^{100}/\epsilon+1} \log m^{(1)} < \mu \leq 6000000e^{-2}\log m^{(1)}$.

We further consider two ranges of $\mu$:

1. When $\frac{1}{2^{100}/\epsilon+1} \log m^{(1)} < \mu \leq \frac{\log m^{(1)}}{10000}$ we have

$$\frac{\log m^{(1)}}{\log 5\delta \mu} + \log \log m^{(1)} \leq \frac{\log m^{(1)}}{(1000/\epsilon)(\log 1/\mu + \log \log m^{(1)})} = \frac{\epsilon}{1000} \cdot \frac{\log m^{(1)}}{\log 1/\mu + \log \log m^{(1)}}$$

(64)
where the first inequality follows from that \( \log \frac{1}{5\delta \mu} + \log \log m = \log \frac{\log m}{\mu} + \log \frac{1}{5\delta} \) and \( \log \frac{1}{\delta \mu} \geq \frac{10^8}{\epsilon^2} \geq (1000/\epsilon)(100/\epsilon + 1) \geq (1000/\epsilon) \log \frac{\log m}{\mu} \). Then since \( m > \frac{\epsilon}{1000} 2^{10^9/\epsilon^2} > 2^{100000} \) we can apply Corollary 4.6 and get

\[
M(m(1), 5\delta \mu) \leq \frac{\epsilon}{10} \cdot M(m(1), \mu). 
\] (65)

2. When \( \delta \mu < \frac{\log m(1)}{10000} \leq \mu \leq \frac{6000000}{\epsilon - 2} \log m(1) \), let \( \mu = \alpha \log m(1) \) where \( \frac{1}{10000} \leq \alpha \leq \frac{6000000}{\epsilon - 2} \), and then we have

\[
\frac{\log m(1)}{\log \frac{1}{5\delta \mu} + \log \log m(1)} = \frac{\log m(1)}{\log \frac{1}{5\delta \alpha}} \leq \frac{\epsilon}{10^7} \log m(1) \leq \frac{\epsilon}{15} \mu. \] (66)

Once again since \( m > \frac{\epsilon}{10000} 2^{10^9/\epsilon^2} > 2^{100000} \) we can apply Corollary 4.6 and get

\[
M(m(1), 5\delta \mu) \leq \frac{2\epsilon}{3} \mu \leq \frac{2\epsilon}{3} \cdot M(m(1), \mu). \] (67)

Therefore the load profile where each machine overflows by at most \( 1 + 5\delta \) comparing to the optimum assignment gives a \( (1 + \frac{2\epsilon}{3}) \)-approximation. \( \square \)

4.4.2 Proof of Lemma 4.5

Proof of Lemma 4.5 Let \( \chi \) be the set of jobs each of which is assigned to a single machine in the first loop of Algorithm 2. Let \( \Lambda \) be the (multi-)set of the sizes of the jobs in \( \chi \). Let \( \chi = \{ \text{Poi}(\lambda) \mid \lambda \in \Lambda \} \) and \( X = \max_{\lambda \in \Lambda} \text{Poi}(\lambda) \). Consider \( \mu \) and \( m(1) \) as same as Algorithm 2 after Line 6. Suppose the algorithm returns an assignment where machine \( j \) has load \( \mu_j \). Let \( \{ \mu_j(1) \}_{j=1}^{m(1)} \) be the machine loads corresponding to the assignment \( \phi' \) obtained before Line 20. That is, if \( \phi' \) is an assignment of the rounded jobs, \( \mu_j(1) \)'s are the corresponding rounded loads.

Without loss of generality, we assume \( \mu_1(1) \geq \mu_2(1) \geq \ldots \geq \mu_{m(1)}(1) \). Let \( \mu(1) = \sum_{j=1}^{m(1)} \mu_j(1) / m(1) \). Note that if Algorithm 2 does not return an assignment using rounded jobs (Line 20), then \( \mu = \mu(1) \).

There are six possible cases:

**Case 0:** The algorithm finishes assigning all jobs before Line 6. Then it must be the case that \( n \leq m \). Therefore, assigning every job solely to a single machine is the optimum solution.

**Case 1:** \( \mu > \frac{\delta \mu}{\delta} \log m(1) \) and the algorithm goes to Lines 7-9. First we state an observation. Note that this is the same observation as Observation 2.1 in [AAWY98] for deterministic case. Both our observation and the one in [AAWY98] follow from the convexity of the objective function with respect to the machine loads.

**Observation 4.9.** There is an optimum assignment such that each job in \( \chi \) is solely assigned to a separate machine.

**Proof.** Suppose in an optimum assignment there is a job in \( \chi \) with size \( \lambda_i \) which is assigned to a machine with another job of size \( \lambda_j \) and suppose \( i \) is the biggest index such that a job with size \( \lambda_i \) shares its machine with another job. Let \( \mu' \) be the value of \( \mu \) in Algorithm 2 just
before assigning $\lambda_i$ to a machine, so we know $\lambda_i > \mu'$. By removing $\lambda_j$ from its machine, all the other machines should have load at least $\mu'$, otherwise we can assign $\lambda_j$ to a machine with load less than $\mu'$ and by Proposition 2.3, the expected maximum decreases which contradicts the optimality. But $\mu'$ is the average load of remaining machines (machines that do not contain any jobs from $\{\lambda_k | k > i\}$) and now each one has load more than $\mu'$, a contradiction. 

Let $L^*_{DET}$ be the optimum answer of deterministic case for job sizes of $\{\lambda_i\}_{i=1}^{n(1)}$ and $m(1)$ machines, where $n(1) = n - |\chi|$. Note that $L^* \geq L^*_{DET}$ and $L^* \geq \max_{\lambda \in A} \lambda$, by Observation 4.9.

Since $\delta = \frac{\epsilon}{1000}$, by Lemma 3.1 we get:

$$L = E \left[ \max \left\{ \frac{m(1)}{\max_j \text{Poi} (\mu_j)}, X \right\} \right]$$

$$\leq E \left[ \max \left\{ \text{Poi} \left( (1 + \epsilon/5)L^*_{DET} \right), \frac{m(1)}{\max_j \text{Poi} (\mu_j)}, X \right\} \right] \quad \text{(by Theorem 4.2)}$$

$$\leq (1 + 5\delta) \max \left\{ (1 + \epsilon/5)L^*_{DET}, \max_{\lambda \in \Lambda} \lambda \right\} \quad \text{(by (8) in Lemma 3.1)}$$

$$\leq (1 + \epsilon/200)(1 + \epsilon/5)L^*$$

$$< (1 + \epsilon)L^*.$$  \hspace{1cm} (68)

**Case 2:** The algorithm goes to Lines 10-12. By calling ILPScheduling multiple times, Algorithm 2 finds the smallest transition point $t_2$ for rounded jobs on $m(1)$ machines as defined in Lemma 3.2. Note that if $\hat{L}$ is the optimum expected maximum for rounded version, then $\hat{L} \leq (1 + 80\delta)L^*$ (by (13) in Lemma 3.2 and Lemma 4.3). Since $\delta = \frac{\epsilon}{1000}$, $m(1) > 2^{200/\epsilon}$ and $\frac{1}{2^{7/8+\epsilon}} \log m(1) < \mu(1) \log m(1) \leq (1 + 5\delta)\frac{6}{\delta^2} \log m(1) < \frac{12}{\delta^2} \log m(1)$ (Lemma 4.3), then by Lemma 3.2 we get:

$$L = E \left[ \max \left\{ \frac{m(1)}{\max_j \text{Poi} (\mu_j)}, X \right\} \right]$$

$$\leq E \left[ \max \left\{ \frac{m(1)}{\max_j \text{Poi} \left( (1 + 5\delta)\mu_j(1) \right)}, X \right\} \right] \quad \text{(Lemma 4.3)}$$

$$\leq (1 + 80\delta)E \left[ \max \left\{ \frac{m(1)}{\max_j \text{Poi} (\mu_j(1))}, X \right\} \right] \quad \text{(by (13) in Lemma 3.2)}$$

$$\leq (1 + 10\delta)(1 + 80\delta)E \left[ \max \{t_2, X\} \right] \quad \text{(by (12) in Lemma 3.2)}$$

$$\leq \frac{(1 + 10\delta)(1 + 80\delta)^2}{1 - 6\delta} \hat{L}$$

$$\leq \frac{(1 + 10\delta)(1 + 80\delta)^2}{1 - 6\delta} L^*$$

$$< (1 + \epsilon)L^*.$$

**Case 3:** $\frac{1}{(m(1))^\theta} < \mu \leq \frac{1}{2^{7/8+\epsilon}} \log m(1)$ and $m(1) \geq 2^{\frac{7}{2}} \log \frac{2}{\delta}$, and the algorithm goes to Lines 7-9. First we state an observation which is the same as Observation 2.2 in [AAWY98] for deterministic case and again follows from the convexity of the objective function with respect to the machine loads.
**Observation 4.10.** If for each \( i \), \( \lambda_i \leq \mu \), then for each load \( \mu_j \) in an optimum assignment we have \( \mu/2 \leq \mu_j \leq 2\mu \).

**Proof.** Suppose a machine has load \( \mu_j > 2\mu \) and a job assigned to the machine is \( \lambda_i \). \( \lambda_i \leq \mu \), so \( \mu_j - \lambda_i > \mu \), so any other machines has load at least \( \mu \), a contradiction. So \( \mu_j \leq 2\mu \).

Suppose a machine has load \( \mu_j < \mu/2 \). So there is a machine with load \( \mu_k > \mu \), so it has at least two jobs assigned. So there is a job \( \lambda_i \) in \( \mu_k \) such that \( \lambda_i \leq \mu_k/2 \), by taking it from \( \mu_k \) its load would be \( \mu_k - \lambda_i \geq \mu_k/2 > \mu_j \), a contradiction as by reassigning \( \lambda_i \) to \( \mu_j \) by Proposition 2.3 the expected maximum decreases. So \( \mu/2 \leq \mu_j \). □

Since \( \delta = \frac{\epsilon}{1000} \) and \( m(1) > 2\delta \log \frac{2}{\delta} \), the conditions of Lemma 3.7 holds and we get:

\[
L = \mathbb{E}\left[ \max \left\{ \max_{j \in [1]} m^{(1)} \text{Poi}(\mu_j), X \right\} \right] \\
\leq \mathbb{E}\left[ \max \left\{ M(m^{(1)}), (1 + \epsilon/5)2\mu, X \right\} \right] \quad \text{(Observation 4.10 and Theorem 4.2)} \\
\leq \mathbb{E}\left[ \max \left\{ M(m^{(1)}), 4\mu, X \right\} \right] \quad \text{(Proposition 2.2)} \\
\leq (1 + 20\delta) \mathbb{E}\left[ \max \left\{ M(m^{(1)}, \mu), X \right\} \right] \quad \text{(by (38) in Lemma 3.7)} \\
< (1 + \epsilon)L^*.
\]

**Case 4:** The algorithm goes to Lines 13–15. By putting \( \delta = \frac{\epsilon}{1000} \) and Observation 4.10, an optimum assignment of rounded jobs satisfies the condition of Lemma 3.8. Let \( W \) be the Bernoulli random variable as described in Lemma 3.8 but with respect to the machine loads of an optimum assignment of the rounded jobs, then \( (1 - 5\delta)\mathbb{E}\left[ \max \left\{ W, X \right\} \right] \leq \hat{L} \) where \( \hat{L} \) is the optimum expected maximum of rounded version. By finding an assignment maximizing \( \prod_{j=1}^{m^{(1)}} \Pr \left[ \text{Poi}(\mu_j) \leq t_4 - 1 \right] \), the value of \( \mathbb{E}\left[ \max \left\{ W, X \right\} \right] \) would be minimized. Let the corresponding optimum \( W \) be \( W^* \). So we have \( (1 - 5\delta)\mathbb{E}\left[ \max \left\{ W^*, X \right\} \right] \leq (1 - 5\delta)\mathbb{E}\left[ \max \left\{ W, X \right\} \right] \leq \hat{L} \). Note that \( \hat{L} \leq (1 + 80\delta)L^* \) by (40) in Lemma 3.8 and Lemma 4.3. As \( \frac{4\log m}{m} < \mu^{(1)} \leq (1 + 5\delta)\frac{m^{(1)}}{m} \leq \frac{2}{m^{*}} \), by Lemma 3.8

\[
L = \mathbb{E}\left[ \max \left\{ \max_{j \in [1]} m^{(1)} \text{Poi}(\mu_j), X \right\} \right] \\
\leq \mathbb{E}\left[ \max \left\{ \max_{j \in [1]} m^{(1)} \text{Poi}\left(1 + 5\delta\mu_j^{(1)}\right), X \right\} \right] \quad \text{(Lemma 4.3)} \\
\leq (1 + 80\delta)\mathbb{E}\left[ \max \left\{ \max_{j \in [1]} m^{(1)} \text{Poi}\left(\mu_j^{(1)}\right), X \right\} \right] \quad \text{(by (40) in Lemma 3.8)} \\
\leq (1 + 16\delta)(1 + 80\delta)\mathbb{E}\left[ \max \left\{ W^*, X \right\} \right] \quad \text{(by (15) in Lemma 3.8)} \\
\leq \frac{(1 + 16\delta)(1 + 80\delta)}{1 - 5\delta} \hat{L} \\
\leq \frac{(1 + 16\delta)(1 + 80\delta)^2}{1 - 20\delta}L^* \\
< (1 + \epsilon)L^*. \quad (69)
\]
**Case 5:** The algorithm returns an assignment on Line 17. With the same argument as the previous case, by maximizing \( \prod_{j=1}^{m^{(1)}} \Pr[\text{Poi}(\mu_j) = 0] \), the value \( E[\max\{W, X\}] \) as defined in Lemma 3.9 would be minimized. Let the corresponding optimum \( W \) be \( W^* \). So we have \( E[\max\{W^*, X\}] \leq \hat{L} \) where \( \hat{L} \) is the optimum expected maximum of rounded jobs with \( m^{(1)} \) machines and \( \hat{L} \leq (1 + 50\delta)L^* \) by (53) in Lemma 3.9 and Lemma 4.3. So we get:

\[
L \leq E\left[ \max\left\{ \max_{j=1}^{m^{(1)}} \text{Poi}\left((1 + 5\delta)\mu_j^{(1)}\right), X \right\} \right] \tag{Lemma 4.3}
\]

\[
\leq (1 + 50\delta)E\left[ \max\left\{ \max_{j=1}^{m^{(1)}} \text{Poi}\left(\mu_j^{(1)}\right), X \right\} \right] \tag{by (53) in Lemma 3.9}
\]

\[
\leq (1 + 10\delta)(1 + 50\delta)E[\max\{W^*, X\}] \tag{by (52)} \text{ in Lemma 3.9}
\]

\[
\leq (1 + 10\delta)(1 + 50\delta)\hat{L}
\]

\[
\leq (1 + 10\delta)(1 + 50\delta)^2L^*
\]

\[
< (1 + \epsilon)L^*.
\]

So for all cases we have \( L < (1 + \epsilon)L^* \) and we are done.

\[\square\]

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A Counterexample for Using Means as Effective Job Sizes

For any odd number \(m\), consider an instance of the problem with \(m+1\) jobs and \(m\) machines, where each job has size \(\text{Poi}(\lambda)\) for \(\lambda = \beta \log(m - 1)\) and \(\beta = \frac{1}{1024}\). An assignment where \(\frac{m+1}{2}\) machines have 2 jobs is an optimum assignment for the deterministic case. We will show that the expected maximum of this assignment is asymptotically larger than an assignment where one machine has 2 jobs and the others have one job each. For estimating the expected maximum, first we state a proposition which we prove later.

**Proposition A.1.** Let \(X_1, X_2, \ldots, X_m \sim \text{Poi}(\lambda)\) be i.i.d. Poissons where

\[
\frac{1}{m} \leq \lambda \leq \frac{\log m}{16}.
\]

Then, for any constant \(\delta > 0\) there exists a constant \(m_0 = m_0(\delta)\) such that for any \(m > m_0\)

\[
(1 - \delta) \cdot \frac{\log m}{\log \frac{1}{\lambda} + \log \log m} \leq \mathbb{E} \left[ \max_{i=1}^{m} X_i \right] \leq (1 + \delta) \cdot \frac{\log m}{\log \frac{1}{\lambda} + \log \log m}.
\]

Consider an arbitrary constant \(\delta \in (0, 1/100)\) and suppose \(m > 2m_0(\delta)\). By the above
proposition we have

\[
\mathbb{E} \left[ \max_{i=1}^{(m+1)/2} \text{Poi}(2\lambda) \right] \geq (1 - \delta) \cdot \frac{\log(m + 1) - 1}{\log \frac{1}{\lambda} + \log \log(m + 1)}
\]

\[
> (1 - \delta) \cdot \frac{\log(m + 1) - 1}{\log \frac{1}{\beta} + \log \frac{1}{\log(m-1)} + \log \log(m - 1) - 1}
\]

\[
= (1 - \delta) \cdot \left\lfloor \frac{\log(m + 1) - 1}{9} \right\rfloor
\]

(72)

On the other hand we have

\[
\mathbb{E} \left[ \max \{ \text{Poi}(2\lambda), \max_{i=2}^{m} \text{Poi}(\lambda) \} \right] \leq \mathbb{E} \left[ \left( \max_{i=2}^{m} \text{Poi}(\lambda) \right) + \text{Poi}(2\lambda) \right]
\]

\[
\leq \left( 1 + \delta \right) \cdot \frac{\log(m - 1)}{\log \frac{1}{\lambda} + \log \log(m - 1)} + 2\lambda
\]

\[
= \left( 1 + \delta \right) \cdot \frac{\log(m - 1)}{10} + \frac{1}{512} \log(m - 1)
\]

(73)

Now for large enough \( m \), \( (1 - \delta) \cdot \left\lfloor \frac{\log(m+1) - 1}{9} \right\rfloor > (1 - \delta)(\log m - 1) \) and \( \left( 1 + \delta \right) \cdot \frac{\log(m - 1)}{10} \) and \( \frac{1}{512} \log(m - 1) < (1 + \delta)(\log m - 9.5) \). So there is a constant gap between (72) and (73).

**Proof of Proposition A.7** Note that

\[
\frac{\log m}{\log \frac{1}{\lambda} + \log \log m} = \lambda \cdot \frac{\log m^{1/\lambda}}{\log \log m^{1/\lambda}}
\]

(74)

We first prove the lower bound. Let \( l \) be defined by (70). If \( l = 0 \), the lower bound trivially holds; otherwise by (70) we have \( l \geq 1 \) and \( l \geq \lambda \). We lower bound the expected maximum by

\[
\mathbb{E} \left[ \max_{i=1}^{m} X_i \right] \geq \Pr \left[ \max_{i=1}^{m} X_i \geq l \right] \cdot l
\]

\[
\geq (1 - (1 - \Pr [X_1 = l])^m) \cdot l.
\]

(75)

We then lower bound the probability that \( X_1 \) equals \( l \):

\[
\Pr [X_1 = l] = e^{-\lambda} \cdot \frac{\lambda^l}{l!}
\]

\[
\geq e^{l - \lambda - 1} \cdot \left( \frac{\lambda}{l} \right)^l
\]

\[
\geq e^{l - \lambda - 1} \cdot \left( \frac{\log \frac{1}{\lambda} + \log m}{\log \frac{1}{\lambda} + \log \log m} \right)
\]

\[
\geq \frac{1}{m - \log \log m^{1/\lambda} - 2 \log \log m^{1/\lambda}}
\]

(76)
where the second line follows from Stirling (Proposition 2.5) and the last line follows from (74). Inserting this into (75) gives

\[
E \left[ \max_{i=1}^{m} X_i \right] \geq \left( 1 - \left( 1 - \frac{1}{m} \right)^{\frac{\log \log \log m^{1/\lambda}}{2 \log \log m^{1/\lambda}}} \right)^{m} \cdot l \\
\geq \left( 1 - e^{-m \frac{\log \log \log m^{1/\lambda}}{2 \log \log m^{1/\lambda}}} \right) \cdot l \\
\geq (1 - \delta) l
\]

(77)

where the last line holds for \( m \) large enough.

We then prove the upper bound. Let \( r \overset{\text{def}}{=} \left( 1 + \delta \right) \cdot \frac{\log m}{\log \frac{1}{\lambda} + \log \log m} \). By (70), \( r \geq 2\lambda \).

\[
E \left[ \max_{i=1}^{m} X_i \right] \leq r - 1 + \sum_{k=r}^{\infty} \Pr \left[ \max_{i=1}^{m} X_i = k \right] \cdot k \\
\leq r - 1 + O(1) \cdot \Pr \left[ \max_{i=1}^{m} X_i = r \right] \cdot r
\]

(78)

where the second inequality follows from that the sum is bounded by a geometric series with constant ratio. We then upper bound the probability that the maximum is \( r \) by a union bound:

\[
\Pr \left[ \max_{i=1}^{m} X_i = r \right] \leq m \cdot \Pr \left[ X_1 = r \right] \\
= m \cdot e^{-\lambda} \cdot \frac{\lambda^r}{r!} \\
\leq m \cdot e^{-\lambda} \cdot \left( \frac{\lambda}{r} \right)^r \\
\leq m \cdot e^{-\lambda} \cdot \left( \frac{\log \frac{1}{\lambda} + \log \log m}{\lambda \log m} \right)^{(1+\delta)} \frac{\log m}{\log \frac{1}{\lambda} + \log \log m} \\
\leq \frac{1}{m^{\frac{\log \log \log m^{1/\lambda}}{2 \log \log m^{1/\lambda}}}}
\]

(79)

where the third line follows from Stirling (Proposition 2.5) and the last line follows from (74). Inserting this into (78) gives

\[
E \left[ \max_{i=1}^{m} X_i \right] \leq r - 1 + O(1) \cdot \frac{1}{m^{\frac{\log \log \log m^{1/\lambda}}{2 \log \log m^{1/\lambda}}}} \cdot r \leq r
\]

(80)

where the last inequality holds for \( m \) large enough.