Einstein synchronisation by quantum teleportation

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Abstract

We present a protocol of Einstein’s synchronization of two spatially separated clocks on the basis of quantum teleportation in which the Einstein-Podolsky-Rosen (EPR) state plays a central role. Our scheme does not contain the frame defining problem nor the phase contamination problem for the EPR state which appeared in the quantum clock synchronization (Jozsa et al 2000 Phys. Rev. Lett. 85, 2010).

1. Introduction

In the seminal paper on the special relativity published in 1905 [1], Einstein considered a thought experiment for two spatially separated clocks to synchronize by exchanging light signals back and forth between them. The first light signal starts from one of the observers Alice at the time \( t_{A0} \) as her clock indicates, reaches the other observer Bob at the time \( t_{B1} \) measured by his clock and then the information of the numerical value \( t_{B1} \) is sent back to Alice by the light signal, who receives the information of \( t_{B1} \) at the time \( t_{A1} \) by her clock. By this communication Alice obtains all of the necessary data \( t_{A0} \), \( t_{B1} \) and \( t_{A2} \) to check the synchronization criterion as shown in figure 1.

\[
\frac{t_{A0} + t_{A2}}{2} = t_{B1}.
\]

Einstein then demanded that the clock synchronization above should hold also in an arbitrary inertial frame with a relative velocity from the rest frame assuming the frame independence of the light velocity and arrived at the Lorentz transformation between the coordinates of the inertial and the rest frames. He chose the criterion (1) on the basis of the equality of the light velocity in both ways.

Nowadays, the Einstein clock synchronization is also practically important to implement the GPS system.

We notice that, although Einstein did not explicitly mention, Bob has to send back the information of \( t_{B1} \) in some way to convince Alice that her clock is synchronized to Bob’s by checking the criterion (1). Some people (e.g., [2, 3]) use the terminology ‘Einstein synchronization’ to broadly mean the date adjustment of the two spatially separated clocks by exchanging signals even without the criterion (1). In this paper the meaning of Einstein synchronization is closer to the original one in the sense that the criterion (1) is required as the validation. We take the criterion (1) as an operational one leaving aside its physical meaning. We stress that checking that the two classical clocks are synchronized is conceptually different from creating the two synchronized clocks, or quantum synchronization if they were quantum clocks. For the latter we will quickly review the recent development.

2. Quantum Clock Synchronization (QCS)

Jozsa et al [2] proposed a quantum mechanical synchronization using a prior entangled state. Let us briefly review the original protocol of the QCS to elucidates its drawbacks which are potentially harmful also to our own protocol of the validation of the clock synchronization. For elaborations and other quantum protocols, see e.g., [4, 5]. Suppose Alice and Bob have identical two-level quantum systems, i.e., a qubit consisting of the basis \( |0\rangle \) and \( |1\rangle \) with the energy levels 0 and \( \hbar \omega \). To make the model definite we choose a Hamiltonian \( H = \hbar \omega (1 - \sigma_z)/2 \) and \( \sigma_z |0\rangle = |0\rangle \), \( \sigma_z |1\rangle = -|1\rangle \). Here \( \sigma_z \), \( \sigma_y \) and \( \sigma_z \) are the standard Pauli matrices.
Initially Alice and Bob share an EPR state:

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B),$$

which remains the same up to an overall phase factor in the time evolution. This choice is important because the protocol is time-blind i.e., insensitive to the time when it starts. The singlet state $|EPR\rangle$ can also be written as

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|-\rangle_A|+\rangle_B - |+\rangle_A|-\rangle_B),$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}[|0\rangle \pm |1\rangle]$ is the eigenstate of the observable $\sigma_x$ with the eigenvalue $\pm 1$. Suppose Alice measures her $\sigma_x$ to obtain $|\pm\rangle_A$. Then the state of Bob jumps to $|\mp\rangle_B$, correspondingly. The state $|\mp\rangle_B = \frac{1}{\sqrt{2}}[|0\rangle_B \mp |1\rangle_B]$ evolves in time duration $t$ as,

$$|\mp\rangle_B = e^{-i\omega t}|1\rangle_B, \tag{4}$$

Alice sends a classical message which asks Bob to do the Hadamard transformation $H_B$. After that Bob will have a clock state:

$$\frac{1}{\sqrt{2}}[|+\rangle_B \mp e^{-i\omega t}|-\rangle_B] = \frac{1 \mp e^{-i\omega t}}{2}|0\rangle_B + \frac{1 \pm e^{-i\omega t}}{2}|1\rangle_B. \tag{5}$$

when Bob’s clock indicates that the time is $t = t_1$.

Alice correspondingly has her clock state:

$$\frac{1 \pm e^{-i\omega t}}{2}|0\rangle_A + \frac{1 \mp e^{-i\omega t}}{2}|1\rangle_A \tag{6}$$

at $t_1$. Alice’s state is identical to Bob’s state if she flips her bit.

To implement the creation of the clock states of Alice and Bob, Alice sends the classical information $\pm$ of the outcome when she has performed the first measurement of $\sigma_x$ to Bob, then Bob can see the time indicated by his own clock. Indeed the QCS proposal is not the check of the synchronization of the two clocks but rather an automatic synchronization of the two quantum clocks.

However, unfortunately there are at least two defects in the original QCS protocol.

(i) As remarked in their own paper [2] and briefly pointed out in the above, the spatial axis $x$ at Bob cannot be definitely arranged relative to the $x$-axis of Alice so that the Hadamard transformation of Bob in the protocol is ambiguous and so is the resultant state by Alice’s observation$^2$.

$^1$ At this stage we have to be careful about the coordinate frame of Bob. The outcome state $|\mp\rangle_B$ may not be the $x$-component of B’s state, because the ‘$x$-direction’ can be any direction perpendicular to the $z$-axis. As we shall see later, this frame arbitrariness becomes the drawback of the original QCS [2].

$^2$ The authors of [2] proposed a modification of the protocol using two different two-level systems with different excitation energies. This might work in practice but the complication would be a severe shortcoming as a fundamental protocol for an advanced theory, say, relativistic quantum information.
The QCS has a drawback similar to Eddinton’s synchronization [6]. Namely, the shared entangled state is contaminated by an uncontrollable phase $\delta$ in general,

$$-\frac{1}{\sqrt{2}}[\ket{1}_A\ket{0}_B - e^{i\delta}\ket{0}_A\ket{1}_B],$$

(7)

because the environments of spatially separated Alice and Bob are different in general and therefore have different interactions [3]. This extra phase factor invalidates its isotropy, i.e., the equation (3) and therefore the break-down of whole scheme. Recently, Ilo-Okeke et al [5] proposed a modification of QCS to resolve the frame ambiguity problem via entanglement purification but not the phase contamination problem.

The motivation of the present work is to propose a protocol for the validation of synchronization of classical clocks in the same sense of Einstein’s original paper but a transmission of the time information from Bob to Alice’s is done by using quantum teleportation, which does not have the two drawbacks of QCS mentioned above in spite of the use of the EPR pair state. The key to the resolution is the Bell measurements.

### 3. The protocol

Here we present a precise protocol of clock synchronization by the quantum teleportation.

(I) Prepare the initial state

$$\ket{\psi(0)} = \ket{EPR}\ket{0}_B^{\text{clock}},$$

(8)

where the $\ket{EPR}$ is the state of EPR defined as before in equation (2).

(II) At an arbitrarily chosen and recorded time $t_B^0$ Bob makes the Hadamard transformation $H_B$ to the clock state $\ket{0}_B^{\text{clock}}$ in equation (8). The state changes to

$$\ket{\psi'(0)} = H_B \ket{\psi(0)} = \ket{EPR}\ket{+}_B^{\text{clock}},$$

(9)

and the state $\ket{+}_B^{\text{clock}} = \frac{1}{\sqrt{2}}[\ket{0}_B^{\text{clock}} + \ket{1}_B^{\text{clock}}]$ starts its time evolution, $\ket{+}_B^{\text{clock}} \rightarrow \frac{1}{\sqrt{2}}[\ket{0}_B^{\text{clock}} + e^{-i\omega(t-t_B^0)}\ket{1}_B^{\text{clock}}]$ which plays a role of clock state.

(III) At an arbitrary time $t_{A0}$, Alice sends a classical signal to ask Bob to do another Hadamard transformation $H_B$, which Bob receives at time $t_B^1$. After receiving it, Bob immediately makes the controlled unitary transformation by $e^{-i\omega\theta_1}$ to cancel the $t_{B0}$ dependence of the state. Bob can do this transformation because he knows the value $t_{B0}$ from the classical record of his clock. The total state becomes at $t = t_B^1$

$$\ket{\psi''(t_B^1)} = \frac{1}{\sqrt{2}}[\ket{1}_A\ket{0}_B - \ket{0}_A\ket{1}_B] \frac{1}{\sqrt{2}}[\ket{0}_B^{\text{clock}} + e^{-i\theta_1}\ket{1}_B^{\text{clock}}],$$

where $\theta_1 = \omega t_{B1}$. At this stage we notice that the resultant state $\ket{\psi''(t_{B1})}$ can be rewritten as (7)

$$\ket{\psi''(t_{B1})} = \frac{1}{2\sqrt{2}} (\ket{1}_A + e^{-i\theta_1}\ket{0}_A)\ket{\Psi_1} + (\ket{1}_A - e^{-i\theta_1}\ket{0}_A)\ket{\Psi_2} + (-\ket{0}_A - e^{-i\theta_1}\ket{1}_A)\ket{\Psi_3} + (-\ket{0}_A + e^{-i\theta_1}\ket{1}_A)\ket{\Psi_4},$$

(10)

where $\Psi_1, \Psi_2, \Psi_3$ and $\Psi_4$ are the Bell states defined by

$$\ket{\Psi_1} = \frac{1}{\sqrt{2}} (\ket{0}_B\ket{0}_B^{\text{clock}} - \ket{1}_B\ket{1}_B^{\text{clock}}),$$

(11)
\[ |\Psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle_B |0\rangle_{\text{clock}} + |1\rangle_B |1\rangle_{\text{clock}}), \]
\[ |\Psi_3\rangle = \frac{1}{\sqrt{2}} (|1\rangle_B |0\rangle_{\text{clock}} - |0\rangle_B |1\rangle_{\text{clock}}), \]
\[ |\Psi_4\rangle = \frac{1}{\sqrt{2}} (|0\rangle_B |1\rangle_{\text{clock}} + |1\rangle_B |0\rangle_{\text{clock}}). \]

(IV) Bob performs the measurements of the Bell states. If he gets the state \(|\Psi_1\rangle\), the state of Alice at the time \(t_{A2}\) would be \(\frac{1}{\sqrt{2}}(|1\rangle_A + e^{-i\theta}|0\rangle_A\) and similar results for the cases \(|\Psi_2\rangle, |\Psi_3\rangle\) and \(|\Psi_4\rangle\).

(V) Suppose that Alice gets the clock state
\[ \frac{1}{\sqrt{2}} [|1\rangle_A + e^{-i\theta}|0\rangle_A], \]
teleported by Bob as well as the classical information that the result of the Bell measurement by Bob was \(|\Psi_1\rangle\). Then she waits for the tuned time duration \(\frac{t_{A0} + t_{A2}}{2}\) knowing the dates \(t_{A0}\) and \(t_{A2}\) by her own clock. Then her clock state becomes
\[ |\psi\rangle_A = \frac{1}{\sqrt{2}} [e^{-i\theta}|1\rangle_A + e^{-i\theta}|0\rangle_A], \]
where \(\theta = \omega \frac{t_{A0} + t_{A2}}{2}\). Alice carries out the Hadamard transformation \(H_A\)
\[ |\psi\rangle_A = H_A|\psi\rangle_A = \frac{1}{\sqrt{2}} [e^{-i\theta}|1\rangle_A + e^{-i\theta}|0\rangle_A] = e^{-i\theta}(|0\rangle_A + \beta |1\rangle_A], \]
where
\[ \alpha = \frac{1 + e^{-i(\theta_2 - \theta_1)}}{2}, \quad \beta = \frac{1 - e^{-i(\theta_2 - \theta_1)}}{2}. \]

Here we have tentatively assumed that the Hadamard transformation \(H_A\) of Alice is the same as the Hadamard transformation \(H_B\) of Bob for the simplicity of explanation. Later we will generalise it taking the frame ambiguity into account.

(VI) Finally Alice measures her qubit. The probability to obtain \(|0\rangle_A\) by the bit measurement under the condition that the Bell measurement resulted in \(|\Psi_1\rangle\) is
\[ |\alpha|^2 = \frac{1 + \cos(\theta_2 - \theta_1)}{2}, \]
which is equal to unity if the synchronization condition
\[ \theta_2 - \theta_1 = \omega \left[ \frac{t_{A0} + t_{A2}}{2} - t_{B2} \right] = 0 \]
is met. Namely, if Alice gets the state \(|0\rangle_A\) with a high probability she can convince herself that her clock is synchronized with Bob’s. If the result of the Bell measurement was different from the \(|\Psi_1\rangle\), Alice can arrange different unitary transformation to check the synchronization criterion by the bit measurement \(|0\rangle_A\) and then check similarly as before.

4. A solution to the phase contamination and the frame adjustment problems

The different environments of Alice and Bob induces the uncontrollable phase \(\delta\) to the EPR state to give
\[ |EPR\rangle \rightarrow |\text{ENT}\rangle = \frac{1}{\sqrt{2}} [|1\rangle_A |0\rangle_B - e^{i\delta}|0\rangle_A |1\rangle_B]. \]
Correspondingly the total state becomes
\[ |\psi\rangle(t_{B2}) = (|1\rangle_A + e^{-i\delta} + i\delta|0\rangle_A)|\Psi_1\rangle + (|1\rangle_A - e^{-i\delta} + i\delta|0\rangle_A)|\Psi_2\rangle \\
+ (-e^{i\delta}|0\rangle_A - e^{-i\delta}|1\rangle_A)|\Psi_3\rangle + (-e^{i\delta}|0\rangle_A + e^{-i\delta}|1\rangle_A)|\Psi_4\rangle, \]
\[ \text{(22)} \]

If the Bell measurement gives \(|\Psi_1\rangle\) state, the state of Alice would be \(\frac{1}{\sqrt{2}}(|1\rangle_A + e^{-i\delta} + i\delta|0\rangle_A). After waiting for the tuned time duration \(\frac{t_{A0} + t_{A2}}{2}\) she performs the Hadamard transformation and then make the bit observation just as explained before in the case that \(\delta = 0\). The probability for Alice to obtain \(|0\rangle_A\) is now
\[ P_1 = |\alpha|^2 = \frac{1 + \cos(\theta_2 - \theta_1 + \delta)}{2}, \tag{23} \]

instead of (19), where the \( \delta \) is the unknown phase contamination. The case of the Bell state |\Psi_2\rangle gives the same probability so that the subsequent bit measurement does not add new information. However, for the case of |\Psi_3\rangle the probability for Alice to obtain |0\rangle_A is

\[ P_3 = \frac{1 + \cos(\theta_2 - \theta_1 - \delta)}{2}. \tag{24} \]

Note the sign difference in front of \( \delta \) from the case of \( P_1 \). For |\Psi_4\rangle, the result \( P_4 \) is the same as \( P_3 \).

Eliminating the unwanted \( \delta \) in Eqs. (23, 24), we obtain an expression for \( \theta_2 - \theta_1 \) as

\[ \theta_2 - \theta_1 = \frac{1}{2} [\cos^{-1}(2P_1 - 1) + \cos^{-1}(2P_3 - 1)]. \tag{25} \]

The right hand side is experimentally accessible. If its value is close to unity, the synchronization is validated. Therefore, the phase contamination problem has been overcome by the combination of the more than two Bell measurements.

Let us turn to the frame adjustment problem. The Hadamard transformation \( H_A \) by Alice can in general be different from \( H_B \) of Bob by unadjustable rotation around z-axis as noted below equation (18). Let the unitary transformation be \( U(\phi) = e^{i\phi z/2} \) and the Hadamard transformation be \( \tilde{H}_A \) explicitly given by

\[ \tilde{H}_A = U(\zeta) H_A U^\dagger(\zeta), \tag{26} \]

so that

\[ \tilde{H}_A |0\rangle_A = \frac{1}{\sqrt{2}} [e^{i\zeta} |1\rangle_A + e^{-i\zeta} |0\rangle_A], \]

\[ \tilde{H}_A |1\rangle_A = \frac{1}{\sqrt{2}} [e^{i\zeta} |0\rangle_A - e^{-i\zeta} |1\rangle_A]. \]

The teleported state \( \frac{1}{\sqrt{2}} \{ (|1\rangle_A + e^{-i\theta_2 + i\theta_1} |0\rangle_A) \} \) in (22) for the Bell result \( \Psi_1 \) is Hadamard transformed to

\[ \tilde{H}_A \frac{1}{\sqrt{2}} \{ (|1\rangle_A + e^{-i\theta_2 + i\theta_1} |0\rangle_A) \} = \left[ \frac{1}{2} (e^{i\zeta} + e^{-i\theta_2 + i\theta_1}) |0\rangle_A + \frac{1}{2} (-1 + e^{-i\theta_2 + i\theta_1}) |1\rangle_A \right]. \tag{27} \]

The waiting procedure explained around (16) replaces \( \theta_1 \) by \( \theta_1 - \theta_2 \).

To sum up, the probability to get \( \Psi_1 \) then \( |0\rangle_A \) is

\[ P_1(0) = \frac{1}{2} [1 + \cos(\theta_2 - \theta_1 + \delta - \zeta)]. \tag{28} \]

Similarly the probability to get \( \Psi_3 \) then \( |0\rangle_A \) is

\[ P_3(0) = \frac{1}{2} [1 + \cos(\theta_2 - \theta_1 - \delta + \zeta)]. \tag{29} \]

Note that the uncontrollable phases \( \zeta \) and \( \delta \) appear only in the combination of \( \delta - \zeta \). Therefore, the formula (25) remains to work for the check of the Einstein synchronisation. This is not a coincidence, because the selective spin rotation of Alice only effectively induces the phase contamination \( \delta \).

We admit that the Bell measurement is idealized in the sense that it can be done instantaneously. More practically, it needs many repeated measurements, during which the uncontrollable phase \( \delta \) and \( \zeta \) might change. We have not estimated the inaccuracy of the synchronization due to this effect.

5. Summary

To summarize, we have presented a quantum protocol to transmit the time information by teleportation for the Einstein synchronisation of two distant clocks, which is free from the frame adjustment and the phase contamination problems. However, there remain points to improve the QCS further. We add a comment on the Eddington synchronisation [6], an alternative of the Einstein synchronisation. A quantum version of the Eddington synchronisation in which a clock state is physically exchanged has also been proposed [7].

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