POINTWISE ROTATION
FOR HOMEOMORPHISMS WITH INTEGRABLE DISTORTION
AND CONTROLLED COMPRESSION

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Abstract

We obtain sharp rotation bounds for homeomorphisms $f : \mathbb{C} \to \mathbb{C}$ whose distortion is in $L^p_{\text{loc}}, p \geq 1$, and whose inverse have controlled modulus of continuity. The interest in this class is partially motivated by examples arising from fluid mechanics. We also present examples proving sharpness in a strong sense.

1 Introduction

We call $f : \mathbb{C} \to \mathbb{C}$ a mapping of finite distortion if it belongs to the Sobolev space $f \in W^{1,1}_{\text{loc}}(\mathbb{C}; \mathbb{C})$, its Jacobian determinant $\det(Df) = J(\cdot, f)$ is locally integrable, and there exists a positive measurable function $K(\cdot, f) : \mathbb{C} \to [1, +\infty]$ such that

$$|Df(z)|^2 \leq K(z, f) \cdot J(z, f)$$

at almost every point $z \in \mathbb{C}$. Here $|Df(z)|$ stands for the operator norm of the differential matrix $Df(z)$ of $f$ at the point $z$. If $K(\cdot, f) \in L^\infty$, then $f$ is said to be a mapping of bounded distortion or $K$-quasiregular, where $K = \|K(\cdot, f)\|_\infty$. Furthermore, $f$ is called $K$-quasiconformal if $f$ is a homeomorphism. Partially motivated by questions in nonlinear elasticity, mappings of finite distortion came into existence as a generalization of quasiregular maps. The interested reader should refer to the monographs [1] and [11] for a background in quasiregular maps and mappings of finite distortion.

Lately there has been a significant upsurge of interest in understanding the pointwise rotational properties of planar homeomorphisms of finite distortion, along with the spiraling rate of these maps, see [2,4,7,8,9,10]. To be precise, given such a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ normalized by $f(0) = 0$ and $f(1) = 1$, one is interested in the maximal growth of $|\arg(f(r))|$ as $r \to 0$. This growth represents the winding number of the image $f([r, 1])$ around the origin as $r \to 0^+$. It is known that this quantity admits several speeds of growth depending on the class of maps under study. For example, it was shown in [2] that if $f$ is $K$-quasiconformal then

$$|\arg(f(r))| \leq \frac{1}{2} \left( K - \frac{1}{K} \right) \log \left( \frac{1}{r} \right) + c_K, \quad \text{for all } 0 < r < 1. \quad (1)$$

On the other hand, for homeomorphisms of finite distortion situation changes drastically and the order of spiraling depends on integrability of the distortion function. Namely, the first named author discovered in [8] that if $e^{K(\cdot, f)} \in L^p_{\text{loc}}$ for some $p > 0$ then

$$|\arg(f(z))| \leq \frac{c}{p} \log^2 \left( \frac{1}{|z|} \right), \quad \text{for small enough } |z|,$$
and moreover this is sharp up to the constant $c > 0$. In other words, there is a certain payoff to transit from boundedness to exponential integrability of $K(\cdot, f)$. Precisely, it is an increment in the exponent of the logarithmic term. Further optimal results were obtained later on in [9] for homeomorphisms with integrable distortion, that is, when $K(\cdot, f) \in L^p_{loc}$ for some $p > 1$,

$$|\arg(f(z))| \leq \frac{c}{|z|^\frac{p}{2}}, \quad \text{for small enough } |z|$$

(2)

or even when $K(\cdot, f) \in L^1_{loc}$,

$$\limsup_{|z| \to 0} |z|^2 |\arg(f(z))| = 0.$$

(3)

The above estimates show, as one would expect, that more spiraling can be allowed by relaxing the degree of integrability of $K(\cdot, f)$.

One of the most important applications of mappings of finite distortion is their prominent role in the study of fluid dynamics. Recently it has been proven that there is an important class of homeomorphisms, arising from fluid mechanics, all of which have distortion function in some $L^p_{loc}$ space and are bi-H"{o}lder. This class, first found at [6], contains as an example the flow maps of any Yudovich solution from boundedness to exponential integrability of $K$.

Moreover, for each such small values of $t > 0$, each of the flow homeomorphisms $X_t = X(t, \cdot) : \mathbb{C} \to \mathbb{C}$ is indeed a mapping of finite distortion. Furthermore, for each such small values of $t > 0$ there is a number $p(t) > 1$ such that the distortion function $K(\cdot, X_t)$ belongs to $L^p_{loc}$ whenever $p < p(t)$. The Hölder continuity of both $X_t$ and $X_t^{-1}$ was proven in [14].

Inspired by this connection between mappings of finite distortion and fluid dynamics the authors reformulated the bound [2] in [5] to fit this situation. More precisely, it was shown that if $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism with finite distortion such that $K(\cdot, f) \in L^p_{loc}$ for some $p > 1$ and $f$ has an $\alpha$-H"{o}lder continuous inverse, then

$$|\arg(f(z))| \leq C \sqrt{\alpha} |z|^{-\frac{1}{2}} \log^+ \left( \frac{1}{|z|} \right), \quad \text{for small enough } |z|,$$

(5)

which drastically improves the bound [2] obtained without the Hölder assumption. This is not surprising as the local rotational properties depend not just on the integrability of $K(\cdot, f)$ but also on the local stretching properties of the mapping, see [2] [8] [9]. In this article, we extend our previous work from [5] to a more general class of homeomorphisms, which have $L^p_{loc}$ distortion for $p \geq 1$ and the inverse having predetermined modulus of continuity.
Theorem 1. Let \( f : \mathbb{C} \to \mathbb{C} \) be a homeomorphism of finite distortion such that \( f(0) = 0 \), \( f(1) = 1 \), and assume that \( \mathcal{K}(\cdot, f) \in L^1_{\text{loc}} \); \( p > 1 \). Then
\[
\left| \arg(f(z)) \right| \leq C |z|^{-\frac{1}{p}} \log^\frac{1}{p} \left( \frac{1}{\min_{|\omega|=|z|} |f(\omega)|} \right) \quad \text{when } |z| \text{ is small. (6)}
\]
Furthermore, if we assume that \( \mathcal{K}(\cdot, f) \in L^1_{\text{loc}} \), then
\[
\limsup_{|z| \to 0} \frac{|z|}{\log \left( \frac{1}{\min_{|\omega|=|z|} |f(\omega)|} \right)} |\arg(f(z))| = 0. \quad \text{(7)}
\]
Towards the optimality of Theorem 1 we can show the following.

Theorem 2. Let \( \varphi \) be a radially increasing homeomorphism with \( p \)-integrable distortion, \( p \geq 1 \), such that
\[
e^{-g_{\varphi,p}|z|^{1+\delta}} \leq |\varphi(z)| < |z|^4 \quad \text{when } |z| \text{ is small, (8)}
\]
where \( g_{\varphi,p} : \mathbb{R} \to \mathbb{R} \) is an increasing continuous function with \( g(r) \to 0 \) when \( r \to 0 \). Then we can choose an increasing onto homeomorphism \( h : [0, +\infty) \to [0, +\infty) \), which can converge to zero as slow as we want, and find a homeomorphism \( \tilde{f} : \mathbb{C} \to \mathbb{C} \) with the following properties:

- \( \tilde{f} \) is a homeomorphism of finite distortion, with \( \mathcal{K}(\cdot, \tilde{f}) \in L^p_{\text{loc}} \).
- \( \tilde{f}(0) = 0 \), \( \tilde{f}(1) = 1 \).
- There exists a decreasing sequence \( \{r_n\} \), such that
  \[
  |\tilde{f}(r_n)| = |\varphi(r_n)| \quad \text{(9)}
  \]
  and
  \[
  |\arg(\tilde{f}(r_n))| \geq r_n^{-\frac{1}{p}} \log^\frac{1}{p} \left( \frac{1}{|\varphi(r_n)|} \right) h(r_n). \quad \text{(10)}
  \]
Furthermore, this sequence \( \{r_n\} \) can be chosen to be a subsequence for an arbitrary decreasing sequence \( \{\lambda_n\} \to 0 \) of positive numbers.

Note that the homeomorphism \( \tilde{f} \) in Theorem 2 is radial and hence \( \min_{|\omega|=|z|} |f(\omega)| = |f(z)| \). Since \( h \) can be chosen to approach zero at any speed and sequence \( \lambda_n \) can be chosen freely, Theorem 2 shows that the upper bound provided in Theorem 1 is essentially sharp when we restrict modulus to satisfy (8).

Let us now briefly give some explanation for the bounds (5). The one on the right specifies that we are studying mappings that compress stronger than Hölder maps, which were studied in [5], and thus have faster maximal spiraling rate than given in (5). On the other hand, the bound on the left is always satisfied when \( p = 1 \), see [4], and when \( p > 1 \) it is exact up to the gauge function \( g_{\varphi,p} \), see [12]. Studying rotation under extremal compression leads to the extremal pointwise spiraling as shown in [9]. Thus Theorem 2 together with examples in [9] proving optimality of the extremal spiraling rate (2), show that whenever mapping \( \tilde{f} \) is compressing we have essentially sharp spiraling rates.

As a Corollary to Theorem 1 we can extend the case with Hölder bounds and \( p > 1 \), see [5] Theorem 1, to the borderline situation \( p = 1 \).

Corollary 3. Let \( f : \mathbb{C} \to \mathbb{C} \) be a homeomorphism of finite distortion such that \( f(0) = 0 \) and \( f(1) = 1 \), and assume that \( \mathcal{K}(\cdot, f) \in L^1_{\text{loc}} \). Moreover, let us suppose that
\[
|f(x) - f(y)| \geq C |x - y|^p \quad \text{if } |x - y| \text{ is small,}
\]
for some $\alpha \geq 1$. Then

$$\limsup_{|z| \to 0} \frac{|z|}{\log \left(\frac{1}{|z|}\right)} |\arg(f(z))| = 0.$$  \hfill (11)

Note that in the case $p = 1$ we get an improvement in the form of vanishing $\limsup$ compared to the case $p > 1$, which is described by the bound $[5]$. This is analogous to the maximal spiraling bounds $[2]$ and $[3]$, where the exact same improvement happens.

Finally, we prove the optimality of the above result in a strong sense.

**Theorem 4.** Given an increasing, onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, an arbitrary $\epsilon > 0$ and a real number $\beta \geq 1$, there exists a homeomorphism $\bar{f} : \mathbb{C} \to \mathbb{C}$ with the following properties:

(a) $\bar{f}$ is a mapping of finite distortion, with $K(\cdot, \bar{f}) \in L^1_{\text{loc}}$

(b) $\bar{f}(0) = 0$, $\bar{f}(1) = 1$

(c) If $\alpha \geq 2(\beta + 2) + \epsilon$, then $|\bar{f}(x) - \bar{f}(y)| \geq C|x - y|^\alpha$ whenever $|x - y| < 1$.

(d) There exists a decreasing sequence $\{r_n\}$, with limit $r_n \to 0^+$ as $n \to \infty$, for which

$$|\arg(\bar{f}(r_n))| \geq \frac{h(r_n)}{r_n} \left(\beta \log \left(\frac{1}{r_n}\right)\right)^{\frac{1}{2}}$$

and again the sequence $r_n$ can be chosen to be a subsequence for an arbitrary decreasing sequence $\{\lambda_n\} \to 0$ of positive numbers.

Towards the proof of Theorem 4, we modify the construction from [5, Theorem 3] giving optimality for the bound $[5]$. However, this construction as written in [5] does not cover the case $K \in L^1_{\text{loc}}$, and thus some changes in the argument are necessary. Also we note that Corollary 3 is extremely sharp as the homeomorphism $h$ in Theorem 4 can go to zero as slow as we wish.

The paper is structured as follows. In Section 2 we introduce the basic preliminaries. In Section 3 we prove the positive results. Finally, we prove optimality in Section 4.

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## 2 Preliminaries

A mapping $f : \mathbb{C} \to \mathbb{C}$ is said to be $\alpha$-Hölder continuous, or simply Hölder from above, if there exist constants $C, d, \alpha > 0$ such that for any two points $x, y \in \mathbb{C}$, with $|x - y| < d$, one has

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}.$$  

Similarly, we say $f$ is $\beta$-Hölder from below if there are constants $C, d, \beta > 0$ such that for any two points $x, y \in \mathbb{C}$, with $|x - y| < d$, one has

$$|f(x) - f(y)| \geq C|x - y|^\beta.$$  

A mapping $f : \mathbb{C} \to \mathbb{C}$ is called bi-Hölder if it is both Hölder from above and from below.

Let us next define pointwise spiraling. Assume $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism of finite distortion.
and fix a point $z_0 \in \mathbb{C}$. In order to study the pointwise rotation of $f$ at the point $z_0$ we start by fixing an argument $\theta \in [0, 2\pi)$ and looking at how the quantity
\[
\arg(f(z_0 + te^{i\theta}) - f(z_0))
\]
changes as the parameter $t$ goes from 1 to a small $r$. This can also be understood as the winding of the path $f([z_0 + re^{i\theta}, z_0 + e^{i\theta}])$ around the point $f(z_0)$. As we are interested in the maximal pointwise spiraling we naturally need to find the maximum over all directions $\theta$,
\[
\sup_{\theta \in [0, 2\pi)} |\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|.
\] (12)
The maximal pointwise rotation is precisely the behavior of the above quantity (12) when $r \to 0$. That is, we say that the map $f$ spirals at the point $z_0$ with a rate $g$, where $g : [0, \infty) \to [0, \infty)$ is a continuous decreasing function, if
\[
\lim_{r \to 0} \sup_{\theta \in [0, 2\pi)} \frac{|\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|}{g(r)} = C
\] (13)
for some constant $C > 0$. Finding maximal pointwise rotation for a given class of mappings is equivalent to find maximal growth of function $g$ when $r \to 0$ for this class. Note that in (13) we must use limit superior as the limit itself might not exist. Furthermore, for a given mapping $f$ there might be many different sequences $r_n \to 0$ along which it has profoundly different rotational behavior.

Our proof of Theorem 1 relies heavily on the modulus of path families. We give here the main definitions, see, for example, [13] for a closer look at the topic. An image of a line segment $I$ under a continuous mapping is called a path, and we denote by $\Gamma$ a family of paths. Given a path family $\Gamma$, we say that a Borel measurable function $\rho$ is admissible for $\Gamma$ if any rectifiable $\gamma \in \Gamma$ satisfies
\[
\int_{\gamma} \rho(z) dz \geq 1.
\]
The modulus of the path family $\Gamma$ is then defined by
\[
M(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z) dA(z),
\]
where $dA(z)$ denotes the Lebesgue measure on $\mathbb{C}$. Intuitively, the modulus is big if the family $\Gamma$ has lots of short paths, and it is small if the paths are long and there are not many of them.

We will also need a weighted version of the modulus. Any measurable, locally integrable function $\omega : \mathbb{C} \to [0, \infty)$ will be called weight function. In our case the weight function $\omega$ will always be the distortion function $K(\cdot, f)$. Then, we define the weighted modulus $M_\omega(\Gamma)$ by
\[
M_\omega(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z) \omega(z) dA(z).
\]
Finally, we need the modulus inequality
\[
M(f(\Gamma)) \leq M_{K(\cdot, f)}(\Gamma)
\] (14)
which holds for any homeomorphism $f$ of finite distortion for which the distortion $K(\cdot, f)$ is locally integrable, proven by the first named author in [9].
3 Spiraling bounds

In this section, we prove Theorems 1 and 3 following closely the proof for extremal spiraling rate (2) in [9].

**PROOF OF THEOREM 1**

Proof. Let \( f \) satisfy the hypothesis of Theorem 1, and let \( z \in \mathbb{C} \setminus \{0\} \) be such that \(|z| < 1\). Our goal is to estimate the winding number \( n(z) \) of the image set \( f([z, z|z]) \) around the origin (recall that \( f(0) = 0 \)). We will bound \( n(z) \) using the modulus inequality (14). More precisely, we will prove that

\[
n(z) \leq C |z|^{\frac{1}{p}} \log^{\frac{1}{p-1}} \left( \frac{1}{\min_{|z_0|=|z|} |f(z_0)|} \right)
\]

which is equivalent to Theorem 1 when \( p > 1 \), and

\[
\limsup_{|z| \to 0} \frac{|z|}{\log \left( \frac{1}{\min_{|z_0|=|z|} |f(z_0)|} \right)} n(z) = 0
\]

for \( p = 1 \).

Let us first prove \( p > 1 \) case. To this end, choose an arbitrary point \( z_0 \in \mathbb{C} \setminus \{0\} \) such that \(|z_0| < 1\). Without loss of generality we can assume that \( z_0 \) is real and positive. Next, fix line segments \( E = [z_0, 1] \) and \( F = (-\infty, 0) \), and let \( \Gamma \) be the family of paths connecting them. Then we estimate the modulus term \( M_{\rho(f)}(\Gamma) \) from above. Fix balls \( B_j = (2^j z_0, 2^j z_0) \), \( j \in \{0, 1, ..., n\} \) and let \( n \) be the smallest integer such that \( 2^n z_0 \geq 1 \). Let us define

\[
\rho_0(z) = \begin{cases} 
2^{\frac{1}{2}}(z) & \text{if } z \in B_0 \\
2^{\frac{1}{2}}(z) & \text{if } z \in B_1 \setminus B_0 \\
\vdots \\
2^{\frac{1}{2}}(z) & \text{if } z \in B_n \setminus B_{n-1} \\
0 & \text{otherwise}
\end{cases}
\]

It is clear that any \( z \in E \) belongs to some ball \( \frac{1}{2} B_j \) and that \( \rho_0(z) \geq \frac{2^{\frac{1}{2}}(z)}{\rho(B_j)} \), whenever \( z \in B_j \). Moreover, since \( B_j \cap F = \emptyset \) for every \( j \), this implies that \( \rho_0(z) \) is admissible with respect to \( \Gamma \). Therefore, one can estimate the modulus from above as follows:

\[
M_{\rho(f)}(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \|\mathbb{K}(\cdot, f)\rho^2(z)\,dA(z)
\]

\[
\leq \int_{\mathbb{C}} \|\mathbb{K}(\cdot, f)\rho_0^2(z)\,dA(z)
\]

\[
\leq \|\mathbb{K}(\cdot, f)\|_{L^p(B(0,4))} \left( \int_{B(0,4)} \rho_0^{-\frac{2}{p-1}}(z)\,dA(z) \right)^{\frac{p-1}{p}}
\]

\[
\leq c_{f,p} \left( \int_{B(0,4)} \rho_0^{-\frac{2}{p-1}}(z)\,dA(z) \right)^{\frac{p-1}{p}}
\]

\[6\]
Let us now estimate the integral term by using the definition of $\rho_0$.

\[
\int_{B(0,4)} \rho_0^{2p}(z) dA(z) \leq \sum_{j=0}^{n} \int_{B_j} \left( \frac{2}{r(B_j)} \right)^{2p} dA(z) \\
= \sum_{j=0}^{n} |B_j| \left( \frac{2}{r(B_j)} \right)^{2p} \\
= c_p \sum_{j=0}^{n} \frac{(r(B_j))^2}{(r(B_j))^2} \\
= c_p \sum_{j=0}^{n} \frac{1}{2^{2p}} \\
= c_p z_0^{-2p} \sum_{j=0}^{n} \frac{1}{2^{2p}} 
\]

The series $\sum_{j=0}^{n} \frac{1}{2^{2p}}$ converges to a constant depending on $p$ for any fixed $p > 1$, and hence

\[
M_{\hat{G}(.,f)}(\Gamma) \leq c_{f,p} z_0^{-\frac{2p}{p}}. \tag{15}
\]

Next, we would like to estimate the modulus term $M(f(\Gamma))$ from below for $p \geq 1$. Let us recall that $f(0) = 0$ and define $M(f(\Gamma))$ in polar coordinates as follows:

\[
M(f(\Gamma)) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z) dA(z) \\
= \inf_{\rho \text{ admissible}} \int_{0}^{2\pi} \int_{0}^{\infty} \rho^2(r,\theta) r dr d\theta
\]

and provide a lower bound for

\[
\int_{0}^{\infty} \rho^2(r,\theta) r dr
\]

for an arbitrary direction $\theta \in [0,2\pi)$ and an arbitrary admissible $\rho$. To this end, let us fix a direction $\theta$ and consider the half line $L_\theta$ starting from the origin to the direction $\theta$. Assume that the image set $f(E)$ winds once around the origin when $z$ moves from a point $t_0$ to a point $t_2$ along $E$ and $f(t_0), f(t_2) \in L_\theta$. Since $f$ is a homeomorphism and the image $f(E)$ contains the origin and points with big modulus, we can deduce that $f(F)$ must intersect the line segment $(f(t_0), f(t_0))$ at least once, say at a point $f(t_1)$, with $t_1 \in F$. Moreover, we can choose $t_1$ in such a way that either the line segment $(f(t_1), f(t_0))$ or the line segment $(f(t_2), f(t_1))$ belongs to the path family $f(\Gamma)$. It is evident that $f(E)$ cycles around the origin $n(z_0) = \left\lfloor \frac{\arg(f(z_0)) - \arg(f(1))}{2\pi} \right\rfloor$ times. So, it is possible to find at least

\[
n(z_0) = \left\lfloor \frac{\arg(f(z_0)) - \arg(f(1))}{2\pi} \right\rfloor - 1
\]

disjoint line segments belonging to the path family $f(\Gamma)$ using this argument. Note that $n(z_0)$ does not depend on the direction $\theta$. Since we are interested in extremal rotation, it can be assumed that $f(E)$ winds around the origin at least once, which makes it clear that $n(z_0)$ is non-negative. Now, the $n(z_0)$ disjoint line segments can be written in the form $(x_j e^{i\theta}, y_j e^{i\theta}) \subset L_\theta$, where $j \in \{1, 2, ..., n(z_0)\}$ and $x_j, y_j$ are positive real numbers satisfying

\[
0 < r_f \leq x_1 < y_1 < ... < x_{n(z_0)} < y_{n(z_0)} \leq c_f
\]
where \( c_f = \sup_{z \in E} |f(z)| \) and \( r_f = \inf_{z \in E} |f(z)| \). Hence we can write

\[
\int_0^\infty \rho^2(r, \theta) r dr \geq \sum_{j=1}^{n(z_0)} \int_{x_j}^{y_j} \rho^2(r, \theta) r dr.
\]

Next, let us consider the Hölder inequality with the functions \( f(r) = \rho \sqrt{r} \) and \( g(r) = \frac{1}{\sqrt{r}} \), which after squaring both sides gives

\[
\int_{x_j}^{y_j} \rho^2(r, \theta) r dr \geq \left( \int_{x_j}^{y_j} \rho(r, \theta) dr \right)^2 \left( \int_{x_j}^{y_j} \frac{1}{r} dr \right)^{-1} \geq \frac{1}{\log \left( \frac{y_j}{x_j} \right)}.
\]

The last inequality holds true as \( \rho \) is admissible with respect to \( f(\Gamma) \) and the line segments \( (x_j e^{i\theta}, y_j e^{i\theta}) \) belong to the path family \( f(\Gamma) \). Therefore,

\[
\int_0^\infty \rho^2(r, \theta) r dr \geq \sum_{j=1}^{n(z_0)} \log \left( \frac{y_j}{x_j} \right).
\]

It is quite clear from the definition of \( c_f \) that

\[
\sum_{j=1}^{n(z_0)} \frac{1}{\log \left( \frac{y_j}{x_j} \right)} \geq \sum_{j=1}^{n(z_0)-1} \frac{1}{\log \left( \frac{y_{j+1}}{x_j} \right)} + \frac{1}{\log \left( \frac{c_f}{\min_{\Gamma} f(x)} \right)}.
\]

Next, let us consider the AM-HM inequality. For every positive integer \( a_j \),

\[
\sum_{j=1}^{n} a_j \geq \frac{n^2}{\sum_{j=1}^{n} \frac{1}{a_j}}.
\]

At this point, we would like to use AM-HM with the precise choices

\[
a_j = \frac{1}{\log \left( \frac{x_j+1}{x_j} \right)} \quad \text{if } j \in \{1, 2, ..., n(z_0) - 1\}, \quad \text{and} \quad a_{n(z_0)} = \frac{1}{\log \left( \frac{c_f}{\min_{\Gamma} f(x)} \right)},
\]

which gives

\[
\sum_{j=1}^{n(z_0)} \frac{1}{\log \left( \frac{y_j}{x_j} \right)} \geq \frac{n^2(z_0)}{\log \left( \frac{c_f}{\min_{\Gamma} f(x)} \right)} \geq \frac{n^2(z_0)}{\log \left( \frac{c_f}{\min_{\Gamma} f(x)} \right)}.
\]

Therefore,

\[
\int_0^\infty \rho^2(r, \theta) r dr \geq \frac{n^2(z_0)}{\log \left( \frac{c_f}{\min_{\Gamma} f(x)} \right)}.
\]

(16)

The constant \( c_f \) is finite and does not depend on either \( \theta \) or \( z_0 \), at least for small \( z_0 \). So, it is irrelevant at the limit \( z_0 \to 0 \). Hence the estimate (16) implies that

\[
M (f(\Gamma)) \geq \frac{cn^2(z_0)}{\log \left( \frac{1}{\min_{|z|=|z_0|} |f(z)|} \right)}.
\]

(17)

Next, use the modulus inequality (14) to get

\[
\frac{n^2(z_0)}{\log \left( \frac{1}{\min_{|z|=|z_0|} |f(z)|} \right)} \leq c_{f,p} z_0^{-\frac{\pi}{2}},
\]

(14)
which implies the desired estimate (6).

**To prove** \( p = 1 \) **case**, we will again use the modulus inequality (14). Note that we have already lower bound for \( M(f(\Gamma)) \) from (17) for any \( p \geq 1 \). Therefore, we just need to estimate modulus term \( M_{K(\cdot, f)}(\Gamma) \) from above. To this end, let us define the function

\[
\rho_0(z) = \begin{cases} \frac{1}{z_0} & \text{if } \text{dist}(z, E) < z_0 \\ 0 & \text{otherwise} \end{cases}
\]

Note that \( \rho_0 \) is admissible with respect to the path family \( \Gamma \). Therefore,

\[
M_{K(\cdot, f)}(\Gamma) \leq \frac{1}{z_0} \int_{\{z : \text{dist}(z, E) < z_0\}} K(\cdot, f)(z) dA(z).
\]

Denote

\[
\int_{\{z : \text{dist}(z, E) < z_0\}} K(\cdot, f)(z) dA(z) = C_f(z_0)
\]

and note that since \( K(\cdot, f)(z) \in L^1_{\text{loc}}(\mathbb{C}) \) and

\[
|\{z : \text{dist}(z, E) < z_0\}| \to 0
\]

it follows that \( C_f(z_0) \to 0 \) as \( z_0 \to 0 \), and thus

\[
M_{K(\cdot, f)}(\Gamma) \leq \frac{C_f(z_0)}{z_0^2} \tag{18}
\]

Next, we use the modulus inequality (14), bounds (17) and (18) to get

\[
\frac{n^2(z_0)}{\log \left( \frac{1}{\min |f(z)|} \right)} \leq \frac{C_f(z_0)}{z_0^2}
\]

which implies the desired estimate (7). Hence, Theorem 1 is proved.

**PROOF OF THEOREM 3**

Proof. As in Theorem 1 our aim is to estimate the \emph{winding number} \( n(z) \) of the image set \( f \left( \left[ z, \frac{1}{|z|} \right] \right) \) around the origin (recall that \( f(0) = 0 \)), when \( f \) satisfies the hypothesis of Theorem 3 and \( z \in \mathbb{C} \setminus \{0\} \) such that \( |z| < 1 \). We will estimate \( n(z) \) using the modulus inequality for homeomorphisms with integrable distortion (14). More precisely, we will show that

\[
\limsup_{|z| \to 0} \frac{|z|}{\sqrt{\log \left( \frac{1}{|z|} \right)}} n(z) = 0,
\]

which is equivalent to Theorem 3.

To this end, let us choose an arbitrary point \( z_0 \in \mathbb{C} \setminus \{0\} \) such that \( |z_0| < 1 \) and which we can again assume to be positive and real. Next, as in the proof of Theorem 1 we fix line segments \( E = [z_0, 1] \) and \( F = (-\infty, 0] \), and let \( \Gamma \) be the family of paths connecting them. We have already estimated modulus term \( M_{K(\cdot, f)}(\Gamma) \) in (18) and thus we can concentrate on estimating \( M(f(\Gamma)) \) from below.
To this end we use the exact same steps as in the proof of Theorem 1 until the lower bound (16), where we now estimate the constant $r_f$ using the Hölder modulus of continuity assumption on the inverse of our map $f$. That is, we estimate

$$|f(z_0)| \geq C|z_0|^\alpha$$

for sufficiently small $z_0$, and obtain

$$M(f(\Gamma)) \geq \frac{n^2(z_0)}{\alpha \log \left( \frac{1}{|z_0|} \right)}.$$ 

The estimates for moduli combined with the modulus inequality (14) results in

$$\frac{n^2(z_0)}{\alpha \log \left( \frac{1}{|z_0|} \right)} \leq \frac{C_f(z_0)}{z_0^2},$$

which provides the desired estimate (11). Hence Theorem 3 is proved.

\section{Optimal Results}

We are going to prove Theorems 2 and 4 in this section.

\textbf{PROOF OF THEOREM 2} 

\textit{Proof.} We prove Theorem 2 in two steps. In the first step, we construct a map which only rotates. This map will have the correct spiraling rate but the distortion of the map will not belong to the desired space. To overcome this barrier we compose it with a radial stretching map, which gives us better control over distortion.

Given an arbitrary annulus $A = B(0,R) \setminus B(0,r)$ let us define the corresponding rotation map as

$$\phi_A(z) = \begin{cases} 
  z & |z| > R \\
  z e^{i\alpha \log \frac{|z|}{R}} & r \leq |z| \leq R \\
  z e^{i\alpha \log \frac{R}{|z|}} & |z| < r 
\end{cases} \quad (19)$$

Here $0 < r < R$, and $\alpha \in \mathbb{R}$. It is clear that $\phi_A : \mathbb{C} \to \mathbb{C}$ is bilipschitz, hence quasiconformal (its quasiconformality constant depends only on $\alpha$), and moreover it is conformal outside the annulus $A$. Moreover, $|\phi_A(te^{i\theta})| = t$ for each $t > 0$ and $\theta \in \mathbb{R}$. This means that $\phi_A$ leaves fixed all circles centered at 0. It is easy to check that the jacobian determinant $J(z,\phi_A) = 1$ for each $z$.

Next, let us consider sequence $\{r_n\}$ such that $0 < r_{n+1} < \frac{r}{2e}$ and $r_1 < \frac{1}{e}$. Also, let $R_n = cr_n$, which ensures that $2r_{n+1} < R_{n+1} < \frac{R}{3}$. Let us now construct disjoint annuli $A_n = B(0,R_n) \setminus B(0,r_n)$, and set $\{f_n\}_n$ to be a sequence of maps, constructed in an iterative way as follows. For $n = 1$, we set

$$f_1(z) = \phi_{A_1}(z) = \begin{cases} 
  z & |z| > R_1 \\
  z e^{i\alpha_1 \log \frac{|z|}{R_1}} & r_1 \leq |z| \leq R_1 \\
  z e^{-i\alpha_1} & |z| < r_1 
\end{cases} \quad (20)$$

where $\alpha_1 \in \mathbb{R}$, $\alpha_1 \geq 1$, is to be determined later. We then define $f_n$ for $n \geq 2$ as

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)$$

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again for some values $\alpha_n \in \mathbb{R}$, $\alpha_n \geq 1$, to be determined later. Clearly, each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli $A_i$, $i = 1, \ldots, n$. It is also clear that $f_n(z) = f_{n-1}(z)$ on the unbounded component of $\mathbb{C} \setminus f_{n-1}(A_n)$ (i.e. outside of $B(0, R_n)$). This proves that the sequence $f_n$ is uniformly Cauchy and hence it converges to a map $f$, that is,

$$f = \lim_{n \to \infty} f_n$$

which is again a homeomorphism by construction. Now, since $f_n$ is quasiconformal for every $n$ and $f_n(z) = f_{n-1}(z)$ everywhere except inside the ball $B(0, R_n)$, where $R_n \to 0$ as $n \to \infty$, the limit map $f$ is absolutely continuous on almost every line parallel to the coordinate axes and differentiable almost everywhere.

It is helpful to note that each $f_n$ leaves fixed all circles centered at the origin, so in particular we have $f_n(A_j) = A_j$ for each $j$, and therefore $\phi_{f_{n-1}(A_n)} = \phi_{A_n}$. Direct calculation shows that

$$|D\phi_{A_n}(z)| = |\partial\phi_{A_n}(z)| + |\overline{\partial}\phi_{A_n}(z)| = \begin{cases} \frac{|z|^2 + \alpha_n^2 + |\alpha_n|^2}{2} & \text{if } |z| > R_n \\ \frac{\alpha_n}{2} & \text{if } r_n \leq |z| \leq R_n \\ 1 & \text{if } |z| < r_n \end{cases}$$

which allows us to estimate that

$$|\partial f(z)| + |\overline{\partial} f(z)| \leq 2\alpha_n \quad \text{whenever } z \in A_n,$$

and $|D f(z)| \leq 1$ otherwise. Therefore, in order to have $D f(z) \in L_{1loc}(\mathbb{C})$ it suffices that

$$\sum_{n} \alpha_n r_n^2 < +\infty. \quad (21)$$

This, together with the absolute continuity, guarantees $f \in W^{1,1}_{loc}(\mathbb{C})$. Also, since $f$ is a homeomorphism, we have that $J_f(z) \in L_{1loc}(\mathbb{C})$, and in fact $J_f(z, f) = 1$ at almost every $z \in \mathbb{C}$. Therefore, $f$ is a homeomorphism of finite distortion, with distortion function

$$\mathbb{K}(z, f) = \frac{|D f(z)|^2}{J_f(z, f)} \leq \begin{cases} 4\alpha_n^2 & z \in A_n, \\ 1 & \text{otherwise.} \end{cases}$$

Especially, in order to have $\mathbb{K}(\cdot, f) \in L^p_{loc}$, it suffices to ensure the convergence of the series

$$\sum_{n=1}^{\infty} |A_n| (4\alpha_n^2)^p \lesssim \sum_{n=1}^{\infty} \alpha_n^{2p} r_n^2 \quad (22)$$

which can be done by choosing $\alpha_n$ properly. Note that if (22) holds, then also (21) holds, because our choice of $\alpha_n$ will guarantee $\alpha_n \geq 1$. The last restriction to choose our $\alpha_n$ comes from rotational behavior of $f$. It is clear from the above construction that $f(0) = 0$, $f(1) = 1$ and

$$|\arg (f(r_n))| \geq \left| \arg \left( \frac{1}{\alpha_n} \right)^{1+\alpha_n} \right| = \alpha_n$$

for every $r_n$. Let us choose $\alpha_n = r_n^{-1/p} \log^{1/2}(1/\varphi(r_n))$. This implies that

$$|\arg (f(r_n))| \geq r_n^{-1/p} \log^{1/2}(1/\varphi(r_n)),$$

which shows that this map would be optimal for Theorem. However, with this particular choice of $\alpha_n$,

$$\sum_{n=1}^{\infty} \alpha_n^{2p} r_n^2 = \sum_{n=1}^{\infty} \log^p (1/\varphi(r_n))$$
which is certainly not finite. This means that $K(\cdot , f) \notin L^p_{loc}$.

Hence we need to modify the construction by adding a stretching factor to our building blocks, which lets us reduce the distortion while preserving spiraling rate. This is precisely done by substituting the logarithmic spiral map $z |z|^{i\alpha} = ze^{i\alpha \log |z|}$ by a complex power $z |z|^{q+i\alpha} = z |z|^{q} e^{i\alpha \log |z|}$ at each iterate. Let us explain this process in detail.

Similarly as in the previous construction, we consider a rapidly decreasing sequence $\{r_n\}$ such that $r_{n+1} < \frac{r_1}{2}$, $r_1 < \frac{1}{2}$ and set $R_n = cr_n$. Given an arbitrary annulus $A = B(0, R) \setminus B(0, r)$ we define the corresponding composition map as follows:

$$
\phi_A(z) = \begin{cases} 
z & |z| > R \\
\left(\frac{2r}{R}\right)^{q-1} e^{i\alpha \log \frac{2r}{R}} & r \leq |z| \leq R \\
\left(\frac{2r}{r_1}\right)^{q-1} e^{i\alpha \log \frac{2r}{r_1}} & |z| < r 
\end{cases}
$$

(23)

Note that we will always choose $q \geq 1$. Direct calculation shows that

$$|\partial \phi_A(z)| + |\overline{\partial} \phi_A(z)| = \begin{cases} 
1 & |z| > R \\
R^{1-q} |z|^{q-1} \left(1 + \frac{q-1}{q+1+|\alpha|}\right) & r \leq |z| \leq R \\
R^{1-q} & |z| < r 
\end{cases}
$$

(24)

and also that

$$J(z, \phi_A) = \begin{cases} 
\frac{2(q-1)}{q} & |z| > R \\
\left(\frac{q}{q}\right)^{2(q-1)} & r \leq |z| \leq R \\
\left(\frac{q}{q}\right)^{2(q-1)} & |z| < r 
\end{cases}
$$

(25)

whence

$$K(z, \phi_A) = \begin{cases} 
\frac{1}{4} & |z| > R \\
\frac{(q+1+|\alpha|)|q-1+|\alpha|)}{2q} & r \leq |z| \leq R \\
\frac{1}{4} & |z| < r 
\end{cases}
$$

(26)

In particular, if $2 \leq q + 1 \leq \alpha$, which will be satisfied for our choices of $\alpha$ and $q$, then one may estimate $\|K(\cdot, \phi_A)\|_{\infty} \leq \frac{\alpha q}{2q}$. Next, let us construct the sequence of maps $f_n$ in an iterative way as follows. For $n = 1$, we set

$$f_1(z) = \phi_{A_1}(z) = \begin{cases} 
z & |z| < R_1 \\
\left(\frac{2r}{R_1}\right)^{q-1} e^{i\alpha_1 \log \frac{2r}{R_1}} & R_1 \leq |z| \leq R_1 \\
\left(\frac{2r}{r_1}\right)^{q-1} e^{-i\alpha_1} & |z| < r_1 
\end{cases}
$$

(27)

where $q_1$ and $\alpha_1$ are to be determined later. Next, assuming we have $f_1, \ldots , f_{n-1}$, we define $f_n$ for $n \geq 2$ as:

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)
$$

Note that $\phi_{f_{n-1}(A_n)}$ is determined by the inner and outer radii of $\phi_{f_{n-1}(A_n)}$ (which are already available since $f_1, \ldots , f_{n-1}$ are known) as well as for the parameters $q_n$ and $\alpha_n$, which will be determined later. Clearly, each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli $A_i$, $i \in \{1, \ldots , n\}$. Moreover, one can easily show that

$$K(\cdot, f_n) = \prod_{j=1}^{n} K(\cdot, f_{n-j} \circ f_{n-j+1}(A_{n-j+1})) = \prod_{j=1}^{n} K(\cdot, \phi_{A_{n-j+1}})
$$

so that $K(z, f_n) \leq C(\alpha)^{-\frac{n}{q}}$ whenever $z \in A_j$, $j = 1 \ldots n$ while $K(\cdot, f_n) = 1$ otherwise. In a similar way, we can use that $|Df_n(z)| \leq C\alpha$ when $z \in A$ (and $|Df_n(z)| \leq 1$ at all other points) to obtain that $|f_n| \leq C\alpha$.
Next, we show that $q_r$ assumption on $\phi$ on $A$.

Again, as it was the case for the pure rotation example, when $p > q_1$, $h$ is absolutely continuous on almost every line parallel to the coordinate axis. For almost every fixed $z_0$ there is a neighbourhood of $z_0$ such that the sequence $\{f_n(z)\}_n$ remains constant for $n$ very large and $z$ in that neighbourhood. Therefore the same happens to the sequences $Df_n(z)$, $J(z, f_n)$ and $K(z, f_n)$, and so their limits are precisely $Df(z)$, $J(z, f)$ and $K(z, f)$. Especially, in order to have $D\bar{f} \in L^{1}_{loc}$ it suffices that

$$\sum_{n=1}^{\infty} |A_n| \alpha_n < +\infty.$$  

(28)

In case this holds true, then $\bar{f}$ is a homeomorphism in $W^{1,1}_{loc}$, and as a consequence its jacobian determinant $J(\cdot, \bar{f}) \in L^{1}_{loc}$. Moreover, in order to have $K(\cdot, \bar{f}) \in L^{p}_{loc}$; ($p \geq 1$) one needs to require that

$$\sum_{n=1}^{\infty} |A_n| \frac{2p}{q_n} < +\infty.$$  

(29)

Again, as it was the case for the pure rotation example, when $p > 1$ condition (29) implies (28) if $q_n \leq \alpha_n$, and for $p = 1$ case we must verify $q_n \leq \alpha_n$. So, our parameters $\alpha_n$ and $q_n$ need to be chosen according to these constrains as well as the purpose of $\bar{f}$ to be optimal for Theorem 1. To this end, note that $f(0) = 0$, $f(1) = 1$ and

$$|\arg(f(r_n))| \geq \left| \arg\left(\frac{1}{e} \right) \right| = |\alpha_n|,$$

(30)

which motivates us to choose

$$q_n = \begin{cases} \log \frac{|\phi(r_n)|}{|\phi(\hat{r}_n)|} & n = 1, \\ \log \left(\frac{e^{-r_n} (\hat{z})^{n_{n-1} + n_{n-2} + \ldots + n_1 - (n-1)}}{|\phi(r_n)|}\right) & n \geq 2 \end{cases}$$

(31)

and

$$\alpha_n = h(r_n) \left(\log \frac{1}{|\phi(r_n)|}\right)^{1/2} r_n^{-\frac{1}{2}}$$

(32)

where $h$ is a monotone non-increasing gauge function such that $h(r) \to 0$ as $r \to 0$ which we specify later.

Next, we show that $q_n \leq \alpha_n$ for all $p \geq 1$, from which $q_n \leq \alpha_n$ then also follows. At this point, we impose an ansatz on $r_n$:

$$r_n < \left(\frac{1}{e}\right)^{q_1 + q_2 + \ldots + q_n - (n-1)},$$

(33)

which is feasible as the radii $r_n$ can be assumed to decrease as fast as we want. Let us then recall our assumption on $\phi$ to satisfy compression bound:

$$|\phi(z)| \geq e^{-g_{\phi, p}(|z|) |z|^{-\frac{1}{2}}}.$$  

(34)
where \( g_{\varphi,p} : \mathbb{R} \to \mathbb{R} \) is some increasing gauge function such that \( |g_{\varphi,p}| \to 0 \) as \( |z| \to 0 \). Now, let us proceed with the calculations.

\[
q_n = \log \left( e \cdot r_n \cdot \left( \frac{1}{|\varphi(r_n)|} \right)^{q_{n-1} + q_{n-2} + \ldots + q_1 - (n-1)} \right)
\]

\[
\leq \log \frac{1}{|\varphi(r_n)|}
\]

\[
= \log^{\frac{1}{q}} \frac{1}{|\varphi(r_n)|} \cdot \log^{\frac{1}{q}} |\varphi(r_n)|
\]

\[
\leq \log^{\frac{1}{q}} \frac{1}{|\varphi(r_n)|} \cdot \sqrt{g_{\varphi,p}(r_n)} \cdot \frac{1}{r_n^q}
\]

\[
\leq \log^{\frac{1}{q}} \frac{1}{|\varphi(r_n)|} \cdot \frac{h(r_n)}{r_n^q} = \alpha_n
\]

where the last inequality holds for \( h \) converging to zero slowly enough. Note that, from [3] Theorem 1.6, we see that if \( p = 1 \) then the compression bound (34) is always satisfied with some \( g_{\varphi,p} \). Thus our choices for \( q_n \) and \( \alpha_n \) satisfy technical constrains.

Next, we show that estimate (29) governing integrability of the distortion holds true for \( p \geq 1 \). We start by estimating

\[
|A_n|^{\frac{2p}{q_n}} = C \cdot r_n^p \cdot \frac{h^{2p} \cdot r_n^{-q}}{\log^{p} \left( \frac{1}{|\varphi(r_n)|} \right) \cdot \left( e \cdot (\frac{1}{|\varphi(r_n)|})^{q_{n-1} + q_{n-2} + \ldots + q_1 - (n-1)} \right)}
\]

\[
\leq C \cdot r_n^{2p} \cdot \frac{\log^{p} \left( \frac{1}{|\varphi(r_n)|} \right)}{\log^{p} \left( \frac{r_n^2}{|\varphi(r_n)|} \right)}
\]

It is easy to check that \( 1 < \frac{\log^{p} \left( \frac{r_n^2}{|\varphi(r_n)|} \right)}{\log^{p} \left( \frac{1}{|\varphi(r_n)|} \right)} \leq 2^p \), using the condition (35), and therefore, up to constants, condition (29) is equivalent to

\[
\sum_n h(r_n)^{2p} < +\infty
\]

which we can always satisfy by choosing \( r_n \) small enough. Having (29) fulfilled, our map \( \bar{f} \) is a mapping of finite distortion with \( E(\cdot, \bar{f}) \in L^p_{loc} \).

Next we must show that our mapping \( \bar{f} \) has right compression and spiraling behaviour. Let us start with modulus and show that \( |\bar{f}(r_n)| = |\varphi(r_n)| \) by calculating

\[
|\bar{f}(r_n)| = \left( \frac{1}{e} \right)^{q_{n-1} + q_{n-2} + \ldots + q_1 - (n-1)} \cdot r_n \cdot \left( \frac{r_n}{R_n} \right)^{q_{n-1}}
\]

\[
= \left( \frac{1}{e} \right)^{q_{n-1} + q_{n-2} + \ldots + q_1 - (n-1)} \cdot r_n \cdot \left( \frac{1}{e} \right)^{q_{n-1}}
\]

\[
= |\varphi(r_n)|,
\]

where the last line follows from the penultimate due to our choice of \( q_n \).

For the spiraling part we must show that the rotation bound (10) holds true. But this follows directly from (30), (32) and from the above modulus equation. Finally we note that in the proof we only need to assure that the sequence \( r_n \) decreases fast enough, and thus we can choose it to be a subsequence for an arbitrary predefined sequence \( \{\lambda_n\} \). This concludes the proof of Theorem 2.
PROOF OF THEOREM 4

Proof. We prove Theorem 4 in two steps similarly to Theorem 2. In the first step we construct a map which only rotates. This map already provides the optimal result in the exponent scale. Then, as in the previous construction, we compose this map with radial stretching mapping and finish the proof.

Given an arbitrary annulus $A = B(0, R) \setminus B(0, r)$ we define the corresponding rotation map $\phi_A$ as in (19). It is clear that $\phi_A : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and moreover it is conformal outside the annulus $A$. Furthermore, $\phi_A$ leaves fixed all circles centered at 0, and the Jacobian determinant $J(z, \phi_A) = 1$ for each $z$.

Next, we again consider sequence $\{r_n\}$ such that $0 < r_{n+1} < \frac{r_n}{2}$, $r_1 < \frac{1}{2}$, and fix $R_n = er_n$. We then construct disjoint annuli $A_n = B(0, R_n) \setminus B(0, r_n)$ and a sequence of maps $\{f_n\}_n$ iteratively as before. That is, set $f_1$ as in (20) and define $f_n$ for $n \geq 2$ as

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)$$

for some $\alpha_n \geq 1$, to be determined later. We can use word by word the same arguments as before to deduce that the limit

$$f = \lim_{n \to \infty} f_n$$

is a homeomorphism with integrable distortion if

$$\sum_n \alpha_n r_n^2 < +\infty$$

and

$$\sum_{n=1}^{\infty} |A_n| \alpha_n^2 \simeq \sum_{n=1}^{\infty} \alpha_n^2 r_n^2 < +\infty.$$  

(35)

Moreover, as we will choose $\alpha_n > 1$, we see that in fact (35) implies (36). Hence we only need to choose $\alpha_n$ so that (36) is satisfied.

Furthermore, it is clear from the above construction that $f(0) = 0$, $f(1) = 1$ and

$$|\arg(f(r_n))| \geq |\arg\left(\left(\frac{1}{e}\right)^{1+i\alpha_n}\right)| = \alpha_n$$

(37)

for every $r_n$. Since we want our map to be optimal for Theorem 3, we may be tempted to choose $\alpha_n = \frac{\log^{1/2}(1/r_n)}{r_n}$. Unfortunately such a choice does not meet the requirement (36) and instead we are forced to choose

$$\alpha_n = \frac{h(r_n)}{r_n},$$

where $h : [0, \infty) \to [0, \infty)$ is a monotonically decreasing gauge function such that $\lim_{r \to 0^+} h(r) = 0$. With this choice, (36) is fulfilled if

$$\sum_{n=1}^{\infty} (h(r_n))^2 < +\infty,$$

which we can ensure by choosing small enough $r_n$. Note that this does not provide optimality for Theorem 4 in full generality, but it already gives the right order in the exponent scale.

Finally, we show that $f^{-1}$ is Hölder continuous with exponent $\frac{1}{2}$. To this end, let us recall that our map $f$ is actually a limit of iterates of logarithmic spiral maps inside the annuli $A_n = B(0, R_n) \setminus B(0, r_n)$. In particular, as shown in [2], if $\gamma \in \mathbb{R}$ then the basic logarithmic spiral map $g(z) = ze^{\gamma \log |z|}$ is $L$-bilipschitz for a constant $L$ such that $|\gamma| = L - \frac{1}{2}$. And thus for large $|\gamma|$ one roughly has $|\gamma| \simeq L$. Since
Let us now start the proof. We first consider the case where \( x, y \in A_n \), and hence \( f(x) = f_n(x), f(y) = f_n(y) \). Since \( r_n > C|x - y| \), we have

\[
|f(x) - f(y)| = |f_n(x) - f_n(y)| \geq \frac{r_n}{h(r_n)}|x - y|
\]

where we have used the bilipschitz nature of \( f_n \) on \( A_n \). The fact that \( f \) is Hölder from below inside the annuli \( A_n \) with exponent 2 implies that in these sets \( f^{-1} \) is Hölder continuous with exponent \( \frac{1}{2} \). Here we note, that \( f \) and \( f^{-1} \) are essentially the same mapping, only the direction of rotation is changed, and hence \( f \) is also Hölder continuous with exponent \( \frac{1}{2} \) inside \( A_n \).

Then we assume that \( x, y \in D_n = B(0, r_n) \setminus B(0, R_{n+1}) \). In this case \( f \) is of the form \( ze^{i\beta} \), where \( \beta \in \mathbb{R} \setminus \{0\} \), which is clearly an isometry and hence Hölder estimate inside \( D_n \) is trivial.

Next, we take \( x \in A_n \) and \( y \in D_n \). In particular, \( |x| \geq |y| \). Then let \( w \) be the point on the outer boundary of \( D_n \) joining \( x \) and \( y \). We have

\[
|f(x) - f(y)| \leq |f(x) - f(w)| + |f(w) - f(y)|
\]

\[
\leq C|x - w|^{\frac{1}{2}} + |w - y|
\]

\[
\leq 2C|x - y|^{\frac{1}{2}}
\]

The same happens if \( x \in D_{n-1} \) and \( y \in A_n \).

So it just remains to see what happens when points are further apart from each other. Let us first cover the case \( x \in A_n = B(0, r_n) \setminus B(0, r_n) \) and \( y \in B(0, R_{n+1}) \). Let \( L \) be the line joining \( x \) and \( y \). We divide it into three parts, viz., \( L_1, L_2 \) and \( L_3 \). Fix \( L_1 \) so that it connects \( x \) to a point \( a \) on the inner boundary of \( A_n \), giving estimate

\[
|f(x) - f(a)| = |f_n(x) - f_n(a)| \leq C|x - a|^{\frac{1}{2}}
\]

Next, \( L_2 \) connects \( a \) to the crossing point of the line \( L \) and the inner boundary of \( D_n \), which we denote by \( b \). And since \( f \) is an isometry in \( D_n \) an estimate for line segment \( L_2 \) is trivial.

For \( L_3 \) part we note that from \( 2R_{n+1} < r_n < \frac{r_n}{2} \) we get that \(|f(a)| > 2|f(b)| \) and hence

\[
|f(b) - f(y)| \leq 2|f(b)| \leq 2|f(b) - f(a)| = 2|b - a|.
\]

Combining these estimates we get

\[
|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(b)| + |f(b) - f(y)|
\]

\[
\leq C|x - a|^{\frac{1}{2}} + |a - b| + |b - y|^{\frac{1}{2}}
\]

The case \( x \in D_n \) and \( y \in B(0, r_{n+1}) \) can be proved in a similar manner. Thus \( f \) is Hölder continuous with exponent \( \frac{1}{2} \).
Here we again note that $f$ and $f^{-1}$ are essentially the same mapping modulo the direction of rotation, and hence $f^{-1}$ is also Hölder continuous with the exponent $\frac{1}{2}$. Thus $f$ is Hölder from below with the exponent $2$.

As we discussed before, the above example approaches the borderline stated in Theorem 3 but it does not attain full optimality yet. To this end, we need to modify it by adding a stretching factor to our building blocks, which lets us increase rotation without increasing the distortion. This is done by replacing, at each iterate, the logarithmic spiral map $z|z|^\alpha = z e^{i \alpha \log |z|}$ by a complex power $z|z|^{\alpha + 1} = z|z|^\alpha e^{i \alpha \log |z|}$. Let us proceed with the details.

So, similarly as in the previous construction, we consider a rapidly decreasing sequence $\{r_n\}$ such that $r_{n+1} < \frac{r_n}{2}$, $r_1 < \frac{1}{2}$ and fix $R_n = cr_n$. Given an arbitrary annulus $A = B(0, R) \setminus B(0, r)$ we define the corresponding radial stretching combined with rotation map as in (23). As before we will choose $q \geq 1$.

The values of the differential, Jacobian and distortion of $\phi_A$ are already known from (24), (25) and (26). In particular, if $2 \leq q + 1 < \alpha$ then one may estimate $\|K(\cdot, \phi_A)\| \leq \frac{4\alpha}{q}$. Next, we construct the sequence of maps $f_n$ in an iterative way as before. Let us set $f_1$ as in (27) and $f_n$ for $n \geq 2$ as:

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z).$$

Each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli $A_i$, $i \in \{1, \ldots, n\}$. Moreover, we still calculate distortion by

$$K(\cdot, f_n) = \prod_{j=1}^n K(\cdot, f_{n-j} \circ \phi_{f_{n-j}(A_{n-j+1})}) = \prod_{j=1}^n K(\cdot, \phi_{A_{n-j+1}})$$

so that $K(z, f_n) \leq C_n^2 q_n$ whenever $z \in A_j$, $j = 1 \ldots n$ while $K(\cdot, f_n) = 1$ otherwise. As before we also use $|D\phi_A(z)| \leq C_n$ when $z \in A_j$ and $|D\phi_A(z)| \leq 1$ at all other points) to obtain that $|Df_n| \leq C \alpha_j$ on $A_j$, $j = 1 \ldots n$, and $|Df_n| \leq 1$ otherwise.

Using the exact same arguments as before we see that for the limit

$$\bar{f} = \lim_{n \to \infty} f_n$$

to be a homeomorphism of integrable distortion it is enough to check that

$$\sum_{n=1}^\infty |A_n| \alpha_n < +\infty \quad (38)$$

and

$$\sum_{n=1}^\infty |A_n| \frac{\alpha_n^2}{q_n} < +\infty. \quad (39)$$

Note that as in the case of $f$ (39) implies (38) when $q_n < \alpha_n$ and so our parameters $\alpha_n$ and $q_n$ need to be chosen such that (39) is satisfied as well as the purpose of $\bar{f}$ to be optimal for Theorem 3. Thus we choose

$$\alpha_n = \frac{h(r_n)}{r_n} \left( \frac{\log 1}{r_n} \right)^{1/2}, \quad q_n = \beta \log \frac{1}{r_n}, \quad \beta \geq 1 \quad (40)$$

where $h$ is any gauge function such that $h(r) \to 0$ as $r \to 0$ and the condition $q_n < \alpha_n$ is satisfied. Indeed, with these choices (39) becomes

$$\sum_n (h(r_n))^2 < +\infty$$

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which, as before, may always be satisfied by choosing small enough \( r_n \). Having \( \mathcal{K}(\cdot, f) \in L_{\text{loc}}^1 \) fulfilled, our map \( f \) is a mapping of finite distortion with \( \mathcal{K}(\cdot, f) \in L_{\text{loc}}^1 \). Furthermore, since we can bound spiraling from below by \( \alpha_n \) at the points \( r_n \) using the same estimate \( \mathcal{K}(\cdot, f) \) as before, the resulting map \( f \) attains the optimal rotational behavior stated at Theorem \( 3 \) modulo the gauge function \( h \) which can be chosen to converge to 0 as slowly as desired.

Therefore, Theorem 4 will be proven once we show that \( f \) is Hölder from below.

To this end, we first observe that the composition of \( z \mapsto z e^{i\alpha \log |z|} \) followed by \( z \mapsto z |z|^{q-1} e^{i\alpha \log |z|} \) is precisely \( z \mapsto z |z|^q e^{i\alpha \log |z|} \). This observation suggests us to decompose \( f = g \circ f \), where \( f \) is essentially the first example in this section (with slightly different choices for the constants \( \alpha_n \)) and \( g \) is constructed by building blocks \( \mathcal{K}(\cdot, f) \) with \( \alpha = 0 \) at each step. Morally, \( f \) leaves fixed all circles centered at 0 and only rotates inside the annuli \( A_n \), while \( g \) conveniently stretches each circle.

The Hölder nature of \( f^{-1} \) has already been proven when \( \alpha_n = \frac{\log r_n}{r_n} \). We need to show that our map \( f^{-1} \) is still Hölder continuous with our new choices for \( \alpha_n \), which we can estimate by

\[
\alpha_n = \frac{h(r_n)}{r_n} \left( \beta \log \frac{1}{r_n} \right)^{1/2} \leq \sqrt{\beta h(r_n) r_n^{1/(1-\epsilon)}}
\]

for an arbitrary \( \epsilon > 0 \) and small enough \( r_n \). This can be done by exactly the same proof as before once we check that \( f \) is Hölder from below inside the annuli \( A_n \). To this end, let us consider two points \( x, y \in A_n \) and note that \( f(x) = f_n(x) \) and \( f(y) = f_n(y) \). Since \( r_n > C|x - y| \), using the estimate \( 41 \) gives

\[
|f(x) - f(y)| = |f_n(x) - f_n(y)| \geq \frac{f_n^{1/(1-\epsilon)}}{h(r_n)} |x - y| \\
\geq \frac{C}{h(r_n)} |x - y|^{1+\frac{1}{2-\epsilon}} \\
\geq C|x - y|^{2+\epsilon}
\]

where we are using the bilipschitz nature of \( f_n \) in \( A_n \). Therefore, in order to prove Theorem 4 it remains to prove that \( g \) is Hölder from below.

To this end, given any two points \( x, y \in B(0, 1) \), we can without loss of generality assume that \( |y| \geq |x| \) and let \( w \) be the point for which \( |w| = |x| \) and \( \arg(w) = \arg(y) \). Now, as \( g \) is a radial stretching map, it follows that

\[
|g(x) - g(y)| \geq \max\{|g(x) - g(w)|, |g(y) - g(w)|\}.
\]

Moreover,

\[
\max\{|x - w|, |y - w|\} \geq \frac{1}{2} |x - y|.
\]

Therefore, it is enough to show that both \( |g(x) - g(w)| \) and \( |g(y) - g(w)| \) satisfy Hölder bounds from below. Note that if \( x = 0 \) then clearly \( w = 0 \) and we have only the radial part \( |g(y) - g(w)| \).

Let us first check the term \( |g(x) - g(w)| \). Since \( g \) maps radially circles centered at the origin to similar circles we see that \( |g(x) - g(w)| \) gets contracted the same amount as the modulus \( |g(x)| \) is contracted under \( g \). Now we must consider two possibilities, either \( x, w \in A_n \) or \( x, w \in D_n \) for some \( n \). Let us first assume \( x, w \in A_n = B(0, R_n) \setminus B(0, r_n) \) for some \( n \). Here we recall the ansatz (33) on \( r_n \). Then we can
estimate
\[|g(x)| = \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot |x| \left(\frac{|x|}{R_n}\right)^{q_{n-1}}\]
\[\geq r_n \cdot |x| \left(\frac{|x|}{r_n}\right)^{q_{n-1}}\]
\[\geq r_n \cdot |x| \left(\frac{1}{e}\right)^{q_{n-1}} = e \cdot r_n^{1+\beta} \cdot |x|\]
for any \(x \in A_n\), where in the last step we use (40). Therefore,
\[|g(x) - g(w)| \geq e \cdot r_n^{1+\beta} \cdot |x - w|\]
\[\geq C \cdot |x - w|^{2+\beta}\]
since \(|x - w| < C \cdot r_n\) for some fixed constant \(C > 0\) when \(x, w \in A_n\).

Next, let \(x, w \in D_n = B(0, r_n) \setminus B(0, R_{n+1})\) for some \(n\). Using (33) we get
\[|g(x)| \geq c \cdot \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot r_n^{1+\beta} \cdot |x|\]
Thus we can use similar argument as in the previous case to estimate
\[|g(x) - g(w)| \geq c \cdot r_n^{1+\beta} \cdot |x - w|\]
\[\geq C \cdot |x - w|^{2+\beta}\]
since \(|x - w| < c \cdot r_n\) for some fixed constant \(c > 0\) when \(x, w \in D_n\).

Since the set \(D \setminus \{0\}\) is partitioned by separated annuli \(A_n\) and \(D_n\) we have thus proven that \(|g(x) - g(w)|\) satisfies Hölder estimates from below.

Finally, let us prove the Hölder estimates from below for the term \(|g(y) - g(w)|\). As the mapping \(g\) is radial, we can assume that \(y\) and \(w\) are real. We intend to use the Fundamental Theorem of Calculus, and thus have to estimate the differential from below. Using (33), as well as the facts that \(q_n > 1\) and \(R_n = err_n\), we can estimate for any real number \(t \in [r_n, R_n]\) that
\[g'(t) = \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot r_n \cdot \left(\frac{t}{R_n}\right)^{q_{n-1}}\]
\[\geq r_n \cdot q_n \cdot \left(\frac{r_n}{R_n}\right)^{q_{n-1}}\]
\[= e q_n r_n^{1+\beta}\]
\[\geq c \beta \cdot t^{1+\beta} \log \left(e + \frac{1}{t}\right).\]
Next, if \(t \in [R_{n+1}, r_n]\), we have
\[g'(t) = \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot \left(\frac{1}{e}\right)^{q_{n-1}}\]
\[\geq c \cdot r_n^{1+\beta}\]
\[\geq c \cdot t^{1+\beta}\]
Thus, as before, since \((0, 1)\) is partitioned by the intervals \([r_n, R_n), [R_{n+1}, r_n]\) and \([R_1, 1)\), we end up getting that
\[
g'(t) \geq c \cdot t^{1+\beta}
\]
for every \(t \in (0, 1)\). Now, we use the fundamental theorem of calculus to get
\[
|g(y) - g(w)| = \int_{w}^{y} g'(t) dt \\
\geq \int_{w}^{y} c \cdot t^{1+\beta} dt \\
= C(\beta) \left(y^{2+\beta} - w^{2+\beta}\right) \\
\geq C|y - w|^{2+\beta}
\]
This proves that the second term is Hölder from below as well, which in turn proves that \(g\) is Hölder from below with exponent \((2 + \beta)\). This finishes the proof of Theorem 4. \(\square\)

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