An Efficient $\varepsilon$-BIC to BIC Transformation and Its Application to Black-Box Reduction in Revenue Maximization

Yang Cai*
Yale University, USA
yang.cai@yale.edu

Argyris Oikonomou
Yale University, USA
argyris.oikonomou@yale.edu

Grigoris Velegkas
Yale University, USA
grigoris.velegkas@yale.edu

Mingfei Zhao
Yale University, USA
mingfei.zhao@yale.edu

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Abstract

We consider the black-box reduction from multi-dimensional revenue maximization to virtual welfare maximization. Cai et al. [12, 13, 14, 15] show a polynomial-time approximation-preserving reduction, however, the mechanism produced by their reduction is only approximately Bayesian incentive compatible ($\varepsilon$-BIC). We provide a new polynomial time transformation that converts any $\varepsilon$-BIC mechanism to an exactly BIC mechanism with only a negligible revenue loss. Our transformation applies to a very general mechanism design setting and only requires sample access to the agents’ type distributions and query access to the original $\varepsilon$-BIC mechanism. Other $\varepsilon$-BIC to BIC transformations exist in the literature [23, 35, 18] but all require exponential time to run. As an application of our transformation, we improve the reduction by Cai et al. [12, 13, 14, 15] to generate an exactly BIC mechanism.

Our transformation builds on a novel idea developed in a recent paper by Dughmi et al. [24]: finding the maximum entropy regularized matching using Bernoulli factories. The original algorithm by Dughmi et al. [24] can only handle graphs with nonnegative edge weights, while our transformation requires finding the maximum entropy regularized matching on graphs with both positive and negative edge weights. The main technical contribution of this paper is a new algorithm that can accommodate arbitrary edge weights.

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1 Introduction

Mechanism design is the study of optimization algorithms with the additional constraint of incentive compatibility. A central theme of algorithmic mechanism design is thus to understand how much this extra constraint hinders our ability to optimize a certain objective efficiently. In the best scenario, one may hope to establish an equivalence between a mechanism design problem and an algorithm design problem, manifested via a black-box reduction that converts any algorithm to an incentive compatible mechanism. In this paper, we study the black-box reduction of a central problem in mechanism design: multi-dimensional revenue maximization.

The problem description is simple: an auctioneer is selling a collection of items to one or more strategic bidders. We follow the standard Bayesian assumption, that is, each bidder’s type is drawn independently from a distribution known to all other bidders and the auctioneer. The auctioneer’s goal is to design a Bayesian incentive compatible (BIC) mechanism that maximizes the expected revenue.

In the special case of single-item auction, Myerson provides an elegant characterization of the optimal mechanism. Indeed, Myerson’s solution can be viewed as a black-box reduction from revenue maximization to the algorithmic problem of (virtual) welfare maximization [32]. A long-standing open question is whether the black-box reduction can be generalized to multi-dimensional settings. In a recent breakthrough, Cai et al. [12, 13, 14, 15] show that there is a polynomial-time approximation-preserving black-box reduction from multi-dimensional revenue maximization to the algorithmic question of (virtual) welfare optimization. However, this result still has the following two caveats: (i) the revenue of the mechanism is only guaranteed to be within an additive $\varepsilon$ of the optimum; and (ii) the mechanism is only approximately Bayesian incentive compatible. Thus, an immediate open problem following their result is whether these two compromises are inevitable. In this paper, we show that approximately Bayesian incentive compatibility is unnecessary through our first main result:

Result I: There is a polynomial-time approximation-preserving black-box reduction from multi-dimensional revenue maximization to the algorithmic question of (virtual) welfare optimization that generates an exactly Bayesian incentive compatible mechanism.

Result I is enabled by a new polynomial time $\varepsilon$-BIC to BIC transformation, which is our second main result:

Result II: There is a polynomial-time $\varepsilon$-BIC to BIC transformation that converts any approximately Bayesian incentive compatible mechanism to an exactly Bayesian incentive compatible mechanism with a negligible revenue loss for any downward-closed environment.

The transformation is fully general and applicable to any downward-closed mechanism design setting. We believe the transformation is of independent interest and would have numerous applications in mechanism design. Indeed, our black-box reduction follows straightforwardly from applying the transformation to the mechanism of Cai et al. [12, 13, 14, 15]. Note that other $\varepsilon$-BIC to BIC transformations have been proposed in the literature [23, 35, 18], however, all of the existing transformations require solving a $\#P$-hard problem repeatedly [27] and therefore cannot be made computationally efficient.

1.1 Our Results and Techniques

We first fix some notations to facilitate our discussion of the results. We consider a general mechanism design environment where there is a set of feasible outcomes denoted by $O$, which we assume to be downward-closed. There are $n$ agents, and each agent $i$ has a type $t_i$ drawn from distribution $D_i$ independently. We use $T_i$ to denote the support of $D_i$, and for every $t_i \in T_i$, $v_i(t_i, \cdot)$ is a valuation function that maps every outcome to a real number in $[0,1]$. A mechanism $\mathcal{M}$ consists of an allocation rule $x(\cdot) : \times_{i \in [n]} T_i \mapsto \Delta(O)$ and a payment rule $p(\cdot) : \times_{i \in [n]} T_i \mapsto \mathbb{R}^n$. We slightly abuse notation to define

\footnote{\textnormal{Roughly speaking the setting is downward-closed if the agents have the choice to not participate in the mechanism. See Section 2 for the formal definition.}}
To overcome the difficulty, we turn to another important problem in mechanism design: even a tiny bit of estimation error can cause the algorithm to violate the property, making the whole mechanism not incentive compatible. The replica-surrogate matching is again the central piece in the reduction. Indeed, the issue is that no matter how many samples we receive, we can construct another mechanism \( \mathcal{M'} \) that is exactly BIC and IR with respect to \( \times_{i \in [n]} \mathcal{D}_i \), and its revenue is at most \( O(n/\sqrt{\epsilon}) \) worse than the revenue of \( \mathcal{M} \). Moreover, for any bid profile \( b = (b_1, \ldots, b_n) \), \( \mathcal{M'} \) computes an outcome \( o \sim x(b) \) and payments \( p_1(b), \ldots, p_n(b) \) in expected running time \( \text{poly}(\sum_{i \in [n]} |T_i|, 1/\epsilon) \) and makes in expectation at most \( \text{poly}(\sum_{i \in [n]} |T_i|, 1/\epsilon) \) queries to \( \mathcal{M} \).

Previous transformations can produce a \( \mathcal{M'} \) with similar guarantees but require time \( \text{poly}(\prod_{i \in [n]} |T_i|) \) to run [23, 35, 18]. Our result achieves an exponential speedup. To illustrate our new ideas, we first briefly review the construction in the literature. In the heart of all the previous constructions lies the problem called replica-surrogate matching.

**Replica-Surrogate Matching** For each agent \( i \), form a bipartite graph \( G_i \). The left hand side nodes are called replicas, which are types sampled i.i.d. from \( D_i \). In particular, the true type \( t_i \) of agent \( i \) is one of the replicas. On the right hand side, the nodes are called surrogates, which are also types sampled from \( D_i \). The edge between a replica with type \( t^{(j)} \) and a surrogate with type \( t^{(k)} \) is assigned weight \( w_{jk} \equiv \mathbb{E}_{t_{-i} \sim D_{-i}} [v_i(t^{(j)}, x(t^{(k)}, t_{-i})) - p_i(t^{(k)}, t_{-i})] \), which is the interim utility of agent \( i \) when her true type is \( t^{(j)} \) but reports \( t^{(k)} \) to \( \mathcal{M} \). Compute the maximum weight matching on \( G_i \). The true type \( t_i \) selects a surrogate using the matching to compete in \( \mathcal{M} \). Agent \( i \) competes in \( \mathcal{M} \) using the type of the surrogate she is matched to in the maximum weight matching.

The intuition is that since \( \mathcal{M} \) is not BIC, the true type \( t_i \) may prefer the outcome and payment from reporting some different type. The matching is set up to allow the true type \( t_i \) to pick a more favorable type to compete in \( \mathcal{M} \) for it. But why wouldn’t the agent misreport in the matching? After all, the edge weights depend on the agent’s report. As it turns out, to guarantee incentive compatibility, one needs to find a matching with a maximal-in-range algorithm. Namely, the matched surrogate is selected to maximize the agent’s induced utility less some cost that only depends on the outcome. It is not hard to verify that the maximum weight matching is indeed maximal-in-range, and therefore the agent has no incentive to lie.

But why does the maximum weight matching take exponential time to find? The problem is that we are not given the edge weights. For each edge, we can only sample from a distribution whose mean is the weight of the edge. Simply sample \( t_{-i} \) from \( D_{-i} \) and compute \( v_i(t^{(j)}, x(t^{(k)}, t_{-i})) - p_i(t^{(k)}, t_{-i}) \). Even if we assume that we know the distributions \( (\mathcal{D}_i)_{i \in [n]} \), it still takes time \( \text{poly}(\prod_{i \neq i} |T_i|) \) to compute the weight of a single edge exactly. But why can’t we first estimate the edge weights with samples and find the maximum matching using the estimated weights? The issue is that no matter how many samples we take, the empirical mean will be off by some estimation error. The maximal-in-range property is so fragile that even a tiny bit of estimation error can cause the algorithm to violate the property, making the whole mechanism not incentive compatible. See Example 1 in Appendix A for a more detailed explanation.

**Black-box Reduction for Welfare Maximization** To overcome the difficulty, we turn to another important problem in mechanism design: black-box reduction for welfare maximization, for inspiration. A line of beautiful results [28, 8, 27, 24] initiated by Hartline and Lucier shows that the mechanism design problem of welfare maximization in the Bayesian setting can be black-box reduced to the algorithmic problem of welfare maximization. The replica-surrogate matching is again the central piece in the reduction. Indeed, the idea of replica-surrogate matching was first proposed by Hartline et al. [26, 27], and later introduced by Daskalakis and Weinberg [23] to the study of \( \epsilon \)-BIC to BIC transformation. The main difference of the two scenarios is the way the edge weights are defined. For welfare maximization, the edge weight between

\[ v_i(t_i, x(b)) \equiv \mathbb{E}_{o \sim x(b)} [v_i(t_i, o)]. \]

If we have query access to \( \mathcal{M} \), then on any query bid profile \( b = (b_1, \ldots, b_n) \), we receive an outcome \( o \sim x(b) \) and payments \( p_1(b), \ldots, p_n(b) \).

Equipped with the notations, we are ready to discuss our \( \epsilon \)-BIC to BIC transformation.

**Informal Theorem 1** (\( \epsilon \)-BIC to BIC transformation). Given sample access to a collection of distributions \( (\mathcal{D}_i)_{i \in [n]} \), and query access to an \( \epsilon \)-BIC and individually rational (IR) mechanism \( \mathcal{M} = (x, p) \) with respect to \( \times_{i \in [n]} \mathcal{D}_i \). We can construct another mechanism \( \mathcal{M'} \) that is exactly BIC and IR with respect to \( \times_{i \in [n]} \mathcal{D}_i \), and its revenue is at most \( O(n/\sqrt{\epsilon}) \) worse than the revenue of \( \mathcal{M} \). Moreover, for any bid profile \( b = (b_1, \ldots, b_n) \), \( \mathcal{M'} \) computes an outcome \( o \sim x(b) \) and payments \( p_1(b), \ldots, p_n(b) \) in expected running time \( \text{poly}(\sum_{i \in [n]} |T_i|, 1/\epsilon) \) and makes in expectation at most \( \text{poly}(\sum_{i \in [n]} |T_i|, 1/\epsilon) \) queries to \( \mathcal{M} \).

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\footnote{The true weight \( w_{jk} \equiv \mathbb{E}_{t_{-i} \sim D_{-i}} [v_i(t^{(j)}, x(t^{(k)}, t_{-i})) - (1 - \eta)p_i(t^{(k)}, t_{-i})] \) is computed using a discounted price, but we can ignore the difference for now.}
a replica \( t^{(i)} \) and a surrogate \( t^{(k)} \) is \( v_{jk} \equiv \mathbb{E}_{t \sim \mathcal{D}_{-i}}[v_i(t^{(i)}, x(t^{(k)}, t_{-i}))] \), namely, the interim value for agent \( i \) when her true type is \( t^{(i)} \) but reports \( t^{(k)} \) to \( \mathcal{M} \). To distinguish the two settings, we will refer to the one with interim utilities as edge weights the U-replica-surrogate matching and the one with interim values as edge weights the V-replica-surrogate matching. Clearly, it also takes exponential time to compute the exact maximum weight V-replica-surrogate matching due to the same reason discussed above.

A striking result by Dughmi et al. [24] shows how to circumvent this barrier for welfare maximization. Their solution has the following two main components: (i) a polynomial time maximal-in-range algorithm to solve the maximum entropy regularized matching problem; (ii) the fast exponential Bernoulli race, a new Bernoulli factory, that allows them to execute the algorithm in (i) exactly with only sample access to distributions whose means are the edge weights \( 3 \). They use the algorithm to find a maximum entropy regularized V-replica-surrogate matching, and argue that this matching has approximately maximum weight, which allows them to conclude that their new mechanism loses at most a negligible fraction of the welfare.

**Our Technical Contributions** Our result is directly inspired by [24], but differs in several major ways. Our plan is to design an algorithm with similar guarantees for U-replica-surrogate matchings. A subtle but crucial difference between our problem and theirs is that the V-replica-surrogate matching only contains positive edge weights\(^4\), so their algorithm only needs to search for a perfect matching, while a U-replica-surrogate matching may contain negative edge weights. Directly applying their algorithm to our problem produces an optimum perfect matching, however, the matching could be far away from the true maximum due to the negative edges. One may try to remove the negative edges using samples. However, removing edges based on the empirical means from samples could easily violate the maximal-in-range property. See example 1 in Appendix A.

We provide a surprising reduction from the case of arbitrary edge weights to the case with only positive edge weights. Indeed, our reduction can be succinctly summarized by the following formula, if an edge has weight \( w_{jk} \), set the new weight by applying the \( \delta \)-softplus function to \( w_{jk} \):\(^5\)

\[
\zeta_\delta(w_{jk}) = \delta \cdot \log \left( \frac{\exp(w_{jk}/\delta)}{\exp(w_{jk}/\delta) + 1} \right),
\]

where \( \delta > 0 \) is a parameter of our algorithm. Note that for any value of \( w_{jk} \), \( \zeta_\delta(w_{jk}) \) is always nonnegative! Moreover, the maximum entropy regularized matching on weights \( (\zeta_\delta(w_{jk}))_{jk} \) can be shown to be close to the maximum weight matching on \( (w_{jk})_{jk} \). So it seems that we only need to run the algorithm from [24] on the new weights \( (\zeta_\delta(w_{jk}))_{jk} \). An astute reader may have already realized that being able to run the algorithm on \( (w_{jk})_{jk} \) does not imply that one can run the algorithm on \( (\zeta_\delta(w_{jk}))_{jk} \), as we can only sample from distributions whose means are \( (w_{jk})_{jk} \) but not \( (\zeta_\delta(w_{jk}))_{jk} \). One idea is to construct a Bernoulli factory to simulate a \( \zeta_\delta(w_{jk}) \)-coin using a \( w_{jk} \)-coin. To the best of our knowledge, no such construction exists. We take a different approach and make use of a crucial property of the algorithm from [24]. Namely, if we run their algorithm with the same parameter \( \delta \), the algorithm only needs to sample from the softmax function over the weights. More specifically, with weights \( (w_{jk})_{jk} \), it suffices to have the ability to sample an edge \( (j, k) \) with probability exactly \( \frac{\exp(w_{jk}/\delta)}{\exp(w_{jk}/\delta) + 1} \). Despite the fact that we cannot directly sample from distributions with means \( (\zeta_\delta(w_{jk}))_{jk} \), we can indeed sample edge \( (j, k) \) with exactly the right probability, as

\[
\frac{\exp \left( \zeta_\delta(w_{jk})/\delta \right)}{\sum_{k'} \exp \left( \zeta_\delta(w_{jk'})/\delta \right)} = \frac{\exp \left( w_{jk}/\delta \right) + 1}{\sum_{k'} \left( \exp(w_{jk'}/\delta) + 1 \right)},
\]

which can be sampled efficiently using the fast exponential Bernoulli race given only sample access to distributions with means equal to the original edge weights \( (w_{jk})_{jk} \).

\(^3\)A Bernoulli factory is an algorithm that with sample access to a \( p \)-coin to simulate a \( f(p) \)-coin. Please see [29, 33] and the references therein for more details.

\(^4\)Dughmi et al. [24] make the assumption that for any agent \( i \), type \( t_i \) and any outcome \( o \in \mathcal{O} \), \( v_i(t_i, o) \geq 0 \).

\(^5\)The function \( \log(\exp(x) + 1) \) is known as the soft plus function.
Our second contribution is to show that an approximately maximum U-replica-surrogate matching suffices to guarantee only a small loss in revenue. Previous results [23, 35, 18] only prove the statement for the exactly maximum matching. We provide a more delicate analysis that allows us to extend the statement to approximately maximum matchings. Finally, as the agent may receive negative utility from certain surrogates, we sometimes need to subsidize the agent to ensure individually rationality. We provide a new careful treatment of the payment rule to guarantee that such payments are small compared to the revenue.

**Multi-dimensional Revenue Maximization** We next apply the ε-BIC to BIC transformation to obtain our black-box reduction for revenue maximization. We first introduce the problem formally.

| Multi-Dimensional Revenue Maximization (MRM): Given as input n type distributions \(D_1, \ldots, D_n\) and a set of feasible outcomes \(O\), output a BIC and IR mechanism \(M\) which chooses outcomes from \(O\) with probability 1 and whose expected revenue is optimal relative to any other, possibly randomized, BIC, and IR mechanism with respect to \(D = \times_{i \in [n]} D_i\). |

To state our black-box reduction, we also need to introduce the virtual welfare optimization problem.

| Virtual Welfare Optimization (VWO): Given as input n functions \(C_i(\cdot) : T_i \mapsto \mathbb{R}\) and a set of feasible outcomes \(O\), output an outcome \(o = \arg\max_{x \in O} \sum_{t \in T_i} C_i(t_i) \cdot v_i(t_i, x)\). We refer to the sum \(\sum_{i} \sum_{t \in T_i} C_i(t_i) \cdot v_i(t_i, x)\) as the virtual welfare of outcome \(x\). |

**Informal Theorem 2.** Given distribution \(\times_{i=1}^n D_i\) and oracle access to an \(\alpha\)-approximation Algorithm \(G\) for VWO, we can construct an exactly BIC and IR mechanism \(M = (x, p)\) with respect to \(\times_{i \in [n]} D_i\), that has expected revenue \(\alpha \cdot \text{OPT} - O(n\sqrt{\varepsilon})\), where \(\text{OPT}\) is the optimal revenue over all BIC and IR mechanisms with respect to \(\times_{i \in [n]} D_i\). The running time is \(\text{poly}\left(\sum_{i \in [n]} |T_i|, \frac{1}{\varepsilon}, b, \text{rt}_G\left(\text{poly}\left(\sum_{i \in [n]} |T_i|, \frac{1}{\varepsilon}, b\right)\right)\right)\), where \(\text{rt}_G(\cdot)\) is the running time of \(G\), and \(b\) is an upper bound on the bit complexity of \(v_i(t_i, o)\) for any agent \(i\), any type \(t_i\), and any outcome \(o\).

Our reduction also holds even when we only have sample access to the distribution \(\times_{i=1}^n D_i\) (Theorem 6).

### 1.2 Further Related Work

Multi-dimensional revenue maximization has recently received lots of attention from computer scientists. Significant progress has been on the computational front [19, 20, 1, 11, 2, 12, 13, 17, 15, 3, 9, 22, 30]. On the structural front, a family of simple mechanisms, i.e., variants of sequential posted price and two-part tariff mechanisms, have been shown to achieve constant factor approximations of the optimal revenue in quite general settings [5, 36, 35, 16, 21, 18]. ε-BIC to BIC transformation has been an instrumental tool in obtaining both the computational and structural results [23, 35, 18, 30].

There has also been significant interest in understanding the sample complexity for learning an almost revenue-optimal auction in multi-item settings. Last year, Gonczarowski and Weinberg [25] show that one can learn an almost revenue-optimal ε-BIC mechanism using \(\text{poly}(n, m, 1/\varepsilon)\) samples under the item-independence assumption, where \(n\) is the number of bidders and \(m\) is the number of items. Brustle et al. [10] generalize the result to settings where the item values are drawn from correlated but structured distributions that can be modeled by either Markov random fields or Bayesian Networks. The mechanism they produce is still ε-BIC. Our transformation can certainly convert these mechanisms from [25, 10] into exactly BIC mechanisms, and the transformation requires poly \(\left(\sum_{i \in [n]} |T_i|, 1/\varepsilon\right)\) many samples. Unfortunately, each \(|T_i|\) is already exponential in \(m\) in their settings. The dependence on \(|T_i|\) is unavoidable for us, as our goal is to provide a transformation that is applicable to a general mechanism design setting. Nonetheless, we suspect the techniques we develop in this paper can be combined with special structure of the distribution to provide more sample-efficient ε-BIC to BIC transformations.

### 1.3 Organization of the Paper

In Section 2, we provide the notations that we use throughout the paper. In Section 3, we describe two powerful tools from the literature, the Replica-Surrogate Matching Mechanism and the Entropy Regularized
Matching, that are useful for us. In Section 4, we provide our new algorithm that solves the entropy regularized matching problem for graphs with arbitrary edge weights. In particular, we will show the arbitrary edge weight case can be reduced to the nonnegative edge weight case. In Section 5, we describe our \(\varepsilon\)-BIC to BIC transformation. In Section 6, we show how to use our \(\varepsilon\)-BIC to BIC transformation to improve the black-box reduction for multi-dimensional revenue maximization.

2 Preliminaries

We specify a general mechanism design setting by the tuple \((n, \mathcal{V}, \mathcal{D}, v, \mathcal{O})\). There are \(n\) agents participating in the mechanism. We use \(\mathcal{O}\) to denote the set of all possible outcomes. We assume that each \(o \in \mathcal{O}\) can be written as a vector \(o = (o_1, ..., o_n)\) where \(o_i\) is the outcome for agent \(i\). We also assume that the all outcome \(\bot\) is available to each agent. One can think of \(\bot\) as the option of not participating in the mechanism. In the paper, we assume that the outcome space \(\mathcal{O}\) is downward-closed, that is, for every \(o = (o_1, ..., o_n) \in \mathcal{O}\), any \(o' = (o'_1, ..., o'_n)\) with \(o'_i = o_i\) or \(o'_i = \bot\) for every \(i\) is also in \(\mathcal{O}\). An example of this setting is the combinatorial auction, where the outcome set contains all possible ways to allocate items to agents, and the null outcome represents allocating nothing to the agent.

Each agent \(i\) has a type \(t_i\) from type space \(\mathcal{V}_i\), which is drawn independently from some distribution \(\mathcal{D}_i\). We use \(T_i \subseteq \mathcal{V}_i\) to denote the support of \(\mathcal{D}_i\). We use \(\mathcal{D}\) to denote \(\times_{i \in [n]} \mathcal{D}_i\). In the paper we consider discrete type spaces, we assume that every \(|T_i| \leq T\) for some finite \(T\). Note that our results can easily be extended to the continuous case using similar techniques as in [24]. For every \(t_i \in \mathcal{V}_i\), \(v_i(t_i, \cdot)\) is a valuation function that maps every outcome to a real number in \([0, 1]\). In particular, \(v_i(t_i, \bot) = 0\) for all agent \(i\) and type \(t_i\). Every agent is risk-neutral and has quasi-linear utility.

For any mechanism \(\mathcal{M}\), denote \(\text{REV}(\mathcal{M}, \mathcal{D}) = E_{t \sim \mathcal{D}} \left[ \sum_{i \in [n]} p_i(t) \right]\) the expected revenue of \(\mathcal{M}\). We use \(\text{REV}(\mathcal{M})\) for short when the agents’ distributions and valuation functions are clear. We use the standard definitions of BIC, \(\varepsilon\)-BIC, IR, and \(\varepsilon\)-IR. We include their definitions in Appendix B for completeness. Finally, we use \(\log(\cdot)\) to denote the natural logarithm and \(\Delta^\ell\) to denote the set of all distributions over \(\ell\) elements.

Definition 1 (Gibbs Distribution). For any integer \(\ell\), define the Gibbs distribution \(z \in \Delta^\ell\) over \(\ell\) states with temperature \(\beta\) as \(z_i = \frac{\exp(E_i/\beta)}{\sum_{j \in [\ell]} \exp(E_j/\beta)}\) for all \(i \in [\ell]\), where \(E_i\) is the energy of element \(i\).

3 Tools from the Literature

In this section, we review two crucial tools for us: the replica-surrogate matching mechanism and the online entropy regularized matching.

3.1 Replica-Surrogate Matching Mechanism

We provide a detailed description of the the replica-surrogate matching mechanism used in [23, 35, 18]. For each agent \(i\), the mechanism generates a number of replicas and surrogates from \(\mathcal{D}_i\), and maps the agent’s type \(t_i\) to one of the surrogates via a maximum weight replica-surrogate matching, and charges the agent the corresponding VCG payment. Then let the matched surrogate participate in the mechanism for the agent. Formally, suppose we are given query access to a mechanism \(\mathcal{M} = (x, p)\), we construct a new mechanism \(\mathcal{M}'\) using the following two-phase procedure:

Phase 1: Surrogate Selection. For each agent \(i\),

1. Given her reported type \(t_i \in \mathcal{D}_i\), create \(\ell - 1\) replicas sampled i.i.d. from \(\mathcal{D}_i\) and \(\ell\) surrogates sampled i.i.d. from \(\mathcal{D}_i\). The value of \(\ell\) is specified in Corollary 1.

2. Construct a weighted bipartite graph between replicas (and agent \(i\)’s true type \(t_i\)) and surrogates. The weight between the \(j\)-th replica \(r^{(j)}\) and the \(k\)-th surrogate \(s^{(k)}\) is the interim value of agent \(i\) when
her true type is \( r^{(j)} \) but reported \( s^{(k)} \) to \( M \) less the interim payment for reporting \( s^{(k)} \) multiplied by 
\((1 - \eta)\):

\[
W_i(r^{(j)}, s^{(k)}) = \mathbb{E}_{t_i \sim D_{-i}} \left[v_i(r^{(j)}, x(s^{(k)}, t_{-i})) \right] - (1 - \eta) \cdot \mathbb{E}_{t_i \sim D_{-i}} \left[p_i(s^{(k)}, t_{-i}) \right].
\]  

(1)

3. Treat \( W_i(r^{(j)}, s^{(k)}) \) as the value of replica \( r^{(j)} \) for being matched to surrogate \( s^{(k)} \). Run the VCG mechanism among the replicas, that is, compute the maximum weight matching w.r.t. edge weight \( W_i(\cdot, \cdot) \) and the corresponding VCG payments. If a replica (or type \( t_i \)) is unmatched in the maximum matching, match it to a random unmatched surrogate.

**Phase 2: Surrogate Competition**

Let \( s_i \) be the surrogate matched with the agent \( i \)'s true type \( t_i \). Run mechanism \( M \) under input \( s = (s_1, \ldots, s_n) \). Let \( o = (o_1, \ldots, o_n) \) be a the outcome generated by \( x(s) \). If agent \( i \) is matched in the maximum matching, her outcome is \( o_i \) and her expected payment is \((1 - \eta) \cdot p_i(s)\) plus the VCG payment for winning surrogate \( s_i \) in the first phase; Otherwise the agent gets the null outcome \( \perp \) and pays 0.

![Figure 1: With \( \bullet \) we denote the replicas and with \( ■ \) the surrogates](image)

The following lemma shows that \( M' \) is BIC and IR. We include the proof in Appendix D for completeness.

**Lemma 1.** [26, 8, 23, 35, 18] \( M' \) is BIC and IR.

Moreover, when \( \ell \) is sufficiently large, the revenue of \( M' \) is close to the revenue of \( M \).

**Corollary 1.** [23, 35, 18] If \( M \) is an \( \epsilon \)-BIC and IR mechanism w.r.t. \( D \), then for any \( \eta \in (0, 1) \) and any \( \ell > \frac{T}{2\delta} \), \( \text{REV}(M', D) \geq (1 - \eta)\text{REV}(M, D) - \Theta(n^2\epsilon) \).

Corollary 1 follows from a special case of Lemma 13 when \( \Delta = 0 \) and \( d = 1 \). The main takeaway of this corollary is that the above mechanism \( M' \) indeed satisfies the requirement of an \( \epsilon \)-BIC to BIC transformation. However, as we discussed in Section 1.1, the mechanism runs in exponential time.

### 3.2 Online Entropy Regularized Matching

We introduce another crucial tool, the online entropy regularized matching algorithm, developed by Dughmi et al. [24]. The original application is to find a matching close to the maximum weight V-replica-surrogate matching, but the algorithm is general and can be applied to any \( d \)-to-1 bipartite matching with positive edge weights.
**d-to-1 Matching**  For every integer $\ell$, and $d$, consider the complete bipartite graph between $d\ell$ left hand side nodes (called LHS-nodes) and $\ell$ right hand side nodes (called RHS-nodes). Let $\omega_{jk}$ be the edge weight between LHS-node $j$ and RHS-node $k$ for $j \in [d\ell], k \in [\ell]$. For ease of notation, let $\omega_j = (\omega_{jk})_{k \in [\ell]}, \omega = (\omega_{j})_{j \in [d\ell]}$, and $\omega_{-j} = (\omega_{j'})_{j' \neq j}$. A matching is called a $d$-to-$1$ matching if every LHS-node is matched to at most one RHS-node, and every RHS-node is matched to at most $d$ LHS-nodes. A $d$-to-$1$ matching is called perfect if every LHS-node is matched to one RHS-node, and every RHS-node is matched to exactly $d$ LHS-nodes.

In this section, we focus on the case where all edge weights $\omega$ are nonnegative, and we refer to this case as the nonnegative weight $d$-to-$1$ matching. In Section 4, we generalize the results to arbitrary weights.

The optimal $d$-to-$1$ matching is simply a maximum weight bipartite matching problem. The challenge is that the weights are not given. For every edge $(j,k)$, we only have sample access to a distribution $F_{jk}$ whose expectation is $\omega_{jk}$. To the best of our knowledge, none of the algorithms for finding a maximum weight bipartite matching can be implemented exactly with such sample access to the edge weights. Moreover, as we require the replica-surrogate matching mechanism to be incentive compatible, the algorithm should be maximal-in-range. Therefore, finding the maximum weight matching using the empirical means is also not an option, as it violates the maximal-in-range property (see the discussion in Section 1.1).

Dughmi et al. [24] provide a polynomial time maximal-in-range algorithm to compute an approximately maximum weight perfect $d$-to-$1$ matching. Note that the weight of the optimal perfect matching is the same as the optimal matching for nonnegative edge weights. The first key idea is to find a "soft maximum weight matching" instead of the maximum weight matching by adding an entropy function as a regularizer to the total weight.

**Definition 2.** Given parameter $\delta > 0$, the (offline) entropy regularized matching program (P) is:

\[
\max \sum_{j,k} z_{jk} \cdot \omega_{jk} - \delta \sum_{j,k} z_{jk} \log(z_{jk}) \\
subject to \sum_{j} z_{jk} \leq d, \forall k \in [\ell] \tag{2}
\]

Take the Lagrangian dual of (P) by Lagrangifying the constraints $\sum_{j} z_{jk} \leq d, \forall k \in [\ell]$:

\[
L(z, \alpha) = \sum_{j,k} z_{jk} \omega_{jk} - \delta \sum_{j,k} z_{jk} \log(z_{jk}) - \sum_{k} \alpha_k (d - \sum_{j} z_{jk}).
\]

The following lemma follows from the first-order condition: for any dual variables $\alpha$, the optimal solution for the Lagrangian is given by a collection of Gibbs distribution $z^* = (z^*_{jk})_{j \in [d\ell]}$.

**Lemma 2.** [24] For every dual variables $\alpha \in [0, h]^\ell$, the optimal solution $z^*$ maximizing the Lagrangian $L(z, \alpha)$ subject to constraints $\sum_{k} z^*_{jk} = 1, \forall j \in [d\ell]$ is $z^*_{jk} = \frac{\exp(\frac{\omega_{jk} - \alpha_k}{\delta})}{\sum_{j'} \exp(\frac{\omega_{j'} - \alpha_k}{\delta})}$, $\forall j \in [d\ell], k \in [\ell]$.

If for every edge $(j,k)$, we are given sample access to a distribution $F_{jk}$ whose mean is $\omega_{jk} \in [0,1]$, we can use the fast exponential Bernoulli race [24] to sample from the Gibbs distribution $z^*_{jk}$ for all $j \in [d\ell]$. In particular, each sample from distribution $z^*_{j} = (z^*_{j1}, \ldots, z^*_{j\ell})$ only requires in expectation poly$(h, \ell, 1/\delta)$ many samples from $(F_{jk})_k$ (Corollary 5).

If the optimal dual variables $\alpha^*$ are known, by complementary slackness, the corresponding $z^*$ in Lemma 2 is the optimal solution of (P). The gap between the expected weight of $z^*$ and the maximum weight is at most the value of the maximum entropy $\delta \cdot d\ell \log \ell$, so we can simply use the matching sampled according to the distribution $z^*$. However, as the optimal dual is unknown, the wrong dual variables $\alpha$ may cause a loss of $\sum_k \alpha_k (d - \sum_j z_{jk})$, which may be too large when $z$ is not computed based on the optimal dual
variables. To resolve this difficulty, Dughmi et al. [24] introduce the second key idea – Online Entropy Regularized Matching algorithm (Algorithm 1). The online algorithm gradually learns a set of dual variables close to the optimum \( \alpha \). When the algorithm terminates, it is guaranteed to find a close to optimal solution to program (P).

Algorithm 1 Online Entropy Regularized Matching with Non-negative Edge Weights (with parameters \( \delta, \eta', \gamma \))

Require: Sample access to the distribution \( F_{j\ell} \) whose expectation is \( \omega_{jk} \), for every \( j \in [d\ell], k \in [\ell] \).

1: for \( j \in [d\ell] \) do
2: Let \( d_k^{(j-1)} \) be the number of LHS-nodes matched to RHS-node \( k \) in the current matching and \( K = \{ k : d_k^{(j-1)} < d \} \).
3: Set \( \alpha^{(j)} \) according to the Gibbs distribution with energy \( d_k^{(j-1)} \) for RHS-node \( k \in K \) and temperature \( \frac{1}{\eta'} \), and \( \alpha_k^{(j)} = 0 \) for all \( k \not\in K \).
4: Match LHS-node \( j \) to a RHS-node \( k \in K \) according to the Gibbs distribution \( \hat{z}_j \) over RHS-nodes in \( K \), where the temperature is \( \delta \) and the energy for matching to a RHS-node \( k \in K \) is \( \omega_{jk} - \gamma \alpha_k^{(j)} \). We can generate a sample from \( \hat{z}_j \) via the fast exponential Bernoulli race with \( \text{poly}(\gamma, \ell, 1/\delta) \) sample from \((F_{\ell j})_k \) in expectation (See Corollary 5 for details).
5: end for

Clearly, the algorithm always returns a perfect \( d \)-to-1 matching.

Definition 3 (Maximal-in-Range Algorithms). An algorithm is maximal-in-range, if for every \( j \in [d\ell] \), there exists a cost function \( c(\cdot) \), which may depend on \( \omega_{\cdot j} \), such that the allocation \( z_j = \arg\max_{z' \in F} \sum_{j \in \ell} \hat{z}_{jk} \cdot \omega_{jk} - c(z') \) for any \( \omega_{\cdot j} \), where \( F \) is a set of all feasible allocations.

From Lemma 3, the algorithm is also maximal-in-range for any choice of the parameters \( \delta, \eta', \gamma \).

Lemma 3. [24] For every \( j, \alpha^{(j)} \) and parameter \( \gamma \), the Gibbs distribution \( \hat{z}_j \) (specified in step 4) is maximal-in-range, as

\[
\hat{z}_j = \arg\max_{z' \in \Lambda^K} \sum_{k \in K} z'_{jk} \omega_{jk} - \delta \sum_{k \in K} z'_{jk} \log(z'_{jk}) - \sum_{k \in K} \gamma \alpha_k^{(j)} \cdot z'_{jk}.
\]

How about the performance of Algorithm 1? Dughmi et al. [24] show that for any choice of \( \delta, \eta' > 0 \) and \( \ell \), if \( d \geq \ell \log \ell / \eta'^2 \) and \( \gamma \in \left[ \frac{\text{OPT}(P)}{d}, \frac{O(1) \cdot \text{OPT}(P)}{d} \right] \), where \( \text{OPT}(P) \) is the optimum of program (P), the solution of Algorithm 1 is a \( (1 - O(\eta')) \) multiplicative approximation to \( \text{OPT}(P) \), which implies that the expected weight of the solution is close to the weight of the maximum matching. Theorem 1 summarize these guarantees.

Theorem 1. [24] When \( \omega_{jk} \in [0, 1] \) for all \( j \in [d\ell], k \in [\ell] \), Algorithm 1 satisfies the following properties:

1. For any choice of the parameters, it always returns a perfect \( d \)-to-1 matching.
2. For any choice of the parameters, the algorithm is maximal-in-range. The expected running time and sample complexity of Algorithm 1 is \( \text{poly}(d, \ell, \gamma, 1/\delta) \).
3. For every \( \delta, \eta' > 0 \), if \( d \geq \ell \log \ell / \eta'^2 \) and \( \gamma \in \left[ \frac{\text{OPT}(P)}{d}, \frac{O(1) \cdot \text{OPT}(P)}{d} \right] \), where \( \text{OPT}(P) \) is the optimum of program (P), the expected value (over the randomness of the Algorithm 1) of the total weight of the matching output by the algorithm is at most \( O(d\ell \psi \eta') \) less than the maximum weight matching.

Notice that \( \alpha^{(j)} \) only depends on the weights incident to the LHS-nodes 1 to \( j - 1 \).

The theorem applies to any bounded edge weights \( \omega_{jk} \in [0, R] \). For simplicity we normalize the edge weights to lie between \([0, 1]\).
The only part that we have not yet explained is how to choose a $\gamma$ that is a constant factor approximation to $\frac{\text{OPT}(P)}{d}$. Dughmi et al. [24] show a polynomial time randomized algorithm that produces a $\gamma$ that falls into $[\frac{\text{OPT}(P)}{d}, \frac{O(1)\cdot \text{OPT}(P)}{d}]$ with high probability, which suffices to find a close to optimum $V$-replica-surrogate matching. Please see Appendix C for details.

4 \hspace{1em} \text{\textit{d-to-1 Matching with Arbitrary Edge Weights}}

To obtain an $\epsilon$-BIC to BIC transformation, we need to find a near-optimal $U$-surrogate-replica matching, where edge weights may be negative. Motivated by this application, we provide a generalization of Theorem 1 to general $d$-to-1 matchings with arbitrary edge weights. We design a new algorithm (Algorithm 2) with guarantees summarized in Theorem 2.

Before stating our algorithm, we first point out several issues of directly applying Algorithm 1 to the general $d$-to-1 matching problem. As Algorithm 1 always produces a perfect matching, we will find a perfect matching whose weight is close to the maximum perfect matching, which unfortunately could be far less than the true maximum $^8$.

A tempting way to fix the issue may be to first remove all edges with negative weights then run Algorithm 1. With only sample access to $F_{jk}$, one way to achieve this is to remove edges with negative empirical means. In fact, with a sufficiently large number of samples, with high probability, all edges with strictly positive weights will remain and all edges with strictly negative weights will be removed. However, with non-zero probability, some edges will either be kept or removed incorrectly causing the algorithm to violate the maximal-in-range property. See Example 1 for a concrete construction.

An alternative way is to relax the constraint $\sum_k z_{jk} = 1$ to $\sum_k z_{jk} \leq 1$, so the algorithm no longer needs to find a perfect matching. However, Lemma 2 fails to hold as the optimal solution is no longer a Gibbs distribution and it is unclear how to sample efficiently from it with only sample access to $F_{jk}$ $^9$. A similar attempt is to add a slack variable $y$ to $(P)$, modifying the constraint $\sum_k z_{jk} = 1$ to $\sum_k z_{jk} + y = 1$. It is equivalent to adding one dummy RHS-node, with weight 0 on every incident edge. Now for every dual variable, the optimal solution for the Lagrangian follows from a Gibbs distribution. However, the program differs from $(P)$, in particular the new dummy RHS-node has no capacity constraint, and as a result there is no dual variable that corresponds to this dummy node. It is not clear how to modify Algorithm 1 to accommodate the new dummy node and to produce a close to maximum matching.

4.1 Reduction from Arbitrary Weights to Non-Negative Weights

In this section, we provide a reduction from the $d$-to-1 matching with arbitrary edge weight case to the non-negative edge weight case.

\textbf{Definition 4.} For arbitrary edge weights $(\omega_{jk})_{jk}$ and parameter $\delta > 0$, define the $\delta$-softplus function:

$$\zeta_\delta(\omega_{jk}) = \delta \cdot \log(\exp(\omega_{jk}/\delta) + 1)$$

\footnote{If all the edge weights are negative and arbitrarily small, the optimal perfect matching has arbitrarily negative weight, while the weight of the optimal matching is always non-negative.}$^8$

\footnote{The issue is that $\sum z_{jk}$ may be strictly less than 1 and has a complex expression. It is not clear whether we can sample efficiently from $z_{jk}$ with only sample access to $(F_{jk})_{jk}$. Moreover, even if we can sample from the distribution, the guarantees in Theorem 1 may no longer hold.}$^9$
Consider the entropy regularized matching program \((P')\) w.r.t. weights \((\zeta_\delta(\omega_{jk}))_{jk}\):

\[
\max \sum_{j,k} z_{jk} \cdot \zeta_\delta(\omega_{jk}) - \delta \cdot \sum_{j,k} z_{jk} \log(z_{jk})
\]

subject to \(\sum_{j} z_{jk} \leq d, \quad \forall k \in [\ell]\)

\[
\sum_{k} z_{jk} = 1, \quad \forall j \in [d\ell]
\]

\[
z_{jk} \in [0, 1], \quad \forall j \in [d\ell], \forall k \in [\ell].
\]

Note that \(\zeta_\delta(x) > 0\) for any \(x\), so the program \((P')\) is exactly a \(d\)-to-1 matching with positive edge weights. Let \(\hat{z}\) be the solution produced by Algorithm 1 on \((P')\). We know that

\[
G(\hat{z}) = \sum_{j,k} \hat{z}_{jk} \cdot \zeta_\delta(\omega_{jk}) - \delta \cdot \sum_{j,k} \hat{z}_{jk} \log(\hat{z}_{jk})
\]

is close to the optimal solution of \((P')\). However, if we match a LHS-node \(j\) to a RHS-node according to the Gibbs distribution \(\hat{z}_j\), it is not yet clear how the expected weight of the matching relates to \(G(\hat{z})\), as the real edge weight is \(\omega_{jk}\) instead of \(\zeta_\delta(\omega_{jk})\). Also, we do not know whether the optimum of \((P')\) is close to the maximum weight matching.

To address these two issues, we introduce an auxiliary convex program \((P'')\).

**Definition 5.** For any parameter \(\delta > 0\), we define the following auxiliary convex program \((P'')\):

\[
\max \sum_{j,k} x_{jk} \omega_{jk} - \delta \cdot \sum_{j,k} (x_{jk} \log(x_{jk}) + y_{jk} \log(y_{jk}))
\]

s.t. \(\sum_{j} (x_{jk} + y_{jk}) \leq d, \quad \forall k \in [\ell]\)

\[
\sum_{k} (x_{jk} + y_{jk}) = 1, \quad \forall j \in [d\ell]
\]

\[
x_{jk}, y_{jk} \in [0, 1], \quad \forall j, k
\]

We prove in Lemma 4 that the optimum of \((P'')\) is exactly the same as the optimum of \((P')\). The proof of Lemma 4 is postponed to Appendix E.

**Lemma 4.** For all \(j \in [d\ell]\) and \(k \in [\ell]\), if \(\frac{x_{jk}}{y_{jk}} = \exp(\omega_{jk}/\delta), x_{jk} + y_{jk} = z_{jk}\), then

\[
z_{jk} \cdot \zeta_\delta(\omega_{jk}) - \delta \cdot z_{jk} \log(z_{jk}) = x_{jk} \cdot \omega_{jk} - \delta \cdot x_{jk} \log(x_{jk}) - \delta \cdot y_{jk} \log(y_{jk}).
\]

This implies that the optimal objective values of \((P')\) and \((P'')\) are equal.

Next, we present our algorithm (Algorithm 2), and show that

\[
G(\hat{z}) = F(\widehat{x}, \hat{y}) = \sum_{j,k} \hat{x}_{jk} \omega_{jk} - \delta \sum_{j,k} (\hat{x}_{jk} \log(\hat{x}_{jk}) + \hat{y}_{jk} \log(\hat{y}_{jk})),
\]

where \((\hat{x}_{jk}, \hat{y}_{jk})_{jk}\) is the solution produced by Algorithm 2. In other words, running Algorithm 1 on \((P')\) is equivalent to running Algorithm 2 on \((P'')\).
Algorithm 2 Online Entropy Regularized Matching with Arbitrary Edge Weights (with parameters $\delta, \eta', \gamma$)

**Require**: Sample access to $F_{jk}$ whose mean is $\omega_{jk}$, for every $j, k$.

1. For each RHS-node $k$, add a 0-RHS-node to the bipartite graph with edge weight 0 to every LHS-node. We refer to the $k$-th original RHS-node the $k$-th normal-RHS-node.

2. for $j \in [d\ell]$ do

3. Let $d_k^{(j-1)}$ be the number of LHS-nodes matched to either the $k$-th normal-RHS-node or the $k$-th 0-RHS-node in the current matching and $K = \{k : d_k^{(j-1)} < d\}$.

4. Set $\alpha^{(j)}$ according to the Gibbs distribution over RHS-nodes in $K$, where the energy for any RHS-node $k \in K$ is $d_k^{(j-1)}$ and the temperature is $1/\eta'$. Set $\alpha_k^{(j)} = 0$ for all $k \notin K$.

5. Match LHS-node $j$ to a normal RHS-node (or a 0-RHS-node) $k \in K$ according to the Gibbs distribution over the $2|K|$ RHS-nodes in $K$, where the temperature is $\delta$ and the energy for matching to a normal-RHS-node $k$ is $(\omega_{jk} - \gamma a_k^{(j)})$ and the energy for matching to a 0-RHS-node $k \in K$ is $(-\gamma a_k^{(j)})$. More specifically, match to the normal-RHS-node $k$ with probability

$$
\tilde{x}_jk = \frac{\exp((\omega_{jk} - \gamma a_k^{(j)})/\delta)}{\sum_{k' \in K} \exp((\omega_{jk'} - \gamma a_{k'}^{(j)})/\delta) + \exp((-\gamma a_k^{(j)})/\delta)}
$$

and match to the 0-RHS-node $k$ with probability $\tilde{y}_jk = \frac{\exp((-\gamma a_k^{(j)})/\delta)}{\sum_{k' \in K} \exp((\omega_{jk'} - \gamma a_{k'}^{(j)})/\delta) + \exp((-\gamma a_k^{(j)})/\delta)}$.

We can generate a sample from $(\tilde{x}_j, \tilde{y}_j)$ via the fast exponential Bernoulli race with $\text{poly}(\gamma, \ell, 1/\delta)$ sample from $(F_{jk})_k$ in expectation (See Corollary 5 for details).

6. end for

If we execute Algorithm 1 on a $d$-to-1 matching with weights $(\zeta, \delta(\omega_{jk}))_k$ and Algorithm 2 over weights $(\omega_{jk})_k$ with the same parameters $\delta, \eta', \gamma$, we can couple the two executions so that the dual variables $\alpha_k^{(j)}$ and the remaining capacities $(d_k^{(j)})_k$ are the same for every $j$. Please see a concrete description of the coupling in the proof of Theorem 2 in Appendix E. An important consequence of the coupling is that

$$
\tilde{x}_jk + \tilde{y}_jk = \hat{x}_jk \text{ for every } j \text{ and } k.
$$

To verify this, simply observe that

$$
\tilde{z}_jk = \exp\left(\frac{\zeta, \delta(\omega_{jk}) - \gamma a_k^{(j)}}{\delta}\right) = (\exp(\omega_{jk} + 1) \cdot \exp(-\gamma a_k^{(j)}) = \exp(\frac{\omega_{jk} - \gamma a_k^{(j)}}{\delta}) + \exp(-\gamma a_k^{(j)}) = \tilde{x}_jk + \tilde{y}_jk.
$$

Combining the observation with Lemma 4, we conclude that $G(\hat{z}) = F(\tilde{x}, \tilde{y})$. Therefore, if we choose the correct parameters, $F(\tilde{x}, \tilde{y})$ is a multiplicative approximation to both the optimum of $(P')$ and $(P'')$ as guaranteed by Theorem 1. Since the original $d$-to-1 matching instance does not contain the 0-RHS-nodes, we just ignore those edges in the matching produced by Algorithm 2, and it is not hard to argue that the matching has total weight close to the maximum weight matching. We summarize our result in the following Theorem. The proof is postponed to Appendix E.

**Theorem 2.** When $\omega_{jk} \in [-1, 1]$ for all $j \in [d\ell], k \in [\ell]$, Algorithm 2 satisfies the following properties:

1. For any choice of the parameters, dropping all the edges incident to any 0-RHS-nodes in the matching, the algorithm produces a feasible $d$-to-1 matching (not necessarily perfect).

2. For any choice of the parameters, the algorithm is **maximal-in-range**. The expected running time and sample complexity is $\text{poly}(d, \ell, 1/\delta, \gamma)$.

3. For every $\delta, \eta' > 0$, if $d \geq \ell \log \ell / \eta'^2$ and $\gamma \in \left[\frac{\text{OPT}(P')}{d}, \frac{O(1)\cdot\text{OPT}(P')}{d}\right]$, where $\text{OPT}(P')$ is the optimum of program $(P')$, then the expected value (over the randomness of the Algorithm 2) of $\sum_{j \in [d\ell], k \in [\ell]} \hat{x}_{jk} \omega_{jk}$$ - \delta \sum_{j \in [d\ell]} \hat{x}_{jk} \log(\hat{x}_{jk}) - \delta \sum_{j \in [d\ell]} \hat{y}_{jk} \log(\hat{y}_{jk})$ is at least $(1 - O(\eta')) \cdot \text{OPT}(P')$.

Moreover, for every $\psi \in (0, 1)$, if we set $\delta = \Theta\left(\frac{\psi}{\log^2}\right), \eta' = \Theta(\psi)$, and $d$ and $\gamma$ satisfy the conditions above, then the expected value of $\sum_{j \in [d\ell], k \in [\ell]} \hat{x}_{jk} \omega_{jk}$, the expected total weight of the matching output by the algorithm

---

\[10\] Again the theorem applies to any bounded edge weights in $[-R, R]$. For simplicity we normalize the edge weights to lie in $[-1, 1]$. 

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Mechanism 3 ε-BIC to BIC Transformation (Mechanism $\mathcal{M}'$)

**Require:** Query access to an IR mechanism $\mathcal{M} = (x, p)$ w.r.t. $\mathcal{D} = \bigtimes_{i \in [n]} \mathcal{D}_i$; sample access to the type distribution $\mathcal{D}_i$ and $\mathcal{D}'_i$ for every $i \in [n]$; Parameters $\eta, \eta', \delta, \ell$, and $d \geq 32 \log(8\eta' - 1) / \delta^2 \log^2(\ell)$

**Phase 1: Surrogate Selection**
1: for $i \in [n]$ do
2: Sample $\ell$ surrogates i.i.d. from $\mathcal{D}_i$. We use $s$ to denote all surrogates.
3: Estimate $\gamma$ with parameters $\eta'$ and $\delta$ using the algorithm in Lemma 5.
4: Agent $i$ reports her type $t_i$. Create $d\ell - 1$ replicas sampled i.i.d. from $\mathcal{D}'_i$ and insert $t_i$ into the replicas at a uniformly random position. We use $r$ to denote all the $d\ell$ replicas.
5. For each normal surrogate $k$, also create a 0-surrogate with a special type $\phi$. Create a bipartite graph $G_i$ between the $d\ell$ replicas and $2\ell$ surrogates. Define the weight between the $j$-th replica $r^{(j)}(t_i)$ is also a replica) and the $k$-th normal surrogate $s^{(k)}$ using

\[
W_i(r^{(j)}, s^{(k)}) = \mathbb{E}_{t_{-i} \sim \mathcal{D}_{-i}}[v_i(r^{(j)}, x(s^{(k)}, t_{-i}))] - (1 - \eta) \cdot \mathbb{E}_{t_{-i} \sim \mathcal{D}_{-i}}[p_i(s^{(k)}, t_{-i})].
\]

A 0-surrogate has edge weight 0 to every replica, that is $W_i(r^{(j)}, \phi) = 0$ for all $j$.
6. Run Algorithm 2 on $G_i$ with parameters $\delta, \eta'$, and $\gamma$. For any edge between a replica $r^{(j)}$ and a surrogate $s^{(k)}$, we can sample the edge weight by first sampling $t_{-i}$ from $\mathcal{D}_{-i}$, then query $\mathcal{M}$ on input $(s^{(k)}, t_{-i})$, and compute $v_i(r^{(j)}, x(s^{(k)}, t_{-i})) - (1 - \eta) \cdot p_i(s^{(k)}, t_{-i})$.
7. Suppose the reported type $t_i$ of agent $i$ is matched to the $k$-th normal surrogate or the $k$-th 0-surrogate. Let $s_k$ be the type of the $k$-th normal surrogate.
8. Sample $\lambda$ from $U[0, 1]$ and charge the agent $q_i(t_i, \lambda)$, which is her payment for Phase 1. $q_i(t_i, \lambda)$ is computed via a modified implicit payment (Definition 6).
9. end for

**Phase 2: Surrogate Competition**
10. Run mechanism $\mathcal{M}$ on input $s = (s_1, \ldots, s_n)$. Let $o = (o_1, \ldots, o_n)$ be a random outcome sampled from $x(s)$. If agent $i$ is matched to a normal surrogate in Phase 1, her outcome is $o_i$ and her payment for Phase 2 is $(1 - \eta) \cdot p_i(s)$; otherwise the agent gets the outcome $\perp$ and pays 0 for Phase 2.

(dropping all the edges incident to any 0-RHS-nodes in the matching), has weight at most $O(d\ell\psi)$ less than the maximum weight matching.

5 ε-BIC to BIC Transformation

In this section, we present our ε-BIC to BIC transformation. In Theorem 3, we prove a more general statement where the given mechanism $\mathcal{M}$ is ε-BIC with respect to $\mathcal{D} = \bigtimes_{i \in [n]} \mathcal{D}_i$, while we construct an exactly BIC mechanism $\mathcal{M}'$ with respect to a different distribution $\mathcal{D}' = \bigtimes_{i \in [n]} \mathcal{D}'_i$. If $\mathcal{D} = \mathcal{D}'$, the problem is the ε-BIC to BIC transformation problem. We show that the revenue of $\mathcal{M}'$ under $\mathcal{D}'$ decreases gracefully with respect to the Wasserstein Distance of the two distributions. For every $i$, we denote $d_w(\mathcal{D}_i, \mathcal{D}'_i)$ the $\ell_\infty$-Wasserstein Distance of distributions $\mathcal{D}_i, \mathcal{D}'_i$. We slightly abuse notation and let $d_w(\mathcal{D}, \mathcal{D}') = \sum_{i=1}^n d_w(\mathcal{D}_i, \mathcal{D}'_i)$. We provide the formal definition of the $\ell_\infty$-Wasserstein Distance in Appendix B.

**Theorem 3.** Given sample access to distributions $\mathcal{D} = \bigtimes_{i \in [n]} \mathcal{D}_i$ and $\mathcal{D}' = \bigtimes_{i \in [n]} \mathcal{D}'_i$ and query access to an ε-BIC and IR mechanism $\mathcal{M}$ w.r.t. distribution $\mathcal{D}$. We can construct an exactly BIC and IR mechanism $\mathcal{M}'$ w.r.t. distribution $\mathcal{D}'$, such that

\[
\text{REV}(\mathcal{M}', \mathcal{D}') \geq \text{REV}(\mathcal{M}, \mathcal{D}) - O(n\sqrt{\epsilon}) - O\left(\sqrt{n \cdot d_w(\mathcal{D}, \mathcal{D}')}\right).
\] (4)
On any input bid $b = (b_1, \ldots, b_n)$, $\mathcal{M}'$ computes the outcome and payments in expected running time $\text{poly}(n, T', 1/\epsilon)$ and makes in expectation at most $\text{poly}(n, T', 1/\epsilon)$ queries to $\mathcal{M}$, where $T'$ is the support of $\mathcal{D}'_t$ and $T = \max_{i \in [n]} |T'_i|$.

Furthermore, for any coupling $c_i(\cdot)$ between $\mathcal{D}_t$ and $\mathcal{D}'_t$ such that $v_i$ is non-increasing w.r.t. $c_i(\cdot)$ \(^{11}\) (see Section B for the formal definition), the error bound can be improved as follows:

$$\text{REV}(\mathcal{M}', \mathcal{D}') \geq \text{REV}(\mathcal{M}, \mathcal{D}) - n\sqrt{\epsilon} - O \left( n\eta + \frac{n\epsilon}{\eta} \right) - \frac{\sum_{t} E_{t \sim \mathcal{D}'} \left[ E_{c_i(t)} [v_i(t_i, x'(t)) - v_i(c_i(t), x'(t))] \right]}{\eta},$$

where $x'(\cdot)$ is the allocation rule of $\mathcal{M}'$ and $\eta$ can be chosen to be an arbitrary constant in $(0, 1)$.

Inequality (4) is our main result, and provides a strong guarantee in very general settings. Even though the difference between Inequality (5) and (4) seems small, we like to point out that the difference can be substantial sometimes and there were indeed cases where one needed a sharper version similar to Inequality (5). In particular, one common application of bounds similar to Inequality (5) is when the coupling simply rounds values down. For example, the main results in [18, 30] heavily rely on inequalities similar to Inequality (5), and these results may not be possible if only an Inequality (4) type bounds are used.

The proof of Theorem 3 is postponed to Appendix F.2. When $\mathcal{D} = \mathcal{D}'$, $d_w(\mathcal{D}, \mathcal{D}') = 0$, the following corollary states the $\epsilon$-BIC to BIC transformation.

**Corollary 2.** If $\mathcal{D} = \mathcal{D}'$, $\text{REV}(\mathcal{M}', \mathcal{D}) \geq \text{REV}(\mathcal{M}, \mathcal{D}) - O(n\sqrt{\epsilon})$.

Another useful corollary is when we choose $\mathcal{M}$ to be the optimal BIC and IR mechanism for $\mathcal{D}$, then we can conclude that the optimal revenue under $\mathcal{D}'$ is not far away from the optimal revenue under $\mathcal{D}$.

**Corollary 3.** If $d_w(\mathcal{D}_i, \mathcal{D}'_i) \leq \kappa$ for all $i \in [n]$, let $\text{OPT}(\mathcal{D})$ and $\text{OPT}(\mathcal{D}')$ be the optimal revenue achievable by any BIC and IR mechanism w.r.t. $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then $|\text{OPT}(\mathcal{D}) - \text{OPT}(\mathcal{D}')| \leq O(n \cdot \sqrt{\kappa})$.

Our transformation is described in Mechanism 3. To make our description complete, we present Lemma 5 that specifies an algorithm to estimate $\gamma$ and Definition 6 that defines the payment of Phase 1. We postpone the proof of Lemma 5 to Appendix C. The approach is similar to Dughmi et al. [24].

**Lemma 5.** For any agent $i$, given parameters $\ell$, $\delta$, $\eta'$, and $d \geq \frac{32 \log(8\eta'/\epsilon)}{\delta / \log(\ell)}$, fix $s$ to be the $\ell$ surrogates, first draw $d\ell$ fresh samples from $\mathcal{D}'_i$, which we denote using $r'$. We use $\text{OPT}(\omega)$ to denote the optimum of $\mathcal{D}'$ when the edge weight between the $j$-th replica/LHS-node and the $k$-th normal surrogate/normal RHS-node is $\omega_{jk}$. There exists a randomized algorithm based only on $r'$ and $s$ that computes a $\gamma$ that lies in $\left[ \frac{2 \text{OPT}(\omega(r'))}{d}, \frac{3 \text{OPT}(\omega(r'))}{2d} \right]$ with probability at least $1 - \eta' / 2$, where $\omega_{jk}(r') = W_i(r(i), s(k))$ as defined in Mechanism 3. Moreover, $\gamma$ is at most $O(\max\{\ell, \delta, \log(\ell)\})$ and the algorithm has poly($d, \ell, 1/\eta', 1/\delta$) running time and makes poly($d, \ell, 1/\eta', 1/\delta$) queries to mechanism $\mathcal{M}$.

Furthermore, if $r$ are i.i.d. samples from $\mathcal{D}'_i$, then $\text{OPT}(\omega(r'))$ lies in $\left[ \frac{\text{OPT}(\omega(r))}{2d}, \frac{3 \text{OPT}(\omega(r))}{2d} \right]$ with probability at least $1 - \eta'/2$ over the randomness of $r$ and $r'$, where $\omega_{jk}(r) = W_i(r(i), s(k))$. In this case, $\gamma$ also lies in $\left[ \frac{\text{OPT}(\omega(r))}{d}, \frac{3 \text{OPT}(\omega(r))}{2d} \right]$ with probability at least $1 - \eta'$.

How do we compute the payment of Phase 1? Note that if any agent $i' \in [n]$ reports truthfully, then the surrogate $s_{i'}$ who participates for agent $i'$ in Phase 2 \(^{12}\) is exactly drawn from distribution $\mathcal{D}_{i'}$. Therefore, if all the other agents report truthfully, agent $i'$'s value for winning a normal surrogate $s$ is exactly $W_i(t_i, s)$ and 0 otherwise. In other words, Mechanism 3 is equivalent to a competition among replicas to win surrogates, and the edge weight between a replica and a surrogate is exactly the replica’s value for the surrogate. To

\(^{11}\)Roughly speaking, $v_i$ is non-increasing w.r.t. a coupling $c_i$ if the coupling always couples a “higher” type to a “lower” type. Namely, for all $t_i$ outcome $o \in O$, if the coupling produces type $t_i$ and $c_i(t_i)$, then $v_i(t_i, o) = v_i(c_i(t_i), o)$.

\(^{12}\)Agent $i'$ may be matched to a 0-surrogate, then $s_{i'}$ is the type of the corresponding normal surrogate.
show that Mechanism 3 is BIC, it suffices to prove that the payment of Phase 1 incentivizes the replicas to submit their true edge weights. As Algorithm 2 is maximal-range, such payment rule indeed exists.

If the true type is the $j$-th replica, and the reported type $t_i$ induces edge weights $(\omega_{jk})_{k \in [\ell]}$ charge the agent

$$\delta \sum_{k \in K} x_{jk} \log(x_{jk}) + \delta \sum_{k \in K} y_{jk} \log(y_{jk}) + \sum_{k \in K} \gamma\alpha_k^{(j)} \cdot (x_{jk} + y_{jk}),$$

where $\alpha^{(j)}$ is the set of dual variables in the $j$-th iteration of Algorithm 2, $x_{jk} = \frac{\exp((\omega_{jk} - \gamma\alpha_k^{(j)}/\delta))}{\sum_{k \in K} \exp((\omega_{jk} - \gamma\alpha_k^{(j)}/\delta))}$, and $y_{jk} = \frac{\exp((-\gamma\alpha_k^{(j)}/\delta))}{\sum_{k \in K} \exp((\omega_{jk} - \gamma\alpha_k^{(j)}/\delta)) + \exp(-\gamma\alpha_k^{(j)}/\delta)}$. Observation 1 implies that the payment rule is BIC. However, direct implementation of the payment requires knowing the edge weights which we only have sample access to. We use a procedure called the implicit payment computation [4, 28, 6, 7, 24] to circumvent this difficulty.

**Definition 6 (Implicit Payment Computation).** For any fixed parameters $\delta, \eta, \eta'$ and $\gamma$, let $(\omega_{jk})_{jk}$ be the edge weights on a $[d] \times [2\ell]$ size bipartite graph, we use $A_i(\omega)$ to denote $(\tilde{x}_1, \ldots, \tilde{x}_j, \tilde{y}_j, \ldots, \tilde{y}_{2\ell})$, the allocation of the $j$-th LHS-node/replica to the surrogates computed by Algorithm 2 on the bipartite graph. Now, fix $r$ and $s$, we use $u_i(t_i, (x, y))$ to denote $\sum_{k \in [\ell]} x_k \cdot W_i(t_i, s^{(k)})$. Suppose agent $i$'s reported type $t_i$ is in position $\pi$, that is, $r(\pi) = t_i$. To compute price $q_i(t_i, \lambda)$, let surrogate $s'$ be the surrogate sampled from $A_{\pi}(W)$ by Algorithm 2 in step 6, where $W$ is the collection of edge weights in graph $G_i$ as defined in step 5 of Mechanism 3, and we sample a surrogate $s''$ from $A_{\pi}(\lambda W_{\pi}, W_{-\pi})$, where $W_{\pi}$ contains all weights of the edges incident to the $\pi$-th replica, and $\lambda W_{\pi}$ is simply multiplying each weight in $W_{\pi}$ by $\lambda$. Then we sample $t_{-i}$ from $D_{-i}$, the price $q_i(t_i, \lambda)$ is

$$weight(t_i, s', t_{-i}) - weight(t_i, s'', t_{-i}) - \sqrt{\delta}(\log(2\ell) + 1),$$

where $weight(t_i, s, t_{-i}) = v_i(t_i, x(s, t_{-i})) - (1 - \eta) \cdot \log(p_i(s, t_{-i})$ if $s \neq \emptyset$, otherwise $weight(t_i, s, t_{-i}) = 0$.

In expectation over $s'$, $s''$ and $t_{-i}$,

$$\mathbb{E}[q_i(t_i, \lambda)] = u_i(t_i, A_{\pi}(W)) - u_i(t_i, A_{\pi}(\lambda W_{\pi}, W_{-\pi})) - \sqrt{\delta}(\log(2\ell) + 1),$$

if we also take expectation over $\lambda$,

$$\mathbb{E}_{\lambda \sim U[0,1]} [q_i(t_i, \lambda)] = u_i(t_i, A_{\pi}(W)) - \int_0^1 u_i(t_i, A_{\pi}(\lambda W_{\pi}, W_{-\pi})) d\lambda - \sqrt{\delta}(\log(2\ell) + 1).$$

With Definition 6, our mechanism is fully specified. We proceed to prove that the mechanism is BIC and IR. Our transformation is quite robust. Even if the original mechanism $M$ is not $\varepsilon$-BIC or the $\gamma$ estimated in step 3 is not a constant factor approximation of $\frac{\text{OPT}(\omega(t))}{\text{OPT}(\omega(t))}$, the mechanism is still BIC and IR. The proof for truthfulness is similar to the one in Dughmi et al. [24]. However, as our edge weights may be negative, it is more challenging to establish the individually rationality compared to Dughmi et al. [24]. To make sure the mechanism is IR, we sometimes need to use negative payments to subsidize the agents, and at the same time guarantee that the total subsidy is negligible compared to the overall revenue. Note that this is also different from the previous $\varepsilon$-BIC to BIC transformations [23, 35, 18], as they essentially use the VCG mechanism to match surrogates to replicas, their mechanisms are clearly individually rational and use non-negative payments. The proof of Lemma 6 is postponed to Appendix F.1.

**Lemma 6.** For any choice of the parameters $\ell$, $d$, $\eta$, $\eta'$, $\delta$ and any IR mechanism $M$, $M'$ is a BIC and IR mechanism w.r.t. $D'$. In particular, we do not require $M$ to be $\varepsilon$-BIC. Moreover, each agent $i$'s expected Phase 1 payment $\mathbb{E}[q_i(t_i, \lambda)]$ is at least $-\sqrt{\delta}(\log(2\ell) + 1)$. Finally, on any input bid $b = (b_1, \ldots, b_n)$, $M'$ computes the outcome in expected running time $\text{poly}(d, \ell, 1/\eta', 1/\delta)$ and makes in expectation at most $\text{poly}(d, \ell, 1/\eta', 1/\delta)$ queries to $M$.

\footnote{The difference between $\mathbb{E}_{\lambda \sim U[0,1]} [q_i(t_i, \lambda)]$ and Equation (6) is indeed a fixed constant, hence our mechanism is BIC.}
5.1 Bounding the Revenue Loss

Now we sketch the proof of Theorem 3, it suffices to lower bound the revenue of \( M' \) from the second phase due to Lemma 6. In the previous transformations [23, 35, 18], the mechanism computes an exact maximum weight replica-surrogate matching, which allows them to bound the revenue from the second phase directly. Our mechanism only computes an approximately maximum weight replica-surrogate matching. As a result, we need to use a more delicate analysis to lower bound the revenue from Phase 2. See Appendix F.2 for details. To facilitate our discussion, we define some new notations.

**New Notations:** For every agent \( i \), and the corresponding bipartite graph \( G_i \), we define a new bipartite graph \( \hat{G}_i \) whose left hand side nodes are the replicas/LHS-nodes of \( G_i \). For each normal surrogate/RHS-node of \( G_i \), we duplicate it \( d \) times to form the set of right hand side nodes of \( \hat{G}_i \). For the \( k \)-th surrogate in \( G_i \), the \((a\ell + k)\)-th surrogate in \( \hat{G}_i \) is one of its copies for all \( 0 \leq a \leq d - 1 \). We do not copy the 0-surrogates to \( \hat{G}_i \). The edge weights in \( \hat{G}_i \) are still defined using \( W_i(\cdot, \cdot) \). Clearly, every \( d \)-to-1 matching in \( G_i \) corresponds to a 1-to-1 matching in \( \hat{G}_i \). If replica \( r \) is matched to a surrogate \( s \) in \( G_i \), simply match \( r \) to the first available copy of \( s \) in \( \hat{G}_i \). We use \( \ell' \) to denote \( d\ell \), and \( \hat{G}_i \) has \( 2\ell' \) nodes. When we say the matching in \( \hat{G}_i \) produced by Algorithm 2, we mean the matching in \( \hat{G}_i \) that corresponds to the matching produced by Algorithm 2 in \( G_i \). We follow the convention to use \( \ell^{(j)} \) to denote the type of the \( j \)-th replica and \( s^{(k)} \) to denote the type of the \( k \)-th surrogate in \( \hat{G}_i \). We further simplify the notation and use \( p_i(t_i) \) to denote \( E_{t_i \sim D_i}[p_i(t_i, t_{-i})] \) for any type \( t_i \in T_i \).

Given the replica profile \( r \) and surrogate profile \( s \), for any matching \( L(r, s) \) in \( \hat{G}_i \), we slightly abuse notation to use \( W_i(L(r, s)) \) to denote \( \sum_{(r, s) \in L(r, s)} W_i(r, s) \). When the replica profile \( r \) and surrogate profile \( s \) are clear from context, we simply use \( W_i(L) \) to denote the total weight of the matching \( L \). Since the analysis mainly concerns the set of surrogates that are matched in a matching, we use \( s \in L(r, s) \) to denote that the surrogate \( s \) is matched in \( L(r, s) \). Let \( O(r, s) \) be the (randomized) matching obtained by Algorithm 2 on \( \hat{G}_i \), \( V(r, s) \) be the maximum weight matching in \( \hat{G}_i \).

We first provide a Lemma that relates the expected revenue of \( M' \) to the size of the matchings.

**Lemma 7.** Let \( \text{REV-SECOND}_i(M', D') \) the expected revenue of \( M' \) from agent \( i \) in Phase 2, \( \text{REV}_i(M, D) \) be the expected revenue of \( M \) from agent \( i \),

\[
\text{REV-SECOND}_i(M', D') \geq (1 - \eta) \cdot E_{r, s} \left[ \sum_{s^{(k)} \in O(r, s)} p_i(s^{(k)}) / \ell' \right],
\]

and

\[
\text{REV}_i(M, D) = E_s \left[ \sum_{k \in [\ell']} p_i(s^{(k)}) / \ell' \right].
\]

**Proof.** For every agent \( i \), only when the agent \( i \) is matched to a surrogate in \( O(r, s) \), she pays the surrogate price. We can again first sample \( r \) and \( s \), and run Algorithm 2 on the corresponding graph \( \hat{G}_i \) to find the matching \( O(r, s) \), then choose a replica uniformly at random to be agent \( i \). Since each replica has exactly probability \( 1 / \ell' \) to be agent \( i \), each surrogate in \( O(r, s) \) is selected with probability \( 1 / \ell' \), the expected revenue paid by agent \( i \) is exactly \((1 - \eta) \cdot E_{r, s} \left[ \sum_{s^{(k)} \in O(r, s)} p_i(s^{(k)}) \right] / \ell' \). The expected payment from agent \( i \) in \( M \) is \( E_{t_i \sim D_i}[p_i(t_i)] \). Since each \( s^{(k)} \) is drawn from \( D_i \), this is exactly the same as \( E_s \left[ \sum_{k \in [\ell']} p_i(s^{(k)}) \right] / \ell' \).

In Lemma 8, we bound the gap between \( \text{REV-SECOND}_i(M', D') \) and \( \text{REV}_i(M, D) \). Indeed, we prove a stronger result that holds for any matching \( K(r, s) \) that has close to maximum total weight. The proof of Lemma 8 is postponed to Appendix F.2.
Lemma 8. Recall that $V(r, s)$ is the maximum weight matching in $\tilde{G}_i$. Let $E_{r,s} [W_i (K (r, s))] = E_{r,s} [W_i (V (r, s))] - \Delta$. We have

$$(1 - \eta) \frac{1}{E} \left[ \sum_{s(k) \in K(r,s)} p_i(s(k)) / \ell' \right] \geq \frac{1}{E_s} \left[ \sum_{s(k) \in \mathcal{C}} p_i(s(k)) / \ell' \right] - n \left( \frac{n + \sqrt{\frac{n}{\eta}} \cdot \frac{\Delta}{\eta} + \frac{\Delta}{\eta} \cdot \frac{1}{\ell} \cdot \frac{1}{\ell'} \cdot \frac{1}{\ell'} \cdot \frac{1}{\ell'} \right) - 2 \frac{\Delta}{\eta} \beta_w (D_{r,s}).$$

To prove Theorem 3, one only needs to choose $K(r, s)$ to be the matching $O(r, s)$ produced Algorithm 2, and combine the guarantees in Lemma 6 and 7. Please see Appendix F.2 for more details.

6 Black-box Reduction for Multi-Dimensional Revenue Maximization

In this section, we apply Theorem 3 to the multi-dimensional revenue maximization problem.

Cai et al. [15] provide a reduction from MRM to VWO. More formally:

Theorem 4 (Rephrased from Theorem 2 of Cai et al. [15]). Given the bidders’ type distributions $D = \prod_i D_i$. Let $b$ be an upper bound on the bit complexity of $v_i(t_i, o)$ and $Pr(t_i)$ for any agent $i$, any type $t_i$, and any outcome $o$, and $OPT$ be the optimal revenue achievable by any BIC and IR mechanisms. We further assume that types are normalized, that is, for each agent $i$, type $t_i$ and outcome $o$, $v_i(t_i, o) \in [0, 1]$.

Given oracle access to an $\alpha$-approximation algorithm $G$ for VWO with running time $rt_G(x)$, where $x$ is the bit complexity of the input, there is an algorithm that terminates in time $\text{poly} \left( n, T, \frac{1}{\varepsilon}, b, rt_G \left( \text{poly} \left( n, T, \frac{1}{\varepsilon}, b \right) \right) \right)$, and outputs a mechanism with expected revenue $\alpha \cdot OPT - \varepsilon$ that is $\varepsilon$-BIC with probability at least $1 - \exp(-n/\varepsilon)$. Recall that $T = \max_{\ell \in \mathbb{N}} |T_\ell|$. On any input bid, the mechanism computes the outcome and payments in expected running time $\text{poly} \left( n, T, \frac{1}{\varepsilon}, b, rt_G \left( \text{poly} \left( n, T, \frac{1}{\varepsilon}, b \right) \right) \right)$.

We can apply Theorem 3 to the final mechanism produced by Theorem 4 and obtain an exactly BIC mechanism with almost the same revenue.

Theorem 5. Given the bidders’ type distributions $D = \prod_i D_i$. Let $b$ be an upper bound on the bit complexity of $v_i(t_i, o)$ and $Pr(t_i)$ for any agent $i$, any type $t_i$, and any outcome $o$, and $OPT$ be the optimal revenue achievable by any BIC and IR mechanisms. We further assume that types are normalized, that is, for each agent $i$, type $t_i$ and outcome $o$, $v_i(t_i, o) \in [0, 1]$.

Given oracle access to an $\alpha$-approximation algorithm $G$ for VWO with running time $rt_G(x)$, where $x$ is the bit complexity of the input, there is an algorithm that terminates in time $\text{poly} \left( n, T, \frac{1}{\varepsilon}, b, rt_G \left( \text{poly} \left( n, T, \frac{1}{\varepsilon}, b \right) \right) \right)$, and outputs an exactly BIC and IR mechanism with expected revenue

$$\text{Rev}(M, D) \geq \alpha \cdot OPT - O \left( n \sqrt{\varepsilon} \right),$$

where $T = \max_{\ell \in \mathbb{N}} |T_\ell|$. On any input bid, $M$ computes the outcome and payments in expected running time $\text{poly} \left( n, T, \frac{1}{\varepsilon}, b, rt_G \left( \text{poly} \left( n, T, \frac{1}{\varepsilon}, b \right) \right) \right)$.

The proof of Theorem 5 follows from Theorem 3 and 4. Details are postponed to Appendix G.

Since our Theorem 3 allows us to construct a close to optimal mechanism $M'$ w.r.t. the type distribution $D'$, if $D'$ is not too far away from the distribution $D$ that $M$ is designed, we can approximate the optimal revenue even when we only have sample access to the bidders’ type distributions. A byproduct of this result is that the running time of our algorithm no longer depends on the bit complexity of the probability that a particular type shows up.

Theorem 6. Given sample access to bidders’ type distributions $D = \prod_i D_i$. Let $b$ be an upper bound on the bit complexity of $v_i(t_i, o)$ and $Pr(t_i)$ for any agent $i$, any type $t_i$, and any outcome $o$, and $OPT$ be the optimal revenue achievable by any BIC and IR mechanisms. We further assume that types are normalized, that is, for each agent $i$, type $t_i$ and outcome $o$, $v_i(t_i, o) \in [0, 1]$.  

16
Given oracle access to an $\alpha$-approximation algorithm $G$ for VWO with running time $rt_G(x)$, where $x$ is the bit complexity of the input, there is an algorithm that terminates in time $\text{poly} \left( n, T, \frac{1}{\varepsilon}, b, rt_G \left( \text{poly} \left( n, T, \frac{1}{\varepsilon}, b \right) \right) \right)$, and outputs an exactly BIC and IR mechanism with expected revenue

$$\text{Rev}(M, D) \geq \alpha \cdot \text{OPT} - O \left( n \sqrt{\varepsilon} \right),$$

where $T = \max_{i \in [n]} |T_i|$. On any input bid, $M$ computes the outcome and payments in expected running time $\text{poly} \left( n, T, \frac{1}{\varepsilon}, b, rt_G \left( \text{poly} \left( n, T, \frac{1}{\varepsilon}, b \right) \right) \right)$.

The complete proof of Theorem 6 can be found in Appendix G.

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A \ Examples

**Example 1.** Let $N$ be the number of samples that the algorithm uses to calculate the empirical expectation. Choose $\sigma > 0$ such that $\frac{\sigma}{1 - 2\sigma} < \frac{1}{N}$. Consider the following example with 1 node on each side. There are two instances. For the first instance, the random variable $\mathcal{F}^{(1)}$ attached to this edge is $1$ w.p. $2\sigma$, and $-\frac{\sigma}{1 - 2\sigma}$ with probability $1 - 2\sigma$. The edge weight $\omega^{(1)} = \sigma$. For the second instance, $\mathcal{F}^{(2)}$ is $\sigma$ w.p. $1$ and $\omega^{(2)} = \sigma$.

For the above example, both instances have the same edge weight and any maximal-in-range allocation will always output the same matching. However in the first instance, with probability $(1 - 2\sigma)^N < 1$ the empirical expectation is negative and the two nodes are not matched. While in the second instance the algorithm will always match the two nodes. Thus the output matching is not maximal-in-range. It is well-known that if the allocation is maximal-in-range, there must exist a payment rule such that the agent is incentive-compatible. Thus the algorithm will violate the incentive-compatibility when applied to the replica-surrogate matching.

B \ Additional Preliminaries

**Bayesian Incentive Compatibility and Individually Rationality**

- **Bayesian Incentive Compatible (BIC):**
  $$\mathbb{E}_{t_i \sim D_i} [v_i(t_i, x(t_i, t_{-i})) - p_i(t_i, t_{-i})] \geq \mathbb{E}_{t_i \sim D_i} [v_i(t_i, x(t_i', t_{-i})) - p_i(t_i', t_{-i})], \quad \forall i \in [n], t_i, t_i' \in T_i.$$

- **Individual Rational (IR):**
  $$\mathbb{E}_{t_i \sim D_i} [v_i(t_i, x(t_i, t_{-i})) - p_i(t_i, t_{-i})] \geq 0, \quad \forall i \in [n], t_i \in T_i.$$

- **$\varepsilon$-BIC:**
  $$\mathbb{E}_{t_i \sim D_i} [v_i(t_i, x(t_i, t_{-i})) - p_i(t_i, t_{-i})] \geq \mathbb{E}_{t_i \sim D_i} [v_i(t_i, x(t_i', t_{-i})) - p_i(t_i', t_{-i})] - \varepsilon, \quad \forall i \in [n], t_i, t_i' \in T_i.$$

- **$\varepsilon$-IR:**
  $$\mathbb{E}_{t_i \sim D_i} [v_i(t_i, x(t_i, t_{-i})) - p_i(t_i, t_{-i})] \geq -\varepsilon, \quad \forall i \in [n], t_i \in T_i.$$

**Coupling between Type Distributions:** In order to measure the difference between the two distributions, we will introduce the following definition. Fix every agent $i$. A coupling $c_i(\cdot, \cdot)$ of distribution $D'_i$ and $D_i$ is a joint distribution on the probability space $T'_i \times T_i$ such that the marginal of $c_i$ coincide with $D'_i$ and $D_i$.

In the paper we slightly abuse the notation, denoting $c_i(b)$ a random variable that is distributed according to the conditional distribution of type $t_i$ over $T_i$ when $t'_i = b$. According to the definition of the coupling, when $t'_i \sim D'_i$, $c_i(t'_i) \sim D_i$.

We say $v_i$ is non-increasing w.r.t. the coupling $c_i$ if for all $t_i \in T'_i$, outcome $o \in O$, and every realized type $c_i(t_i), v_i(t_i, o) \geq v_i(c_i(t_i), o)$. Intuitively, the coupling always maps a “higher” type to a “lower” type. Such coupling is common, for example in a combinatorial auction, rounding agent $i$’s value for each bundle of items down to the closest multiples of $\delta$ can be viewed as such a coupling.

**Wasserstein Distance:** For any $t_i, t'_i \in V_i$, let $\text{dist}_i(t_i, t'_i) = \max_{o \in O} |v_i(t_i, o) - v_i(t'_i, o)|$. The $\ell_\infty$-Wasserstein Distance between distribution $D_i$ and $D'_i$ w.r.t. $\text{dist}_i$ is defined as the smallest expected distance among all couplings. Formally,

$$d_w(D_i, D'_i) = \min_{c_i(\cdot, \cdot)} \int \text{dist}_i(t_i, t'_i) dc_i(t_i, t'_i)$$
C Estimating $\gamma$: Approximating the Offline Optimum of the Regularized Matching

In this section, we show how to estimate the parameter $\gamma$ so that it is a constant factor approximation to optimum of program $(P')$ (see Definition 5) on the replica-surgeon matching in Mechanism 3 with high probability. Importantly, the estimate is completely independent from the agent's reported type. Here is the basic idea. We sample the edge weights between $r'$ and $s$, and use the empirical mean to compute the optimal solution of program $(P')$. We show that with polynomially many samples, the optimum of $(P')$ computed based on the empirical means is a constant approximation to the optimum of $(P')$ on the true edge weights with probability almost 1.

Proof of Lemma 5: We prove our statement in two steps. In the first step, we show that if we take polynomially many samples, we can obtain a sufficiently accurate estimate of $\omega(r')_{jk}$ for each edge ($j,k$). We prove that the optimum of $(P')$ on the estimated weights is close to $\text{OPT}(\omega(r'))$. We use $F_{jk}$ to denote the distribution of the random variable $v_j(r''(l), x(s(k), t_{-i})) - (1 - \eta) \cdot p_j(s(k), t_{-i})$, where $t_{-i}$ is distributed according to $D_{-i}$.

Lemma 9 (adapted from [24]). For each edge $(j,k)$ between the $j$-th replica $r''(l)$ and the $k$-th normal surrogate $s(k)$, if we take $N \geq \frac{22\log(4^{d^2d+1})}{d^2\log(n)}$ samples from $F_{jk}$, and use $\hat{\omega}_{jk}(r')$ to denote the empirical mean of these $N$ samples, then with probability at least $1 - \frac{\eta'}{2}$,

$$\frac{\text{OPT}(\omega(r'))}{2} \leq \text{OPT}(\hat{\omega}(r')) \leq 2\text{OPT}(\omega(r')).$$

Proof. By the Chernoff bound, we know that $\text{Pr} \left[ |\omega_{jk}(r) - \hat{\omega}_{jk}(r)| \geq \frac{\delta \log(\ell)}{2} \right] \leq \frac{\eta'}{2}$ for each edge $(j,k)$.

Since there are $d\ell^2$ many edges, by the union bound, we have that with probability at least $1 - \frac{\eta'}{2}$ for each edge $(j,k)$:

$$|\omega_{jk}(r) - \hat{\omega}_{jk}(r)| \leq \frac{\delta \log(\ell)}{2}.$$

Let $(x^*, y^*)$ be the optimal solution of the $(P')$ with edge weights $(\omega_{jk}(r'))_{jk}$ and $(x^{**}, y^{**})$ be the optimal solution of the $(P')$ with edge weights $(\hat{\omega}_{jk}(r'))_{jk}$. Then

\[
\text{OPT}(\omega(r')) = \sum_{j,k} \left( x^*_{jk} \omega_{jk}(r') - \delta \cdot \left( x^*_{jk} \log(x^*_{jk}) + y^*_{jk} \log(y^*_{jk}) \right) \right) \\
\geq \sum_{j,k} \left( x^{**}_{jk} \omega_{jk}(r') - \delta \cdot \left( x^{**}_{jk} \log(x^{**}_{jk}) + y^{**}_{jk} \log(y^{**}_{jk}) \right) \right) \\
\geq \text{OPT}(\hat{\omega}(r')) - \frac{d\ell \delta \log(\ell)}{2} \\
\geq \frac{\text{OPT}(\hat{\omega}(r'))}{2}.
\]

The last inequality holds since a valid assignment is to set $y_{jk} = 1/\ell$ and $x_{jk} = 0$ for each $j, k$, which has objective value $d\ell \log(\ell)$.

The other direction can be proved similarly. 

Let $A$ be the total weight of the maximum weight matching with edge weights $(\hat{\omega}_{jk}(r'))_{jk}$. It is clear that $A$ lies in $[\text{OPT}(\hat{\omega}(r')) - \delta d \ell \log(2\ell), \text{OPT}(\hat{\omega}(r'))]$. Note that if we set $y_{jk} = 1/\ell$ and $x_{jk} = 0$ for each $j, k$, the objective of $(P')$ has value $\delta d \ell \log(\ell)$. Hence, $\max \{ A, \delta d \ell \log(\ell) \}$ is guaranteed to lie in $\left[ \frac{\text{OPT}(\hat{\omega}(r'))}{4}, \text{OPT}(\hat{\omega}(r')) \right]$. If we choose $\gamma$ to be $\frac{12\max \{ A, \delta d \ell \log(\ell) \}}{\delta}$, $\gamma$ is guaranteed to lie in $\left[ \frac{4\text{OPT}(\hat{\omega}(r'))}{\delta}, \frac{12\text{OPT}(\hat{\omega}(r'))}{\delta} \right]$. 

21
Due to Lemma 9, \( \gamma \) lies in \([\frac{3}{2} \cdot \text{OPT}(\omega(r')), \frac{24}{24} \cdot \text{OPT}(\omega(r'))\] with probability at least \( 1 - \eta'/2 \). As \( A \) can be computed in time \( \text{poly}(d, \ell, 1/\eta', 1/\delta) \), \( \gamma \) can also be computed in time \( \text{poly}(d, \ell, 1/\eta', 1/\delta) \).

In the second step of the proof, we show that \( \text{OPT}(\omega(r)) \) and \( \text{OPT}(\omega(r')) \) are close with high probability. We first need the following Lemma to prove \( \text{OPT}(\omega(r)) \) has bounded difference of 2.

**Lemma 10.** For any \( j \in [d\ell] \), any type \( r^{(j)} \) and replica profile \( r \),

\[
\left| \text{OPT}(\omega(r)) - \text{OPT}\left(\omega\left(r^{(j)}, r^{(-j)}\right)\right) \right| \leq 2,
\]

where \( \omega_{jk}\left(r^{(j)}, r^{(-j)}\right) = W_i(r^{(j)}, s^{(k)}) \) and \( \omega_{jk}\left(r^{(j)}, r^{(-j)}\right) = \omega_{jk}(r) \) for any \( j' \neq j \).

**Proof.** Let \((x^*, y^*), (x^{**}, y^{**})\) be the optimal solutions under replica profile \( r \) and \( (r^{(j)}, r^{(-j)}) \) for \( (p') \) respectively. Then

\[
\text{OPT}(\omega(r)) = \sum_{j' \in [d\ell], k \in [\ell]} \left( x^{*}_{jk} \log(x^{*}_{jk}) + y^{*}_{jk} \log(y^{*}_{jk}) \right)
\geq \sum_{j' \in [d\ell], k \in [\ell]} \left( x^{**}_{jk} \log(x^{**}_{jk}) + y^{**}_{jk} \log(y^{**}_{jk}) \right)
\geq \sum_{j' \neq j, k \in [\ell]} \left( x^{**}_{jk} \log(x^{**}_{jk}) + y^{**}_{jk} \log(y^{**}_{jk}) \right)
+ \sum_{k \in [\ell]} \left( x^{*}_{jk} \log(x^{*}_{jk}) + y^{*}_{jk} \log(y^{*}_{jk}) \right) - 2
\]

The last inequality is because both \( \omega_{jk}(r) \) and \( \omega(r^{(j)}, r^{(-j)}) \) lie in \([-1, 1]\). The other direction follows similarly. \( \square \)

Next, we apply McDiarmid’s inequality to the function \( \text{OPT}(\omega(r)) \).

**Lemma 11.** When \( d \geq \frac{32}{\delta d^2 \log(\ell)} \), if both \( r \) and \( r' \) are collections of \( d\ell \) i.i.d. samples from \( D'_l \), then with probability at least \( 1 - \frac{\eta'}{2} \),

\[
\frac{1}{2} \cdot \text{OPT}(\omega(r)) \leq \text{OPT}(\omega(r')) \leq \frac{3}{2} \cdot \text{OPT}(\omega(r)).
\]

The probability is over the randomness of both \( r \) and \( r' \).

**Proof.** Due to Lemma 11, we can apply McDiarmid’s inequality on the function \( \text{OPT}(\omega(r)) \), and we have

\[
\Pr_r \left[ \left| \text{OPT}(\omega(r)) - \mathbb{E}_r[\text{OPT}(\omega(r))] \right| \geq \frac{\delta d \ell \log(\ell)}{4} \right] \leq \frac{\eta'}{4}
\]

Similarly, we have

\[
\Pr_{r'} \left[ \left| \text{OPT}(\omega(r')) - \mathbb{E}_{r'}[\text{OPT}(\omega(r))] \right| \geq \frac{\delta d \ell \log(\ell)}{4} \right] \leq \frac{\eta'}{4}
\]

Hence, with probability at least \( 1 - \frac{\eta'}{2} \),

\[
\left| \text{OPT}(\omega(r)) - \text{OPT}(\omega(r')) \right| \leq \frac{\delta d \ell \log(\ell)}{2}
\]
Since $\delta \ell \log(\ell)$ is a lower bound on both $OPT(\omega(r'))$ and $OPT(\omega(r))$ (by setting $y_{jk} = 1/\ell$ and $x_{jk} = 0$ for each $j, k$) we have that with probability at least $1 - \frac{\eta'}{2}$,

$$
\frac{1}{2} OPT(\omega(r)) \leq OPT(\omega(r')) \leq \frac{3}{2} OPT(\omega(r)).
$$

\[ \square \]

Our statement follows from Lemma 9 and 11 \[ \square \]

**D Missing Details from Section 3**

*Proof of Lemma 1:*

We prove this in two parts, similarly to [18]. First we argue that the distribution of the surrogate $s_i$ that represents the agent, when the agent reports truthfully, is $D_i$. Since we have a perfect matching, an equivalent way of thinking about the process is to draw $\ell$ replicas, produce the perfect matching (the VCG matching plus the uniform matching between the unmatched replicas and surrogates) and then pick one replica uniformly at random to be the agent. These two processes produce the same joint distribution between replicas, surrogates and the agents $i$. So we can just argue about the second process of sampling.

Since the agent is chosen uniformly at random between the replicas in the second process, the surrogate $s_i$ that represents the agent, will also be chosen uniformly at random between all the surrogates. Thus, the distribution of $s_i$ is $D_i$.

We need to argue that for every agent $i$ reporting truthfully is a best response, if every other agent is truthful. In the VCG mechanism, agent $i$ faces a competition with the replicas to win a surrogate. If agent $i$ has type $t_i$, then her value for winning a surrogate with type $s_i$ in the VCG mechanism is exactly the edge weight $W_i(t_i, s_i) = \mathbb{E}_{\tilde{t}_i \sim D_{\tilde{t}_i}}[v_i(t_i, x(s_i, t_{-i}))] - (1 - \eta) \cdot \mathbb{E}_{\tilde{t}_i \sim D_{\tilde{t}_i}}[v_i(t_i, t_{-i})]$.

Clearly, if agent $i$ reports truthfully, the weights on all incident edges between her and all the surrogates will be exactly her value for winning those surrogates. Since agent $i$ is in a VCG mechanism to compete for a surrogate, reporting the true edge weights is a dominant strategy for her, therefore reporting truthfully is also a best response for her assuming the other agents are truthful. It is critical that the other agents are reporting truthfully, otherwise agent $i$’s value for winning a surrogate with type $s_i$ may be different from the weight on the corresponding edge.

\[ \square \]

**E Missing Details from Section 4**

**Observation 1.** For every $j$, $\alpha^{(j)}$ and parameter $\gamma$, match $j$ according to the Gibbs distribution $(\hat{x}_j, \hat{y}_j)$ to the available $2|K|$ RHS-nodes in $K$,

\[
\hat{x}_{jk} = \frac{\exp\left((\omega_{jk} - \gamma \alpha^{(j)}_k)/\delta\right)}{\sum_{k \in K} \left(\exp((\omega_{jk} - \gamma \alpha^{(j)}_k)/\delta)+\exp(-\gamma \alpha^{(j)}_k/\delta)\right)}, \quad \hat{y}_{jk} = \frac{\exp\left(-\gamma \alpha^{(j)}_k/\delta\right)}{\sum_{k \in K} \left(\exp((\omega_{jk} - \gamma \alpha^{(j)}_k)/\delta)+\exp(-\gamma \alpha^{(j)}_k/\delta)\right)}
\]

maximizes

\[
\sum_{k \in K} x_{jk} \omega_{jk} - \delta \sum_{k \in K} x_{jk} \log(x_{jk}) - \delta \sum_{k \in K} y_{jk} \log(y_{jk}) - \sum_{k \in K} \gamma \alpha^{(j)}_k \cdot (x_{jk} + y_{jk})
\]

subject to the constraint $\sum_k (x_{jk} + y_{jk}) = 1$.

**Observation 2.** For every dual variables $\alpha \in [0, h]^{\ell}$ the optimal solution $x^*, y^*$ maximizing the Lagrangian $L((x, y), \alpha)$ of program (P") subject to the constraints $\sum_k (x_{jk} + y_{jk}) = 1, \forall j \in [d\ell]$ is
So the LHS of Equation (7) equals to \( \omega \)

**Lemma 12.**

\[ x_{jk}^* = \frac{\exp\left(\frac{\omega_{jk} - \alpha_k}{\delta}\right)}{\sum_{\ell'} \left(\exp\left(\frac{\omega_{j\ell'} - \alpha_{\ell'}}{\delta}\right) + \exp\left(-\frac{\alpha_{\ell'}}{\delta}\right)\right)}, \quad \forall j \in [d], \forall k \in [\ell] \]

\[ y_{jk}^* = \frac{\exp\left(-\frac{\alpha_k}{\delta}\right)}{\sum_{\ell'} \left(\exp\left(\frac{\omega_{j\ell'} - \alpha_{\ell'}}{\delta}\right) + \exp\left(-\frac{\alpha_{\ell'}}{\delta}\right)\right)}, \quad \forall j \in [d], \forall k \in [\ell] \]

Hence

\[ \frac{x_{jk}^*}{y_{jk}^*} = \exp(\omega_{jk}/\delta), \quad \forall j, k \]

Proof of Lemma 4: For every \( j, k \), we observe that \( \frac{x_{jk}}{z_k} = \frac{\exp(\omega_{jk}/\delta)}{1 + \exp(\omega_{jk}/\delta)} \), hence \( z_k = \frac{x_{jk}(1 + \exp(\omega_{jk}/\delta)}{\exp(\omega_{jk}/\delta))} \). We simplify the equality that we need to prove to Equation (7).

\[ z_k \cdot \delta \omega_{jk} - \delta \cdot z_k \log(z_k) = x_{jk} \cdot \omega_{jk} - \delta \cdot x_{jk} \log(x_{jk}) - \delta \cdot y_{jk} \log(y_{jk}) \]

\[ \iff \quad \delta \omega_{jk} - \delta \log(z_k) = \frac{x_{jk}}{z_k} \omega_{jk} - \frac{x_{jk}}{z_k} \log(x_{jk}) - \frac{y_{jk}}{z_k} \log(y_{jk}) \]

Since:

\[ \delta \log(z_k) = \delta \log\left(\frac{x_{jk}(1 + \exp(\omega_{jk}/\delta))}{\exp(\omega_{jk}/\delta)}\right) = \delta \log(x_{jk}) - \delta \log(\exp(\omega_{jk}/\delta)) + \delta \log(1 + \exp(\omega_{jk}/\delta)) \]

\[ = \delta \log(x_{jk}) - \omega_{jk} + \zeta(\omega_{jk}) \]  

(8)

So the LHS of Equation (7) equals to \( \omega_{jk} - \delta \log(x_{jk}) \). How about the RHS of Equation (7)?

\[ \frac{x_{jk}}{z_k} \omega_{jk} - \delta \frac{x_{jk}}{z_k} \log(x_{jk}) - \frac{y_{jk}}{z_k} \log(y_{jk}) \]

\[ = \frac{\exp(\omega_{jk}/\delta)}{1 + \exp(\omega_{jk}/\delta)} \omega_{jk} - \frac{\exp(\omega_{jk}/\delta)}{1 + \exp(\omega_{jk}/\delta)} \log(x_{jk}) - \delta \frac{1}{1 + \exp(\omega_{jk}/\delta)} \log \left( \frac{x_{jk}}{\exp(\omega_{jk}/\delta)} \right) \]

\[ = \frac{\exp(\omega_{jk}/\delta)}{1 + \exp(\omega_{jk}/\delta)} \omega_{jk} - \frac{\exp(\omega_{jk}/\delta)}{1 + \exp(\omega_{jk}/\delta)} \log(x_{jk}) - \frac{1}{1 + \exp(\omega_{jk}/\delta)} \log(x_{jk}) + \delta \frac{1}{1 + \exp(\omega_{jk}/\delta)} \log(\exp(\omega_{jk}/\delta)) \]

\[ = \frac{\exp(\omega_{jk}/\delta)}{1 + \exp(\omega_{jk}/\delta)} \omega_{jk} - \exp(\omega_{jk}/\delta) \log(x_{jk}) - \frac{1}{1 + \exp(\omega_{jk}/\delta)} \log(x_{jk}) + \frac{1}{1 + \exp(\omega_{jk}/\delta)} \omega_{jk} \]

\[ = \omega_{jk} - \delta \log(x_{jk}) \]

Hence, Equation (7) holds. Since the optimal values \( x_{jk}^*, y_{jk}^* \) satisfy the requirements by Observation 2, we have that the optimum of \( (P') \) is at least as large as the optimum of \( (P'') \). On the other hand, let \( z^* \) be the optimal solution of \( (P') \), we can choose \( x_{jk}^*, y_{jk}^* \) so that \( x_{jk}^* + y_{jk}^* = z_{jk}^* \) and \( \frac{y_{jk}^*}{z_{jk}^*} = \exp(\omega_{jk}/\delta) \). Clearly, \( (x_{jk}^*, y_{jk}^*) \) is a feasible solution to \( (P'') \), therefore the optimum of \( (P') \) is at most as large as the optimum of \( (P'') \). Combining the two claims, we prove that \( (P') \) and \( (P'') \) have the same optimal objective values. \( \square \)

**Lemma 12.** With parameter \( \delta \geq 0 \), let \( (x^*, y^*) \) be the optimal solution of \( (P') \). The optimum of \( (P'') \), \( \sum_{j,k} x_{jk}^* \omega_{jk} - \delta \cdot \sum_{j,k} (x_{jk}^* \log(x_{jk}^*) + y_{jk}^* \log(y_{jk}^*)) \), is no smaller than the weight of the maximum weight matching.
Proof. Let \( x' \) be the maximum weight matching. It is not hard to see that we can construct a 0 – 1 vector \( y' \) so that \( (x', y') \) is a feasible solution of \( (P') \). As both \( x' \) and \( y' \) only take values in 0 or 1, the entropy term – \( \sum_{jk} x'_j \log(x'_j) - \sum_{jk} y'_j \log(y'_j) = 0 \). Hence, the optimum of \( (P') \) is at least as large as the weight of the maximum weight matching \( \sum_{jk} x'_j \omega_{jk} \).

\[ \square \]

Proof of Theorem 2: As the algorithm always produces a matching that respects the constraints of \( (P') \), the first property clearly holds. As the set of available RHS-nodes \( K \) and the dual variables \( a^{(i)} \) only depend on the first \( j - 1 \) LHS-nodes but not the LHS-node \( j \), the maximal-in-range property follows from Observation 1. The algorithm runs in \( d\ell \) rounds, step 3 and 4 both take \( O(\ell) \) time. Step 5 takes expected time poly(\( \gamma, \ell, 1/\delta \)) many samples from distributions \( (F_{jk})_k \) to complete. Hence, the running time and sample complexity as stated in the second property.

If we execute Algorithm 1 on a \( d \)-to-1 matching with weights \( (\zeta_{\delta}(\omega_{jk}))_k \) and Algorithm 2 over weights \( (\omega_{jk})_k \) with the same parameters \( \delta, \eta', \gamma \), we can couple the two executions so that the dual variables \( a^{(i)} \) and the remaining capacities \( (d^{(i)}_k)_k \) are the same for every \( j \). We introduce the new notation \( K^{(j)} \) which is exactly the set of available RHS-nodes \( K \) in step 2 of both algorithm in round \( j \). Note that \( K^{(j)} \) is deterministically determined by \( (d^{(j-1)}_k)_k \). If \( a^{(j-1)} \) and \( K^{(j)} \) are the same in both algorithms for every \( j \), then \( \hat{x}_{jk} + \hat{y}_{jk} = \hat{z}_{jk} \) for every \( j \in [d\ell] \) and \( k \in K^{(j)} \). To verify this, simply observe that

\[
\hat{z}_{jk} = \exp(\frac{\zeta_{\delta}(\omega_{jk}) - \gamma a^{(j)}_k}{\delta}) = (\exp(\frac{\omega_{jk}}{\delta}) + 1) \cdot \exp(-\gamma a^{(j)}_k) = \exp(\frac{\omega_{jk} - \gamma a^{(j)}_k}{\delta}) + \exp(-\gamma a^{(j)}_k) = \hat{x}_{jk} + \hat{y}_{jk}.
\]

How does the coupling work? We construct it by induction. In the base case where \( j = 1 \), clearly everything is the same in both algorithms. Suppose both dual variables \( a^{(1)}, \ldots, a^{(j)} \) and the remaining capacities \( (d^{(1)}_{jk}, \ldots, d^{(j)}_{jk})_k \) are all the same for the first \( j \) rounds, we argue that we can couple the two executions in round \( j + 1 \) so that \( a^{(j+1)} \) and \( (d^{(j+1)}_k)_k \) remain the same in both algorithms. First, the set \( K^{(j+1)} \) is the same, which implies that the dual variables \( a^{(j+1)} \) are also the same. Next, Algorithm 1 samples a RHS-node \( k \) according to distribution \( \hat{z}_{j+1} \) and Algorithm 2 samples a RHS-node according to distribution \( (\hat{x}_{j+1} k, \hat{y}_{j+1}) \). Note that \( \hat{x}_{(j+1)k} + \hat{y}_{(j+1)k} = \hat{z}_{(j+1)k} \), so wherever Algorithm 1 matches the LHS-node \( j + 1 \) to a RHS-node \( k \), we choose the LHS-node \( j + 1 \) to the normal RHS-node \( k \) with probability \( \frac{\hat{x}_{(j+1)k}}{\hat{z}_{(j+1)k}} \) and to the 0-RHS-node with probability \( \frac{\hat{y}_{(j+1)k}}{\hat{z}_{(j+1)k}} \). Clearly, this coupling makes sure the new remaining capacities \( (d^{(j+1)}_k)_k \) also remain the same. Combining the coupling with Lemma 4, we conclude that

\[
G(\hat{z}) = \sum_{jk} \hat{z}_{jk} \cdot \zeta_{\delta}(\omega_{jk}) - \delta \cdot \sum_{jk} \hat{z}_{jk} \log(\hat{z}_{jk}) = \sum_{jk} \hat{x}_{jk} \omega_{jk} - \delta \cdot \sum_{jk} (\hat{x}_{jk} \log(\hat{x}_{jk}) + \hat{y}_{jk} \log(\hat{y}_{jk})) = F(\hat{x}, \hat{y}).
\]

By Theorem 1, the expected value of \( G(\hat{z}) \) is a \( (1 - O(\eta')) \) multiplicative approximation to \( \text{OPT}(P') \), if we choose the parameters according to the third property of the statement. Therefore, the expected value of \( F(\hat{x}, \hat{y}) \) is a \( (1 - O(\eta')) \) multiplicative approximation to \( \text{OPT}(P') \). Since the optimum of \( (P') \), \( \text{OPT}(P') \), is the same as \( \text{OPT}(P') \) (Lemma 4), the expected value of \( F(\hat{x}, \hat{y}) \) is also a \( (1 - O(\eta')) \) multiplicative approximation to \( \text{OPT}(P') \). Now, invoke Lemma 12, we know that the expected value of \( F(\hat{x}, \hat{y}) \) is at least a \( (1 - O(\eta')) \) multiplicative approximation to the weight of the maximum weight matching, which we denote as \( \text{OPT} \). Note that the entropy term – \( \delta \cdot (\sum_{jk} \hat{x}_{jk} \log(\hat{x}_{jk}) + \sum_{jk} \hat{y}_{jk} \log(\hat{y}_{jk})) \) is non-negative and at most \( \delta d\ell \log(2\ell) \), hence the expected weight of the matching produced by Algorithm 2, the expected value of \( \sum_{jk} \hat{x}_{jk} \omega_{jk} \), is at least \( (1 - O(\eta')) \cdot \text{OPT} - \delta d\ell \log(2\ell) \).

If we choose \( \delta = \Theta(\frac{\psi}{\log\eta}) \), \( \eta' = \Theta(\psi) \), then \( \delta d\ell \log(2\ell) = \Theta(d\ell \psi) \) and \( O(\eta') \cdot \text{OPT} = O(d\ell \psi) \) as \( \text{OPT} \leq d\ell \). Thus, the expected weight of the matching produced by Algorithm 2 is within an additive error of \( \Theta(d\ell \psi) \) from the weight of the maximum weight matching. This completes our proof for the third property.

\[ \square \]
F Missing Details from Section 5

F.1 BIC, IR, and Implicit Payment Computation

Proof of Lemma 6:

\[ \mathcal{M}' \text{ is BIC:} \] We prove the Bayesian Incentive Compatibility in two parts. The first part is similar to the proof of Lemma 1. We argue that the distribution of the normal surrogate \( s_i \) that represents agent \( i \) in Phase 2, when the agent \( i \) reports truthfully, is \( D_i \). Note that for any matching Algorithm 2 produces, the \( k \)-th normal surrogate and the \( k \)-th 0-surrogate together are matched to exactly \( d \) replicas for every \( k \in [\ell] \). As the \( d\ell - 1 \) replicas and the agent’s type are all drawn from the same distribution \( D_i \), we can simply treat all of them as replicas and uniformly choose one to be the agent reported type after Algorithm 2 terminates. Therefore, the surrogate \( s_i \) that represents the agent, will also be chosen uniformly at random between all the normal surrogates. Thus, the distribution of \( s_i \) is \( D_i \).

If all the other agents report truthfully, agent \( i \)'s value for winning a surrogate \( s \) is exactly \( W_i(t_i, s) \) if her true type is \( t_i \). In other words, under the assumption that all other agents report truthfully, Mechanism 3 for agent \( i \) is equivalent to a competition among replicas to win surrogates, and the edge weight between a replica and a surrogate is exactly the replica’s value for winning the surrogate. To show that Mechanism 3 is BIC, it suffices to prove that at any position \( \pi \),

\[
\pi \text{ is BIC, it suffices to prove that at any position } \pi,
\]

\[ u_i(t_i, A_\pi(W(t'_i))) - \mathbb{E}_\lambda[q_i(t_i, \lambda)] \]

is maximized when the reported type \( t'_i \) equals to the true type \( t_i \). Here \( W(t'_i) \) is simply the collection of the edge weights when \( r(\pi) = t'_i \), and the function \( u_i(\cdot) \) is defined in Definition 6. A result by Rochet [34] implies that this is indeed the case. Interested readers can find a modern restatement of the result in Theorem 2.1 of [6].

\[ \mathcal{M}' \text{ is IR:} \] The expected utility for agent \( i \) with type \( t_i \) at position \( \pi \) is

\[
u_i(t_i, A_\pi(W)) - \mathbb{E}_\lambda[q_i(t_i, \lambda)] = \int_0^1 u_i(t_i, A_\pi(\lambda W_\pi, W_{-\pi})) d\lambda + \sqrt{\delta}(\log 2 \ell + 1), \tag{11} \]

where \( W \) is the collection of weights in \( G_i \) when agent \( i \) reports truthfully. We will first prove that for any \( \lambda \in [0, 1] \), \( u_i(t_i, A_\pi(\lambda W_\pi, W_{-\pi})) \) is at least \( -\frac{\lambda \log 2(\ell)}{\lambda} \). Denote \( H(x, y) = -\sum x_k \log(x_k) + \sum y_k \log(y_k) \) as the entropy for distribution \( (x, y) \).

Let \( (x''_\pi, y''_\pi) \) be \( A_\pi(\lambda W_\pi, W_{-\pi}) \). By Observation 1,

\[
(x''_\pi, y''_\pi) = \arg\max_{(x, y)} \sum_k x_{\pi k} \cdot \lambda W_{\pi k} + \delta \cdot H(x_{\pi k}, y_{\pi k}) - \sum_k \gamma a_k^{(\pi)} \cdot (x_{\pi k} + y_{\pi k})
\]

By considering an alternative solution \((0, x''_\pi + y''_\pi)\), we have

\[
\sum_k x''_{\pi k} \cdot \lambda W_{\pi k} + \delta \cdot H(x''_\pi, y''_\pi) - \sum_k \gamma a_k^{(\pi)} \cdot (x''_{\pi k} + y''_{\pi k}) \\
\ge 0 - \delta \cdot \sum_k (x''_{\pi k} + y''_{\pi k}) \log(x''_{\pi k} + y''_{\pi k}) - \sum_k \gamma a_k^{(\pi)} \cdot (x''_{\pi k} + y''_{\pi k})
\]

Since \( -\sum_k (x''_{\pi k} + y''_{\pi k}) \log(x''_{\pi k} + y''_{\pi k}) \ge 0 \),

---

To apply Theorem 2.1 of [6] to our setting, one should think of each surrogate as an outcome, and the corresponding edge weight as the value for the outcome. In other words, a replica’s type is the weights on the incident edges. As the matching is computed by a maximal-in-range algorithm, we can allow the edge weights to be arbitrary numbers, and the induced allocation rule will still be implementable in an incentive compatible way. As a result, we can apply Theorem 2.1 of [6] to our setting. Note that the incentive compatible payment rule it gives is off by an absolute constant compared to our payment rule in Definition 6.
Another lower bound for $E_W$ weight complexity. According to Lemma 5, Step 3 has poly \( \lambda \) time and query complexity. All steps except Step 3, 6, and 8 clearly has poly time and query complexity. Since $\gamma$ is guaranteed to be at most $\max\{\ell, \delta \ell \log \ell\}$, Algorithm 2 in Step 6 has time and query complexity $\text{poly}(d, \ell, 1/\delta)$ according to Theorem 2. From Definition 6, it is clear that Step 8 also has time and query complexity at most $\text{poly}(d, \ell, 1/\delta)$. Hence, the mechanism $M'$ has time and query complexity $\text{poly}(d, \ell, 1/\eta', 1/\delta)$.

\[ u_i(t_i, A_\pi(\lambda W_{\pi}, W_{-\pi})) = \sum_k x'_{nk} \cdot W_{nk} \geq -\delta \log(2\ell) - \frac{\delta}{\lambda} H(x''_{n}, y''_{n}) \geq -\frac{\delta \log(2\ell)}{\lambda}. \]

Next, we prove that $M'$ does not lose too much revenue by subsidizing the agents in Phase 1.

\[ \mathbb{E}_\lambda[q_i(t_i, \lambda)] \text{ is at least } -\sqrt{\delta}(\log(2\ell) + 1): \] It suffices to show that

\[ u_i(t_i, A_\pi(W)) \geq u_i(t_i, A_\pi(\lambda W_{\pi}, W_{-\pi})) \]

for any $\lambda \in [0, 1)$. We still use $(x'_n, y'_n)$ to denote $A_\pi(\lambda W_{\pi}, W_{-\pi})$ and $(x''_n, y''_n)$ to denote $A_\pi(W)$.

By Observation 1, both allocations are maximal-in-range for the same dual variables $\alpha$. Hence, the following two inequalities are true.

\[ \sum_k \hat{x}_{nk} W_{nk} + \delta \cdot H(\hat{x}_{\pi, \hat{y}_{\pi}}) \geq \sum_k \gamma \alpha_k^{(\pi)} \cdot (x'_{nk} + y'_nk) \]

\[ \sum_k x''_{nk} \lambda W_{nk} + \delta \cdot H(x''_{n}, y''_{n}) \geq \sum_k \gamma \alpha_k^{(\pi)} \cdot (x''_{nk} + y''_{nk}) \]

Summing up the two inequalities together, we have

\[ \sum_k (\hat{x}_{nk} - x''_{nk}) W_{nk}(1 - \lambda) \geq 0. \]

Since $\lambda \in [0, 1)$,

\[ u_i(t_i, A_\pi(W)) - u_i(t_i, A_\pi(\lambda W_{\pi}, W_{-\pi})) = \sum_k (\hat{x}_{nk} - x''_{nk}) W_{nk} \geq 0. \]

Finally, we analyze the time and query complexity of the mechanism.

**Time and Query Complexity:** All steps except Step 3, 6, and 8 clearly has $\text{poly}(d, \ell)$ time and query complexity. According to Lemma 5, Step 3 has $\text{poly}(d, \ell, 1/\eta', 1/\delta)$ time and query complexity. Since $\gamma$ is guaranteed to be at most $\max\{\ell, \delta \ell \log \ell\}$, Algorithm 2 in Step 6 has time and query complexity $\text{poly}(d, \ell, 1/\delta)$ according to Theorem 2. From Definition 6, it is clear that Step 8 also has time and query complexity at most $\text{poly}(d, \ell, 1/\delta)$. Hence, the mechanism $M'$ has time and query complexity $\text{poly}(d, \ell, 1/\eta', 1/\delta)$.

\[ \square \]

### F.2 Missing Details of Section 5.1

Instead of proving Lemma 8, we prove the following strengthened version of the statement.
Lemma 13. Let \( \mathbb{E}_{r,s} [W_i (K (r, s))] = \mathbb{E}_{r,s} [W_i (V (r, s))] - \Delta \). We have

\[
(1 - \eta) \frac{1}{r \cdot s} \mathbb{E}_{r,s} \left[ \sum_{s(k) \in K (r, s)} p_i (s(k)) / \ell' \right] \geq \mathbb{E}_{s} \left[ \sum_{k \in [\ell']} p_i (s(k)) / \ell' \right] - \left( \eta + \sqrt{\frac{\mathbb{E}_{r,s} \left[ |T_i'| \right]}{\ell} + \frac{\epsilon}{\eta} + \frac{\Delta}{d \ell \eta} \right) - \frac{2 \cdot d_w (D_i, D_i')}{{\eta} / \ell'}.
\]

Moreover, for any coupling \( c_i (\cdot) \) that \( v_i \) is non-increasing w.r.t. \( c_i (\cdot) \), the last term can be improved to

\[
- \frac{1}{\eta \ell'} \mathbb{E}_{r,s} \left[ \sum_{(r(j), s(k)) \in K (r, s)} c_i (r, r(j), s(k), t_{i-j}) \left| s(k) \right| \right] \geq \mathbb{E}_{r,s} \left[ c_i (r(j), s(k), t_{i-j}) \right] - \sqrt{\frac{d |T_i'| \cdot \ell}{d \ell \eta}}.
\]

Proof of Lemma 13: To prove the statement, we consider an arbitrary coupling \( c_i (\cdot, \cdot) \) of distribution \( D'_i \) and \( D_i \) (see Section B for our definition for coupling between type distributions). For every replica \( r \in T_i' \), \( c_i (r) \) is a random type from \( T_i \). For every realization of the types \( c_i (r) = \left( c_i (r(j)) \right)_{j \in [\ell']} \), we consider the maximal matching that matches a replica \( r(j) \) with a surrogate \( s(k) \) only if \( c_i (r(j)) = s(k) \). We denote the matching as \( L(c_i (r), s) \) and refer to it as the maximal coupled same-type matching. In the next Lemma, we argue that in expectation of \( r, s \) and the realization of \( c_i (r) \), the expected size of \( L(c_i (r), s) \) is close to \( \ell' \).

Lemma 14. For any \( r, s \), and realization of \( c_i (r) = \left( c_i (r(j)) \right)_{j \in [\ell']} \) let \( L(c_i (r), s) \) be a maximal coupled same-type matching, then

\[
\mathbb{E}_{r,s,c_i (r)} \left[ |L(c_i (r), s)| \right] \geq \ell' - \sqrt{d |T_i'| \cdot \ell}.
\]

Proof. To prove the result, we first invoke the following Lemma.

Lemma 15 (Adapted from [26]). Let \( r' \) be \( N \) replicas drawn i.i.d. from distribution \( D'_i \) and \( s' \) be \( N \) surrogates drawn i.i.d. from distribution \( D_i \). For any coupling \( c_i (\cdot, \cdot) \) between \( D'_i \) and \( D_i \), the expected cardinality of a maximal matching that only matches a replica \( r \) and a surrogate \( s \) when \( c_i (r) = s \) is at least \( N - \sqrt{d |T_i'| \cdot N} \). The expectation is over the randomness of \( r' \), \( s' \), and the coupling \( c_i (r') \).

Although we have \( \ell' \) replicas and \( \ell' \) surrogates, we cannot directly apply Lemma 15, as the surrogates are not i.i.d. samples from \( D_i \). Instead, we partition \( \hat{G}_i \) into \( d \) subgraphs. The \( a \)-th subgraph contains all replicas \( r(j) \) and surrogates \( s(k) \) with \( j \) and \( k \) lie in \( [a \ell + 1, (a + 1) \ell] \). If we only consider the \( a \)-th subgraph, due to our construction of \( \hat{G}_i \), the replicas are all sampled i.i.d. from \( D'_i \) and the surrogates are also sampled i.i.d. from \( D_i \). Therefore, Lemma 15 implies that a maximal coupled same-type matching in the \( a \)-th subgraph has expected size at least \( \ell - \sqrt{d |T_i'| \ell} \). Since there are \( d \) subgraphs, so the expected size of a maximal coupled same-type matching is at least \( \ell' - \sqrt{d |T_i'| \cdot \ell'} \). \[ \square \]

Now, it suffices to argue that the total payment from surrogates that are in \( L(c_i (r), s) \) but not in \( K(r, s) \) is small. Indeed, when \( K(r, s) \) is the maximal weight matching, one can directly prove the claim. However, \( K(r, s) \) only has approximately maximum weight, and it appears to be difficult to directly compare \( K(r, s) \) with \( L(c_i (r), s) \). Instead, we construct an auxiliary matching based on both \( K(r, s) \) and \( L(c_i (r), s) \). For any \( r, s \) and realization of types \( \left( c_i (r(j)) \right)_{j \in [\ell']} \) we decompose the union of these two matchings into a set of disjoint alternating paths and cycles. Every surrogate that appears in \( L(c_i (r), s) \) but not in \( K(r, s) \) must be an endpoint of some alternating path. These alternating paths have the following two forms:

(a). It starts with a surrogate in \( L(c_i (r), s) \setminus K(r, s) \) and ends with a surrogate in \( K(r, s) \setminus L(c_i (r), s) \) with the form \( \left( s(1), r(1), s(2), r(2), \ldots, r(a), s(a + 1) \right) \).

\[ \text{For the rest of the proof, when we use the notation } \mathbb{E}_{c_i (r) (\cdot)} \text{, we are taking the expectation over the randomness of the coupling. The } c_i (r) = \left( c_i (r(j)) \right)_{j \in [\ell']} \text{ inside the expectation is the realized type after coupling.} \]

28
(b). It starts with a surrogate in \( L(c_i(r), s) \setminus K(r, s) \) and ends with a replica with the form 
\[
(s_1, r_1, s_2, r_2, \ldots, s_a, r_a).
\]

We use \( P \) to denote the set of all alternating paths of form (a) and (b). We construct a new matching \( K'(c_i(r), s) \) as follows: start with the matching \( K(r, s) \), for any alternating path \( P \) of form (a) and (b), swap the edges in \( K(r, s) \) with the ones in \( L(c_i(r), s) \), that is, replace all edges in \( P \cap K(r, s) \) with edges in \( P \cap L(c_i(r), s) \). Since all the alternating paths are disjoint, \( K'(c_i(r), s) \) is indeed a matching.

**Corollary 4.**
\[
\mathbb{E}_{r, s, c_i(r)} \left[ \sum_{s(k) \in K'(c_i(r), s)} p_i(s(k)) / \ell' \right] \geq \mathbb{E}_{s} \left[ \sum_{k \in \ell'} p_i(s(k)) / \ell' \right] - \sqrt{|T'| / \ell}.
\]

**Proof.** Fix \( r, s \) and types \( \{ c_i(r^{(j)}) \}_j \subseteq \ell' \). For any alternating path \( P \) of form either (a) or (b), \( P \cap L(c_i(r), s) \) is the same as \( P \cap K'(c_i(r), s) \). For other alternating paths, the matched surrogate in \( P \cap L(c_i(r), s) \) is a subset of \( P \cap K(r, s) \). Thus the number of the matched surrogates in \( K'(c_i(r), s) \) is at least \( |L(c_i(r), s)| \). By Lemma 14, \( \mathbb{E}_{r, s, c_i(r)} \left[ |\{ k : s(k) \not\in K'(c_i(r), s) \}| \right] \leq \sqrt{d |T'| / \ell'} \). As \( M \) is IR, \( p_i(s) \leq 1 \) for any surrogate \( s \in T_i \). Therefore,
\[
\mathbb{E}_{r, s, c_i(r)} \left[ \sum_{s(k) \in K'(c_i(r), s)} p_i(s(k)) / \ell' \right] \geq \mathbb{E}_{s} \left[ \sum_{k \in \ell'} p_i(s(k)) / \ell' \right] - \sqrt{|T'| / \ell}.
\]

Equipped with Corollary 4, we only need to compare \( K(r, s) \) with \( K'(c_i(r), s) \).

**Lemma 16.**
\[
\mathbb{E}_{r, s} \left[ \sum_{s(k) \in K(r, s)} p_i(s(k)) / \ell' \right] \geq \mathbb{E}_{r, s, c_i(r)} \left[ \sum_{s(k) \in K'(c_i(r), s)} p_i(s(k)) / \ell' \right] - \frac{1}{\eta} (\varepsilon + \Delta) - \frac{2}{\eta} d_{\omega}(D_i, D_i')
\]

Moreover, for any coupling \( c_i \) such that \( v_i \) is non-increasing w.r.t. \( c_i \), the last term can be improved to
\[-\frac{1}{\eta} \mathbb{E}_{r, s, D_i}[v_i(t, x'(t)) - v_i(c_i(t), x'(t))] \]

**Proof.** Fix any \( r, s \) and realization of \( c_i(r) \). Observe that if we decompose the union of \( K(r, s) \) and \( K'(c_i(r), s) \) into alternating path and cycles, we will end up with many length 2 cycles and all the alternating paths in \( P \). Hence, we only need to consider the paths in \( P \).

Consider any \( k \in [a] \) if the path has form (a) (or \( k \in [a - 1] \) if the path has form (b)), note that \( c_i(r_k) = s_k(r_k) \), as this is also an edge in the matching \( L(c_i(r), s) \). Since \( M \) is \( \varepsilon \)-BIC, we have
\[
\mathbb{E}_{t_i \sim D_i} \left[ v_i(c_i(r_k), x(s_k, t_{-i})) - p_i(s_k) \right] \geq \mathbb{E}_{t_i \sim D_i} \left[ v_i(c_i(r_k), x(s_{k+1}, t_{-i})) - p_i(s_{k+1}) \right] - \varepsilon,
\]

which is equivalent to
\[
W_i(r_k, s_k) - W_i(r_k, s_{k+1}) \geq -\varepsilon - \eta \cdot (p_i(s_{k+1}) - p_i(s_k)) + \Delta_{i,c_i}(r_k, s_k) - \Delta_{i,c_i}(r_k, s_{k+1})
\]

where
\[
\Delta_{i,c_i}(r, s) = \mathbb{E}_{t_i \sim D_i} [v_i(r, x(s, t_{-i})) - v_i(c_i(r), x(s, t_{-i}))].
\]

By summing up Inequality (12) for each \( k \), we are able to relate the difference of the total weight between \( K(r, s) \) and \( K'(c_i(r), s) \) with the total payment from surrogates in \( K(r, s) \) and \( K'(c_i(r), s) \).
• For any form (a) path,

\[ \sum_{k=1}^{a} \left( W_{i}(r_{(k)}, s_{(k+1)}) - W_{i}(r_{(k)}, s_{(k)}) \right) \]

\[ \leq a \cdot \epsilon + \eta \cdot (p_{i}(s_{(a+1)}) - p_{i}(s_{(1)})) - \sum_{k=1}^{a} \left( \Delta_{i, c_{i}}(r_{(k)}, s_{(k)}) - \Delta_{i, c_{i}}(r_{(k)}, s_{(k+1)}) \right) \]

• For any form (b) path

\[ \sum_{k=1}^{a-1} \left( W_{i}(r_{(k)}, s_{(k+1)}) - W_{i}(r_{(k)}, s_{(k)}) \right) - W_{i}(r_{(a)}, s_{(a)}) \]

\[ \leq (a - 1) \cdot \epsilon + \eta \cdot \sum_{k=1}^{a-1} (p_{i}(s_{(k+1)}) - p_{i}(s_{(k)})) - W_{i}(r_{(a)}, s_{(a)}) - \sum_{k=1}^{a-1} \left( \Delta_{i, c_{i}}(r_{(k)}, s_{(k)}) - \Delta_{i, c_{i}}(r_{(k)}, s_{(k+1)}) \right) \]

\[ \leq (a - 1) \cdot \epsilon - \eta \cdot p_{i}(s_{(1)}) - \sum_{k=1}^{a} \Delta_{i, c_{i}}(r_{(k)}, s_{(k)}) + \sum_{k=1}^{a-1} \Delta_{i, c_{i}}(r_{(k)}, s_{(k+1)}) \]

The last inequality is because \( \eta \cdot p_{i}(s_{(a)}) - W_{i}(r_{(a)}, s_{(a)}) \leq -\Delta_{i, c_{i}}(r_{(a)}, s_{(a)}) \), which is implied by the fact that \( M \) is IR.

To sum up, for any alternating path \( P \in \mathcal{P} \),

\[ \sum_{(r^{(i)}, s^{(i)}) \in P \cap K(r, s)} W_{i}(r^{(i)}, s^{(i)}) - \sum_{(r^{(i)}, s^{(i)}) \in P \cap K'(c_{i}(r), s)} W_{i}(r^{(i)}, s^{(i)}) \]

\[ \leq |P \cap K(r, s)| \cdot \epsilon + \eta \left[ \sum_{s^{(i)} \in P \cap K(r, s)} p_{i}(s^{(i)}) - \sum_{s^{(i)} \in P \cap K'(c_{i}(r), s)} p_{i}(s^{(i)}) \right] + \text{DIFF}(P), \quad (13) \]

where \( \text{DIFF}(P) = \sum_{(r^{(i)}, s^{(i)}) \in P \cap K(r, s)} \Delta_{i, c_{i}}(r^{(i)}, s^{(i)}) - \sum_{(r^{(i)}, s^{(i)}) \in P \cap K'(c_{i}(r), s)} \Delta_{i, c_{i}}(r^{(i)}, s^{(i)}) \).

Since \( V(r, s) \) is the maximum weight matching, we have

\[ \sum_{P \in \mathcal{P}} \left[ \sum_{(r^{(i)}, s^{(i)}) \in P \cap K(r, s)} W_{i}(r^{(i)}, s^{(i)}) - \sum_{(r^{(i)}, s^{(i)}) \in P \cap K'(c_{i}(r), s)} W_{i}(r^{(i)}, s^{(i)}) \right] \]

\[ = W_{i}(K(r, s)) - W_{i}(K'(c_{i}(r), s)) \]

\[ \geq W_{i}(K(r, s)) - W_{i}(V(r, s)) \]

(14)

Note that if we are using the matching \( L(c_{i}(r), s) \) instead of \( K'(c_{i}(r), s) \), we can no longer prove Inequality (14). The reason is quite subtle. It is possible that \( L(c_{i}(r), s) \) has much higher weight than \( K(r, s) \) on paths in \( \mathcal{P} \), but much smaller weight on the rest alternating path and cycles. In that case, the first equal sign will be replaced by a less equal sign, which makes the inequality meaningless. By comparing to \( K'(c_{i}(r), s) \), we can avoid this issue.
Combining Inequality (13) and (14), we have

\[
W_i(K(r, s)) - W_i(V(r, s)) \leq \sum_{p \in P} \left[ \sum_{(r', s') \in p \cap K(r, s)} W_i(r', s') - \sum_{(r', s') \in p \cap K'(c_i(r), s)} W_i(r', s') \right] \\
\leq \sum_{p \in P} \left[ |P \cap K(r, s)| \cdot \epsilon + \eta \cdot \left[ \sum_{s' \in P} p_i(s') - \sum_{s' \in P} p_i(s') \right] + \text{DIFF}(P) \right] \\
\leq \ell' \cdot \epsilon + \eta \cdot \left[ \sum_{s' \in P} p_i(s') - \sum_{s' \in P} p_i(s') \right] + \sum_{p \in P} \text{DIFF}(P)
\]

Finally, we take expectation over \( r, s, \) and \( c_i(r) \).

\[
\mathbb{E}_{r, s} \left[ \sum_{s' \in K(r, s)} p_i(s') \right] - \mathbb{E}_{r, s, c_i(r)} \left[ \sum_{s' \in K'(c_i(r), s)} p_i(s') \right] \\
\geq \frac{1}{\eta} \left( -\ell' \cdot \epsilon + \mathbb{E}_{r, s} [W_i(K(r, s))] - \mathbb{E}_{r, s} [W_i(V(r, s))] - \mathbb{E}_{r, s, c_i(r)} \left[ \sum_{P \in P} \text{DIFF}(P) \right] \right) \\
\geq -\frac{1}{\eta} (\ell' \cdot \epsilon + \Delta) - \frac{1}{\eta} \mathbb{E}_{r, s, c_i(r)} \left[ \sum_{P \in P} \text{DIFF}(P) \right]
\]

For every type \( r, s, \) and realized type \( c_i(r) \), \( \Delta_{i,c_i}(r, s) = \mathbb{E}_{t_{-i} \sim D_{-i}} [v_i(r, x(s, t_{-i})) - v_i(c_i(r), x(s, t_{-i}))] \in [-\text{dist}_i(r, c_i(r)), \text{dist}_i(r, c_i(r))] \) (recall that \( \text{dist}_i(r, c_i(r)) = \max_{o \in \mathcal{O}} |v_i(r, o) - v_i(c_i(r), o)| \)). Thus

\[
\sum_{p \in P} \text{DIFF}(P) \leq 2 \sum_{j=1}^\ell' \text{dist}_i(r^{(j)}, c_i(r^{(j)})),
\]

and

\[
\mathbb{E}_{r, s, c_i(r)} \left[ \sum_{P \in P} \text{DIFF}(P) \right] \leq 2\ell' d_w(D, D').
\]

Therefore,

\[
\mathbb{E}_{r, s} \left[ \sum_{s' \in K(r, s)} p_i(s') \right] \geq \mathbb{E}_{r, s, c_i(r)} \left[ \sum_{s' \in K'(c_i(r), s)} p_i(s') \right] - \frac{1}{\eta} (\epsilon + \Delta) - \frac{2}{\eta} d_w(D, D').
\]

If \( v_i \) is non-increasing w.r.t. \( c_i \), then \( \Delta_{i,c_i}(\cdot, \cdot) \) is a non-negative function. Then

\[
\sum_{p \in P} \text{DIFF}(P) \leq \sum_{p \in P} \sum_{(r', s') \in p \cap K(r, s)} \Delta_{i,c_i}(r^{(j)}, s^{(k)}) \leq \sum_{(r', s') \in K(r, s)} \Delta_{i,c_i}(r^{(j)}, s^{(k)}),
\]

and

\[
\mathbb{E}_{r, s, c_i(r)} \left[ \sum_{P \in P} \text{DIFF}(P) \right] \leq \mathbb{E}_{r, s} \left[ \sum_{(r', s') \in K(r, s)} \mathbb{E}_{t_{-i} \sim D_{-i}} [v_i(r^{(j)}, x(s^{(k)}, t_{-i})) - v_i(c_i(r^{(j)}), x(s^{(k)}, t_{-i}))] \right].
\]
Finally, we are ready to prove Lemma 13. Note that for every \( s^{(k)} \), \( p_i(s^{(k)}) \leq 1 \) since \( \mathcal{M} \) is \(-\mathcal{I}R\). We have

\[
(1 - \eta) \cdot \mathbb{E}_{r,s} \left[ \sum_{s^{(k)} \in K(r,s)} p_i(s^{(k)}) / \ell' \right] \geq \mathbb{E}_{r,s} \left[ \sum_{s^{(k)} \in K(r,s)} p_i(s^{(k)}) / \ell' \right] - \eta.
\]

The lemma follows from Lemma 16 and Corollary 4. □

**Proof of Theorem 3:** First, by Lemma 7, we can lower bound the revenue of \( \mathcal{M}' \) under \( \mathcal{D}' \) from agent \( i \) in Phase 2 \( \text{REV-SECOND}_i(\mathcal{M}', \mathcal{D}') \) by \((1 - \eta) \mathbb{E}_{r,s} \left[ \sum_{s^{(k)} \in O(r,s)} p_i(s^{(k)}) / \ell' \right] \), where \( O(r,s) \) is the matching produced by Algorithm 2 on \( \hat{G}_i \). Lemma 7 also provides an equivalent expression for the revenue of \( \mathcal{M} \) under \( \mathcal{D} \) from agent \( i \): \( \text{REV}_i(\mathcal{M}, \mathcal{D}) = \mathbb{E}_s \left[ \sum_{s^{(k)} \in \mathcal{D}} p_i(s^{(k)}) / \ell' \right] \).

We choose the parameters according to Theorem 2, that is, for any \( \eta \in (0,1) \), we set \( \delta = \Theta(\frac{\psi}{\log^2 \ell}), \eta' = \Theta(\psi) \) and \( d \geq \frac{\log \ell}{\eta'^2} \). Theorem 2 implies that \( \mathbb{E}_{r,s}[W_i(O(r,s))] = \mathbb{E}_{r,s}[W_i(V(r,s))] - \Theta(d \ell' \psi) \), that is in expectation \( O(r,s) \) has close to maximum weight. We will specify the choice of the other parameters \( \ell, \eta, \) and \( \psi \) later. By Lemma 13, we know that

\[
\text{REV-SECOND}_i(\mathcal{M}', \mathcal{D}') \geq \text{REV}_i(\mathcal{M}, \mathcal{D}) - \left( \eta + \sqrt{\frac{T'}{\ell}} \right) + \frac{\epsilon}{\eta} + \frac{O(\psi)}{\eta} - \frac{2d_w(D_i, D'_i)}{\eta} \tag{15}
\]

Combining Inequality (15) with Lemma 6, we can obtain the following lower bound on \( \text{REV}(\mathcal{M}', \mathcal{D}') \).

\[
\text{REV}(\mathcal{M}', \mathcal{D}') \geq \text{REV}(\mathcal{M}, \mathcal{D}) - \sum_{i \in [n]} \left( \eta + \sqrt{\frac{T'}{\ell}} \right) + \frac{\epsilon}{\eta} + \frac{O(\psi)}{\eta} - \frac{2d_w(D_i, D'_i)}{\eta} - n\sqrt{\delta(\log(2\ell) + 1)} \tag{16}
\]

Now we set \( \ell = \frac{T'}{T}, \psi = \frac{\epsilon^2}{\log^2 \ell} \), and we can choose \( \eta \) to be \( O\left( \frac{\sqrt{T} + \frac{d_w(D_i, D'_i)}{n}}{\epsilon} \right) \) so that \( \text{REV}(\mathcal{M}', \mathcal{D}') \geq \text{REV}(\mathcal{M}, \mathcal{D}) - O\left( \sqrt{n \cdot d_w(D_i, D'_i)} \right) \). 

Plugging in our choice of the parameters to Lemma 6, we can conclude that both the computational and query complexity of \( \mathcal{M}' \) is \( \text{poly}(n, T', 1/\epsilon) \).

If \( c_i(\cdot) \) that \( v_i \) is non-increasing w.r.t. \( c_i(\cdot) \), we can replace the last term \(-\frac{2d_w(D_i, D'_i)}{\eta}\) in Inequality (15) by

\[
-\frac{1}{\eta \ell} \mathbb{E}_{r,s} \left[ \sum_{(r^{(i)}, s^{(k)}) \in O(r,s)} c_i(r^{(i)}, t_{r^{(i)}}, s^{(k)}, t_{s^{(k)}}) \left[ v_i(r^{(i)}, x(s^{(k)}, t_{-s^{(k)}})) - v_i(c_i(r^{(i)}), x(s^{(k)}, t_{-s^{(k)}})) \right] \right] .
\]

Note that this quantity is the same as

\[
-\frac{1}{\eta} \mathbb{E}_{t \sim \mathcal{D}'} \left[ \mathbb{E}_{c_i(t_i)} \left[ v_i(t_i, x'(t)) - v_i(c_i(t_i), x'(t)) \right] \right] .
\]

Hence, for any \( \eta \in (0,1) \), we can improve the result to

\[
\text{REV}(\mathcal{M}', \mathcal{D}') \geq \text{REV}(\mathcal{M}, \mathcal{D}) - n\sqrt{\epsilon} - O\left( n\eta + \frac{n\epsilon}{\eta} \right) - \frac{\sum_{i \in [n]} \mathbb{E}_{t \sim \mathcal{D}'} \left[ \mathbb{E}_{c_i(t_i)} \left[ v_i(t_i, x'(t)) - v_i(c_i(t_i), x'(t)) \right] \right]}{\eta} .
\]

□

32
G Missing Details of Section 6

Proof of Theorem 5: When the mechanism computed by Theorem 4 is $\epsilon$-BIC, our transformation converts it into a BIC mechanism with at most $O(n\sqrt{\epsilon})$ less revenue. The important property of our transformation as stated in Lemma 6 is that even if the initial mechanism is not $\epsilon$-BIC, our transformation still produces an exactly BIC mechanism. In this case, we can still treat the given mechanism as 1-BIC and IR, and use the corresponding revenue guarantees provided by Theorem 3. Since the probability that the mechanism computed by Theorem 4 is not $\epsilon$-BIC is exponentially small, we can absorb the loss from this exponentially small event in the error term $O(n\sqrt{\epsilon})$. The time complexity follows from Theorem 3 and 4. \qed

Proof of Theorem 6: We can create an empirical distribution $\tilde{D}_i$ for each bidder $i$, such that $d_{TV}(D_i, \tilde{D}_i) \leq \epsilon', \forall i$, with probability at least $1 - \theta$ using $O(\sum_i \frac{TV_2(\tilde{D}_i)}{\epsilon'} \ln \frac{\sum_i |D_i|}{\theta})$ samples.

We first consider the case where $d_{TV}(D_i, \tilde{D}_i) \leq \epsilon', \forall i$. Then, $d_w(D_i, \tilde{D}_i) \leq d_{TV}(D_i, \tilde{D}_i) \leq \epsilon'$, as the highest value for any outcome is at most 1. Apply Theorem 4 on $\tilde{D} = X_{i=1}^n \tilde{D}_i$ and let $\mathcal{M}$ be the produced mechanism, $\tilde{OPT}$ be the optimal revenue achievable by any BIC and IR mechanism w.r.t. $\tilde{D}$. Clearly, Theorem 4 guarantees that $\text{REV}(\tilde{M}, \tilde{D}) \geq \alpha \cdot \tilde{OPT} - \epsilon$. According to Corollary 3, $|\text{OPT} - \tilde{OPT}| \leq O(n\sqrt{\epsilon})$.

We set $\theta = \epsilon$ and $\epsilon' = \epsilon$, we apply Theorem 3 to $\tilde{M}$, that is, the replicas are sampled from $D$ and the surrogates are sampled from $\tilde{D}$. Let $\mathcal{M}$ be the constructed mechanism, and Theorem 3 guarantees that $\text{REV}(\mathcal{M}, D) \geq \text{REV}(\tilde{M}, \tilde{D}) - O(n\sqrt{\epsilon}) \geq \alpha \cdot \tilde{OPT} - O(n\sqrt{\epsilon}) \geq \alpha \cdot \text{OPT} - O(n\sqrt{\epsilon})$, if $\tilde{M}$ is a $\epsilon$-BIC and IR mechanism. Otherwise, we know that $\tilde{M}$ is a 1-BIC and IR mechanism, and this happens with exponentially small probability according to Theorem 4, so we can absorb the loss from this case in $O(n\sqrt{\epsilon})$. To sum up, if $d_{TV}(D_i, \tilde{D}_i) \leq \epsilon', \forall i$, then $\text{REV}(\mathcal{M}, D) \geq \alpha \cdot \text{OPT} - O(n\sqrt{\epsilon})$.

With probability $\epsilon$ we may get unlucky and $d_{TV}(D_i, \tilde{D}_i)$ may be larger than $\epsilon$ for some $i$. In that case we still construct $\mathcal{M}$ in the same way, and we can apply Theorem 3 by upper bounding $d_w(D, \tilde{D})$ by $n$ and treating $\tilde{M}$ as a 1-BIC and IR mechanism, which shows $\text{REV}(\mathcal{M}, D) \geq -O(n)$.

Therefore, in expectation of the randomness of the samples used to estimate $\tilde{D}$, $\text{REV}(\mathcal{M}, D) \geq (1 - \epsilon) \cdot (\alpha \cdot \text{OPT} - O(n\sqrt{\epsilon})) - O(n\epsilon) = \alpha \cdot \text{OPT} - O(n\sqrt{\epsilon})$,

as $\text{OPT} \leq n$. Note that even though mechanism $\mathcal{M}$ depends on $\tilde{D}$ and $\tilde{M}$, it is always BIC and IR w.r.t. $D$. The time complexity follows from Theorem 3 and 4. \qed

H A Brief Introduction to Bernoulli Factories

Suppose we are given a coin with bias $\mu$, can we construct another coin with bias $f(\mu)$ using the original coin? If the answer is yes, then how many flips do we need from the original coin to simulate the new coin? A framework that tackles this problem is called Bernoulli Factories. We refer the reader to [31] for a survey on this topic.

Definition 7 (Keane and O’Brien [29]). Given some function $f : (0, 1) \mapsto (0, 1)$ and black-box access to independent samples of a Bernoulli random variable with bias $p$, the Bernoulli factory problem is to generate a sample from a Bernoulli distribution with bias $f(p)$.

A useful generalization of the previous model is the following\textsuperscript{16}: given sample access to distributions $D_1, D_2, \ldots, D_m$ with expectations $\mu_1, \mu_2, \ldots, \mu_m \in (0, 1)$, and a function $f : (0, 1)^m \mapsto \Delta(X)$, where $X$ is a set

\textsuperscript{16}The model is called Expectations from Samples in [24].
of feasible outcomes and $\Delta(X)$ is a set of probability distributions over these outcomes, how can we want

generate a sample from $f(\mu_1, \ldots, \mu_m)$?

Below we state an important result from [24], which we use in this paper. It proposes an algorithm
called Fast Exponential Bernoulli Race with $X = [m]$. For every $\lambda > 0$, it produce a sample from the Gibbs
distribution with temperature $\frac{1}{\lambda}$ and energy $\mu_i$ for each outcome $i$, given only sample access to distributions
$D_1, D_2, \ldots, D_m$.

**Theorem 7.** [24] Given any parameter $\lambda > 0$ and sample access to distributions $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ with expectations
$\mu_1, \mu_2, \ldots, \mu_m \in (0, 1)$, there exists an algorithm that can sample from a Gibbs distribution in $\Delta^m$, where

$$z_i = \frac{\exp(\lambda \mu_i)}{\sum_{j \in [m]} \exp(\lambda \mu_j)},$$

using $O(\lambda^4 m^2 \log(\lambda m))$ samples in expectation.

In both Algorithm 1 and 2, every LHS-node is matched to a random RHS-node according to some Gibbs
distribution. The following corollary from Theorem [24] states that such a sample can be generated using (in
expectation) polynomially many samples in $\ell, \frac{1}{\lambda}$ and $\gamma$, which is the maximum value of the dual variables.

**Corollary 5.** For any integer $m$, any $\delta > 0$, and any $(\alpha_k)_{k \in [m]} \in [0, h]^m$, given sample access to distributions
$\mathcal{F}_1, \ldots, \mathcal{F}_m$ with expectations $w_1, \ldots, w_m \in [-1, 1]$, a sample from the following Gibbs distribution in $\Delta^m$:

$$z_k = \frac{\exp((w_k - \alpha_k) / \delta)}{\sum_{j \in [m]} \exp((w_j - \alpha_j) / \delta)},$$

can be drawn with $(\frac{4 + h}{\delta})^4 m^2 \log (\frac{4 + h m}{\delta})$ samples from $\mathcal{F}_k_{k \in [m]}$ in expectation.

**Proof.** First, notice that $(z_k)_{k \in [m]}$ can also be represented as the Gibbs distribution with temperature $\frac{\delta}{h+4}$
and energy

$$\Omega_k = \frac{\omega_k - \alpha_k + h + 2}{h + 4}, \quad k \in [m].$$

Note that since $-1 \leq w_k \leq 1$ and $\alpha_k \leq h$, then:

$$0 < \frac{w_k - \alpha_k + h + 2}{h + 4} < 1.$$

Thus, by Theorem 7 with $\lambda = \frac{h+4}{\delta}$, we can generate a sample according to $(z_k)_{k \in [m]}$ with $(\frac{4 + h}{\delta})^4 m^2 \log (\frac{4 + h m}{\delta})$
samples in expectation. 

\[\square\]