Static, spherically symmetric solutions of Yang-Mills-Dilaton theory.

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Abstract
Static, spherically symmetric solutions of the Yang-Mills-Dilaton theory are studied. It is shown that these solutions fall into three different classes. The generic solutions are singular. Besides there is a discrete set of globally regular solutions further distinguished by the number of nodes of their Yang-Mills potential. The third class consists of oscillating solutions playing the role of limits of regular solutions, when the number of nodes tends to infinity. We show that all three sets of solutions are non-empty. Furthermore we give asymptotic formulae for the parameters of regular solutions and confront them with numerical results.

1 Introduction
The dilaton may be considered as a kind of scalar graviton sharing with it a universal coupling to matter. From this point of view it may be not too surprising that the static, spherically symmetric solutions of the Yang-Mills-Dilaton (YMD) theory share many properties with their Einstein-Yang-Mills (EYM) relatives. In fact, numerical studies \[1,2\] have revealed a great similarity between a family of ‘gravitational sphalerons’ – the Bartnik-McKinnon (BK) solutions \[3\] – and a corresponding family of dilaton solutions. Also an existence proof of these solutions running exactly along the lines of the one for the BK solutions \[4\] could be given \[5\].

Introducing a ‘stringy’ radial variable rescaled with a dilaton factor the field equations of the YMD theory resemble very much those of the EYM theory. At first sight the only difference is the larger number of gravitational degrees of freedom, which is however upset by the radial diffeomorphism constraint of the EYM theory. Thus there is practically no difference between the two theories concerning the number of degrees of freedom. In \[6\] a classification of all static, spherically symmetric solutions of the EYM theory with a regular origin was given; our aim is to prove a corresponding classification for the YMD theory. Naively one could expect that this should be an easier task for the YMD theory due to its simpler structure. However, life is not so simple and, although ultimately the result is the same, the proof for the YMD theory appears to be more difficult. The main reason is that in the EYM theory the ‘area variable’ \(r\) can have at most one maximum, its counterpart in YMD theory can oscillate. Apart from this subtlety
things in the EYM and YMD theory turn out to be largely the same. In both cases there are
three different types of solutions with a regular origin. The generic one develops a singularity
of the gravitational resp. dilaton field for a finite value of the corresponding autonomous radial
variable. The second type is a countably infinite family of globally regular solutions differing by
the number \( n \) of nodes of the YM potential \( W \). Finally there is an oscillating limiting solution for
\( n \to \infty \). Based on this classification an existence proof for the three different types of solutions
for the EYM theory was given \([6]\), which can be easily adapted to the YMD theory. Futhermore
the asymptotic scaling law of the parameters of the BK solutions for large \( n \) derived in \([6]\) can
be straightforwardly translated to the YMD theory.

For the \( SU(2) \) Yang-Mills field \( W_\mu^a \) we use the standard minimal spherically symmetric
(purely ‘magnetic’) ansatz

\[
W_\mu^a T_a dx^\mu = W(R)(T_1 d\theta + T_2 \sin \theta d\varphi) + T_3 \cos \theta d\varphi ,
\]

where \( T_a \) denote the generators of \( SU(2) \) and \( R \) the radial coordinate.

The action of YMD theory has the form

\[
S_{YMD} = \int \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4g^2} e^{2\kappa \phi} F_\mu^a F^{a\mu} \right) d^4x ,
\]

where \( g \) and \( \kappa \) are the gauge resp. dilaton coupling. Inserting the ansatz Eq. (1) into the action
and making use of the spherical symmetry we obtain the reduced YMD action

\[
S = \int dR \left( \frac{R^2 \phi'^2}{2} + \frac{1}{g^2} e^{2\kappa \phi} \left( W'^2 + \frac{(1 - W^2)^2}{2R^2} \right) \right) .
\]

The dependence on \( g \) and \( \kappa \) can be removed by the rescaling \( \phi \to \phi/\kappa \), \( R \to R\kappa/g \) and \( S \to Sg\kappa \).

The resulting Euler-Lagrange equations are

\[
(R^2 \phi')' = 2e^{2\phi} \left( W'^2 + \frac{(1 - W^2)^2}{2R^2} \right) ,
\]

\[
W'' = \frac{W(W^2 - 1)}{R^2} - 2\phi'W' .
\]

Using \( \tau = \ln R \) as a coordinate and introducing

\[
r \equiv Re^{-\phi} , \quad N \equiv 1 - R\phi' , \quad U \equiv e^\phi W' \quad T \equiv \frac{W^2 - 1}{r} ,
\]

we obtain the autonomous first order system of Riccati type (the dot denoting a \( \tau \) derivative)

\[
\dot{r} = rN , \tag{6a}
\]

\[
\dot{W} = rU , \tag{6b}
\]

\[
\dot{N} = 1 - N - 2U^2 - T^2 , \tag{6c}
\]

\[
\dot{U} = WT + (N - 1)U , \tag{6d}
\]

\[
\dot{T} = 2WU - NT , \tag{6e}
\]

supplemented by the constraint \( W^2 - rT = 1 \). These equations have a great similarity to the
Eqs. (50) of \([6]\) for the Einstein-YM theory. Clearly there is no analogue of the diffeomorphism
constraint here.

There is a kind of \( \tau \)-dependent ‘energy’ \( E = 2\dot{W}^2 - (W^2 - 1)^2 \) obeying

\[
\dot{E} = 4(2N - 1)\dot{W}^2 , \tag{7}
\]
which will be useful as a ‘Lyapunov Function’. In addition we introduce some other useful auxiliary quantities $e$, $f$ and $g$ defined as

$$e \equiv \frac{E}{r^2} = 2U^2 - T^2, \quad f \equiv (1 - N)^2 + e \quad \text{and} \quad g \equiv 1 - N - f,$$

(8)

obeying the equations

$$\dot{e} = 4(2N - 1)U^2 - 2Ne, \quad \text{(9a)}$$

$$\dot{f} = -2f + 4U^2, \quad \text{(9b)}$$

$$\dot{g} = -g + (1 - N)^2. \quad \text{(9c)}$$

(9d)

In general it does not seem possible to solve Eqs. (6) in closed form, yet there are some simple exeptions. One is the trivial vacuum solution $W^2 = 1$, $\phi = \text{const.}$; besides there is an analogue of the extremal magnetically charged Reissner-Nordstrøm (RN) solution of the EYM theory

$$W \equiv 0 \quad N = 1 - \frac{1}{r} \quad \phi = -\ln(1 + ce^{-r}). \quad \text{(10)}$$

2 Singular Points

Before trying to explore the global behaviour of solutions with a regular origin, it is important to know the singular points of the system (6). Actually, there are two types of singular points, those attained for finite $\tau$ and the fixed points for $\tau \to \pm \infty$. The first type of singularity occurs, if the r.h.s. of Eqs. (6) blows up at some finite value of $\tau$. As we shall prove later, the only such possibility is that $N \to -\infty$ and $r \to 0$. Due to the simple Riccati form of Eqs. (6) it is easy to find all their fixed points. There are first the f.p.s $r = 0$, $\infty$ and $N = 1, W = \pm 1, U = T = 0$. Furthermore there is the f.p. $N = 0, r = T = 1, W = U = 0$ like for the EYM theory. All these f.p.s are of hyperbolic type and thus the application of the theory of dynamical systems [7, 9] provides theorems on local existence and the asymptotic behaviour near the singular points.

Introducing suitable auxiliary variables it is also possible to treat the singularity with $N \to -\infty$ (and $r \to 0$) as a fixed point. For that purpose it turns out to be convenient to use $r$ as the independent variable and two auxiliary dependent variables

$$\kappa \equiv r(1 - N), \quad \lambda \equiv W T + (N - 1) U$$

(11)

obeying the equations

$$\frac{d}{dr} W = -\frac{rU}{\kappa - r}, \quad \text{(12a)}$$

$$\frac{d}{dr} U = -\frac{\lambda}{\kappa - r}, \quad \text{(12b)}$$

$$\frac{d}{dr} \kappa = \frac{(f - 4U^2)r}{\kappa - r}, \quad \text{(12c)}$$

$$\frac{d}{dr} \lambda = \frac{(1 - 3W^2 + 4U^2 - f)U + \lambda}{\kappa - r}, \quad \text{(12d)}$$

$$\frac{d}{dr} f = \frac{2f - 4U^2}{\kappa - r}. \quad \text{(12e)}$$

(12f)

Applying Prop. 4 of [8] assuming $\kappa \neq 0$ we obtain finite limits $W_0$, $U_0$, $\kappa_0$, $\lambda_0$, $\mu_0$ and $f_0$ for the dependent variables. The finiteness of $f$ implies $\kappa_0 = |W_0^2 - 1|$ and thus we have to require
Putting \( W_0^2 \neq 1 \). From the behaviour \( N \sim 1/r \) it follows that \( \tau \) resp. \( r \) stay finite as \( r \to 0 \) and hence \( \phi \to \infty \). The original dependent variables behave as

\[
W = W_0 + \frac{W_0}{2(1 - W_0^2)} r^2 + W_3 r^3 + O(r^4),
\]

(13a)

\[
U = \pm W_0 - \left(3|W_0^2 - 1|W_3 + \frac{W_0}{W_0^2 - 1}\right) r + O(r^3),
\]

(13b)

\[
N = -\frac{|W_0^2 - 1|}{r} + 1 + N_1 r + O(r^2).
\]

(13c)

with arbitrary parameters \( W_0, W_3 \) and \( N_1 \). Actually \( r = 0 \) is a regular point of Eqs. \([12]\) as long as \( \kappa > 0 \) and thus this type of singular behaviour with \( N \to -\infty \) is of a generic type.

The case \( W_0^2 = 1 \) requires special treatment. Unlike the EYM theory there are no solutions with a ‘Schwarzschild type’ singularity \( W_0 = \pm 1, \kappa \sim |W_0^2 - 1|/r + N_0 \) with \( N_0 \neq 1 \) and solutions with \( W_0 = \pm 1 \) automatically have a regular origin with \( U = 0 \) and \( N = 1 \). The f.p. treatment of this case proceeds like for the Bartnik-McKinnon solutions given in \([6]\) and we don’t repeat it here. The linearization at the f.p. gives the solutions

\[
W = \pm (1 - be^{2\tau} + ce^{-\tau}),
\]

(14a)

\[
N = 1 - 4b^2 e^{2\phi_0} e^{2\tau},
\]

(14b)

\[
\phi = \phi_0 + 2b^2 e^{2\phi_0} e^{2\tau} + de^{-\tau},
\]

(14c)

where \( b, c, \phi_0 \) and \( d \) are free parameters. For regular solutions, i.e. those running into the f.p. for \( \tau \to -\infty \) the coefficients \( c \) and \( d \) have to vanish. Without restriction one may choose the + sign for \( W \) and \( \phi_0 = 0 \); then one gets \( W = 1 - br^2 + c/r + O(r^4) \) etc.

Similarly there is the fixed point \( W = \pm 1, U = 0 \) and \( N = 1 \) with \( r \to \infty \) corresponding to asymptotically regular solutions. The asymptotic behaviour close to the f.p. is again dominated by the linear approximation

\[
W = \pm (1 + ce^{-\tau} + be^{2\tau}),
\]

(15a)

\[
N = 1 + de^{-\tau},
\]

(15b)

\[
\phi = \phi_\infty + de^{-\tau}.
\]

(15c)

Regular solutions require now \( b = 0 \) and normalizing \( \phi \) to \( \phi_\infty = 0 \) one obtains \( W = \pm (1 + c/r + br^2 + O(r^{-2})) \) etc.

There is one more fixed point analogous to the ‘RN fixed point’ of the EYM theory given by \( W = U = N = 0, r = 1 \) and hence \( \phi \to \infty \). For simplicity we call it again the RN fixed point. Putting \( \bar{r} = r - 1 \) the linearization at this f.p. is

\[
\dot{\bar{r}} = N, \tag{16a}
\]

\[
\dot{N} = -N + 2\bar{r}, \tag{16b}
\]

\[
\dot{W} = U, \tag{16c}
\]

\[
\dot{U} = -W - U, \tag{16d}
\]

with the solution

\[
W = C_1 e^{-\bar{r}} \sin(\frac{\sqrt{3} \pi}{2} + \theta), \tag{17a}
\]

\[
U = C_1 e^{-\bar{r}} \sin(\frac{\sqrt{3} \pi}{2} + \frac{2\pi}{3} + \theta), \tag{17b}
\]

\[
\bar{r} = C_2 e^{\bar{r}} - C_3 \frac{1}{2} e^{-2\bar{r}}, \tag{17c}
\]

\[
N = C_2 e^{\bar{r}} + C_3 e^{-2\bar{r}}. \tag{17d}
\]
The oscillating solutions running into the f.p. for \( \tau \to \infty \) require \( C_2 = 0 \). For these solutions the term \( C_3 e^{-2\tau} \) does not describe the actual asymptotic behaviour of \( r \) and \( N \), because the true behaviour is determined by nonlinear terms involving \( W \) and \( U \). Integrating the resulting inhomogeneous linear equations for \( \bar{r} \) and \( N \) one finds (neglecting the non-leading term \( C_3 e^{-2\tau} \))

\[
\bar{r} = C_2 e^\tau + C_1^2 e^{-\tau} \left( -\frac{9}{28} \cos(\sqrt{3}\tau + 2\theta) + \frac{\sqrt{3}}{28} \sin(\sqrt{3}\tau + 2\theta) \right),
\]

\[
N = C_2 e^\tau + C_1^2 e^{-\tau} \left( \frac{3}{7} \cos(\sqrt{3}\tau + 2\theta) + \frac{2\sqrt{3}}{7} \sin(\sqrt{3}\tau + 2\theta) \right).
\]

In order to run into the f.p. for \( \tau \to -\infty \) it is necessary that \( C_1 \) and \( C_3 \) vanish. From the form of Eqs. (6) one sees that the vanishing of \( C_1 \) implies \( W \equiv 0 \). Since the ‘exterior’ (\( r > 1 \)) part of the limiting solution, when the number of nodes goes to infinity, runs into the RN f.p. for \( \tau \to -\infty \) it must correspond to \( W \equiv 0 \).

3 Classification

The classification of solutions with a regular origin proceeds very similar to the one for the EYM system in [6]. We distinguish three different classes. The first one, which we call singular (Sing) are solutions becoming singular according to Eqs. (13) with \( r \to 0 \) for some finite value \( \tau_0 \). This class may be further subdivided into the classes Sing\(_{\tau=0} \) \((n = 0, \ldots)\) containing singular solutions with \( W_0^2 > 1 \) and \( W \) having \( n \) nodes and the class Sing\(_{\infty} \) for singular solutions with \( W_0^2 < 1 \). The second class are the globally regular solutions (Reg) reaching the f.p. with \( r = \infty \) described in Eqs. (13). Again the class Reg can be subdivided according to the number of zeros of \( W \). Finally there is a third class, the oscillating solutions (Osc), running into the RN f.p. with \( r = 1 \). Our aim is to prove that any solution of Eqs. (6) with a regular origin belongs to one of the classes Sing, Reg or Osc. This is made plausible by the observation that the dilaton \( \phi \) is a monotonically increasing function of \( \tau \). Thus it can either stay bounded or diverge to \(+\infty\); the latter can either happen for some finite \( \tau_0 \) or for \( \tau \to \infty \). The difficulty is to show that these alternatives lead precisely to the singular points described in Eqs. (13), (15) and (17) corresponding to the three classes introduced above. The essential part of this claim are the equivalences stated in Props. (1) to (3), whose proof will be based on a series of Lemmas (some of which can be found in the Appendix). In what follows we always consider solutions of Eqs. (6) starting from a regular origin with \( W = 1 \).

Lemma 1:

i) The function \( W \) can have neither maxima if \( W > 1 \) or \( 0 > W > -1 \) nor minima if \( W < -1 \) or \( 0 < W < 1 \).

ii) The functions \( \phi = 1 - N, f \) and \( g \) are bounded below and if any of them is positive for some \( \tau_0 \) then it stays positive for all \( \tau > \tau_0 \). In particular they are non-negative for solutions with a regular origin.

iii) For finite \( \tau \) the functions \( U \) and \( W \) are finite as long as \( N \) is finite.

Proof: i) and ii) are trivial consequences of Eqs. (13) and (15) and the fact that \( 1 - N, f \) and \( g \) are positive close to the origin, if they are regular at \( r = 0 \).

In order to prove iii) we remark that ln\( r \) and \( U \) obey linear equations and stay obviously finite as long as \( W \) and \( N \) are finite. Suppose now that \( N \) is bounded and \( W \) is unbounded, then because of property i) \( |W| \) actually tends to \( \infty \). Following closely an argument put forward in [6] Prop. 5 one finds that \( W \) diverges for some finite \( \tau_0 \) like \( W = \pm \sqrt{2}/(\tau_0 - \tau) + O(1) \). Plugging this into Eq. (6) leads to a divergence of \( N \) contrary to our assumption.
Lemma 2: Suppose $N \to -\infty$ at $\tau_0$ then $W$ and $\kappa \equiv r(1-N)$ have finite limits $W_0$ and $\kappa_0$ at $\tau_0$ with $W_0 \neq \pm 1$ and $\kappa_0 > 0$ (and as a consequence $r(\tau_0) = 0$).

Proof: First we want to show that $W$ is bounded. Assume the contrary and without restriction $W > 1$ and $U \geq 0$. If we can show that $U/W$ is bounded we get that also $W/W = rU/W$ is bounded and thus $\ln W$ is bounded. Putting $z = W(1-N) - UT$ one gets

$$\dot{z} = r(1-N)U - z \geq -z,$$

(19)

This implies that $z$ is bounded below and therefore since $T \to +\infty$ we may assume $-z/T < 1$ and estimate

$$\frac{d}{dt} \left( \frac{U}{W} \right) = T + (N-1) \frac{U}{W} - r \frac{U^2}{W^2} < T \left( 1 - \frac{U^2}{W^2} - \frac{z}{W \cdot T} \right),$$

(20)

The r.h.s. is negative, if $U/W > 2$ and thus $U/W$ is bounded implying the boundedness of $W$. Then also $U$ and $W$ are bounded implying the finiteness of $W(\tau_0)$.

Next we want to show that $\kappa$ has a finite positive limit. Putting $\bar{f} \equiv r^2 f$ we get from Eq. (20)

$$\bar{f} = 2(N-1)\bar{f} + 4\dot{W}^2.$$

(21)

Since $\int_0^{\tau_0} N d\tau < 0$ and $\dot{W}^2$ is integrable we get from Lemma 17 that $\bar{f}$ is bounded and hence also $\kappa$. The latter implies $r \to 0$ and thus $\int_0^{\tau_0} N d\tau = -\infty$. Using once more Lemma 17 we get $\bar{f} \to 0$. Since the boundedness of $U = \dot{W}/r$ yields $\dot{W} \to 0$ we get $\kappa^2 \to (W_0^2 - 1)^2$.

In order to show that $\kappa$ stays away from zero we use again $g \geq 0$ to estimate

$$\dot{\kappa} = (N-1)\kappa + 2rU^2 + rT^2 \geq -\kappa + 4rU^2 \geq -\kappa.$$  

(22)

This inequality shows that $\ln \kappa$ is bounded below and consequently $W_0^2 \neq \pm 1$.

Lemma 3: If $N < -1$ at some point $\tau_1$ then $N$ tends to $-\infty$ for some finite $\tau_0 > \tau_1$.

Proof: From $g \geq 0$ we get $T^2 \geq 2U^2 - N(1-N)$ and from Eq. (20)

$$\dot{N}(\tau) \leq 1 - N - 4U^2 + N - N^2 \leq 1 - N^2 < 0 \quad \text{for} \quad \tau \geq \tau_1$$

(23)

and thus $N$ stays below $-1$. Suppose $N(\tau) < -1 - \epsilon$ for $\tau > \tau_1$ with $\epsilon > 0$ then $\dot{N} < -2\epsilon$. Hence $N$ is unbounded from below. Using Lemma 18 we get $N \to -\infty$ for some finite $\tau_0$.

Lemma 4: If $W^2 > 1$ and $WU \geq 0$ at some point then the function $N$ tends to $-\infty$ for some finite $\tau_0$.

Proof: The proof follows essentially the one given in [5]. From

$$\dot{T} = 2WU - NT, \quad \dot{U} = WT + (N-1)U,$$

(24)

we get

$$\frac{d}{dt} \ln |TU| \geq 2\sqrt{2}|W| - 1 \geq 2\sqrt{2} - 1,$$

(25)

It follows that for any $c > 0$ we can find some $\tau_1$ such that $2\sqrt{2}|TU| \geq c$ for $\tau > \tau_1$ and hence $T^2 + 2U^2 \geq c$. Eq. (20) then yields $\dot{N} \leq 1 - c - N$ implying that $N$ eventually becomes arbitrarily negative for large enough $\tau$ and thus Lemma 3 may be applied.

Lemma 5: $r$ cannot have a maximum with $r_{\text{max}} > \sqrt{2}$ for $W^2 \leq 1$.

Proof: At a maximum of $r$ we have $N = 0$ and $\dot{N} \leq 0$. On the other hand $g \geq 0$ implies $T^2 \geq 2U^2$ for $N = 0$ and thus

$$\dot{N} \geq 1 - 2T^2 > 1 - \frac{2}{r^2} > 0.$$  

(26)
Lemma 6: If \( \ln r \) is bounded for \( \tau \to \infty \) then \( N \to 0 \).

Proof: Lemmas 1 and 3 imply \( |N| \leq 1 \). Suppose \( \lim \sup N > 0 \) and a sequence of points \( \tau_i \to \infty \) with \( N(\tau_i) > \epsilon \). Since \( \tilde{N} \leq 2 \) we get \( N(\tau) > \epsilon/2 \) for \( \tau_i - \epsilon/4 < \tau < \tau_i \), implying \( \int^\infty N^2 d\tau = \infty \). Similarly the assumption \( \lim \inf N < 0 \) leads to the divergence of \( \int^\infty N^2 d\tau \). On the other hand we have \( 0 \leq g \leq (1 - N)^2 < 4 \) and thus

\[
\int^\infty (\dot{g} - N)d\tau = 2 \int^\infty (N^2 + 2U^2 - N)d\tau .
\]

Since \( \int^\infty Nd\tau = \ln r + \text{const.} \) is bounded this implies the boundedness of \( \int^\infty N^2 d\tau \) contradicting the assumptions \( \lim \sup N > 0 \) and \( \lim \inf N < 0 \) and hence \( \lim N = 0 \).

Lemma 7: \( r \to \infty \) for \( \tau \to \infty \) implies \( N \to 1 \) for \( \tau \to \infty \).

Proof: According to Lemma 4 we may assume \( W^2 \leq 1 \) and thus \( T \to 0 \) for \( \tau \to \infty \). Thus for any \( \epsilon > 0 \) there is some \( \tau_\epsilon \) such that \( |T| < \epsilon \) for \( \tau > \tau_\epsilon \). From \( g \geq 0 \) we get

\[
\frac{d}{d\tau} (1 - N) = N - 1 + 2U^2 + T^2 \leq 2T^2 - (1 - N)^2 \leq 2\epsilon^2 - (1 - N)^2 .
\]

(28)

Together with \( 1 - N \leq 1 \) implied by Lemma 5 this shows that \( 0 < 1 - N < 2\epsilon \) for \( \tau > \tau_\epsilon + \frac{1}{2\epsilon} \). Hence \( N \to 1 \) for \( \tau \to \infty \).

Proposition 8: The following are equivalent

i) \( \phi \to \infty \) for some finite \( \tau_0 \)

ii) \( r \to 0 \) for some finite \( \tau_0 \)

iii) \( N \to -\infty \) for some finite \( \tau_0 \)

iv) The solution belongs to \( \text{Sing} \)

Proof: Obviously iv) implies i) and i) implies ii).

ii) \( \Rightarrow \) iii): Since \( \ln r \) diverges \( N \) must be unbounded from below and thus iii) follows from Lemma 4.

iii) \( \Rightarrow \) iv): We have to show that \( r \to 0 \) and the functions \( W, U, \kappa, \lambda \) and \( \mu \) have a finite limit at \( \tau_0 \). From Lemma 4 we know that \( r \to 0, W \to W_0 \neq \pm 1 \) and \( \kappa \to \kappa_0 > 0 \). Thus we have only to prove that the r.h.s. of Eqs. 12b,d,e) are integrable. For that reason we have to show that \( U^2 \) and \( U^3 \) are integrable, which will be achieved using arguments put forward in 6.

The boundedness of \( r \) and \( 1/\kappa \) implies that \( |rN|^{-1} \) and \( r^\epsilon \) are bounded for any \( \epsilon > 0 \) and consequently

\[
\int^{\tau_0}_\tau N r^\epsilon d\tau = \frac{1}{\epsilon} r^\epsilon < \infty
\]

(29)

and hence

\[
\int^{\tau_0}_\tau \frac{1}{\tau^{1-\epsilon}} d\tau < \infty .
\]

(30)

Applying Lemma 17 we get that \( r^\epsilon U \) obeying the linear equation

\[
(r^\epsilon U) = \frac{W(W^2 - 1)}{r^{1-\epsilon}} + (N - 1 + \epsilon) r^\epsilon U
\]

(31)

is bounded for \( \epsilon > 0 \). As a consequence \( |U^n| \) is integrable for any \( n > 0 \). According to Lemma 4 this implies that \( \mu \) and consequently \( \lambda \) and \( U \) have a limit.

Proposition 9: The following are equivalent
i) $\phi \to \phi_\infty < \infty$ for $\tau \to \infty$

ii) $r \to \infty$ for $\tau \to \infty$

iii) The solution belongs to $\text{Reg}$

Proof: Obviously iii) implies i) and i) implies ii).

ii) $\Rightarrow$ iii): From Lemma 7 we know that $N \to 1$ for $\tau \to \infty$. Thus $E$ is asymptotically monotonously increasing. Suppose $E \to \infty$, then $|W| \to \infty$. Yet, this is not compatible with the boundedness of $W$. Hence $\dot{W}$ must be bounded and in fact tend to zero for $\tau \to \infty$ according to Lemma 19. Since $E$ has a limit also $W$ has a limit, which must be a f.p. of Eqs. (6). The f.p. with $W = 0$ is, however, excluded, because $E$ is asymptotically increasing and thus cannot tend to its infimum $-1$. Therefore the solution belongs to $\text{Reg}$.

Proposition 10: The following are equivalent

i) $\phi \to \infty$ for $\tau \to \infty$

ii) $|\ln r| < c < \infty$ for all $\tau$

iii) The solution belongs to $\text{Osc}$

Proof: Obviously iii) implies i).

i) $\Rightarrow$ ii): From Lemma 5 and Prop. 9 we conclude that $r$ must be bounded and thus $\int N d\tau$ is bounded from above. Eq. 27 also shows that $\int N d\tau$ is bounded from below and thus $\ln r$ has also a positive lower bound.

ii) $\Rightarrow$ iii): Lemma 6 implies $N \to 0$ for $\tau \to \infty$. Now we can use Lemma 19 to conclude that $E$ has a limit and $\dot{W} \to 0$. Thus also $W$ has a limit, which must be a f.p. of Eqs. (6). Eq. (6c) is only compatible with $N \to 0$, if the f.p. is $W = 0$ and $r = 1$.

Putting together the Props. 8, 9 and 10 with the possible behaviours of $\phi$ discussed at the beginning of this section we obtain the ‘Classification Theorem’

Theorem 11: Any solution of Eqs. (6) with a regular origin belongs to one of the three classes $\text{Sing}$, $\text{Reg}$ or $\text{Osc}$.

4 Topology of ‘Moduli Space’ and Existence Theorem

The method to prove the existence of at least one globally regular solution for each number of nodes and a corresponding limiting solution with infinitely many nodes used in [6] can be almost literally translated to the present case. The proof is based on an analysis of the phase space as a function of the parameter $b$ determining the solutions with a regular origin. While the generic singular solutions (i.e. solutions in $\text{Sing}$) correspond to open intervals of $b$ space, the $b$ values for regular solutions are isolated points accumulating at the value(s) for the limiting solution(s). As a first step we will study, what happens for very small and very large values of $b$. The result is the same as for the EYM system.

4.1 Small $b$

Proposition 12: If $b \neq 0$ is small enough the solution with $W|_{r=0} = 1$ belongs to the singular class $\text{Sing}_0$ for $b < 0$ or to $\text{Sing}_1$ for $b > 0$.

1The numerical analysis yields exactly one regular solution for each node number and correspondingly one single limiting solution
Remark: In view of Lemma \ref{lemma:small}, the restriction for \(b\) to be small is unnecessary for \(b < 0\).

Proof: The proof runs completely along the lines of the one in \cite{6}. Rescaling \(r \to |b|^{-\frac{3}{8}}r\) and \(U \to |b|^{-\frac{3}{8}}U\) we obtain from Eq. (33):

\[
\dot{N} = 1 - N - |b|(2U^2 + T^2) \tag{32}
\]

and the \(b\) independent boundary condition \(\lim_{r \to -\infty} \frac{U}{r} = \mp 2\). For \(b = 0\) the solution is \(N \equiv 1\) and thus \(r = e^\tau\). As was shown the resulting solution \(W\) of the pure YM system diverges like \(W \sim \pm \sqrt{2}/(\bar{r} - \tau)\) for some finite \(\bar{r}\) resp. \(\bar{r}_\pm\) depending on the sign of \(b\). The values of \(\bar{r}_\pm\) have been determined in \cite{3} numerically as \(\bar{r}_+ = \approx 5.317\) and \(\bar{r}_- = \approx 1.746\). For small \(b\) we obtain a small perturbation of this solution as long as \(|b| \int_{-\infty}^\tau (2U^2 + T^2)dr' \ll 1\). This condition still holds for \(|W| \gg 1\) and \(|b|^{-\frac{3}{8}}r \gg 1\), if \(b\) is small. Thus the Prop. follows from Lemma \ref{lemma:small}.

Before we proceed to the case of large values of \(b\) we shall derive the asymptotic behaviour of \(W_0\) as \(r\) runs back to zero, similar to the one given in \cite{11}. This proceeds in several steps. First we integrate the Eqs. (6) for \(N\) from \(N = 1\) to \(N = 0\) using the pure YM solution becoming singular at \(\tau = \bar{r}\). We obtain some value \(\tau_e < \bar{r}\) from

\[
1 = \frac{8|b|}{r^2} \int_{-\infty}^{\tau_e} \frac{d\tau'}{(\bar{r} - \tau')^4} = \frac{8|b|}{3\bar{r}_+^2(\bar{r} - \tau_e)^3}. \tag{33}
\]

This gives \(|W_e| \approx \sqrt{\frac{2}{(\bar{r} - \tau_e)}} \approx 3\bar{r}_+^\frac{3}{8}|b|^{-\frac{3}{8}}/2\). Here we have neglected the term \(1 - N\) in Eq. (33) and pulled \(r\) out of the integral; this yields an error of \(O(1)\) to \(W_e\), small by one order in \(|b|^{-\frac{3}{8}}\). Correspondingly we get \(r_e = \bar{r}_\pm\).

The next step is to integrate Eqs. (6) from \(N = 0\) to \(N = -\infty\) and to determine the value of \(W_0\). Since \(rN\) is \(O(1)\) for \(N \to -\infty\) and \(r_\pm = O(|b|^{-\frac{3}{8}})\) we rescale \(\tau \to \tau |b|^{-\frac{1}{8}}\) and similarly the dependent variables \(W \to W|b|^{-\frac{1}{8}}\) etc. Keeping only leading terms as \(b \to 0\) we obtain from Eqs. (6)

\[
\dot{r} = rN, \tag{34a}
\]
\[
\dot{W} = rU, \tag{34b}
\]
\[
\dot{N} = -2U^2 - T^2, \tag{34c}
\]
\[
\dot{U} = WT + NU, \tag{34d}
\]
\[
\dot{T} = 2WU - NT \tag{34e}
\]

with the constraint \(rT - W^2 = 0\). Due to the scaling the boundary conditions at \(\tau_e\) are changed to \(\tau \to -\infty\) and \(r = \bar{r}_\pm, W = U = T = N = 0\). In order to perform the limit \(r \to 0\) we introduce the analogue of the variables used in Eq. (12) putting \(\kappa = -r\dot{N}, \lambda = WT + NU\) and \(f = N^2 + 2U^2 - T^2\). The Eqs. (34) imply \(\dot{f} = 0\) and hence \(f \equiv 0\). Using the new variables we obtain

\[
\dot{\kappa} = -\lambda, \tag{35a}
\]
\[
\dot{\lambda} = (3W^2 - 4U^2)U, \tag{35b}
\]
\[
\dot{\bar{U}} = \bar{U}, \tag{35c}
\]
\[
\dot{\bar{\lambda}} = 4rU^2. \tag{35d}
\]

These equations have to be integrated from the highly degenerate f.p. \(W = U = \kappa = \lambda = 0, r = \bar{r}_\pm\) attained at \(\tau = -\infty\). In order to lift the degeneracy we use a ‘blow up’ in the direction \(W\) introducing

\[
\bar{U} = \frac{U}{W^2}, \quad \bar{\kappa} = \frac{\kappa}{W^3} \quad \text{and} \quad \bar{\lambda} = \frac{\lambda}{W^3} \tag{36}
\]
and using \( \ln|W| \) as new independent variable. Thus we obtain the equations

\[
\begin{align*}
W \frac{dr}{dW} &= - \frac{W^2 \bar{\kappa}}{r U}, \\
W \frac{d\bar{U}}{dW} &= \frac{\bar{\lambda}}{r U} - 2 \bar{U}, \\
W \frac{d\bar{\lambda}}{dW} &= \frac{3 - 3r\bar{\lambda} - 4W^2 \bar{U}^2}{r}, \\
W \frac{d\bar{\kappa}}{dW} &= -(4\bar{U} + 3\bar{\kappa}).
\end{align*}
\]

(37a)

(37b)

(37c)

(37d)

The f.p. has now moved to the point

\[
W = 0, \quad r = \bar{r}_\pm, \quad \bar{U} = \pm \frac{1}{\sqrt{2} \bar{r}_\pm}, \quad \bar{\kappa} = \pm \frac{2\sqrt{2}}{3 \bar{r}_\pm} \quad \text{and} \quad \bar{\lambda} = \frac{1}{\bar{r}_\pm}.
\]

(38)

Linearisation at this f.p. yields exclusively negative eigenvalues and thus the degeneracy has been removed. At the same time this shows that there are no adjustable parameters at the f.p. as to be expected.

Numerical integration of Eqs. (37) from the f.p. to \( r = 0 \) results in a finite value of \( W \) proportional to \( \bar{r}_\pm \). Taking into account the scaling factor \( |b|^{-\frac{1}{2}} \) we find

\[
W_0 \approx 0.89369 |b|^{-\frac{1}{2}} \bar{r}_\pm.
\]

(39)

Fig. 1 shows a plot of numerically obtained data for \( W_0 \); the dashed lines represent the values of Eq. (39) for the two different signs of \( b \).

Next we turn to solutions with large values of \( b \). Again the situation resembles very much the EYM case.

4.2 Large \( b \)

**Proposition 13:** If \( b \gg 0 \) is large enough the solution with \( W|_{r=0} = 1 \) belongs to the singular class \( \text{Sing}_\infty \).

**Proof:** We put \( r = \bar{r}/b \) and \( W = 1 + \bar{W}/b \). Keeping only leading terms we obtain from Eqs. (40)

\[
\begin{align*}
\dot{\bar{r}} &= N \bar{r}, \\
\dot{\bar{W}} &= \bar{r} \bar{U}, \\
\dot{\bar{T}} &= 2\bar{U} - NT, \\
\dot{\bar{U}} &= T + (N - 1)U, \\
\dot{N} &= 1 - N - 2U^2 - T^2
\end{align*}
\]

(40a)

(40b)

(40c)

(40d)

(40e)

with the constraint \( 2\bar{W} - \bar{r}T = 0 \) and the \( b \) independent boundary condition \( \lim_{r \to -\infty} \frac{U}{\bar{r}} = \mp 2 \).

The combination \( z = N - 1 + T U \) obeys the simple equation \( \dot{z} = -z \). In view of the initial condition \( \lim_{r \to -\infty} z = 0 \) we get \( z \equiv 0 \). This allows us to remove \( N \) from the \( T, U \) system to obtain

\[
\begin{align*}
\dot{\bar{T}} &= U(2 + T^2) - T, \\
\dot{\bar{U}} &= T(1 - U^2).
\end{align*}
\]

(41a)

(41b)

with the boundary condition \( \lim_{r \to -\infty} U = \lim_{r \to -\infty} T = 0 \) such that \( \lim_{r \to -\infty} \frac{U}{\bar{r}} = 1 \). It is straightforward to analyze the flow of this 2d system (Fig. 2 shows a plot). The point \( U = T = 0 \)
Figure 1: Numerical data for \( W_0(b) \); the dashed lines represent the values of Eq. (39) for the two signs of \( b \).

Figure 2: The phase space of Eqs. (41); on the dashed curve \( U = \frac{T}{2\sqrt{T}} \), the orbits are vertical.
is a hyperbolic f.p. with the eigenvalues $-2$ and $1$. No orbits can cross the lines $U = \pm 1$. The solution with the relevant boundary condition from above corresponds to the separatrix for the eigenvalue $1$. For $1 > |U| > |T|/(2 + T^2)$ the orbits are monotonously approaching one of the lines $U = \pm 1$ for $|T| \to \infty$. This shows that eventually $N < -1$ even taking into account the correction terms of $O(1/b)$ in Eqs. (6). Applying Lemma together with $W = 1 - O(1/b)$ proves the proposition.

4.3 Existence Theorem

As for the EYM theory one can characterize the neighbourhood of the sets $\text{Reg}_n$ and $\text{Osc}$.

**Proposition 14:** Given $b_n \in \text{Reg}_n$ for any $n$ then all $b \neq b_n$ sufficiently close to $b_n$ are either in $\text{Sing}_n$ or $\text{Sing}_{n+1}$.

**Proposition 15:** Given $b_\infty \in \text{Osc}$ and some $n_0$ then all $b \neq b_\infty$ sufficiently close to $b_\infty$ are either in $\text{Sing}_\infty$ or in $\bigcup_{n \geq n_0} (\text{Reg}_n \cup \text{Sing}_n)$.

Equipped with the knowledge, what happens for large and small values of $b$ one proves

**Theorem 16:**

i) The sets $\text{Reg}_n$ and $\text{Osc}$ are all nonempty, i.e., for each $n = 0, 1, 2, \ldots$ there exists a globally regular solution with $n$ zeros of $W$ for at least one $b_n \in \text{Reg}_n$ and there exists an oscillating solution with $N > 0$ for all $\tau$ and $r \to 1$ for $\tau \to \infty$ for at least one $b_\infty \in \text{Osc}$.

ii) The union $\bigcup_{n \geq 0} \text{Reg}_n$ has accumulation points that are contained in $\text{Osc}$, i.e., there exists at least one sequence of globally regular solutions and one oscillating solution $W_\infty$ such that $W_n(r) \to W_\infty(r)$ for $r < 1$ and $W_n(r) \to 0$ for $r \geq 1$ for $n \to \infty$.

The proofs of the Props. 14 and 15 and of the Theorem can be literally taken from [6] and will not be repeated here.

**Remark:** The existence of at least one regular solution for any $n$ was already proven in [5]. As mentioned above, the numerical results show that there is exactly one regular solution for any $n$ and correspondingly only one limiting solution. As in the EYM case there is however no uniqueness proof available or in view.

5 Scaling law for large $n$

In [6] a remarkably well satisfied asymptotic scaling law for the parameters of the regular solutions with a large number $n$ of nodes of $W$ was formulated. The derivation was based on the observation that for solutions with many nodes three distinctive regions could be observed. In an inner region the solutions are well approximated by the limiting oscillating solution with infinitely many zeros. This region extends between the origin and $r = 1$. Furthermore there is an asymptotic region for $r > r_n \gg 0$, where the solutions are close to the flat solution connecting the f.p.s with $W = \pm 1$, $U = 0$ and $W = U = 0$. In the intermediate region extending between $r = 1$ and $r = r_n$ the functions $W$ and $U$ stay small and thus the equations for $W$ and $U$ can be linearized on the metric background given by the extremal Reissner-Nordstrom solution. Boundary conditions for these linearized YM equations are obtained by matching with the solutions obtained in the inner and outer regions. The same type of scaling law can be obtained here through more or less identical reasoning.

On region I defined by $0 \leq r \leq 1$ the solutions are approximated by the limiting solution running into the f.p. $W = U = N = 0$. The corresponding behaviour near $r = 1$ is given by Eq. (12) neglecting the $C_3$ term. The $(n$ dependent) coefficient $C_2$ has to be positive to allow
the solution to reach region II with \( r > 1 \). By a suitable shift in \( \tau \) we can always achieve \( C_1 = 1 \). The phase \( \theta \) is adjusted such that \( \sqrt{3}/2 \tau + \theta = m\pi \) at the \( m \)th zero of \( W \).

In region II the Eqs. (6b,d) for \( W \) and \( U \) are linearized in the background of the solution Eq.(10). Surprisingly the solution is identical to that of [6]

\[
\begin{align*}
    r_{II}(\tau) &= 1 + C_{2,n}e^{\tau}, \\
    N_{II}(\tau) &= \frac{C_{2,n}e^{\tau}}{1 + C_{2,n}e^{\tau}}, \\
    W_{II}(\tau) &= e^{-1/2\tau} \sin(\sqrt{3}/2\tau + \theta) + C_{2,n}e^{1/2\tau} \sin(\sqrt{3}/2\tau + \pi/3 + \theta).
\end{align*}
\]

In region III, where we have \( r \gg 1 \) and \( 1 - N \ll 1 \), we take the flat solution connecting the f.p.s with \( W = \pm 1 \), \( U = 0 \) and \( W = U = 0 \). In the region where \( W \) is small this solution can be approximated by

\[
\begin{align*}
    r_{III}(\hat{\tau}) &= c_ne^{\hat{\tau}}, \\
    W_{III}(\hat{\tau}) &= \pm \hat{C}_{1,n}e^{\frac{1}{\sqrt{3}}\hat{\tau}} \sin(\frac{\sqrt{3}}{2}\hat{\tau} + \frac{\pi}{3} + \hat{\theta}),
\end{align*}
\]

with the normalization \( W_{III} \to \pm (1 - e^{-\hat{\tau}}) \). Again the phase \( \hat{\theta} \) is adjusted such that \( \sqrt{3}/2\hat{\tau} + \frac{\pi}{3} + \hat{\theta} = m\pi \) at the last but \( m \)th zero of \( W \). Matching \( r_{II}, W_{II} \) with \( r_{III}, W_{III} \) we obtain

\[
\begin{align*}
    C_{2,n}e^{\tau} &= c_ne^{\hat{\tau}}, \\
    C_{2,n}e^{1/2\tau} &= \hat{C}_{1,n}e^{1/2\hat{\tau}}, \\
    \sqrt{3}/2\tau + \theta &= \sqrt{3}/2\hat{\tau} + \hat{\theta} + n\pi,
\end{align*}
\]

where \( n \) is the total number of zeros of \( W \). Eliminating \( \tau \) and \( \hat{\tau} \) we obtain

\[
C_{2,n} = \hat{C}_{1,n}e^{1/\sqrt{3}(\theta - \hat{\theta} - n\pi)} = C_{2,0}e^{-n\frac{\alpha}{\sqrt{3}}}
\]

and

\[
c_n = C_{2,n}^{-1}\hat{C}_{1} \equiv c_0e^{n\frac{\alpha}{\sqrt{3}}}.
\]

Since the coefficient \( C_2(b) \) has to vanish for the limiting solution, i.e. \( b = b_\infty \), we get

\[
C_2(b) = \frac{\partial C_2}{\partial b}(b) = O((b - b_\infty)^2).
\]

Numerical integration of the limiting solution and its variation with respect to \( b \) yields

\[
\theta \approx 1.562209 \quad \text{and} \quad C_2 \approx 0.835060 \cdot (b_\infty - b),
\]

while numerical integration of the flat YM equations yields [6]

\[
\hat{\theta} \approx 0.339811 \quad \text{and} \quad \hat{C}_1 \approx 0.432478.
\]

From Eqs. (45,46) we obtain

\[
b_n = b_\infty - 1.04894 \cdot e^{-n\alpha}, \quad \text{and} \quad c_n = 0.213530 \cdot e^{n\alpha},
\]

with \( \alpha = \frac{\pi}{\sqrt{3}} \approx 1.81380 \) and \( e^{\alpha} \approx 6.13371 \).
Table 1: Parameters $b$ and $c$ of regular solutions; numerical results versus asymptotic formula Eq. (50)

| $n$ | $b_{\text{num}}$ | $b_{\text{asy}}$ | $c_{\text{num}}$ | $c_{\text{asy}}$ |
|-----|------------------|-------------------|------------------|------------------|
| 1   | 0.2608301456037  | 0.208481717       | 7.525748e - 01   | 1.309730e + 00   |
| 2   | 0.353518098051   | 0.351613357       | 7.320406e + 00   | 8.033504e + 00   |
| 3   | 0.3750018038731  | 0.374948614       | 4.852149e + 01   | 4.927516e + 01   |
| 4   | 0.3787544658699  | 0.378753043       | 3.014792e + 02   | 3.022394e + 02   |
| 5   | 0.379373291287   | 0.379373299       | 1.853097e + 03   | 1.853848e + 03   |
| 6   | 0.3794744134274  | 0.379474414       | 1.137028e + 04   | 1.137096e + 04   |
| 7   | 0.3794908985808  | 0.379490900       | 6.974585e + 04   | 6.974616e + 04   |
| 8   | 0.3794935863472  | 0.379493586       | 4.278045e + 05   | 4.278025e + 05   |

Table 2: Masses of regular solutions; numerical results versus asymptotic formula Eq. (51)

| $n$ | $M_{\text{num}}$ | $M_{\text{asy}}$ |
|-----|------------------|------------------|
| 1   | 0.80380777208    | 0.79578582       |
| 2   | 0.96559851724    | 0.96670624       |
| 3   | 0.99432009439    | 0.99457200       |
| 4   | 0.99907210998    | 0.99911505       |
| 5   | 0.99984867329    | 0.99985572       |
| 6   | 0.999975327358   | 0.99997648       |
| 7   | 0.9999959774969  | 0.99999617       |
| 8   | 0.99999934419598 | 0.999999374      |

Table 3: Quotients of parameters of regular solutions

| $n$ | $\Delta b_n$ | $\Delta c_n$ | $\Delta M_n$ |
|-----|--------------|--------------|--------------|
| 1   | 4.56821367   | 9.72714739   | 5.70301662   |
| 2   | 5.78233316   | 6.62825122   | 6.0569973    |
| 3   | 6.07360547   | 6.21331291   | 6.12131340   |
| 4   | 6.12385583   | 6.14668276   | 6.13170021   |
| 5   | 6.13210160   | 6.13582559   | 6.13338085   |
| 6   | 6.13345216   | 6.13404859   | 6.13365394   |
| 7   | 6.13370605   | 6.13376280   | 6.13369695   |
In [6] also the asymptotic formula for the mass $M_n = 1 - \frac{3}{2}C_{2,n}$ was derived. The same formula is supported by our numerical data in the present case, but in contrast to the EYM theory we were not able to find a simple derivation. Putting in numbers yields

$$M_n = 1 - 1.25259 \cdot e^{-n\alpha}.$$  (51)

Tables 1 and 2 contain a comparison of the numerically determined parameters $b_n, c_n$ and $M_n$ with the asymptotic values computed with the formulas from above. Table 3 displays the quotients $\Delta b_n = \frac{b_{n+1} - b_n}{b_n - b_{n+1}}, \Delta c_n = \frac{c_{n+1} - c_n}{c_n}$ and $\Delta M_n = \frac{1 - M_n}{1 - M_{n+1}}$ of the numerical data. All of them approach rapidly the value $e^{\pi/\sqrt{3}}$.

Appendix:

**Lemma 17:** Consider a solution $y$ of the linear differential equation $\dot{y} = ay + by$ in some interval $\tau_0 \leq \tau < \tau_1$ with $|a|$ integrable. If

$$c(\tau', \tau) = \int_{\tau'}^{\tau} b(\tau'')d\tau'',$$  (52)

is bounded from above for $\tau_0 \leq \tau' \leq \tau < \tau_1$ then $y$ is bounded; if $c(\tau', \tau)$ has a limit as $\tau \to \tau_1$ then $y(\tau)$ has a limit; if $c(\tau', \tau_1) = -\infty$ then $y(\tau_1) = 0$.

**Proof:** All properties are implied by the explicit form

$$y(\tau) = y(\tau_0)e^{c(\tau_0, \tau)} + \int_{\tau_0}^{\tau} a(\tau')e^{c(\tau', \tau)}d\tau'.$$  (53)

**Lemma 18:** Suppose $y$ obeys the inequality $\dot{y} \leq ay + by - y^2$. If $a$ is bounded from above and $b$ is bounded for $\tau \geq \tau_0$, then $y$ is either bounded for all $\tau \geq \tau_0$ or diverges to $-\infty$ for some finite $\tau_1 > \tau_0$.

**Proof:** Let $A, B$ be positive constants such that $a < A$ and $|b| < B$. We can estimate $\dot{y} < 0$ for $|y| > C = \sqrt{2A + 2B}$ and therefore $y$ is bounded from above. Furthermore $y$ monotonically decreases and $\left(\frac{1}{y}\right)' > 1/2$ for $y < -C$, and thus $y \to -\infty$ for some finite $\tau_1$.

**Lemma 19:** Suppose there is some open invariant subset $I$ of the phase space of the system Eq. (6) such that $2N - 1$ has a definite sign in $I$. Then for any trajectory $(2N - 1)r^2U^2$ vanishes on all its limit points for $\tau \to \infty$ in $I$.

**Proof:** Eq. (7) shows that the function $E$ is monotonous along trajectories in $I$ and hence serves as a ‘Lyapunov Function’. According to Lemma 11.1 of [8], $\dot{E} = (2N - 1)r^2U^2$ vanishes on the limit points of the solution.

**Corollary:** Suppose $N \to 0, r \to r_0 \neq 0$ for $\tau \to \infty$ and $W$ and $U$ stay bounded, then the solution tends to a f.p. of Eqs. (6f) with $U = 0$ and $W = 0$ or $W^2 = 1$.

**Proof:** From Lemma 19 we know that $\dot{W} \to 0$ for $\tau \to \infty$. Since $E$ is bounded it has a limit for $\tau \to \infty$ and thus also $W$ has a limit, which must be a f.p. of Eq. (6f) otherwise $\dot{W}$ would not tend to zero.

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