JACOB’S LADDERS AND LAWS THAT CONTROL CHAOTIC BEHAVIOR OF THE MEASURES OF REVERSELY ITERATED SEGMENTS

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Abstract. The main subject to study in this paper are properties of the sequence of reversely iterated segments. Especially, we will examine properties of chaotic behavior of the sequence of measures of corresponding segments. Our results are not accessible within current methods in the theory of Riemann zeta-function.

1. Introduction

1.1. Let us start with some notions and formulae to be reminded:

(A) the sequence

\[ \{T\}_{k=1}^{k_0} \]

is defined by (see [3], (5.1))

\[ \varphi_1(T) = \varphi_{k-1}(T), \quad k = 1, \ldots, k_0, \quad T = T_0[\varphi_1], \]

where \( k_0 \in \mathbb{N} \) is an arbitrary fixed number and \( \varphi_1(t) \) is the Jacob’s ladder;

(B) next (see [3], (1.3))

\[ \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt} + \frac{Z^2(t)}{2\Phi'[\varphi(t)]} = \frac{|\zeta(\frac{1}{2} + it)|^2}{\omega(t)}, \]

\[ \omega(t) = \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right) \right\} \ln t. \]

(1.1)

where

\[ Z(t) = e^{i\vartheta(t)} \zeta\left(1 + it\right), \]

\[ \vartheta(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma\left(\frac{1}{4} + it\right), \]

(1.2)

1.2. We have proved the following theorem (see [3], (2.1) – (2.7)): for every \( L_2 \)-orthogonal system

\[ \{f_n(t)\}_{n=1}^{\infty}, \quad t \in [0, 2l], \quad l = o\left(\frac{T}{\ln T}\right), \quad T \to \infty \]

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there is a continuum set of $L_2$-orthogonal systems
\[
\{F_n(t; T, k, l)\}_{n=1}^{\infty} = \left\{ f_n(\varphi_1(t) - T) \prod_{r=0}^{k-1} |\tilde{Z}[\varphi_1^r(t)]| \right\}_{n=1}^{\infty}, \quad t \in [T, T + 2l],
\]
where
\[
\varphi_1^{k-1} = \left[ T, T + 2l \right], \quad k = 1, \ldots, k_0,
\]
i.e. the following formula is valid
\[
\int_k^T f_n(\varphi_1^r(t) - T)f_n(\varphi_1^s(t) - T) \prod_{r=0}^{k-1} |\tilde{Z}[\varphi_1^r(t)]| dt = \left\{ \begin{array}{ll}
0 & , \quad m \neq n, \\
A_n & , \quad m = n,
\end{array} \right.
A_n = \int_0^{2l} f_n^2(t) dt.
\]

Remark 1. It is clear that the base of above mentioned result is new notion of reverse iterations (comp. (1.3)) in the theory of Riemann $\zeta \left( \frac{1}{2} + it \right)$-function.

In this paper we will study the sequence of reverse iterations
\[
\left\{ \left[ T, T + H \right] \right\}_{r=0}^{k}, \quad k = 1, \ldots, k_0
\]
alone. Namely, we will focus on properties of the sequence of real numbers (measures of corresponding segments)
\[
\left\{ \left\| [T, T + H] \right\| \right\}_{r=0}^{k}.
\]

Remark 2. Results of this paper are not accessible by current methods of the theory of Riemann zeta-function. We mention explicitly that our results are valid also in the microscopic case
\[
H \in \left( 0, \frac{A}{\ln T} \right], \quad T \to \infty.
\]

2. THEOREM 1 AND MOTIVATION BEHIND IT

2.1. Let us remind that the segments
\[
\left[ T, T + H \right], \quad r = 0, 1, \ldots, k
\]
are components of disconnected set
\[
\Delta(T, H, k) = \bigcup_{r=0}^{k} \left[ T, T + H \right], \quad k = 1, \ldots, k_0,
\]
Properties of the set (2.1) are listed below (see [3], (2.5) – (2.7)):

\[ H = o \left( \frac{T}{\ln T} \right) \Rightarrow \]

\[ ||[T, T + H]| = T + H - T = o \left( \frac{T}{\ln T} \right), \]

\[ \text{(2.2)} \]

\[ ||[T + H, T]| = \frac{k-1}{k} \left( T - T + H \right) \sim (1 - c)\pi(T); \pi(T) \sim \frac{T}{\ln T}. \]

\[ \text{(2.3)} \]

\[ [T, T + H] \prec \left[ \frac{1}{T}, \frac{1}{T + H} \right] \prec \cdots \prec \left[ \frac{k}{T}, \frac{k}{T + H} \right] \prec \cdots, \]

where \( c \) is the Euler’s constant and \( \pi(T) \) is the prime-counting function.

**Remark 3.** Consequently, the asymptotic behavior of our disconnected set (2.1) is as follows (see (2.2), (2.3)): if \( T \to \infty \) then the components of the set (2.1) recede unboundedly each from other and all together are receding to infinity. Hence, the set (2.1) behaves as a kind of one-dimensional Friedmann-Hubble expanding universe.

Furthermore, we notice explicitly that the distance \( \rho_l \) of the two consecutive segments

\[ \left[ \frac{l-1}{T}, \frac{l-1}{T + H} \right], \left[ \frac{l}{T}, \frac{l}{T + H} \right], \quad l = 1, 2, \ldots, k \]

is extremely big one, namely (see (2.3))

\[ \rho_l \sim (1 - c)\frac{T}{\ln T} \to \infty, \quad T \to \infty. \]

\[ \text{(2.5)} \]

**Remark 4.** Since the sequence

\[ \left\{ \left[ \frac{r}{T}, \frac{r}{T + H} \right] \right\}_{r=0}^k \]

is extremely sparse one (see (2.5)) then we may assume that the behavior of the measures

\[ \left\{ ||[T, T + H]|\right\}_{r=0}^k \]

is chaotic one.

Consequently, in correspondence with Remark 3, we wish to obtain some law controlling this chaotic behavior. In this direction, the following theorem holds true.

**Theorem 1.** Let

\[ 1 \leq n \leq k_0, \quad \bar{H} = o \left( \frac{T}{\ln T} \right), \]

and let the inequality

\[ ||[T, T + H]| = T + H - T \geq T^{1/3 + \epsilon}, \quad T \to \infty \]

hold true for \( \epsilon > 0 \) - an arbitrary small fixed number. Then we have that

\[ \text{(2.8)} \]

\[ T + H - T \sim \bar{H}, \quad T \to \infty. \]
2.2. Next, let us remind

(A) the Hardy-Littlewood-Ingham formula

\[
\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = T \ln T + (2c - 1 - \ln 2\pi)T + R(T),
\]

with the Balasubramanian’s estimate (for example)

\[
R(T) = \mathcal{O}(T^{1/3})
\]

of the error term in (2.9);

(B) the Good’s Ω-theorem that states

\[
R(T) = \Omega(T^{1/4}), \ T \to \infty;
\]

(C) our almost exact formula (see [1], (2.1), (2.2), \( \frac{\pi}{2} \to \varphi_1(t) \))

\[
\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi) \varphi_1(T) + c_0 + \mathcal{O} \left( \frac{\ln T}{T} \right), \ T \to \infty,
\]

where \( c \) is the Euler’s constant and \( c_0 \) is the constant from the Titchmarsh-Kober-Atkinson formula (see [4], p. 141).

Our discussion concerning formulae (2.10) – (2.12) see in [1], pp. 416, 417.

Remark 5. Consequently, we have proved in [1] that classical Hardy-Littlewood integral (1918)

\[
\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt
\]

has – in addition to the Hardy-Littlewood (and other similar) expressions possessing unbounded errors (as \( T \to \infty \)), (comp. (2.10), (2.11)) – infinite set of almost exact expressions (2.12).

Remark 6. It is clear – in context of (2.10), (2.11) – that our Theorem 1 will be true for every improvement of the exponent \( \frac{1}{3} \):

\[
\frac{1}{3} \to a \in \left( \frac{1}{4}, \frac{1}{3} \right).
\]

3. PROOF OF THEOREM 1

First of all, it follows from (2.9) and (2.10) that

\[
\int_T^{T+U} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim U \ln T, \ T \to \infty,
\]

\[
T^{1/3+\epsilon} \leq U = o \left( \frac{T}{\ln T} \right).
\]

Next, we use, together with (3.1), our formula

\[
\int_T^{T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{k-1}{(T+H - kT) \ln T}
\]

\[
\int_T^{T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim (T+H - kT) \ln T
\]
that follows from [3], (1.1) – (1.3), (7.4) with
\[ [T, T+H] \rightarrow k^{-1} \frac{T}{T+H}, \quad \frac{1}{T}, \frac{1}{T+H} \rightarrow \frac{k}{k-1} \frac{1}{T} + H. \]
Of course, we have (see (2.2))
\[ H = \frac{\sigma}{\ln T} \Rightarrow k \frac{\dot{T}}{T} + H = \frac{\sigma}{\ln T}, \quad T \rightarrow \infty, \quad k = 1, \ldots, k_0. \]
Further, if \( n, \bar{H} \) fulfill the conditions (2.6) and (2.7) then we have (see (3.1), (3.2))
\[ \int \frac{n}{T} \frac{\zeta}{(1/2 + it)} \, dt \sim (T + \bar{H} - T) \ln T, \]
and
\[ \int_{T}^{n} \frac{n-1}{T+\bar{H}} \frac{\zeta}{(1/2 + it)} \, dt \sim (T + \bar{H} - T) \ln T, \]
i.e.
\[ \frac{n}{T+\bar{H} - T} \sim \frac{n-1}{T} \ln T, \quad T \rightarrow \infty, \]
and, consequently,
\[ \frac{n-1}{T+\bar{H} - T} \sim \frac{n-2}{T} \ln T \quad \cdots \quad \sim T + \bar{H} - T = \bar{H}, \]
(see also (3.3)). Thus, we have that
\[ \frac{n}{T+\bar{H} - T} \sim \bar{H}, \quad T \rightarrow \infty, \]
i.e. the assertion (2.8) is verified.

4. CONSEQUENCES OF THEOREM 1

4.1. Corollary 1. Let
\[ H_1 = A(T)T^{1/3+\epsilon}, \quad 0 < A(T) < 1, \]
for example
\[ H_1 = \frac{1}{2} T^{1/3+\epsilon}, \quad \frac{1}{\ln \ln T} T^{1/3+\epsilon}, \ldots \]
Then
\[ \frac{k}{T+H_1} - \frac{k}{T} < T^{1/3+\epsilon}, \quad k = 1, \ldots, k_0. \]
Remark 7. Hence, in the case (4.1) we have that all members of the sequence
\[ \left\{ \left\lfloor \frac{k}{T+H_1} \right\rfloor \right\} \]
are lying below the level \( T^{1/3+\epsilon} \), (see (4.2)).
4.2. Next, as a consequence of Corollary 1, we have

**Corollary 2.** If

\[ H_2 = B(T)T^{1/3+\epsilon}, \quad B(T) > 1, \]

for example

\[ H_2 = 2T^{1/3+\epsilon}, \quad T^{1/3+\epsilon} \ln T, \ldots \]

and there is some

\[ n: \quad 1 \leq n < k_0 \]

such that

\[ T + H_2 - T < A(T)T^{1/3+\epsilon} \]

(see (4.1)), then

\[ \frac{k}{T} + H_2 - T < T^{1/3+\epsilon}, \quad k = n + 1, \ldots, k_0. \]

**Remark 8.** Consequently, in the case (4.3), (4.4) the second jump of the sequence

\[ \left\{ \frac{k}{T}, \frac{H_2}{T} \right\}_{k=1}^{k_0} \]

over the segment

\[ [(1 - \epsilon)T^{1/3+\epsilon}, (1 + \epsilon)T^{1/3+\epsilon}] \]

is forbidden. In other words, the oscillations of the sequence of measures about the measure of the segment (4.6) are forbidden.

5. **AN ESTIMATE FROM BELOW**

5.1. We will use the following in this section:

(A) the estimate

\[ H = o \left( \frac{T}{\ln T} \right) \Rightarrow \]

\[ \int_{1/T}^{1/T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt > (1 - \epsilon)(T + H - H_2) \ln T, \quad k = 1, \ldots, k_0, \]

that follows from the asymptotic formula (3.2), i.e. we have that

\[ \int_{1/T}^{1/T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt > (1 - \epsilon)H \ln T, \]

\[ \int_{1/T}^{1/T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt > (1 - \epsilon)(T + H - 1) \ln T, \]

\[ \vdots \]

\[ \int_{1/T}^{1/T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt > (1 - \epsilon)(T + H - T) \ln T; \]
(B) the property (see (2.1) and [3], sec. 4.1)

\[(5.2) \quad \tau \in \Delta(T, H, k) \Rightarrow \tau \in \left[ T, T + O\left(\frac{T}{\ln T}\right) \right] \subset [T, 2T].\]

5.2. Since (comp. [4], p. 99)

\[\left| \zeta \left(\frac{1}{2} + it\right) \right| < t^{1/6}, \quad t \to \infty \Rightarrow (5.3) \quad \left| \zeta \left(\frac{1}{2} + it\right) \right|^2 < t^{1/3}, \quad t \to \infty, \ t \to \infty,\]

then we have (see (5.2), (5.3)), that

\[t \in \Delta(T, H, k) \Rightarrow (5.4) \quad \left| \zeta \left(\frac{1}{2} + it\right) \right|^2 < \sqrt{2}T^{1/3} < 2T^{1/3}, \ T \to \infty,\]

without any hypothesis. Consequently, we have (see (5.1), (5.4)) following estimates

\[\left[ T, T + H \right] > \frac{1}{2} - \epsilon HT^{-1/3} \ln T,\]

\[\left[ T, T + H \right] > \left(\frac{1}{2} - \epsilon HT^{-1/3} \ln T \right)^2 H,\]

\[\vdots\]

\[\left[ k_0 T, k_0 T + H \right] > \left(\frac{1}{2} - \epsilon HT^{-1/3} \ln T \right)^{k_0} H > \left(\frac{1}{4}T^{-1/3} \ln T \right)^{k_0} H = \left(\frac{\ln T}{64T}\right)^{k_0/3} H, \ \epsilon \in (0, 1/2).\]

Since

\[0 < \frac{\ln T}{4T^{1/3}} < 1, \ T \to \infty,\]

then we have the following

**Theorem 2.**

\[H = o \left(\frac{T}{\ln T}\right) \Rightarrow (5.6) \quad \left[ k_0 T, k_0 T + H \right] > \left(\frac{\ln T}{64T}\right)^{k_0/3} H, \ k = 1, \ldots, k_0, \ T \to \infty.\]

**Example.** If \( H = 1, \ k_0 = 3000 \)

then (see (5.6))

\[\left[ k_0 T, k_0 T + H \right] > \left(\frac{\ln T}{64T}\right)^{1000} , \ k = 1, \ldots, 3000.\]

**Remark 9.** It appears that only advantage of the estimate (5.6) is, probably, its non-triviality.
6. RIEMANN HYPOTHESIS AND OUR ESTIMATE FROM BELOW

The following estimate
\[ \left| \zeta \left( \frac{1}{2} + it \right) \right| < B e^{A \frac{\ln t}{\ln \ln t}}, \quad t \to \infty \]
holds true on the Riemann hypothesis (see [4], p. 300). We use this estimate in the form
\[ \left| \zeta \left( \frac{1}{2} + it \right) \right| < t^{\frac{1}{6}}, \quad t \to \infty. \]
Thus we have (comp. (5.2), (5.4)) that
\[ t \in \Delta(T, H, k) \Rightarrow \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 < (2T)^{\frac{2C}{\ln \ln T}} < (2T)^{\frac{2C}{\ln T}} < \\
< 2^{\frac{2C}{\ln T}} T^{\frac{2C}{\ln T}} < (1 + \varepsilon) T^{\frac{2C}{\ln T}}, \quad \varepsilon \in (0, 1/2), \quad T \to \infty. \]
Now, we obtain from (5.1), (6.1), (comp. (5.2), (5.5)) that
\[ \frac{1}{3k_0} H T^{-k_0 \frac{2C}{\ln T}} \ln k_0 T > \\
= T^{-k_0 \frac{1}{\ln T}} H T^{-k_0 \frac{2C}{\ln T}} T^{k_0 \frac{\ln T}{\ln \ln T}} > HT^{-k_0 \frac{2D}{\ln \ln T}}. \]
Hence, the following theorem holds true.

**Theorem 3.** On Riemann hypothesis we have
\[ H = o \left( \frac{T}{\ln T} \right) \Rightarrow \]
\[ \frac{k}{\Delta(T, H, k)} > HT^{-k_0 \frac{2C}{\ln T}}, \quad k = 1, \ldots, k_0, \quad T \to \infty. \]

**Remark 10.** The conditional estimate (6.2) is effective particularly in the case
\[ H = T^{\Delta}, \quad 0 < \Delta < 1. \]
Namely, in this case we obtain from (6.2)
\[ \frac{k}{\Delta(T, H, k)} > T^{\Delta-o(1)}, \quad T \to \infty. \]

**Remark 11.** It was expected that the Riemann hypothesis has essential influence on that estimate from below. Actually, we have in the case (6.3) that:
(A) without any hypothesis (see (5.6))
\[ \frac{k}{\Delta(T, H, k)} > \left( \frac{1}{4} \ln T \right)^{k_0} T^{\Delta-k_0}, \]
where
\[ k_0 \geq 3 \Rightarrow \Delta - \frac{k_0}{3} < 0; \]
(B) on the Riemann hypothesis (see (6.4))

\[(6.7) \quad \| [T, T + T^\Delta] \| > T^{\Delta - o(1)}, \]

where

\[(6.8) \quad 0 < \Delta - o(1) \to \Delta \text{ as } T \to \infty. \]

**Example.** In the case

\[ \Delta = \frac{1}{3} + \epsilon \]

(see Theorem 1) we have, on Riemann hypothesis, that (see (6.4))

\[ \| [T, T + T^{1/3+\epsilon}] \| > T^{1/3+\epsilon-o(1)}, \quad k = 1, \ldots, k_0, \quad T \to \infty. \]

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