Equivalence of the Path Integral for Fermions in Flat Spacetime in Cartesian and Spherical Coordinates

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Abstract

The path-integral calculation for the free energy of a fermion gas in flat spacetime is performed in spherical coordinates, and its equivalence with Cartesian coordinates is established. An appropriate generalization of spherical harmonics for fermion fields is used. The latter technique has been developed by the authors of this paper to perform the calculation of the free energy of a fermion gas in the presence of a Schwarzschild background, using a near-horizon expansion approach. The robustness of this equivalence in the flat-spacetime limit lends support to the black hole background computation leading to black hole thermodynamics.
I. INTRODUCTION

The authors of this paper have recently advanced a near-horizon (NH) expansion technique for the calculation of black-hole entropy in the framework of 't Hooft’s brick-wall approach [1]. This NH technique has provided a clear origin for the leading divergence of the entropy calculation, as well as revealed a NH conformal symmetry that may have an important role in the understanding of the Bekenstein-Hawking area law [2–4]. In Ref. [2] an expansion in spherical harmonics was used that led to a radial one-dimensional semiclassical analysis for the expression of the density of modes.1 As in 't Hooft’s work, this was done for spin-zero fields.

We have recently performed a similar calculation for Dirac fermion fields [4], in both the canonical and path-integral frameworks. The NH method in spherical coordinates proved to be a critical technique to tackle the technical challenges posed by the fermionic path integral in a spherically symmetric black hole background. The Bekenstein-Hawking area law was verified within the framework of the brick-wall approach appealing to the relevant fermionic spherical harmonics expansion. In this paper we develop for flat spacetime the corresponding expansion in fermionic spherical harmonics, and present further evidence for the technical correctness of the subtle fermionic NH approach. While the path-integral calculation of the free energy for a free-fermion gas in flat spacetime is well known using Cartesian coordinates [6, 7], its counterpart in spherical coordinates is not. Through a series of steps leading to the free-energy path integral in spherical coordinates, we show the equivalence with Cartesian coordinates, and provide a strong check on the correctness of the corresponding fermionic path-integral in a black-hole background. In essence, this calculation highlights some conceptual and technical properties that are crucial for similar computations in curved spacetime, and lends itself to generalizations to higher spin fields.

Consider the Euclideanized action for free fermion fields in flat spacetime

\[ S_E = \int_0^\beta d\tau \int d^3x \bar{\psi} (\gamma^\mu E_\mu \partial_\mu + m) \psi \]  

(1.1)

(the relevant definitions and conventions are summarized in Appendix A). The partition

1 The authors of reference [4] have also developed NH methods without the need for an expansion in spherical harmonics, and obtained the same results [3, 5].
function for this system is

\[ Z = \int D\bar{\psi}(\tau, x) D\psi(\tau, x) \exp \{-S_E\} . \quad (1.2) \]

As is well known, the fermionic fields satisfy anti-periodic boundary conditions in imaginary time \( \tau \),

\[ \bar{\psi}(0, x) = -\bar{\psi}(\beta, x) \]
\[ \psi(0, x) = -\psi(\beta, x) . \quad (1.3) \]

This implies that the frequencies associated with the Euclidean time \( \tau \) are discrete, and of the form

\[ \omega_n = \frac{2\pi}{\beta} \left( n + \frac{1}{2} \right) , \quad n \in \mathbb{Z} . \quad (1.4) \]

In flat spacetime, one can always introduce the momentum space Fourier representation for the spatial part of the fermion field

\[ \psi(\tau, x) = \sum_n \frac{\exp(-i\omega_n \tau)}{\sqrt{\beta}} \int \frac{d^3\vec{p}}{(2\pi)^{3/2} \sqrt{2\omega_p}} \exp \left( i \vec{p} \cdot \vec{x} \right) \psi_n(\vec{p}) . \quad (1.5) \]

The Euclideanized action then becomes

\[ S_E = \sum_n \int \frac{d^3\vec{p}}{(2\pi)^{3/2} 2\omega_p} \bar{\psi}_n(\vec{p}) (i\gamma^\mu P_\mu + m) \psi_n(\vec{p}) , \quad (1.6) \]

where the four vector \( P = (\omega_n, \vec{p}) \). Therefore, using the well-known result for anti-commuting path integrals,

\[ \int \left\{ \prod_i d\bar{a}_i a_i \right\} \exp \left( -a_i^* M_{ij} a_j \right) = \det (M) , \quad (1.7) \]

the partition function (1.2) gives

\[ Z = C \prod_{n, \vec{p}} \det (i\gamma_E^\mu P_\mu + m) \]
\[ = C \left[ \prod_{n, \vec{p}} \det (i\gamma_E^\mu P_\mu + m) \det (-i\gamma_E^\mu P_\mu + m) \right]^{1/2} , \quad (1.8) \]

with \( C \) a constant. Here we used

\[ |\det D| = \sqrt{\det D \det D^\dagger} . \quad (1.9) \]

Thus, up to a phase, we get

\[ Z = \prod_{n, \vec{p}} \det^{1/2} \left[ (\omega_n^2 + \vec{p}^2 + m^2) I_4 \right] , \quad (1.10) \]
where $I_4$ is the $4 \times 4$ unit matrix arising from the Dirac algebra. Notice that the determinant involves products with respect to the Dirac matrices in addition to its functional nature; here, due to the simple factorization above, the matrix part leads to an overall exponent of $4$ for the functional determinant,
\begin{equation}
Z = \prod_{n, \vec{p}} \left[ \det^{1/2} \left( \omega_n^2 + \vec{p}^2 + m^2 \right) \right]^4 = \prod_{n, \vec{p}} \det^2 \left( \omega_n^2 + \vec{p}^2 + m^2 \right) .
\end{equation}
From Eq. (1.11), it is straightforward to get the free energy using $F = -\ln Z/\beta$.

II. PATH INTEGRAL IN SPHERICAL COORDINATES

In spherical coordinates a simple change of coordinates from cartesian to spherical coordinates changes the Dirac operator in the action to
\begin{equation}
\Theta = \gamma_{E}^0 \partial_0 + \gamma^1 \partial_r + \gamma^2 \frac{1}{r} \partial_\theta + \gamma^3 \frac{1}{r \sin \theta} \partial_\phi + m ,
\end{equation}
where
\begin{equation}
\gamma^i = \gamma^i_{E} \epsilon_{\mu i}, \quad \epsilon_{\mu i} = \left( 0, \hat{e}_i \right), \quad i = r, \theta, \phi .
\end{equation}
The operator $\Theta$, defined above in the original field representation $\psi(\tau, x)$, is not convenient for our calculational purposes. Instead, a more “friendly” geometrical version results if we perform a unitary transformation on $\psi$ defined by\(^2\)
\begin{equation}
\begin{aligned}
\psi(\tau, x) &= U \tilde{\psi}(\tau, x) \\
\tilde{\psi}(\tau, x) &= \tilde{\psi}(\tau, x) U^\dagger ,
\end{aligned}
\end{equation}
with the rotation
\begin{equation}
U \left( R_{z} (\phi) R_{y} (\theta) \right) = U \left( R_{z} (\phi) \right) U \left( R_{y} (\theta) \right)
= \exp \left[ -i \phi \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \right] \exp \left[ -i \theta \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \right] .
\end{equation}
Geometrically, this amounts to a realignment of the coordinate axes with the chosen curvilinear coordinates $(r, \theta, \phi)$. Under this field redefinition, the Euclideanized Lagrangian becomes
\begin{equation}
\tilde{\psi}(\tau, x) \Theta \psi(\tau, x) = \tilde{\psi}(\tau, x) U^\dagger \Theta U \tilde{\psi}(\tau, x) \equiv \tilde{\psi}(\tau, x) \tilde{\Theta} \tilde{\psi}(\tau, x) ,
\end{equation}
\(^2\)This transformation leaves the path integral measure invariant.
\[ \tilde{\Theta} = \gamma_0^E \partial_0 + \gamma_3^E \left( \partial_r + \frac{1}{r} \right) + \gamma_1^E \frac{1}{r} \left( \partial_\theta + \frac{1}{2} \cot \theta \right) + \gamma_2^E \frac{1}{r \sin \theta} \partial_\phi + m , \]  
(2.6)

and we used the fact that spatial rotations [in our case, \( U \) in Eq. (2.4)] commute with \( \gamma_0^E \). Effectively, this choice leads to the selection of self-adjoint operators associated with the given coordinates. With our representation of the Dirac matrices (Appendix A), we have

\[
\tilde{\Theta} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \partial_0 - \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} i \left( \partial_r + \frac{1}{r} \right) \\
-\frac{i}{r} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \left( \partial_\theta + \frac{1}{2} \cot \theta \right) - \frac{i}{r} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \frac{1}{\sin \theta} \partial_\phi + m .
\]

(2.7)

It is now necessary to attempt a separation of variables to reduce the problem to one dimension. In particular, owing to the symmetry of the problem in angular variables, suggest an expansion of the form

\[
\psi \propto e^{-\frac{i \omega_n \tau}{\sqrt{\beta}}} \begin{pmatrix} A_n(r) Y_1(\theta, \phi) \\
B_n(r) Y_2(\theta, \phi) \\
C_n(r) Y_1(\theta, \phi) \\
D_n(r) Y_2(\theta, \phi) \end{pmatrix} .
\]

(2.8)

Following Appendix B we introduce the “generalized spherical harmonic” eigenfunctions \( Y_1 \) and \( Y_2 \),

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\theta + \frac{1}{2} \cot \theta \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sin \theta} \partial_\phi \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \lambda_+ Y_1 \\ \lambda_- Y_2 \end{pmatrix} ,
\]

(2.9)

where

\[
\lambda_\pm = \pm (j + 1/2) .
\]

(2.10)

Then, the effect of the Euclidean Dirac operator on the fields (2.8) is

\[
\tilde{\Theta} \psi = \begin{pmatrix} -i \left( \partial_r + \frac{1}{r} \right) C + \frac{\lambda_+}{r} D \right) Y_1 + (-i \omega_n + m) AY_1 \\
-i \left( \partial_r + \frac{1}{r} \right) D + \frac{\lambda_-}{r} C \right) Y_2 + [-i \omega_n + m] BY_2 \\
i \left( \partial_r + \frac{1}{r} \right) A + \frac{\lambda_+}{r} B \right) Y_1 + (i \omega_n + m) CY_1 \\
i \left( - \left( \partial_r + \frac{1}{r} \right) B + \frac{\lambda_-}{r} A \right) Y_2 + (i \omega_n + m) BY_2 \end{pmatrix} e^{-\frac{i \omega_n \tau}{\sqrt{\beta}}} .
\]

(2.11)

\[ ^3 \text{For notational simplicity we are omitting the } j \text{ and } m_j \text{ labels on the spherical functions } Y_{(1,2)} \text{ in Eqs. (2.8)-(2.9).} \]
We use the above harmonics to expand the spatial part of the fermion field, leaving the Euclidean time as before,

\[
\psi = \sum_{j,m,n} e^{-i\omega_n \tau} \begin{pmatrix} A_{nj}(r) Y_{1jm_j} \\ B_{nj}(r) Y_{2jm_j} \\ C_{nj}(r) Y_{1jm_j} \\ D_{nj}(r) Y_{2jm_j} \end{pmatrix},
\]

(2.12)

\[
\bar{\psi} = \sum_{j,m,n} e^{-i\omega_n \tau} \begin{pmatrix} A_{nj}^*(r) Y_{1jm_j}^* \\ B_{nj}^*(r) Y_{2jm_j}^* \\ -C_{nj}^*(r) Y_{1jm_j}^* \\ -D_{nj}^*(r) Y_{2jm_j}^* \end{pmatrix}.
\]

Using

\[
\iint Y_{1jm_j}^* Y_{1jm_j'} \sin \theta d\theta d\phi = \delta_{jj} \delta_{m_j m_j'},
\]

(2.13)

the Euclideanized action becomes

\[
S_E = -i \sum_{n,j,m} \int dr r^2 \left\{ \omega_n \left( A_{nj}^* A_{nj} + B_{nj}^* B_{nj} + C_{nj}^* C_{nj} + D_{nj}^* D_{nj} \right) \\
+ \left[ A_{nj}^* \left( \partial_r + \frac{1}{r} \right) C_{nj} - B_{nj}^* \left( \partial_r + \frac{1}{r} \right) D_{nj} \\
+ C_{nj}^* \left( \partial_r + \frac{1}{r} \right) A_{nj} - D_{nj}^* \left( \partial_r + \frac{1}{r} \right) B_{nj} \right] \\
+ \frac{(j+1/2)}{r} \left( A_{nj}^* D_{nj} - B_{nj}^* C_{nj} + C_{nj}^* B_{nj} - D_{nj}^* A_{nj} \right) \\
+ im \left( A_{nj}^* A_{nj} + B_{nj}^* B_{nj} - C_{nj}^* C_{nj} - D_{nj}^* D_{nj} \right) \right\}.
\]

(2.14)

This expression can be written as

\[
S_E = \sum_{n,j,m_j} \int dr r^2 \bar{\psi}_{njm_j} \Omega_{njm_j} \psi_{njm_j},
\]

(2.15)

where

\[
\psi_{njm_j} = \begin{pmatrix} A_{nj} \\ B_{nj} \\ C_{nj} \\ D_{nj} \end{pmatrix},
\]

\[
\bar{\psi}_{njm_j} = \begin{pmatrix} A_{nj}^* \\ B_{nj}^* \\ -C_{nj}^* \\ -D_{nj}^* \end{pmatrix},
\]

\[
\Omega_{njm_j} = -i \left[ \gamma^{(0)} \omega_n + \gamma^{(3)} \left( \partial_r + \frac{1}{r} \right) + i \gamma^{(2)} \frac{(j+1/2)}{r} + im \right],
\]

(2.16)
leading, up to an irrelevant constant, to

\[
Z = \prod_{j,n} \det \left( \omega_n - i\gamma^{(0)} - i\gamma^{(3)} \left( \partial_r + \frac{1}{r} \right) + \gamma^{(2)} \frac{(j + 1/2)}{r} + m \right). \tag{2.17}
\]

We again use

\[
\det (D) = \sqrt{\det (D) \det (D^T)} \tag{2.18}
\]

(up to a phase factor) and the properties of the Dirac matrices to get

\[
Z = \prod_{j,n} \det \left( \omega_n^2 + m^2 - \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{(j + 1/2)^2}{r^2} \right.
- \left. \frac{(j + 1/2)}{r^2} \left[ \sigma_3 \ 0 \\
0 \ \sigma_3 \right] \right). \tag{2.19}
\]

Notice “det” in Eq. (2.17) refers to both functional \((r)\) and matrix sense. The operator in Eq. (2.19) can be written as

\[
\Lambda_4^{(j,r)} = \begin{bmatrix}
\Omega_{(r,j)} - \frac{(j + 1/2)}{r^2} \sigma_3 & 0 \\
0 & \Omega_{(r,j)} - \frac{(j + 1/2)}{r^2} \sigma_3
\end{bmatrix}, \tag{2.20}
\]

with

\[
\Omega_{(r,j)} = \omega_n^2 - \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{(j + 1/2)^2}{r^2} + m^2. \tag{2.21}
\]

Therefore,

\[
\det \Lambda_4^{(j,r)} = \det^2 \left[ \begin{array}{cc}
\Omega_{(r,j)} - \frac{(j + 1/2)}{r^2} \sigma_3 & 0 \\
0 & \Omega_{(r,j)} - \frac{(j + 1/2)}{r^2} \sigma_3
\end{array} \right] = \det^2 \left[ \begin{array}{cc}
\Omega_{-(r,j)} & 0 \\
0 & \Omega_{+(r,j)}
\end{array} \right] \equiv \det^2 (\Lambda_2), \tag{2.22}
\]

where

\[
\Omega_{\pm(r,j)} = \Omega_{(r,j)} \pm \frac{(j + 1/2)}{r^2} 
\Lambda_2^{(j,r)} = \begin{bmatrix}
\Omega_{-(r,j)} & 0 \\
0 & \Omega_{+(r,j)}
\end{bmatrix}. \tag{2.23}
\]

As \(j\) takes only semi-integer values \((j = 1/2, 3/2, 5/2, \ldots)\), let \(l = j + 1/2 = 1, 2, 3 \ldots\). With this notation,

\[
\Omega_{\pm} \equiv \Omega_{(l, l-1)} = \omega_n^2 - \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l + 1)}{r^2} + m^2. \tag{2.24}
\]

We can then write Eq. (2.19) as

\[
\prod_{n,l} \det 2^l \Lambda_2^{(l,r)} = \prod_{n,l=1} (\det \Omega_l)^{2^l} \prod_{n,l=1} (\det \Omega_{l-1})^{2^l}. \tag{2.25}
\]
Now,
\[ \prod_{l=1}^{\infty} (\det \Omega_{l-1})^{2i} = \prod_{l=0}^{\infty} (\det \Omega_i)^{2i+2} = \prod_{l=0}^{\infty} (\det \Omega_i)^{2i+1} \prod_{l=0}^{\infty} \det \Omega_i, \tag{2.26} \]
\[ \prod_{l=1}^{\infty} (\det \Omega_i)^{2l} = \prod_{l=0}^{\infty} (\det \Omega_i)^{2l}. \tag{2.27} \]

Therefore,
\[ Z = \left( \prod_{n,l=0} (\det \Omega_i)^{2l+1} \right)^2. \tag{2.28} \]

As is well known, \( \prod_{n,l=0} (\det \Omega_i)^{2l+1} \) is the spherical-coordinate version (up to a constant) of \( \det \Box_E \), where \( \Box_E = \partial^2 + \vec{\nabla}^2 \). In momentum space, Eq. (2.28) then becomes, up to a constant, equal to the expression of Eq. (1.11), Q.E.D.

### III. CONCLUSIONS

We have shown the equivalence of the calculation for the partition function of a free gas of fermions in flat spacetime in Cartesian and in spherical coordinates. The latter involved an expansion of the fermion fields in “generalized harmonics” as detailed above. While the result is not surprising, the treatment of the path integral in spherical coordinates in this paper presents novelties and subtleties hitherto not seen or emphasized. It is technically remarkable for instance, to see the emergence of the zero mode \( (l = 0) \) for the scalar case in the process of “squaring” the fermionic determinant. Beyond its own merits, this calculation lends strong support to the correctness of the similar calculation in the case of a black hole background performed recently by the authors [4]. In that case, the result similar to Eq. (2.19) in this paper after following the same procedure is

\[ Z = \prod_{j,n\geq 0} \det^{j+1/2} \left[ \omega_n^2 + \frac{2ff'' - f'^2}{4} - f^2 \frac{d}{dr} \left( \frac{r^2}{r} \frac{d}{dr} \right) + (j + \frac{1}{2})^2 - f + m^2f \right] 
+ (j + \frac{1}{2}) f \frac{d}{dr} \left[ \begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right] + 2imf \frac{d}{dr} \left[ \begin{array}{cc} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{array} \right]. \tag{3.1} \]

In Eq. (3.1) the function \( f \equiv f(r) \) is defined by the Schwarzschild metric
\[ ds^2 = f(r) [dx^0]^2 - g(r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{3.2} \]
where
\[ f(r) = 1 - \frac{2M}{r} = \frac{1}{g(r)}, \tag{3.3} \]
in Minkowski spacetime with signature $(+,-,-,-)$. As we saw above, the determinant in Eq. (2.19) has a very tight structure that allowed for a full diagonalization into products of determinants over subspaces of the original one. The presence of the last term in Eq. (3.1) makes this process considerably more difficult than the flat spacetime case. The authors’ NH expansion method made it possible to systematically calculate and isolate the leading contribution to the divergent part of the free energy—hence of the entropy—of the fermionic thermal atmosphere surrounding the black hole, which led to the Bekenstein-Hawking law within the framework of ‘t Hooft’s brick-wall approach. Therefore, while the black-hole calculation is consistent with the expected fundamental result of black hole thermodynamics, it is important to establish the full validity of the technical aspects of the NH expansion, and this paper lends further credibility to our approach.

**Appendix A: Euclidean Dirac Matrices**

We define the Euclidean Dirac matrices to be

\[
\begin{align*}
\gamma^0_E &= \gamma^0 \\
\gamma^1_E &= -i\gamma^1 \\
\gamma^2_E &= -i\gamma^2 \\
\gamma^3_E &= -i\gamma^3,
\end{align*}
\]  

where, in our convention, the $\gamma^\mu$’s are the Dirac matrices for Minkowski spacetime with $(+,-,-,-)$. Of course, the signature of Euclidean space is $(+,+,+,+)$, with the matrices satisfying

\[
\{\gamma^\mu_E, \gamma^\nu_E\} = 2\delta^{\mu\nu}. 
\]  

(A2)
Explicitly, in our representation,

\[
\gamma^0_E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1_E = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\
\gamma^2_E = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3_E = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}
\]

(Appendix B: Reduction in Spherical Coordinates)

Due to the structure of the Dirac operator, the separation into radial and angular components in this case is more complicated than in the one-component scalar case. One way to proceed is as follows: Eq. (2.7) shows that the problem reduces to a two-dimensional one, so we postulate the “generalized” eigenvalue problem

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta \right) + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \lambda + Y_1 \\ \lambda - Y_2 \end{pmatrix}. \tag{B1}
\]

Solving for \(Y_2\) one can then obtain an equation for \(Y_1\):

\[
\left( \partial^2 + \frac{1}{2} \cot \theta \right) Y_1 = \lambda_+ \lambda_- Y_1. \tag{B2}
\]

Writing

\[
Y_1(\theta, \phi) = e^{im\phi} f(\theta), \tag{B3}
\]

yields the following

\[
\frac{d^2 f}{d\theta^2} + \cot \theta \frac{df}{d\theta} - \left( \frac{1}{4} + \frac{1}{4 \sin^2 \theta} + \frac{m^2}{\sin^2 \theta} - \frac{m \cos \theta}{\sin^2 \theta} \right) f = \lambda_+ \lambda_- f. \tag{B4}
\]

With the further change of variables,

\[
x = \cos \theta \\
f = (1 - x)^{m - \frac{1}{2}} (1 + x)^{m + \frac{1}{2}} P(x). \tag{B5}
\]

\(^4\) Notice that \(\lambda_+\) cannot be equal to \(\lambda_-\).
one obtains the following equation for $P(x)$:

$$
(1 - x^2) \frac{d^2 P}{dx^2} + \left[1 - (2m + 2)x\right] \frac{dP}{dx} + \left[-\frac{1}{4} - \lambda_+ \lambda_- - m(m + 1)\right] P = 0. \quad (B6)
$$

This equation describes the Jacobi polynomials, for which $\lambda_+ \lambda_- = -(j + 1/2)^2$ has to be a negative integer: $\lambda_+ \lambda_- = -1, -2, -3, \ldots$ (and hence $j$ a semi-integer: $j = 1/2, 3/2, 5/2, \ldots$). We can then choose $\lambda_\pm = \pm (j + 1/2)$. Solving for $Y_2$ would give a similar equation to Eq. (B6) but with the coefficient of the first derivative replaced now by $-1 - (2m + 2)x$.

Therefore, we get

$$
\begin{pmatrix}
Y_{1jm} \\
Y_{2jm}
\end{pmatrix} = C_{lm} e^{im\phi} \begin{pmatrix}
(1 - x)^{m - \frac{1}{2}} (1 + x)^{-\frac{1}{2}} I_j^{(m - \frac{1}{2}, m + \frac{1}{2})} (x) \\
(1 - x)^{m + \frac{1}{2}} (1 + x)^{-\frac{1}{2}} I_j^{(m + \frac{1}{2}, m - \frac{1}{2})} (x)
\end{pmatrix} \quad (B7)
$$

In addition, these eigenfunctions $Y$’s are chosen to be orthonormal,

$$
\begin{align*}
\int_Y \int Y_{1jm}^{*} Y_{1jm'}^* \sin \theta d\theta d\phi &= \delta_{jj'} \delta_{m,m'} \\
\int_Y \int Y_{2jm}^{*} Y_{2jm'}^* \sin \theta d\theta d\phi &= \delta_{jj'} \delta_{m,m'} \quad (B8)
\end{align*}
$$

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