Correlation and topology are two of central topics in modern condensed matter physics. The Hubbard model of spin-1/2 fermions with on-site interaction is one of the most famous models that give rise to highly nontrivial correlation effects, such as antiferromagnetic order at half-filling. This model is relevant to many strongly correlated materials such as High-Tc cuprate superconductors. Recently, this model has also been simulated by using ultracold fermions in optical lattices [1–3]. In 1988, Haldane proposed a model of noninteracting fermions in the honeycomb lattice, which can give rise to topological band structure and quantized Hall conductance without trivial correlation effects, such as antiferromagnetic order. Both terms break time-reversal symmetry, and the Hamiltonian described by the Haldane model is given by

\[ \hat{H}_{\text{HH}} = \hat{H}_{\text{H}} + U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} \]

where \( s = \pm \) refers to spin up and down, respectively, the \( t_2 \)-term represents next nearest hopping with a nontrivial phase \( \phi_{ij} = \pm \phi \) for different sublattices, and the \( M \)-term adds a potential imbalance between A and B sublattices, as \( \epsilon_i = \pm 1 \) for \( i \) belonging to A or B sublattices. The \( t_2 \)-term breaks time-reversal symmetry, and the \( M \)-term breaks the inversion symmetry. Both terms open up the gap at Dirac points, and for half-filling, a phase diagram (without the interaction term) including a topological transition from trivial insulator to topologically nontrivial insulator is shown in Fig. 1(a), across which the gap at one of the Dirac point is closed. For the topologically nontrivial insulator, each spin compo-

\[ H = -t_1 \sum_{\langle ij \rangle,s} (\hat{c}_{i,s}^{\dagger} \hat{c}_{j,s} + \text{h.c.}) - t_2 \sum_{\langle (ij) \rangle,s} \left( e^{i\phi_{ij}} \epsilon_{i,s}^{\dagger} \epsilon_{j,s} + \text{h.c.} \right) - M \sum_i \epsilon_{i,s}^{\dagger} \epsilon_{i,s} \]
imbalance

Due to the spin rotational symmetry, we can
in the Haldane model.

ceeding to the self-consistent mean-field calculation, we
would like to first discuss the relation between the follow-

Fig. 1(b). The solid line means continuous evolution while the
dashed line means discontinuous jump. Panels (b) and (c)
illustrate two types of AF magnetic order: (b) is collinear AF
order and (c) is canted AF order with nonzero spin chirality.

nent fills the lower band with Chern number equalling 1
and the total Chern number \( C = 2 \). The interaction term
can be decoupled as

\[
U \sum_i \tilde{n}_{i\uparrow} \tilde{n}_{i\downarrow} = \frac{1}{2} U \tilde{N} - \frac{2}{3} U \sum_i S_i^2
\approx \frac{1}{2} U \tilde{N} + \sum_i \left( -m_i \cdot S_i + \frac{3m_i^2}{8U} \right),
\]

where we have introduced an on-site magnetic order pa-
parameter \( 4U \langle \tilde{S}_i \rangle / 3 = m_i \). Thus the mean-field Hamiltonian \( H_{MF} \) is given by

\[
\hat{H}_{MF} = \hat{H}_H - \sum_i m_i \cdot S_i.
\]

Relation between AF order and Topology. Before pro-
ceeding to the self-consistent mean-field calculation, we
would like to first discuss the relation between the follow-
ing two types of possible AF order and the parameters in
the Haldane model.

(A) Collinear AF order, i.e. \( \mathbf{m}_i = \mathbf{m} \) on the sub-
lattice \( A \) and \( \mathbf{m}_i = -\mathbf{m} \) on the sublattice \( B \), as shown in
Fig. 1(b). Due to the spin rotational symmetry, we can
always choose \( \mathbf{m} = m \hat{z} \). Thus, it adds a spin-dependent
contribution on \( M \) in the single-particle Haldane model
of Eq. 1, i.e. \( M \to M + sm \), where \( s = \pm \) denotes spin.

(B) Canted AF order. For simplicity, we consider the
situation that \( \mathbf{m} \) are different among the three \( A \) sites
(denoted by \( i, j \) and \( k \)) of one honeycomb, as shown in
Fig. 1(c), which leads to finite “scalar spin chirality”
order \( \mathbf{S} = \langle \hat{S}_i \rangle \cdot \langle \hat{S}_j \times \hat{S}_k \rangle \) ( Since we are concerned
with magnetically ordered state, we do not use the usual
definition, i.e. \( \langle \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k) \rangle \) ). Within each unit cell, \( \mathbf{m}_i \), at \( A \) site is approximately opposite to \( \mathbf{m}_j \), at \( B \) site.

Then we can apply an on-site spin rotation \( U_t \) so
that \( U_t^\dagger (\mathbf{m}_i \cdot \mathbf{S}_i) U_t = -\frac{1}{2} \mathbf{m}_i \cdot s_{iz} \) for \( A \) sublattices and
\( U_t^\dagger (\mathbf{m}_i \cdot \mathbf{S}_i) U_t = -\frac{1}{2} \mathbf{m}_i \cdot s_{iz} \), where \( s_{iz} \) is the Pauli matrix
associated to spin. Qualitatively speaking, this local spin
rotation introduces an additional Berry phase factor \( \pm \phi \)
in the next nearest hopping term for different sublattices,
where \( \phi \) is approximately one sixth of the solid angle ex-
expanded by \( \mathbf{m}_i, \mathbf{m}_j, \mathbf{m}_k \), that is to say, \( \phi \) in the original
Haldane Hamiltonian Eq. 2 should be replaced by an
effective phase \( \phi_{\text{eff}} = \phi + \phi \).

Therefore, the mean-field Hamiltonian with a collinear
AF order corresponds to a free-fermion Hamiltonian \( \hat{H}_H \)
with a modified spin-dependent effective \( M \), and the
mean-field Hamiltonian with a canted AF order corre-
sponds to a free-fermion Hamiltonian with both \( M \) and \( \phi \)
in \( \hat{H}_H \) modified. Thus, the noninteracting phase diagram
in Fig. 1(a) is helpful for understanding the mean-field
phases, with \( M \) and \( \phi \) replaced by effective parameters
determined by magnetic order. There emerge two differ-
ent scenarios about how magnetic orders drive transition
between topological nontrivial and trivial insulators.

(i) If AF order increases continuously as interaction strength $U$ increases, the mean-field Hamiltonian will evolve continuously cross the phase boundary from $C = 2$ to $C = 0$ insulator. Inevitably, there will be a topological transition at which gapless fermions and a finite AF order coexist.

(ii) A first-order transition occurs as $U$ increases, at which a jump of AF order brings the system from $C = 2$ regime in the phase diagram to $C = 0$ regime.

**Mean-field Phase Diagram.** A self-consistent mean-field calculation is conducted to determine the phase diagram. For simplicity, we first consider the situation with $M = 0$ and $\phi = \pi/2$ in $H_{\text{MF}}$ in Eq. [2]. In our calculation, we enlarge the unit cell to six sites of each honeycomb, and no further assumption for order parameter $m$, at these six sites are imposed. (i.e. totally 18 parameters are determined from self-consistent iterations.) Enlarging the unit cell in the magnetic ordered phase turns out to be crucial for obtaining the state with lower energy and establishing the correct picture as discussed below. After we obtain the self-consistent solution, we can straightforwardly compute the single-particle gap for fermions, scalar spin chirality order and the Chern number for mean-field ground state [12]. The resulting phase diagram is shown in Fig. 2, which contains both two scenarios of phase transition, depending on the ratio $t_2/t_1$, as well as four different phases: I. topological band insulator with no AF order ($|m| = 0, C = 2$); II. topological AF insulator with collinear AF order ($|m| \neq 0, C = 2$ and $S = 0$); III. trivial AF insulator with collinear AF order ($|m| \neq 0, C = 0$ and $S = 0$), and IV. trivial AF insulator with canted AF order ($|m| \neq 0, C = 0$ and $S \neq 0$).

First, for small $t_2/t_1$, such as the trajectory labelled by (a) in Fig. 3, as $U/t_1$ increases, the system first undergoes a second-order phase transition across which a collinear AF order develops continuously (Fig. 3(a1)). As such a magnetic order increases, the mean-field Hamiltonian $H_{\text{MF}}$ acquires a $M(-M)$ term for spin-up (down), which suppresses the single particle gap at $K$ ($K'$) point (Fig. 3(a2)). Thus, $H_{\text{MF}}$ undergoes a trajectory as labeled by (a) in Fig. 1(a). As $|m|$ increases to a certain value, the single particle gap closes at $K$ ($K'$) point, beyond which the mean-field ground becomes a topological trivial one (i.e. $C = 0$, Fig. 3(a3)). Along this trajectory, the spin chirality $S$ is always zero (Fig. 3(a4)).

Secondly, for intermediate $t_2/t_1$, such as the trajectory labelled by (b) in Fig. 3 as $U/t_1$ increases, the system first develops a collinear AF order (Fig. 3(b1)). Then, instead of reaching a topological transition, the system undergoes a first-order transition across which the magnetic order becomes canted. This is accompanied by a jump of spin chirality order $S$ (Fig. 3(b4)). This corresponds to a discontinuous change of effective $M$ and $\phi$ in $H_{\text{MF}}$, as shown by trajectory labeled by (b) in Fig. 1(a). Consequently, the system jumps from a topological phase to a topological trivial phase (Fig. 3(b3)), and the gap closing point is avoided (Fig. 3(b2)). This canted AF order can also be understood by a ring-exchange spin model in term of local spin picture. As $U/t_1$ further increases, the effective ring exchange is suppressed, $S$ gradually decreases and the system returns to a collinear AF insulator (Fig. 3(b4)).

Finally, for large $t_2/t_1$, such as the trajectory labelled by (c) in Fig. 3 as $U/t_1$ increases, a first-order transition directly brings the system from a topological band insulator to a trivial canted AF insulator, across which $|m|, C, S$ all display discontinuity (Fig. 3(c1), (c3), (c4)), and the fermion gap $\Delta$ remains finite all through (Fig. 3(c2)).

**Fluctuations Beyond Mean-field.** Here we focus on the topological transition from Phase II to Phase III, at which the gapless fermions coexist with gapless Goldstone spin-wave mode of AF order. This invites the question whether the spin-wave fluctuation will change the critical behavior. To answer this question, we introduce the following low-energy theory with action

$$S = \int dt d^2r \left( \mathcal{L}_n + \mathcal{L}_t + \mathcal{L}_I \right)$$

(5)

$$\mathcal{L}_n = \frac{1}{2g} \left( \left( \partial_t \mathbf{n} \right)^2 - \mathbf{c}^2 (\nabla \mathbf{n})^2 \right)$$

(6)

$$\mathcal{L}_t = \Psi^\dagger \left[ i \partial_t + iv \tau_z \sigma_x \partial_x + iv \tau_y \sigma_y \partial_y - m \tau_z \sigma_z \right] \Psi$$

(7)

$$\mathcal{L}_I = -\lambda \Psi^\dagger \left[ \sigma_z \otimes (\mathbf{n} \cdot \mathbf{s}) \right] \Psi.$$  

(8)

where $\mathcal{L}_n$ is a nonlinear sigma model that describes the low-energy fluctuation of AF Néel order $\mathbf{n}$, and the eight-component object $\Psi = \Psi_{\alpha \sigma \sigma}$ describes the Dirac fermion...
therefore the spin-wave-fermion interaction given by \( \Delta \chi \xi \), point, \( \Psi \). The mass term becomes \(-\sigma I \chi \xi \), clear that when \( \chi \), respectively. Parameters \( c \) and \( v_F \) are the spin-wave and fermion velocities, respectively. Parameters \( g \) and \( \lambda \) are coupling constants of spin fluctuation and coupling between spin and fermions, respectively. Here, \( c \), \( v_F \), \( g \), \( m \) and \( \lambda \) can all be given by microscopic parameters. In particular, in Eq.7 the mass \( m \) is given by \( m = 3\sqrt{3}d_2 \).

With collinear AF order, we can assume that \( \langle n \rangle \) is ordered along \( \hat{z} \) direction, and then take \( n_z = 1 \) and expand the action to the linear order of \( n_z \) and \( n_y \), \( \varphi = n_x + in_y \) representing a complex gapless boson field. To bring \( \mathcal{L} \) into a more convenient form, we implement a local spin rotation \( \Psi(x,t) \rightarrow U(x,t)\Psi(x,t) \), with \( U = \exp[i(n_y s_z - n_z s_y)/2 + \cdots \cdots] \), such that \( U^\dagger(n,s)U = s_z \). Since this rotation involves only fermions, the \( \mathcal{L}_n \) term is unchanged, while \( \mathcal{L}_t + \mathcal{L}_1 \) becomes

\[
\mathcal{L}_t + \mathcal{L}_1 = \Psi^\dagger [iD_t + iv_F \tau_z \sigma_D + \sigma_D D_y - m\sigma_z \tau_z - \lambda \sigma_z s_z] \Psi,
\]

where the covariant derivative \( iD_\mu = i\partial_\mu - \frac{i}{2}(s_x \partial_\mu \varphi^* - s_\mu \varphi) \), in which \( s_\pm = \frac{1}{2}(s_x \pm is_y) \) and \( \mu = t, x, y \). Thus, the mass term becomes \( -\Psi^\dagger \sigma_y \otimes (m\sigma_z \otimes I + \Lambda \otimes s_z) \Psi \), where \( I \) denotes 2 \( \times \) 2 identity matrix. It is therefore clear that when \( \lambda = m \), namely, at the mean-field critical point, \( \Psi_{11} \) and \( \Psi_{21} \) become gapless (the sublattice index \( \sigma \) is suppressed). Hereafter we shall define \( \xi_1 = \Psi_{11} \) and \( \xi_2 = \Psi_{21} \), which are gapless fermions, and \( \chi_1 = \Psi_{1\uparrow} \) and \( \chi_2 = \Psi_{2\uparrow} \), which are gapped fermions.

The low-energy spin waves have small momentum, therefore the spin-wave-fermion interaction given by \( D_\mu \) terms does not change the valley index, i.e. by interacting with spin waves, \( \xi \), can only turn into \( \chi \), with same \( \kappa \) (Fig. 4a,b)). After integrating out the spin-wave and the gapped fermions \( \chi \), we can obtain a self-energy correction for low-energy fermions \( \xi \), as well as effective interactions among the massless fermions [12], with corresponding diagrams shown in Fig. 4 (d-e). To the lowest order the self-energy takes the form of \( \Sigma \tau_2 \sigma_2 \). Thus, it merely shifts the phase boundary. The induced interaction reads

\[
\hat{V} = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} V_q \left[ \sum_{\kappa=1,2} (\epsilon_{\kappa,k_1,q}^\ell \sigma_2 \xi_{\kappa,k_2}) - 2(\epsilon_{1,k_1,k_2}^\sigma \sigma_2 \xi_{1,k_1}) (\epsilon_{2,k_1,k_2}^\sigma \sigma_2 \xi_{2,k_2}) \right]
\]

where \( q = (\omega, \mathbf{q}) \), \( d^3q \) is a shorthand notation for \( \int d\mathbf{q} d\omega \), and similarly for \( d^3k \). Neglecting the \( q \) dependence of \( V(q) \), we have \( V(q) = -u \equiv -(1/2)g^2c[\Lambda^3/6\pi^2m^2] \), where \( \Lambda \) is a momentum cutoff.

This spin-wave-induced interaction, if sufficiently strong, can open up a gap at the nominal critical point. To see this fact, we only need to do a mean-field approximation of \( \hat{V} \). We find that if \( u > u_c \equiv \pi v_F(\Lambda \sqrt{3}) \), the gapless “ground-state” at \( \lambda = m \) is unstable towards the dynamical generation [11] of a mass term \( \pm \Delta \xi \sigma_2 s_\xi \), with \( \Delta = \pi v_F(1/u_c - 1/u) \). Away from the \( \lambda = m \) point, the sign \((\pm\text{sign})\) in \( \pm \Delta \xi \sigma_2 s_\xi \) is selected at \( \lambda = m + 0^+ \left(\lambda = m - 0^-\right) \). That is to say, the generated mass jumps by \( 2\Delta \) across the mean-field transition point \( \lambda = m \), thus, the nominal gap closing of fermions is avoided, and the transition becomes a first-order one.

Finally, we remark that this physics triggered by massless spin-wave has no counterpart in the Kane-Mele-Hubbard model, because the SU(2) spin rotational symmetry is explicitly broken there, thus the Goldstone mode is absent therein.

**Final remarks.** Recent cold atom realization of the Haldane model can be naturally described by this HH model. In fact, in the experiment reported in Ref. [3], Mott insulator with suppressed double-occupancy sites has been observed. Our theoretical predictions can be directly verified in this setup. In this realization, since the most crucial next nearest hopping term is generated by periodically shaking optical lattices, the periodic driving will also modify the interaction term, in the order of \( 1/\omega \) (\( \omega \) is shaking frequency). The thermal fluctuation of magnetic order may also be important. These effects will be left for future investigations.

We would like to thank Yi-Zhuang You, Hong Yao, Fa Wang and Shou-Cheng Zhang for discussions. This work is supported by Tsinghua University Initiative Scientific Research Program, NSFC under Grant No. 11304175 (ZW), No. 11325418 (HZ), No. 11174176 (HZ), and NKBRSFC under Grant No. 2011CB921500 (HZ).

Note added: Upon finishing this work, we became aware of Ref. [13], in which the same model is studied.
In the following we will only consider the $M=0$ case. At the mean-field level, we decompose the on-site interaction term as

$$U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} = \frac{1}{2} U \hat{N} - \frac{2}{3} \sum_i S_i^z$$

$$\approx \frac{1}{2} U \hat{N} + \sum_i \left( -m_i \cdot S_i + \frac{3m_i^2}{8U} \right),$$

where $\hat{N}$ is the total particle number operator, and $m_i$ are the momenta. The ansatz (11) and the mean-field approximation (12) are employed here. The Hamiltonian of the Haldane-Hubbard model is given by

$$H = -t \sum_{\langle ij \rangle, s} \left( \hat{c}_{i,s}^\dagger \hat{c}_{j,s} + h.c. \right) - t_2 \sum_{\langle ij \rangle, s} \left( e^{i\phi_{ij}} \hat{c}_{i,s}^\dagger \hat{c}_{j,s} + h.c. \right) - U \sum_i \mathcal{N}_{i,\uparrow} \mathcal{N}_{i,\downarrow} + \sum_i \mathcal{N}_{i,\uparrow} \mathcal{N}_{i,\downarrow}.$$
where $S_i = \frac{1}{2} \sum_{s,s'} \epsilon_{i,s} \epsilon_{i,s'} \hat{c}_{i,s'}$ is the spin operator and $m_i = 4U \langle S_i \rangle / 3$ is the on-site magnetic order parameter. The mean-field Hamiltonian can be constructed as

$$\hat{H}_{\text{MF}} = -t_1 \sum_{(ij),s} \left( \hat{c}_{i,s}^\dagger \hat{c}_{j,s} + \text{h.c.} \right) - t_2 \sum_{\langle (ij),s \rangle} \left( e^{i\phi_{ij}} \hat{c}_{i,s}^\dagger \hat{c}_{j,s} + \text{h.c.} \right) - \sum_i \left\{ m_i^x \left( \hat{c}_{i,\uparrow} \hat{c}_{i,\uparrow} - \hat{c}_{i,\downarrow} \hat{c}_{i,\downarrow} \right) + m_i^y \left( \hat{c}_{i,\uparrow} \hat{c}_{i,\downarrow} + \hat{c}_{i,\downarrow} \hat{c}_{i,\uparrow} \right) - \frac{U}{2} \hat{c}_{i,\uparrow}^\dagger \hat{c}_{i,\uparrow} \hat{c}_{i,\downarrow}^\dagger \hat{c}_{i,\downarrow} \right\} \right).$$

(14)

This Hamiltonian is quadratic form and can be directly diagonalized. As explained in the main text, we enlarge the unit cell to a full hexagon containing six sites. We do a Fourier transformation

$$\hat{c}_{\alpha,s}(k) = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}} e^{-i\mathbf{k} \cdot \mathbf{R}} \hat{c}_{\alpha,s}(\mathbf{R}),$$

where $\mathbf{R}$ is the position of the unit cell, $\alpha = A1, B1, A2, B2, A3, B3$ denote the sublattices and $N$ is the total number of unit cells. The Hamiltonian can be transformed into momentum space:

$$\hat{H}_{\text{MF}} = \sum_{\mathbf{k} \in \text{BZ}} \hat{\Psi}^\dagger(\mathbf{k}) \hat{H}(\mathbf{k}) \hat{\Psi}(\mathbf{k}),$$

(15)

where $\hat{\Psi}^\dagger(\mathbf{k})$ is a 12-component spinor:

$$\hat{\Psi}(\mathbf{k}) = \left( \hat{c}_{A1,\uparrow}^\dagger, \hat{c}_{B1,\uparrow}^\dagger, \hat{c}_{A2,\uparrow}^\dagger, \hat{c}_{B2,\uparrow}^\dagger, \hat{c}_{A3,\uparrow}^\dagger, \hat{c}_{B3,\uparrow}^\dagger, \hat{c}_{A1,\downarrow}^\dagger, \hat{c}_{B1,\downarrow}^\dagger, \hat{c}_{A2,\downarrow}^\dagger, \hat{c}_{B2,\downarrow}^\dagger, \hat{c}_{A3,\downarrow}^\dagger, \hat{c}_{B3,\downarrow}^\dagger \right).$$

(16)

Diagonalizing the matrix $\hat{H}(\mathbf{k})$, one can obtain the energy band for the mean-field Hamiltonian

$$\sum_{\mu \nu} U_{ij}^\dagger H_{ij}(\mathbf{k}) U_{j\nu} = \delta_{\mu\nu} E_{\mu}(\mathbf{k}).$$

(17)

Therefore the diagonalized mean-field Hamiltonian is

$$\hat{H}_{\text{MF}} = \sum_{\mathbf{k} \in \text{BZ}} \sum_{\mu} \Phi_{\mu}^\dagger(\mathbf{k}) E_{\mu}(\mathbf{k}) \Phi_{\mu}(\mathbf{k}),$$

where $\Phi_{\mu}(\mathbf{k}) = \sum_{i} U_{\mu i} \hat{\Psi}_i(\mathbf{k})$ is the fermion operator of each band. For the half-filling case, the ground state is the full-filling of the lowest 6 bands,

$$|\text{GS}\rangle = \prod_{\mu=1}^{6} \prod_{\mathbf{k} \in \text{BZ}} \Phi_{\mu}^\dagger(\mathbf{k}) |0\rangle.$$

(18)

The magnetization $m_\alpha$ can be calculated from the ground state as

$$m_\alpha^x = \frac{1}{2N} \sum_{\mathbf{k} \in \text{BZ}} \left\langle \hat{c}_{\alpha,\uparrow}(\mathbf{k}) \hat{c}_{\alpha,\uparrow}(\mathbf{k}) - \hat{c}_{\alpha,\downarrow}(\mathbf{k}) \hat{c}_{\alpha,\downarrow}(\mathbf{k}) \right\rangle_{\text{GS}},$$

$$m_\alpha^y = \frac{1}{2N} \sum_{\mathbf{k} \in \text{BZ}} \left\langle \hat{c}_{\alpha,\uparrow}(\mathbf{k}) \hat{c}_{\alpha,\downarrow}(\mathbf{k}) + \hat{c}_{\alpha,\downarrow}(\mathbf{k}) \hat{c}_{\alpha,\uparrow}(\mathbf{k}) \right\rangle_{\text{GS}},$$

$$m_\alpha^z = \frac{1}{2N} \sum_{\mathbf{k} \in \text{BZ}} \left\langle \hat{c}_{\alpha,\uparrow}(\mathbf{k}) \hat{c}_{\alpha,\downarrow}(\mathbf{k}) \right\rangle_{\text{GS}}.$$ 

Since there are six sites in each unit cell, one has to self-consistently calculate 18 parameters of $m_\alpha$. We numerically calculate the ground state and iterate until the magnetization at each site converges. We then calculate the spin chirality order defined by

$$\mathcal{S} = \left\langle \hat{S}_i \right\rangle \cdot \left( \left\langle \hat{S}_j \right\rangle \times \left\langle \hat{S}_k \right\rangle \right).$$

The Chern number of interacting quantum anomalous Hall insulators has been defined in Ref. [1]. At the mean-field level it is reduced to the Chern number of the mean-field wavefunction. We numerically calculate this Chern number using the algorithm given in Ref. [2].
Effective Field Theory

Near the phase boundary between Phase II to Phase III, we introduce the following low-energy theory with action

\[ S = \int dt d^2 \mathbf{r} (L_n + L_t + L_4), \]

\[ L_n = \frac{1}{2g} \left[ (\partial_t n_s)^2 - c^2 (\nabla n_s)^2 \right], \]

\[ L_t = \Psi^\dagger [i\partial_t + v_F \tau_z \sigma_i \partial_x + v_F \sigma_y \partial_y - m \tau_z \sigma_z] \Psi, \]

\[ L_4 = -\lambda \Psi^\dagger (\sigma_z \otimes (\mathbf{n} \cdot \mathbf{s})) \Psi. \]

The parameters are explained in the text. With the collinear AF order, we can assume that the \( \mathbf{n} \) is ordered along \( \hat{z} \), direction, namely, \( \mathbf{n} \approx (0, 0, 1) \). We expand \( L_n \) to the linear order of \( n_x \) and \( n_y \) as

\[ L_n = \frac{1}{2g} \sum_{x,y,i} \left[ (\partial_t n_i)^2 - c^2 (\nabla n_i)^2 \right]. \]  (19)

Then we make a spin rotation, \( \Psi(x) \rightarrow U(x)\Psi(x) \), with \( U(x) = \exp \left[ i \left( n_y(x) s_x - n_x(x) s_y \right) / 2 + \cdots \right] \), such that

\[ U^\dagger(x) (\mathbf{n} \cdot \mathbf{s}) U(x) = s_z. \]

Then the last term of the Lagrangian becomes

\[ L_4 = -\lambda \Psi^\dagger (\sigma_z \otimes s_z) \Psi. \]  (20)

It is a mass term of the fermions, so it could be absorbed into the action of fermions. We have \( L_t + L_4 \)

\[ L_t = \Psi^\dagger [iD_t + v_F \tau_z \sigma_x iD_x + v_F \sigma_y iD_y - m \tau_z \sigma_z - \lambda \sigma_z s_z] \Psi, \]  (21)

where the covariant derivative is given by:

\[ D_\mu = \partial_\mu + U^\dagger \partial_\mu U = \partial_\mu + \frac{1}{2} \left( s_- \partial_\mu \varphi - s_+ \partial_\mu \varphi^* \right), \]

where \( s_\pm = \frac{1}{2} (s_x \pm is_y) \), and \( \varphi = n_x + is_y \). This \( L_t + L_4 \) can be written explicitly as

\[ L_t = \Psi^\dagger_1 [i\partial_t + v_F \sigma_x \partial_x + v_F \sigma_y \partial_y - (m + \lambda) \sigma_z] \Psi^\dagger_1 + \Psi^\dagger_4 [i\partial_t + v_F \sigma_x \partial_x + v_F \sigma_y \partial_y - (m - \lambda) \sigma_z] \Psi^\dagger_4 \]

\[ + \Psi^\dagger_1 [i\partial_t - v_F \sigma_x \partial_x + v_F \sigma_y \partial_y + (m - \lambda) \sigma_z] \Psi^\dagger_2 + \Psi^\dagger_4 [i\partial_t - v_F \sigma_x \partial_x + v_F \sigma_y \partial_y + (m + \lambda) \sigma_z] \Psi^\dagger_2 \]

\[ + \frac{1}{2} \Psi^\dagger_1 (i\partial_\mu \varphi + v_F \sigma_x i\partial_\mu \varphi + v_F \sigma_y i\partial_\mu \varphi) \Psi^\dagger_1 - \frac{1}{2} \Psi^\dagger_2 (i\partial_\mu \varphi^* - v_F \sigma_x i\partial_\mu \varphi^* + v_F \sigma_y i\partial_\mu \varphi^*) \Psi^\dagger_2. \]  (22)

When \( \lambda = m \), namely, at the mean field critical point, \( L_t \) becomes

\[ L_t = \Psi^\dagger_1 [i\partial_t + v_F \sigma_x \partial_x + v_F \sigma_y \partial_y - 2m \sigma_z] \Psi^\dagger_1 + \Psi^\dagger_4 [i\partial_t + v_F \sigma_x \partial_x + v_F \sigma_y \partial_y] \Psi^\dagger_4 \]

\[ + \Psi^\dagger_1 [i\partial_t - v_F \sigma_x \partial_x + v_F \sigma_y \partial_y + 2m \sigma_z] \Psi^\dagger_4 \]

\[ + \frac{1}{2} \Psi^\dagger_1 (i\partial_\mu \varphi + v_F \sigma_x i\partial_\mu \varphi + v_F \sigma_y i\partial_\mu \varphi) \Psi^\dagger_1 - \frac{1}{2} \Psi^\dagger_2 (i\partial_\mu \varphi^* - v_F \sigma_x i\partial_\mu \varphi^* + v_F \sigma_y i\partial_\mu \varphi^*) \Psi^\dagger_2. \]  (23)

We can see that \( \Psi^\dagger_4 \) and \( \Psi^\dagger_2 \) become gapless, while \( \Psi^\dagger_1 \) are gapped. We redefine

\[ \xi_1 = 2\Psi^\dagger_1 \sigma_z, \xi_2 = 2\Psi^\dagger_1 \sigma_z, \chi_1 = \Psi^\dagger_1 \sigma_z, \chi_2 = \Psi^\dagger_1 \sigma_z \]

\[ \varphi = n_x + in_y, \varphi^* = n_x - in_y. \]
then the total action becomes

\[ \mathcal{L}_a = \frac{1}{2g} \left( |\partial \chi|^2 - c^2 |\nabla \phi|^2 \right) \]

\[ \mathcal{L}_t = \bar{\xi}_1 (\gamma_0 \partial_t + v_F \gamma_1 \partial_x + v_F \gamma_2 \partial_y) \xi_1 + \bar{\xi}_2 (\gamma_0 \partial_t - v_F \gamma_1 \partial_x + v_F \gamma_2 \partial_y) \xi_2 + \bar{\chi}_1 (\gamma_0 \partial_t + v_F \gamma_1 \partial_x + v_F \gamma_2 \partial_y - 2m) \chi_1 + \bar{\chi}_2 (\gamma_0 \partial_t - v_F \gamma_1 \partial_x + v_F \gamma_2 \partial_y + 2m) \chi_2, \]

\[ \mathcal{L}_1 = \frac{1}{2} \bar{\xi}_1 (\gamma_0 \partial_t \varphi + v_F \gamma_1 \partial_x \varphi + v_F \gamma_2 \partial_y \varphi) \chi_1 - \frac{1}{2} \bar{\chi}_1 (\gamma_0 \partial_t \varphi^* + v_F \gamma_1 \partial_x \varphi^* + v_F \gamma_2 \partial_y \varphi^*) \xi_1 + \frac{1}{2} \bar{\chi}_2 (\gamma_0 \partial_t \varphi^* - v_F \gamma_1 \partial_x \varphi^* + v_F \gamma_2 \partial_y \varphi^*) \chi_2. \]

Note that we have mapped the low energy theory into four Dirac fields interacting with a complex scalar field. In the following, we will integrate out the massive Dirac fields and the complex scaler field to obtain an effective theory for the massless fermions.

The partition function is given by

\[ Z = \int D\xi D\chi D\varphi e^{iS_\xi + iS_\chi + iS_\varphi + iS_\chi}. \] (24)

Integrating out the \( \chi \) and \( \varphi \) field gives rise to an effective action for \( \xi \) field

\[ e^{iS_{\text{eff}}} = e^{iS_\xi} \int D\chi D\varphi e^{iS_\chi + iS_\varphi + iS_\chi} = e^{iS_\xi} \langle e^{iS_\chi} \rangle_0, \]

Here \( \langle e^{iS_\chi} \rangle_0 \) is the average over free \( \chi \) and \( \varphi \) field, which reads

\[ \langle e^{iS_\chi} \rangle_0 = 1 - \frac{1}{2!} \langle S_\xi^2 \rangle + \frac{1}{4!} \langle S_\xi^4 \rangle + \cdots \]

\[ = \exp \left( -\frac{1}{2!} \langle S_\xi^2 \rangle + \frac{1}{4!} \left( \langle S_\xi^4 \rangle - 3 \langle S_\xi^2 \rangle^2 \right) \right) \]

The effective action has the form of

\[ S_{\text{eff}} = S_\xi + \frac{i}{2!} \langle S_\xi^2 \rangle - \frac{i}{4!} \left( \langle S_\xi^4 \rangle - 3 \langle S_\xi^2 \rangle^2 \right). \]

The second term in Eq. (27) generates a self-energy for the gapless fermion, which is illustrated in fig.4 (c).

\[ \frac{i}{2!} \langle S_\xi^2 \rangle = \int \frac{d^3p}{(2\pi)^3} \xi_\kappa (p) \Sigma_\kappa (p) \xi_\kappa (p). \]

This self-energy has the form of

\[ \Sigma_{1,2} (p) = i \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_0 k_0 - v_F \gamma_1 k_1 - v_F \gamma_2 k_2}{2} iD (k) \frac{\gamma_0 k_0 - v_F \gamma_1 k_1 - v_F \gamma_2 k_2}{2} iK_{1,2} (p - k), \]

\[ = i \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \left( k_0^2 - v_F^2 k_1^2 - v_F^2 k_2^2 \right) D (k) K_{1,2} (p - k), \]

where \( D (k) \) is the propagator of the complex scaler field, and \( K_{1,2} (k) \) is the propagator of the massive Dirac fermions:

\[ D (k) = \frac{2g}{k_0^2 - c^2 k_1^2 - c^2 k_2^2 + i\epsilon} \]

\[ K_{1,2} (k) = \frac{1}{\gamma_0 k_0 + v_F \gamma_1 k_1 - v_F \gamma_2 k_2 + 2m + i\epsilon}. \]

For low energy processes, we approximate \( K_{1,2} (k) \approx \mp \frac{1}{2m} \), so that the self energy is given by:

\[ \Sigma_{1,2} (p) = \mp \frac{ig}{4m} \int \frac{d^3k}{(2\pi)^3} \frac{k_0^2 - v_F^2 k_1^2 - v_F^2 k_2^2}{k_0^2 - c^2 k_1^2 - c^2 k_2^2 + i\epsilon}, \] (32)
One can see that it merely shifts the mean-field phase boundary without qualitatively changing its physical properties.

The third term in the effective action as illustrated in fig.4 (d) and (e), generates an effective interaction between the gapless fermions:

\[-\frac{i}{4!} \left( \langle S_4 \rangle - 3 \langle S_2 \rangle^2 \right) = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} V_q \left( \sum_{\kappa=1,2} \tilde{\xi}_{\kappa,k_1} - q \tilde{\xi}_{\kappa,k_2} + q \xi_{\kappa,k_1} - 2 \xi_{1,k_1} - q \tilde{\xi}_{2,k_2} + q \xi_{2,k_2} \xi_{1,k_1} \right),\]

where the \( V_q \) is given by:

\[ V_q = \frac{i}{2^4 4m^2} \int \frac{d^3k}{(2\pi)^3} \times (k_0^2 - v_F^2 k_1^2 - v_F^2 k_2^2) D(k) \left[ (q_0 + k_0)^2 - v_F^2 (q_1 + k_1) - v_F^2 (q_2 + k_2) \right] D(q + k). \]

When \( q = 0 \), we have:

\[ V_0 = \frac{i}{2^4 4m^2} \int \frac{d^3k}{(2\pi)^3} (k_0^2 - v_F^2 k_1^2 - v_F^2 k_2^2)^2 D^2(k). \]

\[ = \frac{ig^2}{2^4 m^2} \int \frac{d^3k}{(2\pi)^3} \left( k_0^2 - c^2 k_1^2 - c^2 k_2^2 \right)^2. \]

\[ = -\frac{g^2 c \Lambda^3}{2^4 \pi^2 m^2} \left[ \frac{1}{6} - \left( 1 - \frac{v_F^2}{c^2} \right) \frac{2}{9} + \left( 1 - \frac{v_F^2}{c^2} \right)^2 \frac{4}{45} \right]. \]

where \( \Lambda \) is a momentum cutoff. This is the effective interaction between the fermions.

\* Electronic address: wangzhongemail@gmail.com
\† Electronic address: hzhai@mail.tsinghua.edu.cn

[1] Z. Wang and S.-C. Zhang, Phys. Rev. X 2, 031008 (2012)
[2] Takahiro Fukui, Yasuhiro Hatsugai, and Hiroshi Suzuki, J. Phys. Soc. Jpn. 74 (2005) pp. 1674-1677