Explicit Salem Sets in $\mathbb{R}^2$

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Abstract

We construct explicit (i.e., non-random) examples of Salem sets in $\mathbb{R}^2$ of dimension $s$ for every $0 \leq s \leq 2$. In particular, we give the first explicit examples of Salem sets in $\mathbb{R}^2$ of dimension $0 < s < 1$. This extends a theorem of Kaufman.

1 Introduction

1.1 Basic Notation

For $x \in \mathbb{R}^d$, $|x| = |x|_{\infty} = \max_{1 \leq i \leq d} |x_i|$ and $|x|_2 = (\sum_{i=1}^d |x_i|^2)^{1/2}$. For $x, y \in \mathbb{R}^d$, $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$ is the Euclidean inner product. If $A$ is a finite set, $|A|$ is the cardinality of $A$. The expression $a \lesssim b$ stands for “there is a constant $c > 0$ such that $a \leq cb$.” The expression $a \gtrsim b$ is analogous.

1.2 Background

If $\mu$ is a finite Borel measure on $\mathbb{R}^d$, then the Fourier transform of $\mu$ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle} d\mu(x) \quad \forall \xi \in \mathbb{R}^d.$$ 

It is a classic result essentially due to Frostman [14] that the Hausdorff dimension of any Borel set $A \subseteq \mathbb{R}^d$ can be expressed as

$$\dim_H A = \sup \left\{ s \in [0, d] : \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{d-s} d\xi < \infty \text{ for some } \mu \in \mathcal{P}(A) \right\},$$

where $\mathcal{P}(A)$ denotes the set of all Borel probability measures with compact support contained in $A$.

The Fourier dimension of a set $A \subseteq \mathbb{R}^d$ is defined to be

$$\dim_F A = \sup \left\{ s \in [0, d] : \sup_{0 \neq \xi \in \mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{d-s} < \infty \text{ for some } \mu \in \mathcal{P}(A) \right\}.$$ 

As general references for Hausdorff and Fourier dimension, see [12], [26], [27], [31]. Recent papers by Ekström, Persson, and Schmeling [10] and Fraser, Orponen, and Sahlsten [13] have revealed some interesting subtleties about Fourier dimension. Plainly, for every Borel set $A \subseteq \mathbb{R}^d$,

$$\dim_F A \leq \dim_H A.$$
Every $k$-dimensional plane in $\mathbb{R}^d$ with $k < d$ has Fourier dimension 0 and Hausdorff dimension $k$. The middle-thirds Cantor set in $\mathbb{R}$ has Fourier dimension 0 and Hausdorff dimension $\ln 2 / \ln 3$. Körner [24] has shown that for every $0 \leq s \leq t \leq 1$ there is a compact set $A \subseteq \mathbb{R}$ with Fourier dimension $s$ and Hausdorff dimension $t$.

Sets $A \subseteq \mathbb{R}^d$ with

$$\dim F A = \dim H A$$

are called Salem sets.

Every ball in $\mathbb{R}^d$ is a Salem set of dimension $d$. Every countable set in $\mathbb{R}^d$ is a Salem set of dimension zero. Less trivially, every sphere in $\mathbb{R}^d$ is a Salem set of dimension $d - 1$. Salem sets in $\mathbb{R}^d$ of dimension $s \neq 0, d - 1, d$ are more exotic.

There are many random constructions of Salem sets. Using Cantor sets with randomly chosen contraction ratios, Salem [28] was the first to show that for every $s \in (0, 1)$ there is a Salem set in $\mathbb{R}$ of dimension $s$. Kahane showed that images of compact subsets of $\mathbb{R}^d$ under certain stochastic processes (namely, Brownian motion, fractional Brownian motion, and Gaussian Fourier series) are almost surely Salem sets (see [20], [21], [22], Ch.17,18)). Through these results, Kahane established that for every $s \in (0, d)$ there is a Salem set in $\mathbb{R}^d$ of dimension $s$. Ekström [11] has shown that the image of any Borel set in $\mathbb{R}$ under a random diffeomorphism is almost surely a Salem set.

Other random constructions of Salem sets have been given by Bluhm [4], Łaba and Pramanik [25], Shmerkin and Suomala [29], and Chen and Seeger [8].

These random constructions give collections of sets where each individual set is “almost surely” or “with positive probability” a Salem set. But they don’t provide any explicit examples of Salem sets.

Explicit Salem sets are much more rare. Kaufman [23] gave the first explicit examples of Salem sets in $\mathbb{R}$ of arbitrary dimension $s \in (0, 1)$. Kaufman showed that set of $\tau$-well-approximable numbers

$$E(\tau) = \{ x \in \mathbb{R} : |qx - r| \leq |q|^{-\tau} \text{ for infinitely many } (q, r) \in \mathbb{Z}^2 \}$$

is a Salem set of dimension $2/(1 + \tau)$ when $\tau > 1$. The Hausdorff dimension of $E(\tau)$ was known to be $2/(1 + \tau)$ by the classic theorem of Jarník [18] and Besicovitch [3]. Kaufman showed that the Fourier dimension of $E(\tau)$ is also $2/(1 + \tau)$. Note that Dirichlet’s approximation theorem easily gives $E(\tau) = \mathbb{R}$ when $\tau \leq 1$. Körner [24] combined Kaufman’s construction and a Baire category argument to prove that for every $0 \leq s \leq t \leq 1$ there is a compact set $A \subseteq \mathbb{R}$ with Fourier dimension $s$ and Hausdorff dimension $t$. Hambrook [16] generalized Kaufman’s argument to show that many sets in $\mathbb{R}$ closely related to $E(\tau)$ are also Salem sets.

Bluhm [5] gave a detailed account of what is essentially Kaufman’s proof and also pointed out that (as a consequence of a theorem of Gatesoupe [15]) the radial set $\{ x \in \mathbb{R}^d : |x|_2 \in E(\tau) \}$ is a Salem set in $\mathbb{R}^d$ of dimension $d - 1 + 2/(1 + \tau)$ whenever $\tau > 1$. However, explicit Salem sets in $\mathbb{R}^d$ of dimension $0 < s < d - 1$ were unknown until now.

From the point of view of Diophantine approximation, the natural multi-dimensional generalization of $E(\tau)$ is

$$E(m, n, \tau) = \{ x \in \mathbb{R}^{mn} : |qx - r| \leq |q|^{-\tau} \text{ for infinitely many } (q, r) \in \mathbb{Z}^n \times \mathbb{Z}^m \} ,$$

where we identify $\mathbb{R}^{mn}$ with the set of $m \times n$ matrices with real entries. By Minkowski’s theorem on linear forms, $E(m, n, \tau) = \mathbb{R}^{mn}$ when $\tau \leq n/m$. Bovey and Dodson [6] showed the Hausdorff dimension of $E(m, n, \tau)$ is $m(n - 1) + (m + n)/(1 + \tau)$ if $\tau > n/m$. The $n = 1$ case was done earlier by Jarník [18] and Eggleston [9]. The mass transference principle and slicing technique of Beresnevich and Velani [1], [2] may also be used to compute the Hausdorff dimension of $E(m, n, \tau)$.
Hambrook [16] proved the Fourier dimension of $E(m, n, \tau)$ is at least $2n/(1 + \tau)$ if $\tau > n/m$. However, it is unclear whether $E(m, n, \tau)$ is a Salem set when $\tau > n/m$ and $mn > 1$.

1.3 Statement of Results

In the present paper, we extend Kaufman’s method [23] and give explicit examples of Salem sets in $\mathbb{R}^2$ of every dimension $s \in [0, 2]$. In particular, we give the first explicit examples of Salem sets in $\mathbb{R}^2$ of dimension $0 < s < 1$.

The key idea is to identify $\mathbb{R}^2$ with $\mathbb{C}$. Then $\mathbb{R}^2$ is a field (so we can multiply and divide elements of $\mathbb{R}^2$), and $\mathbb{Z}^2$ is identified with the ring of Gaussian integers $\mathbb{Z} + i\mathbb{Z}$. This allows us to basically follow Kaufman’s argument.

As a reference for the Gaussian integers, see for example [17]. It will be important that the divisor bound for the Gaussian integers has the same shape as the divisor bound for the integers.

Let $\tau \in \mathbb{R}$. Define

$$E_s(\tau) = \{ x \in \mathbb{R}^2 : |qx - r| \leq |q|^{-\tau} \text{ for infinitely many } (q, r) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \}.$$ 

We identify $\mathbb{R}^2$ and $\mathbb{C}$, so $qx$ is a product of complex numbers. Note $\langle \xi, x \rangle$, which appears in the definition of the Fourier transform, is still $\langle \xi, x \rangle = \xi_1 x_1 + \xi_2 x_2$ for $\xi, x \in \mathbb{R}^2$.

**Theorem 1.** For every closed ball $B \subseteq \mathbb{R}^2$, there exists a Borel probability measure with support contained in $E_s(\tau) \cap B$ such that

$$|\hat{\mu}(\xi)| \lesssim |\xi|^{-2/(1 + \tau)} \exp(\ln |\xi| / \ln \ln |\xi|) \quad \forall \xi \in \mathbb{R}^2, |\xi| > e.$$ 

The proof of Theorem 1 is an extension of Hambrook’s variation [16] on Kaufman’s argument.

By a standard covering argument, we have $\dim_H E_s(\tau) \leq \min \{ 4/(1 + \tau), 2 \}$. Therefore Theorem 1 implies

**Theorem 2.** $E_s(\tau)$ is a Salem set with

$$\dim_H E_s(\tau) = \dim_F E_s(\tau) = \min \{ 4/(1 + \tau), 2 \}.$$ 

The decay rate in Theorem 1 can actually be improved slightly by adhering more closely to Kaufman’s original argument. Let $P$ denote the set of Gaussian primes. That is, $P$ is the set of prime elements in the Gaussian integers. Define

$$E_s(P, \tau) = \{ x \in \mathbb{R}^2 : |qx - r| \leq |q|^{-\tau} \text{ for infinitely many } (q, r) \in P \times \mathbb{Z}^2 \}.$$ 

Then $E_s(P, \tau) \subseteq E_s(\tau)$ and we have

**Theorem 3.** For every closed ball $B \subseteq \mathbb{R}^d$, there exists a Borel probability measure with support contained in $E_s(P, \tau) \cap B \subseteq E_s(\tau) \cap B$ such that

$$|\hat{\mu}(\xi)| \lesssim |\xi|^{-2/(1 + \tau)} \ln |\xi| \ln \ln |\xi| \quad \forall \xi \in \mathbb{R}^d, |\xi| > e.$$ 

By adapting some ideas of Hambrook [16] to the present setting, one readily obtains a more general result than Theorem 1. The statement requires some preparation. Let $Q$ be an infinite subset of $\mathbb{Z}^2$, let $\Psi : \mathbb{Z}^2 \to [0, \infty)$ be positive on $Q$, and let $\theta \in \mathbb{R}^2$. Define

$$E_s(Q, \Psi, \theta) = \{ x \in \mathbb{R}^2 : |qx - r - \theta| \leq \Psi(q) \text{ for infinitely many } (q, r) \in Q \times \mathbb{Z}^2 \}.$$ 

Evidently, $E_*(\tau) = E_*(\mathbb{Z}^2, q \mapsto |q|^{-\tau}, 0)$. For $M > 0$, define
\[
Q(M) = \{q \in Q : M/2 < |q| \leq M\}, \quad \epsilon(M) = \min_{q \in Q(M)} \Psi(q).
\]

A function $h : (0, \infty) \to \mathbb{R}$ will be called slowly growing if there is an $M > 0$ such that $h$ is positive and non-decreasing on $[M, \infty)$ and $\lim_{x \to \infty} \ln h(x) = 0$; the limit is often abbreviated as $h(x) = x^{o(1)}$.

There always exists a number $a \geq 0$, a slowly growing function $h : (0, \infty) \to \mathbb{R}$, and an unbounded set $M \subseteq (0, \infty)$ such that
\[
|Q(M)|\epsilon(M)^ah(M) \geq M^a \quad \forall M \in M.
\]

**Theorem 4.** For every closed ball $B \subseteq \mathbb{R}^d$, there exists a Borel probability measure with support contained in $E_*(Q, \Psi, \theta) \cap B$ such that
\[
|\hat{\mu}(\xi)| \lesssim |\xi|^{-a} \exp(\ln |\xi|/\ln \ln |\xi|)h(4|\xi|) \quad \forall \xi \in \mathbb{R}^2, |\xi| > e.
\]

The proof of Theorem 1 is divided over sections 2, 3, and 4. In section 5, we explain how to modify the proof of Theorem 1 to obtain Theorem 3. We leave the proof of Theorem 4 as an exercise for the reader. It is a simple modification of the proof of Theorem 1 using the ideas of 16.

## 2 Proof of Theorem 1: The Function $F_M$

For $f : \mathbb{R}^2 \to \mathbb{C}$, we will abuse the notation $\hat{f}$ as follows. If $\int_{\mathbb{R}^2} |f(x)|dx < \infty$, then
\[
\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \langle \xi, x \rangle} f(x)dx \quad \forall \xi \in \mathbb{R}^2.
\]
If $\int_{[0,1]^2} |f(x)|dx < \infty$ and $f$ is $\mathbb{Z}^2$-periodic, then
\[
\hat{f}(\xi) = \int_{[0,1]^2} e^{-2\pi i \langle \xi, x \rangle} f(x)dx \quad \forall \xi \in \mathbb{R}^d.
\]
There is no ambiguity because if $\int_{\mathbb{R}^2} |f(x)|dx < \infty$ and $f$ is $\mathbb{Z}^2$-periodic, then $\hat{f} = 0$ under either definition. Remember $\langle \xi, x \rangle = \xi_1x_1 + \xi_2x_2$ for $\xi, x \in \mathbb{R}^2$, even though we have identified $\mathbb{R}^2$ and $\mathbb{C}$.

Define $a = 2/(1 + \tau)$. Fix a positive integer $K > 2 + a$. Fix an arbitrary non-negative $C^K$ function on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} \phi(x)dx = 1$ and $\text{supp}(\phi) \subseteq [-1,1]^2$. Since $\phi \in C^K_c(\mathbb{R}^2)$,
\[
|\hat{\phi}(\xi)| \lesssim (1 + |\xi|)^{-K} \quad \forall \xi \in \mathbb{R}^2. \tag{2.1}
\]

For $\epsilon > 0$, define
\[
\phi^\epsilon(x) = \epsilon^{-2}\phi(\epsilon^{-1}x) \quad \forall x \in \mathbb{R}^2,
\]
\[
\Phi^\epsilon(x) = \sum_{\tau \in \mathbb{Z}^2} \phi^\epsilon(x - \tau) \quad \forall x \in \mathbb{R}^2.
\]

Then $\Phi^\epsilon$ is $\mathbb{Z}^2$-periodic, non-negative, $C^K$, and
\[
\hat{\Phi^\epsilon}(k) = \hat{\phi^\epsilon}(k) = \hat{\phi}(\epsilon k) \quad \forall k \in \mathbb{Z}^2.
\]
Therefore
\[ \Phi^\ell(x) = \sum_{k \in \mathbb{Z}^2} \hat{\phi}(\epsilon k) e^{2\pi i (k, x)} \]
uniformly for all \( x \in \mathbb{R}^2 \). For \( q \in \mathbb{Z}^2 \), define
\[ \Phi^\ell_q(x) = \Phi^\ell(qx) \quad \forall x \in \mathbb{R}^2. \]

**Lemma 5.** For all \( \ell \in \mathbb{Z}^2 \),
\[ \hat{\Phi}^\ell_q(\ell) = \begin{cases} \hat{\phi}(\epsilon \ell/\overline{q}) & \text{if } q \in D(\overline{\ell}) \\ 0 & \text{otherwise} \end{cases} \]
where
\[ D(\ell) = \{ q \in \mathbb{Z}^2 : \ell/q \in \mathbb{Z}^2 \}. \]

**Proof.** We have
\[ \hat{\Phi}^\ell_q(\ell) = \int_{[0,1]^2} e^{-2\pi i (\ell, x)} \sum_{k \in \mathbb{Z}^2} \hat{\phi}(\epsilon k) e^{2\pi i (k, qx)} dx = \sum_{k \in \mathbb{Z}^2} \hat{\phi}(\epsilon k) \int_{[0,1]^2} e^{2\pi i ((k, qx) - (\ell, x))} dx. \]

But
\[
\langle k, qx \rangle - \langle \ell, x \rangle = k_1(qx)_1 + k_2(qx)_2 - \ell_1 x_1 - \ell_2 x_2 \\
= k_1(q_1 x_1 - q_2 x_2) + k_2(q_1 x_2 + q_2 x_1) - \ell_1 x_1 - \ell_2 x_2 \\
= (k_1 q_1 + k_2 q_2 - \ell_1) x_1 + (k_2 q_1 - k_1 q_2 - \ell_2) x_2 \\
= (k^\ell - \ell_1) x_1 + ((k^\overline{\ell})_2 - \ell_2) x_2 \\
= (k^\ell - \ell_1) x_1 + (k^\overline{\ell} - \ell_2) x_2.
\]

Therefore
\[ \hat{\Phi}^\ell_q(\ell) = \sum_{k \in \mathbb{Z}^2} \hat{\phi}(\epsilon k) \int_{[0,1]} \exp(2\pi i ((k^\overline{\ell} - \ell) x_1)) dx_1 \int_{[0,1]} \exp(2\pi i ((k^\overline{\ell} - \ell) x_2)) dx_2. \]

The product of the integrals is 1 if \( \ell/\overline{q} = k \) and 0 otherwise. So \( \hat{\Phi}^\ell_q(\ell) = \hat{\phi}(\epsilon \ell/\overline{q}) \) if \( \ell/\overline{q} \in \mathbb{Z}^2 \) and \( \hat{\Phi}^\ell_q(\ell) = 0 \) otherwise. Note that \( \ell/\overline{q} \in \mathbb{Z}^2 \) if and only if \( \overline{\ell}/q \in \mathbb{Z}^2 \).

For \( M > 0 \), define
\[ \mathbb{Z}^2(M) = \{ q \in \mathbb{Z}^2 : M/2 < |q| \leq M \} , \quad \epsilon(M) = \frac{1}{2} M^{-\tau}, \]
and
\[ F_M(x) = \frac{1}{|\mathbb{Z}^2(M)|} \sum_{q \in \mathbb{Z}^2(M)} \hat{\Phi}^\ell_q(M)(x) \quad \forall x \in \mathbb{R}^2. \]

Then \( F_M \) is \( \mathbb{Z}^2 \)-periodic, non-negative, \( C^K \), and (by Lemma 5)
\[ \hat{F}_M(\ell) = \frac{1}{|\mathbb{Z}^2(M)|} \sum_{q \in \mathbb{Z}^2(M) \cap D(\overline{\ell})} \hat{\phi}(\epsilon(M) \ell/\overline{q}) \quad \forall \ell \in \mathbb{Z}^2. \tag{2.2} \]

Since \( \hat{\phi}(0) = \int_{\mathbb{R}^d} \phi(x) dx = 1 \), we have
\[ \hat{F}_M(0) = 1, \tag{2.3} \]
and consequently
\[ |\hat{F}_M(\ell)| \leq 1 \quad \forall \ell \in \mathbb{Z}^2. \] (2.4)

Suppose \( \ell \in \mathbb{Z}^2 \) with \( \ell \neq 0 \). If \( q \in \mathbb{Z}^2(M) \cap D(\ell) \), then \( M/2 < |q|_2 \) and \( |\ell/q|_2 \geq 1 \), which implies \( |\ell|_2 > M/2 \). So if \( |\ell|_2 \leq M/2 \), then the sum in (2.2) is empty and \( \hat{F}_M(\ell) = 0 \). Note \( |\ell| \leq M/4 \) implies \( |\ell|_2 \leq M/2 \). Therefore
\[ \hat{F}_M(\ell) = 0 \quad \forall \ell \in \mathbb{Z}^2, 0 < |\ell| \leq M/4. \] (2.5)

**Lemma 6.** For every \( \zeta > \ln 2 \) there exists \( L_{\zeta} \in \mathbb{N} \) such that
\[ |\hat{F}_M(\ell)| \lesssim |\ell|^{-\alpha} \exp(\zeta \ln |\ell|/ \ln \ln |\ell|) \quad \forall \ell \in \mathbb{Z}^2, |\ell| \geq L_{\zeta}. \]

The proof of Lemma 6 relies on the following divisor bound for the Gaussian integers (see for example [17]).

**Lemma 7.** For every \( \zeta > \ln 2 \) there exists \( L_{\zeta} \in \mathbb{N} \) such that
\[ |D(\ell)| \leq \exp(\zeta \ln |\ell|/ \ln \ln |\ell|) \quad \forall \ell \in \mathbb{Z}^2, |\ell| \geq L_{\zeta}. \]

**Proof of Lemma 6** Fix non-zero \( \ell \in \mathbb{Z}^2 \). By (2.1) and (2.2),
\[
|\hat{F}_M(\ell)| \leq \frac{1}{|Z^2(M)|} \sum_{q \in Z^2(M) \cap D(\ell)} |\hat{\phi}(\epsilon(M)\ell/q)|
\]
\[
\lesssim \frac{1}{|Z^2(M)|} \sum_{q \in Z^2(M) \cap D(\ell)} (1 + \epsilon(M)|\ell/q|)^{-K}
\]
\[
\leq \frac{|Z^2(M) \cap D(\ell)|}{|Z^2(M)|} (1 + (2M)^{-1} \epsilon(M)|\ell|)^{-K}.
\]

We estimate each factor in the last sum separately. Evidently, \( |Z^2(M)| \gtrsim M^2 \). Since \( K \geq a = 2/(1 + \tau) \) and \( \epsilon(M) = \frac{1}{2} M^{-\tau} \), we have
\[
(1 + (2M)^{-1} \epsilon(M)|\ell|)^{-K} \leq 4^{-a} M^2 |\ell|^{-a}.
\]

Obviously, \( |Z^2(M) \cap D(\ell)| \leq |D(\ell)| = |D(\ell)|. \) So applying Lemma 7 finishes the proof. \( \square \)

**Lemma 8.**
\[
supp(F_M) \subseteq \left\{ x \in \mathbb{R}^2 : |q x - r| \leq |q|^{-\tau} \text{ for some } (q, r) \in Z^2(2) \times Z^2 \right\}. \] (2.6)

For any sequence of positive real numbers \( (M_k)_{k=1}^\infty \) with \( M_k \leq M_{k+1}/2 \) for all \( k \in \mathbb{N} \), we have
\[
\bigcap_{k=1}^\infty supp(F_{M_k}) \subseteq E_\ast(\tau). \] (2.7)

**Proof.** Rewrite \( F_M \) as
\[
F_M(x) = \frac{1}{|Z^2(M)|} \sum_{q \in Z^2(M)} \sum_{r \in Z^2} \epsilon(M)^{-2} \phi(\epsilon(M)^{-1}(qx - r)) \quad \forall x \in \mathbb{R}^2.
\]
Suppose \( x \in \mathbb{R}^2 \) satisfies \( F_M(x) > 0 \). Since \( \phi \) is non-negative and \( \text{supp}(\phi) \subseteq [-1, 1] \), we must have some \( q \in \mathbb{Z}^2(M) \) and \( r \in \mathbb{Z}^2 \) such that
\[
|qx - r| \leq \epsilon(M) = \frac{1}{2}M^{-\tau} \leq \frac{1}{2}|q|^{-\tau}.
\]
More generally, suppose \( x \in \text{supp}(F_M) \). Then we can find \( x' \in \mathbb{R}^2 \) such that \( F_M(x') > 0 \) and \( |x - x'|_2 \leq \frac{1}{4}M^{-(1+\tau)} \). Therefore, by the argument above, there is some \( q \in \mathbb{Z}^2(M) \) and \( r \in \mathbb{Z}^2 \) such that
\[
|qx - r| \leq |qx - qx'| + |qx' - r| \leq \frac{1}{4}M^{-(1+\tau)}|q|_2 + \frac{1}{2}|q|^{-\tau} \leq |q|^{-\tau}.
\]
This proves (2.6).

If \( x \in \text{supp}(F_{M_k}) \) for every \( k \in \mathbb{N} \), we obtain for every \( k \in \mathbb{N} \) a pair \( (q^{(k)}, r^{(k)}) \in \mathbb{Z}^2(M_k) \times \mathbb{Z}^2 \) with \( |q^{(k)}x - r^{(k)}| \leq |q^{(k)}|^{-\tau} \). The pairs must be distinct because
\[
|q^{(k)}| \leq M_k \leq M_{k+1}/2 < |q^{(k+1)}| \quad \forall k \in \mathbb{N}.
\]
This proves (2.7).

\[\square\]

3 Proof of Theorem 1: A Lemma For Recursion

Lemma 9. For every \( \delta > 0 \), \( M_0 > 0 \), and \( \chi \in C^K_c(\mathbb{R}^2) \), there is an \( M_* = M_*(\delta, M_0, \chi) \in \mathbb{N} \) such that \( M_* \geq M_0 \) and
\[
|\hat{\chi F_{M_*}}(\xi) - \hat{\chi}(\xi)| \leq \delta g(\xi) \quad \forall \xi \in \mathbb{R}^2,
\]
where
\[
g(\xi) = \begin{cases} 
|\xi|^{-a} \exp(\ln|\xi|/\ln \ln|\xi|) & \text{if } \xi \in \mathbb{R}^2, |\xi| > e \\
1 & \text{if } \xi \in \mathbb{R}^2, |\xi| \leq e.
\end{cases}
\]

The proof will show \( M_* \) can be taken to be any sufficiently large positive number.

Proof. We begin by recording two auxiliary estimates. Since \( \chi \in C^K_c(\mathbb{R}^2) \),
\[
|\hat{\chi}(\xi)| \lesssim (1 + |\xi|)^{-K} \quad \forall \xi \in \mathbb{R}^2. \tag{3.1}
\]
For every \( p > 2 \), we have
\[
\sup_{\xi \in \mathbb{R}^2} \sum_{\ell \in \mathbb{Z}^2} (1 + |\xi - \ell|)^{-p} < \infty. \tag{3.2}
\]

Fix \( \xi \in \mathbb{R}^2 \). We will write \( \widehat{\chi F_M}(\xi) - \hat{\chi}(\xi) \) in another form. Since \( F_M \) is \( C^K \) and \( \mathbb{Z}^2 \)-periodic, we have
\[
F_M(x) = \sum_{\ell \in \mathbb{Z}^2} \hat{F_M}(\ell) e^{2\pi i \ell \cdot x} \quad \forall x \in \mathbb{R}^2
\]
with uniform convergence. Since \( \chi \in L^1(\mathbb{R}^2) \), multiplying by \( \chi \) and taking the Fourier transform yields
\[
\widehat{\chi F_M}(\xi) = \sum_{\ell \in \mathbb{Z}^2} \hat{F_M}(\ell) \int_{\mathbb{R}^2} \chi(x) e^{2\pi i (\ell - \xi) \cdot x} dx = \sum_{\ell \in \mathbb{Z}^2} \hat{F_M}(\ell) \hat{\chi}(\xi - \ell).
\]
Then, by (2.3) and (2.5), we have
\[
\hat{\chi} F_M(\xi) - \hat{\chi}(\xi) = \sum_{\ell \in \mathbb{Z}^2} \hat{\chi}(\xi - \ell) \hat{F}_M(\ell) - \hat{\chi}(\xi) = \sum_{|\ell| > M/4} \hat{\chi}(\xi - \ell) \hat{F}_M(\ell). \tag{3.3}
\]

Define \( \eta = (K - 2 - a)/2 \), which is positive by our choice of \( K \). To estimate \( \hat{\chi} F_M(\xi) - \hat{\chi}(\xi) \), we use (3.3) and consider two cases.

**Case 1:** \( |\xi| < M/8 \).
If \( |\ell| > M/4 \), then \( |\xi - \ell| > M/8 > |\xi| \). Hence by (2.4), (3.1), (3.2), and (3.3) we have
\[
|\hat{\chi} F_M(\xi) - \hat{\chi}(\xi)| \lesssim \sum_{|\ell| > M/4} (1 + |\xi - \ell|)^{-K} = \sum_{|\ell| > M/4} (1 + |\xi - \ell|)^{-a - \eta - (2 + \eta)} \leq (1 + |\xi|)^{-a} (1 + M/8)^{-\eta} \sum_{|\ell| > M/4} (1 + |\xi - \ell|)^{-2 + \eta} \leq \delta g(\xi)
\]
for all sufficiently large \( M \).

**Case 2:** \( |\xi| \geq M/8 \).
Using (3.3), write
\[
\hat{\chi} F_M(\xi) - \hat{\chi}(\xi) = \sum_{|\ell| > M/4} \hat{\chi}(\xi - \ell) \hat{F}_M(\ell) + \sum_{|\ell| > M/4} \hat{\chi}(\xi - \ell) \hat{F}_M(\ell) = S_1 + S_2.
\]
If \( |\ell| \leq |\xi|/2 \), then \( |\xi - \ell| \geq |\xi|/2 \geq M/16 \). Hence by (2.4), (3.1), and (3.2) we have
\[
|S_1| \lesssim \sum_{|\ell| > M/4} \sum_{|\ell| \leq |\xi|/2} (1 + |\xi - \ell|)^{-K} = \sum_{|\ell| > M/4} (1 + |\xi - \ell|)^{-a - \eta - (2 + \eta)} \leq (1 + |\xi|/2)^{-a} (1 + M/16)^{-\eta} \sum_{|\ell| > M/4} (1 + |\xi - \ell|)^{-2 + \eta} \leq \frac{1}{2} \delta g(\xi)
\]
for all sufficiently large \( M \).

Fix \( \ln 2 < \zeta < 1 \). By Lemma 6 (3.1), and (3.2) we have
\[
|S_2| \lesssim \sum_{|\ell| > M/4} |\ell|^{-a} \exp (\zeta \ln |\ell| / \ln \ln |\ell|) (1 + |\xi - \ell|)^{-K} \lesssim (|\xi|/2)^{-a} \exp (\zeta \ln(|\xi|/2) / \ln \ln(|\xi|/2)) \leq \frac{1}{2} \delta g(\xi)
\]
for all sufficiently large \( M \).

\[\square\]

### 4 Proof of Theorem 1: The Measure \( \mu \)

Given any closed ball \( B \subseteq \mathbb{R}^2 \), fix an arbitrary non-negative \( C^K \) function \( \chi_0 \) on \( \mathbb{R}^2 \) with \( \text{supp}(\chi_0) \subseteq B \) and \( \int_{\mathbb{R}^2} \chi_0(x) dx = 1 \). Using Lemma 9 define
\[
M_1 = M_*(2^{-2}, 1, \chi_0), \quad M_{k+1} = M_*(2^{-k-2}, 2M_k, \chi_0 F_{M_k} \cdots F_{M_k}) \quad \forall k \in \mathbb{N}.
\]
Define measures $\mu_k$ by
\[ d\mu_0 = \chi_0 dx, \quad d\mu_k = \chi_0 F_{M_1} \cdots F_{M_k} dx \quad \forall k \in \mathbb{N}. \]

By Lemma 9, $M_k \leq M_{k+1}/2$ for all $k \in \mathbb{N}$ and
\[ |\hat{\mu}_k(\xi) - \hat{\mu}_{k-1}(\xi)| \leq 2^{-k-1} g(\xi) \quad \forall \xi \in \mathbb{R}^2, k \in \mathbb{N}. \quad (4.1) \]

Since $g$ is bounded, (4.1) implies $(\hat{\mu}_k)_{k=0}^\infty$ is Cauchy, hence convergent, in the supremum norm. Therefore, since each $\hat{\mu}_k$ is a continuous function, $\lim_{k \to \infty} \hat{\mu}_k$ is a continuous function. By (4.1), we have
\[ |\lim_{k \to \infty} \hat{\mu}_k(\xi) - \hat{\mu}_0(\xi)| \leq \sum_{k=1}^{\infty} |\hat{\mu}_k(\xi) - \hat{\mu}_{k-1}(\xi)| \leq \frac{1}{2} g(\xi) \quad \forall \xi \in \mathbb{R}^2. \quad (4.2) \]

Since $\hat{\mu}_0(0) = \int_{\mathbb{R}^2} \chi_0(x) dx = 1$ and $g(0) = 1$, it follows from (4.2) that
\[ 1/2 \leq |\lim_{k \to \infty} \hat{\mu}_k(0)| \leq 3/2. \]

Therefore, by Lévy’s continuity theorem, $(\mu_k)_{k=0}^\infty$ converges weakly to a non-zero finite Borel measure $\mu$ with $\hat{\mu} = \lim_{k \to \infty} \hat{\mu}_k$ and
\[ \text{supp}(\mu) = \text{supp}(\chi_0) \cap \bigcap_{k=1}^{\infty} \text{supp}(F_{M_k}). \]

By Lemma 8 and $\text{supp}(\chi_0) \subseteq B$, we have
\[ \text{supp}(\mu) \subseteq B \cap E_*(\tau). \]

Since $\chi_0 \in C^K_c(\mathbb{R}^2)$, we have $\hat{\mu}_0(\xi) \lesssim (1 + |\xi|)^{-a}$ for all $\xi \in \mathbb{R}^2$. Combining this with (4.2) gives
\[ |\hat{\mu}(\xi)| \lesssim g(\xi) \quad \forall \xi \in \mathbb{R}^2. \]

By multiplying $\mu$ by a constant, we can make $\mu$ a probability measure. This completes the proof of Theorem 1.

5 Outline of Proof of Theorem 3

The proof of Theorem 3 is obtained by modifying the proof of Theorem 1 in a few places, as we now describe.

Throughout the proof, we replace $\mathbb{Z}^2$ by the set of Gaussian primes $P$, and we replace
\[ \mathbb{Z}^2(M) = \{ q \in \mathbb{Z}^2 : M/2 < |q| \leq M \} \]
by
\[ P(M) = \{ q \in P : M/2 < |q| \leq M \}. \]

Lemma 6 is replaced by
\[ |\hat{F}_M(\ell)| \lesssim |\ell|^{-a} \ln |\ell| \quad \forall \ell \in \mathbb{Z}^2, |\ell| \geq 2, M \geq 4. \]

Lemma 10.
The proof of Lemma 10 is a modification of the proof of Lemma 6. Instead of estimating $|Z_2(M)|$ and $|Z_2(M) \cap D(\ell)|$, we estimate $|P(M)|$ and $|P(M) \cap D(\ell)|$. By the prime number theorem in the Gaussian integers (which is a consequence of Landau’s prime ideal theorem), we have

$$|P(M)| \gtrsim \frac{M^2}{\ln M}.$$  

By unique factorization in the Gaussian integers, we have

$$|P(M) \cap D(\ell)| \lesssim \frac{\ln |\ell|}{\ln M}.$$  

We assume $|\ell| \geq 2$ and $M \geq 4$ to avoid technicalities.

Finally, the function $g$ appearing in Lemma 9 is changed to

$$g(\xi) = \begin{cases} |\xi|^{-a} \ln |\xi| \ln \ln |\xi| & \text{if } \xi \in \mathbb{R}^2, |\xi| > e \\ 1 & \text{if } \xi \in \mathbb{R}^2, |\xi| \leq e. \end{cases}$$

The estimate for $S_2$ in the proof of Lemma 9 now goes like this: By Lemma 10 (3.1), and (3.2) we have

$$|S_2| \lesssim \sum_{|\ell| > M/4 \atop |\ell| > |\xi|/2} |\ell|^{-a} \ln |\ell| (1 + |\xi - \ell|)^{-K} \lesssim (|\xi|/2)^{-a} \ln (|\xi|/2) \leq \frac{1}{2} \delta g(\xi)$$

for all sufficiently large $M$.

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