Evolution of the spectral index after inflation

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Abstract. In this article we investigate the time evolution of the adiabatic (curvature) and isocurvature (entropy) spectral indices after inflation era for all cosmological scales with two different initial conditions. For this purpose, we first extract an explicit equation for the time evolution of the comoving curvature perturbation (which may be known as the generalized Mukhanov-Sasaki equation). It would be cleared that the evolution of adiabatic spectral index severely depends on the initial conditions moreover, as expected it is constant only for the super-Hubble scales and adiabatic initial conditions. Additionally, the adiabatic spectral index after recombination approaches a constant value for the isocurvature perturbations. Finally, we re-investigate the Sachs-Wolfe effect and show that the fudge factor $\frac{1}{3}$ in the adiabatic ordinary Sachs-Wolfe formula must be replaced by 0.4.

Keywords: cosmological perturbation theory, inflation

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1 Introduction

Inflation was first produced to solve three classical cosmological problems: the horizon, flatness and monopole problems [1]. It also explains the origin of the CMB anisotropy and structure formation [2, 3], indeed, during inflation the quantum vacuum fluctuations of the scalar field(s) on the scales less than the Hubble horizon are magnified into the classical perturbations in the scalar field(s) on scales larger than the Hubble horizon [4, 5]. It can be shown that these perturbations have a nearly scale-invariant spectrum and can explain the origin of the inhomogeneities in the recent universe such as large structures and CMB anisotropies as well [2, 6]. These classical perturbations can be described by some perturbative cosmic potentials which are related to the FLRW metric perturbations and maybe cosmic fluid perturbations. One of these perturbative cosmic potentials is the comoving curvature perturbation $\mathcal{R}$ which is significant in cosmology due to the following reasons

- It is conserved for the adiabatic perturbations when the scales of the perturbations are extremely longer than the Hubble horizon [6]
- Gauge-invariance and resemblance to the physical observables [7]
- Sasaki-Stewart $\delta N$-formula expressing the perturbation of e-folding number in terms of $\mathcal{R}$ [8]
- The scalar primordial power spectrum usually refers to the power spectrum of $\mathcal{R}$ which characterizes the adiabatic mode [5, 9, 10]

Furthermore, it can be shown that the linear perturbation of the scale factor and signature of the spatial curvature of the universe in the comoving gauge merely depends on $\mathcal{R}$ [11–13]

$$
\delta a(t, x) = a(t) \mathcal{R}(t, x)
$$

$$
\delta K(t, x) = -\frac{2}{3} \nabla^2 \mathcal{R}(t, x).
$$

Finally, $\mathcal{R}$ can be used to connect the observed cosmological perturbations in the adiabatic mode with quantum fluctuations at a very early stage [6]. The dynamic of $\mathcal{R}$ during inflation is described by the well-known Mukhanov-Sasaki equation [14, 15]. This equation yields the power spectrum and spectral index of $\mathcal{R}$ at the inflation era. In this paper we generalize the Mukhanov-Sasaki equation to include the entire history of the universe and then by invoking
a simple model, show how $\mathcal{R}$ evolves after inflation. Additionally, a discussion of the spectral index evolution after the inflation is presented.

The outline of this paper is as follows. In section 2, we present an explicit equation for time evolution of $\mathcal{R}$ which can be used for all history eras of the universe then investigate its solutions for some very simple cases. In section 3, we investigate a universe containing a mixture of radiation and matter then show that the $\mathcal{R}$-evolution equation can be solved after coupling of the Kodama-Sasaki equation [16, 17]. We consider two adiabatic and isocurvature initial conditions and present the numerical solutions. We also supply an analytic method helping us to approximate solutions. Moreover, we present the numerical results of the curvature spectral index and also entropy spectral index evolution in the post-inflationary universe. In section 4, we re-investigate the Sachs-Wolfe effect and point out that $\frac{1}{3}$ factor in the Sachs-Wolfe formula must be enhanced as will be discussed later. A conclusion will be presented in section 5.

2 A general equation for evolution of $\mathcal{R}_q$

We may consider the metric of the universe as [6, 18, 19]

$$ds^2 = a^2 \left\{ (1 + E) d\tau^2 + 2\partial_i F d\tau dx^i + [(1 + A) \delta_{ij} + \partial_i \partial_j B] dx^i dx^j \right\},$$

(2.1)

which is the FLRW metric with $K = 0$ in the comoving quasi-Cartesian coordinates accompanied by the most general scalar linear perturbations. Similarly, the energy-momentum tensor of the cosmic fluid can be written as

$$T_{00} = a^2 \left[ \bar{\rho} (1 + E) + \delta \rho \right],$$

(2.2)

$$T_{i0} = a^2 \left[ \bar{p} \partial_i F - \left( \frac{\bar{\rho} + \bar{p}}{a} \right) \partial_i (\delta u) \right],$$

(2.3)

$$T_{ij} = a^2 \left[ \bar{\rho} (1 + A) \delta_{ij} + \delta p \delta_{ij} + \bar{p} \partial_i \partial_j B + \partial_i \partial_j \Pi^S \right],$$

(2.4)

where $\delta u$ and $\Pi^S$ stand for the scalar velocity potential and scalar anisotropic inertia of the cosmic fluid, respectively. $\Pi^S$ represents departures of the cosmic fluid from perfectness. Furthermore, $\rho = \bar{\rho} + \delta \rho$ and $p = \bar{p} + \delta p$ are the energy density and pressure of the cosmic fluid respectively. Notice that bar over every quantity stands for its unperturbed value. It can be shown that except $\Pi^S$ all of the perturbative scalars in equations (2.1) to (2.4) are not gauge-invariant [6], so we may invoke combinational gauge-invariant scalars like Bardeen potentials [7]

$$\Psi = \frac{A}{2} - \mathcal{H} \sigma,$$

(2.5)

$$\Phi = \frac{E}{2} + \mathcal{H} \sigma + \sigma'.$$

(2.6)

Here the prime symbol stands for the derivative with respect to the $\tau$ and $\mathcal{H} = Ha$ is the comoving Hubble parameter. Furthermore, $\sigma = F - \frac{1}{2} B'$ is the shear potential of the cosmic fluid.
fluid. Some other gauge-invariant scalars are

\[ R = \frac{A}{2} + H \delta u, \]  
\[ \zeta = \frac{A}{2} - \mathcal{H} \frac{\delta \rho}{\bar{\rho}}, \]  
\[ \Delta = a \delta \rho + \delta u, \]  
\[ \Gamma = \delta p - c_s^2 \delta \rho, \]

(2.7)  
(2.8)  
(2.9)  
(2.10)

where \( c_s^2 \) is the adiabatic sound speed in the cosmic fluid. \( R \) is known as comoving (intrinsic) curvature perturbation. According to the perturbative form of the field equations and also energy-momentum conservation law we can write

\[ R_q = -\frac{2\mathcal{H}^2 + \mathcal{H}'}{\mathcal{H}' - \mathcal{H}} \Psi_q - \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}} \Psi'_q + 8\pi \mathcal{G} \mathcal{H}^2 a^2 \frac{q^2}{\mathcal{H}' - \mathcal{H}} \Pi^S_q, \]  
\[ R'_q = \frac{\mathcal{H}c_s^2}{\mathcal{H}' - \mathcal{H}} q^2 \Psi_q - \frac{4\pi \mathcal{G} \mathcal{H}a^2}{\mathcal{H}' - \mathcal{H}} \Lambda_q, \]

(2.11)  
(2.12)

where \( R_q \) denotes the Fourier transformation of \( R \) with the comoving wave number \( q \) and \( \Lambda_q = \Gamma_q - q^2 \Pi^S_q \). Notice that we have taken \( K = 0 \) which corresponds to the observational data [20]. By combination of equations (2.11) and (2.12) after some tedious but straightforward calculation we can derive an explicit equation in terms of \( R_q \)

\[ R'_q = \frac{4\pi \mathcal{G} \mathcal{H}a^2}{\mathcal{H}' - \mathcal{H}} \Lambda_q, \quad c_s^2 = 0 \]  

(2.13)

and

\[ R''_q + 2 \frac{\zeta'_q}{\mathcal{Z}} R'_q + c_s^2 q^2 R_q = \frac{4\pi \mathcal{G} a^2}{\mathcal{H}' - \mathcal{H}} \left[ -4\mathcal{H}^2 + \frac{(\mathcal{H} c_s^2)'}{c_s^2} \right] \Lambda_q - \mathcal{H} \Lambda'_q - 2q^2 \mathcal{H}^2 c_s^2 \Pi^S_q, \quad c_s^2 \neq 0 \]  

(2.14)

where \( \mathcal{Z} = \frac{a}{\mathcal{H}} \sqrt{\frac{\mathcal{H}' - \mathcal{H}}{c_s^2}} \). Both equations (2.13) and (2.14) together include the most general case of the fluid even the scalar field, so the Mukhanov-Sasaki equation is a special case of this equation. For the pure dust universe equation (2.13) yields \( R_q = \text{const} \). It means \( R \) is conserved if the cosmic fluid is dust regardless of the comoving wave number scale. On the other hand, for the pure radiation case equation (2.14) reduces to \( R''_q + \frac{2 q^2}{\mathcal{Z}} R_q = 0 \) and consequently, \( R_q \propto \cos \left( q \sqrt{\frac{3}{2}} \tau \right) \). Besides, if we suppose the radiation era starts immediately after the inflation, it may apply the following initial condition [6]

\[ \tau \rightarrow 0 \quad : \quad R_q \rightarrow N q^{-2 + \frac{n_{s0}}{2}} \left( N \simeq 10^{-5} \quad \text{and} \quad n_{s0} \simeq 0.96 \right) \]  

(2.15)

(\( \tau = 0 \) is supposed to be the end of inflation epoch and the starting time of the radiation epoch.) Thus

\[ R_q (\tau) = N q^{-2 + \frac{n_{s0}}{2}} \cos \left( \frac{q}{\sqrt{3}} \tau \right). \]
Now let’s turn to the inflaton case. Every scalar field can be treated as a perfect fluid. For the homogeneous inflaton field $\bar{\phi}(t)$ with potential $V(\bar{\phi})$ we have

$$\bar{\rho} = \frac{1}{2a^2} \bar{\phi}'^2 + V(\bar{\phi}), \quad \bar{p} = \frac{1}{2a^2} \bar{\phi}'^2 - V(\bar{\phi}).$$

(2.16) (2.17)

It can be shown that under the linear perturbations of the metric i.e. equation (2.1)

$$\delta \rho = \frac{1}{2a^2} E \bar{\phi}'^2 + \frac{1}{a^2} \bar{\phi}' \delta \bar{\phi}' \bar{\phi}' + \frac{\partial V}{\partial \bar{\phi}} \delta \bar{\phi},$$

$$\delta p = \frac{1}{2a^2} E \bar{\phi}'^2 - \frac{1}{a^2} \bar{\phi}' \delta \bar{\phi}' \bar{\phi}' - \frac{\partial V}{\partial \bar{\phi}} \delta \bar{\phi}. \quad (2.18) \quad (2.19)$$

Now let’s confine ourselves to the comoving gauge which indicates $\delta \bar{\phi} = 0$, thus $\delta \rho = \delta p$, consequently $c_s^2 = 1$ and $\Gamma_q = \Lambda_q = 0$. Note that anisotropic inertia $\Pi_S^q$ for the scalar fields vanishes. So equation (2.14) reduces to

$$R_q'' + 2 \frac{Z'}{Z} R_q' + q^2 R_q = 0.$$ 

(2.20)

On the other hand, according to the Friedmann equation $\mathcal{H}^2 - \mathcal{H}' = 4\pi G \bar{\rho}$ Thus $Z = \frac{a \bar{\phi}'}{H}$. Now by introducing the Sasaki-Mukhanov variable as $v_q = Z R_q$, equation (2.20) reduces to

$$v_q'' + \left( q^2 - \frac{Z''}{Z} \right) v_q = 0,$$

which is the famous Mukhanov-Sasaki equation.

### 3 Evolution of $R_q$ in a simplified universe

In this section we consider the case where the cosmic fluid has been constructed from two perfect fluids: matter and radiation which don’t interact with each other, i.e., there is no energy or momentum transfer between them. This model was first introduced by Peebles and Yu [21] and was later used by Seljak [22] in order to approximate CMB anisotropy. Compared to the real universe, this is a simplification since, contrary to the CDM, the baryonic matter does interact with photons. Indeed, the radiation components of the real universe i.e. neutrinos and photons behave like a perfect fluid only until their decoupling [17]. We now have two fluid components, so $\rho = \rho_R + \rho_M$ where $\bar{\rho}_M \propto \frac{1}{a^3}$ and $\bar{\rho}_R \propto \frac{1}{a^4}$. Let’s define the normalized scale factor as

$$y = \frac{a}{a_{eq}} = \frac{\bar{\rho}_M}{\bar{\rho}_R},$$

(3.1)

where $a_{eq}$ is the scale factor in the time of matter-radiation equality. It is clear that

$$\bar{\rho}_M = \frac{y}{y + 1} \bar{\rho},$$

$$\bar{\rho}_R = \frac{1}{y + 1} \bar{\rho}.$$
Consequently,
\[ \omega = \frac{1}{3(y+1)}, \quad c_s^2 = \frac{4}{3(3y+4)}. \]  

Also
\[ H = \frac{y'}{y} = \frac{H_{eq} \sqrt{y+1}}{\sqrt{2} y}, \]
where \( H_{eq} \) is the comoving Hubble parameter of matter-radiation equality. Simply, it can be shown that
\[ \Gamma = -\bar{\rho}_M c_s^2 \mathcal{S}, \]
where \( \mathcal{S} = \delta_M - \delta_R = \frac{4\rho_M}{\bar{\rho}_M} - \frac{3}{4} \frac{\delta \rho_R}{\bar{\rho}_R} \) is the entropy perturbation between matter and radiation. Thus
\[ \Lambda_q = \Gamma_q = -\bar{\rho}_{M,eq} c_s^2 \mathcal{S} q = -\frac{H_{eq}^2}{4\pi Ga_{eq}^2 y^3(3y+4)}, \]
By substituting equations (3.2), (3.3) and (3.5) in equation (2.14) we find
\[ (y+1) R_{qq}^{**} + \frac{21y^2 + 36y + 16}{2y(3y+4)} R_q^* + \frac{8}{3(3y+4)} \left( \frac{q}{H_{eq}} \right)^2 R_q = \frac{2}{y(3y+4)} S_q + \frac{4(y+1)}{(3y+4)^2} S_q^*, \]
where "\( \ast \)" stands for the partial derivative with respect to \( y \).

The adiabatic solution of equation (3.6) for the super-Hubble scales may be found by putting \( S_q = 0 \) and ignoring the term \( q H_{eq} \) i.e.
\[ R_q = C_1 q \left( \ln \frac{\sqrt{y+1}}{\sqrt{y}} - \frac{y+1}{y} \frac{3y+2}{3y+4} \right) + C_2 q. \]
The first term in the equation (3.7) is decaying through the time and has no significance in the late times, so we conclude the conservation of \( R \) as was previously expected [6]. In order to generally solve equation (3.6), we have to determine \( S_q \). \( S_q \) for a mixture of matter and radiation may be obtained from Kodama-Sasaki equation [16]
\[ \frac{1}{H^2} S_q'' + \frac{3 c_s^2}{H} S_q' = -q^2 \left[ \frac{\Delta_q}{3} - \frac{1}{3} \left( 3c_s^2 - 1 \right) S_q \right], \]
where \( \Delta = H \Delta \). By re-writing equation (3.8) in terms of \( y \) we find
\[ y S_q'' + \left( 1 + \frac{4}{3y+4} - \frac{y+2}{2(y+1)} \right) S_q' = -q^2 \left( \frac{2y}{H_{eq}^2} + \frac{y}{y+1} \right) \left( \Delta_q + \frac{y}{3y+4} S_q \right). \]
On the other hand, we have
\[ R' = -\frac{Hc_s^2}{H^2 - H'} \nabla^2 \Psi - \frac{4\pi G H a^2}{H^2 - H'} \Lambda. \]
Furthermore, from Poisson’s equation we can write
\[ \nabla^2 \Psi = -12\pi G (\bar{\rho} + \bar{p}) a^2 \Delta. \]
Combination of equations (3.10) and (3.11) yields
\[ R_q = \frac{4}{y(3y+4)} \left( \Delta_q + \frac{y}{3y+4} S_q \right). \]
Substituting equation (3.12) in equation (3.9) yields

\[(y + 1) S_q^{**} + \frac{3y^2 + 12y + 8}{2y(3y + 4)} S_q^* = -\frac{1}{2} \left( \frac{a}{H_{\text{eq}}} \right)^2 y (3y + 4) R_q^*. \tag{3.13}\]

Equations (3.6) and (3.13) are coupled and must be solved simultaneously

\[
\begin{cases}
(y + 1) S_q^{**} + \frac{3y^2 + 12y + 8}{2y(3y + 4)} S_q^* = -\frac{1}{2} \epsilon y (3y + 4) R_q^*, \\
(y + 1) R_q^{**} + \frac{21y^2 + 36y + 16}{2y(3y + 4)} R_q^* + \frac{8\epsilon}{3 (3y + 4)} R_q = \frac{2}{y (3y + 4)} S_q + \frac{4(y + 1)}{(3y + 4)^2} S_q^*,
\end{cases}
\]

where \( \epsilon = \left( \frac{a}{H_{\text{eq}}} \right)^2. \) At early times, all cosmologically intersecting scales are outside the Hubble horizon, so we may find the behavior of the solutions in the early stage by setting \( \epsilon = 0. \) In this case we will have two independent solutions

- **Solution 1**

\[
\begin{align*}
S_q &= 0, \\
R_q &= \text{const}.
\end{align*}
\]

- **Solution 2**

\[
\begin{align*}
S_q &= \text{const}, \\
R_q &= \frac{y}{3y + 4} S_q = \frac{1}{3} \left( 1 - 3\epsilon_s^2 \right) S_q.
\end{align*}
\]

Solutions 1 and 2 are called adiabatic and isocurvature initial conditions, respectively. Note that they are solutions of the equations (3.6) and (3.13) only in the early times when the scales of the perturbations have been extremely longer than the Hubble horizon. The “initial condition” expression refers to this point.

From equation (3.6) it is clear that the entropy perturbation is a source for the curvature perturbation, so if \( S \neq 0, \) \( R \) is never conserved. In order to analytically solve equations (3.6) and (3.13), we may expand \( S_q \) and \( R_q \) in terms of \( \epsilon \) by the Frobenius method \([23]\)

\[
\begin{align*}
S_q (y) &= \epsilon^\alpha \sum_{n=0}^{\infty} \epsilon^n S_n \left( y \right), \tag{3.14} \\
R_q (y) &= \epsilon^\beta \sum_{n=0}^{\infty} \epsilon^n R_n \left( y \right), \tag{3.15}
\end{align*}
\]

Where \( \alpha \) and \( \beta \), in general, are two arbitrary complex numbers. After substituting equations (3.14) and (3.15) in equations (3.6) and (3.13) and also setting \( \alpha = \beta \) we have

\[
\begin{cases}
(y + 1) S_0^{**} + \frac{3y^2 + 12y + 8}{2y(3y + 4)} S_0^* = 0, \\
(y + 1) R_0^{**} + \frac{21y^2 + 36y + 16}{2y(3y + 4)} R_0^* = \frac{2}{y (3y + 4)} S_0 + \frac{4(y + 1)}{(3y + 4)^2} S_0^*.
\end{cases} \tag{3.16}
\]

- 6 –
And also for \( n \geq 1 \) we find two recursive equations as follows

\[
\begin{align*}
(y + 1) S_{n+1}^* + \frac{3y^2 + 12y + 8}{2y (3y + 4)} S_n^* &= \frac{1}{2} y (3y + 4) R_{n-1}^*, \\
(y + 1) R_{n+1}^* + \frac{21y^2 + 36y + 16}{2y (3y + 4)} R_n^* &= -\frac{8}{3} (3y + 4) R_{n-1} + \frac{2}{y (3y + 4)} S_n + \frac{4(y + 1)}{(3y + 4)^2} S_n^*.
\end{align*}
\]

(3.17)

After fixing the initial conditions, we can solve equations (3.16) and (3.17). On the other hand, the adiabatic initial condition according to the inflationary theory may be written as [6]

\[
\epsilon \rightarrow 0 \quad \text{or} \quad y \rightarrow 0 : \quad R_q \rightarrow N q^{-2 + \frac{n_{s0}}{2}} \quad \text{and} \quad S_q \rightarrow 0.
\]

(3.18)

Where according to the observational data \( N \simeq 10^{-5} \) and \( n_{s0} \simeq 0.96 \) [20]. So under the adiabatic initial condition, we have

\[
\begin{align*}
\alpha &= \beta = -1 + \frac{n_{s0}}{4}, \\
S_0 (y) &= S_1 (y) = 0, \\
R_0 (y) &= N \mathcal{H}_{eq}^{-2 + \frac{n_{s0}}{2}}, \\
R_1 (y) &= \frac{16}{45} N \left[ \frac{1}{y} + \frac{13}{3 (3y + 4)} - \frac{2\sqrt{y + 1} (3y + 2)}{y (3y + 4)} + 2 \ln \frac{1 + \sqrt{y + 1}}{2} - \frac{13}{12} \right] q^{\frac{n_{s0}}{2}}.
\end{align*}
\]

So, when the scales of the perturbations are not extremely smaller than the Hubble horizon, under the adiabatic initial condition the following approximation is appropriate

\[
\begin{align*}
\mathcal{S}_0 (y) &\simeq 0, \\
\mathcal{R}_q (y) &\simeq N q^{-2 + \frac{n_{s0}}{2}} - \frac{16}{45} N \left[ \frac{1}{y} + \frac{13}{3 (3y + 4)} - \frac{2\sqrt{y + 1} (3y + 2)}{y (3y + 4)} + 2 \ln \frac{1 + \sqrt{y + 1}}{2} - \frac{13}{12} \right] q^{\frac{n_{s0}}{2}}.
\end{align*}
\]

On the other hand, the isocurvature initial condition may be written as

\[
\epsilon \rightarrow 0 \quad \text{or} \quad y \rightarrow 0 : \quad S_q \rightarrow M q^{-2 + \frac{n_{iso}}{2}} \quad \text{and} \quad R_q \rightarrow \frac{y}{3y + 4} S_q.
\]

(3.19)

where in accordance with the Liddle and Mazumdar model [24] \( n_{iso} \simeq 4.43 \). It may be also considered \( M \simeq 10^{-5} \) similar to the domain of adiabatic perturbation. Unfortunately, in this case solving the recursive equations results in calculation of some enormous integrals, so we leave it. Nevertheless, it is possible to numerically solve equations (3.6) and (3.13). The results are presented in the following figures. Figures 1 and 2 illustrate the curve of \( R_q \) for the adiabatic and isocurvature initial conditions have also been plotted.

The spatial indices of the adiabatic and isocurvature perturbations respectively are defined as

\[
\begin{align*}
n_s (q) &= 4 + \frac{q}{\mathcal{P}_R (q)} \frac{\partial \mathcal{P}_R (q)}{\partial q} = 4 + \frac{q}{R_q} \frac{\partial R_q}{\partial q}, \\
n_{iso} (q) &= 4 + \frac{q}{\mathcal{P}_S (q)} \frac{\partial \mathcal{P}_S (q)}{\partial q} = 4 + \frac{q}{S_q} \frac{\partial S_q}{\partial q}.
\end{align*}
\]

(3.20)  (3.21)

where \( \mathcal{P}_R (q) \) and \( \mathcal{P}_S (q) \) are the power spectrums of \( R_q \) and \( S_q \), respectively. The curves of \( n_s (q) \) and \( n_{iso} (q) \) in terms of \( y \) have been plotted in figures 3, 4, 5 and 6 for the adiabatic and isocurvature initial conditions respectively. It is clear that \( n_s \) and \( n_{iso} \) depend severely on \( q \), so we may say that the spectral indices are running.
Figure 1. Evolution of the comoving curvature perturbation $R_q$ in a universe constructed from dust and radiation for the comoving wave number $q = 10^5$ i.e. sub-horizon scales (up) and $q = 10^{-10}$ namely severe super-horizon modes (down) providing the adiabatic initial condition. We suppose $n_{s0} = 0.96$ and $N$, the amplitude of $R_q$ at the end of inflation is roughly $10^{-5}$. It is clear that $R_q$ for the super-horizon scales is conserved. Notice that both $q$ and $R_q$ are dimensionless.

4 The Sachs-Wolfe effect: a new survey

Cosmic Microwave Background (CMB) was discovered in a study of noise backgrounds in a radio telescope detected by Penzias and Wilson in 1965 [25]. Two years later, Sachs and Wolfe pointed out that the CMB must show the temperature anisotropy as a result of photon traveling in the perturbed universe [26]. An important contribution to temperature anisotropy of CMB is called the ordinary Sachs-Wolfe effect originating from the intrinsic temperature inhomogeneities on the last scattering surface and also the inhomogeneities of the metric at the time of last scattering. It can be shown that [27–29]

$$\Delta T(\hat{n})T_0 = \zeta_R(t_L, \hat{\mathbf{n}} r_L) + \Psi(t_L, \hat{\mathbf{n}} r_L) + \Phi(t_L, \hat{\mathbf{n}} r_L),$$

where $\zeta_R$ denotes the curvature perturbation of the radiation in the uniform density slices. Moreover, $\hat{n}$ is the unit vector standing for the direction of observation. Notice that all quantities are evaluated in last scattering surface. The corresponding scales well outside the Hubble horizon have dominant imprint on the ordinary Sachs-Wolfe effect. In the multipole
space, the ordinary Sachs-Wolfe effect is responsible for $l \lesssim 30$ [5]. Moreover, its angular power spectrum has a plateau which is known as Sachs-Wolfe plateau. In a mixture of radiation and dust equation (4.1) reduces to

$$
\left[ \frac{\Delta T (\hat{n})}{T_0} \right]_{\text{S.W.}} = \zeta_R (t_L, \hat{\mathbf{n}} r_L) + 2 \Psi (t_L, \hat{\mathbf{n}} r_L).
$$

(4.2)

Calculation of $\zeta_R$ for a radiation-dust mixture is straightforward, because $S = 3 (\zeta_M - \zeta_R)$. On the other hand, $\zeta$ is the weighted average of $\zeta_R$ and $\zeta_M$ i.e.

$$
\zeta = 4 \frac{y}{3y + 4} \zeta_R + 3 \frac{y}{3y + 4} \zeta_M.
$$

(4.3)

Thus

$$
\zeta_R = \zeta - \frac{y}{3y + 4} S.
$$

(4.4)

Moreover, we have

$$
\zeta = \mathcal{R} + \frac{1}{3 (\mathcal{H}^2 - \dot{\mathcal{H}})} \nabla^2 \Psi.
$$

(4.5)

Consequently

$$
\zeta_R = \mathcal{R} + \frac{4y^2}{3 \mathcal{H}_{eq}^2 (3y + 4)} \nabla^2 \Psi - \frac{y}{3y + 4} S.
$$

(4.6)
Figure 3. Evolution of the adiabatic spectral index (spectral index of $R_q$) in a universe constructed from dust and radiation for the comoving wave number $q = 10^5$ i.e. sub-horizon scales (up) and $q = 10^{-10}$ namely severe super-horizon modes (down) providing the adiabatic initial condition. We suppose $n_{s0} = 0.96$ and $N$, the amplitude of $R_q$ at the end of inflation is roughly $10^{-5}$.

Returning equation (4.6) to equation (4.2) we find that

$$
\left[ \frac{\Delta T (\hat{n})}{T_0} \right]_{\text{S.W.}} = \mathcal{R} (t_L, \hat{n}r_L) + 2\Psi (t_L, \hat{n}r_L) - \frac{y_L}{3y_L + 4} S (t_L, \hat{n}r_L).
$$

Note that we omitted the term $\nabla^2 \Psi$ in equation (4.7), because in the ordinary Sachs-Wolfe effect the small scales have subdominant contribution. In the case of adiabatic initial condition equation (4.7) reduces to

$$
\left[ \frac{\Delta T (\hat{n})}{T_0} \right]_{\text{S.W.}}^{\text{Ad}} = \mathcal{R} (t_L, \hat{n}r_L) + 2\Psi (t_L, \hat{n}r_L).
$$

On the other hand,

$$
\frac{2}{3 (\omega + 1)} \mathcal{H} \Psi' + \frac{3\omega + 5}{3 (\omega + 1)} \Psi = -\mathcal{R}.
$$

or

$$
\Psi^* + \frac{5y + 6}{2y (y + 1)} \Psi = -\frac{3y + 4}{2y (y + 1)} \mathcal{R}.
$$
Figure 4. The same as figure 3, except the initial condition is supposed to be isocurvature. We assumed the amplitude of $S_q$ at the end of inflation is about $10^{-5}$ and $n_{iso_0} = 4.43$. Notice that $n_s$ seems to be constant for $y \gtrsim 4$ even under isocurvature initial condition.

Here $R$ is independent of $y$, so equation (4.10) yields

$$\Psi = \frac{-9y^3 - 2y^2 + 8y + 16 - 16\sqrt{y + 1}}{15y^3} R.$$  \hspace{1cm} (4.11)

Notice that

$$\lim_{y \to 0} \Psi = \frac{-2}{3} R, \quad \lim_{y \to \infty} \Psi = \frac{-3}{5} R.$$  

which is expected for the pure radiation and pure dust respectively. By substituting equation (4.11) into equation (4.8) we find that

$$\left[ \frac{\Delta T}{T_0} \right]^{Ad}_{S.W.} = \left[ 2 - \frac{15y_L^2}{9y_L^3 + 2y_L^2 - 8y_L - 16 + 16\sqrt{y_L + 1}} \right] \Psi (t_L, \vec{n}_rL).$$  \hspace{1cm} (4.12)

In addition, we have

$$y_L = \frac{1 + z_{eq}}{1 + z_L} = \frac{1 + 3263}{1 + 1091} = 2.98.$$  \hspace{1cm} (4.13)
Figure 5. Evolution of the isocurvature spectral index (spectral index of $S_q$) in a universe constructed from dust and radiation for the comoving wave number $q = 10^5$ i.e. sub-horizon scales (up), $q = 10^{-5}$ (middle) and $q = 10^{-10}$ (down) namely super-horizon modes providing the adiabatic initial condition. We suppose $n_{s0} = 0.96$ and $N$, the amplitude of $R_q$ at the end of inflation is roughly $10^{-5}$. It is clear that $n_{iso}$ for the adiabatic initial condition is roughly constant but only for severe super-Hubble scales for which $q \ll \mathcal{H}$ and $y \gtrsim 2$.

So

$$\left[ \Delta T \left( \hat{n} \right) \right]_{Ad}^{S,W.} = 0.4 \Psi \left( t_L, \hat{n} r_L \right).$$

(4.14)
Figure 6. Like figure 5; however it is applied to isocurvature initial condition. We supposed the amplitude of $S_q$ at the end of inflation is about $10^{-5}$ and $n_{isqo} = 4.43$. It is clear that $n_{iso}$ for the severe super-Hubble scales and isocurvature initial condition is constant.

So the fudge factor of Sachs-Wolfe effect should be substituted by 0.4 which is greater than the ordinary 1/3 factor [29–31].

Finally, let’s turn to the isocurvature initial condition. In this case we have $R = \frac{y}{3y + 4} S$, so

$$\left[ \frac{\Delta T(n)}{T_0} \right]_{S,W.}^{iso} = 2\Psi(t_L, \tilde{n}r_L),$$

(4.15)
which totally coincides with the previous results [29, 30].

5 Discussion

We have derived a neat equation for the evolution of comoving curvature perturbation and then have found its solutions for some simple cases. It was shown that the Mukhanov-Sasaki equation is a special case of this equation. We also found its numerical solution for the radiation and matter mixture under two different adiabatic and isocurvature initial conditions. As we have seen, in this case the equation cannot be solved alone due to the presence of another unidentified quantity: entropy perturbation. So we coupled the equation to the Kodama-Sasaki equation. We also investigated the time evolution of the adiabatic and isocurvature spectral indices for both initial conditions separately. We showed that not only the curvature spectral index for adiabatic initial condition and severe super-Hubble scales is constant, but also the entropy spectral index for isocurvature initial condition and severe super-Hubble scales is also constant. It seems \( n_{\text{iso}} \) for \( y \gtrsim 2 \) and severe super-Hubble scales is approximately flat regardless of the initial condition. Moreover, we found that \( n_s \) is roughly constant for \( y \gtrsim 4 \) regardless of the scale of perturbations even though the isocurvature initial condition has been chosen. Finally, we re-investigated the ordinary Sachs-Wolfe effect and showed that the factor \( \frac{1}{3} \) must be increased to 0.4 by considering a more real situation. This is consistent with White and Hu pedagogical derivation of Sachs-Wolfe effect.

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