Analytical solution for multi-singular vortex Gaussian beams: the mathematical theory of scattering modes

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Abstract

We present a novel procedure for solving the Schrödinger equation, which in optics is the paraxial wave equation, with an initial multisingular vortex Gaussian beam. This initial condition has a number of singularities in a plane transversal to propagation embedded in a Gaussian beam. We use scattering modes, which are solutions to the paraxial wave equation that can be combined straightforwardly to express the initial condition and therefore allow the problem to be solved. To construct the scattering modes one needs to obtain a particular set of polynomials, which play an analogous role to Laguerre polynomials for Laguerre–Gaussian modes. We demonstrate here the recurrence relations needed to determine these polynomials. To stress the utility and strength of the method we solve first the problem of an initial Gaussian beam with two positive singularities and a negative one embedded in it. We show that the solution permits one to obtain analytical expressions. These can used to obtain mathematical expressions for meaningful quantities, such as the distance at which the positive and negative singularities merge, closing the loop of a vortex line. Furthermore, we present an example of the calculation of an specific discrete-Gauss state, which is the solution of the diffraction of a Laguerre–Gauss state showing definite angular momentum (that is, a highly charged vortex) by a thin diffractive element showing certain discrete symmetry. We show that this problem is therefore solved in a much simpler way than by using the previous procedure based on the integral Fresnel diffraction method.

Keywords: multisingular vortex beams, vortex line, discrete Gaussian beams, quantum knots, electron vortex beams, singular optics, laser speckle

(Some figures may appear in colour only in the online journal)

1. Introduction

The paraxial scalar wave equation for an optical field, which is formally a two-dimensional (2D) linear Schrödinger equation, admits solutions with quantized orbital angular momentum (OAM) [1–5]. These solutions present an on-axis phase singularity in the 2D transversal plane, which is a zero of the intensity of the complex scalar wave, with an undefined phase. When considering a small circuit around a phase singularity, the phase increases in an integer multiple of $2\pi$. This integer is known as the topological charge. The Laguerre–Gauss (LG) modes are the mathematical solutions of such a paraxial equation in free space showing OAM and, therefore, an on-axis phase singularity. They are eigenstates of the OAM operator and, consequently, also of the $O(2)$ continuous rotation group operator [6], and are usually termed as optical vortices.

However, the paraxial wave equation admits solutions with more intricate phase profiles, such as solutions showing many singularities. The whole trajectory of a phase
singularity along the evolution variable is a vortex line, which can have very different geometries when considered as independent entities, showing that the singularities present non-trivial dynamics. Many such solutions, built by combining LG modes, have been reported [7–18]. For example, solutions with an intricate vortex line structure, forming knots and loops, can be obtained by superposition of LG modes [19–22]. An specific formalism for the propagation of multi-singular vortex beams, also named polynomial Gaussian beams, was presented in [23, 24].

Here, we present an easy, general and systematic procedure for constructing the solution to the paraxial wave equation with an initial condition showing any multi-singular structure, that is, a multi-singular vortex beam. The key point is to express the initial condition, showing a combination of singularities embedded in a Gaussian beam, in terms of the so-called scattering modes introduced in [25]. The scattering modes are solutions to the paraxial wave equation in free space that allow us to solve this equation straightforwardly.

We offer two examples to illustrate the procedure: the first one exemplifies the procedure for a general initial condition with a symmetric or non-symmetric structure of singularities. To this end, we detail the particular case of an initial condition with two single, positively charged off-axis singularities and a negative charge located in the axis. We show that the expressions that solve the problem along the whole evolution allows us to obtain analytical expressions for some meaningful quantities associated with the intricate form of the corresponding vortex lines. For example, we show that it is possible to obtain an expression for the value of the distance at which the singularities merge, that is, the distance at which the vortex line loop closes. This can be obtained when initially the two off-axis singularities are at the same distance from the origin. This procedure represents a new strategy for investigating vortex knots and loops in singular optics with analytical tools. In the second example we consider an initial condition showing a definite discrete rotational symmetry. This can be obtained as a diffraction of an LG mode showing definite angular momentum (i.e. a highly charged vortex) using a thin diffractive element showing certain discrete symmetry, as described in [25, 26]. The particular generation of these solutions—termed discrete Gaussian beams—out of LG modes can show the phenomenon known as vortex transmutation, that is the inversion of topological charge as a consequence of discrete symmetry [27–32].

The procedure introduced in this paper is of interest in many different problems where phase singularities play an important role. In particular, in the study of optical waves showing phase singularities, which is a large field in optics, called singular optics or, in the presence of nonlinearity, nonlinear singular optics [33, 34]. In three dimensions, a random light field is known as a laser speckle, where the vortex lines form complicated tangles and/or knots [35]. This can be treated mathematically with superpositions of planes waves [36, 37]. Alternatively, they can also be analyzed by means of the basis of Laguerre–Gauss modes [38]. In this sense the procedure introduced here should be considered an alternative to these approaches. One of the interesting fields in which we expect the procedure introduced here to be fruitful is the study of optical knots in light beams [39–43]. Also, it is of applicability in superfluids, particularly in Bose–Einstein condensates (BEC), as the system is described by the Gross–Pitaevskii equation which is formally identical to the nonlinear Schrödinger equation. The recent realization of quantum knots in a BEC [44] makes this direction a very interesting avenue to explore. In many other systems the concept of knots is of crucial importance and we believe that our method can be of great interest (see, e.g., examples in nematic colloids or water [45, 46]). In addition, the direct experimental creation of electron-vortex beams carrying OAM proven in recent years shows the feasibility of also using our analytical tools in the context of electron quantum mechanics [47].

The paper is organized as follows. In section 2 we introduce the scattering modes and describe the general procedure to use them to solve the paraxial wave equation. In section 3 we discuss the F-polynomials that are necessary to construct the scattering modes, playing indeed an analogous role to Laguerre polynomials for LG states. In section 4 we demonstrate the recurrence relations necessary to construct these polynomials, and therefore, any scattering mode. Sections 5 and 6 deal with the two aforementioned examples. We end this article by offering our conclusions in section 7.

2. Scattering modes

Scattering modes are solutions of the paraxial diffraction equation

\[ -2ik_0 \frac{\partial \phi}{\partial z} + \nabla_z^2 \phi = 0, \]  

(1)

where \( \nabla_z \equiv (\partial/\partial x, \partial/\partial y) \) is the transverse gradient operator and \( k_0 \) is the light wavenumber, verifying the following initial condition at \( z = 0 \):

\[ \Phi_{\psi}(r, \theta, 0) = r^{|l|+2p} \exp(i\ell \phi) \exp \left[ -\frac{k_0 r^2}{2z_R} \right], \]  

(2)

where \( z_R \) is the Rayleigh length. The radial exponential in (2) is nothing but \( \phi_{00}(x, y, 0) \), the fundamental Laguerre–Gauss mode evaluated at \( z = 0 \). In the complex variables \( w = x + iy \) and \( \bar{\pi} = x - iy \) this initial condition has the form:

\[ \Phi_{\psi}(w, \bar{\pi}, 0) = w^{|l|}|w|^{2p} \exp \left[ -\frac{k_0 |w|^2}{2z_R} \right] l \geq 0, \]  

\[ \Phi_{\psi}(w, \bar{\pi}, 0) = \bar{\pi}^{|l|}|w|^{2p} \exp \left[ -\frac{k_0 |w|^2}{2z_R} \right] l < 0. \]  

(3)

Let us briefly discuss how to construct solutions of equation (1) with initial condition (2). Let \( \psi \) and \( \bar{\pi} \) be complex position operators with associated momenta \( \hat{p} = -i\partial/\partial w \) and \( \hat{\bar{p}} = -i\partial/\partial \bar{\pi} \). These operators obey standard commutation relations \([\psi, \hat{p}] = [\bar{\pi}, \hat{\bar{p}}] = i \). They allow us to write the Hamiltonian associated with equation (1) as \( \hat{H} = \hat{p} \hat{\bar{p}} \). Then, the evolution operator,
\[ \hat{U}(z) = \exp[i(2\pi/k_0)\hat{H}], \]
fulfills
\[ [\hat{w}, \hat{U}(z)] = -(2\pi/k_0)\hat{p} \hat{U}(z) \]
and
\[ [\hat{\pi}, \hat{U}(z)] = -(2\pi/k_0)\hat{p} \hat{U}(z). \]

Let us define the operators
\[ \hat{I}_+(z) \equiv w - \frac{2\pi}{k_0} \frac{\partial}{\partial \pi} \]
and
\[ \hat{I}_-(z) \equiv \pi - \frac{2\pi}{k_0} \frac{\partial}{\partial w}. \]

Hence, by using the commutation relations (4), the solution of equation (1) is obtained by replacing \( w \) and \( \pi \) in equation (3) by \( \hat{I}_+ \) and \( \hat{I}_- \), respectively. One obtains
\[ \Phi_+(w, \pi, z) = \hat{I}_+^{(l)+} \hat{I}_+^{(l)} \phi_{00}(w, \pi, z) \quad l \geq 0 \]
\[ \Phi_-(w, \pi, z) = \hat{I}_-^{(l)+} \hat{I}_-^{(l)} \phi_{00}(w, \pi, z) \quad l < 0 \]
where the explicit expression for \( \phi_{00} \) in complex coordinates is
\[ \phi_{00}(w^2, z) = \left( \frac{i z_R}{q(z)} \right) \exp \left( -\frac{i n w \pi}{q(z)} \right). \]

Note that \( \phi_{00} \) is the fundamental Laguerre–Gauss mode which solves equation (1) [6]. The solution (7) can be written as
\[ \Phi_+(w, \pi, z) = \hat{I}_+^{(l)+} \hat{I}_+^{(l)} \phi_{00}(w, \pi, z) \]
\[ = \hat{I}_+^{(l)+} \hat{I}_+^{(l)} \phi_{00}(w, \pi, z), \]
where we have introduced the ‘diagonal’ operator \( \hat{\Delta} \equiv \hat{I}_+ \hat{I}_- \). We anticipate here, and justify below, that \( \hat{I}_+ \) and \( \hat{I}_- \) are the raising and lowering operators for the angular momentum quantum number \( l \), respectively, while \( \hat{\Delta} \) is the raising operator for the radial quantum number \( p \). Repeated application of these operators to \( \phi_{00} \) generates a solution with the corresponding values of \( l \) and \( p \).

Alternatively, one can write equations (7) as
\[ \Phi_{\pi}(w, \pi, z) = \hat{I}_+^{(l)+} \hat{I}_+^{(l)} \phi_{00}(w, \pi, z), \]
\[ = \left( w - \frac{2\pi}{k_0} \frac{\partial}{\partial \pi} \right)^n \left( \pi - \frac{2\pi}{k_0} \frac{\partial}{\partial w} \right)^n \phi_{00}(w^2, z), \]
where the angular momentum carried by the scattering mode \( \phi_{\pi} \) is given by
\[ l = n - \pi \]
whereas
\[ p = \min(n, \pi). \]

The form of solution (9) is that of the scattering mode introduced in [25]. The scattering mode \( \phi_{\pi} \) is obtained by applying \( n \) times the \( \hat{I}_+ \) operator and \( \pi \) times the \( \hat{I}_- \) one onto the fundamental LG mode \( \phi_{00} \).

In the following two sections we will justify that a general closed expression—valid for any \( z \)—for the scattering mode \( \Phi_\pi \) can be obtained. After introducing the Gaussian beam parameter \( q(z) \equiv z + iz_R \), this expression is
\[ \Phi_\pi(r, \theta, z) = \left( \frac{iz_R}{q(z)} \right)^{|l|+1} \left( \frac{2z^2 z_R}{k_0 q(z)} \right)^p F_p[\gamma(z)|r^2] \exp(i\theta) \]
\[ \times \exp \left[ -\frac{1}{2} \frac{k_0}{q(z)} r^2 \right] \]
where \( \gamma(z) = (k_0/2)z_R [z_R(z)]^{-1} \) and \( F_p[\gamma(z)|r^2] \) is a polynomial of \( p \)th order. The expression for the scattering mode \( \Phi_\pi \) is therefore fully determined by the polynomial of \( p \)th order \( F_p[\gamma(z)] \). Explicit expressions for these polynomials can be obtained using the recurrence relations detailed in section 3. Scattering modes also admit a representation in terms of the complex variables \( w \) and \( \pi \). For \( l \geq 0 \),
\[ \Phi_+(w, \pi, z) = \Phi_+^{(l)+} \Phi_+^{(l)} \phi_{00}(w, \pi, z) \]
\[ = \left( w - \frac{2\pi}{k_0} \frac{\partial}{\partial \pi} \right)^n \left( \pi - \frac{2\pi}{k_0} \frac{\partial}{\partial w} \right)^n \phi_{00}(w^2, z), \]
\[ \times \exp \left( -\frac{1}{2} \frac{k_0}{q(z)} w^2 \right) \]
and for \( l < 0 \)
\[ \Phi_-(w, \pi, z) = \Phi_-^{(l)+} \Phi_-^{(l)} \phi_{00}(w, \pi, z) \]
\[ = \left( w - \frac{2\pi}{k_0} \frac{\partial}{\partial \pi} \right)^n \left( \pi - \frac{2\pi}{k_0} \frac{\partial}{\partial w} \right)^n \phi_{00}(w^2, z), \]
\[ \times \exp \left( -\frac{1}{2} \frac{k_0}{q(z)} \pi^2 \right) \]

Recurrence relations for \( F \)-polynomials are demonstrated from the definition of scattering modes in terms of the differential operators in the complex plane.

Scattering modes have a particular simple form at \( z = 0 \) when using the complex coordinates \( w \) and \( \pi \):
\[ \Phi_\pi(w, \pi, 0) = w^{n\pi^2} \phi_{00}(w^2, 0). \]

This property provides a simple method for calculating the diffracted field of any field whose expression at \( z = 0 \) can be given as a product of a series (finite or infinite) in powers of \( w \) and \( \pi \) times a Gaussian. Let \( \phi \) be such a field. Then
\[ \phi(w, \pi, 0) = t(w, \pi) \phi_{00}(w, \pi, 0) \]
\[ = \sum_{n,\pi} t(n, \pi) \phi_{n\pi}(w, \pi, 0). \]

According to [25], the value of the field at arbitrary \( z \) is obtained by the simple substitution rule: \( w \rightarrow \hat{I}_+(z) \) and \( \pi \rightarrow \hat{I}_-(z) \). After substitution in (16), we immediately recognize that the term in brackets in this expression becomes nothing but the scattering mode \( \phi_{n\pi} \). Thus, the coefficients of the expansion of the function \( t \) at \( z = 0 \) are also the coefficients of the diffracted field in terms of the scattering
modes valid in the entire space:
\[
\phi(w, \nu, z) = \sum_{n, \nu} f_{n \nu} \Phi_{n \nu}(w, \nu, z).
\] (17)

Or, alternatively, using \( l \) and \( p \) as ‘quantum numbers’
\[
\phi(w, \nu, z) = \sum_{p} f_{p} \Phi_{p}(w, \nu, z),
\]
in which \( l \) and \( p \) are calculated using equations (10) and (11).

3. Recurrence relations for \( F \)-polynomials

The \( F \)-polynomials in equations (12)–(14) play an analogous role to Laguerre polynomials for LG states. The initial condition (2) is, however, different from that fulfilled for LG states in the \( p = 0 \) case. Therefore, the scattering mode \( \Phi_{p} \) with \( p = 0 \) is necessarily different from any LG mode and so is its expression (12). This includes the \( F \)-polynomials, which are not Laguerre polynomials. However, \( F \)-polynomials can be similarly provided by specific recurrence relations as are Laguerre ones.

We distinguish between fundamental \( F \)-polynomials, characterized by \( l = 0 \) and thus described exclusively by the \( p \) index, and generalized \( F \)-polynomials, for which \( l \neq 0 \). We denote a polynomial of the former type as \( F_{p}(x) \equiv F_{p}^{0}(x) \), whereas we reserve the full notation \( F_{p}(x) \) for the generalized form of the polynomial.

3.1. Recurrence relation for fundamental \( F \)-polynomials

The fundamental \( F \)-polynomials satisfy the following recurrence set of differential equations:
\[
F_{p+1}(x) = (1 - x)F_{p}(x) - (1 - 2x) \frac{dF_{p}(x)}{dx} - \lambda \frac{d^{2}F_{p}(x)}{dx^2}, \quad p = 0, 1, 2, \ldots
\] (18)

with
\[
F_{0}(x) = 1,
\]
which allows us to solve the recurrence set (18) univocally.

An equivalent construction of the fundamental \( F \)-polynomials can be obtained by writing the polynomials in an explicit manner:
\[
F_{p}(x) = \sum_{j=0}^{p} c_{j}^{p} x^{j}.
\]

In this way, the set of differential equations (18) turns into the following system of algebraic recurrence relations for the coefficients \( c_{j}^{p} \):
\[
c_{j}^{p+1} = (1 + 2j) c_{j}^{p} - (1 + j)^{2} c_{j+1}^{p} - c_{j-1}^{p},
\] (19)

together with the conditions
\[
c_{j}^{p} = 0, \quad j < 0 \text{ or } j > p
\]
\[
c_{p}^{p} = (-1)^{p}, \quad p = 0, 1, 2, \ldots,
\]
which act as initialization conditions for the recurrence chain.

We provide in table 1 the explicit expressions of the lower order fundamental polynomials—up to sixth order—obtained using the previous recurrence relations.

| \( F_{p}(x) \) | \( F_{2}(x) \) | \( F_{3}(x) \) | \( F_{4}(x) \) | \( F_{5}(x) \) | \( F_{6}(x) \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| \( F_{0}(x) = 1 \) | \( F_{2}(x) = 2 - 4x + x^{2} \) | \( F_{3}(x) = 6 - 18x + 9x^{2} - x^{3} \) | \( F_{4}(x) = 24 - 96x + 72x^{2} - 16x^{3} + x^{4} \) | \( F_{5}(x) = 120 - 600x + 600x^{2} - 200x^{3} + 25x^{4} - x^{5} \) | \( F_{6}(x) = 720 - 4320x + 5400x^{2} - 2400x^{3} + 450x^{4} - 36x^{5} + x^{6} \) |

3.2. Recurrence relation for generalized \( F \)-polynomials

The generalized \( F \)-polynomials \( F_{p}^{l}(x) \) fulfill the following recurrence set of differential equations:
\[
F_{p}^{l+1}(x) = F_{p}^{l}(x) - \frac{dF_{p}^{l}(x)}{dx}, \quad |l| = 0, 1, 2, \ldots,
\]

which together with the initial condition
\[
F_{p}^{0}(x) = F_{p}(x),
\]
allow us to obtain recursively the generalized \( F \)-polynomial of order \( p \) and angular momentum \( |l| \), \( F_{p}^{l}(x) \) from the corresponding fundamental \( F \)-polynomial \( F_{p}(x) \) previously determined by means of the recurrence relations (18). It is simple to deduce that \( F_{p}^{0}(x) = F_{p}(x) = 1 \).

An equivalent recurrence relation for the coefficients of the generalized \( F \)-polynomial
\[
F_{p}^{l}(x) = \sum_{j=0}^{p} c_{j}^{l|p} x^{j}
\]
is
\[
c_{j}^{l|p+1} = c_{j}^{l|p} - (j + 1)c_{j+1}^{l|p}
\] (20)
with
\[
c_{j}^{0|p} = c_{j}^{p},
\]
c\( j \) being the \( j \)th coefficient of the corresponding fundamental polynomial \( F_{p}(x) \) previously evaluated using the recurrence relations (19). The recurrence relations (20) allow us to obtain eventually all the coefficients \( c_{j}^{l|p} \) out of the fundamental ones \( c_{j}^{0|p} \) in a recursive way.

In table 2 we include the lower order generalized polynomials \( F_{p}^{l}(x) \) for angular momenta \( |l| = 1, 2, \) and \( 3 \).
The operators \( \hat{l}_+ \) and \( \hat{l}_- \) as well as the diagonal operator \( \hat{\Delta} \) can be interpreted as lowering and raising operators for the quantum numbers \( l \) and \( p \). The interpretation is clearer when working in the \( z = 0 \) plane since the expressions for \( \hat{l}_+ \) and \( \hat{\Delta} \) in definition (9) take the simple multiplicative form:

\[
\Phi_{pm}(w, \pi, 0) = w^n \pi^m \phi_{00}(w, \pi, 0),
\]

so that

\[
\hat{l}_+(0) \Phi_{pm}(w, \pi, 0) = w^{n+1} \pi^m \phi_{00}(w, \pi, 0) = \Phi_{n+1,m}(w, \pi, 0).
\]

Analogously,

\[
\hat{l}_-(0) \Phi_{pm}(w, \pi, 0) = \Phi_{n,m+1}(w, \pi, 0).
\]

Therefore, \( \hat{l}_+ \) and \( \hat{\Delta} \) raise the value of the index \( n \) and \( \pi \) by one unit, respectively. If we use ket notation to represent the scattering mode \( \Phi_{pm} \) as \( |SM(n, \pi)\rangle \) this means:

\[
\hat{l}_+ |SM(n, \pi)\rangle = |SM(n + 1, \pi)\rangle \quad \hat{l}_- |SM(n, \pi)\rangle = |SM(n, \pi + 1)\rangle.
\]

In terms of the angular momentum \( l = n - \pi \), it is evident that \( \hat{l}_+ \) increases its value by one unit, whereas \( \hat{l}_- \) decreases it by the same amount. Therefore \( \hat{l}_+ \) and \( \hat{\Delta} \) are the angular momentum raising and lowering operators for scattering modes. It is not difficult to check that the action of \( \hat{l}_+ \) and \( \hat{\Delta} \) on a generic scattering mode written in terms of \( l \) and \( p \) is given by:

\[
\hat{l}_+ |SM(l, p)\rangle = |SM(l + 1, p)\rangle \quad \hat{l}_- |SM(l, p)\rangle = |SM(l - 1, p + 1)\rangle \quad l > 0,
\]

\[
\hat{l}_+ |SM(l, p)\rangle = |SM(l + 1, p + 1)\rangle \quad \hat{l}_- |SM(l, p)\rangle = |SM(l - 1, p)\rangle \quad l \leq 0.
\]

Note that, depending on the sign of \( l \), the action of the angular momentum raising and lowering operators can also affect the \( p \) index. However, the particular combination of these operators given by the diagonal operator \( \hat{\Delta} = \hat{l}_+ \hat{l}_- \) acts systematically as a raising operator for the \( p \) index since an increase in one unit for \( n \) and \( \pi \) implies

\[
p' = \min(n + 1, \pi + 1) = \min(n, \pi) + 1 = p + 1.
\]

Therefore

\[
\hat{\Delta} |SM(l, p)\rangle = |SM(l, p + 1)\rangle.
\]

Expressions (22) and (23) are the key elements for determining the recurrence relations for fundamental and generalized \( F \)-polynomials.

4.2. Derivation of the recurrence relation for fundamental \( F \)-polynomials

Our starting point is equation (23) for the diagonal operator \( \hat{\Delta} \) applied to a scattering mode with \( l = 0 \). In terms of the functions \( \Phi_{0p} \) and \( \Phi_{0p+1} \), this equation reads

\[
\Phi_{0p+1}(w, \pi, z) = \hat{l}_-(z) \hat{l}_+(z) \Phi_{0p}(w, \pi, z)
\]

\[
= \left( w - \frac{z}{\pi} \frac{\partial}{\partial \pi} \right) \left( \pi - \frac{z}{\pi} \frac{\partial}{\partial w} \right) \Phi_{0p}(w, \pi, z).
\]
Let us consider this equation for the lower order scattering modes.

An explicit calculation of the derivatives for \( p = 0 \) provides the following result (taking into account that \( \Phi_{00} = \tilde{\phi}_{00} \)):

\[
\Phi_{00}(w, \bar{w}, z) = \left( \frac{i z R}{q(z)} \right) \left( w - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \left( \bar{w} - i \frac{z}{\pi} \frac{\partial}{\partial \bar{w}} \right) \exp \left( -i \pi w \bar{w} \right) q(z) \]

\[
= \left( \frac{z R}{\pi q(z)} \right) (1 - x) \tilde{\phi}_{00}(|w|^2, z),
\]

where \( x = \pi z R [z q(z)]^{-1} |w|^2 \).

For \( p = 1 \),

\[
\Phi_{0p}(w, \bar{w}, z) = \left( w - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \left( \bar{w} - i \frac{z}{\pi} \frac{\partial}{\partial \bar{w}} \right) \Phi_{00}(w, \bar{w}, z) = \left( \frac{z R}{\pi q(z)} \right)^2 (2 - 4x + x^2) \tilde{\phi}_{00}(|w|^2, z).
\]

Thus, for arbitrary \( p \) we expect the following structure:

\[
\Phi_{0p}(w, \bar{w}, z) = \left( \frac{z R}{\pi q(z)} \right)^p \left( \frac{z R}{\pi q(z)} \right)^{-1} \left( \frac{z R}{\pi q(z)} \right) \cdot \Phi_{00}(w, \bar{w}, z),
\]

where \( F_p(x) \) is a polynomial of order \( p \) in \( x = \gamma(z)|w|^2 \), in which we have defined \( \gamma(z) = \pi z R [z q(z)]^{-1} \).

After substituting the ansatz (25) in equation (24), we obtain an explicit expression for the fundamental F-polynomial of order \( p + 1 \) in terms of \( F_p \) and its derivatives:

\[
F_{p+1}[\gamma(z)|w|^2] = \left( \frac{z R}{\pi q(z)} \right)^{-1} \left[ \tilde{\phi}_{00}(|w|^2, z) \right]^{-1}
\]

\[
\times \left( w - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \left( \bar{w} - i \frac{z}{\pi} \frac{\partial}{\partial \bar{w}} \right) \cdot \Phi_{00}(w, \bar{w}, z).
\]

Developing the differential operators in terms of the \( \partial / \partial w \) and \( \partial / \partial \bar{w} \) derivatives and acting on the product \( F_p \tilde{\phi}_{00} \) provides the following result:

\[
F_{p+1}[\gamma(z)|w|^2] = \left( \frac{z R}{\pi q(z)} \right)^{-1} \left( \frac{z R}{\pi q(z)} \right) \Phi_{00}(w, \bar{w}, z),
\]

\[
\times \left( \gamma(z) - 2 \pi \gamma R \frac{w \bar{w}}{|w|^2} \right) F_p[\gamma(z)|w|^2] - \left( \gamma(z) - 2 \pi \gamma R \frac{w \bar{w}}{|w|^2} \right) F_p[\gamma(z)|w|^2].
\]

After reintroducing the argument of the F-polynomial as \( x = \gamma(z)|w|^2 = \pi z R [z q(z)]^{-1} |w|^2 \), we obtain the desired recurrence relation in terms of the single variable \( x \):

\[
F_{p+1}(x) = (1 - x) F_p(x) - (1 - 2x) \frac{dF_p(x)}{dx} - x \frac{d^2 F_p(x)}{dx^2}.
\]

Since according to equation (25) \( \Phi_{00} = \tilde{\phi}_{00} \), we additionally have that \( F_0(x) = 1 \), which is the initial condition for the recurrence chain. Note how the F-polynomials of the lower order scattering modes we used to set the ansatz (25) are recovered using the previous recurrence relation, thus showing the consistency of the procedure.

### 4.3. Derivation of the recurrence relation for generalized F-polynomials

We start by proving the relation for \( l > 0 \). We consider the first relation in equation (22) relating the scattering mode \( |SM(l + 1, p)\rangle \) with \( |SM(l, p)\rangle \) by means of the angular momentum raising operator \( \hat{I}_+ \). In terms of the functions \( \Phi_{l+1,0} \) and \( \Phi_{00} \), this equation reads:

\[
\Phi_{l+1,0}(w, \bar{w}, z) = \hat{I}_+(z) \Phi_{l,0}(w, \bar{w}, z) = (w - i \frac{z}{\pi} \frac{\partial}{\partial w}) \Phi_{l,0}(w, \bar{w}, z). \quad (26)
\]

On the other hand, the general form of \( \Phi_{lp} \) in terms of \( w \) and \( \bar{w} \) is given by symmetry considerations and by the action of derivatives, as explained in [25]. In this reference, it was shown that a scattering mode with \( l > 0 \) has well-defined angular momentum \( l \) and, thus, it transforms as \( \Phi_{lp} \rightarrow (\exp il\theta) \Phi_{lp} \) under the continuous rotation \( w \rightarrow (\exp il\theta) w \). Therefore, \( \Phi_{lp} \) has to be proportional to \( w^l \) times a function dependent exclusively on the \( O(2) \) invariant \( |w|^2 \). Since both \( \partial / \partial w \) and \( \partial / \partial \bar{w} \) derivatives of any order acting on the fundamental Gaussian mode \( \tilde{\phi}_{00} \) yield terms proportional to \( \tilde{\phi}_{00} \), this \( O(2) \) invariant function has to be also proportional to \( \tilde{\phi}_{00} \). Thus, the general form for the scattering mode in terms of \( w \) and \( \bar{w} \) is given by \( \Phi_{lp} \sim w^l F_p \tilde{\phi}_{00} \), where \( F_p \) is, up to this point, a function (to be determined) dependent on \( |w|^2 \). The normalization factor is, in general, \( z \)-dependent. In the same way as we did for the fundamental F-polynomials, a calculation of the lower order scattering modes using equation (26) helps us to find an ansatz consistent with this symmetry argument. The calculation provides the following results. For \( p = 1 \) and \( l = 0 \):

\[
\Phi_{11}(w, \bar{w}, z) = \hat{I}_+(z) \Phi_{01}(w, \bar{w}, z) = (w - i \frac{z}{\pi} \frac{\partial}{\partial w}) \Phi_{01}(w, \bar{w}, z) = (w - i \frac{z}{\pi} \frac{\partial}{\partial w}) \left( \frac{z R}{\pi q(z)} \right) \tilde{\phi}_{00}(|w|^2, z).
\]

For \( p = 1 \) and \( l = 1 \):

\[
\Phi_{21}(w, \bar{w}, z) = \hat{I}_+(z) \Phi_{11}(w, \bar{w}, z) = (w - i \frac{z}{\pi} \frac{\partial}{\partial w}) \Phi_{11}(w, \bar{w}, z) = w^2 \left( \frac{z R}{\pi q(z)} \right)^2 \left( \frac{z R}{\pi q(z)} \right) \tilde{\phi}_{00}(|w|^2, z).
\]
For $p = 1$ and $l = 2$:

\[
\Phi_{22}(w, \overline{\sigma}, z) = \hat{\Delta}(z)\Phi_{21}(w, \overline{\sigma}, z) = \left( w - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \Phi_{21}(w, \overline{\sigma}, z) = w^2 \left( \frac{i z_R}{q(z)} \right)^2 \left( \frac{z z_R}{\pi q(z)} \right)^2 (12 - 8x + x^2) \phi_{00} \times (|w|^2, z).
\]

This calculation sets the following ansatz for the scattering mode with $l = |l| > 0$ represented by $\phi_{00}$:

\[
\Phi_{|l|p}(w, \overline{\sigma}, z) = w^{|l|} \left( \frac{i z_R}{q(z)} \right)^{|l|} \left( \frac{z z_R}{\pi q(z)} \right)^{|l|} F_{|l|p}(\gamma(z)|w|^2) \phi_{00}(|w|^2, z).
\]

Note that this equation provides a simple reduction to the $l = 0$ case—equation (25)—by naturally assuming that $F_p(x) = F_p^0(x)$.

Now, the equation for the angular momentum raising operator $\hat{L}$, (26) allows us to write an explicit expression for the generalized $F$-polynomial $F_{|l|p}^{|l|+1}$ in terms of $F_{|l|p}$ and its derivative:

\[
F_{|l|p}^{|l|+1}[\gamma(z)|w|^2] = w^{-1} \left( \frac{i z_R}{q(z)} \right)^{-1} \left( \frac{z z_R}{\pi q(z)} \right)^{|l|} F_{|l|p}(\gamma(z)|w|^2) \phi_{00}(|w|^2, z).
\]

Developing the derivative of the $F_{|l|p}^{|l|+1}$ product and introducing $x = \gamma(z)|w|^2 = \pi z_R [z q(z)]^{-1}|w|^2$ in the previous equation provides the following relation:

\[
F_{|l|p}^{|l|+1}(x) = F_{|l|p}(x) - \frac{z q(z)}{\pi z_R} \frac{d F_{|l|p}(x)}{dx} = F_{|l|p}(x) \frac{d F_{|l|p}(x)}{dx}.
\]

An analogous calculation applies to a scattering mode with $l = -|l| < 0$. Now, we use, instead of equation (26), the second relation for the lowering operator $\hat{L}$, in equation (22) valid for $l < 0$

\[
\Phi_{-|l|p}(w, \overline{\sigma}, z) = \hat{L}(z) \Phi_{|l|p}(w, \overline{\sigma}, z) = \left( \overline{\sigma} - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \Phi_{|l|p}(w, \overline{\sigma}, z).
\]

A straightforward calculation of the lower order scattering modes provides the following results. For $p = 1$ and $l = 0$:

\[
\Phi_{-11}(w, \overline{\sigma}, z) = \hat{L}(z) \Phi_{01}(w, \overline{\sigma}, z) = \left( \overline{\sigma} - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \Phi_{01}(w, \overline{\sigma}, z) = \overline{\sigma} \left( \frac{i z_R}{q(z)} \right) \left( \frac{z z_R}{\pi q(z)} \right) (2 - x) \phi_{00}(|w|^2, z).
\]

For $p = 1$ and $l = -1$:

\[
\Phi_{-21}(w, \overline{\sigma}, z) = \hat{L}(z) \Phi_{-11}(w, \overline{\sigma}, z) = \left( \overline{\sigma} - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \Phi_{-11}(w, \overline{\sigma}, z) = \overline{\sigma} \left( \frac{i z_R}{q(z)} \right) \left( \frac{z z_R}{\pi q(z)} \right) (3 - x) \phi_{00}(|w|^2, z).
\]

For $p = 1$ and $l = -2$:

\[
\Phi_{-22}(w, \overline{\sigma}, z) = \hat{L}(z) \Phi_{-21}(w, \overline{\sigma}, z) = \left( \overline{\sigma} - i \frac{z}{\pi} \frac{\partial}{\partial w} \right) \Phi_{-21}(w, \overline{\sigma}, z) = \overline{\sigma} \left( \frac{i z_R}{q(z)} \right) \left( \frac{z z_R}{\pi q(z)} \right) (3 - x) \phi_{00}(|w|^2, z).
\]

We immediately recognize that this structure is identical to that of the $l > 0$ scattering modes with the exception of the dependence on $|w|^2$ instead of on $w|w|^2$. Thus, the general form for $\Phi_{|l|p}$ for $l = -|l| < 0$ is:

\[
\Phi_{-|l|p}(w, \overline{\sigma}, z) = w^{|l|} \left( \frac{i z_R}{q(z)} \right)^{|l|} \left( \frac{z z_R}{\pi q(z)} \right)^{|l|} F_{|l|p}(\gamma(z)|w|^2) \phi_{00}(|w|^2, z).
\]

Thereby $F_{|l|p}^{|l|} = F_{|l|p}^{|l|}$ and the recurrence relation for the generalized $F$-polynomial is also given by equation (28).

Note, as before, that the polynomials of the lower order scattering modes obtained by direct derivation applying the definitions (26) and (29) are identical to those in table 2 obtained using the recurrence relations.

5. Example 1: multi-singular Gaussian beam

Now let us exemplify the procedure of how to use the scattering modes to solve the Schrödinger equation (1) with a general initial condition showing a combination of singularities embedded in a Gaussian beam. That is, we assume that the initial condition is of the form

\[
\phi(w, \overline{\sigma}, 0) = \prod_{i=1}^{N_0} (w - a_i) \prod_{i=1}^{N_0} (\overline{\sigma} - b_i) \phi_{00}(w, \overline{\sigma}, 0).
\]

where $a_i$ ($b_i$) is the location of the $N_0$ ($N_0$) positive (negative) singularities at $z = 0$. We choose as an example the case with two positive singularities out of the origin of the transverse $(x, y)$ plane—$a_1, a_2 \neq (0, 0)$—and one negative singularity at the origin

\[
\phi(w, \overline{\sigma}, 0) = (w - a_1)(w - a_2)\overline{\sigma} \phi_{00}(w, \overline{\sigma}, 0).
\]

This is expanded as

\[
\phi(w, \overline{\sigma}, 0) = |w||w|^2 - (a_1 + a_2)|w|^2 + a_1a_2\overline{\sigma} \phi_{00}.
\]

According to equations (16) and (17), and expressing $(n, \tilde{n})$ as the quantum numbers $(l, p)$ by means of the relations (10)
and (11) one can obtain the solution of equation (1) valid for all \( z \) in terms of the scattering modes as
\[
\phi(w, \varpi, z) = \Phi_{11}(w, \varpi, z) - (a_1 + a_2)\Phi_{01}(w, \varpi, z) + a_1a_2\Phi_{-10}(w, \varpi, z). \tag{32}
\]

Note the following simple rules in the procedure used to obtain solution (32):

1. The coefficients of the scattering modes in the solution valid for all \( z \) are identical to those in the expansion in powers of \( w \) and \( \varpi \) in the initial condition.
2. Every power of \( w \) or \( \varpi \) gives the value of the angular momentum quantum number \( l \) in the corresponding scattering mode.
3. Every power of \( |w|^2 \) gives the value of the radial quantum number \( p \).

The scattering modes \( \Phi_p \) are obtained from equations (13) and (14) upon substitution of the corresponding values of \( (l, p) \). The ones needed to evaluate equation (32) are
\[
\Phi_{11}(w, \varpi, z) = -w\left( \frac{z_R}{q(z)} \right)^2 \left[ 2z_R \frac{2z_R}{k_0q(z)} \right] \times (2 - \gamma(z)|w|^2) \exp\left[-i\frac{k_0|w|^2}{2q(z)} \right], \tag{33}
\]
\[
\Phi_{01}(w, \varpi, z) = \frac{iz_R}{q(z)} \left[ 2z_R \frac{2z_R}{k_0q(z)} \right] \times (1 - \gamma(z)|w|^2) \exp\left[-i\frac{k_0|w|^2}{2q(z)} \right], \tag{34}
\]
\[
\Phi_{-10}(w, \varpi, z) = -\varpi\left( \frac{z_R}{q(z)} \right)^2 \exp\left[-i\frac{k_0|w|^2}{2q(z)} \right]. \tag{35}
\]

where \( q(z) = z + iz_R, \gamma(z) = (k_0/2)z_R|zq(z)|^{-1} \) and we have restored the wavenumber \( k_0 \). For each value of \( z \) one can find analytically the zeros of equation (32), after substituting (33)-(35). That is, one has to solve
\[
\begin{align*}
& w\left( \frac{z_R}{q(z)} \right)^2 [2 - \gamma(z)|w|^2] + i(a_1 + a_2)[1 - \gamma(z)|w|^2] + a_1a_2\varpi \left( \frac{k_0}{2z} \right) = 0.
\end{align*}
\]

This equation completely determines the trajectories of the singularities for all values of \( z \). Let us get a deeper insight by analyzing first a symmetric case, that is, for example, when \( a_1 = -a_2 \) we can get from the previous expression
\[
\begin{align*}
& wz_R[-4z(\z + iz_R) + k_0z_Rw\varpi] + a_1^2k_0\varpi(z + iz_R)^2 = 0, \tag{36}
\end{align*}
\]

where we substituted the expressions for \( q(z) \) and \( \gamma(z) \) and took only its numerator. This has to be solved together with its complex conjugate (we assume \( a_1 \) on the x-axis)
\[
\begin{align*}
& w\z_R[-4\z - iz_R] + k_0\z_Rw\varpi] + a_1^2k_0w(z - iz_R)^2 = 0. \tag{37}
\end{align*}
\]

It is easy to check that, corresponding to the initial condition, there is a root of these equations at \( z = 0 \) at the origin with charge \(-1\) and two positive roots at \( w = \pm a_1 \). We can combine the previous equations and find the \( z \) at which the two positive roots merge at the origin, obtaining that this occurs for
\[
\begin{align*}
& z_m = \pm \frac{a_1^2k_0z_R}{\sqrt{16\z_R^2 - a_1^4k_0^2}}. \tag{38}
\end{align*}
\]

Indeed, as shown in figure 1(a), when the positive singularities are located close to the origin (\( a_1 = 1 \)), the two positive charges located out of the origin merge with the central, negative one, after a short evolution. Beyond this value of \( z_m \) there is only a single positive charge in the origin (the same is valid for negative \( z \)). This merging occurs at
\[
\begin{align*}
& z_m = \pm 1.571 \approx \pi/2, \quad \text{for the parameters used in the figure (} a_1 = 1, z_R = 100 \text{). Note that when } z_R > a_1 \text{ the expression for } z_m \text{ simplifies to approximately } \pm a_1^2(k_0/4), \text{ which equals } \pm a_1^2(\pi/2) \\
& \text{when using } k_0 = 2\pi \text{ normalization, elegantly explaining this behavior. A similar situation occurs when the initial condition fixes the off-axis singularities at a larger distance from the origin (see figures 1(c) and (e)). Note the non-trivial trajectories followed by the off-axis singularities in the interval between the negative and positive values of } z_m. \quad \text{For the cases represented in figures 1(c) and (e) one observes that, close to the merging point, the central negative charge produces two negative ones which move outwards and cancel with the two positive off-axis ones, while leaving a central one with changed sign. We also note that if } a_1^2k_0^2 > 16z_R^2 \text{ no merging occurs as we get an imaginary denominator.}
\end{align*}

Let us now analyze a slightly asymmetric case, that is, \( a_1 + a_2 = \epsilon \). Then we get that
\[
\begin{align*}
& w\left( \frac{2z_R}{k_0q(z)} \right)[2 - \gamma(z)|w|^2] + i\epsilon \left( \frac{2z}{k_0} \right) \times [1 - \gamma(z)|w|^2] + a_1(\epsilon - a_0)\varpi = 0.
\end{align*}
\]

This equation, together with its complex conjugate determine the position of the singularities for all values of \( z \). We have included in the right column of figure 1 the same cases as in the left column, but introducing a small asymmetry. The positive charge located initially closer to the negative central one merges with it at a distance approximately equal to \( z_m \). The other positive one bends and occupies a position close to the origin of the \( x, y \) plane after the merging point. In all cases the total charge is conserved for all values of \( z \) and equals \(+1\).

6. Example 2: discrete-Gauss state

A discrete-Gauss (DG) state is a solution of the paraxial diffraction equation (1), which, in polar coordinates, verifies the initial condition
\[
\phi_{\text{DG}}(r, \theta, 0) = \exp[i2rN\cos(N\theta)]\phi_{\text{LG}}(r, \theta, 0). \tag{38}
\]
\( \phi_{LG} \) being the mathematical expression of the Laguerre–Gauss state \( \ell q \).

Physically, the exponential factor in equation (38) corresponds to the amplitude transmittance function of a diffractive optical element (DOE) located at \( z = 0 \). The form of its transmittance function in complex coordinates (\( w = x + iy = re^{i\theta}, \bar{w} = x - iy = re^{-i\theta} \))

\[
t = \exp\{i2r^N \cos(N\theta)\} = \exp\{iw^N + \bar{w}^N\}
\]

is the one of a DOE owning discrete rotational symmetry of order \( N \). It represents the most general form of \( t \) for a pure discrete-symmetry DOE close to the origin in the absence of lensing effects (no dependence on \( r^2 = w\bar{w} \)).

Explicit analytical expressions can be found for DG states at first order in the \( v \) parameter—known as the deformation parameter [25]. At this order, any DG state can be written as a superposition of three scattering modes (we consider here \( q = 0 \) for the LG state in equation (38)):

\[
\phi_{DG} \sim \Phi_{l0} + iv\Phi_{l+N,0} + iv\Phi_{l-N,\pi} \quad l \geq 0
\]

\[
\phi_{DG} \sim \Phi_{l0} + iv\Phi_{l+N,\pi} + iv\Phi_{l-N,0} \quad l < 0,
\]

where \( N = \min(|l|, N) \). The scattering modes \( \Phi_{lp} \) have the analytical expression (12).

In this section we present an explicit example of the construction of a DG state as a linear combination of scattering modes. This can be considered a particular case of the general formula (39). Nevertheless, in order to appreciate the simplicity of the construction in terms of the scattering modes, it is instructive to provide an explicit example solved from the very beginning. We consider the case of the DG state \( \phi_{DG} \) with \( l = +3 \) and \( N = 4 \). The election of this state is not coincidental since this was the case analytically solved in [26] using a completely different method based on the Fresnel diffraction integral. Moreover, this case was also the one
experimentally demonstrated in [32] showing an excellent agreement with the theoretical results.

From an experimental perspective, the $\phi_{3,4,\varphi}^{DG}$ state is obtained from the LG mode $\phi_{3,0,0}^{LG}$ by acting at its waist (plane $z = 0$) with a pure discrete-symmetry diffractive element owning discrete rotational symmetry of order $N = 4$ [32]. From a mathematical perspective, this operation defines the condition fulfilled by the optical field at $z = 0$, which close to the origin can be written in the form (38). In complex notation and to leading order in the deformation parameter $\varphi$, the field at $z = 0$ is given by:

$$\phi_{3,4,\varphi}^{DG}(w, \varpi, 0) = [1 + iv(w^2 + \varpi^2)]w^3\phi_{00}(|w|^2, 0)$$

$$= [w^3 + ivw^2 + ivw^3\varpi^2]\phi_{00}(|w|^2, 0)$$

$$= \Phi_{30}(w, \varpi, 0) + iv\Phi_{31}(w, \varpi, 0) + iv\Phi_{33}(w, \varpi, 0).$$

We see how the simple observation of the polynomial in brackets directly provides both the quantum numbers $n$ and $\pi$ of the three scattering modes involved and the three components of the linear combination. In fact, the problem has already been automatically solved since the same linear combination provides also the solution for the whole space:

$$\phi_{3,4,\varphi}^{DG}(w, \varpi, z) = \Phi_{30}(w, \varpi, z) + iv\Phi_{31}(w, \varpi, z) + iv\Phi_{33}(w, \varpi, z),$$

where in the last step we have changed the scattering mode representation in terms of the $(n, \pi)$ quantum numbers into the one in terms of the $(l, p)$ numbers, according to the relations (10) and (11). The latter result provides an analytical representation of the DG state using the explicit equations for the scattering modes in complex coordinates (13) and (14) along with the expressions for F-polynomials given in previous sections.

$$\phi_{3,4,\varphi}^{DG}(w, \varpi, z) = [w^3\alpha(z) + ivw^2\beta(z) + iv\pi_3\gamma(z)|w|^2F_3(\gamma|w|^2)]$$

$$\times \exp\left[-\frac{k_0 |w|^2}{2q(z)}\right].$$

In order to compare with the result obtained for the same case in [26], we rewrite the previous expression as:

$$\phi_{3,4,\varphi}^{DG}(w, \varpi, z) = \alpha^8[w^3\alpha^{-4} + ivw^2 + iv\pi_3\beta^2F_3(\gamma|w|^2)]$$

$$\times \exp\left[-\frac{k_0 |w|^2}{2q(z)}\right].$$

$$= \alpha^8[A_0(z)|w|^3 + A_1(w^2) + A_2(|w|^2, z)\varpi]$$

$$\times \exp\left[-\frac{k_0 |w|^2}{2q(z)}\right].$$

(40)

in which

$$A_0(z) = \alpha^{-4} = \frac{q(z)^4}{\zeta_4^4},$$

and

$$A_1(|w|^2, z) = iv\pi_3\gamma^2F_3(\gamma|w|^2)$$

$$= iv\pi_3\gamma^2[24\gamma^{-3} - 36\gamma^{-2}|w|^2 + 12\gamma^{-1}|w|^4 - |w|^6]$$

$$= iv[-24\gamma^3 + 36\gamma^2|w|^2 - 12\gamma^3|w|^4 + |w|^6].$$

where in the last step we have used the identity $\gamma = -\alpha^2/\beta$ and the definition $\varpi \equiv \gamma^{-1}$ used in [26] for comparison purposes. Our final result (40) is identical to that obtained in the aforementioned reference using a completely different method based on the non-trivial calculation of the Fresnel diffraction integral.

7. Conclusions

In this article we have presented the mathematical theory for the systematic and analytical calculation of the so-called scattering modes previously defined in [25]. Additionally, this mathematical construction allows us to provide a simple and efficient procedure to analytically determine discrete-Gauss states, also defined and introduced in the same reference. We have introduced a systematic construction of the so-called fundamental and generalized F-polynomials, which determine completely the analytical form of any scattering mode and, thus, indirectly, of any discrete-Gauss state or of any superposition of scattering modes. Using the concepts of raising and lowering operators we have found the recurrence relations that F-polynomials satisfy. These recurrence relations, both directly for polynomials themselves or for their coefficients, allow us access to analytical expressions for the scattering modes in an analogous way as for standard mode sets, such as Laguerre–Gauss modes. In this sense, F-polynomials play a similar role as Laguerre polynomials do for Laguerre–Gauss modes. As already pointed out in [25], scattering modes constitute a basis for the expansion of any paraxial optical free field. However, they are not as orthogonal nor biorthogonal as Laguerre–Gauss or discrete-Gauss modes. Despite this apparent drawback, scattering modes are an excellent basis for expanding free propagating optical fields
hosting phase singularities embedded in Gaussian envelopes. As shown in the explicit examples provided in this article, knowledge of the singularity structure of a beam, basically its polynomial structure in terms of the complex variables \( w \) and \( \varphi \) at a given propagation plane, determines in a simple and direct form the components of the beam in the scattering mode basis. This linear combination is the solution in the entire space and its construction does not require any type of projection operation. The determination of the whole solution is thus extremely simple and it is obtained in an analytical form with the help of the provided explicit construction of the \( F \)-polynomials. In this direction, in section 5 we provide a neat example of the analytical calculation of a multi-singular vortex Gaussian beam in terms of scattering modes starting exclusively from the knowledge of its multi-singular structure at a given plane. This analytical resolution allows us to obtain closed analytical expressions for meaningful quantities such as the merging axial distances at which vortex-anti vortex loops are created and annihilated. In addition, in section 6 we present an example of calculation of an specific discrete-Gauss state previously obtained using the integral Fresnel diffraction method. Remarkably, here the same problem is solved in a simple and elegant way without the need to resort to any integration or projection operation. The method presented here can help to bring a different perspective to relevant optical problems such as those related to random light fields, as in the phenomenon of laser speckle. In laser speckle fields, phase singularities lines form intricate tangles and/or knots. This context is promising for application to the present procedure since the initial condition of the speckle field is given by a random distribution of singularities, which can be easily modeled by allowing the positions of the singularities, given by the coefficients \( b_i \) and \( a_i \) in equation (31), to be distributed randomly. The main drawback is that the resulting set of equations may not be tractable analytically. The advantage is that it translates the problem of solving the equation in the problem of finding the zeros of a high order polynomial, which can always be computed numerically.

We finally emphasize here that, even though we presented this method in the context of optics, it can be equally used in any field in which it is required to solve the Schrödinger equation with an initial condition formally identical to a multisingular vortex Gaussian beam. This is the case already mentioned of ultra-cold atoms forming a BEC, described by an equation formally identical to the nonlinear Schrödinger equation. In this BEC system, the authors have already discussed how an initially highly charged vortex is annihilated by the action of a discrete symmetry potential, exactly as in the system described in [26], with the main difference being here the presence of an additional parabolic trapping and a Kerr-defocusing nonlinearity [48]. Another alternative application of this method can be found in the framework of electron quantum mechanics. Very recent experiments with electron-vortex beams in interaction with apertures of different discrete symmetries unveil a very rich multisingular electron phase structure completely analogous to that found for light beams in optics [49]. The mathematical mapping between the Schrödinger equation and the paraxial wave equation permits to establish a direct connection between the present formalism and experimental results of this type.

Finally, as an outlook, while in its present form the method cannot be applied to random field statistical mechanics systems, such as in the random \( x-y \) or the Ising model, where the concept of a vortex line is also of fundamental importance [50–52], we believe that future research following the lines discussed here will give valuable insight in this context.

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