LIFTING OF VECTOR-VALUED AUTOMORPHIC FORMS

JITENDRA BAJPAL AND SUBHAM BHAKTA

Abstract. Recently, the first author [1] showed that the admissible vector-valued automorphic forms lift to the admissible ones. In this article, we study the lifts for the logarithmic vector-valued automorphic forms and explicitly compute the Fourier coefficients of the lifted vector-valued automorphic forms.

1. Introduction

Let $G$ be a Fuchsian group of the first kind, and $\rho : G \to \text{GL}_n(\mathbb{C})$ be its any finite dimensional representation. A vector-valued automorphic form of $G$ of weight $k \in 2\mathbb{Z}$ with respect to $\rho$ is a meromorphic function $X : \mathbb{H} \to \mathbb{C}^n$ which has a certain functional and cuspidal behaviour. For details and explanation see Section 3. The theory of vector-valued modular forms has many important applications not only in mathematics but also proven to be useful in physics. For more details on the importance of vector-valued automorphic forms see the introduction of [1, 3].

We call a representation $\rho$ admissible, if $\rho(\gamma)$ is diagonalizable for every parabolic element $\gamma \in G$. In that case, we call the associated vvaf to be an admissible vvaf. Now if we take $H$ to be a finite index subgroup of $G$, then any representation $\rho$ of $H$ can be induced to a representation $\tilde{\rho}$ of $G$. It turns out that, a vvaf $X(\tau)$ of $H$ associated to $\rho$ can also be induced to a vvaf $\tilde{X}(\tau)$ of $G$ associated to $\tilde{\rho}$. In [1] the first author showed that, the induction of an admissible representation is admissible and hence the induced function $\tilde{X}(\tau)$ is also a weakly holomorphic admissible vvaf associated to the induced representation. We are mainly interested in the case when $\rho$ is not necessarily admissible and all eigenvalues of $\rho(\gamma)$ are unitary for every parabolic element $\gamma \in G$. In [2], the authors explained the necessity behind assuming the eigenvalues to be unitary. In the literature, such a representation is called logarithmic representation, and any associated vvaf $X(\tau)$ is called as logarithmic vector-valued automorphic form. Knopp and Mason [4, 5] and Gannon [3] studied this case when $G$ is the modular group. Recently, the authors studied them for any Fuchsian group of the first kind in [2]. In this article, we want to say what happens to logarithmic vector-valued automorphic forms after lifting them.

In this article, a vector-valued meromorphic function on $\mathbb{H}$ is called a weakly holomorphic vector-valued automorphic form (abbreviated as vvaf in what follows) if it satisfies certain functional and cuspidal properties with respect to associated representation.

Date: March 29, 2022.

2010 Mathematics Subject Classification. 11F03, 11F55, 30F35.

Key words and phrases. Fuchsian groups, automorphic forms, Fourier coefficients.
Theorem 1.1. Let $G \subseteq \text{PSL}_2(\mathbb{R})$ be any Fuchsian group of the first kind, and $H$ be any finite index subgroup of $G$. Let $G'$ and $H'$ be respectively the preimages of $G$ and $H$ in $\text{SL}_2(\mathbb{R})$, $\rho : H' \rightarrow \text{GL}_m(\mathbb{C})$ be any finite dimensional representation, and $X(\tau)$ be a weakly holomorphic vvf for $H'$ associated to $\rho$. Then we have the following.

(a) Let $\epsilon$ be an arbitrary cusp of $G$ and \{ $c_i | 1 \leq i \leq n_c$ \} be a set of inequivalent cusps of $H$ for which

$$G \cdot \epsilon = \bigcup_{i=1}^{n_c} H \cdot c_i.$$ 

If $\rho(t_i)$ is not diagonalizable for some $i$, then $\rho'(t_i)$ is not diagonalizable, where $t_i$ and $t_{c_i}$ are the generators of the stabilizer subgroups of the cusp $c_i$ in $H$ and $\epsilon$ in $G$ respectively.

(b) Let $X(\tau)$ be a weakly holomorphic logarithmic vvf associated to $\rho$. Then the lifted form $\tilde{X}(\tau)$ is a weakly holomorphic logarithmic vvf, associated to the induced representation $\tilde{\rho}$ of same weight as $X(\tau)$.

The first author [1] showed that if all of the $\rho(t_i)$ are diagonalizable, then $\tilde{\rho}(t_i)$ is also diagonalizable. In part (a) of the theorem above, we are showing that the converse is true. For any even integer $k$, it has been shown in [1, Page 8] that there exists a bijection between the set all admissible vvf of weight $k$ associated to $\rho$ and the set of all admissible vvf associated to $\tilde{\rho}$ of weight $k$. In this article, we shall define this isomorphism as a lift, and work with this. Furthermore, we also have an analogous isomorphism for the logarithmic vector-valued automorphic forms, and we note this in Proposition 5.4.

As a consequence of the lifting result for admissible vector-valued automorphic forms, the first author showed the existence of weakly holomorphic admissible vector-valued automorphic forms. In the same spirit, we shall discuss about the existence of weakly holomorphic logarithmic vector-valued automorphic forms in the form of Proposition 6.3 and Corollary 6.4. There are some technical issues arising in this process, and we address them in Sections 5 and 6.

In the second half of the article, we study the relations between the Fourier coefficients of $X(\tau)$ and its lifted vvf $\tilde{X}(\tau)$. To get a Fourier expansion of $X(\tau)$ at the cusps of $H$, we need to assume that $\rho(\gamma)$ has only unitary eigenvalues for every parabolic element $\gamma$ in $H'$. In particular, to have Fourier expansion of $\tilde{X}(\tau)$ at the cusps of $G$, we need to ensure that $\tilde{\rho}(\gamma)$ has only unitary eigenvalues for every parabolic element $\gamma$ of $G'$. This is indeed true, and proved by the authors in [2, Sect. 6]. Moreover, the first author gave an explicit description of them in the case of admissible vector-valued automorphic forms. Here we do the same for logarithmic vector-valued automorphic forms, and as a consequence, we deduce the following.

Theorem 1.2. Let $H, G$ and $\rho$ be as in Theorem 1.1. Let $X(\tau)$ be a weakly holomorphic vvf associated to $(H, \rho)$ of weight 0 and $\tilde{X}(\tau)$ be the lifted form associated to $(G, \tilde{\rho})$. Then there exists a family of weakly holomorphic vvf $\{X^{(i)}(\tau) | 1 \leq i \leq d\}$
associated to some $SL_2(\mathbb{R})$ conjugates of $(H, \rho)$ with the following property: let $\mathcal{X}^{(i)}[n]$ to be the $n$th Fourier coefficient of $\mathcal{X}(i)(\tau)$, and $\mathcal{X}[n]$ be the the $n$th Fourier coefficient associated to the lifted form $\mathcal{X}(\tau)$. Then there exists a set of integers $\{m_i, n_i \mid 1 \leq i \leq d\}$, and a set of rationals $\{r_i \mid 1 \leq i \leq d\}$ such that

$$\mathcal{X}[n] = \left(r_i \mathcal{X}^{(i)}[m_in + n_i]\right)_{1 \leq i \leq d}.$$  

1.1. Overview of the article. In Section 2, we recall the basic notions and discuss some key facts from this vast area. In Section 3, we discuss some properties of vaf in the settings of both admissible and logarithmic. The two sections are kind of expository, and serves as preliminary for our work. In Section 4 we recall the results of [1] along with the techniques involved, and in Section 5, we use it to study the lift of logarithmic vector-valued automorphic forms. In Section 6 we discuss about their existence and some examples. In the end, we compute their Fourier coefficients in Section 7.

1.2. Notations. We denote $SL_2(\mathbb{R})$ to be the group of all $2 \times 2$ real matrices with determinant one, and $PSL_2(\mathbb{R})$ to be $SL_2(\mathbb{R})/\{\pm I\}$. We shall write $\tau = x + iy$ a point in $\mathbb{H}$, and $\text{im}(\tau)$ be the corresponding imaginary part. For any unitary $\lambda$, we always denote $\mu(\lambda)$ to be the unique real number such that $\lambda = \exp(2\pi i \mu(\lambda))$ and $0 \leq \mu(\lambda) < 1$. Every constant appearing in the draft depends on the Fuchsian group, unless otherwise specified. We write $f \ll g$ for $|f| \leq c|g|$ where $c$ is a constant irrespective of the domains of $f$ and $g$. Given a vector $v := (v_1, v_2, \cdots, v_n)$ in $\mathbb{C}^n$, we shall always denote $v^t$ to be its transpose, that is the column vector $\left( \frac{v_1}{v_2} \cdots \frac{v_n}{v_n} \right)$.

2. Fuchsian groups

Let $\mathbb{H}$ be the upper half plane and $\mathbb{H}^+ = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ be the extended upper half plane. It is well known that, $PSL_2(\mathbb{R})$ acts on $\mathbb{H}^+$ (widely known as the Möbius action). For any $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL_2(\mathbb{R})$, the action of $\gamma$ is defined as follows:

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$  

A Fuchsian group is a discrete subgroup $G$ of $PSL_2(\mathbb{R})$ for which $G \setminus \mathbb{H}$ is a Riemann surface with finitely many punctures. Because of the discreteness, there exists a fundamental domain for the action on $\mathbb{H}$. When it has finite area, the corresponding Fuchsian group is of the first kind.

An element of $\gamma \in PSL_2(\mathbb{R})$ is called a parabolic element, if $|\text{tr}(\gamma)| = 2$. A point $\tau \in \mathbb{H}^+$ is said to be a fixed point of $\gamma \in PSL_2(\mathbb{R})$ if $\gamma \cdot \tau = \tau$. If $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ is a parabolic element then its fixed point $\tau = \frac{b}{c}$ when $a + d = \pm 2$ and $c \neq 0$, in addition $\tau = \infty$ when $c = 0$.

Let $G$ be a subgroup of $PSL_2(\mathbb{R})$. A point $\tau \in \mathbb{H}^+$ is called a cusp of $G$ if it is fixed by some nontrivial parabolic element of $G$. Let $C_G$ denote the set of all cusps of $G$ and we define $\mathbb{H}^*_G = \mathbb{H} \cup C_G$ to be the extended upper half plane of
G. For example: if $G = \text{PSL}_2(\mathbb{R})$ then $\mathcal{E}_G = \mathbb{R} \cup \{\infty\}$ and if $G = \text{PSL}_2(\mathbb{Z})$ then $\mathcal{E}_G = \mathbb{Q} \cup \{\infty\}$, is the $G$-orbit of cusp $\infty$.

For any $\tau \in \mathbb{H}_G$, let $G_\tau = \{\gamma \in G | \gamma \cdot \tau = \tau\}$ be the stabilizer subgroup of $\tau$ in $G$. For any $c \in \mathcal{E}_G$, $G_c$ is an infinite order cyclic subgroup of $G$. If $c = \infty$ then $G_\infty$ is generated by $t_\infty = \begin{pmatrix} 1 & h_\infty \\ 0 & 1 \end{pmatrix} = t^{h_\infty}$ for a unique real number $h_\infty > 0$ called the cusp width of the cusp $\infty$, where $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In case of $c \neq \infty$, $G_c$ is generated by $t_c = A_c t^{h_c} A_c^{-1}$ for some smallest real number $h_c > 0$, called the cusp width of the cusp $c$ such that $t_c \in G$ where $A_c = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$ so that $A_c(\infty) = c$.

Given a Fuchsian group $G$, we define $G'$ to be its pre-image in $\text{SL}_2(\mathbb{R})$. For example if $G = \text{PSL}_2(\mathbb{Z})$, then $G' = \text{SL}_2(\mathbb{Z})$. An element $\gamma' \in G'$ is called a parabolic element if it is preimage of a parabolic element in $G$. Given an element $\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'$, we denote $j(\gamma', \tau) = c \tau + d$. However, if we take an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$, we can not define $j(\gamma, \tau)$ in this way, because of one possible sign factor. In [1] and [2], this sign factor was redundant as the authors worked with even weights vector-valued automorphic forms and therefore it always comes with an even power.

3. Vector-valued automorphic forms

This section reviews the basics of vector-valued automorphic forms that we need to understand and prove the main results of the article. In [1] and [2], the authors studied vvaf associated to Fuchsian groups of the first kind of even weight. The even weight is a necessary condition to be able to define $j(\cdot, \cdot)$, as we mentioned in the previous section.

We shall study vvaf associated to pre-image of the Fuchsian groups in $\text{SL}_2(\mathbb{R})$. This will allow us a slight improvement, i.e. we will be able to work with any integer weight, not just the even ones. Our treatment to vector-valued automorphic forms in this section closely follow [3].

Definition 3.1 (Stroke operator). If $X : \mathbb{H} \rightarrow \mathbb{C}^m$ is a vector-valued meromorphic function, $\gamma \in \text{SL}_2(\mathbb{R})$ and $k$ be a complex number, we define a vector-valued meromorphic function $X|_k \gamma$ on $\mathbb{H}$ by setting $X|_k \gamma(\tau) = j(\gamma, \tau)^{-k} X(\gamma \tau)$, where $j(\gamma, \tau) = c \tau + d$ when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

It is needless to mention that the function $j(\gamma, \tau)^k$ is well defined because $j(\gamma, \cdot) : \mathbb{H} \rightarrow \mathbb{C}$ is a non-vanishing function. One can show that the Stroke operator induces a right group action of $\text{SL}_2(\mathbb{R})$ on the space of vector-valued meromorphic functions on $\mathbb{H}$. For this article, we shall restrict to integer $k$. Now, we are interested in the vector-valued meromorphic functions having finite order poles at the cusps. More precisely, let us first make the following definition.

Definition 3.2. Let $X : \mathbb{H} \rightarrow \mathbb{C}^m$ be a vector-valued meromorphic function. Then
We say that $X(\tau)$ has moderate growth at $\infty$ when there exists $c \in \mathbb{R}$ and $Y > 0$ such that $\|X(\tau)\| < \exp(cy)$ when $y > Y$, where $y = \text{Im}\tau$.

We say that $X(\tau)$ has moderate growth at $c \in \mathbb{R}$ with respect to $k \in \mathbb{Z}$ when $X|_kA_c$ has moderate growth at $\infty$, where we set $A_c$ to be $\begin{pmatrix} e & 1 \\ 1 & 0 \end{pmatrix}$ as an element of $SL_2(\mathbb{R})$.

Now, to introduce vector-valued automorphic forms, we first discuss the representations of the Fuchsian group $G$ (or of $G'$). For us, these representations are divided into two types. Namely, admissible, and logarithmic (non-admissible) ones. Let us describe them in the following form.

**Definition 3.3.** Let $\rho : G' \to \text{GL}_m(\mathbb{C})$ be a representation such that $\rho(\gamma)$ is equipped with only unitary eigenvalues, for every parabolic element $\gamma \in G$. We say that $\rho$ is an admissible representation of $G$ if $\rho(\gamma)$ is diagonalizable for every parabolic element $\gamma \in G'$. Otherwise, we say that $\rho$ is a logarithmic representation.

Any parabolic element in $G$ is power of $t_c$, for some cusp $c \in \mathcal{C}_G$. Therefore, we are really talking about diagonalizability and the unitary condition, at the finitely many parabolic elements. We shall now separately treat the vector-valued meromorphic functions associated to them.

**Definition 3.4.** Let $G$ be a Fuchsian group of the first kind, $k$ be an integer, $\rho : G' \to \text{GL}_m(\mathbb{C})$ be an admissible representation satisfying $\rho(I) = e^{-i\pi k}\rho(-I)$, and $X : \mathbb{H} \to \mathbb{C}^m$ be a vector-valued meromorphic function. Then, we say that $X(\tau)$ is a weakly homomorphic admissible vvaf of weight $k$ with respect to $\rho$ provided:

1. it is a holomorphic function on $\mathbb{H}$, with moderate growth at any cusp of $G$,
2. it has finitely many poles in the closure of a fundamental domain of $G'$, and
3. $X|_k\gamma(\tau) = \rho(\gamma)X(\tau)$, $\forall \gamma \in G'$.

**Remark 3.5.**

(a) Of course, a representation $\rho$ satisfies $\rho(-I) = e^{i\pi k}\rho(I)$, only if $k$ is an integer. Moreover, $\rho$ can be thought of as a representation of $G$ if and only if, $\rho(-I) = \rho(I)$. Equivalently when the associated vvaf has even weight.

(b) If $\rho$ is any irreducible representation, it turns out that $\rho(-I)$ must be one of the $\pm I$. This is a simple fact from linear algebra.

Recall that $t_c \in \text{PSL}_2(\mathbb{R})$ (up-to conjugation) generates $G_c$. We can naturally think of $t_c$ as an element of $\text{SL}_2(\mathbb{R})$, and since $\rho$ is admissible, we can write $\rho(t_c) = P_c \text{diag}(\lambda_1,c,\lambda_2,c,\cdots,\lambda_m,c)P_c^{-1}$ for any cusp $c$ of $G$. To get the Fourier expansion around the cusps, we denote

$$q_c = \begin{cases} e^{\frac{2\pi i c}{h_c}} & \text{if } c \in \mathcal{C}_G, c \neq \infty, \\ e^{\frac{2\pi i c}{h_c}} & \text{if } c = \infty. \end{cases}$$
Following Proposition 3.11 in [2], at the cusp $\infty$ we have the following Fourier expansion,
\[
\mathcal{X}(\tau) = P_\infty q^{\lambda} P_\infty^{-1} \sum_{n=-M}^{\infty} \mathcal{X}_|n| q^n.
\]
For any other cusp $c \neq \infty$, consider $\mathcal{Y}(\tau) = \mathcal{X}|kA_c(\tau)$, and then
\[
\mathcal{X}(\tau) = \mathcal{Y}|kA_c^{-1}(\tau) = (\tau - c)^{-k} \mathcal{X}_0(A_c^{-1} \tau).
\]
Following this, we have the Fourier expansion
\[
\mathcal{X}(\tau) = (\tau - c)^{-k} P_c q_c^\lambda P_c^{-1} \sum_{n=-M_c}^{\infty} \mathcal{X}_c|n| q_c^n, \quad \mathcal{X}_c^\epsilon \in \mathbb{C}^d,
\]
where we denote $q_c^\lambda$ to be the diagonal matrix $(q_1^\mu(\lambda_1, c), q_2^\mu(\lambda_2, c), \ldots, q_1^\mu(\lambda_m, c))$. In particular, note that the Fourier coefficients do not depend on the choice of the diagonalizing matrix. Note that $\mathcal{X}(\tau)$ is a weakly holomorphic admissible vvaf if $M$ and $M_c$ are always finite.

A weakly holomorphic admissible vvaf $\mathcal{X}(\tau)$ of weight $k$ is called holomorphic if it has no poles in $\mathbb{H}$ and, for any cusp $c$ of $G$, the function $\mathcal{X}|kA_c(\tau)$ is bounded in some half-plane (contained in $\mathbb{H}$), that is to say simply that $\mathcal{X}(\tau)$ is holomorphic everywhere in $\mathbb{H}_G$. It is called a weight $k$ cusp form if, for any cusp $c$, the function $\mathcal{X}|kA_c(\tau)$ approaches to 0 as $y \to \infty$. This is same as saying that $\mathcal{X}(\tau)$ is a holomorphic admissible vvaf if $M_c \leq 0$, and an admissible cusp form if $M_c < 0$ for any cusp $c$ of $G$.

Now we are interested in the representation $\rho$ of $G$ which is not admissible. In other words, $\rho(\gamma)$ is not diagonalizable for some parabolic element $\gamma \in G'$. We call such a representation logarithmic. In this case we can write $\rho(t_c)$ is in the Jordan canonical form as
\[
P_c \begin{pmatrix}
J_{m(\lambda_1, c), \lambda_1, c} & & \\
& J_{m(\lambda_2, c), \lambda_2, c} & \\
& & \cdots & J_{m(\lambda_m, c), \lambda_m, c}
\end{pmatrix} P_c^{-1},
\]
for any cusp $c$ of $G$, where the Jordan block $J_{m, \lambda}$ is defined to be
\[
\begin{pmatrix}
\lambda \\
& \ddots \\
& & \lambda
\end{pmatrix}_{m \times m},
\]
which is conjugate to the canonical Jordan block. We also set $R(t_c)$ to be the collection of the eigenvalues of $\rho(t_c)$, which may not be necessarily pairwise distinct.

Let $\mathcal{X} : \mathbb{H} \to \mathbb{C}^m$ be a vector-valued meromorphic function, and $\rho : G' \to GL_m(\mathbb{C})$ be a logarithmic representation. Suppose that for some integer $k$, $\mathcal{X}(\tau)$ satisfies
\begin{enumerate}
\item [(a)] $\mathcal{X}|k\gamma = \rho(\gamma)\mathcal{X}$, $\forall \gamma \in G'$,
\item [(b)] $\rho(I) = e^{i\pi k} \rho(-I)$.
\end{enumerate}
Now let \( c \) be any arbitrary cusp of \( G \). Then for each eigenvalue \( \lambda \) of \( \rho(t_c) \) there are \( \tilde{q}_c \)-expansions

\[
(2) \quad h_{\lambda, j, c}(\tau) = P_c \tilde{q}_c^{\mu(\lambda)} P_c^{-1} \sum_{n=-M_c}^{\infty} \mathbb{X}^c[\lambda, j, n] \tilde{q}_c^n, \quad 0 \leq j \leq m(\lambda) - 1
\]

such that, at the cusp \( \infty \)

\[
(3) \quad \mathbb{X}(\tau) = \sum_{\lambda \in R(t_c)} m(\lambda) \sum_{j=0}^{m(\lambda)-1} (\log \tilde{q})^j h_{\lambda, j, \infty}(\tau),
\]

and at the cusp \( c(\neq \infty) \),

\[
(4) \quad \mathbb{X}(\tau) = (\tau - c)^{-k} \sum_{\lambda \in R(t_c)} m(\lambda) \sum_{j=0}^{m(\lambda)-1} (\log \tilde{q})^j h_{\lambda, j, c}(\tau),
\]

where \( m(\lambda) \) is denoted to be the size of one of the Jordan block associated to the eigenvalue \( \lambda \).

**Definition 3.6.** Let \( \mathbb{X} : \mathbb{H} \to \mathbb{C}^m \) be a vector-valued holomorphic function and there exists an integer \( k \) such that \( \mathbb{X}(\tau) \) satisfies (3) and (4) above with respect to logarithmic representation \( \rho : G' \to \text{GL}_m(\mathbb{C}) \). Then, we say that \( \mathbb{X}(\tau) \) is a weakly holomorphic logarithmic \( \nu \)vaf of weight \( k \), if for any cusp \( c \) of \( G' \), all of the \( \tilde{q}_c \) expansions \( h_{\lambda, j, c}(\tau) \) start from some finite \( n \).

In particular, we say that \( \mathbb{X}(\tau) \) is holomorphic at the cusp \( \infty \) when \( M = 0 \), and holomorphic at the cusp \( c \) when \( \mathbb{X}_c | A_c(\tau) \) is holomorphic at the cusp \( \infty \), or equivalently when \( M_c = 0 \). If \( \mathbb{X}(\tau) \) is holomorphic at all cusps, then we say that \( \mathbb{X}(\tau) \) is a holomorphic logarithmic \( \nu \)vaf. In addition, we say that \( \mathbb{X}(\tau) \) is a logarithmic vector-valued cusp form, if \( M < 0 \) and \( M_c < 0 \) for any cusp \( c \) of \( G \).

**4. Properties of the Lifted Vector-Valued Automorphic Forms**

In this section, we shall recall the induction of representations and introduce \( \nu \)vaf associated to them. We closely follow [1] throughout this section, and develop the main tools to prove the main results of this article. Let us first recall our usual set up: \( G \) is a Fuchsian groups of the first kind and \( H \) is a finite index subgroup, say \( d \), of \( G \).

**4.1. A special choice of the representatives.** Fix any cusp \( c \in \hat{\mathbb{C}}_G \) and let \( c_1, \ldots, c_n \) be the representatives of the \( H \)-inequivalent cusps which are \( G \)-equivalent to the cusp \( c \), so we have

\[
G \cdot c = \bigcup_{i=1}^{n_c} H \cdot c_i.
\]

Therefore, for each \( i \) we get an \( A_i \in G \) with \( A_i(c) = c_i \). Let us denote \( k_i \) be the cusp width of \( c_i \) in \( G \) and \( h_i \) be the cusp width of \( c_i \) in \( H \). Then a set of coset representatives of \( H \) in \( G \) can be taken to be \( g_{ij} = h_i A_i^{-1} \) for all \( 1 \leq i \leq n_c \) and \( 0 \leq j < h_i \), where \( h_i = \frac{h_i}{k_i} \in \mathbb{Z} \). It turns out that, \( \sum_{1 \leq i \leq n_c} h_i = d \).
Let $\rho$ be a representation of rank $m$ associated to $H'$, and denote $\tilde{\rho} := \text{Ind}_{H}^{G'}(\rho)$ to be the induction of $\rho$. With the choice of coset representatives $\{g_{i,j}\}$ of $H$ in $G$ as described above, we can write $\tilde{\rho}(t_{i})$ in the block diagonal form, where each block is of size $mh_i \times mh_i$, $\forall 1 \leq i \leq n_c$. Moreover, these blocks are in the lower-diagonal form whose right top block is $\rho(t_i)$, and all other blocks are in the lower diagonal entry is $I_{m \times m}$. More precisely, it is of the form

$$
\begin{pmatrix}
0 & \rho(t_i) \\
\vdots & \vdots \\
0 & \rho(t_i)
\end{pmatrix}
$$

Here $t_i = A_{d_i} A_i^{-1}$ is the generator of the stabilizer $H_i$ in $H$.

4.2. Lifting of vector-valued automorphic forms. Let $H, G$ and $\rho$ be as in the previous section, and $\mathcal{X}(\tau)$ be a weakly holomorphic vVAF associated to $(\rho, H')$. Since $H$ has finite index in $G$, one can note that $H'$ has finite index in $G'$ as well. Moreover, one can take a similar set of representatives $\{g_{i,j}\}$, realizing them as elements of $\text{SL}_2(\mathbb{R})$.

Now one may ask for an induced form $\tilde{\mathcal{X}}(\tau)$, which is also a weakly holomorphic vVAF associated to $\tilde{\rho}$. Fix a cusp $c$ of $G$ and let $\{g_{i,j}\}$ be the set of coset representatives of $H$ in $G$, as described earlier. We then define the induced function $\tilde{\mathcal{X}}^{(c)} : \mathbb{H} \to \mathbb{C}^{dm}$ by setting,

$$
\tau \mapsto \left( \mathcal{X}(g_{i,j}^{-1} \tau) \right)^{t}_{1 \leq i \leq n_c}^{0 \leq j < h_i}
$$

The reader can note that, given any cusp $c$ of $G$, we are uniquely lifting the vVAF $\mathcal{X}(\tau)$ to $\tilde{\mathcal{X}}^{(c)}(\tau)$, because $g_{i,j}$ are well defined. Of course, $\tilde{\mathcal{X}}^{(c)}(\tau)$ is just a vector-valued holomorphic function on $\mathbb{H}$ right now, because $\mathcal{X}(\tau)$ is a weakly holomorphic vVAF. To make sure that $\tilde{\mathcal{X}}^{(c)}(\tau)$ is a weakly holomorphic vVAF (be it admissible, or logarithmic), we first need to check the functional equation with respect to the induced representation $\tilde{\rho}$. Note that,

$$
\tilde{\mathcal{X}}^{(c)}(\gamma \tau) = \begin{pmatrix}
\mathcal{X}(\gamma_1^{-1} \gamma_{j_1} \gamma_1^{-1} \tau) \\
\mathcal{X}(\gamma_2^{-1} \gamma_{j_2} \gamma_2^{-1} \tau) \\
\vdots \\
\mathcal{X}(\gamma_d^{-1} \gamma_{j_d} \gamma_d^{-1} \tau)
\end{pmatrix}
= \begin{pmatrix}
\mathcal{X}(\gamma_1^{-1} \gamma_{j_1} \gamma_1^{-1} \tau) \\
\mathcal{X}(\gamma_2^{-1} \gamma_{j_2} \gamma_2^{-1} \tau) \\
\vdots \\
\mathcal{X}(\gamma_d^{-1} \gamma_{j_d} \gamma_d^{-1} \tau)
\end{pmatrix}
\begin{pmatrix}
j(\gamma_1^{-1} \gamma_{j_1} \tau)^{k} \rho(\gamma_1^{-1} \gamma_{j_1}) \mathcal{X}(\gamma_1^{-1} \tau) \\
j(\gamma_2^{-1} \gamma_{j_2} \tau)^{k} \rho(\gamma_2^{-1} \gamma_{j_2}) \mathcal{X}(\gamma_2^{-1} \tau) \\
\vdots \\
j(\gamma_d^{-1} \gamma_{j_d} \tau)^{k} \rho(\gamma_d^{-1} \gamma_{j_d}) \mathcal{X}(\gamma_d^{-1} \tau)
\end{pmatrix},
$$

where $\{\gamma | 1 \leq i \leq d\}$ is the set $\{g_{i,j} | 1 \leq i \leq n_c, 0 \leq j < h_i\}$. To satisfy the functional equation property, we need that

$$
j(\gamma_1^{-1} \gamma_{j_1} \tau)^{k} = j(\gamma, \tau)^{k}, \quad \forall 1 \leq i \leq d, \gamma \in G'.
$$
We can make sure this if, \( r_2(y^{-1} \gamma y_b) = r_2(\gamma), \) \( \forall \ 1 \leq i \leq d, \ \gamma \in G', \) or simply if the weight \( k = 0, \) where \( r_2(\cdot) \) denotes the second row of the corresponding matrix. Of course, it is unlikely that \( r_2(y^{-1} \gamma y_b) = r_2(\gamma), \) \( \forall \ 1 \leq i \leq d, \ \gamma \in G' \) would always hold. So to be on the safer side, we simply stick to the weight \( k = 0 \) case to make sure that the functional equation being satisfied. Before discussing the moderate growth condition, let us first define a suitable lift for any arbitrary weight case. For this, let us first recall a reduction trick introduced in [1]. The idea is to find a scalar-valued cusp form \( \Delta_{G'}(\tau) \) of non-zero weight, which is holomorphic on \( \mathbb{H}_G^* \) and nonzero everywhere, except at \( \infty. \) For instance if \( G' \) is given by the modular group \( SL_2(\mathbb{Z}), \) then one can take
\[
\Delta_{G'}(\tau) = (\eta(\tau))^{24} = q \prod_{n \geq 1} (1 - q^n)^{24},
\]
where \( \eta(\tau) \) is the Dedekind eta function. For the existence in the general case, the reader may look at the exposition in Section 4 of [1].

Now given a weakly holomorphic admissible or logarithmic vWAF \( \mathcal{X}(\tau) \) of weight \( k, \) let us denote \( \mathcal{X}_0(\tau) = \Delta_{H}^{-k} \mathcal{X}(\tau), \) where \( \Delta_{H} \) is the weight of \( \Delta_{H}(\tau). \) It is clear that, \( \mathcal{X}_0(\tau) \) is a weakly holomorphic vWAF of weight 0, associated to the representation \( \rho \otimes v_{H}^{-k}, \) where \( v_{H} \) is the rank 1 representation associated to \( \Delta_{H}^{-k} \mathcal{X}(\tau). \) One can then consider a reduction to a weight 0 automorphic form by \( \mathcal{X}(\tau) \mapsto \mathcal{X}_0(\tau). \) In particular, we now have a recipe to lift to a vWAF of arbitrary weight, by considering the map
\[
\mathcal{X}(\tau) \mapsto \mathcal{X}_0(\tau) \mapsto \tilde{\mathcal{X}}_{0}^{(c)}(\tau)\Delta_{G'}^{k/\nu(\cdot)}(\tau) := \Delta_{G'}^{k/\nu(\cdot)}(\tau)(\mathcal{X}_0(g_{i,j})\tau)_{1 \leq i \leq n, 0 \leq j < h}.
\]
Note that \( \tilde{\mathcal{X}}_{0}^{(c)}(\tau)\Delta_{G'}^{k/\nu(\cdot)}(\tau) \) satisfies the functional equation with respect to the representation \( (\rho \otimes v_{H}^{-k} \otimes v_{G'})^{-c} = \tilde{\rho}. \) Therefore, we refine our definition of lift by setting
\[
\tilde{\mathcal{X}}^{(c)}(\tau) := \tilde{\mathcal{X}}_{0}^{(c)}(\tau)\Delta_{G'}^{k/\nu(\cdot)}(\tau).
\]
If \( \infty \) is a cusp of \( G, \) then we set \( \tilde{\mathcal{X}}(\tau) := \tilde{\mathcal{X}}^{(c)}(\tau) \) as the definition of lifted form.
If not, we pick any cusp \( \Gamma \) of \( G' \) and set \( \tilde{\mathcal{X}}(\tau) := \tilde{\mathcal{X}}^{(c)}|_{\mathcal{A}_{\Gamma}(\tau)}, \) which satisfies the required functional equation with respect to the representation \( \mathcal{A}_{\Gamma}^{-1}\gamma A_{\Gamma} \mapsto \tilde{\rho}(\gamma). \)

4.3. **Preservation of the cuspidal properties.** Before studying the behavior at the cusps, we first need to have a better understanding of the representatives of \( H \) in \( G. \) Suppose that \( \infty \) is a cusp of \( G \) and \( \{c_i|1 \leq i \leq n_{\infty}\} \) are the cusps of \( H \) lying under \( \infty, \) and \( \{g_{i,j}\} \) be the set of coset representatives of \( H \) in \( G \) as described before, with the important property that, \( g_{i,j}(c_i) = \infty. \) The following lemma is about a comparison with the set of all coset representatives \( \{g_{i,j}|1 \leq i \leq n_{\infty}, 0 \leq j < h_{i}\} \) inside \( G' \) and the set \( \{A_{\alpha_{i,j}}|1 \leq i \leq n_{\infty}\} \) inside \( SL_2(\mathbb{R}). \)

**Lemma 4.1.** There exists \( a_{i,j} \) and \( \alpha_{i,j} \in \mathbb{R} \) such that
\[
g_{i,j}\tau = a_{i,j}^{2}A_{-1}\tau + jh_{i} \alpha_{i,j}, \ \forall \ \tau \in \mathbb{H}.
\]
Proof: Recall that, \( g_{i,j} = t^i_\omega A^{-1}_i \) where \( A_i(\infty) = c_i \). We also know that \( A_i(\infty) = c_i \), in particular, \( A^{-1}_1A_1(\infty) = \infty \). On the other hand, \( A^{-1}_1A_1 \in \text{SL}_2(\mathbb{R}) \). In particular, \( A^{-1}_1A_1 = \begin{pmatrix} a \alpha \\ 0 1/a \end{pmatrix} \) for some \( a, \alpha \in \mathbb{R} \). We then have,

\[
g_{i,j}(\tau) = t^i_\omega A^{-1}_i \tau = t^i_\omega \begin{pmatrix} a \alpha \\ 0 1/a \end{pmatrix} A^{-1}_i \tau = a^2 A^{-1}_i(\tau) + j\omega + \alpha a.
\]

This completes the proof by taking \( a_{i,j} := a \) and \( \alpha_{i,j} := \alpha a \). \( \square \)

We recall that any classical holomorphic modular form does not have weight 0, unless it is a constant function. In those cases, the representation under consideration has finite image. However, the same may not be true if the representation does not have finite image. Consider the representation \( \Pi_0 : \text{SL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{C}) \), given by \( \gamma \mapsto \gamma \). Now consider the holomorphic function \( \mathcal{Y}_\gamma : \mathbb{H} \to \mathbb{C}^2 \) given by \( \tau \mapsto (\tau, 1) \). We know that \( \mathcal{Y}_\gamma(\tau) \) is a holomorphic logarithmic vVaf of weight \(-1\). Then \( \mathcal{Y}_\gamma(\tau) \) is a non-constant holomorphic logarithmic vVaf of weight 0 associated to the representation \( \rho' := \Pi_0 \otimes \nu_{\text{SL}_2(\mathbb{Z})} \). In fact, following [3], we know that the space of holomorphic logarithmic vVaf is a free module of rank two over the polynomial ring \( \mathbb{C}[E_4, E_6] \), generated by \( \mathcal{Y}_\gamma(\tau) \) and its modular derivative.

Moreover, the representation \( \rho' : \text{SL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{C}) \) indeed have infinite image, because

\[
\|\rho'(t^n)\| = \|\Pi_0 \otimes \nu_{\text{SL}_2(\mathbb{Z})}(t^n)\| = \|\Pi_0(\tau)\| = n,
\]

which follows from the explicit description of \( \nu_{\text{SL}_2(\mathbb{Z})} \), see [1, Page 7]. Moreover, Gannon in [3] gave an explicit description of the logarithmic representations of rank 2 associated to \( \text{SL}_2(\mathbb{Z}) \), and they all differ from \( \rho' \) by some character of \( \text{SL}_2(\mathbb{Z}) \).

We are now ready to prove the required cuspidal properties of the lifted forms.

**Lemma 4.2.** Let \( c \) be an arbitrary cusp of \( \text{G} \) and \( \{c_i|1 \leq i \leq n_c\} \) be the set of all inequivalent cusps of \( \text{H} \) lying under \( c \). Then we have the following.

1. If \( \mathcal{X}(\tau) \) has moderate growth at all the cusps \( \{c_i|1 \leq i \leq n_c\} \), then the lifted form \( \mathcal{X}^{(i)}(\tau) \) has moderate growth at \( c \) as well.

2. If \( \mathcal{X}(\tau) \) is holomorphic (or vanished) at all the cusps \( \{c_i|1 \leq i \leq n_c\} \), then the lifted form \( \mathcal{X}^{(i)}(\tau) \) has same properties at the cusp \( c \), provided that the weight of \( \mathcal{X}(\tau) \) is 0.

**Proof:** For both of the parts, it is enough to prove that \( \mathcal{X}^{(i)}(\tau) \) satisfy the required cuspidal properties at the cusp \( \infty \).

Let us first prove (1). It is clear that \( \mathcal{X}_0(\tau) \) has moderate growth at all the cusps \( \{c_i|1 \leq i \leq n_\omega\} \), because any power of \( \Delta_F(\tau) \) has the same property. Now we shall show that all the components of \( \mathcal{X}_0(\tau) \) has moderate growth at \( \infty \). Let \( \mathcal{Y}(\tau) := \mathcal{X}_0(g^{-1}_i \tau) \) be such a component. Since \( \mathcal{X}_0(\tau) \) has moderate growth at all the cusps \( \{c_i|1 \leq i \leq n_\omega\} \) there exists a constant \( c \in \mathbb{R} \) such that \( \|\mathcal{X}_0(\tau)\| \ll |e^{2\pi ic A_i^{-1}}| \) as \( \text{im}(\tau) \to \infty \). It follows from Lemma 4.1 that, \( \|\mathcal{Y}(\tau)\| \ll |e^{2\pi ic \tau}| \), for some constant \( c \in \mathbb{R} \), as \( \text{im}(\tau) \to \infty \). This shows that, \( \mathcal{X}_0(\tau) \) has moderate growth at \( \infty \). On
the other hand, any power of $\Delta_G(\tau)$ has moderate growth at $\infty$ as well, and this completes the proof of (1).

Let us now prove (2). In order to show that $\overline{X}(\tau)$ is holomorphic (or vanishes) at the cusp $\infty$, we need to show that all the components are bounded (or vanishes) as $\text{im}(\tau) \to \infty$. The constant $c$ appearing in the previous paragraph is 0 when $\overline{X}(\tau)$ is holomorphic and negative when $\overline{X}(\tau)$ is a cuspidal form. Moreover, it can also be seen from Lemma 4.1 that, the constants $c$ and $c'$ from the previous paragraph are a positive multiple of each other. Therefore $\overline{X}(\tau)$ has the similar cuspidal properties as $X(\tau)$. This shows that the lifted form also shares the same cuspidal properties when the weight is 0.

$\square$

5. LIFTING OF LOGARITHMIC VECTOR-VALUED AUTOMORPHIC FORMS

It was proved, by the first author in [1], that the induction of an admissible representation is admissible. In this section, we shall study about the induction of non-admissible, i.e. logarithmic representations. In this regard, we have the following.

**Proposition 5.1.** Let $c$ be an arbitrary cusp of $G$ and $\{c_i|1 \leq i \leq n_c\}$ be the set of inequivalent cusps of $H$ for which $G \cdot c = \bigcup H \cdot c_i$. Then,

(1) If $\rho(t_i)$ is not diagonalizable for some $i$, then $\overline{\rho}(t_i)$ is not diagonalizable.

(2) In particular if $\rho$ is a logarithmic representation, then $\overline{\rho}$ is a logarithmic representation as well.

**Proof.** Let us start with considering the coset representatives $\{g_{i,j}\}$ of $H'$ in $G'$ from the previous paragraph. We first claim that, for any pair $(i, j)$, some non-trivial power of $g_{i,j} = g_{i,j}^{-1} t_i g_{i,j}$ is in $H'$. To prove this, we start with noting that, there exists $n_{i,j,1}$ and $n_{i,j,2}$ such that $g_{i,j}^{-n_{i,j,1}} H' = g_{i,j}^{-n_{i,j,2}} H'$. This is because, $H'$ has finite index in $G'$. In particular, for each pair $(i, j)$, there exists some integer $n_{i,j}$ such that $g_{i,j}^{-1} t_i^n g_{i,j} \in H'$. Let us now consider $n = \text{lcm}\{n_{i,j}|1 \leq i \leq n_c, 1 \leq j \leq h_i\}$. In particular, $g_{i,j}^{-1} t_i^n g_{i,j} \in H$ for each pair $(i, j)$. Therefore, $\overline{\rho}(t_i^n)$ is a block diagonal matrix where each block is of the form $\rho(g_{i,j}^{-1} t_i^n g_{i,j})$. Note that $g_{i,j}^{-1} t_i^n g_{i,j} \in H'$ and fixes the cusp $c_i$, hence it is some non-trivial power of $t_i$. Let us write $g_{i,j}^{-1} t_i^n g_{i,j} = t_i^m$, where $m \neq 0$ is an integer. By the assumption, there exists some $i \in \mathbb{N}$ for which $\rho(t_i)$ can be written in Jordan normal form, with a Jordan block, say $J_{\lambda}$, of size greater than 1. In particular, $\rho(t_i^n)$ can be written in a block diagonal form, where one of the block is $J_{\lambda}$, which is not diagonalizable for any integer $m \neq 0$.

For the proof of part (2), let $c_i$ be a cusp of $H$ for which $\rho(t_i)$ is not diagonalizable. Now $c_i$ is a cusp of $G$ as well, with $c_i$ itself lying under it as a cusp of $H$. It follows from the part (1) that $\overline{\rho}(t_i)$ is not diagonalizable, which completes the proof. $\square$

**Remark 5.2.** It is clear that $n_{i,j} \leq d$, for each pair $(i, j)$, where $d$ is the index of $H$ in $G$. In other words, $n$ is crudely bounded by $d^d$. However, if $H'$ is normal in $G'$, one can always take $n_{i,j}$ to be $d$. In particular, one can take $n = d$. In fact, the discussion in Section 4.1 allows us to take $n = \text{lcm}\{h_i|1 \leq i \leq n_c\}$. 


5.1. Eigenvalues of the induced representation. Let us first recall the usual set up. Fix a cusp $c$ in $G$, and consider the decomposition

$$G \cdot c = \bigcup_{i=1}^{n_c} H \cdot c_i.$$ 

If we know all the eigenvalues of $\rho(t_i)$, then do we know about the eigenvalues of $\tilde{\rho}(t_i)$? The first author studied this when $\rho$ is an admissible representation and gave an explicit description of the corresponding eigenspaces. Of course, now the question arises if $\rho$ is not admissible. In other words, if one of the $\rho(t_i)$ is not diagonalizable. We shall answer this in the following corollary.

Corollary 5.3. Let $c$ be any arbitrary cusp of $G$, and $\{c_i|1 \leq i \leq n_c\}$ be the set of inequivalent cusps of $H$ lying under $c$. Let $\{\lambda_{i(k)}|1 \leq k \leq e_i\}$ be the set of distinct eigenvalues of $\rho(t_i)$ for each $1 \leq i \leq n_c$. Then, the following statements are true.

(a) The eigenvalues of $\tilde{\rho}(t_i)$ are precisely given by the set

$$\{\zeta \lambda_{i(k)}^{1/h}|1 \leq i \leq n_c, 1 \leq k \leq e_i, \zeta \in R_{h_i}\},$$

where $e_i$ is the number of distinct eigenvalues of $\rho(t_i)$, and $R_{h_i}$ is the set of all $h_i^{th}$ root of unity.

(b) If there is a Jordan block of size $m(\lambda_{i(k)})$ in $\rho(t_i)$ associated to the eigenvalue $\lambda_{i(k)}$, then there is a Jordan block in $\tilde{\rho}(t_i)$ of size

$$m\left(\zeta \lambda_{i(k)}^{1/h}\right) = m(\lambda_{i(k)}), \ \forall \ 1 \leq i \leq n_c, 1 \leq k \leq e_i, \zeta \in R_{h_i}.$$ 

Proof. Let $v_{i(k)}$ be an eigenvector of $\rho(t_i)$ associated to the eigenvalue $\lambda_{i(k)}$. Then it follows from the the description of the block form given at (6) that, there exists a set of vectors $\{v_{i(k,\xi)}|1 \leq k \leq e_i, \xi \in R_{h_i}\}$, whose $i^{th}$ block is of the form

$$\left(\lambda_{i(k)}^{1-1/h} \xi^{-1} v_{(i,k)}, \lambda_{i(k)}^{1-2/h} \xi^{-2} v_{(i,k)}, \cdots, v_{(i,k)}\right)^T,$$

and all other entries are zero, satisfying that

$$\tilde{\rho}(t_i)(v_{i(k,\xi)}) = \zeta \lambda_{i(k)}^{1/h_i} v_{i(k,\xi)}, \ \forall \ 1 \leq i \leq n_c, 1 \leq k \leq e_i, \zeta \in R_{h_i}.$$ 

This shows that the set of eigenvalues of $\tilde{\rho}(t_i)$ contains

$$\{\zeta \lambda_{i(k)}^{1/h_i}|1 \leq i \leq n_c, 1 \leq k \leq e_i, \zeta \in R_{h_i}\}.$$ 

On the other hand, let us recall from Section 4.2 that, $\tilde{\rho}(t_i)$ is in a block diagonal form, where each block is of the form

$$\begin{pmatrix} 0 & \rho(t_i) \\ \cdot & \cdot \\ \cdot & \cdot & \ddots & \ddots \\ \cdot & \ddots & \cdot & 0 \end{pmatrix}_{m h_i \times m h_i}.$$ 

Note that if $\lambda'$ is any eigenvalues of such a block, we must have that $(\lambda')^{h_i} = \lambda_{i(k)}$ for some $1 \leq i \leq n_c$ and $1 \leq k \leq e_i$. This gives the complete description of the eigenvalues of $\tilde{\rho}(t_i)$, as desired.
For a proof of (b), let us first assume that $\rho(t_i)$ has a single Jordan block $J_{m,\lambda}$. We then want to study the Jordan blocks of

\[
\begin{pmatrix}
0 & J_{m,\lambda} \\
I & & \\
& \ddots & I \\
& & 0
\end{pmatrix}_{mh_i \times mh_i}
\]

Taking $h = h_1 h_2 \cdots h_{m_\infty}$, we see that $\tilde{\rho}(t^h_i)$ is in the block diagonal form, whose each block is of the form $J^h_{m,\lambda}$, and each such block appears $h_i$ times. It is known that $J^h_{m,\lambda}$ is a Jordan block of the size $m$ with eigenvalue $\lambda^h$. From the proof of part (a) we know that the set of eigenvalues of any such block in (6) is precisely $\left\{ \frac{\zeta \lambda^h}{h_i} \mid \zeta \in R_{h_i} \right\}$, and clearly they all have size $m := m(\lambda)$.

Now for the general case when $\rho(t_i)$ does not necessarily have a single Jordan block, writing $\rho(t_i) \sim \text{diag} (J_1, J_2, \cdots , J_t)$, we have

\[
\begin{pmatrix}
0 & \rho(t_i) \\
I & \cdots & I \\
\cdots & \ddots & I \\
I & & 0
\end{pmatrix}_{mh_i \times mh_i} \sim \bigoplus_{s=1}^t \begin{pmatrix}
0 & J_s \\
I & \cdots & I \\
\cdots & \ddots & I \\
I & & 0
\end{pmatrix}_{s_i h_i \times s_i h_i},
\]

where $s_i$ is the size of $J_s$ with $\sum_{i=1}^t s_i = m$. This completes the proof because, from the previous paragraph we know sizes of the Jordan blocks of each component of the direct sum above.

We now have all the required tools to prove the main result of this article.

5.2. Proof of Theorem 1.1. The part (a) follows from (1) of Lemma 4.2 and (1) of Proposition 5.1. It follows from (2) of Proposition 5.1 that, part (b) is proved.

5.3. Space of the lifted vector-valued automorphic forms. Let us consider $R_{G'}$ to be the ring of weakly holomorphic scalar-valued automorphic forms on $G'$. Define $M^k(\rho)$ (resp. $M^0_{k,\text{hol}}(\rho)$ and $M^0_{k,\text{cusp}}(\rho)$) be the set of all weakly holomorphic logarithmic vdaf (resp. holomorphic logarithmic vector-valued automorphic forms and logarithmic vector-valued cusp forms) of weight $k$, associated to representation $\rho$ of $G'$. It is clear that $R_{G'}$ acts on those spaces by means of the map $(f, X) \mapsto f^X$, for any $f \in R_{G'}$ and $X(\tau)$ from one of those spaces. As a consequence to the results proved earlier, we have the following consequence.

Proposition 5.4. Let $G$, $H$, $\rho$ be as in Theorem 1.1 and $k$ be an integer. Then there is a $R_{G'}$-module isomorphism between the following spaces.

1. $M^k(\rho)$ and $M^k(\tilde{\rho})$,
2. $M^0_{0,\text{hol}}(\rho)$ and $M^0_{0,\text{hol}}(\tilde{\rho})$, and
3. $M^0_{0,\text{cusp}}(\rho)$ and $M^0_{0,\text{cusp}}(\tilde{\rho})$.

1This is a fact from linear algebra. For a proof, the reader may refer to the answer given by Oscar Cunningham at https://math.stackexchange.com/questions/1849839/jordan-form-of-a-power-of-jordan-block.
Proof. Given a weakly holomorphic vvf $X(\tau)$ from one of the $M_k^1(\rho), M_{0,\text{hol}}(\rho)$ or $M_{0,\text{cusp}}(\rho)$, we consider its lift $\overline{X}(\tau)$. It follows from Theorem 1.1 and (2) of Lemma 4.2 that, indeed this map is a possible candidate for the isomorphism in any of (1), (2) or (3). Clearly this is an injection. On the other, consider a vvf $\mathcal{Y} := (\mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_d)$ in one of the spaces $M_k^1(\rho), M_{0,\text{hol}}(\hat{\rho})$ or $M_{0,\text{cusp}}(\hat{\rho})$. Recall that the weight 0 form $\mathcal{Y}_0(\tau) := \Delta_{G'}^{-k/wd}(\tau)\mathcal{Y}(\tau)$, obtained from $\mathcal{Y}(\tau)$. Without loss of generality we may assume that the weight of $\mathcal{Y}(\tau)$ is 0, if not, we can simply work with $\mathcal{Y}_0(\tau)$.

Let us now fix an element $\gamma \in G$. Then for each $i$, there exists a unique $j := j(i)$ such that $\mathcal{Y}_i(\gamma \tau) = \rho(\gamma^{-1} \gamma_i) \mathcal{Y}_j(\tau)$. Moreover, any such $\gamma_i^{-1} \gamma_i$ is in $H$. In particular, if $\gamma_i = 1$ and $\gamma \in H$, then $\gamma_i = 1$ and we have that $\mathcal{Y}_1(\gamma \tau) = \rho(\gamma) \mathcal{Y}(\tau)$. This implies that, $\mathcal{Y}_1 \in M_k^1(\rho), M_{0,\text{hol}}(\rho)$ or $M_{0,\text{cusp}}(\rho)$. On the other hand if we take $\gamma = \gamma_i$, then we must have that $j(i) = 1$, and hence $\mathcal{Y}_i(\tau) = \gamma(1^{-1}(\tau))$. Therefore, $\mathcal{Y}(\tau) = \overline{\mathcal{Y}}_1(\tau)$. This gives the required isomorphism between the given spaces. 

6. Existence of the logarithmic vector-valued automorphic forms

In this section, we shall discuss about the existence of vvf associated to a given representation. To begin with, let us first discuss about existence, regardless of the representation.

Lemma 6.1. Let $G$ be any Fuchsian group of the first kind, and $k$ be any integer. There exists a weakly holomorphic logarithmic vvf of weight $k$, associated to some representation of $G'$.

Proof. Consider the representation $\mathbb{I} : G' \to GL_2(\mathbb{C})$, given by $\gamma \mapsto \gamma$. Now consider the vector-valued holomorphic function $X_0 : \mathbb{H} \to \mathbb{C}^2$ given by $\tau \mapsto (\tau, 1)$. Note that

$$X_0 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{-1} X_0(\tau), \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G', \tau \in \mathbb{H}.$$ 

Of course, we can not conclude about the weight from here if $c = 0$ and $d = 1$ for every element in $G'$. However, that is not the case because $G$ is a Fuchsian group of the first kind. Therefore, $X(\tau)$ is a weakly holomorphic logarithmic vvf of weight $-1$ associated to $\mathbb{I}$. Then $X_k(\tau) := X_0 \Delta_{G'}^{k/2}(\tau)$ is a logarithmic vvf of weight $k$ associated to the representation $\rho := \mathbb{I} \otimes \chi_{G'}^{k+1}$. 

Now given any representation $\rho : G \to GL_m(\mathbb{C})$ can we find a non-trivial weakly-holomorphic vvf associated to $\rho$ whose components are linearly independent? Of course, here the interesting question is when the components are linearly independent, otherwise one could take a zero function. The first author confirmed this for admissible representations having finite image. The main goal of this section is to study this for logarithmic forms. More precisely, here we consider the following question.
Question 6.2. Let $G$ be any Fuchsian group of the first kind. Given a logarithmic representation $\rho$ of $G'$, can we find a non-trivial \( \text{vvaf} \) associated to $\rho$ whose components are linearly independent?

Before proving our main result of this section, let us quickly recall the result of [1]. As mentioned, the author assumed that the image of $\rho$ has finite image and all the eigenvalues of $\rho(\gamma)$ are unitary, for any parabolic element $\gamma \in G'$. In this case we must have that, $\mu(\cdot)$ of all such eigenvalues are rational.

Now for the logarithmic case, we do not have a luxury to assume that the representation has finite image. If $\rho$ is not admissible, then there exists some parabolic element $\gamma \in G'$ such that $\rho(\gamma)$ has at least one non-trivial Jordan block, say $J_{\lambda}$. Suppose this block is of size $m(\lambda) \times m(\lambda)$. Since it is non-trivial, we may assume that $1 < m(\lambda) \leq m$. If $\rho$ has finite image, then the set \( \{ J_\lambda^n \}_{n \in \mathbb{N}} \) must also be finite. Note that,

$$J_\lambda^n = \lambda^n (I_{m(\lambda)} + N)^n = \lambda^n \sum_{0 \leq i \leq m(\lambda)} \binom{n}{i} N^i,$$

where $N$ is the corresponding nilpotent matrix. In particular for any large $n$, we note that there is an entry in $J_\lambda^n$ of modulus $\gg \left( \frac{n}{m(\lambda)} \right)$. Therefore, for the logarithmic case, we can not assume that the representation has finite image. Here we would like to give a partial answer to Question 6.2 in the following form.

Proposition 6.3. Let $H, G$ and $\rho$ be as in Theorem 1.1. Suppose that $\psi$ be a representation associated to $G'$ such that $\tilde{\rho} = \psi$. Then there exists a weakly holomorphic logarithmic \( \text{vvaf} \) $\tilde{\mathcal{X}}(\tau)$ associated to $(\psi, G')$ with linearly independent components if and only if there exists a weakly holomorphic logarithmic $\text{vvaf}$ $\mathcal{X}(\tau)$ associated to $(\rho, H')$ with linearly independent components.

Proof: It is enough to work with only weight 0 forms, because a multiplication by a scalar-valued non-zero function do not change the linear indepenedency. Suppose that there exists $(\rho, H')$ with an associated weakly holomorphic logarithmic \( \text{vvaf} \) $\mathcal{X}(\tau)$. It follows from Theorem 1.1 that $\tilde{\mathcal{X}}(\tau)$ is a weakly holomorphic logarithmic \( \text{vvaf} \) for Ind$_{H'}^G(\psi) = \rho$. We are now done for the if part, provided that the set $\{ \mathcal{X}(g_i^{-1}\tau) | 1 \leq i \leq n, 0 \leq j < h_i \}$ is a linearly independent set.

If not, we fix a point $\tau_0 \in \mathbb{H}$ such that $g_i^{-1}\tau_0$ are pairwise distinct, and denote $m_{i,j} = \text{ord}_{\tau = \tau_0}(\mathcal{X}(g_i^{-1}(\tau)))$. In fact, we can do this for all but finitely many $\tau_0 \in \mathbb{H}$. Now for some set of positive integers $\{ n_{i,j} \}$ to be specified later, let us consider the function,

$$F(\tau) = \prod_{i=1}^{n} \prod_{j=0}^{h_i-1} \left( f(\tau) - f(g_i^{-1}\tau_0) \right)^{n_{i,j}},$$

for some non-constant modular function $f$. Note that $F(\tau)$ is a modular function for $H$, with

$$\text{ord}_{\tau = \tau_0} F(g_i^{-1}\tau) = \sum n_{i,j} \text{ord}_{\tau = \tau_0} f(g_i^{-1}\tau).$$

Now we choose $\{ n_{i,j} \}$ in such a way, so that $m_{i,j} + \text{ord}_{\tau = \tau_0} F(g_i^{-1}\tau)$ are pairwise distinct, and then $\text{ord}_{\tau = \tau_0} F\mathcal{X}(g_i^{-1}\tau)$ are pairwise distinct. In particular, the set
\[ \left\{ F\mathbb{X}(g_{i,j}) \mid 1 \leq i \leq n, 0 \leq j < h_i \right\} \] is linearly independent. In particular, now \( F\mathbb{X}(\tau) \) is a weakly holomorphic logarithmic vVAF for \( H \) associated to \( \rho \) and \( \tilde{\mathbb{X}}(\tau) := F\mathbb{X}(\tau) \) is a weakly holomorphic logarithmic vVAF associated to \( \tilde{\rho} := \psi \), with linearly independent components, as desired.

On the other hand let \( \tilde{\mathbb{X}}(\tau) \) be a weakly holomorphic logarithmic vVAF for \((\psi, G')\) with linearly independent coefficients. In this case, the proof immediately follows from (1) of Proposition 5.4.

Combining Lemma 6.1 and Proposition 6.3, we record the following consequence.

**Corollary 6.4.** The answer to Question 6.2 is positive, provided that \( \rho \) is one of the representations from the set

\[ \left\{ \mathbb{I} \otimes \nu^k \mid k \in \mathbb{Z}, H \subseteq G, [G/H] < \infty \right\}. \]

### 7. Fourier Coefficients of the Lifted Forms

In the previous section, we discussed about cuspidal properties of the lifted automorphic forms. In this section, we shall study their Fourier coefficients closely and prove Theorem 1.2.

#### 7.1. On the growth of the Fourier coefficients

In [2], the authors studied the growth of any holomorphic vVAF associated to Fuchsian groups of the first kind. More precisely, they showed that, there exists a constant \( \alpha \) (depending on the associated representation) such that \( \| \mathbb{X}[n] \| \ll H, \rho \ n^{k+2\alpha} \), where \( \mathbb{X}(\tau) \) is a holomorphic vVAF of weight \( k \in 2\mathbb{Z} \) associated a representation \( \rho \) of \( H \). Moreover, the constant \( \alpha \) depends only on \( H \) and the exponent \( k + 2\alpha \) can be divided by 2 for cuspforms.

In this section, we shall study the change of this exponent under lifting.

In general we show that, the constant \( \alpha \) is multiplied by at most the index of \( H' \) in \( G' \). In particular, the exponent do not change when \( \alpha = 0 \). In [2], the authors remarked that \( \alpha \) can be taken to be 0 when \( \rho \) is a unitary representation. To prove the main result of this section, let us start with the following.

**Lemma 7.1.** Let \( \rho \) be a representation of \( H' \) such that \( \| \rho(h) \| \ll_H \|h\|^{\alpha} \), \( \forall h \in H' \). Then the induced representation \( \tilde{\rho} \) associated to a finite extension \( G' \) of \( H' \) has the following growth:

\[ \| \tilde{\rho}(\gamma) \| \ll_{G'} \|\gamma\|^{\alpha}, \forall \gamma \in G'. \]

**Proof.** It follows from the definition of induced representations that,

\[ \| \tilde{\rho}(\gamma) \| \leq \max_{1 \leq i, j \leq d} \left\| \rho(\gamma \gamma_f^{-1}) \right\|, \]

On the other hand, it follows from the assumption on \( \rho \), and the semi multiplicative property of \( \| - \| \) that \( \left\| \rho(\gamma \gamma_f^{-1}) \right\| \ll_{G'} \|\gamma\|^{\alpha} \). This completes the proof.
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Let us now recall from Lemma 4.2 that, if \( \mathcal{X}(\tau) \) is a holomorphic vvf of weight 0, then the lifted form \( \tilde{\mathcal{X}}(\tau) \) is a holomorphic vvf as well. As a consequence to Lemma 7.1, we deduce that

**Corollary 7.2.** Let \( H, G \) and \( \rho \) be as in Theorem 1.1, and \( \mathcal{X}(\tau) \) be an admissible holomorphic vvf associated to \( (H', \rho) \) of weight 0. Then there exists a constant \( \alpha \) depending only on \( H' \) and \( \rho \) such that, the Fourier coefficients of the lifted holomorphic vvf \( \tilde{\mathcal{X}}(\tau) \) have the following growth

\[
\| \tilde{\mathcal{X}}[n] \| \ll H,\rho \ n^\alpha,
\]

In particular, the growth of the Fourier coefficients of the lifted vector-valued automorphic forms do not depend on \( G \). However, if \( \tilde{\mathcal{X}}(\tau) \) is a holomorphic logarithmic vvf, then the exponent increases by at most rank(\( \tilde{\rho} \)) := rank(\( \rho \))\( |G/H| \).

**Proof.** When \( \rho \) is admissible, we can write \( \| \tilde{\rho}(\gamma) \| \ll (c^2 + d^2)^\alpha \) due to Lemma 4.1 in [2]. Since \( \mathcal{X}(\tau) \) has even integer weight, it is evident that \( \rho(I) = \rho(-I) \). Therefore \( \rho \) can be thought of as a representation of \( H \). One can then follow the proof for the admissible case in the same article.

For the logarithmic case, one can essentially follow the argument at page 21 of [2]. The increase in the exponent is coming because of the extra logarithmic terms in the Fourier expansion, and they come with a power at most rank(\( \tilde{\rho} \)), which is precisely rank(\( \tilde{\rho} \))|\( G/H | = md. \)

### 7.2. Computations of the Fourier coefficients.

In this section, we shall study the relation between the Fourier coefficients of a weakly holomorphic vvf and its lift. As a consequence, we shall draw some conclusions on the unbounded denominator problem.

Let us start by recalling the coset representatives \( \{ g_{i,j} | 1 \leq i \leq n, 0 \leq j < h_i \} \) of \( H \) in \( G \) from Section 5. Let \( \mathcal{X}(\tau) \) be a weakly holomorphic vvf of weight 0, then by definition

\[
\tilde{\mathcal{X}}(\tau) = \left( \mathcal{X}(g_{i,j}^{-1} \tau) \right)_{1 \leq i \leq n, 0 \leq j < h_i}.
\]

To get the Fourier coefficients of \( \tilde{\mathcal{X}}(\tau) \), say, at the cusp \( \infty \), we need to study the Fourier expansion of each component \( \mathcal{X}(g_{i,j}^{-1} \tau) \) at \( \infty \).

Let \( H, G \) be as in Theorem 1.1 and \( \rho \) be an admissible representation of \( H \). Let \( \infty \) be a cusp of \( G \) and \( \{ c_i | 1 \leq i \leq n_\infty \} \) be the set of all inequivalent cusps of \( H \) lying under \( \infty \). For each such \( i \), we write \( \rho(t_i) = P_i J_i P_i^{-1} \), and \( \tilde{\rho}(t_\infty) = P_\infty J_\infty P_\infty^{-1} \), where \( J_i \) and \( J_\infty \) are the corresponding diagonal matrices. It follows from the proof of Theorem 4.2 in [1] that, we can write \( P_\infty \) in a block diagonal form \( \text{diag}(P'_1, P'_2, \cdots P'_{n_\infty}) \), where each block \( P'_i \) is of the size \( mh_i \times mh_i \), which can be described by the proof of Theorem 4.2 in [1]. In particular, when \( P'_i \) are all trivial, then all the entries of any such block \( P'_i \) either 0 or \( \lambda^{1-j/h_i} \zeta^j \), for some \( \zeta \in R_h \).

When \( \rho \) is a logarithmic representation, \( J_i \) are in a Jordan normal form. Even in this case, we can write each \( P_i \) in a block diagonal form as before. This follows from the discussion at Section 4.1. We do not have an exact description of them in
terms of the eigenvalues and some roots of unity as in the admissible case, however, it is enough for us to have such a block diagonal form.

7.3. Proof of Theorem 1.2. Let us first consider the admissible case. In this case, \( \tilde{X}[n] \) is denoted to be the \( n \)th Fourier coefficient of \( \tilde{X}(\tau) \) at \( \infty \), which is a \( md \times 1 \) vector. For any cusp \( c \) of \( H \) lying under \( \infty \), write \( \rho(t_i) = P_i e^{2\pi i \lambda_i} P_i^{-1} \) and

\[
\rho(t_i) = P_i \text{diag} \left( e^{\frac{2\pi i \lambda_i}{n_{c_i}}} \right) \begin{cases} 1 \leq i \leq n_c, \\ 0 \leq j < h_i \end{cases}
\]

It follows from Corollary 4.7 of [1], equation (1), and the discussion in the previous paragraph that, for each \( 1 \leq i \leq n_c \), we can write

\[
\left( X(g_{i,j}^{-1} \tau) \right)_{0 \leq j < h_i} = P_i' \text{diag} \left( \frac{h_i^{-1}}{q_i^{-1}} \right)_{0 \leq j < h_i} P_i'^{-1} \left( \sum_{n=-M_i}^\infty \tilde{X}_{i,j}^t[n] q_i^{-n} \right)_{0 \leq j < h_i}
\]

where \( \tilde{X}_{i,j}[n] \) is a \( m \times 1 \) vector. Denoting \( \tilde{X}'_{i,j}[n] = P_i'^{-1} \left( \tilde{X}_{i,j}[n] \right)_{0 \leq j < h_i} \), and using the identity that \( h_{c_i} = h_i h_{\infty} \), we have

\[
\left( X(g_{i,j}^{-1} \tau) \right)_{0 \leq j < h_i} = \left( \frac{h_i^{-1}}{q_i^{-1}} \sum_{n=-M_i}^\infty \tilde{X}'_{i,j}^t[n] q_i^{-n} \right)_{0 \leq j < h_i}
\]

\[
= \left( e^{\frac{2\pi i \lambda_i}{n_{c_i}}} \sum_{n=-M_i}^\infty \tilde{X}'_{i,j}^t[n] e^{\frac{2\pi i \tau}{n_{c_i}}} \right)_{0 \leq j < h_i}
\]

\[
= \left( e^{\frac{2\pi i \lambda_i}{n_{c_i}}} \sum_{n=-M_i}^\infty \tilde{X}'_{i,j}^t[n] e^{\frac{2\pi i \gamma_{c_i}}{n_{c_i}}} \right)_{0 \leq j < h_i}
\]

On the other hand, \( X|_{g_{i,j}^{-1}}(\tau) := X|_{0} g_{i,j}^{-1}(\tau) = X(g_{i,j}^{-1} \tau) \) is a weakly holomorphic vvf associated to \( H_{i,j} := g_{i,j} H g_{i,j}^{-1} \), with respect to the representation \( \rho_{i,j} := g_{i,j} \gamma_{i,j} \rightarrow \gamma \). Moreover, \( \infty \) is a cusp of \( H_{i,j} \) because \( g_{i,j}^{-1}(\infty) = c_i \), and \( c_i \) is a cusp of \( H \). In other words, \( g_{i,j} t_{i} g_{i,j}^{-1} \) is the generator of the stabilizer subgroup of \( \infty \) in \( H_{i,j} \), with \( \rho_{i,j}(g_{i,j} t_{i} g_{i,j}^{-1}) = \rho(t_i) \). Therefore, we can write

\[
P_i^{-1} X(g_{i,j}^{-1} \tau) = e^{\frac{2\pi i \lambda_i}{n_{c_i}}} \sum_{n=-M_i}^\infty \tilde{X}'_{i,j}(n) e^{\frac{2\pi i \gamma_{c_i}}{n_{c_i}}},
\]

where \( \tilde{X}'_{i,j}(n) = P_i^{-1} \tilde{X}_{i,j}(n) \). Denoting \( P_i' = P_i^{-1} \text{diag} (P_1, P_2, \ldots, P_{h_i}) \), we have

\[
P_i' \left( e^{\frac{2\pi i \lambda_i}{n_{c_i}}} \sum_{n=-M_i}^\infty \tilde{X}'_{i,j}(n) e^{\frac{2\pi i \gamma_{c_i}}{n_{c_i}}} \right)_{0 \leq j < h_i} = \left( e^{\frac{2\pi i \lambda_i}{n_{c_i}}} \sum_{n=-M_i}^\infty \tilde{X}'_{i,j}(n) e^{\frac{2\pi i \gamma_{c_i}}{n_{c_i}}} \right)_{0 \leq j < h_i}
\]

In particular, \( M_i = h_i M_{i,j} \) and \( \tilde{X}_{i,j}[n] = X(i,j)[n h_i + j] \).
Now, suppose that $X(\tau)$ is a weakly holomorphic logarithmic vvaf, i.e., let $\rho$ be a logarithmic representation of $H'$. Let $\{\lambda_{i,k}\}_{1 \leq k \leq e_i}$ be the set of eigenvalues of $\rho(t_i)$ for each $1 \leq i \leq n_{\infty}$, where $e_i$ is number of distinct eigenvalues of $\rho(t_i)$. From Corollary 5.3, we have an explicit description of the eigenvalues (with multiplicity) of $\rho(\tau_i)$, in terms of the $\lambda_{i,k}$. The argument for this case is essentially same as in the admissible case, therefore we shall omit some details. Following (3), and recalling the identity from Corollary 5.3 that $m(\lambda) = m(\xi \lambda^{1/h_i})$, we can write

\[
\left( X(g_{i,j}^{-1} \tau) \right)^{\prime}_{0 \leq j < h_i} = P_i \sum_{\lambda \in R(R)} \sum_{j = 0}^{m(\lambda) - 1} (\log \tilde{q}_c)^j h_i^j q_c^{\mu(\lambda)} \sum_{n = -M_{i,j}, \lambda, j}^{\infty} \tilde{X}_{i,j}[\lambda, j', n] \tilde{q}_{c_i}^{n h_i + j},
\]

where we denote $\tilde{X}_{i,j}[\lambda, j', n] = P_i^{h_i} \left( \tilde{X}_{i,j}[\lambda, j', n] \right)^{\prime}_{0 \leq j < h_i}$. On the other hand, realizing $X(g_{i,j}^{-1} \tau)$ as a weakly holomorphic vvaf associated to $H_{i,j}$, we are able to write

\[
\left( X(g_{i,j}^{-1} \tau) \right)^{(i,j)} = P_i \sum_{\lambda \in R(R)} \sum_{j = 0}^{m(\lambda) - 1} (\log \tilde{q}_c)^j h_i^j q_c^{\mu(\lambda)} \sum_{n = -M_{i,j}, \lambda, j}^{\infty} X^{(i,j)}[\lambda, j', n] \tilde{q}_{c_i}^{n},
\]

where we write $X^{(i,j)}[\lambda, j', n] = P_i^{h_i} X^{(i,j)}[\lambda, j', n]$. Now combining (9) and (10) we have $h_i M_{i,j}, \lambda, j' = M_{i,j}, \lambda, j'$ and

\[
\tilde{X}_{i,j}[\lambda, j', n] = h_i^{-j'} X^{(i,j)}[\lambda, j', nh_i + j].
\]

Moreover, for the both admissible or logarithmic case, we have

\[
X^{(i,j)}[\lambda, j', nh_i + j''] = 0, \text{ for any } 0 \leq j'' < h_i \text{ such that } j'' \neq j.
\]

Remark 7.3.

1. One can also recover Lemma 4.2 from Theorem 1.2.
2. More generally let $k$ be an arbitrary integer and $X(\tau)$ be a weakly holomorphic vvaf of weight $k$. Recall that, $\tilde{X}(\tau) := \tilde{X}_0^{(\infty)}(\tau) \Delta_{H'}^{k/w(\tau)}(\tau)$, where $X_0(\tau) = \Delta_{H'}^{k/w(\tau)}(\tau) X(\tau)$. In the proof of Theorem 1.2, we basically saw how to compute the Fourier coefficients of $X_0(\tau)$. By the same argument, the Fourier coefficients of $\tilde{X}(\tau)$ can be written in terms of the coefficients of $X_0 g_{i,j}^{-1}(\tau)$ and $\Delta_{H'}(\tau)$.
3. In certain cases, it turns out that $A_i = A_c$. In those cases, we are basically looking at the expansion at the cusp $c_i$ while considering the vvaf $\tilde{X}_{k A_c}$. For example, if $G = \Gamma(1)$ and $H = \Gamma_0(2)$, then one of the cusps is 0 and one of the $A_i$ is precisely $S$. However, in general it may not always be possible that any $A_{c_i}$ is in $G$. In those cases, they differ by an element of the form \( \left( \begin{array}{cc} a & \alpha \\ 0 & 1/a \end{array} \right) \), as already explained in the proof of Lemma 4.1.
ACKNOWLEDGEMENTS

The authors would like to thank for their hospitality the Georg-August Universität Göttingen, and Max Planck Institut für Mathematik, Bonn, where much of the work on this article was accomplished. SB is supported by ERC Consolidator grant 648329 (codename GRANT, with H. Helfgott as PI).

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