FREE BOUNDARY PROBLEM FOR A GAS BUBBLE IN A LIQUID, AND
ASYMPTOTIC STABILITY OF THE MANIFOLD OF SPHERICALLY
SYMMETRIC EQUILIBRIA

CHEN-CHIH LAI AND MICHAEL I. WEINSTEIN

Abstract. We consider the dynamics of a gas bubble immersed in an incompressible fluid of fixed
temperature, and focus on the relaxation of an expanding and contracting spherically symmetric
bubble due to thermal effects. We study two models, both systems of PDEs with a free boundary:
the full mathematical model as well as an approximate model, arising for example in the study
of sonoluminescence. For fixed physical parameters (surface tension of the gas-liquid interface,
liquid viscosity, thermal conductivity of the gas, etc.), both models share a family of spherically
symmetric equilibria, smoothly parametrized by the mass of the gas bubble. Our main result is
the nonlinear asymptotic stability of the manifold of equilibria with respect to small spherically
symmetric perturbations within the approximate system.

We also study the uniqueness of the family of spherically symmetric equilibria within each
model. The family of spherically symmetric equilibria captures all spherically symmetric equilib-
ria of the approximate system. However within the full model, this family is embedded in a larger
family of spherically symmetric solutions. For the approximate system, we prove that all equi-
librium bubbles are spherically symmetric, by an application of Alexandrov’s theorem on closed
surfaces of constant mean curvature.

1. Introduction

This paper considers a free boundary problem for the dynamics of a gas bubble immersed in
a liquid. The bubble occupies a bounded and simply connected subset of $\mathbb{R}^3$, denoted $\Omega(t)$. The
gas within the bubble is a compressible fluid characterized by its density, velocity, pressure and
temperature, as well as constitutive relations relating these variables and the specific entropy. The
surrounding liquid is assumed to be incompressible and is described by its velocity, pressure and
temperature. The gas inside the bubble and liquid outside the bubble are coupled at the boundary
by kinematic and stress-balance equations. Section 2 contains the full mathematical formulation of
the liquid / gas model. We are interested in the long time evolution of the coupled bubble / liquid
system for initial conditions which are near a spherically symmetric equilibrium.

Energy dissipation plays an important role in the bubble / fluid dynamics. Generally, there are
three mechanisms for energy dissipation of bubbles [27, p. 175]: radiation damping (of sound waves
toward infinity for the compressible fluid case), thermal damping (transfer of energy from the gas
into the fluid via thermal conduction) and viscous damping. We consider an approximation to the
full liquid / gas dynamics in which thermal damping is the dominant dissipation mechanism; viscous
damping is comparatively negligible, and there is no radiation damping due to sound wave emission
because the liquid surrounding the gas bubble is assumed incompressible.

The model we study is an asymptotic model by Prosperetti [42]. In [42] the linearized problem
was studied by means of Laplace transform, and it was believed that equilibrium solutions are
asymptotically stable. When the liquid is inviscid on the liquid-gas interface, the Prosperetti model
coincides with the approximate model derived by Biro and Velázquez in [5] based on the parameter
regimes of sonoluminescence experiments [2, 3, 4]. We present the asymptotic model of [42, 5] in
Section 3. In this model, the gas pressure, gas density and gas temperature all vary and are related
via the ideal gas equation of state. Solutions which are spherically symmetric are determined by a
reduced free boundary problem [5.1a]-[5.1c]: a quasilinear parabolic PDE (nonlinear diffusion) for
the density $\rho_g(r, t)$ in the gas bubble region, $0 \leq r \leq R(t)$, coupled to a second-order nonlinear ODE
for the bubble radius, $R(t)$; see Section 5. Local-in-time well-posedness in Hölder spaces was proved
for the initial value problem in [5].
1.1. Main results on asymptotic stability of the spherical bubble. The system \((5.1a)-(5.1c)\), in which spherical symmetry is imposed, has an equilibrium solution for any prescribed gas bubble mass. In [5] these spherically symmetric equilibria were proved to be Lyapunov stable in the following conditional sense. That is, a small spherically symmetric perturbation of the spherical bubble of the same mass will evolve, under the dynamics \((5.1a)-(5.1c)\), as a spherically symmetric solution which remains near the equilibrium bubble for all \(t > 0\).

Our main result builds on the analytical work of Biro-Velázquez [5]. The collection of all such equilibrium bubbles forms a smooth manifold of equilibria parameterized by the bubble mass. We prove asymptotic stability of the manifold of spherically symmetric equilibria: for sufficiently small spherically symmetric initial data perturbations of any spherically symmetric equilibrium bubble, the evolving bubble shape and surrounding liquid relax, as time advances, toward an equilibrium state, of typically different bubble radius and uniform density. Moreover, the equilibrium bubble radius and uniform gas density, which emerge as \(t \to \infty\), are determined by the initial data.

These results are stated in detail in Proposition 8.1 (asymptotic stability of a fixed equilibrium relative to small mass preserving perturbations) and Theorem 6.5 (asymptotic stability of the manifold of equilibria relative to arbitrary small perturbations). Theorem 6.5 is a consequence of Proposition 8.1 and the continuity of functionals.

Result on symmetry of all equilibrium bubbles of the asymptotic model. We also study the general, not necessarily spherically symmetric, version of \((5.1a)-(5.1c)\). That is, the asymptotic model \((3.1)-(3.3)\). We show that all equilibrium bubbles of \((3.1)-(3.3)\) must be spherically symmetric. The detailed result is presented in Part (1) of Proposition 4.3.

1.2. General context of our work and relation to other physical models. The dynamics of gas bubbles immersed in a liquid play an important role in fundamental and applied physics and in engineering applications. Examples include underwater explosion bubbles [21], bubble jetting [30], seismic wave-producing bubbles in magma [46], bubbles at the ocean surface [32], sonochemistry [71], sonoluminescence [7, 44]. Engineering and industrial examples include microfluidics [50], ultrasonic cavitation cleaning [37, 50], and applications of ultrasound cavitation bubbles such as medical imaging [15], shock wave lithotripsy (ESWL) [29, 28], tissue ablation [47, 10, 11], oncology and cardiology [29]. For a discussion of these and other applications of bubble dynamics, see the excellent review articles [26, 43, 31] and the book [27], and references cited therein.

The study of bubble dynamics was initiated in 1917 by Lord Rayleigh [45] during his work with Royal Navy to investigate cavitation damage on ship propellers. He derived an equation for the radial oscillations of a spherically symmetric gas bubble in an incompressible, inviscid liquid with surface tension and examined the pressure prediction during the collapse of a spherical bubble. Over several decades his work was refined and developed by numerous researchers. The Rayleigh-Plesset equation [39] is a second-order nonlinear ODE for the bubble radius. J.B. Keller and collaborators [21, 12, 22] incorporated the effect of liquid compressibility on the bubble dynamics and incorporated sound radiation from the oscillating bubble. These models have been extensively used in modeling and studied by asymptotic analytical and numerical and methods; see, for example, [11, 52, 54, 14, 43, 57, 59, 16, 17, 35] and references therein. This models all impose isothermal or adiabatic approximations in which the gas obeys polytropic equation of state (pressure \(\times\) a power of the volume is equal to a constant). Over the course of bubble oscillations, there are periods where the isothermal assumption and hence an adiabatic pressure volume law is valid (expansion), and periods over which the isothermal approximation is violated; strong compression, as in sonoluminescence experiments. Numerous works compare the two approximations and find a balance between them, e.g., [42, 55, 58].

The model we study \((5.1)-(5.3)\), or more specifically its spherically symmetric reduction, \((5.1a)-(5.1c)\), was introduced by Prosperetti [42], as an asymptotic approximation of the full liquid / gas-bubble system \((2.1)-(2.4)\), in which the gas pressure, density and temperature are related by an ideal gas law. Neither an isothermal nor adiabatic assumption is made. The model studied by Biro-Velázquez in [5] reduces to that of Prosperetti [42] when the liquid viscosity, \(\mu_l\), is assumed to be zero on the bubble interface. The article [5] studies, in the spherically symmetric setting: (i) local well-posedness in the space-time Hölder space, (ii) global well-posedness for initial data...
near a spherically symmetric equilibrium, and (iii) Lyapunov stability of the equilibrium relative to small mass-preserving perturbations. At the heart of their stability result is an energy dissipation identity and a coercivity estimate (lower bound) on the energy around the equilibrium, showing that spherically symmetric equilibria are constrained local minimizers.

Our work extends the result of [5] in the following directions.

1. We consider a more general model, i.e., the Prosperetti model [42], which incorporates liquid viscosity, $\mu_l \geq 0$, on the liquid-bubble interface.
2. We construct a manifold of spherically symmetric equilibria parametrized by the mass of gas bubble (Proposition 4.1).
3. We show that equilibrium bubbles of the approximate model are spherically symmetric provided $\mu_l \neq 0$ (Part (1) of Proposition 4.3).
4. We extend the conditional Lyapunov stability result of [5] to Lyapunov stability relative to arbitrary spherically symmetric perturbations which are small.
5. Most significantly, we prove asymptotic stability of the manifold of equilibria (Theorem 6.5). Our analysis demonstrates that the equilibrium gas-bubble, which emerges as $t \to \infty$, is determined by the initial data (and prescribed parameters of the model); it is the equilibrium bubble on the manifold of spherical states having the same mass as the initial (perturbed) bubble data.
6. We also study the persistence (structural stability) of the above results under a far-field time-dependent pressure, $p(x)(t)$; see Corollary 6.4 and Corollary 6.6.

Finally, we remark that there is an analogy of the present study with the asymptotic stability of coherent structures that the equilibrium state is determined by initial data in other nonlinear diffusive dynamical systems, e.g. smoothed out shock profiles in viscous perturbations of hyperbolic conservation laws [19], traveling front solutions in nonlinear reaction diffusion dynamics [34, 35, 6], and the spatial uniform equilibrium in two dimensional chemotaxis-fluid model [55].

1.3. Some future directions and open problems in the context of the current and closely related models.

1. **Time-decay rates.** Our results establish convergence, as time advances, to the manifold of equilibria. It is natural to study the rate of convergence. We expect that the equilibrium solutions are, at least linearly, exponentially stable.

2. **Time-periodically expanding and contracting bubble oscillations.** The far-field liquid pressure $p_x(t)$, is an external forcing term in our free boundary problem. In this present work, it is prescribed to be either the constant ($p_x(t) = p_{x,*}$) or such that $p_x(t) - p_{x,*}$ is small and decaying to zero sufficiently rapidly as $t \to \infty$. It is also of interest to study the bubble dynamics when the far-field pressure $p_x(t)$ is time-periodic, for example of the form $p_x(t) = 1 + A \cos(\omega t)$, corresponding to far-field periodic acoustic forcing as in physical experiments [2, 3, 4]. We expect that for sufficiently small forcing amplitude, $A$, there exist $2\pi/\omega$-time periodic spherically symmetric pulsating bubble solutions.

3. **Uniqueness of the spherically symmetric equilibria.** Spherically symmetric equilibria of the asymptotic model (3.1)-(3.3) are uniquely characterized in Part (2) of Proposition 4.3. In the context of the general evolution for the asymptotic model, any equilibrium bubble is necessarily spherical (Part (1) of Proposition 4.3). However, there exist non-trivial (rotational) equilibrium gas flows inside the equilibrium spherical bubble (Remark 4.4). In other words, the spherically symmetric equilibria are not unique within the asymptotic model. Under what circumstances is the family spherically symmetric equilibria are unique? Certainly, the above rotational equilibrium gas flows are ruled out if only seek gas flows which are irrotational. But are the spherically symmetric equilibria are unique within another closely related model? We expect that adding a gas viscosity term in the stress balance equation (3.3b) can help us exclude the case of non-trivial equilibrium gas flow in an equilibrium spherical bubble.

4. **Nonspherically symmetric dynamics.** Are spherically symmetric bubbles stable against small perturbations, unconstrained by symmetry? The main asymptotic stability result of the present paper requires spherically symmetric perturbations which are small. It is then natural to ask: Is the...
manifold of spherically symmetric equilibria asymptotically stable relative to small arbitrary (non-spherically symmetric) perturbations in the approximate system (3.1)-(3.3) (or in the full system (2.1)-(2.4))? We expect that surface tension plays an important role in the rounding out bubbles during the evolution.

A related question was studied in a model of a spherical polytropic gas bubble in a compressible liquid [48, 9]. In this case, the damping mechanism is acoustic radiation of waves to spatial infinity, rather than thermal diffusion. In [48] it is proved that the spherically symmetric bubble is linearly asymptotically stable relative to general (not necessarily spherical) perturbations. In the weakly compressible regime, the sharp exponential decay rate of perturbations was determined to spatial infinity, rather than thermal diffusion. In [48] it is proved that the spherically symmetric liquid [48, 9]. In this case, the damping mechanism is acoustic radiation of waves of the bubble. The energy of these shape modes is transferred very slowly to the surrounding compressible liquid and is radiated to infinity via acoustic waves.

Finally, we list some related broader open questions:

1. Well-posedness of the full liquid / gas model (2.1)-(2.4) for general data appears to be open.
2. Nonuniqueness of spherical equilibria of the full liquid / gas model. Even the classification of equilibrium solutions of (2.1)-(2.4) appears to be non-trivial. It is not known, for example, whether there are non-spherical equilibria. And, while in the approximate system (3.1)-(3.3), all symmetric equilibria are spatially uniform, this is not the case for (2.1)-(2.4): within the class of spherically symmetric equilibria of (2.1)-(2.4) there are solutions with spatially non-uniform temperature profiles (Remark 4.2). Some of these equilibria have a singularity in the gas temperature at the origin. It would be an interesting and challenging mathematical problem to investigate the dynamics for perturbations of such equilibria and to see whether they may participate in singularity formation for some classes of smooth initial conditions.
3. Radiation condition for the full liquid / gas model. We require a radiation condition (1.2) for the liquid temperature in order to prove the uniqueness of the spherical equilibrium solutions in (3.3) of a specified mass. Can one ensure that this radiation condition holds for the time-evolution if it is imposed on the initial data? The proof would require an a priori regularity of solutions to a free boundary problem of parabolic equations with time-dependent boundary condition.
4. Global energy minimizer. For the asymptotic model we have shown that the equilibrium \((\rho_*, R_*, \dot{R}_* = 0)\) is a conditional local energy minimizer of the total energy \(E_{\text{total}}\) (Definition 4.1), constrained to fixed mass. Our analysis relies on the Taylor expansion of the energy near an equilibrium. Is the equilibrium bubble solution of mass \(M\) a global minimizer of \(E_{\text{total}}\) relative to arbitrarily large spherically symmetric deformations of mass \(M\)?

1.4. Notation and conventions.

1. \(B_R = \{ x \in \mathbb{R}^3 : |x| < R \}\)
2. For a function \(f(r)\) defined for \(0 < r < R\), we set \(\tilde{f}(x) = f(|x|)\) for \(x \in B_R\) and denote
\[
\|f\|_{C^{2+2\alpha}} = \|\tilde{f}\|_{C^{2+2\alpha}} = \max_{|\beta| \leq 2} \sup_{x \in B_R} |D^\beta \tilde{f}(x)| + \sup_{x \neq y, x,y \in B_R} \frac{|D^2 \tilde{f}(x) - D^2 \tilde{f}(y)|}{|x-y|^{2\alpha}}.
\]
3. \(\Delta_r f = \frac{1}{r^2} \partial_r (r^2 \partial_r f)\) denotes the radial part of the Laplace operator in \(\mathbb{R}^3\).
4. For a state variable, such as the density \(\rho\), if it corresponds to value of a constant equilibrium solution, then we denote it by \(\rho_*\), and similarly for the values of other equilibrium state variables.
5. If \(u, v\) and \(w\) are vector fields, then \(u \cdot \nabla v \cdot w = [(u \cdot \nabla) v] \cdot w\).
6. If \(A = (A_{ij})\) and \(B = (B_{ij})\), then \(A : B = \text{tr}(A B^T) = \sum_{i,j} A_{ij} B_{ij}\).
7. \(\nabla u = (\partial_i u)\); \(|\nabla u|^2 \equiv \sum_{i,j} \partial_i u_j \partial_i u_j = \text{tr}((\nabla u)(\nabla u)^T)\).

Acknowledgements. The authors thank Juan J. L. Velázquez for detailed discussions concerning the article [5], which motivated the present work. We also thank Christophe Josserand and Qiang Du for very stimulating discussions. C. L. acknowledges support by the Simons Foundation as well
as support from the Department of Mathematics at Columbia University. MIW was supported in part by National Science Foundation Grant DMS-1908657 and Simons Foundation Math + X Investigator Award #376319 (Michael I. Weinstein).

2. GAS BUBBLE IN AN INCOMPRESSIBLE LIQUID: THE COMPLETE MATHEMATICAL FORMULATION

In this section we first discuss the complete mathematical description of a gas bubble immersed in an incompressible liquid with constant surface tension. We then present in Section 3 the asymptotic approximation studied in [4] in which thermal diffusion is the key dissipation mechanism. In Section 4 we derive an explicit family of our spherically symmetric equilibrium solutions of both of the full and approximate systems.

Equations for the liquid. Let \( v_l(x, t) \) denote the liquid velocity, \( p_l(x, t) \) the liquid pressure and \( T_l(x, t) \) is the liquid temperature. We assume that the dynamics of the liquid outside the bubble is described by the incompressible (constant density) Navier–Stokes equations

\[
\begin{align*}
(2.1a) & \quad \partial_t(p_l v_l) + \text{div}(p_l v_l \otimes v_l) = \text{div} T_l, \\
(2.1b) & \quad \text{div} v_l = 0, \\
(2.1c) & \quad \rho_l \partial_t(T_l + v_l \cdot \nabla T_l) = \text{div}(\kappa_l \nabla T_l) + S_l : \nabla v_l.
\end{align*}
\]

The stress tensor, \( T_l \) and viscous stress tensor, \( S_l \), are given, respectively, by:

\[
T_l = -p_l \mathbb{I} + S_l(v_l) \quad \text{and} \quad S_l(v_l) = 2\mu_l \mathbb{D}(v_l), \quad \mathbb{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top).
\]

Here, \((u \otimes v)_{ij} = u_i v_j\) denotes the tensor product of vectors and \( \mathbb{A} : \mathbb{B} = \sum_{i,j=1}^{3} A_{ij} B_{ij} = tr(AB^\top)\), where \( \mathbb{A} = (A_{ij})_{i,j=1}^{3}, \mathbb{B} = (B_{ij})_{i,j=1}^{3} \). The equations depend on parameters: \( p_l > 0 \), the density of the liquid, \( \mu_l \geq 0 \), the dynamic viscosity of the liquid, \( \epsilon_l \), the specific heat of the liquid, and \( \kappa_l \), the thermal conductivity of the liquid. Equations (2.1a) and (2.1b) express, respectively, balance of momentum and conservation of mass. These are coupled to equation (2.1c), which governs the temperature field in the liquid.

Equations for the gas. The gas within the bubble is assumed to be a compressible fluid, characterized by its velocity \( v_g(x, t) \), pressure \( p_g(x, t) \), density \( \rho_g(x, t) \), temperature \( T_g(x, t) \), and entropy per unit mass (specific entropy) \( s(x, t) \), with the assumption of the ideal gas law relating \( p_g, T_g \) and \( \rho_g \). The governing equations are the viscous, compressible Navier–Stokes equations

\[
\begin{align*}
(2.2a) & \quad \partial_t \rho_g + \text{div}(\rho_g v_g) = 0, \\
(2.2b) & \quad \partial_t(p_g v_g) + \text{div}(\rho_g v_g \otimes v_g) = \text{div} T_g, \\
(2.2c) & \quad \rho_g T_g \partial_t s + v_g \cdot \nabla s = \text{div}(\kappa_g \nabla T_g) + S_g : \nabla v_g, \\
(2.2d) & \quad \rho_g = \mathbb{R}_g T_g \rho_g, \\
(2.2e) & \quad s = c_v \log \left( \frac{p_g}{\rho_g^\gamma} \right),
\end{align*}
\]

where

\[
\begin{align*}
T_g & = -p_g \mathbb{I} + 2\mu_g \left( \mathbb{D}(v_g) - \frac{1}{3} (\text{div} v_g) \mathbb{I} \right) + \zeta_g (\text{div} v_g) \mathbb{I}, \\
S_g & = 2\mu_g \left( \mathbb{D}(v_g) - \frac{1}{3} (\text{div} v_g) \mathbb{I} \right) + \zeta_g (\text{div} v_g) \mathbb{I}
\end{align*}
\]

is the stress tensor of the gas in which \( \mu_g > 0 \) and \( \zeta_g \) are the dynamic viscosity and the bulk viscosity for the gas, respectively, and

\[
\mathbb{R}_g = \frac{\kappa_g}{c_v} = \frac{c_p}{c_v} > 1
\]

is the viscous tensor of gas. The constant \( \kappa_g \) is the thermal conductivity of the gas. The constant \( \mathbb{R}_g \) is the specific gas constant, the ratio of the ideal gas constant to the molar mass. The constant \( \gamma \equiv 1 + \frac{\mathbb{R}_g}{c_v} \).
is called the adiabatic constant. Here, \( c_p \) denotes the heat capacity at constant pressure and \( c_v \) denotes the heat capacity at constant volume. Equations (2.2a) and (2.2b) are the equations of motion and continuity, respectively, of a compressible fluid. Equation (2.2c) is the entropy equation. Equation (2.2d) is the equation of state (Boyle’s law) for ideal gases. Equation (2.2e) is a consequence of the second law of thermodynamics, (2.2d), and Joule’s second law for ideal gases.

**Boundary conditions at the liquid / gas interface.** Let the bubble surface, \( \partial \Omega(t) \), be given in spherical coordinates

\[ \mathbf{\omega} = \mathbf{\omega}(\theta, \varphi, t) \]

and let \( \mathbf{n} \) denote the outward pointing unit outer normal on \( \partial \Omega(t) \). The boundary conditions on \( \partial \Omega(t) \) are

\[
\begin{align*}
(2.4a) & \quad \mathbf{v}_l(\mathbf{\omega}, t) \cdot \mathbf{n} = \mathbf{v}_g(\mathbf{\omega}, t) \cdot \mathbf{n} = \mathbf{\nabla}_l \mathbf{\omega} \cdot \mathbf{n}, \\
(2.4b) & \quad \mathbf{n} \cdot \mathbf{T}_l - \mathbf{n} \cdot \mathbf{T}_g = \sigma \mathbf{n} (\nabla_S \cdot \mathbf{n}), \\
(2.4c) & \quad T_g = T_l,
\end{align*}
\]

where \( \nabla_S \cdot \mathbf{n} \) denotes the surface divergence, and \( \sigma > 0 \) is the surface tension of the liquid - gas interface, here assumed to be a constant. Equation (2.4a) is the kinematic boundary condition; the normal velocity of the material point on the bubble surface moves with the normal velocity of both the gas and the liquid. Equation (2.4b) is the stress balance equation. Equation (2.4c) means the temperature is continuous across the interface. A detailed derivation of the fundamental equations of fluid dynamics is presented, for example, in [30, 13].

The system (2.1)-(2.4) depends on the physical parameters: \( \nu_l = \mu_l/\rho_l, \rho_l, c_l, \kappa_l, \mu_g, \kappa_g, \alpha, \gamma, c_v, \zeta_g, \sigma, \)

where \( \nu_l \), the kinematic viscosity, is a nonnegative constant and all other parameters are all strictly positive constants. We assume these parameters to be prescribed and fixed. Furthermore, one prescribes:

the far-field liquid pressure \( p_{\infty}(x,t) := \lim_{|x| \to \infty} p_l(x,t) \), and

the far-field liquid temperature \( T_{\infty}(x,t) := \lim_{|x| \to \infty} T_l(x,t) \).

Here, we assume that the far-field pressure in the liquid, \( p_{\infty}(x,t) \), is spatially uniform and is a small perturbation of a positive constant \( p_{\infty,0} \):

\[ p_{\infty} \in C^{1+\alpha}_1(\mathbb{R}_+^3), \quad |p_{\infty}(t) - p_{\infty,0}| + \| \partial_t p_{\infty} \|_{L^1(\mathbb{R}_+^3)} \leq \eta_0, \quad p_{\infty}(t) \to p_{\infty,0} \text{ as } t \to \infty, \]

where \( \eta_0 > 0 \) is some small number to be chosen later. We also assume that the far-field liquid temperature is a constant, \( T_{\infty} \):

\[ T_{\infty}(x,t) \equiv T_{\infty}. \]

For fixed physical parameters (2.5), \( p_{\infty}(x,t) \), and \( T_{\infty} \), the system (2.1)-(2.4) governs the time-evolution of the state variables in the liquid: \( \mathbf{v}_l(x,t), p_l(x,t), T_l(x,t), \)

state variables in the gas: \( \rho_g(x,t), \mathbf{v}_g(x,t), p_g(x,t), T_g(x,t), s(x,t), \) and

the gas bubble region, \( \Omega(t) \subset \mathbb{R}^3. \)

To this we add constitutive relations (2.2d)-(2.2e), which enable us to express \( T_g \) and \( s \) in terms of \( \rho_g, \mathbf{v}_g \) and \( p_g \). Moreover, since the liquid pressure \( p_l \) solves the exterior Dirichlet boundary-value problem of Poisson equation

\[ -\Delta p_l = \rho_l \nabla \cdot (\nabla \mathbf{v}_l) \] in \( \mathbb{R}^3 \setminus \Omega(t), \]

with the boundary conditions

\[ p_l = \mathbf{n} \cdot \mathbf{S}_l \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{T}_g \cdot \mathbf{n} - \sigma \nabla_S \cdot \mathbf{n} \text{ on } \partial \Omega(t), \quad \lim_{|x| \to \infty} p_l(x,t) = p_{\infty}(x,t), \]
\( p_l - p_x(t) \) satisfies
\[-\Delta (p_l - p_x(t)) = \rho_l \nabla v_l : (\nabla v_l)^T \text{ in } \mathbb{R}^3 \setminus \Omega(t), \]
\[ p_l - p_x(t) = \dot{\mathbf{n}} \cdot S_l \cdot \mathbf{n} - \dot{\mathbf{n}} \cdot \nabla S_g \cdot \mathbf{n} - \sigma \nabla S \cdot \mathbf{n} - p_{x}(t) \text{ on } \partial \Omega(t), \]
\[ \lim_{|x| \to \infty} p_l(x,t) - p_x(t) = 0. \]

For suitable \( \nabla v_l \) with sufficient decay at spatial infinity, \( p_l - p_x(t) \) can be expressed by means layer potentials as
\[ p_l(x,t) - p_x(t) = \int_{\mathbb{R}^3 \setminus \Omega(t)} G(x,y,t) \left[ \rho_l \nabla v_l : (\nabla v_l)^T \right](y) \, dy \]
\[- \int_{\partial \Omega(t)} \nabla_y G(x,y,t) \cdot \mathbf{n} \left[ \dot{\mathbf{n}} \cdot S_l \cdot \mathbf{n} - \dot{\mathbf{n}} \cdot \nabla S_g \cdot \mathbf{n} - \sigma \nabla S \cdot \mathbf{n} - p_x(t) \right](y) \, dS_y, \]
where \( G(x,y,t) \) is the Green’s function for the exterior domain \( \mathbb{R}^3 \setminus \Omega(t) \). Then \( p_l \), for \( x \notin \Omega(t) \), can be expressed in terms \( v_l \) by
\[ p_l(x,t) = p_x(t) + \int_{\mathbb{R}^3 \setminus \Omega(t)} G(x,y,t) \left[ \rho_l \nabla v_l : (\nabla v_l)^T \right](y) \, dy \]
\[- \int_{\partial \Omega(t)} \nabla_y G(x,y,t) \cdot \dot{\mathbf{n}} \left[ \dot{\mathbf{n}} \cdot S_l \cdot \mathbf{n} - \dot{\mathbf{n}} \cdot \nabla S_g \cdot \mathbf{n} - \sigma \nabla S \cdot \mathbf{n} - p_x(t) \right](y) \, dS_y. \]
Therefore, (2.1) - (2.4) can be reduced to a problem for the unknown liquid and gas state variables, and the region filled with gas:
\[ v_l(x,t), T_l(x,t), \rho_g(x,t), v_g(x,t), p_g(x,t), \Omega(t). \]

**Initial data.** To solve for the evolution given by the full liquid / gas bubble system (2.1) - (2.4), we must prescribe initial conditions for the state variables:
\[ (2.6) \quad v_l(\cdot,0), T_l(\cdot,0), \rho_g(\cdot,0), v_g(\cdot,0), p_g(\cdot,0), \]
and for the bubble shape at time \( t = 0 \):
\[ (2.7) \quad \Omega(t) \bigg|_{t=0} = \Omega(0). \]
We assume the compatibility conditions for the initial data, i.e., they satisfy (2.1) - (2.4) at \( t = 0 \). In particular, \( \text{div } v_l(\cdot,0) = 0 \).

### 3. An asymptotic approximation to (2.1) - (2.4)

In this paper we work with the following approximation of Prosperetti [12] (see also Biro-Velázquez [5] Appendix A) to the full liquid - bubble system (2.1) - (2.4):
\[ (3.1a) \quad \partial_t v_l = \nu_l \Delta v_l - v_l \cdot \nabla v_l - \frac{1}{\rho_l} \nabla p_l, \]
\[ (3.1b) \quad \text{div } v_l = 0, \]
\[ (3.1c) \quad T_l(x,t) = T_{x}, \quad \text{a prescribed constant}, \]
where \( \nu_l = \frac{\mu_l}{\rho_l} \geq 0 \) is the kinematic viscosity of the liquid,
\[ (3.2a) \quad \partial_t \rho_g + \text{div}(\rho_g v_g) = 0, \]
\[ (3.2b) \quad p_g = p_g(t), \]
\[ (3.2c) \quad \rho_g T_g (\partial_t s + v_g \cdot \nabla s) = \text{div}(\kappa_g \nabla T_g), \]
\[ (3.2d) \quad p_g = \beta_g T_g p_g, \]
\[ (3.2e) \quad s = c_v \log \left( \frac{p_g}{p_g} \right), \]
where \( \beta_g, \kappa_g \) are constant.
and

\begin{align}
\text{(3.3a)} & \quad \mathbf{v}_l(\omega, t) \cdot \hat{n} = \mathbf{v}_g(\omega, t) \cdot \hat{n} = \partial_t \omega \cdot \hat{n}, \\
\text{(3.3b)} & \quad \rho_g \hat{n} - p \hat{n} + 2\mu \hat{n} \cdot \nabla (\mathbf{v}_l) = \sigma \hat{n} (\nabla S \cdot \hat{n}), \quad \text{on } \partial \Omega(t), \ t > 0, \\
\text{(3.3c)} & \quad T_g = T_{x \Omega},
\end{align}

This model reduces to that of [7] Appendix A for the special case when \( \mu_1 = 0 \) in (3.3b). Equation (3.3b) is the Young–Laplace boundary condition; the jump in pressure at the liquid-gas interface is equal to the surface tension, \( \sigma \), times the mean curvature \( H = \frac{1}{2} \nabla S \cdot \hat{n} \). The approximate system (3.1)-(3.3) depends on the physical parameters: \( \nu_l, \rho_l, \kappa_g, \beta_g, \gamma, c_v, \sigma \).

For fixed physical parameters (3.4), \( p_x(t) \), and \( T_{x \Omega} \), the approximate system (3.1)-(3.3) governs the time-evolution of the state variables in the liquid:

\[ \mathbf{v}_l(x, t), p_l(x, t) \]

and in the gas

\[ \rho_g(x, t), \mathbf{v}_g(x, t), p_g(t), T_g(x, t), s(x, t) \]

and \( \Omega(t) \). We show below in Appendix B that the system (3.2) can be reduced to a single equation (3.6) for \( \rho_g \) depending only on \( \Omega \) and \( p_g \). Thus, (3.1)-(3.3) can be reduced to a problem with unknowns

\[ \mathbf{v}_l(x, t), p_l(x, t), p_g(t), \Omega(t). \]

As for the initial conditions to prescribe for the approximate system (3.1)-(3.3), we need

\[ \mathbf{v}_l(\cdot, 0), p_l(\cdot, 0) \]

for state variables, and (2.7) for the bubble shape at time \( t = 0 \). We do not prescribe initial data for \( p_g \) since it can be derived from \( p_l(\cdot, 0) \) and \( \Omega(0) \) via (3.3b).

In this article the approximate system (3.1)-(3.3) is considered under the assumption of spherical symmetry. More precisely, we assume for the system (3.1)-(3.3) that \( \Omega(t) \) is a sphere, \( \mathbf{v}_l, \mathbf{v}_g \) are spherically symmetric, \( p_l, \rho_g, T_g, s \) are radial. Recall that a vector field \( \mathbf{u} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is spherically symmetric if \( \mathbf{u} = u(r)\hat{r}, \ r = |x| \) and \( \hat{r} = x/|x| \), and a scalar function \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) is radial if \( f = f(r) \). In this setting, (3.1)-(3.3) reduces to the system (5.1a)-(5.1c) for \( \rho(r, t), R(t) \). Under the assumption of spherical symmetry, (5.1a)-(5.1c) is well-posed locally in time (Theorem 5.3), and well-posed globally in time for initial data which is close to the equilibrium (see [5] Theorem 4.1) and Theorem 6.3.

4. SPHERICALLY SYMMETRIC EQUILIBRIUM SOLUTIONS

Both the full liquid / gas model (2.1)-(2.4) and the asymptotic model (3.1)-(3.3) share a family of spherically symmetric equilibrium (time-independent) solutions. Let \( B_R \) denote the open ball in \( \mathbb{R}^3 \) of radius \( R \) which is centered at the origin. Suppose a gas of density \( \rho_g(x) \) occupies the region \( B_R \). Then, the mass of the bubble is given by

\[ \text{(4.1)} \quad \text{Mass}[\rho_g, R] = \int_{B_R} \rho_g(x) \, dx. \]

Below we investigate spherical symmetric equilibrium solutions of both (2.1)-(2.4) and the approximation system (3.1)-(3.3). We prove that the spherically symmetric equilibrium gas bubble of the approximate system (3.1)-(3.3), and of the original system (2.1)-(2.4) with additional conditions, is, up to spatial translation of its center, uniquely determined by its total mass. Moreover, we prove that equilibrium bubbles of the approximate system (3.1)-(3.3) are spherical by applying the Alexandrov’s theorem on closed constant-mean-curvature (CMC) surfaces. In equilibrium spherical bubbles of the approximate system (3.1)-(3.3), there exists nontrivial, e.g., rotational, equilibrium gas flow (see Remark 4.4).
Proposition 4.1 (Spherically symmetric equilibria of the original system (2.1)-(2.4)). Fix a constant $p_{x,*} > 0$. Assume the radiation condition for liquid temperature:

$$T_l(|x|) = T_\infty + o(|x|^{-1}), \quad |x| \to \infty.$$  

Then, there is a smooth map from values of the bubble mass to equilibrium radii, $R_*$:

$$M \in (0, \infty) \mapsto R_*[M],$$

such that any regular (non-singular) spherical equilibrium solution of (2.1)-(2.4) (for fixed parameters (2.5)) of bubble mass $M$ is expressible as:

(4.3a) $$v_{l,*} = 0, \quad p_{l,*} = p_{x,*}, \quad \Omega_* = B_{R_*[M]};$$

(4.3b) $$\rho_{g,*}[M] = \frac{1}{\mathcal{A}_g T_\infty} \left( p_{x,*} + \frac{2\sigma}{R_*[M]} \right), \quad v_{g,*} = 0, \quad p_{g,*}[M] = p_{x,*} + \frac{2\sigma}{R_*[M]};$$

(4.3c) $$T_{g,*} = T_{l,*} = T_\infty, \quad s_* = c_0 \log \left( \gamma \left( p_{x,*} + \frac{2\sigma}{R_*[M]} \right)^{-\gamma} \right).$$

The proof of Proposition 4.1 is given in Appendix A.

Remark 4.2. The radiation condition (4.2) and the regularity assumption in part (1) are necessary for the uniqueness of the spherical equilibrium solutions $T_{l,*}$ and $T_{g,*}$ in (4.3). In fact, without such hypotheses there exists a two parameter family of spherical equilibrium solutions $T_{l,*}$ and $T_{g,*}$ of (2.1)-(2.4) given by:

$$T_{l,*}(r) = T_\infty - a_1/r, \quad r \in [R_*, \infty), \quad T_{g,*}(r) = T_\infty - a_1/R_* + a_2(1/R_* - 1/r), \quad r \in [0, R_*),$$

where $a_1, a_2 \in \mathbb{R}$ are arbitrary.

Proposition 4.3 (Equilibria of the approximate system (3.1)-(3.3)). Fix a constant $p_{x,*} > 0$.

1. Equilibrium bubbles of (3.1)-(3.3) are spherical: Let $(v_{l,*}, p_{l,*}, p_{g,*}, \Omega_*)$ be a $C^2$ steady-state solution of (3.1a), (3.1b), (3.2a), (3.3a), (3.3b) with $\lim_{|x| \to \infty} p_{l,*}(x) = p_{x,*}$. Assume $\mu_1 \neq 0$ and $\sigma \neq 0$ in (3.3). Suppose that $\lim_{|x| \to \infty} v_{l,*}(x) = O(|x|^{-2})$ and that $\nabla v_{l,*}$ is bounded. Then $v_{l,*} = 0$, $p_{l,*} = p_{x,*}$ and $\Omega_*$ is a sphere.

2. Spherically symmetric equilibria of (3.1)-(3.3): The reduced / asymptotic model (3.1)-(3.3) shares the family of spherically symmetric equilibria displayed in (4.3). Furthermore, any spherical equilibrium solution of (3.1)-(3.3) is uniquely determined by its total mass as (4.3). No radiation condition (4.2) or regularity assumption is required.

3. The mappings $M \in (0, \infty) \mapsto R_*[M]$ and $\rho_*[M]$, where $\rho_* := \rho_{g,*}$, arising in Proposition 4.1 are continuous (even smooth).

Remark 4.4. Part (1) of Proposition 4.3 is the uniqueness of (4.3a) for the gas phase. Indeed, replacing $v_{g,*} = 0$ in (4.3b) with any non-trivial solenoidal vector field in $\Omega_* \setminus B_{R_*}$ yields another steady state solution to the approximate system (3.1)-(3.3). Recall that a vector field $u$ is solenoidal if $\text{div} u = 0$ and $u \cdot \hat{n}|_{\partial \Omega_*} = 0$. One can choose, for example, $v_{g,*}(x_1, x_2, x_3) = (-x_2, x_1, 0)$. The example is ruled out by the spherically symmetric assumption in Part (1) of Proposition 4.3. Other possible way to show the uniqueness of (4.3a)-(4.3c) is to impose irrotational assumption.

Proof of Proposition 4.3: We first prove Part (1) concerning the uniqueness of the equilibrium (4.3a) of the approximate system (3.1)-(3.3). Note that steady-state solutions of (3.1)-(3.3) solve

(4.4a) $$0 = v_{l,*} \Delta v_{l,*} - v_{l,*} \cdot \nabla v_{l,*} - \frac{1}{\rho_*} \nabla p_{l,*},$$

(4.4b) $$\text{div} v_{l,*} = 0,$$

(4.4c) $$T_{l,*}(x) = T_\infty, \quad \text{a prescribed constant},$$

in $\mathbb{R}^3 \setminus \Omega_*$. 

Using (4.10), (4.9) can be written as
\begin{align*}
\text{(4.10)} & \quad \text{div}(\rho_{g,*} v_{g,*}) = 0, \\
\text{(4.11)} & \quad p_{g,*}(x) = p_{g,*}, \quad \text{a constant}, \\
\text{(4.12)} & \quad \rho_{g,*} T_{g,*} (v_{g,*} \cdot \nabla s_{g,*}) = \text{div}(\kappa_{g} \nabla T_{g,*}), \\
\text{(4.13)} & \quad p_{g,*} = \beta_{g} T_{g,*} p_{g,*}, \\
\text{(4.14)} & \quad s_{g,*} = c_{v} \log \left( \frac{p_{g,*}}{p_{g,*}} \right),
\end{align*}
in \Omega_{*},

and the far-field pressure is
\begin{equation}
\lim_{|x| \to \infty} p_{l,*}(x) = p_{x,*}.
\end{equation}

For \( r > 0 \) sufficiently large, multiplying the equation (4.4a) by \( v_{l,*} \), integrating over \( B_{r} \setminus \Omega_{*} \), using integration by parts formula and \( \text{div} v \) is solenoidal: \( \text{div} v \) is uniformly bounded. Finally, the last term in (4.8) also tends to zero as \( r \to \infty \). Indeed, since \( \lim_{|x| \to \infty} v_{l,*}(x) = O(|x|^{-2}) \),
\begin{equation}
\frac{1}{\rho_{l}} \lim_{r \to \infty} \int_{B_{r}} \rho_{l} v_{l,*} \cdot \hat{n} dS = \frac{1}{\rho_{l}} \lim_{r \to \infty} \int_{B_{r}} v_{l,*} \cdot \hat{n} dS + \int_{B_{r}} (p_{l,*} - p_{x,*}) v_{l,*} \cdot \hat{n} dS
= \left( \frac{1}{\rho_{l}} \lim_{r \to \infty} \int_{B_{r}} \text{div} v_{l,*} dS - \int_{B_{r}} \frac{1}{\rho_{l}} \text{div} v_{l,*} \cdot (-\hat{n}) dS \right) + 0 = 0
\end{equation}
since \( v_{l,*} \) is solenoidal: \( \text{div} v_{l,*} = 0 \) and \( v_{l,*} \cdot \hat{n}|_{\partial \Omega_{*}} = 0 \). Thus, by taking \( r \to \infty \), (4.8) becomes
\begin{equation}
0 = -\mu_{l} \int_{\mathbb{R}^{3} \setminus \Omega_{*}} \left| \nabla v_{l,*} \right|^{2} dx - \nu_{l} \int_{\partial \Omega_{*}} \hat{n} \cdot \nabla v_{l,*} \cdot v_{l,*} dS.
\end{equation}

Multiplying the stress balance equation (4.6b) by \( v_{l,*} \) and using \( v_{l,*} \cdot \hat{n} = 0 \) yield \( \hat{n} \cdot \text{D}(v_{l,*}) \cdot v_{l,*} |_{\partial \Omega_{*}} = 0 \) since \( \mu_{l} \neq 0 \). Using the expression of the deformation tensor \( \text{D}(v_{l,*}) = (\nabla v_{l,*} + (\nabla v_{l,*})^{T})/2 \),
\begin{equation}
0 = 2 \hat{n} \cdot \text{D}(v_{l,*}) \cdot v_{l,*} = \hat{n} \cdot (\nabla v_{l,*} + (\nabla v_{l,*})^{T}) \cdot v_{l,*}
= \hat{n} \cdot \nabla v_{l,*} \cdot v_{l,*} + v_{l,*} \cdot \nabla v_{l,*} \cdot \hat{n}.
\end{equation}
Using (4.10), (4.9) can be written as
\begin{equation}
0 = -\mu_{l} \int_{\mathbb{R}^{3} \setminus \Omega_{*}} \left| \nabla v_{l,*} \right|^{2} dx + \nu_{l} \int_{\partial \Omega_{*}} v_{l,*} \cdot \nabla v_{l,*} \cdot \hat{n} dS.
\end{equation}
Since $-\mathbf{n}$ is the outward normal of $\mathbb{R}^3 \setminus \Omega_*$ on $\partial \Omega_*$, by (4.11) and the divergence theorem,
\begin{align*}
0 &= -\nu \int_{\mathbb{R}^3 \setminus \Omega_*} |\nabla v_{l,*}|^2 \, dx - \nu \int_{\partial \Omega_*} v_{l,*} \cdot \nabla v_{l,*} \cdot (-\mathbf{n}) \, dS \\
&= -\nu \int_{\mathbb{R}^3 \setminus \Omega_*} |\nabla v_{l,*}|^2 \, dx - \nu \int_{\mathbb{R}^3 \setminus \Omega_*} \text{div}(v_{l,*} \cdot \nabla v_{l,*}) \, dx \\
&= -\nu \int_{\mathbb{R}^3 \setminus \Omega_*} |\nabla v_{l,*}|^2 \, dx - \nu \int_{\mathbb{R}^3 \setminus \Omega_*} \nabla v_{l,*} : (\nabla v_{l,*})^T \, dx, \quad \text{where } A : B := \sum_{i,j} A_{ij} B_{ij} = \text{tr}(AB^T), \\
&= -\nu \int_{\mathbb{R}^3 \setminus \Omega_*} \nabla v_{l,*} : (\nabla v_{l,*} + (\nabla v_{l,*})^T) \, dx = -\nu \int_{\mathbb{R}^3 \setminus \Omega_*} \text{tr}(\nabla v_{l,*}(\nabla v_{l,*} + (\nabla v_{l,*})^T) \, dx.
\end{align*}

For any square matrix $A$, decomposing into symmetric and anti-symmetric parts we have:
\[
A(A + A^T) = \frac{1}{2}(A + A^T)^2 + \frac{1}{2}(A - A^T)(A + A^T). \quad \text{Linearity of the trace and the identities } \text{tr}(AC) = \text{tr}(CA) \text{ and } \text{tr}(A) = \text{tr}(A^T), \text{ then imply } \text{tr}[A(A + A^T)] = \frac{1}{2}\text{tr}[(A + A^T)^2] = \frac{1}{2}|A + A^T|^2 \text{ since } A + A^T \text{ is symmetric. Hence,}
\]
\[
\frac{1}{2} \int_{\mathbb{R}^3 \setminus \Omega_*} |\nabla v_{l,*} + (\nabla v_{l,*})^T|^2 \, dx = 0.
\]
Therefore, $\nabla v_{l,*} + (\nabla v_{l,*})^T = 0$ in $\mathbb{R}^3 \setminus \Omega_*$. 

Let $\mathbb{R}(v_{l,*}) = (\nabla v_{l,*} - (\nabla v_{l,*})^T)/2$ be the rotation matrix, i.e., the antisymmetric part of $\nabla v_{l,*}$. Then $\mathbb{R}(v_{l,*}) = \nabla v_{l,*}$ since $\nabla v_{l,*} = -(\nabla v_{l,*})^T$. Recall that $\mathbb{R}(v_{l,*})h = \frac{1}{2}\text{curl } v_{l,*} \times h$ for all $h \in \mathbb{R}^3$ (see, e.g., [33] (1.21)). Taking $h = v_{l,*}$,
\begin{align}
(4.12) \quad \mathbb{R}(v_{l,*})v_{l,*} &= \frac{1}{2}\text{curl } v_{l,*} \times v_{l,*}.
\end{align}

Using $\mathbb{R}(v_{l,*}) = \nabla v_{l,*}$, the left hand side of (4.12) becomes
\[
\nabla v_{l,*} \cdot v_{l,*} = \nabla \left(\frac{|v_{l,*}|^2}{2}\right).
\]

For the right hand side of (4.12), we use the vector calculus identity
\[
\nabla(A \cdot B) = (A \cdot \nabla)B + (B \cdot \nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)
\]
with $A = B = v_{l,*}$ to get
\[
\text{curl } v_{l,*} \times v_{l,*} = -\nabla \left(\frac{|v_{l,*}|^2}{2}\right) + (v_{l,*} \cdot \nabla) v_{l,*}.
\]

So (4.12) yields
\[
\nabla \left(\frac{|v_{l,*}|^2}{2}\right) = -\nabla \left(\frac{|v_{l,*}|^2}{4}\right) + \frac{1}{2}(v_{l,*} \cdot \nabla) v_{l,*},
\]
or, equivalently,
\[
3\nabla \left(\frac{|v_{l,*}|^2}{2}\right) = (v_{l,*} \cdot \nabla) v_{l,*}.
\]

Using $\nabla v_{l,*} = -(\nabla v_{l,*})^T$, we have
\[
3\nabla \left(\frac{|v_{l,*}|^2}{2}\right) = -v_{l,*} \cdot (\nabla v_{l,*})^T = -\nabla \left(\frac{|v_{l,*}|^2}{2}\right).
\]

This implies $v_{l,*} \equiv 0$ since $v_{l,*}(x) \to 0$ as $|x| \to 0$. Since $v_{l,*} \equiv 0$, (4.11) implies that $\nabla p_{l,*} \equiv 0$, and so $p_{l,*} \equiv p_0$ is a constant. Since $\mathbb{D}(v_{l,*}) = (\nabla v_{l,*} + (\nabla v_{l,*})^T)/2 = 0$, the stress balance equation (4.6b) becomes
\[
p_g - p_{l,*} = \sigma \nabla S \cdot \mathbf{n} \quad \text{on } \partial \Omega_*.
\]

Since both $p_g$ and $p_{l,*}$ are constant, $\partial \Omega_*$ is a closed constant-mean-curvature (CMC) surface. By Alexandrov’s Theorem [3], $\Omega_*$ must be a sphere. This asserts Part (1) of Proposition (3.3).

To derive the spherically symmetric equilibria of the approximate system (3.1)–(3.3), we make use of Proposition (5.1) (below), which presents a reduction of (3.1)–(3.3) in the spherically symmetric case, to an equivalent system for $\rho$ and $R$, where $\rho = \rho_g$; see (5.1a)–(5.1b) below. All other state
variables may be derived from these; see Remark \[5.3\] It therefore suffices to seek time-independent solutions of \[5.1\]a-\[5.1\]c. Setting \(\partial_t \rho = \partial_t R = 0\) we obtain from \[5.1\]a and \[5.1\]c that \(R(t) \equiv R_\ast\) (constant equilibrium radius) and

\[\Delta \log \rho = 0 \quad \text{in } B_{R_\ast}, \quad \text{and } \partial_r \rho(R_\ast) = 0.\]

Therefore, \(\rho(r) \equiv \rho_\ast\) for \(0 \leq r \leq R_\ast\) (constant equilibrium density). Evaluating \[5.1\]c at \(r = R_\ast\) and using that \(\rho(R_\ast) = \rho_\ast\) we conclude

\[\rho_\ast = \frac{1}{\mathcal{R}_g T_\ast} \left( p_{x,\ast} + \frac{2\sigma}{R_\ast} \right).\]

The mass of the equilibrium gas bubble of density \(\rho_\ast\) and radius \(R_\ast\) is given by

\[M = \int_{B_{R_\ast}} \rho_\ast \, dx = \frac{4\pi}{3} \rho_\ast R_\ast^3.\]

Therefore, for fixed mass \(M\), the steady state \((\rho_\ast, R_\ast)\) is determined by the simultaneous algebraic equations:

\begin{align}
4\pi R_\ast^3 & = M, \\
\mathcal{R}_g T_\ast \rho_\ast &= p_{x,\ast} + \frac{2\sigma}{R_\ast}.
\end{align}

Therefore, the equilibrium radius \(R_\ast\) is given by a solution to the cubic equation

\[p_{x,\ast} R_\ast^3 + 2\sigma R_\ast^2 - \frac{3\mathcal{R}_g T_\ast M}{4\pi} = 0.\]

It is readily seen that for each fixed \(M > 0\), the cubic \[4.14\] has a unique positive root \(R_\ast\). This choice of \(R_\ast\) determines the equilibrium gas density and, via the relation \(p_\ast = \mathcal{R}_g T_\ast \rho_\ast\), the gas pressure:

\[\rho_\ast = \frac{1}{\mathcal{R}_g T_\ast} \left( p_{x,\ast} + \frac{2\sigma}{R_\ast} \right), \quad p_\ast = \mathcal{R}_g T_\ast \rho_\ast = p_{x,\ast} + \frac{2\sigma}{R_\ast}.\]

Once we obtain the equilibrium \((\rho_\ast, R_\ast, p_\ast)\), we can recover, using the formulas in \[4.14\], the corresponding steady state solution to the system \[5.1\]a-\[5.1\]c for the gas velocity \(v_\ast\), the gas temperature \(T_\ast\), the specific entropy \(s\) of the gas, the liquid velocity \(v_\ast\), and the liquid pressure \(p_\ast\):

\[v_\ast = v_\ast, s = s, s = c_v \log \left( \frac{p_\ast}{\rho_\ast} \right), \quad p_\ast = p_{x,\ast}.\]

Summarizing, we have derived the spherically symmetric equilibrium stated in \[4.13\]. This proves Part (2). Part (3) of Proposition \[4.3\] follows from the smooth dependence of the simple roots of a given polynomial on its coefficients. This completes the proof of Proposition \[4.3\].

\[\square\]

Remark 4.5. The equilibrium radius \(R_\ast\) can be expressed explicitly in terms of \(p_{x,\ast}, \sigma, \mathcal{R}_g, T_\ast, M\) by using the solution formula for the cubic equation \[4.14\].

Below, in Proposition \[5.4\] we shall reduce the study of spherically symmetric solutions to a closed system of equations for the gas density, \(\rho(r, t)\) and the bubble radius \(R(t)\), together with a condition on \(\rho(R(t), t)\), the gas density at the free boundary. Our proof of asymptotic stability is carried out in this setting. We shall use the following continuity result:

Proposition 4.6. Fix a constant \(p_{x,\ast} > 0\). Fix a density-radius pair \((\rho_0(x), R_0) \in L^\infty \times \mathbb{R}\) and let \(M_0 = \text{Mass}[\rho_0, R_0]\) denote the mass of the corresponding bubble. Let \((\hat{R}_\ast, \hat{\rho}_\ast)\) denote the equilibrium radius and density, given by Proposition \[4.3\] for which \(\text{Mass}[\hat{\rho}_\ast, \hat{R}_\ast] = M_0 = \text{Mass}[\rho_0, R_0]\). Then, any equilibrium \((\rho_\ast, R_\ast)\) close to \((\rho_0(x), R_0)\) in \(L^\infty \times \mathbb{R}\) is also close to \((\hat{R}_\ast, \hat{\rho}_\ast)\). Even more strongly, there is a constant \(C = C(R_\ast, \rho_0, R_0) > 0\) such that

\[|\hat{R}_\ast(\rho_0, R_0) - R_\ast| + |\hat{\rho}_\ast(\rho_0, R_0) - \rho_\ast| \leq C \left( |R_0 - R_\ast| + \|\rho_0 - \rho_\ast\|_{L^\infty(B_{R_0})} \right).\]
Taking the difference of these two equations gives:

\[ p_{x, s} R_s^3 + 2\sigma R_s^2 - \frac{3\mathcal{A}_s M T}{4\pi} = 0 \quad \text{and} \quad p_{x, s} \dot{R}_s^3 + 2\sigma \dot{R}_s^2 - \frac{3\mathcal{A}_s M \dot{T}}{4\pi} = 0. \]

Taking the difference of these two equations gives:

\[ p_{x, s}(R_s - \dot{R}_s)(R_s^2 + R_s R_s' + \dot{R}_s^2) + 2\sigma(R_s - \dot{R}_s)(R_s + \dot{R}_s) - \frac{3\mathcal{A}_s T}{4\pi}(M_0 - \dot{M}_0) = 0, \]

and therefore

\[ |\dot{R}_s - R_s| = \frac{3\mathcal{A}_s T}{4\pi} \left| p_{x, s}(R_s^2 + R_s R_s' + \dot{R}_s^2) + 2\sigma(R_s + \dot{R}_s) \right| |M_0 - M_0|. \]  

Bounding \(|\dot{R}_s - R_s|\) therefore reduces to bounding \(|M_0 - M_0|\). Expanding \(M_0\) about \(M_0\) we have:

\[
\dot{M}_0 = \int_{B_{R_0}} \rho_0 \, dx = \int_{B_{R_0}} \rho_{\ast} \, dx + \int_{B_{R_0}} (\rho_0 - \rho_\ast) \, dx = \frac{4\pi}{3} R_0^3 \rho_{\ast} + \int_{B_{R_0}} (\rho_0 - \rho_\ast) \, dx.
\]

Therefore,

\[ |\dot{M}_0 - M_0| \leq \frac{4\pi}{3} \rho_{\ast} (R_0^2 + R_0 R_\ast + \dot{R}_0^2) |R_0 - R_\ast| + \frac{4\pi}{3} R_0^3 |\rho_0 - \rho_\ast| \|_{L^\infty(B_{R_0})}. \]

The bounds (4.17) and (4.16) imply that \(|\dot{R}_s[\rho_0, R_0] - R_\ast|\) satisfies the bound (4.15).

Finally, we bound the difference \(|\dot{\rho}_s[\rho_0, R_0] - \rho_\ast|\). Taking the difference of the relations \(\frac{4\pi}{3} \rho_s \dot{R}_s^3 = \tilde{M}_0\) and \(\frac{4\pi}{3} \rho_s \dot{R}_s^2 = M_0\), we have \(\frac{4\pi}{3} \dot{\rho}_s \dot{R}_s^3 - \frac{4\pi}{3} \dot{R}_s^2 \rho_\ast = \dot{M}_0 - M_0\). Therefore,

\[ \frac{4\pi}{3} \dot{R}_s^3 (\dot{\rho}_s - \rho_\ast) = (\dot{M}_0 - M_0) + \frac{4\pi}{3} \left( \dot{R}_s^2 + \ddot{R}_s R_s + \dot{R}_s^2 \right) (\dot{R}_s - R_\ast). \]

The bound on \(|\dot{\rho}_s[\rho_0, R_0] - \rho_\ast|\) now follows by estimating (4.18) using the bounds (4.17) and (4.16).

5. Reduction of the asymptotic model to a system for \(\rho(r, t)\) and \(R(t)\)

The main purpose of this article is to study the asymptotic stability of the spherically symmetric equilibrium (3.3) of the approximation system (3.1)-(3.3). The perturbations we consider are spherically symmetric and hence we work with the following reduction of the initial value problem:

**Proposition 5.1.** Any sufficiently regular spherically symmetric solution of (3.1)-(3.3) can be constructed from a solution of the following reduced system of equations for the gas density \(\rho_\ast(r, t) \equiv \rho(r, t)\), for \(0 \leq r \leq R(t)\), and the bubble radius \(R(t)\), together with a boundary condition on \(\rho(r, t)\) at the free boundary \(r = R(t)\):

\[
\begin{align*}
\dot{\rho}_s(r, t) &= \frac{\rho}{\gamma c_v} \Delta_r \log \rho(r, t) + \frac{1}{\gamma} \frac{\partial p(t)}{p(t)} \left( \frac{1}{3} r \partial_r \rho(r, t) + \rho(r, t) \right), \quad 0 \leq r \leq R(t), \quad t > 0, \\
\dot{R}(t) &= -\frac{\rho}{\gamma c_v (\rho R(t), t)^2} \frac{R(t) \partial_p(t)}{3 \gamma} p(t), \quad t > 0, \\
\rho(R(t), t) &= \frac{1}{\mathcal{A}_{s T}} \left[ p_x(t) + \frac{2\sigma}{R(t)} + 4 \mu R + \rho_l \left( R(t) \dot{R}(t) + \frac{3}{2} (\dot{R}(t))^2 \right) \right], \quad t > 0,
\end{align*}
\]
with initial data \( \rho(\cdot, 0), R(0), \dot{R}(0) \). Here, \( p = p(t) = p_g(t) \) (gas pressure) and \( \rho(R(t), t) \) are related through the constitutive relation
\[
(5.2) \quad p(t) = \mathcal{A}_g T x \rho(R(t), t), \quad t > 0,
\]
Note that (5.1c) and (5.2) imply
\[
(5.3) \quad p(t) - p_x(t) - \frac{2\sigma}{R(t)} - 4\mu \frac{\dot{R}}{R} = \rho_1 \left( R(t) \dot{R}(t) + \frac{3}{2}(\dot{R}(t))^2 \right).
\]
The proof of Proposition 5.1 is given in Appendix B. The calculations also yield the following expressions for all state variables:

Theorem 5.3
Denote the radial components of the gas and liquid velocities by \( v_g(r, t) \) and \( v_l(r, t) \), respectively. Given a solution \( (\rho(r, t), R(t)) \) to the system (5.1a)–(5.1c), we can reconstruct a spherically symmetric solution \( (v_l, p_l, \rho_g, v_g, p_g, T_g, s) \) to the system (3.1)–(3.3) in terms of \( \rho \) and \( R \) by
\[
(5.5) \quad \Omega(t) = B_{R(t)}, \quad t > 0,
\]
\[
\rho_g(r, t) = \rho(r, t), \quad 0 \leq r \leq R(t), \quad t > 0,
\]
\[
p_g(t) = \mathcal{A}_g T x \rho(R(t), t), \quad t > 0,
\]
\[
v_g(r, t) = \frac{\kappa}{\gamma c_v} \left( \frac{1}{\rho(r, t)} \right) - \frac{\hat{c}_p p_g(t)}{p_g(t)} \frac{r}{3\gamma}, \quad 0 \leq r \leq R(t), \quad t > 0,
\]
\[
T_g(r, t) = \frac{p_g(t)}{\mathcal{A}_g \rho(r, t)}, \quad 0 \leq r \leq R(t), \quad t > 0,
\]
\[
s(r, t) = c_v \log \left( \frac{p_g(t)}{(\rho(r, t))^\gamma} \right), \quad 0 \leq r \leq R(t), \quad t > 0,
\]
\[
v_l(r, t) = \frac{(R(t))^2 \dot{R}(t)}{r^2}, \quad r \geq R(t), \quad t > 0,
\]
\[
p_l(r, t) = p_x(t) + \rho_1 \left( \frac{2R(t)(\dot{R}(t))^2 + (R(t))^2 \dot{R}(t)}{r} - \frac{(R(t))^4 (\dot{R}(t))^2}{2r^4} \right), \quad r \geq R(t), \quad t > 0.
\]
When \( \mu_1 = 0 \) in (5.1c), to study well-posedness [5], Biro and Velázquez mapped, by a change of variables, the free boundary problem on \( B_{R(t)} \): (5.1a)–(5.1c) to a problem on the fixed domain \( B_1 \). In this setting they proved the local well-posedness for the free boundary problem (5.1a)–(5.1c). The proof is based on the derivation of a priori Schauder estimates, application of a Leray–Schauder fixed point argument and the classical regularity theory for quasilinear parabolic equations; see, for example, [25] Chapter V. Theorem 6.1. We extend their result to the general case involving liquid viscosity on the free boundary, which implies local well-posedness of the system (3.1)–(3.3) in the spherically symmetric case.

Theorem 5.3
(Local in time well-posedness). Consider the initial value problem for (5.1a)–(5.1c) with initial radius \( R(0) = R_0 > 0 \) and initial density \( \rho_0 \in C^{2+2\alpha}([0, R(0)]) \), \( 0 < \alpha < \frac{1}{2} \). Suppose also that for some \( \eta > 0 \), \( \rho_0(r) \geq \eta \) for \( 0 \leq r \leq R_0 \). Then, there exists \( \delta = \delta(\|\rho_0\|_{C^{2+2\alpha}}) \) such that the free boundary problem (5.1a)–(5.1c) has a unique solution satisfying
\[
\rho \in C^{1+\alpha}([0, \delta]; C^{2+2\alpha}([0, R(t)])], \quad R \in C^{3+\alpha}([0, \delta]).
\]
Proof. The proof follows the same argument in the proof of [5, Theorem 3.1]. Indeed, the extra viscous term $4\mu_t\hat{R}(t)/\hat{R}(t)$ in the Young-Laplace boundary condition does not destroy the analyticity of $f_2$ in [5, (3.18)]. We omit the proof and refer the reader to [5].

6. Dynamic stability of spherical bubble

In Section 6.1 we recall the results in [5] on conditional Lyapunov stability of spherical symmetric equilibria, that is Lyapunov stability relative to small perturbations of the same bubble mass. We then, in Theorem 6.3 extend this result to Lyapunov stability relative to arbitrary small perturbations. Then, in Section 6.2 we state Theorem 6.5 our main result of asymptotic stability. The proof is presented in subsequent sections.

6.1. Lyapunov stability. In [5] Theorem 4.1, Biro and Velázquez established the global well-posedness of the free boundary problem (5.1a)-(5.1c), $\mu_1 = 0$ in (5.1c), when the initial data is sufficiently close to a given spherically symmetric equilibrium and has the same mass as the mass of the equilibrium solution.

When $\mu_1 > 0$ in (5.1c) the extra viscous term on the boundary leads to the extra term: $-16\pi\mu_t(R(t)(\hat{R}(t))^2$ on the right hand side of the energy dissipation law (7.4). Hence, the key bound [5, (4.41)] still holds, and thus, their proof also applies. In other words, we have that the spherical equilibrium are Lyapunov stable relative to mass preserving perturbations. Introduce the norm

$$
\left| \left( p_1(\cdot, t) - p_2(\cdot, t), R_1(t) - R_2(t), \dot{R}_1(t) - \dot{R}_2(t) \right) \right|
$$

(6.1)

$$
\equiv \| \tilde{p}_1(\cdot, t) - \tilde{p}_2(\cdot, t) \|_{C^{2+\alpha_0}(B_1)} + \| R_1(t) - R_2(t) \| + \| \dot{R}_1(t) - \dot{R}_2(t) \|, \quad \tilde{p}_i(y, t) = \rho_i(R(t)y, t), \quad i = 1, 2.
$$

In [5] it is shown that given $\varepsilon_0 > 0$, there exist $\eta_0 = \eta_0(\varepsilon_0) > 0$ such that

$$
\left| \left( \rho_0 - \rho_\ast[M_0], R_0 - R_\ast[M_0], \dot{R}_0 \right) \right| \leq \eta_0,
$$

(6.2)

where $M_0 = \text{Mass}[\rho_0, R_0]$, then for all $t > 0$

$$
\left| \left( \rho(\cdot, t) - \rho_\ast[M_0], R(t) - R_\ast[M_0], \dot{R}(t) \right) \right| \leq \varepsilon_0.
$$

(6.3)

Remark 6.1. We note that the smallness of initial radial velocity, $\hat{R}(0)$, is not explicitly assumed in [5, Theorem 4.1]. The smallness is needed to control the kinetic energy, KE, and higher derivatives of $R$.

Remark 6.2. The proof of [5, Theorem 4.1] gives a better regularity and control than $\| R(t) - R_\ast \| \leq \varepsilon_0$ and $\| \dot{R}(t) \| \leq \varepsilon_0$ which was stated in [5, Theorem 4.1]. In fact, we obtain

$$
\| R - R_\ast \|_{C^{2+\alpha}(R_\ast)} \leq \varepsilon_0.
$$

(6.4)

Note further from (5.2), that

$$
\| p_\theta \|_{C^{1+\alpha}(R_\ast)} \leq \varepsilon_0.
$$

(6.5)

Using the continuity of functionals, we now extend the conditional Lyapunov stability result [5, Theorem 4.1] to Lyapunov stability relative to arbitrary small perturbations. Specifically, we prove the Lyapunov stability of the manifold of equilibria to the system (5.1a)-(5.1c)

$$
\mathcal{M}_\ast = \left\{ (\rho_\ast[M], R_\ast[M], \dot{R}_\ast = 0) : 0 < M < \infty \right\},
$$

where $\rho_\ast[M]$, $R_\ast[M]$ are given in Proposition 4.3

Introduce the distance of the state defined by $(\rho(\cdot, t), R(t), \dot{R}(t))$ to the manifold of equilibria:

$$
\text{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_\ast) = \inf \left\{ \left| \left( \rho(\cdot, t) - \rho_\ast[M], R(t) - R_\ast[M], \dot{R}(t) - \dot{R}_\ast \right) \right| : (\rho_\ast, R_\ast, \dot{R}_\ast) \in \mathcal{M}_\ast \right\}
$$

$$
= \inf_{0 < M < \infty} \left| \left( \rho(\cdot, t) - \rho_\ast[M], R(t) - R_\ast[M], \dot{R}(t) \right) \right|.
$$
Theorem 6.3 (Lyapunov stability). Consider the time evolution equation (5.1a)-(5.1c) with $p_x(t) = p_{x,\ast}$. Let $\varepsilon_0 > 0$ be arbitrary. There exists $\eta_0 > 0$ such that if the initial data $\rho_0(r), R_0,$ and $\dot{R}_0$ satisfies

$$\text{dist}((\rho_0, R_0, \dot{R}_0), \mathcal{M}_\ast) \leq \eta_0,$$

then $(\rho(t), R(t))$, the global in time solution of the initial value problem (5.1a)-(5.1c), satisfies

$$\text{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_\ast) \leq \varepsilon_0, \quad \text{for all } t > 0.$$

Proof. The proof is a consequence of [5, Theorem 4.1] and Proposition 6.6. Namely, Proposition 6.6 implies that there is a unique $(\tilde{\rho}_\ast, \tilde{R}_\ast)$ such that Mass[$\tilde{\rho}_\ast, \tilde{R}_\ast] = \text{Mass}[\rho_0, R_0]$, and

$$|\tilde{R}_\ast - R_\ast| + |\tilde{\rho}_\ast - \rho_\ast| \leq C \left( |R_0 - R_\ast| + \|\rho_\ast - \rho_0\|_{L^\infty(B_{R_0})} \right),$$

(6.6)

Hence,

$$\|\rho_\ast - \tilde{\rho}_\ast\|_{C^{2+2\alpha}} + |R_0 - R_\ast| = \|\rho_\ast - \rho_\ast + \rho_\ast - \tilde{\rho}_\ast\|_{C^{2+2\alpha}} + |R_0 - R_\ast + R_\ast - \tilde{R}_\ast|$$

$$\leq \|\rho_\ast - \rho_\ast\|_{C^{2+2\alpha}} + |\rho_\ast - \tilde{\rho}_\ast| + |R_0 - R_\ast| + |R_\ast - \tilde{R}_\ast|$$

$$\leq C' \left( \|\rho_\ast - \rho_\ast\|_{C^{2+2\alpha}} + |R_0 - R_\ast| \right).$$

Therefore, choosing $\|\rho_\ast - \rho_0\|_{C^{2+2\alpha}} + |R_0 - R_\ast|$ and $\dot{R}_0$ sufficiently small we conclude from [5, Theorem 4.1] that

$$\text{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_\ast) \leq \|\rho(\cdot, t) - \tilde{\rho}_\ast\|_{C^{2+2\alpha}} + |R(t) - R_\ast| + |\dot{R}(t)| \leq \varepsilon_0.$$

This completes the proof.

The proof of the Lyapunov stability in [5, Theorem 4.1] relies on a coercive energy estimate ([5, Lemma 4.2]), for the case of constant external far-field pressure. In Appendix C we prove an extension of this energy estimate, Theorem 6.3, which enables us to generalize Theorem 6.3.

Corollary 6.4. The conclusions of Theorem 6.3 hold provided we choose $\eta_0$ sufficiently small and so that the following additional conditions are satisfied:

$$|p_x(t) - p_{x,\ast}| \leq \eta_0, \quad \|\hat{p}_t p_x\|_{L^1(\mathbb{R}^+)} \leq \eta_0.$$

6.2. Main result: asymptotic stability. The main goal of this article is to study the asymptotic stability of the family of spherically symmetric equilibria of the approximate system (5.1a)-(5.1c) against small spherically symmetric perturbations. This is a consequence of the following result on asymptotic stability for the reduced system (5.1a)-(5.1c).

Theorem 6.5 (Asymptotic stability of the manifold of spherically symmetric equilibria). Fix parameters (5.4) and set $p_x(t) = p_{x,\ast}$ in the system (5.1a)-(5.1c).

(1) There exists a constant $\eta > 0$ such that if $\text{dist}((\rho_0, R_0, \dot{R}_0), \mathcal{M}_\ast) \leq \eta$, then

$$\text{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_\ast) \to 0 \quad \text{as } t \to +\infty.$$

(2) More precisely, let $M_0 > 0$ and $(\rho_\ast[M_0], R_\ast[M_0])$ as Proposition 6.1 there exist constants $\eta_1 > 0$ and $C_1 > 0$ such that the following holds for all $0 < \eta \leq \eta_1$: Consider initial data $\rho_0(r), R_0, \dot{R}_0$, an arbitrary small perturbation of $(\rho_\ast[M_0], R_\ast[M_0], \dot{R}_\ast) = 0$:

$$\left( |\rho_0 - \rho_\ast[M_0], R_0 - R_\ast[M_0], \dot{R}_0 \right) \leq \eta.$$

Let $\tilde{M}_0 = \int_{B_{R_0}} \rho_0$ denote the initial bubble mass. In general, $\tilde{M}_0 \neq M_0$, however by Proposition 6.6 the corresponding points on the manifold of equilibria are close:

$$|\rho_\ast[\tilde{M}_0] - \rho_\ast[M_0]| \leq C_1 \eta \quad \text{and} \quad |R_\ast[\tilde{M}_0] - R_\ast[M_0]| \leq C_1 \eta.$$

Let $(\rho(t), R(t))$ denote the global in time solution of the free boundary problem (5.1a)-(5.1c) with initial data satisfying (6.7). Then, as $t \to +\infty$

$$\text{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), (\rho_\ast[\tilde{M}_0], R_\ast[\tilde{M}_0], \dot{R}_\ast = 0)) \to 0,$$

(6.8)
and $|\tilde{R}(t)| + |\tilde{R}(t)| \to 0$ as $t \to +\infty$.

(3) The convergence of $(\rho, R)$ in (6.8) is sufficient to imply the convergence of the quantities in Remark 5.9 to their equilibrium values. Therefore, the spherical equilibrium (6.3) of the system (5.1) - (5.3) is asymptotically stable.

It is simple to generalize Theorem 6.5, the asymptotic stability for the model of constant external far-field pressure $p_x$, to the following result for the case that $p_x(t)$ is a small perturbation of a constant.

**Corollary 6.6.** The conclusions of Theorem 6.5 hold provided we choose the constant $\eta_0 > 0$ sufficiently small and such the following conditions on the asymptotically constant far-field pressure hold:

$$(6.9) \quad p_x \in C^{1+\alpha}_t(\mathbb{R}_+), \quad |p_x(t) - p_x, t| + \|\tilde{c}_t p_x\|_{L^1_t(\mathbb{R}_+)} \leq \eta_0, \quad p_x(t) \to p_{x, t} \text{ as } t \to \infty.$$ 

**Sketch of the proof of Theorem 6.5** By a continuity argument, the proof of asymptotic stability relative to arbitrary small perturbations, can be reduced to Proposition 8.1 on asymptotic stability relative to perturbations of a spherical equilibrium which have the same bubble mass. At the heart of the proof of Proposition 8.1 is

(1) the time-integrability over $[0, \infty)$ of the energy dissipation rate: $\int_{B_R(t)} |T_g(\cdot, t)|^{-2} |\nabla T_g(\cdot, t)|^2$

and

(2) the coercive energy estimate, Theorem 7.5 which expresses that the spherical equilibrium of an arbitrary specified mass is a local minimizer of the total energy relative to spherically symmetric perturbations of the same mass; see [5]Lemma 4.2].

By the equation of state, (3.2d), $\int_{B_R(t)} |\rho_g(\cdot, t)|^{-2} |\nabla \rho_g(\cdot, t)|^2$ is time-integrable over $[0, \infty)$. This implies convergence of $\rho(\cdot, t)$, to an equilibrium density. With control of the density, $\rho(r, t)$, and in particular $\rho(R(t), t)$, we obtain that $\dot{R}(t) \to 0$ as $t \to \infty$, from (5.11), the equation for the motion of the boundary. The limiting constant values of $\rho_g$ and $R$ satisfy the system (4.13) and it follows that these correspond to the unique spherically symmetric equilibrium of the given initial mass. We present the detailed proofs in Section 8.

7. **Conservation of mass and energy dissipation**

Solutions of the free boundary problem (3.1) - (3.3) satisfy conservation of mass and an energy dissipation law. These play a central role in the Lyapunov stability theory of [5] and in our asymptotic stability theory. To derive these statements one makes use of the following technical results.

Given a smooth velocity field, $v : (x, t) \mapsto v(x, t) \in \mathbb{R}^3$, define $X^t(\alpha)$ to be the particle trajectory map given by the solution of the initial value problem: $X^t(\alpha) = v(X^t(\alpha), t), X^0(\alpha) = \alpha$. The mapping $\alpha \mapsto X^t(\alpha)$ is smooth and invertible for all $t$ sufficiently small. For an open subset $\Omega \subset \mathbb{R}^2$, let $X^t(\Omega) = \{X^t(\alpha) : \alpha \in \Omega\}$. We first recall the transport formula that gives the rate of change of a function in a domain transported with the fluid.

**Proposition 7.1** ([33 Proposition 1.3]). Let $\Omega$ denote an open, bounded domain with a smooth boundary. Then for any smooth function $f(x, t)$,

$$\frac{d}{dt} \int_{X^t(\Omega)} f \, dx = \int_{X^t(\Omega)} [\hat{c}_t f + \text{div}(f v)] \, dx.$$ 

With the aid of this transport formula, we have the following lemma which is used to compute the time-evolution of the mass and energies on a time-varying spatial domain, $\Omega(t)$.

**Lemma 7.2.** If $\hat{c}_t \rho + \text{div}(\rho v) = 0$ in $\Omega(t)$ and $v(\omega(t), t) = \hat{\omega}(t)$ for $\omega(t) \in \hat{c}\Omega(t)$, then

$$\frac{d}{dt} \int_{\Omega(t)} \rho \phi \, dx = \int_{\Omega(t)} \rho \frac{\partial \phi}{\partial t} \, dx$$

for any smooth function $\phi$, where $\frac{\partial}{\partial t}$ is the material derivative of $f$ given by

$$\frac{Df}{Dt} = \hat{c}_t f + v \cdot \nabla f.$$
Proof of Lemma \[7.2\] Since the boundary $\partial \Omega(t)$ moves along the particle-trajectory mapping $X$ of the velocity field $v$, $\Omega(t) = X(\Omega(0), t)$. By the transport formula Proposition \[7.1\]

$$\frac{d}{dt} \int_{\Omega(t)} \rho \phi \, dx = \int_{\Omega(t)} \left[ \partial_t (\rho \phi) + \text{div} (\rho v \phi) \right] \, dx.$$ 

Using the continuity equation, the right hand side of above equation becomes

$$\int_{\Omega(t)} \left[ \rho \partial_t \phi + \rho v \cdot \nabla \phi \right] \, dx.$$ 

This proves the lemma. \[\square\]

7.1. **Conservation of mass.** By taking $\phi \equiv 1$ in Lemma \[7.2\] we have

**Proposition 7.3** (Bubble mass conservation). Let $X$ be the particle-trajectory mapping associated with $v$. Denote by $\Omega_0 \subset \mathbb{R}^3$ the bubble region at time $t = 0$, assumed to have a smooth boundary, and $\Omega(t) = X^t(\Omega_0)$. Let $\rho = \rho_g$ denote a $C^{1+\alpha}([0, \infty); C^{2+2\alpha}_x(\Omega(t)))$, $0 < \alpha < \frac{1}{2}$ solution of (2.2a) (or equivalently (3.2a)) with $v$ satisfying (2.3a) (or equivalently (3.3a)) and initial data $\rho_0 \in C^{2+2\alpha}(\Omega_0)$. Then, the mass of the bubble is constant in time:

$$\int_{\Omega(t)} \rho(x, t) \, dx = \int_{\Omega_0} \rho_0(x) \, dx \quad t > 0.$$  

7.2. **Energy dissipation law.**

**Definition 7.1** (The total energy). Consider the case of spherically symmetric solutions of (3.1)-(3.4). The total energy of the system is given by

$$\mathcal{E}_{\text{total}}(t) = FE(t) + KE(t) + U_{g-1}(t) + PV_{p_g}(t),$$

where the total energy is made up of the following components:

1. **$FE(t)$**, the Helmholtz free energy:

$$FE(t) = c_v \int_{B_R} \rho_g T_g \, dx - T_c \int_{B_R} \rho_g s \, dx$$

$$= \frac{4\pi c_v}{3\mathcal{M}_g} \rho_g R^3 - T_c \int_{B_R} \rho_g s \, dx$$

$$= \frac{4\pi c_v}{3\mathcal{M}_g} \rho_g R^3 - c_v T_c M_0 \log \rho_g + c_v \gamma T_c \int_{B_R} \rho_g \log \rho_g \, dx,$$

where $M_0 = \text{Mass} \{\rho_g, R\}$, and the second and the last equalities hold by (3.2d) and (3.2e), respectively.

2. **$KE(t)$**, the kinetic energy of the liquid:

$$KE(t) = \frac{1}{2} \int_{\mathbb{R}^3 \setminus B_{R(t)}} \rho |v|^2 \, dx = 2\pi \rho_t [R(t)]^3 \left[ \dot{R}(t) \right]^2,$$

3. **$U_{g-1}(t)$**, the surface energy of the liquid-gas interface

$$U_{g-1}(t) = \sigma \int_{\partial B_{R(t)}} dS = 4\pi \sigma [R(t)]^2,$$

4. **$PV_{p_g}(t)$**, the energy contributed by the work done by the external sound field.

$$PV_{p_g}(t) = |B_{R(t)}| p_x(t) = \frac{4\pi}{3} [R(t)]^3 p_x(t).$$

The energy functional \[7.2\] is at the heart of the stability analysis. Its importance is clear from the following result on energy dissipation, proved in [5] (using Lemma 7.2 for the system 5.1) (equivalently 3.1-3.3) under the assumption of spherical symmetry) and $p_x = 1$. We state and prove a mild generalization to the case of a time-dependent pressure at $p_x(t)$. We shall use the abbreviated notation: $\rho = \rho_g, p = p_g, T = T_g$ and $\kappa = \kappa_g$. 


Proposition 7.4 (Energy dissipation law). Assume that \((\rho(r,t), R(t), p(t) = \mathcal{R}_g T(r,t) \rho(r,t))\) is a solution of \((8.1)\), or equivalently \((8.1) - (8.3)\) under the assumption of spherical symmetry. Then,

\[
\frac{d}{dt} \mathcal{E}_{\text{total}}(t) = -\kappa T_g \int_{B_R(t)} \frac{|\nabla T_g(r,t)|^2}{T_g(r,t)} \, dx - 16\pi \mu \dot{R}(t) \dot{R}(t)^2 + \frac{4\pi}{3} R^3(t) \dot{c}_t p_x(t). \tag{7.4}
\]

Proof. By Lemma \[\overline{7.2}\] differentiating \(\mathcal{E}_{\text{total}}\) with respect to \(t\) using the third line of \((7.3)\) yields

\[
\frac{d}{dt} \mathcal{E}_{\text{total}} = 4\pi c_v \left( \frac{\partial p_g R^3}{3 \mathcal{R}_g} + \dot{R} \right) - T_g \int_{B_R} \frac{DS}{Dt} \, dx \\
+ 4\pi \rho R^2 \dot{R} \left( \frac{R R^2}{2} + \frac{\dot{R}}{2} \right) + 8\pi \sigma R \dot{R} + 4\pi R^2 \dot{R} p_x(t) + \frac{4\pi}{3} R^3 \dot{c}_t p_x. \tag{7.5}
\]

Consider the second term on the right hand side of \((7.5)\). Using \((3.2c)\), integrating by parts, and \((3.3c)\), we obtain

\[
\kappa_g \int_{B_R} \frac{\partial T_g}{\partial t} \, dx = \kappa_g \int_{B_R} \frac{|\nabla T_g|^2}{T_g} \, dx + \frac{4\pi \kappa_g R^2}{T_g} \dot{c}_t p_g(R(t), t) \\
- \frac{4\pi \kappa_g R^2 p_g(R(t), t) \dot{c}_r p_g(R(t), t)}{\mathcal{R}_g T_g \rho_g} \tag{7.6}
\]

where the last equality follows from the constitutive relation \(T_g = p_g(\mathcal{R}_g \rho_g)^{-1}\).

For the third term on the right hand side of \((7.6)\) we use \((5.3)\)

\[
\rho_l \left( R(t) \dot{R}(t) + \frac{3}{2} \dot{R}(t)^2 \right) = \rho_g(t) - p_x(t) - \frac{2\sigma}{R(t)} - 4\mu \dot{R}. \tag{7.7}
\]

Substituting \((7.6)\) and \((7.7)\) into \((7.5)\) we obtain

\[
\frac{d}{dt} \mathcal{E}_{\text{total}} = 4\pi c_v \left( \frac{\partial p_g R^3}{3 \mathcal{R}_g} + \dot{R} \right) + 4\pi \rho R^2 \dot{R} \left( \frac{R R^2}{2} + \frac{\dot{R}}{2} \right) + 8\pi \sigma R \dot{R} + 4\pi R^2 \dot{R} p_x(t) + \frac{4\pi}{3} R^3 \dot{c}_t p_x. \tag{7.8}
\]

or

\[
\frac{d}{dt} \mathcal{E}_{\text{total}} = -T_g \kappa_g \int_{B_R} \frac{|\nabla T_g|^2}{T_g} \, dx - 16\pi \mu R(t) \dot{R}(t)^2 + \frac{4\pi}{3} R^3 \dot{c}_t p_x \\
+ 4\pi R^2 p_g \left( \frac{c_v}{3 \mathcal{R}_g} \frac{\partial p_g}{\partial R} R + \left( 1 + \frac{c_v}{\mathcal{R}_g} \right) \dot{R} + \frac{\kappa_g}{\mathcal{R}_g \rho_g} \frac{\partial c_g}{\partial p_g} \right). \tag{7.9}
\]

Finally we claim that the expression in the square brackets in \((7.10)\) vanishes:

\[
\mathcal{I}(r, t) = \frac{c_v}{\mathcal{R}_g} \frac{\partial p_g}{\partial R} R + \left( 1 + \frac{c_v}{\mathcal{R}_g} \right) \dot{R} + \frac{\kappa_g}{\mathcal{R}_g \rho_g} \frac{\partial c_g}{\partial p_g} = 0, \tag{7.10}
\]

from which Proposition \(7.3\) follows. To prove \((7.10)\) note that the relation \(\gamma = 1 + \frac{\mathcal{R}_g}{c_v}\) (see 2.3) implies

\[
\mathcal{I}(r, t) = \frac{1}{\gamma - 1} \left( \frac{1}{R} \frac{\partial p_g}{\partial R} R + \frac{\gamma - 1}{\gamma - 1} \dot{R} + \frac{\kappa_g}{\mathcal{R}_g \rho_g} \frac{\partial c_g}{\partial p_g} \right) \\
= \frac{\gamma}{\gamma - 1} \left( \frac{1}{3\gamma} \frac{\partial p_g}{\partial R} R + \frac{1}{\gamma} \dot{R} + \frac{\kappa_g}{\mathcal{R}_g \rho_g} \frac{\partial c_g}{\partial p_g} \right). \tag{7.11}
\]

Next, we use \((5.11)\) to simplify \((7.12)\). This yields

\[
\mathcal{I}(r, t) = \frac{\kappa_g}{\gamma - 1} \left( -\mathcal{R}_g \frac{c_v}{\gamma - 1} + \frac{c_v}{\gamma - 1} \right) \frac{\partial c_g}{\partial p_g} = 0 \tag{7.12}
\]

The proof of Proposition \(7.4\) is now complete. \(\square\)
7.3. Coercivity energy estimate. To prove the global existence of solutions and Lyapunov stability, the authors in [5] considered the energy $\mathcal{E}_{\text{total}}$ defined in (7.2) for $p_x(t) \equiv 1$, and used the energy dissipation formula (7.4). By expanding the energy $\mathcal{E}_{\text{total}}$ at the steady state energy up to quadratic terms, they derived the coercivity estimate of the perturbed energy from the steady state energy in [5] Lemma 4.2. We generalize [5] Lemma 4.2 to the following result for the case of general (nonstationary) external far-field pressure $p_x(t)$ whose proof is similar to that of [5] Lemma 4.2 and is in Appendix C for the reader’s convenience.

**Theorem 7.5.** Given positive constants $\rho_*, R_*, p_{x,*}$. Assume that there exists a constant $\nu > 1$ such that

\begin{align}
(7.13) & \quad \nu^{-1} \leq \rho(r) \leq \nu, \\
(7.14) & \quad \nu^{-1} \leq R \leq \nu, \\
(7.15) & \quad \text{Mass}[\rho, R] = \text{Mass}[\rho_*, R_*],
\end{align}

where $\text{Mass}[\rho, R]$ is given in (4.11). Then, denoting $\tilde{\rho}(y) = \rho(Rr)$, we have

\begin{align}
(7.16) & \quad \mathcal{E}_{\text{total}} - \mathcal{E}_x \geq \frac{c_T R_*^3 |B_1|}{2 \rho_*} \left( \frac{1}{|B_1|} \int_{B_1} \tilde{\rho}^2 + 2 \pi \rho_1 R_*^3 \tilde{R}^2 + \frac{R_*^3}{\rho_*} \left( \frac{p_{x,*}}{2} + \frac{2 \sigma}{3 R_*} \right) \int_{B_1} \tilde{\rho}^2 \right. \\
& \quad \quad - 4 \pi R_*^2 \left| p_x - p_{x,*} \right| R - R_* \left. \right| \right) \int_{B_1} \tilde{\rho}^2 \\
& \quad \quad + O \left( |R|^3 + |\tilde{\rho}|^3 + \left( \int_{B_1} |\tilde{\rho}| \right)^3 \right),
\end{align}

where $\mathcal{E}_x$ is $\mathcal{E}_{\text{total}}$ evaluated at $(\rho_*, R_*, \tilde{R}_*) = 0$. Furthermore, (2) there exist constants $\Theta > 0$, $\delta_0 \in (0, 1]$ depending only on $\text{Mass}[\rho_*, R_*]$, $T_x$, and $\nu$ such that if $\|\rho - \rho_*\|_{L^\infty(B_R)} + |p_x - p_{x,*}| \leq \delta_0$, then

\begin{align}
(7.17) & \quad \mathcal{E}_{\text{total}} - \mathcal{E}_x \geq \Theta \left( \int_{B_R} (\rho(x) - \rho_*)^2 dx \right).
\end{align}

**Comments on Theorem 7.5:**

(1) Fix $p_x = p_{x,*}$. Then, the coercive energy estimate of Theorem 7.5 implies that, relative to perturbations of the same bubble mass, the total energy $\mathcal{E}_{\text{total}}$ is locally convex around the equilibrium $(\rho_*, R_*, \tilde{R}_*) = 0$ and that the equilibrium $(\rho_*, R_*, \tilde{R}_*) = 0$ is the unique local minimizer of the total energy $\mathcal{E}_{\text{total}}$.

(2) The estimate (7.17) is a lower bound for the functional $(\rho, R, \tilde{R}) \mapsto \mathcal{E}_{\text{total}}[\rho, R, \tilde{R}]$. It does not depend on $(\rho, R, \tilde{R})$ being a solution of the evolution equations.

(3) Theorem 7.5 applies to all positive solutions $(\rho_*, R_*)$ of the algebraic system (4.13).

**Remark 7.6** (Surface tension versus thermal diffusion). In the spherically symmetric approximate model we study, the surface tension $\sigma$ does not play a role in the relaxation of the bubble to equilibrium. One expects it to play a role in the rounding out of non-spherical bubble deformations, which are not under consideration here. Although typically surface tension $\sigma$ is positive for liquid / gas interface, our analysis for asymptotic stability applies to $\sigma = 0$ or even some negative range of $\sigma$. In fact, the cubic equation (4.14) admits a unique positive solution for all $\sigma \in \mathbb{R}$. Besides, the equilibrium energy $\mathcal{E}_x$ remains the conditional minimizer as long as the coefficient of the third term on the right of (7.16) is positive by the coercive energy estimate (7.17). It is equivalent to $\sigma > -3R_{x,*}^3/4$.

On the other hand, the thermal conductivity of gas, $\kappa_g$, and far-field liquid temperature, $T_x$, both play a role in energy dissipation (7.4). Our analysis fails when either of these two parameters vanishes. However, the case when $T_x = 0$ is excluded since it would lead to a solution that is singular everywhere: $p_g(t) = \mathcal{O}(T_x, \rho_g(R(t), t)) = 0$, and $s = c_v \log(p_g/\rho_g^\gamma) = -\infty$. Therefore, the only physical parameter that plays a role in the damping mechanism is the thermal conductivity of
gas, $\kappa_g$. Indeed, $\kappa_g/(\gamma c_v)$ is the diffusion coefficient of the parabolic PDE (5.1a) that forces the gas density to distribute uniformly inside the bubble and causes the energy dissipation. This then leads to the thermal damping mechanism of the bubble radius by the ODE of bubble radius.

8. Asymptotic stability: Proof of Theorem 6.5

In this section, we prove Theorem 6.5. We show that family (manifold) of spherically symmetric equilibrium states is asymptotically stable. Our first step is to prove

8.1. Asymptotic stability of equilibria with respect to small mass-preserving perturbations. We begin with proving the asymptotic stability of a fixed equilibrium relative to mass preserving perturbations. To this end, we make use of Theorem 7.5 and the temporal integrability of the right hand side of (7.4).

Proposition 8.1 (Asymptotic stability of a fixed equilibrium relative to mass preserving perturbations). Fix parameters (5.2) and set $p_x = p_{x,*}$ in the system (5.1a)-(5.1c). For arbitrary fixed $M_0 > 0$, let $(\rho_*[M_0], R_*[M_0])$ denote the unique spherically symmetric equilibrium with bubble mass $M_0$ given in Proposition 4.3, i.e. $M_0 = \frac{4}{3}\pi \rho_*[M_0] (R_*[M_0])^3$. There exists $\eta_0 > 0$, such that the following holds for all $0 < \eta \leq \eta_0$:

Let $(\rho(r, t), R(t))$ be the global in time solution of the free boundary problem (5.1a)-(5.1c) with initial data $\rho(r, 0) = \rho_0(r)$, $R(0) = R_0$, $\dot{R}(0) = \dot{R}_0$ which is a mass preserving and small perturbation of $(\rho_*[M_0], R_*[M_0])$, i.e.

$$M_0 = \int_{R_0} \rho_0(x) \, dx, \quad \text{(mass preserving perturbation)}$$

and

$$\left| \left( \rho_0 - \rho_*[M_0], R_0 - R_*[M_0], \dot{R}_0 \right) \right| \leq \eta.$$ (8.1)

Then, as $t \to +\infty$

$$\left| \left( \rho(\cdot,t) - \rho_*[M_0], R(t) - R_*[M_0], \dot{R}(t) \right) \right| \to 0,$$ (8.2)

where the norm $\cdot \ | \cdot$ is defined in (6.1). Moreover, $|\dot{R}(t)| + |\ddot{R}(t)| \to 0$ as $t \to +\infty$.

Proof. Consider a fixed equilibrium $(\rho_*, R_*, \dot{R}_*) = 0$ and a nearby (non-constant) initial condition $(\rho_0, R_0, \dot{R}_0)$ (see the hypothesis (8.1)) and such that

$$\int_{B_{R_0}} \rho_0 = \int_{B_{R_*}} \rho_* = \frac{4\pi}{3} R_*^3 \rho_* = M_0.$$ (8.3)

Hence,

$$\int_{B_{R(t)}} \rho(x,t) \, dx = \frac{4\pi}{3} R_*^3 \rho_*, \quad \text{for all } t \geq 0.$$ (8.4)

First recall that $p_g = \mathcal{A}_g T g_0 \rho_g$ (see (2.2d)) and hence

$$\frac{\nabla T}{T} = \nabla \log T = \nabla \left( \log p - \log(\mathcal{A}_g \rho) \right) = -\frac{\nabla \rho}{\rho}, \quad T = T_g, \ p = p_g, \ \rho = \rho_g.$$ (8.4)

Together with the energy dissipation relation (7.4) we have

$$\frac{d}{dt} \mathcal{E}_{\text{total}}(t) = -\kappa T \int_{B_{R(t)}} \frac{\nabla T^2}{T^2} \, dx - 16\pi \mu R \dot{R}^2 = -\kappa T \int_{B_{R(t)}} \frac{\nabla \rho^2}{\rho^2} \, dx - 16\pi \mu R \dot{R}^2,$$ (8.4)

where $\mathcal{E}_{\text{total}}(t)$ is given in Definition 7.1.

$$\mathcal{E}_{\text{total}}(t) = FE(t) + 4\pi \sigma (R(t))^2 + \frac{4\pi}{3} R^3(t) + 2\pi \rho_1 \dot{R}^2(t) R^3(t).$$
Integrating (8.4) with respect to time we obtain
\begin{equation}
(8.5) \quad \kappa T \int_0^t \int_{B_{R(t)}} \frac{|
abla \rho(x, \tau)|^2}{\rho(x, \tau)^2} \, dx \, d\tau + 16\pi \mu_1 \int_0^t R(\tau)(\dot{R}(\tau))^2 \, d\tau = \mathcal{E}_{\text{total}}(0) - \mathcal{E}_{\text{total}}(t).
\end{equation}

Applying now the key coercive lower bound on $\mathcal{E}_{\text{total}}$ (Theorem 7.5 and in particular (7.17)) gives
\begin{equation}
(8.6) \quad \kappa T \int_0^t \int_{B_{R(t)}} \frac{|
abla \rho(x, \tau)|^2}{\rho(x, \tau)^2} \, dx \, d\tau + 16\pi \mu_1 \int_0^t R(\tau)(\dot{R}(\tau))^2 \, d\tau \leq \mathcal{E}_{\text{total}}(0) - \mathcal{E}_{\text{total},*} - (\mathcal{E}_{\text{total}}(t) - \mathcal{E}_{\text{total},*})
\end{equation}

By the regularity of $\rho(x, t)$ and $R(t)$, we have that $\int_{B_{R(t)}} \frac{|
abla \rho(x, \tau)|^2}{\rho(x, \tau)^2} \, dx$ and $R(t)(\dot{R}(t))^2$ are uniformly continuous. Recall the following alternative form of Barbalat’s lemma:

Suppose $\int_0^t f(\tau) \, d\tau$ has a finite limit as $t \to \infty$.

If $f(t)$ is uniformly continuous function, then $\lim_{t \to \infty} f(t) = 0$.

By the above Barbalat’s lemma, we conclude that
\begin{equation}
(8.7) \quad \lim_{t \to \infty} \int_{B_{R(t)}} \frac{|
abla \rho(x, t)|^2}{\rho(x, t)^2} \, dx = 0,
\end{equation}
and, if $\mu_1 > 0$,
\begin{equation}
(8.8) \quad \lim_{t \to \infty} R(t)(\dot{R}(t))^2 = 0.
\end{equation}

We next change variables, via $x = Ry$, to transform integrals over $B_{R(t)}$ into integrals over $B_1$. Noting that $\hat{\rho}(y, t) = \rho(Ry, t)$, we get
\begin{equation}
\int_{B_{R(t)}} \frac{|
abla \rho(x, t)|^2}{\rho(x, t)^2} \, dx = R(t) \int_{B_1} \frac{|
abla_y \hat{\rho}(y, t)|^2}{\hat{\rho}(y, t)^2} \, dy.
\end{equation}
Since that $R(t)$ is bounded away from zero and that $\hat{\rho}(y, t)$ is bounded from above (see (6.3), (7.13)) we have
\begin{equation}
(8.9) \quad \lim_{t \to \infty} \int_{B_1} |\nabla_y \hat{\rho}(y, t)|^2 \, dy = 0.
\end{equation}

Using the interpolation Lemma D.1 with $k = 0$, $n = 3$, $p = 2$, $m = 1$, $\gamma = 2\alpha$, and $s = 2$,
\begin{equation}
|\nabla_y \hat{\rho}|_{L^\infty(B_1)} \leq C_1 \|\nabla_y \hat{\rho}\|_{L^4(B_1)} \|\nabla_y \hat{\rho}\|_{C^{1+2\alpha}(B_1)} + C_2 \|\nabla_y \hat{\rho}\|_{L^2(B_1)}
\end{equation}
where $|\nabla_y \hat{\rho}|_{C^{1+2\alpha}(B_1)} \sim |\rho|_{C^{2+2\alpha}(B_R)}$ is uniformly bounded by (6.3). Therefore,
\begin{equation}
\nabla_y \hat{\rho}(\cdot, t) \to 0 \text{ uniformly in } B_1 \text{ as } t \to \infty.
\end{equation}
and by (6.3)
\begin{equation}
(8.10) \quad \nabla \rho(x, t) \to 0 \text{ uniformly in } B_{R(t)} \text{ as } t \to \infty.
\end{equation}

Furthermore, by the Poincaré inequality and (8.7)
\begin{equation}
(8.11) \quad \int_{B_1} \left( \hat{\rho}(y, t) - \frac{1}{|B_1|} \int_{B_1} \hat{\rho}(z, t) \, dz \right)^2 \, dy \leq \int_{B_1} |\nabla_y \hat{\rho}|^2 \, dy \to 0 \text{ as } t \to \infty.
\end{equation}
Moreover, by (8.3) (conservation of mass)
\begin{equation}
(8.12) \quad \frac{1}{|B_1|} \int_{B_1} \hat{\rho}(z, t) \, dz = \left( \frac{R_\ast}{R(t)} \right)^3 \rho_\ast.
\end{equation}
and hence,
(8.11) \[
\int_{B_1} \left( \tilde{\rho}(y, t) - \left( \frac{R_*}{R(t)} \right)^3 \rho_* \right)^2 \to 0 \text{ as } t \to \infty.
\]

In fact we claim that
(8.12) \[
\tilde{\rho}(y, t) - \left( \frac{R_*}{R(t)} \right)^3 \rho_* \to 0 \text{ uniformly on } B_1.
\]

Indeed, applying interpolation Lemma \[\text{(D.1)}\] with \(k = 0, n = 3, p = 2, m = 2, \gamma = 2\alpha, \) and \(s = 2, \) we have
\[
\left\| \tilde{\rho}(\cdot, t) - \left( \frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{L^p(B_1)} \leq C_1 \left\| \tilde{\rho}(\cdot, t) - \left( \frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{L^q(B_1)} + C_2 \left\| \tilde{\rho}(\cdot, t) - \left( \frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{L^2(B_1)}.
\]

Since by \[\text{(6.3)}\], we have that
\[
\left\| \tilde{\rho}(\cdot, t) - \left( \frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{C^{2+2\alpha}(B_1)} \sim \left\| \tilde{\rho}(\cdot, t) - \left( \frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{C^{2+2\alpha}(B_1)} \text{ is uniformly bounded, we have (8.12) and hence}
\]
(8.13) \[
\tilde{\rho}(y, t)(R(t))^3 \to \rho_* R_*^3 \quad \text{uniformly in } y \in B_1 \text{ as } t \to \infty.
\]

We next show that as \(t \to \infty, \) we have \(\tilde{R}, \tilde{\tilde{R}}, \) and \(\tilde{R} \to 0.\) Since \(p(t) = \mathcal{A}_g T_x \tilde{p}(y = 1, t)\) (using \[\text{(3.2)}\] and \(T(r = 1, t) = T_\infty), \) we have \(p(t)(R(t))^3 \to \mathcal{A}_g T_x \rho_* R_*^3 \) as \(t \to \infty. \) By the regularity of \(p(t)\) and \(R(t), \) \[\text{(6.1)}\] and \[\text{(6.5)}\], we have that \(\frac{\partial}{\partial t}(p(t)(R(t))^3)\) is uniformly continuous. Recall Barbalat’s lemma:
\[
\text{Suppose } f(t) \in C^1(a, \infty) \text{ and } \lim_{t \to \infty} f(t) = \alpha, \text{ with } |\alpha| < \infty.
\]
If \(f'(t)\) is uniformly continuous, then \(\lim_{t \to \infty} f'(t) = 0.\)

By the Barbalat’s lemma, \(\frac{\partial}{\partial t}(p(t)(R(t))^3) \to 0 \text{ as } t \to \infty.\) That is, \(\tilde{\gamma} p(t)(R(t))^3 + 3p(t)(R(t))^2 \tilde{R}(t) \to 0 \text{ as } t \to \infty,\) and therefore since \(\rho(t)\) and \(R(t)\) are uniformly bounded away from zero,
\[
\frac{\tilde{\gamma} p(t)}{3p(t)} R(t) + \tilde{R} \to 0 \text{ as } t \to \infty,
\]

Sending \(t \to \infty\) in \[\text{(5.1b)}\] yields
\[
\lim_{t \to \infty} \tilde{R}(t) = \lim_{t \to \infty} \left[ -\frac{\kappa}{\gamma c_v} \frac{\rho(R(t), t)}{\rho(R(t), t)^2} - \frac{R \tilde{\gamma} \rho}{3 \gamma} \frac{p}{p} \right]
\]
\[
= \lim_{t \to \infty} \left[ -\frac{\kappa}{\gamma c_v} \frac{\rho(R(t), t)}{\rho(R(t), t)^2} - \frac{1}{\gamma} \left( \frac{R \tilde{\gamma} p}{3} + \tilde{R} \right) \right] = \gamma^{-1} \lim_{t \to \infty} \tilde{R}(t).
\]

Hence, \((1 - \gamma^{-1}) \lim_{t \to \infty} \tilde{R}(t) = 0,\) and since \(\gamma \neq 1\) we have \(\lim_{t \to \infty} \tilde{R}(t) = 0. \) Further application of Barbalat’s lemma yields \(\tilde{R}(t), \tilde{R}(t) \to 0 \text{ as } t \to \infty.\)

We next prove \(R(t) \to R_*\) along a sequence of time \(t_k \to \infty.\) Since \(R(t)\) is bounded in \(t, \) there is a sequence \(\{t_k\}\) along which \(R(t_k) \to R_*\) as \(k \to \infty\) for some \(R_* > 0.\) Furthermore, by \[\text{(8.12)}\]
\(\tilde{p}(y, t_k)\) converges uniformly on \(B_1\) to a limit \(\rho_{**}\) as \(k \to \infty.\) By \[\text{(5.1c)}\], since \(\tilde{R}(t)\) and \(\tilde{R}(t)\) tend to zero as \(t \to \infty,\) we obtain
\[
\rho_{**} = \frac{1}{\mathcal{A}_g T_\infty} \left( p_{x,*} + \frac{2 \sigma}{R_*} \right).
\]

Further, passing to the limit in \[\text{(8.3)}\], we obtain
\[
\rho_{**} R_*^3 = \rho_* R_*^3.
\]

Hence, \((\rho_{**}, R_{**})\) satisfies the algebraic system \[\text{(4.13)}\] characterizing the unique spherically symmetric equilibrium of with bubble mass \((4\pi/3)R_*^3 \rho_* \). We conclude that \((\rho_{**}, R_{**}) = (\rho_*, R_*)\). Furthermore, \(p(t_k) \to p_* \text{ as } k \to \infty,\) where \(p_* = \mathcal{A}_g T_\infty \rho_* = p_{x,*} + \frac{2 \sigma}{R_*} \).
We are in the position to prove the main result, namely equation (8.14) holds for $t \rightarrow \infty$ by the identity

$$
\mathcal{E}_{\text{total}}(t) = -\kappa T \int_0^t \int_{B_R(t)} \frac{\nabla \rho(x,t)}{\rho(x,t)} \cdot \nabla \rho(x,t) \, dx \, dt - 16\pi \int_0^t R(\tau)(\dot{R}(\tau))^2 \, d\tau + \mathcal{E}_{\text{total}}(0),
$$

which follows from rearranging terms in (8.13), that $\lim_{t \rightarrow \infty} \mathcal{E}_{\text{total}}(t)$ exists. Since $\mathcal{E}_{\text{total}}(t_k) \rightarrow \mathcal{E}_{\text{total},\ast}$ as $k \rightarrow \infty$, we have $\mathcal{E}_{\text{total}}(t) \rightarrow \mathcal{E}_{\text{total},\ast}$ as $t \rightarrow \infty$. By Theorem 7.3,

$$
\lim_{t \rightarrow \infty} \int_{B_R(t)} (\rho(x,t) - \rho_\ast)^2 \, dx \leq \Theta^{-1} \lim_{t \rightarrow \infty} (\mathcal{E}_{\text{total}}(t) - \mathcal{E}_{\text{total},\ast}) = 0.
$$

Using the interpolation Lemma 3.1 with $k = 0$, $n = 3$, $p = 2$, $m = 2$, and $\gamma = 2\alpha$, we obtain

$$
\|\rho(\cdot,t) - \rho_\ast\|_{L^p(B_R(t))} \leq C_1 \|\rho(\cdot,t) - \rho_\ast\|_2^{\frac{2}{3}} \|\rho(\cdot,t) - \rho_\ast\|_2^2 \|B_{\hat{R}}(t)\| \leq C_2 \|\rho(\cdot,t) - \rho_\ast\|_2^2 \rightarrow 0 \text{ as } t \rightarrow \infty.
$$

Thus, $\rho(x,t) \rightarrow \rho_\ast$ uniformly on $B_R(t)$.

From the uniform convergence of $\rho(x,t)$ to $\rho_\ast$, we will finally conclude that $R(t) \rightarrow R_\ast$. By conservation of mass,

$$
\int_{B_{\hat{R}}(t)} (\rho(x,t) - \rho_\ast) \, dx = \int_{B_{R_0}} \rho_0(x) \, dx - \frac{4\pi \rho_\ast}{3} R_\ast^3(t) = \frac{4\pi \rho_\ast}{3} (R_\ast^3 - R(t)),
$$

which implies

$$
|R(t) - R_\ast| \leq \frac{3}{4\pi \rho_\ast (R_\ast^2(t) + R(t)R_\ast + R_\ast^2)} \left( \int_{B_{\hat{R}}(t)} |\rho(\cdot,t) - \rho_\ast|^2 \, dx \right)^{\frac{1}{2}} \leq C \left( \int_{B_{\hat{R}}(t)} |\rho(\cdot,t) - \rho_\ast|^2 \, dx \right)^{\frac{1}{2}}
$$

It follows from (8.15) and (8.17) that $R(t) \rightarrow R_\ast$ as $t \rightarrow \infty$.

Finally, since we have $\|\rho(\cdot,t) - \rho_\ast\|_L^2 = \|\nabla \rho(\cdot,t)\|_L^2 + |R(t) - R_\ast| + |\dot{R}(t)| + |\ddot{R}(t)| + |\dddot{R}(t)| \rightarrow 0$ as $t \rightarrow +\infty$, the same bootstrap argument in the end of the proof of [5] Theorem 4.1 yields

$$
\|\rho(\cdot,t) - \rho_\ast\|_{C^{2+2\alpha}} \rightarrow 0 \text{ as } t \rightarrow \infty,
$$

completing the proof of Proposition 8.1.

\[\square\]

8.2. Proof of Theorem 6.5: Asymptotic stability of the family of spherically symmetric equilibria relative to arbitrary small perturbations. We are in the position to prove the main theorem, Theorem 6.5. The proof is similar to that of Theorem 6.3. We now assume that $(\rho_0, R_0, \hat{R}_0)$ is an arbitrary sufficiently small perturbation of some fixed equilibrium $(\rho_\ast, R_\ast, \hat{R}_\ast = 0)$, in the sense of (6.7). By Proposition 4.6 there is a unique $(\hat{\rho}_\ast, \hat{R}_\ast)$ such that $\text{Mass}[\hat{\rho}_\ast, \hat{R}_\ast] = \text{Mass}[\rho_0, R_0]$, and

$$
|\hat{R}_\ast - R_\ast| + |\hat{\rho}_\ast - \rho_\ast| \leq C \left( |R_0 - R_\ast| + \|\rho_0 - \rho_\ast\|_{L^\infty(B_{R_0})} \right).
$$

Hence,

$$
\|\rho_0 - \hat{\rho}_\ast\|_{C^{2+2\alpha}} + |R_0 - \hat{R}_\ast| = \|\rho_0 - \rho_\ast + \rho_\ast - \hat{\rho}_\ast\|_{C^{2+2\alpha}} + |R_0 - R_\ast + R_\ast - \hat{R}_\ast| \leq \|\rho_0 - \rho_\ast\|_{C^{2+2\alpha}} + |\rho_\ast - \hat{\rho}_\ast| + |R_0 - R_\ast| + |R_\ast - \hat{R}_\ast| \leq C' \left( \|\rho_0 - \rho_\ast\|_{C^{2+2\alpha}} + |R_0 - R_\ast| \right)
$$

where
Therefore, choosing \( \|\rho_0 - \rho_*\|_{C^{2,\alpha}} + |R_0 - R_*| \) and \( \hat{R}_0 \) sufficiently small we conclude from Part 1 that \( \rho(\cdot, t) \) converges uniformly to \( \hat{\rho}_* \) and \( R(t) - \hat{R}_* \to 0 \) as \( t \to \infty \). This completes the proof of Theorem 6.3.

### Appendix A. Spherically Symmetric Equilibria of the Full Liquid / Gas Model

In this appendix, we prove Proposition 4.1. That is, we show that (4.3) is the unique regular spherically symmetric equilibrium solution to the system (2.1)-(2.4) under the radiation condition (4.2) for \( T_l \).

**Proof of Proposition 4.1.** We consider the full liquid / gas model (2.1)-(2.4) and prove Proposition 4.1. Steady-state solutions of (2.1)-(2.4) solve

\[
\begin{align*}
0 &= \nu_l \Delta v_{l_*} - v_{l_*} \cdot \nabla v_{l_*} - g_{l_*} \nabla p_{l_*}, \\
\text{div} v_{l_*} &= 0, \\
\rho_l c_l v_{l_*} \cdot \nabla T_{l_*} &= \kappa_l \Delta T_{l_*} + 2\mu_l \nabla \cdot (v_{l_*} : \nabla v_{l_*}), \\
\text{div}(\rho_g v_{g_*}) &= 0, \\
\rho_g T_{T_g} v_{g_*} \cdot \nabla s_* &= \kappa_g \Delta T_{g_*} + 2\mu_g \nabla \cdot (v_{g_*} : \nabla v_{g_*}) - \frac{2}{3} \left( 2\mu_g - \zeta_g \right) (\text{div} v_{g_*})^2, \\
p_{g_*} &= \mathcal{R}_g T_{T_g} \rho_{g_*}, \\
s_* &= c_g \log \left( \frac{p_{g_*}}{p_{g_*}} \right),
\end{align*}
\]

(A.1a) \quad (A.1b) \quad (A.1c) \quad (A.2a) \quad (A.2b) \quad (A.2c) \quad (A.2d) \quad (A.2e)

For the spherically symmetric case, (A.1b) we have \( v_{l_*}(t) = v_{l_*}(r) \frac{r}{r} \). Therefore, \( 0 = \text{div} v_{l_*} = \partial_r v_{l_*} + \frac{1}{r} \partial_r (r^2 v_{l_*}(r)) \). Hence

\[
\frac{1}{r^2} \partial_r (r^2 v_{l_*}(r)) = 0, \quad r \geq R_*.
\]

Therefore,

\[
v_{l_*}(r) = \frac{a}{r^2}, \quad r \geq R_*,
\]

for constant \( a \). But the boundary condition (A.3a) implies \( v_{l_*}(R_*) = 0 \). So \( a = 0 \), and thus \( v_{l_*} \equiv 0 \). Therefore, (A.1a) becomes \( \nabla p_{l_*} = 0 \). So the pressure \( p_{l_*} \) is a constant and equal to its value at infinity, \( p_{\infty} \).

For the gas velocity \( v_{g_*} \), (A.2a) becomes

\[
\frac{1}{r^2} \partial_r (r^2 v_{g_*}) = 0, \quad 0 \leq r \leq R_*,
\]

which implies \( \rho_{g_*} v_{g_*} \) is a constant. Again, the boundary condition (A.3a) implies \( v_{g_*}(R_*) = 0 \). So \( \rho_{g_*} v_{g_*} \equiv 0 \). But \( \rho_{g_*} \neq 0 \) since otherwise \( s_* \) is singular in (A.2c). Therefore, \( v_{g_*} \equiv 0 \) and thus (A.2b) becomes \( \nabla p_{g_*} = 0 \). So \( p_{g_*} \) is a constant. Moreover, by \( v_{l_*} = v_{g_*} \equiv 0 \), (A.3a) yields \( -p_{l_*} + p_{g_*} = \frac{\mu_g}{R_g} \). So \( p_{g_*} = p_{\infty} - \frac{\mu_g}{R_g} \).
For the equations of the temperatures, due to \( v_{l,*} = 0 \) \( (A.1c) \), becomes \( \Delta T_{l,*} = 0 \) in \( \mathbb{R}^3 \setminus B_{R_*} \), or, in spherical coordinates, 
\[
\frac{1}{r^2} \partial_r (r^2 \partial_r T_{l,*}) = 0, \quad r \geq R_*,
\]
which implies 
\[
\partial_r T_{l,*} = \frac{a_1}{r^2}, \quad r \geq R_*
\]
for some constant \( a_1 \). Integrating over \((r, \infty)\) we get 
\[
T_{l,*}(r) = T_x - \frac{a_1}{r}, \quad r \geq R_*.
\]
By the radiation condition \( (4.2) \), \( T_{l,*}(r) = T_x + o(r^{-1}) \) as \( r \to \infty \). This gives \( a_1 = 0 \) and thus \( T_{l,*} \equiv T_x \). On the other hand, \( (A.2c) \) becomes \( \Delta T_{g,*} = 0 \) in \( B_{R_*} \) since \( v_{g,*} = 0 \). Since \( T_{g,*} \) is regular, \( T_{g,*} \equiv T_{g,*}(R_*) = T_x \) by the maximum principle.

For the gas density \( \rho_{g,*} \), by \( (A.2d) \) 
\[
\rho_{g,*} = \frac{\rho_{g,x,*}}{\mathcal{A}_g T_x} = \frac{1}{\mathcal{A}_g T_x} \left( p_{x,*} + \frac{2\sigma}{R_*} \right).
\]
Due to the conservation of mass \( (1.1) \), 
\[
M := \int_{B_{R_0}} \rho_0(x) \, dx = \lim_{t \to \infty} \int_{B_{R(t)}} \rho_g(x, t) \, dx = \frac{4\pi}{3} \rho_{g,*} R_*^3,
\]
where \( \rho_0(x, R_0), \rho_0(x) \geq 0, R_0 > 0 \), is the initial data. Therefore, the steady state \( (\rho_{g,*}, R_*) \) can be obtained by solving
\[
\begin{align*}
(A.6a) & \quad \frac{4\pi}{3} \rho_{g,*} R_*^3 = M, \\
(A.6b) & \quad \rho_{g,*} = \frac{1}{\mathcal{A}_g T_x} \left( p_{x,*} + \frac{2\sigma}{R_*} \right).
\end{align*}
\]
In particular, the stationary radius \( R_* \) satisfies the cubic equation
\[
(A.7) \quad p_{x,*} R_*^3 + 2\sigma R_*^2 - \frac{3\mathcal{A}_g T_x M}{4\pi} = 0.
\]
It is readily seen that for any \( M > 0 \), the cubic equation has a unique positive root \( R_*[M] \). This proves Proposition \( 4.1 \). \( \square \)

**Appendix B. Derivation of the reduced system for \( \rho(r, t) \) and \( R(t) \): Proof of Proposition 5.1**

Considering the uniformity of the pressure \( p_g \) in \( (3.2d) \), we can eliminate \( T_g \) by plugging \( (3.2d) \) into \( (3.2c) \) and deduce
\[
(B.1) \quad \partial_t s + v_g \cdot \nabla s = \kappa_g \Delta \left( \frac{1}{\rho_g} \right).
\]
Plugging \( (3.2c) \) into the left hand side of \( (B.1) \) and using \( (3.2b) \), we have
\[
(B.2) \quad c_v \left\{ \partial_t p_g \rho_g - \frac{\gamma}{\rho_g} \left[ \partial_t \rho_g + v_g \cdot \nabla \rho_g \right] \right\} = \kappa_g \Delta \left( \frac{1}{\rho_g} \right).
\]
Using \( (3.2a) \) in \( (B.2) \), we obtain
\[
(B.3) \quad c_v \left\{ \partial_t \rho_g \rho_g + \gamma \text{div} \, v_g \right\} = \kappa_g \Delta \left( \frac{1}{\rho_g} \right).
\]
Therefore, the system \( (3.2) \) is reduced to
\[
\begin{align*}
(B.4a) & \quad \partial_t \rho_g + \text{div} (\rho_g v_g) = 0, \\
(B.4b) & \quad \frac{\partial_t \rho_g}{\rho_g} = \frac{\kappa}{c_v} \Delta \left( \frac{1}{\rho_g} \right) - \gamma \text{div} v_g,
\end{align*}
\]
in \( \Omega(t), t > 0 \).
Expanding the term \( \text{div}(\rho_g \mathbf{v}_g) \) in \((\text{B.4a})\) and substituting \( \text{div} \mathbf{v}_g \) using \((\text{B.4b})\), and using the elementary identity

\[
(\text{B.5}) \\
\rho_g \Delta \left( \frac{1}{\rho_g} \right) = -\Delta \log \rho_g + \frac{|\nabla \rho_g|^2}{\rho_g^2},
\]

we get

\[
(\text{B.6}) \\
\partial_t \rho_g = \frac{\kappa}{\gamma c_v} \Delta \log \rho_g - \frac{\kappa}{\gamma c_v} \frac{|\nabla \rho_g|^2}{\rho_g^2} - \mathbf{v}_g \cdot \nabla \rho_g + \frac{\mathbf{\varepsilon}_r \rho_g}{\gamma p_g} \rho_g.
\]

Assuming the bubble is a sphere \( B_{R(t)} \) and solutions are spherically symmetric, and recalling we denoted the radial components of the gas and liquid velocity by \( v_g(r,t) \) and \( v_l(r,t) \), respectively, the systems \((\text{B.1})\) and \((\text{B.4})\) become

\[
(\text{B.7a}) \\
\partial_t v_l = v_l \left( \Delta r v_l - \frac{2v_l}{r^2} \right) - v_l \partial_r v_l - \frac{1}{\rho_l} \partial_r \rho_l, \\
\frac{1}{r^2} \partial_r (r^2 v_l) = 0,
\]

for \( r \geq R(t), t > 0 \),

and

\[
(\text{B.8a}) \\
\partial_t \rho_g + \frac{1}{r^2} \partial_r (r^2 \rho_g v_l) = 0,
\]

\[
(\text{B.8b}) \\
\frac{\partial_t \rho_g}{\rho_g} = \frac{\kappa}{c_v} \frac{1}{r^2} \partial_r \left( \frac{r^2 \partial_r (1/\rho_g)}{2} \right) - \frac{1}{r^2} \partial_r (r^2 v_l),
\]

and the boundary condition \((\text{B.3})\) becomes

\[
(\text{B.9a}) \\
v_l(R(t),t) = v_g(R(t),t) = \dot{R}(t),
\]

\[
(\text{B.9b}) \\
p_g(t) - p_l(R(t),t) + 2 \mu_l \partial_r v_l(R(t),t) = \frac{2\sigma}{R(t)},
\]

\[
(\text{B.9c}) \\
T(R(t),t) = T_{\infty},
\]

The liquid velocity and pressure \((v_l,p_l)\) can be directly solved in terms of \( R(t), \dot{R}(t) \), the liquid pressure \( p_l(R(t),t) \) on the bubble wall, and the far-field liquid pressure \( p_{\infty}(t) := p_l(r = \infty, t) \). In fact, the incompressibility condition \((\text{B.7b})\) and the kinematic boundary condition \((\text{B.9a})\) imply

\[
(\text{B.10}) \\
v_l(r,t) = \frac{(R(t))^2 \dot{R}(t)}{r^2}, \quad r \geq R(t), t > 0.
\]

Plugging Equation \((\text{B.10})\) into Equation \((\text{B.7a})\), we have

\[
(\text{B.11}) \\
\frac{2R \dot{R}^2 + R^2 \ddot{R}}{r^2} = v_l \left( \frac{2R^2 \dddot{R}}{r^4} - 2 \frac{R^2 \ddot{R}}{r^4} \right) + 2 \frac{R^4 \ddot{R}^2}{r^6} - \frac{1}{\rho_l} \partial_r \rho_l, \quad r \geq R(t), t > 0.
\]

Note that the diffusion term in \((\text{B.11})\) vanishes. So the reduction using spherical symmetry assumption also works for Euler equation, i.e., we can take \( v_l = 0 \) in \((\text{B.10})\). Integrating Equation \((\text{B.11})\) over \( r > R(t) \), we deduce

\[
(\text{B.12}) \\
p_l(r,t) = p_{\infty}(t) + \rho_l \left( \frac{2}{r} \frac{R^2 \ddot{R}}{r^4} + \frac{(R(t))^2 \dot{R}(t)}{r^2} - \frac{(R(t))^4 (\dot{R}(t))^2}{2r^4} \right), \quad r \geq R(t), t > 0.
\]

In particular, on the boundary the liquid pressure is

\[
p_l(R(t),t) = p_{\infty}(t) + \rho_l \left( \frac{3}{2} R^2 + R \ddot{R} \right), \quad t > 0.
\]

Moreover, \((\text{B.10})\) implies \( \partial_r v_l(r,t) = -2(R(t))^2 \dot{R}(t)/r^3 \) so that

\[
\partial_r v_l(R(t),t) = -2 \frac{\dot{R}(t)}{R(t)}.
\]
This implies
\begin{equation}
R\ddot{R} + \frac{3}{2} \dot{R}^2 = \frac{1}{\rho_t} \left( p_g(t) - p_{\infty}(t) - \frac{2\sigma}{R} - 4\mu_t \frac{\dot{R}}{R} \right), \quad t > 0,
\end{equation}
where the Young–Laplace boundary condition \(B.9b\) has been used.

For the gas dynamics in the bubble, by integrating \(B.8b\) in \(r\), the radial component of the gas velocity \(v_g\) can be expressed in terms of \(\rho_g(r, t), \dot{c}_r\rho_g(r, t), p_g(t),\) and \(\dot{c}_r p_g(t)\). To be more precise,
\begin{equation}
v_g(r, t) = \frac{\kappa}{\gamma c_v} \dot{c}_r \left( \frac{1}{\rho_g(r, t)} \right) - \dot{c}_r p_g(t) \frac{r}{p_g(t)} 3\gamma, \quad 0 \leq r \leq R(t), \quad t > 0.
\end{equation}
Using \(B.14\) we can eliminate \(v_g\) in \(B.9\) and obtain
\begin{equation}
\dot{c}_r \rho_g = \frac{\kappa}{\gamma c_v} \Delta \log \rho_g + \frac{\dot{c}_r p_g}{3\gamma p_g} \dot{c}_r \rho_g + \frac{\dot{c}_r p_g}{\gamma p_g} \rho_g, \quad 0 \leq r \leq R(t), \quad t > 0,
\end{equation}
where \(\Delta f = \frac{1}{r^2} \partial_r (r^2 \partial_r f)\) for spherically symmetric functions \(f\). From the boundary condition \(B.9c\) for the gas temperature and the equation of state \(3.2d\),
\begin{equation}
p_g(t) = \mathcal{A}_g \rho_g(R(t), t) T_{\infty}.
\end{equation}
Taking time derivative of \(B.16\) we obtain
\begin{equation}
\dot{c}_r p_g = \frac{\dot{c}_r p_g(R(t), t)}{\rho_g(R(t), t)} + \frac{\dot{c}_r p_g(R(t), t)}{\rho_g(R(t), t)} \dot{R}(t).
\end{equation}
Evaluating \(B.14\) at \(r = R(t)\) and using the kinematic boundary condition \(B.9a\) it follows that
\begin{equation}
\dot{R}(t) = -\frac{\kappa}{\gamma c_v} \frac{\dot{c}_r p_g(R(t), t)}{\rho_g(R(t), t)}^2 - \frac{R(t) \dot{c}_r p_g}{3\gamma p_g}.
\end{equation}
For the boundary data for the gas density, we use \(B.16\) and \(B.13\) to deduce
\begin{equation}
\rho_g(R(t), t) = \frac{1}{\mathcal{A}_g T_{\infty}} \left[ \rho_{\infty} + \frac{2\sigma}{R} + \rho_t \left( R\ddot{R} + \frac{3}{2} (\dot{R})^2 \right) \right].
\end{equation}
Collecting the results \(B.15, B.16, B.18, B.19\), we conclude that, under the spherical symmetry assumption, the system \(3.1-3.3\) is reduced to a system of \((\rho(r, t), R(t)):\)
\begin{equation}
\dot{c}_r \rho = \frac{\kappa}{\gamma c_v} \Delta \log \rho + \frac{\dot{c}_r p_g}{3\gamma p_g} \dot{c}_r \rho + \frac{\dot{c}_r p_g}{\gamma p_g} \rho, \quad 0 \leq r \leq R(t), \quad t > 0,
\end{equation}
\begin{equation}
p(t) = \mathcal{A}_g T_{\infty} \rho(R(t), t), \quad t > 0,
\end{equation}
\begin{equation}
\dot{R}(t) = -\frac{\kappa}{\gamma c_v} \frac{\dot{c}_r p_g(R(t), t)}{\rho_g(R(t), t)}^2 - \frac{R(t) \dot{c}_r p_g}{3\gamma p_g}, \quad t > 0,
\end{equation}
\begin{equation}
\rho(R(t), t) = \frac{1}{\mathcal{A}_g T_{\infty}} \left[ \rho_{\infty} + \frac{2\sigma}{R} + 4\mu_t \frac{\dot{R}}{R} + \rho_t \left( R\ddot{R} + \frac{3}{2} (\dot{R})^2 \right) \right], \quad t > 0,
\end{equation}
where \(\rho \equiv \rho_g, \quad p \equiv p_g, \quad \kappa = \kappa_g\). This is the reduced system \(5.1a-5.1c\).

Appendix C. A perspective on coercive energy estimate of Biro-Velázquez, and an extension.

In this appendix, we prove Theorem \(C.5\) which extends the coercivity estimate of Biro-Velázquez to the case where \(p_{\infty} - p_{\infty, a}\) is small in norm.
Proof of Theorem 7.5. Let us recall the total energy
\[ \mathcal{E}_{\text{total}} = FE + KE_l + U_{g-l} + PV_{p_x}, \]
where
\begin{align*}
\text{(C.1a)} & \quad FE = \frac{4\pi c_v}{3\mathcal{A}_g} \rho R^3 - c_v T_x M_0 \log \rho + c_v \gamma T_x \int_{B_R} \rho \log \rho \, dx, \quad M_0 = \text{Mass}[\rho, R], \\
\text{(C.1b)} & \quad KE_l = 2\pi \rho R^3 \dot{R}^2, \\
\text{(C.1c)} & \quad U_{g-l} = 4\pi \sigma R^2, \\
\text{(C.1d)} & \quad PV_{p_x} = \frac{4\pi}{3} R^3 p_x.
\end{align*}

The energy is a functional of state variables, which are defined on a deforming regime, \( B_R \). We fix the region to be \( B_1 \) by setting \( x = Ry \), where \( y \in B_1 \). Defining \( \bar{\rho}(y) = \rho(Ry) \) and using the constitutive relation \( p = \mathcal{A}_g T_x \rho(R) = \mathcal{A}_g T_x \bar{\rho}(1) \) we have that
\[ FE = \frac{4\pi c_v T_x}{3} \bar{\rho}(1) \dot{R}^3 - c_v T_x M_0 \log(\mathcal{A}_g T_x) - c_v T_x M_0 \log \bar{\rho}(1) + c_v \gamma T_x R^3 \int_{B_1} \bar{\rho}(y) \log \bar{\rho}(y) \, dy. \]

Thus, \( \mathcal{E}_{\text{total}} \) is a functional of \( (\bar{\rho}, R, \dot{R}) \):
\[ \mathcal{E}_{\text{total}} = \mathcal{E}_{\text{total}}[\bar{\rho}, R, \dot{R}] = \frac{4\pi c_v T_x}{3} \bar{\rho}(1) \dot{R}^3 - c_v T_x M_0 \log(\mathcal{A}_g T_x) - c_v T_x M_0 \log \bar{\rho}(1) + c_v \gamma T_x R^3 \int_{B_1} \bar{\rho}(y) \log \bar{\rho}(y) \, dy \\
+ 2\pi \rho R^3 \dot{R}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_x. \]

We set \( \bar{\rho} = \rho_* + \bar{\rho}, R = R_* + \bar{R} \) and expand the total energy \( \mathcal{E}_{\text{total}}(\bar{\rho}, R, \dot{R}) \) at \( (\rho_*, R_*, \dot{R}_*) = 0 \) along the mass preserving hypersurface \( M_0 = \text{Mass}[\rho, R] \):
\[ \mathcal{E}_{\text{total}}[\bar{\rho}, R, \dot{R}] = \mathcal{E}_* + d\mathcal{E}_*[\bar{\rho}, R, \dot{R}] + \frac{1}{2} d^2\mathcal{E}_*[\bar{\rho}, R, \dot{R}] + O(|(\bar{\rho}, R, \dot{R})|^3), \]
where
\[ \mathcal{E}_* = \mathcal{E}_{\text{total}}[\rho_*, R_*, \dot{R}_* = 0]. \]

To expand along the mass preserving hypersurface, we first use \( M_0 = \int_{B_R} \rho \, dx \) to rewrite
\[ R^3 \int_{B_1} \bar{\rho} \log \bar{\rho} \, dy = \int_{B_R} \rho \log \rho \, dx = \int_{B_R} \rho \log \rho_* \, dx + \int_{B_R} \rho \log \frac{\rho}{\rho_*} \, dx \]
\[ = \log \rho_* \int_{B_R} \rho \, dx + \int_{B_R} \rho \, dx + \int_{B_R} \rho \left( \log \frac{\rho}{\rho_*} - 1 \right) \, dx \]
\[ = M_0 \log \rho_* + M_0 + R^3 \int_{B_1} \bar{\rho} \left( \log \frac{\rho}{\rho_*} - 1 \right) \, dy, \]
giving the following expression for the total energy:
\[ \mathcal{E}_{\text{total}}[\bar{\rho}, R, \dot{R}] = \frac{4\pi c_v T_x}{3} \bar{\rho}(1, t) R^3 - c_v T_x M_0 \log(\mathcal{A}_g T_x) - c_v T_x M_0 \log \bar{\rho}(1, t) \\
+ c_v \gamma T_x M_0 \log \rho_* + c_v \gamma T_x M_0 + c_v \gamma T_x R^3 \int_{B_1} \bar{\rho} \left( \log \frac{\rho}{\rho_*} - 1 \right) \, dy \\
+ 2\pi \rho R^3 \dot{R}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_x. \]
To expand the logarithmic terms we note that for \( z_* \neq 0 \) and \( |z - z_*| < \frac{1}{2} |z_*| \):

\[
\|z \left( \log \left( \frac{z}{z_*} \right) - 1 \right) - \left( -z_* + \frac{1}{2}(z - z_*)^2 \right) \| \leq \frac{2}{|z_*|} |z - z_*|^3
\]

\[
\| \log z - \left( \log z_* + \frac{1}{z_*}(z - z_*) - \frac{1}{2z_*^2}(z - z_*)^2 \right) \| \leq \frac{2}{3|z_*|^3} |z - z_*|^3
\]

Applying (C.3) and (C.4) we have

\[
\mathcal{E}_{\text{total}}[\bar{\rho}, R, \bar{R}] = \frac{4\pi c_v T_\infty}{3} \tilde{\rho}(1,t) R^3 - c_v T_\infty M_0 \log(\mathcal{R}_g T_\infty)
\]

\[
- c_v T_\infty M_0 \left( \log \rho_* + \frac{1}{\rho_*} \tilde{\varrho}(1) - \frac{1}{2\rho_*^2} \tilde{\varrho}(1)^2 \right) + O \left( \tilde{\varrho}(1)^3 \right)
\]

\[
+ c_v \gamma T_\infty R^3 \left( -\frac{4\pi}{3} \rho_* + \frac{1}{2} \int_{B_1} \tilde{\varrho}^2 \right) + O \left( R^3 \int_{B_1} |\tilde{\varrho}|^3 \right)
\]

\[
+ 2\pi \rho_t R^3 \tilde{\rho}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_\infty.
\]

Rearranging and simplifying gives

\[
\mathcal{E}_{\text{total}}[\bar{\rho}, R, \bar{R}] = -c_v T_\infty M_0 \log(\mathcal{R}_g T_\infty) + c_v (2\gamma - 1) T_\infty M_0 \log \rho_* + c_v \gamma T_\infty M_0
\]

\[
+ \frac{4\pi c_v T_\infty}{3} \tilde{\rho}(1,t) R^3 + c_v T_\infty M_0 \left( -\frac{1}{\rho_*} \tilde{\varrho}(1) + \frac{1}{2\rho_*^2} \tilde{\varrho}(1)^2 \right) + O \left( \tilde{\varrho}(1)^3 \right)
\]

\[
+ c_v \gamma T_\infty R^3 \left( -\frac{4\pi}{3} \rho_* + \frac{1}{2} \int_{B_1} \tilde{\varrho}^2 \right) + O \left( R^3 \int_{B_1} |\tilde{\varrho}|^3 \right)
\]

\[
+ 2\pi \rho_t R^3 \tilde{\rho}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_\infty.
\]

Verification that \( d\mathcal{E}_s[\tilde{\varrho}, \mathcal{R}, \tilde{\mathcal{R}}] = 0 \) when \( p_\infty = p_{\infty,*} \). Starting with (C.5) we calculate:

\[
d\mathcal{E}_s[\tilde{\varrho}, \mathcal{R}, \tilde{\mathcal{R}}] = \frac{4\pi c_v T_\infty}{3} \left( 3 \rho_* R^2 \mathcal{R} + R^3 \tilde{\rho}(1) \right) - \frac{c_v T_\infty}{\rho_*} \left( \frac{4\pi}{3} \rho_* R^3 \right) \tilde{\rho}(1)
\]

\[
- 4\pi c_v \gamma T_\infty R^2 \rho_* \mathcal{R} + 8\pi \sigma R^2 \mathcal{R} + 4\pi R^2 p_\infty \mathcal{R}
\]

\[
= 4\pi R^2 \left( c_v T_\infty \rho_* (1 - \gamma) + \frac{2\sigma}{R_*} + p_\infty \right) \mathcal{R}
\]

\[
= 4\pi R^2 \left( -\mathcal{R}_g T_\infty \rho_* + \frac{2\sigma}{R_*} + p_\infty \right) \mathcal{R}
\]

\[
= 4\pi R^2 \mathcal{P}_\infty \mathcal{R} \quad \text{by (4.131)}.
\]

where \( \mathcal{P}_\infty = p_\infty - p_{\infty,*} \). It is readily to see that \( d\mathcal{E}_s[\tilde{\varrho}, \mathcal{R}, \tilde{\mathcal{R}}] = 0 \) when \( p_\infty = p_{\infty,*} \).

Computation of \( \frac{1}{2} d^2\mathcal{E}[\tilde{\varrho}, \mathcal{R}, \tilde{\mathcal{R}}] \). From (C.5) we compute the quadratic terms:

\[
\frac{1}{2} d^2\mathcal{E}[\tilde{\varrho}, \mathcal{R}, \tilde{\mathcal{R}}] = 4\pi c_v T_\infty \left( \rho_* R^2 + R^2 \tilde{\rho}(1) \mathcal{R} \right) + \frac{c_v T_\infty M_0}{2\rho_*^2} \tilde{\rho}(1)^2
\]

\[
+ \frac{c_v \gamma T_\infty R^3}{2} \int_{B_1} \tilde{\varrho}^2 - 4\pi c_v \gamma \rho_* T_\infty R^2
\]

\[
+ 2\pi \rho_t R^3 \tilde{\rho}^2 + 4\pi \sigma R^2 + 4\pi R^2 p_\infty \mathcal{R}^2.
\]
Next, using that the perturbed bubble is assumed to have mass equal to \(M_0 = \text{Mass}(\rho_*, R_*)\), we express the cross-term just above in terms of a quadratic expression in \(\tilde{\rho}\) as follows:

\[
M_0 = R^3 \int_{B_1} \tilde{\rho} = (R_* + R)^3 \int_{B_1} (\rho_* + \tilde{\rho})
\]

\[
= M_0 + 4\pi R_*^2 \rho_* R + R^3 \int_{B_1} \tilde{\rho} + O(R^2 + \left(\int_{B_1} \tilde{\rho}\right)^2)
\]

and therefore

\[
R = -\frac{R_*}{4\pi \rho_*} \int_{B_1} \tilde{\rho} + O(R^2 + \left(\int_{B_1} \tilde{\rho}\right)^2).
\]

Substitution of (C.8) into (C.7) we obtain a leading expression entirely in terms of the perturbed density \(\tilde{\rho}\). We list the various terms that we rewrite exclusively in terms of \(\tilde{\rho}\):

\[
4\pi c_v T_x \rho_* R_* R^2 = \frac{c_v T_x \rho_* R_*^3}{4\pi \rho_*^2} \left(\int_{B_1} \tilde{\rho}\right)^2 + O \left(|R|^3 + \left(\int_{B_1} |\tilde{\rho}|\right)^3\right)
\]

\[
4\pi c_v T_x R_*^2 \tilde{\rho}(1) R = -\frac{c_v T_x R_*^3}{\rho_*} \tilde{\rho}(1) \int_{B_1} \tilde{\rho} + O \left(|R|^3 + |\tilde{\rho}(1)|^3 + \left(\int_{B_1} |\tilde{\rho}|\right)^3\right)
\]

\[
-4\pi c_v \gamma \rho_* T_x R_* R^2 = -\frac{c_v \gamma T_x R_*^3}{4\pi \rho_*} \left(\int_{B_1} \tilde{\rho}\right)^2 + O \left(|R|^3 + \left(\int_{B_1} |\tilde{\rho}|\right)^3\right)
\]

\[
4\pi (\sigma + R_* p_x) R^2 = \frac{1}{4\pi \rho_*} \left(\frac{\sigma}{R_*} + p_x \right) R_*^3 \left(\int_{B_1} \tilde{\rho}\right)^2 + O \left(|R|^3 + \left(\int_{B_1} |\tilde{\rho}|\right)^3\right).
\]

Inserting these expressions into (C.7), we obtain

\[
\frac{1}{2} \dot{\mathcal{E}}[\tilde{\rho}, R, \tilde{R}] = \frac{c_v T_x M_0}{2 \rho_*^2} \tilde{\rho}(1)^2 + \frac{c_v \gamma T_x R_*^3}{2} \int_{B_1} \tilde{\rho}^2 + 2\pi \rho_1 R_*^3 \tilde{R}^2
\]

\[
+ \frac{R_*^3}{4\pi \rho_*} \left(c_v (1 - \gamma) T_x \rho_* + \frac{\sigma}{R_*} + p_x \right) \left(\int_{B_1} \tilde{\rho}\right)^2 - \frac{c_v T_x R_*^3}{\rho_*} \tilde{\rho}(1) \int_{B_1} \tilde{\rho}
\]

\[
+ O \left(|R|^3 + |\tilde{\rho}(1)|^3 + \left(\int_{B_1} |\tilde{\rho}|\right)^3\right)
\]

The coefficient of the third term in (C.9) can be simplified using the relation \(1 - \gamma = -\mathcal{R}_g/c_v\) and the relation \(\mathcal{R}_g T_x \rho_* = p_{x,*} + 2\sigma/R_*\) between the equilibrium density and bubble radius:

\[
c_v (1 - \gamma) T_x \rho_* + \frac{\sigma}{R_*} + p_x = -\mathcal{R}_g T_x \rho_* + \frac{\sigma}{R_*} + p_x = -\frac{\sigma}{R_*} + \mathcal{P}_x,
\]

where \(\mathcal{P}_x = p_{x,*} - p_{x,*}\). Thus,

\[
\frac{1}{2} \dot{\mathcal{E}}[\tilde{\rho}, R, \tilde{R}] = \frac{c_v T_x M_0}{2 \rho_*^2} \tilde{\rho}(1)^2 + \frac{c_v \gamma T_x R_*^3}{2} \int_{B_1} \tilde{\rho}^2 + 2\pi \rho_1 R_*^3 \tilde{R}^2
\]

\[
- \frac{\sigma R_*^3}{4\pi \rho_*} \left(\int_{B_1} \tilde{\rho}\right)^2 + \frac{R_*^3}{4\pi \rho_*} \mathcal{P}_x \left(\int_{B_1} \tilde{\rho}\right)^2 - \frac{c_v T_x R_*^3}{\rho_*} \tilde{\rho}(1) \int_{B_1} \tilde{\rho}
\]

\[
+ O \left(|R|^3 + |\tilde{\rho}(1)|^3 + \left(\int_{B_1} |\tilde{\rho}|\right)^3\right)
\]
Using that $M_0 = \rho_* R^3_0 |B_1|$, we may rewrite (C.10) as

\[
\frac{1}{2} d^2 \mathcal{E} [\acute{\rho}, \mathcal{R}, \mathcal{R}] = \frac{c_v T_x R^3_0 |B_1|}{2 \rho_*} \left( \frac{\hat{\rho}(1)^2}{\rho_*} - \frac{1}{|B_1|} \int_{B_1} \hat{\rho} + \frac{c_v \gamma T_x R^3_0}{2} \int_{B_1} \hat{\rho}^2 + 2 \pi \rho_1 R^3_* \hat{\rho}^2 \right)
- \frac{\sigma R^2_0}{4 \pi \rho_*} \left( \int_{B_1} \hat{\rho} \right)^2 + \frac{R^3_0}{4 \pi \rho_*} \mathcal{P}_\infty \left( \int_{B_1} \hat{\rho} \right)^2 + O \left( |\mathcal{R}|^3 + |\hat{\rho}(1)|^3 + \left( \int_{B_1} |\hat{\rho}| \right)^3 \right)
\]

or

\[
\frac{1}{2} d^2 \mathcal{E} [\acute{\rho}, \mathcal{R}, \mathcal{R}] = \frac{c_v T_x R^3_0 |B_1|}{2 \rho_*} \left( \frac{\hat{\rho}(1) - \frac{1}{|B_1|} \int_{B_1} \hat{\rho}}{\rho_*} \right)^2 + \frac{c_v \gamma T_x R^3_0}{2} \int_{B_1} \hat{\rho}^2 + 2 \pi \rho_1 R^3_* \hat{\rho}^2
- \left[ \frac{\sigma R^2_0}{4 \pi \rho_*} + \frac{c_v T_x R^3_0}{2 \rho_*} \right] \left( \int_{B_1} \hat{\rho} \right)^2 + \frac{R^3_0}{4 \pi \rho_*} \mathcal{P}_\infty \left( \int_{B_1} \hat{\rho} \right)^2
+ O \left( |\mathcal{R}|^3 + |\hat{\rho}(1)|^3 + \left( \int_{B_1} |\hat{\rho}| \right)^3 \right)
\]

By the Cauchy-Schwarz inequality $\left( \int_{B_1} \hat{\rho} \right)^2 \leq |B_1| \int_{B_1} \hat{\rho}^2$ and therefore

\[
\frac{1}{2} d^2 \mathcal{E} [\acute{\rho}, \mathcal{R}, \mathcal{R}] \geq \frac{c_v T_x R^3_0 |B_1|}{2 \rho_*} \left( \frac{\hat{\rho}(1) - \frac{1}{|B_1|} \int_{B_1} \hat{\rho}}{\rho_*} \right)^2 + 2 \pi \rho_1 R^3_* \hat{\rho}^2
+ \left[ \frac{c_v \gamma T_x R^3_0}{2} - \left( \frac{\sigma R^2_0}{4 \pi \rho_*} + \frac{c_v T_x R^3_0}{2 \rho_*} \right) \right] \int_{B_1} \hat{\rho}^2 + \frac{R^3_0}{4 \pi \rho_*} \mathcal{P}_\infty \left( \int_{B_1} \hat{\rho} \right)^2
+ O \left( |\mathcal{R}|^3 + |\hat{\rho}(1)|^3 + \left( \int_{B_1} |\hat{\rho}| \right)^3 \right)
\]

Finally, we find obtain for the constant in (C.13) that

\[
\frac{c_v \gamma T_x R^3_0}{2} - \left[ \frac{\sigma R^2_0}{4 \pi \rho_*} + \frac{c_v T_x R^3_0}{2} \right] = \frac{R^3_0}{3 \rho_*} \left( \frac{p_{x,*}}{2} + \frac{2 \sigma}{3 R_*} \right).
\]

This follows, yet again, from the relations $\gamma - 1 = \mathcal{A}_x / c_v$ and $\mathcal{B}_x T_x \rho_* = p_{x,*} + 2 \sigma / R_*$. For the term involving $\mathcal{P}_\infty$, using the Cauchy-Schwarz inequality $\left( \int_{B_1} \hat{\rho} \right)^2 \leq |B_1| \int_{B_1} \hat{\rho}^2$,

\[
\mathcal{P}_\infty \left( \int_{B_1} \hat{\rho} \right)^2 \geq -|\mathcal{P}_\infty| |B_1| \int_{B_1} \hat{\rho}^2.
\]

Summarizing

\[
\mathcal{E}_{total} - \mathcal{E}_* \geq \frac{c_v T_x R^3_0 |B_1|}{2 \rho_*} \left( \frac{\hat{\rho}(1) - \frac{1}{|B_1|} \int_{B_1} \hat{\rho}}{\rho_*} \right)^2 + 2 \pi \rho_1 R^3_* \hat{\rho}^2
- \frac{4 \pi R^2_0 |\mathcal{P}_\infty| |\mathcal{R}|}{\rho_*} - \frac{R^3_0}{3 \rho_* |\mathcal{P}_\infty| |B_1|} \int_{B_1} \hat{\rho}^2 + O \left( |\mathcal{R}|^3 + |\hat{\rho}(1)|^3 + \left( \int_{B_1} |\hat{\rho}| \right)^3 \right),
\]

where all explicit terms are non-negative except for the terms involving $\mathcal{P}_\infty$ which can be made small since $|\mathcal{P}_\infty| = |p_{x,*} - p_{x,*}| \leq \delta_0$.

We now conclude the proof by bounding the error term in (C.15) from above by a sufficiently small constant times $\int_{B_1} \hat{\rho}^2$.

For the first term on the right hand side of (C.15), using (C.8), in terms of the perturbed density $\hat{\rho}$,

\[
-4 \pi R^2_0 |\mathcal{P}_\infty| |\mathcal{R}| \geq \frac{R^3_0}{\rho_* |\mathcal{P}_\infty|} \int_{B_1} |\hat{\rho}| - C_0 |\mathcal{P}_\infty| \left( \int_{B_1} \hat{\rho} \right)^2
\]
for some constant $C_0 > 0$. Since $|\ddot{\phi}| \leq \delta_0 \leq 1$ and $|\mathcal{P}_x| = |p_{x} - p_{x,*}| \leq \delta_0$, by the Cauchy-Schwarz inequality $\left(\int_{B_1} \ddot{\phi}^2\right)^2 \leq |B_1| \int_{B_1} \ddot{\phi}^2$

$$-4\pi R^2 _* |\mathcal{P}_x| |\mathcal{R}| \geq -\frac{R^2 _*}{\rho_*} |\mathcal{P}_x| \int_{B_1} \ddot{\phi}^2 - C_0 |\mathcal{P}_x| |B_1| \int_{B_1} \ddot{\phi}^2 \geq -C_1 \delta_0 \int_{B_1} \ddot{\phi}^2$$

for some constant $C_1 > 0$.

Now we estimate the cubic term in the third line on the right hand side of (C.15). Since $M_0 = \text{Mass} [\rho, R]$, 

$$\int_{B_R} (\rho - \rho_*) \, dx = \int_{B_R} \ddot{\phi} \, dx = M_0 - \frac{4\pi R^3}{3} \rho_* = \frac{4\pi R^3}{3} \rho_* - \frac{4\pi R^3}{3} \rho_*, \quad \text{or}$$

(C.16) 

$$\frac{4\pi}{3} (R^3 - R_*^3) = -\int_{B_R} (\rho - \rho_*) \, dx,$$

which implies

(C.17) 

$$|\mathcal{R}| = |R - R_*| \leq \frac{3}{4\pi \rho_*(R^2 + RR_* + R_*^2)} |B_R| \left( \int_{B_R} |\rho - \rho_*|^2 \, dx \right)^{\frac{1}{2}} \leq C_2 \left( \int_{B_R} |\rho - \rho_*|^2 \, dx \right)^{\frac{1}{2}},$$

where $C_2 > 0$ depends only on $\nu, M_0, T_x$. We now control $|\ddot{\phi}(1)|^3$ by the first and the third terms on the right hand side of (C.15). Indeed,

$$|\ddot{\phi}(1)|^3 = |\ddot{\phi}(1)||\ddot{\phi}(1)|^2 \leq \delta_0 C_3 \left\{ \left| \ddot{\phi}(1) - \frac{1}{|B_1|} \int_{B_1} \ddot{\phi} \right| + \left( \int_{B_1} \ddot{\phi}^2 \right)^{\frac{1}{2}} \right\}^2$$

for some $C_3 > 0$ depending only on $\nu, M_0, T_x$. Since $|\ddot{\phi}(1)| = |\rho(R) - \rho_*| \leq \delta_0$, 

(C.18) 

$$|\ddot{\phi}(1)|^3 = |\ddot{\phi}(1)||\ddot{\phi}(1)|^2 \leq \delta_0 C_3 \left\{ \left| \ddot{\phi}(1) - \frac{1}{|B_1|} \int_{B_1} \ddot{\phi} \right| + \left( \int_{B_1} \ddot{\phi}^2 \right)^{\frac{1}{2}} \right\}^2$$

$$\leq 2\delta_0 C_3 \left| \ddot{\phi}(1) - \frac{1}{|B_1|} \int_{B_1} \ddot{\phi} \right|^2 + 2\delta_0 C_3 \int_{B_1} \ddot{\phi}^2.$$

Using (C.17) and (C.18), one has

$$O \left( |\mathcal{R}|^3 + |\ddot{\phi}(1)|^3 + \left( \int_{B_1} |\ddot{\phi}| \right)^3 \right)$$

$$\geq -C \left( \int_{B_R} (\rho - \rho_*)^2 \, dx \right)^{\frac{3}{2}} - C \delta_0 \left| \ddot{\phi}(1) - \frac{1}{|B_1|} \int_{B_1} \ddot{\phi} \right|^2 - C \delta_0 \int_{B_1} \ddot{\phi}^2 - C \int_{B_1} |\ddot{\phi}|^3$$

for some $C > 0$ depending only on $\nu, M_0, T_x$.

Consequently, using $|\mathcal{P}_x| = |p_{x} - p_{x,*}| \leq \delta_0$, (C.15) can be further computed as 

$$\mathcal{E}_{\text{total}} - \mathcal{E}_* \geq -C \delta_0 \int_{B_R} \ddot{\phi}^2 + \left( \frac{c_{T_x} R^3 _* |B_1|}{2\rho_*} - C \delta_0 \right) \left( \ddot{\phi}(1) - \frac{1}{|B_1|} \int_{B_1} \ddot{\phi} \right)^2 + \frac{R^3 _*}{\rho_*} \left( \frac{p_{x,*}}{2} + \frac{2\pi}{3 R_*} \right) \int_{B_1} \ddot{\phi}^2$$

$$- C \delta_0 |B_R| \left( \int_{B_R} (\rho - \rho_*)^2 \, dx - 2C \delta_0 \int_{B_1} \ddot{\phi}^2 - \delta_0 \frac{R^3 _*}{4\pi \rho_*} |B_1| \int_{B_1} \ddot{\phi}^2 \right)$$

$$\geq \Theta \left( \int_{B_R} (\rho - \rho_*)^2 \, dx \right)$$

for some constant $\Theta > 0$, provided $\delta_0 > 0$ is sufficiently small. Note that we’ve used $\int_{B_1} \ddot{\phi}^2 \, dy = R^{-3} \int_{B_R} (\rho - \rho_*)^2 \, dx$ in which $R^{-3} \geq \nu^3$ above. This completes the proof of Theorem 7.5.
APPENDIX D. AN INTERPOLATION LEMMA

Lemma D.1. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $k < m$, and $0 < \gamma \leq 1$. For $u \in C^\infty(\Omega)$,
\[ \|\nabla^k u\|_{L^\infty(\Omega)} \leq C_1 \|u\|_{L^p(\Omega)}^{1-\lambda} \|u\|_{L^m(\Omega)}^{\lambda} + C_2 \|u\|_{L^s(\Omega)} \]
for arbitrary $s \geq 1$, where $\frac{1}{m} = \frac{1}{p} - (1 - \lambda) \frac{1}{n}$, and the constants $C_1$, $C_2$ depend on the domain $\Omega$ and on $s$ in addition to the other parameters.

Proof. By Gagliardo–Nirenberg interpolation inequality,
\[ \|\nabla^k u\|_{L^\infty(\Omega)} \leq C_1 \|u\|_{L^p(\Omega)}^{\lambda} \|\nabla u\|_{L^m(\Omega)}^{1-\lambda} + C_2 \|u\|_{L^s(\Omega)} \]
for arbitrary $s \geq 1$, where
\[ 0 = \frac{k}{n} - \frac{m}{n}(1 - \lambda) + \frac{\lambda}{p} \]
and the constants $C_1$, $C_2$ depend on the domain $\Omega$ and on $s$ in addition to the other parameters. The lemma then follows since $\|\nabla^m u\|_{L^\infty(\Omega)} \leq \|u\|_{C^{m-\gamma}(\Omega)}$.

REFERENCES

[1] A. D. Alexandrov. A characteristic property of spheres. Ann. Mat. Pura Appl. (4), 58:303–315, 1962.
[2] B. P. Barber and S. J. Putterman. Observation of synchronous picosecond sonoluminescence. Nature, 352(6333):318–320, 1991.
[3] B. P. Barber and S. J. Putterman. Light scattering measurements of the repetitive supersonic implosion of a solonulinescent bubble. Phys. Rev. Lett., 69(26):3839, 1992.
[4] B. P. Barber, C. Wu, R. L{"o}fstedt, P. H. Roberts, and S. J. Putterman. Sensitivity of solonulinescence to experimental parameters. Phys. Rev. Lett., 72(9):1380, 1994.
[5] Z. Biro and J. J. L. Velazquez. Analysis of a free boundary problem arising in bubble dynamics. SIAM J. Math. Anal., 32(1):142–171, 2000.
[6] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. Mem. Amer. Math. Soc., 44(285):iv+190, 1983.
[7] M. P. Brenner, D. Lohse, and T. F. Dupont. Bubble shape oscillations and the onset of sonoluminescence. Phys. Rev. Lett., 75(5):975–980, 1995.
[8] M. Calvi, J. Iloreta, and A. Szeri. Dynamics of bubbles near a rigid surface subjected to a lithotripter shock wave. part 2. reflected shock intensifies non-spherical cavitation collapse. J. Fluid Mech., 316:63–97, 2008.
[9] O. Costin, S. Tanveer, and M. I. Weinstein. The lifetime of shape oscillations of a bubble in an unbounded, inviscid, and compressible fluid with surface tension. SIAM J. Math. Anal., 45(5):2924–2936, 2013.
[10] C. C. Coussios and R. A. Roy. Applications of acoustics and cavitation to noninvasive therapy and drug delivery. Annu. Rev. Fluid Mech., 40:395–420, 2008.
[11] G. Curtiss, D. Leppinen, Q. Wang, and J. Blake. Ultrasonic cavitation near a tissue layer. J. Fluid Mech., 730:245–272, 2013.
[12] D. Epstein and J. Keller. Expansion and contraction of planar, cylindrical and spherical underwater gas bubbles. J. Acoust. Soc. Amer., 52:975–980, 1972.
[13] E. Feireisl. Dynamics of viscous compressible fluids, volume 26 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2004.
[14] Z. C. Feng and L. G. Leal. Nonlinear bubble dynamics. In Annual review of fluid mechanics, Vol. 29, volume 29 of Annu. Rev. Fluid Mech., pages 201–243. Annual Reviews, Palo Alto, CA, 1997.
[15] K. Ferrara, R. Pollard, and M. Borden. Ultrasound microbubble contrast agents: fundamentals and application to gene and drug delivery. Annu. Rev. Biomed. Eng., 9:415–447, 2007.
[16] T. Fourest, E. Deletombe, V. Faucher, M. Arrigoni, J. Dupas, and J.-M. Laurens. Comparison of Keller-Miksis model and finite element bubble dynamics simulations in a confined medium. Application to the hydrodynamic ram. Eur. J. Mech. B Fluids, 68:66–75, 2018.
[17] T. Funaki, M. Ohnawa, Y. Suzuki, and S. Yokoyama. Existence and uniqueness of solutions to stochastic Rayleigh–Plesset equations. J. Math. Anal. Appl., 425(1):20–32, 2015.
[18] G. H. Goldshtein. Collapse and rebound of a gas bubble. Stud. Appl. Math., 112(2):101–132, 2004.
[19] J. Goodman. Nonlinear asymptotic stability of viscous shock profiles for conservation laws. Arch. Rational Mech. Anal., 95(4):325–344, 1986.
[20] J. Iloreta, N. Fung, and A. Szeri. Dynamics of bubbles near a rigid surface subjected to a lithotripter shock wave. part 1. consequences of interference between incident and reflected waves. J. Fluid Mech., 616:43–61, 2008.
[21] J. B. Keller and I. I. Kolodner. Damping of underwater explosion bubble oscillations. J. Appl. Phys., 27(10):1152–1161, 1956.
[22] J. B. Keller and M. Miksis. Bubble oscillations of large amplitude. J. Acoust. Soc. Am., 68(2):628–633, 1980.
[23] E. Klaseboer, S. W. Fong, C. K. Turangan, B. C. Khoo, A. J. Szeri, M. L. Calvisi, G. N. Sankin, and P. Zhong. Interaction of lithotripter shockwaves with single inertial cavitation bubbles. J. Fluid Mech., 593:33–56, 2007.
