Solving the problem of unilateral contact of the slab with the strengthening beams by the method of variational approximations

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Abstract. The work is devoted to the numerical-analytical solution of the contact problem when reinforcing the floor slab with beams using the mathematical apparatus of generalized functions. Such tasks arise during the reconstruction of existing buildings, when reinforcement of floor slabs is required to increase their bearing capacity under load. The contact of a slab with reinforcing beams is considered for various types of one-sided local restrictions on its deflections. To solve the structurally nonlinear problem of contact interaction between the slab and the reinforcement beams, a stepwise loading method is used in combination with the method of variational approximations. At each loading step, to ensure the convergence of the desired function of the plate deflections from the action of local contact forces of the beams resistance, the procedure of the method of variational approximations is used. Based on the use of the mathematical apparatus of generalized functions and the use of the method of variational approximations with the help of a stepwise algorithm, numerical-analytical solutions for the problem of contact interaction of an elastic plate with reinforcement beams are obtained. Two types of elastic one-way connections were considered: discrete supports and support along rectilinear segments. Due to the rational choice of approximating functions that describe with a high degree of accuracy the irregular parameters of the considered constructive-nonlinear system while keeping a small number of members of the series, a higher calculation accuracy is achieved in comparison with computational complexes that implement numerical methods of strength calculation of the indicated building structures.

1. Introduction
Structurally nonlinear systems with one-way connections with a previously unknown contact zone are often found in construction practice and other areas of engineering. In particular, this problem arises when it is required to reinforce floor slabs in existing buildings in order to increase their bearing capacity under load. In this case, the slab can have various types of support on the reinforcement beams – both at individual points and in sections of finite length. The state of the contact area can be decisive in assessing the stress-strain state, strength and reliability of structures and structures [1–6].

For the calculation of structures with one-way connections, numerical methods are often used, in which the problem of continuous contact is usually reduced to finite-dimensional problems with discrete one-way connections. A large number of studies have been devoted to the development of methods and algorithms for solving contact problems based on known methods of numerical analysis, among which one can note [7–14]. Meanwhile, for a more accurate assessment of the state of the contact, it is required to construct a solution in a numerical-analytical form. For this purpose, the mathe-
mathematical apparatus of generalized functions [15–17] can be used, which makes it possible to correctly take into account the irregular (discontinuous) parameters of the constructively nonlinear system under consideration.

To calculate constructively nonlinear systems, both iterative [7–10] and stepwise [11–14] methods can be used. The advantage of stepwise algorithms is that they can be used to obtain results at each stage of loading. In addition, stepwise design schemes are more efficient under complex contact conditions and the nature of the load, where the solution to the problem depends on the loading history [13, 14]. Constructive nonlinearity in a stepwise process will manifest itself in a sequential change in the operating contours of the system depending on the magnitude of the load (change in the size of the contact zones and separation of the boundary surfaces of interacting bodies). It is assumed that the deformation of the system is linear between successive stages of loading.

In this paper, to construct a numerical-analytical solution to the problem of one-sided contact of a plate with reinforcement beams, it is proposed to use a stepwise loading method in combination with the method of variational approximations (MVA) [17]. At each step, from the condition of compatibility of the deformations of the contacting elastic elements, deflection functions, the number of contact points of the lumped supports, the length of the contact segments of the linear supports, the values of the rebound are determined, and the working diagram of the "plate – reinforcement beams" system is refined. A numerical-analytical approximation to the exact solution of a contact problem with discontinuous parameters is sought in the class of piecewise continuous generalized functions. To improve the convergence of the desired function of slab deflections from the action of local contact forces of resistance of the reinforcement beams, the procedure of the method of variational approximations is used. The use of the mathematical apparatus of generalized functions allows one to create very reliable mathematical models of deformation of structures with discontinuous parameters and, in particular, to correctly take into account various types of one-way connections.

On the basis of the proposed approach, numerical solutions are obtained for the problem of contact interaction of a floor slab with one-sided constraints of two different types: discrete supports and support along straight sections of finite length. Due to the two-dimensional approximation of the slab deflection function by discontinuous functions, the proposed approach shows satisfactory convergence, stability and accuracy of the solution while maintaining a small number of terms in the series.

2. Methods
Consider an elastic isotropic rectangular plate with an elastic modulus $E$, dimensions $a$ on $b$, arbitrarily supported at $x = 0$, $x = a$ and hinged at the other edges. Along the $x = \text{const}$ lines, the slab is reinforced with several elastic beams. Each beam can have arbitrary support, cross-section and material. In this case, the beams differ in the type of contact interaction with the slab.

Type 1. The slab supports on reinforcement beams installed along the lines $x = x_i$, where $i = 1, 2, \ldots, N$, through narrow spacers.

Type 2. The slab supports directly on the upper flanges of the beams installed along the lines $x = x_j$, where $j = 1, 2, \ldots, M$. The flexural stiffness of each beam is $EI_i$, where $E_i$ is the elastic modulus; $I_i$ is the moment of inertia in the plane of transverse bending; $t = i, j$.

The width of the flange of the beam on which the slab supports is many times less than the span of the slab. It is assumed that the contact of these structures is carried out at a point for a beam of the first type and along a straight line segment for beams of the second type. Each beam in the middle of the $b/2$ span may have an initial span $\Delta_0 \geq 0$ between the contact line of the top of the beam and the undeformed plane $z=+h/2$ of the slab.

The use of the apparatus of generalized functions allows creating reliable mathematical models of deformation of building structures with discontinuous parameters. With its help, you can, for example, take into account various types of one-way relationships. In this contact problem, the slab has two different types of support on the reinforcement beams.

A beam having a support spacer with a very small contact area with the slab can be considered as a concentrated elastic support (type 1). The action of this support on the plate when the one-way con-
connection is turned on is represented in the form of a concentrated force $R^l \delta^l_i \delta^l_j$, where $R^l$ is the unknown resistance of the $i$-th beam of the first type; $\delta^l_x = \delta(x-x_i), \delta^l_y = \delta(y-y_i)$ are Dirac delta functions; $x_i, y_i$ are coordinates of the support point of the slab on the beam.

If the slab supports on an elastic beam along the line $x = x_i$ on a contact segment (type 2) with long $\bar{y}_j = y^j_2 - y^j_1$, then its action on the slab is represented as a load $r^j(y) \delta^j_x H^j_{yy}$ unevenly distributed along the line. Here $r^j(y)$ is the unknown repulse function acting on the plate along the length of the contact segment with the $j$-th beam of the second type; $\delta^j_y = \delta(x-x_j)$ are Dirac delta functions; $H^j_{yy} = H^j(y-y^j_1) - H^j(y-y^j_2)$ is stepwise function; $H^j(y-y^j_i)$ is Heaviside unit function; $y^j_i$ is ordinate, respectively, of the beginning and end of the contact segment; $l = 1, 2$.

Suppose that under the action of an arbitrary transverse load $p(x, y)$ on the slab, a certain working scheme of the considered structurally nonlinear system "slab – reinforcement beams" is found. In this case, the slab supports on $n \leq N$ beams of the first type and $m \leq M$ beams of the second type. Dividing the plate and reinforcement beams along the lines of their contact interaction, we write down the equilibrium equations for each structure

$$
DV^4w(x, y) = p(x, y) - \sum_{i=1}^m R^i \delta^i_x \delta^i_y - \sum_{j=1}^m r^j(y) \delta^j_x H^j_{yy};
$$

$$
E_i I_i \psi^i_{xy}(y) = R^i \delta^i_y; \quad E_j I_j \psi^j_{xy}(y) = r^j(y) H^j_{yy},
$$

where $w(x, y)$ is the function of slab deflections; $u_j(y), v_j(y)$ are the functions of deflections, respectively, of the $i$-th and $j$-th beams in contact with the slab.

The system of equations (1) is nonlinear, since the unknown rebound of the reinforcement beams $R^i, r^j(y)$ depends on the deflection functions of the slab and beams that are included in the work. In addition, with an arbitrary load on the slab, it is practically impossible to predict in advance the actual operating scheme corresponding to the applied load.

To solve the constructively nonlinear problem of contact interaction between the slab and the reinforcement beams, we use the stepwise loading method. First, we determine the function of the slab deflections $w_0(x, y)$ under the action of a given load and find the maximum slab deflections in the middle of the span on the lines of possible contact with the beams. Then we introduce a reduction factor for the load

$$
K_L = \max \left[ w_0(x_i, y_j) / \Delta_i, w_0(x_j, y_i) / \Delta_j \right]
$$

where $\Delta_i, \Delta_j$ are the initial gaps between the slab and the corresponding reinforcement beams in the middle of the span at the point of their possible contact with the slab.

Let us establish the initial loading level $\bar{q} = p / K_L$ and set the loading step $\Delta q$. Increasing the load step by step, we determine the functions of the slab deflection in the middle of the span at points of the lines of possible contact $x = x_i$ and $x = x_j$ until the reinforcement beams are switched on.

Suppose that at some $L$-th loading step, a working scheme was found in which the slab is supported by $n$ beams of the first type and $m$ beams of the second type. Then the function of slab deflections will be sought in the form

$$
w_L = \sum_{k=1}^\infty \left( w^L_k - \sum_{i=1}^n \bar{R}^i_k \psi^i_{sk} - \sum_{j=1}^m \bar{R}^j_k \psi^j_{sk} \right) \sin \beta_k y,
$$

where $w^L_k$ and $R^l_k$ are the coefficients of the expansion in the Fourier series, respectively, the functions of the slab deflections and the resistance of the $i$-th beam at the point of contact with the slab;
function
\[ \psi_{jk} = (Z_{1k} + \beta_k xZ_{3k}) \cosh \beta_k x + (Z_{2k} + \beta_k xZ_{4k}) \sinh \beta_k x + \frac{1}{\beta_k} (\beta_k x \cosh \beta_k x - \sinh \beta_k x) H(x - x_j) \]
is a solution to the differential equation
\[ \left( \frac{d^2}{dx^2} - \beta_k^2 \right) \psi_{jk} = \delta_k^j. \]

Here \( x_j = x - x_j \); \( Z_{1k}, Z_{2k}, Z_{3k}, Z_{4k} \) are arbitrary constants determined from the boundary conditions of the slab at \( x = 0, x = a; \) \( r^j_k \) is coefficient of expansion into a series of the unknown function of the resistance of the \( j \)-th beam along the length of the segment of its contact with the plate;
\[
\bar{r}_k^j = \sum_{k=1}^\infty r_k^j \sin \beta_k y = \sum_{k=1}^\infty \frac{\beta_k}{b} \int_0^b H_{yy}^j \sin \beta_k y \sin \beta_k y dy,
\]
where \( \bar{r}_l^j = \frac{l \pi}{y_j}, l = 1, 2, 3, \ldots; \beta_k = \frac{k \pi}{b}; k = 1, 2, 3, \ldots. \)

The unknown coefficients \( R_k^l \) and \( t_k^l \) are determined at each loading step from the solution of the system of algebraic equations
\[
\begin{align*}
\sum_{k=1}^\infty \left\{ \sum_{i=1}^n \left[ B_{ik}^{jl} \right] + \sum_{j=1}^m \left[ A_{ik}^{jl} \right] \right\} [R_k^l] = w_k^l(x_j) - A_i s_k^j,
\end{align*}
\]
where
\[
A_{ik}^{jl} = A_k^{jl} s_k^{jl} \psi_{sk}^j(x_j) + \bar{s}_k^j t_k^j c_j \bar{y}_j;
\]
\[
s_k^{jl} = \frac{2}{b} \int_0^b H_{yy}^j \sin \beta_k y \sin \beta_k y dy;
\]
\[
\bar{s}_k^{jl} = \frac{2}{b} \int_0^b H_{yy}^j \sin \beta_k y \bar{y} dy;
\]
\[
t_k^j = \frac{2}{b} \int_0^b H_{yy}^j \sin \beta_k y dy;
\]
\[
\bar{A}_{ik}^{jl} = A_k^{jl} s_k^{jl} \psi_{sk}^j(x_j);
\]
\[
\bar{B}_{ik}^{jl} = \bar{s}_k^{jl} \psi_{sk}^j(x_j); \quad c_j \text{ are the stiffness coefficient of the } j \text{-th beam, determined at the previous iteration from the unit load applied on the segment of contact with the plate.}
\]

At each loading step, in order to improve the convergence of the series of the sought-for function of plate deflections from the action of local contact forces of resistance of the reinforcement beams, the procedure of the method of variational approximations is used [17]. Let us illustrate it below by the example of solving the differential equation of equilibrium of a plate under the action of a concentrated force \( R = 1 \) applied at a point with coordinates \( x_1, y_1 \).

The solution to the equilibrium equation for the slab \( D \nabla^4 w_R = \delta_x \delta_y \) at the first iteration is sought in the form
\[
w_R^{(1)} = \chi(y) \psi(x),
\]
where
\[
\chi(y) = C_1 \frac{y^3}{3!} + C_2 \frac{y^2}{2!} + C_3 y + C_4 + \frac{(y - y_1)^3}{3! \alpha} H(y - y_1);
\]
\( C_1, C_2, C_3, C_4 \) are arbitrary constants determined from the boundary conditions of the slab at \( y = 0; \ y = b; \alpha = \pi/a. \)

Substituting expression (3) into the boundary conditions of the slab, we find a function \( \psi(x) \) similar to that of the function \( \psi_{sk} \) and calculate the residual
where the function \( i \varphi \) represents the repulse function in the form

\[ \bar{y} = \frac{y}{y_1} \text{ for } y_1 < y < y_2. \]

and is sought in the form

\[ w_R^{(2)} = \psi(x) \Phi(y), \]

where \( \Phi(y) \) are spline functions on the contact segment. At the second iteration, the deflection function is represented in the form

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In the particular case of the support of the slab on the beam through a narrow spacer of given dimensions, when the length of the contact segment is known, the repulse function is represented in the form

\[ r(y) = \sum_{m=1}^{M} r_m s_m(y), \]

where \( s_m(y) \) are spline functions on the contact segment.
Having constructed the deflection functions of the slab and reinforcement beams at the \( L \)-th step of loading with a given accuracy, we refine the working scheme of the structurally nonlinear system, determine the deflections, internal forces, and proceed to the next stage of loading the slab.

The algorithm for calculating slabs with one-sided support on reinforcement beams is built in the form of nested cycles:

- cycle in steps of loading;
- determination of possible points and lengths of segments of contact between the slab and the beams;
- calculation of the function of the rebound of beams at points and on the contact segment;
- clarification of the working scheme "plate – reinforcement beams";
- MVA procedure for constructing deflection functions with a given accuracy;
- determination of the components of the stress-strain state of the slab and reinforcement beams.

According to the developed algorithm, a program was compiled for calculating the considered constructive nonlinear systems in the Mathcad environment.

3. Results and discussion
Based on the use of the mathematical apparatus of generalized functions and the use of the method of variational approximations according to the developed algorithm, numerical solutions are obtained for the problem of contact interaction of an elastic plate with reinforcement beams. Depending on the type of support of the slab on the beam, their contact is taken into account in the form of one-way connections of two different types: lumped connections (type 1) and connections distributed over rectilinear segments of limited length (type 2).

As an example, a steel square plate hinged along the contour \( E = 2.1 \times 10^{5} \) kN/m\(^2\), \( \mu = 0.3 \) with dimensions \( a = b = 1.2 \) m and thickness 0.01 m is considered (Figure 1 \( a \)). First, the slab is loaded over the entire area with a uniformly distributed load \( p = 10 \) kN/m\(^2\) and the deflection in the center of the slab \( w_A(p) \) is determined. Then, under the slab along the line \( x = a/2 \), a rigid beam is installed with a support platform of a small area at point \( A (x_A = a/2; y_A = b/2) \). In this case, the initial gap between the slab and the support on the beam is \( \Delta_A = w_A(p) = 2.119 \times 10^{-3} \) m. In the next step, as shown in Figure 1 \( b, c \), the slab is loaded on the left and right thirds of the span with a uniformly distributed load \( \Delta q = 10 \) kN/m\(^2\) at \( 0 \leq x \leq a/3; 2a/3 \leq x \leq a; 0 \leq y \leq b \). Deflections are determined at points \( B(x_B = a/3) \) and \( C (x_C = 2a/3) \), located on the axis of symmetry of the slab \( y_B = b/2 \). At this stage of loading, under the slab along the lines \( x = a/3 \) and \( x = 2a/3 \), rigid beams with a support area of a small area are installed at points \( B \) and \( C \) (support type 1). In this case, the initial spans are, respectively, \( \Delta_B = w_B(p + \Delta q); \Delta_C = w_C(p + \Delta q) \).

Then the slab is sequentially loaded only on the left and right thirds of the span with a uniformly distributed load with a step \( \Delta q = 10 \) kN/m\(^2\) until the beam is turned off from the work in the middle of the slab. A similar scheme of stepwise loading is also considered in the case of the support of the slab on the reinforcement beams along short segments at 0.5 m \( \leq y \leq 0.7 \) m along the lines passing through points \( B(x_B = a/3) \) and \( C (x_C = 2a/3) \) (type of support 2). The main results of calculations of the slab for two different types of support on the reinforcement beams at a load \( p + 12\Delta q \) (type 1) and a load \( p + 3\Delta q \) (type 2) are shown in Table. 1.

Under the action of a uniformly distributed load \( p \), the plate hinged along the contour receives initial deflections and forces. With different variants of additional loading, the slab supports on various one-way connections, which leads to a change in its design scheme. At the initial stages of loading with a step \( \Delta q \), the slab first rests only on the middle beam at point \( A \), and then on three beams at points \( A, B \) and \( C \).

With a subsequent stepwise increase in the load on the left and right thirds of the slab span, a transition to a new design scheme occurs. The type of this design scheme significantly depends not only...
on the achieved loading level of the slab, but also on the type of its support on the beams. So, for example, when the slab is supported on three beams through narrow spacers, the middle support \( A \) is completely disconnected from work under a load of \( p+12\Delta q \). In the case of one-sided contact of the slab with the reinforcement beams installed along the lines passing through points \( B \) (\( x_B = a/3 \)) and \( C \) (\( x_C = 2a/3 \)), on short intervals at \( 0.5 \text{ m} \leq y \leq 0.7 \text{ m} \), the middle beam is completely turned off from work already at a load \( p+3\Delta q \).

![Design scheme of a floor slab with two types of support on reinforcement beams](image)

**Figure 1.** Design scheme of a floor slab with two types of support on reinforcement beams.

**Table 1.** Results of calculating a floor slab with two types of support on reinforcement beams.

| Point | \( x \) | \( y \) | \( w_i \) | \( M_{x_i} \) | \( M_{y_i} \) | \( w_i \) | \( M_{x_i} \) | \( M_{y_i} \) |
|-------|------|------|--------|-------|-------|--------|-------|-------|
| \( B_1 \) | 0.7 m | a/3 | 1.763 | 0.303 | 0.383 | 1.677 | 0.407 | 0.340 |
| \( B \) | b/2 | 1.866 | 0.189 | 0.298 | 1.853 | 0.426 | 0.405 |
| \( B_2 \) | 0.5 m | a/3 | 1.763 | 0.303 | 0.383 | 1.677 | 0.407 | 0.348 |
| \( A_1 \) | 0.7 m | a/2 | 2.150 | 0.416 | 0.409 | 1.940 | 0.440 | 0.435 |
| \( A \) | b/2 | 2.102 | 0.465 | 0.345 | 2.121 | 0.464 | 0.460 |
| \( A_2 \) | 0.5 m | a/2 | 2.150 | 0.416 | 0.409 | 1.940 | 0.440 | 0.435 |
| \( C_1 \) | 0.7 m | 1.763 | 0.303 | 0.383 | 1.677 | 0.407 | 0.348 |
| \( C \) | 2a/3 | b/2 | 1.866 | 0.189 | 0.298 | 1.853 | 0.426 | 0.405 |
| \( C_2 \) | 0.5 m | 1.763 | 0.303 | 0.383 | 1.677 | 0.407 | 0.340 |
Thus, under different loading conditions, a given system has four different design schemes for each type of support of the slab on the reinforcement beams, which leads to a significant change in the components of the stress-strain state of the slab, especially in the contact zones (see Table 1). This circumstance must be taken into account, for example, when solving the problem of strengthening floor slabs during the reconstruction of buildings and structures.

4. Conclusion
Based on the use of the apparatus of generalized functions, a mathematical model has been created for calculating elastic plates with various types of support on the reinforcement beams, considered as one-way connections. To construct a numerical-analytical solution, a stepwise loading method is used in combination with the method of variational approximations.

A rational choice of approximating functions that correctly take into account the irregular parameters of the considered constructive nonlinear system ensures high accuracy while keeping a small number of terms in the series. According to the proposed method, an effective algorithm was developed and a program of strength calculation was compiled in the Mathcad environment to analyze the stress-strain state of these systems.

Numerical experiments carried out according to the developed program showed that it is characterized by the simplicity of inputting initial data, a significantly smaller number of unknowns, and also the computation time compared to the finite element method. At the same time, thanks to the obtained analytical dependencies, a higher accuracy of calculations is provided in comparison with computational complexes that implement numerical methods of strength calculation of the indicated building structures.

The examples show that, under different loading conditions, the structurally nonlinear system has different design schemes for each type of support of the slab on the reinforcement beams, which leads to a significant change in the components of the stress-strain state of the slab, especially in the contact zones.

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