Monogamy of Bell correlations and Tsirelson’s bound

Benjamin F. Toner and Frank Verstraete

1 Institute for Quantum Information, California Institute of Technology, Pasadena, CA 91125, USA
2 Centrum voor Wiskunde en Informatica, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands
3 Fakultät für Physik, Universität Wien, Austria

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We consider three parties, A, B, and C, each performing one of two local measurements on a shared quantum state of arbitrary dimension. We characterize the trade-off between the nonlocality of the Bell correlations observed by AB and of those observed by AC. This generalizes Tsirelson’s bound on the quantum value of the CHSH inequality, the latter being recovered when C is completely uncorrelated with AB. We also discuss the trade-off between Bell violations and local expectation values of observables that anticommute with the ones used in the Bell test.

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The existence of Bell inequalities [1, 2] and their observed violation in experiments has had a very deep impact on the way we look at quantum mechanics. On the one hand, it has led to a study of the precise meaning of nonlocality and opened up the field of entanglement theory. On the other, it has led to the observation that Bell violations can be exploited in the design of cryptographic protocols [3]. In that case, an eavesdropper (C) tries to gain access to some quantum correlations shared by Alice (A) and Bob (B). If the Bell correlations between A and B are strong, it can happen that C’s outcomes will be almost uncorrelated with them, and A and B will be able to execute a purification protocol so as to create private randomness. In the present paper, we will make a precise quantitative statement about the following monogamy property: Suppose A and B violate a Bell inequality by a certain amount. How does that bound the possible Bell correlations between A and C? This is also interesting from the point of view of entanglement theory, as it provides monogamy relations independent of the size of the local Hilbert spaces. For the Clauser-Horne-Shimony-Holt (CHSH) inequality, the region of accessible Bell correlations between AB and AC turns out to be very simple (see Figure 1).

In the setting where two parties, A and B, share a quantum state $\rho$, and each has the choice of two local measurements, there is just one relevant Bell inequality, the CHSH inequality [2]. Define the CHSH operator

$$B_{AB} = A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2),$$

where $A_1$ and $A_2$ ($B_1$ and $B_2$) are A’s (B’s) observables and are Hermitian operators with spectrum in $[-1, +1]$. For particular measurements and a particular state $\rho$, the quantum value of the CHSH inequality is defined as $\langle B_{AB} \rangle = \text{tr} (B_{AB} \rho)$. All correlations described by local hidden variable (LHV) models satisfy the CHSH inequality, $|\langle B_{AB}\rangle_{\text{LHV}}| \leq 2$, but in the case of entangled quantum systems, this bound can be violated. For example, on the singlet state of two qubits there exist operators $A_i, B_i$ such that $\langle B_{AB} \rangle = \langle \psi^- | B_{AB} | \psi^- \rangle = 2\sqrt{2}$.

We do not yet know how to calculate a bound on the maximum quantum value of an arbitrary Bell inequality, but a number of ad hoc techniques have been developed [4, 5, 6, 7]. In the case of the CHSH inequality, Tsirelson has proved that $|\text{tr} (B_{AB} \rho)| \leq 2\sqrt{2}$ for all observables $A_1, A_2, B_1, B_2$, and all states $\rho$ [4]. This Tsirelson bound can itself be violated if we consider more general hypothetical no-signalling theories: a nonlocal box violates the CHSH inequality maximally, $\langle B_{AB}\rangle_{\text{NL}} = 4$ [3]. Tsirelson’s bound is a simple mathematical consequence of the axioms of quantum theory, but is there some deeper reason why a violation greater than $2\sqrt{2}$ is unphysical? For example, a violation greater than $\sqrt{32/3} \approx 3.27$ would imply that any communica-

FIG. 1: Accessible values of $\langle B_{AB}\rangle$ and $\langle B_{AC}\rangle$ for classical theories (interior of square), quantum theory (interior of circle), and no-signalling theories (interior of diamond). Note that both quantum and no-signalling theories obey monogamy constraints; classical local hidden variable theories do not.
tion complexity problem can be solved using a constant amount of communication \(^2\). As will be clear from the following results, the bound \(2\sqrt{2}\) is very natural if one considers the possible Bell violations in a three-party setup.

We establish the following monogamy trade-off:

**Theorem 1.** Suppose that three parties, A, B, and C, share a quantum state (of arbitrary dimension) and each chooses to measure one of two observables. Then

\[
\langle B_{AB} \rangle^2 + \langle B_{AC} \rangle^2 \leq 8. \tag{2}
\]

Here, \(B_{AC}\) is defined as in Eq. (1), but with B’s observables replaced by C’s. Note that we obtain Tsirelson’s bound, \(\langle B_{AB} \rangle^2 \leq 8\), as a simple corollary. Note also that A’s measurements are the same in \(\langle B_{AB} \rangle\) and \(\langle B_{AC} \rangle\); otherwise we could have \(\langle B_{AB} \rangle = \langle B_{AC} \rangle = 2\sqrt{2}\) and there would be no trade-off. Theorem 1 is analogous to the Coffman-Kundu-Wootters theorem that describes the trade-off between how entangled A is with both B and C. Eq. (2) is the best possible bound: there are states and measurements achieving any values of \(\langle B_{AB} \rangle\) and \(\langle B_{AC} \rangle\) that satisfy it. Previously the best bound known was \(|\langle B_{AB} \rangle| + |\langle B_{AC} \rangle| \leq 4\), which is tight for correlations that arise from no-signalling theories \(^4\). We illustrate the monogamy trade-offs for various theories in Figure 1.

We prove Theorem 1 in two parts. We first show that A’s measurements are the same in \(\langle B_{AB} \rangle\) and \(\langle B_{AC} \rangle\), maximizing over the measurements in \(\langle B_{AB} \rangle\) and \(\langle B_{AC} \rangle\) separately, but keeping the state fixed. Our proof suggests a connection between anticommutation and Bell inequality violation, which we then explore more deeply.

**Dimensional reduction.**—We start by establishing a bound on the dimension of the quantum state required to maximally violate certain Bell inequalities. A similar result was proved by Masanes \(^12\). The main ingredient—a canonical decomposition for a pair of subspaces of \(\mathbb{C}^n\)—is described in more detail in, e.g., Ref. \(^13\).

**Lemma 2.** Consider any Bell inequality in the setting where \(m\) parties each choose from two two-outcome measurements. Then the maximum quantum value of the Bell inequality is achieved by a state that has support on a qubit at each site. Furthermore, we can assume this state has real coefficients and that the observables are real and traceless.

**Proof.** For \(i \in \{1, 2\}\), assume party \(k\) has observables \(M_{k,i}\) acting on a Hilbert space \(\mathcal{H}_k\). By extending the local Hilbert spaces \(\mathcal{H}_k\), we can assume for all \(k\) and for all \(i = 1, 2\) that (i) \(\mathcal{H}_k = \mathbb{C}^{2d}\) for some fixed \(d\), (ii) \(M_{k,i}\) has eigenvalues \(\pm 1\), and (iii) \(\text{tr} M_{k,i} = 0\). The first condition states that all local spaces have the same dimension \(2d\), the latter two that each observable corresponds to a projective measurement onto a \(d\)-dimensional subspace and its complement. We also define \(M_{0,0} = \mathbb{I}_{2d}\), the identity operator on \(\mathcal{H}_k\). We can write a generic Bell operator in the setting stated in the lemma as

\[
B = \sum_{i_1=0}^2 \sum_{i_2=0}^2 \cdots \sum_{i_m=0}^2 c_{i_1 i_2 \cdots i_m} \prod_{k=1}^m M_{k,i_k}, \tag{3}
\]

where the coefficients \(c_{i_1 i_2 \cdots i_m}\) are arbitrary real numbers. Our goal is find the quantum value of this Bell operator, which is maximum of \(B \equiv \langle \psi | B | \psi \rangle\) over states \(|\psi\rangle\) and measurements \(M_{k,i}\).

We now choose a local basis for each \(\mathcal{H}_k\) such that party \(k\)’s observables have a simple form. We start by taking \(M_{k,1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\). This leaves us the freedom to specify the basis within the \(2 \times 2\) block on which \(M_{k,1}\) is constant. Let \(M_{k,2} = 2PP^T = 2\mathbb{I}_{2d}\) (we suppress the dependence on \(k\)), where \(P\) is a \(2d \times 2d\) matrix with orthonormal columns, which span the \(+1\)-eigenspace of \(M_{k,2}\). Write \(P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}\), where \(P_1\) and \(P_2\) are \(d \times d\) matrices. The rows of \(P\) are orthonormal, which implies \(PP^T = P_1^2 + P_2^2 = 2\mathbb{I}_d\), so \(P_1^2 P_1 + P_2^2 P_2\) are simultaneously diagonalizable. This means there is a singular value decomposition of the form \(P_1 = U_1^T D_1 V_1\), \(P_2 = U_2^T D_2 V_2\), where \(U_1\) and \(U_2\) are unitary matrices and \(D_1\) and \(D_2\) are \(\sqrt{I_d - D_1^2}\) and \(\sqrt{I_d - D_2^2}\) are nonnegative (real) diagonal matrices. Changing basis according to the unitary \(U_1 \oplus U_2\), which leaves \(M_{k,1}\) invariant, it follows that

\[
M_{k,2} = \begin{bmatrix} 2D_1^2 - I_d & 2D_1 D_2 \\ 2D_1 D_2 & 2D_2^2 - I_d \end{bmatrix},
\]

where each of the \(d \times d\) blocks is diagonal. We relabel our basis vectors so that \(M_{k,1} = \bigoplus_{j=1}^d Z\), \(M_{k,2} = \bigoplus_{j=1}^d (\cos \theta_j Z + \sin \theta_j X)\), where \(2D_1^2 - I_d = \text{diag} \left( \cos \theta_1, \cos \theta_2, \ldots, \cos \theta_d \right)\) and \(X\) and \(Z\) are the usual Pauli operators. Hence our operators are real and represent a \(d^-1\) of \(\mathbb{C}^2\) subspace of \(\mathcal{H}_k\). They are traceless on each \(\mathbb{C}^2\) space.

We wish to maximize \(B = \langle \psi | B | \psi \rangle\) over states \(|\psi\rangle\) and the measurements \(M_{k,i}\). Fix \(k\), and let \(\rho_{k,j}\) be the reduced density matrix obtained by projecting \(|\psi\rangle\) onto the \(j\)th \(\mathbb{C}^2\) factor of the \(\bigoplus_{j=1}^d \mathbb{C}^2\) subspace induced by \(M_{k,1}\) and \(M_{k,2}\) at site \(k\). Then \(B = \sum_{j=1}^d \text{tr} B \rho_{k,j}\) is a convex sum over the \(\mathbb{C}^2\) factors, whereupon it follows that the maximum is achieved by a state with support on a qubit at site \(k\). Since this argument works for all \(k\), the maximum of \(B\) is achieved by a state that has support on a qubit on each site.

Finally, write \(|\psi\rangle = |\psi_1\rangle + i|\psi_2\rangle\), where \(|\psi_1\rangle\) and \(|\psi_2\rangle\) are real. Then \(\langle \psi | B | \psi \rangle = \langle \psi_1 | B | \psi_1 \rangle + \langle \psi_2 | B | \psi_2 \rangle\) since \(B\) is real, which is the same expression we would obtain if the state were a real mixture of \(|\psi_1\rangle\) and \(|\psi_2\rangle\). Hence the maximum of \(B\) is achieved by a state with real coefficients. \(\square\)
Monogamy trade-off relation.—The region \( \mathcal{R} \) of allowed values of \( \langle B_{AB} \rangle, \langle B_{AC} \rangle \) is convex and can therefore be described by an (infinite) family of half-space inequalities,

\[
c_{AB} \langle B_{AB} \rangle + c_{AC} \langle B_{AC} \rangle \leq d, \tag{4}
\]

with \( c_{AB}, c_{AC}, d \in \mathbb{R} \). The left-hand side of Eq. \( \ref{eq:4} \) is a Bell operator, as defined in Eq. \( \ref{eq:3} \), which means we can apply Lemma \ref{lem:2} to conclude that extreme points of \( \mathcal{R} \) are achieved by real states on three qubits, with measurements of the form \( M = \cos \theta Z + \sin \theta X \). Theorem 1 will emerge as a corollary of:

**Lemma 3.** Let \( |\psi\rangle \) be a pure state in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) with real coefficients. Then the maximum of \( \langle B_{AB} \rangle \) over real traceless observables \( A_1, A_2, B_1, B_2 \) is

\[
2\sqrt{1 + \langle Y_A Y_B \rangle^2 - \langle Y_A Y_C \rangle^2 - \langle Y_B Y_C \rangle^2}, \tag{5}
\]

where \( Y \) is the usual Pauli operator, \( \langle Y_A Y_B \rangle = \text{tr} (Y_A \otimes Y_B \otimes \mathbb{1} \rho) \), and so on. Cyclic permutations of Eq. \( \ref{eq:5} \) hold for \( \langle B_{AC} \rangle \) and \( \langle B_{BC} \rangle \).

Proof. We consider \( \rho_{AB} = \text{tr}_C |\psi\rangle \langle \psi| \), which is a real state on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \). Horodecki and family have calculated the maximum quantum value of the CHSH operator for a state on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) \cite{Horodecki1998}. Their analysis simplifies in our case because the state and measurements are real. Define

\[
T_{AB} = \left[ \frac{(X_A X_B) \langle X_A Z_B \rangle}{\langle Z_A X_B \rangle \langle Z_A Z_B \rangle} \right]. \tag{6}
\]

For \( i = 1, 2 \), write \( A_i = \hat{a}_i \cdot \hat{a}_r, B_i = \hat{b}_i \cdot \hat{a}_r \), where \( \hat{a}_i \) and \( \hat{b}_i \) are two-dimensional unit vectors and \( \hat{a}_r = (X, Z) \). Define

\[
\hat{b}_1 + \hat{b}_2 = 2 \cos \theta \hat{a}_1, \quad \hat{b}_1 - \hat{b}_2 = 2 \sin \theta \hat{a}_2, \tag{7}
\]

where \( \theta \in [0, \pi/2] \) and \( \hat{a}_1 \) and \( \hat{d}_1 \) are orthogonal unit vectors. Then

\[
\frac{1}{2} \max_{A_i, B_j} \langle B_{AB} \rangle = \max_{\hat{d}_1, \theta, \hat{a}_i} \cos \theta \hat{a}_1^t T_{AB} \hat{d}_1 + \sin \theta \hat{b}_2^t T_{AB} \hat{d}_2
\]

\[
= \max_{\hat{d}_1, \theta} \left\| T_{AB} \hat{d}_1 \right\|^2 + \left\| T_{AB} \hat{b}_2 \right\|^2 \tag{8}
\]

\[
= \left\| T_{AB} \hat{b}_1 \right\|^2, \tag{9}
\]

This is just the Frobenius norm of \( T_{AB} \) and it is straightforward to check that, for pure states on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) with real coefficients, it is equal to half of Eq. \( \ref{eq:5} \). \hfill \Box

**Lemma 4.** For a pure state \( |\psi\rangle \) with real coefficients in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \),

\[
\max_{A_i, B_j, C_k} \langle B_{AB} \rangle^2 + \langle B_{AC} \rangle^2 = 8 \left( 1 - \langle Y_B Y_C \rangle^2 \right). \tag{10}
\]

Proof. Lemma \ref{lem:3} applied to \( \langle B_{AB} \rangle \) and \( \langle B_{AC} \rangle \) separately, immediately implies:

\[
\max_{A_i, B_j, C_k} \langle B_{AB} \rangle^2 + \langle B_{AC} \rangle^2 \leq \max_{A_i, B_j} \langle B_{AB} \rangle^2 + \max_{A_i, C_k} \langle B_{AC} \rangle^2
\]

\[
= 8 \left( 1 - \langle Y_B Y_C \rangle^2 \right). \tag{11}
\]

The reason we do not have equality is that the measurements \( A_i \) achieving the maximum in \( \langle B_{AB} \rangle \) and \( \langle B_{AC} \rangle \) may be different. We have to show they can be chosen to be the same. Define \( T_{AC} \) in analogy with Eq. \( \ref{eq:6} \) and write the vectors corresponding to \( C \)’s measurements as

\[
\hat{c}_1 + \hat{c}_2 = 2 \cos \theta \hat{e}_1, \quad \hat{c}_1 - \hat{c}_2 = 2 \sin \theta \hat{e}_2, \tag{12}
\]

in analogy with Eq. \( \ref{eq:7} \) for \( B \)’s observables. One can check that \( [T_{AB} T_{AB}^t, T_{AC} T_{AC}^t] = 0 \) for all pure states \( |\psi\rangle \) with real coefficients. Hence there are orthonormal vectors \( \hat{a}_i \) and \( \hat{a}_j \) that are simultaneous eigenvectors of \( T_{AB} T_{AB}^t \) and \( T_{AC} T_{AC}^t \). Next, note that the term being maximized in Eq. \( \ref{eq:5} \), \( \| T_{AB} \hat{d}_1 \|^2 + \| T_{AB} \hat{d}_2 \|^2 \), is actually independent of \( d_i \) (recall that \( d_1 \cdot d_2 = 0 \)), so we are free to choose the \( d_i \) as we please. Take \( \hat{d}_1 = T_{AB}^t \hat{a}_i \) for \( i = 1, 2 \) and, similarly, take \( \hat{d}_2 = T_{AC}^t \hat{a}_j \). Alice’s measurement vector \( \hat{a}_i \) in the \( AB \) maximization of the previous lemma was taken to be the unit vector along \( T_{AB} \hat{d}_i \), but this is \( T_{AB} T_{AB}^t \hat{a}_i \propto \hat{a}_i \) so \( \hat{a}_i = \hat{a}_i \). The same will hold in the \( AC \) maximization. Hence we can choose \( A \)’s measurement vectors to be the same in both cases, and we have equality in Eq. \( \ref{eq:10} \). \hfill \Box

Combining Lemmas \ref{lem:2} and \ref{lem:3} we obtain Theorem 1.

The monogamy trade-off is tight.—Lemma \ref{lem:3} also implies that any \( \langle B_{AB} \rangle \) and \( \langle B_{AC} \rangle \) compatible with Eq. \( \ref{eq:10} \) are achievable. In particular, the state

\[
|\psi\rangle = c_- (|010\rangle + |101\rangle) + c_+ (|100\rangle + |101\rangle), \tag{13}
\]

where

\[
c_\pm = \frac{1}{2} \sqrt{1 \pm \sqrt{2} \sin t}, \tag{14}
\]

and \( 0 \leq t \leq \pi/4 \), gives \( \langle B_{AB} \rangle = 2 \sqrt{2} \cos t, \langle B_{AC} \rangle = 2 \sqrt{2} \sin t \).

Extensions.—In the case of the CHSH inequality we can, in principle, obtain monogamy trade-offs when there are more than three parties via Lemma \ref{lem:2} which converts the problem into a finite optimization problem. In the three-party setting, if we are interested in \( \langle B_{BC} \rangle \) as well as \( \langle B_{AB} \rangle \) and \( \langle B_{AC} \rangle \) then we can obtain the trade-off surface numerically. The technique of Lemma \ref{lem:3} to allow \( A \)’s measurements to be different in \( \langle B_{AB} \rangle \) and \( \langle B_{AC} \rangle \) and then show that they could be the same anyway—does not work. It predicts that the trade-off surface be the intersection of the three cylinders, \( \langle B_{AB} \rangle^2 + \langle B_{AC} \rangle^2 \leq 8, \langle B_{AB} \rangle^2 + \langle B_{BC} \rangle^2 \leq 8, \) and \( \langle B_{AC} \rangle^2 + \langle B_{BC} \rangle^2 \leq 8, \) but one
can, for example, by using the multipartite generalization of Navascues, Pironio and Acín’s semidefinite programming bounds \(^1\), that there are points on this surface that are not achievable. It would be interesting to extend the semidefinite programming technique to obtain monogamy inequalities for other Bell inequalities.

**Bell inequality violation and anticommutation.**—The precise form of Eq. (10) suggests a general connection between the trade-off of Bell inequality violation and the expectation values of anticommuting observables; indeed, the operator \(Y_B Y_C\) anticommutes with all observables \(B_j, C_k\) measured in the Bell test. We now give a more general result, restricting for simplicity to the two party case.

**Theorem 5.** Let \(B = \sum_{j=1}^n p_{ij} A_i \otimes B_j\) be a 2-party 2-outcome correlation Bell operator such that \(\text{tr} \rho B \leq Q\) for all shared states \(\rho\) and all observables \(A_i, B_j\) (with spectrum in \([-1, +1]\]). Let \(W\) be an observable (with spectrum in \([-1, +1]\]) on Bob’s Hilbert space that anticommutes with \(B_j\) for all \(j\). Then

\[
\text{tr} \rho B \leq Q \sqrt{1 - (\text{tr} \rho W)^2}. \tag{15}
\]

**Proof.** We start by noting that it is sufficient to restrict to the case where \(\rho\) is pure, i.e., \(\rho = |\psi\rangle \langle \psi|\). The general case then follows by applying Jensen’s inequality to the concave function \(f(x) = \sqrt{1 - x^2}\).

If \(|w\rangle\) is an eigenvector of \(W\) with eigenvalue \(w\), \(B_j|w\rangle\) is either 0 or an eigenvector of \(W\) with eigenvalue \(-w\), since \(B_j\) and \(W\) anticommute. This means that there is a decomposition \(H_B = W_0 \oplus W_1 \oplus W_2\), where \(W_0\) annihilates vectors in \(W_0\), every \(B_j\) annihilates vectors in \(W_1\), and \(W_2\) is spanned by eigenvectors of \(W\) with nonzero eigenvalues, which occur in positive/negative pairs.

Denote the distinct positive eigenvalues associated with eigenvectors of \(W\) in \(W_2\) as \(0 < w_2 < w_3 < \cdots < w_m \leq 1\) and let \(V_i^\pm\) be the subspace in \(W_2\) corresponding to eigenvalue \(\pm w_i\). Decompose \(|\psi\rangle\) as \(|\psi\rangle = \sum_{i=0}^m \sqrt{p_i} |\psi_i\rangle\), where \(\sum_i p_i = 1\), \(|\psi_0\rangle\) is in \(W_0\), \(|\psi_1\rangle\) lies in \(W_1\), \((\psi_0|\psi_0\rangle = |\psi_1\rangle|\psi_1\rangle = 1\), and for \(i \geq 2\), \(|\psi_i\rangle = (\cos \theta_i |w_i^+\rangle + \sin \theta_i |w_i^-\rangle)\), \(|w_i^\pm\rangle\) in \(V_i^\pm\), and \((w_i^+ |w_i^-\rangle = 1\). Then \(B|w_i^\pm\rangle\) in \(V_i^\pm\) for \(i \geq 2\). It follows that

\[
\langle \psi|B|\psi\rangle = p_0 \langle \psi_0|B|\psi_0\rangle + \sum_{i=2}^m p_i \text{Re} \left(\langle w_i^+|B|w_i^-\rangle\right) \sin 2\theta_i,
\]

\[
\langle \psi|W|\psi\rangle = p_1 \langle \psi_1|W|\psi_1\rangle + \sum_{i=2}^m p_i w_i^2 \cos 2\theta_i. \tag{16}
\]

But these are the same expressions we would obtain with the mixed state \(\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|\). It therefore follows from our initial remark that it is sufficient to prove the claim for each state \(|\psi_i\rangle\). For \(i = 0, 1\), the result is trivial. Fix \(i \geq 2\), set \(\chi = \text{Re} \left(\langle w_i^+|B|w_i^-\rangle\right)\) and let \(x \in \{\pm 1\}\) be the sign of \(\chi\). Set \(|\phi\rangle = \frac{1}{\sqrt{|\chi|}} |w_i^-\rangle + x \frac{1}{\sqrt{|\chi|}} |w_i^+\rangle\). Then \(\langle \phi|B|\phi\rangle = \chi < Q\) by assumption, while

\[
\langle \psi_i|B|\psi_i\rangle \leq \chi \sin 2\theta_i \leq \chi \sqrt{1 - \langle \psi_i|W|\psi_i\rangle^2} \leq \sqrt{Q^2 - \langle \psi_i|W|\psi_i\rangle^2}. \tag{17}
\]

This completes the proof. \(\square\)

We now apply Theorem 5 to the CHSH inequality. If \(B_1\) and \(B_2\) are observables with \(\pm 1\) eigenvalues, then they both anticommute with their commutator \(i[B_1, B_2]/2\) (the factor of \(i/2\) makes this an observable). Applying Theorem 5 to both Alice and Bob’s observables, it follows that

\[
\langle B_{AB}\rangle \leq 2 \sqrt{2 - |\langle [A_1, A_2] [B_1, B_2]\rangle|}. \tag{20}
\]

These are local analogues of Tsirelson’s bound \(^2\),

\[
\langle B_{AB}\rangle \leq \sqrt{4 + |\langle [A_1, A_2] [B_1, B_2]\rangle|}. \tag{21}
\]

In particular, for maximal quantum violation of the CHSH inequality, the local observables corresponding to the commutators must be locally random \(|\langle [A_1, A_2] [B_1, B_2]\rangle| = 0\) but perfectly correlated \(|\langle [A_1, A_2] [B_1, B_2]\rangle| = 4\). This is a clear manifestation of the fact that entanglement goes hand in hand with local randomness.

In conclusion, we investigated Bell correlations in a tripartite setting and obtained tight monogamy bounds on the trade-off between them. The main message is depicted in Figure 1, which gives a universal picture of nonlocal correlations valid for quantum systems of any dimension.

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\(^1\) Electronic address: Ben.Toner@ewi.nl
\(^2\) Electronic address: frank.verstraete@univie.ac.at

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