Precise Estimates for the Subelliptic Heat Kernel on H-type Groups

Nathaniel Eldredge

Department of Mathematics, University of California, San Diego, 9500 Gilman Drive, Dept. 0112, La Jolla, CA 92093-0112
USA

Abstract
We establish precise upper and lower bounds for the subelliptic heat kernel on nilpotent Lie groups $G$ of H-type. Specifically, we show that there exist positive constants $C_1, C_2$ and a polynomial correction function $Q_t$ on $G$ such that

$$C_1 Q_t e^{-\frac{d^2}{4t}} \leq p_t \leq C_2 Q_t e^{-\frac{d^2}{4t}}$$

where $p_t$ is the heat kernel, and $d$ the Carnot-Carathéodory distance on $G$. We also obtain similar bounds on the norm of its subelliptic gradient $|\nabla p_t|$. Along the way, we record explicit formulas for the distance function $d$ and the subriemannian geodesics of H-type groups.

Key words: heat kernel, subelliptic, hypoelliptic, Heisenberg group

1. Introduction

Nilpotent Lie groups have long been of interest as a natural setting for the study of subelliptic operators; indeed, as shown in [24], they model, at least locally, a general class of hypoelliptic operators on manifolds. Perhaps the simplest example is the classical Heisenberg group of dimension 3, followed by the higher-dimensional Heisenberg or Heisenberg-Weyl groups of dimension $2n+1$ having 1-dimensional centers. Beyond this, a natural generalization of the Heisenberg groups is given by the H-type (or Heisenberg-type) groups, which were introduced in [15]; these have a greater variety of possible dimensions while retaining some fairly strong algebraic structure.

The main result of this paper is found in Corollary 4.3, in which we establish precise upper and lower pointwise estimates on the subelliptic heat kernel $p_t$ for an H-type group $G$, of the form

$$C_1 Q_t e^{-\frac{d^2}{4t}} \leq p_t \leq C_2 Q_t e^{-\frac{d^2}{4t}}$$

Email address: neldredge@math.ucsd.edu (Nathaniel Eldredge)
URL: http://www.math.ucsd.edu/~neldredg/ (Nathaniel Eldredge)

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for some positive constants \( C_1, C_2 \) and an explicit function \( Q_t \), where \( d \) is the Carnot-Carathéodory distance on \( G \). Additionally, in Theorem 4.4 we obtain similar bounds for the subriemannian gradient of \( p_1 \), namely

\[
C_1 Q' e^{-\frac{d^2}{4}} \leq |\nabla p_1| \leq C_2 Q' e^{-\frac{d^2}{4}}
\]  

(1.2)

for another explicit function \( Q' \), where the inequality is valid at points sufficiently far from the identity of \( G \).

Estimates of the form (1.1) for the classical Heisenberg group first appeared in [19], in the context of a gradient estimate for the heat semigroup, as did an estimate equivalent to the upper bound in (1.2). A proof for Heisenberg groups in all dimensions followed in [20]. Our proof is similar in spirit to the latter, in that it relies on the analysis of an explicit formula for \( p_t \) using steepest descent methods and elementary complex analysis.

Less precise versions of the inequalities (1.1) are known to hold in more general settings. Using Harnack inequalities one can show that for general nilpotent Lie groups,

\[
C_1 R_1(t) e^{-\frac{d^2}{4}} \leq p_t \leq C_2(\epsilon) R_2(t) e^{-\frac{d(g)^2}{4t}}
\]

(1.3)

for some constants \( c, C_1, C_2 \) and functions \( R_1, R_2 \), where \( C_2 \) depends on \( \epsilon > 0 \); see chapter IV of [27]. [6], among others, improves the upper bound to

\[
p_t(g) \leq CR_3(g,t) e^{-\frac{d(g)^2}{4t}}
\]

(1.4)

with \( R \) a polynomial correction, using logarithmic Sobolev inequalities, whereas [26] improves the lower bound to

\[
p_t \geq C(\epsilon) R_4(t) e^{-\frac{d\tilde{g}}{4t}}
\]

(1.5)

Similar but slightly weaker estimates were shown for more general sum-of-squares operators satisfying Hörmander’s condition in [17] by means of Malliavin calculus, and in [14] by more elementary methods involving homogeneity and the regular dependence of \( p_t \) on \( t \).

In the specific case of the classical Heisenberg group, asymptotic results similar to (1.1) had been previously obtained in [9] and [11], but without the necessary uniformity to translate them into pointwise estimates. A precise upper bound equivalent to that of (1.1) was given in [1] for Heisenberg groups of all dimensions. All three of these works, like [20] and the present article, were based on an explicit formula for \( p_t \) and involved steepest descent type methods. In [8], similar techniques were used to obtain a Li-Yau-Harnack inequality for the heat equation on Heisenberg groups.

The proof we shall give here is largely self-contained, except for the formula (4.2) for \( p_t \), which has been derived many times in the literature by many different techniques. We have also tried to err to the side of including relevant details.

2. H-type groups

H-type groups were first introduced in [15]. Chapter 18 of [2] contains an extended development of their fundamental properties; we follow its definitions here, and refer the reader there for further details.

**Definition 2.1.** Let \( \mathfrak{g} \) be a finite dimensional real Lie algebra with center \( z \neq 0 \). We say \( \mathfrak{g} \) is of H-type (or Heisenberg type) if \( \mathfrak{g} \) is equipped with an inner product \( \langle \cdot, \cdot \rangle \) such that:

1. \( [z^+, z^+] = z \); and
2. For each \( z \in \mathfrak{z} \), define \( J_z : \mathfrak{z}^+ \rightarrow \mathfrak{z}^+ \) by
\[
\langle J_z x, y \rangle = \langle z, [x, y] \rangle
\]
where \( x, y \in \mathfrak{z}^+ \). Then \( J_z \) is an orthogonal map whenever \( \langle z, z \rangle = 1 \).

An H-type group is a connected, simply connected Lie group whose Lie algebra is of H-type.

Some authors use instead of item 2 the equivalent property that for \( x \in \mathfrak{z}^+ \) with \( \|x\| = 1 \), the map \( \text{ad}_x : (\ker \text{ad}_x) \rightarrow \mathfrak{z} \) is an isometric isomorphism.

We record some algebraic properties of the maps \( J_z \) which will be useful later. We use \( |z| := \sqrt{\langle z, z \rangle} \) to denote the norm associated to the inner product on \( \mathfrak{g} \). The proofs are elementary and are omitted.

**Proposition 2.2.** If \( \mathfrak{g} \) is a H-type Lie algebra, then the maps \( J_z : \mathfrak{z}^+ \rightarrow \mathfrak{z}^+ \) defined in Definition 2.1 enjoy the following properties:

1. For each \( z \), \( J_z \) is a well-defined linear map, and \( z \mapsto J_z \) is also linear.
2. \( J_z^2 = -J_z \).
3. \( J_z^2 = -\|z\|^2 I \). Thus for \( z \neq 0 \), \( J_z \) is invertible and \( J_z^{-1} = -\|z\|^{-2} J_z \).
4. \( J_z J_w + J_w J_z = -2\langle z, w \rangle I \).
5. \( \langle J_z x, J_w x \rangle = \langle z, w \rangle \|x\|^2 \).
6. \( [x, J_z x] = \|x\|^2 z \).

Note that items 2 and 3 say that for \( z \neq 0 \), \( J_z \) is an invertible skew-symmetric linear transformation of \( \mathfrak{z}^+ \). Thus \( \dim \mathfrak{z}^+ \) must be even. We will write \( \dim \mathfrak{z}^+ = 2n \) and \( \dim \mathfrak{z} = m \).

Item 4 says that the subalgebra of \( \text{End}(\mathfrak{z}^+) \) generated by the maps \( J_z \) is a Clifford algebra. In fact, it is a \( 2n \)-dimensional representation of \( C\ell_{0,m}(\mathbb{R}) \), the Clifford algebra generated by a real vector space of dimension \( m \) with a negative definite quadratic form (whose signature is \( (0, m) \)). So in order for an H-type algebra with \( \dim \mathfrak{z} = m \), \( \dim \mathfrak{z}^+ = 2n \) to exist, it is necessary that \( C\ell_{0,m}(\mathbb{R}) \) have such a representation. This condition is also sufficient: given such a representation, let \( V \) be the \( m \)-dimensional generating subspace of \( C\ell_{0,m}(\mathbb{R}) \), and let \( \mathfrak{g} = \mathbb{R}^{2n} \oplus V \), with the maps \( J_z \) defined by the representation. Then the bracket on \( \mathfrak{g} \) can be recovered in terms of the \( J_z \) from 2.1 and \( \mathfrak{g} \) is an H-type Lie algebra.

The Hurwitz-Radon-Eckmann theorem, as found in [7], gives necessary and sufficient conditions on \( n \) and \( m \) for such a representation to exist. The corresponding theorem for H-type algebras appears as Corollary 1 of [15], which we quote here.

**Theorem 2.3.** For any nonnegative integer \( k \), we can uniquely write \( k = a2^{4p+q} \) where \( a \) is odd and \( 0 \leq q \leq 3 \); let \( \rho(k) := 8p + 2q \). (\( \rho \) is sometimes called the Hurwitz-Radon function.) There exists an H-type Lie algebra of dimension \( 2n + m \) with center of dimension \( m \) if and only if \( m < \rho(2n) \). In particular, for every \( m \in \mathbb{N} \) there exists an H-type Lie algebra with center of dimension \( m \).

The special case \( m = 1 \) gives the so-called isotropic Heisenberg groups (also called the Heisenberg-Weyl groups) of real dimension \( 2n+1 \); the very special case \( n = m = 1 \) is the classic Heisenberg group of dimension 3.

A Lie algebra \( \mathfrak{g} \) is said to be nilpotent of step \( k \) if \( k \) is the smallest integer such that all \( k \)-fold brackets of elements of \( \mathfrak{g} \) vanish. A nilpotent Lie algebra is stratified if we can write \( \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \) where \([\mathfrak{g}_i, \mathfrak{g}_{i-1}] = \mathfrak{g}_i \) and \([\mathfrak{g}_1, \mathfrak{g}_k] = 0 \). An H-type Lie algebra is obviously stratified nilpotent of step 2, with \( \mathfrak{g}_1 = \mathfrak{z}^+, \mathfrak{g}_2 = \mathfrak{z} \).

We recall that given a nilpotent Lie algebra \( \mathfrak{g} \), there exists a connected, simply connected Lie group \( G \) whose Lie algebra is \( \mathfrak{g} \), and \( G \) is unique up to isomorphism. Indeed, we can, and will, take \( G \) to be \( \mathfrak{g} \) equipped with the group operation \( \circ \) given by the Baker-Campbell-Hausdorff formula, which for \( \mathfrak{g} \) nilpotent of step 2 reads
\[
x \circ y := x + y + \frac{1}{3}[x, y],
\]
In this case the exponential map \( g \to G \) is just the identity. It is obvious, then, that if \( g, g' \) are isomorphic as Lie algebras, then \((g, \circ), (g', \circ')\) as defined above are isomorphic as Lie groups.

On the other hand, \( g \) can be identified as an inner product space with Euclidean space \( \mathbb{R}^{2n+m} \), identifying \( \mathfrak{j}^1 \) with the first \( 2n \) coordinates and \( \mathfrak{j} \) with the last \( m \). Therefore we can handle H-type groups concretely as follows.

**Proposition 2.4.** If \( G \) is an H-type group, then there exist integers \( n, m \) and a bracket operation \([\cdot, \cdot]\) on \( \mathbb{R}^{2n+m} \) such that \((\mathbb{R}^{2n+m}, [\cdot, \cdot])\) is an H-type Lie algebra whose center is \( \mathbb{R}^m \) and \( G \) is isomorphic to \((\mathbb{R}^{2n+m}, \circ)\), where \( \circ \) is defined by \((2.2)\).

Henceforth we shall assume that any H-type group \( G \) is of this form. We shall use the notation \( g = (x, z) = (x^1, \ldots, x^{2n}, z^1, \ldots, z^m) \) to refer to points of \( G \). The identity of \( G \) is \((0, 0)\), and the inverse operation is given by \((x, z)^{-1} = (-x, -z)\). Because of the identification of \( G \) with its Lie algebra, we will view \([\cdot, \cdot]\) as a bracket on \( G \). By a slight abuse of notation, we will also use \([\cdot, \cdot]\) to refer to the restriction of \([\cdot, \cdot]\) to \( \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \subset G \oplus G \), which is a bilinear skew-symmetric mapping from \( \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \) to \( \mathbb{R}^m \). The maps \( \{J_z : z \in \mathbb{R}^m\} \) are identified with \( 2n \times 2n \) skew-symmetric matrices which are orthogonal when \(|z| = 1\).

We let \( \{e_1, \ldots, e_{2n}\} \) denote the standard basis for \( \mathbb{R}^{2n} \), and \( \{u_1, \ldots, u_m\} \) denote the standard basis for \( \mathbb{R}^m \).

Note that the group operation on \( G \) does not preserve the inner product, and the vector space operations \( g \mapsto g + h, g \mapsto cg \) are not group homomorphisms of \( G \). However, the *dilation*

\[
\varphi_\alpha(x, z) := (\alpha x, \alpha^2 z) \quad (2.3)
\]
is both a group and a Lie algebra automorphism for all \( \alpha \neq 0 \).

We can now identify \( g \) with the set of left-invariant vector fields on \( G \), where \( X_i(0) = \frac{\partial}{\partial x^i}, Z_j(0) = \frac{\partial}{\partial z^j} \); then \( \text{span}\{X_1, \ldots, X_{2n}\} = \mathfrak{j}^1 \), \( \text{span}\{Z_1, \ldots, Z_m\} = \mathfrak{j} \). We can compute

\[
(X_i f)(x, z) = \frac{d}{dt}|_{t=0} f((x, z) \circ (te_i, 0)) \\
= \frac{d}{dt}|_{t=0} f(x + te_i, z + \frac{1}{2} t[x, e_i]) \\
= \frac{d}{dt}|_{t=0} f(x + te_i, z + \frac{1}{2} t \sum_j \langle J_{u_j} x, e_i \rangle u_j) \\
= \left( \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_j \langle J_{u_j} x, e_i \rangle \frac{\partial}{\partial z^j} \right) f
\]

So we have

\[
X_i = \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_j \langle J_{u_j} x, e_i \rangle \frac{\partial}{\partial z^j} \quad (2.4)
\]

\[
Z_j = \frac{\partial}{\partial z^j} \quad (2.5)
\]
The (sub-)gradient on \( G \) is given in these coordinates by

\[
\nabla f(x, z) = \sum e_i X_i f(x, z) = \nabla_x f(x, z) + \frac{1}{2} J_{\nabla_x f(x, z)} x. \quad (2.6)
\]

Note in particular, if \( f \) is radial, so that \( f(x, z) = f(|x|, |z|) \), this becomes

\[
\nabla f(x, z) = f_{|x|}(|x|, |z|) \dot{x} + \frac{1}{2} f_{|z|}(|x|, |z|) |x| J_{\dot{x}} \quad (2.7)
\]

where we use the notation \( \dot{u} := \frac{u}{|u|} \) to denote the unit vector in the \( u \) direction. We draw attention to the fact that \( \dot{x} \) and \( J_{\dot{x}} \dot{x} \) are orthogonal unit vectors in \( \mathbb{R}^{2n} \) for any nonzero \( x, z \).
3. Subriemannian geometry

Our desired estimate for the heat kernel \( p_t \) is in terms of the Carnot-Carathéodory distance \( d \), which is best described in the language of subriemannian geometry. The goal of this section will be to obtain an explicit formula for \( d \), and along the way we record formula for the geodesics of \( G \). The computation is a straightforward application of Hamiltonian mechanics, but we have not seen it appear in the literature in the case of H-type groups. The corresponding computation for the Heisenberg groups (where the center has dimension \( m = 1 \)) appeared in \([1] \) as well as \([2] \); a computation for \( m \leq 7 \), which could be extended without great difficulty, can be found in the preprint \([4] \).

Definition 3.1. A subriemannian manifold is a smooth manifold \( Q \) together with a subbundle \( \mathcal{H} \) of \( TQ \) (the horizontal bundle or horizontal distribution, whose elements are horizontal vectors) and a metric \( \langle \cdot, \cdot \rangle_q \) on each fiber \( \mathcal{H}_q \), depending smoothly on \( q \in Q \). \( \mathcal{H} \) is bracket-generating at \( q \) if there is a local frame \( \{ X_i \} \) for \( \mathcal{H} \) near \( q \) such that span\( \{ X_i(q), [X_i, X_j](q), [X_i, [X_j, X_k]](q), \ldots \} = T_q Q \).

An H-type group \( G \) can naturally be equipped as a subriemannian manifold, by letting \( \mathcal{H}_g := \{ X(g) : X \in \mathfrak{z}^\perp \} \), and using the inner product on \( \mathfrak{g} \) as the metric on \( \mathcal{H} \). In other words, \( \mathcal{H}_g \) is spanned by \( \{ X_1(g), \ldots, X_{2n}(g) \} \), which give it an orthonormal basis. The bracket generating condition is obviously satisfied, since \( g = \mathfrak{z}^\perp \oplus [\mathfrak{z}^\perp, \mathfrak{z}^\perp] \).

Definition 3.2. Let \( \gamma : [0, 1] \to Q \) be an absolutely continuous path. We say \( \gamma \) is horizontal if \( \dot{\gamma}(t) \in H_{\gamma(t)} \) for almost every \( t \in [0, 1] \). In such a case we define the length of \( \gamma \) as \( \ell(\gamma) := \int_0^1 \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} \, dt \). The Carnot-Carathéodory distance \( d : Q \times Q \to [0, \infty] \) is defined by

\[
d(q_1, q_2) = \inf \{ \ell(\gamma) : \gamma(0) = q_1, \gamma(1) = q_2, \gamma \text{ horizontal} \}. \tag{3.1}
\]

Under the bracket generating condition, the Carnot-Carathéodory distance is well behaved. We refer the reader to Chapter 2 and Appendix D of \([22] \) for proofs of the following two theorems.

Theorem 3.3 (Chow). If \( \mathcal{H} \) is bracket generating and \( Q \) is connected, then any two points \( q_1, q_2 \in Q \) are joined by a horizontal path whose length is finite. Thus \( d(q_1, q_2) < \infty \), and \( d \) is easily seen to be a distance function on \( Q \). The topology induced by \( d \) is equal to the manifold topology for \( Q \).

Theorem 3.4. If \( Q \) is complete under the Carnot-Carathéodory distance \( d \), then the infimum in the definition of \( d \) is achieved; that is, any two points \( q_1, q_2 \in Q \) are joined by at least one shortest horizontal path.

For an H-type group, we obtain the following explicit formula for the distance. Note that by its definition, \( d \) is left-invariant, i.e. \( d(g, h) = d(kg, kh) \), so it is sufficient to compute distance from the identity. By an abuse of notation, we write \( d(x, z) \) to mean \( d((0, 0), (x, z)) \).

Theorem 3.5. Define the function \( \nu : \mathbb{R} \to \mathbb{R} \) by

\[
\nu(\theta) = \frac{2\theta - \sin 2\theta}{1 - \cos 2\theta} = \frac{\theta}{\sin \theta} - \cot \theta = -\frac{d}{d\theta}(\theta \cot \theta) \tag{3.2}
\]

where the alternate form comes from the double-angle identities. Then

\[
d(x, z) = \begin{cases} 
|x| \frac{d}{d\sin \theta}, & z \neq 0, x \neq 0 \\
|x|, & z = 0 \\
\sqrt{\frac{4\pi}{|x|} |z|}, & x = 0 
\end{cases} \tag{3.3}
\]

where \( \theta \) is the unique solution in \([0, \pi]\) to \( \nu(\theta) = \frac{4|x|}{|x|^2} \).
We note that it is apparent from \[3.3\] that we have the scaling property
\[
d(\varphi_\alpha(x, z)) = \alpha d(x, z) \tag{3.4}
\]
with \(\varphi\) as in \[22\].

One way to compute the Carnot-Carathéodory distance is to find such a shortest path and compute its length. To find a shortest path, we use Hamiltonian mechanics, following Chapters 1 and 5 of \[22\]. Roughly speaking, it can be shown that a length minimizing path also minimizes the energy \(\frac{1}{2} \int_0^t \|\dot{\gamma}(t)\| \, dt\), and as such should solve Hamilton’s equations of motion. The argument uses the method of Lagrange multipliers, and requires that the endpoint map taking horizontal paths to their endpoints has a surjective differential. This always holds in the Riemannian setting, but is not generally true in subriemannian geometry; the Martinet distribution (see Chapter 3 of \[22\]) is a counterexample in which some shortest paths do not satisfy Hamilton’s equations. Additional assumptions on \(H\) are needed. One which is sufficient (but certainly not necessary) is that the distribution be fat:

**Definition 3.6.** Let \(\Theta\) be the canonical 1-form on the cotangent bundle \(T^*Q\), \(\omega = d\Theta\) the canonical symplectic 2-form, and let \(\mathcal{H}^0 := \{p_q \in T^*Q : p_q(H_q) = 0\}\) be the annihilator of \(\mathcal{H}\). (Note \(\mathcal{H}^0\) is a subbundle, and hence also a submanifold, of \(T^*Q\).) We say \(\mathcal{H}\) is fat if \(\mathcal{H}^0\) is symplectic away from the zero section. That is, if \(p_q \in \mathcal{H}^0\) is not in the zero section, \(v \in T_{p_q}\mathcal{H}^0\), and \(\omega(v, w) = 0\) for all other \(w \in T_{p_q}\mathcal{H}^0\), then \(v = 0\).

**Definition 3.7.** If \((Q, \mathcal{H}, \langle , \rangle)\) is a subriemannian manifold, the subriemannian Hamiltonian \(H : T^*Q \to \mathbb{R}\) is defined by
\[
H(p_q) = \sum_i p_q(v_i)^2 \tag{3.5}
\]
where \(\{v_i\}\) is an orthonormal basis for \((\mathcal{H}_q, \langle , \rangle_q)\). It is clear that this definition is independent of the chosen basis. Let the Hamiltonian vector field \(X_H\) on \(T^*Q\) be the unique vector field satisfying \(dH + \omega(X_H, \cdot) = 0\) (as elements of \(T^*T^*Q\)). \(X_H\) is well defined because \(\omega\) is symplectic. Hamilton’s equations of motion are the ODEs for the integral curves of \(X_H\).

The following theorem summarizes (a special case of) the argument of Chapters 1 and 5 of \[22\].

**Theorem 3.8.** If \(\mathcal{H}\) is fat, then any length minimizing path \(\sigma : [0, 1] \to Q\), when parametrized with constant speed, is also energy minimizing and is the projection onto \(Q\) of a path \(\gamma : [0, 1] \to T^*Q\) which satisfies Hamilton’s equations of motion: \(\dot{\gamma}(t) = X_H(\gamma(t))\).

We now verify explicitly that this theorem applies to H-type groups. We first adopt a coordinate system for the cotangent bundle \(T^*G\).

**Notation 3.9.** Let \((x, z, \xi, \eta) : T^*G \to \mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^{2n} \times \mathbb{R}^m\) be the coordinate system on \(T^*G\) such that \(x^i(p_g) = x^i(g)\), \(z^j(p_g) = z^j(g)\), \(\xi_i(p_g) = p(\frac{\partial}{\partial x^i})\), \(\eta_j(p_g) = p(\frac{\partial}{\partial z^j})\). That is,
\[
p_g = \begin{pmatrix} x(g), z(g), \sum_i \xi_i dx^i + \sum_j \eta_j dz^j \end{pmatrix}.
\]
In these coordinates, the canonical 2-form \(\omega\) has the expression \(\omega = \sum_i d\xi_i \wedge dx^i + \sum_j d\eta_j \wedge dz^j\).

**Proposition 3.10.** If \(G\) is an H-type group with horizontal distribution \(\mathcal{H}\) spanned by the vector fields \(X_i\), then \(\mathcal{H}\) is fat.

**Proof.** For an H-type group \(G\), we have \(p_g \in \mathcal{H}^0\) iff \(p_g(X_i(g)) = 0\) for all \(i\). We can thus form a basis for \(\mathcal{H}^0_q \subset T^*_q G\) by
\[
w^j = dz^j - \sum_i dz^i(X_i(g)) dx^i = dz^j - \frac{1}{2} \sum_i (J_{u^i} x(g), e_i) dx^i.
\]
Expressing \( p_g \) in this basis as \( p_g = \sum_j \theta_j w^j \) yields a system of coordinates \((x,z,\theta)\) for \( \mathcal{H}^0 \), where \( \theta \) can be identified with the element \((\theta^1, \ldots, \theta^m)\) of \( \mathbb{R}^m \). In terms of the coordinates \((x,z,\xi,\eta)\) for \( T^*G \), we have \( \eta = \theta, \xi = -\frac{1}{2} J_\theta x \).

So let \( \gamma : (-\epsilon, \epsilon) \to \mathcal{H}^0 \) be a curve in \( \mathcal{H}^0 \) which avoids the zero section. \( \gamma(0) \) is thus a generic element of \( T\mathcal{H}^0 \). We write \( \gamma(t) \) in coordinates as \((x(t),z(t),\theta(t))\), where \( \theta(t) \neq 0 \). In terms of the coordinates \((x,z,\xi,\eta)\) on \( T^*G \), we have \( \eta(t) = \theta(t), \xi(t) = -\frac{1}{2} J_{\theta(t)} x(t) \). Differentiating the latter gives

\[
\dot{\xi}(t) = -\frac{1}{2} (J_{\theta(t)} x(t) + J_{\theta(t)} \dot{x}(t)).
\]

Suppose that for all other such curves \( \gamma' \) with \( \gamma'(0) = \gamma(0) \), we have \( \omega(\gamma(0), \gamma'(0)) = 0 \). In terms of coordinates,

\[
0 = \omega(\gamma(0), \gamma'(0)) = \sum_i (\dot{\xi}_i(0) \dot{x}^i(0) - \dot{\xi}'_i(0) \dot{x}'^i(0)) + \sum_j (\dot{\eta}_j(0) \dot{z}^j(0) - \dot{\eta}'_j(0) \dot{z}'^j(0))
\]

\[
= (\dot{\xi}(0), \dot{x}'(0)) - (\dot{\xi}'(0), \dot{x}(0)) + (\dot{\eta}(0), \dot{z}'(0)) - (\dot{\eta}'(0), \dot{z}(0))
\]

\[
= -\frac{1}{2} J_{\theta(0)} \dot{x}(0) + J_{\theta(0)} \dot{x}'(0) + \frac{1}{2} J_{\theta'(0)} \dot{x}'(0) - \langle \dot{\theta}(0), \dot{z}'(0) \rangle - \langle \dot{\theta}'(0), \dot{z}(0) \rangle
\]

\[
= \frac{1}{2} (\dot{x}(0), J_{\theta(0)} \dot{x}'(0) + J_{\theta'(0)} \dot{x}'(0)) + \langle \dot{\theta}(0), \dot{z}'(0) \rangle - \langle \dot{\theta}'(0), \dot{z}(0) \rangle
\]

For arbitrary \( u \in \mathbb{R}^m \), take \( \gamma'(t) = (x(0), z(0) + tu, \theta(0)) \); then \( 0 = \omega(\gamma(0), \gamma'(0)) = \langle \dot{\theta}(0), u \rangle \), so we must have \( \dot{\theta}(0) = 0 \). Next, for arbitrary \( v \in \mathbb{R}^n \), take \( \gamma'(t) = (x(0) + tv, z(0), \theta(0)) \); then we have \( 0 = \langle J_{\theta(0)} u, \dot{x}(0) \rangle \). But \( \theta(0) \neq 0 \) by assumption, so \( J_{\theta(0)} \) is nonsingular and we must have \( \dot{x}(0) = 0 \). Finally, take \( \gamma'(t) = (x(0), z(0), \theta(0) + tu) \); then \( \langle u, \dot{z}(0) \rangle = 0 \), so \( \dot{z}(0) = 0 \). Thus we have shown that if \( \omega(\gamma(0), \gamma'(0)) = 0 \) for all \( \gamma' \), we must have \( \dot{\gamma}(0) = 0 \), which completes the proof.

We now proceed to compute and solve Hamilton’s equations of motion for an H-type group \( G \).

The subriemannian Hamiltonian on \( T^*G \) is defined by (c.f. \( 3.6 \))

\[
H(p_g) := \frac{1}{2} \sum_{i=1}^{2n} p_g (X_i(g))^2, \quad p_g \in T^*_g G.
\]

In terms of the above coordinates, we may compute

\[
p_g (X_i(g)) = p_g \left( \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_j (J_{u_j} x, e_i) \frac{\partial}{\partial z^j} \right) = \xi_i(p_g) + \frac{1}{2} (J_{\eta(p_g)} x(g), e_i)
\]

so that

\[
H(p_g) = \frac{1}{2} \left| \xi(p_g) + \frac{1}{2} J_{\eta(p_g)} x(g) \right|^2.
\]

Recall that a path \( \gamma : [0,T] \to T^*Q \) satisfies Hamilton’s equations iff \( \dot{\gamma}(t) = X_H(\gamma(t)) \), i.e. \( dH_{\gamma(t)} + \omega(\gamma(t), \cdot) = 0 \).

In an H-type group \( G \), we write \( \gamma \) in coordinates as \( \gamma(t) = (x(t), z(t), \xi(t), \eta(t)) : [0,T] \to T^*G \), so that we have

\[
\omega(\gamma(t), \cdot) = \sum_i (\dot{\xi}_i(t) dx^i - \dot{x}^i(t) d\xi_i) + \sum_j (\dot{\eta}_j(t) dz^j - \dot{z}^j(t) d\eta_j).
\]

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Thus Hamilton’s equations of motion read

\[ \dot{x}^i = \frac{\partial H}{\partial \xi_i}, \quad \dot{\xi}_i = -\frac{\partial H}{\partial x^i}, \quad \dot{z}_j = \frac{\partial H}{\partial \eta_j}, \quad \dot{\eta}_j = -\frac{\partial H}{\partial z^j}. \] (3.7)

To compute the derivatives, we note that \( \frac{1}{2} \nabla_x |Ax + y|^2 = A^*Ax + A^*y. \) If we write \( B_x \eta = J_x \xi, \) then \( (B_x \eta, y) = (\eta, [x, y]), \) so \( B_x^* B_x = |x|^2 I. \) So for a path \( \gamma(t) = (x(t), z(t), \xi(t), \eta(t)) : [0, T] \to T^*G, \) Hamilton’s equations of motion read

\[ \dot{x} = \nabla_\xi H = \xi + \frac{1}{2} |\eta|^2 J \eta, \] (3.8)

\[ \dot{z} = \nabla_\eta H = \frac{1}{2} |x|^2 \eta + \frac{1}{2} |x, \xi| \] (3.9)

\[ \dot{\xi} = -\nabla_x H = -\frac{1}{4} |\eta|^2 x + \frac{1}{2} |x, \xi| \] (3.10)

\[ \dot{\eta} = -\nabla_z H = 0. \] (3.11)

**Theorem 3.11.** \((x(t), z(t))\) is the projection of a solution to Hamilton’s equations with \( x(0) = 0, z(0) = 0 \) and \( x(1), z(1) \) given if and only if:

1. If \( z(1) = 0, \) we have

\[ x(t) = tx(1), \quad z(t) = 0. \] (3.12)

2. If \( z(1) \neq 0, \) we have

\[ x(t) = \frac{1}{|\eta_0|^2} J_{\eta_0} (I - e^{t J_{\eta_0}}) \xi_0 \] (3.13)

\[ z(t) = \frac{|\xi_0|^2}{2 |\eta_0|^2} (|\eta_0| t - \sin(|\eta_0| t)) \eta_0 \] (3.14)

where, if \( x(1) \neq 0 \) we have

\[ \eta_0 = 2\theta \frac{z(1)}{|z(1)|} \] (3.15)

\[ \xi_0 = -\frac{1}{4} |\eta_0|^2 (J_{\eta_0} (e^{t J_{\eta_0}} - I))^{-1} x(1). \] (3.16)

where \( \theta \) is a solution to

\[ \nu(\theta) = 4 \frac{|z(1)|}{|x(1)|^2}; \] (3.17)

and if \( x(1) = 0 \) we have

\[ \eta_0 = 2\pi k \frac{z(1)}{|z(1)|} \]

\[ |\xi_0| = \sqrt{4k\pi |z(1)|} \]

for some integer \( k \geq 1. \)

**Proof.** We solve (3.8–3.11), assuming \( x(0) = 0, z(0) = 0. \) By (3.11) we have \( \eta(t) \equiv \eta(0) = \eta_0. \) If \( \eta_0 = 0, \) we can see by inspection that the solution is

\[ \eta(t) = 0, \quad \xi(t) = \xi_0, \quad x(t) = t \xi_0, \quad z(t) = 0, \] (3.18)
namely, a straight line from the origin, whose length is clearly \(|x(1)|\). This is \((3.12)\), which we shall see is forced when \(z(1) = 0\).

Otherwise, assume \(\eta_0 \neq 0\). We may solve \((3.8)\) for \(\xi\) to see that
\[
\xi = \dot{x} - \frac{1}{2} J_{\eta_0} x.
\]  
(3.19)

Notice that substituting \((3.19)\) into \((3.9)\) shows that
\[
\dot{z} = \frac{1}{2} [x, \dot{x}],
\]  
(3.20)
from which an easy computation verifies that \((x(t), z(t))\) is indeed a horizontal path.

Substituting \((3.19)\) into the right side of \((3.10)\) shows that
\[
\dot{\xi} = -\frac{1}{4} |\eta_0|^2 x + \frac{1}{2} J_{\eta_0} (\dot{x} - \frac{1}{2} J_{\eta_0} x) = \frac{1}{2} J_{\eta_0} \dot{x}
\]  
since \(J_{\eta_0} x = -|\eta_0|^2 x\). Thus
\[
\xi = \frac{1}{2} J_{\eta_0} x + \xi_0
\]  
(3.21)

where \(\xi_0 = \xi(0)\). If \(\xi_0 = 0\), it is easily seen that we have the trivial solution \(x(t) = 0, z(t) = 0, \xi(t) = 0, \eta(t) = \eta_0\), so we assume now that \(\xi_0 \neq 0\). \((3.21)\) may be substituted back into \((3.8)\) to get
\[
\dot{x} = J_{\eta_0} x + \xi_0
\]  
(3.22)
so that
\[
x = (J_{\eta_0})^{-1} (e^{t J_{\eta_0}} - I) \xi_0 = -\frac{1}{|\eta_0|^2} J_{\eta_0} (e^{t J_{\eta_0}} - I) \xi_0.
\]  
(3.23)

Differentiation (or substitution) shows
\[
\dot{x} = e^{t J_{\eta_0}} \xi_0.
\]  
(3.24)

Note that
\[
|x|^2 = \frac{1}{|\eta_0|^2} |(e^{t J_{\eta_0}} - I) \xi_0|^2 = \frac{2}{|\eta_0|^2} (1 - \cos(|\eta_0| t)) |\xi_0|^2.
\]  
(3.25)

It is easy to see from \((3.23)\) that \(x(t)\) lies in the plane spanned by \(\xi_0\) and \(J_{\eta_0} \xi_0\), and \(x(t)\) sweeps out a circle centered at \(\frac{1}{|\eta_0|^2} J_{\eta_0} \xi_0\) and passing through the origin. In particular, the radius of the circle is \(|\xi_0|/|\eta_0|\).

Now substituting \((3.23)\) and \((3.24)\) into \((3.20)\), we have
\[
\dot{z} = -\frac{1}{2} \frac{|\xi_0|^2}{|\eta_0|^2} \left([J_{\eta_0} e^{t J_{\eta_0}} \xi_0, e^{t J_{\eta_0}} \xi_0] - [J_{\eta_0} \xi_0, e^{t J_{\eta_0}} \xi_0]\right)
\]  
\[
= -\frac{1}{2} \frac{|\xi_0|^2}{|\eta_0|^2} \left(|e^{t J_{\eta_0}} \xi_0|^2 \eta_0 + [J_{\eta_0} \xi_0, e^{t J_{\eta_0}} \xi_0]\right)
\]  
\[
= \frac{|\xi_0|^2}{2 |\eta_0|^2} (1 - \cos(|\eta_0| t)) \eta_0
\]

By integration,
\[
z = \frac{|\xi_0|^2}{2 |\eta_0|^2} (|\eta_0| t - \sin(|\eta_0| t)) \eta_0.
\]  
(3.26)

In particular,
\[
|z| = \frac{|\xi_0|^2}{2 |\eta_0|^2} (|\eta_0| t - \sin(|\eta_0| t)).
\]  
(3.27)
We note that inspection of (3.27) shows that \( z(t) \neq 0 \) for \( t > 0 \). Thus the only solution with \( z(1) = 0 \) is that of (3.12).

To make more sense of this, let \( r = |\xi_0|/|\eta_0| \) be the radius of the arc swept out by \( x(t) \), and \( \phi = |\eta_0| t \) be the angle subtended by the arc. Then

\[
|z| = \frac{1}{2} \frac{r^2 \phi - 1}{2} \sin \phi
\]

which is the area of the region between an arc of radius \( r \) subtending an angle \( \phi \) and the chord which spans it.

We must determine \( \xi_0, \eta_0 \) in terms of \( x(1), z(1) \). We have already ruled out the case \( z(1) = 0 \). If \( x(1) = 0 \), then (3.25) shows we must have \( |\eta_0| = 2k\pi \) for some integer \( k \geq 1 \). (3.26) then shows \( \eta_0 = 2k\pi z(1)/|z(1)| \), and \( |\xi_0| = \sqrt{4k\pi |z(1)|} \), as desired. In this case the direction of \( \xi_0 \) is not determined and \( \xi_0 \) may be any vector with the given length.

On the other hand, if \( x(1) \neq 0 \), then \( |\eta_0| \) is not an integer multiple of \( 2\pi \), so we may divide (3.27) by (3.25) to obtain

\[
\frac{|z(1)|}{|x(1)|^2} = \frac{|\eta_0| - \sin |\eta_0|}{4(1 - \cos |\eta_0|)} = \frac{1}{4} \nu(\theta)
\]

(3.28) taking \( \theta = \frac{1}{2} |\eta_0| \), where \( \nu \) is as in (3.2). Then by (3.25) we have

\[
|\xi_0|^2 = \frac{1}{2} |x(1)|^2 \frac{|\eta_0|^2}{1 - \cos |\eta_0|} = |x(1)|^2 \frac{\theta^2}{\sin^2 \theta}.
\]

(3.29)

Note that once the magnitudes of \( \eta_0, \xi_0 \) are known, their directions are determined: \( \eta_0 = z(1)|\eta_0|/|z(1)| \) by (3.26), while \( \xi_0 \) can be recovered from (3.28): \( \xi_0 = -\eta_0^2(J_{\eta_0}(e^{j\eta_0} - I)^{-1}x(1)). \)

So \( \eta_0, \xi_0 \) and hence \( x(t), z(t) \) are all determined by a choice of \( |\eta_0| \) satisfying (3.25). Writing \( \theta = |\eta_0| \) gives (3.13) (3.14).

The “if” direction of the theorem requires verifying that the given formulas in fact satisfy Hamilton’s equations, which is routine. □

To prove Theorem 3.5 we must now decide which of the solutions given in Theorem 3.11 is the shortest, and compute its length. We collect, for future reference, some facts about the function \( \nu \) of (3.2).

**Lemma 3.12.** There is a constant \( c > 0 \) such that \( \nu'(\theta) > c \) for all \( \theta \in [0, \pi) \).

**Proof.** By direct computation, \( \nu'(\theta) = \frac{2(\sin \theta - \theta \cos \theta)}{\sin^3 \theta} \). By Taylor expansion of the numerator and denominator we have \( \nu'(0) = 2/3 > 0 \). For all \( \theta \in (0, \pi) \) we have \( \sin^3 \theta > 0 \), so it suffices to consider \( y(\theta) := \sin \theta - \theta \cos \theta \).

Now \( y(0) = 0 \) and \( y'(\theta) = \theta \sin \theta > 0 \) for \( \theta \in (0, \pi) \), so \( y(\theta) > 0 \) for \( \theta \in (0, \pi) \). Thus \( \nu'(\theta) > 0 \) for all \( \theta \in [0, \pi) \), and continuity and the fact that \( \lim_{\theta \uparrow \pi} \nu'(\theta) = +\infty \) establishes the existence of the constant \( c \). □

**Corollary 3.13.** \( \nu(\theta) \geq \theta \) for all \( \theta \in [0, \pi) \), where \( c \) is the constant from Lemma 3.12.

**Proof.** Integrate the inequality in Lemma 3.12. Note that \( \nu(0) = 0 \). □

**Proof of Theorem 3.5** We compute the lengths of the paths given in Lemma 3.11. The \( z = 0 \) case is obvious. Observe that for a horizontal path \( \sigma(t) = (x(t), z(t)) \), we have \( \dot{\sigma}(t) = \sum_{i=1}^{2n} \dot{x_i}(t)X_i(\gamma(t)) \), so that \( \|\dot{\sigma}(t)\| = |\dot{x}(t)| \). For paths solving Hamilton’s equations, (3.24) shows that \( |\dot{x}(t)| = |\xi_0| \), so \( \ell(\gamma) = |\xi_0| \). In the case \( x = 0 \), we have \( |\xi_0| = \sqrt{4k\pi |z(1)|} \), where \( k \) may be any positive integer; clearly this is minimized by taking \( k = 1 \).

Now we must handle the case \( x \neq 0, z \neq 0 \). In this case we have \( \ell(\gamma) = |\xi_0| = |x| \frac{\phi}{\sin \theta} \), by (3.29), where \( \theta \) solves (3.17) (recall \( \theta = \frac{1}{2} |\eta_0| \)). The function \( \nu \) has \( \nu(0) = 0, \nu(\pi) = +\infty \), and by Lemma 3.12 \( \nu \) is strictly
increasing on \([0, \pi]\). Thus among the solutions of \([3.17]\) there is exactly one in \([0, \pi]\). We show this is the solution that minimizes \((\frac{\partial}{\sin \theta})^2\) and hence also minimizes \(\ell(\gamma)\).

For brevity, let \(y = \frac{4\ell'(\gamma)}{|\gamma|}\). If \(y \in [0, \pi/2]\) then \(y = \nu(\theta)\) for a unique \(\theta \in [0, \infty)\). This is because \(\nu(\theta) > \nu(\pi/2) = \pi/2\) for \(\theta > \pi/2\). Since \(\theta\) is increasing on \([0, \pi]\) it suffices to show this for \(\theta > \pi\). But for such \(\theta\) we have

\[
\nu(\theta) = \frac{\theta - \sin \theta \cos \theta}{\sin^2 \theta} \geq \frac{\theta - \frac{1}{2}}{\sin^2 \theta} \geq \frac{\theta - \frac{1}{2}}{\frac{ \pi}{2}} > \pi - \frac{1}{2} > \frac{\pi}{2}
\]

since \(\sin \theta \cos \theta \leq \frac{1}{2}\) for all \(\theta\).

Otherwise, suppose \(y > \pi/2\). Let

\[
F(\theta) := \left(\frac{\theta}{\sin \theta}\right)^2 = \frac{\theta^2}{\theta - \sin \theta \cos \theta}
\]

which is smooth on \((\pi/2, \infty)\) after removing the removable singularities. We will show that if \(\pi/2 < \theta_1 < \pi < \theta_2\), then \(F(\theta_1) < F(\theta_2)\). Thus if \(\theta_1\) is the unique solution to \(y = \nu(\theta)\) in \((\pi/2, \pi)\) and \(\theta_2 > \pi\) is another solution, we will have

\[
\left(\frac{\theta_1}{\sin \theta_1}\right)^2 = \nu(\theta_1)F(\theta_1) = yF(\theta_1) < yF(\theta_2) = \nu(\theta_2)F(\theta_2) = \left(\frac{\theta_2}{\sin \theta_2}\right)^2
\]

Toward this end, we compute

\[
F'(\theta) = \frac{2\theta(\theta - \sin \theta \cos \theta) - \theta^2(1 - \cos^2 \theta + \sin^2 \theta)}{(\theta - \sin \theta \cos \theta)^2} = \frac{2\theta \cos \theta(\theta - \sin \theta - \cos \theta)}{(\theta - \sin \theta \cos \theta)^2}.
\]

For \(\theta \in (\pi/2, \pi)\) we have \(\cos \theta < 0, \sin \theta > 0\) and thus \(F'(\theta) > 0\). So \(F(\pi) = \pi < F(\theta_2)\). We have \(F'(\pi) = 2 > 0\) so this is true for \(\theta_2 > \pi\), and \(F(+\infty) = +\infty\) so it is also true for large \(\theta_2\). To complete the argument we show that it holds at critical points of \(F\). Suppose \(F'(\theta_c) = 0\) where \(\theta_c > \pi\); then either \(\cos \theta_c = 0\) or \(\theta_c \cos \theta_c - \sin \theta_c = 0\). If the former then \(F(\theta_c) = \pi > \theta_c > \pi\). If the latter, then \(\theta_c = \tan \theta_c\), so

\[
F(\theta_c) = \frac{\theta_c^2}{\theta_c - \sin \theta_c \cos \theta_c} = \frac{\theta_c^2}{\theta_c - \tan \theta_c \cos \theta_c} = \frac{\theta_c^2}{\theta_c(1 - \cos^2 \theta_c)} \geq \theta_c > \pi
\]

which completes the proof.

\textbf{Notation 3.14.} If \(f, h : G \to \mathbb{R}\), we write \(f(g) \asymp h(g)\) to mean there exist finite positive constants \(C_1, C_2\) such that \(C_1 h(g) \leq f(g) \leq C_2 h(g)\) for all \(g \in G\), or some specified subset thereof.

\textbf{Corollary 3.15.} \(d(x, z) \asymp |x| + |z|^{1/2}\). Equivalently, \(d(x, z) \asymp |x|^2 + |z|\).

\textbf{Proof.} By continuity we can assume \(x \not= 0, z \not= 0\). If \(\theta\) is the unique solution in \([0, \pi]\) to \(\nu(\theta) = \frac{4|x|}{|x|^2}\), we have \(d(x, z)^2 = |x|^2 \left(\frac{\theta}{\sin \theta}\right)^2\), so if we let

\[
F(\theta) := \left(\frac{\theta}{\sin \theta}\right)^2 = \frac{d(x, z)^2}{|x|^2 + 4|z|} \tag{3.30}
\]

it will be enough to show there exist \(D_1, D_2\) with \(0 < D_1 \leq F(\theta) \leq D_2\) for all \(\theta \in [0, \pi]\). \(F\) is obviously continuous and positive on \((0, \pi)\). We can simplify \(F\) as

\[
F(\theta) = \frac{\theta^2}{\sin^2 \theta + \theta - \sin \theta \cos \theta}
\]
from which it is obvious that \( \lim_{\theta \to \pi} F(\theta) = \pi > 0 \), and easy to compute that \( \lim_{\theta \to 0} F(\theta) = 1 > 0 \), which is sufficient to establish the corollary.

Results of this form apply to general stratified Lie groups. A standard argument, paraphrased from [2], where many more details can be found, is as follows. Once it is known that \( d(0, z) \) is a continuous function which is positive except at \((0, 0), d'(x, z) := |x| + |z|^{1/2} \) is another such function, so the conclusion obviously holds on the unit sphere of \( d' \). Now \( d'(\varphi_\alpha(x, z)) = \alpha d'(x, z) \), and inspection of \( (3.3) \) shows that the same holds for \( d \), so for general \((x, z)\) it suffices to apply the previous statement with \( \alpha = d'(x, z)^{-1} \).

4. The sublaplacian and heat kernel estimates

**Definition 4.1.** The sublaplacian \( L \) for \( G \) is the operator given by

\[
L = \sum_i X_i^2
\]

where \( X_i \) are as given in (2.1). The heat kernel \( p_t \) for \( G \) is the unique fundamental solution to the corresponding heat equation \((L - \frac{\partial}{\partial t})u = 0\); that is, \( p_t = e^{tL} \delta_0 \), where \( \delta_0 \) is the Dirac delta distribution supported at 0.

L is obviously left-invariant. L is not strictly elliptic at any point of \( G \), but it is subelliptic everywhere.

If we view the left-invariant vector fields \( \{X_i\} \) as elements of the Lie algebra \( g \) of \( G \), they are an orthonormal basis for \( \mathfrak{g} \), which generates \( g \); that is, \( \text{span}\{X_i, [X_j, X_k] : i, j, k = 1, \ldots, 2n\} = g \). (It is easy to see that \( L \) does not actually depend on the choice of orthonormal basis \( \{X_i\} \) for \( \mathfrak{g} \), but only on the inner product \( \langle \cdot, \cdot \rangle \) on \( g \).) We thus have \( \text{span}\{X_i(g), [X_j, X_k](g) : i, j, k = 1, \ldots, 2n\} = T_g G \) for each \( g \in G \) (it is obvious for \( g = 0 \), and for other \( g \) it follows by left invariance). Thus the collection of vector fields \( \{X_i\} \) is bracket generating. By a famous theorem of Hörmander ([10]), \( L \) is hypoelliptic; that is, if \( Lu \) is \( C^\infty \) on some open set, then so is \( u \). Another case of Hörmander’s theorem applies to the operator \( L - \frac{\partial}{\partial t} \) on \( G \times (0, \infty) = \{(g, t)\}; \) thus, since \((L - \frac{\partial}{\partial t})p_t = 0 \) is \( C^\infty \), \( p_t \) itself is \( C^\infty \) on \( G \times (0, \infty) \).

Our next step is to record an explicit formula for \( p_t(x, z) \). Various derivations of this formula appear in the literature. For general step 2 nilpotent groups, [9] derived such a formula probabilistically from a formula in [10] regarding the Lévy area process. Another common approach, worked out in [2], involves expressing \( p_t \) as the Fourier transform of the Mehler kernel. [23] has a similar computation. [23] obtains the formula for H-type groups as the Radon transform of the heat kernel for the Heisenberg group. Other approaches have involved complex Hamiltonian mechanics ([1]), magnetic field heat kernels ([12]), and approximation of Brownian motion by random walks ([12]). In our notation, we find that

\[
p_t(x, z) = (2\pi)^{-m}(4\pi)^{-n} \int_{\mathbb{R}^m} e^{i\lambda x} e^{-\frac{i}{4}\lambda|\coth(t|\lambda|)z|^2} \left( \frac{|\lambda|}{\sinh(t|\lambda|)} \right)^{n} \, d\lambda.
\]

We can see directly by making the change of variables \( \lambda = \alpha^2 \lambda' \) (among other means) that

\[
p_t(x, z) = \alpha^{2(m+n)}p_{\alpha^2 t}(\alpha x, \alpha^2 z) = \alpha^{2(m+n)}p_{\alpha^2 t}(\phi_\alpha(x, z)).
\]

In particular, taking \( \alpha = t^{-1/2} \),

\[
p_t(x, z) = t^{-m-n}p_1(t^{1/2}x, tz) = t^{-m-n}p_1(\phi_{1/3}(x, z)).
\]

Therefore an estimate on \( p_1 \) will immediately give an estimate on \( p_t \) for all \( t \), and we study \( p_1 \) from this point onward.
We immediately note that the integrand in (4.2) has even real part and odd imaginary part, so that \( p_1 \) is indeed real. Moreover, being the Fourier transform of a radial function, \( p_1 \) is radial, i.e. \( p_1(x, z) \) depends only on \(|x|\) and \(|z|\). So we can apply (2.7) and differentiate under the integral sign to get

\[
\nabla p_1(x, z) = -\frac{1}{2}(2\pi)^{-m} (4\pi)^{-n} |x| (q_1(x, z)\dot{x} + q_2(x, z) \dot{z})
\]

where

\[
q_1(x, z) = -\frac{2}{|x|} \frac{\partial p_1(x, z)}{\partial |x|} = \int_{\mathbb{R}^n} e^{i(\lambda, z) - \frac{1}{4}|\lambda| \coth |\lambda||x|^2} \left( \frac{|\lambda|}{\sinh(|\lambda|)} \right)^{n+1} \cosh(|\lambda|) \, d\lambda
\]

\[
q_2(x, z) = \frac{\partial p_1(x, z)}{\partial |z|} = \int_{\mathbb{R}^n} e^{i(\lambda, z) - \frac{1}{4}|\lambda| \coth |\lambda||x|^2} \left( \frac{|\lambda|}{\sinh(|\lambda|)} \right)^n (-i) \langle \lambda, \dot{z} \rangle \, d\lambda
\]

As before, (4.6) and (4.7) do not really depend on \( \dot{z} \) but only on \(|x|, |z|\).

We now state the main theorem of this paper: the precise estimates on \( p_t \) and its gradient. The proofs will occupy the remainder of the paper.

**Theorem 4.2.** There exists \( d_0 > 0 \) such that

\[
p_t(x, z) \asymp \frac{d(x, z)^{2n-m-1}}{1 + (|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{1}{2}d(x, z)^2}
\]

for \( d(x, z) \geq d_0 \).

**Corollary 4.3.**

\[
p_t(x, z) \asymp t^{-m-n} \frac{1 + (t^{1/2}d(x, z))^{2n-m-1}}{1 + (t|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{1}{2}d(x, z)^2}
\]

for \((x, z) \in G, t > 0, \) with the implicit constants independent of \( t \) as well as \( x, z \).

**Proof.** Theorem 4.2 establishes (4.9) for \( t = 1 \) and \( d(x, z) \geq d_0 \). For \( d(x, z) \leq d_0 \) the estimate follows from continuity and the fact that \( p_t(x, z) > 0 \). Although the positivity of \( p_t \) is not obvious from inspection of (4.2), it is well known. A proof of this fact could be assembled from the fact that the semigroup \( e^{tL} \) is positive and hence \( p_t \geq 0 \) (see, for instance, Theorem 5.1 of [13]) together with a Harnack inequality such as Theorem III.2.1 of [27] (which is written about positive functions but easily extends to cover those which are nonnegative).

Once (4.9) holds for all \((x, z)\) and \( t = 1 \), (4.3) and (4.4) show that it holds for all \( t \), with the same constants.

We also obtain precise upper and lower estimates on the gradient of the heat kernel. Again we work only on \( d(x, z) \geq d_0 \), and since \( \nabla p_t \) vanishes for \( x = 0 \), it is not as clear how to extend to all of \( G \). However, the upper bound is sufficient to establish (4.11), which is of interest itself.

**Theorem 4.4.** There exists \( d_0 > 0 \) such that

\[
|\nabla p_t(x, z)| \asymp |x| \frac{d(x, z)^{2n-m+1}}{1 + (|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{1}{2}d(x, z)^2}
\]

for \( d(x, z) \geq d_0 \). In particular, we can combine this with the lower bound of Theorem 4.2 to see that there exists \( C > 0 \) such that

\[
|\nabla p_t(x, z)| \leq C(1 + d(x, z))p_t(x, z).
\]

The function \( q_2 \) is of interest in its own right, because it gives the norm of the “vertical gradient” of \( p_t \): \( |q_2| = |[Z_t p_1, \ldots, Z_m p_1]| \). The proof of Theorem 4.3 includes estimates on \( q_2 \); we record here the upper bound.
Theorem 4.5. There exists $d_0 \geq 0$ and a constant $C > 0$ such that
\[
|\langle Z_1 p_1, \ldots, Z_m p_1 \rangle(x, z)| = |q_2(x, z)| \leq C \frac{d(x, z)^{2n-m-1}}{1 + (|x| d(x, z))^{n+m}} e^{-\frac{1}{2}d(x, z)^2}. \quad (4.12)
\]
whenever $d(x, z) \geq d_0$. In particular, for all $(x, z) \in G$ we have
\[
|\langle Z_1 p_1, \ldots, Z_m p_1 \rangle(x, z)| \leq C p(x, z). \quad (4.13)
\]

Remark. Since our estimate is based on analysis of the formula (4.2), we will henceforth treat (4.2) as the definition of a function $p_1$ on $\mathbb{R}^{2n+m}$. In particular, it makes sense for all $n, m$, whether or not an H-type group of the corresponding dimension actually exists (which can be ascertained via Theorem 2.3). The proofs of Theorems 4.2 and 4.4 do not depend on the values of $n$ and $m$, so they likewise remain valid for all $n, m$. The estimates given are in terms of the distance function $d$, which likewise should be taken as a function defined by the formula (3.3). Indeed, the only place where we need $p_1$ to be a heat kernel is in the proof of Corollary 4.3 where we use the positivity of $p_1$ which follows from the general theory.

In particular, in Section 7 we shall make use of estimates on $p_1$ for values of $n, m$ not necessarily corresponding to H-type groups.

The proofs of these two theorems are broken into two cases, depending on the relative sizes of $|x|$ and $|z|$. Section 5 deals with the case when $|z| \lesssim |x|^2$; here we apply a steepest descent type argument to approximate the desired function by a Gaussian. Section 6 handles the case $|z| \gg |x|^2$ by a transformation to polar coordinates and a residue computation which only works for odd $m$. The result for $m$ even can be deduced from that for $m$ odd by a Hadamard descent approach, which is contained in Section 7.

5. Steepest descent

We first handle the region where $|z| \leq B_1 |x|^2$ for some constant $B_1$. If $\theta = \theta(x, z)$ is as in Theorem 3.5 this implies $\nu(\theta) \leq 4B_1$; since $\nu$ increases on $[0, \pi)$ we have $0 \leq \theta \leq \theta_0$ in this region. Note also that by Corollary 3.15 we have $d(x, z)^2 \leq D_2(1 + B_1) |x|^2$, as well as $d(x, z)^2 \geq |x|^2$ which is clear from (3.3). Thus for this region the bounds of Theorems 4.2, 4.4 and 4.5 are implied by the following:

Theorem 5.1. For each constant $B_1 > 0$ there exists $d_0 > 0$ such that
\[
p_1(x, z) \approx \frac{1}{|x|} e^{-\frac{1}{2}d(x, z)^2} \quad (5.1)
\]
\[
|q_i(x, z)| \leq C_i \frac{1}{|x|} e^{-\frac{1}{2}d(x, z)^2}, \quad i = 1, 2 \quad (5.2)
\]
\[
C_i \frac{1}{|x|} e^{-\frac{1}{2}d(x, z)^2} \leq \max\{|q_1(x, z)|, |q_2(x, z)|\} \quad (5.3)
\]
for all $x, z$ with $d(x, z) \geq d_0$ and $|z| \leq B_1 |x|^2$.

Our approach here will be a steepest descent argument. Very informally, the motivation is as follows: given a function $F(x) = \int_{\mathbb{R}} e^{-x^2 f(\lambda)} a(\lambda) d\lambda$, move the contour of integration to a new contour $\Gamma$ which passes through a critical point $\lambda_c$ of $f$, so that $f(\lambda) \approx f(\lambda_c) + \frac{1}{2} f''(\lambda_c)(\lambda - \lambda_c)^2$. Then we have
\[
F(x) \approx e^{-x^2 f(\lambda_c)} \int_{\Gamma} e^{-x^2 f''(\lambda_c)(\lambda - \lambda_c)^2/2} a(\lambda) d\lambda.
\]
For large $x$ the integrand looks like a Gaussian concentrated near $\lambda_c$, so $F(x) \approx e^{-x^2 f(\lambda_c)} \frac{a(\lambda_c)}{x \sqrt{f''(\lambda_c)}}$. Our proof essentially follows this line, in $\mathbb{R}^m$ instead of $\mathbb{R}$, but more care is required to establish the desired uniformity.
Our first task is to extend the integrand to a meromorphic function on \( \mathbb{C}^m \), so that we may justify moving the contour of integration.

Let \( \cdot \) denote the bilinear (not sesquilinear) dot product on \( \mathbb{C}^m \), and for \( \lambda \in \mathbb{C}^m \) write \( \lambda^2 := \lambda \cdot \lambda \); this defines an analytic function from \( \mathbb{C}^m \) to \( \mathbb{C} \), and \( \lambda^2 = |\lambda|^2 \) iff \( \lambda \in \mathbb{R}^m \). For \( w \in \mathbb{C} \), let \( \sqrt{w} \) denote the branch of the square root function satisfying \( \text{Im} \sqrt{w} \geq 0 \) and \( \sqrt{w} > 0 \) for \( w > 0 \) (so the branch cut is the positive real axis). Thus if \( g : \mathbb{C} \to \mathbb{C} \) is an analytic even function, \( \lambda \mapsto g(\sqrt{\lambda^2}) \) is analytic as well, and satisfies \( g(\sqrt{\lambda^2}) = g(|\lambda|) \) for \( \lambda \in \mathbb{R}^m \). This holds in particular for the function \( \frac{\sinh w}{\sinh \sqrt{\lambda^2}} \) and \( \sqrt{\lambda^2} \coth \sqrt{\lambda^2} \) are analytic away from points with \( \sqrt{\lambda^2} = ik\pi, \ k = 1, 2, \ldots \).

Using this notation, we let

\[
a_0(\lambda) := \left( \frac{\sqrt{\lambda^2}}{\sinh \sqrt{\lambda^2}} \right)^n \quad a_1(\lambda) := \cosh \sqrt{\lambda^2} \left( \frac{\sqrt{\lambda^2}}{\sinh \sqrt{\lambda^2}} \right)^{n+1} \quad a_2(\lambda) := -i \left( \frac{\sqrt{\lambda^2}}{\sinh \sqrt{\lambda^2}} \right)^{n} \lambda \cdot \hat{\lambda} \in \mathbb{C}^{2n}.
\]

As mentioned previously, \( \hat{\lambda} \) may be any unit vector in \( \mathbb{R}^m \) without affecting the computation. Therefore we shall treat it as fixed, while \( |\lambda| \) is allowed to vary.

Also, for \( \lambda \in \mathbb{C}^m, \theta \in [0, \theta_0], \hat{\lambda} \in \mathbb{S}^{m-1} \subset \mathbb{R}^m \), we define

\[
f(\lambda, \theta, \hat{\lambda}) := -i\nu(\theta) \lambda \cdot \hat{\lambda} + \sqrt{\lambda^2} \coth \sqrt{\lambda^2} \quad (5.4)
\]

so that

\[
\frac{|x|^2}{4} f(\lambda, \theta(x, z), \frac{z}{|z|}) = -i \lambda \cdot z + \frac{1}{4} \sqrt{\lambda^2} \coth \sqrt{\lambda^2} |x|^2.
\]

We henceforth write \( \theta \) for \( \theta(x, z) \). Thus we now have

\[
p_1(x, z) = (4\pi)^{-m-n} \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4} f(\lambda, \theta, \hat{\lambda})} a_0(\lambda) \, d\lambda \quad (5.5)
\]

\[
q_i(x, z) = (4\pi)^{-m-n} \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4} f(\lambda, \theta, \hat{\lambda})} a_i(\lambda) \, d\lambda, \quad i = 1, 2 \quad (5.6)
\]

Written thus, the integrands have obvious meromorphic extensions to \( \lambda \in \mathbb{C}^m \), analytic away from the set \( \{ \sqrt{\lambda^2} = ik\pi, \ k = 1, 2, \ldots \} \).

A simple calculation verifies that \( \frac{d}{d\lambda} w \coth w = i\nu(-i\nu) \), so we can compute the gradient of \( f \) with respect to \( \lambda \) as

\[
\nabla_{\lambda} f(\lambda, \theta, \hat{\lambda}) = -i\nu(\theta) \hat{\lambda} + i\nu(-i\nu) \hat{\lambda} \quad (5.7)
\]

which vanishes when \( \lambda = i\theta \hat{\lambda} \). Thus \( i\theta \hat{\lambda} \) is the desired critical point. We observe that

\[
f(i\theta \hat{\lambda}, \theta, \hat{\lambda}) = \theta \nu(\theta) + i\theta \coth(i\theta) = \theta(\nu(\theta) + \cot(\theta)) = \frac{\theta^2}{\sin^2 \theta} \quad (5.8)
\]

so by (5.8),

\[
|x|^2 f(i\theta \hat{\lambda}, \theta, \hat{\lambda}) = d(x, z)^2. \quad (5.9)
\]

Thus we define

\[
\psi(\lambda, \theta, \hat{\lambda}) := f(\lambda, \theta, \hat{\lambda}) - f(i\theta \hat{\lambda}, \theta, \hat{\lambda}) = -i\nu(\theta) \lambda \cdot \hat{\lambda} + \sqrt{\lambda^2} \coth \sqrt{\lambda^2} - \frac{\theta^2}{\sin^2 \theta} \quad (5.10)
\]
We then have
\[ p_t(x, z) = (4\pi)^{m-n} e^{-d(x,z)^2/4} \int_{\mathbb{R}^m} e^{-i\frac{\lambda^2}{4}} \psi(\lambda, \theta, \hat{z}) a_0(\lambda) \, d\lambda \] (5.11)
and analogous formulas for \( q_1, q_2 \). Thus let
\[ h_t(x, z) := \int \mathbb{R}^m e^{-i\frac{\lambda^2}{4}} \psi(\lambda, \theta, \hat{z}) a_t(\lambda) \, d\lambda. \] (5.12)
It will now suffice to estimate \( h_t \).

The first step in the steepest descent method is to move the “contour” of integration to pass through \( i\theta \hat{z} \). Some preliminary computations are in order.

**Lemma 5.2.** For \( a, b \in \mathbb{R}^m \), we have
\[ |a| - |b| \leq \left| \text{Re} \sqrt{(a+bi)^2} \right| \leq |a|, \quad 0 \leq \text{Im} \sqrt{(a+bi)^2} \leq |b|. \] (5.13)
Equality holds in the upper bounds if and only if \( a \) and \( b \) are parallel, i.e. \( a = rb \) for some \( r \in \mathbb{R} \).

**Proof.** First note that \((a+bi)^2 = |a|^2 - |b|^2 + 2ia \cdot b \). So by the Cauchy-Schwarz inequality,
\[ |(a+bi)^2|^2 = (|a|^2 - |b|^2)^2 + (2a \cdot b)^2 \]
\[ \leq (|a|^2 - |b|^2)^2 + 4|a|^2|b|^2 \]
\[ = (|a|^2 + |b|^2)^2 \] (5.14)
so that \(|(a+bi)^2| \leq |a|^2 + |b|^2 \). Equality holds in the Cauchy-Schwarz inequality iff \( a \) and \( b \) are parallel. On the other hand,
\[ |(a+bi)^2| \geq \text{Re}(a+bi)^2 = |a|^2 - |b|^2. \] (5.15)
Now we can write
\[ \left( \text{Re} \sqrt{(a+bi)^2} \right)^2 = \frac{1}{4} \left( \sqrt{(a+bi)^2} + \sqrt{(a+bi)^2} \right)^2 \]
\[ = \frac{1}{4} \left( (a+bi)^2 + (a+bi)^2 + 2 \sqrt{(a+bi)^2} \right) \]
\[ = \frac{1}{2} (|a|^2 - |b|^2 + |(a+bi)^2|). \]
The upper bound for \( \left| \text{Re} \sqrt{(a+bi)^2} \right| \) then follows from (5.14). The lower bound is trivial if \( |a| \leq |b| \), and otherwise we have by (5.15) that
\[ \left( \text{Re} \sqrt{(a+bi)^2} \right)^2 \geq |a|^2 - |b|^2 \geq (|a| - |b|)^2. \]
The lower bound for \( \text{Im} \sqrt{(a+bi)^2} \) holds by our definition of \( \sqrt{\cdot} \), and the upper bound is similar to the previous one.

**Lemma 5.3.** For each \( \theta_0 \in [0, \pi) \) there exists \( c(\theta_0) > 0 \) such that if \( a, b \in \mathbb{R}^n \) with \( |a| \geq c(\theta_0), |b| \leq 2\pi \), we have
\[ \text{Re} \psi(a+ib, \theta, \hat{z}) \geq |a|/2 \] (5.16)
and
\[ |a_t(a+ib)| \leq 1 \] (5.17)
for all \( \theta \in [0, \theta_0] \), \( \hat{z}, \hat{\hat{z}} \in S^{m-1} \subset \mathbb{R}^m \).
Lemma 5.3. We have
\[ \left| \frac{\nu(b \cdot \hat{z} - \Re f(i\theta \hat{z}, \theta, \hat{z}) + \Re \sqrt{(a + bi)^2 \coth (a + bi)^2}}{a} \right| \geq \frac{2}{3} |a|. \] (5.18)
By continuity, \( \nu(b \cdot \hat{z} - \Re f(i\theta \hat{z}, \theta, \hat{z}) \) is bounded below by some constant independent of \( a \) for all \( \theta \in [0, \theta_0], |b| \leq 2\pi \). Thus it suffices to show that for sufficiently large \( |a| \),

\[ \Re \left[ \sqrt{(a + bi)^2 \coth (a + bi)^2} \right] \geq \frac{2}{3} |a|. \] (5.19)
Now for \( \alpha \in \mathbb{R}, \beta \in [-2\pi, 2\pi] \) we have
\[
\Re((\alpha + i\beta) \coth(\alpha + i\beta)) = \frac{\alpha \sinh \alpha \cosh \alpha + \beta \sin \beta \cos \beta}{\cosh^2 \alpha - \cos^2 \beta} \geq 0 \cosh \alpha - \frac{\beta}{\cosh^2 \alpha} \geq 0 \cosh \alpha - \frac{2\pi}{\cosh^2 \alpha} \geq \frac{3}{4} |\alpha|
\]
for sufficiently large \( |\alpha| \). (Recall that \( \lim_{\alpha \to \pm \infty} \coth \alpha = \pm 1 \).) Thus, since
\[
\left| \Re \sqrt{(a + bi)^2} \right| \geq |a| - |b| \geq |a| - 2\pi
\]
and
\[
\left| \Im \sqrt{(a + bi)^2} \right| \leq 2\pi,
\]
it is clear that \( (5.19) \) holds for sufficiently large \( |a| \).

For the bound on \( a_i \), note that the sinh factor in the denominator of each \( a_i \) can be estimated by
\[
|\sinh(\alpha + i\beta)| = \left| \frac{e^{\alpha + i\beta} - e^{-\alpha + i\beta}}{2} \right| \geq \left| \frac{e^{\alpha + i\beta} - |e^{-\alpha + i\beta}|}{2} \right| = |\sinh \alpha|
\]
so that \( |\sinh \sqrt{(a + bi)^2}| \geq |\sinh \Re \sqrt{(a + bi)^2}| \geq |\sinh(|a| - 2\pi)| \) for \( |a| \geq 2\pi \). This grows exponentially with \( |a| \), so it certainly dominates the polynomial growth of the numerator, and we have \( |a_i(a + ib)| \leq 1 \) for large enough \( |a| \).

Lemma 5.4. Let \( F(\lambda) := e^{-\frac{|x|^2}{2} \psi(\lambda, \theta, \hat{z})} a_i(\lambda) \) be the integrand in \( (5.12) \), where \( x, z \) are fixed. If \( \tau \in \mathbb{R}^m \) with \( |\tau| < \pi \), then
\[
h_i(x, z) = \int_{\mathbb{R}^m} F(\lambda) \, d\lambda = \int_{\mathbb{R}^m} F(\lambda + i\tau) \, d\lambda. \] (5.20)
Proof. Note first that \( F \) is analytic at \( \lambda + ib \) when \( |b| < \pi \), by the second inequality in Lemma 5.3. Also, by Lemma 5.3 we have
\[
|F(\lambda + ib)| \leq e^{-|x|^2|\lambda|^2/8} \] (5.21)
as soon as \( |\lambda| > c(\theta) \).
We view \( \int_{\mathbb{R}^m} F(\lambda) \, d\lambda \) as \( m \) iterated integrals and handle them one at a time. For \( 1 \leq k \leq m \), suppose we have shown that
\[
\int_{\mathbb{R}^m} F(\lambda) \, d\lambda = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} F(\lambda_1 + i\tau_1, \ldots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \ldots, \lambda_m) \, d\lambda_1 \ldots d\lambda_m. \] (5.22)
Continuity of $F$ and (5.21) show that $F$ is integrable, so we may apply Fubini’s theorem and evaluate the $d\lambda_k$ integral first:

$$\int_{\mathbb{R}^m} F(\lambda) \, d\lambda = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k, \ldots, \lambda_m) \, d\lambda_k \, d\lambda_1 \ldots d\lambda_m.$$  

Now

$$\int_{\mathbb{R}} F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k, \ldots, \lambda_m) \, d\lambda_k = \lim_{a \to \infty} \int_{-a}^{a} F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k, \ldots, \lambda_m) \, d\lambda_k.$$  

Since $\lambda_k \mapsto F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k, \ldots, \lambda_m)$ is analytic for $|\text{Im} \lambda_k| \leq \tau_k$ (which holds because $|\tau_1, \ldots, \tau_k| \leq |\tau| < \pi$), we have

$$\int_{-a}^{a} F(\ldots, \lambda_k, \ldots) \, d\lambda_k = \int_{-a}^{-a+i\tau_k} F + \int_{-a+i\tau_k}^{a+i\tau_k} F + \int_{a+i\tau_k}^{a} F$$

where the contour integrals are taken along straight (horizontal or vertical) lines. But as soon as $a$ exceeds $c(\theta)$ from Lemma 5.3 (5.21) gives

$$\int_{-a}^{-a+i\tau_k} |F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k, \ldots, \lambda_m)| \, d\lambda_k \leq \tau_k e^{-|x|^2|\alpha, \lambda_1, \ldots, \lambda_m|/8} \leq C e^{-|x|^2|\alpha|/8} \to 0 \text{ as } a \to \infty.$$  

A similar argument shows the same for $\int_{a+i\tau_k}^{a}$, so we have

$$\int_{a+i\tau_k}^{\infty} F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k, \ldots, \lambda_m) \, d\lambda_k = \int_{\infty+i\tau_k}^{\infty} F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k, \ldots, \lambda_m) \, d\lambda_k$$

Thus applying Fubini’s theorem again, we have shown

$$\int_{\mathbb{R}^m} F(\lambda) \, d\lambda = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} F(\lambda_1 + i\pi_1, \ldots, \lambda_{k-1} + i\pi_{k-1}, \lambda_k + i\pi_k, \ldots, \lambda_m) \, d\lambda_1 \ldots d\lambda_m.$$ (5.23)

Applying this argument successively for $k = 1, 2, \ldots, m$ establishes the lemma. □

For the remainder of this section, we assume that $|z| \leq B_1 |x|^2$, so that $\theta \leq \theta_0(B_1)$. We next show that the contribution from $\lambda$ far from the origin is negligible.

**Lemma 5.5.** There exist $r > 0$ and a constant $C > 0$ such that

$$\left| \int_{B(0,r)^c} e^{-|x|^2/4} \psi(\lambda+i\theta x, z) \alpha_i(\lambda+i\theta \hat{z}) \, d\lambda \right| \leq \frac{C}{|x|^m}.$$ (5.24)

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Lemma 5.6. Let \( \rho \) be measurable. Define \( C, B \) be positive constants such that for all \( x \in \mathbb{R}^n \), \( x, \lambda, \sigma \in \Sigma \) we have

\[
F(x, \lambda, \sigma) := \inf_{\rho \in (0, r)} e^{-|x|^2 \lambda^2} k(x, \lambda, \sigma) d\lambda.
\]

Suppose:

1. There exists a positive constant \( b_1 \) such that \( g(\lambda, \sigma) \geq b_1 |\lambda|^2 \) for all \( \lambda \in B(0, r), \sigma \in \Sigma \);
2. \( k \) is bounded, i.e. \( k_2 := \sup_{x \in \mathbb{R}^n, \lambda \in B(0, r), \sigma \in \Sigma} |k(x, \lambda, \sigma)| < \infty \).

Then there exists a positive constant \( C_2' \) such that

\[
|F(x, \sigma)| \leq \frac{C_2'}{|x|^{m/2}}.
\]

for all \( x > 0, \sigma \in \Sigma \).

If additionally we have:

3. There exists a positive constant \( b_2 \) such that \( g(\lambda, \sigma) \leq b_2 |\lambda|^2 \) for all \( \lambda \in B(0, r), \sigma \in \Sigma \);
4. There exists a function \( \epsilon : \mathbb{R}^+ \rightarrow [0, r] \) such that \( \lim_{\rho \rightarrow +\infty} \rho \epsilon(\rho) = +\infty \), and

\[
k_1 := \inf_{x \in \mathbb{R}^n, \lambda \in B(0, \epsilon(|x|)), \sigma \in \Sigma} \text{Re} \, k(x, \lambda, \sigma) > 0.
\]

Then there exist positive constants \( C_3' \) and \( x_0 \) such that for all \( |x| \geq x_0 \) and \( \sigma \in \Sigma \) we have

\[
\text{Re} \, F(x, \sigma) \geq \frac{C_3'}{|x|^{m/2}}.
\]

Proof. The upper bound is easy, since

\[
|F(x, \sigma)| \leq k_2 \int_{B(0, r)} e^{-|x|^2 b_1 |\lambda|^2} d\lambda
\]

\[
= \frac{k_2}{|x|^m} \int_{B(0, r)} e^{-b_1 |\lambda|^2} d\lambda
\]

\[
\leq \frac{k_2}{|x|^m} \int_{\mathbb{R}^m} e^{-b_1 |\lambda|^2} d\lambda
\]

\[
= \frac{k_2(\pi/b_1)^{m/2}}{|x|^{m/2}}.
\]

where \( \omega_{m-1} \) is the hypersurface measure of \( S^{m-1} \).

We can now apply a steepest descent argument. As a similar argument will be used later in this paper (see Proposition 5.7), we encapsulate it in the following lemma.
For the lower bound, let

\[ F_1(x, \sigma) := \int_{B(0,r) \setminus B(0,\epsilon|x|)} e^{-|x|^2 g(\lambda, \sigma)} k(x, \lambda, \sigma) d\lambda \]

\[ F_2(x, \sigma) := \int_{B(0,\epsilon|x|)} e^{-|x|^2 g(\lambda, \sigma)} k(x, \lambda, \sigma) d\lambda \]

so that \( F = F_1 + F_2 \). Now we have

\[
\begin{align*}
|F_1(x, \sigma)| &\leq k_2 \int_{B(0,r) \setminus B(0,\epsilon|x|)} e^{-|x|^2 b_1 |\lambda|^2} d\lambda \\
&\leq k_2 \int_{\mathbb{R}^m \setminus B(0,\epsilon|x|)} e^{-|x|^2 b_1 |\lambda|^2} d\lambda \\
&\leq \frac{k_2}{|x|^m} \int_{\mathbb{R}^m \setminus B(0,|x|\epsilon(|x|))} e^{-b_1 |\lambda'|^2} d\lambda'.
\end{align*}
\]

where we make the change of variables \( \lambda' = |x| \lambda \). For \( F_2 \) we have

\[
\text{Re} \ F_2(x, \sigma) \geq k_1 \int_{B(0,\epsilon|x|)} e^{-|x|^2 b_1 |\lambda|^2} d\lambda = \frac{1}{|x|^m} k_1 \int_{B(0,|x|\epsilon(|x|))} e^{-b_1 |\lambda'|^2} d\lambda'.
\]

So we have

\[
|x|^m \text{ Re} \ F(x, \sigma) \geq |x|^m \text{ Re} \ F_2(x, \sigma) - |x|^m |F_1(x, \sigma)|
\]

\[
\begin{align*}
&\geq k_1 \int_{B(0,|x|\epsilon(|x|))} e^{-b_1 |\lambda'|^2} d\lambda' - k_2 \int_{\mathbb{R}^m \setminus B(0,|x|\epsilon(|x|))} e^{-b_1 |\lambda'|^2} d\lambda' \\
&\to k_1 (\pi/b_2)^{m/2} - 0 > 0
\end{align*}
\]

as \( |x| \to \infty \). So there exists \( x_0 \) so large that for all \( |x| \geq x_0 \),

\[
\text{Re} \ F(x, \sigma) \geq \frac{1}{2} k_1 (\pi/b_2)^{m/2} \frac{1}{|x|^m}
\]

(5.29)

as desired. \( \square \)

We need another computation before being able to apply this lemma.

**Lemma 5.7.** \( \Re \sqrt{(\lambda + i\theta \bar{z})^2 \coth \sqrt{(\lambda + i\theta \bar{z})^2}} \geq \theta \cot \theta, \) with equality iff \( \lambda = 0 \).

**Proof.** We first note that the function \( \beta \cot \beta \) is strictly decreasing on \([0, \pi]\). To see this, note \( \frac{d}{d\beta} \beta \cot \beta = -\nu(\beta) \). By Corollary 3.13, \( \nu(\beta) > 0 \). In particular, \( \beta \cot \beta \leq 1 \).

Next we observe that for \( \alpha \in \mathbb{R}, \beta \in [0, \pi) \) we have

\[
\text{Re}((\alpha + i\beta) \coth(\alpha + i\beta)) \geq \beta \cot \beta
\]

(5.30)

with equality iff \( \alpha = 0 \). This can be seen by verifying that

\[
\text{Re}((\alpha + i\beta) \coth(\alpha + i\beta)) - \beta \cot \beta = \frac{\sinh^2 \alpha (\alpha \coth \alpha - \beta \cot \beta)}{\cosh^2 \alpha - \cos^2 \beta}
\]

(5.31)

which is a product of positive terms when \( \alpha \neq 0 \), since \( \alpha \coth \alpha > 1 \geq \beta \cot \beta \) and \( \cosh^2 \alpha > 1 \geq \cos^2 \beta \).
Therefore, we have
\[
\text{Re} \sqrt{(\lambda + i\theta \hat{z})^2} \coth \sqrt{(\lambda + i\theta \hat{z})^2} \geq \left( \text{Im} \sqrt{(\lambda + i\theta \hat{z})^2} \right) \cot \left( \text{Im} \sqrt{(\lambda + i\theta \hat{z})^2} \right)
\]
(5.32)
\[
\geq \theta \cot \theta
\]
(5.33)
because \(0 \leq \text{Im} \sqrt{(\lambda + i\theta \hat{z})^2} < \pi\) by Lemma 5.2.

If equality holds in (5.33), it must be that \(\text{Im} \sqrt{(\lambda + i\theta \hat{z})^2} = \theta\). By Lemma 5.2 \(\lambda\) and \(\hat{z}\) are parallel, so \(\sqrt{(\lambda + i\theta \hat{z})^2} = \pm |\lambda| + i\theta\). If equality also holds in (5.32), we have
\[
\text{Re}(\pm |\lambda| + i\theta) \coth(\pm |\lambda| + i\theta) = \theta \cot \theta
\]
so by (5.30) it must be that \(|\lambda| = 0\). This proves the claim. \(\square\)

**Lemma 5.8.** Given \(r > 0\), there exist constants \(b_1, b_2, b_3 > 0\) depending only on \(r\) and \(\theta_0\) such that
\[
b_1 |\lambda|^2 \leq \text{Re} \psi(\lambda + i\theta \hat{z}, 0, \hat{z}) \leq b_2 |\lambda|^2
\]
(5.34)
and
\[
|\text{Im} \psi(\lambda + i\theta \hat{z}, 0, \hat{z})| \leq b_3 |\lambda|^3
\]
(5.35)
for all \(\lambda \in B(0, r) \subset \mathbb{R}^m, \theta \in [0, \theta_0], \hat{z} \in S^{m-1} \subset \mathbb{R}^m\).

**Proof.** Note first that \(\psi(\lambda + i\theta \hat{z}, \theta, \hat{z})\) is smooth for \(\theta \in [0, \theta_0]\) since \(\text{Im} \sqrt{(\lambda + i\theta \hat{z})^2} \leq \theta \leq \theta_0 < \pi\), so that we are avoiding the singularities of \(w \coth w\).

We have \(\psi(i\theta \hat{z}, \theta, \hat{z}) = 0\) and \(\nabla_{\lambda} \psi(i\theta \hat{z}, \theta, \hat{z}) = 0\). We now show the Hessian \(H(i\theta \hat{z})\) of \(\psi\) at \(i\theta \hat{z}\) is real and uniformly positive definite.

By direct computation, we can find
\[
\frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \psi(\lambda, \theta, \hat{z}) = \nu'(-i\sqrt{\lambda^2}) \frac{\lambda_i \lambda_j}{\lambda^2} + i \frac{\nu'(-i\sqrt{\lambda^2})}{\sqrt{\lambda^2}} \left( \delta_{ij} - \frac{\lambda_i \lambda_j}{\lambda^2} \right)
\]
(5.36)
so that for \(u \in \mathbb{R}^m\),
\[
H(\lambda)u \cdot u = \nu'(-i\sqrt{\lambda^2}) \frac{(\lambda \cdot u)^2}{\lambda^2} + i \frac{\nu'(-i\sqrt{\lambda^2})}{\sqrt{\lambda^2}} \left( |u|^2 - \frac{(\lambda \cdot u)^2}{\lambda^2} \right)
\]
(5.37)
and in particular
\[
H(i\theta \hat{z})u \cdot u = \nu'(\theta)(\hat{z} \cdot u)^2 + \frac{\nu(\theta)}{\theta} \left( |u|^2 - \hat{z} \cdot u \right) \frac{\nu'(\theta)}{\theta} \left( |u|^2 - \hat{z} \cdot u \right)
\]
\[
= |u|^2 \left( s \nu'(\theta) + \frac{\nu(\theta)}{\theta} (1 - s) \right)
\]
where \(s := \left( \frac{\hat{z} \cdot u}{|u|^2} \right)^2\), so \(0 \leq s \leq 1\). Note this is a real number whenever \(u \in \mathbb{R}^m\). Thus we have \(H(i\theta \hat{z})u \cdot u\) written as a convex combination of two real functions of \(\theta\), so
\[
H(i\theta \hat{z})u \cdot u \geq |u|^2 \min \left\{ \frac{\nu(\theta)}{\theta}, \nu'(\theta) \right\} \geq c |u|^2
\]
(5.38)
where \(c\) is the lesser of the two constants provided by Lemma 5.12 and Corollary 5.13 respectively. This is valid for \(\theta > 0\) and hence by continuity also for \(\theta = 0\).
By Taylor’s theorem, this shows that \((5.34)\) and \((5.35)\) hold for small \(\lambda\). The upper bounds thus automatically hold for all \(\lambda \in B(0, r)\) by continuity. To obtain the lower bound on \(\Re \psi\), it will suffice to show \(\Re \psi > 0\) for all \(\lambda \neq 0\). But we have

\[
\Re \psi(\lambda + i\theta \hat{z}, \hat{z}) = \theta \nu(\theta) - \Re f(i\theta \hat{z}, \hat{z}) + \Re \left[ \sqrt{(\lambda + i\theta \hat{z})^2} \coth \left( (\lambda + i\theta \hat{z})^2 \right) \right]
\]

\[
= \theta \nu(\theta) - \frac{\theta^2}{\sin^2 \theta} + \Re \left[ \sqrt{(\lambda + i\theta \hat{z})^2} \coth \left( (\lambda + i\theta \hat{z})^2 \right) \right]
\]

\[
= -\theta \cot \theta + \Re \left[ \sqrt{(\lambda + i\theta \hat{z})^2} \coth \left( (\lambda + i\theta \hat{z})^2 \right) \right]
\]

\[
\geq 0
\]

by Lemma \(5.7\) with equality iff \(\lambda = 0\). \(\Box\)

The proof of Theorem \(5.1\) can now be completed.

**Proof of Theorem \(5.1\)**. We establish \((5.1)\) first. We can apply Lemma \(5.6\) with \(\Sigma := [0, \theta_0] \times S^{m-1}\), \(\sigma = (\theta, \hat{z})\), \(r\) the value from Lemma \(5.5\) and

\[
g(\lambda, (\theta, \hat{z})) := \frac{1}{4} \Re \psi(\lambda + i\theta \hat{z}, \hat{z})
\]

\[
k(x, \lambda, (\theta, \hat{z})) := e^{i\lambda^2} \Im \psi(\lambda + i\theta \hat{z}, \hat{z}) a_0(\lambda + i\theta \hat{z}).
\]

The necessary bounds on \(g\) come from \((5.34)\). For an upper bound on \(k\), we have \(|k(x, \lambda, (\theta, \hat{z}))| = |a_0(\lambda + i\theta \hat{z})|\), which is bounded by the fact that \((\lambda, \theta, \hat{z})\) ranges over the bounded region \(B(0, r) \times [0, \theta_0] \times S^{m-1}\) which avoids the singularities of \(a_0\).

Now for the lower bound on \(k\). By direct computation, we have \(a_0(i\theta \hat{z}) = \left( \frac{\theta}{\sin \theta} \right)^n \geq 1\); by continuity there exists \(\delta\) such that \(\Re e^{i\lambda^2} a_0(\lambda + i\theta \hat{z}) \geq \frac{1}{2}\) for all \(|\lambda| \leq \delta\) and \(|s| \leq \delta\), where \(s \in \mathbb{R}\). If \(|\lambda| \leq |x|^{-2/3} \delta / b_3\), where \(b_3\) is as in \((5.36)\), we will have \(|x|^2 |\Im \psi(\lambda + i\theta \hat{z})| \leq \delta\). Thus set \(\epsilon(x) := \min \{ \delta, |x|^{-2/3} \delta / b_3 \}\), so that \(\Re k(x, \lambda, (\theta, \hat{z}) \geq \frac{1}{2}\) for all \(|\lambda| \leq \epsilon(x)\) and all \((\theta, \hat{z}) \in \Sigma\), and \(\lim_{x \to \infty} \epsilon(x) = \lim_{x \to \infty} \rho(x) = +\infty\).

Thus Lemma \(5.6\) applies, and so combining it with Lemmas \(5.2\) and \(5.5\) we have that there exist positive constants \(C, C_1, C_2, x_0\) such that

\[
\left( \frac{C_1}{|x|^{2m}} - \frac{C}{|x|^{2m}} \right) e^{-\frac{1}{4} \epsilon(x)^2} \leq p_1(x, \theta) \leq \left( \frac{C_1'}{|x|^{2m}} + \frac{C'}{|x|^{2m}} \right) e^{-\frac{1}{4} \epsilon(x)^2}.
\]

whenever \(|x| \geq x_0\). We can choose \(x_0\) larger if necessary so that \(|x|^{-m} \gg |x|^{-2m}\). Then taking \(d_0 = x_0\) will establish \((5.1)\).

For \(q_1\), the upper bound is similar; \(|a_1|\) is bounded above just like \(|a_0|\), establishing \((5.2)\).

For \((5.3)\), we cannot necessarily bound both \(|q_1|\) below simultaneously, but it suffices to take them one at a time. For \(0 \leq \theta(x, z) \leq \frac{\pi}{2}\), we have \(a_1(i\theta \hat{z}) = \cos \theta \left( \frac{\theta}{\sin \theta} \right)^{n+1} \geq \frac{1}{\sqrt{2}}\), so by the above logic we obtain the desired lower bound on \(|q_1|\) for such \(\theta\). If \(\frac{\pi}{2} \leq \theta \leq \theta_0\), we estimate \(q_2\) in the same way, since we have \(a_2(i\theta \hat{z}) = \left( \frac{\theta}{\sin \theta} \right)^n \theta \geq \frac{\pi}{2}\). \(\Box\)

**6. Polar coordinates**

In this section, we obtain estimates for \(p_1(x, z)\) and \(|\nabla p_1(x, z)|\) when \(|z| \geq B_1 |x|^2\), where \(B_1\) is sufficiently large. This means that \(\theta(x, z) \geq \theta_0\) for some \(\theta_0\) near \(\pi\). Note that by Corollary \(3.1\) we have \(d(x, z) \approx \sqrt{|z|}\) in this region.

We first consider \(p_1\) and show the following.
Theorem 6.1. For $m$ odd, there exist constants $B_1, d_0$ such that

$$p_1(x, z) = \frac{|x|^2 - m^2}{1 + (|x| \sqrt{|z|})^{n-m/2}} e^{-\frac{1}{2}d(x, z)^2}$$

(6.1)

or, equivalently,

$$p_1(x, z) = \frac{d(x, z)^{2n-m-1}}{1 + (|x| d(x, z))^n} e^{-\frac{1}{2}d(x, z)^2}$$

(6.2)

for $|z| \geq B_1 |x|^2$ and $|z| \geq d_0$ (equivalently, $d(x, z) \geq d_0$).

The effect of the requirement that $|z| \leq B_1 |x|^2$ in the previous section was to ensure that the critical point $i \theta \hat{z}$ stayed away from the singularities of the integrand. As $B_1 \to \infty$, the critical point approaches the set of singularities, and the change of contour we used is no longer effective; the constants in the estimates of Theorem 6.1 blow up. In the case of the Heisenberg groups, where the center of $G$ has dimension $m = 1$, the singularity is a single point, and the technique used in [11] and [1] is to move the contour past the singularity and concentrate on the resulting residue term. For $m > 1$, the singularities form a large manifold and this technique is not easy to use directly. However, by making a change to polar coordinates, we can reduce the integral over $\mathbb{R}^m$ to one over $\mathbb{R}$; this replaces the Fourier transform by the so-called Hankel transform. (A similar approach is used in [23] in the context of $L^p$ estimates for the analytic continuation of $p_1$.) When $m$ is odd, we recover a formula very similar to that for $m = 1$, and the above-mentioned technique is again applicable.

For the rest of this section, we assume that $m$ is odd.

For $m \geq 3$, we write (6.2) in polar coordinates to obtain

$$p_1(x, z) = \frac{(2\pi)^{-m} (4\pi)^{-n}}{2} \int_{S^{m-1}} e^{i\sigma \cdot z} d\sigma e^{-\frac{|z|^2}{4} \rho \coth \rho} \left( \frac{\rho}{\sinh \rho} \right)^n \rho^{m-1} d\rho$$

(6.3)

$$= \frac{(2\pi)^{-m} (4\pi)^{-n}}{2} \int_{\mathbb{R}} e^{i\rho \cdot z} d\rho e^{-\frac{|z|^2}{4} \rho \coth \rho} \left( \frac{\rho}{\sinh \rho} \right)^n \rho^{m-1} d\rho$$

(6.4)

since the integrand is an even function of $\rho$. (To see this, make the change of variables $\sigma \to -\sigma$ in the $d\sigma$ integral. It is not true when $m$ is even.)

The $d\sigma$ integral can be written in terms of a Bessel function. Using spherical coordinates, we can write, for arbitrary $\hat{v} \in S^{m-1}$ and $w \in \mathbb{C}$,

$$\int_{S^{m-1}} e^{i\nu \sigma \cdot \hat{v}} d\sigma = \frac{2 \pi^{m-1}}{\Gamma \left( \frac{m-1}{2} \right)} \int_{0}^{\pi} e^{i\nu \cos \varphi} \sin^{m-2} \varphi d\varphi$$

$$= \frac{4 \pi^{m-1}}{\Gamma \left( \frac{m-1}{2} \right)} \int_{0}^{\pi} \cos(w \cos \varphi) \sin^{m-2} \varphi d\varphi$$

(by symmetry)

$$= \frac{(2\pi)^{m/2}}{w^{m/2-1}} J_{m/2-1}(w)$$

(see page 79 of [21])

$$= \text{Re} \left( \frac{(2\pi)^{m/2}}{w^{m/2-1}} H_{m/2-1}^{(1)}(w) \right)$$

where $H_{\nu}(w)$ is the Hankel function of the first kind, defined by $H_{\nu}(w) = J_{\nu}(w) + iY_{\nu}(w)$, with $Y_{\nu}$ the Bessel function of the second kind. Page 72 of [21] has a closed-form expression for $H_{\nu}$ which yields

$$S_m(w) = 2(2\pi)^{m-1} \text{Re} \left[ \frac{e^{iw}}{w^{m-1}} \sum_{k=1}^{m-1} c_{m,k} (-iw)^k \right]$$

(6.5)
where the coefficients are
\[ c_{m,k} = \frac{(m-k-2)!}{2^{m-k-1} (m-k)! (k-1)!} > 0. \]

The reason for the use of the Hankel function is the appearance of the \( e^{i\omega} \) factor, which gives us an integrand looking much like that for \( p_t \) when \( m = 1 \). This will allow us to apply similar techniques to those which have been used previously for \( m = 1 \). We have
\[
p_1(x, z) = \text{(Re)} \sum_{k=1}^{(m-1)/2} c_{m,k} |z|^{k-m+1} \int_{-\infty}^{\infty} e^{i\rho|z| - |z|^2/\rho} \frac{\rho^n}{\sinh^n \rho} (-i\rho)^k \, d\rho \quad (6.6)
\]
\[
= \sum_{k=1}^{(m-1)/2} c_{m,k} |z|^{k-m+1} e^{-|d(x,z)|^2/2} \int_{-\infty}^{\infty} e^{-|z|^2/\rho} \psi(\rho, \theta) a_k(\rho) \, d\rho \quad (6.7)
\]
where, using similar notation as before,
\[
\psi(\rho, \theta) := -i\nu(\theta)\rho + \rho \coth \rho - i\frac{\rho^2}{\sin^2 \theta} \quad (6.8)
\]
\[
a_k(\rho) := \left( \frac{\rho}{\sinh \rho} \right)^n (-i\rho)^k \quad (6.9)
\]

The constants and coefficients have all been absorbed into the \( c_{m,k} \); we note that \( c_{1,0} > 0, c_{m,k} > 0 \) for \( k \geq 1 \), and \( c_{m,0} = 0 \) for \( m > 1 \). We dropped the (Re) because the imaginary part vanishes, being the integral of an odd function.

For \( m = 1 \), we can write
\[
p_1(x, z) = (4\pi)^{-n} e^{-|d(x,z)|^2/2} \int_{-\infty}^{\infty} e^{-|z|^2/\rho} \psi(\rho, \theta) a_0(\rho) \, d\rho \quad (6.10)
\]

The integrals appearing in the terms of the sum in (6.7), as well as in (6.10), are all susceptible to the same estimate, as the following theorem shows.

**Theorem 6.2.** Let \( D \subset \mathbb{C} \) be the strip \( D = \{0 \leq \text{Im} \rho \leq 3\pi/2\} \). Suppose \( a(\rho) \) is a function analytic on \( D\setminus\{i\pi\} \), with a pole of order \( n \) at \( \rho = i\pi \), \( a(i\theta) \geq 1 \) for \( \theta_0 < \theta < \pi \), and \( \int |a(\rho + 3i\pi/2)| \, d\rho < \infty \). Let
\[
h(x, z) := \int_{-\infty}^{\infty} e^{-|z|^2/\rho} \psi(\rho, \theta) a(\rho) \, d\rho. \quad (6.11)
\]

There exist \( B_1, d_0 \) such that
\[
\text{Re} h(x, z) \geq \frac{|z|^{n-1}}{1 + (|z|^{1/2})^{n-2}} \quad (6.12)
\]
for all \((x, z) \) with \(|z| \geq B_1 |x|^2 \) and \(|z| \geq d_0 \).

The proof of Theorem 6.2 occupies the rest of this section. Theorem 6.1 follows, since Theorem 6.2 applies to each term of (6.7) (note each \( a_k \) satisfies the hypotheses), and the \( k = (m-1)/2 \) term will dominate for large \(|z| \).

An argument similar to Lemma 5.3, using the fact that Lemma 5.3 applies for \(|b| \leq 2\pi \), will allow us to move the contour to the line \( \text{Im} \rho = 3\pi/2 \), accounting for the residue at \( i\pi \):
\[
h(x, z) = \int_{-\infty}^{\infty} e^{-|z|^2/\rho} \psi(\rho + 3i\pi/2, \theta) a(\rho + 3i\pi/2) \, d\rho + \text{Res} \left( e^{-|z|^2/\rho} \psi(\rho, \theta) a(\rho); \rho = i\pi \right) . \quad (6.13)
\]

The following lemma shows that \( h_t(x, z) \), the integral along the horizontal line, is negligible.
Lemma 6.3. There exists $\theta_0 < \pi$ and a constant $C > 0$ such that for all $(x, z)$ with $\theta(x, z) \in [\theta_0, \pi)$ we have
\[ |h_l(x, z)| \leq Ce^{-d(x, z)^2/8}. \] (6.14)

Proof. Observe that $\coth(\rho + 3i\pi/2) = \tanh \rho$. So
\[ \Re \psi(\rho + 3i\pi/2, \theta) = \rho \tanh \rho + \frac{3\pi}{2} \nu(\theta) - \frac{\theta^2}{\sin^2 \theta}. \]

Therefore we have
\[
|h_l(x, z)| \leq e^{-\frac{|s|}{4\pi}\nu(\theta) - \frac{3\pi}{8}\theta^2} \int_{\mathbb{R}} e^{-\frac{|s|}{4\pi}\rho \tanh \rho} |a(\rho + 3i\pi/2)| \, d\rho
\]
\[
\leq e^{-\frac{|s|}{4\pi}\nu(\theta) - \frac{3\pi}{8}\theta^2} \int_{\mathbb{R}} |a(\rho + 3i\pi/2)| \, d\rho
\]
as $\tau \tanh \tau \geq 0$. The integral in the last line is a finite constant, since $a(\cdot + 3i\pi/2)$ is integrable by assumption.

However, for $\theta$ sufficiently close to $\pi$, we have $\nu(\theta) \geq \frac{1}{\pi} \frac{\theta^2}{\sin^2 \theta}$. (If $\beta(\theta) := \nu(\theta) \left( \frac{\theta^2}{\sin^2 \theta} \right)^{-1}$, we have $\lim_{\theta \uparrow \pi} \beta(\theta) = 1/\pi$ and $\lim_{\theta \uparrow \pi} \beta'(\theta) = -2/\pi^2 < 0$. Indeed, $\theta > 0.51$ suffices.) Thus for such $\theta$ we have
\[ |h_l(x, z)| \leq Ce^{-\frac{|s|}{4\pi}\frac{\theta^2}{\sin^2 \theta}} = Ce^{-d(x, z)^2/8}. \] (6.15)

To handle the residue term $h_r$, write it as
\[ h_r(x, z) = \oint_{\partial B(\pi, r)} e^{-\frac{|s|}{4\pi}\psi(\rho, \theta)/4}(\rho) \, d\rho. \] (6.16)

We can choose any $r \in (0, \pi)$ because the integrand is analytic on the punctured disk. To facilitate dealing with the singularity at $\theta = \pi$, we adopt the parameters
\[
s := \pi - \theta(x, z)
\]
\[
y := \pi |x|^2 / s. \] (6.17)

Note that
\[ y/s \asymp |z|, \quad y \asymp |x| \sqrt{|z|}. \] (6.18)

If we let (compare (5.10))
\[
\phi(w, s) := \frac{1}{4\pi} s\psi(i(\pi - w), \pi - s)
\]
\[ = \frac{s}{4\pi} \left( \nu(\pi - s)(\pi - w) + (\pi - w)\cot(\pi - w) - \frac{(\pi - s)^2}{\sin^2 s} \right), \] (6.19)
\[ F(y, s) := s^{n-1} \oint_{\partial B(0, r)} e^{-y\phi(w, s)} a(i(\pi - w))(-i) \, dw \] (6.20)
we have
\[ h_r(x, z) = s^{-(n-1)} F(y, s). \] (6.21)

Note we have made the change of variables $\rho = i(\pi - w)$ from (6.16) to (6.20).

Observe that $F$ is analytic in $y$ and $s$ for $s \neq k\pi$, $k \in \mathbb{Z}$, so we shall now consider $y$ and $s$ as complex variables. The factor of $s^{n-1}$ in $F$ was inserted to clear a pole of order $n - 1$ at $s = 0$, whose presence will be apparent later.
Computing a Laurent series for $\phi$ about $(i\pi, \pi)$, which converges for $0 < |s| < \pi$, $0 < |w| < \pi$, we find
\[ \phi(w, s) = \frac{1}{2} - \frac{w}{4s} - \frac{s}{4w} - sU(w, s) \] (6.22)
with $U$ analytic for $|s| < \pi$, $|w| < \pi$. Also, by the hypotheses on $a$,
\[ a(i\pi - w)) = w^{-n}V(w) \] (6.23)
where $V$ is analytic for $|w| < \pi/2$ and $V(0) > 0$. Thus we have
\[ F(y, s) = s^{n-1} \int_{\partial B(0, r)} e^{-y(x^2 + y^2/sU(w, s))} w^{-n}V(w)(-i)\,dw \] (6.24)
The constant term in the expansion of $\psi$ is slightly inconvenient, so let $G(y, s) = e^{y/2}F(y, s)$. Then:
\[ G(y, s) = s^{n-1} \int_{\partial B(0, r)} e^{y(x^2 + y^2/sU(w, s))} w^{-n}V(w)(-i)\,dw \]
\[ = s^{n-1} \int_{\partial B(0, r)} \sum_{k=0}^{\infty} \frac{y^k}{k!} \left( \frac{w}{4s} + \frac{s}{4w} + sU(w, s) \right)^k w^{-n}V(w)\,dw(-i) \]
\[ = s^{n-1} \sum_{k=0}^{\infty} \frac{y^k}{k!} \int_{\partial B(0, r)} \left( \frac{w}{4s} + \frac{s}{4w} + sU(w, s) \right)^k w^{-n}V(w)(-i)\,dw \]
\[ =: \sum_{k=0}^{\infty} \frac{y^k g_k(s)}{k!} \] (6.26)
where we let
\[ g_k(s) := s^{n-1} \int_{\partial B(0, r)} \left( \frac{w}{4s} + \frac{s}{4w} + sU(w, s) \right)^k w^{-n}V(w)(-i)\,dw. \] (6.27)
The interchange of sum and integral in (6.25) is justified by Fubini’s theorem, since for fixed $s$ $U(s, \cdot)$ and $V$ are bounded on $B(0, r)$, and thus
\[ \sum_{k=0}^{\infty} \int_{B(0, r)} \left| \frac{y^k}{k!} \left( \frac{w}{4s} + \frac{s}{4w} + sU(w, s) \right)^k \left( \frac{\pi}{w} + V(w) \right)^n \right| \,dw \]
\[ \leq \sum_{k=0}^{\infty} \frac{|y|^k}{k!} 2\pi r \left( \frac{r}{4|s|} + \frac{|s|}{4r} + |s| \sup_{|w|=r} |U(w, s)| \right) \left( \frac{\pi}{r} + \sup_{|w|=r} |V(w)| \right)^n \]
\[ = 2\pi r \left( \frac{\pi}{r} + \sup_{|w|=r} |V(w)| \right)^n \exp \left( \frac{|y|}{4|s|} + \frac{|s|}{4r} + |s| \sup_{|w|=r} |U(w, s)| \right) < \infty. \]

We now examine more carefully the terms $g_k$ in (6.26, 6.27).

**Lemma 6.4.** If $g_k$ is defined by (6.27), then:
1. $g_k$ is analytic for $|s| \leq s_0$;
2. There exists $C = C(s_0) \geq 0$ independent of $k$ such that $|g_k(s)| \leq C^k$ for each $k$ and all $|s| \leq s_0$;
3. For $k \leq n - 1$, $g_k(s) = s^{n-1-k}h_k(s)$, where $h_k$ is analytic for $|s| \leq s_0$. In particular, $g_k(0) = 0$ for $k < n - 1$.
4. For $k \geq n - 1$, $g_k(0) > 0$ when $k + n$ is odd, and $g_k(0) = 0$ when $k + n$ is even.
Proof. By the multinomial theorem,
\[ g_k(s) = \sum_{a+b+c=k} \binom{k}{a,b,c} s^{n-1} \int_{\partial B(0,r)} \left( \frac{w}{4s} \right)^a \left( \frac{s}{4w} \right)^b (sU(w,s))^c w^{-n} V(w)(-i) \, dw \]  
(6.28)

= \sum_{a+b+c=k} \binom{k}{a,b,c} 4^{-(a+b)} \int_{\partial B(0,r)} w^{a-b-n} s^{-(a-b-n)-1} (sU(w,s))^c V(w)(-i) \, dw \]  
(6.29)

= \sum_{a+b+c=k} \binom{k}{a,b,c} 4^{-(a+b)} \int_{\partial B(0,r)} w^{a-b-n} s^{-(a-b-n)-1} (sU(w,s))^c V(w)(-i) \, dw \]  
(6.30)

since for terms with \(a-b-n \geq 0\), the integrand is analytic in \(w\) and the integral vanishes. Now the integrand of each term of (6.30) is clearly analytic in \(s\), hence so is \(g_k\) itself, establishing item [1].

For item [2] let \(U_0 := \sup_{|w|=r,|s| \leq s_0} |U(w,s)|\), and \(V_0 := \sup_{|w|=r} |V(w)|\). Then for \(|s| \leq s_0\),
\[ |g_k(s)| \leq \sum_{a+b+c=k} \binom{k}{a,b,c} 4^{-(a+b)} (2\pi r)^{n} s_0^{-a-b-n} (s_0 U_0)^c V_0 \]

\[ \leq 2\pi r V_0 s_0^{n-1} \sum_{a+b+c=k} \binom{k}{a,b,c} \left( \frac{r}{4s_0} \right)^a \left( \frac{s_0}{4r} \right)^b (s_0 U_0)^c \]

\[ \leq 2\pi r V_0 s_0^{n-1} \sum_{a+b+c=k} \binom{k}{a,b,c} \left( \frac{r}{4s_0} \right)^a \left( \frac{s_0}{4r} \right)^b (s_0 U_0)^c \]

\[ \leq 2\pi r V_0 s_0^{n-1} \left( \frac{r}{4s_0} + \frac{s_0}{4r} + s_0 U_0 \right)^k \]

so that a constant \(C\) can be chosen with \(g_k(s) \leq C^k\), establishing item [2].

For item [3] suppose \(k \leq n-1\) and let \(h_k(s) = s^{k-n+1} g_k(s)\), so that
\[ h_k(s) = \sum_{a+b+c=k} \binom{k}{a,b,c} 4^{-(a+b)} \int_{\partial B(0,r)} w^{a-b-n} s^{-(a-b-k)} (sU(w,s))^c V(w)(-i) \, dw \]

But \(a-b-k \leq a-k \leq 0\) since \(a \leq k\) by definition, so only positive powers of \(s\) appear, and \(h_k\) is analytic in \(s\).

For item [4] we see that when \(s = 0\), each term of (6.30) will vanish unless \(c = 0\) and \(a-b-n = -1\), i.e. \(a+b = k\) and \(a+b = n-1\). If \(k\) and \(n\) have the same parity, this happens for no term, so \(g_k(0) = 0\). If \(k\) and \(n\) have opposite parity, this forces \(a = (k+n-1)/2\), \(b = (k+n+1)/2\), both of which are nonnegative integers. In this case
\[ g_k(s) = \binom{k}{(k+n-1)/2} 4^{-k} \int_{\partial B(0,r)} w^{-1} V(w)(-i) \, dw \]

\[ = \binom{k}{(k+n-1)/2} 4^{-k} 2\pi V(0) > 0 \]

since \(V(0) > 0\).

From this we derive corresponding properties of the function \(F\).

**Corollary 6.5.** Let \(F(y,s)\) be defined as in (6.20). Then for all \(s_0 < 2\pi\):

1. \(F\) is analytic for all \(y\) and all \(0 \leq s \leq s_0\).
2. We may write

\[ F(y, s) = e^{-y/2} \left[ \sum_{k=0}^{n-1} \frac{y^k s^{n-1-k}}{k!} h_k(s) + y^n H(y, s) \right] \]  

(6.31)

with \( h_k, H \) analytic for all \( y \) and all \( 0 \leq s \leq s_0 \). Furthermore, \( h_{n-1}(0) > 0 \).

3. \( F(0, y) > 0 \) for all \( y > 0 \).

**Proof.** We prove the corresponding facts about \( G = e^{y/2} F \). By items 1 and 2 of Lemma 6.4, we have that \( G \) is analytic for \( |s| \leq s_0 \) and all \( y \), since the sum in (6.25) is a sum of analytic functions and converges uniformly. By item 3 we have that

\[ G(y, s) = \sum_{k=n-1}^{\infty} \frac{y^k s^{n-1-k}}{k!} h_k(s) + y^n \sum_{k=0}^{\infty} \frac{y^k}{(n+k)!} g_{n+k}(s). \]

And by items 3 and 4, \( G(y, 0) = \sum_{k=n-1}^{\infty} \frac{y^k g_{k}(0)}{k!} > 0 \) for all \( y > 0 \). \( \square \)

**Proposition 6.6.** For all \( y > 1 \), there exist \( \delta > 0 \), and \( 0 < C'_1 \leq C'_2 < \infty \) such that

\[ C'_1 y^{n-1} \leq \text{Re} \ F(y, s) \leq |F(y, s)| \leq C'_2 y^{n-1} \]  

(6.32)

for all \( 0 \leq y < y_1, 0 \leq s < \delta y \). (Here we are treating \( y \) and \( s \) as real variables.)

**Proof.** Let \( K \) be a positive constant so large that \( |h_k(s)| \leq K \) and \( |H(y, s)| \leq K \) for all \( 0 \leq y < y_1, 0 \leq s < y_1, k \leq n - 1 \). For any \( \delta < 1 \) and all \( s \leq \delta y < y_1 \), we have

\[
\text{Re} \ G(y, s) = \frac{y^{n-1}}{(n-1)!} \text{Re} \ h_{n-1}(s) + \sum_{k=0}^{n-2} \frac{y^k s^{n-1-k}}{k!} \text{Re} \ h_k(s) + y^n \text{Re} \ H(y, s)
\]

\[
= y^{n-1} \left[ \frac{\text{Re} \ h_{n-1}(s)}{(n-1)!} - K \sum_{k=0}^{n-2} \frac{\delta^{n-1-k}}{k!} \right] - y^n K.
\]

Since \( h_{n-1}(0) > 0 \), we may now choose \( \delta \) so small that the bracketed term is positive for all \( 0 \leq s \leq \delta y_1 \). Then there exists \( y_0 > 0 \) so small that for all \( 0 \leq y \leq y_0 \), we have \( \text{Re} \ F(y, s) \geq e^{-y_0/2} \text{Re} \ G(y, s) \geq C'_1 y^{n-1} \) for some \( C'_1 > 0 \). On the other hand,

\[
|F(y, s)| \leq |G(y, s)|
\]

\[
\leq \sum_{k=0}^{n-1} \frac{y^k s^{n-1-k}}{k!} |h_k(s)| + y^n \text{Re} \ H(y, s)
\]

\[
\leq y^{n-1} \sum_{k=0}^{n-1} \frac{K \delta^{n-1-k}}{k!} + y^n K.
\]

Again, for small \( y \) (take \( y_0 \) smaller if necessary), we have \( |F(y, s)| \leq C'_2 y^{n-1} \).

It remains to handle \( y_0 \leq y \leq y_1 \). But this presents no difficulty; as \( F(0, 0) > 0 \) for all \( y > 0 \), and \( F \) is continuous, there exists \( \delta \) so small that

\[
\inf_{y_0 \leq y \leq y_1, 0 \leq s \leq \delta y_1} \text{Re} \ F(y, s) > 0.
\]

This completes the proof. \( \square \)
Proposition 6.7. There exists \( y_1 > 0 \), \( s_0 > 0 \) and constants \( C_1, C_2 > 0 \) such that

\[
\frac{C_1}{\sqrt{y}} \leq \text{Re} F(y, s) \leq |F(y, s)| \leq \frac{C_2}{\sqrt{y}}
\]  
(6.33)

for all \( y > y_1 \), \( 0 < s < s_0 \).

Proof. Here the Gaussian approximation technique of Section 5 is again applicable. We will fix the contour in (6.20) as a circle of radius \( r = s \), parametrize it, and examine the integrand directly. Thus let \( w = se^{i\gamma} \) in (6.20) to obtain

\[
F(y, s) = s^{n-1} \int_{-\pi}^{\pi} e^{-y\phi(se^{i\gamma}, s)}a(i(\pi - se^{i\gamma}))se^{i\gamma} d\gamma.
\]  
(6.34)

We shall apply Lemma 5.6 with \( m = 1 \), \( \lambda = \gamma \), \( r = \pi \), \( x = \sqrt{y} \). Let

\[
g(\gamma, s) = \text{Re} \phi(se^{i\gamma}, s)
\]  
(6.35)

\[
k(\sqrt{y}, \gamma, s) = e^{-i\sqrt{y}} \text{Im} \phi(se^{i\gamma}, s)s^n a(i(\pi - se^{i\gamma}))e^{i\gamma}
\]  
(6.36)

Since \( \phi(s, s) = 0 \) and \( w = s \) is a critical point of \( \phi(w, s) \), we have

\[
\frac{\partial^2}{\partial s^2} \phi(se^{i\gamma}, s)_{|\gamma=0} = \frac{s}{4\pi} \phi''(s, s)(is)^2 = \frac{s^3 \nu'(\pi - s)}{4\pi}
\]  
(6.37)

which is bounded and positive for all small \( s \) (recall \( \nu(\pi - s) \sim s^{-2} \)). Thus there exists \( s_0, \epsilon \) small enough and constants \( b_1, b_2 \) such that

\[
b_1 \gamma^2 \leq g(\gamma, s) \leq b_2 \gamma^2
\]  
(6.38)

for \( s < s_0 \), \( |\gamma| < \epsilon \). Also, we have from (6.22) that

\[
\phi(se^{i\gamma}, s) = \frac{1}{2} - \frac{1}{2} \cos \gamma - sU(se^{i\gamma}, s)
\]  
(6.39)

so that by taking \( s_0 \) smaller if necessary, we can ensure \( g(\gamma, s) > 0 \) for all \( s < s_0 \) and \( \epsilon \leq |\gamma| \leq \pi \). Thus (6.38) holds for \( s < s_0 \) and all \( \gamma \in [-\pi, \pi] \), with possibly different constants \( b_1, b_2 \).

Boundedness of \( k \) follows from the fact that \( a \) has a pole of order \( n \) at \( i\pi \), so \( s^n a(i(\pi - se^{i\gamma})) = V(se^{i\gamma}) \) is bounded for small \( s \). Finally, since \( \frac{\partial^2}{\partial s^2} \phi(se^{i\gamma}, s)_{|\gamma=0} > 0 \) and \( V(0) > 0 \), the argument used in the proof of Theorem 5.1 shows that the necessary lower bound on \( k \) also holds. Then an application of Lemma 5.6 completes the proof. \( \square \)

Proof of Theorem 6.2. Choose \( y_1, s_0 \) so that Proposition 6.7 holds, and take \( B_1 \) large enough so that \( \theta(x, z) \geq \pi - s \) when \( |z| \geq B_1 |x|^2 \). Use this value of \( y_1 \) and choose a \( \delta \) such that Proposition 6.6 holds, and take \( d_0 \) large enough that \( s < \delta y \) when \( |z| \geq d_0 \) (see (6.18)). So for such \( (x, z) \), either (6.22) or (6.33) holds; which one depends on the value of \( y = y(x, z) \). We can combine them to get

\[
C_1' \frac{y^{n-1}}{1 + y^{n-\frac{2}{2}}} \leq \text{Re} F(y, s) \leq |F(y, s)| \leq C_2' \frac{y^{n-1}}{1 + y^{n-\frac{2}{2}}}
\]  
(6.40)

Inserting this into (6.21) and using (6.18), we have (in more compact notation)

\[
h_r(x, z) \asymp \left( \frac{y}{s} \right)^{n-1} \frac{1}{1 + y^{n-\frac{2}{2}}} \asymp \frac{|z|^{n-1}}{1 + (|x| \sqrt{|z|})^{n-\frac{2}{2}}}
\]  
(6.41)

By Lemma 6.3, \( h_l \) is clearly negligible by comparison, so Theorem 6.2 is proved. \( \square \)
A similar argument will give us the estimates on $\nabla p_1$ and $q_2$ which correspond to Theorems 4.4 and 4.5.

**Theorem 6.8.** For $m$ odd, there exist constants $B_1, d_0, C$ such that

$$|\nabla p_1(x, z)| \lesssim \frac{|x| d(x, z)^{2n-m+1}}{1 + (|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{i}{4}d(x, z)^2}$$

(6.42)

and

$$|q_2(x, z)| \lesssim C \frac{d(x, z)^{2n-m-1}}{1 + (|x| d(x, z))^{n-\frac{1}{2}}} e^{-\frac{i}{4}d(x, z)^2}$$

(6.43)

whenever $|z| \geq B_1 |x|^2$ and $d(x, z) \geq d_0$.

**Proof.** Applying (4.5) to (6.6), we have

$$\nabla p_1(x, z) = -\frac{1}{2}(2\pi)^{-m}(4\pi)^{-n} |x| (q_1(x, z) \dot{x} + q_2(x, z) J_z \dot{x})$$

where

$$q_1(x, z) = -\frac{2}{|x|} \frac{\partial p_1(x, z)}{\partial |x|}$$

$$q_2(x, z) = \frac{\partial p_1(x, z)}{\partial |x|}$$

Each integral can be estimated by Theorem 6.2. For $q_1$, each integral is comparable to $e^{-\frac{i}{4}d(x, z)^2} \frac{|x|^n}{1 + (|x| d(x, z))^{n+\frac{1}{2}}}$, and the $k = (m - 1)/2$ term dominates, so

$$|q_1(x, z)| \lesssim \frac{|x|^{n-(m-1)/2}}{1 + (|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{i}{4}d(x, z)^2}.$$  

(6.44)

The appearance of the extra minus sign in $q_1$ is to account for the fact that $\cosh(i\pi) = -1$, but Theorem 6.2 requires that $a(\lambda)$ be positive near $\lambda = i\pi$.

For $q_2$, each integral is comparable to $\frac{|x|^{n-1}}{1 + (|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{i}{4}d(x, z)^2}$, and the $k = (m - 1)/2$ term of the second sum dominates, so

$$|q_2(x, z)| \lesssim \frac{|x|^{n-1-(m-1)/2}}{1 + (|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{i}{4}d(x, z)^2}.$$  

(6.45)

which in particular implies (6.43). To combine (6.44) and (6.45), note that for $|x|^2 |z|$ bounded we have

$$|q_1(x, z)| \lesssim |z|^{n-(m-1)/2} e^{-\frac{i}{4}d(x, z)^2}; \quad |q_2(x, z)| \lesssim |z|^{n-1-(m-1)/2} e^{-\frac{i}{4}d(x, z)^2}$$

(6.46)

so that the $q_1$ term dominates, and

$$|\nabla p_1(x, z)| \lesssim |x| |z|^{n-(m-1)/2} e^{-\frac{i}{4}d(x, z)^2}.$$  

(6.47)
For $|x|^2 |z|$ bounded away from 0 we have

$$q_1(x, z) \asymp |x|^{-n-\frac{1}{2}} |z|^{\frac{m}{2} - \frac{1}{2}} e^{-\frac{1}{8} d(x, z)^2}$$

$$q_2(x, z) \asymp |x|^{-n+\frac{1}{2}} |z|\frac{m}{2} - \frac{1}{2} e^{-\frac{1}{8} d(x, z)^2} \asymp \frac{|x|}{|z|^{\frac{1}{2}}} q_1(x, z)$$

(6.48)

so that the $q_1$ term dominates again ($\frac{|x|}{|z|^{\frac{1}{2}}}$ is bounded by assumption). Thus

$$|\nabla p_1(x, z)| \asymp |x| \frac{|z|^{n-(m-1)/2}}{1 + (|x|/|z|)^{n+\frac{1}{2}}} e^{-\frac{1}{8} d(x, z)^2}$$

(6.49)

which is equivalent to the desired estimate.

\[\square\]

7. Hadamard descent

In this section, we obtain estimates for $p_1(x, z)$ and $|\nabla p_1(x, z)|$ for $|z| \geq B_1 |x|^2$, $|z| \geq d_0$, in the case where the center dimension $m$ is even. The methods of the previous section are not directly applicable, but we can deduce an estimate for even $m$ by integrating the corresponding estimate for $m+1$. As discussed in the remark at the end of Section 4, this is valid even though there may not exist an $H$-type group of dimension $2n + m + 1$ with center dimension $m + 1$, since the estimates we use are derived from the formula (4.2) and hold for all values of $n, m$.

We continue to assume that $|z| \geq B_1 |x|^2$ and $|z| \geq d_0$ for some sufficiently large $B_1, d_0$. To emphasize the dependence on the dimension, we write $p^{(n,m)}$ for the function $p_1$ in (4.2).

In order to estimate $p^{(n,m)}$ for even $m$, we consider $p^{(n,m+1)}$. We can observe that

$$p^{(n,m)}(x, z) = \int_R p^{(n,m+1)}(x, (z, z_{m+1})) dz_{m+1}$$

(7.1)

since $\int_R e^{i\lambda z_{m+1}} f(\lambda_{m+1}) d\lambda_{m+1} dz_{m+1} = 2\pi f(0)$. Note that $|(\lambda, 0)|_{\mathbb{R}^{m+1}} = |\lambda|_{\mathbb{R}^m}$. Now $p^{(n,m+1)}$ can be estimated by means of Theorem 6.1 Using the fact that $|(z, z_{m+1})| \geq |z|$, we have that for $m$ even, there exist constants $B_1, d_0$ such that

$$p^{(n,m)}(x, z) \asymp Q^{(2n-m-2, n-\frac{1}{2})}(x, z)$$

(7.2)

whenever $|z| \geq B_1 |x|^2$ and $|z| \geq d_0$, where

$$Q^{(\alpha, \beta)}(x, z) := \int_R \frac{d(x, (z, z_{m+1}))^\alpha}{1 + (|x|/d(x, (z, z_{m+1})))^2} e^{-\frac{1}{8} d(x, (z, z_{m+1}))^2} dz_{m+1}$$

(7.3)

Thus it suffices to estimate the integrated bounds given by $Q^{(\alpha, \beta)}$.

Lemma 7.1. For $|z| \geq B_1 |x|^2$ and $|z| \geq d_0$, we have

$$Q^{(\alpha, \beta)}(x, z) \asymp \frac{d(x, z)^{\alpha+1}}{1 + (|x|/d(x, z))^2} e^{-\frac{1}{8} d(x, z)^2}.$$  

(7.4)

We will require two preliminary computations. Since $d(x, z)$ depends on $z$ only through $|z|$, we will occasionally treat $d$ as a function on $\mathbb{R}^{2n} \times [0, \infty)$.

Lemma 7.2. There exist positive constants $c_1, c_2, B_1$ such that for all $x \in \mathbb{R}^{2n}, u \in \mathbb{R}$ with $u \geq B_1 |x|^2$, we have $0 < c_1 \leq \frac{\partial}{\partial u} d(x, u) \leq c_2 < \infty$.  

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Proof. Let \( \mu(\theta) = \frac{\theta^2}{2\sin^2 \theta} \), so that
\[
d(x,u)^2 = |x|^2 \mu(\theta) \quad \text{with} \quad \theta = \theta(x,z) = \nu^{-1} \left( \frac{2w}{|x|^2} \right).
\]
Then
\[
\frac{\partial}{\partial u} d(x,u)^2 = 2 \frac{\mu'(\theta)}{\nu'(\theta)}.
\] (7.5)

It is easily verified that \( \mu'(\theta) > 0 \), \( \nu'(\theta) > 0 \) for all \( \theta \in (0, \pi) \), and \( \frac{\mu'(\theta)}{\nu'(\theta)} \to \pi > 0 \) as \( \theta \to \pi \). \( \square \)

**Lemma 7.3.** For any \( \alpha \in \mathbb{R} \), there exists \( C_\alpha > 0 \) such that for all \( w_0 \geq 1 \) we have
\[
\int_{w_0}^{\infty} w^\alpha e^{-w} dw \leq C_\alpha w_0^\alpha e^{-w_0}.
\] (7.6)

**Proof.** For \( \alpha \leq 0 \), \( w^\alpha \) is decreasing for \( w \geq 1 \), so
\[
\int_{w_0}^{\infty} w^\alpha e^{-w} dw \leq w_0^\alpha \int_{w_0}^{\infty} e^{-w} dw = w_0^\alpha e^{-w_0}
\] (7.7)
and this holds with \( C_\alpha = 1 \). Now, for a nonnegative integer \( n \), suppose the lemma holds for all \( \alpha \leq n \). Then if \( n < \alpha \leq n + 1 \), we integrate by parts to obtain
\[
\int_{w_0}^{\infty} w^\alpha e^{-w} dw = w_0^\alpha e^{-w_0} + \alpha \int_{w_0}^{\infty} w^{\alpha-1} e^{-w} dw \leq (1 + \alpha C_{\alpha-1}) w_0^\alpha e^{-w_0}
\]
so that the lemma also holds for all \( \alpha \leq n + 1 \). By induction the proof is complete. \( \square \)

**Proof of Lemma 7.1.** We make the change of variables \( u = |(z, z_m+1)| \) so that \( z_m+1 = \sqrt{u^2 - |z|^2} \). By our previous abuse of notation, we can write \( d(x, (z, z_m+1)) = d(x,u) \). Thus
\[
Q^{(\alpha,\beta)}(x,z) = \int_{|z|}^{\infty} \frac{d(x,u)^\alpha}{1 + (|x| d(x,u))^\beta} e^{-\frac{u}{4d(x,u)^2}} du
\]
\[
\approx \int_{|z|}^{\infty} \frac{1}{\sqrt{u - |z|}} \frac{1}{\sqrt{u + |z|}} \frac{1}{1 + (|x| d(x,u))^\beta} e^{-\frac{u}{4d(x,u)^2}} du.
\]
We used the fact that \( u \approx d(x,u)^2 \) where \( |z| \geq B_1 |x|^2 \), by Corollary 3.15.

Now, noting that \( u \mapsto d(x,u) \) is an increasing function, and \( u \mapsto w^{\alpha+2} e^{-\frac{u}{4w^2}} \) is decreasing for large enough \( w \), the lower bound can be obtained by
\[
Q^{(\alpha,\beta)}(x,z) \geq \int_{|z|}^{\sqrt{2|z| + 1} + (|x| d(x,z))^\beta} \frac{d(x,u)^\alpha}{1 + (|x| d(x,u))^\beta} e^{-\frac{u}{4d(x,u)^2}} du
\]
\[
\geq 2 \sqrt{2|z| + 1 + (|x| d(x,z))^\beta} - \frac{d(x,z)^{\alpha+2}}{1 + (|x| d(x,z))^\beta} e^{-\frac{4d(x,z)^2}{4}}
\]
\[
= C \sqrt{2|z| + 1 + (|x| d(x,z))^\beta} - \frac{d(x,z)^{\alpha+2}}{1 + (|x| d(x,z))^\beta} e^{-\frac{4d(x,z)^2}{4}}
\]
where the last line follows because \( u \mapsto d(x,u)^2 \) is Lipschitz, as shown by Lemma 7.2 with a constant independent of \( x \).

Since \( |z| \approx d(x,z)^2 \), we have that
\[
Q^{(\alpha,\beta)}(x,z) \geq C' \frac{d(x,z)^{\alpha+1}}{1 + (|x| d(x,z))^\beta} e^{-\frac{4d(x,z)^2}{4}}.
\] (7.8)
For an upper bound, we have
\[ Q^{(\alpha, \beta)}(x, z) \leq C \left[ \int_{|z|}^{|z|+1} \frac{1}{\sqrt{|u-|z|}} \frac{1}{\sqrt{|u+|z|}} \frac{d(x, u)^{\alpha+2}}{1 + (|x| d(x, u))^{\beta}} e^{-\frac{1}{4} d(x, u)^2} du + \int_{|z|+1}^{\infty} \right] \]

Now
\[
\int_{|z|}^{|z|+1} \frac{1}{\sqrt{|u-|z|}} \frac{1}{\sqrt{|u+|z|}} \frac{d(x, u)^{\alpha+2}}{1 + (|x| d(x, u))^{\beta}} e^{-\frac{1}{4} d(x, u)^2} du 
\leq \left( \int_{|z|}^{|z|+1} \frac{1}{\sqrt{|u-|z|}} \frac{1}{\sqrt{|u+|z|}} \frac{d(x, z)^{\alpha+2}}{2 |z|} (1 + (|x| d(x, z))^{\beta}) e^{-\frac{1}{4} d(x, z)^2} du \right)
\leq 2 \frac{1}{\sqrt{2 |z|}} (1 + (|x| d(x, z))^{\beta}) e^{-\frac{1}{4} d(x, z)^2} 
\leq C \frac{d(x, z)^{\alpha+1}}{1 + (|x| d(x, z))^{\beta}} e^{-\frac{1}{4} d(x, z)^2}.
\]

For the other term, we observe
\[
\int_{|z|}^{\infty} \frac{1}{\sqrt{|u-|z|}} \frac{1}{\sqrt{|u+|z|}} \frac{d(x, u)^{\alpha+2}}{1 + (|x| d(x, u))^{\beta}} e^{-\frac{1}{4} d(x, u)^2} du \leq \int_{|z|}^{\infty} \frac{1}{\sqrt{|u-|z|}} \frac{1}{\sqrt{|u+|z|}} \frac{d(x, u)^{\alpha+2}}{1 + (|x| d(x, u))^{\beta}} e^{-\frac{1}{4} d(x, u)^2} du 
\leq \int_{|z|}^{\infty} \frac{1}{\sqrt{2 u}} (1 + (|x| d(x, u))^{\beta}) e^{-\frac{1}{4} d(x, u)^2} du 
\leq C \int_{|z|}^{\infty} \frac{d(x, u)^{\alpha+1}}{1 + (|x| d(x, u))^{\beta}} e^{-\frac{1}{4} d(x, u)^2} du
\]

We now make the change of variables \( w = \frac{1}{4} d(x, u)^2 \). By the above lemma, \( du/dw \) is bounded, so
\[
\int_{|z|}^{\infty} \frac{d(x, u)^{\alpha+1}}{1 + (|x| d(x, u))^{\beta}} e^{-\frac{1}{4} d(x, u)^2} du \leq C \int_{\frac{1}{4} d(x, z)^2}^{\infty} \frac{(4w)^{(\alpha+1)/2}}{1 + (2 |x| \sqrt{w})^{\beta}} e^{-w} dw,
\]

If \( d(x, z) \leq 1/|x| \), we have
\[
\int_{\frac{1}{4} d(x, z)^2}^{\infty} \frac{(4w)^{(\alpha+1)/2}}{1 + (2 |x| \sqrt{w})^{\beta}} e^{-w} dw \leq C d(x, z)^{\alpha+1} e^{-\frac{1}{4} d(x, z)^2} 
\leq 2C \frac{d(x, z)^{\alpha+1}}{1 + (|x| d(x, z))^{\beta}} e^{-\frac{1}{4} d(x, z)^2}
\]

where we have used Lemma 7.3.

On the other hand, when \( d(x, z) \geq 1/|x| \), we have
\[
\int_{\frac{1}{4} d(x, z)^2}^{\infty} \frac{(4w)^{(\alpha+1)/2}}{1 + (2 |x| \sqrt{w})^{\beta}} e^{-w} dw \leq (2 |x|)^{-\beta} \int_{\frac{1}{4} d(x, z)^2}^{\infty} (4w)^{(\alpha+1-\beta)/2} e^{-w} dw 
\leq C |x|^{-\beta} d(x, z)^{\alpha+1-\beta} e^{-\frac{1}{4} d(x, z)^2} 
\leq 2C \frac{d(x, z)^{\alpha+1}}{1 + (|x| d(x, z))^{\beta}} e^{-\frac{1}{4} d(x, z)^2}
\]

Combining all this, we have as desired that
\[
Q^{(\alpha, \beta)}(x, z) \leq \frac{d(x, z)^{\alpha+1}}{1 + (|x| d(x, z))^{\beta}} e^{-\frac{1}{4} d(x, z)^2}.
\]

(7.9)
Corollary 7.4. Theorems 6.1 and 6.8 also hold for \( m \) even.

Proof. The heat kernel estimate of Theorem 6.1 is immediate, given (7.2) and Lemma 7.1. To obtain an estimate on \( \nabla p_1 \), we define \( q_1^{(n,m)} := -\frac{\partial}{\partial x} p_1^{(n,m)}(x, z) \), \( q_2^{(n,m)} := \frac{\partial}{\partial z} p_1^{(n,m)}(x, z) \), as in (4.35).

For \( q_1 \), we simply differentiate (7.1) to see

\[
q_1^{(n,m)}(x, z) = \int_{\mathbb{R}} q_1^{(n,m+1)}(x, (z, z_{m+1})) dz_{m+1} \\
\approx Q^{2n-m+\frac{1}{2}}(x, z) \\
\approx \frac{d(x, z)^{2n-m+1}}{1 + (|x| d(x, z))^{n+\frac{1}{2}}} e^{-\frac{1}{2}d(x, z)^2} \\
\text{by Lemma 7.1}
\]

For \( q_2 \), we again differentiate (7.1). Here we obtain

\[
q_2^{(n,m)}(x, z) = \int_{\mathbb{R}} q_2^{(n,m+1)}(x, (z, z_{m+1})) \frac{|z|}{(z, z_{m+1})} dz_{m+1} \\
\approx |z| Q^{2n-m-4, n-\frac{1}{2}} \\
\approx d(x, z)^2 \frac{d(x, z)^{2n-m-3}}{1 + (|x| d(x, z))^{n-\frac{1}{2}}} e^{-\frac{1}{2}d(x, z)^2} \\
\approx \frac{d(x, z)^{2n-m-1}}{1 + (|x| d(x, z))^{n-\frac{1}{2}}} e^{-\frac{1}{2}d(x, z)^2}.
\]

Repeating the computation from Theorem 6.8 we have the desired estimates on \( |\nabla p_1| \) and \( |q_2| \).

8. Conclusion

An obvious extension of this result would be to obtain precise estimates for the heat kernel in more general nilpotent Lie groups. For step-2 nilpotent groups, a formula for the heat kernel along the lines of (4.2) can be found in [5], among others. However, the additional algebraic structure enjoyed by H-type groups has played a major part in the analysis presented here, and its absence complicates matters considerably. A key difficulty is that the exponent in the formula for \( p_t \) now contains expressions like \( J_\lambda \cot J_\lambda \), which are awkward to work with when \( J_\lambda \) may not commute with its derivatives with respect to \( \lambda \).

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