Exponential Separations Between Turnstile Streaming and Linear Sketching

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Abstract

Almost every known turnstile streaming algorithm is implementable as a linear sketch. Is this necessarily true, or can there exist turnstile streaming algorithms that use much less space than any linear sketch?

It was shown in [LNW14] that, if a turnstile algorithm works for arbitrarily long streams with arbitrarily large coordinates at intermediate stages of the stream, then the turnstile algorithm can be implemented as a linear sketch. Our results have the opposite form: if either the stream length or the maximum value of the stream are substantially restricted, there exist problems where linear sketching is exponentially harder than turnstile streaming.
1 Introduction

The study of streaming algorithms is concerned with the following question: given a very large dataset that appears over time, what questions can one answer about it without ever storing it in its entirety? Formally, one receives \( x \in \mathbb{Z}^n \) (e.g., the indicator vector for the set of edges in a graph) as a series of updates \( x_i \leftarrow x_i + \Delta \) (e.g., edge insertions and deletions). One would like to estimate properties of the final vector \( x \) while only ever storing \( o(n) \) space, ideally \( \text{poly}(\log n) \). The space used by the algorithm is the primary quantity of interest; other parameters such as update or recovery time are typically well-behaved as a matter of course for small-space algorithms. In this paper we focus on ‘turnstile’ streams, where \( \Delta \) can be negative, as opposed to insertion-only streams, where it must be positive.

The study of turnstile streaming has been very successful at revealing new algorithmic techniques and insights. It has found wide applicability, with algorithms for a huge variety of problems. Examples include norm estimation in \( \ell_2 \) [AMS96] or other \( \ell_p \) [Ind06, CDIM03]; \( \ell_0 \) sampling [FIS08]; heavy hitters [CCF02, CM05]; coresets for \( k \)-median [FS05, IP11]; and graph problems such as finding spanning forests [AGM12], spectral sparsifiers [KLM+14], matchings [AKLY16], and triangle counting [TKMF09, PT12, KP17].

Remarkably, for every single problem described above, the best known algorithm is a linear sketch, where the state of the algorithm at time \( t \) is given by a linear function of the updates seen to \( x \) before time \( t \). And for most of these problems, we know that the linear sketch is optimal.

Linear sketches have a number of other nice properties. Their additivity means that one can, for example, split a data stream across multiple routers and sketch the pieces independently. This has also made such sketches useful in non-streaming applications such as distributed computing [KKM13]. Their output depends only on the final value of \( x \), so they will work regardless of the length of the stream, the order in which the stream arrives, and the intermediate states reached by the stream. Their indifference to stream order means the randomness they use can usually be implemented with Nisan’s PRG [Nis92, Ind06].

They are also easier to prove lower bounds against, either using the simultaneous message passing (SMP) model (e.g., [Kon15, AKLY16, KKP18]) or additional properties of linearity [PW12].

So it would be nice if every turnstile streaming algorithm could be implemented as a linear sketch. And this is true, as shown in [LNW14], but only subject to fairly strong conditions on the algorithm. Most notably, in order for a turnstile streaming algorithm to be implementable as a linear sketch, [LNW14] requires the streaming algorithm to be able to tolerate extremely long streams (longer than \( 2^{2^n} \)) that reach correspondingly large intermediate states. In [AHLW16], it was shown that this equivalence can be extended to ‘strict’ turnstile streams, where the intermediate states never become negative but must still be allowed to become extremely large in the positive direction. However, the result still leaves open the possibilities as problems that require poly\((n)\) space in linear sketching, but in turnstile streaming can be solved in \( O(\text{poly}(\log n, \log \log L)) \) space for length-\( L \) streams, or in \( O(\text{poly}(\log n, \log M)) \) space for streams whose intermediate state never leave \( [-M, M]^n \) (a ‘box constraint’).

Such a box constraint is particularly natural in graph streaming: if the stream represents insertions and deletions of edges in a graph, then the intermediate states \( x \) should lie in \( \{0, 1\}^{n(2)} \). Ironically, graph streaming may be where a theorem on equivalence between streaming and sketching would be most useful: the problems for which we have lower bounds on linear sketches but not turnstile streaming tend to involve graphs. The [LNW14] equivalence gives lower bounds for
these problems, but only for turnstile algorithms that are indifferent to stream length and tolerate multigraphs at intermediate stages.

In this paper, we show that these limitations in the [LNW14] are inherent, presenting natural assumptions on adversarially ordered turnstile streams for which we can prove the first exponential separations between turnstile streaming and linear sketching. We give four different settings in which there are problems that can be solved with a $O(\log n)$ space streaming algorithm, but for which any linear sketch requires $\tilde{\Omega}(n^{1/3})$ space.

Our first setting involves binary streaming: the data stream can have arbitrary length, but must lie in $\{0,1\}^n$ at all times. We present a problem that can be solved over such streams in $O(\log n)$ space, but requires $\Omega(n^{1/3}/\log^{1/3} n)$ space to solve in linear sketching. In our second setting we extend this result to boxes of larger size, applying if the input vector $x$ is in $[-M,M]^n$ for some constant $M$ and intermediate states are allowed to have magnitude as large as $2^M - 1$.

For our first and second results, the problem was designed to produce the separation for a natural ‘box constraint’ on the stream. In our other two results, we focus on a more natural problem: counting triangles in a bounded degree graph on $n$ vertices. Even in degree 2 graphs, distinguishing between $\Theta(n)$ triangles and 0 triangles requires $\Omega(n^{1/3})$ space in linear sketching [KKP18].

Our third result is that an $O(\log n)$ space streaming algorithm for this problem is possible if the stream represents a graph with constant max degree at all intermediate stages, i.e., the graph has the same restriction at intermediate states as it does at the end.

The above results all involve restrictions on the intermediate stream states. Our final result instead focuses on the stream length. We show that an $O(\log n)$ space streaming algorithm is possible in a strict turnstile stream of length $O(n)$, even if the intermediate states are arbitrary multigraphs.

Our two triangle counting algorithms have polynomial dependence on their parameters (maximum degree $d$, length ratio $L/n$, and triangle density $n/T$). Therefore the separations from linear sketching hold even if these grow to small polynomials in $n$ rather than remaining constant.

We now present the definitions required to state our results more formally.

1.1 Definitions

Definition 1. A data stream problem is defined by a relation $P_n \subseteq \mathbb{Z}^n \times \mathbb{Z}$. A turnstile data stream $\sigma$ of length $L$ is a sequence of updates $\sigma_1, \ldots, \sigma_L \in [n] \times \mathbb{Z}$. The state of a stream at time $t$ is given by

$$x^{(t)} := \text{freq}(\sigma^{(t)}) := \sum_{(i,\Delta) \in \{\sigma_1, \ldots, \sigma_t\}} \Delta \cdot e_i.$$ 

and the final state is $x = \text{freq}(\sigma^{(L)})$.

Definition 2. A data stream algorithm $A$ is defined by a random distribution on initial states $y$; a transition function that takes a state $y$ and a stream update $\sigma$ and returns a new state $y'$; and a post-processing function that takes the final state $y$ and returns an output $A(\sigma)$.

We say that $A$ solves a problem $P_n$ in space $S$ under condition $C$ if, for all streams $\sigma \in C$, with $2/3$ probability, $(\text{freq} \sigma, A(\sigma)) \in P_n$ and all states reached by $A$ while processing $\sigma$ can be represented in $S$ bits of space.

Note that this definition, in order to accommodate the [LNW14] reduction, considers only the
space needed in between updates to the stream. It therefore does not rule out the possibility that the transition function between states or the post-processing function requires $\omega(S)$ space to compute. It is also non-uniform, as the algorithms are defined for each $n$. However, every algorithm we present will be uniform and have update and post-processing functions computable in $O(S)$ space.

One very common stream condition considered in the literature is being a ‘strict’ turnstile streams, where $x_i(t) \geq 0$ for all $i$ and $t$. The goal of this paper is to describe relatively mild stream conditions under which turnstile streaming is much easier than linear sketching.

For the problems we consider, which are decision and counting problems, the set of valid outputs for each input forms an interval. Therefore the success probability can always be amplified to $1 - \delta$ by taking the median of $O(\log \frac{1}{\delta})$ repetitions.

**Definition 3.** In a linear sketching algorithm, the state consists of a matrix $A \in \mathbb{Z}^{m \times n}$, a modulus vector $q \in \mathbb{Z}^m$, and the ‘sketch’ $y$ that is, at all times $t$, $Ax(t) \mod q$.

The space required by a linear sketching algorithm is the amount of randomness needed to generate $(A,q)$ plus the $\sum_i \log q_i$ space required to store $y$. For typical algorithms, this is $O(m \log n)$.

**Definition 4 (Box constraint).** $\Gamma_m$ is the set of streams such that for all times $t$, $\|x(t)\|_{\infty} \leq m$. $\Gamma_{0,1}$ is the set of streams such that for all times $t$, $x(t) \in \{0,1\}^n$.

### 1.2 Our Results

**Box-constrained streams.** Our first result concerns binary streams, in which we are promised that the partial stream states $x(t)$ lie in $\{0,1\}^n$ at all times.

**Theorem 5.** For every $n \in \mathbb{N}$, there exists a data stream problem $P_n \subseteq \{0,1\}^n \times \{0,1\}$ such that:

1. Any linear sketching algorithm solving $P_n$ requires $\Omega(n^{1/3} / \log^{1/3} n)$ bits of space.
2. There exists a turnstile streaming algorithm that solves $P_n$ on $\Gamma_{0,1}$ in $O(\log n)$ space.

One property of binary streams is that every update uniquely identifies the coordinate being updated. Over larger domains, this is no longer true. We can still show a similar result for inputs of size $m$, as long as intermediate results never exceed $2m - 1$:

**Theorem 6.** For every $m,n \in \mathbb{N}$, there exists a data stream problem $P_n \subseteq \{-m,\ldots,m\}^n \times \{0,1\}$ such that:

1. Any linear sketching algorithm solving $P_n$ requires $\Omega(n^{1/3} / \log^{1/3} n)$ bits of space.
2. There exists a turnstile streaming algorithm that solves $P_n$ on $\Gamma_{2m-1}$ in $O(\log n \log m)$ space.

Interestingly, this $2m$ threshold matches one of the results in [AHLW16]. Recall that one requirement for [LNW14] to show an equivalence between linear sketching and streaming is that the streaming algorithm tolerate intermediate states of (more than) doubly exponential size, i.e., $\Gamma_{2^{2^n}}$. One result in [AHLW16] shows that this can be relaxed to $\Gamma_{2m}$—as long as $m > 2^{nS}$, where $S$ is the algorithm space. That additional requirement is very strong (e.g., one cannot store a single coordinate of the input) but if it did not exist, the result would imply that our $2m - 1$ threshold cannot be increased.
Graph streams Our separations for binary and box-constrained streams are based on a somewhat unnatural problem. We also present separations for a more natural problem, that of counting triangles in bounded-degree graphs.

In this problem, the final state \( x \in \{0, 1\}^{\binom{n}{2}} \) represents a graph of maximum degree \( d \). In the counting version of the problem, one would like to estimate the number of triangles \( T \) in the graph to within a multiplicative \( 1 \pm \varepsilon \) factor with probability \( 2/3 \); in the decision version, one would like to determine whether the number of triangles is zero or at least \( T \).

In the insertion-only model of computation, the counting problem can be solved in \( O(d^2 m/\varepsilon^2 T \log n) \) space [JG05], while in the linear sketching model it requires \( \Omega(n/T^{2/3}) \) space even for the decision version with \( d = 2 \) [KKP18]. This leaves a natural question: for constant \( d \) and linear \( T \), do turnstile streaming algorithms require \( \log n \) or \( n^{1/3} \) space?

The equivalence between linear sketching and turnstile streaming due to [LNW14, AHLW16] shows that any turnstile algorithm that works for extremely long (more than doubly exponential) streams where the intermediate states may be multigraphs (with doubly exponential multiplicity) must use \( n^{1/3} \) space. We show logarithmic space algorithms under more restrictive assumptions on the stream.

In our first result on this problem, we suppose that the stream represents a bounded degree graph at all times, not just at the end of the stream. In this model, we can match the best known complexity in the insertion-only model [JG05].

**Theorem 7.** There is a streaming algorithm for triangle counting in max-degree \( d \) graphs, over streams with intermediate states of max degree \( d \), that uses \( O\left(\frac{d^2 m}{\varepsilon^2 T} \log n\right) \) bits.

When \( T \) is \( \Theta(n) \), this is \( O(d^3 \log n) \): exponentially smaller than the \( \Omega(n^{1/3}) \) lower bound for linear sketching for constant degree graphs, and still separable up to small polynomial degrees.

In our second result on this problem, we suppose that the total length of the stream is \( L \), but allow the intermediate states to be arbitrary multigraphs.

**Theorem 8.** There is a streaming algorithm for triangle counting in max-degree \( d \) graphs of length-
\( L \) streams using \( O\left(\frac{d^2 L^2}{\varepsilon^2 T} \log n\right) \) bits of space.

For constant degree graphs with \( L \) and \( T \) both \( \Theta(n) \), this is again \( O(\log n) \) rather than the \( \Omega(n^{1/3}) \) required by linear sketching. Note that \( L = O(n) \) is equivalent to saying that at least a constant fraction of the insertions in the stream are never followed by a corresponding deletion; this is a reasonable assumption for real world graph streams such as the Facebook friends graph.

## 2 Related Work

**Equivalences between streaming and linear sketching.** As described above, [LNW14], building on [Gan08], proved that any turnstile streaming algorithm can be implemented as a linear sketch, assuming the streaming algorithm can tolerate arbitrarily long streams that feature arbitrarily complicated intermediate states. The followup work [AHLW16] removed or relaxed some of the restrictions on this equivalence: for example, they show that it still holds if the algorithm only works in the ‘strict’ turnstile model where all intermediate states are non-negative. They also show that it holds if the algorithm only tolerates exponentially large (in the space usage of the algorithm
and the dimension of the problem) intermediate values, rather than doubly exponentially large ones.

Another line of work on the problem has considered XOR streams or other modular updates [KMY16, HLY18]. XOR streams are like binary streams, except that insert and delete updates are indistinguishable. For such streams, [HLY18] shows that for total functions (as opposed to more general relations, as considered in [LNW14] and this paper) the equivalence between streaming and linear sketching holds under much more mild assumptions: as long as the algorithm works on streams of length $\tilde{O}(n^2)$.

**Lower bounds for linear sketches.** The most common lower bound technique in streaming algorithms is the construction of reductions to one-way communication complexity. One encodes a hard one-way communication complexity problem into a stream by encoding Alice's input into the first half of the stream, and Bob's input into the second half. If a solution to the streaming problem yields a solution to the communication problem, this yields a lower bound on the streaming algorithm's space. The hard instances created by this approach tend to be fairly nice: the stream length is never more than $2^n$, for example.

For linear sketching, lower bounds may also be proved by reductions to the more restrictive simultaneous message passing (SMP) model. Rather than Alice sending a short message to Bob, Alice and Bob must both send a short message to a referee, who adds their sketches to solve the problem. (One may also have more than two parties, which is typically more fruitful in the SMP model than in the one-way communication model.)

These lower bounds translate into turnstile streaming lower bounds using [LNW14, AHLW16], but the instances become horrible, leading to weak implications. In particular, this approach can never rule out algorithms using either $O(\log \log L)$ or $O(\log M)$ space, for length-$L$ streams with intermediate states that never leave the $[-M,M]^n$ box.

Still, for a number of problems we only know how to get strong lower bounds via linear sketching. Examples include finding approximate maximum matchings [Kon15, AKLY16], estimating the size of the maximum matching [AKL17], and subgraph counting [KKP18]. Most such problems are graph problems, but the translation of the lower bound from linear sketching to streaming only applies if intermediate states are allowed to be multigraphs.

**Non-linear turnstile algorithms.** We are aware of two turnstile streaming algorithms that are not known to be implementable in linear sketching. They are analogous to our Theorems 7 and 8, in that one assumes all intermediate states of the graph have a bounded parameter, and the other essentially assumes a bound on the length of the stream. But they do not provide a large separation from linear sketching: in one case an exponential separation likely exists but is not known; in the other case a separation is known but is $O(\log n)$.

Chitnis et al [CCHM14] consider the problem of finding a vertex cover of a graph, parameterized by the size of that cover. They show that if the graph has a vertex cover of size $k$ at all times, one can be found in $O(k^2)$ space; by contrast, the best known linear sketching algorithm uses $\tilde{O}(nk)$ space. But it is not known whether an $O(k^2)$ space linear sketching algorithm exists for this problem.

Jayaram and Woodruff [JW18] consider $\ell_p$ estimation of data streams with a bounded number of deletions. The precise result depends on $p$, but roughly speaking: if at least an $\varepsilon < 1/2$ fraction of the updates are insertions, the space complexity can be improved over linear sketches by a factor
of $\log_{1/\varepsilon} n$. So this does prove a separation from linear sketching under similar assumptions to our Theorem 8, but it is a small one.

3 Overview of Techniques

3.1 Binary and Box-Constrained Streams

**Binary streams.** To prove Theorem 5, we embed a hard communication problem from [KKP18] into a binary stream. In this communication problem, which we call TrianglePromise($n$) and illustrate in Figure 1, there are three players and $O(n)$ vertices, each of which is shared between two players. Each player receives a set of $O(n)$ edges, connecting the two sets of vertices shared with the other two players, and a label in $\{0, 1\}$ for each edge. These edges form $n$ disjoint triangles, with each player having one edge from each triangle; every other edge is isolated. The players do not know which of their edges are in triangles. The promise is that for every triangle, the XOR of the associated bits has the same value $\tau \in \{0, 1\}$; the goal is to find $\tau$. In [KKP18] this was shown to take $\Omega(n^{1/3})$ bits of communication in the SMP model.

Each player’s input can be represented in $k = O(n \log n)$ bits. We can define a data stream problem $P \subset \{0, 1\}^3 \times \{0, 1\}$ as follows: for any input $x \in \{0, 1\}^3$, split $x$ into three pieces $x_A, x_B, x_C$, one for each player. If $(x_A, x_B, x_C)$ represents a valid set of inputs to TrianglePromise($n$), let $\tau$ be the corresponding answer and place $(x, \tau)$ in $P$; otherwise, place both $(x, 0)$ and $(x, 1)$ in $P$. Since the players’ inputs are placed in separate coordinates, a linear sketch could solve the SMP communication problem, giving an $\Omega(n^{1/3}) = \Omega(k^{1/3} / \log^{1/3} k)$ lower bound for linear sketches. But how can we solve this problem more efficiently with an arbitrary turnstile streaming algorithm?

This problem is hard in the SMP model because it is difficult for all three players to simultaneously coordinate to sample the same triangle. Any two players can coordinate: they can use shared randomness to sample a shared vertex, and each keep their edge incident to that vertex. But they can’t tell the third player which edge to keep. The idea behind our algorithm is that for any stream, for each triangle some player’s input will finish updating last. As soon as the first two players’ inputs have finished updating, the algorithm will know which of their edges it sampled, and therefore know what parts of the third player’s input are interesting. If the third player’s input hasn’t finished yet, the algorithm will learn at least one bit when it is updated. And to solve TrianglePromise($n$), we only need one bit.

For this to work, we need an encoding of the players’ inputs that satisfies a few properties. We need to be able to sample a vertex, and learn the incident edges if we pay attention for the whole stream. If this vertex is incident to two edges of a triangle, then once we learn one of these edges, we need to know where in the vector to find the encoding of the third edge, and if we learn at least one bit of the third edge’s encoding, we need to be able to compute its bit label $z$ at the end of the stream. This last point might seem tricky, but at the end of the stream the sampled edges tell us both endpoints of the third edge, so $z$ is the only bit we don’t know; it will therefore suffice to store an edge $(u, v, z)$ as $(u \oplus z^B, v \oplus z^B)$ for a slightly larger word size $B$. The precise encoding and recovery algorithm are presented in Sections 4 and 4.3, respectively.

**Box-constrained streams.** For Theorem 6, we take the same instance as for binary streams but place it on $\{-m, m\}^3$. It is no longer the case that, once we start tracking a given coordinate, we can learn its value after a single update. But we can still track the coordinate relative to its
(a) The player’s instance ignoring the permutations. The $x_e$ are the indices of red edges, read from inside out: $x_1 = [1, 0, 1, 1, 0, 1]$, $x_2 = [1, 1, 1, 0, 1]$, $x_3 = [0, 1, 0, 1, 1]$.

| Player 1 | Player 2 | Player 3 |
|----------|----------|----------|
| u  | v  | x  | u  | v  | x  | u  | v  | x  |
| A  | J  | 1  | B  | B  | 1  | C  | D  | 1  |
| C  | A  | 0  | C  | F  | 0  | D  | B  | 1  |
| D  | E  | 0  | D  | I  | 1  | E  | H  | 1  |
| F  | H  | 1  | E  | C  | 1  | G  | A  | 0  |
| G  | B  | 0  | D  | J  | 1  | I  | J  | 0  |

(b) The hard distribution permutes each set of vertices. The players see their edges and associated labels, but not the vertex colors (which represent the pre-permutation identities).

| Player 1 | Player 2 | Player 3 |
|----------|----------|----------|
| u  | v  | u  | v |
| J  | A  | E  | H  | 1  | C  | D  | G  |
| D  | B  | H  | E  | C  | J  | D  |
| G  | A  | J  | I  | J  | 0  |

(c) Each player’s input consists of their edges in (b). $u$ represents the vertex counterclockwise of the player, and $v$ represents the vertex clockwise.

(d) The encoding into $\Sigma^m$. For Theorem 5, each character in $\Sigma$ is encoded into binary; for Theorem 6, the encoding is instead in $\{-m, m\}$.

Figure 1: Illustration of TrianglePromise(4) instance; the true instance would have 36 isolated edges per player, not 2.
initial value, and if the coordinate’s final value is \( m \) more than the smallest value seen, or \( m \) less than the largest value seen, then we will know the coordinate’s value at the end of the stream, as there will be only one of \( \{-m, m\} \) for which this is consistent with staying within \( \Gamma_{2m-1} \).

Now, optimistically decoding based on the sign pattern of each word, we define the ‘last’ player for a triangle as being the player whose input’s decoding achieves its final value last, i.e. the last player to have every coordinate of their input within \( m - 1 \) of its final value. At the time the first two players’ inputs’ decodings achieve their final value, these players will know their sampled edges, and there will be at least one coordinate of the third player’s input that can be learned with the remaining stream.

### 3.2 Bounded Degree Triangle Counting

At a high level, both of our algorithms for bounded-degree triangle counting seek to emulate the insertion-only algorithm of [JG05]. The insertion-only algorithm is as follows: sample edges with probability \( p \), and keep all edges incident to sampled edges. Count the number of triangles using sampled edges (with multiplicity if multiple edges of a triangle are sampled), and divide by \( 3p \). This is an unbiased estimator, using \( O(pm d \log n) \) space, in a graph with \( m \) edges, \( n \) vertices, and max degree \( d \). The expected number of triangles sampled is \( 3pT \). If all the triangles were disjoint, the triangles would be sampled independently and so one could set \( p = O(1/(\epsilon^2 T)) \) and get a \((1 + \epsilon)\)-approximation with 2/3 probability. Even though the triangles are not disjoint, the degree bound keeps the estimator’s variance small; one only needs \( p = O(d/(\epsilon^2 T)) \).

So what happens in turnstile streams? One can run essentially the same algorithm, dealing with edge deletions by removing both the edge deleted and any neighbors that were tracked on its account. This works, but can use too much space if not done carefully.

**Bounded-degree intermediate states.** If every intermediate state is a bounded-degree graph, then the expected amount of space used at any point in the stream is still \( O(pm d \log n) \). However, if the stream is extremely long, the maximum amount of space used will be too large. The natural solution is to have a hard cap of \( O(pm) \) on the number of edges sampled, and to stop sampling edges when at the cap. One might worry that this creates a bias in the estimator. However, the only times this can affect the output of the algorithm are the \( m \) points in time when edges in the final graph are inserted for the last time. At each such time, with high probability, the hard cap will not have been reached. The output of the algorithm will thus be the same as in the insertion-only case.

**Length-constrained streams.** In this model, the intermediate states may be multigraphs with very high degree; call the maximum degree a vertex ever reaches its ‘stream degree.’ One cannot, in general, keep the entire neighborhood of a sampled edge. However, the \( \Omega(T/d) \) edges involved in triangles in the final graph have average stream degree at most \( O(Ld^2/T) \). Therefore we can restrict to considering edges of stream degree \( O(Ld^2/T) \); this loses us at most an \( \epsilon/3d \) fraction of triangle-involved edges, which are involved in at most an \( \epsilon \) fraction of triangles.

Using the same \( p = O(d/(\epsilon^2 T)) \) as in the insertion-only case, we get an algorithm with space

\[
p \cdot L \cdot \frac{Ld^2}{\epsilon T} \cdot \log n = O\left(\frac{d^3 L^2}{\epsilon^3 T^2} \log n\right).
\]
**TrianglePromise(\(n\))**

**Parties:** Let \(V^\Delta\) and \(E^\Delta\) be the vertex and edge sets, respectively, of a triangle \(K_3\). There are three players, one associated with each edge \(e \in E^\Delta\). There is one referee, who receives messages from the three players. No other communication takes place.

**Constants:** Let \(N = 30n\). We define \(N\) vertices \(V_a\) associated with each of the three vertices \(a \in V^\Delta\).

**Inputs:** Each player \(e = ab\) receives a list of \(N/3\) triples \((u, v, z_{uv}) \in V_a \times V_b \times \{0, 1\}\).

**Promise:** The instance satisfies the following promise:

1. No \(u\) or \(v\) appears more than once in any single player’s input. Thus the set of all edges \((u, v)\) in player inputs can be viewed as a graph \(G\) over \(\bigcup_{a \in V^\Delta} V_a\), and this graph has \(N\) edges and \(3N\) vertices.

2. \(G\) contains \(n\) triangles. All \(27n\) other edges are isolated.

3. There exists a \(\tau \in \{0, 1\}\) such that for every triangle \(uvw\) in \(G\),
   \[
   z_{uv} \oplus z_{vw} \oplus z_{wu} = \tau.
   \]

**Goal:** Given the messages received from the players, the referee’s task is to determine whether \(\tau = 0\) or \(\tau = 1\).

Figure 2: Definition of a TrianglePromise instance.

### 4 Box-Constrained Streaming: Problem and lower bound

#### 4.1 Streaming Triangle Game

Our problem is based on encoding an instance of the PromiseCounting\((H, n, T, \varepsilon)\) communication problem from [KKP18] as a binary vector. We will only use the special case where \(H\) is the triangle \(K_3\), \(T = n/10\), and \(\varepsilon = 1\). We refer to this PromiseCounting\((K_3, n, n/10, 1)\) instance as TrianglePromise\((n)\), which we describe in Figure 2 and illustrate in Figure 1.

**Theorem 9** (Implication of Corollary 15 of [KKP18]). Let \(n \geq 1\). Suppose that, for every instance of TrianglePromise\((n)\), no player sends a message of more than \(c\) bits. There exists a universal constant \(\gamma\) such that, if \(c \leq \gamma n^{1/3}\), the probability the referee succeeds is at most 51%.

We note that our TrianglePromise problem is written somewhat differently from the PromiseCounting problem as defined in [KKP18]. Our description is equivalent, however, as suggested in Figure 2 of [KKP18].

Both Theorem 5 and Theorem 6 involve encoding the player’s inputs to TrianglePromise\((n)\) as a frequency vector. The outer encoding, from instances of TrianglePromise\((n)\) to strings from an alphabet \(\Sigma\), is the same for both. The inner encoding will differ, taking strings from \(\Sigma\) to strings
from \{0,1\} and \{-m,m\} for Theorem 5 and Theorem 6 respectively.

For both, the frequency vector will have dimension \(\Theta(n \log n)\). Theorems 5 and 6 then follow by considering an encoding of \textsc{TrianglePromise}(\Theta(n/\log n)).

**Outer Encoding.** We define the alphabet \(\Sigma = ([N] \times \{0,1\}) \cup \{\bot\}\). We encode an instance of \textsc{TrianglePromise}(n) into \(\Sigma^{6N}\) as follows. For each \(e \in E^\Delta\) and \(a \in e\), we create a vector \(y_{e,a}^{e,a} \in \Sigma^N\); the full encoding is the concatenation of the six \(y_{e,a}^{e,a}\).

As illustrated in Figure 1c, the input of player \(e = ab\) consists of a list of \(N/3\) edges \((u,v,z_{uv})\), where each \(u \in V_a\) and \(v \in V_b\). Since \(|V_a| = |V_b| = N\), we can define a canonical bijection from each of \(V_a\) and \(V_b\) into \([N]\); call these \(f_a, f_b\).

Then for every \((u,v,z_{uv})\) in player \(e\)’s list, we set
\[
y_{f_a(u)}^{e,a} := (f_b(v), z_{uv})
y_{f_b(v)}^{e,b} := (f_a(u), z_{uv})
\]
Since each \(u\) appears at most once in \(e\)’s list, this is well defined. This sets \(N/3\) of the \(N\) coordinates in each of \(y_{e,a}^{e,a}\) and \(y_{e,b}^{e,b}\); every other coordinate is set to \(\bot\).

This encoding of the players’ inputs is injective; in fact, either one of \(y_{e,a}^{e,a}\) or \(y_{e,b}^{e,b}\) suffices to recover player \(e\)’s input.

**Inner Encoding.** Let \(B = 1 + \lceil \lg N + 1 \rceil\). For Theorem 5, we encode \(\Sigma\) into \(\{0,1\}^B\). We encode \(\bot\) as \(0^B\). To encode \((l,z) \in [N] \times \{0,1\}\) we first take the standard binary encoding \(l^{\text{bin}}\) of \(l\) into \(\{0,1\}^B\). This is nonzero, since \(l > 0\); and its highest bit is zero, since \(l \leq N\). Then we output the bitwise XOR \(x = l^{\text{bin}} \oplus z^B\).

This encoding is injective, because the highest bit will equal \(z\), after which \(z\) can be removed and \(l\) recovered. Concatenating the outer and inner code gives an injection from the players’ inputs to \(\{0,1\}^{6NB}\).

For Theorem 6, we use the same encoding, and then replace every instance of 1 with \(m\), and every instance of 0 with \(-m\).

**The streaming problem.** We can now define the streaming problem \(P_n\). For any vector \(x\) such that \(x\) is not an encoding of an instance of \textsc{TrianglePromise}(n), \((x,0)\) and \((x,1)\) are in \(P_n\), i.e., any output is acceptable on such an input. For any vector \(x\) such that \(x\) is an encoding of an instance with \(\tau = 0\), \((x,0) \in P_n\), and for any vector \(x\) such that \(x\) is an encoding of an instance with \(\tau = 1\), \((x,1) \in P_n\).

**4.2 Linear Sketching Lower Bound**

By Theorem 9, any protocol for the communication problem that succeeds with probability at least 2/3 requires \(\Omega(n^{1/3})\) bits of communication by at least one player. Now suppose we have a linear sketching algorithm for \(P_n\). Note that the outer code encodes each player’s input into separate coordinates. The inner code, of course, preserves this property. Therefore player \(e\) could encode their part of the problem with the other coordinates set to zero, sketch it, and send it to the referee. The referee can add up these sketches to get a sketch for the full vector \(x\), then determine \(\tau\). Since
each player only sends a message of size equal to the space usage of the linear sketching algorithm, the space used must be $\Omega(n^{1/3})$.

Therefore, $P_n$ satisfies criterion 1 of Theorems 5 and 6. To prove that it satisfies criterion 2, we construct a turnstile algorithm for $P_n$.

### 4.3 Algorithm for TrianglePromise over $\Gamma_{0,1}$

This section will describe an algorithm that either outputs the correct answer or ⊥, and outputs the correct answer with a small positive constant probability. Straightforward probability amplification then can increase the success probability to 2/3.

We start by noting that, for any coordinate $i$, we can establish $x_i$ given any non-empty postfix of the updates to $x_i$, as any increase proves it was previously 0 and any decrease proves it was previously 1.

Recall that any player $e \in E^\Delta$, side $a \in e$, and vertex $u \in V_a$ has an associated symbol $y_{Ja(u)}^{e,a} \in \Sigma$. We use $x^{e,a,u} \in \{0,1\}^B$ to denote the inner encoding of this symbol. The final frequency vector $x$ has $x^{e,a,u}$ placed in a contiguous block, at a position that is easy to find from $(e,a,u)$.

We state the algorithm in Algorithm 1.

**Algorithm 1: Low-probability TrianglePromise over $\{0,1\}$**

1. Let $(a,b,c)$ be a uniformly chosen random labeling of $V^\Delta$. Choose $u \in V_a$ uniformly at random.

2. While passing through the stream:
   (a) Track all updates to $x^{ab,a,u}$ and $x^{ac,a,u}$.
   (b) While doing so, keep checking whether $x^{ab,a,u}$ is a valid inner encoding of $\Sigma$; if it is, and it doesn’t decode to ⊥, then it is an encoding of $(f_b(v'),z)$ for some $v' \in V_b$ and $z'$. Let $(v',z')$ be those values, if they exist.
   (c) As soon as $(v',z')$ is set, track all updates to $x^{bc,b,v'}$. Discard these updates whenever $(v',z)$ changes.

3. After the stream finishes:
   (a) Decode $x^{ab,a,u}$ and $x^{ac,a,u}$ to $\Sigma$.
   (b) If either is ⊥, output ⊥.
   (c) Otherwise, let their decodings be $(f_b(v), z_{uv})$ and $(f_c(w), z_{uw})$ for $v \in V_b$ and $w \in V_c$.
   (d) If the algorithm has not tracked any updates to $x^{bc,b,v}$, output ⊥.
   (e) Otherwise, it knows $x_i^{bc,b,v}$ for some index $i \in [B]$. Let $z_{uv} = x_i^{bc,b,v} \oplus f_c(w)^{(bin)}_i$.
   (f) Output $z_{uv} \oplus z_{uv} \oplus z_{uv}$.

Lemma 10. The space complexity of Algorithm 1 is $O(\log n)$ bits.
Proof. The randomness in step 1 uses $\log(6N)$ bits. After that, the algorithm tracks three length-$B$ vectors; the total space usage is $O(\log n)$. □

**Lemma 11.** Algorithm 1 outputs either $\bot$ or $\tau$. If $u$ is part of a triangle in the underlying $\text{TrianglePromise}(n)$ graph $G$, and the last stream update to $x^{ab,a,u}$ is before the last stream update to $x^{bc,b,v}$, then the algorithm outputs $\tau$.

Proof. Note that $x^{ab,a,u}$ and $x^{ac,a,u}$ are tracked completely, so their final decodings into $\Sigma$ are correct. If $u$ is not part of a triangle, at most one edge is incident to $u$ in the full graph $G$, so at least one of the decodings is $\bot$ and the algorithm returns $\bot$.

Otherwise, if $u$ is part of a triangle, the algorithm correctly deduces $(v,z_{uv})$ and $(w,z_{uw})$. If the algorithm has not seen an update to $x^{bc,b,v}$, it will output $\bot$; otherwise, since it tracks a postfix of the stream, it correctly identifies $x^{bc,b,v}$. Since $uvw$ is a triangle, we know player $bc$ has the input $(v,w,z_{vw})$ for some $vw$, and the inner encoding is

$$x^{bc,b,v}_i = z_{vw} \oplus f_c(w_i^{(\text{bin})}).$$

Thus the algorithm correctly identifies $z_{vw}$, and the $\text{TrianglePromise}(n)$ promise says

$$\tau = z_{uv} \oplus z_{vw} \oplus z_{uw}.$$

Hence the algorithm outputs either $\bot$ or $\tau$. Moreover, it will have deduced $v$ correctly upon the last update to $x^{ab,a,u}$; if this is before the last update to $x^{bc,b,v}$ then it will see at least one update there and output $\tau$. □

**Lemma 12.** Algorithm 1 outputs $\tau$ with at least $\frac{1}{180}$ probability.

Proof. There is a $n/N = 1/30$ chance that $u$ lies in a triangle, independent of the choice of $(a,b,c)$. Furthermore, if it does, which triangle it lies in is independent of the choice of $(a,b,c)$.

Suppose $u$ lies in the triangle $uvw$ with $u \in V_{a'}, v \in V_{b'}, w \in V_{c'}$. One of the three blocks

$$x^{a'b',a',u}, \quad x^{b',b',v}, \quad x^{a'c',c',w}$$

will be the first to finish being updated in the stream. WLOG this is $a'$. Then Lemma 11 says that if $(a,b,c) = (a',b',c')$, Algorithm 1 will output $\tau$. This choice happens with $1/6$ probability; combined with the $1/30$ chance that $u$ lies in a triangle, we get at least a $1/180$ chance of outputting $\tau$. □

**Lemma 13.** There is a turnstile streaming algorithm that solves $P_n$ on $\Gamma_{0,1}$ with probability $2/3$ using $O(\log n)$ bits of space.

Proof. Run Algorithm 1 in parallel 360 times and output any non-$\bot$ result. By Lemma 11 any non-$\bot$ result will be correct. By Lemma 12 the failure probability is at most $(1 - 1/180)^{360} < 1/e^2 < 1/3$. □
4.4 Algorithm for TrianglePromise over $\Gamma_{2m-1}$

We write $\sigma^{(t)}$ for the prefix of $\sigma$ consisting of its first $t$ updates. Define the error correction function $\kappa$ by

$$
\kappa(z)_i = \begin{cases} 
    m & z_i > 0 \\
    -m & z_i < 0 \\
    0 & z_i = 0
\end{cases}
$$

and define the decoding function $\eta : \{-m, m\}^* \to \{0, 1\}$ by:

$$
\eta(z)_i = \begin{cases} 
    1 & z_i = m \\
    0 & z_i = -m
\end{cases}
$$

We will use the following decoding lemma in our algorithm:

**Lemma 14.** Let $\sigma$ be a stream in $\Gamma_{2m-1}$ such that $\text{freq } \sigma \in \{-m, -m\}^*$. Then for any $i$, and for any split of the stream $\sigma = \sigma_1 \cdot \sigma_2$,

1. $\min_i(\text{freq } \sigma_2^{(t)})_i \leq (\text{freq } \sigma_2)_i - m \Rightarrow \eta(\text{freq } \sigma)_i = 1$
2. $\max_i(\text{freq } \sigma_2^{(t)})_i \geq (\text{freq } \sigma_2)_i + m \Rightarrow \eta(\text{freq } \sigma)_i = 0$

and one of these conditions holds iff $\exists t$ such that $\kappa(\text{freq } \sigma_1 \cdot \sigma_2^{(t)})_i \neq \kappa(\text{freq } \sigma)_i$.

**Proof.** Suppose $\min_i(\text{freq } \sigma_2^{(t)})_i \leq (\text{freq } \sigma_2)_i - m$. Then if $\eta(\text{freq } \sigma)_i = 0$, $(\text{freq } \sigma)_i = -m$. Let $t$ be a minimizer of $(\text{freq } \sigma_2^{(t)})_i$, so

$$
(\text{freq } \sigma^{(|\sigma|+t)})_i = (\text{freq } \sigma_1)_i + (\text{freq } \sigma_2^{(t)})_i \\
\leq (\text{freq } \sigma_1)_i + (\text{freq } \sigma_2)_i - m \\
= (\text{freq } \sigma)_i - m \\
= -2m
$$

but by the box constraint $(\text{freq } \sigma^{(t)})_i \geq -2m + 1$, giving a contradiction. So $\eta(\text{freq } \sigma)_i = 1$.

Likewise, if $\max_i(\text{freq } \sigma_2^{(t)})_i \geq (\text{freq } \sigma_2)_i + m$, there exists $t$ such that if $\eta(\text{freq } \sigma)_i = 1$, $(\text{freq } \sigma^{(|\sigma|+t)})_i \geq 2m$, so it must be the case that $\eta(\text{freq } \sigma)_i = 0$.

For the final part of the lemma, note that one of the conditions holds iff

$$
\max_t |(\text{freq } \sigma_2^{(t)})_i - (\text{freq } \sigma_2)_i| \geq m \Leftrightarrow \max_{t \geq |\sigma|} |(\text{freq } \sigma^{(t)})_i - (\text{freq } \sigma)_i|
$$

which as $(\text{freq } \sigma)_i = \pm m$, holds iff there is a $t \geq |\sigma|$ such that either $(\text{freq } \sigma^{(t)})_i \leq 0$ and $(\text{freq } \sigma)_i = m$, or $(\text{freq } \sigma^{(t)})_i \geq 0$ and $(\text{freq } \sigma)_i \leq m$, which in turn holds iff $\kappa(\text{freq } \sigma^{(t)}) \neq \kappa(\text{freq } \sigma)$. □

The algorithm is described in Algorithm 2.

**Lemma 15.** The space complexity of Algorithm 2 is $O(\log n \log m)$ bits.

**Proof.** The randomness in step 1 uses $\log(6N)$ bits. After that, the algorithm tracks three length-$B$ vectors with entries in $\{-m, m\}$; the total space usage is $O(\log n \log m)$. □
Algorithm 2: Low-probability TrianglePromise over $\Gamma_{2m-1}$

1. Let $(a, b, c)$ be a uniformly chosen random labeling of $V^\Delta$. Choose $u \in V_a$ uniformly at random.

2. While passing through the stream:
   
   (a) Track all updates to $x^{ab,a,u}$ and $x^{ac,a,u}$.
   
   (b) While doing so, keep checking whether $\kappa(x^{ab,a,u})$ is a valid inner encoding of $\Sigma$; if it is, and it doesn’t decode to $\bot$, then it is an encoding of $(f_b(v'), z)$ for some $v' \in V_b$ and $z'$. Let $(v', z')$ be those values, if they exist.
   
   (c) As soon as $(v', z')$ is set, track all updates to $x^{bc,b,v'}$, recording the current, minimum, and maximum value of each of its coordinates. Discard these updates whenever $(v', z)$ changes.

3. After the stream finishes:
   
   (a) Decode $\kappa(x^{ab,a,u})$ and $\kappa(x^{ac,a,u})$ to $\Sigma$.
   
   (b) If either is $\bot$, output $\bot$.
   
   (c) Otherwise, let their decodings be $(f_b(v), z_{uv})$ and $(f_c(w), z_{uw})$ for $v \in V_b$ and $w \in V_c$.
   
   (d) If the final observed value for $x^{bc,b,v}$ is within $m-1$ of all the values the algorithm has observed for it, output $\bot$.
   
   (e) Otherwise, by Lemma 14 it knows $\eta(x^{bc,b,v})_i$ for some index $i \in [B]$. Let $z_{uw} = \eta(x^{bc,b,v})_i \oplus f_c(w)_i^{(bin)}$.
   
   (f) Output $z_{uv} \oplus z_{vw} \oplus z_{uw}$.
Lemma 16. Algorithm 2 outputs either $\bot$ or $\tau$. If $u$ is part of a triangle in the underlying TrianglePromise$(n)$ graph $G$, and the last time $\kappa(x^{ab,a,u})$ differs from its final value is before the last time $\kappa(x^{bc,b,v})$ differs from its final value, then the algorithm outputs $\tau$.

Proof. Note that $x^{ab,a,u}$ and $x^{ac,a,u}$ are tracked completely, so their final decodings into $\Sigma$ are correct. If $u$ is not part of a triangle, at most one edge is incident to $u$ in the full graph $G$, so at least one of the decodings is $\bot$ and the algorithm returns $\bot$.

Otherwise, if $u$ is part of a triangle, the algorithm correctly deduces $(v, z_{uv})$ and $(w, z_{uw})$. If the last time $\kappa(x^{ab,a,u})$ differs from its final value is after the last time $\kappa(x^{bc,b,v})$ differs from its final value, then at the time the algorithm starts tracking $x^{bc,b,v}$, $\kappa(x^{bc,b,v})$ has already its final value, and so by Lemma 14, the final observed value for $x^{bc,b,v}$ is within $m - 1$ of all the values observed for it, and so the algorithm outputs $\bot$. Otherwise, by Lemma 14, the algorithm correctly identifies $\eta(x^{bc,b,v})$.

Since $uvw$ is a triangle, we know player $bc$ has the input $(v, w, z_{vw})$ for some $vw$, and we know

$$\eta(x^{bc,b,v})_i = z_{vw} \oplus f_c(w)_i^{(bin)}.$$ 

Thus the algorithm correctly identifies $z_{vw}$, and the TrianglePromise$(n)$ promise says

$$\tau = z_{uv} \oplus z_{vw} \oplus z_{uw}.$$ 

Hence the algorithm outputs either $\bot$ or $\tau$, and the last time $\kappa(x^{ab,a,u})$ differs from its final value is before the last time $\kappa(x^{bc,b,v})$ differs from its final value, then the algorithm outputs $\tau$.

Lemma 17. Algorithm 2 outputs $\tau$ with at least $\frac{1}{180}$ probability.

Proof. There is a $n/N = 1/30$ chance that $u$ lies in a triangle, independent of the choice of $(a, b, c)$. Furthermore, if it does, which triangle it lies in is independent of the choice of $(a, b, c)$.

Suppose $u$ lies in the triangle $uvw$ with $u \in V_a', v \in V_b', w \in V_c'$. WLOG, let $\kappa(x^{a'b',a',u})$ stop changing before $\kappa(x^{b'c',b',v})$ or $\kappa(x^{c'a',c',w})$.

Then Lemma 16 says that if $(a, b, c) = (a', b', c')$, Algorithm 2 will output $\tau$. This choice happens with 1/6 probability; combined with the 1/30 chance that $u$ lies in a triangle, we get at least a 1/180 chance of outputting $\tau$.

Lemma 18. There is a turnstile streaming algorithm that solves $P_n$ on $\Gamma_{2m-1}$ with probability $2/3$ using $O(\log n \log m)$ bits of space.

Proof. Run Algorithm 2 in parallel 360 times and output any non-$\bot$ result. By Lemma 16 any non-$\bot$ result will be correct. By Lemma 17 the failure probability is at most $(1 - 1/180)^{360} < 1/e^2 < 1/3$.

5 Restricted Intermediate State Triangle Counting

5.1 Problem

Valid inputs to our problem will be as follows (for invalid inputs, any output is accepted): $x$ will be a binary string indexed by $E(K_n)$, the set of all possible edges on an $n$-vertex graph. We will
associate it with a graph $G$ on $n$ vertices with edge set \{$e \in E(K_n) : x_e = 1$\}. Finally, $G$ has max degree $d$.

Instead of bounding the length of the stream, we will require that $x^{(t)}$ correspond to a graph $G$ with max degree $d$ for all $t$. One consequence of this is that all updates will be in $[-1, 1]$.

Our problem will be to estimate $T$, the number of triangles in the graph, up to some multiplicative precision $\varepsilon$. Our algorithm will succeed in doing this if the space allocated to it is large enough in terms of $T$. This space requirement is decreasing in $T$, so we may express this as a data stream problem in the sense of Definition 1 by choosing a lower bound $T'$ and making any answer acceptable for an input vector $x$ that does not correspond to a valid input or results in $T < T'$, and making all outputs in $[(1 - \varepsilon)T, (1 + \varepsilon)T]$ acceptable for input vectors that correspond to a valid graph with $T \geq T'$.

5.2 Linear Sketching Lower Bound

By Theorem 7 of [KKP18], any sketching algorithm for this problem requires $\Omega(m/T^{1/3})$ bits. The requirement that $d$ be constant does not affect this, as the [KKP18] reduction is on graphs of max degree 2. Neither does the intermediate state requirement, as the output of a sketching algorithm depends only on the final state of the stream.

5.3 Algorithm

1. Initialize our set of seed edges $S = \emptyset$. Let $h : E \rightarrow \{0, 1\}$ be a threewise independent hash function where $h(e) = 1$ with probability $p$.

2. While passing through the stream:

   (a) On receiving an update $(e, +1)$:
       • If $h(e) = 1$ and $|S| \leq 2pm$, add $e$ to $S$, and initialize $S_e$ as $\emptyset$.
       • If $\exists f \in S$ such that $e$ is incident to $f$, add $e$ to $S_f$.

   (b) On receiving an update $(e, -1)$:
       • Remove it from any of $S$ and the sets $S_f$ that contain it.
       • Delete the set $S_e$ if it exists.

3. For each $e = uv$, set
   $$\tilde{T}_e = \begin{cases} p^{-1}|\{w : uw, vw \in S_e\}| & \text{if } e \in S \\ 0 & \text{otherwise.} \end{cases}$$

4. Return $\tilde{T} = \sum_e \tilde{T}_e$.

5.4 Space Complexity

Lemma 19. This algorithm requires $O(pdm \log n)$ bits of space.

Proof. The set $S$ has size at most $2pm$ at any point in time, and for each element $e$ in $S$ at most $2d - 1$ edges are kept (as each endpoint of $e$ has degree at most $d$ at all times), and each edge takes $O(\log n)$ bits of space to store. \qed
5.5 Correctness

**Definition 20.** $G^{(t)}$ and $S^{(t)}$ denote the state of $G$ and $S$ respectively after the first $t$ updates, so that $G^{(L)} = G$ and $S^{(L)} = S$.

**Definition 21.** For any edge $e \in E$, let $t_e$ denote the time of the last update made to $e$. For any triangle $\tau \in G$, let $\rho(\tau)$ denote the edge $e \in \tau$ that minimizes $t_e$. Then:

$$T_e = |\{\tau : \rho(\tau) = e\}|$$

Note that as each triangle $\tau$ has exactly one $e$ such that $\rho(\tau) = e$, $T = \sum_e T_e$.

**Definition 22.** Let $Q^{(t)} = \{e \in E(G^{(t)}) : h(e) = 1\}$, $Q = Q^{(L)}$, and $Q_e = \{f \text{ incident to } e : t_f > t_e\}$. Then:

$$\tilde{T}_e^+ = \begin{cases} p^{-1}|\{w : uw, vw \in Q_e\}| & \text{if } e \in Q \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{T}^+ = \sum_e \tilde{T}_e^+$$

**Lemma 23.**

$$\mathbb{E}[\tilde{T}^+] = T$$

$$\text{Var}(\tilde{T}^+) \leq p^{-1}dT$$

**Proof.** For each $e \in E(G)$, $\tilde{T}_e^+ = p^{-1}T_e$ if $h(e) = 1$ and 0 otherwise. So

$$\mathbb{E}[\tilde{T}_e^+] = T_e$$

$$\text{Var}(\tilde{T}_e^+) \leq T_e^2/p$$

$$\leq dT_e/p$$

and as $h$ is threewise independent:

$$\mathbb{E}[\tilde{T}^+] = \sum_e T_e$$

$$= T$$

$$\text{Var}(\tilde{T}^+) = \sum_e \text{Var}(\tilde{T}_e^+)$$

$$\leq dT/p.$$

**Lemma 24.** For any $e \in Q$, if $|S^{(t_e-1)}| < 2pm$, $\tilde{T}_e = \tilde{T}_e^+$. Otherwise, $\tilde{T}_e = 0$.

**Proof.** If $e \in Q$, it will be in $S$ unless $S$ is size $2pm$ at the final time it would be added (if it is added earlier, it will be deleted before time $t_e$, so only the size of $S^{(t_e)}$ matters). Furthermore, if it is added, the edges in $S_e$ will be precisely those edges of $G$ that have their final update after $S_e$ is created for the last time, that is, after $t_e$. So if $|S^{(t_e-1)}| < 2pm$, $\tilde{T}_e = \tilde{T}_e^+$.

On the other hand, if $|S^{(t_e-1)}| = 2pm$, then $e \not\in S^{(t_e-1)}$, as it will have been deleted since the last time it might have been added, $e \not\in S^{(t_e)}$, as it will not be added, and so $e \not\in S$, as there are no more updates to $e$. 

\[ \Box \]
Lemma 25. For all $e \in E(G)$:

$$P \left[ |S^{(t_e-1)}| = 2pm \mid h(e) = 1 \right] \leq 1/pm$$

Proof. By the intermediate state condition on $G^{(t_e-1)}$, it has at most $m$ edges. Then as $S^{(t_e-1)} \subseteq Q^{(t_e-1)}$, and as $h$ is threewise independent and $h(e) = 1$ with probability $p$,

$$E \left[ |Q^{(t_e-1)}| \mid h(e) = 1 \right] \leq pm$$

$$\operatorname{Var} \left( |Q^{(t_e-1)}| \mid h(e) = 1 \right) \leq (p - p^2)m$$

so by Chebyshev’s inequality:

$$P \left[ |S^{(t_e-1)}| = 2pm \mid h(e) = 1 \right] \leq P \left[ |Q^{(t_e-1)}| \geq 2pm \mid h(e) = 1 \right] \leq 1/pm$$

Lemma 26.

$$E \left[ |\bar{T} - \bar{T}^+| \right] \leq T/pm$$

Proof. By Lemma 24, $|\bar{T}_e - \bar{T}^+_e| = p^{-1}T_e$ if $h(e) = 1$ and $|S^{(t_e-1)}| = 2pm$, and 0 otherwise. So, Lemma 25:

$$E \left[ |\bar{T} - \bar{T}^+| \right] \leq \sum_e E \left[ |\bar{T}_e - \bar{T}^+_e| \right] \leq \sum_e p^{-1}T_e P \left[ |S^{(t_e-1)}| = 2pm \wedge h(e) = 1 \right] \leq \sum_e (T_e/p^2m) P \left[ h(e) = 1 \right] = T/pm$$

Theorem 7. There is a streaming algorithm for triangle counting in max-degree $d$ graphs, over streams with intermediate states of max degree $d$, that uses $O \left( \frac{d^2 m}{\epsilon^2 T} \log n \right)$ bits.

Proof. Let the algorithm be run with $p = 32d/\epsilon^2 T$. Then by Lemma 26,

$$E \left[ |\bar{T} - \bar{T}^+| \right] \leq T^2/32dm$$

$$\leq T/32 \quad \text{as } T \leq dm.$$ 

Therefore, by Markov’s inequality:

$$P \left[ |\bar{T} - \bar{T}^+| \geq \epsilon T/2 \right] \leq 1/16$$

18
Then, by Lemma 23,
\[ E[\tilde{T}^+] = T \]
\[ \text{Var}(\tilde{T}^+) \leq T^2/8 \]
and so by Chebyshev’s inequality,
\[ P[|\tilde{T}^+ - T| \geq \epsilon T/2] \leq 1/4 \]
so:
\[ P[|\tilde{T} - T| \geq \epsilon T] \leq 5/16 \]
Therefore, by running \( O(\log 1/\delta) \) copies of the algorithm in parallel and taking the median, we can output a \((1 \pm \epsilon)\) multiplicative approximation to \( T \) with probability \( 1 - \delta \).

6 Bounded-Length Triangle Counting

6.1 Problem

We will work in the strict turnstile model, so our input vector \( x = \text{freq}\sigma(L) \) is non-negative at all intermediate steps.

Valid inputs to our problem will be as follows (for invalid inputs, any output is accepted): \( x \) will be indexed by \( E(K_n) \), the set of all possible edges on an \( n \)-vertex graph. We will associate it with a graph \( G \) on \( n \) vertices with edge set \( \{e \in E(K_n) : x_e = 1\} \). \( x \) is binary, but its intermediate states may not be. Finally, \( G \) has max degree \( d \).

Our problem will be to estimate \( T \), the number of triangles in the graph, up to some multiplicative precision \( \epsilon \). Our algorithm will succeed in doing this if the space allocated to it is large enough in terms of \( T \). This space requirement is decreasing in \( T \), so we may express this as a data stream problem in the sense of Definition 1 by choosing a lower bound \( T' \) and making any answer acceptable for an input vector \( x \) that does not correspond to a valid input or results in \( T < T' \), and making all outputs in \([(1 - \epsilon)T, (1 + \epsilon)T]\) acceptable for input vectors that correspond to a valid graph with \( T \geq T' \).

6.2 Linear Sketching Lower Bound

By Theorem 7 of [KKP18], any sketching algorithm for this problem requires \( \Omega(m/T^{1/3}) \) bits. The requirement that \( d \) be constant does not affect this, as the [KKP18] reduction is on graphs of max degree 2, and neither do the stream length and strict turnstile requirements, as they will not affect the output of any linear sketch.

6.3 Algorithm

1. Initialize our set of seed edges \( S = \emptyset \). Let \( h : E \to \{0, 1\} \) be a pairwise independent hash function where \( h(e) = 1 \) with probability \( p \).

2. While passing through the stream, on receiving an update \((e, \chi)\):

19
• If \( h(e) = 1 \), and there is no tuple \((e, \gamma) \in S\), add \((e, \chi)\) to \(S\).
• If \( h(e) = 1 \), and \((e, \gamma) \in S\), replace it with \((e, \chi + \gamma)\).
• If \((e, \chi)\) has been added to \(S\) for some \(\chi > 0\), initialize the set \(S_e = \emptyset\).
• If \((e, 0)\) is now in \(S\), delete \(S_e\).

Then, for each \(f\) incident to \(e\) such that \((f, z) \in S\):

– If \((e, \gamma) \in S_f\), replace it with \((e, \max(\chi + \gamma, 0))\).
– Otherwise, insert \((e, \max(\chi, 0))\) into \(S_f\), unless \(|S_f| \geq \frac{2pL}{\varepsilon T}\).

3. For each edge \(e = uv\), set:

\[
\tilde{T}_e = \begin{cases} 
    p^{-1}|\{w : (uw, 1), (vw, 1) \in S_e\}| & \text{If } (e, 1) \in S, \\
    0 & \text{Otherwise.}
\end{cases}
\]

4. Return \(\tilde{T} = \sum_e \tilde{T}_e\).

### 6.4 Space Complexity

**Lemma 27.** The expected space complexity of this algorithm is at most \(O\left(\frac{pd^2L^2}{\varepsilon T} \log n\right)\) bits.

**Proof.** Each edge in the stream is independently included in \(S\) with probability \(p\), so the expected maximum size of \(S\) is at most \(pL\). For each element of \(S\) we keep an integer of size \(\text{poly}(n)\), requiring \(O(\log n)\) bits, and a set of size no more than \(\frac{2pL}{\varepsilon T}\). The elements of these sets are edges of an \(n\)-vertex graph, and integers of size \(\text{poly}(n)\), and therefore require \(O(\log n)\) bits each to represent. \(\square\)

### 6.5 Correctness

Consider some fixed (strict) turnstile stream of length \(L\). Let \(G\) be the graph with vertex set \([n]\) and edge set \(\{e \in E : x_e = 1\}\), and let \(T\) be the number of triangles in \(G\). We will seek to show that this algorithm can approximate \(T\).

**Definition 28.** For any edge \(e \in G\), let \(t_e\) be the largest \(t \in [L]\) such that:

\[
x_e^{(t-1)} = 0
\]

\[
x_e^{(t)} > 0
\]

For any triangle \(\tau \in G\), let \(\rho(\tau) \in \tau\) be the edge of \(\tau\) that maximizes \(t_{\rho(\tau)}\). Then, define:

\[
T_e = |\{\tau \in G : \rho(\tau) = e\}|
\]

Note that as each triangle \(\tau\) has exactly one edge \(e\) such that \(\rho(\tau) = e\), \(\sum_e T_e = T\).

**Definition 29.** For any \(t \geq t_e\), \(Q_e^{(t)}\) is the set generated by the following procedure:

• For \(t' = t_e, \ldots, t\), and \((f, \chi) = \sigma_{t'}\), if \(f\) is incident to \(e\):
  – If \((f, \gamma) \in Q_e^{(t)}\), replace it with \((e, \max(\chi + \gamma, 0))\).
Lemma 30. For any \( e \) such that \( h(e) = 1 \),
\[
Q_e^{(L)} \supseteq S_e
\]
with equality when
\[
|Q_e^{(L)}| \leq \frac{2d^2L}{\varepsilon}.
\]
Proof. As \( h(e) = 1 \), \( S_e \) will be deleted and recreated for the final time at \( t_e \). After this point, the procedures for creating \( S_e \) and \( Q_e^{(L)} \) are identical as long as \( |S_e| \) (and therefore \( Q_e^{(L)} \)) never reaches size \( \frac{2d^2L}{\varepsilon} \). If it does, the only difference is that some edges may be excluded from \( S_e \).

For any \((f, z)\) such that \( f \in Q_e^{(t)} \) we will also write \( f \in Q_e^{(t)} \), and \( Q_e^{(t)}[f] = z \). Note that \( Q_e^{(r)}[f] \) is well-defined whenever \( f \in Q_e^{(t)} \) (as no edge is added to \( Q_e^{(t)} \) more than once) and \( f \in Q_e^{(t)} \Rightarrow f \in Q_e^{(t+1)} \) (as no edges are ever removed from \( Q_e^{(r)} \).

Lemma 31. For all edges \( f \) incident to \( e \) and integers \( t \in [t_e, L] \),
\[
Q_e^{(t)}[f] = x_f^{(t)} - \min_{r=t_e,\ldots,t} x_f^{(r)}
\]
Proof. We proceed by induction on \( t \). If \( t = t_e \), as the update at time \( t_e \) was to \( e \), \( Q_e^{(t)}[f] = 0 \) and so the result holds. Now suppose \( t > t_e \) and \( Q_e^{(t-1)}[f] = x_f^{(t-1)} - \min_{r=t_e,\ldots,t-1} x_f^{(r)} \).

Then, let \( \sigma_t = (f', \chi) \). If \( f' \neq f \) both sides of the equation are unchanged and we are done. So suppose the update is \((t, f, \chi)\). We will consider two cases.

\( Q_e^{(t-1)}[f] + \chi \geq 0 \) Then \( Q_e^{(t)} = Q_e^{(t-1)}[f] + \chi \) and \( x_f^{(t)} = x_f^{(t-1)} + \chi \). Furthermore, \( \chi \geq -Q_e^{(t-1)}[f] \), so we have:
\[
\begin{align*}
x_f^{(t)} &= x_f^{(t-1)} + \chi \\
&\geq x_f^{(t-1)} - Q_e^{(t-1)}[f] \\
&= \min_{r=t_e,\ldots,t} x_f^{(r)}
\end{align*}
\]

So \( \min_{r=t_e,\ldots,t} x_f^{(r)} = \min_{r=t_e,\ldots,t-1} x_f^{(r)} \), completing the proof.

\( Q_e^{(t-1)}[f] + \chi < 0 \) Then \( Q_e^{(t)} = 0 \), and:
\[
\begin{align*}
x_f^{(t)} &= x_f^{(t-1)} + \chi \\
&= x_f^{(t-1)} - Q_e^{(t-1)} \\
&= \min_{r=t_e,\ldots,t} x_f^{(r)}
\end{align*}
\]

So \( \min_{r=t_e,\ldots,t} x_f^{(r)} = x_f^{(t)} \), and so \( x_f^{(t)} - \min_{r=t_e,\ldots,t} x_f^{(r)} = 0 \), completing the proof.

\( \square \)
Definition 32. For any vertex $x$, let the ‘stream degree’ $l_v$ be the number of edges $e$ incident to $x$ such that there is some update $\sigma_t = (e, \chi)$, regardless of whether $e$ is in the final graph $G$.

Lemma 33. Let $e = uv$ be an edge. Then

$$\tilde{T}_e = \begin{cases} \frac{T_e}{p} & \text{with probability } p \\ 0 & \text{otherwise.} \end{cases}$$

where $\tilde{T}_e = T_e$ if $l_u + l_v \leq \frac{2d_L}{\varepsilon T}$, and $\tilde{T}_e \in [0, T_e]$ otherwise.

Proof. Let $e$ be an edge. If $h(e) = 0$, $(e, 1) \not\in S$, and so $\tilde{T}_e = 0$. This event happens with probability $1 - p$. If $h(e) = 1$ but $e \not\in G$, $x_e = 0$, and so $(e, 1) \not\in S$, so $\tilde{T}_e = 0 = T_e = \tilde{T}_e$.

Now consider the case where $h(e) = 1$ and $e$ in $G$. Then $x_e = 1$, so $(e, 1) \in S$. $\tilde{T}_e$ will then be $p^{-1}$ times the number of triangles $uvw$, where $e = uv$ and $(uw, 1), (vw, 1) \in S_e$. If $l_u + l_v \leq \frac{2d_L}{\varepsilon T}$, then $|Q_e(L)| \leq \frac{2d_L}{\varepsilon T}$ and so by Lemma 30, $Q_e(L) = S_e$, and otherwise $Q_e(L) \supsetneq S_e$.

So it will suffice to show that

$$|\{(w : (uw, 1), (vw, 1) \in Q_e(L))| = |\{\tau \in G : \rho(\tau) = e\}|$$

which implies our result, as it means that $w \in \{w : (uw, 1), (vw, 1) \in Q_e(L)\}$ iff the triangle $uvw$ has $t_{uv} < t_{uw}, t_{uw}$.

For any $f \in E$ incident to $e$, by Lemma 31, $(f, 1) \in Q_e(L)$ iff $x^{(L)}_f - \min_{r = t_e, \ldots, L} x^{(r)}_f = 1$. If $f \not\in G$, then $x^{(L)}_f = 0$ and so this cannot hold, as $x^{(r)}_f \geq 0$ for all $r$. If $f \in G$, then $x^{(L)}_f = 1$ and so this holds iff $\min_{r = t_e, \ldots, L} x^{(r)}_f = 0$, that is, iff $t_f > t_e$. So $(f, 1) \in Q_e(L)$ iff $f \in G$ and $t_f > t_e$, concluding the proof.

\[ \]
Furthermore, as the final graph has max degree \( d \), at most \( \left( \frac{d}{2} \right) \leq d^2 / 2 \) triangles use any vertex. So we have:

\[
L \geq \frac{1}{2} \sum_v l_v \\
\geq \frac{1}{d^2} \sum_{\tau,uv: \rho(\tau) = uv} l_u + l_v \\
\geq \frac{1}{d^2} T - \frac{2d^2 L}{\varepsilon T}
\]

So \( T^- \leq \varepsilon T / 2 \), and the result follows. \( \square \)

**Lemma 35.**

\[ \text{Var}(\tilde{T}) \leq p^{-1}dT \]

**Proof.** For any fixed stream \( \Sigma \), each \( \tilde{T}_e \) depends only on whether \( h(e) = 1 \), and so as \( h \) is pairwise independent, so are the \( \tilde{T}_e \), and so:

\[
\text{Var}(\tilde{T}) = \sum_e \text{Var}(\tilde{T}_e) \\
\leq \sum_e \mathbb{E} \left[ \tilde{T}_e^2 \right] \\
\leq \sum_e \mathbb{P} [h(e) = 1] p^{-2}T_e^2 \\
\leq \sum_e p^{-1}dT_e \\
= p^{-1}dT
\]

**Theorem 8.** There is a streaming algorithm for triangle counting in max-degree \( d \) graphs of length-\( L \) streams using \( O \left( \frac{d^3L^2}{\varepsilon^2T} \log n \right) \) bits of space.

**Proof.** By Lemma 35, we may set \( p \) in the above algorithm to be \( \frac{16d}{\varepsilon T} \), so that the algorithm requires \( O \left( \frac{d^3L^2}{\varepsilon^2T} \log n \right) \) space and \( \text{Var}(\tilde{T}) = \frac{\varepsilon^2T^2}{16} \). Then, by Chebyshev’s inequality, the probability that \( |\tilde{T} - \mathbb{E} [\tilde{T}]| \geq \varepsilon T / 2 \) is at most \( 1/4 \).

We may then repeat the algorithm \( O(\log 1/\delta) \) times in parallel, taking the median, so that our final output is within \( \varepsilon T / 2 \) of \( \mathbb{E} [\tilde{T}] \) with probability \( 1 - \delta \). By Lemma 34, this implies it is within \( \varepsilon T \) of \( T \). \( \square \)

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