Symmetries of conformal correlation functions

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A program of wide interest in modern conformal bootstrap studies is to numerically solve general conformal field theories, based on a critical assumption that the dynamics is encoded in the conformal four-point crossing equations and positivity condition. In this letter we propose and verify a novel algebraic property of the crossing equations which provides strong restriction for this program. We show for various types of symmetries $\mathcal{G}$, the crossing equations can be linearly converted into the $SO(N)$ vector crossing equations associated with the $SO(N) \to \mathcal{G}$ branching rules and the transformations satisfy positivity condition. The dynamics constrained by the $\mathcal{G}$-symmetric crossing equations combined with positivity condition degenerates to the $SO(N)$ symmetric cases, while the non-$SO(N)$ symmetric theories are not directly solvable without introducing the $SO(N)$ symmetry breaking assumptions on the spectrum.

I. INTRODUCTION

The bootstrap approach has been proposed in the early stage of quantum field theory studies, which aims to solve theories using general consistency criteria, particularly the crossing symmetry and unitarity. The bootstrap method for conformal field theories (CFTs) has been proposed in \[1, 2\] with remarkable successes in the classification of 2D rational CFTs \[3\] and the reviving of modern conformal bootstrap \[4, 5\], see e.g. \[6–9\] for part of the significant developments. Stimulated by the success of modern conformal bootstrap, it is widely interested whether general CFTs can be solved using bootstrap approach, as also emphasised in \[3\]: “can we use the bootstrap to fully classify the space of critical CFTs with a given symmetry...?”. For a majority of CFTs with global symmetries, this project can be rephrased as: can the CFTs with a symmetry $\mathcal{G}$ be solved using the $\mathcal{G}$-symmetric crossing equations combined with positivity and few general assumptions on the spectrum? The key assumption is that the dynamics of general CFTs is encoded in the crossing equations and positivity condition. In this work, we will propose and verify a novel algebraic property that the conformal four-point crossing equations with different global symmetries are actually endowed with an $SO(N)$ symmetric positive structure, which leads to strong challenges for the above project.

The motivation of this work originates from rapid developments in conformal bootstrap studies revived since the breakthrough work \[1\]. Among several remarkable developments \[4, 5\], there is a mysterious phenomenon: the bootstrap bounds from crossing equations with different global symmetries can coincide with each other although the bootstrap equations have distinct forms for different symmetries. Such bound coincidences were firstly observed in \[10\], which shows the singlet bounds from the $SO(2N)$ vector and $SU(N)$ fundamental bootstrap are the same. More examples for the bootstrap bound coincidences with different symmetries have been observed in \[11–17\]. The bound coincidences indicate intriguing connections between the bootstrap algorithm and symmetry properties of the crossing equations. This puzzle has been solved recently in \[17\], which discovered a subtle algebraic relation between the $SU(N)$ fundamental and $SO(2N)$ vector crossing equations: the two sets of crossing equations are connected by a linear transformation consistent with the positivity constraint in the bootstrap algorithm. Due to this relation, the linear functionals for the $SO(2N)$ vector bootstrap can be used to construct the linear functionals for the $SU(N)$ fundamental bootstrap, which further leads to the bootstrap bound coincidences. In this work, we will provide convincing evidence for similar algebraic relations of general global symmetries. We will show that this algebraic structure has surprising applications on the symmetries of conformal four-point correlation functions and the conformal bootstrap program.

II. SYMMETRIES IN CROSSING EQUATIONS

Crossing equations are obtained by taking operator product expansions (OPEs) of the conformal four-point correlator $(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4)$ in either $(\mathcal{O}_1 \mathcal{O}_2) - (\mathcal{O}_3 \mathcal{O}_4)$ channel (s-channel) or $(\mathcal{O}_1 \mathcal{O}_3) - (\mathcal{O}_2 \mathcal{O}_4)$ channel (t-channel), which lead to different conformal partial wave (CPW) expansions of the same correlator. They introduce dynamical constraints for the bootstrap computations. For CFTs with a global symmetry $\mathcal{G}$, the crossing equations possess an algebraic structure related to the $6j$ symbols of the group $\mathcal{G}$. A systematic approach for crossing equations with global symmetries has been provided in \[18\]. Considering a scalar $\mathcal{O}_\alpha$ which constructs an irreducible (complex) representation $\pi$ of the group $\mathcal{G}$,

\[1\] The crossing equations strongly depend on the symmetries and one has to compute the crossing equations case by case. In \[19\] a Mathematica program has been developed to compute the crossing equations with global symmetries automatically.
its four-point correlator can be expanded as

\[ \langle O_{\alpha}(x_1)O_{\beta}^\dagger(x_2)O_{\gamma}(x_3)O_{\delta}^\dagger(x_4) \rangle = x_{12}^{-2\Delta} x_{34}^{-2\Delta} \mathcal{G}_0(u, v), \]

where the \( \mathcal{G} \) are the \( \mathcal{G} \)-invariant four-point tensors obtained by contracting two Clebsch-Gordan (CG) coefficients \( (\pi \otimes \pi') \otimes (\pi \otimes \pi') \). Superscripts in \( G^{\pm}_{\alpha\beta} \) denote the correlation functions even/odd under the interchange of external coordinates \( x_1 \leftrightarrow x_2 \) or \( x_3 \leftrightarrow x_4 \). In the CPW expansions, they respectively relate to even/odd spin selection rules. Variables \( u, v \) are the conformal invariant cross ratios \( u \equiv x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2, v \equiv x_{14}^2 x_{23}^2 / x_{13}^2 x_{24}^2 \). This correlator can be alternatively expanded in the t-channel by exchanging \( x_{\alpha}, \dot{\alpha} \) \( \leftrightarrow \) \( \{x_4, \dot{\beta}\} \) in (1). The two expansions lead to a tensor equation

\[ u^{\Delta_0} \sum_{i,j} T^{R_i}_{\alpha \dot{\alpha}} G^R_{\dot{i}}(u, v) + T^{R_j}_{\alpha \dot{\beta}} G^R_{\dot{j}}(u, v) = \]

\[ u^{\Delta_0} \sum_{i,j} T^{R_i}_{\alpha \dot{\alpha}} G^R_{\dot{i}}(v, u) + T^{R_j}_{\alpha \dot{\beta}} G^R_{\dot{j}}(v, u). \]

Constraints from crossing symmetry are completed by considering another configuration of the four-point correlator \( \langle O_{\alpha}(x_1)O_{\beta}^\dagger(x_2)O_{\gamma}^\dagger(x_3)O_{\delta}(x_4) \rangle \). Crossing equations can be obtained by decomposing the \( \mathcal{G} \)-invariant four-point tensors in one channel to another channel, which is given by the inner products of s-channel and t-channel \( \mathcal{G} \)-invariant four-point tensors, namely the 6j symbols of group \( \mathcal{G} \). In conformal bootstrap studies, the correlation functions \( G^{\pm}_{\alpha\beta} \) are further decomposed into infinite sums of conformal blocks, which are reminiscent to the \( \mathcal{G} \)-invariant four-point tensors but with CG coefficients replaced by the conformal three-point correlation functions, and their crossing symmetric decomposition is given by the 6j symbols of the conformal group \([23, 24]\).

The \( SO(N) \) vector four-point correlator \( \langle \phi_i \phi_j \phi_k \phi_l \rangle \) contains three \( SO(N) \)-invariant four-point tensors

\[ \mathcal{G}_0(u, v) = \delta_{ij} \delta_{kl} G^+_{S}(u, v) + \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right) G^-_{A}(u, v) \]

\[ + \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl} / N \right) G^0_{T}(u, v), \]

where \( S/T/A \) denote \( SO(N) \) singlet, traceless symmetric and anti-symmetric representations. Crossing symmetry requires the three functions \( G_{S/T/A}(u, v) \) satisfying \([18]\):

\[ \left( \begin{array}{c} 0 \\ F^+_S \\ H^+_S \end{array} \right) \left( \begin{array}{c} 1 \\ 1 - \frac{1}{N} \\ -1 \end{array} \right) \left( \begin{array}{c} F^+_T \\ F^+_A \\ H^+_T \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \]

where the functions \( H_{\pi}^\pm \) are (anti-)symmetric linear superposition of \( G^\pm_{\pi\pi}(u, v) \) and \( G^0_{\pi\pi}(v, u) \).

Surprisingly, for a large class of examples in previous bootstrap studies \([3]\), we find the crossing equations can be linearly transformed into the \( SO(N) \) vector's, although they have rather distinct forms. Let us take the four-point crossing equations of \( SU(N) \) fundamental scalars as an example. The \( SU(N) \) fundamental crossing equations can be written in a matrix form \([18]\):

\[ M_{SU(N)} \cdot 1_{6 \times 1} = 0 \]

where the \( S \) (singlet) and \( Adj \) (adjoint) sectors appear

\[ M_{SU(N)} \cdot 1_{6 \times 1} = \left( \begin{array}{ccccccc} 0 & 0 & F^+_{Adj} & F^+_{AdJ} \\ 0 & 0 & H^+_{AdJ} & F^+_{AdJ} \\ F^+_S & F^+_S & H^+_S & F^+_{AdJ} \\ H^+_S & H^+_S & (1 - \frac{1}{N}) H^+_S & F^+_{AdJ} \\ F^+_S & -F^+_S & -H^+_S & F^+_T & -H^+_T & H^+_T & \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{array} \right), \]

in \( \Phi \times \Phi^\dagger \sim S^\pm + Adj^\pm \) while the \( T \) (symmetric) and \( A \) (anti-symmetric) sectors appear in \( \Phi \times \Phi \sim T^+ + A^- \).

Crossing equations of the \( SO(2N) \) vector scalars \([4]\) and \( SU(N) \) fundamental scalars \([7]\) have quite different forms. Remarkably, the two sets of crossing equations are actually related through a linear transformation \( \mathcal{F}_{SU(N)} \)

\[ \Phi \times \Phi^\dagger \sim S^\pm + Adj^\pm \]
are all positive for $N > 3$ combined with the function $G_{SU}$ on the $3 \times 3$ matrix in (9) is equivalent to the $SO(2N)$ vector crossing equations (4) \[ \begin{pmatrix} 0 & -F^+_{A_{ij}} & -F^+_{A_{ij}} & -F^+_{A_{ij}} & -F^+_{A_{ij}} & -F^+_{A_{ij}} \\ F^+_{S_{ij}} & F^+_{S_{ij}} & (1 - \frac{1}{N}) F^+_{A_{ij}} & F^+_{S_{ij}} & F^+_{S_{ij}} & F^+_{A_{ij}} \end{pmatrix} \begin{pmatrix} 1 \\ y_1 \\ x_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}, \] (8)

We call the constants $x_i, y_j$ “recombination coefficients”. The $3 \times 6$ matrix in (9) is equivalent to the $3 \times 3$ matrix in the $SO(2N)$ vector crossing equations combined with the $SO(2N) \rightarrow SU(N)$ branching rules

\[ \begin{array}{ll}
SO(2N) & SU(N) \\
S & S, \quad (11) \\
T & Adj \oplus T, \quad (12) \\
A & S \oplus Adj \oplus A. \quad (13)
\end{array} \]

Besides, it is critical that the coefficients $x_i, y_j$ in (10) are all positive for $N > 1$, which guarantees that the positivity in the CPW expansions will not be changed by the transformation.

Now we define a new $SO(2N)$-symmetric correlation function $G^\varphi$ of a presumed $SO(2N)$ vector field $\varphi$ based on the $SU(N)$ fundamental correlation functions $G^\psi_i$ and the recombination coefficients $x_i, y_j$:

\[ G^\varphi_S(u, v) = G^\psi_S(u, v), \quad (14) \]
\[ G^\varphi_T(u, v) = x_1 G^\psi_{A_{ij}}(u, v) + x_2 G^\psi_T(u, v), \quad (15) \]
\[ G^\varphi_A(u, v) = y_1 G^\psi_S(u, v) + y_2 G^\psi_{A_{ij}}(u, v) + y_3 G^\psi_A(u, v). \quad (16) \]

Because of the constraints (9), the above correlation functions $G^\varphi$ satisfy the $SO(2N)$ crossing equations (4)! Moreover, since the recombination coefficients $x_i, y_j$ are positive, as long as the original $SU(N)$ correlator admits CPW expansions with unitary coefficients, so does the new $SO(2N)$ symmetric correlator. Such construction will be dubbed $SO(N)$-ization\(^4\). We present some examples for $SO(N)$-ization in Section VI and VII.

We have obtained linear transformations $T_G$ similar to $T_{SU}$ in (9) for plenty of symmetry groups appeared in previous bootstrap studies, including the discrete symmetries, classical and exceptional Lie groups with fundamental or higher rank representations, see a review in (16) for related bootstrap studies. Part of the interesting examples are presented in Section VII. Due to the transformation $T_G$, the $SO(N)$ symmetric correlation function $G^\varphi$ can be constructed based on the original $G$-symmetric correlation functions $G^\psi$ similar to (14) and (16). The diversified examples suggest the linear transformation $T_G$ and $SO(N)$-ization are true for general $G$, which can be summarized as follows:

Consider the four-point correlator of a scalar $O$ with $N$ real components which forms an irreducible representation $\pi$ of a group $G$. The correlation function consists of components $G^\pm_{G_G}$ multiplied with invariant four-point tensors of $G$ irreducible representations $\pi_i$, which can be classified into three parts: the $G$ singlet $S_G$, and the $G$ non-singlet representations $R_i(r_j)$ which are parity even (odd) under the interchange of external coordinates $x_1 \leftrightarrow x_2$. The components $G^\pm_{G_G}$ satisfy the $G$-symmetric crossing equations $M_G$. We conjecture: a, there is a linear transformation (unique up to normalization) $T_G$, which maps the crossing equations $M_G$ to the $SO(N)$-ization.

\[^4\text{The transformation } G_{SU(N)} \text{ can be solved by imposing the branching rules (11)-(13) and another constraint that there is no mixing between the functions } F \text{ and } H, \text{ which have opposite symmetries under } u \leftrightarrow v. \text{ These constraints lead to overdetermined equations, which surprisingly can be solved uniquely up to normalization.}\]

\[^5\text{We thank David Poland and Slava Rychkov for discussions and suggestions on the terminology.}\]
symmetric crossing equations \(^{3,4}\)

\[
\mathcal{F}_{3 \times n} \cdot \mathcal{M}_n \cdot 1_{n \times 1} =
\begin{pmatrix}
0 & F^+_S & \cdots & -F^-_S \\
F^+_S & (1 - \frac{2}{N}) F^+_R & \cdots & -F^-_R \\
H^+_S & - (1 + \frac{2}{N}) H^+_R & \cdots & -H^-_R
\end{pmatrix}
\begin{pmatrix}
x_i R_i \\
y_j R_j
\end{pmatrix}
= 0_{3 \times 1}
\tag{17}
\]

associated with the SO\((N) \rightarrow \mathcal{G}\) branching rules

\[
S_{SO(N)} \rightarrow S_{\mathcal{G}}, \quad T_{SO(N)} \rightarrow \oplus_i R_i, \quad A_{SO(N)} \rightarrow \oplus_j R_j,
\tag{18}
\]

and \(b_i\) all the recombination coefficients \(x_i, y_j\) are positive. Consequently, a new SO\((N)\)-symmetric correlation function \(\mathcal{F}_\mathcal{G}\) can be constructed from correlation functions \(G^\pm\), and the recombination coefficients \(x_i, y_j\): \(^{5}\)

\[
G^+_S(u, v) = G^+_S(u, v), \quad G^+_T(u, v) = \sum_i x_i G^+_R_i(u, v),
\tag{19}
\]

\[
G^-_A(u, v) = \sum_j y_j G^-_{R_j}(u, v).
\tag{20}
\]

This is strongly against our intuition about symmetries in physics: generically it requires restrictive conditions to have enhanced symmetries in physical systems, while above proposal suggests that due to a dedicate positive structure in the crossing equations, the conformal four-point correlation function can be linearly transformed into a form with a maximal symmetry allowed by its degree of freedom. \(^{6}\)

III. SYMMETRIES IN CONFORMAL BOOTSTRAP

The algebraic relations \(^{17,18}\) associated with positive recombination coefficients have substantial applications in conformal bootstrap studies. To bootstrap the four-point correlator \(\langle QQ'O'O' \rangle\), one takes the conformal block expansions of the correlation functions. The bootstrap crossing equations have the same matrix form as \(\mathcal{M}_\mathcal{G}\) but with the correlation functions \(G^\pm\) replaced by conformal blocks. The bootstrap algorithm \(^{22,23}\) aims to find \(n\)-component linear functions \(\tilde{\alpha}_\mathcal{G}\) whose actions on \(\mathcal{M}_\mathcal{G}\) satisfy the positive conditions

\[
\tilde{\alpha}_{\mathcal{G}_{1 \times n}} \cdot \mathcal{M}_n \cdot 1_{n \times 1} \geq 0_{1 \times n}, \quad \forall \Delta_{x_i, \ell} \geq \Delta^*_{x_i, \ell},
\tag{21}
\]

where \(\Delta^*_{x_i, \ell}\) is the lowest allowed scaling dimension of spin \(\ell\) operators in the \(\pi_i\) representation given by either the unitary bound or certain assumption on the CFT spectrum. In particular the SO\((N)\) vector bootstrap linear functionals satisfy the positive conditions

\[
\tilde{\alpha}_{SO(N)_{1 \times 3}} \cdot \mathcal{M}_{SO(N)_{3 \times 3}} = (h_S, h_T, h_A) \geq 0_{1 \times 3}, \quad \forall \Delta_{S/T/A, \ell} \geq \Delta^*_{S/T/A, \ell}
\tag{22,23}
\]

Due to the algebraic relation \(^{17}\), one can construct the linear functions \(\tilde{\alpha}_{\mathcal{G}_{1 \times n}}\) in \(^{21}\) from the SO\((N)\) linear functionals \(\tilde{\alpha}_{SO(N)_{1 \times 3}}\): \(^{24}\)

\[
\tilde{\alpha}_{\mathcal{G}_{1 \times n}} = \tilde{\alpha}_{SO(N)_{1 \times 3}} \cdot \mathcal{F}_{3 \times n}
\tag{24}
\]

whose actions on the crossing equations \(\mathcal{M}_\mathcal{G}\) are

\[
\tilde{\alpha}_{\mathcal{G}_{1 \times 3}} \cdot \mathcal{M}_n \cdot 1_{n \times 3} = (\tilde{\alpha}_{SO(N)_{1 \times 3}} \cdot \mathcal{F}_{3 \times n}) \cdot \mathcal{M}_n \geq 0_{3 \times 3}, \quad \forall \Delta_{SO(N)_{3 \times 3}} \geq \Delta^*_{SO(N)_{3 \times 3}},
\tag{25}
\]

\[
= (h_S x_i, h_T y_j, h_A, \ldots) \geq 0_{1 \times n}, \quad \forall \Delta_{SO(N)_{3 \times 3}} \geq \Delta^*_{SO(N)_{3 \times 3}}
\tag{25}
\]

where \(M'_{SO(N)_{3 \times 3}}\) is the \(3 \times n\) matrix in \(^{17}\) related to \(M_{SO(N)_{3 \times 3}}\) with replicated columns of \(T, A\) sectors. The positivity of the recombination coefficients is critical for above actions being positive.

To summarize, due to the algebraic relation \(^{17}\) with positive recombination coefficients, the \(\mathcal{G}\)-symmetric crossing equations actually have the same positive algebraic structures of the SO\((N)\) vector’s. Bounds from the \(\mathcal{G}\)-symmetric bootstrap equations are saturated by the SO\((N)\)-symmetric solutions unless the SO\((N)\) symmetric “boundary conditions” in \(^{20}\) are violated explicitly. \(^{8}\)

IV. EXAMPLES OF CROSSING EQUATIONS WITH DIFFERENT GLOBAL SYMMETRIES

CFTs with various types of global symmetries have been studied using modern conformal bootstrap approach \(^{3}\). A comprehensive review on these studies can be found in \(^{4}\). The bootstrap crossing equations computed in these studies provide abundant examples to verify the proposed algebraic relation between crossing equations with different global symmetries and SO\((N)\)-ization. We have found the relation \(^{17}\) and the positivity of recombination coefficients \(x_i, y_j\) are supported by all these examples, providing compelling evidence for the main proposal \(^{17,24}\) of this work. Some of the physically interesting examples are presented in this section, while the algebraic relation of other theories and groups can be obtained similarly. More examples can be found in an attached Mathematica file.

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\(^6\) It has been proved in \(^{13}\) that the single four-point correlator crossing equations \(\mathcal{M}_G\) can always be given by an \(n \times n\) square matrix

\(^7\) The SO\((N)\)-ized correlators in \(^{19,20}\) are not necessarily related to locally interacting theories. A counter example is provided by the SO\((N)\)-ization of the 3D cubic model with \(Z_2^N \times S_N\) symmetry, see \(^{17,24}\) The theory does not have a spin 1 conserved current, nor does its SO\((N)\)-ized correlator. Therefore the later is not of SO\((N)\) symmetric theories with local interactions.

\(^8\) The Eq. \(^{25}\) proves any spectrum excluded by the SO\((N)\) vector bootstrap is also excluded by the \(\mathcal{G}\)-symmetric bootstrap, on the other hand, the SO\((N)\)-symmetric solution can be directly decomposed into the \(\mathcal{G}\)-symmetric solution. Therefore the two bootstrap setups should have the same excluded and allowed parameter spaces, i.e., they are identical.
A. Crossing equations of $SU(N)$ adjoint four-point correlators

We show the relation between the crossing equations of $SU(N)$ adjoint ($\mathcal{M}_{SU(N)}^{Adj}$) and $SO(N^2 - 1)$ vector scalars. Consider an $SU(N)$ adjoint scalar $O$, its OPE $O \times O$ contains 6 different sectors

$$\text{Adj} \otimes \text{Adj} \rightarrow S^+ \oplus \text{Adj}^+ \oplus \text{Adj}^- \oplus \text{Adj}^- \oplus A\bar{A}^+ \oplus S\bar{S}^+,$$

where $S, \text{Adj}$ denotes $SU(N)$ singlet and adjoint representations, while $A/S$ ($\bar{A}/\bar{S}$) denote representations with anti-symmetric/symmetric of the $SU(N)$ fundamental (anti-fundamental) indices. The superscripts in $SU(N)$ representations correspond to even/odd spin selection rules. We follow the convention in [28]. Crossing equations $\mathcal{M}_{SU(N)}^{Adj}$ can be written in following matrix form

$$\mathcal{M}_{SU(N)}^{Adj} = \begin{pmatrix}
0 & 0 & 0 & -F & F & F \\
0 & 0 & -F & F & \frac{2F}{N} & \frac{2F}{N} \\
F & -\frac{36F}{N} & 0 & \frac{2N}{N(N+2)} & \frac{2N}{N(N+2)} & \frac{2N}{N(N+2)} \\
H & -\frac{16}{N} & 0 & -H & -H & H \\
0 & H & -H & -H & H & H
\end{pmatrix},$$

in which the columns are in the order

$$(S^+, \text{Adj}^+, \text{Adj}^-, (A\bar{S} + S\bar{A})^-, A\bar{A}^+, S\bar{S}^+).$$

In [29] and below, we will omit the sub- and superscripts of correlation functions $F/H$ for simplicity. The corresponding symmetry representations and parity charges under the interchange $x_1 \leftrightarrow x_2$ of each channel (columns in the matrix) are shown in a vector below the crossing equation, e.g. [29].

The transformation matrix $\mathcal{F}_{SU(N)}^{Adj}$ can be solved by imposing the $SO(N^2 - 1) \rightarrow SU(N)$ branching rules and another constraint that there is no mixing between the functions $F$ and $H$ when mapping the $SU(N)$ adjoint crossing equations $\mathcal{M}_{SU(N)}^{Adj}$ to the $SO(N^2 - 1)$ vector crossing equations $\mathcal{M}_{SO(N^2-1)}$. The later constraint is justified as the functions $F$ and $H$ have opposite symmetries under $u \leftrightarrow v$.

These constraints lead to over-determined equations, which can be solved uniquely up to normalization

$$\mathcal{F}_{SU(N)}^{Adj} = \begin{pmatrix}
1 & \frac{2(N^2 - 2N + 2)}{N^2 - 2} & 0 & 0 & 0 \\
-1 & \frac{2N}{N^2 - 2} & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{2N}{N^2 - 2}
\end{pmatrix},$$

and the recombination coefficients are given by

$$x_{\text{Adj}^+} = \frac{2(N^4 - 5N^2 + 4)}{N(N^4 - N^2 - 2)}, \quad x_{A\bar{A}^+} = \frac{(N - 3)(N + 1)N^2}{(N^2 - 2)(N^2 + 1)},$$

$$x_{A\bar{S}^+} = \frac{(N - 1)(N + 3)N^2}{(N^2 - 2)(N^2 + 1)}, \quad y_{\text{Adj}^-} = \frac{2N}{N^2 - 2},$$

$$y_{A\bar{S}^-} = \frac{N^2 - 4}{N - 2}.$$

B. Crossing equations of cubic model

The cubic model has discrete symmetry $C_N = \mathbb{Z}_N^2 \times S_N$ with scalars in the fundamental representation. Symmetry $C_N$ has close relation with $O(N)$. In particular the singlet and anti-symmetric representations of $O(N)$ construct the same irreducible representations of $C_N$, while $O(N)$ traceless symmetric representation decomposes into two irreducible representations of $C_N$, denoted by $V$ and $Y$ in [12]. Such $SO(N) \rightarrow C_N$ relation suggests the $SO(N)$-ized four-point correlator has the same spectrum as the cubic model in $S$ and $A$ sectors. Since the cubic model only has discrete global symmetry without a spin 1 conserved current. Consequently there is no conserved current in the SO(N)-ized four-point correlator neither, though it is $SO(N)$ symmetric. Following the convention in [12], the crossing equations of a fundamental scalar can be written in a matrix form

$$\mathcal{M}_{C_N} = \begin{pmatrix}
0 & 0 & F & -F \\
F & -\frac{2F}{N} & F & F \\
H & -\frac{16}{N} & H & H \\
(2 - \frac{2}{N})F & 0 & 0 & 0
\end{pmatrix},$$

Columns in above matrix are in the order

$$(S^+, V^+, Y^+, A^-).$$

The matrix $\mathcal{F}_{C_N}$ which transforms $\mathcal{M}_{C_N}$ to $\mathcal{M}_{SO(N)}$ is

$$\mathcal{F}_{C_N} = \begin{pmatrix}
\frac{N + 1}{N^2 + 2} & -\frac{1}{N^2 + 2} & 0 & \frac{1}{N^2 + 2} \\
-\frac{1}{N^2 + 2} & \frac{N + 1}{N^2 + 2} & 0 & \frac{1}{N^2 + 2} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

and the recombination coefficients are

$$x_{V^+} = \frac{2}{2 + N}, \quad x_{Y^+} = \frac{N}{2 + N}, \quad y_{A^-} = 1.$$

C. Crossing equations of $SU(N) \times SU(N)$ bifundamental four-point correlators

Many interesting 4D gauge theories have flavor symmetries $SU(N) \times SU(N)$, so this group could play an important role in our 4D bootstrap studies. The fermion bilinears $O^a \equiv \psi_i \bar{\psi}_R^a$ furnish bifundamental representation of the flavor symmetry. The $SU(N) \times SU(N)$ representations appearing in the OPE $O \times O$ are a direct product of the two $SU(N)$ representations $\{S, \text{Adj}, T, A\}$:

$$O \otimes O \rightarrow SS^+ \oplus SS^- \oplus AdjAdj^+ \oplus AdjAdj^- \oplus TA^- \oplus AA^+ \oplus TT^+ \oplus AdjS^+ \oplus AdjS^-.$$

There is a permutation symmetry between the two $SU(N)$ groups so we will not distinguish representations $LR$ and $RL$, e.g. $TA$ and $AT$. The crossing equations of bifundamental scalars are provided in [12]. Here we write the crossing equations into a compact matrix form
in which the columns are in the order

\[
(SS^+, \ SS^-, \ Adj \ Adj^+, \ Adj \ Adj^-, \ TA^-, \ AA^+, \ TT^+, \ AdjS^+, \ AdjS^-),
\]

where \(XY\) denotes representation \(X(Y)\) for the left (right) \(SU(N)\). The crossing equations \(\mathcal{M}_{SU(N) \times SU(N)}\) can be transformed into \(SO(2N^2)\) crossing equations by a linear transformation \(\mathcal{F}_{SU(N) \times SU(N)}\):

\[
\mathcal{M}_{SU(N) \times SU(N)} = \begin{pmatrix}
F & -\frac{2F}{2N} & 0 & 0 & 0 & 0 & -\frac{2F}{2N} & -\frac{2F}{2N} \\
H & -\frac{2H}{2N} & 0 & 0 & 0 & 0 & -\frac{2H}{2N} & -\frac{2H}{2N} \\
0 & 0 & F & -F & -\frac{2F}{2N} & -\frac{2F}{2N} & 0 & 0 \\
0 & 0 & -H & H & -\frac{2H}{2N} & -\frac{2H}{2N} & 0 & 0 \\
F & -F & -\frac{2F}{2N} & -\frac{2F}{2N} & 2F & F & -\frac{2F}{2N} & -\frac{2F}{2N} \\
H & -H & -\frac{2H}{2N} & -\frac{2H}{2N} & -2H & -H & -\frac{2H}{2N} & -\frac{2H}{2N} \\
0 & 0 & -\frac{2F}{N} & -\frac{2F}{N} & 0 & -F & F & -F \\
0 & 0 & -\frac{2H}{N} & -\frac{2H}{N} & 0 & H & -H & -H
\end{pmatrix},
\]

and the recombination coefficients are

\[
x_{\text{Adj}\text{Adj}^+} = \frac{(N^2 - 1)^2}{2N^4 + 2N^2 - 1}, \quad x_{TT^+} = \frac{N^2(N + 1)^2}{2N^4 + 2N^2 - 1}, \quad x_{\text{Adj}\text{Adj}^+} = \frac{2N(N^2 - 1)}{2N^4 + 2N^2 - 1}, \quad x_{AA^+} = \frac{(N - 1)^2N^2}{2N^4 + 2N^2 - 1},
\]

\[
y_{SS^-} = \frac{1}{2N^2 - 1}, \quad y_{\text{Adj}\text{Adj}^-} = \frac{(N^2 - 1)^2}{N^2(2N^2 - 1)}, \quad y_{TA^-} = \frac{2(N^2 - 1)}{2N^2 - 1}, \quad y_{\text{Adj}\text{Adj}^-} = \frac{2 - 2N^2}{N - 2N^2}.
\]

D. Crossing equations of \(SO(N) \times SO(M)\)
bifundamental four-point correlators

CFTs with \(SO(N) \times SO(M)\) symmetry can be used to describe the phase transitions in frustrated spin models [29, 30], in which the order parameters construct bi-fundamental representations of the symmetry. Conformal bootstrap study of this theory has been initiated in [31]. The OPE of an \(SO(N) \times SO(M)\) bifundamental scalar \(O_{\alpha, \beta}\) is given by:

\[
\mathcal{O} \otimes \mathcal{O} \rightarrow SS^+ \oplus ST^+ \oplus SA^- \oplus TS^+ \oplus TT^+ \oplus TA^- \oplus AS^- \oplus AT^- \oplus AA^+,
\]

where the \(S, T, A\) are the singlet, traceless symmetric and anti-symmetric representations of \(SO(N)\) and \(SO(M)\), and in the representation \(XY\), \(X(Y)\) denotes the representation for the \(SO(N)\) (\(SO(M)\)). The crossing equations of \(SO(N) \times SO(M)\) bifundamental scalars are provided in [11], which can be written in a compact matrix form

\[
\mathcal{M}_{SO(N) \times SO(M)} = \begin{pmatrix}
F & -\frac{2F}{2N} & 0 & -\frac{2F}{2N} & \left(\frac{4}{2N^2} + 1\right)F & F & 0 & F & F \\
H & -\frac{2H}{2N} & 0 & -\frac{2H}{2N} & \left(\frac{4}{2N^2} + 1\right)H & -H & 0 & -H & -H \\
0 & 0 & F & -F & \left(1 - \frac{2}{2N}\right)F & F & -F & \left(\frac{2}{2N} - 1\right)F & -F \\
0 & 0 & -H & H & \left(\frac{2}{2N} + 1\right)H & H & H & \left(-\left(\frac{2}{2N} - 1\right)\right)H + H \\
0 & F & -F & 0 & \left(1 - \frac{2}{2N}\right)F & \left(-\left(1 - \frac{2}{2N}\right)\right)F & 0 & F & -F \\
0 & -H & H & 0 & \left(\frac{2}{2N} + 1\right)H & \left(-\left(\frac{2}{2N} + 1\right)\right)H & 0 & H & -H \\
0 & -F & -F & -F & \left(\frac{2}{2N} + \frac{2}{2N}\right)F & \frac{2F}{2N} & -F & \frac{2F}{2N} & 0 \\
0 & 0 & -H & -H & \left(\frac{2}{2N} - \frac{2}{2N}\right)H & \frac{2F}{2N} & -H & \frac{2F}{2N} & 0 \\
0 & 0 & 0 & 0 & F & -F & 0 & -F & F
\end{pmatrix},
\]
in which the columns are in the order

\[(SS^+, ST^+, SA^-, TS^+, T T^+, TA^-, AS^-, AT^-, AA^+),\]  

(43)

Note with \(N = M\) the two \(SO(N)\)'s are interchangeable and the representations like \(TA\) and \(AT\) are indistinguishable, which can result in different crossing equations. The crossing equations \(\mathcal{M}_{SO(N) \times SO(M)}\) can be transformed into \(SO(MN)\) crossing equations by a linear transformation \(\mathcal{F}_{SO(N) \times SO(M)}\):

\[
\mathcal{F}_{SO(N) \times SO(M)} = \begin{pmatrix}
0 & 0 & (N-1)(M+N+1) & 0 & (M-1)(N+M+1) & 0 & (M-1)(N+M+1) & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & (M-1)(M+N+2) & 0 & (M-1)(M+N+2) & 0 & (M-1)(M+N+2) & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & (MN-1)(M+N+2) & 0 & (MN-1)(M+N+2) & 0 & (MN-1)(M+N+2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M^{-1} & 0 & M^{-1} & 0 & (M^2-1) & M^{-1} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & (MN-1)(M+N+2) & 0 & (MN-1)(M+N+2) & 0 & (MN-1)(M+N+2) & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & (MN-1)(M+N+2) & 0 & (MN-1)(M+N+2) & 0 & (MN-1)(M+N+2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M^{-1} & 0 & M^{-1} & 0 & (M^2-1) & M^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M^{-1} & 0 & M^{-1} & 0 & (M^2-1) & M^{-1} & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

(44)

and the recombination coefficients are

\[
x_{ST^+} = \frac{(M^2+M-2)N}{M^2N^2+MN-2}, \quad x_{TS^+} = \frac{M(N^2 + N - 2)}{M^2N^2 + MN - 2}, \quad x_{TT^+} = \frac{(M^2 + M - 2)(N^2 + N - 2)}{M^2N^2 + MN - 2}, \quad y_{SA^-} = \frac{M-1}{MN-1} \quad y_{TA^-} = \frac{M-1}{MN-1}.
\]

(45, 46)

V. \(SO(N)\)-IZATION OF GENERALIZED FREE THEORIES

In this section we provide two examples for the \(SO(N)\)-ization: the four-point correlators of the \(SU(N)\) fundamental generalized free scalars and the \(SU(N)\) adjoint generalized free fermion bilinears.

Let us consider an \(SU(N)\) fundamental scalar \(\Phi^i\), whose four-point correlator can be solved in generalized free field theory

\[
\langle \Phi^i(x_1)\Phi^j(x_2)\Phi^k(x_3)\Phi^l(x_4) \rangle = \frac{1}{x_{12}^{\delta_{ij}^l}x_{34}^{\delta_{jk}^l}}(\delta_{ij}^l + \delta_{jk}^l)u^{\Delta_{ij}^l}.
\]

(47)

There are 6 invariant tensors in the \(SU(N)\) crossing equations \(\mathcal{W}\) and the associated correlation functions \(G_i(u, v)\) can be solved through the four-point correlation function \(\mathcal{W}\) and its modified configurations \(\langle \Phi^i\Phi^j\Phi^k\Phi^l \rangle\), \(\langle \Phi^i\Phi^j\Phi^k\Phi^l \rangle\):

\[
G_S^+ = 1 + \frac{1}{2N}u^{\Delta_{ij}^l}(v^{-\Delta_{ij}^l} + 1),
\]

(48)

\[
G_A^+ = \frac{1}{2}u^{\Delta_{ij}^l}(v^{-\Delta_{ij}^l} + 1),
\]

(49)

\[
G_A^- = G_S^- = \frac{1}{2}u^{\Delta_{ij}^l}(v^{-\Delta_{ij}^l} - 1).
\]

(50)

Applying the above solutions in \(\mathcal{W}\) we can reproduce the exact \(SO(2N)\) correlation functions \(G_i(u, v)\)

\[
G_S^+ = 1 + \frac{1}{N}u^{\Delta_{ij}^l}(1 + v^{-\Delta_{ij}^l}),
\]

(51)

\[
G_T^+ = \frac{1}{2}u^{\Delta_{ij}^l}(1 + v^{-\Delta_{ij}^l}),
\]

(52)

\[
G_A^- = \frac{1}{2}u^{\Delta_{ij}^l}(-1 + v^{-\Delta_{ij}^l}),
\]

(53)

which can be obtained from Wick contractions of the \(SO(N)\) vector four-point correlator. This is consistent but trivial – the \(SO(2N)\) symmetry enhancement of the \(SU(N)\) fundamental generalized free field theory is obvious.

A less trivial example is provided by the four-point correlator of the 3D fermion bilinear operator \(O_m^a \equiv \bar{\psi}_1v^m\frac{1}{\tau_3}\bar{\psi}_k\psi^k\), which has scaling dimension \(2\Delta\) and furnishes the adjoint representation of the \(SU(N)\) flavor symmetry. This is a real representation of \(SU(N)\), therefore it suffices to consider the single correlator \(\langle O_m^aO_m^aO_m^aO_m^a \rangle\). The crossing equations of this correlator have been studied using conformal bootstrap \([14, 28, 32]\). For \(N \geq 4\) there are 6 sectors in the crossing equations \(\mathcal{M}_{SU(N)_{\text{Adj}}}\), which are shown in Section \(\mathcal{V}_{\text{Adj}}\) and explicit formulas of the \(SU(N)\)-invariant four-point tensors \(T_{\text{Adj}}\) are provided in \([28, 32]\). The fermion bilinear four-point correlator is

\[
\mathcal{G}_{SU(N)_{\text{Adj}}} = \left(T_S^+G_S^+ + T_{\text{Adj}}^+G_{\text{Adj}}^+ + T_{\text{Adj}}^-G_{\text{Adj}}^- + (T_{\text{Adj}} + T_{\text{Adj}}^-)G_{\text{Adj}}^- + T_{\text{Adj}}^+G_{\text{Adj}}^+ + T_{\text{Adj}}^+G_{\text{Adj}}^+\right).
\]

(54)
In the generalized free fermion theory, contributions to the correlator from disconnected diagrams are

\[ G^+_S = 1 + \frac{u^{2\Delta}(v^{-2\Delta}+1)}{N(v^2-1)}, \quad G^+_{AA} = \frac{N u^{2\Delta}(v^{-2\Delta}+1)}{2(N^2-4)}, \quad G^+_{AA} = \frac{u^{2\Delta}(v^{-2\Delta}+1)}{2N}, \quad (55) \]

\[ G^+_{AS} = \frac{u^{2\Delta}(v^{-2\Delta}-1)}, \quad G^+_{SS} = \frac{u^{2\Delta}(v^{-2\Delta}+1)}, \quad (56) \]

and the contributions from connected diagrams are

\[ G^+_S = \frac{2u^{\Delta}v^{-\Delta}}{N(N^2-1)} \left( N^2(v^2-1)(v^{\Delta+\frac{1}{2}}-1) - u(v^{\Delta+\frac{1}{2}}+1) \right) - u^{\Delta+\frac{1}{2}} + (v+1)u^{\Delta+\frac{1}{2}} + ut^{\Delta+\frac{1}{2}} + (v-1)u^{\Delta+\frac{1}{2}} + v-1 \), \quad (57) \]

\[ G^+_{Ad} = -\frac{u^{\Delta}v^{-\Delta}}{2(N^2-4)} \left( N^2 - 4 \right) \left( u(v^{\Delta+\frac{1}{2}}+1) - (v-1)(v^{\Delta+\frac{1}{2}}+1) \right) + 4u^{\Delta+\frac{1}{2}} - 4(v+1)u^{\Delta+\frac{1}{2}} \), \quad (58) \]

\[ G^+_{Ad} = \frac{1}{2} u^{\Delta} v^{-\Delta} \left( u(v^{\Delta+\frac{1}{2}}+1) - (v-1)(v^{\Delta+\frac{1}{2}}+1) \right), \quad (59) \]

\[ G^+_{AA} = -G^+_{SS} = \frac{u^{2\Delta}v^{-\Delta}}{(u+v+1)}, \quad G^+_{AS} = 0. \quad (60) \]

The recombination coefficients solved from the crossing equations \( M_{SU(N)A_d} \) are provided by (11). Using the formulas (10,20), we can construct \( SO(N^2-1) \)-symmetric correlation functions straightforwardly. In particular the disconnected correlation functions (56) reproduce the \( SO(N^2-1) \) correlator of generalized free field theory (53). The connected correlation functions (57,60) lead to more complicated results

\[ G^+_S = \frac{u^{\Delta}v^{-\Delta}}{N(N^2-1)} \left( N^2 \left( 2u^{\Delta+\frac{1}{2}} - 2(v+1)u^{\Delta+\frac{1}{2}} - u(v^{\Delta+\frac{1}{2}}+1) + (v-1)(v^{\Delta+\frac{1}{2}}-1) \right) \right) + 4 \left( u^{\Delta+\frac{1}{2}} + (v+1)u^{\Delta+\frac{1}{2}} + ut^{\Delta+\frac{1}{2}} + (v-1)u^{\Delta+\frac{1}{2}} + v-1 \right) \right) + N^2 \left( -4u^{\Delta+\frac{1}{2}} + 4(v+1)u^{\Delta+\frac{1}{2}} + 5u(v^{\Delta+\frac{1}{2}}+1) - 5(v-1)(v^{\Delta+\frac{1}{2}}-1) \right), \quad (61) \]

\[ G^+_T = \frac{2u^{\Delta}v^{-\Delta}}{N(N^2-1)} \left( -u^{\Delta+\frac{1}{2}} + (v+1)u^{\Delta+\frac{1}{2}} + ut^{\Delta+\frac{1}{2}} + (v-1)u^{\Delta+\frac{1}{2}} + v-1 \right) + N^2 \left( (v-1)(v^{\Delta+\frac{1}{2}}-1) - u(v^{\Delta+\frac{1}{2}}+1) \right), \quad (62) \]

\[ G^+_A = \frac{Nu^{\Delta}v^{-\Delta}}{N^2-2} \left( u(v^{\Delta+\frac{1}{2}}+1) - (v-1)(v^{\Delta+\frac{1}{2}}+1) \right). \quad (63) \]

The above correlation functions do satisfy the \( SO(N^2-1) \) crossing equations (41)!

VI. \( SO(N) \)-IZATION OF WZW MODEL

A more non-trivial test of the \( SO(N) \)-ization mechanism is provided by the 2D Wess-Zumino-Witten (WZW) model [34, 35]. Let us consider the four-point correlator of the fundamental field \( g^a_{\kappa} \) in the \( SU(N_f)_k \) WZW model:

\[ \langle g^a_{\kappa}(z_1, \bar{z}_1)g^b_{\kappa}(z_2, \bar{z}_2)g^c_{\alpha}(z_3, \bar{z}_3)g^d_{\beta}(z_4, \bar{z}_4) \rangle^{-1} = |z_{12}|^{-2\Delta_c} |z_{34}|^{-2\Delta_c} \langle g^a_{\kappa}g^b_{\alpha}g^c_{\beta}g^d_{\rho} \rangle (\eta, \bar{\eta}), \quad (64) \]

where \( z = x^0 + ix^1, z_{ij} = z_i - z_j, \eta = z_{12}z_{34}/z_{13}z_{24} \) and \( \bar{z}, \bar{\eta} \) are their complex conjugates. The conformal primary field \( g^\alpha_{\kappa} \) forms bi-fundamental representation of the symmetry \( SU(N_f)_k \times SU(N_f)_k \). The correlation function \( \mathcal{G} \) has been solved in [35] using the Knizhnik-Zamolodchikov equation.
and crossing symmetry which is:

\[ |1 - \eta|^2 \delta_{\alpha}^{a \alpha} \delta_{\beta}^{\beta \rho} (\eta, \bar{\eta}) = |\eta|^2 \delta_{\alpha}^{a \alpha} \delta_{\beta}^{\beta \rho} (1 - \eta, 1 - \bar{\eta}), \]

\[ \delta_{\alpha}^{a \alpha} \delta_{\beta}^{\beta \rho} (\eta, \bar{\eta}) = \left( \delta_{a}^{\alpha} \delta_{\beta}^{\rho} P_{1}(\eta) + \delta_{a}^{\alpha} \delta_{\beta}^{\rho} P_{2}(\eta) \right) \left( \delta_{\beta}^{\alpha} \delta_{a}^{\rho} P_{1}(\bar{\eta}) + \delta_{\beta}^{\alpha} \delta_{a}^{\rho} P_{2}(\bar{\eta}) \right) + h \left( \delta_{\beta}^{\alpha} \delta_{a}^{\rho} Q_{1}(\eta) + \delta_{\beta}^{\alpha} \delta_{a}^{\rho} Q_{2}(\eta) \right) \left( \delta_{a}^{\alpha} \delta_{\beta}^{\rho} Q_{1}(\bar{\eta}) + \delta_{a}^{\alpha} \delta_{\beta}^{\rho} Q_{2}(\bar{\eta}) \right), \]

and the functions \( P_{1}, Q_{1} \) are given by

\[ P_{1}(\eta) = (1 - \eta)^{h_{A} - 2h_{g}} 2F_{1}\left( \frac{1}{N_{f} + k}, \frac{1}{N_{f} + k}; 1 - \frac{N_{f}}{N_{f} + k}, \eta \right), \]

\[ P_{2}(\eta) = \frac{1}{k} \eta (1 - \eta)^{h_{A} - 2h_{g}} 2F_{1}\left( 1 + \frac{1}{N_{f} + k}, 1 - \frac{1}{N_{f} + k}; 2 - \frac{N_{f}}{N_{f} + k}, \eta \right), \]

\[ Q_{1}(\eta) = \eta^{h_{A}} (1 - \eta)^{h_{A} - 2h_{g}} 2F_{1}\left( \frac{N_{f} - 1}{N_{f} + k}, \frac{N_{f} + 1}{N_{f} + k}, 1 + \frac{N_{f}}{N_{f} + k}, \eta \right), \]

\[ Q_{2}(\eta) = -N_{f} \eta^{h_{A}} (1 - \eta)^{h_{A} - 2h_{g}} 2F_{1}\left( \frac{N_{f} - 1}{N_{f} + k}, \frac{N_{f} + 1}{N_{f} + k}, \frac{N_{f}}{N_{f} + k}, \eta \right). \]

The scaling dimensions \( h_{g}, h_{A} \) and the coefficient \( h \) are

\[ h_{g} = \frac{N_{f}^{2} - 1}{2N_{f}(N_{f} + k)} \quad h_{A} = \frac{N_{f}}{N_{f} + k} \]

\[ h = \frac{1}{N_{f}^{2}} \Gamma \left[ \frac{k - 1}{N_{f} + k} \right] \Gamma \left[ \frac{k + 1}{N_{f} + k} \right] \Gamma \left[ \frac{k}{N_{f} + k} \right] \Gamma \left[ \frac{k}{N_{f} + k} \right] \]

The four-point correlation function consists of two parts, in which the \( SU(N_{f})_{L} \) and \( SU(N_{f})_{R} \) indices are factorized. It is straightforward to decompose the correlation function in terms of \( SU(N_{f})_{L} \) and \( SU(N_{f})_{R} \) representations \( (\pi_{L} \text{ and } \pi_{R}) G_{^{\pi_{L}}_{\pi_{R}}} \) with definite parity charges under the interchange of external coordinates \( x_{1} \leftrightarrow x_{2} \). Here \( \pi_{L,R} \) are those appearing in \( \mathcal{M}_{SU(N)} \). Since the conformal primary operator \( g \) constructs bi-fundamental representation of \( SU(N_{f})_{L} \times SU(N_{f})_{R} \), its crossing equations are related to the crossing equations in \( \text{IV C} \). However, there is a subtle difference between the crossing equations of 2D WZW model and higher dimensional CFTs. In the latter case the CPWs and conformal blocks are symmetric between variables \( \eta, \bar{\eta} \), while several \( SU(N_{f})_{L} \times SU(N_{f})_{R} \) representations in \( \text{IV C} \) are not, thought such symmetry is restored in the whole four-point correlator, associated with exchanging of \( SU(N_{f})_{L} \) and \( SU(N_{f})_{R} \) indices. Due to an algebraic relation similar to \( \text{IV C} \), the \( SU(N_{f})_{L} \times SU(N_{f})_{R} \) bifundamental crossing equations can be linearly transformed to the \( SO(2N_{f}^{2}) \) vector crossing equations associated with the \( SO(2N_{f}^{2}) \rightarrow SU(N_{f})_{L} \times SU(N_{f})_{R} \) branching rules

\[ SO(2N_{f}^{2}) \rightarrow SU(N_{f})_{L} \times SU(N_{f})_{R} \]

\[ S \rightarrow (S, S), \]

\[ T \rightarrow (\text{Adj, Adj}) \oplus (\text{Adj, S}) \oplus (S, \text{Adj}) \oplus (T, T) \oplus (A, A), \]

\[ A \rightarrow (S, S) \oplus (\text{Adj, Adj}) \oplus (\text{Adj, S}) \oplus (S, \text{Adj}) \oplus (T, A) \oplus (A, T), \]

and \( SO(2N_{f}^{2}) \)-symmetric correlation functions \( G'_{S/T/A} \) can be constructed following \( \text{IV C} \) with recombination coefficients \( x_{1}, y_{1} \) given in Section \( \text{IV C} \). The correlation functions with general \( N_{f}, k > 1 \) are rather complicated. For the \( k = 1 \) case, which corresponds to the IR fixed point of 2D Quantum Electrodynamics (QED_{2}), the factor \( h \) in \( \text{IV C} \) vanishes and the four-point function is simplified significantly.

\[ 9 \] The crossing equations of 2D WZW models have a subtle difference comparing with higher dimensional crossing equations. In 2D WZW model, some of the components in the four-point correlator \( \text{IV C} \), such as \( G'_{T,A}, G'_{\text{Adj,S}} \) are not symmetric between variables \( \eta \) and \( \bar{\eta} \). In contrast, the higher dimensional conformal blocks are symmetric between the two variables \( \text{IV C} \).
The $SO(2N_f^2)$-ization correlation function is

\[
G^\prime_S(u, v) = \frac{u^{-\frac{N_f+1}{2N_f}}} {2N_f} \left( u^{2/N_f} (-N_f u + (N_f - 1) N_f v + N_f u + v(N_f (N_f - u + v - 1) + u) \right),
\]

\[
G^\prime_T(u, v) = \frac{1}{4 (2N_f^2 + N_f - 1)} v \left( N_f^2 (N_f + 1)^2 u^2 - \frac{N_f}{N_f} v^2 \right) + 2 (N_f^2 - 1) u \left( v^{\frac{N_f}{N_f}} + v^{1-\frac{N_f}{N_f}} \right) - (N_f - 1)^2 N_f^2 u^{1-\frac{N_f}{N_f}} \times v^{\frac{N_f}{N_f}} (u - 2(v+1)) + 2 (N_f^2 - 1) \left( v^{\frac{N_f}{N_f}} (2u - N_f (u + v - 1)) + v^{1-\frac{N_f}{N_f}} (N_f (u + v - 1) + 2u) \right),
\]

\[
G^\prime_A(u, v) = \frac{u \frac{N_f+1}{2N_f} v^{\frac{N_f}{N_f}} (N_f (u (N_f - 1) + v) - 1) + v^{\frac{N_f}{N_f}} (-u (v - 1) (N_f^2 - 1) + u^{\frac{N_f}{N_f}} (u + v - 1) + u N_f^2 + v) \right).}
\]

In the large $N_f$ limit, the above correlation functions contain leading $u, v$ analytical terms:

\[
G^\prime_S(u, v) = 1 + \frac{-uv - u + v^2 - 2v + 1}{2N_f v},
\]

\[
G^\prime_T(u, v) = \frac{u + uv}{2v} + \frac{2u^2 - 3uv - 3u + (v - 1)^2}{4N_f v},
\]

\[
G^\prime_A(u, v) = \frac{u - uv}{2v} + \frac{uv - u - v^2 + 1}{4N_f v}.
\]

consistent with the large $N$ limit of the $SO(N)$-ization of the 2D free fermion bilinear four-point correlators \[61\]. This is expected as the conformal primary operator $g_S$ in the $SU(N_f)_{k=1}$ WZW model is equivalent to the free fermion bilinears in the large $N_f$ limit. The four-point function $g^\prime_S$ can also be truncated into subgroups of $SU(N_f)_{k} \times SU(N_f)_{R}$ to realize different $SO(N)$-ization. For instance, one may take the diagonal part of $SU(N_f)_{L} \times SU(N_f)_{R}$ in \[64\] and construct an $SO(2N_f)$-symmetric four-point correlator. Overall our result suggests there are rich $SO(N)$-symmetric four-point correlation functions from the $SO(N)$-ization of the WZW model.

An interesting question is whether certain $SO(N)$-ized four-point correlation functions can play roles in determining the $SO(N)$ vector bootstrap bounds. In \[14\] a new family of bootstrap kinks have been discovered in the 3D $SO(N)$ vector bootstrap bounds besides the well-known kinks \[32\] corresponding to the classical Wilson-Fisher $O(N)$ fixed points. These kinks also appear in the $SO(N)$ vector bootstrap bounds in general dimensions \[17\] and they approach free fermion theory in the large $N$ limit. There are evidence indicating they may correspond to gauge theories coupled with fermions, such as the IR fixed points of QEDs \[57, 58\] and the Caswell-Banks-Zaks fixed points of 4D Quantum Chromodynamics \[53, 10\]. While the conformal gauge theories are non-$SO(N)$ symmetric and it was unclear how such theories could appear in the $SO(N)$ vector bootstrap bounds. The $SO(N)$-ization provides a possible explanation for this puzzle. The $SU(N_f)_{k}$ WZW model can be considered as dimensional continuations of the IR fixed points of higher dimensional $U(k)$ gauge theories coupled to $N_f$ flavors of Dirac fermions \[11, 12\]. We hope the $SO(N)$-ization of WZW model could provide a key to decode the $SO(N)$ vector bootstrap results.

### VII. Outlooks

The algebraic property of the crossing equations studied in this work clarifies a fundamental barrier for the widely interested modern conformal bootstrap program, which aims to numerically solve or classify the CFTs using crossing equations and positivity condition. This ambitious project relies on the assumption that the dynamics of CFTs is encoded in the crossing equations and positivity condition. This ambitious project relies on the assumption that the dynamics of CFTs is encoded in the crossing equations and positivity condition. However, our results show that for various types of symmetries $\mathcal{G}$, the $\mathcal{G}$-symmetric four-point crossing equations are actually equipped with an $SO(N)$ symmetric positive structure. According to the conformal bootstrap algorithm, without $SO(N)$ symmetry breaking assumptions on the CFT data, the $\mathcal{G}$-symmetric conformal bootstrap degenerates to the $SO(N)$ vector bootstrap, which provides strong restrictions to numerically solve the non-$SO(N)$ symmetric theories. The conclusion is that due to a dedicate algebraic relation of the conformal four-point crossing equations, the $\mathcal{G}$-symmetric crossing equations and positivity condition alone do not contain specific information for the non-$SO(N)$ symmetric theories, and it needs to introduce new ingredients or specific assumptions on the spectrum to move further \[10\].

\[10\] These restrictions should not be considered as a no-go theorem for the bootstrap program. One may introduce extra assumptions on the CFT data of the target CFTs in the bootstrap implementations, which can break the $SO(N)$ symmetry explicitly.
We discuss two interesting implications of our results:

6\textit{j}-symbols, crossing symmetry and positivity: Our results only uncover a tip of the iceberg for a deep connection between representation theory and CFTs. The conformal bootstrap approach exploits constraints from the conformal blocks, crossing symmetry and positivity, and its remarkable success suggests there are unknown positive structures in the conformal blocks associated with crossing symmetry, i.e. the 6\textit{j} symbols of the conformal group $SO(D + 1, 1)$ \cite{Belavin:1984vu, Belavin:1993ts}. The symmetry properties of the conformal four-point functions provide reminiscent while much simpler examples on the positive structures in the 6\textit{j} symbols. We have verified this symmetry property for a variety of groups, and it would be very instructive to obtain a mathematically general proof for the existence of these linear transformations and the positivity of the recombination coefficients. This can clarify how the positivity arises in the 6\textit{j} symbols of the compact global symmetry groups, which can provide insights to study the positive structures in the 6\textit{j} symbols of the non-compact local conformal group $SO(D + 1, 1)$.

Constraints on the RG flows: In 3D the $SO(N)$ vector bootstrap bounds are saturated by the critical $O(N)$ vector models \cite{Polyakov:1974gs}. Considering there are abundant $SO(N)$-symmetric four-point correlators \cite{Rattazzi:2008pe, Rychkov:2014qqa} constructed from the scalar CFTs with symmetries $G \subseteq SO(N)$ \cite{Rattazzi:2008pe, Rychkov:2014qqa}, the bootstrap results indicate these non-$SO(N)$ symmetric scalar CFTs contain certain factors which make their $SO(N)$-ized correlators located in the bulk of the $SO(N)$ vector bootstrap bounds. This would further suggest the RG flows from the critical $O(N)$ vector models to non-$SO(N)$ symmetry fixed points driven by the $SO(N)$ symmetry breaking couplings are restricted within certain directions.

Acknowledgements

It is a pleasure to thank David Poland for stimulating discussions and collaborations on recent projects. The author is grateful to David Poland and Slava Rychkov for valuable comments and suggestions on the draft. The author would like to thank Soner Albayrak and Rajeev Erramilli for discussions. The work of ZL is supported by Simons Foundation grant 488651 (Simons Collaboration on the Non-perturbative Bootstrap) and DOE grant no. DE-SC0020318.

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