The warped product approach to magnetically charged GMGHS spacetime

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Abstract

In the framework of Lorentzian multiply warped products we study the magnetically charged Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) interior spacetime in the string frame. We also investigate geodesic motion in various hypersurfaces, and compare their solutions of geodesic equations with the ones obtained in the Einstein frame.

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1 Introduction

The spherically symmetric static charged black holes in the four-dimensional heterotic string theory has been extensively investigated ever since the discovery of the solutions by Gibbons, Maeda [1] and by Garfinkle, Horowitz, Strominger [2]. Null geodesics and hidden symmetries in the Sen black hole was investigated by Hioki and Miyamoto [3], and Gad [4] studied both geodesic equations and geodesic deviation of the GMGHS black hole solution. Fernando [5] investigated null geodesic motions of the same solution both in the Einstein and string frame. In a Lorentzian multiply warped product spacetime, by exchanging timelike and spacelike coordinates, we have also studied geodesic equations of the GMGHS interior spacetime in the Einstein frame [6].

Recently, we have obtained the (5 + 1)/(5 + 2) dimensional global flat embeddings of the GMGHS spacetime [7] according to the Einstein/string frames, respectively. In the Einstein frame where the solution is obtained from the action in form of the Einstein-Hilbert, we need the (5 + 1) dimensional embedding. The result is similar to the embedding of the Schwarzschild spacetime [8]. On the other hand, in the string frame where strings directly couple to the metric of \( e^{2\phi} g_{\mu\nu} \), the global flat embeddings of the GMGHS spacetime need one more time dimension like the Reiman-Nordström spacetime [9, 10]. Even though the solutions in the two frames are known to be related by a conformal transformation so that they are mathematically isomorphic to each other [11], there are differences in some of the physical properties of the black hole solutions in the two frames [12] including the above mentioned different global flat embeddings. This difference motivates us to study geodesic equations of the GMGHS interior spacetime in the string frame and compare with the ones obtained in the Einstein frame.

In this paper, we study the magnetically charged GMGHS interior spacetime and investigate how different the Lorentzian multiply warped products are in the sting frame from the Einstein one. We also investigate the geodesic motion near hypersurfaces of this interior spacetime in the string frame and compare their solutions of geodesic equations with the ones obtained in the Einstein frame. We shall use geometrized units, i.e., \( G = c = 1 \), for notational convenience.
2 Magnetically charged GMGHS black hole in the framework of warped products

In the string frame, the GMGHS action is described by

\[ S = \int d^4x \sqrt{-g} e^{-2\phi} (R + 4(\nabla \phi)^2 - F_{\mu\nu}F^{\mu\nu}), \]  

(2.1)

where \( R \) is the scalar curvature, \( \phi \) is a dilation, and \( F_{\mu\nu} \) is the Maxwell’s field strength. Through the conformal transformation of \( g_{\mu\nu}^S = e^{2\phi} g_{\mu\nu}^E \), it is related to the Einstein frame action [1, 2]. The Einstein metric does not change when we go from an electrically to a magnetically charged black hole, but since the dilaton \( \phi \) changes sign, the string metric does change. As a result, we get the GMGHS solution of the Einstein field equations which represents the geometry exterior to a spherically symmetric static charged black hole.

In the Schwarzschild coordinates, the line element for the magnetically charged GMGHS black hole metric in the exterior region \((r > 2m)\) has the form as follows

\[ ds^2 = -\frac{\left(1 - \frac{2m}{r}\right)}{\left(1 - \frac{Q^2}{mr}\right)} dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)\left(1 - \frac{Q^2}{mr}\right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(2.2)

Here, the parameters \( m \) and \( Q \) are mass and charge, respectively. When \( Q \to 0 \), it is reduced to the Schwarzschild spacetime, and as like in the Schwarzschild spacetime, the magnetically charged GMGHS solution has an event horizon at \( r = 2m \). The surface area of the sphere of the magnetically charged GMGHS black hole, defined by \( \int d\theta d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} \), is also the same as the Schwarzschild spacetime.

On the other hand, the line element for the magnetically charged GMGHS metric for the proper interior region can be described by

\[ ds^2 = -\frac{dr^2}{\left(\frac{2m}{r} - 1\right)\left(1 - \frac{Q^2}{mr}\right)} + \frac{\left(\frac{2m}{r} - 1\right)}{\left(1 - \frac{Q^2}{mr}\right)} dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(2.3)

where \( r \) and \( t \) are now new temporal and spacial variables, respectively.

A multiply warped product manifold, denoted by \( M = (B \times F_1 \times ... \times F_n, g) \), consists of the Riemannian base manifold \((B, g_B)\) and fibers \((F_i, g_i)\)
(i = 1, ..., n) associated with the Lorentzian metric [13]. In particular, for the specific case of \((B = R, \ g_B = -d\mu^2)\), the magnetically charged GMGHS metric (2.3) can be rewritten as a multiply warped products \((a, b) \times f_1 R \times f_2 S^2\) by making use of a lapse function

\[ N^2 = \left( \frac{2m}{r} - 1 \right) \left( 1 - \frac{Q^2}{mr} \right). \]  

(2.4)

This lapse function is well defined in the region of \(\frac{Q^2}{m} < r < 2m\) to rewrite it as a multiply warped products spacetime by defining a new coordinate \(\mu\) as

\[ \mu = \int_{r_0}^{r} dx \frac{x}{\sqrt{(2m - x)(x - \frac{Q^2}{m})}} = F(r). \]  

(2.5)

Setting the integration constant zero as \(r \to \frac{Q^2}{m}\), we have

\[ \mu = -\sqrt{(2m - r)(r - \frac{Q^2}{m})} \]

\[ -(m + \frac{Q^2}{2m}) \tan^{-1} \left( \frac{2m^2 + Q^2 - 2mr}{2\sqrt{m(2m - r)(mr - Q^2)}} \right) + \frac{\pi(2m^2 + Q^2)}{4m}. \]

(2.6)

We have plotted \(\mu\) as a function of \(r\) in Fig. 1. The new coordinate \(\mu(r)\) explicitly depends on the charge, while the same coordinate in the Einstein frame [7] does not. We also note that in the Einstein frame the GMGHS interior spacetime is defined in \(0 < r < 2m\) with the surface area singularity at \(r = \frac{Q^2}{m}\). However, the magnetically charged GMGHS interior spacetime in the string frame is defined in \(\frac{Q^2}{m} < r < 2m\) with no surface area singularity. Thus, in Fig. 1, we see that as the charge increases, the proper range of \(r\) is narrower.

Now, let us notice \(\frac{dF}{dr} > 0\) which implies \(F^{-1}\) is a well-defined function. We can thus rewrite the magnetically charged GMGHS metric (2.3) with the lapse function (2.4) and the two warping functions \(f_1(\mu)\) and \(f_2(\mu)\) as follows

\[ ds^2 = -d\mu^2 + f_1^2(\mu) dr^2 + f_2^2(\mu)(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(2.7)
Figure 1: $\mu$ as a function of $r$: Fig. (a) is for $m = 1$ and from the top curve near $r = 2$, $Q = 1.2$, $0.9$, $0.6$, $0.3$, while Fig. (b) is for $m = 2$ and from the top curve near $r = 4$, $Q = 2.4$, $1.8$, $1.2$, $0.6$. The bottom curves denote the Schwarzschild limit of $Q = 0$.

where the warping functions are given by

$$f_1(\mu) = \left( \frac{2m}{F^{-1}(\mu)} - 1 \right)^{1/2}, \quad f_2(\mu) = F^{-1}(\mu), \quad (2.8)$$

and $F^{-1}(\mu)$ is defined in Eq. (2.5). We note that the magnetically charged GMGHS interior spacetime written in the multiply warped product spacetime has the same form with the Kantowski-Sachs solution [14].

As a result, in the case of the interior region $\frac{Q^2}{m} < r < 2m$, the magnetically charged GMGHS metric has been rewritten as a multiply warped product spacetime having the warping functions in terms of $f_1(\mu)$ and $f_2(\mu)$ in Eq. (2.8). Moreover, we can write down the Ricci curvature on the multiply warped products as

$$R_{\mu\mu} = -\frac{f''_1}{f_1} - \frac{2f''_2}{f_2},$$

$$R_{rr} = f_1 f''_1 + 2 \frac{f_1 f'_1 f'_2}{f_2},$$

$$R_{\theta\theta} = \frac{f'_1 f_2 f'_2}{f_1} + f''_2 + f_2 f''_2 + 1.$$


\[ R_{\phi\phi} = \left( \frac{f_1 f_2 f_2'}{f_1} + f_2'^2 + f_2 f_2'' + 1 \right) \sin^2 \theta, \]
\[ R_{mn} = 0, \text{ for } m \neq n, \]  
(2.9)

which have the same form with the Ricci curvature of the multiply warped interior Schwarzschild metric [13]. However, it is differently parameterized due to the different functional dependence as like in Eq. (2.8).

3 Geodesic motion near hypersurface

In this section, we are interested in investigating the geodesic curves of the magnetically charged GMGHS interior spacetime.

In local coordinates \( \{x^i\} \) the line element corresponding to this metric (2.3) will be denoted by

\[ dS^2 = g_{ij} dx^i dx^j. \]  
(3.1)

Consider the equations of geodesics in the magnetically charged GMGHS spacetime with affine parameter \( \lambda \) given by

\[ \frac{dx^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0. \]  
(3.2)

Let a geodesic \( \gamma \) be given by \( \gamma(\tau) = \left( \mu(\tau), r(\tau), \theta(\tau), \phi(\tau) \right) \) of the magnetically charged GMGHS interior spacetime of \( \frac{Q^2}{m} < r < 2m \) from Eq. (2.3), then the orbits of the geodesics equation are given as follows

\[ \frac{d^2 \mu}{d\tau^2} + f_1 \frac{df_1}{d\mu} \left( \frac{dr}{d\tau} \right)^2 + f_2 \frac{df_2}{d\mu} \left( \frac{d\theta}{d\tau} \right)^2 + f_2 \frac{df_2}{d\mu} \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0, \]  
(3.3)
\[ \frac{d^2 r}{d\tau^2} + \frac{2}{f_1} \frac{df_1}{d\tau} \frac{dr}{d\tau} = 0, \]  
(3.4)
\[ \frac{d^2 \theta}{d\tau^2} + \frac{2}{f_2} \frac{df_2}{d\tau} \frac{d\theta}{d\tau} - \sin \theta \cos \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0, \]  
(3.5)
\[ \frac{d^2 \phi}{d\tau^2} + \frac{2}{f_2} \frac{df_2}{d\tau} \frac{d\phi}{d\tau} + 2 \cot \theta \frac{d\theta}{d\tau} \frac{d\phi}{d\tau} = 0 \]  
(3.6)
with a following constraint along the geodesic
\[-\left(\frac{d\mu}{d\tau}\right)^2 + f_1^2 \left(\frac{dr}{d\tau}\right)^2 + f_2^2 \left(\frac{d\theta}{d\tau}\right)^2 + f_2^2 \sin^2 \theta \left(\frac{d\phi}{d\tau}\right)^2 = \varepsilon. \tag{3.7}\]

Note that a timelike (nulllike) geodesic is taken as $\varepsilon = -1 \ (\varepsilon = 0)$.

Hereafter, without loss of generality, suppose the geodesic
\[\gamma(\tau_0) = \left(\mu(\tau_0), r(\tau_0), \theta(\tau_0), \phi(\tau_0)\right) \tag{3.8}\]
for some $\tau_0$ and the equatorial plane of $\theta = \frac{\pi}{2}$, thus $\frac{d\theta}{d\tau} = 0$. Then, the geodesic equations are reduced to
\[
\frac{d^2 \mu}{d\tau^2} + f_1 \frac{df_1}{d\mu} \left(\frac{dr}{d\tau}\right)^2 + f_2 \frac{df_2}{d\mu} \left(\frac{d\phi}{d\tau}\right)^2 = 0, \tag{3.9}
\]
\[
\frac{d^2 r}{d\tau^2} + 2 \frac{df_1}{f_1} \frac{dr}{d\tau} = 0, \tag{3.10}
\]
\[
\frac{d^2 \theta}{d\tau^2} = 0, \tag{3.11}
\]
\[
\frac{d^2 \phi}{d\tau^2} + 2 \frac{df_2}{f_2} \frac{d\phi}{d\tau} = 0 \tag{3.12}
\]
with a constraint
\[-\left(\frac{d\mu}{d\tau}\right)^2 + f_1^2 \left(\frac{dr}{d\tau}\right)^2 + f_2^2 \left(\frac{d\phi}{d\tau}\right)^2 = \varepsilon. \tag{3.13}\]

These geodesic equations can be further simplified to give
\[
\frac{dr}{d\tau} = \frac{c_1}{f_1^2}, \tag{3.14}
\]
\[
\frac{d\phi}{d\tau} = \frac{c_2}{f_2^2}, \tag{3.15}
\]
\[
\frac{d^2 \theta}{d\tau^2} = 0 \tag{3.16}
\]
\[
(3.17)
\]
with a constraint

$$ - \left( \frac{d\mu}{d\tau} \right)^2 + \frac{c_1^2}{f_1^2} + \frac{c_2^2}{f_2^2} = \varepsilon. \quad (3.18) $$

The constant $c_1$ represents the total energy per unit rest mass of a particle as measured by a static observer \([4, 15, 16]\), and $c_2$ represents the angular momentum in the GMGHS spacetimes. The equations for $r$ and $\phi$ are obtained from Eqs. (3.10) and (3.12), respectively. Making use of these $r$, $\phi$ equations, we can show that Eq. (3.9) is the exactly same with Eq. (3.13) when we take the integration constant as $-\frac{\varepsilon}{2}$.

Now, we consider the null geodesics in the $r$-direction, which is defined by the hypersurface $\Sigma_r$ by taking $d\theta = d\phi = 0$. Then, we have $c_2 = 0$ in Eq. (3.15). Two equations (3.14) and (3.18) are now reduced to give

$$ dr = \frac{d\mu}{f_1(\mu)}. \quad (3.19) $$

We have plotted in Fig. 2 the null geodesic curve on the hypersurface $\Sigma_r$. We see the radial coordinate $r(\mu)$ is a monotonic function of $\mu$. The curves on the far left in Fig. 2 (a), (b) correspond to the Schwarzschild limit. Comparing with the curve, Fig. 1 in Ref. [6], obtained in the Einstein metric, which has no charge dependence, we have explicit charge dependence on $r(\mu)$.

Let us consider the geodesic in the $\phi$-direction, which lies on the hypersurface $\Sigma_\phi$ at $\theta = \frac{\pi}{2}$ with $dr = 0$. Then, we have $c_1 = 0$ in Eq. (3.14). Two equations (3.15) and (3.18) are reduced to give

$$ d\phi = \frac{d\mu}{f_2(\mu)}, \quad (3.20) $$

where $f_2(\mu)$ is given by Eq. (2.8). In Fig. 3, we have numerically drawn the azimuth angle. The top curve in Fig. 3 corresponds to the zero charged pure Schwarzschild limits. We note all the other curves depending on the charges start from the origin. This contrasts with the corresponding curves in Ref. [6] obtained from the geodesic in the $\phi$-direction in the Einstein frame, where the curves start from the points determined by the charges.
Figure 2: Geodesic curves on the hypersurface $\Sigma_r$: Fig. (a) is for $m = 1$ and, from the left curve, $Q = 0$, the Schwarzschild limit, and $Q = 0.3, 0.6, 0.8, 1.2$, while Fig. (b) is for $m = 2$ and, from the left curve, $Q = 0$, the Schwarzschild limit, and $Q = 0.6, 1.2, 1.8, 2.4$.

Finally, let us find the geodesic in the $\mu$-direction, which is defined by the
hypersurface $\Sigma_\mu$, eliminating $\mu$ in Eqs. (3.19) and (3.20), leading to

$$\frac{d\phi}{dr} = \frac{1}{r} \sqrt{\frac{2m - r}{r - \frac{Q^2}{m}}},$$

which is the exactly same as the one obtained in Ref. [6]. In Fig. 4, we have drawn the geodesic curve $\phi(r)$ for different masses and charges.

4 Conclusions

In this paper, we have studied the magnetically charged GMGHS interior spacetime in associated with a multiply warped product manifold. In the multiply warped product manifold, the magnetically charged GMGHS spacetime has been characterized by two warping functions $f_1(\mu)$ and $f_2(\mu)$, compared with the Schwarzschild spacetime which has one warping function of $f_1(\mu)$ and the GMGHS interior spacetime in the Einstein frame which have two $f_1(\mu)$, $f_2(\mu)$ but different warping functions. We have also investigated the geodesic motion near hypersurfaces in the interior of the event horizon. Due to the charge term on the lapse function (2.4), the two warping functions
are also charge dependent, and as a result, the geodesic curves in the magnetically charged GMGHS interior spacetime have been drawn by charges. This contrast with the geodesic curves in Ref. [7] where the lapse function has no charge dependent term.

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