Well-posedness and Blowup of the Geophysical Boundary Layer Problem

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Abstract. The proposal of this paper is to study the local existence of analytic solutions, and blowup of solutions in a finite time for the geophysical boundary layer problem. In contrast with the classical Prandtl boundary layer equation, the geophysical boundary layer equation has an additional integral term arising from the Coriolis force. Under the assumption that the initial velocity and outer flow velocity are analytic in the horizontal variable, we obtain the local well-posedness of the geophysical boundary layer problem by using energy method in the weighted Chemin-Lerner spaces. Moreover, when the initial velocity and outer flow velocity satisfy certain condition on a transversal plane, for any smooth solution decaying exponentially in the normal variable to the geophysical boundary layer problem, it is proved that its $W^{1,\infty}$ norm blows up in a finite time. Comparing with the blowup result obtained in Kukavica et al. (Adv Math 307:288–311, 2017) for the classical Prandtl equation, we find that the integral term in the geophysical boundary layer equation triggers the formulation of singularities earlier.

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1. Introduction

In this paper, we consider the following initial boundary value problem in the domain $Q_T = \{0 < t < T, x \in \mathbb{R}, y > 0\}$,

\[
\begin{align*}
\partial_t u + u\partial_x u + v\partial_y u - \int_y^{+\infty} (u - U)dy' - \partial_y^2 u &= \partial_t U + U\partial_x U, \\
\partial_x u + \partial_y v &= 0, \\
|_{t=0} u &= u_0(x, y), \\
|_{y=0} (u, v) &= (0, 0), \quad \lim_{y \to +\infty} u(t, x, y) = U(t, x),
\end{align*}
\]

(1.1)

where $(u, v)$ is the velocity field, and $U(t, x)$ is the tangential velocity of the outer flow.

The problem (1.1) describes the oceanic current near the western coast, it can be derived from the beta plane approximation model of the oceanic current motion at midlatitudes under the action of wind and the Coriolis force in the large Reynolds number and beta parameter limit. By properly scaling in certain geophysical regime, and omitting the bottom friction and topography, the beta plane approximation of the oceanic current can be described by the following two dimensional homogeneous model (see [4,14]) in $\{x \in \mathbb{R}, Y > 0\}$,

\[
\begin{align*}
\partial_t U + U \cdot \nabla U - \beta x U^\perp + \nabla\Pi - Re^{-1}\Delta U &= \beta\tau, \\
\text{div } U &= 0, \\
|_{Y=0} U &= 0,
\end{align*}
\]

(1.2)

where the Cartesian-like coordinates $(x, Y)$ represent latitude and longitude respectively, and $U = (U, V)^T$, $\Pi$, $Re$ and $\beta$ are the velocity, the pressure of fluid, the Reynolds number and the beta-plane parameter respectively, $\tau = (\tau_1, \tau_2)^T$ is the shear tensor created by wind, and $-x U^\perp$ represents the effect of the Coriolis force created by rotation with $U^\perp = (-V, U)^T$. When $Re = \beta^2$, for which the inertial force,
the Coriolis force and viscous friction have the same order in boundary layer, by multi-scale analysis it is known that as $\epsilon = \text{Re}^{-1} \to 0$, the solution of (1.2) near $\{Y = 0\}$ behaves as

$$
\begin{align*}
U(t,x,Y) &= u(t,x,\frac{Y}{\sqrt{\epsilon}}) + o(1) \\
V(t,x,Y) &= \sqrt{\epsilon} v(t,x,\frac{Y}{\sqrt{\epsilon}}) + o(\sqrt{\epsilon})
\end{align*}
$$

with $(u(t,x,y), v(t,x,y))$ satisfying the geophysical boundary layer problem (1.1). More detail of the derivation can be found in [7,16].

For the two-dimensional incompressible Navier-Stokes equations with non-slip boundary condition, Prandtl introduced in [15] that in the small viscosity limit, the flow near the boundary is described by a problem similar to (1.1) without the integral term, which is called the Prandtl equation nowadays. Under the monotonicity assumption, $u_y > 0$ of the tangential velocity, the well-posedness of the two-dimensional Prandtl equation is established in [1,12,13,19] and references therein. On the other hand, without the monotonicity condition, the well-posedness of the Prandtl equation was obtained in the analytic class and the Gevery class, cf. [2,5,10,11,20], and the blowup of the Sobolev norm of solutions in a finite time was given in [6,9] for certain class initial data.

Certain formal discussion on the boundary layers of geophysical fluids can be found in [14]. For the system (1.2) with an additional bottom friction term, the behavior of the Munk layers and Stommel layers was given in [4]. Recently, the well-posedness of a two-dimensional steady geophysical boundary layer problem was studied in [3].

The aim of this paper is to study the local well-posedness and blowup of solutions to the unsteady geophysical boundary layer problem (1.1). Comparing with the classical Prandtl equation, there exists an additional integral term in (1.1) arising from the Coriolis force. To deal with this integral term, we need to study the tangential velocity $u$ having certain decay rate in the normal variable $y$, thus we shall consider this problem in a weighted function space. By developing an energy method inspired from [17,20] and estimating the additional integral term, we shall obtain the well-posedness of the geophysical boundary layer problem (1.1) when both of the initial data and the outer flow velocity are analytic in the tangential variable. On the other hand, by using an idea inspired by [9], we shall construct a Lyapunov functional associated with $\partial_x u$ to deduce that the $W^{1,\infty}$-norm of the solution to the geophysical boundary layer problem (1.1) blows up in a finite time for certain class of initial data and outer flow. This shows that in general, the analytic solution of (1.1) exists only locally in time. It is interesting to see that the integral term coming from the Coriolis force has a sensitive effect on the formation of singularity when we study the blowup mechanism of the solution to the unsteady geophysical boundary layer problem (1.1). Especially, in the case of the outer flow velocity vanishing identically on the boundary, we get that the blowup of the boundary layer velocity profile always occurs in a finite time for any nonzero initial data, but, for which the classical Prandtl equation may have an almost global solution ([8,20]).

To state our main results, as in [20], we introduce

$$\phi(t,y) = \text{Erf} \left( \frac{y}{\sqrt{4(t+1)}} \right) \quad \text{with} \quad \text{Erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-z^2} dz$$

to homogenize the data of $u$ at infinity given in (1.1). Obviously, $\phi(t,y)$ is a solution to the problem

$$
\begin{align*}
\partial_t \phi - \partial_y^2 \phi &= 0, \\
\phi_{\mid y=0} &= 0, \quad \lim_{y \to +\infty} \phi(t,y) = 1, \\
\phi_{\mid t=0} &= \text{Erf} \left( \frac{y}{2} \right).
\end{align*}
$$
Let \( u^s(t, x, y) = U(t, x)\phi(t, y) \) and \( w = u - u^s \). From (1.1), we know that \( w \) satisfies the following problem in \( \{ t > 0, x \in \mathbb{R}, y > 0 \} \),

\[
\begin{aligned}
\partial_t w + (w + u^s) \partial_x w + w \partial_x u^s - \int_0^y \partial_x (w + u^s) dy' \partial_y (w + u^s) - \int_y^{+\infty} w dy' - \partial_y^2 w \\
= (1 - \phi) (\partial_t U + (1 + \phi) U \partial_x U) - \int_y^{+\infty} U (1 - \phi) dy',
\end{aligned}
\]

(1.3)

By applying the Littlewood-Paley theory, we shall obtain the following existence and uniqueness of a solution to the problem (1.3) in the weighted Chemin-Lerner spaces, when the initial data and outer flow velocity are analytic in \( x \in \mathbb{R} \).

**Theorem 1.1.** For a given \( T_0 > 0 \), assume that the initial velocity \( w_0(x, y) \) and the outer flow velocity \( U(t, x) \) are analytic in \( x \in \mathbb{R} \), and

\[
e^{(D)}w_0 \in B^{1,0}_{\psi_0},
\]

and

\[
e^{(D)}U \in \tilde{L}_0^\infty \left( B^{2} \right) \cap \tilde{L}_0^\infty (B^{1}) \cap \tilde{L}_0^\infty \left( B^{2} \right), \quad e^{(D)}U_t \in \tilde{L}_0^\infty \left( B^{2} \right),
\]

where the spaces given above will be defined in Definition 2.1 with the Fourier multiplier \( D \) and the weight \( \psi_0 \) being given in (2.1). Then, there exists \( 0 < T^* \leq T_0 \) such that the problem (1.3) has a unique solution \( e^{(D)}w \in \tilde{L}_T^\infty (B^{1,0}) \), where the weight \( \psi(t, y) \) is given in (2.1), and \( \Phi(t, D) \) is the operator associated with the symbol \( \Phi(t, \xi) \) being given in (2.5).

Moreover, we shall have the following blowup result:

**Theorem 2.** For a given \( T > 0 \), assume that the initial data and outer flow satisfy

\[
u_0(0, y) = U(t, 0) = 0, \quad U_x(t, 0) \geq 0, \quad u_{0x}(0, y) \leq U_x(0, 0)
\]

(1.4)

and there is \( M \geq 0 \) depending on \( T \) and \( \|U_x(\cdot, 0)\|_{L^\infty([0, T])} \) such that

\[
\int_0^{+\infty} \rho(y) (U_x(0, 0) - u_{0x}(0, y)) dy > M
\]

(1.5)

for a weight function \( \rho(y) \) given in (3.11)–(3.13), if the smooth solution \( u \in C^3(Q_T) \) of (1.1) satisfies

\[
\sup_{0 \leq t \leq T} \int_0^{+\infty} (u(t, 0, y))^2 e^{2\mu y} dy < +\infty
\]

(1.6)

and

\[
\lim_{y \to +\infty} \sup_{0 \leq t \leq T} (|u_x(t, 0, y) - U_x(t, 0)| e^{\mu y}) = 0
\]

(1.7)

for some \( \mu > 0 \), then the \( W^{1,\infty} \)-norm of \( u \) blows up in a finite time.

**Remark 1.3.** (1) When the outer flow velocity vanishes identically, i.e., \( U(t, x) \equiv 0 \), the constant \( M \) in the condition (1.5) can be zero. In fact, from Lemma 3.3 we shall see that the blowup result of the problem (1.1) always holds for any nonzero initial data satisfying

\[
u_0(0, y) = 0, \quad u_{0x}(0, y) \leq 0.
\]

(1.8)

This result is different from the one obtained in [9] for the classical Prandtl equation. Moreover, if the classical Prandtl equation with the trivial outer flow, \( U(t, x) = 0 \), admits a smooth solution \( u^p(t, x, y) \) in the time interval \([0, T]\), with the initial data \( u^p_0(x, y) \) satisfying the same condition as given in (1.8)
and \( \| \partial_x u_0^\mu(t, \cdot) \|_{L^1(\mathbb{R}^+)} = 1 \), from the scaling invariant of the Prandtl equation, it is easy to know that \( u_0^\mu(t, x, y) = \epsilon^2 u^\mu(\epsilon^2 t, x, ey) \) is also a solution to the classical Prandtl equation in \( [0, \frac{T_2}{\epsilon^2}] \), with the initial data \( u_0^\mu(x, y) = \epsilon^2 u_0^\mu(x, ey) \) satisfying \( \| \partial_x u_0^\mu(0, \cdot) \|_{L^1(\mathbb{R}^+)} = \epsilon \) for any fixed \( \epsilon > 0 \). However, from the inequality given in (3.19), we will see that for the classical solution of the problem (1.1), \( \partial_x u(t, x, y) \) shall blow up at most at the order \( O(\epsilon^{-1}) \) of time when initially \( \| \partial_x u_0(0, \cdot) \|_{L^1(\mathbb{R}^+)} = \epsilon \). Therefore, in contrast with the classical Prandtl equation, the integral term contained in (1.1) triggers the formulation of singularity of the solution earlier.

(2) If the conditions (1.6) and (1.7) hold for some \( \mu = \mu_0 > 0 \), it is clear to see that they also hold for any fixed \( \mu \in (0, \mu_0) \), thus the parameter \( \mu \) in (1.6)–(1.7) can be an arbitrary small positive constant. Under the hypothesis (1.4), the assumption (1.6) will be used in Lemma 3.1 to obtain that the classical solution \( u \) of (1.1) vanishes identically on \( \{ x = 0 \} \), while in Lemma 3.2, the condition (1.7) is to guarantee \( u_x(t, 0, y) \leq U_x(t, 0) \) holding always in the time interval of classical solutions.

Before the end of this section, let us briefly explain certain key points in the proofs of Theorems 1.1 and 1.2. As we mentioned before, we shall develop an energy approach inspired from [17,20] in a weighted space to study the existence and uniqueness of a solution to the problem (1.1) with analyticity in the tangential variable \( x \). In establishing the apriori estimates of solutions of (1.1) in the weighted space in Sect. 2.1, we need to study the solution \( u \) exponentially decaying in \( y \) with the weight

\[
\psi(t, y) = \frac{1 + y^2}{C(1 + t)\gamma}
\]

(see the solution space \( B_{\psi, L}^{s, l} \) defined in Sect. 2), for large \( \gamma \) in order to control the growth in \( y \) arising from the integral term of (1.1), different from the classical Prandtl problem studied in [20], and we also need to introduce a new weight \( \Phi^\delta(t, \xi) = (\delta - \lambda \theta(t))(\xi) \) in (2.3) to measure the analyticity radius of solutions, with \( \theta(t) \) being determined by the problem (2.4), in which there is an additional term \( \langle \xi \rangle^\gamma \), this comes from the need of controlling the estimates on the integral term in the equation of (1.1), e.g. see Lemma 2.3. This fact together with the requirement \( \delta - \lambda \theta(t) > 0 \) may imply that probably there is only a local analytic solution to the problem (1.1) even for small data, which is quite different from the almost global result obtained in [20] for the classical Prandtl equation with small data. Secondly, in the proof of the blowup result given in Theorem 1.2, to verify that the tangential velocity \( u(t, x, y) \) of (1.1) vanishes identically on the plane \( \{ x = 0 \} \) when it is true initially under the hypothesis (1.4), we use the energy method for the problem (3.1) of \( u(t, 0, y) \) by imposing the decay assumption (1.6) on the solution, which is used to control the integral term of (1.1), see the proof of Lemma 3.1. However, for the classical Prandtl equation problem studied in [9], the fact \( u(t, 0, y) \equiv 0 \) holds trivially by the uniqueness of the associated problem. Moreover, in order to have the important fact of the non-negativity of

\[
w(t, y) := \partial_x U(t, 0) - \partial_x u(t, 0, y) \geq 0,
\]
a crucial one for introducing the Lyapunov functional on \( w \), in the time interval of classical solutions of (1.1) when it is true at \( t = 0 \), we impose the constraint (1.7) in order to have the boundedness of the integral term when we prove (1.9) by using the maximal principle for the problem of \( w \) in Lemma 3.2. Finally, when we study the behavior of the Lyapunov functional

\[
G(t) = \int_0^{+\infty} \rho(y)w(t, y)dy
\]

with certain non-negative weight \( \rho(y) \), comparing with [9], the property \( \int_0^y \rho(y')dy' + \rho''(y) \geq 0 \) for a.e. \( y \geq 0 \), given in (F2) below (3.12), implies that one can control the term from viscosity by the integral term of (1.1), and another key point is to have

\[
\int_0^{+\infty} \rho'' \left( \int_0^y wdy' \right)^2 dy \leq 4\alpha \int_0^{+\infty} \rho w^2 dy
\]

for some \( \alpha \in (0, 1) \), which plays an important role in controlling the terms in (3.6) arising from the convection term of (1.1), see the proof of Lemma 3.3 for the detail.
The remainder of this paper is arranged as follows: In Sect. 2, we apply the Littlewood-Paley theory to obtain the existence and uniqueness of an analytic solution to the problem (1.3). In Sect. 3, we analyze the blowup of a smooth solution to the problem (1.1) under the hypotheses given in (1.4)–(1.7). The construction of the weight \( \rho(y) \) is given in the “Appendix”.

2. Local Well-posedness

First, let us recall some basic knowledge on the Littlewood-Paley theory and introduce the function spaces, one can refer to [17,20] for the related definitions and properties.

Let \((\varphi, \chi)\) be smooth functions such that

\[
\text{supp } \varphi \subset \left\{ \tau \in \mathbb{R} \mid \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\}, \quad \text{supp } \chi \subset \left\{ \tau \in \mathbb{R} \mid |\tau| \leq \frac{4}{3} \right\}
\]

satisfying

\[
\sum_{k \in \mathbb{N}} \varphi(2^{-k} \tau) = 1 \quad (\forall \tau \neq 0), \quad \chi(\tau) + \sum_{k \geq 0} \varphi(2^{-k} \tau) = 1 \quad (\forall \tau \in \mathbb{R}).
\]

For any given \( f \in S'(\mathbb{R}) \), denote by \((S_k f)(x) = \mathcal{F}^{-1}_{\xi \to x} [\chi(2^{-k} |\xi|) \mathcal{F}[f](\xi)]\), and

\[
(\Delta_k f)(x) = \begin{cases} 
\mathcal{F}^{-1}_{\xi \to x} [\varphi(2^{-k} |\xi|) \mathcal{F}[f](\xi)], & k \geq 0, \\
\mathcal{F}^{-1}_{\xi \to x} [\chi(|\xi|) \mathcal{F}[f](\xi)], & k = -1, \\
0, & k \leq -2,
\end{cases}
\]

where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier transform and the inverse Fourier transform respectively.

Introduce the following function spaces with parameters \( s > 0, l \in \mathbb{N}_+ \) and \( p \in [1, +\infty) \).

**Definition 2.1.** (i) The space \( B^s \) is the set of functions \( u \in S'(\mathbb{R}) \) such that

\[
\|u\|_{B^s} := \sum_{k \in \mathbb{Z}} 2^{ks} \|\Delta_k u\|_{L^2(\mathbb{R})} < +\infty.
\]

(ii) The space \( B_{\psi}^{s,l} \), with a positive function \( \psi(y) \), is the set of functions \( u \in S'(\mathbb{R}^2_+) \) such that

\[
\|u\|_{B_{\psi}^{s,l}} := \sum_{j=0}^I \sum_{k \in \mathbb{Z}} 2^{ks} \|e^{\psi(y)} \Delta_k \partial^j_y u\|_{L^2(\mathbb{R}_x \times \mathbb{R}_+)} < +\infty.
\]

(iii) The space \( \tilde{L}_{t}^{p}(B^s) \) is defined as the completion of \( C([0, t]; S(\mathbb{R})) \) with the norm

\[
\|u\|_{\tilde{L}_{t}^{p}(B^s)} := \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_0^t \|\Delta_k u(t', \cdot)\|_{L^2(\mathbb{R})}^p dt' \right)^{\frac{1}{p}}.
\]

(iv) For any positive function \( \psi(t', y) \) and nonnegative \( f(t') \in L^1_{\text{loc}}(\mathbb{R}^+) \), the space \( \tilde{L}_{t,f}^{p}(B_{\psi}^{s,l}) \) is defined as the completion of \( C([0, t]; S(\mathbb{R}^2_+)) \) with the norm

\[
\|u\|_{\tilde{L}_{t,f}^{p}(B_{\psi}^{s,l})} := \sum_{j=0}^I \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_0^t \|f(t') e^{\psi(t', y)} \Delta_k \partial^j_y u(t', \cdot)\|_{L^2(\mathbb{R}_x \times \mathbb{R}_+)}^p dt' \right)^{\frac{1}{p}}.
\]

Denote \( \tilde{L}_{t}^{p}(B_{\psi}^{s,l}) \) by \( \tilde{L}_{t}^{p}(B_{\psi}^{s,l}) \) for simplicity when \( f(t') \equiv 1 \). The above notations can be properly modified to the case \( p = +\infty \).
2.1. Apriori Estimates

The main step for proving Theorem 1.1 is to establish apriori estimates of solutions to the problem (1.3), then the existence of solutions can be obtained in a usual way by constructing approximate solution sequence through linearization and a proper iteration scheme, and proving the convergence of approximate solutions via the apriori estimates. Firstly, we study the apriori estimates for the problem (1.3).

To obtain energy estimates for solutions of (1.3), we introduce the weights
\[ \psi(t, y) = \frac{1 + y^2}{16(1 + t)^\gamma} \quad \text{and} \quad \psi_0(y) = \frac{1 + y^2}{16}, \]
with \( \gamma \geq 2 \), then there holds
\[ \sqrt{-\psi_t + 2\psi^2_{yy}} \geq (1 + y)\sqrt{\frac{\gamma - 1}{32(1 + t)^{\gamma+1}}}, \]
which is a key to study the solution of (1.3) in the space \( B^{s,l}_\psi \), in order to control the growth in the \( y \)-variable arising from the additional integral term \( \int_y^{+\infty} wdy' \) and the convection term \( -(\int_y^y \partial_x u^s dy')\partial_y w \) of (1.3) in the weighted energy estimates by choosing \( \gamma \) properly large, as shown in (2.12)–(2.13).

In particular, some linear weight \( \psi \) in \( y \) can be chosen for the case in the Prandtl equation with the constant outer flow, as shown in [20], but for the problem we studied here, it can not be used to control the growth in the \( y \)-variable arising from the convection term \( (\int_0^y \partial_x u^s dy')\partial_y w = (\int_0^y \partial_x U \phi(y')dy')\partial_y w \) for the non-constant outer flow.

Denote by \( \hat{w}(t, \xi, y) \) the Fourier transform of \( w(t, x, y) \) in the \( x \)-variable, and
\[ w_\Phi(t, x, y) = \mathcal{F}^{-1}_{\xi \rightarrow x}[e^{\Phi(t, \xi)} \hat{w}(t, \xi, y)] \]
for a given locally bounded function \( \Phi(t, \xi) \). To deal with the loss of derivatives in the \( x \)-variable in (1.3), for any given \( \delta > 0 \), we take
\[ \Phi^\delta(t, \xi) = (\delta - \lambda \theta(t))(\xi) \]
with \( \langle \xi \rangle = 1 + |\xi| \) and a parameter \( \lambda \), in which \( \theta(t) \) is mainly used to study energy estimates for the nonlinear terms and the integral term of (1.4), and it is determined by the following problem
\[
\begin{cases}
\dot{\theta} = \langle t \rangle^\frac{7}{2} \| \partial_y w_\Phi \|_{B^{\frac{1}{2}, 0}_\psi} + \langle t \rangle^\frac{3}{2} \| U_\Phi \|_{B^{\frac{1}{2}, 0} \psi} + \langle t \rangle^\frac{2}{2} \| w_\Phi \|_{B^{\frac{1}{2}, 0}_\psi} + \| w_\Phi \|_{B^{\frac{1}{2}, 0}_\psi}^2 \\
+ \langle t \rangle^\frac{2}{2} \| U_\Phi \|_{B^{1, 0}_\psi} + \langle t \rangle \| U_\Phi \|_{B^{1, 0}_\psi}^2 + \langle t \rangle^\gamma,
\end{cases}
\]
with
\[ \Phi(t, \xi) := \Phi^1(t, \xi) = (1 - \lambda \theta(t))(\xi). \]

If \( w(t, x, y) \) is a classical solution of the problem (1.3), then we know that \( w_\Phi = \mathcal{F}^{-1}_{\xi \rightarrow x}[e^{\Phi(t, \xi)} \hat{w}(t, \xi, y)] \) satisfies the following equation
\[
\partial_t w_\Phi + \lambda \dot{\theta}(D)w_\Phi + ([w + u^s] \partial_x w) \Phi + [w \partial_x u^s] \Phi \\
+ \left[ -\int_0^y \partial_x (w + u^s)dy' \right] \partial_y (w + u^s) - \left[ \int_y^{+\infty} wdy' \right] \Phi \\
= \partial_y^2 w_\Phi + (1 - \phi)[\partial_t U + (1 + \phi)U \partial_x U] \Phi - \int_y^{+\infty} U_\Phi (1 - \phi)dy'
\]
with \( \Phi(t, \xi) \) being given in (2.5).
Acting the dyadic operator $\Delta_k$ on (2.6) and taking $L^2(Q_T)$ inner product with $e^{2\psi}\Delta_k w_\Phi$ for $\psi(t,y)$ given in (2.1), it follows
\[
\left(e^{\psi}\partial_t \Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right) + \lambda\left(\hat{\theta}(D) e^{\psi}\Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right) - \left(e^{\psi}\Delta_k \partial_y^2 w_\Phi | e^{\psi}\Delta_k w_\Phi \right) \\
= - \left(e^{\psi}\Delta_k \left(\left(\int_0^y \partial_x \left( w + u^* \right) dy' \right) \partial_y \left( w + u^* \right) \right) | e^{\psi}\Delta_k w_\Phi \right) + \left(e^{\psi}\Delta_k \left[ \int_{-\infty}^y w dy' \right] | e^{\psi}\Delta_k w_\Phi \right) + \left(e^{\psi}(1 - \phi) \Delta_k \left[ \partial_t U + (1 + \phi)U \partial_x U \right] | e^{\psi}\Delta_k w_\Phi \right) - \left(e^{\psi} \int_{-\infty}^y \Delta_k U \left(1 - \phi \right) dy' | e^{\psi}\Delta_k w_\Phi \right) \\
:= \sum_{i=1}^6 J_i,
\]
where $\langle \cdot , \cdot \rangle$ represents the inner product in $L^2(Q_T)$.

In the following calculation, for convenience we shall denote $a \leq Cb$ ($a \geq Cb$) by $a \lesssim b$ ($a \gtrsim b$), for a generic constant $C$ may change from line to line. Let us estimate each term given in (2.7).

First, the terms on the left hand side of (2.7) can be estimated as follows:
\[
\left( e^{\psi}\partial_t \Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right) = \frac{1}{2} \left\| e^{\psi}\Delta_k w_\Phi \right\|_{L^2_y(T)}^2 - \frac{1}{2} \left\| e^{\psi}\Delta_k e^{\theta(D)} w_0 \right\|_{L^2_y(T)}^2 - \left( \psi \partial_y e^{\psi}\Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right),
\]
\[
- \left(e^{\psi}\Delta_k \partial_y^2 w_\Phi | e^{\psi}\Delta_k w_\Phi \right) = \left( 2\psi \partial_y e^{\psi}\Delta_k \partial_y w_\Phi | e^{\psi}\Delta_k w_\Phi \right) + \left( e^{\psi}\Delta_k \partial_y w_\Phi | e^{\psi}\Delta_k \partial_y w_\Phi \right) \gtrsim \frac{1}{2} \left(e^{\psi}\Delta_k \partial_y w_\Phi | e^{\psi}\Delta_k \partial_y w_\Phi \right) - \left( 2\psi \partial_y e^{\psi}\Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right)
\]
and
\[
\lambda \left( \hat{\theta}(D) e^{\psi}\Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right) \gtrsim (1 + 2^k) \lambda \left( \hat{\theta} e^{\psi}\Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right).
\]
Thus, we get

**Lemma 2.1.** If we denote by
\[
I(k) = \left(e^{\psi}\partial_t \Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right) + \lambda \left( \hat{\theta}(D) e^{\psi}\Delta_k w_\Phi | e^{\psi}\Delta_k w_\Phi \right) - \left(e^{\psi}\Delta_k \partial_y^2 w_\Phi | e^{\psi}\Delta_k w_\Phi \right) + \frac{1}{2} \left\| e^{\psi}\Delta_k e^{\theta(D)} w_0 \right\|_{L^2_y(T)}^2,
\]
then one has
\[
\sum_{k \in \mathbb{Z}} 2^\frac{3}{2} \sqrt{I(k)} \gtrsim \| w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \sqrt{ \left( \psi + 2\psi_0^2 \right) | w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \| \partial_y w_\Phi \|_{L^p_T(B^1_{\psi,0})} \\
+ \sqrt{\text{A}} \left( \| w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \| w_\Phi \|_{L^p_T(B^1_{\psi,0})} \right).
\]

Now it remains to control the right hand side of (2.7) term by term. First, we have the following estimates.

**Lemma 2.2.** For any given $\sigma > 0$, there is a constant $C_\sigma > 0$ such that the terms $J_1$, $J_2$, $J_3$ and $J_5$ given in (2.7) can be bounded as follows:
\[
\sum_{k \in \mathbb{Z}} 2^\frac{3}{2} \sqrt{|J_1(k)|} \lesssim C_\sigma \| w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \sigma \| \partial_y w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \sigma \left( \langle T \rangle^\frac{1}{2} - 1 \right)^\frac{1}{2} \| U_\Phi \|_{L^p_T(B^1_{\psi,0})},
\]
\[
\sum_{k \in \mathbb{Z}} 2^\frac{3}{2} \sqrt{|J_2(k)|} \lesssim \| w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \sigma T^\frac{1}{2} \| U_\Phi \|_{L^p_T(B^1_{\psi,0})},
\]
\[
\sum_{k \in \mathbb{Z}} 2^\frac{3}{2} \sqrt{|J_3(k)|} \lesssim \| w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \sigma T^\frac{1}{2} \| U_\Phi \|_{L^p_T(B^1_{\psi,0})},
\]
\[
\sum_{k \in \mathbb{Z}} 2^\frac{3}{2} \sqrt{|J_5(k)|} \lesssim \| w_\Phi \|_{L^p_T(B^1_{\psi,0})} + \sigma T^\frac{1}{2} \| U_\Phi \|_{L^p_T(B^1_{\psi,0})}.
\]
\[ \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \sqrt{|J_3(k)|} \leq C_\sigma \left\| w_{\Phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} + C_\sigma \left\| U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} \left\| yw_{\Phi} \right\|_{L^2_\beta(B_{\phi}^{1,0})} + \sigma \left\| \partial_y w_{\Phi} \right\|_{L^2_\beta(B_{\phi}^{1,0})} \]
\[ + C_\sigma \left( \langle T \rangle^{\frac{1}{2}} - 1 \right) \left\| U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} \left\| yw_{\Phi} \right\|_{L^2_\beta(B_{\phi}^{1,0})} + \sigma \left( \langle T \rangle^{\frac{1}{2}} - 1 \right) \left\| U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} \]
and
\[ \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \sqrt{|J_5(k)|} \leq \left\| w_{\Phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} + T^{\frac{1}{2}} \left( \left\| U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} + \left\| U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} \right) + \left\| w_{\phi} \right\|_{L^2_\beta(B_{\phi}^{1,0})}
\[ + \left( \langle T \rangle^{\frac{1}{2}} - 1 \right) \left\| \partial_t U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T}. \]

The proof of these estimates can be found in [17, Lemmas 3.2-3.5, Lemma 3.7] with the multiplier \(2^{\frac{k}{2}}\) being changed as \(2^k\) in the summations. Now, the main task is to estimate \(J_4\) and \(J_6\) in the following lemma.

**Lemma 2.3.** For \(J_4(k)\) and \(J_6(k)\) given in (2.7), we have
\[ \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \sqrt{|J_4(k)|} \leq \left\| yw_{\Phi} \right\|_{L^2_\beta(B_{\phi}^{1,0})} + \left\| w_{\phi} \right\|_{L^2_\beta(B_{\phi}^{1,0})} \quad (2.8) \]
and
\[ \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \sqrt{|J_6(k)|} \leq \left( \langle T \rangle^{\frac{1}{2}} - 1 \right) \left\| U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} + \left\| w_{\phi} \right\|_{L^2_\beta(B_{\phi}^{1,0})}. \quad (2.9) \]

**Proof.** The term \(J_4\) given in (2.7) can be controlled as follows:
\[ |J_4(k)| = \left| \left( e^{\psi} \Delta_k \left[ \int_y^{+\infty} wdy \right] \right) \Phi e^{\psi} \Delta_k w_{\Phi} \right| \]
\[ \leq \int_0^T \left\| e^{\psi} \Delta_k w_{\Phi} \right\|_{L^2_\beta} \left( \int_{\mathbb{R}^+} dy \int_{\mathbb{R}} dx \left( e^{\psi} \int_y^{+\infty} e^{\psi} \Delta_k w_{\phi} \right)^2 dy' \int_y^{+\infty} e^{-2\psi} dy' \right)^{\frac{1}{2}} dt \]
\[ \lesssim \int_0^T (t)^{\frac{3}{2}} \left\| e^{\psi} \Delta_k w_{\Phi} \right\|_{L^2_\beta} \left( \int_{\mathbb{R}^+} dy \int_{\mathbb{R}} dx \int_y^{+\infty} e^{\psi} \left( \Delta_k w_{\phi} \right)^2 dy' \right)^{\frac{1}{2}} dt \]
\[ \leq \int_0^T (t)^{\frac{3}{2}} \left( \left\| y e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta} + \left\| e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta} \right) \left\| e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta} dt \]
\[ \leq \frac{1}{2} \int_0^T \left\| y e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta}^2 dt + \int_0^T \left( (t)^{\frac{3}{2}} + \frac{1}{2} \langle t \rangle^\gamma \right) \left\| e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta}^2 dt, \]
which implies the estimate (2.8) immediately by using the definitions of \(\tilde{L}^p_T(B_{\phi}^{1,0})\), \(\tilde{L}^p_{T,\beta}(B_{\phi}^{1,0})\) and \(\hat{\theta}\).

For the terms \(J_6\), it yields
\[ |J_6(k)| = \left| \left( e^\psi \int_y^{+\infty} \Delta_k U_{\phi}(1 - \phi) dy' \right) \left| e^{\psi} \Delta_k w_{\phi} \right| \right| \]
\[ \lesssim \int_0^T \int_{\mathbb{R}^+} e^\psi \int_y^{+\infty} (1 - \phi) dy' \left\| \Delta_k U_{\phi} \right\|_{L^2_\beta} \left\| e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta} dy dt \]
\[ \lesssim \int_0^T \left\| e^\psi \int_y^{+\infty} (1 - \phi) dy' \left\| \Delta_k U_{\phi} \right\|_{L^2_\beta} \left\| e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta} dt \]
\[ \lesssim \left( \langle T \rangle^{\frac{1}{2}} - 1 \right) \left\| \Delta_k U_{\phi} \right\|_{L^{2,(B_{\phi}^{1,0})}_T} \left( \int_0^T \left\| e^{\psi} \Delta_k w_{\phi} \right\|_{L^2_\beta}^2 dt \right)^{\frac{1}{2}}, \]
where \(\left\| e^\psi \int_0^{+\infty} (1 - \phi) dy' \right\|_{L^2_\beta} \lesssim \langle t \rangle^{\frac{3}{4}}\) has been used. By using the Cauchy inequality, we conclude the estimate (2.9). \(\square\)
Now the apriori estimates are given in the following theorem.

**Theorem 2.1.** Suppose that $w(t,x,y)$ is a classical solution of the problem (1.3), then there exist $T_2 > 0$ and a positive constant $G$ such that there holds

\[
\|w_\Phi\|_{L^\infty_T\left(B^{\frac{1}{2}}_\psi\right)} + \sqrt{-\psi_t + 2\psi^2_y} \|w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} + \|\partial_y w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} \\
+ \sqrt{\lambda}\left(\|w_\Phi\|_{L^2_{T,\sigma}\left(B^{\frac{1}{2}}_\psi\right)} + \|w_\Phi\|_{L^2_{T,\sigma}(B^{1}_{\psi})}\right) \\
\leq G\left(\|e^{(D)}w_0\|_{B^{\frac{1}{2}}_\psi} + \left(\sigma\left(T^\frac{1}{2} - 1\right)^\frac{1}{2} + T^\frac{1}{2}\right)\|e^{(D)}U\|_{L^\infty_T\left(B^\frac{1}{2}\right)} \right) \\
+ \left((1 + \sigma)T^\frac{3}{2} + C_\sigma\left(T^\frac{1}{2} - 1\right)^\frac{1}{2}\|e^{(D)}U\|_{L^\infty_T(B^1)}\right)\|e^{(D)}U\|_{L^\infty_T(B^1)} \\
+ \left((T)^\frac{3}{2} - 1\right)^\frac{1}{2}\|e^{(D)}\partial_t U\|_{L^\infty_T\left(B^\frac{1}{2}\right)} + \left((T)^\frac{3}{2} - 1\right)^\frac{1}{2}\|e^{(D)}U\|_{L^\infty_T\left(B^\frac{1}{2}\right)} \right) \right) \tag{2.10}
\]

for any $0 < T \leq T_2$, and the weight $\Phi(t,\xi)$ is positive in $[0,T_2]$.

**Proof.** Combining Lemmas 2.1, 2.2 and 2.3, there is a constant $G > 0$, such that

\[
\|w_\Phi\|_{L^\infty_T\left(B^{\frac{1}{2}}_\psi\right)} + \sqrt{-\psi_t + 2\psi^2_y} \|w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} + \|\partial_y w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} \\
+ \sqrt{\lambda}\left(\|w_\Phi\|_{L^2_{T,\sigma}\left(B^{\frac{1}{2}}_\psi\right)} + \|w_\Phi\|_{L^2_{T,\sigma}(B^{1}_{\psi})}\right) \\
\leq G\left(\|e^{(D)}w_0\|_{B^{\frac{1}{2}}_\psi} + C_\sigma\|w_\Phi\|_{L^2_{T,\sigma}(B^{1}_{\psi})} + \sigma\|\partial_y w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} + \|y w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} \right) \\
+ \left(\sigma\left(T^\frac{1}{2} - 1\right)^\frac{1}{2} + T^\frac{1}{2}\right)\|U_\Phi\|_{L^\infty_T\left(B^\frac{1}{2}\right)} + C_\sigma\|U_\Phi\|_{L^\infty_T\left(B^\frac{1}{2}\right)}\|y w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} \\
+ \left((1 + \sigma)T^\frac{3}{2} + C_\sigma\left(T^\frac{1}{2} - 1\right)^\frac{1}{2}\|U_\Phi\|_{L^\infty_T(B^1)}\right)\|U_\Phi\|_{L^\infty_T(B^1)} + \left((T)^\frac{3}{2} - 1\right)^\frac{1}{2}\|\partial_t U_\Phi\|_{L^\infty_T\left(B^\frac{1}{2}\right)} \\
+ \left((T)^\frac{3}{2} - 1\right)^\frac{1}{2}\|U_\Phi\|_{L^\infty_T\left(B^\frac{1}{2}\right)} + \|w_\Phi\|_{L^2_{T,\sigma}\left(B^{1}_{\psi}\right)} \right) \tag{2.11}
\]

for any given $\sigma > 0$.

Firstly, we choose $\sigma$ small enough such that $G\sigma < 1$, and take $\lambda$ to be large such that $\sqrt{\lambda} > G(C_\sigma + 2)$. Then, in view of (2.2), one can choose $\gamma$ large enough, such that

\[
2G \left( C_\sigma\|U_\Phi\|_{L^\infty_T\left(B^\frac{1}{2}\right)} + 1 \right) y < \left( y + 1 \right) \sqrt{\frac{\gamma - 1}{32(1 + T_1)^{\gamma + 1}}} \tag{2.12}
\]

holds for some fixed $T_1 > 0$, which follows that

\[
\sqrt{-\psi_t + 2\psi^2_y} \|w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} \geq 2G \left( C_\sigma\|U_\Phi\|_{L^\infty_T\left(B^\frac{1}{2}\right)} + 1 \right)\|y w_\Phi\|_{L^2_T\left(B^{\frac{1}{2}}_\psi\right)} \tag{2.13}
\]

for any $T \in [0,T_1]$. Therefore, for the fixed parameters $\sigma$, $\lambda$ and $\gamma$ in the above, by using the inequalities (2.2) and (2.12), the estimate (2.10) is derived from (2.11) for any $0 < T \leq T_1$. 

On the other hand, in view of (2.4) and (2.10), there exists a constant $C(w_0, U, t)$ depending on $w_0$, $U$ and $t$ such that

$$
\theta(t) = \int_0^t \delta dt'
$$

$$
= \int_0^t \left( \left( t' \right)^{\frac{3}{2}} \| \partial_y w \|_{B_{\frac{1}{\theta}}^{\frac{1}{2}}} + \langle t' \rangle^{\frac{3}{2}} \| U \|_{B_{\frac{1}{\theta}}^{\frac{1}{2}}} + \langle t' \rangle^{\frac{3}{2}} \| U \|_{B_{\frac{1}{\theta}}^{1}} + \langle t' \rangle^{\gamma} \right) dt'
$$

$$
\lesssim \left( (t)^{\frac{3}{2}+1} - 1 \right) \| \partial_y w \|_{L_2^2(B_{\frac{1}{\theta}}^{1,0})} + \left( (t) \right)^{\frac{3}{2}+1} - 1 \| e(D) U \|_{L_{\infty}^2(B_{\frac{1}{\theta}}^{1})} + (t) \| w \|_{L_2^2(B_{\frac{1}{\theta}}^{1,0})}
$$

$$
+ \left( (t)^{\frac{3}{2} - 1} \right) \| e(D) U \|_{L_{\infty}^2(B_{\frac{1}{\theta}}^{1})} + \left( (t)^{2} - 1 \right) \| e(D) U \|_{L_{\infty}^2(B_{1}^{1})} + (t) \gamma - 1
$$

$$
\lesssim C(w_0, U, t).
$$

Therefore one can choose $0 < T_2 \leq T_1$ properly small such that

$$
0 < T_2 \leq \sup_{t > 0} \left\{ \theta(t) < \frac{1}{\lambda} \right\},
$$

(2.14)

which guarantees the weight $\Phi(t, \xi)$ defined in (2.3) is positive on $[0, T_2]$. Thereby we obtain the apriori estimate (2.10) for $0 < T \leq T_2$.

\[ \square \]

Remark 2.2. If the weight $\Phi$ is replaced by $\Phi^\delta$, there exists a time $T_\delta$ such that the apriori estimate in Theorem 2.1 is still valid in $[0, T_\delta]$, with $e(D)$ being replaced by $e^{\delta(D)}$.

2.2. Existence of a Solution

To obtain the existence of a solution to the problem (1.3), similar to that given in [17], consider the approximation of (1.3) as follows for any integer $n \geq 1$,

$$
\begin{align*}
\partial_t w_n + (w_n + u^a)\partial_x w_n + w_n \partial_x u^a - \int_0^y \partial_x (w_n + u^a) dy' \partial_y (w_n + u^a) \\
- \int_0^y w_n dy' - \partial_y^2 w_n - \frac{1}{n^2} \partial_x^2 w_n \\
= (1 - \phi)(\partial_t U + (1 + \phi) U \partial_x U) - \int_0^y U(1 - \phi) dy',
\end{align*}
$$

(2.15)

The well-posedness of the problem (2.15) can be obtained from the classical theory of the parabolic equations, and $w_n(n \geq 1)$ satisfies the same apriori estimate (2.10) on $[0, T_2]$ as given in Theorem 2.1.

For any fixed $\delta \in (0, 1)$ in $\Phi^\delta$ given in (2.3), by using

$$
\| (w_n)_{\Phi^\delta} \|_{L_2^2(B_{\frac{1}{\theta}}^{1,0})} \lesssim \| (w_n)_{\Phi^1} \|_{L_2^2(B_{\frac{1}{\theta}}^{1,0})} \quad (n \geq 1),
$$

in a way similar to the proof of uniqueness given in the next subsection, we can get that there exists $0 < T^* \leq T_2$ such that $V = w_{n+1} - w_n$ satisfies the following estimate,

$$
\| V_{\Phi^\delta} \|_{L_{\infty}^2(B_{\frac{1}{\theta}}^{1,0})} + \| \partial_y V_{\Phi^\delta} \|_{L_2^2(B_{\frac{1}{\theta}}^{1,0})} \lesssim \left( \frac{1}{n^2} + \frac{1}{(n+1)^2} \right) R, \quad \text{for } 0 < T \leq T^*
$$
where $\mathcal{R}$ represents the right hand side of (2.10). Therefore for any fixed $0 < \delta \leq 1$ in the weight $\Phi^\delta$, \{$(w_n)_{\Phi^\delta}$\}$_{n \geq 1}$ is a Cauchy sequence in $L_T^\infty(B^\frac{1}{2}, B^0_\infty)$, which follows the existence of a solution to the problem (1.3).

### 2.3. Uniqueness of the Solution

In this subsection, we study the uniqueness of the solution to the problem (1.3). Suppose that the problem (1.3) has two solutions $w^1$ and $w^2$, obviously $V = w^1 - w^2$ satisfies the following problem,

$$
\begin{aligned}
\partial_t V + \int_{-\infty}^{y} V dy' - \partial_y^2 V &= -(w^2 + u^*)\partial_x V - V\partial_x w^1 + V\partial_x u^* \\
+ \int_0^y \partial_x (w^2 + u^*) \partial_y V + \int_0^y \partial_x V dy' \partial_y (w^1 + u^*), \\
V|_{y=0} &= 0, \lim_{y \to \pm \infty} V = 0, V|_{t=0} = 0.
\end{aligned}
$$

(2.16)

Denote the corresponding weights by $\theta^1(t), \Phi_1^\delta(t, \xi)$ and $\theta^2(t), \Phi_2^\delta(t, \xi)$ given in (2.4) and (2.3) with respect to $w^1$ and $w^2$ for $\delta = 1$, we introduce

$$
\Theta = \theta^1 + \theta^2 \quad \text{and} \quad \tilde{\Phi}^\delta = (\delta - \lambda \Theta(t)) |\xi|, \quad \text{for any given } \delta \in (0, 1).
$$

From (2.16), one has that

$$
\begin{aligned}
\partial_t \Phi_{\delta}^\delta + \lambda \hat{\Theta} V \Phi_{\delta}^\delta - \int_{-\infty}^{y} \Phi_{\delta}^\delta dy' - \partial_y^2 \Phi_{\delta}^\delta &= -\left[(w^2 + u^*)\partial_x \Phi_{\delta}^\delta - [V\partial_x w^1]_{\Phi_{\delta}^\delta} - [V\partial_x u^*]_{\Phi_{\delta}^\delta}ight] \\
&+ \left[\int_0^y \partial_x (w^2 + u^*) dy' \partial_y V\right]_{\Phi_{\delta}^\delta} \\
&+ \left[\int_0^y \partial_x V dy' \partial_y (w^1 + u^*)\right]_{\Phi_{\delta}^\delta}.
\end{aligned}
$$

By acting the dyadic operator $\Delta_k$ on the above equation and taking $L^2(Q_T)$ inner product with $e^{2\psi} \Delta_k \Phi_{\delta}^\delta$, it yields that

$$
\begin{aligned}
\left( e^{\psi} \partial_t \Delta_k \Phi_{\delta}^\delta e^{\psi} \Delta_k \Phi_{\delta}^\delta \right) + \lambda \left( (\hat{\Theta} \langle D \rangle e^{\psi} \Delta_k \Phi_{\delta}^\delta) \right) - \left( e^{\psi} \Delta_k \partial_y^2 \Phi_{\delta}^\delta \right) e^{\psi} \Delta_k \Phi_{\delta}^\delta &= -\left( e^{\psi} \Delta_k [(w^2 + u^*)\partial_x \Phi_{\delta}^\delta] e^{\psi} \Delta_k \Phi_{\delta}^\delta \right) \\
&- \left( e^{\psi} \Delta_k [V\partial_x w^1]_{\Phi_{\delta}^\delta} e^{\psi} \Delta_k \Phi_{\delta}^\delta \right) \\
&+ \left( e^{\psi} \Delta_k \left[ \left( \int_0^y \partial_x (w^2 + u^*) dy' \right) \partial_y V\right]_{\Phi_{\delta}^\delta} e^{\psi} \Delta_k \Phi_{\delta}^\delta \right) \\
&+ \left( e^{\psi} \Delta_k \left[ \left( \int_0^y \partial_x V dy' \right) \partial_y (w^1 + u^*) \right]_{\Phi_{\delta}^\delta} e^{\psi} \Delta_k \Phi_{\delta}^\delta \right) \\
&+ \left( e^{\psi} \Delta_k \left[ \left( \int_{-\infty}^{y} V dy' \right) \partial_y (w^1 + u^*) \right]_{\Phi_{\delta}^\delta} e^{\psi} \Delta_k \Phi_{\delta}^\delta \right)
\end{aligned}
$$

(2.17)

:= \sum_{i=1}^{6} J_i.

In a way similar to that given in Lemmas 2.1 and 2.3, one has the following result:

**Lemma 2.4.** Set

$$
I(k) \triangleq \left( e^{\psi} \partial_t \Delta_k \Phi_{\delta}^\delta e^{\psi} \Delta_k \Phi_{\delta}^\delta \right) + \lambda \left( (\hat{\Theta} \langle D \rangle e^{\psi} \Delta_k \Phi_{\delta}^\delta) \right) - \left( e^{\psi} \Delta_k \partial_y^2 \Phi_{\delta}^\delta \right) e^{\psi} \Delta_k \Phi_{\delta}^\delta.
$$

For $J_6$ and $I(k)$ given at above, there hold

$$
\sum_{k \in \mathbb{Z}} 2^{\frac{\alpha}{2}} \|J_6(k)\|_{L_T^1(B^\frac{1}{2}_\infty)} \leq \|y V\|_{L_T^1(B^\frac{1}{2}_\infty)} + \|V\|_{L_T^1(B^\frac{1}{2}_\infty)}
$$
\[
\sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \sqrt{I(k)} \gtrsim \|V_\Phi\|_{L_T^2(B^\frac{1}{2}_0)} + \|\sqrt{-(\psi_\varepsilon + 2\psi_\psi^2)}V_\Phi\|_{L_T^2(B^\frac{3}{2}_0)} + \|\partial_y V_\Phi\|_{L_T^2(B^\frac{1}{2}_0)} \\
+ \sqrt{\lambda} (\|V_\Phi\|_{L^2_{T,\alpha}(B^\frac{1}{2}_0)} + \|V_\Phi\|_{L^2_{T,\alpha}(B^\frac{1}{2}_0)}).
\]

As shown in [17, 20], for \(0 < \delta < 1\), there hold

\[
\|w^i_{\Phi^i}\|_{L_T^2(B^\frac{1}{2}_0)} \lesssim \|w^i_{\Phi^i}\|_{L_T^2(B^\frac{1}{2}_0)}, \quad \|w^i_{\Phi^i}\|_{L_T^2(B^\frac{3}{2}_0)} \lesssim \|w^i_{\Phi^i}\|_{L_T^2(B^\frac{1}{2}_0)} \quad (i = 1, 2).
\]

The remaining terms given on the right hand side of (2.17) can be controlled as given in [17, Lemmas 4.2-4.7] with the multiplier \(2^\frac{k}{2}\) being changed as \(2^k\) in the summations, and one concludes:

**Lemma 2.5.** For any \(\sigma > 0\), the terms \(J_i\) \((1 \leq i \leq 5)\) given in (2.17) satisfy the following estimate:

\[
\sum_{i=1}^5 \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \sqrt{|J_i(k)|} 
\leq \left( \|\partial_y w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} + \|e(D)U\|_{L_T^2(B^\frac{1}{2}_0)} \right)^\frac{1}{2} \|V_\Phi\|_{L_T^2(B^\frac{1}{2}_0)} (\langle T \rangle^\frac{2}{2} + 1 - \frac{1}{2}) \\
+ \left( \|\partial_y w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} + \|e(D)U\|_{L_T^2(B^\frac{1}{2}_0)} \right)^\frac{1}{2} \|V_\Phi\|_{L_T^2(B^\frac{3}{2}_0)} (\langle T \rangle^\frac{2}{2} + 1 - \frac{1}{2}) \\
+ T^\frac{1}{2} \|e(D)U\|_{L_T^2(B^\frac{1}{2}_0)} \|V_\Phi\|_{L_T^2(B^\frac{3}{2}_0)} + C_\sigma \langle T \rangle^\frac{2}{2} - 1 - \frac{1}{2} \|w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} \|V_\Phi\|_{L_T^2(B^\frac{1}{2}_0)} \\
+ \sigma \|\partial_y V_\Phi\|_{L_T^2(B^\frac{3}{2}_0)} + C_\sigma \langle T \rangle^\frac{2}{2} - 1 - \frac{1}{2} \|w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} \|V_\Phi\|_{L_T^2(B^\frac{3}{2}_0)} \\
+ C_\sigma \|U_\Phi\|_{L_T^2(B^\frac{3}{2}_0)} \|y V_\Phi\|_{L_T^2(B^\frac{1}{2}_0)} + C_\sigma \|V_\Phi\|_{L_T^2(B^\frac{1}{2}_0)}.
\]

Based on the above lemmas, we have:

**Proof of the uniqueness part of Theorem 1.1.** Combining Lemmas 2.4 and 2.5, from (2.17) we obtain that

\[
\|V_\Phi^i\|_{L_T^\infty(B^\frac{1}{2}_0)} + \|\sqrt{-(\psi_\varepsilon + 2\psi_\psi^2)}V_\Phi^i\|_{L_T^2(B^\frac{3}{2}_0)} + \|\partial_y V_\Phi^i\|_{L_T^2(B^\frac{1}{2}_0)} \\
+ \sqrt{\lambda} (\|V_\Phi^i\|_{L^2_{T,\alpha}(B^\frac{1}{2}_0)} + \|V_\Phi^i\|_{L^2_{T,\alpha}(B^\frac{1}{2}_0)} \\
\leq \left( \|\partial_y w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} + \|e(D)U\|_{L_T^2(B^\frac{1}{2}_0)} \right)^\frac{1}{2} \|V_\Phi^i\|_{L_T^2(B^\frac{1}{2}_0)} (\langle T \rangle^\frac{2}{2} + 1 - \frac{1}{2}) \\
+ \left( \|\partial_y w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} + \|e(D)U\|_{L_T^2(B^\frac{1}{2}_0)} \right)^\frac{1}{2} \|V_\Phi^i\|_{L_T^2(B^\frac{3}{2}_0)} (\langle T \rangle^\frac{2}{2} + 1 - \frac{1}{2}) \\
+ T^\frac{1}{2} \|e(D)U\|_{L_T^2(B^\frac{1}{2}_0)} \|V_\Phi^i\|_{L_T^2(B^\frac{3}{2}_0)} + C_\sigma \langle T \rangle^\frac{2}{2} - 1 - \frac{1}{2} \|w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} \|V_\Phi^i\|_{L_T^2(B^\frac{1}{2}_0)} \\
+ \sigma \|\partial_y V_\Phi^i\|_{L_T^2(B^\frac{3}{2}_0)} + C_\sigma \langle T \rangle^\frac{2}{2} - 1 - \frac{1}{2} \|w^2_{\Phi^1}\|_{L_T^2(B^\frac{1}{2}_0)} \|V_\Phi^i\|_{L_T^2(B^\frac{1}{2}_0)} \\
+ C_\sigma \|U_\Phi^i\|_{L_T^2(B^\frac{3}{2}_0)} \|y V_\Phi^i\|_{L_T^2(B^\frac{1}{2}_0)} + C_\sigma \|V_\Phi^i\|_{L_T^2(B^\frac{1}{2}_0)}.
\]
In this section, we are interested in whether the smooth solution of the problem (1.1) exists globally in time. Under the assumption that the initial data $u_0$ and outer flow $U$ satisfy the condition (1.4), we shall prove that the norm $\|\partial_x u(t', 0, y)\|_{L^\infty([0, t] \times \mathbb{R}^+)}$ of solution to the problem (1.1) will blow up in $(0, T)$. This shall be obtained by developing a Lyapunov functional approach inspired from [9] and a contradiction argument.

For functions $f(t, x, y)$ and $g(t, x)$ defined in $[0, T] \times \mathbb{R} \times \mathbb{R}^+$ and $[0, T] \times \mathbb{R}$ respectively, denote by $\dot{f}(t, y) = f(t, 0, y)$ and $\dot{g}(t) = g(t, 0)$ the traces on $\{x = 0\}$.

Restricting the problem (1.1) on the plane $\{x = 0\}$, we get that $\bar{u}(t, y) = u(t, 0, y)$ satisfies the following problem in $H_T = [0, T] \times \mathbb{R}^+$,

$$
\begin{aligned}
\partial_t \bar{u} + \bar{u} \partial_x \bar{u} + \bar{v} \partial_y \bar{u} - \int_y^{+\infty} \bar{u} dy' - \partial_y^2 \bar{u} &= 0, \\
\bar{u}|_{t=0} = 0, \\
(\bar{u}, \bar{v})|_{y=0} = (0, 0), & \lim_{y \to +\infty} \bar{u} = 0,
\end{aligned}
$$

(3.1)

by using the assumption (1.4), where $\partial_x \bar{u} + \partial_y \bar{v} = 0$.

We shall prove Theorem 1.2 by a contradiction argument. Assume that $\partial_x u(t, 0, y)$ does not blow up in $H_T$, and there is a constant $M_T > 0$ such that

$$
\|\partial_x u(t, 0, y)\|_{L^\infty(H_T)} \leq M_T.
$$

(3.2)

First, we have the following result:

**Lemma 3.1.** Assume that $\partial_x \bar{u} \in C^2(H_T)$ satisfies the assumption (3.2). Then the problem (3.1) only has the trivial solution, $u(t, 0, y) \equiv 0$ in $H_T$, in the class that satisfies the integrability condition given in (1.6).

**Proof.** We introduce $\psi(t, y) = \lambda y (1 - \gamma t)$ for a fixed $0 < \lambda < \mu$, where $\mu$ is the parameter given in (1.6), and $\gamma > 0$ shall be determined later. Multiplying the equation in (3.1) by $\bar{u} e^{2\psi}$ and integrating in $y$, it follows

$$
\frac{1}{2} \frac{d}{dt} \int_0^{+\infty} \bar{u}^2 e^{2\psi} dy - \int_0^{+\infty} \partial_t \psi \bar{u}^2 e^{2\psi} dy + \int_0^{+\infty} (\partial_y \bar{u})^2 e^{2\psi} dy = I_1 + I_2 + I_3 + I_4
$$

(3.3)

where

$$
\begin{align*}
I_1 &= - \int_0^{+\infty} 2 \psi \partial_y \bar{u} \bar{u} e^{2\psi} dy, \\
I_2 &= - \int_0^{+\infty} \partial_x \bar{u}^2 e^{2\psi} dy, \\
I_3 &= - \int_0^{+\infty} \bar{v} \partial_y \bar{u} \bar{u} e^{2\psi} dy, \\
I_4 &= \int_0^{+\infty} \left( \int_y^{+\infty} \bar{u}(t, y') dy' \right) \bar{u} e^{2\psi} dy.
\end{align*}
$$

By taking $\lambda$ and $\gamma$ large and $T > 0$ being small properly, the above inequality implies

$$
\|V_{\bar{u}}\|_{L^\infty_T \left( B^{\frac{1}{2}, 0}_\psi \right)} = 0.
$$

Thus, we get $V \equiv 0$ in $0 \leq t \leq T$, this uniqueness can be extended to the whole time interval of the existence of the solution given in Sect. 2.1 with the aid of continuation argument. \qed
By using integration by parts, the assumption (1.6) and Young’s inequality, we have
\[ |I_1| \leq \int_0^{+\infty} -2\psi\partial_y \bar{u}^2 e^{2\psi} dy + \frac{1}{2} \int_0^{+\infty} (\partial_y \bar{u})^2 e^{2\psi} dy, \]
\[ |I_3| = \frac{1}{2} \int_0^{+\infty} 2\partial_y \psi \bar{u}^2 e^{2\psi} dy + \int_0^{+\infty} \partial_y \bar{v} \bar{u}^2 e^{2\psi} dy \]
\[ \leq \||\partial_x u||_{L^\infty(H_T)} \int_0^{+\infty} y|\partial_y \psi| \bar{u}^2 e^{2\psi} dy + \frac{1}{2} \||\partial_x u||_{L^\infty(H_T)} \int_0^{+\infty} \bar{u}^2 e^{2\psi} dy \]
and
\[ |I_4| \leq C_{T_0} \left( \int_0^{+\infty} \left( \int_y^{+\infty} \bar{u}^2 e^{2\psi} dy' \right) \frac{1}{2} |\bar{u}| e^{\psi} dy \leq \int_0^{+\infty} y\bar{u}^2 e^{2\psi} dy + \frac{C_{T_0}^2}{4} \int_0^{+\infty} \bar{u}^2 e^{2\psi} dy, \right. \]
for all \( 0 \leq t \leq T_0 \) with \( 0 < T_0 \leq T \) satisfying \( 1 - \gamma T_0 > 0 \), where
\[ C_{T_0} = \left\| \left( \int_y^{+\infty} e^{-2\psi(t,y')} dy' \right)^{\frac{1}{2}} e^{\psi(t,y)} \right\|_{L^\infty([0,T_0] \times \mathbb{R}^+)} \]
Thus, from (3.3) we obtain that
\[ \frac{1}{2} \frac{d}{dt} \int_0^{+\infty} \bar{u}^2 e^{2\psi} dy + \lambda \int_0^{+\infty} y\bar{u}^2 e^{2\psi} dy + \frac{1}{2} \int_0^{+\infty} (\partial_y \bar{u})^2 e^{2\psi} dy \]
\[ \leq \left( \frac{3}{2} \||\partial_x u||_{L^\infty(H_T)} + \frac{C_{T_0}^2}{4} + 2\lambda^2 \right) \int_0^{+\infty} \bar{u}^2 e^{2\psi} dy + (1 + \lambda \||\partial_x u||_{L^\infty(H_T)} \int_0^{+\infty} y\bar{u}^2 e^{2\psi} dy, \right. \]
for all \( 0 \leq t \leq T_0 \).
Thus, by choosing
\[ \gamma \geq \frac{1}{\lambda} + \||\partial_x u||_{L^\infty(H_T)} \]
from (3.4) we get \( u(t,0,y) \equiv 0 \) for all \( 0 \leq t \leq T_0 \) by using the Gronwall inequality in (3.4). By a continuation argument, we conclude \( u(t,0,y) \equiv 0 \) for all \( 0 \leq t \leq T \). \( \square \)

The second step is to verify that under assumptions given in Theorem 1.2, \( \partial_x u(t,0,y) \leq \partial_x U(t,0) \) holds in the time interval of classical solution of (1.1) if it is true initially. With the aid of the condition (1.4) and Lemma 3.1, we know from (1.1) that \( w(t,y) = \partial_x U(t,0) - \partial_x u(t,0,y) \) satisfies the problem in \( H_T = [0,T] \times \mathbb{R}^+ \),
\[ \begin{cases} 
\partial_t w - w^2 + \partial_y^{-1}(w + \bar{U})\partial_y w - 2\bar{U}w - \int_y^{+\infty} wdy' - \partial_y^2 w = 0, \\
w|_{t=0} = u_0(t) \triangleq \tilde{u}_0(y) - \bar{U}(0), \\
w|_{y=0} = -\bar{U}(t), \quad \lim_{y \to +\infty} w = 0, 
\end{cases} \]
where \( \partial_y^{-1} f(y) := \int_y^y f(y')dy' \), \( \bar{U}(t) = -\partial_x U(t,0) \) and \( \tilde{u}_0(y) = -(\partial_x u_0)(0,y) \).

For the problem (3.6), we have the following result.

**Lemma 3.2.** Under the assumptions (1.4), (1.7) and (3.2), any classical solution of the problem (3.6) is non-negative.
Proof. Let \( V = \omega e^{\lambda y - \lambda t} \) with the positive parameter \( \lambda \) being determined later, where \( \mu \) is the one given in (1.7). From (3.6), we know that \( V \) satisfies the following problem

\[
\begin{aligned}
\frac{\partial V}{\partial t} + (2\mu + \partial_y^{-1}(V e^{\lambda y}) + \tilde{U})) \partial_y V - \mu \partial_y^{-1}(V e^{\lambda y}) + \tilde{U}) V \\
+ (\lambda - 2\tilde{U} - \mu^2) V - \partial_y^2 V = V^2 e^{\lambda y} + \epsilon V \int_0^y e^{-\lambda y'} V dy',
\end{aligned}
\]

(3.7)

For any fixed \( \epsilon > 0 \), we consider \( V^\epsilon = V + \epsilon \), which satisfies

\[
\begin{aligned}
\frac{\partial V^\epsilon}{\partial t} + (2\mu + \partial_y^{-1}((V^\epsilon - \epsilon)e^{\lambda y} + \tilde{U})) \partial_y V^\epsilon - \mu \partial_y^{-1}((V^\epsilon - \epsilon)e^{\lambda y} + \tilde{U}) V^\epsilon \\
+ (\lambda - 2\tilde{U} - \mu^2) V^\epsilon - \partial_y^2 V^\epsilon = -\epsilon \mu \partial_y^{-1}((V^\epsilon - \epsilon)e^{\lambda y} - \epsilon \mu \partial_y^{-1} \tilde{U} \\
+ \epsilon (\lambda - 2\tilde{U} - \mu^2 - \frac{1}{\mu}) + (V^\epsilon - \epsilon)^2 e^{\lambda y} + \epsilon V \int_0^y e^{-\lambda y'} V^\epsilon dy',
\end{aligned}
\]

(3.8)

As \( V^\epsilon \geq \epsilon > 0 \) at both of \( t = 0 \) and \( y = 0 \), we claim that \( V^\epsilon \geq 0 \) in \( H_T \). Otherwise, let \( t^* \in (0, T] \) be the first time such that \( V^\epsilon = 0 \) at an interior point \((t^*, y^*)\), then one has

(i) \( V^\epsilon \geq 0 \) in \( H_t^* = [0, t^*] \times \mathbb{R}^+ \),
(ii) \( V^\epsilon \) attains its minimum in \( H_t^* \) at the point \((t^*, y^*)\).

In addition, in view of the condition (1.7), one has that

\[
\|\omega e^{\lambda y}\|_{L^\infty(H_T)} \leq \tilde{M}_{T, \mu},
\]

(3.9)

for a positive constant \( \tilde{M}_{T, \mu} > 0 \), then there holds

\[
\epsilon \partial_y^{-1}((V^\epsilon - \epsilon)e^{\lambda y}) = \epsilon \int_0^y wdy' \leq \epsilon \frac{\tilde{M}_{T, \mu}}{\mu}.
\]

By noting that at \((t^*, y^*)\), \( \partial_t V^\epsilon \leq 0 \), \( \partial_y^2 V^\epsilon \geq 0 \), \( V^\epsilon = 0 \) and \( \partial_y V^\epsilon = 0 \), and plugging these information into the first equation given in (3.8), it leads to that the left hand side of (3.8)\_1 is non-positive at \((t^*, y^*)\), and its right hand side is positive at \((t^*, y^*)\) when

\[
\lambda > 2\|\tilde{U}\|_{L^\infty([0, T])} + \mu^2 + \frac{1 + \tilde{M}_{T, \mu}}{\mu},
\]

thus it is a contradiction. As a consequence, it deduces that \( V^\epsilon \geq 0 \) in \( H_T \). Moreover, in virtue of the arbitrariness of \( \epsilon \), we conclude that \( V \geq 0 \) in \( H_T \), which implies

\[
w \geq 0 \text{ in } H_T.
\]

Thereby we complete the proof of this lemma. \( \square \)

Next, we shall prove that under certain condition, the solution \( w \) of (3.6) will tend to infinity in a finite time by constructing a Lyapunov functional.

For this, we define the Lyapunov functional as follows:

\[
G(t) = \int_0^y e^{-\lambda y'} V (t, y) dy.
\]

(3.10)

Here the nonnegative weight

\[
\rho(y) \in W^{2, \infty}(\mathbb{R}^+) \cap C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)
\]

(3.11)
satisfies

\[
\rho(0) = \lim_{y \to +\infty} \rho(y) = \lim_{y \to +\infty} \rho'(y) = 0 \tag{3.12}
\]

and the following properties:

(F1) \(y\rho'\rho''(y) \leq C_1 \rho(y),\) for any \(y \in [0, +\infty).\)

(F2) \(\int_0^y \rho(y')dy' + \rho''(y) \geq 0,\) a.e. on \(\mathbb{R}^+.\)

(F3) There exists \(A > 0\) such that \(\rho'(A) > 0, \rho''(y) \leq 0,\) a.e. on \([0, A],\) and the inequality

\[
\frac{1}{\rho'(A)} \int_A^{+\infty} \frac{(\rho'(y))^2}{\rho(y)} dy + \sup_{y \geq A} \int_y^{+\infty} (y' - A) \rho''(y') |dy' - \rho(y)| \leq 4\alpha \tag{3.13}
\]

holds for some \(\alpha \in (0, 1).\)

The weight \(\rho(y)\) satisfying the above properties shall be constructed in the Appendix.

For the Lyapunov functional \(G(t)\) defined in (3.10) for the solution \(w\) of the problem (3.6), we have the following result.

**Lemma 3.3.** Under the condition given in (1.4), and the weight function \(\rho\) satisfying (3.12) and (F1)–(F3), the functional \(G(t)\) has the following estimate,

\[
\frac{dG}{dt} \geq \frac{2(1 - \alpha)}{\|\rho\|_{L^1(\mathbb{R}^+)}^2} G^2 - \|\tilde{U}\|_{L^\infty([0, T])} (3 + C_1) G, \tag{3.14}
\]

where \(\alpha\) and \(C_1\) are given in (3.13) and (F1) respectively.

**Proof.** By using (3.12) and integration by parts, one deduces from (3.6) that

\[
\frac{dG}{dt} = \int_0^{+\infty} \rho w^2 dy - \int_0^{+\infty} \rho \partial_y^{-1} w \partial_y w dy - \int_0^{+\infty} \rho \partial_y^{-1} \tilde{U} \partial_y w dy + 2 \int_0^{+\infty} \rho \tilde{U} w dy
\]

\[
+ \int_0^{+\infty} \rho \left( \int_y^{+\infty} w dy' \right) dy + \int_0^{+\infty} \rho \partial_y^2 w dy
\]

\[
= 2 \int_0^{+\infty} \rho w^2 dy - \frac{1}{2} \int_0^{+\infty} \rho''(y)(\partial_y^{-1} w)^2 dy + \tilde{U} \int_0^{+\infty} yp'(y) w dy + 3\tilde{U} \int_0^{+\infty} p w dy + \int_0^{+\infty} \rho(y) dy' w dy + \int_0^{+\infty} \rho''(y) w dy - \rho'(0) \tilde{U}
\]

\[= \sum_{k=1}^7 J_k. \tag{3.15}\]

Let us estimate the right hand side of (3.15) term by term. The crucial part is to bound the term \(J_2\) from below. By noting the property (F3) and using Holder’s inequality, Cauchy’s inequality and Fubini’s theorem, one can deduce that

\[
\int_0^{+\infty} \rho''(\partial_y^{-1} w)^2 dy \leq \int_A^{+\infty} \rho'' \left( \int_0^A w dy' + \int_0^y w dy \right)^2 dy
\]

\[
= \int_A^{+\infty} \rho'' \left( \int_0^A w dy' \right)^2 dy - \rho'(A) \left( \int_0^A w dy' \right)^2
\]

\[+ 2 \left( \int_0^A w dy' \right) \int_A^{+\infty} \rho'' \left( \int_0^A w dy' \right) dy
\]

\[
\leq \int_A^{+\infty} \rho'' \left( \int_0^A w dy' \right)^2 dy + \frac{1}{\rho'(A)} \left( \int_A^{+\infty} \rho'' \left( \int_0^A w dy' \right) dy \right)^2.
\]
\[ \int_{A}^{+\infty} \rho''(y - A) \left( \int_{A}^{y} w^2 dy' \right) dy + \frac{1}{\rho'(A)} \left( \int_{A}^{+\infty} \rho' wdy' \right)^2 \leq \int_{A}^{+\infty} \left( \int_{y'}^{+\infty} (y - A) \rho'' |dy| \right) w^2 dy' + \frac{1}{\rho'(A)} \left( \int_{A}^{+\infty} \rho' |dy| \right) \left( \int_{A}^{+\infty} \rho w^2 dy' \right) \]
\[ \leq 4\alpha \int_{A}^{+\infty} \rho w^2 dy \]

by using (3.13). Therefore, we have

\[ J_1 + J_2 \geq 2(1 - \alpha) \int_{0}^{+\infty} \rho w^2 dy \geq \frac{2(1 - \alpha)}{\|\rho\|_{L^1(\mathbb{R}_+)} G^2}. \]  

(3.16)

For the terms \( J_l \) \((l = 3, 4, 5, 6)\), by using the properties (F1)–(F2) and Lemma 3.2, these terms can be bounded as below:

\[ J_3 + J_4 = \tilde{U} \int_{0}^{+\infty} y\rho' wdy + 3\tilde{U} \int_{0}^{+\infty} \rho wdy \geq -\|\tilde{U}\|_{L^\infty([0,T])} (3 + C_1) G \]

(3.17)

and

\[ J_5 + J_6 = \int_{0}^{+\infty} \left( \int_{0}^{y} \rho dy' + \rho'' \right) wdy \geq 0. \]

(3.18)

Combining (3.16), (3.17) and (3.18), it concludes the inequality (3.14) by noting that \( J_7 \geq 0 \) from the condition (1.4).

**Proof of Theorem 1.2.** From the inequality (3.14), we know that there exists a time \( 0 < t^* \leq T \) such that

\[ \lim_{t \to t^*^-} G(t) = +\infty \]

when

\[ G(0) > \frac{\|\rho\|_{L^1(\mathbb{R}_+)} \|\tilde{U}\|_{L^\infty([0,T])} (3 + C_1)}{2(1 - \alpha)} \frac{1}{1 - e^{-\|\tilde{U}\|_{L^\infty([0,T])} (3 + C_1) T}}, \]

which implies that \( \lim_{t \to t^*^-} \|\tilde{w}\|_{L^\infty(H_t)} = +\infty \) with the aid of the construction of \( \rho(y) \).

In particular, when \( U \equiv 0 \), the inequality (3.14) is simplified into

\[ \frac{dG}{dt} \geq \frac{2(1 - \alpha)}{\|\rho\|_{L^1(\mathbb{R}_+)} G^2}, \]

(3.19)

which implies that \( G(t) \) always blows up in a finite time for any given nonzero initial data \( u_0(x,y) \) satisfying \( u_0(0, y) = 0 \) and \( \partial_x u_0(0, y) \leq 0 \).

In summary, we conclude that the solution \( w \) to the problem (3.6) must blow up in a finite time under the conditions (1.4), (1.5), (1.6) and (1.7), which is a contradiction with the assumption (3.2). Thereby we complete the proof of Theorem 1.2.

\[ \square \]
4. Appendix: The Construction of the Weight $\rho(y)$

We construct $\rho(y) \in W^{2,\infty}(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ satisfying the assumptions (3.11), (3.12) and (F1)–(F3) by the profile

$$\rho(y) = \begin{cases} ky, & 0 \leq y < A, \\ ay^2 + by + c, & A \leq y < B, \\ \frac{1}{(y+h)^\gamma}, & y \geq B, \end{cases} \quad (4.1)$$

where parameters $A, B, a, b, c, k, h$ and $\gamma$ will be specified later, with $k, h$ and $\gamma$ being positive.

To guarantee $\rho \in C^1(\mathbb{R}_+)$, by requiring the first derivatives of $\rho(y)$ being continuous at $y = A$ and $y = B$ respectively, we need the following compatibility conditions:

$$a = -\frac{1}{B^2 - A^2} \frac{B(1 + \gamma) + h}{(B+h)^{\gamma+1}}, \quad b = -\frac{\gamma}{(B+h)^{\gamma+1}} + \frac{2B}{B^2 - A^2} \frac{B(1 + \gamma) + h}{(B+h)^{\gamma+1}},$$

$$c = -\frac{A^2}{B^2 - A^2} \frac{B(1 + \gamma) + h}{(B+h)^{\gamma+1}}, \quad k = -\frac{\gamma}{(B+h)^{\gamma+1}} + \frac{2}{B + A} \frac{B(1 + \gamma) + h}{(B+h)^{\gamma+1}}. \quad (4.2)$$

Now it remains to choose the proper parameters $A, B, \gamma$ and $h$ such that the properties given in (3.11), (3.12) and (F1)–(F3) hold.

First, when $\gamma > 1$, it is easy to see that the weight $\rho \in W^{2,\infty}(\mathbb{R}_+) \cap C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ satisfies the properties given in (3.12) and (F1) for any fixed $C_1 \geq 1$.

**Verification of (F2).**

When $y \in [0, +\infty) \setminus [A, B]$, as $\rho''(y) \geq 0$, so (F2) holds obviously.

When $y \in (A, B)$, we know that

$$\int_{0}^{y} \rho(y')dy' + \rho''(y) \geq \int_{0}^{A} \rho(y')dy' + \rho''(y) = \frac{1}{2}kA^2 + 2a.$$

On the other hand, by using (4.2), one has that

$$\frac{1}{2}kA^2 + 2a = \frac{1}{(B+h)^{\gamma+1}} \left( \left( \frac{A^2}{B+A} - \frac{2}{B^2-A^2} \right) (B(1 + \gamma) + h) - \frac{1}{2}A^2\gamma \right) > 0$$

provided that

$$\begin{cases} \frac{A^2}{B+A} \geq \frac{4}{B^2-A^2}, \quad \text{i.e.} \quad B \geq A + \frac{4}{A^2} \quad (4.3) \\ h > A\gamma \end{cases}.$$

Thus the weight $\rho(y)$ obeys the property (F2).

**Verification of (F3).**

By construction of $\rho$ given in (4.1), we know that $\rho'(A) > 0$, $\rho''(y) = 0$ for any $y \in [0, A)$. Thus, it remains to verify the inequality (3.13) given in (F3).

By a direct calculation, it follows that

$$\frac{\rho(A)}{\rho(B)} = \frac{A}{B+h} \left( \frac{2}{B+A} (B(1 + \gamma) + h) - \gamma \right), \quad \frac{\rho'(A)}{\rho(A)} = \frac{1}{A}$$

and

$$\frac{|\rho'(B)|}{\rho'(A)} = \frac{\gamma}{\frac{2}{B+A} (B(1 + \gamma) + h) - \gamma},$$

which implies that

$$\lim_{h \to +\infty} \frac{\rho(A)}{\rho(B)} = \frac{2A}{B+A} < 1, \quad \lim_{h \to +\infty} \frac{|\rho'(B)|}{\rho'(A)} = \lim_{h \to +\infty} \frac{|\rho'(B)|}{\rho(A)} = 0. \quad (4.4)$$
Since for any \( y \in [A, B] \), we have that

\[
\int_{y}^{+\infty} (y' - A)|\rho''(y')|dy' \leq \int_{A}^{+\infty} (y' - A)|\rho''(y')|dy' = 2\rho(B) + 2(B - A)|\rho'(B)| - \rho(A).
\]

Thus, by using (4.4) we get that for any fixed \( \epsilon_0 > 0 \), there is \( h_1 > 0 \) such that as \( h \geq h_1 \), for all \( y \in [A, B] \), we have that

\[
\frac{\int_{y}^{+\infty} (y' - A)|\rho''(y')|dy'}{\rho(y)} \leq \frac{2\rho(B) + 2(B - A)|\rho'(B)| - \rho(A)}{\rho(A)} \leq \frac{B}{A} + \epsilon_0.
\]  

(4.5)

On the other hand, for any \( y \geq B \), one has that

\[
\frac{\int_{y}^{+\infty} (y' - A)|\rho''(y')|dy'}{\rho(y)} = 1 + \gamma \frac{y - A}{y + h} \leq 1 + \gamma.
\]  

(4.6)

Therefore, from (4.5) and (4.6) one gets that as \( h \geq h_1 \),

\[
\sup_{y \geq A} \frac{\int_{y}^{+\infty} (y' - A)|\rho''(y')|dy'}{\rho(y)} \leq \max \left\{ 1 + \gamma, \frac{B}{A} + \epsilon_0 \right\}.
\]  

(4.7)

Noting that \( \rho'(A) > 0 \) and \( \rho''(y) < 0 \) on \([A, B]\), we get

\[
\max_{y \in [A, B]} |\rho'(y)| = \max \{ \rho'(A), |\rho'(B)| \}, \quad \text{and} \quad \min_{y \in [A, B]} \rho(y) = \min \{ \rho(A), \rho(B) \}
\]

which implies by using (4.4) that there is \( h_2 \geq h_1 > 0 \) such that when \( h \geq h_2 \),

\[
\rho'(A) = \max_{y \in [A, B]} |\rho'(y)|, \quad \text{and} \quad \rho(A) = \min_{y \in [A, B]} \rho(y).
\]  

(4.8)

Therefore, when \( h \geq h_2 \) we have

\[
\int_{A}^{+\infty} \frac{(\rho'(y))^2}{\rho} dy = \int_{B}^{+\infty} \frac{(\rho'(y))^2}{\rho} dy + \int_{A}^{B} \frac{(\rho'(y))^2}{\rho} dy
\]

\[
\leq \frac{\gamma^2}{\gamma + 1} \frac{1}{(B + h)^{\gamma + 1}} + (B - A) \frac{(\rho'(A))^2}{\rho(A)}
\]

\[
= \left( \frac{\gamma^2}{\gamma + 1} \frac{B + A}{2(B(1 + \gamma) + h) - (B + A)\gamma} + \frac{B - A}{A} \right) \rho'(A),
\]

which implies that for the above \( \epsilon_0 > 0 \), there is \( h_3 \geq h_2 > 0 \), such that as \( h \geq h_3 \), we get that

\[
\frac{1}{\rho'(A)} \int_{A}^{+\infty} \frac{(\rho'(y))^2}{\rho} dy \leq \frac{B - A}{A} + \epsilon_0.
\]  

(4.9)

Thus, we conclude that for any fixed \( \epsilon_0 > 0 \), there is \( h_3 > 0 \) such that as \( h \geq h_3 \), one has

\[
\frac{1}{\rho'(A)} \int_{A}^{+\infty} \frac{(\rho'(y))^2}{\rho} dy + \sup_{y \geq A} \frac{\int_{y}^{+\infty} (y' - A)|\rho''(y')|dy'}{\rho(y)}
\]

\[
\leq \max \{ 1 + \gamma, \frac{B}{A} + \epsilon_0 \} + \frac{B - A}{A} + \epsilon_0.
\]  

(4.10)

To bound the right hand side of (4.10) by \( 4\alpha \) for some \( \alpha \in (0, 1) \), we first fix \( 1 < \gamma < 3, B \geq A + \frac{1}{A^\gamma} \) from (4.3), and let \( A > 0 \) large, and \( \epsilon_0 > 0 \) small such that

\[
\begin{cases}
1 + \gamma + \frac{B - A}{A} + \epsilon_0 < 4 \\
\frac{2B}{A} + 2\epsilon_0 < 5
\end{cases}
\]  

(4.11)

then by choosing \( h > \max(A\gamma, h_3) \) with \( h_3 > 0 \) being determined in (4.10), we get

\[
\frac{1}{\rho'(A)} \int_{A}^{+\infty} \frac{(\rho'(y))^2}{\rho(y)} dy + \sup_{y \geq A} \frac{\int_{y}^{+\infty} (y' - A)|\rho''(y')|dy'}{\rho(y)} \leq 4\alpha
\]
where

\[ \alpha = \frac{1}{4} \max \left( 1 + \gamma + \frac{B - A}{A} + \epsilon_0, \frac{2B}{A} + 2\epsilon_0 - 1 \right) \]

belongs to \((0, 1)\). Till now, we complete the construction of the weight \(\rho(y)\).

For example, if we can take \(\gamma = 2\), \(A = 3\), \(B = \frac{11}{3}\); \(h = 600\), \(\alpha = \frac{8}{9}\) and \(C_1 = 1\), and the parameters \(k, a, b\) and \(c\) are given in (4.2), then the weight \(\rho(y)\) given in (4.1) satisfies all properties listed in (3.11), (3.12) and (F1)–(F3) in Sect. 3.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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