Dynamics of an inhomogeneous quantum phase transition

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Abstract. We argue that, in a second-order quantum phase transition driven by an inhomogeneous quench, the density of quasi-particle excitations is suppressed when velocity at which a critical point propagates across a system falls below a threshold velocity equal to the Kibble–Zurek correlation length times the energy gap at freeze-out divided by \( \hbar \). This general prediction is supported by an analytic solution in the quantum Ising chain. Our results suggest, in particular, that adiabatic quantum computers can be made more adiabatic when operated in an ‘inhomogeneous’ way.

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1. Introduction

A quantum phase transition is a qualitative change in the ground state of a quantum system when one of the parameters in its Hamiltonian passes through a critical point. In a second-order transition, a continuous change is accompanied by a diverging correlation length and vanishing energy gap. The vanishing gap implies that no matter how slowly a system is driven through the transition its evolution cannot remain adiabatic near the critical point. As a result, after the transition the system is excited to a state with a finite correlation length $\hat{\xi}$ whose size shrinks with an increasing rate of the transition. This scenario, known as Kibble–Zurek (KZ) mechanism (KZM), was first described in the context of finite temperature transitions [1, 2]. Although originally motivated by cosmology [1], KZM at the finite temperature was confirmed by numerical simulations of the time-dependent Ginzburg–Landau model [3] and successfully tested by experiments in liquid crystals [4], superfluid helium 3 [5], both high-$T_c$ [6] and low-$T_c$ [7] superconductors and convection cells [8]. Most recently, spontaneous appearance of vorticity during Bose–Einstein condensation (BEC) driven by evaporative cooling was observed in [9]. However, the quantum zero temperature limit, which is in many respects qualitatively different, remained unexplored until recently, see e.g. [10]–[20]. The recent interest is motivated in part by adiabatic quantum computation or adiabatic quantum state preparation, where one would like to cross a quantum critical point as adiabatically as possible, and in part by condensed matter physics of ultracold atoms, where it is easy to manipulate parameters of a Hamiltonian in time and which, unlike their solid state physics counterparts, are fairly well isolated from their environment. In fact, an instantaneous quench to the ferromagnetic phase in a spinor BEC resulted in finite-size ferromagnetic domains whose origin was attributed to KZM [21]. However, since the transition rate was effectively infinite in that experiment, the KZ scaling relation between the average domain size $\hat{\xi}$ and the quench rate has not been verified.

The KZM argument is briefly as follows [2, 12]. When a transition is driven by varying a parameter $g$ in the Hamiltonian across an isolated critical point at $g_c$, then we can define a dimensionless distance from the critical point as

$$\epsilon = \frac{g - g_c}{g_c}. \quad (1)$$

When $\epsilon \to 0$ the correlation length $\xi$ in the ground state diverges as $\xi \sim |\epsilon|^{-\nu}$, and the energy gap $\Delta$ between the ground state and the first excited state vanishes as $\Delta \sim |\epsilon|^{-\nu}$. Setting $\hbar = 1$ from now on, a diverging $\Delta^{-1} \sim |\epsilon|^{-\nu}$ is the shortest timescale on which the ground state can adjust adiabatically to varying $\epsilon(t)$. A generic $\epsilon(t)$ can be linearized near the critical point $\epsilon = 0$ as

$$\epsilon(t) \approx -\frac{t}{\tau_Q} + O(t^2). \quad (2)$$

where the coefficient $\tau_Q$ is called a quench time. Assuming that the system was initially prepared in its ground state, its adiabatic evolution fails at a $\hat{\epsilon}$ when the time $\hat{t}$ left to crossing the critical point equals the shortest timescale $\Delta^{-1}$ on which the ground state can adjust. Solving this equality, we obtain

$$\hat{\epsilon} \sim \tau_Q^{-1/(\nu+1)}, \quad (3)$$

$$\hat{t} \sim \tau_Q^{\nu/(\nu+1)}. \quad (4)$$
From $\epsilon$ the evolution becomes impulse, i.e. the state does not evolve but remains frozen in the ground state at $\epsilon$, until $-\epsilon$ when the evolution becomes adiabatic again. In this way, the ground state at $\epsilon$ with a KZ correlation length

$$\xi \sim \epsilon^{-v} \sim \tau_Q^{\nu/(\nu+1)}$$

becomes the initial excited state for the adiabatic evolution after $-\epsilon$. In particular, $\xi^{-1}$ determines the density of quasiparticles excited during the phase transition

$$d \sim \tau_Q^{-D\nu/(\nu+1)}$$

in $D$ dimensions. Note that when $\tau_Q$ is large, $\epsilon$ is small and the linearization in equation (2) is self-consistent because the KZM physics happens very close to the critical point between $-\epsilon$ and $+\epsilon$.

### 2. Inhomogeneous transition

As pointed out in the finite temperature context [22], in a realistic experiment it is difficult to make $\epsilon$ exactly homogeneous throughout a system. For instance, in the superfluid $^3$He experiments [5] the transition was caused by neutron irradiation of helium 3. Heat released in each fusion event, $n + ^3$He $\rightarrow ^4$He, created a bubble of normal fluid above the superfluid critical temperature $T_c$. As a result of quasi-particle diffusion, the bubble was expanding and cooling with a local temperature $T(t, r) = \exp(-r^2/2Dt)/(2\pi Dt)^{3/2}$, where $r$ is a distance from the center of the bubble and $D$ is a diffusion coefficient. Since this $T(t, r)$ is hottest in the center, the transition back to the superfluid phase, driven by an inhomogeneous parameter

$$\epsilon(t, r) = \frac{T(t, r) - T_c}{T_c},$$

proceeded from the outer to the central part of the bubble with a critical front $r_c(t)$, where $\epsilon = 0$, shrinking with a finite velocity $v = dr_c/dt < 0$.

A similar scenario is likely in the ultracold atom gases in magnetic/optical traps. The trapping potential results in an inhomogeneous density of atoms $\rho(\vec{r})$ and, in general, a critical point $g_c$ depends on atomic density $\rho$. Thus even a transition driven by a perfectly uniform $g(t)$ will be effectively inhomogeneous,

$$\epsilon(t, \vec{r}) = \frac{g(t) - g_c[\rho(\vec{r})]}{g_c[\rho(\vec{r})]},$$

with the surface of critical front, where $\epsilon = 0$, moving with a finite velocity.

According to KZM, in a homogeneous symmetry-breaking transition, a state after the transition is a mosaic of finite ordered domains of average size $\xi$. Within each finite domain orientation of the order parameter is constant but uncorrelated to orientations in other domains. In contrast, in an inhomogeneous symmetry-breaking transition [22], the parts of the system that cross the critical point earlier may be able to communicate their choice of orientation of the order parameter to the parts that cross the transition later and bias them to make the same choice. Consequently, the final state may be correlated at a range longer than $\xi$, or even end up being a ground state. In other words, the final density of excited quasi-particles may be lower than the KZ estimate in equation (6) or even zero.
From the point of view of testing KZM, this inhomogeneous scenario, when relevant, may sound like a negative result: an imperfect inhomogeneous transition suppresses KZM. However, from the point of view of adiabatic computation or adiabatic state preparation it is the KZM itself that is a negative result: no matter how slow the homogeneous transition is there is a finite density of excitations (6) which decays only as a small power of transition time \( \tau_Q \). From this perspective, the inhomogeneous transition may be a way to suppress KZ excitations and prepare the desired final ground state adiabatically.

To estimate when the inhomogeneity may actually be relevant, in a similar way as in equation (2), we linearize the parameter \( \epsilon(t, n) \) in both \( n \) and \( t \) near the critical front where \( \epsilon(t, n) = 0 \):

\[
\epsilon(t, n) \approx \alpha(n - vt).
\]  

Here \( n \) is the position in space, e.g. lattice site number, \( \alpha \) is a slope of the quench and \( v \) is the velocity of the critical front. When observed locally at a fixed \( n \), the inhomogeneous quench in equation (9) looks like the homogeneous quench in equation (2) with

\[
\tau_Q = \frac{1}{\alpha v}.
\]  

The part of the system where \( n < vt \), or equivalently \( \epsilon(t, n) < 0 \), is already in the broken symmetry phase. The orientation of the order parameter chosen in this part can be communicated across the critical point no faster than a threshold velocity

\[
\hat{v} \sim \frac{\xi}{\tau}.
\]  

When \( v \gg \hat{v} \), the communication is too slow for the inhomogeneity to be relevant, but when \( v \ll \hat{v} \), we can expect the final state to be less excited than predicted by KZM.

Given the relation (10), the condition (11) can be solved either as

\[
\hat{v} \sim \tau_Q^{-[\nu(z-1)/z+1]},
\]  

\[
\hat{v} \sim \alpha^{[\nu(z-1)/(1+\nu)]},
\]  

or as a relation between the threshold transition time and the slope,

\[
\hat{\tau}_Q \sim \alpha^{-[(z+1)/(1+\nu)]}.
\]  

This relation means that, for a given inhomogeneity \( \alpha \), the transition is effectively homogeneous when \( \tau_Q \ll \hat{\tau}_Q \), but the inhomogeneity becomes relevant when the transition is slow enough, \( \tau_Q \gg \hat{\tau}_Q \). In the homogeneous limit \( \alpha \rightarrow 0 \), the threshold transition time \( \hat{\tau}_Q \rightarrow \infty \).

The threshold velocity in equation (11) appeared for the first time in the context of finite temperature classical phase transitions [22] where it looks formally the same, but the underlying physics is qualitatively different: the scales \( \xi \) and \( \tau \) are determined not by the gap of a quantum Hamiltonian, but the relaxation time of an open classical system. Nevertheless, the key mechanism that nonzero-order parameter penetrates from the symmetry broken phase into the symmetric phase ahead of the critical front seems to be the same.

In the following section, we rederive results (12)–(14) from a different perspective.
3. Kibble–Zurek mechanism in space

References [23, 24] considered a ‘phase transition is space’ where $\epsilon(n)$ is inhomogeneous but time independent. In the same way as in equation (9), this parameter can be linearized in $n - n_c$,

$$\epsilon(n) \approx \alpha(n - n_c), \quad (15)$$

near the static critical front at $n = n_c$ where $\epsilon = 0$. The system is in the broken symmetry phase where $n < n_c$ and in the symmetric phase where $n > n_c$. In the first ‘local approximation’, we would expect that the order parameter behaves as if the system were locally uniform: it is nonzero for $n < n_c$ only, and tends to zero as $(n_c - n)^\beta$ when $n \to n_c^-$ with the critical exponent $\beta$. However, this first approximation is in contradiction with the basic fact that the correlation (or healing) length $\xi$ diverges as $\xi \sim |\epsilon|^{-\nu}$ near the critical point and the diverging $\xi$ is the shortest length scale on which the order parameter can adjust to (or heal with) the changing $\epsilon(n)$. Consequently, when approaching $n_c^-$ the local approximation $(n_c - n)^\beta$ must break down when a local correlation length $\xi \sim [\alpha(n_c - n)]^{-\nu}$ equals the distance remaining to the critical point $(n_c - n)$. Solving this equality with respect to $\xi$, we obtain

$$\hat{\xi} \sim \alpha^{-\nu/(1+\nu)}. \quad (16)$$

From $n - n_c \simeq -\hat{\xi}$ the evolution of the order parameter in $n$ becomes ‘impulse’, i.e. the order parameter does not change until $n - n_c \simeq +\hat{\xi}$ in the symmetric phase where it begins to follow the local $\epsilon(n)$ again and quickly decays to zero on the same length scale of $\hat{\xi}$. This ‘KZM in space’ predicts that a nonzero-order parameter penetrates into the symmetric phase to a depth

$$\delta n \sim \hat{\xi} \sim \alpha^{-\nu/(1+\nu)}.$$

(17)

The critical point is effectively ‘rounded off’ on the length scale of $\hat{\xi}$. As a consequence, we expect a nonzero gap scaling as

$$\hat{\Delta} \sim \hat{\xi}^{-\nu} \sim \alpha^{z\nu/(1+\nu)}, \quad (18)$$

as opposed to the local approximation, where we would expect gapless quasi-particles near the critical point.

We expect the finite gap in equation (18) to prevent excitation of the system even when the critical point $n_c$ in equation (24) moves with a finite velocity, $n_c(t) = vt$, up to a threshold velocity

$$\hat{v} \sim \frac{\hat{\xi}}{\Delta^{-1}} \sim \alpha^{[\nu(z-1)/(1+\nu)]}, \quad (19)$$

which is identical with the $\hat{v}$ in equation (13).

In the following sections, we test these predictions in the quantum Ising chain.

4. Quantum Ising chain

The model is

$$H = -\sum_{n=1}^{N} g_n \sigma_n^x - \sum_{n=1}^{N-1} \sigma_n^z \sigma_{n+1}^z. \quad (20)$$
For $N \to \infty$, a uniform system with $g_n = g$ has two critical points at $g = \pm 1$ separating a ferromagnetic phase, when $|g| < 1$, from two paramagnetic phases, when $|g| > 1$. We focus on the critical point at $g = 1$ when $\epsilon = g - 1$. Given $z = 1$ and $\nu = 1$, we expect $
abla \simeq 1$ independent of either $\tau_Q$ or $\alpha$. The quench is exactly solvable for homogeneous $g$ [13, 15], but even in an inhomogenous case some useful analytic insights can be obtained as follows.

After Jordan–Wigner transformation to spinless fermionic operators $c_n$, $\sigma_n^+ = 1 - 2c_n^\dagger c_n$ and $\sigma_n^- = -\left(\sigma_n^+ + 1\right)$, equation (20) becomes

$$H = 2 \sum_{n=1}^{N} g_n c_n^\dagger c_n - \sum_{n=1}^{N-1} \left( c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n + \text{h.c.} \right).$$

This quadratic $H$ is diagonalized to $H = \sum_m \omega_m \gamma_m^\dagger \gamma_m$ by a Bogoliubov transformation $c_n = \sum_m (u_m + i v_m) \gamma_m$ with $m$ numerating $N$ eigenmodes of stationary Bogoliubov–de Gennes equations

$$\omega_m u_m = 2g_m u_m - 2u_{m+1}^\mp \partial_n u_m^\mp$$

with $\omega_m \geq 0$. Here, $u_m^\pm = u_m \pm v_m$.

### 5. Ising chain: Kibble–Zurek mechanism in space

To begin with, we consider the ground state of the quantum Ising chain in a static inhomogeneous transverse field $g_n$, which can be linearized near the critical point $g = 1$ as

$$\epsilon(n) = g_n - 1 \approx \alpha(n - n_c),$$

compare with equation (15). The chain is in the (broken symmetry) ferromagnetic phase where $n < n_c$ and in the (symmetric) paramagnetic phase where $n > n_c$. We want to know if the nonzero ferromagnetic magnetization $Z_n = \langle \sigma_n^z \rangle$ in the ferromagnetic phase penetrates across the critical point into the paramagnetic phase and what is the depth $\delta n$ of this penetration.

Since in a homogeneous system quasi-particle spectrum is gapless at the critical point only, we expect low-energy quasi-particle modes $u_{n,m}^\pm$ to be localized near the critical point at $n_c$ where we can use the linearization in equation (24). We also expect that these low-energy modes are smooth enough to treat $n$ as continuous and make a long-wavelength approximation

$$u_{n+1,m}^\mp \approx u_{n,m}^\mp + \frac{\partial}{\partial n} u_{n,m}^\mp$$

in equation (23). Under these assumptions, we obtain a long-wavelength equation

$$\omega_m u_m^\mp = 2\alpha(n - n_c) u_m^\mp \pm 2\partial_n u_m^\mp.$$

After some algebra, its eigenmodes can be found as

$$u_m(n) \propto \psi_{m-1}(x) + \psi_{m}(x),$$

$$v_m(n) \propto \psi_{m-1}(x) - \psi_{m}(x),$$

$$\omega_m = \sqrt{8m\alpha}.$$
where

\[ x = \sqrt{\alpha} (n - n_c), \]  

is a rescaled position, \( \psi_{m \geq 0}(x) \) are eigenmodes of a harmonic oscillator satisfying

\[ \frac{1}{2} (-\partial_x^2 + x^2) \psi_m(x) = (m + 1/2) \psi_m(x), \]

and \( \psi_{-1}(x) = 0 \). As expected, the modes in equation (27) are localized near \( n = n_c \) where \( x = 0 \). A typical width of the lowest energy eigenmodes is \( \delta x \simeq 1 \), or equivalently

\[ \delta n \simeq \alpha^{-1/2}. \]  

When \( \alpha \ll 1 \), then \( \delta n \gg 1 \) and the long-wavelength approximation in equations (25) and (26) is self-consistent. Thus \( \delta n \) in equation (30) is the relevant scale of length near \( n = n_c \) and we expect that this \( \delta n \) determines the penetration depth of the spontaneous ferromagnetic magnetization into the paramagnetic phase.

We test this prediction by a numerical solution for an inhomogeneous transverse magnetic field

\[ g_n = 1 + \tanh[\alpha(n - n_c)], \]  

which is shown in figure 1 with a variable slope \( \alpha \). This field can be self-consistently linearized near \( n = n_c \) as in equation (24) because, when the slope \( \alpha \ll 1 \), the predicted \( \delta n \simeq \alpha^{-1/2} \) is much shorter than the width \( \alpha^{-1} \) of the tanh.

Figures 2(A) and (B) show how the spontaneous ferromagnetic magnetization \( Z_n = \langle \sigma_n^z \rangle \) from the ferromagnetic phase, where \( n < n_c \), penetrates into the paramagnetic phase, where \( n > n_c \). In particular, the collapse of the rescaled plots in figure 2(B) demonstrates that the penetration depth is \( \delta x \simeq 1 \) equivalent to \( \delta n \simeq \alpha^{-1/2} \), as predicted in equations (17) and (30) and [23]. Paramagnetic spins near the critical point are biased towards the direction of spontaneous magnetization chosen in the ferromagnetic phase.

Moreover, the analytic solution (27) implies a finite (relevant) gap

\[ \hat{\Delta} = \omega_0 + \omega_1 = \sqrt{8\alpha}, \]  

Figure 1. The critical front in equations (31) and (46).
Figure 2. In (A) and (B), exact numerical spontaneous magnetization as a function of \( n - n_c \) and \( x = \sqrt{\alpha(n - n_c)} \), respectively. In (C) and (D), ferromagnetic correlation between the site 100 in the ferromagnetic phase and a site \( n \) when \( n_c = vt = 150 \) as a function of \( n - vt \) and \( x_v \) in equation (36), respectively. Results in (C) and (D) were obtained with the Vidal algorithm [25] for \( \alpha = 2^{-5} \) and \( N = 256 \). When \( v \gg 2 \) there is no ferromagnetic correlation across the critical point at \( n - vt = 0 \), see (C), and when \( v \ll 2 \) the correlation penetrates into the paramagnetic phase to a depth of \( \delta x_v \simeq 1 \), see (D).

in accordance with the scaling \( \sim \alpha^{1/2} \) predicted by the general equation (18) and the numerics in [23]. This gap is the energy of the lowest relevant (even parity) excitation of two quasi-particles.

6. Ising chain: inhomogeneous transition

Let the critical front in equation (24) and figure 1 move with a velocity \( v > 0 \):

\[
n_c(t) = vt.
\]  

(33)

A \( t \)-dependent version of the long-wavelength equation (26),

\[
i\partial_t \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} 2\alpha(n - vt)\sigma^x + 2i\sigma^y\partial_n \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix},
\]

(34)

can be solved exactly for both \( v < 2 \) and \( v > 2 \) with qualitatively different solutions in the two regimes. Not incidentally, \( v = 2 \) is the maximal velocity of quasi-particles at the critical point whose dispersion is \( \omega = 2|k| \) for small \( |k| \ll \pi \).
6.1. Case of $v < 2$

When $v < 2$ equation (34) has solutions

$$ u_m(t, n) \propto e^{-i \omega_m t} \left[ \psi_{m-1}(x_v) + e^{i \varphi} \psi_m(x_v) \right] e^{i x_v \sqrt{m/2}}, $$

$$ v_m(t, n) \propto e^{-i \omega_m t} \left[ e^{i \varphi} \psi_{m-1}(x_v) - \psi_m(x_v) \right] e^{i x_v \sqrt{m^2/2}}, $$

where $m = 0, 1, 2, \ldots$, the phase $\varphi = \arcsin(v/2)/2$, and

$$ x_v = \left(1 - \frac{v^2}{4}\right)^{-1/4} \sqrt{\alpha(n - vt)} $$

is a rescaled position. When $v \to 0$ we recover the static solutions (27). In the reference frame of $x_v$, which is co-moving with the critical point, the solutions (35) are stationary modes with $\omega_m \geq 0$ so there are no quasi-particles in the system,

$$ d(v < 2) = 0, $$

and, in particular, no kinks where $g_n = 0$.

As shown in figures 2(C) and (D), ferromagnetic correlations penetrate across the critical point into the paramagnetic phase to a depth $\delta x_v \simeq 1$ equivalent to

$$ \delta x_v = \left(1 - \frac{v^2}{4}\right)^{1/4} \sqrt{\alpha(n - vt)} $$

The penetration depth $\delta x_v$ shrinks to 0 when $v \to 2^-$ suggesting communication problems across the critical point when $v > 2$.

The same $\delta x_v$ is a typical width of the lowest eigenmodes in the spectrum (35). As it shrinks to 0 when $v \to 2^-$, the eigenmodes become inconsistent with the long-wavelength approximation in equation (34).

6.2. Case of $v > 2$

When $v > 2$ then equation (34) can be mapped to a homogeneous transition. Indeed, we replace

$$ \tilde{t} = \left(1 - \frac{4}{v^2}\right)^{-1} \left(t - \frac{n}{v}\right), \quad \tilde{n} = n, $$

introducing local time $\tilde{t}$ measured from the moment the critical point passes through $n$, and simultaneously make a transformation

$$ \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \frac{4}{v^2}} & 2i \\ \frac{v^2}{v} & -1 \end{pmatrix} \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} $$

bringing equation (34) to a new form

$$ i \partial_{\tilde{t}} \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} = \left[ -\frac{2}{\tilde{t}Q} \sigma^+ + 2i \sigma^x \partial_{\tilde{n}} + \frac{4}{iv} \partial_{\tilde{n}} \right] \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix}. $$
Here $\sigma^v = \sigma^v \sqrt{1 - \frac{4}{v^2}} + \frac{2}{v} \sigma^z$. Up to an unimportant rotation of a Pauli matrix $\sigma^v \rightarrow \sigma^v$ and the momentum-dependent energy shift $\frac{4}{i v} \partial_n$, the new equation (41) is a homogeneous version of the old equation (34), but with a longer effective quench time $\tilde{\tau}_Q = \tau_Q \left(1 - \frac{4}{v^2}\right)^{-3/2} > \tau_Q$.

Consequently, a quasi-momentum representation $\tilde{\mathbf{u}}^+ = (a_k, b_k) \exp(ik\tilde{n} - 4ik\tilde{t}/v)/\sqrt{2\pi}$ brings the homogeneous equation (41) to the Landau–Zener form:

$$i \frac{d}{ds} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{2} \left[ -\delta_k s \sigma^x + \sigma^v \right] \begin{pmatrix} a_k \\ b_k \end{pmatrix},$$

where $s = k\tilde{t}$ is a new time variable and $\delta_k = 1/4k^2\tilde{\tau}_Q$ is a new transition rate. The Landau–Zener formula $p_k = \exp(-\pi/2\delta_k)$ gives excitation probability for a quasi-particle $k$ and density of excited quasi-particles is

$$d(v > 2) = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} p_k = \frac{1 - \frac{4}{v^2}}{2\pi \sqrt{2\tau_Q}},$$

where $\Lambda \simeq 1$ is an ultraviolet cut-off. The integral is accurate for $\tilde{\tau}_Q \gg 1$. When $v \gg 2$ the density

$$d(v \gg 2) \approx \frac{1}{2\pi \sqrt{2\tau_Q}} \equiv d_{\text{KZM}}$$

is the same as the density after a homogeneous quench with the same $\tau_Q$, see [13], but when $v \to 2^+$ then $d$ is suppressed below the ‘homogeneous’ density $d_{\text{KZM}}$ by the factor $(1 - 4/v^2)^{3/4}$.

6.3. Numerical results

Since the long-wavelength equation (25) does not give self-consistent long-wavelength solutions when $v \to 2$, we simulated the exact time-dependent version of the Bogoliubov–de Gennes equations

$$i \frac{du_{n,m}^\pm}{dt} = 2g_n(t)u_{n,m}^\mp - 2u_{n+1,m}^\mp,$$

on a finite lattice of $N$ sites for a time-dependent transverse magnetic field

$$g_n(t) = 1 + \tanh[\alpha(n - vt)]$$

with a moving critical point at $n_c = vt$, compare equation (31) and figure 1. Results are shown in figures 3 and 4. Figure 4 demonstrates good quantitative agreement between equation (43) and numerical results, despite the breakdown of the long-wavelength approximation near $v = 2$.

7. Conclusion

We made the general estimate equations (11)–(14) when an inhomogeneous quench cannot be considered homogeneous with respect to KZM. Then we solved the problem in detail in the particular case of the quantum Ising chain where $z = 1$ and the threshold velocity $\tilde{v} = 2$ is equal to velocity of quasi-particles at the critical point. Excitation of kinks is dramatically suppressed when a critical front propagates slower than $\tilde{v}$ and the ferromagnetic phase is able
Figure 3. Numerical simulations of $N = 400$ spins. In (A), final density of kinks $d(\tau_Q)$ for different slopes $\alpha$, and in (B) a rescaled final density $\alpha^{-1/2}d(v)$. The solid lines are guide to the eye. The fit in panel (A) shows that when $v \gg 2$ then $d \simeq \tau_Q^{-1/2}$ like in the homogeneous KZM, and when $v \ll 2$ then $d$ is suppressed below the homogeneous KZM density.

Figure 4. Comparison between equation (43) (solid blue), the homogeneous KZM (dotted red), and numerical simulations on a lattice of $N = 1000$ spins (crosses) at a fixed slope $\alpha = 2^{-6}$.

to communicate its choice of ferromagnetic polarization to the paramagnetic phase ahead of the front. In contrast, when the front is much faster than $\hat{v}$ the communication across the front is not efficient enough and kinks are excited as in the homogeneous KZM. However, even above
density of excited kinks is suppressed below the ‘homogeneous’ KZM density by a factor \( \left(1 - \frac{\hat{v}^2}{v^2}\right)^{3/4} \) which is significantly less than 1 when \( \hat{v} \) is close to \( \hat{v}^* \).

Thus, the general estimates (11)–(14) are confirmed by our solution of the quantum Ising chain, but we leave their interesting implications when \( z \neq 1 \) or in more than one dimension for future exploration.

The estimates and the solution suggest that ‘inhomogeneous’ adiabatic quantum computers can be more adiabatic than their ‘homogeneous’ counterparts.

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