Measuring time with stationary quantum clocks

Sergii Strelchuk\textsuperscript{1} and Mischa P. Woods\textsuperscript{2}

\textsuperscript{1}DAMTP, University of Cambridge, UK
\textsuperscript{2}Institute for Theoretical Physics, ETH Zurich, Zurich, Switzerland

Time plays a fundamental role in our ability to make sense of the physical laws in the world around us. The nature of time has puzzled people — from the ancient Greeks to the present day — resulting in a long running debate between philosophers and physicists alike to whether time needs change to exist (the so-called relational theory), or whether time flows regardless of change (the so-called substantival theory). One way to decide between the two is to attempt to measure the flow of time with a stationary clock, since if time were substantival, the flow of time would manifest itself in the experiment. Alas, conventional wisdom suggests that in order for a clock to function, it cannot be a static object, thus rendering this experiment seemingly impossible. Here we show, with the aid of counterfactual measurements, the surprising result that a quantum clock can measure the passage of time even while being switched off, thus lending constructive support for the substantival theory of time.

I. INTRODUCTION

Time is an essential ingredient in the world we inhabit. Not so surprisingly, it has played a central role in all of our physical theories. Yet this role was rather mundane until the advent of modern physics. In particular, out of the three pillars of modern physics — quantum mechanics, special and general relativity — it was only the latter two which forced us to change our preconceptions about the nature of time. Quantum mechanics, on the other hand, while very strange and mysterious in many ways, did not bring any novel insights as far as time is concerned: time is just a parameter which increases in line with any mundane classical clock — just like in Newton’s laws. Even more recently, it has been proven, in the context of quantum mechanics, that the time-analogue of Bell’s inequality always has a perfectly sound classical explanation [1, 2].

Conversely, in the philosophy of physics domain, the notion of time occupied a prominent role in debates dating back millennia which are still very active today. One of the ongoing debates is whether time necessitates change to exist. The substantival theory of time says that time exists and provides an invisible container in which matter lives, regardless of whether the matter is moving. On the other hand, the relational theory of time says that time is a set of relationships among the events of physical material in space — that is to say, time ceases to exist if matter ceases to change.

Views have continuously shifted through the ages: Many believe that Greek atomists such as Democritus thought time was substantival. The first written account dates back to Aristotle, who advocated for a relational theory: “But neither does time exist without change;...” [3]. Newton and Leibniz had opposed views, with Newton on the substantival camp [4], and Leibniz on the relational camp [5]. Ernst Mach attacked Newton’s arguments in favour of a more relative theory [6]. Einstein credited Mach’s views as being highly influential in his guiding principles when developing his theory of general relativity; although later shifted his stance to a more substantival interpretation of his theory [7].

Arguments for and against either theories are still ongoing [8–11]. One of the biggest problems in this debate is the apparent inability to distinguish experimentally between the two scenarios — indeed, it is widely accepted that any clock (or physical matter used as a rudimentary clock), would need to change its state in order to measure the passing of time, hence rendering a clock useless for detecting the substantival’s hypothesised passage of time without change.

Here we show that nonrelativistic quantum mechanics renders this widely held belief wrong. This is to say, we demonstrate by designing special clocks and adhering to well established interpretations of quantum theory, that it is possible to measure the passage of time between two events even when said clock has been always off, in other words, never evolving. Our result thus allows one to precisely detect the presence of the flow of time without evolution, which we argue provides strong and experimentally testable evidence for the substantival theory of time.

II. RESULTS

Overview: We will start by considering the simplest systems naturally occurring in nature which can serve as elementary clocks when dynamically evolving. We then show how they can be used to tell the time while being switched off via a more advanced protocol and measurement scheme. This is followed by an analysis of the interpretations used and a demonstration of how this strange phenomena cannot be explained if one assumes an underlying “real” description of nature, using classical variables. Finally, in section III, we explore the implications of our results in relation to the long standing historical debates on the nature of time.

Elementary timekeeping systems: While we tend to think of timekeeping devices as engineered systems, we can use the dynamics of naturally occurring processes as elementary clocks. The simplest of which can
be thought of as an arbitrary state $|\psi\rangle$ at time $t = 0$ whose evolution takes it through a sequence of mutually orthogonal states $|\tau_0\rangle, |\tau_1\rangle, \ldots, |\tau_{N_T}\rangle$ at times $\tau_0, \tau_1, \ldots, \tau_{N_T}$ consecutively. If we know that the system is initially in the state $|\psi\rangle$, and we let it evolve for an unknown time $t$ given the promise that $t$ belongs to the set $\{\tau_0, \tau_1, \tau_2, \ldots, \tau_{N_T}\}$, then we can determine precisely what the elapsed time is by measuring in the basis $|\tau_0\rangle, |\tau_1\rangle, |\tau_2\rangle, \ldots, |\tau_{N_T}\rangle$, since at said times, the measurement will deterministically allow one to distinguish between said states. To use this timekeeping system in practice, consider two external events, which we will call 1st event and 2nd event, with some unknown elapsed time $t \in \{\tau_0, \tau_1, \tau_2, \ldots, \tau_{N_T}\}$ between them. One initialises the system in the state $|\psi\rangle$ when the 1st event occurs and then waits until the 2nd event occurs, at which point the measurement would be performed; revealing the elapsed time. When a clock is revealed to have been dynamically evolving upon measurement as in this example, we say it is operating in standard fashion.

This is arguably one of the most elementary types of timekeeping devices one can conceive of. It has no inherently quantum features and should be quite ubiquitous in nature since the states of systems tend to become completely distinguishable if one waits long enough. As a simple illustration of how it could be used, imagine being on a boat at sea heading in to port. There is a lighthouse whose pulsing flashes of light reach you every 3 seconds, but thick fog rolls in obscuring the flashes for some time before clearing. By setting the times $\{\tau_n\}$ to coincide with the flashes and choosing the 1st and 2nd events to be flashes before and after the presence of the fog, the clock can deduce the duration of the foggy period.

Suppose that, in addition, there is a state denoted $|E\rangle$ which is invariant under time evolution, in other words, it remains in the state $|E\rangle$ at all times. Furthermore, suppose it is orthogonal to $|\psi\rangle$ and its time evolved state at all times. We refer to this state as an off state, in contrast to any state which evolves in time, which we refer to as an on state. The state $|E\rangle$ could be, for example, an energy eigenstate of the system. If one starts in the state $|E\rangle$ and then measures the state later, clearly there is no possible measurement one can perform which would reveal any information about the elapsed time — it would be analogous to taking the batteries out of a wall clock, and then trying to use it to tell the time.

**Telling the time when the clock is off:** We will now show, with the aid of some additional stationary ancilla qubits, how quantum mechanics allows one to determine precisely which time $\tau_n$ it is, even though the timekeeping device was never on (in the runs of the experiment in which we determined $\tau_n$). For the purpose of illustration, consider the most basic example of $N_T = 1$, in which the system in question merely has two times $\tau_0, \tau_1$. We relegate the $N_T \geq 2$ version to appendix A. In this case we will only need one ancillary state, which we denote $|A\rangle$. It could be, for example, chosen to be another energy eigenstate which is orthogonal to the other states, or if such a state is not available, we can use an ancilla qubit with a trivial Hamiltonian so as to ensure its states do not evolve. In the latter case, denoting a basis for the ancilla by $|\uparrow\rangle, |\downarrow\rangle$, the states would then be associated with $|\psi(t\rangle \equiv |\psi(t\rangle |\uparrow\rangle$, $|E\rangle \equiv |E\rangle |\uparrow\rangle$, and the new ancilla state with $|A\rangle = |E\rangle |\downarrow\rangle$ (Here $|\psi(t\rangle$ denotes $|\psi\rangle$ after evolving for a time $t$).

We will use the same retrodictive arguments as in other famous experiments, such as the Elitzur-Vaidman bomb tester or Hardy’s paradox. However, let us first describe the protocol, before discussing the interpretation: Initially the system is set to the off state $|E\rangle$. Then, when the 1st event occurs, we apply a unitary $U_0$ such that the system is now in a superposition of on and off, namely, of $|E\rangle$ and $|\psi\rangle$. We use branching notation to indicate the orthogonal branches of the superposition associated with static and dynamical terms (Upper branch is static, lower branch is dynamic). The state reads:

$$
|E\rangle \sim \begin{cases} 
 c |E\rangle \\
 s \langle\psi|, \end{cases}
$$

where $c = \cos(\theta), s = \sin(\theta)$. One then waits until the 2nd event occurs, at either time $\tau_0$ or $\tau_1$, at which point one applies a judiciously chosen unitary $U_m$ followed by immediately measuring in the $|E\rangle, |\tau_0\rangle, |\tau_1\rangle, |A\rangle$ basis, which we will call the measurement basis for short. This completes the protocol — modulo specification of the required constraints on $U_0$ and $U_m$. Diagrammatically, up to the point of measurement, this protocol and unitary $U_m$ have the form:

$$
|E\rangle \xrightarrow{c |E\rangle \sim \tau_0 \xrightarrow{\tau_0} \tau_1} cA_1^0 |E\rangle + cA_2^0 |A\rangle + cA_3^0 |\tau_0\rangle + cA_4^0 |\tau_1\rangle
$$

$$
|E\rangle \xrightarrow{s |\psi\rangle \sim \tau_0 \xrightarrow{\tau_0} \tau_1} -cA_2^0 |A\rangle + sA_3^0 |\tau_0\rangle + sA_4^0 |\tau_1\rangle
$$

(1)
at time $\tau_0$ and

$$
|E\rangle \xrightarrow{\tau_0} c|E\rangle \xrightarrow{U_m} cA_0^0|E\rangle + cA_0^0|A\rangle + cA_0^0|\tau_0\rangle + cA_0^0|\tau_1\rangle
$$

$$
|E\rangle \xrightarrow{\tau_1} s|\psi\rangle \xrightarrow{U_m} s|\tau_1\rangle \xrightarrow{U_m} -cA_0^0|E\rangle + sA_1^0|\tau_0\rangle + sA_1^0|\tau_1\rangle
$$

(2)

at time $\tau_1$, where $|cA_0^0| > 0$, $|cA_0^0| > 0$ and other amplitudes are arbitrary. Squiggly arrows represent the passing of an amount of time $\tau_0$ or $\tau_1$, while straight arrows the application of $U_m$.

Suppose the $E$ outcome is obtained.\(^1\) At time $\tau_0$, a non zero amplitude associated with $|E\rangle$ exists only for the off branch, while for time $\tau_1$, this amplitude from the off branch cancels with an on branch amplitude due to destructive interference. As such, when $E$ is obtained, we can deduce that the clock has collapsed to a branch of the wave function that was always off and that the unknown time $t$ must be $\tau_0$. Similarly, when outcome $A$ is obtained, we deduce that the clock was collapsed to a branch of the wave function that was always off and that the time $t$ must be $\tau_1$. However, if outcome $\tau_0$ or $\tau_1$ is obtained, we cannot conclude anything useful: the clock may have been on and we cannot deduce whether the time is $\tau_0$ or $\tau_1$.

Hence whenever outcome $E$ or $A$ is obtained, we can deduce what the time is, even though the system was always off, that is to say, always in a stationary state (since $|E\rangle$ and $|A\rangle$ are orthogonal to all dynamical branches whenever $E$, $A$ have a non zero probability of being obtained).

**Interpretation:** One may object that the system has certainly been in a superposition of on and off, and in this sense, has actually been evolving in time. However, the outcomes $E$ and $A$ can only arise via an always off branch of the wave function so if it is seen then we have been confined to a part of the total quantum state in which the system is always stationary. Furthermore, this interpretation is independent of the basis used to represent the state during the different stages of the protocol — we represented it in the measurement basis purely for convenience.

Our explanation of the results has relied on a certain kind of retrodictive interpretation of quantum mechanics. This interpretation can be most readily captured by Schrödinger’s proverbial cat experiment: one starts with an alive cat, denoted $|\text{alive}\rangle$, which when put in the closed box, takes on the form $\frac{1}{\sqrt{2}}(|\text{alive}\rangle + |\text{dead}\rangle)$. Suppose that upon opening the box, we make a measurement of alive versus dead, and obtain the outcome “alive”. Then in the conventional collapse of the wave function formalism, the state of the cat changes discontinuously into $|\text{alive}\rangle$ and the $|\text{dead}\rangle$ component ceases to have any further physical existence or further consequence — see fig. 1. This is a form of retrodiction, since it says that upon observing an alive cat we collapse to a state in which the cat was always alive, since the dead and alive branches were orthogonal at all times. In our setup, this retrodiction takes on the form of a so-called *counterfactual measurement*,\(^2\) since when outcome $E$ or $A$ is obtained, it allows one to deduce properties of a dynamical clock by reasoning counterfactually, while collapsing the wave function to a state where those eventualities never actually took place. Counterfactual measurements were first discovered by Elitzur and Vaidman in their famous bomb tester experiment [12–14], and later applied to other scenarios including counterfactual computation [15–18], and experimentally verified [19–21]. Prior examples have played out in space, while ours is the first protocol to do so in time. Another example of retrodiction and counterfactual measurement in quantum mechanics is the celebrated Hardy paradox [22]. Weakly measuring the distinct branches can provide theoretical and experimental evidence for the validity of such retrodictive interpretations [23–25].

We will call our clock when operated in this retrodictive fashion a *counterfactual clock*. Likewise, the measurement outcomes of interest in a counterfactual measurement ($E$ and $A$), are referred to as counterfactual outcomes. If the above protocol is implemented and a counterfactual outcome is obtained, we say that the clock was always off. In fig. 2 we compare one instance of the Elitzur and Vaidman bomb test with one instance of the counterfactual clock.

Our analysis thus far has been presented *forward in time*, namely starting from the initial state $|E\rangle$ and analysing the different steps of the protocol until post-selecting on a counterfactual outcome. We can also examine the *backwards in time* process where we start with one of the post-selected states, and examine how it evolves backwards in time to the initial state $|E\rangle$. Whenever analysing paradoxes involving post-selection, it is prudent to check that the paradox still exists when the problem is analysed backwards in time,

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1 We use the convention in which “outcome $x$” means that the post-measurement state is $|x\rangle$.  

2 Also known as “interaction-free measurement” or “quantum interrogation”.

since this can reveal any “hidden” assumptions inadvertently made due to post-selecting. It is important to do so in the case of counterfactual experiments, as shown in [26]. We prove in supplementary section IV A that when the counterfactual outcomes are analysed backwards in time, the pre-selected wave function is collapsed onto an always off branch. This guarantees that our counterfactual measurements have been properly implemented.

Of course, one can apply alternative interpretations of quantum mechanics. In the many-worlds interpretation of Schrödinger’s cat, upon measurement, reality splits into two parallel worlds — one in which the component |alive⟩ prevails and the cat is alive, the other, in which the component |dead⟩ prevails and the cat is dead. Conversely, the explanation of the counterfactual clock also changes if one uses this latter interpretation. Specifically, in the language of many-worlds, whenever the outcome $E$ or $A$ is obtained, we will be living in a world in which the system never dynamically evolved (i.e. never in an on state) yet in this world we learn information about the dynamical states (i.e. the on branch), namely what the time is. The fact that the system was evolving in another world is of no consequence to us.

The role of the ancilla is more subtle than may appear at first sight: indeed, one might envisage a situation with no ancilla and where we interpret outcomes $E$ and $\tau$ as counterfactual outcomes by choosing the unitary $U_m$ differently to as presented. Notwithstanding, we prove in appendix A 5, that there does not exist a counterfactual clock for which there is no ancilla and the probabilities of the two counterfactual outcomes are both non zero. This implies a more fundamental role for the ancilla state: by including it, we can reach a larger class of unitary transformations which are necessary for the counterfactual clock to function. This is

FIG. 1: Schrödinger’s cat. A1) Schrödiger’s cat starts out alive. At the 1st dotted line, the box is closed and the cat is in a superposition of dead and alive until just before the box is opened (2nd dotted line). A2) The box has now been opened, causing the cat to be measured. Here we show the case in which we see an alive cat. The history of the cat from before the box was closed to present is now set in stone, with the dead cat branch of the wave function ceasing to have ever existed.

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FIG. 2: Comparison between the Elizur-Vaidman bomb test and the counterfactual clock

A1) Mach-Zehnder interferometer with 50/50 beam splitters and a photosensitive bomb placed in the bottom arm. Photon trajectories are black lines. A photon enters at a) and splits into a superposition travelling down the upper and lower arms upon interacting with the 1st beam splitter. The photon in the bottom arm is coherently absorbed by the bomb, hence exploding. The photon in the upper arm encounters the 2nd beam splitter and coherently splits into trajectories b), c) without bomb exploding. A2) Continuing on from A1), a measurement is now performed via the photodetectors D1 and D2. There are 3 possible outcomes: If photon took path c), D1 clicks. If photon took path b), D2 clicks. If bomb exploded, photon is lost and neither photodetector clicks. Fig A2 depicts the scenario that D2 clicks (the counterfactual outcome).

Upon clicking, the wave function collapses into an eventuality where the bomb never exploded. B1) The counterfactual clock starts out off. At the 1st dotted line, the 1st event occurs and the clock is put into a superposition of on and off until just before the 2nd event occurs (2nd dotted line). B2) The 2nd event now occurs, causing the clock to be measured. Here we show the case in which we see a counterfactual outcome. The history of the clock from before the 1st event occurred to present is not set in stone, with the on clock branch of the wave function ceasing to have ever existed. Comparison between A and B: Both use a counterfactual measurement to avoid the eventualities of branch 1 of the wave function occurring when obtaining certain outcomes.
important for understanding what is the minimal model for making the appropriate comparison with classical descriptions, which we will do shortly.

In this set-up, the optimal protocol is the one which maximises the probabilities of obtaining either measurement outcomes \( E \) or \( A \) over unitaries \( U_m \) and branching amplitudes \( c, s \). In the case in which both outcomes are equally likely, \( |cA|^2 = |cA|^2 \); we prove in appendix A 5 that there exists a protocol which assigns probability 1/6 to obtaining the outcome \( E \) at time \( \tau_0 \) and probability 1/6 to outcome \( A \) at time \( \tau_1 \). For later reference, we will refer to the sum of the probabilities corresponding to the counterfactual outcomes, the total counterfactual probability. In this case, it is 1/3.

One may wonder whether a way out of this conundrum is via the existence of an underlying description respecting realism, that is, a theory where the relevant physical properties have well-defined values at all times prior to the measurement in our protocol. For example, such a theory could have hidden variables which evolve dynamically when the clock is supposedly off. We prove that there cannot exist a non-contextual ontic model for the relevant degrees of freedom for the above counterfactual clock with probability 1/6 for each counterfactual outcome. This rules out all classical models under natural assumptions; see supplementary IV B for details.

**Engineered clocks:** in this section we will discuss a more advanced type of engineered clock. This is more of an aside, and one can skip this section heading straight to the discussion in section III, if only interested in interpretation and consequences for the nature of time.

While the protocol thus far discussed allows one to use the most elementary of systems as a timekeeping device, it is limited in that it can only distinguish between times \( \tau_0, \tau_1, \ldots, \tau_N \), given the guarantee that the elapsed time between the two events coincides with one of these times. This limitation is due to an elementary clock design — indeed, this feature was present even when the clock was run in standard fashion. This limitation means that while it can be used for some applications (recall, e.g., the lighthouse example), it cannot be used for others, such as determining whether the winner of a race set a new record, because there in no reason to believe that the interval between starting the race and finishing it, will belong to any prior chosen set \( \{\tau_0, \tau_1, \ldots, \tau_N\} \). To overcome these restrictions, we demonstrate the existence of a counterfactual clock which has counterfactual outcomes with a protocol identical to the one described prior, modulo a few distinctions: 1) The elapsed time between the two events can be arbitrary. 2) The unitaries \( U_0 \) and \( U_m \) are chosen differently. 3) The projective measurement is in a different basis.

The physically significant difference in the protocol is clearly 1). Furthermore, it enjoys the analogous physical interpretation when the counterfactual outcomes are obtained to that described in the previous case above. Moreover, the Hamiltonian we use in the construction is time independent. This is important since otherwise the protocol would likely require an external timekeeping device for its implementation — rendering the entire experiment pointless. Another physically relevant feature of the engineered clock, is that when the clock is on, it cannot change instantaneously between distinct states representing the distinct ticks of the clock. This introduces a small error — regardless of whether it is run counterfactually or in standard fashion. The error is analogous to the situation we face with a wall clock: the second hand (i.e., the hand the records the passing of seconds) cannot instantaneously transition between one clock face marking and the next; therefore, if you happen to read the clock around the milliseconds interval in which the second hand is transitioning, you will likely be off by a second.

In particular, when the clock is run counterfactually, and one of the counterfactual outcomes is obtained, this error means that there is a small probability of a false positive: the quantum destructive interference is not perfect, leading to the possibility that the clock was actually on after all. Luckily, this error can be made arbitrarily small — although at the expense of a decrease in the probability of obtaining a counterfactual outcome. Notwithstanding, reasonable probabilities can be obtained. For example, in the case of one tick, and a total counterfactual probability of 1/12, the probability that the clock was on when one of the counterfactual outcomes is obtained is of order \( 10^{-14} \). See Supplementary IV C for details.

Physically speaking, this tiny error is due to imperfect destructive interference between on and off breaches. Such events are key to all quantum experiments using counterfactual measurements and due to engineering imperfections, will always be present. For example, in the quantum counterfactual experiments involving photons [19–21, 27, 28], small deviations in the reflectivity of the beam splitters lead to ports with photodetectors which were not completely dark.

### III. Discussion

With the aid of counterfactual measurements, we have shown that quantum mechanics allows one to predict the time passed between two events, even though no dynamics has taken place between the two events. The most elementary form of this clock presented in the main text should be obtainable from ubiquitous systems in nature (a four dimensional system with two stationary and two non-stationary states), while the more advanced clocks providing a larger number of distinguishable times (large \( N_T \)) require more ancillas and should be realisable with state of the art current technologies such as those developed for quantum computation.

While it is readily clear that a classical clock can-
not tell the time without dynamical evolution, it could have been that our quantum clock had hidden variables which were changing when the clock was supposedly on — we ruled out such a possibility under reasonable assumptions.

We will now discuss the implications for the relational and substantival theories of time which were presented in the introduction. As explained previously, one of the reasons why the debate has been ongoing for more than two thousand years, is due to the lack of the possibility to experimentally distinguish between the two theories due to an apparent inability to measure the flow of time without evolution. Our results can remedy this dilemma by the following experiment:

Alice has a random number generator which outputs heads or tails. If she finds heads (tails), she sends two light pulses to Bob separated by an elapsed time $\tau_0$ ($\tau_1$). Bob possesses a counterfactual clock in the energy eigenstate $|E\rangle$, which can distinguish between two elapsed times $\tau_0$ and $\tau_1$, but does not know the outcome of the random number generator. Alice tunes her two light pulses so that when they come into contact with Bob’s clock, the 1st pulse implements $U_0$ while the 2nd implements $U_m$. Bob measures his clock in the basis $|E\rangle, |\tau_0\rangle, |\tau_1\rangle, |A\rangle$ upon seeing the second light pulse. Alice repeats this experiment until Bob reports to her that he found a counterfactual outcome. There is a second system near Bob which remains in an energy eigenstate during the entire experiment and does not interact with the light pulses; Bob will use it for reference purposes only. Consider the last run of the experiment: the clock did not change between the two light pulses (relative to the second system) but was nevertheless responsible for the detection and quantification of an elapsed time which is identical to what a clock which was dynamically evolving would have predicted (had one been present). We argue that in a relative theory of time, from the perspective of the clock relative to the second system, any value for the elapsed time would have to be arbitrary, since time would not exist from its perspective. Contrarily, the experiment is perfectly consistent with a substantival theory of time, since in this case, from the clock’s perspective, there is a well-defined elapsed time between the events, as time passes independently of the clock’s state of motion.

We can formalise this observation via the following assumptions and theorem:

(A) Counterfactual measurements exist.

(B) If time can be measured via a clock which is always off, then time is substantival.

Theorem 1. Time is substantival if assumptions (A) and (B) hold.

We prove the theorem in supplementary IV D. If (A) is false, then the other phenomena which uses counterfactual measurements, such as counterfactual computing or the bomb test, would require serious re-evaluation. Similarly, if (A) holds yet (B) is false, then arguably, the very meaning of a substantival theory of time requires substantial review.

Finally, we conclude with a discussion how one might circumvent or modify our conclusions. Our results relied on the validity of the counterfactual measurement interpretation in quantum mechanics, and we discussed how in the alternative interpretation of many-worlds, when one of the counterfactual outcomes is obtained, while we would be in a world where the clock was always off, there would be another world where it would be on. Regarding the theory of time debate, this interpretation would still lead to difficulties for a relativistic theory of time, since in our thought experiment the second system remains in Bob’s world, so when he obtains the counterfactual outcome, there would have been no relative change between the clock and the second system in Bob’s world, yet Bob would know how much time had passed in his world from the measurement outcome. Another well-known alternative interpretation of quantum theory is Bohmian mechanics. Here there is always a real state associated with the wave function even when not observed. It is a contextual interpretation of quantum mechanics, and thus is not ruled out via our no-go results regarding a classical interpretation.

As alluded to in the introduction, our beliefs about the nature of time have been highly influential in the development of our physical theories. Going forward, perhaps one of the key ingredients to finding a correct theory of quantum gravity is contingent of asserting an appropriate belief about the nature of time. We hope that our work will bring much needed attention to this debate.

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AUTHOR CONTRIBUTIONS

M.W. derived the protocols and wrote the paper. S.S. implemented the numerical proof for the non existence of a noncontextual ontic model. Both authors contributed to ideas and discussions.

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IV. SUPPLEMENTARY

Here we provide more details concerning the claims made in the main text. There are three subsections. The first, section IV A, describes the two state formalism analysis in which the states evolve both forwards and backwards in time. The second, section IV B, describes the setup and how we performed our calculations which led us to conclude that our counterfactual clock has no noncontextual ontic description. The third, section IV C, provides more details about the engineered clocks. The fourth, section IV D, contains a proof of theorem 1.

A. Backwards in time analysis

Here we analyse our protocol backwards in time for the post-selected outcomes of interest. The objective is to show that the clock was off when viewed from this perspective. For simplicity, we analyse the case \( N_T = 1 \) presented in the main text where the probability of obtaining either counterfactual outcome is 1/6. The general case is relegated to appendix A 4. The physical interpretation is the same in all cases.

The amplitudes associated with obtaining the counterfactual outcomes are given by

\[
\langle E|U_{m}U(\tau_0)U_0|E\rangle \quad \text{and} \quad \langle A|U_{m}U(\tau_1)U_0|E\rangle,
\]

for times \( t = \tau_0 \) and \( t = \tau_1 \) respectively, where \( U(\tau_j) \) is the unitary dynamics of the clock (when it is on) for a time \( \tau_j \). In the main text we interpreted these processes going forwards in time, namely starting in state \( |E\rangle \) and analysing the state after applying \( U_0, U(\tau_j), U_m \) and finally post-selecting on measurement outcomes \( |E\rangle \) and \( |A\rangle \).

Here we will analyse the reverse process: starting with the post-selected state \( |E\rangle \) or \( |A\rangle \), evolving backwards in time by analysing the affect of the sequence of unitaries \( U_m, U(\tau_j), U_0 \), followed by pre-selecting onto the initial state \( |E\rangle \). This process is known as the two-state vector formalism. This formalism, and many other variants on this idea, permit time-symmetric interpretations of quantum mechanics (see review [29] for more details).

For the example under consideration, \( U_m \) is provided at the end of appendix A 3. Using it we find that the backwards evolved post-selected counterfactual outcome state just prior to dynamics take on the form:

\[
\langle E|U_{m} = \frac{1}{\sqrt{3}}(\langle E| - \langle \tau_1| - \langle A|) \quad \text{and} \quad \langle A|U_{m} = \frac{1}{\sqrt{3}}(\langle E| - \langle \tau_0| - \langle A|).
\]

We now have to evolve backwards in time for an amount \( \tau_0 \). Recall that \( U(\tau_0) \) induces the following transitions: \( U(\tau_0)|E\rangle = |E\rangle, U(\tau_0)|A\rangle = |A\rangle, U(\tau_0)|\psi\rangle = |\tau_0\rangle \). Consider the ansatz \( \langle \tau_1|U(\tau_0) = c_0 |\psi_\perp\rangle + c_2 |E\rangle + c_3 |\psi\rangle \), where \( |\psi_\perp\rangle \) is some state which is perpendicular to \( |E\rangle \) and \( |\psi\rangle \). We have \( 0 = \langle \tau_1|E\rangle = \langle \tau_1|U(\tau_0)U(\tau_0)|E\rangle = c_2 \).

Similarly we find that \( c_3 = 0 \) and thus that \( \langle \tau_1|U(\tau_0) = |\psi_\perp\rangle \). Furthermore, since \( U(\tau_1)|E\rangle = |E\rangle, U(\tau_1)|A\rangle = |A\rangle, U(\tau_1)|\psi\rangle = |\tau_1\rangle \), it follows in an analogous fashion that \( \langle \tau_0|U(\tau_1) = |\psi_\perp\rangle \), where \( |\psi_\perp\rangle \) is some state perpendicular to \( |E\rangle \) and \( |\psi\rangle \). Therefore, from eq. (4), the backwards in time dynamics yields the states

\[
\langle E|U_{m}U(\tau_0) = \frac{1}{\sqrt{3}}(\langle E| - \langle \psi_\perp| - \langle A|) \quad \text{and} \quad \langle A|U_{m}U(\tau_1) = \frac{1}{\sqrt{3}}(\langle E| - \langle \psi_\perp| - \langle A|).
\]

The unitary \( U_0 \) only acts non trivially in the subspace spanned by \( |E\rangle \) and \( |\psi\rangle \): \( U_0|E\rangle = c|E\rangle + s|\psi\rangle, U_0|\psi\rangle = -s|E\rangle + c|\psi\rangle \), and thus it acts trivially on the dynamical branches \( \langle \psi_\perp| \) and \( \langle \psi_\perp| \). We thus have

\[
\langle E|U_{m}U(\tau_0)U_0 = \frac{1}{\sqrt{3}}(c\langle E\rangle - s\langle \psi\rangle - \langle \psi_\perp| - \langle A|) \quad \text{and} \quad \langle A|U_{m}U(\tau_1)U_0 = \frac{1}{\sqrt{3}}(c\langle E\rangle - s\langle \psi\rangle - \langle \psi_\perp| - \langle A|).
\]

Finally, by post-selecting on \( |E\rangle \), and recalling that \( c = 1/\sqrt{2} \), we find that the amplitudes associated with the counterfactual outcomes is \( 1/\sqrt{6} \) in both cases, hence leading to a probability of 1/6 for each counterfactual outcome as stated in the main text. Importantly, during the entire backwards evolution, we see that the dynamical branches of the wave function has not contributed to the pre-selected state \( |E\rangle \). Another way to see this is to note that had we measured whether the clock was on or off during the evolution in either of the above pre and post selected cases, we would have found that the probability of finding the clock on was zero. Namely, we can define the binary outcome projective measurement \( P_{on} := |\psi\rangle\langle \psi| + |\tau_0\rangle\langle \tau_0| + |\tau_1\rangle\langle \tau_1| \) and \( P_{on} := |E\rangle\langle E| + |A\rangle\langle A| \), where \( P_{on} \) corresponds to a projection onto the on subspace. From [29], it follows that the probability of the clock being on at any time during the dynamical phase, that is any \( t \in (0, \tau_0) \), for the run of the experiment corresponding to post-selecting on \( |E\rangle \) is

\[
\text{Prob} (\text{on}, E) := \frac{|\langle E|U_{m}U(\tau_0 - t)P_{on}U(t)U_0|E\rangle|^2}{|\langle E|U_{m}U(\tau_0 - t)P_{on}U(t)U_0|E\rangle|^2 + |\langle E|U_{m}U(\tau_0 - t)P_{on}U(t)U_0|E\rangle|^2}.
\]
Note that since the ancilla and energy eigenstate are stationary, we have that $U(t) = U_{\perp E, A}(t) + \mathbb{1}_{E, A}$, where $\mathbb{1}_{E, A}$ is the identity operator on the subspace spanned by $|E\rangle$ and $|A\rangle$, and that $U_{\perp E, A}(t)$ is orthogonal to said subspace. Therefore, if we assume that the dynamics is Markovian, namely $U(t_1)U(t_2) = U(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}$, it follows that $U(\tau_0 - t)P_{on}U(t) = U_{\perp E, A}(\tau_0)$ and hence that
\[
\text{Prob}(\text{on}, E) = 0
\]
for all $t \in (0, \tau_0)$. Similarity, it follows that the probability of the clock being on at any time during the dynamical phase, that is any $t \in (0, \tau_1)$, for the run of the experiment corresponding to post-selecting on $|A\rangle$ is
\[
\text{Prob}(\text{on}, A) := \frac{|\langle A|U_mU(\tau_1 - t)P_{on}U(t)U_0|E\rangle|^2}{|\langle A|U_mU(\tau_1 - t)P_{on}U(t)U_0|E\rangle|^2 + |\langle A|U_mU(\tau_1 - t)P_{on}U(t)U_0|E\rangle|^2} = 0.
\]

**B. No classical analogue**

The interpretation of measuring time when the clock was off using counterfactual reasoning, relied on a basic concept in quantum mechanics, namely the superposition principle, in which if a system is in a superposition of two states, it cannot be regarded as being in either until measured. As we discussed, Schrödinger famously popularised this point with his thought experiment concerning a cat. However, what if underlying our counterfactual clock protocol, there was a "real" description; that is to say if one could associate the states in our clock protocols with non-contextual measurements? While such a result would not invalidate the current interpretation provided, it would, at least in-principle, provide for an alternative interpretation in which the clock might have been dynamically evolving even when a measurement collapsing the clock to the off branch is obtained. We will now consider such a possibility for the simplest elementary timekeeping systems.

To start with, lets consider the case where we only run the clock when it is on; that is, the clock is never in a superposition of on and off. Then the question is, can we find an ontic state space on which there exists appropriate probability distributions that represent our states $\rho$ and also provide for an alternative interpretation in which the clock is prepared and evolved in a meaningful way? While such a result would not invalidate the current interpretation provided, it would, at least in-principle, provide for an alternative interpretation in which the clock might have been dynamically evolving even when a measurement collapsing the clock to the off branch is obtained. We will now consider such a possibility for the simplest elementary timekeeping systems.

Furthermore, we require the usual convexity relationships of mixtures of quantum states and measurements hold at the ontic level: for all $\lambda \in \Lambda$, for all $p \in [0, 1]$, for all $\rho_1, \rho_2 \in s_{on}$: $P[\lambda|p\rho_1 + (1-p)\rho_2] = pP[\lambda|\rho_1] + (1-p)P[\lambda|\rho_2]$ and similarly, that for all $\lambda \in \Lambda$, for all $p \in [0, 1]$, for all $E_1, E_2 \in e_{on}$: $P[pE_1 + (1-p)E_2|\lambda] = pP[E_1|\lambda] + (1-p)P[E_2|\lambda]$. Finally, we require that the ontic states can reproduce the measurement statistics of our protocol: for all $\rho \in s_{on}$, for all $E \in e_{on}$,
\[
\text{tr}[\rho E] = \int_{\Lambda} d\lambda P[\lambda|\rho] P[E|\lambda],
\]
where $\text{tr}[:]$ denotes the trace, and the r.h.s. is the probability of obtaining outcome associated with $E$ for quantum state $\rho$, according to quantum mechanics.

Notice how we have assumed that the probability distributions $P[\lambda|\rho]$ do not depend on how the quantum state $\rho$ was prepared nor do $P[E|\lambda]$ depend on how the measurements were implemented. Different preparation procedures which lead to the same quantum state and different measurement procedures which lead to the same POVM elements, are called different contexts. We are thus looking for a non-contextual ontic variable model. This notion of contextuality was pioneered by Spekkens [30]. Prior notions of classicality based on local realism [31] or its generalisation to non-contextual realism [32], cannot be tested in our case since we do not dispose of a local structure, nor relevant commuting measurements in our clock protocols. However, in keeping with this philosophy of preparation and measurement non-contextuality, our current formulation of an ontic variable model requires some refinement: since the set $e_{on}$ is not tomographically complete, nor the set $s_{on}$ span all quantum states in the Hilbert space, neither all the d.o.f. of the density matrices nor those in the POVM elements, contribute to the measurement statistics. Indeed, $\text{tr}[\rho E] = \text{tr}[P_\mathcal{R}(\rho)P_\mathcal{R}(E)]$ for all $\rho \in s_{on}$, $E \in e_{on}$, where $P_\mathcal{R}$ is a projection onto the vector space $\mathcal{R}$, generated by projecting the span of $s_{on}$ onto the span of $e_{on}$. We should thus make
the replacements \( P[λ] \cdot \] \( \mapsto P[λ|P_R(\cdot)] \) and \( P[\cdot | λ] \mapsto P[|P_R(\cdot)|λ] \) in the above theory. This completes the description of our would-be non-contextual ontic variable model. Finally, if a classical description exists, one would also need a stochastic map describing the dynamics of the ontic variables throughout the protocols. This imposes an additional constraint on our would-be ontic description, which we will not need to consider.

In this case, it can readily be seen that such a model does exist; for example, one may simply choose the vectors \( \{|E\}, |\tau_0\rangle, |\tau_1\rangle, |A\rangle \) as a basis for the ontic state space \( \Lambda \) with each step and measurement allocated to deterministic distributions \( λ \) on it.

We now turn to the case in which the clock can be also be used to tell the time when off via counterfactual measurements. We need to supplement the sets \( s_{\text{on}}, e_{\text{on}} \) with the other relevant elements which are now required, namely, for states,

\[
s_{\text{cf}} = \text{conv}(s_{\text{on}} \cup \{|cf_0⟩\langle cf_0|, |cf_1⟩\langle cf_1|, U_m|cf_0⟩\langle cf_0|U_m^†, U_m|cf_1⟩\langle cf_1|U_m^†\}),
\]

(11)

where \( |cf_0⟩ := c |E⟩ + s |τ_0⟩ \), \( |cf_1⟩ := c |E⟩ + s |τ_1⟩ \), since these are the states which appear in our protocol. When it comes to the measurements, in addition to those required in the final measurement, namely \( e_{\text{on}} \), we want to include measurements corresponding to ontic degrees of freedom which are able to describe the paradoxical aspects of our protocol. Before applying \( U_m \), at times \( τ_0, τ_1 \), when we measure in the measurement basis, the measurement determined which branch (off or on) we collapsed to, but we can only deduce the time when we happen to collapse onto an off branch. This situation is analogous to a statistical mixture over on and off. What is surprising, is that after \( U_m \) is applied, we can deduce not only which branch we were on, but also the time when we collapse to an off branch. So our would-be ontic model should have a variable which determines whether we are in the off branch, or the on branch at the times \( τ_0, τ_1 \). In other words, a variable which, after the application of \( U_m \), plays the same role that the measurement basis played before the application of \( U_m \). Including this in the set of things which are measurable in our would-be non-contextual theory, gives us

\[
e_{\text{cf}} = \text{conv}(e_{\text{on}} \cup \{U_m^†|E⟩⟨E|U_m, U_m^†|τ_0⟩⟨τ_0|U_m, U_m^†|τ_1⟩⟨τ_1|U_m, U_m^†|A⟩⟨A|U_m\}).
\]

(12)

An alternative motivation for the inclusion of the additional ontic degrees of freedom, is that, quantum mechanically, the unitary channel generated by \( U_m \) could have been applied to rotate the measurement basis, rather than being applied to the state. This alternative protocol, is equivalent to the one we study here, from the perspective of quantum mechanics. We can thus think of these two alternative implementations of our protocol as different contexts which are indistinguishable according to the laws of quantum mechanics, in an analogous way to how would-be ontic variable model is preparation and measurement non-contextual by design.

In [33] an algorithm was developed which, given a set \( s \) of states and set \( e \) of POVM elements, can determine whether an ontic variable model as described here exists. When said sets have finitely many extremal points, as in the case for \( s_{\text{cf}}, e_{\text{cf}} \), it provably runs in finite time. For the simple case where the probability of each counterfactual outcome is 1/6 as described in the main text, the unitary \( U_m \) is given by eq. (A7a) and \( c, s \) by eq. (A7b). For this algorithm, we used the computer architecture assisted proof that the sets \( s_{\text{cf}}, e_{\text{cf}} \) — and hence our counterfactual clock — do not admit a non-contextual ontic variable model as per the above description; see appendix B for details.

A crucial aspect of our protocol for using the clock to tell the time when off, via counterfactual measurements, was the existence of negative amplitudes allowing for destructive interference between the on and off branches. Since probabilities are nonnegative, one might have thought that this aspect of our protocol automatically rules out any non contextual realistic theory. However, this is definitely not the case since other experiments using counterfactual measurements, such as the Elitzur-Vaidman bomb test [12], have been shown to admit a non contextual ontic model description analogous to the one ruled out here for our setup; see [34].

C. Engineered clocks

Here we give a more detailed account of the engineered clocks outline in the main text. As before, we will discuss the simplest case here of just one tick while relegating the full details of the construction and multiple tick scenario, which is qualitatively the same, to the appendix (appendix C). As with the elementary clock, we have a stationary state, \( |ψ_{\text{on}}⟩ = e^{-iHt} |ψ_{\text{off}}⟩ \), and a dynamical one, \( |ψ_{\text{on}}(t)⟩ = e^{-iHt} |ψ_{\text{on}}⟩ \), which are mutually orthogonal at all times: \( ⟨ψ_{\text{off}}|ψ_{\text{on}}(t)⟩ = 0 \). We can run the clock in standard fashion by applying a unitary to \( |ψ_{\text{off}}⟩ \) which maps it to \( |ψ_{\text{on}}⟩ \) when the 1st event occurs, followed by measurement via an appropriate projection-valued measure when the 2nd event occurs. We use the convention that when the clock is on, it ticks at time \( t_1 > 0 \), and that the elapsed time between the 1st and 2nd events is at most \( 2t_1 \). Parameter \( t_1 \) can be chosen to be any value in our construction. Unlike with the elementary clock, we now have that the state \( |ψ_{\text{on}}(t)⟩ \) before the tick
takes place at time $t_1$, will not be exactly orthogonal to the state after the tick occurs. Therefore, in order to unambiguously predict whether the clock has ticked, one must perform an unambiguous quantum discrimination measurement [35]. This is a projective measurement with three possible outcomes: clock has not ticked yet, it has ticked, or I do not know. The last outcome can be thought of as an error, since when it is obtained we cannot say what time it is. This setting allows for more flexibility than in the prior clock setups. In the counterfactual case, this aspect will not be detrimental to its functioning, since, as with previous cases, the counterfactual outcomes will only occur with some probability.

We will now explain how to run this clock in a counterfactual manner. The protocol proceeds similarly to in prior cases: we start the clock in $|\psi_{\text{off}}\rangle$ and apply a unitary which maps it to a suitable superposition of off and on when the 1st event occurs, namely to $c|\psi_{\text{off}}\rangle + s|\psi_{\text{on}}\rangle$. When the 2nd event occurs at some unknown time $t \in [0,2t_1)$, we apply a unitary $U_m$ and measure using the projection-valued measure $|\psi_{\text{off}}\rangle\langle\psi_{\text{off}}|$, $|A\rangle\langle A|$, $\mathbb{1} - |\psi_{\text{off}}\rangle\langle\psi_{\text{off}}|$ $|A\rangle\langle A|$, where $|A\rangle$ is stationary under the Hamiltonian evolution. After applying $U_m$ to the on and off clock states at time $t$, the states take on the form

$$U_m |\psi_{\text{off}}\rangle = \frac{A_1}{N} |\psi_{\text{off}}\rangle + \frac{A_1}{N} |A\rangle + A_2 |A_{\text{off}}\rangle,$$

$$U_m |\psi_{\text{on}}(t)\rangle = -\frac{c}{s} A_1 |\psi_{\text{off}}(t)\rangle - \frac{c}{s} A_1 |A(t)\rangle + A_3 |A_{\text{on}}(t)\rangle,$$

where $A_2 = \sqrt{1 - 2(A_1/N)^2}$, $A_3 = \sqrt{1 - 2(c^2/N^2)}$ are normalisation parameters and the other parameters will be discussed shortly. All kets in the superpositions are orthonormal except for the overlaps $\langle \psi_{\text{off}} | \psi_{\text{off}}(t) \rangle$, $\langle A | \bar{A}(t) \rangle$ and $\langle A_{\text{on}} | A_{\text{on}}(t) \rangle$. The kets $|A_{\text{off}}\rangle$, $|A_{\text{on}}(t)\rangle$ play the role of ancilla states that are chosen to guarantee $\langle \psi_{\text{off}} | \psi_{\text{on}}(t) \rangle = 0$ holds at all times. The overlaps $\langle \psi_{\text{off}} | \psi_{\text{off}}(t) \rangle$, $\langle A | \bar{A}(t) \rangle$ are chosen such that the counterfactual clock functions properly: consider the state of the clock just before the measurement

$$U_m e^{-it\hat{H}} (c|\psi_{\text{off}}\rangle + s|\psi_{\text{on}}\rangle) = cU_m |\psi_{\text{off}}\rangle + sU_m |\psi_{\text{on}}(t)\rangle,$$

there the latter quantities are provided by eqs. (13a) and (13b). If the 2nd event occurs in interval $t \in [0, t_1)$, we require

$$\langle \psi_{\text{off}} | \bar{\psi}_{\text{off}}(t) \rangle = \begin{cases} 0 & \text{for } t \in [0, t_1) \\ 1/N & \text{for } t \in [t_1, 2t_1) \end{cases},$$

in order to be sure that the clock was off and $t \in [0, t_1)$, when the measurement outcome $\psi_{\text{off}}$ is obtained (This can readily be seen from eqs. (13a), (13b), (14) and (15a) and the same counterfactual measurement reasoning presented in the analysis of the elementary timekeeping systems). Similarly, if the 2nd event occurs in time interval $t \in [t_1, 2t_1)$, we require

$$\langle A | \bar{A}(t) \rangle = \begin{cases} 1/N & \text{for } t \in [0, t_1) \\ 0 & \text{for } t \in [t_1, 2t_1) \end{cases},$$

in order to be sure that the clock was off and $t \in [t_1, 2t_1)$, when the outcome $A$ is obtained, according to a counterfactual measurement. The role of $N$ now becomes apparent: it quantifies the overlap between the projectors associated with the counterfactual outcomes and the dynamical kets $|\psi_{\text{off}}(t)\rangle$ and $|A(t)\rangle$.

It readily follows that the probability of knowing whether the 2nd event occurred in time interval $[0, t_1)$ or $[t_1, T_0)$ when the clock was off, that is to say, the probability of obtaining measurement outcomes $\psi_{\text{off}}$ or $A$ in our protocol, is $P_{t_1} = 2c^2 A_1^2 / N^2$. One would want to choose $A_1$, $s$ and $N$ such as to maximise this probability while satisfying all the constraints. It happens that one can find states $|\psi_{\text{off}}\rangle$, $|\psi_{\text{on}}\rangle$, and a Hamiltonian $\hat{H}$, such that all the constraints can be satisfied other than eqs. (15a) and (15b), which appears to lead to unnormalisable states $|\psi_{\text{off}}\rangle$, $|A\rangle$. From a physical perspective, the problem appears to be related to requiring the overlaps to transition from zero to a finite constant instantaneously while having a time evolution governed by a time independent Hamiltonian. However, a minor modification resolves the conflict: One can replace the r.h.s. of eqs. (15a) and (15b) with approximate version using the error function erf. Specifically, $\langle \psi_{\text{off}} | \psi_{\text{off}}(t) \rangle = f_0(t)$, $\langle A | \bar{A}(t) \rangle = f_1(t)$, with

$$f_i(t) = \frac{1}{2N} \left[ \text{erf} \left( \frac{t/t_1 + l - 1}{\sqrt{2}\sigma} \right) - \text{erf} \left( \frac{t/t_1 + l - 2}{\sqrt{2}\sigma} \right) \right],$$

and where $\sigma > 0$ controls the approximation. In the limit $\sigma \to 0$ the above overlaps are equal to eqs. (15a) and (15b). We can now find solutions for all $\sigma > 0$. We plot eq. (16) in fig. 3 for varying approximation values $\sigma > 0$. We can quantify the error by the difference in probabilities associated with the counterfactual outcomes,
FIG. 3: Plot of the overlap $\langle x_0 | x_o(t) \rangle$ in eq. (16) for different values of $\sigma$ (note that $\langle x_1 | x_1(t) \rangle$ is identical up to a shift to the left by an amount $t_1$). Observe how the difference between the ideal case (black) and that of $\sigma = 0.01$ (red) are practically indistinguishable. The small deviation between the black and colored plots induces a small error. Note how the error is centred around $t_1$, which is the time at which the clock would tick if it were on, and $2t_1$, which is the time interval corresponding to the 1st tick would end if the clock were on.

between the actual clock state which is measured, and an “idealised” clock state: $\text{Dif}_p(\sigma, t) := \left| \langle x_p | (c | \psi_{\text{off}} + s | \psi_{\text{on}}(t)) \rangle \right|^2 - \left| \langle x_p | (c | \psi_{\text{off}} + s | \psi_{\text{on}}(t)) \rangle \right|^2$, where we have used the short-hand $x_0 = \psi_{\text{off}}$ for $p = 0$, $x_1 = A$ for $p = 1$. Here the “idealised on state”, namely $| \psi_{\text{on}}(t) \rangle$, is identical to $| \psi_{\text{on}}(t) \rangle$ given by eq. (13b), except that the kets $| \psi_{\text{off}}(t) \rangle, | A(t) \rangle$, satisfy eqs. (15a) and (15b) rather than eq. (16). Importantly, the quantification of the error is meaningful, since if one were to use this idealised state in our protocol, it would result in zero false predictions: it would always predict the correct time and the clock would have always been off when a counterfactual outcome had been obtained. Additionally, since the time $t$ at which one measures the clock could be any time in $[0, 2t_1]$, we time-average $\text{Dif}_p(\sigma, t)$ over the time interval $[p t_1, (p + 1) t_1]$ in which the outcome $x_p$ should have occurred, resulting in what we call the type-1 error (This error is thus due to the clock predicting the correct time, but being on); and we time-average over the remaining time, in which the outcome $x_p$ should not have occurred, resulting in what we call the type-2 error (This error is thus due to the clock predicting the incorrect time). In the case considered here, in which the clock ticks once when turned on, one only needs to consider the type-1 and type-2 errors for the $p = 0$ case, since the error types are identical for the $p = 1$ case. Hence we will forgo the label $p$ in the following discussion.

We show in the appendix C3 that the difference in fidelities can be made arbitrarily small for both error types, while always having a non zero probability of obtaining the counterfactual outcome. In particular, if we choose $\sigma = 0.019$, we can obtain a total counterfactual probability of half what it was in the previous optimal case, namely to $1/6$, and only incur type one and two errors of $1.6 \times 10^{-3}$ and $3.7 \times 10^{-4}$ respectively. If we further reduce $\sigma$ to $\sigma = 0.0012$, the total counterfactual probability only drops by half again, to $1/12$, and the type one and two error probabilities are now merely $7.3 \times 10^{-4}$ and $2.8 \times 10^{-4}$ respectively.

D. Proof of Theorem 1

Here we prove the theorem from the main text:

**Theorem 1.** Time is substantival if assumptions (A) and (B) hold.

Note that while we will only need to consider the counterfactual clock explained in the main text (i.e. the $N_T = 1$ case), the proof is analogous in the case one uses the clocks generalised for $N_T \geq 2$ found in appendix A. In the latter case, the definitions of a counterfactual measurement and of an always off clock which are found in the main text and used in assumptions (A) and (B), would be substituted for the more general definitions found in the appendix; i.e. definitions 1 and 2.

**Proof.** Assumption (A) together with the example of a counterfactual clock for $N_T = 1$ provided in the main text (the values for $U_m$ and $c, s$ which fully determine the protocol are provided at the end of A3 in the appendix), prove that for any two elapsed times $\tau_0, \tau_1$ between two events, a counterfactual clock exists and can determine said elapsed time with the clock always off. Therefore, the theorem follows by invoking assumption (B).
Appendices

Appendix A: Elementary time keeping devices

In subsections A 1 and A 2, we detail the protocol one implements when using the clock to tell the time counterfactually. In subsection A 3, we characterise the measurement basis used at the end of the protocol from subsections A 1 and A 2. In subsection A 4 we prove some consequences of the protocols and definitions made in the previous subsections. Finally in subsection A 5, we provide two structure theorems which prove that ancillas are necessary and that counterfactual clocks exist for arbitrary $N_T \in \mathbb{N}_{>0}$.

1. Setup and protocol

In this subsection, we explain the full protocol which allows one to determine the time $t \in \{\tau_0, \tau_1, \tau_2, \ldots, \tau_{N_T}\}$ using counterfactual measurements. For the purpose of performing general projective measurements, we append an $m$-dimensional ancilla space $\mathcal{H}_A$ to the Hilbert $\mathcal{H}_S$ of the system, leading to a total Hilbert space $\mathcal{H}_S \oplus \mathcal{H}_A$. We will require the number of ancillas to be greater or equal to the number of distinguishable time intervals, namely $m \geq N_T$. The ancilla states are always stationary, namely they do not change in time. When the clock is tuned on, it is initialised to another state denoted $|\psi\rangle$. Thus in total, we have the orthonormal basis states $\{|\psi\rangle, |E\rangle, |\tau_0\rangle, |\tau_1\rangle, \ldots, |\tau_{N_T}\rangle, |A_1\rangle, |A_2\rangle, \ldots, |A_m\rangle\}$, which we call the measurement basis (Since the clock never has support on the basis state $|\psi\rangle$ when measured, it will often be omitted for simplicity). Observe that the $m$ ancillas could be other energy eigenstates of the system which are orthogonal to the states $|\tau_0\rangle, |\tau_1\rangle, \ldots, |\tau_{N_T}\rangle, |E\rangle$, or they could be produced via a separate $m + 1$ dimensional system via the identification $|\tau_0\rangle \equiv |\tau_0\rangle \otimes |d_0\rangle, |\tau_1\rangle \equiv |\tau_1\rangle \otimes |d_0\rangle, \ldots, |\tau_{N_T}\rangle \equiv |\tau_{N_T}\rangle \otimes |d_0\rangle$, and $|E\rangle \equiv |E\rangle \otimes |d_0\rangle$, for original states of the system and $|A_1\rangle \equiv |E\rangle \otimes |d_1\rangle, |A_2\rangle \equiv |E\rangle \otimes |d_2\rangle, \ldots, |A_m\rangle \equiv |E\rangle \otimes |d_m\rangle$, for the ancillary states, where $\{|d_0\rangle, |d_1\rangle, \ldots, |d_m\rangle\}$, are orthonormal states of the separate $m + 1$ dimensional system.

Initially we set the system to be in the energy eigenstate, $|E\rangle$. Then, when the 1st event occurs, a unitary is applied to take the state to a superposition of $|E\rangle$ and $|\psi\rangle$. Using the branching notation from the main text to distinguish between the two orthogonal states, this has the form:

$$
\begin{array}{ccc}
\bullet & c |E\rangle \\
|E\rangle & \xrightarrow{c} & c |E\rangle \\
\bullet & s |\psi\rangle \\
\end{array}
$$

(A1)

where $c := \cos \theta$ and $s := \sin \theta$. We then wait until the 2nd event occurs at unknown time $\tau_1$. The system now is of the form

$$
\begin{array}{ccc}
\bullet & c |E\rangle \xrightarrow{\tau_1} c |E\rangle \\
|E\rangle & \xrightarrow{c} & c |E\rangle \\
\bullet & s |\tau_1\rangle \\
\end{array}
$$

(A2)

where $\tau_1 \in \{\tau_0, \tau_1, \ldots, \tau_{N_T}\}$. We now perform a unitary $U_m$ over the system. The quantum states have support on the ancillas after the application of the unitary. The system now takes on the form:

$$
\begin{array}{ccc}
\bullet & c |E\rangle \xrightarrow{\tau_1} c |E\rangle & \xrightarrow{U_m} & c A_0^0 |E\rangle + \sum_{j=1}^{N_T} c A_1^0 |A_j\rangle + \sum_{k=N_T+1}^{m} c B_{-1,k} |A_k\rangle + \sum_{k=1}^{N_T+1} c B_{-1,k} |\tau_{k-1}\rangle \\
|E\rangle & \xrightarrow{c} & c |E\rangle \\
\bullet & s |\tau_1\rangle & \xrightarrow{U_m} & s A_0^1(\tau_1) |E\rangle + \sum_{j=1}^{N_T} s A_1^1(\tau_1) |A_j\rangle + \sum_{k=N_T+1}^{m} s B_{1,k} |A_k\rangle + \sum_{k=1}^{N_T+1} s B_{1,k} |\tau_{k-1}\rangle \\
\end{array}
$$

(A3)

where the amplitudes $A_0^0, A_1^1(\tau_1)$ are to be characterised in appendix A 3. To finalise the protocol, we immediately measure in the measurement basis after $U_m$ is applied. Here we will use the convention that the counterfactual outcomes are associated with the post-measurement states $\{|E\rangle, |A_1\rangle, \ldots, |A_{N_T}\rangle\}$. 

2. Counterfactual measurement (Formal definition)

Given the presentation of the general protocol in appendix A 1, we can now formally define the counterfactual measurement which is appropriate for our setting. These form a generalisation of the main text to the \( N_T > 1 \) case. In the following, we label the energy \( E \) by \( A_0 := E \) for simplicity of notation (since the symbol \( A_0 \) has not been used previously, this should not lead to confusion).

**Definition 1** (Counterfactual measurement for clocks for \( N_T \in \mathbb{N}_{>0} \)). We call a projective measurement a counterfactual measurement if it has outcomes \( A_0, A_1, A_2, \ldots, A_{N_T} \) (e.g. post-measurement states \( |A_0\rangle, |A_1\rangle, |A_2\rangle, \ldots, |A_{N_T}\rangle \) respectively) of which at least one of these outcomes is a counterfactual outcome. Furthermore, we say that the outcome \( A_k \) is a counterfactual outcome if the following two equations are both satisfied.

\[
A_k^1(\tau_k) = 0, \quad (A4)
\]

and

\[
A_k^1(\tau_l) = -\frac{c}{s} A_k^0
\]

for all \( l = 0, 1, 2, \ldots, k-1, k+1, \ldots, N_T \) and where \( k \in \{0, 1, 2, \ldots, N_T\} \).

**Definition 2** (Clock always off for \( N_T \in \mathbb{N}_{>0} \)). When a counterfactual outcome \( A_k \) is obtained, we say that the elapsed time between the two events was \( \tau_k \), and that the clock was always off.

The reasoning behind these definitions is based on the usual arguments of counterfactual measurements: eq. (A4) allows us to conclude that if one obtains measurement outcome \( A_k \) and \( t = \tau_k \), the clock must have been always off, since the amplitude in the on branch associate with the ket \( |A_k\rangle \) is zero during said time interval (see eq. (A3)). Meanwhile, eq. (A5) guarantees that if \( \tau_k \) is obtained, then \( t = \tau_k \), since the amplitude \( A_k^1(\tau_l) \) cancels out with the amplitude \( A_k^0 \) for all times \( t \neq \tau_k \) (see eq. (A3)). In appendix A 5 we show that these definitions are not vacuous, specifically, that clocks exist for any \( N_T \in \mathbb{N}_{>0} \) with the probabilities of obtaining their counterfactual outcomes all non-zero. In appendix A 4 we will see that these definitions have other interpretational implications.

3. Characterisation of unitary \( U_m \).

We consider the case in which we associate all of the measurement outcomes \( E, \tau_1, \tau_2, \ldots, \tau_{N_T} \) with the elapsed time being \( \tau_0, \tau_1, \tau_2, \ldots, \tau_{N_T} \) respectively, and the clock being off (this is to say, we associate \( E, \tau_1, \tau_2, \ldots, \tau_{N_T} \) with counterfactual outcomes). The constraints imposed by eqs. (A4) and (A5) constrain the matrix coefficients of the unitary matrix \( U_m \). Specifically, by writing them out explicitly, we have that \( U_m \) is of the form \( U_m = U_{ex} U_m^\dagger \) where \( U_{ex} \) implements a simple change of basis, defined via the relations: \( U_{ex} |E\rangle = |E\rangle, U_{ex} |\tau_0\rangle = |A_1\rangle, U_{ex} |\tau_1\rangle = |A_2\rangle, \ldots, U_{ex} |\tau_{N_T-1}\rangle = |A_{N_T}\rangle, U_{ex} |A_1\rangle = |\tau_0\rangle, U_{ex} |A_2\rangle = |\tau_1\rangle, \ldots, U_{ex} |A_{N_T}\rangle = |\tau_{N_T-1}\rangle \), and \( U_{ex} |\tau_{N_T}\rangle = |\tau_{N_T}\rangle \), \( U_{ex} |A_{N_T+1}\rangle = |A_{N_T+1}\rangle \), \( U_{ex} |A_{N_T+2}\rangle = |A_{N_T+2}\rangle \), \ldots, \( U_{ex} |A_m\rangle = |A_m\rangle \). The other matrix \( U_m \) has the following matrix representation:

\[
\begin{array}{cccccccc}
|E\rangle & |\tau_0\rangle & |\tau_1\rangle & |\tau_2\rangle & \cdots & |\tau_{N_T-1}\rangle & |\tau_{N_T}\rangle & |A_1\rangle & \cdots & |A_m\rangle \\
|E\rangle & A_0^0 & A_1^0 & A_2^0 & \cdots & A_{N_T}^0 & \tau_0 & B_{-1,1} & \cdots & B_{-1,m} \\
|\tau_0\rangle & 0 & -A_0^0 r & A_1^0 r & A_2^0 r & \cdots & -A_{N_T}^0 r & \gamma_1 & B_{1,1} & \cdots & B_{1,m} \\
|\tau_1\rangle & -A_0^0 r & 0 & -A_1^0 r & A_2^0 r & \cdots & -A_{N_T}^0 r & \gamma_2 & B_{2,1} & \cdots & B_{2,m} \\
|\tau_2\rangle & -A_0^0 r & -A_1^0 r & 0 & -A_2^0 r & \cdots & -A_{N_T}^0 r & \gamma_3 & B_{3,1} & \cdots & B_{3,m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
|\tau_{N_T-1}\rangle & -A_0^0 r & -A_1^0 r & -A_2^0 r & \cdots & -A_{N_T}^0 r & 0 & \gamma_{N_T} & B_{N_T,1} & \cdots & B_{N_T,m} \\
|\tau_{N_T}\rangle & -A_0^0 r & -A_1^0 r & -A_2^0 r & \cdots & -A_{N_T}^0 r & 0 & \gamma_{N_T+1} & B_{N_T+1,1} & \cdots & B_{N_T+1,m} \\
|A_1\rangle & B_{N_T+2,0} & B_{N_T+2,1} & B_{N_T+2,2} & B_{N_T+2,3} & \cdots & B_{N_T+2,N_T} & B_{N_T+2,N_T+1} & \cdots & B_{N_T+2,N_T+m+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
|A_m\rangle & B_{N_T+m+1,0} & B_{N_T+m+1,1} & B_{N_T+m+1,2} & B_{N_T+m+1,3} & \cdots & B_{N_T+m+1,N_T} & B_{N_T+m+1,N_T+1} & \cdots & B_{N_T+m+1,N_T+m+1} \\
\end{array}
\]

(A6)

where we have denoted \( r := c/s = \cos(\theta)/\sin(\theta) \), and all matrix entries are arbitrary complex numbers such that \( U_m^\dagger U_m = I \). In writing matrix eq. (A6), we have used the convention that kets are row vectors and bras are column vectors. We will used this matrix representation convention throughout. The horizontal solid inner line and the vertical dotted inner line are visual aids only.

The probability of finding the register off and the time to be \( \tau_l \) upon measurement is \( P_{cf}^{(l)} = |c A_l^0|^2 \), \( l \in \mathbb{N}_{\geq 0} \).
The example in the main text where there are just two distinguishable times (i.e., \( N_T = 1 \)) and requires one ancilla state corresponds to the unitary matrix \( U_m = U_{ex} U_m' \), with \( U_{ex} \) determined by \( U_{ex} | E \rangle = | E \rangle \), \( U_{ex} | \tau_0 \rangle = | A \rangle \), \( U_{ex} | A \rangle = | \tau_0 \rangle \), \( U_{ex} | \tau_1 \rangle = | \tau_1 \rangle \). The unitary \( U_m' \) is given by

\[
\begin{array}{c|cccc}
| & |E\rangle & |\tau_0\rangle & |\tau_1\rangle & |A\rangle \\
\hline
|E\rangle & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & 0 \\
|\tau_0\rangle & 0 & -\sqrt{1/3} & \sqrt{1/3} & -\sqrt{1/3} \\
|\tau_1\rangle & -\sqrt{1/3} & 0 & \sqrt{1/3} & \sqrt{1/3} \\
|A\rangle & \sqrt{1/3} & -\sqrt{1/3} & 0 & \sqrt{1/3} \\
\end{array}
\]  

(A7a)

and coefficients \( c \) and \( s \) by

\[
c = s = 1/\sqrt{2}.
\]  

(A7b)

### 4. Backwards in time analysis in the \( N_T \geq 2 \) scenario

Here we extend the analysis presented in section IV A to the general case of multiple distinguishable times (\( N_T \in \mathbb{N}_{>0} \)). Since the results are quantitatively the same as in section IV A, we will not discuss at length the interpretation to avoid repetition. The amplitude corresponding to the \( k \)th counterfactual outcome is

\[
\langle A_k|U_m U(\tau_k)U_0|A_0 \rangle,
\]  

(A8)

for \( k = 0, 1, 2, \ldots, N_T \) and where we are again using the convention \( A_0 := E \). Using \( U_m = U_{ex} U_m' \) and eq. (A6), we have

\[
\langle A_k|U_m = A_k^0 \langle A_0| - A_k^0 r \langle \tau_0| - A_k^0 \langle \tau_1| - \ldots - A_k^0 r \langle \tau_{k-1}| - A_k^0 r \langle \tau_{k+1}| - A_k^0 r \langle \tau_{k+2}| - \ldots - A_k^0 r \langle \tau_{N_T}| \\
+ B_{N_T+2,k} \langle A_1| + B_{N_T+2,k} \langle A_2| + B_{N_T+3,k} \langle A_3| + \ldots + B_{N_T+m,k} \langle A_m|
\]  

(A9)

where \( \langle \tau_{-1} := 0 \). Recall that \( U(\tau_k)|A_0 \rangle = |A_0 \rangle \), \( U(\tau_k)|A_1 \rangle = |A_1 \rangle \), \( U(\tau_k)|A_2 \rangle = |A_2 \rangle \), \ldots, \( U(\tau_k)|A_m \rangle = |A_m \rangle \), and \( U(\tau_k)|\psi \rangle = |\tau_k \rangle \). Therefore, analogously to section IV A, it follows that \( \langle \tau_{l}|U(\tau_k)|\psi \rangle = \langle \psi_{l,k} \rangle \) for all \( k, l = 0, 1, \ldots, N_T \) with \( k \neq l \), where \{\( \langle \psi_{l,k} \rangle \}_{l,k} \) are orthogonal to \( |A_0 \rangle \) and \( |\psi \rangle \). This follows from writing the ansatz \( \langle \tau_{l}|U(\tau_k) = c_0^{(l,k)} \langle \psi_{l,k} \rangle \rangle + c_2^{(l,k)} \langle A_0| + c_3^{(l,k)} \langle \psi \rangle \rangle \), followed by observing the two equalities \( \delta_{k,l} = \langle \tau_{l}|U(\tau_k)|\psi \rangle = c_0^{(l,k)} \langle \psi_{l,k} \rangle \) and \( c_0^{(l,k)} = c_0^{(l,k)} \langle \psi_{l,k} \rangle \rangle + c_2^{(l,k)} \langle A_0| + c_3^{(l,k)} \langle \psi \rangle \rangle = c_0^{(l,k)} \), where \( \delta_{k,l} \) is the Kronecker delta, and

\[
0 = \langle \tau_l|U(\tau_k)|A_0 \rangle = \langle \tau_l|U(\tau_k)\langle A_0 = \langle \psi_{l,k} \rangle \rangle + c_2^{(l,k)} \langle A_0| + c_3^{(l,k)} \langle \psi \rangle \rangle \rangle \langle A_0 = c_0^{(l,k)} \].
\]  

Thus from eq. (A9) we arrive at

\[
\langle A_k|U_m U(\tau_k) = A_k^0 \langle A_0| - A_k^0 r \langle \psi_{0,k} \rangle \rangle - A_k^0 r \langle \psi_{1,k} \rangle \rangle - \ldots - A_k^0 r \langle \psi_{k,k-1} \rangle \rangle - A_k^0 r \langle \psi_{k+1,k} \rangle \rangle - A_k^0 r \langle \psi_{k+2,k} \rangle \rangle \]

\[
- \ldots - A_k^0 r \langle \psi_{N_T,k} \rangle \rangle + B_{N_T+2,k} \langle A_1| + B_{N_T+2,k} \langle A_2| + B_{N_T+3,k} \langle A_3| + \ldots + B_{N_T+m,k} \langle A_m|
\]

\[
= A_k^0 \langle A_0| + \langle \psi_{m,k} \rangle \rangle,
\]  

(A10)

where \( \langle \psi_{m,k} \rangle \rangle \) is another (unnormalised) state orthogonal to \( |A_0 \rangle \) and \( |\psi \rangle \). Thus recalling that \( U_0 \) only acts nontrivially on the subspace spanned by \( |A_0 \rangle \) and \( |\psi \rangle \), we find that

\[
\langle A_k|U_m U(\tau_k) = A_k^0 \langle c_0 \rangle \rangle - s \langle \psi \rangle \rangle + \langle \psi_{m,k} \rangle \rangle.
\]  

(A11)

Thus upon pre-selecting onto the initial energy eigenstate \( |A_0 \rangle \), we observe that the only terms contributing to \( \langle A_k|U_m U(\tau_k)|A_0 \rangle \) come from non dynamical branches of the wave function.

As with the case in which \( N_T = 1 \) presented in supplementary IV A, we can analyse the probability that a measurement of whether the clock was dynamically evolving during the dynamical stage of our pre and post selected setup. For the case of post-selecting on counterfactual outcome \( |A_k \rangle \), following [29], we have the probability of the clock being on at time \( t \in (0, \tau_k) \) is given by

\[
\text{Prob(on, } A_k := \frac{|\langle A_k|U_m U(\tau_k - t)P_{on} U(t)U_0|A_0 \rangle|^2}{|\langle A_k|U_m U(\tau_k - t)P_{on} U(t)U_0|A_0 \rangle|^2 + |\langle A_k|U_m U(\tau_k - t)P_{on} U(t)U_0|A_0 \rangle|^2}.
\]  

(A12)
where $P_{on} := |\psi\rangle\langle\psi| + |\tau_0\rangle\langle\tau_0| + |\tau_1\rangle\langle\tau_1| + \ldots + |\tau_{N_f}\rangle\langle\tau_{N_f}|$ is the projection onto the on subspace and $P_{on} := |A_0\rangle\langle A_0| + |A_1\rangle\langle A_1| + \ldots + |A_{N_m}\rangle\langle A_{N_m}|$ projects onto the complementary space (the subspace where no dynamics occurs). Note that since the ancillas and energy eigenstate are stationary, we have that $U(t) = U_{\perp A_0 \ldots A_{N_m}}(t) + I_{A_0 \ldots A_{N_m}}$, where $I_{A_0 \ldots A_{N_m}}$ is the identity operator on the subspace spanned by $|A_0\rangle, |A_1\rangle, \ldots, |A_{N_m}\rangle$, and that $U_{\perp A_0 \ldots A_{N_m}}(t)$ is orthogonal to said subspace. Therefore, if we assume that the dynamics is Markovian, namely $U(t_1)U(t_2) = U(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}$, it follows that $U(\tau_k - t)P_{on}U(t) = U_{\perp A_0 \ldots A_{N_m}}(\tau_k)$ and hence that

$$\text{Prob(on, } A_k) = 0$$

(A13)

for all $t \in (0, \tau_k)$ and for all $k = 0, 1, 2, \ldots, N_T$.

### 5. Structure and achievability proofs

In this subsection we present two propositions. The first one shows that ancillary states are necessary in order for the counterfactual clock to function, while the second shows that counterfactual clocks exist which can distinguish between an arbitrary number of times (i.e. for all $N_T \in \mathbb{N}_{>0}$).

To start with, we need to motivate why it could have been that the ancillary states were not strictly necessary. The most straightforward way to see this is by recalling eq. (A14) from the previous section, and noting that the unitary $U_m$ has the form $U_m = U_e U_m'$. It follows that

$$\langle A_k | U_m U(\tau_k) U_0 | E \rangle = \begin{cases} \langle E | U_m' U(\tau_0) U_0 | E \rangle & \text{if } k = 0 \\ \langle \tau_{k-1} | U_m' U(\tau_k) U_0 | E \rangle & \text{if } k = 1, 2, 3, \ldots, N_T. \end{cases}$$

(A14)

The physical interpretation of this is that rather than post-selecting on $|E\rangle = |A_0\rangle$ and ancillas $|A_1\rangle, \ldots, |A_{N_f}\rangle$, we could have implemented the unitary $U_m'$, in stead of $U_m$, and post-selected on $|E\rangle$ and $|\tau_0\rangle, \ldots, |\tau_{N_f-1}\rangle$. In this latter method, since the states $|\tau_k\rangle$ evolve in time, upon implementing the counterfactual measurement, the clock would start evolving unless it was immediately turned off (note that this does not happen when post-selecting onto the ancillas, since these are by definition stationary states). However, in the latter method, we see from the r.h.s. of eq. (A14) that one could have used unitaries in which the number of ancillas is zero, i.e. $m = 0$, since the ancillas only appear implicitly (no pre or post selection on them is required). The following proposition proves that in the case of $N_T = 1$, the unitary $U_m'$ needs at least one ancilla. This result implies that the ancillas are necessary.

**Proposition 1** (Ancillas are necessary). Consider the counterfactual clock protocol described by the two state formalism on the r.h.s. of eq. (A14) for the case where the clock can distinguish between two times ($N_T = 1$) and the number of ancilla states is zero ($m = 0$), namely

$$\langle E | U_m' U(\tau_0) U_0 | E \rangle \quad \text{and} \quad \langle \tau_0 | U_m' U(\tau_1) U_0 | E \rangle.$$

(A15)

There is no solution for which the probability of the elapsed time being $\tau_0$, namely $P_{cf}^{(0)} := |\langle E | U_m' U(\tau_0) U_0 | E \rangle|^2$, and the probability of the elapsed time being $\tau_1$, namely $P_{cf}^{(1)} := |\langle \tau_0 | U_m' U(\tau_1) U_0 | E \rangle|^2$, are both non-zero.

**Proof.** For the case $N_T = 1, m = 0$, the matrix representation of $U_m'$ in eq. (A6) reduces to

$$\begin{array}{ccc}
| E \rangle & | \tau_0 \rangle & | \tau_1 \rangle \\
| A_0^\dagger \rangle & A_0^\dagger A_1 & \gamma_0 \\
| 0 \rangle & -A_0^\dagger r & \gamma_1 \\
| -A_0^\dagger r \rangle & 0 & \gamma_2
\end{array}$$

(A16)

Unitary matrices require their row and column vectors to be orthogonal. Applying this constraint to the 1st two columns, one finds $A_0^\dagger A_1^\dagger = 0$, which implies $|A_0^\dagger||A_1^\dagger| = 0$ and hence $A_0^\dagger = 0$ and/or $A_1^\dagger = 0$. Therefore, since $P_{cf}^{(0)} = |c A_0^\dagger|^2$, and $P_{cf}^{(1)} = |c A_1^\dagger|^2$, there is no solution for which $P_{cf}^{(0)} > 0$ and $P_{cf}^{(1)} > 0$. $\square$

We now prove that for all $N_T \in \mathbb{N}_{>0}$, there exits a finite dimensional ancilla system and unitary $U_m$ such that there is a non zero probability of finding the clock off at all measurement times.

**Proposition 2** (Counterfactual clocks with arbitrarily many distinguishable times exist). Let $N_T \in \mathbb{N}_{>0}$. Then, for any $\{A_0^0, A_0^1, A_0^2, \ldots, A_0^{N_T}, \gamma_0, \gamma_1, \ldots, \gamma_{N_T+1}\} \in \mathbb{R}^{2N_T+3}$, and $r \neq 0$, there exists a $\gamma > 0$ for which a unitary matrix of the form eq. (A6) exists, with the amplitudes corresponding to counterfactual outcomes given by $A_0^0 = \gamma A_0^0, A_0^1 = \gamma A_0^1, A_0^2 = \gamma A_0^2, \ldots, A_0^{N_T} = \gamma A_0^{N_T}$, and gamma coefficients given by $\gamma_0 = \gamma \gamma_0, \gamma_1 = \gamma \gamma_1, \gamma_2 = \gamma \gamma_2, \ldots, \gamma_{N_T+1} = \gamma \gamma_{N_T+1}$, and $m = 2(N_T + 2)$ ancillary states.
Proof. It is by construction, and hence can be used to work out particular values of \( \gamma \) for any instance of the problem. Denote by \( \vec{F}_j = \gamma \vec{f}_j \) \((j = 1, 2, 3, \ldots, m)\) the \( j \)th column vector of eq. (A6). As is well known, since eq. (A6) is a square matrix, it follows that if it is a unitary matrix if the vectors \( \{ \vec{F}_j \}_{j=1}^m \) form an orthonormal family. We therefore need to prove that the vectors \( \vec{F}_j \) can be made orthonormal.

To start with, we will decompose the vector \( \{ \vec{f}_j \}_{j=1}^{N_T+2} \) into a direct sum of three other vectors, namely for \( j = 1, 2, 3, \ldots, N_T + 2 \), let \( \vec{f}_j = \vec{c}_j \oplus \vec{X}_j \oplus \vec{Y}_j \), where \( \vec{c}_j, \vec{X}_j, \vec{Y}_j \in \mathbb{R}^{N_T+2} \). Since the matrix in eq. (A6) is a square matrix, this choice fixes the number of ancillas to be \( m = 2(N_T + 2) \). Note that the vectors \( \{ \vec{c}_j \}_{j=1}^{N_T+2} \) are completely fixed, up to the constant \( \gamma \), vectors \( \{ \vec{X}_j \}_{j=1}^{N_T+2} \). Furthermore, set \( \vec{Y}_j = \delta_{j,k} \vec{c}_j \), where \( \delta_{j,k} \) is the Kronecker delta, and the coefficients \( \{ c_j \}_{j=1}^{N_T+2} \in \mathbb{R}^{N_T+2} \) are to be determined. We will now fix the \( \vec{X}_j \) vectors. We will use them for so-called dimensional lifting of the vectors \( \vec{c}_j \) to an orthogonal set.

The algorithm in [36] shows how to choose vectors \( \{ \vec{c}_j \}_{j=1}^{N_T+2} \) so that the vectors \( \{ \vec{c}_j \oplus \vec{X}_j \}_{j=1}^{N_T+2} \) form an orthogonal family for any given set \( \{ \vec{c}_j \}_{j=1}^{N_T+2} \). It thus follows due to the form of the vectors \( \vec{Y}_j \), that the vectors \( \vec{F}_j = \gamma \vec{e}_j \oplus \vec{X}_j \oplus \vec{Y}_j \) form an orthogonal family for all \( \vec{c}_j \in \mathbb{R}^{N_T+2} \) and for all \( \gamma > 0 \). Imposing normalisation on the vectors \( \{ \vec{F}_j \}_{j=1}^{N_T+2} \), we find for \( j = 1, 2, 3, \ldots, N_T + 2 \):

\[
\frac{1}{\gamma^2} - |c_j|^2 = (\vec{X}_j)^\dagger \vec{X}_j + (\vec{Y}_j)^\dagger \vec{Y}_j.
\]

Now denote by \( D := \max_{j \in \{1, 2, \ldots, N_T+2\}} \left\{ (\vec{X}_j)^\dagger \vec{X}_j + (\vec{Y}_j)^\dagger \vec{Y}_j \right\} \), and the value of \( j \) which solves the maximization by \( j^* \). Furthermore, set \( c_{j^*} = 0 \) so that it follows from eq. (A17), that \( \gamma = 1/\sqrt{D} \). We can solve eq. (A17) for all \( j \neq j^* \), with a solution for the coefficients \( c_j \) satisfying \( 0 \leq |c_j|^2 \leq 1/\gamma^2 = D \).

We now have an orthonormal set of vectors \( \{ \vec{F}_j \}_{j=1}^{N_T+2} \), and all that remains is to find the remaining vectors \( \{ \vec{F}_j \}_{j=N_T+3}^{N_T+2+m} \), with \( m = 2(N_T + 2) \). Since this latter set of vectors contains elements without any constraint on them, other than those which allow eq. (A17) to be a unitary matrix, we can simply apply the Gram-Schmidt orthonormalisation procedure on the input sequence \( \{ \vec{F}_j \}_{j=1}^{N_T+2} \sim (\vec{e}_j)_{j=N_T+3}^{N_T+2+m} \), where \( (\vec{e}_j)_{j=N_T+3}^{N_T+2+m} \) is an arbitrary set of vectors linearly independent of \( \{ \vec{F}_j \}_{j=1}^{N_T+2} \), and \( \sim \) denotes sequence concatenation. The output of the Gram-Schmidt orthonormalisation procedure is then the orthonormal family \( \{ \vec{F}_j \}_{j=1}^{3(N_T+2)} \) where the 1st \( N_T + 2 \) elements are identical to the first \( N_T + 2 \) orthonormal vectors of the input sequence to the Gram-Schmidt orthonormalisation procedure. This completes the proof. \( \Box \)

Appendix B: Implementation of the proof of no classical model

Here we describe the numerical implementation of the proof that there does not exist a non contextual ontic model for the counterfactual clock described in supplementary IV.B. The entire problem is fully determined by the set \( \mathbf{s}_c \) of states (eq. (11)) and the set \( \mathbf{s}_e \) of effects (eq. (12)), with \( U_m \) and \( c, s \) given by eqs. (A7a) and (A7b).

We will follow the outline of the algorithm [33] and perform the following steps. The inner product is taken to be the Hilbert-Schmidt inner product:

1. Project the set of extreme points of the set \( \mathbf{s}_e \) onto \( \operatorname{span}(\mathbf{s}_e) \), were span denotes the linear span. Let \( P\_e(\mathbf{s}) \) denote the resulting set vectors. We call \( \operatorname{span}(P\_e(\mathbf{s})) \) the Reduced space and denote it by \( \mathcal{R} \). It is the effective vector space of our clock.

2. Using the Gram-Schmidt orthogonalization procedure, construct an orthonormal basis for \( \mathcal{R} \).

3. Project the extreme points of the sets \( \mathbf{s}_c \) and \( \mathbf{s}_e \) onto \( \mathcal{R} \). Denote the corresponding new sets as \( P\_R(\mathbf{s}) \) and \( P\_R(\mathbf{s}) \). The elements of the latter sets represent rays (also known as half-lines) emanating from the origin (which is the point \( \mathbf{0} := (0, \ldots, 0) \) with dimension \( \dim(\mathcal{R}) \) entries in the orthonormal basis for \( \mathcal{R} \)).

4. Run Vertex Enumeration algorithm [37] twice. Once for rays \( P\_R(\mathbf{s}) \) and again for the rays \( P\_R(\mathbf{s}) \). The input to the algorithm is given using the V-representation which has the format (list of vertices, list of rays) and in our case it simplifies to \( (\mathbf{0}, \text{list of rays}) \). Note that it differs from the H-representation which is set of linear inequalities corresponding to the intersection of halfspaces. We used Matlab wrapper GeoCalcLib (http://worc4021.github.io/) and performed all the computations using Matlab R2020a. The two runs of the algorithm on inputs \((\mathbf{0}, P\_R(\mathbf{s})) \) and \((\mathbf{0}, P\_R(\mathbf{s})) \) produce the following sets of extreme rays: \( \mathcal{R}_S \), \( \mathcal{R}_E \) respectively.
5. Form a new set $\text{Ray}_{\text{final}} = \{a \otimes b \mid \forall a \in \text{Ray}_S \text{ and } \forall b \in \text{Ray}_E\}$.

6. Run Vertex Enumeration algorithm on $(0 \otimes 0, \text{Ray}_{\text{final}})$ using the V-representation to obtain a set of extreme rays $W$. The set $W$ contains the list of potential non-classicality witnesses.

7. To verify that the pair of sets $\text{s}_{\text{cf}}, \text{e}_{\text{cf}}$ does not admit a non contextual ontic model, it suffices to check whether there exists $w \in W$ such that

$$\left\langle \sum_{j=1}^{\dim(R)} R_j \otimes R_j , w \right\rangle < 0,$$

where $(R_j)_{j=1}^{\dim(R)}$ is the sequence of orthonormal basis elements for $R$ calculated in step 2, and $\langle \cdot, \cdot \rangle$ represent the Hilbert-Schmidt inner product.

Note that computation times for the last instance of vertex enumeration (with input $(0 \otimes 0, \text{Ray}_{\text{final}})$ runs for a long time, so we employed a slight optimization by preprocessing the sets Ray$_S$, Ray$_E$ by applying the GeoCalcLib vertex Reduction routine which removed non-extremal (redundant) rays of the polyhedrons in V-representation. This allowed us to run the Vertex Enumeration algorithm with input $(0, T)$, where $T \subset \text{Ray}_{\text{final}}$.

We were able to successfully identify an element of $w \in W$ which provides a violation which is two orders of magnitude larger than any error due to rounding and approximations that stem from using floating-point arithmetic.

Appendix C: Engineered counterfactual clock

Here we will explain in detail how the counterfactual engineered clock works. The material is divided into four subsections to aid comprehension.

1. Preliminaries

Here we will describe the required dynamics of the on and off states of the clock, and show that such dynamics is indeed achievable. In particular, we will need to consider two orthonormal states $|\psi_{\text{off}}\rangle$ and $|\psi_{\text{on}}\rangle$ on an infinite dimensional Hilbert space, whose dynamics is generated via a Hamiltonian $\hat{H}$ of the form

$$\hat{H} = \hat{H}_{\text{on}} - \hat{H}_{\text{on}}|\psi_{\text{off}}\rangle\langle\psi_{\text{off}}|\hat{H}_{\text{on}}/r_0,$$

where $\hat{H}_{\text{on}}$ and $|\psi_{\text{off}}\rangle$ are arbitrary so long as $r_0 := \langle\psi_{\text{off}}|\hat{H}_{\text{on}}|\psi_{\text{off}}\rangle \neq 0$. We require that the dynamics of $|\psi_{\text{on}}\rangle$ under $\hat{H}_{\text{on}}$ is orthogonal to $|\psi_{\text{off}}\rangle$ at all times relevant to the experiment:

$$\langle\psi_{\text{off}}|\psi_{\text{on}}(t)\rangle = 0, \quad \forall t \in [0, x_0T_0]$$

where

$$|\psi_{\text{on}}(t)\rangle := e^{-i\hat{H}_{\text{on}}t} |\psi_{\text{on}}\rangle,$$

and $[0, x_0T_0]$ is the time during which the clock will function and will be detailed later at the beginning of appendix C2. We now demonstrate a simple proposition which shows how $|\psi_{\text{off}}\rangle$ and $|\psi_{\text{on}}(t)\rangle$ evolve under the total Hamiltonian $\hat{H}$:

**Proposition 3.** For all $t \in [0, x_0T_0]$,

$$e^{-it\hat{H}} |\psi_{\text{off}}\rangle = |\psi_{\text{off}}\rangle,$$

$$e^{-it\hat{H}} |\psi_{\text{on}}(t)\rangle = |\psi_{\text{on}}(t)\rangle,$$

where $|\psi_{\text{on}}(t)\rangle$ is given by eq. (C2).

**Proof.** From eq. (C3) it follows

$$0 = \frac{d}{dt} \langle\psi_{\text{off}}|\psi_{\text{on}}(t)\rangle = -i \langle\psi_{\text{off}}|\hat{H}_{\text{on}}|\psi_{\text{on}}(t)\rangle.$$
We thus conclude \(\langle \psi_\text{off} | \hat{H}_\text{on} | \psi_\text{on}(t) \rangle = 0\) for all \(t \in [0, x_0 T_0]\). Therefore, using eq. (C1) we find
\[
\hat{H} | \psi_\text{on}(t) \rangle = \hat{H}_\text{on} | \psi_\text{on}(t) \rangle, \quad t \in [0, x_0 T_0]
\]
where \(0\) is the zero vector. We can use eq. (C7) to obtain eq. (C4):
\[
e^{-i\hat{H}} | \psi_\text{on} \rangle = \lim_{N \to \infty} \left(1 - i\hat{H}/N\right)^N | \psi_\text{on} \rangle = \lim_{N \to \infty} \left(1 - i\hat{H}/N\right)^{N-1} \left(| \psi_\text{on}(t/N) \rangle + \mathcal{O}(t/N)^2\right)
\]
\[
= \lim_{N \to \infty} \left(1 - i\hat{H}/N\right)^{N-2} \left(| \psi_\text{on}(2t/N) \rangle + \mathcal{O}(t/N)^2\right) = \lim_{N \to \infty} \left(| \psi_\text{on}(t) \rangle + N\mathcal{O}(t/N)^2\right)
\]
Likewise, eq. (C5) follows from eq. (C8).

\[\square\]

## 2. Protocol and model derivation

We first state the general dynamical properties of the clock at a qualitative level when turned on (i.e. operated in standard fashion). This will allow for a mental picture which will aid comprehension of the quantitative study which is to follow. When the 1st event occurs, the clock is turned to the on state, \(| \psi_\text{on}(0) \rangle\), and it starts ticking at elapsed times \(t_1, 2t_1, 3t_1, \ldots, N_T t_1\). Then, at time \(T_0 = (N_T + 1)t_1\), the dynamics of the clock repeats itself. The periodic behaviour is repeated \(x_0 \in \mathbb{N}_{>0}\) times. Evidently, this clock can only tell the time modulo the period \(T_0\). Importantly, the time at which the 2nd event occurs can be \textit{any} time in the interval \([0, x_0 T_0]\), with the answer being an estimate on the number of ticks which have occurred between the 1st and 2nd events.

One starts with the clock in the off state \(| \psi_\text{off} \rangle\) and applies a unitary to turn it to \(| \psi_\text{on}(t) \rangle\) when the 1st event occurs. The clock will then evolve unitarily until the 2nd event occurs at some time \(t\), at which point we measure the state \(| \psi_\text{on}(t) \rangle\) using an appropriately chosen measurement. The engineered clock’s on state, \(| \psi_\text{on}(t) \rangle\), will not be orthogonal to itself after ticking: \(\langle \psi_\text{on}(t) | \psi_\text{on}(t') \rangle \neq 0\) for \(t \in [t_1, (l+1)t_1), t' \in [t_1, (l'+1)t_1), l \neq l', t_1, l, l' \in \{0, 1, 2, \ldots, N_T\}\). Since quantum measurements cannot perfectly distinguish non-orthogonal states, this implies that the clock will not be able to tell the time perfectly when used. However, the overlaps will decrease with increasing \(|l - l'|\), and hence a well chosen measurement can still provide a good estimate of the number of ticks occurred. This point will not be of much importance when using the clock counterfactually, since in this modus operandi, the clock can only achieve the counterfactual outcomes with a probability less than one anyway (analogously to the elementary clocks from appendix A) and we will effectively be performing a form of unambiguous quantum state discrimination.

While the quantum system used as a counterfactual clock is much more complex in the engineered quantum clock case, the actual protocol is very similar to the one we have seen already in the elementary clock case, namely, starting with the clock in the off state, \(| \psi_\text{off} \rangle\), a unitary \(U_0\) is applied when the 1st event occurs transforming the clock to \(c | \psi_\text{off} \rangle + s | \psi_\text{on}(0) \rangle\), followed by applying a unitary \(U_m\) when the 2nd event occurs and measuring in the measurement basis. The main physically important new feature is that the elapsed time between the two events can be \textit{any} time in the interval \([0, x_0 T_0]\). It is useful to introduce a set of orthonormal ancillary states, \(\{|A_i\}_{i=1}^{N_T}\}\), which are orthogonal to the Hamiltonian, namely \(\hat{H} | A_i \rangle = 0\). These form part of the basis in which we measure. In this section, the measurement basis, is the set of orthogonal projectors \(\{|\Psi_\text{off}\rangle \langle \Psi_\text{off}|, |\hat{A}_1\rangle \langle \hat{A}_1|, |\hat{A}_2\rangle \langle \hat{A}_2|, \ldots, |\hat{A}_{N_T}\rangle \langle \hat{A}_{N_T}|\}\), \(1 - |\Psi_\text{off}\rangle \langle \Psi_\text{off}| - |\hat{A}_1\rangle \langle \hat{A}_1| - |\hat{A}_2\rangle \langle \hat{A}_2| - \ldots - |\hat{A}_{N_T}\rangle \langle \hat{A}_{N_T}|\}\), where \(1\) denotes the identity operator.\(^3\)

(In the case considered in main text, for which \(N_T = 1\), we denoted \(\hat{A}_1\) by \(A\) for simplicity). Diagrammatically, just before the measurement, the protocol is as follows:

\[
\begin{align*}
\begin{array}{ccc}
|\psi_\text{off}\rangle & \xrightarrow{t & \xrightarrow{U_m} & \frac{A_1}{N} |\psi_\text{off}\rangle + \frac{A_1}{N} |\hat{A}_1\rangle + \ldots + \frac{A_1}{N} |\hat{A}_{N_T}\rangle + A_2 |\Psi_\text{off}\rangle \\
|\psi_\text{off}(0)\rangle & \xrightarrow{s & \xrightarrow{U_m} & -(\xi) A_1 |\psi_\text{off}(t)\rangle - (\xi) A_1 |\hat{A}_1(t)\rangle - \ldots - (\xi) A_1 |\hat{A}_{N_T}(t)\rangle + A_3 |\psi_\text{on}(t)\rangle,
\end{array}
\end{align*}
\]

\(^3\) Strictly speaking, we are not projecting onto a basis for the entire Hilbert space, but only onto the relevant space for us.
where $N > 0$, $A_i > 0$, are to be determined and we have defined the amplitudes

$$A_2 = \sqrt{1 - (N_T + 1) \left( \frac{A_1}{N} \right)^2}, \quad A_3 = \sqrt{1 - (N_T + 1) \left( \frac{C}{s} \right)^2 A_1^2},$$

(C13)

and where all kets are normalised, and we will show that $\langle \tilde{A}_i | \psi_{\text{off}} \rangle = \langle A_{\text{off}} | \psi_{\text{off}} \rangle = \langle \tilde{A}_i(t) | \psi_{\text{off}} \rangle = \langle A_{\text{off}}(t) | \psi_{\text{off}} \rangle = \langle \tilde{A}_i | A_i \rangle = \langle A_{\text{off}} | A_i \rangle = \langle \tilde{\psi}_{\text{off}} | A_i \rangle = \langle \tilde{A}_i(t) | A_i(t) \rangle = \langle \tilde{\psi}_{\text{off}}(t) | A_i(t) \rangle = \langle \tilde{A}_i(t) | \psi_{\text{off}}(t) \rangle = \langle A_{\text{off}}(t) | \psi_{\text{off}}(t) \rangle = \langle \tilde{A}_i(t) | A_i(t) \rangle (A_{\text{on}}(t) | A_i(t) \rangle = 0$ for all $t \geq 0$, and $l, r \in 0, 1, 2, \ldots, N_T - 1$ such that $l \neq r$.

The remaining overlaps, namely $\langle \tilde{\psi}_{\text{off}}(t) | \psi_{\text{off}} \rangle$, $\langle \tilde{A}_i(t) | A_i \rangle$, $\langle A_{\text{on}}(t) | A_i \rangle$, are to be determined.

As we will soon see, we will associate the counterfactual outcomes with measurement outcomes $\psi_{\text{off}}, \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{N_T}$, resulting from a measurement in the measurement basis performed immediately after $U_m$ is applied. To make the analysis a bit easier, we can consider the mathematically equivalent scenario in which, rather than applying the unitary $U_m$ to the state before measuring in the measurement basis, we can apply $U_m$ to the projectors onto the measurement basis instead, rotating the basis in which me measure to:

$$|\tilde{x}_0\rangle = U^\dagger_m |\psi_{\text{off}}\rangle |U_m, |\tilde{x}_1\rangle = U^\dagger_m |\tilde{A}_1\rangle |U_m, |\tilde{x}_2\rangle = U^\dagger_m |\tilde{A}_2\rangle |U_m, \ldots, |\tilde{x}_{N_T}\rangle = U^\dagger_m |\tilde{A}_{N_T}\rangle |U_m,\hspace{1mm}I - \sum_{l=0}^{N_T} |\tilde{x}_l\rangle\langle \tilde{x}_l|.$$

We additionally chose $U_m$ to act trivially on $|A_{\text{off}}\rangle$ and $|A_{\text{on}}(t)\rangle$, for all $t \in [0, x_0 T_0)$. Writing the states $|\tilde{\psi}_{\text{off}}\rangle$ and $|\tilde{\psi}_{\text{on}}(t)\rangle$ in the new basis, we find

$$|\tilde{\psi}_{\text{off}}\rangle = \frac{A_1}{N} |\tilde{x}_0\rangle + \frac{A_1}{N} |\tilde{x}_1\rangle + \ldots + \frac{A_1}{N} |\tilde{x}_{N_T}\rangle + \sqrt{1 - (N_T + 1) \left( \frac{A_1}{N} \right)^2} |A_{\text{off}}\rangle,$$

(C14)

$$|\tilde{\psi}_{\text{on}}(t)\rangle = e^{-i t \hat{H}_{\text{on}}} \left( -\left( \frac{C}{s} \right) A_1 |\tilde{x}_0(0)\rangle - \left( \frac{C}{s} \right) A_1 |\tilde{x}_1(0)\rangle - \ldots - \left( \frac{C}{s} \right) A_1 |\tilde{x}_{N_T}(0)\rangle + \sqrt{1 - (N_T + 1) \left( \frac{C}{s} \right)^2 A_1^2} |A_{\text{on}}(0)\rangle \right)$$

(C15)

$$= -\left( \frac{C}{s} \right) A_1 |\tilde{x}_0(t)\rangle - \left( \frac{C}{s} \right) A_1 |\tilde{x}_1(t)\rangle - \ldots - \left( \frac{C}{s} \right) A_1 |\tilde{x}_{N_T}(t)\rangle + \sqrt{1 - (N_T + 1) \left( \frac{C}{s} \right)^2 A_1^2} |A_{\text{on}}(t)\rangle,$$

(C16)

where $|\tilde{x}_0(t)\rangle := U^\dagger_m |\tilde{\psi}_{\text{off}}\rangle, |\tilde{x}_1(t)\rangle := U^\dagger_m |\tilde{A}_1\rangle, |\tilde{x}_2(t)\rangle := U^\dagger_m |\tilde{A}_2\rangle, \ldots, |\tilde{x}_{N_T}(t)\rangle := U^\dagger_m |\tilde{A}_{N_T}\rangle$.

Hence $\langle \tilde{\psi}_{\text{off}}|\tilde{\psi}_{\text{off}}\rangle = \langle \tilde{x}_0|\tilde{x}_0(t)\rangle$ and $\langle \tilde{\psi}_{\text{on}}|\tilde{\psi}_{\text{on}}\rangle = \langle \tilde{x}_1|\tilde{x}_1(t)\rangle$. We impose the constraint that the overlaps must satisfy

$$\langle \tilde{x}_l|\tilde{x}_l(t)\rangle = \frac{1}{N} \sum_{q=0}^{x_0-1} G_0, \sigma \left( x - q T_0 \right)$$

(C17)

where $\theta = -1$ if $N_T = 1$, and $\theta = 1$ otherwise; and where

$$G_0, \sigma(x) := \sum_{q=-(\theta+1)/2}^{x_0-1} G_0, \sigma \left( x - q T_0 \right),$$

(C18)

with $T_0 = (N_T + 1) t_1$ the clock period, $x_0 \in \mathbb{N}_{>0}$ the number of cycles through the period the clock performs and $G_0, \sigma$ is an approximation to the top hat function; namely

$$G_0, \sigma(t) := \frac{1}{2} \left[ \text{erf} \left( \frac{t/t_1 + 1/2}{\sqrt{2} \sigma} \right) - \text{erf} \left( \frac{t/t_1 - 1/2}{\sqrt{2} \sigma} \right) \right],$$

(C19)

where erf is the error function and $\sigma > 0$ controls the quality of the approximation. Approximations to the top hat function are know as “bells”; see [38] and references in the introduction for other possibilities. The overlap $\langle \tilde{x}_l|\tilde{x}_l(t)\rangle$ is plotted in fig. 4 in the limit of small positive $\sigma$. By considering the conditions for a counterfactual measurement, we see that this is precisely the necessary form of the function $G_0, \sigma$. Namely, it guarantees that for $t \notin [t_1, (l + 1) t_1)$, the probability of obtaining outcome $\tilde{x}_l$ is zero, while for $t \in [t_1, (l + 1) t_1)$, the probability of obtaining outcome $\tilde{x}_l$ is non zero but the overlap with the on branch is zero, namely $\langle \tilde{x}_l|\tilde{x}_{l,T}(t)\rangle = \langle \tilde{x}_l|\tilde{x}_l(t)\rangle = 0$.

Expanding the kets $|\tilde{x}_l\rangle, |\tilde{x}_l\rangle$ in the energy basis, $|\tilde{x}_l\rangle = \int dE \tilde{x}_l(E) |E\rangle$, $|\tilde{x}_l\rangle = \int dE \tilde{x}_l(E) |E\rangle$, the overlap $\langle \tilde{x}_l|\tilde{x}_l(t)\rangle$ becomes

$$\langle \tilde{x}_l|\tilde{x}_l(t)\rangle = \int dE \tilde{x}_l^* (E) \tilde{x}_l(E) e^{-i 2 \pi E t},$$

(C20)
FIG. 4: Plot of $\langle \tilde{x}_l | \bar{x}_l(t) \rangle$ over one period: the function is zero in the time interval $[lt_1, (l+1)t_1)$ and $1/N$ at all other times.

which we identify as the Fourier Transform of $\tilde{x}_l^*(E)\bar{x}_l(E)$. Denoting the Fourier Transform and its inverse by $\mathcal{F}[f(t)](E) := \int dE f(E)e^{-i2\pi Et}$ and $\mathcal{F}^{-1}[f(E)](t) := \int dt f(t)e^{i2\pi Et}$ respectively, and using the shift and rescaling properties of the Fourier Transform, we find

$$\tilde{x}_l^*(E)\bar{x}_l(E) = \mathcal{F}^{-1}[\langle \tilde{x}_l | \bar{x}_l(t) \rangle](E) = \frac{1}{N} \mathcal{F}^{-1}\left[ G_{\sigma} \left( \frac{t - (\theta l + 1)t_1}{N_T} - \frac{t_1}{2} \right) \right](E)$$

(C21)

$$= \frac{N_T}{N} e^{i\pi E(2(\theta l + 1) + N_T)t_1} \mathcal{F}^{-1}[G_{\sigma}(t)](EN_T)$$

(C22)

Using Lemma 1, we find the inverse Fourier Transform of $G_{\sigma}(t)$:

$$\mathcal{F}^{-1}[G_{\sigma}(t)](EN_T) = \mathcal{F}^{-1}[G_{0,\sigma}(t)](EN_T) \frac{\sin(\pi(x_0 + (\theta + 1)/2)ET_0)}{\sin(\pi ET_0)} e^{-i\pi ET_0(\theta + 1)(x_0 + 2)/2}$$

(C23)

$$= \mathcal{F}^{-1}[G_{0,\sigma}(t)](EN_T) \frac{\sin(\pi(N_T + 1)(x_0 + (\theta + 1)/2)ET_1)}{\sin(\pi(N_T + 1)ET_1)} e^{-i\pi ET_1(N_T + 1)(\theta + 1)(x_0 + 2)/2},$$

(C24)

where by direct calculation one finds

$$\mathcal{F}^{-1}[G_{0,\sigma}(t)](x) = t_1 e^{-2\pi^2\sigma^2(xt_1)^2} \text{sinc}(\pi xt_1),$$

(C25)

where $\text{sinc}(x) = \sin(x)/x$ is the $\text{sinc}$ function. We now make the trial solutions

$$\tilde{x}_l^*(E) = \sqrt{\frac{N_T}{N} \mathcal{F}^{-1}[G_{\sigma}(t)](EN_T) \frac{\sin(\pi(x_0 + (\theta + 1)/2)ET_0)}{\sin(\pi ET_0)} e^{i\lambda ET_1}}$$

$$\tilde{x}_l(E) = \sqrt{\frac{N_T}{N} \mathcal{F}^{-1}[G_{\sigma}(t)](EN_T) \frac{\sin(\pi(x_0 + (\theta + 1)/2)ET_0)}{\sin(\pi ET_0)} e^{-i\lambda ET_1} e^{i\pi E(2(\theta l + 1) + N_T)t_1} e^{-i\pi ET_0(\theta + 1)(x_0 + 2)/2}}$$

(C26)

for $l = 0, 1, 2, \ldots, N_T$; and where $\sqrt{}$ denotes the principle square root and $\lambda \in \mathbb{R}$ is to be determined. We can now calculate the normalisation constant $N$:

$$1 = \langle \tilde{x}_l | \tilde{x}_l \rangle = \int dE \tilde{x}_l^*(E)\tilde{x}_l(E) = \frac{N_T}{N} \int dE \left| \sqrt{\mathcal{F}^{-1}[G_{\sigma}(t)](EN_T) \frac{\sin(\pi(x_0 + (\theta + 1)/2)ET_0)}{\sin(\pi ET_0)}} \right|^2,$$

(C27)
hence using the identity $|\sqrt{z}|^2 = |z|$ for all $z \in \mathbb{C}$,

$$N = \int dE \frac{F^{-1} \left[ G_{\sigma}(t) \right] (EN_T)}{\sin (\pi ET_0)} \sin \left( \pi (x_0 + (\theta + 1)/2)ET_0 \right)$$

$$= \int dE t_1 e^{-2\pi^2 \sigma^2 (EN_T t_1)^2} \sin (\pi ET_0) \sin \left( \pi (x_0 + (\theta + 1)/2)ET_0 \right)$$

$$= \int dy e^{-2(\pi N_T \sigma)^2 y^2} \sin (\pi N_T y) \sin \left( \pi (N_T + 1)(x_0 + (\theta + 1)/2) y \right)$$

Observe that this choice for $N$ also implies $(\tilde{x}_t(t)|\tilde{x}_t(t)) = 1$ for all $t \in \mathbb{R}$. We now examine the implications of the requirement that the states $|\psi_{\text{off}}\rangle$ and $|\psi_{\text{on}}(t)\rangle$ have to be orthogonal at all times:

$$0 = \langle \psi_{\text{off}}|\psi_{\text{on}}(t)\rangle = -\left(\frac{\xi}{s}\right) \frac{A_1^2}{N} \sum_{l=0}^{N_T} \langle \tilde{x}_l|\tilde{x}_l(t)\rangle + \sqrt{\left(1 - (N_T+1) \left(\frac{A_1}{N}\right)^2 \right) \left(1 - (N_T+1) \left(\frac{\xi}{s}\right)^2 A_1^2 \right)} \langle A_{\text{off}}|A_{\text{on}}(t)\rangle.$$

Therefore, $A_1$ must satisfy the equation

$$c_0 = \frac{\left(\frac{\xi}{s}\right) \frac{A_1^2}{N}}{\sqrt{\left(1 - (N_T+1) \left(\frac{A_1}{N}\right)^2 \right) \left(1 - (N_T+1) \left(\frac{\xi}{s}\right)^2 A_1^2 \right)}} \in \mathbb{R},$$

where $c_0$ satisfies

$$\langle A_{\text{off}}|A_{\text{on}}(t)\rangle = c_0 \sum_{l=0}^{N_T} \langle \tilde{x}_l|\tilde{x}_l(t)\rangle.$$

We will now fix a value of $c_0$ which satisfies eq. (C33) and come back to eq. (C32) later. Expanding $|A_{\text{off}}\rangle$ and $|A_{\text{on}}(t)\rangle$ in the energy basis, $|A_{\text{off}}\rangle = \int dE A_{\text{off}}(E)|E\rangle$, $|A_{\text{on}}(0)\rangle = \int dE A_{\text{on}}(E)|E\rangle$, we have that $\langle A_{\text{off}}|A_{\text{on}}(t)\rangle = \int dE A_{\text{off}}(E)A_{\text{on}}(E)e^{-i2\pi E t}$ and hence

$$A_{\text{off}}^*(E)A_{\text{on}}(E) = c_0 F^{-1} \sum_{l=0}^{N_T} \langle \tilde{x}_l|\tilde{x}_l(t)\rangle (E).$$

We calculate the summation before proceeding with taking the inverse Fourier transform. Taking into account eqs. (C17) and (C18) one finds

$$\sum_{l=0}^{N_T} \langle \tilde{x}_l|\tilde{x}_l(t)\rangle = \sum_{l=0}^{N_T} \frac{1}{N} G_{\sigma} \left( \frac{t - (\theta l + 1)T_1}{N_T} - \frac{t_1}{2} \right) = \sum_{l=0}^{N_T} \sum_{q = -\left(\theta + 1\right)/2}^{x_0 - 1} G_{0,\sigma} \left( \frac{t - (\theta l + 1)T_1}{N_T} - \frac{t_1}{2} - q \frac{T_0}{N_T} \right),$$

$$= \frac{1}{N} \sum_{l=0}^{N_T} \sum_{q = -\left(\theta + 1\right)/2}^{x_0 - 1} G_{0,\sigma} \left( \frac{t}{N_T} - \frac{t_1}{2} - (\theta l + 1) + q(N_T + 1) \frac{T_1}{N_T} \right),$$

$$= \frac{1}{N} \sum_{q = -N_T - \left(\theta - 1\right)/2}^{N_T + (x_0 - 1)(N_T + 1) - \left(\theta + 1\right)/2} G_{0,\sigma} \left( \frac{t}{N_T} - \frac{t_1}{2} - q \frac{T_1}{N_T} \right).$$

where in going from line C36 to line C37, we have used the identity $\sum_{n=0}^{d-1} \sum_{m=a}^{b} f(n + md) = \sum_{n=a}^{b} f(n)$, which holds for arbitrary function $f$ and $d \in \mathbb{N}_{>0}$, $a \leq b$, $a, b \in \mathbb{Z}$ and we have fined

$$G_{\sigma}^{(2)}(x) := \sum_{q = -N_T - \left(\theta - 1\right)/2}^{N_T + (x_0 - 1)(N_T + 1) - \left(\theta + 1\right)/2} G_{0,\sigma} \left( x - q \frac{T_1}{N_T} \right).$$
We now return to the task of taking the inverse Fourier transform in eq. (C34):

\[
A_{\text{off}}^* (E) A_{\text{on}} (E) = c_0 \mathcal{F}^{-1} \left[ \sum_{i=0}^{N_T} \langle \tilde{x}_i | \tilde{x}_i (t) \rangle \right] (E) = c_0 \frac{N}{N_T} \mathcal{F}^{-1} \left[ G_{\sigma}^{(2)} \left( \frac{t}{N_T} - \frac{1}{2} \right) \right] (E) \tag{C39}
\]

\[
= c_0 \frac{N_T}{N} e^{i\pi E t_N} \mathcal{F}^{-1} \left[ G_{\sigma}^{(2)} (t) \right] (E N_T), \tag{C40}
\]

where by direct calculation using lemma 1, one has

\[
\mathcal{F}^{-1} \left[ G_{\sigma}^{(2)} (t) \right] (E N_T) = \mathcal{F}^{-1} \left[ G_{0,\sigma} (t) \right] (E N_T) \frac{\sin \left( \pi (x_0 + 1) E t_0 \right)}{\sin \left( \pi E t_1 \right)} e^{-i\pi E t_1 \left( (N_T+1)(x_0+1)-1 \right) \left( N_T+1-1/2 \right)}, \tag{C41}
\]

\[
= \mathcal{F}^{-1} \left[ G_{0,\sigma} (t) \right] (E N_T) \frac{\sin \left( \pi (N_T + 1)(x_0 + 1) E t_1 \right)}{\sin \left( \pi E t_1 \right)} e^{-i\pi E t_1 \left( (N_T+1)(x_0+1)-1 \right) \left( N_T+1-1/2 \right)}, \tag{C42}
\]

where recall that \( \mathcal{F}^{-1} \left[ G_{0,\sigma} (t) \right] (E N_T) \) is given by eq. (C25). We thus make the following trial solutions for \( A_{\text{off}}^* (E) \) and \( A_{\text{on}} (E) \):

\[
A_{\text{off}}^* (E) = \sqrt{c_0} \frac{N_T}{N} \mathcal{F}^{-1} \left[ G_{0,\sigma} (t) \right] (E N_T) \frac{\sin \left( \pi (N_T + 1)(x_0 + 1) E t_1 \right)}{\sin \left( \pi E t_1 \right)} e^{i\lambda E t_1/2}
\]

\[
A_{\text{on}} (E) = \sqrt{c_0} \frac{N_T}{N} \mathcal{F}^{-1} \left[ G_{0,\sigma} (t) \right] (E N_T) \frac{\sin \left( \pi (N_T + 1)(x_0 + 1) E t_1 \right)}{\sin \left( \pi E t_1 \right)} e^{-i\lambda E t_1/2} e^{-i\pi E t_1 \left( (N_T+1)(x_0+1)-1 \right) \left( N_T+1-1/2 \right)}, \tag{C43}
\]

where, as before, we use the principle square root. Normalisation of \( |A_{\text{off}}| \) and \( |A_{\text{on}} (t)| \) imply

\[
1 = \frac{N_T}{N} \int \mathcal{F}^{-1} \left[ G_{0,\sigma} (t) \right] (E N_T) \frac{\sin \left( \pi (N_T + 1)(x_0 + 1) E t_1 \right)}{\sin \left( \pi E t_1 \right)} dE \tag{C44}
\]

from which, using the identity \( |\sqrt{z}|^2 = |z| \) for all \( z \in \mathbb{C} \), we determine the value of \( 1/|c_0| \) to be

\[
1/|c_0| = \frac{N_T}{N} \int \mathcal{F}^{-1} \left[ G_{0,\sigma} (t) \right] (E N_T) \frac{\sin \left( \pi (N_T + 1)(x_0 + 1) E t_1 \right)}{\sin \left( \pi E t_1 \right)} \quad \left( C45 \right)
\]

\[
= \frac{N_T}{N} \int dy e^{-2(\pi N_T y)^2 \theta^2} \left| \frac{\sin \left( \pi (N_T + 1)(x_0 + 1) y \right)}{\sin \left( \pi y \right)} \right|, \tag{C46}
\]

and where the sign of \( c_0 \in \mathbb{R} \) is given by the sign of \( c/s \); this follows from eq. (C32). Now that we have a value for \( c_0 \), we can determine \( A_1^2 \) using eq. (C32), which is a quadratic equation in \( A_1^2 \). We thus find

\[
\frac{A_1^2}{N^2} = \frac{(N_T + 1) \left( \left( \frac{x_0}{x} \right)^2 N^2 + 1 \right) + \Gamma_{\pm} \sqrt{(N_T + 1)^2 \left( \left( \frac{x_0}{x} \right)^2 N^2 + 1 \right)^2 - 4 \left( \frac{x_0}{x} \right)^2 \left( (N_T + 1)^2 - 1/c_0^2 \right)}}{2 \left( \frac{x_0}{x} \right)^2 N^2 \left( (N_T + 1)^2 - 1/c_0^2 \right)}, \tag{C47}
\]

where \( \Gamma_{\pm} \) is -1 or 1 depending on which solution to the equation we choose. Since we have assumed the amplitudes associated with the kets \( |A_{\text{off}}| \) and \( |A_{\text{on}}| \) to be both real in eqs. (C14) and (C16) respectively, we need to take the \( \Gamma_{\pm} = -1 \) solution. Since for small \( \sigma \), the probability of measuring a tick counterfactually is \( P_{c_f} = c^2 A_1^2 / N^2 \) for any one of the \( N_T \) ticks, we have that

\[
P_{c_f} = s^2 \frac{(N_T + 1) \left( \left( \frac{x_0}{x} \right)^2 N^2 + 1 \right) - \sqrt{(N_T + 1)^2 \left( \left( \frac{x_0}{x} \right)^2 N^2 + 1 \right)^2 - 4 \left( \frac{x_0}{x} \right)^2 \left( (N_T + 1)^2 - 1/c_0^2 \right)}}{2N^2 \left( (N_T + 1)^2 - 1/c_0^2 \right)}, \tag{C48}
\]

where recall that \( c = \cos(\theta), s = \sin(\theta), \) and \( N = N(\sigma), c_0 = c_0(\sigma) \) are given by eqs. (C30) and (C46) respectively. The numerical values of \( P_{c_f} \) for a given \( \sigma \) presented in the main text and are calculated by setting \( N_T = x_0 = 1 \) and evaluating \( N \) and \( c_0 \) numerically for this \( \sigma \), followed by maximising \( P_{c_f} \) numerically over \( \theta \in [0, 2\pi] \).
Finally, in order for the trial solutions for the wave functions \( \tilde{x}_1(E), \tilde{x}_1(E), A_{\text{off}}(E) \) and \( A_{\text{on}}(E) \) to be valid, we must verify their orthogonality relations; recall that we have so far assumed \( \langle \tilde{x}_1 | \tilde{x}_1 \rangle = \langle \tilde{x}_1 | \tilde{x}_r \rangle = \langle \tilde{x}_1 | A_{\text{on}}(t) \rangle = \langle \tilde{x}_1 | A_{\text{off}} \rangle = 0 \) for all \( t \geq 0 \), and \( l, r \in \{0, 1, 2, \ldots, N_T \} \) such that \( l \neq r \). This is where the phase factors \( \lambda \) come in to play. From eqs. (C26) and (C43), we see that these overlaps are proportional in absolute value, to integrals of the two following different forms:

Case 1:

\[
F_1(\lambda) := \int dx \, e^{-2(\pi A_\sigma)^2 x^2} \sin(\pi B_1 x) \sin(\pi C_1 x) \sin(\pi D_1 x) e^{iH_1 x} e^{iH_2 \lambda x},
\]

where \( A, B_1, C_1, D_1 \in \mathbb{N}_{>0}, H_1 \in \mathbb{Z} \), while \( H_2 \) is either a non zero integer or half integer.

Case 2:

\[
F_2(\lambda) := \int dx \, e^{-2(\pi A_\sigma)^2 x^2} \left| \sin(\pi B_1 x) \sin(\pi C_1 x) \sin(\pi D_1 x) \right|^* e^{iH_1 x} e^{iH_2 \lambda x},
\]

where \( C_1, D_1, H_1, H_2 \) satisfy the same constraints as in Case 1, while \( B_2, C_2, D_2, C_2 / D_2 \in \mathbb{N}_{>0} \).

We will now verify that we can make \( F_1(\lambda), F_2(\lambda) \) arbitrarily small by choosing \( \lambda \gg 0 \) large enough.

For case 1, since the integrand of \( F_1(\lambda) \) is smooth and converges absolutely, it follows from the principle of stationary phase, that \( F_1(\lambda) \) tends to zero as \( \lambda \) tends to infinity.

For case 2, the integrand is not absolutely continuous, due to the changes in sign of the functions under the square roots. As such, the principle of stationary phase does not directly apply. However, a variant of the usual stationary phase type argument still applies. For this we will make use of lemma 2. We start by choosing a consistent expression for \( c, f, G, \) and \( g \):

\[
c(x) = \frac{\text{sign} \left( \sin(\pi B_1 x) \sin(\pi C_1 x) \sin(\pi D_1 x) \right)}{\left| \sin(\pi B_1 x) \sin(\pi C_1 x) \sin(\pi D_1 x) \right|^*},
\]

\[
f(x) = \frac{\sin(\pi B_1 x) \sin(\pi C_1 x) \sin(\pi D_1 x)}{\left| \sin(\pi B_1 x) \sin(\pi C_1 x) \sin(\pi D_1 x) \right|^*}
\]

\[
G(x) = e^{-2(\pi A_\sigma)^2 x^2},
\]

\[
g(x) = e^{iH_1 x} e^{iH_2 \lambda x},
\]

where \( \text{sign}(\cdot) \) is the sign function. Lemma 2 allows us to upper bound \( F_2(\lambda) \). Since \( \sum_{n \in \mathbb{Z}} G(nT) = \sum_{n \in \mathbb{Z}} e^{-2(\pi A_\sigma)^2 T^2 n^2} < \infty \), the right hand side of eq. (C73) is finite in this case. Moreover, the only \( \lambda \) dependency enters in the \( v_0 \) term. Since \( v(x) = \int dx \, e^{iH_1 x} e^{iH_2 \lambda x} = e^{iH_1 x + iH_2 \lambda x} / (iH_1 + iH_2 \lambda) \), we can choose \( v_0 = 1 / |H_1 + H_2 \lambda| \). Therefore, \( F_2(\lambda) \) tends to zero as \( \lambda \) tends to infinity.

3. Precision quantification

We now show how to calculate the fidelity between the prescribed protocol in appendix C and that of a hypothetical "idealised" version of the clock which satisfies both of the following:

1) The probability that the clock had been on when one of the outcomes associated with the projectors \( \{|\tilde{x}_l]| 0 \leq l \leq N_T \} \) is obtained, is exactly zero.

2) The probability that the clock would have ticked \( n \) times, had it been on, when the outcome associated with projector \( |\tilde{x}_n]| \) is obtained, is exactly one.
The only difference between the idealised version of the clock and that of appendix C, will be the dynamics of the \(|\psi_{\text{on}}(t)\rangle\) state. Specifically, we replace eq. (C16), namely,

\[
|\psi_{\text{on}}(t)\rangle = - \left( \frac{c}{s} \right) A_1 |\bar{x}_0(t)\rangle - \left( \frac{c}{s} \right) A_1 |\bar{x}_1(t)\rangle - \ldots - \left( \frac{c}{s} \right) A_1 |\bar{x}_{N_T}(t)\rangle + \sqrt{1 - (N_T + 1) \left( \frac{c}{s} \right)^2} |A_{\text{on}}(t)\rangle,
\]

with

\[
|\psi_{\text{on}}I(t)\rangle := - \left( \frac{c}{s} \right) A_1 |\bar{x}_0(t)\rangle - \left( \frac{c}{s} \right) A_1 |\bar{x}_1(t)\rangle - \ldots - \left( \frac{c}{s} \right) A_1 |\bar{x}_{N_T}I(t)\rangle + \sqrt{1 - (N_T + 1) \left( \frac{c}{s} \right)^2} |A_{\text{on}I}(t)\rangle,
\]

where \(I\) stands for idealised, and the kets \(|\bar{x}_mI(t)\rangle\) satisfy \(|\bar{x}_i\rangle|\bar{x}_mI(t)\rangle = \frac{1}{l} \delta_{i,m} \delta(t)\), where \(\delta_{i,m}\) is the Kronecker delta, and where \(\delta_i(t) = 0\) if \(t \in \cup_{q=0}^{p_{-1}} [lt_1 + qt_0, (l + 1)t_1 + qt_0]\) and \(\delta_i(t) = 1\) otherwise. The ket \(|A_{\text{on}I}(t)\rangle\) is orthogonal to the kets \(|\bar{x}_i\rangle\) \(\cup 1\) and obeys \(|A_{\text{off}}|A_{\text{on}I}(t)\rangle = c_0 \sum_{t=0}^{N_T} \langle \bar{x}_t|\bar{x}_I(t)\rangle\) with \(c_0\) given by eq. (C32), in analogy with eq. (C33) satisfied by \(|A_{\text{on}}(t)\rangle\).

Note that while if one performs the protocol associated with the counterfactual clock laid out in appendix C 2, one finds that the clock works “perfectly”, that is to say, satisfies 1) and 2) above, it is unclear whether such dynamics are achievable with a time independent Hamiltonian. This is the justification for calling it idealised. Its purpose is to show that our protocol which is realised via a time independent Hamiltonian, approximates the one finds that the clock works “perfectly”, that is to say, satisfies 1) and 2) above, it is unclear whether such dynamics are achievable with a time independent Hamiltonian. This is the justification for calling it idealised. Its purpose is to show that our protocol which is realised via a time independent Hamiltonian, approximates the idealised clock in fidelity up to an arbitrary precision, by choosing \(\delta > 0\) sufficiently small. In the special case in which the clock ticks just once \((N_T = 1)\), this description of an idealised clock is the same as eqs. (15a) and (15b) in the main text. In particular, we are only concerned with the differences in fidelity when obtaining one of the outcomes \(|\bar{x}_i\rangle\) \(\cup 1\), that is to say, the outcomes which are relevant for the counterfactual operation of the clock.

We start by evaluating the difference in fidelities at time \(t\) between what is actually obtained with our protocol, and what would have been obtained in the idealised case:

\[
\text{Diff}_{\text{p}}(\sigma, t) := \left| \langle \bar{x}_p | (c|\psi_{\text{off}}) + s|\psi_{\text{on}}(t)\rangle \right|^2 - \left| \langle \bar{x}_p | (c|\psi_{\text{off}}) + s|\psi_{\text{on}}I(t)\rangle \right|^2
\]

\[
= \frac{A_0^2 c^2}{N^2} \left( |1 - N \langle \bar{x}_p|\bar{x}_p(t)\rangle|^2 - |1 - \delta_p(t)|^2 \right),
\]

for \(p = 0, 1, \ldots, N_T\). Furthermore, since the clock could be measured at any time, it is instructive to consider the time averaged error rate over the total operation time of the clock, namely the interval \([0, x_0T_0)\). We can furthermore subdivide this interval into two disjoint parts. The first is the interval in which \(p\) ticks should have occurred (and thus the overlaps should be close to one), namely \(\cup_{q=0}^{\rho_{-1}} [pt_1 + qt_0, (p + 1)t_1 + qt_0]\). The second consists in the remaining intervals, namely \(\cup_{q=0}^{p-1} [qT_0, pt_1 + qt_0) \cup ([p + 1)t_1 + qt_0, (q + 1)T_0)\), for which the probability of the clock ticking \(p\) times, had it been on, should be very small. For the 1st type of error, we find

\[
\text{Diff}^{(1)}_{\text{p}}(\sigma) := \frac{1}{x_0 t_1} \sum_{q=0}^{x_0-1} \int_{pt_1 + qt_0}^{(p+1)t_1 + qt_0} \text{Diff}^{(1)}_{\text{p}}(\sigma, t)\]

\[
= \frac{A_0^2 c^2}{N^2 x_0 t_1} \sum_{q=0}^{x_0-1} \int_{pt_1 + qt_0}^{(p+1)t_1 + qt_0} dt \left| 1 - G_\sigma \left( \frac{l - (\theta p + 1)t_1}{N_T} - t_1 \right) \right|^2 - 1 \right)
\]

\[
= \frac{A_0^2 c^2}{N^2} \left( -1 + \sum_{q=0}^{x_0-1} \int_0^1 dx \left| 1 - G_\sigma \left( \frac{x - 1 + (1 - \theta)p + q(N_T + 1)}{N_T} - \frac{1}{2} t_1 \right) \right|^2 \right),
\]

which, recalling the definition of \(G_\sigma\) (eqs. (C17) and (C18)), is observed to be \(t_1\) independent. For the case studied in the main text, \(x_0 = 1\), \(\theta = -1\), \(N_T = 1\), \(p = 0, 1\), and we find

\[
\text{Diff}^{(1)}_{\text{p}}(\sigma) = \frac{A_1^2 c^2}{N^2} \left( -1 + \int_0^1 dx \left| 1 - G_\sigma \left( \frac{x - 1 + 2p - \frac{1}{2} t_1}{2} \right) \right|^2 \right)
\]

\[
= \frac{A_1^2 c^2}{N^2} \left( -1 + \int_0^1 dx \left| 1 - G_\sigma \left( \frac{x - 1 - \frac{1}{2} t_1}{2} \right) \right|^2 \right),
\]
which holds for both $p = 0, 1$ since $G_\sigma$ is a symmetric function. Similarly, we have

$$\text{Dif}_p^{(2)}(\sigma) := \frac{1}{x_0(T_0 - t_1)} \sum_{q=0}^{x_0-1} \int_{x_0 \cdot q \cdot T_0}^{x_0 \cdot (q+1) \cdot T_0} \text{Dif}_p^{(2)}(\sigma, t) + \int_{(p+1) \cdot t_1 \cdot q \cdot T_0}^{(q+1) \cdot T_0} \text{Dif}_p^{(2)}(\sigma, t)$$

(C65)

$$= \frac{1}{x_0 N_T} \sum_{q=0}^{x_0-1} \left[ \int_0^p dx \left| 1 - G_\sigma \left( \frac{x - (\theta p + 1) + q(N_T + 1) - \frac{1}{2}}{N_T} \frac{1}{t_1} \right) \right|^2 \right]$$

(C66)

$$+ \int_0^{N_T - p} dx \left| 1 - G_\sigma \left( \frac{x + (1 - \theta)p + q(N_T + 1) - \frac{1}{2}}{N_T} \frac{1}{t_1} \right) \right|^2 ,$$

(C67)

which, similarly to $\text{Dif}_p^{(1)}(\sigma)$, is observed to be $t_1$ independent. For the special case of one tick described in the main text ($x_0 = 1$, $\theta = -1$, $N_T = 1$, $p = 0, 1$), the expression reduces to

$$\text{Dif}_p^{(2)}(\sigma) = \left[ \int_0^p dx \left| 1 - G_\sigma \left( \frac{x + p - 1 - \frac{1}{2}}{t_1} \right) \right|^2 + \int_0^{1-p} dx \left| 1 - G_\sigma \left( \frac{x + 2p - 1}{t_1} \right) \right|^2 \right]$$

(C68)

$$= \int_0^1 dx \left| 1 - G_\sigma \left( \frac{x - 1}{2} \frac{1}{t_1} \right) \right|^2 ,$$

(C69)

for both $p = 0$ and $p = 1$.

### 4. Technical lemmas

For the following lemma, recall the definition of the inverse Fourier transform: $\mathcal{F}^{-1}[f(x)](y) := \int dx f(x)e^{i2\pi xy}$.

**Lemma 1.** Consider a function $f(x) = \sum_{m=a}^b f_0(x - mT)$, generated by some Schwartz space function $f_0 : \mathbb{R} \to \mathbb{R}$, with parameters $a \leq b$, $a, b \in \mathbb{Z}$, $T \in \mathbb{R}$. Its inverse Fourier transform takes on the form

$$\mathcal{F}^{-1}[f(x)](y) = \mathcal{F}^{-1}[f_0(x)](y) \frac{\sin((b-a+1)\pi y T)}{\sin(\pi y T)} e^{i\pi yaT(2+b-a)}$$

(C70)

for all $y \in \mathbb{R}$ and where the singular points are assigned by continuity.

**Proof.** First observe that $f$ can be written in terms of convolution of $f_0$ with a Dirac comb:

$$f(x) = f_0(x) \otimes \sum_{q=a}^b \delta(x - qT),$$

(C71)

where $\otimes$ denotes convolution. Now apply the inverse Fourier transform to both sides of the equation and invoke the convolution theorem. This gives

$$\mathcal{F}^{-1}[f(x)](y) = \mathcal{F}^{-1}[f_0(x)](y) \mathcal{F}^{-1} \left[ \sum_{q=a}^b \delta(x - qT) \right](y) = \mathcal{F}^{-1}[f_0(x)](y) \sum_{q=a}^b e^{i2\pi qyT}.$$  

(C72)

Finally, take real and imaginary parts of $\sum_{q=a}^b e^{i2\pi qyT}$ followed by applying the sums of cosines and sines arithmetic progressions formulas from [39].

**Lemma 2.** Consider a convergent integral of the form $I := \int_{-\infty}^{\infty} dx c(x)f(x)G(x)g(x)$, where $c : \mathbb{R} \to \mathbb{C}$, $f : \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}_{>0}$, $g : \mathbb{R} \to \mathbb{C}$ and $G, g \in C^1$ with $\frac{d}{dx}G(x) \geq 0$ for $x \in (-\infty, 0]$ and $\frac{d}{dx}G(x) \leq 0$ for $x \in [0, \infty)$. Furthermore, let $f$ be periodic with a countable number of zeros greater or equal to one in $[0, T]$, where $T$ is its period; and let $f \in C^1$ on the intervals $\mathbb{R} \setminus \{a_1, a_2, a_{-1}, a_{1}, a_2, \ldots \}$ where $\ldots, a_{-2}, a_{-1}, a_1, a_2, \ldots$ is the sequence of zeros of $f$ in ascending order. In addition, let there exist a sequence of constants $(c_l)_l$ such that $c(x) = c_l$ for all $x \in (a_l, a_{l+1})$, $l \in \mathbb{Z}$, where $a_0 = 0$. Let both $c(x)$ and $v(x) := \int dx g(x)$ be uniformly bounded: $|c(x)| = 1$ and $|v(x)| \leq v_0$ for some $v_0 > 0$, for all $x \in \mathbb{R}$. The following bound holds:

$$|I| \leq 2N_Z(1 + 2N_{TP}) \left( \max_{x \in [0, T]} |f(x)| \right) \left( 2G(0) + \sum_{n \in \mathbb{Z}} G(nT) \right) v_0,$$

(C73)

where $N_{TP} \in \mathbb{N}_{>0}$ is the number of tuning points of $f$ in $[0, T] \setminus \{\ldots, a_{-2}, a_{-1}, a_1, a_2, \ldots \}$, while $N_Z \in \mathbb{N}_{>0}$ is number of zeros of $f$ in interval $[0, T]$. 

[39]
Proof. Since $f$ is periodic and has a countable number of zeros, without loss of generality, let $a_l < 0$ for $l < 0$ and $a_l \geq 0$ for $l > 0$. Define $c_0 := c_1$ unless 0 is a zero of $f$, in which case $c_0$ is already defined. We start by observing

$$I = \sum_{l \in \mathbb{Z}} c_l \int_{a_l}^{a_{l+1}} dx f(x) G(x) g(x).$$  

(C74)

Since $f, G$ and $g$ are smooth on the intervals $(a_l, a_{l+1})$, we can now integrate by parts:

$$I = \sum_{l \in \mathbb{Z}} c_l [f(x) G(x) v(x)]_{a_l}^{a_{l+1}} + c_l \int_{a_l}^{a_{l+1}} dx \left( \frac{d}{dx} f(x) G(x) \right) v(x).$$  

(C75)

Taking absolute values, employing the triangle inequality and rearranging the summation, we arrive at

$$|I| \leq \sum_{l \in \mathbb{Z}} \left| c_l \int_{a_l}^{a_{l+1}} dx \left| f(x) G(x) \right| v(x) \right| + 2 \sum_{l \in \mathbb{Z}} \left| f(a_l) G(a_l) v(a_l) \right| .$$  

(C76)

$$\leq v_0 \sum_{l \in \mathbb{Z}} \int_{a_l}^{a_{l+1}} dx \left| \frac{d}{dx} f(x) G(x) \right| v(x),$$  

(C77)

$$\leq v_0 \sum_{l \in \mathbb{Z}} \int_{a_l}^{a_{l+1}} dx \left| f(x) G(x) \right| + \int_{a_l}^{a_{l+1}} dx \left| f(x) \right| \frac{d}{dx} G(x) \right|,$$  

(C78)

$$\leq v_0 \left( \sum_{l \in \mathbb{Z}} \int_{a_l}^{a_{l+1}} dx \left| f(x) \right| \frac{d}{dx} G(x) \right) + \sum_{l \in \mathbb{Z}} G(a_l) \int_{a_l}^{a_{l+1}} dx \left| f(x) \right| + J \sum_{l \in \mathbb{Z}} \int_{a_l}^{a_{l+1}} dx \left| \frac{d}{dx} G(x) \right|,$$  

(C79)

where we have denoted $J := \max_{x \in [0,T]} |f(x)|$. Observing that the last term simplifies to

$$\sum_{l \in \mathbb{Z}} \int_{a_l}^{a_{l+1}} dx \left| \frac{d}{dx} G(x) \right| = \sum_{l \in \mathbb{Z}} \int_{a_l}^{a_{l+1}} dx \frac{d}{dx} G(x) = \sum_{l \in \mathbb{Z}} |G(a_{l+1}) - G(a_l)| = 2 \sum_{l \in \mathbb{Z}} G(a_l),$$  

(C80)

we conclude

$$|I| \leq v_0 \left( \sum_{l \in \mathbb{Z}} \left| G(a_{l+1}) \right| \int_{a_l}^{a_{l+1}} dx \left| \frac{d}{dx} f(x) \right| + \sum_{l \in \mathbb{Z}} G(a_l) \int_{a_l}^{a_{l+1}} dx \left| \frac{d}{dx} f(x) \right| + 2J \sum_{l \in \mathbb{Z}} G(a_l) \right).$$  

(C81)

We now further decompose the integral $\int_{a_l}^{a_{l+1}} dx \left| \frac{d}{dx} f(x) \right|$ into a summation over the interval where the derivative of $f$ does not change sign, namely let $b_{r(l)}, b_{r(l)+1}, b_{r(l)+2}, \ldots, b_{r(l)+m(l)}$ be the sequence of the locations of the turning points of $f$ in interval $(a_l, a_{l+1})$ ordered in ascending order. We thus have:

$$\int_{a_l}^{a_{l+1}} dx \left| \frac{d}{dx} f(x) \right| = \int_{a_l}^{b_{r(l)}} dx \left| \frac{d}{dx} f(x) \right| + \int_{b_{r(l)}}^{b_{r(l)+1}} dx \left| \frac{d}{dx} f(x) \right| + \int_{b_{r(l)+1}}^{b_{r(l)+2}} dx \left| \frac{d}{dx} f(x) \right| + \ldots + \int_{b_{r(l)+m(l)}} dx \left| \frac{d}{dx} f(x) \right|$$  

(C82)

$$\leq 2 \left( |f(b_{r(l)})| + |f(b_{r(l)+1})| + |f(b_{r(l)+2})| + \ldots + |f(b_{r(l)+m(l)})| \right) \leq 2N_{TP} J$$  

(C83)

for all $l \in \mathbb{Z}$. We can now plug eq. (C82) into eq. (C81), yielding:

$$|I| \leq v_0 \left( \sum_{l \in \mathbb{Z}} G(a_{l+1}) + \sum_{l \in \mathbb{Z}} G(a_l) \right) 2N_{TP} J + 2J \sum_{l \in \mathbb{Z}} G(a_l)$$  

(C84)

$$= 2J v_0 \left( 2N_{TP} G(0) + (2N_{TP} + 1) \sum_{l \in \mathbb{Z}} G(a_l) \right),$$  

(C85)
thus observing that $\sum_{l \in \mathbb{Z}} G(\alpha_l) \leq NZ \left( \sum_{n \in \mathbb{N}_0} G(nT) + G(-nT) \right) = NZ \left( G(0) + \sum_{n \in \mathbb{Z}} G(nT) \right)$ we conclude the proof.