REFINED SIGN-BALANCE ON 321-AVOIDING PERMUTATIONS

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version of May 22, 2003

Abstract. The number of even 321-avoiding permutations of length $n$ is equal to the number of odd ones if $n$ is even, and exceeds it by the $\frac{n-1}{2}$th Catalan number otherwise. We present an involution that proves a refinement of this sign-balance property respecting the length of the longest increasing subsequence of the permutation. In addition, this yields a combinatorial proof of a recent analogous result of Adin and Roichman dealing with the last descent. In particular, we answer the question how to obtain the sign of a 321-avoiding permutation from the pair of tableaux resulting from the Robinson-Schensted-Knuth algorithm. The proof of the simple solution bases on a matching method given by Elizalde and Pak.

1 Introduction

Let $T_n$ be the set of 321-avoiding permutations in the symmetric group $S_n$. (A permutation is called 321-avoiding if it has no decreasing subsequence of length three.) Simion and Schmidt \cite{SimionSchmidt} proved the following sign-balance property of $T_n$: the number of even permutations in $T_n$ is equal to the number of odd permutations if $n$ is even, and exceeds it by the Catalan number $C_{\frac{n-1}{2}}$ otherwise. Very recently, Adin and Roichman \cite{AdinRoichman} refined this result by taking into account the maximum descent. We give an analogous result for a further important permutation statistic, the length of the longest increasing subsequence. In a recent paper, Stanley \cite{Stanley} established the importance of the sign-balance.

For $\pi \in T_n$, let $\text{lis}(\pi)$ be the length of the longest increasing subsequence in $\pi$. By \cite[Thm. 4]{SimionSchmidt}, the number of permutations $\pi \in T_n$ for which $\text{lis}(\pi) = k$ is just the square of the ballot number

$$b(n,k) = \frac{2k - n + 1}{n+1} \binom{n+1}{k+1}$$

where $\lfloor \frac{n+1}{2} \rfloor \leq k \leq n$. This is an immediate consequence of the Robinson-Schensted-Knuth correspondence which gives a bijection between permutations and pairs of standard Young tableaux of the same shape (see, e.g., \cite{Skand})}. It is well-known that the length of the longest increasing (decreasing) subsequence of a permutation is just the length of the first row (column) of its associated tableaux.
Consequently, 321-avoiding permutations correspond to pairs of standard Young tableaux having at most two rows. Such tableaux can be identified with ballot sequences in a natural way. (We call a sequence \( b_1 b_2 \cdots b_n \) with \( b_i = \pm 1 \) a ballot sequence if \( b_1 + b_2 + \cdots + b_j \geq 0 \) for all \( j \).

Given a standard Young tableau with at most two rows, set \( b_i = 1 \) if \( i \) appears in the first row and \( b_i = -1 \) otherwise. By the Ballot theorem (see, e.g., [4]), there are \( b(n, k) \) ballot sequences of length \( n \) having exactly \( k \) components equal 1.

Elizalde and Pak [3] already made use of the simplicity of the RSK correspondence for 321-avoiding permutations for their bijection between refined restricted permutations.

Our main result is the following.

**Theorem 1.1** For all \( n \geq 1 \), we have

\[
\sum_{\pi \in T_{2n+1}} \text{sign}(\pi) \cdot q^{\text{lis}(\pi)} = \sum_{\pi \in T_n} q^{2\text{lis}(\pi)+1}
\]

\[
\sum_{\pi \in T_{2n+2}} \text{sign}(\pi) \cdot q^{\text{lis}(\pi)} = (q - 1) \sum_{\pi \in T_n} q^{2\text{lis}(\pi)+1}.
\]

To prove this theorem, we establish an involution on \( T_n \) which is defined in terms of the corresponding tableau pairs. While it is clear how to see the length of the longest increasing subsequence from the tableaux, for the sign of a permutation this connection is unknown until now. In Section 2, we solve this problem for 321-avoiding permutations.

The description of the main bijection \( \Phi \) is done in Section 3. Modifying \( \Phi \) slightly yields a combinatorial proof of Adin-Roichman’s result which is given in Section 4. We conclude with a simple combinatorial proof of the equidistribution of the last descent and \( \pi^{-1}(n) - 1 \) over \( T_n \), Adin and Roichman asked for.

## 2 How to obtain the sign from the tableaux

The key problem we are confronted with is to figure out how to see the sign of a permutation by looking at its associated pair of tableaux. For 321-avoiding permutations, there is a simple answer.

Given \( \pi \in T_n \), let \( p \) and \( q \) be the ballot sequences defined by the tableaux \( P \) and \( Q \) which we obtain by applying the RSK algorithm to \( \pi \). (As usual, we write \( P \) to denote the insertion tableau, and \( Q \) for the recording tableau.) Define the statistic \( srs \) to be the sum of the elements of the second row of \( P \) and \( Q \), that is,

\[
srs(\pi) = \sum_{p_i = -1} i + \sum_{q_i = -1} i.
\]

**Proposition 2.1** For any \( \pi \in T_n \) with \( \text{lis}(\pi) = k \), we have \( \text{sign}(\pi) = (-1)^{srs(\pi) + n - k} \).
For the proof we use a method to generate a matching between exceedances and anti-exceedances described by Elizalde and Pak. For \( \pi \in S_n \), an exceedance (anti-exceedance) of \( \pi \) is an integer \( i \) for which \( \pi_i > i \) (\( \pi_i < i \)). Here the element \( \pi_i \) is called an exceedance letter (anti-exceedance letter). It is characteristic for 321-avoiding permutations that both the subword consisting of all exceedance letters and the subword consisting of the remaining letters are increasing. Due to this condition, we say that the permutation is bi-increasing. In particular, the fixed points of a permutation \( \pi \in T_n \) are just such integers \( i \) for which \( \pi_j < \pi_i \) for all \( j < i \) and \( \pi_j > \pi_i \) for all \( j > i \). Consequently, each longest increasing subsequence of \( \pi \) contains all the fixed points of \( \pi \).

Given a permutation \( \pi \in T_n \), let \( i_1 < i_2 < \ldots < i_s \) be its exceedances and \( j_1 < j_2 < \ldots < j_t \) the anti-exceedances. Construct the matching as follows.

1) Initialize \( a = b = 1 \).
2) While \( a \leq s \) and \( b \leq t \) repeat the procedure:
   - If \( i_a > j_b \), then increase \( b \) by 1.
   - If \( \pi_{i_a} < \pi_{j_b} \), then increase \( a \) by 1.
   - If \( i_a < j_b \) and \( \pi_{i_a} > \pi_{j_b} \), then match \( i_a \) with \( j_b \) and increase both \( a \) and \( b \) by 1.

For more clarity, we represent \( \pi \in T_n \) by an \( n \times n \) array with a dot in each of the squares \((i, \pi_i)\). We will identify the integer \( i \) with the dot \((i, \pi_i)\), and use the terms “exceedance” and “anti-exceedance” correspondingly for the dots as well.

\[ \begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & & & & & & & & & \\
\bullet & & & & & & & & & & & \\
& & & & & & & & & & & \\
\end{array} \]

Figure 1 Example of the matching for \( \pi = 4 \ 1 \ 2 \ 5 \ 7 \ 8 \ 3 \ 6 \ 9 \ 12 \ 10 \ 11 \) \( \in T_{12} \)

It follows immediately from the descriptions of the RSK algorithm and the matching that the element \( \pi_i \) appears in the second row of the tableau \( P \) if and only if \((i, \pi_i)\) is a matched exceedance. In particular, the number of matched pairs is just the length of the second row of \( P \), see Lemma 5. Because of the symmetry of the RSK correspondence – if \((P, Q)\) is the pair of tableaux associated with \( \pi \in S_n \), then \((Q, P)\) corresponds to the inverse permutation \( \pi^{-1} \) – the integer \( j \) is contained in the second row of \( Q \) if and only if \((j, \pi_j)\) is a matched anti-exceedance.
For the permutation \( \pi \in T_{12} \) considered in the above figure, we obtain the tableaux

\[
P = \begin{array}{ccccccccccc}
1 & 2 & 3 & 6 & 8 & 9 & 10 & 11 \\
4 & 5 & 7 & 12 \\
\end{array}
\quad Q = \begin{array}{ccccccccccc}
1 & 3 & 4 & 5 & 6 & 9 & 10 & 12 \\
2 & 7 & 8 & 11 \\
\end{array}
\]

and hence \( \text{rsrs}(\pi) = 56 \).

The connection between the matched pairs and the inversions of the permutation is not difficult to see. We call a pair \( (i, j) \) an inversion of \( \pi \) if \( i < j \) and \( \pi_i > \pi_j \), and write \( \text{inv}(\pi) \) to denote the number of inversions of \( \pi \).

Let \([i, j]\) be a matched pair where \( i < j \). Denote by \( c(i, j) \) the number of integers \( l \) for which \( i < l < j \) and \( \pi_l > \pi_i \) or \( j < l \) and \( \pi_j < \pi_l < \pi_i \). Graphically, the dots \((l, \pi_l)\) are contained in the regions 2 and 3 of the scheme:

\[
\begin{array}{c}
\pi_j \\
\pi_i \\
1 \\
2 \\
3 \\
4
\end{array}
\]

Clearly, the first mentioned ones are excedances while the last mentioned are anti-excedances.

For \( r = 1, \ldots, 4 \), let \( c_r(i, j) \) be the number of dots in the region \( r \).

**Lemma 2.2** Let \( \pi \in T_n \) with \( \text{lis}(\pi) = k \). Furthermore, let \([i_1, j_1], [i_2, j_2], \ldots, [i_{n-k}, j_{n-k}]\) be the matched pairs where \( i_l < j_l \) for all \( l \). Then we have:

a) \( c(i, j) \) is even if and only if \( \pi_i + j \) is even.

b) \( \text{inv}(\pi) = c(i_1, j_1) + c(i_2, j_2) + \ldots + c(i_{n-k}, j_{n-k}) + n - k \).

**Proof.**

a) Since \( \pi \in T_n \), we have \( c_1(i, j) = 0 \). Moreover, there is no dot northeast of \((i, \pi_i)\) or southwest of \((j, \pi_j)\). Hence \( c_2(i, j) + c_3(i, j) + 2c_4(i, j) = n - \pi_i + n - j \). Thus we have \( c(i, j) = c_2(i, j) + c_3(i, j) \equiv \pi_i + j \mod 2 \).

b) Since \( \pi \) is bi-increasing, any inversion of \( \pi \) has to be a pair \((i, j)\) with an excedance \( i \) and an anti-excedance \( j \).

First assume that \( i \) is a matched excedance, and let \( j \) be its match. Furthermore, let \((i, j')\) be an inversion. If \( j \leq j' \) we have \( \pi_j \leq \pi_j' < \pi_i \). There are exactly \( c_3(i, j) + 1 \) such integers \( j' \). In case \( j' < j \), the anti-excedance \( j' \) is matched with an excedance \( i' \) for which \( i' < i \). (Otherwise, \( j' \) would be matched with \( i \) by the definition of the matching.) Thus \((i, \pi_i)\) belongs to the dots which are counted by \( c_2(i', j') \).

Now let \( i \) be an unmatched excedance. If \((i, j)\) is an inversion, then \( j \) must be matched. (Otherwise, we could match \( i \) with \( j \).) By the construction, the match \( i' \) of \( j \) satisfies \( i' < i \) (and
hence \( \pi' < \pi_i \). Thus \((i, \pi_i)\) is contained in the region whose dots are counted by \( c_2(i', j) \). \( \square \)

By the definition, we have \( \text{srs}(\pi) = \pi_{i_1} + \pi_{i_2} + \ldots + \pi_{i_n-k} + j_1 + j_2 + \ldots + j_{n-k} \). Therefore, from the lemma we immediately obtain the assertion of Proposition 2.1: a 321-avoiding permutation is even if and only if the sum of the elements of the second row of the two tableaux, increased by the length of this row, is even.

### 3 Proof of Theorem 1.1

In this section, we construct a bijection on \( T_n \) which proves the main result. Its essential part is an involution on the ballot sequences of length \( n \) having a given number \( k \) of 1’s.

We define the \textit{sign} of a ballot sequence \( b \) to be 1 if the sum of all integers \( i \) with \( b_i = -1 \) is even and \(-1\) otherwise, and write \( \text{sign}(b) \) to denote it.

For a ballot sequence \( b \), let \( \varepsilon(b) \) be the smallest \textbf{even} integer \( i \) for which \( b_i = -b_{i+1} \). If there is no such integer set \( \varepsilon(b) = 0 \). By \( A_{n,k} \) we denote the set of all ballot sequences \( b \) of length \( n \) having \( k \) 1’s and satisfying \( \varepsilon(b) > 0 \), and by \( A^*_{n,k} \) the set of all sequences \( b \) with \( \varepsilon(b) = 0 \).

Consider now the following map \( \phi \) on \( A_{n,k} \). Let \( b \in A_{n,k} \) satisfy \( \varepsilon(b) = j \). Then define \( c = \phi(b) \) to be the sequence which is obtained from \( b \) by exchanging the elements \( b_j \) and \( b_{j+1} \), i.e., \( c_j = -b_j \), \( c_{j+1} = -b_{j+1} \), and \( c_i = b_i \) otherwise.

**Proposition 3.1**

1. The map \( \phi \) is a \textbf{sign-reversing} involution on \( A_{n,k} \).
2. For odd \( n \), we have \( \left| A^*_{n,k} \right| = b\left(\frac{n-1}{2}, \frac{k-1}{2}\right) \) if \( k \) is odd, and \( \left| A^*_{n,k} \right| = 0 \) otherwise.
3. For even \( n \), we have \( \left| A^*_{n,k} \right| = b\left(\frac{n}{2} - 1, \left\lfloor \frac{k-1}{2} \right\rfloor \right) \) for all \( k \).

**Proof.**

a) If \( b \in A_{n,k} \) with \( \varepsilon(b) = j \), then the sequence \( c = \phi(b) \) belongs to \( A_{n,k} \) as well. Note that \( b_1 + \ldots + b_{j-1} \geq 1 \) since \( j \) is even, and hence \( c_1 + \ldots + c_j = b_1 + \ldots + b_{j-1} - b_j \geq 0 \). Obviously, \( \phi \) is an involution. By the construction, the sum of the positions of the \(-1\)’s in \( b \) and \( c \) differs by 1. Hence \( \text{sign}(b) = -\text{sign}(c) \).

b), c) A sequence \( b \) belongs to \( A^*_{n,k} \) if and only if \( b_{2i} = b_{2i+1} \) for \( 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

It is easy to see that there is a bijection between the sequences in \( A^*_{n,k} \) and all the ballot sequences of length \( \left\lfloor \frac{n-1}{2} \right\rfloor \) having a certain number of 1’s. In case of odd \( n \), set \( d = b_2 b_4 \ldots b_{n-1} \). (Note that \( b_2 = b_3 = 1 \); otherwise, we would have \( b_1 + b_2 + b_3 = -1 \).) The number \( k \) of 1’s in \( b \) is always an odd integer since \( b_1 = 1 \). Therefore, \( d \) is a ballot sequence with \( \frac{k-1}{2} \) components equal 1. If \( n \) is even, then the sequence \( d \) is defined as \( d = b_2 b_4 \ldots b_{n-2} \). Because we have \( b_n = 1 \) if \( k \) is even and \( b_n = -1 \) otherwise we may omit the final element. Hence the number of 1’s in \( d \) is
equal to \( \lfloor \frac{k - 1}{2} \rfloor \).

It should be clear now how to construct the involution for proving Theorem 1.1. Given \( \pi \in T_n \) with \( \text{lis}(\pi) = k \), let \( p \) and \( q \) be the ballot sequences defined by the tableaux \( P \) and \( Q \) resulting from the RSK algorithm. Define \( \sigma = \Phi(\pi) \) to be the 321-avoiding permutation whose associated pair \((P', Q')\) of tableaux is given by the pair

\[
(p', q') = \begin{cases} 
(\phi(p), q) & \text{if } p \in A_{n,k} \\
(p, \phi(q)) & \text{if } p \in A^*_{n,k} \text{ and } q \in A_{n,k} \\
(p, q) & \text{otherwise}
\end{cases}
\]
on of ballot sequences.

Since \( \phi \) preserves the number of 1’s in a ballot sequence we have \( \text{lis}(\sigma) = \text{lis}(\pi) \). By the definition, \( \text{srs}(\pi) \) is even if and only if \( p \) and \( q \) have the same sign. Consequently, \( \Phi \) reverses the sign of the permutation if \( p \in A_{n,k} \) or \( q \in A_{n,k} \). (This follows from 2.1 and 3.1a.)

Let now \( \pi \) be a fixed point of \( \Phi \), that is, \( \varepsilon(p) = \varepsilon(q) = 0 \). Clearly, the number of such permutations \( \pi \) is just \( |A^*_{n,k}| \). In case of odd \( n \) (and hence odd \( k \)), we have \( \text{sign}(\pi) = (-1)^{\text{srs}(\pi)} \).

By the proof of Proposition 3.1b, \( p_i = -1 \) for any even integer \( i \) if and only if \( p_{i+1} = -1 \). The same relation holds for the sequence \( q \), too. Therefore, \( \sum p_i = -1 i \) and \( \sum q_i = -1 i \), respectively, are the sum of \( \frac{n-k}{2} \) odd numbers. In particular, \( p \) and \( q \) are of the same sign. Thus \( \text{srs}(\pi) \) is even, and \( \text{sign}(\pi) = 1 \). By similar reasoning, we obtain \( \text{sign}(\pi) = (-1)^k \) if \( n \) is even.

Summarized, \( \Phi \) yields the relations

\[
e(2n + 1, 2k + 1) - o(2n + 1, 2k + 1) = b(n, k)^2
\]
\[
e(2n + 1, 2k) - o(2n + 1, 2k) = 0
\]
\[
e(2n + 2, 2k + 1) - o(2n + 2, 2k + 1) = -b(n, k)^2
\]
\[
e(2n + 2, 2k + 2) - o(2n + 2, 2k + 2) = b(n, k)^2
\]

for all \( n \) and \( k \). Here let \( e(n, k) \) and \( o(n, k) \) denote the number of even and odd permutations in \( T_n \), respectively, whose longest increasing subsequence is exactly of length \( k \).

### 4 A Combinatorial Proof of Adin-Roichman’s Result

In [1], Adin and Roichman proved the sign-balance on the set of 321-avoiding permutations having a given last descent by a recursion formula for the generating function and asked for a combinatorial proof. By modifying \( \Phi \) slightly, we obtain the desired involution. Moreover, this involution even proves a further refinement of our main result and Theorem 4.1.

For \( \pi \in S_n \), a descent of \( \pi \) is an integer \( i \) for which \( \pi_i > \pi_{i+1} \). The set of the descents of \( \pi \) we
Lemma 4.2

Let $p$ associated pair of tableaux. Furthermore, let $b_1, b_2, \ldots, b_n$ respectively. (The set $A_n$ and $B_n, B_{n,k}^1, B_{n,k}^2$ to denote the set of all ballot sequences $b$ of length $n$ having $k$ 1’s and satisfying $0 < \varepsilon(b) < \delta(b) - 1$, $\varepsilon(b) = \delta(b) - 1$, and $\varepsilon(b) \geq \delta(b)$, respectively. (The set $A_{n,k}$ is just the union of $B_{n,k}, B_{n,k}^1$, and $B_{n,k}^2$.)

Theorem 4.1 (Π Thm. 4.1) For all $n \geq 1$, we have

$$
\sum_{\pi \in T_{2n+1}} \text{sign}(\pi) \cdot q^{\text{ldes}(\pi)} = \sum_{\pi \in T_n} q^{2\text{ldes}(\pi)}, \quad \sum_{\pi \in T_{2n}} \text{sign}(\pi) \cdot q^{\text{ldes}(\pi)} = (1-q) \sum_{\pi \in T_n} q^{2\text{ldes}(\pi)}.
$$

By the description of the RSK algorithm, the last descent is just the maximum element $i$ in the first row of the tableau $Q$ for which $i+1$ appears in the second row of $Q$. (Moreover, an integer $i$ is a descent of $\pi \in T_n$ if and only if $i$ appears in the first row while $i+1$ is in the second row of $Q$.) Thus the bijection $\Phi$ also preserves the last descent of $\pi$ if $\varepsilon(p) > 0$ or $\varepsilon(q) \leq \text{ldes}(\pi) - 2$. Consequently, the involution we look for will nearly coincide with $\Phi$; we only have to modify the cases $\varepsilon(p) = 0$, $\varepsilon(q) \geq \text{ldes}(\pi) - 1$ and $\varepsilon(p) = \varepsilon(q) = 0$.

For a ballot sequence $b$, let $\delta(b)$ be the greatest integer $i$ for which $b_i = -b_{i+1} = 1$. (As mentioned above, we have $\delta(q) = \text{ldes}(\pi)$ if $q$ is the sequence defined by the insertion tableau of $\pi$.) In addition to $A_{n,k}$ and $A_{n,k}^*$, we use $B_{n,k}, B_{n,k}^1, B_{n,k}^2$ to denote the set of all ballot sequences $b$ of length $n$ having $k$ 1’s and satisfying $0 < \varepsilon(b) < \delta(b) - 1$, $\varepsilon(b) = \delta(b) - 1$, and $\varepsilon(b) \geq \delta(b)$, respectively. (The set $A_{n,k}$ is just the union of $B_{n,k}, B_{n,k}^1$, and $B_{n,k}^2$.)

Lemma 4.2 Let $\pi \in T_n$ be a permutation for which $\text{lis}(\pi) = k$ and $\text{ldes}(\pi) = d$, and $(P, Q)$ its associated pair of tableaux. Furthermore, let $p$ and $q$ be the ballot sequences defined by $P$ and $Q$, respectively. Assume that $p \in A_{n,k}^*$ and $q \in A_{n,k}^* \cup B_{n,k}^1 \cup B_{n,k}^2$. Then we have:

a) If $d$ is even, then $\pi$ is even.

b) If both $n$ and $d$ are odd, then $\pi$ is even if and only if $q \in A_{n,k}^*$. Moreover, there are as many sequences $q \in A_{n,k}^*$ as sequences $q \in B_{n,k}^*$ satisfying $\delta(q) = d$.

c) If $n$ is even and $d$ is odd, then $\pi$ is even if and only if $q \in A_{n,k}^*$ and $k$ is even. Moreover, for any even $k$ there are as many sequences $q \in A_{n,k}^*$ as sequences $q \in B_{n,k}^*$ with $q_n = 1$ satisfying $\delta(q) = d$.

Proof. By the definition of $\delta$, we have $q = q_1 q_2 \cdots q_{d-1} (-1)^a 1^b$ where $a, b \geq 0$. (Here exponentiation denotes repetition.)

a) If $d$ is even, then we have $\varepsilon(q) = d$. In particular, all the components $q_i = -1$ with $i < d$ appear in pairs. First let $n$ be odd. By Proposition 3.1, then $k$ must be odd as well since $p \in A_{n,k}^*$. As already discussed in the previous section, $\sum_{h=-1}^n i$ is the sum of $\frac{n-k}{2}$ odd numbers. Since $q$ contains the element $-1$ exactly $n-k$ times, the exponent $a$ is odd. Thus $\sum_{q_i = -1} i$ is
also the sum of \( \frac{n-k}{2} \) odd numbers. Hence \( p \) and \( q \) have the same sign, and \( \text{rs}(\pi) \) is even. In case of even \( n \), we have to distinguish whether \( k \) is even or not. By reasoning in a similar way as done for odd \( n \), we can show that \( \text{rs}(\pi) \) is even if and only if \( k \) is even. Thus \( \text{sign}(\pi) = 1 \).

**b)** Let both \( n \) (and hence \( k \)) and \( d \) be odd. By the definition, we have \( q \in A^*_{n,k} \) if \( q_{d-1} = 1 \), and \( q \in B^*_{n,k} \) if \( q_{d-1} = -1 \). (Note that the exponent \( a \) must be odd in the first case.) Consider the following map \( \psi \) that takes \( q \in A^*_{n,k} \) to a sequence \( q' \in B^*_{n,k} \). For \( q \in A^*_{n,k} \), let \( j \) be the last position of \(-1\) in \( q \). Define \( q' \) to be the sequence which is obtained from \( q \) by exchanging the elements \( q_{d-1} \) and \( q_j \). Since \( d-1 \) is even, we have \( q_1 + \ldots + q_{d-2} \geq 1 \), and the exchange is permitted. Obviously, \( q' \in B^*_{n,k} \) and \( \delta(q') = d \). (Note that \( j \) must be odd. Thus we may assume that \( j > d+1 \).) Moreover, \( \psi \) reverses the sign. It is easy to see that \( \psi \) is bijective. The inverse map just takes \( q \in B^*_{n,k} \) to the sequence \( q' \in A^*_{n,k} \) which is defined by \( q'_{d-1} = q_j = 1 \), \( q'_{d-1} = q_{d-1} = -1 \) and \( q'_i = q_i \) otherwise where \( j \) is the position of the first \( 1 \) to the right of \( q_d \).

Similarly to part a), we can prove that the signs of \( p \) and \( q \) coincide if \( q \in A^*_{n,k} \). Applying \( \psi \) yields the conversion.

**c)** Let \( n \) be even now and \( d \) odd. For even \( k \) we can prove the assertion by the same arguments as used in part b). Note that the map \( \psi \) is not defined if \( q \in B^*_{n,k} \) ends with an element \(-1\). In particular, we have \( \text{sign}(p) = \text{sign}(q) \) (and hence \( \text{sign}(\pi) = 1 \)) if and only if \( q \in A^*_{n,k} \). If \( k \) is odd, then \( p \) and \( q \) are of the same sign. On the one side, we have \( p_n = -1 \), on the other side either there is an even integer \( i \) for which \( q_i = -1 \) and \( q_{i+1} = 1 \) or \( d = n-1 \). Hence \( \text{sign}(\pi) = -1 \). \( \square \)

Let \( \pi \in T_n \) satisfy \( \text{lis}(\pi) = k \) and \( \text{ldes}(\pi) = d \). As before, let \( p \) and \( q \) be the ballot sequences defined by the tableaux \( P \) and \( Q \) obtained from the RSK correspondence. Define \( \sigma = \Psi(\pi) \) to be the 321-avoiding permutation whose associated tableaux \( P' \) and \( Q' \) are given by the sequences

\[
(p', q') = \begin{cases} 
(\phi(p), q) & \text{if } p \in A_{n,k} \\
(p, \phi(q)) & \text{if } p \in A^*_{n,k} \text{ and } q \in B_{n,k} \\
(p, \psi(q)) & \text{if } p \in A^*_{n,k} \text{ and } q \in A^*_{n,k} \cup B^*_{n,k}; \text{ } n-k \text{ is even and } d \text{ is odd} \\
(p, q) & \text{otherwise}
\end{cases}
\]

where \( \psi \) is defined in the proof of Lemma 4.2b, and \( B^*_{n,k} \) denotes the set of all sequences \( q \in B^*_{n,k} \) for which \( q_n = 1 \). By the previous discussion, \( \Psi \) is an involution on \( T_n \) which preserves both \( \text{lis} \) and \( \text{ldes} \), and reverses the sign if \( \pi \) is not a fixed point.

Finally, we enumerate the fixed points of \( \Psi \) regarding the last descent.

**Proposition 4.3** Let \( f(n, d) \) be the number of permutations \( \pi \in T_n \) for which \( \Psi(\pi) = \pi \) and \( \text{ldes}(\pi) = d \).

**a)** For odd \( n \), we have \( f(n, d) = b(\frac{n+d-3}{2}, \frac{n-3}{2}) \) if \( d \) is even, and \( f(n, d) = 0 \) otherwise.

**b)** For even \( n \), we have \( f(n, d) = b(\lfloor \frac{n+d-2}{2} \rfloor, \frac{n-2}{2}) \) for all \( d \).
Proof. a) First let \( n \) be odd. In this case, \( \pi \) is a fixed point if and only if \( p \in A^n_{n,k} \) and \( \varepsilon(q) = d \). (On condition that \( d \) is even, this is the only possibility for \( q \in A^n_{n,k} \cup B^n_{n,k} \cup B^n_{n,k^+} \).) As shown in the proof of Proposition 3.1, the sequence \( p \) corresponds in a one-to-one fashion to a ballot sequence of length \( \frac{n-1}{2} \) with \( \frac{k-1}{2} \) components equal 1. (Note that \( k \) has to be odd.) On the other side, there is a bijection between ballot sequences \( q \) of length \( n \) with an odd number \( k \) of 1’s satisfying \( \varepsilon(q) = d \) and ballot sequences \( q' \) of length \( \frac{n-1}{2} \) having \( \frac{k-1}{2} \) 1’s and satisfying \( \delta(q') = \frac{d}{2} \). Set \( q' = q_2q_4 \cdots q_{d-2}q_dq_{d+2} \cdots q_n \). Note that we have \( q_i = q_{i+1} \) for all even \( i \leq d \) and \( q_i = 1 \) for \( j+1 \leq i \leq n \) where \( j \) must be an even integer because \( n-k \) is even. In particular, \( q_{d+2} = -1 \). Hence we have \( \delta(q') = \frac{d}{2} \). Consequently, \((p, q)\) corresponds to a pair of ballot sequences that is associated with a permutation \( \tau \in T_{n-1}^{\ast} \) for which \( \text{lides}(\tau) = \frac{d}{2} \) (and \( \text{lis}(\tau) = \frac{k-1}{2} \)). Thus \( f(n, d) = b\left(\frac{n-1}{2} + \frac{d}{2} - 1, \frac{n-1}{2} - 1\right) \).

b) Now consider the case of \( n \) even. If \( d \) is even, we have \( \varepsilon(q) = d \) again. For even \( k \), the situation is similar to part a). The sequence \( q \) corresponds to a ballot sequence \( q' \) of length \( \frac{n}{2} \) with \( \frac{k}{2} \) elements equal 1 and \( \delta(q') = \frac{d}{2} \); set \( q' = q_2q_4 \cdots q_{d-2}q_dq_{d+2} \cdots q_n \). If \( k \) is odd, we can not assume that \( q_{d+2} = -1 \). Thus we must add the element \( q_{d+1} \) to \( q' \) to obtain \( \delta(q') = \frac{d}{2} \).

On the other side, we may omit the final element since we have \( q_n = 1 \). Hence \( q \) can be identified with \( q' = q_2q_4 \cdots q_{d+2}q_{d+2} \cdots q_{n-2}; \) a ballot sequence of length \( \frac{n}{2} \) with \( \frac{k-1}{2} \) 1’s and \( \delta(q') = \frac{d}{2} \). For any fixed point \( \pi \) of \( \Psi \), the sequence \( p \) belongs to \( A^n_{n,k} \). By Proposition 3.1, \( p \) corresponds to a ballot sequence of length \( \frac{n}{2} - 1 \) with \( \left\lfloor \frac{k-1}{2} \right\rfloor \) 1’s. Define \( p' \) to be the sequence obtained from \( p \) by this bijection and adding an element 1 if \( k \) is even and \(-1 \) otherwise. This yields a one-to-one correspondence between \( (p, q) \) and a pair \((p', q')\) of ballot sequences of length \( \frac{n}{2} \) with \( \left\lfloor \frac{k}{2} \right\rfloor \) 1’s satisfying \( \delta(q') = \frac{d}{2} \). Consequently, \( f(n, d) \) counts the number of permutations in \( T_{\frac{n}{2}}^{\ast} \) having the last descent \( \frac{d}{2} \).

It is not difficult to see that for any even \( d \) there are as many sequences \( q \) with \( \delta(q) = d \) as such ones with \( \delta(q) = d + 1 \) among the ballot sequences arising from the insertion tableau of a fixed point of \( \Psi \).

By Lemma 4.2 the fixed points of \( \Psi \) are odd permutations if and only if \( n \) is even and \( d \) is odd. This completes the proof of Theorem 4.1.

As a by-product we obtain the following refinement of the theorems 1.1 and 4.1

**Corollary 4.4** For all \( n \geq 1 \), we have

\[
\sum_{\pi \in T_n} q^{2\text{lis}(\pi)+1} t^{2\text{des}(\pi)} = \sum_{\pi \in T_{2n+1}^*} \text{sign}(\pi) \cdot q^{\text{lis}(\pi)} t^{\text{des}(\pi)}
\]

\[
\sum_{\pi \in T_n} q^{2\text{lis}(\pi)+1} t^{2\text{des}(\pi)} = \sum_{\pi \in T_{2n}^*} \text{sign}(\pi) \cdot q^{\text{lis}(\pi)} t^{\text{des}(\pi)} + q \sum_{\pi \in T_{2n}^*} \text{sign}(\pi) \cdot q^{\text{lis}(\pi)} t^{\text{des}(\pi)}.
\]

where \( T_n^\ast \) (resp. \( T_n^\ast \)) denotes the set of all permutations \( \pi \in T_n \) whose last descent is even and whose longest increasing subsequence is of odd (resp. even) length.
In their paper, Adin and Roichman investigate a further permutation statistic on $T_n$: the position of the letter $n$ in $\pi$, denoted by $l_{\text{ind}}$. Using generating functions they prove the equidistribution of $\text{ldes}$ and $l_{\text{ind}} - 1$. We give a simple bijective proof.

**Theorem 5.1 ([II Thm. 3.1])** The statistics $\text{ldes}$ and $l_{\text{ind}} - 1$ are equidistributed over $T_n$. Moreover, for any $B \subseteq [n - 2]$ they are equidistributed over the set

$$\{ \pi \in T_n : D(\pi^{-1}) \cap [n - 2] = B \}$$

where $[n]$ means the set $\{1, \ldots, n\}$.

**Proof.** Let $\pi \in T_n$ be a permutation for which $\text{ldes}(\pi) = d$. Define $\sigma$ to be the permutation which is obtained from $\pi$ by deleting the letter $n$, followed by inserting the element $n$ between the positions $d$ and $d + 1$.

Obviously, this map is a bijection on $T_n$ since the construction of $\sigma$ preserves the order of the excedance letters and non-excedance letters, respectively. Clearly, $l_{\text{ind}}(\sigma) = d + 1$.

Furthermore, we have $D(\pi^{-1}) \cap [n - 2] = D(\sigma^{-1}) \cap [n - 2]$. By the definition, the descent set $D(\pi^{-1})$ of the inverse of $\pi$ contains all the integers $i$ for which the element $i + 1$ appears to the left of the element $i$ in $\pi$. (If $n - 1 \in D(\pi^{-1})$, then we have $\text{ldes}(\pi) = l_{\text{ind}}(\pi) = d$. In case $\pi_{d+1} = n - 1$, the set $D(\sigma^{-1})$ does not contain $n - 1$.)

Consequently, we also obtain sign-balance on $T_n$ concerning the statistic $l_{\text{ind}}$.

**Acknowledgement** I would like to thank Herb Wilf for the interest taken in this work.
1. R. M. Adin and Y. Roichman, Equidistribution and Sign-Balance on 321-Avoiding Permutations, preprint, 2003, math.CO/0304429.
2. E. Deutsch, A. J. Hildebrand, and H. S. Wilf, Longest increasing subsequences in pattern-restricted permutations, preprint, 2003, math.CO/0304126.
3. S. Elizalde and I. Pak, Bijections for refined restricted permutations, preprint, 2002, math.CO/0212328.
4. W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. I, Wiley, New York, 1968.
5. A. Reifegerste, The excedances and descents of bi-increasing permutations, preprint, 2002, math.CO/0212247.
6. R. Simion and F. W. Schmidt, Restricted Permutations, *Europ. J. Combinatorics* 6 (1985), 383–406.
7. R. P. Stanley, *Enumerative Combinatorics*, vol. II, Cambridge University Press, 1999.
8. R. P. Stanley, Some Remarks on Sign-Balanced and Maj-Balanced Posets, preprint, 2002, math.CO/0211113.