BOUNDED TOPOLOGICAL GROUPS

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Abstract. In this note for a topological group \( G \), we introduce a bounded subset of \( G \) and we find some relationships of this definition with other topological properties of \( G \).

1. preliminaries and Introduction

Suppose that \( G \) is a topological group and \( E \subseteq G \). In this paper, we want to know when \( E \) is bounded or unbounded subset of \( G \) and if \( G \) is metrizability, we show that \( E \subseteq G \) is bounded with respect to topology if and only if it is bounded with respect to metric. Let \( E \subseteq G \) be bounded and closed. Then \( E \) is compact subset of \( G \). Conversely if \( E \) is a component of \( e \) and compact, then \( E \) is bounded. We investigated some topological property for bounded subset of \( G \).

Now we introduce some notations and definitions that we used throughout this paper.

For topological group \( G \), \( e \) is identity element of \( G \) and for \( E \subseteq G \), \( E^{-} \) is closure of \( E \) and for every \( n \in \mathbb{N} \),

\[
E^n = \{x_1x_2x_3...x_n : x_i \in E, 1 \leq i \leq n\}.
\]

A topological space \( X \) is \( O - \)dimensional if the family of all sets that are both open and closed is open basic for the topology.

2. Bounded Topological Groups

Definition 2-1. Let \( G \) be topological group and \( E \subseteq G \). We say that \( E \) is bounded subset of \( G \), if for every neighborhood \( V \) of \( e \), there is natural number \( n \) such that \( E \subseteq V^n \).

It is clear that if \( E \) is bounded subset of \( G \) and \( H \) is subgroup of \( G \), then \( E/H \) is bounded subset of \( G/H \).

Theorem 2-2. Let \( G \) be topological group and metrizable with respect to a left invariant metric \( d \). Then \( G \) is bounded with respect to topology if and only if \( G \) is bounded with respect to metric \( d \).

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Proof. Let $G$ be a bounded topological group and $\varepsilon > 0$. Take $d([0,\varepsilon)) = U \times V$ where $U$ and $V$ are neighborhoods of $e$. Suppose that $W$ is symmetric neighborhood of $e$ such that $W \subseteq U \cap V$. Then there is natural number $n$ such that $W^n = G$. Since $d(W \times W) < \varepsilon$, we show that $d(W^2 \times W^2) < 2\varepsilon$, and so $d(W^n \times W^n) < n\varepsilon$. Assume that $x, y, x', y' \in W$. Then we have

$$d(xy, x'y') \leq d(xy, e) + d(e, x'y') = d(y, x^{-1}) + d(x^{-1}, y') < 2\varepsilon.$$ 

Then $d(G \times G) = d(W^n \times W^n) < n\varepsilon$.

Conversely, suppose that $G$ is bounded with respect to metric $d$. Then there is $M > 0$ such that $d(G \times G) < M$. Let $U$ be a neighborhood of $e$. Choose $\varepsilon > 0$ such that $d^{-1}([0,\varepsilon)) \subseteq U \times U$. Then there is $\varepsilon > M$. Then we have

$$G \times G = d^{-1}([0,M)) = d^{-1}([0,n\varepsilon)) \subseteq V^n \times V^n.$$ 

It follows that $G = V^n$, and so that $G$ is bounded.

\[ \square \]

Theorem 2-3. Let $G$ be topological group and let $H$ be a normal subgroup of $G$. If $H$ and $G/H$ are bounded, then $G$ is bounded.

Proof. Let $U$ be a neighborhood of $e$. Put $V = U \cap H$. Then there are natural numbers $m$ and $n$ such that

$$(U/H)^n = G/H \text{ and } V^m = H.$$ 

We show that $U^{n+m} = G$.

Let $x \in G$. Then if $x \in H$, we have

$$x \in V^m \subseteq U^m \subseteq U^{n+m}.$$ 

Now let $x \notin H$. Then $xH \in (U/H)^n$. Assume that $x_1, x_2, ..., x_n \in U$ such that

$$xH = x_1x_2...x_nH.$$ 

Consequently there is $h \in H$ such that $xh \in U^n$, and so $x \in U^nH \subseteq U^nV^m \subseteq U^nU^m = U^{n+m}$. We conclude that $U^{n+m} = G$, and so $G$ is bounded. \[ \square \]

Theorem 2-4. If $G$ is a locally compact O-dimensional topological group, then $G$ is unbounded.

Proof. Let $U$ be a neighborhood of $e$ such that $U^{-} \text{ is compact and } U^{-} \neq G$. Since $G$ is a O-dimensional topological group, $U$ contains an open and closed neighborhood as $V$. Then $V$ is a compact neighborhood of $e$. By apply [1, Theorem 4.10] to obtain a neighborhood $W$ of $e$ such that $WV \subseteq V$. Take $W_0 = W \cap V$. Then $W_0^2 \subseteq WV \subseteq V \subseteq U^{-}$. By finite induction, we have

$$W_0^n \subseteq W_0W_0^{n-1} \subseteq WV \subseteq V \subseteq U^{-},$$ 

for every natural number $n$. It follows that $W_0^n \not\subseteq G$ for every natural number $n$, and so $G$ is unbounded. \[ \square \]

Theorem 2-5. Suppose that $G$ is a locally compact, Hausdorff, and totally disconnected topological group. Then $G$ is unbounded.

Proof. By using [1, Theorem 3.5] and Theorem 2-4, proof is hold. \[ \square \]

Theorem 2-6. Let $G$ be topological group. Then we have the following assertions.
(1) If $E \subseteq G$ is bounded, then $E^-$ is bounded subset of $G$.
(2) If $G$ is bounded, then $G$ is connected and moreover $G$ has no proper open subgroups.

**Proof.** 1) Let $U$ be a neighborhood of $e$ and suppose that $V$ is a neighborhood of $e$ such that $V^- \subset U$. Since $E$ is bounded subset of $G$, there is natural number $n$ such that $E \subset V^n$. Then $E^- \subset (V^n)^- \subset (V^-)^n \subset U^n$. It follows that $E^-$ is a bounded subset of $G$.

2) Since $G$ is bounded, there is a natural number $n$ such that $G = V^n$ where $V$ is neighborhood of $e$. By using [1, Corollary 7.9], proof is hold.

□

**Corollary 2-7.** Assume that $G$ is a locally compact topological group. Then every bounded and closed subset of $G$ is compact, moreover if $E \subseteq G$ is bounded, then $E^-$ is compact.

Every bounded topological group $G$, in general, is not compact, for example $\mathbb{R}/\mathbb{Z}$ is bounded, but is not compact.

**Theorem 2-8.** Let $G$ be topological group and suppose that $E \subseteq G$ is the component of $e$. If $E$ is compact, then $E$ is bounded.

**Proof.** Since $E$ is the component of $e$, by using [1, Theorem 7.4], for every neighborhood $U$ of $e$, we have $E \subseteq \bigcup_{k=1}^{\infty} U^k$. Since $E$ is compact there is natural number $n$ such that $E \subseteq U^n$. Then $E$ is bounded subset of $G$.

In general, every compact subset $E$ of a topological group $G$ is not bounded and in above Theorem, it is necessary that $E$ must be a component of $e$. For example $Z_n = \{0, 1, 2, ..., n\}$ for every $n \geq 1$, with discrete topology is not bounded, but it is compact.

**Corollary 2-9.** If $G$ is a locally compact topological group, then the component of $e$ is bounded.

**Theorem 2-10.** Let $G$ and $G'$ be topological group and suppose that $\pi : G \rightarrow G'$ is group isomorphism. If $\pi$ is continuous and $E \subseteq G$ is a bounded subset of $G$, then $\pi(E)$ is bounded subset of $G'$.

**Proof.** Let $V'$ be a neighborhood of $e' \in G'$. Then $\pi^{-1}(V')$ is a neighborhood of $e$. Since $E$ is a bounded subset of $G$, there is a natural number $n$ such that $E \subseteq (\pi^{-1})^n(V') \subseteq \pi^{-1}(V'^n)$ implies that $\pi(E) \subseteq V'^n$. Thus $\pi(E)$ is a bounded subset of $G'$.

□

**Definition 2-11.** Let $G$ and $G'$ be topological group. We say that the mapping $\pi : G \rightarrow G'$ is compact, if for every bounded subset $E \subseteq G$, $\pi(E)$ is relatively compact.
Theorem 2-12. Let $G$ and $G'$ be topological group and suppose that $\pi : G \to G'$ is continuous and group isomorphism. Then if $G'$ is locally compact, then $\pi$ is compact.

Proof. Let $E \subseteq G$ be bounded. By using Theorem 2.10, $\pi(E)$ is bounded subset of $G'$ and by using Theorem 2.6, $\pi(E)^-$ is compact, and so that $\pi$ is compact.

References

1. E. Hewitt, K. A. Ross, Abstract harmonic analysis, Springer, Berlin, Vol I 1963.