Algebraic Renormalization of $N = 1$ Supersymmetric Gauge Theories with Supersymmetry Breaking Masses\textsuperscript{1}

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ABSTRACT

We provide $N = 1$ Super Yang-Mills theory in the Wess-Zumino gauge with mass terms for the supersymmetric partners of the gauge fields and of the matter fields, together with a supersymmetric mass term for the fermionic matter fields. All mass terms are chosen in such a way to induce soft supersymmetry breakings at most, while preserving gauge invariance to all orders of perturbation theory. The breakings are controlled through an extended Slavnov-Taylor identity. The renormalization analysis, both in the ultraviolet and in the infrared region, is performed.

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1 Introduction

In a previous paper [1] we dealt with the problem of renormalizing $N = 1$ super Yang-Mills (SYM) in the Wess-Zumino (WZ) gauge, to all orders of perturbation theory. We restricted ourselves to the massless case with exact supersymmetry. Our main results were the expression of the most general counterterm and of the supersymmetric extension of the Adler-Bardeen-Bell-Jackiw gauge anomaly. From the structure of the counterterm, one can infer a nonvanishing $\beta$-function for the physical parameters of the theory: the gauge and the Yukawa couplings constants. Actually, it is known that nonrenormalization properties hold in some cases. In particular it has been shown that a class of $N = 1$ SYM theories satisfying certain criteria have vanishing $\beta$-function [2].

It is of primary importance to investigate whether the mentioned nonrenormalization properties are maintained in realistic supersymmetric models, i.e. in theories in which the supersymmetry is broken by mass terms for the superpartners of the presently known particles, namely for the gauginos, the squarks and the sleptons. This program entails as a necessary preliminary step the construction at the quantum level of such a massive theory with broken supersymmetry, which is the aim of the present work. As explained in [1], the main benefits of superspace are lost if supersymmetry is broken. We therefore prefer to work in the WZ gauge, which has the advantage of depending on a minimal number of fields and which also turns out to be the preferred choice in the phenomenological applications.

The masses being physical quantities, the corresponding mass terms in the action must be chosen gauge invariant. On the other hand some of them will necessarily break supersymmetry. Our task is to control the supersymmetry breakings while preserving gauge invariance to all orders of perturbation theory, by means of a renormalizable quantum field theory.

As in the massless case, the whole information is contained in a generalized Slavnov-Taylor operator, from which the symmetries of the theory, such as BRS and supersymmetry for instance, can be recovered by making a filtration in some global ghosts (see the next section or [1] for details). The masses which break supersymmetry, are introduced through constant shifts of suitable external fields, following a procedure [3] which allows us to control easily the supersymmetry breakings via a Slavnov-Taylor identity, with quantum gauge invariance – expressed by the usual BRS invariance – guaranteed by a simple functional identity.

We restrict ourselves to the mass terms selected by Girardello and Grisaru (GG) [4] as yielding “soft” supersymmetry breakings, i.e. not inducing ultraviolet (UV) divergences worse than logarithmic.

An important property of $N = 1$ supersymmetric theories is their invariance under a phase transformation, not commuting with the supersymmetry generators, called $R$-invariance [5]. It is at the basis of the supercurrent multiplet [6] which plays a crucial role in the proof of the nonrenormalization of the coupling constants of some supersym-
metry gauge models \cite{2}. However $R$-invariance is broken by the mass terms. But this breakdown is also controlled by the Slavnov-Taylor identity which takes into account the transformation properties under $R$ of the same external fields already introduced for controlling the supersymmetry breakings. Moreover, we also consider supersymmetric masses in order to keep in with the GG conditions, which would be violated by a supersymmetry breaking matter fermion mass. Since they break $R$-invariance as well, these masses are also introduced via similar external fields.

Besides the gauginos, all matter fields are made massive. Although this is not necessary it helps in avoiding infrared problems. Only the gauge vector field and the Faddeev-Popov ghosts remain massless.

Gauge field and fermion masses produced by a Higgs mechanism are not considered in the present paper. Gauge invariance is assumed exact, without spontaneous breakdown.

Due to the presence both of massless and of massive fields the proof of the absence of infrared (IR) singularities needs a careful discussion. This will be done in an appendix in order not to charge the main text with technicalities. As we shall see it turns out that the construction based only on the ultraviolet (UV) power counting, yields a model free of IR singularity.

The paper is organized as follows. In Section 2 we give a brief summary of \cite{1}. In Section 3 we present our method for introducing masses and write the general classical action for massive $N = 1$ SYM theory. Section 4 is devoted to the quantum extension of the model, by proving its renormalizability. The Callan-Symanzik equation is derived in Section 5. Finally we give an outlook in the concluding Section 6. The absence of IR singularities is shown in the Appendix.

2 The Massless Case

2.1 The Massless Action

The massless theory \cite{1} is described by an effective action – or vertex functional –

$$\hat{\Gamma}(A^i_\mu, \lambda^i_\alpha, \phi_a, \psi_{aa}, c^i, \bar{c}^i, b^i; A^{*i}_\mu, \lambda^{*i}_\alpha, \phi_a^*, \psi_{aa}^*, c^{*i}, \bar{c}^{*i}, \xi^{\mu}, \varepsilon_\alpha, \eta) = \hat{\Sigma} + O(h),$$  \hspace{1cm} (2.1)
where the superscript on \( \Gamma \) and \( \Sigma \) means “massless”, and \( \hat{\Sigma} \) is the corresponding classical action, equivalent to the effective action at the tree level:

\[
\hat{\Sigma} = \int d^4x \left( \frac{1}{g^2} \left( -\frac{1}{4} F^{i\mu} F_{i\mu} - i \lambda^{i\alpha} \sigma^\mu_{\alpha\beta}(D_\mu \bar{\lambda}^\beta)^i \right) - \frac{1}{8} g^2 |\bar{\phi}_a T^a_{ab} \phi_b|^2 + \frac{1}{2} |D_\mu \phi|^2 - i \bar{\psi}_a^{\alpha} \sigma^\mu_{\alpha\beta}(D_\mu \bar{\psi}^{\beta})_a - 2\lambda_{(abc)} \bar{\phi}_b \phi_c \bar{\lambda} \bar{\phi}_e - i \bar{\psi}_a^{\alpha} T^a_{ab} \phi_b \bar{\lambda}_{\beta} \right)
\]

Here \( s \) is an extended BRS operator which puts together the gauge, the supersymmetry and \( R \)-transformations as well as the translations. We refer the reader to \([1]\) for the expression of the \( s \)-variation of the various fields and global ghosts. The latter are the following:

- the gauge and gaugino fields \( A \) and \( \lambda \), in the adjoint representation of the gauge group, with running index \( i \),
- the scalar and fermion matter fields \( \phi \) and \( \psi \), in some (reducible) representation, with running index \( a \),
- the ghost, antighost and Lagrange multiplier fields \( c, \bar{c} \) and \( b \), in the adjoint representation,
- the “antifields” \( A^* \), \( \lambda^* \), \( \phi^* \) and \( \psi^* \), i.e. the external fields coupled to the BRS transformations of the fields which transform nonlinearly under BRS,
- the coordinate independent infinitesimal parameters \( \xi \), \( \varepsilon \) and \( \eta \) of the translations, supersymmetry transformations and \( R \)-transformations, respectively, promoted to the rank of “global ghosts”, their Grassmann parity being chosen as opposite to the natural one.

The spinor fields are in the Weyl representation, with spinor index \( \alpha = 1,2 \) (and dotted index \( \dot{\alpha} = 1,2 \) for their conjugates).

The dimensions, Grassmann parities, ghost numbers and \( R \)-weights of all the fields and global ghosts are shown in Table \([1]\). More details on our conventions and notations can be found in Appendix C of \([1]\).

The extended BRS operator \( s \) in (2.2) is nilpotent only on-shell, but the terms in the classical action which are quadratic in the external fields \( \lambda^* \) and \( \psi^* \) allow the off-shell
extension of the latter property. The action (2.2) is indeed the most general solution of a Slavnov-Taylor identity, which has been shown in [1] to hold for the effective action $\hat{S}$ to all orders of perturbation theory:

$$\hat{S}(\hat{\Gamma}) = 0,$$

(2.3)

where the Slavnov-Taylor operator of the massless theory is defined, for any functional $F$, by

$$\hat{S}(F) :=$$

$$\int d^4x \left( \frac{\delta F}{\delta A^{\mu}} \frac{\delta}{\delta A^{\mu}} + \frac{\delta F}{\delta A^{\mu}} \frac{\delta}{\delta A^{\mu}} + \frac{\delta F}{\delta \lambda^{i\alpha}} \frac{\delta}{\delta \lambda^{i\alpha}} + \frac{\delta F}{\delta \lambda^{i\alpha}} \frac{\delta}{\delta \lambda^{i\alpha}} - \frac{\delta F}{\delta \lambda^i} \frac{\delta}{\delta \lambda^i} - \frac{\delta F}{\delta \lambda^{i\alpha}} \frac{\delta}{\delta \lambda^{i\alpha}} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} \right) - 2 \varepsilon^{\alpha \beta} \varepsilon^{\beta \gamma} \frac{\partial F}{\partial \varepsilon^{\alpha \beta}} - \eta^{\alpha \beta} \frac{\partial F}{\partial \varepsilon^{\alpha \beta}} - \eta^{\beta \alpha} \frac{\partial F}{\partial \varepsilon^{\beta \alpha}} \right).$$

(2.4)

The corresponding “linearized” Slavnov-Taylor operator reads

$$\hat{B}_F :=$$

$$\int d^4x \left( \frac{\delta F}{\delta A^{\mu}} \frac{\delta}{\delta A^{\mu}} + \frac{\delta F}{\delta A^{\mu}} \frac{\delta}{\delta A^{\mu}} + \frac{\delta F}{\delta \lambda^{i\alpha}} \frac{\delta}{\delta \lambda^{i\alpha}} + \frac{\delta F}{\delta \lambda^{i\alpha}} \frac{\delta}{\delta \lambda^{i\alpha}} - \frac{\delta F}{\delta \lambda^i} \frac{\delta}{\delta \lambda^i} - \frac{\delta F}{\delta \lambda^{i\alpha}} \frac{\delta}{\delta \lambda^{i\alpha}} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} + \frac{\delta F}{\delta \delta F} \frac{\delta}{\delta \delta F} \right) - 2 \varepsilon^{\alpha \beta} \varepsilon^{\beta \gamma} \frac{\partial F}{\partial \varepsilon^{\alpha \beta}} - \eta^{\alpha \beta} \frac{\partial F}{\partial \varepsilon^{\alpha \beta}} - \eta^{\beta \alpha} \frac{\partial F}{\partial \varepsilon^{\beta \alpha}} \right).$$

(2.5)

The fulfillment of the Slavnov-Taylor identity (2.3) implies the nilpotency property

$$(\hat{B}_F)^2 = 0.$$
By filtrating the linearized Slavnov-Taylor operator with the operator

\[ \mathcal{N} = \varepsilon^\alpha \frac{\partial}{\partial \varepsilon^\alpha} + \varepsilon^\dot{\alpha} \frac{\partial}{\partial \varepsilon^{\dot{\alpha}}} + \xi^\mu \frac{\partial}{\partial \xi^\mu} + \eta \frac{\partial}{\partial \eta}, \]

(2.7)

which counts the number of global ghosts, we can extract each quantum symmetry by means of an algebraic identification. Writing

\[ \mathcal{B}_\Gamma = \sum_{n \geq 0} \mathcal{B}_\Gamma^{(n)}, \quad \text{with} \quad [\mathcal{N}, \mathcal{B}_\Gamma^{(n)}] = n \mathcal{B}_\Gamma^{(n)}, \]

(2.8)

we identify \( \mathcal{B}_\Gamma^{(0)} \) as the usual quantized gauge BRS operator, whereas \( \mathcal{B}_\Gamma^{(1)} \) contains the functional operators ("Ward identity operators") \( W_\alpha, W_\mu, \) and \( W_{R} \) of supersymmetry, translation and \( R \)-invariance, respectively:

\[ \mathcal{B}_\Gamma^{(1)} = \varepsilon^\alpha W_\alpha + \varepsilon^{\dot{\alpha}} W_{\dot{\alpha}} + \xi^\mu W_\mu + \eta W_{R} - 2 \varepsilon^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \varepsilon^{\dot{\alpha}} \frac{\partial}{\partial \xi^\mu} - \eta \varepsilon^\alpha \frac{\partial}{\partial \varepsilon^\alpha} + \eta \varepsilon^{\dot{\alpha}} \frac{\partial}{\partial \varepsilon^{\dot{\alpha}}}. \]

(2.9)

This means that, at any order of perturbation theory, we can skip from the language represented by the generalized Slavnov-Taylor operator \( (2.4) \) to the description in terms of the single Ward operators \( \mathcal{W} \). Let us recall that the algebra obeyed by the latter is not closed (see Section 4 of [1]).

### 2.2 Gauge Condition, Ghost Equation and Global Ghost Equations

Since the identities written in this and in the following subsection are not altered by the introduction of the masses we drop here the superscript \( ^0 \) on \( \Gamma \) and \( \Sigma \).

The gauge fixing condition is of the Landau type:

\[ \frac{\delta \Sigma}{\delta b^i} = \partial^\mu A^i_\mu. \]

(2.10)

The ghost equation, peculiar to the Landau gauge, reads

\[ \mathcal{F}_i \Sigma := \int d^4 x \left( \frac{\delta}{\delta c^i} - f^{ijk} \frac{\delta}{\delta b^k} \right) \Sigma = \Delta^i_g, \]

(2.11)

where

\[ \Delta^i_g := \int d^4 x \left( f^{ijk} \left( - A^* j^\mu A^j_\mu + \lambda^* j^\alpha \lambda^k_\alpha + \bar{\lambda}^* j^{\dot{\alpha}} \bar{\lambda}^k_{\dot{\alpha}} + c^* j^k \right) \\
+ T^{ia}_{ab} \left( \phi^a_\alpha \phi^b_\alpha + \bar{\phi}^a_{\dot{\alpha}} \bar{\phi}^b_{\dot{\alpha}} - \psi^* a_\alpha \psi_{b\alpha} - \bar{\psi}^* a_{\dot{\alpha}} \bar{\psi}_{b_{\dot{\alpha}}}, \right) \]

(2.12)

is a classical breaking, i.e. is linear in the dynamical fields.
The global ghost equations are
\[
\frac{\partial \Sigma}{\partial \xi^\mu} = \Delta^t_{\mu}, \quad \frac{\partial \Sigma}{\partial \eta} = \Delta_R,
\]
where $\Delta^t_{\mu}$ and $\Delta_R$ are classical breakings given by
\[
\Delta^t_{\mu} := -i \int d^4x \left( -A^{*i}_\nu \partial_\mu A_i^{\nu} + \lambda^{*i} \partial_\mu \lambda^i + \bar{\lambda}^{*i}_\beta \partial_\mu \bar{\lambda}^{i\beta} + c^{*i} \partial_\mu c^i - \phi^*_a \partial_\mu \phi_a + \bar{\psi}^{*a}_\alpha \partial_\mu \bar{\psi}^{a\alpha} - \bar{\bar{\psi}}^{*a}_\alpha \partial_\mu \bar{\bar{\psi}}^{a\alpha} \right),
\]
\[
\Delta_R := \int d^4x \left( -\lambda^{*i}_\alpha \lambda^i_\alpha + \bar{\bar{\lambda}}^{*i}_\beta \bar{\lambda}^{i\beta} + \frac{2}{3} \phi^*_a \phi_a - \frac{2}{3} \bar{\phi}^*_a \bar{\phi}_a + \frac{1}{3} \psi^{*a}_a \psi_{a\alpha} - \frac{1}{3} \bar{\psi}^{*a}_a \bar{\psi}_{a\alpha} \right).
\]
The two global ghost equations express the linearity of the translations and of the $R$-transformations, respectively.

### 2.3 Algebra, Antighost Equation and Rigid Invariance

The Slavnov-Taylor operator $S$, the functional derivative $\delta/\delta b$, the ghost operator $\mathcal{F}^i$ and the partial derivatives $\partial/\partial \xi^\mu$, $\partial/\partial \eta$ obey an algebra together with the following operators:

The antighost operator
\[
\mathcal{F}^i := \frac{\delta}{\delta \bar{c}^i} + \partial_\mu \frac{\delta}{\delta A^{*i}_\mu} + i \xi^\mu \partial_\mu \frac{\delta}{\delta b^i},
\]
the Ward operator for the rigid transformations:
\[
\mathcal{W}_{\text{rig}} := \int d^4x \left( \sum_{\varphi} \delta_{\varphi} \frac{\delta}{\delta \varphi} \right),
\]
the translation Ward operator:
\[
\mathcal{W}_\mu := -i \int d^4x \left( \sum_{\varphi} \partial_\mu \varphi \frac{\delta}{\delta \varphi} \right)
\]
and the Ward operator for the $R$-transformations:
\[
\mathcal{W}_R := \int d^4x \left( \sum_{\varphi} R_\varphi \varphi \frac{\delta}{\delta \varphi} \right).
\]
The summation over $\varphi$ in (2.17) includes all the fields listed in Table 1. Here $\delta_{\text{rig}} \varphi$ is the infinitesimal rigid transformation of $\varphi$, the gauge, gauginos and (anti)ghost fields transforming in the adjoint representation of the gauge group and the matter fields in their own representation. In (2.18) and (2.19) the summation also includes the external doublets $(u,v)$, $(U,V)$ to be introduced in the next section. $R_\varphi$ is the $R$-weight of the field $\varphi$. 

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The algebra reads

\[ B_F S(F) = 0, \]  
\[ \frac{\delta}{\delta \delta_b} S(F) - B_F \left( \frac{\delta F}{\delta \delta_b} - \partial_\mu A^{i\mu} \right) = \bar{F}^i F, \]  
\[ \bar{F}^i S(F) + B_F \bar{F}^i F = 0, \]  
\[ F^i S(F) + B_F \left( F^i F - \Delta^i_\xi \right) = W^{i}_{\text{rig}} F, \]  
\[ W^{i}_{\text{rig}} S(F) - B_F W^{i}_{\text{rig}} F = 0, \]  
\[ \frac{\partial}{\partial \xi^\mu} S(F) + B_F \left( \frac{\partial F}{\partial \xi^\mu} - \Delta^i_\mu \right) = W^\mu F, \]  
\[ W^\mu S(F) - B_F W^\mu F = 0, \]  
\[ \frac{\partial}{\partial \eta^R} S(F) + B_F \left( \frac{\partial F}{\partial \eta^R} - \Delta^F_\eta \right) = W^R F, \]  
\[ W^R S(F) - B_F W^R F = 0, \]

where \( F \) is an arbitrary functional.

Setting \( F = \Sigma \) and using the Slavnov-Taylor identity and the identities (2.10), (2.11) and (2.13) satisfied by the classical action \( \Sigma \), we get the antighost equation and the Ward identities for the linear symmetries, i.e. for the rigid, translation and \( R \)-symmetries, respectively:

\[ \bar{F}^i \Sigma = 0, \quad W^{i}_{\text{rig}} \Sigma = 0, \quad W^\mu \Sigma = 0, \quad W^R \Sigma = 0. \]  

3 Massive \( N = 1 \) SYM Theory

3.1 Generalities

Let us begin with a general description of the procedure we shall follow for introducing a generic mass term.

Masses are introduced, in the classical theory, in two ways: the first one breaks supersymmetry, the second one does not. The aim of the former is to give masses to the gauginos and to the scalar matter fields. These breakings are known as soft in the classification of [4]. The aim of the latter is to give masses to the fermion matter fields without introducing supersymmetry breakings which would not be “soft”. Only the gauge fields remain massless. Let us remark that all these masses break \( R \)-symmetry. However, the addition of the mass terms to the massless action (2.2) has to preserve gauge invariance, also at the quantum level.
3.1.1 Supersymmetry Breaking Masses

A generic supersymmetry breaking mass term is denoted by

$$\Sigma_m = \int d^4x \, M(x) ,$$  \hspace{1cm} (3.1)

where \(M(x)\) is some quadratic expression in the matter fields or in the gaugino field. Its gauge invariance is expressed by the invariance under the gauge BRS operator \(\hat{\mathcal{B}}^{(0)}_{\Sigma}\) defined by the zeroth order term of the filtration (2.8) – with \(\hat{\Gamma}\) replaced by the massless action \(\hat{\Sigma}\) since we are dealing here with the classical theory. Thus

$$\hat{\mathcal{B}}^{(0)}_{\Sigma} \Sigma_m = 0 .$$  \hspace{1cm} (3.2)

This can equivalently be written as

$$\hat{\mathcal{B}}^{(0)}_{\Sigma} \Sigma_m = O(\varepsilon, \eta) ,$$  \hspace{1cm} (3.3)

the terms in \(\varepsilon\) of the right-side being due to the breakdown of supersymmetry and those in \(\eta\) to the breakdown of \(R\)-invariance.

Adding the mass terms (3.1) to the massless action \(\hat{\Sigma}\), we demand that the new action \(\Sigma\) obeys a new, exact, Slavnov-Taylor identity:

$$\mathcal{S}(\Sigma) = 0 .$$  \hspace{1cm} (3.4)

The idea is to couple each mass term (3.1) to a doublet \((u, v)\) in a BRS invariant way, i.e. in a such a way that the new Slavnov-Taylor identity (3.4) holds, with \(\mathcal{S}\) defined by

$$\mathcal{S}(F) = \hat{\mathcal{S}}(F) + \sum_{u,v} \int d^4x \left( su \frac{\delta F}{\delta u} + sv \frac{\delta F}{\delta v} \right) ,$$  \hspace{1cm} (3.5)

the summation being performed over all doublets \((u, v)\) – and their complex conjugates – associated to each mass term. The Grassmann parities of \(u\), resp. \(v\) are negative, resp. positive. The nilpotent BRS variations \(su\) and \(sv\) are given by

$$su(x) = v(x) + \kappa_u - i \xi^\mu \partial_\mu u(x) + \eta R_u u(x) , \hspace{1cm} s\bar{u} = -\bar{s}u ,$$

$$sv(x) = -2i\varepsilon\sigma^\mu \bar{\varepsilon} \partial_\mu v(x) - i \xi^\mu \partial_\mu v(x) + \eta R_u (v(x) + \kappa_u) , \hspace{1cm} s\bar{v} = \bar{s}v .$$  \hspace{1cm} (3.6)

The external fields \(v\) are shifted by the dimensionful constants \(\kappa_u\) which parameterize the masses. The numbers \(R_u\) are the \(R\)-weights of the doublets \((u, v)\), chosen opposite to those of the mass terms (3.1) in order to keep the formal \(R\)-invariance.

The associated linearized Slavnov-Taylor operator is

$$\mathcal{B}_F = \hat{\mathcal{B}}_F + \sum_{u,v} \int d^4x \left( su \frac{\delta F}{\delta u} + sv \frac{\delta F}{\delta v} \right) ,$$  \hspace{1cm} (3.7)
being defined by (2.5). Again the validity of the Slavnov-Taylor identity (3.4) implies the nilpotency condition
\[(B_{\Sigma})^2 = 0 .\] (3.8)
The identity (3.4) taken at \(u = v = 0\) reads
\[
\delta S(\Sigma)|_{u=v=0} = -\sum_{u,v} \kappa_u \int d^4 x \left( \frac{\delta \Sigma}{\delta u} + \eta R_u \frac{\delta \Sigma}{\delta v} \right)|_{u=v=0} .
\] (3.9)
This is the Slavnov-Taylor identity explicitly broken by the mass terms. The first term in the right-hand side represents the supersymmetry breaking, whereas the second one represents the breaking of \(R\)-invariance.

As we said above we want to keep the usual gauge invariance. We therefore impose the supplementary condition
\[
X\Sigma^{(0)} := \int d^4 x \frac{\delta \Sigma}{\delta u}\bigg|_{\epsilon=\xi=\eta=0} = 0 ,
\] (3.10)
which ensures the validity of the gauge Slavnov-Taylor identity
\[
\delta S(\Sigma)|_{\epsilon=\xi=\eta=0} = 0 .
\] (3.11)
The general solution of the full Slavnov-Taylor identity (3.4) – the gauge fixing condition, ghost equation, etc. being taken into account – has the structure
\[
\Sigma = \ddot{\Sigma} + \sum_{u,v} \left( B_{\dot{\Sigma}} \int d^4 x \ u M_u(x) + \Delta_u \right)
\]
\[
= \ddot{\Sigma} + \sum_{u,v} \left( \int d^4 x \ \left( (v + \kappa_u) M_u(x) - u B_{\dot{\Sigma}} M_u(x) \right) + \Delta_u \right) ,
\] (3.12)
where the \(M_u(x)\) are “mass terms” such as the one introduced in (3.1), gauge invariant due to the condition (3.10). The \(\Delta_u\) are corrections needed due to the nonlinearity of the operator \(\delta S\).

### 3.1.2 Supersymmetric Masses

Mass terms
\[
\Sigma_{m,\text{SUSY}} = \int d^4 x \ M_{\text{SUSY}}(x) ,
\] (3.13)
which do not break supersymmetry, are also considered. They are introduced in the same way as the nonsupersymmetric ones, i.e. through a doublet of external fields, now denoted by \((U, V)\), with the same transformation laws (3.6) as the generic doublet \((u, v)\). The Slavnov-Taylor operator (3.5), as well as its linearized form (3.7), are completed accordingly. The \(V\) fields are shifted, too, by amounts \(\kappa_U\) which parameterize the supersymmetric masses.
We need a condition ensuring the invariance under supersymmetry of this kind of mass terms. Taking into account the fact that the supersymmetry invariance of a mass term such as (3.13) is expressed by the fact that its integrand $M_{\text{SUSY}}(x)$ transforms as a total derivative under the supersymmetry transformations, we can write this condition as

$$Y^\Sigma := \int d^4x \frac{\delta \Sigma}{\delta U} = 0 .$$

(3.14)

Indeed, the Slavnov-Taylor identity (3.9), taken at $U = V = 0$ shows only the $R$-breaking term in its right-hand side. Moreover a gauge invariance condition such as (3.10) is not needed in this case since it is already implied by the condition (3.14).

### 3.2 The Massive Action

Two types of masses are introduced:

i) **Supersymmetry breaking masses belonging to the GG class:** i.e. masses which do not generate UV divergences worse than logarithmic. Their exhaustive list, given in [4], consists of a gaugino mass term and of two scalar field mass terms:

$$M^{(\lambda)} = \lambda^i \lambda^i ,$$
$$M^{(\phi \phi)}_{ab} = \phi_a \bar{\phi}_b ,$$
$$M^{(\phi \phi)}_{ab} = \phi_a \phi_b .$$

(3.15)

ii) **Supersymmetric masses:** i.e. the supersymmetry invariant terms

$$M_{ab}^{\text{SUSY}} = \psi^\alpha_a \psi^\alpha_b - \bar{\lambda}_{(acd)} \phi_d \bar{\phi}_c \bar{\phi}_d - \bar{\lambda}_{(bcd)} \phi_d \bar{\phi}_c \bar{\phi}_d + \bar{\epsilon}_{\dot{\alpha}} \psi^{\dot{\alpha}}_a \phi_b + \bar{\epsilon}_{\dot{\alpha}} \psi^{\dot{\alpha}}_b \phi_a ,$$

(3.16)

In order to control the breakings of supersymmetry and $R$-invariance induced by the mass terms, the latter are coupled to doublets of shifted external fields

$$(u, \bar{v} := v + \kappa), \quad (U_{(ab)}, \bar{V}_{(ab)} := V_{(ab)} + K_{(ab)}) ,$$

(3.17)

following the scheme defined in Subsection 3.1. Because of the gauge invariance of the mass terms, $u$ and $v$ are singlets of the gauge group, whereas $U_{ab}$ and $V_{ab}$ are symmetric invariant tensors. The shifts $K_{ab}$ are in general nondiagonal if some matter supermultiplets belong to the same irreducible representation of the gauge group. The dimensions, $R$-weights, Grassmann parities and ghost numbers are shown in Table 2. The doublet $(u, \bar{v})$ is coupled to the gaugino mass term $M^{(\lambda)}$ and the doublet $(U_{ab}, \bar{V}_{ab})$ to the supersymmetric mass term $M_{ab}^{\text{SUSY}}$. We did not introduce a doublet of external fields for each of the mass terms listed above. The reason, as we shall see, is that the mass terms $M^{(\phi \phi)}_{ab}$ and $M^{(\phi \phi)}_{ab}$ will appear coupled to the doublets already introduced.
Ultraviolet dimensions $d$, infrared dimensions $r$ (see the Appendix) and R-weights of the doublets of complex external fields. The Grassmann parity of the fields $u$ and $U_{ab}$ is odd, that of the fields $v$ and $V_{ab}$ is even. The formers have ghost number $-1$, the latter have ghost number 0.

The complete classical action defined accordingly to the scheme defined by (3.12), taking into account the possible mixing compatible with power counting and $R$-invariance, reads

$$
\Sigma = \hat{\Sigma} + B_\Sigma \int d^4x \left( u \left( M^{(\lambda)} + \rho_{1(abc)} \phi_a \phi_b \phi_c + \rho_{2ab} \phi_a \varepsilon^a \psi_{\alpha a} + \rho_{3ab} \hat{\nu} M^{(\phi \phi)} \right) + \rho_{4ab} \bar{u} \varepsilon^a \psi_{\alpha a} \tilde{\phi}_b + \rho_{5(ab)(cd)} \hat{V}_{ab} M^{(\phi \phi)} + \rho_7 \varepsilon^a \sigma_{\alpha a} \tilde{\lambda} \varepsilon^a A_i^\lambda \right) + U_{ab} \left( M_{ab}^{\text{SUSY}} - \rho_{6(ab)c} \theta^\mu (\tilde{\phi}_c A_i^\mu) \right) + \text{c.c.} + \Delta,
$$

(3.18)

where the coefficients $\rho$ have symmetry properties of their indices which are explicitly shown by the parentheses. Moreover the matrix $\rho_3$ can be chosen hermitian: $\rho_{3ab} = \rho_{3ba}$. Indeed, one can see that its antihermitian part may be absorbed in $\rho_4$. Besides the Slavnov-Taylor invariance, we have also imposed on $\Sigma$ the gauge invariance condition (3.10) as well as the supersymmetry condition (3.14) for $U = U_{ab}$. The last term $\Delta$, whose presence is due to the nonlinearity of the Slavnov-Taylor identity, reads

$$
\Delta = \int d^4x \left( 4g^2 \bar{u} \varepsilon^a \lambda_i^\alpha \varepsilon_{\beta} \tilde{\lambda}^\beta \Delta_{3} + \frac{1}{2} \left( \rho_{2ba} \left( \bar{u} \phi_b - 2u \varepsilon^a \psi_{\alpha a} \right) - 2\hat{V}_{ab} \bar{\phi}_b \right) \left( \rho_{2ca} \left( \bar{u} \phi_c - 2u \varepsilon^a \psi_{\alpha a} \right) - 2\hat{V}_{ac} \phi_c \right) \right)
$$

$$+ \left[ 2\rho_7 \bar{\rho}_7 \bar{u} \varepsilon^a \lambda_i^\alpha \varepsilon_{\beta} \Delta_{3} + i \frac{\bar{\rho}_5 \hat{V}_{ab} \bar{\phi}_b}{\rho_{2ca}} \left( \bar{u} \phi_c - 2u \varepsilon^a \psi_{\alpha a} \right) \right]
$$

(3.19)

Setting to zero the external fields $u, v,$ etc., we get the massive action

$$
\Sigma|_{u=v=U_{ab}=V_{ab}=0} = \hat{\Sigma} + \int d^4x \left( \frac{1}{2g^2} m^{(\lambda)} M^{(\lambda)} + \frac{1}{2} m^{(\psi)} M_{ab}^{\text{SUSY}} - \frac{1}{4} m^{(\phi \phi)} M_{ab}^{(\phi \phi)} \right)
$$

$$+ \frac{1}{4} m^{(\phi \phi)} M_{ab}^{(\phi \phi)} + \lambda_{(3)abc} \phi_a \phi_b \phi_c \right) + \text{c.c.} + O(\varepsilon),
$$

(3.20)
where the masses and the trilinear coupling are given by

\[
m^{(\lambda)} = 2\kappa g^2 ,
\]
\[
m_{ab}^{(\psi)} = 2K_{ab} ,
\]
\[
m_{ab}^{2(\phi\bar{\phi})} = -4\kappa\bar{\kappa}\rho_{3ab} + \kappa\bar{\kappa}\rho_{2ac}\bar{\rho}_{2bc} + 4K_{ac}\bar{K}_{bc} ,
\]
\[
m_{ab}^{2(\phi\bar{\phi})} = 4\kappa K_{cd}\bar{\rho}_{5(cd)(ab)} + 2\kappa(K_{ca}\rho_{2bc} + K_{cb}\rho_{2ac}) ,
\]
\[
\lambda^{(3)(abc)} = \kappa\rho_{1(abc)} .
\]

(3.21)

We see that, besides the supersymmetric mass, all the mass terms belonging to the GG class are present. A coupling trilinear in the scalar matter field \(\phi\) appears, too, thus completing the list of the GG “soft” breakings.

4 Renormalization

The quantization of the theory runs according to the usual procedure of studying its stability under radiative corrections and the determination of the possible anomalies. This analysis has been carried out in \([3]\) for the massless case, and it basically remains unaltered in the massive case thanks to the particular way of introducing masses, which does not change the cohomological sector of the theory.

4.1 Stability of the Classical Theory

The stability amounts to show that all possible perturbations of the classical action (3.18) can be absorbed by a redefinition of its parameters. These perturbations give all possible invariant counterterms which can be freely added to the action at the quantum level, at each order in \(\hbar\).

By imposing that the perturbed action \(\Sigma + \zeta \Sigma_c\) satisfies all the constraints defining the theory, at first order in the infinitesimal parameter \(\zeta\) we find

\[
\Sigma_c = \Sigma_{\text{nontriv}} + \Sigma_{\text{triv}},
\]

(4.1)

where the nontrivial counterterm can be written as

\[
\Sigma_{\text{nontriv}} = Z_g \frac{\partial \Sigma}{\partial g} + Z_{(abc)} \frac{\partial \Sigma}{\partial \lambda_{(abc)}} + \bar{Z}_{(abc)} \frac{\partial \Sigma}{\partial \bar{\lambda}_{(abc)}} ,
\]

(4.2)

\(^{3}\)To which belongs actually the term in \(K_{ac}\bar{K}_{bc}\), which corrects for the fact that the supersymmetric mass term (3.16) was defined by the supersymmetry transformation rules of the massless theory.
and the trivial one as
\[ \Sigma_{\text{triv}} = B_{\Sigma} \left( \hat{\Sigma}_1 + \hat{\Sigma}_2 \right) , \]
\[ \hat{\Sigma}_1 = \int d^4x \left( Z^A(A_{i\mu} + \partial_{\mu}c^i)A^i_{\mu} + Z^\lambda \lambda^i_{\alpha} \lambda^i_{\beta} \right. \\
\[ + \left( Z_{ab}^\phi \phi_a \phi_b + Z_{ab}^\psi \psi^*_a \psi_{b \alpha} + Z^u u \frac{\delta \Sigma}{\delta v} + Z^{U_{(ab)(cd)}} U_{ab} \frac{\delta \Sigma}{\delta V_{cd}} - \text{c.c.} \right) \right) , \]
\[ \hat{\Sigma}_2 = \int d^4x \left( u \left( Z^1_{(abc)} \phi_a \phi_b \phi_c + Z^2_{ab} \phi_a \epsilon^a \psi^*_b \psi_{b \alpha} + Z^3_{ab} \epsilon \phi_{M_{ab}} \right. \\
\[ + Z^4_{(ab)(cd)} \epsilon_{M_{cd}} \phi_a \phi_c + Z^5 \epsilon \phi \phi_{M_{ab}} + Z^6 \epsilon \phi \phi_{M_{cd}} \right) \right) . \]

As it can be seen from (4.2), the structure of the counterterm \( \Sigma_{\text{nontriv}} \) is the same as in the massless case. It corresponds to renormalizations of the gauge and Yukawa coupling constants. On the other hand, the trivial cocycle \( \Sigma_{\text{triv}} \) consists of two parts. The first one, \( B_{\Sigma} \hat{\Sigma}_1 \), is the same as in the massless case, and contains the renormalizations of the fields, i.e. it corresponds to anomalous dimensions. The second contribution is new. It yields the renormalization of the additional parameters \( \rho \) appearing in the most general solution (3.18) of the Slavnov-Taylor identity, hence in particular of the masses through the definitions (3.21) – except for the renormalization of \( m^{(\lambda)} \) and \( m^{(\psi)}_{ab} \), which is already given by that of the doublets of external fields in (4.4). However an independent renormalization of the latter masses is given by a possible redefinition \( \kappa \rightarrow \kappa' \) and \( K_{ab} \rightarrow K'_{ab} \) of the shifts in the expressions (3.17) which appear explicitly in the Slavnov-Taylor operator.

4.2 The Symmetries at the Quantum Level

For the quantum extension of the symmetries of the theory, one can easily prove that the introduction of the shifted external doublets does not affect the validity to all orders of perturbation theory of the gauge condition (2.10), of the ghost equation (2.11) and of the global ghost equations (2.13). In the following we shall prove that it is possible to define a quantum vertex functional \( \Gamma \) such that the condition of gauge invariance (3.10), the condition of supersymmetry (3.14) and the Slavnov-Taylor identity (3.4) hold to all orders:
\[ \delta X \Gamma^{(0)} = \int d^4x \frac{\delta \Gamma^{(0)}}{\delta u} = 0 , \]
\[ \delta Y_{ab} \Gamma = \int d^4x \frac{\delta \Gamma}{\delta U_{ab}} = 0 . \]
where the superscript \( (0) \) means that the corresponding quantity must be taken at vanishing global ghosts.
Let us start with the first identity (4.6). According to the quantum action principle (QAP) [9], the quantum extension of the corresponding classical identity is

$$X \Gamma^{(0)} = \Delta^{(0)} + O(\hbar \Delta^{(0)}), \quad (4.9)$$

where the right-hand side is a breaking whose lowest nonvanishing order in $\hbar$ is an integrated local functional $\Delta^{(0)}$. Due to the anticommutativity of the operator $\delta/\delta u$, $\Delta^{(0)}$ must satisfy the consistency condition

$$X \Delta^{(0)} = 0. \quad (4.10)$$

One easily checks that the most general solution of the latter equation has the form

$$\Delta^{(0)} = X \Delta^{(0)}_X, \quad (4.11)$$

where $\Delta^{(0)}_X$ is an integrated local functional with zero ghost number and $R$-weight, of dimension $\leq 4$. This implies that, after absorption of $\Delta^{(0)}_X$ as a counterterm, the functional $\Gamma$ will obey (4.6) at each order in $\hbar$.

Coming to the identity (4.7), we get from the QAP:

$$Y_{ab} \Gamma = \Delta_{ab} + O(\hbar \Delta_{ab}), \quad (4.12)$$

where $\Delta_{ab}$ is an integrated local functional. Due to the anticommutativity of the operators $\delta/\delta U_{ab}$, $\Delta_{ab}$ must obey the consistency condition

$$Y_{ab} \Delta_{cd} + Y_{cd} \Delta_{ab} = 0. \quad (4.13)$$

Because of the condition (4.6) it must satisfy the additional constraint

$$X \Delta^{(0)}_{ab} = 0. \quad (4.14)$$

It is straightforward to find that the most general solution of the two latter conditions reads

$$\Delta_{ab} = Y_{ab} \Delta_Y, \quad (4.15)$$

where $\Delta_Y$ has the right dimension and quantum numbers. Moreover, it may be chosen such as to obey the condition

$$X \Delta^{(0)}_Y = 0. \quad (4.16)$$

This means that (4.7) can be made to hold, too, without spoiling (4.6).

Finally, the Slavnov-Taylor identity acquires a quantum breaking:

$$S(\Gamma) = \Delta_{ST} + O(\hbar \Delta_{ST}), \quad (4.17)$$

$\Delta_{ST}$ being an integrated local functional of ghost number 1, $R$-weight 0 and dimension bounded by 4, constrained by the consistency conditions

$$B_\Sigma \Delta_{ST} = 0, \quad X \Delta^{(0)}_{ST} = 0, \quad Y_{ab} \Delta_{ST} = 0. \quad (4.18)$$
The first condition is the Wess-Zumino consistency condition, which is solved by

$$\Delta_{ST} = \sum_i r_i A_i + B_{\Sigma} \hat{\Delta}$$ \hspace{1cm} (4.19)

where the “anomalous” terms $A_i$ form a basis of the cohomology of the nilpotent operator $B_{\Sigma}$. With respect to the massless case $[1]$, no new anomaly of this kind appears in the present massive case. In order to see this, let us introduce an expansion in the number $n$ of doublet fields $u, \hat{v}, U_{ab}$ and $\hat{V}_{ab}$, i.e. such that each term of the expansion is homogeneous of degree $n$ in these fields, and let us denote by $b_0$ the term of $B_{\Sigma}$ of degree 0:

$$b_0 = \hat{B}_{\Sigma} + \int d^4x \left( v' \frac{\delta}{\delta u} + V'_{ab} \frac{\delta}{\delta U_{ab}} \right)$$ \hspace{1cm} (4.20)

where we have set (c.f. (3.6))

\[ v' = \hat{v} - i\xi^\mu \partial_\mu u + \eta R_{a} u, \quad V'_{ab} = \hat{V}_{ab} - i\xi^\mu \partial_\mu U_{ab} + \eta R_{U_{ab}} U_{ab}. \] \hspace{1cm} (4.21)

$b_0$ is nilpotent and does not change the degree. We have

$$b_0 u = v' , \quad b_0 v' = 0 , \quad b_0 U_{ab} = V'_{ab} , \quad b_0 V'_{ab} = 0$$ \hspace{1cm} (4.22)

This characterizes the pairs $(u, v')$ and $(U_{ab}, V'_{ab})$ as “$b_0$-doublets”. It is known$^4$ that the $b_0$-cohomology does not depend on such doublets, and also that the cohomology of $B_{\Sigma}$ is isomorphic to a subspace of the $b_0$-cohomology.

The only element $A_i$ in (1.13) is thus the supersymmetric extension of the usual gauge anomaly $[1]$, whose coefficient will be assumed to vanish through a suitable choice of the matter field representation. Therefore the Slavnov-Taylor identity can be established by absorption of $\Delta$ as a counterterm. It can be checked that $\Delta$ can be chosen in such a way that it has the correct quantum numbers (ghost number 0, $R$-weight 0) and satisfies the conditions

$$X \Delta^{(0)} = 0 , \quad Y_{ab} \Delta = 0$$ \hspace{1cm} (4.23)

which are necessary for absorbing $\Delta$ as a counterterm without spoiling the two identities (4.6) and (4.7).

The absence of anomalies, together with the stability of the model, ends the proof of the renormalizability of the model.

## 5 Callan-Symanzik Equation

From now on we shall change from the parametrization

$$\{ g, \lambda_{abc}, \kappa, K_{ab}, \rho_2^{ab}, \rho_3^{ab}, \rho_1^{abc}, \rho_2^{Aab}, \rho_4^{ab}, \rho_5^{(ab)(cd)}, \rho_6^{i(ab)c}, \rho_7 \}$$ \hspace{1cm} (5.1)

$^4$See e.g. Section 5.2 of $[8]$. 

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to the new parametrization
\[ \{ g, \lambda_{abc}, m^{(\lambda)}, m_{ab}^{(\psi)}, m_{ab}^{2(\phi\phi)}, m_{ab}^{2(\phi\bar{\phi})}, \lambda_{(3)(abc)}, \rho_{2ab}^{A}, \rho_{4ab}, \rho_{5(ab)(cd)}, \rho_{6(ab)c}, \rho_{7} \} \]  \tag{5.2}

with the change of variables given by \((3.21)\). Taking into account the fact that \(m_{ab}^{2(\phi\phi)}\) is symmetric, we have separated the matrix \(\rho_{2}^{A}\) in its symmetric and antisymmetric parts:
\[ \rho_{2ab} = \rho_{S}^{2} + \rho_{A}^{2} \cdot [ab] \]  \tag{5.3}

The parameters \((5.2)\) constitute the set of physical parameters of the theory. The remaining parameters correspond to the unphysical field amplitude renormalizations compatible with the symmetries shown in \((4.4)\).

All the masses in particular being defined by normalization conditions, the shifts \(\kappa\) and \(K_{ab}\) are no longer independent parameters. They are are thus renormalized in \(\kappa' = \kappa + (\bar{h})\) and \(K_{ab}' = K_{ab} + O(h)\). The latter quantities replace the shifts \(\kappa\) and \(K_{ab}\) defined in \((3.17)\) and appearing explicitly in the Slavnov-Taylor operator (see \((3.5)\)).

The Callan-Symanzik equation describes the behaviour of the quantum theory under scale transformations. It may be obtained as follows [8]. First, apply to the effective action the generator of scale transformations
\[ D := \mu \frac{\partial}{\partial \mu} + \sum_{m} m \frac{\partial}{\partial m} \]  \tag{5.4}

where \(\mu\) is the mass scale at which the normalization conditions defining the parameters of the quantum theory are taken, and where the summation over \(m\) extends over all masses and dimensionful coupling constants. This defines, through the QAP, an insertion of dimension \(\leq 4\). Second, expand the insertion \(D\Gamma\) in a suitable basis with the same dimensional constraints.

In order to keep with the symmetries of the problem it is convenient to work with invariant insertions. An invariant insertion is by definition an insertion \(\Delta\) obeying the conditions
\[ B_{\Gamma}(\Delta \cdot \Gamma) = 0 , \quad \mathcal{F}^{i}(\Delta \cdot \Gamma) = 0 , \quad \partial_{\xi}^{\mu}(\Delta \cdot \Gamma) = 0 , \quad \partial_{\eta}(\Delta \cdot \Gamma) = 0 , \quad X(\Delta \cdot \Gamma)^{(0)} = 0 , \quad Y_{ab}(\Delta \cdot \Gamma) = 0 . \]  \tag{5.5}

Such an invariant insertion is generated through the QAP by the application to the effective action of a symmetric operator \(\nabla\):
\[ \nabla \Gamma = \Delta \cdot \Gamma \]  \tag{5.6}

i.e. of an operator \(\nabla\) which obeys the conditions
\[ \nabla \mathcal{S}(F) = B_{\Gamma} \nabla F , \quad \forall F , \]  
\[ [\nabla, \mathcal{F}^{i}] = 0 , \quad [\nabla, \partial_{\xi}^{\mu}] = 0 , \quad [\nabla, \partial_{\eta}] = 0 , \quad [\nabla, X]^{(0)} = 0 , \quad [\nabla, Y_{ab}] = 0 . \]  \tag{5.7}
\[ \nabla \Delta_i^g = 0, \quad \nabla \Delta_i^t = 0, \quad \nabla \Delta_R = 0, \quad (5.8) \]

where \( \Delta_i^g, \Delta_i^t \) and \( \Delta_R \) are the classical insertions appearing in the right-hand sides of the identities (2.11) and (2.13). The invariance of the insertion (5.8) then follows from the effective action \( \Gamma \) fulfilling the quantum extensions of the identities (3.4), (2.11), (2.13) (3.10) and (3.14).

Let us look for a basis of invariant insertions. The renormalized shifts \( \kappa' \) and \( K'_{ab} \) depending on the parameters of the list (5.2), the partial derivatives of the latter are made symmetric by redefining them according to

\[
\nabla_{p_I} := \frac{\partial}{\partial p_I} - \int d^4x \left( \frac{\partial \kappa'}{\partial p_I \delta v} \frac{\delta}{\delta v} + \frac{\partial K'_{ab}}{\partial p_I} \frac{\delta}{\delta v_{ab}} + \frac{\partial \bar{K}'_{ab}}{\partial p_I} \frac{\delta}{\delta v_{ab}} \right),
\]

\[
\{p_I\} := \{g, \lambda_{abc}, m_a^{2(\phi^2)}, m_b^{2(\phi^2)}, \lambda_3^{(3)_{(abc)}}, \rho_{2a}^{A_{ab}}, \rho_{4a}^{ab}, \rho_{5(a)_{(cd)}}, \rho_{6(i)_{(ab)c}}, \rho_7\}.
\]

The mass parameters \( m^{(\lambda)} \) and \( m^{(\psi)}_{ab} \) are not included in the latter list. Indeed, in the classical approximation they are identical to the shifts \( \kappa \) and \( K_{ab} \) (c.f. (3.21)), and thus the corresponding \( \nabla \) operators, giving zero when applied to the classical action, will not generate independent symmetric insertions.

Further symmetric operators are the generators of the field amplitude renormalizations compatible with the symmetries (c.f. (4.4)):

\[
N^A := N^A - N^{A*} - N^b - N^c, \quad N^\lambda := N^\lambda - N^{\lambda*} \quad \text{and c.c.} \; ,
\]

\[
N^{\phi}_{ab} := (N^\phi - N^{\phi*})_{ab} \quad \text{and c.c.} \; , \quad N^{\psi}_{ab} := (N^\psi - N^{\psi*})_{ab} \quad \text{and c.c.} \; ,
\]

\[
N^u := N^u + N^\bar{\phi}, \quad N^{U}_{(ab)(cd)} := (N^U + N^{\bar{V}})_{(ab)(cd)} \quad \text{and c.c.} \; ,
\]

where

\[
N^\varphi = \int d^4x \varphi \frac{\delta}{\delta \varphi}, \quad \varphi = A, \lambda, \bar{A}, \lambda^*, \bar{\lambda}^*, b, \bar{c}, u, \bar{v}, \bar{u}, \bar{v} ,
\]

\[
N^{\phi}_{ab} = \int d^4x \varphi_a \frac{\delta}{\delta \varphi_b}, \quad N^{\phi*}_{ab} = \int d^4x \varphi_a \frac{\delta}{\delta \varphi^*_{ab}}, \quad \varphi = \phi, \bar{\psi}, \bar{\phi}, \bar{\psi} ,
\]

\[
N^{U}_{(ab)(cd)} = \int d^4x \varphi_{ab} \frac{\delta}{\delta \varphi_{cd}}, \quad \varphi = U, \bar{V}, \bar{U}, \bar{V} .
\]

The operators \( \mathcal{N} \) are symmetric according to the definition (5.7) and (5.8), except \( N^u \) and \( N^{U}_{(ab)(cd)} \) for which

\[
[\mathcal{N}^u, X] = -X, \quad [\mathcal{N}^{U}_{(ab)(cd)}, Y_{ef}] = -\delta_{(ab)(ef)}Y_{cd}.
\]

where

\[
\delta_{(ab)(cd)} = \begin{cases} 1 & \text{if } ab = cd \text{ or } dc \\ 0 & \text{otherwise} \end{cases}.
\]

But this does not prevent them, together with the other ones, to define symmetric insertions through (5.6).
The set
\[ \{ \nabla_{p_I}, N_K \} , \]
with \[ \{ N_K \} = \{ N^A, N^\lambda, N^\phi_{ab}, N^\psi_{ab}, N^u_{(ab)(cd)} \} . \]
forms a basis for the symmetric operators of the theory. Their application to the effective action yields a basis for the invariant insertions of dimension \( \leq 4 \).

The set of invariant insertions thus constructed represents the quantum extension of the classical counterterms (4.1).

On the other hand the symmetrized form
\[ \nabla_D := D - \int d^4x \left( \kappa' \frac{\delta}{\delta v} + K'_{ab} \frac{\delta}{\delta V_{ab}} + \text{c.c.} \right) \]
of the scale operator (5.4), where we have used the dimensional analysis identities
\[ D \kappa' = \kappa' , \quad DK'_{ab} = K'_{ab} , \]
gives rise to an invariant insertion which may be expanded in the basis (5.13), yielding the Callan-Symanzik equation:
\[ \left( \nabla_D + \sum_I \beta_I \nabla_{p_I} - \sum_K \gamma_K N_K \right) \Gamma = 0 . \]

The latter can be rewritten in a more explicit form
\[ C \Gamma := \left( D + \sum_I \beta_I \frac{\partial}{\partial p_I} - \sum_K \gamma_K N_K^{\text{hom}} \right) \Gamma \]
\[ = \int d^4x \left( \alpha_v \frac{\delta \Gamma}{\delta v} + \alpha_{\bar{v}} \frac{\delta \Gamma}{\delta \bar{v}} + \alpha_{V_{ab}} \frac{\delta \Gamma}{\delta V_{ab}} + \alpha_{\bar{V}_{ab}} \frac{\delta \Gamma}{\delta \bar{V}_{ab}} \right) , \]
where
\[ \alpha_v = (C + \gamma_u) \kappa' , \quad \alpha_{\bar{v}} = (C + \gamma_{\bar{u}}) \bar{K}' , \]
\[ \alpha_{V_{ab}} = CK'_{ab} + \gamma_{(ab)(cd)} K'_{cd} , \quad \alpha_{\bar{V}_{ab}} = \bar{C} \bar{K}'_{ab} + \gamma_{(ab)(cd)} \bar{K}'_{cd} , \]
and \( N_K^{\text{hom}} \) are the unshifted counting operators.

Remark: The \( \beta_I \)-terms for \( p_I = m_{ab}^{2(\phi)} \), \( m_{ab}^{2(\phi_\hat{a})} \) correspond to renormalizations of these mass parameters. The renormalizations of the masses \( m^{(\lambda)} \) and \( m^{(\psi)}_{ab} \) are expressed by the terms in the right-hand side of (5.17): they depend on the anomalous dimensions of the external field doublets \( (u, \hat{v}) \) and \( (\hat{U}_{ab}, \hat{V}_{ab}) \). This fact can already be seen by inspecting the structure of the classical invariant counterterms (4.4), (4.5), the masses \( m^{(\lambda)} \) and \( m^{(\psi)}_{ab} \) being given by the shifts of \( v \) and \( V_{ab} \), respectively.
6 Conclusions

A realistic supersymmetric gauge field theory needs masses for the supersymmetric partners of the particles known up to now. In this paper, we introduced those mass terms which, according to [4], induce supersymmetry breakings leading to less than quadratic divergences. In addition, we completed the theory by adding a mass term for the fermionic matter fields, preserving supersymmetry, besides classical gauge invariance. The main idea has been to introduce masses through a constant shift of some external fields (3.6). This allowed us to control the breakings due to the non-supersymmetric mass terms through the generalized Slavnov-Taylor identity (4.8), collecting all the symmetries of the theory. The issue of quantum gauge invariance of the mass terms has been characterized by the functional relation (4.6). The quantum extension of the model has been performed according to the lines dictated by the Quantum Action Principle [9] (see (A.7)), both in the ultraviolet and in the infrared regions, this latter analysis being mandatory because of the presence of both massive and massless fields. Moreover, the behaviour of the quantum theory under scale transformations has been characterized by the Callan-Symanzik equation (5.16). The massive quantum theory constructed in this paper will be the starting point for the analysis of the nonrenormalization properties of a realistic supersymmetric theory [7].

Appendix: Infrared Power Counting

A.1 Ultraviolet and Infrared Subtractions

In order to assure the existence of the Green functions, subtractions have to be made. The momentum space subtraction scheme of Zimmermann and Lowenstein [10, 11] assures their existence as tempered distributions, free from UV and IR singularities, provided the UV dimensions \( d \) and IR dimensions \( r \) of the fields fulfill certain conditions spelled in [11] (see below). The method consists in subtracting off the first few terms of the Taylor expansion of the integrand of a divergent integral, corresponding to a Feynman graph or subgraph, in its external momenta \( p \) and a certain parameter \( s \) around \( p = s = 0 \). The parameter \( s \) appears through the introduction of an auxiliary mass \( M(s - 1) \) in the denominator of every massless propagator – here the gauge field and ghost propagators. These subtractions, performed recursively on the integrands of the graph and of all its divergent one-particle-irreducible (1PI) subgraphs, make the integral UV-finite without introducing spurious IR singularities thanks to the subtraction terms involving only massive propagators. At the end \( s \) is set to its physical value 1. The Taylor expansion of the integrand associated to a divergent 1PI graph or subgraph \( \gamma \) ends at the order defined by the “UV degree of subtraction”

\[
\delta(\gamma) = 4 - \sum_E d_E ,
\]  

(A.1)
where the sum is performed on all the external lines of $\gamma$ and $d_E$ is the UV dimension of the corresponding quantum or external field.

However IR post-subtractions have to be performed as well, in order that subtracted subgraphs, such as self-energy subgraphs of the massless fields, be kept vanishing at zero momentum. The latter subtractions are done according to a Taylor expansion around $p = 0, s = 1$, terminating at the order $\rho(\gamma) - 1$, where $\rho(\gamma)$ is the “IR degree of subtraction”

$$\rho(\gamma) = 4 - \sum_E r_E,$$  \hspace{1cm} (A.2)

$r_E$ being the IR dimension of the corresponding field.

The conditions on the UV and IR dimensions are the following.

- If the propagator $\Delta_{AB}(k, s)$ between two quantum fields $\varphi_A$ and $\varphi_B$ behaves asymptotically as
  $$\Delta_{AB}(\lambda p, \lambda s) \sim \lambda^{d_{AB}}, \quad \lambda \to \infty, \ s \to \infty,$$
  $$\Delta_{AB}(\lambda p, \lambda(s - 1)) \sim \lambda^{r_{AB}}, \quad \lambda \to 0, \ s \to 1,$$
  then
  $$d_A + d_B \geq d_{AB} + 4, \quad r_A + r_B \leq r_{AB} + 4.$$

- For any (quantum or external) field $\varphi_A$,
  $$r_A \geq d_A.$$  \hspace{1cm} (A.4)

- Some of the external momenta entering the graph or subgraph along external lines corresponding to the fields $\varphi_A$ may be set to zero if and only if
  $$\rho_A \leq 0,$$
  with the exception:
  $$\rho_A \leq \frac{3}{2}, \quad \text{if only one external momentum is set to zero.}$$  \hspace{1cm} (A.5)

**A.2 Quantum Action Principle**

Let us define the UV dimension $d$ and the IR dimension $r$ of the functional derivative with respect to a field $\varphi$ of dimensions $d_\varphi$ and $r_\varphi$ as

$$d \left( \frac{\delta}{\delta \varphi(x)} \right) = 4 - d_\varphi, \quad r \left( \frac{\delta}{\delta \varphi(x)} \right) = 4 - r_\varphi.$$  \hspace{1cm} (A.6)

More generally the UV dimension of a functional operator, e.g. the operator of a Ward or Slavnov-Taylor identity, is defined as the maximum of the UV dimensions of the individual terms occurring in it, whereas its IR dimension is given by the minimum of the individual
IR dimensions. The quantum action principle (QAP) \[8\] states that applying a functional operator \(F(x)\) to the vertex functional yields the generating functional of the vertex functions with a local insertion \(Q(x)\) of bounded UV and IR dimensions. Similarly for the partial derivative with respect to a parameter \(p\) of the theory. More precisely
\[
F(x)\Gamma = Q(x) \cdot \Gamma, \quad d(Q) \leq d(F), \quad r(Q) \geq r(F),
\]
where \(Q(x)\) and \(R(x)\) are local field polynomials.

An insertion characterized by the bounds “UV dimension \(\leq d\), IR dimension \(\geq r\)” is called an \(N^r_d\) normal product. One notes, from (A.5), that integrated insertions must be \(N^r_d\), but also that an \(N^r_d\) integrated insertion with \(4 > r \geq 5/2\) is allowed if it occurs only once. Moreover the insertions corresponding to the interaction terms of the classical action must all be \(N^4_4\), and such must be the case of all counterterms.

### A.3 Absence of IR Singularities

The UV and IR dimension assignments given in Tables \(\ref{table1}\) and \(\ref{table2}\) fulfill the conditions (A.3), (A.4) and the \(N^4_4\) prescription for the interaction terms of the classical action.

However, IR problems, namely IR anomalies, may be encountered if insertions with \(r < 4\) have to be absorbed in order to assure the validity of the functional identities which define the symmetries of the theory, according to the procedure followed in Section \(\ref{section4}\). Let us review the various cases at hand.

**Identities (4.6) and (4.7):** In both cases an inspection of the possible candidate counterterms \(\Delta_X^{(0)}\) (see (4.11)) and \(\Delta_Y\) (see (4.15)), respectively, shows that all of them are \(N^4_4\).

**Slavnov-Taylor identity (4.8):** The situation is slightly more tricky since, due to the inhomogeneity of the Slavnov-Taylor operator the breaking \(\Delta_{ST}\) in (4.17) is \(N^{5/2}_4\). An inspection shows that all the possible counterterms \(\hat{\Delta}\) (see (4.19)) have IR dimension \(\geq 4\), but one:
\[
\hat{\Delta}^3 = \int d^4x \hat{V}_{ab}\psi_a^{\gamma\alpha}\sigma_{\alpha\dot{\alpha}}^\mu \gamma^{\dot{\alpha}} A_{\mu}, \quad (A.8)
\]
which has IR dimension 3 due to the presence of a shifted external field. However, being linear in the quantum fields, it can be absorbed as a counterterm since it will not appear in any 1PI graph.

In conclusion the theory presents no infrared anomaly, excepted the harmless counterterm \(\hat{\Delta}^3\).

**Callan-Symanzik equation (5.16):** We finally come to the Callan-Symanzik equation, and doing so we shall explain the strange prescription for the IR dimension of the external
field $\lambda^*$ (see Table [4]). Although the Callan-Symanzik equation does not belong to the defining properties of the theory, its validity belongs to the current “renormalization group” understanding of a renormalizable theory, according to which any change of the energy-momentum scale is compensable by a finite renormalization of the parameters of the theory.

We first observe that the application of the symmetric operator $\nabla_D$ (5.14) to the effective action yields an $N_3^4$ insertion. The same holds for the basis of symmetric operators (5.13). The Callan-Symanzik equation we have derived in Section 5 is thus perfectly compatible with the renormalization scheme based on the dimensional prescriptions of Tables [1] and [2].

**Remark:** Corresponding to the *apriori* more natural choice for the IR dimension 5/2 of the external field $\lambda^*$, the term of coefficient $\rho_7$ is absent from the classical action (3.18) since it contains a part with IR dimension 3 due to the shift in $B_{\Sigma u} = \hat{v}$, and for the same reason it does not appear as a counterterm: this prescription for the IR dimension of $\lambda^*$ amounts to normalize $\rho_7$ to the value zero. But a quantum extension of this same term appears in the expansion of $\nabla_D \Gamma$ in a basis of invariant insertions. Since in the absence of the parameter $\rho_7$ it cannot be written as a partial derivative of the action, the renormalization group interpretation of the Callan-Symanzik equation is lost. We therefore have assigned the value 7/2 to the IR dimension of $\lambda^*$, thus allowing for the presence of the parameter $\rho_7$ on the same footing as all the other ones and solving this apparent paradox.

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