THE RIESZ BASIS PROPERTY OF THE EIGENVECTORS CONNECTED TO THE EXPONENTIAL STABILITY PROBLEM OF A BOUNDARY DAMPED TUBE CARRYING THE STATIONARY FLOW OF A FLUID

MAHYAR MAHINZAEIM$^{1,*}$, GEN QI XU$^2$, AND XIAO XUAN FENG$^3$

Abstract. In the present paper we study the stability problem for a stretched tube conveying an ideal fluid with boundary damping. The spectral problem concerns operator functions of the forms

$$\mathcal{M}(\lambda) = \lambda^2 G + \lambda D + C \quad \text{and} \quad \mathcal{P}(\lambda) = \lambda I - T$$

taking values in different Hilbert spaces. Thorough analysis is made of the location and asymptotics of eigenvalues in the complex plane and Riesz basis property of the corresponding eigenvectors. Well-posedness of the initial-value problem for the abstract equation

$$\dot{x}(t) = T x(t)$$

is established as well as expansions of the solutions in terms of eigenvectors and exponential stability of the corresponding $C_0$-semigroup.

1. Introduction, statement of the problem, and preliminary results

The partial differential equation governing the motion of a thin homogeneous horizontal tube of unit length, subjected to an external tensile force proportional to a parameter $\gamma > 0$, and carrying the stationary flow of an incompressible fluid (not necessarily inviscid) can be written in terms of the transverse deflection $w(s,t)$ for $s \in [0,1]$, $t \in \mathbb{R}_+$ as

$$\frac{\partial^4 w(s,t)}{\partial s^4} - (\gamma - \eta^2) \frac{\partial^2 w(s,t)}{\partial s^2} + 2\beta \eta \frac{\partial^2 w(s,t)}{\partial s \partial t} + \frac{\partial^2 w(s,t)}{\partial t^2} = 0. \quad (1.1)$$

Here $\eta \geq 0$ represents the velocity of the fluid and $\beta \in (0,1)$ is a parameter depending only on the tube and fluid masses per unit length. Several variants of (1.1) can be obtained. For example, when there is no flow, $\eta = 0$, the equation reduces to the governing partial differential equation for the incompressible fluid (not necessarily inviscid) can be written

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Here $\eta \geq 0$ represents the velocity of the fluid and $\beta \in (0,1)$ is a parameter depending only on the tube and fluid masses per unit length. Several variants of (1.1) can be obtained. For example, when there is no flow, $\eta = 0$, the equation reduces to the governing partial differential equation for the transverse motion of a stretched Euler–Bernoulli beam. On the other hand, if the tube is made of a material modelled by the Kelvin–Voigt model for linear viscoelasticity and is situated in a viscous surrounding medium, then the equation analogous to (1.1) includes an extra term

$$\alpha \frac{\partial^5 w(s,t)}{\partial s^5 \partial t} + \delta \frac{\partial w(s,t)}{\partial t},$$

where $\alpha > 0$ is the viscoelastic damping coefficient and the parameter $\delta \geq 0$ corresponds to the viscous damping due to friction from the surrounding medium. Referring the reader to [2], we will not consider this situation further here and will, in fact, suppose damping to be present, through the boundary conditions, only at the ends of the tube; we will discuss this formally later. (Extensive information and historical references on the derivation of (1.1) and its variants may be found in the books by Païdoussis [14, 15].)

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Let there be associated with (1.1) given initial conditions

\[ w(s, 0) = g(s), \quad \frac{\partial w(s, t)}{\partial t} \bigg|_{t=0} = h(s), \quad (1.2) \]

where \( g, h \) are assumed sufficiently smooth, and standard undamped boundary conditions at \( s = 0 \) and \( s = 1 \) which might typically be hinged,

\[ w(0, t) = 0, \quad \frac{\partial^2 w(s, t)}{\partial s^2} \bigg|_{s=0} = 0, \quad (1.3) \]

or clamped,

\[ w(1, t) = 0, \quad \frac{\partial w(s, t)}{\partial s} \bigg|_{s=1} = 0, \quad (1.4) \]

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\[ w(1, t) = 0, \quad \frac{\partial w(s, t)}{\partial s} \bigg|_{s=1} = 0. \quad (1.6) \]

By “sufficiently smooth” one means that (for example) \( g \) and \( h \) are elements of \( C^4[0, 1] \). To avoid any confusion we shall use in this section the acronym IBVP (initial/boundary-value problem) to denote the problem posed by (1.1) with \( \gamma = 0 \), the initial conditions (1.2), and either of the boundary conditions (1.3), (1.4) or (1.5), (1.6). We are concerned with the stability of (solutions of) the initial/boundary-value problem with damped boundary conditions and for this it will be instructive first to consider the IBVP in the context of the existing literature.

There is a long history as regards the stability properties of the IBVP, which, in general, is investigated by studying the eigenvalues of the corresponding spectral problem under variations in the parameter pair \( \{\beta, \eta\} \). In the engineering literature, there have appeared many articles on this topic from both the numerical and the analytical point of view, for example \( [1, 4, 5, 12, 16] \) to name but a few. Movchan’s classic result in \( [12] \) is that while solutions of the IBVP are stable in the interval \( 0 \leq \eta < \pi \), they are unstable for \( \eta = \pi n, n \in \mathbb{N}^+ \). We will shortly specify what we mean by stability.

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The first operator-theoretic account of the IBVP goes back to Miloslavskii in \( [7] \), and more refined results were developed in \( [8, 9, 24] \) (see also the paper \( [21] \) by Röhr). In \( [24] \), the authors also provided some interesting numerical results which explore the possibility of having regions in the \( \eta \beta \)-plane for so-called “gyroscopic stabilisation”, meaning that although solutions of the IBVP are unstable for some \( \eta \) and \( \beta = \beta_0 \), they can be stable for the same \( \eta \) and some \( \beta > \beta_0 \). Pivovarchik also took up this this stabilisation idea in the papers \( [17, 19, 20] \). His more recent work \( [17] \) on the subject consists of computing estimates of the size of the stability regions in the \( \eta \beta \)-plane. However, in \( [24] \) it was shown that gyroscopic stabilisation is not possible in the intervals \( \pi (2n - 1) \leq \eta \leq 2\pi n \) \( (n \in \mathbb{N}^+) \), regardless of the value of \( \beta \).

The emphasis in the above mentioned works dealing with the operator formulation of the IBVP is usually placed on the stability of the spectral problem for the quadratic operator function, called pencil in this paper,

\[ \mathcal{M}(\lambda) = \lambda^2 I + \lambda D + C, \quad \lambda \in \mathbb{C}, \]

wherein \( D \) and \( C \) (for definitions, see \( [19] \) or \( [24] \) are closed linear operators acting in the Hilbert space \( \mathbb{L}_2(0, 1) \), and \( I \) is the identity operator. The operator \( C \) is selfadjoint and uniformly positive, and \( D \) is skewsymmetric. The \( \lambda \)-independent domain \( D(C) \) of \( C \) is dense in \( \mathbb{L}_2(0, 1) \) and is contained in the domain \( D(D) \) of the operator \( D \), such that, by definition, \( D(\mathcal{M}(\lambda)) = D(D) \cap D(C) = D(C) \). The stability of the IBVP is then equivalent (see \( [24] \) to the condition that the spectrum of \( \mathcal{M} \) must be purely imaginary and semisimple, the latter
meaning that \( \mathcal{M} \) has only eigenvectors and no associated vectors. The notion of “stability of the pencil” is then sometimes used.

Alternatively, using a semigroup formulation of the IBVP, one could study the stability of the spectral problem associated with the linear pencil

\[
\mathcal{P}(\lambda) = \lambda I - T, \quad \lambda \in \mathbb{C},
\]

in the product space \( \mathbb{X} := \mathcal{D}(C^{1/2}) \times L_2(0,1) \), where the operator \( T \) – the system operator – is defined by

\[
T := \begin{pmatrix} 0 & I \\ -C & -D \end{pmatrix}
\]

with domain \( \mathcal{D}(T) = \mathcal{D}(C) \times (\mathcal{D}(D) \cap \mathcal{D}(C^{1/2})) \). It is easy to see that \( T \) is closed. Moreover it is also seen readily that the spectrum of \( \mathcal{P} \) or, equivalently, of \( T \) coincides with that of \( \mathcal{M} \). So, to conclude, stability of the IBVP is equivalent to stability of the pencil \( \mathcal{M} \) or that of \( \mathcal{P} \).

However, differences arise when it comes to stating conclusions about the exponential stability of the IBVP, since the spectrum \( \sigma(T) \) of the system operator \( T \) does not, in general, determine the decay constant of the semigroup. In order to see this, let \( T \) be the infinitesimal generator of a strongly continuous semigroup – abbreviated \( C_0 \)-semigroup – of bounded linear operators, \( S(t), t \geq 0, \) on \( \mathbb{X} \). Formally writing the IBVP in the form of the abstract initial-value problem

\[
\dot{x}(t) = T x(t), \quad x(t) = \begin{pmatrix} w(\cdot, t) \\ v(\cdot, t) \end{pmatrix}, \quad x(0) = x_0 = \begin{pmatrix} g(\cdot) \\ h(\cdot) \end{pmatrix}
\]

(1.8)

gives the solution

\[
x(t) = S(t) x_0
\]

(1.9)

for all sufficiently smooth \( x_0 \), for example, for \( x_0 \in \mathcal{D}(T) \); this holds if and only if \( w(\cdot, t) \) is a solution of the IBVP and \( v(\cdot, t) = (\partial w/\partial t)(\cdot, t) \). Consider the inequality

\[
\|x(t)\| \leq M e^{-\varepsilon t} \|x_0\|,
\]

(1.10)

for a certain positive constant \( M \), where \( \| \cdot \| \) is the norm in \( \mathbb{X} \) induced by the “appropriate” inner product denoted by \( \langle \cdot, \cdot \rangle \). (Throughout this paper, we will use the same symbol \( \| \cdot \| \) to denote any one of several norms when it is perfectly clear from the usage which norm is intended.) The necessary and sufficient condition for the solution \( x(t) \) to decay exponentially to zero as \( t \to \infty \) is that \( \varepsilon > 0 \), and in this case we say that \( S(\cdot) \) is exponentially stable. This is equivalent to the condition that the type

\[
\omega(T) := \lim_{t \to \infty} \frac{1}{t} \log \|S(t)\| \leq -\varepsilon;
\]

but while there always holds, by the Hille–Yoshida theorem,

\[
\sup \{ \Re \lambda \mid \lambda \in \sigma(T) \} \leq \omega(T),
\]

(1.11)

equality in \([13,14]\) unfortunately does not always hold (and so the spectral mapping theorem does not apply). It is a consequence, therefore, of the aforementioned characterisation that it is not generally justified to require for exponential stability only that the spectrum of \( T \) be confined to the open left half-plane in the sense that

\[
\sup \{ \Re \lambda \mid \lambda \in \sigma(T) \} \leq -\varepsilon
\]

(1.12)

when \( \varepsilon > 0 \).

There are however several conditions for the inequality \([13,14]\) to allow for equality. For example, equality holds for bounded generators, or equivalently for uniformly continuous semigroups, for holomorphic semigroups, and for compact semigroups (see \([22,23]\)). A condition of particular interest to us, however, is formulated in terms of the Riesz basis property of the root vectors (eigen- and associated vectors). To illustrate this, let \( \{ \lambda_k \}_{k=1}^{\infty} \) be a sequence of normal eigenvalues (isolated eigenvalues having finite algebraic multiplicities) of \( T \), assumed simple for simplicity of exposition. Suppose the corresponding sequence of eigenvectors \( \{ x_n \}_{n=1}^{\infty} \) forms
a Riesz basis for $X$. There exists then a unique sequence of vectors $\{z_n\}_{n=1}^{\infty}$ such that the eigenvectors $x_n$ are biorthogonal to the $z_n$ in $X$. Since $x_0$ can be expanded in the Riesz basis of eigenvectors,

$$x_0 = \sum_{n=1}^{\infty} \langle x_0, z_n \rangle x_n,$$

the solution (1.9) can be represented in series form, that is,

$$x(t) = S(t)x_0 = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x_0, z_n \rangle x_n.$$

By using the standard estimate (see [3, Section VI.2.(2.4)])

$$M_1 \sum_{n=1}^{\infty} |\langle x_0, z_n \rangle|^2 \leq \|x_0\|^2 \leq M_2 \sum_{n=1}^{\infty} |\langle x_0, z_n \rangle|^2$$

for some positive constants $M_1$, $M_2$, and assuming that (1.12) holds, we compute

$$\|x(t)\|^2 = \|S(t)x_0\|^2 \leq M_2 \sum_{n=1}^{\infty} |e^{\lambda_n t} \langle x_0, z_n \rangle|^2 \leq \frac{M_2}{M_1} e^{-2\epsilon t} \|x_0\|^2.$$

This is just (1.10) with $M := \sqrt{M_2/M_1} > 0$, so that solutions of the IBVP are exponentially stable for $\epsilon > 0$ and stable when $\epsilon = 0$. For the purposes of this paper, the conditions of stability and exponential stability may thus be defined as follows.

**Definition 1.1.** Let $T$ be the generator of a $C_0$-semigroup of bounded linear operators, $S(t)$, $t \geq 0$, on the Hilbert space $X$. Let the spectrum $\sigma(T)$ of $T$ consist entirely of normal eigenvalues with corresponding root vectors (all of which may be eigenvectors) which form a Riesz basis for $X$. Then $S(\cdot)$ is said to be exponentially stable (respectively, stable) if and only if $\sup \{\Re \lambda \mid \lambda \in \sigma(T)\} \leq -\epsilon$ with $\epsilon > 0$ (respectively, $\epsilon = 0$).

As we have remarked earlier, stability of the IBVP, and hence of the semigroup $S(\cdot)$, is really the same as stability of the pencil $\mathcal{M}$. The question remains: Can we conclude exponential stability of $S(\cdot)$ from that of the pencil $\mathcal{M}$; and if so, what does exponential stability of $\mathcal{M}$ actually mean? Ultimately, we would like to know if the eigenvectors of $T$ also form a Riesz basis for $X$, when those of $\mathcal{M}$ form a Riesz basis for $L_2(0,1)$. This appears to be a worthwhile question, the answer to which is not obvious. It should be pointed out that if Definition 1.1 is applied to the problem of exponential stability associated with $S(\cdot)$, then the analytical work of verification of the Riesz basis property is perhaps the most important (but most difficult) of the whole work.

So far we have limited our attention to the simplest class of tubes with undamped boundary conditions. As will be seen, the underlying spaces and operators in the above considerations are modified in a natural way if damping is involved in the boundary conditions. The stability problem related to such tubes has been raised explicitly and implicitly many times over the years but most recently in our paper [2]. In the present paper we shall pose for (1.1) the same initial/boundary-value problem as we did in this introductory section for the case of hinged boundary conditions but replace the boundary conditions (1.3) by

$$\frac{\partial^2 \mathbf{w} (s,t)}{\partial s^2} \bigg|_{s=1} = -\kappa \frac{\partial \mathbf{w} (s,t)}{\partial s \partial t} \bigg|_{s=1}, \quad \frac{\partial^3 \mathbf{w} (s,t)}{\partial s^3} \bigg|_{s=1} = (\gamma - \eta^2) \frac{\partial \mathbf{w} (s,t)}{\partial s} \bigg|_{s=1}. \quad (1.13)$$

In the former condition in (1.13) we take account of viscous damping in the bending moment which is proportional to a parameter $\kappa \geq 0$. The latter condition assumes that the tube
undergoes conservative tension (that is, acting in a fixed direction along its axis). To obtain exponential stability results, we must assume

\[ \gamma > \eta^2, \]

which means that the tension in the tube is very much bigger than the fluid velocity. From a technical viewpoint, the reason for this restriction will become clear later.

A well-known observation is that the requirement of stability of the kind of problem we are interested in here places rather severe restrictions on the admissibility of damping. Roughly speaking, and this may seem surprising at first glance, arbitrarily small damping can lead to destabilisation (see [11, Section 4.4] for a brief historical review and [10, 18] for a formal interpretation). So in the case of the initial/boundary-value problem (1.1)–(1.3), (1.13) one expects that for \( \eta > 0 \) (when there is flow) stability can be destroyed by any \( \kappa > 0 \), no matter how small. We will show this is not the case here. In fact, we prove that under the condition \( \gamma > \eta^2 \) solutions of (1.1)–(1.3), (1.13) decay exponentially as time progresses for \( \eta > 0 \), even in the presence of damping represented by \( \kappa > 0 \).

Using separation of variables, by letting \( w (s, t) = e^{\lambda t} w (\lambda, s) \), it is easily verified that the boundary-eigenvalue problem corresponding to (1.1)–(1.3), (1.13) consists of the fourth-order differential equation

\[ w^{(4)} (\lambda, s) - (\gamma - \eta^2) w'' (\lambda, s) + 2\lambda \beta \eta w' (\lambda, s) + \lambda^2 w (\lambda, s) = 0 \]  

(1.14)

together with the boundary conditions

\[ w (\lambda, 0) = 0, \]  

(1.15)
\[ w'' (\lambda, 0) = 0, \]  

(1.16)
\[ w'' (\lambda, 1) + \lambda \kappa w' (\lambda, 1) = 0, \]  

(1.17)
\[ w^{(3)} (\lambda, 1) - (\gamma - \eta^2) w' (\lambda, 1) = 0. \]  

(1.18)

This problem is equivalent to the abstract spectral problem for the quadratic pencil

\[ \mathcal{M} (\lambda) = \lambda^2 G + \lambda D + C, \quad \lambda \in \mathbb{C}, \]  

(1.19)

where the operators \( C, D, G \) act in the Hilbert space \( L^2 (0, 1) \times \mathbb{C} \) and are defined by

\[ Cy = \left( \begin{array}{c}
 w^{(4)} - (\gamma - \eta^2) w'' \\
 w'' (1)
\end{array} \right), \quad Dy = \left( \begin{array}{c}
 2\beta \eta w' \\
 \kappa w' (1)
\end{array} \right), \quad Gy = \left( \begin{array}{c}
 w \\
 0
\end{array} \right) \]

on the domains

\[ D (C) = \left\{ y = \left( \begin{array}{c}
 w \\
 w'' (1)
\end{array} \right) \left| \begin{array}{c}
 w (0) = 0, \ w'' (0) = 0, \ w^{(3)} (1) - (\gamma - \eta^2) w' (1) = 0
\end{array} \right\} \subseteq W^4_2 (0, 1), \right. \]

\[ D (D) = D (C), \quad D (G) = L^2 (0, 1) \times \mathbb{C}, \]

the space \( W^m_2 (0, 1), m \in \mathbb{N}^+, \) denoting the usual Sobolev–Hilbert space. It is easily verified that \( C \) is selfadjoint and, under the condition \( \gamma > \eta^2 \), uniformly positive; the operator \( D \) for \( \eta = 0 \) is nonnegative and of rank 1, and \( G \) is positive definite when restricted to \( D (C) \). The domain of \( \mathcal{M} (\lambda) \) is \( \lambda \)-independent and is given by \( D (\mathcal{M} (\lambda)) = D (C) \). For every \( y \in D (C) \), there holds \( \mathcal{M} (\lambda) y = 0 \) if and only if \( (1.14) \) and the boundary conditions \( (1.15)–(1.18) \) hold. So the pencil \( \mathcal{M} (\lambda) \) represents the boundary-eigenvalue problem \( (1.14)–(1.18) \).

Alternatively \((1.14)–(1.18)\) may be cast into the framework of the spectral problem for the linear pencil \( P (\lambda) \), which is of exactly the same form as \( (1.17) \), with \( T \) (definition in Section 2) acting in the space

\[ X = \left\{ x = \left( \begin{array}{c}
 w \\
 v
\end{array} \right) \left| \begin{array}{c}
 w \in \tilde{W}^2_2 (0, 1) \times L^2 (0, 1)
\end{array} \right\} \right. \]

(1.20)
here

\[ W_2^m(0, 1) = \{ w \in W_2^m(0, 1) \mid w(0) = 0 \}, \quad m \in \mathbb{N}^+. \]

The eigenvalues of \( \mathcal{M} \) coincide (including multiplicities) with those of \( T \) and are the same as those of the boundary-eigenvalue problem (1.14)–(1.16). Moreover, the root functions \( w_0(\lambda, s), w_1(\lambda, s), \ldots, w_k(\lambda, s) \) of (1.14)–(1.16) and root vectors \( y_0, y_1, \ldots, y_k \) of \( \mathcal{M} \) for the eigenvalue \( \lambda \) are in one-to-one correspondence. So, returning to the problem of Riesz basis in \( L_2(0, 1) \times \mathbb{C} \), we will see in the sequel that it reduces in the end to that in the space \( L_2(0, 1) \), where it is the root functions that we are concerned with. The important question of course is as to what may be inferred from their Riesz basis property as far as what the Riesz basis property of the root vectors \( x_0, x_1, \ldots, x_k \) of \( T \) in \( X \) might be. We give a careful analysis of this question, not attempted in any of the papers we are aware of.

The organisation of the paper is as follows. In Section 2 we give the semigroup formulation of the initial/boundary-value problem (1.1)–(1.3), (1.13) in terms of the operator \( T \), and then use these in Section 4 to prove that there exists a sequence of eigenfunctions or eigenvectors of the boundary-eigenvalue problem (1.14)–(1.18) forming a Riesz basis for the spaces \( L_2(0, 1) \) and \( L_2(0, 1) \times \mathbb{C} \), and then prove that the corresponding sequence of eigenvectors of \( T \) have the property of being a Riesz basis for \( X \) too. A key tool in our proofs will be a verification that the sequence of eigenfunctions or eigenvectors is quadratically close to some orthonormal basis for the space considered and then to appeal to a well-known theorem due to Bari (see [3, Section VI.2] for details). In the case where \( \eta, \kappa > 0 \), as will be seen in Section 3 the eigenvalues approach a vertical asymptote in the left half-plane a finite distance away from the imaginary axis. As a consequence we have by Definition 1.1 that \( T \) is the generator of an exponentially stable \( C_0 \)-semigroup, which constitutes the principal result of Section 5. Finally, the implications of our study for future research are discussed in the conclusions, Section 6.

Throughout this paper, we use the following standard notions from the spectral theory of pencils in a Hilbert space. Let \( \lambda \mapsto \mathcal{L}(\lambda) \) be a mapping from \( \mathbb{C} \) into the set of closed linear operators in \( X \). The resolvent set of \( \mathcal{L} \) is the set of \( \lambda \) for which \( \mathcal{L}(\lambda) \) is boundedly invertible (that is \( \lambda \in \mathbb{C} \) if and only if \( \mathcal{L}(\lambda) \) is closed and bounded) and is denoted by \( \varrho \mathcal{L} \). The spectrum of \( \mathcal{L} \) is the set of \( \lambda \in \mathbb{C} \) that \( \mathcal{L}(\lambda) \) is closed and bounded and is denoted by \( \sigma \mathcal{L} \). If a number \( \lambda \in \mathbb{C} \) has the property that \( \ker \mathcal{L}(\lambda) \neq \{ 0 \} \) then it is called an eigenvalue of \( \mathcal{L} \) and there exists an eigenvector \( x \neq 0 \) corresponding to such a number \( \lambda \). The set of all eigenvalues of \( \mathcal{L} \) is denoted by \( \sigma_p \mathcal{L} \). If an eigenvalue \( \lambda_0 \in \sigma_p \mathcal{L} \) is isolated and \( \mathcal{L}(\lambda_0) \) is a Fredholm operator (see [3, Section IV.5.1] for definition), then we call \( \lambda_0 \) a normal eigenvalue. The set of all such eigenvalues forms the discrete spectrum. Obviously these definitions coincide with the familiar definitions for the spectrum of a closed operator \( A \) in \( X \) when \( \mathcal{L}(\lambda) = \lambda I - A \).

2. Semigroup formulation and well-posedness

In the space \( X \) defined by (1.20), the initial/boundary-value problem (1.11)–(1.13), (1.13) is equivalent to an initial-value problem of the form (1.3): that is,

\[ \dot{x}(t) = Tx(t), \quad x(0) = x_0, \]

wherein \( x(t), x_0 \) are defined formally as in (1.3) and where the operator \( T := A + B \) with

\[ Ax = \begin{pmatrix} v \\ -w(4) + (\gamma - \eta^2)w'' \end{pmatrix}, \quad Bx = \begin{pmatrix} 0 \\ -2\beta\eta v' \end{pmatrix}, \]

(2.2)
Integrating the differential equation in (2.5) twice, making use of the boundary conditions where

\[ w'' (0) = 0, \quad w'' (1) + \kappa v' (1) = 0, \]

\[ w^{(3)} (1) - (\gamma - \eta^2) w' (1) = 0 \]

while \( B \) has domain

\[ D (B) = \left\{ x = \begin{pmatrix} w \\ v \end{pmatrix} \right| w \in W^2_2 (0, 1) \cap W^3_2 (0, 1), \quad v \in \hat{W}^2_2 (0, 1) \right\} \]

Under the condition \( \gamma > \eta^2 \) the space \( \mathbb{X} \) is a Hilbert space when equipped with the inner product

\[ \langle x, \bar{x} \rangle := (w, \bar{w})_2 + (w, \bar{w})_1 + (v, \bar{v}), \quad x = \begin{pmatrix} w \\ v \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} \bar{w} \\ \bar{v} \end{pmatrix} \in \mathbb{X}, \]

where

\[ (w, \bar{w})_2 = \int_0^1 w'' (s) \bar{w}'' (s) \, ds, \quad (w, \bar{w})_1 = (\gamma - \eta^2) \int_0^1 w' (s) \bar{w}' (s) \, ds, \]

and \( (\cdot, \cdot) \) denotes the usual inner product in \( L^2 (0, 1) \). The basic result ensuring the well-posedness of (2.1) is given by the following lemma.

**Lemma 2.1.** Under the condition \( \gamma > \eta^2 \), the system operator \( T \) has a compact inverse and is maximal dissipative for \( \kappa > 0 \). For \( \eta = \kappa = 0 \) the operator is skewadjoint.

**Proof.** Consider the equation \( Ax = \bar{x} \) with \( \bar{x} \in \mathbb{X} \) and \( x \in D (A) \). Equivalently

\[ \begin{align*}
   v &= \bar{w}, \\
   w^{(4)} - (\gamma - \eta^2) w'' &= -\bar{v}, \\
   w (0) &= w'' (0) = 0, \\
   w'' (1) + \kappa v' (1) &= 0, \\
   w^{(3)} (1) - (\gamma - \eta^2) w' (1) &= 0.
\end{align*} \]

Integrating the differential equation in (2.5) twice, making use of the boundary conditions \( w (0) = w'' (0) = 0 \) and \( w^{(3)} (1) - (\gamma - \eta^2) w' (1) = 0 \), we get

\[ w'' (s) - (\gamma - \eta^2) w (s) = \int_0^s dt \int_t^1 \tilde{v} (r) \, dr. \]

So

\[ w (s) = a \sinh \sqrt{\gamma - \eta^2} s + \frac{1}{\sqrt{\gamma - \eta^2}} \int_0^s \sinh \sqrt{\gamma - \eta^2} (s - r) \tilde{V} (r) \, dr \]

with the integral term \( \tilde{V} (s) = \int_0^s dt \int_t^1 \tilde{v} (r) \, dr \) is the solution of the differential equation satisfying the aforementioned three boundary conditions. Application of the remaining boundary condition gives

\[ a = -\frac{\sqrt{\gamma - \eta^2} \int_0^1 \sinh \sqrt{\gamma - \eta^2} (1 - r) \tilde{V} (r) \, dr - \tilde{V} (1) - \kappa \bar{v}' (1)}{\sqrt{\gamma - \eta^2} \sinh \sqrt{\gamma - \eta^2} \, (\gamma - \eta^2)} =: b (\bar{w}, \bar{v}), \]

where we have used that \( v = \bar{w} \). Thus the inverse operator

\[ (A^{-1} \bar{x}) (s) = \begin{pmatrix} b (\bar{w}, \bar{v}) \sinh \sqrt{\gamma - \eta^2} s + \frac{1}{\sqrt{\gamma - \eta^2}} \int_0^s \sinh \sqrt{\gamma - \eta^2} (s - r) \tilde{V} (r) \, dr \end{pmatrix} \]
exists, and it is apparent that $A^{-1} \dot{x} = x \in D(\lambda I)$. The space $(W_2^4(0,1) \cap W_2^2(0,1)) \times W_2^2(0,1)$ is compactly embedded in $X$ and therefore $A^{-1}$ is a compact operator on $X$. Furthermore, we have that

$$BA^{-1} \left( \begin{array}{c} \bar{w} \\ \bar{\bar{v}} \end{array} \right) = \left( \begin{array}{c} 0 \\ -2\beta \eta \bar{v} \end{array} \right)$$

and so, $\bar{w}$ being an element of $W_2^2(0,1)$, we have that $BA^{-1}$ is compact. Using the compactness of $A^{-1}$ together with that of $BA^{-1}$ we see that $T$ is a relatively compact perturbation of $A$. Therefore $T^{-1}$ is compact (see [6, Theorem IV.5.26]).

Now let us verify dissipativeness of $T$ by showing that $\text{Re} \langle Tx, x \rangle \leq 0$ for all $x \in D(\lambda I)$. We compute

$$2\text{Re} \langle Tx, x \rangle = \langle Tx, x \rangle + \langle x, Tx \rangle = -2\beta \eta |v(1)|^2 - 2\kappa |v'(1)|^2, \quad (2.6)$$

and that $T$ is dissipative can be easily verified using that $\beta \in (0,1)$ and $\eta, \kappa \geq 0$, by definition. The fact that $T$ is maximal dissipative follows from the fact (just proven) that it is closed. From (2.6) it is clear that $T$ is skewsymmetric for $\eta = \kappa = 0$. In fact using (2.6) we can show that $T$ is skewadjoint. If $\eta = \kappa = 0$, we have with $Tx = y$, where $x \in D(\lambda I)$,

$$\langle y, T^{-1}y \rangle = \langle TT^{-1}y, T^{-1}y \rangle = -\langle T^{-1}y, TT^{-1}y \rangle = -\langle T^{-1}y, y \rangle.$$

As $T^{-1}$ is bounded, it is skewadjoint, whence the skewadjointness of $T$ follows.

With the fact that $T$ is, by Lemma 2.1, a densely defined linear operator in $X$ (because $X$ is a Hilbert space), we have the following result by the Lumer–Phillips theorem.

**Theorem 2.1.** Suppose $T$ is maximal dissipative. Then $T$ is the infinitesimal generator of a contractive $C_0$-semigroup of bounded linear operators, $S(t), t \geq 0, X$.

An immediate consequence of Theorem 2.1 is that the problem (2.1) is well posed in the sense that it has the unique solution $x \in C^1([0, \infty); X) \cap C([0, \infty); D(\lambda I))$ of the form (1.9) for all $x_0 \in D(\lambda I)$.

3. **Spectrum and eigenvalue asymptotics**

Consider the abstract spectral problem for $T$ in $X$,

$$P(\lambda) x = (\lambda I - T) x = 0, \quad x \in D(\lambda I), \quad \lambda \in \mathbb{C}. \quad (3.1)$$

We know already from Lemma 2.1 that $T$ has a compact resolvent on $X$, and $0 \in \rho(T)$. Therefore its spectrum is purely discrete, $\sigma(T) = \sigma_p(T)$, consisting of normal eigenvalues accumulating only at infinity (see [6, Theorem III.6.29]). Important further information about the location of the eigenvalues are obtained in the next result.

**Theorem 3.1.** The spectrum of $T$ is symmetric with respect to the real axis and lies in the closed left half-plane, excluding the origin when $\gamma > \eta^2$. In the case when additionally $\eta, \kappa > 0$ the spectrum is confined to the open left half-plane.

**Proof.** Let $x \neq 0$ be an eigenvector of $T$ corresponding to an eigenvalue $\lambda$. Then, from (3.1),

$$\langle (\lambda I - T) x \rangle = 0. \quad (\lambda I - T) x = 0.$$

With $T := A + B$ as defined by (2.2)–(2.4) we have

$$\langle (\lambda I - T) x \rangle = \langle (\lambda I - T) \varpi \rangle = 0,$$

which means that $\varpi$ is an eigenvector corresponding to the eigenvalue $\lambda$. This proves that the spectrum of $T$ is symmetric with respect to the real axis.

We now take the inner product of $(\lambda I - T) x$ with $x$ to obtain

$$\langle (\lambda I - T) x, x \rangle = \lambda \|x\|^2 - \langle Tx, x \rangle = 0.$$

The real part of this equation is

$$\text{Re} \lambda \|x\|^2 - \text{Re} \langle Tx, x \rangle = 0. \quad (3.2)$$
By Lemma 2.1
\[ \text{Re } \lambda = \frac{\text{Re} \langle Tx, x \rangle}{\|x\|^2} \leq 0, \]
proving that the spectrum of \( T \) lies in the closed left half-plane. That the origin does not belong to the spectrum is an immediate consequence of Lemma 2.1.

For the proof of the final statement, namely, that when \( \eta, \kappa > 0 \) then \( \text{Re } \lambda < 0 \), suppose \( \text{Re } \lambda = 0 \), that is, \( \lambda \) is a purely imaginary eigenvalue. Then from (3.2) along with (2.6) we have
\[ \text{Re} \langle Tx, x \rangle = -\beta \eta |v(1)|^2 - \kappa |v'(1)|^2 = 0. \]
This implies, for \( \eta, \kappa > 0 \), that \( |v'(1)|^2 = 0, |v(1)|^2 = 0 \)
and, as
\[ x = \begin{pmatrix} w \\ v \end{pmatrix} \]
is an eigenvector, so \( v = \lambda w \),
\[ |\lambda|^2 |w'(1)|^2 = 0, |\lambda|^2 |w(1)|^2 = 0. \]
Therefore \( w(1) = w'(1) = 0 \) since \( |\lambda| > 0 \) by Lemma 2.1. In this case, \( w \) solves the boundary-eigenvalue problem
\[
\begin{cases}
  w^{(4)} - (\gamma - \eta^2) w'' + 2\lambda \beta \eta w' + \lambda^2 w = 0, \\
  w(0) = w''(0) = 0, \\
  w(1) = w'(1) = w''(1) = w^{(3)}(1) = 0.
\end{cases}
\]
From the well-known uniqueness of the solutions of the differential equation in (3.3) it follows that only the trivial solution can result from \( w(1) = w'(1) = w''(1) = w^{(3)}(1) = 0 \). However, this contradicts the assumption that
\[ x = \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} w \\ \lambda w \end{pmatrix} \neq 0 \]
is an eigenvector. Consequently, we must have \( \text{Re } \lambda < 0 \) as \( \eta, \kappa > 0 \). The theorem is thus proven. \( \square \)

**Remark 3.1.** The relevance of Theorem 3.1 is remarkable (for our purposes at least). It says that as long as \( \gamma > \eta^2 \), the eigenvalues lie in the open left half-plane for \( \eta > 0 \) even when damping is admitted – that is, when \( \kappa > 0 \).

3.1. Asymptotics of eigenvalues. Our goal in this subsection is to investigate the asymptotic behaviour of the boundary-eigenvalue problem (1.14)–(1.18) when \( |\lambda| \) is sufficiently large and also to find explicit asymptotic expressions for its eigenvalues. By Theorem 3.1 all eigenvalues with nonzero imaginary part occur in pairs, so we need only consider the boundary-eigenvalue problem in the second quadrant of the complex plane. As usual, we use the standard substitution \( \lambda = i\rho^2 \) and define the sector \( S \) in the complex plane by
\[ S := \{ \rho \in \mathbb{C} \mid 0 \leq \arg \rho \leq \pi/4 \} . \]
So for \( \rho \in S \) we have the ordering
\[ \text{Re} (-\rho) \leq \text{Re} (i\rho) \leq \text{Re} (-i\rho) \leq \text{Re} (\rho) ; \]
obviously the inequalities \( \text{Re} (-\rho) \leq -c |\rho| \) and \( \text{Re} (i\rho) \leq 0 \) then hold with \( c \) a positive constant. Moreover, asymptotically, that is for large \( |\rho| \),
\[ |e^{i\rho}| \leq 1, \quad |e^{-\rho}| = \mathcal{O}(e^{-c|\rho|}), \]
which we shall use subsequently.
As mentioned above, one of our main tasks is to obtain asymptotic expressions for the eigenvalues of the boundary-eigenvalue problem (1.14)-(1.18). First we prove a lemma.

**Lemma 3.1.** In the sector $S$, the differential equation (1.14) with $\lambda = i\rho^2$ has four linearly independent solutions $w_r(\rho, s)$, $r = 1, 2, 3, 4$, which are analytic functions of $\rho \in S$ for $|\rho|$ large and whose asymptotic formulae are as follows:

\[
\begin{align*}
    w_1(\rho, s) &= e^{ips} \left[ 1 + W_1(s) + \frac{W_{11}(s)}{\rho} + O(\rho^{-2}) \right], \\
    w_2(\rho, s) &= e^{-ips} \left[ 1 + W_2(s) + \frac{W_{21}(s)}{\rho} + O(\rho^{-2}) \right], \\
    w_3(\rho, s) &= e^{-ips} \left[ 1 + W_3(s) + \frac{W_{31}(s)}{\rho} + O(\rho^{-2}) \right], \\
    w_4(\rho, s) &= e^{ips} \left[ 1 + W_4(s) + \frac{W_{41}(s)}{\rho} + O(\rho^{-2}) \right],
\end{align*}
\]

where

\[
\begin{align*}
    W_1(s) &= -1 + e^{\frac{i\beta\eta s}{2}}, \\
    W_2(s) &= -1 + e^{-\frac{i\beta\eta s}{2}}, \\
    W_3(s) &= -1 + e^{\frac{i\beta\eta s}{2}}, \\
    W_4(s) &= -1 + e^{-\frac{i\beta\eta s}{2}}.
\end{align*}
\]

and

\[
\begin{align*}
    W_{11}(s) &= -\frac{i}{4} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 \right) e^{\frac{i\beta\eta s}{2}}, \quad W_{21}(s) = -\frac{1}{4} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 \right) e^{-\frac{i\beta\eta s}{2}}, \\
    W_{31}(s) &= \frac{i}{4} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 \right) e^{\frac{i\beta\eta s}{2}}, \quad W_{41}(s) = \frac{1}{4} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 \right) e^{-\frac{i\beta\eta s}{2}}.
\end{align*}
\]

**Proof.** If in (1.14) we make the trivial change of the eigenvalue parameter from $\lambda$ to $i\rho^2$ associated with $\rho \in S$, then (1.14) may be rewritten as

\[
w^{(4)}(\rho, s) - (\gamma - \eta^2) w''(\rho, s) + 2i\rho^2 \beta\eta w'(\rho, s) - \rho^4 w(\rho, s) = 0. \tag{3.5}
\]

Take \(\{w_r(\rho, s)\}_{r=1}^4\) to be the fundamental system of solutions to (3.5) having the asymptotic expansions (see [14, Section II.4.6])

\[
w_r(\rho, s) = e^{\rho \omega_r s} \left[ 1 + W_r(s) + \frac{W_{1r}(s)}{\rho} + \frac{E_r(s, \rho)}{\rho^2} \right], \quad r = 1, 2, 3, 4, \tag{3.6}
\]

for $|\rho|$ large, where $\omega_r = i\tau$ and the functions $s \mapsto E_r(s, \rho)$ are sufficiently often continuously differentiable on $[0, 1]$. Substituting (3.6) and their derivatives (with respect to $s$) in (3.5) and collecting terms according to powers of $\rho$, we obtain after some simplification

\[
\begin{align*}
    0 &= \rho^3 \left[ 4\omega_r^3 W_r'(s) + 2i\beta\eta \omega_r \left( 1 + W_r(s) \right) \right] + \rho^2 \left[ 4\omega_r^3 W_r''(s) + 2i\beta\eta \omega_r W_r'(s) - (\gamma - \eta^2) \omega_r^2 \left( 1 + W_r(s) \right) \right] \\
    &\quad + 6\omega_r^2 W_r'''(s) + 2i\beta\eta W_r'(s) - (\gamma - \eta^2) \omega_r^2 \left( 1 + W_r(s) \right) \right] \\
    &\quad + \rho \left[ 4\omega_r^3 E_r(s, \rho) + 2i\beta\eta \omega_r E_r(s, \rho) + 6\omega_r^2 W_r''(s) + 2i\beta\eta W_r'(s) - (\gamma - \eta^2) \omega_r^2 W_r(s) \right] \\
    &\quad + 4\omega_r W_r^{(3)}(s) - 2(\gamma - \eta^2) \omega_r W_r''(s) + 6\omega_r^2 E_r''(s, \rho) + 2i\beta\eta E_r'(s, \rho) \\
    &\quad - (\gamma - \eta^2) \omega_r^2 E_r(s, \rho) + 4\omega_r W_r^{(3)}(s) - 2(\gamma - \eta^2) \omega_r W_r''(s) + W_r^{(4)}(s) - (\gamma - \eta^2) W_r''(s) \\
    &\quad + \rho^{-1} \left[ W_r^{(4)}(s) - (\gamma - \eta^2) W_r''(s) + 4\omega_r E_r^{(3)}(s, \rho) - 2(\gamma - \eta^2) \omega_r E_r'(s, \rho) \right] \\
    &\quad + \rho^{-2} \left[ E_r^{(4)}(s, \rho) - (\gamma - \eta^2) E_r''(s, \rho) \right].
\end{align*}
\]
Let us now consider from this the system of differential equations
\[ 4\omega_r^3 W_r''(s) + 2i\beta\eta\omega_r (1 + W_r(s)) = 0, \quad (3.7) \]
\[ 4\omega_r^4 W_r' (s) + 2i\beta\eta\omega_r W_r(s) + 6\omega_r^2 W_r''(s) + 2i\beta\eta W_r'(s) - (\gamma - \eta^2) \omega_r^2 (1 + W_r(s)) = 0. \quad (3.8) \]
Solving (3.7), (3.8) with \( W_r(0) = W_r'(0) = 0 \) we have
\[ W_r(s) = -1 + e^{-\frac{\beta}{2}\omega_r^2 s}, \quad W_r'(s) = \frac{\omega_r^3}{4} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 \right) e^{-i\frac{\beta}{2}\omega_r^2 s}. \quad (3.9) \]
Combining (3.7) and (3.8), along with the fact that \( W_r'(s) = -i\frac{\beta}{2}\omega_r^2 e^{-i\frac{\beta}{2}\omega_r^2 s} \), we can compute
\[ W_r'(s) + i\frac{\beta}{2}\omega_r^2 W_r(s) - \frac{\beta^2 \eta^2}{8} \omega_r^3 e^{-i\frac{\beta}{2}\omega_r^2 s} - (\gamma - \eta^2) \omega_r^2 e^{-i\frac{\beta}{2}\omega_r^2 s} = 0. \]
Since \( (W_r'(s) + i\frac{\beta}{2}\omega_r^2 W_r(s)) e^{i\frac{\beta}{2}\omega_r^2 s} = (W_r(s) e^{i\frac{\beta}{2}\omega_r^2 s})' \), it follows that
\[ (W_r(s) e^{i\frac{\beta}{2}\omega_r^2 s})' = \frac{\omega_r^3}{4} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 \right). \]
Using this expression it is readily verified that the functions \( E_r(s, \rho) \) are uniformly bounded with respect to \( s \in [0, 1] \) and \( \rho \in S \) for large \( |\rho| \). Noting that \( \omega_r^2 = (1)^r, \omega_r^3 = (-i)^r \) then gives the desired result from (3.9) for \( r = 1, 2, 3, 4 \). \( \square \)

With Lemma 3.1 established, we can now determine the asymptotic expressions for the eigenvalues of (1.14)–(1.18).

**Theorem 3.2.** Supposing \( \rho \in S \), eigenvalues of the boundary-eigenvalue problem (1.14)–(1.18) are given by \( \lambda_n = i\rho_\nu \), the \( \rho_n \) being expressed asymptotically by
\[ \rho_n = (n + 1/2) \pi + \frac{\beta^2 \eta^2}{8} + \gamma - \eta^2 + 2i\beta\eta + \frac{\beta}{2} \pi + O(n^{-2}). \]

**Proof.** Let \( \rho \in S \) and \( w(\rho, s) \) be an eigenfunction of (1.14)–(1.18) (again \( \lambda \) is replaced by \( i\rho^2 \)). Applying the result of Lemma 3.1, the linear combination
\[ w(\rho, s) = a_1 w_1(\rho, s) + a_2 w_2(\rho, s) + a_3 w_3(\rho, s) + a_4 w_4(\rho, s) \]
for constants \( a_r, r = 1, 2, 3, 4 \), is inserted into the boundary conditions. Then we have that the boundary-eigenvalue problem has a nonzero solution if and only if
\[
\begin{pmatrix}
  w_1(0, \rho) & w_2(0, \rho) \\
  w_1'(0, \rho) & w_2'(0, \rho) \\
  w_1''(1, \rho) + i\kappa^2 w_1'(1, \rho) & w_2''(1, \rho) + i\kappa^2 w_2'(1, \rho) \\
  w_1'''(1, \rho) - (\gamma - \eta^2) w_1'(1, \rho) & w_2'''(1, \rho) - (\gamma - \eta^2) w_2'(1, \rho)
\end{pmatrix}
= 0.
\]
\[ (3.10) \]
Let, as before in the proof of Lemma 3.1,
\[ w_r(\rho, s) = e^{i\omega_r s} \left[ 1 + W_r(s) + \frac{W_r'(s)}{\rho} + \frac{E_r(s, \rho)}{\rho^2} \right], \quad \omega_r = i^r, \quad r = 1, 2, 3, 4. \]
We may directly compute
\[
\begin{align*}
    w' (\rho, s) &= \rho \omega_r e^{\rho \omega_r s} \left[ 1 + W_r (s) + \frac{\omega_r W_r (s)}{\rho \omega_r} + O (\rho^{-2}) \right], \\
    w'' (\rho, s) &= (\rho \omega_r)^2 e^{\rho \omega_r s} \left[ 1 + W_r (s) + \frac{\omega_r W_r (s) + 2 W_r' (s)}{\rho \omega_r} + O (\rho^{-2}) \right], \\
    w''' (\rho, s) &= (\rho \omega_r)^3 e^{\rho \omega_r s} \left[ 1 + W_r (s) + \frac{\omega_r W_r (s) + 3 W_r' (s)}{\rho \omega_r} + O (\rho^{-2}) \right].
\end{align*}
\]

Hence, recalling (3.9), we obtain after some straightforward computations
\[
\begin{align*}
    w_r (0, \rho) &= 1, \\
    w''_r (0, \rho) &= (\rho \omega_r)^2 \left[ 1 - i \frac{\beta \eta}{\rho} \omega_r + O (\rho^{-2}) \right], \\
    w''_r (1, \rho) + i \kappa \rho^2 w'_r (1, \rho) &= \frac{\omega_r}{i \kappa \rho} e^{-i \frac{\beta \eta}{2}} \left[ 1 + \frac{1}{4 \rho \omega_r} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 - 2 i \beta \eta \omega_r \right) + O (\rho^{-2}) \right], \\
    w'''_r (1, \rho) - (\gamma - \eta^2) w'_r (1, \rho) &= (\rho \omega_r)^3 e^{i \frac{\beta \eta}{2}} e^{-i \frac{\beta \eta}{2}} \left[ 1 + \frac{1}{4 \rho \omega_r} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 - 6 i \beta \eta \omega_r \right) + O (\rho^{-2}) \right].
\end{align*}
\]

Using these relations in (3.10) together with (3.4) shows that, for large $|\rho|$,\[
\begin{align*}
\det \left( \begin{array}{ccc}
    e^{-i \beta \eta} e^{-i \frac{\beta \eta}{2}} & e^{i \beta \eta} e^{-i \frac{\beta \eta}{2}} \\
    - \left[ 1 + \frac{\beta \eta}{\rho} + O (\rho^{-2}) \right] e^{-i \beta \eta} e^{-i \frac{\beta \eta}{2}} & 0 \end{array} \right) & = 0, \\
\begin{array}{ccc}
    0 & -i + \frac{1}{4 \rho} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 + 2 i \beta \eta \right) + \frac{1}{\kappa \rho} + O (\rho^{-2}) \\
    -i - \frac{1}{4 \rho} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 + 6 i \beta \eta \right) + O (\rho^{-2}) & 0
\end{array} \\
0 & 0 & 0 \\
1 + \frac{1}{4 \rho} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 - 2 i \beta \eta \right) + \frac{1}{\kappa \rho} + O (\rho^{-2}) & 1 + \frac{1}{4 \rho} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 - 6 i \beta \eta \right) + O (\rho^{-2})
\end{array} \right) + O (e^{-c |\rho|}) = 0,
\end{align*}
\]
where we have used that $\omega_1 = -\omega_3 = i$ and $\omega_2 = -\omega_4 = -1$. We find, after reducing the determinant, that
\[
\begin{align*}
    -4 i \cos \frac{\rho}{\rho} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 + 2 i \beta \eta \right) + \frac{2 i}{\kappa} \\
    -i \sin \frac{\rho}{\rho} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 + 2 i \beta \eta \right) + \frac{2 i}{\kappa}
\end{align*}
\]
\[
\end{align*}
\]
\]

(3.11)
Now, set
\[ \rho_n = (n + 1/2) \pi + \varepsilon_n, \quad n \in \mathbb{N}^+, \]
where the \( \varepsilon_n \) are complex parameters to be determined shortly. Substituting \( \rho_n \) for \( \rho \) in (3.11) and noting that
\[
\begin{align*}
\cos \rho_n &= \cos (n + 1/2) \pi \cos \varepsilon_n - \sin (n + 1/2) \pi \sin \varepsilon_n = -(-1)^n \sin \varepsilon_n, \\
\sin \rho_n &= \sin (n + 1/2) \pi \cos \varepsilon_n + \cos (n + 1/2) \pi \sin \varepsilon_n = (-1)^n \cos \varepsilon_n,
\end{align*}
\]
we obtain that the \( \varepsilon_n \) satisfy
\[
\sin \varepsilon_n = \frac{\cos \varepsilon_n}{4 \rho_n} \left( \frac{\beta^2 \eta^2}{2} + \gamma - \eta^2 + 2i\beta\eta + \frac{2i}{\kappa} \right) + \mathcal{O}(\rho_n^{-2})
\]
for \( |\rho_n| \) large. Thus, for small \( \varepsilon_n \),
\[
\varepsilon_n = \frac{\beta^2 \eta^2 + \gamma - \eta^2 + 2i\beta\eta + \frac{2i}{\kappa}}{4(n + 1/2) \pi} + \mathcal{O}(n^{-2}),
\]
which, together with (3.12), completes the proof of the theorem. \( \square \)

**Remark 3.2.** Using the asymptotic formula in the theorem for the \( \rho_n \) we can calculate
\[
\lambda_n = i \rho_n^2 = - \left( \beta \eta + \frac{1}{\kappa} \right) + i \left[ \frac{i}{\tau_n} + \frac{\beta^2 \eta^2 + \gamma - \eta^2}{2} \right] + \mathcal{O}(\tau_n^{-2}), \quad \tau_n = (n + 1/2) \pi,
\]
from which it is then clear that the eigenvalues line up along vertical asymptotes in the left half-plane.

4. **Riesz basis properties**

So far in the paper we have shown that for \( \gamma > \eta^2 \) all eigenvalues of the boundary-eigenvalue problem (1.14)–(1.18) are normal, distributed symmetrically with respect to the real axis in the left half of the complex plane, and, asymptotically, they lie in a strip of finite width parallel to the imaginary axis in the open left half-plane when \( \eta, \kappa > 0 \). It follows that all eigenvalues with nonzero imaginary part occur in pairs \( \{\lambda_n, \overline{\lambda_n}\} \), and that we can index all such \( \lambda_n \) so that \( \lambda_{n+1} = \overline{\lambda_n} \). Moreover, the eigenvalues can be ordered \( 0 \leq \text{Im} \lambda_n \leq \text{Im} \lambda_{n+1} \).

Our ultimate goal, of course, is to prove exponential stability of the semigroup \( S(\cdot) \) in the space \( X \) for which one will have to verify that the conditions of Definition 3.1 are satisfied. In this section, we prove two principal results. We show that the eigenfunctions of the boundary-eigenvalue problem (1.14)–(1.18) form a Riesz basis for \( L_2(0,1) \), and, furthermore, the eigenvectors of the pencil \( M \) (given by (1.19)) form a Riesz basis for the space \( L_2(0,1) \times \mathbb{C} \). The second of these two results is to show that the eigenvectors of the system operator \( T \) form a Riesz basis for \( X \).

**Theorem 4.1.** Suppose there is a sequence \( \{\lambda_n\}_{n=1}^{\infty} \) with \( \lambda_n = i \rho_n^2 \) whose asymptotic behaviour is specified as in Theorem 3.2. Assume, without loss of generality, that there are no purely real eigenvalues of \( T \). Then the sequence \( \{\lambda_{n+1}\}_{n=1}^{\infty} \) with \( \lambda_{n+1} = \overline{\lambda_n} \) represents all eigenvalues of the boundary-eigenvalue problem (1.14)–(1.18), or the pencil \( M \) generated by (1.14)–(1.18). Corresponding to these eigenvalues is a sequence \( \{y_{\pm n}\}_{n=1}^{\infty} \) of eigenvectors of \( M \),
\[
y_n = \begin{pmatrix} w(\rho_n, \cdot) \\ w'(\rho_n, 1) \end{pmatrix}, \quad y_{-n} = \begin{pmatrix} w(\rho_n, \cdot) \\ w'(\rho_n, 1) \end{pmatrix}, \quad n \in \mathbb{N}^+,
\]
and a sequence \( \{w(\rho_n, s), w'(\rho_n, s)\}_{n=1}^{\infty} \) of eigenfunctions with \( \|w(\rho_n, \cdot)\| = 1 \), such that
(i) the sequence \( \{w(\rho_n, \cdot), w'(\rho_n, \cdot)\}_{n=1}^{\infty} \) forms a Riesz basis for the space \( L_2(0,1) \); and
(ii) the sequence \( \{y_{\pm n}\}_{n=1}^{\infty} \) forms a Riesz basis for the space \( L_2(0,1) \times \mathbb{C} \).

**Proof.** We prove assertions (i) and (ii). The proof of assertion (i) will proceed in three steps.
Step 1. We begin by considering the boundary-eigenvalue problem
\[
\begin{cases}
\hat{w}^{(4)} - (\gamma - \eta^2) \hat{w}'' + \lambda^2 \hat{w} = 0, \\
\hat{w}(0) = \hat{w}''(0) = 0, \\
\hat{w}''(1) = 0, \\
\hat{w}^{(3)}(1) - (\gamma - \eta^2) \hat{w}'(1) = 0,
\end{cases}
\] (4.1)
wherein we have \(\lambda\)-independent boundary conditions. Let \(\hat{\lambda}_n = i\tilde{\tau}_n^2\) be an eigenvalue of (1.1) with eigenfunction \(\hat{w}(\tilde{\tau}_n, s)\). It should be observed that (1.1) is a selfadjoint problem. Then, using a standard spectral theory result (see [13, Section III.9.1], for example) we have that the sequence \(\{\hat{w}(\tilde{\tau}_n, \cdot), \hat{w}(\tilde{\tau}_n, \cdot)\}_{n=1}^{\infty}\) forms an orthonormal basis for \(L_2(0, 1)\), and by the proof of Theorem 3.2 we note that the following holds:
\[
\tilde{\tau}_n = \tau_n + \frac{\gamma - \eta^2}{\tilde{\tau}_n} + O(\tau_n^{-2}), \quad \tau_n = (n + 1/2)\pi.
\]

Step 2. Consider now the boundary-eigenvalue problem
\[
\begin{cases}
\hat{w}^{(4)} - (\gamma - \eta^2) \hat{w}'' + \lambda^2 \hat{w} = 0, \\
\hat{w}(0) = \hat{w}''(0) = 0, \\
\hat{w}''(1) + \lambda\kappa \hat{w}'(1) = 0, \\
\hat{w}^{(3)}(1) - (\gamma - \eta^2) \hat{w}'(1) = 0,
\end{cases}
\] (4.2)
Let \(\hat{\lambda}_n = i\tilde{\rho}_n^2\) be an eigenvalue of (4.2) with eigenfunction \(\hat{w}(\tilde{\rho}_n, s)\). It can be shown, via the same arguments as applied in the proof of Theorem 3.2 that
\[
\tilde{\rho}_n = \tilde{\tau}_n + i\frac{\kappa}{2\gamma \tilde{\tau}_n} + O(\tilde{\tau}_n^{-2}),
\]
and
\[
\hat{w}(\tilde{\rho}_n, s) = \hat{w}(\tilde{\tau}_n, s) \sum_{k=0}^{\infty} \frac{U_k(s)}{\tilde{\tau}_n^k},
\]
where the functions \(U_k(s)\) are uniformly bounded. Hence there is a positive constant \(M_0\) such that
\[
\|\hat{w}(\tilde{\tau}_n, \cdot) - \hat{w}(\tilde{\rho}_n, \cdot)\|^2 \leq \frac{M_0}{|\tilde{\tau}_n|^2}, \quad n \in \mathbb{N}^+,
\]
so that the gaps \(\hat{w}(\tilde{\tau}_n, \cdot) - \hat{w}(\tilde{\rho}_n, \cdot)\) are uniformly bounded in norm.

Step 3. Let \(w(\rho_n, s)\) be an eigenfunction of the boundary-eigenvalue problem (1.14)–(1.18) corresponding to an eigenvalue \(\lambda_n = i\rho_n^2\). Then, in view of Theorem 3.2 it can be shown by direct calculation that the asymptotic formula for the \(\rho_n\) may be written as
\[
\rho_n = \tilde{\rho}_n + \frac{\beta^2 \rho_n^2 + 2i \beta \eta}{4(n + 1/2)\pi} + O(n^{-2})
\]
or, equivalently, as
\[
\rho_n = \tilde{\rho}_n + \frac{\beta^2 \rho_n^2 + 2i \beta \eta}{4\tilde{\rho}_n} + O(\tilde{\rho}_n^{-2}), \quad \tilde{\rho}_n = (n + 1/2)\pi.
\]
We have
\[
w(\rho_n, s) = \hat{w}(\tilde{\rho}_n, s) \sum_{k=0}^{\infty} \frac{F_k(s)}{\tilde{\rho}_n^k}.
\]
where the functions \( F_k(s) \) are uniformly bounded. Thus, again, we have that there exists a positive constant \( M_1 \) such that
\[
\left\| w(\rho_n, \cdot ) - \hat{w}(\tilde{\rho}_n, \cdot ) \right\|^2 \leq \frac{M_1}{|\tilde{\rho}_n|^2}, \quad n \in \mathbb{N}^+.
\]

Note that
\[
\lim_{n \to \infty} \frac{\rho_n}{\tau_n} = \lim_{n \to \infty} \frac{\tilde{\rho}_n}{\tau_n} = \lim_{n \to \infty} \frac{\tilde{\tau}_n}{\tau_n} = 1
\]
and there exists a positive constant \( M \) such that
\[
\left\| \hat{w}(\tilde{\tau}_n, \cdot ) - w(\rho_n, \cdot ) \right\|^2 \leq \frac{M}{\tilde{\tau}_n^2}, \quad n \in \mathbb{N}^+.
\]

Hence the sequence \( \{ w(\rho_n, s), w(\rho_n, s) \}_{n=1}^\infty \) is quadratically close to the orthonormal sequence \( \{ \hat{w}(\tilde{\tau}_n, s), \hat{w}(\tilde{\tau}_n, s) \}_{n=1}^\infty \), that is
\[
\sum_{n=1}^\infty \left\| \hat{w}(\tilde{\tau}_n, \cdot ) - w(\rho_n, \cdot ) \right\|^2 < \infty.
\]

The desired result now follows from Bari’s theorem.

Assertion [iii] can be reversed. Let the sequence \( \{ y_{\pm n} \}_{n=1}^\infty \) form a Riesz basis for \( L_2(0, 1) \times \mathbb{C} \). Then it forms a Riesz basis for \( D(C) \), equipped with the norm
\[
\| y \|_{D(C)} = \| \mathcal{M}(\lambda) y \|_{L_2(0,1) \times \mathbb{C}}, \quad y \in D(C),
\]
and there exists a bounded and boundedly invertible operator \( S \) in \( L_2(0, 1) \times \mathbb{C} \) such that
\[
Sy = S \left( \begin{array}{c} w \\ w'(1) \end{array} \right) = w
\]
with \( y \in D(C) \) and \( w \in L_2((0,1)) \). Hence the sequence \( \{ w(\rho_n, \cdot ), w(\rho_n, \cdot ) \}_{n=1}^\infty \) forms a Riesz basis for \( L_2(0, 1) \) which indeed is true by assertion [i]. This in turn implies that the sequence \( \{ y_{\pm n} \}_{n=1}^\infty \) does indeed form a Riesz basis for \( L_2(0, 1) \times \mathbb{C} \), and the theorem is proven. \( \square \)

**Theorem 4.2.** Let the \( \lambda_n \) in the sequence \( \{ \lambda_n \}_{n=1}^\infty \) be given as in Theorem 3.2, and let \( \{ w(\rho_n, \cdot ), w(\rho_n, \cdot ) \}_{n=1}^\infty \) be the sequence of eigenfunctions of the boundary-eigenvalue problem (1.11) - (1.12) corresponding to these eigenvalues. Set
\[
x_n = \left( \begin{array}{c} w(\rho_n, \cdot ) \\ \lambda_n \end{array} \right), \quad x_{-n} = \left( \begin{array}{c} w(\rho_n, \cdot ) \\ \lambda_n \end{array} \right), \quad n \in \mathbb{N}^+.
\]

Then the following statements hold:

1. \( \sigma(T) = \sigma_p(T) = \{ \lambda_n, \lambda_n \}_{n=1}^\infty \) is discrete.

2. \( (\lambda_n I - T) x_n = 0 \) and \( (\lambda_n I - T) x_{-n} = 0 \), and
\[
\| x_n \| < \sqrt{2}, \quad n \in \mathbb{N}^+.
\]

3. The sequence \( \{ x_{\pm n} \}_{n=1}^\infty \) forms a Riesz basis for \( \mathcal{X} \).

**Proof.** Statement [1] is an immediate consequence of Lemma 2.1 and Theorem 3.1.

For the proof of statement [2] note that, for every \( x_n \in D(A) \), we have
\[
Tx_n = \left( \begin{array}{c} w(\rho_n, \cdot ) \\ \lambda_n \end{array} \right) + (\gamma - \eta^2) \frac{w''(\rho_n, \cdot )}{\lambda_n} - 2\beta \eta w'(\rho_n, \cdot ) = \lambda_n x_n
\]
and, since \( (\lambda_n I - T) x_n = 0 \), we see that
\[
(\lambda_n I - T) x_{-n} = (\lambda_n I - T) x_n = 0.
\]
To verify the remaining statements (2) and (3) of the theorem, we first compute for any \( n \), using integration by parts,
\[
\|x_n\|^2 = -\frac{\Re \lambda_n}{|\lambda_n|^2} \left( \kappa |w'(\rho_n, 1)|^2 + \beta \eta |w(\rho_n, 1)|^2 \right) \\
- \frac{\Re \lambda_n^2}{|\lambda_n|^2} \int_0^1 |w(\rho_n, s)|^2 \, ds + \int_0^1 |w(\rho_n, s)|^2 \, ds
\]
and therefore verify that
\[
\|x_n\|^2 \leq M_0 \|w(\rho_n, \cdot)\|^2, \quad n \in \mathbb{N}^+,
\]
for some positive constant \( M_0 \). Set now
\[
\hat{x}_n = \left( \frac{\hat{w}(\tilde{\tau}_n, \cdot)}{i\hat{\tau}_n^2} \right), \quad \hat{x}_{-n} = \left( -\frac{\hat{w}(\tilde{\tau}_n, \cdot)}{i\hat{\tau}_n^2} \right), \quad n \in \mathbb{N}^+.
\]
For any \( m, n \in \mathbb{N}^+ \), it follows that for every \( \hat{x}_m, \hat{x}_n \in \mathbb{X} \),
\[
\langle x_n, x_m \rangle = \frac{\tilde{\tau}_n^2}{|\tilde{\tau}_n^2 + i\omega|} \int_0^1 \hat{w}(\tilde{\tau}_n, s) \overline{\hat{w}(\tilde{\tau}_m, s)} \, ds + \int_0^1 \hat{w}(\tilde{\tau}_n, s) \overline{\hat{w}(\tilde{\tau}_m, s)} \, ds.
\]
Thus for \( m = n \), when \( \|\hat{w}(\tilde{\tau}_n, \cdot)\| = 1 \), we have
\[
\|\hat{x}_n\| = 2
\]
and \( \{\hat{x}_{\pm n}\}_{n=1}^\infty \) forms an orthonormal basis for \( \mathbb{X} \).

Now, there exists a positive constant \( M \) such that
\[
\|x_n - \hat{x}_n\|^2 = \int_0^1 \left| \frac{w''(\rho_n, s)}{\lambda_n} - \frac{\hat{w}''(\tilde{\tau}_n, s)}{i\hat{\tau}_n^2} \right|^2 \, ds + (\gamma - \eta^2) \int_0^1 \left| \frac{w'(\rho_n, s)}{\lambda_n} - \frac{\hat{w}'(\tilde{\tau}_n, s)}{i\hat{\tau}_n^2} \right|^2 \, ds
\]
\[
+ \int_0^1 |w(\rho_n, s) - \hat{w}(\tilde{\tau}_n, s)|^2 \, ds
\]
\[
\leq \frac{M}{|\tilde{\tau}_n|^2} \int_0^1 |\hat{w}(\tilde{\tau}_n, s)|^2 \, ds.
\]
Taking into account the earlier assumption \( \|\hat{w}(\tilde{\tau}_n, \cdot)\| = 1 \), we have
\[
\|x_n - \hat{x}_n\|^2 \leq \frac{M}{|\tilde{\tau}_n|^2}, \quad n \in \mathbb{N}^+.
\]
With this, again applying Bari’s theorem, the proof of the theorem is complete. \( \square \)

5. Series expansions in the space \( \mathbb{X} \) and exponential stability

We begin by characterising the series solution to the initial-value problem (2.1). The following theorem is a direct consequence of Theorem 1.2.

**Theorem 5.1.** Given \( x_0 \in D(A) \), the series solution to (2.1) is given by
\[
x(t) = S(t)x_0 = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x_0, x_n^* \rangle x_n + \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x_0, x_{-n}^* \rangle x_{-n}
\]
where the sequence \( \{x_n^*\}_{n=1}^\infty \) is the biorthogonal sequence for \( \{x_{\pm n}\}_{n=1}^\infty \).

Combining the above theorem with what has been said at the beginning of the previous section about the location and asymptotics of eigenvalues of \( T \), we obtain as a corollary exponential stability of the \( C_0 \)-semigroup corresponding to the initial/boundary-value problem (1.1)–(1.3), (1.13) by appealing to Definition 1.1.
Corollary 5.1. The semigroup $S(\cdot)$ generated by the system operator $T$ is exponentially stable (in the sense of Definition 1.1) for any $\eta, \kappa > 0$.

6. Conclusions

This paper has considered the problem of exponential stability associated with a stretched tube conveying an ideal fluid with damped boundary conditions. The problem has been resolved by using a spectral approach involving a detailed analysis of the spectral properties of the corresponding boundary-eigenvalue problem (1.14)-(1.18) and subsequent analysis of the Riesz basis property of the corresponding eigenvectors. We have shown that the eigenvectors constitute a Riesz basis for the space $\mathbb{X}$. We have also established, somewhat independently, the Riesz basis properties of the eigenfunctions or eigenvectors in the spaces $L^2(0,1)$ and $L^2(0,1) \times \mathbb{C}$. Under these circumstances we have proven a positive result on exponential stability of the semigroup $S(\cdot)$ corresponding to the initial/boundary-value problem (1.1)-(1.3), (1.13).

In the case where $\gamma > \eta^2$ and $\eta, \kappa > 0$, all eigenvalues of (1.14)-(1.18) lie in the open left half-plane and approach a vertical asymptote a finite distance away from the imaginary axis. Specifically, by the formula in Remark 3.2, the eigenvalues obey the asymptotic relation

$$\text{Re} \lambda_n \sim -\left(\beta \eta + \frac{\kappa}{2}\right)$$

where $\beta \in (0,1)$, by definition. This leads to the physically interesting conclusion that, when the tension in the tube is very much bigger than the fluid velocity, $\gamma > \eta^2$, the addition of small damping represented by $\kappa > 0$ does not destroy (and actually improves) the exponential stability – in the sense of Definition 1.1 – of the semigroup $S(\cdot)$, even when there is flow in the tube represented by $\eta > 0$.

The boundary-eigenvalue problem considered in this paper belongs to the general class of boundary-eigenvalue problems with eigenvalue-dependent boundary conditions. Associated with the boundary-eigenvalue problem has been the nonmonic pencil

$$\mathcal{M} (\lambda) = \lambda^2 G + \lambda D + C,$$

with $G$ not being regular. So there arises the more fundamental question as to the appropriate abstract formulation of such boundary-eigenvalue problems and on conditions on the operators $C, D, G$ under which the eigenvectors will form a Riesz basis for the space $\mathbb{X}$. This problem will be addressed in a future paper.

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