MASS TRANSPORTATION FUNCTIONALS ON THE SPHERE WITH APPLICATIONS TO THE LOGARITHMIC MINKOWSKI PROBLEM

ALEXANDER V. KOLESNIKOV

Abstract. We study the transportation problem on the unit sphere $S^{n-1}$ for symmetric probability measures and the cost function $c(x, y) = \log \frac{1}{\langle x, y \rangle}$. We calculate the variation of the corresponding Kantorovich functional $K$ and study a naturally associated metric-measure space on $S^{n-1}$ endowed with a Riemannian metric generated by the corresponding transportational potential. We introduce a new transportational functional which minimizers are solutions to the symmetric log-Minkowski problem and prove that $K$ satisfies the following analog of the Gaussian transportation inequality for the uniform probability measure $\sigma$ on $S^{n-1}$:

$$\frac{1}{n} \text{Ent}(\nu) \geq K(\sigma, \nu).$$

It is shown that there exists a remarkable similarity between our results and the theory of the Kähler-Einstein equation on Euclidean space. As a by-product we obtain a new proof of uniqueness of solution to the log-Minkowski problem for the uniform measure.

1. Introduction

We start with explanations and representation of some related results in the Euclidean case. Let $\mu = e^{-V} dx$, $\nu = e^{-W} dx$ be probability measures on $\mathbb{R}^n$ and $x \rightarrow \nabla \Phi(x)$ be the optimal transportation mapping pushing forward $\mu$ onto $\nu$. The potential $\Phi$ is a convex function solving the dual Kantorovich problem with quadratic cost. If the densities of measures are sufficiently regular, $\Phi$ solves the related Monge–Ampère equation

$$e^{-V} = e^{-W(\nabla \Phi)} \det D^2 \Phi$$

(see [36], [6] for details). The related Kantorovich functional

$$W_2^2(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\pi$$

induces a metric on the space of probability measures with finite second moments and satisfies $W_2^2(\mu, \nu) = \int |x - \nabla \Phi(x)|^2 d\mu$; here $\Pi(\mu, \nu)$ is the space of measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals $\mu, \nu$.

Numerous powerful applications of the optimal transportation theory are based on the use of differential structures on the space $\mathcal{P}_2$ of probability measures endowed with metric $W_2$. It is by now a classical fact that many evolutionary equations can be interpreted as gradients flows on $\mathcal{P}_2$. The reader can find a comprehensive representation in [2] (see also [36], [37], [6]). An important related notion which
was introduced by R. McCann is the displacement convexity property, which means
convexity along the geodesics in the Kantorovich metric.

To show displacement convexity of a functional one has to compute its second
order derivatives along the geodesics. The corresponding calculus relies on the use
of the Monge-Ampère operator and its linearized versions. One of such versions is
given by the following formula:

\[ Lf = \text{Tr}(D^2\Phi)^{-1} D^2 f - \langle \nabla f, \nabla W(\nabla \Phi) \rangle. \] (1.2)

This operator naturally appears with differentiation of (1.1). In particular, the
simple linear variation of the source measure \( \mu_\varepsilon = \mu(1 + \varepsilon v) \) by a function \( v \) with
zero mean corresponds to the variation of the potential \( \Phi_\varepsilon = \Phi + \varepsilon u + o(\varepsilon) \), where
\[ Lu = v. \]

Connection of this formula to differential calculus on \( P_2 \) is explained in Section 2.
It was observed in [25] that \( L \) is the generator of the symmetric Dirichlet form
\[ \mathcal{E}(f, g) = \int \langle (D^2\Phi)^{-1} \nabla f, \nabla g \rangle d\mu = -\int Lfgd\mu. \]

Let us endow \( \mathbb{R}^n \) with the Riemannian metric \( D^2\Phi \). The related metric-measure
space \( (\mu, D^2\Phi) \) is a natural geometric and probabilistic object, it has been studie d
in [13], [14], [20], [21], [22], [23], [24], [25], [28].

Of particular interest is the following special case:
\[ \mu = e^{-\Phi} dx. \] (1.3)

Following the terminology from [13] we say that \( \nu \) is a moment measure if there
exists another probability measure \( \mu \) of the form (1.3) such that \( \nu \) is the image of
\( \mu \) under \( \nabla \Phi \). The most general sufficient condition for \( \nu \) to be a moment measure
was established by B. Klartag and D. Cordero-Erausquin in [13].

It is known that \( \Phi \) is the unique maximum point of the following functional:
\[ J(f) = \log \int e^{-f^*} dx - \int f \nu, \] (1.4)
where \( f^* \) is the Legendre transform of \( f \). This fact was used in [13] to establish
well-posedness of the moment measure problem. Another natural functional which
minimizers solve the same problem was suggested by F. Santambrogio in [33]. The
following Gaussian version of this functional was studied in [26]:
\[ \mathcal{F}(g \cdot \gamma) = \text{Ent}_\gamma(g) - \frac{1}{2} W_2^2(g \cdot \gamma, \nu), \] (1.5)
where \( \gamma \) is the standard Gaussian measure, \( \text{Ent}_\gamma(g) = \int g \log g d\gamma \) is the Gaussian
entropy, and \( W_2(g \cdot \gamma, \nu) \) is the Kantorovich distance between \( g \cdot \gamma \) and \( \nu \). In the
particular case \( \nu = \gamma \) the minimum of \( \mathcal{F} \) equals zero, it is attained at \( g = 1 \).
The positivity of the functional \( \mathcal{F}(g \cdot \gamma) \) for \( \nu = \gamma \) is equivalent to the Talagrand
transportation inequality
\[ \text{Ent}_\gamma(g) \geq \frac{1}{2} W_2^2(g \cdot \gamma, \gamma) \] (1.6)
(see [3]). See [26], [15] for further information on relations between (1.5) and sta-
bility estimates for Gaussian inequalities and [17] for variational approach to other
transportation inequalities. The main result of [15] is the following estimate:
\[ \mathcal{F}(g \cdot \gamma) \geq -\text{Ent}_\gamma(\nu) \]
provided either $\nu$ or $g \cdot \gamma$ has zero mean.

In this work we develop a similar formalism on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Instead of Riemannian analog of $W_2$ on $S^{n-1}$ we shall work with the functional

$$K(\mu, \nu) = \min_{\pi \in \Pi(\mu, \nu)} \int_{S^{n-1}} c(x, y) d\pi,$$

where

$$c(x, y) = \begin{cases} \log \frac{1}{\langle x, y \rangle}, & \langle x, y \rangle > 0 \\ +\infty, & \langle x, y \rangle \leq 0. \end{cases}$$

and $\mu, \nu$ are probability measures on $S^{n-1}$.

The importance of this functional for convex geometry was revealed by V. Oliker in [32]. He proved that the Kantorovich problem with the cost function $c$ is in a sense equivalent to a classical problem from convex geometry, the so-called Aleksandrov problem. See also [4], [19], [27]. The corresponding optimal transportation mapping has the form

$$T(x) = \frac{h(x) \cdot x + \nabla_{S^{n-1}} h(x)}{\sqrt{h^2(x) + |\nabla_{S^{n-1}} h(x)|^2}}, \quad (1.7)$$

where $h$ is a support function of a convex body containing the origin.

In Section 3 we prove that $(1.2)$ admits the following spherical analog for a couple of probability measures with densities

$$\mu = e^{-V} \cdot \sigma, \quad \nu = e^{-W} \cdot \sigma,$$

and the corresponding optimal transportation mapping $(1.7)$:

$$L_{\mu, \nu}(\frac{u}{h}) = \text{Tr}(D^2 h)^{-1} D^2 u - \langle \nabla_{S^{n-1}} W(T) + nT, \frac{u x + \nabla_{S^{n-1}} u}{\sqrt{h^2 + |\nabla_{S^{n-1}} h|^2}} \rangle + \frac{u}{h}.$$  

Here $\sigma$ is the probability uniform measure on $S^{n-1}$,

$$D^2 f = f \cdot \text{Id} + \nabla^2_{S^{n-1}} f,$$

$\nabla_{S^{n-1}}$ is the gradient on $S^{n-1}$, and $\nabla^2_{S^{n-1}}$ is the Hessian operator on $S^{n-1}$. The associated Dirichlet form:

$$\mathcal{E}_{\mu, \nu}(f, g) = \int_{S^{n-1}} h \langle (D^2 h)^{-1} \nabla_{S^{n-1}} f, \nabla_{S^{n-1}} g \rangle d\mu = -\int_{S^{n-1}} f L_{\mu, \nu} g d\mu.$$

A particular case of $L_{\mu, \nu}$ has been studied by E. Milman and the author in [31] in respect to the log-Brunn-Minkowski conjecture (see works of Böröczky, Lutwak, Yang, and Zhang [7], [8]). It is conjectured that a reinforcement of the Brunn–Minkowski inequality holds within the class of all symmetric convex bodies. We don’t give a complete list of references concerning this problem, because it is too long. The readers are advised to consult the papers [7], [8], [10], [12], [31], [34] and the references therein. The log-Brunn–Minkowski inequality implies that the log-Minkowski problem has a unique solution.

**Log-Minkowski problem:** Given a (symmetric) probability measure $\mu$ on $S^{n-1}$ find a (symmetric) convex body $\Omega \ni 0$ of volume 1 such that $\mu$ is the cone measure of $\Omega$.

Note that under additional assumption that $\mu$ has a (sufficiently regular) density $\mu = \rho_\mu dx$ the support function $h$ of $\Omega$ must satisfy the following equation of the Monge–Ampère type:

$$\rho_\mu = \frac{1}{n} h \det D^2 h. \quad (1.8)$$
The main result of [31] states that a local version of the even log-Brunn–Minkowski conjecture is equivalent to the second eigenvalue problem for the operator $L_{\mu,\nu}$, where $\mu$ is given by (1.8) and $\nu = \mu \circ T^{-1}$, where $T$ is given by (1.7). It this case $L_{\mu,\nu}$ has the form

$$L_{\mu,\nu}(u) = \text{Tr}(D^2h)^{-1}D^2u - (n-1)\frac{u}{h}.$$ 

From now let us restrict ourselves to the symmetric case: it is assumed that all the sets are symmetric and functions are even. A necessary and sufficient condition (subspace concentration condition) for existence of a solution to the even log-Minkowski problem has been established in [8]. It turns out that any minimum point of the functional

$$h \to \int_{S^{n-1}} \log h d\mu$$

with the constraint $|\Omega_h| = 1$ is a solution to the log-Minkowski problem for $\mu$.

Following the idea from [33] we introduce another functional which minimizers are solutions to the even log-Minkowski problem. The spherical entropy of a measure $m$ is defined as follows:

$$\text{Ent}(m) = \begin{cases} \int \rho \log \rho d\sigma, & \text{if } m = \rho \cdot \sigma \\ +\infty, & \text{otherwise.} \end{cases}$$

**Theorem 1.1.** The minimizers of the functional

$$F(\nu) = \frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu),$$

are solutions to the log-Minkowski problem for $\mu$.

Note that (1.9) and (1.10) are in remarkable correspondence with (1.4) and (1.5). Some other connections between the moment-measure problem and the log-Minkowski problem has been mentioned in [13]. However, we should stress that unlike Gaussian or Euclidean case (see [33], [26]) it is not clear whether $F$ is displacement convex. Note that the strong displacement convexity of $F$ would imply uniqueness of the solution to the log-Minkowski problem.

On the other hand, we were able to prove the following analog of the Gaussian transportation inequality (1.6).

**Theorem 1.2.** Every symmetric measure $\nu$ on $S^{n-1}$ satisfies

$$\frac{1}{n} \text{Ent}(\nu) \geq K(\sigma, \nu).$$

This result seems to be a natural generalization of (1.6) for the sphere. Other transportational and functional inequalities on the sphere has been studied in [5], [11] (see also [3]). We emphasize that the standard proofs of transportation inequalities usually involve displacement convexity arguments (or some equivalent constructions).

The proof of this inequality follows the classical arguments with an additional ingredient: with the help of the Blaschke-Santaló inequality we establish the following estimate which compensates the lack of displacement convexity (for a more general statement see Proposition 7.2)

$$\frac{1}{n-1} \int_{S^{n-1}} \text{Tr}(D^2h)^{-1} dx \geq \int_{S^{n-1}} \frac{dx}{h},$$

(1.11)
where $h \in C^2(S^{n-1})$ is a symmetric support function of a convex body. The tightness of this inequality immediately implies uniqueness for the log-Minkowski problem for the case $\mu = \sigma$, which was shown first by W. Firey [16] (see explanations in the last section). Finally, we conjecture the following inequality generalizing (1.11):

$$\frac{1}{n - 1} \int_{S^{n-1}} \text{Tr}(D^2 f)^{-1}(D^2 h) d\mu \geq \int_{S^{n-1}} h f d\mu,$$

(1.12)

where $\mu = \frac{1}{n |\Omega|} h \det D^2 h \cdot H^{n-1}$ and $h, f \in C^2(S^{n-1})$ are symmetric support functions of convex bodies. This inequality implies uniqueness of a solution to the general even log-Minkowski problem, provided the equality case holds if and only if $f = ch$.

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2. Preliminaries: variation of the Euclidean Kantorovich distance

In our work we deal with optimal transportation on the sphere and related variational problems. Before we consider the spherical case let us briefly explain the relevant Euclidean technique.

Let $\nabla \Phi$ be the optimal transportation of $\mu = \rho dx = e^{-V} dx$ onto $\nu = e^{-W} dx$. By the change of variables formula

$$V = W(\nabla \Phi) - \log \det D^2 \Phi.$$

We will calculate the variation of $W_2$. The formula we get is a particular case of a well-known result ([36], Theorem 8.3). We include it for completeness of the picture.

Given a function $v$ with zero $\mu$-mean $\int v \rho \, dx = 0$ consider the variation of $\mu$:

$$\rho_\varepsilon = (1 + \varepsilon v) \rho.$$

Let $\nabla \Phi_\varepsilon$ be the optimal transportation of $\rho_\varepsilon dx$ onto $\nu = e^{-W} dx$. One has

$$\Phi_\varepsilon = \Phi + \varepsilon u + o(\varepsilon),$$

where

$$V - \log(1 + \varepsilon v) = W(\nabla \Phi_\varepsilon) - \log \det(D^2 \Phi_\varepsilon).$$

This relation immediately implies

$$Lu := \text{Tr}[(D^2 \Phi)^{-1} D^2 u] - \langle \nabla W(\nabla \Phi), \nabla u \rangle = v.$$  

(2.1)

One can check by direct computations that $L$ is a generator of the Dirichlet form on the metric-measure space $(D^2 \Phi, \mu)$ (see [23]):

**Lemma 2.1.**

$$\int v f \rho \, dx = -\int \langle (D^2 \Phi)^{-1} \nabla u, \nabla f \rangle \rho \, dx.$$

**Proof.** By the change of variables formula for any smooth function $g$

$$\int g(\nabla \Phi_\varepsilon)(1 + \varepsilon v) \rho \, dx = \int g \, d\mu.$$

Expanding in $\varepsilon$ at zero one gets

$$\int g(\nabla \Phi) v \rho \, dx + \int \langle \nabla g(\nabla \Phi), \nabla u \rangle \rho \, dx = 0.$$

Setting $f = g(\nabla \Phi)$ one gets the claim. □
Proposition 2.2. Let $F(\rho_\varepsilon) = \int \langle x, \nabla \Phi_\varepsilon \rangle d\rho_\varepsilon$. One has
\[ \frac{d}{d\varepsilon} F(\rho_\varepsilon)|_{\varepsilon=0} = \int \Phi v \rho dx. \]

Proof.
\[ \frac{d}{d\varepsilon} F(\rho_\varepsilon)|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int \langle x, \nabla \Phi_\varepsilon \rangle \rho_\varepsilon dx|_{\varepsilon=0} = \int \langle x, \nabla u \rangle \rho dx + \int \langle x, \nabla \Phi \rangle v \rho dx. \]

By the previous Lemma
\[ \int \langle x, \nabla \Phi \rangle v \rho dx = - \int \langle (D^2 \Phi)^{-1} \nabla u, \nabla (\langle x, \nabla \Phi \rangle) \rangle \rho dx = - \int \langle x, \nabla u \rangle \rho dx \]
\[ - \int \langle (D^2 \Phi)^{-1} \nabla u, \nabla \Phi \rangle \rho dx = - \int \langle x, \nabla u \rangle \rho dx + \int \Phi v \rho dx. \]

The proof is complete. \( \square \)

Proposition 2.2 can be used to compute the variation of
\[ W^2_2(\mu, \nu) = \int (x - \nabla \Phi)^2 d\mu. \]

Corollary 2.3. The variation of the Kantorovich distance $W_2$ can be computed as follows:
\[ \frac{d}{d\varepsilon} W^2_2(\rho_\varepsilon, \nu)|_{\varepsilon=0} = \int (x^2 - 2\Phi) v \rho dx = -2 \int \langle x - \nabla \Phi, (D^2 \Phi)^{-1} \nabla u \rangle d\mu. \]

The above computation is a particular case of the general expression for derivative of $W_2$ on the space $P_2$ (see [36], Theorem 8.3). According to this result
\[ \frac{d}{dt} W^2_2(\rho_t, \nu)|_{t=0} = 2 \int \langle x - \nabla \Phi, \xi_0 \rangle d\mu, \]
where $\rho_t$ is a family of probability densities satisfying
\[ \frac{\partial \rho_t}{\partial t} + \text{div}(\rho_t \cdot \xi_t) = 0 \quad (2.2) \]
for some given velocity field $\xi_t$. The reader can check that Corollary 2.3 follows from these formulae for a field $\xi_t$ with initial velocity $\xi_0 = -(D^2 \Phi)^{-1} \nabla u$ and (2.2) is equivalent to another representation of $v$:
\[ v = e^V \text{div}((D^2 \Phi)^{-1} \nabla u \cdot e^{-V}). \]

3. Spherical logarithmic Kantorovich functional

Notations. In what follows $|\Omega|$ is the volume of a convex body $\Omega \subset \mathbb{R}^n$, $|\partial \Omega|$ is the $(n-1)$-dimensional Hausdorff measure of the boundary $\partial \Omega$ of $\Omega$, $B$ is the unit ball in $\mathbb{R}^n$ with center at the origin and $S^{n-1}$ is the boundary of $B$. The $(n-1)$-dimensional Hausdorff measure is denoted by $\mathcal{H}^{n-1}$, the normalized probability uniform measure on $S^{n-1}$ is denoted by $\sigma$. Note that
\[ \sigma = \frac{\mathcal{H}^{n-1}|_{S^{n-1}}}{n|B|}. \]

Finally, the integral of a function $f$ over $\partial \Omega$ with respect to the $(n-1)$-dimensional Hausdorff measure on the boundary $\partial \Omega$ will be denoted either by
\[ \int_{\partial \Omega} f dx \]
or by
\[ \int_{\partial \Omega} f \, d\mathcal{H}^{n-1}. \]

Given a support function \( h \), the corresponding convex body will be denoted by \( \Omega_h \).

Everywhere in this section \( \Omega \) is a compact convex body containing the origin. For any given couple of probability measures \( \mu, \nu \) on \( S^{n-1} \) we define the Kantorovich functional

\[ K(\mu, \nu) = \min_{\pi \in \Pi(\mu, \nu)} \int_{S^{n-1}} c(x, y) \, d\pi, \]

where we work with the following cost function:

\[ c(x, y) = \begin{cases} 
\log \frac{1}{\langle x, y \rangle}, & \langle x, y \rangle > 0 \\
+\infty, & \langle x, y \rangle \leq 0.
\end{cases} \]

This function naturally appears in convex geometry. This can be explained by the fact that the radial

\[ r(y) = \sup \{ t : ty \in \Omega, t > 0 \}, \ y \in S^{n-1} \]

and the support function

\[ h(x) = \sup_{z \in \Omega} \langle x, z \rangle, \ x \in S^{n-1} \]

of a convex body \( \Omega \) are related by a Legendre-type transform

\[ h(x) = \sup_{y \in S^{n-1}} r(y) \langle x, y \rangle, \]

\[ \frac{1}{r(y)} = \sup_{x \in S^{n-1}} \frac{\langle x, y \rangle}{h(x)} \]

(see [35]). In particularly, \( \log h \) and \( \log r \) satisfy a Kantorovich-type duality relation

\[ \log r(y) - \log h(x) \leq \log \frac{1}{\langle x, y \rangle}. \]

Applying this observation, V. Oliker [32] has shown that the solution to this transportation problem solves a classical problem in convex geometry known as Aleksandrov problem. Other applications of the cost function \( c \) see in [19], [27].

**Aleksandrov problem:** Given a probability measure \( \nu \) on \( S^{n-1} \) find a convex body \( \Omega \) with \( 0 \in \Omega \) such that \( \nu \) is the image of the uniform measure \( \sigma \) of \( S^{n-1} \) under the mapping

\[ T(y) = n_{\partial \Omega} \circ r(y)y, \]

where

\[ n_{\partial \Omega} : \partial \Omega \to S^{n-1} \]

in the Gauss map.

Aleksandrov gave in [11] the following sufficient condition for existence of a solution to this problem.

**Theorem 3.1** ([11]). **Aleksandrov problem admits a unique solution provided \( \nu \) satisfies the following assumption:** for every spherically convex set \( A \subset S^{n-1}, A \neq S^{n-1} \)

\[ \nu(A) < \sigma(A_{\pi/2}), \quad (3.1) \]

where \( A_{\pi/2} = \{ y : \text{dist}(A, y) < \frac{\pi}{2} \} \) and \( \text{dist} \) is the standard distance on \( S^{n-1} \).
A transportational solution to this problem was obtained by V. Oliker in [32]. He also proved well-posedness of the dual problem and the absence of the duality gap. His result was generalized later by J. Bertrand in [4]. In particular, Bertrand constructed a transportational solution for a couple of probability measures \( \mu = f \cdot \sigma, \nu \) under the generalized Alexandrov-type assumption:

\[
\nu(A) < \mu(A_{\pi/2})
\]

(3.2)

(see Remark 4.9 in [4]).

In what follows \( \nabla_{S^{n-1}}, \nabla^2_{S^{n-1}} \) denote the spherical gradient and the spherical Hessian accordingly.

We will also use the following operator acting on the tangent space \( TS^{n-1}(x) \) at the point \( x \)

\[
D^2f(x) = f(x) \cdot \text{Id} + \nabla^2_{S^{n-1}}f(x),
\]

where \( \text{Id} \) is the identical mapping on \( TS^{n-1}(x) \). In particular

\[
D^2h(n_{\partial \Omega}) = I_{\partial \Omega}^{-1},
\]

where \( I_{\partial \Omega} \) is the second fundamental form of \( \partial \Omega \) and \( h \) is the support functional of \( \Omega \) (see [35]).

The optimal transportation mapping corresponding to \( c \) and pushing forward \( \mu \) onto \( \nu \) has the form

\[
T(x) = \frac{h(x) \cdot x + \nabla_{S^{n-1}}h(x)}{\sqrt{h^2(x) + |\nabla_{S^{n-1}}h(x)|^2}}.
\]

(3.3)

Assuming smoothness of \( h \) one can verify the following change of variables formula.

**Lemma 3.2.** Assume that \( T \) pushes forward a probability measure \( \rho_1 \cdot \sigma \) onto another probability measure \( \rho_2 \cdot \sigma \). Then the following change of variables formula holds

\[
\rho_1 = \rho_2(T) \frac{h \cdot \det D^2h}{(h^2 + |\nabla_{S^{n-1}}h|^2)^{\frac{n}{2}}}. \tag{3.4}
\]

It is important to have in mind the following relation between \( r \) and \( h \):

\[
r(T) = \sqrt{h^2 + |\nabla_{S^{n-1}}h|^2}
\]

(see [34]). In particularly, (3.4) reads also as

\[
\rho_1 = \frac{\rho_2(T)}{r^n(T)} h \cdot \det D^2h. \tag{3.5}
\]

The inverse mapping \( S = T^{-1} \) has the form (see [32])

\[
S(y) = \frac{-\nabla_{S^{n-1}}r(y) + r(y)y}{\sqrt{|\nabla_{S^{n-1}}r(y)|^2 + r^2(y)}}.
\]

To see that \( S \) satisfy the same equation (3.3) for an appropriate choice of the potential, one needs to pass to the inverse radial function

\[
f(y) = \frac{1}{r(y)}. \tag{3.6}
\]

One has

\[
S(y) = \frac{f(y)y + \nabla_{S^{n-1}}f(y)}{\sqrt{|\nabla_{S^{n-1}}f(y)\|^2 + f^2(y)}}.
\]
In our work we will mainly concentrate on the symmetric case, meaning that \( \mu \) and \( \nu \) are invariant with respect to \( x \to -x \). This assumption implies that the body \( \Omega \) is symmetric and \( h \) is even. In the symmetric case the Aleksandrov sufficient condition is automatically satisfied except of some degenerate situations.

**Lemma 3.3.** Assume that \( \mu \) and \( \nu \) are symmetric, \( \mu = f \cdot \sigma \), \( f \) is positive \( \sigma \)-a.e., and \( \nu(S^{n-1} \cap L) = 0 \) for every hyperplane \( L \) passing through the origin. Then (3.2) is satisfied and there exists a unique solution to the Aleksandrov problem.

**Proof.** Let us check (3.2). For every spherically convex subset \( A \neq S^{n-1} \) one can find a hemisphere \( S_l = S^{n-1} - \{ l \geq 0 \} \), where \( l \) is a linear functional such that \( A \subset S_l \). The case \( \nu(A) = 0 \) is obvious, so one can assume \( \nu(A) > 0 \). Since \( \nu(S^{n-1} - \{ l = 0 \}) = 0 \), one has \( \nu(A) = \nu(A') \), where \( A' = A \setminus \{ l = 0 \} \). By the symmetry assumption one gets that \( \nu(A') = \nu(-A') \). Since \( A' \cap -A' = \emptyset \), one gets \( \nu(A) = \nu(A') \leq 1/2 \). Next we note that \( \nu(A) > 0 \), hence one can find a two-points set \( M = \{ a, b \} \subset A \) but the set \( M_{\pi/2} \) is a union of two distinct hemispheres. Since \( \mu \) is symmetric and admits positive density, one immediately gets \( \mu(A_{\pi/2}) \geq \mu(M_{\pi/2}) > 1/2 \geq \nu(A) \). \( \square \)

**Remark 3.4.** Unlike the Aleksandrov problem, a solution to the Monge-Kantorovich problem for \( c \) always exists, because the cost function \( c \) is lower semicontinuous. It may happen that the dual solution \( (h, r) \) does not define any compact convex body \( \Omega \) and the total cost function equals \( +\infty \). For instance, consider \( n = 3 \), \( \mu \) is the symmetric measure which gives value \( 1/2 \) to every pole and \( \nu \) is concentrated on the equator. Clearly, in this case any transportation plan \( \Pi \) is supported on the set \( \{c = +\infty\} \).

If the Aleksandrov problem admits a solution and \( \mu = f \cdot \sigma \), then \( c \in L^\infty(\Pi) \) for the corresponding optimal transport plan \( \Pi \) (see [4]).

4. **Variation of the spherical log-functional**

Let

\[
\mu = \rho_\mu \cdot \sigma = e^{-V} \cdot \sigma
\]

and

\[
\nu = \rho_\nu \cdot \sigma = e^{-W} \cdot \sigma
\]

be probability measures on \( S^{n-1} \) and \( T \) be the optimal transportation mapping for the cost function \( c(x, y) \). It will be assumed that \( \rho_\mu, \rho_\nu \) and \( T \) are sufficiently smooth. Moreover, we assume that \( h \) satisfies

\[
D^2 h(x) > 0, \quad \forall x \in S^{n-1}.
\]

For the fixed target measure \( \nu \) let us consider the variation of the source measure

\[
\mu_\varepsilon = (1 + \varepsilon v)\mu, \quad \int v d\mu = 0.
\]

It corresponds to the following variation of the Kantorovich potential :

\[
h_\varepsilon = h + \varepsilon u + o(\varepsilon).
\]

The variation of the mapping \( T \) looks as follows :

\[
T_\varepsilon = T + \frac{\varepsilon}{\sqrt{h^2 + |\nabla_{S^{n-1}} h|^2}} \Pr_{T_S^{n-1} T(\varepsilon)} (ux + \nabla_{S^{n-1}} u) + o(\varepsilon).
\]
Here

\[ \Pr_{T S_{n-1}^p} \]

is the projection of \( \mathbb{R}^n \) onto the tangent space of \( S^{n-1} \) at the point \( y \).

By the change of variables formula (3.4):

\[
(1 + \varepsilon v) \rho_\mu = \rho_\nu(T_\varepsilon) \frac{h_\varepsilon \cdot \det D^2 h_\varepsilon}{(h_\varepsilon^2 + |\nabla_{S_{n-1}} h_\varepsilon|^2)^{\frac{3}{2}}}.\]

Performing the Taylor expansion of the right-hand side and applying, in particular, the relation

\[ \det(A + \varepsilon B) = \det A \cdot (1 + \varepsilon \text{Tr}[A^{-1}B] + o(\varepsilon)) \]

one obtains

\[ v = \text{Tr}(D^2 h)^{-1} D^2 u - \left\langle \nabla_{S_{n-1}} W(T) + n T, \frac{u x + \nabla_{S_{n-1}} u}{\sqrt{h^2 + |\nabla_{S_{n-1}} h|^2}} \right\rangle + \frac{u}{h}. \quad (4.1) \]

Let us denote the right-hand side of (4.1) by

\[ L_{\mu, \nu}(\frac{u}{h}). \]

This operator is the spherical analog of the Euclidean operator \( L \) (2.1). As in the Euclidean case \( L_{\mu, \nu} \) is associated with a Dirichlet form on \( S^{n-1} \).

**Theorem 4.1.** \( L_{\mu, \nu} \) generates the following symmetric Dirichlet form:

\[
\int_{S_{n-1}} h \left\langle \nabla_{S_{n-1}} g, (D^2 h)^{-1} \nabla_{S_{n-1}} \left( \frac{u}{h} \right) \right\rangle d\mu = - \int_{S_{n-1}} g L_{\mu, \nu}(\frac{u}{h}) d\mu.
\]

**Proof.** Expanding the left-hand side of the change of variables formula in \( \varepsilon \)

\[
\int_{S_{n-1}} f(T_\varepsilon) \ d\mu_\varepsilon = \int_{S_{n-1}} f(T) \ dv
\]

one obtains

\[
\int_{S_{n-1}} f(T) v d\mu + \int_{S_{n-1}} \frac{\left\langle \nabla_{S_{n-1}} f \circ T, \Pr_{T S_{n-1}^p}(u x + \nabla_{S_{n-1}} u) \right\rangle}{\sqrt{h^2 + |\nabla_{S_{n-1}} h|^2}} d\mu = 0.
\]

Set:

\[ g = f(T). \]

One has \( \nabla_{S_{n-1}} g = (DT)^* \nabla_{S_{n-1}} f \circ T \), hence

\[ \nabla_{S_{n-1}} f \circ T = [(DT)^*]^{-1} \nabla_{S_{n-1}} g. \]

Consequently

\[
\int_{S_{n-1}} g v d\mu + \int_{S_{n-1}} \frac{\left\langle \nabla_{S_{n-1}} g, (DT)^{-1} \Pr_{T S_{n-1}^p}(u x + \nabla_{S_{n-1}} u) \right\rangle}{\sqrt{h^2 + |\nabla_{S_{n-1}} h|^2}} d\mu = 0.
\]

It is easy to verify that

\[ DT|_{TS_{n-1}^p} = \frac{1}{\sqrt{h^2 + |\nabla_{S_{n-1}} h|^2}} \Pr_{T S_{n-1}^p} D^2 h. \]
This formula implies
\[
\frac{\langle \nabla_{S^{n-1}g}, (DT)^{-1}\Pr_{TS^\mathbb{T}(x)}(ux + \nabla_{S^{n-1}u}) \rangle}{\sqrt{h^2 + |\nabla_{S^{n-1}} h|^2}} &= \langle \nabla_{S^{n-1}g}, [\Pr_{TS^\mathbb{T}(x)} D^2 h]^{-1}\Pr_{TS^\mathbb{T}(x)}(ux + \nabla_{S^{n-1}u}) \rangle.
\]

We need to compute
\[
v_1 = [\Pr_{TS^\mathbb{T}(x)} D^2 h]^{-1}\Pr_{TS^\mathbb{T}(x)}(ux),
v_2 = [\Pr_{TS^\mathbb{T}(x)} D^2 h]^{-1}\Pr_{TS^\mathbb{T}(x)}(\nabla_{S^{n-1}} u).
\]

Since \( \nabla_{S^{n-1}} u \in TS_x^{n-1} \), it is easy to check that
\[
v_2 = (D^2 h)^{-1}\nabla_{S^{n-1}} u.
\]

Let us compute \( v_1 = u[D^2 h]^{-1}\Pr_{TS^\mathbb{T}(x)} x \). To this end we note that \( v_1 \) is the unique vector from \( TS_x \) such that \( \omega = \frac{1}{u} \cdot D^2 h \cdot v_1 \) belongs to \( TS_x \) and satisfies
\[
\omega = x + \alpha T
\]
for some \( \alpha \in \mathbb{R} \). The condition \( \langle T(x), \omega \rangle = 0 \) implies
\[
\omega = x - \frac{T(x)}{\langle x, T(x) \rangle} = - \frac{\nabla_{S^{n-1}} h}{h}
\]
and
\[
v_1 = -u(D^2 h)^{-1}\nabla_{S^{n-1}} h.
\]

Finally
\[
\int_{S^{n-1}} gvd\mu + \int_{S^{n-1}} \langle \nabla_{S^{n-1}} g, (D^2 h)^{-1}\left( \nabla_{S^{n-1}} u - u \frac{\nabla_{S^{n-1}} h}{h} \right) \rangle d\mu = 0.
\]
Equivalently
\[
\int_{S^{n-1}} gvd\mu = - \int_{S^{n-1}} h\langle \nabla_{S^{n-1}} g, (D^2 h)^{-1}\nabla_{S^{n-1}} \left( \frac{u}{h} \right) \rangle d\mu.
\]

The proof is complete. \( \square \)

Thus we obtain that \( L_{\mu,\nu} \) is the generator of
\[
\mathcal{E}_{\mu,\nu}(f, g) = \int_{S^{n-1}} h\left( (D^2 h)^{-1}\nabla_{S^{n-1}} f, \nabla_{S^{n-1}} g \right) d\mu.
\]

The quadratic form
\[
g_h = \frac{D^2 h}{h} = Id + \frac{\nabla^2_{S^{n-1}} h}{h}
\]
is a Riemannian metric on \( S^{n-1} \) naturally associated with the couple \( (\mu, \nu) \). For this metric \( \mathcal{E}_{\mu,\nu} \) is the standard weighted energy form:
\[
\mathcal{E}_{\mu,\nu}(f) = \int \| \nabla_{g_h} f \|_{g_h}^2 d\mu
\]
and \( L_{\mu,\nu} \) is the weighted Laplacian.
Lemma 4.2. The variation
\[ \delta_v K(\mu, \nu) = \lim_{\varepsilon \to 0} \frac{K(\mu_\varepsilon, \nu) - K(\mu, \nu)}{\varepsilon}, \]
where \( \mu_\varepsilon = (1 + \varepsilon v)\mu, \int_{S^{n-1}} v \, d\mu = 0, \) of the functional \( K(\mu, \nu) \) satisfies
\[ \delta_v K(\mu, \nu) = -\int_{S^{n-1}} \log h \cdot v d\mu. \]

Proof. Note that
\[ K(\mu, \nu) = \int_{S^{n-1}} \log \frac{h^2 + |\nabla_{S^{n-1}} h|^2}{h} \, d\mu. \]
One gets
\[ \delta_v K(\mu, \nu) = -\int_{S^{n-1}} \log h \cdot v d\mu + \int_{S^{n-1}} \frac{uh + \langle \nabla_{S^{n-1}} u, \nabla_{S^{n-1}} h \rangle}{h^2 + |\nabla_{S^{n-1}} h|^2} \, d\mu - \int_{S^{n-1}} \log h \cdot v d\mu \]
\[ + \int_{S^{n-1}} \log h^2 + |\nabla_{S^{n-1}} h|^2 v d\mu. \]
Note that by (4.1)
\[ \int_{S^{n-1}} \log \frac{h^2 + |\nabla_{S^{n-1}} h|^2}{h} \, d\mu = -\int_{S^{n-1}} \log f \cdot v d\mu + \int_{S^{n-1}} \langle \nabla_{S^{n-1}} h, \nabla_{S^{n-1}} u \rangle \, d\mu + \int_{S^{n-1}} u |\nabla_{S^{n-1}} h|^2 \, d\mu. \]
Substituting this into the expression for \( \delta_v K(\mu, \nu) \) one gets the claim. \( \square \)

This result applied to the mapping \( S = T^{-1} \) gives the following formula (see (3.6)):

Corollary 4.3. The variation \( \delta_\omega K(\mu, \nu) = \lim_{\varepsilon \to 0} \frac{K(\mu_\varepsilon, \nu_\varepsilon) - K(\mu, \nu)}{\varepsilon}, \)
where \( \nu_\varepsilon = (1 + \varepsilon \omega)\nu, \int_{S^{n-1}} \omega d\nu = 0, \)
satisfies
\[ \delta_\omega K(\mu, \nu) = -\int_{S^{n-1}} \log f \cdot \omega d\nu + \int_{S^{n-1}} \log r \cdot \omega d\nu. \]

5. Variational formulations of the log-Minkowski problem

Given a convex body \( \Omega \) containing the origin let us denote by \( m \) the measure on \( \partial \Omega \) which is the image of the Lebesgue measure on \( \Omega \) under the mapping \( x \to \frac{x}{\|x\|_{\Omega}}, \)
where \( \| \cdot \|_{\Omega} \) is the associated norm. This measure can be expressed as follows:
\[ m = \frac{1}{n} \langle x, n_{\partial \Omega} \rangle \cdot \mathcal{H}^{n-1}|_{\partial \Omega}, \]
where \( \mathcal{H}^{n-1} \) is the \( (n-1) \)-dimensional Hausdorff measure.

Definition 5.1. The image \( C_\Omega \) of \( m \) under the Gauss map is called the cone measure of \( \Omega \).
Note that
\[ C_\Omega = \frac{1}{n} h \det D^2 h \cdot \mathcal{H}^{n-1}|_{S^{n-1}}, \]
provided \( h \) is sufficiently regular.

**Log-Minkowski problem.** Given a probability measure \( \mu \) on \( S^{n-1} \) find a convex set \( \Omega \subset \mathbb{R}^n \) with \( |\Omega| = 1 \) such that \( \mu \) is the cone measure of \( \Omega \).

The analytical formulation of the log-Minkowski problem looks as follows: given a probability measure with density
\[ \mu = \rho \mu \cdot \sigma = \frac{\rho_\mu}{n|B|} \cdot \mathcal{H}^{n-1}|_{S^{n-1}} \]
on the unit sphere find a convex set \( \Omega \) of volume 1 which support function \( h \) satisfies
\[ \rho_\mu = h \det D^2 h |B|. \]

Note that \( \rho_\mu \) is indeed a probability density:
\[ \int_{S^{n-1}} \rho_\mu \, d\sigma = \frac{1}{n|B|} \int_{S^{n-1}} \rho_\mu \, dx = \frac{1}{n} \int_{S^{n-1}} h \det D^2 h \, dx = \frac{1}{n} \int_{\partial \Omega} \langle x, n_{\partial \Omega} \rangle \, dx = |\Omega| = 1. \]

**Assumption.** In what follows we assume that \( \mu, \nu \) are symmetric measures and \( h \) is an even function.

A variational approach to the log-Minkowski problem was suggested in [8] (Lemma 4.1). It was shown that a solution \( h \) to the following variational problem
\[ \int_{S^{n-1}} \log h \, d\mu \mapsto \min, \]
considered in the class of symmetric support functions of convex bodies with volume 1, is a support function of a body \( \Omega \) solving the log-Minkowski problem for \( \mu \).

In our work we propose another variational functional for the symmetric log-Minkowski problem and defined with the help of mass transportation. Our approach partially motivated by the results of [33].

Another important relation of the log-Minkowski problem to the mass transportation problem implicitly appeared in [31]. Let \( h \) be a support function of some convex set \( \Omega \) of volume 1. The second-order elliptic operator \( L_\Omega \) defined by the following integration by parts formula
\[ \frac{1}{n-1} \int_{S^{n-1}} \langle (D^2 h)^{-1} \nabla_S f, \nabla_S g \rangle h^2 \det D^2 h \, dx = - \int_{S^{n-1}} g (L_\Omega f) h \det D^2 h \, dx \]
is closely related to the log-Minkowski problem. It has been shown in [31] that the (infinitesimal) log-Minkowski problem is a spectral problem for \( L_\Omega \).

It is easy to see that \( L_\Omega \) is a particular case of the operator \((n-1)L_{\mu,\nu}\). It corresponds to the following couple of measures:
\[ \mu = \frac{1}{n} h \det D^2 h \cdot \mathcal{H}^{n-1}|_{S^{n-1}}, \quad \nu = \frac{1}{n} r^n \cdot \mathcal{H}^{n-1}|_{S^{n-1}}, \]
where \( h \) is the support and \( r \) is the radial function of \( \Omega \). Clearly, \( \nu \) is the push-forward image of \( \mu \) under the mapping
\[ T = (hx + \nabla h)/\sqrt{h^2 + |\nabla_{S^{n-1}} h|^2}. \]
The exact expression for the operator \( L_{\mu,\nu} \) takes the form
\[ L_{\mu,\nu} \left( \frac{u}{h} \right) = \text{Tr}(D^2 h)^{-1} D^2 u - (n-1) \frac{u}{h}, \tag{5.1} \]
Operator (5.1) has been studied already by D. Hilbert in his work on the Brunn-Minkowski inequality. The original motivation of E. Milman and the author was to study infinitesimal versions of the Brunn-Minkowski inequality, which are inequalities of the Poincaré type. See in this respect [9], [29], [30].

Recall that the entropy of the probability measure
\[ \nu = \rho \cdot \sigma \]
is given by
\[ \text{Ent}(\nu) = \int \rho \log \rho \, d\sigma \]
and \[ \text{Ent}(\nu) = +\infty \] if \( \nu \) has no density.

The variational formulae proved in the previous section immediately give a formal proof that the minimal points of the following entropic/transportational functional are precisely the solutions to the log-Minkowski problem. We will give later a rigorous justification of this fact.

**Proposition 5.2.** Let \( \nu \) be a stationary point of the functional
\[ F(\nu) = \frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu). \]
Then \( \mu = Ch \det D^2 h \cdot \sigma \), where \( h \) is the potential of the transportation mapping pushing forward \( \mu \) onto \( \nu \).

**Proof.** Set: \( \nu_\epsilon = (1 + \epsilon \omega) \nu \), \( \int \omega d\nu = 0 \). Then the variation of \( K(\mu, \nu) \) (see Lemma 4.3) is equal to \( \int \omega \log r d\nu \) and the variation of \( \text{Ent}(\nu) \) is equal to \( \int \omega \log \rho d\nu \). If \( \nu \) is a stationary point, there holds:
\[ n \log r = \log \rho + c. \]
Thus \( \nu = \rho r^{n} - c \cdot \sigma \). The expression for \( \mu \) follows from the change of variables formula and the normalization assumption: \( \mu(S^{n-1}) = 1 \).

Our observation has the following Euclidean companion. The functional \( F \) is a spherical analog of the functional (5.3), while \( h \mapsto \int \log h d\mu \) seems to be similar to (5.2).

Given a probability measure \( \nu = \rho \, dx \) on \( \mathbb{R}^n \) one can try to find a log-concave measure \( \mu = e^{-\Phi} \, dx \) (i.e., \( \Phi \) is a convex function) satisfying the following remarkable property: \( \nu \) is the image of \( \mu \) under the mapping \( T \) generated by the logarithmic gradient of \( \mu \):
\[ T(x) = \nabla \Phi(x), \quad \nu = \mu \circ T^{-1}. \]

Following the terminology from [13], we say that \( \nu \) is a moment measure if such a function \( \Phi \) exists. The associated Monge–Ampère equation looks as follows:
\[ e^{-\Phi} = \rho(\nabla \Phi) \det D^2 \Phi. \]

It is known that \( \Phi \) is a maximum point of the following functional:
\[ J(f) = \log \int_{\mathbb{R}^n} e^{-f^*} \, dx - \int_{\mathbb{R}^n} f \, d\nu, \]
where \( f^* \) is the Legendre transform of \( f \).

An alternative viewpoint was suggested in [33]: it was shown that \( \rho = e^{-\Phi} \) gives minimum to the functional
\[ \mathcal{F}(\rho) = -\frac{1}{2} W_2^2(\nu, \rho dx) + \frac{1}{2} \int \rho x^2 \, dx + \int \rho \log \rho \, dx. \]

(5.3)
Other remarks on relations between the log-Minkowski problem and the moment measures can be found in [13]. Further developments related to transportational and stability inequalities see in [17], [26], [15].

6. Minimizers of the variational functional

It is known that functional (5.3) admits certain displacement convexity properties (see [33]). Unfortunately, we don’t know whether \( K(\mu, \nu) \) is displacement convex. To see that the standard arguments fail, let us assume that \( h \) is sufficiently regular and apply the change of variables formula

\[
\rho_\mu = \rho_\nu(T) \frac{h \cdot \det D^2 h}{(h^2 + |\nabla S_{n-1} h|^2)^{\frac{n}{2}}}.
\]

Take logarithm

\[
\log \rho_\mu = \log \rho_\nu(T) + n \log \left( \frac{h}{\sqrt{h^2 + |\nabla S_{n-1} h|^2}} \right) + \log \det \left( \text{Id} + \frac{\nabla^2 S_{n-1} h}{h} \right)
\]

and integrate with respect to \( \mu \):

\[
\frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu) = \frac{1}{n} \text{Ent}(\mu) - \frac{1}{n} \int_{S_{n-1}} \log \det \left( \text{Id} + \frac{\nabla^2 S_{n-1} h}{h} \right) d\mu.
\]

It is clear that the function

\[-\log \det \left( \text{Id} + \frac{\nabla^2 S_{n-1} h}{h} \right)\]

is not convex with respect to the natural interpolation \( t \to th_1 + (1-t)h_2 \).

Another representation can be obtained from the duality principle. Let \((h, r)\) be the solution to the dual Kantorovich problem. Applying Kantorovich duality one gets

\[
K(\mu, \nu) = - \int_{S_{n-1}} \log h d\mu + \int_{S_{n-1}} \log r d\nu.
\]

Hence

\[
\frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu) = \frac{1}{n} \text{Ent}(\nu) + \int_{S_{n-1}} \log h d\mu - \int_{S_{n-1}} \log r d\nu
\]

\[
= \int_{S_{n-1}} \log h d\mu + \frac{1}{n} \int_{S_{n-1}} \log \left( \frac{\rho_\nu}{r^n} \right) d\sigma.
\]

Set \( m = \frac{1}{r^n} r^n \cdot \sigma \), where the normalization constant equals

\[
C = \int_{S_{n-1}} r^n d\sigma = \int_{S_{n-1}} h \det D^2 h d\sigma = \frac{1}{n|B|} \int_{\partial \Omega h} h(n_{\partial \Omega}) dx = \frac{|\Omega h|}{|B|}.
\]

Finally,

\[
\frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu) = \int_{S_{n-1}} \log h d\mu - \frac{1}{n} \log C + \frac{1}{n} \int_{S_{n-1}} \log \left( \frac{\rho_\nu}{r^n/C} \right) d\nu.
\]

\[
= \int_{S_{n-1}} \log h d\mu - \frac{1}{n} \log \int_{S_{n-1}} r^n d\sigma + \frac{1}{n} \text{Ent}_m(\nu),
\]

where \( \text{Ent}_m(\nu) = \int_{S_{n-1}} \frac{d\mu}{dm} d\mu \geq 0 \).
Theorem 6.1. Assume that \( \mu(L \cap S^{n-1}) = 0 \) for every hyperplane \( L \) containing the origin. Then the functional
\[
F(\nu) = \frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu)
\]
considered on the space of measures with \( \rho_\nu \in L^\infty(\sigma) \) attains its minimum at some point.

For every such point the related transportational potential \( h \) is a support function of a symmetric convex body which after a suitable renormalization gives minimum to
\[
F_0(h) = \int_{S^{n-1}} \log h d\mu, \ |\Omega_h| = 1.
\]
In particular, \( h \) is a solution to the log-Minkowski problem.

Proof. Set \( m_F = \inf_\nu F(\nu) \). We want to estimate \( m_F \) from below. Clearly, it is sufficient to estimate \( F(\nu) \) from below for a positive density \( \rho_\nu \). In this case the Aleksandrov problem admits a solution \( (h, r) \) by Lemma 3.3 and, as we have already seen,
\[
\frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu) = \int_{S^{n-1}} \log h d\mu - \frac{1}{n} \log \int_{S^{n-1}} r^n d\sigma + \frac{1}{n} \text{Ent}_m(\nu).
\]

Note that potential \( h \) which defines optimal transportation \( T \) is uniquely defined up to multiplication by a constant \( \lambda > 0 \). Thus without loss of generality one can normalize \( h \) is such a way that \( |\Omega_h| = 1 \). Under this normalization
\[
\frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu) = \int_{S^{n-1}} \log h d\mu + \frac{1}{n} \log |B| + \frac{1}{n} \text{Ent}_m(\nu). \tag{6.1}
\]
Using that the term \( \text{Ent}_m(\nu) \) is non-negative, one gets
\[
m_F \geq \min_{h: |\Omega_h| = 1} \int_{S^{n-1}} \log h d\mu + \frac{1}{n} \log |B|.
\]
Now let \( h \) be a minimizer of \( F_0 \). It follows from the result of [8] that \( h \) exists and \( h \) is a solution to the log-Minkowski problem, hence the push-forward measure \( \mu \circ T^{-1} \) of \( \mu \) under \( T \) satisfies \( \rho_\nu = cr^n \) and \( \text{Ent}_m(\nu) = 0 \). Thus for this \( \nu \) one gets
\[
\frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu) = \min_{h: |\Omega_h| = 1} \int_{S^{n-1}} \log h d\mu + \frac{1}{n} \log |B|.
\]
We obtain that \( \nu \) is a minimizer for \( F \) and
\[
m_F = \min_{h: |\Omega_h| = 1} \int_{S^{n-1}} \log h d\mu + \frac{1}{n} \log |B|. \tag{6.2}
\]

Now let \( \tilde{\nu} \) be any minimizer of \( F \). From the relations \( \text{(6.1)} \) and \( \text{(6.2)} \) we infer that \( \text{Ent}_m(\tilde{\nu}) = 0 \), where \( m = cr^n \cdot \sigma \) and \( (h, r) \) is a solution to the dual Kantorovich problem for \( (\mu, \tilde{\nu}) \). We obtain from \( \text{(6.2)} \) that \( h \) is a minimizer of \( F_0 \). This completes the proof. \( \square \)

7. Uniqueness for log-Minkowski Problem and Transportation Inequalities

In what follows we are given a symmetric probability measure \( \mu \) on \( S^{n-1} \). We assume throughout the section that \( \mu(L \cap S^{n-1}) = 0 \) for every hyperplane containing the origin. In particular, \( \mu \) satisfies the subspace concentration property. According
to [8] there exists a solution to the log-Minkowski problem for $\mu$. The uniqueness of
a solution is an open problem. In this paper we propose a weaker conjecture.

**Conjecture:** The functional

$$F(\nu) = \frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu)$$

has a unique global minimizer.

Clearly, if $F$ has distinct global minimizers, then they all are solutions to the
log-Minkowski problem according to Theorem 6.1. The representation

$$\frac{1}{n} \text{Ent}(\nu) - K(\mu, \nu) = \frac{1}{n} \text{Ent}(\mu) - \frac{1}{n} \int_{S^{n-1}} \log \det \left(Id + \nabla^2 h \right) d\mu.$$  \hspace{1cm} (7.1)

obtained above can be used to compute the second variation of $F$. Assuming that $h$
is sufficiently regular and using the second-order necessary condition for minimum
one can easily get.

**Theorem 7.1.** (Infinitesimal uniqueness). Assume that the log-Minkowski problem
admits a unique solution $\Omega_h$ for $\mu$, and $h$ is twice continuously differentiable. Then
every even function $u \in C^2(S^{n-1})$ satisfies the following inequality

$$\int_{S^{n-1}} \| (D^2 h)^{-1}D^2 u \|_{HS}^2 d\mu \geq (n - 1) \int_{S^{n-1}} \left( \frac{u}{h} \right)^2 d\mu.$$  \hspace{1cm} (7.2)

**Proposition 7.2.** For any given couple of support functionals $h, f \in C^2(S^{n-1}),$
$n > 2$, the probability measure

$$\mu = \frac{h \det D^2 h}{n|\Omega_h|} \cdot \mathcal{H}^{n-1}|_{S^{n-1}}$$

satisfies the following inequality

$$\frac{1}{n - 1} \int_{S^{n-1}} \text{Tr}(D^2 f)^{-1}(D^2 h) d\mu \geq \left( \frac{|\Omega_h|}{|\Omega_f|} \right)^{\frac{1}{n-1}} \left( \int_{S^{n-1}} \frac{h}{f} d\mu \right)^{-\frac{1}{n-1}}.$$  \hspace{1cm} (7.3)

**Proof.** We start with the Hölder inequality

$$\int_{S^{n-1}} \text{Tr}(D^2 f)^{-1}(D^2 h) d\mu \cdot \int_{S^{n-1}} \frac{d\mu}{\text{Tr}(D^2 f)^{-1}(D^2 h)} \geq 1.$$  \hspace{1cm} (7.4)

Applying the arithmetic-geometric inequality and Hölder inequality again one gets

$$\int_{S^{n-1}} \frac{1}{\text{Tr}(D^2 f)^{-1}(D^2 h)} d\mu \leq \int_{S^{n-1}} \frac{1}{(n - 1) \det \frac{1}{n-1} [(D^2 f)^{-1}(D^2 h)]} d\mu = \frac{1}{(n - 1)n|\Omega_h|} \int_{S^{n-1}} \det D^2 f \left( \frac{1}{n-1} \det D^2 h \right)^{\frac{n-2}{n-1}} h \, dx.$$  \hspace{1cm} (7.5)

$$\leq \frac{1}{(n - 1)n|\Omega_h|} \left( \int_{S^{n-1}} f \, dx \right)^{\frac{1}{n-1}} \left( \int_{S^{n-1}} \frac{h^{\frac{n-1}{n-2}}}{f^\frac{1}{n-2}} \det D^2 h \, dx \right)^{\frac{n-2}{n-1}}.$$  \hspace{1cm} (7.6)

The integral relation $\int_{S^{n-1}} f \, dx = n|\Omega_f|$ implies

$$\int_{S^{n-1}} \frac{d\mu}{\text{Tr}(D^2 f)^{-1}(D^2 h)} \leq \left( \frac{|\Omega_f|}{|\Omega_h|} \right)^{\frac{1}{n-1}} \left( \int_{S^{n-1}} \frac{h^{\frac{n-1}{n-2}}}{f^\frac{1}{n-2}} d\mu \right)^{\frac{n-2}{n-1}}.$$  \hspace{1cm} (7.7)

$$= \frac{1}{n - 1} \left( \frac{|\Omega_f|}{|\Omega_h|} \right)^{\frac{1}{n-1}} \left( \int_{S^{n-1}} \frac{h^{\frac{n-1}{n-2}}}{f^\frac{1}{n-2}} d\mu \right)^{\frac{n-2}{n-1}} \leq \frac{1}{n - 1} \left( \frac{|\Omega_f|}{|\Omega_h|} \right)^{\frac{1}{n-1}} \left( \int_{S^{n-1}} \frac{h}{f} d\mu \right)^{\frac{1}{n-1}}.$$  \hspace{1cm} (7.8)
This claim follows.

Let us consider the case of the uniform probability measure on $S^{n-1}$:

$$\mu = \sigma.$$

The representation (7.1) takes the form

$$\frac{1}{n} \text{Ent}(\nu) - K(\sigma, \nu) = -\frac{1}{n} \int_{S^{n-1}} \log \det \left( Id + \frac{\nabla^2 h}{h} \right) d\sigma.$$

**Theorem 7.3.** The following inequality holds for every symmetric probability measure $\nu$:

$$\frac{1}{n} \text{Ent}(\nu) \geq K(\sigma, \nu). \quad (7.2)$$

Equivalently, every even twice continuously differentiable support function $h$ satisfies

$$\int_{S^{n-1}} \log \det \left( Id + \frac{\nabla^2 h}{h} \right) d\sigma \leq 0. \quad (7.3)$$

Moreover, a stronger inequality holds:

$$\frac{1}{n-1} \int_{S^{n-1}} \text{Tr}(D^2 h)^{-1} d\sigma \geq \int_{S^{n-1}} \frac{d\sigma}{h}. \quad (7.4)$$

Equalities in (7.3), (7.3), and (7.4) hold only for constant $h$.

**Remark 7.4.** Inequality (7.2) is a spherical variant of the Gaussian Talagrand transportation inequality

$$\text{Ent}_\gamma(\nu) \geq \frac{1}{2} W_2^2(\gamma, \nu),$$

where $\gamma$ is the standard Gaussian measure.

In order to compare both inequalities let us note that

$$\log \frac{1}{\langle x, y \rangle} = \log \frac{2}{2 - |x - y|^2}.$$

Using convexity of the function $\log \frac{2}{2-t}$ and the Jenssen’s inequality one gets

$$\frac{1}{n} \text{Ent}(\nu) \geq K(\nu, \sigma) \geq \log \frac{2}{2 - W_2^2(\sigma, \nu)}.$$

Equivalently

$$\frac{1}{2} W_2^2(\sigma, \nu) \leq 1 - e^{\frac{1}{n} \text{Ent}(\nu)}.$$

Here

$$W_2^2(\sigma, \nu) = \inf_{\pi \in \Pi(\sigma, \nu)} \int_{S^{n-1} \times S^{n-1}} |x - y|^2 d\pi$$

(note that this is not the standard Kantorovich functional $W_2$ because the cost function is equal to the squared Euclidean distance).

Analogously, for the Riemannian $l^1$-transportation cost

$$W_1(\sigma, \nu) = \inf_{\pi \in \Pi(\sigma, \nu)} \int_{S^{n-1} \times S^{n-1}} d(x, y) d\pi,$$
where \( d(x, y) \) is the standard Riemannian distance on \( S^{n-1} \), one gets
\[
\frac{1}{n} \text{Ent}(\nu) \geq -\int_{S^{n-1}} \log \cos(d(x, T(x)))d\nu \geq -\log \cos \int_{S^{n-1}} d(x, T(x))d\nu \geq -\log \cos W_1(\sigma, \nu).
\]

Hence
\[
W_1(\sigma, \nu) \leq \arccos e^{-\frac{1}{n} \text{Ent}(\nu)}.
\]

**Remark 7.5.** Another transportation inequality for the sphere has been obtained by D. Cordero-Erausquin in \([11]\) for general (non-symmetric) measures:
\[
W_c(f \cdot \sigma, \sigma) \leq n - n \int f^{1-\frac{1}{n}}d\sigma,
\]
where
\[
c(x, y) = n - \frac{\sin^{n-1}(d)}{S_{n-1}^{n-1}(d)} - (n - 1) \frac{S_n(d)}{\tan(d)}
\]
and \( S_n(d) = \left(n \int_0^d \sin^{n-1}(s)ds\right)^\frac{1}{n} \).

**Proof.** Clearly, (7.2) follows from (7.3) by the standard approximation arguments. Let us derive (7.3) from (7.4). Set:
\[
f(t) = \int_{S^{n-1}} \left[ \log \det((1 + t(h-1))I + t\nabla^2 h) - (n-1) \log(1 + t(h-1)) \right]d\sigma.
\]
Note that (7.3) is equivalent to \( f(1) \geq 0 \). Since \( f(0) = 0 \), to prove (7.3) it is sufficient to show that \( f'(t) \leq 0 \), equivalently
\[
\int_{S^{n-1}} \text{Tr}((1 + t(h-1))I + t\nabla^2 h)^{-1}[(h-1)I + \nabla^2 h]d\sigma \leq (n-1) \int_{S^{n-1}} \frac{h - 1}{1 + t(h-1)}d\sigma.
\]
Set \( h_t = 1 - t + th \). Then the above inequality can be rewritten as follows:
\[
\frac{1}{t} \int_{S^{n-1}} \text{Tr}(D^2 h_t)^{-1}[D^2 h_t - I]d\sigma \leq \frac{(n-1)}{t} \int_{S^{n-1}} \frac{h_t - 1}{h_t}d\sigma,
\]
which is equivalent to (7.4).

Let us prove (7.4). Assume that \( n > 2 \). By Proposition 7.2,
\[
\frac{1}{n-1} \int_{S^{n-1}} \text{Tr}(D^2 h)^{-1}d\sigma \geq \left(\frac{|B|}{|\Omega_h|}\right)^\frac{1}{n-1} \left(\int_{S^{n-1}} \frac{d\sigma}{h}\right)^{-\frac{1}{n-1}},
\]
It remains to prove that
\[
\frac{|B|}{|\Omega_h|} \geq \left(\int_{S^{n-1}} \frac{d\sigma}{h}\right)^n.
\]
Applying the volume formula for polar body \( \Omega_h^c \) one gets
\[
\left(\int_{S^{n-1}} \frac{d\sigma}{h}\right)^n \leq \int_{S^{n-1}} \frac{d\sigma}{h^n} = \frac{|\Omega_h^c|}{|B|}.
\]
The result follows from the Blaschke-Santaló inequality.

In the case \( n = 2 \) inequality (7.4) (after change of variables) reads as follows:
\[
\int_{\partial \Omega} k^2 dx \geq \int_{\partial \Omega} \frac{k}{\langle x, n_{\partial \Omega}\rangle} dx,
\]
where \( k \) is curvature of \( \partial \Omega \). According to a result of M.E. Gage \[18\]
\[
\int_{\partial \Omega} k^2 \, dx \geq \frac{\pi |\partial \Omega|}{\Omega}.
\]
By the Blaschke-Santaló inequality
\[
\int_{\partial \Omega} \frac{k}{h} \, dx = \int_{S^{n-1}} \frac{dx}{h} \leq |S^1|^{1/2} \sqrt{\int_{S^{n-1}} \frac{dx}{h^2}} = |S^1|^{1/2} \sqrt{2|\Omega|} \leq \frac{2|\pi|^2}{\sqrt{\Omega}}, \tag{7.5}
\]
The result follows from the isoperimetric inequality. \[\square\]

It was shown by W. Firey \[16\] that the log-Minkowski problem has the unique solution for the case \( \mu = \sigma \). We give an independent proof of this result here.

**Theorem 7.6.** Assume that \( h \) is a \( C^2 \)-solution to the log-Minkowski problem for \( \mu = \sigma \):
\[
h \det D^2 h = \frac{1}{|B|}, \quad |\Omega_h| = 1.
\]
Then \( \Omega_h \) is the ball of volume one.

**Proof.** Set \( \nu = r^n |B| \cdot \sigma \), where \( r \) is the radial function of \( \Omega_h \). According to (5.1) and Theorem 4.1
\[
0 = \int_{S^{n-1}} L_{\mu, \nu} h d\mu = \int_{S^{n-1}} \text{Tr}(D^2 h)^{-1} d\mu - (n-1) \int_{S^{n-1}} \frac{d\mu}{h}.
\]
On the other hand, according to Theorem 7.3 equality in (7.4) holds only for constant \( h \). This implies the claim. \[\square\]

**Remark 7.7.** In the same way as for the uniform measure uniqueness of solution to the log-Minkowski problem for arbitrary regular \( \mu \) can be derived from the following conjectured inequality
\[
\frac{1}{n-1} \int_{S^{n-1}} \text{Tr}(D^2 f)^{-1} (D^2 h) d\mu \geq \int_{S^{n-1}} \frac{h}{f} d\mu, \tag{7.6}
\]
where
\[
\mu = \frac{1}{n |\Omega_h|} h \det D^2 h \cdot \mathcal{H}^{n-1}|_{S^{n-1}},
\]
h, \( f \) are arbitrary even \( C^2(S^{n-1}) \) support functions, provided the equality case holds if and only if \( \frac{h}{f} \) is constant.

Inequality (7.6) is reminiscent of several inequalities appearing in relation to Minkowski-type problems. First, note that the log-Brunn–Minkowski inequality is equivalent to the following inequality for support functions \( h, f \) (see [27]):
\[
\int_{S^{n-1}} \log \left( \frac{f}{h} \right) d\mu \geq \frac{1}{n} \log \left( \frac{\Omega_f}{\Omega_h} \right), \tag{7.7}
\]
In addition, it was shown in [10] that the log-Brunn–Minkowski inequality has the following local version (see Proposition 4.4):
\[
\int_{S^{n-1}} \frac{1 + \text{Tr}(D^2 h)^{-1} h}{h^2} u^2 d\mu - n \left( \int_{S^{n-1}} \frac{u}{h} d\mu \right)^2 \leq \int_{S^{n-1}} \frac{((D^2 h)^{-1} \nabla_{S^{n-1}} u, \nabla_{S^{n-1}} u)}{h} d\mu,
\]
u \in C^1(S^{n-1}) (see [31] for a more general version for \( p \)-Brunn–Minkowski inequality).
Remark 7.8. One can conjecture a stronger inequality which by Proposition 7.2 implies (7.6) (and 7.7):

\[
\frac{|\Omega_h|^{\frac{1}{n}}}{|\Omega_f|^{\frac{1}{n}}} \geq \int_{S^{n-1}} \frac{h}{f} d\mu. \tag{7.8}
\]

But this inequality does not hold even for \( f = 1 \). Indeed, in this case (7.8) is equivalent to

\[
\int_{S^1} h^2 \det D^2 h \, dx \leq \frac{2}{\sqrt{\pi}} |\Omega_h|^{\frac{3}{2}}
\]

(for \( n = 2 \)). But this inequality fails for a thin long cylinder \([-1, 1] \times [-R, R]\) for sufficiently large \( R \).

Note, however, that (7.6) holds for \( n = 2, f = 1 \). Indeed, in this case (7.6) reads as

\[
\int_{S^1} h(h + h'')^2 \, dx \geq \int_{S^1} h^2(h + h') \, dx.
\]

Changing variables one gets that the latter is equivalent to

\[
\int_{\partial \Omega} \frac{\langle x, n_{\partial \Omega} \rangle}{k} \, dx \geq \int_{\partial \Omega} \langle x, n_{\partial \Omega} \rangle^2 \, dx,
\]

where \( k \) is curvature of \( \partial \Omega \). To prove it let us use a Bonnessen-type inequality (see [18]):

\[
\langle x, n_{\partial \Omega} \rangle |\partial \Omega| \geq |\Omega| + \pi \langle x, n_{\partial \Omega} \rangle^2.
\]

Integrating over \( \partial \Omega \) and applying \( \int_{\partial \Omega} \langle x, n_{\partial \Omega} \rangle \, dx = 2|\Omega| \) one gets

\[
\int_{\partial \Omega} \langle x, n_{\partial \Omega} \rangle^2 \, dx \leq \frac{|\Omega| |\partial \Omega|}{\pi}.
\]

Using (7.5) and Cauchy inequality one gets

\[
\frac{2|\pi|^\frac{3}{2}}{|\Omega|^\frac{1}{2}} \int_{\partial \Omega} \frac{\langle x, n_{\partial \Omega} \rangle}{k} \, dx \geq \int_{\partial \Omega} \frac{\langle x, n_{\partial \Omega} \rangle}{k} \, dx \int_{\partial \Omega} \frac{k}{\langle x, n_{\partial \Omega} \rangle} \, dx \geq |\partial \Omega|^2.
\]

Finally

\[
\int_{\partial \Omega} \frac{\langle x, n_{\partial \Omega} \rangle}{k} \, dx \geq \frac{|\partial \Omega|^2 \sqrt{|\Omega|}}{2|\pi|^\frac{3}{2}}.
\]

The result follows from the isoperimetric inequality.

REFERENCES

[1] Aleksandrov A.D., Existence and uniqueness of a convex surface with a given integral curvature., C. R. (Dokl.) Acad. Sci. URSS, n. Ser. 35 (1942), 131–134 (English).
[2] Ambrosio L., Gigli N., Savaré G., Gradient flows in metric spaces and in the Wasserstein spaces of probability measures. 2nd ed. Birkhäuser, Basel, 2008; x+334 p.
[3] Bakry D., Gentil I., Ledoux M., Analysis and geometry of Markov diffusion operators, Springer, 2014.
[4] Bertrand J., Prescription of Gauss curvature using optimal mass transport, Geom. Dedic., 183 (2016), 81–99.
[5] Bobkov S.G., Chistyakov G.P., Götze F., Second order concentration on the sphere, Commun. Contemp. Math. 19, 1650058 (2017).
[6] Bogachev V.I., Kolesnikov A.V., The Monge–Kantorovich problem: achievements, connections, and perspectives, Russian Math. Surveys, 67 (2012), N 5, 785–890.
[7] Böröczky K.J., Lutwak E., Yang D., Zhang G., The log-Brunn-Minkowski-inequality, Advances in Mathematics, 231 (2012), 1974–1997.
[8] Böröczky K.J., Lutwak E., Yang D., Zhang G. The logarithmic Minkowski problem. J. Amer. Math. Soc., 26(3):831–852, 2013.
[9] Colesanti A., From the Brunn-Minkowski inequality to a class of Poincaré type inequalities, Comm. Cont. Math, 10, (2008), 765–772.
[10] Colesanti A., Livshyts G., Marsiglietti A., On the stability of Brunn-Minkowski type inequalities, 273 (3), (2017), 1120–1139
[11] Cordero-Erausquin D., A transport inequality on the sphere obtained by mass transport. Pacific Journal of Mathematics 268(1), 2014.
[12] Cordero-Erausquin D., Fradelizi M., Maurey B., The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems, J. Functional Analysis, 214 (2004) 410–427.
[13] Cordero-Erausquin D., Klartag B., Moment measures. J. Functional Analysis, 268 (12), (2015), 3834–3866.
[14] Fathi M., Stein kernels and moment maps. arXiv: 1804.04699.
[15] Fathi M., A sharp symmetrized form of Talagrand’s transport-entropy inequality for the Gaussian measure. arXiv: 1806.06389.
[16] Firey W., Shapes of worn stones, Mathematika 21 (1974), 1–11.
[17] Fontbona J., Gozlan N., Jabir J.-F., A variational approach to some transport inequalities. Ann. Inst. H. Poincaré Probab. Statist. 53(4), (2017), 1719–1746.
[18] Gage M. E., An isoperimetric inequality with applications to curve shortening, Duke Math. J. 50(1983), 1225-1229.
[19] Gangbo W., Oliker V., Existence of Optimal Maps in the Reflector-type Problems. ESAIM: COCV 13(1), 2007, 93–106.
[20] Klartag B., Poincaré inequalities and moment maps. Ann. Fac. Sci. Toulouse Math., 22(1), (2013), 1–41.
[21] Klartag B., Logarithmically-concave moment measures I. Geometric Aspects of Functional Analysis, Vol. 2116 of the series Lecture Notes in Mathematics, (2014), 231–260.
[22] Klartag B., Kolesnikov A.V., Eigenvalue distribution of optimal transportation. Analysis & PDE, 8(1) (2015), 33–55.
[23] Klartag B., Kolesnikov A.V., Remarks on curvature in the transportation metric, Analysis Mathematica 43(1) (2017), 67–88.
[24] Klartag B., Kolesnikov A.V., Extremal Kähler-Einstein metric for two-dimensional convex bodies. https://arxiv.org/abs/1710.04618, (to appear in Jour. Geom. Anal.)
[25] Kolesnikov A.V., Hessian metrics, CD(K,N)-spaces, and optimal transportation of log-concave measures. Discrete and Continuous Dynamical Systems - Series A. 34(4) (2014), 1511–1532.
[26] Kolesnikov A.V., Kosov E.D., Moment measures and stability for Gaussian inequalities, Theory of Stochastic Processes, 22 (38), no. 2, (2017), 47–61.
[27] Kolesnikov A.V., Klartag B., Kudryavtseva O.V., Nagapetyan T., Remarks on Aatri’s theorem and the Monge-Kantorovich problem. Journal of Mathematical Economics 49 (2013) 501–505
[28] Kolesnikov A.V., Klartag B., Kosov E.D., Riemannian metrics on convex sets with applications to Poincaré and log-Sobolev inequalities, Calc. Var. & PDE’s (2016) 55–77.
[29] Kolesnikov A.V., Milman E., Brascamp–Lieb-Type Inequalities on Weighted Riemannian Manifolds with Boundary. The Journal of Geometric Analysis, 2017, 27(2), 1680–1702.
[30] Kolesnikov A.V., Milman E., Poincaré and Brunn–Minkowski inequalities on the boundary of weighted Riemannian manifolds (in press, Amer. Journal of Math.)
[31] Kolesnikov A.V., Milman E., Local $L^p$-Brunn–Minkowski inequalities for $p < 1$. https://arxiv.org/abs/1711.01089
[32] Oliker V., Embedding $S^n$ into $R^{n+1}$ with given integral Gauss curvature and optimal mass transport on $S^n$, Adv. Math. 213 (2007), no. 2, 600–620.
[33] Santambrogio F., Dealing with moment measures via entropy and optimal transport, J. Funct. Anal., 271 (2016), 418–436.
[34] Saroglou C., Remarks on the conjectured log-Brunn-Minkowski inequality, Geom. Dedicata 177 (2015), 353–365.
[35] Schneider R., Convex Bodies: The Brunn–Minkowski Theory (Encyclopedia of Mathematics and its Applications, Book 151) 2nd Edition, Cambridge University Press, 2013.
[36] Villani C., Topics in optimal transportation. Amer. Math. Soc. Providence, Rhode Island, 2003;
[37] Villani C., Optimal transport, old and new. Springer, New York, 2009.
