Fractional Poisson Processes of Order $K$

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Abstract

In this article, we introduce and study time- and space-fractional Poisson processes of order $k$. These processes are defined in terms of fractional compound Poisson processes. Time-fractional Poisson process of order $k$ naturally generalizes the Poisson process and Poisson process of order $k$ to a heavy tailed waiting times counting process. The space-fractional Poisson process of order $k$, allows on average infinite number of arrivals in any interval. We derive the marginal probabilities, governing difference-differential equations of the introduced processes.

Key Words: Time-fractional Poisson process, Poisson process of order $k$, Space-fractional Poisson process, Infinite divisibility.

1 Introduction

The classical Poisson process is the most useful and popular counting process. The waiting times in classical Poisson process are exponentially distributed and the mean arrival rate is constant over time. The non-homogeneous Poisson process is obtained by taking the mean arrival rate as a function of time $t$. The classical Poisson process is a Lévy process and hence doesn’t possess long-range dependence property. The time-fractional Poisson process (TFPP) was introduced by Laskin (2003) [16] by taking Mittag-Leffler distributed waiting times instead of exponential waiting times. Mittag-Leffler distribution is a heavy tailed distribution due to which TFPP has less number of arrivals than the classical Poisson process on average for sufficiently large $t$. Further TFPP has long-range dependence property (see [20]). The Poisson distribution of order $k$ was obtained as a limiting distributions of a sequence of shifted negative binomial distribution [2, 3]. Poisson process of order $k$ (PPoK) was analyzed as compound Poisson process representation and pure birth process (see e.g. [2, 3]). The PPoK has applications in standard risk model, where the arrivals of claims in a group of size $k$ could occur [10]. The space-fractional Poisson process (SFPP) was introduced by [5]. Further, they argue that TFPP and the SFPP are specific cases of the same generalized complete model and hence might be useful in the study of transport of charge carriers in semiconductors [22] or applications related to fractional quantum mechanics [17].

In this article, we introduce and study time- and space-fractional Poisson processes of order $k$. These processes generalize the Poisson process, Poisson process of order $k$, TFPP and SFPP in several directions. The rest of the article is organize as follows. In Section 2, we provide all the relevant definitions and results which are used in subsequent sections. In Section 3, we introduce and study time-fractional Poisson process of order $k$ (TFPPoK). The space-fractional Poisson process of order $k$ (SFPPoK) is introduced in Section 4. The last section concludes.
2 Preliminaries

In this section, we recall some relevant definitions and properties for analyzing the fractional Poisson processes of order \( k \).

2.1 Poisson process of order \( k \) (PPoK)

In this section, we provide important properties of Poisson process of order \( k \) (PPoK) denoted by \( N^k(t) \) defined as a compound Poisson process, introduced in \([10]\). The PPoK \( N^k(t) \) defined on a fixed probability space \((\Omega, \mathcal{F}, P)\) and given by

\[
N^k(t) = \sum_{i=0}^{N(t)} X_i,
\]

where \( X_i, i = 1, 2, \ldots \) are independent identically distributed (iid) discrete uniform random variables with support \( \{1, 2 \cdots, k\} \), which are independent to the Poisson process \( N(t) \) having intensity \( k\lambda \).

The probability mass function (pmf) \( p^{N^k}_n(t) = \mathbb{P}(N^k(t) = n) \) is given by

\[
p^{N^k}_n(t) = \sum_{X=\Omega(k,n)} e^{-k\lambda t} \left( \frac{(\lambda t)^{\zeta_k}}{\Pi_k!} \right), \tag{2.1}
\]

where \( x_1, x_2, \ldots, x_k \) be non-negative integers and \( \zeta_k = x_1 + x_2 + \cdots + x_k \), \( \Pi_k! = x_1!x_2!\ldots x_k! \) and

\[
\Omega(k,n) = \{X = (x_1, x_2, \ldots, x_k) | x_1 + 2x_2 + \cdots + kx_k = n\}. \tag{2.2}
\]

The pmf of \( N^k(t) \) satisfies the following differential-difference equations (see \([10]\))

\[
\frac{d}{dt} p^{N^k}_n(t) = -k\lambda p^{N^k}_n(t) + \lambda \sum_{j=1}^{n\wedge k} p^{N^k}_{n-j}(t), \quad n = 1, 2, \ldots
\]

\[
\frac{d}{dt} p^{N^k}_0(t) = -k\lambda p^{N^k}_0(t), \tag{2.3}
\]

with initial condition \( p^{N^k}_0(0) = 1 \) and \( p^{N^k}_n(0) = 0 \) and \( n \wedge k = \min\{k,n\} \). The probability generating function \( G_{N^k}(s,t) \) is given by (see \([2]\))

\[
G_{N^k}(s,t) = e^{-\lambda t(1-\sum_{j=1}^{k} s^j)}. \tag{2.4}
\]

The mean, variance and covariance function of the PPoK are given by

\[
\mathbb{E}[N^k(t)] = \frac{k(k+1)}{2} \lambda t
\]

\[
\text{Cov} \left( N^k(t), N^k(t) \right) = \frac{k(k+1)(2k+1)}{6} \lambda (t \wedge s). \tag{2.5}
\]
2.2 The $Z$-Transform and Its Inverse

The bilateral or two-sided $z$-transform is defined by (see e.g., [12])

$$F(z) = Zf(k) = \sum_{k=-\infty}^{\infty} f(k)z^{-k}, \ z \in \mathbb{C}.$$  

Alternatively, in case where $f(k)$ is defined only for $k \geq 0$, the unilateral $z$-transform is defined as

$$F(z) = Zf(k) = \sum_{k=0}^{\infty} f(k)z^{-k}, \ z \in \mathbb{C}. \quad (2.6)$$

The inverse $z$-transform is also defined by the complex integral

$$Z^{-1}\{F(z)\} = f(k) = \frac{1}{2\pi i} \oint_C F(z)z^{-k-1} \, dz,$$

where $C$ is simple closed contour enclosing the origin and lying outside the circle $|z| = R$. The existence of the inverse imposes restrictions on $f(k)$ for the uniqueness. The one-sides $z$-transform relates to probability generating function $G(u^{-1}) = F(u)$ is defined by

$$G(u) = \sum_{k=0}^{\infty} u^k f(k), \ |u| \leq 1,$$

where $f(k) \geq 0$, $k \in \mathbb{N} \cup \{0\}$, is probability distribution. Further the following operational property of $z$-transform is used for the solution of initial value problem involving difference equations

$$Z(f(k-m)) = z^{-m}[F(z) + \sum_{r=-m}^{-1} f(r)z^{-r}]. \quad (2.7)$$

2.3 Fractional Poisson processes

In this section, we recall important properties related to time- and space-fractional Poisson processes discussed in [5, 16]. The time-fractional Poisson process (TFPP) is the generalization of standard Poisson process with Mittag-Leffler waiting times. TFPP can also be obtained by time-changing the standard Poisson process $N(t)$ by an independent inverse stable subordinator $Y_\beta(t)$ see in [15] such that

$$N_\beta(t) = N(Y_\beta(t)), \quad (2.8)$$

where $Y_\beta(t) = \inf\{s > 0 : S_\beta(s) > t\}$ and $S_\beta(t)$ is the stable subordinator with Laplace transform

$$\mathbb{E}\left(e^{-zS_\beta(t)}\right) = e^{-tz^\beta}, \ z > 0, \ t \geq 0, \ \beta \in (0,1). \quad (2.9)$$

The process $Y_\beta(t)$ is non-Markovian with non-stationary increments [18] and also its marginals are not infinitely divisible [1]. The TFPP $N_\beta(t)$ has the PMF (see e.g. [6, 11, 13, 16])

$$P_\beta(k,t) = \mathbb{P}\{N_\beta(t) = k\} = \frac{(\lambda \beta)^k}{k!} \sum_{r=0}^{\infty} \frac{(k+r)!}{r!} \frac{(-\lambda \beta)^r}{\Gamma(\beta(k+r)+1)}, \ k = 0, 1, 2, \ldots \quad (2.10)$$
The PMF of \( N_\beta(t) \) satisfy the following fractional differential-difference equations with Caputo-Djrbashain fractional derivative in time

\[
\frac{d^\beta}{dt^\beta} P_\beta(k, t) = -\lambda^\beta (P_\beta(k, t) - P_\beta(k - 1, t)) = -\lambda^\beta (1 - B) P_\beta(k, t),
\]

(2.11)

where \( P_\beta(k, t) = 0, \text{ where } k < 0, \) (2.12)

\[
P_\beta(k, 0) = \delta_{k,0}, \text{ } k = 0, 1, 2, \ldots, \beta \in (0, 1], \)

(2.13)

where \( \frac{d^\beta}{dt^\beta} \) is Caputo-Djrbashain (CD) fractional derivative of order \( \beta \in (0, 1] \), for a function \( g(t), t \geq 0 \) is defined (see (\[14\]), Sections 2.2, 2.3)) as

\[
\frac{d^\beta}{dt^\beta} g(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t \frac{dg(\tau)}{d\tau} \frac{d\tau}{(t - \tau)\beta}, \beta \in (0, 1].
\]

(2.14)

The Laplace transform (LT) of CD fractional derivative is given by (see e.g., [14], p.39)

\[
\mathcal{L} \left( \frac{d^\beta}{dt^\beta} g(t) \right) = \int_0^\infty e^{-st} \frac{d^\beta}{dt^\beta} g(t) dt = s^\beta \tilde{g}(s) - s^{\beta-1} g(0^+), \quad 0 < \beta \leq 1,
\]

(2.15)

where \( \tilde{g}(s) \) is the LT of the function \( g(t), t \geq 0, \) such that

\[
\mathcal{L}(g(t)) = \tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt.
\]

Next we provide the important properties of SFPP. Let \( S_\alpha(t), t \geq 0, \alpha \in (0, 1), \) be a stable subordinator and \( N(t), t \geq 0, \) is homogenous Poisson process with parameter \( \lambda > 0, \) independent of \( S_\alpha(t). \) The SFPP introduced by [5] is defined by

\[
N^\alpha(t) = \begin{cases} N(S_\alpha(t)), t \geq 0, & 0 < \alpha < 1, \\ N(t), t \geq 0, & \alpha = 1. \end{cases}
\]

(2.16)

Let

\[
P^\alpha(k, t) = \mathbb{P}\{N^\alpha(t) = k\}, \text{ } k = 0, 1, 2, \ldots,
\]

(2.17)

be the PMF of the SFPP, which satisfies the following fractional differential-difference equations (see [5])

\[
\frac{d}{dt} P^\alpha(k, t) = -\lambda^\alpha (1 - B)^\alpha P^\alpha(k, t), \text{ } \alpha \in (0, 1], \text{ } k = 1, 2, \ldots
\]

(2.18)

\[
\frac{d}{dt} P^\alpha(0, t) = -\lambda^\alpha P^\alpha(0, t),
\]

(2.19)

with initial condition

\[
P^\alpha(k, 0) = \delta_{k,0}.
\]

(2.20)

The fractional difference operator \( (1 - B)^\alpha \) is defined in [8], where \( B \) is backward shift operator.
3 Time-Fractional Poisson process of order $k$ (TFPPoK)

In this section, we introduce the time-fractional Poisson process of order $k$ (TFPPoK) and discuss its main properties.

**Definition 3.1** (TFPPoK). Let $N_{\beta}(t,k\lambda)$ be the TFPP with rate parameter $k\lambda > 0$ and $X_i$, $i = 1, 2, \cdots$ be the iid discrete uniform random variables such that $\mathbb{P}(X_i = j) = \frac{1}{k}$, $j = 1, 2, \cdots, k$. Then the process defined by

$$N_{\beta}^k(t) = \sum_{i=1}^{N_{\beta}(t,k\lambda)} X_i,$$  \hspace{1cm} (3.21)

is called the TFPPoK.

The probability generating function (pgf) of $X_1$,

$$G_{X_1}(u) = \mathbb{E}(u^{X_1}) = \frac{u(1-u^k)}{k(1-u)},$$  \hspace{1cm} (3.22)

The pgf of TFPP $N_{\beta}(t,k\lambda)$ (see e.g. [16])

$$G_{N_{\beta}}(u,t) = M_{\beta,1}(-k\lambda t^\beta(1-u)), \quad u \in (0,1),$$  \hspace{1cm} (3.23)

where

$$M_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad z \in \mathbb{C}, \quad a,b > 0,$$  \hspace{1cm} (3.24)

is two parameter Mittag-Leffler function (see [19]). Using (3.22) and (3.23), the pgf of TFPPok $N_{\beta}^k(t)$ defined in (3.21) is

$$G_{N_{\beta}^k}(u,t) = M_{\beta,1}(-k\lambda t^\beta(1 - G_{X_1}(u)))$$  \hspace{1cm} (3.25)

Using a conditional argument it follows that

$$\mathbb{E}[N_{\beta}^k(t)] = \frac{k(k+1)}{2} \frac{t^\beta}{\Gamma(1+\beta)}.$$

Further, the covariance function of $N_{\beta}^k(t)$ for $s \leq t$ is

$$\text{Cov}\left(N_{\beta}^k(t), N_{\beta}^k(s)\right) = \frac{k(k+1)(2k+1)}{6} \lambda \mathbb{E}[Y_\beta(s)] + \frac{k^2(k+1)^2\lambda^2}{4} \text{Cov}(Y_\beta(s), Y_\beta(t)).$$

**Theorem 3.1.** The pmf $p_{\beta}^k(n,t) = \mathbb{P}(N_{\beta}^k(t) = n)$ of TFPPoK satisfies the following differential-difference equations

$$\frac{d^\beta}{dt^\beta} p_{\beta}^k(n,t) = -k\lambda p_{\beta}^k(n,t) + \lambda \sum_{j=1}^{n\wedge k} p_{\beta}^k(n-k,t), \quad n = 1,2,\ldots$$

$$\frac{d^\beta}{dt^\beta} p_{\beta}^k(0,t) = -k\lambda p_{\beta}^k(0,t),$$  \hspace{1cm} (3.26)

with initial condition $p_{\beta}^k(0,0) = 1$ and $p_{\beta}^k(n,0) = 0$ and $n \wedge k = \min\{k,n\}$. 

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Proof. Using $z$-transform (2.6) in both hand sides, leads to
\[
\frac{d^{\beta}}{dt^{\beta}} \{ \mathcal{Z}p^{k}_{\beta}(n, t) \} = -k\lambda \left( 1 - \sum_{j=1}^{n\wedge k} \frac{z^{-j}}{k} \right) \{ \mathcal{Z}p^{k}_{\beta}(n, t) \}.
\]
Further, using the Laplace transform of CD fractional derivative (2.15) with respect to the time variable $t$ and the condition $\mathcal{Z}\{P_{n}(0)\} = 1$, it follows
\[
s^{\beta} \mathcal{L} \{ \mathcal{Z}p^{k}_{\beta}(n, t) \} - s^{\beta-1} = -k\lambda \left( 1 - G_{X_{1}}(z^{-1}) \right) \mathcal{L} \{ \mathcal{Z}p^{k}_{\beta}(n, t) \}.
\]
By some manipulation, it follows
\[
\mathcal{L} \{ \mathcal{Z}p^{k}_{\beta}(n, t) \} = \frac{s^{\beta-1}}{s^{\beta} - k\lambda \left( 1 - G_{X_{1}}(z^{-1}) \right)}.
\]
Using the LT of Mittag-Leffler function $\mathcal{L}(M_{\beta,1}(-ut^{\beta})) = \frac{s^{\beta-1}}{u + s^{\beta}}$ (see e.g. [14], p.36), it follows
\[
\mathcal{Z}p^{k}_{\beta}(n, t) = \mathcal{L}^{-1} \left\{ \frac{s^{\beta-1}}{s^{\beta} - k\lambda \left( 1 - G_{X_{1}}(z^{-1}) \right)} \right\} = M_{\beta,1}(-k\lambda \left( 1 - G_{X_{1}}(z^{-1}) \right) t^{\beta}),
\]
which is same as (3.25) by putting $u = z^{-1}$ and hence the result.

**Proposition 3.1.** The pmf of TFPPoK is given by
\[
p^{k}_{\beta}(n, t) = M^{(n)}_{\beta,1}(-k\lambda t^{\beta}) \sum_{X=\Omega(k,n)} \frac{(\lambda t^{\beta})^{\zeta_{k}}}{\Pi_{k}!}.
\]

**Proof.** Note that
\[
\mathcal{Z}p^{k}_{\beta}(n, t) = \sum_{n=0}^{\infty} \frac{(-1)^{n}(k\lambda)^{n}t^{\beta}}{\Gamma(1 + n\beta)} (1 - G_{X_{1}}(z^{-1}))^{n}.
\]
To find $p^{k}_{\beta}(n, t)$, invert the $z$-transform, which is equivalent to finding the coefficient of $z^{-n}$, which leads to
\[
p^{k}_{\beta}(n, t) = \sum_{m=0}^{\infty} \frac{(-k\lambda m^{\beta})^{m}}{m!} \frac{(n + m)!}{\Gamma(1 + r\beta(n + m))} \sum_{X=\Omega(k,n)} \frac{(\lambda t^{\beta})^{\zeta_{k}}}{\Pi_{k}!},
\]
where $M_{\beta,1}(z)$ is the Mittag-Leffler function (3.24) evaluated at $z = -k\lambda t^{\beta}$, and $M^{(n)}_{\beta,1}(z)$ is the nth derivative of $M_{\beta,1}(z)$ evaluated at $z = -k\lambda t^{\beta}$.

**Remark 3.1.** For $k = 1$, equation (3.27) reduces to the pmf of TFPP given in (2.10).
\[
P_{\beta}(k, t) = \frac{((\lambda t^{\beta})^{n}}{n!} \sum_{m=0}^{\infty} \frac{(-\lambda m^{\beta})^{m}}{m!} \frac{(n + m)!}{\Gamma(1 + \beta(n + m))}.
\]
Remark 3.2. For the particular case $\beta = 1$, the pmf of TFPPoK reduces to the PPoK see in equation (2.1).

**Proposition 3.2.** Let $Y_\beta(t)$, $\beta \in (0,1)$, be a right-continuous inverse of stable subordinator and $N^k(t)$, $t \geq 0$, is PPoK with parameter $k\lambda > 0$, independent of $Y_\beta(t)$. The subordination representation is defined, as follows

$$Z_\beta(t) = \begin{cases} N^k(Y_\beta(t)), & 0 < \alpha < 1, \\ N^k(t), & \alpha = 1. \end{cases}$$

Then

$$Z_\beta(t) \overset{d}{=} N^k_\beta(t).$$

**Proof.** The pgf of $Z_\beta(t)$ is given by

$$G_{Z_\beta}(u, t) = \mathbb{E}[u^{Z_\beta(t)}] = M_{\beta,1}(-k\lambda t^\beta (1 - G_{X_1}(u))$$

which is equal to the pgf of TFPPoK given in (3.25). Hence by uniqueness of pgf the result follows.

**Proposition 3.3.** The marginal distributions of TFPPoK are not infinitely divisible.

**Proof.** Note that $N_\beta(t,k\lambda) \overset{d}{=} N(t^\beta Y_\beta(1))$, where $N(\cdot)$ is Poisson process with parameter $k\lambda$. Thus

$$N^k_\beta(t) \overset{d}{=} \sum_{j=1}^{N(t^\beta Y_\beta(1))} X_j.$$ 

We have

$$\frac{\sum_{j=1}^{N(t^\beta Y_\beta(1))} X_j}{t^\beta} = \frac{\sum_{j=1}^{N(t^\beta Y_\beta(1))} X_j}{N(t^\beta Y_\beta(1))} \frac{N(t^\beta Y_\beta(1))}{t^\beta Y_\beta(1)} Y_\beta(1).$$

Using (Th. 3.1.5, p. 81 [21]), the first term converges to $\mathbb{E}X_1$ almost surely (a.s) as $t \to \infty$. By an application of the renewal theorem, the second term converges to $k\lambda$ almost surely. Thus

$$\lim_{t \to \infty} \frac{\sum_{j=1}^{N(t^\beta Y_\beta(1))} X_j}{t^\beta} \overset{a.s.}{\rightarrow} k\lambda \mathbb{E}(X_1) Y_\beta(1).$$

Hence,

$$\lim_{t \to \infty} \frac{\sum_{j=1}^{N(t^\beta Y_\beta(1))} X_j}{t^\beta} \rightarrow k\lambda \mathbb{E}(X_1) Y_\beta(1) \text{ in distribution.}$$

This further implies that

$$\frac{N^k_\beta(t)}{t^\beta} = \frac{\sum_{j=1}^{N_\beta(t,k\lambda)} X_j}{t^\beta} \overset{d}{=} \frac{\sum_{j=1}^{N(t^\beta Y_\beta(1))} X_j}{t^\beta} \overset{d}{\rightarrow} k\lambda \mathbb{E}(X_1) Y_\beta(1), \text{ as } t \to \infty.$$ 

It is known that the distribution of $Y_\beta(1)$ is not infinitely divisible [1]. Suppose that $N^k_\beta(t)$ has infinitely divisible distribution then $\frac{N^k_\beta(t)}{t^\beta}$ will also have an infinitely divisible distribution (see, e.g., [7], Prop. 2.1, p. 94). It is known that the limit in distribution of a sequence of random variables with infinitely divisible distributions has an infinitely divisible distribution (see, e.g., [7, 9]), we have that the distribution of $Y_\beta(1)$ is infinitely divisible, and which is a contradiction. Hence the marginal distributions of $N^k_\beta(t)$ are not infinitely divisible.
4 Space-Fractional Poisson process of order $k$ (SFPPoK)

In this section, we study the space fractional Poisson process of order $k$ and its properties.

**Definition 4.1.** Let $N^{\alpha}(t,k\lambda)$ be the space-fractional Poisson process and $X_i$, $i = 1, 2, \ldots$ be the iid discrete uniform random variables such that $P(X_i = j) = \frac{1}{k}$, $j = 1, 2, \ldots, k$. Then the process defined by

$$R^k_{\alpha}(t) = \sum_{i=1}^{N^{\alpha}(t,k\lambda)} X_i,$$  \hspace{1cm} (4.29)

is called the space-fractional Poisson process of order $k$ (SFPPoK).

The pgf of space-fractional Poisson process (SFPP) $N^{\alpha}(t)$ see in \[5\]

$$G_{N^{\alpha}}(u,t) = e^{-k^{\alpha}\lambda^{\alpha}t(1-u)^{\alpha}}, \ |u| < 1, \ \alpha \in (0,1).$$

The pgf of SFPPoK $R^k_{\alpha}(t)$, given by

$$G_{R^k_{\alpha}}(u,t) = e^{-k^{\alpha}\lambda^{\alpha}t(1-G_{X_1}(u))^{\alpha}}, \ u \in (0,1) \hspace{1cm} (4.30)$$

where $G_{X_1}(u)$ is the pgf of the compounding distribution, given by (3.22).

**Remark 4.1.** We introduced time-changed PPoK, which can be obtained by subordinating PPoK $N^k(t)$ with independent $\alpha$–stable subordinator $S_{\alpha}(t)$, is defined by

$$Z^{\alpha}(t) = N^k(S_{\alpha}(t)), \ t \geq 0.$$  

The pgf of $Z^{\alpha}(t)$ can be calculated easily, given by

$$\mathbb{E}[u^{Z^{\alpha}(t)}] = e^{-k^{\alpha}\lambda^{\alpha}t(1-\frac{1}{k}\sum_{j=0}^{n\wedge k} w^j)^{\alpha}}$$

which is the same as the pgf of SFPPoK $N^{\alpha}_k(t)$. Then

$$Z^{\alpha}(t) \overset{d}{=} R^k_{\alpha}(t).$$

**Theorem 4.1.** The pmf $q^{\alpha,k}(n,t) = \mathbb{P}(R^k_{\alpha}(t) = n)$ of SFPPoK $R^k_{\alpha}(t)$ satisfies the following differential-difference equations

$$\frac{d}{dt}q^{\alpha,k}(n,t) = -k^{\alpha}\lambda^{\alpha}\left(1 - \frac{1}{k}\sum_{j=1}^{n\wedge k} B^j\right)^{\alpha} q^{\alpha,k}(n,t), \ n = 1, 2, \ldots$$

$$\frac{d}{dt}q^{\alpha,k}(0,t) = -k^{\alpha}\lambda^{\alpha} q^{\alpha,k}(0,t),$$  \hspace{1cm} (4.31)

with initial condition $q^{\alpha,k}(0,0) = 1$ and $q^{\alpha,k}(n,0) = 0$.

**Proof.** Taking the $z$-transform in both side, it follows

$$\frac{d}{dt}\{Zq^{\alpha,k}(n,t)\} = -k^{\alpha}\lambda^{\alpha}\left(1 - \frac{1}{k}\sum_{j=1}^{n\wedge k} z^{-j}\right)^{\alpha} Z\{q^{\alpha,k}(n,t)\}$$

where $Z\{q^{\alpha,k}(n,t)\} = \sum_{n=0}^{\infty} q^{\alpha,k}(n,t) z^n$.
solve the above equation for $Z^{\alpha,k}(n,t)$ and using initial condition, leads to

$$\{Z^{\alpha,k}(n,t)\} = \sum_{r=0}^{\infty} \frac{(-1)^r (k\lambda)^{\alpha r}}{r!} \left(1 - \frac{1}{k} \sum_{j=1}^{n/k} z^{-j}\right)^{\alpha r}$$

Further

$$\{Z^{\alpha,k}(n,t)\} = \sum_{r=0}^{\infty} \frac{(-1)^r (k\lambda)^{\alpha r}}{r!} \sum_{\sigma=0}^{\infty} \frac{(\alpha r)}{n^\sigma} \left(\sum_{j=1}^{n/k} z^{-j}\right)^n.$$ 

For finding $q^{\alpha,k}(n,t)$, we take inverse of $z$-transform, we get

$$q^{\alpha,k}(n,t) = \sum_{r=0}^{\infty} \frac{(-1)^{r+n} (k\lambda)^{\alpha r}}{k^{n r!}} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r - n + 1)} \sum_{X=\Omega(k,n)} \frac{1}{\Pi_k!}. \quad (4.32)$$

**Remark 4.2.** For $k = 1$, the pmf of SFPPoK reduces to pmf of SFPP (see [3]), given by

$$P^{\alpha}(n,t) = \frac{(-1)^n}{n!} \sum_{r=0}^{\infty} \frac{(-1)^r (\lambda)^{\alpha r}}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r - n + 1)}.$$ 

5 Simulation

- Fix the values for the parameters $\beta, \lambda$ and $k$. Fix the numbers of arrivals $N$.
- Let $U_1, U_2$, and $U_3$, be iid uniformly distributed random variables on [0,1]. Then $N$ Mittag-Leffler random numbers are generated by using (see [4])

$$T = \frac{d}{(k\lambda)^{1/\beta}} \frac{\sin(\beta \pi U_2)[\sin((1 - \beta)\pi U_2)]^{1/\beta - 1}}{[\sin(\pi U_2)]^{1/\beta} \log(U_3)^{1/\beta - 1}}.$$ 

- Generate $N$ discrete uniform random numbers between [1, k].
- Calculate the cumulative sum of Mittag-Leffler random variables and discrete uniform random variables.
- Now make a times series of these cumulative sums by using Mittag-Leffler cumulative sum as the index.
- Plot the series to get the sample paths.
Figure 1: Sample paths of TFPPOK for parameters $\beta = 0.5, \lambda = 5, k = 3$

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