Research Article
On Nil-Symmetric Rings

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The concept of nil-symmetric rings has been introduced as a generalization of symmetric rings and a particular case of nil-semicommutative rings. A ring \( R \) is called right (left) nil-symmetric if, for \( a,b,c \in R \), where \( a,b \) are nilpotent elements, \( abc = 0 \) (\( cab = 0 \)) implies \( a\cdot b = 0 \). A ring is called nil-symmetric if it is both right and left nil-symmetric. It has been shown that the polynomial ring over a nil-symmetric ring may not be a right or a left nil-symmetric ring. Further, it is also proved that if \( R \) is right (left) nil-symmetric, then the polynomial ring \( R[x] \) is a nil-Armendariz ring.

1. Introduction

Throughout this paper, all rings are associative with unity. Given a ring \( R \), \( \text{nil}(R) \) and \( R[x] \) denote the set of all nilpotent elements of \( R \) and the polynomial ring over \( R \), respectively. A ring \( R \) is called reduced if it has no nonzero nilpotent elements; \( R \) is said to be Abelian if all idempotents of \( R \) are central; \( R \) is symmetric \([1]\) if \( abc = 0 \) implies \( a\cdot b = 0 \) for all \( a,b,c \in R \). An equivalent condition for a ring to be symmetric is that whenever product of any number of elements of the ring is zero, any permutation of the factors still gives the product zero \([2]\). \( R \) is reversible \([3]\) if \( ab = 0 \) implies \( ba = 0 \) for all \( a,b \in R \); \( R \) is called semicommutative \([4]\) if \( ab = 0 \) implies \( a\cdot bR = 0 \) for all \( a,b \in R \). In \([5]\), Rege-Chhawchharia introduced the concept of an Armendariz ring. A ring \( R \) is called Armendariz if whenever polynomials \( f(x) = a_0 + a_1x + \cdots + a_nx^n \), \( g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x] \) satisfy \( f(x)g(x) = 0 \), then \( a_ib_j = 0 \) for each \( i,j \). Liu-Zhao \([6]\) and Antoine \([7]\) further generalize the concept of an Armendariz ring by defining a weak-Armendariz and a nil-Armendariz ring, respectively. A ring \( R \) is called weak-Armendariz if whenever polynomials \( f(x) = a_0 + a_1x + \cdots + a_nx^n \), \( g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x] \) satisfy \( f(x)g(x) = 0 \), then \( a_i,b_j \in \text{nil}(R) \) for each \( i,j \). A ring \( R \) is called nil-Armendariz if whenever \( f(x) = a_0 + a_1x + \cdots + a_nx^n \), \( g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x] \) satisfy \( f(x)g(x) \in \text{nil}(R)[x] \), then \( a_i,b_j \in \text{nil}(R) \) for each \( i,j \). Mohammadi et al. \([8]\) initiated the notion of a nil-semicommutative ring as a generalization of a semicommutative ring. A ring \( R \) is nil-semicommutative if \( ab = 0 \) implies \( a\cdot bR = 0 \) for all \( a,b \in \text{nil}(R) \). In their paper it is shown that, in a nil-semicommutative ring \( R \), \( \text{nil}(R) \) forms an ideal of \( R \). Getting motivated by their paper we introduce the concept of a right (left) nil-symmetric ring which is a generalization of symmetric rings and a particular case of nil-semicommutative rings. Thus all the results valid for nil-semicommutative rings are valid for right (left) nil-symmetric rings also. We also prove that if a ring \( R \) is right (left) nil-symmetric and Armendariz, then \( R[x] \) is right (left) nil-symmetric. In the context, there are also several other generalizations of symmetric rings (see \([9,10]\)).

2. Right (Left) Nil-Symmetric Rings

For a ring \( R \), \( M_n(R) \) and \( T_n(R) \) denote the \( n\times n \) full matrix ring and the upper triangular matrix ring over \( R \), respectively. We observe that if \( R \) is a ring, then

\[
\text{nil}\left(T_n(R)\right) = \begin{pmatrix}
\text{nil}(R) & R & \cdots & R \\
0 & \text{nil}(R) & R & \cdots & R \\
0 & 0 & \text{nil}(R) & R & \cdots & R \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \text{nil}(R)
\end{pmatrix}.
\]
Definition 1. A ring $R$ is said to be right (left) nil-symmetric if whenever, for every $a, b \in \text{nil}(R)$ and for every $c \in R$, $abc = 0$ ($cab = 0$), then $acb = 0$. A ring $R$ is nil-symmetric if it is both right and left nil-symmetric.

Example 2. Let $k$ be a field, and let $R$ be the path algebra of the quiver
\[ 1 \xrightarrow{x} 2 \xrightarrow{y}, \]
over $k$, modulo the relation $y^2 = 0$. Let $e_1$ and $e_2$ be the paths of length 0 at vertices 1 and 2, respectively. Composing arrows from left to right, $xy$ is a non-zero path, while $yx$ is not.

Then any nilpotent element is a linear combination of $x$, $y$, and $xy$.

Let $(ax + by + cz)xy$ and $(dx + ey + fx)yx$ be two such elements and let $(ge_1 + he_2 + ix + jy + ky)yx$ be an arbitrary element. We have
\[
(ax + by + cz)xy = (aeh)yx,
\]
\[
(dx + ey + fx)yx = (aeh)yx.
\]
Thus $R$ is a right nil-symmetric ring. However, we have that $e_2xy = 0$, while $xe_2y = xy \neq 0$. Hence, $R$ is not a left nil-symmetric ring.

Similarly by considering the opposite ring of $R$, one can have a left nil-symmetric ring which is not right nil-symmetric.

Clearly every symmetric ring is nil-symmetric but the converse is not true by Example 3 and that every subring of a right (left) nil-symmetric ring is right (left) nil-symmetric.

Example 3. For a reduced ring $R$, $T_2(R)$ is a nil-symmetric ring which is not symmetric. This can be verified as follows.

Let
\[
\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \text{nil}(T_2(R)); \quad \text{let} \quad \begin{pmatrix} c & d \\ 0 & e \end{pmatrix} \in T_2(R).
\]
Then
\[
\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & e \end{pmatrix} = 0.
\]
Also
\[
\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & e \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = 0.
\]
Thus $T_2(R)$ is a right nil-symmetric ring. Similarly it can be shown that $T_2(R)$ is a left nil-symmetric ring. But
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0
\]
whereas
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.
\]
Thus $T_2(R)$ is not symmetric.

From the above example we observe that a nil-symmetric ring need not be Abelian, as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an idempotent in $T_2(R)$, but
\[
\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Remark 4. An Abelian ring also need not be either a right nil-symmetric or a left nil-symmetric ring as shown by the following example.

Example 5. We consider the ring in [11, Example 2.2]
\[
R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \ a - d \equiv b \equiv c \equiv 0 \ (\text{mod} \ 2) \right\}.
\]

$R$ is an Abelian ring as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are the only idempotents. Again we have
\[
\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \in \text{nil}(R),
\]
\[
\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = 0.
\]
but
\[
\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \neq 0.
\]

Hence, $R$ is neither right nil-symmetric nor left nil-symmetric.

Proposition 6. Let $R$ be a reduced ring. Then
\[
S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\}
\]
is a nil-symmetric ring.

Proof. Let
\[
\begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 & c_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(S), \quad \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_1 & d_3 \\ 0 & 0 & a_2 \end{pmatrix} \in S
\]
be such that
\[
\begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b_2 & c_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_3 & c_3 \\ 0 & 0 & d_3 \\ 0 & 0 & 0 \end{pmatrix} = 0.
\]
This implies
\[
\begin{pmatrix} 0 & 0 & b_1d_2a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad \text{that is} \quad b_1d_2a_3 = 0.
\]
Since $R$ is reduced, $b_1a_3d_2 = 0$. Thus
\[
\begin{pmatrix}
0 & b_1 & c_1 \\
0 & 0 & d_1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_3 & b_3 & c_3 \\
0 & a_5 & d_3 \\
0 & 0 & a_3
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & b_1a_3d_2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}= 0.
\]

Hence, $S$ is a right nil-symmetric ring. Similarly it can be shown that $S$ is a left nil-symmetric ring.

Let $S$ be a reduced ring and we define a new ring as follows:
\[
R_n = \left\{ \begin{pmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{23} & a_{24} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,n} & a_{n-1,n} & \cdots & a_{nn} \end{pmatrix} : a_{ij} \in S \right\},
\]
where $n \geq 2$. Based on Proposition 6, one may think that $R_n$ may also be nil-symmetric for $n \geq 4$, but the following example nullifies that possibility.

Example 7. Let $R$ be a reduced ring and let
\[
R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} : a, a_{ij} \in R \right\}.
\]

Now
\[
\begin{pmatrix}
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = 0,
\]
but
\[
\begin{pmatrix}
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \neq 0. \tag{21}
\]

Thus $R_4$ is neither a right nil-symmetric ring nor a left nil-symmetric ring.

For a ring $R$, let
\[
V(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} : a, a_{ij} \in R \right\}.
\]
Then $V(R)$ forms a subring of $R_4$.

Example 8. For every reduced ring $R$, $V(R)$ is nil-symmetric. Let
\[
\begin{pmatrix}
a_1 & a_{12} & a_{13} & a_{14} \\
0 & 0 & a_{23} & a_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
b_1 & b_{12} & b_{13} & b_{14} \\
0 & 0 & b_{23} & b_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in \text{nil}(V(R))
\]
and let
\[
\begin{pmatrix}
c & c_{12} & c_{13} & c_{14} \\
0 & c & c_{23} & c_{24} \\
0 & 0 & c & 0 \\
0 & 0 & 0 & c
\end{pmatrix} \in V(R)
\]
be such that
\[
\begin{pmatrix}
a_1 & a_{12} & a_{13} & a_{14} \\
0 & 0 & a_{23} & a_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
b_1 & b_{12} & b_{13} & b_{14} \\
0 & 0 & b_{23} & b_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
c & c_{12} & c_{13} & c_{14} \\
0 & c & c_{23} & c_{24} \\
0 & 0 & c & 0 \\
0 & 0 & 0 & c
\end{pmatrix} = 0.
\]
This gives
\[
\begin{pmatrix}
0 & 0 & a_{12}b_{23}c & a_{12}b_{24}c \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = 0. \tag{26}
\]
Thus $a_{12}b_{23}c = 0$, $a_{12}b_{24}c = 0$. Since $R$ is reduced, we have $a_{12}c_{23} = 0$, $a_{12}c_{24} = 0$. Therefore,
\[
\begin{pmatrix}
0 & 0 & a_{12}c_{23} & a_{12}c_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
0 & 0 & a_{23} & a_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
c & c_{12} & c_{13} & c_{14} \\
0 & c & c_{23} & c_{24} \\
0 & 0 & c & 0 \\
0 & 0 & 0 & c
\end{pmatrix} = 0.
\]
Hence, $V(R)$ is a right nil-symmetric ring. Similarly, it can be shown that $V(R)$ is a left nil-symmetric ring.
We also observe that every right (left) nil-symmetric ring is nil-semicommutative.

**Proposition 9.** Every right (left) nil-symmetric ring is nil-semicommutative.

*Proof.* Let \( R \) be a right nil-symmetric ring and \( a, b \in \text{nil}(R) \) such that \( ab = 0 \). Let \( c \in R \) be arbitrary; then \( abc = 0 \). By right nil-symmetric property of \( R \), \( abc = 0 \). Thus \( aRb = 0 \). Hence, \( R \) is nil-semicommutative. Proceeding similarly one can show that every left nil-symmetric ring is nil-semicommutative.

**Remark 10.** The converse is however not true, as shown by the following example.

**Example 11.** For every reduced ring \( R \), \( T_3(R) \) is a nil-semicommutative ring which is neither a right nil-symmetric ring nor a left nil-symmetric ring. This can be verified as follows.

We have

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \in \text{nil}(T_3(R)),
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \neq 0.
\]

Let \( a_{i_1} a_{i_2} \cdots a_{i_n} \in \text{nil}(T_3(R)) \), then \( (a_{i_1} a_{i_2} \cdots a_{i_n}) = 0 \).

**Proposition 14.** For a reduced ring \( R \) and for \( n \geq 2 \),

\[
V_n(R) = \left\{ \begin{pmatrix}
a_1 & a_2 & a_3 & \cdots & a_n \\
0 & a_1 & a_2 & \cdots & a_{n-1} \\
0 & 0 & a_1 & \cdots & a_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_2 \\
0 & 0 & 0 & \cdots & a_1
\end{pmatrix} : a_1, \ldots, a_n \in R \right\} \text{ is a nil-symmetric ring.}
\]

**Example 13.** Let \( Q_8 = \{1, x_1, x_2, x_3, x_2 x_1, x_2 x_3, x_2 x_1 x_3, x_2 x_1 x_3 x_2 \} \) be the quaternion group and let \( \mathbb{Z}_2 \) be the ring of integers modulo 2. Consider the group ring \( R = \mathbb{Z}_2 Q_8 \). By [14, Corollary 2.3], \( R \) is reversible and so semicommutative. Let \( a = 1 + x_1, b = 1 + x_2, c = 1 + x_1 + x_2 + x_3 \). Then \( a, b \in \text{nil}(R) \) and \( c \in R \) such that \( abc = cab = 0 \), but \( abc \neq 0 \). Hence, \( R \) is neither a right nil-symmetric ring nor a left nil-symmetric ring.

**Proposition 16.** Finite product of right (left) nil-symmetric rings is right (left) nil-symmetric.

*Proof.* It comes from the fact that \( \text{nil}(\prod_{i=1}^{n} R_i) = \prod_{i=1}^{n} \text{nil}(R_i) \) [8, Proposition 2.13]. Let \( (a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \text{nil}(\prod_{i=1}^{n} R_i) \) and \( (c_1, c_2, \ldots, c_n) \in \prod_{i=1}^{n} R_i \) such that \( a_1 a_2 \cdots a_n b_1 b_2 \cdots b_n c_1 c_2 \cdots c_n = 0 \). Thus, for each \( i = 1, 2, \ldots, n \), \( a_i b_i c_i = 0 \). Since \( R_i \) is right nil-symmetric, \( a_i b_i c_i = 0 \) for each \( i = 1, 2, \ldots, n \). So, we get \( a_1 a_2 \cdots a_n (c_1 c_2 \cdots c_n) b_1 b_2 \cdots b_n = 0 \). The result can be similarly proved for left nil-symmetric rings.

**Example 15.** Let \( R \) be a reduced ring and let

\[
R_4 = \left\{ \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 \\
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_1 & a_3 \\
0 & 0 & 0 & a_1
\end{pmatrix} : a_1, a_2, a_3, a_4 \in R \right\}.
\]

By [6, Example 2.4], \( R_4 \) is weak-Armendariz. By Example 7, \( R_4 \) is neither a right nor a left nil-symmetric ring.

**Proposition 17.** Let \( R \) be a ring and let \( \Delta \) be a multiplicatively closed subset of \( R \) consisting of central nonzero-divisors. Then \( R \) is right (left) nil-symmetric if and only if \( \Delta^{-1} R \) is right (left) nil-symmetric.

*Proof.* It suffices to prove the necessary condition because subrings of right (left) nil-symmetric rings are also right (left)
Corollary 18. For a ring \( R \), \( R[x] \) is a right (left) nil-symmetric ring if and only if \( R[x; x^{-1}] \) is a right (left) nil-symmetric ring.

Proof. It directly follows from Proposition 17. If \( \Delta = \{1, x, x^2, \ldots\} \), then \( \Delta \) is clearly a multiplicatively closed subset of \( R[x] \) and \( R[x; x^{-1}] = \Delta^{-1} R[x] \). □

Proposition 19. Let \( R \) be a ring. Then \( eR \) and \( (1 - e)R \) are right (left) nil-symmetric for some central idempotent \( e \) of \( R \) if and only if \( R \) is right (left) nil-symmetric.

Proof. It suffices to prove the necessary condition because subrings of right (left) nil-symmetric rings are also right (left) nil-symmetric. Let \( eR \) and \( (1 - e)R \) be right (left) nil-symmetric rings for some central idempotent \( e \) of \( R \). Since, \( R \equiv eR \oplus (1 - e)R \), \( R \) is right (left) nil-symmetric by Proposition 16. □

Since the class of right (left) nil-symmetric rings is closed under subrings, therefore, for any right (left) nil-symmetric ring \( R \) and for any \( e^2 = e \in R \), \( eRe \) is a right (left) nil-symmetric ring. The converse is, however, not true, in general as shown by the following example.

Example 20. Let \( S \) be any reduced ring. Then by Example 21, \( R = T_3(S) \) is neither a right nil-symmetric nor a left nil-symmetric ring. But for

\[
e^2 = e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R, \quad eRe = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in S
\]

(32)

is a reduced ring and so a nil-symmetric ring.

For any nonempty subsets \( A, B, C \) of a ring \( R \), \( ABC \) denotes the set of all finite sums of the elements of the type \( abc \), where \( a \in A, b \in B, c \in C \).

Proposition 21. A ring \( R \) is right (left) nil-symmetric if and only if \( ABC = 0 \) implies \( ACB = 0 \) (\( CAB = 0 \) implies \( ACB = 0 \)) for any two nonempty subsets \( A, B \) of \( \text{nil}(R) \) and any subset \( C \) of \( R \).

Proof. Let \( R \) be a right nil-symmetric ring and let \( A, B \) be nonempty subsets of \( \text{nil}(R) \); let \( C \) be a nonempty subset of \( R \) such that \( ABC = 0 \). Then \( abc = 0 \) for all \( a \in A, b \in B, c \in C \). Right nil-symmetric property of \( R \) gives \( abc = 0 \) for all \( a \in A, b \in B, c \in C \). Thus \( ACB = 0 \). Similar proof can be given for left nil-symmetric rings. The converse is straightforward. □
Then by Proposition 6, $R$ is a nil-symmetric ring. Let $S$ be the trivial extension of $R$ by itself. Then $S$ is not a right nil-symmetric ring. Note that

\[
\begin{pmatrix}
0 & i & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & i & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & i & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Thus $S = T(R, R)$ is not a right nil-symmetric ring.

Example 25. Let $R$ be a ring and let $I$ be an ideal of $R$ such that $R/I$ is nil-symmetric. Then $R$ may not be nil-symmetric. This can be verified as follows. Let $S$ be any reduced ring. Then by Example II, $R = T_4(S)$ is not nil-symmetric but nil-semicommutative. Thus

\[
I = \text{nil}(R) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \end{pmatrix} : b, c, d \in S \right\}
\]

is an ideal of $R$ and $R/I$ is reduced, so nil-symmetric.

Homomorphic image of a right (left) nil-symmetric ring need not be a right (left) nil-symmetric ring. This is discussed after Example 26.

### 3. Polynomial Extension of Nil-Symmetric Rings

Anderson-Camillo [17] proved that a ring $R$ is Armendariz if and only if $R[x]$ is Armendariz; Huh et al. [12] have shown that polynomial rings over semicommutative rings need not be semicommutative; Kim-Lee [16] showed that polynomial rings over reversible rings need not be reversible. Recently Mohammadi et al. [8] have given an example of a nil-semicommutative ring $R$ for which $R[x]$ is not nil-semicommutative. Based on the above findings, it is natural to check whether the polynomial ring over a nil-symmetric ring is nil-symmetric. However, the answer is given in the negative through the following example.

Example 26. Let $\mathbb{Z}_2$ be the field of integers modulo 2 and let $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2$, and $c$ over $\mathbb{Z}_2$. Consider an ideal of the ring $\mathbb{Z}_2 + A$, say $I$, generated by the following elements: $a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_2b_0, a_0b_2 + a_1b_1 + a_2b_2, a_0b_1a_0, a_0b_2a_0, b_0a_0 + b_0a_1 + b_0b_1 + b_0b_2, b_0b_1 + b_0b_2, b_2b_0 + b_2b_1, b_2b_1 + b_2b_2, (a_0 + a_1 + a_2)(r_0b_0 + b_0a_1 + b_0b_2), (b_0 + b_1 + b_2)(a_0 + a_1 + a_2)$, and $r_1r_2r_3r_4$, where $r, r_1, r_2, r_3, r_4 \in A$. Now $R = (\mathbb{Z}_2 + A)/I$ is...
symmetric by [9, Example 3.1] and so a nil-symmetric ring. By [8, Example 3.6], we have $a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2 \in \text{nil}(R[x]).$ Now $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) = c(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) \in I[x],$ but $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) \notin I[x]$ because $a_0cb_1 + a_1cb_0 \notin 1.$ Hence $R[x]$ is neither a right nil-symmetric ring nor a left nil-symmetric ring.

Remark 27. The above example also helps in showing that homomorphic image of a right (left) nil-symmetric ring need not be a right (left) nil-symmetric ring. This is verified as follows.

Example 28. In Example 26, $(Z_2 + A)[x]$ is a domain [16] and so a nil-symmetric ring. But the quotient ring $(Z_2 + A)[x]/I[x] \cong R[x]$ is neither a right nil-symmetric ring nor a left nil-symmetric ring.

Now we study some conditions under which the answer may be given positively. Since every right (left) nil-symmetric ring is nil-semicommutative by Proposition 9, therefore, by [8, Theorem 3.3] for each right (left) nil-symmetric ring $R,$ $\text{nil}(R[x]) = \text{nil}(R)[x].$ The converse is, however, not true, in general. Now we give an example of a ring $R$ which satisfies $\text{nil}(R[x]) = \text{nil}(R)[x],$ but $R$ is neither a right nil-symmetric ring nor a left nil-symmetric ring.

Example 29. We use the ring in [7, Example 4.8]. Let $K$ be a field, $n \geq 2$ and $R = K(a,b|b^2 = 0).$ Then $\text{nil}(R)$ is not an ideal of $R.$ Thus $R$ is neither a right nil-symmetric nor a left nil-symmetric ring by Proposition 9 and [8, Theorem 2.5]. But $R$ is a nil-Armendariz ring and hence by [7, Corollary 5.2], $\text{nil}(R[x]) = \text{nil}(R)[x].$

Proposition 30. If $R$ is a right (left) nil-symmetric and Armendariz ring, then the polynomial ring $R[x]$ is right (left) nil-symmetric.

Proof. Let $R$ be a right nil-symmetric and Armendariz ring and let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in \text{nil}(R[x])$ and $h(x) = \sum_{k=0}^{p} c_k x^k \in R[x]$ such that $f(x)g(x)h(x) = 0.$ Since $R$ is right nil-symmetric, $\text{nil}(R[x]) = \text{nil}(R)[x]$ by Proposition 9 and [8, Theorem 3.3]. Thus $a_i b_j c_k \in \text{nil}(R)$ for $i = 0, 1, 2, \ldots, m; j = 0, 1, 2, \ldots, n.$ Since $R$ is Armendariz, therefore, $a_i b_j c_k = 0$ by [17, Proposition 1]. Thus by right nil-symmetric property of $R,$ $a_i b_j c_k = 0.$ Therefore, $f(x)h(x)g(x) = 0.$ Hence, $R[x]$ is a right nil-symmetric ring. Similarly it can be shown that $R[x]$ is a left nil-symmetric ring if $R$ is a left nil-symmetric and Armendariz ring.

Proposition 31. If $R$ is a right (left) nil-symmetric ring, then $R[x]$ is nil-Armendariz.

Proof. Let $R$ be a right (left) nil-symmetric ring. Thus by Proposition 9, $R$ is nil-semicommutative. By [8, Corollary 2.9], $R$ is a nil-Armendariz ring. Again by [8, Theorem 3.3], $\text{nil}(R[x]) = \text{nil}(R)[x].$ Thus by [7, Theorem 5.3], $R[x]$ is nil-Armendariz.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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