Inverse Boundary-Value Problem for the Equation of Longitudinal Wave Propagation with Non-self-adjoint Boundary Conditions

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Abstract. We study the inverse coefficient problem for the equation of longitudinal wave propagation with non-self-adjoint boundary conditions. The main purpose of this paper is to prove the existence and uniqueness of the classical solutions of an inverse boundary-value problem. To investigate the solvability of the inverse problem, we carried out a transformation from the original problem to some equivalent auxiliary problem with trivial boundary conditions. Applying the Fourier method and contraction mappings principle, the solvability of the appropriate auxiliary inverse problem is proved. Furthermore, using the equivalency, the existence and uniqueness of the classical solution of the original problem are shown.

1. Introduction

In this paper we study the unique solvability of the nonlocal inverse boundary-value problem for the longitudinal wave propagation equation [8]

\[ u_{tt}(x,t) - u_{tttx}(x,t) - u_{xx}(x,t) = q(t)u(x,t) + f(x,t), \quad (x,t) \in D_T, \]  

with the nonlocal initial conditions

\[ u(x,0) = \int_0^T P_1(t)u(x,t)dt + \varphi(x), \]
\[ u_t(x,0) = \int_0^T P_2(t)u(x,t)dt + \psi(x), \quad 0 \leq x \leq 1, \]  

non-self-adjoint boundary conditions

\[ u(0,t) = \beta u(1,t), \quad u_x(0,t) = u_x(1,t), \quad 0 \leq t \leq T, \]  

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and the overdetermination condition
\[ u\left(\frac{1}{2}, t\right) = h(t), \quad 0 \leq t \leq T, \] (4)
where \( D_T := \{(x, t) : 0 \leq x \leq 1, 0 \leq t \leq T\} \) is a rectangular domain, \( \beta \neq \pm 1 \) is given number, \( f(x, t), P_1(t), P_2(t), \varphi(x), \psi(x), h(t) \) are given functions, and \( u(x, t), q(t) \) are unknown functions.

The problem of finding a pair \( \{u(x, t), q(t)\} \) in (1)-(4) will be called an inverse problem.

Most generally, the inverse coefficient problems arises in many different areas of mathematical modeling, such as mineral exploration, biology, medicine, seismology, etc. Problems for the solvability of inverse problems for various types of partial differential equations were discussed in papers and monographs by Belov [2, 3], Kozhanov [16], Ivanov [13], Prilepko [24], Pyatkov [25], Kabanikhin [14], and many others.

In some instances, the equation of propagation of longitudinal waves can arise in the theory of long waves, plasma physics, problems of hydrodynamics and so on [1, 7, 9, 11, 19, 30]. The investigation of the equation of propagation of longitudinal waves is of interest both for specialists in mathematical modeling and for mathematicians. In fundamental science, the equation for propagation of longitudinal waves is considered as a Sobolev type equation and many works have been devoted to it \cite{4, 28, 29}. But, the inverse problems for the equation of longitudinal wave propagation equation have been studied relatively little, and one can get acquainted with them in the articles \cite{20, 21, 23}.

It will be noted that the statement of the problem and the proof technique used in this article are different from those of the above-mentioned articles, and the conditions in the theorems significantly differ from those therein.

We introduce the following set of functions
\[ C^{(2,2)}(D_T) = \{u(x, t) | u(x, t) \in C^2(D_T), u_{xx}(x, t), u_{txtx}(x, t), u_{ttttxx}(x, t) \in C(D_T)\}. \]

**Definition 1.1.** The pair \( \{u(x, t), q(t)\} \) is said to be a classical solution of the problem (1)-(4), if the functions \( u(x, t) \in C^{(2,2)}(D_T) \) and \( q(t) \in C[0, T] \) satisfy Equation (1) in \( D_T \), the condition (2) on \( 0, 1 \), and the statements (3)-(4) on the interval \( [0, T] \).

In order to investigate the problem (1) - (4), we first consider the following auxiliary problem
\[ y''(t) = q(t)y(t), \quad 0 \leq t \leq T, \] (5)
\[ y(0) = \int_0^T P_1(t)y(t)dt, \quad y'(0) = \int_0^T P_2(t)y(t)dt, \] (6)
where \( P_1(t), P_2(t), q(t) \in C[0, T] \) are given functions, and \( y = y(t) \) is unknown function. By a solution of the problem (5), (6), we understand a function \( y(t) \in C^2[0, T] \) satisfying the conditions (5) and (6) in the usual sense.

**Lemma 1.2.** \cite{21} Suppose that \( P_i(t) \in C[0, T] \) \( (i = 1, 2) \), \( q(t) \in C[0, T], \|q(t)\|_{C[0, T]} \leq R = \text{const}, \) and the condition
\[ \left( T \|P_2(t)\|_{C[0, T]} + \|P_1(t)\|_{C[0, T]} + \frac{T}{2}R \right) T < 1 \]
hold. Then the problem (5), (6) has a unique trivial solution.

Now along with the inverse boundary-value problem (1) - (4), we consider the following auxiliary inverse boundary-value problem. It is required to determine a pair \( \{u(x, t), q(t)\} \) of functions \( u(x, t) \in C^{(2,2)}(D_T) \) and \( q(t) \in C[0, T] \) from relations (1)-(3), and
\[ u''\left(\frac{1}{2}, t\right) - u_{txtx}\left(\frac{1}{2}, t\right) = q(t)h(t) + f\left(\frac{1}{2}, t\right), \quad 0 \leq t \leq T. \] (7)
Theorem 1.3. Assume that \( \phi(x), \psi(x) \in C[0,1], P_1(t), P_2(t) \in C[0,T], h(t) \in C^2[0,T], h(t) \neq 0, \)
\(0 \leq t \leq T, f(x,t) \in C(D_T), \) and the compatibility conditions
\[
h(0) = \int_0^T P_1(t)h(t)dt + \phi\left(\frac{1}{2}\right), \quad h'(0) = \int_0^T P_2(t)h(t)dt + \psi\left(\frac{1}{2}\right)
\]
hold. Then the following assertions are valid:

(i) each classical solution \( \{u(x,t), q(t)\} \) of the problem (1)-(4) is a solution of problem (1)-(3), (7), and

(ii) each solution \( \{u(x,t), q(t)\} \) of the problem (1)-(3), (7), satisfying the condition
\[
\left( T \|P_2(\cdot)\|_{C[0,T]} + \|P_1(\cdot)\|_{C[0,T]} + \frac{T}{2} \|q(\cdot)\|_{C[0,T]} \right) T < 1
\]
is a classical solution of problem (1)-(4).

Proof. Let \( \{u(x,t), q(t)\} \) be a classical solution of the problem (1)-(4). Taking into consideration \( h(t) \in C^2[0,T] \)
and differentiating (4), we get
\[
u_t\left(\frac{1}{2}, t\right) = h'(t), \quad u_t\left(\frac{1}{2}, t\right) = h''(t), \quad 0 \leq t \leq T.
\]

Setting \( x = \frac{1}{2} \) in Equation (1), we find
\[
u_t\left(\frac{1}{2}, t\right) - u_{12x}\left(\frac{1}{2}, t\right) - u_{xx}\left(\frac{1}{2}, t\right) = q(t)u\left(\frac{1}{2}, t\right) + f\left(\frac{1}{2}, t\right), \quad 0 \leq t \leq T.
\]

By (4) and (10), we conclude that the relation (7) is fulfilled.

Now, assume that \( \{u(x,t), q(t)\} \) is a solution of problem (1)-(3), (7) and the condition (9) is fulfilled. Then
from (7), taking into account (11), we have
\[
\frac{d^2}{dt^2}\left(u\left(\frac{1}{2}, t\right) - h(t)\right) = q(t)u\left(\frac{1}{2}, t\right) - h(t), \quad 0 \leq t \leq T.
\]

Furthermore, from (2) and (8), we obtain
\[
u\left(\frac{1}{2}, 0\right) - h(0) - \int_0^T P_1(t)u\left(\frac{1}{2}, t\right) - h(t)dt = \phi\left(\frac{1}{2}\right) - \left(h(0) - \int_0^T P_1(t)h(t)dt\right) = 0,
\]
\[
u_t\left(\frac{1}{2}, 0\right) - h'(0) - \int_0^T P_2(t)\left(u\left(\frac{1}{2}, t\right) - h(0)\right)dt = \psi\left(\frac{1}{2}\right) - \left(h'(0) - \int_0^T P_2(t)h(t)dt\right) = 0.
\]

Since by Lemma 1.2, problem (12), (13) has only a trivial solution, it follows that
\[
u\left(\frac{1}{2}, t\right) - h(t) = 0, \quad 0 \leq t \leq T,
\]
i.e. the condition (4) holds. \( \square \)

2. Auxiliary facts and notations

Now, in order to investigate the problem (1)-(3), (7), we cite some known facts.

Consider the following spectral problem [12, 15]
\[
X''(x) + \lambda X(x) = 0, \quad 0 \leq x \leq 1, \quad \lambda > 0,
\]

(14)
Theorem 2.1. [22] The system of functions (18) forms a Riesz basis in the space \( L_2(0, 1) \) and the estimates

\[
r \|g(x)\|_{L_2(0, 1)} \leq \sum_{k=0}^{\infty} g_k^2 \leq R \|g(x)\|_{L_2(0, 1)},
\]

where

\[
g_k = (g(x), Y_k(x)) = \int_0^1 g(x)Y_k(x)dx, \quad k = 0, 1, ...
\]

\[
r = \left\{ \frac{1}{2}\left(\left(a + \frac{3}{2}b\right)^2 + \frac{3}{4}b^2\right) + \frac{1}{2}\left(1 + \|(ax+b)^2\|_{L_2(0,1)}\right) \right\}^{-1},
\]

\[
R = 8\left(1 + \|[1 - b - ax]^2\|_{C^{[0,1]}}\right),
\]

are valid for any function \( g(x) \in L_2(0, 1) \).

Under the assumptions

\[
g(x) \in C^{2s-1}[0, 1], \quad g^{(2)}(x) \in L_2(0, 1),
\]

\[
g^{(2)}(0) = \beta g^{(2)}(1), \quad g^{(2s+1)}(0) = g^{(2s+1)}(1), \quad s = \frac{i}{2} - 1, \quad i \geq 1,
\]

we establish the validity of the inequalities

\[
\left(\sum_{k=1}^{\infty} (\lambda_k^2 g_k^{[2k-1]})^2\right) \leq 2 \sqrt{2} \|g^{(2)}(x)\|_{L_2(0,1)},
\]
we prove the validity of the estimates

\[ \left( \sum_{k=1}^{\infty} (\lambda_{2k+1}^2 g_{2k-1})^2 \right)^{\frac{1}{2}} \leq 2 \sqrt{2} \left\| g^{(2)}(x)(1 - b - ax) - 2ang^{(2i-1)}(x) \right\|_{L^2(0,1)}. \]  

(23)

Further, under the suppositions

\[ g(x) \in C^2[0,1], \quad g^{(2i+1)}(x) \in L_2(0,1), \]

\[ g^{(2i)}(0) = \beta g^{(2i+1)}, \quad g^{(2i-1)}(0) = g^{(2i-1)}(1), \quad s = 0, i, \]

we prove the validity of the estimates

\[ \left( \sum_{k=1}^{\infty} (\lambda_{2k+1}^2 g_{2k-1})^2 \right)^{\frac{1}{2}} \leq 2 \sqrt{2} \left\| g^{(2i+1)}(x) \right\|_{L^2(0,1)}. \]  

(24)

\[ \left( \sum_{k=1}^{\infty} (\lambda_{2k+1}^2 g_{2k})^2 \right)^{\frac{1}{2}} \leq 2 \sqrt{2} \left\| g^{(2i+1)}(x)(1 - b - ax) - a(2i + 1)g^{(2i)}(x) \right\|_{L^2(0,1)}. \]  

(25)

Now, denote by \( B^2_{2,T} \) [17] a set of all the functions of the form

\[ u(x, t) = \sum_{k=0}^{\infty} u_k(t)X_k(x), \]

classical solution of the problem (1)-(3), (7) in the form

\[ \int_{0}^{T} f(t) dt = V(u, t) = \sum_{k=1}^{\infty} (\lambda_{2k+1}^3 \left\| u_{2k-1}(t) \right\|_{C[0,T]}^2)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_{2k+1}^3 \left\| u_{2k}(t) \right\|_{C[0,T]}^2)^{\frac{1}{2}} < +\infty. \]

The norm in this set is defined as follows

\[ \left\| u(x, t) \right\|_{B^2_{2,T}} = f(u). \]

That is, we denote by \( E^3_T \) the space \( B^3_{2,T} \times C[0,T] \) of vector-functions \( z(x, t) = (u(x, t), q(t)) \) with the norm

\[ \left\| z \right\|_{E^3_T} = \left\| u(x, t) \right\|_{B^2_{2,T}} + \left\| q(t) \right\|_{C[0,T]}. \]

It is known that \( B^2_{2,T} \) and \( E^3_T \) are the Banach spaces.

### 3. Classical Solvability of Inverse Boundary-Value Problem

The system (18) is Riesz basis in \( L_2(0,1) \) and the systems (18), and (20) are biorthogonal in \( L_2(0,1) \). We’ll seek the first component \( u(x, t) \) of classical solution \( [u(x, t), q(t)] \) of the problem (1)-(3), (7) in the form

\[ u(x, t) = u_0(t)X_0(x) + \sum_{k=1}^{\infty} u_{2k-1}(t)X_{2k-1}(x) + \sum_{k=1}^{\infty} u_{2k}(t)X_{2k}(x), \]  

(26)

where

\[ u_0(t) = \int_{0}^{1} u(x, t)Y_0(x)dx, \quad u_{2k-1}(t) = \int_{0}^{1} u(x, t)Y_{2k-1}(x)dx, \quad u_{2k}(t) = \int_{0}^{1} u(x, t)Y_{2k}(x)dx. \]  

(27)
Here, $X_k(x)$ ($k = 0, 1, 2, ...$) and $Y_k(x)$ ($k = 0, 1, 2, ...$) are expressed by (18) and (19), respectively. Then by separation of variables, from (1) and (2), we have

$$u'_0(t) = q(t)u_0(t) + f_0(t), \quad 0 \leq t \leq T, \quad (28)$$

$$(1 + \lambda_k^2)u'_{2k-1}(t) + \lambda_k^2u_{2k-1}(t) = q(t)u_{2k-1}(t) + f_{2k-1}(t), \quad k = 1, 2, ..., \quad 0 \leq t \leq T, \quad (29)$$

$$(1 + \lambda_k^2)u'_{2k}(t) + \lambda_k^2u_{2k}(t) = q(t)u_{2k}(t) + f_{2k}(t) - 2a\lambda_k(u''_{2k-1}(t) + u_{2k-1}(t)), \quad k = 1, 2, ..., \quad 0 \leq t \leq T, \quad (30)$$

$$u_k(0) = \varphi_k + \int_0^T P_1(t)u_k(t)dt, \quad u'_k(0) = \psi_k + \int_0^T P_2(t)u_k(t)dt, \quad k = 0, 1, 2, ... \quad (31)$$

where

$$f_k(t) = \int_0^1 f(x, t)Y_k(x)dx, \quad \varphi_k = \int_0^1 \varphi(x)Y_k(x)dx, \quad \psi_k = \int_0^1 \psi(x)Y_k(x)dx, \quad k = 0, 1, ....$$

Solving the problem (28)-(31), we get

$$u_0(t) = \left(q_0 + \int_0^T P_1(t)u_0(t)dt\right) + t\left(\psi_0 + \int_0^T P_2(t)u_0(t)dt\right) + \int_0^t (t - \tau)F_{01}(\tau; u, q)dt, \quad 0 \leq t \leq T, \quad (32)$$

$$u_{2k-1}(t) = \left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt\right) + \frac{1}{\beta_k} \left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt\right) \sin \beta_k t$$

$$u_{2k}(t) = \left(\varphi_{2k} + \int_0^T P_1(t)u_{2k}(t)dt\right) + \frac{1}{\beta_k} \left(\psi_{2k} + \int_0^T P_2(t)u_{2k}(t)dt\right) \sin \beta_k t$$

$$+ \frac{1}{\beta_k(1 + \lambda_k^2)} \int_0^t F_{2k-1}(\tau; u, q)\sin \beta_k(t - \tau)d\tau, \quad k = 1, 2, ..., \quad 0 \leq t \leq T, \quad (33)$$

$$u_{2k}(t) = \left(\varphi_{2k} + \int_0^T P_1(t)u_{2k}(t)dt\right) + \frac{1}{\beta_k} \left(\psi_{2k} + \int_0^T P_2(t)u_{2k}(t)dt\right) \sin \beta_k t$$

$$+ \frac{1}{\beta_k(1 + \lambda_k^2)} \int_0^t F_{2k}(\tau; u, q)\sin \beta_k(t - \tau)d\tau$$

$$- \frac{a\lambda_k(1 - \beta_k^2)}{\beta_k(1 + \lambda_k^2)} \left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt\right) \sin \beta_k t$$

$$+ \left(\frac{1}{\beta_k} \sin \beta_k t - t \cos \beta_k t\right) \frac{1}{\beta_k} \left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt\right) \right]$$

$$- \frac{2a\lambda_k(1 - \beta_k^2)}{\beta_k^2(1 + \lambda_k^2)^2} \int_0^t \left[\int_0^t F_{2k-1}(\xi; u, q)\sin \beta_k(t - \xi)d\xi\right] \sin \beta_k(t - \tau)d\tau$$

$$- \frac{2a\lambda_k}{\beta_k(1 + \lambda_k^2)^2} \int_0^t F_{2k-1}(\tau; u, q)\sin \lambda_k(t - \tau)d\tau, \quad k = 1, 2, ..., \quad 0 \leq t \leq T, \quad (34)$$
where

\[ F_k(t; u, q) = f_k(t) + q(t)u_k(t) \] (\( k = 0, 1, 2, \ldots \))

and

\[ \beta_k = \frac{\lambda_k}{\sqrt{1 + \lambda_k^2}}. \]

After substitution of the expressions \( u_0(t) \), \( u_{2k-1}(t) \), \( u_{2k}(t) \) described by (32), (33), and (34) in (26), respectively, we obtain

\[
\begin{aligned}
    u(x, t) &= \left\{ \phi_0 + \int_0^T P_1(t)u_0(t)dt \right\} + t \left\{ \psi_0 + \int_0^T P_2(t)u_0(t)dt \right\} \\
    &+ \int_0^T (t - \tau)F_0(\tau; u, q)d\tau \left\{ X_0(x) + \sum_{k=1}^\infty \left\{ \varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt \right\} \cos \beta_k t \right. \\
    &+ \frac{1}{\beta_k} \left( \psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt \right) \sin \beta_k t \\
    &+ \frac{1}{\beta_k(1 + \lambda_k^2)} \int_0^T F_{2k-1}(\tau; u, q) \sin \beta_k(t - \tau)d\tau \right\} X_{2k-1}(x) \\
    &+ \sum_{k=1}^\infty \left\{ \varphi_{2k} + \int_0^T P_1(t)u_{2k}(t)dt \right\} \cos \beta_k t + \frac{1}{\beta_k} \left( \psi_{2k} + \int_0^T P_2(t)u_{2k}(t)dt \right) \sin \beta_k t \\
    &+ \frac{1}{\beta_k(1 + \lambda_k^2)} \int_0^T F_{2k}(\tau; u, q) \sin \beta_k(t - \tau)d\tau - \frac{a\lambda_k(1 - \beta_k^2)}{\beta_k(1 + \lambda_k^2)} \\
    &\times \left[ t \left( \varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt \right) \sin \beta_k t + \frac{1}{\beta_k} \sin \beta_k t \right] \\
    &- t \cos \beta_k t \frac{1}{\beta_k} \left\{ \psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt \right\} \left[ \right] \\
    &- \frac{2a\lambda_k(1 - \beta_k^2)}{\beta_k^2(1 + \lambda_k^2)^2} \int_0^T \int_0^T F_{2k-1}(\xi; u, q) \sin \beta_k(t - \xi)d\xi \sin \beta_k(t - \tau)d\tau \right\} \sin \beta_k(t - \tau)d\tau \\
    &- \frac{2a\lambda_k}{\beta_k(1 + \lambda_k^2)^2} \int_0^T F_{2k-1}(\tau; u, q) \sin \lambda_k(t - \tau)d\tau \right\} X_{2k}(x). \tag{35}
\end{aligned}
\]
Now, in order to find the equation for the component \( q(t) \) of \([u(x, t), q(t)]\), from (7) by virtue of (26), we have

\[
q(t) = [h(t)]^{-1} \left\{ \frac{h''(t)}{b} - \left( \frac{1}{2a} \right)^{n} - \left( \frac{1}{2a} \right) b \sum_{k=1}^{\infty} (-1)^{k} \lambda_{k}^{2} (u_{2k-1}(t) + u_{2k-1}'(t)) \right\}.
\]  

(36)

By (33), from (29) we get

\[
\begin{align*}
\lambda_{k}^{2} (u_{2k-1}(t) + u_{2k-1}'(t)) &= -u_{2k-1}''(t) + F_{2k-1}(t; u, q) \\
&= \beta_{k}^{2} u_{2k-1}(t) + \left( 1 - \frac{1}{1 + \lambda_{k}^{2}} \right) F_{2k-1}(t; u, q) = \beta_{k}^{2} u_{2k-1}(t) + \beta_{k}^{2} F_{2k-1}(t; u, q) \\
&= \beta_{k}^{2} \left( \varphi_{2k-1} + \int_{0}^{T} P_{1}(t) u_{2k-1}(t) dt \right) \cos \beta_{k} t + \beta_{k} \left( \psi_{2k-1} + \int_{0}^{T} P_{2}(t) u_{2k-1}(t) dt \right) \sin \beta_{k} t \\
&+ \frac{\beta_{k}}{1 + \lambda_{k}^{2}} \int_{0}^{T} F_{2k-1}(\tau; u, q) \sin \lambda_{k}(t - \tau) d\tau + \beta_{k}^{2} F_{2k-1}(t; u, q).
\end{align*}
\]

(37)

Then from (36), taking into account (37), we have

\[
q(t) = [h(t)]^{-1} \left\{ \frac{h''(t)}{b} - \left( \frac{1}{2a} \right)^{n} - \left( \frac{1}{2a} \right) b \sum_{k=1}^{\infty} (-1)^{k} \right. \]

\[
\times \left( \beta_{k}^{2} \left( \varphi_{2k-1} + \int_{0}^{T} P_{1}(t) u_{2k-1}(t) dt \right) \cos \beta_{k} t + \beta_{k} \left( \psi_{2k-1} + \int_{0}^{T} P_{2}(t) u_{2k-1}(t) dt \right) \sin \beta_{k} t \\
+ \frac{\beta_{k}}{1 + \lambda_{k}^{2}} \int_{0}^{T} F_{2k-1}(\tau; u, q) \sin \lambda_{k}(t - \tau) d\tau + \beta_{k}^{2} F_{2k-1}(t; u, q) \right) \right\}.
\]

(38)

Thus, the problem of finding the solution of (1) - (3), (7) reduces to finding a solution of system (35) and (38) with respect to functions \( u(x, t) \) and \( q(t) \).

Based on the definition of the solution of problem (1) - (3), (7), the following lemma was proved.

**Lemma 3.1.** If \([u(x, t), q(t)]\) is any solution of (1)-(3), (7), then the functions are defined by formulas (27) satisfy system (32), (33), and (34).

From Lemma 3.1 it follows

**Corollary 3.2.** Suppose that system (35) and (38) has a unique solution. Then the problem (1)- (3), (7), can’t have more than one solution.

In other words, if the problem (1) - (3), (7) has a solution, then it is unique.

Next, in the space \( E_{1}^{2} \) consider the operator

\[
\Phi(u, q) = [\Phi_{1}(u, q), \Phi_{2}(u, q)],
\]

where

\[
\Phi_{1}(u, q) = \bar{u}(x, t) \equiv \sum_{k=1}^{\infty} \bar{u}_{k}(t) X_{k}(x), \quad \Phi_{2}(u, q) = \bar{q}(t),
\]

\[
\bar{u}_{k}(t) = \int_{0}^{T} \left( \beta_{k}^{2} \left( \varphi_{2k-1} + \int_{0}^{T} P_{1}(t) u_{2k-1}(t) dt \right) \cos \beta_{k} t + \beta_{k} \left( \psi_{2k-1} + \int_{0}^{T} P_{2}(t) u_{2k-1}(t) dt \right) \sin \beta_{k} t \\
+ \frac{\beta_{k}}{1 + \lambda_{k}^{2}} \int_{0}^{T} F_{2k-1}(\tau; u, q) \sin \lambda_{k}(t - \tau) d\tau + \beta_{k}^{2} F_{2k-1}(t; u, q) \right) d\tau.
\]
and the functions $\tilde{u}_0(t), \tilde{u}_{2k-1}(t), \tilde{u}_{2k}(t)$, and $\tilde{q}(t)$ are equal to the right-hand sides of (32), (33), (34), and (38), respectively.

It is easy to see that

$$1/\sqrt{2} < \beta_k < 1, \quad 0 < 1 - \beta_k^2 < \frac{1}{2}.$$  

Taking into account these relations, we have

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq \|\varrho_0\| + T\|\varphi_0\| + T(\|P_1(t)\|_{C[0,T]} + T\|P_2(t)\|_{C[0,T]})\|\mu_0(t)\|_{C[0,T]}$$

$$+ T\sqrt{T} \left(\int_0^T |f_0(\tau)|^2 d\tau\right)^{\frac{1}{2}} + T^2 \|q(t)\|_{C[0,T]} \|\mu_0(t)\|_{C[0,T]} + T^2 \|q(t)\|_{C[0,T]} \|\mu_0(t)\|_{C[0,T]}$$

$$\left(\sum_{k=1}^\infty (\lambda_k^3 \|\tilde{u}_{2k-1}(t)\|_{C[0,T]}^2)\right)^{\frac{1}{2}} \leq \sqrt{6} \left(\sum_{k=1}^\infty (\lambda_k^3 \|\varphi_{2k-1}\|)^2\right)^{\frac{1}{2}}$$

$$+ 2\sqrt{3} \left(\sum_{k=1}^\infty (\lambda_k^3 \|\varphi_{2k}\|)^2\right)^{\frac{1}{2}} + T \sqrt{6}(\|\mu(t)\|_{C[0,T]}$$

$$+ \sqrt{2}(\|P_2(t)\|_{C[0,T]} \left(\sum_{k=1}^\infty (\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]}^2)\right)^{\frac{1}{2}} + 2T\sqrt{T} \left(\int_0^T (\sum_{k=1}^\infty (\lambda_k |f_{2k-1}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$

$$\left(\sum_{k=1}^\infty (\lambda_k^3 \|\tilde{u}_{2k}(t)\|_{C[0,T]}^2)\right)^{\frac{1}{2}} \leq \sqrt{T}\left(\sum_{k=1}^\infty (\lambda_k^3 \|\varphi_{2k}\|)^2\right)^{\frac{1}{2}} + 2\sqrt{T} \left(\sum_{k=1}^\infty (\lambda_k^3 \|\varphi_{2k}\|)^2\right)^{\frac{1}{2}}$$

$$+ T \sqrt{T}(\|P_1(t)\|_{C[0,T]} + \sqrt{2}(\|P_2(t)\|_{C[0,T]} \left(\sum_{k=1}^\infty (\lambda_k^3 \|u_{2k}(t)\|_{C[0,T]}^2)\right)^{\frac{1}{2}}$$

$$+ 2\sqrt{T} \left(\int_0^T (\sum_{k=1}^\infty (\lambda_k |f_{2k}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + 2\sqrt{T} \|q(t)\|_{C[0,T]} \left(\sum_{k=1}^\infty (\lambda_k^3 \|u_{2k}(t)\|_{C[0,T]}^2)\right)^{\frac{1}{2}}$$

$$+ 2\sqrt{3} \left(\sum_{k=1}^\infty (\lambda_k^3 \|\varphi_{2k-1}\|)^2\right)^{\frac{1}{2}} + \sqrt{T}\left(\sum_{k=1}^\infty (\lambda_k^3 \|\varphi_{2k}\|)^2\right)^{\frac{1}{2}}$$

$$+ \sqrt{2}(\|P_2(t)\|_{C[0,T]} \left(\sum_{k=1}^\infty (\lambda_k^3 \|u_{2k}(t)\|_{C[0,T]}^2)\right)^{\frac{1}{2}}$$

$$+ 2\sqrt{T} \left(\int_0^T (\sum_{k=1}^\infty (\lambda_k |f_{2k-1}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$
+2\sqrt{14}\alpha(T + \sqrt{2})T^2 \left\| \varphi(t) \right\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^2 \left\| u_{2k-1} \right\|_{C[0,T]}^2) \right)^{\frac{1}{2}}.

\left\| \varphi(t) \right\|_{C[0,T]} \leq \left\| h(t) \right\|_{C[0,T]}^{-1} \left( \sum_{k=1}^{\infty} (\lambda_k^2 \left\| u_{2k-1} \right\|_{C[0,T]}^2) \right)^{\frac{1}{2}}.

\left\| \varphi(t) \right\|_{C[0,T]} \leq \left( \sum_{k=1}^{\infty} (\lambda_k^2 \left\| u_{2k-1} \right\|_{C[0,T]}^2) \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_k^2 \left\| u_{2k-1} \right\|_{C[0,T]}^2) \right)^{\frac{1}{2}}.

Assume that the data for the problem (1)-(3), (7) satisfy the following conditions:

(A1) \varphi(x) \in C^2[0, 1], \quad \varphi''(x) \in L^2(0, 1), \quad \varphi(0) = \beta \varphi(1), \quad \varphi'(0) = \varphi'(1), \quad \varphi''(0) = \beta \varphi''(1);

(A2) \psi(x) \in C^2[0, 1], \quad \psi''(x) \in L^2(0, 1), \quad \psi(0) = \beta \psi(1), \quad \psi'(0) = \psi'(1), \quad \psi''(0) = \beta \psi''(1);

(A3) f(x, t) \in C(D_T), \quad f(x, t) \in L^2(D_T), \quad f(0, t) = \beta f(1, t), \quad 0 \leq t \leq T;

(A4) \beta \neq \pm 1, \quad P_i(t) \in C[0, T] \quad (i = 1, 2), \quad h(t) \in C^2[0, T], \quad h'(t) \neq 0, \quad 0 \leq t \leq T.

Then from (39)-(41), we find that

\left\| \varphi(t) \right\|_{C[0,T]} \leq A_1(T) + B_1(T) \left\| \varphi(t) \right\|_{C[0,T]} \left\| u(x, t) \right\|_{L^2} + C_1(T) \left\| u(x, t) \right\|_{L^2}.

\left( \sum_{k=1}^{\infty} (\lambda_k^2 \left\| u_{2k-1} \right\|_{C[0,T]}^2) \right)^{\frac{1}{2}} \leq A_2(T) + B_2(T) \left\| \varphi(t) \right\|_{C[0,T]} \left\| u(x, t) \right\|_{L^2} + C_2(T) \left\| u(x, t) \right\|_{L^2}.

\left( \sum_{k=1}^{\infty} (\lambda_k^2 \left\| u_{2k} \right\|_{C[0,T]}^2) \right)^{\frac{1}{2}} \leq A_3(T) + B_3(T) \left\| \varphi(t) \right\|_{C[0,T]} \left\| u(x, t) \right\|_{L^2} + C_3(T) \left\| u(x, t) \right\|_{L^2}.

\left\| \varphi(t) \right\|_{C[0,T]} \leq A_1(T) + B_1(T) \left\| \varphi(t) \right\|_{C[0,T]} \left\| u(x, t) \right\|_{L^2} + C_1(T) \left\| u(x, t) \right\|_{L^2}.

where

A_1(T) = 2 \left\| \varphi(x) \right\|_{L^2(0, 1)} + 2T \left\| \psi(x) \right\|_{L^2(0, 1)} + 2T \sqrt{T} \left\| f(x, t) \right\|_{L^2(D_T)},

B_1(T) = T^2,

C_1(T) = T(||P_1(t)||_{C[0,T]} + T||P_2(t)||_{C[0,T]}),

A_2(T) = 4 \sqrt{3} \left\| \varphi''(x) \right\|_{L^2(0, 1)} + 4 \sqrt{3} \left\| \psi''(x) \right\|_{L^2(0, 1)} + 4 \sqrt{6}T \left\| f(x, t) \right\|_{L^2(D_T)},

B_2(T) = 2 \sqrt{3}T,

C_2(T) = T \sqrt{6}||P_1(t)||_{C[0,T]} + \sqrt{3}||P_2(t)||_{C[0,T]}).
where the conditions
\[ = \text{right-hand side of equations (35) and (38)}. \]

Finally, from inequalities (46) and (47), we deduce
\[ \|\tilde{u}(x, t)\|_{\mathbb{L}_2} + \|\tilde{q}(t)\|_{\mathbb{L}_2} \leq A(T) + B(T) \|\tilde{q}(t)\|_{\mathbb{L}_2} + C(T) \|\tilde{u}(x, t)\|_{\mathbb{L}_2^2}, \]

where
\[ A(T) = A_4(T) + A_5(T), B(T) = B_4(T) + B_5(T), C(T) = C_4(T) + C_5(T). \]

Finally, from inequalities (46) and (47), we deduce
\[ \|\tilde{u}(x, t)\|_{\mathbb{L}_2} + \|\tilde{q}(t)\|_{\mathbb{L}_2} \leq A(T) + B(T) \|\tilde{q}(t)\|_{\mathbb{L}_2} + C(T) \|\tilde{u}(x, t)\|_{\mathbb{L}_2^2}, \]

where
\[ A(T) = A_4(T) + A_5(T), B(T) = B_4(T) + B_5(T), C(T) = C_4(T) + C_5(T). \]

**Theorem 3.3.** Let the conditions \((A_1) - (A_4), \) and the condition
\[ (A(T) + 2)(B(T)(A(T) + 2) + C(T)) < 1 \]
be fulfilled. Then the problem \((1)-(3), (7)\) has a unique solution in the ball \(K = K_R(\|z\|_{\mathbb{L}_2} \leq A(T) + 2)\) of the space \(E^2.\)

**Proof.** We represent the system of equations (35), (38) in the operator form:
\[ z = \Phi z, \]

where \(z = \{u, q\}, \) the components \(\Phi i(u, q)\) \((i = 1, 2)\) of operator \(\Phi = \{\Phi_1(u, q), \Phi_2(u, q)\}, \) defined by the right-hand side of equations (35) and (38).
Consider the operator $\Phi(u, q)$ in the ball $K = K_R$ of the space $E^3_T$. Let's show that the operator $\Phi$ mappings the elements of the ball $K = K_R$ into itself. Similarly to (48) we obtain that for any $z \in K_R$ the following inequalities hold

$$
\|\Phi z\|_{E^3_T} \leq A(T) + B(T) \|u(x, t)\|_{E^3_T} + C(T) \|q(t)\|_{C[0, T]} + C(T).
$$

Hence, by (49), we get that the operator $\Phi(u, q)$ acts in the ball $K = K_R$. It is easy to show that the operator $\Phi(u, q)$ is contractive. Indeed, for any $z_1, z_2 \in K_R$ the following estimation is true

$$
\|\Phi z_1 - \Phi z_2\|_{E^3_T} \leq (B(T)(A(T) + 2) + C(T))(\|u_1(x, t) - u_2(x, t)\|_{E^3_T} + \|q_1(t) - q_2(t)\|_{C[0, T]}).
$$

Since $B(T)(A(T) + 2) + C(T) < (A(T) + 2)[B(T)(A(T) + 2) + C(T)]$, by virtue of (49), from the latter relation we obtain the operator $\Phi(u, q)$ is contractive. For that reason, the operator $\Phi(u, q)$ has a unique fixed point $z = [u, q]$ in the ball $K = K_R$, which is the solution of Equation (50) in the ball $K = K_R$, i.e. in the ball $K = K_R$ is a unique solution of the system (35), (38). Then the function $u(x, t)$ and $q(t)$ is continuous and has continuous derivatives $u_t(x, t)$ and $u_{ttt}(x, t)$ in $D_T$.

It is easy to see that the function $u_t(x, t)$ is continuous in $D_T$.

Further, it is possible to verify that Equation (1) and conditions (2), (3), (7) are satisfied in the usual sense. Consequently, $[u(x, t), q(t)]$ is a solution of (1) - (3), (7), and by Corollary 3.2, it is unique in the ball $K = K_R$. The proof is complete. \qed

From Theorem 1.3 and Theorem 3.3, it follows directly the following assertion.

**Theorem 3.4.** Suppose that all conditions of Theorem 3.3, and the compatibility conditions

$$
h(0) = \int_0^T P_1(t)h(t)dt + \varphi \left(\frac{1}{2}\right), \quad h'(0) = \int_0^T P_2(t)h(t)dt + \psi \left(\frac{1}{2}\right),
$$

and

$$
\left( T\|P_2(t)\|_{C[0, T]} + \|P_1(t)\|_{C[0, T]} + \frac{T}{2}(A(T) + 2) \right) T < 1
$$

hold. Then the problem (1)-(4) has a unique classical solution in the ball $K = K_R$ of the space $E^3_T$.

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