Z₃-graded analogues of Clifford algebras and generalization of supersymmetry

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Abstract

We define and study the ternary analogues of Clifford algebras. It is proved that the ternary Clifford algebra with N generators is isomorphic to the subalgebra of the elements of grade zero of the ternary Clifford algebra with N + 1 generators. In the case N = 3 the ternary commutator of cubic matrices induced by the ternary commutator of the elements of grade zero is derived. We apply the ternary Clifford algebra with one generator to construct the generalization of the simplest algebra of supersymmetries.

1 Introduction

For the last few years there have appeared the generalizations of supersymmetry. It is well known that from mathematical point of view the concept of supersymmetry is based on the Z₂-graded structures. The generalizations of supersymmetry are usually constructed with the help of Z_n-graded structures, where n = 3, 4, .... Therefore the Z_n-graded generalizations of the Grassmann and Clifford algebras may be useful in constructing the generalized supersymmetries.

In this paper we define and study the analogues of Clifford algebras which have a natural Z₃-grading. Although the generators of these algebras are subjected to the binary commutation relations [4], the analogy with the classical
Clifford algebras appears when we derive the relations (7) involving three generators. This is the reason why we use the term "ternary Clifford algebra" (TCA).

We show that the generalizations of Clifford algebras proposed in this paper have the properties which are very similar to those of the classical Clifford algebras. Particularly we prove that the TCA with \( N \) generators is isomorphic to the subalgebra of the elements of grade zero of the TCA with \( N + 1 \) generators. In section 3 we explore the structure of the TCA with respect to ternary commutator. It is proved that in the case \( N = 3 \) the space spanned by the monomials of grade zero is closed with respect to ternary commutator and this fact is used to derive the formula for ternary commutator of cubic matrices. In section 4 we propose the \( Z_3 \)-graded version of the ternary commutator and we use it to construct the generalization of the simplest algebra of supersymmetries. This algebra is closely related to the concept of fractional supersymmetry [3]-[5].

2 Ternary Clifford algebra (TCA)

1. Definition of TCA. Let \( q = \exp(2\pi/3) \) be the cube root of unit and \( G = \| q_{ij} \| \) be the \( N \times N \)-matrix whose entries are defined as follows

\[
q_{ij} = \begin{cases} 
1, & i = j \\
q, & i > j \\
q^2, & i < j 
\end{cases}
\]  

(1)

It is clear that \( G \) is a Hermitian matrix, i.e. \( G^t = G \). For each triple of indices \( 1 \leq j, k, l \leq N \) such that

\[
|j - k| + |k - l| + |j - l| \neq 0,
\]  

(2)

the entries of the matrix \( G \) satisfy the identity

\[
1 + q_{kl} + q_{jk} + q_{ji} q_{kl} + q_{jl} q_{jk} + q_{kl} q_{jk} q_{ji} = 0,
\]  

(3)

which will play an essential role in what follows.

The ternary Clifford algebra (TCA) is an associative algebra over the field \( \mathbb{C} \) with the identity element 1 generated by the symbols \( Q_1, Q_2, \ldots, Q_N \) such that

\[
Q_i Q_j = q_{ij} Q_j Q_i,
\]  

(4)
and
\[ Q^3_i = 1, \quad i = 1, 2, \ldots, N. \] (5)

Let us denote the above defined TCA by \( T_q(N) \). It is clear that replacing the relations (5) in the above definition by \( Q^3_i = -1, \quad i = 1, 2, \ldots, N \) one does not change the structure of the algebra. In section 4 we shall demonstrate that TCA with \( Q^3_i = -1, \quad i = 1, 2, \ldots, N \) is closely related to the fractional supersymmetry and we shall denote it by \( T^-_q(N) \).

Let us define the analogue of the Kronecker symbol by the formula
\[ \delta_{ijk} = \sum_l \delta^i_l \delta^j_l \delta^k_l. \] (6)

From (3) and (4) it follows then that the generators of the TCA satisfy the ternary relations
\[ \{Q_i, Q_j, Q_k\} = 6 \delta_{ijk}, \] (7)

where the braces at the left-hand side stand for the sum of products of all permutations of the generators \( Q_i, Q_j, Q_k \) which can be called ternary anticommutator. The relations (7) have the form which is similar to that of the commutation relations usually assumed as a basis for the definition of the classical Clifford algebras. Particularly from (7) it follows that the generators of the TCA are ternary anticommutative that is if the indices \( i, j, k \) satisfy the condition (6) then \( \{Q_i, Q_j, Q_k\} = 0 \).

2. General structure and dimension. In order to give a more precise description of the structure of the TCA it is useful to introduce the generators with bars on subscripts. Let us define \( Q\bar{i} = Q^{2}_{i} \). From (3) it follows then that generators with bars on subscripts satisfy the relations
\[ Q_i Q_{\bar{k}} = q_{ik} Q_k Q_{\bar{i}}, \quad Q_i Q_k = q_{ik} Q_{\bar{k}} Q_{\bar{i}}, \quad Q_i Q_k Q_j = q_{ik} Q_{\bar{k}} Q_{\bar{i}}, \] (8)

where \( q_{ik} = q_{ik} = q_{\bar{i}k} \) and \( q_{\bar{ik}} = q_{ik} \). In order to combine the above commutation relations and the relations (6) into a single formula we shall use the subscripts \( A, B, C, \ldots \) running both sets \( \mathcal{N} = \{1, 2, \ldots, N\} \) and \( \bar{\mathcal{N}} = \{\bar{1}, 2, \ldots, \bar{N}\} \). We define the matrix \( G = \|q_{AB}\| \) to be
\[ G = \begin{pmatrix} q_{ik} & q_{i\bar{k}} \\ q_{\bar{i}k} & q_{\bar{i}\bar{k}} \end{pmatrix}, \]
where \( i, k \) run from 1 to \( N \). Now the commutation relations (4) and (8) can be combined into a single formula
\[
Q_A Q_B = q_{AB} Q_B Q_A. \tag{9}
\]
It is easy to see that the matrix \( \mathcal{G} \) which is an extension of \( G \) is a Hermitian matrix.

The structure of the TCA becomes more transparent if we use the notations which are similar to the Kostant ones usually used in classical Grassmann and Clifford algebras. Let \( I = \{ i_1, i_2, \ldots, i_k \} \), \( J = \{ j_1, j_2, \ldots, j_l \} \), \( i_1 < i_2 < \ldots < i_k \), \( j_1 < j_2 < \ldots < j_l \) be two subsets of \( \mathcal{N} \) such that \( I \cap J = \emptyset \). Let us denote \( Q_I = Q_{i_1} Q_{i_2} \ldots Q_{i_k} \) and \( Q_J = Q_{j_1} Q_{j_2} \ldots Q_{j_l} \). Then any element \( f \) of \( \mathcal{T}_q(\mathcal{N}) \) can be expressed in terms of the monomials \( Q_I Q_J \) as follows
\[
f = \sum_{I,J} \alpha_{IJ} Q_I Q_J, \tag{10}\]
where \( \alpha_{IJ} \) are complex numbers and the sum is taken over all possible pairs of subsets \( (I, J) \). Thus the monomials \( Q_I Q_J \) constitute the basis of the vector space underlying the algebra \( \mathcal{T}_q(\mathcal{N}) \). Counting of these monomials yields the dimension of the vector space associated to the TCA which is \( 3^N \). This result points to the analogy with the classical Clifford algebra since the dimension of the vector space of the classical Clifford algebra with \( n \) generators is \( 2^n \).

3. **\( Z_3 \)-grading and tensor product.** TCA has a natural \( Z_3 \)-grading. There are two ways to define the \( Z_3 \)-grading of TCA which are conjugate to each other. The first one is to associate grade 0 to the identity element and grade 1 to the generators \( Q_i \). The second one is to associate as before the grade 0 to the identity element and grade 2 to the generators \( Q_i \). Let us denote the grade of an element \( f \) by \( gr(f) \). As usual we define the grade of any monomial \( Q_I \) as the sum of the grades of its generators modulo 3. Clearly that \( gr(Q_i) = 2 \) in the first case and \( gr(Q_i) = 1 \) in the second. We shall use the grading defined by the first way though the second grading could be used equally well. Then the algebra \( \mathcal{T}_q(\mathcal{N}) \) splits into the direct sum of its subspaces
\[
\mathcal{T}_q(\mathcal{N}) = T_q^0(\mathcal{N}) + T_q^1(\mathcal{N}) + T_q^2(\mathcal{N}), \tag{11}\]
each consisting of the elements respectively of grade 0,1 and 2. The subspace \( T_q^0(\mathcal{N}) \) of the elements of grade 0 is a subalgebra of \( \mathcal{T}_q(\mathcal{N}) \). It can be shown that the dimension of this subalgebra is \( 3^{N-1} \).
In analogy with superstructures we define the $\mathbb{Z}_3$-graded $q$-tensor product of two TCA. Given two algebras $T_q(N), T_q(N')$ generated respectively by $Q_1, Q_2, \ldots, Q_N$ and $Q'_1, Q'_2, \ldots, Q'_{N'}$ we form the $\mathbb{Z}_3$-graded $q$-tensor product $T_q(N) \otimes_q T_q(N')$ which is the tensor product of the underlying vector spaces equipped with the multiplication

$$(f \otimes_q f') (h \otimes_q h') = q^{gr(f') - gr(h)} f f' \otimes_q hh'.$$  

If we identify $Q_i \equiv Q_i \otimes_q 1$ and denote $Q_{N+1} = 1 \otimes_q Q'_1, \ldots, Q_{N+N'} = 1 \otimes_q Q'_{N'}$ then $T_q(N) \otimes_q T_q(N') \cong T_q(N + N')$.

4. Isomorphism $T_q(N) \cong T_q^0(N + 1)$. It is well known that if $Cl(n)$, $Cl(n + 1)$ are two classical Clifford algebras over the field $\mathbb{C}$ generated by $\gamma_1, \gamma_2, \ldots, \gamma_{n+1}$ such that $\{\gamma_i, \gamma_j\} = 2 \delta_{ij}$ then there is the isomorphism

$$r : Cl(n) \rightarrow Cl^0(n + 1),$$

where $Cl^0(n + 1)$ means the even subalgebra of $Cl(n + 1)$. This isomorphism is defined by $r(a_0 + a_1) = a_0 + i a_1 \gamma_{n+1}$ where $a_0, a_1$ are respectively the even and odd parts of an element. It is worth mentioning that this isomorphism plays an essential role in the theory of Dirac operator [6].

It turns out that a similar isomorphism can be constructed for the ternary Clifford algebras as well. Given two TCA’s $T_q(N)$ and $T_q(N + 1)$ we define the mapping $\zeta : T_q(N) \rightarrow T_q^0(N + 1)$ by the formula

$$\zeta(f) = f_0 + p f_1 Q_{N+1} + p^{-1} q f_2 Q_{N+1},$$

where $p^3 = 1$ and $f_0, f_1, f_2$ are the parts of an element $f$ respectively of the grades 0, 1, 2. This mapping is the isomorphism of the algebras that can be proved by means of the relations

$$Q_{N+1} f_k = q^k f_k Q_{N+1}, \quad Q_{N+1} f_k = q^k f_k Q_{N+1}, \quad k = 0, 1, 2.$$

5. Involution. Let $T_q(N)$ be the ternary Clifford algebra generated by $Q_1, Q_2, \ldots, Q_N$. One can define an involution on the algebra $T_q(N)$. Since any element $f$ of the algebra $T_q(N)$ can be expressed in terms of the monomials $Q_I, Q_J$, $I = (i_1, i_2, \ldots, i_k)$, $J = (j_1, j_2, \ldots, j_l)$, $I \cap J = \emptyset$ it is sufficient to define the involution for these monomials and then extend it by linearity to an
arbitrary element of the TCA. Let us define the mapping \( * : \mathcal{T}_q(N) \to \mathcal{T}_q(N) \) by the formula

\[
(\alpha Q_I Q_J)^* = \bar{\alpha} Q^m Q_J Q_I,
\]

(15)

where \( \alpha \in \mathbb{C} \). Using the commutation relations (4) of TCA one can put the right-hand side of the above formula into the form

\[
(\alpha Q_I Q_J)^* = \bar{\alpha} q^m Q_J Q_I,
\]

(16)

where \( m = 1/2 (k(k-1) + l(l-1)) \). The mapping defined by (15) enjoys the following properties:

\[
(\alpha f_1 + \beta f_2)^* = \bar{\alpha} f_1^* + \bar{\beta} f_2^*, \quad (f_1 f_2)^* = f_2^* f_1^*, \quad (f^*)^* = f
\]

(17)

where \( \alpha, \beta \in \mathbb{C}, \ f, f_1, f_2 \in \mathcal{T}_q(N) \). From (17) it follows that the mapping \( * : \mathcal{T}_q(N) \to \mathcal{T}_q(N) \) is an involution of the algebra \( \mathcal{T}_q(N) \). This involution does not change the grades of the elements of the subalgebra \( T^0_q(N) \) and transforms the elements of grade 1 to the elements of grade 2 and vice versa.

## 3 Ternary commutator and the Lie structure of TCA.

1. Ternary commutator and its group properties. It is well known that the classical Clifford algebras can be used to construct the important examples of Lie algebras. The construction of these Lie algebras is based on the notion of commutator. We shall use the notion of the ternary commutator, which can be viewed as a generalization of the ordinary commutator, to study the similar structures of the ternary Clifford algebras. Let \( \mathcal{T}_q(N) \) be the ternary Clifford algebra and \( f_1, f_2, f_3 \in \mathcal{T}_q(N) \). We define the ternary \( p \)-commutator by the formula

\[
[f_1, f_2, f_3]_p = f_1 f_2 f_3 + p f_2 f_3 f_1 + p f_3 f_1 f_2 + \bar{p} f_3 f_1 f_2 + p f_2 f_3 f_1 + p f_1 f_3 f_2 + \bar{p} f_2 f_1 f_3,
\]

(18)

where \( p^3 = 1, p \neq 1 \). The above formula can be given a more shorter form if we define \( f_1 * f_2 * f_3 = f_1 f_2 f_3 + (f_1 f_2 f_3)^* \). Then (18) can be written in the form

\[
[f_1, f_2, f_3]_p = f_1 * f_2 * f_3 + p f_2 * f_3 * f_1 + p f_3 * f_1 * f_2.
\]

(19)
It is easy to see that just as in the case of the classical commutator the ternary \( p \)-commutator equals zero as soon as one of its arguments is the identity element, i.e.

\[
[1, f, h]_p = [f, 1, h]_p = [f, h, 1]_p = 0, \quad f, h \in T_q(N).
\]  

(20)

It is obvious that there are two choices in the above definition: \( p = q \) and \( p = \overline{q} \). Using involution (15) we define the conjugate ternary commutator by the formula

\[
[f_1, f_2, f_3]^\dagger_p = [f_1^*, f_2^*, f_3^*]_p.
\]  

(21)

The straightforward computation shows that

\[
[f_1, f_2, f_3]^\dagger_p = [f_1, f_2, f_3]_{\overline{p}}.
\]  

(22)

The important property of the classical binary commutator is its anticommutativity, i.e. \([a_2, a_1] = -[a_1, a_2]\). Let \( S_2 \) be the group of permutations of two elements and \( w : S_2 \to \{ -1, 1 \} \) its representation. Then the anticommutativity can be written in the form

\[
[a_{\sigma(2)}, a_{\sigma(1)}] = w(\sigma)[a_1, a_2], \quad \sigma \in S_2.
\]  

(23)

It turns out that the ternary \( p \)-commutator has the similar properties with respect to the group \( S_3 \). It is well known that the group \( S_3 \) has two conjugate representations by cubic roots of unit and complex conjugation. Replacing the ordinary complex conjugation by the conjugation (21) and denoting the above mentioned representations by \( s, \overline{s} \) we get

\[
[f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}]_q = s(\sigma)([f_1, f_2, f_3]_q),
\]

\[
[f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}]_{\overline{q}} = \overline{s}(\sigma)([f_1, f_2, f_3]_{\overline{q}}),
\]

where \( \sigma \in S_3 \). We have denoted by \( s \) the representation of \( S_3 \) such that \( s(\tau) = q, s(\chi) = \dagger \), where

\[
\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \chi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.
\]

Then the conjugate representation \( \overline{s} \) takes on the form \( \overline{s}(\tau) = q^2, \overline{s}(\chi) = \dagger \).

2. The structure of the algebra \( T_0^q(3) \). In this section our main concern is the structure of the vector space \( T_0^q(3)/\{1\} \) with respect to ternary commutator (18) with \( p = q \). Let us remind that if \( Cl(n) \) is the classical Clifford
algebra generated by $\gamma_1, \gamma_2, \ldots, \gamma_n$ then the vector space of its even elements of the form $\varsigma(\omega) = 1/4 \gamma^t \omega \gamma$, where $\omega \in so(n, \mathbb{C})$ is a skew-symmetric $n \times n$-matrix, is the Lie algebra. Moreover there is the identity

$$[\varsigma(\omega), \varsigma(\omega')] = \varsigma([\omega, \omega']), \quad \omega, \omega' \in so(n, \mathbb{C}) \quad (24)$$

which clearly shows that $\varsigma : so(n, \mathbb{C}) \to Cl^0(n)$ is the homomorphism of the Lie algebras.

If $N = 1$ then the vector space $T_0^0(N)/\{1\}$ consists only of zero vector. If $N = 2$ then the same vector space is spanned by the monomials $Q_1 Q_2$, $Q_1 Q_2$. Since these monomials commute with each other the space $T_0^0(2)/\{1\}$ is trivial with respect to ternary commutator.

The case $N = 3$ provides us with the first example of a non-trivial structure with respect to ternary commutator. In this case the dimension of the vector space underlying the algebra $T_q(N)$ is $3^N = 27$. The vector space $T_0^0(N)/\{1\}$ has dimension 8 and it is spanned by the monomials $\Sigma_\mu, \Sigma_\bar{\mu}, \mu = 0, 1, 2, 3$ expressed in terms of generators as follows

$$\begin{align*}
\Sigma_0 &= Q_1 Q_2 Q_3 & \Sigma_0 &= Q_1 Q_2 Q_3 \\
\Sigma_1 &= Q_1 Q_2 & \Sigma_1 &= Q_1 Q_2 \\
\Sigma_2 &= Q_2 Q_3 & \Sigma_2 &= Q_2 Q_3 \\
\Sigma_3 &= Q_1 Q_3 & \Sigma_3 &= Q_1 Q_3 
\end{align*}$$

Each of these monomials can be written in the form $Q_1^{\alpha_1} Q_2^{\alpha_2} Q_3^{\alpha_3}$, where $\alpha_i, i = 1, 2, 3$ take the values 0, 1, 2, at least one of them is different from zero and $\alpha_1 + \alpha_2 + \alpha_3 \equiv 0 \pmod{3}$. Then the ternary commutator of any three monomials is expressed in terms of the same monomials as follows

$$[Q^{(\alpha)}, Q^{(\beta)}, Q^{(\gamma)}] = C_{\alpha \beta \gamma} Q_1^{\delta_1} Q_2^{\delta_2} Q_3^{\delta_3}, \quad (25)$$

where $Q^{(\alpha)} = Q_1^{\alpha_1} Q_2^{\alpha_2} Q_3^{\alpha_3}$, $\delta_i = \alpha_i + \beta_i + \gamma_i \pmod{3}$ and the structure constants $C_{\alpha \beta \gamma} = \bar{q}^l (q^{m(\alpha, \beta, \gamma)} + q\bar{q}^{m(\beta, \gamma, \alpha)} + q\bar{q}^{m(\gamma, \alpha, \beta)} + q^{m(\gamma, \beta, \alpha)} + q\bar{q}^{m(\beta, \alpha, \gamma)}),$

where $l = \alpha_3 \beta_3 + \alpha_3 \gamma_3 + \beta_3 \gamma_3$, $m(\alpha, \beta, \gamma) = \alpha_2 \beta_1 + \alpha_2 \gamma_1 + \beta_2 \gamma_1$. The formula (25) proves that the space $T_q^0(3)$ is closed with respect to the ternary commutator. Indeed, the grade of the monomial at the right-hand side of
\( \delta_1 + \delta_2 + \delta_3 \equiv 0 \pmod{3} \). Moreover, the monomial at the right-hand side of (25) can not be equal to the identity element of the algebra since if \( \delta_1 = \delta_2 = \delta_3 = 0 \) then

\[
m(\alpha, \beta, \gamma) = m(\beta, \gamma, \alpha) = m(\gamma, \alpha, \beta) = m(\alpha, \gamma, \beta) = m(\beta, \alpha, \gamma)
\]

and these relations imply that \( C_{\alpha|\beta|\gamma} = 0 \).

3. **Representation by cubic matrices.** The fact proved in the previous section that the space \( T_0^q(3) \) is closed with respect to the ternary commutator can be used to construct the ternary commutator of cubic matrices. In analogy with the classical identity (24) we define the ternary commutator of cubic matrices by the formula

\[
\zeta([R_1, R_2, R_3]) = [\zeta(R_1), \zeta(R_2), \zeta(R_3)],
\]

where \( R^a, a = 1, 2, 3 \) are cubic matrices whose entries we denote respectively by \( \rho^a_{ABC} \) and \( \zeta(R^a) \) are the elements of \( T_0^q(3) \) expressed in terms of generators as follows

\[
\zeta(R^a) = \frac{1}{6} \rho^a_{ABC} Q_A Q_B Q_C.
\]

In order to get the linear combination of the monomials \( \Sigma_\mu, \Sigma_{\bar{\mu}} \) at the right-hand side of the above formula we impose the following requirements on the entries of cubic matrices \( R^a \): i) \( \rho^a_{ABC} = 0 \) if the triple \( \{A, B, C\} \) contains both the elements of the set \( \{1, 2, 3\} \) and the set \( \{\bar{1}, \bar{2}, \bar{3}\} \); ii) \( \rho^a_{ABC} = q_{AB} \rho^a_{BAC} \); iii) \( \rho^a_{kkl} = \rho^a_{k\bar{l}\bar{l}} = \rho^a_{k\bar{l}l} \) for each pair of indices \( (k, l) \). These requirements lead to the relation

\[
\sum_{\sigma \in S_3} \rho^a_{\sigma(A)\sigma(B)\sigma(C)} = 0,
\]

which shows that the cubic matrices \( R^a \) can be viewed as a cubic analogues of skew-symmetric square matrices.

In order to derive the explicit formula for ternary commutator of cubic matrices we write the element \( \zeta(R^a) \) by means of a slightly different notations. The independent entries of the cubic matrix \( R^a \) can be written in the form \( \rho^a_{A(\alpha_1, \alpha_2, \alpha_3)} \), where \( A(\alpha_1, \alpha_2, \alpha_3) = (A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}), \alpha_i = 0, 1, 2, \alpha_1 + \alpha_2 + \alpha_3 \equiv 0 \pmod{3} \) and

\[
A_{\alpha_i} = \begin{cases} \hat{i}, & \alpha_i = 0 \\ \hat{i}, & \alpha_i = 1 \\ \hat{i}, & \alpha_i = 2 \end{cases}
\]

(28)
where the hat over the index means that it is omitted. For instance \( \rho^\alpha_{3(0,1,2)} = \rho^\alpha_{23} \). Then the ternary commutator of cubic matrices is the cubic matrix whose entries are expressed as follows

\[
[R^1, R^2, R^3]_{D(\delta_1, \delta_2, \delta_3)} = \sum_{\alpha, \beta, \gamma} C_{\alpha \beta \gamma} \rho^1_{A(\alpha_1, \alpha_2, \alpha_3)} \rho^2_{B(\beta_1, \beta_2, \beta_3)} \rho^3_{C(\gamma_1, \gamma_2, \gamma_3)}, \tag{29}
\]

and the sum at the right-hand side is taken over all triple \( \alpha, \beta, \gamma \) such that \( \alpha_i + \beta_i + \gamma_i \equiv \delta_i \). Obviously the above formula gives only the independent entries of the cubic matrix \([R^1, R^2, R^3]\). All others entries can be found by means of the requirements i)-iii). We expect that the ternary commutator (29) written in the terms of the entries \( \rho^a_{ABC} \) will induce the ternary multiplication of cubic matrices \( R^a \).

4 Applications: generalization of the algebra of supersymmetries

1. Ternary \( Z_3 \)-graded commutator. It is well known that the classical Clifford algebras can be used to construct the Lie superalgebras. The main tool of the construction is the notion of the \( Z_2 \)-graded commutator which includes both the ordinary commutator and anticommutator. The simplest example is the classical Clifford algebra with one generator \( \gamma \) such that \( \gamma^2 = -1 \). This algebra is the Lie superalgebra with even part consisting of the identity element 1 and the odd part consisting of the generator \( \gamma \). The commutation relations of this Lie superalgebra have the form

\[
[1, 1]_{Z_2} = [1, \gamma]_{Z_2} = 0, \quad [\gamma, \gamma]_{Z_2} = -2. \tag{30}
\]

We shall use the ternary Clifford algebras to construct the \( Z_3 \)-graded generalizations of Lie superalgebras. The main tool to be used is the notion of the \( Z_3 \)-graded ternary \( p \)-commutator which we define by the formula

\[
[f_1, f_2, f_3]_{p, Z_3} = f_1 \ast f_2 \ast f_3 + p^{\pi(a, b, c)} f_2 \ast f_3 \ast f_1 + \bar{p}^{\pi(a, b, c)} f_3 \ast f_1 \ast f_2, \tag{31}
\]

where \( f_1, f_2, f_3 \) are elements of \( T_q(N) \) whose gradings respectively are \( a, b, c \) and \( \pi(a, b, c) = abc(a + b)(b + c)(a + c) \) and as before \( p^3 = 1, \ p \neq 1 \). Throughout what follows we shall fix \( p = q \) in (31) omitting the symbol
Let us consider the TCA with one generator $Q$ such that $Q^3 = -1$. We define its $Z_3$-grading associating grade 0 to the identity element and grade 2 to its generator $Q$. Then this TCA provides us with the simplest example of what may be called the $Z_3$-graded generalization of Lie superalgebra. It can be easily verified that the commutation relations of this algebra have form

\[
\begin{align*}
\{Q, Q, Q\}_{Z_3} &= -6 \\
\{Q, Q, Q^2\}_{Z_3} &= 0 \\
\{Q^2, Q^2, Q^2\}_{Z_3} &= -6 \\
\{Q^2, Q^2, Q^2\}_{Z_3} &= 0
\end{align*}
\]

and all commutators containing the identity element are equal to zero.

2. $Z_3$-graded generalization of the algebra of supersymmetries. We begin this section by reminding how one can construct the simplest algebra of supersymmetries making use of the Lie superalgebra (30). This can be done by replacing the identity element 1 by the operator $P = i \partial_t$ acting on the one dimensional space with the real coordinate $t$ and the generator $\gamma$ by the operator $S = \partial_\xi - i \xi \partial_t$, where $\xi$ is the anticommuting coordinate or the generator of the Grassmann algebra. Then one obtains the simplest algebra of supersymmetries

\[
\begin{align*}
\{P, P\}_{Z_2} &= \{P, S\}_{Z_2} = 0 \\
\{S, S\}_{Z_2} &= -2P
\end{align*}
\]

(33)

In order to construct the $Z_3$-graded generalization of the above algebra by means of the algebra (32) we have to realize the generator $Q$ by the operator acting on the ternary analogue of the classical Grassmann algebra. This analogue with one generator is easily constructed and it is an associative algebra over $\mathbb{C}$ generated by $\theta$ such that $\theta^3 = 0$. There are few different ways to construct the $N$-extended version of this algebra and they can be found in [1],[2]. The derivative with respect to the generator $\theta$ is defined as follows

\[
\partial_b(1) = 0, \quad \partial_b(\theta) = 1, \quad \partial_b(\theta^2) = -q\theta.
\]

(34)

Then the generator $Q$ can be represented by the operator $q \partial_b + \theta^2$. Finally defining the operators

\[
\mathcal{P} = \partial_t, \quad Q = q \partial + \theta^2 \partial_t.
\]

(35)

we get the algebra

\[
\begin{align*}
\{Q, Q, Q\}_{Z_3} &= -6\mathcal{P} \\
\{Q^2, Q^2, Q^2\}_{Z_3} &= -6\mathcal{P}^2
\end{align*}
\]
The commutators containing either the operator $\mathcal{P}$ or $\mathcal{P}^2$ are equal to zero. The remarkable peculiarity of this algebra is that it contains the second derivative with respect to time.

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