On the phase transition in the sublattice TASEP with stochastic blockage

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Abstract. We revisit the defect-induced nonequilibrium phase transition from a largely homogeneous free-flow phase to a phase-separated congested phase in the sublattice totally asymmetric simple exclusion process (TASEP) with local deterministic bulk dynamics and a stochastic defect that mimics a random blockage. Exact results are obtained for the compressibility and density correlations for a stationary grandcanonical ensemble given by the matrix product ansatz. At the critical density the static compressibility diverges while in the phase separated state above the critical point the compressibility vanishes due to strong non-local correlations. These correlations arise from a long range effective interaction between particles that appears in the stationary state despite the locality of the microscopic dynamics.

Keywords: Driven diffusive systems, totally asymmetric simple exclusion process with blockage, defect-induced nonequilibrium phase transition, correlation functions

1. Introduction

Non-equilibrium phase transitions in one-dimensional driven diffusive systems caused by a single static defect bond have a long history of study [1, 2, 3, 4, 5, 6, 7] and continue to intrigue not only from a statistical physics and probabilistic perspective [8, 9, 10, 11, 12] but also because of their recently recognized significance for biological transport by molecular motors [13, 14, 15, 16]. The general picture is that at a critical density \( \rho_c \) of driven particles there is a defect-induced nonequilibrium phase transition from a spatially homogeneous “free-flow” phase for \( \rho < \rho_c \) to a “congested phase” for \( \rho > \rho_c \) with two coexisting low density and high-density segments, corresponding to the formation of a macroscopic “traffic jam” upstream of the blockage bond.

Thus this phenomenon can be regarded as a nonequilibrium analog of phase separation [17, 18, 19]. In the phase separated state the stationary particle current becomes independent of the total conserved particle density. Increasing the total density enlarges the size of the high-density segment rather than changing the current whose maximally attainable value is limited by the blockage strength. The high-density
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segment and the low-density segment are separated by a domain wall which is sharp even on microscopic scale and represents a microscopic realization of what on macroscopic scale constitutes a shock, i.e., a discontinuity in the macroscopic density profile along the system. Such a stable domain wall is generally believed to perform a random walk, see e.g. analytical results for the continuous-time asymmetric simple exclusion process obtained by a variety of different methods [20, 21, 22, 23, 24, 25, 26].

Due to particle number conservation, the shock position in a finite system is confined to a region compatible with the conserved total density. Moreover, long-range correlations between the upstream and downstream regions to the left and right of the blockage respectively were postulated to explain numerically observed fluctuations of the shock position around its stationary mean [2]. To elucidate the phase transition and associated long range correlations further, we consider the sublattice totally asymmetric simple exclusion process (dsTASEP) with deterministic bulk dynamics and local dynamical randomness introduced by a defect [3], informally defined for the one-dimensional lattice with \( L \) sites as follows.

We use the convention to denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{N}_0 \) the set of nonnegative integers. The set \( \mathbb{T}_L = \{1, \ldots, L\} \) refers to the lattice. The state \( \eta = (\eta_1, \ldots, \eta_L) \) of the dsTASEP is represented at any given time by the local occupation numbers \( \eta_k \in \{0, 1\} \). For \( \eta_k = 1 \), we say that site \( k \in \mathbb{T}_L \) is occupied by a particle, thus encoding hard-core repulsion that forbids double occupancy of a site. For \( \eta_k = 0 \) we say that site \( k \) is empty or, alternatively, occupied by a hole. Correspondingly,

\[
\bar{\eta}_k := 1 - \eta_k
\]  

are the hole occupation numbers.

The dsTASEP is a stochastic cellular automaton evolving in discrete time steps \( t \in \mathbb{N}_0 \) and is described by the sequence \( \eta_k(t) \) of the occupation numbers. A full update cycle \( \eta(t) \to \eta(t+2) \) consists of two consecutive time steps. In the first step \( t \to t+1 \), a particle on site \( 2k-1 \) in an odd bond \((2k-1, 2k)\) moves from \( 2k-1 \) to \( 2k \), provided that site \( 2k \) is empty. Otherwise nothing happens in bond \((2k-1, 2k)\). This jump rule is applied to all odd bonds simultaneously, corresponding to a deterministic sublattice version of totally asymmetric random hopping like e.g. in the standard continuous-time TASEP [27] [28]. In the second part \( t+1 \to t+2 \) of an update cycle the same rule is applied to the even bonds \((2k, 2k+1)\) except for the blockage bond \((L, 1)\) on which a particle on site \( L \) jumps randomly to site \( 1 \) with probability \( p \) provided that site \( 1 \) is empty.

The invariant measure for a canonical ensemble of \( N \) particles was derived in [3] in terms of a set of selection rules and probability ratios \( p/q \). It was shown that at the critical density \( \rho_c = p/2 \) there is a phase transition in the thermodynamic limit \( L \to \infty \) from a free flow phase for \( \rho < \rho_c \) to a congested phase for \( \rho > \rho_c \) with two coexisting regimes of different densities, as described in the introductory discussion in the context † We mention that the sublattice property of the update dynamics has an equivalence with parallel update schemes without sublattice structure [29].
of phase separation. Later a grandcanonical invariant measure – where the conserved total particle number is a random variable – was obtained in [30] from a matrix product ansatz (MPA) [31], but not further investigated for this process.

This matrix product approach is used in the present treatment to study the defect-induced nonequilibrium phase transition to the phase separated state rigorously and in considerably more detail than previously. In section 2 we express the invariant measure in a matrix product form similar to that of [30] and point out the presence of a long-range effective interaction in the stationary distribution. Further properties are presented, discussed, and proved in sections 3 (nonequilibrium phase transition), 4 (density profiles), and 5 (correlation functions) where a long-range correlation resulting from the long-range effective interaction is explored. In the appendices we list the properties of various functions used in the proofs (Appendix A) and we show how the matrix product representation of section 2 follows from the MPA established in [30] (Appendix B).

A remark on the presentation: All mathematical results are exact. Their derivation is either elementary – based solely on matrix multiplications and evaluations of geometrical series – or uses well-established properties of convergence of slowly varying discrete functions to continuous functions. No probabilistic or further advanced mathematical concepts are used. However, these derivations are lengthy, involve many case distinctions, and require precise statements concerning the range of validity of various mathematical functions appearing in the treatment. For clarity, we have therefore opted in most sections for an explicit separation between a statistical physics discussion of the results and their mathematical presentation in form of theorems and propositions which are followed by essentially rigorous computational proofs.

2. Stationary matrix product measure

With the i.i.d. random variables $\zeta(t)$ with bimodal distribution $f(\cdot) = (1 - p)\delta_{0,0} + p\delta_{-.1}$ the dsTASEP described informally above is defined for $t \in \mathbb{N}_0$ by the update rules

\[
\begin{align*}
\eta_{2k-1}(t + 1) &= \eta_{2k-1}(t)\bar{\eta}_{2k}(t) \\
\eta_{2k}(t + 1) &= 1 - \bar{\eta}_{2k-1}(t)\bar{\eta}_{2k}(t) \\
\end{align*}
\]

$\text{t even, } 1 \leq k \leq \frac{L}{2}$ (2.1)

and

\[
\begin{align*}
\eta_{2k}(t + 1) &= \eta_{2k}(t)\eta_{2k+1}(t) \\
\eta_{2k+1}(t + 1) &= 1 - \eta_{2k}(t)\bar{\eta}_{2k+1}(t) \\
\eta_{L}(t + 1) &= \eta_{L}(t)\left[1 - \xi(t + 1)\bar{\eta}_{1}(t)\right] \\
\eta_{1}(t + 1) &= \eta_{1}(t) + \xi(t + 1)\bar{\eta}_{1}(t)\eta_{L}(t) \\
\end{align*}
\]

$\text{t odd, } 1 \leq k \leq \frac{L}{2} - 1$ (2.2)

In terms of the instantaneous currents

\[
\begin{align*}
\dot{j}_{2k-1}(t) := \eta_{2k-1}(t)\bar{\eta}_{2k}(t), & \quad 1 \leq k \leq \frac{L}{2} \\
\dot{j}_{2k}(t) := \left[1 - \bar{\eta}_{2k-1}(t)\bar{\eta}_{2k}(t)\right]\left[1 - \eta_{2k+1}(t)\eta_{2k+2}(t)\right], & \quad 1 \leq k \leq \frac{L}{2} - 1 \\
\end{align*}
\]

(2.3)

(2.4)
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\[ j_L(t) := \zeta(t)[1 - \bar{\eta}_{L-1}(t)\eta_L(t)][1 - \eta_1(t)\eta_2(t)], \quad (2.5) \]
a full two-step update cycle is therefore expressed by the discrete continuity equation

\[ \eta_k(t+2) = \eta_k(t) + j_{k-1}(t) - j_k(t), \quad k \in \mathbb{T}_L, t \in \mathbb{N}_0 \quad (2.6) \]

with the definition \( j_0(t) := j_L(t) \).

Under this jump dynamics the total particle number

\[ N(\eta) = \sum_{k=1}^{L} \eta_k \quad (2.7) \]
is conserved, but not the sublattice particle numbers

\[ N^\pm(\eta) = \frac{1}{2} \sum_{k=1}^{L} (1 + (-1)^k)\eta_k. \quad (2.8) \]

The process is invariant under the particle-hole reflection symmetry \( \eta_k \mapsto \bar{\eta}_{L+1-k} \) applied jointly to all \( k \). We take \( M = L/2 \) even and focus on configurations \( \eta \) with \( 0 \leq N \leq L/2 \) particles, corresponding to density \( N/L \leq 1/2 \). The properties of the model for \( N/L > 1/2 \) follow straightforwardly from the particle-hole symmetry.

For \( p = 0 \), particles cannot jump from site \( L \) to site \( 1 \), corresponding to the trivial case of complete blockage where after a finite number of time steps all \( N \) particles of a configuration \( \eta \) pile up on the block of sites \( L - N + 1, \ldots, L \). Also for \( p = 1 \) (no blockage) the dsTASEP becomes trivial after a finite number of steps as it reduces to deterministic translations of all particles by one site per time step. Hence we restrict ourselves to the non-trivial range \( 0 < p < 1 \) of the blockage parameter where translation invariance of the dynamics is broken.

To study the model in a grandcanonical ensemble we slightly modify the matrix product ansatz for the invariant measure developed in [30]. To this end, we define the two-dimensional matrices

\[ D := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (2.9) \]

and

\[ A_0 := p \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_1 := p \mathbb{1}, \quad A_2 := (1 - p) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \quad (2.10) \]

Furthermore, for \( z \in \mathbb{R} \) we define

\[ A := A_0 + z(A_1 + A_2) = \begin{pmatrix} z & p + z(1 - p) \\ 0 & p + pz \end{pmatrix}. \quad (2.11) \]

With these matrices and the function

\[ Y_K(p, z) := \text{Tr}(DA^K) \quad (2.12) \]
the MPA of \([30]\) becomes
\[
P_{L,p,z}(\eta) = \frac{1}{Y_L(p,z)} \text{Tr} \left\{ D \left[ \bar{\eta}_1 \bar{\eta}_L A_0 + z \eta_1 \bar{\eta}_L A_1 + z \bar{\eta}_1 \eta_L A_2 \right] \right.
\]
\[
\times \bar{\eta}_2 \left[ \bar{\eta}_{L-1} A_0 + z \eta_{L-1} (A_1 + A_2) \right] 
\times \ldots 
\times \left( \bar{\eta}_{2k-1} \bar{\eta}_{L+2-2k} A_0 + z \eta_{2k-1} \bar{\eta}_{L+2-2k} A_1 + z \bar{\eta}_{2k-1} \eta_{L+2-2k} A_2 \right) 
\times \bar{\eta}_{2k} \left[ \bar{\eta}_{L+1-2k} A_0 + z \eta_{L+1-2k} (A_1 + A_2) \right] 
\times \ldots 
\times \left( \bar{\eta}_{L-1} \bar{\eta}_{L+2} A_0 + z \eta_{L-1} \bar{\eta}_{L+2} A_1 + z \bar{\eta}_{L-1} \eta_{L+2} A_2 \right) 
\times \bar{\eta}_L \left[ \eta_{L+1} A_0 + z \eta_{L+1} (A_1 + A_2) \right] \}.
\] (2.13)

We say that the measure \(P_{L,p,z}(\eta)\) is a stationary matrix product measure (SMPM). The normalization factor \(Y_L(p,z)\) plays the role of a grandcanonical nonequilibrium partition function in which the total particle number \(N(\eta)\) has a distribution determined by the parameter \(z\) as can be seen by noting that \(\sum_{\eta} z^{N(\eta)} P_{L,p,1}(\eta) = Y_L(p,1) / Y_L(p,z)\). Thus it becomes evident that \(z\) plays the role of a fugacity. Below we drop the dependence of the SMPM on the blockage parameter \(p\) and the fugacity \(z\).

One notices in the structure of the SMPM a fundamental difference between the region to the right of the blockage and the region to the left. To capture this phenomenon it is convenient to introduce lattice sectors.

**Definition 2.1** A site \(k \in \mathbb{T}_L\) is said to belong to sector \(1\), denoted by \(\mathbb{T}_{L,1}\), if \(k \in \{1, \ldots, L/2\}\) and to sector \(2\), denoted by \(\mathbb{T}_{L,2}\), if \(k \in \{L/2 + 1, \ldots, L\}\).

Some other properties of the invariant measure that can be read off directly from of the structure of the SMPM \((2.13)\) and have analogs already found in \([3]\) in terms of a set of rules for the canonical ensemble with fixed particle number \(N\). We generalize these rules here to the grandcanonical case.

**Proposition 2.2** For any measurable function \(f : \{0,1\}^L \to \mathbb{R}\) the SMPS has the projection properties
\[
\langle \eta_k \eta_{L+1-k} f \rangle_L = 0 \quad k \in \mathbb{T}_L \quad \text{(2.14)}
\]
\[
\langle \eta_{2k} f \rangle_L = 0 \quad 2k \in \mathbb{T}_{L,1} \quad \text{(2.15)}
\]
where \(\langle f \rangle_L\) denotes the expectation of a function \(f(\eta)\) w.r.t. \((2.13)\).

**Remark 2.3** The projection property \((2.14)\) demonstrates that the invariant measure incorporates an long-range effective interaction between a site \(k\) in the left segment \(\mathbb{T}_{L,1}\) and the reflected site \(L + 1 - k\) in the right segment \(\mathbb{T}_{L,2}\), no matter how far (in lattice units) the two sites are apart.

The appearance of a stationary effective long-range interaction is somewhat counterintuitive since the microscopic dynamics is one-dimensional, completely local and has finite local state space. An immediate consequence are long-range anticorrelations...
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\[ \langle \eta_k \eta_{L+1-k} \rangle_L - \langle \eta_k \rangle \langle \eta_{L+1-k} \rangle_L = -\langle \eta_k \rangle \langle \eta_{L+1-k} \rangle_L. \]

A long-range reflection property of correlations reminiscent of this anticorrelation was conjectured for the continuous-time TASEP with blockage [2]. We also find it intriguing that the SMPM is similar to a class of probability distributions for annihilating random walks [32].

For explicit computations one needs to know the normalization \( Y_{L/2} \). In terms of the critical fugacity

\[ z_c := \frac{p}{1 - p} \]  

the \( K^{th} \) power of the matrix \( A \) can be written as

\[
A^K = \begin{cases} 
\left( z^K \frac{z + z_c}{z - z_c} \left( z^K - p^K(1 + z^K) \right) \right) & z \neq z_c \\
p^K(1 - p)^{-K} \left( \begin{array}{cc} 1 & 2(1 - p)K \\ 0 & 1 \end{array} \right) & z = z_c.
\end{cases}
\]  

which is proved easily by induction. Therefore,

\[ Y_K = \begin{cases} 
z_c^{K+1} + z_c (1 + z) \left( \frac{1 + z}{1 + z_c} \right)^K - \left( \frac{z}{z_c} \right)^{K+1} & z \neq z_c \\
z_c^K [(1 - p)K + 1] & z = z_c.
\end{cases}
\]  

We point out that the limit \( z \to z_c \) and the thermodynamic limit \( L \to \infty \) may not commute in expectation values.

Furthermore, we recall the quadratic relations [30]

\[ A_0^2 = pA_0, \quad A_2^2 = (1 - p)A_2 \]  

\[ A_0A_2 = 0 \]  

\[ A_0A = p(1 + z)A_0, \quad AA_2 = zA_2 \]  

\[ A_0D = pD, \quad DA_2 = (1 - p)D. \]

From (2.19) and (2.20) together with the trivial relations \( A_\alpha A_1 = A_1 A_\alpha = pA_\alpha \) one obtains the reduction formula

\[ (A_0 + zA_1)(A_1 + A_2) = pA. \]  

Iterating the quadratic relations (2.21) - (2.22) yields for \( n \in \mathbb{N}_0 \)

\[ DA^n A_2 = (1 - p)z^n D \]  

\[ A_0 A^n D = p^{n+1}(1 + z)^n D \]

From (2.24) one reads off the commutator property

\[ DA^n A_2 A_0 = DA^n [A_2, A], \quad n \in \mathbb{N}_0. \]  

These matrix identities, in particular the reduction formulas (2.23), (2.24), and the commutator property (2.26), will be used frequently in computations below. The quadratic relation (2.20) leads to a further long range effective interaction inside sector \( T_{L,2} \) as it implies for any measurable function \( f \) the projection property

\[ \langle \eta_{2k} \eta_{2k+2p-1} \eta f \rangle_L = 0, \quad 2k \in T_{L,2}, \quad 1 \leq p \leq L/2 - k \]  

noticed in [3] for the canonical ensemble.
3. Particle number fluctuations and stationary current

The dynamics conserves the particle number, but the matrix product measure is a mixture of canonical invariant measures with particle number $N$ that, as shown below, has a non-trivial distribution as a function of the blockage parameter $p$ and the fugacity $z$. In particular, it turns out that there is a critical density below which the variance of the particle number is proportional to the system size $L$ – corresponding to a non-zero thermodynamic compressibility – while above the critical density there is a phase separated regime where the variance reaches a constant for $L \to \infty$ so that the thermodynamic compressibility vanishes. This implies that the two coexisting phases are not subcritical bulk phases at two different densities, as one might expect from equilibrium phase separation e.g. in the two-dimensional Ising model. Also the stationary current changes it behaviour at the critical point.

3.1. Critical point and density fluctuations

It was shown in [3] for the canonical ensemble that a non-equilibrium phase transition occurs at a critical density $\rho_c = p/2$. Here we establish an analogous result $\rho_c := \rho(p, z_c) = p/2$ for the grandcanonical SMPM (2.13) in terms of the critical fugacity $z_c$ (2.16) and discuss in detail the variance of the particle number.

**Theorem 3.1** The particle density $\rho(p, z)$

$$\rho(p, z) := \lim_{L \to \infty} \frac{1}{L} \langle N \rangle_L$$

has a jump discontinuity at the critical point given by

$$\rho(p, z) = \begin{cases} 
\frac{1}{2} \frac{z}{1 + z} & z < z_c \\
\frac{1}{4} \frac{1 + p}{z} & z = z_c \\
\frac{1}{2} & z \geq z_c.
\end{cases}$$

**Theorem 3.2** The compressibility $C(p, z)$

$$C(p, z) := \lim_{L \to \infty} \frac{1}{L} \left( \langle N^2 \rangle_L - \langle N \rangle_L^2 \right)$$

diverges at the critical point and is given by

$$C(p, z) = \begin{cases} 
\frac{1}{2} \frac{z}{(1 + z)^2} & z < z_c \\
\infty & z = z_c \\
0 & z > z_c.
\end{cases}$$

Moreover, for the critical regime $z \geq z_c$ one has

$$\lim_{L \to \infty} \frac{1}{2} \left( \langle N^2 \rangle_L - \langle N \rangle_L^2 \right) = \frac{(1 - p)^2}{48}, \quad z = z_c,$$

$$\lim_{L \to \infty} \left( \langle N^2 \rangle_L - \langle N \rangle_L^2 \right) = \frac{zz_c}{(z - z_c)^2}, \quad z > z_c.$$
Remark 3.3 The supercritical particle variance \((3.6)\) is also the amplitude of the finite-size correction to \(C(p, z)\) for \(z < z_c\) to leading order in \(1/L\).

**Proof:** Both theorems are naturally proved together. For notational simplicity we suppress the dependence on \(p\) and \(z\) in all functions considered below.

Since the expectation of the total particle number can be written
\[
\langle N \rangle_L = \sum_{k=1}^{M} \langle \eta_k + \eta_{L+1-k} \rangle_L,
\]
the SMPM yields
\[
\langle N \rangle_L = z \frac{d}{dz} \ln Y_{\frac{L}{2}} \tag{3.8}
\]
Moreover, from (2.14) in Proposition 2.2 one gets
\[
\langle (\eta_k + \eta_{L+1-k})^2 \rangle_L = \langle \eta_k + \eta_{L+1-k} \rangle_L \tag{3.9}
\]
and it follows that
\[
C_L := \left( z \frac{d}{dz} \ln Y_{\frac{L}{2}} \right)^2 = \frac{1}{Y_{\frac{L}{2}}} \left( \frac{1}{Y_{\frac{L}{2}}} \frac{d}{dz} Y_{\frac{L}{2}} \right)^2
\]
is the variance of the particle number in a finite system of length \(L\).

For computing the derivatives w.r.t. \(z\) and then taking the thermodynamic limit it is convenient to introduce
\[
\tilde{Y}_M := z_c^{-M} Y_M = \left( \frac{z}{z_c} - 1 \right)^{-1} \left[ \left( \frac{z}{z_c} \right)^{M+1} - \left( \frac{1+z}{1+z_c} \right)^M \right].
\]
so that one can replace \(p^{-M} Y_M\) in (3.8) and (3.9) by \(\tilde{Y}_M\). One obtains
\[
z \frac{d}{dz} \ln \tilde{Y}_M = \frac{z}{z_c} \left( 1 - \frac{z}{z_c} \right)^{-1}
+ (M + 1) \left( \frac{z}{z_c} \right)^{M+1} - M \frac{z}{1+z} \left( \frac{1+z}{1+z_c} \right)^M
- \left( \frac{z}{z_c} \right)^M - \left( \frac{1+z}{1+z_c} \right)^M.
\]
Setting \(M = L/2\) and taking the limit \(L \to \infty\) immediately gives (3.2) for \(z \neq z_c\).

Next we consider the density fluctuations. The second term in the second equality in (3.9) is \(\langle N \rangle_L^2\) and it remains to compute the first term. Taking the derivative and dividing by \(Y_M\) gives
\[
\frac{1}{Y_M} \left( z \frac{d}{dz} \right)^2 \tilde{Y}_M = 2 \left( \frac{z}{z_c} \right)^2 \left( 1 - \frac{z}{z_c} \right)^{-2} + \left( \frac{z}{z_c} \right) \left( 1 - \frac{z}{z_c} \right)^{-1}
+ 2 \frac{z}{z_c} \left( 1 - \frac{z}{z_c} \right)^{-1} M \frac{z}{1+z_c} \left( \frac{1+z}{1+z_c} \right)^M
- (M + 1) \left( \frac{z}{z_c} \right)^{M+1}
\]
\[
\frac{1}{Y_M} \left( z \frac{d}{dz} \right)^2 \tilde{Y}_M = \left( \frac{Mz}{(1+z)^2} + \frac{M^2z^2}{(1+z)^2} \right) \left( \frac{1+z}{1+z_c} \right)^M
- (M + 1)^2 \left( \frac{z}{z_c} \right)^{M+1}
\]
\[
\left( \frac{1+z_c}{1+z_c} \right)^M - \left( \frac{z}{z_c} \right)^{M+1} \tag{3.12}
\]
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and therefore, after some rearrangement of terms,

\[
C_L = \frac{1}{2} \frac{z}{(1+z)^2} \frac{L/2}{(1 - z/z_c)(1+z)} + \frac{1}{L} \frac{zz_c}{(z - z_c)^2} \\
- \frac{1}{L} \left( \frac{L}{2(1+z)} + 1 \right)^2 \frac{z}{z_c} \left( \frac{z(1+z_c)}{z_c(1+z)} \right)^{L/2} \\
\left( \frac{1 - z}{z_c} \left( \frac{z(1+z_c)}{z_c(1+z)} \right)^{L/2} \right)^2.
\]

(3.13)

Taking the thermodynamic limit one arrives at (3.4) for \( z \neq z_c \). In the same way, by taking appropriate limits, one obtains (3.6) and the claim made in Remark 3.3.

To study the critical point we set \( z = z_c(1 + \epsilon) \) so that

\[
\ln \tilde{Y}_M = \ln \left[ (1 + \epsilon)^{M+1} - (1 + p\epsilon)^M \right] - \ln \epsilon \\
= \ln \left[ \sum_{n=0}^{M} \binom{M}{n+1} \epsilon^n - p \sum_{n=0}^{M-1} \binom{M}{n+1} \epsilon^n \right]
\]

(3.14)

and

\[
z \frac{d}{dz} = (1 + \epsilon) \frac{d}{d\epsilon}.
\]

(3.15)

Expanding in \( \epsilon \) gives to the required second order

\[
\ln \tilde{Y}_M = \ln (M + 1 - pM) + a_M \epsilon + \frac{\epsilon^2}{2} (2b_M - a_M^2) + O(\epsilon^3)
\]

(3.16)

with

\[
a_M = \binom{M+1}{2} - p^2 \binom{M}{2} \\
b_M = \binom{M+1}{3} - p^3 \binom{M}{3}
\]

(3.17)

(3.18)

Therefore

\[
(1 + \epsilon) \frac{d}{d\epsilon} \ln \tilde{Y}_M = a_M + O(\epsilon)
\]

(3.19)

\[
\left( (1 + \epsilon) \frac{d}{d\epsilon} \right)^2 \ln \tilde{Y}_M = a_M + 2b_M - a_M^2 + O(\epsilon)
\]

(3.20)

which yields

\[
\langle N \rangle_L = a_M, \quad C_L = a_M + 2b_M - a_M^2.
\]

(3.21)

Taking the thermodynamic limit for \( \epsilon = 0 \) yields (3.2) and (3.4) for \( z = z_c \) as well as (3.5). □
The stationary current is the space-independent expectation

\[
j_L(p, z) = \langle \eta_{2k-1} \bar{\eta}_{2k} \rangle_L, \quad 1 \leq k \leq L/2
\]

\[
= \langle [1 - \bar{\eta}_{2k-1} \bar{\eta}_{2k}][1 - \eta_{2k+1} \eta_{2k+2}] \rangle_L, \quad 1 \leq k < L/2
\]

\[
= p \langle (1 - \bar{\eta}_{L-1} \bar{\eta}_L)(1 - \eta \eta_2) \rangle_L
\]

of the instantaneous currents (2.3) - (2.5).

**Theorem 3.4** The macroscopic current \( j(p, z) := \lim_{L \to \infty} j_L(p, z) \) is continuous at the critical point and given by

\[
j(p, z) = \begin{cases} 
\frac{z}{1 + z} & z < z_c \\
p & z \geq z_c
\end{cases}
\]

**Proof:** One obtains from the MPA (2.13) for \( \langle \eta_{2k-1} \bar{\eta}_{2k} \rangle_L \) in the range \( 1 \leq k \leq L/4 \)

\[
j_L(p, z) = \frac{1}{Y^*_L} \sum_{\eta} \text{Tr} \{ D [\bar{\eta}_{L} \bar{\eta}_{A_0} + z \eta_{L} A_1 + z \bar{\eta}_{L} A_2] \\
\times \bar{\eta}_2 [\bar{\eta}_{L-1} A_0 + z \eta_{L-1} (A_1 + A_2)] \\
x \ldots \\
x z \eta_{2k-1} \bar{\eta}_{L+2-2k} A_1 \\
x \bar{\eta}_{2k} [\bar{\eta}_{L+1-2k} A_0 + z \eta_{L+1-2k} (A_1 + A_2)] \\
x \ldots \\
x (\bar{\eta}_{L-1} \bar{\eta}_{A_0} + z \eta_{L-1} \bar{\eta}_{A_1} + z \bar{\eta}_{L-1} \eta_{A_2}) \\
\times \bar{\eta}_{L+1} [\bar{\eta}_{L+1} A_0 + z \eta_{L+1} (A_1 + A_2)] \}
\]

\[
= \frac{1}{Y^*_L} \sum_{\eta} \sum_{\eta_{L+2-2k}} \text{Tr} \{ D A^{2k-2} z \eta_{2k-1} \bar{\eta}_{L+2-2k} A_1 \bar{\eta}_{L+2-2k}^{-1} \}
\]

\[
= zp \frac{Y^*_L^{-1}}{Y^*_L}.
\]

For any nonnegative integer \( K \) the normalization ratio is obtained from (2.18) as

\[
\frac{Y_K}{Y^*_L} = \begin{cases} 
[p (1 + z)]^{K - \frac{L}{2}} \frac{1 - \frac{z}{L} (\frac{2(1 + z)}{z_c(1+z)})^K}{1 - \frac{z}{L} (\frac{2(1 + z)}{z_c(1+z)})^L} & z \neq z_c \\
\frac{z_c^{K - \frac{L}{2}} (1 + (1-p)L) K}{1 + (1-p)\frac{L}{2}} & z = z_c
\end{cases}
\]

With the effective length

\[
L_{\text{eff}} := L + 2(1 - p)^{-1}
\]

one gets the exact result

\[
\dot{j}_L(p, z) = \begin{cases} 
\frac{z}{1 + z} \frac{1 - \frac{z}{L} (\frac{2(1 + z)}{z_c(1+z)})^{L/2-1}}{1 - \frac{z}{L} (\frac{2(1 + z)}{z_c(1+z)})^{L/2}} & z \neq z_c \\
p \left(1 - \frac{2}{L_{\text{eff}}} \right) & z = z_c
\end{cases}
\]
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Taking the limit $L \to \infty$ yields (3.25).

**Remark 3.5** The current $j(p, z)$ as function of the particle density is given by

$$j(p, \rho) := j(p, z(\rho)) = \begin{cases} 2\rho & \text{if } \rho < \rho_c \\ p & \text{if } \rho \geq \rho_c \end{cases}$$

which was observed already in [3] for the canonical ensemble.

4. Sublattice density profiles

The density profile

$$\rho_L(k) := \langle \eta_k \rangle_L, \quad k \in \mathbb{T}_L$$

for the odd and even sublattices was computed in [3] for the canonical ensemble. Here we consider the grandcanonical case and provide a full discussion of the limit $L \to \infty$. Guided by the canonical results, we introduce to this end also the shifted lattice defined by the set $\tilde{T}_L = \{-L/2 + 1, \ldots, L/2\}$ of shifted lattice sites and occupation variables for nonpositive $k \in \{-L/2 + 1, \ldots, 0\}$ by $\eta_k = \eta_{k+L}$. We recall that $L$ is an integer multiple of 4 so that $L/2$ is even.

4.1. Synopsis

By equivalence of ensembles one expects in the free flow phase below the critical density similar results for the canonical and the grandcanonical measure (2.13) when taking the thermodynamic limit. Indeed, as shown below, for both ensembles one has an essentially flat density profile except for a boundary layer to the left of the blockage inside sector 2 whose width is proportional to a constant $\xi$, i.e., does not grow as $L \to \infty$. In the domain wall picture of the density profile, the probability of finding the domain wall away from the blockage decays exponentially with parameter $\xi$. Therefore we call $\xi$ the localization length.

The behaviour in the grandcanonical ensemble at and above the critical point is different from the canonical case and clarified below. The linear density profile that we obtain at the critical point indicates that the domain wall position is uniformly distributed over the whole second lattice sector $\mathbb{T}_{L,2}$. In contrast, in the canonical ensemble the domain wall is confined to a region of size $\sqrt{L}$ to the left of the blockage inside sector 2.

In the phase separated state above the critical point, the grandcanonical density profile has essentially two regions of homogeneous density, viz., of low density in sector 1 and of high density in sector 2, separated by a boundary layer to the right of the lattice center in sector 2. On the other hand, in the canonical ensemble the domain wall fluctuates inside the lattice segment 2 around a mean position that is determined by the fixed excess density $\rho - \rho_c$. The fluctuations remain confined to a region of size $\sqrt{L}$ around the mean position.
4.2. Exact results

To state and further discuss these results in precise form we recall the definitions of the floor function

$$|x| = \max \{n \in \mathbb{Z} \mid n \leq x\}, \quad x \in \mathbb{R}$$ \hspace{1cm} (4.2)

and the Heaviside indicator function and its complement

$$\Theta_x = \begin{cases} 
1 & \text{for } x > 0 \\
0 & \text{for } x \leq 0
\end{cases}, \quad \bar{\Theta}_x := 1 - \Theta_x, \quad x \in \mathbb{R}$$ \hspace{1cm} (4.3)

where for $n \in \mathbb{Z}$ one has $\bar{\Theta}_n = \delta_{n,0} + \Theta_{-n}$. We also introduce for $n \in \mathbb{Z}$ the sublattice indicator function

$$Q_n^\pm := \frac{1}{2} (1 \pm (-1)^n).$$ \hspace{1cm} (4.4)

and the open intervals

$$I_1 := (0, \frac{1}{2}), \quad I_2 = (\frac{1}{2}, 1)$$ \hspace{1cm} (4.5)

that are continuum analogs of the lattice sector sets $T_{L,\alpha}$.

To avoid heavy notation, the dependence of most functions on the parameters $p$ and $z$ will be suppressed.

4.2.1. Asymptotic sublattice density profiles on lattice scale

To probe the bulk density we fix a reference position deep inside each lattice sector and study the density profile around this bulk position in the thermodynamic limit.

**Theorem 4.1 (Offcritical bulk density)** Let $z \neq z_c$ be offcritical. The bulk density profile in sector $\alpha$ defined by

$$\rho^{\text{bulk}}_{\alpha}(k) := \lim_{L \to \infty} \langle \eta_{2(\lfloor Lu/2 \rfloor + k)} \rangle_L, \quad \text{for } u \in I_\alpha, k \in \mathbb{Z}$$ \hspace{1cm} (4.6)

depends only on the sublattice and is given by

$$\rho^{\text{bulk}}_1(k) = jQ^-, \quad \rho^{\text{bulk}}_2(k) = \rho^{\text{bulk}}_1(k) + \begin{cases} 
0 & \text{if } \frac{z}{z_c} < \frac{1}{2} \\
\frac{j}{z_c} & \text{if } \frac{z}{z_c} > \frac{1}{2}
\end{cases}$$ \hspace{1cm} (4.7)

with the stationary current $j$ \hspace{1cm} (3.25).

To see the boundary layers announced in the synopsis, the density profile in the thermodynamic limit $L \to \infty$ needs to be studied on lattice scale. This analysis has to be done separately around the blockage bond $(L, 1)$ on the one hand and around the central bond $(L/2, L/2 + 1)$ on the other hand.

**Theorem 4.2 (Boundary layer profiles)** Let $z \neq z_c$. In terms of the localization length

$$\xi := |\ln \frac{z_c(1 + z)}{z(1 + z_c)}|^{-1}$$ \hspace{1cm} (4.8)
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\[
\sigma(k) := \frac{j}{z_c} e^{-\frac{|k|}{\xi}}
\]

(4.9)

\[
\tilde{\sigma}(k) := \frac{j}{z_c} \frac{1 + z_c}{1 + z} e^{-\frac{|k|}{\xi}}
\]

(4.10)

with the critical fugacity \(z_c\) (2.16) and the current \(j\) (3.25), the density profile as seen from the blockage bond defined in the thermodynamic limit by

\[
\rho_\infty(k) := \begin{cases} 
\lim_{L \to \infty} \rho_L(k) & k > 0 \\
\lim_{L \to \infty} \rho_L(L - |k|) & k \leq 0 
\end{cases}
\]

(4.11)

and the central density profile

\[
\tilde{\rho}_\infty(k) := \lim_{L \to \infty} \rho_L(L/2 + k)
\]

(4.12)

as seen in the thermodynamic limit from the central bond are given by

\[
\rho_\infty(k) = \begin{cases} 
\rho_1^{\text{bulk}}(k) + \sigma(k)\Theta_k & z < z_c \\
\rho_1^{\text{bulk}}(k)\Theta_k + \rho_2^{\text{bulk}}(k)\tilde{\Theta}_k & z > z_c 
\end{cases}
\]

(4.13)

\[
\tilde{\rho}_\infty(k) = \begin{cases} 
\rho_1^{\text{bulk}}(k) & z < z_c \\
\rho_1^{\text{bulk}}(k)\tilde{\Theta}_k + \rho_2^{\text{bulk}}(k)\Theta_k - \tilde{\sigma}(k) & z > z_c 
\end{cases}
\]

(4.14)

with the bulk density functions \(\rho_\alpha^{\text{bulk}}(k)\) (4.7) for all \(k \in \mathbb{Z}\).

We point out that seen from the blockage bond, nonpositive values \(k\) correspond to sector 2 of the lattice whereas seen from the central bond, nonpositive values \(k\) correspond to sector 1 of the lattice. Theorems 4.1 and 4.2 thus essentially assert that below the critical point the density profile is homogeneous with sublattice-dependent (but otherwise constant) amplitude \(\rho_1^{\text{bulk}}(k) = jQ_k^+\), except for a boundary layer (4.9) to the left of the blockage inside sector 2, while above the critical point the density “jumps” on each sublattice at the central bond and at the blockage bond between \(\rho_1^{\text{bulk}}(k)\) and \(\rho_2^{\text{bulk}}(k)\) = 1 − \(p + \rho_1^{\text{bulk}}(k)\), except for a central boundary layer (4.10) inside sector 2 that interpolates to the right of the central bond between \(\rho_1^{\text{bulk}}(k)\) and \(\rho_2^{\text{bulk}}(k)\).

4.2.2. Critical density profile As one approaches the critical point, the localization length diverges and the notion of boundary layer looses its meaning. To explore the density profile at the critical point we employ a hydrodynamic scaling \(L \to \infty\) with lattice sites seen from the center and taken as \(k = \lfloor uL \rfloor\) with constant \(u \in (-1/2, 1/2]\). Here \(u\) has the meaning of a macroscopic position \(u\) on a circle of unit length \(\ell = 1\) with the blockage at \(u = 1/2\). Due to the finite sublattice alternation of the local density coming from the term \(jQ_k^-\) in the microscopic density profiles, this limit has to be taken separately for each sublattice. This is achieved by the sublattice decomposition

\[
\rho_L(k) = \rho_L^+(k/2)Q_k^+ + \rho_L^-(\lfloor k/2 \rfloor + 1)Q_k^-
\]

(4.15)
of the density profile with the sublattice density profiles
\[ \rho^+_L(k) := \rho_L(2k), \quad \rho^-_L(k) := \rho_L(2k - 1), \quad 1 \leq k \leq L/2. \] (4.16)

Analogously, and for reference, we define the supercritical sublattice bulk densities
\[ \rho_{\text{bulk},1}^{-} := p, \quad \rho_{\text{bulk},1}^{+} := 0 \] (4.17)
\[ \rho_{\text{bulk},2}^{-} := 1, \quad \rho_{\text{bulk},2}^{+} := 1 - p \] (4.18)

obtained from (4.7) for \( z > z_c \).

**Theorem 4.3 (Macroscopic density profile)** For \( u \in (-1/2, 1/2] \) the macroscopic sublattice density profiles under hydrodynamic scaling
\[ \tilde{\rho}^\pm(u) := \lim_{L \to \infty} \rho^\pm_L(L/4 + \lfloor uL/2 \rfloor), \] (4.19)
are piece-wise linear and given by
\[ \tilde{\rho}^+(u) = \frac{j}{z_c} \Theta(u) \times \begin{cases} 0 & z < z_c \\ 2u & z = z_c \\ 1 & z > z_c \end{cases} \] (4.20)

\[ \tilde{\rho}^-(u) = \tilde{\rho}_c^+(u) + j \] (4.21)
with the macroscopic current \( j \) of Theorem 3.4.

One sees that at the critical point the boundary layer becomes “infinately” wide in the sense that inside sector 2 it interpolates smoothly on each sublattice between the supercritical bulk densities \( \rho_{\text{bulk},1}^{\pm} \) (everywhere inside sector 1) and \( \rho_{\text{bulk},2}^{\pm} \) (attained only as one reaches the blockage) that characterize the phase-separated state.

### 4.3. Proofs

All results follow from exact computation of the density profile for the finite lattice using the SMPM (2.13) and then taking the limit \( L \to \infty \) as defined in each theorem.

**Proof:** We define for \( n \in \{0, \ldots, L/2\} \) the functions
\[ H_L(n) := p(1 - p)z^n \frac{Y_{L/2} - n}{Y_{L/2}}, \quad H_L(n) := H_L(L/2 + 1 - n) \] (4.22)
related to the ratio of partition functions (3.27). In particular, we note that (2.24) yields
\[ \text{Tr}(DA^{n-1}A_2A_{L/2}^{1-n}) = (1 - p)z^{n-1}Y_{L/2}^{-n} \] (4.23)
and therefore
\[ \frac{z}{Y_{L/2}^{1-n}} \text{Tr}(DA^{n-1}A_2A_{L/2}^{1-n}) = \frac{1}{p}H_L(n). \] (4.24)

From Proposition 2.2 one has for sector 1
\[ \langle \eta_{2k-1} \rangle_L = pz \frac{Y_{L/2}^{L-1}}{Y_{L/2}} = j_L, \quad 1 \leq k \leq L/4 \] (4.25)
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while for sector 2 the SMPM yields

\[ \langle \eta_{L+1-2k} \rangle_L = \frac{z^2}{Y_2} \text{Tr}(DA^{2k-1}(A_1 + A_2)A_{L-2k}) \]

\[ = j_L + \frac{1}{p}H_L(2k), \quad 1 \leq k \leq L/4. \] (4.26)

In the last equality (4.24) was used.

For even sites one has from Proposition 2.2 for sector 1

\[ \langle \eta_{2k} \rangle_L = 0, \quad 1 \leq k \leq L/4 \] (4.27)

and for sector 2 one gets from the SMPM

\[ \langle \eta_{L+2-2k} \rangle_L = \frac{z^2}{Y_2} \text{Tr}(DA^{2k-2}A_2A_{L-2k+1}) \]

\[ = \frac{1}{p}H_L(2k-1), \quad 1 \leq k \leq L/4. \] (4.28)

The results (4.25) - (4.28) can be written compactly as

\[ \rho_L(n) = j_LQ_n + \frac{1}{p}H_L(L + 1 - n)\Theta_{L,n}^{(2)}, \quad n \in \mathbb{T}_L \] (4.29)

which yields

\[ \rho_L(L-n) = j_LQ_n + \frac{1}{p}H_L(n+1)\Theta_{L,n+1}^{(1)}, \quad 0 \leq n \leq L-1 \] (4.30)

\[ \rho_L(L/2+n) = j_LQ_n + \frac{1}{p}\tilde{H}_L(n)\Theta_{L,n}^{(1)}, \quad -L/2 + 1 \leq n \leq L/2 \] (4.31)

\[ \rho_L(L/4+n) = \frac{1}{p}\tilde{H}_L(2n-1)\Theta_{L,2n}^{(1)}, \quad -L/4 < n \leq L/4 \] (4.32)

\[ \rho_L(L/4+n) = j_L + \frac{1}{p}H_L(2n)\Theta_{L,2n-1}^{(1)}, \quad -L/4 < n \leq L/4. \] (4.33)

To take the thermodynamic limit \( L \to \infty \) as indicated in each theorem one uses

Theorem 3.4 for the current, the property of the floor function \( \lfloor uL \rfloor = uL + R(u) \)

with \( 0 \leq R(u) < 1 \) uniformly bounded in \( L \), and the asymptotic properties of the

function \( H_L(n) \) detailed in Appendix A.2.
5. Correlations

Due to the absence of translation invariance, the density correlation function

\[ S_L(k, l) := \langle \eta_k \eta_l \rangle_L - \langle \eta_k \rangle_L \langle \eta_l \rangle_L \] (5.1)
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depends on both space coordinates $k, l$. By construction,

$$S_L(l, k) = S_L(k, l)$$

(5.2)

for all $k, l \in \mathbb{T}_L$.

We define the dynamical structure function as the space average

$$S_L(r) := \frac{1}{L} \sum_{k=1}^{L} S_L(k, (k + r) \mod L)$$

(5.3)

of the two-point density correlation function (5.1). From the symmetry (5.2) one deduces that $S_L(r) = S_L(-r)$. To avoid heavy notation we omit the dependence on $(p, z)$ of the functions appearing in this section.

5.1. Density correlation function

For the canonical ensemble some properties of the density correlation function (5.1) were computed in [3] with emphasis on the behaviour near the blockage. It was found that below the critical point the amplitude of correlations decays exponentially with parameter $\xi$ with increasing distance from the blockage while at the critical point there are long-range correlations that extend over a region proportional to $\sqrt{L}$. Here we provide a full discussion in the grandcanonical ensemble defined by the SMPM (2.13).

5.1.1. Synopsis

The main results concern the critical point and the phase separated regime: (i) For $z > z_c$ we identify short-range correlations near the center of the lattice that arise from the presence of the central boundary layer (4.10). (ii) For $z \geq z_c$ we find a long-range anticorrelation between site $k$ and its reflected site $L + 1 - k$. (iii) At the critical point the system is shown to exhibit further long-range correlations with amplitude of order 1, extending over the whole sector 2. These correlations are indicative of a fluctuating microscopic shock as typical stationary configuration of the dsTASEP in the grandcanonical ensemble, with the domain wall position uniformly distributed over sector 2. These correlations are in contrast to those found for the canonical ensemble which extend only over a region of order $\sqrt{L}$ inside sector 2 and they also differ from those observed in the deterministic sublattice TASEP with open boundaries [33, 34, 35] where the long-range correlation at criticality extends over the whole lattice and where the reflective contribution to the correlations is absent.

5.1.2. Preparatory remarks and definitions

On the diagonal $k = l$ in the $(k, l)$-plane the correlation function trivially has a non-vanishing term

$$S_{hc}^L(k, l) := A_{hc}^L(k) \delta_{k,l}, \quad A_{hc}^L(k) = \langle \eta_k \rangle_L - \langle \eta_k \rangle_L^2$$

(5.4)

‡ For translation invariant lattice systems this definition reduces to the usual one $S_L(r) = S_L(k, (k + r) \mod L)$ which is independent of $k$. 

due to hard core exclusion. On the other hand, the reflective projection property (2.14) induces an non-trivial anticorrelation

$$S_L^{\text{bl}}(k, l) := A^{\text{bl}}_L(k) \delta_{k-l,0} + A^{\text{bl}}_L(k) \delta_{k+l,1} + S_L^{\text{bl}}(k, l)$$

between site $k$ and the site $L+1-k$ reflected at the blockage bond $(L, 1)$, i.e., along the perpendicular diagonal $k = L+1-l$. Therefore we decompose the correlation function into the three parts

$$S_L(k, l) = A^{\text{hc}}_L(k) \delta_{k-l,0} + A^{\text{ref}}_L(k) \delta_{k+l,1} + S_L^{\text{bl}}(k, l)$$

with the off-diagonal contribution

$$S_L^{\text{bl}}(k, l) := S_L(k, l) (1 - \delta_{k-l,0} - \delta_{k+l,1})$$

that, as it will turn out, has its origin in the boundary layers. Similarly, we decompose the density correlation function in the thermodynamic limit defined for fixed values of $k, l \in \mathbb{Z}$ by

$$S_\infty(k, l) := \lim_{L \to \infty} [S_L(k, l) \Theta_k \Theta_l + S_L(k, L - |l|) \Theta_k \Theta_l]$$

$$+ S_L(L - |k|, l) \Theta_k \Theta_l + S_L(L - |k|, L - |l|) \Theta_k \Theta_l]$$

$$\tilde{S}_\infty(k, l) := \lim_{L \to \infty} S_L(L/2 + k, L/2 + l)$$

to study correlations around the blockage and around the lattice center respectively. With

$$A^{\text{hc}}_\infty(k) := \lim_{L \to \infty} A^{\text{hc}}_L(k) \Theta_k + A^{\text{hc}}_L(L - |k|) \Theta_k$$

$$A^{\text{ref}}_\infty(k) := \lim_{L \to \infty} A^{\text{ref}}_L(k) \Theta_k + A^{\text{ref}}_L(L - |k|) \Theta_k$$

$$\bar{A}^{\text{hc}}_\infty(k) := \lim_{L \to \infty} A^{\text{hc}}_L(L/2 + k)$$

$$\bar{A}^{\text{ref}}_\infty(k) := \lim_{L \to \infty} A^{\text{ref}}_L(L/2 + k)$$

$$S_L^{\text{bl}}(k, l) := \lim_{L \to \infty} [S_L^{\text{bl}}(k, l) \Theta_k \Theta_l + S_L^{\text{bl}}(k, L - |l|) \Theta_k \Theta_l]$$

$$+ S_L^{\text{bl}}(L - |k|, l) \Theta_k \Theta_l + S_L^{\text{bl}}(L - |k|, L - |l|) \Theta_k \Theta_l]$$

$$\tilde{S}_\infty^{\text{bl}}(k, l) := \lim_{L \to \infty} S_L^{\text{bl}}(L/2 + k, L/2 + l)$$

the corresponding decompositions read

$$S_\infty(k, l) = A^{\text{hc}}_\infty(k) \delta_{k-l,0} + A^{\text{ref}}_\infty(k) \delta_{k+l,1} + S_\infty^{\text{bl}}(k, l)$$

$$\tilde{S}_\infty(k, l) = \bar{A}^{\text{hc}}_\infty(k) \delta_{k-l,0} + \bar{A}^{\text{ref}}_\infty(k) \delta_{k+l,1} + \tilde{S}_\infty^{\text{bl}}(k, l)$$

for $k, l \in \mathbb{Z}$. We point out the symmetries

$$A^{\text{ref}}_L(k) = A^{\text{ref}}_L(L + 1 - k), \quad A^{\text{ref}}_\infty(k) = A^{\text{ref}}_\infty(1 - k)$$

$$\bar{A}^{\text{ref}}_L(k) = \bar{A}^{\text{ref}}_L(1 - k), \quad \bar{A}^{\text{ref}}_\infty(k) = \bar{A}^{\text{ref}}_\infty(1 - k)$$

that follow from the definitions of these quantities.
We also introduce the constant
\[
\kappa_L := j_L(1 - j_{L-2}) \tag{5.20}
\]
\[
= \begin{cases} 
  z \frac{1 - \frac{z}{1+z}e^{-L/(2\xi_s)}}{(1+z)^2} & \text{if } z \neq z_c \\
  p(1 - p) - \frac{2p(1 - 2p)}{L_{\text{eff}}} & \text{if } z = z_c
\end{cases} \tag{5.21}
\]
which has the limiting behaviour
\[
\kappa := \lim_{L \to \infty} \kappa_L = \begin{cases} 
  \frac{z}{1+z} & \text{if } z < z_c \\
  p(1 - p) & \text{if } z \geq z_c
\end{cases} \tag{5.22}
\]
Away from the critical point, finite-size corrections to the asymptotic result are exponentially small in \(L\). For \(z < z_c\) one has \(\kappa = 2C\) with the subcritical compressibility \(C\) established in Theorem 3.2.

### 5.1.3. Main results

As a reference, we begin with the offcritical bulk correlations. To this end, we fix inside the bulk of the sectors an arbitrary reference pair of lattice points \((m, n) = (2\lfloor Lu/2 \rfloor, 2\lfloor Lv/2 \rfloor)\) and study the correlations in the thermodynamic limit at an arbitrary but finite distance around these points.

**Theorem 5.1** For \(z \neq z_c\) and fixed \(k, l \in \mathbb{Z}\) the bulk correlations
\[
S_{\alpha\beta}^{\text{bulk}}(k, l) := \lim_{L \to \infty} S_L(2\lfloor Lu/2 \rfloor + k, 2\lfloor Lv/2 \rfloor + l) \quad u \in \mathbb{I}_\alpha, v \in \mathbb{I}_\beta \tag{5.23}
\]
are given by
\[
S_{\alpha\beta}^{\text{bulk}}(k, l) = \begin{cases} 
  A^\text{hc}_\alpha(k)\delta_{k,l} & \alpha = \beta, \ u = v \\
  A^\text{refl}_\alpha(k)\delta_{k+l,1} & \alpha \neq \beta, \ u = 1 - v \\
  0 & \text{else}
\end{cases} \tag{5.24}
\]
with the sector-dependent bulk amplitudes
\[
A^\text{hc}_1(k) = \kappa Q^-_k \quad z \neq z_c, \quad A^\text{hc}_2(k) = \kappa \begin{cases} 
  Q^-_k & z < z_c \\
  Q^+_k & z > z_c
\end{cases} \tag{5.25}
\]
\[
A^\text{refl}_1(k) = -\kappa \begin{cases} 
  0 & z < z_c \\
  Q^-_k & z \geq z_c
\end{cases}, \quad A^\text{refl}_2(k) = -\kappa \begin{cases} 
  0 & z < z_c \\
  Q^+_k & z > z_c
\end{cases} \tag{5.26}
\]
proportional to \(\kappa\) as given in (5.22).

**Remark 5.2** In the free-flow phase below the critical point the bulk correlations reduce to the hard-core onsite correlations with amplitude \(\kappa = 2C\) proportional to the compressibility (3.4). In the phase separated state above the critical point, the theorem asserts that in addition to the hard-core contribution there are bulk anticorrelations with negative amplitude proportional to \(\kappa = p(1 - p)\). These correlations are long ranged since the correlated occupation numbers \(\eta_m\) and \(\eta_n\) at the lattice points \(m = 2\lfloor Lu/2 \rfloor + k\) and \(n = 2\lfloor L(1 - u)/2 \rfloor + 1 - k\) have a nonzero macroscopic distance \(r = |1 - 2u|\) in the thermodynamic limit.
Next we investigate the offcritical correlations arising from the existence of the boundary layers. We recall that seen from the blockage (center), sector 2 (sector 1) of the finite lattice corresponds to negative lattice points in the thermodynamic limit.

**Theorem 5.3 (Offcritical correlations near the blockage)** Seen from the blockage, the off-critical density correlation function has the hard-core part

\[
A_{\infty}^{hc}(k) = \begin{cases} 
\kappa Q_k^- + \frac{\kappa}{z_c} e^{-|k|/\xi} \left( 1 + (-1)^k z - \frac{z}{z_c} e^{-|k|/\xi} \right) \Theta_k & z < z_c \\
\kappa \left[ Q_k^- \Theta_k + Q_k^+ \Theta_k \right] & z > z_c,
\end{cases}
\]  

(5.27)

the reflective anticorrelations

\[
A_{\infty}^{refl}(k) = \begin{cases} 
-\frac{z_c}{(1 + z_c)^2} e^{-\left(-\frac{|k-\frac{1}{2}|}{\xi}+\frac{1}{2}\right) / \xi} \left( Q_k^- \Theta_k + Q_k^+ \Theta_k \right) & z < z_c \\
-\kappa \left[ Q_k^- \Theta_k + Q_k^+ \Theta_k \right] & z > z_c,
\end{cases}
\]  

(5.28)

and the boundary layer correlations

\[
S_{\infty}^{bl}(k, l) = \begin{cases} 
\frac{z_c - z}{1 + z_c z_c} \Theta_{k+l} \left[ e^{-|k|/\xi} Q_l^- \Theta_l + e^{-|l|/\xi} Q_k^- \Theta_k \Theta_l \right] \\
+ \frac{\kappa}{z_c} e^{-|k|/\xi} \left[ \left( 1 - \frac{z}{z_c} e^{-|k|/\xi} \right) Q_l^- \Theta_{l-k} \Theta_l \right] \\
+ \left( 1 + z \right) \left( \frac{1}{1 + z_c} - \frac{z}{z_c} e^{-|l|/\xi} \right) Q_l^- \Theta_{l-k} \Theta_l \\
+ \frac{\kappa}{z_c} e^{-|l|/\xi} \left[ \left( 1 - \frac{z}{z_c} e^{-|l|/\xi} \right) Q_k^- \Theta_{k-l} \Theta_l \right] \\
0 & z < z_c \\
\end{cases}
\]  

(5.29)

**Remark 5.4** Above the critical point, the correlations are equal to the bulk values of Theorem [5.7] for all \( k, l \in \mathbb{Z} \). Below the critical point, they come arbitrarily close to the bulk values as \( \max \{ k, l \} \) becomes large compared to the localization length \( \xi \).

**Theorem 5.5 (Offcritical correlations near the center)** Seen from the center, the off-critical density correlation function has the hard core part

\[
\tilde{A}_{\infty}^{hc}(k) = \begin{cases} 
\kappa Q_k^- & z < z_c \\
\kappa \left[ Q_k^+ \Theta_k + Q_k^- \Theta_k \right] & z > z_c,
\end{cases}
\]  

(5.30)

the reflective contribution

\[
\tilde{A}_{\infty}^{refl}(k) = \begin{cases} 
0 & z < z_c \\
-\kappa \left( 1 - \frac{1 + z_c z}{1 + z} e^{-\left(-\frac{|k-\frac{1}{2}|}{\xi}+\frac{1}{2}\right) / \xi} \right) \left[ Q_k^+ \Theta_k + Q_k^- \Theta_k \right] & z > z_c,
\end{cases}
\]  

(5.31)
and the boundary layer part

$$S^{bl}_\infty(k,l) = \begin{cases} 
0 & z < z_c \\
-\frac{z - z_c \kappa}{1 + z_c z} \Theta_{k+1} \left( e^{-l/\xi} Q^-_k \Theta_l + e^{-k/\xi} Q^-_l \Theta_k \right) \\
+ \frac{\kappa}{z} e^{-l/\xi} \left[ \left( \frac{1 + z}{1 + z_c} - e^{-k/\xi} \right) Q^+_k \right] \\
+ \left( 1 - e^{-k/\xi} \right) Q^-_k \Theta_{l-k} \Theta_l \\
+ \left( 1 - e^{-l/\xi} \right) Q^-_l \Theta_{k-l} \Theta_l \\
+ \left( 1 - e^{-l/\xi} \right) Q^-_l \Theta_{k-l} \Theta_l & z > z_c.
\end{cases}$$

(5.32)

**Remark 5.6** The picture is reverted compared to the behaviour near the blockage: Correlations are equal to the bulk values of Theorem 5.1 below the critical point and come arbitrarily close to these bulk values above the critical point as \( \max \{ k, l \} \) becomes large compared to the localization length \( \xi \). The boundary layer contribution to the correlation function is restricted to sector 2 due to particle number conservation and the choice \( \rho \leq 1/2 \).

To get insight about the critical point we consider hydrodynamic scaling for the sublattice correlations defined for \( 1 \leq m, n \leq L/2 \) by

$$S^{++}_L(m,n) := S_L(2m,2n), \quad S^{+-}_L(m,n) := S_L(2m,2n-1)$$

(5.33)

$$S^{-+}_L(m,n) := S_L(2m-1,2n), \quad S^{-}\quad_L(m,n) := S_L(2m-1,2n-1)$$

(5.34)

This yields the sublattice decomposition

$$S_L(k,l) = S^{++}_L(k/2,l/2)Q^+_k Q^+_l + S^{-+}_L([k/2] + 1, [l/2] + 1)Q^-_k Q^-_l$$

$$+ S^{--}_L(k/2, [l/2] + 1)Q^-_k Q^-_l + S^{-+}_L([k/2] + 1, [l/2] + 1)Q^-_k Q^-_l$$

(5.35)

of the density correlation function.

**Theorem 5.7 (Correlations on macroscopic scale)** Let \( u, v \in (-1/2, 1/2) \)

$$\tilde{A}^{hc,\pm}(u) := \lim_{L \to \infty} A^{hc,\pm}_L(L/4 + [uL/2])$$

(5.36)

$$\tilde{A}^{refl,\pm}(u) := \lim_{L \to \infty} A^{refl,\pm}_L(L/4 + [uL/2])$$

(5.37)

$$\tilde{S}^{bl,\pm}(u,v) := \lim_{L \to \infty} \tilde{S}^{bl,\pm}_L(L/4 + [uL/2], L/4 + [vL/2])$$

(5.38)

be the centered hardcore, reflective, and boundary layer contributions to the sublattice density correlation function under hydrodynamic scaling. With the scaling functions

$$\tilde{a}^{hc}(u) := \frac{2u}{1 + z_c} \left( 1 - \frac{2u}{1 + z_c} \right)$$

(5.39)

$$\tilde{a}^{refl}(u) := 2|u|$$

(5.40)

$$\tilde{s}^{bl}(u,v) := 2u (1 - 2v)$$

(5.41)
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the non-vanishing contributions are the hardcore correlations

\[
A^{hc,-}(u) = \begin{cases} 
\kappa & z < z_c \\
\kappa - \tilde{a}^{hc}(u)\Theta(u) & z = z_c \\
\kappa\Theta(u) & z > z_c 
\end{cases}
\]

(5.42)

\[
A^{hc,+}(u) = \begin{cases} 
0 & z < z_c \\
\tilde{a}^{hc}(u)\Theta(u) & z = z_c \\
\kappa\Theta(u) & z > z_c 
\end{cases}
\]

(5.43)

the long-range reflective anticorrelations

\[
A^{refl,-}(u) = \begin{cases} 
0 & z < z_c \\
-\kappa\tilde{a}^{refl}(u)\Theta(u) & z = z_c \\
-\kappa\Theta(u) & z > z_c 
\end{cases}
\]

(5.44)

\[
A^{refl,+}(u) = \begin{cases} 
0 & z < z_c \\
-\kappa\tilde{a}^{refl}(u)\Theta(u) & z = z_c \\
-\kappa\Theta(u) & z > z_c 
\end{cases}
\]

(5.45)

and the boundary layer correlations

\[
S^{bl\pm\pm}(u, v) = \begin{cases} 
0 & z < z_c \\
\frac{\kappa}{z_c} s^{bl}(u, v)\Theta(v - u)\Theta(u)\Theta(v) & z = z_c \\
+\frac{\kappa}{z_c} s^{bl}(v, u)\Theta(u - v)\Theta(u)\Theta(v) & z > z_c 
\end{cases}
\]

(5.46)

independently of the sublattices.

Remark 5.8 Since the width \(\xi\) of the boundary layer diverges as one approaches the critical point, correlations extend over the full lattice sector 2. As worked out already in Theorem 5.1, the boundary layer contribution vanishes away from the critical point on hydrodynamic scale since the macroscopic width \(\lim_{L \to \infty} \xi/L\) of the boundary layer is zero for \(z \neq z_c\).

5.1.4. Exact finite-size density correlation function The proofs of all four theorems 5.1 - 5.7 are based on taking appropriate limits \(L \to \infty\) of the exact finite-size expression of the density correlation function established in Proposition 5.9 below.

To express the functional dependence of finite-size density correlation function on the parameters \(p, z, L, k, l\) we introduce as the auxiliary constants

\[
\Gamma_L := j_L(j_{L-2} - j_L) \quad (5.47)
\]

\[
\Delta_L := 1 - \frac{j_L}{p} \quad (5.48)
\]
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written out in explicit form in (A.21), (A.22), and the auxiliary functions

\[
\Psi_L(m) := H_L(m + 1) - H_L(m) \tag{5.49}
\]

\[
\tilde{\Psi}_L(m) := \tilde{H}_L(m - 1) - \tilde{H}_L(m) \tag{5.50}
\]

\[
F_L(m, n) := \frac{H_L(m)}{p} \left[ 1 - p - \frac{H_L(n)}{p} \right] \tag{5.51}
\]

\[
\tilde{F}_L(m, n) := \frac{\tilde{H}_L(m)}{p} \left[ 1 - p - \frac{\tilde{H}_L(n)}{p} \right] \tag{5.52}
\]

given in explicit form in (A.25), (A.33). The SMPM then yields the following exact expressions.

**Proposition 5.9** For \( m, n \in \{1, \ldots, L/2\} \) the density correlation functions for any system size \( L = 4K, K \in \mathbb{N} \) is given by

\[
S_L(m, n) = \Gamma_L Q_m^- Q_n^- + \kappa_L Q_m^- \delta_{m,n}
\]

\[
S_L(m, L + 1 - n) = \Gamma_L Q_m^- Q_n^+ - H_L(m) Q_m^- \delta_{m,n} + \Delta_L H_L(n) Q_m^- + \Psi_L(n) Q_m^- \Theta_{m-n}
\]

\[
S_L(L + 1 - m, n) = \Gamma_L Q_n^- Q_m^+ - H_L(m) Q_m^- \delta_{m,n} + \Delta_L H_L(m) Q_n^- + \Psi_L(m) Q_n^- \Theta_{n-m}
\]

\[
S_L(L + 1 - m, L + 1 - n) = \Gamma_L Q_m^+ Q_n^+ + \kappa_L Q_m^+ \delta_{m,n} + F_L(m, m) \delta_{m,n} + H_L(m) (Q_m^- - Q_m^+) \delta_{m,n} + F_L(m, n) \Theta_{m-n} + F_L(n, m) \Theta_{n-m} + \Psi_L(n) Q_m^+ \Theta_{m-n} + \Psi_L(m) Q_n^+ \Theta_{n-m} + H_L(n) Q_m^+ \Delta_L + \Delta_L H_L(m) Q_n^+.
\]

**Proof:** We prove the proposition with a case-by-case computation of the sublattice correlation functions using the SMPM (2.13) and the properties (2.19) - (2.26) of the matrix algebra. In particular, we note that from (2.21) one gets

\[
\frac{p^z}{Y_L^2} \text{Tr}(D^n A_{2^{L/2} - n}) = H_L(n), \quad 1 \leq n \leq \frac{L}{2}, \tag{5.54}
\]

\[
\frac{z^2}{Y_L^2} \text{Tr}(D^{n-1} A^m A^{\frac{L}{4} - m}) = \frac{H_L(m)}{z_c}, \quad 1 \leq n < m \leq \frac{L}{2}. \tag{5.55}
\]

We also note that for \( m, n \in \mathbb{Z} \)

\[
\delta_{m,|n|} = \delta_{m,n} \Theta_n + \delta_{m,-n} \bar{\Theta}_n, \tag{5.56}
\]

\[
|m - 1/2| + 1/2 = m \Theta_m + (|m| + 1) \bar{\Theta}_m. \tag{5.57}
\]

In the following sublattice computations we assume throughout \( k, l \in \{1, \ldots, L/4\} \).
Odd-even correlations: For the joint expectations one finds

\[
\langle \eta_{2k-1}\eta_{2l-1} \rangle_L = \begin{cases} 
    j_{L-2}j_L & k \neq l \\
    j_L & k = l
\end{cases}
\]

\[
\langle \eta_{2k-1}\eta_{L+1-2l} \rangle_L = \begin{cases} 
    j_{L-2}j_L + H_L(2l) & k \leq l \\
    j_{L-2}j_L + H_L(2l+1) & k \geq l + 1
\end{cases}
\]

\[
\langle \eta_{L+1-2k}\eta_{2l-1} \rangle_L = \begin{cases} 
    j_{L-2}j_L + H_L(2k) & l \leq k \\
    j_{L-2}j_L + H_L(2k+1) & l \geq k + 1
\end{cases}
\]

\[
\langle \eta_{L+1-2k}\eta_{L+1-2l} \rangle_L = \begin{cases} 
    j_{L-2}j_L + \frac{1}{p}H_L(2l) + H_L(2k+1) & k < l \\
    j_L + \frac{1}{p}H_L(2l) & k = l \\
    j_{L-2}j_L + \frac{1}{p}H_L(2k) + H_L(2l+1) & k > l
\end{cases}
\]

With the exact expression (4.29) for the density profile it follows that

\[
S_L(2k-1, 2l-1) = \begin{cases} 
    \Gamma_L & k \neq l \\
    \Gamma_L + \kappa_L & k = l
\end{cases}
\]

\[
S_L(2k-1, L+1-2l) = \begin{cases} 
    \Gamma_L + \Delta_L H_L(2l) & k \leq l \\
    \Gamma_L + \Delta_L H_L(2l) + \Psi_L(2l) & k > l
\end{cases}
\]

\[
S_L(L+1-2k, 2l-1) = \begin{cases} 
    \Gamma_L + \Delta_L H_L(2k) + \Psi_L(2k) & k < l \\
    \Gamma_L + \Delta_L H_L(2k) & k \geq l
\end{cases}
\]

\[
S_L(L+1-2k, L+1-2l) = \begin{cases} 
    \Gamma_L + F_L(2l, 2k) + \Psi_L(2k) + \Delta_L H_L(2k) + \Delta_L H_L(2l) & k < l \\
    \Gamma_L + \kappa_L + F_L(2l, 2k) + (2\Delta_L - 1) H_L(2l) & k = l \\
    \Gamma_L + F_L(2k, 2l) + \Psi_L(2l) + \Delta_L H_L(2k) + \Delta_L H_L(2l) & k > l
\end{cases}
\]

Odd-even correlations: From (2.27) in Proposition 2.7 one obtains

\[
\langle \eta_{L+1-2k}\eta_{L+2-2l} \rangle_L = \langle \eta_{L+2-2l} \rangle_L.
\]

With this and (4.28) the SMPM yields

\[
\langle \eta_{2k-1}\eta_{2l} \rangle_L = 0
\]

\[
\langle \eta_{2k-1}\eta_{L+2-2l} \rangle_L = \begin{cases} 
    H_L(2l-1) & k < l \\
    0 & k = l \\
    H_L(2l) & k > l
\end{cases}
\]
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\[
\langle \eta_{L+1-2k}\eta_{2l} \rangle_L = 0
\]

\[
\langle \eta_{L+1-2k}\eta_{L+2-2l} \rangle_L = \begin{cases} 
\frac{1}{p} H_L(2l - 1) & k < l \\
\frac{H_L(2k)}{z_c} H_L(2l) & k \geq l
\end{cases}
\]

It follows that

\[
S_L(2k - 1, 2l) = 0
\]

\[
S_L(2k - 1, L + 2 - 2l) = \begin{cases} 
\Delta_L H_L(2l - 1) & k < l \\
(\Delta_L - 1) H_L(2l - 1) & k = l \\
\Delta_L H_L(2l - 1) + \Psi_L(2l - 1) & k > l
\end{cases}
\]

\[
S_L(L + 1 - 2k, 2l) = 0
\]

\[
S_L(L + 1 - 2k, L + 2 - 2l) = \begin{cases} 
F_L(2l - 1, 2k) & k < l \\
\Delta_L H_L(2l - 1) + \Psi_L(2l - 1) & k = l \\
F_L(2k, 2l - 1) + \Delta_L H_L(2l - 1) & k > l
\end{cases}
\]

Similarly, by the symmetry (5.2) one has

\[
S_L(2k, 2l - 1) = 0
\]

\[
S_L(2k, L + 1 - 2l) = 0
\]

\[
S_L(L + 2 - 2k, 2l - 1) = \begin{cases} 
\Delta_L H_L(2k - 1) + \Psi_L(2k - 1) & k < l \\
(\Delta_L - 1) H_L(2k - 1) & k = l \\
\Delta_L H_L(2k - 1) & k > l
\end{cases}
\]

\[
S_L(L + 2 - 2k, L + 1 - 2l) = \begin{cases} 
F_L(2l, 2k - 1) + \Psi_L(2k - 1) & k \leq l \\
\Delta_L H_L(2k - 1) & k > l
\end{cases}
\]

**Even-even correlations:** From the projection property (2.15) and (5.55) one finds

\[
\langle \eta_{2k}\eta_{2l} \rangle_L = 0
\]

\[
\langle \eta_{2k}\eta_{L+2-2l} \rangle_L = 0
\]

\[
\langle \eta_{L+2-2k}\eta_{2l} \rangle_L = 0
\]

\[
\langle \eta_{L+2-2k}\eta_{L+2-2l} \rangle_L = \begin{cases} 
\frac{1}{p} H_L(2k - 1) & k < l \\
\frac{H_L(2k - 1)}{z_c} & k \geq l
\end{cases}
\]

Therefore

\[
S_L(2k, 2l) = 0
\]
From Proposition 5.9 one finds that one has for \( k, l \)
the sector indicator functions
\[
S_L(2k, L + 2 - 2l) = 0
\]
\[
S_L(L + 2 - 2k, 2l) = 0
\]
\[
S_L(L + 2 - 2k, L + 2 - 2l) = \begin{cases} 
F_L(2l - 1, 2k - 1) & k < l \\
F_L(2k - 1, 2l - 1) & k = l \\
+ H_L(2k - 1) & k > l .
\end{cases}
\]

Adding up the parts of the correlation function according to the sublattice decomposition (5.35) proves the proposition.

5.1.5. **Proof of the theorems 5.7, 5.8, 5.9, 5.7** It is convenient to introduce for \( n \in \mathbb{Z} \) the sector indicator functions
\[
\Theta^{(1)}_{L,n} := \sum_{k=1}^{L/2} \delta_{k,n}, \quad \Theta^{(0)}_{L,n} := \Theta^{(1)}_{L,n+L/2}, \quad \Theta^{(2)}_{L,n} := \Theta^{(1)}_{L,n-L/2} \quad (5.59)
\]
From Proposition 5.9 one finds that one has for \( k, l \in \mathbb{T}_L \)
\[
S_L(k, l) = \left[ \Gamma_L Q_k^- Q_l^- + \kappa_L Q_k^- \delta_{k,l} \right] \Theta^{(1)}_{L,k} \Theta^{(1)}_{L,l} \\
+ \left[ \Gamma_L Q_k^- Q_l^- - H_L(k) Q_k^- \delta_{k+l,L+1} \\
+ \Delta_L H_L(L + 1 - l) Q_k^- \\
+ \Psi_L(L + 1 - l) Q_k^- \Theta(k + l - L - 1) \right] \Theta^{(1)}_{L,k} \Theta^{(2)}_{L,l} \\
+ \left[ \Gamma_L Q_k^- Q_l^- - H_L(L + 1 - k) Q_k^+ \delta_{k+l,L+1} \\
+ \Delta_L H_L(L + 1 - k) Q_l^- \\
+ \Psi_L(L + 1 - k) Q_l^- \Theta(k + l - L - 1) \right] \Theta^{(2)}_{L,k} \Theta^{(1)}_{L,l} \\
+ \left[ \Gamma_L Q_k^- Q_l^- + \kappa_L Q_k^- \delta_{k,l} \\
+ F_L(L + 1 - k, L + 1 - k) \delta_{k,l} \\
+ H_L(L + 1 - k)(Q_k^+ - Q_k^-) \delta_{k,l} \\
+ \Psi_L(L + 1 - l) Q_k^- \Theta_{l-k} + \Psi_L(L + 1 - k) Q_l^- \Theta_{k-l} \\
+ \Delta_L H_L(L + 1 - l) Q_k^- + \Delta_L H_L(L + 1 - k) Q_l^- \\
+ F_L(L + 1 - k, L + 1 - l) \Theta_{l-k} \\
+ F_L(L + 1 - l, L + 1 - k) \Theta_{k-l} \right] \Theta^{(2)}_{L,k} \Theta^{(2)}_{L,l} . \quad (5.60)
\]

One reads off
\[
A^{hc}_L(k) = (\kappa_L + \Gamma_L) Q_k^- + H_L(L + 1 - k)(Q_k^+ - Q_k^-) \Theta^{(2)}_{L,k} \\
+ F_L(L + 1 - k, L + 1 - k) \Theta^{(2)}_{L,k} \quad (5.61)
\]
\[
A^{red}_L(k) = (\Delta_L - 1) \left[ H_L(k) Q_k^+ \Theta^{(1)}_{L,k} + H_L(L + 1 - k) Q_k^+ \Theta^{(2)}_{L,k} \right] \quad (5.62)
\]
\[
S_L^{pl}(k, l) = \Gamma_L Q_k^- Q_l^- (1 - \delta_{k,l}) \\
+ \Delta_L H_L(L + 1 - l) Q_k^- \Theta^{(1)}_{L,k} \Theta^{(2)}_{L,l} 
\]
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\[ + \Psi_L(L + 1 - l)Q_k^- \Theta(k + l - L - 1)\Theta_{L,k}^{(1)}\Theta_{L,l}^{(2)} \]
\[ + \Delta_L H_L(L + 1 - k)Q_l^- \Theta_{L,k}^{(2)}\Theta_{L,l}^{(1)} \]
\[ + \Psi_L(L + 1 - k)Q_l^- \Theta(k + l - L - 1)\Theta_{L,k}^{(2)}\Theta_{L,l}^{(1)} \]
\[ + [\Psi_L(L + 1 - l)Q_k^- \Theta_{L-k} + \Psi_L(L + 1 - k)Q_k^- \Theta_{k-l} \]
\[ + \Delta_L(H_L(L + 1 - l)Q_k^- + H_L(L + 1 - k)Q_l^-)(1 - \delta_{k,l}) \]
\[ + F_L(L + 1 - k, L + 1 - l)\Theta_{L-k} \]
\[ + F_L(L + 1 - l, L + 1 - k)\Theta_{k-l} \][:]

which yields (i) for the hard core part

\[ A_{hc}^{(1)}(k)\Theta_{L,k}^{(1)} = (\kappa_L + \Gamma_L)Q_k^- \Theta_{L,k}^{(1)} \]
\[ A_{hc}^{(0)}(L - |k|)\Theta_{L,k}^{(0)} = [(\kappa_L + \Gamma_L)Q_k^- + F_L(|k| + 1, |k| + 1)]\Theta_{L,k}^{(0)} \]
\[ + H_L(|k| + 1)(Q_k^+ + (2\Delta_L - 1)Q_k^-)\Theta_{L,k}^{(0)} \]
\[ A_{hc}^{(2)}(L/2 + k) = (\kappa_L + \Gamma_L)Q_k^- + \tilde{F}_L(|k|, |k|)\Theta_{L,k}^{(1)} \]
\[ + \tilde{H}_L(|k|)(Q_k^+ + (2\Delta_L - 1)Q_k^-)\Theta_{L,k}^{(1)} \]

(ii) for the reflective part

\[ A_{ref}^{(1)}(k)\Theta_{L,k}^{(1)} = (\Delta_L - 1)H_L(|k|)Q_k^- \Theta_{L,k}^{(1)} \]
\[ A_{ref}^{(0)}(L - |k|)\Theta_{L,k}^{(0)} = (\Delta_L - 1)H_L(|k| + 1)Q_k^+ \Theta_{L,k}^{(0)} \]
\[ A_{ref}^{(2)}(L/2 + k) = (\Delta_L - 1)\tilde{H}_L(|k| + 1)Q_k^- \Theta_{L,k}^{(0)} \]
\[ + (\Delta_L - 1)\tilde{H}_L(|k|)Q_k^+ \Theta_{L,k}^{(1)} \]

and (iii) for the boundary layer part

\[ S_{bl}^{(1)}(k, l)\Theta_{L,k}^{(1)}\Theta_{L,l}^{(1)} = \Gamma_L Q_k^- Q_l^- \Theta_{L,k}^{(1)}\Theta_{L,l}^{(1)}(1 - \delta_{k,l}) \]
\[ S_{bl}^{(1)}(k, L - |l|)\Theta_{L,k}^{(1)}\Theta_{L,l}^{(0)} = \Delta_L H_L(|l| + 1)Q_k^- \Theta_{L,k}^{(1)}\Theta_{L,l}^{(0)} \]
\[ + \Psi_L(|l| + 1)Q_k^- \Theta(k + l - 1)\Theta_{L,k}^{(1)}\Theta_{L,l}^{(0)} \]
\[ + \Gamma_L Q_k^- Q_l^- \Theta_{L,k}^{(1)}\Theta_{L,l}^{(0)} \]
\[ S_{bl}^{(2)}(L - |k|, l)\Theta_{L,k}^{(0)}\Theta_{L,l}^{(1)} = \Psi_L(|k| + 1)Q_l^- \Theta(k + l - 1)\Theta_{L,k}^{(0)}\Theta_{L,l}^{(1)} \]
\[ + \Gamma_L Q_k^- Q_l^- \Theta_{L,k}^{(0)}\Theta_{L,l}^{(1)} \]
\[ S_{bl}^{(0)}(L - |k|, L - |l|)\Theta_{L,k}^{(0)}\Theta_{L,l}^{(0)} = \Delta_L H_L(|l| + 1)Q_k^- + \Delta_L H_L(|k| + 1)Q_l^- \]
\[ + \Psi_L(|l| + 1)Q_k^- \Theta_{L,k}^{(0)}\Theta_{L,l}^{(0)} \]
\[ + F_L(|k| + 1, |l| + 1)\Theta_{L-k} \]
\[ + F_L(|l| + 1, |k| + 1)\Theta_{L-k} \]
\[ + \Gamma_L Q_k^- Q_l^- (1 - \delta_{k,l})\Theta_{L,k}^{(0)}\Theta_{L,l}^{(0)} \]
\[ S_{bl}^{(2)}(L/2 + k, L/2 + l) = \Delta_L \tilde{H}_L(l)Q_k^- (1 - \delta_{k+l+1})\Theta_{L,k}^{(0)}\Theta_{L,l}^{(1)} \]
\[ + \Delta_L \tilde{H}_L(k)Q_l^- (1 - \delta_{k+l+1})\Theta_{L,k}^{(1)}\Theta_{L,l}^{(0)} \]
The offcritical thermodynamic limits relevant for theorems 5.1 - 5.5 are readily computed from these exact finite-size expressions by using the asymptotic values (5.22) and those derived in Appendix A.

At the critical point where \( \kappa = p(1-p) \) and \( (1-p)^2 = \kappa/z_c \) one gets from Proposition 5.9 and the large-\( L \) results derived in Appendix A to leading order in \( 1/L \)

\[
S_L(L/2 + k, L/2 + l) = \kappa Q_k^\pm \Theta_{L,k}^{(0)} \delta_{k,l} \\
+ (1-p) \left( 1 - \frac{2k}{L} \right) \left( p - (1-p) \frac{2k}{L} \right) Q_k^\pm \Theta_{L,k}^{(1)} \delta_{k,l} \\
+ (1-p) \frac{2k}{L} \left( 1 - (1-p) \frac{2k}{L} \right) Q_k^\pm \Theta_{L,k}^{(1)} \delta_{k,l} \\
- \kappa \frac{2|k|}{L} \left[ Q_k^\pm \Theta_{L,1}(-k) + Q_k^\pm \Theta_{L,k}^{(1)} \right] \delta_{k+l,1} \\
+ \frac{\kappa 2k}{z_c} \left( 1 - \frac{2l}{L} \right) \Theta_{l-k} \Theta_{L,k}^{(1)} \Theta_{L,l}^{(1)} \\
+ \frac{\kappa 2l}{z_c} \left( 1 - \frac{2k}{L} \right) \Theta_{k-l} \Theta_{L,k}^{(1)} \Theta_{L,l}^{(1)}.
\] (5.75)

One reads off

\[
\tilde{A}_L^{\text{hc}}(k) = \kappa Q_k^\pm \Theta_{L,k}^{(0)} \\
+ (1-p) \left( 1 - \frac{2k}{L} \right) \left( p - (1-p) \frac{2k}{L} \right) Q_k^\pm \Theta_{L,k}^{(1)} \\
+ (1-p) \frac{2k}{L} \left( 1 - (1-p) \frac{2k}{L} \right) Q_k^\pm \Theta_{L,k}^{(1)} \delta_{k,l}.
\] (5.76)

\[
\tilde{A}_L^{\text{refl}}(k) = -\kappa \frac{2|k|}{L} \left[ Q_k^\pm \Theta_{L,k}^{(0)} + Q_k^\pm \Theta_{L,k}^{(1)} \right] \\
\tilde{S}_L^\text{bl}(k, l) = \frac{\kappa 2k}{z_c} \left( 1 - \frac{2l}{L} \right) \Theta_{l-k} \Theta_{L,k}^{(1)} \Theta_{L,l}^{(1)} \\
+ \frac{\kappa 2l}{z_c} \left( 1 - \frac{2k}{L} \right) \Theta_{k-l} \Theta_{L,k}^{(1)} \Theta_{L,l}^{(1)}.
\] (5.77, 5.78)

Projecting on the sublattices and taking the scaling limit with \( k = \lfloor uL \rfloor \) and \( l = \lfloor vL \rfloor \) where \( u, v \in (-1/2, 1/2) \) yields (5.42) - (5.46). \( \square \)
5.2. Static structure function

The static structure function (5.3) has recently turned out to be of interest in the context of hydrodynamic scaling \[36\]. Here we use it to shed light on the behaviour of the variance established in Theorem 3.2.

5.2.1. Synopsis  
It is shown that the reflective anticorrelations are responsible for the vanishing compressibility in the phase-separated regime \( z > z_c \). Above the critical point, these anticorrelations exactly cancel the hard core contribution. Nevertheless, locally the offcritical static structure function reduces in the thermodynamic limit to its hard-core contribution, as if (erroneously) correlations produced by the blockage were irrelevant. At the critical point the static structure function has a non-trivial scaling form due to the macroscopic size of the critical boundary layers which also leads to the divergent critical compressibility. In the free flow phase below the critical point the compressibility is fully determined by the hard core part of the static structure function. Both at and off criticality, the static structure function has no sublattice dependence in the limit \( L \to \infty \).

5.2.2. Main results  
We recall that \( S_L(r) = S_L(-r) \) so that is sufficient to consider \( 0 \leq r \leq L/2 \). Guided by the results on the two-point density correlation function we decompose the static structure function (5.3) as

\[
S_L(r) = S_{hc}^L(r) + S_{refl}^L(r) + S_{bl}^L(r) 
\]

and define the limits

\[
S_\infty(r) := \lim_{L \to \infty} S_L(r), \quad S_{hc}^\infty(r) := \lim_{L \to \infty} S_{hc}^L(r) 
\]

\[
S_{refl}^\infty(r) := \lim_{L \to \infty} S_{refl}^L(r), \quad S_{bl}^\infty(r) := \lim_{L \to \infty} S_{bl}^L(r). 
\]

**Theorem 5.10 (Offcritical static structure function)**  
For \( z \neq z_c \) the reflective contribution \( S_{refl}^\infty(r) \) and the boundary layer contribution \( S_{bl}^\infty(r) \) to static structure function \( S_\infty(r) \) vanish in the thermodynamic limit and one has

\[
S_\infty(r) = S_{hc}^\infty(r) = \frac{\kappa}{2} \delta_{r,0}, \quad r \in \mathbb{Z} 
\]

with amplitude \( \kappa \) given in (5.22).

**Remark 5.11**  
As shown in Appendix A, the contributions of the reflective long-range anticorrelation (5.62) and of the boundary layer part (5.63) to the dynamical structure function \( S_{\infty}(r) \) vanish in the thermodynamic limit.

Next we consider hydrodynamic scaling and define for \( |u| \in (0,1/2) \) the limits

\[
S^+(u) := \lim_{L \to \infty} S_L(2[uL/2]), \quad S^-(u) := \lim_{L \to \infty} S_L(2[uL/2] - 1) 
\]

and analogously \( S_{hc}^{\pm}(u), S_{refl}^{\pm}(u), \) and \( S_{bl}^{\pm}(u) \).
Theorem 5.12 (Critical static structure function) At the critical point \( z = z_c \) the hard-core contribution \( S^{hc\pm}(u) \) and the reflective contribution \( S^{refl\pm}(u) \) to the static structure function \( S^\pm(u) \) vanish under hydrodynamic scaling for any macroscopic distance \( u \neq 0 \) and \( |u| \in (0, 1/2) \). One has

\[
S^\pm(u) = S^{bl\pm}(u) = \frac{1}{12} (1 - p)^2 (1 - 2|u|)^3
\]

(5.84) independently of the sublattice.

Remark 5.13 The hard core contribution \( S^{hc\pm}(u) \) vanishes by definition for \( u \neq 0 \) while the reflective long-range contribution \( S^{refl\pm}(u) \) vanishes since in a finite system its contribution to the static structure function is, like in the off-critical case, of order \( 1/L \).

By definition, the particle variance (3.3) is given in terms of the dynamical structure by \( C_L = \sum_{r=-L/2}^{L/2} S_L(r) \). Hence the decomposition \( C_L = C^{hc}_L + C^{refl}_L + C^{bl}_L \) and the corresponding limit

\[
C = C^{hc} + C^{refl} + C^{bl}
\]

(5.85) provides insight into the origin of the fluctuations of the total particle number in the grandcanonical ensemble.

Theorem 5.14 (Particle number fluctuations) The compressibility (3.3) has hard core, reflective, and boundary layer contributions given by

\[
C^{hc} = \begin{cases} 
0 & z < z_c \\
\frac{p(1-p)}{2} + \frac{1}{12} (1 - p)^2 & z = z_c \\
\frac{p(1-p)}{2} & z > z_c
\end{cases}
\]

(5.86)

\[
C^{refl} = \begin{cases} 
0 & z < z_c \\
-\frac{p(1-p)}{4} & z = z_c \\
-\frac{p(1-p)}{2} & z > z_c
\end{cases}
\]

(5.87)

\[
C^{bl} = \begin{cases} 
0 & z < z_c \\
\infty & z = z_c \\
0 & z > z_c
\end{cases}
\]

(5.88)

At the critical point, the scaled variance of the particle number has the limiting behaviour

\[
\lim_{L \to \infty} \frac{1}{L} C^{bl} = \frac{(1 - p)^2}{48}.
\]

(5.89)

Remark 5.15 The limit (5.89) of the scaled variance which arises from the boundary layer contribution alone is equal to the scaled total variance (3.5) established in Theorem 3.2, thus showing that the origin of the divergence of compressibility \( C \) comes from the unbounded fluctuations of the domain wall position in the thermodynamic limit.
5.2.3. Proofs  To deal with the sector dependence of the density correlation function we split the sum \(S_{L}(r)\) defining static structure function as

\[
S_{L}(r) = \sum_{k=1}^{L/2-r} S_{L}(k,k+r) + \sum_{k=L/2-r+1}^{L/2} S_{L}(k,k+r)
+ \sum_{k=L/2+1}^{L-r} S_{L}(k,k+r) + \sum_{k=L-r+1}^{L} S_{L}(k,k+r-L).
\]

(5.90)

We also define for \(0 \leq n \leq L/2\) further auxiliary functions

\[
G_{L}(n) := \frac{2}{L} \sum_{m=1}^{L/2-n} F_{L}(m-n,m)
\]

(5.91)

\[
\Phi_{\pm}^{L}(n) := \frac{1}{L} \sum_{m=1}^{n} \Psi(m)Q_{m}^{\pm}.
\]

(5.92)

Thermodynamic limits below are computed using the results of Appendix A.

Proof of Theorems 5.10 and 5.12:  The hard core part trivially vanishes for \(r = 0\) and with

\[
B_{hc}^{L} := \frac{1}{L} \sum_{k=1}^{L} A_{hc}^{L}(k)
\]

(5.93)

one gets the exact finite-size result

\[
S_{hc}^{L}(r) := B_{hc}^{L} \delta_{r,0}.
\]

(5.94)

From (5.61) one finds

\[
B_{hc}^{L} = \frac{1}{2} \left[ \Gamma_{L} + \kappa_{L} + H_{L} + (2\Delta_{L} - 1)H_{L}^{+} + G_{L}(0) \right].
\]

(5.95)

Taking the limit \(L \to \infty\) yields

\[
B^{hc} = \begin{cases} 
\frac{\kappa}{2} & z < z_{c} \\
\frac{\kappa}{2} + \frac{1}{12} (1-p)^{2} & z = z_{c} \\
\frac{\kappa}{2} & z > z_{c}
\end{cases}
\]

(5.96)

which proves the second equality of (5.82) in Theorem 5.10 and (5.86) in Theorem 5.14.

For the reflective part one has trivially \(S_{r}^{\text{refl}}(2n) = 0\). Proposition 5.9 yields for odd distances \(r = 2n - 1\) after a brief calculation

\[
S_{r}^{\text{refl}}(2n - 1) = \frac{1}{L} \left[ A_{r}^{\text{refl}}(L/2 + 1 - n) + A_{r}^{\text{refl}}(L + 1 - n) \right].
\]

(5.97)

With (5.62) we get

\[
S_{r}^{\text{refl}}(r) = \frac{1}{L} (\Delta_{L} - 1) \left( H_{L}(\|r/2\|)Q_{\|r/2\|}^{+} + H_{L}(\|r/2\|)Q_{\|r/2\|}^{-} \right) Q_{r}^{-}.
\]

(5.98)
and therefore in the limit $L \to \infty$ one has, independently of the sublattice,

$$S^{\text{refl}}(r) = 0, \quad S^{\text{refl} \pm}(u) = 0$$ (5.99)

as stated in Theorem 5.10 and in Theorem 5.12.

Next we compute the sublattice parts of the boundary layer distribution. For even distance $r = 2n > 0$ one gets from Proposition 5.9

$$S^\text{bl}_L(k, k + 2n) = \Gamma_L Q_k^− \quad \sum\text{boundary layer part is the sum} \quad \sum_{k=1}^{\infty} S^\text{bl}_L(k, k + 2n)$$

Taking the thermodynamic limit yields (5.87). The leading contribution from the boundary layer part is the sum $\sum_{r=1}^{L/2} G_L(r)$ which vanishes for for $z \neq z_c$ in the thermodynamic limit and for $z = z_c$ is proportional to $L$ since each term in the sum is of order 1. Hence the limit $\lim_{L \to \infty} C^\text{refl}_L$ diverges for $z = z_c$ which proves (5.88).

From the scaling form (5.84) one calculates

$$\lim_{L \to \infty} C^\text{bl}_L / L = 2 \int_0^{r_{1/2}} S^\text{bl}(u)du$$ (5.104)

which yields (5.89).
Appendix A. Auxiliary constants and functions

We collect exact finite-size expressions, the asymptotic behaviour for large $L$, and the thermodynamic limits $L \to \infty$ of constants and functions that are used in the proofs throughout this paper. The symbol $\epsilon(L)$ denotes unspecified corrections exponentially small in $L$ that may differ from formula to formula.

For $z \neq z_c$ we define the quantity

$$\xi_s(p, z) := \left( \ln \frac{z_c(1 + z)}{z(1 + z_c)} \right)^{-1}$$

that appears in the normalization ratio (3.27). For $z > z_c$ one has $\xi_s < 0$ which needs to be borne in mind when taking thermodynamic limits. For finite size we use the parameter $\xi_s$ in the expressions below, while for large $L$ and for the thermodynamic limit we express all results in terms of the localization length $\xi = |\xi_s|$.

Appendix A.1. The effective length $L_{\text{eff}}$

For $p \neq 1$ one reads off from the definition (3.28)

$$\frac{1}{L_{\text{eff}}} = \frac{1}{L} - \frac{2}{1 - p} \frac{1}{L^2} + O(L^{-3}).$$

(A.2)

For the limiting case $p = 1$ we note that $\lim_{p \to 1} (1 - p)L_{\text{eff}} = 2$ which implies that taking the thermodynamic limit $L \to \infty$ and the limit $p \to 1$ cannot be interchanged whenever $L_{\text{eff}}$ appears in finite-size expressions.

Appendix A.2. The functions $H_L(n)$ and $\tilde{H}_L(n)$

Exact finite size expression: According to the definition (4.8) one has

$$e^{-1/\xi_s} = \frac{z(1 + z_c)}{z_c(1 + z)}$$

(A.3)

where

$$e^{-1/\xi_s} = \begin{cases} 
< 1 & z \neq z_c \\
1 & z = z_c \\
> 1 & z \neq z_c.
\end{cases}$$

(A.4)

From the definitions (4.22) one gets

$$H_L(n) = p(1 - p) \begin{cases} 
\frac{e^{-n/\xi_s} - \frac{z_c}{z}e^{-L/(2\xi_s)}}{1 - \frac{z_c}{z}e^{-L/(2\xi_s)}} & z \neq z_c \\
1 - \frac{2n}{L_{\text{eff}}} & z = z_c
\end{cases}$$

(A.5)

$$\tilde{H}_L(n) = p(1 - p) \begin{cases} 
1 - \frac{1 + z}{1 + z_c} & z \neq z_c \\
\frac{1 - \frac{z_c}{z}e^{L/(2\xi_s)}}{L_{\text{eff}}} & z = z_c
\end{cases}$$

(A.6)
and for the lattice sums

\[ H_L^\pm := \frac{2}{L} \sum_{k=1}^{\frac{L}{2}} Q_k^\pm H_L(k) \]  

one finds

\[
H_L^+ = p(1 - p) \left\{ \begin{array}{ll}
\frac{2}{L} e^{-L/(2\xi_s)} - \frac{1}{z_c} e^{-L/(2\xi_s)} & z \neq z_c \\
(1 + 2(2p - 1)) & z = z_c,
\end{array} \right.
\]

\[
H_L^- = p(1 - p) \left\{ \begin{array}{ll}
\frac{2}{L} e^{-L/(2\xi_s)} - \frac{1}{z_c} e^{-L/(2\xi_s)} & z \neq z_c \\
(1 + \frac{2}{(1 - p)L_{eff}}) & z = z_c.
\end{array} \right.
\]

We remark that according to (4.29) these lattice sums are related to the sublattice densities

\[ \frac{2}{L} \langle N^+ \rangle_L = \frac{1}{p} H_L^- \]  

\[ \frac{2}{L} \langle N^- \rangle_L = j_L + \frac{1}{p} H_L^+ . \]

**Asymptotic behaviour for large L:** For fixed \( n \) one obtains

\[
H_L(n) = p(1 - p) \left\{ \begin{array}{ll}
e^{-n/\xi} + \epsilon(L) & z < z_c \\
1 - \frac{2n}{L} + O(L^{-2}) & z = z_c \\
1 + \epsilon(L) & z > z_c.
\end{array} \right.
\]

\[
\tilde{H}_L(n) = p(1 - p) \left\{ \begin{array}{ll}
\epsilon(L) & z < z_c \\
\frac{2n + 2z_c}{L} + O(L^{-2}) & z = z_c \\
1 - \frac{1 + z_c e^{-n/\xi} + \epsilon(L)}{1 + z} & z > z_c.
\end{array} \right.
\]

The lattice sums behave asymptotically as

\[
H_L^+ = p(1 - p) \left\{ \begin{array}{ll}
\frac{2e^{-2/\xi}}{1 - e^{-2/\xi} \xi} + \epsilon(L) & z < z_c \\
1 - \frac{2p}{2(1 - p)L} + O(L^{-2}) & z = z_c \\
1 - \frac{2}{1 - e^{-2/\xi} \xi} + \epsilon(L) & z > z_c,
\end{array} \right.
\]
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\[ H_L^- = p(1-p) \begin{cases} 
\frac{2e^{-1/\xi}}{1 - e^{-2/\xi}} \frac{1}{L} + \epsilon(L) & z < z_c \\
\frac{1}{4} + \frac{1}{2(1-p)} \frac{1}{L} + O(L^{-2}) & z = z_c \\
1 - \frac{2e^{-1/\xi}}{1 - e^{-2/\xi}} \frac{z_c}{z} \frac{1}{L} & z > z_c.
\end{cases} \] (A.15)

Thermodynamic limit for fixed \( n \in \mathbb{Z} \): In the thermodynamic limit one has for \( z \neq z_c \) and \( n \in \mathbb{N}_0 \)

\[ H(n) := \lim_{L \to \infty} H_L(n) = p(1-p) \begin{cases} 
e^{-n/\xi} & z < z_c \\
1 & z \geq z_c.
\end{cases} \] (A.16)

\[ \tilde{H}(n) := \lim_{L \to \infty} \tilde{H}_L(n) = p \begin{cases} 
0 & z \leq z_c \\
1 - p - \frac{e^{-n/\xi}}{1+z} & z > z_c.
\end{cases} \] (A.17)

At the critical point one gets for \( u \in [0,1/2] \)

\[ H_c(u) := \lim_{L \to \infty} H_L(\lfloor uL \rfloor) \bigg|_{z=z_c} = p(1-p)(1-2u). \] (A.18)

\[ \tilde{H}_c(u) := \lim_{L \to \infty} \tilde{H}_L(L/2 + 1 - \lfloor uL \rfloor) \bigg|_{z=z_c} = p(1-p)2u \] (A.19)

For the lattice sums one gets

\[ H^\pm := \lim_{L \to \infty} H^\pm_L = p(1-p) \begin{cases} 
0 & z < z_c \\
\frac{1}{4} & z = z_c \\
1 & z > z_c.
\end{cases} \] (A.20)

Appendix A.3. The constants \( \Gamma_L \) and \( \Delta_L \)

Using (3.29) one obtains from the definitions (5.47) and (5.48) the exact expressions

\[ \Gamma_L = \begin{cases} 
-\left( \frac{z}{1+z} - \frac{z_c}{1+z_c} \right)^2 \frac{z}{z_c} e^{-L/(2\xi_s)} & z \neq z_c \\
-\frac{4p^2}{L_{\text{eff}}^2} & z = z_c
\end{cases} \] (A.21)

\[ \Delta_L = \begin{cases} 
\frac{1 - e^{-1/\xi_s}}{2} e^{-L/(2\xi_s)} & z \neq z_c \\
\frac{1 - \left( \frac{z}{z_c} \right) e^{-L/(2\xi_s)}}{L_{\text{eff}}} & z = z_c
\end{cases} \] (A.22)
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It follows that
\[
\Delta := \lim_{L \to \infty} \Delta_L = \begin{cases} 
1 - \frac{z}{p(1+z)} & z < z_c \\
0 & z \geq z_c \end{cases} \tag{A.23}
\]
\[
\Gamma := \lim_{L \to \infty} \Gamma_L = 0. \tag{A.24}
\]

Appendix A.4. The functions \(\Psi_L(n)\) and \(\tilde{\Psi}_L(n)\)

From the definition (5.49) one finds with (3.28) and (A.22) the exact expressions
\[
\Psi_L(m) = -p(1-p)\Delta_L e^{-m/\xi_s} \tag{A.25}
\]
\[
\tilde{\Psi}_L(m) = -p(1-p)\Delta_L e^{-L/(2\xi_s)} e^{(m-1)/\xi_s} \tag{A.26}
\]
and, for \(m \in \mathbb{N}\) fixed, the limits
\[
\Psi(m) := \lim_{L \to \infty} \Psi_L(m) = \begin{cases} 
-p \left(1 - e^{-1/\xi_s}\right) \frac{e^{-m/\xi_s}}{1+z_c} & z < z_c \\
0 & z \geq z_c \end{cases} \tag{A.27}
\]
\[
\tilde{\Psi}(m) := \lim_{L \to \infty} \tilde{\Psi}_L(m) = \begin{cases} 
0 & z \leq z_c \\
-p \left(e^{1/\xi_s} - 1\right) \frac{e^{-m/\xi_s}}{1+z} & z > z_c. \end{cases} \tag{A.28}
\]
with \(\Delta\) obtained in (A.23).

Away from the critical point the finite-size corrections to these asymptotic values are exponentially small in \(L\). For the critical point we note that
\[
\Psi(m) = -\tilde{\Psi}(m) = -2p(1-p)\frac{1}{L} + O(L^{-2}) \tag{A.29}
\]

Evaluating the sums \(\Phi^\pm(n)\) defined by (5.92) yields
\[
\Phi_L^+(m) = \frac{1}{L} H_L(2 \lfloor m/2 \rfloor) - H_L(2) \tag{A.30}
\]
\[
\Phi_L^-(m) = \frac{1}{L} H_L(2 \lfloor (m+1)/2 \rfloor - 1) - H_L(1) \tag{A.31}
\]
and \(\lim_{L \to \infty} \Phi_L^\pm(m) = 0\) with corrections of order \(1/L\) to the asymptotic result.

Appendix A.5. The function \(F_L(m, n)\) and \(\tilde{F}(m, n)\)

From the definition (5.51) and the exact expression (A.5) one gets
\[
F_L(m, n) = \begin{cases} 
(1-p)^2 \left( \frac{e^{-m/\xi_s}}{z} \frac{e^{L/(2\xi_s)}}{z_c} \right)^2 \left(1 - \frac{e^{-L/(2\xi_s)}}{z_c} \right)^2 & z \neq z_c \\
(1-p)^2 \left(1 - \frac{2m}{L_{\text{eff}}} \right) \frac{2n}{L_{\text{eff}}} & z = z_c \end{cases} \tag{A.33}
\]
\[
\tilde{F}_L(m, n) = \begin{cases} 
(1-p)^2 \left(1 + \frac{1+z_c e^{-m/\xi_s}}{1+2z_c e^{-L/(2\xi_s)}} \right) \left(1 - \frac{1+z_c e^{L/(2\xi_s)}}{1+2z_c e^{L/(2\xi_s)}} \right)^2 & z \neq z_c \\
(1-p)^2 \left(1 - \frac{2n+2z_c}{L_{\text{eff}}} \right) \frac{2m+2z_c}{L_{\text{eff}}} & z = z_c \end{cases} \tag{A.34}
\]
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and, for \(m, n \in \mathbb{N}\) fixed, the limits

\[
F(m, n) := \lim_{L \to \infty} F_L(m, n) = \begin{cases} 
(1 - p)^2 e^{-m/\xi} (1 - e^{-n/\xi}) & z < z_c \\
0 & z \geq z_c 
\end{cases} \quad (A.35)
\]

\[
\tilde{F}(m, n) := \lim_{L \to \infty} \tilde{F}_L(m, n) = \begin{cases} 
e^{-n/\xi/z} (1 - p - e^{-m/\xi}/z) & z > z_c.
\end{cases} \quad (A.36)
\]

Away from the critical point the finite-size corrections to these asymptotic values are exponentially small in \(L\).

For \(z \neq z_c\) one has

\[
G_L(r) = \frac{2(1 - p)^2}{L} \left( 1 - \frac{z}{z_c} e^{-L/(2\xi_s)} \right) \left[ \frac{e^{-r/\xi_s} - e^{-L/(2\xi_s)}}{1 - e^{-1/\xi_s}} \right] + e^{-1/\xi_s} \frac{e^{-r/\xi_s} - e^{-L-(r)/\xi_s}}{1 - e^{-2/\xi_s}} - \frac{z}{z_c} \frac{e^{-L/(2\xi_s)} - e^{-L-(r)/\xi_s}}{1 - e^{-1/\xi_s}} - \left( 1 - \frac{2r}{L} \right) \left( 1 - \frac{2p}{L} \right) \left( 1 - \frac{2r + 2z_c}{L_{\text{eff}}} \right) \left( 1 - \frac{2r - 4z_c - 2}{L_{\text{eff}}} \right) \quad (A.37)
\]

For \(z = z_c\) one has

\[
G_L(r) = \frac{1}{6} (1 - p)^2 \left( 1 - \frac{2r}{L} \right) \left( 1 - \frac{2r + 2z_c}{L_{\text{eff}}} \right) \left( 1 - \frac{2r - 4z_c - 2}{L_{\text{eff}}} \right) \quad (A.38)
\]

Asymptotically this yields

\[
G_L(r) = \begin{cases} 
\frac{z}{L(1 + z_c)(z_c - z)} \left( 1 + \frac{z(1 + z_c)}{z + z_c + 2zz_c} \right) e^{-r/\xi} + \epsilon_L & z < z_c \\
\frac{1}{6} (1 - p)^2 \left( 1 - \frac{2r}{L} \right)^3 + O(1/L) & z = z_c \\
\frac{2}{L(1 + z_c)(z - z_c)} \left[ 1 - \frac{z^2(1 + z)}{z(z + z_c + 2zz_c)} \right] e^{-r/\xi} + \epsilon_L & z > z_c.
\end{cases} \quad (A.39)
\]

Thus for the limit

\[
G_\infty(r) := \lim_{L \to \infty} G_L(r), \quad r \in \mathbb{N} \quad (A.40)
\]

one gets for fixed \(r\)

\[
G_\infty(r) = \begin{cases} 
0 & z < z_c \\
\frac{1}{6} (1 - p)^2 & z = z_c \\
0 & z > z_c
\end{cases} \quad (A.41)
\]

while for the limit

\[
G(u) := \lim_{L \to \infty} G_L(\lfloor uL \rfloor), \quad u \in [0, 1/2] \quad (A.42)
\]
one finds
\[
G(u) = \begin{cases} 
0 & z < z_c \\
\frac{1}{6} (1 - p)^2 (1 - 2u)^3 & z = z_c \\
0 & z > z_c.
\end{cases}
\] (A.43)

Appendix B. On the matrix product ansatz for the dsTASEP

We briefly explain how the MPA of [30] is related to the SMPM (2.13). We consider \(L/2\) even and recall the notation
\[
\sigma_k := \eta_k + 2\eta_{L+1-k}, \quad 1 \leq k \leq \frac{L}{2}
\] (B.1)
of [30] that represents the occupation pair \((\eta_{L+1-k}, \eta_k)\) as
\[
(0, 0) \mapsto 0, \quad (0, 1) \mapsto 1, \quad (1, 0) \mapsto 2, \quad (1, 1) \mapsto 3.
\] (B.2)
The state \(\eta\) of the dsTASEP can thus be expressed in terms of the state variable \(\sigma = (\sigma_1, \ldots, \sigma_{L/2})\). One has
\[
\delta_{\sigma_k, \sigma} = \frac{1}{4} \sum_{j=0}^{3} e^{i \pi j (\sigma_k - \sigma)}, \quad 1 \leq k \leq \frac{L}{2}
\] (B.3)
and therefore
\[
\eta_k = (\delta_{\sigma_k, 1} + \delta_{\sigma_k, 3}) \Theta_{L,k}^{(1)} + (\delta_{\sigma_{L+1-k}, 2} + \delta_{\sigma_{L+1-k}, 3}) \Theta_{L,k}^{(2)}
\] (B.4)
\[
\tilde{\eta}_k = (\delta_{\sigma_k, 0} + \delta_{\sigma_k, 2}) \Theta_{L,k}^{(1)} + (\delta_{\sigma_{L+1-k}, 1} + \delta_{\sigma_{L+1-k}, 3}) \Theta_{L,k}^{(2)}.
\] (B.5)

According to [30] the MPA is given in terms of vectors \(\langle W \rangle, \langle V \rangle\), matrices \(A_\sigma, B_\sigma\), and a normalization factor \(Y_L\) by
\[
P(\sigma) = \frac{1}{Y_L} \langle W \rangle |A_{\sigma_1} B_{\sigma_2} A_{\sigma_3} B_{\sigma_4} \ldots A_{\sigma_{L/2-1}} B_{\sigma_{L/2}}|V \rangle
\] (B.6)
where the quantum mechanical bra-ket convention for scalar products is used. In terms of the occupation variables \(\eta_k\) this reads
\[
P(\eta) = \frac{1}{Y_L} \langle W \rangle (\tilde{\eta}_1 \eta_L A_0 + \tilde{\eta}_1 \eta_L A_2 + \eta_1 \eta_L A_1 + \eta_1 \eta_L A_3)
\times (\tilde{\eta}_2 \eta_{L-1} B_0 + \tilde{\eta}_2 \eta_{L-1} B_2 + \eta_2 \eta_{L-1} B_1 + \eta_2 \eta_{L-1} B_3)
\times \ldots
\times (\tilde{\eta}_{L-2} \eta_{L-3} + 2 A_0 + \tilde{\eta}_{L-3} \eta_{L-3} + 2 A_2 + \eta_{L-3} \eta_{L-3} + 2 A_1 + \eta_{L-3} \eta_{L-3} + 2 A_3)
\times (\tilde{\eta}_{L-1} \eta_{L-2} B_0 + \tilde{\eta}_{L-1} \eta_{L-2} B_2 + \eta_{L-1} \eta_{L-2} B_1 + \eta_{L-1} \eta_{L-2} B_3) |V \rangle.
\] (B.7)
The vectors are
\[
\langle W \rangle = (1, 1), \quad |V \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\] (B.8)
and one has \(B_1 = B_3 = 0, B_0 = A_0,\) and \(B_2 = A_1 + A_2\). Multiplying the matrices \(A_\sigma\) of [30] by a factor of \(p\), and replacing the scalar product by the trace involving the Kronecker product \(D = \langle |V \rangle \otimes \langle W \rangle \rangle / 2\) one arrives at (2.13).
On the phase transition in the sublattice TASEP with stochastic blockage

References

[1] D.E. Wolf and L.H. Tang, Inhomogeneous growth processes. Phys. Rev. Lett. 65, 1591-1594 (1990).

[2] Janowsky, S.A. and Lebowitz, J.L.: Finite-size effects and shock fluctuations in the asymmetric simple-exclusion process. Phys. Rev. A 45 618–625 (1992)

[3] Schütz, G.: Generalized Bethe ansatz solution of a one-dimensional asymmetric exclusion process on a ring with blockage. J. Stat. Phys. 71, 471–505 (1993)

[4] Tang, L.-H. , Lyuksyutov, I. F.: Directed polymer localization in a disordered medium. Phys. Rev. Lett. 71, 2745–2748 (1993)

[5] M. Henkel and G. Schütz: Boundary induced phase transitions in equilibrium and non-equilibrium systems. Physica A 206 (1994), 187-195

[6] Seppäläinen,T. (2001). Hydrodynamic profiles for the totally asymmetric exclusion process with a slow bond. J. Statist. Phys. 102 69–96.

[7] Bahadoran, C.: Blockage hydrodynamics of one-dimensional driven conservative systems. Ann. Probab. 32, 805–854 (2004)

[8] Schmidt, J., Popkov, V., Schadschneider, A.: Defect-induced phase transition in the asymmetric simple exclusion process EPL 110, 20008 (2015)

[9] R. Basu, V. Sidoravicius and A. Sly, Last passage percolation with a defect line and the solution of the slow bond problem. Unpublished, see arXiv: 1408.3464v3 (2016)

[10] C. Bahadoran and T. Bodineau, Quantitative estimates for the flux of TASEP with dilute site disorder. J. Probab. 23 (2018), paper no. 44

[11] Szwietz-Nossan, Juraj; Romano, M. Carmen; Ciandrini, Luca Power series solution of the inhomogeneous exclusion process Phys. Rev. E 97, 052139 (2018)

[12] P. Neijjar, Transition to Shocks in TASEP and Decoupling of Last Passage Times, Lat. Am. J. Probab. Math. Stat. 15, 1311–1334 (2018)

[13] C. Appert-Rolland, M. Ebbinghaus, L. Santen Intracellular transport driven by cytoskeletal motors: General mechanisms and defects. Phys. Rep 593, 1–59 (2015).

[14] B Mishra, GM Schütz, D Chowdhury, Slip of grip of a molecular motor on a crowded track: Modeling shift of reading frame of ribosome on RNA template EPL 114 68005 (2016)

[15] S. Ghosh, A. Dutta, S. Patra, J. Sato, K. Nishinari, and D. Chowdhury, Biologically motivated asymmetric exclusion process: Interplay of congestion in RNA polymerase traffic and slippage of nascent transcript, Phys. Rev. E 99, 052122 (2019).

[16] Akriti Jindal, Anatoly B Kolomeisky and Arvind Kumar Gupta The role of dynamic defects in transport of interacting molecular motors J. Stat. Mech. (2020) 043206

[17] Ramaswamy, S., Barma, M., Das, D., Basu, A.: Phase Diagram of a Two-Species Lattice Model with a Linear Instability, Phase Transit. 75, 363–375 (2002)

[18] Kafri, Y., Levine, E., Mukamel, D., Schütz, G.M., Willmann R.D.: Phase-separation transition in one-dimensional driven models, Phys. Rev. E 68, 035101(R) (2003)

[19] Chakraborty, S., Pal, S., Chatterjee, S., Barma, M.: Large compact clusters and fast dynamics in coupled nonequilibrium systems, Phys. Rev. E 93, 050102(R) (2016)

[20] Ferrari, P.A., Fontes, L.R.G.: Shock fluctuations in the asymmetric simple exclusion process, Probab. Theory Relat. Fields 99, 305–319 (1994)

[21] Dudziński, M., Schütz, G.M.: Relaxation spectrum of the asymmetric exclusion process with open boundaries. J. Phys. A 33, 8351 - 8364 (2000)

[22] Belitsky, V., Schütz, G.M.: Diffusion and scattering of shocks in the partially asymmetric simple exclusion process. Electron. J. Probab. 7, paper 11, 1-21 (2002)

[23] K. Krebs, F.H. Jafarpour and G.M. Schütz: Microscopic structure of travelling wave solutions in a class of stochastic interacting particle systems, New J. Phys. 5, 145.1-145.14 (2003).

[24] de Gier, J.; Essler, F.H.L.: Exact spectral gaps of the asymmetric exclusion process with open boundaries. J. Stat. Mech. 12011 (2006).
[25] Balázs, M., Farkas, G., Kovács, P., and Rákos, A.: Random walk of second class particles in product shock measures. J. Stat. Phys. 139, 252–279 (2010)

[26] V. Belitsky and G. M. Schütz, Self-duality and shock dynamics in the n-species priority ASEP, Stoch. Proc. Appl. 128, 1165–1207 (2018).

[27] Liggett, T.M.: Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Springer, Berlin (1999).

[28] Schütz, G.M.: Exactly solvable models for many-body systems far from equilibrium, in: Phase Transitions and Critical Phenomena Vol. 19, C. Domb and J. Lebowitz (eds.), 1–251, Academic Press, London (2001).

[29] Rajewsky, N., Santen, L., Schadschneider, A., Schreckenberg, M.: The asymmetric exclusion process: Comparison of update procedures. J. Stat. Phys. 92, 151–194 (1998)

[30] Hinrichsen, H., Sandow, S.: Deterministic exclusion process with a stochastic defect: matrix-product ground states. J. Phys. A: Math. Gen. 30, 2745–2756 (1997)

[31] Blythe, R.A., Evans, M.R.: Nonequilibrium steady states of matrix-product form: a solver’s guide. J. Phys. A: Math. Theor. 40 R333–R441 (2007)

[32] G.M. Schütz, Diffusion-annihilation in the presence of a driving field. J. Phys. A: Math. Gen. 28, 3405–3415 (1995).

[33] G. Schütz, Time-dependent correlation functions in a one-dimensional asymmetric exclusion process, Phys. Rev. E 47, 4265–4277 (1993).

[34] H. Hinrichsen, Matrix product ground states for exclusion processes with parallel dynamics, J. Phys. A: Math. Gen. 29 3659–3667 (1996)

[35] F. H. Jafarpour, S. R. Masharian, Temporal evolution of product shock measures in TASEP with sublattice-parallel update, Phys. Rev. E 79, 051124 (2009)

[36] Karevski, D., Schütz, G.M.: Charge-current correlation equalities for quantum systems far from equilibrium. SciPost Phys. 6, 068 (2019)