The quark vacuum

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Abstract. We conjecture on the structure of the quark vacuum from a viewpoint somewhat different from, but perhaps supplementary to, standard philosophies.

Using a rather simple dynamical Hamiltonian model, vacuum excitations carrying helicities 0 and ±1 are discussed in connection with the dynamical stability of solutions. We speculate on how the rest masses of the light mesons $\pi^0$, $\eta^0$ and $\eta^0'$ could be related to these excitations.

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1. Introduction

We assume in this paper that the physical vacuum, which has acquired a place of honour in contemporary physics because of the great empirical success of the Standard Model, is a physical medium that can be probed just like any other physical medium [1]. This idea has, of course, a long history behind it. In modern physics, it first emerged as a possibility, suggested by detailed QED studies. It seems, however, that Nature has provided us with many alternative ways of empirically testing this idea. We are referring in fact to the physics of chiral and flavour symmetries and symmetry breaking in the world of quarks and leptons, as assumed and described by the Standard Model [2]. In contemporary physics, this is usually tackled by relying on the generally very successful methodology of Effective Field Theories [2, 3, 4, 5].

In this paper, we concentrate on one of the best hints we have about the nature of this presumed medium, through the structure of the lightest self-conjugate mesons. This subject has of course been discussed in detail in the last 40 years or so, but we believe that the case is not definitely closed as yet. The general question this paper studies is: if the vacuum indeed is a mostly unknown physical medium, then how do approximate ansatzes about the nature of this presumed medium manifest themselves on key properties of the predicted particle spectrum.

We begin by reformulating this question in terms of Dirac’s Quantum Field Theory “without dead wood” ([6]; Appendix I). This means that questions about symmetry, spontaneous symmetry breaking and chiral anomalies are formulated in a way that differ somewhat from the standard one [2, 3, 4, 5]. The main technical points are, first we abandon the notion of wave-functionals and vacuum expectation values; this implies among other things, in a perturbative setting and at any stage of the development of the theory, the automatic non-existence of such irrelevant objects as unconnected vac-to-vac diagrams. This non-trivial streamlining of Quantum Field Theory should of course lead to the same results as standard formulations, in so far as these agree with experiments. However, the interpretation differs.

We further emphasise that we always work within momentum space. This feature we share with pure S-matrix theory, although we have no S-matrix here.

Specialized principles that we assume are mutually compatible and that we attempt to obey are:

a) essential ideas of the empirically rather successful quark model for mesons [3, 4], which are defined only in the so-called confinement phase;

b) the reliability of the QCD assumptions (especially chiral-flavour symmetries) for the physics at very short distances (i.e. $d \ll \Lambda_{QCD}^{-1}$ with $\Lambda_{QCD}$ of the order of 0.2 - 0.3 GeV) [2, 3, 4];

c) the presumed moderate reliability of ENJL-like models for the relevant physics of the medium distance quark dynamics (i.e. $d > \Lambda_{x}^{-1}$ but $d < \Lambda_{QCD}^{-1}$ with $\Lambda_{x}$ of the order of 1GeV) [7]. We do not actually use this particular model.

A short preview of this paper would be in order.
In Sections 2 and 3 we present the mathematical machinery used in this and subsequent papers. This is based on what Dirac recommended in his later years [6], and is eminently suitable to our purposes. It may be rather unfamiliar to many particle theorists, but the material in App.A may help.

In Section 4 the master equations previously established are applied to the so-called current quarks, which are our fundamental (Weyl) q-numbers (Dirac’s terminology, [6]). Starting with massless particles in our Model Hamiltonian, we discuss three cases, viz. all quarks massless (case A); next, massless (u,d) quarks, massive s-quark (case B); and finally, three massive species, but obeying the hierarchy $0 < m_u < m_d \ll m_s$ (case C).

We show that all quarks become massive through pure quantum effects. We study the results of self-energy corrections to the quarks energy-momentum relationship, and verify the robustness (stability) of the vacuum exhibiting such effects. Particular features of this calculation (which to some extent are common to all cases) are:

(i) automatic output of massive Dirac particles, even with input Weyl fermions;

(ii) the emergence of Bose-like states (again in Dirac’s sense of this word, Ref 6), one of these a massless state (Goldstone, case A), or a nearly massless one (Pseudo-Goldstone; cases B and C);

(iii) the emergence of another massive highly coherent superposition of $u\bar{u}, d\bar{d}$ and $s\bar{s}$.

The relationship of these vacuum excitations to $\pi^0, \eta^0$ and $\eta'$ is conjectured upon in App.B.

We do not know at the moment how to formulate the problem of confinement in our approach (at least as discussed in the literature), let alone find a solution for it. We have therefore completely by-passed this issue. All gauge fields of the Standard Model (that are assumed to couple to our quarks) provide only some sort of an implicit unanalysed background.

Questions having to do with the electroweak couplings of quarks, although very important to our views, shall be discussed in a future paper.

2. The theoretical framework

The theory used in this paper claims to be yet another example of how to construct a “field theory without dead wood” in practice [6]. Some advantages of this more logical (and simpler!) approach to field theoretical physical problems (in practice treated by standard perturbative methods) were spelled out by Dirac in his later years [6].

2.1. The Model Hamiltonian

We need an effective field theoretic Hamiltonian [2, 3, 4], or a Model Hamiltonian at the level of QCD-quarks that must be restricted by the following conditions:

1) Baryon number and charge conservation;

2) CPT-invariance (we further assume CP-invariance);
3) share with QCD its global colour, flavour and chiral symmetries (we bypass the confinement issue);

4) have a minimal number of special assumptions and parametrizations.

The S-matrix at the quark level is not defined, but this effective Hamiltonian should ensure that one eventually ends up with an hadronic S-matrix with the usual properties [2].

There are of course in principle an infinity of “microscopic” quark Hamiltonians satisfying these conditions appropriate to the vacuum phenomena discussed in this paper, even if we restrict ourselves only to the first two families and to at most quadrilinear terms as effective interactions.

Our model Hamiltonian for QCD-quarks is assumed to have the form:

\[
\hat{H} = U_B + \hat{H}_q + \hat{H}_{ew}
\]

\[
\hat{H}_q = \hat{H}_{0qq} + \hat{V}^{(1)}_{qq} + \hat{V}^{(2)}_{qq}
\]

If \(\Omega\) represents the quantization box volume, then \(U_B/\Omega\) is a “background energy density ” as \(\Omega \to \infty\), supposed to absorb the infinite zero-point energy of the virtual Dirac-Weyl quark seas.

\(\hat{H}_{0qq}, \hat{V}^{(1)}_{qq}, \hat{V}^{(2)}_{qq}\) and \(\hat{H}_{ew}\) are pieces of our model Hamiltonian trying to capture relevant key features of QCD-quarks and their assumed effective interactions as they emerge within our model space \(\mathcal{D}\).

In general we should include in \(\hat{H}_q\) all relevant terms that are at most quadrilinear in the preassigned fundamental fermion representations [7]. We conjecture, however , that the minimal relevant terms for trying to understand the intricate chiral and flavour dynamics of the lightest quarks can be restricted as follows:

\[
\hat{H}_{0qq}[\mathcal{D}] = \sum_{c,\mu} \epsilon_\mu [\alpha_{c\mu}^+ \alpha_{c\mu} - \beta_{c\mu} \beta_{c\mu}^+] : \tag{3}
\]

and

\[
\hat{V}^{(1)}_{qq}[\mathcal{D}] = \sum_{c,\mu} x_\mu [\alpha_{c\mu}^+ \beta_{c\mu}^+ + \beta_{c\mu} \alpha_{c\mu}] : \tag{4}
\]

\[
\hat{V}^{(2)}_{qq}[\mathcal{D}] = \sum_{c,c',\mu,\nu} \sum_{\mu',\nu'} : \alpha_{c\mu}^+ \beta_{c\mu'}^+ < \mu \bar{\mu}' | \hat{V}^{(2)}_{qq}[\mathcal{D}] | \nu \bar{\nu}' > \beta_{c'\nu} \alpha_{c'\nu} : \tag{5}
\]

We assume that the input q-numbers \(\alpha_{c\mu}^+\) etc. are massless colour triplet states carrying quantum numbers \(c\) and \(\mu \equiv (n, \vec{p}, \lambda)\) standing respectively for colour and (flavour, 3-momentum, helicity) ; \(\beta_{c\mu}^+\) are their antiparticle states, here defined to mean their CP-conjugate states. The symbol :: stands for “normal products” to be presently specified .

The coefficients \(x_\mu\) are part of the input parameters defining our model space.

We thus assume that we are dealing with the flavour representations (with usual assignements for \(u, d, s\) quarks)

\[
n = \{(\frac{1}{2}, i_3) \otimes (Q_{\frac{1}{2}}, S_{\frac{1}{2}})\} \oplus \{(0, 0) \otimes (Q_0, S_0)\} \tag{6}
\]
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\[ n = 1 \leftrightarrow (u, d) \]
\[ u = \left( \frac{1}{2}, + \frac{1}{2} \right) \otimes \left( \frac{2}{3}, 0 \right) \quad d = \left( \frac{1}{2}, - \frac{1}{2} \right) \otimes \left( - \frac{1}{3}, 0 \right) \]  

(7)

\[ n = 2 \leftrightarrow (s) \]
\[ s = (0, 0) \otimes \left( - \frac{1}{3}, -1 \right) \]  

(8)

The various relevant model matrix elements (5) are specified in Section 5. The summation symbols stand for ordinary summations over discrete quantum numbers and integrals over 3-momenta.

It is important to note that interactions among our input quasiparticles are defined only within a momentum space \( \mathcal{D} \) extending up to about \( \Lambda_\chi \) from zero momentum (Section 5).

Colour degrees of freedom are not dynamically involved in this model. We have thus nothing to say about confinement as generally understood in the literature.

3. The approximation scheme

3.1. Fermions

The general idea is to introduce a succession of canonical transformations designed to diagonalize as much of this Hamiltonian as possible. The hopefully small and neglected residual contributions can be added perturbatively, if necessary.

Thus, let us begin with the fundamental \( q \)-numbers themselves:

\[ N'_{c\mu} \equiv \begin{pmatrix} b_{c\mu} \\ d^+_{\bar{c}\mu} \end{pmatrix} = \begin{pmatrix} u_{\mu} & v_{\mu} \\ -v^*_{\mu} & u^*_{\mu} \end{pmatrix} \begin{pmatrix} \alpha_{c\mu} \\ \beta^+_{\bar{c}\mu} \end{pmatrix} = U_{\mu} N_{c\mu} \]  

(9)

The following condition upon the transformation matrix is imposed, ensuring its invertibility:

\[ |u_{\mu}|^2 + |v_{\mu}|^2 = 1 \]  

(10)

We furthermore assume (without loss of generality) that these (variational) parameters \( u_{\mu} \) and \( v_{\mu} \) are constrained to satisfy the conditions

\[ u_{\mu} = u_{\mu} = u^*_{\mu} \equiv \cos \frac{\theta_{\mu}}{2} \quad v_{\mu} = v_{\mu} = v^*_{\mu} \equiv \sin \frac{\theta_{\mu}}{2} \]  

(11)

hence

\[ U_{\mu} = \exp(i \frac{\theta_{\mu}}{2} \tau_2) \]  

(12)

where \( \tau_2 \) is one of Pauli matrices.
Our approximation scheme is basically as follows. The q-numbers \( N'_{c\mu} \) generated by the “Dirac-Bogoliubov rotation” (12) should be stationary states (to borrow Dirac’s terminology [6], see also Appendix A)):

\[
[\hat{H}, b^+_{\mu}] = E_{\mu} b^+_{\mu} \tag{13}
\]

\[
[\hat{H}, d^+_{\mu}] = E_{\mu} d^+_{\mu} \tag{14}
\]

We expect that

\[
\hat{H} q = U_{0} + \hat{H}_{I}^{(0)} + \hat{V}_{res}^{(I)}(b^+, b, d^+, d) \tag{15}
\]

\[
\hat{H}_{I}^{(0)} = \sum_{c'} \sum_{\mu} E_{\mu} [b^+_{c'\mu} b_{c\mu} - d^+_{c\mu} d_{c'\mu}] \tag{16}
\]

with \( \hat{V}_{res}^{(I)} \) being our first “residual”, which is at least quadrilinear. We choose the “Dirac-Bogoliubov rotation angles” \( \theta_{\mu} \) so that this residual interaction is as small as possible. We should, however, next check whether or not these residuals could destabilize our zeroth order solutions. We may then continue with this process, pulling more and more out of this \( \hat{V}_{res}^{(I)} \) until ordinary perturbation theory eventually may take over, if desired.

In order to reach this goal, we first compute the commutators, linearize the resulting equations and look for non-trivial solutions. Using the previously defined effective interaction and putting

\[
G_{n\lambda}(\vec{p}) =
- \sum_{c'} \sum_{n'} \sum_{\lambda'} \int d^3 \vec{p}' < n' n'; \lambda' \vec{p}' | \hat{V} | n \bar{n}; \lambda \vec{p} > u_{n'p'} v_{n'p'} \tag{17}
\]

\[
F_{n\lambda}(\vec{p}) \equiv \sum_{c'} \sum_{n'} \sum_{\lambda'} \int d^3 \vec{p}' < n' n'; \lambda' \vec{p}' | \hat{V} | n \bar{n}; \lambda \vec{p} > f_{n'\lambda n\lambda}(\vec{p}, \vec{p}) \tag{18}
\]

where

\[
f_{n'\lambda n\lambda}(\vec{p}, \vec{p}) \equiv \epsilon_{n'} \alpha_{n'p'\lambda} \alpha_{n\bar{p}n\lambda} + \beta_{n'p'\lambda} \beta_{n\bar{p}n\lambda} > \tag{19}
\]

we find Eq.(17) which will be referred to as “the gap equation”. Eqs.(18,19) represent corrections to the single quark/antiquark energies.

We find that the condition for non-trivial solutions is:

\[
|\vec{p}| + F_{n\lambda}(\vec{p}) - E_{n\lambda}(\vec{p}) \quad \epsilon_{n} + G_{n\lambda}(\vec{p})
\]

\[
\epsilon_{n} + G_{n\lambda}(\vec{p}) - \{ |\vec{p}| + F_{n\lambda}(\vec{p}) \} - E_{n\lambda}(\vec{p}) \quad = 0 \tag{20}
\]

and so

\[
E^{2}_{n\lambda}(\vec{p}) = |\vec{p}|^{2} + \Delta^{2}_{n\lambda}(\vec{p}) \tag{21}
\]

where

\[
\Delta^{2}_{n\lambda}(\vec{p}) \equiv 2 |\vec{p}| F_{n\lambda}(\vec{p}) + F^{2}_{n\lambda}(\vec{p}) + [\epsilon_{n} + G_{n\lambda}(\vec{p})]^{2} \tag{22}
\]
Furthermore
\[ v_{n\lambda}^2(\vec{p}) = \frac{1}{2} \left( 1 - \frac{\epsilon_{n\lambda}(\vec{p})}{E_{n\lambda}(\vec{p})} \right) \] (23)
\[ u_{n\lambda}^2(\vec{p}) = \frac{1}{2} \left( 1 + \frac{\epsilon_{n\lambda}(\vec{p})}{E_{n\lambda}(\vec{p})} \right) \] (24)

These are the first set of our master equations. Self-consistency requires that non-trivial solutions of eqs(20) must be positive definite.

Solutions are discussed in Section 5.

3.2. Bosons

We shall next consider another branch of flavourless boson-like vacuum excitation modes. This should give some feeling about the dynamical stability of the above vacuum solutions, at least in these particular channels. We continue to work within the framework of linearization procedures.

The Hamiltonian (15) and (16) is thus rewritten in the form:
\[ \hat{H}[\mathcal{D}] = U_0 + \sum_c \sum_n \sum_{\lambda} \int d^3\vec{p} E_n(p) \left( b_{cn\lambda}^+ b_{cn\lambda} + d_{cn\lambda}^+ d_{cn\lambda} \right) + \] (25)
\[ + \sum_{c,c'} \sum_{n,n'} \sum_{\lambda_1} \sum_{\lambda_2} \sum_{\lambda_3} \sum_{\lambda_4} \int d^3\vec{k} \int d^3\vec{k}' < n'\vec{p}'\lambda_3; \bar{n}' - \vec{p}'\lambda_4 | \hat{V} | n\vec{p}\lambda_1; \bar{n} - \vec{p}\lambda_2 > : \alpha_{c'n'\vec{p}'\lambda_3}^+ \beta_{\vec{p}'\lambda_4}^+ \chi_{\vec{p}\lambda_1} \beta_{\vec{p}\lambda_2} \alpha_{cn\lambda_1} : 
\]

We look for stationary solutions of the equation of motion:
\[ i \frac{\partial}{\partial t} \hat{O} = [\hat{H}[\mathcal{D}], \hat{O}] \] (26)

A key condition for obtaining such stationary solutions is of course that they should obey the conservation laws associated with the symmetries of the effective (model) Hamiltonian \( \hat{H}[\mathcal{D}] \).

We first look for solutions in terms of our primary composites:
\[ \hat{O} = \hat{O}(\Gamma^+, \Gamma) \]

\[ \Gamma^+_{\rho\rho'}(\vec{p}) \equiv \frac{1}{\sqrt{3}} \sum_c b_{c\rho\lambda_\alpha}^+ d_{c\rho'\bar{\rho}\lambda_\beta}^+ \] (27)
\[ \Gamma_{\rho\rho'}(\vec{p}) \equiv \frac{1}{\sqrt{3}} \sum_c d_{c\rho\lambda_\alpha} b_{c\rho'\bar{\rho}\lambda_\beta} \] (28)

In order to ease the notation we conventioned that
\[ \rho\rho' \leftrightarrow f\lambda_\alpha\lambda_\beta(\vec{p}) \leftrightarrow r \quad etc \]

Let us try the ansatz (appropriate momentum integrations understood)
\[ \hat{O} \equiv \Gamma^+_K = \sum_r X_{Kr} \Gamma^+_r - \sum_r Y_{Kr} \Gamma_r \]
In this paper, approximations are limited to linearizing the equation of motion. This is of course consistent with the basic assumption that our zeroth order solutions are truly stable ones, and therefore non-linear corrections are small and expected to contribute only to higher orders. So,

$$\Gamma^+_K(\vec{p}) = \sum_p \int_D d^3\vec{p}' X_K(p, p') \Gamma^+_r(\vec{p}') - \sum_r \int_D d^3\vec{p}' Y_K(p, p') \Gamma_r(\vec{p}')$$  \hspace{1cm} (30)$$

with the condition that

$$\sum_r \int_D d^3\vec{p}' |X_K|^2 - \sum_r \int_D d^3\vec{p}' |Y_K|^2 = 1$$  \hspace{1cm} (31)$$

We start the procedure by trying to satisfy the equations

$$[\hat{H}, \Gamma^+_K(\vec{p})] = \Omega_K(p) \Gamma^+_K(\vec{p})$$  \hspace{1cm} (32)$$

with the condition that

$$\Omega_K(p) \geq 0$$  \hspace{1cm} (33)$$

After some tedious but straightforward calculations we end up with our next set of master equations to be used for studying stability questions and conservation laws (in short-hand notation):

$$X_{K\rho\rho'} = \frac{1}{E_{\rho\rho'} - \Omega_K} \sum_{\mu\mu'} \left\{-X_{K\mu\mu'} M_{\mu\mu'\rho\rho'} + Y_{K\mu\mu'} N_{\mu\mu'\rho\rho'}\right\}$$ \hspace{1cm} (I)  \hspace{1cm} (34)$$

$$Y_{K\rho\rho'} = \frac{1}{E_{\rho\rho'} + \Omega_K} \sum_{\mu\mu'} \left\{-X_{K\mu\mu'} M_{\mu\mu'\rho\rho'} + X_{\mu\mu'} N_{\mu\mu'\rho\rho'}\right\}$$ \hspace{1cm} (II)  \hspace{1cm} (35)$$

with the definitions:

$$\rho^{\rho'} \leftrightarrow f \lambda_a \lambda_b(\vec{p}) \quad \mu^{\mu'} \leftrightarrow n \lambda_1 \lambda_2(\vec{k})$$  \hspace{1cm} (36)$$

$$E_{\rho\rho'}^{\rho\rho'} \equiv$$  \hspace{1cm} (37)$$

$$(E_{\mu} + E_{\mu'}) \delta_{\rho\rho'} + u_{\rho} v_{\rho} < \mu \rho |\hat{V}| \mu' \rho' > u_{\mu'} v_{\mu'} + u_{\rho} v_{\rho} < \rho' \mu' |\hat{V}| \rho \mu' > u_{\mu'} v_{\mu'}$$

$$M_{\rho\rho'\mu\mu'} \equiv 3 \{ u_{\mu} u_{\mu'} < \mu \rho |\hat{V}| \rho \mu' > u_{\rho} u_{\rho'} +$$

$$+ v_{\rho} v_{\rho'} < \rho' \mu' |\hat{V}| \rho \mu > v_{\mu} v_{\mu'} \}$$  \hspace{1cm} (38)$$

$$N_{\rho\rho'\mu\mu'} \equiv 3 \{ v_{\mu} v_{\mu'} < \mu' \rho' |\hat{V}| \rho \mu > v_{\mu} v_{\mu'} +$$

$$+ v_{\rho} v_{\rho'} < \rho' \mu' |\hat{V}| \rho \mu > v_{\mu} v_{\mu'} \}$$  \hspace{1cm} (39)$$

A comment on effects from couplings (25) in the Hamiltonian (not included in Eqs.36 and 37) is in order. Within our linearized theory, it can easily be verified that they do not contribute. Their effects, although numerically unimportant within the context of our assumptions and approximations, could be treated either by including the terms...
that were left out and/or by selective summations, as used in standard perturbative methods. In the latter case, the Feynman diagram rules must be constructed in such a way that one can automatically perform precise corrections for any double countings that may arise, due to this early emergence of free point-like bosons in a pure fermionic theory.

It can also be easily verified that this scheme is a conserving approximation, in the sense that it satisfies symmetries of the Hamiltonian, and thus conservation laws. Thus, for example, let \( F \) be the generator of a global gauge symmetry of the Hamiltonian. Should the approximate zero-th order vacuum solution not be invariant under \( F \), then a zero frequency solution is automatically produced, in agreement with the Goldstone theorem (further comments in Section 4). This approximation scheme respects therefore the symmetries of the Hamiltonian to leading order of small quantities (i.e. those that were left out in the calculations).

We seek non-trivial solutions for this system of equations.

4. Numerical Results and Discussions

We report in this paper the main numerical results based on the theory presented in the previous sections. Explicit expressions for the matrix elements (applicable to our model space) are given in this section.

We assume:

(i) \[
\begin{align*}
< n' \bar{n}' \bar{\lambda}; \bar{\sigma}' | \hat{V} | n \lambda; \bar{n} \lambda; \bar{\sigma} > = & < n' \bar{n'}; \lambda'; \bar{\sigma}' | \hat{V} | n \bar{n}; \lambda; \bar{\sigma} > \\
< n' \bar{n'}; +; \bar{\sigma}' | \hat{V} | n \bar{n}; +; \bar{\sigma} > = & < n' \bar{n'}; +; \bar{\sigma}' | \hat{V} | n \bar{n}; -; \bar{\sigma} > = \\
< n' \bar{n'}; -; \bar{\sigma}' | \hat{V} | n \bar{n}; +; \bar{\sigma} > = & < n' \bar{n'}; -; \bar{\sigma}' | \hat{V} | n \bar{n}; -; \bar{\sigma} > = - \frac{|G|}{4 \pi \Lambda^2} 
\end{align*}
\]

and

(ii) \[
\begin{align*}
< n' +; \bar{n'} -; \bar{\sigma}' | \hat{V} | n +; \bar{n} -; \bar{\sigma} > = & < n' -; \bar{n'} +; \bar{\sigma}' | \hat{V} | n -; \bar{n} +; \bar{\sigma} > = \frac{g_1}{4 \pi \Lambda^2} 
\end{align*}
\]

We have thus introduced a minimum of two further (dimensionless) parameters, viz. \( G \) and \( g_1 \). Note the explicit - sign on the rhs of definition 42. As we shall see, this guarantees positive definite gaps.

This is supposed to represent effective colour singlet q\( \bar{q} \) interactions at momenta below say \( \Lambda \chi \approx 1 \text{ GeV} \). This built-in limit thereby acquires a physical significance in our model.
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4.1. Light quark rest masses

The extensive analysis carried out by Leutwyler et al [5] revealed that the light quark masses obey the condition that their running masses in the $\overline{MS}$ scheme at scale $\mu = 1\,\text{GeV}$ must be

$$m_u = 5.1 \pm 0.9\,\text{MeV} \quad m_d = 9.3 \pm 1.4\,\text{MeV} \quad m_s = 175 \pm 25\,\text{MeV}$$

Using eqs.(20-24) and the effective interaction (42) we find that:

$$\bar{E}_n^2(x) \equiv \bar{e}_n^2(x) + \bar{\Delta}_n^2$$

$$\bar{e}_n(x) = x - \frac{g}{6} \bar{v}_n^2(x)$$

$$\bar{\Delta}_n = x_n + g I(\bar{\Delta}_n)$$

$$I(\bar{\Delta}_n) = \sum_n \int_0^1 x^2 \bar{u}_n(x) \bar{v}_n(x) \, dx$$

and

$$\bar{v}_n^2(x) = \frac{1}{2} (1 - \frac{\bar{e}_n(x)}{\bar{E}_n(x)})$$

$$\bar{u}_n^2(x) = 1 - \bar{v}_n^2(x)$$

with the definitions

$$p = x \Lambda_{\chi}$$

$$\Delta_n = \bar{\Delta}_n \Lambda_{\chi}$$

$$e_n(p) = \bar{e}_n(x) \Lambda_{\chi}$$

$$E_n(p) = \bar{E}_n(x) \Lambda_{\chi}$$

$$\epsilon_n = x_n \Lambda_{\chi}$$

$$g = 12|G| \Lambda_{\chi}$$

$$\Delta_n = \bar{\epsilon}_n + I(\bar{\Delta}_n)$$

2) The parameter space must be such that [5]:

$$m_s(1\,\text{GeV}) = 175 \pm 25\,\text{MeV}$$

$$m_d(1\,\text{GeV}) = 9.3 \pm 1.4\,\text{MeV}$$

$$m_u(1\,\text{GeV}) = 5.1 \pm 0.9\,\text{MeV}$$

A sample of our numerical results can be seen in Fig.1 through Fig.8. All fits satisfy the additional constraints that $|f_n| < 10^{-5}$ where

$$f_n = \bar{\epsilon}_n + I(\bar{\Delta}_n) - \bar{\Delta}_n$$

With these definitions we shall henceforth drop the bars on the symbols in (51).
We shall next examine the dynamical stability of the previous solution. A reasonable criterion for stability, or near stability, is that all solutions \( \Omega_{\kappa \lambda} \) of the master equations (34 and 35) should be positive or zero, assuming that the gap equation (47) is satisfied. The value zero would imply that the chosen vacuum solutions are unstable according to this linearized theory.

### Helicity 0 modes

These channels are non-trivial solutions of the following equations:

\[
X_{\sigma f}(x) = \frac{u_{\sigma}^2(x)}{2E_f'(x) - \Omega_{\sigma}(x)} A_{\sigma}(x) + \frac{g^2_{\sigma}}{2E_f'(x) - \Omega_{\sigma}(x)} B_{\sigma}(x) + \frac{v_{\sigma}^2(x)}{2E_f'(x) - \Omega_{\sigma}(x)} C_{\sigma}(x) - \frac{u_{\sigma}^2(x)}{2E_f'(x) - \Omega_{\sigma}(x)} D_{\sigma}(x)
\]

\[
Y_{\sigma f}(x) = \frac{u_{\sigma}^2(x)}{2E_f'(x) + \Omega_{\sigma}(x)} C_{\sigma}(x) + \frac{v_{\sigma}^2(x)}{2E_f'(x) + \Omega_{\sigma}(x)} D_{\sigma}(x) - \frac{v_{\sigma}^2(x)}{2E_f'(x) + \Omega_{\sigma}(x)} A_{\sigma}(x) - \frac{u_{\sigma}^2(x)}{2E_f'(x) + \Omega_{\sigma}(x)} B_{\sigma}(x)
\]

where

\[
E_f'(x) \equiv 2[E_f(x) - gu_f^2(x)v_f^2(x)]
\]

and the definitions

\[
\sum_n \int_0^1 dx' x'^2 X_{\sigma n}(x, x') u_n^2(x') \equiv A_{\sigma}(x)
\]

\[
\sum_n \int_0^1 dx' x'^2 X_{\sigma n}(x, x') v_n^2(x') \equiv B_{\sigma}(x)
\]

\[
\sum_n \int_0^1 dx' x'^2 Y_{\sigma n}(x, x') u_n^2(x') \equiv C_{\sigma}(x)
\]

\[
\sum_n \int_0^1 dx' x'^2 Y_{\sigma n}(x, x') v_n^2(x') \equiv D_{\sigma}(x)
\]

Putting

\[
u_n^2(x) + v_n^2(x) \equiv 1
\]

\[
u_n^2(x) - v_n^2(x) \equiv \alpha_n(x)
\]

\[K_{\sigma} \equiv \frac{g}{2}(A_{\sigma} + B_{\sigma} - C_{\sigma} - D_{\sigma})
\]
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\[ K'_\sigma \equiv \frac{g}{2} (A_\sigma - B_\sigma + C_\sigma - D_\sigma) \]

we find that

\[ X_{\sigma f}(x) = \frac{g}{2E'_f(x) - \Omega_\sigma(x)} [K_\sigma(x) + \frac{g}{2} \alpha_f(x) K'_\sigma(x)] \]

\[ Y_{\sigma f}(x) = \frac{g}{2E'_f(x) + \Omega_\sigma(x)} [-K_\sigma(x) + \frac{G}{2} \alpha_f(x) K'_\sigma(x)] \]

This leads to the determinantal equation

\[ S(\omega, x) = \begin{vmatrix} S_{11}(\omega, x) & S_{12}(\omega, x) \\ S_{21}(\omega, x) & S_{22}(\omega, x) \end{vmatrix} = 0 \]

where

\[ S_{11}(\omega, x) = 1 - g \sum_n P \int_0^1 dx' x'^2 \frac{2E'_n(x')}{(2E'_n(x'))^2 - \omega(x)^2} \]

\[ S_{12}(\omega, x) = -\frac{g^2}{2} \sum_n P \int_0^1 dx' x'^2 \frac{\alpha_n(x')}{(2E'_n(x'))^2 - \omega(x)^2} \]

\[ S_{21}(\omega, x) = 2\omega \sum_n P \int_0^1 dx' x'^2 \frac{\alpha_n(x')}{(2E'_n(x'))^2 - \omega(x)^2} \]

\[ S_{22}(\omega, x) = -1 + g \sum_n P \int_0^1 dx' x'^2 \frac{2E'_n(x') \alpha_n^2(x')}{(2E'_n(x'))^2 - \omega(x)^2} \]

where \( P \) denotes the principal-value of the integral.

Again, an essential self-consistency condition for the solutions of these equations is that the gap equations must be satisfied. Then it is trivial to show that \( \omega = 0 \) ("the Goldstone") is a solution, if all parameters that explicitly break the chiral-flavour symmetry of the effective Hamiltonian are set to zero. This is evidently a well-known quantum effect.

Graphical solutions of eq(69) for \( x = 0 \) can be found from the plots from Fig.1 through Fig.8 under various conditions. The zeroes of the function \( S(\omega, 0) \) can be interpreted as rest masses of the lowest boson-like vacuum excitations. All but two are situated amongst its poles.

We consider three cases:

(i) A pure Goldstone mode, where

\[ x_u = x_d = x_s = 0 \]

In this case, our effective Hamiltonian is chiral - flavour invariant, but this symmetry is broken by the vacuum. A common mass gap in the quark/antiquark spectrum thus arises out of quantum effects.

The results in Fig.1 and Fig.2 refer to a toy model, but one that correctly reproduces the bare bones of all our calculations, both without self energy corrections and with
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self-energy corrections. These figures illustrate a simplified case in which all q\bar{q} input energies in Eqs(45) to (50) are of the form

\[ \epsilon(i) = \epsilon \quad i = 1, 2, ..., \Omega \]  

One can then calculate everything analytically. Fig.1 illustrates how these mass gaps and g are interrelated. We can thus uniquely renormalize g by trading it with the in principle "observable" gap. The function \( S(\omega) \) (Eq. 69) can thus be expressed as

\[ S(\omega) = \omega^2 R(\omega) \]  

with \( R(0) \neq 0 \). Fig.2 shows both these functions. Besides the root at zero ("the Goldstone") there is another root, roughly 0.7 (arbitrary units). The corresponding state vectors can easily be found from the above relevant equations. The heavy state is made up of a maximally coherent superposition of all q\bar{q} basis states. The interplay between the single-particle energies and the gap is decisive in determining in which frequency range the "heavy" actually shows up.

The remaining \( \Omega - 1 \) q\bar{q} states are orthogonal to this one, and essentially made up of a single q\bar{q} configuration (i.e. some state i).

In Fig.3 the calculations are repeated, but with the input single-particle energy spectrum assumed in this paper.

The quarks energy-momentum relationships within our model space are illustrated in Fig.3 (again, with and without self-energy corrections): no quark species remains massless, but the mass degeneracy is of course not lifted. A feature worth noticing is the numerical influence of the self-energy corrections on the input single particle/antiparticle spectrum. Again, these are computed iteratively. About 10 iterations of the gap variable are sufficient to reach satisfactory convergence.

In Fig.4a and b one can see the relevant branch of vacuum excitations. The quanta of these excitations are boson-like. Note the presence of not only Goldstones (lowest rest mass), but also heavy bosons with the largest rest mass, strongly dependent both on the q,\bar{q} spectrum and on mass gaps. Sandwiched in between the poles of \( S(\omega) \), but not shown in Figs 4a and b, are other massive excitations with an intermediate degree of coherence necessary to make them normalizable and mutually orthogonal. This basic structure of our results can be gleaned from Fig.2.

(ii) A mixed Goldstone-Wigner mode, where

\[ x_u = x_d = 0 \quad ; \quad x_s > 0 \]  

In this case the quark degeneracy is obviously partially lifted. In Fig.5 and Fig.6 one can see the results for a representative g that solves the gap equations. The previous massless Goldstone now becomes a "light" pseudo-Goldstone. The move towards "constituent masses" for low momenta can still be seen.

(iii) A mixed hierarquic Goldstone -Wigner mode, where

\[ x_d > x_u > 0 \quad ; \quad x_s \gg x_d \]  

(ii) A mixed Goldstone-Wigner mode, where

\[ x_u = x_d = 0 \quad ; \quad x_s > 0 \]  

In this case the quark degeneracy is obviously partially lifted. In Fig.5 and Fig.6 one can see the results for a representative g that solves the gap equations. The previous massless Goldstone now becomes a "light" pseudo-Goldstone. The move towards “constituent masses” for low momenta can still be seen.
The relevant results are shown in Figs. 7a,7b and 8.

Fig 7a shows an overall view of the energies-vs-momenta of the three species of quarks (antiquarks). Fig 7b gives a blow-up of this figure for very small momenta. Note that the input mass difference \( x_d - x_s \approx 0.0042 \) has become reduced in the output rest energy difference \( E(d,0) - E(u,0) \approx 0.00078 \). This result however does not take into account virtual electroweak effects.

Finally, Fig.8 shows the corresponding results for bosonic excitation modes. Again, the original Goldstone has been metamorphosed into a light pseudo-Goldstone, which we now show together with all the heavies. One recalls that our \( \lambda = 0 \) basis is in fact a linear combination of spin \( S = 0 \) and \( S = 1 \), \( \lambda = 0 \) and \( I = 0 \) and \( I = 1, I_3 = 0 \) states. To identify these we need sources which inject these spins and isospins into the vacuum. The trends are however unmistakable, even with such relatively unsophisticated effective interactions as those adopted here.

We conjecture that this pseudo-Goldstone would be strongly coupled to the \( \pi^0 \) whereas \( \eta^0 \) would be strongly coupled to the heavy coherent excitation that was previously mentioned (see App.B). The \( \eta^0 \) would correspond to some intermediate superposition of the "original" \( q\bar{q} \) background, but where it eventually would land strongly depends on secondary residuals in the effective Hamiltonian not considered in this paper.

4.2.2. Helicity \( \pm 1 \) modes The equations determining excitations in these channels are formally the same as (56-58). However, this requires a further parameter \( (g_1) \), and thus we have not pursued the matter in this paper.

5. Summary

That there are self-consistent non-trivial stable solutions of the basic equations (20) and (69) suggests that there is indeed support to the conjecture [10] that the quark vacuum may well exist in the same universality class as the vacua (ground states) of some well-understood superconducting Fermi liquid systems (e.g. \( ^3\text{He} \) in its B-phase, [10]).

If so, we have a well-understood effective field theory that could realistically be expected to provide support for the standard Chiral Perturbation Theory of pions and etas [2, 3, 4, 5] in its attempts to account for features that are not determined by chiral symmetry alone. In this connection, we conjecture that Figs 4, 6 and 8 show the rest mass spectrum of boson-like particles, where the lowest mass will be strongly coupled to \( \pi^0 \), the next lowest similarly strongly coupled to \( \eta_0 \) whereas the heaviest (having a maximally coherent superposition of \( u\bar{u}, d\bar{d} \) and \( s\bar{s} \)) could be the famous \( \eta'_0 \). This is a robust feature of these calculations.

The gauge fields of the Standard Model play some sort of a background role not analysed in this paper; neither is the origin of the leading effective interactions defining
our model space (see however Ref. [7]), clearly controlling key features of the lightest mesons.

This conclusion confirms once again Dirac’s belief of the fundamental unity of Physics [6].

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Appendix A.

We sketch here for the benefit of the reader not familiar with the contents of a “Field Theory without dead wood”, the streamlined version of Field Theory proposed by Dirac [6], as the most convenient tool for expressing our ideas.

Given the effective Hamiltonian, our task is to solve the Heisenberg equations of motion [6]:

\[
i \frac{\partial}{\partial t} \hat{K}(t) = [\hat{H}, \hat{K}(t)]
\]  

(A.1)

for q-numbers that we presume to be constants of motion \( \hat{K}(t) \).

We start our approximation scheme by partitioning the Hamiltonian:

\[
\hat{H} = \hat{H}_0' + \hat{V}'
\]  

(A.2)

We first diagonalize \( \hat{H}_0' \) as best as we can. It may however break some of the symmetries of \( \hat{H} \). Approximation schemes for finding stationary solutions of (A.1) must therefore be designed so that they respect the conservation laws associated with these symmetries to any desired accuracy.

Introducing q-numbers in Dirac Picture

\[
\hat{O}_D(t) = e^{i \hat{H}_0' t} \hat{O}(t) e^{-i \hat{H}_0' t}
\]  

(A.3)

\[
\hat{V}_D(t) = e^{i \hat{H}_0' t} \hat{V}' e^{-i \hat{H}_0' t}
\]  

(A.4)

we then obtain:

\[
i \frac{\partial}{\partial t} \hat{O}_D(t) = [\hat{V}_D(t), \hat{O}_D(t)]
\]  

(A.5)

Setting

\[
\hat{O}_D(t) = \sum_{n=0}^{\infty} \hat{O}_D^{(n)}(t)
\]  

(A.6)

we find the conditions:

\[
\frac{\partial}{\partial t} \hat{O}_D^{(0)}(t) = 0
\]  

(A.7)
and the recursion relations
\[ i \frac{\partial}{\partial t} \hat{O}^{(n)}_D(t) = [\hat{V}_D(t), \hat{O}^{(n-1)}_D(t)] \quad n \geq 1 \quad (A.8) \]

The first condition gives
\[ \hat{O}^{(0)}_D(t) \equiv \hat{K} \quad (A.9) \]

where the q-number \( \hat{K} \) is not explicitly time-dependent.

The remaining corrections are given by the recursion relation:
\[ \hat{O}^{(n)}_D(t) = (-i) \int^t dt_1 [\hat{V}_D(t_1), \hat{O}^{(n-1)}_D(t_1)] \quad n \geq 1 \quad (A.10) \]

Before one can interpret the results at any given stage, one must apply the q-numbers thus obtained to the standard (Heisenberg) reference ket \(|0>_t\). This reference ket is not the usual “vacuum state vector”, as we have no such concept here, but simply helps to define normal products at time \( t \) only. We adopt the following procedure [6]:

(i) Compute the commutator and then apply Wick’s theorem [8], using of course all the previously established relevant conditions, if any. This makes significant the need for normal ordering in this q-number theory;

(ii) Carry out the time integrations;

(iii) Pick up the coefficient of the i’th term, and interpret this coefficient as the prediction/postdiction of this theory for what the amplitude for the i’th configuration (state) to emerge at any time \( t \) is, i.e.:
\[ \hat{O}(t)|0>_t = \sum_{N=0}^{\infty} \hat{O}^{(N)}(t)|0>_t = \sum_{N=0}^{\infty} \sum_{\nu} \{|\nu><\nu|\hat{O}^{(N)}(t)|0>_t\}_t \quad (A.11) \]

As emphasized by Dirac, it is not possible within this framework to make predictions, or postdictions, of what can be, or could be, observed at any time other than \( t \), even if one possessed a complete knowledge of all amplitudes at time \( t \). A critical discussion and assessment of this procedure was given by Dirac in the quoted references.

Appendix B.

Let us speculate on the structure of the lightest self-conjugate physical mesons at rest. We rely on an assertion, common to all versions of the Quark Model, that mesons are \( q\bar{q} \) colour singlet composites. So, we define these as the composites
\[ \hat{O} \equiv \Gamma^{+}_{\nu IJ\lambda}(\vec{p}) = \sum_i \int_D d^3p' X_{\nu(iI)J\lambda}(p,p') \Gamma^{+}_{(iI)J\lambda}(p') \\
- \sum_i \int_D d^3p' Y_{\nu(iI)J\lambda}(p,p') \Gamma_{(iI)J\lambda}(p') \quad (B.1) \]

Putting
\[ \nu IJ\lambda \equiv M \]
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\[(iq)\lambda_1\lambda_2 \equiv r; \quad \lambda = \lambda_1 - \lambda_2\]

we get

\[\hat{O} \equiv \Gamma^+_M(\vec{\rho}) = \sum_r \int_D d^3\vec{p}' X_Mr(p,p')\Gamma^+_r(\vec{p}) - \sum_r \int_D d^3\vec{p}' Y_Mr(p,p')\Gamma_r(\vec{p})\] (B.2)

where

\[X_{\nu(iI),J\lambda}(p) \equiv \sum_{q=-i}^{+i} \sum_{\lambda_1,\lambda_2} \left(\frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2\right)|J\lambda)(iqi - q|I0)X_{M(iq)\lambda_1\lambda_2}(p)\] (B.3)

and mutatis mutandis for the \(Y\) amplitudes.

The amplitudes \(X_{\nu(iI),J\lambda}(p, p')\) and \(Y_{\nu(iI),J\lambda}(p, p')\) can be determined after finding non-trivial solutions of the system of equations (34) and (35).

Meson-like particles carrying spin \(J\) and isospin \(I\) at rest would then be describable by wave-packets such as

\[\Gamma^+_M = \int d^3\vec{p}\Psi_M(\vec{p})\Gamma^+_M(\vec{p})\] (B.4)

where

\[\int d^3\vec{p}|\Psi_M(\vec{p})|^2 = 1\] (B.5)

Within these approximations, these composites would appear to be structureless point-like spin 0 and spin 1 \textit{bona fide} free bosons carrying good quantum numbers, in all observations involving small momentum transfers. The amplitudes \(X\) and \(Y\) (B.3) will be explicitly needed, should one wish to further test this conjecture, e.g. by studying electroweak probings of these point-like mesonic quasiparticles at rest. Some residual strong interaction effects, describable within the present scheme, should be treated by first returning to our basic theory (Section 2) and going beyond the leading approximations, but still within the framework of conserving approximations. In any case, this procedure could be a good support to Chiral Perturbation theory as applied to pions and etas [3] beyond its leading approximations.

Specific, but easily handled, field theoretical models such as this one could, among other duties, thus help in testing the widespread suspicion that the non-chiral quark model description of the lightest mesons, as used in more or less sophisticated so-called relativised versions, e.g. [9], is at best too crude to account for these particular mesons and their interactions.

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Figure captions

**Figure 1.** This shows how the “coupling constant $g$” is renormalized in this paper, i.e. how it can be uniquely traded with an (in principle) “observable” mass-gap parameter $\Delta$. $g$ is dimensionless, and the gap parameter is in units of 1 GeV. The lower curve a) does not include self-energy corrections, but the upper curve b) does.

The parameters used in this toy model are: $\epsilon = 0.2$ ; $\Omega = 1000$.

**Figure 2.** This illustrates how certain relevant vacuum Bose modes arise in the toy model: a Goldstone (playing the role of pions) and a heavy mode (playing the role of $\eta^0$?). This is a highly coherent mode, in contrast to the other modes, which remain in nearly the same state as for $g = 0$, and are sandwiched among the poles of $S(\omega)$. The parameters used in making Fig2. are: $\epsilon = 0.2, \Delta = 0.3$.

**Figure 3.** We plot here the energy - momentum relationship in case A (mass-degenerate quarks). Dimensionsful variables are in units of 1 GeV. No self-energy corrections are included in curve $E(1,p)$, but they are fully included in the curve $E(2,p)$. Note the importance (in our case) of the self-energy corrections that tend to push the $p = 0$ masses towards something like ”observed constituent ” values, especially for small momenta. The parameters used in making Fig.3 and Fig.4 are: $\Delta = 0.05 \; (g = 1.34)$.

**Figure 4.** a) We plot here the function $S(\omega,0)$ in case A (massless quarks). Variables having dimensions are in units of 1 GeV. We show only the zeros of $S(\omega)$ which are of interest from the perspective of this paper, i.e. the lightest and the heaviest of the excitations. Note the Goldstone at $\omega = 0$.

b) Note the ”heaviest”, highly coherent excitation at about $\omega \sim 2\text{GeV}$. The self-consistency conditions are fully respected. Intermediate zeroes have not been shown.

**Figure 5.** We plot here the energy - momentum relationship in case B (massless u,d-quarks). Variables with dimensions are in units of 1 GeV. Self-energy corrections are fully included.

The parameters used in making this figure are: $x_u = x_d = 0$; $x_s = 0.2$ and $g = 0.08$.

We find that $E(u,d;0) = 0.008 \; ; E(s;0) = 0.205$. 
Figure 6. We plot here the function $S(\omega, 0)$ in case B. Quantities with dimensions are in units of 1 GeV. Self-energy corrections are fully included. The same parameters as in Fig5.

Note the disappearance of the Goldstone ($S(0, 0) \sim -0.9$). Instead, we get a pseudo-Goldstone with a rest mass somewhere between $\omega = 0.4$ and $\omega = 0.6$.

The “heavy” of special interest to us is the last zero. Its rest mass emerges between $\omega = 1.2$ and $\omega = 1.4$. with our particular choice of parameters. In all cases the details of the $q\bar{q}$ spectrum and the gap parameters are crucial in determining where the masses actually emerge.

Figure 7. a) We plot here the energy - momentum relationship in case C (3 massive quarks). The dimensionfull variables are in units of 1 GeV. Self-energy corrections are fully included. The parameters used in this calculation are:

$x_u = 0.0051$, $x_d = 0.0093$, $x_s = 0.1750$ ; $g = 0.05$

b) We show in this figure for clarity a blow-up of the low momenta part of a)

Figure 8. We plot here the function $S(\omega, 0)$ in case C (massive quarks). Dimensionsful variables are in units of 1 GeV. Only energy corrections on the $q\bar{q}$ energies consistent with the often mentioned self-consistency conditions are fully included. Same parameters as used in Fig.7a. See text for further comments.
Fig. 3

$E(p)$

$E(1,p)$

$E(2,p)$
FIG. 4a)
$S(0) \approx -0.9$
$E(u,d;p) \times 1000$

$E(u,0) = 0.05055$
$E(d,0) = 0.05133$

FIG. 7b)
$S(\omega)$