Lax Representations and Zero Curvature Representations by Kronecker Product

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Abstract

It is showed that Kronecker product can be applied to construct not only new Lax representations but also new zero curvature representations of integrable models. Meantime a different characteristic between continuous and discrete zero curvature equations is pointed out.

Lax representation and zero curvature representation play an important role in studying nonlinear integrable models in theoretical physics. It is based on such representations that the inverse scattering transform is successfully developed (see, say, Ablowitz and Clarkson 1991). They may also provide a lot of information, such as integrals of motion, master symmetries and Hamiltonian formulation. There exist quite many integrable models to possess Lax representation or zero curvature representation (Faddeev and Takhtajan 1987, Das 1989). Two typical examples are Toda lattice (Flaschka 1974) and AKNS systems (Ablowitz, Kaup, Newell and Segur 1974) including KdV equation and nonlinear Schödinger equation.

In this paper, we want to give rise to a kind of new Lax representations and new zero curvature representations by using Kronecker product of matrices, motivated by a recent progress made by Steeb and Heng (Steeb and Heng 1996). Kronecker product itself has nice mathematical properties and important applications in many fields of physics, for example, statistical physics, quantum groups, etc. (Steeb 1991). Our result for zero curvature representation also provides us with a different characteristic between continuous and discrete zero curvature equations.

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Let \( I_M \) denote the unit matrix of order \( M \), \( M \in \mathbb{Z} \). For two matrices \( A = (a_{ij})_{pq}, B = (b_{kl})_{rs} \), Kronecker product \( A \otimes B \) is defined by (Steeb 1991)

\[
A \otimes B = (a_{ij}B)_{(pr) \times (qs)},
\]

or equivalently by (Hoppe 1992)

\[
(A \otimes B)_{ij,kl} = a_{ik}b_{jl}.
\]

Evidently we have a basic relation on Kronecker product (Steeb 1991, Hoppe 1992)

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD),
\]

provided that the matrices \( AC \) and \( BD \) make sense. This relation will be used to show new structure of Lax representation and zero curvature representation of integrable models.

**Theorem 1** (Lax representation) Assume that an integrable model (continuous or discrete) has two Lax representations

\[
L_{1t} = [A_1, L_1], \quad L_{2t} = [A_2, L_2],
\]

where \( L_1, A_1 \) and \( L_2, A_2 \) are \( M \times M \) and \( N \times N \) matrices, respectively. Define

\[
L_3 = \alpha_1 L_1 \otimes L_2 + \alpha_2 (L_1 \otimes I_N + I_M \otimes L_2), \quad A_3 = A_1 \otimes I_N + I_M \otimes A_2,
\]

where \( \alpha_1, \alpha_2 \) are arbitrary constants. Then the same integrable model has another Lax representation \( L_{3t} = [A_3, L_3] \).

**Proof:** First of all, we have

\[
L_{3t} = \alpha_1 (L_{1t} \otimes L_2 + L_1 \otimes L_{2t}) + \alpha_2 (L_{1t} \otimes I_N + I_M \otimes L_{2t}).
\]

On the other hand, using (3) we can calculate that

\[
[L_3, L_3] = \alpha_1 ([A_1, L_1] \otimes L_2 + L_1 \otimes [A_2, L_2])
\]

\[
+ \alpha_2 ([A_1, L_1] \otimes I_N + I_M \otimes [A_2, L_2]).
\]

Now we easily find that the equalities defined by (4) implies \( L_{3t} = [A_3, L_3] \).

When \( \alpha_2 = 0 \), the obtained result is exactly one in Ref. Steeb and Heng 1996. When \( \alpha_1 = 0 \), we get a new Lax representation for a given integrable model, starting
from two known Lax representations. Integrals of motion may also be generated from new Lax representation, because we have

\[
F_{ij} = \text{tr}(\alpha_1 L_i^1 \otimes L_j^2 + \alpha_2 (L_i^1 \otimes I_N + I_M \otimes L_j^2))
\]

\[
= \alpha_1 \text{tr}(L_i^1)\text{tr}(L_j^2) + \alpha_2 (N\text{tr}(L_i^1) + M\text{tr}(L_j^2)),
\]

(7)

where we have used \(\text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B)\) (Steeb 1991) and \((L_i^1)_t = [A_1, L_i^1], (L_j^2)_t = [A_2, L_j^2].\)

**Theorem 2** (Continuous zero curvature representation) Assume that a continuous integrable model has two continuous zero curvature representations

\[
U_{1t} - V_{1x} + [U_1, V_1] = 0, \quad U_{2t} - V_{2x} + [U_2, V_2] = 0,
\]

(8)

where \(U_1, V_1\) and \(U_2, V_2\) are \(M \times M\) and \(N \times N\) matrices, respectively. Define

\[
U_3 = U_1 \otimes I_N + I_M \otimes U_2, \quad V_3 = V_1 \otimes I_N + I_M \otimes V_2.
\]

(9)

Then the same integrable model has another continuous zero curvature representation

\[
U_{3t} - V_{3x} + [U_3, V_3] = 0.
\]

(10)

**Proof:** The proof is also a direct computation. We first have

\[
U_{3t} = U_{1t} \otimes I_N + I_M \otimes U_{2t},
\]

\[
U_{3x} = U_{1x} \otimes I_N + I_M \otimes U_{2x}.
\]

Second, using (3) we can obtain that

\[
[U_3, V_3] = [U_1, V_1] \otimes I_N + I_M \otimes [U_2, V_2].
\]

(11)

Therefore we see that (10) is true once two equalities defined by (8) hold.

We remark that when we choose

\[
U_3 = U_1 \otimes U_2,
\]

the third zero curvature representation (10) is not certain to be true. An example will be displayed later on.
Theorem 3 (Discrete zero curvature representation) Assume that a discrete integrable model has two discrete zero curvature representations

\[ U_{1t} = (EV_1)U_1 - U_1V_1, \quad U_{2t} = (EV_2)U_2 - U_2V_2, \]  

(12)

where \( E \) is the shift operator, \( U_1, V_1 \) are \( M \times M \) matrices, and \( U_2, V_2 \) are \( N \times N \) matrices. Define

\[ U_3 = U_1 \otimes U_2, \quad V_3 = V_1 \otimes I_N + I_M \otimes V_2. \]  

(13)

Then the same integrable model has another discrete zero curvature representation

\[ U_{3t} = (EV_3)U_3 - U_3V_3. \]  

(14)

Proof: Similarly, we first have

\[ U_{3t} = U_{1t} \otimes U_2 + U_1 \otimes U_{2t}. \]  

(15)

On the other hand, we may calculate that

\[
(EV_3)U_3 - U_3V_3 = ((EV_1) \otimes I_N + I_M \otimes (EV_2))(U_1 \otimes U_2) \\
- (U_1 \otimes U_2)(V_1 \otimes I_N + I_M \otimes V_2) \\
= ((EV_1)U_1) \otimes U_2 + U_1 \otimes ((EV_2)U_2) - (U_1V_1) \otimes U_2 - U_1 \otimes (U_2V_2) \\
= ((EV_1)U_1 - U_1V_1) \otimes U_2 + U_1 \otimes ((EV_2)U_2 - U_2V_2)。
\]

In the second equality above, we have used the basic relation (3). Hence we find that (14) holds if we have (12).

We remark that when we choose

\[ U_3 = U_1 \otimes I_M + I_N \otimes U_2, \]

the third discrete zero curvature representation (14) is not certain to be true. An example will also be given later on. This is opposite to the result in the continuous case. It shows us a different characteristic between continuous and discrete zero curvature equations.

In what follows, we would like to display some concrete examples to illustrate the use of the above technique of Kronecker product. Actually once we have a Lax representation or a zero curvature representation, we can obtain a new representation after choosing two required representations to be this known one. Further newer
representation may be constructed by use of this new representation and the process may be infinitely proceeded to. This also tells us that there exist infinitely many Lax representations or zero curvature representations once there exists one representation for a given integrable model. The concrete procedure of construction will be showed in the following examples and can be easily generalized to other integrable models, for example, in Refs. Calogero 1994, Drinfel’d and Sokolov 1984, Ma 1993, Ragnisco and Santini 1990, Tu 1990 etc.

**Example 1:** We consider periodical Toda lattice (Flaschka 1974)

\[ a_{it} = a_{i}(b_{i+1} - b_{i}), \quad b_{it} = 2(a_{i}^2 - a_{i-1}^2), \quad a_{i+N} = a_{i}, \quad b_{i+N} = b_{i}, \]  \hspace{1cm} (16)

which is a Hamiltonian system with Hamiltonian

\[ H(q_1, q_2, \cdots, q_N, p_1, p_2, \cdots, p_N) = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} e^{q_i - q_{i+1}} \]

under the Flaschka’s transformation

\[ a_i = \frac{1}{2}e^{q_i - q_{i+1}}, \quad b_i = -\frac{1}{2} p_i. \]

Toda lattice (16) has a Lax representation with

\[ L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & a_N \\ a_1 & b_2 & a_2 & \cdots & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{N-1} \\ a_N & \cdots & \cdots & a_{N-1} & b_N \end{pmatrix}, \]  \hspace{1cm} (17)

\[ A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & -a_N \\ -a_1 & 0 & a_2 & \cdots & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{N-1} \\ a_N & \cdots & \cdots & -a_{N-1} & 0 \end{pmatrix}. \]  \hspace{1cm} (18)

Through Theorem 1, we obtain a new Lax representation with

\[ L_{new} = \alpha_1 L \otimes L + \alpha_2 (L \otimes I_N + I_N \otimes L), \quad A_{new} = A \otimes I_N + I_N \otimes A. \]  \hspace{1cm} (19)
Here \( \alpha_1, \alpha_2 \) are two arbitrary constants and thus Toda lattice (16) has a lot of different Lax representations. By (7), new integrals of motion may be generated, which are all functions of \( F_i = \text{tr}(L^i) \).

**Example 2:** Nonlinear Schrödinger model (Ablowitz, Kaup, Newell and Segur 1974, Ma and Strampp 1994)

\[
\begin{aligned}
p_t &= -\frac{1}{2} q_{xx} + p^2 q, \\
q_t &= \frac{1}{2} p_{xx} - pq^2 \\
\end{aligned}
\]  

has a continuous zero curvature representation with

\[
U = \begin{pmatrix}
-\lambda & p \\
q & \lambda
\end{pmatrix}, 
\quad V = \begin{pmatrix}
-\lambda^2 + \frac{1}{2} pq & \lambda p - \frac{1}{2} p_x \\
\lambda q + \frac{1}{2} q_x & \lambda^2 - \frac{1}{2} pq
\end{pmatrix}.
\]  

This model has infinitely many symmetries and integrals of motion. According to Theorem 2, we obtain new continuous zero curvature representations with

\[
U_{\text{new}} = U \otimes I_2 + I_2 \otimes U = \begin{pmatrix}
-2\lambda & p & p & 0 \\
q & 0 & 0 & p \\
q & 0 & 0 & p \\
0 & q & q & 2\lambda
\end{pmatrix}, 
\]  

\[
V_{\text{new}} = V \otimes I_2 + I_2 \otimes V = \begin{pmatrix}
-2\lambda^2 + pq & \lambda p - \frac{1}{2} p_x & \lambda p - \frac{1}{2} p_x & 0 \\
\lambda q + \frac{1}{2} q_x & 0 & 0 & \lambda p - \frac{1}{2} p_x \\
\lambda q + \frac{1}{2} q_x & 0 & 0 & \lambda p - \frac{1}{2} p_x \\
0 & \lambda q + \frac{1}{2} q_x & \lambda q + \frac{1}{2} q_x & 2\lambda^2 - pq
\end{pmatrix},
\]

or with

\[
U_{\text{new}} = U \otimes I_4 + I_2 \otimes \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, 
\quad V_{\text{new}} = V \otimes I_4 + I_2 \otimes \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}.
\]

The above spectral operator defined by (22) is similar to one appearing in Khasilev 1992 and we may also discuss its binary nonlinearization (for the cases of 2 \( \times \) 2 and 3 \( \times \) 3 matrices, see Ma and Strampp 1994, Ma, Fuchssteiner and Oevel 1996). However nonlinear Schrödinger model (20) haven’t the continuous zero curvature representation with

\[
U_{\text{new}} = U \otimes U, 
\quad V_{\text{new}} = V \otimes I_2 + I_2 \otimes V.
\]

**Example 3:** We consider Bogoyavlensky lattice (Fuchssteiner and Ma 1996)

\[
u_t = u(u^{-1} - u^1), 
\quad u^{(m)} = E^m u.
\]
More examples of lattice may be found in Refs. Steeb 1991, Ragnisco and Santini 1990 and Tu 1990 etc., for example, one-dimensional isotropic Heisenberg model. The lattice (26) has a discrete zero curvature representation with

\[
U = \begin{pmatrix}
1 & u \\
\lambda^{-1} & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
\frac{1}{2} \lambda - u & \lambda u \\
1 & -\frac{1}{2} \lambda - u^{(-1)}
\end{pmatrix}.
\] (27)

By Theorem 3, we obtain new discrete zero curvature representations with

\[
U_{\text{new}} = U \otimes \begin{pmatrix}
0 & 0 \\
0 & U
\end{pmatrix}, \quad V_{\text{new}} = V \otimes I_4 + I_2 \otimes \begin{pmatrix}
V & 0 \\
0 & V
\end{pmatrix},
\] (28)

or with

\[
U_{\text{new}} = \begin{pmatrix}
U & 0 \\
0 & U
\end{pmatrix} \otimes \begin{pmatrix}
U & 0 \\
0 & U
\end{pmatrix}, \quad V_{\text{new}} = \begin{pmatrix}
V & 0 \\
0 & V
\end{pmatrix} \otimes I_6 + I_2 \otimes \begin{pmatrix}
V & 0 \\
0 & V
\end{pmatrix}.
\] (29)

The latter is two $24 \times 24$ matrices. If we want to directly find these two matrices, we will meet a lot of complicated calculation. It is worth to point out that we haven’t the discrete zero curvature representation with

\[
U_{\text{new}} = U \otimes I_2 + I_2 \otimes U, \quad V_{\text{new}} = V \otimes I_2 + I_2 \otimes V
\] (30)

for Bogoyavlensky lattice (26). This is not strange and shows us a difference between two kinds of zero curvature representations.

Finally we present an open problem. We denote the Gateaux derivative $K'[S]$ by $K'[S] = \frac{\partial}{\partial \varepsilon} K'(u + \varepsilon S)|_{\varepsilon=0}$. We have already established the following result (Ma 1992, Ma 1993, Fuchssteiner and Ma 1996): If $u_t = K(u), \ u_t = S(u)$ have Lax representations

\[
L_t = [A_1, L], \quad L_t = [A_2, L]
\]
or zero curvature representations

\[
U_t - V_{1x} + [U, V_1] = 0 \quad (\text{or } U_t = (EV_1)U - UV_1),
\]

\[
U_t - V_{2x} + [U, V_2] = 0 \quad (\text{or } U_t = (EV_2)U - UV_2),
\]

respectively, then the product model $u_t = [K, S] := K'[S] - S'[K]$ has Lax representation

\[
L_t = [A_3, L], \quad A_3 = A'_1[S] - A'_2[K] + [A_1, A_2]
\]
or zero curvature representation

\[ U_t - V_{3x} + [U, V_3] = 0 \quad (\text{or } U_t = (EV_3)U - UV_3), \quad V_3 = V_1'[S] - V_2'[K] + [V_1, V_2]. \]

Therefore \( u_t = [K, S] \) has Lax representation with the spectral operator and the Lax operator determined by Kronecker product. For example, in the case of Lax representation we have

\[
L_{\text{new}} = \alpha_1 L \otimes L + \alpha_2 (L \otimes I_M + I_M \otimes L),
\]

\[
A_{\text{new}} = (A_1'[S] - A_2'[K] + [A_1, A_2]) \otimes I_M + I_M \otimes (A_1'[S] - A_2'[K] + [A_1, A_2]),
\]

where \( M \) is the order of the matrix \( L \). Product models may be applied to construct symmetries of nonlinear models and thus they are important. Let us now suppose that two models \( u_t = K(u), u_t = S(u) \) have two completely different Lax representations

\[ L_{1t} = [A_1, L_1], \quad L_{2t} = [A_2, L_2] \]

or two zero curvature representations

\[
U_{1t} - V_{1x} + [U_1, V_1] = 0 \quad (\text{or } U_t = (EV_1)U - UV_1),
\]

\[
U_{2t} - V_{2x} + [U_2, V_2] = 0 \quad (\text{or } U_t = (EV_2)U - UV_2).
\]

Here \( L_1 \) and \( L_2 \) or \( U_1 \) and \( U_2 \) are not equal, and sometimes they may have different orders of matrix. The problem is what the corresponding representation for the product model \( u_t = [K, S] \) is. It seems to us that the required spectral operator matrix \( L_{\text{new}} \) or \( U_{\text{new}} \) should be represented by some Kronecker product involving \( L_1, L_2 \) or \( U_1, U_2 \) and \( K, S \).
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