Solving a fractional parabolic-hyperbolic free boundary problem
which models the growth of tumor with drug application using finite
difference-spectral method

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Abstract In this paper, a free boundary problem modelling the growth of tumor is considered. The model includes two reaction-diffusion equations modelling the diffusion of nutrient and drug in the tumor and three hyperbolic equations describing the evolution of three types of cells (i.e. proliferative cells, quiescent cells and dead cells) considered in the tumor. Due to the fact that in the real situation, the subdiffusion of nutrient and drug in the tumor can be found, we have changed the reaction-diffusion equations to the fractional ones to consider other conditions and study a more general and reliable model of tumor growth. Since it is important to solve a problem to have a clear vision of the dynamic of tumor growth under the effect of the nutrient and drug, we have solved the fractional free boundary problem. We have solved the fractional parabolic equations employing a combination of spectral and finite difference methods and the hyperbolic equations are solved using characteristic equation and finite difference method. It is proved that the presented method is unconditionally convergent and stable to be sure that we have a correct vision of tumor growth dynamic. Finally, by presenting some numerical examples and showing the results, the theoretical statements are justified.

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1 Introduction

Cancer is one of the most leading causes of death and many different tumor types in the human body are diagnosed such as Glioblastomas, phyllodes tumors and so on. Glioblastomas are the most common and malignant primary brain tumor, which are very aggressive, with the ability to recur despite extensive treatment [1,2]. Phyllodes tumors are breast tumors and most often are benign (even when they are benign, they can show recurrence), but in some cases they can be malignant [3]. Due to the importance of the treatment of the malignant tumors, scientists including the mathematicians have studied the cancer problem using different tools and techniques. Mathematicians have applied mathematical modelling techniques to model the various aspects of cancer dynamics such as avascular and vascular tumor growth, invasion, metastasis and so on. For instance, the authors of [4,6] studied the mathematical models of vascular tumors while the author of [5] investigated the mathematical models of avascular tumors. Owing to the fact that, low concentrations of glucose and oxygen in the inner regions of spheroids may contribute to the formation of many types of cell subpopulations such as quiescent-, hypoxic-, anoxic- and necrotic cells [8], therefore some tumor growth models have divided alive cells into proliferative and quiescent cells [5,9]. Since, in vitro results show that in the early stages the solid tumors grow approximately spherically symmetric [7], in most of the models it is assumed that the tumor grows radially-symmetric. As we mentioned above, treatment of tumors and destroying them is very important, therefore many researchers have investigated the models of tumor growth in which the treatment of tumor is taken into account [10,11,12]. In [10], a mathematical model for the combined treatment of chemotherapy and radiation in Non-Small Cell Lung Cancer patients is developed to improve the treatment strategies for future clinical trials. The authors of [11] applied a system of nonlinear ordinary differential equations to analyse a mathematical model of the treatment of colorectal cancer. In the model the effects of immunotherapy and chemotherapy on the tumor cells and cancer stem cells is described. In another study [13], applying a system of four coupled partial differential equations,
the interaction between normal, immune and tumor cells in a tumor with a chemotherapeutic drug is described. One of the models with mentioned properties, i.e. in which three types of cells including proliferative, quiescent and dead cells are considered, the tumor is assumed to grow radially symmetric, and the effect of drug on the treatment of the tumor is also taken into account, is presented in [5], which is briefly as follows:

\[
\frac{\partial C}{\partial t} = D_1 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial C}{\partial r} \right) - F(C, P, Q), \quad 0 < r < R(t), \quad t > 0, \quad (1)
\]

\[
\frac{\partial C}{\partial r}(r, t) = 0 \text{ at } r = 0, \quad C(r, t) = C(t) \text{ at } r = R(t), \quad t > 0,
\]

\[
C(r, 0) = C_0(r), \quad 0 \leq r \leq R_0,
\]

\[
\frac{\partial W}{\partial t} = D_2 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial W}{\partial r} \right) - G(W, P, Q), \quad 0 < r < R(t), \quad t > 0, \quad (4)
\]

\[
\frac{\partial W}{\partial r}(r, t) = 0 \text{ at } r = 0, \quad W(r, t) = W(t) \text{ at } r = R(t), \quad t > 0,
\]

\[
W(r, 0) = W_0(r), \quad 0 \leq r \leq R_0,
\]

\[
\frac{\partial P}{\partial t} + \dot{v} \frac{\partial P}{\partial r} = g_{11}(C, W, P, Q, D) P + g_{12}(C, W, P, Q, D) Q + g_{13}(C, W, P, Q, D) D, \quad (7)
\]

\[
\frac{\partial Q}{\partial t} + \dot{v} \frac{\partial Q}{\partial r} = g_{21}(C, W, P, Q, D) P + g_{22}(C, W, P, Q, D) Q + g_{23}(C, W, P, Q, D) D, \quad (8)
\]

\[
\frac{\partial D}{\partial t} + \dot{v} \frac{\partial D}{\partial r} = g_{31}(C, W, P, Q, D) P + g_{32}(C, W, P, Q, D) Q + g_{33}(C, W, P, Q, D) D, \quad (9)
\]

\[
0 \leq r \leq R(t), \quad t > 0,
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \dot{v} \right) = h(C, W, P, Q, D), \quad 0 < r \leq R(t), \quad t > 0, \quad (10)
\]

\[
\dot{v}(0, t) = 0, \quad t > 0,
\]

\[
\frac{dR(t)}{dt} = \dot{v}(R(t), t), \quad t > 0, \quad (12)
\]

\[
P(r, 0) = P_0(r), \quad Q(r, 0) = Q_0(r), \quad D(r, 0) = D_0(r), \quad R(0) = R_0, \quad 0 \leq r \leq R_0,
\]

where \(C\) and \(W\) are the concentration of nutrient and drug, respectively. \(P, Q\) and \(D\) are densities of proliferative cells, quiescent cells and dead cells, respectively and \(R(t)\) is the radius of tumor at time \(t\).
Also, it’s assumed that the initial data satisfies the following conditions

\[ 0 \leq C_0(r) \leq \overline{C}(0), \quad 0 \leq W_0(r) \leq \overline{W}(0), \quad 0 \leq r \leq R_0, \]
\[ \frac{\partial C_0(r)}{\partial r} = 0 \text{ at } r = 0, \quad C_0(R_0) = \overline{C}(0), \]
\[ \frac{\partial W_0(r)}{\partial r} = 0 \text{ at } r = 0, \quad W_0(R_0) = \overline{W}(0), \]
\[ P_0(r) \geq 0, \quad Q_0(r) \geq 0, \quad D_0(r) \geq 0, \quad 0 \leq r \leq R_0, \]
\[ P_0(r) + Q_0(r) + D_0(r) = N, \quad 0 \leq r \leq R_0. \]  

In this model, it is assumed that the nutrient and drug diffuse throughout the tumor with diffusion coefficient \( D_1 \) and \( D_2 \), respectively. But in a real situation, subdiffusion of nutrient and drug in the tumor can be found. Therefore, many papers are devoted to studying and solving the fractional-order mathematical models of the dynamic of cancers and different methods are employed to deal with fractional models. For instance, the application of the homotopy perturbation method to two-point boundary-value problems with fractional-order derivatives of Caputo type is studied in [14]. In another study [15], the fractional-order mathematical model involved three Michaelis Menten nonlinear terms and two types of treatments is numerically solved. In [16], a fractional-order cancer chemotherapy effect model in Caputo sense is formulated, which is studied and analysed by Ansarizadeh et al in [13]. q-HATM is used to solve the system of equations with a chemotherapeutic drug, describing the interaction among tumor cells, immune cells, and normal cells in a tumor.

In this paper, we have considered fractional parabolic equations to deal with a more reliable model of tumor growth. Then, we have solved the obtained problem, which includes two fractional parabolic equations and three hyperbolic equations, employing a combination of finite difference method and spectral method. Solving the problem enables us to have a clear vision of the dynamic of the tumor under the effect of nutrient and drug. For having an accurate vision, it is of great importance to prove the convergence of the method to be sure that the results are trustable. Therefore, we have also proved the unconditional convergence and stability of the method. Finally, by presenting some numerical examples together with the results, the theoretical statements are justified.
2 Fractional model of tumor growth

In this article, it is aimed to solve a fractional parabolic-hyperbolic free boundary problem modelling the growth of tumor with drug application which consists of two fractional parabolic and three hyperbolic differential equations and one ordinary differential equation which are coupled together. The model with integer derivatives (i.e. the model given in (1)–(14)) is presented in [3] but we have changed the problem to a fractional one to consider the subdiffusion of nutrient and drug. Moreover, since it is supposed that the tumor grows radially symmetric with free boundary so we can change the domain of model to the following fixed domain

\[ \{ (\rho, t) \mid 0 \leq \rho \leq 1, \ t \geq 0 \} . \]

Using the following change of variables

\[ \rho = \frac{r}{R(t)} , \ c(\rho, t) = C(r, t) , \ w(\rho, t) = W(r, t) , \ v(\rho, t) = \dot{v}(r, t) , \]

\[ p(\rho, t) = P(r, t) , \ q(\rho, t) = Q(r, t) , \ d(\rho, t) = D(r, t) , \]

and by considering the subdiffusion of nutrient and drug the model becomes

\[ \frac{\partial c}{\partial t} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \left( \frac{D_1}{(R(t)\rho)^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c}{\partial \rho} \right) \right) + \frac{v(1, t)\rho}{R(t)} \frac{\partial c}{\partial \rho} - f(c, p, q) , \ 0 < \rho < 1, \ t > 0 , \]

\[ \frac{\partial c}{\partial \rho} (0, t) = 0 , \ c(1, t) = \pi(t) , \ t > 0 , \]

\[ c(\rho, 0) = c_0(\rho) , \ 0 \leq \rho \leq 1 , \]

\[ \frac{\partial w}{\partial t} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \left( \frac{D_2}{(R(t)\rho)^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial w}{\partial \rho} \right) \right) + \frac{v(1, t)\rho}{R(t)} \frac{\partial w}{\partial \rho} - g(w, p, q) , \ 0 < \rho < 1, \ t > 0 , \]

\[ \frac{\partial w}{\partial \rho} (0, t) = 0 , \ w(1, t) = \overline{\pi}(t) , \ t > 0 , \]

\[ w(\rho, 0) = w_0(\rho) , \ 0 \leq \rho \leq 1 , \]

\[ \frac{\partial p}{\partial t} + \frac{(v - \rho v(1, t))}{R(t)} \frac{\partial p}{\partial \rho} = (g_{11}(c, w, p, q, d) p + g_{12}(c, w, p, q, d) q + g_{13}(c, w, p, q, d) d) , \]

\[ \frac{\partial q}{\partial t} + \frac{(v - \rho v(1, t))}{R(t)} \frac{\partial q}{\partial \rho} = (g_{21}(c, w, p, q, d) p + g_{22}(c, w, p, q, d) q + g_{23}(c, w, p, q, d) d) , \]

\[ \frac{\partial d}{\partial t} + \frac{(v - \rho v(1, t))}{R(t)} \frac{\partial d}{\partial \rho} = (g_{31}(c, w, p, q, d) p + g_{32}(c, w, p, q, d) q + g_{33}(c, w, p, q, d) d) , \]

\[ 0 \leq \rho \leq 1, \ t > 0 , \]
\[ p(\rho,0) = p_0(\rho), \quad q(\rho,0) = q_0(\rho), \quad d(\rho,0) = d_0(\rho), \quad 0 \leq \rho \leq 1, \quad (25) \]

\[
\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{v}{R(t)} \right) = h(c,w,p,q,d), \quad 0 < \rho \leq 1, \quad t > 0,
\]

\[ v(0,t) = 0, \quad t > 0, \quad \frac{dR(t)}{dt} = v(1,t), \quad t > 0, \quad R(0) = R_0, \quad (26) \]

where \( \frac{\partial^\alpha}{\partial \rho^\alpha} = 0D_t^\alpha \) is the Riemann-Liouville fractional derivative and \( 0 < \alpha < 1 \) and

\[
f(c,p,q) = K_1(c)p + K_2(c)q, \quad g(w,p,q) = K_3(w)p + K_4(w)q,
\]

\[
g_{11}(c,w,p,q,d) = [K_B(c) - K_Q(c) - K_A(c) - G_1(w)] - \frac{1}{N}[K_B(c)p - K_Rd],
\]

\[
g_{12}(c,w,p,q,d) = K_P(c), \quad g_{13}(c,w,p,q,d) = 0,
\]

\[
g_{22}(c,w,p,q,d) = -[K_P(c) + K_D(c) + G_2(w)] - \frac{1}{N}[K_B(c)p - K_Rd],
\]

\[
g_{21}(c,w,p,q,d) = K_Q(c), \quad g_{23}(c,w,p,q,d) = 0,
\]

\[
g_{31}(c,w,p,q,d) = K_A(c) + G_1(w), \quad g_{32}(c,w,p,q,d) = K_D(c) + G_2(w),
\]

\[
g_{33}(c,w,p,q,d) = -K_R - \frac{1}{N}[K_B(c)p - K_Rd],
\]

\[
h(c,w,p,q,d) = \frac{1}{N}[K_B(c)p - K_Rd],
\]

\[ g_{ij}(c,w,p,q,d) \geq 0, \quad i \neq j, \]

and the initial data satisfies the following conditions

\[
0 \leq c_0(\rho) \leq \overline{c}(0), \quad 0 \leq w_0(\rho) \leq \overline{w}(0), \quad 0 \leq \rho \leq 1,
\]

\[
\frac{\partial c_0(\rho)}{\partial \rho} = 0 \text{ at } r = 0, \quad c_0(1) = \overline{c}(0),
\]

\[
\frac{\partial w_0(\rho)}{\partial \rho} = 0 \text{ at } r = 0, \quad w_0(1) = \overline{w}(0),
\]

\[
p_0(\rho) \geq 0, \quad q_0(\rho) \geq 0, \quad d_0(\rho) \geq 0, \quad 0 \leq \rho \leq 1,
\]

\[
p_0(\rho) + q_0(\rho) + d_0(\rho) = N, \quad 0 \leq \rho \leq 1.
\]

In this model \( G_1(w) \) and \( G_2(w) \) are the dead rates of the proliferative cells and quiescent cells due to the drug, respectively. \( K_B(c) \) is the mitosis rate of proliferative cells that is dependent on nutrient level \( c \), \( K_A(c) \) and \( K_D(c) \) are death rates of proliferative cells and quiescent cells, respectively. \( K_P(c) \)
and \( K_Q(c) \) are the transferring rate of quiescent cells to proliferative cells and the rate of transferring proliferative cells to quiescent, respectively. \( K_R \) is the constant rate of removing dead cells from the tumor. In the presented model, \((K_1(c)p + K_2(c)q)\) and \((K_3(w)p + K_4(w)q)\) show consumption rate of nutrient and drug, respectively. \( \mathcal{T}(t) \) and \( \mathcal{M}(t) \) are positive functions showing nutrient and drug supply that the tumor receives from its boundary. In the model \((16) - (26)\) it is assumed that

**A.** \((K_1(c), K_2(c), K_3(w), K_4(w), K_A(c), K_B(c), K_P(c), K_Q(c), G_1(w) \) and \( G_2(w) \) are \( C^2 \)-smooth functions.

**B.** \( p_0, q_0 \) and \( d_0 \) are non-negative \( C^1 \)-smooth functions on \([0, R_0] \).

**C.** \( c(|x|, \tau) \) and \( w(|x|, \tau) \) on \( \partial Q_T \setminus (B_1 \times \{ t = T \}) \) are non-negative functions, also \( c(|x|, \tau) = \Psi(x, \tau) \) and \( w(|x|, \tau) = \Psi_1(x, \tau) \) on \( \partial Q_T \setminus (B_1 \times \{ t = T \}) \), where \( Q_T := \{(x, \tau) \in \mathbb{R}^3 \times \mathbb{R} : |x| < 1, 0 < \tau \leq T \} \), \( B_1 = \{x \in \mathbb{R}^3 : |x| \leq 1 \} \) and \( \Psi(x, \tau), \Psi_1(x, \tau) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T) \) for some \( 0 < \alpha < 1 \) \((C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega \times [0, T]) \) for \( 0 < \alpha < 1 \) and \( \Omega \subset \mathbb{R}^3 \) is defined in Appendix).

The main aim of this article is to solve this initial-boundary value problem using the spectral and finite difference method.

### 3 Approximating the solution of the problem

In this section, we approximate the solution of problem \((16) - (26)\) for \( 0 \leq \rho \leq 1 \) and \( 0 \leq t \leq T \).

Let \( t_n := nt^* \) \((n = 0, 1, \cdots, M) \) be mesh points, where \( t^* := \frac{T}{M} \) is the time step and \( M \) is a positive integer. The problem is solved employing spectral method and the following discretization formula \((21)\) for approximating the time fractional derivative

\[
\frac{\partial^\alpha u(\rho, t_n)}{\partial t^\alpha} = u(\rho, 0) \frac{t_n^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\partial u(\rho, s)}{\partial s} \frac{1}{(t_n - s)^\alpha} ds = u(\rho, 0) \frac{t_n^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{u(\rho, t_{k+2}) - u(\rho, t_{k+1})}{t^*} \int_{t_k}^{t_{k+1}} \frac{1}{(t_n - s)^\alpha} ds + E_t^u = u(\rho, 0) \frac{t_n^{-\alpha}}{\Gamma(1-\alpha)} + \frac{(t^*)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \left( u(\rho, t_{k+2}) - u(\rho, t_{k+1}) \right) \left( (n-k)^{1-\alpha} - (n-1-k)^{1-\alpha} \right) + E_t^u = u(\rho, 0) \frac{t_n^{-\alpha}}{\Gamma(1-\alpha)} + \frac{(t^*)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \left( u(\rho, t_{n+1-k}) - u(\rho, t_{n-k}) \right) \left( (k+1)^{1-\alpha} - (k)^{1-\alpha} \right) + E_t^u = u(\rho, 0) \frac{t_n^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{k=0}^{n-1} a_k \left( u(\rho, t_{n+1-k}) - u(\rho, t_{n-k}) \right) + E_t^u,
\]

\((28)\)
From (28) and (16)–(21), we get

\[ a_k = \frac{(k + 1)^{1-\alpha} - (k)^{1-\alpha}}{(\Gamma^*)^2 (2 - \alpha)}, \quad a_{k+1} \leq a_k, \quad \|E^c_t\|_\infty < C_1(t^*)^{2-\alpha}. \quad (29) \]

In the following without loss of generality we can suppose that \( \mathbf{\bar{c}} = \mathbf{\bar{w}} = 0 \) and \( c_0(\rho) = w_0(\rho) = 0 \). We have also assumed that

\[ A_n(\rho) := A(\rho, t_n), \quad B_n := B(t_n). \]

From (28) and (16)–(21), we get

\[ c_{n+1} - c_n + \frac{c_{n-1} - c_n}{3} - \frac{a'_0D_1}{(R_{n+1})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c_{n+1}}{\partial \rho} \right) - t^* \frac{2(2v(1, t_n) - v(1, t_{n-1})))}{3R_{n+1}} \frac{\rho \partial c_{n+1}}{\partial \rho} = \]

\[ -\sum_{k=0}^{n-1} (a'_k - a'_{k+1}) \left( \frac{D_1}{(R_{n-k})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c_{n-k}}{\partial \rho} \right) \right) + \]

\[ \frac{2}{3} \left( -2t^* f (c_n, p_n, q_n) + t^* f (c_{n-1}, p_{n-1}, q_{n-1}) \right) - E^c_t, \quad (30) \]

where \( 0 < \rho < 1, \quad 0 < t \leq T, \quad \frac{\partial c_{n+1}}{\partial \rho} (0) = 0, \quad c_{n+1} (1) = 0, \quad 0 < t \leq T, \quad c_0 (\rho) = 0, \quad 0 \leq \rho \leq 1, \)

\[ w_{n+1} - w_n + \frac{w_{n-1} - w_n}{3} - \frac{a'_0D_2}{(R_{n+1})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial w_{n+1}}{\partial \rho} \right) - t^* \frac{2(2v(1, t_n) - v(1, t_{n-1})))}{3R_{n+1}} \frac{\rho \partial w_{n+1}}{\partial \rho} = \]

\[ -\sum_{k=0}^{n-1} (a'_k - a'_{k+1}) \left( \frac{D_2}{(R_{n-k})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial w_{n-k}}{\partial \rho} \right) \right) + \]

\[ \frac{2}{3} \left( -2t^* g (w_n, p_n, q_n) + t^* g (w_{n-1}, p_{n-1}, q_{n-1}) \right) - E^w_t, \quad (31) \]

where \( 0 < \rho < 1, \quad 0 < t \leq T, \quad \frac{\partial w_{n+1}}{\partial \rho} (0) = 0, \quad w_{n+1} (1) = 0, \quad 0 < t \leq T, \quad w_0 (\rho) = 0, \quad 0 \leq \rho \leq 1, \)

where

\[ a'_n = 0, \quad a'_k = \frac{2(t^*)^{1-\alpha} (k + 1)^{1-\alpha} - (k)^{1-\alpha}}{3\Gamma^* (2 - \alpha)}, \quad a'_{k+1} \leq a'_k. \quad (32) \]

From (29), one can conclude that there exists positive \( C^*_1 \) such that

\[ \max \{ \|E^c_t\|_\infty, \|E^w_t\|_\infty \} < C^*_1 (t^*)^{3-\alpha}. \quad (33) \]

By defining \( \xi(\rho, t) \) as follows

\[ \frac{d}{dt} \xi(\rho_0, t) = v(\xi(\rho_0, t), t), \quad 0 < t \leq T, \quad \xi(\rho_0, 0) = \rho_0, \quad 0 \leq \rho \leq 1, \quad (34) \]
it is easy to conclude that for each \( \rho \in [0, 1] \) there exists \( \rho_0 \in [0, 1] \) such that \( \xi(\rho_0, t) = \rho \). By substituting (34) in (22)–(25), we conclude

\[
\frac{d\rho(t, \rho_0)}{dt} = (g_{11}(c, w, p, q, d) p + g_{12}(c, w, p, q, d) q + g_{13}(c, w, p, q, d) d)\bigg|_{\rho = \xi(\rho_0, t)}, \tag{35}
\]

\[
\frac{dq(t, \rho_0)}{dt} = (g_{21}(c, w, p, q, d) p + g_{22}(c, w, p, q, d) q + g_{23}(c, w, p, q, d) d)\bigg|_{\rho = \xi(\rho_0, t)}, \tag{36}
\]

\[
\frac{dd(t, \rho_0)}{dt} = (g_{31}(c, w, p, q, d) p + g_{32}(c, w, p, q, d) q + g_{33}(c, w, p, q, d) d)\bigg|_{\rho = \xi(\rho_0, t)}, \tag{37}
\]

\[0 \leq \rho \leq 1, \quad 0 < t \leq T,
\]

\[p(\rho, 0) = p_0(\rho), \quad q(\rho, 0) = q_0(\rho), \quad d(\rho, 0) = d_0(\rho), \quad 0 \leq \rho \leq 1.
\]

By considering \( \rho = \xi(\rho_0, t_{n+1}) \), the problem (35)–(37), using the Midpoint rule, becomes

\[
p(\rho, t_{n+1}) = p(\rho - 2t^*\nu(\rho - t^*\nu(\rho_0, t_n), t_n), t_{n-1}) + \underbrace{\text{Forward-difference}}_{\text{Midpoint rule}}
\]

\[
= 2t^*(g_{11}(c_n, w_n, p_n, q_n, d_n) p_n + g_{12}(c_n, w_n, p_n, q_n, d_n) q_n + g_{13}(c_n, w_n, p_n, q_n, d_n) d_n)\bigg|_{(\rho - t^*\nu(\rho_0, t_n), t_n)} + E_t^p,
\tag{38}
\]

\[
q(\rho, t_{n+1}) = q(\rho - 2t^*\nu(\rho - t^*\nu(\rho_0, t_n), t_n), t_{n-1}) +
\]

\[
= 2t^*(g_{21}(c_n, w_n, p_n, q_n, d_n) p_n + g_{22}(c_n, w_n, p_n, q_n, d_n) q_n + g_{23}(c_n, w_n, p_n, q_n, d_n) d_n)\bigg|_{(\rho - t^*\nu(\rho_0, t_n), t_n)} + E_t^q,
\tag{39}
\]

\[
d(\rho, t_{n+1}) = d(\rho - 2t^*\nu(\rho - t^*\nu(\rho_0, t_n), t_n), t_{n-1}) +
\]

\[
= 2t^*(g_{31}(c_n, w_n, p_n, q_n, d_n) p_n + g_{32}(c_n, w_n, p_n, q_n, d_n) q_n + g_{33}(c_n, w_n, p_n, q_n, d_n) d_n)\bigg|_{(\rho - t^*\nu(\rho_0, t_n), t_n)} + E_t^d,
\tag{40}
\]

\[
p(\rho, 0) = p_0(\rho), \quad q(\rho, 0) = q_0(\rho), \quad d(\rho, 0) = d_0(\rho), \quad 0 \leq \rho \leq 1,
\tag{41}
\]

\[
\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 v_n \right) = h(c_n, w_n, p_n, q_n, d_n), \quad v_n(0) = 0, \quad 0 < \rho \leq 1, \quad t > 0,
\]

where \( \rho - 2t^*\nu(\rho - t^*\nu(\rho_0, t_n), t_n) \) is an approximation of \( \xi(\rho_0, t_n) \) using the Midpoint rule and \( \rho - t^*\nu(\rho, t_n) \) is an approximation of \( \xi(\rho_0, t_n) \) using the forward-difference formula. Moreover, from (29) we have

\[
\frac{dR(t)}{dt} = R(t) \int_0^1 \rho^2 h(c, w, p, q, d) d\rho, \quad t > 0.
\tag{42}
\]
that

Now, we approximate the solution of the problem (16)–(26) by (\ref{approximation}) employing Midpoint rule. Also \( \rho - 2t^*\nu(\rho - t^*\nu(\rho, t_n), t_n) \) is an approximation of \( \xi(\rho_0, t_{n-1}) \) using the Midpoint rule and \( \rho - t^*\nu(\rho, t_n) \) is an approximation of \( \xi(\rho_0, t_n) \) using the forward-difference formula. Therefore, we conclude that there exists positive constant \( C^* \) such that

\[
\max\{\|E^p\|_{\infty}, \|E^q\|_{\infty}, \|E^d\|_{\infty}\} < C^* (t^*)^3. \tag{44}
\]

From (33), (43) and (54), we deduce that there exists positive constant \( C^* \) such that

\[
\max\{\|E^p\|_{\infty}, \|E^q\|_{\infty}, \|E^d\|_{\infty}\} < C^* (t^*)^3, \quad \|E^F\|_{\infty} < C^* (t^*)^3, \quad \max\{\|E^c\|_{\infty}, \|E^w\|_{\infty}\} < C^* (t^*)^{3-\alpha}. \tag{45}
\]

Now, we approximate the solution of the problem (16)–(26) by \((e_{n+1}^{ap}, w_{n+1}^{ap}, p_{n+1}^{ap}, q_{n+1}^{ap}, d_{n+1}^{ap})\), which is the approximated solution of the following problem

\[
C_{n+1} - \frac{a^p_n D_1}{(R_{n+1}^{ap})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial C_{n+1}}{\partial \rho} \right) = t^* \frac{2(2v^{ap}(1, t_n) - v^{ap}(1, t_n - 1)) \rho \frac{\partial C_{n+1}}{\partial \rho}}{3R_{n+1}^{ap}} - e_{n+1}^{ap} - \sum_{k=0}^{n-1} (a_k - a_{k+1}) \left( \frac{D_1}{(R_{n-k}^{ap})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial e_{n-k}^{ap}}{\partial \rho} \right) \right) +
\]

\[
\frac{2}{3} \left( -2t^* f(e_{n+1}^{ap}, p_{n+1}^{ap}, q_{n+1}^{ap}) + t^* f(e_{n+1}^{ap}, p_{n+1}^{ap}, q_{n+1}^{ap}) \right),
\tag{46}
\]

\(0 < \rho < 1, \quad 0 < t \leq T;\)

\[
\frac{\partial C_{n+1}}{\partial \rho}(0) = 0, \quad C_{n+1}(1) = 0, \quad 0 < t \leq T;
\]

\(C_0(\rho) = 0, \quad 0 \leq \rho \leq 1,\)

\[
W_{n+1} - \frac{a^p_n D_2}{(R_{n+1}^{ap})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial W_{n+1}}{\partial \rho} \right) = t^* \frac{2(2v^{ap}(1, t_n) - v^{ap}(1, t_n - 1)) \rho \frac{\partial W_{n+1}}{\partial \rho}}{3R_{n+1}^{ap}} - w_{n+1}^{ap} - \sum_{k=0}^{n-1} (a_k - a_{k+1}) \left( \frac{D_2}{(R_{n-k}^{ap})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial w_{n-k}^{ap}}{\partial \rho} \right) \right) +
\]

\[
\frac{2}{3} \left( -2t^* g(w_{n+1}^{ap}, p_{n+1}^{ap}, q_{n+1}^{ap}) + t^* g(w_{n+1}^{ap}, p_{n+1}^{ap}, q_{n+1}^{ap}) \right),
\tag{47}
\]

\(0 < \rho < 1, \quad 0 < t \leq T;\)

\[
\frac{\partial W_{n+1}}{\partial \rho}(0) = 0, \quad W_{n+1}(1) = 0, \quad 0 < t \leq T;
\]

\(W_0(\rho) = 0, \quad 0 \leq \rho \leq 1,\)
We approximate the spectral method. In the following, it is assumed that

\[ (c_{n+1}^{ap}, w_{n+1}^{ap}, p_{n+1}^{ap}, q_{n+1}^{ap}, d_{n+1}^{ap}) = (0, 0, 0, 0, 0) \]

where \((c_{n+1}^{ap}, w_{n+1}^{ap})\) is obtained as an approximation of \((C_{n+1}, W_{n+1})\) by solving \((10)-(17)\) employing the spectral method. In the following, it is assumed that

\[ (f_1, f_2)_{\omega, \beta} = \int_0^1 \rho^2 (1 - \rho)^3 f_1(\rho, t)f_2(\rho, t) d\rho, \quad \|f\|^2_{\omega, \beta} = (f, f)_{\omega, \beta}. \]

We approximate \(C_{n+1}\) employing \(\{P_j(\rho)\}_{j=0}^\infty\), which are chosen such that for each \(k \in \mathbb{N}_0\),

\[ \text{span}\{P_0(\rho), P_1(\rho), \ldots, P_k(\rho)\} = \left\{ u \in \text{span}\{1, \rho, \ldots, \rho^{k+2}\} : \frac{\partial u(\rho)}{\partial \rho} \big|_{\rho=0} = 0, u(1) = 0 \right\}. \]

then, we consider \(c_{n+1}^{ap} = c_{n+1}^{N^1}\) as follows

\[ c_{n+1}^{N^1}(\rho) = \sum_{j=0}^{N^1} a_{n+1}^{N^1} P_j(\rho). \]

We calculate \(\{a_{n+1}^{N^1}\}_{j=0}^{N^1}\) using the spectral method from

\[ H_{N^1, C_{n+1}}^{0,0} - H_{N^1}^{0,0} \left( \frac{a_{n+1}^{N^1} D_1}{(R_{n+1}^{ap})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c_{n+1}^{N^1}}{\partial \rho} \right) \right) - t^* H_{N^1}^{0,0} \frac{2(\nu_{n+1}^{ap}(1, t_n) - \nu_{n+1}^{ap}(1, t_{n-1}))}{3R_{n+1}^{ap}} \frac{\partial c_{n+1}^{N^1}}{\partial \rho} = f_{N^1, g_n}^{0,0}, \]

where \((c_{n+1}^{ap}, w_{n+1}^{ap})\) is obtained as an approximation of \((C_{n+1}, W_{n+1})\) by solving \((10)-(17)\) employing the spectral method. In the following, it is assumed that

\[ (f_1, f_2)_{\omega, \beta} = \int_0^1 \rho^2 (1 - \rho)^3 f_1(\rho, t)f_2(\rho, t) d\rho, \quad \|f\|^2_{\omega, \beta} = (f, f)_{\omega, \beta}. \]

We approximate \(C_{n+1}\) employing \(\{P_j(\rho)\}_{j=0}^\infty\), which are chosen such that for each \(k \in \mathbb{N}_0\),

\[ \text{span}\{P_0(\rho), P_1(\rho), \ldots, P_k(\rho)\} = \left\{ u \in \text{span}\{1, \rho, \ldots, \rho^{k+2}\} : \frac{\partial u(\rho)}{\partial \rho} \big|_{\rho=0} = 0, u(1) = 0 \right\}. \]

then, we consider \(c_{n+1}^{ap} = c_{n+1}^{N^1}\) as follows

\[ c_{n+1}^{N^1}(\rho) = \sum_{j=0}^{N^1} a_{n+1}^{N^1} P_j(\rho). \]

We calculate \(\{a_{n+1}^{N^1}\}_{j=0}^{N^1}\) using the spectral method from

\[ H_{N^1, C_{n+1}}^{0,0} - H_{N^1}^{0,0} \left( \frac{a_{n+1}^{N^1} D_1}{(R_{n+1}^{ap})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c_{n+1}^{N^1}}{\partial \rho} \right) \right) - t^* H_{N^1}^{0,0} \frac{2(\nu_{n+1}^{ap}(1, t_n) - \nu_{n+1}^{ap}(1, t_{n-1}))}{3R_{n+1}^{ap}} \frac{\partial c_{n+1}^{N^1}}{\partial \rho} = f_{N^1, g_n}^{0,0}, \]

where \((c_{n+1}^{ap}, w_{n+1}^{ap})\) is obtained as an approximation of \((C_{n+1}, W_{n+1})\) by solving \((10)-(17)\) employing the spectral method. In the following, it is assumed that

\[ (f_1, f_2)_{\omega, \beta} = \int_0^1 \rho^2 (1 - \rho)^3 f_1(\rho, t)f_2(\rho, t) d\rho, \quad \|f\|^2_{\omega, \beta} = (f, f)_{\omega, \beta}. \]
where \( \Pi_{N_1}^{0,0} \) and \( I_{N_1}^{0,0} \) are the orthogonal projection and Jacobi-Gauss-Lobatto interpolation operator with respect to \( \rho \) on \([0,1]\), respectively, and

\[
g^*_n = \Pi_{N_1}^{0,0}(c^*_{n+1} - c^*_{n}) - \sum_{k=0}^{n-1} (c^*_k - c^*_{k+1})(\frac{D_1}{(R_{n-k}^0)^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho \frac{\partial c^*_{n-k}}{\partial \rho})) + \]

\[
\frac{2}{3} \left( -2t^* f(c^*_n, p^*_n, q^*_n) + t^* f(c^*_n, p^*_n, q^*_n) \right). \tag{56}
\]

Also, we calculate \( w_{n+1}^{N_1} \) similar to \( c_{n+1}^{N_1} \) to approximate \( W_{n+1} \), the solution of \([17]\). Now, using the principle of mathematical induction, we want to show that for \( k = 0, 1, \cdots, M \), there exists positive constants \( c^*, w^*, p^*, q^*, d^* \) and \( R^* \) such that

\[
|p_k^{ap} - p_k| < p^*, \quad |q_k^{ap} - q_k| < q^*, \quad |d_k^{ap} - d_k| < d^*, \quad p_k^{ap} + q_k^{ap} + d_k^{ap} = N,
\]

and

\[
|c_k^{ap} - c_k| \leq c^*, \quad |w_k^{ap} - w_k| \leq w^*, \quad |R_k^{ap} - R_k| \leq R^*,
\]

where \( c, w, p, q, d \) and \( R \) are the exact solutions of \([16] \sim [26]\), respectively. First, we assume that

\[
|p_k^{ap} - p_k| < p^*, \quad |q_k^{ap} - q_k| < q^*, \quad |d_k^{ap} - d_k| < d^*, \quad p_k^{ap} + q_k^{ap} + d_k^{ap} = N, \quad k \leq n < M, \tag{57}
\]

and

\[
|c_k^{ap} - c_k| \leq c^*, \quad |w_k^{ap} - w_k| \leq w^*, \quad |R_k^{ap} - R_k| \leq R^*, \quad k \leq n < M. \tag{58}
\]

**Lemma 3.1** Let \( c \) be the exact solution of \([16]\) on \([0,1] \times [0,T]\), \( c_{n+1}^{ap} = c_{n+1}^{N_1} \) and \( \frac{\partial^2 c}{\partial \rho^2} \) be a \( C^1 \)-smooth function, also for each \( 1 \leq k \leq n \) the conditions presented in \([57] \sim [58]\) be satisfied. Then, there exists a positive constant \( K_4 \) such that

\[
\frac{\partial^2 c}{\partial \rho^2}(c_{n+1}^{N_1} - c_{n+1}^{N_1}) \leq \frac{\partial^2 c}{\partial \rho^2}(c_{n}^{N_1} - c_{n}^{N_1}) + K_4 \left( (t^*)^{1+\alpha} \sum_{k=0}^{n} \frac{1}{3} + \frac{1}{10} \right) \left( \|p_k^{ap} - p_k\|_{\rho \rho} + \|q_k^{ap} - q_k\|_{\rho \rho} + \|d_k^{ap} - d_k\|_{\rho \rho} + \|R_k^{ap} - R_k\|_{\rho \rho} \right) +
\]

\[
K_4 \left( (t^*)^{1+\alpha} \sum_{k=0}^{n} \frac{1}{3} \right) \left( \|c_k^{ap} - c_k\|_{\rho \rho} + \|w_k^{ap} - w_k\|_{\rho \rho} + \|R_k^{ap} - R_k\|_{\rho \rho} \right), \tag{59}
\]

where

\[
\lim_{N_1 \to \infty} K_1^*(N_1) = 0, \quad \|e^*_f\|_{\infty} \leq (t^*)^{2+\frac{1-\alpha}{2}},
\]
and $c_{1}^{N_{1}}$ is a polynomial such that
\[
c_{n,1}^{N_{1}} = c_{1}^{N_{1}}(\rho, t_{n}), \quad \rho^{0,0}_{\mathcal{N}} = c_{1}^{N_{1}}, \quad \frac{\partial c_{n,1}^{N_{1}}}{\partial \rho} (0, t) = 0, \quad c_{1}^{N_{1}} (1, t) = 0, \quad c_{1}^{N_{1}} (\rho, 0) = 0, \quad 0 \leq t \leq T,
\]
and
\[
\lim_{N_{1} \to \infty} \left( \| \frac{\partial c_{n,1}^{N_{1}}}{\partial \rho} - \frac{\partial c_{1}^{N_{1}}}{\partial \rho} \|_{L^{2}(\omega, t)} + \| I_{N}^{0,0} c_{1}^{N_{1}} \|^2_{L^{2}(\omega, t)} + \| I_{N}^{0,0} \partial c_{1}^{N_{1}} \|^2_{L^{2}(\omega, t)} + \| I_{N}^{0,0} \partial^2 c_{1}^{N_{1}} \|^2_{L^{2}(\omega, t)} \right) = 0.
\]
Moreover, if $\frac{\partial^2 c_{1}^{N_{1}}}{\partial \rho^2}$ is a $C^m$-smooth function then
\[
K_{4} \left( (t^*)^{1+\alpha} \sum_{k=0}^{n} \frac{1}{3} \left( \| c_{k,1}^{N_{1}} - c_{k}^{N_{1}} \|^2_{L^{2}(\omega, t)} + \| p_{k} - p_{k}^{0} \|^2_{L^{2}(\omega, t)} + \| q_{k} - q_{k}^{0} \|^2_{L^{2}(\omega, t)} + \| d_{k} - d_{k}^{0} \|^2_{L^{2}(\omega, t)} + \| R_{k} - R_{k}^{0} \|^2_{L^{2}(\omega, t)} \right) \right) \leq K_{5} \left( (t^*)^{1+\alpha} \sum_{k=0}^{n} \frac{1}{3} \left( \| c_{k,1}^{N_{1}} - c_{k}^{N_{1}} \|^2_{L^{2}(\omega, t)} + \| p_{k} - p_{k}^{0} \|^2_{L^{2}(\omega, t)} + \| q_{k} - q_{k}^{0} \|^2_{L^{2}(\omega, t)} + \| d_{k} - d_{k}^{0} \|^2_{L^{2}(\omega, t)} + \| R_{k} - R_{k}^{0} \|^2_{L^{2}(\omega, t)} \right) \right) + \| c_{1}^{N_{1}} \|^2_{L^{2}(\omega, t)} + t^* K_{6} (N^1)
\]
where
\[
\lim_{N_{1} \to \infty} K_{6} (N^1) = 0.
\]

**Theorem 3.1** Let $c_{n+1}^{ap} = c_{n+1}^{N_{1}}$ and $w_{n+1}^{ap} = w_{n+1}^{N_{1}}$. Then, under assumptions of Lemma 3.1, there exist positive constants $M, C_{4}^\ast$ and $C_{5}^\ast$ such that
\[
\max_{k=0,1,\ldots,n+1} \{ \mathcal{E}_{k} \} \leq C_{4}^\ast e^{MT} \left( (t^*)^{2-\frac{1}{2}} + K_{5} (N^1) \right),
\]
and
\[
\max_{k=0,1,\ldots,n+1} \{ \mathcal{E}_{k} \} \leq C_{5}^\ast e^{MT} \left( (t^*)^{2-\frac{1}{2}} + K_{6} (N^1) \right),
\]
where
\[
\lim_{N_{1} \to \infty} K_{6} (N^1) = 0.
\]
\[ E_k = \|p_k^p - p_k\|_\infty + \|q_k^p - q_k\|_\infty + \|d_k^p - d_k\|_\infty + |R_k^p - R_k|, \tag{64} \]

and
\[ E_k = \|\partial (c_k^p - c_k)\|_{\omega,0,0} + \|\partial (w_k^p - w_k)\|_{\omega,0,0}, \tag{65} \]

and if \( \frac{\partial^2 c}{\partial \rho^2} \) and \( \frac{\partial^2 w}{\partial \rho^2} \) are \( C^m \)-smooth functions with respect to \( \rho \), then
\[ K_1^*(N_1) = \frac{1}{(N_1)^m}. \]

**Proof** See Appendix.

Employing the General Sobolev inequalities, there exists a positive constant \( C_2 \) such that for \( 0 \leq t \leq T \), we have
\[ |c_{n+1}^p - c_{n+1}| \leq C_2(\|c_{n+1}^p - c_{n+1}\|_{\omega,0,0} + \|\partial (c_{n+1}^p - c_{n+1})\|_{\omega,0,0}). \tag{66} \]

Now, using the principle of mathematical induction, Theorem 3.1, (57) and (58), we conclude that
\[
|p_k^p - p_k| < p^*, |q_k^p - q_k| < q^*, |d_k^p - d_k| < d^*, p_k^p + q_k^p + d_k^p = N, \quad 0 \leq k \leq M, \tag{67}
\]

and
\[
|c_k^p - c_k| \leq c^*, |w_k^p - w_k| \leq w^*, |R_k^p - R_k| \leq R^*, \quad 0 \leq k \leq M. \tag{68}
\]

Thus from Theorem 3.1 we can conclude that the sequence \( \{(c_n^p, w_n^p, p_n^p, q_n^p, d_n^p, R_n^p)\}_{n=0}^\infty \) converges to the exact solution of problem (16)–(26) on \([0,1] \times [0,T]\).

### 4 Stability

In this section, we want to prove the stability of the presented method. For this aim, first we consider the following problem
\[
\frac{\partial c}{\partial t} = D_1 \frac{\partial^\alpha}{\partial t^\alpha} \left( \frac{1}{(R(t))^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c}{\partial \rho} \right) \right) + \frac{v(1, t)}{R(t)} \frac{\partial c}{\partial \rho} - f(c, p, q) + f_1(\rho, t), \quad 0 < \rho < 1, \ t > 0, \tag{69}
\]
\[
\frac{\partial c}{\partial \rho}(0, t) = 0, \ c(1, t) = \overline{c}(t), \ t > 0, \tag{70}
\]
\[
c(\rho, 0) = c_0(\rho), \quad 0 \leq \rho \leq 1, \tag{71}
\]
\[
\frac{\partial w}{\partial t} = D_2 \frac{\partial^\alpha}{\partial t^\alpha} \left( \frac{1}{(R(t))^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial w}{\partial \rho} \right) \right) + \frac{v(1, t)}{R(t)} \frac{\partial w}{\partial \rho} - g(w, p, q) + f_2(\rho, t), \quad 0 < \rho < 1, \ t > 0, \tag{72}
\]
\[
\frac{\partial w}{\partial \rho}(0, t) = 0, \ w(1, t) = \overline{w}(t), \ t > 0, \tag{73}
\]
\[ w(\rho, 0) = w_0(\rho), \quad 0 \leq \rho \leq 1, \quad (74) \]

\[ \frac{\partial p}{\partial t} + \nu \frac{\partial p}{\partial \rho} = (g_{11}(c, w, p, q, d) p + g_{12}(c, w, p, q, d) q + g_{13}(c, w, p, q, d) d) + f_3(\rho, t), \quad (75) \]

\[ \frac{\partial q}{\partial t} + \nu \frac{\partial q}{\partial \rho} = (g_{21}(c, w, p, q, d) p + g_{22}(c, w, p, q, d) q + g_{23}(c, w, p, q, d) d) + f_4(\rho, t), \quad (76) \]

\[ \frac{\partial d}{\partial t} + \nu \frac{\partial d}{\partial \rho} = (g_{31}(c, w, p, q, d) p + g_{32}(c, w, p, q, d) q + g_{33}(c, w, p, q, d) d) + f_5(\rho, t), \quad (77) \]

\[ 0 \leq \rho \leq 1, \quad t > 0, \]

\[ p(\rho, 0) = p_0(\rho), \quad q(\rho, 0) = q_0(\rho), \quad d(\rho, 0) = d_0(\rho), \quad 0 \leq \rho \leq 1, \quad (78) \]

\[ \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{v}{R(t)} \right) = h(c, w, p, q, d) + f_0(\rho, t), \quad 0 < \rho \leq 1, \quad t > 0, \quad (79) \]

\[ \frac{dR(t)}{dt} = v(1, t), \quad t > 0, \quad R(0) = R_0. \]

In the following theorem, the stability of the proposed method is proved.

**Theorem 4.1** Let \( \epsilon_1 \) be a positive constant and \( |f_i| < \epsilon_1 \) (i = 1, \cdots, 6). Then, under assumptions of Lemma 3.1 there exist positive constants \( M_2, C_6^* \) and \( C_7^* \) such that

\[ \max_{k=0, 1, \cdots, n+1} \{ \mathcal{E}_k^p \} \leq C_6^* e^{M_2 T} \left( (t^*)^{2-\frac{2}{n}} + K_5^*(N^1) + \epsilon_1 \right), \quad (80) \]

and

\[ \max_{k=0, 1, \cdots, n+1} \{ \mathcal{E}_k^q \} \leq C_7^* e^{M_2 T} \left( (t^*)^{2-\frac{2}{n}} + K_5^*(N^1) + \epsilon_1 \right), \quad (81) \]

where

\[ \lim_{N^1 \rightarrow \infty} K_5^*(N^1) = 0, \]

\[ \mathcal{E}_k^p = \| p_k^{ap,p} - p_k \|_\infty + \| q_k^{ap,p} - q_k \|_\infty + \| d_k^{ap,p} - d_k \|_\infty + | R_k^{ap,p} - R_k |, \quad (82) \]

and

\[ \mathcal{E}_k^q = \| \frac{\partial (c_k^{ap,p} - c_k)}{\partial \rho} \|_{\omega_0, \alpha} + \| \frac{\partial (w_k^{ap,p} - w_k)}{\partial \rho} \|_{\omega_0, \alpha}, \quad (83) \]

where \((c_{ap,p}, w_{ap,p}, q_{ap,p}, d_{ap,p}, R_{ap,p})\) is the approximated solution of the perturbed problem (69)-(79) using the presented method and \((c, w, p, q, d, R)\) is the solution of (16)-(26).

**Proof** See Appendix. \(\square\)
5 Numerical experiments

In this section, we solve the model of tumor growth by applying finite difference method for approximating the time derivative using mesh points \( \{ t_i \}_{i=1}^M \) where \( t_i = i \Delta t \) and \( M \) is a positive integer and spectral method in space. To construct trial functions for spectral method, which satisfy the boundary conditions, we use orthogonal Legendre polynomials as trial functions in the form of (53) on (16)-(21) as follows

\[
e^{N+1}_n(\rho) = \sum_{i=0}^{N} c_{i}^{n+1} p_i(\rho), \quad w^{N+1}_n(\rho) = \sum_{i=0}^{N} w_{i}^{n+1} p_i(\rho),
\]

where

\[
p_i(\rho) = L_i(\rho) - \frac{2i + 3}{(i + 2)^2} L_{i+1}(\rho) - \left( \frac{i + 1}{i + 2} \right)^2 L_{i+2}(\rho), \quad i = 0, \ldots, N, \tag{84}
\]

Remark: The scaling factors in the trial functions (84) play the role of precondition factor for the collocation matrices [?].

Also the Gauss quadrature points \( \{ x_i^{0,0} \}_{i=1}^N \) (i.e., the zeros of Legendre polynomial of degree \( N + 1 \)) are considered as collocation points.

In order to verify our numerical results, we need to present the following definition.

**Definition 5.1** A sequence \( \{ x_n \}_{n=1}^{\infty} \) is said to converge to \( x \) with order \( p \) if there exists a constant \( C \) such that

\[
| x_n - x | \leq C n^{-p}, \quad \forall n.
\]

This can be written as \( | x_n - x | = \mathcal{O}(n^{-p}) \). A practical method to calculate the rate of convergence for a discretization method is to use the following formula

\[
p \approx \log_{e}(c_{n_2}/c_{n_1}) / \log_{e}(n_1/n_2), \tag{85}
\]

where \( c_{n_1} \) and \( c_{n_2} \) denote the errors with respect to the step sizes \( 1/n_1 \) and \( 1/n_2 \), respectively [?].

Now, using these trial functions, we want to solve the following example.

**Example**. Consider the following problem:

\[
\frac{\partial c}{\partial t} = \frac{\partial^{n}}{\partial t^{n}} \left( \frac{1}{12(R(t)\rho)^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c}{\partial \rho} \right) \right) + \frac{v(1,t)\rho \partial c}{R(t)} + \frac{c}{16} + \frac{12p}{88}, \quad 0 < \rho < 1, \quad t > 0,
\]

\[
\frac{\partial c}{\partial \rho}(0,t) = 0, \quad c(1,t) = 0, \quad t > 0,
\]

\[
c(\rho,0) = 0, \quad 0 \leq \rho \leq 1,
\]

\[
\frac{\partial w}{\partial t} = \frac{\partial^{n}}{\partial t^{n}} \left( \frac{1}{12(R(t)\rho)^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial w}{\partial \rho} \right) \right) + \frac{v(1,t)\rho \partial w}{R(t)} + \frac{3c}{115} + \frac{12p}{188}, \quad 0 < \rho < 1, \quad t > 0,
\]

\[
\text{etc.}
\]
\[ \frac{\partial w}{\partial \rho}(0,t) = 0, \quad w(1,t) = 0, \quad t > 0, \]
\[ w(\rho,0) = 0(\rho), \quad 0 \leq \rho \leq 1, \quad (87) \]
\[ \frac{\partial p}{\partial t} + \left( v - \rho v(1,t) \right) \frac{\partial p}{R(t)} = (q)p + \left( \frac{C}{2} \right)q + (p)d + f_p, \quad (88) \]
\[ \frac{\partial q}{\partial t} + \left( v - \rho v(1,t) \right) \frac{\partial q}{R(t)} = p + (2p)q + f_q, \quad (89) \]
\[ \frac{\partial d}{\partial t} + \left( v - \rho v(1,t) \right) \frac{\partial d}{R(t)} = p + (2p)q + f_d, \quad 0 \leq \rho \leq 1, \quad t > 0, \quad (90) \]
\[ p(\rho,0) = -((2\rho - 1)^2 + 1), \quad q(\rho,0) = 0, \quad d(\rho,0) = 4\rho, \quad 0 \leq \rho \leq 1, \quad (91) \]
\[ \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{v}{R(t)} \right) = \frac{2p - d}{2} + f_v, \quad 0 < \rho \leq 1, \quad t > 0, \quad (92) \]
\[ v(0,t) = -\frac{\varepsilon \eta^{-1}}{(t + 1)^2}, \quad t > 0, \quad (93) \]
and the exact solutions are as follows
\[ c(t,\rho) = 4t(2\rho + 1)(\rho - 1)^2, \quad w(t,\rho) = -8t(\rho^2 - 1), \quad p(t,\rho) = -\exp(t)((2\rho - 1)^2 + 1), \quad (94) \]
\[ q(t,\rho) = -t((2\rho - 1)^2 - 1), \quad d(t,x) = \exp(t)((2\rho - 1)^2 + 1) + t((2\rho - 1)^2 - 1) + 1, \quad (95) \]

It should be noted that in order to use Legendre polynomials, we map the domain of the problem (86)-(93) to \([-1,1\]). We carried out the numerical computations using the MATLAB 2018a program using a computer with the Intel Core i7 processor (2.90 GHz, 4 physical cores).

In Figures 1-2 we have plotted the graph of error functions
\[ e^{c}_{n,M} = c^\text{ap}_n - c, \quad e^{w}_{n,M} = w^\text{ap}_n - w, \]
\[ e^{p}_{n,M} = p^\text{ap}_n - p, \quad e^{q}_{n,M} = q^\text{ap}_n - q, \]

We have presented the maximum time-error by considering constant \(N = 20\) in Table 1. To better see the time-error of numerical results, Figure 3 is presented. Also, in Table 2 and Figure 4 the computed order of convergence \((p)\) using (85) for numerical finite difference method is shown. It is shown that the finite difference method has almost \(O(h^{2-\alpha/2})\) error, as someone would expect from convergence Theorem 3.1. In Table 3 we have presented the maximum space-errors by constant \(M = 200\) and various values of \(N\). To better see the space-error of numerical results, Figure 5 is presented.
Fig. 1: Error functions $e^c_{n,M,N}$, $e^w_{n,M,N}$, $e^p_{n,M,N}$, $e^q_{n,M,N}$, $e^d_{n,M,N}$, $e^e_{n,M,N}$ for $N=5$ and $M=200$ and $\alpha = 0.1$.

| Error of | M=100   | M=1000  | M=2000  | M=3000  | M=4000  | M=5000  |
|----------|---------|---------|---------|---------|---------|---------|
| c        | 2.22198e-03 | 3.50646e-05 | 9.31974e-06 | 4.25763e-06 | 0.24352e-06 | 1.57668e-06 |
| w        | 2.85697e-03 | 3.04010e-05 | 7.66240e-06 | 3.42437e-06 | 1.93429e-06 | 1.24213e-06 |
| p        | 2.89789e-02 | 2.85568e-04 | 7.14999e-05 | 3.18273e-05 | 1.79269e-05 | 1.14866e-05 |
| q        | 3.08257e-02 | 3.03702e-04 | 7.60399e-05 | 3.38482e-05 | 1.90652e-05 | 1.22160e-05 |

Table 1: Maximum time-errors with $N=20$ and various $M$ and $\alpha = 0.1$.

| Error ratio ($p$) of | c       | w       | p       | q       | d       | $2 - \alpha/2$ |
|---------------------|---------|---------|---------|---------|---------|---------------|
| M=1000,2000         | 1.9116  | 1.9882  | 1.9979  | 1.9978  | 1.9978  |               |
| M=2000,3000         | 1.9321  | 1.9863  | 1.9962  | 1.9962  | 1.9961  |               |
| M=3000,4000         | 1.9420  | 1.9854  | 1.9953  | 1.9953  | 1.9953  | 1.95          |
| M=4000,5000         | 1.9481  | 1.9848  | 1.9947  | 1.9948  | 1.9947  |               |
| M=5000,6000         | 1.9492  | 1.9839  | 1.9944  | 1.9939  | 1.9943  |               |

Table 2: The rate of convergence with respect to the time variable and $\alpha = 0.1$.
Fig. 2: Error functions $e_{c,M,N}^n$, $e_{w,M,N}^n$, $e_{p,M,N}^n$, $e_{n,M,N}^{d,M,N}$, $e_{n,M,N}^{c,M,N}$ for $N=10$ and $M=300$ and $\alpha = 0.1$.

Fig. 3: Maximum time-errors with $N=20$ and various $M$ and $\alpha = 0.1$. 
Fig. 4: The behaviour of time-errors in LogLog scale with $\alpha = 0.1$.

| Error of | N=10  | N=20  | N=40  | N=80  | N=100 |
|----------|-------|-------|-------|-------|-------|
| c        | 1.81574e-05 | 4.72446e-06 | 1.17313e-06 | 2.59655e-07 | 1.48707e-07 |
| w        | 849925e-06 | 2.21145e-06 | 5.49129e-07 | 1.21541e-07 | 6.96080e-08 |

Table 3: Maximum space-error with M=200 and $\alpha = 0.1$.

6 Conclusions

In this paper, we have considered a free boundary problem modelling the growth of tumor including two reaction-diffusion equations describing the diffusion of nutrient and drug in the tumor and three hyperbolic equations describing the evolution of tumor cells. Since in the real situation the subdiffusion of nutrient and drug in the tumor can be found, we have changed the reaction-diffusion equations to the fractional ones to consider this class of anomalous diffusion and deal with a more reliable model of tumor growth. After that in order to have a clear vision of the dynamic of tumor growth and the effect of nutrient and drug on the tumor growth, we have solved the fractional problem. Applying a combination of finite difference method and spectral method, the fractional problem is solved. We have
Fig. 5: Maximum time-error function for $M=200$ and various $N$ and $\alpha = 0.1$.

also proved the method is unconditionally convergent and stable, which leads us to trust the obtained solution. Finally by giving the numerical results, the theoretical statements are justified.

Appendix

We provide here the proofs of theorems and lemmas together with some essential mathematical concepts, lemmas and theorems, which have been used in the mathematical analysis throughout the paper.

**Lemma 1** *(Discrete Gronwall Lemma)* \[17] Let $f_0 \geq 0$ and \{\(h_n\)\} and \{\(g_n\)\} be non-negative sequences.

if the sequence \{\(\psi_n\)\} satisfies

\[
\psi_0 \leq g_0, \\
\psi_n \leq f_0 + \sum_{k=0}^{n-1} g_k + \sum_{k=0}^{n-1} h_k \psi_k, \quad n \geq 1,
\]

then we have

\[
\psi_n \leq (f_0 + \sum_{k=0}^{n-1} g_k e^{\sum_{k=0}^{n-1} h_k}), \quad n \geq 1.
\]

**Definition 1** \[18\] Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $\omega$ be a positive continuous function on $\Omega$. We can define weighted $L^p$-norms as follows:
\[ \|u\|_{p,\omega} := \left( \int_{\Omega} \omega(X)|u(X)|^p dX \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty, \]
\[ \|u\|_{\infty} := \operatorname{ess sup}_{X \in \Omega} |u(X)|, \]
where \( X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and the space of measurable functions on \( \Omega \) for which this norm is finite forms a Banach space, indicated by \( L^p_{w}(\Omega) \). Moreover, for \( L^2_{w}(\Omega) \) the following inner product is defined
\[ (u, v)_\omega := \int_{\Omega} \omega(X)u(X)v(X) dX, \quad \forall u, v \in L^2_{w}(\Omega), \]
and for simplicity we show the norm with \( \|u\|_\omega \).

Now for simplicity, for \( \omega(x) \equiv 1 \), we use \( L^p(\Omega) \) and \( \|u\|_p \) instead of \( L^p_w(\Omega) \) and \( \|u\|_{p,\omega} \), respectively.

**Definition 2**  
Let \( \Omega \subset \mathbb{R}^n \) be open and bounded, \( 1 \leq p < +\infty \) and \( Q_T = \Omega \times [0,T[ \). We can define the space
\[ W^{m,k}_p(Q_T) = \{ u : \partial^l_X u, \partial^r_t u \in L^p(Q_T), \quad 0 \leq |l| \leq m, 0 \leq r \leq k \}, \quad k \geq 1, \]
endowed with the norm
\[ \|u\|_{W^{m,k}_p} := \left( \sum_{1 \leq r \leq k} \|\partial^r_t u\|_{p,\omega}^p + \sum_{0 \leq |l| \leq m} \|\partial^l_X u\|_{p,\omega}^p \right)^{\frac{1}{p}}, \]
\[ \omega(X,t) = \operatorname{sgn}(\max\{\partial^l_t u : t \in [0,T]\})(1 - \frac{1}{T}) + \frac{1}{T}, \]
where \( \operatorname{sgn}(x) \) is the sign function, \( l = (l_1, l_2, \ldots, l_n) \), \( |l| = \sum_{j=1}^n l_j \), and \( \partial^l_X u = \frac{\partial^{l_1+l_2+\cdots+l_n}}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_n^{l_n}} \). Moreover, for \( W^{m,k}_2(Q_T) \), the following inner product can be defined
\[ (u,v)_{W^{m,k}_2} := \int_{Q_T} \left( \sum_{1 \leq r \leq k} \int_{Q_T} \partial^r_t u \partial^r_t v dX dt + \sum_{0 \leq |l| \leq m} \int_{Q_T} \partial^l_X u \partial^l_X v dX dt \right), \quad \forall u,v \in W^{m,k}_2(Q_T). \]

**Definition 3**  
Let \( \Omega \subseteq \mathbb{R}^3 \) be an open set, \( p > \frac{5}{2} \) and \( Q_T = \Omega \times [0,T[ \), then we define \( D_p(\Omega) \) the trace space of \( W^{2,1}_p(Q_T) \) at \( t = 0 \) as follows
\[ D_p(\Omega) = \{ \varphi : \exists u \in W^{2,1}_p(Q_T) \text{ s.t. } u(.,0) = \varphi \}. \]
The norm defined on \( D_p(\Omega) \) is
\[ \|\varphi\|_{D_p} = \inf \{ T^{-\frac{1}{p}} \|u\|_{W^{2,1}_p} : u \in W^{2,1}_p(Q_T), \ u(.,0) = \varphi \}. \]
Lemma 2 For any \( u \in P_N \) on \([a, b]\), we have

\[
\| \partial_t^k u \|_{\omega^{\alpha, \beta}} \lesssim N^k \| u \|_{\omega^{\alpha, \beta}}, \quad k \geq 1, \\
\| \partial_t u \|_{\omega^{\alpha, \beta}} \lesssim N \| u \|_{\omega^{\alpha, \beta}},
\]

where \( \omega^{\alpha, \beta}(t) = (b - t)^\alpha (t - a)^\beta \).

Proof See [19].

Lemma 3 Suppose \( I = ]a, b[ \), max\{\( \alpha + 1, \beta + 1 \)\} \( \leq m \leq N + 1 \) and \( x_j^{\alpha, \beta} \), \( (0 \leq j \leq N) \), are the Jacobi-Gauss-Lobatto quadrature nodes and \( w_j^{\alpha, \beta} \), \( (0 \leq j \leq N) \), are the Jacobi-Gauss-Lobatto weights. The Jacobi-Gauss-Lobatto interpolation operator is denoted by \( I_N^{\alpha, \beta} u \). For any measurable function \( u \) such that \( \partial_t^l u \in L^2_{\omega^{\alpha, \beta}}(I) \), \( (0 \leq l \leq m) \), we have for \( 0 \leq l \leq m \), there exists constant \( C \) independent of \( m, N \), and \( u \), such that

\[
\| \partial_t^l \left( u - I_N^{\alpha, \beta} u \right) \|_{\omega^{\alpha, \beta}} \lesssim C \left( \frac{(N - m + 1)!}{N!} \right)^{\frac{1}{2}} (N)^{l-m} \| \partial_t^m u \|_{\omega^{\alpha, \beta}}.
\] (97)

Proof See [19].

Lemma 4 Let \( I = ]a, b[ \) and \( H_N^{\alpha, \beta} \) be the \( L^2_{\omega^{\alpha, \beta}} \)-orthogonal projection. Then, for any \( u \in H^m_{\omega^{\alpha, \beta}}(I) := \{ u | \partial_t^l u \in L^2_{\omega^{\alpha, \beta}}(I), 0 \leq l \leq m \} \), and \( 0 \leq l \leq m \leq N + 1 \), there exists constant \( C \) such that

\[
\| \partial_t^l \left( u - H_N^{\alpha, \beta} u \right) \|_{\omega^{\alpha, \beta}} \lesssim C \left( \frac{(N - m + 1)!}{(N - l + 1)!} \right)^{\frac{1}{2}} (N + m)^{l-m} \| \partial_t^m u \|_{\omega^{\alpha, \beta}}.
\] (98)

Proof See [19].

For any \( u, v \) on \( I = [a, b] \), we set

\[
(u, v)_N = \sum_{i=0}^N u(x_i)v(x_i)w_i,
\]

where \( x_i \) \( (0 \leq i \leq N) \) are the Gauss, Gauss-Radau or Gauss-Lobatto quadrature nodes and \( w_i \), \( (0 \leq i \leq N) \), are the Gauss, Gauss-Radau or Gauss-Lobatto quadrature weights. The Gauss quadrature formulas imply that

\[
(u, v)_N = (u, v)_\omega, \quad \text{if} \quad u, v \in P_{2N+\delta},
\] (99)

where \( \delta = 1, 0, -1 \) for Gauss, Gauss-Radau, and Gauss-Lobatto quadrature, respectively.
Definition 4 Let $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be bounded. Then, $f \in C^\alpha (\Omega \times [0, T])$ if there exists a positive constant $C$ such that

$$|f(x_1, t_1) - f(x_2, t_2)| \leq C \left( |x_1 - x_2|^\alpha + |t_1 - t_2| \right)^{\alpha}, \forall x_1, x_2 \in \Omega, \forall t_1, t_2 \in [0, T].$$

Furthermore, for any nonnegative integer $k$

$$C^{2k+\alpha,k+\frac{\alpha}{2}} (\Omega \times [0, T]) := \{ f \in C^{\alpha, \frac{\alpha}{2}} (\Omega \times [0, T]) : \partial^\alpha_x \partial^\beta_t f \in C^{\alpha, \frac{\alpha}{2}} (\Omega \times [0, T]), |\beta| + 2i \leq 2k \}.$$

Proof of Lemma 3.7 Since $\frac{\partial^2 c}{\partial \rho^2}$ is a $C^1$-smooth function, therefore for each $N^1 \in \mathbb{N}$, there exists a polynomial $c_{1N}^1$ such that

$$f_{N^1}^{0,0}c_{1N}^1 = c_1^N, \quad \frac{\partial c_{1N}^1}{\partial \rho}(0, t) = 0, \quad c_{1N}^1(1, t) = 0, \quad c_{1N}^1(\rho, 0) = 0, \quad 0 \leq t \leq T,$

and

$$\lim_{N^1 \to \infty} \left( \left\| \frac{\partial c_{1N}^1}{\partial \rho} \right\|_{L^{\infty,0}}^2 + \left\| f_{N^1}^{0,0}c_{1N}^1 \right\|_{L^{\infty,0}}^2 + \left\| f_{N^1}^{0,0} \frac{\partial c_{1N}^1}{\partial \rho} \right\|_{L^{\infty,0}}^2 + \left\| f_{N^1}^{0,0} \frac{\partial^2 c_{1N}^1}{\partial \rho^2} \right\|_{L^{\infty,0}}^2 \right) = 0. \quad (100)$$

Moreover, if $\frac{\partial^2 c}{\partial \rho^2}$ is a $C^m$-smooth function with respect to $\rho$ then it is easy to show that there exist $c_{1N}^1$ and a positive constant $Q^*$ such that

$$\lim_{N^1 \to \infty} \left( \left\| \frac{\partial c_{1N}^1}{\partial \rho} \right\|_{L^{\infty,0}}^2 + \left\| f_{N^1}^{0,0}c_{1N}^1 \right\|_{L^{\infty,0}}^2 + \left\| f_{N^1}^{0,0} \frac{\partial c_{1N}^1}{\partial \rho} \right\|_{L^{\infty,0}}^2 + \left\| f_{N^1}^{0,0} \frac{\partial^2 c_{1N}^1}{\partial \rho^2} \right\|_{L^{\infty,0}}^2 \right) \leq \frac{Q^*}{(N^1)^{2m}}. \quad (101)$$

From (55), we can conclude

$$(\Pi_{N^1}^{0,0}Lc_{n+1}^{N^1} - \Pi_{N^1}^{0,0}Lc_{n+1}^{N^1}) \frac{\partial^2 (c_{n+1}^{N^1} - c_{n+1}^{N^1})}{\partial \rho^2}_{\omega, 0} =$$

$$(f_{N^1}^{0,0} \frac{\partial c_{1N}^1}{\partial \rho} - \frac{\partial \phi}{\partial \rho} \left( 2(2v^{op}(1, t_0) - v^{op}(1, t_{n-1}) - 3v^{op}(1, t_{n-1}) \frac{\partial \phi}{\partial \rho} \right).$$

Therefore, we have

$$\left( c_{n+1}^{N^1} - c_{n+1}^{N^1} \right) \frac{\partial^2 (c_{n+1}^{N^1} - c_{n+1}^{N^1})}{\partial \rho^2}_{\omega, 0} = (f_{N^1}^{0,0} \frac{\partial c_{1N}^1}{\partial \rho} - \frac{\partial \phi}{\partial \rho} \left( 2(2v^{op}(1, t_0) - v^{op}(1, t_{n-1}) - 3v^{op}(1, t_{n-1}) \frac{\partial \phi}{\partial \rho} \right).$$

Therefore, we have

$$(c_{n+1}^{N^1} - c_{n+1}^{N^1}) \frac{\partial^2 (c_{n+1}^{N^1} - c_{n+1}^{N^1})}{\partial \rho^2}_{\omega, 0} = (f_{N^1}^{0,0} \frac{\partial c_{1N}^1}{\partial \rho} - \frac{\partial \phi}{\partial \rho} \left( 2(2v^{op}(1, t_0) - v^{op}(1, t_{n-1}) - 3v^{op}(1, t_{n-1}) \frac{\partial \phi}{\partial \rho} \right).$$
\[ (r^2 \frac{2(v_{ap}(1,t_n) - v_{ap}(1,t_{n-1}))\rho}{3R_{n+1}^p} \partial (c_{n+1}^{N_1} - c_{n+1}^{N_1}), \frac{\partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})}{\partial \rho^2})_{\omega^{0,0}} = \]

\[ \frac{a'_D}{(R_{n+1}^p)^{\frac{3}{2}}} \partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})_{\omega^{0,0}}^2 + \frac{a'_D}{(R_{n+1}^p)^{\frac{3}{2}}} \partial (c_{n+1}^{N_1} - c_{n+1}^{N_1})_{\omega^{0,0}}^2 + \frac{a'_D}{(R_{n+1}^p)^{\frac{3}{2}}} \partial (c_{n+1}^{N_1} - c_{n+1}^{N_1})_{\omega^{0,0}}^2 + \frac{a'_D}{(R_{n+1}^p)^{\frac{3}{2}}} \partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})_{\omega^{0,0}}^2 \]

Thus from (102), one can deduce that

\[ \left| \frac{\partial (c_{n+1}^{N_1} - c_{n+1}^{N_1})}{\partial \rho} \right|_{\omega^{0,0}}^2 \leq \frac{a'_D}{(R_{n+1}^p)^{\frac{3}{2}}} \partial (c_{n+1}^{N_1} - c_{n+1}^{N_1})_{\omega^{0,0}} \left| \frac{\partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})}{\partial \rho^2} \right|_{\omega^{0,0}} \]

In addition, from (56), we have

\[ \left| \frac{a_k - a_{k+1}}{D_1} \frac{1}{(R_{n-k}^p)^{\frac{3}{2}}} \partial \left( \rho^2 \frac{c_{n-k}^{ap} - c_{n-k}^{ap}}{\partial \rho} + \frac{\partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})}{\partial \rho^2} \right) \right|_{\omega^{0,0}} \leq \]

\[ \left| \frac{a_k - a_{k+1}}{D_1} \frac{1}{(R_{n-k}^p)^{\frac{3}{2}}} \partial \left( \rho^2 \frac{c_{n-k}^{ap} - c_{n-k}^{ap}}{\partial \rho} + \frac{\partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})}{\partial \rho^2} \right) \right|_{\omega^{0,0}} \leq \frac{a_k - a_{k+1}}{D_1} \frac{1}{(R_{n-k}^p)^{\frac{3}{2}}} \partial \left( \rho^2 \frac{c_{n-k}^{ap} - c_{n-k}^{ap}}{\partial \rho} + \frac{\partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})}{\partial \rho^2} \right) \]

where

\[ L_2^2 \phi = \phi_{n+1} - \phi_n + \frac{\phi_{n-1} - \phi_n}{3} \]

\[ t^* \frac{2(v_{ap}(1,t_n) - v_{ap}(1,t_{n-1}))\rho}{3R_{n+1}^p} \partial \phi_{n+1} \frac{1}{\rho^2} \partial \left( \rho^2 \frac{\partial \phi_{n+1}}{\partial \rho} \right) + \]

\[ \sum_{k=0}^{n-1} \frac{a_k - a_{k+1}}{D_1} \frac{1}{(R_{n-k}^p)^{\frac{3}{2}}} \partial \left( \rho^2 \frac{c_{n-k}^{ap} - c_{n-k}^{ap}}{\partial \rho} + \frac{\partial^2 (c_{n+1}^{N_1} - c_{n+1}^{N_1})}{\partial \rho^2} \right) \]
Therefore, using [103], Cauchy–Schwarz inequality and (32), for each positive \( \epsilon \) and \( \epsilon_1 \), we can deduce that

\[
\left| \left( \lambda_{n,0}^2 N_0, \frac{\partial^2 (c_{n+1} - c_{n+1,1})}{\partial \rho^2} \right)_{\omega,0} \right| \leq \left| \left( \gamma_{n,0}^2 N_0, \frac{\partial^2 (c_{n+1} - c_{n+1,1})}{\partial \rho^2} \right)_{\omega,0} \right| +
\sum_{k=0}^{n-1} \frac{1}{\epsilon (R^p_{n-k})^2} \| \frac{\partial^2 (c_{n+1} - c_{n+1,1})}{\partial \rho^2} \|_{\omega,0}^2 + \epsilon \sum_{k=0}^{n-1} \frac{(t_k - t_{k+1}) D_1}{(R^p_{n-k})^2} \| \frac{\partial^2 (c_{n+1} - c_{n+1,1})}{\partial \rho^2} \|_{\omega,0}^2 +
\frac{1}{2 \epsilon_1} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 + \epsilon \sum_{k=0}^{n-1} \frac{(t_k - t_{k+1}) D_1}{(R^p_{n-k})^2} \| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \|_{\omega,0}^2 +
\frac{1}{2 \epsilon_1} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 + \frac{1}{3} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 .
\]

Therefore, from (103) and (105), we deduce that there exists a positive constant \( K_2 \) such that

\[
\left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2 \leq \frac{1}{2} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 + \frac{1}{2} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 +\frac{1}{2} \left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2 + K_2 \left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2 +
\sum_{k=0}^{n-1} \frac{(t_k - t_{k+1}) D_1}{(R^p_{n-k})^2} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 +\frac{1}{3} \left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2
\]

Hence, from A. (57) and (58), one can conclude that there exists positive \( K_3 \) such that

\[
\left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2 \leq \frac{1}{2} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 + \frac{1}{4} \left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2 + K_3 \left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2 +
\sum_{k=0}^{n-1} \frac{(t_k - t_{k+1}) D_1}{(R^p_{n-k})^2} \left\| \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right\|_{\omega,0}^2 +\frac{1}{8} \left( \left( \frac{\partial (c_{n+1} - c_{n+1,1})}{\partial \rho} \right)_{\omega,0} \right)^2
\]

where

\[
L^*_c c = c_{n+1} - c_n - \frac{c_n - c_{n-1}}{3} - t \epsilon \frac{2(2v(1,t_n) - v(1,t_{n-1}))}{3R_{n+1}} \frac{\partial c_{n+1}}{\partial \rho} -
\]

(106)
\[
\frac{a'_0 D_1}{(R_{n+1})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c_{n+1}^p}{\partial \rho} \right) + \sum_{k=0}^{n-1} (d_k^* - d_{k+1}^*) \left( \frac{D_k}{(R_{n-k})^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial c_{n-k}^p}{\partial \rho} \right) \right).
\]

Therefore, from Lemma 1 (discrete Gronwall lemma), 13, 52, 57, 58, 100 and 106, we can conclude that there exists positive \( K_4 \) such that

\[
\left\| \frac{\partial (c_{n+1}^p - c_{n+1}^0)}{\partial \rho} \right\|_{\omega,0} \leq \left\| \frac{\partial (c_{n+p}^p - c_{n+1}^1)}{\partial \rho} \right\|_{\omega,0} + K_4 \left( (t^*)^{1+\alpha} \sum_{k=0}^{n} \left( \frac{1}{3} \frac{1}{2} \left( \| e_{k,1}^p - e_k^0 \|_{\omega,0} + \| p_k^p - p_k^p \|_{\omega,0} + \| q_k^p - q_k^p \|_{\omega,0} + \| d_k^p - d_k^p \|_{\omega,0} + \| R_k^{p*} - R_k^{p*} \|_{\omega,0} \right) + \right) \right)\frac{\| e_{k}^p \|_{\omega,0} + (t^*) K^*(N^1)}{N^1}.\]

where

\[
\lim_{N^1 \to \infty} K^*(N^1) = 0, \quad \| e_{k}^p \|_{\infty} \leq (t^*)^{2 + \frac{1}{2m}}.
\]

In addition, if \( \frac{\partial^2}{\partial \rho^2} \) is a \( C^m \)-smooth function with respect to \( \rho \) then from \( 101 \), we can show

\[
\left\| \frac{\partial (c_{n+1}^p - c_{n+1}^0)}{\partial \rho} \right\|_{\omega,0} \leq \left\| \frac{\partial (c_{n+p}^p - c_{n+1}^1)}{\partial \rho} \right\|_{\omega,0} + K_4 \left( (t^*)^{1+\alpha} \sum_{k=0}^{n} \left( \frac{1}{3} \frac{1}{2} \left( \| e_{k,1}^p - e_k^0 \|_{\omega,0} + \| p_k^p - p_k^p \|_{\omega,0} + \| q_k^p - q_k^p \|_{\omega,0} + \| d_k^p - d_k^p \|_{\omega,0} + \| R_k^{p*} - R_k^{p*} \|_{\omega,0} \right) + \right) \right)\frac{\| e_{k}^p \|_{\omega,0} + (t^*) K^*(N^1)}{N^1}.\]

\[
\square
\]

**Proof of Theorem 3.1**

Employing 38–53, 55, and 58–52, one can conclude that there exist positive \( M \) and \( M^* \) such that

\[
\mathcal{E}_{k+1} \leq \mathcal{E}_{k-1} + M^* (\mathcal{E}_k + \epsilon_k) + \| E_t \|_{\infty}, \quad \| E_t \|_{\infty} \leq M^* (t^*)^3, \tag{107}
\]

where

\[
\mathcal{E}_k = \| p_k^p - p_k \|_{\infty} + \| q_k^p - q_k \|_{\infty} + \| d_k^p - d_k \|_{\infty} + \| R_k^{p*} - R_k \|_{\infty}, \tag{108}
\]

and

\[
\epsilon_k = \left\| \frac{\partial(c_k^p - c_k)}{\partial \rho} \right\|_{\omega,0} + \left\| \frac{\partial(w_k^p - w_k)}{\partial \rho} \right\|_{\omega,0}. \tag{109}
\]

Using \( 107 \)–\( 109 \), one can conclude that

\[
\mathcal{E}_{n+1} \leq \mathcal{E}_{n-1} + M^* (\mathcal{E}_n + \epsilon_n) + \| E_t \|_{\infty} \leq (1 + M^*) \max \{ \mathcal{E}_{n-1}, \mathcal{E}_n \} + M^* \epsilon_n + \| E_t \|_{\infty} \leq
\]
\[
\frac{(1 + Mt^*)^{m-1}}{Mt^*} (\|E_t\|_\infty + Mt^* \max_{k=0, 1, \ldots, n} \{\epsilon_k\}).
\]

On the other hand, employing (59)–(61), we get

\[
\max_{k=0, 1, \ldots, n} \{(\epsilon_k)^2\} \leq \frac{M_1}{t} \left( (t^*)^{1+\alpha} \max_{k=0, 1, \ldots, n} \{(\epsilon_k)^2\} + ||\epsilon||\infty^2\right) + M_1 K_3^\dagger (N^1),
\]

where \(M_1\) is a positive constant and

\[
||\epsilon||\infty \leq (t^*)^{2+\frac{1-\alpha}{2}} + K_3^\dagger (N^1) = K_3^\dagger (N^1) + K_2^\dagger (N^1).
\]

From (110) and (111), we deduce that there exists positive \(C_4^\dagger\) such that

\[
\max_{k=0, 1, \ldots, n+1} \{(\epsilon_k)^2\} \leq C_4^\dagger e^{MT} \left( (t^*)^{2-\frac{\alpha}{2}} + (K_3^\dagger (N^1))^{\frac{\alpha}{2}}\right).
\]

Therefore, employing (111) and (112), we conclude that there exists positive constant \(C_5^\dagger\) such that

\[
\max_{k=0, 1, \ldots, n+1} \{\epsilon_k\} \leq C_5^\dagger e^{MT} \left( (t^*)^{2-\frac{\alpha}{2}} + (K_3^\dagger (N^1))^{\frac{\alpha}{2}}\right).
\]

Finally, employing (59) and (60), we can get the desired results. \(\square\)

**Proof of Theorem 4.1** If we solve the perturbed problem (69)-(79) using the presented method, employing (38)-(43), (45), and (48)-(52) for perturbed problem, one can conclude that there exist positive \(M_2\) and \(M_2^\dagger\) such that

\[
\epsilon_{n+1}^p \leq \epsilon_{n-1}^p + M_2 t^* (\epsilon_n^p + \epsilon_k^p) + t^* \epsilon_1 + ||\epsilon||\infty^p < M_2^\dagger (t^*)^3,
\]

where

\[
\epsilon_k^p = \|\epsilon_{k, p}^p - p_k\|\infty + \|q_{k, p}^p - q_k\|\infty + \|\epsilon_{k, p}^p - d_k\|\infty + |R_{k, p}^p - R_k|,
\]

and

\[
\epsilon_k^p = \|\frac{\partial(\epsilon_{k, p}^p - c_k)}{\partial p}\|\omega^{a, a} + \|\frac{\partial(\epsilon_{k, p}^p - w_k)}{\partial p}\|\omega^{a, a}.
\]

Using (114)–(116), one can conclude that

\[
\epsilon_{n+1}^p \leq \epsilon_{n-1}^p + M_2 t^* (\epsilon_n^p + \epsilon_k^p) + t^* \epsilon_1 + ||\epsilon||\infty^p \leq (1 + M_2 t^*) \max\{\epsilon_{n-1}^p, \epsilon_n^p\} + M_2 t^* \epsilon_k^p + t^* \epsilon_1 + ||\epsilon||\infty^p \leq \frac{(1 + M_2 t^*)^{m-1}}{M_2 t^*} (t^* \epsilon_1 + ||\epsilon||\infty^p + M_2 t^* \max_{k=0, 1, \ldots, n} \{\epsilon_k^p\}).
\]

(117)
Also, from (59)–(61), we conclude that
\[
\max_{k=0,1,\ldots,n} \{ e_k^p \} \leq M_3 \left( (t^*)^{\frac{1}{2}} \max_{k=0,1,\ldots,n} \{ e_k^p \} \right) + \epsilon_1 + K_5^*(N^1) + \| E_1^i \|_{\infty},
\]
where \( M_3 \) is a positive constant and
\[
\| E_1^i \|_{\infty} \leq C_6 (t^*)^{\frac{1}{2}} + K_5^*(N^1) = (K_1^*(N^1))^{\frac{1}{2}} + (K_2^*(N^1))^{\frac{1}{2}}.
\]

From (117) and (118), we deduce that there exists positive \( C_6^* \) such that
\[
\max_{k=0,1,\ldots,n+1} \{ e_k^p \} \leq C_6^* e^{M_3 T} \left( (t^*)^{\frac{1}{2}} + K_5^*(N^1) + \epsilon_1 \right).
\]

From (118) and (119), we conclude that there exists positive constant \( C_7^* \) such that
\[
\max_{k=0,1,\ldots,n+1} \{ e_k^p \} \leq C_7^* e^{M_3 T} \left( (t^*)^{\frac{1}{2}} + K_5^*(N^1) + \epsilon_1 \right).
\]

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