A NOETHER-LEFSCHETZ THEOREM FOR SPECTRAL VARIETIES WITH APPLICATIONS

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Abstract. We calculate the Picard group of generic (very general) spectral varieties living in the total space of a very ample line bundle over an algebraically closed field \( k \) of odd characteristics or characteristic 0. We follow the strategy of Ravindra and Srinivas [RS06, RS09] via formal Picard groups. As an application, we calculate the generic fibers of Hitchin systems over a smooth quintic surface of Picard number 1.

1. Introduction

To study the Vafa-Witten theory built by Vafa and Witten in [VW94], Tanaka and Thomas in [TT18, TT20] constructed the Vafa-Witten invariants on a polarised surface \((S, \mathcal{O}_S(1))\) using the moduli space of stable \( \omega_S \)-valued Higgs sheaves. This moduli space is equipped with a Lagrangian fibration called the Hitchin map.

To be more precise, let \( k \) be an algebraically closed field with \( \text{char}(k) = 0 \) or \( \text{char}(k) = p \geq 3, \) \( X \) be a smooth projective variety over \( k \) with \( \dim X \geq 2 \) and \( L \) be a very ample line bundle over \( X \). A Higgs sheaf (with value in \( L \)) in this paper will be referred to as a pair \((E, \theta)\) where \( E \) is a coherent sheaf on \( X \), \( \theta : E \rightarrow E \otimes L \) is an \( \mathcal{O}_X \)-homomorphism. Similarly, as the curve case, the characteristic polynomial of \( \theta \) defines a subvariety in \( \text{Spec Sym}^\bullet_X(\mathcal{L}^\vee) \) (the total space of \( \mathcal{L} \)) which is finite over \( X \), and we call such subvarieties spectral varieties in \( \text{Spec Sym}^\bullet_X(\mathcal{L}^\vee) \).

By the classical Cayley-Hamilton theorem, \( E \) can be treated as a coherent sheaf of generic rank 1 on the corresponding spectral variety. In the particular case that \( X \) is a surface and \( E \) is a vector bundle, \( E \) is indeed a line bundle over the corresponding spectral surface, provided that the spectral surface is smooth. We show this in Proposition 5.6 by the theory of maximal Cohen-Macaulay modules on dimension two normal Noetherian local rings (cf. [BBG97], [BD08]), (see Section 5).

Thus to study the generic fibers of the Hitchin map, we need to know more about Picard groups of generic smooth spectral varieties. Since we are in high dimensional case, the Picard groups, especially their connected component groups, are not as clear as the curve case. Our main result of this paper is to calculate the Picard groups of generic (very general) spectral varieties for the moduli of \( L \)-valued Higgs bundles (with certain ampleness) both in characteristic 0 and characteristic \( p \geq 3 \). In particular, we give a concrete description of generic fibers of such Hitchin systems over surfaces. We hope that our calculation of Picard groups can be used to study Vafa-Witten invariants constructed by Tanaka-Thomas [TT18] via generic fibers of Hitchin systems.

We now introduce some notations. For a locally free coherent sheaf \( \mathcal{E} \) on \( X \), we denote by \( \mathcal{V}\mathcal{E} := \text{Spec}_X \text{Sym}^\bullet_{\mathcal{O}_X} \mathcal{E} \) the associated vector bundle and by \( \mathcal{P}\mathcal{E} := \text{Proj}_X \text{Sym}^\bullet_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\pi} X \) the associated projective bundle with relative \( \mathcal{O}_{\mathcal{P}\mathcal{E}/X}(1) \) such that \( \pi_* \mathcal{O}_{\mathcal{P}\mathcal{E}/X}(1) \cong \mathcal{E} \).

We can see that \( \mathcal{V}\mathcal{E} \subset \widehat{\mathcal{V}\mathcal{E}} := \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \) is an open subvariety and we call \( D_\infty := \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \setminus \mathcal{V}\mathcal{E} \cong \mathbb{P}(\mathcal{E}) \) the infinite divisor. In the particular case that \( \mathcal{E} \) is the line bundle
\( \mathcal{L}, \) denoting \( P(\mathcal{L}^\vee \oplus \mathcal{O}_X) \) by \( Y, \) it is straightforward that:

\[
H^0(Y, \pi^* \mathcal{L}^i \otimes \mathcal{O}_{Y/X}(r)) = \oplus_{i=0}^r H^0(X, \mathcal{L}^i).
\]

Thus there is a natural open immersion

\[
\text{Spec}(\text{Sym}_r \oplus_{i=1}^r H^0(X, \mathcal{L}^i)^\vee) \hookrightarrow P(H^0(Y, \pi^* \mathcal{L}^i \otimes \mathcal{O}(r))^\vee).
\]

**Definition 1.1.** Given \((X, \mathcal{L})\) as before, spectral varieties are divisors corresponds to closed points in \( A := \text{Spec}(\text{Sym}_r \oplus_{i=1}^r H^0(X, \mathcal{L}^i)^\vee). \) And we call \( A \) the Hitchin base.

Here we can see that the spectral varieties are divisors in the linear system \(|\pi^* \mathcal{L}^i \otimes \mathcal{O}_{Y/X}(r)|\) which do not intersect with \( D_\infty. \) For a closed point \( s \in A, \) we denote \( X_s \) the corresponding effective divisor on \( Y \) and we have a finite surjective map \( \pi_s : X_s \to X. \)

We have the following proposition, see also Proposition 2.2:

**Proposition 1.2.** The line bundle \( \pi^* \mathcal{L}^i \otimes \mathcal{O}_{Y/X}(r) \) is big and base point free over \( Y \) for any \( r > 0. \)

In the remaining part of this paper, we fix \( r \) as the rank of Higgs bundles we shall consider in Section 5. To abbreviate the notations, we will use \( \mathcal{V} \) to denote the line bundle \( \pi^* \mathcal{L}^i \otimes \mathcal{O}_{Y/X}(r) \) and \( \mathcal{W} \) to denote its global section \( H^0(Y, \pi^* \mathcal{L}^i \otimes \mathcal{O}_{Y/X}(r)). \)

Our first goal of this paper is to calculate the Picard groups of generic (resp. very general when \( \dim X = 2 \)) spectral varieties, i.e. \( X_s \) for \( s \) in an open subvariety of \( A \) (resp. for a very general \( s \) in \( A \)). The Picard group of \( Y \) can be given by the formula for projective bundles. However, because that \( \pi^* \mathcal{L}^i \otimes \mathcal{O}_{Y/X}(r) \) is not ample on \( Y, \) we can not apply Noether-Lefschetz type theorem directly to calculate the Picard group of \( X_s. \)

In characteristic 0, Ravindra and Srinivas in [RS06, RS09] systematically deal with the Noether-Lefschetz type problem with the very ample line bundle replaced by a big and base point free line bundle which are nontrivial generalizations of the original Noether-Lefschetz type theorem. They modify Grothendieck’s ideas in [Gro68] and [Har06, §IV] and take into consideration of the exceptional locus of the morphism defined by the corresponding big and base point free linear system. We modify their proofs, work over both characteristic 0 and odd characteristics and apply to spectral varieties. In positive characteristics, we always make the following assumption:

**Assumption 1.3.** If \( \text{char}(k) = p > 0, \) we assume that \( X \) admits a \( W_2(k) \) lifting and \( p \geq 3. \)

Now let us state our theorem concerning the Picard group of a generic spectral variety when \( \dim X \geq 3. \)

**Theorem 1.4.** If \( \dim X \geq 3, \) let \( U \) be the open subvariety of the Hitchin base \( A \) parametrizing smooth spectral varieties, then under the assumption 1.3, for any closed point \( s \in U: \pi_s^* : \text{Pic}(X) \to \text{Pic}(X_s) \) is an isomorphism.

In characteristic 0, this is a special case of [RS06]. And following Ravindra and Srinivas’ strategy, since we consider quite special linear systems, we can also prove similar results in positive characteristics by a simple modification of Grothendieck’s ideas in [Gro68] and [Har06, Chapter IV].

In the case that \( \dim X = 2, \) things are more complicated. Thanks to a stronger cohomological result (see Theorem 2.7 and Corollary 2.9 in our case, under the assumption that the relative Picard scheme \( \text{Pic}_X^0 \) is smooth, we prove the following analogous result:

**Theorem 1.5.** Let \( X \) be a surface, we assume \( k \) is uncountable, \( \text{Pic}_X^0 \) is smooth, then there are very general members of \( s \) in \( A \) such that \( \pi_s^* : \text{Pic}(X) \to \text{Pic}(X_s) \) is an isomorphism.

For example, if \( H^1(X, \mathcal{O}_X) = 0 \) by [FGI+05, Corollary 9.5.13] or if \( H^2(X, \mathcal{O}_X) = 0 \) by [FGI+05, Proposition 9.5.19], \( \text{Pic}_X^0 \) is smooth. For more examples, see [Lie09]. As we
mentioned in the beginning of the introduction, this theorem can be used to study generic fibers of Hitchin maps for moduli of $\mathcal{L}$-valued Higgs bundles on $X$. In the last section, we give an application to Hitchin systems over a smooth quintic surface of Picard number one both in characteristic 0 and odd characteristics, see Theorem 5.10. Roughly speaking, we show that

**Theorem 1.6.** Let $X$ be a smooth quintic surface of Picard number 1. We assume the rank of the torsion-free Higgs sheaves is greater than 3. Then generic fibers of Hitchin maps, if non-empty, are connected and isomorphic to Hilbert scheme of points of corresponding spectral surfaces.

Let us now indicate how this relates to previous work.

It is a long-lasting question in algebraic geometry to calculate Picard groups of a divisor known as Noether-Lefschetz type theorem. Grothendieck defines the so-called (effective) Lefschetz condition in [Gro68] to solve Noether-Lefschetz problems systematically by formal geometry. This is also explained in detail by Hartshorne in [Har70]. In our situation, the linear system considered is not ample but big and base-point free, so we can not apply Grothendieck-Lefschetz theorem directly. However, in characteristic 0, Grothendieck’s idea are generalized by Ravindra and Srinivas to big and base-point free linear systems in [RS06] when $\dim X \geq 3$ and in [RS09] when $\dim X = 2$. The assumption of characteristic 0 is essential because the authors use the resolution of singularities, exponential exact sequence for formal Picard groups and also when $\dim X = 2$ the smoothness of Picard varieties is needed which naturally holds in characteristic 0. It is obvious that our calculation of Picard groups follows Ravindra and Srinivas’s strategy. To be more precise, by our assumption of $W_2$-lifting in positive characteristics, when $\dim X \geq 3$, Ravindra and Srinivas’ method can be applied directly, we here present a complete and simpler proof, since the linear system we consider is quite special though not very ample. When $\dim X = 2$, the exponential exact sequence for Picard groups can not hold for arbitrary thickening, we need to use a step by step lifting of line bundles to formal neighbourhood to verify Ravindra and Srinivas’ “Formal Noether-Lefschetz” condition (see [RS09, Definition 1]). To make the step-by-step induction work, we need a stronger cohomological property than “global-generation” in [RS09, Theorem 2] which fortunately holds for the linear systems we consider here, see Theorem 2.7.

We here must mention the very recent and deep results of Lena Ji [Ji21] of Noether-Lefschetz type theory for normal threefolds in positive characteristics. In her thesis [Ji21], Ji uses a quite different method to calculate Picard groups of divisors lying in a linear system with sufficient ampleness on normal threefolds, see [Ji21, Corollary 3.4.2]. Moreover, she does not need to assume the existence of $W_2$ lifting and the smoothness of certain Picard variety. Our results here provided some special cases, i.e., various normal varieties other than $\mathbb{P}^3$, that are not covered by Ji’s [Ji21], see more in Remark 4.14.

As we mentioned in the beginning of the introduction, via the nilpotent cones of Hitchin maps, Tanaka and Thomas [TT18] can construct Vafa-Witten invariants in an algebro-geometric manner. Our original purpose was to study the generic fibers of corresponding Hitchin maps instead of nilpotent cones. By the classical BNR correspondence, Higgs bundles on base varieties can be treated as coherent sheaves on spectral varieties. By the theory of maximal Cohen-Macaulay modules in [BBG97], and the deep theory of maximal Cohen-Macaulay modules (of generic rank 1) over a normal Noetherian local ring of dimension 2, the study of generic fibers turns into a study of Picard groups of spectral surfaces.

We now close this section by briefly describing how the paper is organized. In the second section, we prove certain cohomological results which we will use later. In the third section, we prove our results on Picard groups when $\dim X \geq 3$. In the fourth
section, we prove the results for very general spectral surfaces when \( \dim X = 2 \). In the last section, we give an application to generic fibers of Hitchin systems over a smooth quintic surface of Picard number one.

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2. Vanishing Properties

In this section, we first start with some basic properties of projective bundles. As in the introduction, we denote by \( \overline{\mathcal{V}}_\mathcal{E} := \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \) the associated projective compactification of \( \mathcal{E} \).

The inclusion \( \mathcal{E} \subset \mathcal{E} \cdot e \oplus \mathcal{O}_X \cdot t \) induces the exact sequence of graded sheaves

\[
0 \to (t) \to \text{Sym}^* (\mathcal{E} \cdot e \oplus \mathcal{O}_X \cdot t) \to \text{Sym}^* \mathcal{E} \to 0,
\]

the isomorphism \( \text{Sym}^* \mathcal{E} \cong \text{Sym}^* (\mathcal{E} \cdot \frac{1}{t}) \), and the surjective quotient map \( \mathcal{E} \oplus \mathcal{O}_X \to \mathcal{O}_X \).

Then one has the open immersion \( \mathcal{V} \mathcal{E} \subset \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \) (locally given by \( e \mapsto [e : 1] \)) with its complement \( \mathcal{P} \mathcal{E} \subset \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X \cdot t) \), which is the zero locus of the degree 1 homogeneous section \( t \in \mathcal{O}_{\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)/X}(1) \). We call it the infinity divisor denoted by \( D_\infty \). We also have the zero section \( \sigma : X \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \), which factors as the zero section of the open immersion

\[
\sigma : X \overset{0}{\to} \mathcal{V} \mathcal{E} \subset \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X).
\]

Thus one has \( \mathcal{V} \mathcal{E} \sqcup D_\infty = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \) and \( \sigma(X) \subset \mathcal{V} \mathcal{E} \) via the zero section. Besides there is a (line bundle) projection \( p_{-0} : \mathcal{V} \mathcal{E} - \sigma(X) \to D_\infty \).

Let us consider the case \( \mathcal{E} = L^\vee \). Then \( \pi : Y := \mathbb{P}(L^\vee \oplus \mathcal{O}_X) \to X \) and \( \pi : D_\infty \cong X \). As we specify the line bundle \( L \), we may denote \( \mathcal{O}_{\mathbb{P}(L^\vee \oplus \mathcal{O})/X}(1) \) by \( \mathcal{O}_{Y/X}(1) \) for simplicity. It is easy to see that \( \mathcal{O}_{Y/X}(1)|_{\mathcal{V}L^\vee} = \mathcal{O}_{\mathcal{V}L^\vee}, \mathcal{O}_{Y/X}(1)|_{D_\infty} = (\pi|_{D_\infty})^*L^\vee \).

Remark 2.1. It is straightforward to see that \( \pi^*L \otimes \mathcal{O}_{Y/X}(1)|_{D_\infty} \) is trivial, and thus it is not ample.

Proposition 2.2. The line bundle \( \pi^*L \otimes \mathcal{O}_{Y/X}(1) \) is big and base-point free on \( Y \).

Remark 2.3. This proposition still holds if we only assume \( L \) is big and base point free. Thus in characteristic 0, when \( \dim X \geq 3 \), we can apply [RS06, Theorem 2] to the case that the line bundle \( L \) is only assumed to be big and base point free.

Proof. Since \( L \) is base-point free on \( X \), and \( \mathcal{L} \otimes \pi_*\mathcal{O}_{Y/X}(1) \cong \mathcal{O}_X \oplus \mathcal{L} \), we have the surjective evaluation map:

\[
e : H^0(X, \mathcal{O}_X \oplus \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{L}.
\]

Pullback it via \( \pi^* \), we have the factorization of the evaluation map

\[
\begin{array}{c}
\text{ev}_{\mathcal{L} \oplus \mathcal{O}_{Y/X}(1)} : H^0(Y, \pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)) \otimes \mathcal{O}_Y \rightarrow \pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1) \\
\end{array}
\]

\[
\begin{array}{c}
\pi^*: H^0(X, \mathcal{L} \otimes \pi_*\mathcal{O}_{Y/X}(1)) \otimes \mathcal{O}_X \rightarrow \pi^*\mathcal{L} \otimes \pi^*\pi_*\mathcal{O}_{Y/X}(1)
\end{array}
\]
which shows that $\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)$ is base-point free. Hence $\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)$ is nef. To show that $\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)$ is big, we just have to check that the intersection number $(\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1))^{\dim X+1} > 0$. But $\sigma(X)$ is a zero divisor of $\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)$, and $\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)|_{\sigma(X)} \cong \mathcal{L}$. Then $(\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1))^{\dim X+1} = (\mathcal{L})^{\dim X} > 0$ which follows from the bigness of $\mathcal{L}$ on $X$.

Lemma 2.4. Let $\omega_Y$ be the canonical line bundle of $Y$, then:
$$\omega_Y \cong \pi^*(\omega_X \otimes \mathcal{L}^r) \otimes \mathcal{O}_{Y/X}(-2)$$

Proof. By the relative Euler exact sequence:
$$0 \to \Omega^1_{Y/X} \to \pi^*(\mathcal{L}^r \otimes \mathcal{O}_X) \otimes \mathcal{O}_{Y/X}(-1) \to \mathcal{O}_Y \to 0,$$
we can calculate that $\omega_Y \cong \det(\Omega^1_{Y/X}) \otimes \pi^*\omega_X \cong \pi^*(\omega_X \otimes \mathcal{L}^r) \otimes \mathcal{O}_{Y/X}(-2)$.

Lemma 2.5. $H^i(Y, \pi^*\mathcal{L}^{-n} \otimes \mathcal{O}_{Y/X}(-n)) = 0$ for $i < 3$ and any $n \geq 1$.

Proof. By Proposition 2.2, this follows from Kawamata-Viehweg Vanishing theorem in characteristic 0 for $i < \dim Y$. Now we prove it in positive characteristics.

Recall $\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1) \cong \mathcal{O}_{P(\mathcal{L} \oplus \mathcal{O})/X}(1)$ and $Y = P(\mathcal{L} \oplus \mathcal{O}) \cong P(\mathcal{L} \oplus \mathcal{O})$ is a projective bundle on $X$, thus by the direct image formula of projective bundles, $R^n\pi_*(\pi^*\mathcal{L}^{-n} \otimes \mathcal{O}_{Y/X}(-n)) \cong \oplus_{i=1}^{n-1} \mathcal{L}^{-i}[1]$ for $n > 1$ and 0 for $n = 1$. Thus we have $H^i(Y, \pi^*\mathcal{L}^{-n} \otimes \mathcal{O}_{Y/X}(-n)) \cong H^{i-1}(X, \oplus_{i=1}^{n-1} \mathcal{L}^{-i})$. Since $\mathcal{L}$ is very ample, $X$ can be lift to $W_2(k)$ and $\text{char}(k) \geq 3$, then by [DI87, Corollary 2.8.2(2.8.2)], we have the vanishing $H^i(Y, \pi^*\mathcal{L}^{-n} \otimes \mathcal{O}_{Y/X}(-n)) = 0$ for $i < 3$.

By the construction as in [Gro61a, Corollarie 8.8.4, and Theorem 8.9.1(critère de Grauert)], one has the induced open and closed decomposition:

$$\begin{array}{ccc}
\mathbb{V}\mathcal{L} & \longrightarrow & \mathbb{A}^N_X \\
\text{open} & \downarrow & \\
\mathbb{P}(\mathcal{L}^r \oplus \mathcal{O}_X) & \cong & \mathbb{P}^N_X \\
\sigma & \downarrow & \\
X & \overset{\varphi_{|\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)}}{\longrightarrow} & \mathbb{P}^N_k \\
\varphi_{|\mathcal{L}} & \varphi_{|\mathcal{L}^r} & \varphi_{|\mathcal{L}^r \otimes \mathcal{O}_{Y/X}(1)}
\end{array}$$

By the Grauert’s criterion [Gro61a, 8.9.1], $\mathcal{L}$ is very ample, so $\mathbb{V}\mathcal{L}$ to its image in $\mathbb{A}^N_k$ is the blowing down along $D_\infty$. Its closed complement is the closed immersion defined by $\varphi_{|\mathcal{L}}$. Thus the projection $\varphi_{|\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)}$ is factored as the composition of a blowing down along $D_\infty$ and a closed immersion.

Proposition 2.6. The linear system $|\pi^*\mathcal{L}^r \otimes \mathcal{O}_{Y/X}(r)| = |\mathcal{W}|$ induces a morphism
$$g : Y \to \mathbb{P}H^0(Y, \pi^*\mathcal{L}^r \otimes \mathcal{O}_{Y/X}(r)) = \mathbb{P}(W),$$
which is an immersion over $U$ and $g(D_\infty)$ is a point. The image of $Y$ under $g$ is normal.

Proof. The previous arguments also hold for the linear system $|\pi^*\mathcal{L}^{\otimes r} \otimes \mathcal{O}_{Y/X}(r)|$, denoting the projection by $g : Y \to \mathbb{P}H^0(Y, \pi^*\mathcal{L}^{\otimes r} \otimes \mathcal{O}_{Y/X}(r))$, one has $g$ factored as $g = v_r \circ \varphi_{|\pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)}$, where $v_r$ is the $r$-fold Veronese embedding.

It is easy to see $g(Y)$ is integral. Since $\dim g(Y) \geq 3$ and has a unique isolated singularity, by Serre’s criterion, $g(Y)$ is normal.

Denote by $Z$ the image of $g$. We put $o = g(D_\infty)$ which is the unique singularity of $Z$ and $Z^o = Z - \{o\}$ which is isomorphic to $\mathbb{V}\mathcal{L}^{\otimes r}$. Since $Z$ is also isomorphic to the image of $\varphi_{|\pi^*\mathcal{L} \otimes \mathcal{O}(1)}$ via the $r$-fold Veronese embedding, there is a very ample line bundle $\mathcal{H}$ on $Z$ such that $g^*\mathcal{H} \cong \pi^*\mathcal{L} \otimes \mathcal{O}_{Y/X}(1)$, and $\mathcal{O}_{\mathcal{W}}(1)|_Z = \mathcal{H}^{\otimes r}$. Recall that a coherent sheaf $\mathcal{F}$ on $Z$ is $m$-regular with respect to $\mathcal{H}$, if $H^q(Z, \mathcal{F} \otimes \mathcal{H}^{\otimes (m-q)}) = 0$ for $q \geq 1$. 


**Theorem 2.7.** Let $X$ be a smooth projective surface. Then under the Assumption 1.3, for $r > 3$, we have $g_\ast \omega_Y \otimes \mathcal{O}_{P(W)}(1) = g_\ast (\omega_Y \otimes W)$ is Castelnuovo-Mumford 0-regular with respect to the very ample line bundle $\mathcal{H}$ on $Z$.

**Proof.** We first prove that $Rg_\ast \omega_Y \cong g_\ast \omega_Y$.

Recall that for $f : B \to C$ be a proper morphism between varieties and $\mathcal{F}$ a coherent sheaf on $B$. The following are equivalent, for a proof see [KM98] Proposition 2.69:

- $H^q(B, \mathcal{F} \otimes f^* \mathcal{O}_C(H)) = 0$ for $H$ sufficiently ample,
- $R^q f_\ast \mathcal{F} = 0$.

We take $B = Y, C = Z$, $\mathcal{F} = \omega_Y$, and $H = \mathcal{H}^\ell$ for $\ell \gg 0$. Then by Lemma 2.5, we have $R^q g_\ast \omega_Y = 0$ for $q = 1, 2, 3$. Since dim $Y = 3$, we have $Rg_\ast \omega_Y = g_\ast \omega_Y$.

Since $r > 3$, and $g^* \mathcal{H} \cong \pi^* \mathcal{L} \otimes \mathcal{O}_{Y/X}(1)$, then again by Lemma 2.5, for $q = 1, 2, 3$ we have

$$H^q(Z, g_\ast (\omega_Y \otimes W) \otimes \mathcal{H}^\otimes(q)) = H^{3-q}(Y, \pi^* \mathcal{L}^\otimes q-r \otimes \mathcal{O}_{Y/X}(q-r)) = 0.$$

And the zero-regularity of $g_\ast (\omega_Y \otimes_Y W)$ follows. \hfill $\Box$

**Remark 2.8.** If char($k$) = 0, then by Kollár [Kol86, Theorem 2.1], $Rg_\ast \omega_Y = \omega_Y$ for a generic finite map between proper varieties with $X$ smooth.

In particular, we have:

**Corollary 2.9.** For any $\ell \geq 0$, we have the following surjection:

$$H^0(Z, g_\ast (\omega_Y \otimes W)) \otimes H^0(Z, \mathcal{O}_Z(\ell)) \to H^0(Z, g_\ast (\omega_Y \otimes W) \otimes \mathcal{O}_Z(\ell)).$$

**Proof.** By the 0-regularity of $g_\ast (\omega_Y \otimes W)$ with respect to $\mathcal{H}$, and the Mumford’s theorem, see [FGI^+05, Chapter 5, Lemma 5.1] or [Laz04, Theorem 1.8.5], we have:

$$H^0(Z, g_\ast (\omega_Y \otimes W)) \otimes H^0(Z, \mathcal{H}^\ell) \to H^0(Z, g_\ast (\omega_Y \otimes W) \otimes \mathcal{H}^\ell)$$

is surjective for any $\ell \geq 0$. Since $H^0 = \mathcal{O}_Z(1)$, we are done. \hfill $\Box$

### 3. Higher Dimension Case

In this section, we consider the spectral variety $X_s$ for $s \in A$. First, we check the smoothness of a generic spectral variety. This can be done by considering spectral varieties defined by equations:

$$\lambda' + a_r = 0,$$

where $a_r \in H^0(X, \mathcal{L}^\otimes r) \subset A$. By the very ampleness of $\mathcal{L}$, for generic $a_r$, the zero divisor of $a_r$ is smooth. Then the corresponding spectral variety, as a $r$-cyclic cover of $X$ ramified over zero($a_r$), is smooth. Thus we can see that generic spectral varieties are smooth.

Let $X_s$ be a generic smooth spectral variety with $s \in A$, our goal in this section is to show that the natural map $\pi^* : \text{Pic}(X) \to \text{Pic}(X_s)$ is an isomorphism when dim $X \geq 3$.

Considering the following exact sequence:

$$0 \to \mathbb{Z}[D_\infty] \to \text{Pic}(Y) \to \text{Pic}(U) \cong \pi^* \text{Pic}(X) \to 0,$$

we only need to prove the following exact sequence for generic $s$:

$$0 \to \mathbb{Z}[D_\infty] \to \text{Pic}(Y) \to \text{Pic}(X_s) \to 0. \tag{3.1}$$

Since we also consider positive characteristics and also for the completeness of the paper, we simplified and adapted the proofs in [RS06] to our cases.

Let us denote the formal completion of $Y$ along $X_s$ by $\hat{Y}_s$ and the $\ell$-th thickening of $X_s$ by $X_{s, \ell}$. Then $\hat{Y}_s = \lim X_{s, \ell}$ in the category of locally ringed spaces. We denote the defining ideal $X_s$ by $\mathcal{I}_s \cong \mathcal{W}^{-1}$. One has the exact sequence:

$$0 \to \mathcal{I}_s^\ell \to \mathcal{O}_Y \to \mathcal{O}_{X_{s, \ell}} \to 0.$$
Proposition 3.1. If \( \dim X \geq 3 \), \( \text{Pic}(\hat{Y}_s) \cong \text{Pic}(X_s) \). If \( \dim X = 2 \), the natural map \( \text{Pic}(\hat{Y}_s) \to \text{Pic}(X_s) \) is an injection.

Proof. One has the following exact sequence:

\[
0 \to T^m_s/T^{m+1}_s \to O_{X_s,m}^\times \to O_{X_s,m}^\times \to 0
\]

Since \( \pi^*L^{-m} \otimes O_{Y/X}(-m)|_{X_s} \cong \pi_s^*L^{-m} \), we have

\[
H^i(X_s, T^m_s/T^{m+1}_s) \cong H^i(X_s, \pi^*L^{-m} \otimes O_{Y/X}(-m)|_{X_s})
\]

If \( \dim X \geq 3 \), by the W2(k)-Kodaira Vanishing theorem in positive characteristics and the Kawamata-Viehweg Vanishing theorem for big and base point free line bundle in characteristic 0, \( H^i(X_s, T^m_s/T^{m+1}_s) = 0 \) for \( i = 1, 2 \) and \( m \geq 1 \). Then by [RS06, Exposé XI.1], we get the isomorphism. If \( \dim X = 2 \), similarly, we get \( H^i(X_s, T^m_s/T^{m+1}_s) = 0 \) for \( i = 1 \), thus \( \text{Pic}(\hat{Y}_s) \to \text{Pic}(X_s) \) is an injection. \( \square \)

Now let us recall the modified Grothendieck’s Lefschetz conditions introduced in [RS06, Definition 1], which is weaker than that in [Gro68, Exposé X.2] and fits into our cases well. It is this weaker Lefschetz condition that helps explain why we can not have the results as in [Gro68, Theorem 3.1.8].

Definition 3.2 ([RS06], [Gro68] Exposé X.2). Let \( T \) be a smooth projective variety, and \( D \) an effective divisor in \( T \). We put \( \hat{T} \) the formal completion of \( T \) along \( D \). We say the pair \( (T, D) \) satisfies the weak Lefschetz condition, denoted by \( \text{Lef}^w(T, D) \) if for any any open neighborhood \( V \) of \( D \), and any locally free coherent sheaf \( \mathcal{F} \) on \( V \), there is an open subset \( V' \subset V \), such that the natural map: \( H^0(V', \mathcal{F}|_{V'}) \to H^0(\hat{T}, \hat{\mathcal{F}}) \) is an isomorphism. Here \( \hat{\mathcal{F}} \) is the completion of \( \mathcal{F} \) along \( D \).

We say the pair \( (T, D) \) satisfies the weak effective Lefschetz condition, which we denote by \( \text{Lef}^e(T, D) \), if it satisfies \( \text{Lef}^w(T, D) \) and in addition, for all locally free coherent sheaf \( \mathcal{F} \) on \( \hat{T} \), there exists an open neighbourhood \( V \) of \( D \) and a locally free coherent sheaf \( \mathcal{F} \) on \( V \) such that \( \hat{\mathcal{F}} \cong \mathcal{F} \).

Notice that for any open neighborhood of \( X_s \) and any locally free coherent sheaf \( \mathcal{F} \) on \( V \), it can be extended to a reflexive sheaf on \( Y \). The following proposition shows that the pair \( (Y, X_s) \) satisfies the weak Lefschetz condition in Definition 3.2.

Proposition 3.3. For any reflexive sheaf \( \mathcal{N} \) on \( Y \) which is locally free in an open neighborhood of \( X_s \), there exists an integer \( m \) such that the natural map:

\[
H^0(Y, \mathcal{N}(mD_\infty)) \to H^0(\hat{Y}_s, \hat{\mathcal{N}})
\]

is an isomorphism.

Proof. Since \( X_s \cap D_\infty = \emptyset \), \( H^0(\hat{Y}_s, \hat{\mathcal{N}}) \cong H^0(\hat{Y}_s, \hat{\mathcal{N}}(\ell D_\infty)) \) for any \( \ell \in \mathbb{Z} \).

Recall that, as a divisor, \( X_s \simeq r(D_\infty + \pi^*L) \) and \( X_s - iD_\infty \) is ample for \( 0 < i < r \). In fact, \( D_\infty + (1 + \epsilon)\pi^*L \) is ample for any \( \epsilon > 0 \), i.e., \( r(D_\infty + \pi^*L) + m\pi^*L \) is ample for all \( m > 0 \). This is because \( O_Y(r(D_\infty + \pi^*L) + m\pi^*L) \cong O_{P(L^m \oplus L^{r+m})}(1) \) and \( L^m \oplus L^{r+m} \) is an ample vector bundle.

The map in this proposition is induced by first considering the exact sequence for each thickening \( X_s,n \) (the sequence is exact because we assume \( \mathcal{N} \) is locally free along \( X_s \))

\[
0 \to \mathcal{N}(mD_\infty - nX_s) \to \mathcal{N}(mD_\infty) \xrightarrow{t_n} \mathcal{N}(mD_\infty)|_{X_s,n} \cong \mathcal{N}|_{X_s,n} \to 0
\]

then taking the inverse limit \( \lim_{\leftarrow} H^0(t_n) \) (cf. [FGI+05, Chapter 8, Corollary 8.2.4]). To prove the proposition, we have to show that for \( n \) sufficiently large \( H^0(t_n) \) is both injective and surjective (cf. [FGI+05, 8.2.5.2] or [Gro61b, Ch. 0, 13]). This can be deduced from
the vanishing of the cohomologies
\[ H^0(Y, \mathcal{N}(mD_\infty - nX_s)) = H^1(Y, \mathcal{N}(nX_s - mD_\infty))^\vee = 0, \]
\[ H^1(Y, \mathcal{N}(mD_\infty - nX_s)) = H^{\dim Y - 1}(Y, \omega_Y \otimes \mathcal{N}^\vee(nX_s - mD_\infty))^\vee = 0. \]
This is because \( nX_s - mD_\infty = m(X_s - D_\infty) + (n - m)X_s \), \((X_s - D_\infty)\) is ample and \( X_s \) is nef. Then by the Fujita vanishing theorem [Fuj83, Theorem (1)] (see also [Laz04, Remark 1.4.36]) for \( m \) sufficiently large, and all \( n > m \), we have the desired vanishing of cohomologies, which complete the proof.

**Proposition 3.4.** We have the following exact sequence:
\[ 0 \to \mathbb{Z}D_\infty \to \text{Pic}(Y) \to \text{Pic}(\hat{Y}_s). \]
Since \( \text{Pic}(\hat{Y}_s) \to \text{Pic}(X_s) \) is injective, we also have:
\[ 0 \to \mathbb{Z}D_\infty \to \text{Pic}(Y) \to \text{Pic}(X_s). \]

**Proof.** It is obvious that \( D_\infty \) is trivial when restricts to \( \hat{Y}_s \). Let \( Z' = Y - D_\infty \), the exact sequence is deduced if we can show the injectivity of \( \text{Pic}(Z') \to \text{Pic}(X_s) \). In other words, for line bundle \( \mathcal{M} \) on \( Z' \) such that \( \mathcal{M}|_{X_s} \) is trivial, we have to show \( \mathcal{M} \) is trivial. By the Proposition 3.1, \( \text{Pic}(\hat{Y}_s) \to \text{Pic}(X_s) \), we know that \( \hat{\mathcal{M}} \) is trivial on \( \hat{Y}_s \).

Then there is an invertible section of \( \hat{\mathcal{M}} \). By the Proposition 3.3, there exists an open neighborhood \( V \) such that the isomorphism
\[ H^0(Y, \mathcal{M}(mD_\infty)) \to H^0(\hat{Y}_s, \hat{\mathcal{M}}) \]
factors through \( H^0(V, \mathcal{M}(mD_\infty)|_V) \to H^0(\hat{Y}_s, \hat{\mathcal{M}}) \) which is also an isomorphism (it is injective because of the torsion freeness). Thus \( \mathcal{M}(mD_\infty) \) has an invertible section in \( U \) which means \( \mathcal{M}(mD_\infty) \cong \mathcal{O}_Y(\ell D_\infty) \) for some \( \ell \). We finish the proof.

**Proposition 3.5.** For the pair \((Y, X_s)\), \( \text{Leff}^w(Y, X_s) \) holds.

**Proof.** By [Har70, Chapter IV, Theorem 1.5] (also see [Gro61b, 5.2.4] and [FGI′05, 8.4.3]), since \( W = \pi^* \mathcal{L}' \otimes \mathcal{O}_{Y/X}(r) \) restricts to a very ample line bundle in an open neighborhood of \( X_s \), for any locally free coherent formal sheaf \( \mathcal{F} \) on \( \hat{Y}_s \), we have the exact sequence:
\[ \mathcal{O}_{\hat{Y}_s}(-m_1)^{\oplus M_1} \xrightarrow{\hat{\varphi}} \mathcal{O}_{\hat{Y}_s}(-m_2)^{\oplus M_2} \to \mathcal{F} \to 0. \]

For notation ease, we simply write \( W^m|_{\hat{Y}_s} \) by \( \mathcal{O}_{\hat{Y}_s}(m) \).

By Corollary 3.3, \( \text{Leff}^w(Y, X_s) \) holds and
\[ \hat{\varphi} \in \text{Hom}(\mathcal{O}_{\hat{Y}_s}(-m_1)^{\oplus M_1}, \mathcal{O}_{\hat{Y}_s}(-m_2)^{\oplus M_2}) \cong \text{Hom}(\mathcal{O}_Y(-m_1)^{\oplus M_1}, \mathcal{O}_Y(-m_2)^{\oplus M_2}) \]
is algebizable by \( \varphi \in \Gamma(Y, \text{Hom}_Y(\mathcal{O}_Y(-m_1)^{\oplus M_1}, \mathcal{O}_Y(-m_2)^{\oplus M_2} \otimes \mathcal{O}_Y(m_3D_\infty))) \). Then we have \( \text{Coker} (\varphi) \cong \mathcal{F} \). For any \( y \in X_s \), we have \( I_s \subset m_y \). Then completion along \( I_s \) and then completion along \( m_y^{\infty} \) is equal to directly complete at \( m_y \).

This means \( \text{Coker} (\hat{\varphi}|_{m_y}^{\infty}) \) is locally free at each point \( y \in X_s \). By faithful flatness of the completion along a maximal ideal, \( \text{Coker} (\varphi) \) is locally free after being localized at each closed point \( y \in X_s \). Thus \( \text{Coker} (\varphi) \) is locally free over a neighborhood \( U \) of \( X_s \).

**Theorem 3.6.** We have the following exact sequence:
\[ 0 \to \mathbb{Z}D_\infty \to \text{Pic}(Y) \xrightarrow{\hat{\varphi}} \text{Pic}(\hat{Y}_s) \to 0. \]
in particular, if \( \dim X \geq 3 \), we have \( \pi_s^*: \text{Pic}(X) \to \text{Pic}(X_s) \) is an isomorphism provided \( X_s \) is smooth.
Proof. By Proposition 3.1 and Proposition 3.4, we only need to show that \( c \) is surjective. For any formal line bundle \( M \) on \( \hat{Y}_s \), by Def.\( a(Y, X_s) \) there is an open neighbourhood \( V \) of \( X_s \) and an invertible sheaf \( M_V \) on \( V \) such that \( \hat{M}_V \cong M \). One check that \( M_V \) can always be extend to a line bundle \( M \) over \( Y \) provided \( Y \) is smooth (cf. [Hei10, Corollary 3.4] extend \( M_V \) to a coherent sheaf and take dual). Thus \( c(M) = \hat{M} = M \) and \( c \) is surjective.

If \( \dim X \geq 3 \), \( \text{Pic}(\hat{Y}_s) \to \text{Pic}(X_s) \) is an isomorphism, thus we conclude that \( \pi^*_s : \text{Pic}(X) \to \text{Pic}(X_s) \) is an isomorphism.

4. Surface Case

In this section, we prove a similar result when \( X \) is a smooth surface, assuming that the Picard variety of \( X \) is smooth.

**Notation.** \( \text{Pic}_X/S \) means the relative Picard functor (the fppf sheaf of sets) and the scheme it is represented by (if it is representable), if \( S = \text{Spec} \ k \), we may omit \( S \). \( \text{Pic}(X) \) means the Picard group of \( X \), which is isomorphic to \( H^1(X, \mathcal{O}_X^*) \).

Let \( X \) be a smooth projective surface over an uncountable algebraically closed field \( k \). We assume \( X \) can be lifted to \( W_2(k) \), and \( \mathbf{Pic}^0_X \) is smooth i.e. \( \dim Q, H^1(X_{\text{et}}, Q_\ell) = 2 \dim H^1(X, O_X) \), \( \ell \neq \text{char}(k) \). As before, let \( \mathcal{L} \) be a very ample line bundle on \( X \) and we put \( Y = \text{P}(\mathcal{L}^r \oplus \mathcal{O}) \) with \( \pi : Y \to X \) the natural projection. Since \( \mathcal{L} \) is very ample, we have \( Z = g(Y) \) consisting of a unique singularity \( o = \varphi(D_{\infty}) \) where \( g \) is induced by the complete linear system \( |\pi^* \mathcal{L^r} \otimes \mathcal{O}_Y(r)| = |\mathcal{W}| \).

Let \( X_s \subset Y \) be a smooth spectral surface defined by \( s \in |\pi^* \mathcal{L}^r \otimes \mathcal{O}_Y(r)| \), which does not intersect with \( D_{\infty} \). One can always view \( X_s \) as a very ample divisor in \( Z \) defined by a global section of \( O_Z(1) \). The main goal of this subsection is to prove the following theorem:

**Theorem 4.1** (\( \text{char} > 0 \), \( \dim X = 2 \) case). For very general \( s \in |\pi^* \mathcal{L}^r \otimes \mathcal{O}_Y(r)| \) the map \( \pi^* : \mathbf{Pic}_X \cong \mathbf{Pic}_X \) is an isomorphism.

Since \( Z \) is normal and has a unique singularity, we have:

**Lemma 4.2.** \( \text{Pic}(X) \cong \text{Pic} (\mathcal{V}_X(\mathcal{L}^r)) \cong \text{Pic}(U) = \text{Pic}(Z) \).

Since \( X_s \) can be treated as a closed subvariety of \( Z \), we put \( \hat{Z}_s \) as the formal completion of \( Z \) along \( X_s \). In fact, we have \( \hat{Z}_s \cong \hat{Y}_s \).

**Lemma 4.3.** For any \( s \) parametrizing smooth spectral surface \( X_s \), we have \( \text{Pic}(Z) \leftarrow \text{Pic}(\hat{Z}) \leftarrow \text{Pic}(\hat{Z}_s) \).

**Proof.** In Proposition 3.4, we have proved the exact sequence: \( 0 \to \mathbb{Z}D_{\infty} \to \text{Pic}(Y) \to \text{Pic}(\hat{Y}_s) \). Since \( g : Y \to Z \) is a contraction of \( D_{\infty} \) and \( Z \) is normal, \( 0 \to \mathbb{Z}D_{\infty} \to \text{Pic}(Y) \to \text{Pic}(Z) \to 0 \). And \( \text{Pic}(\hat{Y}_s) \cong \text{Pic}(\hat{Z}_s) \), then by Proposition 3.1, \( \text{Pic}(\hat{Z}) \leftarrow \text{Pic}(\hat{Z}_s) \).

**Lemma 4.4.** For any \( s \) parametrizing smooth spectral surface \( X_s \), then \( \mathbf{Pic}(X_s) \) is also smooth, and we have \( \mathbf{Pic}_X = \mathbf{Pic}_Z \cong \mathbf{Pic}_X^0 \).

**Proof.** We have the composite

\[
\text{Pic}(X) \xrightarrow{\pi^*_\mathcal{V}(\mathcal{L}^r)/X \cong \mathbf{Pic}(\mathcal{V}(\mathcal{L}^r)) \cong \mathbf{Pic}(Z)} \xrightarrow{f_{\mathcal{V}}} \text{Pic}(X_s)
\]

equals to the flat pullback \( \pi^*_s : \text{Pic}(X) \to \text{Pic}(X_s) \), so \( \pi^*_s \) is injective, in particular injective on \( \ell \)-torsion points. i.e. \( \pi^*_s : H^1(X_{\text{et}}, \mu_\ell) \to H^1(X_{\text{et}}, \mu_\ell) \). Thus one has the comparison of \( Z/\ell \)-Betti numbers \( b_1(X) \leq b_1(X_s) \).
Since $\pi_s^*O_{X_s} \cong \bigoplus_{i=0}^{-1} \mathcal L^i$, then by the Kodaira vanishing theorem
\[ \pi_s^*: H^1(X, O_X) \to H^1(X_s, O_{X_s}) \]
is an isomorphism. By our assumption of smoothness of $\text{Pic}^0_X$, we have $b_1(X) = 2h^1(X, O_X)$ (in fact, this also follows from $W_2$ lifting). Combined with the previous results, we have
\[ b_1(X) \leq b_1(X_s) \leq 2h^1(X_s, O_{X_s}) = 2h^1(X, O_X) = b_1(X). \]
Hence we get the smoothness of $\text{Pic}^0_{X_s}$ and thus the smoothness of $\text{Pic}^0_X$ by [FGI+05, Corollary 9.5.13], and isomorphisms $\text{Pic}^0_X = \text{Pic}^0_s \cong \text{Pic}^0_{X_s}$.

Let $\mathcal{Y}$ contained in $Y \times \mathbb{P}(H^0(Y, \pi^* \mathcal{L}' \otimes \mathcal{O}(r))^{\vee}) := \mathbb{P}_Y$ be the universal family of divisors parametrized by $\mathbb{P}(H^0(Y, \pi^* \mathcal{L}' \otimes \mathcal{O}(r))^{\vee})$. Let us denote by the projections restricted to $\mathcal{Y}$ by $p = p_\mathcal{Y} : Y \to Y$ and $q := p_\mathcal{Y}|_Y : Y \to \mathbb{P}(H^0(Y, \pi^* \mathcal{L}' \otimes \mathcal{O}(r))^{\vee})$. By the construction, $\mathcal{Y} \subset \mathbb{P}_Y$ is relatively very ample to $p_Y$. Thus $p_Y^* \mathcal{O}_Y(-\mathcal{Y}) = 0$ for all $i \geq 1$.

For a closed point $s \in A \subset \mathbb{P}(H^0(Y, \pi^* \mathcal{L}' \otimes \mathcal{O}_Y(r))^{\vee})$, we write $Y_s := q^{-1}(s)$ which is isomorphic to $X_s$ under the projection $p$. Similar as before, we put $\mathcal{Y}_{s,\ell}$ the $\ell$-th thickening of $Y_s$, and $\hat{Y}_s$ the formal completion along $Y_s$. We denote the maximal ideal of $s$ by $m_s$, thus the defining ideal of $Y_s$ is $q^*m_s = m_s$. We denote the defining ideal of $X_s$ in $Y$ by $I_s \cong \mathcal{W}^{-1}$. Then we have $p^*I_s \subset q^*m_s$. In particular, we have a map of infinitesimal thickenings $X_{s,\ell} \to Y_{s,\ell}$ induced by $p$. As a result, we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Pic}(\hat{Y}_s) & \longrightarrow & \text{Pic}(X_{s,\ell}) \\
\downarrow & & \downarrow \\
\text{Pic}(Y) & \longrightarrow & \text{Pic}(X_s) \\
\downarrow & & \downarrow \\
\text{Pic}(\hat{Y}_s) & \longrightarrow & \text{Pic}(Y_{s,\ell})
\end{array}
$$

Following [RS09], we introduce the infinitesimal Noether-Lefschetz condition.

**Definition 4.5.** [RS09] We say the pair $(Y, X_s)$ satisfies the $\ell$-th infinitesimal Noether-Lefschetz condition, denoted by INL$\ell$, if the following map is an isomorphism:
\[ \text{Image}(\text{Pic}(X_{s,\ell}) \to \text{Pic}(X_s)) \to \text{Image}(\text{Pic}(Y_{s,\ell}) \to \text{Pic}(X_s)) \]

Similarly, we say the pair $(Y, X_s)$ satisfies the formal Noether-Lefschetz condition, denoted by FNL if the following map is an isomorphism:
\[ \text{Image}(\text{Pic}(\hat{Y}_s) \to \text{Pic}(X_s)) \to \text{Image}(\text{Pic}(\hat{Y}_s) \to \text{Pic}(X_s)) \]

We restrict the universal family to get a smooth family. Without causing ambiguity, we still denote it by $q : Y_U \to U$ where $q$ is smooth with connected fibers. Consider the relative Picard functor $\text{Pic}^q_U$, $q$ is flat, projective with integral geometric fibers, so by [FGI+05, Theorem 9.4.8], $\text{Pic}^q_U$ is representable by a scheme which is separated and locally of finite type over $U$, and represents $\text{Pic}^q_U$ the étale sheaf associated with $\text{Pic}^q_U$.

By the base change property of relative Picard scheme, for any closed point $\xi \in U$, $\text{Pic}^q_{Y_U/\xi} = \text{Pic}^{X_s/\xi}$ is smooth. It means that each fiber of $\text{Pic}^q_U \to U$ is smooth.

Let $\text{Hilb}$ be the set of Hilbert polynomials of line bundles on $Y_U$. $\Phi \subset \text{Hilb}$ be a finite subset. Denote by $\text{Pic}^\Phi_U \subset \text{Pic}_U$ be the components with Hilbert polynomials in $\Phi$, then $\text{Pic}^\Phi_U$ is of finite type over $U$. Denote by $U^\Phi \subset U$ the open subset of $U$ on which $\text{Pic}^\Phi_U$ is flat (hence smooth). This subset is non-empty because it contains the generic point of $U$.

The intersection of $U^\Phi$ for $\Phi$ covering $\text{Hilb}$ is a very general subset $V$ of $U$.

**Lemma 4.6.** If for any $s \in V \subset U$ and any $m_1 > m_2 \in \mathbb{Z}_{>0}$, we have $\pi^* : \text{Pic}(\hat{Y}_{s,m_1}) \to \text{Pic}(\hat{Y}_{s,m_2})$ is surjective.
Proof. For $s \in V \subset U$, we have $\text{Pic}_s$ is smooth over $s$. By the smoothness, we have the following surjective morphisms,

$$\text{Hom}(\text{Spec} \hat{O}_{U,s}, \text{Pic}_s) \twoheadrightarrow \text{Hom}(\text{Spec}(\mathcal{O}_{U,s}/m_s^n), \text{Pic}_s) \twoheadrightarrow \text{Hom}(\text{Spec}(\mathcal{O}_{U,s}/m_s^m), \text{Pic}_s).$$

Consider the following exact sequence, coming from low-degree terms in the Leray spectral sequence [Mil80, Chapter III, Theorem 1.18] for $G_m$ relative to $q_S$ with $S = \text{Spec} \hat{O}_{U,s}$ (see also [FGI’05, Chapter 9, (9.2.11.5)])

$$0 \rightarrow \text{Pic}(S) \rightarrow \text{Pic}(Y_S) \rightarrow \text{Pic}^\ell_\ell(S) \rightarrow H^2(S_{\text{et}}, q_S, G_m).$$

We have $q_{S*}G_m = G_{m,S}$. Thus the previous sequence becomes

$$0 \rightarrow \text{Pic}(S) \rightarrow \text{Pic}(Y_S) \rightarrow \text{Pic}^\ell_\ell(S) \rightarrow H^2(S_{\text{et}}, G_m).$$

By [Mil80, Chapter III, Theorem 3.9], $S$ is strictly local, then $H^2(S_{\text{et}}, G_m) = 0$, we have $\text{Pic}^\ell_\ell(S) \cong \text{Pic}(Y_S)$ for $S = \text{Spec} \hat{O}_{U,s}$. Since $\mathcal{O}_{U,s}/m_s^\ell$ for $\ell \in \mathbb{Z}_{>0}$ is an Artin local ring with algebraically closed residue field, we have $\text{Pic}^\ell_\ell(S) \cong \text{Pic}(Y_S)$ for $S = \text{Spec} \mathcal{O}_{U,s}/m_s^\ell$ for $\ell \in \mathbb{Z}_{>0}$. And the previous surjections show that $\pi^*: \text{Pic}(Y_{s,m_1}) \rightarrow \text{Pic}(Y_{s,m_2})$ is surjective for any $s \in V$ and $m_1 > m_2 \in \mathbb{Z}_{>0}$. □

It is easy to deduce the following corollary,

**Corollary 4.7.** For any closed point $s \in V$, if the pair $(Y, X_s)$ satisfies the FNL, then we have $\pi^*: \text{Pic}(X) \rightarrow \text{Pic}(X_s)$ is an isomorphism.

We here present several cohomological results from [RS09] which we shall need later.

**Lemma 4.8.** [RS09, Lemma 1 and 2.1]

1. $R^p_s(q^*m_s) \cong I_s$;
2. $p_s(q^*m_s^\ell) = I_s^\ell$ for $\ell \geq 1$;
3. $0 \rightarrow \mathcal{O}_X \rightarrow p_*\mathcal{O}_{Y_s} \rightarrow R^1p_*\left(q^*m_s^\ell\right) \rightarrow 0$.
4. $R^j\left(q^*m_s^\ell\right) = 0$ for $j \geq 2$ and $\ell \in \mathbb{Z}_+.$

The following proposition is crucial for us to prove the FNL for $s \in V \subset U$.

**Proposition 4.9.**

$$H^2(X_s, \mathcal{I}_s^\ell/\mathcal{I}_s^{\ell+1}) \rightarrow H^2(Y_s, q^*m_s^\ell / q^*m_s^{\ell+1})$$

is injective for all $\ell > 0$.

**Proof.** Consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{I}_s^\ell & \rightarrow & \mathcal{I}_s & \rightarrow & \mathcal{I}_s^\ell/\mathcal{I}_s^{\ell+1} & \rightarrow & 0 \\
\downarrow & & & & & & & & \\
0 & \rightarrow & q^*m_s^{\ell+1} & \rightarrow & q^*m_s^\ell & \rightarrow & q^*m_s^\ell / q^*m_s^{\ell+1} & \rightarrow & 0
\end{array}
$$

Recall that $\mathcal{I}_s \cong \mathcal{W}^{-1} = \pi^*\mathcal{L}^{-r} \otimes \mathcal{O}_{Y/X}(-r)$. By Lemma 2.5, $H^i(Y, \mathcal{I}_s^\ell) = 0$ for all $\ell > 0, i = 1, 2$. Thus we have:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^2(X_s, \mathcal{I}_s^\ell / \mathcal{I}_s^{\ell+1}) & \rightarrow & H^2(Y, \mathcal{I}_s^\ell) & \rightarrow & H^3(Y, \mathcal{I}_s^\ell) & \rightarrow & 0 \\
\downarrow & & & & & & & & \\
0 & \rightarrow & H^2(Y_s, q^*m_s^{\ell+1} / q^*m_s^\ell) & \rightarrow & H^3(Y, q^*m_s^\ell) & \rightarrow & H^3(Y, q^*m_s^{\ell+1}) & \rightarrow & 0
\end{array}
$$

Then we only need to prove that $H^3(Y, \mathcal{I}_s^\ell) \rightarrow H^3(Y, q^*m_s^\ell)$ is injective for all $\ell > 0$. Since $p_s(q^*m_s^\ell) = I_s^\ell$, and considering the Leray spectral sequence for $p: Y \rightarrow Y$, $H^3(Y, \mathcal{I}_s^\ell) \rightarrow H^3(Y, q^*m_s^\ell)$ is the map $E_2^{0,3} \rightarrow H^3(Y, q^*m_s^\ell)$, thus to show the injectivity, we only need to show the following differential vanishes for all $\ell > 0$:

$$H^1(Y, R^1p_*q^*m_s^\ell) = E_2^{1,1} \rightarrow E_2^{2,0} = H^3(Y, \mathcal{I}_s^\ell).$$
By the Lemma 4.10 below (which is due to Ravindra and Srinivas [RS09]), it amounts to saying that the map:

$$H^0(Y, W^\otimes (\ell - 1)) \otimes H^0(Y, \omega_Y \otimes W) \to H^0(Y, \omega_Y \otimes W^{\otimes \ell})$$

is surjective for all $\ell > 0$. Since $H^0(Y, W^\otimes (\ell - 1)) = H^0(Z, O_Z(\ell - 1))$, this follows from the Corollary 2.9 as a result of the 0-regularity of the sheaf $g_* (\omega_Y \otimes W)$ on $Z = g(Y)$ with respect to the very ample line bundle $H$ on $Z$.

\[\square\]

The following crucial lemma is proved in [RS09, Section 2.1.2.2]. Here for reader’s convenience, we give a sketch of their proof.

**Lemma 4.10.** We denote $\pi^* L^\otimes \otimes O_{Y/X}(r)$ by $W$. The surjectivity of

$$H^0(Y, W^\otimes (\ell - 1)) \otimes H^0(Y, \omega_Y \otimes W) \to H^0(Y, \omega_Y \otimes W^{\otimes \ell})$$

implies the vanishing of $H^1(Y, R^1 p_* q^* m_\ell^s) \to H^3(Y, I_{\ell}^s)$ for $\ell > 1$.

**Proof.** (Completely follows from [RS09, Section 2.1.2.2]). Since $P_Y := Y \times P(H^0(Y, \pi^* L^\otimes \otimes O(r))^{\vee})$ is a trivial projective bundle over $Y$, we have $p_Y s^* O_{P_Y} = 0$ for any $\ell > 0$. Applying $R^1 p_Y s_*$ to the exact sequence:

$$0 \to m_\ell^s O_{P_Y}(-Y) \to m_\ell^s O_{P_Y} \to m_\ell^s O_Y(= q^* m_\ell^s) \to 0$$

since $p_* q^* m_\ell^s = I_{\ell}^s$ and $R^j p_* q^* m_\ell^s = 0$ for all $j \geq 2$, we have

$$0 \to I_{\ell}^s \to R^1 p_Y s_*(m_\ell^s O_{P_Y}(-Y)) \to R^1 p_Y s_*(m_\ell^s O_{P_Y}) \to R^1 p_* s_*(m_\ell^s O_Y) \to 0$$

If we split the above four term exact sequence into two short exact sequences:

$$0 \to I_{\ell}^s \to R^1 p_Y s_*(m_\ell^s O_{P_Y}(-Y)) \to F_\ell \to 0$$

$$0 \to F_\ell \to R^1 p_Y s_*(m_\ell^s O_{P_Y}) \to R^1 p_* s_*(m_\ell^s O_Y) \to 0$$

thus the differential:

$$H^1(Y, R^1 p_* q^* m_\ell^s) = E^{1,1}_2 \to E^{3,0}_2 = H^3(Y, I_{\ell}^s)$$

factors through $H^1(Y, R^1 p_* q^* m_\ell^s) \to H^2(Y, F_\ell) \to H^3(Y, I_{\ell}^s)$. Then it vanishes for all $\ell > 0$, if

$$(4.1) \quad H^3(Y, I_{\ell}^s) \to H^3(Y, R^1 p_Y s_*(m_\ell^s O_{P_Y}(-Y)))$$

is injective.

We denote $\text{Spec}(O_{P,Y}/m_\ell)$ by $s_\ell$. Now applying $R^1 p_{s_\ell}$ to:

$$0 \to m_\ell^s O_{P_Y}(-Y) \to O_{s_\ell \times Y}(-Y_{s_\ell}) \to 0,$$

we have $R^1 p_Y s_{s_\ell}(m_\ell^s O_{P_Y}(-Y)) \cong p_* O_{s_\ell \times Y}(-Y_{s_\ell})$. For notation ease, recall that we put $W = H^0(Y, \pi^* L^\otimes \otimes O_{Y/X}(r)) = H^0(Y, W)$. By Kunneth formula

$$p_* O_{s_\ell \times Y}(-Y_{s_\ell}) \cong O_{s_\ell \times Y} \otimes H^0(P(W), O_{P(W)}(-1)/m_\ell^s) \otimes k I_{s_\ell}.$$

Thus the map (4.1) is:

$$(4.2) \quad H^3(Y, I_{\ell}^s) \to H^3(Y, I_{\ell}^s) \otimes H^0(P(W), O_{P(W)}(-1)/m_\ell^s)$$

which is injective. By Serre duality, it amounts to saying that:

$$(4.3) \quad H^0(Y, \omega_Y \otimes W) \otimes H^0(P(W), O_{P(W)}(-1)/m_\ell^s) \otimes H^0(Y, \omega_Y \otimes W^{\otimes \ell})$$

is surjective.

We can check that:

$$H^0(P(W), O_{P(W)}(-1)/m_\ell^s) \cong \text{Sym}^{\ell - 1}(W) = H^0(P(W), O_{P(W)}(\ell - 1)) = H^0(Y, W^{\otimes (\ell - 1)})$$

and we have the surjective evaluation map:

$$\text{Sym}^{\ell - 1}(W) \otimes O_{P(W)} \to O_{P(W)}(\ell - 1)$$
and its pulling back $\text{Sym}^{\ell - 1}(W) \otimes \mathcal{O}_Y \xrightarrow{\text{surj}} \mathcal{W}^{\otimes (\ell - 1)}$ on $Y$. Then the surjection (4.3) becomes:

$$H^0(Y, \omega_Y \otimes W) \otimes H^0(Y, \mathcal{W}^{\otimes (\ell - 1)}) \twoheadrightarrow H^0(Y, \omega_Y \otimes \mathcal{W}^{\otimes \ell}).$$

The following proposition is an adapted version of that in [RS09, Proposition 1], under the assumption that $\text{char}(\mathcal{O}) = p \geq 3$, $X$ admits a $W_2(k)$-lifting and $\text{Pic}^0_X$ is smooth.

**Proposition 4.11.** When $r > 3$, then for any closed point $s \in V$ (i.e. over $s$ the relative Picard variety is smooth), we have the pair $(Y, X_s)$ satisfies INL-$m$ for all $m > 0$ and then it satisfies FNL.

**Proof.** For any $m > 0$, we have:

$$0 \xrightarrow{\beta_{s+1}} T_s^m / T_s^{m+1} \xrightarrow{T_s^m} \mathcal{O}_{X,s+1}^\times \xrightarrow{\pi_1^s} \mathcal{O}_{X,s}^\times \xrightarrow{\phi_1^s} 0. \quad (4.4)$$

Since $H^1(X_s, T_s^m / T_s^{m+1}) = 0$, we have the following exact sequence:

$$0 \xrightarrow{\text{Pic}(X_s,m+1)} \mathcal{O}_{X,s+1}^\times \xrightarrow{\text{Pic}(X_s,m)} H^2(X_s, T_s^m / T_s^{m+1}) \xrightarrow{\text{Pic}(Y_s,m+1)} H^1(Y_s, q^* m_s^m / q^* m_s^{m+1}) \xrightarrow{\text{Pic}(Y_s,m)} H^2(Y_s, q^* m_s^m / q^* m_s^{m+1}).$$

Notice that $s \in V$, we have the surjection $\text{Pic}(Y_s,m+1) \twoheadrightarrow \text{Pic}(Y_s,m)$, which implies that the map $\text{Pic}(Y_s,m) \rightarrow H^2(Y_s, q^* m_s^m / q^* m_s^{m+1})$ is a zero map. By the injectivity of $H^2(X_s, T_s^m / T_s^{m+1}) \hookrightarrow H^2(Y_s, q^* m_s^m / q^* m_s^{m+1})$ for all $m > 0$, we know that $\text{Pic}(X_s,m) \rightarrow H^2(X_s, T_s^m / T_s^{m+1})$ is a zero map. Thus $\text{Pic}(X_s,m+1) \twoheadrightarrow \text{Pic}(X_s,m)$ is surjective. As a result we have $\text{Pic}(X_s,m+1) = \text{Pic}(X_s,m)$ and the FNL holds.

**Remark 4.12.** Our proof almost follows from [RS09] except that in positive characteristic, we do not have the exponential sequence for $\ell \geq p$:

$$0 \rightarrow T_s^\ell / T_s^{\ell+1} \xrightarrow{\exp} \mathcal{O}_{X,s}^\times \xrightarrow{\pi_1^s} \mathcal{O}_{X,s}^\times \rightarrow 0.$$

To compensate this, we build up the FNL step by step and thus we have to use the stronger property that $g_s(\omega_Y \otimes \pi^* L^{\otimes r} \otimes \mathcal{O}_{Y/X}(r))$ is 0-regular. In characteristic 0, by [RS09, Theorem 2], to satisfy the FNL, we only need $g_s(\omega_Y \otimes \pi^* L^{\otimes r} \otimes \mathcal{O}_{Y/X}(r))$ is globally generated.

To conclude, we have

**Theorem 4.13.** When $r \geq 4$, $\text{Pic}^0_X$ is smooth, for $s \in V$, we have the isomorphism $\pi^*: \text{Pic}(X) \rightarrow \text{Pic}(X_s)$.

Notice that, if $X = \mathbb{P}^2$, $\mathcal{L} = \mathcal{O}(1)$, and $r = 3$, then $X_s$ is a cubic surface and thus its Picard number is different from the Picard number of $X$. However, if the canonical line bundle of $X$ is sufficient ample, we expect that the theorem holds for smaller $r$.

**Remark 4.14.** We can also use the recent deep result of Lena Ji about Noether-Lefschetz theorem on normal threefold in positive characteristics to obtain the theorem for $r = 4$ or $r \geq 6$ without even assuming $W_2$ lifting and smoothness of $\text{Pic}^0_X$. We here introduce the adapted proof of that in [RS09] to provide a self-contained proof. Besides, our results provides various normal varieties, i.e., $Z = \mathcal{G}(Y)$, where $r \geq 4$ is sufficient.
4.1. More on “Formal Noether-Lefschetz” Conditions. In the Proposition 4.11, we show that for \( s \in V \), the FNL holds, and thus we can compute the Picard group of the corresponding spectral surface \( X_s \). In the following lemma, we want to point out that the FNL may holds over \( U \) (possibly much larger than \( V \)) parametrizing smooth spectral surfaces under an extra condition \( H^1(X, \mathcal{O}_X) = 0 \).

**Lemma 4.15.** When \( r > 3 \), and \( H^1(X, \mathcal{O}_X) = 0 \), then for any closed point \( s \in U \) (i.e. over which the spectral surface is smooth), we have the pair \( (Y, X_s) \) satisfies INL for all \( m > 0 \) and then it satisfies FNL.

**Proof.** We still consider the following two exact sequences. For any \( m \), we have:

\[
0 \rightarrow \mathcal{I}_s^m / \mathcal{I}_s^{m+1} \rightarrow \mathcal{O}_{X_s,m+1}^s \rightarrow \mathcal{O}_{X_s,m}^s \rightarrow 0 .
\]

\[
0 \rightarrow q^*m_s^m / q^*m_s^{m+1} \rightarrow \mathcal{O}_{Y_s,m+1}^s \rightarrow \mathcal{O}_{Y_s,m}^s \rightarrow 0
\]

Since \( H^1(X_s, \mathcal{I}_s^m / \mathcal{I}_s^{m+1}) = 0 \), we have the following exact sequence:

\[
(4.5) \quad 0 \rightarrow \text{Pic}(X_{s,m+1}) \rightarrow \text{Pic}(X_{s,m}) \rightarrow H^2(X_s, \mathcal{I}_s^m / \mathcal{I}_s^{m+1}) .
\]

By the assumption \( H^1(X, \mathcal{O}_X) = 0 \), we have \( H^1(X_s, \mathcal{O}_{X_s}) = 0 \), and then

\[
H^1(Y_s, q^*m_s^m / q^*m_s^{m+1}) = 0.
\]

Thus the above diagram takes the following form:

\[
(4.6) \quad 0 \rightarrow \text{Pic}(X_{s,m+1}) \rightarrow \text{Pic}(X_{s,m}) \rightarrow \text{Pic}(Y_{s,m+1}) \rightarrow \text{Pic}(Y_{s,m}) \rightarrow H^2(Y_s, q^*m_s^m / q^*m_s^{m+1}) .
\]

Notice that when \( m = 0 \), \( X_{s,m} \cong Y_{s,m} \). We prove by induction that

\[
\text{Pic}(X_{s,m}) \cong \text{Pic}(Y_{s,m}).
\]

Consider the commutative diagram

\[
(4.7) \quad 0 \rightarrow \text{Pic}(X_{s,m+1}) \rightarrow \text{Pic}(X_{s,m}) \rightarrow \text{Pic}(Y_{s,m+1}) \rightarrow \text{Pic}(Y_{s,m}) \rightarrow H^2(Y_s, q^*m_s^m / q^*m_s^{m+1}) .
\]

where \( \xi_m \) is the composition of the inclusion of \( \text{Pic}(X_{s,m}) \subset H^2(X_s, \mathcal{I}_s^m / \mathcal{I}_s^{m+1}) \) with the map \( H^2(X_s, \mathcal{I}_s^m / \mathcal{I}_s^{m+1}) \rightarrow H^2(Y_s, q^*m_s^m / q^*m_s^{m+1}) \), which is always injective by Proposition 4.9. Since \( \alpha_m \) is an isomorphism, and \( \xi_m \) is injective, then by the snake lemma, \( \alpha_{m+1} \) is an isomorphism. As a result, we have

\[
\text{Image}(\text{Pic}(X_{s,m}) \rightarrow \text{Pic}(X_s)) \rightarrow \text{Image}(\text{Pic}(Y_{s,m}) \rightarrow \text{Pic}(X_s))
\]

is an isomorphism, i.e. we have INL for all \( m \).

\[ \square \]
4.2. "Bigness" is Necessary. Now we assume $f : X \to C$ is a non-isotrivial elliptic surface with a section. We put

$$\Sigma := \{ x \in C | f^{-1}(x) \text{ is singular} \}.$$ 

We assume for the moment that each singular fiber is irreducible (which implies that each fiber is nodal cubic or cuspidal cubic). Then by Kodaira’s canonical bundle formula, $\omega_X \cong f^*(\omega_C \otimes \mathcal{L})$ where $\mathcal{L} := R^1f_*(\mathcal{O}_X)$ with $\deg \mathcal{L} = \chi(\mathcal{O}_X) > 0$. And we have the following lemma:

**Lemma 4.16.** For each $i$, $f^* : H^0(C, \omega_C^{\otimes i}) \to H^0(X, \omega_X^{\otimes i})$ is an isomorphism.

Then for a closed point $s \in \mathcal{A}$, we can associate a spectral surface $X_s$ over $X$ and a spectral curve $C_s$ over $C$.

**Lemma 4.17.** The spectral curve $C_s$ and spectral surface $X_s$ are related in the following Cartesian diagram:

$$
\begin{array}{ccc}
X_s & \xrightarrow{\pi_s} & X \\
\downarrow & & \downarrow f \\
C_s & \longrightarrow & C
\end{array}
$$

If $C_s$ is smooth and ramified outside of $\Sigma$, then $X_s$ is also smooth. In particular, generic spectral surfaces are smooth.

Notice that $\omega_X$ is nef but not big, and $\text{Pic}_X$ is not isomorphic to $\text{Pic}_{\mathcal{X}}$. Actually, we can calculate that $H^0(X_s, \mathcal{O}_{X_s}) = \bigoplus_{i=0}^{r-1} H^0(X, \mathcal{L}^{-i})$. Without the bigness, we can not prove the vanishing of $H^0(X, \mathcal{L}^{-i})$ for $i > 0$, thus $\dim \text{Pic}_{X_s} > \dim \text{Pic}_X$.

5. An Application to Hitchin System over Quintic Surfaces

In this section, we discuss generic fibers of Hitchin fibrations over a generic quintic surface of Picard number 1. We divide this section into two parts dealing with bundle case and torsion free sheaf case. The main result is that if we consider the moduli of torsion free Higgs sheaves, then generic fibers of corresponding Hitchin maps are Hilbert schemes of points of spectral surfaces. And we calculate the number of connected components.

Now, let $X \subset \mathbb{P}^3$ be a smooth quintic surface with Picard number 1 and the base field $k$ will be an algebraically closed field with odd or zero characteristic.

**Basic properties of $X$ and $X_s$.** Let us investigate some basic properties of $X$.

**Lemma 5.1.** $X$ admits a lift on $W_2(k)$.

**Proof.** By the exact sequence of structure sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_X(1) \to 0,$$

we have $h^0(X, \mathcal{O}_X(1)) = 4$ and $h^1(X, \mathcal{O}_X(1)) = 0$. Let us consider the Euler sequence

$$0 \to i^*\Omega^1_{\mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3}^{|4} \xrightarrow{\ev} \mathcal{O}_X(1) \to 0,$$

which shows $h^0(X, i^*\Omega^1_{\mathbb{P}^3}(1)) = 0$. Since $\omega_X = \mathcal{O}_X(1)$, we have

$$0 \to \mathcal{O}_X(-4) \to i^*\Omega^1_{\mathbb{P}^3}(1) \to \Omega^1_{\mathbb{P}^3} \otimes \omega_X \to 0.$$ 

Because $h^1(X, \mathcal{O}_X(-4)) = 0$, $H^2(X, \mathcal{T}_X) \cong H^0(X, \Omega^1_X \otimes \omega_X)^\vee = 0$. Thus $X$ admits a lifting to $W_2(k)$. $\square$

By definition of Todd class, $\text{Td}(X) = 1 + \frac{1}{2}c_1(TX) + \frac{1}{12}(c_1^2(TX) + c_2(TX))$, the following computation shows
Lemma 5.2. The Todd class of $X$ is $\text{Td}(X) = 1 - \frac{1}{12}c_1(\omega_X) + c_1(\omega_X)^2$.

Proof. Let $\mathcal{I}_X$ be the ideal sheaf defining $X$ in $\mathbb{P}^3$ and $i : X \rightarrow \mathbb{P}^3$ is the closed immersion, then

$$0 \rightarrow \mathcal{I}_X / \mathcal{I}_X^2 \rightarrow i^* \Omega^1_{\mathbb{P}^3} \rightarrow \Omega^1_X \rightarrow 0.$$ 

Notice that $\mathcal{I}_X / \mathcal{I}_X^2$ is $i^* \mathcal{O}(-5)$, thus

$$\text{ch}(\Omega^1_X) = i^* \text{ch}(\Omega^1_{\mathbb{P}^3}) - i^* \text{ch}(\mathcal{O}(-5))$$

$$= 4i^* \text{ch}(\mathcal{O}(-1)) - i^* \text{ch}(\mathcal{O}) - i^* \text{ch}(\mathcal{O}(-5))$$

$$= 2 + h + \frac{-21}{2} h^2$$

$$= 2 + c_1(\omega_X) + \frac{1}{2}(c_1(\omega_X)^2 - 2c_2(\Omega^1_X)),$$

where $h = c_1(i^* \mathcal{O}(1)) = c_1(\omega_X)$. We have $c_1(\Omega^1_X) = h$ and $c_2(\Omega^1_X) = 11h^2$. Thus $\text{Td}(X) = 1 + \frac{1}{12} c_1(TX) + \frac{1}{12}(c_1^2(TX) + c_2(TX)) = 1 - \frac{1}{12} h + h^2$. 

We take $\mathcal{L}$ as $\omega_X = \mathcal{O}_X(1)$ and $A$ as Hitchin base in previous sections. Moreover, we have the open subset $U \subset A$ parametrizing the smooth spectral surfaces. Let $s \in U$ and $X_s$ be the corresponding spectral curve.

Lemma 5.3. The canonical line bundle $\omega_{X_s}$ of $X_s$ is isomorphic to $\pi_s^*(\omega_X^{0\cdot r})$.

Proof. By Lemma 2.4, $\omega_Y \cong \mathcal{O}_{Y/X}(-2)$, and thus $\omega_Y|_{X_s}$ is trivial. By the following exact sequence:

$$0 \rightarrow (\pi^* \omega_X^{0\cdot r} \otimes \mathcal{O}_{Y/X}(-r))|_{X_s} \rightarrow \Omega^1_Y|_{X_s} \rightarrow \Omega^1_{X_s} \rightarrow 0,$$

we have $\omega_{X_s} \cong \pi_s^*(\omega_X^{0\cdot r})$. 

We then compute the Todd class of $X_s$. By the exact sequences

$$0 \rightarrow \Omega^1_Y / X \rightarrow \pi^*(\omega_X^0 \oplus \mathcal{O}_X)(-1) \rightarrow \mathcal{O}_Y \rightarrow 0,$$

$$0 \rightarrow \pi^* \Omega^1_X \rightarrow \Omega^1_Y \rightarrow \Omega^1_{Y/X} \rightarrow 0,$$

$$0 \rightarrow \pi^* \omega_X^{0\cdot r}(-r)|_{X_s} \rightarrow \Omega^1_Y|_{X_s} \rightarrow \Omega^1_{X_s} \rightarrow 0,$$

we have

$$\text{ch}(\Omega^1_{X_s}) = i_s^* \text{ch}(\Omega^1_Y) - \text{ch}(\pi_s^*(\omega_X^{0\cdot r}))$$

$$= \pi_s^*(\Omega^1_X) + i_s^* \text{ch}(\Omega^1_{Y/X}) - \text{ch}(\omega_{X_s})$$

$$= \pi_s^* (2 + r \cdot c_1(\omega_X) - \frac{20 + r^2}{2} \cdot c_1(\omega_X)^2)$$

$$= \pi_s^*[2 + r \cdot c_1(\omega_X) + \frac{1}{2} \cdot (r^2 c_1(\omega_X)^2 - 2(10 + r^2) \cdot c_1(\omega_X)^2)].$$

We have $c_1(\Omega^1_Y) = r \cdot \pi_s^* h$ and $c_2(\Omega^1_Y) = (10 + r^2) \cdot \pi_s^* h^2$. Thus $\text{Td}(X_s) = 1 + \frac{1}{2} c_1(TX_s) + \frac{1}{12}(c_1^2(TX_s) + c_2(TX_s)) = 1 - \frac{2}{7} \cdot \pi_s^* h + \frac{(r^2 + 5)}{6} \cdot \pi_s^* h^2.$
Proposition 5.6. Let \( \pi : E \to X \) be a coherent sheaf over \( X \). We assume \( \pi \) is smooth. Let \( \pi_s : X_s \to X \) be the projection, which is finite flat of degree \( r = \text{rank } \pi \). Let us fix \((r, c_1, c_2)\) as before. We assume \( \pi_s \cdot \mathcal{M} = \mathcal{E} \), where \( \mathcal{M} \) is an invertible sheaf over \( X_s \). By Grothendieck-Riemann-Roch, we have:

\[
\pi_s \left( \text{ch}(\mathcal{M}) \cdot \text{Td}(X_s) \right) = \text{ch}(\mathcal{E}) \cdot \text{Td}(X),
\]

(5.1)

\[
\pi_s \left( \text{Td}(X_s) \right) = \text{ch}(\mathcal{E}) \cdot \text{Td}(X).
\]

(5.2)

As \( \mathcal{M} \) is locally free of rank 1, by definition of the Chern character, we have \( \text{ch}(\mathcal{M}) = 1 + c_1(\mathcal{M}) + \frac{c_1(\mathcal{M})^2}{2} \). From the above two equations, we have:

\[
\pi_s \left( (c_1(\mathcal{M}) + \frac{c_1(\mathcal{M})^2}{2}) \cdot \text{Td}(X_s) \right) = \left( \text{ch}(\mathcal{E}) - \text{ch}(\pi_s \cdot \mathcal{O}_{X_s}) \right) \cdot \text{Td}(X).
\]

(5.3)

By our previous computation of Todd classes of \( X \) and \( X_s \), the left hand side of (5.3) becomes

\[
\pi_s \left( (c_1(\mathcal{M}) + \frac{1}{2} c_1(\mathcal{M})^2) \cdot \text{Td}(X_s) \right) = \pi_s c_1(\mathcal{M}) + \pi_s \frac{1}{2} c_1(\mathcal{M})^2 - \frac{r}{2} \pi_s c_1(\mathcal{M}) \cdot h,
\]

and the right hand side of (5.3) becomes

\[
(\text{ch}(\mathcal{E}) - \text{ch}(\pi_s \cdot \mathcal{O}_{X_s})) \cdot \text{Td}(X) = (\text{ch}(\mathcal{E}) - \text{ch}(\pi_s \cdot \mathcal{O}_{X_s})) \cdot (1 - \frac{1}{2} \cdot h + h^2)
\]

\[
= c_1(\mathcal{E}) + \frac{r(r-1)}{2} h + \text{ch}_2(\mathcal{E}) - \frac{r(r-1)(2r-1)}{12} h^2 - \frac{1}{2} c_1(\mathcal{E}) \cdot h
\]

\[
- \frac{r(r-1)}{4} h^2 = c_1(\mathcal{E}) + \frac{r(r-1)}{2} h + \text{ch}_2(\mathcal{E}) - \frac{1}{2} c_1(\mathcal{E}) \cdot h
\]

\[
- \frac{r(r-1)(r+1)}{6} h^2.
\]
To conclude:

**Lemma 5.7.** Denote \( c_1(\omega_X) \) by \( h \), the Chern classes of \( \mathcal{M} \) has to satisfy the following equations:

\[
\begin{align*}
\pi_s(c_1(\mathcal{M})) &= c_1(\mathcal{E}) + \frac{r(r-1)}{2} h, \\
\pi_s\frac{1}{2}(c_1(\mathcal{M})^2 - \frac{r}{2}\pi_s(c_1(\mathcal{M})))h &= ch_2(\mathcal{E}) - \frac{1}{2}c_1(\mathcal{E}) \cdot h - \frac{r(r-1)(r+1)}{6} h^2.
\end{align*}
\]

Now we can compute generic fibers of \( h^\mathfrak{o} \).

**Theorem 5.8.** If the generic fiber of the Hitchin map \( h^\mathfrak{o} \) is not empty, then it is a single point, i.e., the mapping degree of \( h^\mathfrak{o} \) is 1.

**Proof.** As we have proved that \( \pi_+^\mathfrak{s} : \text{Pic}(X) \to \text{Pic}(X_s) \) is an isomorphism for very general \( s \in V \). Now we fix \( s \in V \), then \( \text{Pic}(X_s) = \mathbb{Z} \cdot [\pi_+^\mathfrak{s} \omega_X] \cong \mathbb{Z} \), and then we can assume \( \mathcal{M} \cong \pi_+^\mathfrak{s}(\omega_X^{\otimes \mu}) \) for some integer \( \mu \). Thus we have:

\[
\begin{align*}
r\mu h &= c_1(\mathcal{E}) + \frac{r(r-1)}{2} h
\end{align*}
\]

and

\[
\begin{align*}
\frac{r\mu^2 - r^2 \mu}{2} h^2 &= ch_2(\mathcal{E}) - \frac{1}{2}c_1(\mathcal{E}) + \frac{r(r-1)}{2} h^2 h - \frac{r(r-1)(2r-1)}{12} h^2,
\end{align*}
\]

combined together with equations in Lemma 5.7, we have:

\[
\begin{align*}
c_1(\mathcal{E}) &= (r\mu - \frac{1}{2} r(r-1)) h, \\
ch_2(\mathcal{E}) &= \frac{1}{2} (r\mu + r\mu^2 - r^2 \mu) + \frac{r(r-1)(2r-1)}{12} h^2
\end{align*}
\]

Thus \( \mu \) is uniquely determined by \( c_1(\mathcal{E}) \) and \( c_2(\mathcal{E}) \). The theorem then follows. \( \Box \)

5.2. **Torsion Free Sheaves Case.** Now let us suppose \( (\mathcal{E}, \theta) \) is a Higgs sheaf over \( X \) with image \( s \) in \( A \). Similarly, we have \( (\mathcal{E}, \theta) \) can be identified with torsion free sheaf of rank 1 on \( X_s \). We may denote it by \( \mathcal{M} \otimes \mathcal{I}_\Delta \). Here \( \mathcal{I}_\Delta \) is the ideal sheaf of a closed subscheme of finite length \( \mathfrak{e}_\Delta \) on \( X_s \).

**Lemma 5.9.** \( \text{ch}(\mathcal{M} \otimes \mathcal{I}_\Delta) = \text{ch}(\mathcal{M}) + i_*[\Delta] \).

**Proof.** We denote by \( i : \Delta \to X \) the closed immersion. Again by Grothendieck-Riemann-Roch, we have \( c_1(i_*\mathcal{O}_\Delta) = 0, c_2(i_*\mathcal{O}_\Delta) = -i_*[\Delta] \). \( \Box \)

**Theorem 5.10.** Generic fibers of \( h \) are Hilbert schemes of points of spectral surfaces (if nonempty) and generic fibers are connected.

**Proof.** Now still by Grothendieck-Riemann-Roch, the equations in Lemma 5.7 become:

\[
\begin{align*}
\pi_s(c_1(\mathcal{M})) &= c_1(\mathcal{E}) + \frac{r(r-1)}{2} h, \\
- \frac{r}{2}\pi_s(c_1(\mathcal{M}))h + \pi_s\frac{1}{2}(c_1(\mathcal{M})^2 + [\Delta]) &= - \frac{1}{2}c_1(\mathcal{E})h - \frac{r(r-1)(r+1)}{6} h^2 + ch_2(\mathcal{E}).
\end{align*}
\]

Similarly, we take \( s \) very general and assume \( \mathcal{M} \cong \pi_+^\mathfrak{s}(\omega_X^{\otimes \mu}) \) for some integer \( \mu \), and thus,

\[
\begin{align*}
c_1(\mathcal{E}) &= (r\mu - \frac{1}{2} r(r-1)) h, \\
ch_2(\mathcal{E}) &= \frac{1}{2} (r\mu + r\mu^2 - r^2 \mu) + \frac{r(r-1)(2r-1)}{12} h^2 + \pi_s([\Delta]).
\end{align*}
\]

Thus \( \mu \) and \( [\Delta] \) is uniquely determined by \( c_1(\mathcal{E}) \) and \( c_2(\mathcal{E}) \). In other words, the generic fibers are connected if nonempty. \( \Box \)
