Turbulence scaling laws across the superfluid to supersolid transition

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We investigate quantum turbulence in a two-dimensional trapped supersolid and demonstrate that both the wave and vortex turbulence involve triple rather than dual cascades, as in a superfluid. Because of the presence of a second gapless mode associated with translation symmetry breaking, a new $k^{-13/3}$ scaling law is predicted to occur in the wave turbulence. Simultaneous fast vortex-antivortex creation and annihilation in the interior of the oscillating supersolid results in a $k^{-1}$ scaling law in the vortex turbulence. Numerical simulations based on the Gross-Pitaevskii equation confirmed the predictions.

Turbulence in a superfluid (SF), named quantum turbulence (QT), has recently attracted considerable interest in both liquid helium [1–5] and atomic Bose-Einstein condensates (BEC) [6–16]. Turbulence mostly associated with vortices can be characterized by an energy spectrum following the Kolmogorov $k^{-5/3}$ power law (also known as the K41 law) [17]. This law describes the process wherein energy is conservatively transported through scales, and this process is named an energy-cascade. Richardson [18] provided a useful presentation of the K41 law that indicated that far away from forcing and sink, small eddies gain energy from the break-up of large eddies, after which they themselves are broken into even smaller eddies, and so on. Energy is thus transported conservatively to smaller scales. In a two-dimensional (2D) fluid, in addition to energy, there exists another positive conserved quantity: enstrophy. Thus it typically involves two scaling laws (dual cascades), including the inverse energy-cascade $k^{-5/3}$ spectrum and the forward enstrophy-cascade $k^{-3}$ (or $k^{-4}$) spectrum, for vortex turbulence (VT) in a 2D fluid.

In addition to the hydrodynamic turbulence associated with vortices, turbulence consisting of low-energy waves, named wave turbulence (WT), plays a central role in a system of interactions. WT can be regarded as the out-of-equilibrium statistical mechanics of random nonlinear waves. As a general physical phenomenon, WT is observed in a vast range of nonlinear systems, on quantum to astrophysical scales [21]. Similar to the occurrence of dual cascades in a 2D VT, WT can also exhibit dual cascades in a similar system [22]. Using the four-wave-interaction scheme, it was discovered that WT in an atomic BEC has an inverse wave-action-cascade $k^{-1/3}$ and forward energy-cascade $k^{-1}$ spectra [22]. The latter, named Kolmogorov-Zakharov (KZ) spectrum, is the analog of the Kolmogorov spectrum of hydrodynamic vortex turbulence. The dual-cascade behavior of BEC WT has a precise interpretation: the forward energy and the inverse waveaction cascades correspond to the strongly nonequilibrated process of evaporative cooling and the condensation process, respectively. Recently turbulent cascade has been observed in an oscillating uniform BEC wherein the density momentum distribution $\tilde{n}(k)$ displays a $k^{-7/2}$ scaling behavior (equivalently kinetic energy displays a $k^{-3/2}$ scaling behavior) in the infrared regime associated with the acoustic wave turbulence of a compressible superfluid [23].

Supersolid (SS) is a state of matter that simultaneously possesses superfluidity and solidity and in which both gauge and translational symmetries are broken. The SS state has been verified in Rydberg-dressed BECs [24–28] and such systems are excellent for investigating the quantum hydrodynamics coupled to a coherent structure. Quantized vortex lattices in an atomic SS have been studied in a couple of previous investigations [29, 30]. Exploring how QT, including both VT and WT, behaves in such a system across the SF-to-SS transition is an interesting research topic. For VT, this Letter will demonstrate that an oscillating SS constantly generates vortices and antivortices in the interior and that such a fast vortex-antivortex creation and annihilation cycle can lead to a $k^{-1}$ scaling law [31].

For WT, the dispersion of elementary excitations should play a crucial role in determining scaling laws. For a uniform SF condensate in the absence of an external potential, the elementary excitation behaves according to

$$h\omega(k) = \sqrt{\frac{\hbar^2 k^2}{2m} \frac{\hbar^2 k^2}{2m} + 2n_0 \tilde{U}(k)},$$

where $n_0$ is the uniform density and $\tilde{U}(k) = \tilde{U}(|k|)$ is the Fourier transformation of the isotropic interaction function $U(r)$. If the interaction strength or mean density exceeds the critical value, a modulation instability, named a roton instability, occurs [32]. The corresponding true ground state then undergoes a phase transition from a uniform SF state to a periodic SS state. In Fig. 1(a), the green curve plots the dispersion [31] in an unstable SF state in which imaginary frequencies occur at a range of wavevectors. By contrast, the black curve plots the excitation dispersion of the corresponding true SS ground state in which a hexagonal lattice forms, as illustrated in Fig. 1(b). As displayed.
A numerical simulation was performed to investigate in detail the turbulence dynamics in a 2D SS. We employed the Gross-Pitaevskii equation (GPE), and to study both the WT and VT, the kinetic energy was separated into two mutually orthogonal parts: compressible and incompressible. Of most interest in this study was whether a new $k^{-13/3}$ scaling law exists in the WT. The total energy of the current interacting system can be written as the summation of kinetic, potential, and interaction energies $E(t) = E_{\text{kin}}(t) + E_{\text{pot}}(t) + E_{\text{int}}(t)$, where

$$E_{\text{kin}}(t) = \int \mathcal{E}_{\text{kin}}(r, t) \, dr = \int \frac{\hbar^2 |\nabla \psi(r, t)|^2}{2m} \, dr,$$

$$E_{\text{pot}}(t) = \int V_{\text{pot}}(r) |\psi(r, t)|^2 \, dr,$$

$$E_{\text{int}}(t) = \frac{1}{2} \int U(\mathbf{r}) |\psi(r', t)|^2 |\psi(r, t)|^2 \, dr' \, dr. \quad (5)$$

Here, $V_{\text{pot}}(r) = m\omega_{\text{int}}^2 r^2/2$ is the harmonic trapping potential with a frequency $\omega_{\text{int}}$, $U(\mathbf{r}) = N\tilde{C}/\langle \tilde{R}_e^6 + \tilde{r}^6 \rangle$ is the soft-core interaction kernel with particle number $N$, $\tilde{C}$ is the interaction strength, $R_e$ is the condensate wave function, and $\tilde{r} \equiv r - r'$ is the relative position of two particles. The order parameter $\psi$, satisfying the normalization condition $\int |\psi|^2 \, dr = 1$, is the condensate wave function. Throughout this Letter, $R_e$ and $t_0 \equiv m \tilde{R}_e^2/\hbar$ are used as the units of length and time, respectively. In our simulation, a SS condensate with Thomas-Fermi (TF) radius $R_{\text{TF}} = 6 R_e$ was initially prepared in which circulation-3 vortex and antivortex were imprinted. Time evolution of the wave function followed the GPE: $i\hbar \partial \psi / \partial t = \delta E / \delta \psi$.

When applying the spectral scaling approach to study the turbulence in a trapped BEC, it is useful to express the condensate wave function $\psi(r, t)$ in terms of Madelung transformation, $\psi(r, t) = \sqrt{n(r, t)} \exp[i\varphi(r, t)]$ with $n$ and $\varphi$ the density and phase, respectively. The vector field $\sqrt{n}u$ with the velocity $\mathbf{u} \equiv (h/m) \nabla \varphi$ can then be decomposed into solenoidal and irrotational parts, or correspondingly, incompressible and compressible parts [38–41]: $\sqrt{n}u = (\sqrt{n}u)^{\text{in}} + (\sqrt{n}u)^{\text{c}}$, where $\nabla \cdot (\sqrt{n}u)^{\text{in}} = 0$ and $\nabla \times (\sqrt{n}u)^{\text{c}} = 0$. Because $(\sqrt{n}u)^{\text{in}}$ and $(\sqrt{n}u)^{\text{c}}$ are mutually orthogonal, the kinetic energy density can also be decomposed accordingly: $\mathcal{E}_{\text{kin}} = \mathcal{E}_{\text{kin}}^{\text{in}} + \mathcal{E}_{\text{kin}}^{\text{c}}$, where $\mathcal{E}_{\text{kin}}^{\text{in}} = (m/2) |(\sqrt{n}u)^{\text{in}}|^2$ correspond to the incompressible and compressible kinetic energy density, respectively. Physically $\mathcal{E}_{\text{kin}}^{\text{in}}$ and $\mathcal{E}_{\text{kin}}^{\text{c}}$ correspond to the kinetic energies of the swirls and the sound waves in a superflow, respectively. To study the scaling laws, it is necessary to transform in $k$ space: $E_{\text{kin}}^{\text{in}}(k, t) = E_{\text{kin}}^{\text{in}}(k, t) + E_{\text{kin}}^{\text{c}}(k, t)$ with $E_{\text{kin}}^{\text{in}}(k, t) = \int_0^\infty \tilde{E}_{1D}^{\text{in}}(k, t, d)$. In a 2D space, $\tilde{E}_{1D}^{\text{in}}(k, t)$ is defined as the angle-averaged kinetic-energy spectrum

$$\tilde{E}_{1D}^{\text{in}}(k, t) = \frac{k}{2} \int_0^{2\pi} d\phi_k \tilde{E}_{\text{kin}}^{\text{in}}(k, t). \quad (6)$$

in Fig. 1(a) in an extended zone scheme, there are two gapless modes that appear near $k \to 0$ and $k \to (2\pi/d)^{-}$. (In an isotropic SF, $k = 2\pi/d$ corresponds to the wavevector of the roton before the instability occurred.) The asymptotic behaviors of the two gapless modes are

$$\omega_1(k) = v_1 k, \quad k \to 0$$

$$\omega_2(k) = \frac{\omega_m^2}{v_2} k^{-1} - \omega_m, \quad k \to (2\pi/d)^{-}, \quad (2)$$

where $v_1 = \omega_1'(0)$, $v_2 = |\omega_2'(2\pi/d)|$, and the modulation frequency $\omega_m = 2\pi v_2/d$. The two asymptotic dispersions are indicated by the red and blue curves in Fig. 1(a).

It is intuitively expected that the second gapless mode, which is the signature of a SS state, may result in some new scaling law in WT. Based on the theory of turbulence kinetics, the constant term $\omega_m$ in $\omega_2$ [see (2)] plays no role in determining the scaling law because it will cancel out on both sides of the four-wave ($2 \to 2$) resonance condition. Thus $\omega_2 \sim k^{-1}$ for $k \to (2\pi/d)^-$ so long as scaling law is concerned. Consider wave dispersion that has an approximative form $\omega = \lambda \kappa^\alpha$, where $\lambda$ is a positive parameter and $\kappa$ is the power. By dimension counting, the KZ spectra in an $N$-wave process will possess $k^\alpha$ scaling laws for the waveaction $[21]$. Explicitly,

$$y = D - 6 + \frac{5 - D + 2\alpha (N - 2)}{N - 1}, \quad (3)$$

where $D$ is the dimension $[21]$. With $D = 2$, $N = 4$, and $\alpha = -1$, we predict that a new scaling law with the power

$$y = -\frac{13}{3} \quad (4)$$

will appear in the kinetic energy spectrum of a 2D SS that is associated with the breaking of translational symmetry $[33–37]$. 

![FIG. 1. (a) Excitation spectrum of an unstable uniform SF state (green curve) and true periodic SS ground state [illustrated in (b), black curve] as a function of $k d/\pi$ (where $d$ is the lattice constant). Red and blue curves display the asymptotic behaviors of the black curve near $k = 0$ and $k = (2\pi/d)^-$. (b) 2D hexagonal SS lattice with lattice constant $d$, the gap between adjacent SS droplets $d'$, and droplet radius $R$; the range $a$ in which the fluctuating vortices are mainly located is indicated.](image)
The velocity field of the superflow may change constantly, but the energy spectrum \( E_{i,c}(k) \) takes on a stationary form as time increases.

Figure 2 displays both the compressible (top) and incompressible (bottom) parts of the time-averaged kinetic energy spectrum obtained near \( t = 150 \). Three scaling laws are identified in both cases, with the corresponding powers noted in the legend. The critical wave vectors are identified as \( k_s = 2\pi/s \approx 1.8 \), \( k_d = 2\pi/d \approx 4.5 \), \( k_{d'} = 2\pi/d' \approx 8 \), \( k_a = 2\pi/a \approx 14 \), and \( k_R = 2\pi/R \approx 21.5 \), corresponding to the radius of the eventual condensate \( s \), the 2D hexagonal SS lattice constant \( d \), the gap between adjacent SS droplets \( d' \), the range \( a \) wherein fluctuating vortices mainly reside in a SS, and the radius of the SS droplet \( R \).

Three scaling laws are identified in both cases, with the corresponding powers noted in the legend. The critical wave vector \( k_s \) is identified [Fig. 2(a)] an inverse waveaction-cascade spectrum \( k^{-1/3} \) over which the fluctuating vortices are mainly located [see Fig. 1(a)], the corresponding energy spectrum is an-verse waveaction-cascade spectrum \( k^{-1/3} \), which corresponds to the radius of the eventual condensate \( s \), and the radius of the SS droplet \( R \).

For the first gapless (phonon) mode \( \omega_1 = v_1 k \) at \( k \to 0 \) [see Fig. 1(a)], the corresponding energy spectrum is anticipated to have a forward energy-cascade spectrum \( k^{-3/2} \) in the large-scale (small-\( k \)) limit (21). This \( k^{-3/2} \) spectrum was confirmed both by numerically considering an infrared forcing (10) and by a recent experiment of a uniform system (23). When a SS forms, the \( k \) range required to sustain the phonon mode dispersion is relatively narrow (22), and consequently, this \( k^{-3/2} \) spectrum is not observed in the present system in the infrared regime [see Fig. 2(a)].

In the incompressible VT energy spectra displayed in Fig. 2(b), both Kolmogorov \( k^{-5/3} \) (17) and Saffman \( k^{-4} \) (20) scaling laws can be identified. At the connection between the two cascades at \( k_0 = 2\pi/a \approx 14 \) with \( a \approx 0.45 \), energy is input from the compressible part. When a SS lattice is present, length \( a \) represents the range over which the fluctuating vortices are mainly located [see Fig. 1(b)]. The oscillation of a SS constantly creates vortices and antivortices in the interior, and such a fast vortex-antivortex creation and annihilation cycle can result in a \( k^{-1} \) scaling law in the infrared regime. At the intersection of the spectra \( k^{-1} \) and \( k^{-5/3} \), energy is output to the compressible part.

Because the hexagonal lattice is one of the most critical characteristics of the present system, the motion of the hexagonal structure produces not only vortices but also waves. Thus lattice constant \( d \) should be a crucial length scale in both the compressible and incompressible energies. Taking into account the size of SS droplet, the gap between adjacent SS droplets \( d' = d - 2R \) should be another crucial length scale in both energies. As was confirmed in Fig. 2(a) and 2(b), \( k_{d'} \) is a critical wave vector in both the compressible and incompressible spectra. More
concentrated in a region of length \( a \). Fig. 3(d) clearly shows that the streams are mostly energetically "larger" if it contains more circumfluent particles. With such reasoning, we state that an eddy is energetically "larger" if it contains more circumfluent particles. Inset in (b): size of SS droplet 

(a) Normalized wave function \( \psi \) obtained at \( t = 150 \). Dotted white and dashed red circles respectively indicate the initial and eventual sizes of the condensate at \( s \approx 3.5 \) and \( R_{\text{TF}} = 6 \). (b) Compressible and (c) incompressible kinetic energy densities. Inset in (b): size of SS droplet \( R \) is a crucial length scale for the new \( k^{-13/3} \) scaling law in WT. (d) Stream function \( \phi(r) \) of the corresponding vector field with the length \( a \) indicated (see text).

\[ |\psi(r)| \]

\[ E_{\text{kin}}(r) \]

\[ \phi(r) \]

\[ \psi \]

\[ \Omega \]

explicitly, \( k_d \) represents a scale corresponding to the energy source in the compressible part and the energy sink in the incompressible part. Moreover, energy peak with wave vector \( k_d \) occurs in both spectra.

Figure 3(a) plots the normalized wave function at \( t = 150 \) and displays both the initial and eventual sizes (\( R_{\text{TF}} \) and \( s \)) of the condensate. Fig. 3(b) and 3(c) present the corresponding compressible and incompressible kinetic energy densities. The magnification in the inset of Fig. 3(b) emphasizes that the size of a SS droplet (\( R \)) is an important (short) length scale for the additional new inverse waveaction-cascade \( k^{-13/3} \) spectrum in the ultraviolet region. Plotted in Fig. 3(d) is the stream function \( \phi(r) \) associated with the velocity field \( \sqrt{\ii} \mathbf{u} \), which is the solution of Poisson’s equation \( \nabla^2 \phi = \mathbf{z} \cdot \Omega \), where \( \Omega = \nabla \times (\sqrt{\ii} \mathbf{u}) \) is the 2D vorticity vector. The colors indicate the corresponding values of \( \phi(r) \), and the locations and structures of eddies can be easily recognized by families of closed streamlines. From the definition of the vorticity vector \( \Omega \), we note that the singular and regular parts of \( \Omega \) are modulated by \( \sqrt{\ii} \) and its gradient, respectively. As a result, the magnitude of the vorticity \( \Omega \) of a vortex in high-density area is greater than that of a vortex in low-density area and, with such reasoning, we state that an eddy is energetically "larger" if it contains more circumfluent particles. Fig. 3(d) clearly shows that the streams are mostly concentrated in a region of length \( 2a \), which indicates that the fluctuating vortices mainly reside in this region. The figure also reveals that the streamline bundles form a fluctuating hexagonal lattice, which is consistent with the incompressible VT energy spectra displayed in Fig. 3(c). By comparing the energy spectra of compressible and incompressible parts [Fig. 3(b) and 3(c)], the compressible and incompressible energies are clearly observed to overlap not only in the border area but also in the interior.

In summary, QT including both VT and WT is investigated in a 2D trapped SS. The SS state of an atomic BEC results from a roton instability, a type of modulation instability, in a uniform SF. Because of the breaking of two symmetries, the SS possesses two gapless modes. The two gapless modes behave differently near their zero-frequency wave vectors. The second gapless mode associated with the translational symmetry breaking results in a new \( k^{-13/3} \) scaling law in the WT. Energy following the forward cascade is eventually transferred into another form on small scales. Because of the additional inverse \( k^{-13/3} \) cascade in the present SF/SS system, the kinetic energy is conservatively transported in circles between the compressible and incompressible parts without any short-range dissipation.

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