GRADIENT ESTIMATES FOR DIVERGENCE FORM ELLIPTIC SYSTEMS ARISING FROM COMPOSITE MATERIAL

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Abstract. In this paper, we show that $W^{1, p}$ weak solutions to divergence form elliptic systems are Lipschitz and piecewise $C^1$ provided that the leading coefficients and data are of piecewise Dini mean oscillation, the lower order coefficients are bounded, and interfacial boundaries are $C^{1, \text{Dini}}$. This extends a result of Li and Nirenberg (Comm. Pure Appl. Math. 56 (2003), 892-925). Moreover, under a stronger assumption on the piecewise $L^1$-mean oscillation of the leading coefficients, we derive a global weak type-(1,1) estimate with respect to $A_1$ Muckenhoupt weights for the elliptic systems without lower order terms.

1. Introduction and main results

We consider a composite media with closely spaced interfacial boundaries. The composite media is represented by a bounded domain and divided into a finite number of sub-domains. The physical characteristics of the composite media are smooth in the closure of sub-domains but possibly discontinuous across their boundaries. From the viewpoint of mathematics, these properties are described in terms of a linear second-order divergence type elliptic systems with coefficients which can have jump discontinuities along the boundaries of sub-domains.

To state our main results, we introduce some notation and assumptions. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$ that contains $M$ disjoint sub-domains $\mathcal{D}_1, \ldots, \mathcal{D}_M$ with $C^{1, \text{Dini}}$ boundaries, that is, $\mathcal{D} = (\cup_{j=1}^M \mathcal{D}_j) \setminus \partial \mathcal{D}$. For more details about $C^{1, \text{Dini}}$ boundaries, see Definition 2.2. We assume that any point $x \in \mathcal{D}$ belongs to the boundaries of at most two of the $\mathcal{D}_j$. Hence, if the boundaries of two $\mathcal{D}_j$ touch, then they touch on a whole component of such a boundary. We thus without loss of generality assume that $\partial \mathcal{D} \subset \partial \mathcal{D}_M$.

Consider the following elliptic systems

$$\mathcal{L} u := D_\alpha (A^{\alpha \beta} D_\beta u) + D_\alpha (B^{\alpha} u) + \hat{B}^{\alpha} D_\alpha u + C u = \text{div} g + f, \quad (1.1)$$

where the Einstein summation convention in repeated indices is used,

$$u = (u^1, \ldots, u^n)^T, \quad g_\alpha = (g_\alpha^1, \ldots, g_\alpha^n)^T, \quad f = (f^1, \ldots, f^n)^T$$

are (column) vector-valued functions, $A^{\alpha \beta}, B^{\alpha}, \hat{B}^{\alpha}$ (often denoted by $A, B, \hat{B}$ for abbreviation), and $C$ are $n \times n$ matrices, which are bounded by a positive constant $\Lambda$, and the leading coefficients matrices $A^{\alpha \beta}$ are uniformly elliptic with ellipticity constant $\nu > 0$:

$$\nu |\xi|^2 \leq A^{\alpha \beta}_{ij} \xi^i \xi^j, \quad |A^{\alpha \beta}| \leq \nu^{-1}$$

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for any \( \xi = (\xi^i_n) \in \mathbb{R}^{n \times d} \). We remark that the hypotheses are also satisﬁed by the linear systems of elasticity. Recall that a system is called a system of elasticity if \( n = d \) and the coefficients satisfy \( A^{\alpha \beta}_{ij} = A^\alpha_0 = A_0^\beta \), and for all \( d \times d \) symmetric matrices \( \xi = (\xi^i_n) \),

\[
\forall |\xi|^2 \leq A^{\alpha \beta}_{ij}(x)\xi^i_n\xi^j_n \leq \nu^{-1}|\xi|^2, \quad \forall x \in \mathcal{D}.
\]

From an engineering point of view, one is interested in deriving bounds on the stress which is represented by \( A(x) \) are piecewise constants, given by \( A(x) = A_0 \) for \( x \) inside the sub-domains representing the fibers, and \( A(x) = 1 \) elsewhere in \( \mathcal{D} \). We would like to mention that Babuška et al. [2] studied certain homogeneous isotropic linear systems arising from elasticity. They observed numerically that Babuška et al. [2] studied certain homogeneous isotropic linear systems arising from elasticity. They observed numerically that \( |Du| \) is bounded independently of the distance between the regions. When \( A_0 = 0 \) or \( \infty \), it has been shown in many papers that \( |Du| \) is unbounded as sub-domains get close; for instance, in [3, 7]. When \( A_0 > 0 \) is finite, the scalar case when the sub-domains are circular touching fibers of comparable radii was studied by Bonnetier and Vogelius in [3]. They showed that \( |Du| \) remains bounded by using a Möbius transformation and the maximum principle. A general result concerning the solution to a large class of divergence form elliptic equations with discontinuous coeﬃcients was studied by Li and Vogelius [13]. There the coeﬃcients \( A^{\alpha \beta}_{ij} \) are assumed to be \( C^0 \) \( 0 < \delta < 1 \) up to the boundary in each sub-domain with \( C_{\mu}^1 \) boundary with \( \mu \in (0, 1] \), but may have jump discontinuities across the boundaries of the sub-domains. The authors derived global Lipschitz and piecewise \( C_{\mu}^1 \) estimates of the solution \( u \) for \( 0 < \delta' \leq \min\{\delta, \frac{\mu}{\mu+1}\} \). Li and Nirenberg [13] extended their results to elliptic systems under the same conditions and with \( 0 < \delta' \leq \min\{\delta, \frac{\mu}{\mu+1}\} \). Thus, a natural question is whether it is possible to further improve the range of \( \delta' \). In this paper, we give a deﬁnitive answer to the above question.

Denote by \( \mathcal{A} \) the set of piecewise constant functions in each \( \mathcal{D}_j \), \( j = 1, \ldots, M \). We then further assume that \( A \) is of piecewise Dini mean oscillation in \( \mathcal{D} \); that is,

\[
\omega_A(r) := \sup_{x_0 \in \mathcal{D}} \inf_{A \in \mathcal{A}, \hat{A} \sim A} \int_{B_r(x_0)} |A(x) - \hat{A}| \, dx \tag{1.2}
\]

satisfies the Dini condition, where \( B_r(x_0) \subset \mathcal{D} \). For more details about the Dini condition, see Deﬁnition 2.1. For \( \epsilon > 0 \) small, we set

\[
\mathcal{D}_\epsilon := \{x \in \mathcal{D} : \text{dist}(x, \partial \mathcal{D}) > \epsilon\}.
\]

Now, we state the ﬁrst main result of this paper.

**Theorem 1.1.** Let \( \mathcal{D} \) be deﬁned as above. Let \( \epsilon \in (0, 1) \), \( p \in (1, \infty) \), and \( \gamma \in (0, 1) \). Assume that \( A, B, \) and \( g \) are of piecewise Dini mean oscillation in \( \mathcal{D} \), and \( f, g \in L^\infty(\mathcal{D}) \). If \( u \in W^{1,p}(\mathcal{D}) \) is a weak solution to (1.1) in \( \mathcal{D} \), then \( u \in C^1(\overline{\mathcal{D}_j} \cap \mathcal{D}_\epsilon) \), \( j = 1, \ldots, M \), and \( u \) is Lipschitz in \( \mathcal{D}_\epsilon \). Moreover, for any ﬁxed \( x \in \mathcal{D}_\epsilon \), there exists a coordinate system associated with \( x \), such that for all \( y \in \mathcal{D}_\epsilon \), we have

\[
|(D_x u(x), U(x)) - (D_x u(y), U(y))| \\
\leq N \int_0^{||x-y||} \frac{\omega_A(t)}{t} \, dt + N ||x-y|| \|(D_x U, U)\|_{L^1(\mathcal{D})} + N \int_0^{||x-y||} \frac{\omega_A(t)}{t} \, dt
\]
Let \( D \) be replaced by \( C = (1.1) \) solution to elliptic equations in divergence form is continuously differentiable if the modulus of continuity of leading coefficients in the \( L^1 \)-mean sense satisfies the Dini characteristics of \( D \), and \( \tilde{\alpha}_r(t) \) is a Dini function derived from \( \alpha_r(t) \).

**Corollary 1.2.** Let \( D \) be defined as above and the boundary condition on each sub-domain be replaced by \( C^{1,\mu} \). Let \( \varepsilon \in (0,1) \) and \( p \in (1,\infty) \). Assume that \( A, B, g \in C^{1}(\overline{D}) \) with \( \delta \in (0,1) \), and \( f \in L^\infty(D) \). If \( u \in W^{1,p}(D) \) is a weak solution to \( (1.1) \) in \( D \), then \( u \in C^{1}(\overline{D}) \), \( j = 1, \ldots, M \). Moreover, for any fixed \( x \in D_\varepsilon \), there exists a coordinate system associated with \( x \) such that for all \( y \in \overline{D}_\varepsilon \), we have

\[
|D_{\nu}u(x) - D_{\nu}u(y)| + |U(x) - U(y)| 
\leq N|x - y|^\beta \left( \sum_{j=1}^{M} |g_j|_{L^\infty(D)}^\beta + \|f\|_{L^\infty(D)} + \|u\|_{L^p(D)} \right),
\]

(1.3)

where \( 0 < \delta' = \min(\delta, \frac{\mu}{\mu+1}) \), \( N \) depends on \( n, d, M, \delta, \mu, \beta, \varepsilon, \omega, \) and the \( C^{1,\mu} \) norms of \( D_j \).

**Remark 1.3.** In the above results the gradient estimates are independent of the distance between these sub-domains. In \([8]\), Xiong and Bao derived very general BMO, Dini, and Hölder estimates for \( H^1 \) weak solutions to \( (1.1) \). In particular, for the Hölder estimates, they allowed \( \delta' = \delta = \mu \). However, it appears that the estimates in \([8]\) depends on the distance between sub-domains. Since our estimtes are independent of the distance, by a similar reasoning as in \([15]\) Remark 1.2, we can obtain (1.3) in the case when more than two of the sub-domains \( \overline{D}_j \) touch, by an approximation argument.

**Remark 1.4.** Our result yields \( C^{1,\delta'} \) interior estimates for \( u \). Actually, we observe from (1.3) that for each \( j = 1, \ldots, M, D_{\nu}u, U \in C^0(\overline{D}_j \cap \overline{D}_\varepsilon) \). Moreover, since

\[
D_{\nu}u = (A^{dd})^{-1} \left( U + g_d - B^d u - \sum_{\beta=1}^{d-1} A^{d\beta} D_{\beta} u \right),
\]

we conclude that \( D_{\nu}u \in C^0(\overline{D}_j \cap \overline{D}_\varepsilon) \). Compared to \([15]\), the range of \( \delta' \) is improved.

**Remark 1.5.** The conditions on \( f \) in Theorem 1.1 and Corollary 1.2 can be relaxed. From the proofs below, it is easily seen that we only need \( f \) to be in some weaker Morrey space.

Using the duality argument which is developed in \([3]\), we have the following

**Corollary 1.6.** Under the same conditions as in Theorem 1.1. If \( u \in W^{1,1}(\overline{D}) \) is a weak solution to \( (1.1) \) in \( D \), then \( u \in W^{1,p}_{loc}(D) \) for some \( p \in (1,\infty) \) and the conclusion of Theorem 1.1 still holds.

Our arguments and methods are different from those in \([5, 16]\). The proofs below are based on Campanato’s approach, which was presented in \([6, 14]\) and used previously, for instance, in \([3, 8, 13, 2, 3]\). The authors in \([12]\) showed that any weak solution to elliptic equations in divergence form is continuously differentiable if the modulus of continuity of leading coefficients in the \( L^1 \)-mean sense satisfies the Dini
condition. Recently, Dong, Escauriaza, and Kim \[9\] extended and improved the results in \[12\] to the boundary for solutions satisfying the zero Dirichlet boundary condition.

The main step of such method is to show that the mean oscillation of $Du$ in balls vanish in a certain order as the radii of balls go to zero. However, we cannot employ this method directly because of the following two obstructions. The first one is the discontinuity of $Du$ in one direction, a situation similar to that in \[8\]. For a fixed coordinate system, the author in \[8\] obtained some interior Hölder regularity of $Dx'u$ and $U$ for elliptic systems with coefficients which are Hölder in $x'$ and measurable in $x^d$. Inspired by this, we first choose a coordinate system according to the geometry of the sub-domains, then we consider the elliptic systems with coefficients depending on one variable alone, say $x^d$, and derive some interior Hölder regularity of $Dx'u$ and $U = \bar{A}D^\beta u$, where $\bar{A}$ are piecewise constant matrix-valued functions corresponding to $A$. The second difficulty is that since we only impose the assumptions on the $L^1$-mean oscillation of the leading coefficients and data, we cannot use the usual argument based on $L^p$ ($p > 1$) estimates. To this end, we make use of a duality argument to derive weak type-$(1, 1)$ estimates for solutions to elliptic systems with coefficients depending only on $x^d$. Then, we utilize Campanato’s method in the $L^p$ ($0 < p < 1$) setting and some perturbation arguments on $Dx'u$ and $U$ together with a certain decomposition of $u$ to get the desired results.

In a forthcoming paper, we will study the second-order elliptic equation in non-divergence form under the same assumptions as that in Theorem 1.1.

Throughout this paper, unless otherwise stated, $N$ denotes a constant, whose value may vary from line to line and independent of the distance between sub-domains. We call it a universal constant.

The rest of this paper is organized as follows. In Section 2, we first fix our domain and the coordinate system associated with a fixed point. For reader’s convenience, we introduce some notation, definitions, and lemmas used in this paper. In Section 3, we provide the proofs of Theorem 1.1 and Corollary 1.2. In Section 5, we use the duality argument to prove that $u \in W^{1, p}_{loc}(D)$ for some $p \in (1, \infty)$ under the conditions of Corollary 1.4. Section 6 is devoted to our second purpose, a global weak type-$(1, 1)$ estimates under an additional condition on $\omega_A$ and the Dini function introduced in Definition 2.2.

2. Preliminaries

In this section, we introduce some lemmas which will be used throughout the proofs. Hereafter in this paper, we shall use the following notation and definitions.

2.1. Notation and definitions. We write $x = (x^1, \ldots, x^d) = (x', x^d)$, where $d \geq 2$. We shall denote

$$B_r(x) := \{y \in \mathbb{R}^d : |y - x| < r\}, \quad B_r'(x') := \{y' \in \mathbb{R}^{d-1} : |y' - x'| < r\}.$$ 

We use $B_r := B_r(0)$, $B_r'(0') := B_r'(0')$, and $D_r(x) := D \cap B_r(x)$ for abbreviation, respectively, where $0 \in \mathbb{R}^d$ and $0' \in \mathbb{R}^{d-1}$. We will also use the following notation:

$$Dx'u = u_{x'}, \quad DDx'u = u_{xx'}.$$
For a function $f$ defined in $\mathbb{R}^d$, we set
\[
( f )_D = \frac{1}{|D|} \int_D f(x) \, dx = \int_D f(x) \, dx,
\]
where $|D|$ is the $d$-dimensional Lebesgue measure of $D$. For $\gamma \in (0, 1]$, we denote the $C^\gamma$ semi-norm by
\[
[u]_{\gamma;D} := \sup_{x,y \in D, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{1+\gamma}},
\]
and the $C^\gamma$ norm by
\[
[u]_{\gamma;D} := [u]_{\gamma;D} + |u|_{0;D}, \quad \text{where } |u|_{0;D} = \sup_D |u|.
\]

**Definition 2.1.** We say that a continuous increasing function $\omega : [0, 1] \to \mathbb{R}$ satisfies the Dini condition provided that $\omega(0) = 0$ and
\[
\int_0^t \frac{\omega(s)}{s} \, ds < +\infty, \quad \forall \ t \in (0, 1).
\]

**Definition 2.2.** Let $D \subset \mathbb{R}^d$ be open and bounded. We say that $\partial D$ is $C^{1,\text{Dini}}$ if for each point $x_0 \in \partial D$, there exists $R_0 \in (0, 1/8)$ independent of $x_0$ and a $C^{1,\text{Dini}}$ function (that is, $C^1$ function whose first derivatives are Dini continuous) $\varphi : B_{R_0} \to \mathbb{R}$ such that (upon relabeling and reorienting the coordinates if necessary) in a new coordinate system $(x', x^d)$, $x_0$ becomes the origin,
\[
\mathcal{D}_{R_0}(0) = \{ x \in B_{R_0} : x^d > \varphi(x') \}, \quad \varphi(0') = 0,
\]
and $\nabla_x \varphi$ has a modulus of continuity $\omega_0$, which is increasing, concave, and independent of $x_0$.

2.2. Some properties of the domain, coefficients, and data. Below, we slightly abuse the notation. Consider $D$ to be the unit ball $B_1$ and take $x_0 \in B_{3/4}$. By suitable rotation and scaling, we may suppose that a finite number of sub-domains lie in $B_1$ and that they take the form
\[
x^d = h_j(x'), \quad \forall \ x' \in B_{1}^j, \ j = 1, \ldots, l < M,
\]
with
\[
-1 < h_1(x') < \cdots < h_l(x') < 1 \tag{2.1}
\]
and $h_j(x') \in C^{1,\text{Dini}}(B_1^j)$. Set $h_0(x^d) = -1$ and $h_{l+1}(x^d) = 1$. Then we have $l + 1$ regions:
\[
\mathcal{D}_j := \{ x \in D : h_{j-1}(x') < x^d < h_j(x') \}, \quad 1 \leq j \leq l + 1.
\]
We may suppose that there exists some $\mathcal{D}_{j_0}$, such that $x_0 \in B_{3/4} \cap \mathcal{D}_{j_0}$ and the closest point on $\partial \mathcal{D}_{j_0}$ to $x_0$ is $(x_0', h_{j_0}(x_0'))$, and $\nabla_x h_{j_0}(x_0') = 0'$. We introduce the $l + 1$ “strips”
\[
\Omega_j := \{ x \in D : h_{j-1}(x_0') < x^d < h_j(x_0') \}, \quad 1 \leq j \leq l + 1.
\]
Then we have the following result.

**Lemma 2.3.** There exists a constant $N$, depending on $d, l$ and the $C^{1,\text{Dini}}$ characteristics of $h_j$, $1 \leq j \leq l + 1$, such that
\[
r^{-d}|(\mathcal{D}_j \setminus \Omega_j) \cap B_r(x_0)| \leq N \omega_1(r), \quad 1 \leq j \leq l + 1, \quad 0 < r < r_0 := \frac{2}{3} \int_0^{\text{dist}_{R_0/2}} \omega_0'(s) \, ds,
\]
where $\text{dist}_{R_0}$ is the $d$-dimensional Lebesgue measure of $D$. For $\gamma \in (0, 1]$, we denote the $C^\gamma$ semi-norm by
\[
[u]_{\gamma;\mathbb{R}^d} := \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{1+\gamma}},
\]
and the $C^\gamma$ norm by
\[
[u]_{\gamma;\mathbb{R}^d} := [u]_{\gamma;\mathbb{R}^d} + |u|_{0;\mathbb{R}^d}, \quad \text{where } |u|_{0;\mathbb{R}^d} = \sup_{\mathbb{R}^d} |u|.
\]
where $\mathcal{D}_j \Delta \Omega_j = (\mathcal{D}_j \setminus \Omega_j) \cup (\Omega_j \setminus \mathcal{D}_j)$, $\omega'_j$ denotes the left derivative of $\omega_0$, and $\omega_1(r)$ is a Dini function derived from $\omega_0$ in Definition 2.1.

Proof. Let $j$ be such that $(x', h_j(x')) \in B_j(x_0)$ for some $x' \in B'_j$. We denote the supremum of $|\nabla x h_j(x')|$ in $B'_j(x_0^j)$ by $\lambda$. Now we fix a point $(y', h_j(y')) \in B_r(x_0)$, and without loss of generality, we may assume that $\omega_0(r) \leq 1$ for all $0 < r < 1$. From (2.5) and the fact that $\nabla x h_j(x') = 0$,

$$|h_j(y') - h_j(x_0^j)| \leq \lambda, \quad |\nabla x h_j(y')| \leq \omega_0(2r), \quad |\nabla x h_j| \leq \lambda - \omega_0(2r).$$

Because of (2.1), we have

$$\int_0^R (\lambda - 2\omega_0(2r + s)) \, ds \leq 3r \quad (2.2)$$

for any $R \in (0, 1/8)$. The left-hand side of (2.2) attains its maximum with respect to $R$ when $2\omega_0(2r + R) = \lambda$, which implies that

$$R \omega_0(2r + R) - \int_0^R \omega_0(2r + s) \, ds \leq 3r/2. \quad (2.3)$$

That is,

$$\int_0^R \omega'_0(2r + s) \, ds \leq 3r/2. \quad (2.4)$$

For a fixed $r$, the left-hand side of (2.4) is an increasing function with respect to $R$. To get an upper bound for $\lambda$, we fix the number $R = R(r)(> 2r)$ by (2.4) such that

$$\int_0^R \omega'_0(2r + s) \, ds = 3r/2. \quad (2.5)$$

Denote

$$\omega_1(r) := \omega_0(2r + R). \quad (2.6)$$

Then combined with $|\nabla x h_j(x')| \leq \lambda$ in $B'_j(x_0^j)$, we obtain

$$|(\mathcal{D}_j \Delta \Omega_j) \cap B_r(x_0)| \leq N \lambda r^d = N \omega_1(r) r^d.$$

Next we are left to prove that $\omega_1(r)$ is a Dini function on $(0, r_0)$. For this, it suffices to check the conditions in Definition 2.2. Obviously, (2.5) and (2.6) imply that $\omega_1(r)$ is a continuous increasing function on $(0, r_0)$ and $\omega_1(0) = \omega_0(0) = 0$. Moreover, it follows from the increasing property and concavity of $\omega_0$ that

$$\int_0^R \omega'_0(2r + s) \, ds \geq \frac{R^2}{2} \omega'_0(2r + R). \quad (2.7)$$

From (2.5), we have

$$\omega'_0(2r + R) R \, dR = \frac{3}{2} \, dr \quad \text{for a.e. } r. \quad (2.8)$$

Therefore, in view of (2.7)–(2.8), the increasing property and concavity of $\omega_0$ again, and $2r < R$, we have

$$\int_0^\infty \frac{\omega_1(r)}{r} \, dr = \int_0^\infty \frac{\omega_0(2r + R)}{r} \, dr = \int_0^{\infty/2} \frac{R \omega_0(2r + R) \omega'_0(2r + R)}{\int_0^R \omega'_0(2r + s) \, ds} \, dR.$$
Lemma 2.3 and the boundedness of \( A \) which is also true for \( \hat{\omega} \).

Then using \( \hat{\omega} \) in (1.2), similarly, \( \hat{\omega} \) are defined in \( \Omega \).

The lemma is proved.

\[ \square \]

In the sequel, we extend \( \omega_1(r) \) to be \( \omega_1(r_0) \) when \( r > r_0 \).

**Remark 2.4.** From the proof above, if we assume that the boundaries of \( \mathcal{D}_1, \ldots, \mathcal{D}_M \) are \( C^{1/\mu} \), then \( \omega_0(s) = s^\mu \) and it follows from (2.3) that

\[ R \leq \left( \frac{3(1 + \mu)}{2\mu} \right)^{1/\mu}. \]

Hence,

\[ R^{-d}|(\mathcal{D}_j \Delta \Omega_i) \cap B_r(x_0)| \leq N r^{-\omega}, \]

which is the result in [16, Lemma 5.1].

Let \( \hat{A}^{(i)} \in \mathcal{A} \) be a constant function in \( \mathcal{D}_i \) which corresponds to the definition of \( \omega_A(r) \) in (1.3). Similarly, \( \hat{B}^{(i)} \) and \( \hat{g}^{(i)} \) are defined in \( \mathcal{D}_i \). We define the piecewise constant (matrix-valued) functions

\[ \bar{A}(x) = \hat{A}^{(i)}, \quad x \in \Omega_i. \]

Using \( \hat{B}^{(i)} \) and \( \hat{g}^{(i)} \), we similarly define piecewise constant functions \( \bar{B} \) and \( \bar{g} \). From Lemma 2.3 and the boundedness of \( A \), we have

\[ \int_{B_r(x_0)} |A - \bar{A}| \, dx \leq N r^{-\omega} \sum_{i=1}^{l+1} |(\mathcal{D}_i \Delta \Omega_i) \cap B_r(x_0)| \leq N \omega_1(r), \quad (2.9) \]

which is also true for \( \bar{B} \) and \( \bar{g} \).

### 2.3. Some \( L^p \) estimates and auxiliary lemmas

First, let us recall the variably partially small BMO (bounded mean oscillation) condition (see, for instance, [11]): there exists a sufficiently small constant \( \gamma_0 = \gamma_0(d, n, p, \nu) \in (0, 1/2) \) and a constant \( r_0 \in (0, 1) \) such that for any \( r \in (0, r_0) \) and \( x_0 \in B_1 \) with \( B_r(x_0) \subset B_1 \), in a coordinate system depending on \( x_0 \) and \( r \), one can find a \( \tilde{A}(x') \) satisfying

\[ \int_{B_r(x_0)} |A(x) - \tilde{A}(x')| \, dx \leq \gamma_0. \quad (2.10) \]

The following lemma follows from [11, Theorem 8.6] by a standard localization argument which is similar to that in the proof of [8, Lemma 4], using the Sobolev embedding theorem and a bootstrap argument.

**Lemma 2.5.** Let \( p, q \in (1, \infty) \). Assume \( A \) satisfies (2.10) with a sufficiently small constant \( \gamma_0 = \gamma_0(d, n, p, q, \nu, \Lambda) \in (0, 1/2) \) and \( u \in C^{0,1}_{\text{loc}} \) satisfies (1.1) in \( B_1 \), where \( f, g \in L^q(B_1) \). Then

\[ ||u||_{W^{1,q}(B_{1/2})} \leq N(||u||_{L^q(B_1)} + ||g||_{L^q(B_1)} + ||f||_{L^q(B_1)}). \]
In particular, if \( q > d \), it holds that
\[
|u|_{W^{1,q}(B_1/2)} \leq N\left(\|g\|_{L^q(B_1)} + \|f\|_{L^q(B_1)}\right),
\]
where \( \gamma = 1 - d/q \) and \( N \) depends on \( n, d, \nu, \Lambda, p, q, \) and \( r_0 \).

We next recall the \( W^{1,p}_{loc} \)-solvability for elliptic systems with leading coefficients which satisfy (2.10) in \( B_1 \). To be precise, we choose a cut-off function \( \eta \in C_0^\infty(B_1) \) with
\[
0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } B_{3/4}, \quad |\nabla \eta| \leq 8.
\]
Let \( \tilde{L} \) be the elliptic operator defined by
\[
\tilde{L} u := D_\alpha(\tilde{A}^\alpha D_\beta u),
\]
where \( \tilde{A}^\alpha = \eta A^\alpha(x) + (1 - \eta)\delta_{ij}, \delta_{ij} \) and \( \delta_{ij} \) are the Kronecker delta symbols. Consider
\[
\begin{cases}
\tilde{L} u = \text{div } g + f & \text{in } B_1, \\
u = 0 & \text{on } \partial B_1,
\end{cases}
\tag{2.11}
\]
where \( g, f \in L^q(B_1) \). Then the coefficients \( \tilde{A}^\alpha(x) \) and the boundary \( \partial B_1 \) satisfy the Assumption 8.1 \( (\gamma) \) in [11] for sufficiently small \( \gamma \). Applying [11, Theorem 8.6] to our case, we have

**Lemma 2.6.** For any \( p \in (1, \infty) \), the following hold.

1. For any \( u \in W^{1,p}_0(B_1) \),
\[
\|u\|_{W^{1,p}(B_1)} \leq N\left(\|g\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}\right),
\]
where \( N \) depends on \( d, n, p, \nu, \Lambda, \) and \( r_0 \).

2. For any \( g, f \in L^p(B_1) \), (2.11) admits a unique solution \( u \in W^{1,p}_0(B_1) \).

We note that by (2.10), Lemmas 2.5 and 2.6 are applicable in our case. Next we consider systems with coefficients depending on \( x^\alpha \) alone. We denote
\[
L_0 u := D_\alpha(x^\alpha D_\beta u),
\]
and
\[
\tilde{U} := \tilde{A}^\alpha u, \quad \text{that is,} \quad \tilde{U}^i = \tilde{A}^\alpha_{ij}(x^\alpha)D_\beta u^j, \quad i = 1, \ldots, n.
\]

**Lemma 2.7.** Let \( p \in (0, \infty) \). Assume \( u \in C^{\beta,1}_{loc} \) satisfies \( L_0 u = 0 \) in \( B_1 \). Then for any \( q \in (1, \infty) \), there exists a constant \( N = N(n, d, p, q, \nu, \Lambda) \) such that for any matrix-valued constant \( \xi \in \mathbb{R}^{n\times(d-1)} \),
\[
\|D_x u\|_{L^q(B_1/2)} \leq N\|D_x u - \xi\|_{L^p(B_1)},
\tag{2.12}
\]
and
\[
\|D\tilde{U}\|_{L^q(B_1/2)} \leq N\|D_x u - \xi\|_{L^p(B_1)},
\tag{2.13}
\]

**Proof.** It directly follows from Lemma 2.5 that
\[
\|u\|_{W^{\alpha,q}(B_{1/2})} \leq N\|u\|_{L^2(B_1)}.
\]
Then by the Sobolev embedding theorem for \( q > d \), we have
\[
\|u\|_{L^q(B_{1/2})} \leq N\|u\|_{L^2(B_1)}.
\tag{2.14}
\]
For $0 < p < 1 < \infty$, by using the interpolation inequality, we get
\[
\|u\|_{L^p(B_{0,4})} \leq \|u\|_{L^p(B_{0,3})}^{1/p} \|u\|_{L^\infty(B_{0,3})}^{1/p}.
\]
Thus, combining (2.14) with slightly smaller domain, and Hölder's inequality, we obtain
\[
\|u\|_{L^\infty(B_{0,3})} \leq N\|u\|_{L^p(B_{0,3})}^{1/p} \|u\|_{L^\infty(B_{0,3})}^{1/p/2}.
\]
By a well-known iteration argument (see, for instance, [14, Lemma 3.1 of Ch.V]), we get
\[
\|u\|_{L^\infty(B_{0,3})} \leq \frac{1}{2} \|u\|_{L^\infty(B_{0,3})} + N\|u\|_{L^p(B_{0,3})}, \quad p > 0.
\]
By a well-known iteration argument (see, for instance, [14, Lemma 3.1 of Ch.V]), we get
\[
\|u\|_{L^\infty(B_{0,3})} \leq N\|u\|_{L^p(B_{0,1})}, \quad p > 0.
\]
Now, we define the finite difference quotient
\[
\delta_{h,k} f(x) := \frac{f(x + he_k) - f(x)}{h}
\]
where $k = 1, \ldots, d - 1, 0 < |h| < 1/12$. Since $\tilde{A}^{a\beta}(x')$ are independent of $x'$, we have $L_0(\delta_{h,k} u) = 0$ in $B_1$. We thus use Lemma 2.5 and (2.13) to get
\[
\|\delta_{h,k} u\|_{W^1,q(B_{1/2})} \leq N\|\delta_{h,k} u\|_{L^p(B_{1/2})} \leq N\|D_x u\|_{W^1,q(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})}, \quad p > 0.
\]
Letting $h \to 0$, we obtain
\[
\|D_x u\|_{W^1,q(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})}, \quad p > 0.
\]
Moreover, notice that in $B_1$,
\[
D_d \tilde{U} = -\sum_{a=1}^{d-1} \sum_{\beta=1}^d \tilde{A}_{a\beta} D_{a\beta} u, \quad D_x \tilde{U} = \sum_{\beta=1}^d \tilde{A}_{a\beta} D_{a\beta} D_x u.
\]
Then by using Lemma 2.5, (2.15), (2.16), and the boundedness of $\tilde{A}$, we obtain
\[
\|\tilde{U}\|_{W^1,q(B_{1/2})} = \|\tilde{U}\|_{L^p(B_{1/2})} + \|D \tilde{U}\|_{L^p(B_{1/2})} \leq N\|D u\|_{L^p(B_{1/2})} + N\|D_x \tilde{U}\|_{L^p(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})} + N\|D_x u\|_{L^p(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})} + N\|D_x u\|_{W^1,q(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})}.
\]
Combining (2.16) and (2.18), we get
\[
\|D_x u\|_{W^1,q(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})}, \quad p > 0.
\]
By the Sobolev embedding theorem for $q > d$, we have
\[
\|D_x u\|_{L^{q}(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})}, \quad p > 0.
\]
By using $\tilde{A}^{a\beta}(x') \geq \nu$, we then have
\[
\|D u\|_{L^\infty(B_{1/2})} \leq N\|u\|_{L^p(B_{1/2})}, \quad p > 0.
\]
Similarly, by using the fact that the coefficients of $L_0$ are independent of $x'$ and hence, for any matrix-valued constant $c \in \mathbb{R}^{n \times (d - 1)}$, we have $L_0(D_x u - c) = 0$ in $B_1$. Then
\[
\|D_x u\|_{L^{\infty}(B_{1/2})} \leq N\|D_x u - c\|_{L^p(B_{1/2})}, \quad p > 0.
\]
Thus, (2.12) is proved and (2.13) follows from (2.17) and (2.12). \qed
3.1. Proof of Theorem 1.1. Suppose that for any \( p \in \) a slight modification of the proof there, the result can be easily extended to general constants. Then for \( g \in L^p(D) \), \( \omega \) be a nonnegative bounded function. Suppose there is \( c_1, c_2 > 0 \) and \( 0 < \kappa < 1 \) such that for \( \kappa t \leq s \leq t \) and \( 0 < t < r \),
\[
c_1 \omega(t) \leq \omega(s) \leq c_2 \omega(t).
\]
(2.19)
Then, we have
\[
\sum_{i=0}^{\infty} \omega(\kappa^i r) \leq N \int_0^\infty \frac{\omega(t)}{t} \, dt,
\]
where \( N = N(\kappa, c_1, c_2) \).

Lemma 2.8. Let \( \omega \) be a nonnegative bounded function. Suppose there is \( c_1, c_2 > 0 \) and \( 0 < \kappa < 1 \) such that for \( \kappa t \leq s \leq t \) and \( 0 < t < r \),
\[
c_1 \omega(t) \leq \omega(s) \leq c_2 \omega(t).
\]
(2.19)
Then, we have
\[
\sum_{i=0}^{\infty} \omega(\kappa^i r) \leq N \int_0^\infty \frac{\omega(t)}{t} \, dt,
\]
where \( N = N(\kappa, c_1, c_2) \).

Lemma 2.9. Let \( D \) be a bounded domain in \( \mathbb{R}^d \) satisfying
\[
|\{x \in D(\alpha) : |\{x \in D : |T g(x)| > t| \} | \leq \frac{N}{t} \int_D |g|, \]
where \( N = N(d, c, c, 0, D, A_0, \mu, ||T||_{L^1 \rightarrow L^p}) \) is a constant.

We note that in [9, Lemma 4.1] the exponent \( p \) is assumed to be 2. However, by a slight modification of the proof there, the result can be easily extended to general \( p \in (1, \infty) \). See also Lemma 2.3 below by taking \( w = 1 \).

3. Proofs of Theorem 1.1 and Corollary 1.2

In this section, we give the proofs of Theorem 1.1 and Corollary 1.2.

3.1. Proof of Theorem 1.1. We first show that by using the global \( W^{2,p} \) estimate for the Laplacian operator, Sobolev embedding theorem, and Lemma 2.5, we only need to consider systems without lower order terms. We rewrite (1.1) as
\[
D_t(A^{\alpha_1}D_\beta u) = \text{div}(g - Bu + Dv) \quad \text{in} \quad D = B_1 \subset B_2,
\]
where \( v \in W^{2,p}(B_2) \) is a strong solution to
\[
\begin{cases}
\Delta v = (f - \hat{B}_v D_\alpha u - C u)\chi_B & \text{in} \quad B_2, \\
v = 0 & \text{on} \quad \partial B_2.
\end{cases}
\]
Then by the global \( W^{2,p} \) estimate, we have
\[
\|v\|_{W^{2,p}(B_2)} \leq N \|f - \hat{B}_v D_\alpha u - C u\|_{L^p(B_1)}
\leq N \left( \|u\|_{W^{2,p}(B_1)} + \|f\|_{L^p(B_1)} \right).
\]
(3.1)
By Lemma 2.3, the Sobolev embedding theorem, and (3.1), we have
\[
\|u\|_{W^{1,\kappa}(B_1)} \leq N \left( \|u\|_{L^p(B_1)} + \|g - Bu + Dv\|_{L^p(B_1)} \right)
\]
By Morrey’s inequality, we obtain
inequality, we get
without lower order terms, where
Therefore, we conclude that
Proposition 3.1.
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Dini mean oscillation in
\[ \hat{\omega} \]
\[ r \]
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Lemma 3.2. Let \( p \in (1, \infty) \). Let \( v \in W^{1,p}(B_r(x_0)) \) be a weak solution to the problem
\[
\begin{aligned}
\tilde{L} v &= \text{div}(F \chi_{B_r(x_0)}) \quad \text{in } B_r(x_0), \\
v &= 0 \quad \text{on } \partial B_r(x_0),
\end{aligned}
\]
where \( F \in L^r(B_{r/2}(x_0)). \) Then for any \( t > 0 \), we have
\[
||x \in B_{r/2}(x_0) : |\text{div}(v)| > t|| \leq \frac{N}{t}||F||_{L^1(B_{r/2}(x_0))},
\]
where \( N > 0 \) is a universal constant.

Proof. For simplicity, we set \( x_0 = 0, r = 1, \tilde{A}^{q\beta}(x^d) := \tilde{A}^{q\beta}(0', x^d), \) and \( \tilde{L} := \tilde{L}_0. \) Suppose \( E = (E^{q\beta}(x^d)) \) is a \( d \times d \) matrix with
\[
E^{q\beta}(x^d) = \delta_{\alpha\beta} \text{ for } \alpha, \beta \in \{1, \ldots, d-1\}; \quad E^{q\beta}(x^d) = \tilde{A}^{q\beta}(x^d) \text{ for } \alpha \in \{1, \ldots, d\}; \quad E^{q\beta}(x^d) = 0 \text{ for } \beta \in \{1, \ldots, d-1\}.
\]
For any \( \tilde{F} \in L^r(B_{1/2}) \), let \( F = \tilde{F} \) and solve for \( v \). By Lemma 2.9, we can see that \( T : \tilde{F} \rightarrow Dv \) is a bounded linear operator on \( L^r(B_{1/2}) \). It suffices to show that \( T \) satisfies the hypothesis of Lemma 2.9. We set \( c = 24 \) and fix \( \bar{y} \in B_{1/2} \), \( 0 < r < 1/4 \). Let \( \tilde{b} \in L^r(B_1) \) be supported in \( B_r(\bar{y}) \cap B_{1/2} \) with mean zero, \( b = \tilde{E}\tilde{b} \), and \( v_1 \in W^{1,p}(B_1) \) be the unique weak solution of
\[
\begin{aligned}
\tilde{L} v_1 &= \text{div } b \quad \text{in } B_1, \\
v_1 &= 0 \quad \text{on } \partial B_1.
\end{aligned}
\]
For any \( R \geq cr \) such that \( B_{1/2} \setminus B_R(\bar{y}) \neq \emptyset \) and \( h \in C_0^\infty(B_{2R}(\bar{y}) \setminus B_R(\bar{y})) \cap B_{1/2} \), let \( v_0 \in W^{1,p'}(B_1) \) be a weak solution of
\[
\tilde{L}^* v_0 = \text{div } h \quad \text{in } B_1, \\
v_0 &= 0 \quad \text{on } \partial B_1,
\]
where \( \tilde{L}^* \) is the adjoint operator of \( \tilde{L} \), \( 1/p + 1/p' = 1 \). It follows from the definition of weak solutions and the assumption of \( \tilde{b} \) that
\[
\int_{B_{1/2}} Dv_1 \cdot h = \int_{B_{1/2}} Dv_0 \cdot b = \int_{B_{1/2}} (D_{x^d} v_0, V_0) \cdot \tilde{b} = \int_{B_{1/2}} (D_{x^d} v_0 - D_{x^d} v_0(\bar{y}), V_0 - V_0(\bar{y})) \cdot \tilde{b},
\]
where \( V_0 = \tilde{A}^{q\beta}(x^d)D_{\bar{y}}v_0 \). Therefore, we have
\[
\left| \int_{(B_{2R}(\bar{y}) \setminus B_{r/2}) \cap B_{1/2}} Dv_1 \cdot h \right| \leq ||\tilde{b}||_{L^r(B_{r/2})} ||(D_{x^d} v_0 - D_{x^d} v_0(\bar{y}), V_0 - V_0(\bar{y}))||_{L^p(B_R(\bar{y}) \setminus B_{r/2})}.
\]
Recalling that \( \eta \equiv 1 \) in \( B_{2/3} \) and \( B_{R/12}(\bar{y}) \subset B_{2/3} \), we conclude that \( v_0 \in W^{1,p'}(B_1) \) satisfies
\[
\tilde{L}^* v_0 = 0 \quad \text{in } B_{R/12}(\bar{y})
\]
Then, by using (2.12), (2.13) with a suitable scaling, \( r \leq 24 \), and the \( L^p \) estimate, we have
\[
\|D_x^r v_0 - D_x^r \tilde{v}_0(\tilde{y})\|_{L^p(B_r(\tilde{y}))^n B_{1/2}} + \|V_0 - \tilde{V}_0(\tilde{y})\|_{L^p(B_r(\tilde{y}))^n B_{1/2}} \\
\leq Nr R^{-1-d/p'} \|Dv_0\|_{L^{p'}(B_{1/2}(\tilde{y}))} \\
\leq Nr R^{-1-d/p'} \|H\|_{L^{p'}((B_{2r}(\tilde{y}) \setminus B_r(\tilde{y}))^n B_{1/2})}.
\]

Now, coming back to (2.2) and using the duality, we have
\[
\|Dv_1\|_{L^q((B_{2r}(\tilde{y}) \setminus B_r(\tilde{y}))^n B_{1/2})} \leq Nr R^{-1-d/p'} \|\hat{h}\|_{L^{q'}((B_{1/2}(\tilde{y}))^n B_{1/2})}.
\]

Thus, it follows from Hölder’s inequality that
\[
\|Dv_1\|_{L^q((B_{2r}(\tilde{y}) \setminus B_r(\tilde{y}))^n B_{1/2})} \leq Nr R^{-1-d/p'} \|\hat{h}\|_{L^{q'}((B_{1/2}(\tilde{y}))^n B_{1/2})}.
\]

Let \( N_0 \) be the smallest positive integer such that \( B_{1/2} \subset B_{2N_0 r} \). By taking \( R = cr, 2cr, \ldots, 2^{N_0 - 1} cr \) in (3.3) and summarizing, we have
\[
\int_{B_{1/2}(\tilde{y})^n B_{1/2}} |Dv_1| \, dx \leq N \sum_{k=1}^{N_0} 2^{-k} \|\hat{h}\|_{L^{q'}((B_{1/2}(\tilde{y}))^n B_{1/2})} \leq N \int_{B_{1/2}(\tilde{y})^n B_{1/2}} |\hat{h}| \, dx.
\]

Therefore, \( T \) satisfies the hypothesis of Lemma 2.9. The lemma is proved. \( \square \)

Consider
\[
\phi(x_0, r) := \inf_{q \in \mathbb{R}^{+}} \left( \int_{B_{r}(x_0)} |(D_x u, U)| \, dx \right)^{1/q},
\]
where \( 0 < \alpha < 1 \) is some fixed exponent. First of all, by using Hölder’s inequality, we have
\[
\phi(x_0, r) \leq \left( \int_{B_{r}(x_0)} |(D_x u, U)| \, dx \right)^{1/q} \leq Nr^{-d} \|D_x u, U\|_{L^q(B_{r}(x_0))},
\]
where \( N = N(d) \).

**Lemma 3.3.** For any \( r \in (0, 1) \) and \( 0 < \rho \leq r \leq 1/4 \), we have
\[
\phi(x_0, \rho) \leq N \left[ \frac{\rho}{r} \right]^\gamma \|D_x u, U\|_{L^q(B_{r}(x_0))} + N \tilde{A}(\rho) \|Du\|_{L^q(B_{r}(x_0))} + N \tilde{A}_\gamma(\rho),
\]
where \( N > 0 \) is a universal constant, and \( \tilde{A}(\rho) \) is a Dini function derived from \( \omega_\rho(t) \).

**Proof.** For any \( t > 0 \), by using Lemma 3.2 with \( F = (\tilde{A}(x_0', x') - A(x)) Du + g(x) - \tilde{g}(x_0', x') \), and (2.3), we have
\[
| \{ x \in B_{r/2}(x_0) : |Dv(x)| > t \} | \\
\leq \frac{N}{t} \int_{B_{r}(x_0)} |F| \, dx \\
\leq \frac{N}{t} \left( \int_{B_{r}(x_0)} |g(x) - \tilde{g}(x_0', x')| \, dx + \int_{B_{r}(x_0)} |A(x) - \tilde{A}(x_0', x')| \, dx \right) \\
\leq \frac{N}{t} \left( \int_{B_{r}(x_0)} |g(x) - \tilde{g}| \, dx + \int_{B_{r}(x_0)} |\tilde{g} - \tilde{g}(x_0', x')| \, dx \right) \\
+ \left( \int_{B_{r}(x_0)} |A(x) - \tilde{A}| \, dx + \int_{B_{r}(x_0)} |\tilde{A} - \tilde{A}(x_0', x')| \, dx \right) \|Du\|_{L^q(B_{r}(x_0))}
\]
\[
\int_{B_{r/2}(x_0)} |D\omega^q| \, dx = \int_0^\infty qt^{-1} \mathbb{1}_{[x \in B_{r/2}(x_0) : |D\nu(x)| > t]} \, dt
\]
and \((3.6)\) that
\[
\int_{B_{r/2}(x_0)} |D_x\nu^q| \, dx + \int_{B_{r/2}(x_0)} |V|^q \, dx
\]
\[
\leq N \int_0^\infty qt^{-1} \mathbb{1}_{[x \in B_{r/2}(x_0) : |D\nu(x)| > t]} \, dt
\]
\[
\leq N \left( \int_0^\tau + \int_\tau^\infty \right) qt^{-1} \mathbb{1}_{[x \in B_{r/2}(x_0) : |D\nu(x)| > t]} \, dt
\]
\[
\leq N \tau q |B_r(x_0)| + \frac{Nq}{1-q} \left( t^d \hat{\omega}_q(r) + t^d \hat{\omega}_A(r) \|Du\|_{L^\infty(B_r(x_0))} \right)^q,
\]
where \(V = \hat{A}_0(\nu_0, \nu_0) D_\nu \nu(x)\) and \(\tau = \frac{\nu}{\nu(x_0)}(\hat{\omega}_q(r) + \hat{\omega}_A(r) \|Du\|_{L^\infty(B_r(x_0))})\). Then we have
\[
\int_{B_{r/2}(x_0)} |D_x\nu^q| \, dx + \int_{B_{r/2}(x_0)} |V|^q \, dx
\]
\[
\leq \frac{N}{1-q} \left( |B_r(x_0)|^{1-q} \left( t^d \hat{\omega}_q(r) + t^d \hat{\omega}_A(r) \|Du\|_{L^\infty(B_r(x_0))} \right)^q \right). \tag{3.7}
\]
Let
\[
u_1(x_0) = \int_{x_0}^{x^d} (\hat{A}_0(\nu_0, s))^{-1} g_d(x_0, s) \, ds, \quad \bar{u} = u - u_1, \quad w = \bar{u} - v,
\]
so that \(w\) satisfies \(\hat{\mathcal{L}}_\nu w = 0\) in \(B_{r/2}(x_0)\). Noticing that for any \(q = (q', q_\beta) \in \mathbb{R}^{n_\alpha d}\), the same system is satisfied by \(D_\beta w - q_\beta\) for \(\beta = 1, \ldots, d - 1\). By Lemma 2.7 with a suitable scaling, we have
\[
\|DD_\beta w\|_{L^\infty(B_{r/2}(x_0))} \leq N \tau^{-d+q} \int_{B_{r/2}(x_0)} |D_\nu w - q|^q \, dx
\]
\[
\leq N \tau^{-d+q} \int_{B_{r/2}(x_0)} |(D_x w, W) - q|^q \, dx, \tag{3.8}
\]
\[ \leq N(\kappa r)^{d+q} \left( \|DD_x w\|_{L^p(B_{r/4}(x_0))}^p + \|DW\|_{L^p(B_{r/4}(x_0))}^p \right) \]
\[ \leq N\kappa^{d+q} \int_{B_{r/2}(x_0)} \|D_x w, W\| - q\| dx, \]
which implies
\[ \left( \int_{B_{r/2}(x_0)} \|D_x w - (D_x w)_{B_{r/2}(x_0)}\| dx + \right. \]
\[ \left. \int_{B_{r/2}(x_0)} \|W - (W)_{B_{r/2}(x_0)}\| dx \right)^{1/q} \]
\[ \leq N_0\kappa \left( \int_{B_{r/2}(x_0)} \|D_x w, W\| - q\| dx \right)^{1/q}, \quad (3.9) \]
where \( N_0 > 0 \) is a universal constant. Recalling that \( \tilde{u} = w + v \), we obtain from (3.9) that
\[ \left( \int_{B_{r/2}(x_0)} \|D_x \tilde{u}, \tilde{U}\| - ((D_x w)_{B_{r/2}(x_0)}, (W)_{B_{r/2}(x_0)}) \| dx \right)^{1/q} \]
\[ \leq \left( \int_{B_{r/2}(x_0)} \|D_x \tilde{u} - (D_x w)_{B_{r/2}(x_0)}\| dx + \right. \]
\[ \left. \|W - (W)_{B_{r/2}(x_0)}\| dx \right)^{1/q} \]
\[ \leq 2^{1/q-1} \left( \int_{B_{r/2}(x_0)} \|D_x w - (D_x w)_{B_{r/2}(x_0)}\| dx + \int_{B_{r/2}(x_0)} \|W - (W)_{B_{r/2}(x_0)}\| dx \right)^{1/q} \]
\[ + N \left( \int_{B_{r/2}(x_0)} \|D_x v\| dx + \|V\| dx \right)^{1/q} \]
\[ \leq N_0\kappa \left( \int_{B_{r/2}(x_0)} \|D_x \tilde{u}, \tilde{U}\| - q\| dx \right)^{1/q} + N\kappa^{-d/q} \left( \int_{B_{r/2}(x_0)} \|D_x v\| dx + \|V\| dx \right)^{1/q}, \quad (3.10) \]
where \( \tilde{U} = \tilde{A}^{d\delta}(x', x^d)D_{x'}\tilde{u} \). Recalling that \( D_x \tilde{u} = D_x u, U = A^{d\delta}(x)D_{x'}u - g_d(x) \) and \( \tilde{U} = \tilde{A}^{d\delta}(x', x^d)D_{x'}u - \tilde{g}_d(x', x^d) \), we have for \( x \in B_r(x_0) \),
\[ |U - \tilde{U}| \leq \|Du\|_{L^p(B_{r/2}(x_0))} |A(x) - \tilde{A}(x', x^d)| + |g_d(x) - \tilde{g}_d(x', x^d)|. \]
Thus, coming back to (3.10), using (2.9) and (3.7), we have
\[ \left( \int_{B_{r/2}(x_0)} \|D_x u, U\| - ((D_x w)_{B_{r/2}(x_0)}, (W)_{B_{r/2}(x_0)}) \| dx \right)^{1/q} \]
\[ \leq N_0\kappa \left( \int_{B_{r/2}(x_0)} \|D_x u, U\| - q\| dx \right)^{1/q} + N\kappa^{-d/q} \left( \int_{B_{r/2}(x_0)} \|U - \tilde{U}\| dx \right)^{1/q} \]
\[ + N\kappa^{-d/q} \left( \int_{B_{r/2}(x_0)} \|D_x v\| dx + \|V\| dx \right)^{1/q} \]
\[ \leq N_0\kappa \left( \int_{B_{r/2}(x_0)} \|D_x u, U\| - q\| dx \right)^{1/q} + N\kappa^{-d/q} \left( \|Du\|_{L^p(B_{r/2}(x_0))} \right) \]
\[ \cdot \int_{B_{r/2}(x_0)} |A(x) - \tilde{A}(x', x^d)| dx + \int_{B_{r/2}(x_0)} |g_d(x) - \tilde{g}_d(x', x^d)| dx \]
\[ + N\kappa^{-d/q} \left( \int_{B_{r/2}(x_0)} \|D_x v\| dx + \|V\| dx \right)^{1/q} \]
Since $q \in \mathbb{R}^{n,d}$ is arbitrary, we obtain
\[
\phi(x_0, kr) \leq N_0 \kappa \phi(x_0, r) + N \kappa^{-d/q} \|Du\|_{L^\infty(B_{kr}(x_0))} \bar{\varphi}_A(r) + \bar{\varphi}_S(r).
\]
For any given $\gamma \in (0,1)$, fix a $\kappa \in (0,1/2)$ small enough so that $N_0 \kappa \leq \kappa^\gamma$. Therefore, we have
\[
\phi(x_0, kr) \leq \kappa^\gamma \phi(x_0, r) + N \|Du\|_{L^\infty(B_{kr}(x_0))} \bar{\varphi}_A(\kappa^\gamma r) + \bar{\varphi}_S(\kappa^\gamma r).
\]
Using the fact that $\kappa^\gamma < 1$, by iteration, for $j = 1, 2, \ldots$, we obtain
\[
\phi(x_0, \kappa^j r) \leq \kappa^{j\gamma} \phi(x_0, r) + N \left( \|Du\|_{L^\infty(B_{r}(x_0))} \sum_{i=1}^{j} \kappa^{(i-1)\gamma} \bar{\varphi}_A(\kappa^{i-1} r) + \sum_{i=1}^{j} \kappa^{(i-1)\gamma} \bar{\varphi}_S(\kappa^{i-1} r) \right).
\]
Therefore, we get
\[
\phi(x_0, \kappa^j r) \leq \kappa^{j\gamma} \phi(x_0, r) + N \|Du\|_{L^\infty(B_{r}(x_0))} \bar{\varphi}_A(\kappa^j r) + N \bar{\varphi}_S(\kappa^j r), \quad (3.11)
\]
where
\[
\bar{\varphi}_A(t) = \sum_{i=1}^{\infty} \kappa^{i\gamma} \left( \bar{\varphi}_A(\kappa^{-i} t) \chi_{\kappa^{-i} t \leq 1} + \bar{\varphi}_A(1) \chi_{\kappa^{-i} t > 1} \right). \quad (3.12)
\]
Moreover, $\bar{\varphi}_A(t)$ is a Dini function (see Lemma 1 in [3]) and satisfies (2.19).

Now, for any $\rho$ satisfying $0 < \rho < r \leq 1/4$, we take $j$ to be the integer satisfying $\kappa^{j+1} < \rho/r \leq \kappa^j$. Then, by (3.11) and (2.19), we have
\[
\phi(x_0, \rho) \leq N \left( \rho/r \right)^\gamma \phi(x_0, r) + N \bar{\varphi}_A(\rho) \|Du\|_{L^\infty(B_{r}(x_0))} + N \bar{\varphi}_S(\rho). \quad (3.13)
\]
Hence, (3.5) follows from (3.4) and (3.13). □

**Lemma 3.4.** We have
\[
\|Du\|_{L^\infty(B_{r}(x_0))} \leq N \|\sum_{j=1}^{\infty} \bar{\varphi}_A(t) \chi_{\kappa^{-i} t \leq 1} + \bar{\varphi}_S(1) \chi_{\kappa^{-i} t > 1} \|_{L^\infty(B_{r}(x_0))} \quad (3.14)
\]
where $N > 0$ is a universal constant.

**Proof.** Take $q_{x_0, r} \in \mathbb{R}^{n,d}$ to be such that
\[
\phi(x_0, r) = \left( \int_{B_{kr}(x_0)} |(D_x u, U) - q_{x_0, r}|^q \, dx \right)^{1/q}.
\]
Similarly, we find $q_{x_0, kr} \in \mathbb{R}^{n,d}$, et cetera. Notice that
\[
|q_{x_0, kr} - q_{x_0, r}|^q \leq \|D_x u, U\|_{L^q(B_{kr}(x_0))} + \|D_x u, U\|_{L^q(B_{kr}(x_0))} - \|q_{x_0, kr} - q_{x_0, r}|^q.
\]
By taking average over $x \in B_{kr}(x_0)$ and taking the $q$-th root, we obtain
\[
|q_{x_0, kr} - q_{x_0, r}| \leq N (\phi(x_0, kr) + \phi(x_0, r)).
\]
By iteration, we have
\[
|q_{x_0, k^j r} - q_{x_0, r}| \leq N \sum_{j=0}^{K} \phi(x_0, \kappa^j r). \quad (3.15)
\]
Thus, by using the assumption that $u \in C^{0,1}(B_{3/4})$, we obtain for a.e. $x_0 \in B_{3/4}$,
\[
\lim_{K \to \infty} q_{K,\omega_\nu} = (D_x u(x_0), U(x_0)).
\]
On the other hand, recalling that $\tilde{\omega}_A$ and $\tilde{\omega}_F$ satisfy (2.19). Therefore, by taking $K \to \infty$ in (3.15), using (3.11) and Lemma 2.8 for a.e. $x_0 \in B_{3/4}$, we have
\[
|\langle D_x u(x_0), U(x_0) \rangle - q_{x_0,r}| \\
\leq N \sum_{j=0}^{\infty} \phi(x_0, r^j) \\
\leq N \left( \phi(x_0, r) + \|Du\|_{L^\infty(B_r(x_0))} \int_0^r \frac{\omega_A(t)}{t} \, dt + C_\nu \int_0^r \frac{\omega_F(t)}{t} \, dt \right).
\]
(3.16)

By averaging the inequality
\[
|q_{x_0,r}| \leq |(D_x u, U) - q_{x_0,r}| + |(D_x u, U)|
\]
over $x \in B_r(x_0)$ and taking the $q$-th root, we have
\[
|q_{x_0,r}| \leq 2^{1/q-1} \phi(x_0, r) + 2^{1/q-1} \left( \int_{B_r(x_0)} |(D_x u, U)|^q \, dx \right)^{1/q}.
\]
Therefore, combining (3.16) and (3.4), we obtain for a.e. $x_0 \in B_{3/4}$,
\[
\|D_x u(x_0), U(x_0)\| \leq N r^{-d} \|Du\|_{L^\infty(B_r(x_0))} \\
+ N \left( \|Du\|_{L^\infty(B_r(x_0))} \int_0^r \frac{\omega_A(t)}{t} \, dt + C_\nu \int_0^r \frac{\omega_F(t)}{t} \, dt \right).
\]
For any $x_1 \in B_{1/4}$ and $0 < r < 1/4$, we take the supremum over $B_r(x_1)$ and use $A^{dd} \geq \nu$ to obtain
\[
\|Du\|_{L^\infty(B_r(x_1))} \leq N r^{-d} \|Du\|_{L^\infty(B_r(x_1))} + N \left( \int_0^r \frac{\omega_A(t)}{t} \, dt \right) \\
+ \|Du\|_{L^\infty(B_r(x_1))} \int_0^r \frac{\omega_F(t)}{t} \, dt + \|g\|_{L^\infty(\partial D)}.
\]
We fix $r_0 < 1/4$ such that for any $0 < r \leq r_0$, we have
\[
N \int_0^r \frac{\omega_A(t)}{t} \, dt \leq 4^{-d}.
\]
Then, for any $x_1 \in B_{1/4}$ and $0 < r \leq r_0$, we have
\[
\|Du\|_{L^\infty(B_r(x_1))} \leq 4^{-d} \|Du\|_{L^\infty(B_r(x_1))} + N r^{-d} \|Du\|_{L^\infty(B_r(x_1))} \\
+ N \left( \int_0^r \frac{\omega_A(t)}{t} \, dt + \|g\|_{L^\infty(\partial D)} \right).
\]
For $k = 1, 2, \ldots$, denote $r_k = 3/4 - (1/2)^k$. For $x_1 \in B_{r_k}$ and $r = (1/2)^{k+2}$, we have $B_r(x_1) \subseteq B_{r_{k+1}}$. We take $k_0 \geq 1$ large enough such that $(1/2)^{k_0+2} \leq r_0$. It follows that for any $k \geq k_0$,
\[
\|Du\|_{L^\infty(B_r)} \leq 4^{-d} \|Du\|_{L^\infty(B_{r_{k+1}})} + N2^{kd} \|Du\|_{L^\infty(B_{r_{k+1}})} \\
+ N \left( \int_0^r \frac{\omega_A(t)}{t} \, dt + \|g\|_{L^\infty(\partial D)} \right).
\]
\[
\frac{\omega_A(t)}{t} \leq \frac{\omega_A(1)}{2} \\
\frac{\omega_F(t)}{t} \leq \frac{\omega_F(1)}{2}
\]
for $0 < t < 1/4$. Therefore, we have
\[
\|Du\|_{L^\infty(B_r)} \leq 4^{-d} \|Du\|_{L^\infty(B_{r_{k+1}})} + N2^{kd} \|Du\|_{L^\infty(B_{r_{k+1}})} \\
+ N \left( \int_0^r \frac{\omega_A(t)}{t} \, dt + \|g\|_{L^\infty(\partial D)} \right).
\]
By multiplying the above by $4^{-k\delta}$ and summing over $k = k_0, k_0 + 1, \ldots,$ we have

$$\sum_{k=k_0}^{\infty} 4^{-k\delta}||Du||_{L^\infty(B_{r_k})} \leq \sum_{k=k_0}^{\infty} 4^{-(k+1)\delta}||Du||_{L^\infty(B_{r_{k+1}})} + N||\partial_Du, U||_{L^1(B_{r_{k+1}})} + N \left( \int_0^1 \frac{\tilde{\omega}(t)}{t} dt + ||g||_{L^\infty(\Omega)} \right).$$

It follows from $u \in C^{0,1}(B_{3/4})$ that the summations on both sides are convergent. We thus obtain

$$||Du||_{L^\infty(B_{r_1})} \leq N||\partial_Du, U||_{L^1(B_{r_1})} + N \left( \int_0^1 \frac{\tilde{\omega}(t)}{t} dt + ||g||_{L^\infty(\Omega)} \right).$$

The lemma is proved. \hfill \Box

Finally, we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** By (3.16), we have for $0 < r < 1/8$ that

$$\sup_{x_0 \in B_{1/8}} |(D_{x'}u(x_0), U(x_0)) - q_{x_0,r}|$$

$$\leq N \sup_{x_0 \in B_{1/8}} \phi(x_0, r) + N||Du||_{L^\infty(B_{r_1})} \int_0^r \frac{\tilde{\omega}(t)}{t} dt + N \int_0^r \frac{\tilde{\omega}(t)}{t} dt$$

$$=: N\psi(r).$$

We recall that for each $x_0$, the coordinate system and thus $x'$ are chosen according to $x_0$. By Lemma 3.3, for any $0 < r < 1/8$, we obtain

$$\sup_{x_0 \in B_{1/8}} \phi(x_0, r) \leq N \left( r^\gamma ||D_{x'}u, U||_{L^1(B_{r_1})} + \tilde{\omega}(r)||Du||_{L^\infty(B_{r_1})} + \tilde{\omega}(r) \right). \quad (3.17)$$

Suppose that $y \in B_{1/8} \cap D_{j_1}$ for some $j_1 \in [1, l + 1]$. If $|x_0 - y| \geq 1/32$, combining

$$||D_{x'}u(x_0), U(x)) - (D_{x'}u(y), U(y))|| \leq 2(||Du||_{L^\infty(B_{r_1})} + ||g||_{L^\infty(\Omega)})$$

and (3.14), we have

$$||D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y), U(y))||$$

$$\leq N|x_0 - y|^{\gamma} \left( ||(D_{x'}u, U)||_{L^1(B_{r_1})} + \int_0^1 \frac{\tilde{\omega}(t)}{t} dt + ||g||_{L^\infty(\Omega)} \right), \quad (3.18)$$

where $\gamma \in (0, 1)$ is a constant. On the other hand, if $|x_0 - y| < 1/32$, we set $r = |x_0 - y|$ and discuss it further according to the following dichotomy.

**Case 1.** If

$$r \leq 1/16 \max \{\text{dist}(x_0, \partial D_{j_0}), \text{dist}(y, \partial D_{j_1})\},$$

then $j_0 = j_1$. By using the triangle inequality, we have

$$||D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y), U(y))||$$

$$\leq |(D_{x'}u(x_0), U(x_0)) - q_{x_0,r}| + |q_{x_0,r} - q_{y,r}| + |(D_{x'}u(y), U(y)) - q_{y,r}|$$

$$\leq N\psi^\gamma(r) + |(D_{x'}u(z), U(z)) - q_{x_0,r}| + |(D_{x'}u(z), U(z)) - q_{y,r}| + |(D_{x'}u(z), U(z)) - q_{y,r}|$$

$$+ |(D_{x'}u(y), U(y)) - q_{y,r}|, \quad \forall \ z \in B_r(x_0) \cap B_r(y). \quad (3.19)$$
In order to estimate the last two terms, we define

$$
\varphi(y,r) := \inf_{q \in \mathbb{R}^d} \left( \int_{B_r(y)} |Du - q|^q \, dx \right)^{1/q}.
$$

We use $D_y$ to denote the derivative in the coordinate system associated with $y$. Then for any $q = (q', q_d)$, we have

$$
(D_{y'}u - q', \hat{A}^{d}_{d}D_{\beta}u - \hat{g}_d - q_d)
$$

$$
= (D_{y'}u - q', D_{y'}u - (\hat{A}^{dd}_{d})^{-1}(\hat{g}_d + q_d - \sum_{\beta=1}^{d-1} \hat{A}^{d\beta}_{d}q_{\beta}))E,
$$

(3.20)

where $\hat{A}^{d\beta}_{d}$ and $\hat{g}_d$ are constants corresponding to $A^{d\beta}$ and $g_d$, respectively, and $E = (E^{q\beta})$ is defined by

$$
E^{q\beta} = \delta_{d\beta} \text{ for } \alpha, \beta \in \{1, \ldots, d-1\}; \quad E^{d\beta} = 0 \text{ for } \beta \in \{1, \ldots, d-1\};
$$

$$
E^{\alpha d} = \hat{A}^{\alpha d} \text{ for } \alpha \in \{1, \ldots, d\}.
$$

From (3.20), we get

$$
\varphi(y,r) \leq \left( \int_{B_r(y)} \left| (D_{y'}u - q', D_{y'}u - (\hat{A}^{dd}_{d})^{-1}(\hat{g}_d + q_d - \sum_{\beta=1}^{d-1} \hat{A}^{d\beta}_{d}q_{\beta})) \right|^q \, dx \right)^{1/q}
$$

$$
= \left( \int_{B_r(y)} \left| (D_{y'}u - q', \hat{A}^{d}_{d}D_{\beta}u - \hat{g}_d - q_d)E^{-1}|^q \, dx \right|^{1/q}
$$

$$
= \left( \int_{B_r(y)} \left| (D_{y'}u - q', (0', \hat{A}^{d}_{d}D_{\beta}u - A^{d\beta}D_{\beta}u + g_d - \hat{g}_d)E^{-1}|^q \, dx \right|^{1/q}
$$

$$
\leq 2^{1/q-1} \left( \int_{B_r(y)} |(D_{y'}u, U - q)|E^{-1}|^q \, dx \right)^{1/q} + N \left( \int_{B_r(y)} |(0', g_d - \hat{g}_d)E^{-1}|^q \, dx \right)^{1/q}
$$

$$
+ N \left( \int_{B_r(y)} |(0', A^{d\beta}D_{\beta}u - \hat{A}^{d\beta}D_{\beta}u)E^{-1}|^q \, dx \right)^{1/q}
$$

$$
\leq N \left( \varphi(y, r) \right)^{1/q} + N \left( \omega_A(r)||Du||_{L^\infty(B_r(y))} + \omega_g(r) \right).
$$

(3.21)

Since $q$ is arbitrary, we obtain

$$
\varphi(y, r) \leq N \left( \varphi(y, r) + \omega_A(r)||Du||_{L^\infty(B_r(y))} + \omega_g(r) \right).
$$

(3.22)

In the coordinate system associated with $x_0$, we first notice that

$$
(D_{x'}u, D_{x'}u) = (D_{y'}u, D_{y'}u)X,
$$

(3.23)

where $X = (X^{\alpha\beta})$ is a $d \times d$ matrix, and

$$
X^{\alpha\beta} = \frac{\partial y^{\alpha}}{\partial x^{\beta}} \text{ for } \alpha, \beta = 1, \ldots, d.
$$

(3.24)

By using (3.23), we obtain

$$
\varphi(y, r) \leq \left( \int_{B_r(y)} \left| Du - qX \right|^q \, dx \right)^{1/q}
$$

$$
\leq \left( \int_{B_r(y)} \left| Du - qD_{y'}u \right|^q \, dx \right)^{1/q}.
$$

(3.25)
\[ q \]

Then, by using (3.21), (3.22), and the fact that \( q \) is arbitrary, we have \( \phi(y, r) \leq N \left( \phi(y, r) + \omega_A(r) \| Du \|_{L^\infty(B_{r/4})} + \omega_\gamma(r) \right) \) in the coordinate system associated with \( x_0 \). We thus have proved that the upper bound of \( \phi(y, r) \) is independent of coordinate systems. Now, we denote

\[ \phi_{x_0}(y, r) := \inf_{q \in \mathbb{R}^{n-m}} \left( \int_{B_r(y)} |(D_xu, U) - q| q^{1/q} \, dx \right) \]

Then,

\[
\phi_{x_0}(y, r) \leq \left( \int_{B_r(y)} |(D_xu - q', U - (\hat{A}^{\delta q}_\beta - \hat{g}_\delta))| q^{1/q} \, dx \right)^{1/q} = \left( \int_{B_r(y)} |(D_xu - q', \hat{A}^{\delta q}_\beta D_\beta u - \hat{A}^{\delta q}_\beta) + (0', U - \hat{A}^{\delta q}_\beta D_\beta u + \hat{g}_\delta)| q^{1/q} \, dx \right)^{1/q}
\]

By (3.25), we have

\[ \phi_{x_0}(y, r) \leq N \left( \phi(y, r) + \omega_A(r) \| Du \|_{L^\infty(B_{r/4})} + \omega_\gamma(r) \right) \]

Therefore, by using a similar argument that led to (3.14), we get

\[
| (D_xu(x_0), U(x_0)) - (D_xu(y), U(y)) | \\
\leq N \left( \phi(y, r) + \| Du \|_{L^\infty(B_{r/4})} \int_0^r \frac{\tilde{\omega}_A(t)}{t} \, dt + \int_0^r \frac{\tilde{\omega}_\gamma(t)}{t} \, dt \right).
\]

Now, coming back to (3.15), taking the average over \( z \in B_r(x_0) \cap B_r(y) \), and then taking the \( q \)-th root, we get

\[
| (D_xu(x_0), U(x_0)) - (D_xu(y), U(y)) | \\
\leq N \left( \psi(r) + \phi(x_0, r) + \phi(y, r) \right) \leq N \psi(r).
\]

Therefore, it follows from (3.14) and (3.17) that

\[
| (D_xu(x_0), U(x_0)) - (D_xu(y), U(y)) | \\
\leq N| x_0 - y |^q \| (D_xu, U) \|_{L^q(B_{r/4})} + N \int_0^{|x_0 - y |^{q-1} \frac{\tilde{\omega}_A(t)}{t} \, dt} \\
+ N \int_0^{|x_0 - y |^{q-1} \frac{\tilde{\omega}_\gamma(t)}{t} \, dt} \left( \| (D_xu, U) \|_{L^1(B_{r/4})} + \int_0^1 \frac{\tilde{\omega}_\gamma(t)}{t} \, dt + \| g \|_{L^1(B_{r/4})} \right). \tag{3.26}
\]

**Case 2.** If \( r > 1/16 \max \{ \text{dist}(x_0, \partial \Omega_{B_{r/4}}), \text{dist}(y, \partial \Omega_{B_{r/4}}) \} \), then

\[
| (D_xu(x_0), U(x_0)) - (D_xu(y), U(y)) |^q \\
\leq | (D_xu(x_0), U(x_0)) - q_{x_0, r} |^q + | q_{y_0, r} - q_{y, r} |^q + | (D_yu(y), U(y)) - q_{y, r} |^q \\
+ | (D_xu(y), U(y)) - (D_xu(y), U(y)) |^q \\
\leq N \psi(r) + | (D_xu(z), U(z)) - q_{x_0, r} |^q + | (D_yu(z), U(z)) - q_{y, r} |^q
\]
the Gram-Schmidt process works as follows:

For the last term, on one hand,

$$D_x u(y) - D_y u(y) = (D_x u(y), D_x u(y))(I - X^{-1})l_0,$$

where $l_0 = (l^{\alpha\beta})$ is a $d \times (d - 1)$ matrix with

$$l^{\alpha\beta} = \delta_{\alpha\beta} \text{ for } \alpha, \beta \in \{1, \ldots, d - 1\}; \quad l^{\alpha\beta} = 0 \text{ for } \beta \in \{1, \ldots, d - 1\},$$

$X$ is defined by $(3.11)$, and $I$ is a $d \times d$ identity matrix. On the other hand, we suppose that the closest point on $\partial D_h$ to $y$ is $(y', h_i(y'))$. Then we have

$$\langle \partial |_{\partial D_h} u(y), u(y) \rangle = (D_x u(y), D_x u(y))(I - X^{-1})l_0,$$

$$|D_x u(y) - D_y u(y)| \leq N \|D u\|_{L^\infty(B_{r_0})} \alpha_1(|x_0 - y|).$$
Similarly, we can estimate the difference of $U$ in different coordinate systems. Therefore, we obtain
\[
|\langle D_x u(y), U(y) \rangle - \langle D_x u(y), U(y) \rangle| \leq N\|Du\|_{L^\infty(B_{1/4})} \omega_1(|x_0 - y|).
\] (3.28)

We remark that the penultimate term of (3.27) also satisfies (3.28). Coming back to (3.27), we take the average over $z \in B_r(x_0) \cap B_r(y)$ and take the $q$-th root to get
\[
|\langle D_x u(x_0), U(x_0) \rangle - \langle D_x u(y), U(y) \rangle| \\
\leq N\left(\psi(r) + \phi(x_0, r) + \phi(y, r) + \|Du\|_{L^\infty(B_{1/4})} \omega_1(|x_0 - y|)\right) \\
\leq N\left(\psi(r) + \|Du\|_{L^\infty(B_{1/4})} \omega_1(|x_0 - y|)\right).
\]

Therefore, it follows from (3.14) and (3.17) that
\[
|\langle D_x u(x_0), U(x_0) \rangle - \langle D_x u(y), U(y) \rangle| \\
\leq N|x_0 - y|^r \|\langle D_x u, U \rangle\|_{L^1(B_{1/4})} + N \int_0^{x_0 - y} \frac{A(r)(t)}{t} \frac{\langle D_x u \rangle(t, B_{1/4}) \omega_1(|x_0 - y|)}{t} \\
+ N \int_0^{x_0 - y} \frac{A(r)(t)}{t} \frac{\langle D_x u \rangle(t, B_{1/4}) \omega_1(|x_0 - y|)}{t} \frac{dt + \|\gamma\|_{L^\infty(\Omega)}}{t}.
\] (3.29)

Thus, Proposition 3.1 is proved under the assumption that $u \in C^{0,1}(B_{3/4})$.

We now show that $u \in C^{0,1}(B_{3/4})$ by using the technology of locally flattening the boundaries and an approximation argument. By the interior regularity obtained in [12], it suffices to show that for any $x_0 \in \partial D_j, j = 1, \ldots, M - 1$, there is a neighborhood of $x_0$ in which $u$ is Lipschitz. Recall that $x_0$ belongs to the boundaries of at most two of the subdomains. Thus, we can find a small $r_0 > 0$ and a $C^{1, \text{Dini}}$ diffeomorphism to flatten the boundary $\partial D_j \cap B_{r_0}(x_0)$:
\[
y = \Phi(x) = (\Phi^1(x), \ldots, \Phi^d(x)),
\]
which satisfies $\Phi(x_0) = 0$, $\det D\Phi = 1$, and
\[
\Phi(\partial D_j \cap B_{r_0}(x_0)) = \Phi(B_{r_0}(x_0)) \cap \{y^d = 0\}.
\]

Then $\hat{u}(y) := u(x)$ satisfies
\[
D_u(\hat{A}_{d}^{\alpha\beta} D_{\beta} \hat{u}) = \text{div} \hat{g},
\]
where $\hat{A}^{d\alpha}(y) = D_\alpha \Phi^\alpha D_\beta \Phi^\beta A^{d\beta}(x)$ and $\hat{g}(y) = D\Phi^\top g(x)$. Note that the coefficients $\hat{A}^{d\alpha}$ and $\hat{g}$ are also of piecewise Dini mean oscillation in $\Phi(B_{r_0}(x_0))$. To show that $u$ is Lipschitz near $x_0$, we only need to show that $\hat{u}$ is Lipschitz near 0. Now we take the standard mollification of the coefficients and data in the $y^d$ direction with a parameter $\varepsilon > 0$, apply the result in [8] as well as the a priori Lipschitz estimate in Lemma 3.4 to get a uniform Lipschitz estimate independent of $\varepsilon$, and finally take the limit as $\varepsilon \searrow 0$ by following the proof of [8, Theorem 1]. Theorem 3.1 is proved.

3.2. Proof of Corollary 1.2 Similar to the proof of Theorem 1.1, we take $x_0 \in B_{3/4} \cap \partial D_j$. Let $A^{(i)} \in C^1(\Omega_j), 1 \leq j \leq l + 1$, be matrix-valued functions, $B^{(i)}, g^{(i)}$ be in $C^1(\Omega_j)$. Define the piecewise constant (matrix-valued) functions
\[
\hat{A}(x) = A^{(i)}(x', h_j(x')), \quad x \in \Omega_j.
\]

From $B^{(i)}$ and $g^{(i)}$, we similarly define piecewise constant functions $\hat{B}$ and $\hat{g}$. Using Remark 2.4, we get the following result, which is similar to Lemma 5.2 in [13].
Lemma 3.5. Let $1 < \delta' = \min\{\delta, \frac{\mu}{1 + \mu}\}$.

With $A, \bar{A}, B, g, \text{and } \bar{g}$ be defined as above, there exists a positive constant $N$, depending only on $d, l, \mu, \nu, \lambda, \max_{1 \leq i \leq l} \|A_i\|_{C^0(\Omega)}, \max_{1 \leq i \leq l} \|B_i\|_{C^0(\Omega)}$, and $\max_{1 \leq i \leq l} \|h_i\|_{C^0(\Omega)}$, such that for $0 < r \leq 1$,

$$\int_{B_r(x_0)} |A - \bar{A}| \, dx + \int_{B_r(x_0)} |B - \bar{B}| \, dx + \int_{B_r(x_0)} |g - \bar{g}| \, dx \leq N r^{\delta'}.$$

Thus, Corollary 1.2 directly follows from (3.26), (3.29), and (3.18) by taking $\gamma \in (\delta', 1)$.

4. Proof of Corollary 1.2

We shall make use of the idea in [1, 4], where the $W^{1, p}_{\text{loc}}$-regularity was proved for $W^{1, 1}$ weak solutions to divergence form elliptic equations with Dini continuous coefficients by using a duality argument, $L^p$-regularity property, and bootstrap arguments. In our case, we will use the $W^{1, p}$ estimate in Lemma 2.5 and the interior $W^{1, \infty}$-regularity obtained in Theorem 1.1.

Proof of Corollary 1.2. By the Sobolev embedding theorem, we have $u \in L^{\frac{n}{n-1}}(\Omega)$. Thus, we only need to prove that $Du \in L^p_{\text{loc}}(\Omega)$ for some $p \in (1, \frac{n}{n-1})$. We fix some $1 < p < \frac{n}{n-1}$ so that $2 \leq d < p' < \infty$ with $1/p + 1/p' = 1$. We rewrite (4.1) as

$$\mathcal{L}' u := D_\alpha (A^{ij} D_\beta u) + D_\alpha (B^i u) + B^i u D_\alpha u + (C - \lambda_0) u = \text{div } g + f - \lambda_0 u,$$

where $\lambda_0$ is a fixed large enough number. Denote $f_0 := f - \lambda_0 u \in L^{\frac{n}{n-1}}(\Omega)$. Let $h \in C^0_\infty(\Omega)$ be given, and $v \in H^1_0(\Omega)$ be the solution of

$$\mathcal{L}'' v := D_\beta (A^{ij} D_\alpha v) - D_\beta (B^i v) - B^i D_\alpha v + (C - \lambda_0) v = \text{div } h,$$

where $\mathcal{L}''$ is the adjoint operator of $\mathcal{L}'$. Then, by Theorem 1.3, we obtain $Dv \in L^{\infty}(\Omega)$. By using the definition of weak solutions, uniform ellipticity condition, Hölder’s inequality, $p' > 2$, and the fact that $\lambda_0$ is a large enough number, we get

$$\|v\|_{W^{1, p'}(\Omega)} \leq N \|h\|_{L^2(\Omega)} \leq N \|h\|_{L^{p'}(\Omega)}. \tag{4.1}$$

By Lemma 2.3 and (4.1), we obtain

$$\|v\|_{W^{1, p'}(\Omega)} \leq N \left( \|h\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \right) \leq N \|h\|_{L^{p'}(\Omega)}. \tag{4.2}$$

Since $p' > d$, it follows from Morrey’s inequality that

$$\|v\|_{L^{\infty}(\Omega)} \leq N \|h\|_{L^{p'}(\Omega)}. \tag{4.3}$$

By a density argument, we have for any $\varphi \in W^{1, 1}_0(\Omega)$,

$$\int_{\Omega} A^{ij} D_\alpha v D_\beta \varphi + B^i D_\alpha \varphi - B^i v D_\alpha \varphi + (\lambda_0 - C) v \varphi = \int_{\Omega} h_\alpha D_\alpha \varphi. \tag{4.4}$$

Fix $\zeta \in C^\infty_c(\Omega)$ with $\zeta \equiv 1$ on $\Omega' \subset \subset \Omega$, and we choose $\varphi = \zeta u \in W^{1, 1}_0(\Omega)$ in (4.4). Then

$$\int_{\Omega} A^{ij} D_\alpha v (\zeta D_\beta u + u D_\beta \zeta) + B^i D_\alpha v u \zeta - B^i v (\zeta D_\alpha u + u D_\alpha \zeta) + (\lambda_0 - C) u v \zeta$$
Thus, by Hölder’s inequality, the Sobolev embedding theorem, and (4.1), then by a density argument, for any \( \psi \in W^{1,\infty}_0(D) \), we have

\[
\int_D A^{\alpha \beta} \partial_\alpha u \partial_\beta \psi + B^{\alpha \beta} \partial_\alpha u \partial_\beta \psi + (\lambda_0 - C) u \psi = \int_D g_a \partial_\alpha \psi - f_0 \psi. \tag{4.6}
\]

By taking \( \psi = \zeta v \) in (4.6), we get

\[
\int_D A^{\alpha \beta} \partial_\alpha u (\zeta D_\alpha v + v D_\alpha \zeta) + B^{\alpha \beta} u (\zeta D_\alpha v + v D_\alpha \zeta) - B^{\alpha \beta} u v \zeta + (\lambda_0 - C) u v \zeta,
\]

\[
= \int_D g_a (\zeta D_\alpha v + v D_\alpha \zeta) - f_0 \zeta v. \tag{4.7}
\]

Combining (4.5) and (4.7), we find

\[
\int_D \partial_\alpha D_\alpha u = \int_D -A^{\alpha \beta} \partial_\alpha D_\beta u + \int_D A^{\alpha \beta} u \partial_\alpha v \partial_\beta \zeta - u v B^{\alpha \beta} D_\alpha \zeta - B^{\alpha \beta} u v D_\alpha \zeta - \int_D u h_a \partial_\alpha \zeta + \int_D g_a (\zeta D_\alpha v + v D_\alpha \zeta) - f_0 \zeta v. \tag{4.8}
\]

Thus, by Hölder’s inequality, the Sobolev embedding theorem, \( d < p' \), (4.2), and (4.3), we estimate each term on the right-hand side of (4.8) by

\[
\left| \int_D A^{\alpha \beta} \partial_\alpha D_\beta u \right| \leq N \| u \|_{W^{1,1}(\Omega)} \| v \|_{L^\infty(\Omega)} \leq N \| u \|_{W^{1,1}(\Omega)} \| h \|_{L^p(\Omega)},
\]

\[
\left| \int_D A^{\alpha \beta} u \partial_\alpha v \partial_\beta \zeta - u v B^{\alpha \beta} D_\alpha \zeta - B^{\alpha \beta} u v D_\alpha \zeta \right|
\]

\[
\leq N \| u \|_{L^{p'}(\Omega)} \| v \|_{W^{1,1}(\Omega)} \| h \|_{L^{p'}(\Omega)},
\]

\[
\left| \int_D u h_a \partial_\alpha \zeta \right| \leq N \| u \|_{L^{p'}(\Omega)} \| h \|_{L^p(\Omega)} \leq N \| u \|_{W^{1,1}(\Omega)} \| h \|_{L^{p'}(\Omega)}, \tag{4.9}
\]

and

\[
\left| \int_D g_a (\zeta D_\alpha v + v D_\alpha \zeta) - f_0 \zeta v \right|
\]

\[
\leq N \| g \|_{L^{p'}(\Omega)} \| h \|_{L^p(\Omega)} + N \| f \|_{L^{p'}(\Omega)} \| h \|_{L^p(\Omega)} + N \| f \|_{L^{p'}(\Omega)} \| h \|_{L^p(\Omega)},
\]

\[
\leq N \left( \| g \|_{L^{p'}(\Omega)} + \| f \|_{L^{p'}(\Omega)} + \| u \|_{W^{1,1}(\Omega)} \right) \| h \|_{L^p(\Omega)},
\]

\[
\leq N \left( \| g \|_{L^{p'}(\Omega)} + \| f \|_{L^{p'}(\Omega)} + \| u \|_{W^{1,1}(\Omega)} \right) \| h \|_{L^p(\Omega)}. \]

Thus, we get

\[
\left| \int_D \partial_\alpha D_\alpha u \right| \leq N \left( \| g \|_{L^{p'}(\Omega)} + \| f \|_{L^{p'}(\Omega)} + \| u \|_{W^{1,1}(\Omega)} \right) \| h \|_{L^p(\Omega)}.
\]

It follows from (4.9) that

\[
\left| \int_D h_a \partial_\alpha (u \zeta) \right| \leq N \left( \| g \|_{L^{p'}(\Omega)} + \| f \|_{L^{p'}(\Omega)} + \| u \|_{W^{1,1}(\Omega)} \right) \| h \|_{L^p(\Omega)}.
\]
that the sub-domains, lower-order terms, which is the second purpose of this paper. For that, we assume L for all □ Corollary 1.6 is proved.

Moreover, the linear operator T

We also use the following notation:

The A1 constant [w]A1 of w is defined as the infimum of all such C’s. Moreover, we use the following weighted Sobolev spaces:

We also use the following notation:

Assumption 5.1. (1) A is of piecewise Dini mean oscillation in D, and there exists some constant c0 > 0 such that for any r ∈ (0, 1/2), wA(r) ≤ c0(\ln r)^{-2}.

(2) For some constant c1, c2 > 0, wA(R^-0) ≥ c1 and for any R ∈ (0, R0/2), wA(R) ≤ c2(\ln R)^{-2}.

Theorem 5.2. Let D have a C1,\text{Dini} boundary, p ∈ (1, ∞), w be an A1 Muckenhoupt weight, and Assumption 5.1 be satisfied. For f ∈ L^p_w(D), let u ∈ W^{1,p}_w(D) be a weak solution to

\begin{align*}
Lu &= \text{div } f \quad \text{in } D, \\
u &= 0 \quad \text{on } \partial D.
\end{align*}

Then for any t > 0, we have

\[w\left(\{x ∈ D : |Du(x)| > t\}\right) \leq \frac{N}{t} ||f||_{L^1_w(D)},\]

where N depends on n, d, M, wA, ν, Λ, ε, δ0, [w]A1, the C1,\text{Dini} characteristics of D and D. Moreover, the linear operator T : f ↦ Du can be extended to a bounded operator from L^1_w(D) to weak-L^1_w(D).

We shall use a generalized version of Lemma 2.3 (see the Appendix) since our argument and estimates depend on the coordinate system, as well as the following lemma.
Lemma 5.3 (Lemma 3.4 of [12]). Let \( \omega \) be a nonnegative increasing function such that \( \omega(t) \leq (\ln \frac{1}{r})^{-2} \) for \( 0 < t \leq 1 \), and \( \tilde{\omega} \) be given as in (3.12) with \( \omega \) in place of \( \tilde{\omega} \). Then for any \( r \in (0, 1] \), we have

\[
\int_0^r \frac{\tilde{\omega}(t)}{t} \, dt \leq N \left( \ln \frac{1}{r} \right)^{-1},
\]

where \( N > 0 \) is some positive constant.

Now, we can prove Theorem 5.2.

Proof of Theorem 5.2. To begin with, we note that by using (2.5) in the proof of Lemma 2.3 and Assumption 5.1 (2), we have

\[
\frac{3r}{2} = \int_0^r s \omega'(s + 2r) \, ds \geq \frac{R^2}{2} \omega'((R + 2r)^+) \geq \frac{c_1}{2} R^2.
\]

Then by Assumption 5.3 (2), we obtain for any \( r \in (0, r_0/2) \),

\[
\omega_1(r) = \omega_0(2r + R) \leq 2 \omega_0(R) \leq 2c_2 (\ln R)^{-2} \leq c (\ln r)^{-2}
\]

for some constant \( c > 0 \). We thus conclude that Lemma 5.3 is available in our case by combining Assumption 5.3 (1) and 5.4.

The assumption on \( \partial \mathcal{D} \in C^{1, \text{Dini}} \) implies that (5.4) holds true. Also, the coefficients \( A^{ij} \) satisfy (2.10) in the interior of \( \mathcal{D} \) and near the boundary, \( A^{ij} \) satisfy (5.5). By Lemma 5.6, the \( W_1^{1,p} \)-solvability and estimates for divergence form elliptic systems with \( A_i \) weights are available. Hence, the map \( T : f \mapsto Du \) is a bounded linear map on \( L_0^p(\mathcal{D}) \). Let \( \{Q^k_i\} \) be a collection of dyadic “cubes” as in the proof of [8, Lemma 4.1]. By Remark 5.3, we can assume that each \( Q^k_i \) is small enough so that they do not intersect with \( \bigcup_{j=1}^{M-1} \partial \mathcal{D}_j \) and \( \partial \mathcal{D} \) at the same time. Moreover, for a fixed \( x_k \in Q^k_i \), we associate \( Q^k_i \) with a Euclidean ball \( B_k = B_{r_k}(x_k) \) such that \( x_k \in Q^k_i \subset B_k \), where \( r_k = \text{diam} \ Q^k_i \leq \frac{r}{2} \). Suppose for some \( Q^k_i \) and \( t > 0 \),

\[
t < \frac{1}{w(Q^k_i)} \int_{Q^k_i} |f| w \, dx \leq C_2 t.
\]

Then \( f \) admits a decomposition in a given \( Q^k_i \) according to the following dichotomy.

(i) If \( \text{dist}(x_k, \partial \mathcal{D}) \leq \frac{r_0}{2} \), then \( B_k \) does not intersect with sub-domains \( \mathcal{D}_j \), \( j = 1, \ldots, M-1 \). In this case, we choose the coordinate system according to \( y_k \in \partial \mathcal{D} \), which satisfies \( |x_k - y_k| = \text{dist}(x_k, \partial \mathcal{D}) \). Let

\[
g := \int_{Q^k_i} f \, dx, \quad b = f - g \quad \text{in} \ Q^k_i.
\]

Then

\[
\int_{Q^k_i} b \, dx = 0,
\]

and

\[
|g| \leq \int_{Q^k_i} |f| \, dx \leq \frac{1}{|Q^k_i|} \inf_{Q^k_i} w \int_{Q^k_i} |f| w \, dx \leq \frac{1}{w(Q^k_i)} \int_{Q^k_i} |f| w \, dx \leq C_2 t,
\]

where we used the definition of \( w \) and (5.2). Hence,

\[
\int_{Q^k_i} |g|^p w \, dx \leq C_2 t^p w(Q^k_i).
\]
Let \( u_1 \in W^{1,p}_w(D) \) be the unique weak solution of 
\[
\begin{cases}
L u_1 = \text{div} \ b & \text{in } D, \\
u_1 = 0 & \text{on } \partial D.
\end{cases}
\] 

Set \( c = \frac{4R_0}{\theta_0} \) with \( R_0 = \text{diam } D \). Then for any \( R \geq c \gamma_k \) such that \( D \setminus B_R(x_k) \neq \emptyset \) and \( h \in C^\infty_c(D_2R(x_k) \setminus B_R(x_k)) \), let \( p' = p/(p-1) \), \( L' \) be the adjoint operator of \( L \), and \( u_2 \in W^{1,p'}_w(D) \) be a weak solution of 
\[
\begin{cases}
L' u_2 = \text{div} \ h & \text{in } D, \\
u_2 = 0 & \text{on } \partial D,
\end{cases}
\] 

which satisfies 
\[
\left( \int_D |Du_2|^{p'} w^{-\frac{1}{p'-1}} dx \right)^{\frac{1}{p'}} \leq N \left( \int_D |h|^{p'} w^{-\frac{1}{p'-1}} dx \right)^{\frac{1}{p'}} = N \left( \int_{D_2R(x_k) \setminus B_R(x_k)} |h|^{p'} w^{-\frac{1}{p'-1}} dx \right)^{\frac{1}{p'}}. 
\] (5.3)

See Lemma 5.6. Then we can use the definition of adjoint solutions, the fact that \( b \) is supported in \( Q_0^k \) with mean zero, and \( h \in C^\infty_c(D_2R(x_k) \setminus B_R(x_k)) \) to obtain 
\[
\int_{D_2R(x_k) \setminus B_R(x_k)} Du_1 \cdot h = \int_{Q_0^k} Du_2 \cdot b = \int_{Q_0^k} (Du_2 - Du_2(x_k)) \cdot b. 
\] (5.4)

Since \( R \leq R_0 \), \( B_{\delta_k}(x_k) \) does not intersect with sub-domains \( D_j, j = 1, \ldots, M-1 \). Because \( L'u_2 = 0 \) in \( D_R(x_k) \), by flattening the boundary and using a similar argument that led to an a priori estimate of the modulus of continuity of \( Du_2 \) in the proof of Theorem 1.3 in [9], we have 
\[
|Du_2(x) - Du_2(x_k)| \leq N \left( \frac{|x - x_k|}{R} \right)^\gamma + \omega_A^\gamma(|x - x_k|) R^{-d} \|Du_2\|_{L^1(D_2R(x_k))} 
\] (5.5)

for any \( x \in Q_0^k \subset D_2R(x_k) \), where \( \gamma \in (0, 1) \) is a constant and \( \omega_A^\gamma(t) \) is defined as in [9, (2.34)], which is derived from \( \omega_A(t) \). Then, coming back to (5.4), using Lemma 5.3, Hölder’s inequality, and (5.3), we obtain 
\[
\frac{1}{\inf_{D_2R(x_k)} w} \|b\|_{L^1(Q_0^k)} \|Du_2 - Du_2(x_k)\|_{L^\infty(Q_0^k)} 
\] 
\[
\leq \frac{NR^{-d}}{\inf_{D_2R(x_k)} w} \|b\|_{L^1(Q_0^k)} \|Du_2\|_{L^1(D_2R(x_k))} \left( \frac{r_k^\gamma R^{-\gamma}}{R} + (\ln \frac{4}{r_k})^{-1} \right)
\] 
\[
\leq \frac{N}{\inf_{D_2R(x_k)} w} \|b\|_{L^1(Q_0^k)} \left( \int_D |Du_2|^{p'} w^{-\frac{1}{p'-1}} dx \right)^{\frac{1}{p'}} \left( \int_{D_2R(x_k)} w dx \right)^{\frac{1}{p'}} \left( \frac{r_k^\gamma R^{-\gamma}}{R} + (\ln \frac{4}{r_k})^{-1} \right)
\] 
\[
\leq N \left( \int_{D_2R(x_k)} w dx \right)^{\frac{1}{p'}} \|b\|_{L^1(Q_0^k)} \left( \int_{D_2R(x_k) \setminus B_R(x_k)} |h|^{p'} w^{-\frac{1}{p'-1}} dx \right)^{\frac{1}{p'}} \left( \frac{r_k^\gamma R^{-\gamma}}{R} + (\ln \frac{4}{r_k})^{-1} \right).
\]

By the duality, we have 
\[
\|Du_1\|_{L^\infty(D_2R(x_k) \setminus B_R(x_k))} \leq N \left( \int_{D_2R(x_k)} w dx \right)^{\frac{1}{p'}} \|b\|_{L^1(Q_0^k)} \left( \frac{r_k^\gamma R^{-\gamma}}{R} + (\ln \frac{4}{r_k})^{-1} \right).
\]
Therefore, by Hölder’s inequality, we obtain
\[ ||Du_1||_{L^p_{(D_{2\lambda}(x_k),\partial \Omega))} \leq N||b||_{L^1_{(Q^k_{a})}} \left( r_k^\gamma R^{-\gamma} + \left( \ln \frac{4}{r_k} \right)^{-1} \right). \]

The rest of proof is similar to that of Lemma 3.2. Hence, we obtain
\[ \int_{\partial\Omega(B_{\lambda}(x_k))} |Du_1|w \, dx \leq N \int_{Q^k_{a}} |b|w \, dx \leq N \int_{Q^k_{a}} |f|w \, dx + N \int_{Q^k_{a}} |g|w \, dx \leq Ntw(Q^k_{a}). \]

That is,
\[ \int_{\partial\Omega(B_{\lambda}(x_k))} |Tb_{\chi_{Q^k_{a}}}|w \, dx \leq Ntw(Q^k_{a}). \]

(ii) If dist(x_k, \partial \Omega) \geq \frac{\lambda}{2}, then B_{\lambda} does not intersect with \partial \Omega. In this case, we choose the coordinate system according to x_k. In a given Q^k_{a}, we set
\[ g(x) = E(x) \int_{Q^k_{a}} E^{-1}(y) f(y) \, dy, \quad b = f - g, \]
where E = (E^{\alpha\beta}) is a d \times d matrix with
\[ E^{\alpha\beta} = \delta_{\alpha\beta} \text{ for } 1 \leq \alpha \leq d, 1 \leq \beta \leq d - 1, \quad E^{\alpha d} = A^{\alpha d} \text{ for } 1 \leq \alpha \leq d. \]

By using the boundedness of A, we have
\[ \int_{Q^k_{a}} |g|^p \, w \, dx \leq N^p w(Q^k_{a}). \]

Let \( \tilde{b} = E^{-1}b \), which has mean zero in Q^k_{a}. We now follow the argument as in (i) and get
\[ \int_{\partial\Omega(B_{\lambda}(x_k))} Du_1 \cdot h = \int_{Q^k_{a}} Du_2 \cdot b \leq \int_{Q^k_{a}} \left( D_{x'} u_2, U_2 \right) \cdot \tilde{b} \]
\[ = \int_{Q^k_{a}} \left( D_{x'} u_2 - D_{x'} u_2(x_k), U_2 - U_2(x_k) \right) \cdot \tilde{b}, \quad (5.6) \]
where U_2 = A^{df} D_{x'} u_2. Recalling that cr_k \leq R \leq R_0, B_{r_{\gamma\delta}}(x_k) does not intersect with \partial \Omega. By a similar argument that led to (3.29) (or (3.26), (3.18)), for any x \in Q^k_{a} \subset B_{r_{\gamma\delta}}(x_k), we have
\[ |(D_{x'} u_2(x), U_2(x)) - (D_{x'} u_2(x_k), U_2(x_k))| \]
\[ \leq N \left( \frac{|x - x_k|}{R} \right)^\gamma + \int_0^{\frac{|x - x_k|}{R}} \frac{\omega(A(t))}{t} \, dt + \omega_1 \left( \frac{|x - x_k|}{R} \right)^{d - 1} ||(D_{x'} u_2, U_2)||_{L^1(B_{r_{\gamma\delta}}(x_k))} \]
\[ \leq N \left( \frac{|x - x_k|}{R} \right)^\gamma + \int_0^{\frac{|x - x_k|}{R}} \frac{\omega(A(t))}{t} \, dt + \omega_1 \left( \frac{|x - x_k|}{R} \right)^{d - 1} ||D u_2||_{L^1(B_{r_{\gamma\delta}}(x_k))}. \]

Thus, coming back to (5.6) and using the similar argument as in the case (i), we have
\[ \int_{\partial\Omega(B_{\lambda}(x_k))} |Du_1|w \, dx \leq N \int_{Q^k_{a}} |\tilde{b}|w \, dx \leq N \int_{Q^k_{a}} |b|w \, dx \]
\[ \leq N \int_{Q^k_{a}} |f|w \, dx + N \int_{Q^k_{a}} |g|w \, dx \leq Ntw(Q^k_{a}). \]
Therefore, $T$ satisfies the hypothesis of Lemma 6.3, and for any $t > 0$,

$$w(\{x \in \mathcal{D} : |Du(x)| > t\}) \leq \frac{N}{t} \|f\|_{L^1_\mathcal{D}}.$$

The theorem is proved. □

6. Appendix

In the appendix, we give generalizations of Lemmas 2.9 and 2.6. We first recall the definition of the doubling measure $w$: a nontrivial measure on a metric space $X$ is said to be doubling if the measure of any ball is finite and approximately the measure of its double, more precise, if there is a constant $C > 0$ such that

$$0 < w(B_2(x)) \leq Cw(B_r(x)) < \infty$$

for all $x \in X$ and $r > 0$. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$ satisfying the condition (2.20). We note that $\mathcal{D}$ equipped with the standard Euclidean metric and the doubling measure $w$ (restricted to $\mathcal{D}$) is a space of homogeneous type. By [7, Theorem 11], there exists a collection of “cubes”

$$\{Q^k \subset \mathcal{D} : k \in \mathbb{Z}, \alpha \in I_k\},$$

with $l_k$ at most countable set and constants $\delta \in (0, 1), a_0 > 0$, and $C_1 < \infty$ such that

i) $\mathcal{D} \setminus \bigcup_k Q^k = 0 \; \forall k.$

ii) If $\ell \geq k$ then either $Q^\ell_\mathcal{D} \subset Q^k_\mathcal{D}$ or $Q^\ell_\mathcal{D} \cap Q^k_\mathcal{D} = \emptyset$.

iii) For each $(k, \alpha)$ and each $\ell < k$ there is a unique $\beta$ such that $Q^\ell_\mathcal{D} \subset Q^\beta_\mathcal{D}$.

iv) $\text{diam } Q^k_\mathcal{D} \leq C_1 \delta^k$.

v) Each $Q^k_\mathcal{D}$ contains some “ball” $B_{a_0d^k}(z^k_\alpha) \cap \mathcal{D}$.

From the above and the doubling property of the measure $w$, we can infer that there is a constant $C_2 \geq 1$ such that if $Q^{k-1}_\mathcal{D}$ is the parent of $Q^k_\mathcal{D}$, then we have

$$w(Q^{k-1}_\mathcal{D}) \leq C_2 w(Q^k_\mathcal{D}).$$

Let $p, c \in (1, \infty)$.

Assumption 6.1. i) $T$ is a bounded linear operator on $L^p_w(\mathcal{D})$.

ii) If for some $f \in L^p_w(\mathcal{D})$, $t > 0$, and some cube $Q^k_\mathcal{D}$ we have

$$t < \frac{1}{w(Q^k_\mathcal{D})} \int_{Q^k_\mathcal{D}} |f| w \, dx \leq C_2 t,$$

then $f$ admits a decomposition $f = g + b$ in $Q^k_\mathcal{D}$, where $g$ and $b$ satisfy

$$\int_{Q^k_\mathcal{D}} |g|^p_w \, dx \leq C_1 t^p w(Q^k_\mathcal{D}), \quad \int_{\mathcal{D} \setminus B_{r}(x_0)} |T(b \chi_{Q^k_\mathcal{D}})| w \, dx \leq C_1 t w(Q^k_\mathcal{D})$$

with $x_0 \in Q^k_\mathcal{D}$ and $r = \text{diam } Q^k_\mathcal{D}$.

Remark 6.2. By taking a sufficiently large $c$, Assumption 6.1 satisfies automatically for large cubes (i.e., small $k$). In fact, when $ca_0d^k \geq \text{diam } \mathcal{D}$, we can just take $g = 0$ and (6.1) holds because $\mathcal{D} \setminus B_{cr}(x_0) = \emptyset$. 
Lemma 6.3. Under Assumption 6.1, for any \( f \in L^p_w(\mathcal{D}) \) and \( t > 0 \), we have

\[
w(\{x \in \mathcal{D} : |Tf(x)| > t\}) \leq \frac{N}{t} \int_{\mathcal{D}} |f|w \, dx,
\]

where \( N = N(d, c, D, p, c_1, |T|_{L^p_w \to L^p_w}) \) is a constant. Moreover, \( T \) can be extended to a bounded operator from \( L^p_w(\mathcal{D}) \) to \( \text{weak-} L^p_w(\mathcal{D}) \).

Proof. We fix a \( k_0 \in \mathbb{Z} \) and set \( \theta = \inf_{x \in I_{k_0}} w(Q^k_{\alpha}) > 0 \). See conditions i)–v). Then for any \( t > 0 \), to get (6.2) when

\[
\frac{1}{t} \int_{\mathcal{D}} |f|w \, dx > \theta,
\]

it suffices to choose \( N \geq \theta^{-1}w(\mathcal{D}) \). Otherwise,

\[
\frac{1}{w(Q^k_{\alpha})} \int_{Q^k_{\alpha}} |f|w \, dx \leq t, \quad \forall \alpha \in I_{k_0}.
\]

In this case, let \( \{Q^k_{\alpha}\} \) be a collection of disjoint “cubes” from the Calderón-Zygmund decomposition as those in the proof of Lemma 4.1 in [3], so that we have

\[
t < \frac{1}{w(Q^k_{\alpha})} \int_{Q^k_{\alpha}} |f|w \, dx \leq C_2 t, \quad |f(\alpha)| \leq t \text{ for a.e. } x \in \mathcal{D} \setminus \bigcup_{l} Q^k_{\alpha}.
\]

We associate each \( Q^k_{\alpha} \) with a Euclidean ball \( B_k = B_{r_k}(x_k) \), where \( r_k = \text{diam} Q^k_{\alpha} \) and \( x_k \in Q^k_{\alpha} \subseteq B_k \). We denote \( B_{r_k}(x_k) \) in each \( Q^k_{\alpha} \), we have the decomposition \( f = g + b \) and (6.3). We also define \( g = f \) and \( b = 0 \) in \( \mathcal{D} \setminus \bigcup_{l} Q^k_{\alpha} \). By using the Chebyshev inequality and the assumptions of \( f \) and \( g \), we have

\[
w(\{x \in \mathcal{D} : |Tg(x)| > t/2\}) \leq \frac{N}{p} \int_{\mathcal{D}} |Tg|^p w \, dx
\]

Next by using (6.3), we obtain

\[
\int_{D \setminus B_k} |T(b_{\mathcal{D}', \mathcal{D}})|w \, dx \leq Ntw(Q^k_{\alpha}),
\]

which implies that

\[
\int_{D \setminus \bigcup_{l} B_k} |Tb|w \, dx \leq \sum_{l} \int_{D \setminus B_k} |T(b_{\mathcal{D}', \mathcal{D}})|w \, dx \leq Nt \sum_{l} w(Q^k_{\alpha}) \leq N \int_{\mathcal{D}} |f|w \, dx.
\]

By the Chebyshev inequality, we get

\[
w(\{x \in \mathcal{D} : |Tb(x)| > t/2\} \setminus \bigcup_{l} B_k) \leq \frac{N}{t} \int_{\mathcal{D}} |f|w \, dx.
\]
Clearly, we also have

\[ w\left( \bigcup_l B_l^* \right) \leq \sum_l w(B_l) \leq N \sum_l w(B_l) \leq N \sum_l w(Q_k^* \leq) \leq N \int_D |f||w| \, dx. \]

We thus obtain

\[ w\left( \{ x \in D : |T_n(x)| > t/2 \} \right) \leq N \int_D |f||w| \, dx, \]

which combined with (6.3) finishes the proof of (6.2) since \( T_f = T_g + T_b \). The last assertion follows from the fact that \( L^p_w(D) \) is dense in \( L^1_w(D) \). The lemma is proved. \( \square \)

**Remark 6.4.** If we take \( w = 1 \) and \( g = \frac{1}{W(Q_k^*)} \int_{Q_k^*} fW \, dx \), Lemma 6.3 is then reduced to Lemma 2.9. In the special case when \( g = 1 \), Lemma 6.3 was also essentially used in the proof of [9, Theorem 1.10].

Now we give the statement of a generalization of Lemma 2.6. We first give the definition of \( A_p \) weights: We say \( w : \mathbb{R}^d \rightarrow [0, \infty) \) belongs to \( A_p \) for \( p \in (1, \infty) \) if

\[ \sup_B \frac{w(B)}{|B|} \left( \frac{|w^{-1}(B)|}{|B|} \right)^{p-1} < \infty, \]

where the supremum is taken over all balls in \( \mathbb{R}^d \). The value of the supremum is the \( A_p \) constant of \( w \), and will be denoted by \([w]_{A_p}\).

We next consider the domain \( D \) which is Reifenberg flat and impose the following assumptions on the coefficients \( A^{\alpha \beta} \) and the boundary \( \partial D \), with a parameter \( \gamma_0 \in (0, 1/4) \) to be specified later.

**Assumption 6.5 (\( \gamma_0 \)).** There exists a constant \( r_0 \in (0, 1] \) such that the following conditions hold.

1. In the interior of \( D \), \( A^{\alpha \beta} \) satisfy (2.10) in some coordinate system depending on \( x_0 \) and \( r \).
2. For any \( x_0 \in \partial D \) and \( r \in (0, r_0] \), there is a coordinate system depending on \( x_0 \) and \( r \) such that in this new coordinate system, we have

\[ \{(y', y^d) : x_0^d + \gamma_0 r < y^d \in B_r(x_0) \subset D \cap B_r(x_0) \subset \{(y', y^d) : x_0^d - \gamma_0 r < y^d \in B_r(x_0), \quad (6.4) \} \]

and

\[ \int_{B_r(x_0)} |A(x) - (A)_{B_r(x_0)}| \, dx \leq \gamma_0, \quad (6.5) \]

where \( (A)_{B_r(x_0)} = \int_{B_r(x_0)} A(y', x^d) \, dy'. \)
Lemma 6.6. Let \( p \in (1, \infty) \) and \( w \) be an \( A_p \) weight. There exists a constant \( \gamma_0 \in (0, 1/4) \) depending on \( d, p, \nu, \Lambda, D, \) and \([w]_{A_p}\) such that, under Assumption 5.3, for any \( u \in W^{1,p}_w(D) \) satisfying

\[
\begin{aligned}
D_{\alpha}(A^{\alpha\beta} D_{\beta} u) - \lambda u &= \text{div } f & \text{in } D, \\
u u &= 0 & \text{on } \partial D,
\end{aligned}
\]  

where \( \lambda \geq 0 \) and \( f \in L^p_w(D) \), we have

\[
\|u\|_{W^{1,p}_w(D)} \leq N\|f\|_{L^p_w(D)},
\]

where \( N = N(n, d, p, \nu, \Lambda, [w]_{A_p}, r_0) \). Furthermore, for any \( f \in L^p_w(D) \), (6.6) admits a unique solution \( u \in W^{1,p}_w(D) \).

Proof. For \( \lambda > \lambda_0 \), where \( \lambda_0 > 0 \) is a sufficiently large constant depending on \( n, d, p, \nu, \Lambda, [w]_{A_p}, r_0 \), then the solvability for the operator \( D_{\alpha}(A^{\alpha\beta} D_{\beta} u) - \lambda I \) is proved in [14, Section 8], and [10] Theorem 7.2 implies that (6.7) holds true. For \( 0 \leq \lambda \leq \lambda_0 \), we rewrite (6.6) as

\[
\begin{aligned}
D_{\alpha}(A^{\alpha\beta} D_{\beta} u) - (\lambda + \lambda_0) u &= \text{div } f - \lambda_0 u & \text{in } D, \\
u u &= 0 & \text{on } \partial D.
\end{aligned}
\]  

Then by [10] Theorem 7.2, we have

\[
\|u\|_{W^{1,p}_w(D)} \leq N\left(\|u\|_{L^p_w(D)} + \|f\|_{L^p_w(D)}\right),
\]

where \( N = N(n, d, p, \nu, \Lambda, [w]_{A_p}, r_0) \). Thus by the method of continuity, it suffices to bound \( \|u\|_{L^p_w(D)} \) by \( \|f\|_{L^p_w(D)} \). By using Hölder’s inequality, \( f \in L^p_w(D) \), and the self-improving property of \( A_p \) weights, we have for some small \( \delta_1 > 0 \) depending on \( d, p, \) and \([w]_{A_p}\),

\[
\int_D |f|^{1+\delta_1} \, dx \leq \left( \int_D |f|^p \, dx \right)^{1/p} \left( \int_D w^{-\frac{1+\delta_1}{1-\delta_1}} \, dx \right)^{1-\frac{1}{1+\delta_1}} \leq \left( \int_D |f|^p \, dx \right)^{1/p} \left( \int_D w^{-\frac{1}{r}} \, dx \right)^{-\frac{1}{r}} < \infty.
\]

Next, in view of the reverse Hölder’s inequality for \( A_p \) weights, we have \( w \in L^{1+\delta_2} \) for some \( \delta_2 > 0 \) depending on \( d, p, \) and \([w]_{A_p}\). By using Hölder’s inequality, Young’s inequality, the weighted Sobolev embedding theorems (see [13] Theorem 1.3), and (6.9), we have

\[
\|u\|_{L^p_w(D)} \leq \varepsilon \|u\|_{L^p_w(D)}^{\theta} \|u\|_{L^p_w(D)}^{1-\theta} \leq \varepsilon \|u\|_{L^p_w(D)}^{\theta} \left( \int_D |f|^p \, dx \right)^{1-\frac{1}{r}} \leq \varepsilon \|u\|_{W^{1,p}_w(D)} + N\|u\|_{L^p_w(D)} \leq N\varepsilon \|u\|_{L^p_w(D)} + N\|f\|_{L^p_w(D)} + N\|u\|_{L^p_w(D)}',
\]

which yields

\[
\|u\|_{L^p_w(D)} \leq N\left(\|f\|_{L^p_w(D)} + \|u\|_{L^p_w(D)}'\right)
\]  

(6.11)
if we choose $\varepsilon$ sufficiently small so that $N^2/2 < 1/2$, where
\[ p_1 = \frac{dp}{d-1} > p, \quad \frac{\theta}{p_1} + \frac{1 - \theta}{p} = \frac{1}{p'}, \]
and $\delta_3 \in (0, 1/2)$ is to be determined below. It follows from Hölder’s inequality and $w \in L^{1+\delta_2}$ that
\[ \int_D |u|^{\delta_3} w \, dx \leq \left( \int_D |u|^{1+\delta_3} \, dx \right)^{\frac{\delta_3}{1+\delta_3}} \left( \int_D w^{1+\delta_2} \, dx \right)^{\frac{1}{1+\delta_2}}, \]
where we chose $\delta_3 = \frac{\delta_3(1+\delta_3)}{1+\delta_2}$. Therefore,
\[ \|u\|_{L^{1+\delta_3}(\Omega)} \leq N \|u\|_{L^{1+\delta_4}(\Omega)}. \tag{6.12} \]

Coming back to (6.11), using (6.12), [11, Theorem 8.6 (iii)], and (6.10), we have
\[ \|u\|_{L^{1+\delta_4}(\Omega)} \leq N \left( \|f\|_{L^1(\Omega)} + \|u\|_{L^{1+\delta_2}(\Omega)} \right) \leq N \left( \|f\|_{L^1(\Omega)} + \|f\|_{L^{1+\delta_1}(\Omega)} \right) \leq N \|f\|_{L^1(\Omega)}. \]
Then combining it with (6.9), we get
\[ \|u\|_{W^{1,p}(\Omega)} \leq N \|f\|_{L^1(\Omega)}. \]
Therefore, (6.7) is proved. \hfill $\square$

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