Quantum estimation of parameter in circuit QED by continuous quantum measurement

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Designing high-precision and efficient schemes is of crucial importance for quantum parameter estimation in practice. The estimation scheme based on continuous quantum measurement is one possible type of this, which looks also the most natural choice in case such as continuous dynamical process. In this work we specify the study to the stat-of-the-art superconducting circuit quantum-electrodynamics (QED) system, where the high-quality continuous measurement has been extensively exploited in the past decade. Within the framework of Bayesian estimation and particularly using the quantum Bayesian rule in circuit QED, we numerically simulate the likelihood function as estimator for the Rabi frequency of qubit oscillation. We find that, by proper design of the interaction strength of measurement, the estimate precision can scale with the measurement time beyond the standard quantum limit, which is usually assumed for this type of continuous measurement since no more special quantum resource is involved. We understand this remarkable result by quantum correlation in time between the output signals, and simulate the effect of quantum efficiency of the measurement on the precision scaling behavior.

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I. INTRODUCTION

The problem of accurately estimating unknown parameters is of interest and importance in practice [1, 2]. In order to minimize the estimation uncertainties, a variety of strategies have been developed into the science of quantum metrology over the past decades [3, 4]. In this context, a central topic is how exploiting the quantum technique to achieve parameter estimation with precision beyond that obtainable by any classical scheme [3, 4]. For instance, if a system is initially prepared in a spin coherent state, the precision of frequency estimation [3–5]. For instance, if a system is initially prepared in a spin coherent state, the precision of frequency estimation [3–5].

Owing to practical use and rich physics underlying, in the past years there are considerable interests in the quantum estimation of parameters using the output signals of continuous measurement [12–13]. This is also the most natural choice in cases such as parameter estimation in continuous dynamical process. This scheme has the obvious advantage with high efficiency — unlike the conventional ensemble measurement in quantum theory, it does not need to generate the identical copies of quantum system in order to complete the measurement for extracting meaningful result. Following the seminal work [12], the subsequent studies by Mølmer et al [13–16] formulated the parameter estimation based on continuous measurement as a Bayesian scheme, for the specific example of fluorescence detection of two-level atoms. More recently, the same problem was investigated with a focus on how to speed up the estimation by avoiding numerically integrating the stochastic master equation [17]. This type of parameter estimation has also been considered to include the technique of quantum smoothing [20–21], as a generalization of the classical signal smoothing.

In the present work, we specify the research further to the superconducting circuit quantum-electrodynamics (QED) system [24–26], which is one of the leading platforms for quantum information processing and for quantum measurement and control studies. Particularly, sound studies have been performed for continuously tracking the stochastic evolution of the qubit state in this system, say, tracking the so-called quantum trajectories (QT) [27–28]. On the theoretical aspect, the quantum Bayesian rule has been well developed for circuit QED in the past years [23–27], in some cases which promises the advantage of being more efficient than numerically integrating the quantum trajectory equation [28–30]. Therefore, within the framework of Bayesian parameter esti-
II. BAYESIAN RULE IN CIRCUIT QED

Let us consider a superconducting qubit coupled to a waveguide cavity, i.e., the circuit-QED architecture. In the dispersive regime, the qubit-cavity interaction is well described by the Hamiltonian
\[ H_{\text{int}} = \chi a^\dagger a \sigma_z, \]
where \( \chi \) is the dispersive coupling strength, \( a^\dagger \) and \( a \) are the creation and annihilation operators of the cavity mode, and \( \sigma_z \) is the qubit Pauli operator. Associated with single quadrature homodyne measurement for microwave transmission/reflection, the output current can be reexpressed as (after the so-called polaron transformation to eliminate the degrees of freedom of the cavity photons)
\[ I(t) = -\sqrt{\Gamma_{\text{cl}}(t)} \langle \sigma_z \rangle + \xi(t). \]
In this result, \( \xi(t) \), originated from the fundamental quantum-jumps, is a Gaussian white noise and satisfies the ensemble-average property
\[ E[\xi(t)] = 0 \quad \text{and} \quad E[\xi(t)\xi(t')] = \delta(t - t'). \]
\( \Gamma_{\text{cl}}(t) \) is the coherent information gain rate which, together with the other two, say, the no-information back-action rate \( \Gamma_{ba}(t) \) and the overall measurement decoherence rate \( \Gamma_d(t) \), is given by
\[ \begin{align*}
\Gamma_{\text{cl}}(t) &= k|\beta(t)|^2 \cos^2(\varphi - \theta_\beta), \\
\Gamma_{ba}(t) &= k|\beta(t)|^2 \sin^2(\varphi - \theta_\beta), \\
\Gamma_d(t) &= 4\text{Im}[\alpha_1(t)\alpha_2(t)].
\end{align*} \]
Here \( \varphi \) is the local oscillator’s (LO) phase in the homodyne measurement, \( k \) is the leaky rate of the microwave photon from the cavity, and \( \beta(t) = \alpha_2(t) - \alpha_1(t) \equiv |\beta(t)|e^{i\theta_\beta} \) with \( \alpha_1(t) \) and \( \alpha_2(t) \) the cavity fields associated with the qubit states \([1]\) and \([2]\), respectively. In steady state, the cavity fields read
\[ \tilde{\alpha}_{1,2} = -\epsilon_m[(\Delta_r \mp \chi) - i\kappa/2]^{-1}, \]
where \( \Delta_r \) is the frequency offset between the measuring microwave (with amplitude \( \epsilon_m \)) and the cavity mode. In this work, rather than considering a general set-up of the circuit-QED system \([13, 14, 30] \), we restrict to the bad-cavity and weak-response limits. Under this condition, the transient process of \( \alpha_1(t) \) and \( \alpha_2(t) \) is not important. All the rates shown above can be calculated with the steady-state fields \( \tilde{\alpha}_{1,2} \) given by Eq. (3).

Corresponding to the qubit state \([1]\) \((|2\rangle\rangle\) and after averaging the continuous current over time interval \( \tau \), i.e.,
\[ \mathcal{I}_m = \langle \sqrt{T} \rangle \int_0^{\tau} dt' I(t'), \]
the coarse-grained output current \( \mathcal{I}_m \) is a stochastic variable centered at \( \mathcal{I}_{1(2)} = \mp \sqrt{\Gamma_{\text{cl}}} \) and satisfies the Gaussian distribution with probability
\[ P(\mathcal{I}_{1(2)}(\tau) = (2\pi V)^{-1/2} \exp \left[ -\frac{\left( \mathcal{I}_m - \mathcal{I}_{1(2)} \right)^2}{2V} \right], \]
where \( V = 1/\tau \) is the distribution variance.

Now consider the arbitrary quantum superposed state \( \rho(t) \) (at the moment \( t \)). Based on the subsequent (coarse-grained) current \( \mathcal{I}_m \), the quantum Bayesian rule updates the qubit state as follows \([33, 36]\). For the diagonal elements,
\[ \rho_{jj}(t + \tau) = \rho_{jj}(t) P_j(\tau)/N(\tau), \]
where \( j = 1, 2 \) and \( N(\tau) = \rho_{11}(t) P_1(\tau) + \rho_{22}(t) P_2(\tau) \). This is nothing but the Bayes’ theorem in probability theory. For the off-diagonal elements, which are unique in quantum theory,
\[ \rho_{12}(t + \tau) = \rho_{12}(t) \left[ \sqrt{P_1(\tau)P_2(\tau)/N(\tau)} \right] \times D(\tau) \exp \left\{ -i[\Phi_1(\tau) + \Phi_2(\tau)] \right\}. \]
In this result, the purity factor reads \( D(\tau) = e^{-\left( \Gamma_{ba} - \Gamma_m \right)/2} \), while the measurement rate is given by \( \Gamma_m = \Gamma_{\text{cl}} + \Gamma_{ba} \). Using the steady-state solutions, Eq. (3), one can easily prove \( \Gamma_d = \Gamma_m \). Thus, in the bad-cavity limit (no transient dynamics of the cavity field), the intrinsic D-factor in the successive Bayesian update can be approximated by unity. In order to account for decoherence caused by external origins (such as photon loss and/or amplifier’s noise), one can simply introduce an extra rate \( \Gamma_{\phi} \), thus \( D(\tau) = e^{-\Gamma_{\phi} \tau/2} \).
The first phase factor in Eq. (6), \( e^{-i\Phi_1(\tau)} \), is associated with an ac-Stark-shift modified unitary phase accumulation, i.e., with \( \Phi_1(\tau) = (\Omega_q + B)\tau \) where the ac-Stark-shift of the qubit energy \((\Omega_q)\) reads \( B = 2\chi Re(\delta_1 \delta_2^*) \). Of more interest is the second phase factor \( e^{-i\Phi_2(\tau)} = e^{-i\Delta_\omega(I_n \tau)} \), which is associated with the accumulated random ‘charge’ and reflects the no-information gain backaction on the qubit. More detailed discussion on this stochastic phase factor is referred to Refs. [33–36].

III. METHOD AND DEMONSTRATION

We assume now that the superconducting qubit is subject to a Rabi drive and at the same time subject to continuous measurement. Our goal is to estimate the Rabi frequency from the output current of the continuous measurement. The stochastic evolution of the qubit (the quantum trajectory) is governed by the following iterative rule

\[
\rho(t_j) = U_{ij}M_j[\rho(t_{j-1})], \quad \text{with} \quad j = 1, 2, \cdots N. \tag{6}
\]

Here we have discretized the evolution with time interval \( \tau \), with thus a total measurement time \( T = N \tau \). The superoperator \( M_j \) accounts for the measurement-induced change of the qubit state, whose performance is explicitly given by the quantum Bayesian rule. Notice that, in some sense, the quantum Bayesian rule goes beyond the usual POVM formalism. The POVM formalism is appropriate to update quantum state with purity preserved. In the presence of purity degradation during measurement, one is unable to construct the Kraus operators of the POVM formalism. The superoperator \( U_{ij} \) in Eq. (6) describes the unitary evolution caused by the Rabi drive, i.e., \( U_{ij}(\cdots) = e^{-iH_\tau}(\cdots) = e^{-i\Omega q\tau}(\cdots)e^{iH_0}\tau \) with \( H_q \) the qubit Hamiltonian under Rabi drive and renormalized by the measurement (i.e. with the ac-Stark shift).

As a final remark, for relative small \( \tau \), both operators \( U_{ij} \) and \( M_j \) are commutative for each step.

Based on the rule of Eq. (6), we know the qubit state \( \rho(t_j) \) after the \( j_{th} \) step evolution, conditioned on the coarse-grained current \( I_j \). Meanwhile, for this \( j_{th} \) step of measurement, the probability of getting \( I_j \) is

\[
\mathcal{P}(I_j) = \rho_{11}(t_j-1)P_1(\tau) + \rho_{22}(t_j-1)P_2(\tau), \tag{7}
\]

with \( P_1(\tau) \) and \( P_2(\tau) \) given by Eq. (4). Then, straightforwardly, the joint probability of getting the results \( \{I_1, I_2, \cdots I_N\} \) is simply a product of the individual probabilities

\[
\mathcal{P}(\{I_1, I_2, \cdots I_N\}|\Omega) = \prod_{j=1}^{N} \mathcal{P}(I_j). \tag{8}
\]

Here we explicitly indicate that this probability depends on the parameter \( \Omega \) (the possible Rabi frequency).

We expect, from simple intuition, that the true Rabi frequency \( \Omega_R \) will be most compatible with the output results \( \{I_1, I_2, \cdots I_N\} \), leading thus to maximum probability. Therefore, it is plausible that we get an estimate value \( \Omega_{ML} \) for \( \Omega_R \) from the maximum peak location of the probability function \( \mathcal{P}(\{I_1, I_2, \cdots I_N\}|\Omega) \), in literature which is referred to as likelihood function. Using different \( \Omega \) (rather than \( \Omega_R \)) to calculate \( \mathcal{P}(\{I_1, I_2, \cdots I_N\}|\Omega) \), based on Eqs. (4)−(8), should result in smaller probability. This constitutes the basic idea of the maximum-likelihood-estimation (MLE) method.

Essentially, the MLE method is a Bayesian approach for parameter estimation. One may imagine to start with a uniform distribution \( \mathcal{P}(\Omega) \) over certain range. The uniform distribution means that we have no knowledge about \( \Omega_R \). After getting the data record of measurement and performing the Bayesian inference, the knowledge about \( \Omega_R \) changes to a new probability \( \mathcal{P}(\Omega|I_1, I_2, \cdots I_N) \). The peak of this new distribution can be also an estimate for \( \Omega_R \), which should correspond to the estimated value \( \Omega_{ML} \) from the MLE method.

In practice, the following log-likelihood function is used for parameter estimate

\[
L(\Omega) = \ln \mathcal{P}(\{I_1, I_2, \cdots I_N\}|\Omega), \tag{9}
\]

in order to make the maximum peak more prominent. In Fig. 1, we plot this function to illustrate the MLE method (using dimensionless units here and in remainder of this work). \( L(\Omega) \) is computed using the single realization of continuous measurement current \( I(t) \) over \((0, T)\), by coarse-graining it into \( \{I_1, I_2, \cdots I_N\} \) with \( N = T/\tau \). Notice that this splitting can be rather arbitrary, i.e., with \( L(\Omega) \) not influenced by the choice of \( \tau \). The only requirement is that \( \tau \) should not be too large to violate the precision of the Bayesian update (in the presence of Rabi oscillation). In the whole simulations of this work, we choose \( \tau = 1000 dt = 10^{-3} \), while the time increment \( dt = 10^{-6} \) is used for simulating the quantum trajectory equation \([33, 38]\) to generate the continuous output current. Again, we mention that the \( \Omega \) dependence of \( L(\Omega) \) is introduced through the unitary operator \( e^{-i\Delta_\omega t} \) in each step of state update.

We consider a resonant Rabi drive with true Rabi frequency \( \Omega_R/2\pi = 1 \) (in arbitrary dimensionless units). In the present principle-proof simulation, we assume \( \Delta_\omega = 0 \) (thus \( \theta_0 = 0 \)) and consider the maximal information gain with LO phase \( \varphi = 0 \). Therefore we have \( \Gamma_{ba} = 0, \Gamma_m = \Gamma_{ci} \) and \( \Gamma_d = \Gamma_m \) (owing to the bad-cavity limit). Except Fig. 4, we also do not account for any external decoherence in our simulation (setting \( \Gamma_{\varphi} = 0 \)).

Indeed, as shown in Fig. 1, we get an estimation for the Rabi frequency at \( \Omega_{ML} = 0.992 \), from the maximum peak position of \( L(\Omega) \). In this plot, we only show the log-likelihood function for relatively small range of \( \Omega \), indicating that we already have some prior knowledge about \( \Omega_R \). If we have poor knowledge about \( \Omega_R \), we should calculate \( L(\Omega) \) for wider range. In this case, more peaks may appear in \( L(\Omega) \). Being even worse is that the maximum peak does not occur near \( \Omega_R \). This implies a failure of the estimation and the result should be discarded.
Another point is that, in order to get convergent estimation, one should collect relatively large number of currents \(\{I_1,I_2,\cdots,I_N\}\), i.e., with large \(N\) or more precisely large \(T\) by noting that \(T = N\tau\). Actually, it has been noted that the MLE result can saturate the Cramér-Rao bound when \(N\) is large enough \([13–16]\). However, the classical Cramér-Rao bound is determined by the classical Fisher information which is associated with specific schemes of measurement. It has been well understood that the more sensitive dependence of the output results on the parameter will result in better precision. Searching for optimal measurement protocol in practice is thus of crucial importance but is unclear in general. In the following, in Fig. 2, we will further discuss this point.

A final remark is that quantum correlation may involve in the likelihood function \(L(\Omega)\). This is in some sense similar to the reason of violating the Leggett-Garg inequality (a type of Bell’s inequality in time) \([3,10]\) as demonstrated in this same circuit-QED system via continuous measurements \([1,2]\). We will come back to this point later after displaying the result beyond the standard quantum limit.

**IV. RESULTS AND DISCUSSION**

To characterize the estimation errors, we introduce the root-mean-square (RMS) variance

\[
\delta \Omega = \left( \frac{1}{M} \sum_{k=1}^{M} \left( \Omega_{ML}^{(k)} - \bar{\Omega}_{ML} \right)^2 \right)^{1/2},
\]

where \(\bar{\Omega}_{ML} = \frac{1}{M} \sum_{k=1}^{M} \Omega_{ML}^{(k)}\), with \(\Omega_{ML}^{(k)}\) the estimated result of the \(k\)th realization based on \(\{I_1,\cdots,I_N\}^{(k)}\). To extract the RMS variance, we simulate \(M = 2000\) trajectories for each given measurement time \((T = N\tau)\).

Let us analyze the problem of *appropriate* measurement, in a sense to make the measurement results more sensitive to the parameter under estimation. First, as mentioned above, we should eliminate the “realistic” (no information gain) backaction in order to maximize the information gain rate \((\Gamma_\text{c}\rightarrow \Gamma_m)\) by adjusting the LO phase \(\varphi = \theta_\beta = 0\). Second, we search for an optimal strength for the continuous measurement, which can be characterized by the measurement rate \(\Gamma_m\).

In Fig. 2(a) we show the estimation RMS variance versus the measurement strength. Importantly, we find existing an optimal strength of the continuous measurement. We understand the reason as follows. From the continuous output current Eq. (4), we know that for weak strength of measurement, the noise component (the second term) will be much larger than the information-carrying term (the first one). In other words, the output current carries little information of the qubit state which is governed by the parameter of Rabi frequency. In the other extreme, for strong strength of measurement, while the state-information-carrying component (the first term in Eq. (4)) is enhanced, the Rabi oscillation of the qubit state will be more seriously destroyed by the measurement backaction, making thus the first term of Eq. (4)
In quantum estimation, one of the most important issues in the context of continuous measurement in circuit-QED is how the precision scales with the 'size' of the quantum resource (e.g. the entangled photon number). The precision scales with the measurement time $T$, which depends on the measurement time $T$. In particular, we compare the simulated results (open circles) with the SQL ($\sim 1/\sqrt{T}$, solid line) and HL ($\sim 1/T$, dashed line) scaling behaviors. In (a), by properly choosing the measurement strength (near the optimal one), we find that the precision can evidently exceed the SQL. In (b) and (c), we show that away from the optimal/sub-optimal measurement strength, gradually both the smaller and larger strength $\Gamma_m$ cannot violate the SQL precision.

Let us formally denote the RMS variance as $\delta \Omega = \frac{1}{\Delta}f(T)$, where the specific $M$ dependence is simply from the central-limit-theorem. Our interest is to examine the $T$ dependence, especially, to compare it with the SQL and HL scalings. As a clear comparison, in Fig. 3 we compare the simulated RMS variance with the SQL $\delta \Omega = C_1/\sqrt{T}$ (solid line) and HL $\delta \Omega = C_2/T$ (dashed line). Here we set the constants $C_1$ and $C_2$ by making the SQL and HL curves coincide with the simulated RMS variance at $T = 10$. The two curves simply imply that, if the scaling is governed by SQL (HL), the simulated results should follow the solid (dashed) curve with the increase of $T$.

In Fig. 3, we show results for different measurement strengths. Remarkably, in Fig. 3(a) we find that by properly choosing the measurement strength (near the optimal one), the precision can evidently exceed the SQL. We notice that in the studies by Mølmer et al. [13–16], only the $1/\sqrt{T}$ scaling is obtained for the Fisher information associated with the homodyne detection for the fluorescence radiation. This result was qualitatively understood by the measurement backaction which results in vanished correlation between the output signals. In another work by Jordan et al. [17], the $1/\sqrt{T}$ scaling is also mentioned, while possible support may come from analyzing the estimation using only unitary evolution and periodic projective measurements.

We may understand the result in Fig. 3(a) from different perspectives as follows. First, the ‘inconsistency’ with Refs. [13–16] may originate from the different schemes of measurement. There, the measurement operator $\sigma_z = \cos \varphi \sigma_x - \sin \varphi \sigma_y$ has randomly flipping backaction on the qubit. Compared to $\sigma_z$ measurement, this type of measurement has stronger destructive influence on the qubit, i.e., making the population (superposition) less associated with the Rabi frequency.

Second, for the continuous $\sigma_z$ measurement of the Rabi oscillation, quantum correlation exists between the measurement outcomes. Actually, this type of quantum correlation has inspired the study of the Bell-inequality-in-time, say, the Leggett-Garg inequality [11]. In particular, this quantum correlation has been experimentally demonstrated in the circuit-QED system based on the continuous $\sigma_z$ measurement [12]. Therefore, it seems that the argument of vanished correlation in Refs. [13–14], leading to the $1/\sqrt{T}$ scaling, may not apply to our situation.

Third, for the simple estimation scheme based on continuous measurement (not involving any special techniques), the possibility of reaching the Heisenberg limit is not ruled out (i) For instance, at the end of Ref. [14], it was pointed out that the Fisher information can scale with $T^2$ for undamped system evolution, such as the case
if the system superposition state does not couple to environment and the measurement is performed on the system rather than the emitted radiation. (ii) In Ref. [43], via analyzing the quantum Markov chain defined by a sequence of successive passage of atoms through a cavity and suffering measurement stochastic process, it was found that the quantum Fisher information scales quadratically rather than linearly with the number of atoms, at the limit of weak unitary interaction. (iii) Another example of interest is making the system (e.g., a driven atom under photon emissions) approach to dynamical phase transition [44, 45]. In this case, the quantum Fisher information may become quadratic in times shorter than the correlation time of the dynamics. This becomes valid for all times at the point of dynamical phase transition.

![Diagram](image)

FIG. 4: Further examination of the result in Fig. 3(a). Setting still the sub-optimal measurement strength $\Gamma_m = 0.25$ but introducing extra decoherence ($\Gamma_\phi$) owing to photon loss and/or amplifier’s noise during the measurement, we find that the result can no longer exceed the SQL precision. This supports further the understanding based on quantum correlation to the remarkable result in Fig. 3(a).

Therefore, our result in Fig. 3(a) does not contradict any basic physics, but rather can fall into the criteria of quantum correlation. As a tradeoff of information gain and measurement backaction, proper strength of the continuous measurement is required: shown in Fig. 3(b) and (c) indicate that away from the optimal/sub-optimal measurement strength, gradually either smaller or larger strength ($\Gamma_m$) cannot violate the SQL precision. In addition to the proper measurement strength, sufficient quantum coherence is another condition for the result in Fig. 3(a). In Fig. 4, we further account for the effect of decoherence owing to non-ideal quantum measurement, e.g., photon loss and/or amplifier’s noise during the measurement. From Fig. 4(a) and (b), we observe that the estimate precision becomes worse with the increase of decoherence, and can no longer violate the scaling of SQL by varying the measurement strength. This supports further our quantum-correlation-based understanding to the result in Fig. 3(a), since decoherence indeed suppresses the quantum correlation, as shown in Fig. 4.

V. SUMMARY

We have reexamined the problem of quantum estimation of Rabi frequency of qubit oscillations based on continuous measurement. We specified our research to the superconducting circuit-QED system which may provide an attractive platform for experimental examination of parameter estimation. Our central result is that, by proper design of the measurement strength, the estimate precision can scale with the measurement time beyond the standard quantum limit. We understood this result by quantum correlation between the output signals and supported it by checking the effect of quantum efficiency of the measurement. We expect this preliminary result to inspire further investigations to this interesting problem, including searching for better schemes of continuous measurement and special techniques such as feedback and quantum smoothing.

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