On uniformly $S$-absolutely pure modules

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Abstract

Let $R$ be a commutative ring with identity and $S$ a multiplicative subset of $R$. In this paper, we introduce and study the notions of $u$-$S$-pure $u$-$S$-exact sequences and uniformly $S$-absolutely pure modules which extend the classical notions of pure exact sequences and absolutely pure modules. And then we characterize uniformly $S$-von Neumann regular rings and uniformly $S$-Noetherian rings using uniformly $S$-absolutely pure modules.

Key Words: $u$-$S$-pure $u$-$S$-exact sequences; uniformly $S$-absolutely pure modules; uniformly $S$-von Neumann regular rings; uniformly $S$-Noetherian rings.

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1. introduction and Preliminary

Throughout this paper, $R$ is always a commutative ring with identity, all modules are unitary and $S$ is always a multiplicative subset of $R$, that is, $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$.

The notion of absolutely pure modules was first introduced by Maddox [10] in 1967. An $R$-module $E$ is said to be absolutely pure provided that $E$ is a pure submodule of every module which contains $E$ as a submodule. It is well-known that an $R$-module $E$ is absolutely pure if and only if $\text{Ext}_R^1(N, E) = 0$ for any finitely presented module $N$ ([14, Proposition 2.6]). So absolutely pure modules are also studied with the terminology FP-injective modules (FP for finitely presented), see Stenström [14] and Jain [7] for example. The notion of absolutely pure modules is very attractive in that it is not only a generalization of that of injective modules but also an important tool to characterize some classical rings. A ring $R$ is semi-hereditary if and only if any homomorphic image of an absolutely pure $R$-module is absolutely pure ([11, Theorem 2]). A ring $R$ is Noetherian if and only if any absolutely pure $R$-module is injective ([11, Theorem 3]). A ring $R$ is von-Neumann regular if and only if any $R$-module is absolutely pure ([11, Theorem 5]). A ring
is coherent if and only if the class of absolutely pure \( R \)-modules is closed under direct limits, if and only if the class of absolutely pure \( R \)-modules is a (pre)cover ([14, Theorem 3.2], [1] Corollary 3.5]).

One of the most important methods to generalize the classical rings and modules is in terms of multiplicative subsets \( S \) of \( R \) (see [1, 2, 3, 8, 9] for example). In 2002, Anderson and Dumitrescu [1] introduced \( S \)-Noetherian rings \( R \) in which for any ideal \( I \) of \( R \), there exists a finitely generated sub-ideal \( K \) of \( I \) such that \( sI \subseteq K \). Cohen’s Theorem, Eakin-Nagata Theorem and Hilbert Basis Theorem for \( S \)-Noetherian rings are given in [1]. However, the choice of \( s \in S \) such that \( sI \subseteq K \) in the definition of \( S \)-Noetherian rings as above is not uniform. Hence, Qi et al. [12] introduced the notion of uniform \( S \)-Noetherian rings and obtained the Eakin-Nagata-Formanek Theorem and Cartan-Eilenberg-Bass Theorem for uniformly \( S \)-Noetherian rings. Recently, the first author of the paper [17] introduced the notions of \( u \)-\( S \)-flat modules and uniformly \( S \)-von Neumann regular rings which can be seen as uniformly \( S \)-versions of flat modules and von Neumann regular rings. In this paper, we generalized the classical pure exact sequences and absolutely pure modules to \( u \)-\( S \)-pure \( u \)-\( S \)-exact sequences and \( u \)-\( S \)-absolutely pure modules, and then obtain uniformly \( S \)-versions of some classical characterizations of pure exact sequences and absolutely pure modules (see Theorem 2.2 and Theorem 3.2). Finally, we characterize uniformly \( S \)-von Neumann regular rings and uniformly \( S \)-Noetherian rings using \( u \)-\( S \)-absolutely pure modules (see Theorem 3.5 and Theorem 3.7). As our work involves the uniformly \( S \)-torsion theory, we provide a quick review as below.

Recall from [17], an \( R \)-module \( T \) is said to be \( u \)-\( S \)-torsion (with respect to \( s \)) provided that there exists an element \( s \in S \) such that \( sT = 0 \). An \( R \)-sequence 
\[
\cdots \rightarrow A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \rightarrow \cdots
\]
is \( u \)-\( S \)-exact, if for any \( n \) there is an element \( s \in S \) such that \( s\text{Ker}(f_{n+1}) \subseteq \text{Im}(f_n) \) and \( s\text{Im}(f_n) \subseteq \text{Ker}(f_{n+1}) \). An \( R \)-sequence 
\[
0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
\]is called a short \( u \)-\( S \)-exact sequence (with respect to \( s \)), if \( s\text{Ker}(g) \subseteq \text{Im}(f) \) and \( s\text{Im}(f) \subseteq \text{Ker}(g) \) for some \( s \in S \). An \( R \)-homomorphism \( f : M \rightarrow N \) is an \( u \)-\( S \)-monomorphism (resp., \( u \)-\( S \)-epimorphism, \( u \)-\( S \)-isomorphism) (with respect to \( s \)) provided 
\[
0 \rightarrow M \xrightarrow{f} N
\](resp., \( M \xrightarrow{f} N \rightarrow 0 \), \( 0 \rightarrow M \xrightarrow{f} N \rightarrow 0 \)) is \( u \)-\( S \)-exact (with respect to \( s \)). Suppose \( M \) and \( N \) are \( R \)-modules. We say \( M \) is \( u \)-\( S \)-isomorphic to \( N \) if there exists a \( u \)-\( S \)-isomorphism \( f : M \rightarrow N \). A family \( \mathcal{C} \) of \( R \)-modules is said to be closed under \( u \)-\( S \)-isomorphisms if \( M \) is \( u \)-\( S \)-isomorphic to \( N \) and \( M \) is in \( \mathcal{C} \), then \( N \) is also in \( \mathcal{C} \). One can deduce from the following Proposition [14] that the existence of \( u \)-\( S \)-isomorphisms of two \( R \)-modules is actually an equivalence relation.
**Proposition 1.1.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Suppose there is a $u$-$S$-isomorphism $f : M \to N$ for $R$-modules $M$ and $N$. Then there is a $u$-$S$-isomorphism $g : N \to M$ and $t \in S$ such that $f \circ g = t\text{Id}_N$ and $g \circ f = t\text{Id}_M$.

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker}(f) & \rightarrow & M & \rightarrow & N & \rightarrow & \text{Coker}(f) & \rightarrow & 0 \\
& & & & \downarrow f & & \downarrow & & & & \downarrow \text{Im}(f)
\end{array}
\]

with $s\text{Ker}(f) = 0$ and $sN \subseteq \text{Im}(f)$ for some $s \in S$. Define $g_1 : N \to \text{Im}(f)$ where $g_1(n) = sn$ for any $n \in N$. Then $g_1$ is a well-defined $R$-homomorphism since $sn \in \text{Im}(f)$. Define $g_2 : \text{Im}(f) \to M$ where $g_2(f(m)) = sm$. Then $g_2$ is well-defined $R$-homomorphism. Indeed, if $f(m) = 0$, then $m \in \text{Ker}(f)$ and so $sm = 0$. Set $g = g_2 \circ g_1 : N \to M$. We claim that $g$ is a $u$-$S$-isomorphism. Indeed, let $n$ be an element in $\text{Ker}(g)$. Then $sn = g_1(n) \in \text{Ker}(g_2)$. Note that $s\text{Ker}(g_2) = 0$. Thus $s^2n = 0$. So $s^2\text{Ker}(g) = 0$. On the other hand, let $m \in M$. Then $g(f(m)) = g_2 \circ g_1(f(m)) = g_2(f(sm)) = s^2m$. Set $t = s^2 \in S$. Then $g \circ f = t\text{Id}_M$ and $tm \in \text{Im}(g)$. So $tM \subseteq \text{Im}(g)$. It follows that $g$ is a $u$-$S$-isomorphism. It is also easy to verify that $f \circ g = t\text{Id}_N$. \hfill \qed

**Remark 1.2.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ and $N$ $R$-modules. Then the condition “there is an $R$-homomorphism $f : M \to N$ such that $f_S : M_S \to N_S$ is an isomorphism” does not mean “there is an $R$-homomorphism $g : N \to M$ such that $g_S : N_S \to M_S$ is an isomorphism”.

Indeed, let $R = \mathbb{Z}$ be the ring of integers, $S = R - \{0\}$ and $\mathbb{Q}$ the quotient field of integers. Then the embedding map $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ satisfies $f_S : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism. However, since $\text{Hom}_\mathbb{Z}(\mathbb{Q}, \mathbb{Z}) = 0$, there does not exist any $R$-homomorphism $g : \mathbb{Q} \to \mathbb{Z}$ such that $g_S : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism.

The following two results state that a short $u$-$S$-exact sequence induces long $u$-$S$-exact sequences by the functors “Tor” and “Ext” as the classical cases.

**Theorem 1.3.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $N$ an $R$-module. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence of $R$-modules. Then for any $n \geq 1$ there is an $R$-homomorphism $\delta_n : \text{Tor}_n^R(C, N) \to \text{Tor}_{n-1}^R(A, N)$ such that the induced sequence

\[\cdots \to \text{Tor}_n^R(A, N) \rightarrow \text{Tor}_n^R(B, N) \rightarrow \text{Tor}_n^R(C, N) \xrightarrow{\delta_n} \text{Tor}_{n-1}^R(A, N) \rightarrow \cdots \to \text{Tor}_1^R(C, N) \xrightarrow{\delta_1} A \otimes_R N \rightarrow B \otimes_R N \rightarrow C \otimes_R N \rightarrow 0\]
is $u$-$S$-exact.

**Proof.** Since the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is $u$-$S$-exact at $B$. There are three exact sequences $0 \to \text{Ker}(f) \xrightarrow{\text{Ker}(f)} A \xrightarrow{\text{Im}(f)} \text{Im}(f) \to 0$, $0 \to \text{Ker}(g) \xrightarrow{\text{Ker}(g)} B \xrightarrow{\text{Im}(g)} \text{Im}(g) \to 0$ and $0 \to \text{Coker}(g) \to C \xrightarrow{\text{Coker}(g)} \text{Coker}(g) \to 0$ with $\text{Ker}(f)$ and $\text{Coker}(g)$ $u$-$S$-torsion. There also exists $s \in S$ such that $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$. Denote $T = \text{Ker}(f)$ and $T' = \text{Coker}(g)$.

Firstly, consider the exact sequence

$$\text{Tor}_{n+1}(T', N) \to \text{Tor}_n(\text{Im}(g), N) \xrightarrow{\text{Tor}_n(i_{\text{Im}(g)}, N)} \text{Tor}_n(C, N) \to \text{Tor}_n(T', N).$$

Since $T'$ is $u$-$S$-torsion, $\text{Tor}_{n+1}(T', N)$ and $\text{Tor}_n(T', N)$ is $u$-$S$-torsion. Thus $\text{Tor}_n(i_{\text{Im}(g)}, N)$ is a $u$-$S$-isomorphism. So there is also a $u$-$S$-isomorphism $h_{\text{Im}(g)}^n : \text{Tor}_n(C, N) \to \text{Tor}_n(\text{Im}(g), N)$ by Proposition 1.1. Consider the exact sequence:

$$\text{Tor}_{n-1}(T, N) \to \text{Tor}_{n-1}(A, N) \xrightarrow{\text{Tor}_{n-1}(\pi_{\text{Im}(f)}, N)} \text{Tor}_{n-1}(\text{Im}(f), N) \to \text{Tor}_{n-2}(T, N).$$

Since $T$ is $u$-$S$-torsion, we have $\text{Tor}_{n-1}(\pi_{\text{Im}(f)}, N)$ is a $u$-$S$-isomorphism. So there is also a $u$-$S$-isomorphism $h_{\text{Im}(f)}^{n-1} : \text{Tor}_{n-1}(\text{Im}(f), N) \to \text{Tor}_{n-1}(A, N)$ by Proposition 1.1. We have two exact sequences

$$\text{Tor}_{n+1}(T_1, N) \to \text{Tor}_n(s\text{Ker}(g), N) \xrightarrow{\text{Tor}_n(i_{\text{sKer}(g)}, N)} \text{Tor}_n(\text{Im}(f), N) \to \text{Tor}_{n+1}(T_1, N)$$

and

$$\text{Tor}_{n+1}(T_2, N) \to \text{Tor}_n(s\text{Ker}(g), N) \xrightarrow{\text{Tor}_n(i_{\text{sKer}(g)}, N)} \text{Tor}_n(\text{Ker}(g), N) \to \text{Tor}_{n+1}(T_2, N),$$

where $T_1 = \text{Im}(f)/s\text{Ker}(g)$ and $T_2 = \text{Im}(f)/s\text{Im}(f)$ is $u$-$S$-torsion. So $\text{Tor}_n(i_{\text{Im}(g)}, N)$ and $\text{Tor}_n(i_{\text{sKer}(g)}, N)$ are $u$-$S$-isomorphisms. Thus there is a $u$-$S$-isomorphism $h_{\text{sKer}(g)}^n : \text{Tor}_n(\text{Ker}(g), N) \to \text{Tor}_n(s\text{Ker}(g), N)$. Note that there is an exact sequence

$$\text{Tor}_n(B, N) \xrightarrow{\text{Tor}_n(\pi_{\text{Im}(g)}, N)} \text{Tor}_n(\text{Im}(g), N) \xrightarrow{\text{Tor}_n(i_{\text{Im}(g)}, N)} \text{Tor}_n(\text{Ker}(g), N) \xrightarrow{\text{Tor}_{n-1}(i_{\text{Ker}(g)}, N)} \text{Tor}_{n-1}(B, N).$$

Set $\delta_n = h_{\text{Im}(g)}^n \circ h_{\text{Ker}(g)}^n \circ \text{Tor}_n(i_{\text{sKer}(g)}, N) \circ h_{\text{Ker}(g)}^{n-1} : \text{Tor}_n(C, N) \to \text{Tor}_{n-1}(A, N)$.

Since $h_{\text{Im}(g)}^n, \delta_{\text{Im}(g)}, h_{\text{sKer}(g)}^n$ and $h_{\text{Ker}(g)}^{n-1}$ are $u$-$S$-isomorphisms, we have the sequence

$$\text{Tor}_n(B, N) \to \text{Tor}_n(C, N) \xrightarrow{\delta_n} \text{Tor}_{n-1}(A, N) \to \text{Tor}_{n-1}(B, N)$$

is $u$-$S$-exact.

Secondly, consider the exact sequence:

$$\text{Tor}_{n+1}(T, N) \to \text{Tor}_n(A, N) \xrightarrow{\text{Tor}_n(i_{\text{Im}(f)}, N)} \text{Tor}_n(\text{Im}(f), N) \to \text{Tor}_{n-1}(T, N).$$

Since $T$ is $u$-$S$-torsion, $\text{Tor}_n(i_{\text{Im}(f)}, N)$ is a $u$-$S$-isomorphism. Consider the exact sequences:

$$\text{Tor}_{n+1}(\text{Im}(g), N) \to \text{Tor}_n(\text{Ker}(g), N) \xrightarrow{\text{Tor}_n(i_{\text{Ker}(g)}, N)} \text{Tor}_n(B, N) \to \text{Tor}_n(\text{Im}(g), N)$$
and
\[ \text{Tor}_{n+1}^R(T', N) \to \text{Tor}_n^R(\text{Im}(g), N) \xrightarrow{\text{Tor}^R_{(\text{Im}(g))^{-1}}} \text{Tor}_n^R(C, N) \to \text{Tor}_n^R(T', N). \]

Since \( T' \) is \( u \)-\( S \)-torsion, we have \( \text{Tor}_n^R(\text{Im}(g), N) \) is a \( u \)-\( S \)-isomorphism. Since \( \text{Tor}_n^R(\text{Im}(g), N) \) and \( \text{Tor}_n^R(\text{Im}(g), N) \) are \( u \)-\( S \)-isomorphisms as above, \( \text{Tor}_n^R(A, N) \to \text{Tor}_n^R(B, N) \to \text{Tor}_n^R(C, N) \) is \( u \)-\( S \)-exact at \( \text{Tor}_n^R(B, N) \).

Iterating the above steps, we have the following \( u \)-\( S \)-exact sequence:
\[
\cdots \to \text{Tor}_n^R(A, N) \to \text{Tor}_n^R(B, N) \to \text{Tor}_n^R(C, N) \xrightarrow{\delta_n} \text{Tor}_{n-1}^R(A, N) \to \text{Tor}_{n-1}^R(B, N) \to \cdots \to \text{Tor}_1^R(C, N) \xrightarrow{\delta_1} A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.
\]

Similar to the proof of Theorem 1.3 we can deduce the following result.

**Theorem 1.4.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \) and \( M \) and \( N \) \( R \)-modules. Suppose \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) is a \( u \)-\( S \)-exact sequence of \( R \)-modules. Then for any \( n \geq 1 \) there are \( R \)-homomorphisms \( \delta_n : \text{Ext}_{R}^{n-1}(M, C) \to \text{Ext}_{R}^{n}(M, A) \) and \( \delta_n : \text{Ext}_{R}^{n-1}(A, N) \to \text{Ext}_{R}^{n}(C, N) \) such that the induced sequences
\[
0 \to \text{Hom}_R(M, A) \to \text{Hom}_R(M, B) \to \text{Hom}_R(M, C) \xrightarrow{\delta_n} \text{Ext}_R^1(M, A) \to \cdots \to \\
\text{Ext}_R^{n-1}(M, B) \to \text{Ext}_R^{n-1}(M, C) \xrightarrow{\delta_n} \text{Ext}_R^n(M, A) \to \text{Ext}_R^n(M, B) \to \cdots
\]

and
\[
0 \to \text{Hom}_R(C, N) \to \text{Hom}_R(B, N) \to \text{Hom}_R(A, N) \xrightarrow{\delta_n} \text{Ext}_R^1(C, N) \to \cdots \to \\
\text{Ext}_R^{n-1}(B, N) \to \text{Ext}_R^{n-1}(A, N) \xrightarrow{\delta_n} \text{Ext}_R^n(C, N) \to \text{Ext}_R^n(B, N) \to \cdots
\]
are \( u \)-\( S \)-exact.

2. **\( u \)-\( S \)-pure \( u \)-\( S \)-exact sequences**

Recall from [13] that an exact sequence \( 0 \to A \to B \to C \to 0 \) is said to be pure provided that for any \( R \)-module \( M \), the induced sequence \( 0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0 \) is also exact. Now we introduce the uniformly \( S \)-version of pure exact sequences.

**Definition 2.1.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \). A short \( u \)-\( S \)-exact sequence \( 0 \to A \to B \to C \to 0 \) is said to be **\( u \)-\( S \)-pure** provided that for any \( R \)-module \( M \), the induced sequence \( 0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0 \) is also \( u \)-\( S \)-exact.

Obviously, any pure exact sequence is \( u \)-\( S \)-pure. In [10, 34.5], there are many characterizations of pure exact sequences. The next result generalizes some of these characterizations to \( u \)-\( S \)-pure \( u \)-\( S \)-exact sequences.
Theorem 2.2. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short $u$-$S$-exact sequence of $R$-modules. Then the following statements are equivalent:

1. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a $u$-$S$-pure $u$-$S$-exact sequence;
2. there exists an element $s \in S$ satisfying that if a system of equations $f(a_i) = \sum_{j=1}^{m} r_{ij}x_j$ ($i = 1, \ldots, n$) with $r_{ij} \in R$ and unknowns $x_1, \ldots, x_m$ has a solution in $B$, then the system of equations $sa_i = \sum_{j=1}^{m} r_{ij}x_j$ ($i = 1, \ldots, n$) is solvable in $A$.
3. there exists an element $s \in S$ satisfying that for any given commutative diagram with $F$ finitely generated free and $K$ a finitely generated submodule of $F$, there exists a homomorphism $\eta : F \to A$ such that $s\alpha = \eta i$;
4. there exists an element $s \in S$ satisfying that for any finitely presented $R$-module $N$, the induced sequence $0 \to \text{Hom}_R(N, A) \to \text{Hom}_R(N, B) \to \text{Hom}_R(N, C) \to 0$ is $u$-$S$-exact with respect to $s$.

Proof. (1) $\Rightarrow$ (2): Set $\Gamma = \{(K, R^n) \mid K \text{ is a finitely generated submodule of } R^n \text{ and } n < \infty\}$. Define $M = \bigoplus_{(K, R^n) \in \Gamma} R^n/K$. Then $0 \to M \otimes_R A \xrightarrow{1 \otimes f} M \otimes_R B \to M \otimes_R C \to 0$ by (1). So there exists an element $s \in S$ such that $s\text{Ker}(1_M \otimes f) = 0$. Hence, $s\text{Ker}(1_{R^n/K} \otimes f) = 0$ for any $(K, R^n) \in \Gamma$. Now assume that there exists $b_j \in B$ such that $f(a_i) = \sum_{j=1}^{m} r_{ij}b_j$ for any $j = 1, \ldots, m$. Let $F$ be a free $R$-module with basis $\{e_1, \ldots, e_n\}$, and let $K \subseteq F$ be the submodule generated by $m$ elements $\{\sum_{i=1}^{n} r_{ij}e_i \mid j = 1, \ldots, m\}$. Then, $F/K$ is generated by $\{e_1 + K, \ldots, e_n + K\}$. Note that $\sum_{i=1}^{n} r_{ij}(e_i + K) = \sum_{i=1}^{n} r_{ij}e_i + K = 0 + K$ in $F/K$. Hence, we have

$$\sum_{i=1}^{n} ((e_i + K) \otimes f(a_i)) = \sum_{i=1}^{n} ((e_i + K) \otimes (\sum_{j=1}^{m} r_{ij}b_j)) = \sum_{j=1}^{m} (\sum_{i=1}^{n} r_{ij}(e_i + K)) \otimes b_j = 0$$

in $F/K \otimes B$. And so $\sum_{i=1}^{n} ((e_i + K) \otimes a_i) \in \text{Ker}(1_{F/K} \otimes f)$. Hence, $s\sum_{i=1}^{n} ((e_i + K) \otimes a_i) = \sum_{i=1}^{n} ((e_i + K) \otimes sa_i) = 0$ in $F/K \otimes_R A$. By \cite[Chapter I, Lemma 6.1]{[6]}, there exists $d_j \in A$
and $t_{ij} \in R$ such that $sa_i = \sum_{k=1}^{t} l_{ik}d_k$ and $\sum_{i=1}^{n} l_{ik}(e_i + K) = 0$, and so $\sum_{i=1}^{n} l_{ik}e_i \in K$.

Then there exists $t_{jk} \in R$ such that $\sum_{i=1}^{n} l_{ik}e_i = \sum_{j=1}^{m} t_{jk}(\sum_{i=1}^{n} r_{ij}e_i) = \sum_{j=1}^{m} (\sum_{i=1}^{n} (t_{jk}r_{ij})e_i)$.

Since $F$ is free, we have $l_{ik} = \sum_{j=1}^{m} r_{ij}t_{jk}$. Hence

$$sa_i = \sum_{k=1}^{t} l_{ik}d_k = \sum_{k=1}^{t}(\sum_{j=1}^{m} r_{ij}t_{jk})d_k = \sum_{j=1}^{m} r_{ij}(\sum_{k=1}^{t} t_{jk}d_k)$$

with $\sum_{k=1}^{t} t_{jk}d_k \in A$. That is, $sa_i = \sum_{j=1}^{m} r_{ij}x_j$ is solvable in $A$.

(2) $\Rightarrow$ (1): Let $s \in S$ satisfying (2) and $M$ be an $R$-module. Then we have a $u$-$S$-exact sequence $M \otimes_R A \xrightarrow{1 \otimes f} M \otimes_R B \xrightarrow{r} M \otimes_R C \xrightarrow{0}$ by Theorem [3]. We will show that Ker$(1 \otimes f)$ is $u$-$S$-torsion. Let $\{\sum_{i=1}^{n} u_i^\lambda \otimes a_i^\lambda \mid \lambda \in \Lambda\}$ be the generators of Ker$(1 \otimes f)$. Then $\sum_{i=1}^{n} u_i^\lambda \otimes f(a_i^\lambda) = 0$ in $M \otimes_R B$ for each $\lambda \in \Lambda$. By [6] Chapter I, Lemma 6.1, there exists $r_{ij}^\lambda \in R$ and $b_j^\lambda \in B$ such that $f(a_i^\lambda) = \sum_{j=1}^{m} r_{ij}^\lambda b_j^\lambda$ and $\sum_{i=1}^{n} u_i^\lambda r_{ij}^\lambda = 0$ for each $\lambda \in \Lambda$. So $sa_i^\lambda = \sum_{j=1}^{m} r_{ij}^\lambda x_j^\lambda$ have a solution, say $a_j^\lambda$ in $A$ by (2). Then

$$s(\sum_{i=1}^{n} u_i^\lambda \otimes a_i^\lambda) = \sum_{i=1}^{n} u_i^\lambda \otimes sa_i^\lambda = \sum_{i=1}^{n} u_i^\lambda \otimes (\sum_{j=1}^{m} r_{ij}^\lambda a_j^\lambda) = \sum_{j=1}^{m} (\sum_{i=1}^{n} r_{ij}^\lambda u_i^\lambda) \otimes a_j^\lambda = 0$$

for each $\lambda \in \Lambda$. Hence $s$Ker$(1 \otimes f) = 0$, and $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is $u$-$S$-exact.

(2) $\Rightarrow$ (3): Let $s \in S$ satisfying (2) and $\{e_1, \cdots, e_n\}$ the basis of $F$. Suppose $K$ is generated by $\{y_i = \sum_{j=1}^{m} r_{ij}e_j \mid i = 1, \cdots, m\}$. Set $\beta(e_j) = b_j$ and $a(y_i) = a_i$, then

$$f(a_i) = \sum_{j=1}^{m} r_{ij}b_j.$$ By (2), we have $sa_i = \sum_{j=1}^{m} r_{ij}d_j$ for some $d_j \in A$. Let $\eta : F \to A$ be $R$-homomorphism satisfying $\eta(e_j) = d_j$. Then $\eta_i(y_i) = \eta_i(\sum_{j=1}^{m} r_{ij}e_j) = \sum_{j=1}^{m} r_{ij}\eta(e_j) = \sum_{j=1}^{m} r_{ij}d_j = sa_i = s\alpha(y_i)$, and so we have $s\alpha = \eta_i$.

(3) $\Rightarrow$ (4): Let $s \in S$ satisfy (3). Note that $A$ is $u$-$S$-isomorphic to Im$(f)$ and $C$ is $u$-$S$-isomorphic to Coker$(f)$. Thus, by Proposition [1.1] we have homomorphisms $t_1 : A \to \text{Im}(f)$ with $t_1(a) = f(a)$ for any $a \in A$ and $t'_1 : \text{Im}(f) \to A$ such that $t_1t'_1 = s_1\text{Id}_{\text{Im}(f)}$ and $t'_1t_1 = s_1\text{Id}_A$, and homomorphisms $t_2 : \text{Coker}(f) \to C$ and $t'_2 : C \to \text{Coker}(f)$ such that $f' = t_2\pi_{\text{Coker}(f)}$, $t_2t'_2 = s_2\text{Id}_C$ and $t'_2t_2 = s_2\text{Id}_{\text{Coker}(f)}$.
for some $s_1, s_2 \in S$ where $\pi_{\text{Coker}(f)} : B \to \text{Coker}(f)$ is the natural epimorphism. Let $N$ be a finitely presented $R$-module with $0 \to K \to F \to N \to 0$ exact where $F$ is finitely generated free and $K$ finitely generated. Let $\gamma$ be a homomorphism in $\text{Hom}_R(N, C)$. Considering the exact sequence $0 \to \text{Im}(f) \to B \to \text{Coker}(f) \to 0$, we have the following commutative diagram with rows exact:

$$
\begin{array}{ccc}
0 & \xrightarrow{i_K} & K \\
\downarrow{h} & & \downarrow{g} \\
0 & \xrightarrow{i_{\text{Im}(f)}} & \text{Im}(f) \\
\end{array}
\begin{array}{ccc}
F & \xrightarrow{\pi_N} & N \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
\text{Coker}(f) & & 0 \\
\end{array}
$$

By (3), there exists an homomorphism $\eta : F \to A$ such that $st'h = \eta i_K$. So $ss_1 h = st_1 t_1 h = t_1 \eta i_K$. So the following diagram is also commutative:

$$
\begin{array}{ccc}
0 & \xrightarrow{i_K} & K \\
\downarrow{ss_1 h} & & \downarrow{t_1 \eta} \\
0 & \xrightarrow{i_{\text{Im}(f)}} & \text{Im}(f) \\
\end{array}
\begin{array}{ccc}
F & \xrightarrow{\pi_N} & N \\
\downarrow{s_1 \gamma} & & \downarrow{s_1 \gamma} \\
\text{Coker}(f) & & 0 \\
\end{array}
$$

So by [15, Exercise 1.60], there is an $R$-homomorphism $\delta : N \to B$ such that $ss_1 t_2 \gamma = \pi_{\text{Coker}(f)} \delta$. So $ss_1 s_2 \gamma = ss_1 t_2 t_2 \gamma = t_2 \pi_{\text{Coker}(f)} \delta = f' \delta = f''(\delta)$. Hence $f'' : \text{Hom}_R(N, B) \to \text{Hom}_R(N, C)$ is a $u$-$S$-epimorphism with respect to $ss_1 s_2$. Consequently, one can verify the $R$-sequence $0 \to \text{Hom}_R(N, A) \to \text{Hom}_R(N, B) \to \text{Hom}_R(N, C) \to 0$ is $u$-$S$-exact with respect to $ss_1 s_2$ by Theorem [14].

(4) $\Rightarrow$ (2): Let $s \in S$ satisfying (4) and $0 \to A \xrightarrow{f} B \xrightarrow{f'} C \to 0$ a short $u$-$S$-exact sequence of $R$-modules. Similar with the proof of (3) $\Rightarrow$ (4), we have homomorphisms $t_1 : A \to \text{Im}(f)$ with $t_1(a) = f(a)$ for any $a \in A$ and $t'_1 : \text{Im}(f) \to A$ such that $t_1 t'_1 = s_1 \text{Id}_{\text{Im}(f)}$ and $t'_1 t_1 = s_1 \text{Id}_A$, and homomorphisms $t_2 : \text{Coker}(f) \to C$ and $t'_2 : C \to \text{Coker}(f)$ such that $t_2 t'_2 = \text{Id}_{\text{Coker}(f)}$, $t'_2 t_2 = s_2 \text{Id}_C$ and $t'_2 t_2 = s_2 \text{Id}_{\text{Coker}(f)}$ for some $s_1, s_2 \in S$ where $\pi_{\text{Coker}(f)} : B \to \text{Coker}(f)$ is the natural epimorphism.

Suppose that $f(a_i) = \sum_{j=1}^{m} r_{ij} b_j$ ($i = 1, \cdots, n$) with $a_i \in A$, $b_j \in B$ and $r_{ij} \in R$. Let $F_0$ be a free module with basis $\{e_1, \cdots, e_m\}$ and $F_1$ a free module with basis $\{e'_1, \cdots, e'_n\}$. Then there are $R$-homomorphisms $\tau : F_0 \to B$ and $\sigma : F_1 \to \text{Im}(f)$ satisfying $\tau(e_j) = b_j$ and $\sigma(e'_i) = f(a_i)$ for each $i, j$. Define $R$-homomorphism $h : F_1 \to F_0$ satisfying $h(e'_i) = \sum_{j=1}^{m} r_{ij} e_j$ for each $i$. Then $\tau h(e'_i) = \sum_{j=1}^{m} r_{ij} \tau(e_j) = \sum_{j=1}^{m} r_{ij} b_j = f(a_i) = \sigma(e'_i)$. Set $N = \text{Coker}(h)$. Then $N$ is finitely presented. Thus there exists a homomorphism $\phi : N \to \text{Coker}(f)$ such that the following diagram
commutative:

\[
\begin{array}{c}
F_1 \xrightarrow{h} F_0 \xrightarrow{g} N \xrightarrow{\phi} 0 \\
\sigma \downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
0 \xrightarrow{\text{Im}(f)} B \xrightarrow{\text{Coker}(f)} 0
\end{array}
\]

Note that the induced sequence

\[0 \rightarrow \text{Hom}_R(N, \text{Im}(f)) \rightarrow \text{Hom}_R(N, B) \rightarrow \text{Hom}_R(N, \text{Coker}(f)) \rightarrow 0\]

is \(u\)-\(S\)-exact with respect to \(s_1s_2s\) by (4). Hence there exists a homomorphism \(\delta : N \rightarrow \text{Coker}(f)\) such that \(s_1s_2s\delta = \pi_{\text{Coker}(f)}\delta\). Consider the following commutative diagram:

\[
\begin{array}{c}
F_1 \xrightarrow{h} F_0 \xrightarrow{g} N \xrightarrow{\delta} 0 \\
\sigma \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
0 \xrightarrow{\text{Im}(f)} B \xrightarrow{\text{Coker}(f)} 0
\end{array}
\]

We claim that there exists a homomorphism \(\eta : F_0 \rightarrow \text{Im}(f)\) such that \(\eta f = s_1s_2s\sigma\). Indeed, since \(\pi_{\text{Coker}(f)}\delta g = s_1s_2s\phi g = \pi_{\text{Coker}(f)}s_1s_2s\tau\), we have \(\text{Im}(s_1s_2s\tau - \delta g) \subseteq \text{Ker}(\pi_{\text{Coker}(f)}) = \text{Im}(f)\). Define \(\eta : F_0 \rightarrow \text{Im}(f)\) to be a homomorphism satisfying \(\eta(e_i) = s_1s_2s\tau(e_i) - \delta g(e_i)\) for each \(i\). So for each \(e_i' \in F_1\), we have \(\eta f(e_i') = s_1s_2s\tau f(e_i') - \delta g f(e_i') = s_1s_2s\tau f(e_i')\). Thus \(i_{\text{im}(f)}(s_1s_2s\sigma) = s_1s_2s i_{\text{im}(f)}\sigma = s_1s_2s\tau f = i_{\text{im}(f)}\eta f\). Therefore, \(\eta f = s_1s_2s\sigma\). Hence \(s_1s_2s\sigma(a_i) = s_1s_2s\sigma(e_i') = \eta f(e_i') = \eta(\sum_{j=1}^{m} r_{ij}e_j) = \sum_{j=1}^{m} r_{ij}\eta(e_j)\) with \(\eta(e_j) \in \text{Im}(f)\). So we have \(s_1s_2s a_i = s_1s_2s t_1 f(a_i) = \sum_{j=1}^{m} r_{ij} t_1^j \eta(e_j)\) with \(t_1^j \eta(e_j) \in A\) for each \(i\). \(\square\)

Recall from [18, Definition 2.1] that a short \(u\)-\(S\)-exact sequence \(0 \rightarrow A \xrightarrow{t} B \xrightarrow{g} C \rightarrow 0\) is said to be \(u\)-\(S\)-split provided that there are \(s \in S\) and \(R\)-homomorphism \(t : B \rightarrow A\) such that \(t f(a) = s a\) for any \(a \in A\), that is, \(t f = s \text{Id}_A\).

**Proposition 2.3.** Let \(\xi : 0 \rightarrow A \xrightarrow{t} B \xrightarrow{g} C \rightarrow 0\) be an \(u\)-\(S\)-split short \(u\)-\(S\)-exact sequence. Then \(\xi\) is \(u\)-\(S\)-pure.

**Proof.** Let \(t : B \rightarrow A\) be an \(R\)-homomorphism satisfying \(t f = s \text{Id}_A\). Let \(f(a_i) = \sum_{j=1}^{m} r_{ij} x_j\) be a system of equations with \(r_{ij} \in R\) and unknowns \(x_1, \ldots, x_m\) has a solution, say \(\{b_j \mid j = 1, \ldots, m\}\), in \(B\). Then \(s a_i = t f(a_i) = \sum_{j=1}^{m} r_{ij} t(b_j)\) with \(t(b_j) \in A\). Thus \(s a_i = \sum_{j=1}^{m} r_{ij} x_j\) is solvable in \(A\). So \(\xi\) is \(u\)-\(S\)-pure by Theorem 2.2. \(\square\)
Recall from [17, Definition 3.1] that an $R$-module $F$ is called $u$-$S$-flat provided that for any $u$-$S$-exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the induced sequence $0 \rightarrow A \otimes_R F \rightarrow B \otimes_R F \rightarrow C \otimes_R F \rightarrow 0$ is $u$-$S$-exact. By [17, Theorem 3.2], an $R$-module $F$ is $u$-$S$-flat if and only if $\text{Tor}_1^R(M, F)$ is $u$-$S$-torsion for any $R$-module $M$.

**Proposition 2.4.** An $R$-module $F$ is $u$-$S$-flat if and only if every $(u$-$S$-)exact sequence $0 \rightarrow A \rightarrow B \rightarrow F \rightarrow 0$ is $u$-$S$-pure.

**Proof.** Suppose $F$ is a $u$-$S$-flat module. Let $M$ be an $R$-module and $0 \rightarrow A \rightarrow B \rightarrow F \rightarrow 0$ a short $u$-$S$-exact sequence. Then by Theorem [13], there is a $u$-$S$-exact sequence $\text{Tor}_1^R(M, F) \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R F \rightarrow 0$. Since $F$ is $u$-$S$-flat, $\text{Tor}_1^R(M, F)$ is $u$-$S$-torsion by [17, Theorem 3.2]. Hence $0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R F \rightarrow 0$ is $u$-$S$-exact. So $0 \rightarrow A \rightarrow B \rightarrow F \rightarrow 0$ is $u$-$S$-pure.

On the other hand, considering the exact sequence $0 \rightarrow A \rightarrow P \rightarrow F \rightarrow 0$ with $P$ projective, we have an exact sequence $0 \rightarrow \text{Tor}_1^R(M, F) \rightarrow M \otimes_R A \rightarrow M \otimes_R P \rightarrow M \otimes_R F \rightarrow 0$ for any $R$-module $M$. Since $0 \rightarrow A \rightarrow P \rightarrow F \rightarrow 0$ is $u$-$S$-pure, $\text{Tor}_1^R(M, F)$ is $u$-$S$-torsion. So $F$ is $u$-$S$-flat. □

**Proposition 2.5.** Let $\xi : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short $u$-$S$-exact sequence where $B$ is $u$-$S$-flat. Then $C$ is $u$-$S$-flat if and only if $\xi$ is $u$-$S$-pure.

**Proof.** Suppose $C$ is $u$-$S$-flat. Then $\xi$ is $u$-$S$-pure by Proposition 2.4.

On the other hand, let $M$ be an $R$-module. Then we have a $u$-$S$-exact sequence $\text{Tor}_1^R(M, B) \rightarrow \text{Tor}_1^R(M, C) \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0$. Since $B$ is $u$-$S$-flat, $\text{Tor}_1^R(M, B)$ is $u$-$S$-torsion by [17, Theorem 3.2]. Since $\xi$ is $u$-$S$-pure by assumption, $0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R F \rightarrow 0$ is $u$-$S$-exact. Then $\text{Tor}_1^R(M, C)$ is also $u$-$S$-torsion. Thus $C$ is $u$-$S$-flat by [17, Theorem 3.2] again. □

3. Uniformly $S$-Absolutely Pure Modules

Recall from [10] that an $R$-module $E$ is said to be absolutely pure provided that $E$ is a pure submodule of every module which contains $E$ as a submodule, that is, any short exact sequence $0 \rightarrow E \rightarrow B \rightarrow C \rightarrow 0$ beginning with $E$ is pure. Now we give the uniformly $S$-analogue of absolutely pure modules.

**Definition 3.1.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $E$ is said to be $u$-$S$-absolutely pure (abbreviates uniformly $S$-absolutely pure) provided that any short $u$-$S$-exact sequence $0 \rightarrow E \rightarrow B \rightarrow C \rightarrow 0$ beginning with $E$ is $u$-$S$-pure.
Recall from [12] Definition 4.1] that an $R$-module $E$ is called $u$-$S$-injective provided that the induced sequence

$$0 \to \operatorname{Hom}_R(C,E) \to \operatorname{Hom}_R(B,E) \to \operatorname{Hom}_R(A,E) \to 0$$

is $u$-$S$-exact for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$. Following from [12] Theorem 4.3], an $R$-module $E$ is $u$-$S$-injective if and only if for any short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to \operatorname{Hom}_R(C,E) \to \operatorname{Hom}_R(B,E) \to \operatorname{Hom}_R(A,E) \to 0$ is $u$-$S$-exact, if and only if $\operatorname{Ext}^1_R(M,E)$ is $u$-$S$-torsion for any $R$-module $M$, if and only if $\operatorname{Ext}^n_R(M,E)$ is $u$-$S$-torsion for any $R$-module $M$ and $n \geq 1$. Next, we characterize $u$-$S$-absolutely pure modules in terms of $u$-$S$-injective modules.

**Theorem 3.2.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $E$ an $R$-module. Then the following statements are equivalent:

1. $E$ is $u$-$S$-absolutely pure;
2. any short exact sequence $0 \to E \to B \to C \to 0$ beginning with $E$ is $u$-$S$-pure;
3. $E$ is a $u$-$S$-pure submodule in every $u$-$S$-injective module containing $E$;
4. $E$ is a $u$-$S$-pure submodule in every injective module containing $E$;
5. $E$ is a $u$-$S$-pure submodule in its injective envelope;
6. there exists an element $s \in S$ satisfying that for any finitely presented $R$-module $N$, $\operatorname{Ext}^1_R(N,E)$ is $u$-$S$-torsion with respect to $s$;
7. there exists an element $s \in S$ satisfying that if $P$ is finitely generated projective, $K$ a finitely generated submodule of $P$ and $f : K \to E$ is an $R$-homomorphism, then there is an $R$-homomorphism $g : P \to E$ such that $sf = gi$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (6): Let $I$ be the injective envelope of $E$. Then we have a $u$-$S$-pure exact sequence $0 \to E \to I \to L \to 0$ by (5). Then, by Theorem [22], there is an element $s \in S$ such that $0 \to \operatorname{Hom}_R(N,E) \to \operatorname{Hom}_R(N,I) \to \operatorname{Hom}_R(N,L) \to 0$ is $u$-$S$-exact with respect to $s$ for any finitely presented $R$-module $N$. Since $0 \to \operatorname{Hom}_R(N,E) \to \operatorname{Hom}_R(N,I) \to \operatorname{Hom}_R(N,L) \to \operatorname{Ext}^1_R(N,E) \to 0$ is exact. Hence $\operatorname{Ext}^1_R(N,E)$ is $u$-$S$-torsion with respect to $s$ for any finitely presented $R$-module $N$.

(6) $\Rightarrow$ (1): Let $s \in S$ satisfy (6). Let $N$ be a finitely presented $R$-module and $0 \to E \to B \to C \to 0$ a $u$-$S$-exact sequence with respect to $s_1 \in S$. Then, by Theorem [14], there is a $u$-$S$-exact sequence $0 \to \operatorname{Hom}_R(N,E) \to \operatorname{Hom}_R(N,B) \to \operatorname{Hom}_R(N,C) \to \operatorname{Ext}^1_R(N,E)$ with respect to $s_1$ for any finitely presented $R$-module $N$. By (6), $0 \to \operatorname{Hom}_R(N,E) \to \operatorname{Hom}_R(N,B) \to \operatorname{Hom}_R(N,C) \to 0$ is $u$-$S$-exact.
with respect to $ss_1$ for any finitely presented $R$-module $N$. Hence $E$ is $u$-$S$-absolutely pure by Theorem 2.2.

(6) $\Rightarrow$ (7): Let $s \in S$ satisfy (6). Considering the exact sequence $0 \rightarrow K \xrightarrow{i} P \rightarrow P/K \rightarrow 0$, we have the following exact sequence $\text{Hom}_R(P, E) \xrightarrow{i^*} \text{Hom}_R(K, E) \rightarrow \text{Ext}^1_R(P/K, E) \rightarrow 0$. Since $P/K$ is finitely presented, $\text{Ext}^1_R(P/K, E)$ is $u$-$S$-torsion with respect to $s$ by (6). Hence $i_*$ is a $u$-$S$-epimorphism, and so $s\text{Hom}_R(K, E) \subseteq \text{Im}(i_*)$. Let $f : K \rightarrow E$ be an $R$-homomorphism. Then there is an $R$-homomorphism $g : P \rightarrow E$ such that $sf = gi$.

(7) $\Rightarrow$ (6): Let $s \in S$ satisfy (7). Let $N$ be a finitely presented $R$-module. Then we have an exact sequence $0 \rightarrow K \xrightarrow{i} P \rightarrow N \rightarrow 0$ where $P$ is finitely generated projective and $K$ is finitely generated. Consider the following exact sequence $\text{Hom}_R(P, E) \xrightarrow{i^*} \text{Hom}_R(K, E) \rightarrow \text{Ext}^1_R(N, E) \rightarrow 0$. By (7), we have $s\text{Hom}_R(K, E) \subseteq \text{Im}(i_*)$. Hence $\text{Ext}^1_R(N, E)$ is $u$-$S$-torsion with respect to $s$.  

$\square$

**Proposition 3.3.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then the following statements hold.

1. Any absolutely pure module and any $u$-$S$-injective module is $u$-$S$-absolutely pure.
2. Any finite direct sum of $u$-$S$-absolutely pure modules is $u$-$S$-absolutely pure.
3. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a $u$-$S$-exact sequence. If $A$ and $C$ are $u$-$S$-absolutely pure modules, so is $B$.
4. The class of $u$-$S$-absolutely pure modules is closed under $u$-$S$-isomorphisms.
5. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a $u$-$S$-pure $u$-$S$-exact sequence. If $B$ is $u$-$S$-absolutely pure, so is $B$.

**Proof.** (1) Follows from Theorem 3.2.

(2) Suppose $E_1, \ldots, E_n$ are $u$-$S$-absolutely pure modules. Then there exists $s_1 \in S$ such that $s_i\text{Ext}^1_R(M, E_i) = 0$ for any finitely presented $R$-module $M$ ($i = 1, \ldots, n$). Set $s = s_1 \cdots s_n$. Then $s\text{Ext}^1_R(M, \bigoplus_{i=1}^n E_i) \cong \bigoplus_{i=1}^n s\text{Ext}^1_R(M, E_i) = 0$. Thus $\bigoplus_{i=1}^n E_i$ is $u$-$S$-absolutely pure.

(3) Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a $u$-$S$-exact sequence. Since $A$ and $C$ are $u$-$S$-absolutely pure modules, then, by Theorem 3.2, $\text{Ext}^1_R(N, A)$ and $\text{Ext}^1_R(N, C)$ are $u$-$S$-torsion with respect to some $s_1, s_2 \in S$ for any finitely presented $R$-module $N$. Considering the $u$-$S$-sequence $\text{Ext}^1_R(N, A) \rightarrow \text{Ext}^1_R(N, B) \rightarrow \text{Ext}^1_R(N, C)$ by Theorem 1.4, we have $\text{Ext}^1_R(N, B)$ is $u$-$S$-torsion with respect to $s_1s_2$ for any finitely presented $R$-module $N$. Hence $B$ is $u$-$S$-absolutely pure by Theorem 3.2 again.
(4) Considering the $u$-$S$-exact sequences $0 \to A \to B \to 0 \to 0$ and $0 \to 0 \to A \to B \to 0$, we have $A$ is $u$-$S$-absolutely pure if and only if $B$ is $u$-$S$-absolutely pure by (3).

(5) Let $0 \to A \to B \to C \to 0$ be a $u$-$S$-pure $u$-$S$-exact sequence with respect to some $s \in S$. Then, by Theorem 1.4, there exists a $u$-$S$-sequence $0 \to \text{Hom}_R(N, A) \to \text{Hom}_R(N, B) \to \text{Hom}_R(N, C) \to \text{Ext}^1_R(N, A) \to \text{Ext}^1_R(N, B)$ with respect to $s$ for any finitely presented $R$-module $N$. Note that the natural homomorphism $\text{Hom}_R(N, B) \to \text{Hom}_R(N, C)$ is a $u$-$S$-epimorphism. Since $B$ is $u$-$S$-absolutely pure, it follows that $\text{Ext}^1_R(N, B)$ is $u$-$S$-torsion with respect to some $s_1 \in S$ for any finitely presented $R$-module $N$ by Theorem 3.2. Then $\text{Ext}^1_R(N, A)$ is $u$-$S$-torsion with respect to $ss_1$ for any finitely presented $R$-module $N$. Thus $A$ is $u$-$S$-absolutely pure by Theorem 3.2 again. □

Let $p$ be a prime ideal of $R$. We say an $R$-module $E$ is $u$-$p$-absolutely pure shortly provided that $E$ is $u$-$(R \setminus p)$-absolutely pure.

**Proposition 3.4.** Let $R$ be a ring and $E$ an $R$-module. Then the following statements are equivalent:

1. $E$ is absolutely pure;
2. $E$ is $u$-$p$-absolutely pure for any $p \in \text{Spec}(R)$;
3. $E$ is $u$-$m$-absolutely pure for any $m \in \text{Max}(R)$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) : Trivial.

(3) $\Rightarrow$ (1) : Since $E$ is $m$-absolutely pure for any $m \in \text{Max}(R)$, we have $\text{Ext}^1_R(N, E)$ is uniformly $(R \setminus m)$-torsion for any finitely presented $R$-module $N$. Thus for any $m \in \text{Max}(R)$, there exists $s_m \in S$ such that $s_m \text{Ext}^1_R(N, E) = 0$ for any finitely presented $R$-module $N$. Since the ideal generated by all $s_m$ is $R$, $\text{Ext}^1_R(N, E) = 0$ for any finitely presented $R$-module $N$. So $E$ is absolutely pure. □

Recall from [17, Definition 3.12] a ring $R$ is called uniformly $S$-von Neumann regular provided there exists an element $s \in S$ satisfies that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$. It was proved in [17, Theorem 3.13] that a ring $R$ is uniformly $S$-von Neumann regular if and only if any $R$-module is $u$-$S$-flat.

**Theorem 3.5.** A ring $R$ is uniformly $S$-von Neumann regular if and only if any $R$-module is $u$-$S$-absolutely pure.

**Proof.** Suppose $R$ is an uniformly $S$-von Neumann regular ring. Let $M$ be an $R$-module and $I$ its injective envelope. Then $M/I$ is $u$-$S$-flat by [17, Theorem 3.13]. Hence $M$ is a $u$-$S$-pure submodule of $I$ by Proposition 2.4. So $M$ is $u$-$S$-absolutely pure by Theorem 3.2.
On the other hand, let $M$ be an $R$-module and $\xi : 0 \to K \to P \to M \to 0$ an exact sequence with $P$ projective. Then $P$ is $u$-$S$-flat. Since $K$ is $u$-$S$-absolutely pure, the exact sequence $\xi$ is $u$-$S$-pure. By Proposition 2.5, $M$ is also $u$-$S$-flat. Hence $R$ is uniformly $S$-von Neumann regular by [17, Theorem 3.13]. □

It follows from Proposition 3.3 that every absolutely pure module is $u$-$S$-absolutely pure. The following example shows that the converse is not true in general.

Example 3.6. [17, Example 3.18] Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2$ be a semi-simple ring and $s = (1, 0) \in T$. Then any element $a \in T$ satisfies $a^2 = a$ and $2a = 0$. Let $R = T[x]/\langle sx, x^2 \rangle$ with $x$ the indeterminate and $S = \{1, s\}$ be a multiplicative subset of $R$. Then $R$ is an uniformly $S$-von Neumann regular ring, but $R$ is not von Neumann regular. Thus there exists a $u$-$S$-absolutely pure module $M$ which is not absolutely pure by Theorem 3.5.

Recall from [12] that a ring $R$ is called a uniformly $S$-Noetherian ring provided that there exists an element $s \in S$ such that for any ideal $J$ of $R$, $sJ \subseteq K$ for some finitely generated sub-ideal $K$ of $J$. Following from Theorem [12, Theorem 4.10] that if $S$ is a regular multiplicative subset of $R$ (i.e., the multiplicative set $S$ is composed of non-zero-divisors), then $R$ is uniformly $S$-Noetherian if and only if any direct sum of injective modules is $u$-$S$-injective. Now we give a new characterization of uniformly $S$-Noetherian rings.

Theorem 3.7. Let $R$ be a ring, $S$ a regular multiplicative subset of $R$. Then the following statements are equivalent:

1. $R$ is a uniformly $S$-Noetherian ring;
2. any $u$-$S$-absolutely pure module is $u$-$S$-injective;
3. any absolutely pure module is $u$-$S$-injective.

Proof. (1) $\Rightarrow$ (2): Suppose $R$ is a uniformly $S$-Noetherian ring. Let $s$ be an element in $S$ such that for any ideal $J$ of $R$, $sJ \subseteq K$ for some finitely generated sub-ideal $K$ of $J$. Let $E$ be a $u$-$S$-absolutely pure module. Then there exists $s_2 \in S$ such that $s_2\text{Ext}^1_R(N, E) = 0$ for any finitely presented $R$-module $N$. Let $s_1$ be an element in $S$. Consider the induced exact sequence $\text{Hom}_R(R, E) \to \text{Hom}_R(Rs_1, E) \to \text{Ext}^1_R(R/Rs_1, E) \to 0$. Since $R/Rs_1$ is finitely presented, $s_2\text{Ext}^1_R(R/Rs_1, E) = s_2(E/s_1E) = 0$ since $s_1$ is a non-zero-divisor. Then $s_2E = s_1s_2E$, and thus $s_2E$ is $u$-$S$-divisible. Since $s_2E$ is $u$-$S$-isomorphic to $E$, $s_2E$ is also $u$-$S$-absolutely pure by Proposition 3.3. Hence there exists $s_3 \in S$ such that $s_3\text{Ext}^1_R(N, E) = 0$ for any finitely presented $R$-module $N$. Consider the induced $u$-$S$-exact sequence...
\[
\text{Hom}_R(J/K, s_2E) \rightarrow \text{Ext}^1_R(R/J, s_2E) \rightarrow \text{Ext}^1_R(R/K, s_2E). \]
Since \( R/K \) is finitely presented, we have \( s_2\text{Ext}^3_R(R/K, s_2E) = 0. \) Note that \( s\text{Hom}_R(J/K, s_2E) = 0. \) Then \( s\text{Ext}^4_R(R/J, s_2E) = 0. \) Since \( s_2E \) is \( u \)-\( S \)-divisible, we have \( s_2E \) is \( u \)-\( S \)-injective by [12, Proposition 4.9]. Since \( s_2E \) is \( u \)-\( S \)-isomorphic to \( E, \) \( E \) is also \( u \)-\( S \)-injective by [12, Proposition 4.7].

(2) \( \Rightarrow \) (3): Trivial.

(3) \( \Rightarrow \) (1): Let \( \{I_{\lambda} \mid \lambda \in \Lambda\} \) be a family of injective modules, then \( \bigoplus_{\lambda \in \Lambda} I_{\lambda} \) is absolutely pure, and thus is \( u \)-\( S \)-injective by assumption. Consequently, \( R \) is a uniformly \( S \)-Noetherian ring by [12, Theorem 4.10]. \( \square \)

It is well-known that any direct sum and any direct product of absolutely pure modules are also absolutely pure. However, it does not work for \( u \)-\( S \)-absolutely pure modules.

**Example 3.8.** Let \( R = \mathbb{Z} \) be the ring of integers, \( p \) a prime in \( \mathbb{Z} \) and \( S = \{p^n \mid n \geq 0\}. \) Then an \( R \)-module \( M \) is \( u \)-\( S \)-absolutely pure module if and only if it is \( u \)-\( S \)-injective by Theorem 3.7. Let \( \mathbb{Z}/(p^k) \) be cyclic group of order \( p^k \) \((k \geq 1)\). Then each \( \mathbb{Z}/(p^k) \) is \( u \)-\( S \)-torsion, and thus is \( u \)-\( S \)-absolutely pure. However, the product \( M := \prod_{k=1}^{\infty} \mathbb{Z}/(p^k) \) is not \( u \)-\( S \)-injective by [12, Remark 4.6], so it is also not \( u \)-\( S \)-absolutely pure.

We claim that the direct sum \( N := \bigoplus_{k=1}^{\infty} \mathbb{Z}/(p^k) \) is also not \( u \)-\( S \)-absolutely pure. Indeed, consider the following exact sequence induced by the short exact sequence \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0: \)

\[
0 = \text{Hom}_\mathbb{Z}(\mathbb{Q}, N) \rightarrow \text{Hom}_\mathbb{Z}(\mathbb{Z}, N) \rightarrow \text{Ext}^1_\mathbb{Z}(\mathbb{Q}/\mathbb{Z}, N) \rightarrow \text{Ext}^1_\mathbb{Z}(\mathbb{Q}, N) \rightarrow 0.
\]

Since the submodule \( N = \text{Hom}_\mathbb{Z}(\mathbb{Z}, N) \) is not \( u \)-\( S \)-torsion, \( \text{Ext}^1_\mathbb{Z}(\mathbb{Q}/\mathbb{Z}, N) \) is also not \( u \)-\( S \)-torsion. Then \( N \) is not \( u \)-\( S \)-injective by [12, Theorem 4.3]. So the direct sum \( N := \bigoplus_{k=1}^{\infty} \mathbb{Z}/(p^k) \) is also not \( u \)-\( S \)-absolutely pure.

We also note that, in Theorem 3.2, the element \( s \in S \) in the statement (6) (similar in the statement (7)) is uniform for all finitely presented \( R \)-modules \( N \).

**Example 3.9.** Let \( R = \mathbb{Z} \) be the ring of integers, \( p \) a prime in \( \mathbb{Z} \) and \( S = \{p^n \mid n \geq 0\}. \) Let \( J_p \) be the additive group of all \( p \)-adic integers (see [5] for example). Then \( \text{Ext}^1_R(N, J_p) \) is \( u \)-\( S \)-torsion for any finitely presented \( R \)-modules \( N \). However, \( J_p \) is not \( u \)-\( S \)-absolutely pure.

**Proof.** Let \( N \) be a finitely presented \( R \)-module. Then, by [3] Chapter 3, Theorem 2.7, \( N \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{m} (\mathbb{Z}^n / (p^i))^{n_i} \oplus T, \) where \( T \) is a finitely generated torsion \( S \)-divisible
torsion-module. Thus
\[ \text{Ext}^1_R(N, J_p) \cong \bigoplus_{i=1}^m \text{Ext}^1_R(\mathbb{Z}^n / \langle p^i \rangle, J_p) \cong \bigoplus_{i=1}^m (J_p/p^i J_p) \cong \bigoplus_{i=1}^m \mathbb{Z}^n / \langle p^i \rangle \]
by [5, Chapter 9, section 3 (G)] and [5, Chapter 1, Exercise 3(10)]. So \( \text{Ext}^1_R(N, J_p) \) is obviously \( u \)-\( S \)-torsion. However, \( J_p \) is not \( u \)-\( S \)-injective by [12, Theorem 4.5]. So \( J_p \) is not \( u \)-\( S \)-absolutely pure by Theorem 3.7. \( \Box \)

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