The Area Operator in the Spherically Symmetric Sector of Loop Quantum Gravity

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Abstract

Utilizing the previously established general formalism for quantum symmetry reduction in the framework of loop quantum gravity the spectrum of the area operator acting on spherically symmetric states in 4 dimensional pure gravity is investigated.

The analysis requires a careful treatment of partial gauge fixing in the classical symmetry reduction and of the reinforcement of $SU(2)$-gauge invariance for the quantization of the area operator. The eigenvalues of that operator applied to the spherically symmetric spin network states have the form

$$A_n \propto \sqrt{n(n+2)}, \quad n = 0, 1, 2, \ldots,$$

giving $A_n \propto n$ for large $n$.

The result clarifies (and reconciles!) the relationship between the more complicated spectrum of the general (non-symmetric) area operator in loop quantum gravity and the old Bekenstein proposal that $A_n \propto n$.

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1 Introduction

In a preceding paper [1] we proposed a framework for implementing symmetry reductions for gravitational systems quantized within the loop quantum gravity approach (see the reviews [2]). In the present note we apply that framework to the spectrum of the operator associated with 2-dimensional areas. The problem is of considerable physical interest because the 2-dimensional horizon of a Schwarzschild black hole constitutes such a system the area of which is, up to a factor, a measure for the entropy of the black hole.

The quantum area spectrum of the horizon of (Schwarzschild) black holes in 4-dimensional space-time has a longer history: Already in 1974 Bekenstein, using Bohr-Sommerfeld type arguments [3], suggested that the area \( A = 4\pi R_S^2 \), \( R_S = \frac{2GM^2}{c^2} \) of a (spherically symmetric) Schwarzschild black hole of mass \( M \) has an angular momentum like quantum area spectrum, \( A(n) \propto n, n \in \mathbb{N} \equiv \{n = 1, 2, \ldots \} \), yielding an energy spectrum \( E_n \propto \sqrt{n} \). In the meantime such a spectrum has been argued for by many authors (for details and the corresponding literature see Refs. [4, 5, 6]).

A very recent group theoretical quantization based on the classical canonical structure of the Schwarzschild system in \( D (\geq 4) \) space-time dimensions and the group \( SO^+(1, 2) \) yields the spectrum [7]

\[
A_{D-2}(k; n) \propto (k + n), \quad n \in \mathbb{N}_0 \equiv \{n = 0, 1, 2, \ldots \}, \tag{1}
\]

where \( k \) characterizes the irreducible unitary representation of \( SO^+(1, 2) \) or its covering groups: For \( SO^+(1, 2) \) itself we have \( k \in \mathbb{N} \), for its two-fold coverings \( SU(1, 1) \cong SL(2, \mathbb{R}) \) \( k \in (1/2)\mathbb{N} \) and for the universal covering group \( k \) may be any real number \( > 0 \).

On the other hand the spectrum of the general (non-symmetric) area operator in loop quantum gravity is more complicated [8, 9, 10]: Possible eigenvalues of the area operator in this theory are

\[
A \propto \sum_p \sqrt{j_p(j_p + 1)}, \tag{2}
\]

where \( p \) labels points at which the surface is intersected by a spin network and \( j_p \in \frac{1}{2}\mathbb{N}_0 \) is the spin of the edge intersecting the surface in \( p \). Here we have ignored the singular case that the surface is intersected in a vertex of the spin network.

There is an important difference between the spectra (1) and (2): Whereas for the former the distance between successive eigenvalues remains the same...
for any \( n \) that distance becomes smaller and smaller with increasing area for the spectrum (2) \([10]\). This result has led to expressions of doubts \([12, 13, 2, 14]\) as to the physical validity of the spectrum (1) and its possible implications for the structure of the semi-classical Hawking radiation \([15]\).

Using the framework of Ref. \([1]\) we shall show how the spectra (1) and (2) are related and how the two approaches are to be reconciled:

An observation to start with is that the spectrum (2) contains the other spectrum (1) as a subset: If there are \( n \) edges intersecting the surface, all labeled with the same spin, e.g. \( j = \frac{1}{2} \), then the area eigenvalue will be \( A_n \propto n \). In Ref. \([16]\) the horizon has been treated in loop quantum gravity as a boundary with appropriate boundary conditions which fix a direction in the internal \( SU(2) \)-space thereby breaking \( SU(2) \) to \( U(1) \). This effects a replacement of the Casimir operator \( J^2 \) with eigenvalues \( j(j + 1) \) labeling irreducible representations of \( SU(2) \) by the parameter \( n \) labeling irreducible representations of \( U(1) \). According to Ref. \([11]\) this leads to a spectrum \( A \propto \sum_p j_p \in \frac{1}{2} \mathbb{N}_0 \) which is again of a form of the old area spectrum (1).

Furthermore, it has been argued in Ref. \([11]\) that the loop quantum gravity area operator of Refs. \([8, 10]\) measures the area of surfaces including all fluctuations, with the restricting boundary condition, however, that transversal fluctuations are suppressed at the horizon: The direction of the normal to the surface cannot fluctuate.

Such a boundary condition is analogous to the situation in a symmetry reduced model: By imposing, e.g. spherical symmetry one allows only for symmetric solutions and therefore only for fluctuations respecting that symmetry.

In the present note we, accordingly, analyze properties of the area operator in a spherically symmetric sector of loop quantum gravity along the lines suggested in \([1]\). Analogously to corresponding boundary conditions at the horizon, the classical symmetry reduction involves a reduction of the gauge group from \( SU(2) \) to \( U(1) \). This is means a partial gauge fixing, which will be undone in the quantum theory.

In the next section we shall summarize the essential steps characterizing the classical spherical symmetry reduction and the corresponding setup for the quantum theory. Section 3 contains an analysis of the spectrum of the area operator respecting spherical symmetry. It has the form

\[
A \propto \sqrt{j(j + 1)} \quad \text{(no sum over punctures)}.
\] (3)

Properties of that spectrum are discussed in the final section.
2 Symmetry Reduction and Partial Gauge Fixing

The quantum symmetry reduction of Ref. [1] uses the classification of symmetric principal fibre bundles and invariant connections thereon \([17, 18, 19]\).

In this framework a symmetry group \(S\) acts on a principal fibre bundle \(P(\Sigma, G, \pi)\) with compact structure group \(G\) over the base manifold \(\Sigma\) which is a spacelike hypersurface of the space-time used to carry out the canonical formalism. The symmetry group is a finite-dimensional subgroup of the group of bundle automorphisms, i.e. it commutes with the right action of \(G\) on \(P\). Of special importance is the isotropy subgroup which here is assumed to be the same for all points in \(P\) (if not, the base manifold can be decomposed into components all having the same isotropy subgroup. This amounts to cutting out symmetry centers and axes).

The isotropy subgroup \(F < S\) of points in \(P\) acts on each fibre and therefore determines a homomorphism \(\lambda_p: F \to G\) by \(f(p) =: p \cdot \lambda_p(f)\) for all \(f \in F\) and \(p \in P\). Homomorphisms \(\lambda_p\) and \(\lambda_{p'g}\) related to different points in the same fibre differ only by conjugation: \(\lambda_{p'g} = \text{Ad}_{g^{-1}} \circ \lambda_p\) for \(g \in G\).

By the action of \(S\) the base manifold \(\Sigma\) becomes an orbit bundle \(\Sigma \cong B \times S/F\) with base manifold \(B \cong \Sigma/S\) and orbits \(S/F\). By choosing an analytic section in this orbit bundle the base \(B\) can be embedded analytically in \(\Sigma\). The bundle \(P\) can be restricted to a principal fibre bundle \(P|_B\) over \(B\), which can be further reduced by defining \(Q_\lambda := \{p \in P|_B : \lambda_p = \lambda\}\). This reduction uses a fixed homomorphism \(\lambda: F \to G\), and the reduced bundles are principal fibre bundles over \(B\) with structure group \(Z_\lambda := Z_G(\lambda(F))\), the centralizer in \(G\) of \(\lambda(F)\). All symmetric principal fibre bundles \(P\) are classified by a conjugacy class \([\lambda]\) of homomorphisms and a reduced bundle \(Q\). As noted above, the homomorphisms \(\lambda_p\) get conjugated by changing the point \(p\) in the fibre. Therefore, all homomorphisms in the conjugacy class \([\lambda]\) are equivalent for classifying the symmetric bundle \(P\); selecting one of them amounts to a partial gauge fixing breaking the structure group \(G\) down to \(Z_\lambda\).

Analogously an invariant connection \(\omega\) on a symmetric fibre bundle, classified by \(\lambda\) and \(Q_\lambda\), leads to a \(Z_\lambda\)-connection on \(Q_\lambda\) by restriction. To see that fix a point \(p \in P\) and a vector \(v\) in \(T_pP\). Then the pull back of \(\omega\) by \(f \in F\) applied to \(v\) is by definition \(f^*\omega_p(v) = \omega_{f(p)}(df(v))\). If we now use the fact that \(f\) acts as gauge transformation in the fibres and observe the definition of \(\lambda_p\) and the adjoint transformation of \(\omega\), we obtain
\[\omega_{f(p)}(df(v)) = \text{Ad}_{\lambda_p(f)}^{-1} \omega_p(v) \quad (\omega_p \text{ annihilates by definition the horizontal part of } v, \text{ which only is changed by } df). \]

By assumption the connection \(\omega\) is \(S\)-invariant implying \(f^* \omega_p(v) = \text{Ad}_{\lambda_p(f)}^{-1} \omega_p(v) = \omega_p(v)\). This shows that \(\omega_p(v) \in Z_G(\lambda_p(F))\), and \(\omega\) can be restricted to a connection on the bundle \(Q_\lambda\) with structure group \(Z_\lambda\).

Besides the reduced connection on \(Q_\lambda\) there are several scalar fields, jointly denoted as “Higgs” field in the following, which stem from the components of \(\omega\) tangential to the \(S\)-orbits. The reduced connection together with the Higgs field suffice to classify the invariant connection completely: \(\omega\) can be reconstructed out of this data. This observation was exploited in Ref. [1] by using the reconstruction map to pull back a cylindrical function on the space of connections over \(\Sigma\), i.e. a function in the auxiliary Hilbert space of the unreduced gauge theory, to a function on the space of connections and Higgs fields over \(B\). However, the reduction of the structure group had to be undone to carry out the quantization procedure in the general framework. In the following we will describe this in more detail for the example of spherical symmetry.

We saw that the classification of symmetric bundles and invariant connections makes use of a partial gauge fixing by choosing \(\lambda \in [\lambda]\). However, the full \(G\)-transformations are implemented by acting on the classifying structure: \(Z_\lambda\)-gauge transformations are gauge transformations on the reduced bundle \(Q_\lambda\); they constitute the reduced gauge group. All elements of \(G\) which are not contained in \(Z_\lambda\) change \(\lambda\) by conjugation. They change the reduced bundles in an equivalence class, all of which yield the same symmetric bundle after reconstruction.

From now on we specialize to \(S = SU(2) = G, F = U(1)\) in order to describe spherically symmetric solutions of general relativity in the real Ashtekar formulation. In this case selecting one \(\lambda \in [\lambda]\) amounts to fixing an internal axis (a point in \(S^2\)) and \(Z_\lambda\)-gauge transformations are rotations around this axis, whereas a conjugation of \(\lambda\) rotates the axis. The group \(G = SU(2) \cong S^3\) acting on \(\lambda\) by conjugation has the isotropy subgroup \(Z_\lambda \cong U(1)\). Factoring out this subgroup leads to the Hopf map \(S^3 \to S^2\). The spacelike section \(\Sigma\) of space-time is by the action of \(S\) an orbit bundle with \(S^2\)-fibres over the one-dimensional base manifold \(B\), which can be embedded in \(\Sigma\) as a radial manifold by choosing a section in the orbit bundle.

We can represent \(F = \exp(\tau_3)\) (\(\langle \cdot \rangle\) denotes the linear span) as subgroup of \(S\), where \(\tau_j = -\frac{i}{2} \sigma_j\) are \(SU(2)\)-generators with the Pauli matrices \(\sigma_j\). Then all conjugacy classes of homomorphisms \(\lambda: F \to G\) are given by their
representatives \( \lambda_k : \exp(t\tau_3) \mapsto \exp(kt\tau_3), k \in \mathbb{N}_0 \). For \( k \neq 1 \) they lead to degenerate sectors of vanishing volume (see Ref. [1] for details) and we will be mainly concerned with the \((k = 1)\)-sector in the following. This is the only sector exhibiting a non-vanishing Higgs field; the symmetry reduction leads to the well known connections which are symmetric up to gauge transformations [21].

The only \( SU(2) \)-gauge transformations fixing \( \lambda_1 \) are generated by \( \tau_3 \), all other gauge transformations change \( \lambda \) in its conjugacy class. In this \( \tau_3 \)-gauge we have \( d\lambda_1(\tau_3) = \tau_3 \), whereas in an arbitrary \( \lambda \)-gauge, \( \lambda \in [\lambda_1] \), we have \( d\lambda(\tau_3) = g^{-1}\tau_3 g, g \in SU(2) \). In order to characterize this general gauge by coordinates we parameterize \( SU(2) \) by Euler angles: \( g = g_3(\psi)g_1(\vartheta)g_3(\varphi) \) where \( g_i(t) := \exp t\tau_i \). This yields

\[
d\lambda(\tau_3) = \sin \vartheta \sin \varphi \tau_1 + \sin \vartheta \cos \varphi \tau_2 + \cos \vartheta \tau_3 =: n^i\tau_i
\]

with \( n^i n_i \equiv \vec{n}^2 = 1, n_i = \delta_{ij}n^j \).

Fixed \( \vec{n} \) corresponding to a fixed \( \lambda \in [\lambda_1] \) is analogous to a fixed direction in \( SU(2) \) introduced in [11]. However, as shown above, a chosen \( \vec{n} \) represents pure gauge and physical states and observables should, of course, be independent of the choice.

The classical phase space of the symmetry reduced theory consists of fields \((A_I, E^I), 1 \leq I \leq 3\), where \( A_1 \) is the component of the reduced connection over \( B \), \( E^1 \) is its conjugate momentum, and \( A_2, A_3 \) are Higgs field components with conjugate momenta \( E^2, E^3 \). We will be concerned only with \( A_1, E^1 \) because the 2-dimensional area \( A \) is classically given by

\[
A = 4\pi|E^1| . \tag{4}
\]

In the \( \lambda \)-gauge we have the \( U(1) \)-connection form

\[
A_1 n^i\tau_i \, dx ,
\]

where \( x \) is a (local) coordinate on \( B \), and the \( LU(1) \)-valued field \( E^1 n^i\tau_i \). Their symplectic structure is given by

\[
\{A_1(x), E^1(y)\} = \frac{\kappa l}{4\pi}\delta(x, y) \tag{5}
\]

with \( \kappa = 8\pi G \) and the Immirzi parameter \( \iota \).
Without partial gauge fixing the fields would be $SU(2)$-valued and given by
\[ A^i \tau_i \, dx \quad \text{and} \quad E_i \tau^i \]
with
\[
A^1 = A \sin \vartheta_A \sin \varphi_A, \quad A^2 = A \sin \vartheta_A \cos \varphi_A, \quad A^3 = A \cos \vartheta_A, \\
E_1 = E \sin \vartheta_E \sin \varphi_E, \quad E_2 = E \sin \vartheta_E \cos \varphi_E, \quad E_3 = E \cos \vartheta_E,
\]
in spherical coordinates and with symplectic structure
\[
\{ A^i(x), E^j(y) \} = \frac{\kappa}{4\pi} \delta^i_j \delta(x, y). \tag{6}
\]
(The indices $I = 1$ in Eq. (5) denote space indices whereas the indices $i, j$ in Eq. (6) are $SU(2)$ indices which are lowered or raised in terms of the Killing metric $(\delta_{ij})$.) By using the spherical coordinates we can symplectically embed the phase space $(A_1, E^1)$ in $\lambda$-gauge as a ray in the phase space of $SU(2)$-valued fields: $A_1 \mapsto A_1 n^i = A^i, E^1 \mapsto E^1 n_i = E_i$ with $\vartheta_A = \vartheta_E$ and $\varphi_A = \varphi_E$ fixed such that the direction of $n^i$ is given by the angles $\vartheta_A, \varphi_A$.

If starting quantization from the phase space $(A_1, E^1)$ we would arrive at $U(1)$-spin networks. However, this renders the partial gauge fixing manifest and even worse, a Higgs field cannot be included easily in Higgs vertices using the framework of Ref. [21] because it transforms in the adjoint representation of $SU(2)$, not of $U(1)$ which is, of course, the trivial representation. As described in Ref. [1] we can undo the partial gauge fixing in the quantum theory by extending the spin networks by $SU(2)$-gauge invariance to spin network functions on the space of $SU(2)$-connections and appropriate Higgs fields over $B$. The spin network functions then depend not only on $A_1$ (and Higgs field components) but on all $SU(2)$-components $A^i$ introduced above. Accordingly, the $SU(2)$-components $E_i$ get quantized to functional derivatives
\[
\hat{E}_i(x) = \frac{\hbar \kappa i}{4\pi} \delta \frac{\delta}{\delta A^i(x)}.
\]
This will be crucial for the quantization of the area operator.

Before addressing that problem we mention that the extension to $SU(2)$-invariant spin networks can be understood as integrating the partial gauge fixings $\vec{n}$ over $S^2$ for any edge in the graph underlying the spin network state. In order to make this precise we use coherent states on $SU(2)$ introduced in
Ref. [22]: These are defined for a fixed irreducible representation of $SU(2)$ with weight $j$ and a state $|j, m\rangle$ therein by

$$|m, \vec{n}\rangle_j := \pi^{(j)}(g_3(\varphi)g_1(\vartheta)) |j, m\rangle, \quad \text{for all } \vec{n} \in S^2. \quad (7)$$

Here $\pi^{(j)}$ is the irreducible $SU(2)$-representation of weight $j$, and $\vartheta, \varphi$ are coordinates of $\vec{n}$ in $S^2$. These coherent states are (over-)complete for any fixed $j, m$, namely

$$\frac{2j + 1}{4\pi} \int_{S^2} d^2n |m, \vec{n}\rangle_j \langle m, \vec{n}|_j = \pi^{(j)}(1) \quad (8)$$

is the identity operator in the $j$-representation. We can now project each edge holonomy in an $SU(2)$-spin network to a $U(1)$-holonomy by inserting the projector $|m, \vec{n}\rangle_j \langle m, \vec{n}|_j$ in each edge with spin $j$, where $m; -j \leq m \leq j$ is arbitrary but fixed (for each $j$). At this point there arises an arbitrariness because any $SU(2)$-representation splits into several representations (labeled by $m$) of a $U(1)$-subgroup. This yields the holonomies of a $U(1)$-spin network in the $\lambda$-gauge where $\lambda$ can be chosen arbitrarily for each edge (in the classification one uses only $\lambda$ which are constant along $B$ for simplicity. Such a choice is always possible by choosing an appropriate section in $P|_B$. However, $\lambda$ is defined by the action of $F$ in each point of $P$ and can, of course, vary in different points). Note, however, that we have no such projection procedure for Higgs vertices; a projection of a full spin network and, therefore, a partial gauge fixing in the quantum theory can completely be done only in the degenerate sectors which have no Higgs vertices $\square$.

Arrived at a $U(1)$-spin network (and disregarding Higgs vertices), we can multiply the corresponding states for each edge with $(4\pi)^{-1}(2j + 1)$ and integrate $\vec{n}$ over $S^2$. Using the completeness relation (8) we see that we get back the original $SU(2)$-spin network.

3 Area Operator

Before quantizing the (spherically symmetric) area operator without gauge fixing let us first rephrase the results of Ref. [11] in terms of our framework by using the coherent states [3] and quantize the area in its $\lambda$-gauge fixed form:

The angular component of the metric tensor is given by $|E^1|d\Omega^2$, which leads to the classical expression $A(x) = 4\pi |E^1(x)|$ for the area of a $S^2$-orbit
intersecting the radial manifold $B$ in the point $x$ (in a spherically symmetric theory these are the only surfaces whose area can be defined). Writing

$$A(x) = 4\pi |E^1(x) n_i n^i| = 4\pi |E_i(x) n^i|$$

we can readily quantize it on a gauge fixed edge (projected from a $SU(2)$-edge of spin $j$) containing $x$ by using

$$n^i \hat{E}_i(x) := \frac{\hbar \kappa l}{4\pi} |m, \vec{n})_j n^i J_i \langle m, \vec{n}| = \frac{l_p^2}{4\pi} m |m, \vec{n})_j \langle m, \vec{n}|_j$$

where $J_i$ is the angular momentum operator acting on the coherent state.

However, this quantization depends on what quantum number $m$ we choose for the projection by means of the coherent state. We can justify the choice $m = \pm j$ by demanding that we should be able to recover the spin of the edge uniquely from the projected data. The simplest way of doing so is given by such a selection of $m(j)$, namely choosing $m = j$ is analogous to the extremization used in Ref. [11]. In this way we get the spectrum

$$\frac{1}{2} l_p^2 N_0$$

for the operator, analogously to Ref. [11].

However, in doing so we have used a partial gauge fixing which, as discussed above, is inappropriate in the non-degenerate sector because of the Higgs vertices. If $x$ is a Higgs vertex then we cannot use this operator because we cannot project at that point to a $U(1)$-spin network. This quantization is appropriate only in the degenerate sectors discussed in Ref. [1].

We now (finally!) quantize the area in the non-degenerate sector by using $SU(2)$-gauge invariant spin networks and at the same time undoing any $\lambda$-gauge fixing. We begin by rewriting the area into the form

$$A(x) = 4\pi |E^1(x)| = 4\pi \sqrt{(E^1)^2 n_i n^i} = 4\pi \sqrt{E^i E_i}$$

(9)

which is similar to the area

$$A(S_x) = \int_{S_x} d^2\sigma \sqrt{E^i E_i}$$

of the orbit $S_x$ intersecting $B$ in $x$ in the non-symmetric theory.
From now on we can proceed analogously to the quantization of the area operator in the non-symmetric theory [10]. As discussed in the last section, $E_i$ gets quantized to

$$\frac{i l_P^2}{4 \pi i} \frac{\delta}{\delta A^i}$$

which acts on the $SU(2)$-holonomy $h_e = \mathcal{P} \exp \int_e dx A^i \tau_i$ along the edge $e: [0,1] \to B$.

In order to consider a general point $x$ which can be a Higgs vertex we assume that each edge containing $x$ starts in $x$. We then have two outgoing radial edges, one oriented like $B$ itself which we denote as $e_+$ and one oriented oppositely to $B$ which we denote as $e_-$, and possibly a Higgs vertex in $x$ which does not depend on $A^i$ and which can be understood as representing edges tangential to the surface $S_x$. By applying the functional derivative $\delta/\delta A_i(x)$ to a cylindrical function $f_\gamma$ with $\gamma$ containing the edges $e_+, e_-$ and the Higgs vertex $x$ we get

$$\hat{E}_i(x)f_\gamma = \frac{il_P^2}{4\pi i} \sum_{\epsilon \in \{+,-\}} \int_{e_\epsilon} dy \delta(x,y) \text{tr} \left( (\tau_i h_{e_\epsilon})^T \frac{\partial}{\partial h_{e_\epsilon}} \right) f_\gamma \quad (10)$$

Here $J^i_{e_\epsilon} = -iX^i_{e_\epsilon}$ is given by the $i$-th component of the right invariant vector field on $SU(2)$. This leads to the area operator

$$\hat{A}(x) = \frac{1}{2} l_P^2 \sqrt{(J_{e_+} - J_{e_-})^2} = \frac{1}{2} l_P^2 \sqrt{2J^2_{e_+} + 2J^2_{e_-} - (J_{e_+} + J_{e_-})^2} \quad , \quad (11)$$

with eigenvalues

$$\frac{1}{2} l_P^2 \sqrt{2j_+(j_+ + 1) + 2j_-(j_- + 1) - j_v(j_v + 1)} \quad . \quad (12)$$

Here the edges $e_+$ and $e_-$ carry the spin $j_+$ and $j_-$, respectively, and $j_v$ labels the vertex contractor: If $x$ is a Higgs vertex the associated Higgs point holonomy labeled by a spin $j$ can be visualized as a loop with spin $j$ based in $x$. This is in accordance with the Gauß constraint which can be regularized to a sum of invariant vector fields containing a left ($J^{(L)}_H$) and a right ($J^{(R)}_H$) invariant one for the Higgs field [11]:

$$J_{e_+} + J_{e_-} =: J^v_{e_v} = J^{(L)}_H - J^{(R)}_H.$$
Thus $x$ becomes a 4-vertex whose contractor can be determined by splitting the vertex into two 3-vertices with a new edge $e_v$ connecting the edges $e_+$ and $e_-$ with the Higgs loop. It is labeled by the spin $j_v$ appearing in the eigenvalue of the area operator. Of course, the Higgs loop as well as the $j_v$-edge have no spatial extension in the manifold $B$. Because the Higgs field contributes by left and right invariant vector fields leading to the loop the spin $j_v$ can only be integer valued. This fact has an immediate consequence on the topology-dependence of the area operator discussed in the next section. Here we note that in an appropriate topology of $\Sigma$ the area spectrum is given by all values of the form (12) where $j_v \in \mathbb{N}_0$ and $j_+ \in \frac{1}{2}\mathbb{N}_0$ are arbitrary whereas $j_-$ is restricted by $|j_+ - j_v| \leq j_- \leq j_+ + j_v$. In general, however, the topology can impose restrictions on the possible values of the form (12) occurring in the area spectrum.

4 Discussion

The above area operator in the spherically symmetric sector which we obtained by restoring the full $SU(2)$-gauge invariance resembles the one in the non-symmetric theory. The only, however crucial, difference comes from the simpler topology of one-dimensional graphs. Therefore we have no sum over vertices lying on the surface, but only one vertex which represents the whole surface. This difference influences the area spectrum considerably: Disregarding vertex contributions we get the spectrum

$$A(j) = \iota l_P^2 \sqrt{j(j+1)} , \quad j \in \frac{1}{2}\mathbb{N}_0 , \quad (13)$$

which is only a small subset of the corresponding spectrum (2) in the non-symmetric theory. In particular, for large $j$ the spectrum becomes not dense, but equidistant, and in the large $j$ limit we obtain the spectrum (1) of the horizon area described in the introduction.

Thus we have shown that loop quantum gravity in its spherically symmetric sector reproduces for large spins, i.e. in the assumed semiclassical regime, the older results while it leads to corrections for small $j$, i.e. at the Planck scale.

As in the case of the area operator in the non-symmetric theory the spectrum of the spherically symmetric one depends on the topology of space. Here any surface whose area we can measure in a spherically symmetric theory has, of course, the topology of $S^2$. However, there are essentially two
possible space topologies: A topology with two (or more) boundary components and second homology $H_2(\Sigma) = \mathbb{Z}$, and one with a single boundary component and $H_2(\Sigma) = 0$ (we regard spatial infinity as a boundary). In the first case we have two physical realizations: The wormhole topology $B = \mathbb{R}$, $\Sigma = \mathbb{R} \times S^2$ represents a spacelike hypermanifold in the Kruskal extension of Schwarzschild (or Reissner-Nordstrom) space-time and has two boundary components at $\pm \infty$, whereas the topology $B = \mathbb{R}_+, \Sigma = \mathbb{R}^3\setminus\{0\}$ can be seen as simulating an external, non-dynamical gravitational source sitting in the origin, i.e. in one of the two boundary components of $\Sigma$.

The topology with only one boundary component is given by $B = \mathbb{R}_+ \cup \{0\}, \Sigma = \mathbb{R}^3$ and has only the boundary at spacelike infinity. Here we have to treat the symmetry center in 0 along the general framework of symmetry reduction. The isotropy subgroup is $F = S = SU(2)$ and the homomorphism $\lambda: F \to G$ is either $\overline{\lambda}_0: g \mapsto 1$, $\overline{\lambda}_1: g \mapsto g \cdot \{\pm 1\} \in G\setminus\{\pm 1\} \cong SO(3)$ or $\overline{\lambda}_1: g \mapsto g$ (up to conjugacy) for all $g \in S$. This can be seen from the fact that the kernel of $\lambda$ is an invariant subgroup of $S$ which can only be $S$, $\{\pm 1\}$ or $\{1\}$. By continuity, in the rest of $B$ we must use the homomorphism $\overline{\lambda}_k$ if in 0 we use $\overline{\lambda}_k$ ($\overline{\lambda}_0$ and $\overline{\lambda}_1$ make no difference). This shows that we have only the possibilities $k = 0, 1$ if the symmetry center is contained in $\Sigma$.

The two possibilities lead to manifestly invariant connections ($k = 0$) and to connections invariant up to gauge ($k = 1$). In all these cases there can be no Higgs field in 0, which is in accordance with the fact that the Higgs field represents components of an invariant connection tangential to the $S$-orbits which is a single point in the case of 0. An immediate consequence of this fact is that $F = S$, which implies $LF \perp = \{0\}$ (in the Cartan-Killing metric) and therefore the Higgs field which is a map $\phi: LF \perp \to LG$ vanishes.

We can now consider the implications of these considerations as to gauge invariant spin networks. The crucial observation is that in a Higgs vertex the spins of the neighboring edge holonomies can differ only by an integer value because the spin $j_v$ mentioned above is integer-valued due to the Higgs loop in the vertex. If $\Sigma$ has two boundary components we do not have to enforce $SU(2)$-gauge invariance in the two boundary points of $B$ and the edges can be either all integer valued or all half-integer valued. This leads to the full spectrum (1) given above. However, if $0 \in B$ corresponds to the symmetry center, i.e. an inner point of $\Sigma$ implying that there can lie no Higgs vertex, we have to impose $SU(2)$-gauge invariance in 0. The edge incident in 0 can only have spin 0 which implies that the other edge spins can only be integer
valued. This fact allows only a subset of (12) as area spectrum.

As a last remark we note that the spectroscopy for spherically symmetric black holes (cf. e.g. [4]) is unaltered by our area spectrum (13) because it becomes uniformly spaced for large $j$, like the spectrum (1). This is not the case for the full area spectrum (2) of the non-symmetric theory which becomes almost continuous.

The large discrepancy between these two spectra may be understood as a line splitting due to a broken symmetry. Because of the discrete structure of space made explicit by a spin network spherical symmetry is strongly broken by a state in the non-symmetric theory. As is well known in quantum theory, breaking a symmetry can lead to a splitting of levels which are degenerate before breaking the symmetry. In our case the degeneracy of the levels of a black hole is expected to be very huge, growing exponentially with $j$ (see section 3 of Ref. 3 and the literature mentioned there). Splitting of these strongly degenerate levels by a broken symmetry may lead to an almost continuous spectrum as observed in the non-symmetric theory. This observation may also open up a new way to calculate the degeneracy of the energy levels of black holes in loop quantum gravity.

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