STABILITY, SHARDS, AND PREPROJECTIVE ALGEBRAS

HUGH THOMAS

The goal of this note two-fold. First, I would like to draw attention to the way that stability gives us a geometrical picture of (some of) the extension-closed abelian subcategories of a finite-dimensional algebra. Second, I will describe Nathan Reading’s shards of a hyperplane arrangement, and explain their relevance to understanding the stability picture for finite-type preprojective algebras.

1. Semistable subcategories

Let $A$ be a finite-dimensional algebra over a field $k$. We will work with the category of left $A$-modules. Suppose that $A$ has $n$ pairwise non-isomorphic simple modules $S_1, \ldots, S_n$.

The Grothendieck group of $A$ can be defined as the free abelian group on a set of generators corresponding to the simple modules:

$$K_0(A) = \bigoplus_{i=1}^n \mathbb{Z}[S_i].$$

For any $A$-module $M$, there is a corresponding class in $K_0(A)$, which we denote $[M]$, and which is equal to $\sum_i c_i [S_i]$, where $c_i$ is the number of times $S_i$ appears in a composition series for $M$.

We will be interested in linear functionals on $K_0(A)$. For convenience in drawing pictures, we will extend scalars to consider real-valued functionals.

$$K_0^*(A)_\mathbb{R} = \text{Hom}_\mathbb{Z}(K_0(A), \mathbb{R})$$

Let $\phi \in K_0^*(A)_\mathbb{R}$. An $A$-module $M$ is called semistable with respect to $\phi$ if $\phi([M]) = 0$ and $\phi([N]) \leq 0$ for any submodule $N$ of $M$. This definition is due to King [Ki], who showed that it is equivalent to a notion of semistability coming from geometric invariant theory (which we shall not need in this note). There is also another reformulation in terms of semi-invariants (see, for example, [DW]), but we shall not need to refer to this perspective either.

We will write $(A\text{-mod})^{\phi}$ for the full subcategory of $A$-modules semistable with respect to $\phi$. It was shown by King (and it is an easy exercise) that $(A\text{-mod})^{\phi}$ is an extension-closed, exact abelian subcategory of $A\text{-mod}$.

For example, we could consider a path algebra of type $A_2$, with simples $S_1$ being projective and $S_2$ being injective, as shown in Figure 1. This is a picture of $K_0^*(A)_\mathbb{R} \cong \mathbb{R}^2$. Each point of the picture corresponds to a stability condition. There, $[S_1]^\perp$ designates the line consisting of elements of $K_0^*(A)_\mathbb{R}$ which vanish on $[S_1]$, and similarly for $[S_2]^\perp$ and $([S_1] + [S_2])^\perp$. We will consistently orient our stability diagrams so that the region where $\phi$ is positive on all the simples is at the bottom of the diagram. We have marked two lines and one half-line which are regions of semistability for the indecomposable modules of this algebra. $S_1$ is
stable on the whole line $[S_1]^\perp$, and similarly for $S_2$. $P_2$, however, is only stable on half of the line $(S_1 + S_2)^\perp$, the half that is drawn in solidly. On the other half, its submodule $S_1$ has $\phi([S_1]) > 0$, which causes it to become unstable. Note that the labels $S_1$, $S_2$, and $P_2$ in the diagram do not refer to specific points. Rather, they label their corresponding regions of stability (lines, or, in the case of $P_2$, a half-line). Generically, at a point not on any of the lines in the picture, the subcategory of semistable submodules is the zero category. Finally, at the origin, every module is semistable. Pictures like this, in the hereditary case, have been studied by [IOTW, Ch, IT, IPT].

Recall that an $A$-module is called a brick if its endomorphism ring is a division algebra. To understand the semistable subcategories of $A$-mod, it is sufficient to understand semistability of bricks, by the following lemma.

**Lemma 1.** An exact abelian extension-closed subcategory of $A$-mod is determined by the bricks it contains.

**Proof.** Let $\mathcal{C}$ be an abelian extension-closed subcategory of $A$-mod, and let $\mathcal{B}$ be the set of bricks it contains. The statement of the lemma follows from the fact which we shall establish that the objects of $\mathcal{C}$ are exactly those $A$-modules which admit a filtration by modules in $\mathcal{B}$.

Clearly, any module filtered by modules in $\mathcal{B}$ is contained in $\mathcal{C}$, because $\mathcal{C}$ is extension-closed. Conversely, let $X \in \mathcal{C}$. If $X$ is a brick, it is in $\mathcal{B}$ and we are done. Otherwise, $X$ admits a non-invertible endomorphism $\alpha$. Now $X$ is isomorphic to the extension of the image of $\alpha$ by its kernel, both of which have smaller total dimension than $X$, and both of which are in $\mathcal{C}$ because it is an exact abelian subcategory, so we are done by induction. □

Thus, if we want to understand the map from $K^*_n(A)_R$ to semistable subcategories, it suffices to understand, for each brick of $A$-mod, the region of $K^*_n(A)_R$ for which it is semistable. The category $(A$-mod)$^\phi$ will consist of all modules filtered by the bricks that are semistable for $\phi$.

Our goal in this paper is to describe this picture for finite-type preprojective algebras. This will require a detour into the theory of hyperplane arrangements.
2. Preprojective algebras

First, though, we introduce the finite-type preprojective algebras. Let \( Q \) be a simply-laced Dynkin quiver, with a set \( Q_0 = \{1, \ldots, n\} \) of vertices and a set \( Q_1 \) of arrows. Define \( \overline{Q} \) to be the doubled quiver of \( Q \), which is to say, for each arrow \( a : i \to j \), we add an arrow \( a^* : j \leftarrow i \). The preprojective algebra is then defined to be:

\[
\Pi = k\overline{Q} / \sum_{a \in Q_1} (aa^* - a^*a).
\]

This is a finite-dimensional self-injective algebra.

Preprojective algebras were originally introduced by Gelfand and Ponomarev \[GP\], and, in a version closer to the formulation which is now standard, by Dlab and Ringel \[DR\]. They arise naturally in geometric representation theoretic contexts, playing, for example, an essential role in Lusztig’s definition of the semicanonical basis of the enveloping algebra of the positive part of a symmetric Kac-Moody Lie algebra \[Lu\]. For our purposes, we can just take them as an interesting class of algebras with a Dynkin classification; as we shall see, other elements of Dynkin diagram combinatorics will also turn out to be relevant to their analysis.

As a simple example, let us consider the preprojective algebra of type \( A_2 \). We have two vertices 1 and 2, an arrow \( a \) from 1 to 2, an arrow \( a^* \) from 2 to 1, and the relation \( aa^* - a^*a \). Multiplying this relation on both sides by the idempotent at 1, and multiplying on both sides by the idempotent at 2, we deduce that the ideal generated by \( aa^* - a^*a \) actually contains each of \( aa^* \) and \( a^*a \), so in this case, we could have described the ideal of relations as being generated by \( aa^* \) and \( a^*a \).

Either by noticing that this implies that the preprojective algebra of type \( A_2 \) happens to be a gentle algebra, or just by thinking about it, we determine that this algebra has four indecomposable modules: the simples at each vertex and the projectives at each vertex, which are of length two. All four of these modules are bricks. In keeping with the point of view developed in the previous section, we can ask ourselves where these bricks are semistable. The answer is given in Figure 2.

![Figure 2. Regions of semistability for the preprojective algebra of type A_2.](image)

The verification that this picture is correct is essentially the same as for the hereditary example examined above. For clarity, the regions where \( P_1 \) and \( P_2 \) are semistable are drawn as if they don’t quite touch the origin, but in fact they extend up to and include it.
We notice that if we consider the union of the lines and half-lines where at least one brick is semistable, this is a very symmetrical picture. As we shall see, this is no coincidence, but in order to make this notion precise, we shall have to introduce some further technology: specifically, we shall have to introduce the Weyl groups to provide the symmetries we want.

3. Weyl groups

Good general references for Weyl groups are [Hu, BB].

We want to define a bilinear form on $K_0(\Pi)$. This can be defined very explicitly by saying that $\langle [S_i], [S_i] \rangle = 2$, and for $j \neq i$, $\langle [S_i], [S_j] \rangle$ is minus the number of arrows between vertices $i$ and $j$ in $Q$.

A more conceptual definition is to consider the affine-type quiver $\hat{Q}$ which is obtained by adding a single vertex to $Q$. (There is a unique way to do this.) We can then define the corresponding preprojective algebra $\hat{\Pi}$, and the category of nilpotent $\hat{\Pi}$-modules, and consider its Grothendieck group and Euler form: for $V$ and $W$ nilpotent $\hat{\Pi}$-representations,

$$\langle [V], [W] \rangle = \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}^i(V, W) = \dim \text{Hom}(V, W) - \dim \text{Ext}^1(V, W) + \dim \text{Ext}^2(V, W).$$

Restricted to the subspace spanned by the classes of the simple modules of $\Pi$, we recover the form defined in the previous paragraph. Note that we cannot directly take the Euler form for $\Pi$ because its global dimension is not finite, so $\sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}^i(V, W)$ is not well-defined.

The bilinear form on $K_0(\Pi)$ turns out to be positive definite. Since, in particular, it is non-degenerate, we can use it to identify $K_0^*(\Pi) \otimes \mathbb{R}$ with its dual, and thus define a positive definite symmetric bilinear form on $K_0^*(\Pi) \otimes \mathbb{R}$.

We now want to define a group action on $V = K_0^*(\Pi) \otimes \mathbb{R}$. Let $e_1, \ldots, e_n$ be the standard basis for $V$, with $e_i([S_j]) = \delta_{ij}$, where $\delta_{ij}$ is 1 if $i = j$ and 0 otherwise. Define a linear transformation $s_i$ of $V$ by

$$s_i(\phi) = \phi - \langle e_i, \phi \rangle e_i$$

Each $s_i$ acts by reflecting in a hyperplane with respect to the bilinear form on $V$. We call the transformations $s_i$ simple reflections.

The group generated by these $n$ elements forms, by definition, the Weyl group associated to $Q$. We denote it by $W$, and we think of it as acting on $V$ on the right.

For future use, for any $w \in W$, we define $\ell(w)$ to be the length of the shortest possible expression for $w$ as a product of the simple reflections. The identity element is the unique element of length zero, and the simple reflections are exactly the elements of length one.

Define $T$ to be the set of conjugates in $W$ of $s_1, \ldots, s_n$. It is easy to see that all of these elements will also act by reflections. In fact, they are all the reflections in $W$.

By definition, the reflection arrangement $\mathcal{H}_\Pi$ associated to $W$ consists of the collection of reflecting hyperplanes in $V$ associated to the set $T$ of reflections.

We can now state a rough version of our main result: the region in $K_0^*(\Pi) \otimes \mathbb{R}$ where $(\Pi \text{-mod})^0 \neq 0$ consists of exactly the union of the hyperplanes in $\mathcal{H}_\Pi$. This accounts for the symmetrical picture we noticed at the beginning.
for the regularity which we observed in the case of the $A_2$ preprojective algebra.
In order to refine this result to get a picture like Figure 2, which reflects where each brick is semistable, we will need some way to divide the reflecting hyperplanes up into pieces. It turns out that a natural way to do this was developed, for superficially different purposes, by Reading [Re1], as we now explain.

4. Shards

We must now take a detour into the theory of hyperplane arrangements and, in particular, the poset structure on the poset of regions defined by a hyperplane arrangement. The key results we need are to be found in [Re1]. [Re2] is an exposition which provides further context.

Let $H$ be a hyperplane arrangement in $\mathbb{R}^n$, by which we mean a collection of finitely many linear hyperplanes in $\mathbb{R}^n$. $H$ defines a set of chambers, which are the connected components of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{H}} H$.

There is a natural graph structure associated to $H$, which we denote $G(H)$. The vertices are the chambers, and two chambers are adjacent if their closures intersect in a codimension-one region in $\mathbb{R}^n$.

We shall define a poset structure on the set of chambers by specifying its cover relations, that is to say, the pairs $E, F$ such that $E < F$ and there is no $G$ with $E < G < F$. We write $E \lessdot F$ for a cover relation in a poset. The Hasse diagram of a poset is the directed graph on the elements of the poset, whose edges $(E, F)$ are exactly the cover relations $E \lessdot F$ of the poset.

Choose a base chamber and call it $C$. Define the chamber poset $P(H, C)$ on the set of chambers by imposing that $E \lessdot F$ if and only if $E$ and $F$ are adjacent and $E$ lies on the same side as $C$ of the hyperplane defined by the intersection of the closures of $E$ and $F$.

The Hasse diagram of $P(H, C)$ is then an orientation of the graph $G(H)$. The chamber $C$ is the unique source, corresponding to the fact that it is the minimum element of the poset, and the chamber $-C$ is the maximum element of the poset. (For any chamber $E$, note that $-E$ also forms a chamber.) This poset was introduced by Edelman [Ed].

A chamber is called simplicial if it consists of positive linear combinations of $n$ linearly independent vectors. $H$ is called simplicial if all its chambers are simplicial. If $n \leq 2$, all hyperplane arrangements are simplicial, but this is not true for larger $n$.

An important source of simplicial arrangements are the reflection arrangements. The reflection arrangement associated to $\Pi$, which we have already introduced, is an example, but any finite Coxeter group yields a reflection arrangement in the same way. The reader who is unfamiliar with Coxeter groups may simply take our statements about reflection arrangements as applying to the reflection arrangements we have already introduced, with no loss.

Let $\mathcal{H}$ be a reflection arrangement. $W$ acts on the set of chambers simply-transitively, so, after identifying the base chamber $C$ with the identity element of $W$, we can identify the chambers with the elements of $W$. The poset $P(\mathcal{H}, C)$ is then a well-known poset on $W$, known as (right) weak order, in which the cover relations are given by $v \lessdot vs_i$ if $\ell(vs_i) = \ell(v) + 1$. 
A poset is called a lattice if any pair of elements \( E, F \) has a unique greatest lower bound, denoted \( E \land F \) (the meet of \( E \) and \( F \)), and a unique least upper bound, denoted \( E \lor F \) (the join of \( E \) and \( F \)).

**Theorem 1** ([BEZ, Theorem 3.4]). If \( \mathcal{H} \) is a simplicial arrangement, then the poset \( P(\mathcal{H}, C) \) is a lattice.

Simplicialness is not necessary for \( P(\mathcal{H}, C) \) to be a lattice, see [Re2]. However, since our eventual application will be to reflection arrangements, we may as well not seek the greatest possible generality.

A lattice \( L \) is called semi-distributive if for \( E, F, G \) in \( L \) such that \( E \lor F = E \lor G \), it follows that this element also equals \( E \lor (F \land G) \), and dually if \( E \land F = E \land G \), then this element also equals \( E \land (F \lor G) \). We have the following result:

**Theorem 2** ([Re2, Corollary 9-3.9]). If \( \mathcal{H} \) is a simplicial arrangement, then the lattice \( P(\mathcal{H}, C) \) is semidistributive.

It is an immediate consequence of semidistributivity that if \( G > E \) then there is a unique minimum element among all elements \( F \) such that \( E \lor F = G \).

An element \( E \) of a lattice \( L \) is called join-irreducible if it is not the minimum element of the lattice, and it cannot be written as \( E = F \lor G \) with \( F, G < E \). The following lemma is an easy exercise.

**Lemma 2.** If \( L \) is a finite lattice, then \( E \) is join-irreducible in \( L \) if and only if \( E \) covers exactly one element.

If \( E \) is join-irreducible, we write \( E_* \) for the unique element which it covers.

In a finite lattice, every element can be written as a join of join-irreducible elements, so they have an obviously important structural rôle. (This rôle is shared with the meet-irreducible elements, which are defined dually, and can be studied in the same way as we are doing for join-irreducible elements.)

There is a natural labelling of the edges of the Hasse diagram of \( P(\mathcal{H}, C) \) by join-irreducible elements, as follows: define the join-irreducible label \( j(E \triangleright F) \) to be the minimum \( G \) such that \( E \lor G = F \). By semidistributivity, this is well-defined, and it is clear that it must be join-irreducible.

Given the importance of join-irreducible elements of a lattice, it is natural to ask how to see the join-irreducible elements of \( P(\mathcal{H}, C) \) in terms of the geometry of \( \mathcal{H} \).

Each join-irreducible of \( P(\mathcal{H}, C) \) is naturally associated to a particular hyperplane. Namely, if \( E \) is join-irreducible, then by Lemma 2 it covers a unique other chamber \( F \), and by the definition of the cover relations in \( P(\mathcal{H}, C) \), the span of intersection of the closures of \( E \) and \( F \) defines a hyperplane of \( \mathcal{H} \).

This map from join-irreducible elements of \( P(\mathcal{H}, C) \) to \( \mathcal{H} \) is not a bijection, as demonstrated by Figure 3 which show the Hasse diagram of the poset of regions of a two-dimensional hyperplane arrangement superimposed over the hyperplane arrangement. We always draw the base chamber at the bottom. The join-irreducible elements are marked with black dots, and the arrows indicate the map from join-irreducibles to hyperplanes. We see that there are two join-irreducible elements which are associated to the hyperplanes \( H_2 \) and \( H_3 \).

To define a bijection from join-irreducible elements to something geometric, Reading was impelled to split some of the hyperplanes in two, as in Figure 4. Now, each join-irreducible element (black dot) has a distinct hyperplane or half-hyperplane directly below it.
More formally, when $n = 2$, Reading splits in two the hyperplanes which are not adjacent to the base chamber, and calls this set of hyperplanes and half-hyperplanes the shards of $H$. Now, if $E$ is a join-irreducible element of $P(H, e)$, we see that the facet of $E$ corresponding to the unique cover $E \triangleright F$ lies in a well-defined shard of $H$, and this gives us a bijection from join-irreducible elements to shards.

The fact that this works for $n = 2$ is a rather trivial observation. The surprising fact is that this simple strategy of splitting up hyperplanes is exactly what is needed in general.

To define the general strategy, we need to introduce some further notation. Let $H^{(2)}$ be the set of codimension-two intersections of hyperplanes from $H$, i.e.,

$$H^{(2)} = \{ H \cap K \mid H, K \in H, H \neq K \}.$$  

For each $X \in H^{(2)}$, consider the hyperplanes in $H$ containing $X$. Note that since $C$ is a chamber of our original arrangement, and we are now considering the sub-arrangement of just those hyperplanes that contain $X$, the chamber $C$ is located in a particular chamber of this sub-arrangement. Number the hyperplanes containing $X$ cyclically as $H_1, H_2, \ldots, H_r$ so that $C$ is between $H_1$ and $H_r$, as in Figure 5.

The idea is that, around $X$, we will split the hyperplanes as in the $n = 2$ situation previously discussed. We want $X$ to split the hyperplanes which are not adjacent to $C$, so we define $\text{Split}(X) = \{ H_2, \ldots, H_{r-1} \}$. 

Figure 3. Poset of regions of a two-dimensional hyperplane arrangement

Figure 4. Splitting hyperplanes when $n = 2$
Figure 5. Numbering hyperplanes around a codimension 2 intersection

Now, a hyperplane $H \in \mathcal{H}$ is split into a set of shards, which we denote $\mathcal{III}_H$, by defining

$$\mathcal{III}_H = \text{the components of } \left( H \setminus \bigcup_{X \in \text{Split}(X)} X \right)$$

and

$$\mathcal{III}(\mathcal{H}) = \bigcup_{H \in \mathcal{H}} \mathcal{III}_H.$$

Given a cover relation $E \triangleright F$ in $P(\mathcal{H}, C)$, the intersection of the two chambers $E$ and $F$ (which is a cone in the hyperplane separating them) lies entirely in one shard. We can therefore define $\mathcal{III}(E \triangleright F)$ to be this shard.

Reading proved:

**Theorem 3.** [Re1, Proposition 3.3] The map from join-irreducible elements of $P(\mathcal{H}, e)$ to shards, sending a join-irreducible $G$ to $\mathcal{III}(G \triangleright G_*)$, is a bijection.

Further, we have the following theorem (closely related to statements in [Re1, Re2], but expressed in a way that is convenient for us):

**Theorem 4.** The map sending $G$ to $\mathcal{III}(G \triangleright G_*)$, sends the label $j(E \triangleright F)$ to the label $\mathcal{III}(E \triangleright F)$ for any $E \triangleright F$.

**Proof.** Let $G$ be the join-irreducible corresponding to the shard separating $E$ and $F$. Since $E$ is above that shard, we have $G \leq E$ by [Re1] Lemma 3.5. Thus $G \cap F = E$. Any element below $G$ is below the hyperplane separating $E$ and $F$. Thus $G$ is a minimal element among those which join with $F$ to give $E$. Since $P(\mathcal{H}, e)$ is semidistributive, $G$ must be the minimum element, and thus $G = j(E \triangleright F)$, and we have that the shard associated to $j(E \triangleright F)$ is indeed $\mathcal{III}(E \triangleright F)$.

□

5. JOIN-IRREDUCIBLES OF $W$ AND BRICKS OF $\Pi$

In [IRRT], we constructed a bijection between join-irreducible elements of $W$ and bricks of $\Pi$. The simplest way to state it is as follows. Let $e_i$ be the idempotent of $\Pi$ corresponding to the vertex $i$. Define the two-sided ideal $I_i = \Pi(1 - e_i)\Pi$.

Consider a word $\underline{w} = (i_1, \ldots, i_r)$ with each $i_j \in \{1, \ldots, n\}$. Define $I_{\underline{w}} = I_{i_1} \cdots I_{i_r}$. We say that $(i_1, \ldots, i_r)$ is a reduced word for $w \in W$ if $w = s_{i_1} \cdots s_{i_r}$ and this is an expression for $w$ of the minimum possible length.
Proposition 1 \([\text{IR}]\). If \(w_1\) and \(w_2\) are reduced words for \(w\), then \(I_{w_1} = I_{w_2}\).

We can therefore define \(I_w\) to be the ideal \(I_{w_0}\) where \(w_0\) is any reduced word for \(w\).

Let \(w \gg u\) be a cover relation in weak order. Following \([\text{IRRT}]\), we define the brick label for \(B(w \gg u)\) to be \(I_u/I_w\). This module turns out to be, indeed, a brick.

One of the main results of \([\text{IRRT}]\) can be stated as follows:

Theorem 5 \([\text{IRRT}, \text{Theorem } 1.3]\). The map from join-irreducibles of \(W\) to bricks of \(\Pi\) sending \(w\) to \(B(w \gg w^*)\) is a bijection which transforms the join-irreducible labelling into the brick labelling.

We can now state the main theorem of this note:

Theorem 6. For \(w\) a join-irreducible of \(W\), the region where the brick \(B(w \gg w^*)\) is semistable is the closure of the shard \(\Xi(w \gg w^*)\).

6. TECHNICAL LEMMAS

Before we begin the proof of the main theorem, we need a few technical lemmas.

Lemma 3. Let \(M\) be a \(\Pi\)-module such that \(\text{Hom}(S_i, M) = 0\). Then \([I_i \otimes M] = s_i([M])\).

Proof. If \(\text{Hom}(S_i, M) = 0\), then \(I_i \otimes M\) is isomorphic to the result of applying a certain spherical twist functor to \(M\), where \(M\) is thought of in the derived category of the corresponding affine-type preprojective algebra \(\hat{\Pi}\). (This part of the conclusion of \([\text{IRRT}, \text{Proposition } 3.2(\text{b})]\) follows if we assume only that \(\text{Hom}(S_i, M) = 0\), although there, an additional homological assumption on \(M\) is made.) Spherical twists act like reflections on the level of the Grothendieck group. (See for example \([\text{AIRT}, \text{Lemma } 2.6]\).) \(\square\)

Lemma 4. Let \(w \gg u\) be a cover in weak order on \(W\). Let \(i\) be such that \(\ell(s_i w) > \ell(w)\). Then \(\text{Hom}(S_i, B(w \gg u)) = 0\).

Proof. The Weyl group element \(u\) determines a torsion class \(T_u = \text{Fac } I_u\), and a corresponding torsion-free class \(F_u\). Because \(\ell(s_i u) > \ell(u)\), \(I_{s_i u}\) is properly contained in \(I_u\), and thus \(S_i\) is in the top of \(I_u\), and in particular, \(S_i \in T_u\). On the other hand, by \([\text{IRRT}, \text{Theorem } 4.5]\), \(B(w \gg u) \in F_u\). Thus \(\text{Hom}(S_i, B(w \gg u)) = 0\). \(\square\)

We remark that under the hypotheses of Lemma 4 since \(B(s_i w \gg s_i u) \cong I_i \otimes B(w \gg u)\), what Lemma 4 says is that Lemma 3 applies, so that \([B(s_i w \gg s_i u)] = s_i([B(w \gg u)])\). This is part of what is implied by Theorem 6; see also \([\text{AIRT}, \text{Theorem } 2.7(1)]\).

Lemma 5. Let \(N\) be a submodule of \(M\), and suppose that \(\text{Hom}(S_i, M) = 0\). Then the kernel of the induced map from \(I_i \otimes N\) to \(I_i \otimes M\) is a sum of some number (possibly zero) of copies of \(S_i\).

Proof. From the short exact sequence

\[
0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0
\]
we obtain
\[ \text{Tor}_1(I_i, M/N) \to I_i \otimes N \to I_i \otimes M \to I_i \otimes M/N \to 0 \]
To evaluate Tor\(_1(I_i, M/N)\), we can take
\[ 0 \to I_i \to \Pi \to S_i \to 0 \]
and tensor by \( M/N \), obtaining that Tor\(_1(I_i, M/N) \cong \text{Tor}_2(S_i, M/N) \). As a \( \Pi \)-module, Tor\(_2(S_i, M/N) \) is congruent to a sum of some number of copies of \( S_i \). □

7. Proof of Main Theorem

Proof of Theorem \[ \square \] Let \( w = us_i \triangleright u \) be a cover in weak order on \( W \). We will prove by reverse induction on the length of \( w \) that \( B(w \triangleright u) \) is semistable on the (closed) facet of the Coxeter fan corresponding to \( w \triangleright u \).

There is a unique element of \( W \) of maximal length, usually denoted \( w_0 \), and the chamber corresponding to it is \( -C \). The hyperplanes that bound it are perpendicular to the simple roots, and the modules corresponding to the covers are the simple modules, each of which is semistable on its entire perpendicular hyperplane. This establishes the base case of the induction.

Now suppose that \( w < w_0 \). Let \( s_j \) be a simple reflection such that \( \ell(s_jw) > \ell(w) \). Let \( w' = s_jw, u' = s_ju, B' = B(w' \triangleright u') \). \( B' \) is related to \( B \) by \( B' = I_j \otimes B \). By Lemma 4 and Lemma 3, we have that \( [B'] = s_i([B]) \).

Suppose that \( B \) is not semistable for some \( \phi \) in the facet corresponding to \( w \triangleright u \). This must be because of some subobject \( E \) of \( B \) such that \( \phi([E]) \gg 0 \). Define \( \phi' = s_i(\phi) \). It falls on the facet \( s_jw \triangleright s_ju \). We want to conclude that there is a corresponding destabilizing subobject of \( B' \) for \( \phi' \), which would contradict our induction hypothesis.

By Lemma 4, \( \text{Hom}(S_i, B) = 0 \). It therefore follows that \( \text{Hom}(S_i, E) = 0 \), so we can apply Lemma 3 to conclude that \( [I_i \otimes E] = s_i([E]) \). Therefore \( \phi'([I_i \otimes E]) = \phi([E]) \gg 0 \). Let \( E' \) be the image of \( I_i \otimes E \) in \( B' \). The kernel of the natural map from \( I_i \otimes E \) to \( E' \) is a sum of copies of \( S_i \) by Lemma 5. Since \( \ell(s_ju) > \ell(u) \), the chamber of \( u \) lies on the opposite side from \( C \) of the hyperplane perpendicular to \( [S_i] \). Thus, \( \phi'([S_i]) \leq 0 \), so \( \phi'([E']) \geq \phi'([I_i \otimes E]) > 0 \). It follows that \( E' \) is destabilizing for \( B' \) with respect to \( \phi' \), which is contrary to our induction hypothesis. Therefore \( B(w \triangleright u) \) is semistable with respect to weights on the facet corresponding to \( w \triangleright u \), as desired.

Now we prove the opposite direction, namely, that a brick must be unstable outside the closure of the corresponding shard. Let \( w \) be a join-irreducible of \( W \). Let \( \Pi = \Pi(w \triangleright w_s) \), \( B = B(w \triangleright w_s) \). Consider a facet \( X \) of \( \Pi \). By the construction of shards, the span of \( X \) is a codimension two intersection in \( \mathcal{H}(2) \), and around \( X \) we have a picture with four shards, as shown in Figure 6. As always, the base chamber \( C \) is at the bottom.

By \([\text{IRRT}]\) Proposition 4.3, if \( E \) and \( F \) are the bricks associated to the shards as in the picture, then there is a short exact sequence:
\[ 0 \to E \to B \to F \to 0. \]

For \( \phi \in C \), we have that \( \phi([S_i]) > 0 \) for all \( i \). Let \( \theta \in \Pi \). Since \( \Pi \) is on the opposite side of \( [E]^+ \) from \( E \), \( \theta([E]) \leq 0 \). This is consistent with the fact which we have already established that \( B \) is \( \theta \)-semistable. However, if \( \theta \) is strictly on the opposite side of the hyperplane in \( [B]^+ \) defined by the span of \( X \), then \( \theta([E]) > 0 \), so \( B \) is not \( \theta \)-semistable. This establishes the theorem. □
8. Connection to other work

Baumann, Kamnitzer, and Tingley [BKT] study the representation theory of preprojective algebras of affine type. Many of the ideas from this note could also be extracted from their work, but they do not discuss shards, so the combinatorics we present here is less explicitly developed.

Crawley-Boevey establishes a result about the existence of representations of deformed preprojective algebras [CB2, Theorem 1.2] which implies that \((\Pi\text{-mod})^\phi \neq 0\) iff \(\phi\) lies on a reflecting hyperplane by [CB, Lemma 3]. However, the argument to pass from the deformed preprojective algebra to semistable representations of the usual preprojective algebras depends on an assumption that the ground field is the complex numbers.

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