COLLAPSE OF UNIT HORIZONTAL BUNDLES EQUIPPED WITH A METRIC OF CHEEGER-GROMOLL TYPE

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ABSTRACT. We study unit horizontal bundles associated with Riemannian submersions. First we investigate metric properties of an arbitrary unit horizontal bundle equipped with a Riemannian metric of the Cheeger-Gromoll type. Next we examine it from the Gromov-Hausdorff convergence theory point of view, and we state a collapse theorem for unit horizontal bundles associated with a sequence of warped Riemannian submersions.

1. Introduction and preliminaries

1.1. Introduction. Recently in [2], M. Benyounes, E. Loubeau and C. M. Wood introduced a new class of natural metrics of Cheeger-Gromoll type on the vector bundle over a Riemannian manifold. These metrics, \( h_{p,q} \), \( p, q \in \mathbb{R}, q \geq 0 \), called \((p,q)\)-metrics, generalize Sasaki metric [6] and Cheeger-Gromoll metric [5] on \( TM \).

Although \((p,q)\)-metrics have been discovered together with some new harmonics maps, the geometry of \((p,q)\)-geometry of the tangent bundle is of the independent interest [3].

In the present paper we combine a technique of \((p,q)\)-metrics and Riemannian submersions with the Gromov-Hausdorff distance theory (GH-distance theory).

First, we investigate the unit horizontal bundle \( \tilde{E}^1 \), associated with a Riemannian submersion \( \tilde{P} : \tilde{M} \to M \). The total space of \( \tilde{E}^1 \) consists of all unit vectors in \( T\tilde{M} \) which are orthogonal to the fibres of \( \tilde{P} \).

The differential \( \tilde{P} = P_* \) maps \( \tilde{E}^1 \) into \( SM \) - the unit sphere bundle over \( M \).

Let us equip \( \tilde{E}^1 \) and \( SM \) with the \((p,q)\)-metric \( \tilde{h} \) and \( h \). We ask when \( \tilde{P} : (\tilde{E}^1, \tilde{h}) \to (SM, h) \) is a Riemannian submersion. We prove (Proposition 2.1) that \( \tilde{P} : \tilde{E}^1 \to SM \) is a Riemannian submersion iff the horizontal distribution of \( P \) is integrable. This assertion seems to be of independent interest.

Next, we combine the result from Proposition 2.1 with Theorem 2 from [11], and we obtain Collapse Theorem (Theorem 2.2) for the unit horizontal bundle. We prove that the sequence of unit horizontal bundles \( (E^1_n)_{n \in \mathbb{N}} \) associated with the sequence of warped Riemannian
submersions \((P_n : \tilde{M}_n \to M)_{n \in \mathbb{N}}\) converges (in GH-topology) to the
unit sphere bundle \(SM\) iff \((M_n)_{n \in \mathbb{N}}\) converges to \(M\).

In fact, Theorem 2.2 asserts that this natural construction is continuous in the GH-topology. Some other examples showing the continuity of a natural constructions can be found in [7] by H. Li, where the author proves continuity of \(\theta\)-deformations, and in the recent paper of P. G. Walczak [10] where the continuity of spaces of probability measures associated with a Riemannian manifolds is studied.

1.2. Riemannian submersions. We briefly review basic facts of Riemannian submersions. For more details we refer to [4, Ch. 9].

If \(P : (\tilde{M}, \tilde{g}) \to (M, g)\) is a Riemannian submersion then its tangent bundle \(T\tilde{M}\) splits as a direct orthogonal sum \(T\tilde{M} = \mathcal{H} \oplus \mathcal{V}\), where \(\mathcal{V} = \ker P_*\) is the vertical subbundle and \(\mathcal{H} = \mathcal{V}^\perp\) is the horizontal subbundle. If \(W \in T\tilde{M}\) then \(W = \mathcal{H}W + \mathcal{V}W\) denotes the corresponding orthogonal splitting.

To indicate a submersion we work with, we often write \(\mathcal{H}P\) and \(\mathcal{V}P\) instead of \(\mathcal{H}\) and \(\mathcal{V}\).

A vector field \(\tilde{X}\) (resp. \(\tilde{U}\)) is horizontal (resp. vertical) if \(\tilde{X} \in \Gamma(\tilde{M}, \mathcal{H}P)\) (resp. \(\tilde{U} \in \Gamma(\tilde{M}, \mathcal{V}P)\)). The vector field \(\tilde{X}\) is called basic if \(\tilde{X}\) is horizontal and there exists a vector filed \(X\) on \(M\) such that \(P_*\tilde{X} = X\). There is one-to-one correspondence \(X \to \tilde{X}\) between vector fields on \(M\) and basic vector fields on \(\tilde{M}\). If \(X\) is a vector field on \(M\) then the corresponding basic vector field \(\tilde{X}\) is called the horizontal lift of \(X\). If the vector fields \(\tilde{X}, \tilde{Y}\) are basic and the vector field \(\tilde{U}\) is vertical then \(\tilde{g}(\tilde{X}, \tilde{Y})\) is constant along the fibres of \(P\), and \([\tilde{X}, \tilde{U}]\) is also vertical. Moreover, the horizontal part of the Lie bracket \([\tilde{X}, \tilde{Y}]\) coincide with the horizontal lift \([X, Y]\).

Let \(\tilde{\nabla}\) and \(\nabla\) be the Levi-Civita connections of \(\tilde{g}\) and \(g\), respectively. The connection \(\tilde{D}\) in \(\mathcal{H}P\) is simply the horizontal projection \(\mathcal{H}\nabla\). Since \(\tilde{\nabla}\) is Riemannian, so \(\tilde{D}\) is.

**Lemma 1.1.** Suppose that \(\tilde{X}, \tilde{Y}\) are basic and \(\tilde{U}\) is a vertical vector field. Put \(X = P_*\tilde{X}\) and \(Y = P_*\tilde{Y}\). Then

(i) \(P_*\tilde{D}_{\tilde{X}}\tilde{Y} = \nabla_X Y\), and \(\tilde{D}_{\tilde{X}}\tilde{Y} = \tilde{\nabla}_X Y\)

(ii) \(\tilde{g}(\tilde{D}_{\tilde{U}}\tilde{X}, \tilde{Y}) = -\frac{1}{2}\tilde{g}(\tilde{U}, [\tilde{X}, \tilde{Y}]).\)

**Lemma 1.2.** Let us suppose that \(\tilde{M}\) is compact.

(i) Let \(\{x_1, \ldots, x_k\}\) be an \(\varepsilon\)-net in \(M\), and let for every \(i = 1, \ldots, k\), \(\{\tilde{x}_{i1}, \ldots, \tilde{x}_{il(i)}\}\) be an \(\varepsilon\)-net in the fibre \(\tilde{M}_{x_i} = P^{-1}(x_i)\). Then the set \(\{\tilde{x}_{ij} : i = 1, \ldots, k; j = 1, \ldots l(i)\}\) is a \((2\varepsilon)\)-net in \(\tilde{M}\).

(ii) Let \(\{\tilde{x}_1, \ldots, \tilde{x}_s\}\) be an \(\varepsilon\)-net in \(\tilde{M}\). Then the set \(P(\{x_1, \ldots, x_s\})\) is an \(\varepsilon\)-net in \(M\) (notice that \(P(\tilde{x}_i)\) may be equal to \(P(\tilde{x}_j)\) even if \(i \neq j\).
1.3. \textbf{(p,q)-metrics.} Let \((M^n, g)\) be an arbitrary Riemannian manifold with the Levi-Civita connection \(\nabla\).

Let \(E\) be a vector bundle \(\pi : E \to M\) equipped with a fibre metric \(h\) and a Riemannian connection \(D\).

We define the \textit{connection map} \(K = K^D : T E \to E\) related to \(D\) as follows: \(K\) is a smooth map inducing for every \(\zeta \in E\) a \(\mathbb{R}\)-linear map \(T_\zeta E \to E\), \(x = \pi \zeta\) and determined by the condition: \(K(\xi, v) = D_\nu \xi, v \in TM, \xi \in \Gamma(M, E)\). Notice that by the definition follows that \(K|T_\zeta E\) is the canonical isomorphism \(T_\zeta E \to E_x\). For more details on the connection map we refer to \([9]\) and \([6]\).

Following by \([2]\), one can equip \(TE\) with the \((p, q)\)-metric \(h_{p, q}, p, q \in \mathbb{R}, q \geq 0\), that is, a Riemannian metric defined by:

\[
h_{p, q}(A, B) = g(\pi_\ast A, \pi_\ast B) + \frac{1}{1 + |\xi|^2} P(h(K A, K B) + q h(K A, \zeta) h(K B, \zeta)),
\]

where \(A, B \in T_\zeta E\), and \(|\xi|^2 = h(\xi, \xi)\).

Suppose \(E = (TM, g, \nabla)\). If \(p = q = 0\) then \(h_{p, q}\) coincide with the Sasaki metric \([6]\). If \(p = q = 1\) then \(h_{p, q}\) becomes the Cheeger-Gromoll metric \([3]\).

The projection \(\pi : (E, h_{p, q}) \to (M, g)\) is a Riemannian submersion such that \(\mathcal{H}^\pi = \ker K^D\) and \(\mathcal{V}^\pi = \ker \pi_\ast\). Consequently, for every \(w \in T_x M\) and \(\zeta \in E_x\) there exists a unique horizontal (resp. vertical) vector \(w^h \in \mathcal{H}_\zeta\) (resp. \(w^v \in \mathcal{V}_\zeta\)), i.e., \(\pi_\ast w^h = w\) and \(K w^h = 0\) (resp. \(\pi_\ast w^v = 0\) and \(K w^v = w\)).

Let \(\pi^1 : E^1 \to M\) be the \textit{unit bundle induced from} \(\pi : E \to M\), i.e., \(E^1 = \{\xi \in E : \|\xi\| = 1\}\) and \(\pi^1 = \pi|E^1\). If \(E = TM\) then \(\pi^1 : E^1 \to M\) is simply the \textit{unit sphere bundle} \(SM\). If \(Q : M \to N\) is a Riemannian submersion and \(E = \mathcal{H}^Q\) then \(\pi^1 : E^1 \to M\) is called the \textit{unit horizontal bundle}.

We define a \((p, q)\)-metric on \(E^1\) by the restriction of \(h_{p, q}\) to \(T(E^1)\). We denote it also by \(h_{p, q}\). Notice that \(\pi^1 = \pi|E^1 : (E^1, h_{p, q}) \to (M, g)\) is a Riemannian submersion such that, for any \(\xi \in E^1\), \(\mathcal{H}^\xi_{\pi^1} = \mathcal{H}^\xi_{\pi}\) and \(\mathcal{V}^\xi_{\pi^1} = \{A \in \mathcal{V}^\pi : \langle K^D A, \xi \rangle = 0\}\) (cf. \([9]\) \S2, Lemma 1).

Since \(h_{p, q}\) depends on \(E = (\pi : E \to M, h, D)\) it is convenient to identify \(E\) with its total space \(E\) and write \((E, h_{p, q})\) instead of \((E, h_{p, q})\). Moreover, the corresponding unit horizontal bundle with the induced \((p, q)\)-metric is denoted by \((E^1, h_{p, q})\).

2. Results

2.1. \textbf{Submersion theorem.} Let \(P : (\tilde{M}, \tilde{g}) \to (M, g)\) be a Riemannian submersion, \(b = \dim M, a + b = \dim \tilde{M}\) and \(T \tilde{M} = \mathcal{H}^P \oplus \mathcal{V}^P\) be
the corresponding orthogonal splitting of $T\tilde{M}$. Let $\tilde{\nabla}$ and $\nabla$ denote the Levi-Civita connections of $\tilde{g}$ and $g$, respectively.

Let $\tilde{E}$ denote the horizontal bundle $\mathcal{H}^\tilde{D} \to \tilde{M}$ equipped with the fibre metric $\tilde{g}$ and the Riemannian connection $\tilde{D} = \mathcal{H}\tilde{\nabla}$.

Let $\tilde{\tau}$ be the parallel transport in $\tilde{E}$, $K^D$ and $K^\nabla$ be the connection maps corresponding to $\tilde{D}$ and $\nabla$. Moreover, let $\tilde{\pi}$ and $\pi$ denote the restrictions of the natural projections $\tilde{E} \to \tilde{M}$ and $TM \to M$ to $\tilde{E}^1$ and $SM$, respectively.

For $w \in T_xM$ and $\tilde{x} \in \tilde{E}_x^1 = \tilde{E}_x$ be the unique horizontal vector such that $P_\ast \tilde{w} = w$. Similarly, for any $W \in T_x\tilde{M}$ and any $\xi \in \tilde{E}_x^1$ let $W^h \in \mathcal{H}^\tilde{\pi}_\xi$ be the unique horizontal vector such that $\tilde{\pi}_\ast(W^h) = W$.

If $\tilde{x} \in \tilde{M}$, $P(\tilde{x}) = x$ and $\xi \in \tilde{E}_x^1$ then the horizontal lift $\tilde{w}^h_\xi \in \mathcal{H}^\tilde{\pi}_\xi$ of $w \in T_xM$ to the point $\xi$ may be constructed as follows: Let $\gamma$ be a curve in $M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = w$. Next, let $\tilde{\gamma} = \tilde{\gamma}_\xi$ be the horizontal lift of $\gamma$ such that $\dot{\tilde{\gamma}}(0) = \tilde{x}$. Let $\tilde{\gamma}^h(t) = \tilde{\gamma}_t^\ast = \tilde{\tau}_t^\ast \xi$ be the parallel transport of $\xi$ along $\tilde{\gamma}$ from $0$ to $t$. Then $\tilde{w}^h_\xi = \tilde{\gamma}^h(0)$.

Suppose that $p, q \in \mathbb{R}$, $q \geq 0$ are fixed. Let $h = h_{p,q}$ and $\tilde{h} = \tilde{h}_{p,q}$ denote $(p,q)$-metrics on $SM$ and $\tilde{E}^1$, respectively.

We distinguish the following pairwise orthogonal subbundles of $T(\tilde{E}^1)$:

$$H'_\xi(\tilde{E}^1) = T_\xi\tilde{E}_x^1(\tilde{\pi}(\xi)),
H''_\xi(\tilde{E}^1) = \{\tilde{w}^h_\xi : P\tilde{\pi}(\xi) = x, w \in T_xM\},
V_\xi(\tilde{E}^1) = \{W^h \in \mathcal{H}^\tilde{\pi}_\xi : P_\ast\pi_\ast W^h = 0\}.$$

Let $\tilde{\Pi} = P_\ast$. Since $P$ is a Riemannian submersion we see that $\tilde{\Pi} = P_\ast : \tilde{E}^1 \to SM$. We write $H' = H'((\tilde{E}^1)$, $H'' = H''((\tilde{E}^1)$ and $V = V((\tilde{E}^1)$ for simplicity.

**Claim 1.** $\tilde{\Pi}_\ast : H'_\xi \to T_u(S_xM)$, $u = \tilde{\Pi}\xi$, is an isometry.

**Proof.** Clearly, $\tilde{\Pi}_\ast(H'_\xi) \subset T_uS_xM$ and $\dim H'_\xi = \dim T_u(S_xM)$. Thus it suffices to show that $\tilde{\Pi}_\ast$ preserves the length of vectors.

Let $\tilde{\pi}\xi = \tilde{x}$ and $\pi u = x$. Take $A \in H'_\xi$. Since $A \in V^\tilde{\pi}_\xi$ and $\tilde{\Pi}_\ast A \in V^\pi_u$,

$$|A|^2 = \frac{1}{(1 + |\xi|^2)^p} \left(|K^\tilde{D} A|^2 + q(\tilde{g}(K^D A, \xi))^2\right),
|\tilde{\Pi}_\ast A|^2 = \frac{1}{(1 + |u|^2)^p} \left(|K^\nabla \tilde{\Pi}_\ast A|^2 + q(g(K^\nabla \tilde{\Pi}_\ast A, u))^2\right).$$

Since $\tilde{\Pi} : \tilde{E}_\tilde{x} \to T_xM$ is a linear map, $K^\nabla \tilde{\Pi}_\ast = P_\ast K^\tilde{D}$ on $T_\xi\tilde{E}_x^1$. Thus

$$|K^\nabla \tilde{\Pi}_\ast A| = |P_\ast K^\tilde{D} A| = |K^\tilde{D} A|,
g(K^\nabla \tilde{\Pi}_\ast A, u) = g(P_\ast K^\tilde{D} A, P_\ast \xi) = \tilde{g}(K^\tilde{D} A, \xi).$$
Claim 2. \( \tilde{P}_*: H''_\xi \rightarrow \mathcal{H}^\pi_u, u = P_*\xi \) is an isometry.

Proof. Let \( x = \pi u \). By the definition of \( H'' \), \( \dim H'' = \dim T_xM \). On the other hand, \( \dim \mathcal{H}^\pi_u = \dim T_xM \). To prove the assertion it suffices to show that \( \pi_*\tilde{P}_* \) preserves the length of vectors and its image is contained in \( \mathcal{H}^\pi_u \).

Let \( \tilde{w}^h \in H''_\xi \). We can suppose that \( \tilde{w}^h = \tilde{\gamma}^h(0) \) where \( \tilde{\gamma} \) is a curve in \( M \) such that \( w = \dot{\gamma}(0) \). Then
\[
|\tilde{w}^h| = |\tilde{\pi}_*\tilde{w}^h| = |\tilde{w}| = |P_*\tilde{w}| = |w|.
\]
Observe that \( \tilde{P}\tilde{\gamma}^h \) is a vector field along \( \gamma \). Indeed, we have
\[
\pi\tilde{P}\tilde{\gamma}^h = \pi P_*\tilde{\gamma}^h = P\tilde{\pi}\tilde{\gamma}^h = P\tilde{\gamma} = \gamma.
\]
Next we have
\[
|\pi_*\tilde{P}_*\tilde{w}^h| = |(\pi\tilde{P}\tilde{\gamma}^h)(0)| = |\dot{\gamma}(0)| = |w|.
\]
Consequently, we have shown that \( |\pi_*\tilde{P}_*\tilde{w}^h| = |\tilde{w}^h| \). Thus \( \pi_*\tilde{P}_* \) preserves the length of vectors belonging to \( H''_\xi \).

We have to show that the vertical part of \( \tilde{P}_*\tilde{w}^h \) is equal to zero, or equivalently, \( K^\nabla(\tilde{P}_*\tilde{w}^h) = 0 \). Since \( \dot{\gamma}^h(t) \in \mathcal{H}^\pi \), by Lemma 1.1(i) we conclude that \( \nabla_{\tilde{P}_*\dot{\gamma}^h}P_*\tilde{\gamma}^h = P_\pi\dot{\tilde{\gamma}}^h \). Since \( \tilde{P}\tilde{\gamma}^h \) is a vector field along \( \gamma \) we obtain
\[
K^\nabla(\tilde{P}_*\tilde{w}^h) = (\nabla_{\tilde{P}_*\dot{\gamma}^h}P_*\tilde{\gamma}^h)(0) = (P_\pi\dot{\tilde{\gamma}}^h)(0) = 0.
\]
Consequently, \( \tilde{P}_*\tilde{w}^h \in \mathcal{H}^\pi_u \).

Claim 3. The following conditions are equivalent:

(i) \( \tilde{P}_*(V) = 0 \).

(ii) For every fibre \( P^{-1}(x) \) and every curve \( \eta \) in \( P^{-1}(x) \), \( \tilde{P}\pi^\eta = \tilde{P} \).

(iii) \( \mathcal{H}^\pi \) is integrable.

Proof. (i) \( \Leftrightarrow \) (ii). Let \( W^h \in V_\xi, \xi \in \tilde{E}^1, \tilde{\pi}\xi = \tilde{x} \) and let \( \tilde{x} \in P^{-1}(x) \). Then \( W^h = (t \mapsto \tilde{\pi}_t\xi)(0) \) where \( \eta \) is a curve in the fibre \( P^{-1}(x) \) such that \( \eta(0) = \tilde{x} \) and \( \tilde{\eta}(0) = \tilde{\pi}_*W^h \). Thus \( \tilde{P}_*W^h = (t \mapsto P_*\tilde{\pi}_t^\eta\xi)(0) \). Consequently, we see that \( \tilde{P}_*W^h = 0 \) for every \( W^h \in V \) iff the curve \( t \mapsto P_*\tilde{\pi}_t^\eta\xi \) is constant for every \( \xi \) and \( \eta \). The last condition is equivalent to \( P_*\tilde{\gamma}^\eta = P_* \).

(ii) \( \Leftrightarrow \) (iii). By Lemma 1.1(i) \( \mathcal{H}^\pi \) is integrable iff every basic vector field \( \tilde{X} \) is parallel (\( \tilde{D}\tilde{X} = 0 \)). This is equivalent to the condition \( \tilde{\eta}^\eta\tilde{X} = \tilde{X} \), which is equivalent to (ii).

Observe that
\[
\begin{align*}
(2.1) \quad \dim H' + \dim H'' + \dim V &= (b - 1) + b + a = \dim \tilde{E}^1, \\
(2.2) \quad \dim H' + \dim H'' &= \dim SM.
\end{align*}
\]
Lemma 2.3. Suppose $X$ and $Y$ are two finite dimensional Euclidean spaces. Suppose $Z$ is a subspace of $X$ and $X = Z \oplus Z^\perp$ is the corresponding orthogonal splitting. Given a linear operator $A : X \to Y$ such that $A : Z \to Y$ is an isometry and $AZ^\perp = 0$. If $Z'$ is any linear subspace of $X$ such that $A : Z' \to Y$ is an isometry then $Z' = Z$.

Proposition 2.1. $\bar{P} : (\bar{E}^1, \bar{h}) \to (SM, h)$ is a Riemannian submersion iff $\mathcal{H}^P$ is integrable. Then $\mathcal{H}^P = H' \oplus H''$ and $\nabla^P = V$.

Proof. This assertion follows directly by Claims 1-3, (2.1)-(2.2) and Lemma 2.3. 

2.2. Collapse theorem. Let $P : (\bar{M}, \bar{g}) \to (M, g)$ be a Riemannian submersion and let $f : M \to (0, +\infty)$ be a smooth function. Put $\bar{f} = f \circ P$. We modify the Riemannian metric $\bar{g}$ to $\bar{g}^f$ putting:

$$\bar{g}^f = \bar{g}$$ on $\mathcal{H}^P \times \mathcal{H}^P$,

$$\bar{g}^f = \bar{f}^2 \bar{g}$$ on $\mathcal{V}^P \times T\bar{M}$.

Then $P : (\bar{M}, \bar{g}^f) \to (M, g)$ remains a Riemannian submersion. We call it a warped submersion, while the function $f$ is called warping function [11]. Denote by $M_f$ the Riemannian manifold $(M, g_f)$.

Notice that the horizontal distributions of $P : (\bar{M}, \bar{g}) \to (M, g)$ and $P : (\bar{M}, \bar{g}^f) \to (M, g)$ coincide, and are equal to $\mathcal{H}^P$.

The following Collapse Theorem is proved in [11] Thm. 2. In the whole section ‘lim’ denotes the limit in the GH-topology. For basic facts concerning GH-topology we refer to [8] or [1] §12.4.

Theorem 2.1. Let $P : \bar{M} \to M$ be a Riemannian submersion with $\bar{M}$ compact. Suppose that $(f_n : M \to (0, +\infty))_{n \in \mathbb{N}}$ is a uniformly bounded sequence of warping functions. Put $\bar{M}_n = M_{f_n}$. We have $\lim \bar{M}_n = M$ iff for every $\varepsilon > 0$ there exists a positive integer $N$ such that for every $n > N$ there exists an $\varepsilon$-net $A^{(n)}$ on $M$ such that $f_n|A^{(n)} < \varepsilon$.

Suppose a Riemannian submersion $P : (\bar{M}, \bar{g}) \to (M, g)$ and a warping function $f$ are given. Let $p, q \in \mathbb{R}$ and $q \geq 0$.

Let us denote by $\bar{E}$ the vector bundle $\bar{\pi} : \mathcal{H}^P \to \bar{M}$ with the fibre metric $\bar{g}$.

Let $\bar{E}_f$ denote the vector bundle $\bar{\pi} : \mathcal{H}^P \to \bar{M}$ with the fibre metric $\bar{g}^f$.

Let $\bar{D}$ and $\bar{D}^f$ be the connections in $\bar{E}$ and $\bar{E}_f$, respectively. The corresponding connections maps are denoted by $\bar{K}$ and $\bar{K}^f$, respectively.

As in the previous section, let $h$ and $\bar{h}$ denote the $(p, q)$-metrics on $SM$ and $\bar{E}^1$, respectively. Moreover, let $h_f$ denote the $(p, q)$-metric on $\bar{E}_f^1$. We want to find relations between Riemannian manifolds $(\bar{E}^1, \bar{h})$ and $(\bar{E}_f^1, h_f)$. 

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Let $\tilde{P} = P_s$.

**Claim 1.** $\tilde{P} : (\tilde{E}_f, \tilde{h}_f) \rightarrow (SM, h)$ is a Riemannian submersion iff $\tilde{P} : (\tilde{E}_1, \tilde{h}) \rightarrow (SM, h)$ is. Then the vertical distributions of these bundles coincide and are equal to $V(\tilde{E})$.

**Proof.** The first statement of the assertion follows directly from Proposition 2.1 and the fact that the horizontal distribution of a warped submersion and the initial submersion coincide.

The second one follows from the fact that the vertical distribution is equal to the kernel of $\tilde{P}$. □

**Claim 2.** We have $H'^{(\tilde{E}_f)} = H'(\tilde{E})$ and $H''(\tilde{E}_f) = H''(\tilde{E})$. Consequently, if $\mathcal{H}^P$ is integrable then the horizontal distributions of the Riemannian submersions $\tilde{P} : (\tilde{E}_f, \tilde{h}_f) \rightarrow (SM, h)$ and $\tilde{P} : (\tilde{E}_1, \tilde{h}) \rightarrow (SM, h)$ coincide.

**Proof.** The first identity is obvious. The second follows by Lemma 1.1 (i). More precisely, suppose $\gamma$ is a curve in $M$ and $\tilde{\gamma}$ is its horizontal lift. Take any parallel vector field $\sigma$ along $\gamma$. Let $\tilde{\sigma}$ be the unique horizontal vector field along $\tilde{\gamma}$ such that $P, \tilde{\sigma} = \sigma$. Then Lemma 1.1 (i) implies that $\tilde{\sigma}$ is parallel vector field with respect to both connections. It follows that $H''(\tilde{E}_f) = H''(\tilde{E})$. □

**Claim 3.** Suppose that $\mathcal{H}^P$ is integrable. Then $\tilde{P} : (\tilde{E}_f, \tilde{h}_f) \rightarrow (SM, h)$ is a warped submersion whose warping function is $\tilde{f} = f \circ \pi$, where $\pi : SM \rightarrow M$ is the natural projection.

**Proof.** Let $\tilde{\xi} \in \tilde{E}_f$, $\tilde{\pi} \tilde{\xi} = \tilde{x}$, $P_* \tilde{\xi} = \xi$ and $P \tilde{x} = x$, and $A, B \in T_{\tilde{x}}E_f^1$.

Suppose $A, B \in \mathcal{H}^P$. Since $H'$ and $H''$ are orthogonal we may assume that $A, B \in H'$ or $A, B \in H''$.

In the former case $\tilde{\pi}_*A = \tilde{\pi}_*B = 0$ and $K^fA = KA$, $K^fB = KB$, for $A$ and $B$ are tangent to the fibre at $\tilde{x}$. Thus, $\tilde{h}(A, B) = \tilde{h}_f(A, B)$.

In the latter case, $K^fA = 0$, $K^fB = 0$, and $\tilde{\pi}_*A$ and $\tilde{\pi}_*B$ are members of $\mathcal{H}^P$. Thus $\tilde{h}_f(A, B) = \tilde{g}(\pi_*A, \pi_*B) = \tilde{h}(A, B)$. Consequently, $\tilde{h}_f = \tilde{h}$ on $\mathcal{H}^P \times \mathcal{H}^P$.

If $A, B \in V_{\tilde{x}}$ then $\tilde{\pi}_*A$ and $\tilde{\pi}_*B$ are tangent to the fibre $P^{-1}(x)$ at $\tilde{x}$. We have

$$\tilde{h}_f(A, B) = \tilde{g}(\pi_*A, \pi_*B) = f^2(x)\tilde{g}(\pi_*A, \pi_*B) = (f \circ \pi)^2(\xi)\tilde{g}(\pi_*A, \pi_*B) = \tilde{f}^2(\xi)\tilde{h}(A, B).$$

Thus $\tilde{h}_f = \tilde{f}^2\tilde{h}$ on $\mathcal{V}^P \times \mathcal{V}^P$. Since $\mathcal{V}^P$ and $\mathcal{H}^P$ are mutually orthogonal, $\tilde{h}_f = \tilde{f}^2\tilde{h}$ on $\mathcal{V}^P \times T(\tilde{E}_f^1)$. □

After these preparations we can state a collapse theorem for a unit horizontal bundle.
Theorem 2.2. Let $P : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion with $M$ compact and integrable horizontal distribution. Suppose that $(f_n : M \to (0, +\infty))_{n \in \mathbb{N}}$ is an uniformly bounded sequence of warping functions. We equip each unit horizontal bundle $\tilde{E}^1_{f_n}$ and the sphere bundle $SM$ with the $(p, q)$-metric. Let $\tilde{M}_n = (\tilde{M}, \tilde{g}_{f_n})$ and $\tilde{E}^1_n = (\tilde{E}^1_{f_n}, \tilde{h}_{f_n})$. Then $\lim_{n \to \infty} \tilde{E}^1_n = SM$ iff $\lim_{n \to \infty} \tilde{M}_n = M$.

Proof. From Claim 3 it follows that $\tilde{P} : (\tilde{E}^1_{f_n}, \tilde{h}_{f_n}) \to (SM, h)$ is a Riemannian submersion. We put $\hat{f}_n = f_n \circ \pi$, where $\pi : SM \to M$ is the natural projection.

Suppose that $\lim M_n = M$. Take $\varepsilon > 0$. By Theorem 2.1 ($\Rightarrow$) there exist $N > 0$ such that for every $n > N$ there exist an $(\varepsilon/2)$-net $A^{(n)} = \{x_1, \ldots, x_k\} \subset M$ and $f_n|A^{(n)} < \varepsilon/2$. For each $i = 1, \ldots, k$ there exist $\xi_{ij}, j = 1, \ldots, l$ such that $\{\xi_{ij}, \ldots, \xi_{il}\}$ is an $(\varepsilon/2)$-net in the fibre $S_{x_i}$. Put $A^{(n)} = \{\xi_{ij} : i = 1, \ldots, k; j = 1, \ldots l\}$. By Lemma 1.2(i), $\hat{A}^{(n)}$ is an $\varepsilon$-net in $SM$. Since $\hat{f}_n(\xi_{ij}) = f_n(x_i) < \varepsilon$, $\hat{f}_n|A^{(n)} < \varepsilon$. Consequently, $\lim \tilde{E}^1_n = SM$ by Theorem 2.1 ($\Leftarrow$).

Conversely, suppose that $\lim \tilde{E}^1_n = SM$. Let $\varepsilon > 0$. By Theorem 2.1 ($\Rightarrow$) there exist $N > 0$ such that for every $n > N$ there exists an $\varepsilon$-net $\hat{A}^{(n)} = \{\zeta_1, \ldots, \zeta_m\}$ such that $\hat{f}_n|A^{(n)} < \varepsilon$. Then, by Lemma 1.2(ii), $\pi(A^{(n)})$ is an $\varepsilon$-net in $M$. Moreover, for every $j = 1, \ldots, m$, $f(\pi(\zeta_j)) = \hat{f}(\zeta_j) < \varepsilon$. Thus $f|A^{(n)} < \varepsilon$. Consequently, $\lim M_n = M$, by Theorem 2.1 ($\Leftarrow$).

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