ABOUT COMPLEX STRUCTURES IN CONFORMAL TRACTOR CALCULUS

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ABSTRACT. The aim of this paper is to describe the geometry of conformal structures in Lorentzian signature, which admit a lightlike conformal Killing vector field whose corresponding adjoint tractor acts as complex structure on the standard tractor bundle of conformal geometry. Key to the treatment of this problem is CR-geometry and the Fefferman construction. In fact, we will consider here partially integrable CR-structures and a slightly generalised Fefferman construction for these, which we call the ℓ-Fefferman construction.

We show that a certain class of ℓ-Fefferman metrics on partially integrable CR-spaces provides all solutions to our problem concerning the complex structures.

1. INTRODUCTION

A well known construction by Ch. Fefferman (cf. [Pet76], [BDS77]) assigns to any CR-space an invariantly defined conformal structure on a circle bundle. This construction provides a method of investigating CR-invariants in the realm of conformal geometry. The classical Fefferman construction is restricted to the case of integrable, strictly pseudoconvex CR-spaces. However, it is also known that the Fefferman construction applies to more general situations, like that of partially integrable CR-geometry, and beyond that to various other parabolic geometries (cf. [Cap02], [Cap05b]). Partially integrable CR-spaces are those which admit a totally real Levi-form and a non-trivial Nijenhuis torsion tensor.

In this paper we aim to prove a result about conformal geometry in Lorentzian signature, i.e., the metrics in a given conformal class have signature (1, n). The problem that we pose admits a reasonably natural formulation only in conformal tractor calculus. To solve the problem, we will essentially use the Fefferman construction. However, we need to do this for partially integrable CR-spaces. Moreover, we will extent the Fefferman construction in this case slightly to something, which we call the ℓ-Fefferman construction.

To explain our problem, let us consider the adjoint tractor bundle $\mathcal{A}(F)$ on a space $(F, c)$ with conformal structure $c$ of Lorentzian signature. Via the so-called splitting operator $\mathcal{S}$, any conformal vector field $V$ on $F$ corresponds to a uniquely and invariantly defined section $\mathcal{S}(V)$ in $\mathcal{A}(F)$ which solves the conformally covariant tractor equation

$$\nabla^{\text{nor}} \mathcal{S}(V) = -\Omega^{\text{nor}}(V, \cdot),$$

where $\nabla^{\text{nor}}$ denotes the unique normal tractor connection of conformal geometry and $\Omega^{\text{nor}}$ is the corresponding curvature 2-form. On the other side, any adjoint tractor which solves such an equation stems from a conformal vector field on $F$ (cf. [Cap05c]). The adjoint tractor bundle is generated by the adjoint action of the Möbius group $\text{SO}(2, n + 1)$ on $\mathfrak{sl}(2, n + 1)$ and sections of $\mathcal{A}(F)$ can be naturally interpreted as bundle endomorphisms on the standard tractor bundle $\mathcal{T}(F)$, which comes from the standard representation of $\text{SO}(2, n + 1)$ on $\mathbb{R}^{2,n+1}$. We ask the
following question in this paper: Which Lorentzian conformal structures admit a conformal Killing vector $V$ such that the corresponding adjoint tractor $\mathcal{R} = S(V)$ (which necessarily solves the above equation) acts as complex structure on $\mathcal{T}(F)$?

The answer to this question is known in case that the curvature expression $\Omega^{nor}(V, \cdot)$ on the right hand side of the tractor equation is zero. Namely, all spaces providing a solution to this simplified equation are exactly the Fefferman spaces of strictly pseudoconvex and integrable CR-spaces (cf. [Spa85], [Gr87]). A natural idea to extend the description of spaces with solutions is to apply the Fefferman construction for partially integrable CR-spaces. The latter spaces are known to admit canonical normal Cartan connections (cf. [CS00]). The Nijenhuis torsion tensor should then be the (only) contributor to the conformal curvature term on the right hand side of above tractor equation for $S(V)$. We will show that this extension of the classical case is one part of the solution to our problem concerning the complex structures on $\mathcal{T}(F)$. In fact, it turns out that there is still a gap in the geometric description. This is due to the fact that in the curvature term $\Omega^{nor}(V, \cdot)$ on the right hand side of the tractor equation there can be involved another term which should be seen as a correction to the Weyl connection form used in the Fefferman construction. This effect will be incorporated in our $\ell$-Fefferman construction.

We describe shortly the course of our investigations. We will start by introducing partially integrable CR-spaces. To pursue the Fefferman construction we will use pseudo-Hermitian structures and certain preferred linear connections of those. In particular, we will extend the notion of Tanaka-Webster connections to the case of partially integrable CR-geometry (cf. [Lee86], [Miz93]). With the help of the preferred connections we can construct Fefferman metrics, whose conformal classes turn out to be CR-invariants. An important part of our investigation is then the calculation of the relation between Webster scalar curvature and Riemannian scalar curvature in the Fefferman construction (cf. Theorem 2). Furthermore, we are able to find an explicit expression for the Laplacian applied to the fundamental Killing vector field in the Fefferman construction (cf. Proposition 1). It turns out that all the calculations can be conducted for the more general $\ell$-Fefferman metrics without further expenses. The results identify the explicit form of the adjoint tractor that belongs to the fundamental Killing vector in the $\ell$-Fefferman construction (cf. Proposition 3). A method of reconstruction shows that the $\ell$-Fefferman construction provides all possible solutions to our problem concerning the complex structures on $\mathcal{T}(F)$ (cf. Proposition 4). Finally, we summarise our result in Theorem 4.

2. CR-manifolds and pseudo-Hermitian geometry

We recall in this section the definition of CR-structures on smooth manifolds and explain certain notions about their integrability conditions. The basic integrability condition that we assume throughout this paper will be the partial integrability. For convenience, we will introduce two equivalent definitions for CR-structures, a complex and real version, and use both of them in the sequel. Moreover, we will introduce pseudo-Hermitian forms on CR-manifolds. All these concepts are known and for further informations about this subject we refer to e.g. [Lee86], [Bau99] and [Cap02].

To start with, let $M^n$ be a connected smooth manifold of odd dimension $n = 2m + 1$. First, we introduce the notion of a complex almost CR-structure on $M$, which is by definition a complex subbundle $T_{10}$ of the complexified tangent bundle $TM^\mathbb{C} = TM \otimes \mathbb{C}$ such that

$$T_{10} \cap \overline{T_{10}} = \{0\} \quad \text{and} \quad \dim_c T_{10} = m .$$
We set $T_{01} := \overline{T_{10}}$ and denote by $\Gamma(T_{10})$ the space of smooth sections in $T_{10}$ over $M$. All (complex) almost CR-structures that we will consider shall be non-degenerate. To express this condition, we consider the Levi-form $L$ on $T_{10}$, which is defined by

$$L : T_{10} \times T_{10} \rightarrow E := TM^C/T_{10} \oplus T_{01},$$

where $pr_E$ denotes the projection of (complex) vectors onto the quotient $E$. The complex almost CR-structure $T_{10}$ is called non-degenerate if its Levi-form $L$ is non-degenerate. Furthermore, we want to state certain notions of integrability for $T_{10}$.

We say that a non-degenerate complex CR-structure $T_{10}$ is

- partially integrable if $[\Gamma(T_{10}), \Gamma(T_{10})] \subset \Gamma(T_{10} \oplus T_{01})$ and
- integrable if $[\Gamma(T_{10}), \Gamma(T_{10})] \subset \Gamma(T_{10})$.

The real version of an almost CR-structure consists of a real subbundle $H$ of codimension 1 in the tangent bundle $TM$ and an almost complex structure $J$ on $H$, i.e., $J^2 = -id|_H$. The Lie bracket of vector fields on $M$ induces the tensorial map

$$L_H : H \times H \rightarrow Q := TM/H,$$

$$L_H(X,Y) := pr_Q[X,Y]$$

where $X,Y \in \Gamma(H)$ are arbitrary vector fields. The non-degeneracy condition for a real almost CR-structure says that $H$ is a contact distribution in $TM$, or equivalently, the map $L_H$ is non-degenerate. The partial integrability for $(H,J)$ means that the map $L_H$ is totally real, i.e.,

$$L_H(X,Y) = L_H(JX,JY) \quad \text{for all } X,Y \in H.$$

Finally, the integrability of $(H,J)$ is determined by the additional vanishing of the Nijenhuis tensor:

$$N_J(X,Y) := [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY] = 0$$

for all $X,Y \in \Gamma(H)$.

The natural correspondence between complex and real version of almost CR-structures is given in the one direction starting with $T_{10}$ by

$$H := Re(T_{10} \oplus T_{01}) \quad \text{and} \quad J(U + \bar{U}) := i(U - \bar{U}), \quad U \in \Gamma(T_{10}).$$

In the other direction starting with $(H,J)$, the eigenspaces of the extended complex linear map $J$ on $H^C$ to the eigenvalue $i$ define an almost complex CR-structure $T_{10}$. The introduced notions of integrability and non-degeneracy coincide under the natural correspondence. In this paper, we will be concerned with non-degenerate, partially integrable CR-structures. We call a pair $(M,T_{10})$ (resp. a triple $(M,H,J)$) a partially integrable CR-manifold if $T_{10}$ (resp. $(H,J)$) is non-degenerate and partially integrable. Often we will not distinguish between the real and the complex version but use both notions simultaneously.

A nowhere vanishing 1-form $\theta$ on a non-degenerate almost CR-manifold $(M,H,J)$ with $\theta|_H \equiv 0$ is called a pseudo-Hermitian form (resp. pseudo-Hermitian structure) and the data $(M,H,J,\theta)$ (resp. $(M,T_{10},\theta)$) are called a pseudo-Hermitian manifold. A pseudo-Hermitian form is necessarily a contact form on $M$. The pseudo-Hermitian form $\theta$ determines uniquely the Reeb vector field $T \in \mathfrak{X}(M)$ by

$$\theta(T) \equiv 1 \quad \text{and} \quad d\theta(T,\cdot) \equiv 0.$$

The Hermitian form

$$L_\theta : T_{10} \times T_{10} \rightarrow \mathbb{C},$$

$$L_\theta(U,V) := -id\theta(U,\bar{V})$$
is called the Levi-form of $(M, T_{10}, \theta)$. Obviously, it holds \( \theta(L(U, V)) = L_\theta(U, V) \).

The Levi-form \( L_\theta \) can be naturally extended to \( TM^C \) by

\[
L_\theta(U, V) := 0, \quad L_\theta(T, \cdot) = 0,
\]

\[
L_\theta(\bar{U}, \bar{V}) := \overline{L_\theta(U, V)} = L_\theta(V, U).
\]

The real part of the extension of \( L_\theta \) is a symmetric bilinear form on \( TM \), which is non-degenerate on \( H \). We denote this form also by \( L_\theta : TM \times TM \to \mathbb{R} \).

In general, the signature of \( L_\theta \) on \( H \) is \((2p, 2q)\) for some non-negative integers \( p, q \) with \( p + q = m \), where \( 2p \) is the number of timelike vectors in an orthonormal frame of \((H, L_\theta)\). Since two pseudo-Hermitian forms on \((M, H, J)\) differ only by rescaling with a nowhere vanishing function, the definiteness of \( L_\theta \) is an invariant of the almost CR-structure. Hence, in the definite case, we say that the almost CR-space \((M, H, J)\) is strictly pseudoconvex and a pseudo-Hermitian form on such a space is assumed to be positive definite. We will consider in this paper strictly pseudoconvex spaces. However, this restriction is not essential for our considerations.

3. The Tanaka-Webster Connection

We assume from now on that \((M, T_{10})\) is a partially integrable, strictly pseudo-convex CR-manifold equipped with pseudo-Hermitian form \( \theta \), i.e., the Levi-form \( L_\theta \) is positive definite on \( H \). The purpose of this section is to introduce a certain covariant derivative which naturally belongs to the given pseudo-Hermitian structure. We call it the Tanaka-Webster connection, since it generalises the classical case for \((M, H, J)\) isomorphic to \((\mathbb{R}^m, \bar{J})\).

The following Lemma 1 to 4 are known facts, certainly for the case of integrable CR-structures, where its statements and proofs can be found in (cf. [Lee86], [Bau99], [Miz93]). We give here corresponding modified statements for the weaker condition of partial integrability. Thereby, we mainly explain where refinements of the formulae concerning the integrable case have to be taken into consideration due to partial integrability. We will not repeat the parts of the proofs which do not depend on the Nijenhuis tensor.

**Lemma 1.** (cf. [Bau99]) Let \( L_\theta : TM^C \times TM^C \to \mathbb{C} \) be the Levi-form to \( \theta \) on a partially integrable CR-manifold \((M, T_{10})\) (resp. \((M, H, J)\)) and let \( T \) be the Reeb vector field belonging to \( \theta \). Then

\[
[T, Z] \in \Gamma(H^C) \quad \text{for all } Z \in \Gamma(H^C),
\]

\[
L_\theta([T, U], V) + L_\theta(U, [T, V]) = T(L_\theta(U, V)),
\]

\[
L_\theta([T, U], V) = L_\theta([T, V], U),
\]

\[
L_\theta([T, U], \bar{V}) = L_\theta([T, V], \bar{U})
\]

for all \( U, V \in \Gamma(T_{10}) \), and

\[
L_\theta(X, Y) = d\theta(X, JY),
\]

\[
L_\theta(JX, JY) = L_\theta(X, Y), \quad L_\theta(JX, Y) + L_\theta(X, JY) = 0,
\]

\[
L_\theta([T, X], Y) - L_\theta([T, Y], X) = L_\theta((T, JX)] Y) - L_\theta([T, JY], JX)
\]

for all \( X, Y \in \Gamma(H) \).

As next, we state the existence of a particular covariant derivative, which we call the Tanaka-Webster connection, belonging to any pseudo-Hermitian structure \( \theta \) on a partially integrable CR-manifold \((M, T_{10})\) (cf. [Lee86], [Bau99], [Miz93]).
Lemma 2. (cf. [Bau99]) Let $(M, T_{10}, \theta)$ be a pseudo-Hermitian manifold and let $T$ be the Reeb vector field to $\theta$. Then there exists a uniquely determined covariant derivative

$$\nabla^W : \Gamma(T_{10}) \to \Gamma(T^* M^C \otimes T_{10}),$$

such that

\begin{align*}
(1) & \quad \nabla^W_U T = pr_{10}[T, U], \quad \nabla^W_V T = pr_{10}[V, U], \\
(2) & \quad X(L_\theta(U, V)) = L_\theta(\nabla^W_X U, V) + L_\theta(U, \nabla^W_X V)
\end{align*}

for all $U, V \in \Gamma(T_{10})$ and $X \in TM^C$, where $pr_{10}$ denotes the projection onto $T_{10}$. Moreover, $\nabla^W$ satisfies

$$\nabla^W_U V - \nabla^W_V U = pr_{10}[U, V].$$

We note that in case of an integrable CR-structure $T_{10}$ the more special relation

$$\nabla^W_U V - \nabla^W_V U = [U, V]$$

holds. Nevertheless, the proof of Lemma 2 mainly uses Lemma 1 and remains the same as in [Bau97]. We extend now the Tanaka-Webster connection $\nabla^W$ to the complex tangent bundle $TM^C$ by

$$\nabla^W T := 0 \quad \text{and} \quad \nabla^W_X U := \overline{\nabla^W_X \bar{U}} \quad \text{for all} \quad X \in TM^C, \quad U \in \Gamma(T_{10}).$$

The torsion $\text{Tor}^W$ of this connection is defined in the usual manner as

$$\text{Tor}^W(X, Y) := \nabla^W_X Y - \nabla^W_Y X - [X, Y]$$

for $X, Y \in \Gamma(TM^C)$.

Lemma 3. (cf. [Bau99]) The torsion $\text{Tor}^W$ of the Tanaka-Webster connection

$$\nabla^W : \Gamma(TM^C) \to \Gamma(T^* M^C \otimes TM^C)$$

satisfies

$$\text{Tor}^W(U, \bar{V}) = iL_\theta(U, V) : T,$$

$$\text{Tor}^W(U, V) = -pr_{01}[U, V], \quad \text{Tor}^W(\bar{U}, \bar{V}) = -pr_{10}[\bar{U}, \bar{V}],$$

$$\text{Tor}^W(T, U) = -pr_{01}[T, U], \quad \text{Tor}^W(T, \bar{U}) = -pr_{10}[T, \bar{U}]$$

for all $U, V \in \Gamma(T_{10})$, where $pr_{01}$ denotes the projection onto $T_{01}$.

In the integrable case the formulae for the torsion simplify to

$$\text{Tor}^W(U, V) = \text{Tor}^W(\bar{U}, \bar{V}) = 0$$

when $U, V \in T_{10}$. However, by the refinement in Lemma 2, it holds in the partially integrable case only $\nabla^W_U V - \nabla^W_V U - [U, V] = -pr_{01}[U, V]$. This implies the torsion formulae in Lemma 3. Finally, we can restrict the Tanaka-Webster connection to its real part and obtain a linear connection $\nabla^W$ on the (real) tangent bundle $TM$.

Lemma 4. (cf. [Bau99]) Let $\theta$ be a pseudo-Hermitian structure on a partially integrable CR-manifold $(M, H, J)$. The Webster connection

$$\nabla^W : \mathfrak{X}(M) \to \Gamma(T^* M \otimes TM)$$

is uniquely determined by the following properties

\begin{align*}
(1) & \quad X(L_\theta(Y, Z)) = L_\theta(\nabla^W_X Y, Z) + L_\theta(Y, \nabla^W_X Z)
\end{align*}

for all $X, Y, Z \in \mathfrak{X}(M)$, i.e., $\nabla^W$ is metric with respect to $L_\theta$ on $H$. 

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for all \(X, Y\). In addition, the connection \(\nabla^W\) satisfies
\[
\nabla^W T = 0 \quad \text{and} \quad \nabla^W J = 0 .
\]

**Proof.** Again, the same proof as in [Ban97] (p.16) applies. We have only to take into consideration the refined formula \(Tor^W(U, V) = -pr_{01}[U, V]\) and the relation
\[
-\frac{1}{4}N_j(X, Y) = Tor^W(U, V) + Tor^W(U, \bar{V})
\]
for all \(U, V \in T_{10}\), whereby \(X = U + \bar{U}\) and \(Y = V + \bar{V}\). Together with \(Tor^W(U, \bar{V}) = iL_\theta(U, V) \cdot T\) this shows
\[
Tor^W(X, Y) = Tor^W(U, V) + Tor^W(U, \bar{V}) + Tor^W(U, V) + Tor^W(U, \bar{V})
\]
for all \(X, Y \in H\). Since
\[
\nabla^W_Y JX = \nabla^W_Y (i(U - \bar{U})) = i(\nabla^W_Y U - \nabla^W_Y \bar{U})
\]
\[
= J(\nabla^W_Y U + \nabla^W_Y \bar{U}) = J\nabla^W_Y X
\]
for all \(X \in \Gamma(H)\) and \(Y \in TM\), it follows that \(J\) is parallel with respect to \(\nabla^W\). \(\square\)

The important point in Lemma 4 is that the Nijenhuis tensor is part of the torsion of the Tanaka-Webster connection due to partial integrability. The way in which the Nijenhuis tensor occurs in the torsion \(Tor^W\) is essentially implicated by condition (1) in Lemma 2 (The metric condition (2) of Lemma 2 is a rather inevitable choice.) There are other suitable connections, which occur naturally in the framework of CR-geometry resp. pseudo-Hermitian geometry, most notably the so-called Weyl connections in the sense of [CS03]. This type of connections differs from the Tanaka-Webster connections just by its torsion normalisation. We will later meet the Weyl connections on line bundles.

Finally, we define here expressions of curvature for \(\nabla^W\). Thereby, we introduce the following conventions and notations concerning calculations with respect to a local basis or frame. The indices with letters \(i, j\) and \(k\) run from 1 to 2\(m\), whereas the indices with Greek letters \(\alpha, \beta\) and \(\gamma\) run from 1 to \(m\). With \(\{e_i : i = 1, \ldots, 2m\}\) we denote an orthonormal basis (resp. local frame) of \((H, L_\theta)\) such that the additional condition
\[
J(e_{2\alpha - 1}) = e_{2\alpha}, \quad J(e_{2\alpha}) = -e_{2\alpha - 1} \quad \text{for all} \quad \alpha = 1, \ldots, m
\]
is satisfied. Then the complex vectors
\[
Z_\alpha := \frac{1}{\sqrt{2}}(e_{2\alpha - 1} - iJe_{2\alpha - 1}), \quad \alpha = 1, \ldots, m ,
\]
form an orthonormal basis of \((T_{10}, L_\theta)\) and the vectors \(\overline{Z_\alpha} := \overline{Z_\alpha}\) with \(\alpha = 1, \ldots, m\) represent an orthonormal basis of \(T_{01}\). The curvature operator of \(\nabla^W\) is defined by
\[
R^W(X, Y)Z = \nabla^W_X \nabla^W_Y Z - \nabla^W_Y \nabla^W_X Z - \nabla^W_{[X, Y]} Z ,
\]
where \(X, Y, Z \in \Gamma(TM^C)\) are complex vectors. The curvature operator is tensorial in \(X, Y\) and \(Z\). Moreover, we have the curvature tensor given by
\[
R^W(X, Y, Z, V) = L_\theta(R^W(X, Y)Z, \bar{V})
\]
for $X, Y, Z, V \in TM^C$. A straightforward calculation proves that
\[
R^W(X, Y, Z, V) = -R^W(Y, X, Z, V) = -R^W(X, Y, V, Z),
\]
\[
R^W(A, \overline{B}, C, \overline{D}) = R^W(C, \overline{B}, A, \overline{D}) - L_\theta(To^W(Tor^W(C, A)), D),
\]
\[
R^{\nabla W}(A, B)C = (\nabla^W Tor^W)(A, B)C
\]
for all vectors $X, Y, Z, V$ in $TM^C$ and $A, B, C, D$ in $T_{10}$. In case that the Nijenhuis tensor $N$ vanishes the two latter identities simplify to
\[
R^W(A, B, C, D) = R^W(C, B, A, D)
\]
and
\[
R^{\nabla W}(A, B)C = 0.
\]

The Webster-Ricci and scalar curvatures are defined as contractions of $R^W$:
\[
Ric^W := \sum_{\alpha=1}^m R^W(\cdot, \cdot, Z_\alpha, Z_{\overline{\alpha}}),
\]
\[
scal^W := \sum_{\alpha=1}^m Ric^W(Z_\alpha, Z_{\overline{\alpha}}).
\]
These definitions are independent of the choice of orthonormal basis. With respect to the $e_i$'s we have
\[
Ric^W(X, Y) = i \cdot \sum_{\alpha=1}^m R^W(X, Y, e_{2\alpha-1}, Je_{2\alpha-1}),
\]
\[
scal^W = i \cdot \sum_{\alpha=1}^m Ric^W(e_{2\alpha-1}, Je_{2\alpha-1}).
\]
Obviously, the function $scal^W$ on $(M, T_{10}, \theta)$ is real. We set
\[
\omega^\alpha_{\beta} := L_\theta(\nabla^W Z_\alpha, Z_\beta).
\]

With these components of the Tanaka-Webster connection the Ricci and scalar curvature are expressed by
\[
Ric^W(X, Y) = \sum_{\alpha=1}^m L_\theta([\nabla^W_X, \nabla^W_Y]Z_\alpha - \nabla^W_{[X, Y]}Z_\alpha, Z_\alpha)
= \left( \sum_{\alpha=1}^m d\omega^\alpha_{\alpha} - \sum_{\alpha, \beta=1}^m \omega^\alpha_{\beta} \wedge \omega^\alpha_{\beta} \right)(X, Y)
= \sum_{\alpha=1}^m d\omega^\alpha_{\alpha}(X, Y),
\]
\[
scal^W = \sum_{\alpha, \beta=1}^m d\omega^\alpha_{\alpha}(Z_\beta, Z_{\overline{\beta}}).
\]

Note that the Webster-Ricci curvature $Ric^W$ is not symmetric, in general. However, it is symmetric in case of integrable CR-structures.

4. Rescaling of a pseudo-Hermitian structure

We discuss here the transformation rules for the Tanaka-Webster connection and its scalar curvature under rescaling of a given pseudo-Hermitian structure. We remark right at the beginning that calculations and resulting expressions in this section do not differ formally from those in the classical integrable case, since the Nijenhuis torsion does not play any role in the transformation (cf. [Lee86]). The transformation rule will help us to prove the invariance of the Fefferman construction, which happens in the next section.
Let \((M, T_{10}, \theta)\) be a partially integrable, strictly pseudoconvex CR-space with pseudo-Hermitian structure \(\theta\) and corresponding Tanaka-Webster connection \(\nabla^W\). As always, we use local frames \(\{e_i: i = 1, \ldots, 2m\}\) with the property
\[
J(e_{2\alpha-1}) = e_{2\alpha}, \quad J(e_{2\alpha}) = -e_{2\alpha-1} \quad \text{for all } \alpha = 1, \ldots, m,
\]
and we set \(Z_\alpha = \frac{1}{\sqrt{2}}(e_{2\alpha-1} - iJ e_{2\alpha-1})\). Moreover, we set
\[
\theta^\alpha = L_\theta(\cdot, Z_\alpha), \quad \theta^\bar{\alpha} = L_\theta(\cdot, Z_{\bar{\alpha}})
\]
and
\[
\delta^\alpha = L_\theta(Z_\alpha, Z_\beta), \quad \omega^\alpha_\beta := L_\theta(\nabla^W Z_\alpha, Z_\beta).
\]
Then, it is \(\nabla^W Z_\alpha = \sum_{\beta=1}^m \omega^\alpha_\beta \otimes Z_\beta\). Furthermore, for a real smooth function \(f \in C^\infty(M)\) we define
\[
f_\alpha := Z_\alpha(f), \quad f_{\bar{\alpha}} := Z_{\bar{\alpha}}(f), \quad f_o := T(f),
\]
\[
f_{\alpha\bar{\beta}} := (\nabla^W Z_\beta)(f)(Z_\alpha) \quad \text{and} \quad f_{\bar{\alpha}\beta} := (\nabla^W Z_\beta df)(Z_{\bar{\alpha}}).
\]
Eventually, we set
\[
\delta f := \sum_{\alpha=1}^m f_{\alpha} Z_\alpha \quad \text{and} \quad \Delta_b f := -\sum_{\alpha=1}^m (f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}).
\]
The latter definitions are independent of the choice of orthonormal frame. It holds \(\delta f(f) = \sum_{\alpha=1}^m f_{\alpha} \cdot f_{\alpha}\) and the differential operator \(\Delta_b\) is called sublaplacian.

Now let \(f \in C^\infty(M)\) be an arbitrary function and let \(\tilde{\theta} = e^{2f} \theta\). The Hermitian form \(L_{\tilde{\theta}}\) is again positive definite and \(\tilde{\theta}\) is another pseudo-Hermitian structure on \((M, T_{10})\). In fact, any two pseudo-Hermitian forms on \((M, T_{10})\) differ only by multiplication with a smooth positive function. We want to examine the transformation rules for the Tanaka-Webster connections and the corresponding scalar curvatures under such a rescaling. First of all, we notice that
\[
\tilde{\theta}^\alpha = e^f (\theta^\alpha + 2if_{\alpha}\theta) \quad \text{for all } \alpha = 1, \ldots, m,
\]
where \(\tilde{\theta}^\alpha\) is dual to \(\tilde{Z}_\alpha = e^{-f} \cdot Z_\alpha\) with respect to \(L_{\tilde{\theta}}\), and
\[
\tilde{T} = e^{-2f} \cdot (T - 2i\delta f + 2i\delta f).
\]

**Lemma 5.** (cf. [Lee86]) Let \(\tilde{\theta} = e^{2f} \theta\) be a rescaled pseudo-Hermitian structure on a partially integrable CR-space \((M, T_{10})\). Then it holds

1. \(\nabla^W_X U = \nabla^W_X U + 2df(U) \cdot pr_{10} X + 2df(pr_{10} X) \cdot U - 2L_{\theta}(U, X) \cdot \delta f + i\theta(X) \cdot (4df(U) \cdot \delta f + 4U \cdot \delta f(f) + 2 \cdot \nabla^W_\gamma \delta f)\)

2. For the connection components it holds
\[
\omega^\beta_\alpha = \omega^\beta_\alpha + 2(f_\alpha \theta^\beta - f_\beta \theta^\alpha) + \delta^\beta_\alpha \cdot \left(\sum_{\gamma=1}^m f_{\gamma} \theta^\gamma - f_{\gamma} \theta^\gamma\right)
\]
\[
+ i \theta(\cdot) \left(f_{\bar{\beta}\alpha} + f_{\bar{\alpha}\bar{\beta}} + 4f_{\alpha}f_{\bar{\beta}} + 4\delta^\beta_\alpha \cdot \sum_{\gamma=1}^m f_{\gamma} f_{\gamma}\right),
\]
where \(\alpha, \beta = 1, \ldots, m\).
3. The Webster scalar curvature rescales by
\[
\tilde{\text{scal}}^W = e^{-2f} \cdot \left(\text{scal}^W + 2(m+1) \Delta_b f - 4m(m+1) \delta f(f)\right).
\]
Proof. (1) We take the expression for $\hat{\nabla}^W$ in Lemma 3 as definition for a connection and verify that it satisfies the determining properties for the Tanaka-Webster connection of Lemma 2. First, we have

$$\hat{\nabla}^W_V U = \nabla^W_V U - 2L_\theta(U, V)\delta f,$$

and on the other side, it is

$$\tilde{p}r_{10}[\tilde{T}, U] = pr_{10}[\tilde{T}, U] - 2L_\theta(U, V)\delta f$$

for all $U, V \in \Gamma(T_{10})$. This shows that $\hat{\nabla}^W_V U = \tilde{p}r_{10}[\tilde{T}, U]$. Next we see that

$$L_\theta(\hat{\nabla}^W_X U, V) + L_\theta(U, \hat{\nabla}^W_X V)$$

$$= e^{2\tilde{f}} \cdot L_\theta(\nabla^W_X U + 2df(U)pr_{10}X + 2df(pr_{10}X)U - 2L_\theta(U, \tilde{X})\delta f, V)$$

$$+ e^{2\tilde{f}} \cdot L_\theta(U, \nabla^W_X V + 2df(V)pr_{10}\tilde{X} + 2df(pr_{10}\tilde{X})V - 2L_\theta(V, \tilde{X})\delta f)$$

$$= e^{2\tilde{f}} \cdot X(L_\theta(U, V)) + 2e^{2\tilde{f}} df(X) \cdot L_\theta(U, V)$$

$$+ e^{2\tilde{f}} \cdot (2df(U) \cdot L_\theta(X, V) - 2df(V) \cdot L_\theta(U, \tilde{X}))$$

$$+ e^{2\tilde{f}} \cdot (2df(V) \cdot L_\theta(U, \tilde{X}) - 2df(U) \cdot L_\theta(V, \tilde{X}))$$

$$= X(L_\theta(U, V))$$

for all $U, V \in \Gamma(T_{10})$. It remains to show that $\hat{\nabla}^W_{\tilde{T}} U = \tilde{p}r_{10}[\tilde{T}, U]$. For this purpose we calculate

$$e^{2\tilde{f}} \cdot \tilde{p}r_{10}[\tilde{T}, U] = pr_{10}[T, U] + 2ie^{2\tilde{f}} pr_{10}[\epsilon^{-2\tilde{f}}(-\delta f + \delta \tilde{T}), U]$$

$$= pr_{10}[T, U] - 4idf(U)\delta f - 2i \cdot \nabla^W_\delta f U + 2i \cdot \nabla^W_{\delta f} U$$

$$+ 2i \cdot \sum_{\alpha=1}^m U(f_\alpha, \cdot) \cdot Z_\alpha + 2i \cdot \sum_{\alpha=1}^m f_\alpha \cdot \nabla^W_U Z_\alpha,$$

and on the other side,

$$e^{2\tilde{f}} \cdot \hat{\nabla}^W_{\tilde{T}} U = \nabla^W_{\tilde{T}} U - 2i \nabla^W_{\delta f} U + 2i \nabla^W_{\delta f} U - 8i \cdot df(U)\delta f - 4i \sum_{\alpha=1}^m f_\alpha f_\alpha \cdot U$$

$$+ i(4df(U)\delta f + 4\delta f(f) \cdot U + 2 \cdot \nabla^W_U \delta f),$$

which obviously equals the previous expression.

(2) To calculate the connection components $\hat{\omega}^{\alpha}_\beta$, we use the following identity

$$if_\alpha \delta^\beta_\alpha + 2 \cdot L_\theta(\nabla^W_{\alpha} \delta f, Z_\beta)$$

$$= d\theta(Z_\alpha, Z_\beta)T(f) + 2 \cdot Z_\alpha(f_\beta) - 2 \cdot \nabla^W_{\alpha} Z_\beta(f)$$

$$= 2 \cdot Z_\alpha(f_\beta) + [Z_\beta, Z_\alpha](f) - pr_{10}[Z_\beta, Z_\alpha](f) - pr_{01}[Z_\alpha, Z_\beta](f)$$

$$= Z_\alpha(f_\beta) + Z_\beta(f_\alpha) - \nabla^W_{Z_\alpha} Z_\beta(f) - \nabla^W_{Z_\beta} Z_\alpha(f)$$

$$= f_{\beta\alpha} + f_{\alpha\beta}.$$
It is
\[\tilde{\omega}_\alpha^\beta = L_\theta(\nabla^W Z_\alpha, Z_\beta) = L_\theta(\nabla^W Z_\alpha, Z_\beta) - \delta^\beta_\alpha \cdot df\]
\[= \omega^\alpha_\beta - \delta^\beta_\alpha \cdot df + 2 \cdot f_\alpha \theta^\beta - 2f_\beta \theta^\alpha + 2\delta^\beta_\alpha \sum_{\gamma=1}^m f_\gamma \theta^\gamma \]
\[+ i \cdot \left( 4f_\alpha f_\beta + 2 \cdot L_\theta(\nabla^W f, Z_\beta) + 4\delta^\beta_\alpha \sum_{\gamma=1}^m f_\gamma f_\gamma \right) \theta\]
\[= \omega^\alpha_\beta + 2(f_\alpha \theta^\beta - f_\beta \theta^\alpha) + \delta^\beta_\alpha \sum_{\gamma=1}^m (f_\gamma \theta^\gamma - f_\gamma \theta^\gamma) \]
\[+ i \cdot \left( 4f_\alpha f_\beta + i f_\alpha \theta^\beta + 2 \cdot L_\theta(\nabla^W f, Z_\beta) + 4\delta^\beta_\alpha \sum_{\gamma=1}^m f_\gamma f_\gamma \right) \theta\]
\[= \omega^\alpha_\beta + 2(f_\alpha \theta^\beta - f_\beta \theta^\alpha) + \delta^\beta_\alpha \sum_{\gamma=1}^m (f_\gamma \theta^\gamma - f_\gamma \theta^\gamma) \]
\[+ i \cdot \left( 4f_\alpha f_\beta + f_\beta \alpha + f_\alpha \beta + 4\delta^\beta_\alpha \sum_{\gamma=1}^m f_\gamma f_\gamma \right) \theta .\]

(3) We use now the latter formula for the connection components to calculate the Webster scalar curvature. It is
\[\sum_{\alpha=1}^m \tilde{\omega}_\alpha^\beta = \sum_{\alpha=1}^m \omega_\alpha^\beta + (m + 2) \cdot \sum_{\alpha=1}^m (f_\alpha \theta^\alpha - f_\alpha \theta^\alpha) \]
\[+ i \cdot \sum_{\alpha=1}^m ( f_\alpha + f_\beta \alpha + 4(m + 1)f_\alpha f_\alpha \theta .\]

The trace of the exterior differential of this expression is equal to the Webster scalar curvature. We use
\[d(\sum_{\alpha=1}^m f_\alpha \theta^\alpha)(Z_\gamma, Z_\beta) = \sum_{\alpha=1}^m (df_\alpha \wedge \theta^\alpha + f_\alpha d\theta^\alpha)(Z_\gamma, Z_\beta) \]
\[= -Z_\beta(f_\gamma) + \sum_{\alpha=1}^m f_\alpha \theta^\alpha([Z_\beta, Z_\gamma]) \]
\[= -f_\gamma \beta \]
and obtain
\[\sum_{\alpha, \beta=1}^m d\tilde{\omega}_\alpha^\alpha(Z_\beta, Z_\beta) = \sum_{\alpha, \beta=1}^m d\omega_\alpha^\alpha(Z_\beta, Z_\beta) - (m + 2) \cdot \sum_{\beta=1}^m (f_\beta \beta + f_\beta \beta) \]
\[+ i \sum_{\alpha, \beta=1}^m ( f_\alpha + f_\beta \alpha + 4(m + 1)f_\alpha f_\beta \cdot d\theta(Z_\beta, Z_\beta) .\]
As result we have
\[\widetilde{\text{scal}}^W = e^{-2f} \cdot (\text{scal}^W + 2(m + 1)\Delta f - 4m(m + 1)\delta f(f) ) .\]
5. The Fefferman metric

We construct now the Fefferman metric to a pseudo-Hermitian structure $\theta$ on the total space of the canonical $S^1$-principal bundle of a partially integrable, strictly pseudoconvex CR-manifold $(M, T_{10})$. The importance of the Fefferman metric relies on the independence of its conformal class from the chosen pseudo-Hermitian structure, i.e., the Fefferman conformal class is an invariant of the underlying CR-structure. The construction that we describe coincides with the classical Fefferman construction for integrable CR-spaces. In fact, it looks formally the same as in the classical case (cf. [Lee86], [Bau99]). However, at the end of the section we aim to introduce a slightly more generalised class of metrics, which we call the $t$-Fefferman metrics. The reason for this extension shall find its justification in the later sections. 

Let $(M^n, T_{10})$ be a partially integrable, strictly pseudoconvex CR-manifold of dimension $n = 2m + 1$ and let $\theta$ be a pseudo-Hermitian structure on this CR-space. We denote by 

$$\Lambda^{m+1,0}M := \{ \rho \in \Lambda^{m+1}M \otimes \mathbb{C} : X \cdot \rho = 0 \text{ for all } X \in T_{01} = \mathbb{T}_{10} \}$$

the complex line bundle over $M^n$, which consists of all those complex $(m+1)$-forms that vanish when an element of $T_{01}$ is inserted. The bundle $\Lambda^{m+1,0}M$ is called the canonical line bundle of the CR-space $(M, T_{10})$. The positive real numbers $\mathbb{R}^+$ act by multiplication on $K^* := \Lambda^{m+1,0} \setminus \{0\}$, which denotes the canonical line bundle without zero section. We set $F_c := K^*/\mathbb{R}^+$ and the triple 

$$(F_c, \pi, M)$$

denotes the canonical $S^1$-principal bundle of $(M, T_{10})$ whose fibre action is induced by complex multiplication with the elements of the unit circle $S^1$ in $\mathbb{C}$.

Let $\{Z_\alpha : \alpha = 1, \ldots, m\}$ be a local orthonormal frame of $(T_{10}, L_\theta)$ and let $\theta^\alpha$, $\alpha = 1, \ldots, m$, denote the corresponding dual 1-forms. The $(m+1,0)$-form 

$$\tau := \theta \wedge \theta^1 \wedge \ldots \wedge \theta^m$$

is a local section of $\Lambda^{m+1,0}M$. We denote by $[\tau]$ the corresponding local section in $F_c = K^*/\mathbb{R}^+$. With help of the projection $\pi$ every 1-form $\rho$ on $M$ can be lifted to $F_c$. The result is a 1-form $\pi^* \rho$ on $F_c$. For convenience, we shall usually denote the lifted 1-form by $\rho$ again. With help of the section $[\tau]$ we can also pull back 1-forms $\sigma$ to $M$, which is denoted by $[\tau]^* \sigma$ or just $\sigma$ again.

The Tanaka-Webster connection $\nabla^W$ naturally extends to a covariant derivative on sections of the $(m+1,0)$-form bundle $\Lambda^{m+1,0}(M)$. In fact, the covariant derivative $\nabla^W$ acting on $\Lambda^{m+1,0}M$ is induced by a unique connection 1-form on the $S^1$-principal fibre bundle $F_c$, which we denote by 

$$A^W : TF_c \to i\mathbb{R}.$$ 

It is $[\tau]^* A^W = -\sum_\alpha^m \omega^\alpha$ with respect to the local frame forms $\theta^\alpha$, $\alpha = 1, \ldots, m$. Further, we set 

$$A_\theta := A^W - \frac{i}{2(m+1)} \text{scal}^W \theta.$$ 

The latter is a connection 1-form on $F_c$ as well. It is the so-called Weyl connection on the canonical $S^1$-bundle of $(M, T_{10})$, which belongs to the given pseudo-Hermitian structure $\theta$ (cf. [CS03]). The curvature of $A^W$ is the 2-form $\Omega^W = dA^W$. It holds 

$$\Omega^W = -\sum_{\alpha=1}^m d\omega^\alpha = -\text{Ric}^W.$$ 

We denote the curvature of $A_\theta$ by $\Omega_\theta = dA_\theta$. It is 

$$\Omega_\theta = -\text{Ric}^W - \frac{i}{2(m+1)} \text{scal}^W d\theta - \frac{i}{2(m+1)} d(\text{scal}^W) \cdot \theta.$$
We define now the Fefferman metric to $\theta$ on $F_c$ by

$$f_\theta := \pi^* L_\theta - i \frac{4}{m+2} \pi^* \theta \circ A_\theta,$$

or in shorter notation, we often use the expression $f_\theta = L_\theta - i \frac{4}{m+2} \theta \circ A_\theta$. This is, in fact, a symmetric 2-tensor on the real tangent bundle of $F_c$. In case of an underlying strictly pseudoconvex space its signature is Lorentzian (i.e., $(1,2m+1)$).

The Fefferman conformal class $[f_\theta]$ consists of all smooth metrics $\tilde{f}_\theta$ on $F_c$ which arise by conformal rescaling of $f_\theta$, i.e., it is $\tilde{f}_\theta = e^{2\phi} f_\theta$ for some smooth function $\phi$. As we already mentioned the Fefferman conformal class shall be independent of the particular choice of a pseudo-Hermitian structure on $M$. This makes the Fefferman conformal class $[f_\theta]$ an invariant object in a natural manner attached to the CR-structure $T_{10}$ on $M$. We want to prove this invariance property for $f_\theta$. For this purpose we need to find the transformation rule for the Tanaka-Webster connection form $A^W$ resp. for the Weyl connection $A_\theta$ on $F_c$ under rescaling of $\theta$. Thereby, we use the results from the last section. We will see that the transformation rule for the Weyl connection is particular easy.

To start with, let $\tilde{\theta} = e^{2f} \cdot \theta$ be a rescaled pseudo-Hermitian structure on $(M,T_{10})$ and let $\tilde{Z}_\alpha = e^{-f} \cdot Z_\alpha$, where $\alpha = 1, \ldots, m$, be the rescaled basis vectors. It is $\tilde{\theta}_\alpha = e^f (\theta_\alpha + 2if_\alpha)$ and $\tilde{\tau} = e^{(m+2)f} \cdot \tau$, i.e., $[\tilde{\tau}]$ and $[\tau]$ are identical as local sections of $F_c$. Moreover, it is

$$[\tau]^* A^W = - \sum_{\alpha=1}^m \omega_\alpha^a$$

and

$$[\tau]^* \tilde{A}^W = - \sum_{\alpha=1}^m \tilde{\omega}_\alpha^a.$$ 

By using Lemma 5, we calculate

$$[\tau]^* (\tilde{A}^W - A^W) = - (m+2) \cdot \sum_{\alpha=1}^m (f_\alpha \theta_\alpha - f_\alpha \theta^{\bar{\alpha}})$$

$$+ i( \Delta_b f - 4(m+1) \cdot \sum_{\alpha=1}^m f_\alpha f_\bar{\alpha} ) \cdot \theta,$$

and further,

$$[\tau]^* (A_\tilde{b} - A_b) = - (m+2) \cdot \sum_{\alpha=1}^m (f_\alpha \theta_\alpha - f_\alpha \theta^{\bar{\alpha}})$$

$$- i( -\Delta_b f + 4m(m+1) \cdot \sum_{\alpha=1}^m f_\alpha f_\bar{\alpha} ) \cdot \theta$$

$$- \frac{i}{2(m+1)} \left( \text{scal}^W + 2(m+1) \Delta_b f - 4(m+1) \cdot \sum_{\alpha=1}^m f_\alpha f_\bar{\alpha} \right) \cdot \theta$$

$$+ \frac{i}{2(m+1)} \cdot \text{scal}^W \theta$$

$$= - (m+2) \cdot \sum_{\alpha=1}^m (f_\alpha \theta_\alpha - f_\alpha \theta^{\bar{\alpha}})$$

$$- \left( 2i(m+2) \cdot \sum_{\alpha=1}^m f_\alpha f_\bar{\alpha} \right) \cdot \theta.$$
We conclude that

$$A_\delta = A_\theta - (m + 2) \sum_{\alpha=1}^{m} ( f_\alpha \theta^\alpha - f_\alpha \bar{\theta}^\alpha ) - 2i(m + 2) \sum_{\alpha=1}^{m} f_\alpha f_\alpha \theta .$$

Now we can consider the transformation rule for the Fefferman metric $f_\theta$ under rescaling of $\theta$. It is

$$f_{\tilde{\theta}} = 2 \sum_{\alpha=1}^{m} \tilde{\theta}^\alpha \circ \bar{\tilde{\theta}}^\alpha - \frac{4}{m + 2} \tilde{\theta} \circ A_\tilde{\theta}$$

$$= e^{2f} \cdot \left( \sum_{\alpha=1}^{m} 2 \cdot ( \theta^\alpha \circ \bar{\theta}^\alpha + 2i f_\alpha \theta \circ \bar{\theta}^\alpha - 2i f_\alpha \theta^\alpha \circ \theta + 4f_\alpha f_\alpha \cdot \theta \circ \theta ) \right)$$

$$- \frac{4}{m + 2} \theta \circ A_\theta + \sum_{\alpha=1}^{m} ( 4if_\alpha \theta \circ \theta^\alpha - 4if_\alpha \theta \circ \bar{\theta}^\alpha - 8f_\alpha f_\alpha \cdot \theta \circ \theta )$$

$$= e^{2f} \cdot f_\theta .$$

**Theorem 1.** (cf. [Lee86]) Let $(M^n, T_{10})$ be a partially integrable CR-space with a pseudo-Hermitian structure $\theta$ and Fefferman metric $f_\theta$ on $F_c$. Let $\tilde{\theta} = e^{2f} \theta$ be a rescaled pseudo-Hermitian structure. Then the corresponding Fefferman metric rescales by $f_{\tilde{\theta}} = e^{2f} \cdot f_\theta$.

As we have defined Fefferman metrics in this section they depend on the connection $A_\theta$ associated to $\theta$. A more general class of metrics on $F_c$ (which are again Lorentzian for strictly pseudoconvex CR-structures) is given by choosing any connection form $A$. The difference between $A$ and $A_\theta$ is the lift of a 1-form $\ell$ on $M$ with purely imaginary values. We denote

$$A_\theta,\ell := A_\theta + \ell \quad \text{and} \quad f_{\theta,\ell} = L_\theta - i \frac{4}{m + 2} \theta \circ A_{\theta,\ell} .$$

**Definition 1.** Let $(M^n, T_{10})$ be a partially integrable CR-space, $\theta$ a pseudo-Hermitian structure and $\ell \in \Omega^1(M; i\mathbb{R})$ an arbitrary 1-form on $M$. Then we call the metric

$$f_{\theta,\ell} = L_\theta - i \frac{4}{m + 2} \theta \circ A_{\theta,\ell}$$

the (generalised) $\ell$-Fefferman metric with respect to $\theta$ on $(M^n, T_{10})$.

In case that $\ell = 0$ the metric $f_{\theta,0} = f_\theta$ is just the usual Fefferman metric to $\theta$. If $\ell$ is a closed form on $M$ then the metrics $f_\theta$ and $f_{\theta,\ell}$ are locally isometric. The local isometry is just given by a gauge transformation on $F_c$ which transforms the connection form $A_{\theta,\ell}$ into $A_\theta$. In particular, the fibres are preserved. On the other side, a local isometry between $f_\theta$ and $f_{\theta,\ell}$, which preserves the fibres, can only exist if there is a gauge transformation, i.e., if the difference $\ell = A_{\theta,\ell} - A_\theta$ is closed. Moreover, metrics $f_{\theta,\ell}$ and $f_{\bar{\theta},\bar{\ell}}$ are locally isometric only if the rescaling function $f$ is constant zero, i.e., when $\tilde{\theta} = \theta$ holds. Altogether, we can conclude that generalised Fefferman metrics $f_{\theta,\ell}$ and $f_{\bar{\theta},\bar{\ell}}$ are locally isometric with preserved fibre if and only if $\tilde{\theta} = \theta$ and $\bar{\ell} - \ell$ is a closed form on $M$. Finally, we notice that if $\tilde{\theta} = e^{2f} \theta$ and $\ell \in \Omega^1(M; i\mathbb{R})$ is arbitrary then

$$f_{\tilde{\theta},\ell} = e^{2f} \cdot f_{\theta,\ell} .$$

This shows that the conformal class $[f_{\theta,\ell}]$ of the $\ell$-Fefferman metric is an invariant of the pair $(T_{10}, \ell)$, which consists of a partially integrable CR-structure and a 1-form $\ell \in \Omega^1(M; i\mathbb{R})$. We denote the conformal class of the $\ell$-Fefferman metric by $c_\ell := [f_{\theta,\ell}]$, where $\theta$ is some arbitrary pseudo-Hermitian form.
6. THE TORSION TENSOR

We examine here properties of the torsion tensor \( Tor^W \) with respect to the Tanaka-Webster connection on pseudo-Hermitian spaces. The torsion consists essentially of the Nijenhuis tensor \( N_J \) and \( Tor^W (T, \cdot, \cdot) \), where the latter part is the deviation from transversal symmetry along \( T \). The Nijenhuis tensor is a CR-invariant. The discussion in this section will be for use in the next section when the torsion enters our calculations for curvature expressions that arise in connection with the Fefferman construction.

Let \( (M^n, H, J) \) be a strictly pseudoconvex, partially integrable CR-space with dimension \( n = 2m + 1 \) and let \( \theta \) denote a pseudo-Hermitian structure on \( M \). By \( \{ e_i : i = 1, \ldots, 2m \} \) we denote an orthonormal basis of \( L_\theta \) on \( H \) such that

\[ J(e_{2a-1}) = e_{2a} \quad J(e_{2a}) = -e_{2a-1} \quad \text{for all } \alpha = 1, \ldots, m \, . \]

Now we introduce the following conventions. Let \( A \) be a \((2, r)\)-tensor (with \( r = 0 \) or \( 1 \)). We denote its trace (or contraction) by

\[ tr_\theta A := \sum_{i=1}^{2m} A(e_i, e_i) \, . \]

More generally, we use the notation \( tr^{k,l}_\theta A \) for the trace of a \((s, r)\)-tensor \( A \), where the contraction takes place in the \( k \)th and \( l \)th entry of \( A \). In relation with the contraction we also use the convention \( (\nabla A)(X, Y) = (\nabla_X A)(Y) \), where \( \nabla \) denotes some covariant derivative on the tangent space. If \( A \) is a skew-symmetric tensor on \( M \) then we set

\[ L_\theta A(\cdot, \cdot) := iA(\cdot, J \cdot) \, . \]

It is

\[ tr_\theta L_\theta A = i \cdot tr_\theta A(\cdot, J \cdot) = 2 \cdot \sum_{\alpha=1}^{m} A(Z_\alpha, \bar{Z}_\alpha) \, . \]

If \( A \) is a symmetric \((2, 0)\)-tensor then

\[ tr_\theta A = 2 \cdot \sum_{\alpha=1}^{m} A(Z_\alpha, \bar{Z}_\alpha) \, . \]

The Nijenhuis tensor \( N \) on \( M \) is defined as

\[ N_J(X, Y) := [X, Y] - [JX, JY] + J[JX, Y] + J[X, JY] \, , \]

whereby \( X, Y \in \Gamma(H) \). Since \( Tor^W (X, Y) = L_\theta( JX, Y) \cdot T - \frac{1}{2} N(X, Y) \), the Nijenhuis tensor can be considered as the essential part of the torsion restricted to the contact distribution \( H \). It holds

\[ JN(X, Y) = -N(JX, Y) = -N(X, JY) \quad \text{and} \]

\[ tr_\theta L_\theta N(X, \cdot, \cdot) = 0 \]

for all \( X, Y, Z \in H \). We form with \( N \) the \( B_\theta \)-tensor by

\[ B_\theta(X, Y, Z) := \frac{1}{8} \left( L_\theta(N(X, Y), Z) + L_\theta(N(Z, Y), X) + L_\theta(N(Z, X), Y) \right) \, . \]

Moreover, let

\[ B(X, Y) := \sum_{i=1}^{2m} B_\theta(X, Y, e_i) e_i \]

denote the corresponding \((2, 1)\)-tensor. The tensor \( B \) does not depend on the chosen \( \theta \) and the orthonormal frame \( \{ e_i \} \). It holds

\[ B(X, Y) - B(Y, X) = \frac{1}{2} N(X, Y) \]
and $B$ vanishes identically if and only if $N$ vanishes identically. In other words, the tensor $B$ contains the same information as $N$. Moreover, it holds

$$B_{\theta}(X, Y, Z) = -B_{\theta}(X, Z, Y),$$
$$B_{\theta}(X, Y, Z) = -B_{\theta}(JX, JZ, Y) - B_{\theta}(JX, Y, JZ) - B_{\theta}(X, JY, JZ),$$
$$J^a B(Y, X) = -B(JX, Y) - B(X, JY),$$
$$tr B = \sum_{i=1}^{2m} B(e_i, e_i) = 0,$$
$$tr_{1,3} B_{\theta}(X) = \sum_{i=1}^{2m} B_{\theta}(e_i, X, e_i) = 0,$$
$$tr_{2,3} B_{\theta}(X) = \sum_{i=1}^{2m} B_{\theta}(X, e_i, e_i) = 0.$$

Straightforward calculations using essentially the condition of partial integrability show the following identities:

$$tr_{\theta} L_{\theta}(N(N(X, \cdot), Y), \cdot) = tr_{\theta} L_{\theta}(N(N(X, \cdot), \cdot), Y) - tr_{\theta} L_{\theta}(N(X, \cdot), N(Y, \cdot))$$
$$tr_{\theta} B_{\theta}(N(X, \cdot), \cdot, Y) = \frac{1}{4} tr_{\theta} L_{\theta}(N(X, \cdot), N(Y, \cdot))$$
$$tr_{\theta} L_{\theta}(B(X, \cdot), B(Y, \cdot)) = \frac{1}{8} tr_{\theta} L_{\theta}(N(X, \cdot), \cdot, Y)$$
$$tr_{\theta} L_{\theta}(B(X, \cdot), B(\cdot, Y)) = \frac{1}{16} tr_{\theta} L_{\theta}(N(X, \cdot), \cdot, Y)$$
$$\sum_{i,j=1}^{2m} L_{\theta}(N(e_i, e_j), e_j), e_i) = \frac{1}{2} \cdot \sum_{i,j=1}^{2m} L_{\theta}(N(e_i, e_j), N(e_i, e_j)).$$

The third identity above shows that the tensor $tr_{\theta} L_{\theta}(N(X, \cdot), \cdot, Y)$ is symmetric in $X$ and $Y$.

The second part of the torsion is

$$Tor^{W}(T, X) = -\frac{1}{2} \left( [T, X] + J[T, JX] \right),$$

where $X \in H$. We define the tensor

$$\mathcal{T}_{\theta}(X, Y) := -2 \cdot L_{\theta}(Tor^{W}(T, X), Y) \quad \text{for} \quad X, Y \in H.$$

Then it holds

$$Tor^{W}(T, JX) = -J(Tor^{W}(T, X)),$$
$$\mathcal{T}_{\theta}(X, Y) = \mathcal{T}_{\theta}(Y, X) = L_{\theta}([T, X], Y) + L_{\theta}([T, Y], X),$$
$$\mathcal{T}_{\theta}(X, JY) = \mathcal{T}_{\theta}(JX, Y) \quad \text{and}$$
$$tr_{\theta} \mathcal{T}_{\theta} = tr_{\theta} \mathcal{T}_{\theta}(\cdot, J \cdot) = 0.$$

7. The scalar curvature of a Fefferman metric

We calculate in this section parts of the Ricci-curvature tensor and with the help of these the scalar curvature for Fefferman metrics. In fact, we will do this calculation for the more general class of $\ell$-Fefferman metrics. The formulae that we obtain generalise the results of [Lee80]. Due to partial integrability the Nijenhuis tensor $N$ will enter the curvature expressions. This makes our calculations more laborious.
Let \((M^n, H, J)\) be a strictly pseudoconvex, partially integrable CR-space with dimension \(n = 2m + 1\) and let \(\theta\) denote a pseudo-Hermitian structure on \(M\). The \(\ell\)-Fefferman metric on the canonical \(S^1\)-bundle \(F_\ell\) to \(\theta\) on \(M\) is defined as

\[
f_{\theta, \ell} = L_\theta - i \frac{4}{m+2} \theta \circ A_{\theta, \ell},
\]

where \(\ell\) is (the lift of) an arbitrary 1-form on \(M\) with purely imaginary values and 

\[A_{\theta, \ell} := A_\theta + \ell,\]

i.e., \(A_{\theta, \ell}\) takes the form of a generic connection 1-form on \(F_\ell\). In this section, we will sometimes denote the \(\ell\)-Fefferman metric abbreviated by \(f\).

The \(S^1\)-action on the fibres of \(F_\ell\) induces a (vertical) fundamental vector field for each element in the Lie algebra \(iR\) of \(S^1\). We denote by \(S\) the fundamental field which is determined by

\[A_{\theta, \ell}(S) = \frac{m+2}{2} \frac{e_{m+2}}{m+2} .\]

The field \(S\) is lightlike on \((F_\ell, f_{\theta, \ell})\). As usual, we denote by \(\{e_i : i = 1, \ldots, 2m\}\) an orthonormal local frame of \((H, L_\theta)\) satisfying

\[J(e_{2\alpha-1}) = e_{2\alpha} \quad \text{and} \quad J(e_{2\alpha}) = -e_{2\alpha-1} \quad \text{for all } \alpha = 1, \ldots, m .\]

Moreover, let \(T\) be the Reeb vector field to \(\theta\) on \(M\) and let \(X^*\) denote the horizontal lift to \(F_\ell\) of any vector \(X\) on \(M\) with respect to the connection \(A_{\theta, \ell}\). It holds \(f_{\theta, \ell}(S, T^*) = 1\) and

\[\{e_1^*, \ldots, e_n^*, T^*, S\}\]

is a local frame on \((F_\ell, f_{\theta, \ell})\). Throughout this section we use (local) vector fields \(X, Y, Z\) and \(V\) on \(M\) which have constant coefficients with respect to the chosen local frame \(\{e_i : i = 1, \ldots, 2m\}\). In particular, this implies that scalar products of \(X^*, Y^*, Z^*\) and \(V^*\) with each other and respect to \(f_{\theta, \ell}\) are constant. To start with the calculations, we note that

\[
[X^*, S] = 0
\]

\[
[X^*, Y^*]_{vert} = i\frac{2}{m+2} \Omega_{\theta, \ell}(X, Y) \cdot S
\]

\[
[X^*, Y^*]_{horiz} = [X, Y]^*
\]

\[
[T^*, X^*] = [T, X]^* + i\frac{2}{m+2} \Omega_{\theta, \ell}(T, X) \cdot S
\]

\[
[X^*, Y^*] = pr_H[X, Y]^* - d\theta(X, Y) \cdot T^* + i\frac{2}{m+2} \Omega_{\theta, \ell}(X, Y) \cdot S ,
\]

where \(\Omega_{\theta, \ell} = dA_{\theta, \ell}\) is the curvature of the connection form \(A_{\theta, \ell}\) on \(F_\ell\).

**Lemma 6. (cf. [Bau99])** For the \(\ell\)-Fefferman metric \(f\) on \(F_\ell\), it holds

\[
f(\nabla^f_X Y^*, Z^*) = L_\theta(\nabla^W_X Y, Z) + \mathbb{B}_\theta(X, Y, Z)
\]

\[
f(\nabla^f_X Y^*, Z^*) = \frac{1}{2} L_\theta(JY, Z)
\]

\[
f(\nabla^f_X Y^*, S) = -\frac{1}{2} L_\theta(JX, Y)
\]

\[
f(\nabla^f_T Y^*, Z^*) = \frac{1}{2} ( L_\theta([T, Y], Z) - L_\theta([T, Z], Y) - i\frac{2}{m+2} \Omega_{\theta, \ell}(Y, Z) )
\]

\[
f(\nabla^f_X Y^*, T^*) = \frac{1}{2} ( L_\theta([T, X], Y) + L_\theta([T, Y], X) + i\frac{2}{m+2} \Omega_{\theta, \ell}(X, Y) )
\]

\[
f(\nabla^f_T T^*, Z^*) = -i\frac{2}{m+2} \Omega_{\theta, \ell}(T, Z)
\]

\[
f(\nabla^f S, S) = f(\nabla^f S, T^*) = f(\nabla^f T^*, T^*) = 0
\]

\[
f(\nabla^f S, T^*) = f(\nabla^f T^*, S) = f(\nabla^f T^*, Z^*) = 0
\]
for all $X,Y,Z \in \Gamma(H)$, which have pairwise constant scalar products with respect to $L_\theta$.

**Proof.** We apply the Koszul formula for the Levi-Civita connection $\nabla^f$, namely it holds

$$ f(\nabla^f_B C, D) = \frac{1}{2} \left( f([D, B], C) + f([C, B], D) + f([C, D], B) \right) $$

for all vector fields $B, C, D$ on $F_c$, which have constant length and pairwise constant scalar products. In fact, the formulae of Lemma 6 result immediately from this Koszul formula, the above expressions for commutators of vector fields on $F_c$ and replacing the scalar products with respect to $f$ by those with respect to $L_\theta$ after projecting the vectors to $M$. For example, for the first formula we find that

$$ 2f(\nabla^f_X Y, Z) = L_\theta([X, Y], Z) + L_\theta([Z, Y], X) + L_\theta([Z, X], Y) $$

for all sections $X,Y,Z$ in $H$ with pairwise constant scalar products. The other formulae follow in a similar way by applying the Koszul formula and the above expressions for commutators.

Note that in the expressions for the connection components in Lemma 6 the Nijenhuis torsion occurs only in the $H$-part of the horizontal distribution. The 1-form $\ell$ influences the curvature expression that appear.

**Lemma 7.** (cf. [Lee86]) It is

$$ \text{Ric}^{f,\ell}(S, T^*) = \frac{1}{2(m+1)} \text{scal}^W - \frac{i}{2(m+2)} \text{tr}_\theta \ell(J \cdot, \cdot) \quad \text{and} $$

$$ \text{Ric}^{f,\ell}(X^*, V^*) = \frac{\text{scal}^W}{(m+1)(m+2)} \cdot L_\theta(X, V) $$

$$ + \frac{m}{2(m+2)} \left( \text{Ric}^W(X, J V) + \text{Ric}^W(V, J X) \right) $$

$$ - \frac{m}{4} \left( \mathcal{T}_\theta(X, J V) + \mathcal{T}_\theta(V, J X) \right) $$

$$ + \text{tr}_\theta^{1,4}(\nabla^W \mathcal{B}_\theta)(X, V) + \text{tr}_\theta^{1,4}(\nabla^W \mathcal{B}_\theta)(V, X) $$

$$ - \frac{1}{8} \text{tr}_\theta L_\theta(N(X, \cdot), N(V, \cdot)) + \frac{1}{4} \text{tr}_\theta L_\theta(N(N(X, \cdot), \cdot), V) $$

$$ + \frac{i}{m+2} \left( d\ell(X, J V) + d\ell(V, J X) \right) $$

for all vectors $X^*, V^*$ in the horizontal lift of $H$ to $TF_c$.

**Proof.** We will use the connection components of Lemma 6 in order to obtain second covariant derivatives of vector fields on $F_c$ and certain components of the
Riemannian curvature tensor $R^{\ell \nu \tau}$. For convenience, we set

$$G(X, V) := \mathcal{T}_\theta(X, V) + i \frac{2}{m+2} \Omega_{\theta, \ell}(X, V)$$

for all $X, V$ in $H$. First, we have

$$f(\nabla^f_X, \nabla^f_Y, Z^*, V^*) = X^*(f(\nabla^f_Y, Z^*, V^*)) - f(\nabla^f_Y, Z^*, \nabla^f_X, V^*)$$

$$= L_\theta(\nabla^W_X \nabla^W_Y Z, V)$$

$$+ \frac{1}{4} L_\theta(JY, Z) \cdot G(X, V) + \frac{1}{4} L_\theta(JX, V) \cdot G(Y, Z)$$

$$- L_\theta(\nabla^W_Y Z, B(X, V)) + L_\theta(\nabla^W_X (B(Y, Z)), V)$$

$$- L_\theta(B(Y, Z), B(X, V)),$$

$$f(\nabla^f_{[X, Y]} Z^*, V^*) = L_\theta(\nabla^W_{pr_H[X,Y]} Z, V) + \frac{i}{m+2} \Omega_{\theta, \ell}(X, Y) \cdot L_\theta(JZ, V)$$

$$- \frac{1}{2} L_\theta(JX, Y) \cdot$$

$$(L_\theta([T, Z], V) - L_\theta([T, V], Z) - i \frac{2}{m+2} \Omega_{\theta, \ell}(Z, V))$$

$$+ B_\theta(pr_H[X, Y], Z, V) ,$$

which results to the curvature component

$$R^{\ell}(X^*, Y^*, Z^*, V^*) = R^W(X, Y, Z, V)$$

$$- \frac{i}{m+2} L_\theta(JZ, V) \cdot \Omega_{\theta, \ell}(X, Y) - \frac{1}{2} L_\theta(JX, Y) \cdot G(Z, V)$$

$$- L_\theta(JX, Y) \cdot L_\theta(\text{Tor}^W(T, Z), V)$$

$$+ \frac{1}{4} L_\theta(JY, Z) \cdot G(X, V) + \frac{1}{4} L_\theta(JX, V) \cdot G(Y, Z)$$

$$- \frac{1}{4} L_\theta(JX, Z) \cdot G(Y, V) - \frac{1}{4} L_\theta(JY, V) \cdot G(X, Z)$$

$$- (\nabla^W_Y B_\theta)(X, Z, V) + (\nabla^W_X B_\theta)(Y, Z, V)$$

$$- \frac{1}{4} B_\theta(N(X, Y), Z, V)$$

$$- L_\theta(B(Y, Z), B(X, V)) + L_\theta(B(X, Z), B(Y, V))$$

for $X, Y, Z$ and $V$ in $H$. Moreover, it is

$$f(\nabla^f_T, \nabla^f_X, S, V^*) = - f(\nabla^f_X, S, \nabla^f_T, V^*)$$

$$= \frac{1}{4} (L_\theta([T, V], JX) - L_\theta([T, JX], V) - \frac{2i}{m+2} \Omega_{\theta, \ell}(V, JX),$$

$$f(\nabla^f_{[X, T]} S, V^*) = \frac{1}{2} L_\theta([T, X], JV)$$

and we obtain

$$R^{\ell}(X^*, T^*, S, V^*) + R^{\ell}(X^*, S, T^*, V^*) = - \frac{i}{2(m+2)} \left( \Omega_{\theta, \ell}(V, JX) + \Omega_{\theta, \ell}(X, JV) \right)$$

$$- \frac{1}{4} \left( \mathcal{T}_\theta(X, JV) + \mathcal{T}_\theta(V, JX) \right) .$$
With these curvature components we calculate that

\[
\sum_{i=1}^{2m} R^{h,\ell}(X^*, e_i^*, e_i^*, V^*) = \sum_{i=1}^{2m} R^{W}(X, e_i, e_i, V) \\
+ \frac{3i}{2(m+2)} (\Omega_{\theta,\ell}(X, JV) - \Omega_{\theta,\ell}(JX, V)) \\
- \sum_{i=1}^{2m} (\nabla^{W}_{e_i} \mathcal{B}_{\theta})(X, e_i, V) + \frac{1}{4} \mathcal{B}_{\theta}(\mathcal{N}(X, e_i), e_i, V) \\
+ \sum_{i=1}^{2m} L_{\theta}(\mathcal{B}(X, e_i), \mathcal{B}(e_i, V)),
\]

and

\[
\sum_{i=1}^{2m} R^{f_{\ell},\ell}(X^*, e_i^*, Je_i^*,JV^*) = \sum_{i=1}^{2m} R^{W}(X, e_i, e_i, V) \\
- \frac{1}{2(m+1)} \text{scal}^{W}_{\theta}(X, V) + \frac{m + 1}{m + 2} \Omega_{\theta,\ell}(X, JV) \\
- \frac{m - 1}{2} L_{\theta}(\text{Tor}^{W}_{\theta}(T, X), JV) \\
- \frac{m - 1}{2} L_{\theta}(\text{Tor}^{W}_{\theta}(T, V), JX) \\
+ \sum_{i=1}^{2m} (\nabla^{W}_{e_i} \mathcal{B}_{\theta})(X, e_i, V) + \frac{1}{4} \mathcal{B}_{\theta}(\mathcal{N}(X, e_i), e_i, V) \\
- \sum_{i=1}^{2m} L_{\theta}(\mathcal{B}(X, e_i), \mathcal{B}(e_i, V))
\]

and

\[
\sum_{i=1}^{2m} R^{f_{\ell},\ell}(X^*, JV^*, e_i^*, Je_i^*) = \sum_{i=1}^{2m} R^{W}(X, JV, e_i, Je_i) \\
+ \frac{\text{scal}^{W}_{\theta}}{m + 1} L_{\theta}(X, V) - \frac{2m + 2}{m + 2} \Omega_{\theta,\ell}(X, JV) \\
- 2 \cdot \sum_{i=1}^{2m} L_{\theta}(\mathcal{B}(X, e_i), \mathcal{B}(V, e_i)).
\]

By using

\[
\sum_{i=1}^{2m} R^{f_{\ell},\ell}(X^*, e_i^*, Je_i^*, JV^*) = \sum_{\alpha=1}^{m} R^{f_{\ell},\ell}(X^*, e_{2\alpha-1}^*, Je_{2\alpha-1}^*, JV^*) \\
- \sum_{\alpha=1}^{m} R^{f_{\ell},\ell}(X^*, Je_{2\alpha-1}^*, e_{2\alpha-1}^*, JV^*) \\
= - \sum_{\alpha=1}^{m} R^{f_{\ell},\ell}(X^*, JV^*, e_{2\alpha-1}^*, Je_{2\alpha-1}^*)
\]
we obtain

\[
\sum_{i=1}^{2m} R^{f_{\theta,\ell}}(X^*, e_i^*, e_i^*, V^*) = \ i \text{Ric}^W(X, JV) \\
+ \frac{3i}{2(m+2)} \left( \Omega_{\theta,\ell}(X, JV) - \Omega_{\theta,\ell}(JX, V) \right) \\
+ \frac{m-1}{2} \left( L_\theta(T_{\text{Tor}}^W(T, X), JV) + L_\theta(T_{\text{Tor}}^W(T, V), JX) \right) \\
- \sum_{i=1}^{2m} \left( 2(\nabla_{e_i^*}^W \mathcal{B}_\theta)(X, e_i, V) + \frac{1}{2} \mathcal{B}_\theta(N(X, e_i), e_i, V) \right) \\
+ \sum_{i=1}^{2m} \left( 2L_\theta(\mathcal{B}(X, e_i), \mathcal{B}(e_i, V)) + L_\theta(\mathcal{B}(X, e_i), \mathcal{B}(V, e_i)) \right).
\]

Adding the expression for \( R^{f_{\theta,\ell}}(X^*, T^*, S, V^*) + R^{f_{\theta,\ell}}(S, T^*, V^*) \) and using the identities for torsion terms of section 6 we obtain

\[
\text{Ric}^{f_{\theta,\ell}}(X^*, V^*) = \ i \text{Ric}^W(X, JV) + 2 \cdot \text{tr} \frac{1}{4} \left( \nabla_{e_i^*}^W \mathcal{B}_\theta \right)(X, V) \\
+ \frac{i}{m+2} \left( \Omega_{\theta,\ell}(X, JV) + \Omega_{\theta,\ell}(V, JX) \right) \\
- \frac{m}{4} \left( \mathcal{T}_\theta(X, JV) + \mathcal{T}_\theta(V, JX) \right) \\
- \frac{1}{8} \text{tr} L_\theta(N(X, \cdot), N(V, \cdot)) + \frac{1}{4} \text{tr} L_\theta(N(N(X, \cdot), \cdot), V).
\]

After symmetrisation in \( X \) and \( V \) of (the first line of) the right hand side of the latter equation, we obtain the component \( \text{Ric}^{f_{\theta,\ell}}(X^*, V^*) \) as stated. Furthermore, it is

\[
\text{Ric}^{f_{\theta,\ell}}(S, T^*) = \sum_{i=1}^{2m} \left( f_{\theta,\ell}(\nabla_{T^*}^f_s S, e_i^*) - f_{\theta,\ell}(\nabla_{T^*}^f_s V, e_i^*) \right) \\
= \frac{i}{2(m+2)} \sum_{i=1}^{2m} \left( \Omega_{\theta,\ell}(J e_i, e_i) - L_\theta([T, e_i], J e_i) + L_\theta([T, J e_i], e_i) \right) \\
= \frac{1}{2(m+1)} \text{scal}^W - \frac{i}{2(m+2)} \text{tr} \phi \mathcal{L}(\cdot, J) + \frac{1}{m+1} \text{scal}^W + \frac{1}{m+2} \text{tr} \phi \mathcal{L} \phi.
\]

\[\square\]

**Theorem 2.** (cf. [Lee86]) The scalar curvature of the \( \ell \)-Fefferman metric \( f_{\theta,\ell} \) on \( F_e \) is given by

\[
\text{scal}^{f_{\theta,\ell}} = \frac{2m+1}{m+1} \cdot \text{scal}^W + \frac{1}{m+2} \cdot \text{tr} \phi \mathcal{L} \phi.
\]
Proof. It is
\[
\text{scal}_{\theta,\ell} = 2 \cdot \text{Ric}_{\theta,\ell}(T^*, S) + \sum_{i=1}^{2m} \text{Ric}_{\theta,\ell}(e_i^*, e_i^*)
\]
\[
= \left( -\frac{1}{m+1} + \frac{2m}{(m+1)(m+2)} + \frac{2m}{m+2} \right) \cdot \text{scal}^W
\]
\[
- \frac{1}{8} \cdot \sum_{i,j=1}^{2m} L_\theta \langle N(e_i, e_j), e_i \rangle
\]
\[
+ \frac{1}{4} \cdot \sum_{i,j=1}^{2m} L_\theta \langle N(e_i, e_j), N(e_i, e_j) \rangle
\]
\[
+ \frac{2i}{m+2} tr_\theta d\ell(\cdot, J \cdot) - \frac{i}{m+2} tr_\theta d\ell(\cdot, J \cdot)
\]
\[
= \frac{2m+1}{m+1} \cdot \text{scal}^W + \frac{i}{m+2} tr_\theta d\ell(\cdot, J \cdot)
\].

Thereby, we use again the torsion identities from section 6, which hold under the condition of partial integrability. □

Theorem 2 shows that the scalar curvature of the Fefferman metric \( f_\theta \) and the Webster scalar curvature are proportional. For arbitrary \( \ell \in \Omega^1(M; i\mathbb{R}) \) this is not true.

8. The Laplacian of the fundamental Killing vector

We discuss now properties of the fundamental vector field \( S \) which is vertical along the \( S^1 \)-fibre bundle \( F_c \) in the Fefferman construction. It is easy to see that it is a Killing vector with respect to any \( \ell \)-Fefferman metric \( f_{\theta,\ell} \), i.e., \( \mathcal{L}_S f_{\theta,\ell} = 0 \), where \( \mathcal{L} \) denotes the Lie derivative. However, the main goal of this section is to calculate the Laplacian of \( S \). We are able to give an explicit expression for it. The result is a first step in direction of a tractor calculus description for \( \ell \)-Fefferman spaces which come from partially integrable CR-spaces. We will complete the tractor description in the later sections.

The fundamental vector field \( S \) is uniquely determined by \( A_{\theta,\ell}(S) = \frac{i m+2}{2} \) and it is lightlike with respect to the \( \ell \)-Fefferman metric \( f_{\theta,\ell} \), where \( \ell \in \Omega^1(M; i\mathbb{R}) \). Lemma 6 shows that
\[
f_{\theta,\ell}(\nabla f_{\theta,\ell}^* S, B) = -f_{\theta,\ell}(\nabla f_{\theta,\ell}^* S, C)
\]
for all vectors \( B, C \in TF_c \), i.e., \( S \) is a Killing vector for any \( \ell \)-Fefferman metric \( f_{\theta,\ell} \). Equivalently, for the dual 1-form \( \theta \) to \( S \) on \( (F_c, f_{\theta,\ell}) \) holds
\[
\nabla f_{\theta,\ell} \theta = 1/2 \cdot d\theta
\].

Now let \( \Delta f_{\theta,\ell} = d^*d + dd^* \) denote the Laplace-Beltrami operator acting on differential forms, whereby \( d^* \) is the codifferential with respect to \( f_{\theta,\ell} \). By \( tr f_{\theta,\ell} \nabla^2 \) we denote the Bochner-Laplacian acting on arbitrary tensor fields \( \rho \) through
\[
tr f_{\theta,\ell} \nabla^2 \rho = \nabla f_{\theta,\ell}^* \nabla f_{\theta,\ell}^* \rho + \nabla f_{\theta,\ell}^* \nabla f_{\theta,\ell}^* \rho + \sum_{i=1}^{2m} \left( \nabla_{e_i^*} f_{\theta,\ell}^* \nabla_{e_i^*} f_{\theta,\ell}^* \rho - \nabla f_{\theta,\ell}^* \nabla f_{\theta,\ell}^* \rho \right)
\]
(with respect to our special choice of frame on \( F_c \)). In general, for a Killing 1-form \( \theta \) holds

\[
d^*\theta = 0 \quad \text{and} \quad tr\nabla^2\theta = -1/2 \cdot d^*d\theta = -1/2 \cdot \Delta \theta .
\]

By using the formulae of Lemma \( \text{[6]} \) we find a simple expression for the Laplacian applied to \( \theta \) on \((F_c, f_{\theta,\ell})\).

**Proposition 1.** (cf. \[Lei05\]) Let \((F_c, f_{\theta,\ell})\) be the \( \ell \)-Fefferman space of a partially integrable CR-space \((M^n, T_{10})\) of dimension \( n = 2m + 1 \) with pseudo-Hermitian structure \( \theta \) and \( \ell \in \Omega^1(M; i\mathbb{R}) \). Then

(1) the fundamental \( S \) is a Killing vector and the lift \( \theta \) to \( F_c \) of the pseudo-Hermitian form is the dual Killing 1-form. In particular,

\[
\nabla f_{\theta,\ell}\theta = 1/2 \cdot d\theta .
\]

(2) For the Laplace-Beltrami operator applied to \( \theta \) on \((F_c, f_{\theta,\ell})\) holds

\[
\Delta f_{\theta,\ell}\theta = -2i\left(\frac{n-1}{n+3}\right)A_{\theta,\ell} + \left(\frac{\text{scal} f_{\theta,\ell}}{n} - \frac{2(n+1)}{n(n+3)}tr_{\theta}L_{d\ell}\right) \cdot \theta
\]

resp. it holds

\[
P \theta = \frac{i}{n+3} \cdot A_{\theta,\ell} + \frac{(n+1) \cdot tr_{\theta}L_{d\ell}}{n(n-1) \cdot (n+3)} \cdot \theta ,
\]

where \( P = P f_{\theta,\ell} \) is the differential operator \( \frac{1}{n-1} \left( tr f_{\theta,\ell} \nabla^2 + \frac{\text{scal} f_{\theta,\ell}}{2n} \right) \).

**Proof.** Let \( X, Y, Z \) denote sections in \( H \) on \( M \) such that their coordinates with respect to a local frame \( \{ e_i : i = 1, \ldots, 2m \} \) are constant. It holds

\[
f_{\theta,\ell}(\nabla_{X^*} f_{\theta,\ell}, S, S) = -f_{\theta,\ell}(\nabla_{Y^*} f_{\theta,\ell}, S, S) = -\frac{1}{4} f_{\theta,\ell}((JY)^*, (JX)^*)
\]

\[
= -\frac{1}{4} f_{\theta,\ell}(X^*, Y^*) ,
\]

\[
f_{\theta,\ell}(\nabla_{X^*} f_{\theta,\ell}, S, T^*) = -f_{\theta,\ell}(\nabla_{Y^*} f_{\theta,\ell}, S, T^*)
\]

\[
= -\frac{1}{4} \left( L_{\theta}([X, T], JY) + L_{\theta}([JY, T], X) \right) - i\frac{1}{2(m+2)} \Omega_{\theta,\ell}(JY, X),
\]

\[
f_{\theta,\ell}(\nabla_{X^*} f_{\theta,\ell}, S, Z^*) = -f_{\theta,\ell}(\nabla_{Y^*} f_{\theta,\ell}, S, Z^*)
\]

\[
= \frac{1}{2} \left( L_{\theta}(\nabla_{X^*} JZ, Y) + B_{\theta}(X, JZ, Y) \right) .
\]
These formulae show that
\[
\begin{align*}
\phi_\ell(\text{tr} f_{\ell,\theta} \nabla^2 S, S) &= -\frac{m}{2} \\
\phi_\ell(\text{tr} f_{\ell,\theta} \nabla^2 S, Z^*) &= \frac{1}{2} \sum_{i=1}^{2m} \left( L_\theta(\nabla^W_{e_i} JZ, e_i) + L_\theta(\nabla^W_{e_i} JZ, JZ) \right) \\
&\quad + \frac{1}{2} \sum_{i=1}^{2m} \beta_\ell(e_i, JZ, e_i) \\
&= \frac{1}{2} \sum_{i=1}^{2m} e_i \left( L_\theta(e_i, JZ) \right) = 0 \\
\phi_\ell(\text{tr} f_{\ell,\theta} \nabla^2 S, T^*) &= -\frac{i}{2(m+1)} \sum_{i=1}^{2m} \Omega_\ell(\nabla^W_{e_i} JZ, e_i) \\
&= -\frac{i}{2} \text{scal} + \frac{i}{2} \text{tr}_g d\ell(\cdot, J\cdot) \\
&= -\frac{i}{2} \text{scal} + \frac{n+1}{n(n+3)} \text{tr}_g L_d\ell,
\end{align*}
\]
whereby we use the relation
\[
\text{scal} = \frac{2m+1}{m+1} \text{scal} + \frac{1}{m+2} \text{tr}_g L_d\ell.
\]
With the identity \( \phi_\ell(\text{tr} f_{\ell,\theta} \nabla^2 S, \cdot) = \text{tr}_g \nabla^2 \theta(\cdot) = -\frac{1}{2} \Delta f_{\ell,\theta} \theta(\cdot) \) we obtain the stated formula for the Laplace-Beltrami operator applied to \( \text{tr}_g \).

The importance of the next conclusion and the differential operator \( \mathcal{P} \) on a semi-Riemannian space in the context of conformal geometry will be explained in the next sections.

**Corollary 1.** Let \( (F_c, f_{\theta,\ell}) \) be the \( \ell \)-Fefferman space of \( (M, T_{10}) \) with respect to \( \theta, \ell \) on \( M \). It holds
\[
\mathcal{P}_\theta = \frac{i}{n+3} A_{\theta,\ell} \quad \text{if and only if} \quad \text{tr}_g L_d\ell = 0.
\]

Eventually, we define here the vector space
\[
H^1_{tr}(M, T_{10}) := \{ \ell \in \Omega^1(M; i\mathbb{R}) : \text{tr}_g d\ell(\cdot, J\cdot) = 0 \}/\{ \ell \in \Omega^1(M; i\mathbb{R}) : d\ell = 0 \}
\]
to any partially integrable CR-space \( (M, T_{10}) \). This quotient can be understood as the vector space on which the affine space of those connection forms is modelled, which admit the same scalar curvature as the Weyl connection on \( F_c \) belonging to some fixed \( \theta \), modulo gauge transformations. The class \( c_\ell = [f_{\theta,\ell}] \) is locally conformally equivalent to any \( c_\ell = [f_{\theta,\tilde{\ell}}] \) with \( \tilde{\ell} \in [\ell] \), where \( [\ell] \) denotes the class of 1-forms with purely imaginary values which differ from \( \ell \) only by a closed form (i.e. gauge transformation). Hence we can introduce the notion \( c_{[\ell]} \) which denotes (locally) a uniquely determined conformal structure, which we call the \( [\ell] \)-Fefferman conformal class on \( F_c \). The map
\[
\Psi : [\ell] \in H^1_{tr}(M, T_{10}) \mapsto c_{[\ell]} = [f_{\theta,\ell}]
\]
is then bijective onto the space of local conformal structures on \( F_c \) belonging to \( \ell \)-Fefferman metrics, which have the property that
\[
\mathcal{P}_\theta = \frac{i}{n+3} A_{\theta,\ell}
\]
for all pseudo-Hermitian forms \( \theta \) and \( \tilde{\ell} \in [\ell] \).
9. Aspects of conformal tractor calculus

We collect in this section notions and facts from conformal tractor calculus. Thereby, we will restrict ourselves only to the very necessary parts of the apparatus, which we will need for our purposes in the remaining sections. We will omit the use of CR-tractors. For a broader explanation of the topics that are raised we refer to [CG02, Cap05a, CSS00, CSS97, CSS01, Cap05c].

Let \( \mathfrak{g} \) denote the Lie algebra \( \mathfrak{so}(2, n + 1) \) belonging to the special orthogonal group \( \text{SO}(2, n + 1) \), which acts by standard representation on the Euclidean space \( \mathbb{R}^{2n+1} \) with indefinite scalar product

\[
\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1},
\]

where the matrix

\[
J = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}
\]

gives rise to the Minkowski metric on \( \mathbb{R}^{n+1} \). (We choose here the special signature \( (2, n + 1) \) for \( \mathfrak{g} \), since we want to work in the Lorentzian setting, which is related to strictly pseudoconvex CR-geometry.) The Lie algebra \( \mathfrak{g} = \mathfrak{so}(2, n + 1) \) is \([1]-\)graded:

\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,
\]

where \( \mathfrak{g}_0 = \mathfrak{co}(1, n) \), \( \mathfrak{g}_{-1} = \mathbb{R}^{n+1} \) and \( \mathfrak{g}_1 = \mathbb{R}^{n+1*} \). The 0-part \( \mathfrak{g}_0 \) decomposes further to its semisimple part \( \mathfrak{o}(1, n) \) and the center \( \mathbb{R} \). We realise the subspaces \( \mathfrak{g}_0, \mathfrak{g}_{-1} \) and \( \mathfrak{g}_1 \) by matrices of the form

\[
\begin{pmatrix} 0 & 0 & 0 \\ m & 0 & 0 \\ 0 & -t m J & 0 \end{pmatrix} \in \mathfrak{g}_{-1}, \quad \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \in \mathfrak{g}_0, \quad \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & -\bar{J} l \end{pmatrix} \in \mathfrak{g}_1.
\]

The commutators with respect to these matrices are given by

\[
[\cdot, \cdot] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0, \quad [(A, a), (A', a')] = (AA' - A'A, 0)
\]

\[
[\cdot, \cdot] : \mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}, \quad [(A, a), m] = Am + am
\]

\[
[\cdot, \cdot] : \mathfrak{g}_1 \times \mathfrak{g}_0 \to \mathfrak{g}_1, \quad [l, (A, a)] = lA + al
\]

\[
[\cdot, \cdot] : \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0, \quad [m, l] = (ml - J t (ml) J, lm),
\]

where \((A, a), (A', a') \in \mathfrak{o}(1, n) \oplus \mathbb{R}, m \in \mathbb{R}^{n+1}, l \in \mathbb{R}^{n+1*}\).

The subalgebra

\[
\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1
\]

of \( \mathfrak{g} \) is a parabolic, i.e., it contains a maximal solvable subalgebra of \( \mathfrak{g} \). We also denote \( \mathfrak{p}_+ := \mathfrak{g}_1 \). While the grading of \( \mathfrak{g} \) is not \( \mathfrak{p} \)-invariant, it gives rise to an invariant filtration

\[
\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{p}_+.
\]

The subgroup \( P \) of \( G = \text{SO}(2, n + 2) \), whose adjoint action on \( \mathfrak{g} \) preserves this filtration is a parabolic subgroup with Lie algebra \( \mathfrak{p} \). And the subgroup \( G_0 \) which preserves the grading of \( \mathfrak{g} \) is the group \( \text{CSO}(1, n) = \text{SO}(1, n) \times \mathbb{R}^+ \) with Lie algebra \( \mathfrak{g}_0 = \mathfrak{co}(1, n) \). Moreover, the exponential map restricts to a diffeomorphism from \( \mathfrak{p}_+ \) onto a normal subgroup \( P_+ \subset P \) and the parabolic \( P \) is the semidirect product of \( G_0 \) and \( P_+ \). Eventually, we note that the homogeneous space \( G/P \) is the flat model of conformal geometry in Lorentzian signature \( (1, n + 1) \).

The Killing form of \( \mathfrak{g} \) defines a duality between \( \mathfrak{g}/\mathfrak{p} \) and \( \mathfrak{p}_+ \) as \( \mathfrak{p} \)-modules (with respect to the adjoint action). In particular, for each \( k \in \mathbb{N} \) we get a \( \mathfrak{p} \)-module isomorphism

\[
\text{Hom}(\Lambda^k \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \cong \Lambda^k \mathfrak{p}_+ \otimes \mathfrak{g}.
\]
The latter spaces are the groups in the standard complex computing the Lie algebra homology of \( p_+ \) with coefficients in \( \mathfrak{g} \). The differentials in this standard complex are linear maps denoted by

\[
\partial^* : \text{Hom}(\Lambda^k \mathfrak{g}/p, \mathfrak{g}) \to \text{Hom}(\Lambda^{k-1} \mathfrak{g}/p, \mathfrak{g}) .
\]

For \( k = 1 \) the explicit formula for \( \partial^* \) is given by

\[
\phi \in p_+ \otimes \mathfrak{g} \quad \mapsto \quad \partial^* \phi = \sum_{i=1}^{n+1} [\eta_i, \phi(\xi_i)] \in \mathfrak{g} ,
\]

where \( \{\xi_i : i = 1, \ldots, n+1\} \) is some basis of \( \mathfrak{g}_{-1} \cong \mathfrak{g}/p \) and \( \{\eta_i : i = 1, \ldots, n+1\} \) is the dual basis of \( p_+ \). By construction, it is \( \partial^* \circ \partial^* = 0 \) for all \( k \) and we denote the \( k \)th homology group \( \ker(\partial^*)/\text{im}(\partial^*) \) which is a \( p \)-module by \( \mathcal{H}^k_{\mathfrak{g}} \). It is well known that \( p_+ \) acts trivially on \( \mathcal{H}^k_{\mathfrak{g}} \) (cf. [Cap05a]).

Now let \( (F^{n+1}, c) \) be a smooth orientable manifold with conformal structure \( c \) of signature \( (1, n) \). The conformal structure is given by a \( G_0 \)-principal bundle reduction \( \mathcal{G}_0(F) \) of the general linear frame bundle over \( F \). By the process of prolongation we obtain from \( \mathcal{G}_0(F) \) the \( P \)-principal fibre bundle \( \mathcal{P}(F) \) which is a reduction of the second order linear frame bundle on \( F \) and again determines the conformal structure \( c \) on \( F \) uniquely (cf. e.g. [Kob72, CSS97]). A Cartan connection \( \omega \) is a smooth 1-form on \( \mathcal{P}(F) \) with values in \( \mathfrak{g} \) such that

1. \( \omega(\chi_A) = A \) for all fundamental fields \( \chi_A \), \( A \in \mathfrak{p} \),
2. \( R_g^* \omega = \text{Ad}(g^{-1}) \circ \omega \) for all \( g \in P \) and
3. \( \omega|_{T_u \mathcal{P}(F)} : T_u \mathcal{P}(F) \to \mathfrak{g} \) is a linear isomorphism for all \( u \in \mathcal{P}(F) \).

The curvature 2-form \( \Omega \) of a Cartan connection \( \omega \) is given by

\[
\Omega = d\omega + \frac{1}{2}[\omega, \omega] .
\]

The corresponding curvature function \( \kappa : \mathcal{P}(F) \to \text{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \) is given by

\[
\kappa(u)(X,Y) := dw_0(\omega^{-1}_u(X),\omega^{-1}_u(Y)) + [X,Y]
\]

for \( u \in \mathcal{P}(F) \) and \( X, Y \in \mathfrak{g} \). The curvature function \( \kappa \) vanishes if one of its entries lies in \( \mathfrak{p} \subset \mathfrak{g} \). Hence we can view \( \kappa \) as an \( P \)-equivariant smooth function

\[
\kappa : \mathcal{P}(F) \to \text{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) .
\]

An important fact of conformal geometry is that there exists a distinguished Cartan connection \( \omega_{\text{nor}} \) on \( \mathcal{P}(M) \), which we call the normal Cartan connection of conformal geometry (cf. e.g. [Kob72, CSS97]). The normal Cartan connection \( \omega_{\text{nor}} \) is uniquely determined by the condition

\[
\partial^* \circ \kappa = 0 .
\]

Furthermore, we can extend the \( P \)-principal fibre bundle \( \mathcal{P}(F) \) by the structure group \( G \). This gives rise to the \( G \)-principal fibre bundle \( \mathcal{G}(F) = \mathcal{P}(F) \times_G G \) over \( F \). The normal Cartan connection \( \omega_{\text{nor}} \) on \( \mathcal{P}(F) \) extends by equivariance as well and becomes a principal fibre bundle connection on \( \mathcal{G}(F) \) which we denote again by \( \omega_{\text{nor}} \). Now let \( \mathcal{V} \) be a \( G \)-representation. From the representation \( \mathcal{V} \) we obtain the vector bundle \( \mathcal{V} := \mathcal{G}(F) \times_G \mathcal{V} \). We call \( \mathcal{V} \) the tractor bundle which belongs to the \( G \)-representation \( \mathcal{V} \). The normal connection \( \omega_{\text{nor}} \) on \( \mathcal{G}(F) \) induces on each tractor bundle \( \mathcal{V} \) an invariant covariant derivative:

\[
\nabla_{\text{nor}} : \Gamma(TF^*) \otimes \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}) .
\]

Of particular interest for our purpose are the standard and adjoint tractor bundles of a space \((F,c)\) with conformal structure. We denote by \( \mathcal{T}(F) := \mathcal{G}(F) \times_G \mathbb{R}^{2,n+1} \) the standard tractor bundle and by \( \mathcal{A}(F) := \mathcal{G}(F) \times_{Ad(G)} \mathfrak{g} \) the adjoint tractor bundle. In the latter case the adjoint representation of \( G \) on \( \mathfrak{g} \) can
be restricted to $P$, which preserves the filtration of $\mathfrak{g}$. Hence we obtain a natural filtration

$$A(F) \supset A^0(F) \supset A^1(F),$$

whereby $A^0(F) = \mathcal{P}(F) \times_{Ad(P)} \mathfrak{p}$ and the bundle $A^1(F) = \mathcal{P}(F) \times_{P} \mathfrak{p}^+_+$ is the dual $T^*F$ of the tangent bundle of $F$. Notice that with this identification the dual tangent bundle $T^*F$ is canonically contained in $A(F)$. Furthermore, by choosing a Lorentzian metric $g$ in $c$ we can reduce the bundles $\mathcal{S}(F)$ and $\mathcal{P}(F)$ (by using the corresponding Weyl structure of $g$) to the structure group SO$(1,n)$, which is the semisimple part of $G_0$. This reduction gives rise to a grading on the adjoint tractor bundle. It is

$$A(F) \cong_g TF \oplus \mathfrak{co}(TF) \oplus T^*F.$$

Thereby, notice that $TF = S_0(F) \times G_0 \mathfrak{g}_{-1}$ and $\mathfrak{co}(TF)$ is the subset of the endomorphism bundle $End(TF)$ consisting of skew-symmetric maps plus non-zero multiples of the identity map. The concrete identification of $A(F)$ with this graded sum depends very much on the choice of $g$ in $c$. The standard tractor bundle splits as graded sum with respect to a metric $g$ in $c$ by

$$T(F) \cong_g \mathbb{R} \oplus TF \oplus \mathbb{R}.$$

The Lie bracket on $\mathfrak{g}$ is $Ad(P)$-invariant. Hence there is an induced algebraic bracket on the adjoint tractor bundle $A(F)$, which we denote by $\{\cdot,\cdot\}$. This bracket admits with respect to some metric $g$ in the conformal class following expressions. For elements $(\xi,\varphi,\omega), (\tau,\psi,\eta) \in A_p(F)$ at $p \in F$, whereby the components are given with respect to the induced grading of $A(F)$ by $g$, it holds

$$\{ (\xi,\varphi,\omega) , (\tau,\psi,\eta) \} =$$

$$= ( \{ \xi,\varphi \} + \{\varphi,\tau\}, \{\xi,\eta\} + \{\varphi,\psi\} + \{\omega,\tau\}, \{\varphi,\eta\} + \{\omega,\psi\} ).$$

Moreover, it is

$$\{e_i,\psi\} = -\psi(e_i)$$

$$\{e_i^*,\psi\} = e_i^* \circ \psi$$

$$\{e_i,e_j^*\} = e_i^* \otimes e_j^* - \varepsilon_i \varepsilon_j e_j \otimes e_i^* \quad \text{for} \quad i \neq j$$

$$\{e_i,e_i^*\} = id|_{TF}$$

$$\{e_i,\eta\} = e_i \otimes \eta - \sum_{j=1}^{n+1} \varepsilon_i \varepsilon_j \eta(e_j)e_j \otimes e_i^* + \eta(e_i)id$$

$$\{\xi,e_j^*\} = \xi \otimes e_j^* - \varepsilon_j e_j \otimes g(\xi,\cdot) + e_j^*(\xi)id,$$

where $\{e_i : i = 1,\ldots,n+1\}$ denotes an orthonormal frame on $F$ with respect to $g$ and $\varepsilon_i := g(e_i,e_i)$.

Furthermore, the $\mathfrak{g}$-action on $\mathbb{R}^{2,n+1}$ which is compatible with the $P$-action gives rise to a multiplication on $T(F)$ by adjoint tractors, which we denote by

$$\bullet : A(F) \otimes T(F) \rightarrow T(F).$$

$$(A,t) \quad \mapsto \quad A \bullet t$$

In this respect, we can view an adjoint tractor as endomorphism on standard tractors. We also use the notation $A^2 \in End(T(F))$ for the superposition $A \bullet A$ of $A \in A(F)$ acting on standard tractors. Of course, $A^2$ is in general not an adjoint tractor.

Since the differential $\partial^* : \mathfrak{p}^+_+ \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ is $P$-equivariant, it induces a vector bundle map

$$\partial^* : T^*F \otimes A(F) \rightarrow A(F).$$
The image of $\partial^*$ is $\mathfrak{p}$ resp. $A^0(F)$. Let us denote $H^1_F := \mathcal{P}(F) \times_F \mathcal{H}^1$. It holds

$$H^1_F = A(F)/A^0(F).$$

We mentioned above that the group $P_+$ acts trivially on $\mathcal{H}^1_f$. For that reason we can view $H^1_F$ as the associated bundle $S_0 \times G_0 \mathcal{H}^1_f$, which is naturally identified with the tangent bundle of $F$:

$$H^1_F \cong TF.$$

Let $\pi_H : A(F) \to H^1_F$ denote the natural projection. This projection does not depend on the choice of a metric or a Weyl structure in $c$. If $\alpha$ is a section of $A(F)$ then we can consider $\pi_H(\alpha)$ as a tangent vector field, which we often call the ‘first slot’ of the adjoint tractor $\alpha$.

On any space $(F, c)$ with conformal structure there exists a certain (conformally) invariant differential operator, which assigns to any section of the bundle $H^k_F$ of homology groups of $F$ with coefficients in some $G$-module $\mathcal{V}$ an element of $\Omega^k(F, \mathcal{V})$, where the latter is the space of smooth $k$-forms with values in the tractor bundle belonging to the representation $\mathcal{V}$. This operator is called the splitting operator and plays a prominent role in the construction of BGG sequences (cf. Cap05a). We denote this operator by $S$. For the rest of the section, we aim to calculate $S$ explicitly in terms of a metric $g$ in the conformal class $c$ on $F$ for the case when it is a map from vector fields to sections of the adjoint tractor bundle:

$$S : \mathfrak{X}(F) \to \Gamma(A(F)).$$

In general, the splitting operator is determined by the following property (cf. CSS01). Let $d^{nor} : \Omega^k(F, \mathcal{V}) \to \Omega^{k+1}(F, \mathcal{V})$ denote the covariant exterior derivative, which is induced by the normal connection on the $G$-principal fibre bundle $\mathcal{G}(F)$. Then for arbitrary $\tau \in H^k_F$, it holds

1. $\partial^*(S(\tau)) = 0$ and $\pi_H(S(\tau)) = \tau$,

2. $\partial^*(d^{nor} S(\tau)) = 0$.

In the particular situation when $k = 0$ and $\mathcal{V} = \mathfrak{g}$ the covariant exterior derivative is just equal to $\nabla^{nor}$. Hence the splitting operator

$$S : \mathfrak{X}(F) \to \Gamma(A(F))$$

is determined by the property that for all vector fields $\tau$ on $F$, it holds

$$\pi_H(S(\tau)) = \tau$$

and

$$\partial^*(\nabla^{nor} S(\tau)) = 0.$$

Moreover, in this situation the map $S$ admits the property that if $\tau \in \mathfrak{X}(F)$ is a conformal Killing vector field on $(F, c)$ then

$$\nabla^{nor}_\tau S(\tau) = -\Omega^{nor}(\tau, X) \quad \text{for all } X \in TF.$$

Conversely, it is also known that the ‘first slot’ of any adjoint tractor, which solves the latter equation, is a conformal Killing vector field (cf. CD01, Cap05c).

The covariant exterior derivative $d^{nor} = \nabla^{nor}$ acting on a section $(\tau, \psi, \eta)$ in $A(F)$ with respect to a metric $g$ in the conformal class $c$ on $F$ is given by

$$d^{nor}(\tau, \psi, \eta) = (\nabla^g_\tau, \nabla^g_\psi, \nabla^g_\eta) + \{\cdot, 0, P^g(\cdot)\}, (\tau, \psi, \eta).$$

where $\nabla^g$ denotes the Levi-Civita connection and

$$P^g : TF \to T^*F$$

$$X \mapsto \frac{1}{n-1} \left( \frac{\text{scal}}{2n} g(X, \cdot) - \text{Ric}^g(X, \cdot) \right)$$
is the Schouten tensor with respect to \( g \) (cf. \cite{CG02}). Applying the differential \( \partial^* \) results to
\[
\partial^* d(\tau, \psi, \eta) = \begin{pmatrix}
0 \\
\sum_{i=1}^{n+1} (e_i^*, \nabla^g_{e_i} \tau) + \{e_i^*, \{e_i, \psi\}\} \\
\sum_{i=1}^{n+1} (e_i^*, \nabla^g_{e_i} \psi) + \{e_i^*, \{e_i, \eta\}\} + \{e_i^*, \{P^g(e_i), \tau\}\}
\end{pmatrix}.
\]

With the formulae for the bracket \( \{\cdot, \cdot\} \) from above we calculate
\[
\sum_{i=1}^{n+1} (e_i^*, \{e_i, \psi\}) = \psi + tr_g \psi \cdot id - \sum_{i=1}^{n+1} \varepsilon_i g(\psi e_i, \cdot) \\
\sum_{i=1}^{n+1} (e_i^*, \nabla^g_{e_i} \tau) = -\nabla^g \tau - tr_g \nabla^g \tau + \sum_{i=1}^{n+1} \varepsilon_i g(\nabla^g_{e_i} \tau, \cdot) \\
\sum_{i=1}^{n+1} (e_i^*, \{e_i, \eta\}) = (n+1) \cdot \eta \\
\sum_{i=1}^{n+1} (e_i^*, \nabla^g_{e_i} \psi) = \sum_{i=1}^{n+1} e_i^* \circ \nabla^g_{e_i} \psi \\
\sum_{i=1}^{n+1} \{e_i^*, \{P^g(e_i), \tau\}\} = -2P^g(\tau) + tr_g P^g \cdot g(\tau, \cdot),
\]
where \( tr_g P^g = -\frac{\text{scal}^g}{2n(n+1)} \). One can immediately see that the equation \( \partial^* d(\tau, \psi, \eta) = 0 \) is solved for an arbitrary vector field \( \tau \) on \( F \) by setting
\[
\psi = \nabla^g \tau \quad \text{and} \\
\eta = -\frac{1}{n+1} \left( (\sum_{i=1}^{n+1} e_i^* \circ \nabla^g_{e_i} \psi) - 2P^g(\tau) + tr_g P^g \cdot g(\tau, \cdot) \right).
\]

By inserting the first equation into the second one we obtain
\[
\eta = -\frac{1}{n+1} \left( (\sum_{i=1}^{n+1} e_i^* \circ \nabla^g_{e_i} (\nabla^g \tau)(\cdot)) - 2P^g(\tau) + tr_g P^g \cdot g(\tau, \cdot) \right).
\]

We reformulate the latter expression for \( \eta \) in the case when \( \tau \) is a conformal Killing vector field. In this situation, it holds
\[
\sum_{i=1}^{n+1} e_i^* \circ \nabla^g_{e_i} (\nabla^g \tau) = -g(tr_g \nabla^2 \tau, \cdot) + \frac{2}{n+1} d(\text{div}(\tau)) \quad \text{and}
\]
\[
d(\text{div}(\tau)) = -g(tr_g \nabla^2 \tau, \cdot) + \frac{2}{n+1} d(\text{div}(\tau)) - \text{Ric}^g(\tau, \cdot),
\]
where \( \text{div}(\tau) = tr_g g(\nabla^g \tau, \cdot) \). This implies
\[
\frac{n-1}{n+1} \cdot d(\text{div}(\tau)) = -g(tr_g \nabla^2 \tau, \cdot) - \text{Ric}^g(\tau, \cdot),
\]
\[
\sum_{i=1}^{n+1} e_i^* \circ \nabla^g_{e_i} (\nabla^g \tau) = -\frac{n+1}{n-1} g(tr_g \nabla^2 \tau, \cdot) - \frac{2}{n-1} \text{Ric}(\tau, \cdot),
\]
and finally, we obtain
\[
\eta = \frac{1}{n-1} \cdot g(tr_g \nabla^2 \tau, \cdot) + \frac{\text{scal}^g}{2n \cdot (n-1)} g(\tau, \cdot).
\]

We conclude that the splitting operator \( S \) applied to a conformal Killing vector \( \tau \) on \( (F, c) \) is given with respect to some metric \( g \) in \( c \) by
\[
S(\tau) = (\tau, \nabla^g \tau, \mathcal{P}(g(\tau, \cdot)))
\]
where the differential operator \( \mathcal{P} = \mathcal{P}^g \) is defined as in Proposition \( \text{I} \) by
\[
\mathcal{P} := \frac{1}{n-1} \left( tr_g \nabla^2 + \frac{\text{scal}^g}{2n} \right).
\]
10. Conformal tractor calculus for Fefferman space

Now we come back to the realm of Fefferman spaces and study their conformal tractor calculus. It is well known in case of integrable CR-structures that the standard tractor bundle of the Fefferman conformal class admits a complex structure $J_{CR}$, which arises naturally as the lift of the invariant complex structure on the standard CR-tractors. However, the complex structure $J_{CR}$ has another important property, namely it is parallel with respect to the normal conformal tractor connection $\nabla^{nor}$ and the corresponding conformal vector field to $J_{CR}$ (i.e., the 'first slot' of $J_{CR}$) is exactly twice the fundamental Killing field $S$ in fibre direction on $F_c$. We show here that a similar statement is true in the partially integrable case, but for general $\ell$-Fefferman spaces the situation changes.

Let $(F_{c,n+1}^n, f_{\theta,\ell})$ be a $\ell$-Fefferman space over a strictly pseudoconvex, partially integrable CR-space $(M^n, T_{10})$ with pseudo-Hermitian structure $\theta$ and $\ell \in \Omega^1(M; i\mathbb{R})$. Let $\mathcal{T}(F_c)$ be the standard tractor bundle to the conformal structure $[f_{\theta,\ell}]$ on $F_c$. Moreover, let $\mathcal{A}(F_c)$ be the adjoint tractor bundle, which splits to a graded sum by the choice of $f_{\theta,\ell}$. Let us denote $R := 2S$. We define the section $J_{CR}$ in $\mathcal{A}(F_c)$ with respect to the graded sum corresponding to $f_{\theta,\ell}$ as

$$J_{CR} := \begin{pmatrix} R & J_{\theta,\ell} & 2i \frac{A_{\theta,\ell}}{n+3} \end{pmatrix},$$

whereby $J_{\theta,\ell}$ denotes the horizontal lift of the almost complex structure $J$ on $H$ to $TF_c$ (with respect to $A_{\theta,\ell}$).

Proposition 2. (1) The endomorphism

$$J_{CR} = (R, J_{\theta,\ell}, \frac{2i}{n+3}A_{\theta,\ell})$$

defined as section of the adjoint tractor bundle $\mathcal{A}(F_c)$ does not depend on the choice of $\theta$, i.e., $J_{CR}$ is a CR-invariant.

(2) It is

$$J_{CR}^2 = -id|_{\mathcal{T}(F_c)},$$

i.e., $J_{CR}$ is a complex structure on the standard tractor bundle $\mathcal{T}(F_c)$.

Proof. (1) In fact, the complex structure $J_{CR}$ is the lift of the invariant complex structure on the CR-standard tractors of $M$. Since we do not introduce CR-tractors, we can not use this argument here. Alternatively, we could apply the transformation law for the graded sum of adjoint tractors under rescaling of the metric (cf. [CG02]) to show that the definition of $J_{CR}$ does not depend on $\theta$. However, we do not aim to introduce this transformation law either. So we postpone the proof of invariance until Proposition 3. There, we will see that $J_{CR}$ is even a conformally invariant object on $\mathcal{A}(F_c)$.

(2) With respect to the metric $f_{\theta,\ell}$ we can describe the action of $J_{CR}$ on a standard tractor $t = (a, \xi, b)$ with $a \in \mathbb{R}$, $\xi \in TF_c$ and $b \in \mathbb{R}$ as application of the matrix

$$J_{CR} = \begin{pmatrix} 0 & 2i \frac{A_{\theta,\ell}}{n+3} & 0 \\ R & J_{\theta,\ell} & \frac{2i}{n+3}A_{\theta,\ell} \\ 0 & -f_{\theta,\ell}(R, \cdot) & 0 \end{pmatrix},$$

where $A_{\theta,\ell}$ denotes the dual vector to $A_{\theta,\ell}$ with respect to $f_{\theta,\ell}$. For the middle entry of this matrix note that $f_{\theta,\ell}(\nabla_{X}^{f_{\theta,\ell}} R, Y) = d\theta(J_{\theta,\ell}X, Y)$ for all $X, Y \in TF_c$. 


Then it is

\[ J_{CR} \cdot t = \begin{pmatrix} \frac{2i}{n+3} A_{\theta,\ell}(\xi) \\ a \cdot R + J_{\theta,\ell}(\xi) - \frac{2ia}{n+3} A_{\theta,\ell}^b \\ -f_{\theta,\ell}(R,\xi) \end{pmatrix} , \]

where \( J_{\theta,\ell} \) acts trivially on the complement of the horizontally lifted distribution \( H \). Applying \( J_{CR} \) by multiplication again results to

\[ J_{CR} \cdot J_{CR} \cdot t = \begin{pmatrix} \frac{2ia}{n+3} A_{\theta,\ell}(R) \\ \frac{2ia}{n+3} A_{\theta,\ell}(\xi) \cdot R - pr_H(\xi) + \frac{2i}{n+3} f_{\theta,\ell}(R,\xi) A_{\theta,\ell}^b \\ \frac{2ia}{n+3} A_{\theta,\ell}(R) \end{pmatrix} = -t , \]

whereby we use \( \frac{2ia}{n+3} A_{\theta,\ell}(R) = -1 \) and \( pr_H \) denotes the projection of \( TF_c \) to the horizontal lift of \( H \).

We know so far that \( J_{CR} \) is a complex structure on \( T(F_c) \), whose 'first slot' is the (conformal) Killing vector field \( R \), which is vertical in the fibres of the \( \ell \)-Fefferman construction. With the help of the splitting operator \( S \) we obtain the conformal invariant \( S(R) \) on the adjoint tractor bundle \( A(F_c) \). We also set

\[ \mathcal{U} := \frac{n+1}{n \cdot (n-1)(n+3)} (0, 0, tr_{g} L_{dt} \cdot \theta) \in A(F_c) . \]

Note that the 1-form \( tr_{g} L_{dt} \cdot \theta \) does not depend on the pseudo-Hermitian structure \( \theta \), i.e., \( tr_{g} L_{dt} \cdot \theta \) is a uniquely determined section in \( T^* F_c \). Since \( T^* F_c \) is canonically included in the adjoint tractor bundle, we know that \( \mathcal{U} \) is a uniquely defined section in \( A(F_c) \) and does not depend on the choice of \( f_{\theta,\ell} \).

**Proposition 3.** Let \( (F^{n+1}, [f_{\theta,\ell}]) \) be the \( \ell \)-Fefferman space of a strictly pseudoconvex, partially integrable CR-space \( (M^n, T_{10}) \) with pseudo-Hermitian form \( \theta \), correction term \( \ell \in \Omega^1(M; i\mathbb{R}) \) and fundamental Killing vector field \( R \). It holds

\[ \begin{align*}
(1) & \quad S(R) = J_{CR} + \mathcal{U} . \\
(2) & \quad \text{In particular, } J_{CR} \text{ is a conformal invariant on } A(F_c).
\end{align*} \]

It holds

\[ S(R) = J_{CR} \quad \text{if and only if} \quad [\ell] \in H^1_{cr}(M, T_{10}) . \]

In this case it is

\[ \nabla^{nor} J_{CR} = -\Omega^{nor}(R, \cdot) . \]

Proposition 3 follows immediately from Proposition 1 and the definition of \( J_{CR} \) and \( \mathcal{U} \). We note that the curvature expression \( \Omega^{nor}(R, \cdot) \) is entirely determined by the Nijenhuis tensor \( \mathcal{N} \) and the 1-form \( \ell \). The result also shows the independence of \( J_{CR} \) from the choice of a pseudo-Hermitian form. This proves the first statement of Proposition 3.

11. THE RECONSTRUCTION

In Proposition 3 of the previous section we have seen how to obtain through the \( \ell \)-Fefferman construction conformal structures which admit an orthogonal complex structure \( \mathcal{R} \) on the standard tractor bundle whose 'first slot' \( R \) is a conformal vector field. We want to argue now in the reversed direction and discuss a reconstruction result from the existence of such a complex structure \( \mathcal{R} \). Our discussion will be of local nature, since we do not want to assume a \( S^1 \)-fibration on our initial space.

Let \( (F^{n+1}, c) \) be a smooth manifold of dimension \( n+1 \) with conformal structure \( c \). We assume here that there exists an adjoint tractor \( \mathcal{R} \in \Gamma(A(F)) \) which acts as
complex structure on the standard tractors $\mathcal{T}(F)$ such that the 'first slot' $\pi_H(R) = R$ is a conformal vector field on $F$, i.e., it holds
\[
\mathcal{R}^2 = -id|_{\mathcal{T}(F)} \quad \text{and} \quad \nabla^{\text{nor}}\mathcal{R} = -\Omega^{\text{nor}}(R, \cdot).
\]
The splitting operator applied to $R$ reproduces the adjoint tractor: $\mathcal{S}(R) = \mathcal{R}$.

In the following discussion, we will show step by step how to construct from the existence of $\mathcal{R}$ a (local) CR-structure on a quotient manifold. The following Lemma is the first step.

**Lemma 8.** Let
\[
\beta = \begin{pmatrix}
-a & l & 0 \\
m & A & -\mathcal{J} \ i l \\
0 & -\mathcal{J} \ m & a
\end{pmatrix}
\]
be an arbitrary matrix in $\mathfrak{g} = \mathfrak{so}(2, n + 1)$ with $\beta^2 = -id$. Then the vectors $m$ and $\mathcal{J} \ i l \in \mathbb{R}^{n+1}$ are non-zero and lightlike. Moreover, it is $A(m) = am$ and $l \circ A = al$ and the restriction of $A$ to the $\mathcal{J}$-orthogonal complement $W$ of $\text{span}\{m, \mathcal{J} \ i l\}$ in $\mathbb{R}^{n+1}$ is a complex structure, i.e., $A^2|_{W} = -id$.

**Proof.** It is
\[
\beta^2 = \begin{pmatrix}
a^2 + lm & -al + lA & -l \mathcal{J} \ i l \\
aml + Am & m + \mathcal{J} \ i l m \mathcal{J} & -A \mathcal{J} \ i l - a \mathcal{J} \ i l \\
-\mathcal{J} \ m & -\mathcal{J} \ A - a \mathcal{J} & -l \mathcal{J} \ i l + a^2
\end{pmatrix}
\]
This matrix square immediately shows that the condition $\beta^2 = -id$ implies the statements of Lemma.

In general, with respect to a metric $f$ in $c$ and a corresponding orthonormal frame the matrix $\beta \in \mathfrak{g}$ represents an adjoint tractor at some point $p \in F$. In particular, since we have chosen a metric $f$, the vector $m$ which determines the $\mathfrak{g}_{-1}$-part of the matrix $\beta$ represents the 'first slot' of that adjoint tractor. Moreover, the matrix $A$ gives rise to a skew-symmetric endomorphism on $T_p F$ and $l$ is a lightlike 1-form at $p \in F$. For our particular adjoint tractor $\mathcal{R}$ this implies the following objects.

It is $\mathcal{R} = (R, \nabla^I R, \mathcal{P}(f(R, \cdot)))$, where the 'first slot' $R$ is a non-zero, lightlike conformal Killing vector field and the 1-form $\mathcal{P}(f(R, \cdot))$ is non-zero and lightlike as well. The skew-symmetric part of the endomorphism $\nabla^I R$ restricts on the $f$-orthogonal complement $W^f$ of the lightlike vectors $R$ and the dual of $\mathcal{P}(f(R, \cdot))$ in $TF$ to an orthogonal complex structure $J^f$.

Now we denote by $\theta^f$ the dual of $R$ with respect to $f$ in $c$. The subbundles $\mathcal{R} R$ and $E := \ker(\theta^f)$ give rise to a flag in $TF$. The quotient $Q := E/\mathcal{R} R$ is via $f$ identified with the subbundle $W^f$. The image of the tensorial map
\[
\nabla^I R : E \rightarrow TF
\]
again lies in $E$ and $R$ is mapped to $\frac{-\text{div}^f(R)}{n+1} \cdot R$. Hence $\nabla^I R$ can be interpreted as a tensorial map on the quotient bundle $Q$. The skew-symmetric part $J$ of this map corresponds via $f$ to $J^f$ on $W^f \subset E$. From Lemma we know that $J : Q \rightarrow Q$ is a complex structure. This complex structure is orthogonal on $Q$ with respect to the positive definite scalar product $f_Q$ which is induced by $f$ on $Q$, i.e., it holds $f_Q(JX, JY) = f_Q(X, Y)$ for all vectors $X, Y$ in $Q$. Moreover, the 2-form $f_Q(J, \cdot)$ equals the 2-form that is induced by $d\theta$ on the quotient $Q$. Thereby, we note that $d\theta(R, \cdot) = 0$ on $E$.
So far we have constructed from $\mathcal{R}$ with the help of some fixed $f$ in $c$ the data:

$$\mathcal{R}, \ E, \ Q := E/\mathcal{R} \text{ and } J : Q \to Q.$$ 

We want to show now that these data do not depend on the choice of $f$, i.e., they are uniquely determined by the conformal structure $c$. Obviously, the subbundles $\mathcal{R}$ and $E := \ker(\theta^f)$ do not depend on $f$ in $c$, since $H^2_R F$ is invariantly identified with $TF$. Now, let $\tilde{f} = e^{-2\phi}f$ be an arbitrary rescaled metric in the conformal class $c$. From the transformation law for Levi-Civita connections in the conformal class we obtain

$$\nabla^\tilde{f}_X R = \nabla^f_X R - d\phi(X)R - d\phi(R)X \quad \text{for all } X \in E.$$ 

This relation shows that the skew-symmetric part of $\nabla^\tilde{f} R$ induces on $Q$ the complex structure $J$ again, i.e., $J$ on $Q$ does not depend on the choice of metric.

As next, we note that, in general, there exists a naturally defined Lie derivative acting on sections of tractor bundles with respect to conformal Killing vector fields. This derivative is given by differentiation of a given tractor along the flow of the vector, which consists of (local) conformal diffeomorphisms. For example, let $V$ denote any conformal Killing vector on $F$ and let $Q$ be an adjoint tractor then the Lie derivative of $Q$ with respect to $V$ is given by

$$\mathcal{L}_V Q = \{Q, S(V)\},$$

where $S(V)$ is the splitting operator applied to $V$. In case that $V$ is a Killing vector field for some suitable metric $f$ in $c$ this Lie derivative coincides with the usual $\mathcal{L}_V$ applied to the components of $Q$ with respect to the graded sum $TF \oplus \mathfrak{so}(TF) \oplus T^*F$.

In particular, for $\mathcal{R}$ we have

$$\mathcal{L}_R \mathcal{R} = \{\mathcal{R}, \mathcal{R}\} = 0.$$ 

In fact, since the conformal vector field $R$ has no zero, we can find locally a metric $f$ in $c$ such that $R$ is Killing, i.e., $R$ is an infinitesimal isometry of $f$ and $\mathcal{L}_R f = 0$. The adjoint tractor $\mathcal{R}$ takes with respect to $f$ the form $( R, \nabla^f R, \mathfrak{V}(f(R, \cdot)) )$. The property that $R$ is Killing implies that the commutators $[\mathcal{L}_R, \nabla^f]$ and $[\mathcal{L}_R, \mathfrak{V}]$ vanish. Hence it is $[R, R] = 0, \mathcal{L}_R(\nabla^f R) = 0$ and $\mathcal{L}_R(\mathfrak{V}(f(R, \cdot))) = 0$. With this remark we can see that $\mathcal{R} R, E$ and the complex structure $J$ on $Q$ are invariant under the (local) flow of the conformal vector field $R$ on $F$. In fact, let $\Phi^t_R$ denote this flow with time parameter $t$ and let $f$ be a metric such that $R$ is Killing. Then it holds $[R, R] = 0$ and $\mathcal{L}_R \theta = 0$, which proves that $\Phi^t_R(\mathcal{R} R) = \mathcal{R} R$ and $\Phi^t_R(E) = E$ to any time $t$. Moreover, it is $\mathcal{L}_R \nabla^f R = 0$, which shows that $J$ on $Q$ is invariant under the flow $\Phi^t$ as well.

Furthermore, we notice that the algebraic bracket

$$L_Q : \ Q \times Q \to K := TF/E$$

$$(X, Y) \mapsto pr_K[X, Y]$$

on the quotient $Q$ is $\Phi^R$-invariant and non-degenerate. This follows from the fact that $E, Q$ are $\Phi^R$-invariant and from the relation $\theta^f ([X,Y]) = -d\theta^f (X, Y) = f(X, \nabla^f_Y R)$ for all $X, Y$ in $E$ when $\mathcal{L}_R f = 0$. In this situation, it also holds

$$\theta^f ([\nabla^f_X R, \nabla^f_Y R]) = -d\theta^f (\nabla^f_X R, \nabla^f_Y R) = -f(\nabla^f_X R, Y) = f(X, \nabla^f_Y R),$$

which shows that $L_Q(JX, JY) = L_Q(X, Y)$ for all $X, Y$ in $Q$, i.e., the bracket $L_Q$ is totally real.

Now we consider again the conformal Killing vector $R$ on $F$. Since $R$ admits no zero, we can factorise the manifold $F$ locally around every point through the
integral curves $\text{Int}(R)$ of $R$ and obtain a smooth projection onto a $C^\infty$-manifold of dimension $n$: 
\[ \pi_U : U \subset F \to M_U := U/\text{Int}(R). \]

Thereby, we can assume that the open submanifold $U$ of $F$ is diffeomorphic to $M_U \times \mathbb{R}$ and $\pi_U$ is just the natural projection onto the first factor. Since the subbundle $E$ of $TF$ is invariant under the flow $\Phi^R$, it projects by $\pi_U$ to a distribution $H$ in $TM_U$. The subbundle $RR$ of $TF$ vanishes after projection to $M_U$. Hence the distribution $H$ is of codimension one in $TM_U$ and the quotient bundle $Q$ projects naturally onto the distribution $H$. Since the algebraic bracket $L_Q$ is non-degenerate and $\Phi^R$-invariant on $F$, the distribution $H$ is contact in $TM_U$. Moreover, the $\Phi^R$-invariant complex structure $J$ on $Q$ projects to a unique complex structure on the contact distribution $H$ in $TM_U$, which we again denote by $J$. The algebraic bracket which corresponds to the contact distribution $H$ is then real with respect to $J$. At this point we have shown that the adjoint tractor $\mathcal{R}$ on $(F,c)$ generates locally a uniquely determined CR-manifold $(M_U, H, J)$ of dimension $n$, which is partially integrable.

Eventually, we construct pseudo-Hermitian structures on $(M_U, H, J)$. For this purpose, let us consider (locally) any metric $f$ in $c$ on $U \subset F$ with $\mathcal{L}_R f = 0$. Then, it holds $\mathcal{L}_R \theta^f = 0$ and $\theta^f$ projects uniquely to a 1-form $\theta$ on $M_U$ such that $\theta|_U = 0$, i.e., a metric $f$ gives rise to a pseudo-Hermitian form $\theta$ on $(M_U, H, J)$ when $\mathcal{L}_R f = 0$. From such a $\theta$ we obtain the Levi-form $L_\theta$ on the CR-space $(M_U, H, J)$. The Levi-form $L_\theta$ is naturally $\pi_U$-related to $f_Q$ on the quotient bundle $Q$, which shows that $(M_U, H, J)$ and $\theta$ are strictly pseudoconvex.

**Proposition 4.** Let $(F,c)$ be a space with conformal structure $c$ admitting $\mathcal{R} \in \Gamma(A(F))$ which acts as complex structure on $\mathcal{T}(F)$ such that the 'first slot' $R$ is conformal Killing. Then

1. the local factorisation of $F$ through the integral curves $\text{Int}(R)$ to $R$ generates a smooth space admitting a uniquely determined strictly pseudoconvex and partially integrable CR-structure.

2. Any (local) metric $f$ in $c$ with $\mathcal{L}_R f = 0$ generates a pseudo-Hermitian structure on that CR-space.

So far we have not used in our discussion all the information that is encoded in the existence of the complex structure $\mathcal{R}$. And, in fact, there is still an open issue. It is the question how the conformal structure $c$ on $U \subset F$ is related to the induced CR-space $(M_U, H, J)$. To give an answer to this question we consider the 1-form $A^f := \mathcal{P}(f(R, \cdot))$ with respect to any metric $f$ such that $\mathcal{L}_R f = 0$. Then it holds $A^f(R) = -1$ and $\mathcal{L}_R A^f = 0$. These properties show that we can understand $A^f$ as a 'local connection form' on the fibration $\pi_U : U \subset F \to M_U$. From our construction so far, it is clear that $f$ on $U$ takes the form
\[ \pi_U^* L_\theta - 2 \cdot \theta^f \circ A^f, \]
where $L_\theta$ is the Levi-form on $(M_U, H, J)$ belonging to $\theta$, which in turn was induced by $f$ on $U \subset F$.

If we choose $U$ suitably small we can identify it with an open subset of the canonical $S^1$-bundle over $(M_U, H, J)$. (Such an identification can be understood as the choice of a gauge in the canonical $S^1$-bundle.) With such a gauge we can compare $f$ on $U$ with the Fefferman construction to $\theta$ on the canonical $S^1$-bundle of $(M_U, H, J)$. So let $A_\theta$ be the Weyl connection to $(M_U, H, J)$ that we introduced in section 5. We denote by $\ell := A^f - A_\theta$ the difference of the two given connections (via the chosen gauge). Remember that $A_\theta$ is determined by $f$. We can see that $f = L_\theta - 2 \cdot \theta^f \circ A^f$ is just the $\ell$-Fefferman metric on $U$ over $(M_U, H, J)$.
curvature $\Omega$

where $T$ in the canonical complex line tractor bundle, which is defined as

Proposition 5. Let $(F,c)$ be a space with conformal structure $c$ admitting $R \in \Gamma(A(F))$ which acts as complex structure on $\mathcal{T}(F)$ such that the ‘first slot’ $R$ is conformal Killing and let $(M_U,H,J)$ be the (locally) induced CR-space (cf. Proposition 4). Then $c$ on $F$ is (locally) conformally equivalent to some $\ell$-Fefferman metric which is constructed on $(M_U,H,J)$ with suitable $[\ell] \in H^1_{tr}(M_U,H,J)$. 

We remark that the Nijenhuis tensor $N_{J}$ of the CR-space $(M_U,H,J)$ and the 1-form class $[\ell]$ can be recovered from the curvature expression $\Omega^{nor}(R,\cdot)$. It is well known that the classical Fefferman construction over integrable CR-spaces produces conformal classes $c$ of metrics, which admit a parallel complex structure on the standard tractor bundle $\mathcal{T}(F)$, i.e., $\nabla^{nor} J_{CR} = 0$ for $J_{CR} = S(R)$ and $\Omega^{nor}(R,\cdot) = 0$. The latter condition is equivalent to $R \cdot W = 0$ and $R \cdot \mathcal{C} = 0$, where $W$ denotes the Weyl tensor of the conformal structure $c$ and $\mathcal{C}$ denotes the Cotton-York tensor. The Weyl tensor $W$ is determined by the $\mathfrak{g}_0$-part of the curvature $\Omega^{nor}$ and $\mathcal{C}$ comes from the $\mathfrak{g}_1$-part. On the other side, the existence of a lightlike Killing vector $R$ for some Lorentzian metric $f$ satisfying

$$
R \cdot W = 0, \quad R \cdot \mathcal{C} = 0 \quad \text{and} \quad \text{Ric}(R,R) > 0
$$

implies that $f$ is locally isometric to the Fefferman metric of an integrable CR-space (cf. [Gra87]). In the situation when there exists a complex structure $J_{CR}$ with $\nabla^{nor} J_{CR} = 0$, these conditions are automatically satisfied with respect to a metric $f$ such that the ‘first slot’ of $J_{CR}$ is (locally) a Killing vector. The reason is that $\mathcal{P}(f(R,\cdot))(R) = -1$ implies $\text{Ric}(R,R) > 0$ (cf. section 9). Notice once again that such a metric $f$ always exists locally.

In particular, this shows that the existence of a parallel complex structure $J_{CR}$ on the standard tractor bundle $\mathcal{T}(F)$ gives rise at least locally to a parallel section in the canonical complex line tractor bundle, which is defined as

$$
\mathcal{F} := \Lambda^{m+2*}(\mathcal{T}_{10}),
$$

where $\mathcal{T}_{10}$ denotes the $i$-eigenspace bundle to the extended endomorphism $J_{CR}$ on the complexified standard tractor bundle $\mathcal{T}^c(F) = \mathcal{T}(F) \otimes \mathbb{C}$. In fact, the conformal curvature $\Omega^{nor}$ acts on $\mathcal{T}(F)$ by

$$
\Omega^{nor}(X,Y)(t) = \kappa(X,Y) \cdot t
$$

for all $t \in \mathcal{T}(F)$ and vectors $X,Y \in TF \cong H^1_{\rho}F$. Now let $f$ be any metric in $c$, let $U$ denote the $f$-orthogonal complement to the span of $R = \pi_R(J_{CR})$ and the dual vector field of $\mathcal{P}(f(R,\cdot))$ and let $J$ by the antisymmetric part of the endomorphism $\nabla^{J}R : TF \to TF$. Furthermore, let $\{e_{\alpha} : \alpha = 1, \ldots, m\}$ denote an orthonormal basis of $(U,f|_U)$ such that $J(e_{2\alpha-1}) = e_{2\alpha}$ and $J(e_{2\alpha}) = -e_{2\alpha-1}$ for all $\alpha = 1, \ldots, m$. Then it is a straightforward calculation to see that the conformal curvature $\Omega^{nor}$ acts on the line tractor bundle $\mathcal{F}$ by multiplication with the complex number $\rho$, which is given by

$$
i \cdot \sum_{\alpha=1}^{m} e^{*}_{2\alpha}(\kappa_{0}(X,Y)(e_{2\alpha-1})) = i \cdot \sum_{\alpha=1}^{m} W(X,Y,e_{2\alpha-1},J_{\alpha}) \cdot t.$$
The property of $J_{CR}$ to be parallel for $\nabla^{nor}$ implies directly that
\[ W(X,Y,JZ,V) = -W(X,Y,Z,JV) \quad \text{for all } X, Y, Z, V \in TF. \]
Using the Bianchi identity and the fact that $\partial^* \circ \kappa = 0$ (i.e. the fact that the Weyl tensor $W$ has no trace) we find that the complex number $\rho$ is zero, i.e., the line tractor bundle $\mathcal{F}$ has no conformal curvature. Hence $\mathcal{F}$ admits locally parallel sections.

**Theorem 3.** (cf. [Spa85], [Gra87]) Let $(F, c)$ be a manifold with a conformal structure $c$ of Lorentzian signature admitting a conformal Killing vector $R$ whose adjoint tractor $S(R)$ is a $\nabla^{nor}$-parallel complex structure. Then $c$ is locally conformally equivalent to a Fefferman metric $f_\theta$ constructed over an integrable CR-space.

In particular, Theorem 3 says that $\Omega^{nor}(R, \cdot) = 0$ if and only if $N_J = 0$ and $[\ell] = 0$,

where $R$ is a conformal Killing vector and $S(R)$ is a complex structure. In terms of conformal holonomy the statement of Theorem 3 says that if the holonomy algebra of the normal conformal connection $\omega^{nor}$ is reduced to $\mathfrak{u}(1, m)$ then it is already reduced to $\mathfrak{su}(1, m)$. The latter is the Lie algebra of the structure group $SU(1, m)$ of CR-geometry. In particular, if a complex structure $J_{CR}$ with $\nabla^{nor} J_{CR} = 0$ exists, then there exists locally also a solution of the twistor equation for spinors (cf. [Ban99]).

12. Main Theorem

Proposition 3 and 5 establish our main result about complex structures on the conformal standard tractor bundle. We combine them in order to formulate the following Theorem. We remember that $c[\ell]$ denotes the conformal class which is determined by a $[\ell]$-Fefferman metric $f_{\theta, \ell}$, where $[\ell]$ denotes the class of a 1-form $\ell$ modulo closed forms.

**Theorem 4.** Any smooth manifold $(F^{n+1}, c)$ with Lorentzian conformal structure $c$ admitting a conformal Killing vector field $R$ whose corresponding adjoint tractor $S(R)$ is a complex structure on the standard tractor bundle $\mathcal{T}(F)$ is locally conformally equivalent to a $[\ell]$-Fefferman conformal class constructed on some strictly pseudoconvex, partially integrable CR-structure $(M, T_{10})$ with some $[\ell] \in H^1_{tr}(M, T_{10})$, i.e., the map

\[ \Psi : (T_{10}, [\ell]) \rightarrow (c[\ell], S(R)) \]

is surjective for $[\ell] \in H^1_{tr}(M, T_{10})$ onto the local conformal classes with complex structures $S(R)$.

We remark that it is not clear at this point whether the map $\Psi$ is a bijection. In principle, there might be the possibility that a conformal class is $[\ell]$-Fefferman with respect to two different constructions along different fibres (resp. lightlike Killing vector fields), which are not equivalent. We remark also that non-trivial examples for elements in $H^1_{tr}(M, T_{10})$ do exist for CR-structures, in general. However, we do not aim to investigate here the space $H^1_{tr}(M, T_{10})$ and the properties of their elements. We expect that the $\ell$-Fefferman construction is really an extension of the Fefferman construction and produces new sorts of Lorentzian metrics and conformal classes.

There are several more aspects of the relation of CR-geometry and conformal geometry, which are not touched here. We want to mention some of them. This happens also with the intention to set our discussion into a broader context. In fact, an issue that is behind our investigations here, is the relation of the canonical
normal Cartan connections of CR-geometry and conformal geometry in the Fefferman construction. Since the Fefferman construction can also be generalised and applied to various other parabolic geometries, this topic seems to be very basic and of general importance.

For integrable CR-structures it is well known that the equivariant extension of the normal CR-Cartan connection in the Fefferman construction gives rise to the normal Cartan connection of conformal geometry. In fact, this is true if and only if the CR-geometry is integrable. In case that the CR-geometry is only partially integrable, there still exists a unique normal Cartan connection (cf. [CS00]). However, the equivariant extension in the Fefferman construction does not give rise to the normal connection of conformal geometry.

What we have shown in this paper is that the two natural connections in the Fefferman construction (without an ℓ-term) of partially integrable CR-structures coincide at least when they are applied to the natural complex structure $J_{CR}$ that appears in the construction. Of course, this does not mean that they are the same. In fact, it shows only that they are equal up to a 1-form with values in the Lie algebra $\mathfrak{u}(1,m)$. It remains to determine this 1-form in the difference of the two natural Cartan connections. To achieve this, it should be in principle enough to calculate all components of the Ricci curvature of the Fefferman metric with respect to some pseudo-Hermitian structure.

Moreover, if we know the Ricci curvature of the Fefferman metric we can proceed and calculate the conformal curvature in dependence of the Tanaka-Webster curvature. So we would know then the right hand side $-\Omega^{nor}(\pi_H(\mathbb{R}), \cdot)$ of our tractor equation in terms of Webster curvature. This curvature expression would characterise essential properties of the fundamental Killing vector in the Fefferman construction. However, one reason that we did not continue to go this way was that calculations are already rather extensive. And, since the Fefferman construction is invariant in its nature, the idea is that there should be an invariant approach and calculation for the comparison of the two normal connection. This means an approach that does not use a Weyl connection or any other form of preferred linear connection. We do not know yet how this invariant approach works. On the other side, the occurrence of the ℓ-Fefferman construction seems to show that the existence of complex structures on standard tractors should not only be seen as a CR-invariant problem. Maybe this is an indication that an invariant approach to the comparison of the normal Cartan connections is not too obvious in the first place.

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