On bipartite cages of excess 4

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Abstract

The Moore bound $M(k, g)$ is a lower bound on the order of $k$-regular graphs of girth $g$ (denoted $(k, g)$-graphs). The excess $e$ of a $(k, g)$-graph of order $n$ is the difference $n - M(k, g)$. In this paper we consider the existence of $(k, g)$-bipartite graphs of excess 4 via studying spectral properties of their adjacency matrices. We prove that the $(k, g)$-bipartite graphs of excess 4 satisfy the equation $kJ = (A + kI)(H_{d-1}(A) + E)$, where $A$ denotes the adjacency matrix of the graph in question, $J$ the $n \times n$ all-ones matrix, $E$ the adjacency matrix of a union of vertex-disjoint cycles, and $H_{d-1}(x)$ is the Dickson polynomial of the second kind with parameter $k - 1$ and of degree $d - 1$. We observe that the eigenvalues other than $\pm k$ of these graphs are roots of the polynomials $H_{d-1}(x) + \lambda$, where $\lambda$ is an eigenvalue of $E$. Based on the irreducibility of $H_{d-1}(x) \pm 2$ we give necessary conditions for the existence of these graphs. If $E$ is the adjacency matrix of a cycle of order $n$ we call the corresponding graphs graphs with cyclic excess; if $E$ is the adjacency matrix of a disjoint union of two cycles we call the corresponding graphs graphs with bicyclic excess. In this paper we prove the non-existence of $(k, g)$-graphs with cyclic excess 4 if $k \geq 6$ and $k \equiv 1(\text{mod 3})$, $g = 8, 12, 16$ or $k \equiv 2(\text{mod 3})$, $g = 8$, and the non-existence of $(k, g)$-graphs with bicyclic excess 4 if $k \geq 7$ is odd number and $g = 2d$ such that $d \geq 4$ is even.

Keywords: cage problem, bipartite graphs, cyclic excess, bicyclic excess

1 Introduction

A $k$-regular graph of girth $g$ is called a $(k, g)$-graph. A $(k, g)$-cage is a $(k, g)$-graph with the fewest possible number of vertices, among all $(k, g)$-graphs. The order of a $(k, g)$-cage is denoted by $n(k, g)$. The Cage Problem calls for finding cages, and this problem was considered for the first time by Tutte [16]. It is known that a $(k, g)$-graph

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exists for any combination of \( k \geq 2 \) and \( g \geq 3 \), \([7, 14]\). However, the orders \( n(k, g) \) of \((k, g)\)-cages have only been determined for very limited sets of parameters \([9]\). A natural lower bound on the order of a \((k, g)\)-cage is called the Moore bound, and the form of the bound depends on the parity of \( g \), i.e.,

\[
 n(k, g) \geq M(k, g) = \begin{cases} 
 1 + k + k(k - 1) + \ldots + k(k - 1)^{(g - 3)/2}, & g \text{ odd}, \\
 2 \left( 1 + (k - 1) + \ldots + (k - 1)^{(g - 2)/2} \right), & g \text{ even}.
\end{cases}
\] (1)

The graphs whose orders are equal to the Moore bound are called Moore graphs. They are known to exist if \( k = 2 \) and \( g \geq 3 \), \( g = 3 \) and \( k \geq 2 \), \( g = 4 \) and \( k \geq 2 \), \( g = 5 \) and \( k = 2, 3, 7 \), or \( g = 6, 8, 12 \) and a generalized \( n \)-gon of order \( k - 1 \) exists \([11, 14, 9]\).

The existence of a \((57, 5)\)-Moore graph is an open question.

The excess \( e \) of a \((k, g)\)-graph is the difference between its order \( n \) and the Moore bound \( M(k, g) \), i.e., \( e = n - M(k, g) \). Regarding graphs of even girth we will use the following three results:

**Theorem 1.1** ([3]) Let \( G \) be a \((k, g)\)-cage of girth \( g = 2d \geq 6 \) and excess \( e \). If \( e \leq k - 2 \), then \( e \) is even and \( G \) is bipartite of diameter \( d + 1 \).

For the next theorem, let \( D(k, 2) \) denote the incidence graph of a symmetric \((v, k, 2)\)-design.

**Theorem 1.2** ([3]) Let \( G \) be a \((k, g)\)-cage of girth \( g = 2d \geq 6 \) and excess 2. Then \( g = 6 \), \( G \) is a double-cover of \( D(k, 2) \), and \( k \) is not congruent to 5 or 7 \((\text{mod } 8)\).

**Theorem 1.3** ([11]) Let \( k \geq 6 \), \( g = 2d > 6 \). No \((k, g)\)-graphs of excess 4 exist for parameters \( k, g \) satisfying at least one of the following conditions:

1) \( g = 2p \), with \( p \geq 5 \) a prime number, and \( k \not\equiv 0, 1, 2 \text{ (mod } p) \);

2) \( g = 4 \cdot 3^s \) such that \( s \geq 4 \), and \( k \) is divisible by 9 but not by \( 3^{s-1} \);

3) \( g = 2p^2 \) with \( p \geq 5 \) a prime number, and \( k \not\equiv 0, 1, 2 \text{ (mod } p) \) and even;

4) \( g = 4p \), with \( p \geq 5 \) a prime number, and \( k \not\equiv 0, 1, 2, 3, p - 2 \text{ (mod } p) \);

5) \( g \equiv 0 \text{ (mod } 16) \), and \( k \equiv 3 \text{ (mod } g) \).

Motivated by the result in Theorem 1.3 which was obtained through counting cycles in a hypothetical graph with given parameters and excess 4, in this paper we address the question of the existence of \((k, g)\)-graphs of excess 4 using spectral properties of their adjacency matrices. The question of the existence of \((k, g)\)-graphs of excess 4 is wide open, and prior to the publication of [11], no such results were known. The results contained in our paper further extend our understanding of the structure of the potential graphs of excess 4. Throughout, we assume that \( k \geq 6 \), \( g = 2d \geq 6 \) and \( G \) is a \((k, g)\)-graph of excess 4 and order \( n \). Due to Biggs’s result stated in Theorem 1.3, the restriction of the parameters \( k, g \) given above allows us
to conclude that $G$ is a bipartite graph with diameter $d + 1$. For each integer $i$ in the range $0 \leq i \leq d + 1$, we define the $n \times n$ matrix $A_i = A_i(G)$ as follows. The rows and columns of $A_i$ correspond to the vertices of $G$, and the entry in position $(u, v)$ is 1 if the distance $d(u, v)$ between the vertices $u$ and $v$ is $i$, and zero otherwise.

Clearly, $A_0 = I$, $A_1 = A$, the usual adjacency matrix of $G$. The last non-zero matrix is the matrix $A_{d+1}$ which we shall denote by $E$ and refer to it as the excess matrix, i.e., $E$ is the adjacency matrix of the graph with the same vertex set $V$ as $G$ such that two vertices of $V$ are adjacent if and only if they have distance $d + 1$. We will call this graph the excess graph of $G$ and we will denote it by $G(E)$. If $J$ is the all-ones matrix, the sum of the $i$-distance matrices $A_i, 0 \leq i \leq d$, and the matrix $E$ yields $\sum_{i=0}^{d} A_i + E = J$. To apply the last identity we will use Lemma 4 from [11]. Employing the methodology used by Bannai et al. in [1], [2], later by Biggs et al. in [3], Delorme et al. in [5] and Garbe in [10], we will show that the eigenvalues of $G$ other than $\pm k$ are the roots of the polynomials $H_{d-1}(x) + \lambda$. Here, $H_{d-1}(x)$ is the Dickson polynomial of the second kind with parameter $k - 1$ and degree $d - 1$, and $\lambda$ is an eigenvalue of the excess matrix $E$. Furthermore, for odd $k \geq 7$ and $d \geq 4$, we prove that the polynomial $H_{d-1}(x) \pm 2$ is irreducible over $\mathbb{Q}[x]$, which leads to necessary conditions for existence of $(k, g)$-graphs of excess 4, Theorem 2.7.

We say that a graph $G$ has a cyclic excess if the excess graph $G(E)$ is a cycle of length $n$, and a graph $G$ has a bicyclic excess if $G(E)$ is a disjoint union of two cycles. In [6] Delorme et al. considered graphs with cyclic defect and excess 2, proving non-existence of infinitely many such graphs. The paper describes the cycle structure of the excess graphs of the known non-trivial graphs of excess 2:

1) the excess graph of the only $(3, 5)$-graph of excess 2 is a disjoint union of a 9-cycle and a 3-cycle or a disjoint union of an 8-cycle and 4-cycle;

2) the excess graph of the unique $(4, 5)$-graph of excess 2 (the Robertson graph) is a disjoint union of a 3-cycle, a 12-cycle and a 4-cycle;

3) the excess graph of the unique $(3, 7)$-graph of excess 2 (the McGee graph) is a disjoint union of six 4-cycles.

We note that no $(k, g)$-graph of cyclic excess 2 are known, while examples of graphs with bicyclic excess 2 can be found among the $(3, 5)$-graphs of excess 2. Proving that the excess graphs of bipartite graphs of excess 4 form a disjoint union of cycles, while also inspired by the results in [6], in Section 3 we consider the existence of bipartite graphs of excess 4 with cyclic and bicyclic excess 4. Based on the irreducibility of $H_{d-1}(x) \pm 2$ and $H_{d-1}(x) - 1$ over $\mathbb{Q}[x]$, we prove the non-existence of infinitely many such graphs of girths at least 8.
2 Necessary conditions for the existence of graphs of even girth and excess 4

Let \( k \geq 6, \ g = 2d \geq 6 \), and let \( G \) be a \((k, g)\)-graph of excess 4. Then \( G \) is bipartite of diameter \( d + 1 \). Let \( N_G(u, i) \) denote the set of vertices of \( G \) whose distance from \( u \) in \( G \) is equal to \( i \), \( 1 \leq i \leq d + 1 \). The subgraph of \( G \) induced by the set of vertices of \( G \) whose distance from \( u \) is at most \( \frac{g-2}{2} \) and whose distance from \( v \) is by one larger than their distance from \( u \) induces a tree of depth \( \frac{g-2}{2} \) rooted at \( u \) (we will call it \( T_u \)). Also, the subgraph of \( G \) induced by the set of vertices of \( G \) whose distance from \( v \) is at most \( \frac{g-2}{2} \) and whose distance from \( u \) is by one larger than their distance from \( v \) induces a tree of depth \( \frac{g-2}{2} \) rooted at \( v \) (we will call it \( T_v \)). Since \( G \) is of girth \( g \), the trees \( T_u \) and \( T_v \) are disjoint and contain no cycles. Since each vertex of \( G \) is of degree \( k \), the order of \( T_u \cup T_v \) is equal to \( 2(1 + (k - 1) + (k - 1)^2 + \ldots + (k - 1)^{\frac{g-2}{2}}) \). We will call the union of the trees \( T_u, T_v \) with the edge \( f \) Moore tree of \( G \) rooted at \( f \); it is the subtree of \( G \) that is the basis of the Moore bound for even \( g \). The graph \( G \) must contain 4 additional vertices \( w_1, w_2, w_3, w_4 \) which do not belong to either \( T_u \) or \( T_v \), and whose distance from both \( u \) and \( v \) is greater than \( \frac{g-2}{2} \). We will call these vertices the excess vertices with respect to \( f \) and denote this set \( X_f = \{w_1, w_2, w_3, w_4\} \); we call the edges not contained in the Moore tree of \( G \) horizontal edges.

The following lemma restricts the possible ways in which the four excess vertices are attached to the Moore tree.

**Lemma 2.1 ([11])** Let \( k \geq 6, \ g = 2d \geq 6 \). Let \( G \) be a \((k, g)\)-graph of excess 4, \( u, v \) be two adjacent vertices in \( G \), and \( X_f = \{w_1, w_2, w_3, w_4\} \) be the four excess vertices with respect to the edge \( f = \{u, v\} \). The induced subgraph \( G[w_1, w_2, w_3, w_4] \) is isomorphic to \( 2K_2 \) (two disjoint copies of \( K_2 \)) or \( P_3 \) (a path of length 3).

Next, let us define the following polynomials:

\[
F_0(x) = 1, \quad F_1(x) = x, \quad F_2(x) = x^2 - k;
\]

\[
G_0(x) = 1, \quad G_1(x) = x + 1;
\]

\[
H_{-2}(x) = -\frac{1}{k-1}, \quad H_{-1}(x) = 0, \quad H_0(x) = 1, \quad H_1(x) = x;
\]

\[
P_{i+1}(x) = xP_i(x) - (k-1)P_{i-1}(x) \quad \text{for} \quad \begin{cases} i \geq 2, & \text{if } P = F, \\ i \geq 1, & \text{if } P = G, \\ i \geq 1, & \text{if } P = H. \end{cases} \tag{2}
\]

In [13], Singleton gives many relationships between these polynomials. We will use two of them. Given any \( i \geq 0 \),

\[
G_i(x) = \sum_{j=0}^{i} F_j(x) \tag{3}
\]
\[ G_{i+1}(x) + (k-1)G_i(x) = (x+k)H_i(x). \] (4)

The above defined polynomials have a close connection to the properties of a graph \( G \). Namely, for \( t < g \) the element \((F_t(A))_{xy}\) counts the number of paths of length \( t \) joining vertices \( x \) and \( y \) of \( G \). It follows from (3) that \( G_t(A) \) counts the number of paths of length at most \( t \) joining pairs of vertices in \( G \). All of the preceding claims can be found in [3]. The next lemma is based on the structure of \( G \) described in Lemma 2.1.

**Lemma 2.2** Let \( k \geq 6 \), \( g = 2d \geq 6 \) and let \( G \) be a \((k,g)\)-graph of excess 4. If \( A \) is the adjacency matrix of \( G \) and \( E \) is the excess matrix of \( G \), then

\[ F_d(A) = kA_d - AE. \]

**Proof.** Let \( f = \{u, v\} \) be a base edge of the Moore tree and let \( f_1 = \{w_1, w_2\}, f_2 = \{w_3, w_4\} \) be the edges of the subgraph induced by \( X_f \). Also, let us assume that \( d(u, w_1) = d(u, w_3) = d \) and \( d(u, w_2) = d(u, w_4) = d + 1 \). We consider the case when \( G[w_1, w_2, w_3, w_4] \) is isomorphic to \( 2K_2 \) in which case the excess vertices do not share common neighbour. The other cases when \( G[w_1, w_2, w_3, w_4] \) is isomorphic to \( 2K_2 \) and the excess vertices share common neighbour or the subgraph induced by the excess vertices contains \( P_3 \) are analogous. Since there are \( k-1 \) paths of length \( d \) from \( u \) to \( w_1 \) and \( w_3 \), by the definition of \( F_i(x) \) we have \((F_d(A))_{u,w_1} = (F_d(A))_{u,w_3} = k - 1\). Considering the vertices of distance \( d \) from \( u \), there are also the \((k-1)^{d-1}\) leaves of the subtree rooted at \( v \). For \( 2(k-1) \) of these vertices there exists \( k-1 \) paths of length \( d \) from \( u \) to them. Namely, they are the vertices adjacent to \( w_2 \) or \( w_4 \). For all the other leaves, there are \( k \) paths between. Thus, \((F_d(A))_{u,s} = 0 \) if \( d(u, s) \neq d \), \((F_d(A))_{u,s} = k \) if \( s \) is a leaf of a branch rooted at \( v \) and \( d(u, s) = d \), and \((F_d(A))_{u,s} = k - 1 \) if \( s \) is \( w_1, w_3 \) or a leaf of a branch rooted at \( v \) and \( d(u, s) = d \). This yields for the matrix \( kA_d \) that \((kA_d)_{u,s} = k \) if \( d(u, s) = d \) and \((kA_d)_{u,s} = 0 \) if \( d(u, s) \neq d \). Now, let \( s \) be a vertex of \( G \) such that \( d(u, s) = d \) and \( s \) is adjacent to \( w_2 \) or \( w_4 \). If \( s = w_1 \) or \( s = w_3 \) then it is easy to see that \((AE)_{u,s} = 1 \). On the other hand, since \( s \) is adjacent to the subtree rooted at \( u \) through \( k-2 \) different horizontal edges, it follows that between the \( k-1 \) branches of the subtree rooted at \( u \) there exists one sub-branch that is not adjacent to \( s \) though a horizontal edge. Let \( s_1 \) be the root of that sub-branch. Then, \( d(s, s_1) = d + 1, d(u, s_1) = 1 \), which implies \((A)_{u,s_1} = 1 \) and \((E)_{s_1,s} = 1 \). Let \( s_2 \) be the other vertex of distance \( d + 1 \) from \( s \). Because all neighbours of \( u \), except \( s_1 \), are of distance smaller than \( d + 1 \) of \( s \), we have \((A)_{u,s_2} = 0 \) and \((E)_{s_2,s} = 1 \). Thus \((AE)_{u,s} = 1 \). If \( s \) is a vertex of \( G \) such that \( d(u, s) = d \) and \( s \) is not adjacent to \( w_2 \) or \( w_4 \) then the distance between \( s \) and the neighbours of \( u \) is \( d - 1 \). In this case, \((AE)_{u,s} = 0 \). If \( d(u, s) \neq d \) then the distance between \( s \) and the neighbours of \( u \) is different from \( d + 1 \), and therefore \((AE)_{u,s} = 0 \). The required identity follows from summing up the above conclusions. \( q.e.d. \)

**Lemma 2.3** Let \( k \geq 6 \), \( g = 2d \geq 6 \) and let \( G \) be a \((k,g)\)-graph of excess 4. If \( A \) is the adjacency matrix of \( G \), \( E \) is the excess matrix of \( G \) and \( I \) is the all-ones matrix,
then
\[ kJ = (A + kI)(H_{d-1}(A) + E). \]

**Proof.** By the definition of the polynomials \( G_i(x) \) and using the fact that \( G \) has diameter \( d + 1 \) we conclude \( J = G_{d-1}(A) + A_d + E \). The relation (3), setting \( i = d \), asserts \( G_d(A) = G_{d-1}(A) + F_d(A) \). Substituting this identity in (4), where we fix \( i = d - 1 \), we get \( kG_{d-1}(A) + F_d(A) = (A + kI)H_{d-1}(A) \). Due to Lemma 2.2 the last identity is equivalent to \( kG_{d-1}(A) + kA_d + kE = (A + kI)(H_{d-1}(A) + E) \). From \( kJ = kG_{d-1}(A) + kA_d + kE \) follows \( kJ = (A + kI)(H_{d-1}(A) + E) \). q.e.d.

The next theorem gives a relationship between the eigenvalues of the matrices \( A \) and \( E \) (this result is an analogue of Theorem 3.1 in [5]):

**Theorem 2.4** If \( \mu(\neq \pm k) \) is an eigenvalue of \( A \), then
\[ H_{d-1}(\mu) = -\lambda, \]
where \( \lambda \) is an eigenvalue of \( E \).

**Proof.** Let us suppose that \( \mu \) is an eigenvalue of \( A \). Since \( G \) is a \( k \)-regular graph, the all-ones matrix \( J \) is a polynomial in \( A \). This implies that any eigenvector of \( A \) is also an eigenvector of \( J \). From \( kJ = (A + kI)(H_{d-1}(A) + E) \) and since \( H_{d-1}(A) \) is also a polynomial in \( A \), we have that \( E \) is a polynomial in \( A \), and consequently, every eigenvector of \( A \) is an eigenvector of \( E \). Therefore, the eigenvalues of \( kJ \) are of the form \( (\mu + k)(H_{d-1}(\mu) + \lambda) \). As is well-known, the eigenvalues of \( kJ \) are \( kn \) (with multiplicity 1) and 0 (with multiplicity \( n - 1 \)). The eigenvalue \( kn \) corresponds to \( \mu = k \), and so all the remaining eigenvalues, except for \( -k \), satisfy the above equation.

Since the eigenvalues of a disjoint union of cycles are known, we are now in a position to determine the spectrum of \( A \):

**Lemma 2.5** Let \( k \geq 6 \), \( g = 2d \geq 6 \) and let \( G \) be a \((k,g)\)-graph of excess 4. If \( A \) is the adjacency matrix of \( G \) and \( E \) is the excess matrix of \( G \), then:

1) The matrix \( E \) is the adjacency matrix of a graph \( G(E) \), consisting of a disjoint union of \( c \) cycles \( C_i \) of length \( l_i \) with \( 1 \leq i \leq c \). Moreover, if \( d \) is odd and \( V_1 \) and \( V_2 \) are the two partition sets of the bipartite graph \( G \), then every cycle in \( G(E) \) is completely contained either in \( V_1 \) or \( V_2 \).

2) The spectrum of \( A \) consists of:

   2.1) \( \pm k, c - 2 \) many solutions of \( H_{d-1}(x) = -2 \), and one solution of each equation \( H_{d-1}(x) = -2 \cos\left(\frac{2\pi j}{l_i}\right), j = 1, ..., l_i - 1; 1 \leq i \leq c, \) for \( d \) odd;

   2.2) \( \pm k, c - 1 \) many solutions of \( H_{d-1}(x) = -2 \), and one solution of each equation (except one) \( H_{d-1}(x) = -2 \cos\left(\frac{2\pi j}{l_i}\right), j = 1, ..., l_i - 1; 1 \leq i \leq c, \) for \( d \) even;
Proof. 1) Our proof is analogous to that of Kovács for girth 5, \cite{12}, and Garbe’s proof for odd girth $g = 2k + 1 > 5$, \cite{10}. Let $f = \{u, v\}$ be a base edge of a bipartite Moore tree of $G$. Lemma 2.4 asserts that there exist exactly two vertices of $G$ on distance $d + 1$ from $u$. Namely, they are the excess vertices adjacent to the leaves of the subtree rooted at $v$. The excess matrix $E$ is the adjacency matrix for the graph $G(E)$ with same vertex set $V$ as $G$ such that two vertices of $G(E)$ are adjacent if and only if they are of distance $d + 1$. Because for each vertex $u \in V(G)$ there are exactly two vertices on distance $d + 1$ from $u$, every component of $G(E)$ is a cycle. Let $c$ be the number of these cycles and let $l_i, i = 1, .., c$, be the lengths of these cycles ordered in an arbitrary manner. Moreover, if $d$ is an odd number, any two vertices of $G$ with distance $d + 1$ lie in the same partite set. Therefore any connected component of $G(E)$ is entirely contained either in $V_1$ or $V_2$.

2) The eigenvalues of an $n$-cycle are known and are equal to $2 \cos(\frac{2\pi j}{n}), (j = 0, ..., n - 1)$. Therefore the eigenvalues of $G(E)$ are $2 \cos(\frac{2\pi j}{l_i}), j = 0, 1, ..., l_i - 1; 1 \leq i \leq c$, \cite{10}. Since $G$ is a $k$-regular bipartite graph, it has (among others) the eigenvalues $k$ and $-k$. Let $V_1$ and $V_2$ be the partition sets of $G$. Hence the eigenvector of $A$ corresponding to $k$ consist of the all-ones vector $j$, and the eigenvector corresponding to $-k$ is the vector $j'$ with values 1 on $V_1$ and values $-1$ on $V_2$. If $d$ is an odd number then two vertices of $G(E)$ are adjacent if and only if they are in the same partite set. Therefore $E \cdot j' = 2j'$, which implies that from the set of $c$ solutions on $H_{d-1}(x) = -2$ we need to subtract two multiplicities for the eigenvalues $k$ and $-k$ of $G$. If $d$ is an even number then two vertices of $G(E)$ are adjacent if and only if they are in different partite sets. Thus $E \cdot j' = -2j'$. In this case, from the set of $c$ solutions on $H_{d-1}(x) = -2$ we need to subtract one multiplicity for the eigenvalue $k$ and from the set of all solutions on $H_{d-1}(x) = 2$ we need to subtract one multiplicity for the eigenvalue $-k$.

$q.e.d.$

Lemma 2.6 Let $k \geq 6, g = 2d \geq 6$ and let $G$ be a $(k, g)$-graph of excess 4. Furthermore, let $c$ be the number of cycles of $G(E)$ and $c_2$ be the number of cycles of even length. Then:

1) If $H_{d-1}(x) - 2$ is irreducible over $\mathbb{Q}[x]$ then $d - 1 \text{ divides } c - 1$ or $c - 2$;

2) If $H_{d-1}(x) + 2$ is irreducible over $\mathbb{Q}[x]$, then $d - 1 \text{ divides } c_2 - 1$ or $c_2$.

Proof. 1) Combining Theorem 2.4 and part 2) from Lemma 2.5 we obtain that $H_{d-1}(x) - 2$ is an irreducible factor of the characteristic polynomial of $A$. Realizing that the roots of an irreducible factor of a characteristic polynomial of given rational symmetric matrix have the same multiplicities, \cite{12}, from 2) of Lemma 2.5 we have: If $d$ is an even number then the $d - 1$ roots of $H_{d-1}(x) - 2$ have multiplicity $\frac{c - 2}{d - 1}$, which has to be a positive integer. If $d$ is odd then the $d - 1$ roots have multiplicity $\frac{c - 2}{d - 1}$.

2) Part 2) follows along the same lines as part 1).

$q.e.d.$
We can base the testing of irreducibility of \( H_{d-1}(x) \pm 2 \) on the well-known Eisenstein’s criterion that asserts for a polynomial \( f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x] \) and a prime \( p \) that divides \( a_i \) for all \( 0 \leq i < n \), does not divide \( a_n \) and \( p^2 \) does not divide \( a_0 \) Now we are ready for the main result in this section:

**Theorem 2.7** Let \( k \geq 7 \) be an odd number and let \( g = 2d \geq 8 \). Let \( c \) be the number of cycles of \( G(E) \) and \( c_2 \) be the number of cycles with even length. If there exists a \((k, g)\)-graph of excess \(4\) then

1) if \( d \) is an odd number then \( d - 1 \) divides \( c - 2 \) and \( c_2 \);

2) if \( d \) is an even number then \( d - 1 \) divides \( c - 1 \) and \( c_2 - 1 \).

**Proof.** According to Lemma [2.6] it is enough to prove that the polynomials \( H_{d-1}(x) - 2 \) and \( H_{d-1}(x) + 2 \) are irreducible. We will prove using induction on \( d \geq 4 \) that \( H_{d-1}(x) = x^{d-1} + (k - 1) \cdot P_{d-3}(x) \), where \( P_{d-3}(x) \) is an integer polynomial of degree \( d - 3 \). For \( d = 4 \) we calculate \( H_3(x) = x^3 - 2(k - 1)x \). Let us suppose that the above formula holds for \( H_{d-2}(x) \) and \( H_{d-3}(x) \). That yields

\[
H_{d-1}(x) = x(x^{d-2} + (k - 1) \cdot P_{d-4}(x)) - (k - 1)(x^{d-3} + (k - 1) \cdot P_{d-5}(x)) = x^{d-1} + (k - 1) \cdot P_{d-3}(x).
\]

Therefore \( H_{d-1}(x) \pm 2 = x^{d-1} + (k - 1) \cdot P_{d-3}(x) \pm 2 \). By the inducational hypothesis, follows that for an odd \( d \) occurs \( H_{d-1}(0) = (-1)^{d-1} \cdot (k - 1)^{d-1}/2 \) and \( H_{d-1}(0) = 0 \) for an even \( d \). Hence for an odd \( d \geq 5 \) the absolute value \( (-1)^{d-1} \cdot (k - 1)^{d-1}/2 \) is not divisible by \( 2^2 \), and clearly for an even \( d \geq 4 \), \( \pm 2 \) is not divisible by \( 2^2 \). Since \( k - 1 \) is even, it follows that every coefficient on \( H_{d-1}(x) \pm 2 \) except for the coefficient 1 of \( x^{d-1} \) is divisible by 2. Thus, the conditions of the Eisenstein’s criterion are satisfied, and \( H_{d-1}(x) \pm 2 \) is irreducible. \( q.e.d. \)

3 The non-existence of bipartite graphs of cyclic or bicyclic excess

In this section we still deal with the family of graphs considered as in Section 2. Again, let \( k \geq 6, g = 2d \geq 6 \) and let \( G \) be a \((k, g)\)-graph of excess \(4\) and order \( n \). Clearly \( n \) is even number. We have already proved that the excess graph \( G(E) \) consists of a disjoint union of \( c \) cycles \( C_i, 1 \leq i \leq c \). If \( c = 1 \) and \( G(E) \) consists of an \( n \)-cycle, \( G \) is of cyclic excess \(4\), and if \( c = 2 \) and \( G(E) \) consists of a disjoint union of two cycles, \( G \) is of bicyclic excess \(4\). These are the graphs we study in this section. Note that there are no graphs \( G \) with cyclic excess \(4\) if \( d \) is an odd number; in this case we showed that each cycle of \( G(E) \) is completely contained either in \( V_1 \) or \( V_2 \).

Let \( d \) be an even number and let \( L_n \) be an \( n \)-cycle formed by the vertices of \( G(E) \). If \( A' \) is the adjacency matrix of \( L_n \), its characteristic polynomial \( \chi(L_n, x) \) satisfies \( \chi(L_n, x) = (x - 2)(x + 2)(R_n(x))^3 \), where \( R_n \) is a monic polynomial of degree
\frac{n}{2} - 1. Consider the factorization \( x^n - 1 = \prod_{l \mid n} \Phi_l(x) \), where \( \Phi_l(x) \) denotes the \( l \)-th cyclotomic polynomial. In the following paragraph, we summarize the properties of cyclotomic polynomials as listed in [6].

The cyclotomic polynomial \( \Phi_l(x) \) has integral coefficients, it is irreducible over \( \mathbb{Q}[x] \), and it is self-reciprocal \( (x^\phi(l)) \Phi_l(1/x) = \Phi_l(x) \). From the irreducibility and the self-reciprocity of \( \Phi_l(x) \) follows that the degree of \( \Phi_l(x) \) is even for \( l \geq 2 \).

Thus we obtain the following factorization of \( R_n(x) : R_n(x) = \prod_{3 \leq l \mid n} f_i(x) \), where \( f_i \) is an integer polynomial of degree \( \frac{\phi(l)}{2} \) satisfying \( x^{\phi(l)/2} f_i(x + 1/x) = \Phi_l(x) \). Also, \( f_i \) is irreducible over \( \mathbb{Q}[x] \) and \( f_3(x) = x + 1 \), \( f_4(x) = x \), \( f_5(x) = x^2 + x - 1 \), \( f_6(x) = x - 1 \). Substituting \( y = -H_{d-1}(x) \) into \( \frac{\chi(L_n,y)}{(y-2)} \), we obtain a polynomial \( F(x) \) of degree \( (n-1)(d-1) \) which satisfies \( F(A)u = 0 \) for each eigenvector \( u \) of \( A \) orthogonal to the all \(-1\) vector. Setting \( F_{l,k,d-1}(x) = f_i(-H_{d-1}(x)) \) yields

\[
F(x) = (-H_{d-1}(x) + 2) \prod_{3 \leq l \mid n} (F_{l,k,d-1}(x))^2.
\]

**Lemma 3.1** Let \( g = 2d > 6 \) and \( l \geq 3 \) be a divisor of \( n \). If there is a \((k,g)\)-graph with cyclic excess \( 4 \) and order \( n \), then \( F_{l,k,d-1}(x) \) must be irreducible over \( \mathbb{Q}[x] \).

**Proof.** The degree of \( F_{l,k,d-1}(x) \) is equal to \((d-1) \cdot \frac{\phi(l)}{2} \). If \( F_{l,k,d-1}(x) \) is irreducible over \( \mathbb{Q}[x] \), then all its roots must be eigenvalues of \( A \). Employing Observation 3.1. from [6], we conclude that there are at most \( \phi(l) \) roots of \( F_{l,k,d-1}(x) \) that are eigenvalues of \( A \). Thus \((d-1) \cdot \frac{\phi(l)}{2} = \phi(l) \) i.e., \( d = 3 \). This contradicts the assumption that \( 2d > 6 \). q.e.d.

Note that \( \text{deg}(F_{l,k,d-1}(x)) = d - 1 \) if and only if \( \phi(l) = 2 \), i.e., if and only if \( l \in \{3,4,6\} \).

**Lemma 3.2** Let \( k \geq 6, g = 2d > 6 \), and let \( n \) be the order of a \((k,g)\)-graph with cyclic excess \( 4 \). Then

1) if \( n \equiv 0 \pmod{3} \), then \( H_{d-1}(x) - 1 \) must be reducible over \( \mathbb{Q}[x] \);

2) if \( n \equiv 0 \pmod{4} \), then \( H_{d-1}(x) \) must be reducible over \( \mathbb{Q}[x] \);

3) if \( n \equiv 0 \pmod{6} \), then \( H_{d-1}(x) + 1 \) must be reducible over \( \mathbb{Q}[x] \).

**Proof.** Follows directly from Lemma 3.1 with the additional assumption \( f_3(x) = x + 1 \), \( f_4(x) = x \) and \( f_6(x) = x - 1 \). q.e.d.

If \( n \equiv 0 \pmod{4} \), then using the formula for the order of \( G \), \( d - 1 \) must be odd. On the other hand, since \( H_1(x) = x, H_3(x) = x^3 - 2(k-1)x \) and \( H_{d-1}(x) = xH_{d-2}(x) - (k-1)H_{d-3}(x) \) we see that if \( d - 1 \) is an odd number then \( x \) divides \( H_{d-1}(x) \), which implies that \( H_{d-1}(x) \) is reducible. Therefore the second condition from Lemma 3.2 is satisfied.

The irreducibility of the polynomials \( H_{d-1}(x) - 1 \) over \( \mathbb{Q}[x] \) is examined in [5], where it is analytically proven that these polynomials are irreducible for \( d \in \{4,6,8\} \) and the paper contains a conjecture that \( d \geq 10 \), \( H_{d-1}(x) - 1 \) is irreducible. From the irreducibility of \( H_{d-1}(x) - 1 \) we obtain the main non-existence result of our paper.
Theorem 3.3 If \( k \) and \( g \) satisfy one of the following conditions, there exist no \((k, g)\)-graphs of cyclic excess 4:

1) \( k \equiv 1, 2 \pmod{3} \) and \( g = 8 \);
2) \( k \equiv 1 \pmod{3} \) and \( g = 12 \);
3) \( k \equiv 1 \pmod{3} \) and \( g = 16 \).

Proof. Because the order of the graphs is equal to \( 4 + 2 \left( 1 + (k - 1) + \ldots + (k - 1)^{(g - 2)/2} \right) \)
we conclude \( n \equiv 0 \pmod{3} \). Since the polynomial \( H_{d-1}(x) - 1 \) is known to be irreducible for \( d \in \{4, 6, 8\} \), we get contradiction to 1) from Lemma 3.2. q.e.d.

Remark: Since \( d \) is an even number, Theorem 2.7 asserts that \( d - 1 \) divides \( c - 1 \) and \( c_2 - 1 \). This claim is satisfied because \( c = c_2 = 1 \).

Next, let us consider graphs of bicyclic excess 4. In this case, we can assume an arbitrary (even or odd) \( d \), as this case does not depend of the parity of \( d \). So, let \( G(E) \) be a graph consisting of a disjoint union of two cycles \( C_1 \) and \( C_2 \). If \( d \) is an odd number, then the vertex sets of the cycles \( C_1 \) and \( C_2 \) correspond to the partite sets \( V_1 \) and \( V_2 \), respectively.

If \( n \equiv 0 \pmod{4} \), \( d \) is an even, each edge of \( C(E) \) has endpoints in \( V_1 \) and \( V_2 \), and therefore each of the cycles has even length, \( c_2 = 2 \). Furthermore, \( k - 1 \) must be odd. Unfortunately, this will not help us in excluding any family of pairs \((k, g)\) for which \( G \) does not exist. In fact, for an odd \( d - 1 \) and an odd \( k - 1 \) we cannot conclude irreducibility of \( H_{d-1}(x) + 2 \), thus, we cannot employ Lemma 2.6.

If \( n \equiv 2 \pmod{4} \) and \( d \) is odd, then the lengths of \( C_1 \) and \( C_2 \) are equal to \( \frac{n}{2} \) (clearly \( n = 2s + 1 \) is odd). Therefore \( c_2 = 0 \) and clearly \( d - 1 \) divides \( c - 2 \) and \( c_2 \).

The main result about the non-existence of graphs \( G \) with bicyclic excess 4 is given in the following theorem:

Theorem 3.4 If \( k \geq 7 \) is an odd and \( g = 2d \geq 8 \), where \( d \) is an even integer, then there exist no \((k, g)\)-graphs with bicyclic excess 4.

Proof. We have \( c = 2 \). Theorem 2.7 implies that \( d - 1 \) divides \( c - 1 \); a contradiction. q.e.d.

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