A CARTAN FORMULA FOR THE COHOMOLOGY OF POLYHEDRAL PRODUCTS AND ITS APPLICATION TO THE RING STRUCTURE

A. BAHRI, M. BENDERSKY, F. R. COHEN, AND S. GITLER

Abstract. We give a geometric method for determining the cohomology groups and the product structure of a polyhedral product $Z(K; (X, A))$, under suitable freeness conditions or with coefficients taken in a field $k$. This is done by considering first the special case $(X_i, A_i) = (B_i \vee C_i, B_i \vee E_i)$ for all $i$, where $E_i \hookrightarrow C_i$ is a null homotopic inclusion, and then deriving a decomposition for these polyhedral products which resembles a Cartan formula. The result is then generalized to arbitrary $Z(K; (X, A))$. This leads to a direct computation of the Hilbert-Poincaré series for $Z(K; (X, A))$. Other applications are included.

The product structure on $\tilde{H}^*(Z(K; (X, A)))$ is described in terms of the additive generators, labelled via the Cartan decomposition. This is done by combining with the Wedge Lemma, the description in [3] of the $*$-product structure of a polyhedral product which is induced from the stable splitting.

The description given suffices to enable explicit calculations.

Contents

1. Introduction 2
2. The polyhedral product of wedge decomposable pairs 4
3. A filtration 6
4. The proof of Theorem 2.2 7
5. Cohomological wedge decomposability and the general case 8
6. The canonical cofibration for wedge decomposable pairs 11
7. The proof of Theorem 5.4 for the boundary of a simplex 13
8. The proof of Theorem 5.4 for general $K$ 14
9. The Hilbert-Poincaré series for $Z(K; (X, A))$ 14
10. Applications 15
11. Product structure 16
11.1. Background 17
11.2. Partial diagonals, the wedge lemma and the product of links 18
11.3. Product structure for wedge decomposable pairs 21
11.4. The product structure for general CW pairs 25
11.5. An example 32
References 33

2010 Mathematics Subject Classification. Primary: 52B11, 55N10, 14M25, 55U10, 13F55, Secondary: 14F45, 55T10.

Key words and phrases. polyhedral product, cohomology, polyhedral smash product.
1. Introduction

Polyhedral products $Z(K; (X, A))$, [1], are defined for a simplicial complex $K$ on the vertex set $[m] = \{1, 2, \ldots, m\}$, and a family of pointed CW pairs

$$(X, A) = \{(X_i, A_i) : i = 1, 2, \ldots, m\}.$$  

They are natural subspaces of the Cartesian product $X_1 \times X_2 \times \cdots \times X_m$, in such a way that if $K = \Delta^{m-1}$, the $(m-1)$-simplex, then

$$Z(K; (X, A)) = X_1 \times X_2 \times \cdots \times X_m.$$  

More specifically, we consider $K$ to be a category where the objects are the simplices of $K$ and the morphisms $d_{\sigma, \tau}$ are the inclusions $\sigma \subset \tau$. A polyhedral product is given as the colimit of a diagram $D(X, A): K \to CW_*$, where at each $\sigma \in K$, we set

$$(1.1) \quad D(X, A)(\sigma) = \prod_{i=1}^{m} W_i,$$  

where $W_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \in [m] - \sigma. \end{cases}$

Here, the colimit is a union given by

$$Z(K; (X, A)) = \bigcup_{\sigma \in K} D(X, A)(\sigma),$$  

but the full colimit structure is used heavily in the development of the elementary theory. Notice that when $\sigma \subset \tau$ then $D(X, A)(\sigma) \subseteq D(X, A)(\tau)$. In the case that $K$ itself is a simplex,

$$Z(K; (X, A)) = \prod_{i=1}^{m} X_i.$$  

Polyhedral products were formulated first for the case $(X_i, A_i) = (D^2, S^1)$ by V. Buchstaber and T. Panov in [7]; they called their spaces moment-angle complexes.

In a way entirely similar to that above, a related space $\hat{Z}(K; (X, A))$, called the polyhedral smash product, is defined by replacing the Cartesian product everywhere above by the smash product. That is,

$$\hat{D}(X, A)(\sigma) = \bigwedge_{i=1}^{m} W_i \quad \text{and} \quad \hat{Z}(K; (X, A)) = \bigcup_{\sigma \in K} \hat{D}(X, A)(\sigma)$$  

with

$$\hat{Z}(K; (X, A)) \subseteq \bigwedge_{i=1}^{m} X_i.$$  

The polyhedral smash product is related to the polyhedral product by the stable decomposition discussed in [1] and [2]. We denote by $(X, A)_J$ the restricted family of CW-pairs $\{(X_j, A_j)\}_{j \in J}$, and by $K_J$, the full subcomplex on $J \subset [m]$.

This work was supported in part by grant 426160 from Simons Foundation. The authors are grateful to the Fields Institute for a conducive environment during the Thematic Program: Toric Topology and Polyhedral Products. The first author acknowledges Rider University for a Spring 2020 research leave.
Theorem 1.1. [2, Theorem 2.10] Let $K$ be an abstract simplicial complex on vertices $[m]$. Given a family $\{((X_j, A_j))\}_{j=1}^m$ of pointed pairs of CW-complexes, there is a natural pointed homotopy equivalence

$$H : \Sigma(Z(K; (X, A))) \longrightarrow \Sigma(\bigvee_{J \subseteq [m]} \tilde{Z}(K_J; (X, A)_J))$$

In many of the most important cases, the spaces $\tilde{Z}(K_J; (X, A)_J)$ can be identified explicitly, [2]. Moreover, it is shown by the authors in [3] that for based CW pairs $(X, A)$, the product structure on the cohomology of the polyhedral product has a canonical formulation in terms of partial diagonal maps on these spaces, (reviewed in Section 11.1).

Aside from the various unstable and stable splitting theorems, [1, 12, 11, 13, 14], there is an extensive history of computations of the cohomology groups and rings of various families of polyhedral products, [5, Sections 5, 8 and 11], see also [15, 10, 6, 18, 19, 4, 8, 9].

Some very early calculations of the cohomology of certain moment-angle complexes, (the case $(X_i, A_i) = (D^2, S^1)$ for all $i = 1, 2, \ldots, m$), appeared in the work of Santiago López de Medrano [15], though at that time the spaces he studied were not recognized to have the structure of a moment-angle complex. The cohomology algebras of all moment-angle complexes was computed first by M. Franz [10] and by I. Baskakov, V. Buchstaber and T. Panov in [6].

The cohomology of the polyhedral product $Z(K; (X, A))$, for $(X, A)$, satisfying certain freeness conditions, (coefficients in a field $k$ for example), was computed using a spectral sequence by the authors in [4]. A computation using different methods by Q. Zheng can be found in [18, 19].

The description herein of the product structure of $\tilde{H}^*(\tilde{Z}(K; (X, A)))$ is the most explicit of which we know.

As announced in [5, Section 12], one goal of the current paper is to show that for certain pairs $(U, V)$, called wedge decomposable, the algebraic decomposition given by the spectral sequence calculation [4, Theorem 5.4] is a consequence of an underlying geometric splitting. Moreover, the results of this observation extend to general based CW-pairs of finite type.

This paper is partly a revised version of the authors’ unpublished preprint from 2014, which in turn originated from an earlier preprint from 2010. In addition, the results have been extended now to describe the product structure of the cohomology.

We begin in Section 2 by defining wedge decomposable pairs $(U, V)$ and deriving for them an explicit decomposition of the polyhedral product into a wedge of much simpler spaces, (Theorem 2.2 and Corollary 2.5). In particular, this allows us to identify explicit additive generators for $H^*(Z(K; (U, V)))$. The proof in Section 3 is an induction based on a filtration of the polyhedral product which is introduced in Section 3.

A notion of cohomological wedge decomposability is introduced in Section 5 and is used in Sections 6 to 8 to extend the results described above for $H^*(Z(K; (U, V)))$, to the cohomology of arbitrary polyhedral products $Z(K; (X, A))$. Given a collection $(X, A)$, we associate to it wedge decomposable pairs $(U, V)$ such that there is an isomorphism of
groups

\[ \theta_{(U,V)}: \tilde{H}^*(\hat{Z}(K;(U,V))) \longrightarrow \tilde{H}^*(\hat{Z}(K;(X,A))) \]

Effectively, this result (Theorem 5.4) gives an explicit method for labelling additive generators of the group \( H^*(Z(K;(X,A)); k) \), in terms of an appropriate choice of generators for \( H^*(X) \) and \( H^*(A) \), and the link structure of the simplicial complex \( K \).

Applications of the additive results comprise Sections 9 and 10.

A discussion of the ring structure of \( H^*(Z(K;(X,A))) \) occupies Section 11. The main result is Theorem 11.13 which, for two classes \( \theta_{(U,V)}(u) \) and \( \theta_{(U,V)}(v) \) in \( \tilde{H}^*(\hat{Z}(K;(X,A))) \), allows for the explicit determination of the product

\[ \theta_{(U,V)}(u) \cdot \theta_{(U,V)}(v) \in \tilde{H}^*(\hat{Z}(K;(X,A))). \]

(Of course, in general, this will not be the class \( \theta_{(U,V)}(u \cdot v) \).)

The discussion of ring structure begins in subsection 11.1 with a review from [3] of the relationship between the stable splitting of the polyhedral product and the cohomology cup product. The Cartan decomposition of Theorem 2.2 enables access to the wedge lemma which decomposes the polyhedral smash product for a wedge decomposable pair, into spaces which are indexed by links. We observe in subsections 11.2 and 11.3 the way in which this leads to a concise description of cohomology product in the case of wedge decomposable pairs, Theorem 11.10 and Corollary 11.11.

Included also in subsection 11.2, is a new method for computing the \( * \)-product of the links which index summands in the wedge lemma decomposition of the polyhedral smash product. This uses the polyhedral product \( Z(K;(S^0,S^0)) \) where \( S^0 \simeq S^0 \) is defined by (11.12).

Finally, in subsection 11.4 we describe the way in which the results for wedge decomposable pairs can be extended to arbitrary pairs \( (X,A) \). The fact that all pairs are cohomologically wedge decomposable, suffices to do product calculations, the main preoccupation being the tracking of changes to the indexing links generated by a mixing of terms arising from cup products in \( \tilde{H}^*(X_i) \) and \( \tilde{H}^*(A_i) \).

2. The Polyhedral Product of Wedge Decomposable Pairs

We begin with a definition.

**Definition 2.1.** The special family of CW pairs \( (U,V) = (B \vee C, B \vee E) \) satisfying \((U_i,V_i) = (B_i \vee C_i, B_i \vee E_i)\) for all \( i \), where \( E_i \hookrightarrow C_i \) is a null homotopic inclusion, is called **wedge decomposable**.

The fact that the smash product distributes over wedges of spaces, leads to the characterization of the smash polyhedral product in a way which resembles a **Cartan formula**.

**Theorem 2.2.** *(Cartan Formula)* Let \( (U,V) = (B \vee C, B \vee E) \) be a wedge decomposable pair, then there is a homotopy equivalence

\[ \hat{Z}(K;(U,V)) \longrightarrow \bigvee_{I \leq [m]} \left( \hat{Z}(K_I;(C,E)) \wedge \hat{Z}(K_{[m]-I};(B,B)_{[m]-I}) \right) \]
which is natural with respect to maps of decomposable pairs. Of course,
\[ \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}) = \bigwedge_{j \in [m]-I} B_j \]
with the convention that
\[ \hat{Z}(K_{\emptyset}; (B, B)_{\emptyset}), \hat{Z}(K_{\emptyset}; (C, E)_{\emptyset}) \text{ and } \hat{Z}(K_I; (\emptyset, \emptyset)_I) = S^0. \]

We can decompose \( \hat{Z}(K; (U, V)) \) further by applying (a generalization of) the Wedge Lemma. We recall first the definition of a link.

**Definition 2.3.** For \( \sigma \) a simplex in a simplicial complex \( K \), \( \text{lk}_\sigma(K) \) the link of \( \sigma \) in \( K \), is defined to be the simplicial complex for which \( \tau \in \text{lk}_\sigma(K) \) if and only if \( \tau \cup \sigma \in K \).

**Theorem 2.4.** [1, Theorem 2.12], [20, Lemma 1.8] Let \( K \) be a simplicial complex on \([m]\) and \((C, E)\) a family of CW pairs satisfying \( E_i \hookrightarrow C_i \) is null homotopic for all \( i \) then
\[ \hat{Z}(K; (C, E)) \simeq \bigvee_{\sigma \in K} |\Delta(K)_{<\sigma}| \ast \hat{D}^{[m]}_{C, E}(\sigma) \]
where \( |\Delta(K)_{<\sigma}| \cong |\text{lk}_\sigma(K)| \), the realization of the link of \( \sigma \) in the simplicial complex \( K \) and
\[(2.1) \quad \hat{D}^{[m]}_{C, E}(\sigma) = \bigwedge_{j=1}^{m} W_{ij}, \quad \text{with} \quad W_{ij} = \begin{cases} C_{ij} & \text{if } i_j \in \sigma \\ E_{ij} & \text{if } i_j \in [m]-\sigma. \end{cases} \]

Applying this to the decomposition of Theorem 2.2, we get

**Corollary 2.5.** There is a homotopy equivalence
\[ \hat{Z}(K; (U, V)) \rightarrow \bigvee_{I \leq [m]} \left( \left( \bigvee_{\sigma \in K_I} |\text{lk}_\sigma(K_I)| \ast \hat{D}^{[m]}_{C, E}(\sigma) \right) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}) \right). \]

where \( \hat{D}^{[m]}_{C, E}(\sigma) \) is as in (2.1) with \( I \) replacing \([m]\).

Combined with Theorem 1.1, this gives a complete description of the topological spaces \( Z(K; (U, V)) \) for wedge decomposable pairs \((U, V)\).

The case \( E_i \simeq * \) simplifies further by [2, Theorem 2.15] to give the next corollary.

**Corollary 2.6.** For wedge decomposable pairs of the form \((B \lor C, B)\), corresponding to \( E_i \simeq * \) for all \( i = 1, 2, \ldots, m \), there are homotopy equivalences
\[ \hat{Z}(K_I; (C, E)_I) \simeq \hat{Z}(K_I; (C, *)_I) \simeq \hat{C}^I, \]
and so Theorem 2.2 gives \( \hat{Z}(K; (B \lor C, B)) \simeq \bigvee_{I \leq [m]} (\hat{C}^I \wedge \hat{B}^{([m]-I)}) \).

Notice here that the Poincaré series for the space \( \hat{Z}(K; (B \lor C, B)) \) follows easily from Corollary 2.6.
Remark. In comparing these observations with [4, Theorem 5.4], notice that the links appear in the terms $\hat{Z}(K_I; (C; E)_I)$. Also, while Theorem 2.2 and Corollary 2.5 give a geometric underpinning for the cohomology calculation in [4, Theorem 5.4] for wedge decomposable pairs, the geometric splitting does not require that $E$, $B$ or $C$ have torsion-free cohomology.

3. A filtration

We begin by reviewing the filtration on polyhedral products used for the spectral sequence calculation in [4]. Following [4, Section 2], where more details can be found. The length-lexicographical ordering, $(\text{shortlex})$, on the faces of the $(m-1)$-simplex $\Delta[m-1]$ is induced by an ordering on the vertices. This is the left lexicographical ordering on strings of varying lengths with shorter strings taking precedence. The ordering gives a filtration on $\Delta[m-1]$ by

$$F_t(\Delta[m-1]) = \bigcup_{s \leq t} \sigma_s.$$  

In turn, this gives a total ordering on the simplices of a simplicial $K$ on $m$ vertices

$$(3.1) \quad \sigma_0 = \emptyset < \sigma_1 < \sigma_2 < \ldots < \sigma_t < \ldots < \sigma_s$$

via the natural inclusion

$$K \subset \Delta[m-1].$$

This is filtration preserving in the sense that $F_tK = K \cap F_t\Delta[m-1]$.

Example 3.1. Consider $[m] = [3]$ and

$$K = \{ \phi, \{v_1\}, \{v_2\}, \{v_3\}, \{\{v_1\}, \{v_3\}\}, \{\{v_2\}, \{v_3\}\} \}$$

with the realization consisting of two edges with a common vertex. Here the length-lexicographical ordering on the two-simplex $\Delta[2]$ is

$$\phi < v_1 < v_2 < v_3 < v_1v_2 < v_1v_3 < v_2v_3$$

and so the induced ordering on $K$ is

$$\phi < v_1 < v_2 < v_3 < v_1v_3 < v_2v_3.$$  

Remark. Notice that if $t < m$, then $F_tK$ will contain ghost vertices, that is, vertices which are in $[m]$ but are not considered simplices, They do however label Cartesian product factors in the polyhedral product.

As described in [4, Section 2], this induces a natural filtration on the polyhedral product $Z(K; (X, A))$ and the smash polyhedral product $\hat{Z}(K; (X, A))$ as follows:

$$F_tZ(K; (X, A)) = \bigcup_{k \leq t} D(X, A)(\sigma_k) \quad \text{and} \quad F_t\hat{Z}(K; (X, A)) = \bigcup_{k \leq t} \hat{D}_X(\sigma_k).$$

Notice also that the filtration satisfies

$$(3.2) \quad F_t\hat{Z}(K; (X, A)) = \hat{Z}(F_tK; (X, A)).$$
4. The proof of Theorem 2.2

Let the family of CW pairs \((U, V)\) be wedge decomposable as in Definition 2.1. We begin by checking that Theorem 2.2 holds for \(F_0 \hat{Z}(K; (U, V))\). In this case \(F_0 K\) consists of the empty simplex, (the boundary of a point), and \(m - 1\) ghost vertices. So,

\[
\hat{Z}(F_0 K; (U, V)) = V_1 \wedge V_2 \cdots \wedge V_m = (B_1 \lor E_1) \wedge B_2 \lor E_2) \wedge \cdots \wedge (B_m \lor E_m).
\]

Next, fix \(I = (i_1, i_2, \ldots, i_k) \subseteq [m]\) and set \([m] - I = (j_1, j_2, \ldots, j_{m-k})\). Then

\[
\hat{Z}(F_0 K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}) = (E_{i_1} \wedge E_{i_2} \wedge \cdots E_{i_k}) \wedge (B_{j_1} \wedge B_{j_2} \wedge \cdots B_{j_{m-k}}),
\]

is the \(I\)-th wedge term in the expansion of the right hand side of (4.1). This confirms Theorem 2.2 for \(t = 1\).

We suppose next the induction hypothesis that

\[
F_{t-1} \hat{Z}(K; (U, V)) \simeq \bigvee_{I \subseteq [m]} \hat{Z}(F_{t-1} K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}),
\]

with a view to verifying it for \(F_t\). The definition of the filtration gives

\[
F_t \hat{Z}(K; (U, V)) = \tilde{D}_{U, V}(\sigma_t) \cup F_{t-1} \hat{Z}(K; (U, V))
\]

\[
\simeq \tilde{D}_{U, V}(\sigma_t) \cup \bigvee_{I \subseteq [m]} \hat{Z}(F_{t-1} K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}).
\]

The space \(\tilde{D}_{U, V}(\sigma_t)\) is the smash product

\[
\bigwedge_{j=1}^{m} B_j \lor Y_j, \quad \text{with} \quad Y_j = \begin{cases} C_j & \text{if } j \in \sigma_t \\ E_j & \text{if } j \notin \sigma_t. \end{cases}
\]

After a shuffle of wedge factors, the space \(\tilde{D}_{U, V}(\sigma_t)\) becomes

\[
\bigvee_{I \subseteq [m], \sigma_t \in I} \tilde{D}_{U, V}^I(\sigma_t) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}) \wedge \bigvee_{I \subseteq [m], \sigma_t \notin I} \hat{Z}(K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I})
\]

where the space \(\tilde{D}_{U, V}^I(\sigma_t)\) is defined by (2.1).

Remark. Notice here the relevant fact that the number of subsets \(I \subseteq [m]\) is the same as the number of wedge summands in the expansion of (4.3), namely \(2^m\).

The right-hand wedge summand in (4.4) is a subset of

\[
\bigvee_{I \subseteq [m]} \hat{Z}(F_{t-1} K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I})
\]

and so,

\[
\bigvee_{I \subseteq [m], \sigma_t \notin I} \hat{Z}(K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}) \cup \bigvee_{I \subseteq [m]} \hat{Z}(F_{t-1} K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I})
\]

\[
= \bigvee_{I \subseteq [m]} \hat{Z}(F_{t-1} K_I; (C, E)_I) \wedge \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}).
\]

Finally, for each \(I \subseteq [m]\) with \(\sigma_t \in I\), we have
This concludes the inductive step to give
\begin{equation}
F_t \hat{Z}(K; (U, V)) \simeq \bigvee_{I \leq [m]} \hat{Z}(F_tK_I; (C, E)_I) \land \hat{Z}(K_{[m]-I}; (B, B)_{[m]-I}).
\end{equation}
It is straightforward to explicitly check the steps above in the case of $F_0$ and $F_1$. This completes the proof. \qed

5. Cohomological wedge decomposability and the general case

The result of the previous section can be exploited to give information about the groups $\tilde{H}^*(\hat{Z}(K; (X, A)))$ over a field $k$ for pointed, finite, path connected pairs of CW-complexes $(X, A)$ of finite type, which are not wedge decomposable.

**Definition 5.1.** A strongly free decomposition of the homology of $(X, A)$ with coefficients in a ring $k$, is a quadruple of $k$-modules $(E', B', C', W)$ such that the long exact sequence
\begin{equation}
\delta \to \tilde{H}^*(X/A) \xrightarrow{f} \tilde{H}^*(X) \xrightarrow{c} \tilde{H}^*(A) \xrightarrow{\delta} \tilde{H}^{*+1}(X/A) \to \cdots
\end{equation}
satisfies the condition that there exist isomorphisms
\begin{enumerate}
\item $\tilde{H}^*(A) \cong B' \oplus E'$
\item $\tilde{H}^*(X) \cong B' \oplus C'$, where $B' \xrightarrow{c} B'$, $c|_{C'} = 0$
\item $\tilde{H}^*(X/A) \cong C' \oplus W$, where $C' \xrightarrow{\ell} C'$, $\ell|_{B'} = 0$, $E' \xrightarrow{\delta} W$
\end{enumerate}
for free graded $k$-modules $E', B', C'$ and $W$ of finite type.

A pair $(X, A)$ with a strongly free decomposition is said to be strongly free. A morphism of strongly free pairs
\[(f_\ell, f_\iota, f_\delta): (X, A) \to (U, V)\]
is a morphism of long exact sequences
\begin{equation}
\cdots \xrightarrow{\delta(X,A)} \tilde{H}^*(X/A), \xrightarrow{\ell(X,A)} \tilde{H}^*(X), \xrightarrow{\ell(X,A)} \tilde{H}^*(X), \xrightarrow{\delta(X,A)} \tilde{H}^{*+1}(X/A), \xrightarrow{\ell(X,A)} \cdots
\end{equation}
\begin{equation}
\cdots \xrightarrow{\delta(U,V)} \tilde{H}^*(U/V), \xrightarrow{\ell(U,V)} \tilde{H}^*(U), \xrightarrow{\ell(U,V)} \tilde{H}^*(U), \xrightarrow{\delta(U,V)} \tilde{H}^{*+1}(U/V), \xrightarrow{\ell(U,V)} \cdots
\end{equation}
with all diagrams commuting, which restricts to maps of the submodules $E', B', C'$, $W$ corresponding to each pair.

**Remark.** When $k$ is a field, the homology of $(X, A)$ is always strongly free. When $k = \mathbb{Z}$, strong freeness holds if all the spaces are torsion free and of finite type.

Two lemmas about null homotopic inclusions are needed next.

**Lemma 5.2.** Let $\iota: E \hookrightarrow C$ be an inclusion of based CW complexes. then
There is a map $g: cE \to C$ from the reduced cone, so that the diagram below commutes

$$
\begin{array}{c}
C \vee cE \\
\uparrow h \\
E \\
\end{array}
\xrightarrow{1_C \vee g} 
\begin{array}{c}
C \\
\uparrow \\
E \\
\end{array}
$$

where $h(e) = [(e, 0)]$, that is, $h$ includes $E$ into the base of the cone.

There is a homotopy equivalence of colimits

$$
\tilde{Z}(\partial \sigma_i; (C \vee cE, E)) \to \tilde{Z}(\partial \sigma_i; (C, E)).
$$

The vertical maps in (5.3) have homotopy equivalent cofibers.

Proof. Since $\iota: E \hookrightarrow C$ is null homotopic, the exists a homotopy $G: E \times I \to C$ satisfying, for every $e \in E$, $G((e, 0)) = \iota(e)$ and $G((e, 1)) = e_0$, the base point of $E$, which is also the basepoint of $C$. This implies the existence of an extension

$$
\begin{array}{c}
E \times I \\
\downarrow \pi \\
cE \\
\end{array}
\xrightarrow{g} 
\begin{array}{c}
C \\
\downarrow g \\
\end{array}
$$

where $\pi$ projects $E \times \{1\}$ to the cone point. The commutativity of (5.3) follows.

Item (2) follows from the Homotopy Lemma, (20, Lemma 1.7], because the horizontal maps in (5.3) are homotopy equivalences and the commutativity implies that they give a map of diagrams defining the colimits $\tilde{Z}(\partial \sigma_i; (C \vee cE, E))$ and $\tilde{Z}(\partial \sigma_i; (C, E))$.

Finally, the third item is the standard homotopy invariance of cofibers, see for example [17, Theorem 2.3.7].

Given a pair $(X, A)$ which is strongly free, let $B', C'$ and $E'$ be the $k$–modules specified in items (1) and (2) of Definition 5.1 for each pair $(X, A)$. Now, wedges of spheres $B$, $C$ and $E$ exist with cohomology equal to the modules $B', C'$ and $E'$ so that

$$
(U, V) = (B \vee C, B \vee E)
$$

satisfies the criterion for a wedge decomposable pair as in Definition 2.1. In particular, the inclusion $\iota: E \to C$ is a null homotopic inclusion of wedges of spheres. Consider next diagram (5.2) for the pairs $(X, A)$ and $(U, V)$, the latter defined as in (5.6).

Lemma 5.3. For a pair $(U, V)$ derived from $(X, A)$ in the manner above, there is an isomorphism $(f_\ell, f_\iota, f_\delta): (X, A) \to (U, V)$ of strongly free pairs as in (5.1) and (5.2).

Proof. The map $f_\iota$ is the isomorphism $\tilde{H}^*(X) \xrightarrow{\cong} B' \oplus C'' = \tilde{H}^*(U)$ and the map $f_\iota$ is the compatible isomorphism $\tilde{H}^*(A) \xrightarrow{\cong} B' \oplus E' = \tilde{H}^*(V)$, both from Definition 5.1. Since the inclusion $\iota: E \to C$ is null homotopic, we apply Lemma 5.2 to write it as $E \hookrightarrow cE \vee C$ where $cE$ is the unreduced cone and the inclusion is onto the base of the cone. Again
from Lemma 5.2 item (3), it follows that \( C/E \cong \Sigma E \lor C \) and hence \( U/V \cong \Sigma E \lor C \). This gives the isomorphism
\[
f_\delta : \tilde{H}^*(X/A) \xrightarrow{\cong} W \oplus C' \xrightarrow{\cong} \tilde{H}^*(\Sigma E \lor C) \xrightarrow{\cong} \tilde{H}^*(U/V).
\]
Finally, it follows from the definitions that the diagrams 5.2 all commute. \( \square \)

Our goal is to show that under the strong freeness condition of Definition 5.1, an analogue of Theorem 2.2 holds for pairs \((X, A)\), using pairs \((U, V)\) where each \((U_i, V_i)\) has been constructed from \((X_i, A_i)\) via 5.6 above.

**Theorem 5.4.** Under the conditions stated above, there is an isomorphism of cohomology groups with coefficients in a field \( k \)
\[
\theta_{(U,V)} : \tilde{H}^*(\tilde{Z}(K;(U,V))) \longrightarrow \tilde{H}^*(\tilde{Z}(K;(X,A)))
\]
where the right hand side is determined by Corollary 2.3. (This is not necessarily an isomorphism of modules over the Steenrod algebra and does not preserve products in general.)

**Corollary 5.5.** Let \((X, A)\) and \((Y, B)\) be two families of strongly free pairs so that there is an isomorphism \((f_\delta, f_i, f_\delta)\) of strongly free pairs \((X_i, A_i) \rightarrow (Y_i, B_i)\) for each \( i \in [m] \) as in 5.1 and 5.2, then there is an isomorphism of groups for cohomology with coefficients in a field \( k \)
\[
\tilde{H}^*(\tilde{Z}(K;(X,A))) \longrightarrow \tilde{H}^*(\tilde{Z}(K;(Y,B))).
\]

**Proof.** For each \( i \in [m] \), the associated wedge decomposable pairs \((U_i, V_i)\) given by \(5.6\) for both \((X, A)\) and \((Y, B)\) are the same and so the result follows from Theorem 5.4. \( \square \)

The proof of Theorem 5.4 is centered around a result of J. Grbic and S. Theriault [12], as described in [11 Section 3]. In order to keep track of ghost vertices, we introduce further notation. For \( \sigma = \{i_1, i_2, \ldots, i_{n+1}\} \), with complementary vertices \(\{j_1, j_2, \ldots, j_{m-n-1}\}\), set
\[
\tilde{A}^{[m]-\sigma} = A_{j_1} \land A_{j_2} \land \cdots \land A_{j_{m-n-1}}
\]

**Theorem 5.6.** [12] [11] For general pairs \((X, A)\), the diagram below is commutative diagram of cofibrations for each \( t, 0 \leq t \leq s \).
\[
\cdots \xrightarrow{\delta^{(X,A)}_t} F_{t-1}\tilde{Z}(K;(X,A)) \xrightarrow{\iota} F_t\tilde{Z}(K;(X,A)) \xrightarrow{\gamma^{(X,A)}_t} \tilde{C}(X,A) \cdots
\]
\[
\cdots \xrightarrow{\delta^{(U,V)}_t} \tilde{Z}(\partial\tilde{\sigma}_t;(X,A)) \land \tilde{A}^{[m]-\sigma} \xrightarrow{\iota} \tilde{Z}\tilde{\sigma}_t;(X,A)) \land \tilde{A}^{[m]-\sigma} \gamma^{(U,V)}_t \tilde{C}(X,A) \cdots
\]

where here, \(\tilde{\sigma}_t\) and \(\partial\tilde{\sigma}_t\) represents the simplex \(\sigma_t\) and the boundary of \(\tilde{\sigma}_t\) respectively, considered without ghost vertices, that is, as simplicial complexes on \(\{i_1, i_2, \ldots, i_{n+1}\}\), the vertices of the simplex only.

An outline of the proof of Theorem 5.4 is as follows.

1. The lower cofibration in (5.8) is analyzed geometrically in the case that \((X, A)\) is a wedge decomposable pair \((U, V)\). This is done in Section 6.
(2) In Section 7 these results are used then to prove Theorem \[ K = \partial \sigma_t \], the boundary of a simplex.

(3) An inductive argument in Section 8 uses Diagram \((5.8)\) to complete the proof for general \(K\).

6. The canonical cofibration for wedge decomposable pairs

The three spaces in the lower cofibration of \((5.8)\) are now analyzed in the case of a wedge-decomposable pair \((U, V) = (B \lor C, B \lor E)\). For \(\sigma_t\) an \(n\)-simplex on vertices \(\{i_1, i_2, \ldots, i_{n+1}\}\), with no ghost vertices, the identification of the space \(\hat{Z}(\sigma_t; (U, V))\) is straightforward.

\[
\hat{Z}(\sigma_t; (U, V)) = (B_{i_1} \lor C_{i_1}) \land (B_{i_2} \lor C_{i_2}) \land \cdots \land (B_{i_{n+1}} \lor C_{i_{n+1}})
\]

\[
\simeq C_{i_1} \land C_{i_2} \land \cdots \land C_{i_{n+1}} \lor \overline{D}_{\sigma_t}
\]

where \(\overline{D}_{\sigma_t}\) is a wedge of smash products of spaces \(B_{i_j}\) and \(C_{i_k}\).

For general \((X, A)\), the short argument in \([4, \text{Lemma 3.6}]\) shows that the cofiber in Theorem \(5.6\) is given by

\[
\mathcal{C}(X, A) \simeq X_{i_1}/A_{i_1} \land X_{i_2}/A_{i_2} \land \cdots \land X_{i_{n+1}}/A_{i_{n+1}} \land \hat{A}^{[m] - \sigma}
\]

which we write as

\[
\mathcal{C}(X, A) \simeq \overline{\mathcal{C}}(X, A) \land \hat{A}^{[m] - \sigma}
\]

Once again, for \((X, A) = (U, V) = (B \lor C, B \lor E)\), we use Lemma \(5.2\) to replace the inclusion \(E_{i_j} \hookrightarrow C_{i_j}\) with \(E_{i_j} \hookrightarrow cE_{i_j} \lor C_{i_j}\) where \(cE_{i_j}\) is the cone. Again, we have \(C_{i_j}/E_{i_j} \simeq \Sigma E_{i_j} \lor C_{i_j}\) and we get the next lemma.

Lemma 6.1. The cofiber \(\overline{\mathcal{C}}(U, V)\), \((6.3)\), for a wedge decomposable family of pairs, decomposes homotopically into a wedge of spaces as follows.

\[
\overline{\mathcal{C}}(U, V) \simeq (\Sigma E_{i_1} \lor C_{i_1}) \land (\Sigma E_{i_2} \lor C_{i_2}) \land \cdots \land (\Sigma E_{i_{n+1}} \lor C_{i_{n+1}})
\]

\[
\simeq \Sigma (E_{i_1} \land E_{i_2} \land \cdots \land E_{i_{n+1}}) \lor \bigvee_{j=1}^{n+1} C_{i_j} \lor \Sigma (E_{i_1} \land E_{i_2} \land \cdots \land \hat{E}_{i_j} \land \cdots \land E_{i_{n+1}})
\]

\[
\lor \bigvee_{k_1 < k_2}^{n+1} C_{i_{k_1}} \land C_{i_{k_2}} \lor \Sigma (E_{i_1} \land E_{i_2} \land \cdots \land \hat{E}_{i_{k_1}} \land \cdots \land E_{i_{k_2}} \land \cdots \land E_{i_{n+1}})
\]

\[
\lor \cdots \lor \bigvee_{j=1}^{n+1} \Sigma E_{i_j} \land C_{i_1} \land C_{i_2} \land \cdots \land \hat{C}_{i_j} \land \cdots \land C_{i_{n+1}}
\]

\[
\lor C_{i_1} \land C_{i_2} \land \cdots \land C_{i_{n+1}}
\]

where here, the equivalence \(\Sigma(S \lor T) \simeq S \lor T\) has been used iteratively. On the other hand, for the simplicial complex \(\partial \sigma_t\), we have the following lemma.
Lemma 6.2. The polyhedral smash product \( \hat{Z}(\partial \bar{\sigma}_i; (C \lor cE, E)) \) decomposes homotopically into a wedge of spaces as follows.

\[
\hat{Z}(\partial \bar{\sigma}_i; (C, E)) \simeq \hat{Z}(\partial \bar{\sigma}_i; (C \lor cE, E))
\]

\[
\simeq \bigcup_{k=1}^{n+1} (C_{i_1} \lor cE_{i_1}) \land \ldots \land (C_{i_{k-1}} \lor cE_{i_{k-1}}) \land E_{i_k} \land (C_{i_{k+1}} \lor cE_{i_{k+1}}) \land \ldots \land (C_{i_{n+1}} \lor cE_{i_{n+1}})
\]

\[
\simeq E_{i_1} \ast E_{i_2} \ast \cdots \ast E_{i_{n+1}}
\]

\[
\lor \bigvee_{j=1}^{n+1} C_{i_j} \land E_{i_1} \ast E_{i_2} \ast \cdots \ast \hat{E}_{i_j} \ast \cdots \ast E_{i_{n+1}}
\]

\[
\lor \bigvee_{k_1 < k_2} C_{i_{k_1}} \land C_{i_{k_2}} \land (E_{i_1} \ast E_{i_2} \ast \cdots \ast \hat{E}_{i_{k_1}} \ast \cdots \ast \hat{E}_{i_{k_2}} \ast \cdots \ast E_{i_{n+1}})
\]

\[
\lor \cdots
\]

\[
\lor \bigvee_{j=1}^{n+1} E_{i_j} \land C_{i_1} \land C_{i_2} \land \ldots \land \hat{C}_{i_j} \land \ldots \land C_{i_{n+1}}
\]

where this time, the equivalence \( (Y \land cY) \lor Y \land (cY \land Y) \simeq Y \ast Y \) as been used iteratively.

Remark. This decomposition is the same as that given by the Wedge Lemma, [11 Theorem 2.12], in the case \( K = \partial \sigma \), the boundary of a simplex.

Finally, comparing the two decompositions above, we arrive at the identity

\[
(6.4) \quad \bar{c}_{(U,V)} \simeq \Sigma \hat{Z}(\partial \bar{\sigma}_i; (C, E)) \lor (C_{i_1} \land C_{i_2} \land \ldots \land C_{i_{n+1}}).
\]

Definition 6.3. We introduce now notational abbreviations which account for ghost vertices. Here, \((U,V) = (B \lor C, B \lor E)\) and \(\bar{\sigma}_t, \partial \bar{\sigma}_t\) are as in Theorem 5.6

1. \( \hat{Z}(\sigma_t; (Y, Q)) = \hat{Z}(\bar{\sigma}_t; (Y, Q)) \land \hat{Q}[m] - \sigma_t \) for any family of pairs \((Y, Q)\).
2. \( \hat{Z}(\partial \sigma_t; (Y, Q)) = \hat{Z}(\partial \bar{\sigma}_t; (Y, Q)) \land \hat{Q}[m] - \sigma_t \) for any family of pairs \((Y, Q)\).
3. \( E_{\sigma_t} = \hat{Z}(\partial \bar{\sigma}_t; (C, E)) \land \hat{V}[m] - \sigma_t \)
4. \( C_{\sigma_t} = C_{i_1} \land C_{i_2} \land \ldots \land C_{i_{n+1}} \land \hat{V}[m] - \sigma_t \)
5. \( c_{(Y,Q)} = \bar{c}_{(Y,Q)} \land \hat{Q}[m] - \sigma_t \) for any family of pairs \((Y, Q)\).
6. \( D_{\sigma_t} = \bar{D}_{\sigma_t} \land \hat{V}[m] - \sigma_t \) (see (6.1)).

Theorem 6.4. For the pair \((U,V) = (B \lor C, B \lor E)\), the lower sequence corresponding to (5.8)

\[
\rightarrow \hat{Z}(\partial \sigma_t; (U,V)) \xrightarrow{i} \hat{Z}(\sigma_t; (U,V)) \xrightarrow{\gamma^{(U,V)}_{\sigma_t}} \bar{c}_{(U,V)} \xrightarrow{\delta^{(U,V)}_{\sigma_t}} \Sigma \hat{Z}(\partial \sigma_t; (U,V)) \xrightarrow{i}
\]

splits geometrically as:

\[
(6.5) \quad \rightarrow E_{\sigma_t} \lor D_{\sigma_t} \xrightarrow{i} C_{\sigma_t} \lor D_{\sigma_t} \xrightarrow{\gamma^{(U,V)}_{\sigma_t}} \Sigma E_{\sigma_t} \lor C_{\sigma_t} \xrightarrow{\delta^{(U,V)}_{\sigma_t}} \Sigma E_{\sigma_t} \lor \Sigma D_{\sigma_t} \xrightarrow{i}
\]

where the function \( i \) maps \( D_{\sigma_t} \) by the identity and the function \( \gamma^{(U,V)}_{\sigma_t} \) maps \( C_{\sigma_t} \) by the identity.
7. The proof of Theorem 5.4 for the boundary of a simplex

Returning now to (5.8), we begin to examine the consequences of Lemma 5.3

**Lemma 7.1.** There is an isomorphism of groups

\[(7.1) \quad \tilde{H}^*(\tilde{Z}(\sigma_t; (X, A))) \cong \tilde{H}^*(\tilde{Z}(\partial \sigma_t; (U, V))).\]

**Proof.** Applying Lemma 5.3 to Definition 6.3 part (3), and using the Künneth Theorem, we have

\[
\tilde{H}^*(\tilde{Z}(\sigma_t; (X, A))) \cong \tilde{H}^*(\tilde{Z}(\sigma_t; (X, A))) \otimes \tilde{H}^*(\hat{A}^{[m]} - \sigma)
\]

\[
\cong \tilde{H}^*(X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_{n+1}}) \otimes \tilde{H}^*(A_{j_1} \wedge A_{j_2} \wedge \cdots \wedge A_{j_{m-n-1}})
\]

\[
\cong \tilde{H}^*(X_{i_1} \otimes \cdots \tilde{H}^*(X_{i_{n+1}}) \otimes \tilde{H}^*(A_{j_1}) \otimes \cdots \tilde{H}^*(A_{j_{m-n-1}})
\]

\[
\cong \tilde{H}^*(U_{i_1} \otimes \cdots \tilde{H}^*(U_{i_{n+1}}) \otimes \tilde{H}^*(V_{j_1}) \otimes \cdots \tilde{H}^*(V_{j_{m-n-1}})
\]

\[
= \tilde{H}^*(\tilde{Z}(\sigma_t; (U, V))) \quad \square
\]

The strong freeness condition and Lemma 5.3 yield the next lemma in an analogous way.

**Lemma 7.2.** There is an isomorphism of groups

\[(7.2) \quad \tilde{H}^*(\tilde{c}(X, A)) \cong \tilde{H}^*(\tilde{c}(U, V)).\]

**Proof.** Lemma 5.3 (6.2) and the Künneth Theorem give

\[
\tilde{H}^*(\tilde{c}(X, A)) \cong \tilde{H}^*(X_{i_1} / A_{i_1} \wedge X_{i_2} / A_{i_2} \wedge \cdots \wedge X_{i_{n+1}} / A_{i_{n+1}}) \otimes \tilde{H}^*(A_{j_1} \wedge A_{j_2} \wedge \cdots \wedge A_{j_{m-n-1}})
\]

\[
\cong \tilde{H}^*(X_{i_1} / A_{i_1}) \otimes \cdots \otimes \tilde{H}^*(X_{i_{n+1}} / A_{i_{n+1}}) \otimes \tilde{H}^*(A_{j_1}) \otimes \cdots \tilde{H}^*(A_{j_{m-n-1}})
\]

\[
\cong \tilde{H}^*(U_{i_1} / V_{i_1}) \otimes \cdots \tilde{H}^*(U_{i_{n+1}} / V_{i_{n+1}}) \otimes \tilde{H}^*(V_{j_1}) \otimes \cdots \tilde{H}^*(V_{j_{m-n-1}})
\]

\[
\cong \tilde{H}^*(\tilde{c}(U, V)) \quad \square
\]

The next lemma extends the isomorphism (7.1) to the boundary of a simplex.

**Lemma 7.3.** There is an isomorphism of groups

\[
\phi_t: \tilde{H}^*(\tilde{Z}(\partial \sigma_t; (U, V))) = \tilde{H}^* (E_{\sigma_t}) \oplus \tilde{H}^* (D_{\sigma_t}) \longrightarrow \tilde{H}^*(\tilde{Z}(\partial \sigma_t; (X, A))).
\]

**Proof.** Consider the ladder arising from the lower cofibration in (5.8), for both \((X, A)\) and \((U, A)\). We have adopted the notation of (6.5) for the latter.

\[
\cdots \tilde{H}^*(\tilde{Z}(\sigma_t; (X, A))) \xrightarrow{\iota_t} \tilde{H}^*(\tilde{Z}(\partial \sigma_t; (X, A))) \xrightarrow{\delta^{(X, A)}_{\sigma_t}} \tilde{H}^{n+1}(c(X, A)) \cdots
\]

\[
\cong \tilde{H}^*(\tilde{Z}(\partial \sigma_t; (X, A))) \xrightarrow{\phi_t} \tilde{H}^*(E_{\sigma_t}) \oplus \tilde{H}^*(D_{\sigma_t}) \xrightarrow{\delta^{(U, V)}_{\sigma_t}} \tilde{H}^{n+1}(\Sigma E_{\sigma_t, \sigma_t}) \oplus \tilde{H}^{n+1}(C_{\sigma_t}) \cdots
\]

We use next the geometric splitting from (6.5) and the vertical isomorphisms to define a map

\[
\phi_t: \tilde{H}^*(\tilde{Z}(\partial \sigma_t; (U, V))) = \tilde{H}^* (E_{\sigma_t}) \oplus \tilde{H}^* (D_{\sigma_t}) \longrightarrow \tilde{H}^*(\tilde{Z}(\partial \sigma_t; (X, A))).
\]

Theorem 6.4 gives

\[
\tilde{H}^*(\tilde{Z}(\partial \sigma_t; (U, V))) \cong \tilde{H}^* (C_{\sigma_t}) \oplus \tilde{H}^* (D_{\sigma_t}),
\]
and so, working over a field, we consider the splitting
\[ \widetilde{H}^*(\partial \sigma_t; (X, A)) \cong \widetilde{H}^*(\partial \sigma_t; (X, A))/((\iota \circ \beta_t)(\widetilde{H}^*(D_{\sigma_t}))/((\iota \circ \beta_t)(\widetilde{H}^*(D_{\sigma_t})). \]
For \( d \in \widetilde{H}^*(D_{\sigma_t}) \), set \( \phi_t(d) = (\iota \circ \beta_t)(d) \), and for \( e \in \widetilde{H}^*(E_{\sigma_t}) \), set \( \phi_t(e) \) equal to the unique class \( u \) in
\[ \widetilde{H}^*(\partial \sigma_t; (X, A))/((\iota \circ \beta_t)(\widetilde{H}^*(D_{\sigma_t})). \]
such that \( \delta_{\sigma_t}^{(X, A)}(u) = (\kappa_t \circ \delta_{\sigma_t}^{(U, V)})(e) \). The diagram commutes by the construction of the map \( \phi_t \). The Five-Lemma implies now that the map \( \phi_t \) is an isomorphism. \( \Box \)

8. The Proof of Theorem 5.4 for General \( K \)

We consider below a diagram of vector spaces over a field. It is constructed from the commutative diagram \((5.8)\) applied to the pairs \((U, V)\) and \((X, A)\) and incorporating the isomorphisms from Lemma \(7.3\). We assume by way of induction that
\[ \widetilde{H}^*(F_{t-1}Z(K; (U, V))) \cong \widetilde{H}^*(F_{t-1}Z(K; (X, A))), \]
which is true when \( F_{t-1}K \) is a simplex by \((3.2)\) and Theorem \(7.1\)
\[
\begin{array}{cccccc}
\cdots \widetilde{H}^*(\mathcal{E}_{(U, V)}) & \overset{\gamma_t^{(U, V)}}{\longrightarrow} & \widetilde{H}^*(F_tZ(K; (U, V))) & \overset{\delta_t^{(U, V)}}{\longrightarrow} & \widetilde{H}^+(\mathcal{E}_{(U, V)}) & \cdots \\
\downarrow{\cong}^{g_t} & & \downarrow{\cong}^{\alpha_t} & & \downarrow{\cong}^{\gamma_t^{(U, V)}} & \\
\cdots \widetilde{H}^*(\Sigma E_{\sigma_t}) \oplus \widetilde{H}^*(C_{\sigma_t}) & \overset{\gamma_t^{(U, V)}}{\longrightarrow} & \widetilde{H}^*(C_{\sigma_t}) \oplus \widetilde{H}^*(D_{\sigma_t}) & \overset{\delta_t^{(U, V)}}{\longrightarrow} & \widetilde{H}^*(E_{\sigma_t}) \oplus \widetilde{H}^*(D_{\sigma_t}) & \cdots \\
\downarrow{\cong}^{g_t} & & \downarrow{\cong}^{\alpha_t} & & \downarrow{\cong}^{\gamma_t^{(U, V)}} & \\
\cdots \widetilde{H}^*(\mathcal{E}_{(X, A)}) & \overset{\gamma_t^{(X, A)}}{\longrightarrow} & \widetilde{H}^*(F_tZ(K; (X, A))) & \overset{\delta_t^{(X, A)}}{\longrightarrow} & \widetilde{H}^+(\mathcal{E}_{(X, A)}) & \cdots \\
\end{array}
\]
The exactness and the commutativity of the diagram implies that we can choose isomorphisms as follows
\[ \widetilde{H}^*(F_tZ(K; (U, V))) \cong \widetilde{H}^*(C_{\sigma_t}) \oplus L \quad \text{for some } L \]
\[ \widetilde{H}^*(F_{t-1}Z(K; (U, V))) \cong \widetilde{H}^+(\Sigma E_{\sigma_t}) \oplus L \]
\[ \widetilde{H}^*(F_tZ(K; (X, A))) \cong \widetilde{H}^*(C_{\sigma_t}) \oplus L' \quad \text{for some } L' \]
\[ \widetilde{H}^*(F_{t-1}Z(K; (X, A))) \cong \widetilde{H}^+(\Sigma E_{\sigma_t}) \oplus L' \]
The inductive hypothesis implies now that \( L \cong L' \) and so
\[ \widetilde{H}^*(F_tZ(K; (U, V))) \cong \widetilde{H}^*(F_tZ(K; (X, A))) \]
as required. This, together with the fact that result is true for a simplex and its boundary, (Section \(7\)), completes the proof.

9. The Hilbert-Poincaré Series for \( Z(K; (X, A)) \)

We begin by reviewing some of the elementary properties of Hilbert-Poincaré series. Assume now that homology is taken with coefficients in a field \( k \) and all spaces are
pointed, path conected with the homotopy type of CW-complexes. The Hilbert-Poincaré series
\[ P(X, t) = \sum_n (\dim_k H_n(X; k)) t^n \]
and the reduced Hilbert-Poincaré series
\[ \overline{P}(X, t) = -1 + P(X, t) \]
satisfy the following properties.

1. \( P(X, t) P(Y, t) = P(X \times Y, t) \), and
2. \( \overline{P}(X, t) \overline{P}(Y, t) = P(X \land Y, t) \).

For a pair \((X, A)\) satisfying the conditions of Theorem 5.4, we have
\[ \text{(9.1)} \quad \overline{P}(\hat{Z}(K; (X, A)), t) = \overline{P}(\hat{Z}(K; (U, V)), t) \]
where the pair \((U, V)\) is as in Definition 5.1. Next, Theorem 2.2 gives
\[ \text{(9.2)} \quad \overline{P}(\hat{Z}(K; (U, V)), t) = \sum_{I \leq [m]} \left[ \sum_{\sigma \in K_I} [(t) \overline{P}(|lk_\sigma(K_I)|, t) \cdot \overline{P}(\hat{D}_{IC,E}(\sigma))] \right] \cdot \prod_{j \in [m]-I} \overline{P}(B_i, t) \]
We apply now Corollary 2.5 to refine this further and obtain the next theorem.

**Theorem 9.1.** The reduced Hilbert-Poincaré series for \( \hat{Z}(K; (U, V)) \), and hence for \( \hat{Z}(K; (X, A)) \) is given as follows,
\[ \overline{P}(\hat{Z}(K; (U, V)), t) = \sum_{I \leq [m]} \left[ \sum_{\sigma \in K_I} [(t) \overline{P}(|lk_\sigma(K_I)|, t) \cdot \overline{P}(\hat{D}_{IC,E}(\sigma))] \right] \cdot \prod_{j \in [m]-I} \overline{P}(B_i, t) \]
where \( \overline{P}(\hat{D}_{IC,E}(\sigma)) \) can be read off from (2.1).

Finally, Theorem 1.1 gives now the Hilbert-Poincaré series for \( Z(L; (U, V)) \) and for \( Z(L; (X, A)) \), by applying (9.1) for each \( K = L, J \subseteq [m] \).

### 10. Applications

**Example 10.1.** Consider the composite
\[ f: \mathbb{C}P^3 \rightarrow \mathbb{C}P^3/\mathbb{C}P^1 \overset{\iota}{\hookrightarrow} \mathbb{C}P^8/\mathbb{C}P^1 \]
where the map \( \iota \) is the inclusion of the bottom two cells. Denote the mapping cylinder of (10.1) by \( M_f \) and consider the pair \((M_f, \mathbb{C}P^3)\) for which we shall determine \( \hat{Z}(K; (M_f, \mathbb{C}P^3)) \) and \( Z(K; (M_f, \mathbb{C}P^3)) \) for any simplicial complex \( K \) on vertices \([m]\). Here,
\[ (U, V) = \left( \bigvee_{k=2}^3 S^{2k} \lor S^{2k}, \bigvee_{k=4}^8 S^{2k} \lor \bigvee_{k=2}^3 S^{2k} \lor S^2 \right) \]
so that \( B = \bigvee_{k=2}^3 S^{2k}, C = \bigvee_{k=4}^8 S^{2k} \) and \( E = S^2 \). Theorem 5.4 gives now
\[ \tilde{H}^*(\hat{Z}(K; (M_f, \mathbb{C}P^3))) \cong \tilde{H}^*(\hat{Z}(K; (U, V))). \]
Applying Theorem 2.2, we get
\[ \tilde{\mathbb{Z}}(K; (U, V)) \cong \bigvee_{I \leq [m]} \tilde{\mathbb{Z}}(K_I; (\bigvee_{k=4}^{8} S^{2k}, S^2)) \land \tilde{\mathbb{Z}}(K_{[m]-I}; (\bigvee_{k=2}^{3} S^{2k}, \bigvee_{k=2}^{3} S^{2k})) \]
where the last term represents the \((|m| - I)|\)-fold smash product. Finally, Corollary 2.5 determines completely each term
\[ \tilde{\mathbb{Z}}(K_I; (\bigvee_{k=4}^{8} S^{2k}, S^2)) \]
by enumerating all the links \(|lk_{\sigma}(K_I)|\).

Theorem 2.2 applies particularly well in cases where spaces have unstable attaching maps.

**Example 10.2.** The homotopy equivalence \(S^1 \land Y \simeq \Sigma(Y)\) implies homotopy equivalences
\[ \Sigma^m(\tilde{\mathbb{Z}}(K; (X, A))) \longrightarrow \tilde{\mathbb{Z}}(K; (\Sigma^q(X), \Sigma^q(A))) \]
where as usual, \(m\) is the number of vertices of \(K\). Recall now that \(SO(3) \cong \mathbb{R}P^3\) and consider the pair \((X, A) = (SO(3), \mathbb{R}P^2)\), for which there is a well known homotopy equivalence of pairs, [16, Section 1],
\[ (\Sigma^2(SO(3)), \Sigma^2(\mathbb{R}P^2)) \longrightarrow (\Sigma^2(\mathbb{R}P^2) \lor \Sigma^2(S^3), \Sigma^2(\mathbb{R}P^2)), \]
which makes the pair \((SO(3), \mathbb{R}P^2)\) *stably wedge decomposable*. Next, combining (10.2) and (10.3), we get a homotopy equivalence
\[ \Sigma^{2m}(\tilde{\mathbb{Z}}(K; (SO(3), \mathbb{R}P^2))) \longrightarrow \tilde{\mathbb{Z}}(K; (\Sigma^2(\mathbb{R}P^2) \lor \Sigma^2(S^3), \Sigma^2(\mathbb{R}P^2))). \]
Finally, Theorem 5.4 allows us to conclude that \(\tilde{\mathbb{Z}}(K; (SO(3), \mathbb{R}P^2)))\), and hence the polyhedral product \(Z(K; (SO(3), \mathbb{R}P^2)))\), is stably a wedge of smash products of \(S^3\) and \(\mathbb{R}P^2\).

11. **Product structure**

The purpose of this section is to describe the product structure of \(\tilde{H}^*(Z(K; (X, A)))\) in terms of the names of the additive generators given by Theorem 5.4 Corollary 2.5 and (11.9) below. We continue to work under the strong freeness conditions of Definition 5.1, which are satisfied, in particular, if we take coefficients in a field \(k\).
11.1. **Background.** We begin with a brief summary of the properties of partial diagonal maps from [3] and [4]. The main theorem of [3] asserts that the product structure on the algebra $H^*(Z(K; (X, A)))$, where $K$ is a simplicial complex on $[m]$, is determined completely by partial diagonals defined for $P, Q$ subsets of $[m]$, ([3, Section 2]),

$$\tilde{\Delta}^{P,Q}_{P \cup Q} : \tilde{Z}(K_{P \cup Q}; (X, A)) \to \tilde{Z}(K_P; (X, A)_P) \wedge \tilde{Z}(K_Q; (X, A)_Q)$$

inducing

$$\tilde{H}^*(\tilde{Z}(K_P; (X, A)_P)) \otimes \tilde{H}^*(\tilde{Z}(K_Q; (X, A)_Q) \to \tilde{H}^*(\tilde{Z}(K_{P \cup Q}; (X, A)_{P \cup Q})).$$

A sketch of the ideas from [4, Section 6] follows next. A family of pairs $(x, a)^{P,Q}_{P \cup Q}$ is defined by

$$(11.2) \quad [(x, a)^{P,Q}_{P \cup Q}]_i = \begin{cases} (x_i, a_i) & \text{if } i \in (P \cup Q) \setminus (P \cap Q) \\ (x_i \wedge x_i, a_i \wedge a_i) & \text{if } i \in P \cap Q. \end{cases}$$

The map $\tilde{\Delta}^{P,Q}_{P \cup Q}$ factors as

$$(11.3) \quad \tilde{Z}(K_{P \cup Q}; (X, A)) \overset{\tilde{\Delta}^{P,Q}_{P \cup Q}}{\to} \tilde{Z}(K_{P \cup Q}; (X, A)^{P,Q}_{P \cup Q})$$

followed by

$$(11.4) \quad \tilde{Z}(K_{P \cup Q}; (X, A)^{P,Q}_{P \cup Q}) \overset{\tilde{S}}{\to} \tilde{Z}(K_P; (X, A)_P) \wedge \tilde{Z}(K_Q; (X, A)_Q)$$

where $\tilde{S}$ is a shuffle map, originating from a natural rearrangement of smash product factors at the diagram level, ([3, Section 7]), and $\tilde{\psi}^{P,Q}_{P \cup Q} : (x, a) \to (x, a)^{P,Q}_{P \cup Q}$ is induced by the map of pairs

$$(11.5) \quad (x_i, a_i) \mapsto \begin{cases} (x_i, a_i) & \text{if } i \in (P \cup Q) \setminus (P \cap Q) \\ (x_i \wedge x_i, a_i \wedge a_i) & \text{if } i \in P \cap Q. \end{cases}$$

The connection between the partial diagonals and the cup product in $H^*(Z(K; (X, A)))$ is given by the next commutative diagram of diagonals and projections, ([3, Section 1]).

$$\begin{array}{ccc} Z(K; (X, A)) & \overset{\Delta_{Z(K; (X, A))}}{\longrightarrow} & Z(K; (X, A)) \wedge Z(K; (X, A)) \\
\downarrow \tilde{\Pi}_{P \cup Q} & & \downarrow \tilde{f}_P \wedge \tilde{f}_Q \\
\tilde{Z}(K_{P \cup Q}; (X, A)) & \overset{\tilde{\Delta}^{P,Q}_{P \cup Q}}{\longrightarrow} & \tilde{Z}(K_P; (X, A)_P) \wedge \tilde{Z}(K_Q; (X, A)_Q) \end{array}$$

The projection maps $\tilde{\Pi}_I$ are induced from the composition of the two projections:

$$(11.7) \quad \gamma^{|m|} \to \gamma^I \to \hat{\gamma}^I.$$

This leads to the definition of the $*$-product.

**Definition 11.1.** Given cohomology classes

$$u \in \tilde{H}^p(\tilde{Z}(K_P; (X, A)_P)) \quad \text{and} \quad v \in \tilde{H}^q(\tilde{Z}(K_Q; (X, A)_Q)),$$

the star product $*$-product is defined by

$$u * v = (\tilde{\Delta}^{P,Q}_{P \cup Q})^*(u \otimes v) \in \tilde{H}^{p+q}(\tilde{Z}(K_{P \cup Q}; (X, A)_{P \cup Q})).$$
The commutativity of diagram (11.6) implies that
\[(\tilde{\Pi}_{P,Q})^*(u \ast v) = (\tilde{\Pi}_P)^*(u) \sim (\tilde{\Pi}_Q)^*(v)\]
where \(\sim\) denotes the cup product in \(\tilde{H}^*(Z(K; (X, A)))\). Finally, the \(*\)-product endows
\[\bigoplus_{I \subseteq [m]} \tilde{H}^*(\tilde{Z}(K_I; (X, A)_I))\]
with a ring structure, and there is an isomorphism of rings
\[(11.9) \quad \eta : \bigoplus_{I \subseteq [m]} \tilde{H}^*(\tilde{Z}(K_I; (X, A)_I)) \rightarrow \tilde{H}^*(Z(K; (X, A)))\]
induced from the additive isomorphism of Theorem 1.1, as described in [3, Section 1].

11.2. Partial diagonals, the wedge lemma and the product of links. We discuss now the partial diagonal maps given by (11.1) in the context of Theorem 2.4, which asserts that if every \(E_i \hookrightarrow C_i\) is null homotopic then for any \(I \subseteq [m]\),
\[(11.10) \quad \tilde{Z}(K_I; (C, E)_I) \simeq \bigvee_{\sigma \in K_I} |\Delta((K_I)_{<\sigma})| \ast \tilde{D}^I_{C, E}(\sigma) \simeq \bigvee_{\sigma \in K_I} |\operatorname{lk}_\sigma(K_I)| \ast \tilde{D}^I_{C, E}(\sigma)\]
where \(\tilde{D}^I_{C, E}(\sigma)\) is as in (2.1). The next lemma checks that the partial diagonal maps behave as expected on the summands given by the wedge lemma.

**Lemma 11.2.** The proof of the wedge lemma, [20, Section 4], exhibits vertical embeddings of homotopy colimits so that the following diagram commutes for \(I = J \cup L\), \(\tau = \sigma \cap J\) and \(\omega = \sigma \cap L\),
\[(11.11) \quad \tilde{Z}(K_I; (C, E)_I) \xrightarrow{\tilde{\Delta}^{J,L}_I} \tilde{Z}(K_J; (C, E)_J) \wedge \tilde{Z}(K_L; (C, E)_L) \xrightarrow{\tilde{\gamma}(\tau) \wedge \tilde{\gamma}(\omega)} |\operatorname{lk}_\tau(K_J)| \ast \tilde{D}^I_{C, E}(\tau) \wedge |\operatorname{lk}_\omega(K_L)| \ast \tilde{D}^I_{C, E}(\omega)\]

**Proof.** The map \(\tilde{\xi}^{J,L}_I\) is the restriction of \(\tilde{\Delta}^{J,L}_I\). \(\square\)

In particular, (11.11) identifies the target under the partial diagonal \(\tilde{\Delta}^{J,L}_I\), for every wedge summand of \(\tilde{Z}(K_I; (C, E)_I)\). In order to better understand the map \(\tilde{\xi}^{J,L}_I\) from (11.11) and its effect on the cohomology of the links, we need to consider a polyhedral product which contains all the links.

Consider next the space \(S^0 = D^1 \vee S^0\) and let
\[(11.12) \quad \iota : S^0 \hookrightarrow D^1 \hookrightarrow S^0\]
be the basepoint preserving inclusion which takes \(S^0\) to the endpoints of \(D^1\); this gives a pair \((S^0, S^0)\). Next, let \(\mathcal{K}\) be a simplicial complex on \([m]\) and apply Theorem 2.4 to the polyhedral smash product \(\tilde{Z}(\mathcal{K}; (S^0, S^0))\) to get
\[(11.13) \quad \tilde{Z}(\mathcal{K}; (S^0, S^0)) \simeq \bigvee_{\sigma \in \mathcal{K}} |\operatorname{lk}_\sigma(\mathcal{K})| \ast \tilde{D}^{[m]}_{S^0, S^0}(\sigma) \simeq \bigvee_{\sigma \in \mathcal{K}} \Sigma |\operatorname{lk}_\sigma(\mathcal{K})|,\]
since $\tilde{D}_{S^0}^{[m]}(\sigma) \simeq S^0$. For each $\sigma \in \mathcal{K}$, the inclusion of each wedge summand given by Theorem 2.4 can be described as follows

\begin{equation}
|lk_\sigma(\mathcal{K})| * \tilde{D}_{C,E}^{[m]}(\sigma) = \Sigma |lk_\sigma(\mathcal{K})| \wedge \tilde{D}_{C,E}^{[m]}(\sigma) \to \tilde{Z}(\mathcal{K}; (S^0, S^0)) \wedge \tilde{D}_{C,E}^{[m]}(\sigma).
\end{equation}

Combining diagram (11.6) with (11.13), we get the commutative diagram below.

\begin{equation}
\begin{array}{ccc}
Z(\mathcal{K}; (S^0, S^0)) & \xrightarrow{\Delta_{Z(\mathcal{K}; (S^0, S^0))}} & Z(\mathcal{K}; (S^0, S^0)) \wedge Z(\mathcal{K}; (S^0, S^0)) \\
\downarrow \tilde{\pi}_J & & \downarrow \tilde{\pi}_J \wedge \tilde{\pi}_L \\
\tilde{Z}(\mathcal{K}_I; (S^0, S^0)_I) & \xrightarrow{\tilde{\Delta}_{J,L}^{I}} & \tilde{Z}(\mathcal{K}_I; (S^0, S^0)_I) \wedge \tilde{Z}(\mathcal{K}_I; (S^0, S^0)_I) \\
\downarrow \pi_\sigma & & \downarrow \pi_\sigma \wedge \pi_\omega \\
|lk_\sigma(\mathcal{K}_I)| & \xrightarrow{\tilde{\xi}_{I,L}^{J,L}} & |lk_\tau(\mathcal{K}_I)| \wedge |lk_\omega(\mathcal{K}_I)|
\end{array}
\end{equation}

where the vertical maps at the bottom are projections onto wedge summands. This motivates the next definition.

**Definition 11.3.** Let $\alpha \in \tilde{H}^*(|lk_\tau(\mathcal{K}_I)|)$ and $\beta \in \tilde{H}^*(|lk_\omega(\mathcal{K}_I)|)$, we write

$$\alpha * \beta := (\tilde{\xi}_{I,L}^{J,L})^*(\alpha_\tau \otimes \alpha_\omega_L).$$

We need more information about $\tilde{H}^*(Z(K; (S^0, S^0)))$ before undertaking an example.

**Lemma 11.4.** We have the following isomorphism of cohomology groups for the simplicial complex $K$ consisting of the empty simplex and $m$ discrete points

$$\tilde{H}^*(Z(K; (S^0, S^0))) \xrightarrow{\cong} \bigoplus_{I \subseteq [m]} \left( \bigoplus_{\sigma \in K_I} \tilde{H}^*(|lk_\sigma(K_I)| * \tilde{D}_{S^0}^{[m]}(\sigma)) \right).$$

*Proof. Here $K = \{\emptyset, \{1\}, \{2\}, \ldots, \{m\}\}$. Theorems 1.1 and 2.4 give

$$\tilde{H}^*(Z(K; (S^0, S^0))) \cong \bigoplus_{I \subseteq [m]} \tilde{H}^*(\tilde{Z}(K_I; (S^0, S^0)_I))$$

\begin{equation}
\cong \bigoplus_{I \subseteq [m]} \left( \bigoplus_{\sigma \in K_I} \tilde{H}^*(|lk_\sigma(K_I)| * \tilde{D}_{S^0}^{[m]}(\sigma)) \right)
\end{equation}

\begin{equation}
\cong \bigoplus_{I \subseteq [m]} \left( \bigoplus_{\sigma \in K_I} \tilde{H}^*(\Sigma |lk_\sigma(K_I)|) \right)
\end{equation}

\begin{equation}
\cong \bigoplus_{I \subseteq [m]} \left( \bigoplus_{\sigma \in K_I} \tilde{H}^*(\Sigma |lk_\sigma(K_I)|) \right) \oplus \tilde{H}^*(\Sigma \emptyset)
\end{equation}

where $\tilde{D}_{S^0}^{[m]}(\sigma) \simeq S^0$ is as in (2.1). \qed

**Example 11.5.** We shall now analyze diagram (11.15) for the case that $\mathcal{K} = K$ is the simplicial complex consisting of the empty simplex and $m$ discrete points

$$K = \{\emptyset, \{1\}, \{2\}, \ldots, \{m\}\}.$$
Set $I \subset [m]$, $J$ and $L$ subsets of $I$ satisfying $J \cup L = I$. Let $\sigma \in K_I$ be a simplex, we wish to compute

$$\tilde{H}^*(\Sigma |lk_{\tau}(K_I)|) \xleftarrow{\tilde{\varepsilon}_I^{J,L}*} \tilde{H}^*(\Sigma |lk_{\tau}(K_J)|) \otimes \tilde{H}^*(\Sigma |lk_{\omega}(K_L)|)$$

for $\tau = \sigma \cap J$ and $\omega = \sigma \cap L$. There are cases to consider:

1. $\sigma = \{i\} \in K_I$
   
   a. $\tau = \sigma \cap J = \{i\}$, $\omega = \sigma \cap L = \{i\}$, so $lk_{\tau}(K_I) = \emptyset$ and $lk_{\omega}(K_J) = \emptyset$.

   b. $\tau = \sigma \cap J = \emptyset$, $\omega = \sigma \cap L = \{i\}$, so $lk_{\tau}(K_I) = K_J$ and $lk_{\omega}(K_L) = \emptyset$.

2. $\sigma = \emptyset \in K_I$
   
   Here, $\tau = \sigma \cap J = \emptyset$, $\omega = \sigma \cap L = \emptyset$, so $lk_{\tau}(K_I) = K_J$ and $lk_{\omega}(K_L) = K_L$.

For each case, we wish to compute

$$\tilde{H}^*(\Sigma |lk_{\tau}(K_I)|) \xleftarrow{\tilde{\varepsilon}_I^{J,L}*} \tilde{H}^*(\Sigma |lk_{\tau}(K_J)|) \otimes \tilde{H}^*(\Sigma |lk_{\omega}(K_L)|). \tag{11.17}$$

**Case[1(a)]:** In this case the bottom row of \[11.15\], which we wish to compute in cohomology, is

$$\tilde{H}^0(\Sigma \emptyset) \xleftarrow{\tilde{\varepsilon}_I^{J,L}*} \tilde{H}^0(\Sigma \emptyset) \otimes \tilde{H}^0(\Sigma \emptyset) \cong k \cong k \otimes k. \tag{11.18}$$

We denote the unit generators in \eqref{11.18} by $\alpha_{i,J}$, $\alpha_{i,L}$ and $\alpha_{i,L}$ respectively. Applying cohomology to diagram \eqref{11.15}, we see that

$$(\tilde{\Pi}_I^* \circ \pi^*_\sigma)(\alpha_{i,J} \ast \alpha_{i,L}) = (\tilde{\Pi}_J^* \circ \pi^*_\tau)(\alpha_{i,J}) \sim (\tilde{\Pi}_L^* \circ \pi^*_\omega)(\alpha_{i,L}).$$

Corresponding to these classes, the top row of the diagram restricts to the cup product isomorphism

$$\tilde{H}^0(S^0) \xleftarrow{\Delta^*_I^{(K,\emptyset, S^0)}} \tilde{H}^0(\Sigma \emptyset) \otimes \tilde{H}^0(\Sigma \emptyset) \cong k \cong k \otimes k. \tag{11.19}$$

Now, since the map $(\iota_{\tau})^* \otimes (\iota_{\omega})^*$ is an isomorphism in this example, we conclude in this case that

$$\alpha_{i,J} \ast \alpha_{i,L} = \alpha_{i,J}. \tag{11.20}$$

**Case[1(b)]:** In this case the bottom row of \eqref{11.15} in cohomology, is

$$\tilde{H}^0(\Sigma \emptyset) \xleftarrow{\tilde{\varepsilon}_I^{J,L}*} \tilde{H}^0(\Sigma \emptyset) \otimes \tilde{H}^0(\Sigma \emptyset) \cong k \cong k \otimes k. \tag{11.21}$$

This time, we denote the unit generators in \eqref{11.21} by $\alpha_{i,J}$, $\beta_{i,J}$ and $\alpha_{i,L}$ respectively. Applying cohomology to diagram \eqref{11.15}, we see that

$$(\tilde{\Pi}_I^* \circ \pi^*_\sigma)(\beta_{i,J} \ast \alpha_{i,L}) = (\tilde{\Pi}_J^* \circ \pi^*_\tau)(\beta_{i,J}) \sim (\tilde{\Pi}_L^* \circ \pi^*_\omega)(\alpha_{i,L})$$
which is zero for dimensional reasons and we conclude
\[(11.22) \quad \beta_{i,J} \ast \alpha_{i,L} = 0.\]

**Case[2]:** This time the bottom row of \((11.15)\), in cohomology, is
\[(11.23) \quad \tilde{H}^*(K_I) \xrightarrow{(\tilde{\xi}_I^\ast)^*} \tilde{H}^*(K_J) \otimes \tilde{H}^{ast}(K_L) \cong H^*(\bigvee_{|J|-1} S^1) \cong H^*(\bigvee_{|L|-1} S^1).\]

The unit generators in \((11.23)\) are denoted this time by \(\beta_{i,I}\), \(\beta_{i,J}\) and \(\beta_{i,L}\) respectively. Applying cohomology to diagram \((11.15)\), we see that
\[(\tilde{\Pi}_I^* \circ \pi^*_\beta)(\beta_{i,J} \ast \beta_{i,L}) = (\tilde{\Pi}_J^* \circ \pi^*_\beta)(\beta_{i,J}) - (\tilde{\Pi}_L^* \circ \pi^*_\beta)(\alpha_{i,L})\]
which is zero for dimensional reasons and so we conclude
\[(11.24) \quad \beta_{i,J} \ast \beta_{i,L} = 0.\]

This ends the discussion of Example \((11.5)\).

An alternative addition to the toolkit for computing the \(\ast\)-products of links is outlined in the following remark.

**Remark 11.6.** Let \(\mathcal{K}\) be a simplicial complex on \([m]\), \(I \subset [m]\) and \(\sigma \in \mathcal{K}_I\). For a pair of subsets \(J\) and \(L\) of \([m]\) satisfying \(I = J \cup L\), set \(\tau = \sigma \cap J\) and \(\omega = \sigma \cap L\). Then \((\mathcal{K}_I)_J = \mathcal{K}_J, (\mathcal{K}_I)_L = \mathcal{K}_L\) and \((lk_\sigma(\mathcal{K}_I))_J = lk_\sigma(\mathcal{K}_J)\). There are also natural inclusions
\[(11.25) \quad (lk_\sigma(\mathcal{K}_I))_J \xrightarrow{\omega} lk_\tau(\mathcal{K}_J) \quad \text{and} \quad (lk_\sigma(\mathcal{K}_I))_L \xrightarrow{\omega} lk_\omega(\mathcal{K}_L).\]

In cases when these inclusions are equivalences of simplicial complexes, information about link products may be gleaned from the commutative diagram
\[
\begin{array}{ccc}
Z(lk_\sigma(K);(D^1,S^0)) & \xrightarrow{\Delta_{Z(lk_\sigma(K));(D^1,S^0)}} & Z(lk_\sigma(K);(D^1,S^0)) \wedge Z(lk_\sigma(K);(D^1,S^0)) \\
\downarrow \tilde{\n}_I & & \downarrow \tilde{\n}_L \\
\tilde{Z}((lk_\sigma(K));(D^1,S^0)) & \xrightarrow{\Delta_{\tilde{Z}(lk_\sigma(K));(D^1,S^0)}} & \tilde{Z}((lk_\sigma(K));(D^1,S^0)) \wedge \tilde{Z}((lk_\sigma(K));(D^1,S^0)) \\
\downarrow \cong & & \downarrow \cong \\
\Sigma |(lk_\sigma(K))_J| & \xrightarrow{\Delta_{\Sigma}(lk_\sigma(K))_J} & \Sigma |(lk_\sigma(K))_J| \wedge \Sigma |(lk_\sigma(K))_L|.
\end{array}
\]

where the lower vertical isomorphisms are given by Theorem \((2.4)\). In Example \((11.5)\) the inclusions \((11.25)\) are equivalences in cases \([1(a)]\) and \([2]\) but not in case \([1(b)]\). In that case however a dimension argument can be used to get \((11.22)\).

11.3. **Product structure for wedge decomposable pairs.** We consider now pairs of the form \((U, V) = (B \vee C, B \vee E)\) where the inclusion \(E_i \hookrightarrow C_i\) is null homotopic for all \(i \in \{1, 2, \ldots, m\}\), as in Definition \((2.1)\). We begin with the special case \(B = \ast\) (a point), and compute the \(\ast\)-product. In turn, this suffices to determine the ring structure of
\[
\tilde{H}^*(Z(K;(U, V))) = \tilde{H}^*(Z(K;(C, E))).
\]
by the ring isomorphism \( (11.9) \). The \(*\)-product is given by diagram \( (11.11) \) above, based on the decomposition of \( \hat{H}^*(Z(K; (C, E))) \) given by Corollary 2.5

As usual we set \( I \subset [m] \), \( J \) and \( L \) subsets of \( I \) satisfying \( J \cup L = I \). Let \( \sigma \in K_I \) be a simplex. Our goal then is to compute

\[
(11.26) \quad \hat{H}^*(\Sigma|l_kr(K_J)|) \otimes H^*(\hat{D}_{C,E}^J(\tau)) \otimes \hat{H}^*(\Sigma|l_k\omega(K_L)|) \otimes H^*(\hat{D}_{C,E}^L(\omega)) \xrightarrow{\xi^{J,L}_*} \hat{H}^*(\Sigma|l_k\sigma(K_I)|) \otimes H^*(\hat{D}_{C,E}^I(\sigma))
\]

for \( \tau = \sigma \cap J \) and \( \omega = \sigma \cap L \). Consider a class

\[
u \otimes \nu \in \hat{H}^*(\tilde{Z}(K_J; (C, E)_I)) \otimes \hat{H}^*(\tilde{Z}(K_L; (C, E)_L)).
\]

in the left hand side of \( (11.26) \), it has the form

\[
(11.27) \quad u \otimes v = (\alpha \otimes \bigotimes_{i \in \tau} c_i \otimes \bigotimes_{j \in J \setminus \tau} e_j) \otimes (\beta \otimes \bigotimes_{k \in \omega} c_k \otimes \bigotimes_{l \in L \setminus \omega} e_l)
\]

where \( \alpha, \beta \) are classes in \( \hat{H}^*(\Sigma|l_krK_J|) \) and \( \hat{H}^*(\Sigma|lk\omega K_L|) \) respectively. (Here the cohomology grading of the all the classes has been suppressed.) Aside from classes in the cohomology of the links, the only cohomology classes which are multiplied are those specified by the diagonals \( (11.5) \) which arise on the intersection \( J \cap L \). That is

\[
(\hat{A}_{J \cup L}^{IL})^*(u \otimes v) = (\alpha \ast \beta) \otimes \bigotimes_{i \in k \in \tau \wedge \omega} c_i c_k \otimes \bigotimes_{i \neq k \in (\tau \cap \omega) \setminus (\tau' \cap \omega')} c_i \otimes c_k \otimes \bigotimes_{j \in \tau' \cap \omega} e_j e_l \otimes \bigotimes_{j \neq \tau' \cap \omega'} e_j \otimes e_l
\]

in \( \hat{H}^*(\Sigma|lk\sigma K_I|) \otimes \hat{H}^*(\hat{D}_{C,E}^I(\sigma)) \).

where \( \tau' \) and \( \omega' \) denote \( J \setminus \tau \) and \( L \setminus \omega \) respectively. We arrive now at the next lemma.

**Lemma 11.7.** The product structure of the ring \( \hat{H}^*(Z(K; (C, E))) \) is determined by the \(*\)-product. On two classes given as in \( (11.27) \), it is evaluated by taking the \(*\)-product of the link classes, (cf. subsection 11.2), and the ordinary cohomology product, coordinate-wise on the other cohomology factors of \( (11.26) \) which correspond. \( \square \)

**Remark.** Though this formula is concise, the computation of the link product \( \alpha \ast \beta \) can be an obstacle. It is described in more detail in [4 Section 7] from the point of view of L. Cai’s work [8]. From the more practical perspective of subsection 11.2, diagram \( (11.15) \) relates the \(*\)-product of these links to an actual cup product in \( \hat{H}^*(Z(K; (S^0, S^0))) \) via \( (11.6) \), which can often be checked explicitly. We shall take this approach below.

We consider next a full wedge decomposable pair \( (U, V) = (B \vee C, B \vee E) \) beginning with the part of the partial diagonals given by the shuffle maps from \( (11.4) \).

\[
(11.28) \quad \tilde{Z}(K_{P \cup Q}; (U, V)^{PQ}_{P \cup Q}) \xrightarrow{\delta} \tilde{Z}(K_P; (U, V)_P) \land \tilde{Z}(K_Q; (U, V)_Q)
\]

For the pairs \( (U, V), \) \( (11.2) \) becomes

\[
[(U, V)^{PQ}_{P \cup Q}]_i = \begin{cases} (U_i, V_i) = (B_i \vee C_i, B_i \vee E_i) & \text{ if } i \in P \cup Q \setminus P \cap Q \\ (U_i \cup U_i, V_i \cup V_i) = (\hat{B}_i \vee \hat{C}_i, \hat{B}_i \vee \hat{E}_i) & \text{ if } i \in P \cap Q. \end{cases}
\]
where here
\[ \hat{B}_i = B_i \land B_i \]  
\[ (11.29) \]
\[ \hat{C}_i = (C_i \land C_i) \lor (C_i \land B_i) \lor (B_i \land C_i) \]  
\[ \hat{E}_i = (E_i \land E_i) \lor (E_i \land B_i) \lor (B_i \land E_i). \]  
\[ (11.31) \]

In particular, the pairs in \((U, V)^{PQ}_{P \cup Q}\) are all wedge decomposable. This motivates the notational convention following.

**Definition 11.8.** For any subsets \(S, T\) subsets of \([m]\), we set the notation
\[ \tilde{(B \lor C, \tilde{B} \lor \tilde{E})}_{S \cup T} = (U, V)^{ST}_{S \cup T} \]

where
\[ (\tilde{B}_i \lor \tilde{C}_i, \tilde{B}_i \lor \tilde{E}_i) := \begin{cases} (B_i \lor C_i, B_i \lor E_i) & \text{if } i \in (S \cup T) \setminus (S \cap T) \\ (\tilde{B}_i \lor \tilde{C}_i, \tilde{B}_i \lor \tilde{E}_i) & \text{if } i \in S \cap T. \end{cases} \]

In that which follows in this section, we adopt additional notation following:

1. \(P\) and \(Q\) are subsets of \([m]\)
2. \(I \subset P \cup Q\)
3. \(I = J \cup L\) with \(J \subset P\) and \(L \subset Q\),

Notice that in this notation,
\[ (11.30) \quad (P \cup Q) \setminus I = ((P \setminus J) \cup (Q \setminus L)) \setminus (J \cap (Q \setminus L)) \cup (L \cap (P \setminus J)). \]

Generally, the meaning of the notation should become clear from the context. Figure 1 is a Venn diagram illustrating these sets, among other things.

Next, we use
\[ (11.31) \quad \hat{Z}(K_{P \cup Q}; (U, V)^{PQ}_{P \cup Q}) = \hat{Z}(K_{P \cup Q}; (B \lor C, \tilde{B} \lor \tilde{E})_{P \cup Q}), \]

and apply Theorem \[2.2\] to the right hand side, \((P \cup Q\) here plays the role of \([m]\) in the theorem), to exhibit the shuffle map \[(11.28)\] on a wedge summand of \[(11.31)\] as

\[ (11.32) \quad \hat{Z}(K_I; (\tilde{C}, \tilde{E})_I) \land \hat{Z}(K_{P \cup Q - I}; (\tilde{B}, \tilde{B})_{P \cup Q - I}) \]
\[ \xrightarrow{\hat{S}} \hat{Z}(K_J; (C, E)_I) \land \hat{Z}(K_{P - J}; (B, B)_P - J) \land \hat{Z}(K_L; (C, E)_L) \land \hat{Z}(K_{Q - L}; (B, B)_Q - L) \]

where \(J \cup L = I\). In each of the the partial diagonal maps, the shuffle map is preceded by the map \(\hat{\psi}^{1}_{J,L}\) of \[(11.3)\] and \[(11.5)\]

\[ (11.33) \quad \hat{Z}(K_{P \cup Q}; (U, V)^{PQ}_{P \cup Q}) \xrightarrow{\hat{\psi}^{PQ}_{P \cup Q}} \hat{Z}(K_{P \cup Q}; (U, V)^{PQ}_{P \cup Q}). \]

We apply now Theorem \[2.2\] to this and consider the same wedge summand to get
\[ (11.34) \quad \hat{Z}(K_I; (\tilde{C}, \tilde{E})_I) \land \hat{Z}(K_{P \cup Q - I}; (\tilde{B}, \tilde{B})_{P \cup Q - I}) \]
\[ \xrightarrow{\tilde{\chi}^{1}_{I,L}} \hat{Z}(K_I; (\tilde{C}, \tilde{E})_I) \land \hat{Z}(K_{P \cup Q - I}; (\tilde{B}, \tilde{B})_{P \cup Q - I}). \]

where \(\tilde{\chi}^{1}_{I,L}\) is induced from \(\hat{\psi}^{PQ}_{P \cup Q}\) in \[(11.33)\] via Theorem \[2.2\]. In order to analyze this further, we need a lemma.
Lemma 11.9. The diagram below commutes.

\[
\begin{align*}
\tilde{Z}(K_I; (C, E)_I) &\xrightarrow{\tilde{\epsilon}_i^{j,L}} \tilde{Z}(K_I; (\tilde{C}, \tilde{E})_I) \\
\tilde{Z}(K_I; (C, E)_I) &\xrightarrow{\tilde{\varphi}_i^{j,L}} \tilde{Z}(K_I; (C, E)_{J,L})
\end{align*}
\]

(11.35)

where the map \(\tilde{\varphi}_i^{j,L}\) is induced by the map of pairs

\[
(C_i, E_i) \mapsto (C_i, E_i) \quad \text{if } i \in J \cup L \setminus J \cap L, \text{ else if } i \in J \cap L,
\]

\[
(C_i \cap C_i, E_i \cap E_i) \mapsto ((C_i \cap C_i) \vee (B_i \cap C_i) \vee (C_i \cap B_i), \quad (E_i \cap E_i) \vee (B_i \cap E_i) \vee (B_i \cap E_i))
\]

the latter by the inclusion into the first wedge summands.

Proof. This follows from the naturality of Theorem 2.2 for maps of wedge decomposable pairs.

This lemma allow us now to replace (11.34) with the map

\[
\begin{align*}
(\tilde{Z}(K_I; (C, E)_I) \wedge \tilde{Z}(K_{P\cup Q-I}; (B, B)_{P\cup Q-I})) &\xrightarrow{\tilde{\varphi}_i^{j,L} \wedge \tilde{\varphi}_i^{P-J,Q-L}} \\
(\tilde{Z}(K_I; (C, E)_{J,L}) \wedge \tilde{Z}(K_{P\cup Q-I}; (B, B)_{P\cup Q-I}))
\end{align*}
\]

(11.36)

where we have used

\[
(B, B)_{P\cup Q-I}^{P-J,Q-L} = (\tilde{B}, \tilde{B})_{P\cup Q-I}.
\]

The next theorem describes now the product for the cohomology of a wedge decomposable pair.

Theorem 11.10. The partial diagonal map for a wedge decomposable pair \((U, V)\)

\[
\tilde{\Delta}_{P\cup Q} : \tilde{Z}(K_{P\cup Q}; (U, V)_{P\cup Q}) \longrightarrow \tilde{Z}(K_P; (U, V)_P) \wedge \tilde{Z}(K_Q; (U, V)_Q),
\]

is realized on each wedge summand given by Theorem 2.2 as

\[
\begin{align*}
(\tilde{Z}(K_I; (C, E)_I) \wedge \tilde{Z}(K_{P\cup Q-I}; (B, B)_{P\cup Q-I})) &\xrightarrow{\tilde{\Delta}_I^{j,L} \wedge \tilde{\Delta}_{P\cup Q-I}^{P-J,Q-L}} \\
(\tilde{Z}(K_I; (C, E)_I) \wedge \tilde{Z}(K_{P\cup Q-I}(B, B)_{P\cup Q-I})) &\wedge (\tilde{Z}(K_I; (C, E)_I) \wedge \tilde{Z}(K_{Q-L}(B, B)_{Q-L}))
\end{align*}
\]

(11.38)

Proof. The map is obtained in this form by using (11.37) to compose (11.36) with the shuffle map (11.32).

This theorem enables now direct calculation of the \(*\)-product on a wedge decomposable pair by combining the calculation for \((C, E)\) given by Lemma 11.7 with calculation for \((B, B)\) which is described next.

No links appear in the \(\tilde{Z}(K_{P\cup Q-I}; (B, B)_{P\cup Q-I})\) wedge summand and we have have

\[
\tilde{H}^*(\tilde{Z}(K_{P\cup Q-I}; (B, B)_{P\cup Q-I})) \cong \bigotimes_{j \in P\cup Q-I} \tilde{H}^*(B_j).
\]
On this wedge factor, the partial diagonals are

\[(11.39) \quad \overline{\Delta}_{(P \cup Q - I)}^{P - J, Q - L} : \tilde{Z}(K_{P \cup Q - I}; (B, B)_{P \cup Q - I}) \to \tilde{Z}(K_{P - J}; (B, B)_{P - J}) \land \tilde{Z}(K_{Q - L}; (B, B)_{Q - L}),\]

inducing

\[(11.40) \quad \bigotimes_{j \in P - J} \tilde{H}^*(B_j) \otimes \bigotimes_{k \in Q - L} \tilde{H}^*(B_k) \to \bigotimes_{l \in P \cup Q - I} \tilde{H}^*(B_l),\]

As in the case \((C, E)\), classes in the same cohomology factors are multiplied; these are the \(\tilde{H}^*(B_i)\) with \(i \in (P \setminus J) \cap (Q \setminus L)\), cf. (11.30). This is all summarized in the next lemma.

**Corollary 11.11.** The \(*\)-product associated to the ring \(\tilde{H}^*(Z(K; (U, V)))\) for a wedge decomposable pair \((U, V)\), is determined by the smash product of partial diagonal maps (11.38). In cohomology the products are described by Lemma 11.7 and by (11.40). This determines the cup product structure in \(H^*(K; (U, V))\) by the description given in subsection 11.1.

### 11.4. The product structure for general CW pairs.

We wish now to extend these results about products, from wedge decomposable pairs \((U, V) = (B \lor C, B \lor E)\), to the cohomology of \(Z(K; (X, A))\) for general pairs \((X, A)\). Our main tool is Theorem 5.4 which asserts that, given \((X, A)\), we can find wedge decomposable pairs \((U, V) = (B \lor C, B \lor E)\) so that as groups, there is an isomorphism

\[(11.41) \quad \theta_{(U, V)} : \tilde{H}^*(Z(K; (U, V))) \to H^*(Z(K; (X, A))).\]

Here (11.41) is used to label the generators only, the product structure will be in terms of the product structure in the rings \(\tilde{H}^*(X_i)\) and \(\tilde{H}^*(A_i)\) and not in \(\tilde{H}^*(B_i), \tilde{H}^*(C_i)\) or \(\tilde{H}^*(E_i)\).

Unlike the previous example of wedge decomposable pairs, Lemma 11.9 will not hold as the pairs \((X, A)\) are not wedge decomposable in general. In this new situation, products in \(\tilde{H}^*(X_i)\) and \(\tilde{H}^*(A_i)\) will mix the modules \(B'_i, C'_i\) and \(E'_i\) and so upset the links which appear in the partial diagonal maps (11.11) and (11.39). Our task then is to keep track of these changes.

We begin by examining the diagonal maps for the spaces \(X_i\) and \(A_i\) and look at the diagonal maps (11.5) in terms of the notation adopted in Definition 5.1. For the long exact sequence

\[\delta : \tilde{H}^*(X_i \land X_i / A_i \land A_i) \to \tilde{H}^*(X_i \land X_i) \to \tilde{H}^*(A_i \land A_i) \to \tilde{H}^*(X_i \land X_i / A_i \land A_i) \to \]

the appropriate image, kernel and cokernel modules are given as follows.
(1) \( \tilde{H}^*(A_i \wedge A_i) \cong \tilde{B}_i \oplus \tilde{E}_i \)

(2) \( \tilde{H}^*(X_i \wedge X_i) \cong \tilde{B}_i \oplus \tilde{C}_i \), where \( \tilde{B}_i \overset{\sim}{\rightarrow} \tilde{B}_i, \quad \ell|_{\tilde{C}_i} = 0 \)

(3) \( \tilde{H}^*(X_i \wedge X_i / A_i \wedge A_i) \cong \tilde{C}_i \oplus W'_i \), where \( \tilde{C}_i \overset{\ell}{\rightarrow} \tilde{C}_i, \quad \ell|_{\tilde{B}_i} = 0, \quad \tilde{E}_i \overset{\delta}{\rightarrow} W'_i \).

Comparing these to the graded \( k \)-modules we have for the pair \((X, A)\) we get

\[
\begin{align*}
\tilde{B}_i &= B'_i \otimes B'_i \\
\tilde{C}_i &= C'_i \otimes C'_i \oplus C'_i \otimes B'_i \oplus B'_i \otimes C'_i \\
\tilde{E}_i &= E'_i \otimes E'_i \oplus E'_i \otimes B'_i \oplus B'_i \otimes E'_i.
\end{align*}
\]

Note: These cohomology modules are realized now by the cohomology of the spaces \( \tilde{B}_i, \tilde{C}_i \) and \( \tilde{E}_i \) as defined in (11.29), and we continue to adopt the notation from Definition 11.8. Also, in order to keep the notation as simple as possible, we shall suppress explicit mention of the additive isomorphism \( \theta_{(U, V)} \) from (11.41), though its usage will be understood throughout the remainder of this section.

Consider again the part of the partial diagonals given by the shuffle maps from (11.4)

\[
\tilde{Z}(K_{P \cup Q}; (X, A)_{P \cup Q}) \overset{\tilde{S}}{\rightarrow} \tilde{Z}(K_P; (X, A)_P) \wedge \tilde{Z}(K_Q; (X, A)_Q).
\]

Applying Theorem 5.4, the shuffle map for \((X, A)\) can be described \emph{additively} in cohomology, in terms of wedge decomposable pairs, as follows

\[
\tilde{H}^*(\tilde{Z}(K_P; (U, V)_P) \wedge \tilde{Z}(K_Q; (U, V)_Q)) \overset{\tilde{S}}{\rightarrow} \tilde{H}^*(\tilde{Z}(K_{P \cup Q}; (U, V)_{P \cup Q}));
\]

where as in (11.31) we have, (in the notation of Definition 11.8)

\[
\tilde{Z}(K_{P \cup Q}; (U, V)_{P \cup Q}) = \tilde{Z}(K_{P \cup Q}; (\tilde{B} \vee \tilde{C}, \tilde{B} \vee \tilde{E})_{P \cup Q}).
\]
Figure 1 below is a useful aid in visualizing all that follows. It displays all sets, simplices and modules which are relevant to the discussion of the shuffle map for CW pairs and their cohomological wedge decompositions. (The numbers in square brackets label regions of the Venn diagram non-uniquely.)

\[ \text{Figure 1} \]

The arrangement of modules in \( H^*(\hat{Z}(K_{P \cup Q}; (U, V)_{P \cup Q})) \) = \( H^*(\hat{Z}(K_{P \cup Q}; (\tilde{B} \vee \tilde{C}, \tilde{B} \vee \tilde{E})_{P \cup Q})) \)

Next, we apply Theorem 2.2 to the spaces \( \hat{Z}(K_P; (U, V)_P) \) and \( \hat{Z}(K_Q; (U, V)_Q) \) which appear on the left hand side of (11.44), and then take the cohomology of each of the wedge summands resulting, to get the cohomological description of (11.43) below,

(11.46) \[ \tilde{H}^*(\hat{Z}(K_J; (C, E)_J)) \otimes \tilde{H}^*(\hat{Z}(K_{P-J}; (\tilde{B}, \tilde{B})_{P-J})) \]
\[ \otimes \tilde{H}^*(\hat{Z}(K_L; (C, E)_L)) \otimes \tilde{H}^*(\hat{Z}(K_{Q-L}; (\tilde{B}, \tilde{B})_{Q-L})) \]
\[ \xrightarrow{\tilde{s}^*} \tilde{H}^*(\hat{Z}(K_{P \cup Q}; (\tilde{B} \vee \tilde{C}, \tilde{B} \vee \tilde{E})_{P \cup Q})). \]

The next lemma is now apposite. As usual, (Definition 11.8), we set \( I = J \cup L \).
Lemma 11.12. In (11.46) the target of the shuffle map $\hat{S}^*$ is the summand 

(11.47) \[ \tilde{H}^*(\hat{Z}(K_I; (\tilde{C}, \tilde{E})_I)) \otimes \tilde{H}^*(\hat{Z}(K_{P\cup Q-I}; (\tilde{B}, \tilde{B})_{P\cup Q-I})) \]

in the decomposition of $\tilde{H}^*(\hat{Z}(K_{P\cup Q}; (\tilde{B} \lor \tilde{C}, \tilde{B} \lor \tilde{E})_{P\cup Q}))$ given by Theorem 2.2.

Remark. This is not obvious from (11.32) because it is at the cohomology level only that we can replace the shuffle map in which we are interested, (11.43), with one involving wedge decomposable pairs (11.44).

Proof. Consider again the equality of sets (11.30), (cf. Figure 1),

\[ (P \cup Q) \setminus I = ((P \setminus J) \cup (Q \setminus L)) \setminus (J \cap (Q \setminus L)) \cup (L \cap (P \setminus J)). \]

All the factors $H^*(B_i)$ on the left hand side of (11.46) satisfy $i \in (P \setminus J) \cup (Q \setminus L)$. The only ones which can be paired with $H^*(C_i)$ or $H^*(E_i)$ are those for which $i$ is in the disjoint union $(J \cap (Q \setminus L)) \cup (L \cap (P \setminus J)) \subset I$. In this case,

\[ H^*(C_i) \otimes H^*(B_i) \subset H^*(\hat{C}_i) \quad \text{and} \quad H^*(E_i) \otimes H^*(B_i) \subset H^*(\hat{E}_i) \]

So, the factors of $H^*(B_i)$ from the left hand side of (11.46) which are “lost” are precisely the ones for which $i \in I$. All other copies of $H^*(B_i)$ survive to appear in the right hand tensor factor of (11.47). \qed

The main theorem of this section is next.
Theorem 11.13. The additive isomorphism $\theta_{(U,V)}$ of (11.41) suffices to determine explicitly the partial diagonal map (11.1)

$$\tilde{H}^*(\tilde{Z}(K_P; (X,A)_P) \wedge \tilde{Z}(K_Q; (X,A)_Q)) \xrightarrow{\tilde{\Delta}^Q_{P,Q}*} \tilde{H}^*(\tilde{Z}(K_{P,Q}; (X,A)_{P,Q})).$$

and hence the cup product structure in $H^*(K; (X,A))$, by the description given in subsection 11.1.

The rest of this section is devoted to the proof of Theorem 11.13. Consider a class $x_P \otimes x_Q$ in the left hand side of (11.44)

$$(11.48) \quad x_P \otimes x_Q \in \tilde{H}^*(\tilde{Z}(K_P; (U,V)_P) \wedge \tilde{Z}(K_Q; (U,V)_Q)).$$

Theorem 2.2 describes this class as a sum of terms of the form $u \otimes v$ in

$$(11.49) \quad \tilde{H}^*(\tilde{Z}(K_J; (C,E)_J)) \otimes \tilde{H}^*(\tilde{Z}(K_{P-J}; (B,B)_{P-J}))$$

$$\otimes \tilde{H}^*(\tilde{Z}(K_L; (C,E)_L)) \otimes \tilde{H}^*(\tilde{Z}(K_{Q-L}; (B,B)_{Q-L})), $$

each of which has the form below by Corollary 2.5

$$u = \alpha \otimes \bigotimes_{i \in \tau} c_i \otimes \bigotimes_{j \in J \setminus \tau} e_j \otimes \bigotimes_{j \in P \setminus J} b_j \in \tilde{H}^*(\Sigma|lk\tau K_J|) \otimes \tilde{H}^*(\tilde{D}_C^J(\tau)) \otimes \tilde{H}^*(B_j)$$

$$v = \beta \otimes \bigotimes_{i \in \omega} c_i \otimes \bigotimes_{j \in L \setminus \omega} e_j \otimes \bigotimes_{j \in Q \setminus L} b_j \in \tilde{H}^*(\Sigma|lk\omega K_L|) \otimes \tilde{H}^*(\tilde{D}_E^L(\omega)) \otimes \tilde{H}^*(B_j)$$

It will be convenient to call $\alpha$ and $\beta$ indexing links. Next, we apply the shuffle map in cohomology (11.46), ignoring cohomological degree and keeping in mind (11.29) and Figure 1. This gives the description below in which cohomology elements are labelled by the regions in Figure 1 to which they belong. Recall that (11.5) restricts cohomology products to the set $P \cap Q$ only, in particular, to Regions [1], [2], [3], [5] and [5] of Figure 1 only. From (11.42) we see that products on Region [4] are not supported. (Note that some relabelling in the names of classes is necessary to avoid ambiguities.)

$$\tilde{S}^*(u \otimes v) = \tilde{S}^*(\alpha \otimes \beta)$$

$$(11.50) \quad \bigotimes_{i \in [5]} (c_i \otimes \tilde{c}_i) \otimes \bigotimes_{j \in [6]} (c_j \otimes b_j) \otimes \bigotimes_{k \in [2]} (e_k \otimes \tilde{e}_k) \otimes \bigotimes_{l \in [3]} (e_l \otimes b_l)$$

$$\otimes \bigotimes_{s \in [1]} (b_s \otimes \tilde{b}_s) \otimes \bigotimes_{\{t,u,v\}} (c_t \otimes e_u \otimes b_v)$$

in the group

$$(11.51) \quad \tilde{H}^*(\tilde{Z}(K_{P,Q}; (U,V)_{P,Q})) = \tilde{H}^*(\tilde{Z}(K_{P,Q}; (B \lor C, B \lor E)_{P,Q})).$$

Notice that by (11.42) no products arising from Region [4] in Figure 1 can be supported. Next, we compose with the map induced by (11.3),

$$(11.52) \quad \tilde{H}^*(\tilde{Z}(K_{P,Q}; (X,A)_{P,Q})) \xrightarrow{\tilde{\phi}_{P,Q}^*} \tilde{H}^*(\tilde{Z}(K_{P,Q}; (X,A)_{P,Q}))$$

to get the full partial diagonal map induced by (11.1),
\[ \tilde{H}^*(\tilde{Z}(K_{P}; (X, A)_P)) \wedge \tilde{Z}(K_{Q}; (X, A)_Q)) \to \tilde{H}^*(\tilde{Z}(K_{P\cup Q}; (X, A)_{P\cup Q})) \]

and hence a computation of

\[ u \ast v = (\tilde{\Delta}_{P \cup Q}^{P, Q})^*(u \otimes v) = ((\tilde{\psi}_{P \cup Q}^{P, Q})^* \circ \tilde{S}^*)(u \otimes v) \]

which is the part of \( x_P \ast x_Q \) in the summand \((11.49)\). The homomorphism \( \tilde{\psi}_{P \cup Q}^{P, Q} \) multiplies the terms in \( \tilde{S}^*(u \otimes v) \) corresponding to the marked regions. The consequences are discussed below. Recall that the target of the map \( (\tilde{\psi}_{P \cup Q}^{P, Q})^* \), \((11.52)\), is given by Corollary 2.5.

Addition of the modules \( \tilde{\Delta}_{P \cup Q}^{P, Q} \ast (u \otimes v) \), the properties of the modules \( B'_i, C'_i \) and \( E'_i \) from Definition 5.1 are used strongly below. All cup products are either in \( \tilde{H}^*(X_i) \cong B'_i \oplus C'_i \) or in \( \tilde{H}^*(A_i) \cong B'_i \oplus E'_i \).

Below is list of all cup products which can occur when \( (\tilde{\psi}_{P \cup Q}^{P, Q})^* \) is applied to the class \( \tilde{S}^*(u \otimes v) \) in the group \((11.51)\):

1. For \( i \in \text{Region} [5], \) \( c_i \otimes \bar{c}_j \mapsto c_i^{[5]} \in C'_i \).
2. For \( j \in \text{Region} [6], \) \( c_i \otimes b_j \mapsto c_j^{[6]} \in C'_j \).
3. For \( k \in \text{Region} [2], \) \( e_k \otimes \bar{e}_k \mapsto e_k^{[2]} + b_k^{[2]} \in E'_k \oplus B'_k \).
4. For \( l \in \text{Region} [3], \) \( e_l \otimes b_l \mapsto e_l^{[3]} + b_l^{[3]} \in E'_l \oplus B'_l \).
5. For \( s \in \text{Region} [1], \) \( b_s \otimes \bar{b}_s \mapsto b_s^{[1]} + c_s^{[1]} \in B'_s \oplus C'_s \).

The next two lemmas will allow us to keep track of the links indexing the monomials.

**Lemma 11.14.** Let \( K \) be a simplicial complex on vertices \([m], I \subset [m] \setminus \{s\}, \sigma \in K_I \) and \( \sigma \cup \{s\} \in K_{I \cup \{s\}} \). Then there is a natural map

\[ \rho_{I, s}: lK_{\sigma \cup \{s\}}K_{I \cup \{s\}} \to lK_{\sigma}K_I \]

\[ \tau \mapsto \tau \cap I \]

**Proof:** Let \( \tau \in lK_{\sigma \cup \{s\}}K_{I \cup \{s\}}, \) then \( \tau \cup (\sigma \cup \{s\}) \in K_{I \cup \{s\}} \) and so \( \tau \cup \sigma \in K_{I \cup \{s\}} \) implying \( (\tau \cap I) \cup \sigma \in K_I \). \( \square \)
Lemma 11.15. Let $K$ be a simplicial complex on vertices $[m]$, $I \subset [m]$, $\sigma \in K_I$ and $l \in I$. Then there is a natural inclusion,
\begin{equation}
(11.56)\quad \iota_l: \operatorname{lk}_\sigma K_I \setminus \{l\} \longrightarrow \operatorname{lk}_\sigma K_I.
\end{equation}
and the diagram below commutes.
\begin{equation}
(11.57)\quad \begin{array}{c}
\operatorname{lk}_\sigma K_I \setminus \{l\} \\
\uparrow \rho_I (l,s)
\end{array} \quad \xrightarrow{\iota_l} \quad \begin{array}{c}
\operatorname{lk}_\sigma K_I \\
\rho_I (l,s) \uparrow
\end{array} \quad \begin{array}{c}
\operatorname{lk}_{\sigma \cup \{s\}} K_{(I \cup \{s\}) \setminus \{l\}} \\
\iota_l
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\operatorname{lk}_{\sigma \cup \{s\}} K_{I \cup \{s\}}
\end{array}
\end{equation}

Proof: Notice first that \( \iota_l \) exists in \( K_{I \cup \{s\}} \setminus \{l\} = \operatorname{lk}_{\sigma \cup \{s\}} K_{I \setminus \{l\} \cup \{s\}} \),
because \( I \cup \{s\} \setminus \{l\} = (I \setminus \{l\}) \cup \{s\} \). Now let \( \tau \in \operatorname{lk}_{\sigma \cup \{s\}} K_{(I \cup \{s\}) \setminus \{l\}} \), then
\begin{equation}
(\rho_I, l,s) (\iota_l)(\tau) = \rho_I, l,s(\tau) = \tau \cap I.
\end{equation}
On the other hand,
\begin{equation}
(\iota_l \circ \rho_I, l,s)(\tau) = \iota_l(\tau \cap I) = \tau \cap I.
\end{equation}
as well. \hfill \Box

We are in a position now to enumerate and describe the monomials, including their indexing links, which arise from (11.50). Recall that
\((\hat{\psi}_{P,Q})^* \circ \hat{S}^* (\alpha \otimes \beta) = \alpha \ast \beta\) is described in subsection 11.2. The subscripts below are preserved from (11.50).

(a) \( (\alpha \ast \beta) \otimes c_i^{[5]} \otimes c_j^{[6]} \otimes e_k^{[2]} \otimes c_l^{[3]} \otimes b_s^{[1]} \)

There are no changes to the indexing link \( \alpha \ast \beta \) here.

(b) \( \rho_i^*, (\alpha \ast \beta) \otimes c_i^{[5]} \otimes c_j^{[6]} \otimes e_k^{[2]} \otimes c_l^{[3]} \otimes c_s^{[1]} \)

The class \( c_s^{[1]} \) in item (5) above will be zero unless the simplex \( \nu = \sigma \cup \{s\} \) exists in \( K_{I \cup \{s\}} \). In which case, the indexing link \( \alpha \ast \beta \) must change to \( \rho_i^* (\alpha \ast \beta) \). Here, the simplex \( \nu = \sigma \cup \{s\} \) exists in \( K_{I \cup \{s\}} \) and so \( \nu \) is a full subcomplex of \( K \), in which case, \( Z(\nu; (X, A)) = \prod_{i_k \in \nu} X_{i_k} \) retracts off \( Z(K; (X, A)) \) and the product is occurring in the subring \( H^*( \prod_{i_k \in \nu} X_{i_k}) \).

(c) \( \iota_i^*(\alpha \ast \beta) \otimes c_i^{[5]} \otimes c_j^{[6]} \otimes e_k^{[2]} \otimes b_l^{[3]} \otimes b_s^{[1]} \)

In this case the appearance of the class \( b_l^{[3]} \) from item (4) above removes the factor \( \hat{H}^*(E_l) \) from \( \hat{H}^*(\hat{D}_{C,E}(\sigma)) \), which was supporting the class \( c_i^{[3]} \). This changes the set \( I \subset P \cup Q \) to the set \( I \setminus \{l\} \) and so the indexing link \( \alpha \ast \beta \) changes to \( \iota_i^*(\alpha \ast \beta) \).

(d) \( (\iota_l \circ \rho_I, l,s)^* (\alpha \ast \beta) \otimes c_i^{[5]} \otimes c_j^{[6]} \otimes e_k^{[2]} \otimes b_l^{[3]} \otimes c_s^{[1]} \)

The class \( c_s^{[1]} \) in item (5) above will be zero unless the simplex \( \nu = \sigma \cup \{s\} \) exists in \( K_{I \cup \{s\}} \). We have also the appearance of \( b_l^{[3]} \) as in item (c), necessitating a change from \( I \) to \( I \setminus \{l\} \). According to Lemma 11.15, the indexing link changes from \( \alpha \ast \beta \) to \( (\iota_l \circ \rho_I, l,s)^*(\alpha \ast \beta) \).
(e) \( \iota_k^*(\alpha \ast \beta) \otimes c_i^5 \otimes c_j^6 \otimes b_k^2 \otimes e_l^3 \otimes b_k^1 \)

This time the link is altered by the appearance of \( b_k^2 \) replacing a class in \( E_k' \), so \( I \) is replaced with \( I \{ k \} \) and the link \( \alpha \ast \beta \) is changed to \( \iota_k^*(\alpha \ast \beta) \).

(f) \( (\iota_k \circ \rho_{I \setminus k,s})^*(\alpha \ast \beta) \otimes c_i^5 \otimes c_j^6 \otimes b_k^2 \otimes e_l^3 \otimes c_s^1 \)

The class \( c_s^1 \) in item (5) above will be zero unless the simplex \( \nu = \sigma \cup \{ s \} \) exists in \( K_{I \cup \{ s \}} \). We have also the appearance of \( b_k^2 \) as in item (e), necessitating a change from \( I \) to \( I \{ k \} \). According to Lemma 11.15 the indexing link changes from \( \alpha \ast \beta \) to \( (\iota_k \circ \rho_{I \setminus k,s})^*(\alpha \ast \beta) \).

(g) \( (\iota_l \circ \iota_k)^*(\alpha \ast \beta) \otimes c_i^5 \otimes c_j^6 \otimes b_k^2 \otimes b_l^3 \otimes b_k^1 \)

This time we have the terms \( b_k^2 \) and \( b_l^3 \) appearing, requiring a change from \( I \) to \( I \{ l, k \} \). The indexing link changes from \( \alpha \ast \beta \) to \( (\iota_l \circ \iota_k)^*(\alpha \ast \beta) \).

(h) \( c_i^5 \otimes c_j^6 \otimes b_k^2 \otimes b_l^3 \otimes c_s^1 \)

Here, all three link changing terms \( b_k^2 \), \( b_l^3 \) and \( c_s^1 \) appear. In the manner above, this alters the link from \( \alpha \ast \beta \) to \( (\iota_l \circ \iota_k \circ \rho_{I \setminus \{ l, k \}, s})^*(\alpha \ast \beta) \).

This completes the proof of Theorem 11.13.

11.5. **An example.** The methods of subsection 11.4 are used now to compute multiplicative structure in Example 10.1 for the case \( K = K_2 \) the simplicial complex consisting of two discrete points, so that \( |m| = 2 \). This is the polyhedral product \( Z(K_2; (M_f, \mathbb{CP}^3)) \), where \( M_f \) is the mapping cylinder of the map

\[
f: \mathbb{CP}^3 \to \mathbb{CP}^3 / \mathbb{CP}^1 \xrightarrow{\iota} \mathbb{CP}^8 / \mathbb{CP}^1
\]

and the map \( \iota \) is the inclusion of the bottom two cells. Here

\[
\tilde{H}^*(M_f) \cong k\{b^4, b^6, c^8, c^{10}, c^{12}, c^{14}, c^{16}\} \quad \text{and} \quad \tilde{H}^*(\mathbb{CP}^3) \cong k\{c^2, b^4, b^6\},
\]

where **cohomological degree is denoted by a superscript.** According to (Definition 5.1), we have

\[
B'_1 = k\{b^4, b^6\}, \quad B'_2 = k\{b^4, b^6\}
\]

\[
C'_1 = k\{c^8, c^{10}, c^{12}, c^{14}, c^{16}\}, \quad C'_2 = k\{c^8, c^{10}, c^{12}, c^{14}, c^{16}\}
\]

\[
E'_1 = k\{e_2^2\}, \quad E'_2 = k\{e_2^2\}
\]

where the subscript corresponds to the vertex and distinguishes the two different copies of the modules. Note also that in \( \tilde{H}^*(M_f) \) there is the non-trivial cup product \( c^8_i \cup c^8_i = c^i_{16} \) and in \( \tilde{H}^*(\mathbb{CP}^3) \) we have \( e_2^2 \cup b_4^1 = b_4^1 \).
We consider the case $P = Q = \{1, 2\}$, $J = \{1\}$ and $L = \{1, 2\}$, $\sigma = \tau = \omega = \{1\}$ and begin by determining the shuffle map $\tilde{S}^*$ on terms in $H^*(\Sigma k_{\{1\}}(K_{\{1\}}) \otimes \tilde{H}^*(\tilde{D}_{C, E}^{(1)}(\{1\})) \otimes \tilde{H}^*(B_2)$

$$u = \alpha_{1,\{1\}} \otimes c^8_1 \otimes b^2_1 \in \tilde{H}^*(\Sigma k_{\{1\}}(K_{\{1\}}) \otimes \tilde{H}^*(\tilde{D}_{C, E}^{(1)}(\{1\})) \otimes \tilde{H}^*(B_2)$$

$$v = \alpha_{1,\{2\}} \otimes c^8_1 \otimes e^2 \in \tilde{H}^*(\Sigma k_{\{1\}}(K_{\{1,2\}}) \otimes \tilde{H}^*(\tilde{D}_{C, E}^{(1,2)}(\{1\})))$$

where here, the groups are summands of $\tilde{H}^*(\tilde{Z}(K_P; (U, V)_P)$ and $\tilde{Z}(K_Q; (U, V)_Q)$) given by Theorem 2.2. We apply now the shuffle map as in (11.50) to get

$$\tilde{S}^*(u \otimes v) = \tilde{S}^*(\alpha_{1,\{1\}} \otimes \alpha_{1,\{2\}}) \otimes (c^8_1 \otimes c^8_1) \otimes (b^2_1 \otimes e^2_2)$$

$$= \alpha_{1,\{1,2\}} \otimes (c^8_1 \otimes c^8_1) \otimes (b^2_1 \otimes e^2_2)$$

by (11.20). Next we apply the map $(\tilde{\Delta}^P_{P,\{Q\},\{Q\}})^*(u \otimes v) = i_2^*(\alpha_{1,\{1,2\}}) \otimes c^1_1 \otimes b^6_2 = \alpha_{1,\{1\}} \otimes c^1_1 \otimes b^6_2$.

References

1. A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, Decompositions of the polyhedral product functor with applications to moment-angle complexes and related spaces, PNAS, July, 2009, 106:12241–12244.
2. A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler, The polyhedral product functor: a method of computation for moment-angle complexes, arrangements and related spaces. Advances in Mathematics, 225 (2010), 1634–1668.
3. A. Bahri, M. Bendersky, F. Cohen and S. Gitler, Cup products in generalized moment-angle complexes. Mathematical Proceedings of the Cambridge Philosophical Society, 153, (2012), 457–469.
4. A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler, A spectral sequence for polyhedral products. Advances in Mathematics, 308, (2017), 767–814
5. A. Bahri, M. Bendersky and F.R. Cohen, Polyhedral products and features of their homotopy theory. Handbook of Homotopy Theory, Haynes Miller (ed.) Chapman and Hall/CRC, 2019.
6. I. Baskakov, V. Buchstaber and T. Panov, Cellular cochain complexes and torus actions, Uspekhi. Mat. Nauk, 59, (2004), no. 3, 159–160 (russian); Russian Math. Surveys 89, (2004), no. 3, 562–563 (English translation).
7. V. Buchstaber and T. Panov, Actions of tori, combinatorial topology and homological algebra, Russian Math. Surveys, 55 (2000), 825–921
8. L. Cai, On products in a real moment-angle manifold, J. Math. Soc. Japan, 69(2), (2017), 503–528.
9. L. Cai and S. Choi, Integral cohomology groups of real toric manifolds and small covers.
Online at: https://arxiv.org/pdf/1604.06988.pdf
10. M. Franz, On the integral cohomology of smooth toric varieties.
Online at: https://arxiv.org/format/math/0308253
11. J. Grbić and S. Theriault, The homotopy type of the complement of a coordinate subspace arrangement, Topology 46, (2007), 357–396.
12. J. Grbić and S. Theriault, The homotopy type of the polyhedral product for shifted complexes, Advances in Mathematics, 245, (2013) 690-715. DOI: 10.1016/j.aim.2013.05.002
13. K. Iriye and D. Kishimoto, Decompositions of polyhedral products for shifted complexes, Advances in Mathematics, 245 (2013), 716–736.
14. K. Iriye and D. Kishimoto, Fat wedge filtrations and decomposition of polyhedral products, To appear in Kyoto J. Math. (DOI:10.1215/21562261-2017-0038)
15. S. López de Medrano, Topology of the intersection of quadrics in $\mathbb{R}^n$, Algebraic Topology, Arcata California, (1986), Lecture Notes in Mathematics, 1370 Springer-Verlag, (1989), 280–292.
16. J. Mukai, Some homotopy groups of the double suspension of the real projective space $\mathbb{R}P^6$, 10th Brazilian Topology Meeting, (São Carlos, 1996). Mat. Contemp. 13 (1997), 235–249.
[17] B. Munson, *Cubical Homotopy Theory*, online at: https://web.math.rochester.edu/people/faculty/doug/otherpapers/munson-volic.pdf

[18] Q. Zheng, *The homology coalgebra and cohomology algebra of generalized moment-angle complexes*, Journal of Pure and Applied Algebra, (2012).

[19] Q. Zheng, *The cohomology algebra of polyhedral product spaces*, Journal of Pure and Applied Algebra, 220, (2016), 3752–3776

[20] G. M. Ziegler and R. T. ˇZivaljevi´c, *Homotopy types of subspace arrangements via diagrams of spaces*, Math. Ann. 295, 527–548 (1993). Online at: https://doi.org/10.1007/BF01444901

DEPARTMENT OF MATHEMATICS, RIDER UNIVERSITY, LAWRENCEVILLE, NJ 08648, U.S.A.

E-mail address: bahri@rider.edu

DEPARTMENT OF MATHEMATICS CUNY, EAST 695 PARK AVENUE NEW YORK, NY 10065, U.S.A.

E-mail address: mbenders@hunter.cuny.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14625, U.S.A.

E-mail address: cohf@math.rochester.edu