TOPOLOGICAL STABILITY IN SET-VALUED DYNAMICS

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Abstract. We propose a definition of topological stability for set-valued maps. We prove that a single-valued map which is topologically stable in the set-valued sense is topologically stable in the classical sense [14]. Next, we prove that every upper semicontinuous closed-valued map which is positively expansive [15] and satisfies the positive pseudo-orbit tracing property [9] is topologically stable. Finally, we prove that every topologically stable set-valued map of a compact metric space has the positive pseudo-orbit tracing property and the periodic points are dense in the nonwandering set. These results extend the classical single-valued ones in [1] and [14].

1. Introduction. The topological dynamics of set-valued maps has been studied recently in the literature. For instance, [4], [5] and [8] introduced the metric and topological entropies for set-valued maps. In [11] it is defined the specification and topologically mixing properties for set-valued maps. In [6] it is considered the continuum-wise expansivity for set-valued maps.

In this paper we will propose a definition of topological stability for set-valued maps. We prove that a single-valued map which is topologically stable in the set-valued sense is topologically stable in the classical sense [14]. Next, we prove that every upper semicontinuous closed-valued map which is positively expansive [15] and satisfies the positive pseudo-orbit tracing property [9] is topologically stable. Finally, we prove that every topologically stable set-valued map of a compact metric space has the positive pseudo-orbit tracing property and the periodic points are dense in the nonwandering set. These results extend the classical single-valued ones in [1] and [14].

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2. Definitions and results. We start this section by introducing the concept of topologically stable set-valued map. This will require some basic notions of set-valued analysis [2]. Afterwards, we state our results.

Let $X$ denote a metric space. Denote by $2^X$ the set formed by the subsets of $X$. By a set-valued map of $X$ we mean a map $f : X \to 2^X$. We say that $f$ is single-valued if $\text{card}(f(x)) = 1$ for every $x \in X$, where $\text{card}(\cdot)$ denotes cardinality. There is an obvious correspondence between single-valued maps $f : X \to 2^X$ and maps $f : X \to X$. In what follows all set-valued maps will be assumed to be strict, i.e., $f(x) \neq \emptyset$ for every $x \in X$. A set-valued map $f$ is closed-valued if $f(x)$ is closed for every $x \in X$. We say that $f$ is upper semicontinuous if for every $x \in X$ and every neighborhood $U$ of $f(x)$ there is $\eta > 0$ such that $f(x') \subset U$ for every $x' \in X$ satisfying $d(x, x') < \eta$. This definition reduces to the usual continuity in the single-valued case.

The distance between single-valued maps $f$ and $g$ of $X$ is defined by

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Next we present the classical definition of topologically stable single-valued map by Walters [14].

**Definition 2.1.** A continuous single-valued map $f : X \to X$ is topologically stable, in the class of continuous maps (or topologically stable for short), if for every $\epsilon > 0$ there is $\delta > 0$ such that for every continuous map $g : X \to X$ with $d(f, g) < \delta$ there is a continuous map $\hat{h} : X \to X$ such that

$$d(\hat{h}, \text{Id}_X) < \epsilon \quad \text{and} \quad f \circ \hat{h} = \hat{h} \circ g,$$

where $\text{Id}_X : X \to X$ is the identity.

To extend this definition to the set-valued context we require further notations. Given $A, B \subset X$ we define the distance

$$d(A, B) = \inf\{d(a, b) : (a, b) \in A \times B\},$$

and the Hausdorff distance

$$d_H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}.$$  \hspace{1cm} (1)

The distance between the set-valued maps $f$ and $g$ of $X$ is defined by

$$d_H(f, g) = \sup_{x \in X} d_H(f(x), g(x)).$$

Notice that $d_H(f, g)$ reduces to the distance $d(f, g)$ when the involved set-valued maps $f$ and $g$ are single-valued.

In what follows $\mathbb{N}$ will denote the set of nonnegative integers, i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Denote by

$$X^\mathbb{N} = \prod_{n \in \mathbb{N}} X$$

the infinite product of copies of $X$, equipped with the distance

$$d^*((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} 2^{-n-1}d(x_n, y_n).$$  \hspace{1cm} (1)
Another distance to be considered in $X^N$ is

$$D((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sup_{n \in \mathbb{N}} d(x_n, y_n).$$

(2)

We say that $(x_n)_{n \in \mathbb{N}} \subseteq X^N$ is an orbit of a set-valued map $f$ (or an $f$-orbit for short) if

$$x_{n+1} \in f(x_n), \quad \forall n \in \mathbb{N}.$$ 

The set $\lim f$ formed by the $f$-orbits is often called the inverse limit space induced by $f$ (cf. [8]). The name inverse limit system is also used (cf. [1]). Precisely,

$$\lim f = \{ (x_n)_{n \in \mathbb{N}} \in X^N : x_{n+1} \in f(x_n), \forall n \in \mathbb{N} \}.$$ 

It turns out that $f$ induces a map, to be called left shift

$$\sigma_f : \lim f \to \lim f,$$ 

defined by

$$\sigma_f((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}.$$ 

Let $\pi : X^N \to X$ the projection in the first variable, i.e., $\pi((x_n)_{n \in \mathbb{N}}) = x_0$. Define the map $\pi_f : \lim f \to X$ as the restriction of $\pi$ to $\lim f$.

Now we present our definition of topologically stable set-valued map.

**Definition 2.2.** An upper semicontinuous closed-valued map $f$ of $X$ is topologically stable, in the class of upper semicontinuous closed-valued maps (or topologically stable for short), if for every $\epsilon > 0$ there is $\delta > 0$ such that for every upper semicontinuous closed-valued map $g$ with $d_H(f, g) < \delta$ there is a continuous map $h : (\lim \leftarrow g, d^*) \to (\lim \leftarrow f, d^*)$ such that

$$D(h, Id_X) < \epsilon \quad \text{and} \quad \sigma_f \circ h = h \circ \sigma_g,$$ 

where

$$D(h, Id_X) = \sup \{ D(h(x), x) : x \in \lim \leftarrow g \}.$$ 

The following remark holds.

**Remark 2.1.** An important difference between definitions 2.1 and 2.2 is that the domain of the semiconjugacy $h$ in the latter definition depends on the perturbation $g$.

Since every continuous single-valued map is upper semicontinuous and closed valued as a set-valued map, it is natural to compare the definitions 2.1 and 2.2 in the single-valued context. This motivates the following result.

**Theorem 2.1.** Every continuous single-valued map of a metric space which is topologically stable as a set-valued map (Definition 2.2) is topologically stable in the classical sense (Definition 2.1).

Unfortunately we do not know if the converse of Theorem 2.1 holds, namely, if a single-valued map which is topologically stable in the classical sense (Definition 2.1) is also topologically stable when regarded as a set-valued map (Definition 2.2). The next theorem (and Example 2.1 below) give some light to this question.

**Theorem 2.2.** Every topologically stable single-valued map $f$ of a compact metric space $X$ satisfies the following property:
For every $\epsilon > 0$ there is $\delta > 0$ such that for every continuous single-valued map $g : X \to X$ with $d_H(f, g) < \delta$ there is a continuous map $h : \lim \left( \lim g, d^* \right) \to \lim \left( f, d^* \right)$ such that

$$D(h, Id_X) < \epsilon \quad \text{and} \quad \sigma_f \circ h = h \circ \sigma_g.$$ 

In [13] Walters proved that every positively expansive map with the positive pseudo-orbit tracing property of a compact metric space is topologically stable. Now we extend this result to the set-valued context. Previously we recall the concepts of positive expansivity and pseudo-orbit tracing property in the set-valued context.

Definition 2.3 ([15]). A set-valued map $f$ of a metric space $X$ is positively expansive if there is $\epsilon > 0$ (called positive expansivity constant) such that $x = y$ whenever $x, y \in X$ satisfy that there are $f$-orbits $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ such that $x_0 = x, y_0 = y$ and $d(x_n, y_n) \leq \epsilon$ for every $n \in \mathbb{N}$. Sometimes we will say that $f$ is positively expansive with respect to $d$ to emphasize the metric $d$ of $X$.

Definition 2.4 ([9]). We say that a set-valued map $f$ of a metric space $X$ has the positive pseudo-orbit tracing property (abbrev. POTP$_+$) if for every $\epsilon > 0$ there is $\delta > 0$ such that for each sequence $(p_n)_{n \in \mathbb{N}}$ in $X$ satisfying $d(p_{n+1}, f(p_n)) \leq \delta$, $\forall n \in \mathbb{N}$, there is an $f$-orbit $(q_n)_{n \in \mathbb{N}}$ satisfying $d(p_n, q_n) \leq \epsilon$, $\forall n \in \mathbb{N}$.

These definitions extend the classical single-valued ones by Utz [12], Eisenberg [7] and Bowen [3]. Using them we obtain the following set-valued version of Walters stability theorem [13].

Theorem 2.3. Every upper semicontinuous positively expansive closed-valued map with the POTP$_+$ of a compact metric space is topologically stable.

Let us present two examples where Theorem 2.3 applies.

Example 2.1. Let $f : X \to X$ a continuous positively expansive single-valued map with the POTP$_+$ of a compact metric space. Then, $f$ is an upper semicontinuous positively expansive closed-valued map with the POTP$_+$. Hence, by Theorem 2.3 $f$ is topologically stable not only as a single but also as a set-valued map.

A genuine (i.e. not single-valued) example where the theorem applies is as follows.

Example 2.2. Endow the unit interval $[0, 1]$ with the Euclidean metric. Define the set-valued map $f$ of $[0, 1]$ by

$$f(x) = \begin{cases} 
\{2x\}, & \text{if } 0 \leq x < \frac{1}{2} \\
\{0, 1\}, & \text{if } x = \frac{1}{2} \\
\{2x - 1\}, & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}$$

It follows that $f$ is an upper semicontinuous positively expansive closed-valued map with the POTP$_+$ of $[0, 1]$. Therefore, by Theorem 2.3 $f$ is a topologically stable set-valued map of $[0, 1]$. 
Next we present a property of the topologically stable set-valued maps.

Given a set-valued map \( f \) of \( X \), we say that \( x \in X \) is a periodic point if there are an \( f \)-orbit \( (x_n)_{n \in \mathbb{N}} \) and \( m \in \mathbb{N}^+ \) such that \( x_0 = x \) and \( x_{n+m} = x_n \) for every \( n \in \mathbb{N} \). The set of periodic points is denoted by \( \text{Per}(f) \). The nonwandering set of \( f \) is the set \( \Omega(f) \) of those points \( x \in X \) such that for every neighborhood \( U \) of \( x \) there is \( m \in \mathbb{N}^+ \) satisfying \( U \cap f^m(U) \neq \emptyset \). With these definitions we obtain the following result.

**Theorem 2.4.** Every topologically stable upper semicontinuous closed-valued map of a compact metric space has the POTP\(_+\). Moreover, \( \text{Per}(f) \) is dense in \( \Omega(f) \).

A short application of this theorem in the single-valued context is as follows. Recall that, on every compact manifold, every single-valued map \( f \) which is topologically stable in the classical sense has the POTP\(_+\) and \( \text{Per}(f) \) is dense in \( \Omega(f) \). See for instance Theorem 2.4.8 in [1] or [13].

In the following corollary of Theorem 2.4 and Theorem 2.1 we obtain that, on every metric space, every single-valued map \( f \) which is topologically stable as a set-valued map (Definition 1.4) has the POTP\(_+\) and \( \text{Per}(f) \) is dense in \( \Omega(f) \). In other words we have the following result.

**Corollary 2.5.** Every continuous single-valued map \( f \) of a metric space which is topologically stable as a set-valued map (Definition 2.2) has the POTP\(_+\). Moreover, \( \text{Per}(f) \) is dense in \( \Omega(f) \).

3. **Proof of the theorems.** In this section we will prove the theorems stated in the previous section. We start with a lemma about the left shift map for single-valued maps.

**Lemma 3.1.** If \( f \) is a continuous single-valued map of a compact metric space \( X \), then the left shift \( \pi_f : (\lim - \downarrow, d^\star) \to (X, d) \) is a homeomorphism.

**Proof.** Since \( f \) is single-valued, one has \( \pi_f((x_n)_{n \in \mathbb{N}}) = x \) if and only if \( x_n = f^n(x) \) for every \( n \in \mathbb{N} \). Then, \( \pi_f \) is bijective with inverse \( \pi_f^{-1}(x) = (f^n(x))_{n \in \mathbb{N}} \). Also, for fixed \( \gamma > 0 \), if \( d^\star((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) < \frac{\gamma}{2} \), then

\[
d(\pi_f((x_n)_{n \in \mathbb{N}}), \pi_f((y_n)_{n \in \mathbb{N}})) = d(x_0, y_0) \leq 2d^\star((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) < \gamma
\]

proving that \( \pi_f \) is continuous.

On the other hand, for fixed \( \gamma > 0 \) we let \( \text{diam}(X) \) denote the diameter of \( X \) and we let \( n_0 \in \mathbb{N} \) be such that

\[
\sum_{n \geq n_0} 2^{-n-1}\text{diam}(X) < \frac{\gamma}{2}
\]

Since \( f \) is continuous, there is \( \rho > 0 \) such that

\[
\sum_{n=0}^{n_0-1} 2^{-n-1}d(f^n(x), f^n(y)) < \frac{\gamma}{2} \quad \text{whenever } d(x, y) < \rho.
\]

Then,

\[
d^\star(\pi_f^r(x), \pi_f^r(y)) = d^\star((f^n(x))_{n \in \mathbb{N}}, (f^n(y))_{n \in \mathbb{N}})
= \sum_{n \in \mathbb{N}} 2^{-n-1}d(f^n(x), f^n(y))
\]
Proof of Theorem 2.2. Fix $\epsilon > 0$ and let $\delta$ be given by that property. Take $g : X \to X$ continuous such that $d(f,g) < \delta$. Since $f$ and $g$ are single-valued, $d_H(f,g) = d(f,g)$ and so $d_H(f,g) < \delta$. Then, there is $h : (\lim g, d^*) \to (\lim f, d^*)$ continuous such that $D(h, Id_X) \leq \epsilon$ and $\sigma_f \circ h = h \circ \sigma_g$.

By Lemma 3.1 since both $f$ and $g$ are single-valued, we have that the maps $\pi_f : (\lim f, d^*) \to (X, d)$ and $\pi_g : (\lim g, d^*) \to (X, d)$ are homeomorphisms. Then, the composition $\hat{h} = \pi_f \circ h \circ \pi_g^{-1}$ defines a continuous map $\hat{h} : X \to X$. Since $d(\hat{h}(x), x) = d(\pi_f(h(\pi_g^{-1}(x))), x) = d(\pi_f(h((g^n(x))_{n\in\mathbb{N}})), x) \leq D(h, Id_X) \leq \epsilon$

for every $x \in X$, one has $d(\hat{h}, Id_X) \leq \epsilon$.

In addition, since $f \circ \pi_f = \pi_f \circ \sigma_f$, one has

$$(f \circ \hat{h})(x) = f(\hat{h}(x)) = f(\pi_f(h(\pi_g^{-1}(x))))$$

$$= f(\pi_f(h((g^n(x))_{n\in\mathbb{N}})))$$

$$= \pi_f(\sigma_f(h((g^n(x))_{n\in\mathbb{N}})))$$

$$= \pi_f(h(\sigma_g((g^n(x))_{n\in\mathbb{N}})))$$

$$= \pi_f(h(g^{n+1}(x))_{n\in\mathbb{N}}))$$

$$= (\pi_f \circ h \circ \pi_g^{-1})(g(x)) = (\hat{h} \circ g)(x)$$

i.e., $f \circ \hat{h} = \hat{h} \circ g$. Then, $f$ is topologically stable according to Definition 2.1.

Next we prove Theorem 2.2. Fix $\epsilon > 0$ and let $\delta$ be given by the topological stability of $f$. Take $g : X \to X$ continuous such that $d_H(f,g) < \delta$. Then, $d(f,g) < \delta$ and so there is $h : X \to X$ continuous such that $d(h, Id_X) \leq \epsilon$ and $f \circ h = h \circ g$.

On the other hand, by Lemma 3.1 we have that $\pi_f : (\lim f, d^*) \to (X, d)$ and $\pi_g : (\lim g, d^*) \to (X, d)$ are homeomorphisms. Then, since $h$ is continuous, the composition $h = \pi_f^{-1} \circ h \circ \pi_g$ defines a continuous map $h : (\lim g, d^*) \to (\lim f, d^*)$.

Since $g$ is single-valued, $x_n = g^n(x_0)$ for all $(x_n)_{n\in\mathbb{N}} \in \lim g$ and $n \in \mathbb{N}$. Then,

$$D(h((x_n)_{n\in\mathbb{N}}), (x_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} d(\hat{h}(g^n(x_0)), g^n(x_0)) \leq \epsilon$$

for all $(x_n)_{n\in\mathbb{N}} \in \lim g$ proving $D(h, Id_X) \leq \epsilon$.
Moreover,
\[
(\sigma_f \circ h)((x_n)_{n \in \mathbb{N}}) = \sigma_f(h((x_n)_{n \in \mathbb{N}})) \\
= \sigma_f(\pi_f^{-1}(\hat{h}(\pi_f((x_n)_{n \in \mathbb{N}})))) \\
= \pi_f^{-1}(f(\hat{h}(x_0))) \\
= \pi_f^{-1}(\hat{h}(g(x_0))) \\
= (\pi_f^{-1} \circ \hat{h} \circ \pi_f)(\sigma_g((x_n)_{n \in \mathbb{N}})) \\
= (h \circ \sigma_g)((x_n)_{n \in \mathbb{N}})
\]
proving \(\sigma_f \circ h = h \circ \sigma_g\). Since \(\epsilon\) is arbitrary, \(f\) satisfies the required property and the proof follows. \(\square\)

To prove the remainder theorems we need some short preliminaries. The first one is a basic property of the upper semicontinuous closed valued maps (see Proposition 1.4.8 in [2]).

**Lemma 3.2.** Let \(f\) be an upper semicontinuous closed-valued map of a compact metric space \(X\). If \((a^k)_{k \in \mathbb{N}}\) and \((b^k)_{k \in \mathbb{N}}\) are sequences such that \(a^k \to a\), \(b^k \to b\) and \(a^k \in f(b^k)\) for all \(k \in \mathbb{N}\), then \(a \in f(b)\).

Since \(\lim \limits_{\leftarrow} f = \sigma_f(X)\) we obtain the following lemma.

**Lemma 3.3.** The limit inverse space \((\lim \limits_{\leftarrow} f, d^\ast)\) of an upper semicontinuous closed-valued map \(f\) of a compact metric space \(X\) is a compact subset of \((X^\mathbb{N}, d^\ast)\).

For the next lemma we will use an auxiliary definition.

**Definition 3.1.** We say that a set-valued map \(f\) of a metric space \(X\) has the finite shadowing property if for every \(\epsilon > 0\) there is \(\delta > 0\) such that for every finite set \(\{p_0, \ldots, p_m\}\) satisfying \(d(p_{n+1}, f(p_n)) < \delta\) for every \(0 \leq n \leq m-1\) there is a finite set \(\{q_0, \ldots, q_m\}\) such that \(q_{n+1} \in f(q_n)\) and \(d(p_n, q_n) < \epsilon\) for every \(0 \leq n \leq m-1\).

With this definition we obtain the following result.

**Lemma 3.4.** An upper-semicontinuous closed-valued map of a compact metric space has the \(\text{POTP}_+\) if and only if it has the finite shadowing property.

**Proof.** We only need to prove the sufficiency. Let \(f\) be an upper semicontinuous closed-valued map with the finite shadowing property of a compact metric space \(X\). Let \(\epsilon > 0\) be given. Find a corresponding \(\delta > 0\) given by the finite shadowing property. Let \((p_n)_{n \in \mathbb{N}}\) be a sequence satisfying \(d(p_{n+1}, f(p_n)) \leq \delta\) for every \(n \in \mathbb{N}\). Then, by finite shadowing, for every \(m \in \mathbb{N}\) there is a sequence \(\{q_{m_0}^m, \ldots, q_{m_0}^m\}\) such that \(q_{n+1}^m \in f(q_n^m)\) and \(d(p_n^m, q_n^m) \leq \epsilon\) for every \(0 \leq n \leq m\). Since \(X\) is compact, we can assume by passing to subsequences if necessary that there is a sequence \((q_n)_{n \in \mathbb{N}}\) such that \(q_n^m \to q_n\) as \(m \to \infty\) for every \(n \in \mathbb{N}\). Since \(f\) is upper semicontinuous, closed-valued and \(X\) is compact, Lemma 3.2 implies \((q_n)_{n \in \mathbb{N}} \in \lim \limits_{\leftarrow} f\). By fixing \(n\) in \(d(p_n, q_n^m) \leq \epsilon\) and letting \(m \to \infty\) we obtain \(d(p_n, q_n^m) \leq \epsilon\) for every \(n \in \mathbb{N}\). Then, \(f\) has the \(\text{POTP}_+\) proving the result. \(\square\)

The next lemma is about the expansivity of the shift map for positively expansive set-valued maps.
Lemma 3.5. If \( f \) is a positively expansive set-valued map of a metric space \( X \), then the left shift \( \sigma_f : \lim f \rightarrow \lim f \) is positively expansive with respect to the metric \( d^* \) in \([7]\).

Proof. Let \( \epsilon \) be a positive expansivity constant of \( f \). Take \((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \in \lim f \) such that
\[
d^* \left( \sigma_f^k((x_n)_{n \in \mathbb{N}}), \sigma_f^k((x'_n)_{n \in \mathbb{N}}) \right) \leq 2^{-1} \epsilon, \quad \forall k \in \mathbb{N}.
\]
It follows that
\[
\sum_{n \in \mathbb{N}} 2^{-n-1}d(x_{n+k}, x'_{n+k}) \leq 2^{-1} \epsilon, \quad \forall k \in \mathbb{N}.
\]
Since
\[
2^{-1}d(x_k, x'_k) \leq \sum_{n \in \mathbb{N}} 2^{-n-1}d(x_{n+k}, x'_{n+k})
\]
we obtain
\[
d(x_k, x'_k) \leq \epsilon, \quad \forall k \in \mathbb{N}.
\]
Since \( \epsilon \) is a positive expansivity constant of \( f \), \((x_k)_{k \in \mathbb{N}} = (x'_k)_{k \in \mathbb{N}} \) so \( \sigma_f \) is positively expansive.

The following result is the positively expansive version of Lemma 2 in [13] (with similar proof).

Lemma 3.6. Let \( r : Y \rightarrow Y \) be a positively expansive continuous map of a compact metric space \( Y \). Then, for every positive expansivity constant \( \epsilon \) and every \( \Delta > 0 \) there is \( N \geq 1 \) such that \( d(x, y) \leq \Delta \) whenever \( x, y \in Y \) satisfy \( d(r^k(x), r^k(y)) \leq \epsilon \) for every \( 0 \leq k \leq N \).

Next we prove the continuity of the left shift.

Lemma 3.7. For every set-valued map \( g \) of a metric space \( X \), the left shift \( \sigma_g : (\lim g, d^*) \rightarrow (\lim g, d^*) \) is continuous.

Proof. If \((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \in \lim g\), then
\[
d^* \left( \sigma_g((x_n)_{n \in \mathbb{N}}), \sigma_g((x'_n)_{n \in \mathbb{N}}) \right) = d^* \left( (x_{n+1})_{n \in \mathbb{N}}, (x'_{n+1})_{n \in \mathbb{N}} \right) = \sum_{n \in \mathbb{N}} 2^{-n-1}d(x_{n+1}, x'_{n+1}) = \sum_{n \geq 1} 2^{-n}d(x_n, x'_n) \leq 2 \sum_{n \in \mathbb{N}} 2^{-n-1}d(x_n, x'_n) = 2d^* \left( (x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \right)
\]
proving
\[
d^* \left( \sigma_g((x_n)_{n \in \mathbb{N}}), \sigma_g((x'_n)_{n \in \mathbb{N}}) \right) \leq 2d^* \left( (x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \right), \quad \forall (x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \in \lim g.
\]
This completes the proof. □

Proof of Theorem [2.3] Let \( f \) be an upper semicontinuous positively expansive closed-valued map with the POTP \( \ast \) of a compact metric space \( X \). It follows from Lemma 3.5 that the left shift \( \sigma_f : \lim f \rightarrow \lim f \) is positively expansive with respect to the metric \( d^* \) in \([1]\). Let \( \hat{\epsilon} \) be the corresponding positive expansivity constant.

\[\]
Fix $\epsilon > 0$ and let $\delta$ be given from POTP$_+$ for the constant $\epsilon_0 = \min\{\epsilon, e, \hat{e}\}$, where $e$ is the positive expansivity constant of the set-valued map $f$. Fix a set-valued map $g$ such that $d_H(f, g) \leq \frac{\delta}{8}$.

Let $(x_n)_{n \in \mathbb{N}}$ be a $g$-orbit. Since $d_H(g(x_0), f(x_0)) \leq \frac{\delta}{8}$ (by hypothesis) and $x_1 \in g(x_0)$, we have
\[ d(x_1, f(x_0)) < \delta. \]

Similarly, since $d_H(g(x_1), f(x_1)) \leq \frac{\delta}{8}$ and $x_2 \in g(x_1)$, we have
\[ d(x_2, f(x_1)) < \delta. \]

Repeating this argument we conclude that
\[ d(x_{n+1}, f(x_n)) < \delta, \quad \forall n \in \mathbb{N}. \]

Then, by the POTP$_+$ and the choice of $\delta$, there is an $f$-orbit $(y_n)_{n \in \mathbb{N}}$ such that
\[ d(x_n, y_n) \leq \epsilon_0, \quad \forall n \in \mathbb{N}. \tag{3} \]

It turns out that this $f$-orbit is unique. Indeed, any other $f$-orbit $(y'_n)_{n \in \mathbb{N}}$ satisfying
\[ d(x_n, y'_n) \leq \epsilon_0, \quad \forall n \in \mathbb{N}, \]

must satisfy
\[ d(y_n, y'_n) \leq 2\epsilon_0 = \frac{\min\{\epsilon, e, \hat{e}\}}{4} < e, \quad \forall n \in \mathbb{N}, \]

and so $(y_n)_{n \in \mathbb{N}} = (y'_n)_{n \in \mathbb{N}}$ because $e$ is a positive expansivity constant of $f$.

From this uniqueness, we obtain a map $h : \lim g \rightarrow \lim f$ given by $h((x_n)_{n \in \mathbb{N}}) = (y_n)_{n \in \mathbb{N}}$. It follows from (3) that
\[ D(h, Id_X) \leq \epsilon. \]

On the other hand, replacing $n$ by $n + 1$ in (3) we get $d(x_{n+1}, y_{n+1}) \leq \epsilon_0$ for every $n \in \mathbb{N}$. Then, $(y_{n+1})_{n \in \mathbb{N}} = h((x_{n+1})_{n \in \mathbb{N}})$ and so
\[ \sigma_f(h((x_n)_{n \in \mathbb{N}})) = (y_{n+1})_{n \in \mathbb{N}} = h((x_{n+1})_{n \in \mathbb{N}}) = h(\sigma_g((x_n)_{n \in \mathbb{N}})), \quad \forall (x_n)_{n \in \mathbb{N}} \in \lim g. \]

This proves
\[ \sigma_f \circ h = h \circ \sigma_g. \]

It remains to prove that $h$ is continuous.

Fix $\Delta > 0$.

By lemmas 3.3 and 3.5 the map $\sigma_f : \lim f \rightarrow \lim f$ is a positively expansive map of the compact metric space $Y = (\lim f, d^*)$. But $\sigma_f : (\lim f, d^*) \rightarrow (\lim f, d^*)$ is also continuous by Lemma 3.7. Then, we can apply Lemma 3.6 to obtain an integer $N \geq 1$ for the given $\Delta$. Since $\sigma_g : (\lim g, d^*) \rightarrow (\lim g, d^*)$ is continuous and $(\lim g, d^*)$ compact by lemmas 3.7 and 3.3 respectively, there is $\gamma > 0$ such that
\[ d^*(\sigma^k_g((x_n)_{n \in \mathbb{N}}), \sigma^k_g((x'_{n})_{n \in \mathbb{N}})) < \frac{\gamma}{4}, \quad \forall 0 \leq k \leq N, \]

whenever $(x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \in \lim g$ satisfy $d^*((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}}) < \gamma$. 
Then, whenever \((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \in \lim g\) satisfy \(d^*((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}}) < \gamma\), one has for \((y_n)_{n \in \mathbb{N}} = h((x_n)_{n \in \mathbb{N}})\) and \((y'_n)_{n \in \mathbb{N}} = h((x'_n)_{n \in \mathbb{N}})\) that

\[
d^*\left(\sigma_f^k((y_n)_{n \in \mathbb{N}}), \sigma_f^k((y'_n)_{n \in \mathbb{N}})\right) = d^*\left(h\left(\sigma_f^k((x_n)_{n \in \mathbb{N}})\right), h\left(\sigma_f^k((x'_n)_{n \in \mathbb{N}})\right)\right) \\
\leq d^*\left(h\left(\sigma_f^k((x_n)_{n \in \mathbb{N}})\right), \sigma_f^k((x'_n)_{n \in \mathbb{N}})\right) + \\
\leq d^*\left(\sigma_f^k((x_n)_{n \in \mathbb{N}}), \sigma_f^k((x'_n)_{n \in \mathbb{N}})\right) + \\
\leq \frac{\hat{e}}{4} + \frac{\hat{e}}{4} + \frac{\hat{e}}{4} \\
= 3\hat{e} \\
< \hat{e}, \quad \forall 0 \leq k \leq N.
\]

Therefore, by Lemma 3.6

\[
d^*\left(h\left((x_n)_{n \in \mathbb{N}}\right), h\left((x'_n)_{n \in \mathbb{N}}\right)\right) < \Delta,
\]

whenever \((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}} \in \lim g\) satisfy \(d^*((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}}) < \gamma\). This proves the continuity of \(h\) and completes the proof of the theorem.

**Proof of Theorem 2.4.** Let \(f : X \to X\) be a topologically stable upper semicontinuous closed-valued map of a compact metric space \(X\).

First we prove that \(f\) has the finite shadowing property. Fix \(\epsilon > 0\) and let \(\delta > 0\) be given by the topological stability of \(f\). Let \(\{p_0, \cdots, p_m\}\) be a finite set satisfying

\[
d(p_{n+1}, f(p_n)) \leq \frac{\delta}{8}, \quad \forall 0 \leq n \leq m - 1.
\]

Define the set-valued map

\[
g(x) = \begin{cases} 
\{f(x)\}, & \text{if } x \notin \{p_0, p_1, \cdots, p_m\} \\
B[f(p_n), \frac{\delta}{4}], & \text{if } x = p_n \text{ for some } n \in \{0, \cdots, m\}.
\end{cases}
\]

Clearly \(d_H(f, g) \leq \delta\). Moreover, since \(f\) is closed-valued, \(g\) also is. Furthermore, since \(\{p_0, \cdots, p_m\}\) is a finite set and \(f\) is upper semicontinuous, we have that \(g\) is upper semicontinuous. Then, by the choice of \(\delta\), there exists \(h : (\lim g, d^*) \to (\lim f, d^*)\) continuous such that \(D(h, Id_X) \leq \epsilon\) and \(\sigma_f \circ h = h \circ \sigma_g\). On the other hand, it follows from the definition that \(p_{n+1} \in g(p_n)\) for every \(0 \leq n \leq m - 1\). Then, since \(f\) (and so \(g\)) are strict, we can complete \(\{p_0, \cdots, p_m\}\) to a \(g\)-orbit \((p_n)_{n \in \mathbb{N}}\) and so \((q_n)_{n \in \mathbb{N}} = h((p_n)_{n \in \mathbb{N}})\) is a well-defined \(f\)-orbit. Since \(D(h, Id_X) \leq \epsilon\) we have \(d(p_n, q_n) \leq \epsilon\) for every \(n \in \mathbb{N}\). In particular, \(q_{n+1} \in f(q_n)\) and \(d(p_n, q_n) < \epsilon\) for every \(0 \leq n \leq m - 1\) proving the finite shadowing property. Then, \(f\) has the POTP\(_+\) by Lemma 3.4.

Next we prove that \(Per(f)\) is dense in \(\Omega(f)\). Fix \(\epsilon > 0\) and \(x \in \Omega(f)\). For this \(\epsilon\) we let \(\delta > 0\) be given by topological stability. Since \(x \in \Omega(f)\), there are \(m \in \mathbb{N}^+\) and a finite sequence \(\{z_0, z_1, \cdots, z_m\}\) such that \(z_0, z_m \in B(x, \frac{\epsilon}{4})\) and \(z_{n+1} \in f(z_n)\) for every \(0 \leq n \leq m - 1\). Define the set-valued map

\[
g(x) = \begin{cases} 
f(x), & \text{if } x \notin \{z_0, z_1, \cdots, z_m\} \\
B[f(z_n), \frac{\delta}{4}], & \text{if } x = z_n \text{ for some } n \in \{0, \cdots, m\}.
\end{cases}
\]

As before we have that \(g\) is upper semicontinuous, closed-valued and \(d_H(f, g) \leq \delta\). Then, by the choice of \(\delta\), there is \(h : (\lim g, d^*) \to (\lim f, d^*)\) continuous such
that $D(h, Id_X) \leq \epsilon$ and $\sigma_f \circ h = h \circ \sigma_g$. Now define the sequence $(x_n)_{n \in \mathbb{N}}$ by $x_{lm+r} = z_r$ whenever $l \in \mathbb{N}$ and $0 \leq r \leq m - 1$. It follows that $(x_n)_{n \in \mathbb{N}} \in \lim g$. Moreover, since for all $n \in \mathbb{N}$ there are $l \in \mathbb{N}$ and $0 \leq r \leq m - 1$ such that $n = lm + r$, one has $x_{n+m} = x_{(l+1)m+r} = z_r = x_{lm+r} = x_{n+m}$. It follows that $\sigma_{m}^{g}(x_{n})_{n \in \mathbb{N}} = (x_{n})_{n \in \mathbb{N}}$. Therefore, the $f$-orbit $(y_{n})_{n \in \mathbb{N}} = h((x_{n})_{n \in \mathbb{N}})$ is well defined. Since $\sigma_{m}^{g}(x_{n})_{n \in \mathbb{N}} = (x_{n})_{n \in \mathbb{N}}$, one has $\sigma_{m}^{f}(y_{n})_{n \in \mathbb{N}} = (y_{n})_{n \in \mathbb{N}}$ and so $y_0 \in \text{Per}(f)$. Moreover, since $D(h, Id_X) \leq \epsilon$, we have $d(y_0, x) \leq \epsilon$ proving the result.

REFERENCES

[1] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems. Recent Advances*, North-Holland Mathematical Library, 52, North-Holland Publishing Co., Amsterdam, 1994.

[2] J.-P. Aubin and H. Frankowska, *Set-valued Analysis*, Reprint of the 1990 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009.

[3] R. Bowen, *ω-limit sets for axiom A diffeomorphisms*, J. Differential Equations, 18 (1975), 333–339.

[4] D. Carrasco-Olivera, A. R. Metzger and C. A. Morales, Logarithmic expansion, entropy and dimension for set-valued maps, Preprint, (2016), to appear.

[5] D. Carrasco-Olivera, R. Metzger Alvan and C. A. Morales, *Topological entropy for set-valued maps* Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), 3461–3474.

[6] W. Cordeiro and M. J. Pacífico, *Continuum-wise expansiveness and specification for set-valued functions and topological entropy* Proc. Amer. Math. Soc., 144 (2016), 4261–4271.

[7] M. Eisenberg, Expansive transformation semigroups of endomorphisms, Fund. Math., 59 (1966), 313–321.

[8] J. P. Kelly and T. Tennant, Topological entropy for set-valued functions, arXiv:1509.08413.

[9] S. Y. Pilyugin and J. Rieger, Shadowing and inverse shadowing in set-valued dynamical systems. Contractive case, Topol. Methods Nonlinear Anal., 32 (2008), 139–149.

[10] S. Y. Pilyugin, *Shadowing in Dynamical Systems*, Lecture Notes in Mathematics, 1706. Springer-Verlag, Berlin, 1999.

[11] B. E. Raines and T. Tennant, The specification property on a set-valued map and its inverse limit, arXiv:1509.08415.

[12] W. R. Utz, Unstable homeomorphisms Proc. Amer. Math. Soc., 1 (1950), 769–774.

[13] P. Walters, On the pseudo-orbit tracing property and its relationship to stability. The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977), Lecture Notes in Math., Springer, Berlin, 668 (1978), 231–244.

[14] P. Walters, Anosov diffeomorphisms are topologically stable, Topology, 9 (1970), 71–78.

[15] R. K. Williams, A note on expansive mappings Proc. Amer. Math. Soc., 22 (1969), 145–147.

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