F Theory on $K3 \times K3$ and

Instantons on 7-branes

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ABSTRACT

We discuss $N = 2$ supersymmetric compactifications to four dimensions from the point of view of F-theory and heterotic theory. In a relatively simple setup, we illustrate the spectral theory for vector bundles on $K3 \times T^2$ and discuss the heterotic – F-theory map. The moduli space of instantons on the 7-brane wrapped around $K3$ is discussed from the point of view of Higgs mechanism in the effective four-dimensional $N = 2$ theory. This allows us to elaborate on the transitions between various branches of the moduli space of the heterotic/F-theory. We also describe the F-theory compactification on smooth $K3 \times K3$ without the 3-branes in which the anomaly is cancelled by the gauge fields.

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1. Introduction

F-theory has proved to be a very effective tool in studying string theory in $D \geq 5$ dimensions \[1\] \[2\] \[3\]. Partially this can be attributed to relatively simple dynamics of higher-dimensional gauge theories, which can be captured by the geometry of elliptic fibrations. Compactifications down to four dimensions are different in this respect. Various perturbative and nonperturbative phenomena occur in the effective field theory, some of them do not have any straightforward interpretation in terms of elliptic fibrations. The first question that arises is simply whether there is a moduli space, or all flat directions are lifted due to the superpotential. At present, we do not know how to answer this question completely, although the mechanism for generating the superpotential is known and it the superpotential is shown to be generated in some examples \[4\] \[5\].

In this paper we discuss the F-theory compactification on $K3 \times K3$ which possesses $N = 2$ supersymmetry in four dimensions. This allows to avoid the question about the superpotential and makes the discussion of the moduli space much easier. Similarly to a generic F-theory compactification on the Calabi-Yau fourfold, the compactification on $K3 \times K3$ has a three-brane anomaly of $-24$ which can be cancelled by either 24 3-branes \[6\] or by the nontrivial background gauge fields on the 7-branes. Thus there are two kinds of moduli, the ones associated with 3-branes and the others associated with 7-branes. The 3-brane in the bulk (away from 7-branes) carries an $N = 4$ vector multiplet which consists of $N = 2$ vector multiplet and a hypermultiplet in the adjoint. The corresponding gauge theory is in the Coulomb phase with the complex scalar of the vector multiplet measuring the position of the 3-brane with respect to 7-branes. The 7-brane carries an $N = 2$ vector multiplet as well as hypermultiplets in various representations. The background gauge bundle on the compact part of the 7-brane worldvolume breaks the gauge symmetry down from the maximal one dictated by the singularity of elliptic fibration. In terms of the effective field theory in four dimensions this symmetry breaking can be interpreted as Higgs mechanism in the maximal gauge theory, which leads to identification of the Higgs branch of moduli space with the moduli space of instantons. In fact, we will see that this identification is incomplete in the sense that the moduli space of instantons carries a bit more information about the string theory compactification.

As a 3-brane approaches the 7-brane, the other kind of hypermultiplets become massless. They come from strings connecting 3-brane to the 7-brane. The expectation values of these hypermultiplets become a part of coordinates of the Higgs moduli space. Since the
3-brane on top of the 7-brane with these expectation values turned on is indistinguishable from an instanton in the background gauge bundle on the 7-brane, we refer to this process as the 3-brane – instanton transition \[7\] \[8\]. This transition does not change the total value of the 3-brane anomaly.

In general in $N = 2$ supersymmetric gauge theories the (hyperkähler) Higgs branch does not receive quantum corrections while the Coulomb branch does. Thus it is important to understand to what extent the above geometric description accounts for these corrections. As it will become clear, the geometric Coulomb moduli space of 3-branes is not corrected while the geometric moduli space of 7-branes (the complex structure of elliptic fibration) does get quantum corrections. In particular, the geometric singularity of elliptic fiber on the 7-brane predicts certain gauge symmetry. When it is unbroken by the background gauge bundle, i.e. when the gauge theory is in the Coulomb phase, this symmetry is broken by the strong coupling infrared effects which generate the mass gap. The value of the mass gap is related to the Kähler rather than complex structure of $K3 \times K3$ so one may speculate about the existence of some more general geometric theory that would combine both to reproduce the quantum corrections.

There is also another interesting possibility. Consider two 7-branes both equipped with the nontrivial gauge bundles approaching each other so that by the end of the day there are two 7-branes on top of each other. It turns out that in many cases this configuration does not produce a gauge group in four dimensions because of the gauge bundle inside the 7-branes. Naively it looks like that by adjusting the relative positions of the 7-branes (parameters of the Coulomb branch) we end up far in the middle of the Higgs branch. We explain this phenomenon in the Section 4 noticing that the same space (the moduli space of instantons on $K3$) may be viewed as Higgs branch for several different Coulomb branches. More precisely, looking from the Coulomb branch of the gauge theory with maximal gauge group, the Higgs branch is an open patch of the instanton moduli space in the vicinity of small instantons. The instanton moduli space is a way the string theory compactifies the Higgs branch; in doing this some points at infinity may be added. The Coulomb branches of other gauge theories with smaller gauge groups touch the compactified Higgs branch exactly at these points. In the example considered in this paper the $SU(2)$ Higgs branch with the even number of hypermultiplets touches the $U(1)$ Coulomb branch.

The inverse process is also very interesting. At the special points in the Higgs branch where the gauge bundle is reducible one can deform the singularity $I_{G_1 \times G_2} \rightarrow I_{G_1} \times I_{G_2}$. As the result of this process one obtains two 7-branes equipped with the gauge bundles with
non zero first Chern class. The typical example of this situation would be the deformation of $I_2$ singularity into two $I_1$ singularities. Therefore, starting with the compactification with 24 7-branes and 24 3-branes we first can go to the enhanced symmetry locus, in which some of the 7-branes are on top of each other, then replace 3-branes by nonabelian instantons and further deform a theory to a locus in which the gauge bundle is reducible. Repeating this process one can reach the phase with 24 7-branes equipped with nontrivial line bundles and with no 3-branes. In this phase, the fourfold $K3 \times K3$ is smooth.

Another question that should be understood is the heterotic/ F-theory duality in four dimensions. Some progress in understanding the general aspects of duality was recently achieved [9] [10]. It turns out that the complex structure of the elliptic fibration on the F-theory side codes both the complex structure of the heterotic Calabi-Yau threefold and the a part of the information about the heterotic vector bundle known as spectral cover. The gauge bundles inside 7-branes are mapped into the other piece of data specifying the heterotic bundle known as spectral bundle. Together, the spectral cover and the spectral bundle fix the heterotic bundle completely. Also, the F-theory/heterotic duality maps 3-branes into heterotic 5-branes. All this turns out to be strikingly simple for the $K3 \times K3$ compactification. The dual heterotic theory is compactified on $K3 \times T^2$. It can be viewed in two different ways. We can either first compactify down to 6 dimensions on $K3$ and then further on $T^2$, or compactify first down to 8 dimensions on $T^2$ and then further on $K3$. The second point of view appears to be very fruitful. In this case the spectral theory is very simple and the spectral surface is just a collection of $K3$ surfaces. On the F-theory side the set of 7-branes is also a collection of $K3$s. This observation also simplifies the discussion of the map between the gauge bundle on different components of the spectral surface and the gauge bundle inside the 7-branes.

There is another duality which maps F-theory compactification on $K3 \times K3$ to Type IIA compactification on the certain elliptic Calabi-Yau threefolds. The base of fibration is a (generalized) Del Pezzo surface which is $P^1 \times P^1$ blown up in $N$ points. As usual in Type IIA, the Higgs branch is parameterized by the complex structures and the Coulomb branch is parameterized by the Kähler structures of the threefold. It turns out that under the duality the Coulomb branch of vector multiplets on 7-branes is mapped to the Kähler moduli of singular fibers, while the Coulomb branch of vector multiplets on 3-branes is mapped to the Kähler moduli moduli of $N$ blow ups of the base. Thus the number $N$ corresponds in F-theory to the number of 3-branes in the Coulomb phase, i.e. sitting in the bulk away from 7-branes. Preserving the Calabi-Yau condition, one can blow up
as many as 24 points on $\mathbb{P}^1 \times \mathbb{P}^1$ if these points lie on two parallel lines. The resulting Calabi-Yau has Hodge numbers $(43, 43)$ and two $E_8$ singularities along two rational curves. It is dual to F-theory with all 24 3-branes in the bulk.

2. Chain of dualities vs. the anomaly.

We will start by reviewing the chain of dualities associated with the F-theory compactification on $K^3 \times K^3$ [3]. Doing the compactification on one $K^3$ first we get a dual of heterotic theory on $T^2$. Compactification on the second $K^3$ gives heterotic theory on $T^2 \times K^3$. On the other hand, since heterotic on $K^3$ is F-theory on Calabi-Yau threefold $CY_3$, the theory at hand is dual to F-theory on $T^2 \times CY_3$ which is Type IIA on $CY_3$.

Naively, there appears to be a puzzle when one realizes that the F-theory on $K^3 \times K^3$ has an anomalous 3-brane charge 24. This anomaly can be derived directly from the D-brane picture. The base $B_F = \mathbb{P}^1 \times K^3$, and the discriminant locus consists of 24 D-branes represented by 24 points on $\mathbb{P}^1$ and wrapped on $K^3$. Wrapping the 7-brane around $K^3$ produces a 3-brane charge of $-1$, so in total there is $Q(3) = -24$ to be compensated. To cancel the anomaly one can put 24 3-branes into $K^3 \times K^3$. Thus in the dual heterotic picture one needs to turn on 24 5-branes wrapped around $T^2$. (What happens in the Type IIA picture, we will discuss below.) However, there are heterotic compactifications on $T^2 \times K^3$ without 5-branes connected by transitions to compactifications with 5-branes. What these compactifications correspond to in the F-theory description?

In fact, this apparent puzzle can be explained completely within the heterotic description. To obtain a compactification on $T^2 \times K^3$ we can compactify either first on $K^3$ and then on $T^2$, or first on $T^2$ and then on $K^3$. Following the first path, generically we choose a $(12-n, 12+n)$ $K^3$ compactification with 244 massless hypermultiplets, no vectors and one tensor multiplet (for $n = 0, 1, 2$). Compactifying further on $T^2$ we recover the $N = 2$ theory with 244 hypermultiplets and 3 vectors. Since the $E_8 \times E_8$ is already broken completely in 6 dimensions, there are no Wilson lines to put on $T^2$. Thus the corresponding $E_8 \times E_8$ heterotic bundle on $T^2 \times K^3$ is a pullback of the bundle on $K^3$. The heterotic anomaly is cancelled by the second Chern classes of $E_8$ bundles.

If we compactify on $T^2$ first, generically we get an $N = 1$ theory with 18 vectors in eight dimensions. The group $E_8 \times E_8$ is broken to $U(1)^{16}$ by the non trivial Wilson lines around $T^2$ and the Kaluza-Klein modes of $T^2$ provide two more $U(1)$’s. Let us try to compactify this theory on $K^3$, choosing the trivial background $U(1)^{18}$ bundle, so that the
heterotic bundle on $T^2 \times K3$ is a pullback from the bundle on $T^2$. Then to compensate for the second Chern class of $K3$, one needs to turn on 24 heterotic 5-branes wrapped around $T^2$. The resulting theory generically has an unbroken $U(1)^{18}$ from eight dimensions plus $U(1)^{24}$ from the 5-branes. It is this compactification that can be identified with F-theory on $K3 \times K3$ with 24 3-branes [3].

We may also choose a different background for $U(1)^{18}$, picking 18 line bundles $L_i$ on $K3$ such that $\sum c_1(L_i) = 0$ and $\sum c_1^2(L_i) = -48 + 2N_5$. The heterotic anomaly is cancelled by the gauge fields and $N_5$ 5-branes. As the above examples show, there seems to be a discrete set of combinations of a number of 5-branes and a set of the Chern classes for the background gauge bundles to cancel the anomaly. In fact, all these compactifications are connected on the larger moduli space of all heterotic compactifications, as we show below.

3. Bundles on $T^2 \times K3$

Having followed two different paths to get to four dimensions, one ends up with two rather different types of heterotic bundles on $T^2 \times K3$. To put these two examples in a general framework we will consider on $T^2 \times K3$ the bundles with the first Chern class equal to zero and the second Chern class a pullback from $K3$. For simplicity, we will discuss only the $SU(r)$ bundles.

To describe the vector bundles, we use the spectral theory, which dramatically simplifies on the product $T^2 \times K3$. A vector bundle $V$ is described by means of two objects: a spectral surface and a collection of spectral bundles. The spectral surface $\Sigma(V)$ of bundle $V$ consists of several components $\bigcup_i n_i x_i \times S_i$, where $S_i$ is a copy of $K3$ and $x_i$ is a point in the dual torus $\hat{T}^2$, $n_i$ is its multiplicity so that $\sum n_i = r$. Each component $S_i$ carries a spectral bundle $M_i$ of rank $n_i$. The bundle $V$ on $T^2 \times K3$ is fixed by this data as follows:

$$V = \bigoplus_i p_1^* L_{x_i} \otimes p_2^* M_i,$$

where $L_{x_i}$ are the line bundles on $T^2$ corresponding to $x_i \in \hat{T}^2$. We denote by $p_1$ and $p_2$ the projections from the product $T^2 \times K3$ to the factors $T^2$ and $K3$, respectively.

The first Chern class of $V$ is zero which forces $\sum_i n_i x_i = 0$ and $\sum_i c_1(M_i) = 0$, but the individual $c_1(M_i)$’s may be non-trivial. The second Chern class $c_2(V)$ is given by

$$c_2(V) = \sum_i c_2(M_i) - \frac{1}{2} c_1^2(M_i).$$

When all \( x_i = 0 \in \hat{T}^2 \), there is only one component of the spectral surface and the bundle \( V = p_*^* M \) is a pullback of the bundle \( M \) on \( K3 \). Alternatively, when all \( x_i \) are different the bundles \( M_i \) are the \( U(1) \) bundles which appear in the second compactification path above.

Let us study the deformations of \( V \). They come from the deformations of the spectral surface \( \Sigma(V) \) and the deformations of the collection of spectral bundles \( M_i \). If the component \( x_i \times S_i \) of \( \Sigma(V) \) has multiplicity \( n_i = 1 \), it contributes one-parameter family of deformations corresponding to moving \( x_i \) on the dual torus \( \hat{T}^2 \). On the other hand, if \( n_i \geq 2 \) there are three kinds of deformations. First, one can again move \( x_i \) on \( \hat{T}^2 \). Second, one can deform the spectral bundle \( M_i \): the dimension of the moduli space (for \( U(n_i) \) bundles) is given by

\[
\dim \mathcal{M}_{M_i} = 2n_i c_2(M_i) - 2(n_i^2 - 1).
\]

(3.3)

Finally, the deformations of the third kind split the multiple point to several points with smaller multiplicities. Such deformations are only possible if the spectral bundle \( M_i \) is reducible: \( M_i = \oplus_j M_{ij} \). Then one can split the component \( n_i x_i \times S_i \) to \( \cup_j \text{rk}(M_{ij}) x_{ij} S_{ij} \).

As an example, let us consider the splitting of a multiplicity 2 component with an \( SU(2) \) spectral bundle \( M \) on it. The splitting may occur in the (singular) points of the moduli space \( \mathcal{M}_{SU(2)} \) where \( M = L \oplus L^{-1} \). The possible line bundles \( L \) are restricted by the condition \( c_2(M) = -c_2^1(L) \). Note that \( c_2(M) \) has to be even since \( c_2^1(L) \) is always even. The multiplicity 2 component with \( M = L \oplus L^{-1} \) splits into two multiplicity 1 components carrying the spectral bundles \( L \) and \( L^{-1} \) respectively.

This procedure is a regular way to produce spectral bundles with the non-trivial first Chern classes. For instance, one may start with a pullback of a \((12-n, 12+n)\) \( E_8 \times E_8 \) bundle on \( K3 \), go to the point on the moduli space where this bundle splits, then deform and end up with 16 line bundles with non-trivial \( c_1 \)’s. The gauge symmetry is restored to \( U(1)^{18} \). This model describes an alternative compactification of eight dimensional heterotic theory down to four dimensions, where the \( K3 \) anomaly \( \text{tr} R^2 \) is cancelled by \( \sum_i c_2^1(L_i) \) of the background \( U(1) \) bundles and the 5-branes are absent.

Now one may ask if it is possible to arrive at this compactification starting with the compactification with the 5-branes. Here we will outline the answer leaving the detailed discussion for the next section. Let us interpret a 5-brane as a point-like gauge instanton on \( K3 \). In a 5-brane – instanton transition the instanton acquires finite size. However, in the vicinity of the pointlike instantons, the bundle is irreducible. One has to go a
finite distance on the moduli space to reach the point where the bundle splits. Therefore, the transition connecting two heterotic vacua, one with 5-branes and the other without 5-branes, goes via the locus of gauge symmetry enhancement along the moduli space of instantons on $K3$.

4. Bundles on 7-branes

4.1. Heterotic – F-theory map

This model admits a particularly simple map between heterotic and F-theory pictures. On the general grounds, we expect that the 5-branes are mapped into the 3-branes, the spectral surface is mapped into the complex structure of the elliptic fibration and the spectral bundles are mapped into the gauge bundles on 7-branes. In our example, all these maps are straightforward:

The F-theory 3-branes in $\mathbb{P}^1 \times K3$ are mapped to heterotic 5-branes wrapped on $T^2$ in $T^2 \times K3$. The map identifies two $K3$ factors and interprets $\mathbb{P}^1$ as a Coulomb branch of the effective worldvolume theory of the 5-brane wrapped on $T^2$.

The map between the spectral surface $\Sigma(V)$ and the discriminant locus follows from the heterotic/F-theory duality in 8 dimensions. Indeed, $\Sigma(V)$ is determined by the Wilson lines $V |_{T^2}$ which together with the complex and Kähler structure of $T^2$ determine the heterotic compactification on $T^2$. The discriminant locus is on the other hand determined by 24 points on $\mathbb{P}^1$ which describe the dual F-theory compactification on $K3$.

Generically, both the spectral bundles and the bundles on 7-branes have rank one. The heterotic $E_8 \times E_8$ is broken to $U(1)^{16}$ so that the spectral surface consists of 16 copies of $K3$. There are two other $U(1)$ vector multiplets corresponding to the complex and Kähler moduli of $T^2$. In F-theory, there are 24 mutually non-local 7-branes with 18 independent $U(1)$’s. These line bundles are mapped into the 18 line bundles on the heterotic side.

Now consider the process in which two components of the spectral surface join into one component with the multiplicity two: $(x_1S_1) \cup (x_2S_2) \to 2xS$. The above map between the spectral surface and the discriminant tells us that at this very moment two 7-branes with singularities $I_1$ join into one 7-brane with singularity $I_2$, so one can identify the rank two bundle on the component of the spectral surface with the rank two background bundle defined on the 7-brane with singularity $I_2$. It is easy to continue this identification. The spectral bundles $M_i$ should be mapped into the background gauge bundles on the 7-branes. In particular, the $SU(n)$ gauge symmetry enhancement can be either described in terms
of $n$ 7-branes coming together or $n$ components of the spectral surface coming together to form a multiplicity $n$ component. In this situation, there is a one-to-one correspondence between the components of the spectral surface and the spectral bundles on one hand, and the 7-branes with the background gauge bundles on the other hand.\[\text{\cite{10}}\]

Because of that correspondence, everything that was said above about the spectral bundles can also be applied to the background gauge bundles on 7-branes. In particular, a (multiple) 7-brane equipped with the *irreducible* bundle cannot split into a union of 7-branes with smaller charges. If the bundle splits as a direct sum of several sub-bundles $M_i$, the 7-brane can split to several 7-branes, each equipped with its own $M_i$.

4.2. 3-branes and structure of the moduli space

Let us discuss the 3-brane – instanton transition in this context, stressing the new aspects the compactification brings in. Consider an $A_{N-1}$ 7-brane with the background bundle $M$ on it. In four dimensions this gives rise to a pure $N = 2$ supersymmetric $SU(N)$ Yang-Mills theory. A 3-brane on top of the 7-brane contributes a massless hypermultiplet in the fundamental (antifundamental) representation of $SU(N)$. Let us take a number $k > N$ of such 3-branes, so that there are $k$ hypermultiplets and the gauge group can be Higgsed completely. The (baryonic) Higgs branch has the dimension

$$2Nk - 2(N^2 - 1).$$ \[4.1\]

As a space, it is a hyperkähler quotient $\mathcal{H}$ of the total space of matter fields $\mathbb{C}^{2Nk}$ by the action of the complexified gauge group $SL(N, \mathbb{C})$.

On the other hand, a 3-brane on top of the $SU(N)$ 7-brane can be identified with a pointlike instanton (a torsionless sheaf, to be more precise). Higgsing may be interpreted along the lines of the ADHM construction as giving this instanton a finite size. Therefore we are led to an identification of the baryonic Higgs branch with the moduli space of $SU(N)$ instantons on $K3$ with the instanton number $k$. Comparing the dimensions (3.3) and (4.1) one sees they are the same. Also, both spaces are hyperkähler.

However, it would be wrong to claim these spaces are identical. Indeed, the (Gieseker) moduli space $\mathcal{M}$ of $SU(N)$ bundles on $K3$ is a *compact* variety, which cannot be obtained

\[\text{\footnote{There are at least two different compactifications of the moduli space of instantons on $K3$. In the string theory context it is appropriate to consider the Gieseker compactification, which adds *torsionless sheaves* to vector bundles to get the compact space. The torsionless sheaves correspond to pointlike instantons considered as 3-branes.}}\]
by a naive ADHM-like construction. Rather, the field-theoretical Higgs branch $H$ lies in $\mathcal{M}$ as an open neighborhood of the point-like $k$-instantons. String theory provides a natural compactification for $H$ completing it to $\mathcal{M}$. The compactification may add certain singularities which describe the physics missed by the naive field theory description. As a concrete example, let us consider $SU(2)$. For even $k$, the moduli space $\mathcal{M}_{SU(2)}$ has conifold singularities at points corresponding to reducible bundles $L \oplus L^{-1}$, where $L$ are line bundles with $-c_1^2(L) = k$. From the point of view of four dimensional theory, singularities at these points correspond to the fact that the gauge group is restored to $U(1)$. At these points, the $U(1)$ Coulomb branch touches $\mathcal{M}_{SU(2)}$. This Coulomb branch is different from the one attached at the origin of $H$: on it the number of 3-branes is smaller by $-c_1^2(L)$. It corresponds to splitting of the $SU(2)$ 7-brane into two $U(1)$ 7-branes, with the background $U(1)$ bundles $L$ and $L^{-1}$ respectively. The complex scalar parameterizing the Coulomb branch measures the separation between two 7-branes.

It is worthwhile to discuss the effective $U(1)$ theory. At the point where the Coulomb branch touches the Higgs branch, there are massless fields coming from the moduli of the $SU(2)$ instanton bundle on $K3$. If this point were smooth on $\mathcal{M}_{SU(2)}$, these fields would form the tangent space with the dimension equal to $4k - 6$. The tangent space to $\mathcal{M}_{SU(2)}$ at a smooth point corresponding to a vector bundle $M$ is given by the cohomology group $H^1(K3, End(M))$. However, the point $M = L \oplus L^{-1}$ is not smooth, there is a conifold singularity there. The cohomology group

$$H^1(End(L \oplus L^{-1})) = H^1(L^2) \oplus H^1(L^{-2})$$

(4.2)

computes what is called the Zarisski tangent cone. The dimension of this space is $4k - 4$ — i.e. it is bigger by two than the dimension of the moduli space. The fields corresponding to the cohomology groups (4.2) are charged with respect to $U(1)$: the elements of $H^1(L^2)$ have charge +2 and the elements of $H^1(L^{-2})$ have charge −2. The matter should fit into $N = 2$ hypermultiplets, so it is necessary that $\dim H^1(L^2) = \dim H^1(L^{-2})$. This condition is indeed satisfied due to Serre duality on $K3$. Thus the $U(1)$ theory has $2k - 2$ hypermultiplets with charges ±2. On the Coulomb branch away from the Higgs branch these hypermultiplets are massive. The dimension of the $U(1)$ Higgs branch is $2(2k - 2) - 2 = 4k - 6$. It coincides with the dimension of the moduli space of $SU(2)$ instantons, as necessary for consistency.

We conclude that the moduli space of $SU(2)$ bundles on $K3$ with even instanton number $k$ can be described as a baryonic Higgs branch of $SU(2)$ theory with $k$ flavors.
in the vicinity of pointlike instantons and as a Higgs branch of $U(1)$ theory with $2k - 2$ charged hypermultiplets in the vicinity of reducible bundles $L \oplus L^{-1}$, $c_2(L) = -k$.

The situation for $SU(N)$, $N > 2$ bundles is the straightforward generalization of the construction discussed above. In the vicinity of the reducible bundles $V_n = \oplus_i V_{r_i}$ the $SU(n)$ baryonic Higgs branch can also be described as Higgs branch of a different gauge theory with the gauge group being $\otimes SU(r_i)$ times appropriate number of $U(1)$s.

5. $K3 \times K3$ vs $CY_3 \times T^2$ compactifications

5.1. A Type IIA dual

Let us start with a (rather degenerate) situation in which the heterotic bundle on $T^2 \times K3$ is a pullback of the bundle $V_1 \times V_2$ on $K3$. The structure group of this bundle is $H_1 \times H_2$, we assume that $H_i$ are both simple $ADE$ groups. The second Chern class is distributed between $V_1$ and $V_2$ as $(12 + n, 12 - n)$. The unbroken gauge group in four dimensions is $G_1 \times G_2$ (where $H_i \times G_i \subset E_8$ is a maximal embedding). Each spectral surface $\Sigma(V_i)$ has a multiple component equipped with the $H_i$-bundle $V_i$.

The F-theory dual is compactified on $K3 \times K3$, where the first factor is a singular elliptic $K3$. The elliptic fibration has two $E_8$ singularities and generically four $I_1$ singularities. The two 7-branes with $E_8$ singularities are equipped with the $H_i$-bundles $V_i$. The $E_8 \times E_8$ gauge group is broken by the $H_1 \times H_2$ instantons leaving the unbroken gauge group $G_1 \times G_2$ times the appropriate number of $U(1)$s. The moduli space of the gauge fields inside the 7-branes and the gauge fields inside the spectral surface are isomorphic to each other.

The $E_8$ singularity in F-theory can be deformed away, but the smallest possible singularity is fixed by the group $H_i$. The $H_i$ 7-brane is equipped with the irreducible $H_i$-bundle $V_i$, so it cannot split. The deformations destroying the $E_8$ singularities describe the classical Coulomb branch of the $N = 2$ susy theory with the gauge group $G_1 \times G_2$. To obtain the quantum Coulomb branch, one needs to account for the nonperturbative effects.

For example, consider two 7-branes located close to each other on $P^1$ and wrapped on $K3$. Separating the center of mass, we get a Coulomb phase of the $SU(2)$ gauge theory

\[ \text{To be precise, this statement is true only classically. The relative locations of the 7-branes parameterize the Coulomb branch that gets quantum correction.} \]
with no matter in four dimensions. Quantum mechanically, the mass gap \( \Lambda^2 = e^{-V/g_{\text{str}}^2} \) is dynamically generated, and the coupling constant is given by the Witten-Seiberg formula

\[
\frac{1}{g^2} = \frac{V}{g_{\text{str}}^2} + \log(a^2) + \sum_{N=1}^{\infty} c_N e^{-2N(V/g_{\text{str}}^2+\log(a^2))} ,
\]

(5.1)

where \( V \) is a volume of \( K3 \) and \( a \) is a scalar of the \( U(1) \) vector multiplet. The first term in (5.1) gives the perturbative effective tension of a pair of 7-branes wrapped around \( K3 \) and separated by \( a \) on \( \mathbb{P}^1 \). The \( \log a^2 \) term is a contribution of the (Euclidean) gas of massive strings stretched between the 7-branes. The exponential terms in (5.1) come from the Euclidean 3-branes wrapped around \( K3 \). When the 3-brane is wrapped on top of the 7-brane it gets additional zero modes coming from the massless string stretched between the 3-brane and the 7-brane, which make it contribute to the prepotential.

In the semiclassical regime \( |a|^2 >> \Lambda^2 \) the scalar \( a \) measures the separation between the 7-branes. However, in quantum theory there is no point on the Coulomb branch where \( a = 0 \) and the \( SU(2) \) is never restored \[11\].

This F-theory compactification has another interpretation. The \( (12 - n, 12 + n) \) heterotic compactification on \( K3 \) is equivalent to the F-theory compactification on the Calabi-Yau threefold \( CY_3 \). The threefold \( CY_3 \) is an elliptic fibration over the Hirzerbruch surface \( F_n \). Compactifying further on \( T^2 \) we get an F-theory compactification on \( T^2 \times CY_3 \) which can also be interpreted as a Type IIA compactification on \( CY_3 \). Either one has to be dual to the above \( K3 \times K3 \) compactification.

As a result, we arrive to the conclusion that type IIB on \( P^1 \times K3 \) with 7-branes wrapped on \( K3 \) is equivalent to the type IIB on \( F_n \times T^2 \) with 7-branes wrapped on \( C_i \times T^2 \), where \( C_i \) are curves in \( F_n \)! Geometrically, these two compactifications seem to be very different. For example, one manifold is simply connected while the other one has non-contractible paths. It is remarkable that one can trade the topology of the manifold for the non-trivial gauge bundles on 7-branes.

It is instructive to compare the low energy descriptions. Both theories are \( N = 2 \) super YM with the gauge group \( G_1 \times G_2 \) coming from 7-branes. The \( F_n \times T^2 \) compactification has a \( G_1 \) 7-brane wrapped on the zero section of \( F_n \) and a \( G_2 \) 7-brane wrapped at the section at infinity. These 7-branes are also wrapped on \( T^2 \). In this language, the Coulomb

\( ^3 \) Assuming that the 3-branes are located far away from these 7-branes so that the corresponding hypermultiplets are massive.
branch is parameterized by the $G_1 \times G_2$ Wilson lines on the torus. In the Type IIA language, the Coulomb branch corresponds to the Kähler classes of the components of exceptional fibers over the discriminant locus with $G_i$ singularity. Comparing this with the description in terms of Type IIB on $\mathbf{P}^1 \times K3$ we see that the complex deformations of the elliptic fibration over $\mathbf{P}^1$ are mapped to the Kähler moduli of $CY_3$. The quantum Kähler moduli space is corrected by the worldsheet instantons. Translating this back to F-theory we get the “quantum elliptic fibration” which describes the same space in terms of complex deformations of something.

The description of the matter multiplets also differs between two languages. On the one hand, the $\mathbf{F}_n \times T^2$ compactification gets the hypermultiplets from the 7-branes intersecting along tori $T^2$. The Higgs branches corresponding to the nonzero vev’s of these fields describe the complex deformations of $CY_3$ destroying the $G_1 \times G_2$ singularity. On the other hand, in the $\mathbf{P}^1 \times K3$ compactification the matter hypermultiplets come by dimensional reduction from 8 dimensions. The number of matter multiplets $N_i$ in the representation $R_i$ of the unbroken group $G$ is given by the index theorem $N_i = (12 \pm n)\text{index}(S_i) - \text{dim}(S_i)$, where $S_i$ is the representation of $H$ entering into the decomposition $248 = \oplus_i (R_i, S_i)$. This computation is identical to the computation in the heterotic string. The Higgs branch can be described completely in terms of the bundles $V_1$ and $V_2$. Therefore, the moduli of $V_1$ and $V_2$ are mapped into the the complex moduli of $CY_3$.

5.2. 3-branes vs blowups

The theories labeled by different values of $n$ are related to each other by phase transitions. From the heterotic theory point of view one has to shrink one instanton (say of $V_1$) to zero size, reinterpret it as a 5-brane and then “dissolve” it into a finite instanton of the second bundle. Similarly, in the description of F-theory compactified on $K3 \times K3$ the process of changing $n$ consists of three steps. First, one shrinks the instanton inside the first 7-brane down to zero size and replaces it by a 3-brane. The 3-brane can move freely inside $\mathbf{P}^1 \times K3$. Then one puts a 3-brane on top of another 7-brane and finally dissolves it into a finite size instanton.

If one removes sufficiently many instantons from one of the 7-branes, so that the instanton number $k$ is less then 10, the background bundle does not break $E_8$ completely and the enhanced gauge symmetry appears. For $k = 3$, the group is $SU(3)$, for $k = 4$, it is $SO(8)$ etc.
In the F-theory on $T^2 \times CY_3$ (or equivalently, in Type IIA on $CY_3$) this transition can be described as a sequence of one blowup and one blowdown of the base $F_n$. Let us blow up a point on the zero section of $F_n$. Two rational curves pass through this point: one is the zero section $S_0$ and the other is the fiber $F = P^1$ of the projective bundle $F_n$. These curves generate the cohomology ring of $F_n$. They satisfy $F^2 = 0$, $S_0^2 = n$, $S_0 \cdot F = 1$.

After the blowup $\pi : B \to F_n$, the cohomology ring of $B$ is generated by $\pi^* F$, $\pi^* S_0$ and the exceptional divisor $E$ such that $E^2 = -1$, $E \cdot S_0 = E \cdot F = 0$. Two curves $S_0$ and $F$ which pass through the blown up point are transformed into $\hat{S}_0 = \pi^* S_0 - E$ and $\hat{F} = \pi^* F - E$. Their intersections are: $\hat{S}_0^2 = n - 1$, $\hat{F}^2 = -1$, $\hat{F} \cdot \hat{S}_0 = 0$ and $\hat{F} \cdot F = 0$. These relations show that the line $\hat{F}$ can be blown down which results in $B \to F_{n-1}$. The new zero section is $\hat{S}_0$. This process changes the index of the Hirzebruch surface $n \to n - 1$ (or $n \to n + 1$, if we start with a point on $S_\infty$).

Blowing up a point on the curve $S$ with self-intersection $S^2 = -k$ gets this curve properly transformed into a curve $\hat{S} = \pi^* S - E$ with self-intersection $-(k + 1)$. If this number is less than $-2$, the curve $\hat{S}$ has to be a component of the discriminant locus [2], as a consequence of the adjunction formula which tells us that the intersection of the rational curve $S$ with the discriminant $\Delta = -12K$ is given by $\Delta \cdot S = 12(S^2 + 2)$. For $S^2 < -2$, $\Delta \cdot S < 0$ and since $\Delta$ is an effective divisor the curve $S$ has to be one of its components: $\Delta = \Delta' + rS$, where the integer $r$ satisfies

$$r \geq 12 \frac{S^2 + 2}{S^2} \quad (5.2)$$

The type of the singular elliptic fiber over $S$ is also determined by $S^2$. For example, the self-intersection $-3$ corresponds to the $I_3$ fiber and the unhiggsed $SU(3)$ gauge group in space-time and the self-intersection $-4$ corresponds to the $I_0^*$ fiber and the unhiggsed $SO(8)$. The gauge group $E_6$ appears for self-intersections $-5$, $-6$, the group $E_7$ – for self-intersections $-7$, $-8$, and the group $E_8$ – for self-intersections $-9$, $-10$, $-11$, $-12$. In general, the charged matter that could higgs the gauge group in space-time comes from the intersection $\Delta' \cdot S$ of $S$ with the other components of discriminant. So whenever $\Delta' \cdot S = 0$, the gauge symmetry cannot be higgsed. This condition implies that the inequality (5.2) is saturated which can only happen for self-intersections $-3$, $-4$, $-6$, $-8$, $-12$. In the language of the Calabi-Yau elliptic fibration $CY_3$ this means that there are no deformations destroying such a component of the discriminant locus, so everywhere in the moduli space of $CY_3$ the base of the elliptic fibration ought to have a curve with self-intersection $-3$,  

$\text{13}$
−4 etc. We will discuss the consequences of that in the next section. The minimal gauge group which cannot be Higgsed is the same as the minimal subgroup of $E_8$, left unbroken by $12 + S^2$ instantons.

Comparing two pictures, we conclude that the F-theory on $K3 \times K3$ with $N = m + n$ 3-branes and two $E_8$ 7-branes with the instanton numbers $12 - m$ and $12 - n$ is described by the Type IIA compactified on $\text{CY}_3$ elliptically fibered over the base which is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in $m + n$ points. The set of $m$ points belongs to the line $S_0$ and the set of $n$ points belongs to the line $S_\infty$ which does not intersect $S_0$. The blown up surface has two nonintersecting rational curves with with self-intersection $-m$ and $-n$ respectively.

The vector multiplets in Type IIA which come from the exceptional divisors $E_i$ correspond to the vector multiplets on 3-branes. The complexified area of $E$ should be interpreted as the $\mathbb{P}^1$ coordinate of the 3-brane on $\mathbb{P}^1 \times K3$. Let us denote by $\omega$ the Kähler class of the base. Then

$$\omega(E) + \omega(\hat{F}) = \omega(F) = \text{const}.$$ 

(5.3)

The variation of $\omega(E)$ from zero to $\omega(F)$ corresponds to the 3-brane motion from one $E_8$ 7-brane to another.

The $E_8$ hypermultiplet corresponding in the F-theory language to a string connecting a 3-brane with the first $E_8$ 7-brane, in the Type IIA language should correspond to 2-branes wrapped around the rational curves on the exceptional divisor $D_1 \subset \text{CY}_3$ which covers the exceptional curve $E \subset F_n$. Similarly, the hypermultiplet corresponding to the string connecting the 3-brane with the second $E_8$ 7-brane corresponds to the Type IIA 2-branes wrapped around the divisor $D_2$ covering the exceptional curve $\hat{F}$. Both $D_1$ and $D_2$ are almost Del Pezzo surfaces with $\chi(D_i) = 12$, so that the supersymmetric 2-cycles in each of them reproduce the $E_8$ lattice.

It should be emphasized that since the 3-brane description actually gives the quantum Coulomb branch, the above map is actually a mirror map. It translates the worldsheet instanton corrections in Type IIA into the nonperturbative prepotential in the effective $N = 2$ four-dimensional theory on the 3-brane probe.

5.3. Del Pezzo surfaces and elliptic fibrations

Now consider the Type IIA compactification on a Calabi-Yau threefold $\text{CY}_3$ which is an elliptic fibration over the base $B$, where $B$ is obtained by blowing up $N \leq 24$ points on $F_n$. This compactification is dual to the F-theory compactification on $K3 \times K3$ with $N$
3-branes in the bulk. In the previous section we discussed the map between the positions of 3-branes on \( P^1 \) and the (complexified) blowup Kähler moduli of \( CY_3 \). The other Kähler moduli of \( CY_3 \) come from the exceptional fibers of the elliptic fibration. They correspond to the Coulomb branch of the \( N = 2 \) gauge theory on 7-branes.

Let us start by blowing up 1 point \( P \) on \( F_1 \). Depending on whether or not this point lies on the line \( S_\infty \), the result will be different. If \( P \) does not lie on \( S_\infty \), the blown up surface \( B_2 \) will have two \((-1)\) curves: \( S_\infty \) and the exceptional divisor \( E \). Alternatively, if \( P \in S_\infty \), the \((-1)\) curve \( S_\infty \) gets properly transformed into a \((-2)\) curve \( C = S_\infty - E \), so the blowup \( \tilde{B}_2 \) has one \((-1)\) curve \( E \) and one \(-2\) curve \( C \). The surfaces \( B_2 \) and \( \tilde{B}_2 \) have different complex structures. The \((-2)\) curve \( C \) in \( \tilde{B}_2 \) corresponds to the root of \( SU(2) \). If \( C \) is contracted down to zero size, a nodal \( A_1 \) singularity forms on \( \tilde{B}_2 \), and the Calabi-Yau 3-fold elliptically fibered over \( \tilde{B}_2 \) gets an elliptic curve of \( A_1 \) singularities.

Blowing up 2 points \( P_1, P_2 \) on \( F_1 \), we have more possibilities. We are mostly interested in \((-2)\) curves, so the following cases are worth mentioning.

i) Both \( P_1 \in S_\infty \) and \( P_2 \in S_\infty \), \( P_1 \neq P_2 \). Then the curve \( S_\infty \) is transformed into a \((-3)\) curve \( \Sigma = S_\infty - E_1 - E_2 \). There are no \((-2)\) curves on the blowup. The curve \( \Sigma \) becomes a component of the discriminant locus of \( CY_3 \) with the singularity \( A_2 \), so there is \( N = 2 \), \( SU(3) \) Yang-Mills theory in space-time.

ii) \( P_1 \in S_\infty \), \( P_2 \notin S_\infty \) and \( P_1, P_2 \) do not belong to the same fiber \( F \). Then the only \((-2)\) curve on the blowup is \( C = S_\infty - E_1 \).

iii) \( P_1, P_2 \notin S_\infty \) and both \( P_1 \) and \( P_2 \) lie on the same fiber \( F \). Then \( F \) is properly transformed into a \((-2)\) curve \( F - E_1 - E_2 \).

iv) \( P_1 = P_2 \notin S_\infty \). Considering one blowup after another, we end up either with one \((-2)\) curve \( E_1 - E_2 \) if \( P_2 \) approaches \( P_1 \) generically or with two nonintersecting \((-2)\) curves \( E_1 - E_2 \) and \( F - E_1 - E_2 \) if \( P_2 \) approaches \( P_1 \) along the fiber \( F \).

v) \( P_1 = P_2 \in S_\infty \).

   a) If \( P_2 \) approaches \( P_1 \) along \( S_\infty \), there is only one \((-2)\) curve \( E_1 - E_2 \), and a \((-3)\) curve \( S_\infty - E_1 - E_2 \) which has \( A_2 \) singular elliptic fiber over it.

   b) If \( P_2 \) approaches \( P_1 \) along generic direction, there are two \((-2)\) curves \( S_\infty - E_1 \) and \( E_1 - E_2 \) forming the root system of \( SU(3) \).

   c) Finally, if \( P_2 \) approaches \( P_1 \) along the fiber \( F \), one gets the third \((-2)\) curve \( F - E_1 - E_2 \) and the root system of \( SU(2) \times SU(3) \).

All examples (i-v) correspond to two instantons shrinking to zero size. In this interpretation all these cases differ by the relevant orientation of the instantons inside the \( E_8 \).
For example, in the cases (i), (v.b) the $SU(3)$ subgroup is restored, leaving $E_6$ completely broken by the remaining instantons.

The number of possibilities increases with the number of blowups. For example, for 3 blowups the $A_4$ root system (maximal) appears when one makes 3 blowups on $S_\infty$ and top of each other. Concretely, $P_1 \in S_\infty$, $P_2$ approaches $P_1$ along the fiber and $P_3$ approaches $P_2$ from generic direction. The $A_4$ root system is generated by four ($-2$) curves $S_\infty - E_1$, $E_1 - E_2$, $E_2 - E_3$ and $F - E_1 - E_2$. In general, for $N \leq 8$ blowups the maximal root system coincides with the root system of $E_N$ Lie algebra, where $E_3 = A_2 \times A_1$, $E_4 = A_4$ and $E_5 = D_5$ (see the discussion in [13]).

Already these simple examples allow us to learn some lessons. Let us test further the correspondence between 3-branes on $\mathbf{P}^1 \times K3$ and blow-ups on $F_1$. When $k$ 3-branes meet, we expect to see $N = 4$, $SU(k)$ supersymmetric Yang-Mills theory in space-time. Also, if the number of instantons on the $E_8$ 7-brane is less than 10, a part of the full $E_8$ group should be restored. In Type IIA picture this appears to correspond to the situation when one blows up $k$ points $P_i \in S_\infty$ (for $k = 2$, this is the case v.a above). To get $SU(k)$, one needs to bring the points $P_i$ together, moving them along $S_\infty$. The root system of $SU(k)$ is generated by the $-2$ curves $E_i - E_{i+1}$. Shrinking this system of curves to a point one produces an $A_{k-1}$ singularity at that point on $B_{k+1}$ and a whole elliptic fiber of $A_{k-1}$ singularities in the elliptic fibration $CY_3$. In Type IIA compactification such singularity is known to correspond to $N = 4$, $SU(k)$ supersymmetric Yang Mills. Also, the line $S_\infty$ with $k$ points blown up becomes a $(-k - 1)$ curve $S_\infty - \sum E_i$. For $k \geq 2$, it has to be a component of the discriminant with the singular fiber which correctly describes the subgroup of $E_8$ left unbroken by $11 - k$ instantons.

It should be noted that the complex structure of the blow-up $B_{k+1}$ depends on the relative positions of $P_i$. For $k \leq 3$, the blow-ups are rigid\(^4\) and one has a discrete set of different $B_{k+1}$’s. For $k \geq 4$, the generic surface $B_{k+1}$ has $2(k - 3)$ complex deformations. The generic surface has no $(-2)$ curves which appear along certain divisors in the moduli space. The moduli space of the base should be considered as a part of the moduli space of the Calabi-Yau fibration $CY_3$. It turns out that moving along this moduli space, one cannot connect the generic surface $B_{k+1}$ with some blowups. The most important example is the blowup of $F_1$ in $> 1$ points on $S_\infty$ or $> 3$ points on $S_0$, so that there is a curve with

\(^4\) This follows from the fact that there are three different $\mathbf{P}^2$ with four marked points, distinguished by how many points (2, 3 or 4) lie on one line.
self-intersection less than $-2$. As we explained in section 5.2, CY$_3$ has no deformations destroying a $-3$ curve in the base, so we can never reach a generic surface $B_m$ which has no such curves. Similarly, $-4, -6, -8$ and $-12$ curves cannot be destroyed. Thus the theory where $N$ generic points on the base are blown up cannot be reached starting from the theory where the points which are blown up lie on two curves.

Blowing up 24 points on two lines in $\mathbf{P}^1 \times \mathbf{P}^1$ lands us on the (43, 43) Calabi-Yau fibration, which has two rational curves of generic $E_8$ singularities [3], [4]. This compactification is equivalent to F-theory on $K3 \times K3$ with the first $K3$ having two $E_8$ singularities and 24 3-branes in the bulk.

On the other hand, one can blow up as many as 8 generic points on $F_1$ and that would land us on the (19, 19) Calabi-Yau. This threefold is a double elliptic fibration over $\mathbf{P}^1$, so that the fiber is a product of two elliptic curves. The Kähler cone of CY$_3$ is generated by the Kähler cones of $B$ and $B'$, with a single relation coming from the class of $\mathbf{P}^1$ shared by both $B$ and $\bar{B}$. It is clear that this compactification is connected through a series of phase transitions with the model (43, 43). Indeed, one can blow down all 24 spheres in (43, 43) model, get type IIA compactification on Calabi-Yau threefold fibered over $F_1$ and then blow up eight points on $F_1$. It follows that one should be able to see the (19, 19) phase in the F-theory description. We conjecture that some degenerations of gauge bundle inside the 7-branes lead to this phase. The relevant orientation of the instantons shrinking to zero size is crucial: the instantons should be embedded in such a way that by shrinking each of them to zero size we adjust every time exactly 29 parameters. These instantons should break $E_8$ to an abelian subgroup.

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