SUPERMINIMIZERS AND A WEAK CARTAN PROPERTY FOR \( p = 1 \) IN METRIC SPACES

By

PANU LAHTI

Abstract. We study functions of least gradient as well as related superminimizers and solutions of obstacle problems in complete metric spaces that are equipped with a doubling measure and support a Poincaré inequality. We show a standard weak Harnack inequality and use it to prove semicontinuity properties of such functions. We also study some properties of the fine topology in the case \( p = 1 \). Then we combine these theories to prove a weak Cartan property for superminimizers in the case \( p = 1 \), as well as a strong version at points of nonzero capacity. Finally, we employ the weak Cartan property to show that any topology that makes the upper representative \( u^+ \) of every 1-superminimizer \( u \) upper semicontinuous in open sets is stronger (in some cases, strictly) than the 1-fine topology.

1 Introduction

It is well known that solutions \( u \) of the \( p \)-Laplace equation, for \( 1 < p < \infty \), can be characterized as local minimizers of the \( L^p \)-norm of \( |\nabla u| \). This formulation has the advantage that it can be generalized to a metric measure space, by replacing \( |\nabla u| \) with the minimal \( p \)-weak upper gradient \( g_u \); see Section 2 for definitions and notation. The study of such \( p \)-minimizers is a starting point for nonlinear potential theory, which is now well developed even in metric spaces that are equipped with a doubling measure and support a Poincaré inequality; see especially the monograph [2] and, e.g., [3, 8, 9, 44], and also the monographs [37] and [23] for the Euclidean theory and its history in the nonweighted and weighted setting, respectively.

In the case \( p = 1 \), instead of the \( p \)-energy it is natural to minimize the total variation among functions of bounded variation (BV functions), and the resulting minimizers are called functions of least gradient; see, e.g., [11, 38, 39, 46] for previous works in the Euclidean setting, and [20, 32] in the metric setting. More precisely, a function \( u \in BV_{loc}(\Omega) \) is a function of least gradient in an open set \( \Omega \subset X \) if for every \( \varphi \in BV_c(\Omega) \), we have

\[
\| Du \| (\text{spt } \varphi) \leq \| D(u + \varphi) \| (\text{spt } \varphi).
\]
Testing only with nonnegative $\varphi$ leads to the notion of 1-superminimizers, whose study is the main objective of this paper. In the case $p > 1$, much of potential theory deals with superminimizers and the closely related concept of superharmonic functions, which were introduced in the metric setting in [29]. A notion of 1-superharmonic functions has been studied in the Euclidean setting in [42], but especially in the metric setting very little is known about these concepts in the case $p = 1$.

For consistency, we use the term 1-minimizer instead of function of least gradient. In [20, Theorem 4.1], 1-minimizers were shown to be continuous outside their jump sets. In this paper we show that this is a consequence of the fact that for super- and subminimizers $u$, the pointwise representatives $u^\wedge$ and $u^\vee$ are lower and upper semicontinuous, respectively, at every point. This is Theorem 3.11.

We also study some basic properties of solutions of obstacle problems in the case $p = 1$. Such solutions minimize the total variation among functions that lie above a given obstacle function and the solutions are, in particular, 1-superminimizers. In the Euclidean setting, obstacle problems for the BV class have been studied in, e.g., [13, 43, 48], and in the metric setting in [28]. In this paper we first prove standard De Giorgi-type and weak Harnack inequalities for 1-subminimizers and certain solutions of obstacle problems, following especially [2, 19, 25]; these inequalities allow us to control the total variation and essential supremum of solutions locally by their $L^1$-norms. Then we use these inequalities to show the aforementioned semicontinuity property of 1-superminimizers as well as a similar property for solutions of obstacle problems at points where the obstacle is continuous; see Theorem 3.13.

While these results are of some independent interest, our main goal is to consider certain questions of fine potential theory when $p = 1$. In the case $p > 1$ it is known that the so-called $p$-fine topology is the coarsest topology that makes all $p$-superharmonic functions continuous in open sets (alternatively upper semicontinuous, as superharmonic functions are lower semicontinuous already with respect to the metric topology). In Section 4 we define the notion of thinness and the resulting fine topology in the case $p = 1$, following [33], and generalize some properties concerning, in particular, points of nonzero capacity from the case $p > 1$ to the case $p = 1$. Then in Section 5 we prove the main result of this paper, namely the following weak Cartan property for 1-superminimizers.

**Theorem 1.1.** Let $A \subset X$ and let $x \in X \setminus A$ such that $A$ is 1-thin at $x$. Then there exist $R > 0$ and $u_1, u_2 \in \text{BV}(X)$ that are 1-superminimizers in $B(x, R)$ such that $\max\{u_1^\wedge, u_2^\wedge\} = 1$ in $A \cap B(x, R)$ and $u_1^\vee(x) = 0 = u_2^\vee(x)$. 
The analogous property in the case $p > 1$ is well-known and proved in the metric setting in [6]. We also prove a strong version of the property, requiring only one superminimizer, at points of nonzero capacity; this is Proposition 5.4. Then as in the case $p > 1$ we use the weak Cartan property to show that any topology that makes the upper approximate limit $u^\vee$ of every 1-superminimizer $u$ upper semicontinuous in every open set $\Omega$ necessarily contains the 1-fine topology. This is Theorem 5.7. However, we observe that unlike in the case $p > 1$, the converse does not hold, that is, the 1-fine topology does not always make $u^\vee$ upper semicontinuous for 1-superminimizers $u$; see Example 5.8.

Our main results seem to be new even in Euclidean spaces. A key motivation for the work is that a weak Cartan property will be useful in considering further questions such as $p$-strict subsets, fine connectedness, and the relationship between finely open and quasiopen sets for $p = 1$. Such properties have been considered in the case $p > 1$ in [4, 5, 35]; in this case, it seems that the strong version of the Cartan property, involving only one superminimizer function instead of two, is often required. However, we expect that the weak version will be mostly sufficient in the case $p = 1$.

2 Preliminaries

In this section we introduce most of the notation, definitions, and assumptions employed in the paper.

Throughout this paper, $(X, d, \mu)$ is a complete metric space that is equipped with a metric $d$ and a Borel regular outer measure $\mu$ that satisfies a doubling property. The doubling property means that there is a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d\mu(B(x, r)) < \infty$$

for every ball $B(x, r) := \{y \in X : d(y, x) < r\}$ with center $x \in X$ and radius $r > 0$. Sometimes we abbreviate $B := B(x, r)$ and $aB := B(x, ar)$ with $a > 0$; note that in metric spaces, a ball does not necessarily have a unique center point and radius, but we will always consider balls for which these have been specified. We also assume that $X$ supports a $(1, 1)$-Poincaré inequality that will be defined below, and that $X$ consists of at least 2 points. By iterating the doubling condition, we obtain for any $x \in X$ and any $y \in B(x, R)$ with $0 < r \leq R < \infty$ that

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C_d^Q} \left(\frac{r}{R}\right)^Q,$$

where $Q > 1$ only depends on the doubling constant $C_d$. When we want to state that a constant $C$ depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$, and
we understand all constants to be strictly positive. When a property holds outside a set of $\mu$-measure zero, we say that it holds almost everywhere, abbreviated a.e.

A complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact. Since $X$ is proper, for any open set $\Omega \subset X$ we define $\text{Lip}_{\text{loc}}(\Omega)$ to be the space of functions that are Lipschitz in every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of $\Omega$. Other local spaces of functions are defined analogously.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension one is defined to be

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.$$ 

The codimension one Hausdorff measure of $A \subset X$ is then defined to be

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

The measure theoretic boundary $\partial^* E$ of a set $E \subset X$ is the set of points $x \in X$ at which both $E$ and its complement have strictly positive upper density, i.e.,

$$\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$ 

The measure theoretic interior and exterior of $E$ are defined respectively by

(2.2) $$I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\}$$

and

(2.3) $$O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}.$$ 

Note that the space is always partitioned into the disjoint sets $\partial^* E$, $I_E$, and $O_E$. By Lebesgue’s differentiation theorem (see, e.g., [21, Chapter 1]), for a $\mu$-measurable set $E$ we have $\mu(E \Delta I_E) = 0$, where $\Delta$ is the symmetric difference.

All functions defined on $X$ or its subsets will take values in $[\mathbb{R}] := [-\infty, \infty]$. By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into $X$. A nonnegative Borel function $g$ on $X$ is an upper gradient of a function $u$ on $X$ if for all curves $\gamma$, we have

(2.4) $$|u(x) - u(y)| \leq \int_{\gamma} g \, ds,$$

where $x$ and $y$ are the end points of $\gamma$ and the curve integral is defined by using an arc-length parametrization; see [24, Section 2] where upper gradients were
originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|, |u(y)|$ is infinite.

In what follows, let $1 \leq p < \infty$. We say that a family of curves $\Gamma$ is of zero $p$-modulus if there is a nonnegative Borel function $\rho \in L^p(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho \, ds$ is infinite. A property is said to hold for $p$-almost every curve if it fails only for a curve family with zero $p$-modulus. If $g$ is a nonnegative $\mu$-measurable function on $X$ and (2.4) holds for $p$-almost every curve, we say that $g$ is a $p$-weak upper gradient of $u$. By only considering curves $\gamma \in \Gamma_1 \subset X$, we can talk about a function $g$ being a $(p$-weak) upper gradient of $u$ in $\Omega$.

Given an open set $\Omega \subset X$, we let

$$\|u\|_{N^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \inf \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all $p$-weak upper gradients $g$ of $u$ in $\Omega$. The substitute for the Sobolev space $W^{1,p}$ in the metric setting is the Newton-Sobolev space

$$N^{1,p}(\Omega) := \{ u : \|u\|_{N^{1,p}(\Omega)} < \infty \}.$$

We understand every Newton–Sobolev function to be defined at every $x \in \Omega$ (even though $\| \cdot \|_{N^{1,p}(\Omega)}$ is then only a seminorm). It is known that for any $u \in N^{1,p}_{loc}(\Omega)$ (recall the definition of local function spaces from page 58) there exists a minimal $p$-weak upper gradient of $u$ in $\Omega$, always denoted by $g_u$, satisfying $g_u \leq g$ a.e. in $\Omega$, for any $p$-weak upper gradient $g \in L^p_{loc}(\Omega)$ of $u$ in $\Omega$; see [2, Theorem 2.25].

The $p$-capacity of a set $A \subset X$ is given by

$$\text{Cap}_p(A) := \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u \geq 1$ in $A$. We know that $\text{Cap}_p$ is an outer capacity, meaning that

$$\text{Cap}_p(A) = \inf \{ \text{Cap}_p(U) : U \supset A \text{ is open} \}$$

for any $A \subset X$; see, e.g., [2, Theorem 5.31].

If a property holds outside a set $A \subset X$ with $\text{Cap}_p(A) = 0$, we say that it holds $p$-quasi-everywhere, abbreviated $p$-q.e. If $u \in N^{1,p}(\Omega)$, then $\|u - v\|_{N^{1,p}(\Omega)} = 0$ if and only if $u = v$ $p$-q.e. in $\Omega$; see [2, Proposition 1.61]. By [18, Theorem 4.3, Theorem 5.1] we know that if $A \subset X$,

$$\text{(2.5) } \text{Cap}_1(A) = 0 \text{ if and only if } \mathcal{H}(A) = 0.$$
The variational $p$-capacity of a set $A \subset \Omega$ with respect to an open set $\Omega \subset X$ is given by
\[
cap_p(A, \Omega) := \inf \int_X g_u^p \, d\mu,
\]
where the infimum is taken over functions $u \in N^{1,p}(X)$ such that $u \geq 1$ in $A$ (equivalently, $p$-q.e. in $A$) and $u = 0$ in $X \setminus \Omega$; recall that $g_u$ is the minimal $p$-weak upper gradient of $u$. We know that $\cap_p$ is also an outer capacity, in the sense that if $\Omega \subset X$ is a bounded open set and $A \subset \Omega$, then
\[
cap_p(A, \Omega) = \inf \{ \cap_p(U, \Omega) : U \text{ open}, A \subset U \subset \Omega \};
\]
see [2, Theorem 6.19]. It is easy to see that in the definitions of capacities, we can assume the test functions to satisfy $0 \leq u \leq 1$. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see, e.g., [2].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [40]. See also, e.g., [1, 14, 16, 17, 47] for the classical theory in the Euclidean setting. Let $\Omega \subset X$ be an open set. Given a function $u \in L^1_{\text{loc}}(\Omega)$, we define the total variation of $u$ in $\Omega$ by
\[
\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in \text{Lip}_{\text{loc}}(\Omega), u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\},
\]
where each $g_{u_i}$ is again the minimal 1-weak upper gradient of $u_i$ in $\Omega$. (In [40], local Lipschitz constants were used instead of upper gradients, but the properties of the total variation can be proved similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in BV(\Omega)$, if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define
\[
\|Du\|(A) := \inf \{ \|Du\|(U) : A \subset U, U \subset X \text{ is open} \}.
\]
If $u \in L^1_{\text{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$, $\|Du\|\cdot$ is a Radon measure on $\Omega$ by [40, Theorem 3.4]. A $\mu$-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where $\chi_E$ is the characteristic function of $E$. The perimeter of $E$ in $\Omega$ is also denoted by
\[
P(E, \Omega) := \|D\chi_E\|(\Omega).
\]
For any $u, v \in L^1_{\text{loc}}(\Omega)$, it is straightforward to show that
\[
(D(u + v))_\Omega \leq \|Du\|_\Omega + \|Dv\|_\Omega.
\]
We have the following coarea formula from [40, Proposition 4.2]: if $\Omega \subset X$ is an open set and $u \in L^1_{\text{loc}}(\Omega)$, then
\[
\|Du\|_\Omega = \int_{-\infty}^\infty P(\{u > t\}, \Omega) \, dt.
\]
The following Leibniz rule is proved in [20, Lemma 3.2]: if $\Omega \subset X$ is an open set, $u \in \text{BV}(\Omega)$, and $\eta \in \text{Lip}(\Omega)$ with $0 \leq \eta \leq 1$, then $\eta u \in \text{BV}(\Omega)$ with

$$\|D(\eta u)\|((\Omega)) \leq \int_{\Omega} |u| g_\eta \, d\mu + \int_{\Omega} \eta \, d\|Du\|,$$

(2.8)

where $g_\eta$ is the minimal 1-weak upper gradient of $\eta$ in $\Omega$.

We will assume throughout the paper that $X$ supports a $(1, 1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_P r \int_{B(x, \lambda r)} g \, d\mu,$$

where

$$u_{B(x, r)} := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu.$$

Applying the Poincaré inequality to sequences of approximating locally Lipschitz functions in the definition of the total variation, we get the following BV version: for every ball $B(x, r)$ and every $u \in L^1_{\text{loc}}(X)$, we have

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_P r \frac{\|Du\|(B(x, \lambda r))}{\mu(B(x, \lambda r))}.$$

(2.9)

For a $\mu$-measurable set $E \subset X$, the above implies (see, e.g., [31, Equation (3.1)]) the relative isoperimetric inequality

$$\min \{ \mu(B(x, r) \cap E), \mu(B(x, r) \setminus E) \} \leq 2C_P r \text{P}(E, B(x, \lambda r)).$$

Recall the exponent $Q > 1$ from (2.1). From the $(1, 1)$-Poincaré inequality, by [2, Theorem 4.21, Theorem 5.51] we also get the following Sobolev inequality: if $x \in X$, $0 < r < \frac{1}{4} \text{diam} X$, and $u \in N^{1,1}(X)$ with $u = 0$ in $X \setminus B(x, r)$, then

$$\left( \int_{B(x, r)} |u|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \leq C_S r \int_{B(x, r)} |u|^{Q/(Q-1)} \, d\mu$$

(2.10)

for a constant $C_S = C_S(C_d, C_P, \lambda) \geq 1$. Then for any $x \in X$, any $0 < r < \frac{1}{4} \text{diam} X$, and any $u \in L^1_{\text{loc}}(X)$ with $u = 0$ in $X \setminus B(x, r)$, by applying the above to a suitable sequence approximating $u$, we obtain

$$\left( \int_{B(x, r)} |u|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \leq C_S r \frac{\|Du\|(X)}{\mu(B(x, r))}.$$

(2.11)

For any $\mu$-measurable set $E \subset B(x, r)$, this implies by Hölder’s inequality

$$\mu(E) \leq C_S r \text{P}(E, X).$$

(2.12)
Moreover, if $\Omega \subset X$ is an open set with $\text{diam} \, \Omega < \frac{1}{4} \text{diam} \, X$ (meaning $\text{diam} \, \Omega < \infty$ in the case $\text{diam} \, X = \infty$) and $u \in L^1_{\text{loc}}(X)$ with $u = 0$ in $X \setminus \Omega$, then we can take a ball $B(x, r) \supset \Omega$ with $r = \text{diam} \, \Omega$, and so by (2.11) and Hölder’s inequality

\begin{equation}
\int_{\Omega} |u| d\mu \leq C_S \text{diam} \, \Omega \|Du\|(X).
\end{equation}

The lower and upper approximate limits of a function $u$ on $X$ are defined respectively by

\begin{equation}
\hat{u}(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(\{u < t\} \cap B(x, r))}{\mu(B(x, r))} = 0 \right\}
\end{equation}

and

\begin{equation}
\check{u}(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(\{u > t\} \cap B(x, r))}{\mu(B(x, r))} = 0 \right\}.
\end{equation}

Recall that we understand Newton–Sobolev functions to be defined at every point; this is standard practice (see, e.g., the monograph [2]) and is convenient for example in making sense of the upper gradient condition (2.4). By contrast, we understand BV functions to be $\mu$-equivalence classes. This is natural since in order to study the fine properties of BV functions, we need to consider the two different pointwise representatives $u^\wedge$ and $u^\vee$.

Throughout this paper we assume that $(X, d, \mu)$ is a complete metric space that is equipped with a doubling measure $\mu$ and supports a $(1, 1)$-Poincaré inequality.

### 3 Superminimizers and obstacle problems

In this section we consider superminimizers and solutions of obstacle problems in the case $p = 1$. The symbol $\Omega$ will always denote a nonempty open subset of $X$. We denote by $\text{BV}_{c}(\Omega)$ the class of functions $\phi \in \text{BV}(\Omega)$ with compact support in $\Omega$, that is, $\text{spt} \, \phi \subset \subset \Omega$.

#### Definition 3.1.

We say that $u \in \text{BV}_{\text{loc}}(\Omega)$ is a **1-minimizer** in $\Omega$ if for all $\phi \in \text{BV}_{c}(\Omega), \,$

\begin{equation}
\|Du\|((\text{spt} \, \phi)) \leq \|(u + \phi)\|((\text{spt} \, \phi)).
\end{equation}

We say that $u \in \text{BV}_{\text{loc}}(\Omega)$ is a **1-superminimizer** in $\Omega$ if (3.1) holds for all nonnegative $\phi \in \text{BV}_{c}(\Omega)$. We say that $u \in \text{BV}_{\text{loc}}(\Omega)$ is a **1-subminimizer** in $\Omega$ if (3.1) holds for all nonpositive $\phi \in \text{BV}_{c}(\Omega)$, or equivalently if $-u$ is a 1-superminimizer in $\Omega$. 

Equivalently, we can replace spt $\varphi$ by any set $A \subset \Omega$ containing spt $\varphi$ in the above definitions. It is easy to see that if $u$ is a 1-superminimizer and $a \geq 0$, $b \in \mathbb{R}$, then $au + b$ is a 1-superminimizer.

Given a nonempty bounded open set $\Omega \subset X$, a function $\psi : \Omega \to \mathbb{R}$, and $f \in L^1_{\text{loc}}(X)$ with $\|Df\|(X) < \infty$, we define the class of admissible functions

$$
\mathcal{K}_{\psi,f}(\Omega) := \{ u \in BV_{\text{loc}}(X) : u \geq \psi \text{ in } \Omega \text{ and } u = f \text{ in } X \setminus \Omega \}.
$$

The (in)equalities above are understood in the a.e. sense, since BV functions are only defined up to sets of $\mu$-measure zero. For brevity, we sometimes write $\mathcal{K}_{\psi,f}$ instead of $\mathcal{K}_{\psi,f}(\Omega)$. By using a cutoff function, it is easy to show that $\|Du\|(X) < \infty$ for every $u \in \mathcal{K}_{\psi,f}(\Omega)$.

**Definition 3.2.** We say that $u \in \mathcal{K}_{\psi,f}(\Omega)$ is a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem if $\|Du\|(X) \leq \|Dv\|\text{ for all } v \in \mathcal{K}_{\psi,f}(\Omega)$.

**Proposition 3.3.** If $\text{diam} \, \Omega < \frac{1}{4} \text{diam} \, X$ and $\mathcal{K}_{\psi,f}(\Omega) \neq \emptyset$, then there exists a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem.

**Proof.** Pick a sequence of functions $(u_i) \subset \mathcal{K}_{\psi,f}(\Omega)$ with

$$
\lim_{i \to \infty} \|Du_i\|(X) = \inf \{ \|Dv\|(X) : v \in \mathcal{K}_{\psi,f}(\Omega) \} < \infty.
$$

By the Poincaré inequality (2.13) and the subadditivity (2.6), we have for each $i \in \mathbb{N}$

$$
\int_X |u_i - f| d\mu \leq C_5 \text{diam} \, \Omega \|D(u_i - f)\|(X)
\leq C_5 \text{diam} \, \Omega (\|Du_i\|(X) + \|Df\|(X)),
$$

which is a bounded sequence. Thus $(u_i - f)$ is a bounded sequence in BV$(X)$, and so by [40, Theorem 3.7] there exists a subsequence (which we still denote by $(u_i - f)$) and a function $v \in BV(X)$ such that $u_i - f \to v$ in $L^1_{\text{loc}}(X)$. We can select a further subsequence (still denoted by $(u_i - f)$) such that $u_i(x) - f(x) \to v(x)$ for a.e. $x \in X$. Hence, letting $u := v + f$, we have $u \geq \psi$ in $\Omega$ and $u = f$ in $X \setminus \Omega$. Moreover, $u_i \to u$ in $L^1_{\text{loc}}(X)$, and then by lower semicontinuity of the total variation with respect to $L^1$-convergence, we get

$$
\|Du\|(X) \leq \liminf_{i \to \infty} \|Du_i\|(X),
$$

and so $u$ is a solution.

Unlike in the case $p > 1$, solutions are not generally unique. \qed
Example 3.4. Let $X = \mathbb{R}$ (unweighted), let $\Omega := (-1, 1)$ and let $f := \chi_{(0, \infty)}$ (the Heaviside function). Then it is easy to check that $u = f$ is a solution of the $\mathcal{K}_{-\infty, f}$-obstacle problem, but so is any increasing function $0 \leq u \leq 1$ with $u(-1) = 0$ and $u(1) = 1$.

The following fact, which is also in stark contrast to the case $p > 1$, is often useful.

Proposition 3.5. If $A \subset X$ and there exists a solution of the $\mathcal{K}_{A,0}$-obstacle problem, then there exists a set $E \subset X$ such that $\chi_E$ is also a solution.

Proof. Let $u$ be a solution. By the coarea formula (2.7) there exists $t \in (0, 1)$ such that $P(\{u > t\}, X) \leq \|Du\|(X)$. Letting $E := \{u > t\}$, we clearly have $\chi_E \geq \chi_A$ in $\Omega$ and $\chi_E = 0$ in $X \setminus \Omega$. Thus $\chi_E \in \mathcal{K}_{A,0}$ and so it is a solution. □

Whenever the characteristic function of a set $E$ is a solution of an obstacle problem, for simplicity we will call $E$ a solution as well. Similarly, if $\psi = \chi_A$ for some $A \subset X$, we let $\mathcal{K}_{A,f} := \mathcal{K}_{\psi,f}$.

The following simple fact will be of much use to us.

Lemma 3.6. If $x \in X$, $0 < r < R < \frac{1}{8} \text{diam} X$, and $A \subset B(x, r)$, then there exists $E \subset X$ that is a solution of the $\mathcal{K}_{A,0}(B(x, R))$-obstacle problem with

$$P(E, X) \leq \text{cap}_1(A, B(x, R)).$$

Proof. Fix $\epsilon > 0$. Since $A \subset B(x, R)$, clearly $\text{cap}_1(A, B(x, R)) < \infty$. By the definition of the variational capacity, we find $u \in N^{1,1}(X)$ with $u \geq 1$ in $A$, $u = 0$ in $X \setminus B(x, R)$, and

$$\text{cap}_1(A, B(x, R)) + \epsilon \geq \int_X g_u \, d\mu \geq \|Du\|(X),$$

where the last inequality follows from the fact that Lipschitz functions are dense in $N^{1,1}(X)$; see [45] or [2, Theorem 5.1]. Now $u \in \mathcal{K}_{A,0}(\Omega)$. Since $\epsilon > 0$ was arbitrary, by Proposition 3.3 and Proposition 3.5 we conclude that the $\mathcal{K}_{A,0}(B(x, R))$-obstacle problem has a solution $E \subset X$ such that $P(E, X) \leq \text{cap}_1(A, B(x, R))$. □

The following fact follows directly from the definitions.

Proposition 3.7. If $u \in \mathcal{K}_{\psi,f}(\Omega)$ is a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem, then $u$ is a 1-superminimizer in $\Omega$.

Next we prove De Giorgi-type and weak Harnack inequalities for 1-subminimizers and certain solutions of obstacle problems. The arguments we use
are mostly standard and have been employed in the metric setting previously in [2, 19, 25], but only for (quasi)minimizers or in the case \( p > 1 \), so we repeat the entire proofs with small modifications and some simplifications.

We denote \( u_+ := \max\{u, 0\} \) and \( u_- := \max\{-u, 0\} \).

**Proposition 3.8.** Suppose \( k \in \mathbb{R} \) and \( B(x, s_2) \subseteq \Omega \), and assume either that 

(a) \( u \) is a 1-subminimizer in \( \Omega \), or 

(b) \( \Omega \) is bounded, \( u \) is a solution of the \( \mathcal{K}_{\psi,f}(\Omega) \)-obstacle problem, and \( \psi \leq k \) a.e. in \( B(x, s_2) \).

Then if \( 0 < s_1 < s_2 \),

\[
\|D(u - k)_+\|(B(x, s_1)) \leq \frac{2}{s_2 - s_1} \int_{B(x,s_2)} (u - k)_+ \, d\mu.
\]  

**Proof.** Take a Lipschitz function \( 0 \leq \eta \leq 1 \) with compact support in \( B(x, s_2) \), such that \( \eta = 1 \) in \( B(x, s_1) \) and \( g_\eta \leq 2/(s_2 - s_1) \). Using the Leibniz rule (2.8) it can be verified that \( \eta(u - k)_+ \in BV_c(\Omega) \). Now if \( u \) is a 1-subminimizer (alternative (a)), we get

\[
\|Du\|(B(x, s_2)) \leq \|D(u - \eta(u - k)_+)\|(B(x, s_2)).
\]  

In alternative (b), we have at a.e. point in \( \Omega \) either \( u - \eta(u - k)_+ = u \geq \psi \) or \( u - \eta(u - k)_+ = (1 - \eta)u + \eta k \geq \psi \), and moreover \( u - \eta(u - k)_+ \in BV_{loc}(X) \), so then \( u - \eta(u - k)_+ \in \mathcal{K}_{\psi,f}(\Omega) \), and thus (3.3) again holds. Note that

\[
u - \eta(u - k)_+ = \min\{u, k\} + (1 - \eta)(u - k)_+.
\]

Using the coarea formula it can be shown that \( \|Du\| = \|D\min\{u, k\}\| + \|D(u - k)_+\| \) as measures in \( \Omega \) (see [20, Lemma 3.5]), and thus we get

\[
\|D\min\{u, k\}\|(B(x, s_2)) + \|D(u - k)_+\|(B(x, s_2)) = \|Du\|(B(x, s_2)) \leq \|D\min\{u, k\} + (1 - \eta)(u - k)_+\|(B(x, s_2))\quad \text{by (3.3)}
\]

\[
\leq \|D\min\{u, k\}\|(B(x, s_2)) + \|D((1 - \eta)(u - k)_+)\|(B(x, s_2))
\]

by (2.6). Since \( \|D\min\{u, k\}\|(B(x, s_2)) \leq \|Du\|(B(x, s_2)) < \infty \), we get

\[
\|D(u - k)_+\|(B(x, s_2)) \leq \|D((1 - \eta)(u - k)_+)\|(B(x, s_2)).
\]
Here we have by the Leibniz rule (2.8)
\[
\begin{align*}
\|D((1-\eta)(u-k)_+)(B(x, s_2)) & \leq \int_{B(x,s_2)} g \eta (u-k)_+ \, d\mu + \int_{B(x,s_2)} (1-\eta) d\|D(u-k)_+\| \\
& \leq \frac{2}{s_2-s_1} \int_{B(x,s_2)} (u-k)_+ \, d\mu + \|D(u-k)_+\|(B(x, s_2) \setminus B(x, s_1)).
\end{align*}
\]
Noting that also \(\|D(u-k)_+\|(B(x, s_2) \setminus B(x, s_1)) < \infty\), we combine the above with (3.4) to get the result. \(\square\)

In proving the following weak Harnack inequality, we closely follow [2, Proposition 8.2], where the analogous result is proved in the case \(p > 1\). Recall the definition of the exponent \(Q > 1\) from (2.1).

**Proposition 3.9.** Let \(u \in BV(B(x,R))\) such that (3.2) holds for all \(0 < s_1 < s_2 \leq R < \frac{1}{4} \text{diam}X\) and all \(k \geq k^* \in \mathbb{R}\). Let \(k_0 \geq k^*\) and \(r \in (0, R)\).

Then
\[
\text{ess sup}_{B(x,r)} u \leq C_1 \left(\frac{R}{R-r}\right)^Q \int_{B(x,R)} (u-k_0)_+ \, d\mu + k_0
\]
for some constant \(C_1 = C_1(C_d, C_P, \lambda)\).

**Proof.** Choose \(r \leq r_1 < r_2 \leq R\) and let \(\rho := (r_1+r_2)/2\). Let \(\eta\) be a \(2/(r_2-r_1)\)-Lipschitz function such that \(0 \leq \eta \leq 1\), \(\eta = 1\) in \(B(x, r_1)\), and \(\eta = 0\) outside \(B(x, \rho)\). Let \(l_2 > l_1 \geq k^*\), and
\[
A := \{u > l_2\} \cap B(x, \rho).
\]

Now
\[
\mu(A) \leq \frac{1}{l_2-l_1} \int_{B(x,\rho)} (u-l_1)_+ \, d\mu \leq \frac{1}{l_2-l_1} \int_{B(x,r_2)} (u-l_1)_+ \, d\mu.
\]
Let \(v := \eta(u-l_2)_+\). By Hölder’s inequality
\[
\int_{B(x,r_1)} (u-l_2)_+ \, d\mu \leq \int_{B(x,\rho)} v \, d\mu \leq \left(\int_{B(x,\rho)} v^{Q/(Q-1)} \, d\mu\right)^{(Q-1)/Q} \mu(A)^{1/Q}
\]
\[
\leq \left(\int_{B(x,\rho)} v^{Q/(Q-1)} \, d\mu\right)^{(Q-1)/Q} \left(\frac{1}{l_2-l_1} \int_{B(x,r_2)} (u-l_1)_+ \, d\mu\right)^{1/Q}.
\]

(3.5)
By the Sobolev inequality (2.11) (here we need $R < \frac{1}{4} \text{diam } X$)

\[
\left( \int_{B(x, \rho)} v^\frac{Q}{Q-1} d\mu \right)^{(Q-1)/Q} = \left( \int_{B(x, \rho_2)} v^\frac{Q}{Q-1} d\mu \right)^{(Q-1)/Q} \\
\leq \frac{C_{Sr_2}}{\mu(B(x, r_2))^{1/Q}} \|Dv\|((X) \\
\leq \frac{C_{Sr_2}}{\mu(B(x, r_2))^{1/Q}} \left( \int_X \eta \|D(u - l_2)\| + \int_X g_\eta(u - l_2)_+ d\mu \right),
\]

where the last inequality follows from the Leibniz rule (2.8). Note that $g_\eta = 0$ a.e. in $X \setminus B(x, \rho)$; see [2, Corollary 2.21]. By using this and the assumption of the proposition, we can estimate

\[
\int_X \eta \|D(u - l_2)\| + \int_X g_\eta(u - l_2)_+ d\mu \\
\leq \|D(u - l_2)_+(B(x, \rho)) + \int_{B(x, \rho)} g_\eta(u - l_2)_+ d\mu \\
\leq \frac{2}{r_2 - \rho} \int_{B(x, r_2)} (u - l_2)_+ d\mu + \frac{2}{r_2 - r_1} \int_{B(x, \rho)} (u - l_2)_+ d\mu \\
\leq \frac{6}{r_2 - r_1} \int_{B(x, r_2)} (u - l_2)_+ d\mu.
\]

By combining this with (3.5) and (3.6), we get (note that $l_1 < l_2$)

\[
\int_{B(x, r_1)} (u - l_2)_+ d\mu \\
\leq \frac{6C_{Sr_2}}{\mu(B(x, r_2))^{1/Q}(r_2 - r_1)} \int_{B(x, r_2)} (u - l_1)_+ d\mu \left( \frac{1}{l_2 - l_1} \int_{B(x, r_2)} (u - l_1)_+ d\mu \right)^{1/Q}.
\]

Let

\[ u(k, s) := \int_{B(x, s)} (u - k)_+ d\mu. \]

Assuming that $r_2 \leq 2r_1$, we have $\mu(B(x, r_2)) \leq C_d \mu(B(x, r_1))$. Letting $C_0 := 6C_S C_d$, we now get

\[ u(l_2, r_1) \leq \frac{C_0 r_2}{(r_2 - r_1)(l_2 - l_1)^{1/Q}} u(l_1, r_2)^{1+1/Q}. \]

For $i = 0, 1, \ldots$, let $\rho_i := r + 2^{-i}(R - r)$ and $k_i := k_0 + d(1 - 2^{-i})$, where $d > 0$ is chosen below. We show by induction that $u(k_i, \rho_i) \leq 2^{-i(1+Q)} u(k_0, R)$ for $i = 0, 1, \ldots$. This
is clearly true for \( i = 0 \). Assuming the claim is true for \( i \), we have

\[
    u(k_{i+1}, \rho_{i+1}) \leq \frac{C_0 R}{(\rho_i - \rho_{i+1})(k_{i+1} - k_i)^{1/Q}} u(k_i, \rho_i)^{1+1/Q}
\]

\[
    \leq \frac{C_0 R}{2^{-(i+1)(R - r)d^{1/Q}2^{-(i+1)/Q}} u(k_i, \rho_i)^{1+1/Q}}
\]

\[
    \leq \frac{C_0 R}{(R - r)d^{1/Q}2^{(i+1)(1+1/Q)}} (2^{-i(1+Q)}u(k_0, R))^{1+1/Q}
\]

\[
    = 2^{-i(1+Q)}u(k_0, R)
\]

if \( d = 2^{(Q+1)^2}(C_0 R)^Q u(k_0, R)/(R - r)^Q \) (note that we can assume \( u(k_0, R) > 0 \)), and so the claim is true for \( i + 1 \). It follows that

\[
    u(k_0 + d, R) \leq u(k_i, \rho_i) \leq 2^{-i(1+Q)}u(k_0, R)
\]

for all \( i = 0, 1, \ldots \), so that \( u(k_0 + d, R) = 0 \), i.e., \( u \leq k_0 + d \) a.e. in \( B(x, R) \). \( \square \)

We combine the previous two propositions to get the following theorem. Recall that \( \Omega \) always denotes a nonempty open set.

**Theorem 3.10.** Suppose \( k \in \mathbb{R} \) and \( 0 < R < \frac{1}{4} \text{diam} \ X \) with \( B(x, R) \subseteq \Omega \), and assume either either that

(a) \( u \) is a 1-subminimizer in \( \Omega \), or

(b) \( \Omega \) is bounded, \( u \) is a solution of the \( K_{\psi,f}(\Omega) \)-obstacle problem, and \( \psi \leq k \)

a.e. in \( B(x, R) \).

Then for any \( 0 < r < R \),

\[
    \text{ess sup}_{B(x,r)} u \leq C_1 \left( \frac{R}{R - r} \right)^Q \int_{B(x,R)} (u - k)_+ \, d\mu + k.
\]

Unlike \( p \)-harmonic functions for \( p > 1 \), 1-minimizers can be discontinuous (regardless of how we choose the representative), as demonstrated already by the Heaviside function on the real line. However, the following semicontinuity holds. Recall the definitions of the pointwise representatives \( u^\wedge \) and \( u^\vee \) from (2.14) and (2.15).

**Theorem 3.11.** Let \( u \) be a 1-superminimizer in \( \Omega \). Then \( u^\wedge : \Omega \to (-\infty, \infty] \) is lower semicontinuous.

**Proof.** Let \( x \in \Omega \) and \( R > 0 \) with \( B(x, 2R) \subseteq \Omega \). By Theorem 3.10(a), \( u \geq \beta \) a.e. in \( B(x, R) \) for some \( \beta \in \mathbb{R} \). Thus \( u^\wedge(x) \geq \beta > -\infty \). Take a real number \( t \in [\beta, u^\wedge(x)] \); note that we could have \( u^\wedge(x) = \infty \). Clearly \( t - u \) is a
1-subminimizer in $\Omega$. Fix $\varepsilon > 0$. By applying Theorem 3.10(a) with $k = 0$, we get for any $0 < r \leq R$

\[
\text{ess sup}_{B(x, r/2)} (t - u) \leq 2^Q C_1 \int_{B(x, r)} (t - u) \, d\mu
\]

\[
= \frac{2^Q C_1}{\mu(B(x, r))} \left( \int_{B(x, r) \cap \{ t - \varepsilon \leq u < t \}} (t - u) \, d\mu + \int_{B(x, r) \cap \{ u < t - \varepsilon \}} (t - u) \, d\mu \right)
\]

\[
\leq 2^Q C_1 \varepsilon + 2^Q C_1 (t - \beta) \mu(\{ u < t - \varepsilon \} \cap B(x, r)) \mu(B(x, r))
\]

\[
\rightarrow 2^Q C_1 \varepsilon \text{ as } r \rightarrow 0
\]

by the definition of the lower approximate limit $u^\wedge(x)$, and the fact that $t \leq u^\wedge(x)$. Hence for small enough $r > 0$, $u \geq t - 2^Q C_1 \varepsilon - \varepsilon$ a.e. in $B(x, r/2)$. Thus $u^\wedge \geq t - 2^Q C_1 \varepsilon - \varepsilon$ in $B(x, r/2)$. Now if $u^\wedge(x) \in \mathbb{R}$, we can choose $t = u^\wedge(x)$ to establish the lower semicontinuity, whereas if $u^\wedge(x) = \infty$, we can choose $t \in \mathbb{R}$ arbitrarily large to achieve the same.

We conclude that for a 1-minimizer $u$, $u^\wedge$ is lower semicontinuous and $u^\vee$ is upper semicontinuous. From this we immediately get the following corollary, which was previously proved in [20, Theorem 4.1]. We define the jump set $S_u$ of a BV function $u$ as the set where $u^\wedge < u^\vee$.

**Corollary 3.12.** Let $u$ be a 1-minimizer in $\Omega$. Then $u^\vee$ (alternatively $u^\wedge$, or the precise representative $\bar{u} := (u^\wedge + u^\vee)/2$) is continuous at every $x \in \Omega \setminus S_u$.

For obstacle problems in the case $p > 1$, it is well known that continuity of the obstacle implies continuity of the solution; see [15] or [2, Theorem 8.29]. In the case $p = 1$, the best we can hope for is lower semicontinuity of $u^\wedge$ (which holds for superminimizers and thus for solutions of obstacle problems) and upper semicontinuity of $u^\vee$. These we can indeed obtain.

**Theorem 3.13.** Let $u$ be a solution of the $\mathcal{K}_{\psi, f}(\Omega)$-obstacle problem. If $x \in \Omega$ and $\psi^\vee(x) = \text{ess lim sup}_{y \to x} \psi(y)$ (with value in $\mathbb{R}$), then $u^\vee$ is ($\mathbb{R}$-valued) upper semicontinuous at $x$. If $\psi^\vee(x) < \infty$, then $u^\vee$ is real-valued upper semicontinuous at $x$.

Here $\text{ess lim sup}_{y \to x} \psi(y) := \lim_{r \to 0} \text{ess sup}_{B(x, r)} \psi$. In particular, it is enough if $\psi$ is continuous (as an $\mathbb{R}$-valued function) at $x$. In Example 5.8 we will see that without assumptions on $\psi$, $u^\vee$ is not always upper semicontinuous.
Proof. Assume first that $\psi^\vee(x) = \infty$. Then since $u \geq \psi$, clearly

$$u^\vee(x) \geq \psi^\vee(x) = \infty,$$

guaranteeing $\mathbb{R}$-valued upper semicontinuity at $x$.

From now on, assume $\psi^\vee(x) < \infty$. By the fact that $\psi^\vee(x) = \text{ess lim sup}_{y \to x} \psi(y)$, there exist $k_0 \in \mathbb{R}$ and $R_0 > 0$ such that $\psi \leq k_0$ a.e. in $B(x, R_0) \subseteq \Omega$, and so by Theorem 3.10(b),

$$\text{ess sup} u \leq 2^Q C_1 \int_{B(x, R_0)} (u - k_0) d\mu + k_0 =: M < \infty.$$ 

We conclude that $u^\vee(x) < \infty$, and since also $u^\wedge(x) > -\infty$ by Theorem 3.11, we have $u^\vee(x) \in \mathbb{R}$.

Fix $\varepsilon > 0$. Since $u \geq \psi$ in $\Omega$, we have

$$u^\vee(x) \geq \psi^\vee(x).$$

Let $k := u^\vee(x) + \varepsilon \geq \psi^\vee(x) + \varepsilon$. If $\psi^\vee(x) \in \mathbb{R}$ (respectively, $\psi^\vee(x) = -\infty$), by the fact that $\psi^\vee(x) = \text{ess lim sup}_{y \to x} \psi(y)$ we find $R \in (0, R_0/2]$ such that $\psi \leq \psi^\vee(x) + \varepsilon \leq k$ a.e. in $B(x, R)$ (respectively, $\psi \leq k$ a.e. in $B(x, R)$). Thus we can apply Theorem 3.10(b) to get for any $0 < r \leq R$

$$\text{ess sup} u \leq 2^Q C_1 \int_{B(x, r/2)} (u - u^\vee(x) - \varepsilon) d\mu + u^\vee(x) + \varepsilon.$$

Here

$$\int_{B(x, r)} (u - u^\vee(x) - \varepsilon) d\mu \leq (M - u^\vee(x)) \frac{\mu(\{u > u^\vee(x) + \varepsilon\} \cap B(x, r))}{\mu(B(x, r))} \to 0$$

as $r \to 0$ by the definition of the upper approximate limit $u^\vee(x)$. Thus for sufficiently small $r > 0$,

$$\text{ess sup} u \leq u^\vee(x) + 2\varepsilon$$

and thus $u^\vee \leq u^\vee(x) + 2\varepsilon$ in $B(x, r/2)$. We conclude that

$$\limsup_{y \to x} u^\vee(y) \leq u^\vee(x) < \infty,$$

and since $u^\vee \geq u^\wedge > -\infty$ in $\Omega$ by Theorem 3.11, we have established real-valued upper semicontinuity at $x$. □

For general BV functions we have the following result, which follows from [34, Theorem 1.1], and was proved earlier in the Euclidean setting in [12, Theorem 2.5].
Proposition 3.14. Let $u \in BV(X)$ and $\varepsilon > 0$. Then there exists an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $u^\wedge|_{X \setminus G}$ is lower semicontinuous.

This quasi-semicontinuity is to be compared with the quasicontinuity of Newton–Sobolev functions: if $u \in N^{1,p}(X)$ for $1 \leq p < \infty$ and $\varepsilon > 0$, then there exists an open set $G \subset X$ such that $\text{Cap}_p(G) < \varepsilon$ and $u|_{X \setminus G}$ is continuous; see [7, Theorem 1.1] or [2, Theorem 5.29].

4 The 1-fine topology

In this section we consider some basic properties of the 1-fine topology. The following definition is from [33].

Definition 4.1. We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.$$ 

If $A$ is not 1-thin at $x$, we say that it is 1-thick. We also say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open subsets of $X$.

See [33, Section 4] for motivation of the definition of 1-thinness, and for a proof of the fact that the 1-fine topology is indeed a topology. See also Example 5.8 for an example of how a Euclidean set’s thinness can be estimated.

We record the following fact given in [2, Lemma 11.22], and use it to prove two lemmas that will be useful later.

Lemma 4.2. Let $x \in X$, $r > 0$, and $A \subset B(x, r)$. Then for every $1 < s < t$ with $tr < \frac{1}{4}\text{diam}X$, we have

$$\text{cap}_1(A, B(x, tr)) \leq \text{cap}_1(A, B(x, sr)) \leq C_5 \left(1 + \frac{t}{s - 1}\right) \text{cap}_1(A, B(x, tr)),$$

where $C_5$ is the constant in the Sobolev inequality (2.10).

Lemma 4.3. Let $A \subset X$, $x \in X$, $R > 0$, and $M > 1$ such that

$$\lim_{i \to \infty} M^{-i}R \frac{\text{cap}_1(A \cap B(x, M^{-i}R), B(x, 2M^{-i}R))}{\mu(B(x, M^{-i}R))} = 0.$$ 

Then

$$\lim_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.$$
Proof. If \( i \in \mathbb{N} \) such that \( 2M^{-i+1}R < \frac{1}{4} \text{ diam } X \) and \( r \in [M^{-i-1}R, M^{-i}R] \), we have by Lemma 4.2 (with \( s = 2 \) and \( t = 2M \))

\[
\frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} \\
\leq C_S(1 + 2M)\frac{\text{cap}_1(A \cap B(x, r), B(x, 2Mr))}{\mu(B(x, r))} \\
\leq C_S(1 + 2M)\frac{\text{cap}_1(A \cap B(x, r), B(x, 2M^{-i}R))}{\mu(B(x, r))} \\
\leq C_S(1 + 2M)\mu\lceil \frac{\log_2 M}{M} \rceil M^{-i}R \frac{\text{cap}_1(A \cap B(x, M^{-i}R), B(x, 2M^{-i}R))}{\mu(B(x, M^{-i}R))},
\]

where \( \lceil a \rceil \) denotes the smallest integer at least \( a \). From this the claim follows. \( \square \)

The following is a standard result in the case \( p > 1 \) (see, e.g., [23, Lemma 12.11] or [6, Lemma 4.7]) and we prove it similarly for \( p = 1 \).

**Lemma 4.4.** Let \( A \subset X \) and \( x \in X \setminus A \). If \( A \) is 1-thin at \( x \), there exists an open set \( W \supset A \) that is 1-thin at \( x \).

Proof. Let \( B_i := B(x, 2^{-i}) \), \( i \in \mathbb{N} \). By Lemma 4.2, if \( 2^{-i+2} < \frac{1}{4} \text{ diam } X \), then

\[
\text{cap}_1(A \cap \overline{B_i}, 2B_i) \leq 5C_S \text{cap}_1(A \cap \overline{B_i}, 4B_i) \leq 5C_S \text{cap}_1(A \cap 2B_i, 4B_i).
\]

By the fact that \( \text{cap}_1 \) is an outer capacity, for each \( i \in \mathbb{N} \) we find an open set \( W_i \supset A \cap \overline{B_i} \) such that

\[
2^{-i} \frac{\text{cap}_1(W_i, 2B_i)}{\mu(B_i)} \leq 2^{-i} \frac{\text{cap}_1(A \cap \overline{B_i}, 2B_i)}{\mu(B_i)} + 1/i.
\]

Let

\[
W := (X \setminus \overline{B_1}) \cup (W_1 \setminus \overline{B_2}) \cup (W_1 \cap W_2 \setminus \overline{B_3}) \cup (W_1 \cap W_2 \cap W_3 \setminus \overline{B_4}) \cup \cdots.
\]

Now \( W \) is open and \( A \subset W \), and \( W \cap B_i \subset W_i \) for all \( i \in \mathbb{N} \). Thus by combining the two inequalities above, we get for any \( i \in \mathbb{N} \) with \( 2^{-i+2} < \frac{1}{4} \text{ diam } X \),

\[
2^{-i} \frac{\text{cap}_1(W \cap B_i, 2B_i)}{\mu(B_i)} \leq 2^{-i} \frac{\text{cap}_1(W_i, 2B_i)}{\mu(B_i)} \\
\leq 2^{-i} \frac{\text{cap}_1(A \cap \overline{B_i}, 2B_i)}{\mu(B_i)} + 1/i \\
\leq 5C_S C_d 2^{-i+1} \frac{\text{cap}_1(A \cap 2B_i, 4B_i)}{\mu(2B_i)} + 1/i \\
\to 0 \quad \text{as } i \to \infty
\]

by the fact that \( A \) is 1-thin at \( x \). By Lemma 4.3 we conclude that \( W \) is also 1-thin at \( x \). \( \square \)
The analog of the next proposition is again known for $p > 1$ (see [6, Proposition 1.3]) but in this case our proof will be rather different. In the case $p > 1$ the proof relies on the theory of $p$-harmonic functions, but we are able to use a more direct argument that relies on the relative isoperimetric inequality.

**Proposition 4.5.** Let $x \in X$ with $\text{Cap}_1(\{x\}) > 0$. Then $\{x\}$ is 1-thick at $x$.

Towards proving the proposition, we first collect some more facts. According to [2, Proposition 6.16], if $x \in X$, $0 < r < 1/8 \text{ diam} X$, and $A \subset B(x, r)$, then for some constant $C = C(C_d, C_P, \lambda)$,

\begin{equation}
\frac{\text{Cap}_1(A)}{C(1 + r)} \leq \text{cap}_1(A, B(x, 2r)) \leq 2 \left(1 + \frac{1}{r}\right) \text{Cap}_1(A).
\end{equation}

In fact, the proof reveals that the second inequality holds with any $r > 0$. We will need one more estimate for the variational 1-capacity; recall the definition of the measure theoretic interior $I_A$ from (2.2).

**Lemma 4.6.** Let $x \in X$, $0 < r < 1/8 \text{ diam} X$, and $A \subset B(x, r)$ with $x \in I_A$. Then there exists $s_r \in (0, r]$ such that

\[ \frac{\mu(B(x, s_r))}{C_2 s_r} \leq \text{cap}_1(A, B(x, 2r)) \]

for a constant $C_2 = C_2(C_d, C_P)$.

**Proof.** By Lemma 3.6 we find a set $E \subset B(x, 2r)$ such that $A \subset E$ and

\begin{equation}
P(E, X) \leq \text{cap}_1(A, B(x, 2r)).
\end{equation}

By the doubling property of the measure and the fact that $0 < r < 1/8 \text{ diam} X$, there exists $\beta = \beta(C_d) \in (1/2, 1)$ such that

\[ \mu(E) \leq \mu(B(x, 2r)) \leq \beta \mu(B(x, 4r)); \]

see [2, Lemma 3.7, Proposition 4.2]. Now pick the first number $i = 0, 1, \ldots$ such that

\[ \mu(E \cap B(x, 2^{-i+1}r)) \geq \frac{1}{2} \mu(B(x, 2^{-i+1}r)); \]

such $i$ exists by the fact that $x \in I_A \subset I_E$. If $i = 0$, then

\[ \frac{1}{2C_d} \mu(B(x, 4r)) \leq \frac{1}{2} \mu(B(x, 2r)) \leq \mu(E) \leq \beta \mu(B(x, 4r)). \]

If $i \geq 1$, then

\[ \mu(E \cap B(x, 2^{-i+2}r)) \leq \frac{1}{2} \mu(B(x, 2^{-i+2}r)), \]
but also
\[\mu(E \cap B(x, 2^{-i}r)) \geq \mu(E \cap B(x, 2^{-i+i'}r)) \geq \frac{1}{2} \mu(B(x, 2^{-i+i'}r)) \geq \frac{1}{2C_d} \mu(B(x, 2^{-i}r)).\]

Letting \(s := 2^{-i}r\), in both cases
\[\frac{1}{2C_d} \mu(B(x, s)) \leq \mu(E \cap B(x, s)) \leq \beta \mu(B(x, s)).\]

By the relative isoperimetric inequality (2.9),
\[2C_P s P(E, B(x, \lambda s)) \geq \min \left\{1 - \beta, \frac{1}{2C_d}\right\} \mu(B(x, s)).\]

Thus by (4.2),
\[\text{cap}_1(A, B(x, 2r)) \geq P(E, X) \geq P(E, B(x, \lambda s)) \geq (2C_P)^{-1} \min \left\{1 - \beta, \frac{1}{2C_d}\right\} \frac{\mu(B(x, s))}{s} \geq (8C_P)^{-1} \min \left\{1 - \beta, \frac{1}{2C_d}\right\} \frac{\mu(B(x, s/4))}{s/4}.\]

Thus we can choose \(s_r := s/4\). \(\square\)

We also need the following simple lemma.

**Lemma 4.7.** Suppose \(f : (0, \infty) \to (0, \infty)\) such that \(\lim_{r \to 0^+} f(r) = 0\). Pick \(s_r \in (0, r]\) for every \(r > 0\). Then
\[\limsup_{r \to 0} \frac{f(r)}{f(s_r)} \geq 1.\]

**Proof.** Fix \(\varepsilon > 0\). For every sufficiently small \(R > 0\) we find \(0 < r < R\) such that \(f(r) > \sup_{0 < s < R} f(s)/(1 + \varepsilon)\). Then also \(f(r) > f(s_r)/(1 + \varepsilon)\), and letting \(\varepsilon \to 0\) we get the result. \(\square\)

**Proof of Proposition 4.5.** First assume that
\[\limsup_{r \to 0} \frac{r}{\mu(B(x, r))} > 0.\]

By (4.1) we have
\[\limsup_{r \to 0} \frac{\text{cap}_1(\{x\} \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} \geq \limsup_{r \to 0} \frac{\text{Cap}_1(\{x\})}{C(1 + r) \mu(B(x, r))} > 0,\]
so \(\{x\}\) is 1-thick at \(x\).
Then suppose
\[ \lim_{r \to 0} \frac{r}{\mu(B(x, r))} = 0. \] (4.3)
(Note that this possibility needs to be taken into account by the Example below.)

Let \( 0 < r < \frac{1}{8} \text{ diam } X \). By the fact that \( \text{cap}_1 \) is an outer capacity, we find \( 0 < t \leq r \) such that
\[ \text{cap}_1(\{x\}, B(x, 2r)) \geq \text{cap}_1(B(x, t), B(x, 2r)) - 1. \]
By Lemma 4.6 we find \( s_r \in (0, r] \) such that
\[ \text{cap}_1(B(x, t), B(x, 2r)) \geq \frac{\mu(B(x, s_r))}{C_2 s_r}. \]
Combining these,
\[ \frac{r \text{cap}_1(\{x\}, B(x, 2r))}{\mu(B(x, r))} \geq \frac{r}{\mu(B(x, r))} \frac{\mu(B(x, s_r))}{C_2 s_r} - \frac{r}{\mu(B(x, r))}. \]
Letting \( f(r) := r/\mu(B(x, r)) \), we get by (4.3) and Lemma 4.7
\[ \limsup_{r \to 0} \frac{r \text{cap}_1(\{x\}, B(x, 2r))}{\mu(B(x, r))} = \limsup_{r \to 0} \frac{f(r)}{C_2 f(s_r)} \geq \frac{1}{C_2} > 0, \]
so that \( \{x\} \) is 1-thick at \( x \).

**Example 4.8.** Let \( X = \mathbb{R}^n \) equipped with the Euclidean metric and the weighted Lebesgue measure \( d\mu := w d\mathcal{L}^n \), with \( w = |x|^a \) for \( a \in (-n, -n + 1) \).
It is straightforward to check that \( w \) is a Muckenhoupt \( A_1 \)-weight, and thus \( \mu \) is doubling and supports a \((1, 1)\)-Poincaré inequality; see, e.g., [23, Chapter 15] for these concepts. Denoting the origin by 0, we have
\[ \lim_{r \to 0} \frac{r}{\mu(B(0, r))} = 0, \]
demonstrating that this possibility needs to be taken into account.

Now we derive a kind of converse to Lemma 3.6, given in Lemma 4.10 below.

**Lemma 4.9** ([33, Lemma 4.3]). Let \( x \in X, r > 0 \), and let \( E \subset X \) be a \( \mu \)-measurable set with
\[ \frac{\mu(E \cap B(x, 2r))}{\mu(B(x, 2r))} \leq \frac{1}{2C_d^{\lceil \log_2(128\lambda) \rceil}}. \] (4.4)
Then for some constant \( C_3 = C_3(C_d, C_P, \lambda) \),
\[ \text{cap}_1(I_E \cap B(x, r), B(x, 2r)) \leq C_3 P(E, B(x, 2r)). \] (4.5)
We can strengthen this in the following way.

**Lemma 4.10.** Let \( x \in X, r > 0 \), and let \( E \subset X \) be a \( \mu \)-measurable set with
\[
\frac{\mu(E \cap B(x, 2r))}{\mu(B(x, 2r))} \leq \frac{1}{2C_d^{\log_2(128\lambda)}}.
\]
Then
\[
\text{cap}_1((I_E \cup \partial^* E) \cap B(x, r), B(x, 2r)) \leq C_3 P(E, B(x, 2r)),
\]
where \( C_3 \) is the constant from Lemma 4.9.

**Proof.** By Lemma 4.9, (4.5) holds. We can assume that \( P(E, B(x, 2r)) < \infty \). Fix \( \varepsilon > 0 \). By the definition of the variational capacity, we find a function \( v \in N^{1,1}(X) \) with \( v = 1 \) in \( I_E \cap B(x, r) \), \( v = 0 \) in \( X \setminus B(x, 2r) \), and
\[
(4.6) \quad \int_X g_o \, d\mu \leq \text{cap}_1(I_E \cap B(x, r), B(x, 2r)) + \varepsilon.
\]
We know that 1-q.e. point is a Lebesgue point of \( v \) by [26, Theorem 4.1, Remark 4.2]; note that [26] makes the assumption \( \mu(X) = \infty \), but this can be avoided by using [41, Lemma 3.1] instead of [26, Theorem 3.1] in the proof of the Lebesgue point theorem. If \( y \in \partial^* E \cap B(x, r) \) is a Lebesgue point of \( v \), then \( v(y) = 1 \), because if this were not the case, we would have
\[
\limsup_{r \to 0} \int_{B(y, r)} |v - v(y)| \, d\mu \geq \limsup_{r \to 0} \frac{1}{\mu(B(y, r))} \int_{I_E \cap B(y, r)} |v - v(y)| \, d\mu
\]
\[
= \limsup_{r \to 0} \frac{1}{\mu(B(y, r))} \int_{I_E \cap B(y, r)} |1 - v(y)| \, d\mu
\]
\[
= \limsup_{r \to 0} \frac{\mu(I_E \cap B(y, r))}{\mu(B(y, r))} |1 - v(y)|
\]
\[
> 0
\]
by the fact that \( y \in \partial^* E \) and by Lebesgue’s differentiation theorem. Thus we have \( v(y) = 1 \) for 1-q.e. \( y \in \partial^* E \cap B(x, r) \), and now by (4.6) and (4.5),
\[
\text{cap}_1((I_E \cup \partial^* E) \cap B(x, r), B(x, 2r)) \leq \int_X g_o \, d\mu \leq \text{cap}_1(I_E \cap B(x, r), B(x, 2r)) + \varepsilon
\]
\[
\leq C_3 P(E, B(x, 2r)) + \varepsilon.
\]
Letting \( \varepsilon \to 0 \), we get the result. \( \square \)

Finally, we record the following consequence of [33, Theorem 5.2].

**Theorem 4.11.** Let \( u \in BV(X) \). Then \( u^\wedge \) is 1-finely lower semicontinuous at 1-q.e. \( x \in X \).
In other words, for 1-q.e. \( x \in X \), every set \( \{ u^\land > t \} \) (with \( t \in \mathbb{R} \)) that contains \( x \) is a 1-fine neighborhood of \( x \). For Newton–Sobolev functions we have the stronger result that if \( u \in N^{1,p}(X) \) for \( 1 < p < \infty \), then \( u \) is \( p \)-finely continuous at \( p \)-q.e. \( x \in X \) (see [10], [30], or [2, Theorem 11.40]); we do not give the definition of the \( p \)-fine topology for \( p > 1 \) here but it can also be found in the above references.

5 The weak Cartan property

In this section we prove the weak Cartan property, Theorem 1.1, as well as a strong version at points of nonzero 1-capacity. Our proof will rely on breaking the set \( A \) given in the theorem into two subsets that do not intersect certain annuli around \( x \).

Such a separation argument is inspired by the proof of the analogous property in the case \( p > 1 \); see [6], which in turn is based on [22] and [36].

Lemma 5.1. Let \( B = B(x, R) \) be a ball with \( 0 < R < \frac{1}{12} \) diam \( X \), and suppose that \( A \subset B \) with \( A \cap (\frac{4}{20} B \setminus \frac{1}{4} B) = \emptyset \). Let \( E \subset X \) be a solution of the \( \mathcal{K}_{A,0}(\frac{2}{5} B) \)-obstacle problem (as guaranteed by Lemma 3.6). Then for all \( y \in \frac{2}{5} B \setminus \frac{5}{16} B \),

\[
\chi_E^\land (y) \leq C_4 R \frac{\text{cap}_1(A, 2B)}{\mu(B)}
\]

for some constant \( C_4 = C_4(C_d, C_P, \lambda) \).

Proof. By Lemma 3.6 and Lemma 4.2 we know that

\[
P(E, X) \leq \text{cap}_1(A, \frac{3}{2} B) \leq 5 C_S \text{cap}_1(A, 2B),
\]

and thus by the isoperimetric inequality (2.12),

\[
(5.1) \quad \mu(E) \leq 2 C_S R P(E, X) \leq 10 C_S^2 R \text{cap}_1(A, 2B).
\]

For any \( z \in \frac{2}{5} B \setminus \frac{5}{16} B \), letting \( r := R/20 \) we have \( B(z, r) \subset \frac{9}{20} B \setminus \frac{1}{4} B \). Since now \( A \cap B(z, r) = \emptyset \), we can apply Theorem 3.10(b) with \( k = 0 \) to get

\[
\sup_{B(z, r/2)} \chi_E^\land \leq \text{ess sup}_{B(z, r/2)} \chi_E \leq C_1 \left( \frac{r}{r - r/2} \right)^Q \int_{B(z, r)} (\chi_E^+)_+ \, d\mu \\
= \frac{2^Q C_1}{\mu(B(z, r))} \int_{B(z, r)} (\chi_E^+)_+ \, d\mu \\
\leq \frac{2^Q C_1 C_6^d}{\mu(B)} \mu(E) \leq 5 \times 2^Q C_1 C_6^d C_5^2 R \frac{\text{cap}_1(A, 2B)}{\mu(B)}
\]

by (5.1). Thus we can choose \( C_4 = 5 \times 2^Q C_1 C_6^d C_5^2 \).
Now we prove the weak Cartan property, Theorem 1.1. In fact, we give the following formulation containing somewhat more information, which will be useful in future work when considering p-strict subsets and a Choquet property in the case $p = 1$; cf. [5, Lemma 3.3], [35, Lemma 2.6], and [4].

**Theorem 5.2.** Let $A \subset X$ and let $x \in X \setminus A$ be such that $A$ is 1-thin at $x$. Then there exist $R > 0$ and $E_0, E_1 \subset X$ such that $\chi_{E_0}, \chi_{E_1} \in \BV(X)$, $\chi_{E_0}$ and $\chi_{E_1}$ are 1-superminimizers in $B(x, R)$, $\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} = 1$ in $A \cap B(x, R)$, $\chi_{E_0}^\vee(x) = 0 = \chi_{E_1}^\vee(x)$, \{max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\} is 1-thin at $x$, and

$$
\lim_{r \to 0} \frac{P(E_0, B(x, r))}{\mu(B(x, r))} = 0, \quad \lim_{r \to 0} \frac{P(E_1, B(x, r))}{\mu(B(x, r))} = 0.
$$

**Proof.** By Lemma 4.4 we find an open set $W \supset A$ that is 1-thin at $x$. Fix $0 < R < \frac{1}{12} \diam X$ such that

$$
\sup_{0 < s \leq R} \frac{\text{cap}_1(W \cap B(x, s), B(x, 2s))}{\mu(B(x, s))} < \frac{1}{2C_4}.
$$

Let $B_i := B(x, 2^{-i}R)$ and let $H_i := B_i \setminus \frac{9}{10} B_{i+1}$, $i = 0, 1, \ldots$. Then let

$$
D_i := \bigcup_{j=i, i+2, i+4, \ldots} H_j, \quad i = 0, 1, \ldots,
$$

so that $D_0 \cup D_1 = B(x, R)$. Let $W_i := W \cap D_i$, $i = 0, 1, \ldots$, and then by Lemma 3.6 we can let $E_i \subset X$ be a solution of the $\K_{W_i, 0}(\frac{3}{2} B_i)$-obstacle problem; clearly $\chi_{E_i} \in \BV(X)$ for all $i$. Let $F_i := \frac{4}{5} B_i \setminus \frac{5}{4} B_{i+1} \subset H_i, i \in \N$. Fix $i = 0, 1, \ldots$. From Lemma 5.1 we get for all $y \in F_{i+1}$

$$
\chi_{E_i}(y) \leq C_4 2^{-i} R \frac{\text{cap}_1(W \cap B_i, 2B_i)}{\mu(B_i)} \leq \frac{1}{2}.
$$

Since $\chi_{E_i}$ and thus also $\chi_{E_i}^\vee$ can only take the values 0, 1, we conclude that $\chi_{E_i}^\vee = 0$ in $F_{i+1}$, and thus by Lebesgue’s differentiation theorem,

$$
\mu(E_i \cap F_{i+1}) = 0.
$$

Note that $E_{i+2} \cup (E_i \setminus \frac{4}{5} B_{i+1})$ is admissible for the $\K_{W_i, 0}(\frac{3}{2} B_i)$-obstacle problem. Now if we had

$$
P(E_{i+2}, X) < P(E_i \cap \frac{5}{4} B_{i+2}, X),
$$

then by the fact that the sets $E_{i+2} \subset \frac{3}{2} B_{i+2}$ and $E_i \setminus \frac{4}{5} B_{i+1}$ are separated by a strictly positive distance,

$$
P\left(E_{i+2} \cup \left(E_i \setminus \frac{4}{5} B_{i+1}\right), X\right) = P(E_{i+2}, X) + P\left(E_i \setminus \frac{4}{5} B_{i+1}, X\right)
< P(E_i \cap \frac{5}{4} B_{i+2}, X) + P(E_i \setminus \frac{4}{5} B_{i+1}, X) = P(E_i, X)
$$
by (5.3), which would contradict the fact that $E_i$ is a solution of the $\mathcal{K}_{W_i,0}(\frac{3}{2}B_i)$-obstacle problem. Thus $P(E_{i+2}, X) \geq P(E_i \cap \frac{5}{4}B_{i+2}, X)$, and since $E_i \cap \frac{5}{4}B_{i+2}$ is admissible for the $\mathcal{K}_{W_{i+2},0}(\frac{3}{2}B_{i+2})$-obstacle problem, we conclude that it is a solution. We use this inductively as follows. First, $E_0 \cap \frac{5}{4}B_2$ is a solution of the $\mathcal{K}_{W_2,0}(\frac{3}{2}B_2)$-obstacle problem. Assuming that $E_0 \cap \frac{5}{4}B_i$ is a solution of the $\mathcal{K}_{W_i,0}(\frac{3}{2}B_i)$-obstacle problem for a given $i = 2, 4, \ldots$, we find that $E_0 \cap \frac{5}{4}B_i \cap \frac{5}{4}B_{i+2} = E_0 \cap \frac{5}{4}B_{i+2}$ is a solution of the $\mathcal{K}_{W_{i+2},0}(\frac{3}{2}B_{i+2})$-obstacle problem.

In conclusion, $E_0 \cap \frac{5}{4}B_i$ is a solution of the $\mathcal{K}_{W_i,0}(\frac{3}{2}B_i)$-obstacle problem, for any $i = 2, 4, 6, \ldots$. Analogously, $E_1 \cap \frac{5}{4}B_i$ is a solution of the $\mathcal{K}_{W_i,0}(\frac{3}{2}B_i)$-obstacle problem, for any $i = 3, 5, 7, \ldots$. By (5.3), it now also holds that

\begin{equation}
\mu(E_0 \cap F_i) = 0 \text{ for all } i = 1, 3, \ldots \quad \text{and} \quad \mu(E_1 \cap F_i) = 0 \text{ for all } i = 2, 4, \ldots
\end{equation}

By Lemma 3.6 and the fact that $E_0 \cap \frac{5}{4}B_i$ is a solution of the $\mathcal{K}_{W_i,0}(\frac{3}{2}B_i)$-obstacle problem, and by Lemma 4.2, we have

\begin{equation}
P \left( E_0 \cap \frac{5}{4}B_i, X \right) \leq \text{cap}_1 \left( W_i, \frac{3}{2}B_i \right) \leq 5C_S \text{cap}_1(W \cap B_i, 2B_i)
\end{equation}

for every $i = 2, 4, 6, \ldots$, and similarly $P(E_1 \cap \frac{5}{4}B_i, X) \leq 5C_S \text{cap}_1(W \cap B_i, 2B_i)$ for every $i = 3, 5, 7, \ldots$.

Let $0 < \delta < (20C_S^2 \bar{c}_d^{\log(128\bar{c}_d)})^{-1}$. Since $W$ is 1-thin at $x$, for some even $m \in \mathbb{N}$ and every $i = m, m+2, \ldots$, we have

\[2^{-i}R \frac{\text{cap}_1(W \cap B_i, 2B_i)}{\mu(B_i)} \leq \delta.\]

Fix such $m$. Together with (5.5), this gives

\begin{equation}
2^{-i}R \frac{P(E_0 \cap \frac{5}{4}B_i, X)}{\mu(B_i)} \leq 5C_S \delta
\end{equation}

for every $i = m, m+2, \ldots$. By the isoperimetric inequality (2.12), we now have

\[\mu \left( E_0 \cap \frac{5}{4}B_i \right) \leq C_S 2^{i+1}R P \left( E_0 \cap \frac{5}{4}B_i, X \right) \leq 10C_S^2 \delta \mu(B_i).\]

Thus

\begin{equation}
\frac{\mu(E_0 \cap \frac{5}{4}B_i)}{\mu(2B_i)} \leq \frac{\mu(E_0 \cap \frac{5}{4}B_i)}{\mu(B_i)} \leq 10C_S^2 \delta \leq \frac{1}{2C_d^{\log(128\bar{c}_d)}}.
\end{equation}

By the fact that $\chi_{E_0}^\vee = 1 \cap B_i = (I_{E_0} \cup \partial^* E_0) \cap B_i = (I_{E_0 \cap \frac{5}{4}B_i} \cup \partial^* (E_0 \cap \frac{5}{4}B_i)) \cap B_i$
and Lemma 4.10, we get
\[
2^{-iR} \frac{\text{cap}_1(\{\chi_{E_0}^\vee = 1\} \cap B_i, 2B_i)}{\mu(B_i)} = 2^{-iR} \frac{\text{cap}_1((I_{E_0} \cap \frac{5}{4}B_i) \cup \partial^+(E_0 \cap \frac{5}{4}B_i)) \cap B_i, 2B_i)}{\mu(B_i)}
\leq 2^{-iR} C_3 \frac{P(E_0 \cap \frac{5}{4}B_i, X)}{\mu(B_i)} \leq C_3 C_5 \delta
\]
by (5.6). Since this holds for every \(i = m, m + 2, \ldots\), and since \(\delta\) can be made arbitrarily small, by Lemma 4.3 we obtain
\[
\lim_{r \to 0} r \frac{\text{cap}_1(\{\chi_{E_0}^\vee = 1\} \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.
\]

Analogously, we prove the corresponding result for \(E_1\). Since \(\chi_{E_0}^\vee > 0\) exactly when \(\chi_{E_0}^\vee = 1\), we have established that \(\{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}\) is 1-thin at \(x\). Since \(\delta\) can be chosen arbitrarily small also in (5.7), we get \(\chi_{E_0}^\vee(x) = 0\), and similarly \(\chi_{E_1}^\vee(x) = 0\). Moreover, since \(A \cap D_0 \subset W_0 \subset E_0\) and \(W_0\) is open, \(\chi_{E_0}^\vee = 1\) in \(A \cap D_0\). Analogously, \(\chi_{E_1}^\vee = 1\) in \(A \cap D_1\), so that \(\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} = 1\) in \(A \cap B(x, R)\). Finally, (5.2) follows easily from (5.6) (and the corresponding property for \(E_1\)).

It can be noted that in the case \(p > 1\), the proof of the weak Cartan property relies on the comparison principle as well as weak Harnack inequalities for both superminimizers and subminimizers. We only have the last of these three tools available, but we are able to replace the others (and in fact get a simpler argument) with the very powerful fact that the superminimizer functions can be taken to be characteristic functions of sets of finite perimeter; recall especially (5.3).

**Proof of Theorem 1.1.** Let \(R > 0\) and \(E_0, E_1 \subset X\) as given by Theorem 5.2, and choose \(u_1 := \chi_{E_0}\) and \(u_2 := \chi_{E_1}\).

The analog of the following result is again known in the case \(p > 1\); see [2, Lemma 6.2]. Our proof will be similar, but we need to rely on the quasisemicontinuity of BV functions instead of the quasicontinuity that is available in the case \(p > 1\).

**Proposition 5.3.** Let \(A \subset X\) be 1-thin at \(x \in X\) and let \(R_0 > 0\). Then
\[
\lim_{r \to 0} \text{cap}_1(A \cap B(x, r), B(x, R_0)) = 0.
\]

Note that this does not follow directly from the definition of 1-thinness, since it is possible that \(r/\mu(B(x, r)) \to 0\) as \(r \to 0\); recall Example 4.8.
Proof. First assume that $\text{Cap}_1(\{x\}) = 0$. Then $\text{cap}_1(\{x\}, B(x, R_0)) = 0$ by (4.1), and so by the fact that $\text{cap}_1$ is an outer capacity,

$$\limsup_{r \to 0} \text{cap}_1(A \cap B(x, r), B(x, R_0)) \leq \limsup_{r \to 0} \text{cap}_1(B(x, r), B(x, R_0))$$

$$= \text{cap}_1(\{x\}, B(x, R_0)) = 0.$$ 

Then assume that $\text{Cap}_1(\{x\}) > 0$. By Proposition 4.5 we know that $\{x\}$ is 1-thick at $x$, and so $x \notin A$. By Theorem 1.1 we find $R > 0$ and functions $u_1, u_2 \in \text{BV}(X)$ such that $\max\{u_1^+, u_2^+\} = 1$ in $A \cap B(x, R)$ and $u_1^+(x) = u_2^+(x) = 0$. Then also $\max\{u_1^-, u_2^-\} \geq 1$ in $A \cap B(x, R)$. Fix $0 < \varepsilon < \text{Cap}_1(\{x\})$. By Proposition 3.14 there exists an open set $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that $u_1^-(x) = \text{Cap}_1(\{x\})$ is upper semicontinuous. By comparing capacities, we conclude that $x \notin G$. Thus by the upper semicontinuity, we necessarily have $\{u_1^+ \geq 1\} \cap B(x, r) \subset G$ for some $0 < r < R_0/2$. This implies that $\text{Cap}_1(\{u_1^+ \geq 1\} \cap B(x, r)) < \varepsilon$. Analogously, and by making $r$ smaller if necessary, $\text{Cap}_1(\{u_2^+ \geq 1\} \cap B(x, r)) < \varepsilon$, so in total,

$$\text{Cap}_1(A \cap B(x, r)) \leq \text{Cap}_1(\{u_1^+ \geq 1\} \cup \{u_2^+ \geq 1\}) \cap B(x, r)) < 2\varepsilon.$$ 

Then by (4.1),

$$\text{cap}_1(A \cap B(x, r), B(x, R_0)) \leq 2\left(1 + \frac{2}{R_0}\right)\text{Cap}_1(A \cap B(x, r)) < 4\varepsilon\left(1 + \frac{2}{R_0}\right).$$

Since $\varepsilon$ can be chosen arbitrarily small, we have the result. \(\square\)

Just as in the case $p > 1$ (see [6, Proposition 6.3]) at points of nonzero capacity we obtain a strong Cartan property, where we need only one superminimizer.

Proposition 5.4. Suppose that $x \in X$ with $\text{Cap}_1(\{x\}) > 0$, that $A \subset X$ is 1-thin at $x$, and that $0 < R < \frac{1}{8} \text{diam} X$. Then there exists a 1-superminimizer $u$ in $B(x, R)$ such that

$$\lim_{A_{xy} \to x} u^-(y) = \infty > u^+(x).$$

Proof. By Proposition 5.3 we find a decreasing sequence of numbers $0 < r_i < R$ such that

$$\text{cap}_1(A \cap B(x, r_i), B(x, R)) < 2^{-i}, \quad i \in \mathbb{N}.$$ 

Since $\text{cap}_1$ is an outer capacity, there exist open sets $U_i \supset A \cap B(x, r_i)$ such that

$$\text{cap}_1(U_i, B(x, R)) < 2^{-i}.$$ 

By the definition of the variational 1-capacity, we find nonnegative functions $\psi_i \in N^{1,1}(X)$ with $\psi_i = 1$ in $U_i$, $\psi_i = 0$ in $X \setminus B(x, R)$, and

$$\int_X g \psi_i d\mu < 2^{-i},$$
where as usual $g_{\psi_i}$ is the minimal 1-weak upper gradient of $\psi_i$. By the Sobolev inequality (2.10) and Hölder’s inequality, we get $\|\psi_i\|_{L^1(X)} < 2^{-i}C_5R$, for each $i \in \mathbb{N}$. By using the fact that $N^{1,1}(X)/\sim$ is a Banach space with the equivalence relation $u \sim v$ if $\|u - v\|_{N^{1,1}(X)} = 0$ (see [2, Theorem 1.71]) we conclude

$$\psi := \sum_{i=1}^{\infty} \psi_i \in N^{1,1}(X) \subset \text{BV}(X)$$

with $\psi = 0$ in $X \setminus B(x, R)$. Since $\psi \in \mathcal{K}_{\psi,0}(B(x, R))$, by Proposition 3.3 there exists a solution $u$ of the $\mathcal{K}_{\psi,0}(B(x, R))$-obstacle problem. Then $u$ is a 1-superminimizer in $B(x, R)$ by Proposition 3.7, and $u^\wedge \geq k$ in the open set $U_1 \cap \cdots \cap U_k \supset A \cap B(x, r_k)$, for every $k \in \mathbb{N}$. Thus

$$\lim_{A \ni y \to x} u^\wedge(y) = \infty.$$ 

However, by [27, Lemma 3.2] we know that $u^\vee(z) < \infty$ for $\mathcal{H}$-a.e. $z \in X$, and thus $u^\vee(z) < \infty$ for 1-q.e. $z \in X$ by (2.5). Since $\text{Cap}_1(\{x\}) > 0$, necessarily $u^\vee(x) < \infty$. □

In the case $p > 1$, the $p$-fine topology is known to be the coarsest topology that makes all $p$-superharmonic functions on open subsets of $X$ continuous; see [6, Theorem 1.1]. Equivalently, it is the coarsest topology that makes such functions upper semicontinuous, since they are lower semicontinuous already with respect to the metric topology. In the following we consider what the analog of this could be in the case $p = 1$.

**Definition 5.5.** We define the **1-superminimizer topology** to be the coarsest topology that makes the representative $u^\vee$ upper semicontinuous in $\Omega$ for every 1-superminimizer $u$ in $\Omega$, for every open set $\Omega \subset X$.

Note that if $X$ is bounded and thus compact, the only 1-superminimizers in $X$ are constants (for nonconstant $u \in \text{BV}(X)$ it follows from the coarea formula (2.7) that

$$\|D \max\{u, k\}\|(X) < \|Du\|(X)$$

for some $k \in \mathbb{R}$). This is why we want to talk about 1-superminimizers in open sets $\Omega$, and as a result, the metric topology is contained in the 1-superminimizer topology by definition.

**Remark 5.6.** It would not make sense to replace $u^\vee$ by $u^\wedge$ in the definition of the 1-superminimizer topology. To see this, consider $X = \mathbb{R}$ (unweighted), $\Omega = \mathbb{R}$, and the Heaviside function $u = \chi_{(0, \infty)}$. Moreover, let $v := 1 - u$. Now both $u$ and $v$ are clearly 1-minimizers. On the other hand,

$$\{u^\wedge < 1\} \cap \{v^\wedge < 1\} = \{0\}.$$
Hence if the sets \( \{ u^\wedge < t \} \), for \( t \in \mathbb{R} \) and 1-superminimizers \( u \in BV(X) \), are open in some topology, this topology contains all subsets of \( \mathbb{R} \).

**Theorem 5.7.** The 1-superminimizer topology contains the 1-fine topology.

**Proof.** Let \( U \subset X \) be a 1-finely open set, and let \( x \in U \). The set \( X \setminus U \) is 1-thin at \( x \). By Theorem 1.1, there exist \( R > 0 \) and 1-superminimizers \( u_1, u_2 \in B(x, R) \) such that \( \max\{ u_1^\wedge, u_2^\wedge \} \geq \max\{ u_1^\vee, u_2^\vee \} = 1 \) in \( B(x, R) \setminus U \) and \( u_1^\wedge(x) = u_2^\vee(x) = 0 \). Thus \( x \in B(x, R) \cap \{ u_1^\wedge < 1 \} \cap \{ u_2^\vee < 1 \} \), which is a set belonging to the 1-superminimizer topology, and contained in \( U \).

Now it might seem reasonable to postulate that the converse would hold as well, i.e., that the 1-fine topology would make \( u^\vee \) upper semicontinuous for all 1-superminimizers \( u \) in open sets. However, this is not the case.

**Example 5.8.** Let \( X = \mathbb{R}^2 \) equipped with the usual 2-dimensional Lebesgue measure \( \mathcal{L}^2 \), and denote the origin by 0. Let \( 0 < \varepsilon < 1/10 \) and

\[
A := \bigcup_{j=0}^{\infty} A_j
\]
with

\[
A_j := [(1 - 2\varepsilon)10^{-j}, (1 - \varepsilon)10^{-j}] \times [0, 10^{-2j}\varepsilon] ;
\]

note that \( A \subset B(0, 1) \). It is straightforward to check that for every \( j = 0, 1, \ldots \) and every \( R > 0 \) such that \( A_j \subset B(0, R) \),

\[
\text{cap}_1(A_j, B(0, R)) = 2 \times 10^{-j}\varepsilon + 2 \times 10^{-2j}\varepsilon .
\]

Now given any \( 0 < r \leq 1 \), choose \( j = 0, 1, \ldots \) such that \( r \in (10^{-j-1}, 10^{-j}] \). Then by Lemma 4.2,

\[
\frac{1}{11C_S} \text{cap}_1(A \cap B(0, r), B(0, 2r)) \leq \text{cap}_1(A \cap B(0, r), B(0, 10r)) \leq \text{cap}_1 \left( \bigcup_{k=j}^{\infty} A_k, B(0, 10r) \right) \leq 4\varepsilon \sum_{k=j}^{\infty} 10^{-k} \leq 10^{-j+1}\varepsilon \leq 100r\varepsilon
\]

and

\[
\text{cap}_1(A \cap B(0, r), B(0, 2r)) \geq \text{cap}_1(A_{j+1}, B(0, 2r)) \geq 2 \times 10^{-j-1}\varepsilon \geq \frac{1}{5}r\varepsilon ,
\]
and so for any $0 < r \leq 1$,

\[
\frac{1}{5} r \varepsilon \leq \text{cap}_1 (A \cap B(0, r), B(0, 2r)) \leq 1100 C_S r \varepsilon,
\]

which is comparable to $L^2(B(0, r))/r$. Let $E \subset \mathbb{R}^2$ be a solution of the $K_{A,0}(B(0, 2))$-obstacle problem. For any $y \in \mathbb{R}^2$ with $\frac{5}{16} \leq |y| < \frac{2}{5}$, by Lemma 5.1 and (5.8) we find

\[
\chi^\vee_E(y) \leq C_4 \frac{\text{cap}_1 (A \cap B(0, 1), B(0, 2))}{L^2(B(0, 1))} \leq 1100 C_4 C_S \varepsilon \leq \frac{1}{2}
\]

by choosing $\varepsilon \leq 1/(2200 C_4 C_S)$. Thus $\chi^\vee_E(y) = 0$ for $\frac{5}{16} \leq |y| < \frac{2}{5}$, and so

\[
P(E, \mathbb{R}^2) = P(E \cap B(0, \frac{5}{16}), \mathbb{R}^2) + P(E \setminus B(0, \frac{2}{5}), \mathbb{R}^2).
\]

Thus we see that the minimization of the perimeter of $E$ (i.e. solving the obstacle problem) takes place independently in the sets $B(0, \frac{5}{16})$ and $\mathbb{R}^2 \setminus B(0, \frac{2}{5})$. Now it is straightforward to show that we must have $E \setminus B(0, \frac{2}{5}) = A_0$. Inductively, we find $E = A$. Clearly $\chi^\vee_A(0) = 0$, but on the other hand, $A = \{ \chi^\vee_E = 1 \}$ is 1-thick at the origin, by (5.8). Thus $\chi^\vee_E$ is not 1-finely upper semicontinuous at the origin.

Nevertheless, it is perhaps interesting to note that in Theorem 5.2, $\chi^\vee_{E_0}$ and $\chi^\vee_{E_1}$ are 1-finely upper semicontinuous at $x$, since $\chi^\vee_{E_0}(x) = 0 = \chi^\vee_{E_1}(x)$ and the sets $\{ \chi^\vee_{E_0} > 0 \}$ and $\{ \chi^\vee_{E_1} > 0 \}$ are 1-thin at $x$. We expect this fact to be a useful substitute for fine upper semicontinuity in future research.

In Table 1 we compare the properties of Newton–Sobolev and $p$-superharmonic functions (for $1 < p < \infty$) with the analogous properties of BV functions and 1-superminimizers. For the results in the left column, see the comment after Proposition 3.14, the comment after Theorem 4.11, [29, Theorem 5.1] or [2, Theorem 8.22], and [6, Theorem 1.1]. For the results in the right column, see Proposition 3.14, Theorem 4.11, Theorem 3.11, and Theorem 5.7.

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Properties of Newton–Sobolev and \( p \)-superharmonic functions, for \( 1 < p < \infty \):

- Every \( u \in N^{1,p}(X) \) is quasicontinuous.
- Every \( u \in N^{1,p}(X) \) is \( p \)-finely continuous \( p \)-q.e.
- Every \( p \)-superminimizer has a lower semicontinuous representative (a \( p \)-superharmonic function).
- Any topology that makes \( p \)-superharmonic functions (upper semi-)continuous in open sets contains the \( p \)-fine topology.
- The \( p \)-fine topology makes \( p \)-superharmonic functions in open sets continuous.

Properties of BV functions and 1-superminimizers:

- For every \( u \in BV(X) \), \( u^\wedge \) is quasi lower semicontinuous.
- For every \( u \in BV(X) \), \( u^\wedge \) is 1-finely lower semicontinuous 1-q.e.
- For every 1-superminimizer \( u \), \( u^\wedge \) is lower semicontinuous.
- Any topology that makes \( u^\vee \) upper semicontinuous for every 1-superminimizer \( u \) in every open set contains the 1-fine topology.

Table 1. A comparison chart.

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Panu Lahti
DEPARTMENT OF MATHEMATICS
LINKÖPING UNIVERSITY
LINKÖPING SE-581 83, SWEDEN
email: panu.lahti@aalto.fi

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