Superfluidity near Phase Separation in Bose-Fermi Mixtures

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Abstract. We study the transition to fermion pair superfluidity in a mixture of interacting bosonic and fermionic atoms. The fermion interaction induced by the bosons and the dynamical screening of the condensate phonons due to fermions are included using the nonperturbative Hamiltonian flow equations. We determine the bosonic spectrum near the transition towards phase separation and find that the superfluid transition temperature may be increased substantially due to phonon damping.

PACS. 67.85.De Dynamic properties of condensates; excitations, and superfluid flow – 67.85.Fg Multi-component condensates; spinor condensates – 74.20.Fg Mixtures of Bose and Fermi gases – 74.20.Rk BCS theory and its development

1 Introduction

Mixtures of bosonic and fermionic atoms have initially been used for sympathetic cooling of fermions. This allows reaching the degenerate regime in ultracold Fermi gases despite the freezing out of thermalizing collisions between fermions in a single internal state at low temperature \[ T_c. \] From a theory point of view, Bose-Fermi mixtures are of interest in their own right. Indeed, already at the mean-field level, a number of different ground states have been predicted. Depending on the density \( n_F \) of fermions and the value of the Bose-Fermi \( (a_{BF}) \) and Bose-Bose \( (a_{BB}) \) scattering lengths, a phase separation instability is expected for strong repulsive Bose-Fermi interactions \( a_{BF}^2 > a_{BB}/n_F^{1/3} \). For sufficiently strong attractive interactions, in turn, the mixture is unstable. Accordingly, in a trap, a collapse is expected beyond a critical value of the particle number \[ N_c. \] The situation \( a_{BF} < 0 \) applies, for instance, to the case of \(^{40}\text{K}-^{87}\text{Rb} \) mixtures, where signatures for a collapse have been observed experimentally \[ ^{40}\text{K}-^{87}\text{Rb}. \] The scattering length \( a_{BF} \) can be controlled using magnetically tunable Feshbach resonances \[ ^{40}\text{K}-^{87}\text{Rb}. \] This opens the possibility to explore a number of novel phases in Bose-Fermi mixtures with nontrivial many-body correlations. One of the simplest among those is an \( s \)-wave superfluid in a Fermi gas with two internal states, where the attractive interaction is mediated by the exchange of phonons in the condensed bosonic gas \[ ^{40}\text{K}. \] More exotic states like \( p \)-wave pairing \[ ^{40}\text{K}. \] or an odd-frequency \( s \)-wave state \[ ^{40}\text{K}. \] may arise in the case where only a single internal state of the fermionic atoms is present. For Bose-Fermi mixtures on a lattice, a number of nontrivial phases have been predicted, for instance a supersolid phase of bosons in the presence of a fermionic density wave \[ ^{40}\text{K}. \] or phases with composite fermions \[ ^{40}\text{K}. \]

In our present work, we reconsider the rather basic problem of induced pairing in a balanced gas of fermions with two internal states. The bosons are assumed to form a Bose-Einstein condensate (BEC) whose density fluctuations can be described by the standard Bogoliubov theory. Even in the absence of a direct interaction between the fermionic atoms there is an induced interaction mediated by the phonons of the BEC. This situation is analogous to that of phonon-mediated superconductivity of electrons in a solid. As derived in textbooks \[ ^{40}\text{K}. \] second order perturbation theory in the Bose-Fermi coupling leads to an attractive, retarded interaction between the fermions in the \( s \)-wave channel \[ ^{40}\text{K}. \] In weak coupling, the resulting BCS instability appears at a temperature \( T_c \) much smaller than the Fermi temperature \( T_F \). The effects of retardation are then negligible because pairing only affects fermions at the Fermi surface \[ ^{40}\text{K}. \] In the context of cold gases, however, reaching temperatures far below \( T_F \) is impossible. The observation of fermionic superfluidity requires \( T_c/T_F \) to be of order 0.1 or larger, similar to the situation of attractive Fermi gases near a Feshbach resonance \[ ^{40}\text{K}. \]\[ ^{40}\text{K}. \] To study induced pairing in Bose-Fermi mixtures, weak coupling approaches in which only the physics in the vicinity of the Fermi surface is relevant, are therefore not reliable. This applies in particular in the regime close to the instability for phase separation where mean-field calculations predict critical temperatures that are favorable for observation \[ ^{40}\text{K}. \] Indeed, the strong interactions and the fact that—in contrast to the situation in solids—bosons and fermions have comparable masses, lead to a strong renormalization of the phonon modes (dynamical screening) which is due to the excitation of fermionic particle-hole pairs. This effect lowers the phonon frequencies and also gives rise to damping, thus broadening the phonon spectral function. If the dimensionless Bose-Fermi cou-
pling exceeds a critical value, the phonon frequencies become negative and the velocity of sound imaginary. This signals an instability towards phase separation (for repulsive Bose-Fermi interaction) or collapse (for attractive interaction). Since softer phonons induce a stronger fermion interaction, we concentrate on the parameter region near this instability, which is the most favorable for fermion pair superfluidity. Obviously, the presence of strong fluctuations which may increase \( T_c \) requires a technique that goes beyond the mean-field level of a BCS approach, where retardation and the backaction between the fermions and the phonons of the reservoir (in our case the BEC) are negligible.

In conventional superconductors, the effects of retardation are usually treated within Eliashberg theory \[21\]. This approach typically relies on the assumption that the induced interaction between fermions still appears only close to the Fermi surface and also that the Fermi velocity is much larger than the velocity of sound (adiabatic limit). In the context of cold atoms, however, where \( T_c/T_F \) may be of order unity, fluctuations with energies up to the Fermi energy \( E_F \) become important. Moreover, in typical mixtures like \(^{40}\text{K}\) and \(^{87}\text{Rb}\), the mass ratio of fermions and bosons is of order unity. Provided that the healing length of the BEC is of the same order as the average spacing between fermions, the resulting ratio of the sound and Fermi velocities is near unity. A treatment of induced fermion pairing in Bose-Fermi mixtures in terms of Eliashberg theory in such a situation was given by Wang \[22\]. He found that the strong-coupling effects enhance \( T_c \) considerably, at least in the regime \(|a_{BF}|k_F \lesssim 0.1\).

To deal with the problem of strong coupling in Bose-Fermi mixtures, we propose a different route using a renormalization group method which automatically takes into account effects on all energy scales and naturally identifies the elementary excitations of the coupled system. To this end, we follow Wegner’s idea of a continuous unitary transformation of the Hamiltonian \[23,24\]. It may be viewed as a reorganization of perturbation theory in such a way that the new basis describes dressed particles which do not decay and whose effective interactions are regular. In this way, the induced interaction is always attractive, free of singularities and truly retarded (i.e., it vanishes for large energy transfer or short times) \[25,26\]. The method thus properly describes the effective interactions even in a regime where the typical fermion energies exceed the phonon energy. By retaining the full Hilbert space it allows to deal with fluctuations far away from the Fermi surface in a natural manner and provides quantitatively reliable results for the critical temperature near the interesting regime of the phase separation instability. Compared with Eliashberg theory, the merit of the flow equations is that they yield a block-diagonal Hamiltonian from which one can read off the elementary excitations and identify the relevant degrees of freedom. Moreover, in contrast to the Wilson momentum-shell RG where scattering between bare particles depends on both frequencies and momenta, the scattering of renormalized particles in the flow equations depends on momenta alone, which makes them easier to parametrize numerically.

The plan of the paper is as follows: in section 2 we introduce the Hamiltonian flow equation method which allows us to separate energy scales and avoid spurious singularities. In section 3 we present our results for the renormalization of the phonon dispersion relation near the transition towards phase separation (where the effect is largest), and for the change in the induced interaction and of \( T_c \) due to this damping. Finally, we discuss our results and where best to observe pair superfluidity experimentally in section 4.

2 Hamiltonian flow equations

We consider a homogeneous, three-dimensional Bose-Fermi mixture of spin-polarized bosons and fermions in two equally populated hyperfine states. The bosons interact via a repulsive pseudopotential with strength \( g_{BB} = 4\pi\hbar^2a_{BB}/m_B \), where \( a_{BB} > 0 \) is the associated s-wave scattering length and \( m_B \) the boson mass. The interaction \( g_{BF} = 2\pi\hbar^2a_{BF}/m_r \) between bosons and fermions with reduced mass \( m_r \) may be attractive or repulsive, depending on the sign of the interspecies scattering length \( a_{BF} \). For simplicity, we assume that the direct interaction between fermions is negligible.

At low temperatures the bosons are condensed and it is sufficient to consider how the fermions couple to the phononic excitations on top of the condensate. The resulting model Hamiltonian (more precisely \( \mathcal{H} - \mu N \))

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}
\]

\[
\mathcal{H}_0 = \sum_q \omega_q : a_q^\dagger a_q : + \sum_k \epsilon_k : c_k^\dagger c_k :
\]

\[
\mathcal{H}_{\text{int}} = \sum_{kq} M_q \left( a_{k-q}^\dagger + a_q \right) c_{k+q}^\dagger c_k
\]

then coincides with that of an idealized description of the electron-phonon interaction in solids. Here \( \{ k, \sigma \} \) denotes the momentum and effective spin degree of freedom \( \sigma = \uparrow, \downarrow \) to label the different hyperfine states. Similarly, \( a^{(\dagger)} \) and \( c^{(\dagger)} \) are creation and annihilation operators for the phonons and fermions, respectively, while \( : \ldots : \) denotes normal ordering. The single-particle energies of the fermions are \( \epsilon_k = k^2/2m_F - \mu_F \). The phonons of the BEC are described by a Bogoliubov spectrum of the form \( \omega_q = c_s |q| \sqrt{1 + q^2 \xi^2} \) with phonon velocity \( c_s = \sqrt{n_B g_{BB}/m_B} \) and healing length \( \xi = 1/\sqrt{4m_B n_B g_{BB}} \) for the condensate with density \( n_B \). The fermion-phonon coupling is given by \( M_q = g_{BF} \sqrt{2n_B \omega_q^0}/\omega_q \) with bare bosonic dispersion \( \omega_q^0 = q^2/2m_B \).

On the mean-field level the stability of a Bose-Fermi mixture is guaranteed by the condition \[27\]

\[
\lambda_{\text{MF}} = \sqrt{\frac{\gamma_{BF}}{\gamma_{BB}}} < 1
\]

with the dimensionless couplings \( \gamma_{BF} = g_{BF} N(E_F) \) and \( \gamma_{BB} = g_{BB} N(E_F) \), where \( N(E_F) \) is the bare fermionic
density of states at the Fermi level. Even for weak Bose-Fermi coupling $\gamma_{BF}$ the system can become unstable if $\gamma_{BF}$ is also small (for typical values see section 3 below). As we will see below in equation (23), the induced fermionic coupling $\lambda$ at weak-coupling is given by the effective Cooper pair interaction at the Fermi surface averaged over the scattering angle,

$$\lambda = \frac{\gamma_{BF} x \log (1 + \frac{1}{x\gamma_{BB}})}{m}$$

(3)

with $x = n_B m_B / (3 n_F m_F)$ relating the densities and masses of bosonic and fermionic atoms. Note that in the limit $x\gamma_{BB} \gg 1$ (small healing length $\xi \ll k_F^{-1}$) the angular average has no effect and $\lambda \rightarrow \lambda_{MF}$. We are interested in the regime where $\lambda$ is of order unity: in this case the polaronic effects described by an enhanced effective fermion mass [28]

$$\frac{m}{m^*} = 1 + \lambda$$

(4)

are not very large (we will consider them in a future publication [29]). However, at the same time the induced fermionic interaction $\lambda$ is quite large compared to the very weak bare fermion repulsion (see section 3 below).

The standard Fröhlich transformation [16, 21] decouples the fermionic and bosonic degrees of freedom and yields an induced fermion-fermion interaction. In particular, for the BCS channel $k' = -k$ one obtains

$$V_{k,-k,q}^{(Fr)} = -\frac{\omega_q M_q^2}{\omega_q^2 - (\epsilon_{k+q} - \epsilon_k)^2}$$

(5)

which has a singularity if large fermion energies are relevant in the gap equation, as happens in the case of cold atoms. A well-established way to avoid such divergences is to perform a regularization and renormalization that first takes into account scattering processes with large energy transfer (off-shell) and successively proceeds to processes with smaller energy transfer (approaching on-shell scattering). In this way, perturbation theory is reorganized and resummed so as to satisfy energy scale separation and to avoid small energy differences in the denominator of perturbative expressions [24].

Essentially, there is some freedom in choosing the unitary transformation to decouple the fermion and phonon sectors: real physical processes (on-shell) of course have to remain unchanged, but the off-shell interaction (which is important for fermion pairing) can be made regular by choosing an appropriate basis for the fermionic quasiparticles.

Specifically, this change of basis is achieved by a continuous unitary transformation on the Hamiltonian that eliminates scattering processes with successively lower energy transfer [23, 24]. It can be expressed in the form of a differential flow equation

$$\frac{d\mathcal{H}(\ell)}{d\ell} = [\eta(\ell), \mathcal{H}(\ell)]$$

(6)

with a flow parameter $\ell$ going from 0 to $\infty$ that has dimension (energy transfer)$^{-2}$. Here $\eta(\ell)$ is an anti-hermitian operator which generates the unitary transformation

$$U(\ell) = T_\ell \exp \left( \int_0^\ell d\ell' \eta(\ell') \right)$$

(7)

(with $\ell$-ordering defined in the same way as time ordering) such that

$$\mathcal{H}(\ell) = U(\ell)\mathcal{H}(\ell = 0)U(\ell)^\dagger.$$  

(8)

There is some freedom in choosing $\eta$ appropriately; for models where the Hamiltonian can be split into diagonal and interacting parts Wegner [23] suggested the canonical choice

$$\eta(\ell) = [\mathcal{H}_0(\ell), \mathcal{H}_{int}(\ell)]$$

(9)

which makes the Hamiltonian increasingly energy diagonal along the flow and guarantees energy scale separation. For the Fröhlich Hamiltonian [11] we choose the generator that has been used originally by Lenz and Wegner [25]

$$\eta(\ell) = \sum_{kq} \left( M_{kq} \alpha_{kq} a_{-q} + M_{k+q,-q} \beta_{kq} a_q \right) c_{k+q}^\dagger c_k$$

(10)

where

$$\alpha_{kq} = \epsilon_{k+q} - \epsilon_k + \omega_q \quad \beta_{kq} = \epsilon_{k+q} - \epsilon_k - \omega_q$$

(11)

denote the energy gain in the fermion-phonon scattering (on-shell $\alpha_{kq} = \beta_{kq} = 0$). The flow equation (6) leads to a flow of the single-particle energies $\epsilon_k$ and $\omega_k$ and the fermion-phonon coupling $M_{kq}$. Moreover, it generates a new coupling $V_{kk'q}$ between two fermions. Two-phonon terms are absent under the assumption of harmonic density waves [10], and higher couplings between several phonons and fermions would only be generated at higher orders in the fermion-phonon coupling beyond $O(M^2)$. Following [25] we choose to neglect these and consider only the following running couplings in the Hamiltonian:

$$\mathcal{H}(\ell) = \sum_q \omega_q(\ell) : a_{-q}^\dagger a_q :$$

$$+ \sum_k \left( \epsilon_k(\ell) - 2 \sum_q n_{k+q} V_{k,k+q,q}(\ell) : c_{k}^\dagger c_k \right) :$$

$$+ \sum_{kk'q} V_{kk'q}(\ell) : c_{k+q}^\dagger c_{k'} c_k :$$

$$+ \sum_{kq} \left( M_{kq}(\ell)a_{-q}^\dagger + M_{k+q,-q}(\ell)a_q \right) c_{k+q}^\dagger c_k$$

$$+ \text{irrelevant terms.}$$

(12)

The renormalization of these couplings up to second order $O(M^2)$ in the flowing fermion-phonon coupling is given by
the flow equations

$$\frac{dM_{kq}}{dl} = -\alpha_{kq}^2 M_{kq}$$  \hspace{1cm} (13)

$$\frac{d\omega_q}{dl} = 2 \sum_k M_{kq}^2 \alpha_k (n_{k+q} - n_k)$$  \hspace{1cm} (14)

$$\frac{d\omega_{kq}}{dl} = -2 \sum_q (n_q M_{k+q,-q}^2 + (n_q + 1) M_{kq}^2 \alpha_{kq})$$  \hspace{1cm} (15)

$$\frac{dV_{kk'q}}{dl} = M_{kq} M_{k'q} \beta_{k'q} \beta_{k'q} - M_{kq} M_{k'q} - M_{k'q} M_{k'q}$$  \hspace{1cm} (16)

where all couplings on the right-hand side are $l$-dependent and $n_k$ and $n_q$ are the fermionic and bosonic occupation numbers, resp. The initial conditions are $\epsilon_k(l = 0) = \epsilon_k$, $\omega_q(l = 0) = \omega_q$, $M_{kq}(l = 0) = M_q$ and $V_{kk'q}(l = 0) = 0$ (without direct fermion-fermion interaction). The flow equation for the fermion-phonon coupling can be solved immediately,

$$M_{kq}(l) = M_q \exp\left(-\int_0^l dl' \alpha_{kq}(l')\right).$$  \hspace{1cm} (17)

$M_{kq}$ vanishes during the flow as the unitary transformation successively decouples the fermion and boson sectors of the Hamiltonian. Since we assume a weak fermion-phonon coupling we neglect the renormalization of the fermion single-particle energies $\epsilon_k$. Instead, we are mainly interested in the effect of phonon damping on the induced interaction $V$, to see whether already a small damping may change the interaction in the limit of resonant scattering. The flow of the phonon energies in three dimensions then becomes

$$\frac{d\omega_q}{dl} = -\frac{M_{kq}^2}{2\pi^2 q} \int_0^\infty dk k^2 n_k \int_{-1}^1 d(cos\theta)$$

$$\left(\alpha_{kq} e^{-2\int_0^\infty d\epsilon' \alpha_{kq}^2} + \beta_{kq} e^{-2\int_0^\infty d\epsilon' \beta_{kq}^2}\right).$$  \hspace{1cm} (18)

The integration over the angle $\theta$ between $k$ and $q$ is lengthy but straightforward,

$$\frac{d\omega_q}{dl} = -\frac{M_{kq}^2}{2\pi^2 q} \int_0^\infty dk k n_k e^{-2\int_0^\infty d\epsilon' \omega_q^2(\epsilon') d\epsilon' - \epsilon \omega_q^2(\epsilon)}$$

$$\times \left\{\frac{e^+ - e^-}{4l} + \frac{e^+ - e^-}{4\ell} + \frac{e^+ - e^-}{2\sqrt{2l/\pi}}\right\}$$

with $\omega_q(\ell) = (1/\ell) \int_0^\ell \omega_q(\epsilon') d\epsilon'$, $e^+ = \exp[-2\ell(\alpha_{kq}^2)]$, $E^+ = \text{erf}[\sqrt{2\alpha_{kq}^2}]$ and $\omega_q = q^2/(2m_F) + \sigma k q / m_F + \sigma \omega_q$. We use a Sommerfeld expansion for the temperature dependence of the Fermi function $n_k$ which allows the remaining $k$ integrals to be evaluated analytically. The flow equation is then integrated numerically for different $q$ values; we use a logarithmic $q$ grid near the phase transition.

In order to determine the transition temperature for fermion pairing, we concentrate on the BCS channel of the induced interaction, $V_{kq}^{BCS} = V_{kq}$, for which the flow equation simplifies to

$$\frac{dV_{kq}^{BCS}}{dl} = (\beta_{kq} - \alpha_{kq}) M_{kq} M_{k+q,-q}$$  \hspace{1cm} (20)

$$= -2\omega_q M_{kq}^2 \exp(-\int_0^l (\alpha_{kq}^2 + \beta_{kq}^2) d\ell')$$

$$= -2\omega_q M_{kq}^2 \exp(-2l(\epsilon_k - \epsilon_k')^2 - 2\int_0^l \omega_q^2 d\ell')$$

If the phonon frequency is not renormalized (mean field), the flow equation can be integrated to give

$$V_{kq}^{BCS}(l = \infty) = -\frac{\omega_q M_{kq}^2}{\omega_q^2 + (\epsilon_{k+q} - \epsilon_k)^2}$$  \hspace{1cm} (21)

which differs from the Fröhlich result [30] by the + sign in the denominator: this induced interaction is always attractive and vanishes for large energy transfer (retarded). The difference is due to using a different fermionic quasi-particle basis.

For solving the gap equation it will be useful to express the induced interaction between a $k$, $-k$ Cooper pair and a $k'$, $-k'$ pair in terms of energy variables $\epsilon = \epsilon_k$, $\epsilon' = \epsilon_{k'}$ and average over the angle:

$$V(\epsilon, \epsilon') = -\frac{N(E_F)^2}{2k_F^2 N(E_F)} \int_{[k-k']} dq q \frac{\omega_q M_{kq}^2}{\omega_q^2 + (\epsilon' - \epsilon)^2}$$  \hspace{1cm} (22)

with $k = \sqrt{2m_F(\epsilon + \mu_F)}$ and likewise for $k'$. Specializing further to Cooper pairs on the Fermi surface one obtains the phonon-induced coupling strength [31]

$$\lambda = -N(E_F) V(\epsilon = \epsilon' = E_F)$$  \hspace{1cm} (23)

$$= \frac{N(E_F)}{2k_F^2} \int_0^{2k_F} dq q \frac{M_{kq}^2}{\omega_q}$$

$$= \frac{\gamma_{BB}^2}{\gamma_{BB}^2 (2k_F \xi)^2} \ln[1 + (2k_F \xi)^2]$$

which depends only logarithmically on $\gamma_{BB}$ (see equation [3] above). Inserting this expression independent of energy into the gap equation

$$\Delta(\epsilon) = -\int d\epsilon' N(E_F)V(\epsilon, \epsilon') \frac{\Delta(\epsilon')}{2E(\epsilon')} \tanh \left(\frac{E(\epsilon')}{2T}\right)$$  \hspace{1cm} (24)

yields the well-known weak-coupling result

$$T_c = \frac{\gamma}{\pi} \left(\frac{2}{5}\right)^{7/3} \exp\{-1/\lambda\}$$  \hspace{1cm} (25)

where now and in the following we have already included the correction to the prefactor due to the polarization of the fermions [30].
However, the projection onto the Fermi surface in equation (23) \([9]\) is justified only when \(c_s \gg v_F\) and becomes insufficient as \(c_s \lesssim v_F\) and retardation effects become important. Therefore, it is necessary to go beyond the mean-field level and include the damping of the phonons and its influence on the induced interaction by solving the full flow equations (19) and (20), and finally solving the gap equation (24) with this renormalized effective interaction.

### 3 Results

The flow equation (19) yields the renormalized phonon spectrum due to the excitation of fermionic particle-hole pairs. The Bogoliubov spectrum for a given repulsive \(g_{BB} > 0\) is softened upon increasing \(g_{BF}^2\) up to the point where \(\omega_{q=0}\) turns negative; this signals a local instability towards phase separation. Typical \(^{40}\)K-\(^{87}\)Rb systems \([5,6]\) with the K atoms prepared in the \(|9/2, -9/2\rangle, |9/2, -7/2\rangle\) hyperfine states and the Rb atoms in the \(|1, 1\rangle\) state have fermion densities \(n_F = \frac{n_F}{a_F^3} \approx 10^{12} \text{cm}^{-3}\) (such that \(k_F = k_{F, \perp} \approx 1.7 \cdot 10^{-4} a_0^{-1}\)), a boson density \(n_B \approx 10^{14} \text{cm}^{-3}\), and a mass ratio \(\rho = m_F/m_B = 0.46\). The background s-wave scattering lengths are \(a_B = 99 a_0, a_B = -284 a_0, a_{FF} = 174 a_0\) \([6]\). This results in the dimensionless couplings

\[
\gamma_{BB} = N(E_F)g_{BB} = \frac{2r}{\pi} k_F a_{BB} \approx 0.005 \quad (26)
\]

\[
\gamma_{BF} = N(E_F)g_{BF} = \frac{1 + r}{4} k_F a_{BF} \approx -0.02 \quad (27)
\]

\[
\gamma_{FF} = N(E_F)g_{FF} = \frac{2}{\pi} k_F a_{FF} \approx 0.02. \quad (28)
\]

These couplings may be tuned by Feshbach resonances towards the phase transition at \(\gamma_{BB}^{-1} = \gamma_{BB}\) (mean field). Moreover, for the healing length \(\xi\)

\[
k_F \xi = \sqrt{\frac{k_F^2}{4 m_B a_B g_{BB}}} = \sqrt{\frac{3r}{4(n_B/n_F)\gamma_{BB}}} \approx 0.8 \quad (29)
\]

and the ratio of phonon and fermion velocities is

\[
\frac{c_s}{v_F} = \frac{r m_B}{k_F} \sqrt{\frac{n_B g_{BB}}{n_B}} = \frac{r}{2 k_F \xi} \approx 0.3. \quad (30)
\]

In this regime, corrections to the fermion-phonon vertex are small \((\Gamma < 0.1)\) \([22,31]\).

The spectrum is most interesting at this critical point because here we expect the induced fermion interaction to be largest. Fig. 1 shows the phonon spectrum for \(\gamma_{BB} = 0.005\) and \(g_{BF} \rightarrow g_{BF,c}\) approaching the transition towards phase separation. From top to bottom: \(g_{BB} = 0\) (Bogoliubov spectrum, black), \(g_{BF} = 0.98 g_{BF,c}\) (red), \(g_{BF} = 0.9996 g_{BF,c}\) (green), \(g_{BF} = 0.99999 g_{BF,c}\) (blue).

Note that this spectrum of undamped oscillations belongs not to the original phonons but to the elementary bosonic excitations of the interacting Hamiltonian which are phonons dressed with particle-hole excitations. One can perform the unitary transformation backwards to the original basis of physical fermions and phonons to obtain the broadening of the phonon spectral function \([33]\). The induced interaction is obtained from the flow equation (20) by inserting the renormalized phonon dispersion on the right-hand side. Comparison with the interaction due to unrenormalized phonons (22) in Fig. 2 shows that phonon damping leads to an enhanced scattering of Cooper pairs near the Fermi surface. As the transition towards phase transition is approached, this enhancement becomes more pronounced and eventually leads to a logarithmic singularity of the induced interaction as \(t' \to E_F\), and of the peak value of the interaction as \(g_{BF} \to g_{BF,c}\) (see the inset of Fig. 1).

We finally compare solutions of the gap equation (24) with the different forms of the effective interaction between Cooper pairs. By projecting all energies onto the Fermi surface one obtains the weak-coupling result (22) (dashed line in Fig. 3), with \(T_c/T_F \approx 0.05\). Including the dependence of \(V\) on the Cooper pair energy away from the Fermi surface but without phonon damping as in equation (22) yields a somewhat higher \(T_c/T_F \approx 0.08\) (circles in Fig. 3). Finally, the full inclusion of phonon damping using the flow equation (20) leads to a further increase to \(T_c/T_F \approx 0.1\) (diamonds).

### 4 Summary and discussion

We have employed the Hamiltonian flow equation method to derive the induced fermion interaction in a Bose-Fermi mixture near phase separation beyond the mean-field ap-
by phonons near the transition towards phase separation. The inclusion of energies away from the Fermi level but unrenormalized phonon damping according to equation (20), circles represent the full results, while the upper (solid) line includes the effect of phonon damping which leads to a logarithmic singularity. Parameters are $\gamma_{BF} = 0.065$ and $g_{BF} = 0.98 g_{BF,c}$ (upper pair), $g_{BF} = 0.84 g_{BF,c}$ (middle pair), and $g_{BF} = 0.7 g_{BF,c}$ (lower pair). The inset shows the peak value of the interaction (at $\epsilon' = E_F$) as a function of $g_{BF}/g_{BF,c}$, again with the lower (dashed) line representing bare phonons and the upper (solid) line including renormalized phonons.

**Fig. 2.** [color online] Induced fermion interaction in the BCS channel $V(\epsilon, \epsilon')$ approaching the transition towards phase separation. For each pair of curves the lower (dashed) line is without phonon damping, while the upper (solid) line includes the effect of phonon damping which leads to a logarithmic singularity. Parameters are $\gamma_{BF} = 0.065$ and $g_{BF} = 0.98 g_{BF,c}$ (upper pair), $g_{BF} = 0.84 g_{BF,c}$ (middle pair), and $g_{BF} = 0.7 g_{BF,c}$ (lower pair). The inset shows the peak value of the interaction (at $\epsilon' = E_F$) as a function of $g_{BF}/g_{BF,c}$, again with the lower (dashed) line representing bare phonons and the upper (solid) line including renormalized phonons.

**Fig. 3.** [color online] Transition temperature $T_c$ towards fermion pair superfluidity due to the fermion interaction induced by phonons near the transition towards phase separation. The curves are from top to bottom: diamonds represent the full phonon damping according to equation (20), circles represent the inclusion of energies away from the Fermi level but unrenormalized phonons (middle pair), and the dashed line represents the weak-coupling result (upper pair) with the interaction restricted to the Fermi level (lower pair). As before, $\gamma_{BB} = 0.005$.

Transition temperature $T_c$ towards fermion pair superfluidity due to the fermion interaction induced by phonons near the transition towards phase separation. The curves are from top to bottom: diamonds represent the full phonon damping according to equation (20), circles represent the inclusion of energies away from the Fermi level but unrenormalized phonons (middle pair), and the dashed line represents the weak-coupling result (upper pair) with the interaction restricted to the Fermi level (lower pair). As before, $\gamma_{BB} = 0.005$.

proximation, and found an increase in the resulting transition temperature towards fermion pair superfluidity.

This involved going beyond the asymptotic solution of the phonon flow equations to the full solution including high-energy fluctuations; we obtained a dispersion $\omega_q \sim q^0$ for dressed phonons near the transition towards phase separation. While it has been known that the phonon softening asymptotically leads to a logarithmic singularity in the induced interaction, we have computed the form of the induced interaction quantitatively for Cooper pairs also far away from the Fermi level and found that the superfluid transition temperature near the phase separation instability will increase substantially due to phonon damping.

In order to identify the experimental parameters which are most favorable for the observation of superfluidity in Bose-Fermi mixtures, we note that the transition temperature $T_c$ grows monotonically with the induced fermion coupling $\lambda$ in equation (3), both at weak and strong coupling. For the experimental parameters given above, $\lambda \approx 70$. For strong boson repulsion and/or light fermions, $x^2 g_{BB} \gg 1$ and the coupling $\lambda \to \lambda_{MF} = g_{BF}^2/\gamma_{BB}$ agrees with the mean-field value (4). Since $\lambda$ decreases monotonically with $\gamma_{BB}$, one obtains a larger coupling $\lambda$ by reducing $\gamma_{BB}$. However, one eventually crosses over into the regime where $x^2 g_{BB} \lesssim 1$ and $\lambda$ is enhanced only logarithmically when further decreasing $\gamma_{BB}$.

At the same time, $\lambda$ is a monotonically increasing function of $x$, so one can reach a higher critical temperature by increasing the density of bosons relative to fermions, or the mass ratio between bosons and fermions. To conclude, the most direct way to observe superfluidity appears to be to increase the strength of the Bose-Fermi coupling $|\gamma_{BF}|$, for instance by a magnetically tuned Feshbach resonance, to the vicinity of phase separation or collapse. It will also be interesting to study the influence of a fermion mass difference on fermion pairing which is relevant in experiments with $^6$Li-$^{40}$K mixtures immersed in a $^{87}$Rb condensate.

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Vertex corrections will become important at the transition towards phase separation but as in [12] we assume that slightly away from the transition they are still small enough.