Superconductor-insulator duality for the array of Josephson wires

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We propose novel model system for the studies of superconductor-insulator transitions, which is a regular lattice, whose each link consists of Josephson-junction chain of $N \gg 1$ junctions in sequence. The theory of such an array is developed for the case of semiclassical junctions with the Josephson energy $E_J$ large compared to the junctions’s Coulomb energy $E_C = e^2/2C$. Exact duality transformation is derived, which transforms the Hamiltonian of the proposed model into a standard Hamiltonian of JJ array. The nature of the ground state is controlled (in the absence of random offset charges) by the parameter $q \approx N^2 \exp(-\sqrt{8E_J/E_C})$, with superconductive state corresponding to small $q < q_c$. The values of $q_c$ are calculated for magnetic frustrations $f = 0$ and $f = \frac{1}{2}$. Temperature of superconductive transition $T_c(q)$ and $q < q_c$ is estimated for the same values of $f$. In presence of strong random offset charges, the $T = 0$ phase diagram is controlled by the parameter $\bar{q} = q/\sqrt{N}$; we estimated critical value $\bar{q}_c$.

Introduction and model. Quantum phase transitions (QPT) between superconductive and insulative states in Josephson-junctions (JJ) arrays with submicron-sized junctions were intensively studied, both as function of the ratio between Josephson and charging energies $E_J/E_C$, and of the applied transverse magnetic field producing frustration of the Josephson couplings (cf. e.g. review [1]). To a large extent, an approach based upon ”duality” between Cooper pairs and superconductive vortices [2], was used for theoretical description of phase transition and for interpretation of the data. There are several difficulties related with this approach: i) duality transformation to vortex variables cannot be implemented exactly for the standard Hamiltonian of JJ array, and some poorly controlled approximations are necessarily used, ii) comparison of theory with experiments is complicated by the fact that the normal-state resistance of junctions $R_n$ is close to quantum resistance $R_Q = h/4e^2$ in the transition region, thus $E_J \sim E_C \sim \Delta$ and standard approximation of the local in time, ”phase-only” Hamiltonian cannot be justified, iii) randomly frozen ”off-set” charges known to exist in all JJ arrays introduce random frustration into the kinetic energy term for vortices; the role and relative importance of this effect for the S-I transition is barely unknown.

In the present Letter we propose and study modified version of JJ array (shown in Fig.1) which possesses quantum phase transition within parameter range $E_J \gg E_C = e^2/2C$. Each single bond of this novel array contains a chain (referred to as Josephson wire, JW) of $N \gg 1$ identical junctions with Josephson energy $E_J$ and capacitance $C$. We neglect self-capacitances $C_{isl}$ of islands compared to junctions capacitances $C$. Lagrangian of this array ($M \times M$ plackets) is:

$$\mathcal{L} = \sum_j \left[ \frac{1}{16E_C} \left( \frac{d\theta_j}{dt} \right)^2 + E_J \cos \theta_j \right].$$

where sum goes over all junctions shown in Fig.1, and $\theta_j$ is the phase difference on the $j$-th junction. Phase differences $\theta_j$ are subject to the constraints on each lattice placket (counted by dual lattice coordinate $r$):

$$\sum_\square \theta_j = 2\pi f_r = 2\pi \Phi_r/\Phi_0,$$

where $\Phi_r$ is external magnetic flux through the placket $r$. An effective Josephson coupling $E_J^{eff}$ between the nodes of JJ lattice is suppressed as $E_J/N$, whereas effective amplitude of quantum phase slip processes (i.e. amplitude of vortex tunnelling) is enhanced, either $N$ in the absence of off-set charges, or $\propto \sqrt{N}$, if off-set charge disorder is strong. Therefore, at sufficiently large $N$ the whole array will become insulating even if the ratio $E_J/E_C$ is large. Such a model possess two important features which makes theoretical analysis simpler: i) for a long chain of junctions, semiclassical energy-phase relation $E(\phi)$ is piece-wise parabolic, with a period $\phi \in (-\pi, \pi)$, and ii) an amplitude $v$ of an individual quantum phase slip in each
of $N$ junctions is small, $v \ll E_{J}^{\text{eff}}$; therefore the simplest vortex tunnelling Hamiltonian is an adequate description of multiple phase slips. On experimental side, the advantages of the proposed system are: i) an effective Josephson frequency of an array can be made small, allowing for clear separation between collective bosonic excitations of an array and single-electron excitations within superconductive islands, and ii) superconductor-insulator transition can be explored with a set of arrays with exactly same parameters specifying configurations $p$ and $p'$. Below we denote this amplitude as $\Upsilon_{r,r'}$, and will specify its explicit form later. The next step is to perform Fourier transformation from the set of integers $\{p\}$ into the set of phase variables $\phi_{r}$ associated with sites of dual lattice, according to $a_{\{p\}} = \sum_{\{p\}} a_{r} e^{ip_{r}}$, $a_{r} = \int D\phi \; a_{\{\phi\}} e^{-ip_{r}}$. Now the Hamiltonian \textbf{(4)} can be written as

$$H = \prod_{r} d\phi_{r} \left[ \frac{(2\pi)^{2} E_{J}}{2N} a_{\{r\}}^{+} \hat{E} a_{\{r\}} - \frac{1}{2} \sum_{\{r\},\{r'\}} [\Upsilon_{r,r'} \exp (i\phi_{r} - i\phi_{r'}) + h.c.] \right]$$

where $\hat{E} = \sum_{r'} G_{r,r'} \left( -i \frac{\partial}{\partial \phi_{r'}} - f \right) \left( -i \frac{\partial}{\partial \phi_{r'}} - f \right)$ and the sum is taken over all non-directed bonds on the dual lattice. The corresponding first-quantized "dual" Hamiltonian in terms of vortex number and phase operators $\hat{N}_{r}$ and $\phi_{r}$ reads:

$$H^{\text{dual}} = 4 \hat{E}_{C} \sum_{r} (\hat{N}_{r} - f) G_{r,r'} (\hat{N}_{r'} - f) - \frac{1}{2} \sum_{\{r\},\{r'\}} [\Upsilon_{r,r'} e^{i(\phi_{r} - \phi_{r'} + \chi_{r,r'})} + \text{H.c.}]$$

where $\chi_{r,r'} = \text{Arg}\, \Upsilon_{r,r'}$. In Eq. \textbf{(4)} we define "dual charging energy"

$$\tilde{E}_{C} = \pi^{2} E_{J}/2N,$$

uniform "charge" frustration $f \in (0,1)$, frustrated "dual Josephson" couplings with the local strengths $\tilde{E}_{J}(r,r') = |\Upsilon_{r,r'}|$ and "magnetic" frustration parameters $\Gamma_{x} = \frac{1}{N} \sum_{\{x\}} \chi_{x,x}$. The Hamiltonian \textbf{(4)} is of the standard form for the Josephson-junction array with junction-dominated capacitive energy. An important remark in is order: the Hamiltonian defined by \textbf{(4)} was derived neglecting single-electron excitations within each superconducting island; this is legitimate below the parity effect temperature $T^{*} \approx \Delta / \log (\nu \Delta V)$ only \textbf{(4)}; we will assume $T < T^{*}$ below.

If the original array is free from background charges, one finds, following derivation in Ref.\textsuperscript{[5]}, that $\Upsilon_{r,r'} \equiv \Upsilon_{r,r'}^{(1)}$, where $\Upsilon = 2Nv$, and $\gamma_{r,r'}^{(1)} = 1$ for nearest neighbouring cites $r$, $r'$ on the dual square lattice, and zero otherwise.

$$v = \frac{2^{11/4}}{\sqrt{\pi}} \left( E_{J}^{2} E_{C} \right)^{1/4} \exp \left[ -2 \sqrt{\frac{2E_{J}}{E_{C}}} \right]$$

is the amplitude of a tunneling process (quantum phase slip) in each of $N$ junctions which constitute an elementary link of the JJ array. In this case $E_{J} = \Upsilon = 2Nv$ whereas $\Gamma_{x} \equiv 0$. The nature of the ground state is then controlled by the value of

$$q = \tilde{E}_{J}/\tilde{E}_{C} = 4N^{2}v/\pi^{2}E_{J}.$$
The "insulating" (in dual variables) state is realized (at $f = 0$) for $q < q_c \approx 0.5$, according to the lowest-order variational calculation \[\text{[6]}\] and Quantum Monte-Carlo simulations \[\text{[5]}\]. Below we extend the calculation of ref. \[\text{[6]}\] and find $q_c$, for which we will also find $T = 0$ expression for the superconducting density $\rho_s(q)$ of the wire array at $q < q_c$. Insulating state of the wire array is realized at $q > q_c$; here we calculate effective dielectric permeability $\varepsilon(q)$.

Background off-set charges coupled to the "bond" islands modify \[\text{[6]}\] phases of amplitudes of phase slips in different junctions: $\nu_k = ve^{i\chi_k}$. If off-set charge disorder is strong, phases $\chi_k$ are totally random and distributed over the circle $(0,2\pi)$. As a result, tunnelling amplitudes $\Upsilon_{r,r'}$ constitute now Hermitian random matrix with Gaussian statistics:

$$\Upsilon_{r,r'} = \tilde{E}_j \cdot \gamma_{r,r'}^{(1)} \cdot z_{r,r'} \quad \tilde{E}_j = 2\sqrt{N \nu} \cdot |z_{r,r'}|^2 = 1. \quad (10)$$

The strength of "dual Josephson coupling" is suppressed due to charge disorder by the factor $1/\sqrt{N}$, and relevant control parameter is now $q = q/\sqrt{N}$. Moreover, dual array with couplings $\Upsilon_{r,r'}$ is randomly frustrated due to randomness of phases $\chi_{r,r'} = \text{Arg} \Upsilon_{r,r'}$. Critical value $q_c$ for the Hamiltonian \[\text{[6]}\] with random matrix $\Upsilon_{r,r'}$ will be calculated below.

Off-set charges $Q_x$ related to the "node" islands contribute directly to the frustration parameter $\Gamma_x$

$$\Gamma_x = Q_x + \Gamma_x(Q_x = 0) \quad (11)$$

Eq. \[\text{(11)}\] is useful for discussion of the relation \[\text{(13)}\] below.

To complete duality transformation, we need to identify dual partners for the superconducting density $\rho_s$ and dielectric permeability $\varepsilon$ characterising electromagnetic response of an original array. Superconducting density is defined via the energy $E_q(\theta) = \frac{E}{2} \int d^2x (\nabla \theta)^2$ of an inhomogeneous state, it is related to kinetic inductance per square: $L_K = \Phi_0/(4\pi^2 \rho_s)$. We calculate $\rho_s$ by introducing infinitesimal vector potential $\delta A$ modifying magnetic frustration of the original array, which transforms into modification of "charge frustration" $f_r$ in the dual representation. Dielectric permeability $\varepsilon$ of the original array is calculated via an energy response to the introduction of an additional infinitesimal stray charges $Q_{\mathbf{r}} = \delta Q$ and $Q_{\mathbf{r}'} = -\delta Q$ via 2D Coulomb relation $d^2E/d(\delta Q)^2 = [N/\varepsilon C] G_{\mathbf{r}\mathbf{r}'}$. Variation of stray charges transform then into a variation of "magnetic" frustration in the dual representation. Simple calculations lead to the dual relations

$$\rho_s = \frac{E_J}{N} \cdot \varepsilon^{-1} \quad (12)$$

$$\varepsilon^{-1} = \frac{\pi^2}{2N^2 E_C \rho_s^D} \quad (13)$$

$\varepsilon_D$ is the effective dielectric permeability of dual array and $\rho_s^D$ is its effective superconducting density, which are defined in the insulating and superconducting states of the dual array, correspondingly.

**S-J transition point at $T = 0$: variational method.**

Variational method for the Hamiltonian $H = H_0 + H_J$ defined by Eq. \[\text{(6)}\] was developed in \[\text{[3]}\] for the determination of the transition point. The idea of this method (we use it at $T = 0$ and for static case only) is to consider the ground-state energy $E_{\text{var}}$ as a bilinear functional of average values $\psi_i = \langle e^{i\phi_i} \rangle$, i.e. $E_{\text{var}} = \sum_{r_1,r_2} L_{r_1,r_2} \psi^*_{r_1} \psi_{r_2}$, and to determine the condition for the operator $\hat{L}$ to acquire zero mode. This calculation was performed in \[\text{[3]}\] for $f = 0$. We generalized such a calculation for the case $f = \frac{1}{2}$ as well; the matrix $L_{r_1,r_2}$ is presented below:

$$L_{r_1,r_2} = \epsilon_+ \left( \delta_{r_1,r_2} - \frac{\epsilon_f}{\epsilon_+} \delta_{r_1,r_2}^{(1)} \right) \quad (14)$$

where $\epsilon_1 = 2\tilde{E}_C$ is the Coulomb energy of the smallest $(-,\ldots,\ldots,$) dipole residing on nearest-neighbouring sites, and $\epsilon_+ = (2\tilde{E}_C/\pi) \log M$ is the Coulomb energy of a single-charge excitation, $c_0 = 1$ and $\epsilon_2 = \frac{3}{4}$. Eq. \[\text{(14)}\] is derived in the main approximation over small parameter $\epsilon_1/\epsilon_+ \sim \log^{-1} M$. The result for critical values of $q = \tilde{E}_J/\tilde{E}_C$ reads:

$$q_c = \frac{1}{2} \quad \text{for } f = 0 \quad q_c = \frac{3}{8} \quad \text{for } f = \frac{1}{2} \quad (15)$$

**Superconducting density $\rho_s$ and phase diagram without off-set charges.**

At $q < q_c$ and low temperatures $T < T_{\text{sup}}(q)$ the Josephson array is superconductive. Superconducting density $\rho_s$ coincides with $E_J/N$ in the absence of both thermal and quantum fluctuations, $T \rightarrow 0$ and $q \rightarrow 0$. We start from analysing quantum corrections to $\rho_s$ at $T = 0$, making use of the dual relation \[\text{(12)}\]. The ground state of the dual array with the Hamiltonian \[\text{[6]}\] is insulating, its dielectric permeability $\varepsilon_D$ can be expressed in terms of the Fourier-transform $R(p,0)$ of the irreducible zero (Matsubara) frequency-charge-correlation function $R(r,\omega = 0) = \int d\tau (\langle N_r(\tau) N_0(0) \rangle)$:

$$\frac{1}{\varepsilon_D} = 1 - 8\tilde{E}_C \frac{R(p,0)}{p^2} \quad (16)$$

Correlation function $R(p,0)$ can be expanded in series over "dual Josephson" part of the Hamiltonian \[\text{[6]}\], this explanation contains even powers of $q = \tilde{E}_J/\tilde{E}_C$ only. We calculated $R(p,0)$ for the $f = 0$ case up to the 4-th order in $q$. Details of this rather tedious calculations will be presented elsewhere \[\text{[6]}\], the result is

$$\rho_s = \frac{E_J}{N} \left[ 1 - q^2 - (a_p + a_r) q^4 \right], \quad a_p = 0.84 \quad a_r = 2.42 \quad (17)$$

Here coefficient $a_p$ corresponds to the contribution of diagrams which include two couplings $\Upsilon_{r_1,r_2}$ and $\Upsilon_{r_3,r_4}$ with pair-wise equal coordinates $r_1 = r_4$ and $r_2 = r_3$, whereas coefficient $a_r$ corresponds to "ring" diagrams...
with all four different points \(r_{1,2,3,4}\) (all diagrams contributing in the order \(q^2\) contain products \(\langle Y_{r,r'} \rangle^2\) only). The result \((17)\) is reliable as long as 4th-order correction is small compared to the 2nd-order one, i.e. \(q \leq 0.4\). Eqs. \((12)\) and \((17)\) determines reduction of the \(T = 0\) superconducting density due to quantum fluctuations of vortices beyond vortex-free ground-state. Upon temperature increase, superconductivity is destroyed according to Berezinsky-Kosterlitz-Thouless mechanism of vortex depairing, with transition temperature \(T_{\text{BKT}} = A \tilde{\rho}_s(T = 0)\). Suppression factor \(A_0 = 0.87\) was found numerically \([10, 11]\) for classical phase transition in the Gaussian periodic XY model like the one we study here, for \(f = 0\). In the fully frustrated case \(f = \frac{1}{2}\) suppression is stronger \([11, 12]\). \(A_4 = 0.52\). Full line in Fig. 2 presents \(q\)-dependence of the superconducting transition temperature \(T_{\text{sup}}(q) = \pi A_0 E_j/2N \varepsilon_D\).

In presence of magnetic frustration \(f \neq 0\) calculations of quantum corrections to \(\varepsilon_D\) up the 4-th order in \(E_J\) looks complicated, here we present 2-nd order results only:

\[
\rho_s = \frac{E_J}{N} \left(1 - \frac{112}{27} q^2\right).
\]

The corresponding superconducting transition temperature \(T_{\text{sup}} = \pi A_0 E_j/2N \varepsilon_D\) as function of \(q\) is shown in Fig. 2 by the line with crosses.

At \(q > q_c\) the ground state of the dual Hamiltonian contains Bose-condensed vortices. Very deep inside the dual superfluid state \((q \gg q_c)\) the corresponding "dual superfluid density" \(\rho_s^D(q = \infty) = \tilde{E}_J = 2N \nu\), cf. last term of the Hamiltonian \((6)\). Such a state possesses collective excitations with frequency

\[
\omega_j^D = \sqrt{8E_C \tilde{E}_J} = 2^{3/2} \pi \nu E_j,
\]

which is a dual analog of usual Josephson plasma oscillations with much higher frequency \(\omega_j = \sqrt{8E_J E_C} \gg \omega_j^D\). Finite-\(q\) correction to \(\rho_s^D\) in the lowest order over \(q^{-1}\) is due to anharmonicity of the zero-point fluctuations of phases \(\varphi_r\); it can be calculated as \(\rho_s^D = \rho_s^D(q = \infty) \cos(\varphi_r - \varphi_{r+b}) = \rho_s^D(1 - 1/\sqrt{8q})\). Note that perturbative corrections to \(\rho_s^D\) do not depend on density of "dual charges" controlled by \(f\).

According to Eqs. \((8)\) and \((13)\), the original array is then in the insulating ground state with inverse dielectric permeability

\[
\varepsilon^{-1} = 2^{11/4} \pi^{3/2} \left(\frac{E_j}{E_C}\right)^{3/4} e^{-\sqrt{8E_J/E_C}} \left(1 - \frac{1}{\sqrt{8q}}\right).
\]

Interaction of \(2e\) charges in such an array is logarithmic, \(U(x) = (4NE_C/\pi \varepsilon) \log(x)\), the corresponding BKT charge unbinding temperature is \(T_{\text{ins}} = 0.57N E_C/\pi \varepsilon\) \([1]\). Note that dielectric constant \(\varepsilon\) is very large in the whole range of applicability of our theory; this is due to our major assumption of \(E_J \gg E_C\). The line with asterisks marks on Fig. 2 shows the normalized transition temperature \(T_{\text{ins}}(q)/T_{\text{ins}}(0)\). At \(T > T_{\text{ins}}\) Cooper pairs are unbound and array possesses nonzero thermoactivated conductivity. Below \(T_{\text{ins}}\) linear conductivity vanishes (cf. \([13, 14]\) for similar experimental observations in thin amorphous superconductive films).

**Strong off-set charges: superconductor to "Coulomb glass" transition at \(T = 0\).** Now we concentrate on the case of strong random stray charges, but assume no real magnetic field present, \(\gamma = 0\). Then dual "Josephson" couplings in the Hamiltonian \((6)\) are diminished in magnitudes, so the parameter which controls quantum fluctuations is now

\[
\tilde{q} = E_J^d/E_C = 4N^{3/2} \nu/\pi^2 E_j,
\]

and strongly frustrated by random phases, thus all effects related to vortex tunnelling are suppressed. In particular, it concerns reduction of superconducting density \(\rho_s\) due to vortex fluctuations, given (up to the 4-th order in \(E_J\))...
by

\[ \rho_s = \frac{E_J}{N} \left(1 - \bar{q}^2 - 0.84\bar{q}^4\right) \]  

(22)

In comparison with Eq.(17), note the absence of the ring diagram’s contribution \( a_r q^4 \) which vanishes due to averaging over random phases (other terms contain magnitudes \( |Y_{r,r'}| \) only). Eq.(22) provides reasonable accuracy up to \( \bar{q} \approx 0.6 \).

Upon sufficient increase of \( \bar{q} \) the superconductive ground state will be destroyed. In the dual representation it corresponds to formation at \( \bar{q} = \bar{q}_c \approx 1 \) of a gauge glass state (cf. e.g. [15]) with frozen in ”vortex currents”, a la persistent electric currents in magnetically frustrated random Josephson network. Physically it means an appearance of a collective insulating state with local lateral electric fields. At \( \bar{q} \gg 1 \) and \( T = 0 \) the corresponding ”dual superfluid density” \( \rho_s^D \) scales as \( E_J^2 = 2\sqrt{N}v \).

Gauge glass state in 2D nearest-neighbours array is unstable due to thermal fluctuations at any nonzero temperature [16], thus at any \( T > 0 \) our array will possess small but nonvanishing conductivity. The absence of finite- \( T \) charge unbinding transition demonstrates qualitative difference with the same model without random off-set charges, studied above.

Conclusions. We presented exact duality transformations for the JW array, proposed as a novel model system with superconductor-insulator QPT. Our main results are presented by Eqs.(17,18,22) for the array’s macroscopic superconducting density \( \rho_s \), and by Eq.(20) for dielectric permeability \( \varepsilon \) in the insulating state. Collective vortex oscillations with \( N \)-independent frequency [19] are predicted for the deeply insulating state in the model without off-set charges. In the opposite limit of strong charge disorder \( \omega_D \) scales with \( N \) as \( N^{-1/4} \). Variational estimates for QPT locations are presented in Eq.(15).

\( T \neq 0 \) phase diagram is summarized in Fig.2. Low-temperature measurement of kinetic inductance seems to be the most adequate experimental method to study QPT in JW array. We are grateful to E. Cuevas, R. Fazio, L. B. Ioffe, S. E. Korshunov and M. Mueller for useful discussions. This research was supported by RFBR grant # 07-02-00310 and Programm “Quantum Macrophysics” of RAS. I.V.P. acknowledges support from Dynasty Foundation.

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