INTERSECTION PROPERTIES OF RELATIONS

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Abstract. We derive some equalities for relations on the algebra \( A \), under the assumption that every subalgebra of \( A \times A \) is congruence modular.

1. Notations

\( \alpha, \beta \ldots \) denote congruences on some algebra \( A \); \( \Theta, \Gamma \) are used for tolerances (reflexive, symmetric and admissible relations), while we reserve the letters \( R, S \) to denote reflexive (but otherwise arbitrary) binary relations. The words admissible and compatible will be used with the same meaning.

Already [J] realized the importance of dealing with reflexive admissible relations (cf. [J, Theorem 2.16]). [T] presents a very clear discussion of the interplay between congruence identities and identities involving reflexive admissible relations. Notice that our notation mainly comes from [T], and differs from [J].

We shall write \( aRb \) to mean that \( (a, b) \in R \), and we will use chains of the above notation: for example, \( a\Theta bacRD \) means \( (a, b) \in \Theta, (b, c) \in \alpha \) and \( (c, d) \in R \).

Intersection is sometimes denoted by juxtaposition; in particular \( \alpha\beta \) denotes the meet of the congruences \( \alpha \) and \( \beta \).

\( R^* \) denotes the transitive closure of the binary relation \( R \); in particular, \( \Theta^* \) is the smallest congruence which contains the tolerance \( \Theta \). \( \overline{R} \) denotes the smallest compatible relation containing \( R \) (where \( R \) is a binary relation on some algebra which should be clear from the context). In particular, \( \Theta \cup \Gamma \) is the smallest tolerance which contains the tolerances \( \Theta \) and \( \Gamma \).

\( R + S \) denotes \( \bigcup_{n \in \mathbb{N}} R \circ S \circ R \circ S \ldots \). Thus, \( R + S \) is the transitive closure of \( R \cup S \), and even the transitive closure of \( R \circ S \), since \( R \) and \( S \) are supposed to be reflexive. In particular, if \( \alpha, \beta \) are congruences, \( \alpha + \beta \) is the join of \( \alpha \) and \( \beta \) in the lattice of congruences, while, for \( \Theta, \Gamma \) tolerances, \( \Theta + \Gamma \) is the smallest congruence which contains both \( \Theta \) and \( \Gamma \). Notice that \( \Theta + \Gamma \) is far larger than the join of \( \Theta \) and \( \Gamma \) in the lattice of tolerances.

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Suppose that Theorem 2.1. such that for all reflexive relations \( R \) of \( A \) particularly, \( \alpha \) belongs to \(( R \) for all reflexive relations \( R \) are compatible then \( R \circ S \) is compatible; and, by an induction, we get that \( R + S \) is compatible, too. Thus, if \( R \) is compatible, then \( Cg(R) = R + R^- \). In general, for \( R \) not necessarily compatible, \( Cg(R) = Cg(R) = \overline{R + R^-} \).

2. Intersection properties

**Theorem 2.1.** Suppose that \( A \) is an algebra such that every subalgebra of \( A \times A \) generated by 4 elements satisfies \( \beta(\gamma + \delta) \subseteq \beta\gamma + \delta \), for all congruences \( \beta, \gamma, \delta \) with \( \delta \leq \beta \).

Then \( A \) satisfies
\[
\alpha(R+S) \subseteq \alpha(R \cup S^-)+\alpha(R^- \cup S^-) = \alpha(R \cup S^-)+\alpha(R^- \cup S^-) = \alpha(Cg(R)+Cg(S))
\]
for all reflexive relations \( R \) and \( S \) and every congruence \( \alpha \).

**Proposition 2.2.** Under the hypothesis of Theorem 2.1 \( A \) satisfies
\[
\alpha(R \circ S) \subseteq \alpha(R \cup S^-) + \alpha(R^- \cup S^-)
\]
for all reflexive relations \( R \) and \( S \) and every congruence \( \alpha \).

**Proof.** Let \( a, c \in A \) and \((a, c) \in \alpha \cap (R \circ S) \). Thus \( aoc \), and there is \( b \in A \) such that \( aRbSc \).

Consider the subalgebra \( B \) of \( A \times A \) generated by the four elements \((a, a), (a, b), (c, b), (c, c) \).

First, observe that if \((x, y) \in B \) then \((x, y) \in R \cup S^- \), since all the generators of \( B \) are in \( R \cup S^- \), and, by definition, \( R \cup S^- \) is compatible.

We have that \((a, a) \), \((a, b) \), \((c, b) \), \((c, c) \) belong to \( (0 \times 1)_B \) \((0 \times 1)_B \) \((0 \times 1)_B \) \((0 \times 1)_B \) belongs to \((\alpha \times 0)_B \) \((\alpha \times 0)_B \) \((\alpha \times 0)_B \) \((\alpha \times 0)_B \) belongs to \((\alpha \times \alpha)_B \). The above relations imply that \((\alpha \times \alpha)_B \) belongs to
\[
(\alpha \times \alpha)|_B \cap \left( (0 \times 1)|_B \circ (\alpha \times 0)|_B \circ (0 \times 1)|_B \right)
\]
Since \((\alpha \times 0)_B \leq (\alpha \times \alpha)_B \), by the hypothesis of the Theorem,
\[
(\alpha \times \alpha)|_B \cap \left( (0 \times 1)|_B \circ (\alpha \times 0)|_B \circ (0 \times 1)|_B \right) \subseteq
(\alpha \times \alpha)|_B \cap (0 \times 1)|_B + (\alpha \times 0)|_B = (0 \times \alpha)|_B + (\alpha \times 0)|_B
\]
In conclusion, \((\alpha \times \alpha)_B \) belongs to
\[
(0 \times \alpha)|_B + (\alpha \times 0)|_B
\]
This implies that there is some \( n \), and there are pairs \((x_i, y_i) \in B \) \((0 \leq i \leq n) \) such that
\[
(a, a) = (x_0, y_0) \quad (x_n, y_n) = (c, c)
(x_i, y_i) \equiv (x_{i+1}, y_{i+1}) \mod (0 \times \alpha)|_B \quad \text{for } i \text{ even}
(x_i, y_i) \equiv (x_{i+1}, y_{i+1}) \mod (\alpha \times 0)|_B \quad \text{for } i \text{ odd}
In other words,
\[
a = x_0 = y_0 \quad c = x_n = y_n \\
x_i = x_{i+1}, \quad y_i \alpha y_{i+1} \quad \text{for } i \text{ even} \\
x_i \alpha x_{i+1}, \quad y_i = y_{i+1} \quad \text{for } i \text{ odd}
\]

In particular, \(a = x_0 = x_1 \alpha x_2 = x_3 \alpha x_4 \ldots\), and \(a = y_0 \alpha y_1 = y_2 \alpha y_3 = y_4 \ldots\), hence \(x_i \alpha y_j\) for all \(i\)'s and \(j\)'s, since \(\alpha\) is a congruence, and both \(x_i\) and \(y_j\) are congruent to \(a\) modulo \(\alpha\).

Moreover, since \((x_i, y_i) \in B\), then \((x_i, y_i) \in \overline{R \cup S^{-}}\) for all \(i\)'s, by the remark made after the definition of \(B\).

Hence, for all \(i\)'s, \((x_i, y_i) \in \alpha(\overline{R \cup S^{-}})\), and \((y_i, x_i) \in \left(\alpha(\overline{R \cup S^{-}})^{-}\right) = \alpha(\overline{R^{-} \cup S})\).

In conclusion, the sequence
\[
a = x_0 = x_1 \quad y_1 = y_2 \quad x_2 = x_3 \quad y_3 = x_4 \quad \ldots \quad x_n = y_n = c
\]

witnesses that \((a, c) \in \alpha(\overline{R \cup S^{-}}) + \alpha(\overline{R^{-} \cup S})\).

\[\square\]

**Corollary 2.3.** \[\square\] **Under the hypothesis of Theorem 2.1** \(A\) satisfies 
\[\text{(wTIP)} \quad \alpha \Theta^* = (\alpha \Theta)^*
\]
for every tolerance \(\Theta\) and every congruence \(\alpha\).

**Proof.** One inclusion is trivial. By taking \(R = S = \Theta\) in Proposition 2.2, we get \(\alpha(\Theta \circ \Theta) \subseteq (\alpha \Theta)^*\). The conclusion follows by induction: see \[\square\] Lemma 3.3 for details; actually, the argument comes from \[\square\] and \[\square\]. \[\square\]

**Corollary 2.4.** **Under the Hypothesis of Theorem 2.1** \(A\) satisfies 
\[\alpha(R + R^-) \subseteq \alpha(\overline{R + R^-}) = \alpha\overline{R} + \alpha\overline{R^-} = \alpha Cg(R)
\]
for every reflexive relation \(R\).

**Proof.** The first inclusion, as well as the inclusion \(\alpha(\overline{R + R^-}) \supseteq \alpha\overline{R} + \alpha\overline{R^-}\) are trivial.

Since \(\overline{R \circ R^-}\) is a tolerance, we can apply Corollary 2.3 with \(\overline{R \circ R^-}\) in place of \(\Theta\), thus getting \(\alpha(\overline{R + R^-}) = \alpha(\overline{R \circ R^-})^* = (\alpha(\overline{R \circ R^-}))^* \subseteq \alpha\overline{R} + \alpha\overline{R^-}\), where the last inclusion follows from Proposition 2.2 with \(\overline{R}\) in place of \(S\), since \(\overline{R^-} = \overline{R}\), and since \(\alpha\overline{R} + \alpha\overline{R^-})^* = \alpha\overline{R} + \alpha\overline{R^-}\).

\[Cg(R) = \overline{R} + \overline{R^-}\] holds in every algebra, as mentioned at the end of Section 1\[\square\] hence \(\alpha Cg(R) = \alpha(\overline{R + R^-})\).

**Proof of Theorem 2.1** Since \((\overline{R \cup S^-}) = \overline{R^- \cup S}\), we can apply Corollary 2.4 with \(\overline{R \cup S^-}\) in place of \(R\), getting \(\alpha(R + S) \subseteq \alpha((\overline{R \cup S^-}) + (\overline{R^- \cup S})) = \alpha((\overline{R \cup S^-}) + (\overline{R \cup S^-})) = \alpha((\overline{R \cup S^-}) + (\overline{R \cup S^-})^-) = \alpha(\overline{R \cup S^-}) + \alpha(R \cup S^-) = \alpha(R \cup S^-) + \alpha(\overline{R \cup S^-})\

Since \(\overline{R \cup S^-} \subseteq \overline{R \circ S^-}\), \(\alpha(\overline{R \cup S^-}) \subseteq \alpha(\overline{R \circ S^-}) \subseteq \alpha(\overline{R \cup S}) + \alpha(\overline{R^- \cup S})\), by Proposition 2.2 with \(\overline{R}\) in place of \(R\) and \(\overline{S^-}\) in place of \(S\), and since
\[ R \cup S = R \cup S. \] Similarly, \( \alpha(R^{-} \cup S) \subseteq \alpha(S \circ R^{-}) \subseteq \alpha(R \cup S) + \alpha(R^{-} \cup S^{-}) \), hence \( \alpha(R \cup S^{-}) + \alpha(R^{-} \cup S) \subseteq \alpha(R \cup S) + \alpha(R^{-} \cup S^{-}) \). By replacing \( S \) with \( S^{-} \) in the inclusion just obtained, we get the reverse inclusion.

For the last identity, \( \alpha(Cg(R) + Cg(S)) = \alpha(Cg(R \cup S)) = \alpha(R \cup S) + \alpha(R \cup S^{-}) = \alpha(R \cup S) + \alpha(R^{-} \cup S^{-}) \) by the last identity in Corollary 2.4, with \( R \cup S \) in place of \( R \).

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