Graded Entailment for Compositional Distributional Semantics

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Abstract
The categorical compositional distributional model of natural language provides a conceptually motivated procedure to compute the meaning of sentences, given grammatical structure and the meanings of its words. This approach has outperformed other models in mainstream empirical language processing tasks. However, until recently it has lacked the crucial feature of lexical entailment – as do other distributional models of meaning.

In this paper we solve the problem of entailment for categorical compositional distributional semantics. Taking advantage of the abstract categorical framework allows us to vary our choice of model. This enables the introduction of a notion of entailment, exploiting ideas from the categorical semantics of partial knowledge in quantum computation.

The new model of language uses density matrices, on which we introduce a novel robust graded order capturing the entailment strength between concepts. This graded measure emerges from a general framework for approximate entailment, induced by any commutative monoid. Quantum logic embeds in our graded order.

Our main theorem shows that entailment strength lifts compositionally to the sentence level, giving a lower bound on sentence entailment. We describe the essential properties of graded entailment such as continuity, and provide a procedure for calculating entailment strength.

Categories and Subject Descriptors: Artificial Intelligence [Natural Language Processing]: Lexical Semantics

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1. Introduction
Finding a formalization of language in which the meaning of a sentence can be computed from the meaning of its parts has been a long-standing goal in formal and computational linguistics.

Distributional semantics represent individual word meanings as vectors in finite dimensional real vector spaces. On the other hand, symbolic accounts of meaning combine words via compositional rules to form phrases and sentences. These two approaches are in some sense orthogonal. Distributional schemes have no obvious compositional structure, whereas compositional models lack a canonical way of determining the meaning of individual words.

In Coecke et al. (2010), the authors develop the categorical compositional distributional model of natural language semantics. This model leverages the shared categorical structure of pregroup grammars and vector spaces to provide a compositional structure for distributional semantics. It has produced state-of-the-art results in measuring sentence similarity [Kartsaklis et al. 2012] [Grefenstette and Sadrzadeh 2011], effectively describing aspects of human understanding of sentences.

A satisfactory account of natural language should incorporate a suitable notion of lexical entailment. Until recently, categorical compositional distributional models of meaning have lacked this crucial feature. In order to address the entailment problem, we exploit the freedom inherent in our abstract categorical framework to change models. We move from a pure state setting to a category used to describe mixed states and partial knowledge in the semantics of categorical quantum mechanics. Meanings are now represented by density matrices rather than simple vectors. We use this extra flexibility to capture the concept of hyponymy, where one word may be seen as an instance of another. For example, red is a hyponym of colour. The hyponymy relation can be associated with a notion of logical entailment. Some entailment is crisp, for example: dog entails animal. However, we may also wish to permit entailments of differing strengths. For example, the concept dog gives high support to the the concept pet, but does not completely entail it: some dogs are working dogs. The hyponymy relation we describe here can account for these phenomena. We should also be able to measure entailment strengths at the sentence level. For example, we require that Cujo is a dog crisply entails Cujo is an animal, but that the statement Cujo is a dog does not completely entail Cujo is a pet. Again, the relation we describe here will successfully describe this behaviour at the sentence level.

An obvious choice for a logic built upon vector spaces is quantum logic (Birkhoff and Von Neumann 1936). Briefly, this logic represents propositions about quantum systems as projection operators on an appropriate Hilbert space. These projections form an orthomodular lattice where the distributive law fails in general. The logical structure is then inherited from the lattice structure in the usual way. In the current work, we propose an order that embeds the orthomodular lattice of projections, and so contains quantum logic. This order is based on the Löwner ordering with propositions represented by density matrices. When this ordering is applied to density matrices with the standard trace normalization, no propositions compare, and therefore the Löwner ordering is use-less as applied to density operators. The trick we use is to develop an approximate entailment relationship which arises naturally from any commutative monoid. We introduce this in general terms and describe conditions under which this gives a graded measure of entailment. This grading becomes continuous with respect to noise. Our framework is flexible enough to subsume the Bayesian partial ordering of Coecke and Martin (2011) and provides it with a grading.
Most closely related to the current work are the ideas in Balkir et al. [2014, 2016, 2015]. In this work, the authors develop a graded form of entailment based on von Neumann entropy and with links to the distributional inclusion hypotheses developed by Gefet and Dagan [2005]. The authors show how entailment at the word level carries through to entailment at the sentence level. However, this is done without taking account of the grading. In contrast, the measure that we develop here provides a lower bound for the entailment strength between sentences, based on the entailment strength between words. Further, the measure presented here is applicable to a wider range of sentence types than in Balkir et al. [2015]. Some of the work presented here was developed here in the first author’s MSc thesis [Bankova 2015].

Density matrices have also been used in other areas of distributional semantics. They are exploited in [Kartsaklis 2015, Piedeleu 2014, Piedeleu et al. 2015] to encode ambiguity. Blacoe et al. [2013] use density operators to encode the contexts in which a word occurs, but do not use these operators in a compositional structure. Quantum logic has been applied to distributional semantics in [Fiddows and Peters 2009], allowing queries of the form ‘suit NOT lawsuit’. Here, the vector for ‘suit’ is projected onto the subspace orthogonal to ‘lawsuit’. A similar approach, in the field of information retrieval, is described in [Van Rijsbergen 2004]. In this setting, document retrieval is modelled as a form of quantum logical inference.

The majorization preorder on density matrices has been extensively used in quantum information [Nielsen 1999], however it cannot be turned into a partial order and therefore it is of no use as an entailment relation.

1.1 Background

Within distributional semantics, word meanings are derived from text corpora using word co-occurrence statistics [Lund and Burgess 1996, Mitchell and Lapata 2010, Bullinaria and Levy 2007]. Other methods for deriving such meanings may be carried out. In particular, we can view the dimensions of the vector space as attributes of the concept, and experimentally determined attribute importance as the weighting on that dimension as in [Hampton 1987, McAue et al. 2005, Vinson and Vigliocco 2008, Devereux et al. 2014]. Distributional models of language have been shown to effectively model various facets of human meaning, such as similarity judgments [McDonald and Ramsay 2011], word sense discrimination [Schütze 1998, McCarthy et al. 2004] and text comprehension [Landauer and Dumais 1997, Folz et al. 1998].

Entailment is an important and thriving area of research within distributional semantics. The PASCAL Recognising Textual Entailment Challenge [Dagan et al. 2006] has attracted a large number of researchers in the area and generated a number of approaches. Previous lines of research on entailment for distributional semantics investigate the development of directed similarity measures which can characterize entailment [Weeds et al. 2004, Kotlerman et al. 2010, Lenci and Benotto 2012, Gefet and Dagan 2005] introduce a pair of distributional inclusion hypotheses, where if a word \( v \) entails another word \( w \), then all the typical features of the word \( w \) will also occur with the word \( v \). Conversely, if all the typical features of \( v \) also occur with \( w \), \( v \) is expected to entail \( w \). Clarke [2009] defines a vector lattice for word vectors, and a notion of graded entailment with the properties of a conditional probability. Kimell [2014] explores the limitations of the distributional inclusion hypothesis by examining the the properties of those features that are not shared between words. An interesting approach in [Kiel et al. 2015] is to incorporate other modes of input into the representation of a word. Measures of entailment are based on the dispersion of a word representation, together with a similarity measure.

Attempts have also been made to incorporate entailment measures with elements of compositionality. Baroni et al. [2012] exploit the entailment relations between adjective-noun and noun pairs to train a classifier that can detect similar relations. They further develop a theory of entailment for quantifiers.

2. Categorical Compositional Distributional Meaning

Compositional and distributional account of meaning are unified in Coecke et al. [2010], constructing the meaning of sentences from the meanings of their component parts using their syntactic structure.

2.1 Pregroup Grammars

In order to describe syntactic structure we use Lambek’s pregroup grammars [Lambek 1999]. This choice of grammar is not essential, and other forms of categorial grammar can be used, as argued in Coecke et al. [2013]. A pregroup \( (P, \leq, 1, (-)^t, (-)^r) \) is a partially ordered monoid \( (P, \leq, 1) \) where each element \( p \in P \) has a left adjoint \( p'^l \) and a right adjoint \( p'^r \), such that the following inequalities hold:

\[
p'^l \cdot p \leq 1 \leq p \cdot p'^l \quad \text{and} \quad p \cdot p'^r \leq 1 \leq p'^r \cdot p \tag{1}
\]

Intuitively, we think of the elements of a pregroup as linguistic types. The monoidal structure allows us to form composite types, and the partial order encodes type reduction. The important right and left adjoints then enable the introduction of types requiring further elements on either their left or right respectively.

The pregroup grammar \( \text{Preg}_{\text{gs}} \) over an alphabet \( B \) is freely constructed from the atomic types in \( B \). In what follows we use an alphabet \( B = \{ n, s \} \). We use the type \( s \) to denote a declarative sentence and \( n \) to denote a noun. A transitive verb can then be denoted \( n' \cdot s \\ n \\ s \\ 1 \leq s \\ 1 \).

This symbolic reduction can also be expressed graphically, as shown in figure [1]. In this diagrammatic notation, the elimination of types by means of the inequalities \( n' \cdot s \leq 1 \) and \( n' \cdot n \leq 1 \) is denoted by a ‘cup’ while the fact that the type \( s \) is retained is represented by a straight wire.

![Figure 1: A transitive sentence in the graphical calculus](image)

\[
\begin{align*}
\text{John} & \quad \text{kicks} \quad \text{cats} \\
\text{n} & \quad \text{n'} \quad \text{s} \quad \text{n} \quad \text{n}
\end{align*}
\]

2.2 Compositional Distributional Models

The symbolic account and distributional approaches are linked by the fact that they share the common structure of a compact closed category. This compatibility allows the compositional rules of the grammar to be applied in the vector space model. In this way we can map syntactically well-formed strings of words into one shared meaning space.

A compact closed category is a monoidal category in which for each object \( A \) there are left and right dual objects \( A' \) and \( A'' \), and corresponding unit and counit morphisms \( \eta : I \to A \otimes A' \), \( \eta' : I \to A'' \otimes A \), \( \epsilon : A' \otimes A \to I, \epsilon' : A \otimes A'' \to I \) such that...
the following snake equations hold:

\[(1_A \otimes \epsilon') \circ (\eta' \otimes 1_A) = 1_A \quad (\epsilon' \otimes 1_A) \circ (1_A \otimes \eta') = 1_A \]
\[(\epsilon' \otimes 1_{A'}) \circ (1_{A'} \otimes \eta') = 1_{A'}, \quad (1_{A'} \otimes \epsilon') \circ (\eta' \otimes 1_{A'}) = 1_{A'} \]

The underlying poset of a pregroup can be viewed as a compact closed category with the monoidal structure given by the pregroup monoid, and \(\epsilon', \eta', \eta''\) the unique morphisms witnessing the inequalities of \(\mathcal{P}\).

Distributional vector space models live in the category \(\text{FHilb}\) of finite dimensional real Hilbert spaces and linear maps. \(\text{FHilb}\) is compact closed. Each object \(V\) is its own dual and the left and right unit and counit morphisms coincide. Given a fixed basis \(\{|v_i\}\) of \(V\), we define the unit:

\[\eta : \mathbb{R} \to V \otimes V \quad : 1 \mapsto \sum_i |v_i\rangle \otimes |v_i\rangle\]

and counit:

\[\epsilon : V \otimes V \to \mathbb{R} : \sum_{ij} c_{ij} |v_i\rangle \otimes |v_j\rangle \mapsto \sum_i c_{ii}\]

Here we use the physicists bra-ket notation, for details see [Nielsen and Chuang 2010].

### 2.3 Graphical Calculus

The morphisms of compact closed categories can be expressed in a convenient graphical calculus ([Kelly and Laplaza 1980]) which we will exploit in the sequel. Objects are labelled wires, and morphisms are given as vertices with input and output wires. Composing morphisms consists of connecting input and output wires, and the tensor product is formed by juxtaposition, as shown in figure 2.

\[\begin{array}{ccc}
A & \rightarrow & A \\
\downarrow f & & \downarrow f \\
B & \rightarrow & B \\
\end{array} \quad \begin{array}{ccc}
B & \rightarrow & B \\
\downarrow g & & \downarrow g \\
A & \rightarrow & A \\
\end{array} \quad \begin{array}{ccc}
A & \rightarrow & A \\
\downarrow f & & \downarrow f \\
C & \rightarrow & C \\
\end{array}
\]

Figure 2: Monoidal Graphical Calculus

By convention the wire for the monoidal unit is omitted. The morphisms \(\epsilon\) and \(\eta\) can then be represented by ‘cups’ and ‘caps’ as shown in figure 3. The snake equations can be seen as straightening wires, as shown in figure 4.

\[\begin{array}{ccc}
\epsilon' & \circ \ & \eta' \\
\downarrow & & \downarrow \\
\eta' & \circ \ & \epsilon' \\
\end{array}
\]

Figure 3: Compact Structure Graphically

### 2.4 Grammatical Reductions in Vector Spaces

Following [Preller and Sadrzadeh 2011], reductions of the pregroup grammar may be mapped into the category \(\text{FHilb}\) of finite dimensional Hilbert spaces and linear maps using an appropriate strong monoidal functor \(Q\):

\[Q : \text{Preg} \to \text{FHilb}\]

Strong monoidal functors automatically preserve the compact closed structure. For our example \(\text{Preg}_{(n,s)}\), we must map the noun and sentence types to appropriate finite dimensional vector spaces:

\[Q(n) = N \quad Q(s) = S\]

Composite types are then constructed functorially using the corresponding structure in \(\text{FHilb}\). Each morphism \(\alpha\) in the pregroup is mapped to a linear map interpreting sentences of that grammatical type. Then, given word vectors \(|w_i|\) with types \(p_i\), and a type reduction \(\alpha : p_1, p_2, \ldots, p_n \to s\), the meaning of the sentence \(w_1, w_2, \ldots, w_n\) is given by:

\[|w_1 w_2 \ldots w_n| = Q(\alpha)(|w_1| \otimes |w_2| \otimes \ldots \otimes |w_n|)\]

For example, as described in section 2.1, transitive verbs have type \(n'sn'\), and can therefore represented in \(\text{FHilb}\) as a rank 3 space \(N \otimes S \otimes N\). The transitive sentence \(\text{John kicks cats}\) has type \(n(n'sn')n\), which reduces to the sentence type via \(\epsilon' \otimes 1_s \otimes \epsilon'\). So if we represent \(|\text{kicks}|\) by:

\[|\text{kicks}| = \sum_{ijk} c_{ijk} |e_i\rangle \otimes |s_j\rangle \otimes |e_k\rangle\]

using the definitions of the counits in \(\text{FHilb}\) we then have:

\[|\text{John kicks cats}| = \epsilon_N \otimes 1_S \otimes \epsilon_N(\langle \text{John} | \otimes |\text{kicks} \rangle \otimes |\text{cats}\rangle)\]

\[= \sum_{ijk} c_{ijk} \langle \text{John}| e_i \rangle \otimes |s_j\rangle \otimes |e_k|\langle \text{cats}\rangle\]

The category \(\text{FHilb}\) is actually a \(\dagger\)-compact closed category. A \(\dagger\)-compact closed category is a compact closed category with an additional dagger functor that is an identity on objects involution, satisfying natural coherence conditions. In the graphical calculus, the dagger operation “flips diagrams upside-down”. In the case of \(\text{FHilb}\) the dagger sends a linear map to its adjoint, and this allows us to reason about inner products in a general categorical setting.

Meanings of sentences may be compared using the inner product to calculate the cosine distance between vector representations. So, if sentence \(s\) has vector representation \(|s|\) and sentence \(s'\) has representation \(|s'|\), their degree of synonymy is given by:

\[\frac{\langle s|s'\rangle}{\sqrt{\langle s|s\rangle \langle s'|s'|\rangle}}\]

The abstract categorical framework we have introduced allows meanings to be interpreted not just in \(\text{FHilb}\), but in any \(\dagger\)-compact closed category. We will exploit this freedom when we move to density matrices. Detailed presentations of the ideas in this section are given in [Coecke et al. 2010; Preller and Sadrzadeh 2011] and an introduction to relevant category theory given in [Coecke and Paquette 2011].
3. Density Matrices in Categorical Compositional Distributional Semantics

3.1 Positive Operators and Density Matrices

The methods outlined in section 2 can be applied to the richer setting of density matrices. Density matrices are used in quantum mechanics to express uncertainty about the state of a system. For a vector $|v\rangle$, the projection operator $|v\rangle \langle v|$ onto the subspace spanned by $|v\rangle$ is called a pure state. Pure states can be thought of as giving sharp, unambiguous information. In general, density matrices are given by a convex sum of pure states, describing a probabilistic mixture. States that are not pure are referred to as mixed states. Necessary and sufficient conditions for an operator $\rho$ to encode such a probabilistic mixture are:

- $\forall v \in V. \langle v | \rho | v \rangle \geq 0$
- $\rho$ is self-adjoint.
- $\rho$ has trace 1.

Operators satisfying the first two axioms are called positive operators. The third axiom ensures that the operator represents a convex mixture of pure states. However, relaxing this condition gives us different choices for normalization, which we will outline in section 3.2.

In distributional models of meaning, we can consider the meaning of a word $w$ to be given by a collection of unit vectors $\{|w_i\rangle\}$, where each $|w_i\rangle$ represents an instance of the word expressed by the word. Each $|w_i\rangle$ is weighted by $p_i \in [0,1]$, such that $\sum_i p_i = 1$. These weights describe the meaning of $w$ as a weighted combination of exemplars. Then the density operator:

$$\rho = \sum_i p_i |w_i\rangle \langle w_i|$$

represents the word $w$. For example a cat is a fairly typical pet, and a tarantula is less typical, so a simple density operator for the word pet might be:

$$\rho_{\text{pet}} = 0.9 \times |\text{cat}\rangle \langle \text{cat}| + 0.1 \times |\text{tarantula}\rangle \langle \text{tarantula}|$$

3.2 The CPM Construction

Applying Selinger’s CPM construction [Selinger 2007] to FHilb produces a new †-compact closed category in which the states are positive operators. This construction has previously been exploited in a linguistic setting in [Kartsaklis 2015; Piedeleu et al. 2015; Balkir et al. 2016].

Throughout this section $\mathcal{C}$ denotes an arbitrary †-compact closed category.

**Definition 1** (Completely positive morphism). A $\mathcal{C}$-morphism $\varphi : A^* \otimes A \to B^* \otimes B$ is said to be completely positive [Selinger 2007] if there exists $\mathcal{C} \in \text{Ob}(\mathcal{C})$ and $k \in \mathcal{C}(C \otimes A, B)$, such that $\varphi$ can be written in the form:

$$(k_\otimes \otimes k) \circ (1_{A^*} \otimes \eta_C \otimes 1_A)$$

Identity morphisms are completely positive, and completely positive morphisms are closed under composition in $\mathcal{C}$, leading to the following:

**Definition 2.** If $\mathcal{C}$ is a †-compact closed category then $\text{CPM}(\mathcal{C})$ is a category with the same objects as $\mathcal{C}$ and its morphisms are the completely positive morphisms.

The †-compact structure required for interpreting language in our setting lifts to $\text{CPM}(\mathcal{C})$:

**Theorem 1.** $\text{CPM}(\mathcal{C})$ is also a †-compact closed category. There is a functor:

$$\mathcal{E} : \mathcal{C} \to \text{CPM}(\mathcal{C})$$

| $\mathcal{CPM}(\mathcal{C})$ | $\mathcal{C}$ |
|---|---|
| $\epsilon : \mathcal{C} \to \mathcal{C}$ | $\mathcal{C}$ |
| $\varepsilon : A^* \otimes A \otimes A \to \mathcal{I}$ | $\mathcal{A}$ |
| $\varepsilon : |e_i\rangle \otimes |e_j\rangle \otimes |e_k\rangle \otimes |e_l\rangle \mapsto \langle e_i|e_k\rangle \langle e_j|e_l\rangle$ | $\mathcal{A}$ |
| $\eta : \mathcal{I} \to A \otimes A \otimes A^*$ | $\mathcal{A}$ |
| $\eta : 1 \mapsto \sum_{i,j} |e_i\rangle \otimes |e_j\rangle \otimes |e_l\rangle \otimes |e_l\rangle$ | $\mathcal{A}$ |

This functor preserves the †-compact closed structure, and is faithful “up to a global phase” [Selinger 2007].

3.3 Diagrammatic calculus for $\text{CPM}(\mathcal{C})$

As $\text{CPM}(\mathcal{C})$ is also a †-closed category, we can use the graphical calculus described in section 2.5. By convention, the diagrammatic calculus for $\text{CPM}(\mathcal{C})$ is drawn using thick wires. The corresponding diagrams in $\mathcal{C}$ are given in table 1.

| $\text{Table 1: Table of diagrams in } \text{CPM}(\mathcal{C}) \text{ and } \mathcal{C}$ |
|---|---|
| $f_1 \otimes f_2 : A^* \otimes C^* \otimes C \otimes A \to B^* \otimes D^* \otimes D \otimes B$ | $k \to k^* \otimes k$ |

This functor preserves the †-compact closed structure, and is faithful “up to a global phase” [Selinger 2007].

3.3.1 Sentence Meaning in the category $\text{CPM}(\text{FHilb})$

In the vector space model of distributional models of meaning the transition between syntax and semantics was achieved via a strong monoidal functor $\mathcal{S} : \text{Preg} \to \text{FHilb}$. Language can be assigned semantics in $\text{CPM}(\text{FHilb})$ in an entirely analogous way via a strong monoidal functor:

$$\mathcal{S} : \text{Preg} \to \text{CPM}(\text{FHilb})$$

**Definition 3.** Let $w_1, w_2, \ldots, w_n$ be a string of words with corresponding grammatical types $t_i$ in $\text{Preg}$. Suppose that the type reduction is given by $t_1, \ldots, t_n \rightarrow x$ for some $x \in \text{Ob}(\text{Preg})$. Let $[w_i]$ be the meaning of word $w_i$ in $\text{CPM}(\text{FHilb})$, i.e. a state of the form $I \rightarrow S(t_i)$. Then the meaning of $w_1 w_2 \ldots w_n$ is given by:

$$[w_1 w_2 \ldots w_n] = \mathcal{S}(\mathcal{R})([w_1] \otimes \ldots \otimes [w_n])$$

We now have all the ingredients to derive sentence meanings in $\text{CPM}(\text{FHilb})$.

**Example 1.** We firstly show that the results from $\text{FHilb}$ lift to $\text{CPM}(\text{FHilb})$. Let the noun space $\mathcal{N}$ be a real Hilbert space with basis vectors given by $\{|n_i\rangle\}_i$, where for some $i$, $|n_i\rangle = |\text{Clara}\rangle$ and for some $j$, $|n_j\rangle = |\text{beer}\rangle$. Let the sentence space be another space $S$ with basis $\{|s_i\rangle\}_i$. The verb $|\text{likes}\rangle$ is given by:

$$|[\text{likes}]\rangle = \sum_{pq} C_{pqr} |n_p\rangle \otimes |s_q\rangle \otimes |n_r\rangle$$

The density matrices for the nouns Clara and beer are in fact pure states given by:

$$[\text{Clara}] = |n_i\rangle \langle n_i| \quad \text{and} \quad [\text{beer}] = |n_j\rangle \langle n_j|$$
and similarly, \([\text{likes}]\) in CPM(FHilb) is:
\[
[\text{likes}] = \sum_{pqrsv} C_{pqr} C_{rs} \mid n_p \rangle \langle n_q \mid s_v \rangle \langle n_r \mid (n_v)
\]

The meaning of the composite sentence is simply \((\varepsilon N \otimes 1_S \otimes \varepsilon N)\) applied to \(([\text{Clara}] \otimes [\text{likes}] \otimes [\text{beer}])\) as shown in figure 5 with interpretation in FHilb shown in figure 6.

Diagrammatically, this is shown in figure 7.

4. Predicates and Entailment

If we consider a model of (non-deterministic) classical computation, a state of a set \(X\) is just a subset \(\rho \subseteq X\). Similarly, a predicate is a subset \(A \subseteq X\). We say that \(\rho\) satisfies \(A\) if:

\[\rho \subseteq A\]

which we write as \(\rho \models A\). Predicate \(A\) entails predicate \(B\), written \(A \models B\) if for every state \(\rho\):

\[\rho \models A \implies \rho \models B\]

Clearly this is equivalent to requiring \(A \subseteq B\).

4.1 The Löwner Order

As our linguistic models derive from a quantum mechanical formalism, positive operators form a natural analogue for subsets as our predicates. This follows ideas in (D’Hondt and Panangaden 2006) and earlier work in a probabilistic setting in (Kozen 1983). Crucially, we can order positive operators (Löwner 1934).

**Definition 4 (Löwner Order).** For positive operators \(A\) and \(B\), we define:

\[A \sqsubseteq B \iff B - A\] is positive

If we consider this as an entailment relationship, we can follow our intuitions from the non-deterministic setting. Firstly we introduce a suitable notion of satisfaction. For positive operator \(A\) and density matrix \(\rho\), we define \(\rho \models A\) as the positive real number \(\text{tr}(\rho A)\). This generalizes satisfaction from a binary relation to a binary function into the positive reals. We then find that the Löwner order can equivalently be phrased in terms of satisfaction as follows:

**Lemma 1** (D’Hondt and Panangaden 2006). Let \(A\) and \(B\) be positive operators. \(A \subseteq B\) if and only if for all density operators \(\rho\):

\[\rho \models A \iff \rho \models B\]

Linguistically, we can interpret this condition as saying that every noun, for example, satisfies predicate \(B\) at least as strongly as it satisfies predicate \(A\).

4.2 Quantum Logic

Quantum logic (Birkhoff and Von Neumann 1936) views the projection operators on a Hilbert space as propositions about a quantum system. As the Löwner order restricts to the usual ordering on projection operators, we can embed quantum logic within the poset of projection operators, providing a direct link to existing theory.

4.3 A General Setting for Approximate Entailment

We can build an entailment preorder on any commutative monoid, viewing the underlying set as a collection of propositions. We then
write:

\[ A \models B \]
and say \( A \) entails \( B \) if there exists a proposition \( D \) such that:

\[ A + D = B \]

If our commutative monoid is the powerset of some set \( X \), with union the binary operation and unit the empty set, then we recover our non-deterministic computation example from the previous section. If on the other hand we take our commutative monoid to be the positive operators on some Hilbert space, with addition of operators and the zero operator as the monoid structure, we recover the Löwner ordering.

In linguistics, we may ask ourselves does dog entail pet? Naïvely, the answer is clearly no, not every dog is a pet. This seems too crude for realistic applications though, most dogs are pets, and so we might say dog entails pet to some extent. This motivates our need for an approximate notion of entailment.

For proposition \( E \), we say that \( A \) entails \( B \) to the extent \( E \) if:

\[ A \models B + E \]

We think of \( E \) as a term, for instance in our dogs and pets example, \( E \) adds back in dogs that are not pets. Expanding definitions, we find \( A \) entails \( B \) to the extent \( E \) if there exists \( D \) such that:

\[ A + D = B + E \] \hspace{1cm} (2)

From this more symmetrical formulation it is easy to see that for arbitrary propositions \( A, B \), proposition \( A \) trivially entails \( B \) to extent \( A \), as by commutativity:

\[ A + B = B + A \]

It is therefore clear that the mere existence of a suitable error term is not sufficient for a weakened notion of entailment. If we restrict our attention to errors in a complete meet semilattice \( E_{A, B} \), we can take the lower bound on the \( E \) satisfying equation (2) as our canonical choice. Finally, if we wish to be able to compare entailment strengths globally, this can be achieved by choosing a partial order \( K \) of “error sizes” and monotone functions:

\[ E_{A, B} \xrightarrow{k, A, B} K \]

sending errors to their corresponding size.

For example, if \( A \) and \( B \) are positive operators, we take our complete lattice of error terms \( E_{A, B} \) to be all operators of the form \((1 - k)A\) for \( k \in [0, 1] \), ordered by the size of \( 1 - k \). We then take \( k \) as the strength of the entailment, and refer to it as \( k \)-hyponymy.

In the case of finite sets \( A, B \), we take \( E_{A, B} = \mathcal{P}(A) \), and take the size of the error terms as:

\[
\text{cardinality of } E \\
\text{cardinality of } A
\]

measuring “how much” of \( A \) we have to supplement \( B \) with, as indicated in the shaded region below:

```
  B \\
  A
```

In terms of conditional probability, the error size is then:

\[ P(A \mid \neg B) \]

### 4.3.1 \( k \)-hyponymy Versus General Error Terms

We can see that the general error terms are strictly more general than considering the \( k \)-hyponymy case. If we consider positive operators with matrix representations:

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} \\
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Predicate \( A \) cannot entail \( B \) with any positive strength \( k \). We can see \( B - kA \) is never a positive operator as the following expression is always negative:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} - k \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 - k & 0 & 0 \\
0 & 1 - k & 0 \\
0 & 0 & 1 - k
\end{pmatrix}
\]

We can find a more general positive operator \( E \) such that \( A \sqsubseteq B + E \) though, as:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Therefore the general error terms offer strictly more freedom than \( k \)-hyponymy.

## 5. Hyponymy in Categorical Compositional Distributional Semantics

Modelling hyponymy in the categorical compositional distributional semantics framework was first considered in [Balkir 2014]. She introduced an asymmetric similarity measure called representativeness on density matrices based on quantum relative entropy. This can be used to translate hyponym-hypernym relations to the level of positive transitive sentences. Our aim here will be to provide an alternative measure which relies only on the properties of density matrices and the fact that they are the states in CPM(FHH1b). This will enable us to quantify the strength of the hyponymy relationship, described as \( k \)-hyponymy. The measure of hyponymy that we use has two advantages over the representativeness measure. Firstly, it combines with the linear sentence maps so that we can work with sentence-level entailment across a larger range of sentences. Secondly, due to the way it combines with linear maps, we can give a quantitative measure to sentence-level entailment based on the entailment strengths between words, whereas representativeness is not shown to combine in this way.

### 5.1 Properties of hyponymy

Before proceeding with defining the concept of \( k \)-hyponymy, we will list a couple of properties of hyponymy. We will show later that these can be captured by our new measure.

- **Asymmetry.** If \( A \) is a hyponym of \( B \), then this does not imply that \( Y \) is a hyponym of \( X \). In fact, we may even assume that only one of these relationships is possible, and that they are mutually exclusive. For example, *football* is a type of *sport* and hence *football-sport* is a hyponym-hypernym pair. However, *sport* is not a type of *football*.

- **Pseudo-transitivity.** If \( X \) is a hyponym of \( Y \) and \( Y \) is a hyponym of \( Z \), then \( X \) is a hyponym of \( Z \). However, if the hyponymy is not perfect, then we get a weakened form of transitivity. For example, *dog* is a hyponym of *pet*, and *pet* is a hyponym of *things that are cared for*. However, not every dog is well cared-for, so the transitivity weakens. An outstanding question is where the entailment strength reverses. For example, *dog* imperfectly entails *pet*, and *pet* imperfectly entails *mammal*, but *dog* perfectly entails *mammal*.

The measure of hyponymy that we described above and named \( k \)-hyponymy will be defined in terms of density matrices - the con-
tainers for word meanings. The idea is then to define a quantitative order on the density matrices, which is not a partial order, but does give us an indication of the asymmetric relationship between words.

5.2 Ordering Positive Matrices

A density matrix can be used to encode the extent of precision that is needed when describing an action. In the sentence I took my pet to the vet, we do not know whether the pet is a dog, cat, or tarantula and so on. The sentence I took my dog to the vet entails the sentence I took my pet to the vet.

Lemma 2. For positive self-adjoint matrices \(A, B\) and \(\lambda \geq 0\), we write
\[
A \preceq \lambda B \quad \text{if} \quad \langle \psi | (A - \lambda B) | \psi \rangle \leq 0 \quad \forall |\psi\rangle.
\]

We now develop an expression for the optimal \(k\)-max value for which \(k\)-hyponymy holds between two positive operators. This \(k\)-max value quantifies the strength of the entailment between the two operators. In what follows, for operator \(A\) we write \(A^{+}\) for the corresponding Moore-Penrose pseudo-inverse and \(\text{supp}(A)\) for the support of \(A\).

Lemma 2 (Balkir 2014). Let \(A, B\) be positive operators.\n
\[
\text{supp}(A) \subseteq \text{supp}(B) \iff \exists k \geq 0 \text{ and } B - kA \geq 0.
\]

Lemma 3. For positive self-adjoint matrices \(A, B\) such that:
\[
\text{supp}(A) \subseteq \text{supp}(B)
\]
\(B^{+} A\) has non-negative eigenvalues.

We now develop an expression for the optimal \(k\) in terms of the matrices \(A\) and \(B\).

Theorem 2. For positive self-adjoint matrices \(A, B\) such that:
\[
\text{supp}(A) \subseteq \text{supp}(B)
\]
the maximum \(k\) such that \(B - kA \geq 0\) is given by \(1/\lambda\) where \(\lambda\) is the maximum eigenvalue of \(B^{+} A\).

5.3 Properties of \(k\)-hyponymy

Reflexivity \(k\)-hyponymy is reflexive for \(k = 1\). For any operator \(A, A - A = 0\).

Symmetry \(k\)-hyponymy is neither symmetric nor anti-symmetric. For example it is not symmetric since:
\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \preceq 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \preceq 1/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

but it is also not anti-symmetric, since
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \preceq 1/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \preceq 1/2 \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}
\]

Transitivity \(k\)-hyponymy satisfies a version of transitivity. Suppose \(A \preceq_{k} B\) and \(B \preceq_{l} C\). Then \(A \preceq_{kl} C\), since:
\[
B \supseteq kA \text{ and } C \supseteq lB \implies C \supseteq klA
\]

by transitivity of the Löwner order. For the maximal values \(k_{\text{max}}, l_{\text{max}}, m_{\text{max}}\) such that \(A \preceq_{k_{\text{max}}} B, B \preceq_{l_{\text{max}}} C\) and \(A \preceq_{m_{\text{max}}} C\), we have the inequality
\[
m_{\text{max}} \geq k_{\text{max}} l_{\text{max}}
\]

Continuity For \(A \preceq_{k} B\), when there is a small perturbation to \(A\) there is a correspondingly small decrease in the value of \(k\). The perturbation must lie in the support of \(B\), but can introduce off-diagonal elements.

Theorem 3. Given \(A \preceq_{k} B\) and density operator \(\rho\) such that \(\text{supp}(\rho) \subseteq \text{supp}(B)\), then for any \(\varepsilon > 0\) we can choose a \(\delta > 0\) such that:
\[
A' = A + \delta \rho \implies A' \preceq_{k'} B \text{ and } |k - k'| < \varepsilon.
\]

5.4 Scaling

When comparing positive operators, in order to standardize the magnitudes resulting from calculations, it is natural to consider normalizing their trace so that we work with density operators. Unfortunately, this is a poor choice when working with the Löwner order as distinct pairs of density operators are never ordered with respect to each other. Instead we consider bounding our operators as having maximum eigenvalue 1, as suggested in (D’Hondt and Panangaden 2006). With this ordering, the projection operators regain their usual ordering and we recover quantum logic as a suborder of our setting.

Our framework is flexible enough to support other normalization strategies. The optimal choice for linguistic applications is left to future empirical work. More interesting ideas are also possible. For example we can embed the Bayesian order (Coecke and Martin 2011) within our setting via a suitable transformation on positive operators. This is described in more detail in appendix section A.1. Further theoretical investigations of this type are left to future work.

5.5 Examples

In this section we give three simple examples and illustrate the order for 2-dimensional matrices in the Bloch sphere.

Example 3. Consider the density matrix:
\[
|\text{per}| = 1/2 |\text{dog}\rangle \langle \text{dog}| + 1/2 |\text{cat}\rangle \langle \text{cat}|
\]

The entailment strength \(k\) such that \(k |\text{dog}\rangle \langle \text{dog}| \preceq |\text{per}|\) is \(1/2\).

Further, if two mixed states \(\rho, \sigma\) can both be expressed as convex combinations of the same two pure states, the extent to which one state entails the other can also be derived.

Example 4. For states:
\[
\rho = r |\psi\rangle \langle \psi| + (1 - r) |\phi\rangle \langle \phi|
\]
\[
\sigma = s |\psi\rangle \langle \psi| + (1 - s) |\phi\rangle \langle \phi|
\]
the entailment strength $k$ such that $k\sigma \leq \rho$ is given by:

$$k = \begin{cases} 
\frac{2}{r - s} & \text{if } r < s \\
\frac{1}{1 - s} & \text{otherwise}
\end{cases}$$

**Example 5.** Suppose that $[B] = k_j [A] + \sum_{i \neq j} k_i [X_i]$. Then:

$$[A] \preceq_k [B]$$

for any $k \leq k_j$.

From the above example we notice that the value $k_1$ definitely gives us $k_1$-hyponymy between $A$ and $B$, but it is actually possible that there exists a value, say $l$, such that $l > k_1$ and for which we have $l$-hyponymy between $A$ and $B$. Indeed, this happens whenever we have an $l$ for which:

$$(k_1 - l) \langle x \mid [A] \mid x \rangle \geq -\sum_{i \neq 1} p_i \langle x \mid [X_i] \mid x \rangle$$

Thus, $k_1$ may not be the maximum value for hyponymy between $A$ and $B$. In general, however, we are interested in making the strongest assertion we can and therefore we are interested in the maximum value of $k$, which we call the **entailment strength**.

For matrices on $\mathbb{R}^2$, we can represent these entailment strengths visually using the Bloch sphere restricted to $\mathbb{R}^2$ - the ‘Bloch disc’.

### 5.5.1 Representing the order in the ‘Bloch disc’

The Bloch sphere, $\{\text{Bloch}\}[1946]$, is a geometrical representation of quantum states. Very briefly, points on the sphere correspond to pure states, and states within the sphere to impure states. Since we consider matrices only over $\mathbb{R}^2$, we disregard the complex phase which allows us to represent the pure states on a circle. A pure state $\cos(\theta/2) \langle 0 \rangle + \sin(\theta/2) \langle 1 \rangle$ is represented by the vector $(\sin(\theta), \cos(\theta))$ on the circle.

We can calculate the entailment factor $k$ between any two points on the disc. For example, in figure 8 we show contour maps of the entailment strengths for the state with Bloch vector $(\frac{2}{3} \sin(\pi/5), \frac{4}{3} \cos(\pi/5))$, using the maximum eigenvalue normalization.

![Bloch diagram](image)

**Figure 8:** Entailment strengths in the Bloch disc for the state with Bloch vector $(\frac{2}{3} \sin(\pi/5), \frac{4}{3} \cos(\pi/5))$.

### 6. Main Results on Compositionality

We will now consider what happens when we have two sentences such that one of them contains one or more hyponyms of one or more words from the other. We will show in this case that the hyponymy is ‘lifted’ to the sentence level, and that the $k$-values are preserved in a very intuitive fashion. After considering a couple of specific sentence constructions, we will generalise this result to account for a broad category of sentence patterns that work in the compositional distributional model.

#### 6.1 $k$-hyponymy in positive transitive sentences

A positive transitive sentence has the diagrammatic representation in CPM($\text{FHLb}$) given in figure 7. The meaning of the sentence $\text{subj verb obj}$ is given by:

$$(\varepsilon_N \otimes 1_S \otimes \varepsilon_N)([\text{subj}] \otimes [\text{verb}] \otimes [\text{obj}]),$$

where the $\varepsilon_N$ and $1_S$ morphisms are those from CPM($\text{FHLb}$). We will represent the subject and object by:

$$[\text{subj}] = \sum_k a_{ik} \langle n_k \rangle \langle n_k \rangle \text{ and } [\text{obj}] = \sum_{jl} b_{jl} \langle n_j \rangle \langle n_j \rangle.$$

Finally, let the verb be given by:

$$[\text{verb}] = \sum_{pqrtuv} C_{pqrtuv}(n_p \otimes n_q \otimes n_r \otimes s_t \otimes n_v \otimes n_v)$$

**Theorem 4.** Let $n_1, n_2, n_3, n_4$ be nouns with corresponding density matrix representations $[n_1], [n_2], [n_3]$ and $[n_4]$, such that $n_1$ is a $k$-hyponym of $n_2$ and $n_3$ is a $k$-hyponym of $n_4$. Then:

$$\varphi(n_1 \text{ verb } n_3) \preceq_{k_1} \varphi(n_2 \text{ verb } n_4),$$

where $\varphi = \varepsilon_N \otimes 1_S \otimes \varepsilon_N$ is the sentence meaning map for positive transitive sentences.

#### 6.2 General Sentence $k$-hyponymy

We can show that the application of $k$-hyponymy to various phrase types holds in the same way. In this section we provide a general proof for varying phrase types. We adopt the following conventions:

- A **positive phrase** is assumed to be a phrase in which individual words are upwardly monotone in the sense described by [MacCartney and Manning 2007]. This means that, for example, the phrase does not contain any negations, including words like not.
- The **length** of a phrase is the number of words in it, not counting definite and indefinite articles.

**Theorem 5** (Generalised Sentence $k$-Hyponymy). Let $\Phi$ and $\Psi$ be two positive phrases of the same length and grammatical structure, expressed in the same noun spaces $N$ and sentence spaces $S$. Denote the nouns and verbs of $\Phi$, in the order in which they appear, by $A_1, \ldots, A_n$. Similarly, denote these in $\Psi$ by $B_1, \ldots, B_n$. Let their corresponding density matrices be denoted by $[A_1], \ldots, [A_n]$ and $[B_1], \ldots, [B_n]$ respectively. Suppose that $[A_i] \preceq_{k_i} [B_i]$ for $i \in \{1, \ldots, n\}$ and some $k_i \in \{0, 1\}$. Finally, let $\varphi$ be the sentence meaning map for both $\Phi$ and $\Psi$, such that $\varphi(\Phi)$ is the meaning of $\Phi$ and $\varphi(\Psi)$ is the meaning of $\Psi$. Then:

$$\varphi(\Phi) \preceq_{k_1 \cdots k_n} \varphi(\Psi).$$

Intuitively, this means that if (some of) the functional words of a sentence $\Phi$ are $k$-hyponyms of (some of) the functional words of a sentence $\Psi$, then this hyponymy is translated into sentence hyponymy. Upward-monotonicity is important here, and in particular implicit quantifiers. It might be objected that dogs bark should not imply pets bark. If the implicit quantification is universal, then this is true, however the universal quantifier is downward monotone, and therefore does not conform to the convention concerning positive phrases. If the implicit quantification is existential,
then some dogs bark does entail some pets bark, and the problem is averted. Discussion of the behaviour of quantifiers and other word types is given in (MacCartney and Manning 2007). The quantity $k_1 \cdots k_n$, is not necessarily maximal, and indeed usually is not. As we only have a lower bound, zero entailment strength between a pair of components does not imply zero entailment strength between entire sentences. Results for phrases involving relative clauses may be found in appendix C.

Corollary 1. Consider two sentences:

$$\Phi = \bigotimes_i [A_i], \quad \Psi = \bigotimes_i [B_i]$$

such that for each $i \in \{1, \ldots, n\}$ we have $[A_i] \subseteq [B_i]$, i.e. there is strict entailment in each component. Then there is strict entailment between the sentences $\varphi(\Phi)$ and $\varphi(\Psi)$.

We consider a concrete example.

**Compositionality of $\cdot$-hypernymy in a transitive sentence.** More examples may be found in appendix D. Suppose we have a noun space $N$ with basis $\{ |e_i \rangle \}$, and sentence space $S$ with basis $\{ |x_i \rangle \}$. We consider the verbs nibble, scoff and the nouns cake, chocolate, with semantics:

$$\begin{align*}
\text{nibble} &= \sum_{pqru} a_{pqr} a_{tu} |e_p \rangle \langle e_q| \otimes |x_u \rangle \otimes |e_r \rangle \langle e_t| \\
\text{scoff} &= \sum_{pqru} b_{pqr} b_{tu} |e_p \rangle \langle e_q| \otimes |x_u \rangle \otimes |e_r \rangle \langle e_t| \\
\text{cake} &= \sum_i c_i c_j |e_i \rangle \langle e_j| \\
\text{chocolate} &= \sum_i d_i d_j |e_i \rangle \langle e_j|
\end{align*}$$

which make these nouns and verbs pure states. The more general eat and sweets are given by:

$$\begin{align*}
\text{eat} &= \frac{1}{2} (\text{nibble} + \text{scoff}) \\
\text{sweets} &= \frac{1}{2} (\text{cake} + \text{chocolate})
\end{align*}$$

Then

$$\begin{align*}
\text{scoff} &\preceq_{1/2} \text{eat} \\
\text{cake} &\preceq_{1/2} \text{sweets}
\end{align*}$$

We consider the sentences:

$$s_1 = \text{John scoffs cake}$$

$$s_2 = \text{John eats sweets}$$

The semantics of these sentences are:

$$\begin{align*}
[s_1] &= \varphi([\text{Mary}] \otimes [\text{scoffs}] \otimes [\text{cake}]) \\
[s_2] &= \varphi([\text{Mary}] \otimes [\text{eats}] \otimes [\text{sweets}])
\end{align*}$$

and as per theorem 5 we will show that $[s_1] \preceq_{kl} [s_2]$ where $kl = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$. Expanding $[s_2]$ we obtain:

$$\begin{align*}
[s_2] &= \varphi([\text{Mary}] \otimes \frac{1}{2} ([\text{nibbles}] + [\text{scoffs}]) \\
&\quad \otimes \frac{1}{2} ([\text{cake}] + [\text{choc}])) \\
&= \frac{1}{4} (\varphi([\text{Mary}] \otimes [\text{scoffs}] \otimes [\text{cake}]) \\
&\quad + \varphi([\text{Mary}] \otimes [\text{scoffs}] \otimes [\text{choc}]) \\
&\quad + \varphi([\text{Mary}] \otimes [\text{nibbles}] \otimes [\text{cake}]) \\
&\quad + \varphi([\text{Mary}] \otimes [\text{nibbles}] \otimes [\text{choc}]))
\end{align*}$$

Therefore:

$$\begin{align*}
[s_2] - \frac{1}{4} [s_1] &= \varphi([\text{Mary}] \otimes [\text{choc}] \otimes [\text{choc}]) \\
&\quad + \varphi([\text{Mary}] \otimes [\text{nibbles}] \otimes [\text{cake}]) \\
&\quad + \varphi([\text{Mary}] \otimes [\text{nibbles}] \otimes [\text{choc}])
\end{align*}$$

We can see that $[s_2] - \frac{1}{4} [s_1]$ is positive by positivity of the individual elements and the fact that positivity is preserved under addition and tensor product. Therefore:

$$[s_1] \preceq_{kl} [s_2]$$

as required.

7. Conclusion

Integrating a logical framework with compositional distributional semantics is an important step in improving this model of language. By moving to the setting of density matrices, we have described a graded measure of entailment that may be used to describe the extent of entailment between two words represented within this enriched framework. This approach extends uniformly to provide entailment strengths between phrases of any type. We have also shown how a lower bound on entailment strength of phrases of the same structure can be calculated from their components.

We can extend this work in several directions. Firstly, we can examine how narrowing down a concept using an adjective might operate. For example, we should have that red car entails car. Other adjectives should not operate in this way, such as former in former president.

Another line of inquiry is to examine transitivity behaves. In some cases entailment can strengthen. We had that dog entails pet to a certain extent, and that pet entails mammal to a certain extent, but that dog completely entails mammal.

Our framework supports different methods of scaling the positive operators representing propositions. Empirical work will be required to establish the most appropriate method in linguistic applications.

Sentences with non-identical structure must also be taken into account. One approach to this might be to look at the first stage in the sentence reductions at which the elementwise comparison can be made. For example, in the two sentences John runs very slowly, and Hungry boys sprint quickly, we can compare the noun phrases John, and Hungry boys, the verbs runs and sprints, and the adverb phrases very slowly and quickly. Further, the inclusion of negative words like not, or negative prefixes, should be modelled.

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A. Proofs

Proof of Theorem 2. We wish to find the maximum $k$ for which
\[ \forall x \in \mathbb{R}^n \quad (B - pA) x \geq 0 \]
Since $\text{supp}(A) \subseteq \text{supp}(B)$, such a $k$ exists. We assume that for $k = 1$, there is at least one $x$ such that $\langle x \rangle (B - kA) x \leq 0$, since otherwise we're done. For all $x \in \mathbb{R}^n$, $\langle x \rangle (B - kA) x$ increases continuously as $k$ decreases. We therefore decrease $k$ until $\langle x \rangle (B - kA) x \geq 0$, and there will be at least one $x_0$ at which $\langle x_0 \rangle (B - kA) x_0 = 0$. These points are minima so that the vector of partial derivatives $\nabla_{x_0} (B - kA) x_0$ will be zero everywhere.

\[ \nabla (x_0) (B - k_0A) |x_0) = 2(B - k_0A) |x_0) = 0 \]
requires $B$, A self-adjoint.

Therefore $B |x_0) = k_0 A |x_0)$, and so $1/k_0 B + B |x_0) = B^A |x_0)$. Since $B^A B$ is a projector onto the support of $B$ and $\text{supp}(A) \subseteq \text{supp}(B)$, we have:

\[ 1/k_0 |v_0) = B^A |v_0) \]
where $|v_0) = B^A |x_0)$, i.e., $1/k_0$ is an eigenvalue of $B^A$.

Now, $B^A$ has only non-negative eigenvalues, and in fact every pair of eigenvalue $1/k$ and eigenvector $|v)$ will satisfy the condition $B |v) = k A |v)$. We now claim that to secure $\forall x \in \mathbb{R}^n \quad \langle x \rangle (B - kA) x \geq 0$, we must choose $k_0$ equal to the reciprocal of the maximum eigenvalue $\lambda_0$ of $B^A$. For a contradiction, take $\lambda_1 < \lambda_0$, so $1/\lambda_1 = k_1 > k_0 = 1/\lambda_0$. Then we require that $\forall x \in \mathbb{R}^n \quad (x) (B - k_1A) x \geq 0$, and in particular for $|v)$. However:

\[ 1/k_0 |v_0) = B^A |v_0) \quad \text{contradiction, since } k_0 < k_1 \]
We therefore choose $k_0$ equal to $1/\lambda_0$ where $\lambda_0$ is the maximum eigenvalue of $B^A$, and $\langle x \rangle (B - k_0A) x \geq 0$ is satisfied for all $x \in \mathbb{R}^n$.

Proof of Theorem 3. We wish to show that we can choose $\delta$ such that $|k - k'| < \varepsilon$. We use the notation $\lambda(A)$ for the maximum eigenvalue of $A$, and $A^+$ for the Moore-Penrose pseudo-inverse of $A$. $A' = A + \delta p$ satisfies the condition of Theorem 2 that $\text{supp}(A') \subseteq \text{supp}(B)$, since $\langle x \rangle \notin \text{supp}(B)$. $\text{supp}(A) \subseteq \text{supp}(B)$, so $\langle x \rangle \notin \text{supp}(A)$ and $A |x) = 0$. Similarly, $\rho |x) = 0$. Therefore $(A+\rho) |x) = A |x) = 0$, so $\langle x \rangle \notin \text{supp}(A')$.

By theorem 2 we have:

\[ 1/k = \frac{1}{\lambda_0(B^A)} \]

\[ k' = \frac{1}{\lambda_0(B^A)} \]

\[ k - k' = \frac{\lambda_{\text{max}}(B^+) - \lambda_{\text{max}}(B^A)}{\lambda_{\text{max}}(B^+ A^+) \lambda_{\text{max}}(B^A)} \]

\[ (3) \]

The denominator of $\lambda$ we may treat as a constant. We expand the numerator and apply Weyl's inequalities [Weyl 1912]. These inequalities apply only to Hermitian matrices, whereas we need to apply these to products of Hermitian matrices. Note that since $B^+$, $A$, and $p$ are all real-valued positive semidefinite, the products $B^+ A$ and $B^+ \rho$ have the same eigenvalues as the Hermitian matrices $A^+ B^+ A^2$ and $\rho^2 B^+ \rho^2$, which are Hermitian. Now:

\[ \lambda_{\text{max}}(B^+ A) - \lambda_{\text{max}}(B^A) = \lambda_{\text{max}}(B^+ A + \delta B^+ \rho) - \lambda_{\text{max}}(B^+ A) \leq \lambda_{\text{max}}(B^+ A) + \delta \lambda_{\text{max}}(B^+ \rho) - \lambda_{\text{max}}(B^+ A) \]

\[ \delta \lambda_{\text{max}}(B^+) \]

\[ \delta \lambda_{\text{max}}(B^+) \lambda_{\text{max}}(B^A) \]

Therefore:

\[ k - k' \leq \frac{\lambda_{\text{max}}(B^+)}{\lambda_{\text{max}}(B^+ A^+) \lambda_{\text{max}}(B^A)} \]

so that given $\varepsilon$, $A$, $B$, we can always choose a $\delta$ to make $k - k' \leq \varepsilon$.

Proof of Theorem 5. Let the density matrix corresponding to the verb be given by $[Z]$ and the linear map $(\mathbb{E} \otimes \mathbb{E} \otimes \mathbb{E} \otimes \mathbb{E} \otimes \mathbb{E})$ be given by $\varphi$. Then we can write the meanings of the two sentences as:

\[ \varphi([n_1] \otimes [verb] \otimes [n_3]) = \varphi([n_1] \otimes [Z] \otimes [n_3]) \]

\[ \varphi([n_2] \otimes [verb] \otimes [n_4]) = \varphi([n_2] \otimes [Z] \otimes [n_4]) \]

Substituting $[n_2] = k[n_1] + D$ and $[n_4] = l[n_3] + D'$ in the expression for the meaning of “$n_2$ verb $n_4$” gives:

\[ \varphi([n_2] \otimes [Z] \otimes [n_4]) = \varphi(k[n_1] + D) \otimes [Z] \otimes ([l[n_3] + D']) \]

\[ = kl \varphi([n_1] \otimes [Z] \otimes [n_2]) + \varphi(k[n_1] \otimes [Z] \otimes D') \]

\[ + (D \otimes [Z] \otimes I[n_3]) + (D \otimes [Z] \otimes D') \]

Therefore, $\varphi([n_2] \otimes [Z] \otimes [n_4]) - kl \varphi([n_1] \otimes [Z] \otimes [n_2])$ is equal to $\varphi(k[n_1] \otimes [Z] \otimes D') + (D \otimes [Z] \otimes I[n_3]) + (D \otimes [Z] \otimes D')$ which is positive by positivity of $[n_1]$, $[Z]$, $[n_3]$, $D$, $D'$, and the scalars $k$ and $l$. Therefore:

\[ \varphi([n_1] \otimes [Z] \otimes [n_3]) \preceq kl \varphi([n_2] \otimes [Z] \otimes [n_4]) \]

as needed.

Proof of Corollary 2. Since $k_i = 1$ for each $i \in \{1, \ldots, n\}$,

\[ \varphi \preceq k_1 \ldots k_n \varphi \(
\]
We refer to the following simple factor, verified here:

**Lemma 4.** Let \( \sigma, \tau \) be density operators. Then:

\[
\sigma \subseteq \tau \quad \Rightarrow \quad \sigma = \tau
\]

**Proof.** If \( \sigma \subseteq \tau \) then \( \tau - \sigma \) is positive, and clearly:

\[
\sigma + (\tau - \sigma) = \tau
\]

therefore, applying the trace and noting density operators all have trace 1:

\[
\text{tr}(\tau - \sigma) = 0
\]

and as \( \tau - \sigma \) is positive, it must be the zero operator. \( \square \)

### A.1 Incorporating the Bayesian Order

We can work with the Bayesian order on density operators [Coecke and Martin 2011]. In order to do this, we apply the following operations to transform our density operators:

1. Diagonalize the operator, choosing a permutation of the basis vectors such that the diagonal elements are in descending order.
2. Let \( d_i \) denote the \( i^{th} \) diagonal element. We define the diagonal of a new diagonal matrix inductively as follows:

\[
d_0 = d_0 \quad d_{i+1} = d_i^* + d_{i+1}
\]

3. Transform the new operator back to the original basis

### B. Examples

#### B.1 Examples of \( k \)-hyponymy in positive transitive sentences

In this section we give three toy examples of the use of \( k \)-hyponymy in positive transitive sentences. Each example uses a different sentence space.

#### B.1.1 Truth-theoretic sentence spaces

The following example illustrates what happens to positive transitive sentence hyponymy if we take a truth-theoretic approach to sentence meaning. Suppose that our sentence space \( S \) is 1-dimensional, with its single non-trivial vector being \( |1\rangle \). We will take \(|1\rangle\) to stand for True and the origin 0 for False. The sentences we will consider are:

\[
s_1 = \text{Annie enjoys holidays}
\]

\[
s_2 = \text{Students enjoy holidays}
\]

Let the vector space for the subjects of the sentences be \( \mathbb{R}^3 \) with chosen basis \( \{ |e_1\rangle, |e_2\rangle, |e_3\rangle \} \). Let:

\[
[\text{Annie}] = |e_1\rangle \langle e_1|, \quad [\text{Betty}] = |e_2\rangle \langle e_2|, \quad [\text{Chris}] = |e_3\rangle \langle e_3|
\]

Let the object vector space be \( \mathbb{R}^n \) for some arbitrary \( n \in \mathbb{N} \), where we take \( \{|v_i\rangle\} \) to be the standard basis for \( \mathbb{R}^n \), where \( |v_i\rangle \) has 1 in the \( i \)th position and 0 elsewhere. Let \( |\text{holidays}\rangle = |v_1\rangle \). We will treat the word students as being a hyponym of the individual students in our universe.

\[
[\text{students}] = \frac{1}{3} [\text{Annie}] + \frac{1}{3} [\text{Betty}] + \frac{1}{3} [\text{Chris}]
\]

We have the choices for normalization that we outlined in section B.2.1. Since we are viewing the sentence space as truth-theoretic, we keep the normalization to trace 1.

Finally, let the verb be given by:

\[
[\text{enjoy}] = \sum_{(p,q) \in R} |e_p\rangle \langle e_r| \otimes |v_q\rangle \langle v_s|
\]

where \( R = \{(i,j) | |e_i\rangle \text{ enjoys } |v_j\rangle \} \)

Suppose that Annie and Betty are known to enjoy holidays, while Chris does not. Clearly, we have that:

\[
[\text{Annie}] \preceq_{k_{\text{max}}} [\text{students}]
\]

for \( k_{\text{max}} = \frac{1}{3} \) since:

\[
[\text{students}] = \frac{1}{3} [\text{Annie}] + \frac{1}{3} ([\text{Betty}] + [\text{Chris}]) \geq 0
\]

and any higher value than \( \frac{1}{3} \) will no longer be positive.

We will see that the \( k \)-hyponymy for \( k = \frac{1}{3} \) does translate into \( k \)-hyponymy of sentence \( s_1 \) to sentence \( s_2 \). First of all, consider the meanings of the two sentences:

\[
[s_1] = \langle \varepsilon N \otimes 1_S \otimes \varepsilon N | [\text{Annie}] \otimes [\text{enjoys}] \otimes [\text{holidays}] \rangle
\]

\[
= \langle \varepsilon N \otimes 1_S \otimes \varepsilon N | (|e_1\rangle \langle e_1|) \otimes (|v_1\rangle \langle v_1|) \otimes (|v_1\rangle \langle v_1|)
\]

\[
\otimes \sum_{(p,q) \in R} |e_p\rangle \langle e_r| \otimes |v_q\rangle \langle v_s| \otimes |v_1\rangle \langle v_1|
\]

\[
= \sum_{(p,q) \in R} \delta_{1p} \delta_{1r} \delta_{q1} \delta_{s1} = \sum_{(1,1) \in R} 1 = 1
\]

\[
[s_2] = \langle \varepsilon N \otimes 1_S \otimes \varepsilon N | ([\text{students}] \otimes [\text{enjoy}] \otimes [\text{holidays}]) \rangle
\]

\[
= \langle \varepsilon N \otimes 1_S \otimes \varepsilon N | \frac{1}{3} (|e_1\rangle \langle e_1| + |e_2\rangle \langle e_2| + |e_3\rangle \langle e_3|)
\]

\[
\otimes \sum_{(p,q) \in R} |e_p\rangle \langle e_r| \otimes |v_q\rangle \langle v_s| \otimes |v_1\rangle \langle v_1|
\]

\[
= \frac{1}{3} \sum_{(p,q) \in R} (|e_1\rangle \langle e_1| \langle v_1\rangle + |e_2\rangle \langle e_2| \langle v_1\rangle)
\]

\[
+ (|e_3\rangle \langle e_3| \langle v_1\rangle) \langle v_1\rangle \langle v_1\rangle
\]

\[
= \frac{1}{3} \times 2 = \frac{2}{3}
\]

Clearly, we have that \( [s_1] \preceq_k [s_2] \) for \( k = \frac{1}{3} \), as \( \frac{2}{3} - \frac{1}{3} \times 1 \geq 0 \), but this is not the maximum value of \( k \) for which this \( k \)-hyponymy holds. The maximum value for which this works is \( k = \frac{2}{3} \).

#### B.1.2 Simple case of object hyponymy

We now give a simple case with a non-truth-theoretic sentence space. We show that the \( k \)-hyponymy of the objects of two sentences translates into \( k \)-hyponymy between the sentences, and that in this case the maximality of the value of \( k \) is also preserved.

Let \( m \in \mathbb{N} \), \( m > 2 \) be such that \( \{|n_i\rangle\}_{i=1}^m \) is a collection of standard basis vectors for \( \mathbb{R}^m \). We will use the nouns:

\[
[\text{Gretel}] = |n_1\rangle \langle n_1|, \quad [\text{gingerbread}] = |n_2\rangle \langle n_2|
\]

\[
[\text{cake}] = |n_3\rangle \langle n_3|, \quad [\text{pancakes}] = |n_4\rangle \langle n_4|
\]
Let the density matrix corresponding to the hypernym \textit{sweets} be given by:

$$[\text{sweets}] = \frac{1}{10} |n_2\rangle \langle n_2| + \sum_{i=3}^{m} p_i |n_i\rangle \langle n_i|.$$ 

Our object and subject vector space will be $R^m$ and for the sentence space we take $S = R^m \otimes R^m$. Using this sentence space simplifies the calculations needed, as shown in (Grefenstette and Sadrzadeh 2011). For the rest of this example, we will adopt the following notation for the purpose of brevity:

$$|x_{jk}\rangle = |n_j\rangle |n_k\rangle, \quad \langle x_{jk}| = \langle n_j| |n_k\rangle,$$

$$|x_{ij}\rangle \langle x_{kl}| = |n_i\rangle |n_k\rangle \otimes |n_j\rangle \langle n_l|.$$ 

Then the density matrix representation of our verb becomes:

$$[\text{likes}] = \sum_{jklp} C_{jklp} |n_j\rangle |n_k\rangle \otimes |x_{lp}\rangle \langle x_{lp}| \otimes |n_l\rangle \langle n_p|$$

We will consider the following two sentences:

- $s_1 = \text{Gretel likes sweets}$
- $s_2 = \text{Gretel likes gingerbread}$

Let the corresponding sentence meanings be given by:

- $[s_1] = (\varepsilon_N \otimes 1_S \otimes \varepsilon_N) \circ (\text{[Gretel]} \circ \text{[likes]} \circ \text{[sweets]})$
- $[s_2] = (\varepsilon_N \otimes 1_S \otimes \varepsilon_N) \circ (\text{[Gretel]} \circ \text{[likes]} \circ \text{[gingerbread]})$

Observe that:

$$\text{[gingerbread]} \preceq_k \text{[sweets]} \quad \text{for } k \leq \frac{1}{10}.$$ 

In particular, we have $k_{\text{max}}$-hyponymy between \textit{gingerbread} and \textit{sweets} for $k_{\text{max}} = \frac{1}{10}$. We will now show that this hyponymy translates to the sentence level. With $\varphi = \varepsilon_N \otimes 1_S \otimes \varepsilon_N$ and $\rho = \sum_{i=3}^{m} p_i |n_i\rangle \langle n_i|$ we have:

$$[s_1] = \varphi \left( |n_1\rangle \langle n_1| \otimes \text{[likes]} \otimes \left( \frac{1}{10} |n_2\rangle \langle n_2| + \rho \right) \right)$$

$$= \frac{1}{10} \varphi \left( |n_1\rangle \langle n_1| \otimes \text{[likes]} \otimes |n_2\rangle \langle n_2| \right) + \varphi \left( |n_1\rangle \langle n_1| \otimes \text{[likes]} \otimes \rho \right)$$

$$[s_2] = \varphi \left( |n_1\rangle \langle n_1| \otimes \text{[likes]} \otimes |n_2\rangle \langle n_2| \right)$$

We claim that the maximum $k$-hyponymy between $[s_2]$ and $[s_1]$ is achieved for $k = \frac{1}{10}$. In other words, this is the maximum value of $k$ for which we have $[s_2] \preceq_k [s_1]$, i.e. $[s_1] - k [s_2] \succeq 0$. We first show that $[s_1] - \frac{1}{10} [s_2]$ is positive:

$$[s_1] - \frac{1}{10} [s_2] = \varphi \left( |n_1\rangle \langle n_1| \otimes \text{[likes]} \otimes \rho \right)$$

$$= (\varepsilon_N \otimes 1_S \otimes \varepsilon_N) \circ (|n_1\rangle \langle n_1| \otimes \text{[likes]} \otimes \rho)$$

$$= \sum_{i=3}^{m} C_{111i} p_i |n_i\rangle \langle n_i|.$$ 

This is positive by positivity of $C_{111i}$ and $p_i$.

For a value of $k = \frac{1}{10} + \epsilon$, by a similar calculation we obtain:

$$\varphi \left( [s_1] - k [s_2] \right) = \sum_{i=3}^{m} C_{111i} p_i |x_{1i}\rangle \langle x_{1i}| - \epsilon |x_{12}\rangle \langle x_{12}|$$

We then note that:

$$\langle x_{12}| \varphi ([A] - k[B]) |x_{12}\rangle = -\epsilon$$

and therefore $k = \frac{1}{10}$ is maximal.

### B.1.3 \textit{k}-hyponymy for positive transitive sentences

Now suppose that the subject and object vector spaces are two-dimensional with bases $|e_1\rangle$, $|e_2\rangle$ and $|n_1\rangle$, $|n_2\rangle$ respectively. We let:

$$[\text{Hansel}] = |e_1\rangle \langle e_1|, \quad [\text{Gretel}] = |e_2\rangle \langle e_2|$$

$$[\text{gingerbread}] = |n_1\rangle \langle n_1|, \quad [\text{cake}] = |n_2\rangle \langle n_2|$$

The density matrices for the hypernyms \textit{the siblings} and \textit{sweets} are:

$$[\text{the siblings}] = \frac{1}{2} [\text{Hansel}] + \frac{1}{2} [\text{Gretel}]$$

$$[\text{sweets}] = \frac{1}{2} [\text{gingerbread}] + \frac{1}{2} [\text{cake}].$$

The verb \textit{like} is given as before and we assume that Gretel likes gingerbread but not cake and Hansel likes both. The sentence reduction map $\varphi$ is again $\varepsilon_N \otimes 1_S \otimes \varepsilon_N$. Then we have:

$$[s_1] = \varphi ([\text{Gretel}] \otimes \text{[likes]} \otimes \text{[gingerbread]})$$

$$[s_2] = \varphi ([\text{the siblings}] \otimes \text{[like]} \otimes \text{[sweets]})$$

$$= \frac{1}{4} \varphi ([\text{Gretel}] \otimes \text{[likes]} \otimes \text{[gingerbread]})$$

$$+ \frac{1}{4} \varphi ([\text{Gretel}] \otimes \text{[likes]} \otimes \text{[cake]})$$

$$+ \frac{1}{4} \varphi ([\text{Hansel}] \otimes \text{[likes]} \otimes (\text{[gingerbread]} + [\text{cake}]))$$

We then have:

$$[s_2] - \frac{1}{4} [s_1] = \frac{1}{4} \varphi ([\text{Gretel}] \otimes \text{[likes]} \otimes \text{[cake]})$$

$$+ ([\text{Hansel}] \otimes \text{[likes]} \otimes (\text{[gingerbread]} + [\text{cake}]))$$

$$= \frac{1}{4} (|x_{22}\rangle \langle x_{22}| + |x_{11}\rangle \langle x_{11}| + |x_{12}\rangle \langle x_{12}|)$$

which is clearly positive.

Again, $k = \frac{1}{4}$ is maximal, since taking $k' = \frac{1}{4} + \epsilon$ gives us the following:

$$[s_2] - k' [s_1] = \frac{1}{4} \varphi ([\text{Gretel}] \otimes \text{[like]} \otimes \text{[cake]})$$

$$+ ([\text{Hansel}] \otimes \text{[like]} \otimes (\text{[gingerbread]} + [\text{cake}]))$$

$$- \epsilon ([\text{Gretel}] \otimes \text{[like]} \otimes \text{[gingerbread]})$$

$$= \frac{1}{4} (|x_{22}\rangle \langle x_{22}| + |x_{11}\rangle \langle x_{11}| + |x_{12}\rangle \langle x_{12}|)$$

$$- \epsilon (|x_{21}\rangle \langle x_{21}|)$$

Then $[s_2] - k' [s_1]$ is no longer positive, since:

$$\langle x_{21}| ([B] - k'[A]) |x_{21}\rangle = -\epsilon$$

and therefore $\frac{1}{4}$ is maximal.

In these last two examples, the value of $k$ that transfers to the sentence space is maximal. In general this will not be the case. The reason that the maximality of the $k$ transfers in these examples is due to the orthogonality of the noun vectors that we work with.

### C. Applying the theory to Frobenius Algebras

This appendix details techniques that we have not included in the main body of the text.

#### C.1 Frobenius Algebras

We state here how a Frobenius algebra is implemented within a vector space over $R$. For a mathematically rigorous presentation see (Sadrzadeh et al. 2013). A real vector space with a fixed basis $\{\psi_i\}$ has a Frobenius algebra given by:

$$\Delta :: |v_i\rangle \rightarrow |v_i\rangle \otimes |v_i\rangle \quad \iota :: |v_i\rangle \rightarrow 1$$
\[ \mu : |v_i \rangle \otimes |v_i \rangle \mapsto \delta_{ij} |v_i \rangle \quad \zeta : 1 \mapsto \sum_i |v_i \rangle \]

This algebra is commutative, so for the swap map \( \sigma : X \otimes Y \to Y \otimes X \), we have \( \sigma \circ \Delta = \Delta \) and \( \mu \circ \sigma = \mu \). It is also special so that \( \mu \circ \Delta = 1 \). Essentially, the \( \mu \) morphism amounts to taking the diagonal of a matrix, and \( \Delta \) to embedding a vector within a diagonal matrix. This algebra may be used to model the flow of information in noun phrases with relative pronouns.

### C.1.1 An example noun phrase

In [Sadrzadeh et al. (2013)](#), the authors describe how the subject and object relative pronouns may be analyzed. We describe here the subject relative pronoun. The phrase John who kicks cats is a noun phrase; it describes John. The meaning of the phrase should therefore be John, modified somehow. The word who is typed \( n^r \) and the meaning of the relative clause is given by:

\[
|n^r \rangle \otimes |n^r \rangle \otimes |n^l \rangle \otimes |n^l \rangle \mapsto \sum_{ij} |e_{ij} \rangle \otimes |e_{ij} \rangle \otimes |e_{ij} \rangle \otimes |e_{ij} \rangle
\]

\[
|\mu \rangle = \langle 1 | \otimes \langle 1 | \otimes \langle 1 | \otimes \langle 1 | \mapsto \sum_{ij} \langle e_{ij} | \otimes \langle e_{ij} | \otimes \langle e_{ij} | \otimes \langle e_{ij} |
\]

\[
|\zeta \rangle = \langle 1 | \otimes \langle 1 | \otimes \langle 1 | \otimes \langle 1 |
\]

*Sadrzadeh et al. (2013)* analyse the subject-relative pronoun who as having a structure that can be formalised using the Frobenius algebra as follows:

\[
\text{subject} \quad \text{verb} \quad \text{object}
\]

![Figure 9: A noun phrase generated by the subject relative pronoun sentence in CPM(\(C\))](#)

### C.2 \( k \)-hyponymy in relative clauses

Relative clauses are expressions such as John who kicks cats. These are noun phrases, and the diagrammatic representation of such phrases was introduced in section [C.1](#). As for sentences, the diagram in CPM(FHilb) is equivalent to the diagram in FHilb but with thick wires, given in figure [9](#). The diagrammatic representation of subject relative clauses in FHilb is given in figure [10](#).

We assume that the relative pronoun is which. Then the meaning map for the relative clause subj which verb obj in CPM(FHilb) is \( \mu_N \otimes 

[\text{subj}] \otimes [\text{verb}] \otimes [\text{obj}] \) and the meaning of the relative clause is given by:

\[(\mu_N \otimes \iota_S \otimes \iota_N)([\text{subj}] \otimes [\text{verb}] \otimes [\text{obj}]).\]

We can now characterise the relationship between relative clauses 'A which verb C' and 'B which verb D' where \( [A] \preceq_k [B] \) and

### Table 2: Table of diagrams for Frobenius algebras in CPM(\(C\)) and \( C \)

| \( \text{CPM}(C) \) | \( \text{C} \) |
|----------------------|----------------------|
| \( E(\mu) = \mu_+ \otimes \mu_+ \) | \( \mu : A^+ \otimes A^+ \otimes \to A^+ \otimes A^+ \to A^+ \otimes A^+ \) |
| \( \mu : |e_{ij} \rangle \otimes |e_{ij} \rangle \otimes |e_{ij} \rangle \otimes |e_{ij} \rangle \mapsto \langle e_{ij} | \langle e_{ij} | \langle e_{ij} | \langle e_{ij} | \) | \( \Delta : A^+ \otimes A \to A^+ \otimes A \otimes A^+ \otimes A \) |
| \( \Delta = \Delta_+ \otimes \Delta_+ \) | \( \Delta : A^+ \otimes A \to A^+ \otimes A \otimes A^+ \otimes A \) |
| \( E(\iota) = \iota_+ \otimes \iota_+ \) | \( \iota'' : A^+ \otimes A \to I \) |
| \( \iota'' : A^+ \otimes A \to I \) | \( \iota : |e_{ij} \rangle \otimes |e_{ij} \rangle \mapsto 1 \) |
| \( \iota = \iota'' \) | \( \zeta : A^+ \otimes A \to I \) |
| \( \zeta = \iota'' \) | \( \zeta : A^+ \otimes A \to I \) |

### Figure 10: A noun phrase generated by the subject relative pronoun sentence in \( C \)
\[ [C] \preceq_i [D], \text{ and obtain a result very similar to the one we had for the positive semi-definite sentence types, under the same assumptions.} \]

**Theorem 6.** Let \( n_1, n_2, n_3, n_4 \) be nouns with corresponding density matrix representations \( [n_1], [n_2], [n_3] \) and \( [n_4] \), and such that \( [n_2] = \kappa [n_1] + D \) and \( [n_4] = l [n_3] + D' \) for some \( k, l \in (0, 1] \). Then we have that:

\[
\varphi ([n_1] \otimes [\text{verb}] \otimes [n_2]) \preceq_{kl} \varphi ([n_2] \otimes [\text{verb}] \otimes [n_4])
\]

**Proof.** The proof of this result is identical to that of theorem 4 except for the fact that when we consider

\[
\varphi([n_2] \otimes [\text{verb}] \otimes [n_4]) - kl \varphi([n_1] \otimes [\text{verb}] \otimes [n_3])
\]

we get \( \varphi = \mu_N \otimes I_S \otimes \varepsilon_N \) applied to \( (k[n_1] \otimes [Z] \otimes D') + (D \otimes [Z] \otimes l[n_3]) + (D \otimes [Z] \otimes D') \) instead of \( \varphi = (\varepsilon_N \otimes I_S \otimes \varepsilon_N) \) applied to the same. The result is, however, still a positive quantity by the property of the morphisms \( \mu_N, I_S \) and \( \varepsilon_N \) to map density matrices to density matrices. Thus, we can conclude as before that:

\[
\varphi ([n_2] \otimes [\text{verb}] \otimes [n_4]) \preceq_{kl} \varphi ([n_1] \otimes [\text{verb}] \otimes [n_3]).
\]

\( \square \)

### C.2.1 \( k \)-hyponymy applied to relative clauses

We will consider the containment of the sentence:

\[ s_1 = \text{Elderly ladies who own cats} \]

in the sentence:

\[ s_2 = \text{Women who own animals} \]

First of all, let the subject and object space for the vectors corresponding to the subjects and object of our sentences be \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) respectively. Let:

\[ [\text{elderly ladies}] = |e_1\rangle \langle e_1|, \quad [\text{young ladies}] = |e_2\rangle \langle e_2| \]

and the density matrix for the hypernym women be:

\[ [\text{women}] = \frac{1}{3} [\text{elderly ladies}] + \frac{2}{3} [\text{young ladies}] \]

Similarly, let:

\[ [\text{cats}] = |n_1\rangle \langle n_1| \]
\[ [\text{dogs}] = |n_2\rangle \langle n_2| \]
\[ [\text{hamsters}] = |n_3\rangle \langle n_3| \]

and take the density matrix for animals to be:

\[ [\text{animals}] = \frac{1}{2} [\text{cats}] + \frac{1}{4} [\text{dogs}] + \frac{1}{4} [\text{hamsters}] \]

The sentence space will not matter in this case, as it gets deleted by the \( I_S \) morphism, so we just take it to be an unspecified \( S \). Let the verb own be given by:

\[ [\text{own}] = \sum_{ijkl} C_{ijkl} |e_i\rangle \langle e_k| \otimes |s\rangle \otimes |n_j\rangle \langle n_l| \]

and the sentence map \( \psi \) is given by \( \mu_N \otimes I_S \otimes \varepsilon_N \). Then the meaning of sentences \( s_1 \) and \( s_2 \) are given by:

\[ [s_1] = \psi([\text{elderly ladies}] \otimes [\text{own}] \otimes [\text{cats}]) \]
\[ [s_2] = \psi([\text{women}] \otimes [\text{own}] \otimes [\text{animals}]) \]

\[ = \frac{1}{6} \psi([\text{elderly ladies}] \otimes [\text{own}] \otimes [\text{cats}]) \]
\[ + \frac{1}{12} \psi([\text{elderly ladies}] \otimes [\text{own}] \otimes [\text{dogs}]) \]

\[ + \frac{1}{12} \psi([\text{elderly ladies}] \otimes [\text{own}] \otimes [\text{hamsters}]) \]

\[ + \frac{1}{3} \psi([\text{young ladies}] \otimes [\text{own}] \otimes [\text{animals}]) \]

Then \( [s_2] - \frac{1}{6} [s_1] \) is given by:

\[ + \frac{1}{12} \psi([\text{elderly ladies}] \otimes [\text{own}] \otimes [\text{dogs}]) \]
\[ + \frac{1}{12} \psi([\text{elderly ladies}] \otimes [\text{own}] \otimes [\text{hamsters}]) \]
\[ + \frac{1}{3} \psi([\text{young ladies}] \otimes [\text{own}] \otimes [\text{animals}]) \]

which is clearly positive.