Fractional Quantum Hall States in Low-Zeeman-Energy Limit

X.G. Wu and J.K. Jain

Department of Physics, State University of New York at Stony Brook, Stony Brook, New York 11794-3800
(November 15, 2021)

1. INTRODUCTION

It is common to consider the limit of large magnetic field, $B \to \infty$, in the theoretical study of fractional quantum Hall effect (FQHE), since the phenomenon occurs at large magnetic fields. This simplifies the problem for two reasons. One, the Hilbert space is restricted to the lowest Landau level (LL), and consequently has a finite size. Two, the spin degree of freedom is frozen, so the electrons can effectively be taken to be spinless. However, the effective mass and the effective g-factor in GaAs are such that the Zeeman splitting is roughly $1/60$ of the LL spacing. Therefore, while in the true $B \to \infty$ limit the Zeeman splitting would maximally polarize the system, in actual experiments it may actually be negligible. Indeed, it has been found that at sufficiently low fields several incompressible FQHE states are not maximally polarized. Therefore, it is meaningful to consider the limit in which electrons are still restricted to the lowest LL but the Zeeman splitting is zero, which we will call the vanishing-Zeeman-splitting (VZS) limit. This paper reports our investigation of this limit in which electrons are still restricted to the lowest LL, i.e., to the filling factor range $1 < \nu < 2$. In fact, since $2 > \nu > 1$ is related to $\nu < 1$ by an exact particle-hole symmetry in the lowest LL, it is sufficient to consider $\nu < 1$.

The present study is a generalization of an earlier study of spinless electrons. There, it was found that the low-energy Hilbert space of interacting electrons is well described at arbitrary filling factors in terms of non-interacting composite fermions (CF’s). We find that the CF theory provides a consistent picture of the low-energy states in the VZS limit as well, except that here we need to impose a hard-core condition on the CF’s.

We start by giving a brief introduction to the CF theory of the FQHE in Section II. Section III compares the numerical eigenstates with the CF states for systems of six and eight electrons for a range of filling factors. The paper is concluded in Section IV.

II. CORRELATED COMPOSITE FERMION BASIS

The FQHE state is characterized by the formation of a new kind of particle called composite fermion which is an electron carrying two (in general, an even number of) vortices of the wave function. In a mean-field sense, a CF can be thought of as a bound state of an electron and two flux quanta, where a flux quantum is $\phi_0 = hc/e$, since flux quanta produce the same phase factors as vortices as the electrons wind around each other. The strongly correlated liquid of interacting electrons in the fractional quantum Hall state is equivalent to a weakly interacing gas of CF’s, and a good qualitative as well as quantitative understanding is obtained straightforwardly in terms of CF’s. Because of the binding of the flux quanta, the CF’s see an effective magnetic field

$$B^* = B + 2\phi_0\rho,$$

where $B$ is the external field, and $\rho$ is the electron (or CF) density per unit area. The $(-)$ sign corresponds to the case when the flux bound to the CF’s is in the same (opposite) direction as $B^*$. This implies that the CF filling factor, $\nu^* = \phi_0\rho/B^*$, is related to the electron filling factor, $\nu = \phi_0\rho/B$, by

$$\nu = \frac{\nu^*}{2\nu^* \pm 1}.$$
We assume a large LL spacing throughout this work, which allows us to restrict the Hilbert space to the lowest kinetic energy band. For electrons with $\nu < 2$, this implies that the Hilbert space is restricted to the lowest LL. The electron system maps to a CF system which involves, in general, several quasi-LL’s. In this case, the (restricted) Hilbert space consists only of those states in which an integer number of quasi-LL’s is fully occupied, one quasi-LL is partially occupied, and all higher quasi-LL’s are unoccupied. The usefulness of the CF theory lies in the fact that the size of the Hilbert space at $\nu^*$ is much smaller than that at $\nu$, resulting in a simplification of the problem. More specifically, it provides a small correlated basis for the low-energy eigenstates of interacting electrons at $\nu$, called the CF basis, and asserts that, insofar as the low-energy spectrum is concerned, a good approximation of the exact eigenstates is obtained by diagonalizing the Hamiltonian in this much smaller CF basis. The CF basis states at $\nu^*$, which also provide the correlated basis for interacting electrons at $\nu$, are constructed as follows:

(i) First, determine $\nu^*$ from Eq. (3). It is allowed to be negative.

(ii) The CF states at $\nu^*$ are related to the electron states at $\nu^*$. In particular, non-interacting CF’s at $\nu^*$ are related to non-interacting electrons at $\nu^*$. Therefore, consider non-interacting electrons at $\nu^*$. When $\nu^*$ is an even integer $(2n)$, the ground state is unique, and contains an integer number $(n)$ of filled LL’s. In other cases, when $2n < \nu^* < 2(n+1)$, the ground state is highly degenerate, since all possible arrangements of electrons in the partially filled $(n+1)$th LL have the same energy. It is straightforward to write wave functions for all these states. Let us denote these by $\chi_{\alpha,\nu^*}$.

(iii) The wave functions of non-interacting CF’s at $\nu^*$ are now obtained by simply multiplying these states by $\Phi^2$, where

$$\Phi \equiv \prod_{j<k=1}^{N} (z_j - z_k). \quad (3)$$

Here $z_j = x_j - iy_j$ denotes the position of the $j$th electron.

(iv) Finally, we project these product states onto the lowest LL. Calling the projection operator $P$, we get the CF basis states:

$$\chi_{\alpha} = \Phi P \chi_{\alpha,\nu^*}. \quad (4)$$

Note that we choose this projection as opposed to the simple projection $P \Phi^2 \chi_{\alpha,\nu^*}$. This ensures that $\chi_{\alpha}$ contains a factor of $\Phi$, and satisfies the “hard-core” property, i.e., has zero probability of having two electrons at the same point. This builds good correlations in the presence of the repulsive Coulomb interaction.

Multiplication by $\Phi^2$ attaches two vortices to each electron, thus creating a CF. Note that while non-interacting CF’s at $\nu^*$ are related to non-interacting electrons at $\nu^*$, they provide a correlated basis for interacting electrons at $\nu$. The CF theory thus maps the problem of interacting electrons at $\nu$ to non-interacting electrons at $\nu^*$.

We employ the usual spherical geometry for our numerical calculations, in which electrons move on the surface of a sphere, with the magnetic field provided by a magnetic monopole at the center. For a monopole of strength $q$, defined such that the flux through the surface of the sphere is $2|q|/\Phi_0$, the lowest LL single electron states have angular momentum $l = |q|$, and the degeneracy of the lowest LL is $2(2|q|+1)$. Thus the problem is that of $N$ interacting electrons in angular momentum $l = |q|$ shell. The wave functions are a generalization of the spherical harmonics, called the “monopole harmonics”. The eigenstates of the many-body system have well-defined total angular momentum, $L$, and total spin, $S$.

The CF theory is easily translated into the spherical geometry. Now the low-energy states of interacting electrons at $q$ are related to the low-energy states of non-interacting electrons at $q^*$, given by

$$q = q^* + N - 1, \quad (5)$$

which is equivalent to Eq. (2) in the limit of large $N$, since, then, $\nu = N/2|q|$, and $\nu^* = N/2|q^*|$. In the spherical geometry, $\Phi$ is the spatial part of the fully polarized state at monopole flux strength $(N-1)/2$; it is completely antisymmetric, and identical to the wave function of the filled lowest LL of spinless electrons. The CF basis states for interacting electrons at $q$ are then given by

$$\chi_{\alpha}^q = \Phi P \chi_{\alpha}^{q^*}. \quad (6)$$

Since $\chi_{\alpha}^q$ are eigenstates of $L$ and $S$, we choose $\chi_{\alpha}^q$ to be eigenstates of $L$ and $S$. Multiplication by $\Phi$ and projection onto the lowest LL do not change these quantum numbers. Therefore, a state at $q^*$ with a given $L$ and $S$ produces a state at $q$ with the same $L$ and $S$. The states with different $L-S$ are automatically orthogonal, so it is sufficient to diagonalize the Hamiltonian separately in each $L-S$ subspace.

We have studied in the past a large number of filling factors for 6-10 electrons in the large-Zeeman splitting limit, where $S$ takes the largest possible value of 13. Our results convincingly showed the validity of the CF approach in this limit. We found that the low-energy states of interacting electrons at $q$ form a band, well separated from the other higher-energy states. The number of states in this band, as well as their quantum numbers, match perfectly with those of the CF basis states. Furthermore, the actual eigen-functions are very well approximated by the CF wave functions.

III. NUMERICAL RESULTS

In the present work, we set the Zeeman energy to zero. We study a six electron system for $3 \leq q \leq 7$ and an eight electron system for $5 \leq q \leq 6.5$. Due to the symmetry of the problem, it is sufficient to work in the sector
where $L_z = 0$ and $S_z = 0$, where $L_z$ and $S_z$ are the z-components of $L$ and $S$. We restrict our discussion to this sector, with the understanding that when we talk about one state with a given $L$ and $S$ in this sector, there are actually a total of $(2L+1)(2S+1)$ degenerate states with the same energy.

The exact low-energy spectra for electrons interacting via the Coulomb interaction are shown in Fig.1. The low-energy states are expected to be related to the low-energy states at $q^*$. The $L$-$S$ quantum numbers of the degenerate ground states of non-interacting electrons at $q^*$ are shown in Table I.

For six electrons at $q = 4.0$ and $6.0$, and for eight electrons at $q = 5.5$, the electron system maps to a CF system with the filled lowest quasi-LL (i.e., $\nu^* = 2$). In these cases, there is only one CF state, with quantum numbers 0-0, and it is expected to be incompressible, i.e., separated from other states by a gap. This is in agreement with the actual spectra of Fig.1.

In other cases, there is no satisfactory matching between the quantum numbers of the CF’s at $q^*$ and those of the low-energy electron states at $q$. We construct CF basis states according to the above prescription. Many of these states are annihilated upon projection on to the lowest LL; these are marked by $A$ in Table I. Sometimes, there are two states at $q^*$ at a given $L$-$S$, which produce the same $L$-$S$ state at $q$; in such a case, it is possible to construct two linear combinations of these states so that one is annihilated. The annihilation of some states brings the low-energy spectrum in agreement with the CF theory for the six electron system at $q = 3.0$ and 3.5, and for the eight electron system at $q = 5.0$. However, in general, the situation is still unsatisfactory.

We now show that the CF theory explains the low-energy spectrum at $q$, provided the CF’s are themselves taken to be interacting. Just as non-interacting CF’s are related to non-interacting electrons at $q^*$, interacting CF’s are related to interacting electrons at $q^*$. For the present purpose, it is sufficient to incorporate interactions only to the extent of distinguishing hard-core states from other states, where, as mentioned earlier, a hard-core state satisfies the property that its wave function vanishes whenever any two electrons coincide.

The Coulomb interaction is defined by its pseudopotential parameters $V_m^{\ast}$, where $V_m$ is the energy of two electrons in a state of relative angular momentum $m$. Consider a model interaction in which all $V_m$ are set to zero except $V_0$. The hard-core states have zero energy for this model interaction while the non-hard-core states have a finite positive energy. Since $V_0$ is quite large for the Coulomb interaction, we expect this model to be qualitatively reasonable. We have found that the low-energy Coulomb states do indeed satisfy the hard-core property to an excellent approximation. Furthermore, these states form a well defined band, the “hard-core band”, which is well separated from the other non-hard-core states. For example, for the six electron system at $q = 3$ or 3.5, a low-energy band is clearly visible. To make sure that the origin of this band lies in the hard-core part of the Coulomb interaction, we have constructed the true hard-core states by diagonalizing the $V_0$ interaction. These have exactly the same quantum numbers as the states in this band, and have very large overlaps ($> 0.99$) with these states. For most values of $q$ considered here, the hard-core band is not visible in Fig.1 since it is quite large, and all of the states shown belong to this band.

Now let us consider hard-core CF’s. This corresponds to taking the electron states $\chi_{q^*}$ to be hard-core. “Hard-core” is used here in a slightly more general sense. We impose the hard-core condition only on the electrons in the partially filled LL. This assumption is valid when the hard-core interaction is small compared to the gap between the quasi-LL’s of the CF’s. Also, for more than half-filled LL, we impose the hard-core condition on the holes rather than electrons. The quantum numbers of the hard-core states at $q^*$ are marked by an asterisk in Table I. These match quite well with the quantum numbers of the low-energy states of interacting electrons at $q$; the only exception is at $q = 6$ for eight electrons, where a 0-2 state, which is not a part of the CF basis, has a slightly lower energy than the CF state at 2-0. The overlaps of the CF states with the exact Coulomb states are shown in Table II, and provide a more complete confirmation of the CF theory.

Several comments are in order.

(i) In this work, no diagonalization of the Hamiltonian is necessary in the hard-core CF basis. This is because, due to the small size of the system, there is only one hard-core CF state at each $L$-$S$ studied here. Thus, the CF states do not contain any adjustable parameters.

(ii) Because of the factor $\Phi$, even the states of non-interacting CF’s satisfy the hard-core property for electrons (provided they are not annihilated). The hard-core interaction at $q^*$ (between CF’s) corresponds to a long-range part of the inter-electron interaction at $q$. When the CF’s occupy more than one quasi-LL, the hard-core interaction between the CF’s is applicable only to the CF’s in the partially filled quasi-LL, which will translate into a rather complex effective interaction between the electrons.

(iii) Annihilation of a large number of CF states may seem somewhat mysterious. However, in most cases, it is explained quite straightforwardly. As indicated above, the CF basis states are hard-core by construction. The projection operator must annihilate an unprojected CF state when no hard-core state is available at $q$ at the corresponding quantum numbers. This is the reason for most of the annihilations. Moreover, when there is only one hard-core $L$-$S$ state at $q$, all unprojected CF states with these quantum numbers must produce this state, as was found to be case at 0-0 and 1-1 for the six electron system at $q = 3$. However, in some cases, e.g., for the 3-0 state of the six electron system at $q = 3.5$, annihilation of the CF state is non-trivial, since a 3-0 hard-core state does exist here.

(iv) When there is only one hard-core $L$-$S$ state at
$q$, the projected CF state is identical to this state, and the large overlap of this state with the Coulomb state tells us nothing more than that the Coulomb state satisfies the hard-core property to a good approximation. In other cases, there are several hard-core states at $L$-$S$. For example, for the six electron system at $q = 5$, there are eight 1-1 hard-core states. In such cases, the large overlaps provide a more rigorous verification of the CF character of the low-energy states.

(v) The low-energy spectra at $q = N - 1 + q^*$ and $q = N - 1 - q^*$ look strikingly similar, even though there is no exact symmetry relating these two values of $q$. This is, of course, easily explained by the CF theory.

(vi) Note that the hard-core property is satisfied by the CF’s automatically in the case of spinless electrons because of the Pauli principle. Thus, the low-energy spectrum of interacting electrons can be explained both in the large and small Zeeman energy limits provided the CF’s are taken to be hard-core.

IV. CONCLUSION

This work reports an extensive numerical study in the limit of vanishing Zeeman splitting, and shows that the CF theory explains the low-energy spectrum at arbitrary filling factors provided a hard-core condition is imposed upon the CF’s.

This work was supported in part by the Office of Naval Research under Grant no. N00014-93-1-0880, and by the National Science Foundation under Grant No. DMR90-20637.

1. D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
2. B.I. Halperin, Helv. Phys. Acta 56, 75 (1983).
3. R.G. Clark, S.R. Haynes, A.M. Suckling, J.R. Mallett, P.A. Wright, J.J. Harris, and C.T. Foxon, Phys. Rev. Lett. 62, 1536 (1989); J.P. Eisenstein, H.L. Stormer, L.N. Pfeiffer, and K.W. West, Phys. Rev. Lett. 62, 1540 (1989); J.P. Eisenstein, H.L. Stormer, L.N. Pfeiffer, and K.W. West, Phys. Rev. B 41, 7910 (1990); L.W. Engel, S.W. Hwang, T. Sajoto, D.C. Tsui, and M. Shayegan, Phys. Rev. B 45, 3418 (1992).
4. For earlier calculations in the VZS limit, see: F.C. Zhang and T. Chakraborty, Phys. Rev. B 30, 7320 (1984); T. Chakraborty and F.C. Zhang, Phys. Rev. B 29, 7032 (1984); E.H. Rezayi, Phys. Rev. B 36, 5454 (1987); X.C. Xie, Y. Guo, and F.C. Zhang, Phys. Rev. B 40, 3487 (1989); T. Chakraborty, Surf. Sci. 229, 16 (1990); X.C. Xie and F.C. Zhang, Mod. Phys. Lett. B, 471 (1991); P.A. Maksym, J. Phys. Condens. Matter 1, 6299 (1989); T. Chakraborty and P. Pietilainen, Phys. Rev. B 41, 10862 (1990).
5. G. Dev and J.K. Jain, Phys. Rev. Lett. 69, 2843 (1992).
6. J.K. Jain, Phys. Rev. Lett. 63, 199 (1989); Phys. Rev. B 41, 7653 (1990); Adv. Phys. 41, 105 (1992).
7. We use the term “quasi-LL” for the energy levels of composite fermions to emphasize that these are different from the real LL’s of electrons. While the real LL’s of electrons occur as a result of the kinetic energy quantization, quasi-LL’s of CF’s occur as a result of an “effective” quantization of the interaction energy. Of course, CF’s can occupy several quasi-LL’s even when the electrons are completely confined to their lowest real LL.
8. V.J. Goldman, J.K. Jain, and M. Shayegan, Phys. Rev. Lett. 65, 907 (1990); Mod. Phys. Lett. 5, 479 (1991).
9. X.G. Wu, G. Dev, and J.K. Jain, Phys. Rev. Lett. 71, 153 (1993).
10. B.I. Halperin, P.A. Lee, and N. Read, Phys. Rev. B 47, 7312 (1993).
11. R.R. Du, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer, and K.W. West, Phys. Rev. Lett. 70, 2944 (1993); R.L. Willett, R.R. Ruel, K.W. West, and L.N. Pfeiffer, preprint; W. Kang, H.L. Stormer, L.N. Pfeiffer, K.W. Baldwin, and K.W. West, preprint; V.J. Goldman, Bo Su, and J.K. Jain, preprint.
12. F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983).
13. T.T. Wu and C.N. Yang, Nucl. Phys. B 107, 365 (1976); Phys. Rev. D 16, 1018 (1977).
14. The results for the incompressible states have been published earlier, and are repeated here for completeness. Also, the spin-singlet CF state at $q = 6$ for $N = 6$ is identical to one of the states proposed by Halperin.

FIG. 1. This figure shows the low-energy spectra for several values of $q$ for (a) six and (b) eight electrons. The spin quantum numbers of some low-energy states are shown on the figure.

| $q$ | $q^*$ | $L$-$S$ |
|-----|-----|------|
| 3.0 | -2.0 | 2-2, 0-0, 1-1, 2-1(A), 3-0(A), 4-1(A), |
|     |     | 5-1(A), 6-0(A), 0-0(A), 1-1(A), 2-0(A), 2-0(A), |
|     |     | 3-1(A), 3-1(A), 4-0(A), 4-0(A) |
| 3.5 | -1.5 | 0-1, 1-0, 2-1, 3-0 |
| 4.0 | -1.0 | 0-0 |
| 4.5 | -0.5 | 0-1, 2-1, 1-0, 3-0 |
| 5.0 | +0.0 | 1-1, 0-0, 2-0 |
| 5.5 | +0.5 | 0-1, 2-1, 1-0, 3-0 |
| 6.0 | +1.0 | 0-0 |
| 6.5 | +1.5 | 0-1, 1-0, 2-1, 3-0 |
| 7.0 | +2.0 | 2-2, 0-0, 1-1, 2-1, 3-0, 4-1, |
|     |     | 5-1, 6-0, 0-0, 1-1, 2-0, 2-0, |
|     |     | 3-1, 3-1, 4-0, 4-0 |

TABLE I(a)
TABLE I(b)

TABLE I. This table shows the quantum numbers of all states with the lowest kinetic energy at $q^*$ for (a) six and (b) eight particles. The states satisfying the hard-core property are marked by asterisk. The states marked by (A) are annihilated upon the CF transformation, and do not produce any CF state at $q$.

| $q/L-S$ | 3.0/0-0 | 3.0/1-1 | 3.0/2-2 | 3.5/0-1 | 3.5/1-0 |
|---------|----------|----------|----------|----------|----------|
| overlap | 0.9991   | 0.9933   | 0.9988   | 0.9959   | 0.9978   |

TABLE II(a)

| $q/L-S$ | 5.0/0-0 | 5.0/2-0 | 5.0/3-1 | 5.0/1-1 | 5.0/0-0 |
|---------|----------|----------|----------|----------|----------|
| overlap | 0.9930   | 0.9959   | 0.9940   | 0.9919   | 0.9980   |

TABLE II(b)

| $q/L-S$ | 6.0/1-1 | 6.0/3-1 | 6.0/0-0 | 6.0/2-0 | 6.5/0-2 |
|---------|----------|----------|----------|----------|----------|
| overlap | 0.9807   | 0.9906   | 0.9829   | 0.9466   | 0.9918   |

TABLE II. This table gives the overlaps between the hard-core CF states and the corresponding exact Coulomb states for (a) six and (b) eight electrons.