On the existence of monodromies for the Rabi model

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Abstract
We discuss the existence of monodromies associated with the singular points of the eigenvalue problem for the Rabi model. The complete control of the full monodromy data requires the taming of the Stokes phenomenon associated with the unique irregular singular point. The monodromy data, in particular, the composite monodromy, are written in terms of the parameters of the model via the isomonodromy method and the $\tau$ function of the Painlevé V. These data provide a systematic way to obtain the quantized spectrum of the Rabi model.

Keywords: Rabi model, isomonodromy, Painlevé transcendents, Heun equation, scattering theory

1. Introduction
The Rabi model \cite{1, 2} describes the interaction of a single electromagnetic mode—a simple harmonic oscillator—with matter—a two-level quantum system. It is a quite simple model, yet it has an interesting and rich spectrum. Recently, it has attracted vivid attention for experimental and mathematical reasons. From the applied and experimental physics side, there have emerged interesting applications in quantum optics and quantum computation because of good prospects of experimental realization using Josephson junctions, trapped ions, and others (see references in \cite{3}). From the mathematical side, for a long time, it has been a challenge to prove its exact solvability. So, finally, as recently as 2011, Braak \cite{3} solved the model in a Bargmann (coherent state) representation obtaining, in a systematic way, its regular spectrum as zeroes of a transcendental function. The exceptional part of the
spectrum was already known since the late 1970s [4], but it can also be obtained in the framework proposed by Braak [5].

This fact has opened up a whole new set of interesting problems in mathematical physics. One issue regards the notion of integrability of the Rabi model. At face value, the Rabi model could be the first instance of an exactly solvable, yet not integrable, model in mathematical physics. A controversy ensues since the proper definition of integrability in quantum physics—or even in mathematics—seems not to be clear cut [6, 7]. Braak claimed that the Rabi model is integrable in a new quantum integrability criterion coined by himself. A system is dubbed Braak integrable if there are \( f = f_c + f_d \) quantum numbers classifying the eigenstates uniquely, where \( f_c, f_d \) stand for the number of continuous and discrete degrees of freedom, respectively. For the Rabi model, this number is \( f_c = f_d = 1 \), one harmonic oscillator and one two-level system so that \( f = 2 \). These two quantum numbers arise from the \( \mathbb{Z}_2 \) parity symmetry of the Hamiltonian. Even in the lack of a better mathematical formulation of this criterion—still a pending project—it served as the guiding principle to the solution for this outstanding problem.

Controversies aside, it would be desirable to frame this discussion in a more conservative setup. Therefore, Batchelor and Zhou [7] raised the issue of whether the Rabi model is Yang–Baxter integrable (YBI). They succeed to show YBI for two special points in the space of parameters of the Rabi model. For a generic point in these moduli, YBI is still an open question.

Another important issue arising from Braak’s work is that of finding a universal method that applies to a wide variety of models involving coupling of a boson mode with a two-level system in the Bargmann representation. Such a program has been developed by Maciejewski et al [8, 9]. They have used a method framed in terms of the Wronskians, that is, a \( 2 \times 2 \) matrix containing both the wave function and its derivative in the neighborhood of a singular point.

The present work advances some enlightenment in the direction of proving YBI for a generic point in the moduli of parameters of the Rabi model. The departing point is the Bargmann representation of the Rabi model. In this representation, it can be easily shown [5] that the Rabi model is described by a confluent Heun equation. Here, we use the known fact [10, 11] that the monodromy data of these equations can be cast in terms of Painlevé V transcendent \( \tau \) function [12] via isomonodromy equations. The global properties, relevant for the problem of the spectrum and for the YBI, are encoded in the notion of composite monodromies. We, then, present the composite monodromy parameter of the Rabi model. Finally, we discuss, in general terms, how one could use this composite monodromy parameter to obtain the Rabi model’s spectrum.

The novelty of our work consists in the presentation of the monodromies associated with the singular points of the ordinary differential equation (ODE) arising from the eigenvalue problem of the Rabi model. We, then, discuss the relevance of the Stokes phenomenon in order to have a complete monodromy data set. We conjecture that it is the emergence of the Stokes phenomenon and the need of extra parameters in the monodromy data set that rendered extra difficulties in the full demonstration of the YBI of the Rabi model.

This work is organized as follows: we first write the Rabi model as a standard Garnier form (see equation (10)), and, then, we discuss the monodromies around the singular points. A special situation happens for the monodromy around the unique irregular singular point, the point at infinity, giving rise to the Stokes phenomenon, which we discuss in detail. As an outcome, we obtain the general group relation for the monodromies and how it is related with YB equations. We next discuss the isomonodromy method aiming at writing the composite monodromy parameter in terms of the monodromy parameter at the irregular point and the
Stokes parameters. On the other hand, the existence of monodromy matrices for our original system is obtained from the $\tau$ function of the Painlevé V. We, finally, obtain the composite monodromy parameter in terms of the parameters of the Rabi model, which provides, in a systematic way, the quantized spectrum of the Rabi model.

2. Rabi and its monodromies

The quantum Rabi model (QRM) is described by the Hamiltonian,

$$H_R = a^\dagger a + \Delta \sigma_3 + g \sigma_1 (a^\dagger + a),$$

(1)

where the boson mode is described by $[a, a^\dagger] = 1$, the fermion mode is described by the Pauli matrices, $\Delta$ is the level separation of the fermion mode, and $g$ is the boson–fermion coupling.

We consider the Pauli matrices in the standard form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2)

The QRM can be written in terms of two copies of the Jaynes–Cummings model (JCM), each with its appropriate chirality. Indeed, the chiral JCM Hamiltonian reads

$$H_{JC} = a^\dagger a + \Delta \sigma_3 + g (\sigma^+ a + \sigma^- a^\dagger),$$

(3)

and the antichiral one reads

$$H_{JCA} = a^\dagger a + \Delta \sigma_3 + g (\sigma^- a + \sigma^+ a^\dagger),$$

(4)

so that

$$H_R = \frac{1}{2} (H_{JC} + H_{JCA}) = a^\dagger a + \Delta \sigma_3 + g \sigma_1 (a^\dagger + a).$$

(5)

where we have used $\sigma_1 = (\sigma^+ + \sigma^-)/2$. It is important to notice that $[H_{JC}, H_{JCA}] = 0$.

Consider the ansatz

$$|\psi(\alpha)| = f_1 (\alpha)|0\rangle + f_2 (\alpha)|0\rangle,$$

(6)

where the harmonic oscillator ground state is defined by $a|0\rangle = 0$, $\sigma_1 \pm | = \pm |$, and $f_i, i = 1, 2$, are analytic functions of $\alpha$. We can now use Bargmann’s prescription,

$$a^\dagger \mapsto w, \quad a \mapsto \partial_w,$$

(7)

so that $[a, a^\dagger] f(w) = f(w)$. Substituting the ansatz into the stationary Schrödinger equation or the eigenvalue equation $H_0|\psi\rangle = E|\psi\rangle$, we obtain, after setting $f_+ = f_1 \pm f_2$,

$$\partial_w f_+ = \frac{E - gw}{w + g} f_+ - \frac{\Delta}{w + g} f_-, \quad \partial_w f_- = -\frac{\Delta}{w - g} f_+ + \frac{E + gw}{w - g} f_-.$$  

(8)

By eliminating, say, $f_-$ in terms of $f_+$, this results in a second order linear differential equation for $f_+$ (and $f_-$), which could be brought to a confluent Heun equation [5]. Let us define

$$z = -2g(w + g), \quad \Phi(z) = \begin{pmatrix} f_+^{(1)} \\ f_+^{(2)} \end{pmatrix},$$

(9)

where $f_+^{(1,2)}$ are the two linearly independent solutions of the system above. The fundamental matrix $\Phi(z)$ is, then, invertible and unique up to right multiplication of a constant matrix. With
this change in variables, we can bring the model to a standard Garnier form [11]
\[
\frac{d\Phi}{dz^{-1}} = \frac{\sigma_1}{2} + \frac{A_0}{z} + \frac{A_i}{z-t} A_i,
\]
with \( t = -4g^2 \) and
\[
A_0 = \begin{pmatrix} E + g^2 & -\Delta \\ 0 & 0 \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & 0 \\ -\Delta & E + g^2 \end{pmatrix}
\]
(11)

The system (10) has two regular singular points at \( z_i = 0, t, i = 0, t \) and an irregular singular point at \( z_\infty = \infty \) with Poincaré index 1. The analytical structure of the system near the regular singular point is characterized by the monodromy matrices \( M_0 \) and \( M_t \), defined as the effect of an analytical continuation around the corresponding singular point,
\[
\Phi((z - z_i)e^{2\pi i} + z_i) = \Phi(z)M_i, \quad i = 0, t.
\]
(12)

One notes that, since any two sets of solutions \( \Phi(z) \) are related by right multiplication, the monodromy matrices are defined up to an overall conjugation. Moreover, one can choose the initial conditions of (10) so that, near a regular singular point \( z_i \), one has
\[
\Phi(z) \mid_{z=\infty z_i} = \begin{pmatrix} (z - z_i)^{\alpha_i^+} & 0 \\ 0 & (z - z_i)^{\alpha_i^-} \end{pmatrix},
\]
(13)
where \( \alpha_i^\pm \) are solutions of the indicial equation at \( z_i \) associated with (10). Therefore, one can see that, generically, the monodromy matrix \( M_i \) can be written as
\[
M_i = g_i \begin{pmatrix} e^{2\pi i \alpha_i^+} & 0 \\ 0 & e^{2\pi i \alpha_i^-} \end{pmatrix} g_i^{-1},
\]
(14)
where \( g_i \in \text{SL}(2, \mathbb{C}) \) are called the connection matrices (see equation 4.1.4 in [11]). They are also defined up to left multiplication. One also notes that, for algebraic purposes, only the difference \( \alpha_i^+ - \alpha_i^- \) is important. An overall shift of the coefficients \( \alpha_i^\pm \) can be obtained by an \( s \)-transformation of the solutions: \( f_{\pm}(z) \rightarrow (z-z_i)^{s} f_{\pm}(z) \).

The system at the irregular singular point at \( z = \infty \) is slightly more complicated due to the Stokes phenomenon. In order to describe it, let us start by noting that, close to \( z = \infty \), the solutions are of the form \( f_{\pm}(z) \approx e^{2\pi i / 2 \cdot z} \), and the Frobenius series obtained at this point is only formal: its convergence radius is zero. One can see that, generically, near infinity, the system has the form
\[
\frac{d\Phi}{dz^{-1}} = -\frac{1}{2} \sigma_3 + \frac{A_0 + A_i}{z} + O(z^{-2}).
\]
(15)

From the coefficient \( A_\infty = -(A_0 + A_i) \) of the \( z^{-1} \) term, one can define the naive monodromy at \( z = \infty \), given by the difference of the eigenvalues \( \theta_\infty = (\alpha_\infty^+ - \alpha_\infty^-)/2 \). The epithet naive comes in because the monodromy structure around \( z = \infty \) also depends on the first constant term. To describe this structure, we follow [10, 12, 13] and define the sectors of the complex plane,
\[
S_j = \left\{ z \in \mathbb{C} \mid (2j - 5) \frac{\pi}{2} < \arg z < (2j - 1) \frac{\pi}{2} \right\},
\]
(16)
j = 1, 2, \ldots . On each \( S_j \), we have the following asymptotic behavior for the solutions of the system (10) (see equation (3.4) in [10]):
\[ \Phi(z) = G_j(z^{-1}) \exp \left( \frac{1}{2} z \sigma_3 \right) z^{-\frac{1}{2} \theta_j \sigma_3}, \]  
where \( G_j(z^{-1}) = 1 + O(z^{-1}) \) is analytic near \( z = \infty \). The Stokes phenomenon relates the solutions satisfying (17) between different sectors \( S_j \), 
\[ \Phi_{j+1}(z) = \Phi_j(z) S_j, \]  
where \( S_k \) are the Stokes matrices. Now, \( \Phi_j(e^{2\pi i} z) = \Phi_{j+2}(z) e^{-2\pi i \theta_j} \sigma_3 \) — they are defined in the same domain —, so we have that \( S_{j+2} = e^{2\pi i \theta_j} \sigma_3 S_j e^{2\pi i \theta_j} \). Therefore, one can choose a basis where 
\[ S_{2j} = \begin{pmatrix} 1 & s_{2j} \\ 0 & 1 \end{pmatrix}, \quad S_{2j+1} = \begin{pmatrix} 1 & 0 \\ s_{2j+1} & 1 \end{pmatrix}, \]  
where the numbers \( s_k \) are called Stokes parameters. The monodromies \( M_\infty, M_\infty \) define a group with the usual matrix multiplication satisfying the relation, 
\[ M_\infty M_i M_0 = 1. \]  
Note that, once we settle in a sector \( S_j \), say \( j = 1 \), then knowledge of only two consecutive Stokes parameters \( s_1 \) and \( s_2 \) are sufficient to reconstruct the whole series of Stokes matrices \( S_j \).

The outcome of the above analysis is that the whole set of parameters \( \bar{\theta} = \{ \theta_0, \theta_i, \theta_\infty \} \) and \( \bar{\sigma} = \{ s_1, s_2 \} \) is sufficient to determine the monodromy matrices up to an overall conjugation. This set \( (\bar{\theta}, \bar{\sigma}) \) is, thus, called the monodromy data. The existence of the monodromy matrices provides an explicit representation of the three-braid group — in fact, the permutation group \( S_3 \) — acting on \( M_i \) as 
\[ \sigma_j(M_i) = M_j M_i M_j^{-1}, \]
\[ \sigma_j(M_j) = (M_j M_i) M_j (M_j M_i)^{-1}. \]  
These generators satisfy 
\[ \sigma_j \circ \sigma_k \circ \sigma_i = \sigma_k \circ \sigma_i \circ \sigma_j, \]  
which is known as the YB relation [14]. The existence of the monodromy matrices, then, assures that the Rabi model is integrable in the YB sense. Finding the exact form of the monodromy matrices is not a trivial task as the negative results of [7] can attest.

3. The isomonodromy method

The proof of existence of the monodromies described above is deeply tied to the Riemann–Hilbert problem of ODEs. Originally, the Riemann–Hilbert problem consisted of finding an ODE with a prescribed set of monodromies. Our problem is the inverse one: given the Garnier system (10), we want to find the monodromy matrices described in the last section. In order to solve such an inverse Riemann–Hilbert problem, we will first notice that there are many different families of \( A_i \)'s that give the same monodromy. Some of them are trivially related by an overall conjugation by a rational matrix, 
\[ A'_i = GA_i G^{-1} + (\partial_i G) G^{-1}, \]  
where \( \partial_i \) denotes the partial derivatives with respect to \( z_i \).
which corresponds to left multiplication by \( G \) of the fundamental matrix \( \Phi \). Since the entries of \( G \) are rational functions, its contribution to the monodromy will always be trivial. There is, however, still a family of a nontrivial set of Garnier systems with the same monodromy data. This family was first described by Schlesinger—see [11, 12, 15, 16] for reviews—but it is more easily understood in terms of flat holomorphic connections [17]. Suppose we set

\[
A_1(z, t) = \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_i(t)}{z - t} = \frac{\partial \Phi(z, t)}{\partial z} \Phi^{-1}(z, t),
\]

as the ‘\( z \) component’ of a flat connection \( A \). It is straightforward to see that, if we consider the ‘\( t \) component’ as

\[
A_i = -\frac{A_i(t)}{z - t},
\]

then, the associated curvature vanishes

\[
F \equiv \mathbb{d} A + A \wedge A = \partial_i A_i - \partial_z A_i + [A_i, A_i] = 0,
\]

if the \( A_i(t) \) satisfy

\[
\frac{\partial A_0}{\partial t} = \frac{1}{t} [A_i, A_0],
\]

\[
\frac{\partial A_i}{\partial t} = -\frac{1}{t} [A_i, A_0] - \frac{1}{2} [A_i, \sigma_3].
\]

This system is called the Schlesinger equations. Since the ‘field strength’ \( F \) vanishes, then, the monodromy data of the Garnier system (10) will be independent of \( t \) if \( A_0(t) \) and \( A_i(t) \) satisfy equations (28). The matrices \( A_0 \) and \( A_i \) can be thought of as a Lax pair for the isomonodromy flow.

Despite being seemingly more complicated than the ODE itself, the Schlesinger equations (28) have a Hamiltonian structure. The most direct way to illustrate this fact is to consider the ODE associated with the generic Garnier system (25). Let us choose the gauge \((i = 0, t)\),

\[
\text{Tr} A_i = \theta_i \quad \text{and} \quad \text{Tr} \sigma_3 (A_0 + A_i) = -\theta_\infty.
\]

These are constants of motion of the Schlesinger evolution (28), so these are fixed as we vary \( t \). Consider the off-diagonal term \( A_{12} \) of (25). It may be cast in the form

\[
A_{12}(z) = \frac{A_0}{z} + \frac{A_i}{z - t} = \frac{k(z - \lambda)}{z(z - t)},
\]

where \( k, \lambda \) are linear functions of \((A_0)_{12}\) and \((A_i)_{12}\). Now, by writing the solution as

\[
\Phi(z, t) = \begin{pmatrix}
    f^{(1)}_+(z, t) & f^{(2)}_+(z, t) \\
    f^{(1)}_-(z, t) & f^{(2)}_-(z, t)
\end{pmatrix},
\]

one can check that, according to (25), the elements of the first row \( f^{(1,2)}_+(z, t) \) of the fundamental matrix \( \Phi(z, t) \) will satisfy
\[
\frac{d^2 f^{(1,2)}_+}{dz^2} + p(z) \frac{df^{(1,2)}_+}{dz} + q(z) f^{(1,2)}_+ = 0, \\
p(z) = \frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - t} - \frac{1}{z - \lambda}, \\
q(z) = -\frac{1}{4} + \frac{H_0}{z} + \frac{H_1}{z - t} + \frac{\mu}{z - \lambda},
\]

where \(H_0, H_t, \lambda, \) and \(\mu\) are complicated functions of the entries of \(A_0(t)\) and \(A_t(t)\). This ODE has, along with the singular points at \(z = 0, t, \infty\), an extra singularity at \(z = \lambda\). This singularity can be checked to be an apparent one: the solutions of the indicial equation at \(z = \lambda\) gives \(a = l + 0, 2,\) and there will be no logarithm behavior if it applies the following algebraic relation between the parameters,

\[
\mu^2 = \left(\frac{\theta_0 - 1}{\lambda} + \frac{\theta_t - 1}{\lambda - t}\right) \mu + \frac{H_0}{\lambda} + \frac{H_1}{\lambda - t} = \frac{1}{4}.
\]

This relation means that the change in \(t\) has to be accompanied by a change in \(\lambda\) and \(\mu\) so that the relation is maintained. As it was found in [12, 15, 16], the change in \(\lambda, \mu\) with respect to \(t\) is Hamiltonian, generated by the function \(H_0\) seen as a function of \(\mu\) and \(\lambda\) as in the algebraic relation above. One can also check that \(H_0\) and \(H_t\) are related to the behavior of the function near \(z = \infty\),

\[
H_0 + H_t = -\mu + \theta_\infty - \frac{1}{2}.
\]

The Schlesinger system (28) yield the Painlevé V equation for the following function of the entries of \(A_t\):

\[
y(t) = \frac{(A_0)_{11}(A_t)_{12}}{(A_t)_{11}(A_0)_{12}} = \frac{\theta_0 + \theta_t - \theta_\infty - (2\mu - 1)(\lambda - t)}{\theta_0 + \theta_t - \theta_\infty - (2\mu - 1)\lambda}.
\]

The Painlevé V is part of the family of second order differential equations with rational coefficients and the Painlevé property: all the branch points of the solutions are fixed, determined by the ODE itself [11, 18]. These equations define new special functions, and the Painlevé V, in particular, has been useful to compute correlation functions of strongly coupled bosonic systems [19], distribution functions of random matrix theory, certain limits of conformal blocks, and the XY model—see [20] for a (not exhaustive) list of applications. It has also been shown to give exact analytic expressions for the scattering of massless fields in black hole backgrounds [21, 22].

We follow [13] and define the \(\tau\) function,

\[
\frac{d}{dt} \log \tau(t, \theta, \sigma) = -\frac{1}{2} \text{Tr} \sigma_3 A_t - \frac{1}{t} \text{Tr} A_0 A_t,
\]

which satisfies a third order nonlinear ODE—the so-called \(\sigma\) form of the Painlevé V equation [24]. This Painlevé V \(\tau\) function is defined up to a multiplicative constant. Such a \(\tau\) function has the direct interpretation of generating function for correlations in field-theoretic applications of the Painlevé transcendents. Asymptotic expressions for the Painlevé V \(\tau\) function have been derived in [10], and the (irregular) conformal block interpretation was given in [23, 24]. The relevant results are listed in the appendix. In order to describe this \(\tau\) function, we define the composite monodromy parameter,
then, the monodromy data are extended to \( \tilde{\theta} = \{ \theta_0, \theta_i, \theta_\infty, \sigma, s_i \} \). Now, these monodromy data can be written in terms of the parameters of the system \( \Delta, g, \) and the energy \( E \). We will assume generic (i.e., nonmultiples of \( \pi \)) values for the monodromy data \( \theta_\infty \), so these expressions can be locally inverted in terms of the parameters of the Rabi–Garnier system.

In terms of (35), the existence of monodromy matrices for the Rabi–Garnier system (10) amounts to the existence of a solution to the \( \tau \) function given the initial conditions,

\[
\begin{align*}
\frac{d}{dt} \log \tau (t, \tilde{\theta}, \tilde{\sigma}) \big|_{t=-4g^2} &= \frac{E + g^2}{2} + \frac{\Delta^2}{4g^2}, \\
\frac{d^2}{dt^2} \log \tau (t, \tilde{\theta}, \tilde{\sigma}) \big|_{t=-4g^2} &= \frac{1}{t^2} \Tr A_0 A_\tau = \frac{\Delta^2}{16g^4},
\end{align*}
\]

which is guaranteed on general grounds. Note that, in the case of interest, by reading from (10), we have

\[
\theta_0 = \theta_i = E + g^2 \quad \text{and} \quad \theta_\infty = 0.
\]

4. Quantization

The pair of equations (37) provides an implicit transcendental equation for the nontrivial monodromy data—the Stokes parameters \( \tilde{\sigma} \). It should be stressed that the equations above solve the Rabi model in a combinatorial sense: the formulas given in the appendix give an asymptotic expansion for the Painlevé V \( \tau \) function near \( t = 0 \). Also, with the full set of monodromy data \( \tilde{\theta}, \tilde{\sigma} \), one can now fulfill the task of writing an explicit representation of the monodromy matrices and, then, can prove that the model is YB integrable. The same special type of Painlevé V system as above was studied in a series of papers [25–27] where the invariants of the isomonodromy flow were calculated and, in the last of the series, a Toda chain structure was outlined.
\[ \Phi(z) = \begin{cases} G_0 z \{ 0, 0 \} (1 + O(z)) C_0, & z \to 0 \\ G_t (z - t) \{ 0, 0 \} (1 + O(z - t)) C_t, & z \to t, \end{cases} \]

where \( G_i (i = 0, t) \) are the matrices diagonalizing \( A_i \) and \( C_i \) are called the connection matrices. The monodromy matrices are diagonalized by the connection matrices, that is,

\[ M_i = C_i^{-1} e^{i \theta_0 n} C_i, \]

Thus, \( C_i \) are defined up to right multiplication. Without loss of generality we can take \( \det C_i = 1 \). We define the ‘natural solutions’ \( \Phi_i(z) \) as the solutions behaving near \( z_i = 0, t \), such as

\[ \Phi_i(z) = G_i z \{ 0, 0 \} (1 + O(z)). \]

The natural solutions \( \Phi_i(z) \) are the ones obtained by Frobenius method near \( z_i \). One can see that the matrix \( C_0 C_t^{-1} \) connects the natural solutions of the system (10) at \( z = 0 \) and \( z = t \).

We can now use the monodromy matrix to solve for the eigenvalue problem. The solution of (10) is required from physical grounds to be analytic on the whole plane—it will have an essential singularity at \( z = \infty \), but analyticity at \( z = 0, t \) ensures that the quantum state defined by the solution has finite expectation values for the relevant physical quantities (such as the bosonic number operator). This condition is translated to our language by requiring that the matrix \( C_0 C_t^{-1} \) that connects the natural solutions at \( z = 0 \) and \( z = t \) (diagonal: the analytic solution at \( z = 0 \) will also be analytic at \( z = t \). In principle, the connection could be ‘upper triangular’: the second solution at \( z = 0 \), which diverges as \( z^k \), could be connected to a superposition of the divergent and the regular solutions at \( z = t \), but one can easily see that this does not happen: consider the determinant of the fundamental matrix \( \det \Phi \), which satisfies the equation,

\[ \frac{d}{dz} \det \Phi = \left( \frac{\theta_0}{z} + \frac{\theta_1}{z - t} \right) \det \Phi. \]

This equation yields \( \det \Phi = z^{\theta_0} (z - t)^{\theta_1} \), and the result can be used to normalize the solutions, in the sense that now the fundamental matrix has unit determinant. One can convince oneself that this normalization does not change the connection matrices \( C_i \), but, now, the two natural solutions at any particular singular points have a similar behavior: \((z - z_i)^{\theta_0/2}\). This parity, where \( \theta_1 \to -\theta_1 \), is the same \( Z_2 \) symmetry that was fundamental in Braak’s work. For our application, an upper triangular connection matrix \( C_0 C_t^{-1} \) with a nonvanishing off diagonal term would violate this symmetry. Therefore, this symmetry guarantees that the vanishing of one of the off-diagonal elements of \( C_0 C_t^{-1} \) will imply the vanishing of the other off-diagonal term. Hence, \( C_0 C_t^{-1} \) will be diagonal.

Now, a diagonal \( C_0 C_t^{-1} \) implies, for the composite monodromy parameter \( \sigma \), defined in (36), that

\[ \cos \pi \sigma = \cos \pi (\theta_0 + \theta_1). \]

Therefore, the regularity of the solution is expressed as a quantization condition,

\[ \sigma_n = 2n + \theta_0 + \theta_1 = 2(E + g^2 + n), \quad n \in \mathbb{Z}. \]

Since \( \sigma \) is given in terms of the Stokes parameters \( S_{1,2} \), this condition can be fed into the solution (37) to yield the quantized values for the energy \( E_n \). Since \( \sigma \) is a function of \( E, g \), and \( \Delta \), the completion of this task requires the knowledge of the expansion of the \( \tau \) function given
in the appendix. Lastly, we note that the values of \( \sigma \) dictate the eigenvalues of the effective monodromy at \( z = \infty \) as per (20). The behavior of the solution at that point, illustrated by (17) will be composed of one normalizable solution, \( f_1^{(1)} \) in the parametrization given by (9), and a non-normalizable one \( f_1^{(1)} \). Making use of the \( \mathbb{Z}_2 \) symmetry, these are linked by \( \sigma_n \rightarrow -\sigma_n \), so only the solutions with \( \sigma_n > 0 \) are independent in the quantization condition (43).

A series of simple solutions is obtained adjusting \( \Delta \) so that \( \sigma_n = 2m \) is an even integer. By (36), this requires that, at least, one of the Stokes parameters vanishes. The spectrum is given by the formula above (43),

\[ E_n^J = n - g^2, \quad n \in \{1, 2, 3, \ldots\}. \]  

These are called the Judd solutions in [3]. These conditions imply \( \theta_0 = \theta_1 = n \), and their monodromy will be trivial, save perhaps for a single nonvanishing Stokes matrix at infinity. One notes that, for an even integer \( \sigma \), the \( \tau \) function degenerates to a polynomial in \( t \). In terms of the Painlevé V equation, these correspond to rational solutions.

4.1. Small coupling expansion

Given the particular form of \( \theta_{\infty} \) and the quantization condition, we can work a few particular cases. Let us rewrite the solution (37) as

\[ i \frac{d}{dt} \left( \log (t, \tilde{\theta}, \tilde{\sigma}) \right)_{t=-4g^2} = -2g^2(E + g^2) - \Delta^2, \]

\[ \left( i \frac{d}{dt} \right)^2 \log (t, \tilde{\theta}, \tilde{\sigma})_{t=-4g^2} = -2g^2(E + g^2). \]  

In the limit \( g \rightarrow 0 \), the sum (51) in the definition of the \( \tau \) function (50),

\[ \tau(t, \tilde{\theta}, \tilde{\sigma}) = e^{2\theta_{\infty}t-(\sigma-2m)^2/4[f(t)]^{-1}} \]

becomes

\[ f(t) = K (1 + \beta_1 t + \gamma_1 t^{1-(\sigma-2m)} + O(t^2, t^{2(1+(\sigma-2m))}), \]

where \( m \) is the largest integer smaller than \( \sigma/2 \) and the constants \( K, \beta_1, \) and \( \gamma_1 \) are given by the formulas in the appendix,

\[ K = C \left( \tilde{\theta}, \frac{1}{2} \sigma - m \right), \]

\[ \beta_1 = -\frac{\theta_{\infty}}{4(\sigma - 2m)^2} (\theta_1^2 - \theta_0^2 + (\sigma - 2m)^2), \]

\[ \gamma_1 = \frac{(\theta_{\infty} + (\sigma - 2m))(\theta_1 + \sigma - 2m)(\theta_0^2 - \theta_1^2)}{8(\sigma - 2m)^2(1 - \sigma + 2m^2)} s^{-1}. \]  

In terms of \( f(t) \), the solution (37) is written as

\[ \frac{1}{4}(\sigma - 2m)^2 + \left( \frac{\dot{f}(t)}{f(t)} \right)_{t=-4g^2} = \Delta^2, \]

\[ -i \left( \frac{\ddot{f}(t)}{f(t)} + i \frac{\dot{f}^2(t)}{f^2(t)} - \frac{\dot{f}(t)}{f(t)} \right)_{t=-4g^2} = 0. \]  

\[ 10 \]
Now, the second equation just sets the value of \( \hat{s} \) in terms of \( g^2 \) and \( \Delta \). As \( \theta_\infty = 0, \beta_1 \) vanishes and to first order in \( t = -4g^2 \), we have

\[
\sigma - 2m = -2\sqrt{\Delta^2 + 2} + 2 + O(g^{2-\sigma - 2\mu}),
\]

which, along with the quantization condition (43), yields the spectrum perturbatively in \( g \),

\[
E_n \approx n - g^2 \pm (\sqrt{\Delta^2 + 2} - 2),
\]

with \( n \) as an integer chosen so that \( E_n \) is non-negative.

5. Perspectives

The methods described here are useful not only to show the existence of the monodromy matrices for the Rabi model and, hence, YB integrability, but also to provide a solution for the eigenvalue problem in terms of the transcendental equation (37). Given that there is a combinatorial expansion of the Painlevé \( V \) \( \tau \) function in terms of irregular conformal blocks, one can then implement a numerical/symbolic computation to complete the task of finding the eigenvalues using the expansion given in the appendix and the quantization condition (43).

Another interesting direction would be to use the proposed formalism to other similar systems. For instance, the extension to the model with broken parity introduced by Braak. This consists of adding a term of the form \( \gamma \sigma_1 \) to the Rabi Hamiltonian, which seems to be a simple extension and amenable through the methods described here.

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Appendix. Formulas for the Painlevé \( V \) \( \tau \) function

Here, we lift the relevant formulas from [24]. In the following, we consider their definition for the \( \tau \) function:

\[
\tau(t) = t^{(\theta_\infty - \theta_0)/4}[\tau(t)]^{-1}.
\]

The expansion for the \( \tau \) function is of the form

\[
\tau(t, \tilde{\theta}) = \sum_{n \in \mathbb{Z}} C \left( \{\theta\}, \frac{1}{2}\sigma + n \right) t^{\sigma(2\sigma + n)\beta} B \left( \{\theta\}, \frac{1}{2}\sigma + n; \beta \right),
\]

where the irregular conformal block \( B \) is given as a power series over the set of Young tableaux \( \mathbb{Y} \),

\[
B \left( \{\theta\}, \frac{1}{2}\sigma; \beta \right) = e^{-4\beta\beta} \sum_{\lambda, \mu \in \mathbb{Y}} B_{\lambda, \mu} \left( \{\theta\}, \frac{1}{2}\sigma \right) e^{i\lambda + |\mu|},
\]

with \( \lambda, \mu \) as Young tableau.

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\[
B_{\alpha,\beta} = \prod_{(i,j)\in\lambda} \left( \frac{1}{2}\theta_\infty + \frac{1}{2}\sigma + i - j \right) \left( \frac{1}{2}\theta_j + \frac{1}{2}\sigma + i - j \right)^2 - \frac{1}{4}\theta_0^2 \right) / h^2_i(i,j)(\lambda'_{i,\mu} - i - j + 1 + \sigma) \times \prod_{(i,j)\in\mu} \left( \frac{1}{2}\theta_\infty - \frac{1}{2}\sigma + i - j \right) \left( \frac{1}{2}\theta_j - \frac{1}{2}\sigma + i - j \right)^2 - \frac{1}{4}\theta_0^2 \right) / h^2_i(i,j)(\lambda_\mu + \mu'_{j,\gamma} - i - j + 1 + \sigma).
\]

where \(\lambda\) denotes a Young tableau, \(\lambda_i\) is the number of boxes in row \(i\), \(\lambda'_{j,\mu}\) is the number of boxes in column \(j\), and \(h_\mu(i,j) = \lambda_\mu + \mu'_{j,\gamma} - i - j + 1\) is the hook length related to the box \((i,j)\) \(\in\lambda\). The structure constants \(C\) are rational products of Barnes functions,

\[
C((\theta_j),\sigma) = \prod_{c_{\pm}} G\left(1 - \frac{1}{2}\theta_\infty + \epsilon_\frac{1}{2}\sigma\right) G\left(1 + \frac{1}{2}\theta_0 + \epsilon_\frac{1}{2}\sigma\right) \times G\left(1 + \frac{1}{2}\theta_j - \epsilon_\frac{1}{2}\sigma\right) / G(1 + \epsilon\sigma),
\]

where \(G(z)\) is defined by the functional equation \(G(1+z) = \Gamma(z)G(z)\). The parameters \(\sigma\) and \(s\) in (51) are related to the ‘constants of integration’ of the Painlevé V equation. \(\sigma\) is the same monodromy parameter as (36), whereas, \(s\) has a rather lengthy expression in terms of monodromy data that can be read from [10]. The Painlevé V \(\tau\) function was also considered in great detail in [13]. The particular set of parameters considered here were also considered in [25–27].

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