A time-frequency analysis perspective on Feynman path integrals

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Abstract The purpose of this expository paper is to highlight the starring role of time-frequency analysis techniques in some recent contributions concerning the mathematical theory of Feynman path integrals. We hope to draw the interest of mathematicians working in time-frequency analysis on this topic, as well as to illustrate the benefits of this fruitful interplay for people working on path integrals.

Key words: Path integral; modulation spaces; pseudodifferential operators.

1 Introduction

The path integral formulation of non-relativistic quantum mechanics is a paramount contribution by Richard Feynman (Nobel Prize in Physics, 1965) to modern theoretical physics. The origin of this approach goes back to Feynman’s Ph.D. thesis of 1942 at Princeton University (recently reprinted, cf. [8]) but was first published in the form of research paper in 1948 [23]; see also [63] for some historical hints. In rough terms we could say that this approach provides a quantum counterpart to Lagrangian mechanics, while the standard framework for canonical quantization as developed by Dirac relies on the Hamiltonian formulation of classical mechanics. Path integrals and Feynman’s deep physical intuition were the main ingredients of the celebrated diagrams, introduced in the 1949 paper [24], which gave a whole new outlook on quantum field theory.

For a first-hand pedagogical introduction we recommend the textbook [25], where it is clarified how the physical intuition of path integrals comes from a deep understanding of the lesson given by the two-slit experiment. We briefly outline below the main features of Feynman’s approach. Recall that the state of a non-relativistic
particle in the Euclidean space $\mathbb{R}^d$ at time $t$ is represented by the wave function $\psi(t, x), (t, x) \in \mathbb{R} \times \mathbb{R}^d$, such that $\psi(t, \cdot) \in L^2(\mathbb{R}^d)$. The time-evolution of a state $\varphi(x)$ at $t = 0$ is governed by the Cauchy problem for the Schrödinger equation:

$$\begin{cases}
i\hbar \partial_t \psi = (H_0 + V(x))\psi \\
\psi(0, x) = \varphi(x),
\end{cases}$$

where $0 < \hbar \leq 1$ is a parameter (the Planck constant), $H_0 = -\hbar^2 \Delta/2$ is the standard Hamiltonian for a free particle and $V$ is a real-valued potential; we conveniently set $m = 1$ for the mass of the particle. The map $U(t, s) : \psi(x, \cdot) \mapsto \psi(t, \cdot), t, s \in \mathbb{R}$, is a unitary operator on $L^2(\mathbb{R}^d)$ and is known as propagator or evolution operator; we set $U(t)$ for $U(t, 0)$. Since $U(t)$ is a linear operator we can formally represent it as an integral operator, namely

$$\psi(t, x) = \int_{\mathbb{R}^d} u_t(x, y)\varphi(y)dy,$$

where the kernel $u_t(x, y)$ (we also write $u_{t, s}(x, y)$ or $u(t, s)(x, y)$ for the kernel of $U(t, s)$) is interpreted as the transition amplitude from the position $y$ at time 0 to the position $x$ at time $t$. In a nutshell, Feynman’s prescription is a recipe for this kernel, the main ingredients being all the possible paths from $y$ to $x$ that the particle could follow. The contribution of each interfering alternative path to the total probability amplitude is a phase factor involving the action functional evaluated on the path, that is

$$S[\gamma] = S(t, 0, x, y) = \int^t_0 L(\gamma(\tau), \dot{\gamma}(\tau))d\tau,$$

where $L$ is the Lagrangian functional of the underlying classical system. Therefore, the kernel should be formally represented as

$$u_t(x, y) = \int e^{i\pi S[\gamma]}D\gamma,$$

that is a sort of integral over the infinite-dimensional space of paths satisfying the conditions above. This intriguing picture is further reinforced by the following remark: a formal application of the stationary phase method shows that the semiclassical limit $\hbar \to 0$ selects the classical trajectory, hence we recover the principle of stationary action of classical mechanics.

It is well known after Cameron [9] that $D\gamma$ cannot be a Lebesgue-type measure on the space of paths, neither it can be constructed as a Wiener measure with complex variance - it would have infinite total variation. The literature concerning the problem of putting formula (2) on firm mathematical ground is huge; the interested reader could benefit from the monographs [4] [29] [51] as points of departure. We will

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1 We remark that in physics literature the term “propagator” is usually reserved to the integral kernel $u_t$ of $U(t)$, see below. This may possibly lead to confusion since it is in conflict with the traditional nomenclature adopted in the analysis of PDEs.
describe below only two of the several schemes which have been manufactured in order to give a rigorous meaning to (2), the type of techniques involved ranging from geometric to stochastic analysis: these approaches both rely on operator-theoretic strategies and are called sequential approach and time slicing approach. Basically, one is lead to study sequences of operators on $L^2(\mathbb{R}^d)$ which converge to the exact propagator $U(t)$ in a sense to be specified, the strength of convergence competing against the regularity of the potential $V$.

This is the point where time-frequency analysis enters the scene. Techniques of phase space analysis are indeed very well suited to the study of path integrals, the reasons being manifold. First of all, pseudodifferential and Fourier integral operators can be effectively treated from a time-frequency analysis perspective as evidenced by a now vast literature - we highlight [11, 12, 31, 32] among others. Typical function spaces for this purpose are modulation and Wiener amalgam spaces, which may serve as space of symbols as well as background where to investigate boundedness and related properties (algebras for composition, sparsity, diagonalization, etc.). In the same spirit, many results are known on dispersive nonlinear PDEs, in particular on the Schrödinger equation. Notably, function spaces of time-frequency analysis enjoy a fruitful balance between nice properties (Banach spaces/algebras, embeddings, decomposition, etc.) and regularity of their members.

The purpose of this overview is to concisely witness some results of this successful interplay, which has made possible to advance in the quest for a rigorous theory of path integral with remarkable results. In particular, we are going to describe three recent contributions on the topic:

1. convergence of time-slicing approximations in $L^p$ spaces (with loss of derivatives) for $p \neq 2$ - based on [55];
2. convergence of non-smooth time-slicing approximations inspired by the custom in physics and chemistry - based on [57];
3. pointwise convergence of the integral kernels of the sequential approximations - based on [58].

First we provide a concise exposition of the two operator-theoretic approaches to path integrals mentioned above. We also collect some preliminary concepts in a separate section for the sake of clarity.

**Notation.** We denote by $S(\mathbb{R}^d)$ the Schwartz space of rapidly decaying smooth functions on $\mathbb{R}^d$ and by $S'(\mathbb{R}^d)$ the space of temperate distributions. We set $\langle x \rangle = (1 + |x|)^{1/2}$, $x \in \mathbb{R}^d$. The space of smooth bounded functions on $\mathbb{R}^d$ with bounded derivatives of any order is denoted by $C^\infty_b(\mathbb{R}^d)$ (also known as $S^{0}_{0,0}$ in microlocal analysis); it is equipped with the family of seminorms

$$
\|f\|_k = \sup_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty} < \infty, \quad k \in \mathbb{N}_0 = 0, 1, 2, \ldots
$$

The conjugate exponent $p'$ of $p \in [1, \infty]$ is defined by $1/p + 1/p' = 1$. We write $f \lesssim g$ if the underlying inequality holds up to a constant factor $C > 0$, that is $f \leq Cg$. 


2 A few facts on modulation spaces

In this section we set the function space framework for the rest of the paper. The reader is urged to consult \[30, 67, 68, 69\] for more details and the proofs of the mentioned properties.

Modulation spaces were introduced by Feichtinger in the ‘80s [21, 22]. At first, they can be thought of as Besov spaces with cubic geometry, namely characterized by isometric boxes in the frequency domain instead of dyadic annuli. To be precise, fix an integer \(d \geq 1\); for any \(1 \leq p, q \leq \infty\) and \(s \in \mathbb{R}\) we set

\[
M_{p,q}^s(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) : \| f \|_{M_{p,q}^s} = \left( \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \| \Box_k f \|_{L^p}^q \right)^{1/q} < \infty \right\},
\]

where the frequency-uniform decomposition operator \(\Box_k\) is the Fourier multiplier whose symbol is a suitably smoothed version of the characteristic function of the unit cube with center \(k \in \mathbb{Z}^d\). Trivial modifications are needed to cover the cases \(p, q = \infty\); for the unweighted case \(s = 0\) we write \(M_{p,q}^0(\mathbb{R}^d)\), while the case corresponding to \(p = q\) simply becomes \(M_p(\mathbb{R}^d)\). We emphasize that, in heuristic terms, the parameter \(s \geq 0\) can be interpreted as the degree of fractional differentiability of \(f \in M_{p,q}^s(\mathbb{R}^d)\).

An equivalent, insightful definition of \(M_{p,q}^s(\mathbb{R}^d)\) is known as the *phase-space representation* in the jargon of coorbit theory, namely it is given in terms of the global decay of the phase-space concentration of a distribution. To be concrete, given \(f \in S'(\mathbb{R}^d)\) and a non-zero Schwartz window function \(g \in S(\mathbb{R}^d)\), the short-time Fourier transform (STFT) \(V_g f\) is defined as a windowed version of the ordinary Fourier transform:

\[
V_g f(x,\xi) = \mathcal{F} [f g(\cdot - x)](\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot t} f(t) g(t-x) dt, \quad (x,\xi) \in \mathbb{R}^{2d},
\]

where \(\mathcal{F}\) denotes the Fourier transform. Roughly speaking, the STFT can be interpreted as the magnitude of a tight frequency band centered at \(\xi\) in a short time interval centered at \(x\). Therefore \(V_g f\) can be thought of as a “musical score” of the signal \(f\), that is an approximately simultaneous time-frequency representation - a perfect phase-space localization is forbidden by the uncertainty principle. The modulation space \(M_{p,q}^s(\mathbb{R}^d)\), \(1 \leq p, q \leq \infty\) can then be equipped with the norm

\[
\| f \|_{M_{p,q}^s} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x,\xi)|^p dx \right)^{q/p} \langle \xi \rangle^{qs} d\xi \right)^{1/q},
\]

which is proved to be equivalent to the one introduced in (3) - we improperly use the same notation. Furthermore, different window functions for the STFT yield equivalent norms on \(M_{p,q}^s(\mathbb{R}^d)\). We emphasize that several well-known spaces are related with modulation spaces: for instance,

(i) \(M^2(\mathbb{R}^d)\) coincides with the Hilbert space \(L^2(\mathbb{R}^d)\);
(ii) $M^2_2(\mathbb{R}^d)$ coincides with the standard $L^2$-based Sobolev space $H^s(\mathbb{R}^d)$;
(iii) continuous embeddings with Lebesgue spaces hold:

$$M^{p,1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,\infty}(\mathbb{R}^d).$$

A third perspective on modulation spaces is provided by inspecting the definition of the STFT: it may be thought of as a continuous expansion of the function $f$ with respect to the uncountable system $\{\pi(z)g : z = (x, \xi) \in \mathbb{R}^{2d}\}$, where we introduced the \textit{time-frequency shift operator}

$$\pi(z) = \pi(x, \xi) = M_{\xi}T_x, \quad M_{\xi}g(t) = e^{i\xi \cdot t}g(t), \quad T_xg(t) = g(t-x).$$

Notice that $\pi(z)g$ is a wave packet highly concentrated near $z$ in phase space. In short, we have $V_\xi f(x, \xi) = \langle f, \pi(x, \xi)g \rangle$ in the sense of the (extension to the duality $S' - S$ of the) inner product on $L^2$. This remark can be made completely rigorous in the context of \textit{frame theory}, leading to discrete time-frequency representations. In particular, given a non-zero window function $g \in L^2(\mathbb{R}^d)$ and a subset $\Lambda \subset \mathbb{R}^{2d}$, we call \textit{Gabor system} the collection of the time-frequency shifts of $g$ along $\Lambda$, namely $\mathcal{G}(g, \Lambda) = \{\pi(z)g : z \in \Lambda\}$. For the sake of concreteness one may consider regular lattices such as $\Lambda = a\mathbb{Z} \times b\mathbb{Z} = \{(ak, bn) : k, n \in \mathbb{N}\}$, for lattice parameters $a, b > 0$; in that case we write $\mathcal{G}(g, a, b)$ for the corresponding Gabor system. Recall that a \textit{frame} for a Hilbert space $\mathcal{H}$ is a sequence $\{x_j\}_{j \in J} \subset \mathcal{H}$ such that for all $x \in \mathcal{H}$

$$A \|x\|_{\mathcal{H}}^2 \leq \sum_{j \in J} |\langle x, x_j \rangle|^2 \leq B \|x\|_{\mathcal{H}}^2,$$

for some universal constants $A, B > 0$ (frame bounds). In a nutshell, the paradigm of frame theory consists in the following steps: decompose a vector $x$ along the frame; investigate how operators act on those elementary pieces; reconstruct the processed vector. The entire process is encoded by the \textit{frame operator}

$$S : \mathcal{H} \ni x \mapsto \sum_{j \in J} \langle x, x_j \rangle x_j \in \mathcal{H}.$$ 

If a Gabor system $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ it is called \textit{Gabor frame}. Notice that the Gabor frame operator then reads $Sf = \sum_{z \in \Lambda} V_\xi f(z)\pi(z)g$. A remarkable result is that, for a window function $g \in M^1(\mathbb{R}^d)$ such that $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R}^d)$, an equivalent discrete norm for $M^p_{q,q}(\mathbb{R}^d)$ is given by

$$\|f\|_{M^p_{q,q}} = \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |V_\xi f(ak, bn)|^p \right)^{q/p} \langle \beta n \rangle^{qs} \right)^{1/q}. $$
3 Two rigorous approaches to path integrals

We now briefly outline the main features of a pair of mathematical schemes which are in fact two faces of the same philosophy. While in the literature one may easily notice that different names are interchangeably used for them, we consider the classification below for the sake of clarity.

3.1 The sequential approach

The so-called **sequential approach** to path integrals was first introduced by Nelson in [54] and relies on two basic results. First, recall that the free evolution operator for the Schrödinger equation $U_0(t) = e^{-i\frac{\mathcal{H}_0}{\hbar} t}$, $\mathcal{H}_0 = -\hbar^2 \nabla^2/2$, is a Fourier multiplier; routine computation yields the following integral representation [60, Sec. IX.7]:

$$e^{-i\frac{\mathcal{H}_0}{\hbar} t} \varphi(x) = \frac{1}{(2\pi i \hbar)^{d/2}} \int_{\mathbb{R}^d} \exp \left( \frac{i}{\hbar} \frac{|x-y|^2}{2t} \right) \varphi(y) dy, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$  \hspace{1cm} (4)

Notice that the phase factor in the integral actually coincides with the action functional evaluated along the line $\gamma_{cl}(\tau) = y + (x-y)\tau/t$, namely the classical trajectory of a free particle moving from position $y$ at time $\tau = 0$ to position $x$ at time $\tau = t$ in the absence of external forces.

Next, we need a result from the theory of operator semigroups. Provided that suitable conditions on the domain of $\mathcal{H}_0$ and on the potential $V$ are satisfied (see below), the **Trotter product formula** holds for the propagator generated by $\mathcal{H} = \mathcal{H}_0 + V$:

$$e^{-\frac{i}{\hbar} \mathcal{H} t} = \lim_{n \to \infty} \left( e^{-\frac{i}{\hbar} \mathcal{H}_0 n} e^{-\frac{i}{\hbar} V n} \right)^n,$$

where the limit is intended in the strong topology of operators in $L^2(\mathbb{R}^d)$. Combining these two ingredients yields the following representation of the complete propagator $e^{-\frac{i}{\hbar} \mathcal{H} t}$ as limit of integral operators (cf. [60 Thm. X.66]):

$$e^{-\frac{i}{\hbar} \mathcal{H} t} \varphi(x) = \lim_{n \to \infty} \left( 2\pi \hbar \right)^{d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} S_n(t;x_0,...,x_{n-1};x)} \varphi(x_0) dx_0 \ldots dx_{n-1},$$

where we set

$$S_n(t;x_0,...,x_{n-1},x) = \sum_{k=1}^{n} \frac{t}{n} \left[ \frac{|x_k - x_{k-1}|}{t/n} \right]^2 - V(x_k), \quad x_0 = y, \ x_n = x.$$

With the aim of understanding the role of $S_n(t;x_0,...,x_n)$, consider the following argument. Given the points $x_0,...,x_{n-1}, x \in \mathbb{R}^d$, let $\overline{\gamma}$ be the polygonal path (broken line) through the vertices $x_k = \overline{\gamma}(kt/n)$, $k = 0,...,n$, $x_n = x$, parametrized as
\[ \gamma(\tau) = x_k + \frac{x_{k+1} - x_k}{t/n}(\tau - k\frac{t}{n}), \quad \tau \in \left[k\frac{t}{n}, (k+1)(\frac{t}{n})\right], \quad k = 0, \ldots, n-1. \]

Hence \( \gamma \) prescribes a classical motion with constant velocity along each segment. The action for this path is thus given by

\[
S[\gamma] = \sum_{k=1}^{n} \frac{1}{2n} \left( \frac{|x_k - x_{k-1}|}{t/n} \right)^2 - \int_{t_0}^{t} V(\gamma(\tau))d\tau.
\]

According to Feynman’s interpretation formula (5) can be thought of as an integral over all polygonal paths, where \( S_n(x_0, \ldots, x_n, t) \) is a finite-dimensional approximation of the action functional evaluated on them. The limiting behaviour for \( n \to \infty \) is now intuitively clear: the set of polygonal paths becomes the set of all paths and in some sense we recover (2). We remark that the custom in Physics community after Feynman is exactly to employ the suggestive formula (2) as a placeholder for (5) and the related arguments - see for instance [34, 42].

For what concerns the assumptions on the potential perturbation \( V \) under which the Trotter product formula holds, a standard result shows that it is enough to choose \( V \) in such a way that \( H_0 + V \) is essentially self-adjoint on \( D = D(H_0) \cap D(V) \) in \( L^2(\mathbb{R}^d) \), cf. for instance [59, Thm. VIII.31]. The power of Nelson’s perturbative approach is that one can cover wide classes of wild potentials, such as Kato potentials, including finite sums of real-valued functions in \( L^p(\mathbb{R}^d) \) with \( 2p > d \) and \( p \geq 2 \) [54, Thm. 8].

### 3.2 The time-slicing approximation

We now consider another scheme that could be informally called “the Japanese way” to rigorous path integrals, since the leading players in its construction were Fujiwara and Kumano-go, with further developments by Ichinose and Tsuchida. The main references for this approach are the papers [27, 28, 37, 38, 44, 45, 46] and the monograph [29], to which the reader is referred for further details.

Let us briefly reconsider equation (5) and its interpretation in terms of finite-dimensional approximations along broken lines; a similar result can be achieved without recourse to the Trotter formula as detailed below. First, let us specify the class of potentials involved in this approach.

**Assumption (A).** The potential \( V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) satisfies \( \partial^\alpha_t V \in C^0(\mathbb{R} \times \mathbb{R}^d) \) for any \( \alpha \in \mathbb{N}_0^d \) and

\[
|\partial^\alpha_t V(t, x)| \leq C_\alpha, \quad |\alpha| \geq 2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d
\]

for suitable constants \( C_\alpha > 0 \).

For this wide class of smooth, time-dependent, at most quadratic potentials Fujiwara showed [27, 28] that the propagator \( U(t, s) \), \( 0 < s < t \), is an oscillatory integral operator (for short, OIO) of the form.
U(t, s)\varphi(x) = \frac{1}{(2\pi \hbar(t - s))^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}(S(t, x, y) - S(s, x, y))} a(h, t, s)(x, y)\varphi(y)dy, \quad (7)

for some amplitude function \(a(h, t, s) \in C^\infty_0(\mathbb{R}^{2d})\). In concrete situations, except for a few cases, there is no hope to compute the exact propagator in an explicit, closed form. Due to this difficulty and inspired by the free particle operator \[4\], one is lead to consider approximate propagators (parametrices), such as

\[ E^{(0)}(t, s)\varphi(x) = \frac{1}{(2\pi \hbar(t - s))^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}(S(t, x, y) - S(s, x, y))} \varphi(y)dy. \quad (8) \]

In view of the previous remarks, this operator is supposed to provide a good approximation of the \(U(t, s)\) for \(t - s\) small enough. The case of a long interval \([s, t]\) can be treated by means of composition of such operators in the spirit of the time slicing method proposed by Feynman: given a subdivision \(\Omega = t_0, \ldots, t_L\) of the interval \([s, t]\) such that \(s = t_0 < t_1 < \ldots < t_L = t\), we define the operator

\[ E^{(0)}(\Omega, t, s) = E^{(0)}(t_L, t_{L-1})E^{(0)}(t_{L-1}, t_{L-2}) \cdots E^{(0)}(t_1, t_0), \]

whose integral kernel \(e^{(0)}(\Omega, t, s)(x, y)\) can be explicitly computed from \(8\). The parametrix \(E^{(0)}(\Omega, t, s)\) is then expected to converge (in some sense) to the actual propagator \(U(t, s)\) in the limit \(\omega(\Omega) = \max\{t_j - t_{j-1}, \ j = 1, \ldots, L\} \to 0\).

We have not specified the path along which the action functional in \(8\) should be evaluated. A standard choice, inspired by the custom in physics after Feynman [25], is the broken line approximation introduced above in \(6\), namely

\[ \gamma(\tau) = x_j + \frac{x_{j+1} - x_j}{t_{j+1} - t_j}(\tau - t_j), \quad \tau \in [t_j, t_{j+1}], \ j = 0, \ldots, L. \]

A quite complete theory of path integration in this context has been developed by Kumano-go [44]. In fact, the time-slicing approximation shows its full power when straight lines are replaced by classical paths. To be precise, under the assumptions on the potential detailed above, a short-time analysis of the Schrödinger flow reveals that there exists \(\delta > 0\) such that for \(0 < |\tau - s| \leq \delta\) and any \(x, y \in \mathbb{R}^d\) there exists a unique solution \(\gamma\) of the classical equation of motion \(\ddot{\gamma}(\tau) = -\nabla V(\gamma(\tau))\) satisfying the boundary conditions \(\gamma(s) = y, \gamma(t) = x\). In particular, this can be adapted to the subdivision \(\Omega\) by making the separation small enough, namely \(\omega(\Omega) \leq \delta\). A detailed analysis can be found in [29] Chap. 2.

Among the large number of results proved in this context we mention two milestones from forerunner papers by Fujiwara. In [27] he proved convergence of \(E^{(0)}(\Omega, t, s)\) to \(U(t, s)\) in the norm operator topology in \(\mathcal{B}(L^2(\mathbb{R}^d))\) - the space of bounded operators in \(L^2\). Under the same hypotheses convergence at the level of integral kernels in a very strong topology was proved in [28]. It should be emphasized that the aforementioned results are given for the higher order parametrices \(E^{(N)}(t, s), \ N \in \mathbb{N}_0\), also known as Birkhoff-Maslov parametrices [6] [50] and defined by
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\[ E^{(N)}(t, s) \varphi(x) = \frac{1}{(2\pi i(t - s))^{d/2}} \int_{\mathbb{R}^{d}} e^{i S(t, s, x, y)} a^{(N)}(\hbar, t, s)(x, y) \varphi(y) dy, \quad (9) \]

where \( a^{(N)}(\hbar, t, s)(x, y) = \sum_{j=1}^{N} (\frac{\hbar}{\pi})^{1/2} a_j(t, s)(x, y) \) for suitable functions \( a_j(t, s) \in C^\infty_0(\mathbb{R}^{2d}) \) for \( t - s \leq \delta \), with \( a_0(t, s) \equiv 1 \). We remark that \( E^{(N)}(t, s) \) are parametrices in the sense that they satisfy

\[ (i\hbar \partial_t + \hbar^2 \Delta/2 - V(t, x)) E^{(N)} \psi = G^{(N)}(t, s) \psi, \quad (10) \]

where \( G^{(N)}(t, s) \) has the form in (9) but \( a^{(N)} \) is replaced by the amplitude function \( g^{(N)}(\hbar, t, s)(x, y) \) which satisfies \( \| g^{(N)}(\hbar, t, s) \|_m \leq C_m \hbar \delta^{N+1} |t - s|^{N+1}, m \in \mathbb{N}_0 \). As before, the case of a long interval \([s, t]\) can be treated by means of composition over a sufficiently fine subdivision \( \Omega = t_0, \ldots, t_L \) of the interval \([s, t]\) such that \( s = t_0 < t_1 < \cdots < t_L = t \), namely

\[ E^{(N)}(\Omega, t, s) = E^{(N)}(t_L, t_{L-1}) E^{(N)}(t_{L-1}, t_{L-2}) \cdots E^{(N)}(t_1, t_0). \quad (11) \]

The core results of the \( L^2 \) theory for the time slicing approximation read as follows.

**Theorem 1** Let the potential \( V \) satisfy Assumption (A) and fix \( T > 0 \). For \( 0 < t - s \leq T \) and any subdivision \( \Omega \) of the interval \([s, t]\) such that \( \omega(\Omega) \leq \delta \), the following claims hold.

1. There exists a constant \( C = C(N, T) > 0 \) such that

\[ \left\| E^{(N)}(\Omega, t, s) - U(t, s) \right\|_{L^2 \to L^2} \leq Ch^N \omega(\Omega)^{N+1} (t - s), \quad N \in \mathbb{N}_0. \quad (12) \]

2. We have (cf. (7))

\[ \lim_{\omega(\Omega) \to 0} a^{(N)}(\Omega, \hbar, t, s) = a(\hbar, t, s) \quad \text{in} \quad C^\infty_0(\mathbb{R}^{2d}). \]

Precisely, there exists \( C = C(m, N, T) > 0 \) such that

\[ \left\| a(\hbar, t, s) - a^{(N)}(\Omega, \hbar, t, s) \right\|_m \leq Ch^N \omega(\Omega)^{N+1} (t - s), \quad m, N \in \mathbb{N}_0. \]

The proof of these results ultimately relies on fine analysis of OIOs. The underlying overall strategy can be condensed as follows:

1. prove that “time slicing approximation is an oscillatory integral” (cf. (29)), i.e., that the operators arising from (9) are indeed OIOs under suitable assumptions;
2. derive precise estimates for the operator norm of such OIOs;
3. employ the algebra property of \( B(L^2) \) in order to deal with composition in (11).

With reference to the last item, we mention that an aspect to be considered is that composition of OIOs results in an OIO only for short times, due to the occurrence of caustics, and in general one should not expect smoothing effects for long times.

For the sake of completeness we also mention that Nicola showed in (30) how parts of the conclusions in Theorem 1 still hold under weaker regularity assumptions.
for the potential. Assumption (A) is now replaced by the following one.

**Assumption (A').** The potential $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ belongs to $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d)$ and for almost every $t \in \mathbb{R}$ and $|\alpha| \leq 2$ the derivatives $\partial^\alpha_x V(t, x)$ exist and are continuous with respect to $x$. Furthermore

$$\partial^\alpha_x V(t, x) \in L^\infty(\mathbb{R}; H^{d+1}_{ul}(\mathbb{R}^d)), \quad |\alpha| = 2,$$

where $H^n_{ul}(\mathbb{R}^d)$, $n \in \mathbb{N}$, is the Kato-Sobolev space (also known as uniformly local Sobolev space) of functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ satisfying $\|f\|_{H^n_{ul}} = \sup_B \|f\|_{H^n(B)} < \infty$, the supremum being computed on all open balls $B \subset \mathbb{R}^d$ of radius 1.

**Theorem 2** ([55, Thm. 1.1]) Let the potential $V$ satisfy Assumption (A'). For any $T > 0$ there exists $C = C(T) > 0$ such that for any $0 < t - s \leq T$ and any subdivision $\Omega$ of the interval $[s, t]$ with $|\omega(\Omega)| \leq \delta$ and $0 < \delta \leq 1$,

$$\left\| E^0(\Omega, t, s) - U(t, s) \right\|_{L^2 \to L^2} \leq C\omega(\Omega)(t - s).$$

4 Beyond the $L^2$ theory via Gabor analysis

In view of the results recalled above it seems that the analysis of convergence of time slicing approximations of path integrals can be suitably conducted at the level of operators on $L^2(\mathbb{R}^d)$, i.e. in the space $\mathcal{B}(L^2(\mathbb{R}^d))$ (usually) equipped with the norm operator topology.

It is then natural to wonder whether there exists an $L^p$ analogue of Theorem 1 with $p \neq 2$. We cannot expect a naive transposition of the claim for several reasons. First of all, notice that the Schrödinger propagator is not even bounded on $L^p(\mathbb{R}^d)$ for $p \neq 2$. The parabolic geometry of its characteristic manifold implies that a peculiar loss of derivative, ultimately due to dispersion, occurs ([7, [53]):

$$\|e^{it\Delta} f\|_{L^p} \leq C\left\| (1 - \hbar\Delta)^{k/2} f \right\|_{L^p}, \quad k = 2d[1/2 - 1/p], \quad 1 < p < \infty.$$

On the basis of this observation one is lead to consider the following scale of semiclassical $L^p$-based Sobolev spaces: for $1 < p < \infty$ and $k \in \mathbb{R}$ define

$$L^p_k(\mathbb{R}^d) = \{f \in S'(\mathbb{R}^d) : \|f\|_{L^p_k} = \left\| (1 - \hbar\Delta)^{k/2} f \right\|_{L^p} < \infty\}.$$

We set $L^p_k(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ in the case where $\hbar = 1$. This is indeed a suitable setting for the analysis of Schrödinger operators, in particular for Fourier integral operators arising as Schrödinger propagators associated with quadratic Hamiltonians, cf. [16].

We are also confronted with another issue: the space of bounded operators $L^p_k \to L^p$ (or viceversa) is clearly not an algebra under composition. This is a major obstacle
for a proficient time slicing approximation, having in mind the construction of the parametrices \( E^{(N)}(\Omega, t, s) \) in (11) and the role of this feature in the \( L^2 \) setting.

A possible solution comes from time-frequency analysis, since all these issues become manageable as soon as one transfers the problem to the phase space setting. The first key results in this context are due to Nicola [55] and read as follows - the notation has been introduced in the previous section.

**Theorem 3 ([55 Thm. 1.1])** Assume the condition in Assumption (A) and let \( 1 < p < \infty, k = 2d|1/2 - 1/p| \).

1. For any \( T > 0 \) there exists a constant \( C = C(T) > 0 \) such that for all \( f \in S(\mathbb{R}^d) \), \( |t - s| \leq T \) and \( 0 < \hbar \leq 1 \):

\[
\|U(t, s)f\|_{L^p} \leq C \|f\|_{L^p}, \quad 1 < p \leq 2,
\]

\[
\|U(t, s)f\|_{L^p_k} \leq C \|f\|_{L^p}, \quad 2 \leq p < \infty.
\]

2. For any \( T > 0 \) and \( N \in \mathbb{N}_0 \) there exists a constant \( C = C(T) > 0 \) such that for \( 0 < t - s \leq T \) and any subdivision \( \Omega \) of the interval \([s, t]\) with \( \omega(\Omega) \leq \delta \), \( f \in S(\mathbb{R}^d) \) and \( 0 < \hbar \leq 1 \):

\[
\left\| \left( E^{(N)}(\Omega, t, s) - U(t, s) \right) f \right\|_{L^p} \leq C h^N \omega(\Omega)^{N+1} (t - s) \|f\|_{L^p}, \quad 1 < p \leq 2,
\]

\[
\left\| \left( E^{(N)}(\Omega, t, s) - U(t, s) \right) f \right\|_{L^p_k} \leq C h^N \omega(\Omega)^{N+1} (t - s) \|f\|_{L^p}, \quad 2 \leq p < \infty.
\]

For the sake of clarity we organize the discussion of the relevant aspects in separate sections.

### 4.1 The role of modulation spaces

A phase space perspective can be embraced by recasting the problem in terms of modulation spaces. First of all, notice that the characterizing frequency-uniform decomposition in [3] is particularly well-suited for the analysis of the Schrödinger propagator, see [69] for a detailed account. The localized operator \( \Box_k U_0(t) \) is indeed uniformly bounded on \( L^p(\mathbb{R}^d) \), namely

\[
\|\Box_k U_0(t)f\|_{L^p} \leq (1 + |t|)^{d|1/2 - 1/p|} \|\Box_k f\|_{L^p}.
\]

Moreover it does satisfy the following \( L^p - L^{p'} \) estimate:

\[
\|\Box_k U_0(t)f\|_{L^p} \leq (1 + |t|)^{d(1/2 - 1/p)} \|\Box_k f\|_{L^{p'}}, \quad p \geq 2.
\]
This is a remarkable improvement with respect to the corresponding dyadic estimate
\[ \| \mathcal{Q}_k U_0(t) f \|_{L^p} \lesssim |t|^{-d(1/2-1/p)} \| \mathcal{Q}_k f \|_{L^{p'}} , \quad p \geq 2, \]
showing a singularity at \( t = 0 \).

The phase space lifting strategy in [55] crucially relies on the following non-trivial embeddings relating modulation and Sobolev spaces.

**Theorem 4 ([43])** Let \( 1 < p < \infty \) and \( k = 2d(1/2 - 1/p) \). The following embeddings hold:
\[
L^p(\mathbb{R}^d) \hookrightarrow M^p(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad 1 < p \leq 2,
\]
\[
L^p(\mathbb{R}^d) \hookrightarrow M^p(\mathbb{R}^d) \hookrightarrow L^p_{-k}(\mathbb{R}^d), \quad 2 \leq p < \infty.
\]
As a consequence we get that a bounded linear operator \( T \in B(M^p(\mathbb{R}^d)) \) extends to a bounded operator \( T : L^p_k(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d) \) in the case where \( 1 < p \leq 2, \]
\[ k = 2d(1/2 - 1/p), \]
that is
\[ \| T \|_{L^p_k \rightarrow L^p} \lesssim \| T \|_{M^p \rightarrow M^p}. \]
Similar arguments apply to the case \( p \geq 2 \). It is interesting to remark that in the case of the free propagator \( U_0(t) \) the estimate arising from the \( M^p \) operator norm is sharp, that is
\[ \| U_0(t) \|_{L^p_k \rightarrow L^p} \lesssim \| U_0(t) \|_{M^p \rightarrow M^p} \approx (1 + |t|^{d(1/2-1/p)}), \]
where \( 1 < p < \infty \) (the case \( p \geq 2 \) follows by duality arguments) and \( k \) is as before. It is also worth mentioning that moving to phase space has another advantage. Establishing endpoint continuity results for integral operators for \( p \neq 2 \) requires delicate, highly non-trivial arguments usually involving Hardy-type spaces, cf. [64, 66]; these technical aspects are now hidden behind the embeddings in Theorem 4.

### 4.2 Sparse operators on phase space

From the perspective of phase space analysis the next step should be to study how operators transform Gabor wave packets on phase space. A natural object to investigate this feature is the **Gabor matrix** of an operator \( T \), namely
\[ \mathcal{M}(T, g, z, w) = |\langle T \pi(z)g, \pi(w)g \rangle|, \quad g \in S(\mathbb{R}^d) \setminus \{0\}, \quad z, w \in \mathbb{R}^{2d}. \] (13)

We are going to introduce a class of operators characterized by the sparsity of their Gabor matrix. First, we say that a map \( \chi : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d} \) is a *tame canonical*...
transformation if it is a smooth, invertible symplectomorphism such that
\[ |\partial_y^\alpha \partial_{\eta}^\beta \chi(y, \eta)| \leq C_{\alpha, \beta}, \quad |\alpha| + |\beta| \geq 1, \quad y, \eta \in \mathbb{R}^d. \]

**Definition 1** Let \( \chi \) be a tame canonical transformation. We define \( \text{FIO}(\chi) \) to be the collection of operators \( T : S(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) such that for some (hence any) \( g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \) and any \( s \geq 0 \)
\[ \|T\|_{s, \chi} = \sup_{z, w \in \mathbb{R}^{2d}} \langle w - \chi(z) \rangle^s M(T, g, z, w) < \infty. \]
\( \{\|T\|_{s, \chi}, s \geq 0\} \) is a family of seminorms for \( \text{FIO}(\chi) \).

In heuristic terms this definition encodes the property of operators \( T \in \text{FIO}(\chi) \) of being concentrated along the graph of \( \chi \) in phase space. By an abuse of language we could say that \( T \) is almost diagonalized by Gabor systems - see also Section 6.1 for further details.

For future purposes we define a semiclassical version \( \text{FIO}_h(\chi) \) of \( \text{FIO}(\chi) \) as the space of operators \( T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \) such that \( D_{\hbar^{-1/2}} T D_{\hbar^{1/2}} \in \text{FIO}(\chi) \), where we introduced the canonical dilation operator \( D_{\hbar^{1/2}} f(t) = \hbar^{-d/4} f(\hbar^{-1/2} t) \). \( \text{FIO}_h(\chi) \) is equipped with the family of seminorms \( \|T\|_{s, \chi} = \|D_{\hbar^{-1/2}} T D_{\hbar^{1/2}}\|_{s, \chi}, \quad s \geq 0. \)

The key properties of the class \( \text{FIO}(\chi) \) are summarized in the following result.

**Theorem 5**
1. Any operator \( T \in \text{FIO}(\chi) \) extends to a bounded operator on \( M^p(\mathbb{R}^d) \) for \( 1 \leq p \leq \infty \) and in particular on \( L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d) \).
2. The composition \( T^{(1)} T^{(2)} \) of operators \( T^{(j)} \in \text{FIO}(\chi_j), \quad j = 1, 2, \) belongs to \( \text{FIO}(\chi_1 \circ \chi_2) \).
3. Assume the hypotheses and notation introduced in Section 5.2. For \( 0 < |t - s| \leq \delta \) and \( a \in C^m_b(\mathbb{R}^{2d}) \), the oscillatory integral operator
\[ T f(x) = \frac{1}{(2\pi i(t - s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{i \langle S(t, s, x, y) \rangle} a(x, y) f(y) dy \quad (14) \]
belongs to \( \text{FIO}_h(\chi^h(t, s)) \) for a suitable canonical transformation \( \chi^h(t, s) \) satisfying \( \chi^h(t, s) = \chi^h(t, s) \circ \chi^h(s, t) \). Moreover, for any \( m \in \mathbb{N}_0 \) there exist \( m' \in \mathbb{N}_0 \) and a universal constant \( C > 0 \) such that \( \|T\|_{m, x}^h \leq C \|a\|_{m'} \).
4. Let \( T \in \text{FIO}_h(\chi), \quad 1 < p < \infty \) and \( k = 2d[1/p - 1/2] \). Then \( T \) extends to a bounded operator \( T : L^p_k(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) if \( 1 < p \leq 2 \) and \( T : L^p(\mathbb{R}^d) \to L^p_{-k}(\mathbb{R}^d) \) if \( 2 \leq p < \infty \). In particular, for \( m > 2d \) there exists \( C > 0 \) such that
\[ \|T\|_{L^p_k \to L^p} \leq C \|T\|_{m, x}^h \quad (1 < p \leq 2), \quad \|T\|_{L^p \to L^p_{-k}} \leq C \|T\|_{m, x}^h \quad (2 \leq p < \infty). \]

We address the reader to [10, 52] for details and proofs.

It is worthwhile to compare the quite natural proof of the algebra property with the painful arguments concerning the composition of OIOs in [29]. Semiclassical
versions of Theorems 4 and 5 are needed (and proved) in [55]; see also [10]. Nonetheless, the first part of Theorem 3 essentially follows by noticing that the short-time propagator $U(t, s)$ in (7) is an OIO as in (14), that is $U(t, s) \in FIO_h(x^h(t, s))$; a careful management of the algebra property of $FIO_h(x^h(t, s))$ and the estimates (15) yields the claimed result for $0 < t - s < T$.

4.3 An unavoidable dichotomy

The peculiar dichotomy in Theorem 3 cannot be avoided. A simple argument shows indeed that these results are completely sharp and characterize all possible $L^p$ estimates for time-slicing approximations. Fix $\hbar = 1$ for simplicity and consider the standard harmonic oscillator, namely

$$i\partial_t \psi = -\frac{1}{2} \Delta \psi + \frac{1}{2} |x|^2 \psi.$$

The propagator can be explicitly computed, its integral kernel is known as the Mehler kernel [18, 41]: for $k \in \mathbb{Z}$,

$$u_t(x, y) = \begin{cases} c(k) |\sin t|^{-d/2} \exp \left( i\frac{x^2 + y^2}{2\tan t} - i\frac{x \cdot y}{\sin t} \right) & (\pi k < t < \pi(k+1)) \\ c'(k) \delta((-1)^k x - y) & (t = k\pi) \end{cases}$$

for suitable phase factors $c(k), c'(k) \in \mathbb{C}$. Notice then that for $t = \pi/2$ the propagator $U(t)$ coincides with the ordinary Fourier transform up to constant factors. The following result implies the sharpness of Theorem 3.

Theorem 6 ([55, Prop. 7.1]) Let $1 < p < \infty$ and $k_1, k_2 \in \mathbb{R}$. The Fourier transform is bounded $L^p_{k_1}(\mathbb{R}^d) \to L^p_{k_2}(\mathbb{R}^d)$ if and only if

$$k_1 \geq 2d(1/p - 1/2), \quad k_2 \leq 0 \quad (1 < p \leq 2),$$

$$k_1 \geq 0, \quad k_2 \leq -2d(1/2 - 1/p) \quad (2 \leq p < \infty).$$

5 Higher order rough parametrices

Let us come back to Fujiwara’s main result, Theorem 1, to make some important remarks. First, the occurrence of convergence results at two different levels, a coarser one (parametrices in $B(L^2(\mathbb{R}^d))$) and a finer one (OIO amplitudes in $C^\infty_b(\mathbb{R}^{2d})$), suggests that the assumptions may be relaxed in order to preserve convergence in operator norm - we will devote the subsequent section to the convergence problem for integral kernels. A first step in this direction is the aforementioned paper [56] by Nicola, where a delicate analysis of low-regular potentials leads to the desired
A time-frequency analysis perspective on Feynman path integrals

result. We are now going to consider another class of non-smooth potentials as in [57], inspired again by time-frequency analysis function spaces. Assumption (B). \( V(t, x) \) is a real-valued function of \( (t, x) \in \mathbb{R} \times \mathbb{R}^d \) and there exists \( N \in \mathbb{N}, N \geq 1 \), such that

\[
\partial_t^k \partial_x^\alpha V \in C^0_b(\mathbb{R}, M^{\alpha,1}(\mathbb{R}^d)),
\]

for any \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^d \) satisfying \( 2k + |\alpha| \leq 2N \).

The modulation space \( M^{\alpha,1}(\mathbb{R}^d) \) is also known as the Sjöstrand class, since it was first introduced by Sjöstrand in [65] as an exotic class of symbols still yielding bounded pseudodifferential operators on \( L^2(\mathbb{R}^d) \). It was later established that symbols in this space associate with bounded operators on any modulation space and enjoy a rich operator-algebraic structure [31, 32]. As a rule of thumb, a function in \( M^{\alpha,1}(\mathbb{R}^d) \) is bounded and continuous on \( \mathbb{R}^d \); see Section 6 for further details on the regularity of these functions. Roughly speaking, potentials satisfying Assumption (B) are bounded continuous functions together with a certain number of derivatives. Assumptions in the same spirit, or even stronger, are quite popular in scattering theory [52].

In the second place, the estimate (12) reveals other interesting aspects of the parametrices \( E^{(N)} \). In particular, notice that while the approximation power increases with \( N \) from the point of view of semiclassical analysis (positive powers of \( \hbar \)), the rate of convergence with respect to the length of the time interval does not enjoy any improvement. Moreover, sophisticated parametrices like those introduced in (9) have limited applicability to concrete situations and computational problems since the knowledge of the exact action functional is required, the latter being an intractable problem except for a number of simple systems. These remarks lead one to consider short-time approximations for the action by means of the so-called midpoint rules. In short, given the action functional corresponding to the standard Hamiltonian \( H(q, p, t) = p^2/2 + V(q) \), that is

\[
S(t, s, x, y) = \frac{|x - y|^2}{2(t - s)} - V(t, s, x, y), \quad \mathcal{V} = \int_s^t V(\gamma(\tau))d\tau,
\]

the latter integral involving paths with \( \gamma(s) = y \) and \( \gamma(t) = x \), \( \mathcal{V} \) is replaced with approximate expressions such as

\[
\mathcal{V}_1 = \frac{V(x) + V(y)}{2}(t - s), \quad \text{or} \quad \mathcal{V}_2 = V\left(\frac{x + y}{2}\right)(t - s).
\]

A simple test in the case of known models reveals that, in spite of their popularity within the physics literature, these procedures are not sufficiently accurate. For the harmonic oscillator and the corresponding approximate actions \( S_1, S_2 \) one has indeed

\[
S(t, s, x, y) - S_j(t, s, x, y) = O(t - s), \quad j = 1, 2.
\]

\(^3\) We denote by \( C^0_b(\mathbb{R}, X) \) the space of continuous and bounded functions \( \mathbb{R} \to X \).
The quest for a correct short-time approximation was initiated by Makri and Miller \cite{47, 48, 49}, leading to the rule
\[
\overline{V}(s, x, y) = \int_0^1 V(\tau x + (1 - \tau)y, s) d\tau.
\]

This procedure satisfies a correct first-order approximation, i.e. \( S(t, s, x, y) = \overline{S}(t, s, x, y) = O((t - s)^2) \) for small \( t - s \). We refer the interested reader to the aforementioned papers and the recent one \cite{17} by de Gosson.

Inspired by this discussion and by the current practice in physics and chemistry we consider different time slicing approximation operators than \( \ref{9} \), namely
\[
\overline{E}^{(N)}(t, s)\varphi(x) = \frac{1}{(2\pi i(t - s))^{d/2}} \int_{\mathbb{R}^d} e^{i\overline{S}^{(N)}(t, s, x, y)} \varphi(y) dy,
\]
where the approximate action \( S^{(N)} \) is essentially a Taylor-like expansion of the exact action \( S \) at \( t = s \):
\[
S^{(N)}(t, s, x, y) = \frac{|x - y|^2}{2(t - s)} + \sum_{k=1}^{N} W_k(s, x, y)(t - s)^k. \tag{19}
\]

The coefficients \( W_k(s, x, y) \) are recursively constructed after careful analysis of power series solutions for the modified Hamilton-Jacobi equation
\[
\frac{\partial S}{\partial t} + \frac{1}{2} |\nabla_x S|^2 + V(t, x) + \frac{ihd}{2(t - s)} - \frac{ih}{2} \Delta_x S = 0.
\]

The last two terms are tailored to enhance the approximating power of \( \overline{E}^{(N)} \) as parametrix, as showed below. Nevertheless, the “counterterm” is first order in \( \hbar \) and identically vanishes in the free particle case (\( V = 0 \)). Plus, we remark that \( W_1(s, x, y) = \overline{V}(s, x, y) \) as expected.

The main properties of these parametrices are summarized below - proofs can be found in \cite{57}.

**Theorem 7** Let \( V \) satisfy Assumption (B) above and let \( t, s, T > 0 \) be such that \( 0 < t - s \leq Th, 0 < h \leq 1 \).

1. There exists \( C = C(T) > 0 \) such that \( \|\overline{E}^{(N)}(t, s)\|_{L^2 \to L^2} \leq C \). Moreover, \( \overline{E}^{(N)}(t, s) \to I \) (identity op.) for \( t \to s \) in the strong operator topology on \( L^2 \).
2. \( \overline{E}^{(N)}(t, s) \) is a parametrix in the sense that
\[
\left(ih\partial_t + \frac{1}{2}h^2 \Delta - V(t, x)\right)\overline{E}^{(N)}(t, s) = G^{(N)}(t, s),
\]
where
\[
G^{(N)}(t, s)f = \frac{1}{(2\pi i(t - s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{i\overline{S}^{(N)}(t, s, x, y)} g_N(t, s, x, y)f(y) dy,
\]
where the amplitude $g_N$ satisfies
\[
\|g_N(t, s, \cdot)\|_{M^{-1}(\mathbb{R}^{2d})} \leq C (t - s)^N,
\]
for some $C = C(T) > 0$.

3. There exists a constant $C = C(T) > 0$ such that
\[
\|\tilde{E}^{(N)}(t, s) - U(t, s)\|_{L^2 \to L^2} \leq C\hbar^{-1}(t - s)^{N+1}.
\]  
(20)

These estimates should be compared with those appearing in Section 3.2. Similarly, given a subdivision $\Omega = t_0, \ldots, t_L$ of the interval $[s, t]$ such that $s = t_0 < t_1 < \ldots < t_L = t$, we introduce the long-time composition
\[
\tilde{E}^{(N)}(\Omega, t, s) = \tilde{E}^{(N)}(t_L, t_{L-1})\tilde{E}^{(N)}(t_{L-1}, t_{L-2}) \cdots \tilde{E}^{(N)}(t_1, t_0),
\]
and the main result in [57] reads as follows.

**Theorem 8 ([57, Thm. 1])** Let $V$ satisfy Assumption (B) above. For any $T > 0$ there exists a constant $C = C(T) > 0$ such that, for $0 < t - s \leq T\hbar$, $0 < \hbar \leq 1$, and any sufficiently fine subdivision $\Omega$ of the interval $[s, t]$, we have
\[
\|\tilde{E}^{(N)}(\Omega, t, s) - U(t, s)\|_{L^2 \to L^2} \leq C\omega(\Omega)^N.
\]  
(21)

The proof of Theorem 8 is largely inspired by the proof of Theorem 1. In fact, one can isolate a strategy of general interest which can be applied to suitable operators, cf. [57] Thm. 10.

**The role of $\hbar$**

We already remarked that Birkhoff-Maslov parametrices enjoy several nice properties, one of them being an increasing semiclassical approximation power - the exponent of $\hbar$ in (12) increases with $N$. This is of course related to the construction of the parametrices, relying on piecewise classical paths. This desirable property is lost when one considers rougher parametrices as those in (18), where the balance weights in favour of accelerated rate of convergence with respect to time. A cursory glance at the estimates for the operators $\tilde{E}^{(N)}$ in Theorem 7 reveals that negative powers of $\hbar$ are involved, making them completely unfit for semiclassical arguments. Nevertheless, one can also notice that all the estimates are uniform in $\hbar$ as soon as time is measured in units of $\hbar$, which is a particularly interesting feature.
The role of $M^{\infty,1}$

Although being hidden in the details of the proofs, the role of the Sjöstrand class $M^{\infty,1}(\mathbb{R}^d)$ is crucial for the results presented insofar. There is in particular a special feature of this space playing a major role in the arguments, namely the fact that it is a commutative Banach algebra under pointwise product. In general, precise conditions on $p$, $q$, $s$ and $d$ must hold in order for $M^{p,q}(\mathbb{R}^d)$ to be a Banach algebra with respect to pointwise multiplication.

**Proposition 1 ([61] Thm. 3.5 and Cor. 2.10)** Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. The following facts are equivalent.

(i) $M^{p,q}_s(\mathbb{R}^d)$ is a Banach algebra for pointwise multiplication.

(ii) $M^{p,q}_s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.

(iii) Either $s = 0$ and $q = 1$ or $s > d / q'$.

6 Pointwise convergence of integral kernels

A concise way to resume the philosophy behind the operator-theoretic approaches to rigorous path integral discussed in Section 3 could be the following one: design suitable sequences of approximation operators and prove that they are bounded together with their compositions, where the latter should converge to the exact propagator in a suitable topology on $\mathcal{B}(L^2(\mathbb{R}^d))$. There are good reasons for not being completely satisfied with this state of affairs. First of all, looking back at Feynman’s original paper [23] and the textbook [25] one immediately notices that the entire process of defining path integrals in (2) can be read in terms of a sequence of integral operators (finite-dimensional approximation operators as in (5) or (9)); in particular, Feynman’s insight calls for the pointwise convergence of their integral kernels to the kernel $u_\tau$ of the propagator. This remark strongly motivates a focus shift from the operators to their kernels, which may appear as an unaffordable problem in general: approximation operators should be first explicitly characterized as integral operators, at least in the sense of distributions by some version of Schwartz’s kernel theorem, then one should determine if the kernels are in fact functions and finally hope for convergence. Both the approximation schemes discussed insofar are well suited for this purpose, since oscillatory integrals are explicitly involved. A clue in this direction, already mentioned at the beginning of the previous section, is that the regularity assumptions in Theorem 1 imply short-time convergence in a finer topology at the level of integral kernels.

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To be precise, the result provided here concerns conditions under which the embedding $M^{p,q}_s(\mathbb{R}^d) \hookrightarrow M^{p,q}_s(\mathbb{R}^d)$ is continuous; this means that the algebra property eventually holds up to a constant. It is well known that one may provide an equivalent norm for which the boundedness estimate holds with $C = 1$ (cf. [62] Thm. 10.2)]. This condition will be tacitly assumed whenever concerned with Banach algebras from now on.
The solution of this problem in the framework of the sequential approach as presented in Section 3.1 was recently obtained by the author and Nicola in the paper [58], where techniques of time-frequency analysis of functions and operators are heavily used. We need a few preparation in order to state the main result.

**A word of warning about notation.** In this section we restore the “harmonic analysis” normalization of the Fourier transform with \(2\pi\) in the phase factor. This reflects into the definition of Weyl, Wigner and short-time Fourier transforms, in contrast with the “PDE” normalization adopted insofar. We are sorry if this choice may cause confusion but the aim is to clean up the relevant formulas from annoying normalization constants. For the same reason we set \(\hbar = 1\) from now on.

### 6.1 Weyl operators

A summary of the fruitful exchange between analysis of pseudodifferential operators and time-frequency analysis is far beyond the purposes of this note. The crucial point of contact is represented by the Wigner distribution

\[
W(f, g)(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot \xi} f\left(x + \frac{y}{2}\right) g\left(x - \frac{y}{2}\right) dy, \quad f, g \in \mathcal{S}(\mathbb{R}^d),
\]

which is a well-known phase space transform deeply connected with the STFT [30, 18]. We define the Weyl transform

\[
\sigma_w : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)
\]

of the symbol \(\sigma \in \mathcal{S}'(\mathbb{R}^{2d})\) by duality as follows:

\[
\langle \sigma_w f, g \rangle = \langle \sigma, W(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d).
\] (22)

As an elementary example of Weyl operator consider the multiplication by \(V(x)\), whose symbol is trivially given by

\[
\sigma_V(x, \xi) = V(x) = (V \otimes 1)(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d}.
\]

The composition of Weyl transforms induces a bilinear form on symbols, the so-called twisted/Weyl product: \(\sigma_w \circ \rho_w = (\sigma \# \rho)_w\).

Explicit formulas for the twisted product of are known (cf. [70]) but we are more interested in the algebra structure induced on symbol spaces. It turns out that the Sjöstrand \(M^\infty,1(\mathbb{R}^{2d})\), as well as the family of modulation spaces \(M^\infty_s(\mathbb{R}^{2d})\) with \(s > 2d\), enjoy a peculiar double Banach algebra structure:

- a commutative one associated with pointwise multiplication, as a consequence of Proposition 1
- a non-commutative one associated with the Weyl product of symbols ([33, 65]).

The latter algebra structure has been thoroughly investigated in view of its role in the distinctive sparse behaviour satisfied by pseudodifferential operators with symbols in those spaces - the so called almost diagonalization property with respect
to time-frequency shifts. Having in mind the Gabor matrix defined in (13), it can be proved that $\sigma \in M^\infty (\mathbb{R}^d)$ if and only if, for some (hence any) $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$,

$$M(\sigma^w, g, z, w) \leq C (w-z)^{-s}, \quad z, w \in \mathbb{R}^d.$$  

Similarly, $\sigma \in M^{\infty, 1}(\mathbb{R}^d)$ if and only if there exists $H \in L^1(\mathbb{R}^d)$ such that

$$M(\sigma^w, g, z, w) \leq H (w-z), \quad z, w \in \mathbb{R}^d.$$  

The consequences of phase space sparsity have been thoroughly studied in the papers [11, 12, 13, 15, 32, 33], mainly in order to extend Sjöstrand’s theory of Wiener subalgebras of Weyl operators [65] to more general pseudodifferential and Fourier integral operators.

6.2 Main results

In order to state the main results in full generality we need to slightly generalize the free Hamiltonian operator $H_0$ in (1). Let $a$ be a real-valued, time-independent, quadratic homogeneous polynomial on $\mathbb{R}^d$, namely

$$a(x, \xi) = \frac{1}{2} x^T A x + \xi^T B x + \frac{1}{2} \xi^T C \xi,$$

for some symmetric matrices $A, C \in \mathbb{R}^{d \times d}$ and $B \in \mathbb{R}^{d \times d}$. Consider then the Weyl quantization of $a$ as above. A classical result in phase space harmonic analysis (see [18, Sec. 15.1.3] and also [14, 26]) is that the solution of (1) with $H_0 = a^w$ and $V = 0$ is given by

$$\psi(t, x) = e^{-it H_0} \varphi(x) = \mu(\mathcal{A}_t) \varphi(x),$$

where $\mu(\mathcal{A}_t)$ is a metaplectic operator, designed as follows. First, the classical phase space flow governed by the Hamilton equations

$$2\pi \dot{z} = J \nabla_{\xi} a(z) = \mathcal{A}_t \mathcal{A}_t = \begin{pmatrix} B & C \\ -A & -B^T \end{pmatrix} \in sp(d, \mathbb{R}),$$

defines a mapping

$$\mathbb{R} \ni t \mapsto \mathcal{A}_t = e^{it/2\pi J} \mathcal{A}_t e^{it/2\pi J} \in Sp(d, \mathbb{R}).$$

In very sloppy terms, the metaplectic map $\mu$ is a double-valued unitary representation of the symplectic group on $L^2(\mathbb{R}^d)$, hence the classical flow $\mathcal{A}_t$ is “lifted” to a family of unitary operators on $L^2(\mathbb{R}^d)$. Under certain circumstances an explicit characterization for $\mu(\mathcal{A}_t)$ can be provided: for all $t \in \mathbb{R}$ such that $\mathcal{A}_t$ is a free symplectic

\[ \text{The factor } 2\pi \text{ derives from the normalization of the Fourier transform adopted in this section.} \]
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matrix, namely such that the upper-right block $B_t$ is invertible, the corresponding metaplectic operator is a quadratic Fourier transform - cf. [18, Sec. 7.2.2]:

$$\mu(\mathcal{A}_t)\phi(x) = c_t |\det B_t|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_t(x,y)} \phi(y) dy, \quad \phi \in \mathcal{S}(\mathbb{R}^d),$$

(24)

for some $c_t \in \mathbb{C}$, $|c_t| = 1$, where

$$\Phi_t(x,y) = \frac{1}{2} x D_t B_t^{-1} x - y B_t^{-1} x + \frac{1}{2} y B_t^{-1} A_t y, \quad x, y \in \mathbb{R}^d.$$  

(25)

It is known that $H_0$ is a self-adjoint operator on its domain (see [55])

$$D(H_0) = \{\psi \in L^2(\mathbb{R}^d) : H_0 \psi \in L^2(\mathbb{R}^d)\}.$$  

In order for the machinery developed in Section 5.1 to hold in the case where $H_0 = a^w$ as above we need to consider a version of Trotter formula which holds for semigroups in more general frameworks (cf. for instance [19, Cor. 2.7]). For our purposes, it is enough to assume that $V$ is a bounded perturbation of $H_0$, namely $V \in \mathcal{B}(L^2(\mathbb{R}^d))$; notice that $V \in L^\infty(\mathbb{R}^d)$ is then a suitable choice, hence including complex-valued potentials.

Therefore, under the hypotheses on $H_0$ and $V$ discussed insofar, we have

$$e^{-it(H_0+V)} = \lim_{n \to \infty} E_n(t), \quad E_n(t) = \left(e^{-i\frac{t}{t_n}H_0} e^{-i\frac{t}{t_n}V}\right)^n,$$

(26)

with convergence in the strong operator topology in $L^2(\mathbb{R}^d)$. Let us denote by $e_{n,t}(x,y)$ the distribution kernel of $E_n(t)$ and by $u_{n,t}(x,y)$ that of $U(t) = e^{-it(H_0+V)}$.

We assumed $V \in L^\infty(\mathbb{R}^d)$, hence there is some room left for tuning the regularity. A suitable choice is given by the modulation spaces $M^{s,1}(\mathbb{R}^d)$, $s > d$, and $M^{s,1}(\mathbb{R}^d)$, also in view of the rich algebraic structure already discussed.

In order to grasp the regularity of functions in this space, recall the definition of the Fourier-Lebesgue space: for $s \in \mathbb{R}$ we set

$$f \in \mathcal{F}L^1_s(\mathbb{R}^d) \iff \|f\|_{\mathcal{F}L^1_s} = \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)| |\xi|^s d\xi < \infty.$$  

The analytic properties of the involved potentials are briefly collected in the following result.

**Proposition 2**  
1. $\cap_{s>0} M^{s,1}(\mathbb{R}^d) = C^0(\mathbb{R}^d)$.
2. $M^{s,1}(\mathbb{R}^d) \subset M^{s,1}(\mathbb{R}^d)$ for $s > d$.
3. $M^{s,1}(\mathbb{R}^d) \subset (\mathcal{F}L^1)_{\text{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
4. $(M^{s,1})_{\text{loc}}(\mathbb{R}^d) = (\mathcal{F}L^1)_{\text{loc}}(\mathbb{R}^d)$.
5. $\mathcal{F}M(\mathbb{R}^d) \subset M^{s,1}(\mathbb{R}^d)$, where $\mathcal{F}M(\mathbb{R}^d)$ is the space of Fourier transforms of (finite) complex measures on $\mathbb{R}^d$.

Roughly speaking, we have a scale of decreasing regularity spaces.
1. The first is “the best of all possible worlds”, that is \( C_b^\infty(\mathbb{R}^d) \).

2. At an intermediate stage we have the scale of modulation spaces \( M^s(\mathbb{R}^d) \), 
   \( s > d \), populated by bounded continuous functions with decreasing (fractional) regularity as \( s \searrow d \).

3. Finally, we have the maximal space \( M^{s_1,1}(\mathbb{R}^d) \) where fractional differentiability is completely lost. Nevertheless it is a space of bounded continuous functions locally enjoying the mild regularity of a function in \( \mathcal{F}L^1 \).

We first present our main result for potentials in \( M^s(\mathbb{R}^d) \).

**Theorem 9** Let \( H_0 = a^w \) as discussed above and \( V \in M^s(\mathbb{R}^d) \), with \( s > 2d \). Let \( \mathcal{A}_t \) denote the classical flow associated with \( H_0 \) as in (23). For any \( t \in \mathbb{R} \) such that \( \mathcal{A}_t \) is free, that is \( \det B_t \neq 0 \):

1. the distributions \( e^{-2\pi i \Phi_t} e_{n,t}, n \geq 1 \), and \( e^{-2\pi i \Phi_t} u_t \) belong to a bounded subset of \( M^s(\mathbb{R}^{2d}) \);
2. \( e_{n,t} \rightharpoonup u_t \) in \( (\mathcal{F}L^1)_{\text{loc}}(\mathbb{R}^{2d}) \) for any \( 0 < r < s - 2d \), hence uniformly on compact subsets.

The first part of the claim assures that the kernel convergence problem is well posed in this case - the “amplitudes” are bounded continuous functions. The second part precisely characterizes the regularity at which convergence occurs, hence the desired pointwise convergence.

In view of the first item in Proposition 2 and the related characterization \( C^\infty(\mathbb{R}^{2d}) = \cap_{r>0} (\mathcal{F}L^1)_{\text{loc}}(\mathbb{R}^{2d}) \), we expect to improve the convergence result in the smooth scenario.

**Corollary 1** Let \( H_0 = a^w \) as discussed above and \( V \in C_b^\infty(\mathbb{R}^d) \). Let \( \mathcal{A}_t \) denote the classical flow associated with \( H_0 \) as in (23). For any \( t \in \mathbb{R} \) such that \( \mathcal{A}_t \) is free, that is \( \det B_t \neq 0 \):

1. the distributions \( e^{-2\pi i \Phi_t} e_{n,t}, n \geq 1 \), and \( e^{-2\pi i \Phi_t} u_t \) belong to a bounded subset of \( C_b(\mathbb{R}^{2d}) \);
2. \( e_{n,t} \rightharpoonup u_t \) in \( C^\infty(\mathbb{R}^{2d}) \), hence uniformly on compact subsets together with any derivatives.

This result should be compared with the second claim in Theorem 1 by Fujiwara, which motivated our quest. In spite of the different assumptions and approximation schemes, we stress that our result is almost global in time - more on exceptional times below.

We conclude with the analogous convergence result for potentials in the Sjöstrand class.

**Theorem 10** Let \( H_0 = a^w \) as discussed above and \( V \in M^{s_1,1}(\mathbb{R}^d) \). Let \( \mathcal{A}_t \) denote the classical flow associated with \( H_0 \) as in (23). For any \( t \in \mathbb{R} \) such that \( \mathcal{A}_t \) is free, that is \( \det B_t \neq 0 \):

1. the distributions \( e^{-2\pi i \Phi_t} e_{n,t}, n \geq 1 \), and \( e^{-2\pi i \Phi_t} u_t \) belong to a bounded subset of \( M^{s_1,1}(\mathbb{R}^{2d}) \);
2. \( e_{n,t} \to u_t \) in \( (FL^1)_{\text{loc}}(\mathbb{R}^{2d}) \), hence uniformly on compact subsets.

It seems appropriate to highlight that a typical potential setting in the papers by Albeverio and coauthors [1,2,3,5] and Itô [39,40] is “harmonic oscillator plus a bounded perturbation”, the latter in the form of the Fourier transform of a finite complex measure on \( \mathbb{R}^d \). While the cited references rely on completely different mathematical schemes for path integrals (which are manufactured as infinite-dimensional OIOs), in view of the embedding \( FM(\mathbb{R}^d) \subset M^\infty(\mathbb{R}^d) \) mentioned in Proposition 2 we are able to encompass this class of potentials too.

6.3 The proof at a glance

In order to understand why our choice of modulation spaces is suitable for the purpose of pointwise convergence we outline the general strategy of the proof of Theorem 9. The first step is to express the parametrix \( E_n(t) \) in integral form and derive a manageable form of the kernel \( e_{n,t} \). The algebra property of \( M^\infty(\mathbb{R}^d) \) will play a crucial role from now on. First, we are able to write

\[
E_n(t) = \left( e^{-i\frac{t}{n}H_0}e^{-i\frac{t}{n}V_0} \right)^n = \left( \mu(\mathcal{A}_{t/n}) \left( 1 + i\frac{t}{n}V_0 \right) \right)^n
\]

for a suitable \( V_0 = V_{0,n,t} \in M^\infty(\mathbb{R}^d) \) - see [58, Lem. 3.2]. We now expand the (ordered) product and identify multiplication by \( 1 + i\frac{t}{n}V_0 \) with a suitable Weyl operator, then the nice intertwining properties of Weyl and metaplectic operators (the so-called symplectic covariance of Weyl calculus, cf. [18, Thm. 215]) yield

\[
E_n(t) = \left[ \prod_{k=1}^n \left( I + i\frac{t}{n} \left( \sigma_{V_0} \circ \mathcal{A}_{-k\frac{t}{n}} \right) \right)^{w_{k/n}} \right] \mu(\mathcal{A}_{t/n})^n = a^{w}_{n,t} \mu(\mathcal{A}_t),
\]

where the first (ordered) product is understood in the Banach algebra \( (M^\infty(\mathbb{R}^d), \#) \). The symbol of \( a^{w}_{n,t} \) satisfies the estimate \( \|a^{w}_{n,t}\|_{M^\infty(\mathbb{R}^d)} \leq e^{C(t)}t \) for some locally bounded constant \( C(t) > 0 \) independent of \( n \).

Since \( \mathcal{A}_t \) is a free symplectic matrix, the integral formula (24) holds and with the help of some technical lemmas we are able to precisely characterize the integral kernels \( e_{n,t} \) and \( u_t \) as temperate distributions. The non-trivial step is to prove convergence in \( S'(\mathbb{R}^{2d}) \), but it can be handled with Banach algebras techniques and some topological arguments. The assumptions on potentials imply the boundedness of the sequence \( \{e_{n,t}\} \) in \( M^\infty(\mathbb{R}^{2d}) \). Finally, the convergence of \( e_{n,t} \) to \( u_t \) in \( (FL^1)_{\text{loc}}(\mathbb{R}^{2d}) \), \( 0 < r < s - 2d \), essentially follow by dominated convergence arguments.

The proof of Theorem 10 ultimately moves along the same lines but is more involved, so we will not give the details here. The basic ingredient is a “high-cut filter decomposition” of \( M^\infty(\mathbb{R}^d) \), see [58, Lem. 3.3]: a rough function \( f \in M^\infty(\mathbb{R}^d) \) can be split as the sum of a very regular part \( f_1 \in C^\infty(\mathbb{R}^d) \) plus an arbitrarily small
(in norm) rough remainder $f_2 \in M^{\infty,1}(\mathbb{R}^d)$. Theorem 10 essentially follows as a perturbation of Theorem 9 after a careful management of these remainders.

6.4 Why exceptional times?

The occurrence of a set of exceptional times in Theorems 9 and 10 is to be expected from a mathematical point of view: it may happen that the integral kernel of the propagator degenerates into a distribution. A well-known example of this behaviour is provided by the harmonic oscillator, already met in Section 4.3. Mehler’s formula (16) precisely shows the expected degenerate behaviour, which is consistent with the fact that $\mathcal{A}_t$ is free if and only if $t \neq k\pi$, $k \in \mathbb{Z}$, since $B_t = (\sin t)I_{d\times d}$ (up to normalization constants). The physical interpretation of the exceptional values is not entirely clear at the moment, but milder convergence results in the spirit of the theorems above may be proved to hold also at exceptional times. These and other related issues will be object of forthcoming contributions [20].

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