VECTOR SPACES WITH A DENSE-CODENSE GENERIC SUBMODULE

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Abstract. We study expansions of a vector space $V$ over a field $\mathbb{F}$, possibly with extra structure, with a generic submodule over a subring of $\mathbb{F}$. We construct a natural expansion by existentially defined functions so that the expansion in the extended language satisfies quantifier elimination. We show that this expansion preserves tame model theoretic properties such as stability, NIP, NTP$_1$, NTP$_2$ and NSOP$_1$. We also study induced independence relations in the expansion.

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References

1. Introduction

This paper brings together ideas of dense-codense expansions of geometric structures [7, 4, 9] with ideas about generic expansions by groups by D’Elbée [16]. Our base structures are vector spaces over a fixed field $\mathbb{F}$, possibly with extra structure, such that the algebraic closure agrees with the $\mathbb{F}$-linear span. We then fix a subring $R$ of $\mathbb{F}$ and we study expansions by additive $R$-submodules satisfying some form of genericity (for technical details, see the definition of $T_U$ and $T^G$ in Definition 3.3), the main goal is to see how tame model-theoretic properties transfer from the original structure to the expansion.

There are many papers that deal with expansions by predicates and preservation of tame properties. There are general approaches [11, 14] that study stable or NIP
structures expanded by predicates, the main idea being that if the induced structure on the predicate is stable/NIP and if the formulas in the expansion are equivalent to bounded formulas (i.e., where the quantifiers range over the predicate), then the pair is again stable/NIP. On the other hand one can start with a geometric structure \([7, 4, 9]\) and study preservation of properties like NTP\(_2\), strong dependence, supersimplicity or NSOP\(_1\) under expansions by well-behaved dense codense predicates (for example, the predicate being an elementary substructure \([4]\), a collection of algebraically independent elements \([7]\) or a multiplicative substructure with the Mordell-Lang property \([9, 23, 5]\)). This approach shares some ingredients with the previous one, formulas in the expanded language are equivalent to bounded formulas and the density property implies that the induced structure on the predicate is tame. One can also study different generic expansions, a classical example is the generic predicate, which preserves simplicity (see \([12]\)). A more general construction is due to Winkler \([29]\) in his thesis: given a model-complete \(L\)-theory \(T\), and a language \(L' \supset L\), the theory \(T\) can be considered as an \(L'\)-theory which has a model-companion, provided \(T\) has elimination of \(\exists^\infty\). Winkler \([29]\) also considered the expansion of a theory by generic Skolem functions. Both expansions of Winkler were later shown to preserve the property NSOP\(_1\) \(([24], [25])\). One can also consider the expansion of a theory by a predicate for a reduct of this theory, for instance expanding a theory of fields by an additive or multiplicative generic subgroup (see \([16, 18, 8]\)).

We start this paper (Section 2) by studying the expansion of a theory of \(F\)-vector space (in which the algebraic closure is the vector span) by a predicate for a generic \(R\)-submodule, where \(R\) stands for an integral domain such that \(F = \text{Frac}(R) = \text{the fraction field of } R\). After adding predicates for pp-formulas in the language of \(R\)-modules, we characterize the expansions that are existentially closed, prove the existence of a model companion \(T^G\) and in doing so show quantifier elimination for the expansion. As a Corollary of quantifier elimination, we show that \(T^G\) is stable (resp. NIP) whenever \(T\) is stable (resp. NIP).

In Section 3, we study the general case where we drop the assumption \(F = \text{Frac}(R)\). We construct a natural expansion by existentially defined functions so that the expansion in the extended language satisfies quantifier elimination and we prove a model-completeness result. In Section 4 we use our description of definable sets to show that NIP and stability are preserved under these expansions. We would like to point out that these preservation results can be proved using the approach from \([9]\).

In Section 5, we prove the main results of this paper: preservation of NTP\(_1\), NTP\(_2\) and NSOP\(_1\) in the expansion. The approach we follow is to check the property by doing a formula-by-formula analysis, separating the cases when the corresponding definable set is small (algebraic over the predicate) or large. To show the preservation of NSOP\(_1\) we build on ideas presented in \([26]\), the proofs for the preservation of NTP\(_1\), NTP\(_2\) generalize ideas presented in \([3, 20]\). In Section 6, we study independence notions in the expansion, assuming the original theory has a good notion of independence. As a Corollary, we give another proof for the preservation of simplicity assuming \(T\) has \(SU\)-rank one and \(F = \text{Frac}(R)\). In Section 7, we introduce a candidate for an example of a non-simple NSOP\(_1\) pregeometric theory with modular pregeometry.
2. A first example: the case $\mathbb{F} = \text{Frac}(R)$

Let $F$ be a field and let $R$ be a subring of $F$ such that $F = \text{Frac}(R)$ (the fraction field of $R$). Let $L_0 = \{(\lambda)_{\lambda \in \mathbb{F}}, +, 0\}$ be the language of vector spaces over $F$ and let $L = \{+, 0, \{\lambda\}_{\lambda \in \mathbb{F}}, \ldots\}$ be an extension. Let $T$ be a complete $L$-theory that expands the theory of vector spaces over $F$ which has quantifier elimination in $L$ and that satisfies the following properties:

(i) Whenever $M \models T$ and $\bar{a} \in M^n$, $\text{dcl}(\bar{a}) = \text{span}_F(\bar{a})$.

(ii) It eliminates the quantifier $\exists$.

Let $G$ be a unary predicate and for each formula $\phi(\bar{x})$ in the language $L_{R-\text{mod}} = \{+, 0, \langle r \cdot \rangle_{r \in R}\}$ of $R$-modules, let $P_\phi(\bar{x})$ be a new predicate. Let $L_G$ be the expansion of $L$ by $G$ and $P_\phi$ for all formulas $\phi$ in $L_{R-\text{mod}}$. We will consider pairs $(V, G)$ in the language $L_G$ that satisfy the following first order conditions:

(A) $G$ is an $R$-module and for all formulas $\phi(\bar{x})$ in the language of $R$-modules,

$$\forall \bar{x}(P_\phi(\bar{x}) \rightarrow G(\bar{x})) \land \forall \bar{x}(G(\bar{x}) \leftrightarrow (P_\phi(\bar{x}) \leftrightarrow \hat{\phi}(\bar{x}))),$$

where $\hat{\phi}(\bar{x})$ is the relativization of $\phi(\bar{x})$ to $G$ and we interpret the relativization seeing $G$ as an $R$-module.

(B) (For all $r \in R \setminus \{0\}$, $rG$ is dense in $V$). For every $L$-formula $\phi(x, \bar{y})$, the axiom $\exists^G \phi(x, \bar{y}) \rightarrow \exists \bar{x}(\phi(x, \bar{y}) \land rG(\bar{x}))$;

(C) (Extension/co-density property) for any $L$-formulas $\phi(x, \bar{y})$ and $\psi(x, \bar{y}, \bar{z})$ and $n \geq 1$, the axiom

$$(\exists^G \phi(x, \bar{y}) \land \forall \bar{z} \exists \bar{x}^n \psi(x, \bar{y}, \bar{z})) \rightarrow \exists \bar{x}(\phi(x, \bar{y}) \land \forall \bar{z}(G(\bar{z}) \rightarrow \neg \psi(x, \bar{y}, \bar{z}))).$$

Let $T^G$ be the $L_G$ theory satisfying the schemes (A), (B) and (C). In all models $(V, G)$ under consideration, the predicate $G$ will interpret an $R$-module. Since modules have quantifier elimination up to boolean combinations of pp-formulas (see [30]), we can always assume that the predicates $P_\phi(\bar{x})$ are only defined for positive and negative instances of pp-formulas. Throughout this paper we will assume the reader is familiar with basic properties of pp-formulas inside $R$-modules.

**Notation 2.1.** In what follows, whenever $(V, G) \models T^G$ and $A \subseteq V$, we will write $G(A)$ for $A \cap G(V)$.

Our first goal is to show that the theory $T^G$ has quantifier elimination. We will start by proving properties of the divisible elements and the $L$-terms.

**Lemma 2.2** (Density of $G^{\text{div}}$). Let $(V, G)$ be an $|\mathbb{F}|^+$-saturated model of $T^G$. Let $B \subseteq V$ such that $|B| \leq |\mathbb{F}|$ and $p(x)$ be a consistent non-algebraic $L$-type in a single variable over $B$. Let $G^{\text{div}} = \bigcap_{r \in R \setminus \{0\}} rG$. Then $p(V) \cap G^{\text{div}}(V)$ is infinite.

**Proof.** By compactness, it is sufficient to show that for all $r_1, \ldots, r_n \in R$, $p(x)$ has infinitely many realisations in $\bigcap_{i=1}^n r_i G$. Let $r = r_1 \ldots r_n$. Now note that by axiom (B), the type $rG(x) \land p(x)$ has infinitely many realisations. \hfill $\square$

**Lemma 2.3.** Let $t(\bar{x})$ be an $L$-term, then there exists $L$-formulas $\theta_1(\bar{x}), \ldots, \theta_n(\bar{x})$ forming a partition of the universe, (i.e. such that $V \models \forall \bar{x}(\bigvee_i \theta_i(\bar{x})) \land \bigwedge_{i \neq j} \neg \exists \bar{x} \theta_i(\bar{x}) \land \theta_j(\bar{x}))$, and $L_0$-terms $t_1(\bar{x}), \ldots, t_n(\bar{x})$ such that

$$t(\bar{x}) = y \leftrightarrow \bigvee_i t_i(\bar{x}) = y \land \theta_i(\bar{x}).$$

**Proof.** Since $T$ satisfies property $(i)$, the algebraic closure agrees with the $F$-span, hence $\{t(\bar{x}) = y) \cup \{\bar{x} : \bar{x} \neq y | \bar{x} \in F^n\}$ is inconsistent. Thus we can find $\hat{x}^1, \ldots, \hat{x}^n \in F$
\( F^n \) such that \( t(\vec{x}) = y \rightarrow \bigwedge_i \vec{x}^i \cdot \vec{x} = y \). We may assume that \( \vec{x}^i \cdot \vec{x} = y \) define disjoint vector spaces. Now choose \( t_i(\vec{x}) = \vec{x}^i \cdot \vec{x} \) and let \( \theta_i(\vec{x}) \) be the formula \( t_i(\vec{x}) = t(\vec{x}) \). \( \square \)

**Theorem 2.4.** The theory \( T^G \) has quantifier elimination.

**Proof.** We show that the set of partial isomorphisms between two \( |L|^+ \)-saturated models \((V,G),(V',G')\) of \( T^G \) has the back and forth property. Let \( B \subset V, B' \subset V' \) be two small substructures (i.e. \( B = \langle B \rangle \) and \( B' = \langle B' \rangle \), \(|B|,|B'| < |L|^+\) such that there exists a partial isomorphism \( \sigma : B \rightarrow B' \).

Let \( a \in V \setminus B \). Since \( B = \text{span}_{\mathbb{F}}(B) \) and \( T \) satisfies property (i), we have that \( \text{tp}_{\mathbb{E}}(a/B) \) is non-algebraic. Every formula in \( \text{tp}_{\mathbb{E}}^G(a/B) \), the quantifier free type in the extended language, is equivalent to a disjunction of formulas of the form

\[
\phi(x,\vec{b}) \land P_\psi(t_1(x,\vec{b}),\ldots,t_k(x,\vec{b})) \land \lnot P_\psi(t'_1(x,\vec{b}),\ldots,t'_k(x,\vec{b}))
\]

where \( \phi(x,\vec{g}) \) is an \( L \)-formula and each of \( t_i(x,\vec{b}), t_j(x,\vec{b}), t_k(x,\vec{b}), t_l(x,\vec{b}) \) is an \( L \)-term. By Lemma 2.3, up to a finite disjunction, we may assume that each \( t \) is an \( L_0 \)-term, i.e. of the form \( ax + b \) for \( b \in \text{span}_{\mathbb{F}}(B) \) and \( a \in \mathbb{F} \setminus \{0\} \). As \( \mathbb{F} = \mathbb{R} \), conditions of the form \( P_\psi(t_1(x,\vec{b}),\ldots,t_k(x,\vec{b})) \land \lnot P_\psi(t'_1(x,\vec{b}),\ldots,t'_k(x,\vec{b})) \) with \( L_0 \)-terms are equivalent to a single \( P_\psi(\vec{t}(x,\vec{b})) \) (for \( \vec{t}(x,\vec{b}) = (t_1(x,\vec{b}),\ldots,t_k(x,\vec{b})) \)) a tuple of linear combinations of \( x \) and \( \vec{b} \) with coefficients in \( R \) up to a finite disjunction of formulas of the form \( t_j(x,\vec{b}) \notin G \). As \( a \notin B \), conditions of the form \( t_k(x,\vec{b}) = 0 \) do not appear in \( \text{tp}_{\mathbb{E}}^G(a/B) \). It turns out we only need to consider formulas of the form

\[
\phi(x,\vec{b}) \land P_\psi(\vec{t}(x,\vec{b})) \land \bigwedge_i t_i(x,\vec{b}) \in G \land \bigwedge_j t_j(x,\vec{b}) \notin G \land \bigwedge_l t_l(x,\vec{b}) \neq 0.
\]

We will extend the map \( \sigma \) by cases.

**Step 1.** If \( a \in G \). Then, as \( \mathbb{F} = \mathbb{R} \), formulas of the form \( qx + b \in G \), where \( q \in \mathbb{F} \), are equivalent to formulas of the form \( rx + g \in r'G \) for some \( r,r' \in \mathbb{R} \) and \( g \in G(B) \) so conditions of the form \( rx + g \in r'G \land x \in G \) are equivalent to \( P_\psi(x,g) \) for some \( L_\mathbb{R} \)-mod-formula \( \psi \). By quantifier elimination in \( R \)-modules [30], the condition \( P_\psi(\vec{t}(x,\vec{b})) \) is equivalent to a boolean combination of formulas of the form \( rx + \vec{r} \cdot \vec{b} \in s_1G + \cdots + s_nG \). If \( x \in G \), the latter is equivalent to \( rx + \vec{r} \cdot \vec{b} \) and tuple \( \vec{g} \in G(B) \). Let \( q(x) = \{ P_\psi(x,\vec{g}) \mid G \models \psi(a,\vec{g}), \psi \in L_{R-\text{mod}}, \vec{g} \in G(B) \} \), \( q^\sigma(x) = \sigma(q)(x) \) and \( p(x) = \sigma(\text{tp}_{\mathbb{E}}(a/B)) \). To extend \( \sigma \) it is enough to show that there are infinitely many realisations of \( p(x) \cup q^\sigma(x) \) in \( V' \). First using the fact that \( G(V') = |\mathbb{F}|^+ \)-saturated and that \( |B'| < |\mathbb{F}|^+ \), there is \( a'' \models q^\sigma(x) \). Let \( p_{sh}(x) = p(x + a'') \), the type shifted by \( a'' \). The type \( p_{sh} \) is also non-algebraic, hence by density of \( R \)-divisible elements (Lemma 2.2), there exists infinitely many \( d \in G^{\text{div}} \) such that \( d \models p_{sh}(x) \). For any such \( d \), let \( a' = d + a'' \).

**Claim:** \( a' \models q^\sigma(x) \cup p(x) \). First, as \( d \models p_{sh}(x), a' = d + a'' \models p(x) \). By quantifier elimination in \( R \)-modules [30], every formula in \( q^\sigma(x) \) is a boolean combination of conditions of the form \( ra' + g \in r_1G + \cdots + r_nG \) for \( r \in \mathbb{R} \) and \( g \in G(B) \). As \( d \in G^{\text{div}} \), we have that \( rd \in r_1G \) for all \( r \in \mathbb{R} \), hence \( ra' + g \in r_1G + \cdots + r_nG \) if and only if \( ra + g \in r_1G + \cdots + r_nG \), so since \( a'' \models q^\sigma(x) \), we also have \( a' \models q^\sigma(x) \).
It follows that \( q^a(x) \cup p(x) \) has infinitely many realisations.

Step 2. If \( a \in \text{span}_F(G(V)B) \). Then \( a = q_1g_1 + \cdots + q_ng_n + b \) for some \( q_i \in F \) and \( g_i \in G \). Applying step 1 to the elements \( g_1, \ldots, g_n \), we can extend \( \sigma \) to a partial isomorphism (which we still call \( \sigma \)) that includes \( g_1, \ldots, g_n \) in its domain. Now set \( \sigma(a) = q_1\sigma(g_1) + \cdots + q_n\sigma(g_n) + \sigma(b) \).

Step 3. If \( a \notin \text{span}_F(GB) \) then formulas of the form \( qx + b \in G \) do not appear in \( tp_{\mathcal{L}^G}(a/B) \), and the formula \( P_\psi(x, \bar{g}) \) belongs to \( tp_{\mathcal{L}^G}(a/B) \) only when \( \psi \) is the negation of a pp-formula. Thus it is enough to show that \( \sigma(\text{tp}_\mathcal{L}(a/B)) \) have infinitely many realisations in \( V' \setminus \text{span}_F(GB') \), which follows easily from condition (C), compactness, the fact that \( |B'| < |F| \) and the fact that \( (V', G') \) is \(|F|^+-\) saturated. \( \square \)

**Corollary 2.5.** Let \((V, G)\) and \((V', G')\) be two models of \( T^G \). Then whenever \( \bar{a} \in V \), \( \bar{a}' \in V' \) are two tuples of the same length such that

1. \( \text{tp}_{\mathcal{R}^\text{mod}}(G(\text{span}_R(\bar{a}))) = \text{tp}_{\mathcal{R}^\text{mod}}(G(\text{span}_R(\bar{a}'))) \) (the types agree in the sense of \( R \)-modules); \( \text{tp}_\mathcal{L}(\bar{a}) = \text{tp}_\mathcal{L}(\bar{a}') \) (their types agree in the language \( \mathcal{L} \));

Then \( \text{tp}_G(\bar{a}) = \text{tp}_G(\bar{a}') \).

**Proof.** From Theorem 2.4, \( T^G \) has quantifier elimination, hence every formula in \( \text{tp}_G(\bar{a}) \) is equivalent to a disjunction of formulas of the form

\[
\phi(\bar{a}) \land P_\psi(\bar{t}(\bar{a})) \land \bigwedge_i t_i(\bar{a}) \in G \land \bigwedge_j t_j(\bar{a}) \notin G
\]

for some quantifier-free \( \mathcal{L} \)-formula \( \phi(x) \), \( \psi \) an \( \mathcal{R} \)-module formula and \( \mathcal{L} \)-terms \((t_i(\bar{x}), t_j(\bar{x}))_{i,j}, \bar{t}(\bar{x})\). Using Lemma 2.3, we may assume that terms are \( F \)-linear combinations and that \( \bar{t}(\bar{a}) \) is a tuple of \( R \)-linear combinations (as in the proof of Theorem 2.4) in \( G \).

It follows that \( \text{tp}_G(\bar{a}) \) is equivalent to a set of formulas of the form

\[
\phi(\bar{a}) \land P_\psi(\bar{t}(\bar{a})) \land \bigwedge_i q_i \cdot \bar{a} \in G \land \bigwedge_j q_j \cdot \bar{a} \notin G
\]

for \( q_i, q_j \in F \). As \( F = \text{Frac}(R) \), each condition of the form \( q_i \cdot \bar{a} \in G \) is equivalent to a condition \( \bar{r_i} \cdot \bar{a} \in s_iG \) for \( r_i, s_i \in R \). From condition (1), \( \bar{r_i} \cdot \bar{a} \in s_iG \) if and only if \( \bar{r_i} \cdot \bar{a}' \in s_iG \). Also by (1), we have that \( P_\psi(\bar{t}(\bar{a})) \) holds if and only if \( P_\psi(\bar{t}(\bar{a}')) \) holds. Finally by condition (2), for \( \phi(x) \) an \( \mathcal{L} \)-formula, \( \phi(\bar{a}) \) holds if and only if \( \phi(\bar{a}') \) holds. This proves the desired result. \( \square \)

**Corollary 2.6** (to the proof of Theorem 2.4). Let \((V, G)\) be a model of \( T^G \) and let \( B \subseteq V \). Then

\[
\text{acl}_G(B) = \text{dcl}_G(B) = \text{span}_F(B).
\]

**Proof.** We follow the proof of Theorem 2.4. In case 1, \( (a \in G \text{ and } a \notin \langle B \rangle) \), observe that \( a \) is not in \( \text{acl}_{\mathcal{R}^\text{mod}}(G(B)) \) hence \( q = \text{tp}_{\mathcal{R}^\text{mod}}(a/G(B)) \) is a non-algebraic type so neither is \( q^a \), thus \( q^a \cup \text{tp}_\mathcal{L}(a/B) \) has infinitely many realisations. The other two cases are similar. \( \square \)

\[1\] By which we mean that

\[
\{P_\psi(\bar{x}) \mid G \models \psi(\bar{g}), \bar{g} \in G(\text{span}_F(\bar{a})), \psi \in \mathcal{L}_{\mathcal{R}^\text{mod}}\} = \{P_\psi(\bar{x}) \mid G \models \psi(\bar{g}), \bar{g} \in G(\text{span}_F(\bar{a}')), \psi \in \mathcal{L}_{\mathcal{R}^\text{mod}}\}.
\]
Lemma 2.7. Let $T$ be a complete pregeometric theory with quantifier elimination. Then $T$ has SAP.

Proof. We first prove the following claim.

Claim. Let $T$ be a complete theory with quantifier elimination. Then $T$ has AP.

Let $M_0, M_1, M_2 \models T$ and assume there are embeddings $f_i : M_0 \to M_i$ for $i = 1, 2$. By quantifier elimination the maps are elementary embeddings. Let $\kappa$ be the biggest cardinal among $|M_1|, |M_2|$. Then by $\kappa^+$-saturation and $\kappa^+$-strong homogeneity. Then by $\kappa^+$-saturation for each $i = 1, 2$ there is an elementary map $g_i : M_i \to N$. Since $g_i(f_i(M_0)) \equiv g_2(f_2(M_0))$ by strong homogeneity there is an automorphism $h$ of $N$ such that for each $m_0 \in M_0$, $h(g_2(f_2(m_0))) = g_1(f_1(m_0))$. Then $h(g_2(M_2))$ is an elementary copy of $M_2$, $g_1(M_1)$ an elementary copy of $M_1$ and for each $m_0 \in M_0$ we have $h(g_2(M_2)) = g_1(f_1(m_0))$ and the AP holds.

We now prove SAP. Let $M_0, M_1, M_2$ be three models of $T$ and $f_i : M_0 \to M_i$ be embeddings. Since $T$ has the AP there exists a model $M_3$ of $T$ and $g_i : M_i \to M_3$ such that $g_i \circ f_i = g_2 \circ f_2$. Let $M_0' = g_1 \circ f_1(M_0)$, $M_1' = g_1(M_1)$, $M_2' = g_2(M_2)$. The type of $M_1'$ over $M_0'$ has a free extension with respect to $M_2'$ (in the sense of the pregeometry), hence in a monster model $N$ of $T$ containing $M_3$ there exists $M'' \equiv_{M_1'} M_1'$ such that $M'' \subset N$ and $M_1''$ are independent over $M_0'$, in particular, as $M_1''$, $M_1'$ and $M_2'$ are algebraically closed, $M_1'' \cap M_2' = M_0'$. Let $\sigma$ be an automorphism of $N$ over $M_0'$ such that $\sigma(M_1') = M_1''$. Then $g_1' = (\sigma \upharpoonright M_1') \circ g_1$ is an embedding $M_1 \to N$ and $g_1'(M_1') \cap g_2(M_2') = M_0'' = g_1' \circ f_1(M_0)$.

Remark 2.8. The previous lemma actually holds without the assumption of $T$ being pregeometric. One could use for instance [1, Proposition 1.5] to get the existence of such $M''_1$.

Proposition 2.9. Let $T_U$ be the $\mathcal{L}_G$-theory satisfying condition (A). If $T_U$ is inductive, then $T_G$ is the model-completion of $T_U$.

Proof. We first check that every model of $T_U$ extends to a model of $T_G$. This is done by a standard chain argument. For example, if $(V, G) \models T_U$, $\phi(x, \bar{y})$ is an $\mathcal{L}$-formula, $r \in R \setminus \{0\}$ and $\bar{a} \in V$ is such that $V \models \exists x \phi(x, \bar{a})$, we let $V_1$ be a proper extension of $V$ such that $\phi(V_1, \bar{a}) \setminus \phi(V, \bar{a})$ is infinite. Then one chooses $b_1 \in \phi(V_1, \bar{a}) \setminus \phi(V, \bar{a})$ and defines $G_1 = (G, \frac{b_1}{r})$ the smallest $R$-module containing $G$ and $\frac{b_1}{r}$. One may apply this argument for all tuples $\bar{a} \in V$ with $V \models \exists x \phi(x, \bar{a})$ and assume that $\phi(rG_1, \bar{a}) \neq \emptyset$. Then we repeat the process for tuples in $V_1$ and build a chain $\{(V_i, G_i)\}_{i=1}^\infty$ whose union has the desired properties.

Second, one needs to check that $T_U$ has the amalgamation property. In fact, $T_U$ has the strong amalgamation property. Let $(V_0, U_0)$, $(V_1, U_1)$ and $(V_2, U_2)$ be three models of $T_U$ such that there exists embeddings $f_i : (V_0, U_0) \to (V_i, U_i)$ for $i = 1, 2$. By Lemma 2.7, $T$ has SAP hence there exists $V_3 \models T$ and $\mathcal{L}$-embeddings $g_i : V_i \to V_3$ ($i = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$ and $g_1(V_1) \cap g_2(V_2) = g_1 \circ f_1(V_0) = g_2 \circ f_2(V_0)$. Define $U_3 = g_1(U_1) + g_2(U_2)$, then $(V_3, U_3)$ is a model of $T_U$. We have to check that $g_i : (V_i, U_i) \to (V_3, U_3)$ are $\mathcal{L}_U$-embeddings, i.e. $g_i(V_i) \cap U_3 = g_i(U_i)$ for $i = 1, 2$. Without loss of generality, we may assume that $i = 1$. We have that $g_1(V_1) \cap (g_1(U_1) + g_2(U_2)) = g_1(U_1) + g_1(V_1) \cap g_2(U_2)$. Also $g_1(V_1) \cap g_2(V_2) = g_2 \circ f_2(V_0)$ hence $g_1(V_1) \cap g_2(U_2) \subseteq g_2 \circ f_2(V_0)$. Now as $g_2$ is injective and $f_2$ is an $\mathcal{L}_U$-embedding, we have that $g_2(U_2) \cap g_2 \circ f_2(V_0) \subseteq g_2 \circ f_2(U_0)$ hence $g_1(V_1) \cap g_2(U_2) \subseteq g_2 \circ f_2(U_0) = g_1 \circ f_1(U_0) \subseteq g_1(U_1)$.

It follows that $g_1(V_1) \cap (g_1(U_1) + g_2(U_2)) = g_1(U_1)$, which finishes the argument.
Finally note that $T^G$ has quantifier elimination and thus is model complete. □

It follows from Theorem 2.4 that the theory $T^G$ has quantifier elimination in the extended language. We will use this fact below to show that several tameness properties are preserved in the expansion.

We start with a small lemma showing the stability of the new predicate.

**Lemma 2.10.** Let $t(\vec{x}, \vec{y})$ be a $L_0$-term, then the formula $G(t(\vec{x}, \vec{y}))$ is stable.

**Proof.** Assume that $t(\vec{x}, \vec{y}) = \vec{a} \cdot \vec{x} + \vec{\beta} \cdot \vec{y}$ and that there exist sequences $(\vec{a}_i)_{i \in \omega}$ and $(\vec{b}_i)_{i \in \omega}$ such that $t(\vec{a}_i, \vec{b}_j) \in G$ if and only if $i < j$. Now let $c_i = \vec{\alpha} \cdot \vec{a}_i$ and $d_i = -\vec{\beta} \cdot \vec{b}_i$. Then we have $c_i - d_j \in G$ if and only if $i < j$. In particular we get $c_1 \in d_2 + G$, $c_1 \in d_3 + G$ and $c_2 \in d_3 + G$. So the cosets $d_1 + G$, $d_2 + G$ are equal and thus $c_2 \in d_2 + G$, which implies that $c_2 - d_2 \in G$, a contradiction. □

**Corollary 2.11.** If $T$ is stable, then $T^G$ is stable. If $T$ is NIP, then so is $T^G$.

**Proof.** By Theorem 2.4 every formula in the extended language $L_G$ is equivalent to a disjunction of conjunctions of $L$-formulas, $R$-module formulas and formulas of the form $t(x, \vec{y}) \in G$ and their negation. Since the theory of modules over any ring is stable, any $R$-module formula is stable.

Assume now that $T$ is stable, then any $L$-formula is stable. By Lemma 2.3, any formula of the form $t(\vec{x}, \vec{y}) \in G$, where $t(\vec{x}, \vec{y})$ be a $L$-term, is equivalent to a formula of the form $\bigvee_i t_i(x, \vec{y}) \in G \land \theta_i(x, \vec{y})$ where $\theta_i(x, \vec{y})$ is an $L$-formula and each $t_i(x, \vec{y})$ is a $L_0$-term. By Lemma 2.10 the formulas $t_i(x, \vec{y}) \in G$ are stable for $i \leq n$ and by hypothesis $\theta_i(x, \vec{y})$ is also stable, thus $t(x, \vec{y}) \in G$ is stable. Thus if $T$ is stable so is $T^G$. The same argument works for preservation of NIP. □

We even get some easy properties about strong dependence and the stability spectrum based on properties of the subgroups of $G$:

**Corollary 2.12.** Assume that $\text{char}(F) = 0$ and that for infinitely many primes $\{p_i : i \in I\}$ we have that the index of $p_i G$ in $G$ is infinite. Then if $T$ is stable we have that $T^G$ is strictly stable. If $T$ is NIP then $T^G$ is not strongly dependent.

**Proof.** One can build an array using the collection of groups $\{p_i G : i \in I\}$ and their cosets (see for example [5]) to show that the expansion does not preserve strong dependence. Similarly, if we order the primes in the list as $p_1, p_2, \ldots$ then the chain of definable groups $p_1 G, (p_1 p_2) G, \ldots$ is strictly descending and the expansion can not be superstable. □

We end this section with some examples and comments relating our work with the general perspective from [9].

**Example 2.13.** Let $T$ is the theory of a pure vector space over a field $F$ of characteristic zero. This is a strongly minimal theory and thus geometric and it satisfies that $\text{acl} = \text{span}_F$. As we saw before the corresponding theory $T^G$ is stable and we will see below how the stability spectrum of the expansion will vary according to the choice of $R$.

Assume first that $R = F = \mathbb{Q}$, then the corresponding theory $T^G$ is the theory of beautiful pairs and it will be $\omega$-stable of Morley rank equal to 2 (see for example [10] for the value of Morley rank in beautiful pairs of strongly minimal theories). It is the model companion of the theory of pairs of models of $T$ (see for example [27]).
On the opposite end, consider the case $R = \mathbb{Z}$. By Corollary 2.12 the corresponding expansion $T^G$ will be stable, not superstable. It is the model companion of the theory of $T$ expanded by a subgroup. Note that this is vector space-like phenomenon, the theory of a field of characteristic 0 with a predicate for an additive subgroup does not have a model-companion [16].

Assume now that $R = \mathbb{Z}_{2\mathbb{Z}}$, the rationals whose denominator is relatively prime with 2. Then the collection of definable groups $\{2^nG : n \in \omega\}$ is a descending chain and $T^G$ is not superstable. \textbf{Question.} Is $T^G$ dp-minimal?

\textbf{Proposition 2.14.} Let $(V, G) \models T^G$ and assume that $V$, seen as an $L$-structure, is an abelian structure. Then $(V, G)$ is an abelian structure.

\textbf{Proof.} We show that all definable subsets in the pair are again boolean combinations of $\emptyset$-definable groups. By Theorem 2.4 the expansion has quantifier elimination and it suffices to show that atomic formulas in the pair $(V, G)$ give rise to definable sets that also have this property. Since $L$-definable sets have the desired property, we only need to consider definable sets given by $G(t(\vec{x}, \vec{a}))$ where $t(\vec{x}, \vec{y})$ is an $L$-term and for a predicates of the form $P_\phi(x, \vec{b})$. Consider first the case of a formula of the form $G(t(\vec{x}, \vec{a}))$. Assume first that there are $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_k \in F$ such that $t(\vec{x}, \vec{y}) = \lambda_1x_1 + \cdots + \lambda_nx_n + \mu_1y_1 + \cdots + \mu_ky_k$. We now consider two cases.

Case 1: Assume that $\lambda_1, \ldots, \lambda_n = 0$. Then $G(t(\vec{x}, \vec{a}))$ holds if and only if $\mu_1a_1 + \cdots + \mu_k a_k \in G$. If $\mu_1a_1 + \cdots + \mu_k a_k \in G$, then $G(t(V^n, \vec{a})) = V^n$ and if $\mu_1a_1 + \cdots + \mu_k a_k \notin G$ then $G(t(V^n, \vec{a})) = \emptyset$ and both sets are boolean combinations of $\emptyset$-definable groups.

Case 2: Assume that for some $i$, we have $\lambda_i \neq 0$. Choose $\vec{b} = b_1, \ldots, b_n \in V$ such that $\lambda_1b_1 + \cdots + \lambda_nb_n = \mu_1a_1 + \cdots + \mu_k a_k$. Then $G(t(\vec{x}, \vec{a}))$ holds if and only if $\lambda_1(x_1 - b_1) + \cdots + \lambda_k(x_k - b_k) \in G$. Now observe that $H = \{\vec{x} \in V^n : \lambda_1x_1 + \cdots + \lambda_nx_n \in G\}$ is a $\emptyset$-definable group and the coset $H - \vec{b}$ agrees with $G(t(V^n, \vec{a}))$.

Assume now that the term $t(\vec{x})$ agrees with $t_i(\vec{x})$ for some $L_0$-terms $\{t_i(\vec{x}) : i \leq n\}$ and some partition $\{t_i(\vec{x}) : i \leq n\}$ given by $L$-formulas. Then $G(t(\vec{x}))$ holds if and only if $G(t_i(\vec{x}))$ holds when $t_i(\vec{x})$ holds. Since both $\theta(\vec{x})$ and $G(t_i(\vec{x}))$ are boolean combinations of sets, so is their conjunction as well as disjunctions of families of this form.

On the other hand, all definable sets in a module are boolean combination of cosets of $\emptyset$-definable groups, in particular this will be the case for $P_\phi(x, \vec{b})$. \hfill \Box

\textbf{Corollary 2.15.} Let $T$ be the theory of a pure vector space over a field $F$ and let $R$ be a subring. Then $T^G$ is 1-based.

\textbf{Proof.} Since $V$ is a 1-based group, it is an abelian structure. Now apply the previous Proposition. \hfill \Box

\textbf{Example 2.16.} In this example we deal with the base structure $\mathcal{V} = (V, +, 0, <, \{\lambda_r\}_{r \in F})$ and we assume that $F$ is an ordered field and $V$ is an ordered vector space over $F$. The theory of $\mathcal{V}$ is dense $\alpha$-minimal, has quantifier elimination and acl = span$_F$. In particular it is geometric.

We first consider the case where $R = \mathbb{Z}$ and $F = \text{Frac}(\mathbb{Z}) = \mathbb{Q}$. Since $T$ has NIP, so does $T^G$. Moreover since $nG$ has infinite index in $G$ for $n > 1$ the expansion is not strongly dependent.
Now consider the case where $\mathbb{F} = R = \mathbb{Q}$, then the theory $T^G$ agrees with the theory of lovely pairs (dense pairs) and it is strongly dependent of dp-rank two (see for example [21, Lemma 3.5]).

Finally, we point a few connections with the setting introduced in [9] that can also be used to analyze this family of expansions. Following the terminology from [9], we can define the languages $\mathcal{L}_I = \mathcal{L} = \{+, 0, \{\lambda \}_{\lambda \in \mathbb{F}}\}$ and $\mathcal{L}_G = \{+, 0, \{\lambda \}_{\lambda \in \mathbb{R}}\}$ and define $T_I = Th(V, +, 0, \{\lambda \}_{\lambda \in \mathbb{F}})$ and $T_G$ to be the theory of $R$-modules. The theory $T^G$ is geometric and by the results we proved in this section $T^G$ is an example of a Mordell-Lang theory of pairs in the sense of Definition 2.6 [9]. In particular, by [9, Corollary 3.6] the theory $T^G$ is near model complete (we prove that it has quantifier elimination, furthermore we show in Proposition 2.9 that it is the model-completion of $T^G$ when $T^U$ is inductive) and Corollary 2.11 follows from [9, section 4]. There are some other properties of the pair that are studied [9] that are not addressed in this paper; among others, it follows by [9, Theorem 4.5] that the family considered in Example 2.16 has o-minimal open core.

3. V-structures: Back-and-forth and first properties

We now turn to the general case, we do not assume anymore that $\mathbb{F} = Frac(R)$, instead we suppose that $\mathbb{F}$ is a field of characteristic zero and that $R$ is a subring of $\mathbb{F}$. Let $\mathcal{L}_0 = \{+, 0, \{\lambda \}_{\lambda \in \mathbb{F}}\}$ and let $\mathcal{L} \supset \mathcal{L}_0$ be an extension. Let $T$ be an $\mathcal{L}$-theory expanding the theory of vector spaces over $\mathbb{F}$ which has quantifier elimination in $\mathcal{L}$ for which $dcl = acl = span$ for all $L$.

Remark 3.2. Since our ambient structure is a vector space $V$ over $\mathbb{F}$ satisfying $dcl = acl = span$, we can be a little more explicit in axiom scheme $(D)$. Instead of listing all $\phi(x, y, z)$ such that $\exists^\leq_n x \phi(x, y, z)$ holds, we could simply list all finite disjunctions of linear equations with coefficients in $\mathbb{F}$ in the variables $x, y, z$ with nontrivial coefficient in $x$. 

Notation 3.1. Throughout the rest of the paper, we will denote $\hat{R} = Frac(R)$.

Let $G$ be a unary predicate and for each formula $\phi(\bar{x})$ in the language $\mathcal{L}_{R-mod} = \{+, 0, \{\lambda \}_{\lambda \in \mathbb{R}}\}$ of $R$-modules, let $P_\phi(\bar{x})$ be a new predicate. Let $\mathcal{L}_G$ be the expansion of $\mathcal{L}$ by $G$ and $P_\phi$ for all formula $\phi$ in $\mathcal{L}_{R-mod}$. We will consider pairs $(V, G)$ in the language $\mathcal{L}_G$ such that $V \models T$ and that also satisfy the following first order conditions

(A) $G$ is a proper $R$-submodule of the universe, and for all $\bar{a} \in V$, $P_\phi(\bar{a})$ if and only if $\bar{a} \in G$ and $G \models \phi(\bar{a})$ as an $R$-module.

(B) If $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are $\hat{R}$-linearly independent, then for all $g_1, \ldots, g_n \in G$

$$\lambda_1 g_1 + \cdots + \lambda_n g_n = 0 \implies \bigwedge_i g_i = 0.$$

(C) (Density Property) for all $r \in R \setminus \{0\}$, $rG$ is dense in the universe. This is a first order property that can be axiomatized through the scheme: for every $\mathcal{L}$-formula $\phi(x, \bar{y})$, add the sentence $\exists^\infty x \phi(x, \bar{y}) \rightarrow \exists x(\phi(x, \bar{y}) \land rG(x));$

(D) (Extension/co-density property) for any $\mathcal{L}$-formulas $\phi(x, \bar{y})$ and $\psi(x, \bar{y}, \bar{z})$ and $n \geq 1$, the following sentence

$$\exists x \phi(x, \bar{y}) \wedge \forall z \exists^\leq_n x \psi(x, \bar{y}, z) \rightarrow \exists x \phi(x, \bar{y}) \wedge \forall z (G(z) \rightarrow \neg \psi(x, \bar{y}, \bar{z})).$$

Remark 3.2. Since our ambient structure is a vector space $V$ over $\mathbb{F}$ satisfying $dcl = acl = span$, we can be a little more explicit in axiom scheme $(D)$. Instead of listing all $\phi(x, \bar{y}, \bar{z})$ such that $\exists^\leq_n x \phi(x, \bar{y}, \bar{z})$ holds, we could simply list all finite disjunctions of linear equations with coefficients in $\mathbb{F}$ in the variables $x, \bar{y}, \bar{z}$ with nontrivial coefficient in $x$. 

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Definition 3.3. Let $T_U$ be the theory consisting of $T$ together with the schemes $(A)$ and $(B)$ and $T^G$ the theory consisting of adding $(A), (B), (C), (D)$.

In order to understand definable sets in $T^G$, we will consider an expansion by definition of $T_U$ and $T^G$, see Definition 3.5.

Remark 3.4. Assume that $\bar{\lambda} \in F^n$ is $\bar{R}$-independent, and that in a model of $T_U$ we have $a = \lambda_1g_1 + \cdots + \lambda_ng_n$ for some $g_i \in G$. Then, axiom (B) implies that such collection $(g_1, \ldots, g_n)$ is unique and depends only on $a$ and $\bar{\lambda}$, hence the following definition.

Definition 3.5. For each finite $\bar{R}$-independent tuple $\bar{\lambda}$, we add a new unary predicate $G_{\bar{\lambda}}$. For each finite $\bar{R}$-independent tuple $\bar{\lambda} = \lambda_1, \ldots, \lambda_n$ and $1 \leq i \leq n$ we also add a new unary function symbol $f_{\bar{\lambda},i}$. Let

$$L^+_G = L_G \cup \{ G_{\bar{\lambda}} \mid \bar{\lambda} \in F^n \text{ is } \bar{R} \text{-independent} \} \cup \{ f_{\bar{\lambda},i} \mid 1 \leq i \leq n, \bar{\lambda} \in F^n \text{ is } \bar{R} \text{-independent} \}.$$ 

Let $T^{G+}$ be the expansion of $T^G$ to the language $L^+_G$ by the following sentences:

$$\forall x \left( G_{\bar{x}}(x) \leftrightarrow \exists \bar{y} \in G \sum_i \lambda_i y_i = x \right)$$

$$\forall x, y \left[ x = f_{\bar{\lambda},i}(y) \leftrightarrow \left( y \in G_{\bar{\lambda}} \land \exists \bar{x} \in G y = \bar{\lambda} \cdot \bar{x} \land x = y \right) \lor (y \notin G_{\bar{\lambda}} \land x = 0) \right]$$

We will show that $T^{G+}$ has quantifier elimination.

The following notion is a straightforward modification of the corresponding notions from [23] and [9].

Definition 3.6 (Mordell-Lang property). A model $(V,G)$ of $T_U$ has the Mordell-Lang property if for all definable sets $X$ of the form $\lambda_1x_1 + \cdots + \lambda_nx_n = 0$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$, the trace $X \cap G^n$ is equal to the trace $Y \cap G^n$ for $Y$ defined by a conjunction of formulas of the form $r_1x_1 + \cdots + r_nx_n = 0$ where $r_i \in R$. In particular, $X \cap G^n$ is $0$-definable from the $R$-module structure in $G$, i.e., by a formula of the form $P_\psi(x_1, \ldots, x_n)$ for some $L_{R-mod}$ pp-formula $\psi$.

Lemma 3.7. Every model of $T_U$ has the Mordell-Lang property.

Proof. Let $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathbb{F}$ and let $X$ be the definable set given by

$$\lambda_0x_0 + \lambda_1 x_1 + \cdots + \lambda_k x_k = 0.$$ 

We may assume that for some $0 \leq m \leq k$, $\lambda_0, \lambda_1, \ldots, \lambda_m$ are linearly independent over $\bar{R}$ as elements of the field $\mathbb{F}$, while $\lambda_{m+1}, \ldots, \lambda_k \in \text{span}_R(\lambda_0, \lambda_1, \ldots, \lambda_m)$.

Then for each $m + 1 \leq i \leq k$ we have

$$\lambda_i = \sum_{j=0}^m q_{i,j} \lambda_j,$$

where $q_{i,j} \in \bar{R}$. Now let $g_0, \ldots, g_k \in G$ be a realization of $X$. Collecting terms with $\lambda_0, \ldots, \lambda_m$, we have:

$$\lambda_0(g_0 + \sum_{i=m+1}^k q_{i,0} g_i) + \lambda_1 (g_1 + \sum_{i=m+1}^k q_{i,1} g_i) + \cdots + \lambda_m (g_m + \sum_{i=m+1}^k q_{i,m} g_i) = 0.$$

By multiplying by a common multiple of the denominators of the $q_{i,j}$, there are $r_{i,j} \in R$ such that

$$\lambda_0(r_{0,0} g_0 + \sum_{i=m+1}^k r_{0,i} g_i) + \lambda_1 (r_{1,1} g_1 + \sum_{i=m+1}^k r_{1,1} g_i) + \cdots + \lambda_m (r_{m,m} g_m + \sum_{i=m+1}^k r_{i,m} g_i) = 0.$$
Lemma 3.8. We actually proved a stronger version of the Mordell-Lang property: given a definable set $X$ of the form $A_{1}x_{1} + \cdots + A_{n}x_{n} = 0$ with $A_{1}, \ldots, A_{n} \in \mathbb{F}$, the trace $X \cap G^{n}$ is quantifier free definable by a positive formula in the $R$-module $G$. We could have also restated the Model-Lang property as any $F_{m}$-linear dependence in $G$ is witnessed by a linear combination with coefficients in $\hat{R}$.

Recall that $\mathcal{L}_{0} = \{(\lambda \cdot)_{\lambda \in \mathbb{F}} +, 0\}$. We want to express $\mathcal{L}_{0}^{+}$-terms using only the $\mathbb{F}$-vector space language together with the $f_{\lambda,i}$-functions. So we introduce the auxiliary language

$$\mathcal{L}_{0}^{+} := \mathcal{L}_{0} \cup \{(f_{\lambda,i})_{\lambda \in \mathbb{F}, i \leq n}\}.$$

Lemma 3.9. Let $(V, G) \models T^{G^{+}}$ and let $t(\vec{x})$ be an $\mathcal{L}_{0}^{+}$-term. Then there exists $\mathcal{L}$-formulas $\theta_{1}(\vec{x}), \ldots, \theta_{n}(\vec{x})$ forming a partition of $V[\vec{x}]$ and $\mathcal{L}_{0}^{+}$-terms $t_{1}(\vec{x}), \ldots, t_{n}(\vec{x})$ such that

$$t(\vec{x}) = y \Leftrightarrow \bigvee_{i=1}^{n} t_{i}(\vec{x}) = y \land \theta_{i}(\vec{x}).$$

Proof. We prove it by induction on the complexity of $t(\vec{x})$. If $t(\vec{x}) = g(t_{1}(\vec{x}), \ldots, t_{n}(\vec{x}))$ for some $\mathcal{L}$-function $g$, then by induction hypothesis, for each $1 \leq k \leq n$ there exists $\mathcal{L}$-partitions $(\theta_{i}^{k}(\vec{x}))$, and $\mathcal{L}_{0}^{+}$-terms $(t_{i}^{k}(\vec{x}))$, such that for each $1 \leq i \leq n$, $t_{i}^{k}(\vec{x}) = z \Leftrightarrow \bigvee_{i} t_{i}^{k}(\vec{x}) = z \land \theta_{i}^{k}(\vec{x})$. Also, by Lemma 2.3, there exists $\mathcal{L}_{0}$-terms $s_{1}(\vec{z}), \ldots, s_{m}(\vec{z})$ and an $\mathcal{L}$-partition $\psi_{1}(\vec{z}), \ldots, \psi_{m}(\vec{z})$ such that $g(\vec{z}) = y \Leftrightarrow \bigvee_{j} s_{j}(\vec{z}) = y \land \psi_{j}(\vec{z})$. Putting together, we have

$$t(\vec{x}) = y \Leftrightarrow \left( \exists z_{1}, \ldots, z_{n} \bigwedge_{k} z_{k} = t_{k}(\vec{x}) \land g(\vec{z}) = y \right)$$

$$\Leftrightarrow \left( \exists z_{1}, \ldots, z_{n} \bigwedge_{i} \left( \bigvee_{i} t_{i}^{k}(\vec{x}) = z_{k} \land \theta_{i}^{k}(\vec{x}) \right) \land \left( \bigvee_{j} s_{j}(\vec{z}) = y \land \psi_{j}(\vec{z}) \right) \right)$$

By axiom (B), each of the terms in parentheses above is equal to 0, that is

$$r_{0,0}g_{0} + r_{m+1,0}g_{m+1} + r_{m+2,0}g_{m+2} + \cdots + r_{k,0}g_{k} = 0.$$
It is easy to see that for any finite family of sets $A_{i,k}$ such that $A_{i,k} \cap A_{j,k} = \emptyset$ for $i \neq j$, then $\bigcap_k \bigcup_i A_{i,k} = \bigcup_i \bigcap_k A_{i,k}$ hence

$$t(\vec{x}) = y \iff \left( \exists z_1, \ldots, z_n \bigvee_{i,j} \left( t^i_j(\vec{x}) = z_k \wedge \theta^i_k(\vec{x}) \wedge s_j(z) = y \wedge \psi_j(z) \right) \right)$$

$$\iff \left( \exists z_1, \ldots, z_n \bigwedge_{i,j} \left( t^i_j(\vec{x}) = z_k \wedge \theta^i_k(\vec{x}) \wedge s_j(z) = y \wedge \psi_j(z) \right) \right)$$

$$\iff \bigvee_{i,j} \left( s_j(t^i_j(\vec{x}), \ldots, t^n_i(\vec{x})) = y \wedge \theta^i_k(\vec{x}) \wedge \psi_j(t^i_j(\vec{x}), \ldots, t^n_i(\vec{x})) \right)$$

$$\iff \bigvee_{i,j} s_j(t^i_j(\vec{x}), \ldots, t^n_i(\vec{x})) = y \wedge \left( \bigwedge_k \theta^i_k(\vec{x}) \wedge \psi_j(t^i_j(\vec{x}), \ldots, t^n_i(\vec{x})) \right)$$

We now have to check that the family $\left( (\bigwedge_k \theta^i_k(\vec{x})) \wedge \psi_j(t^i_j(\vec{x}), \ldots, t^n_i(\vec{x})) \right)_{i,j}$ forms a partition of $V^{[\vec{x}]}$. It is clear that the union is the whole universe. Let $(i,j) \neq (i',j')$. If $i \neq i'$, then $(\bigwedge_k \theta^i_k(\vec{x})) \wedge \psi_j(t^i_j(\vec{x}), \ldots, t^n_i(\vec{x}))$ and $(\bigwedge_k \theta^{i'}_k(\vec{x})) \wedge \psi_{j'}(t^{i'}_{j'}(\vec{x}), \ldots, t^n_{i'}(\vec{x}))$ are disjoint since $\theta^i_k(\vec{x})$ is a partition for all $k$. Otherwise if $i = i'$ and $j \neq j'$ then the result follows from the fact that $\psi_j(t^i_j(\vec{x}), \ldots, t^n_i(\vec{x}))$ and $\psi_{j'}(t^i_{j'}(\vec{x}), \ldots, t^n_{i'}(\vec{x}))$ are disjoint, because $\psi_j(z)$ defines a partition of $V^{[\vec{x}]}$. □

We now give a description of $L^+_{\bar{\alpha}}$-terms.

**Lemma 3.10.** In a model $(V, G)$ of $T^{G^+}$, any term $t(\vec{a})$ in the language $L^+_{\bar{\alpha}}$ is equivalent to a term of the form:

$$\sum_{i} \lambda_i f_{\bar{\mu},k}(\bar{\alpha}_i \cdot \bar{a}) + \bar{\beta} \cdot \bar{a}$$

where $\bar{a} = \bar{a} \cap (V \setminus G)$ and $\lambda_i \in \mathbb{F}, \bar{\alpha}_i \in \mathbb{F}^{[\bar{a}]}, \bar{\mu}_i \in \mathbb{F}^m, \bar{\beta} \in \mathbb{F}^{[\bar{a}]}$.

**Proof.** First, observe the following:

**Claim:** For every $g \in G$, $b \in V$, $\alpha, \bar{\mu} \in \mathbb{F}$, there exists $s \in R$, $q \in \hat{R}$ and $\bar{\mu}' \in F$ such that

$$f_{\bar{\mu},k}(\alpha g + b) = \frac{1}{s} f_{\bar{\mu}',k}(sb) + qg.$$

We proof the claim by cases.

- **Case 1:** Assume $\alpha \notin \text{span}_R(\bar{\mu})$. Then $f_{\bar{\mu},k}(\alpha g + b) = f_{\bar{\mu}',k}(b)$, for $\bar{\mu}' = \bar{\mu}^-(-\alpha)$, so choose $s = 1$ and $q = 0$.

- **Case 2:** Assume $\alpha \in \text{span}_R(\bar{\mu})$, write $\alpha = q_1\mu_1 + \cdots + q_n\mu_n$ for some $q_j \in \hat{R}$. Let $s$ be the product of denominators of $(q_j)_{i}$, so $sq_i \in R$, for all $i$. Then write $b + \alpha g = \bar{\mu} \cdot \bar{g}$ with $\bar{g} \in G$, so $f_{\bar{\mu},k}(b + \alpha g) = g_k$. On the other hand $b = \sum_i \mu_i(g_i - q_i g)$, hence $sb = \sum_i \mu_i(sq_i + sg_i g)$. As $sg_i + sq_i g \in G$, we have $f_{\bar{\mu},k}(sb) = sg_k + rg$ for $r = sq_k \in R$. We conclude that $f_{\bar{\mu},k}(\alpha g + b) = g_k = \frac{1}{s} f_{\bar{\mu},k}(sb) - sq_k g$, so choose $q = q_k$.

By the claim, the result follows if every term is equivalent to one of the form $\sum_i \lambda_i f_{\bar{\mu},k,i}(\bar{\alpha}_i \cdot \bar{a}) + \bar{\beta} \cdot \bar{a}$, which we prove now by induction. By linearity of the
expression, the only step to check is the one where \( t(\vec{a}) = f_{\vec{\gamma},i}(t'(\vec{a})) \). For convenience, we assume \( i = 1 \). By induction, \( t'(\vec{a}) \) is of the form \( \sum_i \lambda_i f_{\vec{\mu},i,k_i}(\vec{\alpha}_i \cdot \vec{a}) + \vec{\beta} \cdot \vec{a} \). If \( t(\vec{a}) \neq 0 \), then there exists \( g_1, \ldots, g_n \) such that

\[
\sum_i \lambda_i f_{\vec{\mu},i,k_i}(\vec{\alpha}_i \cdot \vec{a}) + \vec{\beta} \cdot \vec{a} = \vec{\gamma} \cdot \vec{a}
\]

As \( \vec{\gamma} \) is linearly independent, we may assume there is \( s < |\vec{\lambda}| \) be such that \( \vec{\gamma} \prec (\lambda_i)_{i \leq s} \) is \( \vec{\beta} \)-independent and \( \lambda_i \in \text{span}_{\vec{R}}(\vec{\gamma}(\lambda_i)_{i \leq s}) \) for \( i > s \). Let \( q_{i,j}, q_i' \in \vec{R} \) be such that for \( j > s \) we have \( \lambda_j = \sum \lambda_i q_{i,j} \vec{\lambda}_i + \sum \lambda_i q_{i,j} \vec{\gamma}_i \). Let \( \vec{\delta} = \vec{\gamma} \prec (\lambda_i)_{i \leq s} \). It follows that

\[
f_{\vec{\gamma},1}(\vec{\beta} \cdot \vec{a}) = g_1 - \sum_{j=s+1}^{13} q_{i,j} f_{\vec{\mu},j,k_j}(\vec{\alpha}_j \cdot \vec{a})
\]

So \( g_1 = t(\vec{a}) \) is of the required form. \( \square \)

We denote by \( \langle B \rangle \) the \( \mathcal{L}_0^+ \)-substructure generated by \( B \). Using Lemma 3.10,

\[
\langle B \rangle = \text{span}_{\vec{F}}(B \cup \{ f_{\vec{\lambda},i}(c) \mid c \in \text{span}_{\vec{F}}(B), \vec{\lambda} \in \mathcal{F}, i \leq |\vec{\lambda}| \}).
\]

**Lemma 3.11.** Let \( (V, G) \) be a model of \( T^\mathcal{G}^+ \), \( B \subset V \) and \( a \in V \).

1. If \( a \in G \) or \( \vec{a} \notin \text{span}_{\vec{F}}(BG) \) then every \( \mathcal{L}_0^+ \)-term \( t(a, \vec{b}) \) with \( \vec{b} \in B \) is equal to a term of the form \( \lambda a + b' \) for some \( b' \in \langle B \rangle \) and \( \lambda \in \mathcal{F} \).
2. If \( a \in \text{span}_{\vec{F}}(BG) \), let \( \vec{c} \in \mathcal{F} \) be \( \vec{\beta} \)-independent and \( c \in \text{span}_{\vec{F}}(B) \) be such that \( a = \sum_{i=1}^n \alpha_i f_{\vec{\alpha},i}(a - c) + c \). Then every \( \mathcal{L}_0^+ \)-term \( t(a, \vec{b}) \) for \( \vec{b} \in B \) is equal to a term of the form \( \sum_{i=1}^{|\vec{\lambda}|} \lambda_i f_{\vec{\alpha},i}(a - c) + b' \) for \( \vec{\lambda}, \vec{\beta} \in \vec{F}, b' \in \langle B \rangle \).

**Proof.** By Lemma 3.10, every \( \mathcal{L}_0^+ \)-term \( t(a, \vec{b}) \) is equal to \( \sum_i \lambda_i f_{\vec{\mu},i,k_i}(\vec{\alpha}_i \vec{c}) + \beta a + \vec{\beta} \cdot \vec{b} \), for \( \vec{c} \in a \sim \vec{b} \cap (V \setminus G) \).

1. Assume first that \( a \in G \). The result is clear since \( \vec{c} \subseteq \vec{b} \), so we can choose \( b' = \sum_i \lambda_i f_{\vec{\mu},i,k_i}(\vec{\alpha}_i \vec{c}) + \vec{\beta} \cdot \vec{b} \in \langle B \rangle \) and \( t(a, \vec{b}) = b' + \beta a \). If on the other hand \( a \notin \text{span}_{\vec{F}}(BG) \), then \( f_{\vec{\mu},i}(\lambda a + \vec{b} \vec{a}) = 0 \) for all \( \vec{a}, \vec{\mu}, \vec{\beta} \), so the result follows.
2. Let \( \vec{a}_1, \ldots, \vec{a}_n \in \mathcal{F} \) be such that for some \( g_1, \ldots, g_n \in G \) one have \( a = \vec{\alpha} \cdot \vec{g} + c \) for \( c \in \text{span}_{\vec{F}}(B) \). By extracting an \( \vec{R} \)-basis of \( \vec{\alpha} \) and replacing \( a \) by \( ra \) for some \( r \in \vec{R} \), we may assume that \( \vec{\alpha} \in \vec{F} \)-independent, so \( a = \sum_{i=1}^n \alpha_i f_{\vec{\alpha},i}(a - c) + c \) and \( g_i = f_{\vec{\alpha},i}(a - c) \).

**Claim:** For all \( \vec{\alpha}, \vec{\beta}, \vec{g} \in \mathcal{F} \) and \( b \in \langle B \rangle \), \( f_{\vec{\beta},k}(\beta a + b) = \sum_{i=1}^n q_i f_{\vec{\alpha},i}(a - c) \) for some \( b' \in \langle B \rangle \), and \( q_i \in \vec{R} \).

For all \( \vec{\alpha}, \vec{\beta}, \vec{g} \in \mathcal{F} \) and \( b \in \langle B \rangle \), \( f_{\vec{\beta},k}(\beta a + b) = f_{\vec{\beta},k}(\alpha_1' \vec{g}_1 + \cdots + \alpha_n' \vec{g}_n + \beta c + b) \) for \( \vec{\alpha}_i = \vec{\beta} \alpha_i \). From the claim in the proof of Lemma 3.10, we have that

\[
f_{\vec{\beta},k}(\beta a + b) = f_{\vec{\beta},k}(\alpha_1' \vec{g}_1 + \cdots + \alpha_n' \vec{g}_n + \beta c + b)
\]

\[
= f_{\vec{\beta}',k}(\beta c + b) + \sum_{i=1}^n q_i g_i
\]

\[
= b' + \sum_{i=1}^n q_i f_{\vec{\alpha},i}(a - c)
\]

for \( b' = f_{\vec{\beta}',k}(\beta c + b) \in \langle B \rangle \).
By Lemma 3.10, for all \( \bar{b} \in B \), every \( L^+_0 \)-term \( t(a, \bar{b}) \) is the sum of an \( F \)-linear combinations of \( a, \bar{b} \) (which is of the desired form since \( a = \sum_{i=1}^{n} \alpha_i f_{\bar{a}, i}(a - c) + c \)) and of an \( F \)-linear combination of \( f_{\bar{a}, k}(\beta a + \gamma \bar{b}) \) which is also of the desired form by the Claim.

\[ \square \]

We will use the term ‘locally’ to mean that something holds on a finite definable partition of the universe.

**Lemma 3.12.** Let \( (V, G) \) be a model of \( T_{G^+} \), let \( \bar{b} \in V \), \( \bar{x} \in \mathbb{F} \). Let \( t(\bar{x}, \bar{y}) \) be an \( L^+_G \)-term. Then every formula of the form

\[ t(\bar{x}, \bar{y}) \in G_{\bar{x}} \land \bar{x} \subset G \]

is locally equivalent to a formula of the form \( P_{\psi}(\bar{x}, \bar{h}) \) for \( \psi(\bar{x}, \bar{z}) \) an \( L_{R-mod} \)-formula and a tuple \( \bar{h} \) in \( G(\langle B \rangle) \). Equivalently, there is a finite \( L \)-partition \( (\theta_i(\bar{x}))_i \) of the universe and \( L_{R-mod} \)-formulas \( \psi_i \) such that

\[ t(\bar{x}, \bar{y}) \in G_{\bar{x}} \land \bar{x} \subset G \iff \bigvee_i P_{\psi_i}(\bar{x}, \bar{h}) \land \theta_i(\bar{x}). \]

**Proof.** By Lemma 3.9, \( t(\bar{x}, \bar{y}) \) is locally an \( L^+_0 \)-term, and by Lemma 3.11 (1), we may assume that \( t(\bar{x}, \bar{y}) \) is locally of the form \( \alpha \cdot \bar{x} + b \) for some \( b \in \langle B \rangle \). We show that the formula \( \alpha \cdot \bar{x} + b \in G_{\bar{x}} \) is equivalent to a formula \( P_{\psi}(\bar{x}, \bar{h}) \) for \( \bar{h} \in G(\langle B \rangle) \) and \( \psi \) an \( L_{R-mod} \)-formula. Let \( \bar{a} \in G \) be a realisation of \( \alpha \cdot \bar{x} + b \in G_{\bar{x}} \) and let \( \bar{g} \in G \) be such that \( \alpha \cdot \bar{a} + b = \bar{x} \cdot \bar{g} \). Then for some basis \( \delta \) of \( \text{span}_R(\bar{\alpha} \bar{X}) \) we have \( b \in G_{\bar{\delta}} \) and hence there exists \( \bar{h} \in G(\langle B \rangle) \) such that \( b = \bar{\delta} \cdot \bar{h} \). It follows that \( \alpha \cdot \bar{a} + \bar{\delta} \cdot \bar{h} = \bar{X} \cdot \bar{g} \). As \( \bar{a} \in G \), by Lemma 3.7, there exists an \( L_{R-mod} \)-formula \( \psi(\bar{x}, \bar{g}) \) such that \( \exists \bar{z} \in G(\bar{x} \cdot \bar{z} + \bar{\delta} \cdot \bar{h} = \bar{X} \cdot \bar{z}) \) is equivalent to \( G \models \psi(\bar{x}, \bar{h}) \), hence to \( P_{\psi}(\bar{x}, \bar{h}) \).

\[ \square \]

**Remark 3.13.** Note that the proof above also proves that whenever \( t(\bar{x}, \bar{y}) \) is a term, there exists a finite \( L \)-partition \( (\theta_i(\bar{x}))_{i \leq n} \) of the universe and \( L_{R-mod} \)-formulas \( (\psi_i)_{i \leq n} \) such that

\[ t(\bar{x}, \bar{y}) = 0 \land \bar{x} \subset G \iff \bigvee_i P_{\psi_i}(\bar{x}, \bar{h}) \land \theta_i(\bar{x}). \]

**Proposition 3.14.** Let \( (V, G) \) be a model of \( T_{G^+} \) and let \( B = \langle B \rangle \subset V \). Assume that \( (V, G) \) is \(|B|-saturated \). Let \( a \in G \) and let

- \( q(x) \) be the set of boolean combinations of formulas of the form \( t(x, \bar{b}) \) in \( G_{\bar{x}} \) satisfied by \( a \), for \( \bar{b} \in B \), \( t(x, \bar{b}) \) an \( L^+_G \)-term and \( \bar{x} \in \mathbb{F} \).
- \( q_1(x) \) be the set of formulas of the form \( P_{\phi}(x, \bar{g}) \) satisfied by \( a \), for \( \bar{g} \in G(B) \), where \( \phi(x, \bar{y}) \) is an \( L_{R-mod} \)-formula.

Then \( t_{p_{\mathcal{L}}}(a/\emptyset) \cup q_1(x) = q(x) \). Moreover, for any non-algebraic \( L \)-type \( p(x) \), \( p(x) \cup q_1(x) \) is consistent with infinitely many realisations.

**Proof.** Let \( \phi(x, \bar{b}) \in q(x) \). By Lemma 3.12, for \( x \in G \), every formula of the form \( t(x, \bar{b}) \in G_{\bar{x}} \) is equivalent to \( \bigvee_i P_{\psi_i}(\bar{x}, \bar{h}) \land \theta_i(\bar{x}) \). As \( L \)-formulas and formulas of the form \( P_{\phi} \) are closed by boolean combinations, every formula \( t(x, \bar{b}) \notin G_{\bar{x}} \) is also equivalent to a finite disjunction of expressions the form \( \bigvee_j P_{\psi_j}(x, \bar{h}) \land \partial_j(x) \) (note
We prove it by induction on $\lambda$. Let $a' \models P_{L}(a/B) \cup q_1(x)$ and let $\phi(x, \vec{b}) \in q(x)$. Then by Lemma 3.12 there is an $L_{R-mod}$-formula $\psi(x, \vec{g})$ and an $L$-formula $\vartheta(x)$ such that $a \models P_{L}(x, \vec{g}) \land (P_{\vartheta}(x, \vec{g}) \land \vartheta(x)) \rightarrow \phi(x, \vec{b})$, for some $g \in G(B)$. It follows that $a'$ satisfies $\phi(x, \vec{b})$.

Let $p(x)$ be a non-algebraic $L$-type. We show that $q_1(x)$ is consistent with $p(x)$, with infinitely many realisations. Let $p_{sh}(x) = p(x + a)$. The type $p_{sh}$ is also non-algebraic, hence by density of $R$-divisible elements (Lemma 2.2), there exists infinitely many $d$’s such that $d \models p_{sh}(x) \cap G^{div}$. Let $a' = d + a$.

Claim: $a' \models q_1(x) \cup p(x)$. First, as $d \models p_{sh}(x)$, $a' = d + a \models p(x)$. By quantifier elimination in $R$-modules, every formula in $q_1(x)$ is a boolean combination of conditions of the form $rd + b \in s_1G + \cdots + s_nG$ for $r, s \in R$ and $b \in G(B)$. As $d \in G^{div}$, we have that $rd \in s_1G + \cdots + s_nG$ for all $r, s \in R$, hence $ra' + b \in s_1G + \cdots + s_nG$ if and only if $ra + b \in s_1G + \cdots + s_nG$, so $a' \models q_1(x)$. It follows that $q_1(x) \cup p(x)$ has infinitely many realisations. □

**Corollary 3.15.** Let $q(\vec{x})$ be an $L$-type over $B = \langle B \rangle$, such that $p(\vec{x}) \models \exists x \in X$. Let $\vec{g}$ be any set of boolean combinations of formula of the form $t(\vec{x}, \vec{b}) \in G_{\vec{g}}$, for $\vec{b} \in B$ and $t(\vec{x}, \vec{g})$ an $L_{G}^{+}$-term, which is consistent in $G$. If $(p(\vec{x}) \cup \emptyset) \cup q(\vec{x})$ is consistent in $G$, then $p(\vec{x}) \cup q(\vec{x})$ is also consistent in $G$.

**Proof.** We prove it by induction on $|\vec{x}|$. For $|\vec{x}| = 1$, it is Lemma 3.14 (since $p(x) \cup q_1(x) = p(x) \cup q(x)$). Assume $|\vec{x}| = n \geq 1$. Let $p_{n-1}(x_1, \ldots, x_{n-1})$ and $q_{n-1}(x_1, \ldots, x_{n-1})$ be the restrictions of $p(\vec{x})$ and $q(\vec{x})$ to the first $n - 1$ variables. By induction hypothesis, there exists $a_1, \ldots, a_{n-1} \models p_{n-1} \land q_{n-1}$. In particular, $a_1, \ldots, a_{n-1}$ are $F$-independent over $B$. Let $p'(x_n)$ and $q'(x_n)$ be completions to $\langle B_{a_1}, \ldots, a_{n-1} \rangle$ of $p(a_1, \ldots, a_{n-1}, x_n)$ and $q(a_1, \ldots, a_{n-1}, x_n)$, so that $p'(x_n) = \exists x_n \in \langle B_{a_1}, \ldots, a_{n-1} \rangle$. Using again Lemma 3.14, $p'(x_n) \cup q'(x_n)$ has a realisation, say $a_n$, then $a_1, \ldots, a_n \models p(x) \cup q(\vec{x})$. □

**Theorem 3.16.** The theory $T_{G}^{+}$ has quantifier elimination.

**Proof.** We show that the set of partial isomorphisms between two $[F]^{+}$-saturated models $(V, G), (V', G')$ of $T_{G}^{+}$ has the back and forth property. Let $B \subset V, B' \subset V'$ be two small substructures (i.e. $B = \langle B \rangle$ and $B' = \langle B' \rangle$) such that there exists a partial isomorphism $\sigma : B \to B'$ (i.e. a function that preserves the quantifier free type in $L_{G}^{+}$, relatively to the ambient model, $tp_{QFL_{G}^{+}}(B) = tp_{QFL_{G}^{+}}(B')$).

Let $a \in V \setminus B$. As $B = \langle B \rangle$, by hypothesis on $T$ we have that $tp_{\mathcal{L}}(a/B)$ is non-algebraic.

Step 1. If $a \in G(V)$. From Proposition 3.14, the quantifier-free type of $a$ over $B$ is implied by $tp_{\mathcal{L}}(a/B)$, the set $q(x)$ of boolean combinations of conditions of the form $t(x, \vec{b}) \in G_{\vec{g}}$ satisfied by $a$ in $V$, for $b \in B$, $t(x, \vec{g})$ an $L_{G}^{+}$ term and $\vec{g} \in G$, and inequations $ax \neq b$ for $b \in B$. Equations do not appear as $a \notin B$ (by Lemma 3.11 (1)). By compactness and Proposition 3.14, it is enough to show that $\sigma[q(x) \cup tp_{\mathcal{L}}(a/B)]$ has infinitely many solutions. Let $q_1(x)$ be as in Proposition 3.14. In particular, for any $\phi(x, \vec{g}) \in q_1(x)$ we have $(V, G) \models P_{\psi}(\vec{g})$ and $\vec{g} \in G(B)$
for $\psi(\vec{g}) = \exists x \phi(x, \vec{y})$, so $(V', G') \models P_\psi(\sigma(\vec{g}))$. By compactness, this shows that $\sigma(q_1) = q_2^*(x)$ is a consistent partial type in $(V', G')$, and $q_1^*$ is the set of boolean combinations of formulas of the form $t(x, \vec{b})$ satisfied by a realisation of $q_2^*$ in $V'$.

Let $p(x) = \sigma(tp_C(a/B))$. The fact that $q_2^*(x) \cup p(x)$ has infinitely many realisations follows from Proposition 3.14, as $p(x)$ is non-algebraic. Using again Proposition 3.14, we have that $q_2^*(x) \cup p(x) \models q_2^*(x)$, so the type $q_2^*(x) \cup p(x)$ is realised and non-algebraic, so we can extend $\sigma$ by $\sigma(a) = a'$ for some $a' \models q_2^*(x) \cup p(x)$.

Step 2. If $a \in \text{span}_F(BG)$, then by Lemma 3.11 (2) there exists $\vec{r}$-independent $\vec{a} \in F$ and $c \in B$ such that $a = \vec{a} \cdot \vec{g} + c$, and so $g_i = f_{\vec{a},i}(a - c)$. Lemma 3.11 (2) also implies that every $\mathcal{L}_G^+$-term $t(a, \vec{b})$ is locally an $F$-linear combination of $g_1, \ldots, g_n$. By Step 1 we can extend $\sigma$ to $g_1, \ldots, g_n$ hence we can extend $\sigma$ to $a$.

Step 3. If $a \in V \setminus \text{span}_F(BG)$. By Lemma 3.11 (1), every term $t(a, \vec{b})$ is locally of the form $aa + b$, for some $b \in B$. So it is enough to find in $V'$ an element $a'$ such that $a' + b' \notin G_{\vec{a}}$ for all $b' \in B'$, $\lambda \in F$, and such that $\sigma(tp_C(a/B)) = tp_C(a'/B')$. As $tp_C(a/B)$ is non-algebraic, such $a'$ exists by axiom $(D)$ and $|F|^+$-saturation. □

Recall that $T_U$ is the theory consisting of $T$ together with the schemes $(A)$ and $(B)$ and $T^G$ the theory consisting of adding $(A), (B), (C), (D)$, hence $T^G$ is the restriction of $T^{G^+}$ to the language $\mathcal{L}_G$.

**Corollary 3.17.** If $T_U$ is inductive, then $T^G$ is model-complete and it is the model-completion of $T_U$.

**Proof.** $T^{G^+}$ has quantifier elimination by Theorem 3.16, and every function $f_{\vec{a},i}$ is existentially definable in the language $\mathcal{L}_G$, to $T^G$ is model-complete. Note that the proof of Lemma 2.7 did not use that $\vec{R} = F$, hence we have that $T_U$ has SAP. It remains to prove that every model of $T_U$ extends to a model of $T^G$, which is similar to the proof of Corollary 2.9. □

We can now characterize the algebraic closure in the extended language.

**Corollary 3.18.** Let $(V, G)$ be a model of $T^{G^+}$ and let $B \subseteq V$. Then

$$\text{acl}_{\mathcal{L}_G}(B) = \text{acl}_{\mathcal{L}_G^+}(B) = \langle B \rangle = \text{span}_F(B \cup \{ f_{\vec{a},i}(c) \mid c \in \text{span}_F(B), \vec{X} \in F, i \leq |\vec{X}| \}).$$

**Proof.** It is clear that $\langle B \rangle \subseteq \text{acl}_{\mathcal{L}_G}(B) \subseteq \text{acl}_{\mathcal{L}_G}(B)$. Let $a \notin \langle B \rangle$.

**Case 1.** If $a \in G$, then from quantifier elimination $tp_G(a/B)$ is determined by $tp_C(a/B) \cup q(x)$ for $q(x)$ as in Proposition 3.14, and is non-algebraic from the conclusion of Proposition 3.14, since $tp_C(a/B)$ is non-algebraic.

**Case 2.** If $a \in \text{span}_F(BG(V))$. By Lemma 3.11 (2) there is an $F$-independent tuple $\vec{a}, \vec{g} \in G$ and $c \in B$ such that $a = \vec{a} \cdot \vec{g} + c$ and $g_i = f_{\vec{a},i}(a - c)$ and such that every term in $\vec{a}b$ is equal to an element in $\text{span}_F(\vec{g}B)$. By quantifier elimination, $tp_C(a/B)$ is determined by $tp_C(\vec{g}/B) \cup \{ g_1 = f_{\vec{a},1}(x - c), \ldots, g_n = f_{\vec{a},n}(x - c) \}$. As $a \notin \langle B \rangle$, there is some $i$ such that $g_i \notin \text{acl}(B)$, hence by case 1 $g_i \notin \text{acl}(B)$ hence $a \notin \text{acl}(B)$.

**Case 3.** If $a \in \text{span}_F(BG(V))$. By Lemma 3.11 and quantifier elimination, $tp_C(a/B)$ is determined by $tp_C(a/B) \cup \{ x \notin G_{\vec{a}} + b \mid b \in \langle B \rangle, \vec{X} \in F, \vec{R}$-independent $\}$). By codensity $(D)$, as $tp_C(a/B)$ is non-algebraic, this type has infinitely many realisations, so $a \notin \text{acl}_{\mathcal{L}_G}(B)$. □

From Theorem 3.16, every $\mathcal{L}_G$-formula $\phi(\vec{x})$ is a boolean combination of quantifier free $\mathcal{L}_G^+$-formulas, so there exists an $\mathcal{L}$-formula $\psi(\vec{x})$, an $\mathcal{L}_{R-mod}$-formula $\theta(\vec{x})$
and $\mathcal{L}_{\omega}^+$-terms $t_i(\vec{x}), t'_i(\vec{x}), t_j(\vec{x}), t'_j(\vec{x})$ such that $\phi(\vec{x})$ is equivalent to a disjunction of formulas of the form
\[
\psi(\vec{x}) \land P_\theta(\vec{x}) \land \bigwedge_i t_i(\vec{x}) \in G_{\vec{\lambda}_i} \land \bigwedge_j t_j(\vec{x}) \notin G_{\vec{\rho}_j} \land \bigwedge_i t_i(\vec{x}) = 0 \land \bigwedge_j t_j(\vec{x}) \neq 0
\]

Note that $P_\theta(\vec{x})$ is equivalent to a formula such that quantifiers only occur in the predicate $G$. Similarly, using the definition of $G_{\vec{\lambda}_i}$ and $f_{\vec{\lambda},i}$, we see that quantification only occurs in the group $G$. This particular instance of formulas is called bounded in the sense of [14], hence every formula in $T^G$ is bounded.

For subsets of $G$ we get a cleaner description.

**Definition 3.19.** Let $A \subset V$. We say that $A$ is $G$-independent if $A \perp_{G(A)} G(V)$.

We now give a characterization of $G$-independent sets.

**Lemma 3.20.** Let $A \subset V$, then $A$ is $G$-independent if and only if $\text{span}_F(A) = \langle A \rangle$.

**Proof.** Clearly $\text{span}_F(A) \subseteq \langle A \rangle$ holds for any set $A$. In order to proof the lemma, we will show that $A \perp_{G(A)} G(V)$ if and only if $\text{span}_F(A) \supseteq \langle A \rangle$.

We may assume that $A = \text{span}_F(A)$. Suppose first that $A \not\perp_{G(A)} G(V)$, then there is $a \in A \setminus \text{span}_F(G(A))$ such that $a \in \text{span}_F(G(V))$. Let $\lambda_1, \ldots, \lambda_n \in F$ and $g_1, \ldots, g_n \in G(V)$ be such that $a = \lambda_1 g_1 + \cdots + \lambda_n g_n$. We may choose $\lambda_1, \ldots, \lambda_n \in F$ to be $R$-independent. Since $a \notin \text{span}_F(G(A))$, then $A$ is not closed under the function $f_{\vec{\lambda},i}$ for some $i$ and thus $\text{span}_F(A) \not\subseteq \langle A \rangle$.

Now assume that $A \perp_{G(A)} G(V)$ and $A = \text{span}_F(A)$. Using Lemma 3.10, we get that $\langle A \rangle = \text{span}_F(A \cup \{ f_{\vec{\lambda},i}(c) \mid c \in \text{span}_F(A), \vec{\lambda} \in F, i \leq |\vec{\lambda}| \})$, so $A = \langle A \rangle$ if and only if $f_{\vec{\lambda},i}(a) \in A$ for all $a \in \text{span}_F(A)$, $\vec{\lambda} \in F$ a $R$-independent tuple and $i \leq |\vec{\lambda}|$.

Let $\lambda_1, \ldots, \lambda_n \in F$ be $R$-independent, let $a \in A$ and consider the function $f_{\vec{\lambda},i}(a)$. If $f_{\vec{\lambda},i}(a) = 0$, since $A = \text{span}_F(A)$, we have that $0 \in A$ as desired. If $f_{\vec{\lambda},i}(a) \neq 0$, then $a \in \text{span}_F(G) \cap A$, which equals $\text{span}_F(G(A))$ by assumption, so $a \in \text{span}_F(G(A))$. Then there exists an $R$-independent tuple $\vec{\alpha}$ such that $a = \sum \alpha_i f_{\vec{\alpha},i}(a)$, with $f_{\vec{\alpha},i}(a) \in G(A)$. By Lemma 3.11 (2) (with $B = 0$), every $\mathcal{L}_{\omega}^+$-term in $\vec{a}$ is a linear combination of $f_{\vec{\alpha},i}(a)$ hence belongs to $\text{span}_F(A)$ so $f_{\vec{\lambda},i}(a) \in G(A)$. \hfill \square

The next corollary shows quantifier elimination for $G$-independent tuples down to $\mathcal{L} \cup \{ G \}$-formulas and $R -$ mod-formulas (the latter restricted to elements of $G$), similar to Proposition 3.4 of [5].

**Corollary 3.21.** Assume $(V, G)$, $(W, G)$ are models of $T^{G+}$ and let $\vec{a} \in V$ and $\vec{b} \in W$ be $G$-independent tuples such that $\text{tp}_\mathcal{L}(\vec{a}, G(\vec{a})) = \text{tp}_\mathcal{L}(\vec{b}, G(\vec{b}))$ and $\text{tp}_{R - \text{mod}}(G(\vec{a})) = \text{tp}_{R - \text{mod}}(G(\vec{b}))$. Then $\text{tp}_{\mathcal{L}_{\omega}^+(G)}(\vec{a}) = \text{tp}_{\mathcal{L}_{\omega}^+(G)}(\vec{b})$.

**Proof.** Note that the $\mathcal{L}$-elementary map taking $\vec{a}$ to $\vec{b}$ extends uniquely to an elementary map

\[
\tau : \text{span}_F(\vec{a}) \rightarrow \text{span}_F(\vec{b}).
\]

We claim that
\[
\tau(G(\text{span}_F(\vec{a}))) = G(\text{span}_F(\vec{b})).
\]

It suffices to show that
\[
\tau(G(\text{span}_F(\vec{a}))) \subseteq G(\text{span}_F(\vec{b})).
\]
Indeed, if \( \bar{a} = (a_1, \ldots, a_n) \), and \( g \in G(\text{span}_F(\bar{a})) \), then \( g \in G(V) \) and
\[
g = \lambda_1 a_1 + \ldots + \lambda_n a_n
\]
for some \( \lambda_i \in \mathbb{F} \). On the other hand, \( \bar{a} \) is \( G \)-independent, so assuming for simplicity that \( G(\bar{a}) = (a_1, \ldots, a_k) \) for some \( k \leq n \), we also have \( g = \alpha_1 a_1 + \ldots + \alpha_k a_k \) for some \( \alpha_i \in \mathbb{F} \). By the Mordell-Lang property, we may assume that \( \alpha_i \in \hat{R} \). The fact that
\[
\alpha_1 a_1 + \ldots + \alpha_k a_k \in G(V)
\]
is witnessed by \( \text{tp}_{R-mod}(G(\bar{a})) \), and since \( \text{tp}_{R-mod}(G(\bar{a})) = \text{tp}_{R-mod}(G(\bar{b})) \), we also have that \( \tau(g) = \alpha_1 \tau(a_1) + \ldots + \alpha_k \tau(a_k) \in G(W) \), as we wanted. Since \( \bar{a} \) and \( \bar{b} \) are \( G \)-independent, Lemma 3.20 implies that \( \text{span}_F(a) = \langle a \rangle \) and \( \text{span}_F(b) = \langle b \rangle \) and by the observation above \( \tau(f_{\lambda,i}(\bar{a})) = f_{\lambda,i}(\bar{b}) \), so \( \tau \) preserves the functions in the extended language \( \mathcal{L}_G^+ \) and by quantifier elimination (Theorem 3.16) we get that \( \tau \) is an elementary map.

**Corollary 3.22.** Let \( \bar{a} \) be a \( G \)-independent tuple, let \( \bar{h} = G(\bar{a}) \) and let \( \varphi(\bar{x}, \bar{y}) \) be an \( \mathcal{L}_G^+ \)-formula. Then there is an \( \mathcal{L} \)-formula \( \psi(\bar{x}, \bar{y}) \) and an \( \mathcal{L}_{R-mod} \)-formula \( \theta(\bar{x}, \bar{y}) \) such that
\[
\forall \bar{x}[\varphi(\bar{x}, \bar{a}) \land G(\bar{x})] \leftrightarrow [\psi(\bar{x}, \bar{a}) \land P_\theta(\bar{x}, \bar{h}) \land G(\bar{x})]
\]

**Proof.** This is a direct consequence of quantifier elimination (Theorem 3.16), Lemma 3.12 and Remark 3.13.

Recall from [6] that a unary expansion \((M, P)\) of a model \(M\) of geometric theory \(T\) satisfies Type Equality Assumption (TEA) if whenever \(\bar{a}, \bar{b}, \bar{c} \in M\) are such that \(\bar{a}\) is \( P \)-independent (i.e. \(\bar{a} \downarrow_{P(\bar{a})} P(M)\)), \(\bar{b} \downarrow_{\bar{a}} P(M)\), \(\bar{c} \downarrow_{\bar{a}} P(M)\), and \(\text{tp}(\bar{b}a) = \text{tp}(\bar{a}\bar{a})\), then \(\text{tp}(\bar{b}a) = \text{tp}(\bar{a}\bar{a})\) (where \(\text{tp}_P\) refers to the type in the language expanded by the unary predicate symbol).

The following corollary follows directly from Corollary 3.21.

**Corollary 3.23.** Let \((V, G)\) be a model of theory \(T^G\). Then the reduct of \((V, G)\) to the language \(\mathcal{L} \cup \{ G \}\) satisfies TEA.

Note that the conclusion of TEA also holds for the full language \(\mathcal{L}_G^+\).

**Remark 3.24.** In the context of our construction, one can define the notion of "\(G\)-basis" analogous to the one used in [5]. Namely, given a \(G\)-independent set \(C\) and tuple \(\bar{a}\) in \(V\), we are looking for a "canonical" subset of \(G(V)\) such that adding it to \(\bar{a}\bar{C}\) makes the set \(G\)-independent. We claim that the appropriate notion of \(GB(\bar{a}/C)\) is given by
\[
\langle \bar{a}\rangle \cap G(V) \setminus \text{span}_R(G(C)).
\]
Indeed, \(GB(\bar{a}/C) \cup \bar{a} \cup C\) is \(G\)-independent, and for any \(\bar{b} \in G(V)\) such that \(\bar{a} \downarrow_{\bar{C}\bar{b}} G(V)\), we have \(GB(\bar{a}/C) \subset \text{span}_R(\bar{b}G(C))\).

We will now explicitly show that \(T^G\) is consistent by showing how to construct a model using the notion of \(H\)-structure [7]. We will assume the reader is familiar with the definition of \(H\)-structure, but we will not require any deeper knowledge of its theory. Let \((V, H)\) be a sufficiently saturated \(H\)-structure and let \(G(V)\) be the \(R\)-submodule of \((V, +, 0)\) generated by \(H(V)\). We will show that \(Th(V, G) = T^G\).
Lemma 3.25. $G(V)$ is dense-codense in $\mathcal{V}$ in the sense of geometric structures, and $(\mathcal{V}, G)$ satisfies axioms (A), (C), (D) of $T^G$.

Proof. It is clear that $H(\mathcal{V}) \subseteq G(\mathcal{V}) \subseteq \text{acl}(H(\mathcal{V}))$. Since $H$ structures satisfy the density and codensity property (see Definition 2.2 in [7]) then so does $G(V)$ and it follows that (A), (C) and (D) hold for $G(V)$. □

Lemma 3.26. Let $\lambda_1, \ldots, \lambda_k \in \hat{F}$ be linearly independent over $\hat{R}$, $g_1, \ldots, g_k \in G(V)$ and assume that $\lambda_1 g_1 + \ldots + \lambda_k g_k = 0$. Then $g_1 = \ldots = g_k = 0$, i.e. $(V, G)$ satisfies axiom (B) of $T^G$.

Proof. Let $h_1, \ldots, h_m \in H(\mathcal{V})$ be distinct elements such that $g_1, \ldots, g_k \in \text{span}_R(h_1, \ldots, h_m)$.

For each $1 \leq i \leq k$ let $r_{ij} \in R$ be such that

$$g_i = \sum_{j=1}^{m} r_{ij} h_j.$$

Then we have:

$$\lambda_1 \sum_{j=1}^{m} r_{ij} h_j + \ldots + \lambda_k \sum_{j=1}^{m} r_{kj} h_j = 0$$

$$\sum_{i=1}^{k} \sum_{j=1}^{m} \lambda_i r_{ij} h_j = 0$$

$$\left( \sum_{i=1}^{k} \lambda_i r_{i1} \right) h_1 + \ldots + \left( \sum_{i=1}^{k} \lambda_i r_{im} \right) h_m = 0.$$

Hence, for any $1 \leq j \leq m$ we have $\sum_{i=1}^{k} r_{ij} \lambda_i = 0$. Since $\lambda_1, \ldots, \lambda_k$ are linearly independent over $\hat{R}$, we conclude that $r_{ij} = 0$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$. Thus, $g_1 = \ldots = g_k = 0$, as needed. □

We can now deduce the following result.

Proposition 3.27. The theory $T^G$ is complete and consistent.

Proof. Consistency follows by Lemmas 3.25 and 3.26. Completeness follows by Theorem 3.16. □

4. Preservation of stability and NIP

Now we will start by proving that NIP is preserved in the expansion by using the ideas of Chernikov and Simon [14]. Let $(V, G)$ be a $G$-structure. Recall [14] that a formula $\varphi(\vec{x}, \vec{y}) \in L_G$ is said to be NIP over $G$ if there is no $L_G$-indiscernible sequence $\{ \vec{a}_i : i \in \omega \}$ of tuples of $G$ and an element $\vec{b} \in M$ such that $\varphi(\vec{a}_i, \vec{b}) \leftrightarrow i$ is even.

Proposition 4.1. Assume that $T$ is NIP. Then no formula $\varphi(\vec{x}, \vec{y}) \in L_G$ is NIP over $G$. 

Proof. Assume otherwise, so there is a $L_{G}$-indiscernible sequence $\{\bar{a}_{i} : i \in \omega\}$ of tuples of $G$ and an element $\bar{b} \in M$ such that $\varphi(\bar{a}_{i}, \bar{b}) \leftrightarrow i$ is even. We may enlarge $\bar{b}$ if necessary and assume that $\bar{b}$ is $G$-independent and let $\bar{h} = G(\bar{b})$. By Corollary 3.22 there are $\theta(\bar{x})$ an $R$-module formula and $\psi(\bar{x}, \bar{y})$ an $L$-formula such that $\forall \bar{x}(\varphi(\bar{x}, \bar{b}) \land G(\bar{x}) \leftrightarrow \theta(\bar{x}, \bar{h}) \land \psi(\bar{x}, \bar{b})))$. But every $R$-module formula is stable, so the cofinal value of $\theta(\bar{a}_{i}, \bar{h})$ is fixed. Similarly, since $T$ is NIP, the cofinal value of $\psi(\bar{a}_{i}, \bar{b})$ is fixed, a contradiction. \hfill \Box

**Proposition 4.2.** Assume that $T$ is NIP, then $T^{G+}$ is also NIP.

*Proof.* By quantifier elimination in $T^{G+}$ every $L_{G}$ formula is equivalent to a bounded formula and by Proposition 4.1 no formula has NIP over $G$, by Theorem 2.4 in [14] we have that $T_{P}$ is also NIP. \hfill \Box

A similar argument studying the induced structure of $(V, G)$ in $G$ and using the ideas of Casanovas and Ziegler [11] shows that stability is also preserved in the pair:

**Proposition 4.3.** Assume that $T$ is stable, then $T^{G+}$ is also stable.

We will now consider some examples.

**Example 4.4.** (Pure vector spaces, example 2.13 revisited)

Let $V$ be a pure vector space over a field $F$, which is strongly minimal. It is proved in [11] that the stability spectrum of the expansion depends on the stability spectrum of $V$ and the stability spectrum induced by the pair in the predicate. In our setting, since $V$ is $\aleph_{0}$-stable, it depends on the relation between $R$ and $F$.

We will concentrate in the case where the pair is $\omega$-stable and we will construct expansions with Morley rank $n$ for every $n \geq 2$. Assume that $R$ and $F$ are fields and $[F : R] = n$. We can see $F$ as a vector space of dimension $n$ over $R$. Choose $\lambda_{1}, \ldots, \lambda_{n}$ a basis of $F$ over $R$. Consider the map $f : G^{n} \rightarrow V$ given by $f(g_{1}, \ldots, g_{n}) = \lambda_{1}g_{1} + \cdots + \lambda_{n}g_{n}$. This map is definable and generically one to one. Since the structure of $G$ is that of a pure vector space over $R$, we have that $MR(G) = 1$ and thus $MR(f(G^{n})) = n$. By the extension property $V$ contains properly the set $f(G^{n})$, so $MR(V, G) = n + 1$. Note that $f(G^{n})$ is a vector space over $F$ and that the structure $(V, f(G^{n}))$ is a lovely pair of the theory vector spaces over $F$.

**Lemma 4.5.** (Lemma 2.15 revisited) Let $V$ be a pure vector space over a field $F$ and let $R$ be a subring of $F$. Then $T^{G+}$ is 1-based.

*Proof.* We follow the same strategy as we did in section 2. Since $V$ is a 1-based group and $T^{G+}$ has quantifier elimination in the extended language, it is enough to show that atomic formulas define boolean combinations of cosets of $0$-definable subgroups. By the proof of Lemma 2.15 it suffices to check that the result is true for formulas of the form $G_{\lambda}(t(\bar{x}, \bar{a}))$ and $t(\bar{x}, \bar{a}) = 0$ where $t(\bar{x}, \bar{a})$ is a term. Since $G_{\lambda}(x)$ defines a group, a similar computation to the one done in Lemma 2.15 gives the desired result. \hfill \Box

**Example 4.6.** (Ordered vector spaces, example 2.16 revisited)

In this example we deal with the base structure $V = (V, +, 0, <, \{\lambda_{r}\}_{r \in P})$ and we assume that $F$ is an ordered field and $V$ is an ordered vector space over $F$. The theory of $V$ is dense o-minimal, so it is geometric.
As in the previous example, we can consider the case where \( R \) and \( \mathbb{F} \) are ordered fields and \( [\mathbb{F} : R] = n \). Then the expansion is strongly dependent and has \( \text{dp} - \text{rk}(V,G) = n + 1 \). This example can also be studied from the perspective of [9]; in particular by Theorem 4.8 [9] it was already known that this expansion is NIP. The authors of [9] also show this expansion has open core and characterize when it is decidable.

5. Preservation of Tree Properties: NTP2, NTP1 and NSOP1

In this section, we prove the expansion preserves other nice properties such as NTP2, NTP1 and NSOP1. To prove the preservation of NTP2 and NTP1, we will follow the approach from [3], [20], but will need to modify several parts of the arguments. The main difference is that in our setting the definable subsets in \( G \) involve not only the induced structure from \( V \) (as is the case in [3, 20]) but also the structure that it carries as an \( R \)-module. We will follow a similar approach to show the preservation of NSOP1 using as a guide the ideas from [26]. Let us start by recalling the basic definitions for NTP2.

**Definition 5.1.** A theory \( T \) has \( k \text{-TP} \) (for some integer \( k \geq 2 \)) if there exist a formula \( \varphi(\vec{x}, \vec{y}) \) and a set of tuples \( \{\vec{a}_{i,j} \mid i, j < \omega \} \) (in some model of \( T \)) such that \( \{\varphi(\vec{x}, \vec{a}_{i,j}(i)) \mid i < \omega \} \) is consistent for every function \( f : \omega \to \omega \) and \( \{\varphi(\vec{x}, \vec{a}_{i,j}) \mid j < \omega \} \) is \( k \)-inconsistent for every \( i < \omega \). A theory has TP2 if it has 2-TP2 and a theory has NTP2 if it does not have TP2.

**Definition 5.2.** Let \( M \) be a structure in a language \( \mathcal{L} \). A set of parameters \( \{\vec{a}_\mu \mid \mu \in \omega \times \omega \} \) in \( M \) is called an indiscernible array if the \( \mathcal{L} \)-type of any finite tuple \( (\vec{a}_{\mu_1}, \ldots, \vec{a}_{\mu_n}) \) is determined by the quantifier-free array-type of the tuple \( (\mu_1, \ldots, \mu_n) \).

Just as sequences can be enlarged in order to extract indiscernible sequences, for arrays we have the following result:

**Fact 5.3.** If a formula \( \varphi(\vec{x}, \vec{y}) \) witnesses \( k \text{-TP} \) then it may do so with an indiscernible array. Moreover, for such any such indiscernible array and any function \( f : \omega \to \omega \), the collection of formulas \( \{\varphi(\vec{x}, \vec{a}_{i,f(i)}) \mid i < \omega \} \) has infinitely many realizations.

Using a result of Chernikov [13] reducing the property \( TP_2 \) to formulas of the form \( \varphi(x, \vec{y}) \) (where \( x \) is a single variable) and the fact that we can witness \( TP_2 \) with indiscernible arrays, we can reduce the problem to:

**Proposition 5.4.** A theory has \( TP_2 \) if and only if there exist a formula \( \varphi(x, \vec{y}) \) (where \( x \) is a single variable) and an indiscernible array \( \{\vec{a}_{i,j} \mid i, j < \omega \} \) such that

1. \( \bigwedge_{i<\omega} \varphi(x, \vec{a}_{i,0}) \) has infinitely many realizations,
2. \( \bigwedge_{j<\omega} \varphi(x, \vec{a}_{0,j}) \) has at most finitely many realizations.

**Proof.** For details, the reader can see [3, Section 3].

Now let us consider the problem in the setting of this paper. As before, we write \( \mathcal{L} \) for the language of the vector space (maybe with extra structure), \( T \) for its theory and \( \mathcal{L}_G \) and \( T_G^{G^+} \) are the language and the theory of the associated \( G \)-structure. As mentioned before, we want to show that \( T \) has NTP2 if and only if \( T_G^{G^+} \) does. Our first result in this direction deals with the induced structure on the predicate.
Proposition 5.5. Assume there exists some $\mathcal{L}_G$-formula $\varphi(x, \vec{y})$ (where $x$ is a single variable) such that $\varphi(x, \vec{y}) \land G(x)$ witnesses $k$-$TP_2$ for some $k \geq 2$. Then $T$ has $TP_2$.

Proof. Assume that there exists such an $\mathcal{L}_G$-formula $\varphi(x, \vec{y})$. By Proposition 5.4 we may assume that $\varphi(x, \vec{y}) \land G(x)$ witnesses $TP_2$ with some indiscernible array $A := \{\vec{a}_{i,j} \mid i, j < \omega\}$ and that for every function $f : \omega \to \omega$, the collection $\{\varphi(\vec{x}, \vec{a}_{i,f(i)}) \mid i < \omega\}$ is infinite. Furthermore, enlarging the indiscernible array if necessary, we may assume that each $\vec{a}_{i,j}$ is $G$-independent (for details see [3, Section 4]). Then, by Corollary 3.22, there exists some $\mathcal{L}$-formula $\psi(x, \vec{y})$ and a $\mathcal{L}_{R}$-$mod$-formula $\theta(x, \vec{y})$ such that for all $i, j < \omega$,

$$\varphi(x, \vec{a}_{i,j}) \land G(x) \leftrightarrow \psi(x, \vec{a}_{i,j}) \land \theta(x, \vec{a}_{i,j}) \land G(x)$$

Since $\bigwedge_{i < \omega} \varphi(x, \vec{a}_{i,0}) \land G(x)$ has infinitely many realizations, the conjunction $\bigwedge_{i < \omega} \theta(x, \vec{a}_{i,0})$ also has infinitely many realizations in $G$.

Claim 1. $\bigwedge_{i < \omega} \theta(x, \vec{a}_{i,0})$ has at infinitely many realizations.

Otherwise it has finitely many realizations and by Proposition 5.4 the $R$-module formula $\theta(x, \vec{y})$ has $TP_2$. Since the theory of $R$-modules is stable, we obtain a contradiction.

Let us now analyze the formula $\psi(x, \vec{y})$. As before, it is easy to see that the conjunction $\bigwedge_{i < \omega} \psi(x, \vec{a}_{i,0})$ also has infinitely many realizations.

Claim 2. $\bigwedge_{i < \omega} \psi(x, \vec{a}_{i,0})$ has finitely many realizations.

Otherwise, it has infinitely many solutions.

By Proposition 3.14 and Claim 1 we have that $\bigwedge_{j < \omega} \psi(x, \vec{a}_{0,j}) \land \bigwedge_{j < \omega} \theta(x, \vec{a}_{0,j})$ also has infinitely many solutions, a contradiction.

Since the conjunction $\bigwedge_{i < \omega} \psi(x, \vec{a}_{i,0})$ has infinitely many realizations, and the conjunction $\bigwedge_{j < \omega} \psi(x, \vec{a}_{0,j})$ has finitely many realizations, by Proposition 5.4 we conclude that $T$ has $TP_2$. $\square$

As in [3, Section 4] we can extend the previous result to several variables:

Corollary 5.6. If there exists some $\mathcal{L}_G$-formula $\varphi(\vec{x}, \vec{y})$ such that $\varphi(\vec{x}, \vec{y}) \land G(\vec{x})$ witnesses $k$–$TP_2$ for some $k \geq 2$ then $T$ has $TP_2$.

Proof. It follows from submultiplicity of burden, for details see [3] $\square$

Definition 5.7. Let $(V, G) \models T^G$ be sufficiently saturated, let $X \subset V'$ and let $A \subset V$ be a set of parameters. We say that $X$ is $A$-small if $X \subset acl(AG)$. If $X$ is $A$-small for some $A$, then we say $X$ is small, otherwise we say the set $X$ is large. Similarly, for $b \in V$, we write $b \in acl(A)$ if $b \in acl(AG)$ and we say that $b$ belongs to the small closure of $A$.

In this section we need to approximate large sets by $\mathcal{L}$-definable sets:

Proposition 5.8. Let $(V, G) \models T^G$ and let $Y \subset V$ be $\mathcal{L}_G$-definable. Then there is $X \subset V$ an $\mathcal{L}$-definable set such that $Y \triangle X$ is small.

Proof. By Theorem 3.16 we may assume $Y = \bigcup_{j \leq n} Y_j$ where each $Y_j$ is the set of realizations of a formula of the form

$$\psi_j(x) \land P_{y_j}(x) \land \bigwedge_k t_k(x) \in G_{\vec{x}_j} \land \bigwedge_l t_l(x) \notin G_{\vec{a}_j} \land \bigwedge_k t_k(x) = 0 \land \bigwedge_k t_l(x) \neq 0$$

Assume for each $Y_j$ we can find a $\mathcal{L}$-definable set $X_j$ such that $Y_j \triangle X_j$ is small. Then $\bigcup_{j \leq n} Y_j \triangle \bigcup_{j \leq n} X_j \subset \bigcup_{j \leq n} (Y_j \triangle X_j)$ is small. So it suffice to prove the result
for the sets $Y_j$. If the set $Y_j$ is small we can choose $X = \emptyset$. Otherwise we can choose $X_j = \psi_j(V)$. 

\[ \square \]

Note that the above proposition also follows from the fact that $T^G$ has TEA, see Corollary 3.23 and [6, Proposition 2.6]. We are ready to prove the first main result.

**Theorem 5.9.** If $T^{G+}$ has TP$_2$ then so does $T$.

**Proof.** Assume $T^{G+}$ has TP$_2$. So there exists some $\mathcal{L}_G$-formula $\varphi(x, \vec{y})$ (where $x$ is a single variable) witnessing TP$_2$ with some indiscernible array $A := \{ \vec{a}_{i,j} \mid i, j < \omega \}$ and we may assume each element $\vec{a}_{i,j}$ is $G$-independent. There are two possible cases:

**Case 1.** $\bigwedge_{i<\omega} \varphi(x, \vec{a}_{i,0})$ is realized by some $b \in \text{scl}(A)$.

Such $b$ is then in the algebraic closure (in the language $\mathcal{L}$) of some tuples $\vec{c} = (c_1, \ldots, c_n) \in A$ and $\vec{h} = (h_1, \ldots, h_k) \in G(M)$. Since in the old language the algebraic closure coincides with the vector space span, we have $b \in \text{dcl}(\vec{c}, \vec{h})$ and there are some coefficients in $\mathbb{P}$ such that $b = \sum_{i=1}^n \lambda_i c_i + \sum_{j=1}^k \lambda'_j h_j$. Let $\hat{\phi}(z_1, \ldots, z_k, \vec{y}; \vec{c})$ be the formula

$$
\varphi\left(\sum_{i=1}^n \lambda_i c_i + \sum_{j=1}^k \lambda'_j z_j, \vec{y}\right) \land \bigwedge_{j=1}^k G(z_j)
$$

Choose a finite $N$ such that $\vec{c}$ is part of the sub-array $\{ \vec{a}_{i,j} \mid i \leq N, j < \omega \}$ and let $A' := \{ \vec{a}_{i,j} \mid j < \omega, N < i < \omega \}$. It is then easy to show that the $\mathcal{L}_G$-formula $\hat{\phi}(z_1, \ldots, z_k, \vec{y}; \vec{c})$ has TP$_2$ with respect to the array $A'$ and the result follows from Proposition 5.5.

**Case 2.** All the realizations of $\bigwedge_{i<\omega} \varphi(x, \vec{a}_{i,0})$ are in $M \setminus \text{scl}(A)$.

By Proposition 5.8 and indiscernibility of the array $A$, there exists a single $\mathcal{L}$-formula $\psi(x, \vec{y})$ such that, for each $i, j < \omega$, $\varphi(x, \vec{a}_{i,j}) \land \psi(x, \vec{a}_{i,j})$ defines an $\vec{a}_{i,j}$-small set. Since the realizations of the conjunction are not small, every realization of $\bigwedge_{i<\omega} \varphi(x, \vec{a}_{i,0})$ is also a realization of $\bigwedge_{i<\omega} \psi(x, \vec{a}_{i,0})$. In particular, $\bigwedge_{i<\omega} \psi(x, \vec{a}_{i,0})$ has infinitely many realizations.

Moreover, $\bigwedge_{j<\omega} \psi(x, \vec{a}_{0,j})$ has only finitely many realizations. (Otherwise, the co-density condition of the predicate $G$ implies that $\psi(x, \vec{a}_{0,0}) \land \psi(x, \vec{a}_{0,1})$ is realized by some $d \in M \setminus \text{scl}(\vec{a}_{0,0}\vec{a}_{0,1})$. But then such $d$ also realizes $\varphi(x, \vec{a}_{0,0}) \land \varphi(x, \vec{a}_{0,1})$, contradiction.)

Hence, $T$ has TP$_2$ by Proposition 5.4. \[ \square \]

Now we deal with NTP$_1$ (also called NSOP$_2$). We will now follow the strategy from [20], emphasizing the main differences that are needed to adapt the arguments to the new setting. We start with an appropriate notion of tree-indiscernability (see [20, Def. 3.3]) that will play the role of indiscernible array in the argument for NTP$_2$.

**Definition 5.10.** Given ordinals $\alpha, \beta$ we see the set $S = \alpha^{<\beta}$ as a tree. We say that a tree $(\vec{a}_n)_{n \in S}$ of compatible tuples of elements of a model $M$ is strongly indiscernible over a set $C \subset M$, if whenever the ordered tree types satisfy $q_{\text{tp}_{\text{tree}}}(\eta_0, \ldots, \eta_{n-1}) = q_{\text{tp}_{\text{tree}}}(\nu_0, \ldots, \nu_{n-1})$ then $\text{tp}(a_{\eta_n}, \ldots, a_{\eta_{n-1}}/C) = \text{tp}(a_{\nu_n}, \ldots, a_{\nu_{n-1}}/C)$ for all $n < \omega$ and all tuples $(\eta_0, \ldots, \eta_{n-1}), (\nu_0, \ldots, \nu_{n-1})$ of elements of $S$. 


Fact 5.11 (Dobrowolski-H.Kim). A theory $T$ has SOP$_2$ if there is a formula $\phi(\vec{x}, \vec{y})$ and a strongly indiscernible tree $(a_\eta)_{\eta \in 2^{<\omega}}$ such that

1. $\{ \phi(\vec{x}, a_\eta) \mid n < \omega \}$ has infinitely many realizations;
2. $\phi(\vec{x}, a_0) \land \phi(\vec{x}, a_1)$ has finitely many realizations.

If $q(\vec{x})$ is a type and if instead of condition (1) we have

1'. $\{ \phi(\vec{x}, a_\eta) \mid n < \omega \} \land q(\vec{x})$ has infinitely many realizations

then we say the theory has SOP$_2$ inside $q(\vec{x})$.

Instead of an argument of burden, in this setting we have.

Fact 5.12 (Dobrowolski-H.Kim (Fact 2.7 [20])). Suppose a theory $T$ has SOP$_2$ inside of some type $q(x_0, ..., x_{n-1}) = \bigcup_{i < n} q_i(x_i)$. Then, for some $i < n$, $T$ has SOP$_2$ inside of $q_i(x_i)$.

Lemma 5.13. Let $\phi(\vec{x}, \vec{y})$ be an $L_G$-formula. If $\phi(\vec{x}, \vec{y}) \land G(\vec{x})$ has SOP$_2$, then $T$ has SOP$_2$.

Proof. By applying Fact 5.12 with the type $q(\vec{x}) = G(\vec{x})$, we may assume that $|\vec{x}| = 1$. By Fact 5.11 there is a strongly indiscernible tree $(a_\eta)_{\eta \in 2^{<\omega}}$ witnessing SOP$_1$ and the following properties hold:

1. $\{ \phi(x, a_\eta) \land G(x) \mid n < \omega \}$ has infinitely many realizations;
2. $\phi(x, a_0) \land \phi(x, a_1) \land G(x)$ has finitely many realizations.

By Corollary 3.22, there exists some $L$-formula $\psi(x, \vec{y})$ and an $R$-module formula $\theta(x, \vec{y})$ such that for all $i, j < \omega$,

$$\varphi(x, \vec{a}_n) \land G(x) \leftrightarrow \psi(x, \vec{a}_n) \land \theta(x, \vec{a}_n) \land G(x)$$

It is clear that $\bigwedge_n \psi(x, a_{n_0})$ and $\bigwedge_n \theta(x, a_{n_0})$ have infinitely many realisations. As $\theta(x, \vec{y})$ is stable, it also follows that $\theta(x, a_0) \land \theta(x, a_1)$ has infinitely many realisations (otherwise using Fact 5.11 we would have that $\theta(x, \vec{y})$ witnesses SOP$_2$).

Claim. $\psi(x, a_0) \land \psi(x, a_1)$ has finitely many realisations.

Assume not, then $\psi(x, a_0) \land \psi(x, a_1)$ has infinitely many realisations and by Proposition 3.14, so does $\psi(x, a_0) \land \psi(x, a_1) \land G(x) \land G(x)$, a contradiction.

Since $\psi(x, \vec{y})$ is an $L$-formula, we conclude that $T$ has SOP$_2$. \(\square\)

Theorem 5.14. If $T^{G+}$ has SOP$_2$ then so does $T$.

Proof. Assume $T^{G+}$ has SOP$_2$. By Fact 5.12, there is a formula $\varphi(x, \vec{y})$ and a strongly indiscernible tree $S = (a_\eta)_{\eta \in 2^{<\omega}}$ that witnesses it. We may also assume each element $\vec{a}_\eta$ of the tree is $G$-independent. There are two possible cases:

Case 1. $\bigwedge_{i < \omega} \varphi(x, \vec{a}_\eta)$ is realized by some $b \in \text{scl}(S)$.

Such $b$ is in the $L$-definable closure of some tuples $\vec{c}$ in $S$ and $\vec{h}$ in $G(M)$, so there are coefficients $(\lambda_i), (\rho_j)$ in $F$ such that $b = \sum \lambda_i h_i + \sum \rho_j c_j$ for some elements $\{s_j\}_j$ in $S$. Let $n < \omega$ be such that $\vec{c}$ belongs to the subtree $(a_\eta)_{\eta \in 2^{<\omega}, |\eta| \leq n}$ and choose $\nu < 2^{<\omega}$ such that $|\nu| > n$. Let $\psi(x, \vec{x}', \vec{y}, \vec{z})$ be the formula $\varphi(x, \vec{y}) \land x = \sum \lambda_i x'_i + \sum \rho_j z_j$. It is then easy to check that the $L_G$-formula $\exists x \psi(x, \vec{x}', \vec{y}, \vec{c}) \land G(\vec{x}')$ has SOP$_2$ with respect to the strongly indiscernible tree $(a_\eta c)_{\eta \in 2^{<\omega}, |\eta| > \nu}$, and the result follows from Lemma 5.13.

Case 2. All the realizations of $\bigwedge_{i < \omega} \varphi(x, \vec{a}_\eta)$ are in $M \setminus \text{scl}(S)$.

By Proposition 5.8, there exists some $L$-formula $\psi(x, \vec{y})$ such that, for each $\eta \in 2^{<\omega}$, $\varphi(x, \vec{a}_\eta) \land \psi(x, \vec{a}_\eta)$ defines an $\vec{a}_\eta$-small set. Since all realizations are not small over the tree $S$, this implies that every realization of $\bigwedge_{i < \omega} \varphi(x, \vec{a}_\eta)$ is also
a realization of $\bigwedge_{i < \omega} \psi(x, \vec{a}_i)$. In particular, $\bigwedge_{i < \omega} \psi(x, \vec{a}_i)$ has infinitely many realizations.

We show that $\psi(x, \vec{a}_0) \land \psi(x, \vec{a}_1)$ has only finitely many realizations. Assume not, then by the codensity property for $G$, there some $d \in M \setminus \text{scl}(\vec{a}_0\vec{a}_1)$ realizing $\psi(x, \vec{a}_0) \land \psi(x, \vec{a}_1)$. As $\varphi(x, \vec{a}_0) \Delta \psi(x, a_0)$ is $a_0$-small, $d$ satisfies $\varphi(x, a_0)$, and similarly, $d$ satisfies $\varphi(x, a_1)$. It follows that $d$ satisfies $\varphi(x, a_0) \land \varphi(x, a_1)$. Repeating this process and using again the codensity property, there is an infinite sequence $(\vec{d}_i : i < \omega)$ of realisations of $\psi(x, \vec{a}_0) \land \psi(x, \vec{a}_1)$ such that $\vec{d}_i \notin \text{scl}(a_0a_1\vec{d}_{<i})$. Hence $\varphi(x, a_0) \land \varphi(x, a_1)$ has infinitely many realisation, a contradiction.

It follows that the $L$-formula $\psi(x, \vec{y})$ witnesses that $T$ has SOP$_2$. \qed

In particular, since simple theories are those that have at the same time NTP$_2$ and NSOP$_2$ we get the following Corollary.

**Corollary 5.15.** Assume $T$ is simple, then so is $T^{G+}$.

Now we deal with NSOP$_1$. We will use the following characterization presented in the work by Ramsey [26]:

**Definition 5.16.** Let $T$ be a complete theory and let $M \models T$ be sufficiently saturated. We say $T$ has SOP$_1$ if there is a formula $\varphi(\vec{x}; \vec{y})$, possibly with parameters in a set $C$, and array $(\vec{c}_{i,j})_{i<\omega,j<2}$ so that

(a) $\vec{c}_{i,0} \equiv_{C \subseteq_i} \vec{c}_{i,1}$ for all $i < \omega$.

(b) $\{ \varphi(x; \vec{c}_{i,0}) : i < \omega \}$ is consistent.

(c) $\{ \varphi(x; \vec{c}_{i,1}) : i < \omega \}$ is 2-inconsistent.

By Lemma 2.5 and Theorem 2.7 [26] it suffices to check the definition above for formulas $\varphi(x; \vec{y})$ where $x$ is a single variable and we may choose an array $(\vec{c}_{i,j})_{i<\omega,j<2}$ that also satisfies

(d) $(\vec{c}_{i,j})_{i<\omega}$ is an indiscernible sequence.

(e) $(\vec{c}_{k,0})_{k<i}$ is $\vec{c}_{<i} \vec{c}_{i,1}$-indiscernible.

Whenever $(\vec{c}_{i,j})_{i<\omega,j<2}$ satisfies these extra conditions, we say the array $(\vec{c}_{i,j})_{i<\omega,j<2}$ is an indiscernible array.

We will need the following modified version of the property that is more suitable to dense pairs. The next result is due to Nicholas Ramsey:

**Proposition 5.17** (Ramsey). A theory $T$ has SOP$_1$ if and only if there exists some set $C$, a formula $\varphi(x; \vec{y})$ (where $x$ is a single variable) and an indiscernible array $(\vec{c}_{i,j})_{i<\omega,j<2}$ such that

(a) $\vec{c}_{i,0} \equiv_{C \subseteq_i} \vec{c}_{i,1}$ for all $i < \omega$;

(b) $\{ \varphi(x; \vec{c}_{i,0}) : i < \omega \}$ is consistent and non-algebraic;

(c) Whenever $i < j$, $\varphi(x; \vec{c}_{i,1}) \land \varphi(x; \vec{c}_{j,1})$ is finite.

**Proof.** If $T$ has SOP$_1$, then by [26, Theorem 2.7], there is a formula in one variable witnessing SOP$_1$, in particular, it satisfies conditions (a), (b) and (c). Conversely assume that $C, (\vec{c}_{i,j})_{i<\omega,j<2}$ and $\phi(x, \vec{y})$ are given. Since the array is indiscernible, there is an $N$ such that for any $i < j$ the set $\varphi(x; \vec{c}_{i,1}) \land \varphi(x; \vec{c}_{j,1})$ has cardinality $N$. Consider the formula

$$
\psi(\vec{x}, \vec{y}) = \psi(x_0, ..., x_N, \vec{y}) = \bigwedge_{0 \leq i \leq N} \phi(x_i, \vec{y}) \land \bigwedge_{0 \leq i < j \leq N} x_i \neq x_j
$$

\[25\]
Using (b), \( \{ \psi(\vec{x}, \vec{c}_i, 0) : i < \omega \} \) is still consistent. However if \( \psi(\vec{x}, \vec{c}_i, 1) \land \psi(\vec{x}, \vec{c}_j, 1) \) were consistent, there would be more than \( N \) different realisations of \( \varphi(x, \vec{c}_i, 1) \land \varphi(x, \vec{c}_j, 1) \), so \( \psi(\vec{x}, \vec{c}_i, 1) \land \psi(\vec{x}, \vec{c}_j, 1) \) is inconsistent, and \( T \) has SOP$_1$. \( \Box \)

We follow the same strategy that as we did earlier in this section, first we check what happens for formulas in \( G(x) \) and then we extend the results to the general case.

**Proposition 5.18.** Assume there exists some \( L_G \)-formula \( \varphi(x, \vec{y}) \) (where \( x \) is a single variable) such that \( \varphi(x, \vec{y}) \land G(x) \) witnesses SOP$_1$. Then \( T \) has SOP$_1$.

**Proof.** Assume that there exists such an \( L_G \)-formula \( \varphi(x, \vec{y}) \) that witnesses SOP$_1$. We may assume that the sequence \( (\vec{c}_i)_{i<\omega} \) witnessing SOP$_1$ is an indiscernible array and enlarging the tuples if necessary we may assume that each \( \vec{c}_i \) is \( G \)-independent. Using Corollary 3.22, there exists some \( L \)-formula \( \psi(x, \vec{y}) \) and a \( L_{R-mod} \)-formula \( \theta(x, \vec{y}) \) such that for all \( i, j < \omega \),

\[
\varphi(x, \vec{a}_{i,j}) \land G(x) \leftrightarrow \psi(x, \vec{a}_{i,j}) \land \theta(x, \vec{a}_{i,j}) \land G(x)
\]

Since \( \bigwedge_{i<\omega} \varphi(x, \vec{c}_i, 0) \land G(x) \) is consistent and non-algebraic, clearly the collection of \( L \)-formulas \( \{ \psi(x, \vec{c}_i, 0) : i < \omega \} \) is consistent and non-algebraic. Similarly, \( \{ \theta(x, \vec{c}_i, 0) : i < \omega \} \) is consistent and non-algebraic. Also \( \vec{c}_{i,0} \equiv_{\vec{c} \subseteq 1} \vec{c}_{i,1} \) for all \( i < \omega \). We just need to show the almost 2-inconsistency of the family \( \{ \psi(x, \vec{c}_i, 1) : i < \omega \} \).

**Claim 1.** \( \theta(x, \vec{c}_i, 1) \land \theta(x, \vec{c}_j, 1) \) has infinitely many realizations whenever \( i < j \).

Otherwise whenever \( i < j \) we have that \( \theta(x, \vec{c}_i, 1) \land \theta(x, \vec{c}_j, 1) \) has infinitely many realizations and by Proposition 5.17 the \( R \)-module formula \( \theta(x, \vec{y}) \) has SOP$_1$. Since the theory of \( R \)-modules is stable, we obtain a contradiction.

Let us now analyze the formula \( \psi(x, \vec{y}) \).

**Claim 2.** \( \psi(x, \vec{c}_i, 0) \land \psi(x, \vec{c}_j, 0) \) has finitely many realizations whenever \( i < j \).

Assume not, then \( \psi(x, \vec{c}_i, 0) \land \psi(x, \vec{c}_j, 0) \) has infinitely many realisations and by Proposition 3.14, so does \( \psi(x, \vec{c}_i, 0) \land \psi(x, \vec{c}_j, 0) \land \theta(x, \vec{c}_i, 0) \land \theta(x, \vec{c}_j, 0) \land G(x) \), a contradiction.

Now apply again Proposition 5.17 to the formula \( \psi(x, \vec{y}) \) to get the desired result. \( \Box \)

**Corollary 5.19.** Assume there exists some \( L_G \)-formula \( \varphi(\vec{x}, \vec{y}) \) (where \( \vec{x} \) is a tuple) such that \( \varphi(\vec{x}, \vec{y}) \land G(\vec{x}) \) witnesses SOP$_1$. Then \( T \) has SOP$_1$.

**Proof.** An easy modification of the proof of Theorem 2.7 in [26] shows that for some single variable \( x_i \) in the tuple \( \vec{x} \) and an appropriate tuple \( \vec{b} \), the formula \( \varphi(x_i, \vec{b}, \vec{y}) \land G(x_i, \vec{b}) \) has SOP$_1$. Now apply Proposition 5.18 to get the desired result. \( \Box \)

With the previous results we are ready to show the last theorem of this section:

**Theorem 5.20.** If \( T^{G+} \) has SOP$_1$ then so does \( T \).

**Proof.** Assume that \( T^{G+} \) has SOP$_1$. Let \( C, \varphi(x, \vec{y}) \) and \( S = (\vec{c}_{i,j})_{i<\omega, j<2} \) be as in Proposition 5.17, in particular \( \bigwedge_{i<\omega} \varphi(x; \vec{c}_{i,0}) \) is consistent and non-algebraic. We may assume that \( (\vec{c}_i)_{i<\omega} \) is an indiscernible sequence and enlarging the tuples if necessary we may assume that each \( \vec{c}_{i,j} \) is \( G \)-independent. Also, to simplify the presentation we will assume that \( C = \emptyset \).

**Case 1.** \( \bigwedge_{i<\omega} \phi(x; \vec{c}_{i,0}) \) is realised by some \( b \in \text{scl}(S) \).
Let \( \vec{d} \in S \) and let \( \vec{g} \in G \) be tuples such that \( b = \sum_i \lambda_i d_i + \sum_j \rho_j g_j \) for some coefficients \( \{\lambda_i\}_i, \{\rho_j\}_j \in \mathbb{F} \). Let \( \psi (x, \vec{x}, \vec{y}, \vec{z}) \) be the formula \( \phi (x, \vec{y}) \wedge x = \sum_i \lambda_i z_i + \sum_j \rho_j x_j \). Let \( n_0 < \omega \) be such that \( \vec{d}' \subset \{ \vec{e} : i < n_0, j < 2 \} \). Then for all \( n_0 < i < \omega, \vec{e} \equiv \vec{c} \) and \( \vec{e}_1. \) It is then easy to see that the formula \( \exists \psi (x, \vec{x}, \vec{y}, \vec{d}) \wedge G (\vec{x}') \) witnesses SOP\(_1\), hence by Corollary 5.19, \( T \) has SOP\(_1\).

Case 2. All realisations of \( \bigwedge_{i<\omega} \phi (x, \vec{c}_i, 0) \) are not in \( \text{scl}(S) \).

By Proposition 5.8, there exists some \( L \)-formula \( \psi (x, \vec{y}) \) such that, for each \( i < \omega \) and \( j < 2 \), \( \phi (x, \vec{c}_i, \vec{y}) \Delta \psi (x, \vec{c}_i, j) \) defines a \( \vec{c}_i, j \)-small set. Since all realisations are not small over \( S \), this implies that every realization of \( \bigwedge_{i<\omega} \phi (x, \vec{c}_i, 0) \) is also a realization of \( \bigwedge_{i<\omega} \psi (x, \vec{c}_i, 0) \). In particular, \( \bigwedge_{i<\omega} \psi (x, \vec{c}_i, 0) \) has infinitely many realizations.

We show that \( \psi (x, \vec{c}_i, 1) \wedge \psi (x, \vec{c}_j, 1) \) has only finitely many realizations, for \( i \neq j \). Assume not, then by the codensity property for \( G \), there is some \( d \in M \setminus \text{scl}(\vec{c}_{i,1}) \) realizing \( \psi (x, \vec{c}_i, 1) \wedge \psi (x, \vec{c}_j, 1) \). As \( \phi (x, \vec{c}_i, 1) \Delta \psi (x, \vec{c}_j, 1) \) is \( \vec{c}_i \)-small, \( d \) satisfies \( \phi (x, \vec{c}_i, 1) \), and similarly, \( d \) satisfies \( \phi (x, \vec{c}_j, 1) \). It follows that \( d \) satisfies \( \phi (x, \vec{c}_i, 1) \wedge \phi (x, \vec{c}_j, 1) \). Repeating this process and using again the codensity property, there is an infinite sequence \( (d_i : i < \omega) \) of realizations of \( \psi (x, \vec{c}_i, 1) \wedge \psi (x, \vec{c}_j, 1) \) such that \( d_i \notin \text{scl}(\vec{c}_i, \vec{c}_j, \vec{d}_{<i}) \). Hence \( \phi (x, \vec{c}_i, 1) \wedge \phi (x, \vec{c}_j, 1) \) has infinitely many realisation, a contradiction. It follows that the \( L \)-formula \( \psi (x, \vec{y}) \) witnesses that \( T \) has SOP\(_1\).

\( \Box \)

**Question.** It may be interesting to check in this setting if other model theoretic properties are preserved. For example, is \( n \)-dependence preserved? Is NSOP\(_n\) preserved for \( n \geq 3 \)?

We end this section with some examples.

**Example 5.21.** Consider structures of the form \( \mathcal{V} = (V, +, 0, S, \{\lambda_r\}_{r \in \mathbb{F}}) \), where \( V \) is a pure vector space over a field \( \mathbb{F} \) and \( S \) is a generic subset of \( V \) in the sense of Chatzidakis-Pillay [12]. To clarify the notation, set \( \mathcal{L}_0 = \{+, 0, \{\lambda_r\}_{r \in \mathbb{F}}\} \) and \( \mathcal{L} = \mathcal{L}_0 \cup \{S\} \). Since for pure vector spaces over \( \mathbb{F} \) we have that acl\(_{\mathcal{L}_0} = \text{dcl}_{\mathcal{L}_0} = \text{span}_\mathbb{F} \), by [12] the structure \( \mathcal{V} \) has quantifier elimination and acl = dcl = span\(_{\mathbb{F}} \).

The completions of the theory depend on the truth value of \( S(0) \) and to simplify the example, assume that \( 0 \in S \). By [12] this structure is simple unstable of SU-rk one and thus geometric.

Regardless how we choose the submodule \( R \), by Corollary 5.15 the theory of the pair \( (\mathcal{V}, G) \) will always be simple, but as before \( \kappa_T \) will depend on how we choose \( \mathbb{F} \) and \( R \).

Assume first that \( R = \mathbb{Z} \) and \( \mathbb{F} = \mathbb{Q} \). Then the theory of the pair \( (\mathcal{V}, G) \) will be simple and not supersimple. Assume now that \( R = \mathbb{Q} = \mathbb{F} \), then the expansion \( T^G \) corresponds to the theory of a lovely pair of models of \( T \) (see [27]), which is known to be supersimple of SU-rank two (see [27]).

For our next example, we will need the following lemmas and proposition showing preservation of NTP\(_2\) under generic predicate expansions, assuming acl is given by linear span. Note that a more general version of this result appears as the Theorem 7.3 in [13]; however the argument written in in [13] seems incomplete (in using singletons rather than tuples), and at this point we are not aware if it had been fixed.

**Lemma 5.22.** Let \( T \) be a complete geometric theory extending the theory of vector spaces over \( \mathbb{F} \) where acl = span\(_\mathbb{F} \), let \( P \) be a new predicate and let \( T_P \) be the theory
$T$ together with the scheme saying that $P$ is a generic predicate. Let $(M, P) \models T_p$. Then

1. For any $\vec{a}, \vec{b} \in M^n$, we have $tp_{P}(\vec{a}) = tp_{P}(\vec{b})$ iff $tp(\vec{a}) = tp(\vec{b})$ and for every $\lambda_1, \ldots, \lambda_n \in F$, $P(\lambda_1 a_1 + \cdots + \lambda_n a_n)$ holds iff $P(\lambda_1 b_1 + \cdots + \lambda_n b_n)$ holds.

2. Every $L_p$-formula $\varphi(x, \vec{y})$ is equivalent modulo $T_p$ to a disjunction of formulas of the form $\psi(x, \vec{y}) \wedge \theta(x, \vec{y})$, where $\psi(x, \vec{y})$ is a $L$-formula and $\theta(x, \vec{y})$ is a conjunction of formulas of the form $\pm P(t(x, \vec{y}))$ and each $t(x, \vec{y})$ is an $F$-linear combination of the elements $x, \vec{y}$.

Proof. (1) The left to right direction is clear. The right direction to left direction follows the argument in Corollary 2.6(b) in [12] together with the fact that by assumption, whenever $\vec{c} \in M$, we have $acl(\vec{c}) = span(\vec{c})$.

(2) Follows from part (1) and compactness. One could also prove this result using Corollary 2.6(d) in [12].

Lemma 5.23. Suppose $\phi(x, \vec{y}) = \theta_1(x, \vec{y}) \lor \ldots \lor \theta_m(x, \vec{y})$ has $TP_2$. Then so does $\theta_i(x, \vec{y})$ for some $i \leq m$.

Proof. It suffices to prove the statement for $m = 2$. By Proposition 5.4, there exists an indiscernible array $\{\vec{a}_{i, j} : i < \omega, j < \omega\}$ such that $\bigwedge_{i < \omega} \varphi(x, \vec{a}_{i, 0})$ has infinitely many realizations, and $\bigwedge_{i < \omega} \varphi(x, \vec{a}_{0, j})$ has at most finitely many realizations. Clearly, each of $\bigwedge_{j < \omega} \theta_1(x, \vec{a}_{0, j})$ and $\bigwedge_{j < \omega} \theta_2(x, \vec{a}_{0, j})$ has at most finitely many realizations. Thus, it remains to show that at least one of $\bigwedge_{i < \omega} \theta_1(x, \vec{a}_{i, 0})$ or $\bigwedge_{i < \omega} \theta_2(x, \vec{a}_{i, 0})$ has infinitely many realizations.

Suppose both $\bigwedge_{i < \omega} \theta_1(x, \vec{a}_{i, 0})$ and $\bigwedge_{i < \omega} \theta_2(x, \vec{a}_{i, 0})$ have finitely many realizations. Then for some $n > 0$, both $\bigwedge_{i < n} \theta_1(x, \vec{a}_{i, 0})$ and $\bigwedge_{i < n} \theta_2(x, \vec{a}_{i, 0})$ have finitely many realizations. We claim that

$$\bigwedge_{i < 2n + 1} \theta_1(x, \vec{a}_{i, 0}) \lor \theta_2(x, \vec{a}_{i, 0})$$

has finitely many realizations, which is a contradiction. Indeed, this formula is equivalent to

$$\bigvee_{\sigma \in \{1, 2\}^{2n+1}} \bigwedge_{i < 2n+1} \theta_{\sigma(i)}(x, \vec{a}_{i, 0}).$$

For any $\sigma \in \{1, 2\}^{2n+1}$, the formula

$$\bigwedge_{i < 2n+1} \theta_{\sigma(i)}(x, \vec{a}_{i, 0})$$

includes a conjunction of the form

$$\theta_1(x, \vec{a}_{i_{1}, 0}) \land \ldots \land (\theta_1(x, \vec{a}_{i_{n}, 0})$$

or

$$\theta_2(x, \vec{a}_{i_{1}, 0}) \land \ldots \land (\theta_2(x, \vec{a}_{i_{n}, 0})$$

for some $0 \leq i_1 < \ldots < i_n < 2n + 1$. By indiscernibility of $(\vec{a}_{i, 0} : i < \omega)$, each such conjunction has finitely many realizations.

Proposition 5.24. Let $T$ be a complete geometric theory extending the theory of vector spaces over $F$ where $acl = span$, let $P$ be a new predicate and let $T_p$ be the theory $T$ together with the scheme saying that $P$ is a generic predicate. Assume
that there is a $\mathcal{L}_p$-formula $\varphi(x, \bar{y})$ (where $x$ is a single variable) such that $\varphi(x, \bar{y})$ witnesses $k$-TP$_2$ for some $k \geq 2$. Then $T$ has TP$_2$.

**Proof.** Assume that there exists such an $\mathcal{L}_p$-formula $\varphi(x, \bar{y})$. By Proposition 5.4 we may assume that $\varphi(x, \bar{y})$ witnesses TP$_2$ with some indiscernible array $A := \{\bar{a}_{i,j} \mid i < \omega, j < \omega\}$. By Lemma 5.22 there exist some $\mathcal{L}$-formulas $\psi_s(x, \bar{y})$, $1 \leq s \leq m$, and a list of linear combinations $t_1(x, \bar{y}), \ldots, t_s(x, \bar{y}) \in \text{span}(x, \bar{y})$ such that $\varphi(x, \bar{y})$ is equivalent to $\bigvee_{s=1}^m \psi_s(x, \bar{y}) \land \theta_s(x, \bar{y})$, where each $\theta_s(x, \bar{y})$ is a conjunction of formulas of the form $\pm P(t_i(x, \bar{y}))$. We split the argument into cases.

**Case 1.** The conjunction $\bigwedge_{k < \omega} (\bigvee_{s=1}^m \psi_s(x, \bar{a}_{0,k}))$ has finitely many realizations. 
Note that for any $f : \omega \rightarrow \omega$ the collection $\{\bigvee_{s=1}^m \psi_s(\bar{x}, \bar{a}_{i,f(i)}) \mid i < \omega\}$ has infinitely many realizations since the collection $\{\varphi(\bar{x}, \bar{a}_{i,f(i)}) \mid i < \omega\}$ already has infinitely many realizations. It follows from Proposition 5.4 that the $\mathcal{L}$-formula $\bigvee_{s=1}^m \psi_s(x, \bar{y})$ has TP$_2$ as desired.

**Case 2.** The conjunction $\bigwedge_{k < \omega} (\bigvee_{s=1}^m \psi_s(x, \bar{a}_{0,k}))$ has infinitely many realizations. 
By Lemma 5.23, we may assume that $m = 1$, i.e., $\phi(x, \bar{y}) = \psi(x, \bar{y}) \land \theta(x, \bar{y})$.

Now, we have $\phi(x, \bar{y}) = \psi(x, \bar{y}) \land \bigwedge_{k=1}^\ell P^{b_k}(t_k(x, \bar{y}))$, where $\delta_k = 0$ or 1. We may assume that $x$ appears non-trivially in each linear combination. We know that $\phi(x, \bar{a}_{0,i})$ has infinitely many realizations for every $i$. We also know that $\bigwedge_{i < \omega} \psi(x, \bar{a}_{0,i})$ has infinitely many realizations. Let $b$ be one such realization, non-algebraic over the sequence. By Erdős-Rado Theorem, there exists an $\mathcal{L}$-indiscernible sequence $(b'_{\bar{a}_i} : i < \omega)$ such that for any $i_1 < \ldots < i_n < \omega$ there exist $f_1 < \ldots < f_n < \omega$ such that 

$$\text{tp}_\mathcal{L}(b'_{\bar{a}_{i_1}}, \ldots, b'_{\bar{a}_{i_n}}) = \text{tp}_\mathcal{L}(b_{\bar{a}_{0,j_1}}, \ldots, b_{\bar{a}_{0,j_n}}).$$

Thus, we may assume that $\bigwedge_{i < \omega} \phi(x, \bar{a}_{0,i})$ is $\mathcal{L}$-indiscernible over $b$. Suppose for some $i < \omega$ and some $k, m \leq \ell$ we have $t_k(b, \bar{a}_{0,i}) = t_m(b, \bar{a}_{0,j})$. Choosing $n > j$, we have $t_m(b, \bar{a}_{0,j}) = t_k(b, \bar{a}_{0,i}) = t_m(b, \bar{a}_{0,n}) = t_k(b, \bar{a}_{0,j})$. Thus, by $\mathcal{L}$-indiscernibility over $b$, $t_k(b, \bar{a}_{0,i}) = t_m(b, \bar{a}_{0,j})$ for all $i < \omega$, and thus, also $t_k(b, \bar{a}_{0,i}) = t_k(b, \bar{a}_{0,j})$ for all $i, j < \omega$. Then, we can also assume that for some $k, \ell$ we have $t_s(b, \bar{a}_{0,i}) = t_r(b, \bar{a}_{0,j})$ for all $1 \leq s \leq k$ and all $i, j < \omega$, $t_s(b, \bar{a}_{0,i}) \neq t_r(b, \bar{a}_{0,i})$ for $s < r \leq k$ (and fixed $i$), and for $k < m_1, m_2 \leq \ell$ and $i, j < \omega$, $t_{m_1}(b, \bar{a}_{0,i}) \neq t_{m_2}(b, \bar{a}_{0,j})$ whenever $(m_1, i) \neq (m_2, j)$. Since $b$ appears non-trivially in each term, none of these elements is algebraic over the sequence of $\bar{a}_{0,i}$'s. Now we assign $P$ to these terms in any way, using the genericity of the predicate, thus, making the conjunction $\bigwedge_{i=1}^\omega \phi(x, \bar{a}_{0,i})$ consistent, a contradiction.

**Example 5.25.** We start with the ordered vector space case (Example 2.16) and add a generic predicate $S$. This expansion eliminates the quantifier $\exists^\infty$ (see section 2 in [22]) and has the same algebraic closure as the original theory (see [12]), so it agrees with the span of the parameters over $\mathbb{F}$. This vector space is NTP$_2$, not simple (because of the order) and not NIP (because of the generic predicate). The expansion by any submodule will be again NTP$_2$. As before, depending on the choice of $R$ and $\mathbb{F}$ we can get examples of theories that are strong and others that are not strong.

6. Independence relations in the expansion

We start this section by analyzing the case when the base theory $T$ is simple of $SU$-rank one. We first show that forking in $T^G$ has local character, and in doing so,
we will extract conditions that give a geometric characterization of independence in the expansion.

**Lemma 6.1.** Let $M$ be a model of an SU-rank one theory, $(D_i : i \in \omega)$ an indiscernible sequence in $M$ (where $D_i$ are small), $\vec{a} \in M$ a finite tuple of elements acl-independent over $D_0$. Then there exists $\vec{a}'' \models tp(\vec{a}/D_0)$ such that $(D_i : i \in \omega)$ indiscernible over $\vec{a}''$ and $\vec{a}''$ is acl-independent over $\bigcup_{i \in \omega} D_i$.

**Proof.** The proof will be by induction on the length of $\vec{a}$. Suppose first that $\vec{a} = b$ is a single element and that $b \notin acl(D_0)$. Fix $n \geq 1$, and let $b_1, \ldots, b_n$ be realizations of $tp(b/D_0)$ which are acl-independent over $D_0$. Let $\vec{b} = (b_1, \ldots, b_n)$. Then $tp(\vec{b}/D)$ does not divide over $\emptyset$, and, thus, we can assume that $(D_i : i \in \omega)$ is indiscernible over $\vec{b}$. Since $n$ was arbitrary, we have that the partial type in variables $x_1, x_2, \ldots$ over $\bigcup_{i \in \omega} D_i$ saying that $x_i$ are distinct realizations of $tp(b/D_0)$ and $(D_i : i \in \omega)$ is indiscernible over each of $x_i$ is consistent. It follows that there exists a realization of $b'$ of $tp(b/D_0)$ such that $(D_i : i \in \omega)$ is indiscernible over $b'$ and $b' \notin acl(\bigcup_{i \in \omega} D_i)$. This finishes the base case. The induction step is done by adding a part of the tuple $\vec{a}$ to $D_0$ and repeating the above construction. \hfill $\square$

**Remark 6.2.** Let $(V_0, G) \models T^{G^+}$ let $C \subseteq V_0$ be closed and let $\vec{a} = (a_1, \ldots, a_n) \in V_0$. We may assume that for some $k \leq n$, we have $dim(a_1, \ldots, a_k/C \cup \{G(V) = k\}$ and for $i > k$, $a_i \in span_{F}(a_1, \ldots, a_k, C \cup \{g_i\})$. Assume $a_i = \sum_{j \leq k} \mu_i a_j + c + \sum_{j > k} \lambda_j g_j$ for some $c \in C$ and $g_j \in G$. Choose $\ell$ minimal with with property, so the tuple $(g_1, \ldots, g_\ell) \in G$ is algebraically independent over $C$. Then $a_i$ is interalgebraic in the extended language (using the functions $f_{\vec{a},i}$ with $(g_1, \ldots, g_\ell)$ and thus we may assume, when considering a tuple over $C$, $\vec{a} = (\vec{a}_1, \vec{a}_2, \vec{a}_3)$, where $\vec{a}_1$ is independent over $CG$, $\vec{a}_2 \in G$ is independent over $C\vec{a}_1$ and $\vec{a}_3 \in dcl(\vec{a}_1, \vec{a}_2, C)$.

**Theorem 6.3.** Assume that $T$ has SU-rank one. Then dividing in $T^{G}$ has local character (and, therefore, $T^{G}$ is simple).

**Proof.** We work inside a sufficiently saturated model $(V, G) \models T^{G}$. Let $D \subseteq V$ be small, $D = (D_i)$.

Consider a tuple of the form $\vec{a} = \vec{a}_1 \vec{a}_2 \vec{a}_3$, where $\vec{a}_1$ is a $D \cup G(V)$-independent tuple, $\vec{a}_2 \in G(V)$ is $F$-linearly independent over $D$ and $\vec{a}_3 \in span_{F}(\vec{a}_1 \vec{a}_2 D)$. Note that under this assumption $dim(\vec{a}_1 \vec{a}_2 / D) = |\vec{a}_1 \vec{a}_2|$. Let $C \subseteq D$ be such that $tP_R(\vec{a}_2 / G(D))$ does not fork over $G(C)$ in the sense of $R$-modules and such that $\vec{a}_3 \in span_{F}(\vec{a}_1 \vec{a}_2 C)$. We will show that $tP_G(\vec{a}/D)$ does not fork over $C$.

Let $(D_i : i \in \omega)$ be an $L_G$-indiscernible sequence over $C$ with $D_0 =$ $\emptyset$. We start with $\vec{a}_1$.

**Claim 1** There exists $\vec{a}_1' \models tp(\vec{a}_1/D)$ such that $\vec{a}_1'$ is $F$-linearly independent over $G(V) \cup \bigcup_{i \in \omega} D_i$ and $(D_i : i \in \omega)$ is $L_G$-indiscernible over $\vec{a}_1' C$.

Let $p(x, D) = tp(\vec{a}_1, D)$. Since $\vec{a}_1$ is independent over $D$, $tp(\vec{a}_1/D)$ does not divide over $C$ (in fact, over $\emptyset$) and $\bigcup_{i \in \omega} p(x, D_i)$ is consistent. By Lemma 6.1 we can find $\vec{a}_1'$ with $\vec{a}_1' \models \bigcup_{i \in \omega} p(x, D_i)$ such that $\{\vec{a}_1' D_i : i \in \omega\}$ is indiscernible over $C$ and $\vec{a}_1'$ is $F$-linearly independent over $\bigcup_{i \in \omega} D_i$. Applying the extension property in each of the components of $\vec{a}_1'$, we may assume that $\vec{a}_1'$ is $F$-linearly independent over $\bigcup_{i \in \omega} D_i, G(V)$. Note that $G(span_{F}(\vec{a}_1 D_i)) = G(D_i)$ and that $G(span_{F}(\vec{a}_1' D_i)) = G(D_i)$ for all $i$. Thus $tp_G(\vec{a}_1' D_i) = tp_G(\vec{a}_1 D_i)$ for all $i$. Applying Erdős-Rado theorem, we can assume that $\{\vec{a}_1' D_i : i \in \omega\}$ is $L_G$-indiscernible over $C$. Since the $L_G$-type of any increasing tuple of the new sequence coincides with that of
one of the increasing tuples of the old sequence, we still have that \( \vec{a}' \) is \( F \)-linearly independent over \( \cup_{i \in \omega} D_i \).

Now we deal with the part that belongs to \( G \). We may assume that \( \vec{a}_1 = \vec{a}'_1 \) satisfies the statement of Claim 1.

**Claim 2** There exists \( \vec{a}_2 \) such that for all \( i \in \omega \), we have \( \text{tp}_G(D_i \vec{a}_1 \vec{a}_2) = \text{tp}_G(D_i \vec{a}_1 \vec{a}_1) \).

Let \( q(x, D) = \text{tp}_R(\vec{a}_2, G(D)) \). Note that \( (G(D_i) : i \in \omega) \) is an \( R \)-module indiscernible sequence over \( G(C) \). Since \( \vec{a}_2 \) is \( R \)-module independent from \( G(D) \) over \( G(C) \) we get that \( \cup_{i \in \omega} q(x, G(D_i)) \) is consistent.

Let \( p_2(x, D\vec{a}_1) = \text{tp}(\vec{a}_2/D\vec{a}_1) \). By construction, the sequence \( (D_i \vec{a}_1 : i \in \omega) \) is an \( L_G \)-indiscernible sequence over \( C \), hence, also \( L \)-indiscernible over \( C \). Since \( \vec{a}_2 \) is a independent tuple over \( D\vec{a}_1 \), \( \text{tp}(\vec{a}_2/D\vec{a}_1) \) does not \( L \)-divide over \( \emptyset \), hence, also over \( C\vec{a}_1 \) and \( \cup_{i \in \omega} p_2(x, D_i \vec{a}_1) \) is consistent.

We need to make the \( R \)-module extension \( q_\omega(x) = \cup_{i \in \omega} q(x, G(D_i)) \) and the \( L \)-extension \( p_{2\omega}(x) = \cup_{i \in \omega} p_2(x, D_i \vec{a}_1) \) compatible. Since \( \dim(\vec{a}_2/D\vec{a}_1) = |\vec{a}_2| \), by Lemma 6.1 we also have \( \dim(p_{2\omega}(x)) = |\vec{a}_2| \) and the consistency of \( q_\omega(x) \cup p_{2\omega}(x) \) follows from Corollary 3.15. Let \( \vec{a}_2' \in G[\vec{a}_2] \) be such that \( \vec{a}_2' \models p_{2\omega}(x) \cup q_\omega(x) \). Note that \( \vec{a}_2' \) belongs to \( G \) and thus for each \( i \in \omega \) \( D_i \vec{a}_1 \vec{a}_2' \) is \( G \)-independent, \( \text{tp}(D_i \vec{a}_1 \vec{a}_2') = \text{tp}(D_i \vec{a}_1 \vec{a}_2) \) and \( \text{tp}_G(\vec{a}_2', G(D_i)) = \text{tp}_G(\vec{a}_2, G(D_i)) \) and thus \( \text{tp}_G(D_i \vec{a}_1 \vec{a}_2') = \text{tp}_G(D_i \vec{a}_1 \vec{a}_2) \), as needed.

Finally let \( \vec{a}_3 \) be such that \( \text{tp}_G(D_i \vec{a}_1 \vec{a}_2 \vec{a}_3) = \text{tp}_G(D_i \vec{a}_1 \vec{a}_2 \vec{a}_3) \). Then \( \vec{a}_3 \in \text{span}_F(\vec{a}_1 \vec{a}_2 C) \), and, therefore, for all \( i \in \omega \), \( \text{tp}_G(D_i \vec{a}_1 \vec{a}_2 \vec{a}_3) = \text{tp}_G(D_i \vec{a}_1 \vec{a}_2 \vec{a}_3) \). Recall that the requirements on the set \( C \) are that \( \text{tp}_R(\vec{a}/G(D)) \) does not fork over \( G(C) \) in the sense of \( R \)-modules and such that \( \vec{a}_3 \in \text{span}_F(\vec{a}_1 \vec{a}_2 C) \), and therefore, there is a uniform bound on the size of \( C \). Hence, dividing in \( T^G \) has local character, and, thus \( T^G \) is simple. Furthermore, the local character associated to \( T^G \) is the local character associated to forking in \( R \)-modules, since for the condition \( \vec{a}_3 \in \text{span}_F(\vec{a}_1 \vec{a}_2 C) \) it suffices to pick a finite subset \( C \subset D \).

The main goal of this section is to study independence relations in the expansion. We start by extracting the information that we needed for the argument above to work. For Claim 1 and Claim 2, we require \( \dim(\vec{a}/C) = \dim(\vec{a}/D) \), that is, algebraic independence should hold. For claim 1 we need \( \dim(\vec{a}/CG(V)) = \dim(\vec{a}/DG(V)) \), so we need algebraic independence after localizing in \( G \). Finally for claim 2 we need \( R \)-module independence in the components that belong to \( G \). This will give us intuition when we set up Definition 6.7 below. In any case, we point out a strong limitation of the argument above:

**Remark 6.4.** In the argument above we used the extension property to guarantee that \( \dim(\vec{a}/G(V)) = \dim(\vec{a}/G(V)C) \). The extension property only guarantees the existence of such tuples for algebraic independence localized in \( G(V) \), not for other forms of independence that may properly extend algebraic independence.

Now we introduce a more general setting for the rest of this section. Assume the structure \( V = (V, +, 0, \{\lambda\}_{\lambda \in \mathbb{R}}, \ldots) \models T \) is sufficiently saturated and assume the models of its theory have a natural independence relation \( \perp^b \) that implies (but may extend properly) algebraic independence. We will reserve the symbol \( \perp \) for algebraic independence, as we did earlier in the text.

Key to the arguments below is the following result of [19] (see also [15, Fact 4.6]):
Fact 6.5. Let $T$ be a first order theory. Then $T$ is NSOP$_1$ if and only if there is an automorphism invariant ternary relation $\downarrow^*$ on small subsets of a monster model, only allowing models in the base, satisfying the following properties:

1. symmetry. If $A \downarrow^*_M B$ then $B \downarrow^*_M A$.
2. existence. $A \downarrow^*_M M$, for all $M$.
3. finite character. If $\bar{a} \downarrow^*_M B$ for all finite $\bar{a} \subseteq A$, then $A \downarrow^*_M B$.
4. monotonicity. If $A \downarrow^*_M B$ then $A \downarrow^*_M B$.
5. transitivity. For $M \subseteq N$ if $A \downarrow^*_N B$ and $N \downarrow^*_M B$ then $AN \downarrow^*_M B$.
6. extension. If $A \downarrow^*_M B$, then for all $D$ there exists $A' \equiv_{MB} A$ and $A' \downarrow^*_M BD$.
7. local character*. Let $\bar{a}$ be a finite tuple. Then whenever $T$ in $\bar{e}$ be a finite tuple. Then whenever $\bar{a} \downarrow^*_M \bar{a}$, we have span$_F(\bar{a}) \downarrow^0_{V_0} \text{span}_F(D)$.

Indeed, by extension, there exists $a' \equiv_{V_0D} a$ with $a' \downarrow^0_{V_0} \text{span}_F(D)$. If $\sigma$ is an $L$-automorphism over $V_0D$ sending $a'$ on $a$, then by invariance of $\downarrow^0$ we also get $\bar{a} \downarrow^0_{V_0} \text{span}_F(D)$, and we conclude by the result by symmetry.

We now introduce an independence relation in the expansion $(V, G) \models T^{G+}$ and assume the expansion is sufficiently saturated.

**Definition 6.7.** Let $V_0 \subseteq D \subseteq V$ be such that $V_0 \models T^{G+}$, $D$ is algebraically closed in $T^{G+}$ and let $\bar{a} \in V^n$. For $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{F}^n$, denote $G_{\bar{\lambda}} = \lambda_1 G + \ldots + \lambda_n G$. Define $\bar{a} \downarrow^G_{V_0} D$ if and only if

1. $\langle \bar{V}_0 \bar{a} \rangle \downarrow_{V_0}^0 D$
2. $G(\langle \bar{a}V_0 \rangle) \downarrow_{G(V_0)}^{R-mod} G(D)$.
3. $G(\langle \bar{a}V_0 \rangle + D) = G(\langle \bar{a}V_0 \rangle) + G(D)$.

In the above definition above we require the set $D$ to be algebraically closed in $T^{G+}$ and in particular it is $G$-independent. Note that for $\bar{a}, \bar{d}$ tuples, we can define $D' = \langle V_0 \bar{d} \rangle$, it satisfies $V_0 \subseteq D'$, so we can extend the definition and say that $\bar{a} \downarrow^G_{V_0} \bar{d}$ holds whenever we have $\bar{a} \downarrow^G_{V_0} D'$. Similarly, for a set $A$ we write $A \downarrow^G_{V_0} D$ if and only if $\bar{a} \downarrow^G_{V_0} D$ for all finite $\bar{a} \subseteq A$. Note that by definition, we have $\bar{a} \downarrow^G_{V_0} D$ if and only if $\bar{a}V_0 \downarrow^G_{V_0} D$.

Let us see how condition (3) above arises naturally in our setting. A key observation is the following strong connection between forking and cosets of definable groups.
**Remark 6.8.** Let $T$ be any theory of abelian group in which there is a predicate $G$ for a subgroup of the ambient theory. Assume that for any $b \notin G$ there exists an infinite sequence $(b_i)_{i<\omega}$ in $\text{tp}(b/C)$ such that $b_i - b_j \notin G$ for all $i \neq j$. Denote by $\downarrow^d$ the dividing independence relation. Then for any subgroups $C \subseteq A \cap B$

$$A \downarrow^d C B \iff G(A + B) = G(A) + G(B)$$

First, we prove that for any $b$ the formula $\phi(x,b)$ 2-divides over $C$, where $\phi(x,y)$ is defined by $x + y \in G \land y \notin G$. If $b \notin G$ the formula $\phi(x,b)$ is inconsistent hence 2-divides over emptyset. Assume that $b \notin G$ and let $(b_i)_{i<\omega}$ be as in the hypotheses, in $\text{tp}(b/C)$. If $a$ is a realisation of $\phi(a,b_i) \land \phi(a,b_j)$, then $a - b_i \in G$ and $a - b_j \in G$ implies $b_i - b_j \in G$ which is a contradiction. So $\{\phi(x,b_i) \mid i < \omega\}$ is 2-inconsistent.

Now we prove the above implication, or rather its contrapositive. Assume that for $C \subseteq A \cap B$ we have $G(A + B) \neq G(A) + G(B)$ then there exists $a \in A$ and $b \in B$ such that $a + b \in G$ and $a + b \notin G(A) + G(B)$. As $a + b \in G$, we have that

$$a \in G \iff b \in G$$

(using that $A$ and $B$ are groups). In particular, if $b \in G$ then $b \in G(B)$ and $a \in G(A)$ hence $a + b \in G(A) + G(B)$, a contradiction. It follows that neither $a$ nor $b$ is in $G$. In particular $\phi(x,b)$ $\in \text{tp}(A/B)$ hence $\text{tp}(A/B)$ divides over $C$.

**Remark 6.9.** The same argument gives that Kim-dividing implies this condition as long as for any $b \notin G$ there exists an invariant/f.s./non-forking-Morley sequence in $\text{tp}(b/C)$ such that $b_i - b_j \notin G$. A case where such sequences does not exist: consider a theory $T$ of abelian group where there is a unary function symbol $f$ which is a group homomorphism and assume that $G = \ker f$ in this theory. Now assume that $b \notin G$ but that $f(b) = c \in C$. Then for any sequence $(b_i)_{i<\omega}$ in $\text{tp}(b/C)$ we have $f(b_i) = c = f(b)$ hence $f(b_i - b_j) = 0$ i.e. $b_i - b_j \in \ker f = G$. In that case the formula $\phi(x,y) = x + y \in G \land y \notin G$ does not necessarily divide over $C$.

**Lemma 6.10.** The independence relation $\downarrow^G$ satisfies invariance and monotonicity.

**Proof.** Invariance follows from the fact that $\downarrow^0$ satisfies invariance, that condition (3) is invariant under automorphisms and the fact that $\downarrow^{R-\text{mod}}$ being forking independence in $R$-modules, also satisfies invariance.

Let us consider now monotonicity. Given two sets $A \subseteq B \subseteq V$, we have that $G(A) \subseteq G((B))$, since monotonicity holds for forking-independence, in stable theories, we have that item (2) in the definition of $\downarrow^G$ is preserved after taking subsets of $C$ and $D$. Since $\downarrow^0$ satisfies monotonicity, it remains to show that monotonicity holds for condition (3), which is easy, and where done in more generality in [16, Section 3].

Now we point out some observations on $F$-linear independence.

**Lemma 6.11.** The relation $\downarrow^G$ is symmetric.

**Proof.** Fix tuples $\vec{a}, \vec{d}$. Clearly items (1) and (2) are symmetric conditions since $\downarrow^0$ satisfies symmetry as well as forking-independence in modules. Condition (3) satisfies symmetry once we change the sets for their $L_G$-algebraic closures with $V_0$.  

Lemma 6.12. Assume that $\downarrow^0$ is transitive, then so is $\downarrow^G$.

Proof. Let $(V_0,G) \subseteq (V_1,G) \models T^G+$ and assume that $A \downarrow^G V_1$ and $V_1 \downarrow^G V_0$. Using the definition and the fact that $\downarrow^0$ is transitive we get $AV_1 \downarrow^G V_0$. Similarly, since transitivity holds for forking-independence in $R$-modules we also get the condition $G((AV_1)) \downarrow^R_{mod} G((BV_0))$ as desired. It remains to show condition (3) which is easy, and is done in more generality in [16, Section 3].

Lemma 6.13. The independence relation $\downarrow^G$ satisfies extension.

Proof. Let $(V_0,G) \models T^G+$, let $B$, $D$ be sets, and let $p(\vec{x})$ be a $\mathcal{L}_{G+}$-type over $V_0B$ such that whenever $\vec{a} = p$ we have $\vec{a} \downarrow^G V_0B$. After adding to $\vec{a}$ elements from $\langle \vec{a}V_0 \rangle$ if necessary, we may assume that if $\vec{a} \models p$, then we can write $\vec{a} = (\vec{a}_1, \vec{a}_2, \vec{a}_3)$, where $\vec{a}_1 \downarrow^G V_0 G(V)$ and it is $\mathcal{F}$-linearly independent over $V_0$, $\vec{a}_2 \in G(V)$ and $\vec{a}_2$ is $\mathcal{F}$-linearly independent over $V_0\vec{a}_1$, and $\vec{a}_3 \in \text{span}_G(\vec{a}_1\vec{a}_2V_0)$. Since $\downarrow^0$ satisfies extension we can find $\vec{a}' \models p$ such that $\vec{a}' = (\vec{a}'_1, \vec{a}'_2, \vec{a}'_3) \downarrow^0 V_0 BD$.

Since $\dim(\vec{a}_1/V_0) = |\vec{a}_1|$ and $\downarrow^G$ extends algebraic independence, we also have $\dim(\vec{a}'_1/V_0BD) = \dim(\vec{a}'_1/V_0) = |\vec{a}'_1|$. By the codensity property in the pair $(V,G)$ we may assume $\dim(\vec{a}'_1/G(V)BD) = |\vec{a}'_1|$, so the tuple $\vec{a}'_1$ can also be chosen to be $\mathcal{F}$-linearly independent over $BDG(V)$ (**).

Now let $p_2(\vec{x}_2) = \text{tp}_G(\vec{a}'_2/V_0BD\vec{a}'_1)$ and let $p_2^R(\vec{x}_2) = \text{tp}_{R-mod}(\vec{a}_2/G((V_0B)))$. Since $\downarrow^R_{mod}$ is a stable independence notion, it satisfies existence and we can find $p_2^R(\vec{x}_2)$ a non-forking extension of $p_2^G(\vec{x}_2)$ over $G((V_0BD))$.

Using Corollary 3.15 we may assume that $\vec{a}'_2 \models p_2(\vec{x}_2) \cup p_2^R(\vec{x}_2)$. Let $\vec{a}' = (\vec{a}'_1, \vec{a}'_2, \vec{a}'_3)$.

We check that condition (3) is satisfied, i.e. $G((\vec{a}') + (BD)) = G((\vec{a}')) + G((BD))$. Let $x = x_1 + x_2 + b + d \in G$ be in the left hand side, for $x_1 \in \langle \vec{a}'_1 \rangle$, $x_2 \in \langle \vec{a}'_2 \rangle$, $b \in B$, $d \in D$. Then $x_1 \in \langle \vec{a}_1 \rangle \cap (G(V)BD) = \{0\}$, by (**). So $x = x_2 + b + d \in G$.

It is clear that $b + d \in G(BD)$ so it follows that $x \in G((\vec{a}'_2)) + G((BD))$.

This shows that $\vec{a}' \downarrow^G V_0 D$. Since both tuples $(\vec{a}_1, \vec{a}_2), (\vec{a}'_1, \vec{a}'_2)$ are $G$-independent over $V_0B$, $G(\vec{a}_1, \vec{a}_2) = \vec{a}_2$ and $G(\vec{a}'_1, \vec{a}'_2) = \vec{a}'_2$ we get by Corollary 3.21 that $\text{tp}_{G+}(\vec{a}/V_0B) = \text{tp}_{G+}(\vec{a}'/V_0B)$ and $\vec{a}'$ is the desired tuple.

Remark 6.14. If $A, B$ are $\mathcal{F}$-vector spaces in a model of $T_G$ then there exists $B_0 \subseteq B$ with $|B_0| \leq |A|$ such that $G(A + B) \subseteq G(A + B_0) + G(B)$. To see this, for each $a \in A$, define $b_a$ to be any element of the set $\{b \in B | a + b \in G\}$ if it is nonempty, or else $b_a = \emptyset$. Set $B_0 = \bigcup_{a \in A} b_a$ and take any $g \in G(A + B)$, then $g = a + b = a + b_a + (b - b_a)$ and $b - b_a \in G$ hence $g \in G(A + B_0) + G(B)$.

Lemma 6.15. Assume that $\downarrow^0$ satisfies local character*, then so does $\downarrow^G$.

Proof. Let $\vec{a}$ be a finite tuple and let $\kappa > |T_G^+| = |T|$ be a regular cardinal. Let $\{(V_j, G_j)\}_{j<\kappa}$ be a continuous chain of models of $T_G^+$ and let $(V,G) = \bigcup_{j<\kappa} (V_j, G_j)$. Then $\{V_j\}_{j<\kappa}$ is also a continuous chain and by hypothesis there is $j_1 < \kappa$ such that $\vec{a} \downarrow^0_{V_{j_1}} V$.

Since forking independence in $R$-modules is stable, let $\kappa_R$ be the corresponding local character cardinal and note that $\kappa_R \leq |T_G^+|$. Let $B \subseteq G(V)$ be of cardinality
\[ \leq \kappa_R \text{ be such that } G(\langle aB \rangle) \downarrow_{G(B)}^{R-mod} G(V). \] Since \( \kappa \) is regular and \( \kappa_R < \kappa \), we may find \( j_2 < \kappa \) such that \( B \subset G(V_{j_2}) \) and thus \( G(\langle aB(\langle V_{j_2} \rangle) \rangle) \downarrow_{G(V_{j_2})}^{R-mod} G(V). \)

Let \( j_3 = \max(j_1, j_2) \), then we have \( a \downarrow_{V_{j_3}}^0 V \) and \( G(\langle aB(\langle V_{j_3} \rangle) \rangle) \downarrow_{G(V_{j_3})}^{R-mod} G(V). \)

Let \( \tau \) be the operator defined by setting
\[
\tau(A) = \operatorname{span}_G(A \cup \bigcup_{\lambda,i} f_{\lambda,i}(A)).
\]

Then \( A = \langle A \rangle \) if and only if \( f_{\lambda,i}(A) \subseteq A \) for all \( \lambda \) and all \( i \).

Using Remark 6.14, we iteratively construct a sequence \( (A_n)_{n<\omega} \) with \( A_0 = \langle aV_{j_3} \rangle \) and such that for some sequence \( (V_{k_n})_{n<\omega} \) we have:

1. \( \tau(A_n) + V_{k_n} \subseteq A_{n+1} \).
2. \( A_n \subseteq \langle aV_{k_n} \rangle \).
3. \( j_3 \leq k_0 \leq k_1 \leq \ldots < \kappa \).
4. \( G(A_n + V) \subseteq G(A_{n+1}) + G(V) \).

(Start with \( A_0 = \langle aV_{j_3} \rangle \) and apply Remark 6.14 to \( \tau(A_0) \) to get \( k_0 \geq j_3 \) and \( V_{k_0} \) such that \( G(\tau(A_0) + V) \subseteq G(\tau(A_0) + V_{k_0}) + G(V) \). Then define \( A_1 = \tau(A_0) + V_{k_0} \subseteq \langle aV_{k_0} \rangle \), we have \( G(A_0 + V) \subseteq G(\tau(A_0) + V) \subseteq G(A_1) + G(V) \). Then apply again Remark 6.14 with \( \tau(A_1) \), etc... An iteration of this process yields the desired sequences \( (A_n)_{n<\omega} \) and \( (V_{k_n})_{n<\omega} \).

Let \( A = \bigcup_{n<\omega} A_n \), so \( G(A + V) = G(A) + G(V) \) by (4). By (3), as \( \kappa \) is regular, there exists \( V_k \) such that \( A = \langle aV_k \rangle \). As \( k \geq j_3 \) we conclude that \( a \downarrow_{V_k}^0 V \), \( G(\langle aV_k \rangle) \downarrow_{G(V_k)}^{R-mod} G(V) \) and \( G(\langle aV_k \rangle + V) = G(\langle aV_k \rangle) + G(V) \) hence \( a \downarrow_{V_k}^G V \). \( \Box \)

**Remark 6.16.** Assume that for some \( A, D \subseteq G \) we have \( A \downarrow_{G(V_0)}^{R-mod} D \). Let \( a \in A \), \( d \in D \) and \( r \in R \) be such that \( a + d + rG \) then there exists \( v_0 \in G(V_0) \) such that \( a + v_0 + rG \).

**Proof.** The theory of \( R \)-modules is stable, hence as \( \operatorname{tp}(a/\operatorname{D}(G(V_0))) \) (in the sense of \( R \)-modules) does not fork over the model \( G(V_0) \) we get that \( \operatorname{tp}(a/\operatorname{D}(G(V_0))) \) is an heir of \( \operatorname{tp}(a/G(V_0)) \), hence there exists \( v_0 \in G(V_0) \) such that the formula \( \exists y(y \in G) x = v_0 + rG \) belongs to \( \operatorname{tp}(a/G(V_0)) \). \( \Box \)

**Theorem 6.17.** Assume that \( T \) has \( SU \)-rank one and that \( \hat{\mathbb{R}} = \mathbb{F} \). Then the independence theorem also holds for \( \downarrow_{G} \).

**Proof.** Let \( (V_0, G) \models T^{G+} \) and let \( B, C \supset V_0 \) be supersets such that \( B \downarrow_{V_0}^G C \). In particular by clause (1) of \( G \)-independence we get \( B \downarrow_{V_0}^0 C \) and, exchanging the sets for their \( L_G \)-algebraic closure, we may assume that \( C = \langle C \rangle \) and \( D = \langle D \rangle \).

Let \( p(x) \) be an \( L_G \)-type over \( V_0 \) and let \( p_B(x) \in S(B) \), \( p_C(x) \in S(C) \) be independent extensions of \( p(x) \) in the sense of \( \downarrow_{G} \). If \( a \models p(x) \), after adding elements of \( \langle aV_0 \rangle \) if necessary, we may assume that any realization of \( p(x) \) is of the form \( a = a_1 a_2 a_3 \), where \( a_1 \) is a \( V_0 \cup \langle G(V) \rangle \)-independent tuple, \( a_2 \in G(V) \) is \( \mathbb{F} \)-linearly independent over \( V_0 \) and \( a_3 \in \operatorname{dcl}_F(a_1 a_2 V_0) \). Note that under this assumption \( a \) is \( G \)-independent over \( V_0 \).
Let $p^\mathcal{L}_B(\vec{x})$ be the restriction of $p_B(\vec{x})$ to the language $\mathcal{L}$. Similarly let $p^\mathcal{C}_C(\vec{x})$ be the restriction of $p_C(\vec{x})$ to the language $\mathcal{L}$. Since $B \models^0_{\mathcal{V}_0} C$, we use the independence theorem for $\models^0$-independence to find $\vec{a}_1, \vec{a}_2, \vec{a}_3 \models p^\mathcal{L}_B(\vec{x}) \cup p^\mathcal{C}_C(\vec{x})$ such that $\vec{a}_1, \vec{a}_2, \vec{a}_3 \models^0_{\mathcal{V}_0} B \cup C$.

Using the codensity property, we can assume that $\vec{a}_1$ is $F$-linearly independent from $G(V)$ over $B \cup C$.

We could use the density property and assume that $\vec{a}_2 \in G$, but the module type of $\vec{a}_2$ over $\mathcal{V}_0$ may not be the desired one. Instead, let $p^\mathcal{L}_B(\vec{x}_2) = tp_G(\vec{a}_2/BC\vec{a}_1)$; we will refine the way we choose the realization of this last type in order to take into account the module structure.

Let $p_B(\vec{x}_2)$ be the restriction of $p_B(\vec{x})$ to the $|\vec{a}_2|$-coordinates considered over the parameters set $B$ and let $q^B_G(\vec{x}_2)$ be its restriction to the language of $R$-modules over the parameter set $G(B)$. Similarly let $p_C(\vec{x}_2)$ be the restriction of $p_C(\vec{x}) \in S(C)$ to the $|\vec{a}_2|$-coordinates considered only over the parameters set $C$ and let $q^C_G(\vec{x}_2)$ be its restriction to the language of $R$-modules to $G(C)$. Finally let $q^R_G(\vec{x}_2)$ be the restriction of $p(\vec{x})$ to the $|\vec{a}_2|$-coordinates considered only over the parameters set $G(\mathcal{V}_0)$ restricted to the language of $R$-modules.

Since $C \models G \models D$ we get using clause (2) that $G(C) \models^0_{G(\mathcal{V}_0)} G(D)$.

Since the theory of $R$-modules is stable, $q^R_G(\vec{x}_2)$ is the unique non-forking extension of $q^R_G(\vec{x}_2)$ to $G(C)$ and $q^R_G(\vec{x}_2)$ is the unique non-forking extension of $q^R_G(\vec{x}_2)$ to $G(B)$ which can be amalgamated to the unique non-forking extension of $q^R_G(\vec{x}_2)$ to $G(B) \cup G(C)$, which we will call $q^R_{G(BC)}(\vec{x}_2)$.

Now we show that both extensions $p^R_{G(BC)}$ and $q^R_{G(BC)}$ are compatible and can be extended to a type over $B \cup C$ that does not fork over $\mathcal{V}_0$ in the sense of $\models^0_G$.

Claim 1. $q^R_{G(BC)}(\vec{x}_2) \cup p^R_{G(BC)}(\vec{x}_2) \models^0_{G(\mathcal{V}_0)} p_B(\vec{x}) \cup p_C(\vec{x})$ and write $\vec{a} = \vec{a}_1, \vec{a}_2, \vec{a}_3$ for short.

Claim 2. $\vec{a} \models p_B(\vec{x}) \cup p_C(\vec{x})$ and it is $\models^0_G$-independent from $B \cup C$ over $\mathcal{V}_0$.

Since $\vec{a}_1, \vec{a}_2, \vec{a}_3 \models^0_{\mathcal{V}_0} B \cup C$, $\vec{a}_1, \vec{a}_2, \vec{a}_3 \models^\mathcal{C}_B$, $\vec{a}_1, \vec{a}_2, \vec{a}_3$ we get that part (1) in the definition of $G$-independence holds. By the way we constructed $\vec{a}_3$ we know that part (2) in the definition of $\models^0_G$-independence also holds.

Let us consider part (3). Let $t \in G(\langle \vec{a} \mathcal{V}_0 \rangle + \langle BC \rangle)$, then $t = d + \sum_{i=1}^n \alpha_i a_i$ for some $d \in \langle BC \rangle$. We may enumerate $\vec{a}_1 = (a_1, \ldots, a_n), \vec{a}_2 = (a_{n+1}, \ldots, a_{n+k})$. By construction of $\vec{a}_1$, we must have $\alpha_i = 0$ for $1 \leq i \leq n$. Thus $t = d + \sum_{n+1 \leq i \leq n+k} \alpha_i a_i$.

In this part of the argument we will use that $F = \hat{R}$. Then we can write $t = d + \sum_{n+1 \leq i \leq n+k} \hat{p}_i q_{n+1} \cdots q_{n+k} a_i'$ for some $p_i, q_i \in \hat{R}$. Then

\[
q_{n+1} \cdots q_{n+k} t = q_{n+1} \cdots q_{n+k} d + \sum_{n+1 \leq i \leq n+k} p_i q_{n+1} \cdots q_{n+k} a_i' \in G.
\]
Example 6.19. gives evidence that our characterization of independence works under more general \( V = T \) elimination. Let \( \| \) hold for \( \sum_{n<i\leq n+k} p_i^t d_i' \in q_{n+1} \cdots q_{n+k} d + q_{n+1} \cdots q_{n+k} G. \)

By Remark 6.16, there is \( v_0 \in V_0 \) such that \( \sum_{n<i\leq n+k} p_i^t d_i' \in q_{n+1} \cdots q_{n+k} v_0 + q_{n+1} \cdots q_{n+k} G, \) so \( q_{n+1} \cdots q_{n+k} (v_0 - d) \in q_{n+1} \cdots q_{n+k} G, \) so \( d = v_0 + g \) for some \( g \in G. \) So \( t = g + v_0 + \sum_{n<i\leq n+k} (p_i/q_i) a_i', \) with \( g \in G((BC)), \) \( v_0 + \sum_{n<i\leq n+k} (p_i/q_i) a_i' \in G(\langle d' V_0 \rangle). \)

□

Corollary 6.18. Assume that \( T \) has SU-rank one and that \( \hat{R} = \mathbb{F}. \) Then \( T^G \) is simple and forking independence is given by \( \| G. \)

Question Assume that \( T \) has SU-rank one. Does the independence theorem hold for \( \| G \) without the assumption \( \mathbb{F} = \hat{R}? \) In the proof above we only use this assumption for proving part (3) in the amalgam. A positive answer would also show that our characterization of forking independence given by \( \| G \) also works in this more general setting.

The following examples give a base theory \( T \) which is geometric, \( MR(T) = 2, \) \( \| 0 \neq \| \) and conditions (1),(2) and (3) still capture forking independence. This gives evidence that our characterization of independence works under more general assumptions.

Example 6.19. Let \( V = (V, \ldots) \) be a supersimple SU-rank 1 expansion of a vector space over a field \( \mathbb{F}, \) and we assume that \( acl = \text{span}_\mathbb{F} \) and \( T = Th(V) \) has quantifier elimination. Let \( T^P \) be the theory of a lovely pair \( (V, P). \) It has been shown in [27] that in this case the algebraic closure in the pair coincides with the algebraic closure in \( V, \) i.e. \( acl_P = acl = \text{span}_\mathbb{F} \) and \( T^P \) is supersimple of SU-rank 2. As shown in [28], for any SU-rank 1 theory \( T, \) the theory of lovely pairs of \( T \) has the weak non-finite cover property. In particular, \( T^P \) eliminates \( \exists^\infty. \) Thus, in our setting, \( T^P \) is geometric, with algebraic closure given by the linear span. Note also that due to the modularity of the pregeometry, \( T^P \) has quantifier elimination (any closed set is \( P\)-independent).

Assume that \( R = \mathbb{F} \) and consider the dense-codense generic submodule expansion \( T^{PG} \) of \( T^P. \) Thus, \( G(V) \) is an \( \mathbb{F} \)-vector subspace of \( V, \) and moreover, by the density property, an elementary submodel of \( V \) viewed as an \( L \)-structure. The fact that it is an elementary submodel follows (from the Tarski-Vaught test) by density and being closed under acl: if a \( 1 \)-formula with parameters in \( G(V) \) is algebraic, then it is realized in \( G(V) \) because \( G(V) \) is algebraically closed; if it is non-algebraic, it is realized by density of \( G(V). \)

Clearly, \( T^{PG} \) is the theory of lovely pairs of \( T^P, \) where \( T^P \) is viewed as a geometric theory. On the other hand, note that \( T^{PG} \) is not the theory of lovely pairs of \( T^P \) in the sense of [2]. In any model \( (V, P, G) \) of \( T^{PG}, \) for any \( a \in V \) the formula \( P(x - a) \) is non-algebraic, and therefore is realized in \( G(V). \) It follows that \( V = P(V) + G(V). \) To compare this behavior with that of a lovely pair expansion \( T^{PQ} \) of \( T^P \) in the sense of [2], recall that forking independence in \( (V, P) \) (denoted
by \( \perp^P \) is given by

\[
A \perp^P B \iff A \perp_{C_\cup P(V)} B, \text{ span}_F(AC) \cap \text{span}_F(BC) = \text{span}_F(C).
\]

The extension property in the sense of a pair \((V,P)\) of models of \(T^P\) then implies that for any \(v_1, \ldots, v_n \in V\), we can find \(w \in V \backslash P(V)\) such that \(w \perp^P wQ(V)\), and, in particular, \(w \perp_{P(V)} wQ(V)\), equivalently, \(w \notin \text{span}_F(wP(V)Q(V))\). It follows that \(V\) has infinite dimension over \(P(V) + Q(V)\). Note also that any pair \((V,P,G) \models T^PG\) fails the extension property (in the sense of \(T^P\)): the \(L_P\)-type of \(a \notin P(V) \cup G(V)\) over \(\emptyset\) has no non-forking extension over \(G(V)\) realized in \(V\). Indeed, for any such \(a\), we have \(a = b + c\) where \(b \in P(V)\) and \(c \in G(V)\). Then \(a \perp_{P(V)} G(V)\), and, thus, a \(\perp^P\)

**Example 6.20.** Let us simplify further the previous example and assume that \(T\) is the theory of pure vector spaces over \(\mathbb{F}\) and \((V,P,G) \models T^PG\). As explained in the previous example, \(V = P(V) + G(V)\) and we can view \((V,P,G)\) as a definable reduct of \((W,P,G)\), a lovely pair of \(T^P\) in the sense of \([2]\) (so \(W\) has infinite dimension over \(V\)). Note that both \((P(V),P(V) \cap G(V))\) and \((G(V),P(V) \cap G(V))\), viewed as lovely pairs of vector spaces, have full quantifier elimination. Each of them can be viewed as the small model inside a lovely pair of \(T^P\) or \(T^G\) and no extra structure is induced on them when viewed in \((W,P,G)\), and so, also in \((V,P,G)\). We claim that \(MR(V,P,G) = 3\). Indeed, \(MR(P(V),P(V) \cap G(V)) = MR(G(V),P(V) \cap G(V)) = 2\) as each is a lovely pair of pure vector spaces. Consider now the definable linear map \(m : G \times P \rightarrow V\) given by \(m(a,b) = a + b\). The map is onto with kernel \(K = \{(a, -a) : a \in G \cap P\}\) and we have \(MR(K) = 1\). Thus, \(MR(V) = 4 - 1 = 3\).

We now show that independence as described in Definition 6.7 agrees with non-forking in \((V,P,G)\). Notice that condition (1) recovers independence in the pair \((V,P)\) and independence in the pair strictly extends algebraic independence. Note also that condition (2) corresponds to algebraic independence inside \(G\). It remains to show we can witness dependence when an element becomes algebraic over \(G\). Let \(D, V_0\) as in Definition 6.7 and consider a singleton \(a\). Suppose we have \(a \in \text{acl}(G \cup D) \setminus \text{acl}(G \cup V_0)\), then \(a = g + d\) for some \(a \in A\) and \(g \in G\), so \(g = a - d \in G(a + d)\) but if \(a - d \in \text{span}_F(aV_0) + G(D)\), then \(a = d + g_1 + g_2\) with \(g_1 + g_2 \in G(\text{span}_F(aV_0)) + G(D)\). Then either \(g_1 = \mu v_0 + v_0\) for \(\mu \neq 0\), \(v_0 \in V_0\) and then \(a \in \text{acl}(G \cup V_0)\), a contradiction, or \(g_1 = v_0\) and thus \(a \in \text{acl}(G \cup V_0)\), again a contradiction. The converse follows from Remark 6.8.

7. Example: Lovely pair of vector spaces with a generic NSOP\(_1\) structure on its factor-space

In this section, we describe a candidate for an example of an NSOP\(_1\) not simple theory which is modular pregeometric. This construction shall serve as a general method for constructing modular pregeometric theories of arbitrary combinatorial complexity (stable, simple, NIP, NSOP\(_1\)). We leave most of the details to the reader.

Let \(\mathbb{F}\) be any field, and let us denote by \(\mathbb{F}\)-vs the theory of infinite dimensional \(\mathbb{F}\)-vector space in the usual one-sorted language. For \(V \models \mathbb{F}\)-vs and \(\bar{a}\) a tuple in \(V\), we write \(\text{tp}^V(\bar{a})\) for it type in the vector space language. Similarly, when we add a
new sort $S$ with extra structure to $V$, we will write $tp^S(\vec{a})$ for the type of the tuple seeing $S$ with the induced structure, provided $\vec{a}$ belongs to the sort $S$.

**Fact 7.1.** Let $V \models \mathbb{F}-\text{vs.}$ Let $H \subseteq V$ such that $(V, H)$ is an $H$-structure. Then $H_{\text{ind}}$, the induced structure on $H$, is trivial. Let $H' \supseteq H_{\text{ind}}$ be any expansion of $H$ which is NSOP. Then $(V, H')$ is NSOP and for all $\vec{a} \subseteq V$, we have
\[
\text{tp}^V(\vec{a}) \cup \text{tp}^{H'}(\text{dcl}_{\text{eq}}(H(\text{span}_\mathbb{F}(\vec{a})))) \vdash \text{tp}^{(V, H')}(\vec{a})
\]
where $\text{dcl}_{\text{eq}}(A)$ is the set of codes for finite subsets of $A$.

**Proof.** Since the theory $\mathbb{F}-\text{vs}$ is strongly minimal the pair $(V, H)$ is $\omega$-stable, $H$ is stably embedded and there is no outside structure induced on $H_{\text{ind}}$. Moreover, this last set just models the theory in the empty language with infinitely many elements. The rest follows from the work in [15, Section 3]. We check the following properties:

1. $\text{acl}^V(\vec{a}) \cap H^\infty = \text{dcl}_{\text{eq}}(H(\text{span}_\mathbb{F}(\vec{a})))$;
2. $H$ is algebraically embedded in $(V, H)$, which means: if $\vec{a} \not\subseteq (V, H) \overset{\mathcal{C}}{\downarrow} \vec{b}$, then
\[
\text{acl}^V(\vec{a} \vec{b} \vec{c}) \cap H^\infty \subseteq \text{acl}^V(\text{acl}^V(\vec{a} \vec{c}) \cap H^\infty, \text{acl}^V(\vec{b} \vec{c}) \cap H^\infty)
\]
(see [15, Definition 4.9]).

(1) follows from the fact that $H_{\text{ind}}$ and $V$ have weak elimination of imaginaries, and the description of algebraic closure in $H$-structures (see Corollary 3.14 in [7]).

**Fact 7.2.** Let $(V, W)$ be a lovely pair of models of the theory $\mathbb{F}-\text{vs}$. Consider the two sorted structure $(V, V/W, \pi)$ with the quotient homomorphism $\pi : V \to V/W$ and let $V^* = V/W$ the quotient vector space.

1. The theory of $(V, V^*, \pi)$ is stable.
2. For all $a \subseteq V$, $a \subseteq V^*$ we have $tp^{V^*}(\alpha/\pi(a)) \vdash tp^{(V, V^*)}(\alpha/a)$.
3. The induced structure on $V^*$ in $(V, V^*, \pi)$ is the one of a pure vector space. In particular $V^*$ is stably embedded and $V_{\text{ind}} = V^*$ has weak elimination of imaginaries.
4. $(V, V^*, \pi)$ has weak elimination of imaginaries.
5. For all $\vec{a} \subseteq V$ and $\vec{a} \subseteq V^*$, $\text{acl}^V(\vec{a}, \vec{a}) = \text{dcl}_{\text{eq}}(\text{span}_\mathbb{F}(\vec{a} \vec{a}))$.

**Proof.** This is left to the reader as an exercise, it is very similar to the work in [17, Section 3].

**Corollary 7.3.** Let $(V, (V^*, H'))$ be the structure obtained by expanding the structure on $V^*$ by an $H$-predicate and an NSOP$_1$ expansion $H' \supseteq H$. Then the expansion $(V, (V^*, H'))$ is NSOP. Furthermore, for each $\vec{a} \vec{a}$ we have
\[
\text{tp}^{(V, V^*)}(\vec{a} \vec{a}) \cup \text{tp}^{H'}(\text{dcl}_{\text{eq}}(H(\text{span}_\mathbb{F}(\pi(\vec{a}), \vec{a})))) \vdash \text{tp}^{(V, (V^*, H'))}(\vec{a} \vec{a})
\]

**Proof.** It will again follow from [15, Section 3]. By Fact 7.1, $(V^*, H')$ is an NSOP$_1$ expansion of the stably embedded set $V^*$ inside $(V, V^*)$. To preserve NSOP$_1$, one has to check that $V^*$ is algebraically embedded in $(V, V^*)$, which is the following: for $\downarrow f$ forking independence in $(V, V^*)$, if $\vec{a} \vec{a} \overset{\mathcal{C}}{\downarrow} f \vec{b} \vec{a}$, and $\vec{c} \vec{c} \subseteq \vec{a} \vec{a} \cap \vec{b} \vec{a}$, then
\[
\text{acl}^V(\vec{a} \vec{b} \vec{c} \vec{b} \vec{a}) \cap V^\infty \subseteq \text{acl}^V(\text{acl}^V(\vec{a} \vec{a}) \cap V^\infty, \text{acl}^V(\vec{b} \vec{b} \vec{a} \vec{a} \cap V^\infty))
\]
This easily follows by Fact 7.2 (5), (regardless of the forking condition):

\[
\text{span}_F(\pi(a\bar{b})\bar{a}\bar{\beta}) = \text{span}_F(\text{span}_F(\pi(a)\bar{a}), \text{span}_F(\pi(b)\bar{\beta})),
\]

so \(\text{dcl}^\text{eq}(\text{span}_F(\pi(a\bar{b})\bar{a}\bar{\beta})) = \text{dcl}^\text{eq}(\text{span}_F(\text{dcl}^\text{eq}(\text{span}_F(\pi(a)\bar{a})), \text{dcl}^\text{eq}(\text{span}_F(\pi(b)\bar{\beta})))),\)

which proves that \(V^*\) is algebraically embedded in \((V, V^*)\).

Now, we consider the structure \(V^\pi\) whose universe is \(V\) with the structure \((V^*, H')\) pulled back under \(\pi\). By this we mean that there is a predicate \(H^\pi \subseteq V\) such that

\[x \in H^\pi \iff \pi(x) \in H,\]

Also, for any relation \(R \subseteq H^n\) in \(H',\) define \(R^\pi \subseteq (H^\pi)^n\) so that

\[x \in R^\pi \iff \pi(x) \in R.\]

For function symbol \(f\) and constant symbol \(c\), define the relations \(R^\pi_f\) and \(R^\pi_c\) as follows:

\[x \in R^\pi_f \iff \pi(x) \in \text{Graph}(f),\]

\[x \in R^\pi_c \iff \pi(x) = c.\]

We also keep the predicate \(W\) in the structure \(V^\pi\).

**Corollary 7.4.** The structure \(V^\pi\) is NSOP\(_1\). If \(H'\) is not simple, neither is \(V^\pi\).

**Proof.** Recall that \((V, W)\) is a lovely pair and \(\pi : V \to V/W\) is the projection and \(W\) is kept as a predicate inside \(V^\pi\). The structure \(V^\pi\) is interdefinable with \((V, (V^*, H'))\) so \(V^\pi\) is NSOP\(_1\). Also if \(H'\) is not simple, \(V^\pi\) is not simple.

Now even if the algebraic closure in \((V^*, H')\) might be larger than the vector space span, the algebraic closure should not change in \(V^\pi\) because \(\pi\) has infinite fibers. Every formula \(\phi(x)\) in \((V^*, H')\) has a dual formula \(\phi^\pi(x)\) in \(V^\pi\) such that \(V^\pi \models \phi^\pi(a) \iff V^* \models \phi(\pi(a))\) so no algebraicity should come from the extra structure in \(V^\pi\). For the same reason, uniform finiteness should be preserved. This leads us to state as a conjecture:

**Conjecture.** In \(V^\pi\), the algebraic closure of a set \(A \subseteq V\) is given by the \(\text{span}_F(A)\), and \(V^\pi\) eliminates \(\exists^\infty\). In particular the theory of \(V^\pi\) is geometric.

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