Isometric group actions on Banach spaces and representations vanishing at infinity

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Abstract

Our main result is that the simple Lie group \( G = Sp(n, 1) \) acts properly isometrically on \( L^p(G) \) if \( p > 4n + 2 \). To prove this, we introduce property (BP\( V_0 \)), for \( V \) be a Banach space: a locally compact group \( G \) has property (BP\( V_0 \)) if every affine isometric action of \( G \) on \( V \), such that the linear part is a \( C_0 \)-representation of \( G \), either has a fixed point or is metrically proper. We prove that solvable groups, connected Lie groups, and linear algebraic groups over a local field of characteristic zero, have property (BP\( V_0 \)). As a consequence for unitary representations, we characterize those groups in the latter classes for which the first cohomology with respect to the left regular representation on \( L^2(G) \) is non-zero; and we characterize uniform lattices in those groups for which the first \( L^2 \)-Betti number is non-zero.

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1 Introduction

The study of affine isometric actions on Hilbert spaces has, in recent years, found applications ranging from geometric group theory, to rigidity theory, to \( K \)-theory of \( C^* \)-algebras. For instance, if \( G \) is \( \sigma \)-compact, \( G \) has Kazhdan’s Property (T) if and only if every affine isometric action on a Hilbert space has a fixed point (see [BHV, Chapter 2]). This is known to have strong group-theoretic consequences on \( G \); for instance, this implies that \( G \) is compactly generated and has compact abelianization (see [BHV, Chap. 2] for a direct proof). On the other hand, a group is said to be a-T-menable if it admits a proper affine action on a Hilbert
space, and a deep result of Higson and Kasparov \cite{HiKa} asserts that an a-Tmenable group satisfies the Baum-Connes conjecture.

These notions naturally extend to affine isometric actions on Banach spaces. Generalizations of Property (T) to the context of uniformly convex Banach spaces have proven to be fruitful, producing new rigidity results \cite{BFGM, FM}; on the other hand, a result of Kasparov and G. Yu \cite{KY} asserts that the Novikov conjecture holds for finitely generated groups embedding uniformly in superreflexive Banach spaces (in particular for those admitting a proper isometric action on such a space).

Guoliang Yu recently proved \cite{Yu} that every Gromov hyperbolic group admits a proper isometric action on the uniformly convex space $\ell^p(\Gamma \times \Gamma)$, for $p \gg 0$. This result contrasts with the existence of Gromov hyperbolic groups with Property (T).

Our main result is the following theorem.

**Theorem 1.1.** Let $k$ be a local field. Let $G$ be a simple algebraic group of rank 1 over $k$. Let $p_0$ be the Hausdorff dimension of the visual boundary of $G$. Then, for every $p > \max\{1, p_0\}$, there exists a proper affine action of $G$ on $L^p(G)$ with linear part $\lambda_{G,p}$.

Note that this result cannot be extended to simple groups of rank $\geq 2$ (see \cite[Theorem B]{BFGM}). We also prove (see Proposition 3.1) that every countable a-T-menable group admits a proper affine action on some $L^p$-space, for every $p \geq 1$.

A key ingredient of the proof of Theorem 1.1 is a result of Pansu \cite{Pa1, Pa2}, who computes the first $L^p$-cohomology for semi-simple Lie groups for $1 < p < \infty$. The first $L^p$-cohomology actually coincides (see Section 3.2) with the first cohomology of the group with values in the right regular representation on $L^p(G)$. The proof of Theorem 1.1 then consists in showing that non-trivial 1-cocycles on such representations are automatically proper. This latter fact is part of a more general phenomenon: the properness of non-trivial 1-cocycles on an isometric $L^p$-representation $\pi$ of a group $G$ is actually true under very general assumptions on $G$ and $\pi$.

Our approach was initially motivated by the following example. The cyclic group $\mathbb{Z}$ acts naturally on $\ell^2(\mathbb{Z})$; the corresponding operator $T$, given by the action of the positive generator of $\mathbb{Z}$ is usually called the bilateral shift. Now take $f \in \ell^2(\mathbb{Z})$, and let us consider the affine isometry $T_f$ of $\ell^2(\mathbb{Z})$ given by $T_f(v) = Tv + f$. It is immediately checked that this isometry has a fixed point if and only if $f \in \text{Im}(T - 1)$. We show that otherwise the corresponding action
is proper, that is, for every/some $v \in \ell^2(\mathbb{Z})$, $\|T^n_f(v)\| \to \infty$ when $|n| \to \infty$. Our aim is to make this observation systematic.

One essential feature in the above context is that the representation of $\mathbb{Z}$ on $\ell^2(\mathbb{Z})$ is $C_0$. In a general context, let $V$ be a Banach space. An isometric linear representation $\pi$ of a locally compact group $G$ is $C_0$ if for every $L \in V^*$ (the topological dual), and every $v \in V$, we have $L(\pi(g)v) \to 0$ when $g$ tends to infinity. In other words, $\pi(g)v$ weakly tends to zero for every $v \in V$.

**Definition 1.2.** Let $V$ be a Banach space. A locally compact group $G$ has **Property (BP$_0^V$)** if, for every $C_0$ isometric linear representation $\pi$ of $G$ on $V$, any affine isometric action of $G$ with linear part $\pi$ either has a bounded orbit or is proper. We say that $G$ has property (BP$_0$) if it has (BP$_0^V$) for every Hilbert space $V$.

The acronym (BP$_0$) stands for “Bounded or Proper with respect to $C_0$-representations”.

Thus the observation above amounts to prove that $\mathbb{Z}$ has Property (BP$_0$). This is part of the following result.

**Proposition 1.3.** (see Proposition 2.10) Let $G$ be a locally compact group, and $V$ a Banach space.

1) Suppose that $G$ has two non-compact normal subgroups centralizing each other. Then $G$ has Property (BP$_0^V$).

2) Suppose that $G$ has a non-compact normal subgroup with Property (BP$_0^V$). Then $G$ has Property (BP$_0^V$).

**Corollary 1.4.** (see Corollary 2.12 and Proposition 2.14) Let $V$ be a Banach space.

1) Every solvable locally compact group has Property (BP$_0^V$).

2) Every connected Lie group or linear algebraic group over a $p$-adic field has property (BP$_0^V$).

As an application of Corollary 1.4, we characterize in Proposition 4.11 those connected Lie groups and linear algebraic groups over a $p$-adic field, such that the first cohomology of $G$ with coefficients in the left regular representation $\lambda_G$ on $L^2(G)$ is non-zero; this generalizes a result of Guichardet (Proposition 8.5 in Chapter III of [Gu2]) for simple Lie groups.
Proposition 1.5. (see Proposition 4.11) Let $G$ be a connected Lie group or $G = G(K)$, the group of $K$-points of a linear algebraic group $G$ over a local field $K$ of characteristic zero. Assume $G$ non-compact. Then the following are equivalent

(i) $H^1(G, \lambda_G) \neq 0$.

(ii) Either $G$ is amenable, or there exists a compact normal subgroup $K \subset G$ such that $G/K$ is isomorphic to $\text{PSL}_2(\mathbb{R})$ (case of Lie groups), or a simple algebraic group of rank one (case of an algebraic group over a $p$-adic field).

We also characterize those uniform lattices $\Gamma$ in a group as above, whose first $L^2$-Betti number $\beta^1_{(2)}(\Gamma)$ is non-zero. For uniform lattices in connected Lie groups, this gives a new proof of Theorem 4.1 in [Eck].

Corollary 1.6. (see Corollary 4.12) Let $G$ be a connected Lie group or $G = G(K)$ where $K$ is a local field of characteristic zero; let $\Gamma$ be a uniform lattice in $G$. If the first $L^2$-Betti number $\beta^1_{(2)}(\Gamma)$ is non-zero, then $\Gamma$ is commensurable either to a non-abelian free group or to a surface group.

In contrast with property (BP$_0$), we have

Proposition 1.7. (see Proposition 5.3) There exists an affine isometric action of $\mathbb{Z}$ on a complex Hilbert space, that is neither proper nor bounded, and whose linear part has no finite-dimensional subrepresentation.

This result can be extended to $\mathbb{R}$ in view of the following result.

Proposition 1.8. (see Proposition 5.3) Every isometric action of $\mathbb{Z}$ on a complex Hilbert space can be extended to a continuous action of $\mathbb{R}$.

While property (BP$_0$) is a rule for connected Lie groups or linear algebraic groups over local fields of characteristic zero, this is certainly not the case for discrete groups:

Proposition 1.9. (see Proposition 4.3 and Corollary 4.7) Non-abelian free groups and surface groups do not have property (BP$_0$).

The proof of Proposition 1.9 follows from a direct and simple construction for free groups. However, for surface groups, our proof relies on von Neumann algebra arguments as well as the analytical zero divisor Conjecture.

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2 Actions on Banach spaces

We define a Banach pair as a pair \((V, \mathcal{L})\) where \(V\) is a Banach space and \(\mathcal{L}\) is a linear subspace of \(V^*\). We call it a Banach-Steinhaus pair if it satisfies the Banach-Steinhaus Property: any subset \(X \subset V\) is bounded if and only if \(L(X)\) is bounded for every \(L \in \mathcal{L}\). For instance, the Banach-Steinhaus Theorem states that \((V^*, V)\) is a Banach-Steinhaus pair for every Banach space \(V\), and in particular \((V, V^*)\) is a Banach-Steinhaus pair. If \((V, \mathcal{L})\) is a Banach-Steinhaus pair, and if \(W\) is a closed subspace of \(V\), then \((W, \mathcal{L}|_W)\) is clearly a Banach-Steinhaus pair, where \(\mathcal{L}|_W\) is the set of restrictions of \(L \in \mathcal{L}\) to \(W\).

If \((V, \mathcal{L})\) is a Banach pair, we say that an isometric linear action \(\pi\) of a locally compact group \(G\) on \(V\) is \(C_0^L\)-representation if \(L(\pi(g)v)\) tends to zero when \(g \to \infty\), for every \(v \in V\) and \(L \in \mathcal{L}\). Note that being \(C_0^L\) definitely depends on \(L\) (see the example below); however when the context is clear we write it \(C_0\).

Example 2.1. Let \(G\) be a discrete, infinite group. Consider its regular \(\ell^1\)-representation. Then it is \(C_0^{c_0}(G)\) but not \(C_0^{\ell^\infty}(G)\). Note that both are Banach-Steinhaus pairs. This example motivates the introduction of Banach-Steinhaus pairs different from \((V, V^*)\).

If \(\pi\) is a \(C_0\) representation as above, and if \(W\) is a closed invariant subspace, defining a subrepresentation \(\pi|_W\), then \(\pi|_W\) is \(C_0^{\mathcal{L}|_W}\).

Definition 2.2. Let \((V, \mathcal{L})\) be a Banach pair. A locally compact group \(G\) has Property \((FH^0_{(V, \mathcal{L})})\) (respectively \((BP^0_{(V, \mathcal{L})})\)) if, for every \(C_0\)-representation \(\pi\) of \(G\), any affine isometric action of \(G\) on \(V\) with linear part \(\pi\) has a bounded orbit (resp. either has a bounded orbit or is proper).

We say that \(G\) has Property \((FH^0_{(W, \mathcal{L})})\) if it has Property \((FH^0_{(W, \mathcal{L}|_W)})\) for every closed subspace \(W\) of \(V\). We define analogously Property \((BP^0_{(W, \mathcal{L})})\).

Similarly, we say that \(G\) has Property \((FH^0_{(V^*)})\) (respectively \((BP^0_{(V^*)})\)) if it has Property \((FH^0_{(V^*, \mathcal{L})})\) (respectively \((BP^0_{(V^*, \mathcal{L})})\)) for \(\mathcal{L} = V^*\).

When the space \(V\) is superreflexive, i.e. isomorphic to a uniformly convex space, it is known that every nonempty bounded subset has a unique circumcenter (also called Chebyshev center, see p. 27 in [BL]). As a consequence, every isometric action with a bounded orbit on \(V\) has a globally fixed point.

Lemma 2.3. Let a compact group \(K\) act by affine isometries on a Banach space. Then it fixes a point.

Proof. Let \(\Omega\) be an orbit. As \(\Omega\) is compact, its closed convex hull \(X\) is also compact (see for instance [Rud, Theorem 3.25]). As \(K\) is amenable and acts on \(X\) by affine transformations, it has a fixed point.

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Lemma 2.4. Let \((V, \mathcal{L})\) be a Banach pair. Let \(K\) be a compact, normal subgroup of \(G\). The following are equivalent.

(i) \(G\) has Property \((\text{FH}_0^{[V], \mathcal{L}])\) (resp. \((\text{BP}_0^{[V], \mathcal{L}})\));

(ii) \(G/K\) has Property \((\text{FH}_0^{[V], \mathcal{L}])\) (resp. \((\text{BP}_0^{[V], \mathcal{L}})\)).

Proof. The implication (i) \(\Rightarrow\) (ii) is clear. Suppose that \(G/K\) has Property \((\text{BP}_0^{[W], \mathcal{L}|_W})\). By Lemma 2.3, the set \(W\) of \(K\)-fixed points is a non-empty closed affine subspace; moreover it is \(G\)-invariant. As \(G\) has Property \((\text{BP}_0^{[W], \mathcal{L}|_W})\), its action on \(W\), and therefore on \(V\), is either bounded or proper. The case of Property \((\text{FH}_0^{[V], \mathcal{L}})\) is similar.

Lemma 2.5. Let \((V, \mathcal{L})\) be a Banach-Steinhaus pair. Let \(H, K\) be closed, non-compact subgroups of the locally compact group \(G\) which centralize each other. Let \(\alpha\) be an affine isometric action of \(G\) on \(V\), whose linear part \(\pi\) is a \(C_0\)-representation. Then either \(\alpha|_H\) and \(\alpha|_K\) are both bounded, or they are both proper.

Proof. Set \(b(g) = \alpha(g)(0)\). We assume that \(\alpha|_H\) is not proper, i.e.

\[ M =: \liminf_{h \to \infty, h \in H} \|b(h)\| < \infty. \]

For \(k \in K, h \in H\), the 1-cocycle relation gives

\[ \pi(k)b(h) + b(k) = b(gh) = b(hk) = \pi(h)b(k) + b(h), \]

which we will use in the following form:

\[ b(k) = \pi(h)b(k) + (1 - \pi(k))b(h). \]

Then, for every \(L \in \mathcal{L}\) we have

\[ L(b(k)) = L(\pi(h)b(k)) + L((1 - \pi(k))b(h)), \]

and thus

\[ |L(b(k))| \leq |L(\pi(h)b(k))| + |L((1 - \pi(k))b(h))| \leq |L(\pi(h)b(k))| + 2\|L\||b(h)||. \]

Taking the inferior limit when \(h \to \infty\) in \(H\), we obtain

\[ |L(b(k))| \leq 2\|L\|M. \]

Thus \(L(b(K))\) is bounded for every \(L\); as \((V, \mathcal{L})\) is a Banach-Steinhaus pair this means that \(b(K)\) is bounded.

Inverting the roles of \(H\) and \(K\), we can easily conclude.
The following proposition is an immediate consequence of Lemma 2.5 by taking \( H = G \) and \( K = Z(G) \).

**Proposition 2.6.** Let \( G \) be a locally compact group with non-compact centre (e.g. a non-compact, locally compact abelian group). Then \( G \) has property \((BP_0^{(V,L)})\) for every Banach-Steinhaus pair \((V,L)\).

In order to enlarge the class of groups for which we are able to prove Property \((BP_0)\), we need the following lemma.

**Lemma 2.7.** Let \((V,L)\) be a Banach-Steinhaus pair. Let \( \alpha \) be an affine isometric action of \( G \) on \( V \), with linear part a \( C_0 \)-representation \( \pi \). Set \( b(g) = \alpha(g)(0) \). Let \( H \) be a closed, non-compact subgroup of \( G \). Assume that there exists a sequence \( (g_k)_{k \geq 1} \) in \( G \), going to infinity, such that

- the sequence \((b(g_k))\) is bounded in \( V \);
- for every \( h \in H \), the set \( \{g_k^{-1}hg_k \mid k \geq 1\} \) is relatively compact in \( G \).

Then \( \alpha|_H \) is bounded.

**Proof.** Fix \( M > 0 \) such that \( \|b(g_k)\| \leq M \) for every \( k \geq 1 \), and, for \( h \in H \) define \( K_h \) as the closure of the set \( \{g_k^{-1}hg_k \mid k \geq 1\} \), which is compact by assumption. Let us show that \( \|b(h)\| \leq 2M \) for every \( h \in H \). Noting that \( hg_k = g_kh_k \), where \( h_k = g_k^{-1}hg_k \), we expand \( b(hg_k) - b(g_k) - b(h) \) in two ways, and we obtain

\[
\pi(g_k)b(h_k) - b(h) = (\pi(h) - 1)b(g_k).
\]

So, for every \( L \in \mathcal{L} \),

\[
|L(b(h))| \leq |L(\pi(g_k)b(h_k))| + |L((\pi(h) - 1)b(g_k))| \\
\leq |L(\pi(g_k)b(h_k))| + 2\|L\|\|b(g_k)\|.
\]

Using the assumption that \( h_k \in K_h \) for every \( k \), and the fact that for a \( C_0 \)-representation the decay of coefficients to 0 is uniform on compact subsets of the ambient Banach space, we get for \( k \to \infty \),

\[
|L(b(h))| \leq 2M\|L\|.
\]

As \((V,L)\) is a Banach-Steinhaus pair, this implies that \( b(H) \) is bounded. □

The following lemma is a kind of a geometric Hahn-Banach statement.
Lemma 2.8. Let $(V, L)$ be a Banach-Steinhaus pair. Then there exists $\eta > 0$ with the following property: for every $v \in V$, there exists $L \in L$ such that $\|L\| \leq 1$ and $L(v) \geq \eta \|v\|$.

Proof. Suppose the contrary. For every $n$, there exists $v_n \in V$ of norm one such that for every $L \in L$, we have $L(v_n) < 2^{-n}\|L\|$. Set $X = \{2^n v_n | n \geq 0\}$. Then $L(X)$ is bounded for every $L \in L$; by the Banach-Steinhaus Property, $X$ is bounded; this is a contradiction. \qed

Lemma 2.9. Let $(V, L)$ be a Banach-Steinhaus pair. Let $G$ be a locally compact group, and $N$ a non-compact, closed normal subgroup. Let $\alpha$ be an affine isometric action of $G$ on $V$ whose linear part is $C_0$.

(1) Suppose that $\alpha|_N$ is bounded. Then $\alpha$ is also bounded.

(2) Suppose that $\alpha|_N$ is proper. Then $\alpha$ is also proper.

Proof. (1) For $M \geq 0$, define $A_M$ as the set of all $x \in V$ whose $N$-orbit has diameter at most $M$. Clearly $A_M$ is $G$-invariant. By assumption, for some $M$ (which we fix now), the set $A_M$ is non-empty. We claim that it is bounded, allowing us to conclude.

Consider $x, y \in A_M$, and set $v = x - y$. Then, for $h \in N$

$$\pi(h)v - v = \alpha(h)x - x - \alpha(h)y + y.$$ 

So

$$\|\pi(h)v - v\| \leq \|\alpha(h)x - x\| + \|y - \alpha(h)y\| \leq 2M.$$ 

Fix $\eta$ and $L$ as in Lemma 2.8. Then

$$\eta \|v\| \leq L(v) \leq |L(v) - L(\pi(h)v)| + |L(\pi(h)v)| \leq 2M + |L(\pi(h)v)|.$$ 

As $N$ is non-compact, letting $h \to \infty$, we obtain $\|v\| \leq 2M/\eta$. Thus the diameter of $A_M$ is bounded by $2M/\eta$.

(2) Suppose by contradiction that $\alpha|_N$ is proper and $\alpha$ is not proper. Then there exists a sequence $(g_k)$ in $G$, tending to infinity, such that $(b(g_k))$ is bounded. As $\alpha|_N$ is unbounded, there exists, by Lemma 2.7, an element $n \in N$ such that the sequence $(g_k^{-1}ng_k)_{k \geq 1}$ is not relatively compact in $N$; extracting if necessary we can suppose that it tends to infinity. Therefore, as $\alpha|_N$ is proper, $\|b(g_k^{-1}ng_k)\|$ tends to infinity. But it is bounded by $2\|b(g_k)\| + \|b(n)\|$, which is bounded, a contradiction. \qed
From this we deduce the following.

**Proposition 2.10.** Let \((V, \mathcal{L})\) be a Banach-Steinhaus pair, and let \(G\) be a locally compact group. Let \(N\) be a non-compact, closed, normal subgroup of \(G\). If either the centralizer \(C_G(N)\) of \(N\) is non-compact, or \(N\) has Property \((BP_0(V, \mathcal{L}))\), then \(G\) also has Property \((BP_0(V, \mathcal{L}))\).

**Proof.** Let \(\alpha\) be an affine isometric action of \(G\), with linear part a \(C_0\)-representation \(\pi\). If \(\alpha\big|_N\) is bounded, then \(\alpha\) is bounded by Lemma \(2.9(1)\). Assume then that \(\alpha\big|_N\) is unbounded. Then \(\alpha\big|_N\) is proper (in case \(C_G(N)\) is non-compact, this follows from lemma \(2.5\)). Accordingly, by Lemma \(2.9(2)\), \(\alpha\) is proper. \(\square\)

**Corollary 2.11.** Let \((V, \mathcal{L})\) be a Banach-Steinhaus pair. Then Properties \((BP_0(V, \mathcal{L}))\) and \((FH_0(V, \mathcal{L}))\) are preserved by extensions.

**Proof.** Let \(1 \to N \to G \to H \to 1\) be an extension of locally compact groups, and suppose that \(N\) and \(H\) have Property \((BP_0(V, \mathcal{L}))\). If \(N\) is compact, then, by Lemma \(2.4\) since \(H\) has Property \((BP_0(V, \mathcal{L}))\), so does \(G\). If \(N\) is not compact, then, since it has Property \((BP_0(V, \mathcal{L}))\), by Proposition \(2.10\) \(G\) has Property \((BP_0(V, \mathcal{L}))\). The case of Property \((FH_0(V, \mathcal{L}))\) is similar (and easier). \(\square\)

**Corollary 2.12.** Locally compact, solvable groups have Property \((BP_0(V, \mathcal{L}))\) for every Banach-Steinhaus pair \((V, \mathcal{L})\).

**Proof.** Since, using Proposition \(2.6\) locally compact abelian groups have Property \((BP_0(V, \mathcal{L}))\), this follows from Corollary \(2.11\). \(\square\)

**Lemma 2.13.** Let \((V, \mathcal{L})\) be a Banach pair. Property \((BP_0(V, \mathcal{L}))\) is inherited from cocompact subgroups.

**Proof.** The proof is straightforward. \(\square\)

**Proposition 2.14.** Connected Lie groups, and linear algebraic groups over \(p\)-adic fields, have Property \((BP_0(V, \mathcal{L}))\) for every Banach-Steinhaus pair \((V, \mathcal{L})\).

**Proof.** This follows from Lemma \(2.13\) and Corollary \(2.12\) since \(G\) contains a solvable cocompact subgroup \(H\): for linear algebraic groups over local fields of characteristic zero, this follows from [BT, Théorème 8.2]; for Lie groups, taking the quotient by the maximal solvable normal subgroup, we can also use [BT, Théorème 8.2]. \(\square\)
3 Proper affine actions of rank 1 groups on $L^p$-spaces

3.1 Spaces with measured walls, and the non-archimedean case

Recall that a locally compact $\sigma$-compact group is $a$-T-menable if it acts properly isometrically on some Hilbert space.

**Proposition 3.1.** Let $\Gamma$ be a countable, discrete group. The following are equivalent:

i) $\Gamma$ is $a$-T-menable;

ii) for every $p \geq 1$, the group $\Gamma$ acts properly isometrically on some $L^p$-space.

**Proof.** We prove the non-trivial implication $(i) \Rightarrow (ii)$. We recall from [CMV] that a space with measured walls is a 4-tuple $(X, \mathcal{W}, \mathcal{B}, \mu)$ where $X$ is a set, $\mathcal{W}$ is a set of partitions of $X$ into 2 classes (called walls), $\mathcal{B}$ is a $\sigma$-algebra of sets on $\mathcal{W}$, and $\mu$ is a measure on $\mathcal{B}$ such that, for every pair $x, y$ of distinct points in $X$, the set $\omega(x, y)$ of walls separating $x$ from $y$ belongs to $\mathcal{B}$ and satisfies $w(x, y) =: \mu(\omega(x, y)) < \infty$.

It was proved in Proposition 1 of [CMV] that a countable group is $a$-T-menable if and only if it admits a proper action on some space with measured walls (by this we mean that $\Gamma$ preserves the measured wall space structure, and that the function $g \mapsto w(gx, x)$ is proper on $\Gamma$).

A half-space in a space with measured walls $X$ is a class of the partition defined by some wall in $\mathcal{W}$. Let $\Omega$ be the set of half-spaces, $\pi : \Omega \to \mathcal{W}$ the canonical map (associating to any half-space the corresponding wall), $\mathcal{A} =: p^{-1}(\mathcal{B})$ the pulled-back $\sigma$-algebra, and $\nu$ the pulled-back measure defined by

$$\nu(A) = \frac{1}{2} \int_{\mathcal{W}} \text{card}(A \cap p^{-1}(x)) \ d\mu(x)$$

for $A \in \mathcal{A}$. Let $\chi_x$ be the characteristic function of the set of half-spaces through $x$. For $x, y \in X$, we define a function $c(x, y) \in L^p(\Omega, \nu)$ by:

$$c(x, y) = \chi_x - \chi_y.$$

Suppose that $\Gamma$ acts properly on $(X, \mathcal{W}, \mathcal{B}, \mu)$. For $p \geq 1$, let $\pi_p$ denote the quasi-regular representation of $\Gamma$ on $L^p(\Omega, \nu)$. Observe that:

- $c(x, y) + c(y, z) = c(x, z)$;
\[ c(gx, gy) = \pi_p(g)c(x, y); \]
\[ \|c(x, y)\|_p^p = w(x, y) \]
for every \( x, y, z \in X, g \in \Gamma \). Fixing a base-point \( x_0 \in X \), the map
\[ b : \Gamma \to L^p(\Omega, \nu) : g \mapsto c(gx_0, x_0) \]
defines a 1-cocycle in \( Z^1(\Gamma, \pi_p) \). Since \( \|b(g)\|_p = w(gx_0, x_0)^{1/p} \), this cocycle is proper, so that the corresponding affine isometric action is proper.

\[ \text{Remark 3.2.} \quad \text{What the above proof really shows is that every locally compact group acting properly on a space with measured walls, admits a proper action on some } L^p\text{-space, for every } p \geq 1. \text{ Several non-discrete examples appear in [CMV].} \]

A tree \( X = (V, E) \) is an example of a space with measured walls (with \( \mathcal{W} = E, \mu = \text{counting measure} \)). The set \( \Omega \) of half-spaces identifies with the set \( E \) of oriented edges. Suppose that a locally compact group \( G \) acts properly cocompactly on a tree \( X \). We choose a reference edge \( e_0 \in E \) and use it to lift the cocycle \( b \in Z^1(G, L^p(E)) \) from the previous proof to a cocycle \( \tilde{b} \in Z^1(G, L^p(G)) \), by the formula \( (\tilde{b}(g))(h) = (b(g))(he_0) \). Then
\[ \|\tilde{b}(g)\|_p^p = \frac{m_0d(gx_0, x_0)}{k}, \]
where \( m_0 \) is the Haar measure of the stabilizer of \( e_0 \) in \( G \), and \( k \) is the number of orbits of \( G \) in \( E \). This shows that \( \tilde{b} \) is a proper cocycle. We have proved:

**Proposition 3.3.** Let \( G \) be a locally compact group. If \( G \) acts properly cocompactly on a tree (e.g. if \( G \) is a rank 1 simple algebraic group over a non-Archimedean local field), then for every \( p \geq 1 \), the group \( G \) admits a proper isometric action on \( L^p(G) \), with linear part the left regular representation \( \lambda_{G,p} \).

### 3.2 The Lie group case

Let \( M \) be a Riemannian manifold equipped with its Riemannian measure \( \mu \). Fix \( p > 1 \). Denote by \( D_p(M) \) the vector space of differentiable functions whose gradient is in \( L^p(TM) \). Equip \( D_p(M) \) with a pseudo-norm \( \|f\|_{D_p} = \|\nabla f\|_p \), which induces a norm on \( D_p(M) \) modulo the constants. Denote by \( D_p(M) \) the completion of this normed vector space. We have \( W^{1,p}(M) = L^p(M) \cap D_p(M) \). Hence, \( W^{1,p}(M) \) canonically embeds in \( D_p(M) \) as a subspace.
The first $L^p$-cohomology of $M$ is the quotient space
$$H^1_p(M) = \mathbf{D}_p(M)/\mathbb{W}^{1,p}(M).$$

The first reduced $L^p$-cohomology of $M$ is the quotient space
$$\overline{H}^1_p(M) = \mathbf{D}_p(M)/\overline{\mathbb{W}^{1,p}(M)},$$
where $\overline{\mathbb{W}^{1,p}(M)}$ is the closure of $\mathbb{W}^{1,p}(M)$ in the Banach space $\mathbf{D}_p(M)$. Note that the two latter spaces coincide if and only if the norm $\| \cdot \|_{D^p}$ on the Sobolev space $W^{1,p}(M)$ is equivalent to the usual Sobolev norm $\| \cdot \|_p + \| \cdot \|_{D^p}$, that is, if $M$ satisfies the strong Sobolev inequality in $L^p$: $\| f \|_p \leq C \| \nabla f \|_p$. If the group of isometries $G$ of $M$ acts cocompactly on $M$, the strong Sobolev inequality in $L^p$ is satisfied if and only if $G$ is either non-amenable or non-unimodular [Pit].

Assume now that $M = G$ is a connected, unimodular Lie group, endowed with a left invariant Riemannian metric. Denote by $\rho_{G,p}$ the right regular representation on $D_p(G)$. Let $g \in G$ and $\gamma : [0, d(1, g)] \to G$ be a geodesic between $1$ and $g$. For any $f \in D_p(G)$ and $x \in G$, we have
$$(f - \rho_{G,p}(g)f)(x) = f(x) - f(xg) = \int_0^{d(1,g)} \nabla f(\gamma_x(t)) \cdot \gamma'_x(t) dt,$$
where $\gamma_x(t) = x\gamma(t)$. Using Hölder’s inequality, we deduce that
$$\| f - \rho_{G,p}(g)f \|_p \leq d(1,g) \| \nabla f \|_p.$$  
Therefore, there is a well defined map from $D_p(G)$ to $Z^1(G, \rho_{G,p})$
$$J : f \mapsto (b_f : g \mapsto f - \rho_{G,p}(g)f).$$
The map $J$ induces an injective map from $\mathbf{D}_p(G)$ to $Z^1(G, \rho_{G,p})$. Moreover, $b_f$ is a coboundary if and only if $f$ is in $L^p(G) + \{\text{constants}\}$, i.e. if and only if the class of $f$ is zero in $H^1_p(G)$. Hence, $J$ induces an injective linear map from $H^1_p(G)$ to $H^1(G, \rho_{G,p})$.

Let $G$ be a simple Lie group of rank 1 equipped with a left-invariant Riemannian metric. Up to taking the quotient by a normal compact subgroup, $G$ is $\text{PO}(n,1)$, $\text{PU}(n,1)$, $\text{PSp}(n,1)$ or $\text{F}_4(-20)$. Let $\partial G$ be the sphere at infinity of $G$, and let $p_0$ be its Hausdorff dimension, so that
$$p_0 = \begin{cases} 
  n - 1 & \text{if } G = \text{PO}(n,1) \\
  2n & \text{if } G = \text{PU}(n,1), n \geq 2 \\
  4n + 2 & \text{if } G = \text{PSp}(n,1) \\
  22 & \text{if } G = \text{F}_4(-20)
\end{cases}.$$
By a result of P. Pansu, \[ H^1_{(p)}(G) \neq 0 \] if and only if \( p > \max\{1, p_0\} \).

From the above discussion, we deduce that \( H^1(G, \rho_{G,p}) \neq 0 \) for those groups as soon as \( p > \max\{1, p_0\} \). Together with the fact that connected Lie groups have Property \((BP_0^p)\) for \(1 < p < \infty\), this yields the following result.

**Theorem 3.4.** Let \( G \) be a simple Lie group of rank 1 over \( k \). Let \( p_0 \) be the Hausdorff dimension of the visual boundary of \( G \). Then, for every \( p > \max\{1, p_0\} \), there exists a proper affine action of \( G \) on \( L^p(G) \) with linear part \( \lambda_{G,p} \).

4 Actions on Hilbert spaces

4.1 Property \((BP_0)\)

Recall that a locally compact group \( G \) has Property (FH) if every affine isometric action of \( G \) on a Hilbert space has a fixed point. For \( G \sigma\)-compact, this is known to be equivalent to the celebrated Kazhdan’s Property (T) (see [BHV, Chapter 2]).

When \( V \) is a Hilbert space (sufficiently large in comparison to \( G \)), we write \((BP_0^V)\) and \((FH_0^V)\) for \((BP_0^V)\) and \((FH_0^V)\).

There is a simple characterization of groups with Property \((FH_0)\) among groups with Property \((BP_0)\).

**Proposition 4.1.** Let \( G \) be a locally compact group.

1) Suppose that \( G \) has Property \((BP_0)\). Then either \( G \) is a-T-menable or has Property \((FH_0)\).

2) If \( G \) is both a-T-menable and has Property \((FH_0)\), then it is compact.

**Proof.** The first statement is clear. Suppose that \( G \) is a-T-menable and is not compact. Then \( G \) is \( \sigma\)-compact (take a proper action \( \alpha \) and consider \( K_n = \{ g \in G : \| \alpha(g)(0) \| \leq n \} \}). Since \( G \) is a-T-menable, it is Haagerup, i.e. it has a \( C_0\)-representation \( \pi \) with almost invariant vectors; since \( G \) is not compact, \( \pi \) has no invariant vector. By Proposition 2.5.3 in [BHV], \( \infty \pi \) has nontrivial 1-cohomology, while it is \( C_0 \). Hence \( G \) does not have Property \((FH_0)\).

Let us mention an application of Property \((BP_0)\) in ergodic theory.

**Proposition 4.2.** Let \( G \) be a locally compact group with Property \((BP_0)\) and \( \text{Hom}(G, \mathbb{R}) = 0 \). Let \( G \) act (on the right), in a measure-preserving way, on a probability space \((X, \mathcal{B}, \mu)\); assume that the action is mixing. Let \( F : X \times G \to \mathbb{C} \) be a measurable function such that

Since \( G \) is unimodular, the representations \( \lambda_{G,p} \) and \( \rho_{G,p} \) are isomorphic.

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\[ \int_X |F(x,g)|^2 \, d\mu(x) < \infty \text{ for every } g \in G; \]
\[ F(x,gh) = F(x,g) + F(x.g,h) \text{ for every } g, h \in G, \text{ almost everywhere in } x. \]

Then the following alternative holds: either there exists \( f \in L^2(X, \mu) \) such that 
\[ F(x,g) = f(x.g) - f(x) \text{ (for every } g \in G, \text{ almost everywhere in } x), \]

or
\[ \lim_{g \to \infty} \int_X |F(x,g)|^2 \, d\mu(x) = \infty. \]

**Proof.** Set \( L^2_0(X, \mu) = \{ f \in L^2(X, \mu) : \int_X f(x) \, d\mu(x) = 0 \} \). As \( \int_X F(x,gh) \, d\mu(x) = \int_X F(x,g) \, d\mu(x) + \int_X F(x,h) \, d\mu(x) \) for every \( g, h \in G \), we have \( F(\cdot,g) \in L^2_0(X, \mu) \) for every \( g \in G \), in view of the assumption \( \text{Hom}(G, \mathbb{R}) = 0 \). Let \( \pi \) be the standard representation of \( G \) on \( L^2_0(X, \mu) \). Then \( F(\cdot,g) \) defines a 1-cocycle with respect to \( \pi \). Since the \( G \)-action on \( X \) is mixing, the representation \( \pi \) is \( C_0 \) (see Theorem 2.9 in [BM]), so that the conclusion follows immediately from Property (BP_0). \( \square \)

### 4.2 Discrete groups without (BP_0)

Proposition 2.14 provides a wealth of groups with Property (BP_0). We now provide examples of groups without Property (BP_0) (in particular the free group \( F_n \) on \( n \) generators, \( n \geq 2 \)).

**Proposition 4.3.** Let \( H \) be an infinite group, \( K \) a non-trivial group, and \( F \) a common finite subgroup of \( H \) and \( K \), which is distinct from \( K \). Let \( G = H \ast_F K \) be the amalgamated product. Then there exists a 1-cocycle with respect to the regular representation \( \lambda_G \) which is neither bounded nor proper. In particular, \( G \) does not have Property (BP_0).

**Proof.** Let \( w \) be a \( \ell^2 \) function on \( G \) which is left \( F \)-invariant, but not left \( K \)-invariant (in particular \( w \neq 0 \)). Define \( \alpha(k) = \lambda_G(k) \) for \( k \in K \), and \( \alpha(h) = t_w \circ \lambda_G(h) \circ t_{-w} \) for \( h \in H \), where \( t_w \) denotes the translation by \( w \) on \( \ell^2(G) \). Then \( \alpha \) is well-defined on \( H \ast_F K \) (by the \( F \)-invariance assumption on \( w \)). The fixed point set of \( K \) is the set of all left \( K \)-invariant functions. The set of fixed points of \( H \) is reduced to \( \{w\} \) (since \( H \) is infinite). Accordingly, the action has no fixed point. On the other hand, since \( H \) is infinite and has a fixed point, the action is not proper. \( \square \)

To obtain other examples of groups without (BP_0), we first establish a connection with a classical conjecture on discrete groups. For a group \( \Gamma \), we denote
by $C\Gamma$ the group algebra over $C$, and by denote again by $\lambda_\Gamma$ the left regular representation of $C\Gamma$ on $\ell^2(\Gamma)$:

$$\lambda_\Gamma(f)\xi = f \ast \xi$$

($f \in C\Gamma$, $\xi \in \ell^2(\Gamma)$). Here is the analytical zero-divisor conjecture:

**Conjecture 1.** If $\Gamma$ is a torsion-free group, then $\lambda_\Gamma(f)$ is injective, for every non-zero $f \in C\Gamma$.

The main result on Conjecture 1 is due to P. Linnell [Lin]: it holds for groups which are extensions of a right-orderable group by an elementary amenable group; in particular, we will use the fact that it holds for non-abelian free groups.

**Lemma 4.4.** Let $\Gamma$ be a group satisfying Conjecture 1. Let $f_1, f_2 \in C\Gamma$ be non-zero elements. There exists non-zero functions $\xi_1, \xi_2 \in \ell^2(\Gamma)$ such that $\lambda_\Gamma(f_1)\xi_1 + \lambda_\Gamma(f_2)\xi_2 = 0$.

**Proof.** We start with a

**Claim:** If $f \in C\Gamma$ is a non-zero element, then $\lambda_\Gamma(f)$ has dense image. To see that, observe that the orthogonal of the image of $\lambda_\Gamma(f)$ is the kernel of $\lambda_\Gamma(f^*)$, which is $\{0\}$ as $\Gamma$ satisfies Conjecture 1.

Let $L(\Gamma)$ be the von Neumann algebra of $\Gamma$, i.e. the bi-commutant of $\lambda_\Gamma(C\Gamma)$ in $B(\ell^2(\Gamma))$. A (non-necessarily closed) subspace of $\ell^2(\Gamma)$ is affiliated with $L(\Gamma)$ if it is invariant under the commutant $\lambda_\Gamma(C\Gamma)'$ of $\lambda_\Gamma(C\Gamma)$. E.g., if $f \in C\Gamma$, the image of $\lambda_\Gamma(f)$ is an affiliated subspace. A result of L. Aagaard [Aag] states that the intersection of two dense, affiliated subspaces is still dense. We apply this with the images of $\lambda_\Gamma(f_1)$ and of $\lambda_\Gamma(-f_2)$, so that there exist non-zero $\xi_1, \xi_2$ such that $\lambda_\Gamma(f_1)\xi_1 = \lambda_\Gamma(-f_2)\xi_2$. \hfill \Box

**Proposition 4.5.** Fix $k \geq 2$. Let $w$ be a non-trivial reduced word in the free group $F_k$. There exists an unbounded 1-cocycle $b_w \in Z^1(F_k, \lambda_{F_k})$, such that $b_w(w) = 0$. In particular, $b_w$ is not proper.

**Proof.** We start with $k = 2$. Write $F_2$ as the free group on 2 generators $s, t$. Write $w$ as a reduced word in $s^{\pm 1}, t^{\pm 1}$:

$$w = x_1^{e_1}x_2^{e_2}...x_n^{e_n}$$

($x_j \in \{s, t\}; e_j \in \{-1, 1\}$). If either $s$ or $t$ does not appear in $w$, then the existence of the desired cocycle follows from the proof of Proposition 4.3 (with
\[ H = K = \mathbb{Z} \). So may assume that both \( s \) and \( t \) appear in \( w \). Set \( \delta_j = \frac{\epsilon_{j-1}}{2} \) and define two elements \( f_{w,s}, f_{w,t} \in CF_2 \) by:

\[
f_{w,s} = \sum_{j:x_j=s} \epsilon_j x_1^{\epsilon_1} ... x_{j-1}^{\epsilon_{j-1}} x_j^{\delta_j};
\]

\[
f_{w,t} = \sum_{j:x_j=t} \epsilon_j x_1^{\epsilon_1} ... x_{j-1}^{\epsilon_{j-1}} x_j^{\delta_j}.
\]

Note that \( f_{w,s} \) and \( f_{w,t} \) are non-zero, as \( s \) and \( t \) appear in \( w \). Since \( F_2 \) satisfies Conjecture \( \mathbb{H} \) (by Linnell’s result already quoted [Lin]), we may appeal to Lemma \( \mathbb{H}.4 \) and find non-zero functions \( \xi_s, \xi_t \in \ell^2(F_2) \) such that \( \lambda_{F_2}(f_{w,s})\xi_s + \lambda_{F_2}(f_{w,t})\xi_t = 0 \).

Set then \( b_w(s) = \xi_s, b_w(t) = \xi_t \) and, using freeness of \( F_2 \), extend uniquely to a 1-cocycle \( b_w \in Z^1(F_2, \lambda_{F_2}) \). Using the relations

\[
b(g_1g_2...g_m) = \sum_{j=1}^m \lambda_{F_2}(g_1...g_{j-1})b(g_j)
\]

and \( b(g^{-1}) = -\lambda_{F_2}(g^{-1})b(g) \) (for a cocycle \( b \) and \( g_1, ..., g_m, g \in F_2 \)), one checks that

\[
b_w(w) = \lambda_{F_2}(f_{w,s})b_w(s) + \lambda_{F_2}(f_{w,t})b_w(t) = 0.
\]

It remains to show that \( b_w \) is unbounded, i.e. that the corresponding affine action \( \alpha_w \) has no fixed point. Let \( H = \langle w \rangle \) be the cyclic subgroup generated by \( w \). As the linear action is \( C_0 \), the only fixed point of \( \alpha_w|_H \) is 0. But 0 is clearly not fixed under \( \alpha_w(s) \) or \( \alpha_w(t) \), which completes the proof in case \( k = 2 \).

Suppose now \( k \geq 2 \). View \( F_k \) as a subgroup of index \( k-1 \) in \( F_2 \). The restriction of \( \lambda_{F_2} \) to \( F_k \) is the direct sum of \( k-1 \) copies of \( \lambda_{F_k} \). Project the cocycle \( b_w \) given by the first part of the proof, to each of these \( k-1 \) summands. This way, get \( k-1 \) cocycles in \( Z^1(F_k, \lambda_{F_k}) \), each of them vanishing on \( w \). At least one of them is unbounded, because \( b|_{F_k} \) is unbounded. \( \square \)

**Corollary 4.6.** For \( k \geq 2 \), let \( \Gamma = F_k * \mathbb{Z} G \) be an amalgamated product over \( \mathbb{Z} \) an infinite cyclic subgroup. Then \( \Gamma \) does not have property \((BP_1)\).

**Proof.** Let \( w \in F_k \) and \( g \in G \) correspond to the positive generators of the copies of \( \mathbb{Z} \) that are amalgamated. Choosing representatives for the left cosets of \( F_k \) in \( \Gamma \), identify \( \lambda_{\Gamma}|_{F_k} \) with \( \infty \lambda_{F_k} =: \lambda_{F_k} \oplus \lambda_{F_k} \oplus .... \). Let \( b_w \in Z^1(F_k, \lambda_{F_k}) \), as in Proposition \( \mathbb{H}.3 \) Define an affine action \( \alpha \) of \( F_k \), with linear part \( \lambda_{\Gamma}|_{F_k} \), by:

\[
\alpha(x)(v_1, v_2, v_3, ...) = (\lambda_{F_k}(x)v_1 + b_w(x), \lambda_{F_k}(x)v_2, \lambda_{F_k}(x)v_3, ...)
\]
On the other hand, view $\lambda|_{G}$ as an affine action of $G$. Since

$$\alpha(w) = \lambda_{\Gamma}(w) = \lambda_{\Gamma}(g),$$

these two affine actions can be "glued together", i.e. extend to an affine action $\tilde{\alpha}$ of $\Gamma$, with linear part $\lambda_{\Gamma}$. By the very construction, $\tilde{\alpha}$ has unbounded orbits and is not proper.

**Corollary 4.7.** The surface groups $\Gamma_g$ ($g \geq 2$) do not have Property (BP$_0$).

**Proof.** The presentation

$$\Gamma_g = \langle a_1, \ldots, a_g, b_1, \ldots, b_g | [a_1, b_1]^{-1} = \prod_{j=2}^{g} [a_j, b_j] \rangle$$

exhibits $\Gamma_g$ as an amalgamated product $F_2 \ast_{Z} F_{2g-2}$ so Corollary 4.6 applies.

Here is an intriguing question, in view of the fact that $\text{PSL}_2(\mathbb{Z})$ contains a free group of finite index:

**Question 1.** Does $\text{PSL}_2(\mathbb{Z}) \simeq C_2 \ast C_3$ have Property (BP$_0$)?

### 4.3 Application to the regular representation

Let us recall that Guichardet [Gu1, Théorème 1] proved that, if $\pi$ is a representation without non-zero fixed vector of a locally compact, $\sigma$-compact group, the space $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$ (endowed with the topology of uniform convergence on compact subsets) if and only if $\pi$ does not almost have invariant vectors. In particular $H^1(G, \pi) \neq 0$ if $\pi$ almost has invariant vectors. This rests on a clever use of the open mapping theorem for Fréchet spaces. Using this, we can reprove the following result, first proved in [AW] (see also [BCV]).

**Proposition 4.8.** Let $G$ be a $\sigma$-compact, locally compact group. If $G$ is Haagerup, then it is $a$-$T$-menable.

**Proof.** Set $H = G \times \mathbb{Z}$; then $H$ is $\sigma$-compact, locally compact, is Haagerup, and has noncompact center. Hence, by Proposition 2.6, it has Property (BP$_0$). Take a $C_0$-representation $\pi$ of $H$, almost having invariant vectors. By Guichardet’s result recalled above, there exists an affine action $\alpha$ of $H$, with linear part $\pi$, and without fixed point. By Property (BP$_0$), the action $\alpha$ is proper. So the restriction $\alpha|_{G}$ is proper too. $\square$
If $G$ is $\sigma$-compact and amenable, the representation $\pi$ in the above proof can be taken as the left regular representation of $G \times \mathbb{Z}$ on $L^2(G \times \mathbb{Z})$. (By way of contrast, if $\Gamma$ is a discrete, non-amenable group, then $H^1(\Gamma \times \mathbb{Z}, \lambda_{\Gamma \times \mathbb{Z}}) = 0$ by Corollary 10 in [BV]).

Concerning affine actions on $L^2(G)$, we have the following

**Conjecture 2.** For an amenable group $G$, every affine action with linear part $\lambda_G$ is either bounded or proper.

Evidence for this conjecture comes from the fact that Proposition 2.10, Corollary 2.12 and Proposition 2.14 establish it in numerous cases: amenable groups with infinite center, solvable groups, amenable Lie groups, etc... More evidence comes from a result proved in [MV]: if $\Gamma$ is a countable amenable group, and $A$ is any infinite subgroup, then the restriction map $H^1(\Gamma, \lambda_{\Gamma}) \rightarrow H^1(A, \lambda_{\Gamma}|_A)$ is injective. If true, our conjecture would provide a conceptual explanation of this fact.

Being more ambitious, one may even ask

**Question 2.** Does every amenable group have Property (BP$_0$)?

We now turn to the study of some groups $G$ for which $H^1(G, \lambda_G) \neq 0$.

**Lemma 4.9.** Let $G$ be a locally compact, second countable group. Suppose that, for some $k \geq 2$, the group $G$ has closed normal subgroups $N_1, \ldots, N_k$ such that $[N_i, N_j] = 1$ whenever $i \neq j$ and $G = N_1 \cdots N_k$. Let $\pi$ be a unitary representation such that $\overline{H^1}(G, \pi) \neq 0$. Then at least one of the $N_i$ has an invariant vector by $\pi$.

**Proof.** There is an obvious map $p$ of $N = \prod_{i=1}^k N_i$ onto $G$. Then $\overline{H^1}(N, \pi \circ p) \neq 0$. This uses the standard fact that every compact subset of $G$ is the image of a compact subset of $N$ (note that we use here $\sigma$-compactness).

Suppose that for some $i$, the group $N_i$ has no invariant vector by $\pi \circ p$. Write $N = N_i \times \prod_{j \neq i} N_j$; by [Sha, Proposition 3.2] (which uses second countability), $\prod_{j \neq i} N_j$ has an invariant vector by $\pi \circ p$, so that for every $j \neq i$, $N_j$ has an invariant vector by $\pi \circ p$.\hfill $\square$

**Proposition 4.10.** Let $G$ be a connected Lie group or $G = G(K)$, the group of $K$-points of a linear algebraic group $G$ over a local field $K$ of characteristic zero. Suppose that $G$ has a $C_0$-representation $\pi$ such that $\overline{H^1}(G, \pi) \neq 0$. Then either $G$ is amenable, or has a compact subgroup $K$ such that $G/K$ is a simple Lie group (resp. a simple linear algebraic group) with trivial centre.
Proof. By Property (BP0), \( G \) has the Haagerup Property. If \( G \) is a connected Lie group, by [CCJJV, Chap. 4], \( G = RS_1 \ldots S_k \) where \( R, S_1, \ldots, S_k \) centralize each other, \( R \) is a connected amenable Lie group, and each \( S_i \) is a simple, noncompact, connected Lie group with the Haagerup Property (with possibly infinite centre).

In the case of an algebraic group, the same conclusion holds [Cor], except that the \( S_i \)'s are simple linear algebraic groups.

If \( G \) is not amenable, then \( k \geq 1 \), and in this case by Lemma 4.9 it follows that \( k = 1 \) and \( R \) is compact. By [Sha, Corollary 3.6], the centre \( Z(G) \) has an invariant vector by \( \pi \) and thus is compact since \( \pi \) is \( C^0 \); since in our situation \( Z(S_1) \subset Z(G) \), we see that \( S_1 \) has finite centre, so that \( K = RZ(S_1) \) is compact and \( G/K \) is a simple group with trivial centre.

**Proposition 4.11.** Let \( G \) be a connected Lie group or \( G = G(K) \) where \( K \) is a local field of characteristic zero. Assume \( G \) non-compact. Then the following are equivalent

(i) \( H^1(G, \lambda_G) \neq 0 \).

(ii) Either \( G \) is amenable, or there exists a compact normal subgroup \( K \subset G \) such that \( G/K \) is isomorphic to \( \text{PSL}_2(\mathbb{R}) \) (case of Lie groups), or a simple algebraic group of rank one (case of an algebraic group over a \( p \)-adic field).

Proof. Suppose (i). If \( G \) is not amenable, then, by the result of Guichardet already mentioned [Gu1, Théorème 1], one has \( \overline{H^1}(\lambda_G) = \overline{H^1}(G, \lambda_G) \neq 0 \). By Proposition 4.10, \( G \) has a compact normal subgroup \( K \) such that \( G/K \) is simple with trivial centre. Moreover, \( G \) does not have Property (T), hence has rank one [DK]. This settles the non-Archimedean case. If \( G \) is a Lie group, then by [Mar, Theorem 6.4], \( \lambda_G \) contains an irreducible subrepresentation \( \sigma \) factoring through \( S \), such that \( \overline{H^1}(G, \sigma) = \overline{H^1}(S, \sigma) \neq 0 \). Then \( \sigma \leq \lambda_S \), as \( S \) is cocompact in \( G \), so that \( H^1(S, \lambda_S) \neq 0 \). By a result of Guichardet (Proposition 8.5 in Chapter III of [Gu2]), this implies that \( S \simeq \text{PSL}_2(\mathbb{R}) \).

Conversely suppose (ii). If \( G \) is amenable, then \( H^1(G, \lambda_G) \) is not Hausdorff, hence is nonzero. Otherwise, suppose \( G \) non-amenable, and consider \( K \) as in (ii). By Proposition 3.3 and Theorem 3.3 (noticing that \( p_0 = 1 \) for \( \text{PSL}_2(\mathbb{R}) \)), we have \( H^1(G/K, \lambda_{G/K}) \neq 0 \). Then \( H^1(G, \lambda_G) \neq 0 \) by the same elementary argument as used in the proof of Proposition 3.3.

**Corollary 4.12.** Let \( G \) be a connected Lie group or \( G = G(K) \) where \( K \) is a local field of characteristic zero; let \( \Gamma \) be a uniform lattice in \( G \). If the first \( L^2 \)-Betti number \( \beta^1_2(\Gamma) \) is non-zero, then \( \Gamma \) is commensurable either to a non-abelian free group or to a surface group (more precisely: \( \Gamma \) has a finite index subgroup \( \Gamma_0 \) with...
a finite normal subgroup $N$ such that $\Gamma_0/N$ is either a non-abelian free group or a surface group).

**Proof.** From $\beta^1_2(\Gamma) > 0$, it follows that $\Gamma$ (and also $G$) is non-amenable: see Theorem 0.2 in [CG]. On the other hand, it was proved in [BV] that, for $\Gamma$ a finitely generated non-amenable group: $\beta^1_2(\Gamma) > 0$ if and only if $H^1(\Gamma, \lambda_\Gamma) \neq 0$. Since $\Gamma$ is uniform in $G$, we have by Shapiro’s lemma (Proposition 4.6 in Chapter III of [Gu2]):

$$0 \neq H^1(\Gamma, \lambda_\Gamma) = H^1(G, \text{Ind}_\Gamma^G \lambda_\Gamma) \simeq H^1(G, \lambda_G).$$

By Proposition 4.11, the group $G$ admits a compact normal subgroup $K$ such that $G/K$ is isomorphic either to $\text{PSL}_2(\mathbb{R})$ or to a simple algebraic group of rank 1 over a $p$-adic group. Let $p : G \to G/K$ be the quotient map. Then $p(\Gamma)$ is a uniform lattice in $G/K$. By Selberg’s lemma, find a finite-index torsion-free subgroup $\tilde{\Gamma}_0$ of $p(\Gamma)$: then $\tilde{\Gamma}_0$ is either a surface group (case of $\text{PSL}_2(\mathbb{R})$) or a non-abelian free group (non-Archimedean case). Set $\Gamma_0 = p^{-1}(\tilde{\Gamma}_0)$, a finite-index subgroup of $\Gamma$. Conclude by observing that the kernel $\text{Ker}(p|_{\Gamma_0})$ is contained in $\Gamma \cap K$, so is finite.

The preceding result overlaps a result of B. Eckmann (Theorem 4.1 in [Eck]), who classified lattices $\Gamma$ (not necessarily uniform) with $\beta^1_2(\Gamma) > 0$ in a connected Lie group.

### 4.4 Some non-$\sigma$-compact groups

Here is a curiosity. Start with the observation from the proof of Proposition 4.1 that a locally compact, non-$\sigma$-compact group cannot be a-T-menable. Accordingly, if it has Property (BP$_0$), then it also has Property (FH$_0$).

The above observation shows that the $\sigma$-compactness assumption is necessary in Guichardet’s result mentioned above. It also provides, in the non-Fréchet case, some explicit counterexamples to the statement of the open mapping theorem.

**Proposition 4.13.** Let $G$ be a non-$\sigma$-compact locally compact amenable group with Property (BP$_0$). Endow $Z^1(G, \lambda_G)$ with the topology of uniform convergence on compact subsets. Then the map

$$\partial : \left\{ \begin{array}{c} L^2(G) \\ \xi \end{array} \right\} \rightarrow \left\{ \begin{array}{c} Z^1(G, \lambda_G) \\ (g \mapsto \lambda_G(g)\xi - \xi) \end{array} \right\}$$

is a continuous linear bijective homomorphism, whose inverse is not continuous.
Proof. The map $\partial$ is linear, injective (as $G$ is not compact) and surjective (since $G$ has property ((FH$_0$))). It is clearly continuous. By amenability of $G$, for every $\varepsilon > 0$ and every compact subset $K \subset G$, there exists a unit vector $\xi \in L^2(G)$ such that

$$\max_{g \in K} \| (\partial \xi)(g) \| < \varepsilon.$$ 

This clearly shows that $\partial^{-1}$ is not continuous. 

Example 4.14. Examples of non-$\sigma$-compact amenable groups with Property (BP$_0$) include

- Uncountable solvable groups (by Corollary 2.12)
- Discrete groups of the form $G = F^I$, where $F$ is a non-trivial finite group and $I$ is any infinite set. Indeed $G$ is amenable, as it is locally finite, and since $G$ is isomorphic to $G \times G$ (as $I$ is infinite), $G$ contains an infinite normal subgroup with infinite centralizer, so Proposition 2.10 applies.

5 Actions of Z and R

5.1 Actions of Z

We have shown that every action of $\mathbb{Z}$ on a Hilbert space with $C_0$ linear part is either bounded or proper.

An example of Edelstein [Ede] shows that the $C_0$ assumption cannot be dropped. Let us briefly recall his example. On $\mathbb{C}$, consider the rotation $r_n$ with centre 1 and angle $2\pi/n!$. Consider the abstract product $\mathbb{C}^N$, and, for $(z_n)_{n \in \mathbb{N}} \in \mathbb{C}^N$, set $r((z_n)_{n \in \mathbb{N}}) = (r_n(z_n))_{n \in \mathbb{N}}$. This self-map is bijective and has the constant sequence 1 as unique fixed point. Moreover, it can be shown that $r(\ell^2(\mathbb{N})) = \ell^2(\mathbb{N})$. Thus $r$ induces an affine isometry of $\ell^2(\mathbb{N})$, which has no fixed point since the constant 1 is not in $\ell^2(\mathbb{N})$. However, the action is not proper; actually 0 is a recurrent point: an easy computation gives $\|r_n(0)\|^2 \leq \sum_{k>n} (2\pi n!/k!)^2$, and this sum clearly tends to zero. Notice that $r$ almost has fixed points: with $v_m$ the characteristic function of $\{1, ..., m\}$, one has $\lim_{m \to \infty} \|r(v_m) - v_m\| = 0$.

Observe that this isometry has diagonalizable linear part. Let us now provide another counter-example with further assumptions on the linear part.

Definition 5.1. A unitary or orthogonal representation of a group is weakly $C_0$ if it has no nonzero finite dimensional subrepresentation.\footnote{$C_0$ (resp. weakly $C_0$) representations are often called mixing (resp. weakly mixing).}
Proposition 5.2. There exists an affine isometric action of \( \mathbb{Z} \) on a complex Hilbert space, which is neither bounded nor proper, and has weakly \( C_0 \) linear part.

Proof. Write \( \sigma \) for the affine action of \( \mathbb{Z} \), and \( \pi \) for its linear part. Let \( \mu \) be a probability measure on \([0,1]\) and write \( H = L^2([0,1], \mu) \). Let \( \pi(1) \) be the multiplication by the function \( e(x) = \exp(2i\pi x) \). Write \( \sigma(1) = \tau_1 \circ \pi(1) \) where \( \tau_1 \) is the translation by the constant function 1. Note that \( \pi \) is weakly \( C_0 \) if and only if the spectrum of \( \pi(1) \) has no atom, i.e. \( \mu \) is nonatomic.

Let \( b \) be the corresponding cocycle and write \( c(n) = \|b(n)\|^2 \). An immediate computation shows that
\[
c(n) = \int \phi_n(x) d\mu(x)
\]
where \( \phi_n(x) = \left| \frac{\sin(\pi n x)}{\sin(\pi x)} \right|^2 \).

Let \( N_n \) a increasing sequence of integers and let \( \varepsilon_n \) be a decreasing sequence in \([0,1]\), such that \( \varepsilon_n \rightarrow 0 \). Moreover, let us assume that \( N_n/N_{n+1} = o(\varepsilon_n) \).

For all positive integer \( n \), write
\[
I_n(k) = \left[ \frac{k - \varepsilon_n}{N_n}, \frac{k + \varepsilon_n}{N_n} \right] \cap [0,1]
\]
and
\[
K_n = K_{n-1} \cap \bigcup_{k=0}^{N_n} I_n(k).
\]

Finally, write
\[
K = \bigcap_n K_n.
\]

One can check easily that \( K \) is homeomorphic to a Cantor space.

Let \( \mu \) be a probability measure on \([0,1]\) such that

- its support is contained in \( K \cap [0,1/2] \);
- There exists a subsequence \( \varepsilon_{k_n} \) such that
\[
\mu \left( [0, \sqrt{\varepsilon_{k_n}}] \right) = \sqrt{\varepsilon_{k_n}}.
\]

We choose the sequence \( k_n \) such that each interval \( I_n = [\sqrt{\varepsilon_{k_{n+1}}}, \sqrt{\varepsilon_{k_n}}] \) intersects \( K \) nontrivially. Take for \( \mu_n \) any nonatomic measure supported by \( K \cap I_n \) such that \( \mu_n(I_n) = \sqrt{\varepsilon_{k_n}} - \sqrt{\varepsilon_{k_{n+1}}} \) and define \( \mu = \sum_n \mu_n \) : clearly, \( \mu \) is nonatomic.

Claim 1. The action \( \sigma \) has no fixed point (so it has unbounded orbits).
If $\sigma$ has a fixed point $f$, then $f(x) = (1 - \exp(2i\pi x))^{-1}$ $\mu$-a.e. Let us show that $f$ does not belong to $L^2([0, 1])$. Indeed, note that $|f|^2 = (1/\sin(\pi x))^2$. For all $x \in [0, \sqrt{\varepsilon_{k_n}}]$, we have

$$\sin(\pi x)^2 \leq \pi^2 x^2 \leq \pi^2 \varepsilon_{k_n}$$

and by (1)

$$\mu([0, \sqrt{\varepsilon_{k_n}}]) = \sqrt{\varepsilon_{k_n}}.$$

It follows that

$$\int |f|^2 d\mu \geq \frac{1}{\pi^2 \sqrt{\varepsilon_{k_n}}}$$

which proves claim 1.

**Claim 2.** If moreover $\varepsilon_{k_n} = o(N_{k_n}^{-4})$ (for instance, $N_n = 2^{n^3}$ and $\varepsilon_n = (N_n)^{-5}$), then $c(N_{k_n})$ tends to 0, so that the action is not proper.

Indeed, let us show that $c(N_{k_n}) = o(1)$.

First, note that for all $x \in K$, the fractional part of $N_{k_n}x$ is less than $\varepsilon_{k_n}$. Thus, for every $x \geq \sqrt{\varepsilon_{k_n}}$ and every $x \in K$, it comes

$$\phi_{N_{k_n}}(x) \leq \left(\frac{\sin(2\pi \varepsilon_{k_n})}{\sin(2\pi \sqrt{\varepsilon_{k_n}})}\right)^2 \leq \pi^2 \varepsilon_{k_n}/4.$$

On the other hand, we have

$$\frac{\sin(2\pi N_{k_n}x)}{\sin(2\pi x)} \leq N_{k_n}$$

and by (1)

$$\mu([0, \sqrt{\varepsilon_{k_n}}]) = \sqrt{\varepsilon_{k_n}}.$$

It follows that

$$c(N_{k_n}) \leq \sqrt{\varepsilon_{k_n}} N_{k_n}^2 + \pi^2 \varepsilon_{k_n}/4.$$

So we get $c(N_{k_n}) = o(1)$. \hfill \Box

### 5.2 Actions of $\mathbb{R}$

Let us now show that the “pathological” actions of $\mathbb{Z}$ described above can be extended to $\mathbb{R}$.

Recall that a group $G$ is said to be exponential if, for every $g \in G$, there is a one-parameter subgroup through $g$ (i.e. a continuous homomorphism $\beta : \mathbb{R} \to G$ such that $\beta(1) = g$). Clearly, an exponential group has to be arc-connected.

Endow the group of affine isometries of a complex Hilbert space, $\mathcal{H} \rtimes \mathcal{U}(\mathcal{H})$, with the product topology, for the natural topology on $\mathcal{H}$ and the norm operator topology on the unitary group $\mathcal{U}(\mathcal{H})$. 23
Proposition 5.3. The group of affine isometries of a complex Hilbert space \( \mathcal{H} \), is exponential.

Proof. Let \( \alpha(v) = Uv + b \) be an affine isometry of \( \mathcal{H} \).

By the spectral theorem, we find a projection-valued measure \( P \) on \( [-\pi, \pi] \) such that \( U = \int_{-\pi}^{\pi} e^{ix} dP(x) \), in the sense that, for every \( \xi, \eta \in \mathcal{H} \)

\[
\langle U\xi | \eta \rangle = \int_{-\pi}^{\pi} e^{ix} d\mu_{\xi,\eta}(x)
\]

where \( \mu_{\xi,\eta}(A) = \langle P(A)\xi | \eta \rangle \) for any Borel subset \( A \subset [-\pi, \pi] \). Consider the one-parameter group of unitary operators

\[
v(s) = \int_{[-\pi,\pi]} e^{ix} dP(x).
\]

For every \( \xi \in \mathcal{H} \) and \( t \in \mathbb{R} \), define

\[
b_\xi(t) = \int_0^t v(s)\xi ds.
\]

It is straightforward that \( b_\xi \in Z^1(\mathbb{R}, v) \). Let us consider the operator \( A = \int_0^1 v(s) ds \). Then \( b_\xi(1) = A\xi \) for every \( \xi \in \mathcal{H} \). Thus, to show that there exists \( \xi \in \mathcal{H} \) such that \( b = b_\xi(1) \), it suffices to establish that \( A \) is invertible.

By Fubini’s Theorem

\[
\int_0^1 v(s) ds = \int_0^1 \left( \int_{-\pi,\pi} e^{ix} dP(x) \right) ds = \int_{[-\pi,\pi]} \frac{e^{ix} - 1}{ix} dP(x).
\]

Since the function \( x \mapsto \frac{ix}{e^{ix} - 1} \) is bounded on \( [-\pi, \pi] \), we obtain that

\[
\int_{[-\pi,\pi]} \frac{ix}{e^{ix} - 1} dP(x)
\]

is a bounded operator on \( \mathcal{H} \), and is the inverse of \( A \); so we may take \( \xi = A^{-1}(b) \).

\[\square\]

In view of Proposition 5.2, we obtain

Corollary 5.4. There exists an affine isometric action of \( \mathbb{R} \) on a complex Hilbert space that is neither bounded nor proper. Moreover, it can be chosen weakly \( C_0 \).

\[\square\]
Remark 5.5. Proposition [5.3] is false for real Hilbert spaces. This follows from the fact that the orthogonal group of a real Hilbert space is not exponential. This is clear in finite dimension (the group $O(n)$ is not connected), and was observed by Putnam and Wintner [PW2] in infinite dimension (although the orthogonal group is then connected [PW1]). An example of an orthogonal transformation which is not in the image of the exponential map is a reflection

$$S = \text{diag}(-1, 1, 1, 1, \ldots);$$

this can be seen by noticing that $S$ is not a square in the orthogonal group: indeed if $S = R^2$, since $R$ commutes with $S$ it stabilizes the $-1$-eigenspace of $S$, which leads to a contradiction.

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