AN OCTONIONIC GEOMETRIC (BALANCED) STATE SUM MODEL

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Abstract We propose a new 4D state sum model, related to the balanced model, which is constructed using the octonions, or equivalently, triality. An effective continuum physical theory constructed from this model coupled to the balanced model would have a non-vanishing cosmological constant, chiral asymmetry, and a gauge group related to the octonions.

1. Introduction

In [1], a discrete state sum model was proposed for the quantum theory of gravity. The model had a very distinct mathematical form, namely it was a constrained version of the topological state sum of [2] for the quantum group $U_Q(SO(4))$, where the constraints had the effect of destroying the topological nature of the theory. The specific nature of the constraint was to restrict the spins appearing in the state sum to those representations of $so(4)=so(3)+so(3)$ which have equal spins for both summands. For this reason, we called the resulting state sum “balanced”.

The derivation of the model in [1] began from a description of the geometry of a Euclidean 4-simplex in terms of the bivectors on its faces. The constraint on spins is a quantization of the classical geometric fact that a bivector in $R^4$ is simple if and only if its self dual and anti self dual parts have equal absolute values. For this reason we also describe the model of [1] as a “geometric” state sum. The significance of this is that terms in this sum can be interpreted as quantum geometries, and the state sum seems to be related to the quantization of general relativity in D=4 (see [3,4].)

The construction of [1] is algebraically special insofar as it utilizes the accidental isomorphism of $so(4)$ with $so(3)+so(3)$. It is therefore natural to ask whether any state sums of the “same type” exist, i.e. are there any other constrained versions of TQFTs around with geometric interpretation?

The purpose of this paper is to construct a second geometric state sum. This sum is related to the TQFT derived from the representation theory of $U_q(so(8))$ and makes use in an essential way of the multiplication table of the octonions [5]. For this reason we refer to it as an octonionic geometric state sum.

The motivation for this construction is to extend the program for constructing quantum general relativity from a state sum to include matter fields. As we will explain below, the new model shares the characteristics of the balanced one which give us some hope of its usefulness.

Let us warn the reader at the outset that this paper is not self contained. A familiarity with [1] is an absolute necessity for reading it.
The geometric interpretation of the state sum we are presenting is a Kaluza-Klein model with $R^8$ as fiber over a four dimensional space time. Thus it would have as a semiclassical limit a Yang-Mills theory with gauge group SO(8). This is reminiscent of dimensionally reduced supergravity. If the state sum model of [1] turns out to be a useful model for quantum gravity, then the model described here may well be an interesting way of introducing matter fields.

It is therefore natural to wonder how many geometric state sums actually exist. The construction of [1] and the present construction both involve special accidents in the theory of Lie algebras and the existence of division algebras, namely the quaternions and octonions respectively. This leads one to conjecture that there are very few geometric state sum models, at least in $D=4$. If this is so, then perhaps they are relevant to the problem of finding a unique fundamental theory. It is worth noting that the geometric models seem to use the same bits of algebra as supergravity and supersymmetric Yang-Mills theory, although in a different way. Of course, it would be much nicer to have a good general definition of “geometric state sum” before attacking such questions.

This paper mostly consists of an exposition of the octonionic model. We begin by presenting the model of [1] in a new way, then show how the new approach has an octonionic version. We close with a few remarks about the physical program of which this is a part.

2. The Balanced Model and the Maximal Torus.

Let us begin by proving in a new way that simple bivectors in $R^4$ correspond to elements in the Lie algebra so(4) with equal absolute values for the projections on the two summands. We can use the inner product on $R^4$ to raise one index of a bivector, changing it into a skew symmetric matrix, ie an element of so(4).

Since every element of the group SO(4) is conjugate to an element of a maximal torus of the group, we can restrict attention to elements of the Lie algebra of a maximal torus, ie. a Cartan subalgebra.

A maximal torus for SO(4) is just the set of rotations around two orthogonal planes. The corresponding Cartan subalgebra is just $R^2$ as a lie algebra. If we write coordinates for $R^4$ as $x_i =1,2,3,4$, then we can pick for a maximal torus all rotations in the 12 and 34 planes. If we introduce the natural basis for the bivectors on $R^4$ $b_{ij} = x_i \wedge x_j$, then the bivectors of the form

$$\alpha b_{12} + \beta b_{34}$$

are the ones which correspond to the given Cartan subalgebra.

Within this restricted set of bivectors it is easy to see which ones are simple: either $\alpha$ or $\beta$ must be zero.

Now we would like to see how this relates to the decomposition so(4) =so(3)+so(3).

This is also very straightforward: the summands correspond to the self dual and anti-self-dual parts. Hence the summands have absolute values $|\alpha + \beta|$ and
It is now an elementary calculation to show that the simple bivectors correspond to the elements of the Lie algebra whose self-dual and anti-self dual projections have equal absolute values.

Now since both simple bivectors and (anti)self-dual Lie algebra elements are carried into themselves by conjugation, we can see that our result is true in general.

Furthermore, since irreducible representations of a Lie algebra are described by a lattice point in a Cartan subalgebra, it is not hard to see that we can “quantize” the self dual bivectors in \( R^4 \) in terms of a subcategory of the representations of so(4).

3. Regge-Ponzano Kaluza-Klein Theory.

The idea of the construction of [1] was to describe the geometry of a 4-simplex in Euclidean 4-space by means of the bivectors on its 2d faces. If we start not with a single 4-simplex but with a triangulated 4-manifold, we end up with a flat metric on each 4-simplex, with compatibility on lower dimensional faces. This is a familiar technique which relativists use to produce an approximation to a Riemannian manifold, known as Regge calculus.

The octonionic model is also a state sum for a triangulated 4-manifold, but with data associated to a larger group than SO(4). The geometric situation which motivates it is a fiber bundle over a four dimensional space-time where the group of the fiber is SO(8).

In a Kaluza-Klein theory, one considers a higher dimensional manifold which is the total space of a fiber bundle over spacetime. Einstein metrics of a restricted type are considered, in which the metric along each fiber is some fixed, symmetric one. The result is that Einstein’s equation for the total space reduces to Einstein’s equation on the base coupled to Yang-Mills. In effect, the off diagonal piece of the metric (or more precisely, the frame field) on the total space comes to play the role of a connection on a vector bundle over the base.

What we are proposing here is a discrete analog of a Kaluza-Klein theory over a 4-dimensional base \( B^4 \). We choose a triangulation of the base manifold, then choose a flat, affine lifting of each 4-simplex to the total space \( T \) of the bundle, identified with \( B^4 \times R^8 \). This is equivalent to picking a flat connection on each 4-simplex. Two liftings related by a translation in \( R^8 \) would describe equivalent geometries. The curvature of the discretized connection so described becomes evident if we try to lift a path which loops around a 2-simplex (bone) in \( B^4 \) (translating each flat lifting if necessary so the lifts of subsequent pieces of the path meet) and note that we do not return to the same point in \( T \).

Actually, this is only a partial analog of Kaluza-Klein theory, since the variables correspond to a choice of a connection, but not of a metric on the base.

As far as we know, no model of this type has appeared anywhere in the literature. We are not going to investigate this procedure here as a general approach to investigating Yang-Mills theory. (Presumably it would be necessary...
to couple it to a metric on the base.) Our purpose is to use it to motivate the construction of a geometric state sum, i.e. to rewrite the geometric data in it in such a way as to allow us to quantize it by a procedure similar to the construction in [1].

In order to do this, we note that each edge in the triangulation of $B^4$ is being assigned a displacement vector in $\mathbb{R}^8$, and each 2-simplex is assigned a simple bivector. Furthermore, since the lifting of each whole 4-simplex is affine, the sum of the 2 bivectors on 2 adjacent 2-simplices is embedded in a hyperplane, and hence is simple.

If we now take the data on 2-simplices and subdivided 3-simplices as the fundamental data for our model, we can see that we get a set of simple bivectors in $\mathbb{R}^8$ adding in the same pattern as the 4d bivectors in the model in [1]. (We remind the reader that this paper makes no pretense of being self contained).

Up to this point we do not have a natural quantization. The astute reader has probably also noticed that there was really no motivation up to this point for the choice of the dimension 8 either. The crucial point in this argument is that simple bivectors in $\mathbb{R}^8$ have a special quantization.

4. Triality, Octonians and Simple Bivectors

Let us attempt to describe simple bivectors in $\mathbb{R}^8$ by a similar process to the one we used in $\mathbb{R}^4$. First, we can use the standard Euclidean metric to identify the bivectors with elements of the Lie algebra so(8), just as in D=4. Next, we can pick a basis and choose a standard Cartan subalgebra, which would be identified with the bivectors of the form

$$\alpha b_{12} + \beta b_{34} + \gamma b_{56} + \delta b_{78}$$

with the obvious notation.

The simple bivectors in this set are still just the points where three of the four coefficients are zero.

Now we do not have a decomposition of the Lie algebra so(8). What we do have is a very special automorphism of the algebra, called triality.

The useful thing here is that the operation of the triality map can be written as a linear map from the Cartan subalgebra to itself. In fact, the natural basis for so(8) can be decomposed into seven copies of the basis for a Cartan subalgebra, and the action of triality is the same on each. The only place where this is explicitly written of which we are aware is [6].

Specifically, the triality transformation can be written as follows:

$$F_{12} = 1/2(b_{12} - b_{34} - b_{56} - b_{78})$$
$$F_{34} = 1/2(-b_{12} + b_{34} - b_{56} - b_{78})$$
$$F_{56} = 1/2(-b_{12} - b_{34} + b_{56} - b_{78})$$
$$F_{78} = 1/2(-b_{12} - b_{34} - b_{56} + b_{78})$$

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(The notation here is not identical to [6].)
Where the Fs are a new basis for the Lie algebra with the same relations as
the natural basis associated to the b’s.
This transformation is a direct translation of the multiplication table of
the octonians, just as the decomposition of so(4) used above is essentially the
multiplication law of the quaternions.
In the transformed basis, it is easy to see that the simple bivectors are
characterized by the relations:
\[ |F_{12}| = |F_{34}| = |F_{56}| = |F_{78}| \quad (1) \]
Thus we can represent the quantum version of a simple bivector as the set of
representations of so(8) which are sums of highest weight representations with
highest weights on the diagonal of the weight lattice.
This leads naturally to the suggestion of a state sum on a triangulated 4-
manifold, which is formally very similar to the one in [1].
We would label each face and each tetrahedron with an irreducible repre-
sentation of \( U_q(\text{so}(8)) \) whose highest weight satisfied (1), join the representations
into a \( 15 J_q \) symbol as in the model in [2], multiply together the evaluations
of the \( 15 J_q \) symbols for all the 4-simplices in the triangulation, normalize with
the product of powers of quantum dimensions as in [2], and sum over labellings.
Unlike the case in [1], but as in [2], we would have to sum over a basis for the ten-
sor operators at each trivalent vertex in our diagrams, since the representation
category of so(8) does not have unique tensor operators like the representation
category of so(3).

5. Interpretation of the Octonionic Model

We have shown that a state sum model can be defined with a mathemat-
ical form very similar to the model in [1], which has at least a plausible link
to a geometrical construction. The fact that it is necessary to use the triality
isomorphism to connect the geometrical condition of simple bivectors to the
categorical algebraic form of restriction to a subcategory is at once interesting
and confusing. Since triality is an isomorphism between the vector and half spin
representations, (which is, in fact, at the heart of most supergravity and super-
string models), it is tempting to think of the model as a Kaluza-Klein model in
superspace, ie to think of the displacements on the edges of the triangulation
as lying in a chiral fermionic direction.
At this point, we have no knowledge of the dynamics of the new model,
but given the relationship between the model of [1] with the Einstein-Hilbert
Lagrangian, [3,4] it is at least worth studying.
In order to use this model as part of a unified theory, it will be necessary to
couple it to the balanced model. We have not yet investigated this.
Nevertheless, it is possible to discern three features which a physical model derived from the octonionic model would have.

**PREDICTION 1** The cosmological constant is nonzero.

Rationale: The model is only finite if we use the representations of $U_q(SO(8))$ for $q$ a root of unity. The value of $q$ is associated with a value for the cosmological constant in the resulting effective theory.

Prediction 1 seemed to be an embarrassment until recently.

**PREDICTION 2** The theory is not chirally symmetric

Rationale: unlike the balanced model, the octonionic model is sensitive to orientation, as in the topological model in [2].

**PREDICTION 3** The effective gauge group of the theory will be one of the groups with an octonionic construction

The exact group we obtain will depend on how we couple the balanced and octonionic models. Probably, some further algebraic coincidences will appear in this picture.

Given the extreme difficulty of progress along any line to a unified theory, this direction is worthy of further study.

6. Programmatic Remarks

This paper is part of a program (the “categorical program”) which aims at solving the fundamental problems of theoretical physics by changing the mathematical structure within which theories are constructed. Instead of using a Lagrangian in a smooth manifold, the theory is to be constructed in a triangulated, or PL manifold, using the structure of a tensor category to construct a state sum model on the triangulation.

The motivating physical ideas are that physical lengths have a natural cutoff (justifying the switch to a PL manifold), and that in quantum theories symmetry is expressed by the presence of a tensor category, making it the most natural choice for a building block for a theory.

The mathematical motivation for the approach is a very profound fit between the axioms of categorical algebra and the rules of combinatorial topology [8,9,10,11].

For a number of years, this program was confined to the field of topological quantum field theory, where it had many successes. In TQFT, the result of a physical calculation is triangulation independent; as a result of the axioms of tensor categories it is possible to construct state sum models with this property.

The model of [1] expanded the categorical program beyond the sphere of TQFT, creating a categorical state sum which seems closely related to the study of the Ashtekar/loop variables for quantum gravity [12,13,14].
However, the model of [1] is not topological, so we need to confront the problem of what happens to the model as the triangulation is refined. In a sense, the old ultraviolet problem recurs in the new setting.

We are setting forth here a conjecture as to how this problem might resolve itself, which we call “the quantum self-censorship conjecture”.

**QSC CONJECTURE** The state sum model of [1] has a natural continuation to lorentzian signature. If a physical experiment is described in this model by choosing fixed labels on the spacelike boundary of a spatially compact PL manifold with boundary and a fixed triangulation of the initial and final hypersurfaces, then there exists a finite triangulation $T$ of the region consistent with the triangulations of the hypersurfaces, with the property that if probability amplitudes are calculated for any initial and final data (labellings along the initial and final hypersurfaces) using the state sum on $T$ then if any refinement of $T$ is used, the same amplitudes will result.

For the purposes of this program, it would actually be almost equally good to replace this conjecture by a weaker conjecture of “asymptotic quantum self censorship” in which the effects suggested in the heuristic discussion below make the physical predictions of amplitudes converge as the triangulation became finer.

**Asymptotic QSC CONJECTURE** The state sum model of [1] has a natural continuation to lorentzian signature. If a physical experiment is described in this model by choosing fixed labels on the spacelike boundary of a spatially compact PL manifold with boundary and a fixed triangulation of the initial and final hypersurfaces, then there exists a finite triangulation $T$ of the region consistent with the triangulations of the hypersurfaces, with the property that if probability amplitudes are calculated for any initial and final data (labellings along the initial and final hypersurfaces) using the state sum on $T$ and then in all triangulations refining $T$, the net of values obtained for the physical quantity will converge.

If either conjecture is true, then the theory constructed from the balanced state sum model is effectively finite, and hence, an interesting candidate model for the quantum theory of gravity.

The reason for believing that one or the other conjecture might be true is that the model in [1] is closely related to a topological theory. If we refine the triangulation on which we calculate the model but allow no curvature along the new bones, then the constrained result on the finer triangulation will be exactly equal to the result on the coarser one because the constraint of flatness makes the theory topological.

The hope is that given fixed labels (=quantum geometry) on the boundary, any curvature on the new bones will appear to be below the Planck scale from
the boundary, hence will disappear into a black hole and not affect the results of the calculation of what is to be observed at the final hypersurface.

Clearly, these are optimistic conjectures. At least, they have the advantage of being capable of investigation.

If the QSC conjecture or its asymptotic version were true, then it would be extremely natural to look for state sum models which extend the balanced model and share its properties, in order to add matter fields to the theory. The octonionic model has the twin virtues of having an extremely similar form to the balanced model, and having a good deal of kinship with the most interesting candidates for grand unified theories. It is this combination of factors which motivated its development. The two properties of the balanced model which motivate the QSS conjecture, namely the close relation to a topological theory and the geometric interpretation, both apply to the octonionic model as well.

7. Further Directions

Although this paper has already gone considerably beyond what can be precisely done at this point, it is tempting to guess one step farther. The most symmetric mathematical object which appears in the algebraic approaches to string theory and supergravity has not yet appeared in this approach. We are referring to the exceptional Jordan algebra, or geometrically viewed, the octonionic projective plane [5].

This object has a natural action of $F_4$ or $E_6$, depending exactly what structure one asks to preserve. Either of these groups are interesting gauge groups for a grand unified theory. It would be interesting (and not terribly difficult) to see if a geometrical state sum can be constructed out of the algebraic accidents relating these two groups and their close relatives.

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