Relations in bounded cohomology

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Abstract
We explain some interesting relations in the degree 3 bounded cohomology of surface groups. Specifically, we show that if two faithful Kleinian surface group representations are quasi-isometric, then their bounded fundamental classes are the same in bounded cohomology. This is novel in the setting that one end is degenerate, while the other end is geometrically finite. We also show that a difference of two singly degenerate classes with bounded geometry is boundedly cohomologous to a doubly degenerate class, which has a nice geometric interpretation. Finally, we explain that the above relations completely describe the linear dependencies between the ‘geometric’ bounded classes defined by the volume cocycle with bounded geometry. We obtain a mapping class group invariant Banach subspace of the reduced degree 3 bounded cohomology with explicit topological generating set and describe all linear relations.

1. Introduction
The cohomology of a surface group with negative Euler characteristic is well understood. In contrast, the bounded cohomology in degree 1 vanishes, in degree 2, it is an infinite-dimensional Banach space with the $\| \cdot \|_\infty$ norm [19, 25] and in degree 3 it is infinite-dimensional but not even a Banach space [35]. In degree 4 and higher, almost nothing is known (see Remark 1.10). In this paper, we study a subspace in degree 3 generated by bounded fundamental classes of infinite volume hyperbolic 3-manifolds homotopy equivalent to a closed oriented surface $S$ with negative Euler characteristic. These manifolds correspond to $\text{PSL}_2 \mathbb{C}$ conjugacy classes of discrete and faithful representations $\rho : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$, but we restrict ourselves to the representations that do not contain parabolic elements to avoid technical headaches. Algebraically, the bounded fundamental class of a manifold will be the pullback, via $\rho$, of the volume class $\text{Vol} \in H_3^{cb}(\text{PSL}_2 \mathbb{C}; \mathbb{R})$. It can also be understood as the singular bounded cohomology class with representative defined by taking the signed volume of a straightened tetrahedron (see Sections 2.1–2.3). When we restrict our attention to bounded fundamental classes with bounded geometry, we will actually give a complete list of the linear dependencies among these bounded classes. As a consequence we obtain a subspace of $H_3^{cb}(\pi_1(S); \mathbb{R})$ on which the seminorm $\| \cdot \|_\infty$ restricts to a norm; this gives us a Banach space and we provide an explicit uncountable topological basis (see Section 8). This subspace is also mapping class group invariant and any linear dependencies among bounded classes have a very nice geometric description in terms of a certain cut-and-paste operation on hyperbolic manifolds (see Theorem 1.2 and the discussion following it).

The positive resolution of Thurston’s ending lamination conjecture for surface groups due to [7, 30] building on work of [23, 24] tells us that the end invariants of a hyperbolic structure on $S \times \mathbb{R}$ determine its isometry class which, in the totally degenerate setting, is a version of the motto, ‘topology implies geometry’. Suppose $\rho_1$ and $\rho_2$ are discrete, faithful, without
parabolics and their quotient manifolds $M_{\rho_i}$ share one geometrically infinite end invariant. Then they are quasi-conformally conjugate, and there is a bi-Lipschitz homeomorphism in the preferred homotopy class of mappings $M_{\rho_1} \to M_{\rho_2}$ inducing $\rho_2 \circ \rho_1^{-1}$ on fundamental groups (see Section 3). The bi-Lipschitz constant depends on the dilatation of the quasi-conformal conjugacy, which is essentially the exponential of the Teichmüller distance between their geometrically finite end invariants. This bi-Lipschitz homeomorphism lifts to covers, inducing an equivariant quasi-isometry. Our first main result in this paper is

**Theorem 1.1.** If $\rho_1$ and $\rho_2$ are discrete, faithful, without parabolics, and their quotient manifolds $M_{\rho_i}$ are singly degenerate and share one geometrically infinite end invariant, then $\rho_1^* \text{Vol} = \rho_2^* \text{Vol} \in H^3_b(\pi_1(S); \mathbb{R})$.

We emphasize that Theorem 1.1 holds even for manifolds with unbounded geometry. In Sections 4–7, we restrict ourselves to the setting of manifolds with bounded geometry. A manifold $M$ has bounded geometry if its injectivity radius $\text{inj}(M)$ is positive. That is, there is no sequence of essential closed curves whose lengths tend to 0. An ending lamination $\lambda \in \mathcal{EL}(S)$ has bounded geometry if any singly degenerate manifold with $\lambda$ as an end invariant has bounded geometry. Note that if some singly degenerate manifold with geometrically infinite end invariant $\lambda$ and no parabolic cusps has bounded geometry, then every singly degenerate manifold with $\lambda$ as an end invariant and no cusps has bounded geometry [29, Bounded Geometry Theorem]. This is closely related to the fact that any Teichmüller geodesic ray tending toward a bounded geometry ending lamination $\lambda$ stays in a compact set [33]. Let $\mathcal{EL}_b(S) \subset \mathcal{EL}(S)$ be the laminations that have bounded geometry. Let $\lambda, \lambda' \in \mathcal{EL}_b(S)$ and $X, Y \in \mathcal{F}(S)$, then we have representations $\rho_{(\nu_-, \nu_+)} : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ with end invariants $\nu_-, \nu_+ \in \{ \lambda, \lambda', X, Y \}$. Let $\hat{\omega}(\nu_-, \nu_+) \in \mathcal{C}_b^3(S; \mathbb{R})$ be the corresponding bounded volume 3-cocycle (see Section 2.3). In Section 8 we prove our main results relating doubly degenerate bounded classes to each other. The following theorem states that the doubly degenerate bounded classes decompose into a sum of singly degenerate bounded classes.

**Theorem 1.2.** Let $\lambda, \lambda' \in \mathcal{EL}_b(S)$ and $X, Y \in \mathcal{F}(S)$ be arbitrary. We have an equality in bounded cohomology

$$[\hat{\omega}(\lambda', \lambda)] = [\hat{\omega}(\lambda', X)] + [\hat{\omega}(Y, \lambda)] \in H^3_b(S; \mathbb{R}).$$

We can think of the singly degenerate bounded classes as ‘atomic’. If we cut a doubly degenerate manifold with end invariants $(\lambda', \lambda)$ along an embedded surface, we are left with 2-manifolds, each of which is bi-Lipschitz equivalent to the convex core of a hyperbolic manifold with end invariants $(\lambda', X)$ or $(X, \lambda)$. This gives a geometric explanation for Theorem 1.2, once we establish that bounded cohomology ignores bounded geometric perturbations. As a corollary, we see that the ‘cohomological shadows’ of geometrically infinite ends vanish under addition in $H^3_b(S; \mathbb{R})$.

**Corollary 1.3.** Suppose $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{EL}_b(S)$ are distinct. Then we have an equality in bounded cohomology

$$[\hat{\omega}(\lambda_1, \lambda_2)] + [\hat{\omega}(\lambda_2, \lambda_3)] = [\hat{\omega}(\lambda_1, \lambda_3)] \in H^3_b(S; \mathbb{R}).$$

We now state some consequences of the results of this paper and the classification theory for finitely generated Kleinian groups. To do so, we restate results of previous work of the author in the case of marked Kleinian surface groups.

**Theorem 1.4** [14, Theorems 6.2 and 7.7]. Fix a closed, orientable surface $S$ of negative Euler characteristic. There is an $\epsilon > 0$ such that if $\{[\rho_\alpha]\}_{\alpha \in \Lambda} \subset \text{Hom}(\pi_1(S), \text{PSL}_2 \mathbb{C})/\text{PSL}_2 \mathbb{C}$
are discrete, faithful and without parabolics such that at least one of the geometrically infinite end invariants of $\rho_\alpha$ is different from the geometrically infinite end invariants of $\rho_\beta$ for all $\beta \neq \alpha \in \Lambda$, then

1. $\{\rho_\alpha^* \text{Vol} \}_{\alpha \in \Lambda} \subset H^3_0(\pi_1(S); \mathbb{R})$ is a linearly independent set;
2. $\|\sum_{i=1}^N a_i \rho_{\alpha_i}^* \text{Vol}\|_\infty \geq \varepsilon \max |a_i|$.

Say that $\rho_1, \rho_2 : \Gamma \to \text{PSL}_2 \mathbb{C}$ are quasi-isometric if there exists a $(\rho_1, \rho_2)$-equivariant quasi-isometry $\mathbb{H}^3 \to \mathbb{H}^3$. Combining Theorems 1.1 and 1.4 with the Ending Lamination Theorem (Theorem 2.2) and the quasi-conformal deformation theory (see the discussion following Theorem 3.2), we obtain

**Corollary 1.5.** The bounded fundamental class is a quasi-isometry invariant of discrete and faithful representations of $\pi_1(S)$ without parabolics. In this setting, $\|\rho_0^* \text{Vol} - \rho_1^* \text{Vol}\|_\infty < \varepsilon_S$ if an only if $\rho_0$ is quasi-isometric to $\rho_1$.

Soma [36, Theorem A and Corollary B] has recently obtained a version of Corollary 1.5 that does not factor through the curve complex machinery that was used in the proof of the Ending Lamination Theorem and instead relies on the almost rigidity properties of hyperbolic tetrahedra with almost maximal volume. Subsequently, Soma provides an alternate approach to the Ending Lamination Theorem [36, Theorem C and Corollary D].

It turns out that the seminorm on bounded cohomology can also detect representations with dense image and faithful representations in the following sense:

**Theorem 1.6 [13, Theorem 1.1; 14, Theorem 1.3].** Suppose $\rho_0 : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ is discrete, faithful, has no parabolic elements, and at least one geometrically infinite end invariant and let $\rho : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ be arbitrary. If $\|\rho_0^* \text{Vol} - \rho^* \text{Vol}\|_\infty < \varepsilon_S/2$, then $\rho$ is faithful. If $\rho$ has dense image, then $\|\rho_0^* \text{Vol} - \rho^* \text{Vol}\|_\infty \geq v_3$, where $v_3$ is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$.

We then obtain the following rigidity theorem as a corollary of Theorem 1.6 and Corollary 1.5.

**Corollary 1.7.** Suppose $\rho_0 : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ is discrete and faithful with no parabolics and at least one geometrically infinite end invariant and $\rho_1 : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ has no parabolics (but is otherwise arbitrary). If $\|\rho_0^* \text{Vol} - \rho_1^* \text{Vol}\|_\infty < \varepsilon_S/2$, then $\rho_1$ is discrete and faithful, hence quasi-isometric to $\rho_0$.

Let $M$ be a topologically tame hyperbolic 3-manifold with incompressible boundary (that is, the compact 3-manifold whose interior is homeomorphic to $M$ has incompressible boundary) and no parabolic cusps. The inclusion of any surface subgroup $i_S : \pi_1(S) \to \pi_1(M) = \Gamma \leq \text{PSL}_2 \mathbb{C}$ corresponding to an end of $M$ induces a seminorm non-increasing map $i_S^* : H^3_0(\pi_1(M); \mathbb{R}) \to H^3_0(\pi_1(S); \mathbb{R})$. If the end corresponding to $S$ is geometrically infinite and $M$ is not diffeomorphic to $S \times \mathbb{R}$, by the Covering Theorem, $i_S : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ is a singly degenerate marked Kleinian surface group. By Theorem 1.1, $i_S^* \text{Vol}$ identifies the geometrically infinite end invariant of $M_{i_S}$ (equivalently, the end invariant of $M$ corresponding to $S$). Say that a hyperbolic manifold is totally degenerate if all of its ends are geometrically infinite. Applying Waldhausen’s Homeomorphism Theorem [39] and the Ending Lamination Theorem [7], we have

**Theorem 1.8.** Suppose $M_0$ and $M_1$ are hyperbolic 3-manifolds without parabolic cusps with holonomy representations $\rho_i : \pi_1(M_i) \to \text{PSL}_2 \mathbb{C}$, $M_1$ is totally degenerate, and $h : M_1 \to M_2$
is a homotopy equivalence. If $M_1$ is topologically tame and has incompressible boundary, then there is an $\epsilon$ depending only on the topology of $M_1$ such that $h$ is homotopic to an isometry if and only if $\|\rho^n_0 \text{Vol} - \rho^n_1 \text{Vol}\|_\infty < \epsilon$.

Apply Theorem 1.4 to see that the singly degenerate classes form a linearly independent set, and they are uniformly separated from each other in seminorm. Fix a base point $X \in \mathcal{T}(S)$, and define $\iota : \mathcal{EL}(S) \to H^n_0(S;\mathbb{R})$ by the rule $\iota(\lambda) = [\omega(X,\lambda)]$. By Theorem 1.1, $\iota$ does not depend on the choice of $X$, and $\iota$ is mapping class group equivariant. We summarize here, and elaborate in Section 8.

**Theorem 1.9.** The image of $\iota$ is a linearly independent set. Moreover, for all $\lambda, \lambda' \in \mathcal{EL}(S)$, if $\|\iota(\lambda) - \iota(\lambda')\|_\infty < \epsilon_S$ then $\lambda = \lambda'$. Finally, $\iota$ is mapping class group equivariant, and $\iota(\mathcal{EL}(S))$ is a topological basis for the image of its closure in the reduced space $\Pi^n_0(S;\mathbb{R})$.

By Corollary 1.3, we know that the $\mathbb{R}$-span of $\iota(\mathcal{EL}_b(S))$ contains all bounded classes of doubly degenerate manifolds with bounded geometry. The results of this paper give a complete characterization of the linear dependencies among elements in the closure of this Banach subspace. Again, see Section 8.

**Remark 1.10.** The bounded fundamental class is a construction that, for surface and free groups, is necessarily uninteresting in dimension 4 (more generally even dimensions at least 4). Let $S$ be a compact, oriented surface and $\rho : \pi_1(S) \to \text{Isom}^+(\mathbb{H}^n)$ be discrete and faithful with $n \geq 4$ even. Then [4] shows that the hyperbolic $n$-manifold $\mathbb{H}^n / \text{im} \rho$ has positive Cheeger constant. Moreover, [20] show that positivity of the Cheeger constant is equivalent to the vanishing of the bounded class $\rho^* \text{Vol}_n \in H^n_0(S;\mathbb{R})$. We arrive at the claim in the beginning of the remark.

The organization of the paper is as follows. In Section 2 we review the definitions of (continuous) bounded cohomology of groups and spaces, some terminology from Kleinian groups, the singular Sol metric on the universal bundle over a Teichmüller geodesic as a model for bounded geometry manifolds, and notions in coarse geometry. In Section 3, we consider singly degenerate classes and prove Theorem 1.1. We exploit Geometric Inflexibility Theorems to obtain volume preserving, bi-Lipschitz maps $\Phi : M_0 \to M_1$ between singly degenerate manifolds that share their geometrically infinity end invariant. Essentially, we use this map to compare $\Phi(\text{str}_0(\tau))$ with $\text{str}_1(\Phi(\tau))$, where $\tau : \Delta_2 \to M_0$. We find a homotopy between these two maps with bounded volume, which allows us to express the difference in the bounded fundamental classes of the 2-manifolds as a bounded coboundary. The proof of Theorem 1.2 is modeled on the same strategy, but we need to make a few technical detours and assume that our manifolds have bounded geometry. Namely, we will need to take a limit of bi-Lipschitz, volume preserving maps to obtain a volume preserving map (up to ‘compact error’) from the convex core of a singly degenerate manifold to a doubly degenerate manifold. The assumption that our limit manifold has bounded geometry allows us to ensure that the convex core boundaries of manifolds further out in the sequence get (linearly) further away from some fixed reference point. We use this to get control over the bi-Lipschitz constants of our maps using geometric in flexibility. Without the bounded geometry assumption, it is possible (generic) that we cannot take bounded quasi-conformal jumps toward some ending lamination $\lambda$ while making uniform progress away from our reference point, because there are large subsurface product regions where distance from the convex core boundary grows only logarithmically instead of linearly. We extract the volume preserving limit map in Section 5.

Since our map is only volume preserving away from a compact subset of the convex core of our singly degenerate manifold (that is, up to ‘compact error’), we need to see that bounded
cohomology does not witness this compact error. We consider functions $f : M \to \mathbb{R}$ that have compact support, and show that when we weight the hyperbolic volume by $f$, that this new bounded class is indistinguishable from the old in bounded cohomology. This we consider in Section 4.

In Section 6, we take a coarse geometric viewpoint. We study hyperbolic ladders in the singular Sol metric on the universal bundle over a Teichmüller geodesic. This viewpoint was inspired by [31], and we use it to understand the behavior of geodesics straightened with respect to two ‘nested metrics’. Essentially, we can choose the geometrically finite end invariant of a singly degenerate manifold so that there is a $B$-bi-Lipschitz embedding of its convex core into the doubly degenerate manifold, allowing us to think of one as a subspace of the other with the path metric. We will straighten a based geodesic loop with respect to both metrics and use the geometry of ladders to show that the two straightenings coarsely agree, when they can. That is, the two paths will fellow travel when it is most efficient to travel in the subspace, and when it is not, one geodesic will stay close to the boundary of the subspace while the other finds a shorter path. We observe that thin triangles mostly track their edges, so to understand where two geodesic triangles live inside our manifolds, it suffices to understand the trajectories of their edges. We will use these observations to ‘zero out’ half of a doubly degenerate manifold with a smooth bump function (on an entire geometrically infinite end) and prove that the resulting bounded class is boundedly cohomologous to that of the singly degenerate class in Section 7. We reiterate that the scheme for the proof there is based on that in Section 3.

Finally, we prove our main results in Section 8; they now follow somewhat easily from the work in previous sections. Throughout the paper, we reserve the right to use several different notations where convenient to hopefully improve the exposition.

2. **Background**

2.1. **Bounded cohomology of spaces**

Given a connected cell complex $X$, we define a norm on the singular chain complex of $X$ as follows. Let $\Sigma_n = \{\sigma : \Delta_n \to X\}$ be the collection of singular $n$-simplices. Write a simplicial chain $A \in C_\bullet(X; \mathbb{R})$ as an $\mathbb{R}$-linear combination

$$A = \sum \alpha_\sigma \sigma,$$

where each $\sigma \in \Sigma_n$. The 1-norm or Gromov norm of $A$ is defined as

$$\|A\|_1 = \sum |\alpha_\sigma|.$$

This norm promotes the algebraic chain complex $C_\bullet(X; \mathbb{R})$ to a chain complex of normed linear spaces; the boundary operator is a bounded linear operator. Keeping track of this additional structure, we can take the topological dual chain complex

$$(C_\bullet(X; \mathbb{R}), \partial, \|\cdot\|_1)^* = (C^\bullet_b(X; \mathbb{R}), d, \|\cdot\|_\infty).$$

The $\infty$-norm is naturally dual to the 1-norm, so the dual chain complex consists of bounded co-chains. Define the bounded cohomology $H^\bullet_b(X; \mathbb{R})$ as the (co)-homology of this complex. For any bounded $n$-co-chain, $\alpha \in C^n_b(X; \mathbb{R})$, we have an equality

$$\|\alpha\|_\infty = \sup_{\sigma \in \Sigma_n} |\alpha(\sigma)|.$$

The $\infty$-norm descends to a pseudo-norm on the level of bounded cohomology. If $A \in H^n_b(X; \mathbb{R})$ is a bounded class, the seminorm is described by

$$\|A\|_\infty = \inf_{\alpha \in A} \|\alpha\|_\infty.$$
We direct the reader to [16] for a systematic treatment of bounded cohomology of topological spaces and fundamental results.

Matsumoto–Morita [25] and Ivanov [19] prove independently that in degree 2, \( \| \cdot \|_\infty \) defines a norm in bounded cohomology, so that the space \( H^2_b(X; \mathbb{R}) \) is a Banach space with respect to this norm. In [35], Soma shows that the pseudo-norm is in general not a norm in degree at least 3. In Section 8, we will consider the quotient \( \overline{H}^2_b(S; \mathbb{R}) = H^2_b(S; \mathbb{R})/Z \) where \( Z \subset H^2_b(S; \mathbb{R}) \) is the subspace of zero-seminorm elements. Then \( \overline{H}^2_b(S; \mathbb{R}) \) is a Banach space with the \( \| \cdot \|_\infty \) norm.

2.2. Continuous bounded cohomology of groups

Let \( G \) be a topological group. We define a co-chain complex for \( G \) by considering the collection of continuous, \( G \)-invariant functions

\[ C^n(G; \mathbb{R}) = \{ G^{n+1} \rightarrow \mathbb{R} \}. \]

The homogeneous coboundary operator \( d \) for the trivial \( G \) action on \( \mathbb{R} \) is, for \( f \in C^n(G; \mathbb{R}) \),

\[ df(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}), \]

where \( \hat{g}_i \) means to omit that element, as usual. The coboundary operator gives the collection \( C^\bullet(G; \mathbb{R}) \) the structure of a (co)-chain complex. An \( n \)-co-chain \( f \) is bounded if

\[ \| f \|_\infty = \sup |f(g_0, \ldots, g_n)| < \infty, \]

where the supremum is taken over all \( n + 1 \) tuples \((g_0, \ldots, g_n) \in G^{n+1}\).

The operator \( d : C^n_b(G; \mathbb{R}) \rightarrow C^{n+1}_b(G; \mathbb{R}) \) is a bounded linear operator with operator norm at most \( n + 2 \), so the collection of bounded co-chains \( C^\bullet_b(G; \mathbb{R}) \) forms a subcomplex of the ordinary co-chain complex. The cohomology of \( (C^\bullet_b(G; \mathbb{R}), d) \) is called the continuous bounded cohomology of \( G \), and we denote it \( H^\bullet_b(G; \mathbb{R}) \). When \( G \) is a discrete group, the continuity assumption is vacuous, and we write \( H^\bullet_b(G; \mathbb{R}) \) to denote the bounded cohomology of \( G \). In the case that it is discrete. The \( \infty \)-norm \( \| \cdot \|_\infty \) descends to a pseudo-norm on bounded cohomology in the usual way. A continuous group homomorphism \( \varphi : H \rightarrow G \) induces a map \( \varphi^* : H^\bullet_b(G; \mathbb{R}) \rightarrow H^\bullet_b(H; \mathbb{R}) \) that is norm non-increasing. Brooks [8], Gromov [16] and Ivanov [18] proved the remarkable fact that for any connected cell complex \( M \), the classifying map \( K(\pi_1(M), 1) \rightarrow M \) induces an isometric isomorphism \( H^\bullet_b(M; \mathbb{R}) \rightarrow H^\bullet_b(\pi_1(M); \mathbb{R}) \). We therefore identify the two spaces \( H^\bullet_b(\pi_1(M); \mathbb{R}) = H^\bullet_b(M; \mathbb{R}) \).

2.3. The bounded fundamental class

Let \( x \in \mathbb{H}^3 \) and consider the function \( \text{vol}_x : (\text{PSL}_2 \mathbb{C})^4 \rightarrow \mathbb{R} \) which assigns to \((g_0, \ldots, g_3)\) the signed hyperbolic volume of the convex hull of the points \( g_0 x, \ldots, g_3 x \). Any geodesic tetrahedron in \( \mathbb{H}^3 \) is contained in an ideal geodesic tetrahedron. There is an upper bound \( v_3 \) on volume that is maximized by a regular ideal geodesic tetrahedron [37], so \( \| \text{vol}_x \|_\infty = v_3 \). One checks that \( d \text{vol}_x = 0 \), so that \( [\text{vol}_x] \in H^3_{\text{ch}}(\text{PSL}_2 \mathbb{C}; \mathbb{R}) \). Moreover, for any \( x, y \in \mathbb{H}^3 \), \( [\text{vol}_x] = [\text{vol}_y] \neq 0 \). Define \( \text{Vol} = [\text{vol}_x] \); the continuous bounded cohomology \( H^3_{\text{ch}}(\text{PSL}_2 \mathbb{C}; \mathbb{R}) = \langle \text{Vol} \rangle_\mathbb{R} \), and in fact \( \| \text{Vol} \|_\infty = v_3 \), as well (see, for example, [10] for a discussion of the hyperbolic volume class in dimensions \( n \geq 3 \)). Let \( \rho : \Gamma \rightarrow \text{PSL}_2 \mathbb{C} \) be any group homomorphism. Then \( \rho^* \text{Vol} \in H^3_{\text{ch}}(\Gamma; \mathbb{R}) \) is called the bounded fundamental class of \( \rho \).

We now specialize to the case that \( \Gamma = \pi_1(S) \), where \( S \) is a closed oriented surface of negative Euler characteristic and give a geometric description of the bounded fundamental class. If \( \rho : \pi_1(S) \rightarrow \text{PSL}_2 \mathbb{C} \) is discrete and faithful, then the quotient \( M_\rho = \mathbb{H}^3/\text{im} \rho \) is a hyperbolic manifold and it comes equipped with a homotopy equivalence \( f : S \rightarrow M_\rho \) inducing \( \rho \) on fundamental groups.
Let \( \omega \in \Omega^3(M_\rho) \) be such that \( \pi^*\omega \) is the Riemannian volume form on \( \mathbb{H}^3 \) under the covering projection \( \mathbb{H}^3 \xrightarrow{\pi} M_\rho \). Suppose \( \sigma : \Delta_3 \to M_\rho \) is a singular 3-simplex. We have a chain map \cite{37}

\[
\text{str} : C_*(M_\rho) \to C_*(M_\rho)
\]
defined by homotoping \( \sigma \), relative to its vertex set, to the unique totally geodesic hyperbolic tetrahedron \( \text{str} \sigma \). The co-chain

\[
\hat{\omega}(\sigma) = \int_{\text{str} \sigma} \omega \equiv \int_{\Delta_3} (\text{str} \sigma)^* \omega
\]
measures the signed hyperbolic volume of the straitening of \( \sigma \). We use the fact that \( \text{str} \) is a chain map, together with Stokes’ theorem to observe that if \( \nu : \Delta_4 \to M_\rho \) is any singular 4-simplex,

\[
d\hat{\omega}(\nu) = \int_{\partial \text{str} \nu} \omega = \int_{\partial \text{str} \nu} \omega = \int_{\text{str} \nu} d\omega = 0,
\]
because \( d\omega \in \Omega^4(M_\rho) = \{0\} \). The class \( \hat{\omega} \in H_3^*(M_\rho; \mathbb{R}) \) is the \textit{bounded fundamental class} of \( M_\rho \). Under the isometric isomorphism \( f^* : H_3^*(M_\rho; \mathbb{R}) \to H_3^*(\pi_1(S); \mathbb{R}) \) induced by \( f : S \to M_\rho \), \( f^* [\hat{\omega}] = \rho^* \text{Vol} \) if \( M_\rho \) has end invariants \( \nu = (\nu_-, \nu_+) \) (see Sections 2.6–2.8), we also use the notation \( [\hat{\omega} (\nu_-, \nu_+)] \) to denote the bounded fundamental class.

### 2.4. Coarse geometry

A general reference for material in this section is \cite{5}. Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces (whenever possible we use subscripts to explain which metric we are calculating distances with with respect to). A \((\lambda, \epsilon)\)-\textit{quasi-isometric embedding} is a not necessarily continuous map \( f : X \to Y \) such that, for all \( x, y \in X \),

\[
\lambda^{-1} d_X(x, y) - \epsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \epsilon.
\]
A \((\lambda, \epsilon)\)-\textit{quasi-geodesic} is a quasi-isometric embedding of \( \mathbb{Z} \) or \( \mathbb{R} \). A \((\lambda, \epsilon)\)-qi-embedding is a \textit{quasi-isometry} if it is coarsely surjective. That is, for all \( y \in Y \) there is an \( x \in X \) such that \( d_Y(f(x), y) < \epsilon \).

Let \( (X, d) \) be a geodesic metric space, and for \( x, y \in X \), denote by \([x, y]\) a geodesic segment (we will often conflate geodesics as maps parameterized by or proportionally to arc-length and their images) joining \( x \) and \( y \). If there is some \( \delta \geq 0 \) such that for any \( x, y, z \in X \), any triangle with geodesic sides \( \Delta xyz \) satisfies the property that any side is in the \( \delta \)-neighborhood of the union of the other two, then \( (X, d) \) is said to be \( \delta \)-\textit{hyperbolic}.

The following result is sometimes referred to as the Morse lemma in the literature.

**Theorem 2.1** (see, for example, \cite{5}). For all \( \delta \geq 0 \), \( \epsilon > 0 \), and \( \lambda \geq 1 \), there is a constant \( D \) such that the following holds: Suppose \( Y \) is a \( \delta \)-hyperbolic metric space. Then the Hausdorff distance between a geodesic and a \((\lambda, \epsilon)\)-quasi-geodesic joining the same pair of endpoints is no more than \( D \).

### 2.5. Teichmüller space

Let \( S \) be a closed, oriented surfaced of genus \( g \geq 2 \). The \textit{Teichmüller space} \( \mathcal{T}(S) \) is formed from the set of pairs \((g, X)\) where \( g : S \to X \) is a homotopy equivalence and \( X \) is a hyperbolic structure on \( S \). Two pairs \((g, X)\) and \((h, Y)\) are said to be equivalent if there is an isometry \( \iota : X \to Y \) such that \( \iota \circ g \sim h \). The equivalence class of a pair is denoted \([g, X]\), and the Teichmüller space of \( S \) is the set of equivalence classes of such pairs, with topology defined by
the Teichmüller metric. Briefly, if \( g \) is a homotopy inverse for \( g \), the Teichmüller distance is defined by

\[
d_{\mathcal{T}(S)}([g, X], [h, Y]) = \inf_{f \sim h \circ g} \frac{1}{2} \log(K_f),
\]

where \( K_f \) is the maximum of the pointwise quasi-conformal dilatation of the map \( f \). Thus the Teichmüller metric measures the difference between the marked conformal structures determined by \( (f, X) \) and \( (g, Y) \). Teichmüller’s theorems imply that there is a distinguished map \( T : X \to Y \), called a Teichmüller map, in the homotopy class of \( h \circ g \), which achieves the minimal quasi-conformal dilatation among all maps homotopic to \( h \circ g \). By abuse of notation, we will often suppress markings and write \( X \in \mathcal{T}(S) \) to refer to the class of a marked hyperbolic structure \( g : S \to X \). We also do not distinguish between pairs and equivalence classes of pairs, but we freely precompose markings with homeomorphisms isotopic to the identity on \( S \) to stay within an equivalence class.

2.6. Ends of hyperbolic 3-manifolds

Tameness of manifolds with incompressible ends was proved by Bonahon [3]. Canary proved that topological tameness implied Thurston’s notion of geometric tameness [11]. For discrete and faithful representations \( \rho : \pi_1(S) \to \PSL_2 \mathbb{C} \), the quotient \( M_\rho \) is diffeomorphic to the product \( S \times \mathbb{R} \). Thus \( \rho \) determines a homotopy class of maps \( [f : S \to M_\rho] \) inducing (the conjugacy class of) \( \rho \) at the level of fundamental groups. \( M_\rho \) has two ends, which we think of as \( S^+ = S \times \{\infty\} \) and \( S^- = S \times \{-\infty\} \). Let \( \Gamma = \im \rho \). The limit set \( \Lambda_\Gamma \) is the set of accumulation points of \( \Gamma \cdot x \subset \partial \mathbb{H}^3 \) for some (any) \( x \in \partial \mathbb{H}^3 \). The domain of discontinuity of \( \Gamma \) is \( \Omega_\Gamma = \hat{\mathbb{C}} \setminus \Lambda_\Gamma \). Denote by \( \mathcal{H}(\Lambda_\Gamma) \subset \mathbb{H}^3 \) the convex hull of \( \Lambda_\Gamma \). The convex core of \( M_\rho \) is \( \text{core}(M_\rho) = \mathcal{H}(\Lambda_\Gamma)/\Gamma \).

Let \( \epsilon \in \{+, -\} \). Say that \( S^e \) is geometrically finite if there is some neighborhood of \( S^e \) disjoint from \( \text{core}(M_\rho) \). Call \( S^e \) geometrically infinite otherwise. By the Ending Lamination Theorem (see Section 2.10) the isometry type of \( M_\rho \) is uniquely determined by the surface \( S \) together with its end invariants \( \nu = (\nu(S^-), \nu(S^+)) \). We describe the end invariants \( \nu(S^e) \) below.

2.7. Geometrically finite ends

When \( S^e \) is geometrically finite, there is a non-empty component \( \Omega_\Gamma^c \subset \Omega_\Gamma \) on which \( \Gamma \) acts freely and properly discontinuously by conformal automorphisms. This action induces a conformal structure \( X = \Omega_\Gamma^c/\Gamma \) on \( S^e \) with marking \( f : S \to X \) inducing \( \rho \) on fundamental groups. Moreover, the end of \( M_\rho \) given by \( S_\epsilon \) admits a conformal compactification by adjoining \( X \) at infinity. If \( S^e \) is geometrically finite, we associate the end invariant \( \nu(S^e) = [f, X] \in \mathcal{T}(S) \).

2.8. Geometrically infinite ends

Suppose \( S^e \) is geometrically infinite, and let \( E_\epsilon \cong S \times [0, \infty) \) be a neighborhood of \( S^e \). Then \( E_\epsilon \) is simply degenerate. That is, there is a sequence \( \{\gamma_i^e\} \) of homotopically essential, closed geodesics exiting \( E_\epsilon \). Each \( \gamma_i^e \) is homotopic in \( E_\epsilon \) to a simple closed curve \( \gamma_i \subset S \times \{0\} \). Moreover, we may find such a sequence such that the length \( \ell_{M_\rho}(\gamma_i^e) \leq L_0 \), where \( L_0 \) is the Bers constant for \( S \). Equip \( S = S \times \{0\} \) with any hyperbolic metric. Find geodesic representatives \( \gamma_i^e \subset S \) with respect to this metric. Then up to taking subsequences, the projective class of the intersection measures \( [\gamma_i^e] \in \mathcal{PML}(S) \) converges to the projective class of a measured lamination \( [\lambda_\epsilon] \in \mathcal{PML}(S) \). Thurston [37], Bonahon [3] and Canary [11] show, in various contexts, that the topological support \( \lambda_\epsilon \subset S \) is minimal, filling and does not depend on the exiting subsequence \( \{\gamma_i\} \). Furthermore, given any two hyperbolic structures on \( S \), the spaces of geodesic laminations are canonically homeomorphic. So \( \lambda_\epsilon \) also does not depend on our choice of metric on \( S \). This ending lamination is the end invariant \( \nu(S^e) = \lambda_\epsilon \). Call \( \mathcal{E}(S) \) the space of minimal, filling laminations.
2.9. Pleated surfaces

A pleated surface is a map $f : S \to M_\rho$ together with a hyperbolic structure $X \in \mathcal{T}(S)$, and a geodesic lamination $\lambda$ on $S$ so that $f$ is length preserving on paths, maps leaves of $\lambda$ to geodesics, and is locally geodesic on the complement of $\lambda$. Pleated surfaces were introduced by Thurston [37]. We insist also that $f$ induces $\rho$ on fundamental groups, that is, it is in the homotopy class of the marking $S \to M_\rho$. We write $f : X \to M_\rho$ for a pleated surface. The pleating locus of $f$ is denoted by pleat$(f)$; it is the minimal lamination for which $f$ maps leaves geodesically.

2.10. The Ending Lamination Theorem

By Thurston’s Double Limit Theorem [38], any pair of end invariants $\nu = (\nu_-, \nu_+) \in (\mathcal{T}(S) \sqcup \mathcal{EL}(S))^2 \setminus \Delta(\mathcal{EL}(S))$ can be realized as the end invariants of a hyperbolic structure on $S \times \mathbb{R}$. It was a program of Minsky to establish the following converse, conjectured by Thurston.

It was a program of Minsky to establish the following converse, conjectured by Thurston.

Theorem 2.2 (Ending Lamination Theorem). Let $S$ be a closed, orientable surface of genus at least 2 and $\rho_0, \rho_1 : \pi_1(S) \to \text{PSL}_2\mathbb{C}$ be discrete and faithful representations. There is an orientation preserving isometry $\iota : M_{\rho_0} \to M_{\rho_1}$ such that $\iota \circ f_0 \sim f_1$ if and only if $\nu(M_{\rho_0}) = \nu(M_{\rho_1})$. Equivalently, there is a $\gamma \in \text{PSL}_2\mathbb{C}$ such that $\rho_0 = \gamma \circ \rho_1 \circ \gamma^{-1}$ if and only if $\nu(M_{\rho_0}) = \nu(M_{\rho_1})$.

In the case that $M_{\rho_1}$ have positive injectivity radius, Theorem 2.2 was proved by Minsky [28] (see Section 2.11). For the general case, Masur and Minsky initiated a detailed study of the geometry of the curve complex of $S$ [23] as well as its hierarchical structure [24]. Given a representation $\rho : \pi_1(S) \to \text{PSL}_2\mathbb{C}$, Minsky extracts the end invariants $\nu = (\nu_-, \nu_+)$ of $M_\rho$, and then uses the hierarchy machinery to build a model manifold $\mathcal{M}_\nu$ and Lipschitz homotopy equivalence $\mathcal{M}_\nu \to M_\rho$ [30]. Brock, Canary and Minsky then promote $\mathcal{M}_\nu \to M_\rho$ to a bi-Lipschitz homotopy equivalence [7]. An application of the Sullivan Rigidity Theorem then concludes the proof of the Ending Lamination Theorem in the case of marked Kleinian surface groups. In this paper, we will use the geometry of the model manifold for bounded geometry ending data from [28] which is built from a Teichmüller geodesic joining $\nu_-$ to $\nu_+$.

2.11. Teichmüller geodesics and models with bounded geometry

Suppose $M$ is a hyperbolic structure on $S \times \mathbb{R}$ with bounded geometry and $f : S \to M$ is a homotopy equivalence. Fix a basepoint $Y_0 \in \mathcal{T}(S)$ so that for each geometrically infinite end $S^\infty$ of $M$ and transverse measure $\mu$ supported on $\lambda_\epsilon = \nu(S^\infty) \in \mathcal{EL}(S)$, there is a unique quadratic differential $Q_\mu$ that is holomorphic with respect to the conformal structure underlying $Y_0$ and whose vertical foliation is measure equivalent to $\mu$ [17]. Since $M$ has bounded geometry, the Teichmüller geodesic ray $t \mapsto Y_t \in \mathcal{T}(S)$, $t \geq 0$ determined by $Q_\mu$ has bounded geometry [33, Theorem 1.5] (see also [29, Bounded Geometry Theorem]), that is, it projects to a compact subset of the moduli space. By Masur’s Criterion [22, Theorem 1.1] and because $\lambda_\epsilon$ is minimal, the set of transverse measures supported on $\lambda_\epsilon$ is equal to $\mathbb{R}_{\geq 0}\mu$. This means that $Q_\mu$ is completely determined by $Y_0$, $\lambda_\epsilon$, and the area of the singular flat metric $|Q_\mu|$.

In summary, since $M$ has bounded geometry, there is a unique projective class of measures supported on $\lambda_\epsilon$ and unique quadratic differential $Q_\mu$ holomorphic on $Y_0$ with area 1 and vertical foliation that is topologically equivalent to $\lambda_\epsilon$. Minsky proved that the pleated surfaces that can be mapped into an end of $M$ are approximated by a Teichmüller geodesic ray in the following sense.
Theorem 2.3 [27, Theorem A; 28, Theorem 5.5]. The Teichmüller geodesic ray \( t \mapsto Y_t \in \mathcal{J}(S) \) determined by \( Q \) satisfies: for every \( t \in [0, \infty) \), there is a pleated surface \( f_t : X_t \to M \) homotopic to \( f \), such that \( d_{\mathcal{J}(S)}(X_t, Y_t) \leq A \), where \( A \) depends only on \( S \) and \( \text{inj}(M) = \epsilon > 0 \).

Assume \( M \) is doubly degenerate with end invariants \( (\lambda_-, \lambda_+) \); using Theorem 2.3 we outline the construction of a model metric \( ds \) on \( S \times \mathbb{R} \), such that \( \mathfrak{M} = (S \times \mathbb{R}, ds) \) approximates the geometry of \( M \). The nature of this approximation is made precise in Theorem 2.4. The details of this construction are carried out in [28, Section 5].

Again, since \( M \) is \( \epsilon \)-thick, there are unique projective classes of measured laminations \( [\lambda_\pm] \in \mathcal{PML}(S) \) with supports equal to \( \lambda_\pm \). There exists a hyperbolic structure \( Y_0 \in \mathcal{J}(S) \) and a quadratic differential \( Q_0 \) of unit area, holomorphic with respect to the conformal structure underlying \( Y_0 \), such that \( Q_+ \) is the (hyperbolic straightening of the) vertical foliation \( Q_0^+ \) of \( Q_0 \), and \( Q_- \) is the horizontal foliation \( Q_0^- \) of \( Q_0 \). The conclusion of Theorem 2.3 holds for \( S^- \) and \( S^+ \), since we can choose \( Y_0 \) as our basepoint.

The quadratic differential \( Q_0 \) gives rise to a singular Euclidean metric \( |Q_0| \) on \( S \), which we can write infinitesimally as

\[
|Q_0| = dx^2 + dy^2
\]

away from the zeros of \( Q_0 \), where \( dx \) is the measure induced by \( Q_0^+ \) and \( dy \) is the measure induced by \( Q_0^- \). Normalizing so that the identity map from \( Y_0 \) to \( Y_t \) is the Teichmüller map, the image quadratic differential \( Q_t \) (holomorphic with respect to \( Y_t \)) induces a metric \( |Q_t| \) given by

\[
|Q_t| = e^{2t}dx^2 + e^{-2t}dy^2
\]

away from the zeros of \( Q_0 \). Define a metric \( ds \) on \( S \times \mathbb{R} \) by

\[
ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2,
\]

where \( t \) denotes arc-length in the Teichmüller metric. The singularities of this metric are exactly \( \Sigma \times \mathbb{R} \), where \( \Sigma \subset S \times \{0\} \) is the set of zeros of \( Q_0 \).

Theorem 2.4 [28, Theorem 5.1]. There is a homotopy equivalence

\[
\Psi : \mathfrak{M} = (S \times \mathbb{R}, ds) \to M
\]

inducing \( f_* \) on fundamental groups that lifts to an \((L, c)\)-quasi-isometry \( \tilde{\Psi} : \tilde{\mathfrak{M}} \to \tilde{M} \) of universal covers, where \( L \) and \( c \) depend only on \( \epsilon \) and \( S \). The map \( \Psi \) satisfies the following properties:

(i) for each \( n \in \mathbb{Z} \), \( \Psi(\cdot, n) = f_n \), where \( f_n : X_n \to M \) is the pleated map as in Theorem 2.3;
(ii) the identity mapping of \( S \) lifts of an \((L, c)\)-quasi-isometry of universal covers with respect to the singular flat metric \( |Q_n| \) and the hyperbolic metric \( X_n \).

If \( M_k \) is the singly degenerate with end invariants \( (Y_k, \lambda_+) \), then we have

\[
\Psi_k : \mathfrak{M}_k = (S \times [k, \infty), ds) \to \text{core}(M_k)
\]
satisfies the same properties as above. In addition, \( \Psi(\cdot, k) \) maps to \( \partial \text{core}(M_k) \).

3. Singly degenerate classes

We will use the Geometric Inflexibility Theorem of Brock–Bromberg [6, Theorem 5.6] that generalizes a result of McMullen [26, Theorem 2.11]. Roughly, Geometric Inflexibility says that the deeper one goes into the convex core of a hyperbolic manifold, the harder it is to
deform the geometry, there. In this section, we use only the volume preserving and global bi-Lipschitz constant. In later sections, we use the pointwise, local bi-Lipschitz estimates. Given a homotopy equivalence \( \phi : M_0 \to M_1 \) of complete hyperbolic manifolds, say that \( M_1 \) is a \( K \)-quasi-conformal deformation of \( M_0 \) if \( \phi \) lifts to universal covers and continuously extends to an equivariant \( K \)-quasi-conformal homeomorphism \( \partial \mathbb{H}^3 \to \partial \mathbb{H}^3 \).

**Theorem 3.1 (Geometric inflexibility [6, Theorem 5.6]).** Let \( M_0 \) and \( M_1 \) be complete hyperbolic structures on a 3-manifold \( M \) so that \( M_1 \) is a \( K \)-quasi-conformal deformation of \( M_0 \), \( \pi_1(M) \) is finitely generated, and \( M_0 \) has no rank-one cusps. There is a volume preserving \( K^{3/2} \)-bi-Lipschitz diffeomorphism

\[
\Phi : M_0 \to M_1
\]

whose pointwise bi-Lipschitz constant satisfies

\[
\log \text{bilip}(\Phi, p) \leq C_1 \epsilon^{-2} \text{d}(p, \text{core}(M_0))
\]

for each \( p \in M_0 \geq \epsilon \), where \( C_1 \) and \( C_2 \) depend only on \( K, \epsilon \), and \( \text{area}(\partial \text{core}(M_0)) \).

Some remarks about the statement of Theorem 3.1 are in order, since the above formulation is rather spread out over the literature. Namely, the result quoted above uses work of Reimann [34] that unifies constructions of Ahlfors [1] and Thurston [37, Chapter 11] relating a quasi-conformal deformation at infinity to the internal geometry of a hyperbolic structure; see also [6, Theorem 5.1] for a summary of the main result of Reimann’s work that includes the \( K^{3/2} \)-bi-Lipschitz constant that is absent in the statement of [6, Theorem 5.6]. A self-contained exposition of Reimann’s work can be found in [26, § 2.4 and Appendices A and B]. From a time-dependent quasi-conformal vector field on \( \mathbb{C} = \partial \mathbb{H}^3 \), Reimann [34] analyzes a nice extension to \( \mathbb{H}^3 \) via a visual averaging procedure that is natural with respect to Möbius transformations and that can be integrated to obtain a path through hyperbolic metrics. The behavior of this path of metrics is controlled by the quasi-conformal constant of the initial vector field at infinity. The visual average of a quasi-conformal vector field is divergence free, and so the flow is volume preserving (c.f. [26, Appendices A and B], especially [26, Theorems B.10 and B.21]). Thus the map \( \Phi \) from Theorem 3.1 is volume preserving (this statement is also absent from [6, Theorem 5.6]).

We consider only marked Kleinian surface groups with no parabolic cusps. We now also restrict ourselves to representations with one geometrically infinite end. Fix a closed, oriented surface \( S \) of negative Euler characteristic. In light of Theorem 2.2, we may supply a pair \( (\nu, \lambda) \) of end invariants to obtain a discrete and faithful representation \( \rho \). The \( \rho \)-conjugacy class of \( \rho(\nu) \) and the homotopy class of \( \rho(\nu) \) are uniquely determined by \( \nu \). Let \( \lambda \in \mathcal{EL}(S) \) and \( X \in \mathcal{T}(S) \). Then we have an orientation reversing isometry \( \iota : M_{(X, \lambda)} \to M_{(\lambda, X)} \) such that \( \iota \circ f_{(X, \lambda)} \sim f_{(\lambda, X)} \). Without loss of generality, we will work with manifolds whose ‘+’ end is geometrically infinite, and whose ‘−’ end invariant is geometrically finite. That is, we will consider manifolds with end invariants \( \nu = (X, \lambda) \in \mathcal{T}(S) \times \mathcal{EL}(S) \). Fix \( \lambda \in \mathcal{EL} \) and take \( \nu_0 = (X_0, \lambda) \) and \( \nu_1 = (X_1, \lambda) \). The goal of this section is to prove

**Theorem 3.2.** With notation as above, we have an equality in bounded cohomology

\[
\rho(\nu_0)^* \text{Vol} = \rho(\nu_1)^* \text{Vol} \in H^1_b(\pi_1(S); \mathbb{R}).
\]

Now that \( \lambda \) and \( X_i \) are fixed, we abbreviate \( M_i = M_{\nu_i} \), \( \rho_i = \rho(\nu_i) \), \( f_i = f_{\nu_i} \), and \( \Gamma_i = \Gamma(\nu_i) \). We claim that there is a \( K \)-bi-Lipschitz volume preserving diffeomorphism
\[ \Phi : M_0 \to M_1, \text{ where } K^{2/3} \text{ is dilatation of the Teichmüller map } X_0 \to X_1, \text{ that is, } K^{2/3} = \exp(2d_{\mathcal{S}}(X_0, X_1)). \]

Indeed, recall that \( X_0 \) is the quotient of the domain of discontinuity \( \Omega_0 \) of \( \rho_0 \) by the action of \( \rho_0 \). By Bers’ theorem [2], there is a \( K^{2/3} \)-quasi-conformal homeomorphism \( \varphi : \hat{C} \to \hat{C} \) so that \( \rho' = \varphi \circ \rho_0 \circ \varphi^{-1} : \pi_1(S) \to \text{PSL}_2 \mathbb{C} \) is discrete, faithful, and such that the quotient of the domain of discontinuity of \( \rho' \) by the action of \( \rho' \) corresponds to \( X_1 \). By Theorem 3.1, we have a \( K \)-bi-Lipschitz volume preserving diffeomorphism \( \Phi : M_0 \to M' = \mathbb{H}^3 / \text{im} \rho' \). By construction, \( X_1 \) is the \( \rho' \)-end invariant of \( M' \), and since \( \Phi \) maps curves of bounded length exiting the geometrically infinite end of \( M_0 \) to curves of bounded length exiting an end of \( M' \), \( \lambda \) is the other end invariant of \( M' \). By Theorem 2.2, \( \rho' \) is conjugate in PSL\(_2 \mathbb{C} \) to \( \rho_1 \). Then \( M_1 = M' \), and so \( \Phi : M_0 \to M_1 \) is desired map. We now wish to consider the difference

\[ \hat{\omega}_0 - \Phi^* \hat{\omega}_1 \in C^0(M_0; \mathbb{R}). \]

We will now construct an explicit 2-cochain \( C \in C^2(M_0) \) such that

\[ dC(\sigma) = \int_{\text{str}_\sigma} \omega_0 - \int_{\text{str}_\sigma} \omega_1 = \hat{\omega}_0(\sigma) - \Phi^* \hat{\omega}_1(\sigma). \]

To this end, let \( \tau : \Delta_2 \to M_0 \) be continuous. We define a homotopy \( H(\tau) : \Delta_2 \times I \to M_1 \) such that \( H_1 = \text{str}_1 \Phi_\tau \tau \) and \( H_0 = \Phi_0 \text{str}_0 \tau \) as follows. First, we consider lifts \( \tau_0 : \Delta_2 \to \mathbb{H}^3 \) of \( \Phi_0 \text{str}_0 \tau \) and \( \tau_1 : \Delta_2 \to \mathbb{H}^3 \) of \( \text{str}_1 \Phi_\tau \tau \) to the universal cover \( \mathbb{H}^3 \overset{\pi}{\to} M_1 \) such that \( \tau_0(v_i) = \tau_1(v_i) \) where \( v_i \) are the vertices of the 2-simplex \( \Delta_2 \).

**Remark 3.3.** We would like to define our homotopy to be the geodesic homotopy joining \( \tau_0(x) \) to \( r(\tau_0(x)) \), where \( r \) is the nearest point projection onto \( \text{im} \tau_1 \). However, it is imperative that edges of \( \tau_0 \) are mapped to edges of \( \tau_1 \), and it is not guaranteed that \( r \) will accomplish this task. The following construction of \( H(\tau) \) fixes this potential problem.

Let \([i, j] \subset \Delta_2\) be the edge joining vertices \( v_i \) and \( v_j \). The image \( \tau_1([i, j]) \) is a geodesic segment, and so there is a nearest point projection \( r_{i,j} : \mathbb{H}^3 \to \text{im} \tau_1([i, j]) \), and since \( \text{im} \tau_1 \subset \mathbb{H}^3 \) is geodesically convex, we have the nearest point projection \( r : \mathbb{H}^3 \to \text{im} \tau_1 \). We note here for use later that by convexity of the distance function on \( \mathbb{H}^3 \), each of the \( r_{i,j} \) and \( r \) are 1-Lipschitz retractions.

For two points \( x, y \in \mathbb{H}^3 \), let \([x, y] : I \to \mathbb{H}^3 \) be the unique geodesic segment, parameterized proportionally to arc-length joining \( x \) to \( y \), that is, \( [x, y](0) = x, \ [x, y](1) = y \) and \( [x, y]'(t) = d(x, y) \). First we define our map on the edges of \( \Delta_2 \). Let \( p \in [i, j] \), and define

\[ \gamma_t(p) = \begin{cases} [\tau_0(p), r_{i,j}(\tau_0(p))](2t), & 0 \leq t \leq \frac{1}{2} \\ [r_{i,j}(\tau_0(p)), \tau_1(p)](2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases} \]

Find regular, convex neighborhood \( N_{i,j} \subset (\Delta_2, d_{\text{str}_0 \tau}) \) of \([i, j]\) that meet only at the vertices of \( \Delta_2 \), such that \( N_{i,j} \subset N_{i,k} \cap N_{j,k} \) where distance is measured in the metric induced by \( \text{str}_0 \tau \). Let \( N = N_{0,1} \cup N_{1,2} \cup N_{2,0} \) and \( \Delta_2' = \Delta_2 \setminus N \).

For \( x \in \Delta_2' \) and \( t \in I \), define

\[ \gamma_t(x) = \begin{cases} [\tau_0(x), \tau(\tau_0(x))](2t), & 0 \leq t \leq \frac{1}{2} \\ [\tau(\tau_0(x)), \tau_1(x)](2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases} \]

Finally, let \( q \in \partial \text{cl}(N_{i,j}) \setminus [i, j] \) and \( p(q) \in [i, j] \) be the closest point to \( q \) in the \( d_{\text{str}_0 \tau} \) metric (Figure 3.1). The geodesic segment joining \( p \) to \( p(q) \) is naturally parameterized proportionally to
Figure 3.1 (colour online). Schematic of the definition of $H(\tau)$ in pieces.

arc-length by convex combinations of $q_s = s \cdot p + (1 - s) \cdot (p(q))$, so that $q_1 = q$ and $q_0 = p(q)$.

Define now

$$
\gamma_1^1(q_s) = [r_{i,j}(p(q)), r(q)](s),
$$

and

$$
\gamma_1(q_s) = \begin{cases} 
[r_0(q_s), \gamma_1^1(q_s)(2t)], & 0 \leq t \leq 1/2 \\
[\gamma_1^1(q_s), \tau_1(q_s)(2t - 1)], & 1/2 \leq t \leq 1.
\end{cases}
$$

Finally, take

$$
H(\tau) : \Delta_2 \times I \rightarrow M_1
$$

$$(x, t) \mapsto \pi(\gamma_t(x)).$$

Note that $H(\tau)$ is almost projection of the straight-line homotopy between $\tau_0$ and $r \circ \tau_0$ concatenated with a homotopy between $r \circ \tau_0$ and $\tau_1$. Instead, we constructed $H(\tau)$ to ensure that the edges of $\Delta_2$ do not land in the interior of $\text{im} \tau_1$ at any stage of the homotopy. We then linearly interpolated between the (potentially different) two maps on the edges over the region $N$. $H(\tau)$ is piecewise smooth, and by convexity of $\text{im} \tau_1$, the image of the later homotopy is contained entirely within $\tau_1$. Define $C \in C^2(M_0)$ by linear extension of the rule

$$
C(\tau) = \int_{\Delta_2 \times I} H(\tau)^* \omega_1.
$$

Lemma 3.4. Let $\sigma : \Delta_3 \rightarrow M_0$. Then

$$(\hat{\omega}_0 - \Phi^* \hat{\omega}_1)(\sigma) = dC(\sigma).$$

Proof. Both $\Phi$ and $\Phi^{-1}$ are volume preserving, so

$$
\int_{\text{str}_0 \sigma} \omega_0 = \int_{\Phi_* \text{str}_0 \sigma} \Phi^{-1}^* \omega_0 = \int_{\Phi_* \text{str}_0 \sigma} \omega_1,
$$

so by linearity of the integral

$$
(\hat{\omega}_0 - \Phi^* \hat{\omega}_1)(\sigma) = \int_{\Phi_* \text{str}_0 \sigma - \text{str}_1 \Phi_* \sigma} \omega_1.
$$

Since $H^3(M_1) = 0$, there is a 2-form $\eta \in \Omega^2(M_1)$ such that $d\eta = \omega_1$. The exterior derivative is natural with respect to pullbacks and $\text{str}_i$ are chain maps. Applying Stokes’ theorem for
manifolds with corners, we have
\[ \int_{\Phi_+ \text{ str}_{t_0} \sigma - \text{ str}_1 \Phi_+ \sigma} \omega_1 = \int_{\Phi_+ \text{ str}_{t_0} \sigma - \text{ str}_1 \Phi_+ \sigma} d\eta \]
\[ = \int_{\partial(\Phi_+ \text{ str}_{t_0} \sigma - \text{ str}_1 \Phi_+ \sigma)} \eta = \int_{(\Phi_+ \text{ str}_{t_0} - \text{ str}_1 \Phi_+)(\partial \sigma)} \eta. \] (2)

We write \( \partial \sigma = \tau_0 - \tau_1 + \tau_2 - \tau_3 \). By the construction of \( H(\tau_k) \), if \( \tau_k |_{[i,j]} = \tau_{k'} |_{[i',j']} \), then the equality \( H(\tau_k) |_{[i,j]}(t) = H(\tau_{k'}) |_{[i',j']} (t) \) holds for all \( t \in I \). Thus,
\[ \int_{(\Phi_+ \text{ str}_{t_0} - \text{ str}_1 \Phi_+)(\partial \sigma)} \eta = \sum_{i=0}^{3} (-1)^i \int_{\partial(\Delta_2 \times I)} H(\tau_i)^* \eta \]
\[ = \sum_{i=0}^{3} (-1)^i \int_{\Delta_2 \times I} H(\tau_i)^* \omega_1 \]
\[ = C(\partial \sigma) = dC(\sigma). \] (5)

This is precisely what we wanted to show.

The following proposition completes the proof of Theorem 3.2.

**Proposition 3.5.** There is a \( c = c(K) \) such that for each \( \tau : \Delta_2 \to M_0 \), we have
\[ \left| \int_{\Delta_2 \times I} H(\tau)^* \omega_1 \right| \leq \pi K^2 c^3. \]
In other words, \( \|C\|_{\infty} \leq \pi K^2 c^3 \), and so \( C \in \mathcal{C}_0^2(M_0) \).

**Proof.** With \( t \in I \) fixed, call \( \Delta_t = \Delta \times \{t\} \), and let \( \tilde{n}_t(x) \) denote the unit normal vector to the image of the surface \( \gamma_t : \Delta_t \to \mathbb{H}^3 \) at \( \gamma_t(x) \). In what follows, \( dA_t \) is the Riemannian area form for the pullback of the hyperbolic metric by \( \gamma_t \), \( d_t \) is the distance function on \( \Delta_t \) induced by this metric, and \( d \) is the distance function on \( \mathbb{H}^3 \). We begin by estimating
\[ \left| \int_{\Delta_2 \times I} H(\tau)^* \omega_1 \right| \leq \int_{\Delta_t} 1 \int_{\Delta_t} \left| \frac{\partial H(\tau)}{\partial t}, \tilde{n}_t \right| dA_t \]
\[ \leq \int_{\Delta_t} \int_{\Delta_t} \left( \frac{\partial H(\tau)}{\partial t}, \tilde{n}_t \right) dA_t dt + \int_{\Delta_t} \int_{\Delta_t} \left( \frac{\partial H(\tau)}{\partial t}, \tilde{n}_t \right) dA_t dt. \] (7)

Note that for \( 1/2 < t \leq 1 \), we have that \( \left( \frac{\partial H(\tau)}{\partial t}, \tilde{n}_t \right) \) are traveling orthogonally to \( \tilde{n}_t = \tilde{n}_1 \) within \( \text{im} \tau_1 \), for all \( x \). By construction, for \( 0 \leq t \leq 1/2 \), \( \left| \frac{\partial H(\tau)}{\partial t}(x,t) \right| = 2|\gamma'_1(x)| = 2d(\gamma_0(x), \gamma_1(x)) \). Thus, by Cauchy–Schwartz we have
\[ \left| \left( \frac{\partial H(\tau)}{\partial t}, \tilde{n}_t \right) \right| \leq \begin{cases} 2d(\gamma_0(x), \gamma_1(x)), & 0 \leq t \leq 1/2 \\ 0, & 1/2 < t \leq 1. \end{cases} \]
Combining the above expression with inequality (7), we have
\[ \left| \int_{\Delta_2 \times I} H(\tau)^* \omega_1 \right| \leq 2 \max_{x \in \Delta_2} \{d(\gamma_0(x), \gamma_1(x))\} \int_{0}^{1/2} \left( \int_{\Delta_t} dA_t \right) dt. \] (8)
We now show that the distance \( d(\gamma_0(x), \gamma_\frac{1}{2}(x)) \) is uniformly bounded, independently of \( \tau \). \( \text{str}_0 \tau_{[i,j]} \) is a geodesic segment parameterized proportionally to its length \( \ell_{i,j} \). Since \( \Phi \) is \( K \)-bi-Lipschitz, \( \tau_{[i,j]} \) is a \((K,0)\)-quasi-geodesic segment (parameterized proportionally to \( \ell_{i,j} \)). Since \( \mathbb{H}^3 \) is \( \log \sqrt{3} \)-hyperbolic, by the Morse lemma (Theorem 2.1) \( \tau_0([i,j]) \) is no more than \( c' = c'(\log \sqrt{3}, K) \) from \( \tau_1([i,j]) \), because \( \tau_1([i,j]) \) is the geodesic segment with the same endpoints as \( \tau_0([i,j]) \). Since hyperbolic triangles are \( \log \sqrt{3} \)-thin, given \( x \in \Delta_2 \), we can find an edge and a point \( p \in [i,j] \) such that \( x \) is distance at most \( \log \sqrt{3} \) from \( p \) in the metric induced by \( \text{str}_0 \tau \). If the \( \text{str}_0 \tau \)-segment between \( x \) and \( p \) passes through \( \partial \text{cl}(N) \setminus [i,j] \), call the point of intersection \( q \). Note that \( p = p(q) \) from our definition of \( H(\tau) \). Now \( d(\gamma_\frac{1}{2}(p), r(\tau_0(p))) \leq c' \), so

\[
d(\gamma_\frac{1}{2}(p), \gamma_\frac{1}{2}(x)) \leq d(\gamma_\frac{1}{2}(p), r(\tau_0(p))) + d(r(\tau_0(p)), r(\tau_0(q))) + d(r(\tau_0(q)), \gamma_\frac{1}{2}(x))
\]

\[
\leq c' + 1 + K \log \sqrt{3}
\]

because \( d(r(\tau_0(q)), \gamma_\frac{1}{2}(x)) \leq K \log \sqrt{3} \) and since \( d_{\text{str}_0 \tau}(p,q) \leq 1/K \) and \( r \) is \( 1 \)-Lipschitz, \( d(r(\tau_0(p)), r(\tau_0(q))) \leq 1 \). More importantly,

\[
d(\gamma_0(x), \gamma_\frac{1}{2}(x)) \leq d(\gamma_0(x), \gamma_0(p)) + d(\gamma_0(p), \gamma_\frac{1}{2}(p)) + d(\gamma_\frac{1}{2}(p), \gamma_\frac{1}{2}(x))
\]

\[
\leq K \log \sqrt{3} + c' + (c' + 1 + K \log \sqrt{3})
\]

\[
= K \log 3 + 2c' + 1.
\]

Since \( x \) was arbitrary, we have shown that

\[
\max_{x \in \Delta_2} \{ d(\tau_0(x), \gamma_\frac{1}{2}(\tau_0(x))) \} \leq c,
\]

where we define \( c := K \log 3 + 2c' + 1 \).

We would like to show that the family of identity maps \( id_t : (\Delta_2, d_{\text{str}_0 \tau}) \to (\Delta_t, d_t) \) are \( cK \)-Lipschitz on \( \Delta_2 \). Let \( x, y \in \Delta_2 \), and consider the \( \text{str}_0 \)-geodesic segment \([x,y] \subset \Delta_2 \). For \( t = 0 \), we have that \( d_0(x,y) \leq K d_{\text{str}_0 \tau}(x,y) \), because \( \tau_0([x,y]) \) is the \( K \)-Lipschitz image of a geodesic. Partition the segment \([x,y]\) into intervals \([x = x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n−1}, x_n = y] \), and approximate \( \tau_0([x,y]) \) as the concatenation of the geodesic paths \( \gamma_0(x_i), \gamma_0(x_{i+1}) \). By convexity of the hyperbolic metric, and since \( r \) is \( 1 \)-Lipschitz, if \([x_i, x_{i+1}] \subset \Delta_2 \) we have

\[
d(\gamma_t(x_i), \gamma_t(x_{i+1})) \leq td(\gamma_0(x_i), \gamma_0(x_{i+1})) + (1−t)d(r(\gamma_0(x_i)), r(\gamma_0(x_{i+1})))
\]

\[
\leq td(\gamma_0(x_i), \gamma_0(x_{i+1})) + (1−t)d(\gamma_0(x_i), \gamma_0(x_{i+1}))
\]

\[
= d(\gamma_0(x_i), \gamma_0(x_{i+1})).
\]

Assume now that \([x_i, x_{i+1}] \subset N \). Then a similar analysis shows that \( d(\gamma_t(x_i), \gamma_t(x_{i+1})) \leq cd(\gamma_0(x_i), \gamma_0(x_{i+1})) \).

Taking a limit over finer partitions, we see that the length \( \ell(\gamma_t([x,y])) \leq c \ell(\gamma_0([x,y])) \), and so \( d_t(x,y) \leq cd_0(x,y) \). Therefore, \( d_t(x,y) \leq cKd_{\text{str}_0 \tau}(x,y) \) and the maps

\[
id_t : (\Delta_2, d_{\text{str}_0 \tau}) \to (\Delta_t, d_t), \ x \mapsto H(\tau)(x,t)
\]

are \( cK \)-Lipschitz for all \( t \in [0,1] \).

We can now estimate

\[
\int_{\Delta_t} dA_t = \int_{id_t(\Delta_2)} dA_t = \int_{\Delta_2} id_t^* dA_t \leq \int_{\Delta_2} (cK)^2 dA_{\text{str}_0 \tau} \leq \pi(cK)^2
\]

(18)
because the Lipschitz constant $cK$ bounds the Jacobian of $id_t$ by $(cK)^2$, and the area of a hyperbolic triangle is no more than $\pi$. Combining estimates (8), (14) and (18), we obtain
\[
\left| \int_{\Delta_2 \times I} H(\tau)^* \omega_1 \right| \leq 2c \int_0^1 (cK)^2 \pi \, dt = \pi K^2 c^3,
\]
which completes the proof of the proposition and Theorem 3.2. \qed

Lemma 3.4 and Proposition 3.5 are slightly more general.

**Corollary 3.6.** Let $\zeta_0 \in \Omega(\mathbb{M}_0)$ and $\zeta_1 \in \Omega(\mathbb{M}_1)$, and suppose $F : \mathbb{M}_0 \to \mathbb{M}_1$ is a bi-Lipschitz homeomorphism satisfying $\int_{\sigma} F^{-1} \zeta_0 = \int_{\sigma} \zeta_1$, for all continuous maps $\sigma : \Delta_3 \to \mathbb{M}_1$. If the co-chains $\zeta_i$ are bounded, then
\[
[F^* \zeta_1] = [\zeta_0] \in H^3(\mathbb{M}_0) = H^3(\mathbb{S}) = H^3(\mathbb{R}).
\]

Next, we show that singly degenerate bounded fundamental classes can be represented by a class defined on the whole manifold $F \times \mathbb{R}$, but which has support only in the convex core. This will come in handy when we prove Theorem 1.2. We remark that the following discussion is essentially standard [9], but we include arguments here for completeness.

We will now work with one marked hyperbolic manifold without parabolic cusps $f : S \to \mathbb{M} = \mathbb{M}_\nu$, and such that $f_* = \rho(\nu) : \pi_1(S) \to \Gamma(\nu) < \text{PSL}_2 \mathbb{C}$. Choose a point $p \in \text{core}(\mathbb{M})$. We define a new straightening operator $\text{str}_p : C_\bullet(\mathbb{M}) \to C_\bullet(\text{core}(\mathbb{M}))$. Let $\sigma : \Delta_n \to \mathbb{M}$ and take any lift of the ordinary straightening $\text{str} \sigma : \Delta_n \to \mathbb{H}^3$ to the universal cover $\pi : \mathbb{H}^3 \to \mathbb{M}$. Fix $\bar{p} \subset \pi^{-1}(p)$, and let $D = \{q \in \mathbb{H}^3 : d(\bar{p}, q) \leq d(\gamma(\bar{p}), q) \text{ for all } \gamma \in \Gamma(\nu) \}$ be the Dirichlet fundamental polyhedron for $\Gamma(\nu)$ centered at $\bar{p}$; delete a face of $D$ in each face-pair $(F, \gamma F)$ to obtain a fundamental domain for $\Gamma(\nu)$, which we still call $D$. Then the vertices $v_0, \ldots, v_n$ of $\text{str} \sigma$ are labeled by group elements $v_i = \gamma_i q_i$, where $\rho(\nu)(v_i) = \gamma_i \in \Gamma(\nu)$, $q_i \in D$, and $(v_i, q_i) \in \pi_1(S) \times D$ is unique. Define $\text{str}_p \sigma = \pi(\sigma_p(\gamma_0, \ldots, \gamma_n))$, where $\sigma_p(\gamma_0, \ldots, \gamma_n)$ is the straightening of any simplex whose ordered vertex set is $(\gamma_0 \bar{p}, \ldots, \gamma_n \bar{p})$. The definition is clearly independent of choices. Since $p$ is in the convex core of $\mathbb{M}$ and all of the edges of $\text{str}_p \sigma$ are geodesic loops based at $p$, the edges of $\text{str}_p \sigma$ (hence of all of $\text{im} \text{str}_p \sigma$) are contained in $\text{core}(\mathbb{M})$; moreover, all maps are chain maps and the operator norm $\|\text{str}_p\| \leq 1$. This is just because some simplices in a chain may collapse and cancel after applying $\text{str}_p$. Define
\[
\tilde{\omega}_p(\sigma) = \int_{\text{str}_p \sigma} \omega.
\]

**Remark 3.7.** It is not hard to see that the co-chain $\tilde{\omega}_p$ is essentially a topological description of the group co-chain $\rho^* \text{vol}_p$.

Now we describe a prism operator on 2-simplices (the definition extends to $n$-simplices, and so it can be shown that the prism operator defines a chain homotopy between $\text{str}_p$ and $\text{str}$, but we will not need this). As before, for $x, y \in \mathbb{H}^3$, $[x, y] : I \to \mathbb{H}^3$ is the unique geodesic segment joining $x$ to $y$ parameterized proportionally to arc-length. Take the lift $\text{str} \tau$ with ordered vertex set $(\gamma_0 q_0, \gamma_1 q_1, \gamma_2 q_2)$ where $q_i \in D$, and define
\[
H_p(\tau) : \Delta_2 \times I \to \mathbb{M}
\]
\[
(x, t) \mapsto \pi([\text{str} \tau(x), \sigma_p(\gamma_0, \gamma_1, \gamma_2)(x)](t)).
\]
Note that this is just projection of the straight-line homotopy between specific lifts of $\text{str} \tau$ and $\text{str}_p \tau$. Recall that the prism operator from algebraic topology decomposes the space $\Delta_n \times I$ combinatorially as a union of $(n + 1)$-simplices glued along their faces in a consistent
way and is used to prove that homotopic maps induce chain homotopic maps at the level of chain complexes. Apply the prism operator to $\Delta_2 \times I$, to obtain a triangulation comprised of three tetrahedra inducing a chain $C = C_0 + C_1 + C_2$. Then $C_p(\tau) = \text{str}(H_p(\tau), C)$ is a straight 3-chain satisfying $dC_p(\sigma) = C_p(\partial \sigma) = \text{str}_p \sigma - \text{str} \sigma$ for any $\sigma \in C_3(M; \mathbb{R})$. Moreover,

$$\int_{\text{str} C_p(\tau)} \omega \leq 3v_3$$

(19)

because $\text{str} C_p(\tau)$ is a sum of three hyperbolic tetrahedra. We have proved

 Lemma 3.8. Let $M$ be a hyperbolic 3-manifold and $p \in \text{core}(M)$. Then 

$$[\hat{\omega}] = [\omega_p] \in H^3_b(M).$$

4. Compactely supported bounded classes

To prove Theorem 1.2, we will construct a bi-Lipschitz embedding of the core of a singly degenerate manifold into a doubly degenerate one that is volume preserving away from a compact set. We would like to ignore what happens near that compact set, a neighborhood. The curves still approach the ending lamination and so the/a limiting sequence will exit an end.

If $M$ is a Riemannian manifold and $\omega \in \Omega^k(M)$ is a smooth $k$-form, we define the norm at a point $x \in M$ by

$$\|\omega\|(x) = \sup\{\omega_x(v_1, \ldots, v_k) : v_i \in T_xM, \|v_i\| \leq 1\},$$

define $\|\omega\|_{\infty} = \sup\{\|\omega\|(x) : x \in M\} \leq \infty$.

Lemma 4.1. Let $M$ be a hyperbolic manifold diffeomorphic to $S \times \mathbb{R}$ with bounded geometry. Let $\omega \in \Omega^3_b(M)$ be a compactely supported differential 3-form. Then there is an $\eta \in \Omega^2(M)$ such that $d\eta = \omega$ and $\|\eta\|_{\infty} < \infty$.

Proof. Identify $M$ diffeomorphically with $S \times \mathbb{R}$. Since $S \times \mathbb{R}$ is an open manifold, there is a differential form $\beta \in \Omega^2(S \times \mathbb{R})$, such that $d\beta = \omega$. Without loss of generality, assume that $S \times (-1, 1)$ contains the support of $\omega$. Let $U = S \times (-\infty, -1)$, $W = S \times (-2, 2)$ and $V = S \times (1, \infty)$, and find a partition of unity $\psi = \{\psi_U, \psi_V, \psi_V\}$ subordinate to this cover. Then $d\beta|_U = 0$ because the support of $\omega$ is contained in the complement of $U$. The de Rham cohomology $H^2(U) \cong \mathbb{R}$, and so $[\beta|_U]$ represents the class of $b_U \in \mathbb{R}$. For example, since $[S \times \{-10\}]$ is a generator of the degree 2 homology group, we have

$$\langle [\beta|_U], [S \times \{-10\}] \rangle = \int_{S \times \{-10\}} \beta|_U = b_U.$$

Since $M$ has bounded geometry, we can find an embedded geodesic $\ell : \mathbb{R} \to M$ such that the injectivity radius $\text{inj}_{\ell(t)}(M) > \epsilon$ for all $t$, and such that $\ell$ exits both ends of $M$. To see this, choose a basepoint $p \in M$ and sequence $q_n \in M$ at distance $n$ from $p$. Find geodesic segments $[p, q_n]$ from $p$ to $q_n$ of length $n$ exiting an end of $M$. By Arzelà–Ascoli, we can extract a limit $\ell^+$, which is a geodesic ray; also find a ray $\ell^-$ based at $p$ exiting the other end of $M$. The concatenation of these rays is a quasi-geodesic and tracks a geodesic $\ell$ closely; $\ell$ is uniformly thick, because $M$ is. Moreover, $\ell$ is minimizing in the sense that there is a constant $C > 0$ such that $d(p, \ell(t)) \geq |t| - C$. This is because $\ell$ is essentially a limit of minimizing segments making linear progress away from $p$ (see [21] for more about minimizing geodesics). Since the injectivity radius of $M$ is bounded away from 0, and $\ell$ is minimizing, $\ell$ only comes within $\epsilon$ of itself finitely many times. Let $m = \max\{n : \{t_1, \ldots, t_n\} \text{ are pairwise distinct and } d(\ell(t_1), \ell(t_i)) < \epsilon \text{ for all } i\}$. 
Then the Poincaré dual of im $\ell$ is a generator for the de Rham cohomology $H^2(S \times \mathbb{R})$; it can be represented by a form $\eta_{U}$ whose support is contained in a tubular $T$ neighborhood of $\ell$. Indeed, by assumption, the $\epsilon$ neighborhood of $\ell$ is such a tubular neighborhood, foliated by planes meeting $\ell$ orthogonally. Let $O$ be an orthonormal section of the 2-frame bundle over $T$ which is tangent to each disk $D_{i}(\ell(s))$, and let $\psi_{U}$ be a bump function on the disk $D_{i}(\ell(s))$ which is constant on $D_{2}^{2}(\ell(s))$, decreases radially to 0, and has integral 1. Then $\eta_{U} = \psi_{U}O^* \in \Omega^{2}(S \times \mathbb{R})$ is our desired representative for the Poincaré dual of $\ell$, and by construction $\|\eta_{U}\|_{\infty} \leq m/(\text{area}(D_{1/2}))$.

Then $b_{U}\eta_{U}|_{U} - \beta|_{U} = \alpha|_{U}$ for some $\alpha|_{U} \in \Omega^{1}(U)$. Construct a new form that interpolates from $b_{U}\eta_{U}|_{U}$ to $\beta|_{U}$ as follows:

$$\gamma_{U} := \psi_{U}b_{U}\eta_{U} + \psi_{W}\beta|_{U}.$$  

Then $\gamma_{U}$ is not closed, but by adding a correction term $d\psi_{U} \wedge \alpha|_{U}$, one checks that $d(\gamma_{U} + d\psi_{U} \wedge \alpha|_{U}) = 0$. Moreover, since $d\psi_{U} \wedge \alpha|_{U}$ has support in $U \cap W$, and $\gamma_{U} = b_{U}\eta_{U}|_{U}$ away from $W$, we have that

$$\langle \gamma_{U} + d\psi_{U} \wedge \alpha|_{U}, [S \times \{-10\}] \rangle = \int_{S \times \{-10\}} b_{U}\eta_{U} = b_{U}.$$  

Thus there is an $\alpha'_{U} \in \Omega^{1}(U)$ such that $\gamma_{U} + d\psi_{U} \wedge \alpha|_{U} + \beta|_{U} = \alpha'_{U}$. Since $\gamma_{U} + d\psi_{U} \wedge \alpha|_{U}$ and $\beta|_{U}$ agree near the boundary of $U$, $\alpha'_{U}$ is zero there, and so it extends to a 2-form on all of $S \times \mathbb{R}$ with support in $U$.

Repeat these steps on $V$, and define

$$\eta := \psi_{U}(\gamma_{U} + d\psi_{U} \wedge \alpha|_{U}) + \psi_{W}(\beta - \alpha'_{U} - d\alpha'_{V}) + \psi_{V}(\gamma|_{V} + d\psi_{V} \wedge \alpha|_{V}).$$  

A tedious but straightforward calculation shows that $d\eta = \omega = d\beta$. Observe that

$$\|\beta|_{W} - \alpha'_{U}|_{W} - \alpha'_{V}|_{W} + d\psi_{U} \wedge \alpha|_{U} + d\psi_{V} \wedge \alpha|_{V}\|_{\infty} = \epsilon < \infty,$$

because $W$ is precompact, the supports of both $d\psi_{U} \wedge \alpha|_{U}$ and $d\psi_{V} \wedge \alpha|_{V}$ are compact, and all functions are smooth. Finally we estimate

$$\|\eta\|_{\infty} \leq m\cdot \max \left\{ \frac{|b_{V}|}{\text{area}(D_{2}^{2})}, \frac{|b_{V}|}{\text{area}(D_{2}^{2})} \right\} + \epsilon,$$

so $\|\eta\|_{\infty} < \infty$ and we are finished.

The previous result was the main technical step for proving the main result from this section, which now follows easily. The idea is that differential forms define cochains by integration on geodesic simplices. If the pointwise norm of the differential form is globally bounded with respect to the ambient hyperbolic metric, then this cochain is bounded. Stokes' theorem then provides the link between bounded primitives of differential forms and bounded primitives of cochains in negative curvature.

**Proposition 4.2.** Let $\rho : \pi_{1}(S) \twoheadrightarrow \text{PSL}_{2}\mathbb{C}$ be Kleinian with bounded geometry and let $f : M_{\rho} \rightarrow \mathbb{R}$ be compactly supported and smooth. Then the class $[f\omega] = 0 \in H_{0}^{2}(M_{\rho}; \mathbb{R})$.

**Proof.** By Lemma 4.1, there is an $\eta \in \Omega^{2}(M)$ such that $d\eta = f\omega$ and $\|\eta\|_{\infty} < \infty$. Since str is a chain map, by Stokes' theorem,

$$\hat{\omega}(\sigma) = \int_{\text{str} \sigma} \omega = \int_{\text{str} \partial \sigma} \eta = \hat{\eta}(\partial \sigma).$$
This means that $\hat{\omega} = d\hat{\eta}$. Since the area of a hyperbolic triangle is bounded by $\pi$,  

$$|\hat{\eta}(\tau)| = \left| \int_{\text{str } \tau} \eta \right| \leq \int_{\text{str } \tau} \|\eta\| \, dA \leq \pi \|\eta\|_\infty.$$  

This shows that $\hat{\eta} \in C^2_b(M)$, and so $[\hat{\omega}] = 0$ in bounded cohomology. \qed

5. Volume preserving limit maps  

In this section, we consider maps of singly degenerate manifolds into doubly degenerate manifolds that have an end invariant in common. Since the doubly degenerate manifold is quasi-conformally rigid, we cannot apply the Geometric Inflexibility Theorem to obtain a bi-Lipschitz, volume preserving map. Instead, we take a limit of such maps. We will only have control on the bi-Lipschitz constant away from the convex core boundary, so we have to modify our map near it.

Let $\lambda_-, \lambda_+ \in \mathcal{E}L_1(S)$, and set $\nu = (\lambda_-, \lambda_+)$; then $\text{inj} \, M_\nu = \epsilon > 0$. Then we have a model manifold $\mathfrak{M}_\nu$ and map

$$\Psi_\nu : \mathfrak{M}_\nu \to M_\nu$$

as in Theorem 2.4, a Teichmüller geodesic $t \to Y_t$, $t \in \mathbb{R}$ and pleated surfaces $f_n : X_n \to M_\nu$ for each $n \in \mathbb{Z}$ as in Theorem 2.3. Since $\Psi_\nu$ is an $(L, c)$-quasi-isometry and $\Psi_\nu(\cdot, n) = f_n$, there is $k \in \mathbb{N}$ such that

$$d_{M_\nu}(\text{im } f_{kn}, \text{im } f_{k(n+1)}) > 1 \quad (20)$$

for all $n$.

Let $\nu_i = (\lambda_-, Y_{ki}), \rho_i = \rho(\nu_i) : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$, and $M_i = M_{\rho_i}$. Since $t \to Y_t$ is geodesic,

$$d_{\mathcal{G}(S)}(Y_{ki}, Y_{k(i+1)}) = \frac{1}{2} \log(e^{2k}),$$

so by Theorem 3.1 we have volume preserving $K = e^{3k}$-bi-Lipschitz diffeomorphisms

$$\phi_i : M_i \to M_{i+1}$$

and constants $C_1, C_2 > 0$ independent of $i$ such that

$$\log \text{bilip}(\phi_i, p) \leq C_1 e^{-C_2(d(p, M_i)_{\text{core}(M_i)})}$$

for all $p \in M_i$. As in Theorem 2.4, there are models $\mathfrak{M}_i$ and maps

$$\Psi_i : \mathfrak{M}_i = (S \times (-\infty, ik], ds) \to \text{core}(M_i).$$

Observe that the inclusion $\mathfrak{M}_i \to \mathfrak{M}_\nu$ is an isometric embedding with respect to the path metric on its image. Note that $[\rho_i]$ converges to $[\rho_\nu]$ algebraically.

Let $S_n = S \times \{n\} \subset S \times \mathbb{R}$. By Theorem 2.4, for $i \in \mathbb{N} \cup \{0\}$ and $n \leq ki$ the maps $\Psi_i(\cdot, n) : X_n \to M_i$ are pleated maps. We have the following analogue of Inequality (20):

$$d_{M_i}(\Psi_i(S_{kn}), \Psi_i(S_{k(n+1)})) > 1 \quad (21)$$

for all $n < i$.

**Proposition 5.1.** For every $\epsilon' > 0$, there is a compact set $K \subset \text{core}(M_0)$ and a volume preserving $(1 + \epsilon')$-bi-Lipschitz embedding $\Phi_\nu : \text{core}(M_0) \setminus K \to M_\nu$ in the homotopy class determined by $\rho_0$ and $\rho_\nu$.

**Proof.** The $M_i$ will converge geometrically to $M_\nu$, and we will extract a limiting map from the sequence $\{\Psi_i = \phi_i \circ \cdots \circ \phi_0|_{\text{core}(M_0) \setminus K}\}$ by passing to universal covers. As long as we can
control the bi-Lipschitz constant of $\Phi_i$, we can apply the Arzelà–Ascoli theorem to obtain a convergent subsequence with limit $\Phi_{i'} : \text{core}(\tilde{M}_0) \setminus K \to \mathbb{H}^3$. By the algebraic convergence of $[\rho_n] \to [\rho_{i'}]$, $\Phi_{i'}$ is $(\rho_0, \rho_{i'})$-equivariant and hence descends to a map $\Phi_{i'} : \text{core}(M_0) \setminus K \to M_{i'}$ of quotients. The limiting map will be volume preserving away from $K$ and $(1 + \epsilon')$-bi-Lipschitz, where $K$ depends on both $\epsilon'$ and $\kappa$.

Now we choose $K$. For $N \in \mathbb{N}$, $\Psi_0(\cdot, -N)$ is an immersed homotopy equivalence. By minimal surface theory [15], for any $\epsilon'' > 0$, there is an embedding $g_{-N} : S_{-N} \to N_{\epsilon''}(\Psi_0(S_{-N}))$ that is homotopic to $\Psi_0(\cdot, -N)$. By the homeomorphism theorem of Waldhausen [39], $\partial \text{core}(M_0)$ and $\text{im} g_{-N}$ bound an embedded submanifold $K_N$ homeomorphic to $S \times I$.

By Inequality (21), since $\partial \text{core}(M_0) = \Psi_0(S_0)$,
\[
d_{M_0}(\Psi_0(S_0), \Psi_0(S_{-N})) > \frac{N}{k}.
\]

Choose $N$ large enough so that, using the above estimate and Theorem 3.1,
\[
bilip(\phi_0, \text{core}(M_0) \setminus K_N) \leq (1 + \epsilon')^{1-e^{-C_2}} < (1 + \epsilon'),
\]
and take $K = K_{N+1}$.

Now we show, by induction, that for $n \geq 0$,
\[
\log \text{bilip}(\Phi_n, \Phi_{n-1}(\Psi_0(S_{-N}))) \leq (1 - e^{-C_2}) \log(1 + \epsilon') e^{-C_2n}.
\]
(22)

From this it will follow that for all $n \geq 0$,
\[
\log \text{bilip}(\Phi_n, \Psi_0(S_{-N})) \leq (1 - e^{-C_2}) \log(1 + \epsilon') \sum_{j=0}^{n} e^{-jC_2}
\]
\[
< \log(1 + \epsilon') \frac{1 - e^{-C_2}}{1 - e^{-C_2}} = \log(1 + \epsilon').
\]
(24)

The base case was covered, above. So assume that (22) holds for all $m < n$, so that $\text{bilip}(\Phi_{n-1}, \Psi_0(S_{-N})) < (1 + \epsilon')$. It suffices to show that there is some constant $e \geq 0$ such that
\[
d_{M_n}(\Phi_{n-1}(\Psi_0(S_{-N})), \partial \text{core}(M_n)) \geq n + \frac{N}{k} - e.
\]

To this end, we would like to show that $d_{M_n}^H(\Phi_{n-1}(\Psi_0(S_{-N})), \Psi_n(S_{-N})) < e$ for some $e$, where $d^H_{M_n}$ denotes the Hausdorff distance on closed subsets of a metric space $X$. We use the following well known fact, which follows from the observation that the projection onto a geodesic in $\mathbb{H}^3$ contracts by a factor of at least $\cosh R$, where $R$ is the distance to the geodesic.

**Lemma 5.2 [28, Lemma 4.3].** Let $M$ be a hyperbolic manifold, $\alpha : S^1 \to M$ a homotopically non-trivial closed curve, and $\alpha^* \subset M$ its geodesic representative. Then $d^H_M(\alpha, \alpha^*) \leq R$, where $R = \cosh^{-1}(\frac{\ell_M(\alpha)}{\ell_M(\alpha^*)}) + \frac{1}{2} \ell_M(\alpha)$.

Let $\alpha \subset S_{-N}$ be a simple closed $X_{-N}^0$-geodesic whose length is at most $L_0$ — the Bers constant for $S$. By property (ii) in Theorem 2.4, the $(S, |Q_0|)$-length of $\alpha$ is at most $LL_0 + c$, where $L$ and $c$ depend only on $S$ and $e$. A second application of property (ii) in Theorem 2.4 shows that the $X_{-N}$-length of $\alpha$ is at most $L^2 L_0 + Lc + c$. Since $\text{bilip}(\Phi_{n-1}, \Psi_0(S_{-N})) < (1 + \epsilon')$, $\ell_{M_n}(\Phi_{n-1}(\Psi_0(\alpha))) \leq (1 + \epsilon') L_0$. The $M_n$-length of the geodesic representative of $\alpha$ is at least $2\epsilon$, so applying Lemma 5.2, there is a constant $L' = L'(S, \epsilon, \epsilon')$ such that
\[ d^H_{M_n} (\Phi_{n-1}(\Psi_0(\alpha)), \Psi_n(\alpha)) \leq 2L'. \]

There is a universal bound \( D = D(\epsilon) \) on the diameter of an \( \epsilon \)-thick hyperbolic surface. Thus
\[ d^H_{M_n} (\Psi_n(S_{-N}), \Phi_{n-1}(\Psi_0(S_{-N}))) \leq D + (1 + \epsilon')D + 2L' = e. \tag{25} \]

Recall that \( \partial \text{core}(M_n) = \Psi_n(S_{kn}) \). We use Inequalities (25) and (21) together with the triangle inequality to estimate
\[ d_{M_n} (\partial \text{core}(M_n), \Phi_{n-1}(\Psi_0(S_{-N}))) \]
\[ \geq d_{M_n} (\Psi_n(S_{kn}), \Psi_n(S_{-N})) - d^H_{M_n} (\Psi_n(S_{-N}), \Phi_{n-1}(\Psi_0(S_{-N}))) \]
\[ \geq \frac{N}{k} + n - e. \]

We (retroactively) assume that \( N/k - e \) was large enough so that \( C_1 e^{-C_2 (\frac{N}{k} - e)} < (1 - e^{-C_2}) \log(1 + \epsilon') \). Then \( \log \text{bilip}(\Phi_n, \Phi_{n-1}(\Psi(S_{-kn}))) \leq (1 - e^{-C_2}) \log(1 + \epsilon') e^{-C_2 n} \), which completes the proof of (22) and also (23).

\[ \square \]

**Remark 5.3.** Later on, we will only use the existence of a bi-Lipschitz embedding \( \Phi_- : \text{core}(M_0) \to M_\nu \) that is volume preserving away from a compact set. We can therefore take \( \epsilon = 1 \) (this is an arbitrary choice) in the statement of Proposition 5.1, and extend \( \Phi_\nu \) to \( \mathcal{K} \) by any bi-Lipschitz homeomorphism onto some subset \( \mathcal{K}' \subset M_\nu \setminus \text{im} \Phi_\nu \) to obtain \( \Phi_- \). The overall bi-Lipschitz constant will be \( B = \max\{2, \text{bilip}(\Phi_-|_{\mathcal{K}})\} \).

6. Ladders

We will want to understand the relationship between based geodesics loops in a doubly degenerate manifold \( M \) with bounded geometry and with end invariants \( (\lambda_-, \lambda_+) \) and the same homotopy class of based geodesic loop in a singly degenerate manifold \( N \) with end invariants \( (Y_0, \lambda_+) \). Here, \( Y_0 \) is a point on the Teichmüller geodesic between \( \lambda_- \) and \( \lambda_+ \), as in Section 2.11. The model manifolds have the form \( \mathcal{N} = (S \times \mathbb{R}_{>0}, d_{0N}) \) and \( \mathcal{M} = (S \times \mathbb{R}, d_{0M}) \), where the infinitesimal formulation of the metrics \( d_{0N} \) and \( d_{0M} \) are both of the form \( ds^2 = |Q_\lambda| + dt^2 \), as explained in Section 2.11. Then the inclusion \( \mathcal{N} \to \mathcal{M} \) of universal covers is isometric with the path metric on the image, but \( \mathcal{N} \subset \mathcal{M} \) is very far from being even quasiconvex. The model manifold \( \mathcal{N} \) has universal cover that is equivariantly quasi-isometric to \( \text{core}(N) \).

Let \( \Sigma \subset S_0 \) be the set of zeros of \( Q_0 \). We pick a point \( p \in \Sigma \subset \mathcal{N} \subset \mathcal{M} \), and choose \( \tilde{p} \) a lift of \( p \) under universal covering projections. Let \( \gamma^*d \) be singular Euclidean metric on \( \tilde{S}_t = \tilde{S} \times \{t\} \) defined by \( |Q_\lambda| \), so that the inclusions \( \gamma_t : \tilde{S}_t \to \mathcal{M} \) as level surfaces are isometries with respect to the path metric on the image. We withhold the right to abbreviate a point \((x,t) \in \tilde{S}_t \) as \( x_t \). Note that the (identity) map \( \tilde{S}_t \to \tilde{S}_t \), \( x_t \mapsto x_t \) is a lift of the Teichmüller map between the conformal structures underlying \( Q_\lambda \) and \( Q_\mu \), by construction (see Section 2.11). That is, \( x_s \mapsto x_t \) is affine, \( e^{s-t}\)-bi-Lipschitz, and it takes \( \gamma^*d \)-geodesics to \( \gamma^*d \)-geodesics. The inclusions \( (\tilde{S}_t, \gamma^*d) \to (\mathcal{M}, d_{0M}) \) are uniformly proper for all \( t \) in the following sense.

**Lemma 6.1.** For all \( t \in \mathbb{R} \) and \( D \geq 0 \), if \( d_{0M}(x_t, y_t) = D \), then \( \gamma^*d(x_t, y_t) \leq D e^{D/2} \).

**Proof.** Let \( \gamma : [0, D) \to \mathcal{M} \) be a \( d_{0M} \)-geodesic joining \( x_t \) and \( y_t \). Then \( \text{im} \gamma \subset \tilde{S} \times [t - D/2, t + D/2] \). Endow \( \tilde{S} \times [t - D/2, t + D/2] \subset \mathcal{M} \) with its path metric; the projection \( p_t : \tilde{S} \times [t - D/2, t + D/2] \to S_t \), mapping \( x_s \mapsto x_t \) is then a \( e^{D/2} \)-Lipschitz retraction that
commutes with the bi-Lipschitz Teichmüller map; they are both identity on the first coordinate. So the length of \( p_1(\gamma) \subset \tilde{S}_t \) is at most \( e^{D/2}D \), which bounds the distance \( i_t^*d(x_i, y_i) \).

Suppose \( \gamma : I \to S_0 \) is a closed \( d_0 \)-geodesic based at \( p \). Find the lift \( \tilde{\gamma} : I \to \tilde{S}_0 \) based at \( \tilde{p} \). We will conflate the map \( \tilde{\gamma} \) with its image \( \gamma \subset \tilde{S} \subset \tilde{M} \). While \( \tilde{M} \) and \( \tilde{S} \) do not have negative sectional curvatures, they are \( \delta \)-hyperbolic, where \( \delta \) depends on both \( S \) and \( \epsilon \) as in Theorem 2.4; the Teichmüller geodesic \( Y_\epsilon \subset \mathcal{T}_\partial \). Straighten \( \tilde{\gamma} \) with respect to the two metrics \( \gamma^* \subset \mathcal{S} \) and \( \gamma^+_\epsilon \subset \mathcal{S} \) rel endpoints. We would like to make precise the notion that \( \gamma^* \) and \( \gamma^+_\epsilon \) fellow travel in \( \mathcal{S} \), and while \( \gamma^* \) spends time in \( \mathcal{S} \), \( \gamma^+_\epsilon \) spends time near \( \partial \mathcal{S} \). The main result from this section is

**Theorem 6.2.** There exists a constant \( D_1 > 0 \) depending only on \( \epsilon \) and \( S \) such that the following holds. Let \( \gamma : I \to \mathcal{S} \) be a loop based at \( p \). With notation as above, let \( \Pi : \mathcal{S} \to \mathcal{I} \). We prove Theorem 6.2 at the end of the section after first describing a family of quasi-geodesics in \( \mathcal{S} \subset \mathcal{M} \). With \( \gamma \) and \( p \) as above, let \( \mathcal{L}(\gamma) = \tilde{\gamma} \times \mathcal{S} \) and \( \mathcal{L}_+ (\gamma) = \gamma \times \mathcal{S} \). The spaces \( \mathcal{L}(\gamma) \subset \mathcal{M} \) and \( \mathcal{L}_+ (\gamma) \subset \mathcal{S} \) can be thought of as the union of the lifts of based \( \iota_t^*d^\mathcal{S} \)-geodesic segments as \( t \) ranges over \( \mathbb{R} \) or \( \mathbb{R}^{>0} \). Each of the spaces \( \mathcal{L}(\gamma) \subset \mathcal{M} \) and \( \mathcal{L}_+ (\gamma) \subset \mathcal{S} \) inherits a path metric that we call \( d_{\mathcal{L}} \). We call the spaces \( \mathcal{L}(\gamma), d_{\mathcal{L}} \) and \( \mathcal{L}_+ (\gamma), d_{\mathcal{L}} \) ladders after [31, 32]. The point of the ladder construction is that we will be able to reduce the problem of understanding based geodesics in \( \mathcal{M} \) or \( \mathcal{S} \) to understanding the way based geodesics behave in two-dimensional quasi-convex subsets.

Ladders also inherit metrics from the ambient spaces \( \mathcal{M} \) and \( \mathcal{S} \). We now show that the inclusion of a ladder with its path metric is undistorted with respect to the ambient model metric, and so ladders are quasi-convex. The strategy of the proof is standard and can be found in [32]. We include details, because we prefer to continue to work with the infinitesimal formulation of the metric as opposed to discretizing our space.

**Proposition 6.3.** There are constants \( L_0 \) and \( L_1 \) depending on \( S \) and \( \epsilon \), such that the inclusion \( (\mathcal{L}(\gamma), d_{\mathcal{L}}) \to (\mathcal{M}, d_{\mathcal{M}}) \) is an \( (L_0, L_0) \)-quasi-isometric embedding. Consequently, \( \mathcal{L}(\gamma) \subset \mathcal{M} \) is \( L_1 \)-quasi-convex. Analogous statements hold for \( \mathcal{L}_+ (\gamma) \subset \mathcal{S} \).

**Proof.** We will construct an \( (L_0, L_0) \)-coarse-Lipschitz retraction \( \Pi^\gamma : \mathcal{M} \to \mathcal{L}(\gamma) \). Once we have done so, we see that for each \( x, y \in \mathcal{L} \), we have \( d_{\mathcal{L}}(x, y) = d_{\mathcal{L}}(\Pi^\gamma(x), \Pi^\gamma(y)) \leq L_0 d_{\mathcal{M}}(x, y) + L_0 \).

By definition of the path metric, for all \( x, y \in \mathcal{L} \), we have \( d_{\mathcal{M}}(x, y) \leq d_{\mathcal{L}}(x, y) \). Combining inequalities, we see that

\[
d_{\mathcal{M}}(x, y) \leq d_{\mathcal{L}}(x, y) \leq L_0 d_{\mathcal{M}}(x, y) + L_0,
\]

which is to say that \( \mathcal{L} \to \mathcal{M} \) is an \( (L_0, L_0) \)-quasi-isometric embedding. Since \( (\mathcal{M}, d_{\mathcal{M}}) \) is a \( \delta \)-hyperbolic metric space, \( \mathcal{L} \) is \( \delta' = \delta' (\delta, L_0) \) hyperbolic, and by the Morse lemma (Theorem 2.1), \( \mathcal{L} \) is \( L_1 = L_1 (\delta, L_0) \)-quasi-convex.

Now we construct \( \Pi^\gamma \). Let \( \pi_t^\gamma : \tilde{S} \to \gamma \) be the nearest point projections with respect to the metric \( \iota_t^*d \); the spaces \( \tilde{S}, \iota_t^*d \) are CAT(0) and \( \gamma_t \) is \( \iota_t^*d \)-geodesically convex, so the projections
\[ \pi_i^\gamma \] are 1-Lipschitz retractions \cite{5}. The global retraction \[ \Pi^\gamma : \mathfrak{M} \rightarrow \mathcal{L}(\gamma) \]

\[ (x, t) \mapsto (\pi_i^\gamma(x), t) \]

leaves \( dt^2 \) invariant. We now need to understand how much the projections \( \pi_i^\gamma \) change as \( t \) changes, because unlike in Lemma 6.1, the projections \( \pi_i^\gamma \) do not commute with \( x_s = (x, s) \mapsto (x, t) = x_t \).

Assume \( d_{\mathfrak{M}}(x_s, y_t) \leq 1 \). We will find \( L_0 \geq 1 \) such that \( d_{\mathcal{L}}(\Pi^\gamma(x_s), \Pi^\gamma(y_t)) \leq L_0 \), and the lemma will follow from this by a repeated application of the triangle inequality. In what follows we abbreviate \( \pi_i^\gamma \) to \( \pi_i \).

\[
\begin{align*}
& d_{\mathcal{L}}(\Pi^\gamma(x_s), \Pi^\gamma(y_t)) \\
= & \quad d_{\mathcal{L}}((\pi_s(x), s), (\pi_t(y), t)) \\
\leq & \quad d_{\mathcal{L}}((\pi_s(x), s), (\pi_s(x), t)) + d_{\mathcal{L}}((\pi_s(x), t), (\pi_t(x), t)) + d_{\mathcal{L}}((\pi_t(x), t), (\pi_t(y), t)) \\
\leq & \quad |s - t| + \iota_t^* d((\pi_s(x), t), (\pi_t(x), t)) + \iota_t^* d((\pi_t(x), t), (\pi_t(y), t)) \\
\leq & \quad 1 + C + \iota_t^* d(x_s, y_t).
\end{align*}
\]

Equality in (26) is the definition of \( \Pi^\gamma \), and (27) is the triangle inequality. In (28), we use the fact that segments of the form \((x, I)\) for any connected interval \( I \subset \mathbb{R} \) are convex and have length \( |I| \), and the fact that \( d_{\mathcal{L}} \leq \iota_t^* d \) restricted to \( S_t \cap \mathcal{L} \). The Teichmüller map \( S_s \rightarrow S_t \) is \( \text{e}^{s-t} \)-bi-Lipschitz, hence in particular an \((\varepsilon, 0)\)-quasi-isometry, because \( |s - t| \leq d_{\mathfrak{M}}(x_s, y_t) \leq 1 \). Since ‘quasi-isometries almost commute with nearest point projections’ in a \( \delta \)-hyperbolic metric space, for example, \cite[Proposition 11.107; 32, Lemma 2.5]{12}, \( \iota_t^* d((\pi_s(x), t), (\pi_t(x), t)) \leq C = C(\delta, \varepsilon) \), which is the constant appearing in (29). We have also used that \( \pi_t : S \rightarrow \gamma \) is 1-Lipschitz in (29).

We assumed that \( d_{\mathfrak{M}}(x_s, y_t) \leq 1 \) so

\[
\begin{align*}
& d_{\mathfrak{M}}(y_t, x_t) \leq d_{\mathfrak{M}}(y_t, x_s) + d_{\mathfrak{M}}(x_s, x_t) \leq 1 + |s - t| \leq 2.
\end{align*}
\]

Using Lemma 6.1, we obtain \( \iota_t^* d(x_s, y_t) \leq 2e^{s/2} = 2e \). Taking \( L_0 = 1 + C + 2e \) completes the proof that \( \Pi^\gamma \) is an \((L_0, L_0)\)-coarse-Lipschitz retraction. The proof of the corresponding statements for \( \mathcal{L}(\gamma) \subset \mathfrak{M} \) are analogous.

Let \( \alpha : [0, a) \rightarrow \tilde{S}_0 \) be a saddle connection, that is, \( \alpha(0), \alpha(a) \in \Sigma, \alpha(t) \notin \Sigma \) for all \( t \in (0, a) \), and \( \alpha \) is a \( d_0 \)-geodesic segment parameterized by arc-length (so \( a \) is the \( d_0 \) length of \( \alpha \)). Then \( \alpha \) is a Euclidean segment, and its slope \( s(\alpha) \) is the ratio of the transverse measures of \( \alpha \) in the vertical and horizontal directions when those are non-zero; in other words, \( \frac{\int_\alpha \, dy}{\int_\alpha \, dx} = |s(\alpha)| \). If \( \int_\alpha \, dy = 0 \), then \( \alpha \) is an expanding saddle connection. If \( \int_\alpha \, dx = 0 \), then \( \alpha \) is a contracting saddle connection. When \( \alpha \) is neither an expanding nor a contracting saddle connection, the pullback of the model metric \( ds^2 \) by inclusion of \( \mathcal{L}(\alpha) \hookrightarrow \mathfrak{M} \) has the form

\[
e^{2t}dx^2 + e^{-2t}s(\alpha)^2 dx^2 + dt^2,
\]

and the value of \( t \) where \( t \mapsto e^{2t} + e^{-2t}s(\alpha)^2 \) is minimized is given by \( b(\alpha) := \frac{1}{2} \log(|s(\alpha)|) \). Call \( b(\alpha) \) the balance time of \( \alpha \); thus we see that the function \( t \mapsto \ell_t(\alpha) \) is convex, and in fact takes the form \( t \mapsto \ell_{b(\alpha)}(\alpha) \cosh(2(t - b(\alpha))) \); \( \ell_{b(\alpha)}(\alpha) \) is the unique minimum value.

For any non-expanding and non-contracting saddle connection \( \alpha \) whose length is at most 1 at its balance time, define two numbers \( b_{\pm}(\alpha) \in \mathbb{R} \) by the rule \( \ell_{b_{\pm}(\alpha)}(\alpha) = 1 \): the two values of \( b_{\pm} \) are given explicitly by the function in the previous paragraph, and are defined such that \( b_- \leq b \leq b_+ \). If \( \alpha \) is expanding, the length of \( \alpha \) in level surfaces is minimized at \( b(\alpha) := -\infty \).
Figure 6.1 (colour online). On the left, \( \ell_{b(x)}(\alpha) < 1 \). The VH-crossings between some points, \( a, \ldots, f \in \partial \mathcal{L}(\alpha) \) are depicted in red. On the right, \( \beta \) is a contracting saddle connection, and \( i_\beta \) maps \( \mathcal{L}(\beta) \) to \( S(0) \subset \mathbb{U} \).

while if \( \alpha \) contracting, it’s length in level surfaces is infimized at \( b(\alpha) := +\infty \). Note however that if \( \alpha \) is expanding, then \( \ell_{\log a}(\alpha) = 1 \); set \( b_+(\alpha) = \log a \) and \( b_-(\alpha) = b(\alpha) = -\infty \). If \( \alpha \) is contracting, then \( \ell_{-\log a}(\alpha) = 1 \); set \( b_-(\alpha) = -\log a \) and \( b_+(\alpha) = b(\alpha) = +\infty \). If \( \alpha \) is not short at its balance time, set \( b_+(\alpha) = b_-(\alpha) \). See Figure 6.1 for a schematic of the definitions of \( b_\pm(\alpha) \) and \( b(\alpha) \).

Say that \( \beta : [0, a] \to \mathcal{L}(\gamma) \) is a horizontal segment if \( \text{im} \beta \subset S_t \) for some \( t \in \mathbb{R} \). Say that \( \beta : [0, a] \to \mathcal{L}(\gamma) \) is a vertical segment if \( \text{im} \beta \subset \{x\} \times \mathbb{R} \) for some \( x \in S_0 \). Say that \( \gamma \) is a VH-path if it decomposes as a concatenation of subpaths \( \gamma = \beta_1 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \beta_{n+1} \) that alternate between vertical and horizontal segments, beginning and ending in vertical segments (that may have length 0).

Notation 6.4. If \( \alpha \subset \tilde{S}_0 \) is a saddle connection and \( h \in \mathbb{R} \), then by abuse of notation \( \alpha_h \) denotes both the geodesic map \( [0, \ell_h(\alpha)] \to \tilde{S}_h, t \mapsto \alpha(t) \times \{h\} \) or \( t \mapsto \alpha(\ell_h(\alpha) - t) \times \{h\} \) and the image of that map \( \alpha \times \{h\} \). If \( I \) is an index set, \( \alpha_i \subset \tilde{S}_0 \) will refer to the \( i \)-th saddle connection in a set \( \{\alpha_i\}_{i \in I} \).

Definition 6.5. Let \( \alpha \subset \tilde{S}_0 \) be a saddle connection, and let \( x_s, y_t \) be points on the two different components of \( \partial \mathcal{L}(\alpha) \). Say that a VH-path \( \gamma = \beta_r \cdot \alpha_h \cdot \beta_l \) joining \( x_s \) and \( y_t \) is a VH-crossing, if \( \beta_r \) and \( \beta_l \) are geodesics parameterized by arc-length, \( \alpha_h \) is a \( d_h \)-geodesic segment, and

(i) if \( \ell_{b(x)}(\alpha) > 1 \), or equivalently, \( b(\alpha) = b_\pm(\alpha) \), then \( h = b(\alpha) \);
(ii) if \( b_-(\alpha) < b(\alpha) < b_+\alpha) \), then
   (a) if \( s \geq b_+(\alpha) \), then \( h = b_+(\alpha) \), and if \( s \leq b_-(\alpha) \), then \( h = b_-(\alpha) \);
   (b) if \( b_-(\alpha) \leq s \leq b_+(\alpha) \), then \( h = s \).

We aim to prove that VH-crossings are efficient, which is partly the content of Lemmas 6.6, 6.10 and 6.13. We now explain how to think about ladders as convex subsets of the hyperbolic plane bound by bi-infinite geodesics; VH-crossings mimic the behavior of geodesics in hyperbolic geometry that cross this convex set.
Let $U$ be the upper half plane and $\rho_U$ its Poincaré metric. For $\ell > 0$, take $S(\ell) = \{1 \leq |z| \leq e^\ell \} \cap U$ the strip of width $\ell$; by direct computation, the map

$$i_\alpha : (L(\alpha), d_L) \to (S(\ell b(\alpha)) \cap U, \rho_U)$$

$$(x, t) \mapsto e^x \tanh(t - b(\alpha)) + ie^x \text{sech}(t - b(\alpha))$$

is $\sqrt{2}$-bi-Lipschitz and takes $\alpha \times \{b(\alpha) - t\}$ to the boundary of the $t$-neighborhood of the unique common perpendicular $O$ to $\partial S(\ell b(\alpha))$. For $\ell = 0$, $S(0) = \{z \in U : 0 \leq \Re(z) \leq 1\}$. Let $a \in \{|z| = 1\} \cap S(\ell)$ and $b \in \{|z| = e^\ell\} \cap S(\ell)$, and

$$\beta' = [a, i] \cdot O \cdot [ie^\ell, b] \subset U,$$  \hspace{1cm} (30)

where $[x, y]$ denotes the $\rho_U$-geodesic segment joining $x$ to $y$ (Figure 6.2). It is a standard fact in hyperbolic geometry that $\beta'$ is a $(K(\ell), 0)$-quasi-geodesic segment, where $K$ can be chosen to depend continuously on $\ell$. By Theorem 2.1,

$$d^H_{\rho_U}(\beta', [a, b]) \leq n(\ell),$$  \hspace{1cm} (31)

where $n(\ell)$ is some decreasing continuous function of $\ell$ that goes to infinity as $\ell$ goes to $0$. The following is almost immediate from our description of $i_\alpha$ and from the observation in (31).

**Lemma 6.6.** Let $\alpha \subset \tilde{S}_0$ be a saddle connection. Let $a, b$ be two points on different components of $\partial L(\alpha)$, and $\beta$ be the $d_L$-geodesic segment joining them. Let $\beta'' = \beta_r \cdot \alpha_h \cdot \beta_l$ be the VH-crossing joining $a$ and $b$. There is a universal constant $N$ such that $d^H_{\rho_U}(\beta, \beta'') \leq N$.

**Proof.** Consider the case that $b(\alpha) = b_\pm(\alpha)$, that is, $\alpha$ is not too short at its balance time. Then, more explicitly, we have $h = b(\alpha)$ and

$$\beta'' = [a, \alpha_{b(\alpha)}(0)] \cdot \alpha_{b(\alpha)} \cdot [\alpha_{b(\alpha)}(\ell b(\alpha)) \cdot b].$$

Then $i_\alpha(\beta)$ is a $(\sqrt{2}, 0)$ quasi-geodesic, so by Theorem 2.1, there is a universal constant $M > 0$ such that $d^H_{\rho_U}(i_\alpha(\beta), [i_\alpha(a), i_\alpha(b)]) \leq M$. Defining $\beta'$ by (30), using (31), and because $\ell_{b(\alpha)}(\alpha) \geq 1$, $d^H_{\rho_U}(\beta', [i_\alpha(a), i_\alpha(b)]) \leq n(1)$. The explicit description of $\beta''$ yields $i_\alpha(\beta'') = \beta'$.  

**Figure 6.2** (colour online). *The ladder $L(\alpha)$ is $\sqrt{2}$-bi-Lipschitz equivalent to a strip in the hyperbolic plane.*
Applying $i^{-1}_\alpha$ to $\beta'$ and collecting constants, we see that $N = \sqrt{2}(M + n(1))$ is sufficient, in this case.

The case that $b(\alpha) \neq \pm (\alpha)$ is modeled on the previous paragraph. However, we have to consider the two relevant alternatives (a) and (b) in Definition 6.9 to construct $\beta''$. The point is that VH-geodesics map to quasi-geodesics $\beta' = i_\alpha(\beta'')$ in the corresponding convex subsets of $U$ bounded by bi-infinite geodesics.

**Definition 6.7.** Let $\gamma \subset \tilde{S}_0$ be a concatenation of saddle connections $\alpha_1 \cdot \ldots \cdot \alpha_n$. A concatenation $\gamma^{VH} = \beta_1 \cdot \ldots \cdot \beta_n$ of VH-crossings $\beta_i = \beta_{i,n} \cdot \alpha_{h_i} \cdot \beta_{r_i}$ is a VH-geodesic tracking $\gamma$ if for $i \neq n$, $\beta'_{r,i}$ is a point and $\gamma^{VH}$ joins $\partial \gamma$.

It is not hard to see that there is only one VH-geodesic $\gamma^{VH}$ tracking $\gamma$.

**Lemma 6.8.** Let $\gamma \subset \tilde{S}_0$ be a concatenation of saddle connections $\alpha_1 \cdot \ldots \cdot \alpha_n$, let $\gamma^{VH}$ the VH-geodesic tracking $\gamma$, and let $\gamma^{VH}_\gamma$ be the $(\mathcal{L},d_{\mathcal{L}})$-geodesic joining $\partial \gamma$. Then $d_H(\gamma^{VH};\gamma^{VH}_\gamma) \leq N$.

**Proof.** The ladder $\mathcal{L}(\gamma) = \mathcal{L}(\alpha_i) \cup \cdots \cup \mathcal{L}(\alpha_n)$ is $\sqrt{2}$-bi-Lipschitz equivalent to a convex region in the hyperbolic plane bounded by bi-infinite geodesics obtained by gluing together translates by hyperbolic isometries of the maps $i_{\alpha_i}$ along their common boundaries. It is well known that the image of $\gamma^{VH}$ mimics the behavior of the geodesic segment in $U$ joining the image of $\partial \gamma$. In particular, Definition 6.7 guarantees that $\gamma^{VH}$ does not backtrack. Collecting constants as in the proof of Lemma 6.6 yields the lemma.

We would now like to understand geodesics in $\mathcal{L}_+(\alpha)$, as in Lemma 6.6. There are two cases: either $\alpha$ is balanced in $\mathfrak{N}$, that is, $b(\alpha) \geq 0$, or else $b(\alpha) < 0$ and $\alpha$ is balanced in $\mathfrak{N} \setminus \mathfrak{N}$.

**Definition 6.9.** Let $\alpha \subset \tilde{S}_0$ be a saddle connection, and let $x_s,y_t$ be points on the two different components of $\partial \mathcal{L}_+(\alpha)$, so that $s,t \geq 0$. Say that a VH-path $\gamma = \beta_r \cdot \alpha_b \cdot \beta_l$ joining $x_s$ and $y_t$ is a VH$_+$-crossing, if $\beta_r$ and $\beta_l$ are geodesics parameterized by arc-length, $\alpha_b$ is a $d_{\mathcal{L}}$-geodesic segment and

(i) if $b_+ (\alpha) \geq 0$, then $\gamma$ is a VH-geodesic;

(ii) if $b_+ (\alpha) < 0$, then $h = 0$.

Given $\hat{b} \in \mathbb{R}$, let $\Theta(\hat{b}) = \{z : 1 > \cos(\arg(z)) \geq \tanh(-\hat{b})\} \subset \mathbb{H}$, so that $\partial \Theta(\hat{b})$ contains a component of $\partial \mathcal{L}_+(\alpha)$; if $\hat{b} \geq 0$, then $\Theta(\hat{b}) \supset \mathcal{O}$, while $\Theta(\hat{b}) \cap \mathcal{O} = \emptyset$, otherwise. Since $i_{\alpha}$ maps $\alpha_t = (\alpha,t)$ to the boundary of the $t$-neighborhood of $\mathcal{O}$, we see that

$$i_{\alpha}(\mathcal{L}_+(\alpha)) : \mathcal{L}_+(\alpha) \to \mathcal{S}(\ell_{b(\alpha)}(\alpha)) \cap \Theta(b(\alpha))$$

is a $\sqrt{2}$-bi-Lipschitz diffeomorphism taking $\tilde{S}_0 \cap \mathcal{L}_+(\alpha)$ into $\partial \Theta(b(\alpha))$.

**Lemma 6.10.** Suppose $\alpha$ is a saddle connection such that $b_+ (\alpha) \geq 0$, and let $a,b$ be points on different components of $\partial \mathcal{L}_+(\alpha)$. Let $\beta$ be the $(\mathcal{L},d_{\mathcal{L}})$-geodesic joining $a$ and $b$, and let $\beta''$ be a $(\mathcal{L}_+,d_{\mathcal{L}})$-geodesic joining $a$ and $b$. There is a constant $N' = N'(e,S)$ such that $d_H(\beta,\beta'') \leq N'$.

Let $\beta''$ be the VH$_+$-crossing joining $a$ and $b$. Then $d_H(\beta'',\beta) \leq N'$, as well.

**Proof.** The VH-crossing and VH$_+$-crossing $\beta''$ joining $a$ and $b$ coincide. Then $i_{\alpha}(\beta'') \subset \mathcal{S}(\ell_{b(\alpha)}(\alpha)) \cap \Theta(b(\alpha))$ is quasi-geodesic in both $\mathcal{S}(\ell_{b(\alpha)}(\alpha)) \cap \Theta(b(\alpha))$ and $\mathcal{S}(\ell_{b(\alpha)}(\alpha))$. The geodesic joining $i_{\alpha}(a)$ and $i_{\alpha}(b)$ in $\mathcal{S}(\ell_{b(\alpha)}(\alpha)) \cap \Theta(b(\alpha))$ has bounded Hausdorff distance from the geodesic $[i_{\alpha}(a),i_{\alpha}(b)] \subset \mathcal{S}(\ell_{b(\alpha)}(\alpha))$. Mapping these geodesics back to $\mathcal{L}$ via $i^{-1}_\alpha$, we again
Figure 6.3 (colour online). On the left is $\gamma^VH \subset \mathcal{L}(\gamma)$ and $\gamma^VH_+ \subset \mathcal{L}_+(\gamma)$; on the right is the corresponding $\sqrt{2}$-bi-Lipschitz convex subset of $\mathbb{H}^2$. The partition from the proof of Theorem 6.2 is $I_1 = [1, 5]$, $I_2 = [6, 8]$, and $I_3 = [9]$.

obtain $(\sqrt{2}, 0)$-quasi-geodesics. The conclusion follows from the proof of Lemma 6.6 and an application of the triangle inequality. □

Definition 6.11. Let $\gamma \subset \tilde{S}_0$ be a concatenation of saddle connections $\alpha_1 \cdot \ldots \cdot \alpha_n$. A concatenation $\gamma^VH = \beta_1 \cdot \ldots \cdot \beta_n$ of VH-crossings $\beta_i = \beta_{i_1} \cdot \alpha_{i_2} \cdot \beta_{i_3}$ is a VH-geodesic tracking $\gamma$ if for $i \neq n$, $\beta_{r,i}$ is a point and $\gamma^VH$ joins $\partial \gamma$.

Again, $\gamma^VH$ is uniquely determined by these conditions and $\gamma$. The following is immediate from Definition 6.11 and Lemma 6.10.

Lemma 6.12. Suppose that $\gamma \subset \tilde{S}_0$ is a concatenation of saddle connections $\alpha_1 \cdot \ldots \cdot \alpha_n$, $b_+(\alpha_i) > 0$ for each $i$, and $\gamma^VH = \beta_1 \cdot \ldots \cdot \beta_n$ is the VH-geodesic tracking $\gamma$. Then $\gamma^VH = \gamma^VH_+ \subset \mathcal{L}_+$ is a uniform $(\mathcal{L}, d_{\mathcal{L}})$-quasi-geodesic and a uniform $(\mathcal{L}_+, d_{\mathcal{L}_+})$-quasi-geodesic. In particular, if $\gamma^VH_+ \subset \mathcal{L}_+$ is a uniform $(\mathcal{L}, d_{\mathcal{L}})$-quasi-geodesic and a uniform $(\mathcal{L}_+, d_{\mathcal{L}_+})$-quasi-geodesic joining $\partial \gamma$, respectively, then $d_{\mathcal{L}}(\gamma^VH, \gamma^VH_+) \leq N'$ and $d_{\mathcal{L}_+}(\gamma^VH, \gamma^VH_+) \leq N'$ (Figure 6.3).

Now we model the situation that the saddle connection $\alpha$ is balanced in $\mathcal{M} \setminus \tilde{\mathcal{N}}$. If $b < 0$, let $\mathcal{O}' = \partial \Theta(b) \cap \tilde{\mathcal{S}}(\ell)$; $\mathcal{O}'$ has $\partial \mathcal{O}' = \{p, q\} \subset \partial \tilde{\mathcal{S}}(\ell)$. Take

$$\beta'_+ = [a, p] \cdot \mathcal{O}' \cdot [q, b] \subset \mathbb{U}. \quad (32)$$

Lemma 6.13. Let $\alpha \subset S_0$ be a saddle connection such that $b_+(\alpha) < 0$, let $a, b$ be two points on different components of $\partial \mathcal{L}_+(\alpha)$, and let $\beta_+$ be a $(\mathcal{L}_+, d_{\mathcal{L}_+})$-geodesic segment joining them. Swapping $a$ and $b$ if necessary, let $\beta''_+$ be the VH$_+$-crossing

$$\beta''_+ = [a, \alpha_0(0)] \cdot \alpha_0 \cdot [\alpha_0(\ell_0(\alpha)), b].$$

There is a universal constant $N''$ such that $d^H_{\mathcal{L}}(\beta'_+, \beta''_+) \leq N''$. 
Proof. This is directly analogous to the proof of Lemma 6.6 when \( \alpha \) is not short at balance time. Define \( \beta'_n \) by (32). The key point is that \( \beta'_n = i_n(\beta''_n) \), and \( \beta'_n \) is uniformly quasi-geodesic in the path metric on \( \Theta(b(\alpha)) \cap S(b(\alpha)) \) (which is no longer convex in \( \mathbb{U} \)). This is independent of whether \( \alpha \) is short at balance time, because \( b_+(\alpha) < 0 \) by assumption. \( \square \)

We are now in a position to prove the main result from this section.

Proof of Theorem 6.2. By our choice of \( \varphi \in \Sigma \) and fixed lift \( \bar{\varphi} \), the lift \( \gamma \subset \widetilde{S}_0 \) based at \( \bar{\varphi} \) of any \( d_0 \)-geodesic loop \( \gamma \subset S_0 \) based at \( \varphi \) decomposes as a concatenation of saddle connections, that is, \( \gamma = \alpha_1 \cdots \alpha_n \subset \widetilde{S}_0 \). Now, build the ladders \( L_+(\gamma) \subset \mathcal{L}(\gamma) \subset \mathbb{R} \). The ladders decompose as the union \( \mathcal{L}(\gamma) = \mathcal{L}(\alpha_1) \cup \cdots \cup \mathcal{L}(\alpha_n) \). Let \( \gamma^*_L \) be a \((\mathcal{L}, d_L)\)-geodesic and \( \gamma^*_L \) a \((\mathcal{L}_+, d_L)\)-geodesic.

For integers \( i \leq j \), let \( \lbrack i, j \rbrack = [i, j] \cap \mathbb{N} \). Starting with \( i_1 = 1 \), we iteratively construct a partition \( 1 = i_1 < t_1 < t_2 < \cdots < i_k \leq t_k = n \) so that the subintervals \( I_j = \lbrack i_j, t_j \rbrack \) satisfy the following three properties.

1. If \( b_+(\alpha_{i_j}) \geq 0 \), then \( b_+(\alpha_m) \geq 0 \) for each \( m \in I_j \), and either \( t_j = n \) or \( b_+(\alpha_{i_{j+1}}) < 0 \).
2. If \( b_+(\alpha_{i_j}) < 0 \), then \( b_-(\alpha_m) < 0 \) for each \( m \in I_j \) and either \( t_j = n \) or \( b_-(\alpha_{i_{j+1}}) \geq 0 \).
3. For \( j \geq 1 \), \( i_{j+1} = t_j + 1 \).

Let \( \alpha_{I_j} = \alpha_{i_j} \cdots \alpha_{t_j} \) or \( \alpha_{I_j} = \alpha_{j} \) if \( i_j = t_j \), and set \( \mathcal{L}(\alpha_{I_j}) = \mathcal{L}(\alpha_{i_j}) \cup \cdots \cup \mathcal{L}(\alpha_{t_j}) \) and \( \mathcal{L}_+(\alpha_{I_j}) = \mathcal{L}(\alpha_{i_j}) \cap \mathcal{L}_+(\gamma) \).

Claim 6.14. There exists a number \( D = D(\epsilon, S) \) such that for each \( j = 1, \ldots, k \), we have

\[
\gamma^*_L \cap \mathcal{L}(\alpha_{I_j}) \subset N_D(\gamma^*_L \cap \mathcal{L}(\alpha_{I_j})) \cup \alpha_{I_j}.
\]

That is, restricted to each subladder \( \mathcal{L}(\alpha_{I_j}) \), \( \gamma^*_L \) is close to \( \gamma^*_L \) or \( \alpha_{I_j} \subset \widetilde{S}_0 \). Moreover, \( \gamma^*_L \cap \mathcal{L}_+(\alpha_{I_j}) \subset N_D(\gamma^*_L) \).

We perform an inductive argument on the number \( k \) of subintervals in this partition. We will prove the claim on the ladder \( \mathcal{L}(\alpha_{I_1}) \) induced by the first interval and show that we may repeat the argument on \( d_L \)-geodesics joining \( \partial(\alpha_{t_1+1} \cdots \alpha_n) \). A reformulation of the claim says that \( d_L \)-geodesics satisfy the conclusion of Theorem 6.2. Since \( d_L \)-geodesics approximate \( d_{\mathbb{R}} \) and \( d_{\mathbb{R}} \) geodesics within bounded Hausdorff distance by Proposition 6.3, after we prove Claim 6.14 and collect constants, the theorem will be proved. In what follows, ‘uniformly close’, or ‘uniformly bounded distance’, etc. means that there is a constant depending at most on \( S \) and \( \epsilon \), such that the objects mentioned are distance within that constant of each other.

Proof of Claim 6.14. Suppose that \( I_1 \) satisfies (1). Then either \( t_1 = n \) or \( b_+(\alpha_{t_1+1}) < 0 \). If \( t_1 = n \), then by Lemma 6.12, \( d_H(\gamma^*_L, \gamma^*_L) \leq 2N' \). If \( b_+(\alpha_{t_1+1}) < 0 \), then VH-geodesic \( \gamma^V_H \) tracking \( \gamma \) contains the vertical segment \( \mathcal{L}(\alpha_{t_1}) \cap \mathcal{L}(\alpha_{t_1+1}) \) joining \( 0 < b_+(\alpha_{t_1}) \) to \( b_+(\alpha_{t_1+1}) < 0 \), so \( \gamma^V_H \) intersects the terminal endpoint \( p_{t_1} \) of \( \alpha_{t_1} \). Then by Lemma 6.8, there is a point \( q_{t_1} \in \gamma^*_L \) passing uniformly close to \( p_{t_1} \). Since \( b_+(\alpha_{t_1+1}) < 0 \), each component of \( \partial \mathcal{L}_+(\alpha_{t_1+1}) \) is union of \( 1 \)-separated convex sets in a \( \delta \)-hyperbolic space. The points \( \partial \alpha_{t_1+1} \) (nearly) achieve the distance between the two components of \( \partial \mathcal{L}_+(\alpha_{t_1+1}) \) by our description of \( \mathcal{L}_+(\alpha_{t_1+1}) \) in Lemma 6.13, so a point \( q_{t_1+1} \in \gamma^*_L \) comes uniformly close to \( \partial \alpha_{t_1+1} \), hence close to \( p_{t_1} \). By hyperbolicity of ladders, the \((\mathcal{L}, d_L)\)-geodesic and \((\mathcal{L}_+, d_L)\)-geodesic segments joining \( \partial(\alpha_{t_1+1} \cdots \alpha_n) \) are uniformly close to the \((\mathcal{L}, d_L)\)-geodesic and \((\mathcal{L}_+, d_L)\) segments joining \( \gamma(0) \) with \( q_{t_1} \) and \( q_{t_1+1} \), respectively. Another application of Lemma 6.12 in this case therefore proves the theorem on \( \mathcal{L}(\alpha_{I_1}) \) when \( I_1 \) satisfies (1).
We now consider the case that $I_1$ satisfies (2). We will see that the case that $t_1 = n$ follows similarly as in the case that $t_1 \neq n$ and $b_-(\alpha_{t_i+1}) \geq 0$. The VH$_+$-geodesic $\gamma^{VH}_{-}$ that tracks $\gamma$ on $\alpha_{t_i}$ is equal to $\alpha_{t_i}$, by construction in Definition 6.11. In the spirit of Lemma 6.12, we may apply Lemma 6.13 to concatenations of saddle connections to see that $\gamma^{VH}_{-}$ is within bounded Hausdorff distance of $\gamma^{L}_{-}$ on $L(\alpha_{t_i})$. By the construction of Definition 6.9 and the conditions of (2), the VH-geodesic $\gamma^{VH}$ that tracks $\gamma$ meets $L_{+}(\alpha_{t_i})$ only in its initial and terminal vertical segments. The initial segment meets $L_{+}(\alpha_{t_i})$ exactly at $\alpha_1(0)$ and the terminal vertical segment of $\gamma^{VH}$ meets $L_{+}(\alpha_{t_i}) \cap L(\alpha_{t_i+1})$ in a segment containing $[0, b_-(\alpha_{t_i+1})]$; in particular $L_{+}(\alpha_{t_i}) \cap \gamma^{VH} = \partial \alpha_{t_i}$. By Lemma 6.8, we can now conclude that $\gamma^{L}_{-}$ is within uniformly bounded Hausdorff distance from this VH-geodesic on $L(\alpha_{t_i})$. Thus, $\gamma^{L}_{-}$ can only be far from $\gamma^{L}_{+}$ in $L(\alpha_{t_i})$ when $I_1$ satisfies (2) and $\gamma^{L}_{+} \subset L(\alpha_{t_i}) \setminus L_{+}(\alpha_{t_i})$.

In both cases, since $\gamma^{L}_{-}$ and $\gamma^{L}_{+}$ both come close to the initial point of $\alpha_{t_i+1}$, we can now replace $\gamma$ with $\alpha_{t_i+1} \ldots \alpha_n$ and repeat the argument, as promised.

This concludes the proof of the theorem.

7. Zeroing out half the manifold

Let us recall the notation and set up from Section 5. We have $\nu = (\lambda_-, \lambda_+)$ where $\lambda_{\pm}$ have bounded geometry, and $Y_0 \in \mathcal{T}(S)$ was a point on the Teichmüller geodesic joining $\lambda_-$ and $\lambda_+$. We defined $M_{-} = M_{(\lambda_-, Y_0)}$ and constructed a map $\Phi_{-} : \text{core}(M_{-}) \rightarrow M_{\nu}$ inducing $\rho(\nu) \circ \rho(\lambda_{-}, Y_0)^{-1}$ on fundamental groups, that was a $B$-bi-Lipschitz embedding, and that was volume preserving away from a compact product region $K$. Take $\text{inj}(M_{\nu}) = \epsilon > 0$. First we would like to use our results from Section 6.

**Lemma 7.1.** There exists a constant $D' > 0$ depending only on $\epsilon, S$, and $B$ such that the following holds. Let $\gamma : I \rightarrow M_{-}$ be a geodesic loop based at $p \in \text{core}(M_{-})$ and $\tilde{\gamma} : I \rightarrow \tilde{M}_{-}$ be a choice of lift. Let $\Pi : \tilde{M}_{\nu} \rightarrow \text{im } \tilde{\Phi}_{-}(\tilde{\gamma})$ be the nearest point projection.

If $d(\tilde{\Phi}_{-}(\tilde{\gamma}(t)), \Pi(\tilde{\Phi}_{-}(\tilde{\gamma}(t)))) > D'$, then $\Pi(\tilde{\Phi}_{-}(\tilde{\gamma}(t))) \in N_{D'}(\tilde{M}_{\nu} \setminus \tilde{\Phi}_{-} (\text{core}(M_{-})))$.

**Proof.** Recall that $\tilde{\Psi}_{\nu} : \tilde{\mathcal{M}}_{\nu} \rightarrow \tilde{M}_{\nu}$ is a quasi-isometry by Theorem 2.4. So given a geodesic segment $\tilde{\gamma}' \subset \tilde{\mathcal{M}}_{\nu}$, we have that $\tilde{\Psi}_{\nu}(\tilde{\gamma}')$ is Hausdorff-close to $\tilde{\Psi}_{\nu}(\tilde{\gamma}')^*$, by the Morse lemma (Theorem 2.1). So statements that are true about geodesics in the model manifold $\tilde{\mathcal{M}}_{\nu}$ are true (within bounded Hausdorff distance) in $\tilde{M}_{\nu}$, via $\tilde{\Psi}_{\nu}$. Analogously, statements about geodesics in core($M_{-}$) are approximated by statements about geodesics in $\mathcal{M}$ via $\Psi_{\nu} : \mathcal{M} \rightarrow \text{core}(M_{-})$.

This means that the conclusion of the lemma will be immediate from Theorem 6.2 once we establish that $\Phi_{-} \circ \Psi_{0} : \mathcal{M} \rightarrow M_{\nu}$ and $\Psi_{\nu} \circ \Phi_{-} : \mathcal{M} \rightarrow M_{\nu}$ are good enough approximations of each other. That is, the conclusion of the lemma is immediate once we show that

$$\sup_{x \in \mathcal{M}} d_{M_{\nu}}(\Phi_{-}(\Psi_{0}(x)), \Psi_{\nu}(x)) < \infty. \quad (33)$$

To this end, for a negative integer $n$ and level surface $S_{n} \subset \mathcal{M}$, the argument given in the paragraph directly preceding (25) in the proof of Proposition 5.1 shows that there is a constant $E = E(S, \epsilon, B)$ such that

$$d_{M_{-}}^{H}(\Psi_{\nu}(S_{n}), \Phi_{-}(\Psi_{0}(S_{n}))) \leq E.$$ 

Now, every $x \in \mathcal{M}$ is within $1/2$ of some level surface $S_{n}$, and $\tilde{\Psi}_{\nu}$ and $\tilde{\Psi}_{0}$ are $(L, c)$ quasi-isometries, so $d_{M_{-}}(\Psi_{0}(S_{n}), \Psi_{0}(x)) \leq L/2 + c$ and $d_{M_{-}}(\Psi_{\nu}(S_{n}), \Psi_{\nu}(x)) \leq L/2 + c$. 

Finally, \( \Phi_- \) is \( B \)-bi-Lipschitz, and the diameter of an \( \epsilon \)-thick hyperbolic surface is at most \( D = D(\epsilon, S) \), so
\[
d_{M_1}(\Phi_-(\Psi_0(x)), \Psi_0(x)) \leq \left( \frac{L}{2} + c \right)(B + 1) + D + E < \infty,
\]
which establishes (33) and completes the proof of the lemma. \( \square \)

Let \( K' = K \cup N_{B,D}(\partial \text{core}(M_-)) \), and let \( g_- : \text{core}(M_-) \to \mathbb{R} \) be a bump function taking values 1 away from \( N_1(K') \) and 0 on \( K' \). Take
\[
f_-(x) = \begin{cases} 
g_-\left( \Phi_-(x) \right), & x \in \text{im} \Phi_-
l, & \text{else}. \end{cases}
\]

**Remark 7.2.** We have constructed \( f_- \) so that \( \text{supp}(f_-) \) is contained in the complement of \( N_{B,D}(M_v \setminus \Phi_-(\text{core}(M_-))) \).

The following is one of the main technical ingredients that will go into the proof of Theorem 1.2 in the next section. We want to show that by ‘zeroing out’ an end of the doubly degenerate manifold with \( f_- \), the bounded fundamental class of the singly degenerate manifold with positive \( f_- \)-volume survives. We write \( [\tilde{\omega}_-] \) to denote the bounded fundamental class of \( M_- \) and \( [\tilde{\omega}] \) for that of \( M_v \).

**Proposition 7.3.** With notation as above, we have an equality
\[
[\tilde{\omega}_-] = [\Phi^* \tilde{f}_- \omega] \in H^3_0(M_-; \mathbb{R})
\]
This equality takes the form
\[
[\tilde{\omega}_-] = [\tilde{f}_- \omega] \in H^3_0(S; \mathbb{R})
\]
when we suppress markings, that is, pass the first equality through the isometric isomorphism \( H^3_0(M_-; \mathbb{R}) \to H^3_0(S; \mathbb{R}) \).

With \( f_+ \) and \( M_+ = M_{(Y_0, \lambda_+)} \) defined analogously,
\[
[\tilde{\omega}_+] = [\tilde{f}_+ \omega] \in H^3_0(S; \mathbb{R}).
\]

**Proof.** By Lemma 3.8, \( [\tilde{\omega}_-] \) is represented by the cocycle \( \tilde{\omega}_{-,p} \), where \( p \in \text{core}(M_-) \). By Proposition 5.2, \( [\tilde{\omega}_{-,p}] \) is represented by the cocycle \( g_- \tilde{\omega}_{-,p} \), because \( 1 - g_- \) has compact support on \( \text{core}(M_-) \).

**Assumption 7.4.** Assume that \( \sigma : \Delta_3 \to M_- \) maps the vertices of \( \Delta_3 \) to \( p \). Then \( \text{str}_p \sigma = \text{str} \sigma \), and we drop the subscript \( p \) from notation.

Restricting to \( \text{im} \Phi_- \) and appealing to the definition of \( f_- \) in (34), we compute
\[
\Phi_-^{-1}*(g_- \omega_-) = f_- \cdot \text{Jac}(\Phi_-^{-1}) \cdot \omega = f_- \omega
\]
because \( \Phi_-^{-1} \) is orientation and volume preserving on the support of \( f_- \) by Proposition 5.1.

We therefore have
\[
(g_- \omega_- - \Phi^* \tilde{f}_- \omega)(\sigma) = \int_{\text{str} \sigma} g_- \omega_- - \int_{\text{str} \Phi_- \sigma} f_- \omega
\]
\[
= \int_{\Phi_- \text{str} \sigma} \Phi_-^{-1}*(g_- \omega_-) - \int_{\text{str} \Phi_- \sigma} f_- \omega
\]
\[
= \int_{\Phi_- \text{str} \sigma - \text{str} \Phi_- \sigma} f_- \omega.
\]
Figure 7.1 (colour online). Trajectories of \(H(\tau)\) have bounded length when intersected with the support of \(f_-\), even though they are generally unbounded in \(M_\nu\).

We define \(H(\tau)\) exactly as in the discussion preceding Lemma 3.4, and take

\[
C_{f_-}(\tau) = \int_{\Delta_2 \times I} H(\tau)^*(f_- \omega).
\]

Apply the proof of Lemma 3.4, using the equality (37) to see that

\[
(\hat{g} - \omega - \Phi^- \hat{f} - \omega)(\sigma) = dC_{f_-}(\sigma).
\]

As in Assumption 7.4, assume \(\tau\) maps the vertices of \(\Delta_2\) to \(p\). We want to show that

\[
|C_{f_-}(\tau)| < M, \text{ for some } M.
\]

We mimic the proof of Proposition 3.5. The proof had two main ingredients, namely (1) that trajectories \(H(\tau)(x, I)\) had uniformly bounded length and (2) that the area of the level triangle \(H(\tau)(\Delta_2, t)\) was uniformly bounded. We proceed exactly as in the proof of Lemma 3.5, except we need to replace the bound on trajectories with something that makes more sense. We have

\[
\left| \int_{\Delta_2 \times I} H(\tau)^* f_- \omega \right| = \left| \int_0^1 \int_{\Delta_t} f_- \left( \frac{\partial H(\tau)}{\partial t}, \vec{n}_t \right) dA_t \ dt \right| 
\leq \int_0^1 \int_{\Delta_t} \sqrt{f_- \left( \frac{\partial H(\tau)}{\partial t}, \vec{n}_t \right)} \sqrt{f_-} dA_t \ dt.
\]

The geodesic triangle \(\text{str } \Phi^*_- \tau\) tracks the B-bi-Lipschitz triangle \(\Phi^*_{-}, \text{str } \tau\) on \(\text{supp}(f_-)\) (see Figure 7.1). More precisely, by Lemma 7.1 and by construction of \(f_-\) (see Remark 7.2), there is a \(D'\) such that if \(p \in [i, j] \subset \Delta_2\) and \(q = \Phi^*_{-}, \text{str } \tau(p) \in \text{supp}(f_-)\), then

\[
d_{M^-}(\Pi(q), q) \leq D',
\]

where \(\Pi : \mathbb{H}^3 \rightarrow \text{str } \Phi^*_- \tau([i, j])\) is closest point projection. Take \(c := (2B \log \sqrt{3} + 2D' + 1)\); then

\[
\left| \sqrt{f_- \left( \frac{\partial H(\tau)}{\partial t}, \vec{n}_t \right)} \right| \leq 2c.
\]

Compare (40) with (9)–(14) in the proof of Proposition 3.5.

Recall that \(\Phi^-\) is B-Lipschitz. On \(\text{supp}(f_-)\), the discussion preceding estimate (18) bounding the area of any level triangle goes through as before

\[
\int_{\Delta_t} \sqrt{f_-} \ dA_t \leq \pi(Bc)^2, \ t \in [0, 1/2].
\]
Therefore, 
\[ |C_{f_{\omega}}(\tau)| = \left| \int_{\Delta_{2} \times I} H(\tau)^{*} f_{\omega} \right| \leq \pi B^2 c^3, \]

which completes the proof of the proposition. \(\square\)

8. The relations and a Banach subspace

Let \(\lambda, \lambda' \in \mathcal{E}L_b(S)\) and \(X, Y \in \mathcal{T}(S)\), then we have representations \(\rho(\nu_{-}, \nu_{+}) : \pi_1(S) \to PSL_2 \mathbb{C}\) where \(\nu_{\pm} \in \{\lambda, \lambda', X, Y\}\). Let \(\hat{\omega}(\nu_{-}, \nu_{+}) \in C^{3}_b(S; \mathbb{R})\) be the corresponding bounded 3-cocycle. We gather the results from the previous sections to conclude that

**Theorem 8.1.** Let \(\lambda, \lambda' \in \mathcal{E}L(S)\) have bounded geometry and \(X, Y \in \mathcal{T}(S)\) be arbitrary. We have an equality in bounded cohomology

\[ \hat{\omega}(\lambda', \lambda) = \hat{\omega}(X, \lambda) + \hat{\omega}(\lambda', Y) \in H^3_b(S; \mathbb{R}). \]

**Proof.** The constant function 1 on \(M(\lambda, \lambda')\) decomposes as a sum \(1 = f_{-} + f_{+}\) where \(f_{\pm} \geq 0\) and \(\{x \in M(\lambda, \lambda') : (f_{-} \cdot f_{+})(x) \neq 0\}\) is precompact. By linearity of the integral, \([f_{-}\hat{\omega}(\lambda', \lambda)] + [f_{+}\hat{\omega}(\lambda', \lambda)] = [\hat{\omega}(\lambda', \lambda)]\). Consider the sequence \(\{Y_t\}_{t \in \mathbb{Z}}\) along the Teichmüller geodesic \(t \mapsto Y_t\) between \(\lambda'\) and \(\lambda\). Applying Proposition 7.3,

\[ [f_{-}\hat{\omega}(\lambda', \lambda)] = [\hat{\omega}(\lambda', Y_{k_{-}})], \]

and

\[ [f_{+}\hat{\omega}(\lambda', \lambda)] = [\hat{\omega}(Y_{k_{+}}, \lambda)], \]

for some \(k_{-}, k_{+} \in \mathbb{Z}\). By Theorem 3.2, \([\hat{\omega}(Y_{k_{+}}, \lambda)] = [\hat{\omega}(X, \lambda)]\) and \([\hat{\omega}(\lambda', Y_{k_{-}})] = [\hat{\omega}(\lambda', Y)]\). Stringing together the equalities, we arrive at the claim. \(\square\)

**Corollary 8.2.** If \(\lambda_1, \lambda_2, \lambda_3 \in \mathcal{E}L_b(S)\) are distinct, then we have an equality in bounded cohomology

\[ \hat{\omega}(\lambda_1, \lambda_3) = \hat{\omega}(\lambda_1, \lambda_2) + \hat{\omega}(\lambda_2, \lambda_3) \in H^3_b(S; \mathbb{R}). \]

**Proof.** By Theorem 8.1 for any \(X \in \mathcal{T}(S)\),

\[ \hat{\omega}(\lambda_1, \lambda_2) + \hat{\omega}(\lambda_2, \lambda_3) = \hat{\omega}(\lambda_1, X) + \hat{\omega}(X, \lambda_2) + \hat{\omega}(\lambda_2, X) + \hat{\omega}(X, \lambda_3). \]

(42)

There is an orientation reversing isometry \(M_{(X, \lambda_2)} \to M_{(\lambda_2, X)}\) respecting markings, so \(\hat{\omega}(X, \lambda_2) = -\hat{\omega}(\lambda_2, X)\). The right-hand side of (42) becomes

\[ \hat{\omega}(\lambda_1, X) + \hat{\omega}(X, \lambda_3) = \hat{\omega}(\lambda_1, \lambda_3), \]

by another application of Theorem 8.1. \(\square\)

The following is an easy consequence of Theorem 1.4 (see [14, Theorems 6.2 and 7.7]).

**Theorem 8.3.** Suppose \(\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathcal{E}L(S)\) are distinct, and suppose \(\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{R}\) is such that \(\sum_{n=1}^{\infty} \alpha_n [\hat{\omega}(X, \lambda_n)] \in H^3_b(S; \mathbb{R})\). Then \(\sum_{n=1}^{\infty} \alpha_n [\hat{\omega}(X, \lambda_n)] = 0\) if and only if \(\alpha_n = 0\) for all \(n\).
Proof. The ‘if’ direction is trivial. For the ‘only if’ direction, we apply Theorem 1.4 to see that there is a number $\epsilon_S > 0$ depending only on $S$ such that any finite sum
\[ \left\| \sum_{n=1}^{N} \alpha_n [\omega(X, \lambda_n)] \right\|_{\infty} \geq \epsilon_S \max \{|\alpha_n|\}. \]

So, if $\alpha_{n_0} \neq 0$ for some $n_0$, then the infinite sum
\[ \left\| \sum_{n=1}^{\infty} \alpha_n [\omega(X, \lambda_n)] \right\|_{\infty} \geq \epsilon_S |\alpha_{n_0}| > 0. \]

Thus, $\sum_{n=1}^{\infty} \alpha_n [\omega(X, \lambda_n)] = 0$ only if $\alpha_n = 0$ for all $n$. \qed

Let $\mathcal{EL}_b(S) \subset \mathcal{EL}(S)$ be the set of bounded geometry ending laminations; $\mathcal{EL}_b(S)$ is a mapping class group invariant subspace. Fix a base point $X \in \mathcal{S}(S)$, and define $\iota : \mathcal{EL}_b(S) \rightarrow H^3_3(S; \mathbb{R})$ by the rule $\iota(\lambda) = [\omega(X, \lambda)]$. By Theorem 3.2, $\iota$ does not depend on the choice of $X$. By Theorem 8.1, $[\omega(X, \lambda)] = \iota(\lambda) - \iota(\lambda')$, so every doubly degenerate bounded fundamental class without parabolics is in the $\mathbb{R}$-linear span of $\im \iota$. Let $Z \subset H^3_3(S; \mathbb{R})$ be the subspace of zero-semi-norm elements. Then $\overline{\Pi}_b^3(S; \mathbb{R}) = H^3_3(S; \mathbb{R})/Z$ is a Banach space with $\| \cdot \|_{\infty}$ norm. Take $\overline{\iota}$ to be the composition of $\iota$ with the quotient $H^3_3(S; \mathbb{R}) \rightarrow \overline{\Pi}_b^3(S; \mathbb{R})$. By Theorem 8.3, $\im \overline{\iota}$ is a topological basis for the $\| \cdot \|_{\infty}$-closure of the span of its image $\mathcal{V} \subset \overline{\Pi}_b^3(S; \mathbb{R})$. By a topological basis, we mean that $\sum_{n=1}^{\infty} \alpha_n \overline{\iota}(\lambda_n) = 0$ only if $\alpha_n = 0$ for all $n$ when $\sum_{n=1}^{\infty} \alpha_n \overline{\iota}(\lambda_n) \in \overline{\Pi}_b^3(S; \mathbb{R})$. We have produced

**Corollary 8.4.** The Banach subspace $\mathcal{V} \subset \overline{\Pi}_b^3(S; \mathbb{R})$ has topological basis $\iota(\mathcal{EL}_b(S))$, $\iota$ is a mapping class group equivariant, and $\mathcal{V}$ contains all of the doubly degenerate bounded volume classes with bounded geometry.

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