Fractional Sobolev regularity for solutions to a strongly degenerate parabolic equation

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Abstract

We carry on the investigation started in [2] about the regularity of weak solutions to the strongly degenerate parabolic equation

$$u_t - \text{div} \left[ \left( (|Du| - 1)^+ \right)^{p-1} \frac{Du}{|Du|} \right] = f \quad \text{in } \Omega_T = \Omega \times (0, T),$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ for $n \geq 2$, $p \geq 2$ and $p'$ stands for the positive part. Here, we weaken the assumption on the right-hand side, by assuming that $f \in L^{p'}_\text{loc} (0, T; B^{\alpha}_{p', \infty, \text{loc}} (\Omega))$, with $\alpha \in (0, 1)$ and $p' = p/(p-1)$. This leads us to obtain higher fractional differentiability results for a function of the spatial gradient $Du$ of the solutions. Moreover, we establish the higher summability of $Du$ with respect to the spatial variable. The main novelty of the above equation is that the structure function satisfies standard ellipticity and growth conditions only outside the unit ball centered at the origin. We would like to point out that the main result of this paper can be considered, on the one hand, as the parabolic counterpart of an elliptic result contained in [1], and on the other hand as the fractional version of some results established in [2].

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1 Introduction and statement of the results

In this paper, we aim to pursue our investigation started in [2] about the regularity properties of weak solutions to the strongly degenerate parabolic equation

$$u_t - \text{div} \left[ \left( |Du| - 1 \right)^{p-1} \frac{Du}{|Du|} \right] = f \quad \text{in } \Omega_T = \Omega \times (0, T),$$

where $p \geq 2$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$), $(\cdot)_+$ stands for the positive part and $f$ is a given function.

The main feature of this PDE is that the structure function satisfies standard growth and ellipticity conditions for an exponent $p \geq 2$, but only outside the unit ball centered at the origin.
A motivation for studying equation (1.1) can be found in gas filtration problems (for a detailed explanation, we refer to [2, Section 1.1]).

The elliptic version of the above equation naturally arises in optimal transport problems with congestion effects, and the regularity properties of its weak solutions have been widely investigated: see, for instance, [1, 3, 4] and [5]. In this regard, we also want to point out the very recent paper [21], whose author examines the higher regularity of weak solutions to the very degenerate elliptic system

$$\text{div} \left[ a(x) \frac{|Dv| - 1}{|Dv|} Dv \right] = 0 \quad \text{in } \Omega,$$

with a growth exponent $p \geq 2$ and Lipschitz continuous coefficients $a : \Omega \to \mathbb{R}$.

To the best of our knowledge, the only parabolic counterpart of the aforementioned works with weaker assumptions on the datum $f$ that is available in the literature is the paper [2]. There, we establish the higher differentiability of integer order and the higher integrability of the spatial gradient of the weak solutions $u$ to equation (1.1), as well as the existence of a weak time derivative $u_t$, by assuming that $f \in L^q(0, T; W^{1,q}(\Omega))$ for a suitable exponent $q > 1$.

Here, instead, we prove local higher differentiability results for a certain function of the spatial gradient $Du$ in the scale of fractional Sobolev spaces, under the assumption that the datum $f$ belongs to the local Bochner space $L^{p'}_{\text{loc}}(0, T; B^{\alpha}_{p',\infty,\text{loc}}(\Omega))$, where $p' = p/(p-1)$ is the conjugate exponent of $p$, $\alpha \in (0, 1)$, while $B^{\alpha}_{p',\infty}(\Omega)$ denotes a particular class of Besov functions (see Section 3.2 below for the definition).

Furthermore, we establish the local higher summability of $Du$ with respect to the spatial variables, under the same assumption on the regularity of $f$.

The distinguishing feature of equation (1.1) is that the principal part behaves like a $p$-Laplace operator only at infinity. Before giving the main result, let us summarize a few previous results on this topic: the regularity of solutions to parabolic problems with asymptotic structure of $p$-Laplacian type has been explored in [14], where a BMO regularity has been proved for solutions to asymptotically parabolic systems in the case $p = 2$ and $f = 0$ (see also [16], where the local Lipschitz continuity of weak solutions with respect to the spatial variable is established). In addition, we want to mention the results contained in [7], where nonhomogeneous parabolic problems involving a discontinuous nonlinearity and an asymptotic regularity over an irregular domain in divergence form of $p$-Laplacian type are considered. There, the authors establish a global Calderón-Zygmund estimate by converting a given asymptotically regular problem to a suitable regular problem.

We would like to notice that our assumption on the datum $f$ is weaker than those considered in the mentioned papers. This prevents us from achieving a (local) higher summability for $Du$ over $\Omega_T$ (i.e., with respect to the space-time variable $z = (x, t)$), contrary to what we were able to do in [2, Theorem 1.3]. That is why here we obtain the higher summability of $Du$ only with respect to the spatial variable (see Corollary 1.2 below). For the same reason, we are led to deal with fractional Sobolev spaces, rather than the traditional Sobolev spaces of integer order.

The main result we prove in this paper is the following theorem. We refer to Sections 2 and 3 for notation and definitions.

**Theorem 1.1.** Let $n \geq 2$, $p \geq 2$, $\alpha \in (0, 1)$ and $f \in L^{p'}_{\text{loc}}(0, T; B^{\alpha}_{p',\infty,\text{loc}}(\Omega))$. Moreover, assume that

$$u \in C^0(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$$

and
is a weak solution of equation (1.1). Then the solution satisfies
\[ H_p^2(Du) \in L^2_{loc}(0,T;W^{s,2}_{loc}(\Omega,\mathbb{R}^n)) \quad \text{for all } s \in \left(0, \frac{\alpha + 1}{2}\right), \]
where
\[ H_p^2(Du) := (|Du| - 1)^{p/2} \frac{Du}{|Du|}. \] (1.2)

Furthermore, the following local estimate holds true for any \( i \in \{1, \ldots, n\} \), for any parabolic cylinder \( Q_\varrho(z_0) \subset Q_{2\varrho}(z_0) \subset \Omega, T \), for any \( h \in \mathbb{R} \) such that \( |h| < \varrho/4 \) and a positive constant \( c \) depending at most on \( p, n \) and \( \varrho \).

The proof of Theorem 1.1 is achieved using the well-known difference quotients technique in the spatial directions (see Section 4 below). Here we will argue as in [10, Lemma 5.1] and [12, Theorem 4.1], but we need to take into account the strong degeneracy of equation (1.1), exactly as we have done in [2]. This is why we obtain the higher fractional differentiability not for the spatial gradient of the solution itself, but for a function of the spatial gradient \( Du \) that vanishes in the set where equation (1.1) becomes degenerate.

Actually, Theorem 1.1 can be considered as the parabolic counterpart of Theorem 4.1 in [1], where however we obtained a higher fractional differentiability result in the scale of Besov spaces, starting from a datum in the local Besov-Lipschitz space \( B^\alpha_{p',\infty,loc}(\Omega) \), for some \( \alpha \in (0,1) \). Furthermore, the above theorem can also be viewed as the fractional version of Theorems 1.1 and 1.4 in [2].

As a consequence of the previous result, using a Sobolev-type embedding theorem for fractional Sobolev spaces, we get a gain of summability for \( Du \) with respect to the spatial variable. More precisely, we have the following

**Corollary 1.2.** Under the assumptions of Theorem 1.1, we obtain that
\[ H_p^2(Du) \in L^2_{loc}(0,T;L^r_{loc}(\Omega,\mathbb{R}^n)) \quad \text{for all } r \in \left[1, \frac{2n}{n - \alpha - 1}\right) \]
and
\[ Du \in L^p_{loc}(0,T;L^q_{loc}(\Omega,\mathbb{R}^n)) \quad \text{for all } q \in \left[1, \frac{np}{n - \alpha - 1}\right). \]

It is worth pointing out that, starting from the assumption \( f \in L^p_{loc}(0,T;B^\alpha_{p',\infty,loc}(\Omega)) \), higher fractional differentiability results such as that of Theorem 1.1 above seem not to have been established yet for the spatial gradient of weak solutions to the strongly degenerate equation (1.1).

However, we would like to mention the result contained in [7, Theorem 2.10], whose authors consider a weak solution \( u \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W^{1,p}_0(\Omega)) \) to the following nonlinear
parabolic problem in divergence form:
\[
\begin{aligned}
    u_t - \text{div } a(x, t, Du) &= \text{div } (|F|^{p-2} F) \quad \text{in } \Omega_T, \\
    u &= 0 \quad \text{on } \partial_{\text{par}} \Omega_T,
\end{aligned}
\]
where \( \frac{2n}{n+2} < p < \infty \), \( a \) is a discontinuous nonlinearity with an asymptotic regularity, \( \Omega \) is a bounded domain whose boundary \( \partial \Omega \) is nonsmooth and \( F = F(x, t) = (f_1(x, t), \ldots, f_n(x, t)) \in L^p(\Omega_T, \mathbb{R}^n) \) is a given vector-valued function. In fact, the just mentioned result gives an affirmative answer as to what are both the weakest regularity requirement on \( a \) and the lowest level of geometric assumption on \( \partial \Omega \) under which the implication
\[
|F|^p \in L^q(\Omega_T) \implies |Du|^p \in L^q(\Omega_T)
\]
holds true for every \( q \in [1, \infty) \).

1.1 Comparison with a less degenerate elliptic equation

Before describing the structure of this paper, we wish to make some considerations about the interplay between the regularity of the right-hand side of (1.1) and that of the vector field \( H_p^2(Du) \) defined in (1.2), starting from the comparison with a known result for an elliptic equation which is less degenerate than (1.1).

In [6, Theorem 1.1], Brasco and Santambrogio established the sharp assumptions on the datum \( g \) in order to obtain that the \( W^{1,p} \) solutions to the Poisson equation for the \( p \)-Laplace operator
\[
-\Delta_p v := -\text{div}(|Dv|^{p-2} Dv) = g \quad \text{in } \Omega, \tag{1.3}
\]
still satisfy the following Uhlenbeck’s result (see [24, Lemma 3.1])
\[
|Dv|^{p-2} Dv \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n), \tag{1.4}
\]
in the superquadratic case \( p > 2 \), where \( \Omega \subset \mathbb{R}^n \) is an open set. For the sake of completeness, we recall their result below here, by noting that \( g \) must belong at least to a suitable (local) fractional Sobolev space for condition (1.4) to be true:

**Theorem 1.3.** (6, Theorem 1.1). Let \( p > 2 \) and let \( U \in W^{1,p}_{\text{loc}}(\Omega) \) be a local weak solution of equation (1.3). If
\[
g \in W^{s,p}_{\text{loc}}(\Omega) \quad \text{with } \frac{p-2}{p} < s \leq 1, \tag{1.5}
\]
then
\[
V := |DU|^\frac{p-2}{p} DU \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n),
\]
and
\[
DU \in W^{\sigma,p}_{\text{loc}}(\Omega, \mathbb{R}^n), \quad \text{for } 0 < \sigma < \frac{2}{p}.
\]
As for the sharpness of assumption (1.5), we refer the interested reader to [6, Sections 1.2 and 5]. Under the hypotheses of Theorem 1.3, we find that \( DU \) also belongs to some fractional Sobolev space (locally in \( \Omega \)).

The interplay between the regularity of the right-hand side \( g \) and that of the vector field \( V \) has been considered in detail also in [19], where the point of view is slightly different: the
main concern there is to obtain (fractional) differentiability of the vector field $V$ when $g$ is not regular. In particular, in [19] the datum $g$ may not belong to the relevant dual Sobolev space and the concept of solution to (1.3) has to be carefully defined. We refer to [6, Remark 1.3] and the references therein for comparison with other results.

In the case of the strongly degenerate parabolic equation (1.1), the vector-valued function $V$ is replaced by the function $H_{p}(Du)$ defined in (1.2), which belongs to $L^{2}_{loc}(0,T;W^{s,2}_{loc}(\Omega,\mathbb{R}^{n}))$ for all $s \in (0, \frac{\alpha+1}{2})$ under the assumptions of the Theorem 1.1 that we prove in this paper. One could then ask himself what are the optimal (weakest) assumptions to be imposed on the datum $f$ so that the weak solutions of equation (1.1) still satisfy “our” following condition (see [2, Theorems 1.1 and 1.4])

$$H_{p}(Du) \in L^{2}_{loc}(0,T;W^{1,2}_{loc}(\Omega,\mathbb{R}^{n})), \quad (1.6)$$

for $p \geq 2$, since here we cannot achieve the result (1.6) under the hypothesis

$$f \in L^{p'}_{loc}(0,T;B^{\alpha}_{p',\infty,loc}(\Omega)), \quad \text{with} \quad \alpha \in (0,1).$$

However, it is worth emphasizing that equation (1.1) exhibits a more severe degeneracy than equation (1.3) and that the crucial point in [6] that $|DU|^\frac{p}{p-2}DU \in W^{1,2}_{loc} \implies DU \in W^{\sigma,p}_{loc}$ cannot be retrieved in our framework.

The paper is organized as follows. Section 2 is devoted to the preliminaries: after a list of some classic notations and some essentials estimates, we recall the fundamental properties of the difference quotients of Sobolev functions. In Section 3 we recall the basic facts on the functional spaces involved in this paper: subsection 3.1 is entirely devoted to the definition and properties of the fractional Sobolev spaces that will be useful to establish our results, while in subsection 3.2 we give the definition of the Besov spaces $B^{\alpha}_{q,\infty}(\Omega)$ for $0 < \alpha < 1$ and $1 \leq q < \infty$, in order to introduce the Bochner space $L^{q}(0,T;B^{\alpha}_{q,\infty}(\Omega))$ and its local version. Finally, in Section 4 we prove Theorem 1.1 and Corollary 1.2.

## 2 Notations and preliminaries

In this paper we shall denote by $C$ or $c$ a general positive constant that may vary on different occasions. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^{n}$ will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^{n}$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

For points in space-time, we will frequently use abbreviations like $z = (x,t)$ or $z_{0} = (x_{0},t_{0})$, for spatial variables $x$, $x_{0} \in \mathbb{R}^{n}$ and times $t$, $t_{0} \in \mathbb{R}$. We also denote by $B(x_{0}, r) = B^{r}(x_{0}) = \{x \in \mathbb{R}^{n} : |x - x_{0}| < r\}$ the open ball with radius $r > 0$ and center $x_{0} \in \mathbb{R}^{n}$; when not important, or clear from the context, we shall omit to denote the center as follows: $B_{r} \equiv B(x_{0}, r)$. Unless otherwise stated, different balls in the same context will have the same center. Moreover,
we use the notation
\[ Q_\varrho(z_0) := B_\varrho(x_0) \times (t_0 - \varrho^2, t_0), \quad z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}, \quad \varrho > 0, \]
for the backward parabolic cylinder with vertex \((x_0, t_0)\) and width \(\varrho\). We shall sometimes omit the dependence on the vertex when all cylinders occurring in a proof share the same vertex.

We now recall some tools that will be useful to prove our results. For the auxiliary function \(H_\lambda : \mathbb{R}^n \to \mathbb{R}^n\) defined as
\[ H_\lambda(\xi) := \begin{cases} \frac{1}{\lambda} (|\xi| - 1) + \frac{\xi}{|\xi|} & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases} \]
where \(\lambda > 0\) is a parameter, we record the following estimates (see [5, Lemma 4.1]):

**Lemma 2.1.** If \(2 \leq p < \infty\), then for every \(\xi, \eta \in \mathbb{R}^n\) we get
\[
\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \geq \frac{4}{p^2} \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|^2,
\]
\[
|H_{p-1}(\xi) - H_{p-1}(\eta)| \leq (p - 1) \left( \left| H_{\frac{p}{2}}(\xi) \right|^{\frac{p-2}{p}} + \left| H_{\frac{p}{2}}(\eta) \right|^{\frac{p-2}{p}} \right) \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|.
\]

We conclude this first part of the preliminaries by recalling the following

**Definition 2.2.** A function \(u \in C^0(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))\) is a weak solution of equation (1.1) iff for any test function \(\varphi \in C_0^\infty(\Omega_T)\) the following integral identity holds:
\[
\int_{\Omega_T} (u \cdot \partial_t \varphi - \langle H_{p-1}(Du), D\varphi \rangle) \, dz = -\int_{\Omega_T} f \varphi \, dz. \tag{2.1}
\]

### 2.1 Difference quotients

We recall here the definition and some elementary properties of the difference quotients that will be useful in the following (see, for instance, [13]).

**Definition 2.3.** For every vector-valued function \(F : \mathbb{R}^n \to \mathbb{R}^N\) in the direction \(x_i\) is defined by
\[ \tau_{i,h} F(x) = F(x + he_i) - F(x), \]
where \(h \in \mathbb{R}\), \(e_i\) is the unit vector in the direction \(x_i\) and \(i \in \{1, \ldots, n\}\).

The difference quotient of \(F\) with respect to \(x_i\) is defined for \(h \in \mathbb{R} \setminus \{0\}\) by
\[ \Delta_{i,h} F(x) = \frac{\tau_{i,h} F(x)}{h}. \]

When no confusion can arise, we shall omit the index \(i\) and simply write \(\tau_h\) or \(\Delta_h\) instead of \(\tau_{i,h}\) or \(\Delta_{i,h}\), respectively.
Proposition 2.4. Let $F$ be a function such that $F \in W^{1,q}(\Omega)$, with $q \geq 1$, and let us consider the set 

$$\Omega_{|h|} := \{ x \in \Omega : \text{dist}(x, \partial\Omega) > |h| \}.$$ 

Then:

(a) $\Delta_h F \in W^{1,q}(\Omega_{|h|})$ and 

$$D_j(\Delta_h F) = \Delta_h(D_j F), \quad \text{for every } j \in \{1, \ldots, n\}.$$ 

(b) If at least one of the functions $F$ or $G$ has support contained in $\Omega_{|h|}$, then 

$$\int_{\Omega} F \Delta_h G \, dx = - \int_{\Omega} G \Delta_h F \, dx.$$ 

(c) We have 

$$\Delta_h(FG)(x) = F(x + he_i)\Delta_hG(x) + G(x)\Delta_hF(x).$$

The next result about the finite difference operator is a kind of integral version of the Lagrange Theorem.

Lemma 2.5. If $0 < q < R$, $|h| < \frac{R-q}{2}$, $1 < q < +\infty$, and $F \in L^q(B_R, \mathbb{R}^N)$, $DF \in L^q(B_R, \mathbb{R}^{N\times n})$, then 

$$\int_{B_{|h|}} |\tau_h F(x)|^q \, dx \leq c^q(n) |h|^q \int_{B_R} |DF(x)|^q \, dx.$$ 

Moreover 

$$\int_{B_{|h|}} |F(x + he_i)|^q \, dx \leq \int_{B_R} |F(x)|^q \, dx.$$ 

Finally, we recall the following fundamental result, whose proof can be found in [13, Lemma 8.2]:

Lemma 2.6. Let $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^q(B_R, \mathbb{R}^N)$ with $1 < q < +\infty$. Suppose that there exist $q \in (0, R)$ and a constant $M > 0$ such that 

$$\sum_{i=1}^n \int_{B_q} |\tau_{i,h} F(x)|^q \, dx \leq M^q |h|^q$$ 

for every $h$ with $|h| < \frac{R-q}{2}$. Then $F \in W^{1,q}(B_q, \mathbb{R}^N)$. Moreover 

$$\|DF\|_{L^q(B_q)} \leq M$$ 

and 

$$\Delta_{i,h} F \to D_i F \quad \text{in } L^q_{\text{loc}}(B_R), \quad \text{as } h \to 0,$$

for each $i \in \{1, \ldots, n\}$. 
3 Functional spaces

Here we recall some essential facts about the functional spaces involved in this paper, starting with the definition and some properties of the fractional Sobolev spaces that will be useful to prove our results (see, for instance, [5]).

3.1 Fractional Sobolev spaces

Let $\Omega$ be a general, possibly nonsmooth, bounded open set in $\mathbb{R}^n$. For any $s \in (0,1)$ and for any $q \in [1, +\infty)$, we define the fractional Sobolev space $W^{s,q}(\Omega, \mathbb{R}^k)$ as follows

$$W^{s,q}(\Omega, \mathbb{R}^k) := \left\{ v \in L^q(\Omega, \mathbb{R}^k) : \frac{|v(x) - v(y)|}{|x - y|^\frac{n}{q} + s} \in L^q(\Omega \times \Omega) \right\},$$

i.e. an intermediate Banach space between $L^q(\Omega, \mathbb{R}^k)$ and $W^{1,q}(\Omega, \mathbb{R}^k)$, endowed with the norm

$$\|v\|_{W^{s,q}(\Omega)} := \left( \int_\Omega |v|^q \, dx + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^q}{|x - y|^n + sq} \, dx \, dy \right)^\frac{1}{q},$$

where the term

$$[v]_{W^{s,q}(\Omega)} := \left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^q}{|x - y|^n + sq} \, dx \, dy \right)^\frac{1}{q}$$

is the so-called Gagliardo seminorm of $v$.

As in the classic case with $s$ being an integer, the space $W^{s',q}(\Omega)$ is continuously embedded in $W^{s,q}(\Omega)$ when $s \leq s'$, as shown by the next result (see [5, Proposition 2.1]).

**Proposition 3.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$, $q \in [1, +\infty)$ and $0 < s \leq s' < 1$. Then there exists a constant $C \equiv C(n, s, q) \geq 1$ such that, for any $v \in W^{s',q}(\Omega)$, we have

$$\|v\|_{W^{s,q}(\Omega)} \leq C \|v\|_{W^{s',q}(\Omega)}.$$

In particular, $W^{s',q}(\Omega) \subseteq W^{s,q}(\Omega)$.

As is well known when $s \in \mathbb{N}$, under certain regularity assumptions on the open set $\Omega \subset \mathbb{R}^n$, any function in $W^{s,q}(\Omega)$ can be extended to a function in $W^{s,q}(\mathbb{R}^n)$. Extension results are needed to improve some embedding theorems, in the classic case as well as in the fractional one. In this regard, we now give the following

**Definition 3.2.** For any $s \in (0,1)$ and any $q \in [1, \infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,q}$ if there exists a positive constant $C \equiv C(n, q, s, \Omega)$ such that: for every function $v \in W^{s,q}(\Omega)$ there exists $\tilde{v} \in W^{s,q}(\mathbb{R}^n)$ with $\tilde{v}|_{\Omega} = v$ and

$$\|\tilde{v}\|_{W^{s,q}(\mathbb{R}^n)} \leq C \|v\|_{W^{s,q}(\Omega)}.$$

In general, an arbitrary open set may not be an extension domain for $W^{s,q}$. However, the following result ensures that every open Lipschitz set $\Omega$ with bounded boundary is an extension domain for $W^{s,q}$ (a proof can be found in [5, Theorem 5.4]).

**Theorem 3.3.** Let $q \in [1, +\infty)$, $s \in (0,1)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary. Then $W^{s,q}(\Omega)$ is continuously embedded in $W^{s,q}(\mathbb{R}^n)$, namely for any $v \in W^{s,q}(\Omega)$ there exists $\tilde{v} \in W^{s,q}(\mathbb{R}^n)$ such that $\tilde{v}|_{\Omega} = v$ and

$$\|\tilde{v}\|_{W^{s,q}(\mathbb{R}^n)} \leq C \|v\|_{W^{s,q}(\Omega)}.$$
for some positive constant $C \equiv C(n, q, s, \Omega)$.

For more information on the problem of characterizing the class of sets that are extension domains for $W^{s,q}$, we refer the interested reader to Zhou’s paper [25], where an answer to this question has been given (see also [15] and [17, Chapters 11 and 12]).

For further needs, we now recall the following Sobolev-type embedding theorem, whose proof can be found in [8, Theorem 6.7].

**Theorem 3.4.** Let $s \in (0,1)$ and $q \in [1, +\infty)$ be such that $sq < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,q}$. Then there exists a positive constant $C \equiv C(n, q, s, \Omega)$ such that, for any $v \in W^{s,q}(\Omega)$, we have

$$\|v\|_{L^r(\Omega)} \leq C \|v\|_{W^{s,q}(\Omega)}$$

for any $r \in [q, q^*]$; i.e. the space $W^{s,q}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [q, q^*]$, where $q^* := nq/(n - sq)$ is the so-called “fractional critical exponent”.

Moreover, if $\Omega$ is bounded, then the space $W^{s,q}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [1, q^*]$.

**Remark 3.5.** In the critical case $r = q^*$, the constant $C$ in Theorem 3.4 does not depend on $\Omega$ (see Remark 6.8 in [3]).

For the treatment of parabolic equations, the following function space plays an important role. Let $1 \leq q < \infty$ and $0 < s < 1$. A map $g \in L^q(\Omega \times (t_0, t_1), \mathbb{R}^k)$ belongs to the space $L^q(t_0, t_1; W^{s,q}((\Omega), \mathbb{R}^k))$ if and only if

$$\int_{t_0}^{t_1} \int_{\Omega} \int_{\Omega} \frac{|g(x, t) - g(y, t)|^q}{|x - y|^{n+sq}} \, dx \, dy \, dt < \infty.$$

In this paper, we will use the corresponding local version of this space, which will be denoted by the subscript “loc”. More precisely, we write $g \in L_{loc}^q(0, T; W_{loc}^{s,q}(\Omega, \mathbb{R}^k))$ if and only if $g \in L^q(t_0, t_1; W^{s,q}(\Omega', \mathbb{R}^k))$ for all domains $\Omega' \times (t_0, t_1) \subseteq \Omega_T$.

Now we conclude this section with the parabolic version of the well-known result about the relation between Nikol’skii spaces and fractional Sobolev spaces. This result is contained in [22, Lemma 2.4] (see also [10, Proposition 2.19]), and its proof can be obtained by a simple adaptation of the standard elliptic results [9, 11, 13, 19, 20].

**Proposition 3.6.** Let $Q_\sigma(z_0) \subseteq \mathbb{R}^{n+1}$ be a parabolic cylinder. Moreover, assume that $G \in L^q(Q_\sigma(z_0), \mathbb{R}^k)$, where $1 \leq q < \infty$. Then, for any $\theta \in (0, 1)$, the estimate

$$|h|^{-q\theta} \int_{Q_\sigma(z_0)} |G(x + he_1, t) - G(x, t)|^q \, dx \, dt \leq M^q < \infty$$

for a fixed constant $M \geq 0$, every $0 \neq |h| \leq h_0$ and every $i \in \{1, \ldots, n\}$ implies

$$G \in L_{loc}^q(t_0 - \sigma^2, t_0; W_{loc}^{s,q}(B_\sigma(x_0), \mathbb{R}^k))$$

for all $s \in (0, \theta)$. 
3.2 Besov spaces

Here we recall the definition of the Besov space $B^{a}_{q,\infty}(\Omega)$ for $0 < a < 1$ and $1 \leq q < \infty$ (see Section 2.5.12 in [23]). For a function $v \in L^q(\Omega)$ we say that $v \in B^{a}_{q,\infty}(\Omega)$ if

$$[v]_{B^{a}_{q,\infty}(\Omega)} := \sup_{h \in \mathbb{R}^n} \left( \int_{\Omega} \frac{|\Delta_h[v](x)|^q}{|h|^{aq}} \, dx \right)^{\frac{1}{q}} < \infty, \quad (3.1)$$

where $\Delta_h[v](x) := [v(x+h) - v(x)] \cdot 1_{\Omega}(x+h)$. One can define a norm on the space $B^{a}_{q,\infty}(\Omega)$ as follows

$$\|v\|_{B^{a}_{q,\infty}(\Omega)} := \left( \|v\|_{L^q(\Omega)}^q + [v]_{B^{a}_{q,\infty}(\Omega)}^q \right)^{\frac{1}{q}},$$

and with this norm $B^{a}_{q,\infty}(\Omega)$ is a Banach space. Actually, in (3.1) we can simply take the supremum over $|h| < \delta$ for a fixed $\delta > 0$ and obtain an equivalent norm, since

$$\sup_{|h| \geq \delta} \left( \int_{\Omega} \frac{|\Delta_h[v](x)|^q}{|h|^{aq}} \, dx \right)^{\frac{1}{q}} \leq c(n, \alpha, q, \delta) \|v\|_{L^q(\Omega)}.$$

By construction, we have $B^{a}_{q,\infty}(\Omega) \subset L^q(\Omega)$.

For the treatment of parabolic equations, we now give the following

**Definition 3.7.** A map $g \in L^q(\Omega \times (t_0, t_1))$ belongs to the space $L^q(t_0, t_1; B^{a}_{q,\infty}(\Omega))$ if and only if

$$\int_{t_0}^{t_1} \left( \sup_{h \in \mathbb{R}^n} \int_{\Omega} \frac{|g(x+h, t) - g(x, t)|^q}{|h|^{aq}} \cdot 1_{\Omega}(x+h) \, dx \right) \, dt < \infty.$$

In this paper, we use the corresponding local version of this space, which is denoted by the subscript “loc”. Actually, in this framework we write $g \in L^q_{\text{loc}}(0, T; B^{a}_{q,\infty,\text{loc}}(\Omega))$ if and only if $g \in L^q(t_0, t_1; B^{a}_{q,\infty} (\Omega'))$ for all domains $\Omega' \times (t_0, t_1) \Subset \Omega_T$. Furthermore, we shall also use the following notation, which is typical of Bochner spaces:

$$\|g\|_{L^q(t_0, t_1; B^{a}_{q,\infty}(\Omega'))} := \left( \int_{t_0}^{t_1} \|g(\cdot, t)\|_{B^{a}_{q,\infty}(\Omega')}^q \, dt \right)^{\frac{1}{q}}.$$

4 Proofs of the results

We will now use the well-known difference quotients method in the spatial directions, as well as the properties of the function $H_\lambda$ and Proposition 3.6 to establish the

**Proof of Theorem 1.1.** By a slight abuse of notation, for $g \in L^1_{\text{loc}}(\Omega_T, \mathbb{R}^N)$ and $i \in \{1, \ldots, n\}$, $h \neq 0$, we set (when $x + he_i \in \Omega$)

$$\tau_h g(x, t) \equiv \tau_{i,h} g(x, t) := g(x + he_i, t) - g(x, t),$$

$$\Delta_h g(x, t) \equiv \Delta_{i,h} g(x, t) := \frac{g(x + he_i, t) - g(x, t)}{h},$$
where \( c_i \) is the unit vector in the direction \( x_i \).

Since \( u \) is a weak solution of equation (1.1), we have
\[
\int_{\Omega_T} (u \cdot \partial_t \varphi - \langle H_{p-1}(Du), D\varphi \rangle) \, dz = - \int_{\Omega_T} f \varphi \, dz,
\]
for every test function \( \varphi \in C_0^\infty(\Omega_T) \). Replacing \( \varphi \) by \( \tau_h \varphi \), where \( 0 < |h| < \text{dist(supp } \varphi, \partial \Omega_T \) by virtue of the properties of the finite difference operator, we get
\[
\int_{\Omega_T} (\tau_h u \cdot \partial_t \varphi - \langle \tau_h H_{p-1}(Du), D\varphi \rangle) \, dz = - \int_{\Omega_T} \tau_h f \cdot \varphi \, dz.
\]
We now replace \( \varphi \) by \( \varphi_{\varepsilon} \equiv \phi_{\varepsilon} \ast \varphi \) in the previous equation, where \( \{ \phi_{\varepsilon} \} \), \( \varepsilon > 0 \), denotes the family of standard, non-negative, radially symmetric mollifiers in \( \mathbb{R}^{n+1} \). This yields, for \( 0 < \varepsilon < 1 \)
\[
\int_{\Omega_T} (\tau_h u_{\varepsilon} \cdot \partial_t \varphi - \langle (\tau_h H_{p-1}(Du))_{\varepsilon}, D\varphi \rangle) \, dz = - \int_{\Omega_T} (\tau_h f)_{\varepsilon} \cdot \varphi \, dz.
\]
Now, in the last equation we choose the test function \( \varphi \equiv \Phi(\tau_h u)_{\varepsilon} \), where \( \Phi \in C_0^\infty(\Omega_T) \) is a cut-off function which will be specified later. After an integration by parts and then letting \( \varepsilon \searrow 0 \), we obtain
\[
- \frac{1}{2} \int_{\Omega_T} |\tau_h u|^2 \partial_t \Phi \, dz + \int_{\Omega_T} \Phi \langle \tau_h H_{p-1}(Du), D\tau_h u \rangle \, dz = - \int_{\Omega_T} \langle \tau_h H_{p-1}(Du), D\Phi \rangle \tau_h u \, dz + \int_{\Omega_T} \tau_h f \cdot \Phi \cdot \tau_h u \, dz.
\]
Note that an approximation argument yields the same identity for any \( \Phi \in W^{1,\infty}(\Omega_T) \) with compact support in \( \Omega_T \) and any sufficiently small \( h \in \mathbb{R} \setminus \{0\} \). In what follows, we will denote by \( c_k \) some positive constants which do not depend on \( h \).

Now, let us consider a parabolic cylinder \( Q_{2\varepsilon}(z_0) \subseteq Q_{\varepsilon}(z_0) \subseteq \Omega_T \). For a fixed time \( t_1 \in (t_0 - \varepsilon^2, t_0) \) and \( \delta \in (0, t_0 - t_1) \), we choose \( \Phi(t) = \tilde{\chi}(t) \chi(t) \eta^2(x) \) with \( \chi \in W^{1,\infty}(0, T), [0, 1] \), \( \chi \equiv 0 \) on \( (0, t_0 - \varepsilon^2) \) and \( \partial_t \chi \geq 0 \), \( \eta \in C_0^\infty(B_{\varepsilon}(x_0), [0, 1]) \), and with the Lipschitz continuous function \( \tilde{\chi}: (0, T) \rightarrow \mathbb{R} \) defined by
\[
\tilde{\chi}(t) = \begin{cases} 
1 & \text{if } t \leq t_1, \\
al \text{affine} & \text{if } t_1 < t < t_1 + \delta, \\
0 & \text{if } t \geq t_1 + \delta.
\end{cases}
\]
With such a choice of \( \Phi \), equation (4.1) turns into
\[
- \frac{1}{2} \int_{\Omega_T} |\tau_h u|^2 \eta^2(x) \chi(t) \partial_t \tilde{\chi}(t) \, dz - \frac{1}{2} \int_{\Omega_T} |\tau_h u|^2 \eta^2(x) \tilde{\chi}(t) \partial_t \chi(t) \, dz + \int_{\Omega_T} \tilde{\chi}(t) \chi(t) \eta^2(x) \langle \tau_h H_{p-1}(Du), D\tau_h u \rangle \, dz = -2 \int_{\Omega_T} \tilde{\chi}(t) \chi(t) \eta(x) \langle \tau_h H_{p-1}(Du), D\eta \rangle \tau_h u \, dz + \int_{\Omega_T} (\tau_h f) \langle \tau_h u, \tilde{\chi}(t) \chi(t) \eta^2(x) \rangle \, dz.
\]
Letting \( \delta \rightarrow 0 \) in the previous equation, we get
\[
\frac{1}{2} \int_{B_{\rho}(x_0)} \chi(t_1) |\nabla u(x, t_1)|^2 \, dx + \int_{Q^{t_1}} \chi(t) |\nabla f(x)| \, dx
\]
\[
= -2 \int_{Q^{t_1}} \chi(t) \eta(x) \langle \nabla u(\rho, t_1), D\eta \rangle \, dx + \int_{Q^{t_1}} (\nabla f(x) \nabla u(x)) \eta(x) \, dx 
\]
\[
+ \frac{1}{2} \int_{Q^{t_1}} |\partial_t \chi| \eta^2 |\nabla u|^2 \, dx,
\]
for every \( h \in \mathbb{R} \setminus \{0\} \) such that \( |h| < \rho/4 \) and for almost every \( t_1 \in (t_0 - \rho^2, t_0) \), where we have used the abbreviation \( Q^{t_1} = B_{\rho}(x_0) \times (t_0 - \rho^2, t_1) \).

Now, by Lemma [2.1] we have
\[
\frac{4}{p^2} \int_{Q^{t_1}} \chi(t) \eta^2(x) |\nabla u(\rho, t_1)|^2 \, dx \leq \int_{Q^{t_1}} \chi(t) \eta^2(x) \langle \nabla u(\rho, t_1), D\nabla u \rangle \, dx,
\]
and
\[
\left| \int_{Q^{t_1}} \chi(t) \eta(x) \langle \nabla u(\rho, t_1), D\nabla u \rangle \, dx \right|
\]
\[
\leq 2(p-1) \int_{Q^{t_1}} \chi(t) \eta(x) \left( |\nabla u(\rho, t_1)| \left| \nabla u \right|^2 \eta \left| \nabla u \right|^2 \right) \, dx.
\]

Using Young’s inequality with exponents \((2, 2)\) in the right-hand side of the previous estimate, we obtain
\[
\left| \int_{Q^{t_1}} \langle \nabla f(x) \nabla u(x), \chi(t) \eta^2(x) \rangle \, dx \right|
\]
\[
\leq c_1(n) |h| \| Du \|_{L^p(Q_{2\rho})} \left( \int_{Q^{t_1}} |\nabla f|^2 \left| \nabla u \right|^2 \eta \left| \nabla u \right|^2 \right) \, dx 
\]
\[
\leq c_1 |h| \| Du \|_{L^p(Q_{2\rho})} \left[ \int_{t_0 - 4\rho^2}^{t_0} \sup_{|y| < \frac{4}{3} B_{2\rho}(x_0)} \left| \int_{B_{2\rho}(x_0)} \frac{|f(x+y, t) - f(x, t)| \left| \nabla f \right|^2 \left| \nabla u \right|^2 \, dx \right| \, dt \right]^{\frac{1}{p'}} 
\]
\[
\leq c_1 |h| \| Du \|_{L^p(Q_{2\rho})} \left[ \int_{t_0 - 4\rho^2}^{t_0} \left| \int_{B_{2\rho}(x_0)} \frac{|f(x+y, t)| \left| \nabla f \right|^2 \left| \nabla u \right|^2 \, dx \right| \, dt \right]^{\frac{1}{p'}} 
\]
\[
= \frac{c_1 |h|}{\left[ \int_{t_0 - 4\rho^2}^{t_0} \frac{|f(x+y, t)| \left| \nabla f \right|^2 \left| \nabla u \right|^2 \, dx \right]^{\frac{1}{p'}}}.
\]
for every $h \in \mathbb{R} \setminus \{0\}$ such that $|h| < \varrho/4$. Joining estimates (4.2), (4.3), (4.4) and (4.5), and choosing $\sigma = 4/p^2$, we arrive at

$$
\int_{B_{\varrho}(x_0)} \chi(t_1)\eta^2(x) \left| \tau_h u(x, t_1) \right|^2 \, dx + \int_{Q_{t_1}} \chi(t)\eta^2(x) \left| \tau_h H^Z_h(Du) \right|^2 \, dz
\leq c_2(p) \int_{Q_{t_1}} \left[ \chi(t) |D\eta| \left( \left| H^Z_h(Du(x + he_i, t)) \right|^{\frac{n-2}{p}} + \left| H^Z_h(Du) \right|^{\frac{n-2}{p}} \right)^2 \right] \left| \tau_h u \right|^2 \, dz
+ c_3(n, p) |h|^\alpha \left\| Du \right\|_{L^p(Q_{2\varrho})} \left\| f \right\|_{L^p'(t_0 - 4\varrho^2, t_0 ; B^\infty_{p, \infty}(B_{2\varrho}(x_0)))},
$$

which holds for almost every $t_1 \in (t_0 - \varrho^2, t_0)$. We now choose a cut-off function $\eta \in C_0^\infty (B_{\varrho}(x_0))$ with $\eta \equiv 1$ on $B_{\varrho/2}(x_0)$ such that $0 \leq \eta \leq 1$ and $|D\eta| \leq C/\varrho$. For the cut-off function in time, we choose the piecewise affine function $\chi : (0, T) \to [0, 1]$ with

$$
\chi \equiv 0 \quad \text{on} \quad (0, t_0 - \varrho^2), \quad \chi \equiv 1 \quad \text{on} \quad (t_0 - (\varrho/2)^2, T)
$$

and

$$
\partial_t \chi \equiv \frac{4}{3\varrho^2} \quad \text{on} \quad (t_0 - \varrho^2, t_0 - (\varrho/2)^2).
$$

Dividing both sides of the previous estimate by $|h|$ and using the properties of $\chi$ and $\eta$, we obtain

$$
\sup_{t_0 - (\varrho/2)^2 < t < t_0} \int_{B_{\varrho/2}(x_0)} \left| \frac{\tau_{t_0}u(x, t)}{|h|^{1/2}} \right|^2 \, dx + \int_{Q_{t_1}(z_0)} \left| \frac{\tau_{t_0}H_{\varrho/2}(Du)}{|h|^{1/2}} \right|^2 \, dz
\leq c_4(p) |h|^{\varrho^2} \int_{Q_{\varrho}(z_0)} \left( \left| H^Z_h(Du(x + he_i, t)) \right|^{\frac{n-2}{p}} + \left| H^Z_h(Du) \right|^{\frac{n-2}{p}} \right)^2 + 1 \left| \Delta_h u \right|^2 \, dz \quad (4.6)
+ c_3(n, p) |h|^\alpha \left\| Du \right\|_{L^p(Q_{2\varrho})} \left\| f \right\|_{L^p'(t_0 - 4\varrho^2, t_0 ; B^\infty_{p, \infty}(B_{2\varrho}(x_0)))},
$$

Now we set

$$
I := \int_{Q_{\varrho}(z_0)} \left( \left| H^Z_h(Du(x + he_i, t)) \right|^{\frac{n-2}{p}} + \left| H^Z_h(Du) \right|^{\frac{n-2}{p}} \right)^2 + 1 \left| \Delta_h u \right|^2 \, dz \quad (4.7)
$$

and we will assume that $p > 2$. By Hölder’s inequality with exponents $\left( \frac{n-2}{p}, \frac{p}{p-2} \right)$, as well as by the properties of the difference quotients, we can control $I$ as follows

$$
I \leq c_5(n) \left( \int_{Q_{2\varrho}} |Du|^p \, dz \right)^{\frac{2}{p}} \left( \int_{Q_{2\varrho}} \left[ H^Z_h(Du) \right]^{\frac{2(p-2)}{p}} + 1 \right)^{\frac{p-2}{p}} \, dz
\leq c_6(n, p) \left( \int_{Q_{2\varrho}} |Du|^p \, dz \right)^{\frac{2}{p}} \left( \int_{Q_{2\varrho}} (|Du|^p + 1) \, dz \right)^{\frac{p-2}{p}} \quad (4.8)
\leq c_6(n, p) \int_{Q_{2\varrho}} (|Du|^p + 1) \, dz,
$$
provided that $|h| < 2\alpha/3$. Combining estimates (4.6) and (4.8), we then have

$$
\left| \tau_{i,h}(Du) \right|^2 \leq c \left| \frac{\tau_{i,h}(Du)}{|h|^{(\alpha+1)/2}} \right|^2 dx + \int_{Q_{\eta/2}(z_0)} \left| \frac{\tau_{i,h}H_{p/2}(Du)}{|h|^{(\alpha+1)/2}} \right|^2 dz
$$

(4.9)

with $c \equiv c(n, p) > 0$. Since the previous estimate holds for every $h \in \mathbb{R} \setminus \{0\}$ such that $|h| < 2\alpha/3$, from (4.9) we obtain

$$
\int_{Q_{\eta/2}(z_0)} \left| \frac{\tau_{i,h}H_{p/2}(Du)}{|h|^{(\alpha+1)/2}} \right|^2 dz \leq 4^{\alpha-1} c \frac{\eta^{-\alpha}}{\eta} \int_{Q_{\eta/2}(z_0)} (|Du|^p + 1) dz
$$

+ $c \|Du\|_{L^p(Q_{\eta/2}(z_0))} \|f\|_{L^{p'}}(t_0 - 4\eta^2, t_0; B_{p',\infty}^{\eta,n}(B_{2\eta}(x_0)))$.

from which we can deduce

$$
\int_{Q_{\eta/2}(z_0)} \left| \tau_{i,h}H_2(Du) \right|^2 dz \leq M |h|^\alpha + 1,
$$

for some finite positive constant $M$ depending on $p$, $n$, $\eta$, $\alpha$, $\|f\|_{L^{p'}(t_0 - 4\eta^2, t_0; B_{p',\infty}^{\eta,n}(B_{2\eta}(x_0)))}$ and $\|Du\|_{L^p(\Omega_T)}$, but not on $h$. Note that the above estimate holds for every $i \in \{1, \ldots, n\}$ and every sufficiently small $h \in \mathbb{R} \setminus \{0\}$. Therefore, using Proposition 3.6 with the choices $G = H_{p/2}(Du)$, $q = 2$ and $\theta = \frac{\alpha+1}{2}$, as well as a standard covering argument, we infer that

$$
H_2(Du) \in L^2_{\text{loc}}(0, T; W^{\alpha,2}_{\text{loc}}(\Omega, \mathbb{R}^n)) \quad \text{for all } s \in \left(0, \frac{\alpha+1}{2}\right).
$$

Finally, when $p = 2$, arguing in a similar fashion we reach the same conclusions. This completes the proof.

We are now in a position to give the

**Proof of Corollary 1.2.** By virtue of Theorems 1.1, 3.3 and 3.4, for any fixed $s \in \left(0, \frac{\alpha+1}{2}\right)$ we have

$$
H_2(Du) \in L^2(t_0, t_1; L^\gamma(\Omega', \mathbb{R}^n)) \quad \text{for every } \gamma \in \left[1, \frac{2n}{n-2s}\right],
$$

for any open Lipschitz set $\Omega' \subseteq \Omega$ and any $(t_0, t_1) \subset (0, T)$. Now, let us observe that the function $g_n : (0, \frac{\alpha+1}{2}) \rightarrow \mathbb{R}$ defined by

$$
g_n(s) = \frac{2n}{n-2s}
$$

is continuous and strictly increasing, and $g_n(s) \xrightarrow{s \rightarrow \frac{\alpha+1}{2}} \frac{2n}{n-\alpha-1}$ as $s \xrightarrow{s \rightarrow \frac{\alpha+1}{2}}$. Hence, letting $s$ tend to $\frac{\alpha+1}{2}$ from below, we obtain that

$$
H_2(Du) \in L^2_{\text{loc}}(0, T; L^r_{\text{loc}}(\Omega, \mathbb{R}^n)) \quad \text{for all } r \in \left[1, \frac{2n}{n-\alpha-1}\right],
$$
since every open ball $B_\varepsilon \subset \Omega$ is a set of class $C^{0,1}$ with bounded boundary. In particular, for all Lipschitz open subsets $\Omega' \subset \Omega$ and all $(t_0, t_1) \subset (0, T)$ we have

$$\int_{t_0}^{t_1} \left( \int_{\Omega'} |Du(x,t)|^{rp/2} \, dx \right)^{\frac{2}{p}} \, dt = \int_{t_0}^{t_1} \|H_\frac{p}{2}(Du(\cdot,t))\|_{L^r(\Omega')}^2 \, dt < +\infty$$

(4.10)

for any $r \in \left[1, \frac{2n}{n-\alpha-1}\right)$. Now notice that

$$1 \leq r < \frac{2n}{n-\alpha-1} \implies 1 \leq \frac{p}{2} \leq \frac{rp}{2} < \frac{np}{n-\alpha-1},$$

since $p \geq 2$. Therefore, from (4.10) it follows that

$$\int_{t_0}^{t_1} \|(|Du(\cdot,t)|^{r})^{\frac{p}{2}} \|_{L^q(\Omega')} \, dt = \int_{t_0}^{t_1} \left( \int_{\Omega'} |Du(x,t)|^{q} \, dx \right)^{\frac{2}{q}} \, dt < +\infty$$

for all $q \in \left[1, \frac{np}{n-\alpha-1}\right)$, for any open Lipschitz set $\Omega' \subset \Omega$ and any $(t_0, t_1) \subset (0, T)$. This is sufficient to ensure that

$$Du \in L^p_{\text{loc}}(0, T; L^q_{\text{loc}}(\Omega, \mathbb{R}^n)) \quad \text{for all } q \in \left[1, \frac{np}{n-\alpha-1}\right).$$

\[\square\]

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