A study of stationary states of a plasma diode in the presence of a transverse magnetic field.

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Abstract. In this paper we studied the stationary states of plasma diode in the presence of a transverse magnetic field. We considered regimes without and with electron turnaround by the magnetic field. Stationary solutions depending on the electric field strength at the emitter for different values of the electrode gap, the applied voltage and the magnetic field were found. The stability of solutions was studied.

1. Introduction

In recent years, due to the need to create effective environmentally friendly energy sources there is a significantly increased interest in the thermionic energy converters (TIC) [1, 2]. In addition to the experimental work, the research in the field of modelling of appropriate physical processes is actively conducting [3]. One of the important tasks is to study currents flowing in the TIC.

The new generation high-performance thermionic energy converter operates at high temperatures of emitter. Electrons in the inter-electrode gap move without collisions. Such a TIC can produce very strong current that creates a magnetic field. This field is perpendicular to the direction of electron current and can partially turnaround electrons. This leads to a decrease in the current passing. Instead of studying the influence of its own magnetic field, we can consider the problem of the influence of an external magnetic field on the currents of the diode. As shown earlier [4], this problem can be successfully solved by considering the Pierce diode, i.e., diode, in which a beam of electrons moves through a fixed ion background.

2. Problem statement

Consider the Pierce diode. Its scheme is shown in Fig. 1. There are two infinite parallel flat electrodes between which a voltage $U$ is applied. An external magnetic field $B$ is parallel to the electrode surface. A beam of electrons with a density $n_b$ and velocity $v_b$ emitted from the emitter ($z = 0$) and moves without collisions through a fixed ion background of constant density $n_i$ towards the collector ($z = d$).

Let us define the neutralization coefficient as the ratio of the ion density to the beam density $\gamma = n_i/n_b$. We present the results for the case of $\gamma = 1$, because this case better corresponds to the TIC. However, the results for any other values of $\gamma$ may also be obtained. Electrons moving in the diode, reach the collector but they can also be turned around by the magnetic field and return to the emitter. We suggest that in both cases they are completely absorbed by the electrodes. We believe that all quantities depend only on the $z$ coordinate. In the case when
all electrons reach the collector the process is described by the following system of differential equations (which are continuity, momentum, and the Poisson’s equations):

\[ \frac{d}{dz}(nv_z) = 0, \]
\[ v_z \frac{dv_z}{dz} = -\frac{e}{m}E - \omega v_x, \quad v_z \frac{dv_x}{dz} = \frac{eB}{m}v_z, \]
\[ E = -\frac{d\varphi}{dz}, \quad \frac{dE}{dz} = -\frac{e}{\epsilon_0}(n - \gamma). \] (1)

The Larmor frequency and radius are define as:

\[ \omega = \frac{eB}{m} = \left( \frac{2e}{m} \right) \frac{V_b^{1/2}}{\lambda_D} \text{[s}^{-1}], \quad \lambda_L = \frac{mv_b}{eB} \approx 0.3372 \cdot 10^{-3} \frac{V_b^{1/2}}{B} \text{[cm]}. \]

Here \( V_b \) is the beam accelerating voltage, \( e, m \) are charge and mass of the electron, \( \epsilon_0 \) is the free-space permittivity.

It is convenient to switch to the dimensionless coordinate, time, speed, potential and the electric field: \( (\zeta, \chi) = (z,x)/\lambda_D, \quad \tau = t\omega_b, \quad (u_\zeta, u_\chi) = (v_z, v_x)/\sqrt{2W_b/m}, \quad \eta = e\varphi/(2W_b), \quad \varepsilon = eE\lambda_D/(2W_b), \quad \Omega = \omega/\omega_b. \) Here \( \lambda_D = [2\epsilon_0W_b/(e^2n_b)]^{1/2} \) is the beam Debye length, \( W_b = mv_b^2/2 \) is the kinetic energy of electrons at the emitter, \( \omega_b = v_b/\lambda_D \) the characteristic frequency. The corresponding system of dimensionless equations takes the form:

\[ \frac{d}{d\zeta}(nu_\zeta) = 0, \]
\[ u_\zeta \frac{du_\zeta}{d\zeta} = -\varepsilon - \Omega u_\chi, \quad u_\zeta \frac{du_\chi}{d\zeta} = \Omega u_\zeta, \]
The resulting system of equations (2) is solved by introducing a Lagrangian coordinate \( \tau \).

\[
\varepsilon = - \frac{d\eta}{d\zeta}, \quad \frac{d\zeta}{d\tau} = -n + \gamma. \tag{2}
\]

The boundary conditions are of the form: density \( n(\zeta = 0) = 1 \), components of velocity \( u_\zeta(\zeta = 0) = 1, \ u_\chi(\zeta = 0) = 0 \), electric potential \( \eta(\zeta = 0) = 0 \), and electric field \( \varepsilon(\zeta = 0) = \varepsilon_0 \).

The emitter electric field \( \varepsilon_0 \) will be used as a variable parameter for our following analysis.

### 3. Results

The resulting system of equations (2) is solved by introducing a Lagrangian coordinate \( \tau \) and Lagrange transformation

\[
\zeta = \int_0^\tau u_\zeta(\tau')d\tau'.
\]

Thus, \( u_\zeta d/d\zeta = d/d\tau \). Then equations (2) take the form

\[
\frac{d\eta}{d\tau} = -\varepsilon - \Omega u_\chi, \quad \frac{d\zeta}{d\tau} = \Omega u_\zeta,
\]

\[
\frac{d\eta}{d\tau} = -u_\zeta\varepsilon, \quad \frac{d\zeta}{d\tau} = -1 + \gamma u_\zeta. \tag{3}
\]

The following analytical solutions were obtained for this system:

\[
u_\zeta(\tau) = \frac{1}{\alpha^2} + \left(1 - \frac{1}{\alpha^2}\right) \cos(\alpha \tau) - \frac{\varepsilon_0}{\alpha} \sin(\alpha \tau).
\]

\[
\zeta(\tau) = \frac{1}{\alpha^2} \tau + \frac{1}{\alpha} \left(1 - \frac{1}{\alpha^2}\right) \sin(\alpha \tau) + \frac{\varepsilon_0}{\alpha^2} \cos(\alpha \tau) - 1.
\]

\[
u_\chi(\tau) = \Omega \zeta(\tau) = \frac{\Omega}{\alpha} \tau + \frac{\Omega}{\alpha} \left(1 - \frac{1}{\alpha^2}\right) \sin(\alpha \tau) + \frac{\varepsilon_0 \Omega}{\alpha^2} \cos(\alpha \tau) - 1.
\]

\[
\varepsilon(\tau) = \varepsilon_0 - \tau + \gamma \zeta(\tau) = \frac{\Omega^2}{\alpha^2} (\varepsilon_0 - \tau) + \frac{\gamma}{\alpha} \left(1 - \frac{1}{\alpha^2}\right) \sin(\alpha \tau) + \frac{\gamma \varepsilon_0}{\alpha^2} \cos(\alpha \tau).
\]

\[
\eta(\tau) = \frac{1}{2} \left[u_\zeta^2(\tau) + \frac{\Omega^2}{\alpha^2} \right] - 1
\]

here the effective “frequency” \( \alpha = \sqrt{\gamma + \Omega^2} \) is introduced. For every value of magnetic field \( \Omega \) the family of solution of system (4) is obtained. Then the parameter \( \varepsilon_0 \) is varied. Calculations stopped when electron velocity \( u_\zeta \) vanishes at the first time (it corresponds to the moment when electrons turnaround). The threshold value of \( \varepsilon_0 \) equals to \( \pm \sqrt{2 - \alpha^2} \).

Examples of potential and velocity profiles are shown in Fig.2 and 3 respectively. As it can be seen from the figures, in weak magnetic fields, potential profile is an oscillating function, the same way as is the case in the absence of a magnetic field [5]. The oscillation period is equal to \( 2\pi(\gamma + \Omega^2)^{-3/2} \). Desired solutions correspond to points of intersection of potential curves with line \( \eta = V \). The collector position corresponds to the values: \( \tau = T, \ \zeta = \delta, \ \eta(\delta) = V \). Thus we have at the collector:

\[
\delta = \frac{1}{\alpha^2} T + \frac{1}{\alpha} \left(1 - \frac{1}{\alpha^2}\right) \sin(\alpha T) + \frac{\varepsilon_0}{\alpha^2} \cos(\alpha T) - 1,
\]

\[
u_\zeta(T) = \frac{1}{\alpha^2} + \left(1 - \frac{1}{\alpha^2}\right) \cos(\alpha T) - \frac{\varepsilon_0}{\alpha} \sin(\alpha T),
\]

\[
V = \frac{1}{2} \left[u_\zeta^2(T) + \frac{\Omega^2}{\alpha^2} \right] - 1.
\]
Here, $T$ is the time-of-flight of an electron between electrodes. With an increase of magnitude of the magnetic field, the potential distribution becomes higher than $\eta = V$ and all solutions with $\delta > \delta^* = \pi(\gamma + \Omega^2)^{-3/2}$ disappear.

It is convenient to denote the solutions by the points on the plane $(\varepsilon_0, \delta)$, where $\delta$ is coordinate of the collector, and $\varepsilon_0$ is the electric field strength at the emitter. For fixed external parameters, solutions lie on the continuous curves. We call them branches of solutions. Several of these dependencies shown in Figure 4. Branches of solutions corresponding to $\delta < \delta^*$ are Bursian branches. The emitter electric field can take negative values unlike the Bursian diode, i.e. the diode without ion background. We see that Bursian branches could have several solutions. The right point on Bursian branch called space charge limit (SCL) point. It corresponds to maximum current that can go through diode in a certain regime.
For a complete consideration of the solutions we have to take into account a possible turnaround of electrons by the magnetic field. This allows us to supplement the \( \varepsilon_0 - \delta \) diagram shown in Figure 4. We use method proposed in the paper [7]. Initially, we postulated that the electron beam is mono-energetic, so if at some point \( \zeta_r \) the longitudinal electron velocity \( u_\zeta \) vanishes then it should make all electrons turn back. Nevertheless, in real life electrons of the beam always have a finite (even if very small) velocity spread. In such a way we can assume that some electrons with a bit higher velocities could overcome zero velocity point and with lower velocities will be turned. In this case we can talk about beam “splitting”. This situation could be described via introducing reflection coefficient \( r \). This coefficient is the probability that an electron of the beam is turned around by the magnetic field. In such a way, there are direct and reverse flows in the region \( 0 < \zeta < \zeta_r \) and electron density there is \( 1/u_\zeta + r/|u_\zeta| \). To the right of \( \zeta_r \) point the direct electron current will have a weight \( 1 - r \). All solutions with partial electron turn around could be derived by varying the \( r \) coefficient.

In this case, the system of equations takes the form:

\[
\begin{align*}
\nu u_\zeta &= H(\zeta; \zeta_r, r), \\
\nu_\zeta \frac{du_\zeta}{d\zeta} &= -\varepsilon - \Omega u_\chi, \\
\nu_\zeta \frac{dx}{d\zeta} &= \Omega u_\zeta, \\
\varepsilon &= -\frac{d\eta}{d\zeta}, \quad \frac{d\varepsilon}{d\zeta} = -n + \gamma.
\end{align*}
\]

(5)

where \( H(\zeta; \zeta_r, r) = [(1 + r)\Theta(\zeta_r - \zeta) + (1 - r)\Theta(\zeta - \zeta_r)] \), and \( \Theta \) is the Heaviside function. Its solutions have the form

\[
\begin{align*}
\zeta(\tau) &= \frac{1 + r}{\alpha^2} \tau + \frac{1}{\alpha} \left(1 - \frac{1 + r}{\alpha^2}\right) \sin(\alpha \tau) + \frac{\varepsilon_0}{\alpha^2} (\cos(\alpha \tau) - 1), \\
\nu(\tau) &= \frac{1 + r}{\alpha^2} + \frac{1 - r}{\alpha^2} (\tau - \tau_r).
\end{align*}
\]

(6)

(7)

for \( \zeta < \zeta_r \) and

\[
\begin{align*}
\zeta(\tau) &= \zeta + \frac{1 - r}{\alpha^2} (\tau - \tau_r) - \frac{1 - r}{\alpha^3} \sin(\alpha (\tau - \tau_r)), \\
\nu(\tau) &= \frac{1 - r}{\alpha^2} [1 - \cos(\alpha (\tau - \tau_r))].
\end{align*}
\]

(8)

for \( \zeta > \zeta_r \). Here \( \tau_r = \alpha^{-2}[(1 + r)\tau_r - \varepsilon_0] \). Depending on the values of \( \gamma \) and \( \Omega \), the function \( \tau_r(\gamma, \Omega) \) reads for \( \varepsilon_0 \geq 0 \) as

\[
\begin{align*}
\frac{1}{\alpha} \sin^{-1} \frac{\alpha \sqrt{2}(1 + r) - \alpha^2}{1 + r}, \quad &\text{if } \Omega^2 \leq 1 + r - \gamma, \\
\frac{1}{\alpha} \left( \pi - \sin^{-1} \frac{\alpha \sqrt{2}(1 + r) - \alpha^2}{1 + r} \right), \quad &\text{if } \Omega^2 \geq 1 + r - \gamma.
\end{align*}
\]

(8)

and for \( \varepsilon_0 < 0 \) \( \tau_r \) is equal to

\[
\begin{align*}
\frac{1}{\alpha} \left( 2\pi - \sin^{-1} \frac{\alpha \sqrt{2}(1 + r) - \alpha^2}{1 + r} \right), \quad &\text{if } \Omega^2 \leq 1 + r - \gamma, \\
\frac{1}{\alpha} \left( \pi + \sin^{-1} \frac{\alpha \sqrt{2}(1 + r) - \alpha^2}{1 + r} \right), \quad &\text{if } \Omega^2 \geq 1 + r - \gamma.
\end{align*}
\]

(9)
For the potential distribution we have

\[ \eta(\tau) = \frac{1}{2} \left[ u_\zeta^2(\tau) + \Omega^2 \zeta^2(\tau) - 1 \right]. \tag{10} \]

Note that \( \varepsilon_0 \) connected with the parameter \( r \) by the formulae \( \varepsilon_0 = \pm [2(1 + r) - \alpha^2]^{1/2} \).

Solutions of the Eqs. (7)–(10) can also be plotted on \( \varepsilon_0 - \delta \) plane (see Fig. 5). We can see from this Figure that each \( \varepsilon_0 - \delta \)-curve contains an oscillating segment as \( r \) tend to 1, i.e., it is many-valued over a certain range of \( \delta \)'s. It happens because for certain value of \( r(\varepsilon_0) \), the electron velocity \( u_\zeta \) could vanish several times in the region \( \zeta > \zeta_r \) (see Eq. (8)). The Figure also shows that the amplitude of the oscillations diminishes with \( r \) for each zigzag segment. It is also found that the maximum width of the oscillating segment shrinks with the increase of \( \Omega \).

It can also be seen that along with the Bursian branches the non-Bursian branches arise. With an increase in the magnetic field these branches disappear gradually. It happens when the Larmor radius turns out to be about 10 times higher than the Debye length. There are two bifurcation points on the Bursian branches: the left one noted as BF (close circles on Fig. 5) and right one noted as SCL (open circles on Fig. 5).

We studied the stability of solutions upon small perturbations. The \( \eta - \varepsilon \) diagram method [6] were used to investigate stability upon aperiodic perturbations. We have shown that solutions belonging to the Bursian family are stable with respect to aperiodic perturbations at \( \varepsilon_0 < \varepsilon_{0,SCL} \). At the same time, those located between bifurcation points \( SCL \) and \( BF \) (\( \varepsilon_{0,SCL} < \varepsilon_0 < \varepsilon_{0,BF} \)) are unstable. Solutions are aperiodic stable when \( \varepsilon_0 > \varepsilon_{0,BF} \) although they can be oscillatory unstable and it need further study.

Then we have studied the solutions belonging to the non-Bursian family. In the region of negative \( \varepsilon_0 \) values in Fig. 5 all solutions belonging to the leftmost branch which goes down from the \( SCL \)-type bifurcation point are aperiodic stable, and these belonging to the branch which goes to the right of this bifurcation point are unstable. In the region of positive \( \varepsilon_0 \) values the lower segment of the branch located to the left of the \( SCL \)-type bifurcation point contains the aperiodic stable solutions. The segment of the branch located between this \( SCL \)-type bifurcation point and the leftmost \( BF \)-type bifurcation point contains unstable solutions. The next segment of the branch which locates between two \( BF \)-type bifurcation points corresponds to aperiodic

Figure 5. \( \varepsilon - \delta \) diagram. \( \gamma = 1 \).

Case with electron reflection.
stable solutions and so on. We also studied stability of solutions corresponding to the zigzag part of branches (see Fig. 5). We found that each left-to-right segments of the zigzag is aperiodic stable and right-to-left segments are unstable.

In order to study the stability of the solutions for the regime without electron turning around we derived the dispersion equation. It has the following form:

\[
F(\sigma; \delta, T) = -\exp(-\sigma T) \left[ 2\sigma \cos \alpha T + \frac{\sigma^2 - \alpha^2}{\alpha} \sin \alpha T \right] + (\sigma^2 + \alpha^2)^2 \delta - (\sigma^2 + \alpha^2)T + 2\sigma = 0. \tag{11}
\]

Note that in Eq. (11), \(T\) and \(\delta\) are derived from formulas (5). We found that at \(\delta < \delta^+ \approx \frac{3\pi}{(\gamma + \Omega^2)^{3/2}}\) all oscillatory eigen-modes have a negative growth rate, i.e. they are oscillatory stable whereas some solutions are aperiodic unstable. When \(\delta > \delta^+\) a portion of solutions can be oscillatory unstable as well as in the absence of the magnetic field (see, e.g., [5]).

4. Conclusion
We explored in detail stationary states of the Pierce diode \((\gamma = 1)\) in the presence of the magnetic field. Analytical expressions for electron trajectories, potential and electric field distributions were derived. We studied cases with and without electron turning by a magnetic field. The \(\varepsilon_0 - \delta\) diagrams describing states of the diode were plotted for several most distinctive cases. Also we study stability of solutions upon small perturbations.

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