Quantum Ratchets on Maximally Uniform States in Phase Space: Semiclassical Full-Chaos Regime

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Abstract

A generic kind of quantum chaotic ratchet is introduced, based on initial states that are uniform in phase space with the maximal possible resolution of one Planck cell. Unlike a classical phase-space uniform density, such a state usually carries a nonzero ratchet current, even in symmetric systems. This quantum ratchet effect basically emerges from the generic asymmetry of the state quasicoordinates in the Planck cell. It is shown, on the basis of exact results, general arguments, and extensive numerical evidence, that in a semiclassical full-chaos regime the variance of the current over all the states is nearly proportional to the chaotic-diffusion rate and to the square of the scaled Planck constant. Experimental realizations are suggested.

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Understanding quantum transport in generic Hamiltonian systems, which are classically nonintegrable and exhibit chaos, is a problem of both fundamental and practical importance. The study of simple model systems have already led to the discovery of a variety of quantum-transport phenomena [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], several of which have been observed in atom-optics experiments [3, 9, 10] and allow to control and manipulate the quantum motion of cold atoms or Bose-Einstein condensates in different ways.

Recently, classical and quantum “ratchet” transport in Hamiltonian systems has started to attract a considerable interest, both theoretically [4, 5, 6, 7, 8] and experimentally [9, 10]. Ratchets, originally proposed as mechanisms for some kinds of biological motors and as models for nanoscale devices [11], are generally conceived as spatially periodic systems with noise and dissipation in which an unbiased (zero-mean) external force can lead to a directed current of particles due to some spatial/temporal asymmetry. In classical Hamiltonian ratchets [4], dissipation is absent and noise is replaced by deterministic chaos. A basic general result [4] is that the average current of an initially uniform density in phase space is always zero, even in the presence of an asymmetry. This implies that the classical current of a chaotic region may be nonzero only for an asymmetric system with a mixed phase space. However, the corresponding quantized system can feature significant ratchet effects also under full-chaos conditions [3, 6, 7, 8, 9, 10]. An important and apparently still open problem is the nature of these effects in a semiclassical regime, in particular how precisely they vanish, as expected, in the classical limit. A proper approach to this problem should take into account the fact that the quantum current can be quite sensitive to the initial state, as indicated by recent exact results in strong quantum regimes [8]. Thus, the strength of the quantum-ratchet effects should be globally measured by average characteristics of the current over a representative large set of initial states. In order to understand these effects in a semiclassical regime and, at the same time, to exhibit fundamental differences between classical and quantum ratchets in the clearest way, it is natural to choose pure initial states that are analogous as much as possible to a classical phase-space uniform density, for which ratchet effects are totally absent. Initial states used until now to study quantum ratchets do not feature some well-defined uniformity in all phase-space directions. For example, the often used pure momentum state [3, 6, 7] is uniform in position but is infinitely localized in momentum.
In this paper, we introduce a generic kind of quantum chaotic ratchet based on initial states that are \textit{uniform} in phase space with the \textit{maximal possible} resolution of one Planck cell. Such a state is associated with a phase-space lattice whose unit-cell area is $h$ and whose origin is specified by “quasicoordinates” relative to a fixed Planck cell. We derive a general exact expression for the quantum directed current carried by a state as a function of its quasicoordinates. This current is usually \textit{nonzero}, also in \textit{completely symmetric} systems, due to the generic asymmetry of the quasicoordinates in the Planck cell. Averages of the current over quasicoordinates, corresponding to mixed states, are shown to vanish. The quantum ratchet effect is globally measured by the \textit{variance} of the current over all the maximally uniform states. We then obtain from general arguments the following main result in a semiclassical full-chaos regime: The variance of the current is nearly proportional to the chaotic-diffusion coefficient and to the square of the scaled Planck constant [see Eq. (9) below]. This implies, in particular, that a system asymmetry is \textit{not} significant in this regime since it can manifest itself only in the chaotic-diffusion rate. These results and related issues, such as independence on the Planck-cell shape, are supported by extensive numerical evidence. Finally, we briefly discuss possible experimental realizations of the theory.

We start by defining maximally uniform states in a phase space. Let $\hat{x}$ and $\hat{p}$ denote position and momentum operators, $[\hat{x}, \hat{p}] = i\hbar$. Then, $\hat{T}_x(a) = \exp(i\hat{p}a/\hbar)$ and $\hat{T}_p(b) = \exp(-i\hat{x}b/\hbar)$ are translation operators shifting $\hat{x}$ and $\hat{p}$ by $a$ and $b$, respectively. A state $|\psi\rangle$ is uniform on the phase-space lattice with unit cell formed by $a$ and $b$ if it is invariant under application of $\hat{T}_x(a)$ and $\hat{T}_p(b)$, up to constant phase factors: $\hat{T}_x(a) |\psi\rangle = \exp(i\alpha) |\psi\rangle$ and $\hat{T}_p(b) |\psi\rangle = \exp(-i\beta) |\psi\rangle$. This means that $|\psi\rangle$ is a simultaneous eigenstate of $\hat{T}_x(a)$ and $\hat{T}_p(b)$, implying that these operators must commute. It is easy to show that $[\hat{T}_x(a), \hat{T}_p(b)] = 0$ only if the unit-cell area $ab$ is a multiple of $h = 2\pi\hbar$. Maximally uniform states $|\psi\rangle$ correspond to the smallest unit cell, i.e., the Planck cell with $ab = h$, and are given by

$$\langle x |\psi\rangle = \psi_w(x) = \sum_n \exp(2\pi inw_2/b) \delta(x - w_1 - na),$$

where $w = (w_1, w_2)$, with $0 \leq w_1 < a$ and $0 \leq w_2 < b$, is related to the eigenphases $\alpha$ and $\beta$ above by $\alpha = w_2a/\hbar$ and $\beta = w_1b/\hbar$. The quantities $w_1$ and $w_2$ are quasicoordinates.
specifying, respectively, the quasiposition and quasimomentum of $\psi_w(x)$ in the Planck cell: $w_1 = x \mod(a)$, $w_2 = p \mod(b)$. The states (1) form, for all $w$, a complete set and their $p$ representation is also a delta comb, at the points $w_2 + nb$ for all $n \geq 1$. Thus, (1) is associated with the phase-space lattice $(x, p) = w + zn$, where $zn = (n_1 a, n_2 b)$ for all $n = (n_1, n_2)$. Pure momentum states may be viewed as the limit $a \to 0$ ($b = \hbar/a \to \infty$) of the states (1).

We consider the general quantum-ratchet systems described by Hamiltonians $\hat{H}(\hat{x}, \hat{p}, t)$ periodic in phase space $(\hat{x}, \hat{p})$ and in time $t$. Realistic such systems are known to exhibit robust quantum-ratchet effects, see also below. Without loss of generality, we assume that the period of $\hat{H}$ in both $(\hat{x}, \hat{p})$ is $2\pi$, implying a similar periodicity of the evolution operator $\hat{U}(\hat{x}, \hat{p})$ in one time period. We shall derive a general exact expression [Eq. (6) below] for the quantum-ratchet current carried by a state (1) in terms of the quasienergy (QE) eigenvalues and eigenstates of $\hat{U}(\hat{x}, \hat{p})$. First, we present a straightforward generalization of relevant results in Ref. [13] concerning basic QE properties of operators $\hat{U}(\hat{x}, \hat{p})$. A generic value of the scaled Planck constant $\hbar/(2\pi)$ can be approximated to arbitrary accuracy by a rational value, $\hbar/(2\pi) = q/N$ ($q$ and $N$ are coprime integers). We then choose in (1) $a = 2\pi q_1/N_1$ and $b = 2\pi q_2/N_2$ for some given integers $(q_1, q_2)$ and $(N_1, N_2)$ satisfying $q_1 q_2 = q$ and $N_1 N_2 = N$. Clearly, $\hat{U}$ commutes with both $\hat{T}_x^{N_1}(a)$ and $\hat{T}_p^{N_2}(b)$, so that one can find simultaneous eigenstates of these three commuting operators. It is easy to see that the general eigenstates of $\hat{T}_x^{N_1}(a)$ and $\hat{T}_p^{N_2}(b)$ can be expressed in terms of (1) as follows:

$$\Psi_{j, w}(x) = \sum_{m_1=0}^{N_2-1} \sum_{m_2=0}^{N_1-1} \phi_j(m_1, m_2; w) \psi_{w_1+m_1 a, w_2+m_2 b}(x),$$

where the index $j$, $j = 1, \ldots, N$, labels $N$ independent vectors of coefficients, $V_j(w) = \{\phi_j(m; w)\}$, for $m \equiv (m_1, m_2)$ with $m_1 = 0, \ldots, N_2 - 1$ and $m_2 = 0, \ldots, N_1 - 1$. These coefficients are determined by requiring (2) to be QE eigenstates of $\hat{U}$: $\hat{U} \Psi_{j, w}(x) = \exp[-i\omega_j(w)] \Psi_{j, w}(x)$, where $\omega_j(w)$ are $N$ QE bands. This leads to the eigenvalue equation $\hat{M}(w)V_j(w) = \exp[-i\omega_j(w)]V_j(w)$, where $\hat{M}(w)$ is an $N \times N$ unitary matrix whose elements can be expressed in terms of the Fourier coefficients of $\hat{U}(\hat{x}, \hat{p})$, as in Ref. [13].
Now, the quantum-ratchet current in the $p$ direction for the initial state $\text{1}$ is given by

$$I(w) = \lim_{s \to \infty} \frac{1}{s} \int_{0}^{2\pi q_1} dx [\hat{U}^s \psi_w(x)]^* \left(-i\hbar \frac{d}{dx}\right) [\hat{U}^s \psi_w(x)]$$

(s integer), where the integration range $2\pi q_1$ is just the minimum common multiple of the periods $a = 2\pi q_1/N_1$ and $2\pi$ of $\text{1}$ and $\hat{U}$, respectively; the current in the $x$ direction is defined similarly. It is understood that one has to replace $\psi_w(x)$ in $\text{1}$ by a function $\tilde{\psi}_w(x)$ which is normalized in $0 \leq x < 2\pi q_1$ and which tends to $\psi_w(x)$ in some limit. For example,

$$\tilde{\psi}_w(x) = \frac{\exp(iw_2x/\hbar)}{\sqrt{2\pi q_1(2B + 1)}} \sum_{n=-B}^{B} \exp[2\pi in(x - w_1)/a]$$

($B$ integer); essentially, $\tilde{\psi}_w(x) \to \psi_w(x)$ as $B \to \infty$. Let us invert Eq. (2) using the completeness of the (orthonormal) eigenvectors $V_j(w)$, i.e., $\sum_{j=1}^{N} \phi_j^*(m; w)\phi_j(m'; w) = \delta_{m,m'}$:

$$\psi_w(x) = \sum_{j=1}^{N} \phi_j^*(0; w)\Psi_{j,w}(x).$$

After inserting $\text{3}$ in $\text{3}$ and using the eigenvalue equation above for $V_j(w)$, we find that the dominant terms for large $s$ are contributed by the functions $\partial \hat{U}^s(\hat{x}, \hat{p})/\partial \hat{x} \psi_{j,w}(x)$ given by $\sum_m \tilde{\phi}_{j,s}(m; w)\psi_{w+mh}(x)$, where the coefficients $\tilde{\phi}_{j,s}(m; w)$ are the components of the vectors $\partial \hat{M}^s(w)/\partial w_1 V_j(w)$ whose dominant behavior for $s \gg 1$ is $-is\partial \omega_j(w)/\partial w_1 \exp[-is\omega_j(w)]V_j(w)$. We then obtain the exact result:

$$I(w) = -\hbar \sum_{j=1}^{N} |\phi_j(0; w)|^2 \frac{\partial \omega_j(w)}{\partial w_1}.$$  

(6)

Clearly, the current (6) is nonzero for generic values of $w$, even for a symmetric system, in which case $I(w)$ may generally vanish only at symmetry points $w$ of the bands $\omega_j(w)$. Let us assume, for definiteness, that the QE eigenvalues $\exp[-i\omega_j(w)]$ are nondegenerate at fixed $w$. This implies, using (2) and considerations similar to those in Ref. (13), that:

(a) Each QE band $\omega_j(w)$ is periodic in both $w_1$ and $w_2$ with period $2\pi/N_1$ and $2\pi/N_2$, respectively; these periods define a unit cell $C$ that $q$ times smaller than the Planck cell. (b) $|\phi_j(0; w)|$ is periodic in $w_1$ and $w_2$ with periods $2\pi/N_1$ and $2\pi/N_2$, respectively; these periods define a unit cell $C$. (c) $|\phi_j(m; w)| = |\phi_j(0; w + mh)|$. It follows from (a), (b), and (3) that $I(w)$ is periodic in $w$ with unit cell $C$. Using (a), (c), (3), and the normalization of $V_j(w)$, $\sum_m |\phi_j(m; w)|^2 = 1$, we get that $\int_{w_2}^{2\pi/N} dw_1 \sum_m I(w + mh) = 0$ at any fixed $w_2$. This result specifies general mixtures of the pure states (1) carrying a zero
mean current and implies, in particular, that \( \int_C d\mathbf{w} I(\mathbf{w}) = 0 \).

The fluctuations of \( I(\mathbf{w}) \) around its zero mean are measured by the variance:

\[
(\Delta I)^2 = \frac{q}{\hbar} \int_C d\mathbf{w} I^2(\mathbf{w}).
\]

(7)

The behavior of \( \Delta I \) will now be studied in a semiclassical full-chaos regime using arguments based on classical analogues of the states (1). For simplicity, we assume that \( q = 1 \) (\( \hbar = 2\pi/N \)), so that \( C \) is just the Planck cell, \( N \) times smaller than the basic torus of periodicity of the system, \( T^2: 0 \leq x, p < 2\pi \). The natural classical analogue of the state (1) is the phase-space lattice \( \mathbf{w} + \mathbf{z}_n \) above, where now \( \mathbf{z}_n = (2\pi n_1/N, 2\pi n_2/N) \). This can be restricted to a finite lattice of \( N \) points in \( T^2 \) with \( n_1 = 0, \ldots, N_1 - 1 \) and \( n_2 = 0, \ldots, N_2 - 1 \). Denoting by \( p_s(\mathbf{w} + \mathbf{z}_n) \) the momentum evolving from initial condition \( \mathbf{w} + \mathbf{z}_n \) after \( s \) time periods, the corresponding average classical current is \( I_s^{(c)}(\mathbf{w} + \mathbf{z}_n) = \Delta p_s(\mathbf{w} + \mathbf{z}_n)/s \), where \( \Delta p_s(\mathbf{w} + \mathbf{z}_n) = p_s(\mathbf{w} + \mathbf{z}_n) - w_2 - n_2b \). The average current on the finite lattice, analogous to \( I(\mathbf{w}) \), is \( \bar{I}_s^{(c)}(\mathbf{w}) = \sum_n I_s^{(c)}(\mathbf{w} + \mathbf{z}_n)/N \). In analogy to \( \int_C d\mathbf{w} I(\mathbf{w}) = 0 \) above, one has \( \int_C d\mathbf{w} \bar{I}_s^{(c)}(\mathbf{w})/\hbar = \int_{T^2} dz I_s^{(c)}(z)/(4\pi^2) = 0 \), where \( z \equiv (x, p) \) and the latter equality is the basic uniformity result in Ref. [4]. We then obtain the classical analogue of (7) \( (q = 1) \):

\[
(\Delta \bar{I}_s^{(c)})^2 = \frac{1}{\hbar} \int_C d\mathbf{w} \left[ \bar{I}_s^{(c)}(\mathbf{w}) \right]^2 = \frac{2}{N\pi} \sum_n D_s(n),
\]

(8)

where \( D_s(n) = \int_{T^2} dz \Delta p_s(z)\Delta p_s(z + \mathbf{z}_n)/(8\pi^2 s) \) are correlations of \( \Delta p_s \) in phase space. For sufficiently large \( s \), \( D_s(0) \) \( (z_0 = 0) \) is approximately the chaotic-diffusion coefficient \( D \), \( \langle (\Delta p_s)^2 \rangle_{T^2} \approx 2Ds \), while \( D_s(n) \) for \( n \neq 0 \) \( (z_n \neq 0) \) should be negligible since it is expected to decay with \( s \). Thus, \( \bar{I}_s^{(c)}(\mathbf{w}) \) is nearly given by \( 2D/(N\pi) \), showing how the full uniformity limit \( (\Delta \bar{I}_s^{(c)} = 0) \) is approached by increasing the lattice size \( N \) and/or the length \( s \) of a chaotic orbit which will then visit almost uniformly the phase space. The quantum evolution should mimic the classical one only up to a “break time” \( s \sim s^* \sim 2\pi/\Delta \omega = N \), where \( \Delta \omega = 2\pi/N \) is the mean spacing between neighboring QE levels. Then, assuming that the limit in (3) is essentially reached for \( s \sim s^* \sim N \gg 1 \) (semiclassical regime), the variance (7) will be approximately equal to \( (\Delta \bar{I}_s^{(c)})^2 \sim 2D/(N\pi) \) with \( s \sim N \):

\[
(\Delta I)^2 \sim \frac{2D}{N^2} = \frac{D\hbar^2}{2\pi^2}.
\]

(9)

We have extensively checked the result (9) and related issues using the generalized kicked Harper models [2, 13] with \( \hat{U} = \exp[-iL\cos(\hat{p})/\hbar] \exp[-iKV(\hat{x})/\hbar] \), where \( V(\hat{x}) \)
is a $2\pi$-periodic potential and $(K, L)$ are parameters. These are realistic models \cite{2,14} (see below) and were shown recently \cite{7} to exhibit robust quantum ratchet effects. The classical phase space is, in practice, fully chaotic for sufficiently large $(K, L)$ and, as $K, L \to \infty$, $D/D_{ql} \to 1$, where $D_{ql} = \int_0^{2\pi} dx [KdV(x)/dx]^2/2$ is the quasilinear value of $D$. We first verified numerically for several potentials, parameter values, and Planck-cell shapes ($a = 2\pi/N_1, b = 2\pi/N_2$), $N_1N_2 = N$, that $D_s(n)$ in \cite{8} is indeed negligible for $n \neq 0$ and $s = N$ if $D_N(0)$ is well converged to $D$; then, $(\Delta \bar{f}_N^{(c)})^2 \approx 2D/N^2$ to a very good accuracy. The quantum variance \cite{7} was calculated using the exact Eq. \cite{6}. According to Rel. \cite{9}, $R \equiv (N\Delta I)^2/(2D_{ql})$ should be approximately equal to $D/D_{ql}$, independently of the Planck-cell shape. This was supported by much numerical evidence. As representative examples, Figs. 1 and 2 show $R$ versus $K$ in a completely symmetric and strongly asymmetric case, respectively. In both cases, we present results for two distinct factorizations of $N = 121$: The “most uniform” factorization $N_1 = N_2 = 11$ (square Planck cell) and $N_1 = 121, N_2 = 1$ (Planck cell elongated in the $p$ direction). Also shown is $D_N(0)/D_{ql} \approx D/D_{ql}$. We see that there is generally a reasonably good agreement between $R$ and $D_N(0)/D_{ql}$, with only a weak dependence on the factorization, especially in the symmetric case (Fig. 1). In this case, discrepancies arise only around values of $K$ where small accelerator-mode islands exist; these lead to the divergence of $D$ and to enhanced finite values of $R$ and $D_N(0)/D_{ql}$. Fig. 3 shows plots of $\Delta I$ versus $N$. The results agree very well with the $N^{-2}$ or $\hbar^2$ behavior predicted by Rel. \cite{9}. The value of $\Delta I$ for sets of states less uniform than \cite{11} was found to be significantly larger than that in Figs. 1-3. For example, for the set of QE states $\Psi_{j,w}(x)$, with currents $I_j(w) = -\hbar \partial \omega_j(w)/\partial w_1$, $(\Delta I)^2$ is nearly $N$ times larger than \cite{9}.

In conclusion, we have introduced a generic kind of quantum ratchet in fully chaotic systems, based on maximally uniform states in phase space and described by average characteristics of the directed current over these states. The fundamental source of this current is the generic asymmetry of the quasicoordinates of a state in the Planck cell. A quantum ratchet effect then arises also in completely symmetric systems from this single unremovable source. According to Rel. \cite{9}, the average measure $\Delta I$ of the effect decays as $\hbar$ in a semiclassical regime (see Fig. 3) and a system asymmetry can affect $\Delta I$ only in a non-substantial manner, i.e., through the chaotic diffusion rate which does not change.
significantly from completely symmetric to strongly asymmetric systems (compare Figs. 1 and 2). We also found $\Delta I$ to be almost independent on the Planck-cell shape, again in accordance with Rel. (9). The generalized kicked Harper models, used to illustrate our results, are related to realistic systems, such as kicked charges in a magnetic field [2] and modified kicked rotors [14], and may be thus experimentally realizable by, e.g., atom-optics techniques. A state [1] can be well approximated by the finite superposition [4] of plane waves (pure momentum states) whose number is small ($\sim 1$) if the Planck cell is sufficiently elongated in the $p$ direction, as for the second factorization in Figs. 1-3. A superposition of two plane waves was used quite recently [10] in experimental realizations of quantum-resonance ratchets and superpositions of more plane waves may be similarly prepared [15]. The new generic quantum ratchet effect presented here should be then experimentally observable to some extent.

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FIGURE CAPTIONS

FIG. 1. Plots of $R \equiv (N\Delta I)^2/(2D_{ql})$ versus $K$ in the case of the symmetric kicked Harper model, with $V(x) = \cos(x)$ and $L = K$ ($D_{ql} = K^2/4$), for two factorizations of $N = 2\pi/\hbar = 121$: $N_1 = N_2 = 11$ (squares) and $N_1 = 121$, $N_2 = 1$ (filled circles). Crosses joined by line: The scaled diffusion rate $D_N(0)/D_{ql}$ versus $K$ for this model ($N = 121$). In all the figures, the average over $w$ in Eqs. (7) and (8) was made on a grid of $2500N$ points covering the Planck cell.

FIG. 2. Similar to Fig. 1 but for the strongly asymmetric kicked Harper model with $V(x) = \cos(x) + \sin(2x)$ and $L = K/2$ ($D_{ql} = 5K^2/4$).

FIG. 3. Loglog plots of $\Delta I$ versus $N = 2\pi/\hbar$ in the interval $49 \leq N \leq 169$ ($N$ odd) for $N_1 = N_2$ (filled squares) and $N_1 = N$, $N_2 = 1$ (circles) in the case of the symmetric kicked Harper model with $V(x) = \cos(x)$ and $L = K = 15$. The linear fit to the data has slope $-1.04 \pm 0.06$ for $N_1 = N_2$ and slope $-0.99 \pm 0.01$ for $N_1 = N$, $N_2 = 1$. 

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Figure 1
Figure 3