A DESCRIPTION OF \((C_p[L_p(M)], R_p[L_p(M)])_\theta\)

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Abstract. We give a simple explicit description of the norm in the complex interpolation space \((C_p[L_p(M)], R_p[L_p(M)])_\theta\) for any von Neumann algebra \(M\) and any \(1 \leq p \leq \infty\).

Let \(M\) be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace \(\tau\). Let \(L_p(M)\) be the associated noncommutative \(L_p\)-space. Given an integer \(n\) let \(C^n[M]\) (resp. \(R^n[M]\)) be \(M^n\) equipped with the following norm

\[
\| \sum_{k=1}^n x_k^* x_k \|_{M^{1/2}} \quad (\text{resp. } \| \sum_{k=1}^n x_k x_k^* \|_{M^{1/2}}).
\]

We view \((C^n[M], R^n[M])\) as a compatible couple by identifying algebraically both \(C^n[M]\) and \(R^n[M]\) with \(M^n\). Then we can consider the complex interpolation space \((C^n[M], R^n[M])_\theta\) for \(0 < \theta < 1\) (cf. [1] for complex interpolation). Pisier [12] described this interpolation norm by the following simple formula: For any \((x_1, \ldots, x_n) \in M^n\)

\[
\|(x_1, \ldots, x_n)\|_{(C^n[M], R^n[M])_\theta} = \left\| \sum_{k=1}^n L_{x_k^*} R_{x_k} \right\|_{B(L_p(M))},
\]

where \(1/p = \theta\), and where \(L_x\) (resp. \(R_x\)) denotes the multiplication on \(L_p(M)\) by \(x\) from the left (resp. right). Haagerup [3] then extended this formula to any von Neumann algebra \(M\), at least for \(\theta = 1/2\). In this case \(L_2(M)\) can be any Hilbert space at which \(M\) acts standardly.

Pisier used [11] as a tool in his study of the problem when there is a contractive projection from a super von Neumann algebra \(N\) onto \(M\). In particular, when \(N = B(L_2(M))\) \((M\) being then represented by left multiplication on \(L_2(M))\), this problem reduces to the injectivity of \(M\). We refer the interested reader to [13], [12], [11] and [3] for more information. Here we content ourselves only by mentioning the following result from [13]: \(M\) is injective iff

\[
\|(x_1, \ldots, x_n)\|_{(C^n[M], R^n[M])_{1/2}} = \left\| \sum_{k=1}^n x_k \otimes \bar{x}_k \right\|_{M_{\min} \otimes \bar{M}}.
\]

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The purpose of this note is to consider the $L_p$-space version of (1) for any $1 \leq p \leq \infty$, i.e., to give a simple description of the norm of the interpolation space $(C_0^n[L_p(M)], R_p^n[L_p(M)])$, for any von Neumann algebra $M$.

It is well known by now that there are several equivalent constructions of the noncommutative $L_p$-spaces associated with a general von Neumann algebra. In this note we use those constructed by Haagerup [4]. Our reference for these spaces is [13]. In the sequel, $M$ will denote a general von Neumann algebra, unless explicitly stated otherwise. $L_p(M)$ stands for the Haagerup noncommutative $L_p$-space based on $M$. However, whenever $M$ is semifinite, we will always consider $L_p(M)$ as defined by a normal semifinite faithful trace. We refer to the survey [17] for semifinite noncommutative $L_p$-spaces and references therein.

Let $M$ be a von Neumann algebra and $1 \leq p \leq \infty$. Given an integer $n \in \mathbb{N}$ we denote by $C_p^n[L_p(M)]$ (resp. $R_p^n[L_p(M)]$) $L_p(M)^n$ equipped with the following norm

$$\|(\sum_{k=1}^{n} x_k^* x_k)^{1/2}\|_p \quad \text{(resp.} \quad \|(\sum_{k=1}^{n} x_k x_k^*)^{1/2}\|_p\).$$

$C_0^n[L_\infty(M)]$ (resp. $R_\infty^n[L_\infty(M)]$) is, of course, $C_0^n[M]$ (resp. $R_\infty^n[M]$) introduced previously. As before in the case of $p = \infty$ we regard $(C_p^n[L_p(M)], R_p^n[L_p(M)])$ as a compatible couple by identifying $C_0^n[L_p(M)]$ and $R_\infty^n[L_p(M)]$ with $L_p(M)^n$. The main result of this note is the following generalization of (1). $\| \|_p$ denotes the norm in $L_p(M)$.

**Theorem 1.** Let $1 \leq p \leq \infty$ and $0 < \theta < 1$. Let $r, r_0(\theta)$ and $r_1(\theta)$ be determined by

$$\frac{1}{r} = 1 - \frac{2}{\max(p, p')}, \quad \frac{1}{r_0(\theta)} = \frac{\theta}{2r}, \quad \frac{1}{r_1(\theta)} = \frac{1 - \theta}{2r},$$

where $p'$ is the index conjugate to $p$. Let $x = (x_1, ..., x_n) \in L_p(M)^n$.

i) If $p \leq 2$, then

$$\|x\|_{(C_p^n[L_p(M)], R_p^n[L_p(M)])_\theta} = \inf \left\{ \|a\|_{r_0(\theta)} \|b\|_{r_1(\theta)} \left( \sum_{k} \|y_k\|_2^2 \right)^{1/2} \right\},$$

where the infimum runs over all factorizations of $x$ as $x_k = ay_k b$ with $a \in L_{r_0(\theta)}(M), b \in L_{r_1(\theta)}(M)$, and $y_k \in L_2(M)$ ($1 \leq k \leq n$).

ii) If $p \geq 2$, then

$$\|x\|_{(C_p^n[L_p(M)], R_p^n[L_p(M)])_\theta} = \sup \left\{ \left( \sum_{k} \|ax_kb\|_2^2 \right)^{1/2} \right\},$$

where the supremum runs over all $a$ and $b$ respectively in the unit balls of $L_{r_0(\theta)}(M)$ and $L_{r_1(\theta)}(M)$.

After having completed this note, we learnt from Marius Junge that he and Parcet had obtained a result similar to (even more general than) Theorem 1 (see [7]).

Clearly, (3) in the case of $p = \infty$ reduces to (1) for a semifinite $M$. For a general $M$ we get the following extension of Haagerup’s result to all $\theta \in (0, 1)$. 

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Corollary 2. Let $M$ be a von Neumann algebra. Let $0 < \theta < 1$ and $\frac{1}{p} = \theta$. Then for any $x = (x_1, \ldots, x_n) \in M^n$ we have

$$
\left\| (x_1, \ldots, x_n) \right\|_{(C_p^n[M], R^n[M])_\theta} = \left\| \sum_{k=1}^n Lx_k^* R_{x_k} \right\|_{B(L_p(M))}.
$$

It is a routine exercise to extend the theorem above to the case of infinite sequences. Indeed, let $C_p[L_p(M)]$ be the completion (relative to the $w^*$-topology for $p = \infty$) of the family of all finite sequences in $L_p(M)$ with respect to the following norm

$$
\left\| \left( \sum_{n} x_n^* x_n \right)^{1/2} \right\|_p.
$$

It is easy to see that $C_p[L_p(M)]$ consists of all sequences $x = (x_n)$ in $L_p(M)$ such that

$$
\sup_n \left\| \left( \sum_{k=1}^n x_k^* x_k \right)^{1/2} \right\|_p < \infty
$$

and the norm of $x$ is equal to the supremum above. Similarly, we define $R_p[L_p(M)]$ as the space of all sequences $(x_n)$ in $L_p(M)$ such that $(x_n^*) \in C_p[L_p(M)]$ equipped with the norm

$$
\|(x_n)\|_{R_p[L_p(M)]} = \|(x_n^*)\|_{C_p[L_p(M)]}.
$$

It should be pointed out that the two norms in $C_p[L_p(M)]$ and $R_p[L_p(M)]$ are in general not comparable at all. Again, we view $(C_p[L_p(M)], R_p[L_p(M)])$ as a compatible couple by injecting both $C_p[L_p(M)]$ and $R_p[L_p(M)]$ into $\ell_\infty(L_p(M))$. Then Theorem 1 still holds for $(C_p[L_p(M)], R_p[L_p(M)])_\theta$ without any change, except that in the case of $p = \infty$, the norm on the left hand side of (3) should be replaced by that of $(C_\infty[L_\infty(M)], R_\infty[L_\infty(M)])_\theta$, the space constructed by the second complex interpolation method.

Before proceeding to the proof of Theorem 1 let us make some comments for the readers familiar with operator space theory. First, such a reader might have already realized that $C_p[L_p(M)]$ (resp. $R_p[L_p(M)]$) is not only a pure notation but the $p$-column space $C_p$ (resp. the $p$-row space $R_p$) with values in $L_p(M)$ in Pisier’s language [14]. Here, $C_p$ (resp. $R_p$) is defined as the (first) column (resp. row) subspace of the Schatten class $S_p$. All noncommutative $L_p$-spaces are equipped with their natural operator space structure (see [14], [5] and [4]).

Second, let $M$ be an injective von Neumann algebra and $E$ an operator space. We then have the vector-valued noncommutative $L_p$-space $L_p(M; E)$ as defined in [14]. (Note that this is done in [14] with the additional assumption that $M$ is semifinite. However, the type $\text{III}$ case can be dealt with similarly.) In this language we have

$$
C_p[L_p(M)] = L_p[M; C_p] \quad \text{and} \quad R_p[L_p(M)] = L_p[M; R_p].
$$

Then by [14], for any $0 < \theta < 1$

$$
(L_p[M; C_p], L_p[M; R_p])_\theta = L_p[M; (C_p, R_p)_\theta] = L_p[M; C_q],
$$

where $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p'}$. Thus for an injective $M$, Theorem 1 can be restated as follows.
Corollary 3. Keep the notations in Theorem 1 with the additional assumption that 
\( M \) is injective. Let \( q \) be defined by \( \frac{1}{q} = \frac{1-p}{p} + \frac{a}{q} \). Let \( x \) be a finite sequence in 
\( L_p(M) \). Then if \( p \leq 2 \),
\[
\|x\|_{L_p[M:C_q]} = \inf \left\{ \|a\|_{r_0(\theta)} \|b\|_{r_1(\theta)} \left( \sum_k \|y_k\|_2^2 \right)^{1/2} \right\},
\]
where the infimum runs over all factorizations of \( x \) as \( x_k = ay_kb \) with \( a \in L_{r_0(\theta)}(M), \)
\( b \in L_{r_1(\theta)}(M) \) and \( y_k \in L_2(M) \) (\( k \geq 1 \)); if \( p \geq 2 \),
\[
\|x\|_{L_p[M:C_q]} = \sup \left\{ \left( \sum_k \|ax_kb\|_2^2 \right)^{1/2} \right\},
\]
where the supremum runs over all \( a \) and \( b \) respectively in the unit balls of 
\( L_{r_0(\theta)}(M) \) and \( L_{r_1(\theta)}(M) \).

Remark. In the case where \( M = B(\ell_2) \), (5) has been already known before. In fact, the case \( p = \infty \) is [13] Theorem 8.4. On the other hand, the case \( 2 \leq p < \infty \)
is proved in [21]. These results are particularly useful for studying the complete 
boundedness of maps with values in \( C_q \) (see [16], [21]).

The rest of the note is devoted to the proof of Theorem 1. For notational 
simplicity, we will denote the norm of \((C_p^n[L_p(M)], R_p^n[L_p(M)])_b \) by \( \| \|_{p,0} \), and 
given \( x = (x_1, \ldots, x_n) \in (L_p(M))^n \) define \( \alpha_{p,\theta}(x) \) to be the infimum in (2) if \( p \leq 2 \)
and the supremum in (4) if \( p \geq 2 \). In the following, we will also use these notations 
for \( \theta = 0 \) and \( \theta = 1 \). In these extreme cases, \( \alpha_{p,\theta}(x) \) is defined by the same 
formulas; but \( \|x\|_{p,0} \) (resp. \( \|x\|_{p,1} \)) must be, of course, replaced by \( \|x\|_{C_p^n[L_p(M)]} \)
(resp. \( \|x\|_{R_p^n[L_p(M)]} \)), as usual in interpolation theory.

We will need the following lemma.

Lemma 4. Let \( 2 < p < \infty \) and \( p' \) be the conjugate index of \( p \). Then the dual norm 
of \( \alpha_{p,\theta} \) on \( L_p(M)^n \) is equal to \( \alpha_{p',\theta} \).

Proof. Let \( y = (y_1, \ldots, y_n) \in L_{p'}(M)^n \). Define 
\[
\ell_y : L_p(M)^n \rightarrow \mathbb{C} \quad \text{by} \quad \ell_y(x) = \sum_k \text{tr}(y_k^*x_k),
\]
where \( \text{tr} \) is the distinguished tracial functional on \( L_1(M) \). Then it is easy to see that 
\[
|\ell_y(x)| \leq \alpha_{p',\theta}(y) \alpha_{p,\theta}(x).
\]
It follows that \( \ell_y \) is a continuous functional on \( (L_p(M))^n \), \( \alpha_{p,\theta} \) and of norm \( \leq \)
\( \alpha_{p',\theta}(y) \).

Conversely, assume that \( \ell \) is a continuous functional on \( (L_p(M))^n \), \( \alpha_{p,\theta} \). Since 
the restriction of \( \alpha_{p,\theta} \) to each component of \( L_p(M)^n \) coincides with the norm of 
\( L_p(M) \), there is \( y \in L_{p'}(M)^n \) such that \( \ell = \ell_y \) as above. Thus for any \( x \in L_p(M)^n \)
we have
\[
|\ell(x)| = \left| \sum_k \text{tr}(y_k^*x_k) \right| \leq \|\ell\| \alpha_{p,\theta}(x) = \|\ell\| \sup \left\{ \left| \sum_k \text{tr}(z_k^*ax_kb) \right| \right\},
\]
where the supremum is taken over all \( a, b \) and \( z \), respectively, in the unit balls 
of \( L_{r_0(\theta)}(M) \), \( L_{r_1(\theta)}(M) \) and \( L_2(L_2(M)) \). One can further require \( a \) and \( b \) to be 
positive. Since the left and right supports of the \( y_k \) are \( \sigma \)-finite projections, so is 
their supremum \( e \). Replacing \( M \) by the reduced algebra \( eMe \) if necessary, we can
assume $M$ itself $\sigma$-finite, and so $L_p(M)$ can be constructed from a normal faithful state on $M$.

By a typical minimax principle (cf. e.g. 2 Lemma 2.3.1)), or alternatively, a Hahn-Banach separation argument by convexifying 10 (cf. 10), we deduce from 50 that there are $a \geq 0, b \geq 0$ and $z$, respectively, in the unit balls of $L_{r_0(\theta)}(M)$, $L_{r_1(\theta)}(M)$ and $\ell^2_2(L_2(M))$ such that

$$|\ell(x)| \leq \|\ell\| \left(\sum_{k} \text{tr}(z_k^*ax_kb)\right) \leq \|\ell\| \left(\sum_{k} \|ax_kb\|_2^2\right)^{1/2}, \quad \forall x \in L_p(M)^n.$$

Hence $(ax_1, \ldots, ax_n) \mapsto \ell(x)$ extends to a continuous functional on $\ell^n_2(L_2(M))$ of norm $\leq \|\ell\|$. Therefore, there is $u \in \ell^n_2(L_2(M))$ such that

$$\ell(x) = \sum_{k} \text{tr}(u_k^*ax_kb) \quad \text{and} \quad \left(\sum_{k} \|u_k\|_2^2\right)^{1/2} \leq \|\ell\|.$$

Recalling that $\ell = \ell_y$, we have

$$\text{tr}(y_k^*x_k) = \text{tr}(u_k^*ax_kb) = \text{tr}(bu_k^*ax_k), \quad \forall x_k \in L_p(M), \ 1 \leq k \leq n.$$

It then follows that $y_k = au_kb$ for all $k$, and so $\alpha_{q',\theta}(y) \leq \|\ell\|$. Thus the lemma is proved.

**Remarks.**

i) The proof above also applies to normal functionals on $(M^n, \alpha_{q,\theta})$: if $\ell$ is a normal functional on $(M^n, \alpha_{q,\theta})$, then $\ell = \ell_y$ for some $y \in L_1(M)^n$ and $\|\ell\| = \alpha_{q,\theta}(y)$. Consequently, $(L_1(M)^n, \alpha_{1,\theta})$ is the predual of $(M^n, \alpha_{q,\theta})$.

ii) Let $1 \leq q < 2$. It is easier to show that the dual space of $(L_q(M)^n, \alpha_{q,\theta})$ is $(L_{q'}(M)^n, \alpha_{q',\theta})$. Then we can recover Lemma 10 by reflexivity. However, this argument does not seem to yield the previous remark on the normal functionals on $(M^n, \alpha_{\infty,\theta})$, which will be also needed later.

We will further need a well-known unpublished result by Haagerup [3]. To state it, we first recall that a weight $\varphi$ on $M$ is called strictly normal if there is a family $\{\varphi_i\}_{i \in I}$ of normal positive functionals with pairwise disjoint supports such that

$$\varphi = \sum_i \varphi_i.$$

Any von Neumann algebra admits a strictly normal semifinite faithful weight. As usual, $\sigma_t^\varphi$ stands for the modular automorphism group of a weight $\varphi$.

**Theorem 5. (Haagerup)** Let $\varphi$ be a strictly normal semifinite faithful weight on $M$. Then there are a von Neumann algebra $\mathcal{M}$, a strictly normal semifinite faithful weight $\hat{\varphi}$ on $\mathcal{M}$ and an increasing family $\{\mathcal{M}_i\}_{i \in I}$ of $w^*$-closed $*$-subalgebras of $\mathcal{M}$ satisfying the following properties:

i) $M$ is a von Neumann subalgebra of $\mathcal{M}$, $\hat{\varphi} |_M = \varphi$ and $\sigma_t^\varphi |_M = \sigma_t^\hat{\varphi}$ for all $t \in \mathbb{R}$;

ii) there is a normal faithful conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $M$ such that $\hat{\varphi} \circ \mathcal{E} = \varphi$ and $\sigma_t^\varphi \circ \mathcal{E} = \mathcal{E} \circ \sigma_t^\hat{\varphi}$ for all $t \in \mathbb{R}$;

iii) each $\mathcal{M}_i$ is finite and $\sigma$-finite and their union is $w^*$-dense in $\mathcal{M}$;

iv) for every $i \in I$ there is a normal conditional expectation $\mathcal{E}_i$ from $\mathcal{M}$ onto $\mathcal{M}_i$ such that $\mathcal{E}_i \circ \mathcal{E}_j = \mathcal{E}_j \circ \mathcal{E}_i$ whenever $i \leq j$ and $\sigma_t^\varphi \circ \mathcal{E}_i = \mathcal{E}_i \circ \sigma_t^\hat{\varphi}$, $t \in \mathbb{R}$, $i \in I$. 
Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Both \( p = 2 \) and \( p' \) are trivially true for \( p = 2 \). Thus in the following we assume \( p \neq 2 \). The proof is divided into three steps. We will first prove \( \text{2} \) in the case where \( M \) is semifinite (or only finite). The main ingredient of this part is an operator-valued version of the classical Szegő factorization theorem. Then we will use Theorem \( \text{6} \) to reduce the general case to the finite one. Finally, we will get \( \text{3} \) by duality from \( \text{2} \).

**Step 1: The proof of \( \text{2} \) for finite \( M \).** Let \( 1 \leq p < 2 \). In this first part we will prove \( \text{2} \) with the additional assumption that \( M \) is finite and \( \sigma \)-finite. Thus we can assume that \( L_p(M) \) is defined on \( M \) by a normal finite faithful normalized trace \( \tau \).

Let us first prove that \( \|x\|_{p,\theta} \leq \alpha_{p,\theta}(x) \) for any \( x = (x_1, \ldots, x_n) \in (L_p(M))^n \). This is easy for \( \theta = 0 \) and \( \theta = 1 \). Indeed, let \( \alpha_{p,0}(x) < 1 \), and let \( x_k = ay_k b \) be a factorization of \( x \) such that

\[
\|a\|_\infty < 1, \quad \|b\|_{2r} < 1 \quad \text{and} \quad \sum_{k=1}^n \|y_k\|_2^2 < 1.
\]

Then

\[
\sum_k a_k^* x_k \leq \|a\|_\infty^2 \sum_k b_k^* y_k^* y_k b \leq b^* (\sum_k y_k^* y_k) b.
\]

Thus by the Hölder inequality

\[
\|x\|_{p,0} = \|\left( \sum_k x_k^* x_k \right)^{1/2} \|_p \leq \|b^*\|_{2r}^{1/2} \|b\|_{2r}^{1/2} \|\left( \sum_k y_k^* y_k \right)^{1/2}\|_2 < 1.
\]

It follows that \( \|x\|_{p,0} \leq \alpha_{p,0}(x) \). Similarly, \( \|x\|_{p,1} \leq \alpha_{p,1}(x) \). Then by complex interpolation for trilinear maps (cf. [1] Theorem 4.1.1), we deduce \( \|x\|_{p,\theta} \leq \alpha_{p,\theta}(x) \) for all \( 0 < \theta < 1 \).

It is the converse inequality which is non trivial. The following proof is similar to the proof of [12, Theorem 2.1]. Fix an \( x \in (L_p(M))^n \) such that \( \|x\|_{p,\theta} < 1 \). Let \( S = \{ z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1 \} \). Let \( \partial_0 \) and \( \partial_1 \) be respectively the right and left border of \( S \). Then there is a continuous function \( F : S \to (L_p(M))^n \) such that \( F \) is analytic in the interior of \( S \), \( F(\theta) = x \), and such that

\[
\sup_{z \in \partial_0} \|F(z)\|_{C_p[L_p(M)]} < 1, \quad \sup_{z \in \partial_1} \|F(z)\|_{R_p[L_p(M)]} < 1.
\]

Write \( F = (F_1, \ldots, F_n) \) and let \( \varepsilon \) be a fixed (small) positive number. Define

\[
X(z) = (\varepsilon + \sum_k F_k(z)^* F_k(z))^{1/2} \quad \text{for} \quad z \in \partial_0 \quad \text{and} \quad X(z) = (\varepsilon + \sum_k F_k(z) F_k(z)^*)^{1/2} \quad \text{for} \quad z \in \partial_1.
\]

Note that \( X(z) \) is a positive invertible measurable operator with \( X(z)^{-1} \in M \) for every \( z \in \partial_0 \cup \partial_1 \). Since

\[
F_k(z)^* F_k(z) \leq X(z)^2, \quad z \in \partial_0,
\]

there is \( u_k : \partial_0 \to M \) such that

\[
F_k(z) = u_k(z) X(z) \quad \text{and} \quad \sum_k u_k(z)^* u_k(z) \leq 1, \quad z \in \partial_0.
\]

Similarly, there is \( v_k : \partial_1 \to M \) such that

\[
F_k(z) = X(z) v_k(z) \quad \text{and} \quad \sum_k v_k(z)^* v_k(z) \leq 1, \quad z \in \partial_1.
\]
Define
\begin{align}
A(z) &= 1 \text{ for } z \in \partial_0 \quad \text{and} \quad A(z) = \frac{X(z)^{-\frac{1}{2}}}{\Phi(z)} \text{ for } z \in \partial_1; \\
B(z) &= X(z)^{-\frac{1}{2}} \text{ for } z \in \partial_0 \quad \text{and} \quad B(z) = 1 \text{ for } z \in \partial_1;
\end{align}
\begin{align}
W_k(z) &= u_k(z)X(z)^{\frac{1}{2}} \text{ for } z \in \partial_0 \quad \text{and} \quad W_k(z) = X(z)^{\frac{1}{2}}v_k(z) \text{ for } z \in \partial_1.
\end{align}

Then we have the following factorization
\begin{equation}
F_k(z) = A(z)W_k(z)B(z), \quad z \in \partial_0 \cup \partial_1.
\end{equation}

Now we use a well-known Szegő type factorization for operator-valued analytic functions to bring the factorization (10) to an analytic one. The result we need here is [17, Corollary 8.2] applied to the special case of Example (iii) in [17, p.1496]. We should emphasize that this result is a combination (as well as a certain improvement) of several previous results due to notably Nelson-Lowdenslager, Winne-Masani, Devinatz and Sarason. We refer to [17] for more information and more historic references. We also note that for our purpose we can use instead [20] plus an approximation argument.

Thus by [17, Corollary 8.2], there are two functions \(\Phi\) and \(\Psi\) defined on \(S\) with values in \(L_{2r}(M)\) such that \(\Phi\) and \(\Psi\) are analytic in the interior of \(S\), and such that
\begin{equation}
\Phi(z)\Phi(z)^* = A(z)^2 \quad \text{and} \quad \Psi(z)^*\Psi(z) = B(z)^2, \quad z \in \partial_0 \cup \partial_1.
\end{equation}

Moreover, both \(\Phi(z)\) and \(\Psi(z)\) are invertible with bounded inverses in \(M\) for all \(z \in S\). Instead of [17, Corollary 8.2], we can directly use [17, Theorem 8.12] (which is due to Saito). Indeed, let \(w(z) = B(z)^{-1}\). Then \(w(z) \in M\) and \(w(z)^{-1} \in L_{2r}(M)\) for every \(z \in \partial_0 \cup \partial_1\). Since \(r \geq 1\) and \(M\) is finite, \(L_{2r}(M) \subset L_2(M)\). Thus by [17, Theorem 8.12], there are a function \(\tilde{u}\) such that \(\tilde{u}(z) \in M\) is unitary and an invertible analytic function \(\psi\) such that \(w(z) = \tilde{u}(z)\psi(z)\) for every \(z \in \partial_0 \cup \partial_1\). Set \(\Psi(z) = \psi(z)^{-1}\). Then \(\Psi\) is an invertible analytic function and \(B(z) = \psi(z)\Psi(z)\). It then follows that \(\Psi(z)^*\Psi(z) = B(z)^2\) for all \(z \in \partial_0 \cup \partial_1\), as required.

(11) implies that there are \(U : S \to M\) and \(V : S \to M\) such that
\begin{equation}
A(z) = \Phi(z)U(z), \quad B(z) = V(z)\Psi(z), \quad \|U(z)\|_{\infty} \leq 1, \quad \|V(z)\|_{\infty} \leq 1, \quad z \in \partial_0 \cup \partial_1.
\end{equation}

Thus
\begin{equation}
F_k(z) = \Phi(z)[U(z)W_k(z)V(z)] \Psi(z) \equiv \Phi(z)Y_k(z)\Psi(z).
\end{equation}

Since \(\Phi(z)^{-1}\) and \(\Psi(z)^{-1}\) are analytic in the interior of \(S\), so is \(Y_k(z) = \Phi(z)^{-1}F_k(z)\Psi(z)^{-1}\) for every \(k\). Therefore, we have the desired analytic factorization.

Let us estimate the norms of each factor on the border of \(S\). By the choice of \(\Phi(z)\) in (11) and the definition of \(A(z)\) in (5), we have
\begin{equation}
\sup_{z \in \partial_0} \|\Phi(z)\|_{\infty} = \sup_{z \in \partial_0} \|A(z)\|_{\infty} = 1
\end{equation}
and
\begin{equation}
\sup_{z \in \partial_1} \|\Phi(z)\|_{2r}^2 = \sup_{z \in \partial_1} \|A(z)\|_{2r}^2 = \sup_{z \in \partial_1} \|X(z)\|_p^2 < 1,
\end{equation}
provided \(\varepsilon\) is small enough. Therefore, by interpolation
\begin{equation}
\|\Phi(\theta)\|_{r_0(\theta)} \leq 1.
\end{equation}
Similarly, $\|\Psi(\theta)\|_{r_1(\theta)} \leq 1$.

Concerning $Y_k$, for any $z \in \partial_0$ by (12), the definition of $W_k$ in (9) and the inequality in (8), we have

$$\sum_k \|Y_k(z)\|^2_2 \leq \sum_k \|W_k(z)\|^2_2 = \|X(z)\|^2_{\frac{p}{2}} \sum_k u_k(z)^* u_k(z) X(z)\|_1 \leq \|X(z)\|_p^2 < 1.$$  

The same is true for $z \in \partial_1$. Hence, by the maximum principle,

$$\sum_k \|Y_k(\theta)\|^2_2 \leq 1.$$

Set

$$a = \Phi(\theta), \quad y_k = Y_k(\theta), \quad b = \Psi(\theta).$$

Then by the previous discussion we have

$$a y_k b = F_k(\theta) = x_k \quad \text{and} \quad \|a\|_{r_0(\theta)} \|b\|_{r_1(\theta)} \left(\sum_k \|y_k\|^2_2\right)^{\frac{1}{2}} \leq 1.$$  

Thus $\alpha_{p,\theta}(x) \leq 1$. This finishes the proof of Step 1.

**Step 2: The proof of (8) in the general case.** Assume again $1 \leq p < 2$; but now $M$ is a general von Neumann algebra. The inequality $\|x\|_{p,\theta} \leq \alpha_{p,\theta}(x)$ can be proved as before by interpolation. Indeed, the same proof as in Step 1 shows that this inequality still holds for $\theta = 0$ and $\theta = 1$. Then we can use Terp’s interpolation theorem [19] to conclude as in Step 1. Alternatively, instead of using Terp’s theorem, we can also appeal to Kosaki’s interpolation theorem [9], which is clearly applicable to strictly normal semifinite faithful weights. $L_p(M)$ can be, of course, constructed from such a weight on $M$.

We will use Haagerup’s reduction theorem to prove the converse inequality. Keep all notations in Theorem 5. We consider the noncommutative $L_p$-spaces based on $M$, $\mathcal{M}$ and $\mathcal{M}_i$. $L_p(\mathcal{M})$ is constructed with respect to the weight $\varphi$ there. Then i) and iv) of Theorem 5 imply that $L_p(M)$ and $L_p(\mathcal{M}_i)$ can be considered, in a natural way, as (isometric) subspaces of $L_p(\mathcal{M})$. On the other hand, by ii), iv) and [8, Lemma 2.2], $\mathcal{E}$ and $\mathcal{E}_i$ extend to contractive projections from $L_p(\mathcal{M})$ onto $L_p(M)$ and onto $L_p(\mathcal{M}_i)$, respectively ($1 \leq p \leq \infty$). These extensions are still denoted by the same symbols. Finally, by the w*-density of $\bigcup_i \mathcal{M}_i$ in $\mathcal{M}$ and [8, Lemma 1.1], $\bigcup_i L_p(\mathcal{M}_i)$ is dense in $L_p(M)$ for $p < \infty$. Moreover, by the commutation relations $\mathcal{E}_i \circ \mathcal{E}_j = \mathcal{E}_j \circ \mathcal{E}_i$ in Theorem 5 iv), we deduce that the family $\{L_p(\mathcal{M}_i)\}_{i \in I}$ is also increasing, and for any $x \in L_p(M)$ the net $\{\mathcal{E}_i(x)\}$ converges to $x$ in $L_p(M)$ (relative to the w*-topology for $p = \infty$).

Note that $\mathcal{E}$ extends coordinate-wise to a projection from $L_p(\mathcal{M})^n$ onto $L_p(M)^n$, which is still denoted by $\mathcal{E}$. Then $\mathcal{E}$ is contractive on $C_p^n[L_p(\mathcal{M})]$ and on $R_p^n[L_p(\mathcal{M})]$. The same remark applies to each condition expectation $\mathcal{E}_i$ too.

By this complementation of $L_p(M)^n$ and $L_p(\mathcal{M}_i)^n$ in $L_p(\mathcal{M})^n$, we have the following isometric inclusions

$$(C_p^n[L_p(M)], R_p^n[L_p(M)])_{\theta} \subset (C_p^n[L_p(\mathcal{M})], R_p^n[L_p(\mathcal{M})])_{\theta}$$

and

$$(C_p^n[L_p(\mathcal{M}_i)], R_p^n[L_p(\mathcal{M}_i)])_{\theta} \subset (C_p^n[L_p(M)], R_p^n[L_p(\mathcal{M})])_{\theta}.$$
On the other hand, by the complementation of $L_p(\mathcal{M}_i)$ in $L_p(\mathcal{M}_j)$ for $i \leq j$ and the density of $\bigcup_{i} L_p(\mathcal{M}_i)$ in $L_p(\mathcal{M})$, we deduce that the family

$$\left\{ \left( C^*_{p} [L_p(\mathcal{M}_i)], \ R^p_p [L_p(\mathcal{M}_i)] \right) \right\}_{i \in I}$$

is increasing and its union is dense in $\left( C^*_{p} [L_p(\mathcal{M})], \ R^p_p [L_p(\mathcal{M})] \right)_\theta$. Consequently, for any $x \in L_p(\mathcal{M})^n$, $\{ E_i(x) \}$, converges to $x$ with respect to the interpolation norm $\| \|_p,\theta$.

Now fix $x \in L_p(\mathcal{M})^n$ with $\| x \|_p,\theta < 1$. Then $E_i(x) \in L_p(\mathcal{M}_i)^n$, and by the previous discussion, $\| E_i(x) \|_p,\theta < 1$ for every $i \in I$. Since $\mathcal{M}_i$ is finite and $\sigma$-finite, we find, by Step 1, $a_i, b_i$ and $(y_{k,i})_{1 \leq k \leq n}$ in the unit balls of $L_{r_0(\theta)}(\mathcal{M}_i)$, $L_{r_1(\theta)}(\mathcal{M}_i)$ and $\ell^2_2(L_2(\mathcal{M}_i))$, respectively, such that

$$E_i(x_k) = a_i y_{k,i} b_i, \quad 1 \leq k \leq n, \ i \in I.$$ 

Define

$$\ell : L_{p'}(\mathcal{M})^n \to \mathbb{C} \quad \text{by} \quad \ell(z) = \sum_k \text{tr}(x_k^* z_k).$$

Since $E_i(x_k) \to x_k$ in $L_p(\mathcal{M})$ for every $k$, we have

$$\ell(z) = \lim_i \sum_k \text{tr}[E_i(x_k)^* z_k] = \lim_i \sum_k \text{tr}[y_{k,i}^* (a_i^* z_k b_i^*)], \quad \forall z \in L_{p'}(\mathcal{M})^n.$$ 

However, by Cauchy-Schwarz

$$\left| \sum_k \text{tr}[y_{k,i}^* (a_i^* z_k b_i^*)] \right| \leq \left( \sum_k \| y_{k,i} \|_{2}^{1/2} \right)^{1/2} \left( \sum_k \| a_i^* z_k b_i^* \|_{2}^{1/2} \right)^{1/2} \leq \alpha_{p',\theta}(z).$$

It then follows that $\ell$ is a contractive functional on $(L_{p'}(\mathcal{M})^n, \alpha_{p',\theta})$. It is clearly normal in the case $p' = \infty$. Therefore, by Lemma 4 (and the remark following it), we deduce $\alpha_{p,\theta}(x) \leq 1$ for $\ell$ is defined by $x$. This achieves the proof of (2).

**Step 3: The proof of (3).** Let $2 < p \leq \infty$. The inequality $\alpha_{p,\theta}(x) \leq \|x\|_{p,\theta}$ can be easily proved by interpolation. We omit the details. Let us prove the converse by duality using (2). Fix $x \in L_p(\mathcal{M})^n$. Note that

$$\left( C^*_{p'} [L_p(\mathcal{M})], \ R^p_p [L_p(\mathcal{M})] \right)_\theta = \left( C^*_{p'} [L_p(\mathcal{M})], \ R^p_p [L_p(\mathcal{M})] \right)_\theta$$

isometrically.

Thus by (2) already proved

$$\| x \|_{p,\theta} = \sup \left\{ \left| \sum_k \text{tr}(y_k^* x_k) \right| : y \in L_{p'}(\mathcal{M})^n, \| y \|_{p',\theta} \leq 1 \right\}$$

$$= \sup \left\{ \left| \sum_k \text{tr}(y_k^* x_k) \right| : y \in L_{p'}(\mathcal{M})^n, \alpha_{p',\theta}(y) \leq 1 \right\} \leq \alpha_{p,\theta}(x).$$

This is the desired inequality. Therefore, the proof of Theorem 4 is complete. \qed

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