Ambiguities in DoA Estimation With Linear Arrays

Frederic Matter, Tobias Fischer, Marius Pesavento, Senior Member, IEEE, and Marc E. Pfetsch

Abstract—In this paper, we present a novel approach to compute ambiguities in thinned uniform linear arrays, i.e., sparse non-uniform linear arrays, via a mixed-integer program. Ambiguities arise when there exists a set of distinct directions-of-arrival, for which the corresponding steering matrix is rank-deficient and are associated with nonunique parameter estimation. Our approach uses Young tableaux for which a submatrix of the steering matrix has a vanishing determinant, which can be expressed through vanishing sums of unit roots. Each of these vanishing sums then corresponds to an ambiguous set of directions-of-arrival. We derive a method to enumerate such ambiguous sets using a mixed-integer program and present results on several examples.

Index Terms—Ambiguities, array processing, direction-of-arrival estimation, identifiability, kruskal rank, spark, sparse linear arrays, thinned linear arrays, uniqueness, Young tableaux.

I. INTRODUCTION

Sample uniform linear arrays, obtained from thinning uniform linear arrays (ULAs), are widely used in sensor array processing due to their ability to maintain the aperture size of the corresponding full ULA while reducing the number of array elements. This is associated with a significant decrease of the array costs including power consumption, hardware and computational complexity [1] as well as a reduction in the mainbeam width and mutual coupling [2, 3]. A variety of array thinning techniques have been proposed in the literature to control the side and grating lobe levels of the array. These techniques can be classified into three main areas [4]: i) analytic thinning, e.g., based on prime number selection, where the array is formed from a $\lambda/2$ ULA by selecting sensors at prime multiples of the baseline as, e.g., in nested arrays, coprime arrays, and minimum redundancy arrays [5], [6], [7], ii) statistical thinning techniques, where sensors are selected randomly [8], and iii) optimization based thinning in which an appropriate error function is minimized [9], [10], [11], [12], [13], [14].

Manuscript received 20 October 2021; revised 17 April 2022 and 28 June 2022; accepted 7 August 2022. Date of publication 22 August 2022; date of current version 15 September 2022. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Lin Bai. This work was supported by the EXPRESS II Project within the DFG Priority Program CoSIP under Grant DFG-SPP 1798. The work of Marius Pesavento was supported by BMBF project Open6GHub under Grant 16KIS014. (Corresponding author: Marius Pesavento.)

Frederic Matter, Tobias Fischer, and Marc E. Pfetsch are with the Department of Mathematics, TU Darmstadt, 64289 Darmstadt, Germany (e-mail: matter@mathematik.tu-darmstadt.de; tob-fischer@web.de; pfetsch@mathematik.tu-darmstadt.de).

Marius Pesavento is with the Electrical Engineering and Information Technology Department, TU Darmstadt, 64289 Darmstadt, Germany (e-mail: pesavento@nt.tu-darmstadt.de).

Digital Object Identifier 10.1109/TSP.2022.3200548

Low side and grating lobe levels are important indicators for the resolution performance for sparse linear arrays in beamforming and Direction-of-Arrival (DoA) estimation applications. However, a fundamental question in the context of multi-source DoA estimation in thinned arrays is that of the maximum number of sources that can uniquely be determined from an array measurement and the characterization of ambiguities in the measurements. Ambiguities in the array manifold arise when there exists a set of distinct directions-of-arrivals in the field of view, for which the corresponding steering matrix is rank-deficient. In this case it is impossible to uniquely determine the parameters of interest from single snapshot measurements.

Being able to characterize the ambiguities that exist in an array is of great practical importance for a number of reasons. First, with respect to the array design, to achieve good resolution performance at reasonable hardware costs, it is preferable to design sparse arrays with large apertures and a low number of antenna elements. Ambiguities in the array response lead to spurious, i.e., ambiguous, solutions in DoA estimation methods, which cannot be distinguished from the true solutions. In this sense, it is desirable to choose arrays with few, if possible, finitely many ambiguities, and arrays that exhibit ambiguities outside the field of interest for a particular application. Second, if all ambiguity sets of an array are known, then it is possible to distinguish spurious, hence ambiguous solutions from true signal solutions, in a simple post processing step.

The concept and mathematical framework for the ambiguity problem in direction finding for linear arrays dates back to Schmidt [15], who introduced the rank of an ambiguity based on linear dependency in the array steering matrix and thus classified ambiguities based on this notion of rank. In the subsequent years, research focused on specific array geometries which have desirable properties or are free of certain types of ambiguities [16], [17], [18], [19]. For the popular ULA geometry it is known that no ambiguities exist, if the intersensor spacing is at most half wavelength [17], [20], [21], [22]. A linear array that is not uniform is called a non-uniform linear array.

Manikas and Proukakis [20] use tools from differential geometry in order to analyze ambiguities for general linear arrays. They derive a sufficient condition for the presence of ambiguities in a linear array. This sufficient condition implies that every non-uniform linear array with integer positions suffers from ambiguities, see [23]. Moreover, Manikas and Proukakis derive a class of ambiguities that are present in every non-uniform integer linear array, see [20, Theorem 2.2]. We call these ambiguities uniform ambiguities, since they are derived from a uniform partitioning of the array manifold.
For symmetric linear arrays, which can be shifted globally to positions \( r_1, \ldots, r_M \), such that \( \sum_{i=1}^{M} (r_i)^n = 0 \) holds for all odd \( n \in \mathbb{N} \), Manikas and Proukakis identify additional ambiguities, see [20, Theorem 3]. This ambiguity criterion for symmetric linear arrays is generalized in Dowlat [24, Theorem 3] and Manikas [25, Theorem 7.1] to a whole class of so-called non-uniform ambiguities depending on some parameter. Moreover, Manikas [25, Lemma 7.1] states that these non-uniform ambiguities converge to a uniform ambiguity for the corresponding parameter tending to infinity.

Wax and Ziskind [26] derive conditions for which the signal model (1) has a unique solution based on the number of sensors in the array and the spark of the steering matrix. The spark of a matrix \( A \) is defined as the smallest number of linearly dependent columns in \( A \).

In a recent paper, Achanta et al. [27] investigate the spark of DFT matrices and use vanishing sums in order to find rows of a DFT matrix that induce a matrix with full spark. However, they do not draw the connection to generalized Vandermonde matrices and the Schur polynomial. Translated to our setting, the results of [27] can determine which linear arrays do not have any ambiguities within a fixed set of certain DoAs. In contrast, our paper determines ambiguities of a fixed linear array.

### A. Contribution

Which non-uniform integer linear array suffers from ambiguities, besides the ones already known in the literature? In order to answer this question, we need to find steering vectors \( a(\theta_i) \) (or more precisely DoAs \( \theta_i \)) such that the steering matrix for the position vector \( r \) has spark at most \( M \). This means that the \( M \) sensors cannot uniquely localize the signal sources with directions \( \theta_i \). In the following, we present a novel approach for identifying ambiguities in (non-uniform) integer linear arrays. This approach can in theory be used to compute all ambiguities from which a given integer linear array suffers, see Section III-C. However, those computations are too expensive to be of practical use, even for small arrays. By making a (small) structural assumption, we are able to formulate a mixed-integer problem (MIP) that is capable of enumerating ambiguities, see Section IV. At least for non-symmetric integer linear arrays, we show that this approach can find all ambiguities already known in the literature (namely those from [20, Theorem 2.2]), see Proposition IV.5. Moreover, we demonstrate the usefulness of our approach in Section V by presenting examples of non-symmetric integer linear arrays for which we find many previously unknown ambiguities. One major important feature of our approach is that it can detect whole classes of infinitely many ambiguities of the same structure, see Example V.1, whereas the existing method of Manikas and Proukakis [20] can only yield a finite number of ambiguities. We also show one non-uniform linear array which has on average the same intersensor spacing as a uniform linear array, but which suffers from ambiguities in contrast to a uniform linear array.

In this paper, the following notations are used: matrices are denoted by boldface uppercase letters \( A \), vectors are denoted by boldface lowercase letters \( a \), and scalars are denoted by regular letters \( a \). The \( M \times M \) identity matrix is denoted by \( I_M \). Symbols \((\cdot)^T\), \((\cdot)^*\), \((\cdot)^H\) and \((\cdot)^{-1}\) denote the transpose, element-wise complex conjugate Hermitian and inverse of the (matrix) argument. For \( n \in \mathbb{N} \), we define \( [n] := \{1, \ldots, n\} \).

### II. PROBLEM DESCRIPTION

Consider a sparse (thinned) uniform linear array composed of \( M \) sensors located on the \( x \)-axis in \( \mathbb{R}^2 \) at positions corresponding to integer multiples \( r_1 < r_2 < \cdots < r_M \) of a common baseline \( d \in [0, 1] \) measured in half wavelength. We define \( r = (r_1, \ldots, r_M) \in \mathbb{Z}^M \). A superposition of \( L \) narrowband waveforms emitted from sources at azimuth angles \( \Phi = (\theta_1, \ldots, \theta_L) \) is impinging on the array. Throughout the paper, we denote the azimuth angles \( \theta \in \Theta \) as Directions-of-Arrival (DoA). The received signal \( y \) in a noise-free setting is expressed as

\[
y = A(\Theta) x,
\]

where \( x \in \mathbb{C}^L \) is the emitted signal and \( A(\Theta) = [a(\theta_1), \ldots, a(\theta_L)] \in \mathbb{C}^{M \times L} \) the array steering matrix with columns

\[
a(\theta) = [z^{r_1}, \ldots, z^{r_M}]^T,
\]

for \( \theta \in \Theta \) and \( z = e^{-j\pi d \cos(\theta)} \). By using the electrical angle \( \Phi = -\pi d \cos(\theta) \), this simplifies to \( z = e^{j\Phi} \). We then denote the array steering matrix with \( A(\Phi) \).

In the case of linear measurement systems, unique recovery is assessed from the spark of the measurement matrix \( A \). If the spark is large enough, then it is possible to uniquely recover \( x \) from its measurements \( Ax \), even if the linear system is under-determined, see, e.g., the book [28].

Uniqueness of the measurement model (1) is of importance for DoA estimation as well and directly related to the number of sources that can be identified from the measurements without ambiguities [26]. In our setting, the steering matrix \( A(\Theta) \) depends on the unknown directions from which the signals in \( x \) impinge on the linear array \( r \). This leads to a generalized notion of the spark: The ability to uniquely recover a signal \( x \) coming from directions \( \Theta \) depends on the rank of the induced steering matrix \( A(\Theta) \). If the spark of \( A(\Theta) \) for a fixed linear array \( r \) is not full, then there exists a subset of the columns of \( A(\Theta) \), i.e., a subset of steering vectors, which are linearly dependent and thus induce a rank-deficient submatrix of \( A(\Phi) \). This is called an ambiguity, which can be formally defined as follows.

**Definition II.1 ([20]):** Let \( r \in \mathbb{R}^M \) be a linear array with \( M \) sensors and let \( \Theta = [\theta_1, \ldots, \theta_L]^T \) be an ordered vector of DoAs with \( L \leq M \). Then \( \Theta \) is called an ambiguous vector of DoAs, if

\[
\text{rank}(A(\Theta)) = \text{rank}(\{a(\theta_1), \ldots, a(\theta_L)\}) < L.
\]

Furthermore, its rank of ambiguity is defined as \( \rho_a = \text{rank}(A(\Theta)) \in \mathbb{N} \).

An important question is whether the steering matrix \( A(\Theta) \) for a fixed linear array and a vector of directions \( \Theta \) has full spark, i.e., if the spark is given by \( \min \{M, L\} \).

Note that more signal sources than sensor positions \( M \) always produce an ambiguity, since the rank of the corresponding submatrix can be at most \( M \). Thus, we focus on the search of ambiguities with at most \( M \) signal sources.
In order to simplify the presentation, we make some assumptions without loss of generalization.

(A1) We assume that the first sensor of a linear array \( r = (r_1, \ldots, r_M)^\top \) is located in the origin, i.e., \( r_1 = 0 \), since a global shift of the array positions does not change the ambiguities.

(A2) We assume that the electrical angles \( \phi \) lie in \( \Omega = [-\pi d, \pi d] \), since \( a(\theta) = a(2\pi - \theta) \) for \( \theta \in [0, 2\pi] \). For \( d = 1 \), we assume \( \Phi \in [-\pi, \pi] \), since in this case \( a(0) = a(\pi) \). This implies a one-to-one correspondence between electrical angles and DoAs, so that from now on, we will also use electrical angles.

(A3) An appropriate global rotation of the DOAs amounts to multiplying the steering matrix \( \mathbf{A}(\Theta) \) with a constant diagonal matrix with unit complex entries, which retains the ambiguity property. Thus, we assume without loss of generality that \( \theta_1 = 0 \) resulting in an electrical angle \( \phi_1 = -rd \).

Furthermore, we assume that \( r \in \mathbb{N}_M \). Recall that \( r \) denotes the integer multiples of the common baseline \( d \in [0, 1] \), measured in half wavelength. Such sparse uniform sampling is often used in practical systems due to the simplicity of the hardware design and several attractive features associated with it, such as the existence of efficient search-free rooting-based DoA estimation techniques [29], [30], [31], [32], [33] and virtual signal decorrelation procedures [34], [35], [36].

Remark II.2: Our notion of ambiguity and the failure to uniquely resolve source signals can also be used to characterize ambiguities in difference co-arrays corresponding to non-fully augmentable arrays [5], [23]. In this case, the uniqueness of the vectorized array covariance matrix model

\[
\text{vec}(\mathbf{R}_{yy}) = \mathbf{A}(\Theta) \mathbf{p} + \sigma^2 \text{vec}(\mathbf{I}_M) \tag{3}
\]

is considered instead of model (1), where \( \mathbf{R}_{yy} = \mathbb{E}[\mathbf{yy}^\top] \) is the array covariance matrix, \( \mathbf{R}_{xx} = \mathbb{E}[\mathbf{xx}^\top] = \text{diag}(\mathbf{p}) \) denotes the uncorrelated signal covariance matrix with the source powers \( \mathbf{p} \in \mathbb{R}^M \) on the main diagonal, \( \sigma^2 \) is noise power and

\[
\mathbf{A}(\Theta) = \mathbf{A}(\Theta) \circ \mathbf{A}^*(\Theta) \tag{4}
\]

is the difference co-array steering matrix corresponding to the physical sensor array, with \( \circ \) denoting the Kratli-Rao product, hence column-wise Kronecker product. The difference co-array corresponds to a virtually extended array with sensors located at integer positions in the set \( \{ r_i - r_j \} \), for all \( i, j = 0, \ldots, M \) of the common baseline \( d \in [0, 1] \) measured in half wavelength. To investigate unique recovery in the covariance model (3) the spark of the difference co-array steering matrix \( \mathbf{A}(\Theta) \) is investigated instead of the spark of \( \mathbf{A}(\Theta) \).

III. A NOVEL APPROACH FOR DETECTING AMBIGUITIES IN LINEAR ARRAYS

Let Assumptions (A1)–(A3) hold and let \( r \in \mathbb{N}_M \). We first give a compact overview over our procedure to find ambiguities that is presented in the next sections.

1) The array steering matrix \( \mathbf{A}(\Phi) \) for \( r \) induces ambiguities if some of its \( M \times M \) submatrices have a zero determinant. Therefore, we search for \( M \) electrical angles \( \Phi_1, \ldots, \Phi_M \) such that the determinant of their induced submatrix of \( \mathbf{A}(\Phi) \) vanishes.

2) Every submatrix of \( \mathbf{A}(\Phi) \) is a generalized Vandermonde determinant, see Section III-A. This determinant is divisible by the classical Vandermonde determinant, and this quotient is equal to the Schur polynomial \( s(z) \), see Equation (6). Thus, instead of searching for roots of the generalized Vandermonde determinant, we can search for roots of the Schur polynomial.

3) The Schur polynomial \( s(z) \) can be represented using semistandard Young tableaux (SSYTs), see Section III-B and Lemma III.1 therein.

4) In our case, \( z_i = e^{j\Phi_i} \) for \( i \in [M] \), and if \( \Phi_i \in \{2\pi q : q \in \mathbb{Q}\} \), the Schur polynomial is a sum of unit roots. This means, we search unit roots such that their sum vanishes and the relationship to the SSYTs due to Lemma III.1 is satisfied, see Section III-D.

5) We construct these vanishing sums of unit roots by adding up rotated minimal vanishing sums. This can be formulated as a mixed-integer problem, see Section IV.

A. Generalized Vandermonde Matrix

Our goal is to determine whether there exist electrical angles \( \Phi_1 < \cdots < \Phi_M \) and an \( M \times M \) submatrix of the corresponding array steering matrix for an integer linear array \( r \) with zero determinant. Every \( M \times M \) submatrix of \( \mathbf{A}(\Phi) \) is a generalized Vandermonde matrix of the form

\[
\mathbf{B}_r(z) = \begin{pmatrix}
z_1^{r_1} & \cdots & z_M^{r_1} \\
z_1^{r_2} & \cdots & z_M^{r_2} \\
\vdots & \ddots & \vdots \\
z_1^{r_M} & \cdots & z_M^{r_M}
\end{pmatrix},
\]

with \( z_i = e^{j\Phi_i} \), \( i \in [M] \) and \( \Phi_1 < \cdots < \Phi_M \). We define the polynomial

\[
V_r : \mathbb{C}^M \to \mathbb{C}, \quad V_r(z) := \det(\mathbf{B}_r(z)) \tag{5}
\]

as the generalized Vandermonde determinant. Every root of \( V_r \) induces an ambiguity.

If \( r = (0, 1, \ldots, M-1)^\top \), then \( V_r : \mathbb{C}^M \to \mathbb{C} \) is the classical Vandermonde determinant

\[
V(z) = \prod_{1 \leq k < \ell \leq M} (z_k - z_\ell).
\]

In this case, we obtain that \( V(z) = 0 \) if and only if \( z_\ell = z_k \) for indices \( i \neq k \). This means that ambiguities only arise if there are two equal electrical angles \( \Phi_j = \Phi_k \) for \( i \neq k \) [37].

For steering matrices with a non-uniform linear array, \( \mathbf{B}_r(z) \) is a so-called generalized Vandermonde matrix, and its determinant \( V_r(z) \) is a generalized Vandermonde determinant. For more information on generalized Vandermonde matrices and determinants, see, e.g., [38], [39], [40], [41] as well as [42], [43] for results in case that \( r \in \mathbb{R}^M \) and that \( V_r \) is defined over a finite field, respectively.
In the present paper, we have \( z \in \mathbb{C}^M \). In this case, it is well known that \( V_\mathcal{T}(z) \) is divisible by the (classical) Vandermonde determinant \( V(z) \), see, e.g., [44]. The ratio
\[
s_\lambda(z) := \frac{V_\mathcal{T}(z)}{V(z)}
\]
with \( \lambda = \mathbf{r} - \mathbf{d} \), and \( \delta = (0, 1, \ldots, M - 1)^\top \) is the so-called Schur polynomial. Here, \( \lambda \) is sorted non-decreasingly.

An ambiguous vector of electrical angles \( \{\Phi_1, \ldots, \Phi_M\}^\top \) induces a generalized Vandermonde matrix with a zero determinant, and thus forms a root of the Schur polynomial \( s_\lambda(z) \) with \( z_i = e^{i\Phi_i}, i \in [M] \). In order to find such roots, we introduce Young tableaux in the next section, as they can be used to represent the Schur polynomial.

B. Young Tableaux

The notations, definitions and statements in this section are taken from [45].

Given \( \lambda = (\lambda_1, \ldots, \lambda_M)^\top, 0 \leq \lambda_1 \leq \cdots \leq \lambda_M \), the Young diagram of shape \( \lambda \) is defined as a collection of boxes that are arranged in \( M \) left-justified rows such that the number of boxes in row \( M - i + 1 \) is \( \lambda_i \). A semistandard Young tableau (SSYT) of shape \( \lambda \) is obtained by filling the Young diagram with positive integers \( i \in [M] \) such that entries increase weakly along each row and increase strictly down each column.\(^1\)

For example, if \( \lambda = (1, 3, 4)^\top \), we need to fill the Young diagram with integers from the set \{1, 2, 3\} and one possible SSYT is the following:

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 3 & \\
3 & & & \\
\end{array}
\]

For a given \( \lambda \), we define \( \mathcal{T}_\lambda \) as the set of all SSYTs with entries in \([M]\). Each tableau \( \mathcal{T} \in \mathcal{T}_\lambda \) defines a weight vector \( \alpha(T) = (\alpha_1(T), \ldots, \alpha_M(T))^\top \), where \( \alpha_i(T) \) is the number of times \( i \) appears. By definition, \( \sum_{i=1}^M \alpha_i(T) = \sum_{i=1}^M \lambda_i \) holds. In the above example, \( (\alpha_1(T), \alpha_2(T), \alpha_3(T))^\top = (2, 2, 4)^\top \).

SSYTs have the beautiful property that they can be used to represent the Schur polynomial. This is known in the literature under the term “Young’s Rule,” and it plays a key role in our procedure, since it yields a polynomial representation of the quotient of the generalized Vandermonde determinant and the classical Vandermonde determinant.

Lemma III.1 (Stanley [45]): Consider \( \lambda = (\lambda_1, \ldots, \lambda_M)^\top \) with \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_M \). Then,
\[
s_{\lambda}(z) = \sum_{T \in \mathcal{T}_{\lambda}} \prod_{i=1}^M z_i^{\alpha_i(T)}.
\]

Lemma III.1 is the reason why we only allowed integers \( i \in [M] \) to appear in an SSYT in the definition at the beginning of this section. From Lemma III.1, we deduce that the degree of \( s_{\lambda} \) is \( \sum_{i=1}^M \lambda_i \). Moreover, if \( \lambda \neq 0 \), then \( s_{\lambda} \) is homogeneous. For a

\(^1\)Note that in general, a Young diagram can be filled with arbitrary positive integers \( i \in \mathbb{N} \) to obtain an SSYT, but as it will become clear later in this section, in our setting, only integers \( i \in [M] \) are allowed.

C. Algebraic Root Finding

Since the Schur polynomial \( s_{\lambda}(z) \) is a complex polynomial in the variables \( z_i \), we can search roots of the Schur polynomial algebraically by splitting each variable \( z_i = x_i + j \cdot y_i \) with \( x_i = \text{Re}(z_i) \) and \( y_i = \text{Im}(z_i) \). Thus, \( s_{\lambda}(x, y) := \text{Re}(s_{\lambda}(z)) \) and \( s_{\lambda}^j(x, by) := \text{Im}(s_{\lambda}(z)) \) are real polynomials in the variables \( x_i, y_i \). This yields a polynomial equation system, which can be solved using, for instance, elimination theory or multivariate resultants. For more details, see, e.g., [47].

Thus, for very small examples, the roots of the Schur polynomial can be derived algebraically. However, if \( M \) or \( |\mathcal{T}_{\lambda}| \) is large, then a numerical solution is usually difficult. Moreover, the algebraic approach becomes computationally extremely expensive.

As an alternative, the idea in Section IV is to formulate a mixed-integer program (MIP) whose feasible solutions correspond to roots of the Schur polynomial. Before we state this result, we introduce the definition of vanishing sums of unit roots in the next section, as it turns out that these play an important role for the detection of roots of the Schur polynomial.
D. Vanishing Sums of Unit Roots

Let $T_{\lambda} = \{T_1, \ldots, T_N\}$ be the set of all SSYTs of shape $\lambda$ with cardinality $N := |T_{\lambda}|$. In our setting, the variables $z_i$ are given by $z_i = e^{i\Phi_i}$, $\Phi_i \in [-\pi d, \pi d]$, for $i \in [M]$. The Schur polynomial can be written as

$$s_{\lambda}(z) = \sum_{\ell=1}^{N} \prod_{i=1}^{M} z_i^{\alpha_{\ell} i} = \sum_{\ell=1}^{N} e^{i\alpha_{\ell} \Phi_i},$$

(7)

where $\alpha = (\alpha_1, \ldots, \alpha_M)^\top \in \mathbb{Z}^{M \times N}$ is the vector $\alpha(T_{\ell})$ defined in Section III-B. Recall that our goal is to search for roots of the Schur polynomial, which means to check whether there exists a vector $\Phi \in [-\pi d, \pi d]^M$ such that

$$s_{\lambda}(z) = \sum_{\ell=1}^{N} e^{i\sigma_{\ell} \Phi} = 0,$$

(8)

$$\sigma_{\ell} = \sum_{i=1}^{M} \alpha_{\ell i} \Phi_i \pmod{2\pi}, \quad \forall \ell \in [N].$$

(9)

Since $e^{i\sigma_{\ell} \Phi} = e^{i(\sigma_{\ell} \Phi + 2\pi)}$ for $k \in \mathbb{N}$, Equation (9) only needs to hold modulo $2\pi$.

For $m \in \mathbb{N}$, define $\omega_m = e^{i2\pi/m}$. Then, $\omega_m^v$ is called an $m$-th unit root for $v \in [0, 1, \ldots, m-1]$. If all $\sigma_{\ell}/2\pi$ are rational, then the sum $\sum_{\ell=1}^{N} e^{i\sigma_{\ell} \Phi} = 0$ is called a vanishing sum of unit roots.

The most important special case of a vanishing sum of unit roots is the sum of all $m$-th unit roots, that is, $\omega_m + \omega_m^2 + \cdots + \omega_m^m = 0$ holds for an integer $m > 1$, which can be seen using a geometric sum. In the literature these sums are often denoted as trivial. An example of a nontrivial vanishing sum of unit roots is

$$\omega_6 + \omega_6^2 + \omega_6^3 + \omega_6^5 = 0.$$

(10)

This sum can be written as the sum of the trivial sum $\sum_{v=1}^{5} \omega_6^v = 0$ and the rotated trivial sum $\omega_6 + \omega_6^3 + \omega_6^5 + \omega_6^1 = \sum_{v=1}^{3} \omega_6^v$, see Fig. 2.

Remark III.3: An obvious way to find unit roots $e^{i\sigma_{\ell} \Phi}$ and electrical angles $\Phi_i$ corresponding to an ambiguity of a linear array would be a feasibility problem with Constraints (8) and (9).

However, the real-valued formulation of $\sum_{\ell} e^{i\sigma_{\ell} \Phi} = 0$, with $\sigma_{\ell} \in (0, 2\pi)$ is $\sum_{\ell} \cos(\sigma_{\ell}) = 0$ and $\sum_{\ell} \sin(\sigma_{\ell}) = 0$, which is a trigonometric constraint that is hard to handle in most solvers. The same problem emerges, if $e^{i\sigma_{\ell} \Phi}$ is replaced by a complex variable $\omega_{\ell}$ for all $\ell \in [M]$. This would yield $\sum_{\ell} \text{Re}(\omega_{\ell}) = \sum_{\ell} \text{Im}(\omega_{\ell}) = 0$ and $\text{Re}(\omega_{\ell})^2 + \text{Im}(\omega_{\ell})^2 = 1$. In this case, we need to control the argument of the complex variables $\omega_{\ell}$ to model Equation (9), which is nonlinear.

In order to avoid the issue in Remark III.3, we make the following additional assumption.

(A4) For every root of the Schur polynomial, the corresponding exponential sum $\sum_{\ell=1}^{N} e^{i\sigma_{\ell} \Phi}$ is a vanishing sum of roots of unity, possibly rotated by a complex number on the unit circle.

The subsequent Example III.7 demonstrates that there exist linear arrays with ambiguities that violate this assumption. However, the Assumption (A4) will allow us to use a purely linear procedure to find many ambiguities that are present in a linear array. This procedure relies on the following theorem, which reduces the case of general vanishing sums to minimal vanishing sums, i.e., vanishing sums, such that no proper subsum also vanishes.

Theorem III.4 (Mann [48], Corollary 1.1): Let $\sum_{i=1}^{k} a_i \eta_i = 0$ be a vanishing sum with $a_i \in \mathbb{Z} \setminus \{0\}, i \in [k]$, and unit roots $\eta_i, i \in [k]$. Then this sum can be written as

$$\sum_{i=1}^{k} a_i \eta_i = \zeta_1 \sum_{i=1}^{k_1} a_i \nu_i + \cdots + \zeta_u \sum_{i=k_{u-1}+1}^{k_u} a_i \nu_i,$$

with unit roots $\zeta_1, \ldots, \zeta_u, 0 =: k_0 < k_1 < \cdots < k_u := k$ and all $\nu_i$ are $(p_1 p_2 \cdots p_s)$-th unit roots with $0 < p_t \leq \max\{k_j - k_{j-1} : j \in [u]\}$, for prime numbers $p_t, t \in [s]$. Moreover, each vanishing sum $\sum_{i=k_{j-1}+1}^{k_{j}} a_i \nu_i = 0$, $j = 0, \ldots, u-1$ is minimal.

Note that the upper bound for $p_t$ in Theorem III.4 does not depend on $t$. Moreover, the representation in Theorem III.4 is clearly not unique.

Because of Theorem III.4, we only need to consider minimal vanishing sums, as they can be rotated and linearly combined to obtain general rotated vanishing sums and thus ambiguities as roots of the Schur polynomial. If we allow the linear coefficients $\zeta_i$ in Theorem III.4 to be complex numbers on the unit circle, i.e., $\zeta_i = e^{i2\pi \alpha_i}, \alpha_i \in [0, 1)$, instead of roots of unity, we are also able to find slightly more general exponential sums that sum to zero and thus induce ambiguities.

Remark III.5: There are no (nontrivial) vanishing sums of length 1. Furthermore, every vanishing sum of length 4 can be written as the sum of two (rotated) minimal vanishing sums of length 2, and thus, there are no minimal vanishing sums of length 4, which can be deduced from [49, Lemma 3.2 and Lemma 3.3].

Remark III.6 (Limitation of the approach): At least for symmetric linear arrays there exist ambiguities which cannot be represented as a linear combination of minimal vanishing sums with rotation factors of the form $e^{i2\pi \nu}, \nu \in [0, 1)$ and thus violate Assumption (A4). In this case, the approach of this section is not able to detect these ambiguities. The following example shows such an ambiguity, which can be obtained by using [25, Theorem 7.1].

Example III.7: Consider the linear array $r = (0, 1, 3, 4)^\top$ with baseline $d = 1$ and $\lambda = (0, 0, 1, 1)^\top$. The Schur polynomial for $r$ is given by

$$s_{\lambda} = e^{i(\Phi_1 + \Phi_2)} + e^{i(\Phi_1 + \Phi_3)} + e^{i(\Phi_2 + \Phi_4)}$$
\[ + e^{j(\Phi_2 + \Phi_3)} + e^{j(\Phi_2 + \Phi_4)} + e^{j(\Phi_3 + \Phi_4)}. \]

The ambiguity
\[ \Phi = [2 \arctan(\Delta_1) - \pi, 2 \arctan(\Delta_2) - \pi, \]
\[ \pi - 2 \arctan(\Delta_2), \pi - 2 \arctan(\Delta_1)]^\top \]
\[ \approx (43.6^\circ, 77.35^\circ, 102.7^\circ, 136.4^\circ)^\top, \]

with
\[ \Delta_1 = \sqrt{\frac{1}{3} (12 - \sqrt{129})}, \Delta_2 = \sqrt{\frac{1}{3} (12 + \sqrt{129})}, \]
yields \( \Phi_1 + \Phi_4 = \Phi_2 + \Phi_3 \), whereas all other exponents of \( s_\lambda \) are pairwise different. This means, in order to encode the solution \( \Phi \) by summing rotated minimal vanishing sums of roots of unity, two of the six resulting roots must be equal (after a possible rotation). Since there are \( N = 6 \) SSYTs for the linear array \( r \), there are three ways of summing rotated minimal vanishing sums in order to obtain a vanishing sum of length 6, namely, \( 6 = 3 + 3 = 2 + 2 + 2 \). Note that there are no minimal vanishing sums of lengths 1 and 4, see Remark III.5. However, it is easy to see that for all three cases, it is not possible to find rotation factors such that two roots of unity are equal and five are pairwise different. Thus, the ambiguity given by \( \Phi \) cannot be represented using sums of rotated minimal vanishing sums, so that Assumption (A4) is violated.

IV. ENUMERATION USING MINIMAL VANISHING SUMS

Recall that finding ambiguous DoAs for the linear array \( r = \lambda + \delta \) corresponds to finding roots of the Schur polynomial \( s_\lambda \) in (7). The Schur polynomial is given by a sum of exponentials, where the exponents need to fulfill a linear relation modulo \( 2\pi r \), which originates from the SSYTs of shape \( \lambda \) for the linear array \( r \), see (8) and (9). We assumed that \( s_\lambda \) can be represented using linear combinations of minimal vanishing sums with coefficients \( \zeta = e^{j2\pi v}, v \in [0, 1) \). By that, we can find a fairly large subset of roots of the Schur polynomial. If \( N \) is the number of SSYTs of shape \( \lambda \), then the following procedure can be used to find ambiguities in \( r \).

1) Choose a partition \( N = p_1 + \cdots + p_k, k \in [N] \), with \( p_i \in [N] \setminus \{1, 4\} \), i.e., without using the elements 1 and 4, since there are no minimal vanishing sums of unit roots of these lengths, see Remark III.5.
2) For each partition element \( p_i \) choose a minimal vanishing sum \( S_i \) of length \( p_i \).
3) Choose an assignment of the roots of unity appearing in \( S_i \) to the variables \( \sigma_\ell, \ell \in [N] \).
4) Check if there exists a solution to the linear equation system (9), where each \( S_i \) can be rotated by an arbitrary \( e^{j2\pi v_i}, v_i \in [0, 1) \), c.f. Theorem III.4 and the discussion thereafter.

Clearly, for this approach to work, we need to know all possible minimal vanishing sums that can be used to build the desired vanishing sum. For \( N \leq 12 \), all minimal vanishing sums (up to rotations) of length \( N \) are characterized by Poonen and Rubinstein [49, Table 1 and Theorem 3] \(^2\), and their construction can be easily extended to \( N > 12 \).

A. The MIP-Formulation

The approach described above to find ambiguities for a given linear array \( r \) can be formulated as a mixed-integer (linear) program (MIP). Every feasible solution of the MIP then corresponds to an ambiguity for \( r \), such that by enumerating feasible solutions of the MIP, it is possible to obtain many ambiguities for \( r \). The only requirement is that all minimal vanishing sums with length up to \( N \) need to be known in advance. The following parameters are used in the MIP-formulation (11) in Fig. 3:

\[ M := \text{Number of sensors in the linear array} \]
\[ r = \lambda + \delta, \text{i.e., } r \in \mathbb{Z}^M, \]
\[ d := \text{common baseline of the array positions} \]
\[ N := \text{number of SSYTs of shape} \lambda, \]
\[ P := [2, \ldots, 2, 3, \ldots, 3, 5, \ldots, 5, 6, \ldots, 6, \ldots, n], \]
\[ |N/3| \text{many } |N/3| \text{many } |N/3| \text{many } |N/6| \text{many } \]
\[ m_i := \# \text{ minimal vanishing sums of length } P_i \]
\[ u_{i,k}^{(i)} := k\text{-th root of unity of } \text{t-th minimal vanishing sum} \]
\[ \text{of length } P_i, i \in [\|P\|], t \in [m_i], k \in [P_i], \]
\[ I := \{ (i,t,k) : i \in [\|P\|], t \in [m_i], k \in [P_i] \}. \]

The vector \( P \) is needed to model all partitions of \( N \) that can be used in the first step of the procedure above, where \( |P| \) denotes its number of entries.

The variables are:
- \( q_i^{(i)} \in \{0, 1\}^{m_i} \) with \( q_i^{(i)} = 1 \) if and only if \( P_i \) is part of the chosen partition of \( N \) and the \( t \)-th minimal vanishing sum is selected for this \( P_i \).
- \( b_{i,k,\ell}^{(i)} \in \{0, 1\}^{m_i \times P_i \times N} \). We have \( b_{i,k,\ell}^{(i)} = 1 \) if and only if \( \sigma_\ell \) is assigned to the \( k \)-th root of unity in the \( t \)-th minimal vanishing sum chosen for the \( i \)-th element of the partition.
- \( v^{(i)} \in [0, 2\pi)^{m_i} \): rotation factors for the minimal vanishing sums of length \( P_i \).
- \( w^{(i)} \in [0, 2\pi)^{m_i \times P_i \times N} \): auxiliary variables for linearization:
- \( w_{i,k,\ell}^{(i)} = u_{i,k}^{(i)} \cdot b_{i,k,\ell}^{(i)} \).
- \( z \in \mathbb{Z}^N \): models that a rotation is always applied modulo \( 2\pi \). Since the rotation values lie in \([0, 2\pi]\), all \( z_\ell \) can be assumed to be binary.
- \( \sigma \in [0, 2\pi)^N, \Phi \in [-\pi d, \pi d]^M, x \in \mathbb{Z}^N \): model Equations (8) and (9), where \( x \) is needed for the modulo operation. Constraint (11i) implies that each \( x_\ell \) is bounded by
\[ -\frac{1}{2} \sum_{m=1}^M \alpha_{m,\ell} - 1 \leq x_\ell \leq \frac{1}{2} \sum_{m=2}^M (\alpha_{m,\ell} - \alpha_{1,\ell}), \]
for \( \ell \in [N], \) see Constraint (11k).

Constraints (11a) and (11b) ensure that a valid partition of \( N \) together with corresponding minimal vanishing sums is chosen.

\(^2\)Mann [48] already characterized all minimal vanishing sums up to \( N = 7 \), and Conway and Jones [50] characterized all minimal vanishing sums up to \( N = 9 \).
\[
\sum_{i=1}^{P} P_i \left( \sum_{t=1}^{m_i} q_t^{(i)} \right) = N, \quad \forall i \in [P], \quad (11a)
\]
\[
\sum_{t=1}^{m_i} q_t^{(i)} \leq 1, \quad \forall i \in [P], \quad (11b)
\]
\[
\forall (i, t, k, \ell) \in I \times [N], \quad (11c)
\]
\[
\sum_{i=1}^{P} \sum_{t=1}^{m_i} \sum_{k=1}^{P_i} b_{t,k,\ell}^{(i)} = 1, \quad \forall \ell \in [N], \quad (11d)
\]
\[
\sum_{t=1}^{m_i} \sum_{k=1}^{P_i} b_{t,k,\ell}^{(i)} \leq 1, \quad \forall (i, t, k) \in I, \quad (11e)
\]
\[
\left| \sum_{i=1}^{P} \sum_{t=1}^{m_i} \sum_{t=1}^{m_i} \sum_{k=1}^{P_i} w_{t,k,\ell}^{(i)} + b_{t,k,\ell}^{(i)} u_{t,k}^{(i)} - 2\pi z_\ell = \sigma_\ell, \quad \right. \quad \forall \ell \in [N], \quad (11f)
\]
\[
\forall (i, t, k, \ell) \in I \times [N], \quad (11g)
\]
\[
2\pi (-1 + b_{t,k,\ell}^{(i)} + w_{t,k,\ell}^{(i)} u_{t,k}^{(i)}) \leq u_{t,k}^{(i)} \leq 2\pi (-1 - b_{t,k,\ell}^{(i)} + w_{t,k,\ell}^{(i)}), \quad \forall (i, t, k, \ell) \in I \times [N], \quad (11h)
\]
\[
\sum_{m=1}^{M} \sum_{t=1}^{m_i} \sum_{t=1}^{m_i} \sum_{k=1}^{P_i} a_{m,\ell} \Phi_m - 2\pi x_\ell = \sigma_\ell, \quad \forall \ell \in [N], \quad (11i)
\]
\[
-\pi d = \Phi_1 < \Phi_2 < \cdots < \Phi_M \leq \pi d, \quad (11j)
\]
\[
-\frac{1}{2} d \sum_{m=1}^{M} a_{m,\ell} - 1 \leq x_\ell \leq -\frac{1}{2} d \sum_{m=1}^{M} a_{m,\ell} - \alpha_1, \quad \forall \ell \in [N], \quad (11k)
\]

Constraint (11c) models a minimal vanishing sum needs to be selected as part of the chosen partition in order for its unit roots to be used. The subsequent Constraints (11d) and (11e) ensure that each \( \sigma_\ell \) is assigned exactly one unit root and that each unit root is assigned at most one \( \sigma_\ell \). The actual assignment of unit roots to \( \sigma_\ell \) can be modeled by the bilinear constraint
\[
\sum_{i=1}^{P} \sum_{t=1}^{m_i} \sum_{t=1}^{m_i} \sum_{k=1}^{P_i} b_{t,k,\ell}^{(i)} (u_{t,k}^{(i)} + v_{t,k}^{(i)}) - 2\pi z_\ell = \sigma_\ell, \quad \forall \ell \in [N].
\]

Since the terms \( b_{t,k,\ell}^{(i)} \cdot v_{t,k}^{(i)} \) are nonlinear and nonconvex, we replace them with the auxiliary variables \( u_{t,k,\ell}^{(i)} \) leading to Constraint (11f). A linearization of \( u_{t,k,\ell}^{(i)} = b_{t,k,\ell}^{(i)} \cdot v_{t,k}^{(i)} \) is given by Constraints (11g) and (11h). Finally, Constraint (11i) models the linear equations (9). Because of Assumption (A3), we can fix the first electrical angle \( \Phi_1 = -\pi d \) in Constraint (11j). In terms of vanishing sums, this amounts to a global rotation of all appearing roots of unity which does not destroy the sum being 0, as can be seen by
\[
\sum_{\ell=1}^{N} e^{i(2\pi v+\mu)} = \sum_{\ell=1}^{N} e^{i2\pi v} \cdot e^{i\mu}.
\]

In order to remove some symmetric solutions, the variables \( \Phi \) are ordered increasingly in Constraint (11j). This is justified by Theorem IV.1 in the next section. Additionally, the strict inequalities prevent trivial ambiguities consisting of two or more equal electrical angles. These strict inequalities together with the upper bounds for \( \mathbf{v}^{(i)}, \mathbf{w}^{(i)}, \Phi \) and \( \sigma \) are modeled using non-strict inequalities with a small \( \epsilon = 0.001 \), so that (11) is indeed a MIP.

Note that there could exist different feasible solutions that correspond to the same ambiguous vector of electrical angles. There are different reasons for this behavior. First, the decomposition of a vanishing sum of unit roots into minimal vanishing sums is not unique, such that different sums of (rotated) minimal vanishing sums of unit roots can lead to the same vanishing sum and thus, to the same ambiguity. Second, there can be different assignments of the chosen unit roots to the variables \( \sigma_\ell \) that lead to the same electrical angles \( \Phi_m \).

B. Analysis of the MIP Formulation

Theorem IV.1: Consider a feasible solution \( \mathbf{X} = (\mathbf{b}^{(i)}, \mathbf{q}^{(i)}, \mathbf{x}, \mathbf{z}, \mathbf{v}^{(i)}, \mathbf{w}^{(i)}, \Phi, \sigma) \) for (11) and let \( \tau \in S_M \) be an arbitrary permutation of \([M]\). Then there exists a feasible solution \( \tilde{\mathbf{X}} = (\tilde{\mathbf{b}}^{(i)}, \tilde{\mathbf{q}}^{(i)}, \tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \tilde{\mathbf{v}}^{(i)}, \tilde{\mathbf{w}}^{(i)}, \tilde{\Phi}, \tilde{\sigma}) \) with \( \tilde{\Phi} = (\Phi_{\tau(1)}, \ldots, \Phi_{\tau(M)}) \in [-\pi d, \pi d]^M \).

Note that the statement is not trivial: If the generalized Vandermonde determinant is 0, permuting \( \Phi \) does not change this, and it does not change the Schur polynomial. However, it is not clear whether there still exists a sum of rotated minimal...
vanishing sums and assignment of the appearing roots of unity so that Constraint (11i) is satisfied.

Proof: Let \( r = k + \delta \) be a linear array, and let \( \mathcal{T}_k \) be the set of all SSYTs of shape \( \lambda \). We first prove the assertion that if there exists an SSYT \( T_i \in \mathcal{T}_k \) with weight vector \( \alpha(T_i) = (\alpha_1, \ldots, \alpha_M, t) \), then for all permutations \( \tau \in S_M \) there also exists an SSYT \( T_k \in \mathcal{T}_k \) with \( \alpha(T_k) = (\alpha_1, \ldots, \alpha_M, k) \).

Let \( \beta, \beta' \) be two possible weight vectors of an SSYT of shape \( \lambda \) so that \( \beta \) and \( \beta' \) only differ by swapping two consecutive entries. Then there exists a bijection between the SSYTs of shape \( \lambda \) with weight vector \( \beta \) and \( \beta' \). Consider the entries \( \beta_i \) and \( \beta_{i+1} \) of the weight vector \( \beta \), and let \( T \) be an SSYT of shape \( \lambda \) with weight vector \( \beta \). Select all columns of \( T \) that contain exactly one entry equal to \( i \) or \( i + 1 \). All other columns contain either no or two such entries. In each row of \( T \) replace each \( i \) appearing in these columns by \( i + 1 \) and vice versa. After reordering the rows so that these are again sorted nondecreasingly, we obtain an SSYT of shape \( \lambda \) with weight vector \( \beta' \). This yields the desired bijection, see also [45, Proof of Theorem 7.10.2]. Thus, for a given weight vector \( \alpha \) and a given permutation \( \tau \in S_M \) we can find an SSYT with weight vector \( \tau(\alpha) \) by decomposing \( \tau \) into a sequence of transpositions of consecutive entries and using the compositions of the respective bijections.

Consider the solution \( \mathbf{X} \) and permutation \( \tau \). In order to prove the existence of solution \( \mathbf{X} \) with \( \Phi_m = \Phi_{\tau(m)} \), \( m \in [M] \), we show that there exists a permutation \( \gamma \in S_N \) with \( \tilde{\sigma}_\ell = \sigma_{\gamma(\ell)} \):

\[
\tilde{\sigma}_\ell = \alpha_{1,\ell} \tilde{F}_1 + \cdots + \alpha_{M,\ell} \tilde{F}_M - 2 \tilde{x}_\ell
= \alpha_{1,\ell} \Phi_{\tau(1)} + \cdots + \alpha_{M,\ell} \Phi_{\tau(M)} - 2 \tilde{x}_\ell
= \alpha_{\tau^{-1}(1),\ell} \Phi_1 + \cdots + \alpha_{\tau^{-1}(M),\ell} \Phi_M - 2 \tilde{x}_\ell.
\]

By the assertion, there exists an SSYT \( T \) with weight vector \( \tau(\alpha(T_i)) \). Defining the permutation \( \gamma \) such that \( \alpha_{m,\gamma(\ell)} = \alpha_{\tau^{-1}(m),\ell} \) for all \( m \in [M] \), i.e., such that the SSYT \( T_{\gamma(\ell)} \) has weight vector \( \tau(\alpha(T_i)) \) and setting

\[
\tilde{x}_\ell = x_{\gamma(\ell)}(i), \tilde{z}_\ell = z_{\gamma(\ell)}, \tilde{b}_{i,k,\ell} = b_{i,k,\gamma(\ell)},
\]

\[
w_{i,k,\ell}^{(i)} = w_{i,k,\gamma(\ell)}^{(i)}, q_{i,\ell} = q_{i,\ell}^{(i)}, r_{i,\ell}^{(i)} = r_{i,\ell},
\]

for all \( (i, k, \ell) \in I \times [N] \) yields the desired feasible solution \( \mathbf{X} \) with \( \Phi \in [-\pi d, \pi d]^M \) and \( \Phi_m = \Phi_{\gamma(m)} \) for all \( m \in [M] \). \( \square \)

The following two Lemmas state that for a linear array with integer positions, the feasibility problem (11) finds all ambiguities in the array that can be represented using a linear combination of minimal vanishing sums with coefficients on the complex unit circle.

**Lemma IV.2:** For a linear array with positions corresponding to integer multiples \( r \in \mathbb{Z}^M \) of a common baseline \( d \leq 1 \) measured in half wavelength, each feasible solution of the feasibility-MIP (11) corresponds to an ambiguous vector of electrical angles.

Proof: Let \( \mathbf{X} = (b^{(i)}, q^{(i)}, x, z, v^{(i)}, w^{(i)}, \Phi, \sigma) \) be a feasible solution for (11). It is clear by construction that the Schur polynomial \( s_\lambda(z) \) with \( z_i = e^{i\Phi_i} \), satisfies \( s_\lambda(z) = 0 \). By definition, this implies that the generalized Vandermonde determinant \( V_\ell(z) \) vanishes. Thus, the array steering matrix \( A(\Phi) \) is rank-deficient, i.e., \( \Phi_1, \ldots, \Phi_M \) are ambiguous. \( \square \)

The next Lemma is an immediate consequence of the definition of the Schur polynomial and the arguments above.

**Lemma IV.3:** Let \( r \in \mathbb{Z}^M \) be an arbitrary integer linear array where the positions are multiples of a common baseline \( d \leq 1 \) measured in half wavelength. Each ambiguous vector of electrical angles that forms a root of the Schur polynomial and that can be represented using solutions of rotated minimal vanishing sums, corresponds to at least one feasible solution of the feasibility-MIP (11).

Let us now relate our approach to the uniform ambiguities from [20, Theorem 2.2]. For the ease of presentation, we restate this result in terms of electrical angles.

**Theorem IV.4:** Let \( r = (r_1, \ldots, r_M)^T \in \mathbb{R}^M \) be an arbitrary linear array with positions measured in half wavelength and baseline \( d = 1 \). Define the vector \( \Phi_{i,j} \) of electrical angles as

\[
\Phi_{i,j} := \left[ -\pi, -\pi \left( 1 - \frac{2}{|r_i - r_j|} \right), -\pi \left( 1 - \frac{4}{|r_i - r_j|} \right), \ldots, -\pi \left( 1 - \frac{2c}{|r_i - r_j|} \right) \right]^T,
\]

where \( i \neq j \in [M] \) and \( c \in \mathbb{N} \) is the largest integer satisfying \( c < |r_i - r_j| \). Then, if \( \Phi_{i,j} \) contains at least \( M \) elements, any subvector of \( M \) elements from \( \Phi_{i,j} \) is a ambiguous vector.

Note that these are all ambiguities that are currently known in the literature for non-symmetric integer linear arrays.

We can now prove our main result, namely, that for an integer linear array, our approach is able to identify all uniform ambiguities from Theorem IV.4. This directly implies we can find all ambiguities previously known in the literature for non-symmetric integer linear arrays.

**Proposition IV.5:** Let \( r \in \mathbb{Z}^M \) be an arbitrary integer linear array where the positions are multiples of a common baseline \( d \leq 1 \) measured in half wavelength. Let \( \mathbf{r} = d \cdot r \in \mathbb{R}^M \), be the array \( r \) rescaled to a baseline \( d = 1 \), such that \( r \) and \( \mathbf{r} \) are in fact two representations (with different baselines) of the same linear array. Then any ambiguity of the form \( \Phi_{i,j} \) as stated in Theorem IV.4 (for the representation \( \mathbf{r} \)) corresponds to at least one solution of the feasibility-MIP (11) (for the representation \( r \)).

Proof: Let \( \Phi \in \mathbb{R}^M \) be an ambiguity for \( \mathbf{r} \) in the form of Theorem IV.4, and assume w.l.o.g. that the first electrical angle is \( -\pi d \), i.e., \( \Phi_1 = -\pi d \) and for \( m = \{2, \ldots, M\} \):

\[
\Phi_m = -\pi d \left( 1 - \frac{2c_m}{|r_i - r_j|} \right) = -\pi d \left( 1 - \frac{2c_m}{d |r_i - r_j|} \right),
\]

with \( i \neq j \in [M] \), \( c_m \in \mathbb{N} \) and \( c_{m_1} \neq c_{m_2} \) for all \( m_1 \neq m_2 \in [M] \). The variables \( \sigma_\ell \) in (11) are then given by

\[
\sigma_\ell = \sum_{m=1}^{M} \alpha_{m,\ell} \Phi_m \mod 2\pi,
\]
for \( m \in [M] \) and \( \ell \in [N] \). Thus,

\[
\sum_{\ell=1}^{N} e^{j\sigma_{\ell}} = \sum_{\ell=1}^{N} \exp \left( j \sum_{m=1}^{M} \alpha_{m,\ell} (\Phi_{m} + \pi d - \pi d) \right) = e^{-j\pi d} \sum_{\ell=1}^{N} \exp \left( j \sum_{m=1}^{M} \alpha_{m,\ell} (\Phi_{m} + \pi d) \right),
\]

(13)

where \( K = \sum_{m=1}^{M} \alpha_{m,\ell} \) is the same constant for all \( \ell \in [N] \). Since \( \Phi_{m} + \pi d \in \{2\pi w : w \in \mathbb{Q} \} \) for all \( m \in [M] \) it holds that (13) is a sum of roots of unity, rotated by a complex number on the complex unit circle.

Since \( \Phi \) forms an ambiguous vector of electrical angles for the linear array \( r \), \( z_{m} = \exp(j\Phi_{m}) \) form a root of the generalized Vandermonde determinant \( V_{r}(z) \) and thus of the Schur polynomial \( s_{\lambda}(z) \), where \( \lambda = r - (0, 1, \ldots, M - 1) \). Equations (7), (8) and (9) imply \( 0 = s_{\lambda}(z) = \sum_{i=1}^{N} \exp(j\sigma_{i}) \), and thus (13) is a rotated vanishing sum of roots of unity. Theorem III.4 implies that (13) can be written as linear combination of minimal vanishing sums with coefficients on the complex unit circle. By Lemma IV.3, there is at least one feasible solution of the MIP (11) that corresponds to the ambiguity \( \Phi \), which finishes the proof. \( \square \)

Moreover, the computational results in Section V below show that for non-symmetric integer linear arrays, our approach finds many more ambiguities than previously known. In general, it remains an open question, whether there also exist ambiguities for non-symmetric integer linear arrays which cannot be expressed as linear combination of minimal vanishing sums with coefficients on the complex unit circle, or if our approach indeed finds all ambiguities that are present in such an array.

Before presenting computational results, let us shortly discuss some details of enumerating the feasible solutions with our approach in the next section.

C. Enumerating All Feasible Solutions of the MIP

Since it is possible that the MIP (11) has infinitely many feasible solutions, we do not count all feasible solutions, but only all configurations of the integer and binary variables \( b, q, x, z \) such that after fixing all these variables the remaining problem has at least one feasible solution. Because all integer and binary variables are bounded, there are only finitely many configurations that need to be checked.

After all feasible configurations of the integer and binary variables have been found, we apply a post-processing step in order to find the ambiguities corresponding to each configuration. Due to the rotations \( v_{j}^{(i)} \) of the used minimal vanishing sums, there can exist configurations of the integer (and binary) variables with infinitely many feasible solutions for the continuous variables. These solutions form a whole class of ambiguities with a number of parameters, see Example V.1. Altogether, we end up with either finitely many ambiguities, or finitely many classes of ambiguities, each of them depending on a number of parameters. The ambiguities can be converted into DoAs by using \( -\pi d \cos(\theta_{m}) = \Phi_{m} \).

Remark IV.6: In order to reduce the computational effort of enumerating all feasible integer solutions of (11), the problem can be divided into smaller subproblems, one for each possible partition. In each of the smaller MIPs, most of the variables \( q_{j}^{(i)} \in \{0, 1\}^{m} \), can be fixed, according to the corresponding partition. Moreover, this eliminates many symmetric solutions in terms of the variables \( q_{j}^{(i)} \).

V. COMPUTATIONAL RESULTS

In this section, we use the approach of obtaining ambiguities by enumerating the feasible solutions of (11) to identify ambiguities for some exemplary integer linear arrays. To enumerate all feasible solutions we use the counting feature of SCIP 6.0.0 [51]. We use CPLEX 12.7.1.0 as LP solver. The first two Examples V.1 and V.2 were enumerated using the full Problem (11), whereas for Examples V.3 and V.4, we divided Problem (11) into smaller ones, one for each possible partition as described in Remark IV.6. Unless stated otherwise, the baseline is \( d = 1 \). The used model files can be obtained via the website of the last author.

The computation for Examples V.1 to V.3 were performed on a Linux desktop with 3.6 GHz Intel Core i7-7700 Quad-Core CPUs having 16 GB main memory and 8 MB cache, whereas the computations for Example V.4 were done on a Linux cluster with 3.5 GHz Intel Xeon E5-1620 Quad-Core CPUs, having 32 GB main memory and 10 MB cache. All computations were performed single-threaded and, in the case of Example V.4 with a timelimit of 450 000 s. For this Example, we also display the time needed to enumerate all feasible solutions, the total number of solutions that were enumerated, as well as the number of processed nodes in the enumeration process.

Example V.1 shows a linear array with an infinite number of ambiguities, even after fixing the first electrical angle to \( -\pi d \).

Example V.1: Consider the linear array \( r = \lambda + \delta = (0, 1, 3, 4)^{\top} \), where \( \delta = (0, 1, 2, 3)^{\top} \) and \( \lambda = (0, 0, 1, 1)^{\top} \). The possible Young tableaux are shown in Fig. 4.

This results in the Schur polynomial 

\[
s_{\lambda}(z) = z_{1} z_{2} + z_{1} z_{4} + z_{2} z_{3} + z_{2} z_{4} + z_{3} z_{4}.
\]

Since \( r \) has \( N = 6 \) corresponding SYSTs, there are three possible partitions: \( 2 + 2 + 3 + 3 = 6 \).\(^{3}\) For the partitions \( 2 + 2 + 2 + 3 \), there exist infinitely many valid solutions for the feasibility problem (11), and thus infinitely many ambiguities, which can be classified into three classes.

The partition \( 2 + 2 + 2 \) yields the two infinite classes of electrical angles

\[
-\pi \cdot [1, v, 0, -v]^{\top}, \quad -\pi \cdot [1, 1 - v, 1 - 2v, -v]^{\top},
\]

\(^{3}\)Note that there are no minimal vanishing sums of length 1 or 4, see Remark III.5.
for all \( v \in (0, 1) \). The partition 3 + 3 yields the following class of electrical angles for \( v \in (-1, 1) \) and \( \gamma = \arccos(-\pi v) \):

\[
-\pi \cdot [1, \frac{1}{3}, \frac{1}{3}, v] \simeq [0^\circ, 70.53^\circ, 109.47^\circ, \gamma]^	op.
\]

Since already the steering vectors corresponding to the electrical angles \(-\pi \cdot [1, \frac{1}{3}, \frac{1}{3}, v] \) are linearly dependent and thus induce a rank-deficient steering matrix, an arbitrary electrical angle \( v \in (-\pi, \pi) \) can be added to the three electrical angles in order to obtain an ambiguity with four electrical angles.

For the third partition 6, there are finitely many corresponding solutions, namely the eight ambiguities in Table II. Note that all three ambiguities that were already found by using the methods from Proukakis and Manikas [20], are contained in one of the three classes.

Some of the non-uniform ambiguities obtained with the methods for symmetric linear arrays in [25] are contained in \(-\pi \cdot [1, v, 0, -v]^	op \), but there also exist non-uniform ambiguities which cannot be found with our approach, see Example III.7.

Altogether, our approach finds two new classes of ambiguities for this particular linear array that have not been known in the literature. Additionally, all previously known ambiguities which can be expressed as sums of rotated minimal vanishing sums are found by our approach as well.

The next example demonstrates that there exist linear arrays for which the methods from Manikas and Proukakis [20] cannot find any ambiguities. In contrast, enumerating feasible solutions of the MIP (11) yields infinitely many ambiguities. Note that this can only happen with a baseline \( d < 1 \), i.e., the sensor positions are not integral (measured in half wavelength) but integer multiples of the baseline. Otherwise, for a non-uniform linear array with integer positions in half wavelength, the sufficient condition in [20] is satisfied and the corresponding methods always find some ambiguities.

**Example V.2:** Consider the sensor positions \( r = (0, 1, 2, 4) \) together with the baseline \( d = \frac{1}{2} \). This corresponds to a linear array with sensor positions \( \tilde{r} = (0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}) \) measured in half wavelength. There are \( N = 4 \) SSYTs of shape \( \lambda = (0, 0, 0, 1) \), such that we need to search for vanishing sums of length 4. The only possible partition of 4 is given by \( 2 + 2 \). We obtain one class of infinitely many electrical angles that form an ambiguity, namely

\[
-\pi \cdot [\frac{3}{4}, 2 - 2v, -\frac{1}{4}, 1 - 2v],
\]

where \( v \in (\frac{1}{2}, 1) \). The methods from Manikas and Proukakis [20] do not find any ambiguities for this array.

This result is indeed remarkable as this array with the shortened baseline \( d = \frac{1}{2} \) measured in half-wavelength exhibits on average the same half-wavelength inter-sensor spacing as the perfect uniform linear array \((0, 1, 2, 3)\) with baseline \( d = 1 \), which is known to be free of ambiguities. Hence, when choosing an array with 4 sensors and total aperture size of \( \frac{3}{4} \) wavelength, the uniform linear array is preferable to the above mentioned array concerning the desire to prevent ambiguities.

Since the last two examples where uniform linear arrays with only one sensor position missing, we investigate a linear array with a larger gap in the next example.

**Example V.3:** Consider the sensor positions \( r = (0, 1, 2, 5) \). There are \( N = 10 \) SSYTs of shape \( \lambda = (0, 0, 0, 2) \), such that we need to search for vanishing sums of length 4. The possible partitions are

\[
\begin{align*}
2 + 2 + 2 + 2 &= 2 + 2 + 3 + 3 = 2 + 3 + 5 \\
5 + 5 &= 2 + 2 + 6 = 3 + 7 = 2 + 8 = 10.
\end{align*}
\]

The partition \( 2 + 2 + 2 + 2 \) yields the following five ambiguities, expressed in electrical angles,

\[
\begin{align*}
-\pi \cdot [1, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, \frac{1}{4}], & \quad -\pi \cdot [1, \frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}], \\
-\pi \cdot [1, \frac{3}{4}, \frac{1}{4}, 0], & \quad -\pi \cdot [1, 0, -\frac{1}{4}, -\frac{3}{4}], \\
-\pi \cdot [1, \frac{1}{2}, 0, -\frac{1}{2}]. & \quad (14)
\end{align*}
\]

The partition 5 + 5 yields four ambiguities, namely

\[
-\pi \cdot [1, a, b, c], \quad a, b, c \in \{\pm \frac{1}{6}, \pm \frac{1}{5}\}, \quad (15)
\]

with \( a, b, \) and \( c \) pairwise different. The partition 7 + 3 yields eight ambiguities, namely

\[
\begin{align*}
-\frac{\pi}{15} \cdot [15, 8, -2, -9], & \quad -\frac{\pi}{5} \cdot [15, 9, 2, -8], \\
-\frac{\pi}{15} \cdot [15, 8, 2, -5], & \quad -\frac{\pi}{15} \cdot [15, 5, -2, -8], \\
-\frac{\pi}{5} \cdot [15, 14, -4, -5], & \quad -\frac{\pi}{5} \cdot [15, 5, 4, -14], \\
-\frac{\pi}{15} \cdot [15, 14, 4, 3], & \quad -\frac{\pi}{15} \cdot [15, -3, -4, -14].
\end{align*}
\]

The other partitions do not yield any ambiguities. The methods from Manikas and Proukakis [20] only find the five ambiguities in (14) and (15).

As another example of a linear array with a gap larger than one, we also considered the sensor positions \( r = (0, 4, 5) \). We have enumerated nine ambiguities in the resulting linear array, all of which can already be found with the methods from Manikas and Proukakis [20].

**Example V.4:** Consider the linear array \( r = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12) \), which has \( N = 12 \) SSYTs. There are 14 partitions of \( N = 12 \) not using the numbers 1 and 4, namely

\[
\begin{align*}
2 + 2 + 2 + 2 + 2 &= 2 + 2 + 2 + 3 + 3 = 3 + 3 + 3 + 3 \\
2 + 2 + 3 + 5 &= 2 + 5 + 5 = 2 + 2 + 6 = 3 + 3 + 6 \\
6 + 6 &= 2 + 3 + 7 = 5 + 7 = 2 + 8 = 3 + 9 = 2 + 10 = 12.
\end{align*}
\]

The partitions 2 + 3 + 7 and 12 reached the timelimit of 450 000 seconds, so that for these partitions we possibly have enumerated
TABLE III
DEFINITION OF $f_{y}(v)$ AND $f_{y}^{(k)}(v)$ USING ELECTRICAL ANGLES

| $f_{y} = v \{ -1, v \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y} = v \{ -1, v, v + \frac{1}{2} \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y} = v \{ -1, v, v + 2 \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y}^{(1)} = v \{ -1, v - \frac{1}{2}, v - \frac{1}{2}, v - 1 \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y}^{(2)} = v \{ -1, v - 1, v - \frac{1}{2}, v - \frac{1}{2}, v - 1 \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y}^{(3)} = v \{ -1, v - \frac{1}{2}, v - \frac{1}{2}, v - 1 \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y}^{(4)} = v \{ -1, v - 1, v - \frac{1}{2}, v - \frac{1}{2}, v - 1 \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y}^{(5)} = v \{ -1, v - \frac{1}{2}, v - \frac{1}{2}, v - 1 \}$, | $v \in (0, \frac{1}{2})$ |
| $f_{y}^{(6)} = v \{ -1, v - 1, v - \frac{1}{2}, v - \frac{1}{2}, v - 1 \}$, | $v \in (0, \frac{1}{2})$ |

TABLE IV
ALL FOUND AMBIGUITIES IN THE LINEAR ARRAY $r = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12)$, FOR A SUBSET OF ALL POSSIBLE PARTITIONS, EXPRESSED IN ELECTRICAL ANGLES

| Partition | # Solutions | Time (s) | # Nodes |
|-----------|-------------|----------|---------|
| $2+2+2+2+2$ | 160 | 719.43 | 2204798 |
| $2+2+2+3+3$ | 26664 | 189691.62 | 7470545 |
| $3+3+3+3$ | 405 | 57905.55 | 17067233 |
| $2+2+2+3+5$ | 42336 | 300709.50 | 102050134 |
| $2+5+5$ | 1680 | 20150.43 | 45847950 |
| $2+2+2+6$ | 9968 | 85418.02 | 60992369 |
| $3+3+6$ | 2736 | 26180.16 | 56723306 |
| $6+6$ | 130 | 2495.70 | 9615581 |
| $2+3+7$ | 13318 | > 450000.00 | > 157647076 |
| $5+7$ | 1289 | 37766.13 | 68727902 |
| $2+2+8$ | 8046 | 260789.23 | 144653756 |
| $3+9$ | 1188 | 44698.63 | 60887227 |
| $2+10$ | 1830 | 75348.62 | 83449667 |
| $12$ | 315 | > 450000.00 | > 70499108 |

VI. CONCLUSION
We demonstrated that for several integer linear arrays our method is able to find more ambiguities than were known previously by the methods from Manikas and Proukakis [20] (general case) and the methods from Dowlut [24] and Manikas [25] (symmetric case). In Example V.2 we showed that our approach is also applicable for linear arrays for which the methods from Manikas and Proukakis fail to find ambiguities. Even more, after a small post-processing, our approach can detect whole classes of infinitely many ambiguities, whereas the methods from Manikas and Proukakis can only find finitely many single ambiguities.

Concerning the objective to achieve unambiguous DoA estimation arrays with infinitely many ambiguities such as the array $r = (0, 1, 3, 4)$ measured in half wavelength are therefore less recommendable than, e.g., the array $r = (0, 1, 2, 5)$ measured in half wavelength, for which our approach identifies finitely many ambiguities.

Overall, it turns out that arrays with a small number of SSYTs also have a small number of (infinite) classes of ambiguities and the number of SSYTs increases with the number of holes in the corresponding linear array.

Example V.1 shows that at least for symmetric linear arrays there exist non-uniform ambiguities that cannot be found with our approach described in Section IV. Therefore, the interesting question arises, whether there can exist ambiguities in non-symmetric linear arrays that cannot be represented using sums of rotated minimal vanishing sums (c.f. Example III.7 and the discussion thereafter).

ACKNOWLEDGMENTS
The authors thank three reviewers for detailed comments that helped to improve the presentation of the paper.

REFERENCES
[1] G. Oliveri, M. Donelli, and A. Massa, “Linear array thinning exploiting almost difference sets,” IEEE Trans. Antennas Propag., vol. 57, no. 12, pp. 3800–3812, Dec. 2009.
[2] C. Liu and P. P. Vaidyanathan, “Super nested arrays: Linear sparse arrays with reduced mutual coupling–Part I: Fundamentals,” IEEE Trans. Signal Process., vol. 64, no. 15, pp. 3997–4012, Aug. 2016.

only a subset of all feasible solutions. For a subset of the remaining partitions all ambiguities found by enumerating feasible solutions of the corresponding smaller MIP are displayed in Table IV, expressed as electrical angles. Here, $f_{y}(v)$ and $f_{y}^{(k)}(v)$ are vectors of electrical angles depending on a parameter, which are defined in Table III. Thus, the (infinite) classes of ambiguities are given by combining the specified vectors of electrical angles into one large vector of 12 electrical angles depending on a number of parameters.

In Table V the number of feasible solutions, the time needed for enumerating the feasible solutions and the number of processed nodes in the enumeration process are displayed for each partition. Theorem IV.4 yields one uniform ambiguity, namely $-\pi \cdot \left[ \begin{array}{c} 5 \\ 6 \\ 4 \\ 6 \\ 3 \\ 2 \\ 6 \\ 1 \\ 0 \\ 6 \\ 1 \\ 6 \\
-2 \\
-3 \\
-4 \\
-5 \\end{array} \right] ^{T}$ which is contained in the class $[f_{2}(0), f_{2}(v_{1}), f_{2}(v_{2}), f_{2}(v_{3}), f_{2}(v_{4}), f_{2}(v_{5})]$ of partition $2+2+2+2+2+2$. 

| Partition | # Solutions | Time (s) | # Nodes |
|-----------|-------------|----------|---------|
| $2+2+2+2+2+2$ | 160 | 719.43 | 2204798 |

| Partition | # Solutions | Time (s) | # Nodes |
|-----------|-------------|----------|---------|
| $2+2+2+3+3$ | 26664 | 189691.62 | 7470545 |
| $3+3+3+3$ | 405 | 57905.55 | 17067233 |
| $2+2+2+3+5$ | 42336 | 300709.50 | 102050134 |
| $2+5+5$ | 1680 | 20150.43 | 45847950 |
| $2+2+2+6$ | 9968 | 85418.02 | 60992369 |
| $3+3+6$ | 2736 | 26180.16 | 56723306 |
| $6+6$ | 130 | 2495.70 | 9615581 |
| $2+3+7$ | 13318 | > 450000.00 | > 157647076 |
| $5+7$ | 1289 | 37766.13 | 68727902 |
| $2+2+8$ | 8046 | 260789.23 | 144653756 |
| $3+9$ | 1188 | 44698.63 | 60887227 |
| $2+10$ | 1830 | 75348.62 | 83449667 |
| $12$ | 315 | > 450000.00 | > 70499108 |
[3] C. Liu and P. P. Vaidyanathan, “Super nested arrays: Linear sparse arrays with reduced mutual coupling–Part II: High-order extensions,” IEEE Trans. Signal Process., vol. 64, no. 16, pp. 4203–4217, Aug. 2016.

[4] M. Toso, C. Mangenot, and A. G. Roederer, “Sparse and thinned arrays for multiple beam satellite applications,” in Proc. 2nd Eur. Conf. Antennas Propag., 2007, pp. 1–7.

[5] P. Pal and P. P. Vaidyanathan, “Nested arrays: A novel approach to array processing with enhanced degrees of freedom,” IEEE Trans. Signal Process., vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[6] P. P. Vaidyanathan and P. Pal, “Sparse sensing with co-prime samplers and arrays,” IEEE Trans. Signal Process., vol. 59, no. 2, pp. 573–586, Feb. 2011.

[7] A. Moffet, “Minimum-redundancy linear arrays,” IEEE Trans. Antennas Propag., vol. AP-16, no. 2, pp. 172–175, Mar. 1968.

[8] P. Rocca, R. L. Haupt, and A. Massa, “Interference suppression in uniform linear arrays through a dynamic thinning strategy,” IEEE Trans. Antennas Propag., vol. 59, no. 12, pp. 4525–4533, Dec. 2011.

[9] X. Wang, M. Amin, and X. Cao, “Analysis and design of optimum sparsely array configurations for adaptive beamforming,” IEEE Trans. Signal Process., vol. 66, no. 2, pp. 340–351, Jan. 2018.

[10] S. Joshi and S. Boyd, “Sensor selection via convex optimization,” IEEE Trans. Signal Process., vol. 57, no. 2, pp. 451–462, Feb. 2009.

[11] R. L. Haupt, “Thinned arrays using genetic algorithms,” IEEE Trans. Antennas Propag., vol. 42, no. 7, pp. 993–999, Jul. 1994.

[12] T. Isemia, F. J. Ares Pena, O. M.ucci, M. D’urso, J. Fondeve Lopez, and J. A. Rodriguez, “A hybrid approach for the optimal synthesis of pencil beams through array antennas,” IEEE Trans. Antennas Propag., vol. 52, no. 11, pp. 2912–2918, Nov. 2004.

[13] J. Robinson and Y. Rahmat-Samii, “Particle swarm optimization in electromagnetics,” IEEE Trans. Antennas Propag., vol. 52, no. 2, pp. 397–407, Feb. 2004.

[14] V. Murino, A. Trucco, and C. S. Regazzoni, “Synthesis of unequally spaced arrays by simulated annealing,” IEEE Trans. Signal Process., vol. 44, no. 1, pp. 119–123, Jan. 1996.

[15] R. O. Schmidt, “A signal subspace approach to multiple emitter location spectral estimation,” Ph.D. dissertation, Stanford Univ., Stanford, CA, USA, 1981.

[16] J. T. Lo and S. L. Marple, “Observability conditions for multiple signal direction finding and array sensor localization,” IEEE Trans. Signal Process., vol. 40, no. 11, pp. 2641–2650, Nov. 1992.

[17] K.-C. Tan, S. S. Goh, and E. C. Tan, “A study of the rank-ambiguity issues in direction-of-arrival estimation,” IEEE Trans. Signal Process., vol. 44, no. 4, pp. 880–887, Apr. 1996.

[18] K.-C. Tan and Z. Goh, “A construction of arrays free of rank ambiguities,” in Proc. IEEE Int. Conf. Acoust. Speech Signal Process., 1994, pp. IV-545–IV-548.

[19] K.-C. Tan, G.-L. Oh, and M. Er, “A study of the uniqueness of steering vectors in array processing,” Signal Process., vol. 34, no. 3, pp. 245–256, 1993.

[20] A. Manikas and C. Proukakis, “Modeling and estimation of ambiguities in linear arrays,” IEEE Trans. Signal Process., vol. 46, no. 8, pp. 2166–2179, Aug. 1998.

[21] T. T. Puthenpurakal, Introduction to Radar Systems, 2nd ed. New York, NY, USA: McGraw-Hill, 1980.

[22] C. M. Tan, M. A. Beach, and A. R. Nix, “Problems with direction finding using linear array with element spacing more than half wavelength,” in Proc. 1st Annual 273 COST Workshop, pp. 6–21, 2002.

[23] Y. I. Abramovich, N. K. Spencer, and A. Y. Gorokhov, “Identifiability and manifold ambiguity in DOA estimation for nonuniform linear antenna arrays,” in Proc. IEEE Int. Conf. Acoustics, Speech Signal Process., 1999, pp. 2845–2848.

[24] N. D. Sidiropoulos and X. Liu, “Identifiability results for blind beamforming in incoherent multipath with small delay spread,” IEEE Trans. Signal Process., vol. 49, no. 1, pp. 228–236, Jan. 2001.

[25] E. R. Heineman, “Generalized Vandermonde determinants,” Trans. Amer. Math. Soc., vol. 31, no. 3, pp. 464–476, 1929.

[26] C. Liu and P. P. Vaidyanathan, “Super nested arrays: Linear sparse arrays with reduced mutual coupling–Part II: High-order extensions,” IEEE Trans. Signal Process., vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[27] M. W. Buck, R. A. Coley, and D. P. Robbins, “A generalized Vandermonde determinant,” J. Algebraic Comb., vol. 1, no. 2, pp. 105–109, 1992.

[28] T. Ernst, “Generalized Vandermonde determinants,” Dept. Math., Upsala Univ., Sweden, U. D. M. Tech. Rep. 2006:6, 2000.

[29] H. Schlickewei and C. Viola, “Generalized Vandermonde determinants,” Acta Arith., vol. 95, no. 2, pp. 123–137, 2000.

[30] O. H. Mitchell, “Note on determinants of powers,” Amer. J. Math., vol. 4, no. 1, pp. 341–344, 1881.

[31] E. H. Moore, “A two-fold generalization of Fermat’s theorem,” Bull. Amer. Math. Soc., vol. 2, no. 7, pp. 189–199, 1896.

[32] S. De Marchi, Generalized Vandermonde Determinants, Toepol, Matri ces and the Polynomial Division Algorithm (Ser. Ergebnisberichte Ange wandte Mathematik), Dortmund, Fachbereich Mathematik, 1999.

[33] R. Stanley, Enumerative Combinatorics: Volume 2, Cambridge, U.K.: Cambridge Univ. Press, 1997.

[34] C. Fulton and J. Harris, Representation Theory: A First Course. Berlin, Germany: Springer, 2013, vol. 129.

[35] B. Surneflvs, Solving Systems of Polynomial Equations, vol. 97. Providence, RI, USA: AMS, 2002.

[36] H. B. Mann, “On linear relations between roots of unity,” Mathematika, vol. 12, no. 2, pp. 107–117, 1965.

[37] J. Bonen and M. Rabinstein, “The number of intersection points made by the diagonals of a regular polygon,” SIAM J. Discret. Math., vol. 11, no. 1, pp. 135–156, 1998.

[38] J. Conway and A. Jones, “Trigonometric diophantine equations (on vanishing sums of roots of unity),” Acta Arith., vol. 30, no. 3, pp. 229–240, 1976.

[39] A. Gleixner et al., “The SCIP optimization suite 6.0” 2018. [Online]. Available: http://www.optimization-online.org/DB/HTML/2018/07/6692.html

Frederic Matter received the M.Sc. degree in mathematics from Goethe University, Frankfurt, Germany in 2017. Then he joined the Research Group Optimization, Technische Universität Darmstadt, Darmstadt, Germany, as a Doctoral Candidate, where he successfully defended his Ph.D. thesis in April 2022. His research interests include mixed-integer semidefinite programming and compressed sensing, in particular sparse recovery under side constraints.
Tobias Fischer received a Diploma in mathematics and the Ph.D. degree from the Technical University of Darmstadt, Darmstadt, Germany, in 2012 and 2017, respectively. From 2017 to 2021, he was a Postdoctoral Researcher with the Fraunhofer Institute for Industrial Mathematics, Kaiserslautern, Germany. He is currently a Member of the Development Team for production planning and detailed scheduling, SAP SE, Walldorf, Germany. His research interests include mixed-integer programming, constraint programming, and solving technology for production scheduling problems.

Marius Pesavento (Senior Member, IEEE) received the Dipl.-Ing. degree from Ruhr-Universität Bochum, Bochum, Germany, in 1999, the M.Eng. degree from McMaster University, Hamilton, ON, Canada, in 2000, and the Dr.-Ing. degree in electrical engineering from Ruhr-Universität Bochum, Bochum, Germany, in 2005. From 2005 to 2008, he was a Research Engineer in two start-up companies. He became an Assistant Professor in robust signal processing in 2010 and a Full Professor in communication systems in 2013 with the Department of Electrical Engineering and Information Technology, Technische Universität Darmstadt, Darmstadt, Germany. His research interests include robust signal and sensor array processing, multi-antenna and multi-user communication systems, optimization techniques for signal processing, communications, and learning. He was an Associate Editor for the IEEE TRANSACTIONS ON SIGNAL PROCESSING during 2012–2016. He is a Senior Area Editor of the IEEE OPEN JOURNAL OF SIGNAL PROCESSING and a Member of the Editorial Board of the EURASIP Signal Processing Journal. He is the Vice-Chair of the Technical Area Committee Signal Processing for Multisensor Systems of the EURASIP. He was the recipient of the 2003 ITG/VDE Best Paper Award and the 2005 Young Author Best Paper Award of the IEEE Transactions on Signal Processing.

Marc E. Pfetsch received the Diploma in mathematics from the University of Heidelberg, Heidelberg, Germany, in 1997, the Ph.D. degree in mathematics, and the Habilitation degree from the Berlin Institute of Technology, Berlin, Germany, in 2002 and 2008, respectively. From 2008 to 2012, he was a Full Professor for mathematical optimization with Technische Universität Braunschweig, Braunschweig, Germany. Since April 2012, he has been a Full Professor for discrete optimization with Technische Universität Darmstadt, Germany. His research interests include mostly in discrete and nonlinear optimization, in particular compressed sensing, symmetry in integer programs, and algorithms for mixed nonlinear integer programs.