Abstract

I survey the use of the Haag expansion as a technique to solve quantum field theories. After an exposition of the asymptotic condition and the Haag expansion, I report the results of applying the Haag expansion to several quantum field theories, including galilean-invariant theories, matter at finite temperature (using the BCS model of superconductivity as an illustrative example), the Nambu–Jona-Lasinio model and the Schwinger model. I conclude with the outlook for further development of this method.

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1. ADVERTISEMENT

Recently there has been a renewal of interest in analyzing quantum chromodynamics starting from the action of the theory and using continuum methods, rather than the lattice methods that have been extensively pursued. Light-front methods have been emphasized as having the following virtues:

- Simple vacuum structure
- Simple boosts
- Intuitive wave function picture
- Can be systematically improved
- Nonperturbative

Light-front method have some drawbacks:

- Not explicitly Lorentz invariant, so rotations are complicated
- Functions are required in counter terms for renormalization
- The gauge is fixed, so gauge invariance is difficult to check
- Presence of zero modes

The amplitudes in the Haag expansion\cite{1} share the virtues of the light front method, and in addition have the following virtues:

- Have the same number of kinematic variables as Schrödinger amplitudes
- Obey three-dimensional equations that are explicitly covariant
- In other words, are as close to completely on-shell as possible in field theory
- Composite particles are treated in parallel with elementary particles
- Can be made crossing symmetric

This last property holds because, unlike the Tamm-Dancoff expansion, the Haag expansion is made in the fields, rather than in the states. The Haag expansion method also has some drawbacks:

- There are more graphs, because many of the lines are on-shell
- It is not clear in general how to truncate the expansion
- In confining theories, a replacement must be found for asymptotic fields

There are some hints how to do this from the Schwinger model. If you like, you can use the Haag expansion on the light front and thus combine the virtues and drawbacks of the two methods.
2. ASYMPTOTIC FIELDS AND THE HAAG EXPANSION

Asymptotic fields have been part of quantum field theory at least since the work of Lehmann, Symanzik and Zimmermann [2]; however, there are still misconceptions that should be cleared up. The asymptotic fields are free fields, because (at least in theories which have neither massless particles nor confinement) the particles described by the asymptotic fields separate for large magnitude of the time. The free field property of the asymptotic fields does not depend on unphysical “adiabatic switching off” of the interactions. The physical assumption is that for $t \to \pm \infty$, particles either (a) separate widely and thus move freely, since interactions fall off exponentially in space or (b) stay close together and thus form a bound state that itself moves freely. In this case asymptotic fields must be introduced for the bound state. In either case, the exact eigenstates can be labelled by the quantum numbers of free particles. The limits for $t \to \pm \infty$ are the out or in fields that make eigenstates at the corresponding limiting times. The asymptotic fields at finite times are the limiting fields brought back to finite times according to the free equations of motion. The unitary relation between these fields is given by the $S$-operator,

$$S\phi_{\text{out}} S^\dagger = \phi_{\text{in}}.$$  

We need asymptotic fields for those bound states that are stable in the approximation under consideration; for example, in considering strong interactions, pions would be taken to be stable and would receive asymptotic fields.

The in fields all have free commutation or anticommutation relations and free equations of motion and commute or anticommute among themselves. The same is true for the out fields. Given the masses and spins of the fields, each set, in or out, is a completely known set of fields. Thus each set is a convenient set of building blocks for the construction of solutions of quantum field theories. The relation between the in and the out fields is nontrivial, given by the S-operator.

The limits that define the asymptotic fields are subtle. The relations that appears in some books,

$$\phi(x) \to \phi_{\text{out}}, \text{in}(x), x^0 \to \pm \infty$$  

are ill-defined. The proper limit is a weak operator limit that constructs an asymptotic field of a given mass $m$ from the neighborhood of the mass $m$ part of the
relevant (product of) (scalar) Lagrangian field(s)\[3\],

\[\phi^{\text{out, in}}(x) = \lim_{\tau \to \pm \infty} \left[ - \int y_0 = \tau \Delta(x - y; m^2) \partial_{y^\tau} \phi(y) d^3y \right]. \quad (3)\]

The relation to the mass \(m\) part of the Lagrangian field is transparent from the momentum space version,

\[\tilde{\phi}^{\text{out, in}}(k) \delta(k^2 - m^2) = \lim_{\tau \to \pm \infty} \epsilon(k^0) \delta(k^2 - m^2) \int dq^0 (q^0 + k^0) \tilde{\phi}(q^0, k) e^{-i(q^0 - k^0)\tau}. \quad (4)\]

For composite particles, one must use a product of the Lagrangian fields of the elementary constituents. More about that later. When the field strength renormalization diverges, which is generally the case in relativistic theories, one must introduce an averaging over time in the definition of the limit. See\[3\] for details of that.

To motivate the Haag expansion, recall that we expect that a quantum field theory of particles in Hilbert space has three complete and irreducible sets of field operators. Here complete means that any state in the Hilbert space can be approximated by polynomials in the smeared fields acting on a cyclic vector, usually the vacuum state, and irreducible means that any operator that commutes with an irreducible set of operators is a multiple of the identity. The first such complete and irreducible set is the set of Lagrangian fields, i.e., the fields that appear in the Lagrangian and in the action of the theory. The second and third such sets are the two sets of asymptotic fields, including fields for bound states, if there are any. Since by themselves the set of \(\text{in}\) fields are completely known, they are standard building blocks from which the Lagrangian fields can be constructed. The same, of course, is true for the set of \(\text{out}\) fields. The Haag expansion is just the expression of this idea. For a theory with a single (scalar) field and no bound states, the Haag expansion is

\[A(x) = A^{\text{in}}(x) + \sum_{n=2}^{\infty} \int d^4x_1 \cdots d^4x_n f^{(n)}(x - x_1, \ldots, x - x_n) : A^{(\text{in})}(x_1) \cdots A^{(\text{in})}(x_n) :, \quad (5)\]

where the double dots indicate normal ordering of the \(\text{in}\) fields\[1, 4\]. If there are bound states, then \(\text{in}\) fields for the bound states have to be introduced wherever the conservation laws of the theory allow\[5\]. The physical vacuum is the state annihilated by the positive frequency parts of the \(\text{in}\) fields; only the physical vacuum enters and it is a structureless state in this formulation. The equations for the
(scalar) in fields are

\[(\Box + m^2)A^{(in)}(x) = 0,\]  

\[A^{(in)}(x), A^{(in)}(y)\] = i\(\Delta(x - y; m^2),\]  

and the different in fields completely commute or anticommute depending as usual on whether the fields are bosons or fermions. If spontaneous symmetry breaking occurs, the Haag expansion starts with a c-no. term which is the vacuum matrix element of the scalar field[4]. Haag introduced this expansion in 1955 to discuss questions of principle. The present effort is aimed at developing a practical calculational method based on his expansion. The \(f^{(n)}\)'s ("Haag" amplitudes) are multiple retarded commutator functions with all but one leg on-shell[4]. They automatically correspond to connected graphs. They obey the Klein-Gordon equation (for scalar in fields) in each argument. Because of this, the terms in the expansion can be replaced by

\[
\int d^3x_i f^{(n)}(\cdots, x - x_i, \cdots) \partial_{x_i}^+ \cdots \cdot A^{(in)}(x_i) \cdots:
\]

which illustrates that the Haag amplitudes are both three-dimensional and covariant. The asymptotic limit applied to \(A(x)\) gives \(A^{(out)}(x)\) in terms of \(A^{(in)}(x)\). The relation between the (anti)commutators of these two,

\[\[A^{(out)}(x), A^{(out)}(y)\] = [A^{(in)}(x), A^{(in)}(y)]\]

gives unitarity for all processes. The equal-time commutation relations,

\[\[A(x, t), A(y, t)\] = 0, [A(x, t), \hat{A}(y, t)] = iZ_3^{-1}\delta(x, y),\]

give generalizations of unitarity. (The choice of \(\hat{A}\) as the canonical conjugate is valid for theories without derivative coupling.) The Haag amplitudes for bound states are like Schrödinger wave functions for the bound states. There are no relative-time coordinates as in the Bethe-Salpeter amplitudes. The Haag amplitudes for elastic scattering are the scattering amplitudes with one leg off-shell and the other three legs on-shell. The Haag amplitudes for higher processes are related to scattering and production amplitudes in a more complicated way. This formalism is as close to being completely on-shell as is possible in field theory. Among the good features of this approach is the fact that the Haag amplitudes have better ultraviolet behavior than the totally off-shell time-ordered amplitudes. This is because the latter contain phase-space integrals that grow rapidly for large numbers of particles in intermediate
states. The Haag amplitudes for the four-dimensional derivative coupling model, which is exactly solvable, illustrates this difference[8]. I emphasize that there is nothing unorthodox about the Haag expansion; what is surprising about the Haag expansion is that, up to now, it has not been developed into a powerful calculation method. That development is the goal of the present work.

3. NONRELATIVISTIC FIELD THEORY WITH BOUND STATES

3a. The model

Consider a model with two spinless nonrelativistic Fermi fields, $A(x)$ and $B(x)$, with the Hamiltonian

$$ H = m_A \int d^3x \, A^\dagger(x) A(x) + \frac{1}{2m_A} \int d^3x \, \nabla_x A^\dagger(x) \cdot \nabla_x A(x) + (B \text{ terms}) $$

$$ + \int_{x^0 = y^0} d^3x \, d^3y \, B^\dagger(y) A^\dagger(x) V_{AB}(|x - y|) A(x) B(y); \quad (11) $$

for simplicity I assumed an $AB$ interaction, but no $AA$ or $BB$ interaction. The equation of motion for $A(x)$ is

$$ i \partial_x \omega A(x) = (m_A - \frac{1}{2m_A} \nabla_x^2) A(x) + \int_{x^0 = y^0} d^3y \, B^\dagger(y) V_{AB}(|x - y|) B(y) A(x). \quad (12) $$

The asymptotic (in or out) fields for (possibly composite) particles are characterized by their rest energy $E$, kinetic mass $m$ and spin $J$. I suppress the spin in what follows. The kinetic mass is the mass that enters in the kinetic energy, $p^2/2m$; for composite particles, as discussed below, the kinetic mass is the sum of the kinetic masses of the constituents, without the binding energy, because of the Bargmann mass superselection rule described in Sec. 3b. The asymptotic fields obey the following free field equation and anticommutation or commutation relations:

$$ i \partial_{x^0} C^{in(out)}(x) = (E - \frac{1}{2m} \nabla^2) C^{in(out)}(x) \quad (13) $$

$$ [C^{in(out)}(x, t), C^{in(out)}(y, t')]_{\pm} = \mathcal{D}(x - y, t - t'; E, m), \quad (14) $$

$$ \mathcal{D}(x, t; E, m) = \frac{1}{(2\pi)^3} \int d^4k \delta(k^0 - E - \frac{k^2}{2m}) e^{-ik^0t + ik \cdot x}. \quad (15) $$
Note that
\[ D(x, 0; E, m) = \delta(x), \quad \forall E, m. \]  

(16)

Using translation invariance, one can show that the Haag expansion of the interacting field \( A(x) \) in terms of \( \text{in} \) fields takes the following form in position space (with an analogous expansion for the \( B \) field) [9]

\[
A(x_A) = A^{\text{in}}(x_A) + \sum_i \int d^3x_Bd^3x_i \tilde{f}_{B;i}(x_A - x_B; x_A - x_i) B^{\dagger \text{in}}(x_B)(ABi)^{\text{in}}(x_i) \\
+ \int d^3x_Bd^3y_Ad^3y_B \tilde{f}_{B;AB}(x_A - x_B; x_A - y_A; x_A - y_B) B^{\dagger \text{in}}(x_B)A^{\text{in}}(y_A)B^{\text{in}}(y_B) \\
+ \cdots, \tag{17}
\]

where, since both the asymptotic fields and the Haag amplitudes obey free equations, the integrals are independent of the times \( x_0^B, x_0^A, y_B^0, y_A^0 \) because of the translation invariance of the Schrödinger scalar products. Label the Haag amplitude that is the coefficient of a product of (asymptotic) creation and annihilation operators by the labels of the operators. Labels to the left of the semicolon are creation operators, to the right are annihilation operators; the two-body (\( AB \)) bound state in level \( i \) is labeled by \( i \).

Some calculations are simpler in momentum space, therefore define

\[
A(x) = \int d^4k e^{-ik \cdot x} \tilde{A}(k), \tag{18}
\]

\[
V_{AB}(|x - y|) = \frac{1}{(2\pi)^3} \int d^3q e^{i\mathbf{q} \cdot (x - y)} \tilde{V}_{AB}(\mathbf{q}). \tag{19}
\]

From now on, drop the tildes on \( \tilde{A} \) and \( \tilde{V} \). Transforming the equation of motion to momentum space yields

\[
(k_A^0 - m_A - \frac{k_A^2}{2m_A})A(k_A) = \int d^4k_Bd^4p_Bd^4p_A \delta(k_A + k_B - p_B - p_A) \\
\times B^{\dagger}(k_B)V_{AB}(p_B - k_B)B(p_B)A(p_A). \tag{20}
\]

Define

\[
C^{\text{in}}(x) = (2\pi)^{-3/2} \int d^4k \delta(k^0 - E_C - \frac{k^2}{2m_C})e^{-ik^0t + ik \cdot x} \text{e}^{\text{in}}(k), \tag{21}
\]
\[ [c^{in}(k), c^{i\ d}(l)]_+ = \delta(k - l) \]  

\[ f_{B;i}(x; y) = \frac{1}{(2\pi)^3} \int d^3k_1d^3k_2e^{i(m_B + \frac{k_i^2}{2m_B})y^0 - i\mathbf{k}_1 \cdot \mathbf{x} + i(m_{AB} - \epsilon_i + \frac{k_i^2}{2m_{AB}})y^0 + ik_2 \cdot y} \tilde{f}_{B;i}(k_1, k_2) \]  

and similar definitions for other Fourier transforms chosen so that powers of \(2\pi\) are absent from most of the momentum-space formulas. The result is

\[ A(k_A) = (2\pi)^{-3/2} : k_A : \delta(k_A^0 - m_A - \frac{k_A^2}{2m_A}) \]

\[ + \int d^3k_Bd^3\epsilon\delta(k_A^0 + \frac{k_B^2}{2m_B} - m_A + \epsilon_i - \frac{k_i^2}{2m_{AB}})\delta(k_A + k_B - k_i)\tilde{f}_{B;i}(k_B; k_i) : k_B^\dagger k_i : \]

\[ + \int d^3k_Bd^3p_Bd^3p_A\delta(k_A^0 + \frac{k_B^2}{2m_B} - m_A - \frac{p_A^2}{2m_A} - \frac{p_B^2}{2m_B})\delta(k_A + k_B - p_A - p_B) \]

\[ \times \tilde{f}_{B;AB}(k_B; p_A, p_B) : k_B^\dagger p_A p_B : + \cdots, \]  

where I define : \(p_A) \equiv a^{\dagger in}(p), \) with normalization as given in Eq.(22), etc. Note that I am expanding in terms of \(in\) fields; there are analogous expansions in terms of \(out\) fields. In the next section I derive the constraints on the \(f's\) that follow from Galilean invariance.

3b. Galilean Invariance

Bargmann[12] showed that the unitary projective representations (i.e., representations up to a factor) of the Galilean group that occur in the quantum mechanics of nonrelativistic particles cannot be reduced to vector (i.e., true) representations. This contrasts with the corresponding situation for the Poincaré and Lorentz groups (and indeed most other physically interesting groups), where the representations can be reduced to true representations. The explicit mass parameter in the phases leads to the Bargmann superselection rule that the sum of the masses (that appear in the kinetic terms) must be conserved in every process. Nonetheless, bound states can be formed and particles can be created and annihilated, provided the Bargmann superselection rule is obeyed.

Note that, for example, a bound state of particles of masses \(m_A\) and \(m_B\) with binding energy \(\epsilon\) has energy \(E = m_A + m_B - \epsilon + k^2/2m_{AB}\), rather than \(E = \)]
\[ m_A + m_B - \epsilon + \frac{k^2}{2(m_{AB} - \epsilon)} \]
as one might expect from the nonrelativistic limit of a relativistic bound state with rest energy \( m_A + m_B - \epsilon \). (I use the abbreviation \( m_{AB} = m_A + m_B \).) Another manifestation of this effect is that for this same bound state the momentum transforms under Galilean boosts as \( k \rightarrow k + m_{AB}v \), rather than as \( k \rightarrow k + (m_{AB} - \epsilon)v \).

If the projective representation has the form
\[
U(G_2)U(G_1) = \omega(G_2, G_1)U(G_2G_1)
\]
then another projective representation is equivalent to this if the other representation has the factor system \( \omega'(G_2, G_1) = [\phi(G_2)\phi(G_1)/\phi(G_2G_1)]\omega(G_2, G_1) \), where \( \phi \) has modulus one. This arbitrariness allows simplification of some formulas.

Bargmann gives as the Galilean transformation of a nonrelativistic scalar wave function,
\[
(T(G)\psi)(x) = e^{-i\theta(G,x)}\psi(G^{-1}x),
\]
where \( x = (x, t) \), the Galilean transformation is \( Gx = (Rx + vt + a, t + b) \), \( G = (a, b, R, v) \), where \( a \) and \( b \) are space and time translations, \( R \) is a rotation and \( v \) is a boost, and \( \theta(G, x) = m(\frac{1}{2}v^2t - v \cdot x) \). To infer the corresponding transformation for a nonrelativistic scalar field, I require
\[
U(G)A(\psi)U^\dagger(G) = A(\psi_G), \quad \psi_G(x) = (T(G)\psi)(x) = e^{-i\theta(G,x)}\psi(G^{-1}x),
\]
\[
A(\psi) = \int A(x)\psi(x)d^4x.
\]

Then
\[
U(G)A(x)U^\dagger(G) = e^{-i\theta_{A(G,Gx)}(G)}A(Gx), \quad \theta_{A(G,Gx)} = m_A[\frac{1}{2}v^2(t+b) - v \cdot (Rx + vt + a)].
\]

If the field has spin \( s \), then \( A \) on the left hand side is replaced by \( A_i \) and \( A \) on the right hand side is replaced by \( \sum_j A_jD_j^{(s)}(G) \), where \( D^{(s)} \) is a representation of \( SU(2) \), which is the little group in this case. The corresponding transformation holds for \( B \) with \( m_B \) replacing \( m_A \). Asymptotic fields transform the same way. The implications of the transformation law for the Haag amplitudes is found by transforming the interacting field in two ways: (1) insert the right-hand side of Eq.\((29)\) in the Haag expansion, or (2) transform each of the asymptotic fields and then change variables to get the transformation into the amplitudes. The two amplitudes \( f_{B;i} \) and \( f_{B;AB} \) obey
\[
f_{B;i}(G(x_A - x_B); G(x_A - x_i)) =
\]
A momentum space, to which I now turn. The transformation law is satisfied by having a delta function in the space and time coordinates identifying the coordinate \( x \) induced in parallel with the derivation of the position space law. The result is

\[
e^{i\theta_A(G,Gx_A) + i\theta_B(G,Gx_B) - i\theta_{AB}(G,Gx_i)} f_{B;i}(x_A - x_B; x_A - x_i),
\]

(30)

\[
f_{B;AB}(G(x_A - x_B); G(x_A - y_A), (G(x_A - y_B)) =
\]

\[
e^{i\theta_A(G,Gx_A) + i\theta_B(G,Gx_B) - i\theta_{AB}(G,Gx_i)} f_{B;AB}(x_A - x_B; x_A - y_A, x_A - y_B).
\]

(31)

Note that \( \theta_{AB} \) is independent of the bound state \( i \) because of the Bargmann mass superselection rule. The combination of phases in the first of these is

\[
\theta_A(G, Gx_A) + \theta_B(G, Gx_B) - \theta_{AB}(G, Gx_i) =
\]

\[
-\frac{1}{2} \mathbf{v}^2 (m_A x_A^0 + m_B x_B^0 - m_{AB} x_i^0) - \mathbf{v} \cdot \mathbf{R} (m_A x_A + m_B x_B - m_{AB} x_i).
\]

(32)

The transformation law is not satisfied by having a delta function in the space and time coordinates identifying the coordinate \( x_i \) with the center-of-mass of particles \( A \) and \( B \), although at equal times such a delta function does occur for the space coordinates. The way in which the transformation laws are satisfied is best seen in momentum space, to which I now turn.

The corresponding transformations in momentum space are

\[
(V(G)\phi)(k) = e^{-i\Omega(G,k)k} \phi(G^{-1}k),
\]

(33)

\[
\Omega(G, k) = (k - mv) \cdot a - (k^0 - \frac{1}{2}mv^2)b,
\]

(34)

where \( k = (k, E) \), \( Gk = (Rk + m\mathbf{v}, E + \mathbf{v} \cdot Rk + \frac{1}{2}mv^2) \), and \( G^{-1}k = (R^{-1}(k - mv), E - k \cdot \mathbf{v} + \frac{1}{2}mv^2) \). The momentum space transformation law for the field is induced in parallel with the derivation of the position space law. The result is

\[
W(G)A(k)W^\dagger(G) = e^{-i\Omega_A(G, -Gk)}A(Gk),
\]

(35)

where \( \Omega_A(G, -Gk) = (E + \mathbf{v} \cdot Rk)b - Rk \cdot a \). In the transformation law for the Haag amplitudes, all the phase factors cancel and the result for—say—the second term in the Haag expansion is what one would expect naively,

\[
\tilde{f}_{B;i}(k_B; k_i) = \tilde{f}_{B;i}(R(k_B - m_B\mathbf{v}); R(k_i - m_{AB}\mathbf{v})).
\]

(36)

Thus I can choose the \( \mathbf{v} = k_i/m_{AB} \) so that the bound-state momentum vanishes and eliminate the second argument of \( f_{B;i} \),

\[
\tilde{f}_{B;i}(k_B; k_i) = \tilde{f}_{B;i}(k_B - \frac{m_B}{m_{AB}}k_i, 0) \equiv \tilde{F}_{B;i}(k_B - \frac{m_B}{m_{AB}}k_i).
\]

(37)
For the spinless case, $\tilde{F}_{B;i}(k) = \tilde{F}_{B;i}(Rk)$. All these results are exact, valid in any Galilean frame. The extension to fields with spin is straightforward. It is worth noting that the Poincaré transformation law in a relativistic theory is simpler than the Galilean transformation law because the Bargmann phase is absent for the Poincaré group.

Taking account of Galilean invariance, one finds that the position-space Haag amplitude is

$$f_{B;i}(x; y) = (2\pi)^{-3} \int d^3k d^3k_i \exp[i(m_B + \frac{1}{2m_B}(\textbf{k} + \frac{m_B}{m_{AB}}\textbf{k}_i))^2x^0 - i(\textbf{k} + \frac{m_B}{m_{AB}}\textbf{k}_i) \cdot \textbf{x}]$$

$$\times \exp[-i(m_{AB} - \epsilon_i + \frac{k_i^2}{2m_{AB}})y^0 + i\textbf{k}_i \cdot \textbf{y}] \tilde{f}_{B;i}(\textbf{k}; 0).$$

The integral over $\textbf{k}_i$ can be done, but the result is complicated and not useful, except when all times are equal, in which case the result is both simple and useful,

$$f_{B;i}(\textbf{x}_A - \textbf{x}_B; \textbf{x}_A - \textbf{x}_i) = \delta(\textbf{x}_i - \frac{m_A\textbf{x}_A + m_B\textbf{x}_B}{m_{AB}})F_{B;i}(\textbf{x}_A - \textbf{x}_B),$$

$$F_{B;i}(\textbf{x}) = \int d^3ke^{-i\textbf{k} \cdot \textbf{x}}f_{B;i}(\textbf{k}; 0).$$

Using the constraints due to Galilean invariance, the Haag expansion in $x$-space at equal times takes the form

$$A(\textbf{x}_A) = A^{in}(\textbf{x}_A) + \sum_i \int F_{B;i}(\textbf{x}_A - \textbf{x}_B)B^{in}(\textbf{x}_B)(ABi)^{in}(\frac{m_A\textbf{x}_A + m_B\textbf{x}_B}{m_{AB}})d^3\textbf{x}_B$$

$$+ \int d^3r d^3r'F_{B;AB}(\textbf{r}; \textbf{r}')B^{in}(\textbf{x}_A - \textbf{r}')A^{in}(\textbf{x}_A + \frac{m_B(\textbf{r} - \textbf{r}')}{m_{AB}})B^{in}(\textbf{x}_A - \frac{m_A\textbf{r} + m_B\textbf{r}'}{m_{AB}})$$

$$+ \cdots.$$ (41)

In momentum space, the expansion is

$$A(k_A) = \frac{1}{(2\pi)^{3/2}} : k_A : \delta(k^0_A - m_A - \frac{k_A^2}{2m_A})$$

$$+ \int d^3k_B \delta(k^0_A + \frac{k_B^2}{2m_B} - m_A + \epsilon_i - (\frac{k_A + k_B}{2m_{AB}})^2)F_{B;i}(\frac{m_Ak_B - m_Bk_A}{m_{AB}}) : k^0_B(k_A + k_B)_i :$$

$$+ \int d^3k_B d^3p_B d^3p_A \delta(k^0_A + \frac{k_B^2}{2m_B} - m_A - \frac{p_A^2}{2m_A} - \frac{p_B^2}{2m_B})\delta(k_A + k_B - p_A - p_B)$$
\[ \times \tilde{F}_{B;AB}(\frac{m_A k_B - m_B k_A}{m_{AB}}, \frac{m_A p_B - m_B p_A}{m_{AB}}) : k_B^\dagger p_A p_B : + \cdots. \]  

(42)

### 3c. Two-Body Bound State

To derive the equation for the two-body bound state, insert the Haag expansion Eq.(17) in the equation of motion Eq.(12), renormal order, and equate the coefficients of the terms with the operators \( B^\dagger_n (AB)_n \). After commuting or anticommuting with the relevant \( in \) fields, the result is

\[ (i \frac{\partial}{\partial x_A^0} - m_A + \frac{1}{2m_A} \nabla^2_{x_A} - V(x_A - x_B)) f_{B;i}(x_A - x_B; x_A - x_i) = 0 \]  

(43)

It is convenient to eliminate the time derivative by using \( \partial / \partial x_A^0 = -\partial / \partial x_B^0 - \partial / \partial x_i^0 \), the independence of the Schrödinger scalar product on the time and the free equations satisfied by the \( in \) fields to find free equations for the \( x_B^0 \) and \( x_i^0 \) dependences of \( f_{B;i} \). The results are

\[ (i \frac{\partial}{\partial x_B^0} - m_B + \frac{1}{2m_B} \nabla^2_{x_B}) f_{B;i} = 0, \]  

(44)

\[ (i \frac{\partial}{\partial x_i^0} + m_{AB} - \epsilon_i - \frac{1}{2m_{AB}} \nabla^2_{x_i}) f_{B;i} = 0. \]  

(45)

The equation without time derivatives is

\[ [- \frac{1}{2m_A} \nabla^2_{x_A} - \frac{1}{2m_B} \nabla^2_{x_B} + V(x_A - x_B)] f_{B;i} = (\epsilon_i - \frac{1}{2m_{AB}} \nabla^2_{x_i}) f_{B;i}. \]  

(46)

Now using Eq.(39) the usual Schrödinger equation for \( F_{B;i} \) results,

\[ [- \frac{1}{2\mu} \nabla^2_{r_{AB}} + V(r_{AB})] F_{B;i} = -\epsilon_i F_{B;i}, \quad \frac{1}{\mu} = \frac{1}{m_A} + \frac{1}{m_B}. \]  

(47)

where the reduced mass enters. This establishes that \( F_{B;i} \) is the Schrödinger wave function of the bound state. Note that the bound-state amplitude is given exactly in any reference frame in terms of the amplitude in the rest frame of the bound state. (The corresponding statement also holds for other amplitudes, as well as for relativistic theories.)
3d. Two-Body Scattering

Two-body scattering is described in position space at equal times by the amplitude

\[
f_{B;AB}(x_A-x_B,0;x_A-y_A,0,x_B-y_B,0) = F_{B;AB}(x_A-x_B;y_A-y_B)\delta(R'-R),
\]

\[
F_{B;AB}(x;y) = (2\pi)^{-3/2} \int d^3k'd^3k \tilde{f}_{B;AB}(k'-k, k) \exp[i(-k' \cdot (x_A-x_B)+k \cdot (y_A-y_B))],
\]

\[
R' = \frac{m_Ax_A + m_Bx_B}{m_{AB}}, \quad R = \frac{m_Ay_A + m_By_B}{m_{AB}}.
\]

I prefer to discuss two-body scattering in momentum space, using the amplitude \( \tilde{f}_{B;AB}(k_B; p_A, p_B) \) which is the coefficient of the term : \( k_B^+ P_A p_B \) : in the Haag expansion of \( A(k) \). The procedure for finding the equation for \( \tilde{f}_{B;AB} \) is analogous to that for the two-body bound state amplitude. One finds

\[
\left( \frac{p_A^2 - (p_A + p_B - k_B)^2}{2m_A} + \frac{p_B^2 - k_B^2}{2m_B} \right) \tilde{f}_{B;AB}(k_B; p_A, p_B) = \frac{V_{AB}(|k_B - p_B|)}{m_{AB}} + \int d^3k' V_{AB}(|k_B - k_B'|) \tilde{f}_{B;AB}(k_B; p_A, p_B).
\]

Galilean invariance relates \( \tilde{f}_{B;AB} \) at arbitrary momenta to itself in the center-of-mass,

\[
\tilde{f}_{B;AB}(k_B; p_A, p_B) = \tilde{f}_{B;AB}(R(k_B - m_B v); R(p_A - m_A v), R(p_B - m_B v)).
\]

By choosing \( v = (p_A + p_B)/m_{AB} \), I can replace \( \tilde{f}_{B;AB} \) by a function of one fewer variable,

\[
\tilde{f}_{B;AB}(k_B; p_A, p_B) = \tilde{F}_{B;AB}(k; p),
\]

where here and below, \( k = (m_Ak_B - m_B k_A)/m_{AB} \), \( p = (m_A p_B - m_B p_A)/m_{AB} \) and I used conservation of momentum to introduce \( k_A \). The momenta \( p \) and \( k \) are the center-of-mass momenta of particle \( B \) in the initial and the final state, respectively. The elastic scattering equation becomes

\[
\frac{1}{2\mu} (p^2 - k^2) \tilde{F}_{B;AB}(k; p) = V(|k - p|) + \int d^3k' V(k - k') \tilde{F}_{B;AB}(k'; p),
\]

The solution is the Born series,

\[
\tilde{F}_{B;AB}(k; p) = \tilde{G}_R(k; p)V(|k - p|) + \tilde{G}_R(k; p) \int d^3k' V(|k - k'|) \tilde{G}_R(k'; p)V(|k' - p|) + \cdots,
\]
where \( \tilde{G}_R(k; p) = [(p^2 - k^2)/2\mu - i\epsilon]^{-1} \).

The amplitude \( \tilde{F}_{B;AB} \) is closely related to the \( T \)-matrix element for \( AB \) scattering. The \( S \)-matrix element is

\[
S(k_A, k_B; p_A, p_B) \equiv_{out} \langle k_B, k_A|p_A, p_B\rangle_{in} \equiv \langle 0 : k_B^{out} k_A^{out}\rangle_{out} : p_A^{\dagger}p_B^{\dagger} : 0, \tag{54}
\]

where I remind the reader that \( : k_A : \) etc., stands for the \( in \) field. In order to eliminate the \( out \) fields in terms of the \( in \) fields, use the definitions,

\[
A^{in(out)}(x) = \lim_{\tau \to -\infty(\infty)} \int d^3 y \mathcal{D}(x - y; m_A, m_A)A(y), \tag{55}
\]

where \( \mathcal{D} \) was defined in Eq.(53). The nonrelativistic analog of the reduction formula follows from calculating \( \int d^3 y \partial/\partial y \mathcal{D}(x - y; m_A, m_A)A(y) \) in two ways: performing the integral and carrying out the derivative. The result is

\[
A^{out}(x) - A^{in}(x) = \int d^3 y \mathcal{D}(x - y; m_A, m_A)(\partial_{\rho} + i m_A - \frac{i}{2m_A} \nabla^2_x)A(y). \tag{56}
\]

Fourier transforming this one gets, after removing a factor of \( \delta(k^\rho - m_A - k^2/2m_A) \),

\[
\frac{1}{(2\pi)^{3/2}}(a^{out}(k) - a^{in}(k)) = -2\pi i(k^\rho - m_A - \frac{k^2}{2m_A})A(k). \tag{57}
\]

The right-hand-side is non-vanishing (and there is scattering) only when \( A(k) \) has a pole at \( (k^\rho - m_A - k^2/2m_A) = 0 \). Since \( a^{out}(k)|0 = a^{in}(k)|0 \) for stable particles, the only \( out \) operator in the \( S \)-matrix element \( \langle 0 : k_B^{out} :: k_A^{out} :: p_A^{\dagger} :: p_B^{\dagger} : |0 \rangle \) that must be eliminated using Eq.(57) is \( : k_A^{out} : \). The result is

\[
S(k_A, k_B; p_A, p_B) = \delta(k_A - p_A)\delta(k_B - p_B) - 2\pi i\delta\left(\frac{k_A^2}{2m_A} + \frac{k_B^2}{2m_B} - \frac{p_A^2}{2m_A} - \frac{p_B^2}{2m_B}\right)
\]

\[
\times \delta(k_A + k_B - p_A - p_B)(\frac{k_A^2}{2m_A} + \frac{k_B^2}{2m_B} - \frac{p_A^2}{2m_A} - \frac{p_B^2}{2m_B})\tilde{F}_{B;AB}(k; p), \tag{58}
\]

where again \( k \) and \( p \) are defined below Eq.(52). Thus the reduced \( T \)-matrix for elastic scattering on the momentum shell is

\[
t'(k_A, k_B; p_A, p_B) = [\frac{p_A^2}{2m_A} + \frac{p_B^2}{2m_B} - \frac{k_A^2}{2m_A} - \frac{k_B^2}{2m_B}]\tilde{F}_{B;AB}(k; p). \tag{59}
\]
I emphasize that because the Haag amplitude is the scattering amplitude with one leg off shell, it contains the information necessary for calculations in the three-body sector. This contrasts with the on-shell scattering amplitude, which does not suffice for such calculations.

3e. Anticommutation Relations

In this section I show that the canonical (equal time) anticommutation relations of the Lagrangian fields imply general relations among Haag amplitudes, independent of the equations of motion of the specific theory. For example, the vanishing of the canonical anticommutator \([A,B]_+\) at equal times, considered for the coefficient of the bound state in field for state \(i\), gives

\[
F_{A:i}(y-x) = F_{B:i}(x-y) \equiv F_i(x-y)
\]  

(60)

where I took \((AB)i^{in}(R) = -(BA)i^{in}(R)\) because of the Fermi statistics of \(A\) and \(B\). Thus the apparent asymmetry in the treatment of the constituents of the bound state, due to the fact that the Haag amplitude that serves as the two-body wave function of the \((AB)\) bound state in the Haag expansion of the \(A\) field has the \(A\) particle off-shell and the \(B\) particle on-shell, while these roles are interchanged for the amplitude for the same bound state in the Haag expansion of the \(B\) field, is not a real asymmetry. These two amplitudes determine each other uniquely. The analogous result for the off-shell elastic scattering amplitudes is

\[
F_{B:AB}(x-y;r) = F_{A:BA}(y-x;-r) \equiv F_{AB}(x-y;r).
\]  

(61)

Again the two apparently different off-shell amplitudes uniquely determine each other.

The consequence for elastic scattering is

\[
t(k_A,k_B;p_A,p_B) - (t(p_A,p_B;k_A,k_B))^* =
\]

\[
(2\pi)^{5/2} \int d^3q_A d^3q_B \delta\left(\frac{k_A^2}{2m_A} + \frac{k_B^2}{2m_B} - \frac{q_A^2}{2m_A} - \frac{q_B^2}{2m_B}\right)\delta(k_A + k_B - q_A - q_B)
\]

\[
\times t(k_A,k_B;q_A,q_B)(t(p_A,p_B;q_A,q_B))^*,
\]

(62)

where \(k\) and \(p\) are as defined below Eq.(52). This is elastic unitarity.
The canonical anticommutator \([A, A^\dagger]_+\) at equal times leads to a generalization of unitarity,

\[
\frac{1}{(2\pi)^{3/2}}(\tilde{F}_{B;AB}(k; p) + \tilde{F}_{B;AB}^*(p; k))
\]

\[= \sum_i \tilde{F}_{B;i}(k)\tilde{F}_{B;i}^*(p) + \int d^3q \tilde{F}_{B;AB}(k; q)\tilde{F}_{B;AB}^*(p; q), \tag{63}\]

where again \(k\) and \(p\) are as defined below Eq.(52) and I have used momentum conservation, \(k_A + k_B = p_A + p_B\). By taking the appropriate limit, I recover the elastic unitarity relation, Eq.(62). On taking into account the relations between the Haag amplitudes in the expansions of \(A\) and of \(B\), one find that these are all the independent two-body relations obtained from the anticommutation relations.

There are also quadratic relations between the amplitudes for the \((ABi)\) and \((ABj)\) bound states and the amplitudes for the breakup of these bound states due to scattering with the \(A\) or \(B\) particle. Since this involves a higher sector, I do not give this relation here.

3f. Construction of the asymptotic field for the bound state

In this section I show how to construct the asymptotic field for the bound state from a product of Lagrangian fields. My suggestion differs from that proposed by Nishijima[10] and by Zimmermann[11]. The procedure is to multiply the appropriate Lagrangian fields at separated space points, integrate with the bound-state amplitude in the relative coordinate, and take the asymptotic limit. If the \(in\) field expansions of the Lagrangian fields are inserted and the resulting expression normal ordered, then the \(t \to -\infty\) limit gives the \(in\) field bound state operator and the \(t \to \infty\) limit gives a reduction formula for the \(out\) field bound state operator. The result is

\[
(ABi)^{in(out)}(x) = \lim_{\tau \to -\infty(\infty)} \int_{y^\rho = \tau} d^3y D(x - y; m_{AB} - \epsilon_i, m_{AB})F^*(w) \times \frac{1}{2}[B(y - \frac{m_A}{m_{AB}}w), A(y + \frac{m_B}{m_{AB}}w)]_w d^3w. \tag{64}\]

A straightforward calculation shows this limit is \((ABi)^{in}(x)\) for \(\tau \to -\infty\) and the leading term for \(\tau \to \infty\) is \((ABi)^{out}(x)\). Both results are what we expect.
later terms in the Haag expansion for \((ABi)^{\text{out}}(x)\) are in a higher sector that I don’t discuss here.

I derived many results of the nonrelativistic quantum mechanics of two-particle systems in a unified way with particular attention to Galilean invariance, taking into account the fact that the representations of the Galilean group in quantum mechanics are necessarily representations up to a factor, rather than vector representations. The Haag amplitude for the simplest term with the two-body bound-state operator is precisely the Schrödinger wave function of the two-body bound state. The amplitude for the term with three \(in\) fields is the scattering amplitude with one leg off-shell. These interpretations carry over to explicitly covariant relativistic theories, where the corresponding Haag amplitude is defined on three-dimensional manifolds, but is covariant. Of course in the relativistic case, a bound state that is mainly a two-body state also will have amplitudes to be composed of more than two particles.

4. The NAMBU–JONA-LASINIO MODEL

The Haag expansion is effective in treating the Nambu–Jona-Lasinio model in one-loop approximation. In particular, the Haag expansion sums crossed as well as direct graphs\,[13, 14], in contrast to the usual methods which sum only direct graphs. Further, using the Haag expansion one deals directly with the bound-state wavefunction (or amplitude); one does not have to extract the bound-state amplitude as the residue of a pole in the scattering amplitude.

The Lagrangian of the model without isospin is

\[
\mathcal{L} = i \bar{\psi} \gamma^\mu \psi - \frac{1}{2} g_0 [ (\bar{\psi} \gamma_\mu \psi) \gamma^\mu \psi - (\bar{\psi} \gamma_\mu \gamma_5 \psi) \gamma^\mu \gamma_5 \psi ].
\] (65)

To take account of operator ordering, symmetrize or antisymmetrize the operator products. After going to momentum space, the equation of motion is

\[
\hat{q} \psi(q) = -\frac{1}{2} g_0 \int d^4 p_1 d^4 p_2 d^4 p_3 \delta(q - p_1 - p_2 - p_3) \{ [[[\bar{\psi}(p_1), \psi(p_2)]_- \psi(p_3)]_+ - [[[\bar{\psi}(p_1), \gamma_5 \psi(p_2)]_- \gamma_5 \psi(p_3)]_+] \} 
\] (66)

The lowest approximation, to take \(\psi(p) = \psi^{\text{in}}(p) \delta(p^2 - m^2)\), leads to the gap equation,

\[
\mu = \frac{4g_0 \mu}{(2\pi)^3} \int d^4 p \delta(p^2 - \mu^2) .
\] (68)

Surely, this is a simple derivation of this result! Since this model is nonrenormalizable, one must cut off the integral. This can be done covariantly, if desired. For
0 < \pi^2/g_0 \Lambda^2 < 1$, where $\Lambda$ is the cutoff, there are three solutions: $\mu = 0$ and $\mu = \pm m$. The first is the unbroken symmetry solution; the last two are equivalent broken symmetry solutions. To decide which solution is the stable one, calculate the matrix element of the Hamiltonian in the corresponding vacuum state. The result is that the broken symmetry solutions have lower vacuum matrix elements and are thus the correct solutions.

To find the bound states, consider the term in the Haag expansion with the product $: \psi^{in} B^{in} :$, where $B^{in}$ is the bound state in field. The coefficient of this term serves as the bound-state wavefunction; indeed as shown above in the nonrelativistic case it is precisely the Schrödinger wavefunction. Inserting the two terms of the Haag expansion that have been introduced into the equation of motion and renormalizing and keeping the coefficients of the bound-state term leads to a linear integral equation for the bound-state wavefunction. Because the interaction is a contact interaction, this equation can be solved exactly. For the $J^{PC} = 0^{-+}$ state, the analog of the pion, the mass is zero, as expected from the Nambu-Goldstone theorem. For other states, the results agree generally with previous calculations, except in some cases the limits on the masses differ, perhaps because the present calculation includes crossed graphs.

5. FINITE TEMPERATURE FIELD THEORY

5a. Sketch of thermo field theory using the Haag expansion

I illustrate the application of the Haag expansion to the solution of second quantized field theories at finite temperature using the BCS model of superconductivity. In order to have a state annihilated by the annihilation operators, which is necessary for normal ordering, I use the thermo field theory formalism of Umezawa and collaborators\[15,16\]. This formalism uses a doubled set of operators to account for finite temperature. The annihilation and creation operators are subjected to two Bogoliubov transformations: one comes from the dynamics of the electron pair interaction; the other from the thermo field formalism which takes account of the finite temperature. In lowest approximation, the method leads to the usual gap equation. The asymptotic fields whose annihilation parts annihilate the vacuum at zero temperature no longer annihilate the state which is a thermal mixture at finite temperature $T$. Indeed, no set of annihilation operators annihilates the mixed state
at finite $T$. In order to obtain a state which is annihilated by the annihilation parts of a suitable set of asymptotic fields, the Hilbert space of states must be enlarged to include hole states in the thermal equilibrium state at a given temperature and the set of operators must include operators which annihilate and create holes in addition to the operators which annihilate and create particles. Thermo field theory does this, for example for a Hilbert space with a discrete energy eigenstate basis $\{|n\rangle\}$, by replacing the Hilbert space of the system under consideration with the tensor product Hilbert space with basis $\{|n\rangle \otimes |\tilde{n}\rangle\}$. Correspondingly the set of operators with respect to which the vacuum $|0\rangle \otimes |\tilde{0}\rangle$ is cyclic is doubled and includes the particle $\{a_k \otimes 1, a_k^\dagger \otimes 1\}$ and hole $\{1 \otimes \tilde{a}_k, 1 \otimes \tilde{a}_k^\dagger\}$ operators. (To simplify the notation, I will drop the $\otimes$ factors.) A Bogoliubov transformation relates the original annihilation and creation operators for the particles together with the “tilde” operators which describe the holes to another doubled set of operators whose annihilation parts annihilate the pure state (called the “thermal vacuum”) in the enlarged Hilbert space which represents the thermal mixture in the usual theory.

Let the density operator be

$$\rho = Z(\beta)^{-1} e^{-\beta H} = Z(\beta)^{-1} \sum e^{-\beta E_n} |n\rangle \langle n|,$$  \hspace{1cm} (69)

$$Z(\beta) = tr e^{-\beta H} = \sum e^{-\beta E_n}. \hspace{1cm} (70)$$

Here $H$ can either be the Hamiltonian $H$ or $H - \mu N$, where $\mu$ is the chemical potential and $N$ is the number operator. Now consider an enlarged Hilbert space in which the tensor product basis $\{|n\rangle \otimes |\tilde{n}\rangle\}$ replaces the basis $\{|n\rangle\}$ of the original Hilbert space. Let the thermal vacuum be

$$|O(\beta)\rangle = Z(\beta)^{-1/2} \sum e^{-\beta E_n/2} |n\rangle \otimes |\tilde{n}\rangle. \hspace{1cm} (71)$$

Let the doubled set of operators, $c, c^\dagger, \tilde{c}$ and $\tilde{c}^\dagger$ be operators for the normal modes of the total system. The thermal vacuum at finite temperature is the state which satisfies

$$c_{ki}|O(\beta)\rangle = 0, \hspace{0.5cm} \tilde{c}_{ki}|O(\beta)\rangle = 0. \hspace{1cm} (72)$$

The existence of a state $|O(\beta)\rangle$ which is annihilated by the annihilation operators is essential to defining a normal-ordered operator product. The average of an observable $A$ in the thermal mixture at inverse temperature $\beta$ is given by the matrix element of the corresponding operator $A \otimes \tilde{1}$ in the thermal vacuum $|O(\beta)\rangle$,

$$tr(\rho A) = \langle O(\beta) | A \otimes \tilde{1} | O(\beta) \rangle. \hspace{1cm} (73)$$
For the Fermi case of interest for the electrons in superconductivity, the Bogoliubov transformation between the operators, $b$ and $b^\dagger$, for the normal modes of the electrons in the system and the operators, $\tilde{b}$ and $\tilde{b}^\dagger$, for the normal modes of the holes on the one hand and the doubled set of operators, $c$, $c^\dagger$, $\tilde{c}$ and $\tilde{c}^\dagger$ which are the normal modes of the total system on the other hand is

\[
\begin{pmatrix}
    b_{k1}^\dagger \\
    \tilde{b}_{k1}
\end{pmatrix}
= \begin{pmatrix}
    \sqrt{1-n_k} & \sqrt{n_k} \\
    -\sqrt{n_k} & \sqrt{1-n_k}
\end{pmatrix}
\begin{pmatrix}
    c_{k1}^\dagger \\
    \tilde{c}_{k1}
\end{pmatrix},
\]

(74)

\[
\begin{pmatrix}
    b_{k2} \\
    \tilde{b}_{k2}^\dagger
\end{pmatrix}
= \begin{pmatrix}
    \sqrt{1-n_k} & -\sqrt{n_k} \\
    \sqrt{n_k} & \sqrt{1-n_k}
\end{pmatrix}
\begin{pmatrix}
    c_{k2} \\
    \tilde{c}_{k2}^\dagger
\end{pmatrix}
\]

(75)

The requirement that

\[
\langle O(\beta)|b_{k1}^\dagger b_{k1}|O(\beta)\rangle = \langle O(\beta)|b_{k2}^\dagger b_{k2}|O(\beta)\rangle = n_k = 1/(e^{\beta E_k} + 1)
\]

(76)

fixes the coefficients in the Bogoliubov transformation. As indicated in Eq. (76), I assume that $n_k$ is independent of spin polarization.

The electron-electron interaction which leads to superconductivity leads to another Bogoliubov transformation which has the form

\[
\begin{pmatrix}
    a_{k\uparrow}^\dagger \\
    a_{k\downarrow}
\end{pmatrix}
= \begin{pmatrix}
    u_k & v_k \\
    -v_k^* & u_k^*
\end{pmatrix}
\begin{pmatrix}
    b_{k1}^\dagger \\
    b_{k2}
\end{pmatrix}
\]

(77)

and

\[
\begin{pmatrix}
    \tilde{a}_{k\uparrow}^\dagger \\
    \tilde{a}_{k\downarrow}^\dagger
\end{pmatrix}
= \begin{pmatrix}
    u_k & v_k \\
    -v_k^* & u_k^*
\end{pmatrix}
\begin{pmatrix}
    \tilde{b}_{k1}^\dagger \\
    \tilde{b}_{k2}
\end{pmatrix}
\]

(78)

The solution of the operator equations of motion for the $a$ and $\tilde{a}$ operators determine the coefficients $u$ and $v$ in Eq. (77); the operators $b$ and $\tilde{b}$ are then expressed in terms of the $c$ and $\tilde{c}$ set. The final result gives the $a$ and $\tilde{a}$ operators in terms of the
The tilde Hamiltonian, \( \tilde{H} \), of motion is the symbol \( \tilde{H} \) for the new “total Hamiltonian” and the symbol \( H \) for the new “total Hamiltonian.”

The time dependence of any operator is \( \mathcal{O}(t) = e^{i \tilde{H}t} \mathcal{O} e^{-i \tilde{H}t} \). Thus the equation of motion is

\[
- i \partial_t \mathcal{O}(t) = [\tilde{H}, \mathcal{O}(t)].
\] (83)

The equations of motion for creation and annihilation operators are

\[
- i \partial_t a_{k\uparrow}^\dagger = \hat{\epsilon}_k a_{k\uparrow}^\dagger + (1/2) \Sigma_{\ell} [a_{\ell\uparrow}^\dagger, a_{-\ell\downarrow}] V_{k\ell} a_{-k\downarrow},
\] (84)

\[
- i \partial_t a_{-k\downarrow} = -\hat{\epsilon}_k a_{-k\downarrow} + (1/2) \Sigma_{\ell} a_{k\uparrow}^\dagger V_{k\ell} [a_{-\ell\downarrow}, a_{\ell\dagger}],
\] (85)

\[
- i \partial_t \bar{a}_{k\uparrow} = \hat{\epsilon}_k \bar{a}_{k\uparrow} + (1/2) \Sigma_{\ell} \bar{a}_{-k\downarrow}^\dagger V_{k\ell} [\bar{a}_{-\ell\downarrow}, \bar{a}_{\ell\dagger}],
\] (86)
\[-i\partial_t \hat{a}^\dagger_{-k\downarrow} = -\hat{e}_k \hat{a}^\dagger_{-k\downarrow} + (1/2) \Sigma_\ell [\hat{a}^\dagger_{\ell\uparrow}, \hat{a}^\dagger_{-\ell\downarrow}] V_{\ell k} \hat{a}_{k\uparrow}. \tag{87}\]

Use a Haag expansion in the thermal in (or out) fields to get an approximate solution of the operator equations of motion. In the lowest approximation, use Eq. (79), insert it in the equations of motion, re-normal order and keep only the linear terms in the \(c\) operators to get equations for the unknown coefficients \(u_k\) and \(v_k\). The result has the form

\[
\sum_j (F_k + \sum_l G_{kl})_{ij} C_j = 0, \tag{88}\]

where \(C_j = (c^\dagger_{k1}, c_{k2}, \tilde{c}^\dagger_{k1}, \tilde{c}_{k2})\) (as a column vector). Since the \(c\) and \(\tilde{c}\) operators are linearly dependent, each of the 16 equations,

\[
(F_k + \sum_l G_{kl})_{ij} = 0, \quad i, j = 1 \text{ to } 4, \tag{89}\]

must hold. Only two of these equations,

\[
(E_k - \hat{e}_k - V_{kk} w_k) u_k + \Sigma_\ell V_{k\ell} y_\ell v_k = 0, \tag{90}\]

\[
(E_k + \hat{e}_k + V_{kk} w_k) v_k + \Sigma_\ell V_{k\ell} y_\ell u_k = 0, \tag{91}\]

where

\[
w_k \equiv v^2_k (1 - n_k) + u^2_k n_k, \quad y_\ell \equiv u_\ell v_\ell (1 - 2 n_\ell), \tag{92}\]

are linearly independent. It is convenient to write these equations in matrix form and to make the equations appear simpler by introducing new symbols

\[
\epsilon_k \equiv \hat{e}_k + V_{kk} w_k, \quad \Delta_k \equiv \sum_\ell V_{k\ell} y_\ell. \tag{93}\]

Then

\[
\begin{pmatrix} E_k - \epsilon_k & \Delta_k \\ \Delta_k & E_k + \epsilon_k \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = 0. \tag{94}\]

There is a solution only if the determinant of the matrix vanishes, i.e., if

\[
E_k^2 = \epsilon_k^2 + \Delta_k^2. \tag{95}\]

The solution, which is implicit because \(\epsilon\) and \(\Delta\) depend on \(u_k\) and \(v_k\) via Eq. (92) and (93), for \(u_k\) and \(v_k\) is

\[
u_k = \sqrt{(E_k + \epsilon_k)/2 E_k}, \quad v_k = -\Delta_k/\sqrt{2 E_k(E_k + \epsilon_k)}. \tag{96}\]
It is easy to check that this solution satisfies the constraint $u_k^2 + v_k^2 = 1$. When Eq. (96) is inserted in the definition of $\Delta$, the celebrated gap equation results,

$$\Delta_k = \sum_l V_{kl} (\Delta_l/2E_l)(1 - 2n_l).$$  \hspace{1cm} (97)$$

With the choices,

$$V_{kl} = \begin{cases}  -V_0, & |\epsilon_k| \leq \Delta \epsilon, \quad p = k \text{ or } l \\ 0, & \text{otherwise}, \end{cases} \hspace{1cm} (98)$$

$$\Delta_k = \begin{cases}  \Delta(T), & |\epsilon_k| \leq \Delta \epsilon, \\ 0, & \text{otherwise}, \end{cases} \hspace{1cm} (99)$$

using Eq. (76) and making the usual replacement of the sum by an integral, the gap equation becomes

$$V_0 N(0) \int_0^{\Delta \epsilon} d\epsilon_l \tanh(\beta E_l/2)/E_l = 1,$$  \hspace{1cm} (100)$$

where $E_k = \sqrt{\epsilon_k^2 + \Delta^2(T)}$. This is the usual gap equation for this model and the usual results for $\Delta(0)/kT_c = 1.764$, $N(0)V_0 \ln(1.13\Delta \epsilon/kT_c) = 1$, etc., where $T_c$ is the transition temperature, follow.

6. THE SCHWINGER MODEL

The Schwinger model\cite{18, 19} is massless two-dimensional quantum electrodynamics, an exactly soluble model. In the Lorentz gauge\cite{4}

$$\mathcal{L} = \bar{\psi} i D^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial \cdot A)^2,$$  \hspace{1cm} (101)$$

where

$$D^\mu = \partial^\mu - eA^\mu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$  \hspace{1cm} (102)$$

Lowenstein and Swieca\cite{20} found an operator ansatz that yields the matrix elements computed by Schwinger. Their ansatz solution has the following form:

$$A^\mu(x) = -\sqrt{\frac{\pi}{e^2}} (\epsilon^{\mu\nu} \partial_\nu \Sigma + \partial^\mu \eta),$$  \hspace{1cm} (103)$$

$^2$This form of the Lagrangian ignores wavefunction renormalization of the spinor field. Taking account of this wavefunction renormalization, the spinor term in the Lagrangian is $\bar{\psi}_u i D^\mu \psi_u - \langle \bar{\psi}_u i D^\mu \psi_u \rangle_0 = Z \bar{\psi}_u i D^\mu \psi$, where $\psi_u$ is the unrenormalized spinor field and $Z$ is the spinor wavefunction renormalization. We suppress the factor $Z$ below.
\[ \psi(x) = \exp[i\sqrt{\pi}(\gamma_5 \Sigma(x) - \eta(x))] :\psi^{(0)}(x), \]  

(104)

where \( \eta \) is a free neutral massless field with negative metric corresponding to the gauge degrees of freedom satisfying \([\eta(x), \dot{\eta}(y)]_{ET} = -i\delta(x^1 - y^1)\), while \( \Sigma \) is a free neutral massive field with positive metric representing the physical degrees of freedom satisfying \([\Sigma(x), \dot{\Sigma}(y)]_{ET} = i\delta(x^1 - y^1)\) and \( \psi^{(0)} \) is a solution of the free massless Dirac equation. (ET stands for equal time.) This solution displays the main property of the Schwinger model: the only physical state is a free particle of mass \( e/\sqrt{\pi} \), but the solution does not obey the canonical commutation relations for \( A_\mu \). The Haag expansion provides a solution that does obey these relations\(^{[22]}\).

Use the Lagrangian Eq.(101), but drop surface terms so that the gauge field part becomes

\[ -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu). \]  

(105)

Since the Lorentz group (without inversions) is abelian in 1+1, all irreducible representations are one-dimensional; thus the vector and spinor fields as to in the model are composed of one-dimensional irreducibles arbitrarily pasted together. Express the Lagrangian in terms of the irreducible fields in the basis in which

\[ A^0 = \frac{1}{2}(A^+ + A^-), \quad A^1 = \frac{1}{2}(A^+ - A^-), \quad \psi = (\psi_1, \psi_2), \]  

(106)

with

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(107)

In terms of the irreducible fields,

\[ L = \psi_1^\dagger(2i\partial^+ - eA^-)\psi_1 + \psi_2^\dagger(2i\partial^- - eA^+)\psi_2 + \frac{1}{2}(\partial^1 A^+ \partial^1 A^- - \partial^0 A^+ \partial^0 A^-). \]  

(108)

using lightcone coordinates, \( x^+ = x^0 + x^1, \quad x^- = x^0 - x^1 \). The corresponding derivatives are \( \frac{\partial}{\partial x^\pm} = \frac{1}{2}(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}) \). Define these so that \( \frac{\partial}{\partial x^\pm} x^\pm = 1 \). Note that although the fields \( A^\pm \) are lightcone fields, I am not using lightcone quantization, but rather am using equal-time canonical quantization. The naive operator equations of motion are

\[ \Box A^+ - 2e\psi_1^\dagger \psi_1 = 0, \]  

(109)

\[ \Box A^- - 2e\psi_2^\dagger \psi_2 = 0, \]  

(110)

\[ (2i\partial^+ - eA^-)\psi_1 = 0, \]  

(111)
\[(2i\partial^- - eA^+)\psi_2 = 0. \quad (112)\]

As Schwinger pointed out in his original paper, the spinor bilinear products require a line integral of the “vector” potential between the \(\psi^\dagger\) and the \(\psi\) in order to ensure gauge invariance; this is done explicitly below using point-splitting. For example, \(\psi_1^\dagger \psi_1\) is replaced by

\[
\lim_{\epsilon \to 0} \frac{1}{2} \left[ \psi^\dagger_1(x + \epsilon)e^{-ie\int_{x^-}^{x^+} A^\mu(w)dw^\mu} \psi_1(x) + cc. \right]. \quad (113)
\]

The canonical momenta are

\[
\pi_{A^+} = -\frac{1}{2} \partial^0 A^- , \quad \pi_{A^-} = -\frac{1}{2} \partial^0 A^+ , \quad \pi_{\psi_j} = i\psi^\dagger_j. \quad (114)
\]

Solve the Dirac equations by exponentiation,

\[
\psi_1(x) = \mathcal{P} \exp\left[ -\frac{ie}{2} \int_{-\infty}^{x^+} A^-(w^+, x^-)dw^+ \right] \psi_1^{(0)}(x^-) , \quad \partial^+ \psi_1^{(0)} = 0, \quad (115)
\]

\[
\psi_2(x) = \mathcal{P} \exp\left[ -\frac{ie}{2} \int_{-\infty}^{x^-} A^+(x^+, w^-)dw^+ \right] \psi_2^{(0)}(x^+) , \quad \partial^- \psi_2^{(0)} = 0. \quad (116)
\]

The point-splitting vector is taken spacelike, \(\epsilon = (0, \epsilon^1), \quad \epsilon^\pm = \pm \epsilon^1\). Thus, for example, \(\psi_1^\dagger\) must be replaced by

\[
\psi_1^\dagger(x + \epsilon) = \psi_1^{0 \dagger}(x^- - \epsilon^1) \mathcal{P} \exp\left[ \frac{ie}{2} \int_{-\infty}^{x^+ + \epsilon^1} A^-(w^+, x^- - \epsilon^1)dw^+ \right]. \quad (117)
\]

The symbols \(\mathcal{P}\) and \(\mathcal{P}\) stand for path and antipath ordering, respectively. The result of the point-splitting differs from the usual one by having integrated (nonlocal) terms. The equations for \(A^\pm\) become

\[
(\Box + \frac{\epsilon^2}{2\pi}) A^+ - \frac{\epsilon^2}{2\pi} \int_{-\infty}^{x^+} \partial A^- (w^+, x^-)dw^+ = 2e\psi_1^{(0)\dagger}(x^-)\psi_1^{(0)}(x^-) \quad (118)
\]

\[
(\Box + \frac{\epsilon^2}{2\pi}) A^- - \frac{\epsilon^2}{2\pi} \int_{-\infty}^{x^-} \partial A^+ (x^+, w^-)dw^- = 2e\psi_2^{(0)\dagger}(x^+)\psi_2^{(0)}(x^+). \quad (119)
\]

The integrated terms here can be removed by taking derivatives with respect to the upper limit. Combining the resulting equations leads to

\[
\Box \partial \cdot A = 0 \quad (120)
\]
\[ (\Box + \frac{e^2}{\pi}) \epsilon_{\mu\nu} \partial^\mu A^\nu = 0; \] (121)

thus \( \partial \cdot A \equiv \eta \) is a massless field and \( \epsilon_{\mu\nu} \partial^\mu A^\nu \equiv \Sigma \) is a field of mass \( e/\sqrt{\pi} \).

The fields \( \eta \) and \( \Sigma \) (which is the electric field in 1+1 dimensions) are the gauge-variant and gauge-invariant degrees of freedom, respectively. Then \( \Box A^\mu \) must be a linear combination of \( \partial^\mu \eta \) and \( \epsilon^{\mu\nu} \partial_\nu \Sigma \). \( \Box A^\mu \) must be the convolution of the \( \tilde{\Delta}(x) = -\frac{1}{2} \epsilon(x^0) \Delta(x) \) Green’s function with this linear combination plus terms annihilated by \( \Box \). The convolution of \( \tilde{\Delta}(x) \) with \( \eta \) does not exist, because, formally, it is \( \int d^2 y \tilde{\Delta}(x-y) \eta(y) = \int d^2 k \exp(-ik \cdot x) \delta(k^2) \tilde{\eta}(k)/k^2 \), which is ill-defined. Because of this, a new field \( a \) that obeys \( \Box a = \eta \) must be introduced. This was first done by Capri and Ferrari[23]. Thus

\[ A^\mu = c_1 \partial^\mu a + c_2 \epsilon^{\mu\nu} \partial_\nu \Sigma + c_3 \partial^\mu \eta + c_4 \bar{\psi}^{(0)} \gamma^\mu \psi^{(0)}. \] (122)

For the massless case,

\[ \bar{\psi}^{(0)} \gamma^\mu \psi^{(0)} = \partial^\mu \phi, \quad \Box \phi = 0, \] (123)

where \( \phi \) is a free positive-metric scalar field. Using

\[ [\eta, \dot{\eta}]_{ET} = \tau i \delta, \quad [\eta, \dot{a}]_{ET} = c_{\eta a} i \delta, \quad [a, \dot{\eta}]_{ET} = c_{aa} i \delta, \quad [a, \dot{a}]_{ET} = c_{aa} i \delta. \] (124)

with the choices \( c_{\eta a} = c_{aa} = 0 \) and

\[ \tau = -1, \quad c_1 = \mp \sqrt{\frac{e^2}{5\pi}}, \quad c_2 = \pm \sqrt{\frac{\pi}{e^2}}, \quad c_3 = \pm \sqrt{\frac{5\pi}{e^2}}. \] (125)

yields a solution in which the vector potential obeys the canonical commutation relations.

The canonical commutation relations in field theory and their predecessors in classical mechanics and quantum mechanics are important for many reasons. The Poisson brackets in classical mechanics, for example, ensure that the Hamiltonian is the generator of time translations. In quantum mechanics, for example, the relation \( [x, p] = i\hbar \) leads to the uncertainty relation. In quantum field theory, the CCR’s lead to the free field being a collection of quantized oscillators. In nonrelativistic field theories at least, the CCR’s imply unitarity[9]. A new feature of the canonical commutation relations in quantum field theory is that they ensure that the asymptotic fields have the proper free commutation relation. (The renormalized canonical commutation relations will do as well as the original CCR’s for this purpose.) For these reasons, a solution that obeys the canonical commutation relations is important.
7. SUMMARY AND OUTLOOK FOR FUTURE WORK

The N quantum approach using the Haag expansion has met the test of non-gauge theories, including bound states and both spontaneous and dynamical symmetry breaking. Previous work by Amit Raychaudhuri\cite{24} and work presently in progress in collaboration with Eli Hawkins\cite{25} and with Rashmi Ray and Felix Schlumpf\cite{26} in both non-gauge theories and in non-confining gauge theories give promise of using the N quantum approach to treat bound states in a way that has advantages over the Bethe-Salpeter equation. The treatment of confined degrees of freedom in quantum chromodynamics remains a goal for the future.

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