A group arrival retrial $G$ - queue with multi optional stages of service, orbital search and server breakdown

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Abstract: A group arrival feedback retrial queue with $k$ optional stages of service and orbital search policy is studied. Any arriving group of customer finds the server free, one from the group enters into the first stage of service and the rest of the group join into the orbit. After completion of the $i^{th}$ stage of service, the customer under service may have the option to choose $(i+1)^{th}$ stage of service with $\theta_i$ probability, with $p_i$ probability may join into orbit as feedback customer or may leave the system with $q_i = \left\{ \begin{array}{ll} 1-p_i-\theta_i, & i = 1,2,...k-1 \\ 1-p_i, & i = k \end{array} \right.$ probability. Busy server may get to breakdown due to the arrival of negative customers and the service channel will fail for a short interval of time. At the completion of service or repair, the server searches for the customer in the orbit (if any) with probability $\alpha$ or remains idle with probability $1-\alpha$. By using the supplementary variable method, steady state probability generating function for system size, some system performance measures are discussed.

1. Introduction
Nowadays we retry to get the service such as telephone calls, ATMs, good output from the candidates and searching data etc., Retrial is very common in queues. Artalejo[1], Artalejo and Ret al. [2] analyzed, the retrial policy in queueing systems. Sometimes the service station may get down suddenly due to malware attack. This malware is treated as negative customers. These customers disturb the server as well as the customer in service and it does not require any service. The server will be ready to give the service, after the repair process. Negative customers in queueing system are called G-queue. Recently, Wang J et al. [17] and Zhang M et al. [18] discussed the concept of G queue.

After service or repair completion, the idle server may search for the customer in the orbit to provide service. Gao S et al. [10] analyzed the retrial G queue with this concept. The service has many stages in nature. Here a server gives the multi optional stages of service. Authors like, Bagyam JE A et al. [3], Choudhury etal. [4],[6] and Radhaet al. [15] are surveyed the multi stage and two phase service in queueing system. This work has the application in the communication network.
2. Model description

2.1 Arrival process
Units arriving the system in batches with Poisson arrival rate $\lambda$. Let $X_k$, the number of units in the $k^{th}$ batch, where $k = 1, 2, 3, \ldots$ with common distribution $\Pr[X_k = n] = \chi_n$, $n = 1, 2, 3, \ldots$ The PGF (probability generating function) of $X$ is $\chi(z)$. The first and second moments are $E(X)$ and $E(X(X-1))$.

2.2 Retrial process
If there is no space to wait, one from the arriving unit begins service (if the server is free) and rest are waiting in the orbit. If an arriving batch finds the server either busy or on vacation or breakdown, then the batch joins into an orbit. Here Inter-retrial times form an arbitrary distribution $R(s)$ with corresponding Laplace-Stieltjes transform (LST) $R^*(s)$.

2.3 Service process
Here a server gives $k$ stages of service. The First Stage Service (FSS) is followed by $i$ stages of service. The service time $S_i$ for $i = 1, 2, \ldots, k$ has a distribution (general) function $S_i(s)$ having LST $S_i^*(s)$ and first and second moments are $E(S_i)$ and $E(S_i^2)$, $(i = 1, 2, \ldots, k)$.

2.4 Feedback rule
After completion of $i^{th}$ stage of service the customer may go to $(i+1)^{th}$ stage with probability $\theta_i$ or may join into the orbit as feedback customer with probability $p_i$ or leaves the system with probability $1-i(\theta_1 + p_1)$ for $i = 1, 2, \ldots, k-1$. From this model, the service time or the time required by the customer to complete the service cycle is a random variable $S$ is given by $S = \sum_{i=1}^{k} \Theta_{i-1}S_i$ having the LST $S^*(s) = \prod_{i=1}^{k} \Theta_{i-1}S_i^*(s)$ and the expected value is $E(S) = \sum_{i=1}^{k} \Theta_{i-1}E(S_i)$, where $\Theta_i = \theta_1\theta_2\ldots\theta_i$ and $\Theta_0 = 1$.

2.5 Repair process
The negative customers enters the station with Poisson arrival rate $\delta$. The repair time $G_i$ has the distributions function $G_i(y)$ and LST $G_i^*(s)$ for $(i = 1, 2, \ldots, k)$.

2.6 An orbital search
An idle server enters the orbit to search the customer with probability $\alpha$ or remains idle with probability $1-\alpha$. Here, the search time is negligible.

Various stochastic processes involved in the system are assumed to be mutually exclusive. In the steady state, let $R(0) = 0, R(x) = 1, S_i(0) = 0, S_i(x) = 1, i = 1, 2, \ldots, k$ are continuous at $x = 0$ and $G_i(0) = 0, G_i(x) = 1$ are continuous at $y = 0, (1 \leq i \leq k)$. Let $R^0(t), S_i^0(t)$ and $G_i^0(t)$ be the elapsed times for retrial, service on $i^{th}$ stage and repair on $i^{th}$ stage, $(1 \leq i \leq k)$ respectively. Now, a random variable at time $t$,

$$C(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy on } i^{th} \text{ stage,} \\ 2, & \text{if the server is repair on } i^{th} \text{ stage,} \end{cases}$$

The Markov process \{C(t), N(t); t \geq 0\} describes the system state, where \(C(t)\) - the server state and \(N(t)\) - the number in orbit at time \(t\), the functions \(a(x), \mu_i(x)\) and \(\xi_i(y)\) are the conditional completion rates for retrial, service, vacation, delay in repair and repair respectively \((1 \leq i \leq k)\).

\[
a(x)dx = \frac{dR(x)}{1-R(x)} \cdot \mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)} \quad \text{and} \quad \xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)}.
\]

Then define \(B_i^* = S_1^*S_2^*...S_i^*\) and \(B_0^* = 1\). The first moment \(M_{1i}\) and second moment \(M_{2i}\) of \(B_i^*\) are given by

\[
M_{1i} = \lim_{z \to 1} dB_i^*[A_i(z)]/dz = \sum_{j=1}^{k} \lambda E(X)E(S_j)
\]

\[
M_{2i} = \lim_{z \to 1} d^2 B_i^*[A_i(z)]/dz^2 = \sum_{j=1}^{k} \left( \left( \lambda E(X) \right)^2 E(S_j^2) - \left( \lambda \right)^2 E(X)E(X(X-1))E(S_j) \right)
\]

where

\[A_i(z) = \delta + b(z)\] and \(b(z) = \lambda (1-X(z))\)

Let \(\{t_n; n = 1,2,\ldots\}\) be the service period ending time or repair period ending time. In this system, \(Z = \{C(t_n), N(t_n)\}\) forms an embedded Markov chain which is ergodic \(\Leftrightarrow \rho < 1\), where

\[
\rho = E(X)\left(1-R'(\lambda)\right) + \lambda E(X)\left(\frac{1-S_i^*(\delta)}{\delta}\right)(1+\delta g^{(0)}) + \sum_{i=1}^{k} B_i,
\]

\[
\sum_{i=1}^{k} \Theta_{i-1}M_{1i} + \sum_{i=1}^{k} p_i \Theta_{i-1} - \sum_{i=1}^{k} \Theta_{i-1}M_{1i}
\]

and

\[
B_i = \sum_{i=1}^{k} \Theta_{i-1} \left[ M_{1i-1}(1-S_i^*(\delta)) - M_{1i}B_i^*(\delta) \right]
\]

3. Steady state distribution

For \(\{N(t), t \geq 0\}\), define the probabilities functions at time \(t\),

- \(P_0(t)\) - \(Pr\) (the system is empty),

At time \(t\) and \(n\) customers in the orbit,

- \(P_n(x,t)\) - \(Pr\) (an elapsed retrial time \(x\) of the retrial customers),
- \(\Pi_{ij}(x,t), (1 \leq i \leq k)\) - \(Pr\) (elapsed service time \(x\) on \(i^{th}\) stage of the customer under service),
- \(R_{ij}(x,y,t), (1 \leq i \leq k)\) - \(Pr\) (an elapsed times for service is \(x\) and repair is \(y\) on \(i^{th}\) stage),

The stability condition exists for \(t \geq 0, x \geq 0, y \geq 0, n \geq 0\) for \(i = 1,2,\ldots k\).
\[ P_0 = \lim_{t \to \infty} P_0(t), \quad P_n(x) = \lim_{t \to \infty} P_n(x,t), \quad \Pi_{i,n}(x) = \lim_{t \to \infty} \Pi_{i,n}(x,t), \quad R_{i,n}(x,y) = \lim_{t \to \infty} R_{i,n}(x,y,t), \quad \text{for } t \geq 0. \]

a. **Steady state equations**

By the supplementary variable technique (Kelison and Servi. [11]) the following governing equations are obtained for \((i=1,2,...,k)\).

\[ \lambda P_n = \sum_{i=1}^{k} q_i \int_{0}^{\infty} \mu_i(x) \Pi_{i,0}(x,t) dx + \int_{0}^{\infty} \zeta_i(x) R_{i,0}(x) dx \quad (1) \]

\[ \frac{dP_n(x)}{dx} + P_n(x)[\lambda + a(x)] = 0, \quad n \geq 1 \quad (2) \]

\[ \frac{d\Pi_{i,0}(x)}{dx} + \Pi_{i,0}(x)[\lambda + \delta + \mu_i(x)] = 0, \quad n \geq 1 \quad (3) \]

\[ \frac{d\Pi_{i,n}(x)}{dx} + \lambda \sum_{k=1}^{n} \Pi_{i,n-k}(x) = 0, \quad n = 1,2,... \quad (4) \]

\[ \frac{dR_{i,0}(x)}{dx} + R_{i,0}(x)[\lambda + \zeta_i(x)] = 0, \quad n = 0 \quad (5) \]

\[ \frac{dR_{i,n}(x)}{dx} + R_{i,n}(x)[\lambda + \zeta_i(x)] - \lambda \sum_{k=1}^{n} \Pi_{i,n-k}(x) = 0, \quad n = 1,2,... \quad (6) \]

The steady state boundary conditions at \(x = 0\) and \(y = 0\) are

\[ P_n(0) = \sum_{i=1}^{k} q_i \int_{0}^{\infty} \mu_i(x) \Pi_{i,0}(x) dx + \sum_{i=1}^{k} p_i \int_{0}^{\infty} \mu_i(x) \Pi_{i,n-1}(x) dx + \int_{0}^{\infty} \zeta_i(x) R_{i,n}(x) dx, \quad n \geq 1 \quad (7) \]

\[ \Pi_{i,0}(0) = \lambda \chi_i P_0 + \int_{0}^{\infty} a(x) \Pi_{i,1}(x) dx, \quad n = 0 \quad (8) \]

\[ \Pi_{i,n}(0) = \left\{ \begin{array}{l}
\int_{0}^{\infty} a(x) P_n(x) dx + \lambda \sum_{k=1}^{n} \Pi_{i,n-k+1}(x) dx + \beta \lambda P_{n+1} P_0 \\
+ \alpha \end{array} \right\}, \quad n \geq 1 \quad (9) \]

\[ \Pi_{i,n}(0) = \theta \int_{0}^{\infty} \mu_i(x) \Pi_{i-1,n}(x) dx, \quad n = 1,2,... (2 \leq i \leq k) \quad (10) \]

\[ R_{i,n}(x,0) = \delta \int_{0}^{\infty} \Pi_{i,n}(x), \quad n \geq 0, \quad \text{for } (1 \leq i \leq k) \quad (11) \]

The normalizing condition is
\( P_0 + \sum_{n=1}^{\infty} \int P_n(x)dx + \sum_{n=0}^{\infty} \left( \sum_{i=1}^{k} \int \Pi_{i,n}(x)dx + \int R_{i,n}(x)dx \right) = 1 \) \hspace{1cm} (12)

### b. Steady state solutions

To solve the above equations, generating functions defined for \(|z| \leq 1\),

\[
P(x,z) = \sum_{n=1}^{\infty} P_n(x)z^n; \quad P(0,z) = \sum_{n=1}^{\infty} P_n(0)z^n; \quad \Pi_i(x,z) = \sum_{n=0}^{\infty} \Pi_{i,n}(x)z^n; \quad \Pi_i(0,z) = \sum_{n=0}^{\infty} \Pi_{i,n}(0)z^n;
\]

\[
R_i(x,z) = \sum_{n=0}^{\infty} R_{i,n}(x)z^n; \quad R_i(0,z) = \sum_{n=0}^{\infty} R_{i,n}(0)z^n
\]

Now multiplying Eqns. (2) - (11) by \( z^n \) and summing over \( n \), \( n = 0,1,2... \) and \( 1 \leq i \leq k \)

\[
\frac{dP(x,z)}{dx} = -P(x,z)[\lambda + a(x)] \tag{13}
\]

\[
\frac{d\Pi_i(x,z)}{dx} = -\Pi_i(x,z)[\lambda(1-X(z)) + \delta + \mu_i(x)] \tag{14}
\]

\[
\frac{dR_i(x,z)}{dx} = -R_i(x,z)[\lambda(1-X(z)) + \xi_i(y)] \tag{15}
\]

At \( x = 0 \) and \( y = 0 \),

\[
P(0,z) = (1-a) \sum_{i=1}^{k} \left( p_i z + q_i \right) \int_{0}^{\infty} \mu_i(x)\Pi_i(x,z)dx \left( -\lambda P_0 + \int_{0}^{\infty} \xi_0(x) R_i(x,z)dx \right)
\tag{16}
\]

\[
\Pi_i(0,z) = \left\{ \frac{1}{z} \int_{0}^{\infty} a(x)P(x,z)dx + \frac{P_0 X(z)}{z} - \lambda P_0 + \lambda \frac{X(z)}{z} \int_{0}^{\infty} P(x,z)dx \right\}
\tag{17}
\]

\[
+ \frac{\alpha}{z} \left( \sum_{i=1}^{k} \left( p_i z + q_i \right) \int_{0}^{\infty} \Pi_i(x,z)\mu_i(x)dx + \int_{0}^{\infty} R_i(x,z)\xi_i(x)dx \right), \quad n \geq 1.
\]

\[
\Pi_i(0,0) = \theta_i \int_{0}^{\infty} \Pi_{i-1}(0,z)dx, \quad (i = 2,3,...k). \tag{18}
\]

\[
R_i(0,z) = \delta \int_{0}^{\infty} \Pi_i(x,z)dx \tag{19}
\]

Solving Eqns. (13)-(19),

\[
P(x,z) = P(0,z)e^{-\lambda x}[1-R(x)] \tag{20}
\]

\[
\Pi_i(x,z) = \Pi_i(0,z)e^{-A_i(x)z}[1-S_i(x)] \tag{21}
\]

\[
R_i(x,z) = R_i(0,z)e^{-b_i(z)x}[1-G_i(x)] \tag{22}
\]

where

\[
A_i(z) = \delta + b(z) \quad \text{and} \quad b(z) = \lambda \left(1-X(z)\right)
\]
Solving the above system of equations, we obtain the following.

\[
\Pi_i(0, z) = \left( \frac{P(0, z)}{z} \right)^{R^*(\lambda)} + \frac{\lambda X(z)}{z} P_0 - \sum_{k=1}^{i-1} \left( \frac{P(0, z)}{z} \right)^{R^*(\lambda)} + \frac{\lambda X(z)}{z} P_0 \left( \frac{\alpha}{z} \right) \prod_{k=1}^{i-1} \left( \frac{P(0, z)}{z} \right)^{R^*(\lambda)} P_i(0, z) S^*_i(A_i(z)) + R_i(0, z) G^*_i(b(z)) - \lambda P_0 \right) \]

(23)

\[
\Pi_i(0, z) = \Theta_{i-1} \Pi_i(0, z) \left( B_i^* \left[ A_{i-1}(z) \right] \right), \quad (i = 2, 3, \ldots, k) \]

(24)

\[
R_i(0, z) = \delta \Theta_{i-1} \Pi_i(0, z) \left( B_i^* \left[ A_{i-1}(z) \right] \right) \frac{1 - S^*_i(A_i(z))}{A_i(z)} \]

(25)

\[
P(0, z) = \frac{\lambda P_0 (1-\alpha) \left( X(z) - \alpha \right) - (1-\alpha) \left( R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right) \right)}{zA_i(z) - \left( \frac{\alpha}{z} \right) (R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right))} \delta G^*_i \left( b(z) \right) \Theta_{i-1} \left( B_i^* \left[ A_{i-1}(z) \right] \right) \left( 1 - S^*_i(A_i(z)) \right) \]

(26)

Using (26) in (23), we get,

\[
\Pi_i(0, z) = \left( \frac{\lambda P_0 (1-\alpha) \left( X(z) - \alpha \right)}{zA_i(z) - \left( \frac{\alpha}{z} \right) (R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right))} \delta G^*_i \left( b(z) \right) \Theta_{i-1} \left( B_i^* \left[ A_{i-1}(z) \right] \right) \left( 1 - S^*_i(A_i(z)) \right) \right) \]

(27)

\[
\Pi_i(0, z) = \left( \frac{\lambda P_0 A_i(z) \left( X(z) - \alpha \right)}{zA_i(z) - \left( \frac{\alpha}{z} \right) (R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right))} \delta G^*_i \left( b(z) \right) \Theta_{i-1} \left( B_i^* \left[ A_{i-1}(z) \right] \right) \left( 1 - S^*_i(A_i(z)) \right) \right) \]

(28)

Similarly,

\[
R_i(0, z) = \left( \frac{\delta \lambda P_0 A_i(z) \left( B_i^* \left[ A_{i-1}(z) \right] \right) \left( 1 - S^*_i(A_i(z)) \right)}{zA_i(z) - \left( \frac{\alpha}{z} \right) (R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right))} \delta G^*_i \left( b(z) \right) \Theta_{i-1} \left( B_i^* \left[ A_{i-1}(z) \right] \right) \left( 1 - S^*_i(A_i(z)) \right) \right) \]

(29)

Using Eqns. (20) to (22) and Eqns. (26) to (29), the obtained \( P(x, z), \Pi_i(x, z) \) and \( R_i(x, y, z) \) are given, under \( \rho < 1 \),

\[
P(x, z) = \left( \frac{\lambda P_0 (1-\alpha) \left( X(z) - \alpha \right)}{zA_i(z) - \left( \frac{\alpha}{z} \right) (R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right))} \delta G^*_i \left( b(z) \right) \Theta_{i-1} \left( B_i^* \left[ A_{i-1}(z) \right] \right) \left( 1 - S^*_i(A_i(z)) \right) \right) \]

(30)

\[
\Pi_i(x, z) = \left( \frac{\lambda P_0 (1-\alpha) \left( X(z) - \alpha \right) - (1-\alpha) \left( R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right) \right)}{zA_i(z) - \left( \frac{\alpha}{z} \right) (R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right))} \delta G^*_i \left( b(z) \right) \Theta_{i-1} \left( B_i^* \left[ A_{i-1}(z) \right] \right) \left( 1 - S^*_i(A_i(z)) \right) \right) \]

(31)
\[ R_{i}(x, y, z) = \left( 1 - S_{i}(x) \right) e^{-A(z)} \left[ 1 - G_{i}(y) \right] e^{-B(z)} \]

\[
\begin{align*}
\delta P_{i} \Theta_{i+1}(B^{*}_{i+1}(A_{i+1}(z))) \left[ 1 - S_{i+1}(A_{i+1}(z)) \right] \left[ (X(z) - \alpha) \right] - (1 - \alpha) \left[ R'(\lambda) + X(z) \left[ 1 - R'(\lambda) \right] \right] \\
\left( zA_{i}(z) - \alpha + (1 - \alpha) \left[ R'(\lambda) + X(z) \left[ 1 - R'(\lambda) \right] \right] \right) \delta G_{i}^{*}[b(z)] \Theta_{i+1} \left( B_{i+1}^{*}[A_{i+1}(z)] \right) \left[ 1 - S_{i+1}(A_{i+1}(z)) \right]
\end{align*}
\] (32)

**Theorem 3.1.** Under $\rho < 1$, the stationary distributions of the numbers in the system when server being idle, busy during $i$th stage and repair on $i$th stage (for $1 \leq i \leq k$) are given by

\[
P(z) = \frac{P_{0} \left( 1 - (1 - \alpha) \right) E(X) \left( 1 - R'(\lambda) \right) - \lambda E(X) \left( \frac{1 - S_{i}(\delta)}{\delta} \right) (1 + \delta g^{(1)}) + \Sigma_{i} + B_{i}}{zA_{i}(z) - \alpha + (1 - \alpha) \left[ R'(\lambda) + X(z) \left[ 1 - R'(\lambda) \right] \right] \delta G_{i}[b(z)] \Theta_{i+1} \left[ B_{i+1}^{*}[A_{i+1}(z)] \right] \left[ 1 - S_{i+1}(A_{i+1}(z)) \right]}
\] (33)

\[
\Pi_{i}(z) = \frac{1 - S_{i}(A_{i}(z)) \left( 1 - G_{i}(b(z)) \right)}{zA_{i}(z) - \alpha + (1 - \alpha) \left[ R'(\lambda) + X(z) \left[ 1 - R'(\lambda) \right] \right] \delta G_{i}[b(z)] \Theta_{i+1} \left[ B_{i+1}^{*}[A_{i+1}(z)] \right] \left[ 1 - S_{i+1}(A_{i+1}(z)) \right]}
\] (34)

where, $P_{0} = \frac{1}{\beta} \left[ 1 - (1 - \alpha) E(X) \left( 1 - R'(\lambda) \right) - \lambda E(X) \left( \frac{1 - S_{i}(\delta)}{\delta} \right) (1 + \delta g^{(1)}) + \Sigma_{i} + B_{i} \right]$

\[
\beta = \frac{\left\{ 1 - (1 - \alpha) E(X) \left[ 1 - R'(\lambda) \right] - E(X) R'(\lambda) \left[ 1 - (1 - \alpha) \right] \left[ 1 - R'(\lambda) \right] (E(X) R'(\lambda) + E(X)) \right\} + \left\{ E(X) - (1 - \alpha) E(X) \left[ 1 - R'(\lambda) \right] \right\} \lambda \left( \frac{1 - S_{i}(\delta)}{\delta} \right) (1 + \delta g^{(1)})}{\delta}
\]

Proof. Integrating (49) to (53) with respect to $x$ and $y$ (whenever needed), defined the following for ($1 \leq i \leq k$) $P(z) = \int_{0}^{\infty} P(x, z) dx$, $\Pi_{i}(z) = \int_{0}^{\infty} \Pi_{i}(x, z) dx$, $R_{i}(x, z) = \int_{0}^{\infty} R_{i}(x, y, z) dy$, $R_{i}(z) = \int_{0}^{\infty} R_{i}(x, z) dx$. Since, $P_{0}$ can be determined using (12). $P_{0} + P(1) + \sum_{i=1}^{k} (\Pi_{i}(1) + R_{i}(1)) = 1$ is obtained by setting $z = 1$ in (33) to (35).

**Theorem 3.2** Under $\rho < 1$, probability generating function of the system size and orbit size distribution at stationary point of time is

\[
K(z) = \frac{N r(z)}{D r(z)}
\] (37)
\[
\left\{ \begin{aligned}
\mathcal{N}(z) &= P_0 \\
&+ \left[ X(z) - \alpha \left( 1 - R^*(\lambda) \right) \right] \sum_{i=1}^{k} \frac{b(z)\Theta_{i-1}\left(B_{i-1}^{*}(A_{i-1}(z))\right)}{1 - G_i^*\left(b(z)\right)} \left( 1 - S_i^*\left(A_i(z)\right) \right)
\end{aligned} \right.
\]

\[
\mathcal{D}(z) = [1 - X(z)] \left( A_i(z) - \alpha \left( 1 - R^*(\lambda) \right) \right) \sum_{i=1}^{k} \frac{b(z)\Theta_{i-1}\left(B_{i-1}^{*}(A_{i-1}(z))\right)}{1 - G_i^*\left(b(z)\right)} \left( 1 - S_i^*\left(A_i(z)\right) \right)
\]

\[
H(z) = \frac{\mathcal{N}(z)}{\mathcal{D}(z)}
\]

where \(P_0\) is given in Eq. (36).

Proof. The statement is obtained by using

\[
K(z) = P_0 + P(z) + z \sum_{i=1}^{k} \left( \Pi_i(z) + R_i(z) \right)
\]

and

\[
H(z) = P_0 + P(z) + \sum_{i=1}^{k} \left( \Pi_i(z) + R_i(z) \right).
\]

4. Performance measures

**Theorem 4.1** If \(\rho < 1\) is satisfied, then the following probabilities of the server state, that is the server is idle during the retrial, busy during \(i^{th}\) stage, frequency of customer loss and under repair on \(i^{th}\) stage respectively are obtained.

\[
P = P_0 \left( 1 - \alpha \right) \left( 1 - R^*(\lambda) \right) \frac{E(X) \left\{ \frac{1 - S^*_i(\delta)}{\delta} \left( 1 + \delta R^{(k)} \right) + 1 \right\} + \sum_{i=1}^{k} B_i - 1}{1 - \rho}
\]

\[
\Pi_i = \sum_{i=1}^{k} \frac{\lambda P_0 \Theta_{i-1}\left(B_{i-1}^{*}(A_{i-1}(z))\right)\left( 1 - S_i^*\left(A_i(z)\right) \right) E(X) \left( 1 - \alpha \right) \left( 1 - R^*(\lambda) \right)}{\delta (1 - \rho)}
\]
\[ F_{\text{total}} = \delta \Pi_i = \sum_{i=1}^{k} \frac{\lambda P \Theta_i \left( B_i^* \left[ A_i \left( z \right) \right] \right) \left( 1 - S_i^* \left( A_i \left( z \right) \right) \right) E(X) \left( 1 - (1 - \alpha)(1 - R^* \left( \lambda \right)) \right)}{(1 - \rho)} \]

\[ R_i = \sum_{i=1}^{k} \frac{\lambda P g^{(i)} \Theta_i \left( B_i^* \left[ A_i \left( z \right) \right] \right) \left( 1 - S_i^* \left( A_i \left( z \right) \right) \right) E(X) \left( 1 - (1 - \alpha)(1 - R^* \left( \lambda \right)) \right)}{(1 - \rho)} \]

Proof. The stated formula follows by using

\[ P = \lim_{z \to 1} P(z), \quad \sum_{i=1}^{k} \Pi_i = \lim_{z \to 1} \sum_{i=1}^{k} \Pi_i \left( z \right) \quad \text{and} \quad \sum_{i=1}^{k} R_i = \lim_{z \to 1} \sum_{i=1}^{k} R_i \left( z \right). \]

**Theorem 4.2.** Let \( L_a, L_q, W_i \) and \( W_q \) be the average system size, average orbit size, average waiting time in the system and in the orbit respectively, then under \( \rho < 1 \),

\[ L_q = \frac{P_0}{V^{*}(\lambda)} \left[ \frac{N_{r}^{*} \left( 1 \right) D_{r}^{*}(1) - D_{r}^{*}(1) N_{r}^{*}(1)}{3 \left( D_{r}^{*}(1) \right)^2} \right] \]

where

\[ N_{r}^{*}(1) = -2\delta \left[ \frac{E(X)}{\delta} \left( 1 - (1 - \alpha)(1 - R^* \left( \lambda \right)) \right) \left( \delta - \lambda - c^* \right) + \lambda \left( E(X) \right)^2 \left( 1 - (1 - \alpha)(1 - R^* \left( \lambda \right)) \right) \sum_{i=1}^{k} \frac{1 - S_i^* \left( \delta \right)}{\delta} \left( 1 + \delta g^{(i)} \right) \right] \]

\[ E(X) \left( 1 - \sum_{i=1}^{k} B_i + \lambda \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) \right) \]

\[ + \lambda \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) + \lambda \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) \]

\[ N_{r}^{*}(1) = -3\delta \left[ \left( 1 - (1 - \alpha)(1 - R^* \left( \lambda \right)) \right) \sum_{i=1}^{k} \left( 1 - S_i^* \left( \delta \right) \right) + \lambda \delta \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) \right] \]

\[ - \lambda \left( E(X) \right)^2 \left( 1 - (1 - \alpha)(1 - R^* \left( \lambda \right)) \right) \sum_{i=1}^{k} \left( 1 - S_i^* \left( \delta \right) \right) + \lambda \delta \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) \]

\[ D_{r}^{*}(1) = -2\delta E(X) \left( 1 - \rho \right) \]

\[ D_{r}^{*}(1) = 3\delta \left[ \left( 1 - \alpha \right) E(X) \left( 1 - R^* \left( \lambda \right) \right) \right] \sum_{i=1}^{k} \left( 1 - S_i^* \left( \delta \right) \right) + \lambda \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) \]

\[ + \lambda \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) + \lambda \left( E(X) \right)^2 \left( 1 - S_i^* \left( \delta \right) \right) \]

\[ k_i = \left( - \lambda E(X) \right) g^{(2)} \left( 1 - S_i^* \left( \delta \right) \right) + \left( \frac{-2 + \sum_{i=1}^{k} E(X) \left( 1 - \delta \right) - 2 g^{(i)} B_i}{\delta} \right) - E(X) \left( \delta + B_i \right) \]

\[ c^* = \delta \left( \lambda E(X) \right) \left( 1 - S_i^* \left( \delta \right) \right) g^{(0)} + \sum_{i=1}^{k} B_i \]
\[ B_2 = \left( \sum_{i=1}^{k} \Theta_{i-1}(-M_{2i-1}\lambda E(X)(1-S^*_i(\delta)) - M_{2i-1}\lambda E(X(X-1))(1-S^*_i(\delta))) \right. \\
+2(\lambda E(X))^2 E(S^*_i)M_{2i} - (\lambda E(X))^2 E(S^*_1)B^*_i(\delta) - \lambda E(X(X-1))E(S^*_1) \]

and

\[ \Sigma_2 = \sum_{i=1}^{k} \Theta_{i-1}M_{2i} + 2\sum_{i=1}^{k} p_i \Theta_{i-1}M_{2i} - \sum_{i=1}^{k-1} \Theta_i M_{2i} \]

\[ L_q = \frac{P_0}{V^*(\lambda)} \left( \frac{N_r^{*\alpha}(1)Dr^*_q(1) - Dr^*_q(1)N_r^*(1)}{3(Dr^*_q(1))^2} \right) \]

where

\[ N_r^*(1) = N_r^{*\alpha}(1) - 6\sum_{i=1}^{k} \Theta_{i-1}\lambda E(X)^2 \left(1-(1-\alpha)(1-R^*(\lambda))\sum_{i=1}^{k}(1-S^*_i(\delta))(1+\delta g(1)) \right) \]

\[ W_s = \frac{L_q}{\lambda E(X)} \quad \text{and} \quad W_q = \frac{L_q}{\lambda E(X)} \]

Proof. The statement is obtained by using

\[ L_q = \frac{N_r(z)}{Dr(z)} = \lim_{\zeta \to 1} \frac{H(z)}{dz} = H'(1) = \frac{P_0}{V^*(\lambda)} \left( \frac{N_r^{*\alpha}(1)Dr^*_q(1) - Dr^*_q(1)N_r^*(1)}{3(Dr^*_q(1))^2} \right) \]

and

\[ L_q = \frac{N_r(z)}{Dr(z)} = \lim_{\zeta \to 1} \frac{K(z)}{dz} = K'(1) = \frac{P_0}{V^*(\lambda)} \left( \frac{N_r^{*\alpha}(1)Dr^*_q(1) - Dr^*_q(1)N_r^*(1)}{3(Dr^*_q(1))^2} \right) \]

\( W_s \) and \( W_q \) under steady-state condition due to Little’s formula is, \( L_q = \lambda W_s \) and \( L_q = \lambda W_q \).

4.1 Special cases

Case (i): Single phase, No retrial, No balking and reneging and No breakdown

Let \( Pr\{X = 1\} = 1 \), \( R^*(\lambda) \to 1 \), \( Pr\{V = 0\} = 1 \), \( b = 1 \), \( r = 1 \) and \( \alpha_1 = \alpha_2 = 0 \). Our model can be reduced to Multi stage M/G/1 feedback queueing system. The following results agree with Salehirad et al. [16].

\[ K(z) = P_0 \left\{ \left(1-S^*_1(A(z))\right) + \sum_{i=2}^{k} \Theta_{i-1} \left( B^*_i \left[ A_{i-1}(z) \right] \right) \left(1-S^*_i(A(z))\right) \right\} \]

\[ z - \sum_{i=1}^{k} \left( p_i z + q_i \right) \Theta_{i-1} \left( B^*_i \left[ A_{i-1}(z) \right] \right) \]

Case (ii): Single phase, No feedback, No balking & reneging and No breakdown

Let \( Pr\{X = 1\} = 1 \), \( k = 1 \), \( Pr\{S_i = 0\} = 1 \), \( \theta_i = 0 \), \( b = 1 \), \( r = 1 \) and \( \alpha_1 = \alpha_2 = 0 \), the model reduced to M/G/1 retrial queue.
\[ K(z) = \left\{ \begin{array}{c} \frac{R'(-\lambda) - \lambda E(S_0)}{z - [R'(-\lambda)]} S'_0 \left[ \lambda - \lambda z \left[ z - 1 \right] \right] \right\} ; \quad L_q = \frac{\lambda^2 E(S_0^2) + 2\lambda E(S_0) \left( 1 - R'(-\lambda) \right)}{2 \left[ R'(-\lambda) - \lambda E(S_0) \right]} \]

The above results agree with Gomez-Corral [11].

Case (iii): Single phase, No balkig & reneging and No breakdown
Let \( Pr \{X = 1\} = b, a = 1, r = 1 \) and \( \alpha_i = \alpha = 0 \). The model reduced to Multi stage retrial queueing system with Bernoulli feedback. The following results agree with Bagyam et al.[3].

\[ K(z) = Pr R'(-\lambda) \left\{ \sum_{i=1}^{k} \left[ p_i z + \delta_i \right] \Theta_{i-1} \left( B_i(z) \right) \left[ 1 - S_i A_i(z) \right] - 1 \right\} \left[ R'(-\lambda) + X(z) \left( 1 - R'(-\lambda) \right) \right] \sum_{i=1}^{k} \left[ p_i z + \delta_i \right] \Theta_{i-1} \left( B_i(z) \right) \]

where

\[ P_0 = \frac{R'(-\lambda) - \sum_{i=1}^{k} \Theta_{i-1} M_i - \sum_{i=1}^{k} p_i \Theta_{i-1} + \sum_{i=1}^{k} \Theta_{i-1} M_i}{R'(-\lambda) \left[ 1 - \sum_{i=1}^{k} \Theta_{i-1} \lambda E(X) E(S_i) \left( 1 - \alpha_i E(G_i) \right) - \sum_{i=1}^{k} \Theta_{i-1} \left( p_i + M_i \right) + \sum_{i=1}^{k} \Theta_{i-1} M_i \right]} \]

5. Numerical illustration
Here, some numerical examples are given using MATLAB. The retrial times, service times and repair times are exponentially \( f(x) = ve^{-x}, x > 0 \) for Erlang-2 stage \( f(x) = v^2xe^{-x}, x > 0 \) and hyper-exponentially \( f(x) = cve^{-x} + (1-c)v^2e^{-x}, x > 0 \) distributed. And assume the arbitrary values to the parameters satisfies \( x \geq 0 \). The following tables indicate the computed values of \( P_0, P, P_i, F_{Loss} \) and \( R_i \) for \( i = 1, 2, ..., k \) respectively. For the effect of \( a, p, \gamma \) and \( \xi_i \) are retrial rate, feedback probability, customer loss and repair rate on FSS respectively graphs are given in Figure 1 to 6.

Table 1 indicates when \( a \) increases, then \( P_0 \) increases, \( R_q \) and \( P_0 \) decreases for \( \lambda = 0.2; p_i = 0.2; \mu_i = 5; \alpha_i = 0.2; \xi_i = 3; c = 0.8; k = 1 \). Table 2 indicates when \( (p_{i1}) \) increases, \( P_0 \) decreases, \( R_q \) and \( P_0 \) increases for \( \lambda = 2; a = 5; \mu_1 = 10; \mu_2 = 8; \theta_1 = 0.2; \theta_2 = 0.4; \alpha_i = 0.2; \xi_i = 5; c = 0.8; k = 2 \). Table 3 indicates when \( \delta \) increases, then the probability that server is idle \( P_0 \), \( L_q \) and \( F_\gamma \) increases for \( \lambda = 0.5; a = 5; \mu_i = 10; p_i = 0.4; \alpha_i = 0.1; \xi_i = 5; c = 0.8; k = 1 \).

Figure 1-6 are given for \( a, p_i, \gamma \) and \( \xi_i \). Figure 1 indicates \( P_0 \) increases for increasing \( a \). Figure 2 indicates \( L_q \) increasing for increasing \( p_i \). Figure 3 indicates \( L_q \) decreases for increasing \( p_i \). Figure 4 indicates \( P_0 \) increases for increasing \( \xi_i \). Three dimensional Figure 5 indicates \( p_i \) and \( \xi_i \) increases against \( L_q \) increases. \( P_0 \) increasing against increasing \( a \) and \( \gamma \) is shown in Figure 6.
### Table 1. The effect of \( a \) on \( P_0, L_q \) and \( P \)

| \( \alpha \) | \( P_0 \) | \( L_q \) | \( P \) | \( P_0 \) | \( L_q \) | \( P \) | \( P_0 \) | \( L_q \) | \( P \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.20 | 0.9109 | 0.0064 | 0.0011 | 0.8272 | 0.0148 | 0.0055 | 0.9053 | 0.0132 | 0.0033 |
| 0.30 | 0.9200 | 0.0063 | 0.0010 | 0.8446 | 0.0145 | 0.0048 | 0.9132 | 0.0129 | 0.0029 |
| 0.40 | 0.9291 | 0.0062 | 0.0009 | 0.8620 | 0.0141 | 0.0042 | 0.9211 | 0.0127 | 0.0025 |
| 0.50 | 0.9381 | 0.0062 | 0.0007 | 0.8793 | 0.0137 | 0.0035 | 0.9291 | 0.0124 | 0.0021 |
| 0.60 | 0.9472 | 0.0061 | 0.0006 | 0.8967 | 0.0133 | 0.0028 | 0.9370 | 0.0121 | 0.0017 |

### Table 2. The effect of \( p_1 \) on \( P_0, L_q \) and \( P \)

| \( p_1 \) | \( P_0 \) | \( L_q \) | \( P \) | \( P_0 \) | \( L_q \) | \( P \) | \( P_0 \) | \( L_q \) | \( P \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.10 | 0.9007 | 0.0059 | 0.0003 | 0.8076 | 0.0116 | 0.0034 | 0.8988 | 0.0103 | 0.0018 |
| 0.20 | 0.9004 | 0.0067 | 0.0010 | 0.8062 | 0.0145 | 0.0059 | 0.8982 | 0.0113 | 0.0026 |
| 0.30 | 0.9001 | 0.0076 | 0.0017 | 0.8048 | 0.0174 | 0.0084 | 0.8975 | 0.0123 | 0.0035 |
| 0.40 | 0.8997 | 0.0084 | 0.0024 | 0.8033 | 0.0206 | 0.0111 | 0.8968 | 0.0134 | 0.0045 |
| 0.50 | 0.8994 | 0.0093 | 0.0030 | 0.8017 | 0.0238 | 0.0138 | 0.8962 | 0.0145 | 0.0054 |

### Table 3. The effect of \( \delta \) on \( P_0, L_q \) and \( F_f \)

| \( \delta \) | \( P_0 \) | \( L_q \) | \( F_f \) | \( P_0 \) | \( L_q \) | \( F_f \) | \( P_0 \) | \( L_q \) | \( F_f \) |
|---|---|---|---|---|---|---|---|---|---|
| 5.00 | 0.8909 | 0.0177 | 0.0984 | 0.7874 | 0.0481 | 0.1776 | 0.8927 | 0.0240 | 0.1434 |
| 6.00 | 0.8914 | 0.0193 | 0.1156 | 0.7895 | 0.0533 | 0.2081 | 0.8941 | 0.0253 | 0.1649 |
| 7.00 | 0.8920 | 0.0206 | 0.1328 | 0.7906 | 0.0571 | 0.2368 | 0.8955 | 0.0263 | 0.1846 |
| 8.00 | 0.8925 | 0.0217 | 0.1479 | 0.7913 | 0.0612 | 0.2637 | 0.8969 | 0.0271 | 0.2027 |
| 9.00 | 0.8931 | 0.0227 | 0.1620 | 0.7936 | 0.0658 | 0.2899 | 0.8981 | 0.0278 | 0.2195 |
Figure 1. $P_0$ versus $p_f$

Figure 2. $P_0$ versus $\alpha$
Figure 3. $L_q$ versus $\delta$

Figure 4. $F_{\text{loss}}$ versus $\delta$
6. Conclusion
A group arrival feedback retrial queue with $k$ optional stages of service and orbital search policy are meticulously studied. The PGF of the numbers in the system and orbit are found. $L_s, L_q, W_s$ and $W_q$ are obtained. The mathematical results are validated by simulation results.

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