The radiative efficiency of neutron stars at low-level accretion

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ABSTRACT
When neutrons star low-mass X-ray binaries (NS-LMXBs) are in the low-level accretion regime (i.e., $L_X \lesssim 10^{36} \text{ erg s}^{-1}$), the accretion flow in the inner region around the NS is expected to be existed in the form of the hot accretion flow, e.g., the advection-dominated accretion flow (ADAF) as that in black hole X-ray binaries. Following our previous studies in Qiao & Liu 2020a and 2020b on the ADAF accretion around NSs, in this paper, we investigate the radiative efficiency of NSs with an ADAF accretion in detail, showing that the radiative efficiency of NSs with an ADAF accretion is much lower than that of $\dot{E} \sim \frac{2GM}{cR^2} \sim 0.2$ despite the existence of the hard surface. As a result, given a X-ray luminosity $L_X$ (e.g., between 0.5 and 10 keV), $M$ calculated by $M = \frac{L_X}{\dot{E}}$ is lower than the real $M$ calculated within the framework of the ADAF accretion. The real $M$ can be more than two orders of magnitude higher than that of calculated by $M = \frac{L_X}{\dot{E}}$ with appropriate model parameters. Finally, we discuss that if applicable, the model of ADAF accretion around a NS can be applied to explain the observed millisecond X-ray pulsation in some NS-LMXBs (such as PSR J1023+0038, XSS J12270-4859 and IGR J17379-3747) at a lower X-ray luminosity of a few times of $10^{33} \text{ erg s}^{-1}$, since at this X-ray luminosity the calculated $M$ with the model of ADAF accretion can be high enough to drive a fraction of the matter in the accretion flow to be channelled onto the surface of the NS forming the X-ray pulsation.

**Keywords:** accretion, accretion discs – stars: neutron – black hole physics – X-rays: binaries

1 INTRODUCTION

Currently, there are two types of accretion flow around compact objects [black holes (BH) and neutron stars (NS)], i.e., the geometrically thin, optically thick, cold accretion disc (Shakura & Sunyaev 1973), and the geometrically thick, optically thin, hot accretion flow, such as the advection-dominated accretion flow (ADAF) (Yuan & Narayan 2014, for review). The cold accretion disc with a higher mass accretion rate is widely used to explain the optical/UV emission in luminous active galactic nuclei (AGNs), and the X-ray emission of X-ray binaries at the high/soft state (e.g. Mitsuda et al. 1984; Makishima et al. 1986). While the ADAF with a lower mass accretion rate is often used to explain the dominant emission in low-luminosity AGNs, as well as the low/hard and the quiescent state of X-ray binaries (Done et al. 2007, for review).

In general, the cold accretion disc is a kind of radiatively efficient accretion flow in both BH case and NS case. In the approximation of the Newtonian mechanics, for a non-rotating BH, the radiative efficiency of the accretion disc is $\epsilon = \frac{\dot{E}}{\frac{2GM}{cR^2}/Mc^2} \sim 0.1$ (with $G$ being the gravitational constant, $M$ being the mass accretion rate in units of $\text{g s}^{-1}$, $c$ being the speed of light, and $R_0$ being the Schwarzschild radius with $R_0 = \frac{2GM}{c^2} \approx 2.95 \times 10^5 \text{ M}_\odot /\text{cm}$), i.e., half of the gravitational energy will be released out in the form of the electromagnetic radiation in the accretion disc. While for a NS, the radiative efficiency of the accretion is $\epsilon = \frac{\dot{E}}{\frac{2GM}{cR^2}/Mc^2} \sim 0.2$ (taking $M = 1.4M_\odot$, and $R_0 = 12.5 \text{ km}$)$^1$, which is obtained as that half of the gravitational energy will be released out in the accretion disc and the other half of the gravitational energy will be released out in a thin boundary layer between the accretion disc and the surface of the NS in the form of the electromagnetic radiation (Gilfanov & Sunyaev 2014).

In general, the ADAF solution is a kind of radiatively inefficient accretion flow in BH case. In the approximation of the Newtonian mechanics, for a non-rotating BH, the radiative efficiency of the ADAF is $\epsilon \approx \frac{\dot{E}}{\frac{2GM}{cR^2}/Mc^2} \sim 0.1$ (Ichimaru 1977; Rees et al. 1982; Narayan & Yi 1994, 1995; Manmoto et al. 1997; Yuan & Narayan 2014, for review). This is due to the optically thin nature of the ADAF solution, a fraction of the viscously dissipated energy in the ADAF is stored in the gas of the ADAF as the internal energy, and finally advected into the event horizon of the BH with-

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$^1$ We take $R_0 = 12.5$ because we intend to keep the same efficiency of the gravitational energy release between a NS and a non-rotating BH. If the NS mass $m = 1.4$ is taken, the corresponding NS radius is $R_0 = 12.5 \text{ km}$. 

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out radiation. The fraction of the viscously dissipated energy stored in the gas of the ADAF as the internal energy is dependent on $M$, and this fraction increases with decreasing $M$, which means that the radiative efficiency of the ADAF around a BH decreases with decreasing $M$. Specifically, when $M$ is close to the critical mass accretion rate $M_{\text{crit}}$ of the ADAF ($M_{\text{crit}} \sim \alpha^2 M_{\text{Edd}}$, with $\alpha$ being the viscosity parameter, and $M_{\text{Edd}} = L_{\text{Edd}}/0.1c^2 \approx 1.39 \times 10^{38} M/M_8$ g s$^{-1}$ being the Eddington scaled mass accretion rate, where $L_{\text{Edd}}$ is defined as $L_{\text{Edd}} = 1.26 \times 10^{38} M/M_8$ erg s$^{-1}$), the value of $\epsilon$ is close to 0.1 as that of the cold accretion disc around a BH (Xie & Yuan 2012). However, when $M$ is significantly less than $M_{\text{crit}}$, the radiative efficiency $\epsilon$ decreases dramatically with decreasing $M$, and the value of $\epsilon$ is much less than 0.1 (Xie & Yuan 2012, for discussions).

In this paper, we focus on the radiative efficiency of the ADAF solution around a NS (strictly speaking, it is the radiative efficiency of NSs with an ADAF accretion, since a fraction of the ADAF energy released at the surface of the NS can finally radiate out to be observed). The dynamics of the ADAF around a weakly magnetized NS has been investigated by some authors previously (Medvedev & Narayan 2001; Medvedev 2004; Narayan & Yi 1995). In general, one of the most difficult problems for the study of the ADAF around a NS is how to treat the dynamics and radiation of the boundary layer between the surface of the NS and the ADAF. The physics of the boundary layer can significantly affect the global dynamics and radiation of the ADAF (Medvedev & Narayan 2001; Medvedev 2004; D’Angelo et al. 2015). Recently, in a series of papers, i.e., Qiao & Liu (2018), Qiao & Liu (2020a), and Qiao & Liu (2020b), we study the dynamics and the radiation of the ADAF around a weakly magnetized NS in the framework of the self-similar solution of the ADAF by simplifying the physics of the boundary layer. Specifically, we introduce a parameter, $f_{\text{th}}$, which describes the fraction of the ADAF energy released at the surface of the NS as thermal emission to be scattered in the ADAF. Under this assumption, i.e., considering the radiative feedback between the surface of the NS and the ADAF, we self-consistently calculate the structure and the corresponding emergent spectrum of the NS with an ADAF accretion. The value of $f_{\text{th}}$ can affect the radiative efficiency of NSs with an ADAF accretion. Physically, the value of $f_{\text{th}}$ is uncertain. However, it has been shown that the value of $f_{\text{th}}$ can be constrained in a relatively narrow range by comparing with the observed X-ray spectra (typically between 0.5 and 10 keV) of neutron star low-mass X-ray binaries (NS-LMXBs), since the value of $f_{\text{th}}$ can affect both the shape and the luminosity of the X-ray spectra (Qiao & Liu 2020a,b). The results in Qiao & Liu (2020a,b) jointly suggest that the value of $f_{\text{th}}$ is certainly less than 0.1, and a smaller value of $f_{\text{th}} \sim 0.01$ is more preferred.

In this paper, following the results of Qiao & Liu (2020a,b) for the constraints to the value of $f_{\text{th}}$, we investigate the radiative efficiency of NSs with an ADAF accretion for taking two typical values of $f_{\text{th}}$ as that of $f_{\text{th}} = 0.1$ and $f_{\text{th}} = 0.01$ respectively. The radiative efficiency is defined as $\epsilon_{\text{th}} = L_{\text{bol}}/Mc^2$ (with $L_{\text{bol}}$ being the bolometric luminosity). Based on the emergent spectra of NSs with an ADAF accretion for the bolometric luminosity, we find that $\epsilon_{\text{th}}$ is nearly a constant with $M$ for either taking $f_{\text{th}} = 0.1$ or taking $f_{\text{th}} = 0.01$. The value of $\epsilon_{\text{th}}$ for $f_{\text{th}} = 0.1$ is roughly one order of magnitude lower than the previously expected value of $\epsilon \sim \frac{L_{\text{bol}}}{M_{\text{Edd}}}/c^2 \sim 0.2$, and $\epsilon_{\text{th}}$ for $f_{\text{th}} = 0.01$ is roughly two orders of magnitude lower than the value of $\epsilon \sim 0.2$. Then, we suggest that NSs with an ADAF accretion is radiatively inefficient despite the existence of the hard surface. Further, we investigate the radiative efficiency in some specific bands, e.g., $6_{\text{th}}-10_{\text{th}}$ keV and $6_{\text{th}}-50$ keV (defined as $6_{\text{th}}-10_{\text{th}}$ keV $= L_{6_{\text{th}}-10_{\text{th}}}$ keV $/Mc^2$ and $6_{\text{th}}-50$ keV $= L_{6_{\text{th}}-50}$ keV $/Mc^2$). $6_{\text{th}}-10_{\text{th}}$ keV is nearly same with $6_{\text{th}}-10_{\text{th}}$ keV for a fixed $M$ with $f_{\text{th}} = 0.1$ and $f_{\text{th}} = 0.01$ respectively, $6_{\text{th}}-10_{\text{th}}$ keV (or $6_{\text{th}}-50$ keV) decreases very quickly with decreasing $M$. As a result, for a NS-LMXB, if we intend to use the observed X-ray luminosity (e.g., between 0.5 and 10 keV) as the indicator for $M$, $M$ calculated with the formula of $M = L_{\text{bol}}/c^2$ is lower than that of calculated with our model of ADAF accretion around a NS. Obviously, given a X-ray luminosity, the difference between the $M$ calculated with our model of ADAF accretion and the $M$ calculated with the formula of $M = L_{\text{bol}}/c^2$ depends on $f_{\text{th}}$, with the difference of the calculated $M$ increases with decreasing $f_{\text{th}}$.

Finally, in this paper, we argue that if applicable, the model of ADAF accretion around a NS can probably be used to explain the observed millisecond X-ray pulsation in some NS-LMXBs (such as PSR J1023+0038, XSS J12270-4859 and IGR J17379-3747) at a X-ray luminosity (between 0.5 and 10 keV) of a few times of $10^{33}$ erg s$^{-1}$, since at this X-ray luminosity the calculated $M$ with the model of ADAF accretion can be high enough, e.g., more than two orders of magnitude higher than that of calculated with the formula of $M = L_{\text{bol}}/c^2$ for taking $f_{\text{th}} = 0.01$, to drive a fraction of the matter in the accretion flow to be channelled onto the surface of the NS forming the X-ray pulsation. A brief summary on the ADAF model around a NS and the constraints to the value of $f_{\text{th}}$ in Qiao & Liu (2020a,b) are introduced in Section 2. The results are shown in Section 3. The discussions are in Section 4 and the conclusions are in Section 5.

## 2 A SUMMARY ON QIAO & LIU 2020A,B

The structure and the corresponding emergent spectra of the ADAF around a NS are strictly investigated within the framework of the self-similar solution of the ADAF (Qiao & Liu 2018). In Qiao & Liu (2020a,b), we update the code with the effect of the NS spin considered compared with that of in Qiao & Liu (2018). In our model, there are seven parameters, i.e., the NS mass $m$ ($m = M/M_8$), NS radius $R$, NS spin frequency $\nu_{\text{NS}}$, mass accretion rate $\dot{m}$ ($\dot{m} = M/M_{\text{Edd}}$), as well as the viscosity parameter $\alpha$, and the magnetic parameter $\beta$ [with magnetic pressure $p_m = B^2/8\pi = (1 - \beta)p_{\text{tot}}$, $p_{\text{tot}} = p_m + p_{\text{gas}}$] for describing the microphysics of the ADAF. The last parameter is, $f_{\text{th}}$, describing the fraction of the ADAF energy released at the surface of the NS as thermal emission to be scattered in the ADAF to cool the ADAF itself, which consequently controls the feedback between the ADAF and the NS. We always take $m = 1.4$, and $R$ in the range of 10-12.5 km (Degenaar & Suleimanov 2018) [$R = 12.5$ km in Qiao & Liu (2020a), and $R = 10$ km in Qiao & Liu (2020b)]. In general, it has been proven that the effect of the NS spin frequency $\nu_{\text{NS}}$ on the structure and the emergent spectra of the ADAF around a NS is very little, and nearly can be neglected [see Figure 8 of Qiao & Liu (2020a) for taking $\nu_{\text{NS}} = 0$, 200, 500, 700 Hz respectively]. So we fix $\nu_{\text{NS}} = 0$ Hz in Qiao & Liu (2020a,b). The magnetic field in ADAF is very weak as suggested by the magnetohydrodynamic simulations (Yuan & Narayan 2014, for review). We fix $\beta = 0.95$ in Qiao & Liu (2020a,b).

The X-ray spectra of NS-LMXBs in the low-level accretion regime ($L_{6_{\text{th}}-10_{\text{th}}}$ keV $\lesssim 10^{36}$ erg s$^{-1}$) can be described by a single power-law model, or a two-component model, i.e., a thermal soft X-ray component plus a power-law component. In general, if the Swift X-ray data are used, the spectral fitting with a sin-
ingle power-law model can return an accepted fit. And if the high-quality XMM – Newton X-ray data are used, the spectra fitting with a two-component model can significantly improve the fitting results in some X-ray luminosity range, e.g., in the range of \( L_{0.5-10 keV} \approx 10^{34} - 10^{37} \) erg s\(^{-1}\) (e.g., Wijnands et al. 2015). In Qiao & Liu (2020a), we test the effect of \( \alpha \) and \( f_{\text{th}} \) on the X-ray spectra between 0.5 and 10 keV, and explain the fractional contribution of the power-law component \( \eta (\eta \equiv L_{0.5-10 keV}/L_{0.5-10 keV}) \) (with the spectra fitted with the two-component model) as a function of the \( L_{0.5-10 keV} \) for a sample of non-pulsating NS-LMXBs in a wide range from \( L_{0.5-10 keV} \approx 10^{33} - 10^{38} \) erg s\(^{-1}\) and an anticorrelation between \( \eta \) and \( L_{0.5-10 keV} \) for \( L_{0.5-10 keV} >\) a few times of \( 10^{38} \) erg s\(^{-1}\). Observationally, there is a positive correlation between \( \eta \) and \( L_{0.5-10 keV} \) for \( L_{0.5-10 keV} >\) a few times of \( 10^{39} \) erg s\(^{-1}\), and an anticorrelation between \( \eta \) and \( L_{0.5-10 keV} \) for \( L_{0.5-10 keV} >\) a few times of \( 10^{39} \) erg s\(^{-1}\). By comparing with the observed correlation (both the positive correlation and the anticorrelation) between \( \eta \) and \( L_{0.5-10 keV} \), it is found that the effect of \( \alpha \) on the correlation between \( \eta \) and \( L_{0.5-10 keV} \) is very little, and nearly can be neglected. Meanwhile, it is found that the correlation between \( \eta \) and \( L_{0.5-10 keV} \) can be well matched by adjusting the value of \( f_{\text{th}} \). The value of \( f_{\text{th}} \) is constrained to be less than 0.1. Especially, \( f_{\text{th}} > 0.01 \) is more preferred. We define three quantities for the radiative efficiency in different bands, i.e.,

\[
\epsilon_{0.5-10 keV} = L_{0.5-10 keV}/\dot{M} c^2, \\
\epsilon_{0.5-100 keV} = L_{0.5-100 keV}/\dot{M} c^2, \\
\epsilon_{\text{bol}} = L_{\text{bol}}/\dot{M} c^2.
\]

In panel (2) of Fig. 2, we plot \( \epsilon_{0.5-10 keV} \), \( \epsilon_{0.5-100 keV} \) and \( \epsilon_{\text{bol}} \) as a function of \( \dot{m} \) for \( f = 0.1 \) and \( f = 0.01 \) respectively. Specifically, \( \dot{m} = 0.1, \dot{m}_{\text{bol}} \approx 0.02 \) as a comparison, one can refer to the dashed line in panel (1) of Fig. 2 for clarity. It can be seen that all the three luminosities, i.e., \( L_{0.5-10 keV}, L_{0.5-100 keV} \) and \( L_{\text{bol}} \), are lower than the luminosity calculated with the formula of \( L = \frac{\dot{M} c^2}{\Delta M} \) for a fixed \( M \) (or \( \dot{m} \)).

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\[
\epsilon_{0.5-10 keV} \approx L_{0.5-10 keV}/\dot{M} c^2, \\
\epsilon_{0.5-100 keV} \approx L_{0.5-100 keV}/\dot{M} c^2, \\
\epsilon_{\text{bol}} = L_{\text{bol}}/\dot{M} c^2.
\]

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We define three quantities as a comparison, we plot the radiative efficiency in different bands, i.e.,

\[
\epsilon_{0.5-10 keV} \approx L_{0.5-10 keV}/\dot{M} c^2, \\
\epsilon_{0.5-100 keV} \approx L_{0.5-100 keV}/\dot{M} c^2, \\
\epsilon_{\text{bol}} = L_{\text{bol}}/\dot{M} c^2.
\]
Recently, the millisecond X-ray pulsations have been observed in several NS X-ray sources, such as, the transitional millisecond pulsar (tMSP) PSR J1023+0038 (Archibald et al. 2015) and XSS J12270-4859 (Papitto et al. 2015) as they are in the accretion-powered LMXB state, as well as the X-ray transient IGR J17379-3747 (Bult et al. 2019), at a lower X-ray luminosity (between 0.5 and 10 keV) of a few times of $10^{33}$ erg s$^{-1}$. This challenges the traditional accretion disc theory for the formation of the X-ray pulsation at such a low X-ray luminosity, since in general at this low X-ray luminosity (if the mass accretion rate calculated with formula of $M = \frac{L_{X,5}}{E_{\text{bol},5}}$), the corotation radius of the NS accreting system is less than the magnetospheric radius. In this case, the ‘propeller’ effect may work, expelling (a fraction of the) matter in the accretion flow to be leaving away from the NS (Illarionov & Sunyaev 1975). In this paper, we suggest that, if applicable, our model of ADAF accretion may be applied to explain the observed millisecond X-ray pulsation at the X-ray luminosity of a few times of $10^{33}$ erg s$^{-1}$. This is because at this X-ray luminosity, $M$ calculated from our model of ADAF accretion for taking an appropriate value of $f_{\text{bd}}$, such as $f_{\text{bd}} = 0.01$, can be more than two orders of magnitude higher than that of calculated with the formula of $M = \frac{L_{X,5}}{E_{\text{bol},5}}$ to make the magnetospheric radius less than the corotation radius. In this case, as the accretion flow moves inward, if the radius is less than the magnetospheric radius, (a fraction of) the matter in the accretion flow will be magnetically channelled onto the surface of the NS, leading to the formation of the X-ray pulsation.

For clarity, we list the expression of the corotation radius $R_{c}$ and the magnetospheric radius $R_{m}$ respectively as follows. The corotation radius $R_{c}$ is expressed as,

$$R_{c} = \left( \frac{GM}{4\pi^{2}v_{NS}} \right)^{1/3},$$

where $M$ is the NS mass, and $v_{NS}$ is the NS spin frequency. The magnetospheric radius $R_{m}$ is expressed as (Spruit & Taam 1993; D’Angelo & Spruit 2010; D’Angelo et al. 2015),

$$R_{m} = \left( \frac{\eta v_{NS}^{2}}{4\Omega_{NS} M} \right)^{1/5},$$

$$= 26.6 \text{ km} \left( \frac{M}{1.4M_{\odot}} \right)^{1/3} \left( \frac{v_{NS}}{500 \text{ Hz}} \right)^{-2/3},$$

where $M$ is the NS mass, and $v_{NS}$ is the NS spin frequency. The magnetospheric radius $R_{m}$ is expressed as (Spruit & Taam 1993; D’Angelo & Spruit 2010; D’Angelo et al. 2015),

$$R_{m} = \left( \frac{\eta v_{NS}^{2}}{4\Omega_{NS} M} \right)^{1/5},$$

$$\approx 24 \text{ km} \left( \frac{B}{10^{5} \text{ G}} \right)^{2/5} \left( \frac{R_{c}}{10 \text{ km}} \right)^{6/5} \times \left( \frac{v_{NS}}{500 \text{ Hz}} \right)^{-1/5} \left( \frac{M}{10^{5} \odot} \right)^{-1/5},$$

where $\mu = B_{c}R_{c}^{3}$ is magnetic dipole moment (with $B$, being the magnetic field at the surface of the NS, $R_{c}$ being the NS radius), $\Omega_{NS}$ is the rotational angular velocity of the NS (with $\Omega_{NS} = 2\pi v_{NS}$), $\eta \leq 1$ is the dimensionless parameter describing the strength of the toroidal magnetic field induced by the relative rotation between the accretion flow and dipolar magnetic field, and $M$ is the mass accretion rate in units of g s$^{-1}$. We investigate the relation between $R_{c}$ and $R_{m}$ for PSR J1023+0038, XSS J12270-4859 and IGR J17379-3747 respectively as follows.

**PSR J1023+0038:** The millisecond X-ray pulsation of PSR J1023+0038 has been discovered at the X-ray luminosity of $L_{X,5,10keV} \sim 3.0 \times 10^{33}$ erg s$^{-1}$. The spin frequency of PSR J1023+0038 is $v_{NS} = 592$ Hz (Archibald et al. 2015). As we can see from Table 1, at the X-ray luminosity of $L_{X,5,10keV} \sim 3.0 \times 10^{33}$ erg s$^{-1}$, the mass accretion rate $M_{01}$ calculated with the formula of $M_{0} = L_{X,5,10keV} \frac{c}{4\pi} \left( \frac{\nu}{10^{3}} \right) \left( \frac{\rho}{10^{-4}} \right) \left( \frac{g}{10^{5}} \right)$ is $3.36 \times 10^{13}$ g s$^{-1}$, which is ~ 23 times less than $M_{01}$ calculated with our model of ADAF accretion for taking $f_{\text{bd}} = 0.01$, and is ~ 192 times less than the mass accretion rate $M_{001}$ calculated with our model of ADAF accretion for taking $f_{\text{bd}} = 0.01$. The relatively higher $M$ calculated from our model of ADAF accretion has very clear physical meanings, as will be discussed in Section 3.2.

**XSS J12270-4859 and IGR J17379-3747:** The millisecond X-ray pulsation of both XSS J12270-4859 and IGR J17379-3747 are discovered at the X-ray luminosity of $L_{X,5,10keV} \sim 5.0 \times 10^{33}$ erg s$^{-1}$. The spin frequency of XSS J12270-4859 and IGR J17379-3747 are $v_{NS} = 593$ Hz and $v_{NS} = 468$ Hz respectively. As we can see from Table 1, at the X-ray luminosity of $L_{X,5,10keV} = 5.0 \times 10^{33}$ erg s$^{-1}$, the mass accretion rate $M_{0}$ calculated with the formula of $M_{0} = L_{X,5,10keV} \frac{c}{4\pi} \left( \frac{\nu}{10^{3}} \right) \left( \frac{\rho}{10^{-4}} \right) \left( \frac{g}{10^{5}} \right)$ is $3.36 \times 10^{13}$ g s$^{-1}$. At this X-ray luminosity, the mass accretion rate $M_{0}$ calculated with our model of ADAF accretion for $f_{\text{bd}} = 0.01$ is $3.88 \times 10^{13}$ g s$^{-1}$, and mass accretion rate $M_{001}$ calculated with our model of ADAF accretion for $f_{\text{bd}} = 0.01$ is $1.46 \times 10^{14}$ g s$^{-1}$, as we take in Section 3.1 of this paper, a typical value of the magnetic field at the surface of the NS $B_{c} = 10^{8}$ G and $\eta = 0.1$, according to equation (4), $R_{c} \approx 23.8$ km, and according to equation (5), $R_{m} = 66.2$ km, $R_{m1} = 34.2$ km, and $R_{m001} = 23.1$ km (with $R_{m}$, $R_{m1}$ and $R_{m001}$ being the magnetospheric radii calculated by taking the mass accretion rate as $M_{0}$, $M_{01}$ and $M_{001}$ respectively). It can be seen that, if the mass accretion rate, i.e. $M_{0}$, is calculated with the formula of $M_{0} = L_{X,5,10keV} \frac{c}{4\pi} \left( \frac{\nu}{10^{3}} \right) \left( \frac{\rho}{10^{-4}} \right) \left( \frac{g}{10^{5}} \right)$, $R_{m} > R_{c}$, theoretically, in this case the pulsation cannot be formed. If the mass accretion rate, i.e. $M_{0}$, is calculated with our model of ADAF accretion for $f_{\text{bd}} = 0.01$, $R_{m} > R_{c}$, theoretically, the pulsation also cannot be formed. While, if the mass accretion rate, i.e. $M_{001}$, is calculated with our model of ADAF accretion for $f_{\text{bd}} = 0.01$, $R_{m001} < R_{c}$, theoretically, the pulsation can be formed (Illarionov & Sunyaev 1975).
Radiative efficiency at low-level accretion

Figure 1. Panel (1): emergent spectra of NSs with an ADAF accretion for different $m$ with $f_{\text{th}} = 0.1$. Panel (2): emergent spectra of NSs with an ADAF accretion for different $m$ with $f_{\text{th}} = 0.01$.

Figure 2. Panel (1): X-ray luminosity $L_{0.5-10keV}$, $L_{0.5-100keV}$ and bolometric luminosity $L_{\text{bol}}$ as a function of $m$ respectively. The symbols of black ‘+’, blue ‘+’, and red ‘+’ refer to $L_{0.5-10keV}$, $L_{0.5-100keV}$, and $L_{\text{bol}}$ respectively from our model of ADAF accretion with $f_{\text{th}} = 0.1$. The symbols of black ‘△’, blue ‘△’, and red ‘△’ refer to $L_{0.5-10keV}$, $L_{0.5-100keV}$, and $L_{\text{bol}}$ respectively from our model of ADAF accretion with $f_{\text{th}} = 0.01$. The dashed line refers to the luminosity calculated with the formula of $L = \frac{MGM}{R^2}$. Panel (2): radiative efficiency $\epsilon_{0.5-10keV}$, $\epsilon_{0.5-100keV}$ and $\epsilon_{\text{bol}}$ as a function of $m$. The symbols of black ‘+’, blue ‘+’, and red ‘+’ refer to $\epsilon_{0.5-10keV}$, $\epsilon_{0.5-100keV}$ and $\epsilon_{\text{bol}}$ respectively from our model of ADAF accretion with $f_{\text{th}} = 0.1$. The symbols of black ‘△’, blue ‘△’, and red ‘△’ refer to $\epsilon_{0.5-10keV}$, $\epsilon_{0.5-100keV}$ and $\epsilon_{\text{bol}}$ respectively from our model of ADAF accretion with $f_{\text{th}} = 0.01$. The dashed line refers to the radiative efficiency calculated with the formula of $\epsilon = \frac{\dot{M}}{\dot{M}_{\text{Edd}}}$.  

Table 1. Mass accretion rate obtained for a given X-ray luminosity $L_{0.5-10keV}$ based on the curves in Fig. 3. Specifically, $m_0$ and $M_0$ are obtained with the formula of $M_0 = L_{0.5-10keV} \frac{\dot{m}}{\dot{M}_{\text{Edd}}}$, $m_{01}$ and $M_{01}$ are in units of $M_{\text{Edd}}$ and $g$ s$^{-1}$ respectively. $m_{0.1}$ and $M_{0.1}$ are obtained from our model results of NSs with an ADAF accretion for $f_{\text{th}} = 0.1$. $m_{0.01}$ and $M_{0.01}$ are in units of $M_{\text{Edd}}$ and $g$ s$^{-1}$ respectively. $m_{0.01}$ and $M_{0.01}$ are obtained from our model results of NSs with an ADAF accretion for $f_{\text{th}} = 0.01$. $m_{0.01}$ and $M_{0.01}$ are in units of $M_{\text{Edd}}$ and $g$ s$^{-1}$ respectively.

| $L_{0.5-10keV}$ (erg s$^{-1}$) | $m_0$ (M$_{\text{Edd}}$) | $M_0$ (g s$^{-1}$) | $m_{0.1}$ (M$_{\text{Edd}}$) | $M_{0.1}$ (g s$^{-1}$) | $m_{0.01}$ (M$_{\text{Edd}}$) | $M_{0.01}$ (g s$^{-1}$) |
|-----------------------------|--------------------------|-----------------------|-----------------------------|-----------------------|-----------------------------|-----------------------|
| $3.0 \times 10^{31}$       | 1.04 \times 10^{-5}     | 2.02 \times 10^{13}   | 2.82 \times 10^{-4}       | 5.48 \times 10^{14}   | 2.00 \times 10^{-3}       | 3.88 \times 10^{15}    |
| $5.0 \times 10^{31}$       | 1.73 \times 10^{-5}     | 3.36 \times 10^{13}   | 3.98 \times 10^{-4}       | 7.75 \times 10^{14}   | 3.16 \times 10^{-3}       | 6.15 \times 10^{15}    |

4859 and IGR J17379-3747, the results are similar to that of PSR J1023+0038. Specifically, if mass the mass accretion rate, i.e. $M_0$, is calculated with the formula of $M_0 = L_{0.5-10keV} \frac{\dot{m}}{\dot{M}_{\text{Edd}}}$, $R_{\text{bol}} > R_a$, and if the mass accretion rate, i.e. $M_{0.1}$, is calculated with our model of ADAF accretion for $f_{\text{th}} = 0.1$. $R_{\text{bol}} > R_a$. In these two cases, theoretically, the pulsation cannot be formed. If the mass accretion rate, i.e. $M_{0.01}$, is calculated with our model of ADAF accretion for $f_{\text{th}} = 0.01$, $R_{\text{bol}} > R_a$. In this case, theoretically, the pulsation can be formed (Illarionov & Sunyaev 1975). One can refer to Table 2 for the detailed numerical results of $R_a$, $R_{\text{bol}}$, $R_{\text{bol}} > R_a$, and $R_{\text{bol}} > R_a$ for XSS J12270-4859 and IGR J17379-3747 respectively.

Here, we would like to remind that we take a fixed value of $\dot{m}$ as in Fig. 3, and the corresponding calculations for $f_{\text{th}} = 0.01$, $R_{\text{bol}} > R_a$. In this case, theoretically, the pulsation can be formed (Illarionov & Sunyaev 1975). One can refer to Table 2 for the detailed numerical results of $R_a$, $R_{\text{bol}}$, $R_{\text{bol}} > R_a$, and $R_{\text{bol}} > R_a$ for XSS J12270-4859 and IGR J17379-3747 respectively.
4 DISCUSSIONS

4.1 The effect of the large-scale magnetic field of ~ $10^5$ G on the radiative efficiency of NSs with an ADAF accretion

In this paper, we investigate the radiative efficiency of weakly magnetized NSs with an ADAF accretion for taking two typical values of $f_B = 0.1$ and $f_B = 0.01$ as suggested in Qiao & Liu (2020a) and Qiao & Liu (2020b). Then, we show that NSs with an ADAF accretion is radiatively inefficient, with which we further explain the observed millisecond X-ray pulsations for PSR J1023+0038, XSS J12270-4859 and IGR J17379-3747 at the X-ray luminosity of a few times of $10^{33}$ erg s$^{-1}$. However, we should note that, in our model of NSs with an ADAF accretion, we do not consider the effect of the large-scale magnetic field on the emission of the ADAF, which probably will affect the radiative efficiency of the NSs with an ADAF accretion.

In general, accreting millisecond X-ray pulsars (AMXPs) are believed to have a relatively weaker magnetic field of ~ $10^8$ G (e.g. Wijnands & van der Klis 1998; Casella et al. 2008; Patruno & Watts 2012, for review), which is different from the standard X-ray pulsars often with a stronger magnetic field of ~ $10^{12}$ G (e.g. Coburn et al. 2002; Potschmidt et al. 2005; Caballero & Wilms 2012; Revnivtsev & Mereghetti 2015, for review). Due to the relatively weaker magnetic field in AMXPs, it is often suggested that the magnetic field in AMXPs does not significantly affect the X-ray spectra (e.g. Poutanen & Gierliński 2003), which seems to be supported by some observations by comparing the X-ray spectra between the non-pulsating NS-LMXBs and the AMXPs. In general, it is found that there is no systematic difference of the X-ray spectra between the non-pulsating NS-LMXBs and the AMXPs in the range of $L_{0.5-10keV}$ ~ $10^{34} - 10^{36}$ erg s$^{-1}$. For example, in Wijnands et al. (2015), the authors compiled a sample composed of eleven non-pulsating NS-LMXBs, finding that systematically there is an anticorrelation between the X-ray photon index $\Gamma$ (obtained by fitting the X-ray spectra between 0.5 and 10 keV with a single power law) and the X-ray luminosity $L_{0.5-10keV}$ in the range of $L_{0.5-10keV}$ ~ $10^{34} - 10^{36}$ erg s$^{-1}$. Further, the authors added three AMXPs, i.e., NGC 6440 X-2, IGR J00291+5934, and IGR J18245-2452, with well measured $\Gamma$ and $L_{0.5-10keV}$ to compare with the non-pulsating NS sample, showing that at a fixed X-ray luminosity, the X-ray spectra of the AMXPs appear to be slightly harder than that of the non-pulsating NS-LMXBs. More accurately, the authors did 2D KS test to study whether the AMXPs are consistent with the non-pulsating data. It is found that a 90 per cent confidence interval for the probability of 1.2$\times10^{-6}$ $-$ 3.5$\times10^{-4}$ that the AMXPs data and the non-pulsating data have the same distribution. However, given the fact that only three AMXPs are included in this study, actually, the authors also reminded that they cannot draw strong conclusions whether the presence of the magnetic field in AMXPs can alter the X-ray spectra (Wijnands et al. 2015). In a further study of Parikh et al. (2017), the authors combined the data
in Wijnands et al. (2015) and some additional new data in the range of $L_{0.5-10keV} \sim 10^{34}-10^{36}$ erg s$^{-1}$ for the anticorrelation between the X-ray photon index $\Gamma$ and the X-ray luminosity $L_{0.5-10keV}$, the authors showed that they did not find that the X-ray spectra of AMXPs are systematically harder than that of the non-pulsating sources as tested in Wijnands et al. (2015), suggesting that the hardness of the X-ray spectra does not have strict connection with the presence of the dynamic effect of the magnetic field.

As for $L_{0.5-10keV} \lesssim 10^{34}$ erg s$^{-1}$ (generally defined as the quiescent state), the X-ray spectra of non-pulsating NS-LMXBs are very complex and diverse, which can be (1) completely dominated by a thermal soft X-ray component, (2) completely dominated by a power-law component, or (3) described by the two-component model, i.e. a thermal soft X-ray component plus a power-law component (e.g. Wijnands et al. 2015, for discussions). For example, the X-ray spectra of the non-pulsating NS-LMXB Cen X-4 at the X-ray luminosity of $L_{0.5-10keV} \sim 10^{35}$ erg s$^{-1}$ can be well fitted by the two-component model, i.e. a thermal soft X-ray component plus a power-law component, revealing a harder X-ray photon index of $\Gamma \sim 1$ \cite{Chakrabarty et al. 2014; D'Angelo et al. 2015}, while the X-ray spectra of several non-pulsating NS-LMXBs are well fitted by a single power law with a softer X-ray photon index of $\Gamma \sim 3$ \textendash{} 5 at the X-ray luminosity of $L_{0.5-10keV}$ \textendash{} a few times of $10^{35}$ erg s$^{-1}$ \cite{Sobas et al. 2018}. For the three sources, i.e., PSR J1023+0038, XSS J12270-4859 and IGR J17379-3747 with the millisecond X-ray pulsations observed at the X-ray luminosity of $L_{0.5-10keV}$ \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \si...
NS exceeds the research scope in the present paper, and definitely will be carried out in the future.

In Qiao & Liu (2020a) and Qiao & Liu (2020b), the constraint to the value of $f_0$ is based on some statistically observed correlations in non-pulsating NSs, such as the fractional contribution of the power-law component $\eta$ as a function of $L_{4.5-10 keV}$, as well as the X-ray photon index $\Gamma$ as a function of $L_{4.5-10 keV}$. In order more precisely to constrain the value of $f_0$, we expect that the detailed X-ray spectral fittings will be done for some typically single source in the future, such as the study for Cen X-4 (e.g. Chakrabarty et al. 2014; D’Angelo et al. 2015).

As discussed in Section 3.2, if our model of ADAF accretion can be applied to explain the observed millisecond X-ray pulsation at the X-ray luminosity of a few times of $10^{33}$ erg s$^{-1}$ for PSR J1023+0038, XSS J12270-4859 and IGR J17379-3747, a small value of $f_0$, e.g., $f_0 = 0.01$ is required. Based on some related results from the model of ADAF accretion for taking $f_0 = 0.01$, we can further estimate the change rate of the NS spin frequency $\dot{\nu}_{NS}$. If we assume that the change of the NS spin is due to the accretion, according to the conservation of angular momentum, we have

$$I \dot{\Omega}_{NS} = M (\Omega - \Omega_{NS}) R^2 \dot{f},$$

(6)

where $I$ is the moment of inertia of the NS, $\Omega$, is the rotational angular velocity of the ADAF at $R$, with $\Omega = 2 \nu$, (with $\nu$ being the angular frequency at $R$), and $\Omega_{NS}$ is the rotational angular velocity of the NS with $\Omega_{NS} = 2 \nu_{NS}$. Rearranging equation (6), we can express the change rate of the NS spin frequency as follows,

$$\dot{\nu}_{NS} = M (\nu - \nu_{NS}) R^2 / I.$$  

(7)

Given the value of $M$, $\nu$, $\nu_{NS}$ and $I$, we can calculate the change rate of the NS spin frequency $\dot{\nu}_{NS}$. For example, for PSR J1023+0038 the millisecond X-ray pulsation is observed at the X-ray luminosity of $L_{4.5-10 keV} \sim 3 \times 10^{33}$ erg s$^{-1}$, the corresponding $M$ is $3.88 \times 10^{-5}$ $\text{g s}^{-1}$ based on our model of ADAF accretion for $f_0 = 0.01$. With $M = 3.88 \times 10^{-5}$ $\text{g s}^{-1}$, we recalculate the structure of the ADAF for $\nu$. The value of $\nu$ is 253 Hz. The moment of inertia $I \sim 1.75 \times 10^{45}$ g cm$^2$ for taking the typical value of $M = 1.4 M_\odot$ and $R \sim 12.5$ km respectively. The spin frequency $\nu_{NS}$ of PSR J1023+0038 is 592 Hz. Substituting the value of $M$, $\nu$, $\nu_{NS}$ and $I$ into equation (7), we get $\dot{\nu}_{NS} \sim -1.2 \times 10^{-15}$ Hz s$^{-1}$, which is close to (- 2.5 times less than) the observed value of $\dot{\nu}_{NS} \sim -3.04 \times 10^{-15}$ Hz s$^{-1}$ for PSR J1023+0038 at the LMXB state (Jaodand et al. 2016). Here, we should note that in this case, the value of $\nu$, from our model of ADAF accretion is less than $\nu_{NS}$, which means that a negative torque will be exerted on the NS, consequently making the rotational energy of the NS transferred onto the ADAF and the NS to be spin-down, rather than the ADAF energy transferred onto the NS and the NS to be spin-up. Further, since a variable flat-spectrum of radio emission is revealed as PSR J1023+0038 in the LMXB state, it means that the outflow is existed, which physically can further make the NS to be spin-down to match the observed $\dot{\nu}_{NS}$ (Deller et al. 2015).

A similar calculation for $\dot{\nu}_{NS}$ is done for XSS J12270-4859 and IGR J17379-3747 with the millisecond X-ray pulsations observed at the X-ray luminosity of ~ $5 \times 10^{33}$ erg s$^{-1}$. At this X-ray luminosity, the mass accretion rate $\dot{M}$ is $6.15 \times 10^{15}$ g s$^{-1}$ based on our model of ADAF accretion for $f_0 = 0.01$. With $\dot{M} = 6.15 \times 10^{15}$ g s$^{-1}$, we recalculate the structure of the ADAF for $\nu$. The value of $\nu$ is 278 Hz. The spin frequency $\nu_{NS}$ is 593 Hz for XSS J12270-4859, and is 468 Hz for IGR J17379-3747. The moment of inertia $I \sim 1.75 \times 10^{45}$ g cm$^2$ for taking $M = 1.4 M_\odot$ and $R = 12.5$ km respectively. Again substituting the value of $\dot{M}$, $\nu$, $\nu_{NS}$ and $I$ into equation (7), we get $\dot{\nu}_{NS} \sim -1.73 \times 10^{-15}$ Hz s$^{-1}$ for XSS J12270-4859 and $\dot{\nu}_{NS} \sim -1.0 \times 10^{-15}$ Hz s$^{-1}$ for IGR J17379-3747. It is clear that the value of $\dot{\nu}_{NS}$ is negative (i.e., spin-down) for XSS J12270-4859 and IGR J17379-3747 as for PSR J1023+0038, which means that the rotational energy of the NS is transferred onto the ADAF. If our explanation for the formation of the observed millisecond X-ray pulsations for XSS J12270-4859 and IGR J17379-3747 at the X-ray luminosity of $5 \times 10^{33}$ erg s$^{-1}$ are correct, the predicted change rate of the NS spin frequency $\dot{\nu}_{NS}$ is at the level of $\sim -10^{15}$ Hz s$^{-1}$, which we expect can be tested by the observations in the future. Further, if the change rate of the NS spin frequency $\dot{\nu}_{NS}$ predicted by our model of ADAF accretion can be confirmed in the future, which actually in turn supports our idea in the present paper that NSs with an ADAF accretion is radiatively inefficient despite the existence of the hard surface. Finally, we would like to address that the estimation of $\dot{\nu}_{NS}$ in this paper is based on our model of ADAF accretion around a weakly magnetized NSs, which will make the estimated value of $\dot{\nu}_{NS}$ uncertain as applied to the AMXP cases. So the consideration of the effect of the magnetic field (~ $10^4$ G) on the value of $\dot{\nu}_{NS}$ in AMXPs is still very necessary in the future, which however exceeds the scope in the present paper.

5 CONCLUSIONS

Following the paper of Qiao & Liu (2020a) and Qiao & Liu (2020b) for the constraints to the value of $f_0$ controlling the feedback between the surface of the NS and the ADAF, in this paper, we investigate the radiative efficiency of NSs with an ADAF accretion within the framework of the self-similar solution of the ADAF by taking two typically suggested values of $f_0$, i.e., $f_0 = 0.01$ and $f_0 = 0.01$ respectively. Then, we show that the radiative efficiency of NSs with an ADAF accretion is significantly lower than that of $\epsilon \sim \frac{M \nu}{4 \pi M^2 c^2} \sim 0.2$. Specifically, the radiative efficiency of our model of NSs with an ADAF accretion for $f_0 = 0.01$ is roughly one order of magnitude lower than that of $\epsilon \sim \frac{M \nu}{4 \pi M^2 c^2} \sim 0.2$, and the radiative efficiency of our model of NSs with an ADAF accretion for $f_0 = 0.01$ is roughly two orders of magnitude lower than that of $\epsilon \sim \frac{M \nu}{4 \pi M^2 c^2} \sim 0.2$. As a result, we propose that the lower radiative efficiency of our model of ADAF accretion probably can be applied to explain the observed millisecond X-ray pulsation in some NS-LMXBs (such as PSR J1023+0038, XSS J12270-4859 and IGR J17379-3747) at the X-ray luminosity (between 0.5 and 10 keV) of a few times of $10^{33}$ erg s$^{-1}$, since at this X-ray luminosity the real $M$ calculated with our model of ADAF accretion for taking an appropriate value of $f_0$, such as $f_0 = 0.01$, can be more than two orders of magnitude higher than that of calculated with the formula of $M = L_{X} \frac{M \nu}{4 \pi M^2 c^2}$ to ensure a fraction of the matter in the ADAF to be channelled onto the surface of the NS forming the X-ray pulsation.

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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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