COMPUTATIONS ON NONDETERMINISTIC CELLULAR AUTOMATA

Y.I. Ozhigov

Department of Mathematics, Moscow State Technological University "Stankin", Vadkovsky per. 3a, 101472, Moscow, Russia

e-mail: y@oz.msk.ru

Abstract

The work is concerned with the trade-offs between the dimension and the time and space complexity of computations on nondeterministic cellular automata. We assume that the space complexity is the diameter of area in space involved in computation.

It is proved, that

1). Every NCA $A$ of dimension $r$, computing a predicate $P$ with time complexity $T(n)$ and space complexity $S(n)$ can be simulated by $r$-dimensional NCA with time and space complexity $O(T^{1/r}S^{1/r})$ and by $r + 1$-dimensional NCA with time and space complexity $O(T^{1/2} + S)$, where $T$ and $S$ are functions, constructible in time.

2) For any predicate $P$ and integer $r > 1$ if $A$ is a fastest $r$-dimensional NCA computing $P$ with time complexity $T(n)$ and space complexity $S(n)$, then $T = O(S)$.

3). If $T_{r,P}$ is time complexity of a fastest $r$-dimensional NCA computing predicate $P$ then

$$T_{r+1,P} = O((T_{r,P})^{1-r/(r+1)^2}),$$

$$T_{r-1,P} = O((T_{r,P})^{1+2/r}).$$

Similar problems for deterministic CA are discussed.
1. Introduction

It is well known that nondeterministic computations are more powerful than deterministic ones. The interrelation between deterministic and nondeterministic time complexity of computations was established by S. Cook in the work [2] where he showed the existence of NP-complete problems. However, it is still unknown can nondeterministic computations be fulfilled on physical devices or not. In this paper we show how the complexity of computations on a nondeterministic device depend on its dimension. Note that the similar problem for deterministic computers is open (look at the section 6).

Cellular automata (CA) provide a convenient framework for studies on this problem. A cellular automaton is a dynamical system with local interactions operating in discrete space and time, and simultaneously CA may be used as a general model of computational device. CA were introduced by S. Ulam in the work [7] and J. von Neumann in the work [5] and since then various problems pertaining to CA were treated in a great many works (look, for example, at [1], [3], [6], [9], [11], [12]).

Let \( r \geq 1 \) be an integer, \( \mathbb{Z}^r \) be the space, \( \omega \) be a finite alphabet for possible states of any cell \( \bar{i} \in \mathbb{Z}^r \). Cellular automaton of dimension \( r \) in alphabet \( \omega \) is a function of the form: \( A : \omega^{2r+1} \rightarrow \omega \).

A cellular automaton determines the special class of evolutions in \( \mathbb{Z}^r \) so that the states of all cells \( \bar{i} \in \mathbb{Z}^r \) evolve synchronously in discrete time steps according to the states of their nearest neighbours.

It must be mentioned, that a more general approach can be of interest for applications, where the function \( A \) depends on \( \bar{i} \) or \( t \). It is not our intention to regard such possibilities here.

A configuration is an ensemble of states of all cells at some instant of time. It is apparent that an evolution of CA is uniquely determined by the initial configuration.

If we consider a multifunction instead of \( A \), we obtain the definition of a nondeterministic cellular automaton (NCA). Generally speaking, evolutions of NCA are not uniquely determined by initial configurations. A behaviour of NCA may be described by a state transition network (look at [4]). It is a graph, each of whose nodes represents some configuration. Directed arcs join the nodes to represent the transition between configurations. All nodes have out-degree one iff the cellular automaton is deterministic. Moreover, if the cellular automaton at hand is in fact nondeterministic and we consider the configurations in unlimited space \( \mathbb{Z}^r \), then out-degrees of some nodes in the state transition network will be infinitely large.

The difference between one and high dimensional CA has emerged from the solution of the predecessor existence problem (PEP) for CA. It is the problem of existence of a predecessor for the given configuration. S. Wolfram showed in article [8] that PEP is decidable for one dimensional CA, and T. Taku in article [10] showed that PEP is undecidable for some CA of dimension \( r \) where \( r = 2, 3, \ldots \).

The computational equivalence of CA and Turing Machines is a well known fact (look at [2], [11]). The time complexity \( T(n) \) and the space complexity \( S(n) \) can be defined routinely for any CA \( A \).

This brings up the question: given an arbitrary predicate \( P \), how does the minimal complexity of \( r \) dimensional CA, computing \( P \), depend on \( r \)?

More precisely, if some predicate \( P \) is computable on CA of dimension \( r \) with time complexity \( T(n) \geq O(n) \) is it possible to compute \( P \) substantially faster on CA of dimension \( r' > r \)?
This is called TCD-problem. This problem is open for CA.

A different situation arises with TCD-problem for NCA. For example, it is found that if some predicate $P$ can be computed on NCA with time complexity $T(n) = O(n^\alpha)$ and space complexity $O(n^{\alpha/2})$ then the increase of dimension by one unit allows to compute $P$ in time $O(n^{\alpha/2})$. Moreover, given $\beta > 0$, the time complexity $O(n^\beta)$ can be attained for the computation of such predicate if we increase the dimension of NCA to a suitable value. A similar result takes place also for faster increasing functions $T(n)$. It means that multi-dimensional NCA will become the faster instrument for computation as the dimension increases.

We proceed with the exact definitions. All constants are assumed to depend on the dimension $r$.

2. The main definitions and results

Let $\omega = \{c_0, \ldots, c_k\}$ be an alphabet for possible states of cells. Let $t$ takes the values from the set $\mathbb{N} = \{0, 1, \ldots\}$.

An evolution in $\mathbb{Z}^r$ is a function of the form $a : \mathbb{N} \times \mathbb{Z}^r \rightarrow \omega$. A configuration is a function of the form $a^{(t)} : \mathbb{Z}^r \rightarrow \omega$. Any evolution $a$ may be displayed as a sequence

$$a^{(0)}, a^{(1)}, \ldots$$

of configurations at the instants of time $t = 0, 1, \ldots$, where $a^{(t)}(\vec{i}) = a(t, \vec{i})$. $j$th component of $i \in \mathbb{Z}^r$ will be denoted by $i_j$. If $\vec{i} = (i_1, i_2, \ldots, i_r) \in \mathbb{Z}^r$, we shall write $a(t, i_1, \ldots, i_r)$ instead of $a(t, \vec{i})$.

The following notations are fixed for the cells $\vec{i}(j)$ comprising the neighborhood of the cell $\vec{i}$:

$$\vec{i}(0) = \vec{i},$$
$$\vec{i}(1) = (i_1 - 1, i_2, \ldots, i_r),$$
$$\vec{i}(2) = (i_1 + 1, i_2, \ldots, i_r),$$
$$\vec{i}(3) = (i_1, i_2 - 1, i_3, \ldots, i_r),$$
$$\vec{i}(4) = (i_1, i_2 + 1, i_3, \ldots, i_r),$$
$$\cdots$$
$$\vec{i}(2r) = (i_1, i_2, \ldots, i_r + 1).$$

We put : $a_j(t, \vec{i}) = a(t, \vec{i}(j))$.

We’ll consider only such evolutions $a$ that $\exists C : \forall \vec{i} : \|\vec{i}\| > C \ a(t, \vec{i}) = c_0$, therefore all configurations $a^{(t)}$ will be finite objects.

Let $l_r$ be a fixed computable one-to-one mapping $l_r : \mathbb{N} \rightarrow \mathbb{Z}^r$ such that $\|l_r(n)\| = O(n^{1/r})$. This function represents an embedding of one dimensional space into $r$ dimensional with the least norm $\|l_r(n)\|$.

Given an alphabet $\sigma$, the set of all words over $\sigma$ is denoted by $\sigma^*$. If $B = c_{j_1}c_{j_2}\ldots c_{j_s} \in \omega^*$, then the initial configuration corresponding to the word $B$ is defined by

$$a^{(0)}(\vec{i}) = \begin{cases} c_{j_k} & \text{if } 0 \leq l^{-1}_r(\vec{i}) = k \leq s, \\ c_0 & \text{otherwise}, \end{cases}$$

we denote this configuration by $a_B^{(0)}$. 
A nondeterministic cellular automaton of $r$ dimensions is a function of the form

$$A : \omega \times \omega \times \cdots \times \omega \rightarrow 2^\omega.$$  

Let $\bar{b} \in \omega^{2r+1}$, $c \in A(\bar{b})$. Any word of the form $\bar{b} \rightarrow c$ is called a command of this automaton. This command is called trivial if $A(\bar{b}) = \{c\}$ and $\bar{b}$ has the form $(c, u_1, \ldots, u_{2r})$. A set $G$ of commands of $A$ which contains all nontrivial commands is called a program of $A$. The behaviour of $A$ is defined by its program.

If $\forall b \in \omega^{2r+1}$ and $A(\bar{b})$ consists of exactly one element, then in fact $A$ has the form $\omega^{2r+1} \rightarrow \omega$, and we obtain the definition of a deterministic cellular automaton.

We assume that $A(c_0, \ldots, c_0) = c_0$. This letter $c_0$ plays the role of blank, it is denoted by $0(\omega)$.

An evolution of NCA $A$ is a sequence $\bar{a}$ of the form (1) where $\forall t = 0, 1, \ldots$, $\forall \bar{i} \in Z^r$

$$a(t + 1, \bar{i}) \in A(a_0(t, \bar{i}), a_1(t, \bar{i}), \ldots a_{2r}(t, \bar{i})).$$

It is obvious that in deterministic case $a^{(t)}$ depends on $A, t, a^{(0)}$, and in nondeterministic case, in addition, on the choice of elements (2).

The set of all values of a function $f$ is denoted by $\text{Im } f$.

Let the alphabet $\omega$ be divided into two nonintersecting parts: $\omega = \omega' \cup \omega''$, where $\omega'$ is the set of main letters, $\omega''$ is the set of auxiliary letters, and let $E \subset \omega''$ be the set of end letters, where $c_{k-1}, c_k \in E$. We denote $\omega''$ by $\Sigma$, $c_k$ by $\text{succ}(A)$, $E$ by $E(A)$. Let for the evolution (1) of NCA $A$ $\text{Im } a^{(0)} \subseteq \omega' \cup \{c_0\}$. Let $\tau(\bar{a})$ be the least value of $t$ such that there exists one and only one letter $c \in E \cap \text{Im } a^{(t)}$. This letter $c$ is denoted by $\text{res}(\bar{a}, A)$ and is called the result of the operation of $A$ on the initial configuration $a^{(0)}$ in evolution $\bar{a}$. A configuration $\bar{a}_{\tau(\bar{a})}$ is called a resulting configuration for $a^{(0)}$.

In general terms, the result of the operation of $A$ is defined uniquely only in the deterministic case. The set of all results in evolutions which begin with $a^{(0)}$ is denoted by $A[a^{(0)}]$.

A predicate $P$ on the set $\Sigma$ is an arbitrary subset of $\Sigma$.

NCA $A$ computes a predicate $P$ iff $\forall B \in \Sigma$

$$\left\{ \begin{array}{ll} \text{succ}(A) \in A[a_B^{(0)}], & \text{if } B \in P; \\
\text{succ}(A) \notin A[a_B^{(0)}], & \text{if } B \notin P. 
\end{array} \right.$$  

It’s obvious that such a predicate $P$ is defined uniquely for $A$ if it exists. We denote this predicate by $P_A$. A cell $\bar{i} \in Z^r$ is called accessible in evolution $\bar{a}$ iff $\exists t' \leq \tau(\bar{a}) : a(t', \bar{i}) \neq c_0$.

The diameter of the set of all accessible cells is denoted by $D(\bar{a})$.

Given $B \in P$, the least value of $\tau(\bar{a})$ from all evolutions $\bar{a}$, where $a^{(0)} = a_B^{(0)}$, $\text{res}(\bar{a}, A) = \text{succ}(A)$ is denoted by $\tau_A(B)$.

Let $D_A(B)$ denotes the least value of $D(\bar{a})$ from all evolutions $\bar{a}$ where $a^{(0)} = a_B^{(0)}$, $\text{res}(\bar{a}, A) = \text{succ}(A)$, $\tau(\bar{a}) = \tau_A(B)$. 

Y.I. OZHI GOV
The *time complexity* of NCA $\mathcal{A}$ is the function $T_\mathcal{A} : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$T_\mathcal{A}(n) = \max \{ \tau_\mathcal{A}(B) \mid B \in P_\mathcal{A}, |B| \leq n \},$$

where $|B|$ denotes the length of $B$.

The *space complexity* of $\mathcal{A}$ is the function $S_\mathcal{A} : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$S_\mathcal{A}(n) = \max \{ D_\mathcal{A}(B) \mid B \in P_\mathcal{A}, |B| \leq n \},$$

Without loss of generality we may anticipate that $\mathcal{A}(c_1, \ldots) = c_i$ for all $c_i \in E$.

Functions $T_\mathcal{A}(n)$ and $S_\mathcal{A}(n)$ can be very complicated, and it’s convenient to use their best upper approximations $T_\mathcal{A}$, $S_\mathcal{A} \in \mathcal{K}$: $T_\mathcal{A}(n) = O(T_\mathcal{A}(n))$, $S_\mathcal{A}(n) = O(S_\mathcal{A}(n))$ instead of them, where $\mathcal{K}$ is the class of functions constructible in time (see below). Given the space $\mathbb{Z}^r$, a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is called *constructible in time* on $r$-dimensional NCA, if there exists a constant $c$ and NCA $\mathcal{A}$ with time complexity $T(n) = c(f(n) + n^{1/r})$ such that for every word $B \in \omega^*$, $|B| = n$ there exists the single resulting configuration for $a_B(0)$ which has the form

$$a_r(\bar{i}) = \begin{cases} c_1, & \text{if } \bar{i} \in \{1, 2, \ldots, T(n)/2\}^r, \\ c_0, & \text{in the opposite case.} \end{cases}$$

It means that $r$-dimensional cube of side $T(n)/2$ can be isolated in time $T(n)$, where $O(n^{1/r})$ is the size of necessary domain for input word $B$.

For example, the constructibility in time for $n^4$ and $2^{2n}$ is in fact proved in the section 4 (group G1), for the functions $n^\alpha, q^{\alpha n}$, $q, \alpha \in \mathbb{Q}$ and for their combinations with additions, multiplications and superpositions the constructibility in time may be proved along similar lines.

A pair of functions $(T_\mathcal{A}, S_\mathcal{A})$ is called a complexity of NCA $\mathcal{A}$. Thus in what follows $T_\mathcal{A}$, $S_\mathcal{A}$ (or, simply $T$, $S$ with or without indices) will be constructible in time.

The class of predicates $P$, computable on NCA of dimension $r$ with complexity $(T, S)$ is denoted by $\text{NC}(r, T, S)$.

We’ll write $T_1 < O(T)$ instead of the following: $\forall C > 0 \exists N \forall n \geq N T_1(n) \leq CT(n)$.

$r$-dimensional NCA, computing predicate $P$ with complexity $(T, S)$ is called a *fastest* NCA if $P$ can not be computed on $r$-dimensional NCA in time $T_1 < O(T)$.

Let $T_{r,P}$ denote the time complexity of a fastest $r$-dimensional NCA computing predicate $P$.

Here are the main results of this paper.

**Theorem 1.**

$$\text{NC}(r, T, S) \subseteq \text{NC}(r + 1, \sqrt{T} + S, \sqrt{T} + S).$$

**Theorem 2.**

$$\text{NC}(r, T, S) \subseteq \text{NC}(r, T_1, T_1),$$

where $T_1 = T^{1/r} S^{1/r}$

**Theorem 3.** Let $\mathcal{A}$ be a fastest NCA of dimension $r$. Then

$$(3) \quad T_\mathcal{A}(n) = O(S_\mathcal{A}(n)).$$

**Theorem 4.**

1. $T_{r-1,P} = O((T_{r,P})^{1+2/r})$,
2. $T_{r+1,P} = O((T_{r,P})^{1-r/(r+1)^2})$. 
Now we shall give the outline of the following sections. All these results are based on two main methods of speeding up computations: The method of direct simulation in $r + 1$ dimensional space (section 3, Proof of Theorem 1) and the method of optimization of NCA in the same space (section 4, Proof of Theorem 2). Theorem 3 will be simply derived from Theorem 2. Point 1) of Theorem 4 will be proved by the method of simulation in $r - 1$ dimensional space (method of evolvents, section 5). Point 2) of Theorem 4 will be proved in two steps: reduction of space complexity in $r + 1$ space and the following optimization.

Note that the both two method of speeding up require nondeterminism.

3. Simulation in $r + 1$-dimensional space. Direct method

Proof of Theorem 1.

Let $P = P_{\mathcal{A}} \in \text{NC}(r, T, S)$, $\mathcal{A}$ be a cellular automaton of dimension $r$ with alphabet $\omega$ and complexity $(T, S)$, $0(\omega) = c_0$.

In this section we’ll present the direct method of speeding up: we shall construct the nondeterministic cellular automaton NCA1 of dimension $r + 1$, which simulates $\mathcal{A}$ in time $O(T^{1/2} + S)$.

The rough idea is that we expand the alphabet of the cellular automaton $\mathcal{A}$ at hand and use $r + 1$st dimension to code $\mathcal{H}$ state transitions of $\mathcal{A}$ into one big $r + 1$ dimensional state transition of the new automaton NCA1 simulating $\mathcal{A}$, where $\mathcal{H} = O(\sqrt{T} + S)$. The single obstacle which will remain is that the initial configuration for NCA1 is not $a_0^{(0)}$, but the ascending map, corresponding to input word $B$ (the definition is in the next section). This obstacle will be overcome in the last part of this section.

Definition A port is a list $p$ of the form:

\[(4) \quad p = (\text{Mark}(p), \text{con}(p), \text{env}_1(p), \text{env}_2(p), \ldots, \text{env}_{2r+2}(p)),\]

where $A, B$ are the special new letters, $\text{Mark}(p) \in \{A, D\}$, the other members of list (4) are arbitrary letters from $\omega$, and the functions $\text{con}(p), \text{env}_j(p), j = 1, \ldots, 2r + 2$ are defined by equality (4).

Alphabet of NCA1

Let $\omega_0$ be the set of all ports with the exception of $(D, c_0, \ldots, c_0)$. Then the alphabet of NCA1 is $\omega_1 = \omega_0 \cup \{\emptyset, b\}$, where $\emptyset, b$ are new letters, and let $0(\omega_1) = (A, c_0, \ldots, c_0)$.

Commands of NCA1

Since the dimension of NCA1 is $r + 1$, the commands of it have the following form:

\[(p_0, p_1, \ldots, p_{2r+2}) \rightarrow \{p\}.\]

Let all $p_q \in \omega_1$, $q = 0, 1, \ldots, 2r + 2$. The set $\{p\}$ consists of all elements $p \in \omega_1$ to be described below. Let $s^+ = \{0, 2, 3, 4, \ldots, 2r + 2\}, s^- = \{0, 1, 3, 4, 5, \ldots, 2r + 2\}, k(j) = \begin{cases} j + 1, & \text{for } j \text{ odd,} \\ j - 1, & \text{for } j \text{ even.} \end{cases}$

We’ll consider separately five different cases.
Case 1. Let the following conditions be satisfied:

\[
\forall j \in s^+ \ p_j \in \omega_0, \ \text{Mark}(p_j) = A,
\]

\[
\forall j \in s^- - \{0\} \ \text{con}(p_j) = \text{env}_j(p_0), \ \text{con}(p_0) = \text{env}_k(p_j),
\]

where \( k = k(j) \) and

\[
\text{con}(p_2) \in \mathcal{A}(\text{con}(p_0), \text{con}(p_3), \ldots, \text{con}(p_{2r+2})).
\]

In this case \( p \) is such a port that \( \text{Mark}(p) = D \), or \( p \) is \( p_0 \).

Case 2. Let the following conditions be satisfied:

\[
\forall j \in s^- \ p_j \in \omega_0, \ \text{Mark}(p_j) = D \text{ or } p_j = 0(\omega_1),
\]

\[
\forall j \in s^- - \{0\} \ \text{con}(p_j) = \text{env}_j(p_0), \ \text{con}(p_0) = \text{env}_k(p_j),
\]

where \( k = k(j) \) and

\[
\text{con}(p_1) \in \mathcal{A}(\text{con}(p_0), \text{con}(p_3), \ldots, \text{con}(p_{2r+2})).
\]

In this case \( p \) is such a port that \( \text{Mark}(p) = A \), or \( p \) is \( p_0 \).

Case 3. Let \( p_1 = b \), \( \text{Mark}(p_0) = D \) or \( p_2 = b \).

Then \( p \) is \( p_0 \) or \( p \) is obtained from \( p_0 \) by the following redefinition: we put

\[
\text{Mark}(p) = \begin{cases} 
A, & \text{if } \text{Mark}(p_0) = D, \\
D, & \text{if } \text{Mark}(p_0) = A.
\end{cases}
\]

Case 4. If \( \exists j \in s^+ \cap s^- : p_j = b \), we put \( p = b \).

Case 5. In all other cases we put \( p = \emptyset \).

Note that NCA1 is nondeterministic even though \( \mathcal{A} \) may be a CA of deterministic type, because in the cases 1 and 2 the choice of \( p \) is not uniquely defined.

Let \( t \) be time step, \( H \) be some positive integer. We suppose that \( H \) is arbitrary till Lemma 3.

Some peculiar configurations of NCA1 are called \( H, t \)-maps. Maps will be of two sorts.

Definition. An Ascending map of high \( H \) at time step \( t \) (\( A, H, t \)-map ) is such a configuration \( a(t) \) for NCA1 that \( \forall i \in \mathbb{Z}^{r+1} \)

1. If \( i_1 \notin \{-1, H\} \), then \( a(t, i) \in \omega_0 \). If \( a(t, \tilde{i}) \neq 0(\omega_1) \), then \( 1 \leq i_1 \leq H \), and

\[
a(t,-1,i_2,\ldots,i_{r+1}) = a(t,H,i_2,\ldots,i_{r+1}) = b;
\]

2. If \( \tilde{i} : 0 \leq i_1 < H - 1 \), then

\[
\text{con}(a(t,i_1+1,i_2,\ldots,i_{r+1})) = \mathcal{A}(\text{con}(a_0(t,\tilde{i})), \text{con}(a_3(t,\tilde{i})), \ldots, \text{con}(a_{2r+2}(t,\tilde{i})));
\]

3. If

\[
s(j) = \begin{cases} 
2j, & \text{if } \epsilon = 1, \\
2j - 1, & \text{if } \epsilon = -1,
\end{cases}
\]

then \( \forall j : 1 \leq j \leq r+1 \exists \epsilon \in \{1,-1\} \ \text{con}(a(t, i_1, \ldots, i_j + \epsilon, \ldots, i_{r+1})) = \text{env}_k(a(t, \tilde{i})) \),

if the two parts of this equality exist.
Definition. A Descending map of high $H$ at time step $t$ $(D, H, t$-map) is such a configuration $a^{(t)}$ for NCA1, that $\forall \tilde{t} \in \mathbb{Z}^{r+1}$

1. See above.

2. If $0 < i_1 < H$, then

$$\text{con}(a(t, i_1 - 1, i_2, i_3, \ldots, i_{r+1})) = \mathcal{A}(\text{con}(a_0(t, \tilde{t})), \text{con}(a_3(t, \tilde{t})), \ldots, \text{con}(a_{2r+2}(t, \tilde{t})))$$

3. See above.

The following proposition relates evolutions of $\mathcal{A}$ to those of NCA1.

**Lemma 1.** $a^{(0)}, \ldots, a^{(r)}, \ldots$ is evolution of $\mathcal{A}$ iff there exists an evolution

(5)

of NCA1, where for every $\tau = 0, 1, \ldots$ the following condition $C_\tau$ is fulfilled:

$$C_\tau \Leftrightarrow \exists t = t(\tau) \in \mathbb{N} \exists q = q(\tau) : 0 < q \leq H - 1 \forall \tilde{t} \in \mathbb{Z}^r \text{ con}(a(t, q, i_1, i_2, \ldots, i_r)) = \alpha(\tau, \tilde{t}),$$

where for $t$ even: $\tau = t(H - 1) + q$, and $a^{(t)}$ is $A, H, t$-map, for $t$ odd: $\tau = (t + 1)(H - 1) - q$ and $a^{(t)}$ is $D, H, t$-map.

Proof.

1. Necessity. Let $\alpha$ be an evolution of $\mathcal{A}$. An evolution $a$ of NCA1 is called $\tau$-evolution if the condition $C_\tau$ is fulfilled. At first let us prove by induction on $\tau$ that for any $\tau$ there exists $\tau$-evolution. Basis: $\tau = 0$ follows from the definition of $A, H, t$-map. Step: follows from the definition of NCA1. \(\square\)

Now we introduce the order $\prec$ on $\mathbb{N} \times \{0, 1, \ldots, H - 1\}$ by the following: $(t_1, q_1) \prec (t_2, q_2)$ iff $t_1 < t_2$ or $(t_1 = t_2 \text{ - even and } q_1 < q_2)$ or $(t_1 = t_2 \text{ - odd and } q_2 < q_1)$. Note that if $\tau_1 < \tau_2$, then $(t(\tau_1), q(\tau_1)) \preceq (t(\tau_2), q(\tau_2))$. Consequently, in view of the definition of $\tau$-evolution, if $a$ is $\tau_1$-evolution, $d$ is $\tau_2$-evolution, then for every pair $(t, q) \preceq (t(\tau_1), q(\tau_1))$ we have $\forall \tilde{t} \in \mathbb{Z}^r \text{ a}(t, q, \tilde{t}) = d(t, q, \tilde{t})$. Thus we obtain that there exists the evolution $a$ of NCA1 which coincides with every $\tau$-evolution on all lists $(t, q, \tilde{t})$, where $(t, q) \preceq (t(\tau), q(\tau))$. Necessity is proved.

2. Sufficiency. Follows from the definition of NCA1. Lemma 1 is proved.

**Lemma 2.** If $a^{(0)}, a^{(1)}, \ldots, a^{(t)}, \ldots$ is evolution of NCA1 and $a^{(0)}$ is $H, 0$-map, then $a^{(t)}$ is $H, t$-map iff

(6)

$$\forall \tilde{t} \in \mathbb{Z}^{r+1} : 0 \leq i_1 - H - 1 \ a(t, \tilde{t}) \neq \emptyset.$$

Proof.

Induction on $t$. Basis: $t = 0$. Follows from the condition. Step follows from the definition of NCA1 and those of $H, t$-map. Lemma 2 is proved.

Let $[x]$ denote integral part of $x \in \mathbb{R}$.

**Lemma 3.** Let some predicate $P$ be computed on NCA $\mathcal{A}$ with complexity $(T, S)$, $|H - \sqrt{\frac{T(n)}{2}}| < \sqrt{T(n)}$, $B \in \Sigma$, $a^{(0)}$ is the initial configuration of $\mathcal{A}$, corresponding to $B$, $t = 2\sqrt{T(n)}$, $n = |B|$.

If $B \in P$ then for some evolution of NCA1 of the form (5) the condition (6) is fulfilled and

(7)

$$\exists \tilde{t} \in \mathbb{Z}^{r+1} : \text{con}(a(t, \tilde{t})) = \text{succ}(\mathcal{A})$$

and if $A \notin P$ then for any evolution of NCA1 of the form (5) with the condition (6) the following property takes place

(8)

$$\forall \tilde{t} : \text{con}(a(t, \tilde{t})) \neq \text{succ}(\mathcal{A}).$$
Proof.

Follows immediately from the definition of computation on NCA, Lemma 1 and Lemma 2. Lemma 3 is proved.

4. Auxiliary automata

To finish the proof of Theorem 1 we must construct two auxiliary NCA:

1. NCA2 – which begins to operate from $d^{(0)}$ – initial configuration, corresponding to $B \in \Sigma$ in $Z^{r+1}$, so that for some evolution $a$ from Lemma 1 and for some $t' \alpha^{(t')} = a^{(0)}$, $|H - \sqrt{T(n)}| < \sqrt{T(n)/2}$, $n = |B|$.

2. CA3 – deterministic CA, which begins to operate from arbitrary chosen $H$, $t$-map from (5) and has the result only if (6) is fulfilled, and in this case

\[
\begin{cases} 
\text{succ}(CA3) \in CA3[a^{(t)}], & \text{if (7)}, \\
\text{succ}(CA3) \notin CA3[a^{(t)}], & \text{if (8)}. 
\end{cases}
\]

If now we combine the sets of commands of NCA1, NCA2 and CA3, and put $\text{succ}(B) = \text{succ}(CA3)$, $E(B) = E(CA3)$, then in view of Lemma 3 the resulting NCA will satisfy the conditions of Theorem 1.

Now let us describe NCA2 and NCA3. Real work of NCA2 is such that at first the group G1 will operate, and after that, groups G2 and G3 will operate simultaneously.

NCA2.

The program contains 3 groups of commands:

G1: Realization of function $H = O(\sqrt{T_A(n)} + S_A(n))$.

G2: Isolation of area for modeling.

G3: Construction of $A - H - 0$-map.

If, for example, $T_A(n) = 2^{2n}$, $S_A(n) = n^4$, then G1 consists of all commands of the following forms:

\[
\begin{align*}
(c_i, \ldots, y, \tilde{c}_i) & \rightarrow \tilde{c}_i, & (c_0, c_0, \ldots, \tilde{c}_i) & \rightarrow d, \\
(\tilde{c}_i, \ldots) & \rightarrow c_i, & (c_i, \ldots, c_0) & \rightarrow c_i^+, \\
(c_i, \tilde{c}_i, \ldots) & \rightarrow \tilde{c}_i, & (c_i^+, \ldots) & \rightarrow c_i, \\
(c_i^+, \ldots, c_i) & \rightarrow c_i^+, & (c_0, \ldots, \tilde{c}_i, c_i^+) & \rightarrow c_i^+, \\
(d, c_0, \ldots, c_i) & \rightarrow d', & (y, \ldots, c_i^+) & \rightarrow e,
\end{align*}
\]

where $i, j, s$ take all values from $\{1, 2, \ldots\}$, $? \ldots$ means an arbitrary letter, $y \in \{c_0, d, d'\}$, $\tilde{c}_i, c_i^+, c_i^+$ are new different special letters for any $c_i \in \omega$.

Group G2 consists of all commands of the following forms:

\[
\begin{align*}
(c_0, d', \ldots) & \rightarrow d, & (e, \ldots) & \rightarrow c_0, \\
(d, d', \ldots) & \rightarrow d', & (z_1, c_0, z_2, \ldots c_0) & \rightarrow c_0, \\
(d', d, \ldots) & \rightarrow d, & (z_1, c_0, c_0, \ldots c_0) & \rightarrow b, \\
(\ldots, b, \ldots) & \rightarrow b, & (c_0, c_0, e, \ldots) & \rightarrow b,
\end{align*}
\]

where $z_1, z_2 \in \{d, d'\}$.
Group G3 consists of all commands of the following forms:

\[(c_j, b, \ldots) \rightarrow h_j,\]
\[(c_0, h_i, \ldots) \rightarrow h_s,\]
\[(h_i, \ldots) \rightarrow \Gamma_i,\]
\[(h_s, \ldots, \Gamma_j, \ldots) \rightarrow \emptyset,\]
\[(\Gamma_j, \ldots h_s, \ldots) \rightarrow \emptyset,\]

where \(i, j, s\) take all values from \(\{1, 2, \ldots\}\), \(h_i\) are special new letters, corresponding to each \(i\), and \(\Gamma_i\) takes values from the set of all ports \(p\) such that \(\text{con}(p) = c_i\).

For \(T_A(n) = n^4, S_A(n) < O(\sqrt{T})\) the group G1 consists of all commands of the following forms:

\[(c_0, \ldots, c_i) \rightarrow d,\]
\[(c_0, \ldots, c_i, c_j^\prime, c_j^\prime) \rightarrow c_i,\]
\[(c_0, \ldots, c_i, c_j^\prime, c_j) \rightarrow c_i,\]
\[(c_i, \ldots, c_0, c_j^\prime) \rightarrow c_i^\prime,\]
\[(c_i, \ldots, c_0, c_j, c_j^\prime) \rightarrow c_i,\]
\[(c_i, \ldots, c_0) \rightarrow c_i,\]
\[(c_i, \ldots, c_0, c_j^\prime) \rightarrow c_i^\prime,\]
\[(c_i, \ldots, c_0, c_j) \rightarrow c_i,\]
\[(c_i, \ldots, c_0, c_j^\prime) \rightarrow c_i,\]
\[(d, \ldots, c_i) \rightarrow d',\]

where \(z \in \{d, d'\}\). NCA2 is described.

**CA3.**

Put \(E(CA3) = \{c_k', c_{k-1}'\}\).

\[(\Gamma_{e_p}, \neq \emptyset, b, \ldots) \rightarrow \{\rho^0_p\},\]
\[(\rho^i_p, \ldots) \rightarrow \{\rho^{i+1}_p\}, i = 0, 1, \ldots, 2r, p \in \{k, k - 1\},\]
\[(\neq \emptyset, \neq \emptyset, \rho^{2r+1}_p, \ldots) \rightarrow \{\rho^0_p\},\]
\[(b, \rho^i_p, \ldots) \rightarrow c'_p,\]
\[(\ldots, \emptyset, \ldots) \rightarrow \emptyset,\]

where every letter of NCA1, NCA2 may occur in "\ldots".

Theorem 1 is proved.

The general form of "successful" operation of resulting cellular automaton is shown on Figure 1.

The technique evolved allows to derive the following amplification of Theorem 1.
Corollary 1. If \( r_2 > r_1 \), then
\[
\text{NC}(r_1, T, S) \subseteq \text{NC}(r_2, \sqrt{T} + S, (S^{r_1}T^{1/2})^{1/r_2} + S).
\]

Proof
Let \( A \) be a nondeterministic cellular automaton of dimension \( r_1 \) computing predicate \( P_A \) with complexity \((T, S), S(n) < O(T(n))\). Nondeterministic cellular automaton \( B \) of dimension \( r_2 \) simulating \( A \) with complexity \((T_1, S_1), T_1 = O(\sqrt{T} + S), S_1 = O((S^{r_1}T^{1/2})^{1/r_2} + S)\) can be constructed by a modification of method from the proof of Theorem 1.

The area for simulation can be organized so that its diameter will be
\[
O((S^{r_1}T^{1/2})^{1/r_2} + S)
\].

For example, for \( r_1 = 1, r_2 = 2 \) the area from the Figure 1 can be constructed as a spiral so that the operation of \( B \) will have the form, presented on the Figure 2. Here the work of \( B \) in straight strips is the same as in the proof of Theorem 1, and in corner squares \( B \) only verifies the coincidence of the words written on two sides at each time step, like on Figure 3. It is not difficult to understand how \( B \) must be arranged, and I omit awkward details. □

Corollary 2. If \( \ln T/\ln S = O(1) \), then for all \( r = 1, 2, \ldots \)
\[
\text{NC}(r, T, S) \subseteq \text{NC}(\lfloor \ln T/\ln S \rfloor + \lfloor \log_2(\ln T/\ln S) \rfloor + 2 + r, S, S).
\]

Proof
Lemma 4. Let \( A \) be NCA, computing \( P_A \) with complexity \((T, S), S < O(T)\). For every \( k = 0, 1, \ldots \) there exists a sequence
\[
A_0, A_1, A_2, \ldots, A_k,
\]
where \( A = A_0 \), for all \( i = 0, 1, \ldots \). \( A_i \) is NCA of dimension \( r_i \) and complexity \((T_i, S_i)\), computing \( P_A \), where for \( i > 0 \)
\[
\begin{align*}
    r_i &\leq r_{i-1} + \lfloor \ln T_i/2 \ln S \rfloor + 1, \\
    T_i &= O(\sqrt{T_{i-1}} + S), \\
    S_i &= O(S).
\end{align*}
\]

Proof
Induction on \( k \). Basis is evident. Step. Let \( k > 0 \). Applying the inductive hypothesis we obtain the sequence \( A_0, A_1, A_2, \ldots, A_{k-1} \) with the conditions (9). When \( T_{k-1} = O(S_{k-1}) \), we put \( A_k = A_{k-1} \). When \( T_{k-1} > O(S_{k-1}) \), Corollary 1 with \( T_{k-1} \) playing the role of \( T \) yields the required NCA \( A_k \). Lemma 4 is proved.

Let \( h \) be the least number such that \( T_h = O(S) \). In view of \( T_i = O(\sqrt{T_{i-1}} + S) \) we conclude that \( h \leq \lfloor \log_2(\ln T/\ln S) \rfloor + 1 \). Then it follows from (9) that
\[
r_h \leq r + h + 1 + L/2 + L/4 + \cdots + L/2^h \leq r + h + 1 + L,
\]
where \( L = \lfloor \ln T/\ln S \rfloor \). Corollary 2 is proved.
4. The optimization of NCA. Fastest NCA

Proof of Theorem 2

Proposition 1. Given a function $\alpha(n) : \mathbb{N} \rightarrow \mathbb{N}$, constructible in time, every $r$-dimensional NCA $A$ with complexity $(T, S)$ can be simulated in $\mathbb{Z}^r$ with complexity $(C_1T/\alpha + C_2S\alpha^{1/r}, C_3S\alpha^{1/r})$, $C_1, C_2, C_3$ depend on $r$.

Proof.

As above, we shall consider the case $r = 1$ in more detail. NCA $A'$, simulating $A$ has the form

$$A' = A_1 * A_2,$$

where $A_1$ constructs the area for simulating and $A_2$ simulates $A$.

Given an input word $B$, the area for simulating has a form

$$(10) \ldots c_0bB^k(b0^p)^\alpha b_1c_0 \ldots ,$$

where $p = k + |S|$. The configuration (10) can be simply constructed in a time $O(\alpha S)$ by $A_1$. The configuration (10) is an input one for $A_2$. We suppose that $T/\alpha = q(n) \in \mathbb{N}$ and let an evolution $a^{(1)}, a^{(2)}, \ldots, a^{(T)}$ of $A$ be divided into $\alpha$ sequential segments:

$$\Delta_1 : a^{(1)}, \ldots, a^{(q+1)},$$

$$\Delta_2 : a^{(q+1)}, \ldots, a^{(2q+1)},$$

$$\ldots$$

$$\Delta_\alpha : a^{(T-q+1)}, \ldots, a^{(T)},$$

such that input configuration of $\Delta_{j+1}$: $in(j+1)$ and output configuration of $\Delta_j$ : $out(j)$ are identical, $j = 1, \ldots, \alpha - 1$.

$A_2$ simultaneously simulates all possible segments $\Delta_1, \Delta_2, \ldots, \Delta_\alpha$ at the sites of sequential occurrences of the words: $b0^k, b0^p, b0^p, \ldots$ in (10). For every $\Delta_j$, $A_2$ stores $in(j)$ and checks the equality

$$(11) \hspace{1cm} in(j + 1) = out(j)$$

for all $j = 1, \ldots, \alpha - 1$. $A_2$ places a special mark of "flow" in a cell where the violating of equality (11) is detected. At last the letter $b_1$ moves to left through all domains where the equality (11) has been verified and if $b_1$ has not met a mark of "flow" before $c_0$, then $A_2$ achieves "success". All these actions can be fulfilled simultaneously because $A_2$ is nondeterministic and we obtain that $A$ achieves "success" in time $O(T)$ iff $A_2$ achieves "success" in time $O(T/\alpha)$ beginning with (10).
A program of $A_2$ has the following form:

$$(c, ?, ?) \rightarrow (c, c),$$

$$(0, ?, ?) \rightarrow (x, x),$$

$$((c_1, c_2), (e_1, e_2), (g_1, g_2)) \rightarrow (c_1, A(c_2, e_2, g_2)),$$

$$(x, y, ?, ?) \rightarrow (x^+, y^+),$$

$$(x^+, y^+), ? \rightarrow (x^+, 0),$$

$$(b, (x^+, y^+), ?) \rightarrow y^+,$$

$$(x^+, 0), y^+, ?) \rightarrow (x^+, y^-),$$

$$(y^+, ?, ?) \rightarrow b,$$

$$((x^+, 0), (y^+, z^+, ?)) \rightarrow (x^+, z^+),$$

$$((x^+, y^+), ?, (y^+, 0)) \rightarrow (x^+, 0),$$

$$((x^+_1, y^-_1), (x^+_2, y^-_2), (x^+_3, y^-_3)) \rightarrow h,$$

where $h = f$, if $x_1 \neq y_1$, or $x_2 \neq y_2$, or $x_3 \neq y_3$,

else $h = s'$,

$$((x^+, y^+), u, v) \rightarrow f$$

where $(z, w) \in \{u, v\},$

$$(x^+, y^-), (y^-), (x^+, y^-), (x^+, y^-) \rightarrow s,$$

$$(b, s', c_0) \rightarrow s,$$

$$(x^+, y^-) \text{ or } b \text{ or } s' \text{ or } s'', ?, s) \rightarrow s,$$

$$(c, c, x, ?) \rightarrow f,$$

$$(c, c, ?, x) \rightarrow f,$$

$$(s, c_0, ?) \rightarrow !,$$

where succ($A_2$) = !, $f$ denotes "flow". The case $r = 1$ is considered. In the cases $r > 1$ the input configuration $D_0$ for $A_2$ consists of $\alpha$ copies of cube $\{1, \ldots, S(n)\}^r$, disposed sequentially along a spiral, so that the size of $D_0$ does not exceed $O(S\alpha^{1/r})$.

Evident changes must be done in the definition of $A_2$. Proposition 1 is proved. Let us turn to the proof of Theorem 2.

Proposition 1 makes it possible to achieve the time complexity

$$T(\alpha) = C_1 \frac{T}{\alpha} + C_2 \alpha^{1/r}. $$

With the aim of finding the minimum of $T(\alpha)$ we consider the equation:

$$T'(\alpha) = -C_1 \frac{T}{\alpha^2} + C_2 \alpha^{2/r - 1} = 0$$

which yields

(12) $$\alpha_{\text{min}} = C(T/S)^{\frac{r}{r+1}}$$

for some constant $C = C(r)$. The function (12) is constructible in time as $T$ and $S$. Taking this value of $\alpha$ for $A'$ from Proposition 1, we obtain NCA $A'$ simulating $A$ with time and space complexity $O(T^{r/r+1} S^{r/r+1})$. This proves Theorem 2.
Proof of Theorem 3.

The assumption that \( T > O(S) \) for some fastest NCA contradicts to Theorem 2. Theorem 3 is proved.

 Hence, while on the subject of fastest NCA we may talk only about their (time) complexity \( T \), because in view of Theorem 3 \( S = O(T) \).

5. Complex method of simulation. Method of evolvents

Proof of Theorem 4.

Proposition 2. If \( r > 1 \), then \( \text{NC}(r, T(n), S(n)) \subseteq \text{NC}(r−1, T(n)S(n), S^\omega(n)) \).

Proof.

Given \( r \)-dimensional NCA \( A_r \) with complexity \( (T, S) \), computing predicate \( P \), we shall define NCA \( A_{r−1} \) of dimension \( r−1 \) with complexity \( (T_{r−1}, S_{r−1}) \), such that each time step in the evolution of \( A \) will be simulated by \( A_{r−1} \) in \( O(S(n)) \) time steps.

The support of configuration \( a^{(t)} \) is the set \( S_r^{(t)} = \{ i ∈ \mathbb{Z}^r | a(t, i) ≠ c_0 \} \). Without loss of generality we can assume that all supports \( S_r^{(t)} \) for \( A_r \) are \( r \)-dimensional cube \( B^r \), \( B = \{ 1, \ldots, H \} \), where \( H = S(|B|) \) depends on the input word \( B \). All supports \( S_{r−1}^{(t)} \) of corresponding evolution of \( A_{r−1} \) are contained in \( r−1 \)-dimensional cube \( B_1^{r−1} \), \( B_1 = \{ 1, \ldots, H_1 \} \) where \( H_1 = \min \{ p ∈ \mathbb{N} | pH ≥ H^\omega \} \) then \( H_1 = (p+1)H \).

If \( V_x' \) denotes a section of \( B^r \) by hyperplane \( \bar{x}_r = x \), \( x = 1, \ldots, H \), then there exist \( V_1, \ldots, V_H ⊂ B_1^{r−1} \) and fixed isomorphisms \( V_x' \rightarrow V_x \) such that

\[
\bigcup_{x=0}^H V_x ⊆ B_1^{r−1}.
\]

Definition. The set \( \bigcup_{x=0}^H V_x \) is called an evolvent of \( B^r \). An evolution of \( A_{r−1} \) in \( V_x \) will simulate an evolution of \( A_r \) in \( V_x' \).

We shall describe \( A_{r−1} \) in detail only for \( r = 2 \) because for \( r > 2 \) \( A_{r−1} \) can be constructed along similar lines. To facilitate further notations we need to introduce some auxiliary notions. Let \( A \) and \( B \) be one dimensional CA with alphabets \( ω_1 \) and \( ω_2 \), determined by programs \( Π_A \) and \( Π_B \) respectively.

Definition.

1). A composition of \( A \) and \( B \) is a cellular automaton, denoted by \( A * B \) which is determined by program \( Π_A ∪ Π_B \).

2). A direct product of \( A \) and \( B \) is a cellular automaton denoted by \( A × B \) with alphabet \( ω^0 = (ω_1 × ω_2) \cup \{ 0(ω_1) \} \) and program \( Π \), which consists of all commands of the form

\[
((u_1, u_2), (v_1, v_2), (w_1, w_2)) \rightarrow (z_1, z_2),
\]

where \( z_1 = A(u_1, v_1, w_1) \), \( z_2 = B(u_2, v_2, w_2) \), \( 0(ω^0) = 0(ω_1) \).

3). In addition, let \( G \) be some set of words of the form:

\[
((α_1, α_2), (β_1, β_2), (γ_1, γ_2)) \rightarrow (δ_1, δ_2),
\]

where \( α_j, β_j, γ_j, δ_j ∈ ω_j, \ j = 1, 2 \).
NCA $\mathcal{D}$, determined by the program $\Pi \cup G$ is denoted by $\mathcal{A} \times_G \mathcal{B}$. It may be called a semidirect product.

4). Let $\Pi^*_A$ be the set of all commands of the form $(x, y, z) \rightarrow \omega$, where $(x, y, z) \rightarrow \omega$ contains in $\Pi_A$. Then $\mathcal{A}^*$ denotes NCA with program $\Pi^*_A$.

5). At last let $\mathcal{N}_\omega$ denote a standard automaton in alphabet $\omega$: $\mathcal{N}_\omega(x, y, z) = x$.

**Definition of $\mathcal{A}_{r-1}$**.

Put:

$$\mathcal{A}_{r-1} = \mathcal{B}_{r-1} \ast \mathcal{M},$$

where $\mathcal{M}$ is a marker of initial evolvent $a_B$ and $\mathcal{B}_{r-1}$ simulates $\mathcal{A}_{r-1}$ in evolvent.

Let $\omega = \{c_0, c_1, \ldots, c_k\}$ be alphabet of $\mathcal{A}_r$, $\omega^+ = \{c_0^+, c_1^+, \ldots, c_k^+\}$ and $\omega' = \{c_0', c_1', \ldots, c_k'\}$ be new different copies of $\omega$; $0, b$ be new letters. Put $\sigma = \omega \cup \omega^+ \cup \omega' \cup \{0, b\}$.

An auxiliary CA $\mathcal{W}$ in alphabet $\sigma$ is defined by the following program $\Pi$:

$$(c_i, ?, x) \rightarrow 0, \quad (c_i', ?, 0) \rightarrow 0,$$

$$(0, c_i, ?) \rightarrow c_i, \quad (0, y, ?) \rightarrow y,$$

$$(b, c_i, ?) \rightarrow c_i^+, (c_i^+, ?, ?) \rightarrow b,$$

where $x$ takes all values from $\{0, b\}$, $y$ - from $\omega^+ \cup \omega'$, $i = 0, \ldots, k$.

Let $\omega_1 = (\sigma \times \sigma \times \sigma) \cup \{c_0\}$, $G$ be the list of all commands of the following form:

$$((u_1, u_2, u_3), (v_1, v_2, v_3), (w_1, w_2, w_3)) \rightarrow (z, z, z),$$

where $z = \mathcal{A}_r(u_2, v_2, w_2, u_3, u_1)$.

Put

$$\mathcal{B}_{r-1} = (\mathcal{W} \times \mathcal{N}_\sigma) \times_G \mathcal{W}^*,$$

$$\text{succ}(\mathcal{B}_{r-1}) = (c_k, c_k, c_k),$$

$$E(\mathcal{B}_{r-1}) = \{(c, c, c) | c \in E(\mathcal{A}_r)\},$$

$$0(\omega_1) = c_0,$$

where $c_k = \text{succ} \mathcal{A}_r$, $c_0 = 0(\omega)$.

Given an input word $B \in \omega^*$ for $\mathcal{A}_r$, let $d(r, s)$ be the initial configuration of $\mathcal{A}_r$, corresponding to $B$, we define corresponding input word $a_B = h_0h_1\ldots h_{H^2}$, $H = S_{\mathcal{A}_r}([B])$ for $\mathcal{B}_{r-1}$ by the following:

$$h_i = \begin{cases} (b, b, b), & \text{if } i \equiv 0 \pmod{H}, \\
(c_0, c_0, c_0), & \text{if } i \equiv r \pmod{H}, \quad q < r < H, \\
d(r, s), & \text{if } i = sH + r, \quad 1 \leq r \leq q. \end{cases}$$

The following Lemma 5 may be deduced immediately from the definition (13).

**Lemma 5.** $\forall B \in P_{\mathcal{A}_r} \tau_{\mathcal{B}_{r-1}}(a_B) \leq 2H\tau_{\mathcal{A}_r}(B)$.

Now to finish the proof of Proposition 2 it is sufficient to construct CA $\mathcal{M}$ transforming $B$ to $a_B$ in time $O(H^2) = O(H^{1.0})$. It may be done as in the proof of Theorem 1 (look at NCA 2: G1 and G2). Note that here we have $V_x = \{(x-1)H + 1, \ldots, xH\}$.
The case \( r = 2 \) is considered.

Let now \( r > 2 \). This case differs in that the domains \( V_x \) for sections \( i_r = x, \ x = 0, 1, \ldots, H, \ H = O(S) \) are disposed sequentially along a spiral so that \( S_{r-1} = O(S^{\frac{1}{r-1}}), \ T_{r-1} = O(TS) \). Such initial evolvent \( a_R \subset Z^{-1} \) can be isolated from the space and marked out according to the direction of laying of all \( V_x \) into evolvent in time \( O(S) \).

Proposition 2 is proved.

Note that analogous proposition takes place also for deterministic CA.

Now we turn to the proof of Theorem 4.

1. Given a fastest \( r \)-dimensional NCA \( A \) with complexity \( T = T_{r,p}, \ P = P_A \). Proposition 2 yields \( r - 1 \)-dimensional NCA \( A' \) simulating \( A \) with complexity \( (T^2, T^{\frac{1}{r-1}}) \). Then by Theorem 2 we obtain \( r - 1 \)-dimensional NCA simulating \( A' \) with complexity \( O(T^{\frac{2/r}{r}}(T^{\frac{1}{r-1}})^{\frac{1}{r-1}}) = O(T^{1+2/r}) \). Point 1) is proved.

2. We can suppose that \( A \) acts in \( r \)-dimensional cube \( B^r, \ B = \{1, \ldots, S\} \).

Let \( r + 1 \)-dimensional NCA simulating \( A \) in time \( O(T^{1-r/(r+1)^2}) \) will be constructed in two steps.

Step 1. Simulation of \( A \) in \( Z^{r+1} \) with complexity \( (T, T^{\frac{1}{r+1}}) \).

Step 2. Applying of Theorem 2 to NCA obtained in Step 1.

Step 1.

Definition. A set \( A \) consisting of inclusions of the form

\[
L_j : Y_j \longrightarrow Z^r, \ j \in N,
\]

is called a constructible set of inclusions if for some \( q \leq r \) all \( Y_j \subset Z^q \), and for some constants \( c, c_1, c_2 \) the following three conditions are satisfied.

\[
a). \ \forall j \in N, \ \bar{x}, \bar{y} \in Y_j
\]

\[
(14) \quad \rho(L_j(\bar{x}), L_j(\bar{y})) \leq c\rho(\bar{x}, \bar{y}),
\]

where \( \rho \) denotes standard metric in \( Z^q \) or \( Z^r \).

\[
b). \ \text{Diam}(\text{Im} L_j) \leq c_1|L_j|^{1/r}, \ \text{where} \ |L| \ \text{denotes the number of elements in} \ L.
\]

\[
c). \ \text{All domains} \ \text{Im} L_j \ \text{can be marked out by NCA in time} \ c_2|L_j|^{1/r} \ \text{so that every}\n\]

\[
cell \ \bar{z} = L_j(\bar{x}) \in \text{Im} L_j \ \text{will be marked by a label} \ m(\bar{z}) \ \text{which points to the disposition of all such cells} \ L_j(\bar{y}) \ \text{that} \ \rho(\bar{x}, \bar{y}) = 1 \ \text{with respect to} \ \bar{z}. \ (\text{Every label} \ m(\bar{z}) \ \text{has the form} \ (g_0, g_1, \ldots, g_{2q}) \ \text{where} \ g_p = L_j(\bar{x}(p)) - \bar{z} \ \text{in} \ Z^r, \ p = 0, 1, \ldots, 2q; \ \bar{x}(p) \ \text{is defined in the section 1.} \ \text{In view of inequality} \ (14) \ \text{the required number of all such labels} \ m(\bar{z}) \ \text{does not depend on} \ j.)
\]

To fulfill Step 1 it is sufficient to prove the following

Proposition 3. For every \( r \) there exists a constructible set of inclusions \( L^r_S : B^r \longrightarrow Z^{r+1}, \ B = \{1, \ldots, S\}, \ S = 1, 2, \ldots \).

Really, with such inclusions \( L^r_S \) in view of the point b) we can simulate \( A \) with complexity \( (T, O(S^{\frac{1}{r+1}})) \) in \( Z^{r+1} \).

Lemma 6. Given \( r \leq 1 \) there exists a constructible set of inclusions

\[
R^r_S : Y_S \longrightarrow Z^{r+1}, \ S = 1, 2, \ldots
\]

where \( Y_S = B_1^r \times B \subset Z^{r+1}, \ B_1 = \{1, \ldots, S_1\}, \ S_1 = [S^{\frac{1}{r}}] \).
Proof.
A parallelepiped $Y_S$ in $\mathbb{Z}^{r+1}$ can be conceived of as a thread of length $S$ and $S^{r-1}$ thick. This thread can be rolled up into a ball which has the shape of cube of side $4|Y_S|^\frac{1}{r+1}$ in $\mathbb{Z}^{r+1}$, because $S^{r-1} < |Y_S|^\frac{1}{r+1} = S^{\frac{r}{r+1}}$. The construction of NCA performing the required marking of $\text{Im} Y_S$ is evident. □

Proof of Proposition 3.
Induction on $r$. Basis $r = 1$. Proposition 3 follows from Lemma 6. Step: $r > 1$.
It follows from the inductive hypothesis that there exists a constructible set of inclusions:

$$L_S^{r-1} : B^{r-1} \rightarrow B^r_1$$

Let $R_S^{r}$ be inclusions from Lemma 6. Then we obtain a constructible set of inclusions $L_S^r : B^r \rightarrow \mathbb{Z}^{r+1}$, defined by the following:

$$L_S^r(\bar{x}, y) = R_S^r(L_S^{r-1}(\bar{x}), y),$$

where $\bar{x} \in \mathbb{Z}^{r-1}$, $y \in \mathbb{Z}$.
□
Step 1 is fulfilled.

Step 2.
Applying Theorem 3 we obtain a simulation of $A$ in $\mathbb{Z}^{r+1}$ with complexity $T^\frac{1}{r+1}(T^\frac{1}{r+1})^\frac{1}{r+1} = T^{1-r/(r+1)^2}$.

Theorem 4 is proved.
Thus, if $T_A > S_A$, then we have a variety of ways to accelerate the computations of $P_A$ in $r + 1$-dimensional space, for example:

1). By Theorem 1 (or by Corollary 1).
2). Sequential applications of Theorem 2 and point 2) of Theorem 4.

The second way gives the better acceleration for $T_A \gg S_A$. But if $T$ is small, for example $T = S^2$, then Theorem 1 gives the more strong estimate for $T_{r+1,p}$ because $1 < \frac{r+2}{r+1} \frac{1}{(r+1)^2}$.

Remark. For any predicate $P$ computable in time $T = O(n^\alpha)$, $\alpha = \text{const}$ on $d$-dimensional NCA and for any $\beta > 0$ there exists such a number $r \geq d$ that $T_{r,p} = O(n^\beta)$.

Proof.
Applying Theorem 4, point 2), we obtain that for $r > d$

$$T_{r,p}(n) \leq c(R) \exp[\ln(T_{d,p}(n)) \prod_{m=d+1}^{r} (1 - m/(m+1)^2)].$$

We have:

$$\ln \prod_{m=d+1}^{r} (1 - m/(m+1)^2) \sim - \sum_{m=d+1}^{r} m/(m+1)^2 \rightarrow -\infty (r \rightarrow +\infty).$$

Thus, $\prod_{m=d+1}^{r} (1 - m/(m+1)^2) \rightarrow 0 (r \rightarrow \infty)$.
6. Discussion

We see that if cellular automata are nondeterministic, then increase of dimension leads to the substantial acceleration of computations. In addition, programming on nondeterministic CA is more simple, then on ordinary CA. Thus, if it is possible, the realization of many dimensional nondeterministic cellular automata by a physical device would be of great practical consequence. From the other side, TCD-problem for deterministic CA remains unsolved. Let $C(r, T, S)$ be the class of predicates, computable on CA in time $T$ and space $S$. TCD-problem for CA is as follows:

Given $r, S, T$, is there an increasing function $f(n)$ such that

$$C(r, T, S) \subseteq C(r + 1, \frac{T}{f(n)} + S, S_1)$$

for some $S_1$?

At last note that the hypothesis $T = O(f(S))$ for the fastest deterministic CA is open for question for every function $f(n) \geq n$ growing slowly in comparison with exponential.

7. Acknowledgements

I am grateful to Alexander Shen for useful information on the theory of complexity, and Nadia Viktorova for editorial help.

References

1. A.W.Burks, Essays on Cellular Automata, Univ. of Illinois Press, 1970.
2. S.Cook, The complexity of theorem-proving procedures, ACM, NY, Proceedings of the 3rd Annual Symposium on the theory of Computing (1971), 151-158.
3. K.Culik, Sheng Yu., Undecidability of cellular automata. Classification schemes, Complex Systems 2(2) (1988), 177–190.
4. O.Martin, A.Odlyzko, S.Wolfram, Algebraic Properties of Cellular Automata, Communications in Mathematical Physics 93 (1984), 51 – 89.
5. J. von Neumann, Theory of Self-Reproducing Automata (A.Burks, ed.), Univ. of Illinois Press, 1966.
6. K.Sutner, The computational complexity of cellular automata, Lecture Notes in Computer Science, Proceedings of Fundamentals of Computation Theory, vol. 380, Springer, Berlin, 1989, p. 451.
7. S. Ulam, Random Processes and Transformations, Proc. Int. Cong. Mathem. 2 (1952), 264–275.
8. S.Wolfram, Computation theory of cellular automata, Comm.Math.Physics 96(1) (1984), 15–57.
9. S.Wolfram, Cellular Automata and Complexity: Collected Papers, Addison-Wesley, 1994.
10. T.Yaku, The constructibility of a configuration in cellular automata, Journal of Computers and System Science 7 (1973), 481–496.
11. Cellular Automata, Proc. of an Interdisciplinary workshop, Los Alamos, Ne mexic o, USA, Mar. 7–11 1983, vol. 13, Amsterdam, North-Holand physics publ., 1984, p. 247.
12. Cellular automata,Theory and experiment, Proc. of a workshop, Los Alamos, Ne Mexico, USA, vol. 17, Amsterdam, North–Holland, 1990, pp. 483.
Figure 1. Successful operation of CA3

Ascending map

Descending map
Figure 2. Operation of $B$
Figure 3. Operation of $B$ in the corner squares