The $q$-Log-Concavity and Unimodality of $q$-Kaplansky Numbers

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Abstract. $q$-Kaplansky numbers were considered by Chen and Rota. We find that $q$-Kaplansky numbers are connected to the symmetric differences of Gaussian polynomials introduced by Reiner and Stanton. Based on the work of Reiner and Stanton, we establish the unimodality of $q$-Kaplansky numbers. We also show that $q$-Kaplansky numbers are the generating functions for the inversion number and the major index of two special kinds of $(0, 1)$-sequences. Furthermore, we show that $q$-Kaplansky numbers are strongly $q$-log-concave.

Keywords: Inversion number, major index, $q$-log-concavity, unimodality, $q$-Catalan numbers, Foata’s fundamental bijection, integer partitions

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1 Introduction

The main objective of this paper is to give two combinatorial interpretations of $q$-Kaplansky numbers introduced by Chen and Rota [4] and to establish some properties of $q$-Kaplansky numbers. Recall that the Kaplansky number $K(n, m)$ is defined by

$$K(n, m) = \frac{n}{n-m} \binom{n-m}{m},$$

for $n \geq 2m \geq 0$. The combinatorial interpretation of $K(n, m)$ was first given by Kaplansky [14], so we call $K(n, m)$ the Kaplansky number. Kaplansky found that $K(n, m)$ counts the number of ways of choosing $m$ nonadjacent elements arranged on a cycle, which can also be interpreted as the number of dissections of type $1^{n-2k}2^k$ of an $n$-cycle given by Chen, Lih and Yeh [5]. Kaplansky numbers appear in many classical polynomials, such as Chebyshev polynomials of the first kind [17, 18] and Lucas polynomials [15].

$q$-Kaplansky numbers were introduced by Chen and Rota [4]. For convenience, we adopt the following definition: For $n \geq 1$ and $0 \leq m \leq n$,

$$K_q(n, m) = \frac{1-q^{n+m}}{1-q^n} \binom{n}{m}, \quad (1.1)$$
where $\binom{n}{m}$ is the Gaussian polynomial, also called the $q$-binomial coefficient, as given by

$$\binom{n}{m} = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-m+1})}{(1 - q^m)(1 - q^{m-1}) \cdots (1 - q)}.$$  

By the symmetric property of the Gaussian polynomial, it is not hard to show that $K_q(n, m)$ is a symmetric polynomial of degree $m(n - m) + m$ with nonnegative coefficients.

The first result of this paper is to give two combinatorial interpretations of $q$-Kaplansky numbers. Let $w = w_1w_2 \cdots w_n$ be a $(0, 1)$-sequence of length $n$, the number of inversions of $w$, denoted $\text{inv}(w)$, is the number of pairs $(w_i, w_j)$ such that $i < j$ and $w_i > w_j$, and the major index of $w$, denoted $\text{maj}(w)$, is the sum of indices $i < n$ such that $w_i > w_{i+1}$. For example, for $w = 10010110$, we have $\text{inv}(w) = 8$ and $\text{maj}(w) = 1 + 4 + 7 = 12$.

It can be shown that $q$-Kaplansky numbers are related to two sets $\mathcal{K}(m, n - m + 1)$ and $\overline{\mathcal{K}}(m, n - m + 1)$ of $(0, 1)$-sequences. More precisely, for $n \geq m \geq 0$, let $\mathcal{K}(m, n - m + 1)$ denote the set of $(0, 1)$-sequences $w = w_1w_2 \cdots w_{n+1}$ of length $n + 1$ consisting of $m$ copies of 1’s and $n - m + 1$ copies of 0’s such that if $w_{n+1} = 1$, then $w_1 = 0$. For $n \geq m \geq 0$, let $\overline{\mathcal{K}}(m, n - m + 1)$ denote the set of $(0, 1)$-sequences $w = w_1w_2 \cdots w_{n+1}$ of length $n + 1$ consisting of $m$ copies of 1’s and $n - m + 1$ copies of 0’s such that if $w_{n+1} = 1$ and $t := \max\{i : w_i = 0\}$, then $t = 1$ or $w_{t-1} = 0$ when $t \geq 2$. We have the following combinatorial interpretations.

**Theorem 1.1.** For $n \geq m \geq 0$,

$$K_q(n, m) = \sum_{w \in \mathcal{K}(m, n - m + 1)} q^{\text{inv}(w)} \quad (1.2)$$

$$= \sum_{w \in \overline{\mathcal{K}}(m, n - m + 1)} q^{\text{maj}(w)}. \quad (1.3)$$

The second result of this paper is to establish the strong $q$-log-concavity of $K_q(n, m)$. Recall that a sequence of polynomials $(f_n(q))_{n \geq 0}$ over the field of real numbers is called $q$-log-concave if the difference

$$f_m(q)^2 - f_{m+1}(q)f_{m-1}(q)$$

has nonnegative coefficients as a polynomial in $q$ for all $m \geq 1$. Sagan [20] also introduced the notion of the strong $q$-log-concavity. We say that a sequence of polynomials $(f_n(q))_{n \geq 0}$ is strongly $q$-log-concave if

$$f_n(q)f_m(q) - f_{n-1}(q)f_{m+1}(q)$$

has nonnegative coefficients as a polynomial in $q$ for any $m \geq n \geq 1$.

It is known that $q$-analogues of many well-known combinatorial numbers are strongly $q$-log-concave. Butler [2] and Krattenthaler [16] proved the strong $q$-log-concavity of $q$-binomial coefficients, respectively. Leroux [12] and Sagan [20] studied the strong $q$-log-concavity of $q$-Stirling numbers of the first kind and the second kind. Chen, Wang and Yang [8] have shown that $q$-Narayana numbers are strongly $q$-log-concave.
We obtain the following result which implies that \( q \)-Kaplansky numbers are strongly \( q \)-log-concave.

**Theorem 1.2.** For \( 1 \leq m \leq l < n \) and \( 0 \leq r \leq 2l - 2m + 2 \),

\[
K_q(n, m)K_q(n, l) - q^r K_q(n, m - 1)K_q(n, l + 1)
\]

has nonnegative coefficients as a polynomial in \( q \).

**Corollary 1.3.** Given a positive integer \( n \), the sequence \( (K_q(n, m))_{0 \leq m \leq n} \) is strongly \( q \)-log-concave.

It is easy to check that the degree of \( K_q(n, m)K_q(n, l) \) exceeds the degree of \( K_q(n, m - 1)K_q(n, l + 1) \) by \( 2l - 2m + 2 \), so if the difference (1.4) of these two polynomials has nonnegative coefficients, then \( r \leq 2l - 2m + 2 \).

To conclude the introduction, let us say a few words about the unimodality of \( q \)-Kaplansky numbers. We find that \( q \)-Kaplansky numbers are connected to the following symmetric differences of Gaussian polynomials introduced by Reiner and Stanton [19].

\[
F_{n,m}(q) = \binom{n + m}{m} - q^n \binom{n + m - 2}{m - 2}.
\]

The following theorem is due to Reiner and Stanton [19].

**Theorem 1.4** (Reiner-Stanton). When \( m \geq 2 \) and \( n \) is even, the polynomial \( F_{n,m}(q) \) is symmetric and unimodal.

Recently, Chen and Jia [6] provided a simple proof of the unimodality of \( F_{n,m}(q) \) by using semi-invariants. According to the following recursions of Gaussian polynomials [1, p.35, Theorem 3.2 (3.3)],

\[
\binom{n}{m} = \binom{n - 1}{m - 1} + q^m \binom{n - 1}{m},
\]

\[
\binom{n - 1}{m} = \binom{n}{m} - q^{n - m} \binom{n - 1}{m - 1},
\]

we find that

\[
F_{n,m}(q) = \binom{n + m}{m} - q^n \binom{n + m - 2}{m - 2}
\]

\[
\overset{(1.6)}{=} \binom{n + m - 1}{m - 1} - q^n \binom{n + m - 2}{m - 2} + q^m \binom{n + m - 1}{m}
\]

\[
\overset{(1.7)}{=} \binom{n + m - 2}{m - 1} + q^m \binom{n + m - 1}{m}
\]

\[
= \frac{1 - q^{n + 2m - 1}}{1 - q^{n + m - 1}} \binom{n + m - 1}{m}
\]

3
Combining Theorem 1.4 and (1.8), we have the following result.

**Theorem 1.5.** When \( n \geq m \geq 2 \) and \( n - m \) is odd, the \( q \)-Kaplansky number \( K_q(n, m) \) is symmetric and unimodal.

It should be noted that \( K_q(n, m) \) is not always unimodal for any \( n \geq m \geq 2 \). For example,

\[
K_q(6, 2) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10}
\]

is not unimodal.

\( q \)-Kaplansky numbers are also related to \( q \)-Catalan polynomials \( C_n(q) \), defined by

\[
C_n(q) = \frac{1 - q}{1 - q^{n+1}} \binom{2n}{n} = \frac{1 - q}{1 - q^{2n+1}} \binom{2n + 1}{n}.
\]

(1.9)

It is well-known that \( C_n(q) \) is a polynomial in \( q \) with non-negative coefficients [10]. Combining (1.1) and (1.9), it is readily seen that

\[
(1 - q)K_q(2n + 1, n) = (1 - q^{3n+1})C_n(q).
\]

Hence, by Theorem 1.5, we obtain the following result.

**Theorem 1.6.** When \( n \) is even, the polynomial \( \frac{1 - q^{3n+1}}{1 - q} C_n(q) \) is symmetric and unimodal.

Finally, we would like to state a result of Stanley [22, p.523] about the unimodality of the \( q \)-Catalan polynomials and two conjectures on the unimodality of the \( q \)-Catalan polynomials due to Chen, Wang and Wang [7] and Xin and Zhong [24, Conjecture 1.2], respectively. Apparently, Conjecture 1.8 implies Conjecture 1.9 when \( n \geq 16 \).

**Theorem 1.7** (Stanley). For \( n \geq 1 \), the polynomial \( \frac{1 + q}{1 + q^n} C_n(q) \) is symmetric and unimodal.

**Conjecture 1.8** (Chen, Wang and Wang). For \( n \geq 16 \), the \( q \)-Catalan polynomial \( C_n(q) \) is unimodal.

**Conjecture 1.9** (Xin and Zhong). For \( n \geq 1 \), the polynomial \( (1 + q)C_n(q) \) is unimodal.

## 2 Proof of Theorem 1.1

To prove Theorem 1.1, we first recall a result due to MacMahon [13]. For \( n \geq m \geq 0 \), let \( \mathcal{M}(m, n - m) \) be the set of \((0, 1)\)-sequences of length \( n \) consisting of \( m \) copies of 1's and \( n - m \) copies of 0's. The following well-known result is due to MacMahon (see [1, Chapter 3.4]).
Theorem 2.1 (MacMahon). For $n \geq m \geq 0$,

$$\binom{n}{m} = \sum_{w \in \mathcal{M}(m, n-m)} q^{\text{inv}(w)}$$  \hspace{1cm} (2.1)

$$= \sum_{w \in \mathcal{M}(m, n-m)} q^{\text{maj}(w)}. \hspace{1cm} (2.2)$$

Foata’s fundamental bijection [9] can be used to establish the equivalence of (2.1) and (2.2). There are several ways to describe Foata’s fundamental bijection, see, for example, Foata [9], Haglund [11, p.2] and Sagan and Savage [21]. Here we give a description due to Sagan and Savage [21].

Proof of the equivalence between (2.1) and (2.2): Let $w = w_1 w_2 \cdots w_n \in \mathcal{M}(m, n-m)$. We aim to construct a $(0, 1)$-sequence $\tilde{w} = \phi(w) = \tilde{w}_1 \tilde{w}_2 \cdots \tilde{w}_n$ in $\mathcal{M}(m, n-m)$ such that $\text{inv}(\tilde{w}) = \text{maj}(w)$.

Let $w$ be a $(0, 1)$-sequence with $d$ descents, so that we can write

$$w = 0^{m_0} 1^{n_0} 0^{m_1} 1^{n_1} 0^{m_2} \cdots 1^{n_{d-1}} 0^{m_d} 1^{n_d}, \hspace{1cm} (2.3)$$

where $m_0 \geq 0$ and $m_i \geq 1$ for $1 \leq i \leq d$, $n_i \geq 1$ for $0 \leq i \leq d-1$ and $n_d \geq 0$.

Define

$$\tilde{w} = \phi(w) = 0^{m_d-1} 10^{m_{d-1}-1} 1 \cdots 0^{m_1-1} 10^{m_0-1} 1 n_1 \cdots 01^{n_{d-1}-1} 10^{n_d} \hspace{1cm} (2.4)$$

It has been shown in [21] that $\text{inv}(\tilde{w}) = \text{maj}(w)$.

The inverse map $\phi^{-1}$ of $\phi$ can be described recursively. Let $\tilde{w} \in \mathcal{M}(m, n-m)$, we may write $\tilde{w} = 0^a 1u01^b$ for $a, b \geq 0$, define

$$w = \phi^{-1}(\tilde{w}) = \phi^{-1}(u)10^{a+1}1^b. \hspace{1cm} (2.5)$$

It has been proved in [21] that $\phi^{-1}(\phi(w)) = w$ and $\phi(\phi^{-1}(\tilde{w})) = \tilde{w}$. Furthermore, $\text{inv}(\tilde{w}) = \text{maj}(w)$. Hence the map $\phi$ is a bijection. This completes the proof of the equivalence of (2.1) and (2.2). \hfill \Box

For $n \geq m \geq 0$, let $\mathcal{M}_0(m, n-m+1)$ be the set of $(0, 1)$-sequences $w = w_1 w_2 \cdots w_{n+1}$ of length $n+1$ consisting of $m$ copies of 1’s and $n-m+1$ copies of 0’s such that $w_{n+1} = 0$. We have the following result.

Lemma 2.2. For $n \geq m \geq 0$,

$$q^m \binom{n}{m} = \sum_{w \in \mathcal{M}_0(m, n-m+1)} q^{\text{inv}(w)} \hspace{1cm} (2.6)$$

$$= \sum_{w \in \mathcal{M}_0(m, n-m+1)} q^{\text{maj}(w)}. \hspace{1cm} (2.7)$$
Proof. By Theorem 2.1, we see that
\[
\binom{n}{m} = \sum_{w \in \mathcal{M}(m,n-m)} q^{\text{inv}(w)}.
\]

To prove (2.6), it suffices to show that
\[
\sum_{w \in \mathcal{M}(m,n-m)} q^{\text{inv}(w)+m} = \sum_{w \in \mathcal{M}_0(m,n-m+1)} q^{\text{inv}(w)}. \tag{2.8}
\]

We construct a bijection \(\psi\) between the set \(\mathcal{M}(m,n-m)\) and the set \(\mathcal{M}_0(m,n-m+1)\) such that for \(w \in \mathcal{M}(m,n-m)\) and \(\psi(w) \in \mathcal{M}_0(m,n-m+1)\), we have
\[
\text{inv}(w) + m = \text{inv}(\psi(w)). \tag{2.9}
\]

Let \(w = w_1w_2 \cdots w_n\). Define
\[
\psi(w) = w_1w_2 \cdots w_n0.
\]

It is clear that \(\psi(w) \in \mathcal{M}_0(m,n-m+1)\) and (2.9) holds. Furthermore, it is easy to see that \(\psi\) is reversible. Hence we have (2.8).

We proceed to show that (2.6) and (2.7) are equivalent by using Foata’s fundamental bijection \(\phi\). Let \(w = w_1w_2 \cdots w_{n+1}\) be in \(\mathcal{M}_0(m,n-m+1)\), by definition, we see that \(w_{n+1} = 0\). Define
\[
\tilde{w} = \phi^{-1}(w) = \tilde{w}_1\tilde{w}_2 \cdots \tilde{w}_{n+1},
\]
where \(\phi^{-1}\) is defined in (2.5). By (2.5), we see that \(\tilde{w}_{n+1} = 0\) since \(w_{n+1} = 0\). Hence \(\tilde{w} \in \mathcal{M}_0(m,n-m+1)\). Furthermore \(\phi^{-1}\) is reversible and \(\text{inv}(w) = \text{maj}(\tilde{w})\). It follows (2.6) and (2.7) are equivalent, and so (2.7) is valid. \(\blacksquare\)

For \(n \geq m \geq 1\), let \(\mathcal{M}_1(m,n-m+1)\) be the set of \((0,1)\)-sequences \(w = w_1w_2 \cdots w_{n+1}\) of length \(n+1\) consisting of \(m\) copies of \(1\)'s and \(n-m+1\) copies of \(0\)'s such that \(w_1 = 0\) and \(w_{n+1} = 1\). For \(n \geq m \geq 1\), let \(\overline{\mathcal{M}}_1(m,n-m+1)\) be the set of \((0,1)\)-sequences \(w = w_1w_2 \cdots w_{n+1}\) of length \(n+1\) consisting of \(m\) copies of \(1\)'s and \(n-m+1\) copies of \(0\)'s such that \(w_{n+1} = 1\), and if \(t := \max\{i : w_i = 0\}\), then \(t = 1\) or \(w_{t-1} = 0\) when \(t \geq 2\). To wit, for \(w \in \overline{\mathcal{M}}_1(m,n-m+1)\), if \(m \geq 1\) and \(n > m\), then \(w\) can be written as \(u001^{n+1-t}\), where \(2 \leq t \leq n\) and \(u \in \mathcal{M}(m+t-n-1,n-m-1)\), and if \(m \geq 1\) and \(n = m\), then \(w\) can be written as \(01^m\).

Lemma 2.3. For \(n \geq m \geq 1\),
\[
\begin{align*}
\binom{n-1}{m-1} &= \sum_{w \in \mathcal{M}_1(m,n-m+1)} q^{\text{inv}(w)} \tag{2.10} \\
&= \sum_{w \in \overline{\mathcal{M}}_1(m,n-m+1)} q^{\text{maj}(w)}. \tag{2.11}
\end{align*}
\]
**Proof.** By Theorem 2.1, we see that

\[
\begin{bmatrix} n - 1 \\ m - 1 \end{bmatrix} = \sum_{w \in \mathcal{M}(m-1,n-m)} q^{\text{inv}(w)}.
\]

To prove (2.10), it suffices to show that

\[
\sum_{w \in \mathcal{M}(m-1,n-m)} q^{\text{inv}(w)} = \sum_{w \in \mathcal{M}_1(m,n-m+1)} q^{\text{inv}(w)}. \tag{2.12}
\]

We now construct a bijection \( \varphi \) between the set \( \mathcal{M}(m-1,n-m) \) and the set \( \mathcal{M}_1(m,n-m+1) \) such that for \( w \in \mathcal{M}(m-1,n-m) \) and \( \varphi(w) \in \mathcal{M}_1(m,n-m+1) \), we have

\( \text{inv}(w) = \text{inv}(\varphi(w)). \tag{2.13} \)

Let \( w = w_1w_2 \cdots w_{n-1} \). Define

\( \varphi(w) = 0w_1w_2 \cdots w_{n-1}1. \)

It is clear that \( \varphi(w) \in \mathcal{M}_1(m,n-m+1) \) and (2.13) holds. Furthermore, \( \psi \) is reversible. Hence we have (2.12).

We proceed to show that (2.11) holds. By (2.2), it suffices to show that

\[
\sum_{w \in \mathcal{M}(m-1,n-m)} q^{\text{maj}(w)} = \sum_{w \in \mathcal{M}_1(m,n-m+1)} q^{\text{maj}(w)}. \tag{2.14}
\]

We now construct a bijection \( \tau \) between the set \( \mathcal{M}(m-1,n-m) \) and the set \( \overline{\mathcal{M}_1}(m,n-m+1) \) such that for \( w \in \mathcal{M}(m-1,n-m) \) and \( \tau(w) \in \overline{\mathcal{M}_1}(m,n-m+1) \), we have

\( \text{maj}(w) = \text{maj}(\tau(w)). \tag{2.15} \)

Let \( w = w_1w_2 \cdots w_{n-1} \in \mathcal{M}(m-1,n-m) \). If \( n = m \), then \( w = 1^{m-1} \), and so define \( \tau(w) = 01^n \). If \( n > m \), then let \( t = \max\{i : w_i = 0\} \), obviously, \( t \geq 1 \). In this case, we may write \( w = w_1w_2 \cdots w_{t-1}01^{n-t-1} \). Define

\[
\tilde{w} = \tau(w) = \tilde{w}_1\tilde{w}_2 \cdots \tilde{w}_{n+1}
\]

as follows: set \( \tilde{w}_{n+1} = 1 \), and set \( \tilde{w}_j = w_j \) for \( 1 \leq j \leq t \), \( \tilde{w}_{t+1} = 0 \), and set \( \tilde{w}_{j+1} = w_j = 1 \) for \( t+1 \leq j \leq n-1 \).

From the above construction, it is easy to see that \( \tilde{w} \in \overline{\mathcal{M}_1}(m,n-m+1) \) and (2.15) holds. Furthermore, it can be checked that this construction is reversible, so (2.14) is valid.

We are now in a position to give a proof of Theorem 1.1 based on Lemma 2.2 and Lemma 2.3.

**Proof of Theorem 1.1:** By the definition of \( \mathcal{K}(m,n-m+1) \), we see that

\[
\mathcal{K}(m,n-m+1) = \mathcal{M}_0(m,n-m+1) \cup \mathcal{M}_1(m,n-m+1).
\]
Combining (2.6) and (2.10), we derive that for \( n \geq m \geq 1 \),

\[
\sum_{w \in K(m, n-m+1)} q^{\text{inv}(w)} = \sum_{w \in M_0(m, n-m+1)} q^{\text{inv}(w)} + \sum_{w \in M_1(m, n-m+1)} q^{\text{inv}(w)}
\]

\[
= q^m \binom{n}{m} + \binom{n-1}{m-1}
\]

\[
= \frac{1}{1-q^n} \binom{n}{m}
\]

\[
= K_q(n, m).
\]

Similarly, by definition, we see that

\[
\overline{K}(m, n-m+1) = M_0(m, n-m+1) \cup M_1(m, n-m+1).
\]

By (2.7) and (2.11), we find that \( n \geq m \geq 1 \),

\[
\sum_{w \in K(m, n-m+1)} q^{\text{maj}(w)} = \sum_{w \in M_0(m, n-m+1)} q^{\text{maj}(w)} + \sum_{w \in M_1(m, n-m+1)} q^{\text{maj}(w)}
\]

\[
= q^m \binom{n}{m} + \binom{n-1}{m-1}
\]

\[
= \frac{1}{1-q^n} \binom{n}{m}
\]

\[
= K_q(n, m).
\]

Furthermore, it is easy to check that (1.2) and (1.3) are valid when \( m = 0 \). This completes the proof of Theorem 1.1. \( \blacksquare \)

### 3 Proof of Theorem 1.2

Before we prove Theorem 1.2, it is useful to preset the following result.

**Lemma 3.1.** For \( 1 \leq m \leq l < N \) and \( M - m \geq N - l \geq 1 \),

\[
D_q(M, N, m, l) = \binom{M}{m} \binom{N}{l} - \binom{M}{m-1} \binom{N}{l+1}
\]

has nonnegative coefficients as a polynomial in \( q \).

Lemma 3.1 reduces to the strong \( q \)-log-concavity of Gaussian polynomials when \( M = N \). We prove Lemma 3.1 by generalizing Butler’s bijection [2]. To describe the proof, we need to recall some notation and terminology on partitions as in [1, Chapter 1]. A partition \( \lambda \) of a positive integer \( n \) is a finite nonincreasing sequence of positive integers
(λ₁, λ₂, ..., λᵦ) such that \( \sum_{i=1}^{ᵦ} λ_i = n \). Then λᵢ are called the parts of \( \lambda \) and λ₁ is its largest part. The number of parts of \( \lambda \) is called the length of \( \lambda \), denoted by \( l(\lambda) \). The weight of \( \lambda \) is the sum of parts of \( \lambda \), denoted \( |\lambda| \). The conjugate \( \lambda' = (λ₁', λ₂', ..., λᵦ') \) of a partition \( \lambda \) is defined by setting \( λ_i' \) to be the number of parts of \( \lambda \) that are greater than or equal to \( i \). Clearly, \( l(\lambda') = λ₁' \) and \( l(\lambda') = l(\lambda) \).

Let \( \mathcal{P}(m, n - m) \) denote the set of partitions \( \lambda \) such that \( ℓ(\lambda) \leq m \) and \( λ₁ \leq n - m \). It is well-known that the Gaussian polynomial has the following partition interpretation [1, Theorem 3.1]:

\[
\binom{n}{m} = \sum_{\lambda \in \mathcal{P}(m, n-m)} q^{|\lambda|}.
\]  

(3.1)

We are now prepared for the proof of Lemma 3.1 based on (3.1).

Proof of Lemma 3.1: For \( 1 \leq m \leq l < N \) and \( M - m \geq N - l \geq 1 \), by (3.1), it suffices to construct an injection \( \Phi \) from \( \mathcal{P}(m - 1, M - m + 1) \times \mathcal{P}(l + 1, N - l - 1) \) to \( \mathcal{P}(m, M - m) \times \mathcal{P}(l, N - l) \) such that if \( \Phi(\lambda, \mu) = (η, ρ) \), then \( |\lambda| + |\mu| = |η| + |ρ| \).

Let

\[
\lambda = (λ₁, λ₂, ..., λᵦ-1) \in \mathcal{P}(m - 1, M - m + 1)
\]

and

\[
\mu = (μ₁, μ₂, ..., μᵦ+1) \in \mathcal{P}(l + 1, N - l - 1),
\]

where \( λ₁ \leq M - m + 1 \) and \( μ₁ \leq N - l - 1 \).

We aim to construct a pair of partitions

\[
(η, ρ) \in \mathcal{P}(m, M - m) \times \mathcal{P}(l, N - l).
\]

Let \( I \) be the largest integer such that \( λ_I \geq μ₉+₁ + l - m + M - N + 1 \). If no such \( I \) exists, then let \( I = 0 \). In this case, we see that \( λ₁ < M - m \) and set \( γ = λ \) and \( τ = μ \). Obviously, \( γ₁ < M - m \) and \( τ₁ < N - l \). We now assume that \( 1 \leq I \leq m - 1 \) and define

\[
γ = (μ₁ + (l - m + M - N + 1), ..., μ_I + (l - m + M - N + 1), λ₉+₁, ..., λᵦ-1)
\]

(3.2)

and

\[
τ = (λ₁ - (l - m + M - N + 1), ..., λ_I - (l - m + M - N + 1), μ₉+₁, ..., μᵦ+1).
\]

(3.3)

Since \( I \) is the largest integer such that \( λ_I \geq μ₉+₁ + (l - m + M - N + 1) \), we get

\[
λ₉+₁ < μ₉+₂ + (l - m + M - N + 1) \leq μ_I + (l - m + M - N + 1).
\]

It follows that \( γ \) defined in (3.2) and \( τ \) defined in (3.3) are partitions. Furthermore,

\[
γ₁ = μ₁ + (l - m + M - N + 1) \leq M - m
\]

and

\[
τ₁ = λ₁ - (l - m + M - N + 1) \leq N - l.
\]
Let $\gamma'$ and $\tau'$ be the conjugates of $\gamma$ and $\tau$, respectively. We see that
\[
\ell(\gamma') = \gamma_1 \leq M - m \quad \text{and} \quad \ell(\tau') = \tau_1 \leq N - l,
\]
so we can assume that
\[
\gamma' = (\gamma'_1, \gamma'_2, \ldots, \gamma'_{M-m})
\]
and
\[
\tau' = (\tau'_1, \tau'_2, \ldots, \tau'_{N-l}).
\]
Then
\[
\gamma'_1 \leq m - 1 \quad \text{and} \quad \tau'_1 \leq l + 1.
\]
Let $J$ be the largest integer such that $\tau'_J \geq \gamma'_{J+1} + l - m + 1$. If no such $J$ exists, let $J = 0$, then $\tau'_1 < l$ and set $\tilde{\gamma} = \gamma'$, and $\tilde{\tau} = \tau'$. Obviously, $\tilde{\gamma}_1 < m$ and $\tilde{\tau}_1 < l$. We now assume that $1 \leq J \leq N - l$ and define
\[
\tilde{\gamma} = (\tau'_1 - (l - m + 1), \tau'_2 - (l - m + 1), \ldots, \tau'_J - (l - m + 1), \gamma'_{J+1}, \ldots, \gamma'_{M-m}) \quad (3.4)
\]
and
\[
\tilde{\tau} = (\gamma'_1 + (l - m + 1), \gamma'_2 + (l - m + 1), \ldots, \gamma'_J + (l - m + 1), \tau'_{J+1}, \ldots, \tau'_{N-l}). \quad (3.5)
\]
Similarly, since $J$ is the largest integer such that $\tau'_J \geq \gamma'_{J+1} + l - m + 1$, we find that
\[
\tau'_{J+1} < \gamma'_{J+2} + l - m + 1 \leq \gamma'_J + l - m + 1,
\]
so $\tilde{\gamma}$ defined in (3.4) and $\tilde{\tau}$ defined in (3.5) are partitions. By the construction of $\tilde{\gamma}$ and $\tilde{\tau}$, we see that
\[
\tilde{\gamma}_1 = \tau'_1 - (l - m + 1) \leq m
\]
and
\[
\tilde{\tau}_1 = \gamma'_1 + (l - m + 1) \leq l.
\]
Let $\eta$ and $\rho$ be the conjugates of $\tilde{\gamma}$ and $\tilde{\tau}$, respectively. It is easy to check that $\eta \in \mathcal{P}(m, M-m)$ and $\rho \in \mathcal{P}(l, N-l)$. Furthermore, this process is reversible. Thus, we complete the proof of Lemma 3.1.

Combining Lemma 3.1 and the unimodality of Gaussian polynomials, we obtain the following result.

**Lemma 3.2.** For $1 \leq m \leq l < N$, $M-m \geq N-l \geq 1$ and $0 \leq r \leq M-N+2l-2m+2$,
\[
D_q^r(M, N, m, l) = \begin{bmatrix} M \\ m \end{bmatrix} \begin{bmatrix} N \\ l \end{bmatrix} - q^r \begin{bmatrix} M \\ m-1 \end{bmatrix} \begin{bmatrix} N \\ l+1 \end{bmatrix} \quad (3.6)
\]
has nonnegative coefficients as a polynomial in $q$.

**Proof.** Let $A$ denote the degree of the polynomial $\begin{bmatrix} M \\ m \end{bmatrix} \begin{bmatrix} N \\ l \end{bmatrix}$ and let $B$ denote the degree of the polynomial $\begin{bmatrix} M \\ m-1 \end{bmatrix} \begin{bmatrix} N \\ l+1 \end{bmatrix}$. We have
\[
A = m(M - m) + l(N - l),
\]
\[ B = (m - 1)(M - m + 1) + (l + 1)(N - l - 1). \]

Furthermore,
\[ A - B = M - N + 2l - 2m + 2. \]

Let
\[ \begin{bmatrix} M \\ m \end{bmatrix} \begin{bmatrix} N \\ l \end{bmatrix} = \sum_{i=0}^{A} a_i q^i, \quad \begin{bmatrix} M \\ m - 1 \end{bmatrix} \begin{bmatrix} N \\ l + 1 \end{bmatrix} = \sum_{i=0}^{B} b_i q^i \]
and let
\[ D_r^r(M, N, m, l) = \begin{bmatrix} M \\ m \end{bmatrix} \begin{bmatrix} N \\ l \end{bmatrix} - r \begin{bmatrix} M \\ m - 1 \end{bmatrix} \begin{bmatrix} N \\ l + 1 \end{bmatrix} = \sum_{i=0}^{A} c_i q^i, \]
where \( c_i = a_i \) for \( 0 \leq i < r \), \( c_i = a_i - b_i - r \) for \( r \leq i \leq B + r \) and \( c_i = a_i \) for \( B + r + 1 \leq i \leq A \). It is easy to see that \( c_i \geq 0 \) for \( 0 \leq i < r \) and \( B + r + 1 \leq i \leq A \). It remains to show that \( c_i \geq 0 \) for \( r \leq i \leq B + r \).

It is known that the Gaussian polynomial \( \begin{bmatrix} M \\ m \end{bmatrix} \) is symmetric and unimodal, see, for example, [1, Theorem 3.10] and [23, Exercise 7.75], so
\[ a_i = a_{A-i} \text{ for } 0 \leq i \leq A, \quad \text{and} \quad b_i = b_{B-i} \text{ for } 0 \leq i \leq B, \quad (3.7) \]
\[ a_0 \leq a_1 \leq \cdots \leq a_{\lfloor A/2 \rfloor} = a_{\lceil A/2 \rceil} \geq \cdots \geq a_{A-1} \geq a_A, \quad (3.8) \]
and
\[ b_0 \leq b_1 \leq \cdots \leq b_{\lfloor A/2 \rfloor} = b_{\lceil A/2 \rceil} \geq \cdots \geq b_{B-1} \geq b_B. \quad (3.9) \]

By Lemma 3.1, we see that for \( 0 \leq i \leq A \),
\[ a_i - b_i \geq 0. \quad (3.10) \]

We consider the following two cases:

Case 1. If \( r \leq i \leq A/2 \), then
\[ c_i = a_i - b_{i-r} = a_i - a_{i-r} + a_{i-r} - b_{i-r}, \]
which is nonnegative by (3.8) and (3.10).

Case 2. If \( A/2 \leq i \leq B + r \), then
\[ c_i = a_i - b_{i-r} = a_{A-i} - b_{B-i+r} = a_{A-i} - a_{B-i-r} + a_{B-i+r} - b_{B-i+r}, \]
which is nonnegative by (3.8) and (3.10). Thus, we complete the proof of Lemma 3.2.

We conclude this paper with a proof of Theorem 1.2 by using Lemma 3.2.

**Proof of Theorem 1.2**: Recall that
\[ K_q(n, m) = \frac{1 - q^{n+m}}{1 - q^n} \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} + q^n \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}. \]
Hence
\[
K_q(n, m)K_q(n, l) - q^r K_q(n, m - 1)K_q(n, l + 1)
= \left( \left[ \begin{array}{c} n \\ m \end{array} \right] + q^n \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right] \right) \left( \left[ \begin{array}{c} n \\ l \end{array} \right] + q^n \left[ \begin{array}{c} n-1 \\ l-1 \end{array} \right] \right) 
- q^r \left( \left[ \begin{array}{c} n \\ m-1 \end{array} \right] + q^n \left[ \begin{array}{c} n-1 \\ m-2 \end{array} \right] \right) \left( \left[ \begin{array}{c} n \\ l+1 \end{array} \right] + q^n \left[ \begin{array}{c} n-1 \\ l \end{array} \right] \right) 
= \left[ \begin{array}{c} n \\ m \end{array} \right] \left[ \begin{array}{c} n \\ l \end{array} \right] - q^r \left[ \begin{array}{c} n \\ m-1 \end{array} \right] \left[ \begin{array}{c} n \\ l+1 \end{array} \right] 
+ q^n \left( \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right] \left[ \begin{array}{c} n \\ l \end{array} \right] - q^r \left[ \begin{array}{c} n-1 \\ m-2 \end{array} \right] \left[ \begin{array}{c} n \\ l+1 \end{array} \right] \right) 
+ q^n \left( \left[ \begin{array}{c} n \\ m \end{array} \right] \left[ \begin{array}{c} n-1 \\ l-1 \end{array} \right] - q^r \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right] \left[ \begin{array}{c} n-1 \\ l \end{array} \right] \right) 
+ q^{2n} \left( \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right] \left[ \begin{array}{c} n-1 \\ l-1 \end{array} \right] - q^r \left[ \begin{array}{c} n-1 \\ m-2 \end{array} \right] \left[ \begin{array}{c} n-1 \\ l \end{array} \right] \right) .
\]

Using the notation in Lemma 3.2, we see that
\[
K_q(n, m)K_q(n, l) - q^r K_q(n, m - 1)K_q(n, l + 1)
= D_q^r(n, n, m, l) + q^n D_q^r(n - 1, n, m - 1, l) + q^n D_q^r(n, n - 1, m, l - 1) 
+ q^{2n} D_q^r(n - 1, n - 1, m - 1, l - 1).
\]

Applying Lemma 3.2, we find that for \(1 \leq m \leq l < n\) and \(0 \leq r \leq 2l - 2m + 2\),
\[
D_q^r(n, n, m, l), D_q^r(n - 1, n, m - 1, l), \text{ and } D_q^r(n - 1, n - 1, m - 1, l - 1)
\]
have nonnegative coefficients as polynomials in \(q\), respectively, and for \(1 \leq m \leq l < n\)
and \(0 \leq r \leq 2l - 2m + 1\),
\[
D_q^r(n, n - 1, m, l - 1)
\]
has nonnegative coefficients as a polynomial in \(q\). It follows that for \(1 \leq m \leq l < n\) and \(0 \leq r \leq 2l - 2m + 1\),
\[
K_q(n, m)K_q(n, l) - q^r K_q(n, m - 1)K_q(n, l + 1)
\]
has nonnegative coefficients as a polynomial in \(q\). Hence it remains to show that the difference (3.11) has nonnegative coefficients as a polynomial in \(q\) when \(r = 2l - 2m + 2\).

It suffices to show that
\[
q^n D_q^{2l-2m+2}(n - 1, n - 1, m - 1, l - 1) + D_q^{2l-2m+2}(n - 1, n, m, l - 1)
\]
has nonnegative coefficients as a polynomial in \(q\). First, it is easy to check that
\[
q^n D_q^{2l-2m+2}(n - 1, n - 1, m - 1, l - 1) + D_q^{2l-2m+2}(n - 1, n, m, l - 1)
\]
\[ K_q(n, m) \left[ \frac{n - 1}{l - 1} \right] - q^{2l-2m+2} K_q(n, m - 1) \left[ \frac{n - 1}{l} \right]. \]

Using the following relation:

\[ K_q(n, m) = \frac{1 - q^{n+m}}{1 - q^n} \left[ \frac{n}{m} \right] = \left[ \frac{n - 1}{m - 1} \right] + q^m \left[ \frac{n}{m} \right], \]

we find that

\[ q^n D_{q^{2l-2m+2}}(n - 1, n - 1, m - 1, l - 1) + D_{q^{2l-2m+2}}(n, n - 1, m, l - 1) \]

\[ = \left( \left[ \frac{n - 1}{m - 1} \right] + q^m \left[ \frac{n}{m} \right] \right) \left[ \frac{n - 1}{l - 1} \right] - q^{2l-2m+2} \left( \left[ \frac{n - 1}{m - 2} \right] + q^{m-1} \left[ \frac{n}{m - 1} \right] \right) \left[ \frac{n - 1}{l} \right] \]

\[ = \left[ \frac{n - 1}{m - 1} \right] \left[ \frac{n - 1}{l - 1} \right] - q^{2l-2m+2} \left[ \frac{n - 1}{m - 2} \right] \left[ \frac{n - 1}{l} \right] \]

\[ + q^m \left( \left[ \frac{n}{m} \right] \left[ \frac{n - 1}{l - 1} \right] - q^{2l-2m+1} \left[ \frac{n - 1}{m - 1} \right] \left[ \frac{n - 1}{l} \right] \right) \]

\[ = D_{q^{2l-2m+2}}(n - 1, n - 1, m - 1, l - 1) + q^m D_{q^{2l-2m+1}}(n, n - 1, m, l - 1). \]

From Lemma 3.2, we see that

\[ D_{q^{2l-2m+2}}(n - 1, n - 1, m - 1, l - 1), \quad \text{and} \quad D_{q^{2l-2m+1}}(n, n - 1, m, l - 1) \]

have nonnegative coefficients as a polynomial in \( q \), respectively, and so (3.12) has nonnegative coefficients as a polynomial in \( q \). Thus, we complete the proof of Theorem 1.2.

\[ \square \]

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