Multimode theory of single-photon subtraction

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Abstract
We develop a general theory to describe the manipulation of a multimode quantum state of light via the subtraction of a single photon. The theory is applicable for various types of subtraction schemes independent of the physical nature of the light modes, their number or the embedded quantum states. We show that different subtraction schemes can be described in a unified approach through the characterization of their intrinsic subtraction modes. The conditional state of the multimode quantum light after the photon subtraction is defined by the number of subtraction modes and their matching with the light modes. We propose the manipulation of light states by controlling the subtraction modes. Performing a photon subtraction on a multimode quantum resource is promising for the implementation of a number of quantum information protocols in all-optical, multiplexed and scalable way.

1. Introduction

Highly multimode quantum light is a resource achievable nowadays in a number of experimental setups, especially in the continuous variable regime, where many optical modes hosting squeezed states are multiplexed in one beam of light [1–5]. Among distinct applications of the light one of the most promising is the optical implementation of continuous variable cluster states and the realization of scalable measurement based quantum computation [6].

At the same time, the controllable subtraction of a single photon from a beam of light is a practical technique to conditionally manipulate a quantum state of light at the level of single photons. Up to date it has been applied to solve a number of tasks in quantum information processing, particularly, in continuous variables regimes. The successful realization using this technique includes the preparation of various key quantum states for quantum optics such as Fock states [7, 8] and cat-like states [9]. In a similar fashion, single-photon subtraction is used in quantum state engineering to achieve hybrid entanglement [10] and to enhance entanglement between parties through entanglement distillation [11, 12]. More fundamentally, it has allowed the probing of quantum commutation rules along with single-photon addition [13]. In the perspective of quantum computation, single-photon subtraction is meant to allow one to turn a Gaussian state into a non-Gaussian state thus implementing universal non-Gaussian gates such as the cubic gate [14–16]. Also, it was demonstrated more recently that an assembly of photon subtracted squeezed states is suitable to tackle the boson sampling problem and its intrinsic complexity [17].

In this work, we consider the situation where both the quantum resource (multimode quantum light) and the state manipulation technique (photon subtraction) are brought together with the following motivation: an addressable and pure photon subtraction from a multimode light has the potential to engineer a quantum state and perform quantum information protocols in a scalable way. Existing studies on the photon subtraction [7, 18, 19] are focused on the conditional state of a single mode of the light, without considering the overall multimode quantum state and the general multimode theory has never been developed.
Here we develop a general theoretical description of the multimode single-photon subtraction applicable for arbitrary type of light modes and physical mechanism used to subtract a photon. We show in section 2 that the overall subtraction processes, modeled as a multimode light splitting and mode resolving photon detection, can be described with a set of subtraction modes and associated subtraction probabilities intrinsic to the process. Within this general framework, we calculate the conditional multimode quantum state. In section 3 we discuss the effect of matching the subtraction modes to the light modes on the resulting state of the light and investigate how purity of the resulting state depends on the subtraction procedure. We then consider photon subtraction from a multimode squeezed vacuum in section 4. In the weak squeezing limit, the photon subtraction heralds a state similar to a single-photon state and we find a general expression for the mode that maximizes its fidelity. We conclude in section 5 by summing up our results.

2. General framework of multimode single-photon subtraction

2.1. The single-mode case

In order to introduce the general formalism, we recall the results about the single-mode subtraction of a photon from a light field, whose density matrix writes \( \hat{\rho} \). The subtraction procedure is performed through the following steps: light splitting that can be performed via different physical mechanisms (optical beamsplitter [20], weak parametric up-conversion [21]) and conditioning the state of the signal light onto the single-photon detection in the split arm.

In order to calculate the evolution of a quantum state of light along the procedure we consider an enlarged system composed of the signal light and an auxiliary optical bath which is initially in the vacuum state. The light splitting from an optical mode \( \hat{\alpha}(r, t) \) with bosonic amplitude \( \hat{A} \) to a vacuum mode \( \hat{\beta}(r, t) \) with the amplitude \( \hat{B} \) is modelled via a beamsplitter unitary transformation \( \hat{U} \) that acts on the joint state represented by the separable density matrix: \( \hat{\rho} \otimes |0\rangle \langle 0| \). It is reminded that an optical mode is defined as a normalized solution of the wave equation \([22, 23]\) and is thus a vector of a modal Hilbert space with associated inner product \( \langle \alpha, \beta \rangle = \int \hat{\alpha}^\dagger(r, t) \cdot \hat{\beta}(r, t) \, dr \, dt \). This modal Hilbert space contains every physical property of the light such as spatio-temporal distribution, polarization, etc. The aforementioned unitary transformation reads:

\[
\hat{U} = \exp \left[ i \hat{\theta} (\hat{B}^\dagger \hat{A} + \hat{B} \hat{A}^\dagger) \right] \approx \hat{1} + i \hat{\theta} (\hat{B}^\dagger \hat{A} + \hat{B} \hat{A}^\dagger)
\]  

(1)

We have assumed weak interaction (\( \theta \ll 1 \)) so that a first order Taylor development could be performed. On the split arm, a single-photon detector of unit quantum efficiency performs a single-photon detection on an optical mode denoted \( \hat{\beta}(r, t) \) with annihilation operator \( \hat{D} \). The detection is described by a positive operator of measurement (POM) \([24]\) \( \hat{P} = \hat{1} - |0\rangle \langle 0| \approx |1\rangle \langle 1| \), that we intentionally reduce to a single-photon detection operator in mode \( \hat{\beta}(r, t) \). The quantum state of the signal light conditioned on a single-photon detection can then be computed by performing a partial trace over the state of the split arm: \( \hat{\rho}^- \sim \text{Tr}_b (\hat{\rho} \otimes |0\rangle \langle 0| \hat{D}^\dagger \hat{P}) \). To calculate the state we also take into account that: \( \langle 1| \hat{\beta} \rangle = \langle 0| \hat{D} \hat{B} \rangle |0\rangle = \langle \hat{\beta}, \hat{\beta} \rangle \), i.e. the probability amplitude to detect a photon in the splitted beam is defined by the overlap of the photon mode and the detection mode. As a result conditional quantum state of the signal light reads:

\[
\hat{\rho}^- = \theta \langle \hat{\beta}, \hat{\beta} \rangle \hat{1} \hat{\rho} \hat{A} \hat{A}^\dagger / P
\]  

(2)

The constant \( P \) is defined from the normalization condition of the density matrix and reads:

\[
P = \theta \langle \hat{\beta}, \hat{\beta} \rangle \text{ Tr} (\hat{1} \hat{A} \hat{A}^\dagger \hat{\rho})
\]  

(3)

This constant gives a probability to subtract and detect a single photon and consequently depends on the number of photons in the signal light, the splitting efficiency and the overlap between modes.

2.2. General multimode case

Here, we develop the general framework to describe the conditional multimode subtraction of a single photon from a quantum light in an arbitrary multimode state \( \hat{\rho} \). To do so, we model splitting of the signal light (see figure 1) via a multimode beamsplitter transformation \( \hat{U} \) as follows:

\[
\hat{U} = \exp \left[ i \sum_n \theta_n (\hat{B}_n^\dagger \hat{A}_n + \hat{B}_n \hat{A}^\dagger_n) \right]
\]

\[
\approx \hat{1} + i \sum_n \theta_n (\hat{B}_n^\dagger \hat{A}_n + \hat{B}_n \hat{A}^\dagger_n)
\]  

(4)

The transformation describes light splitting from multiple signal modes \( \{ \hat{\alpha}_n \} \) to the corresponding vacuum modes \( \{ \hat{\beta}_n \} \), where we omitted their spatial and temporal arguments. The associated annihilation operators are
\[ q_n = 1 \]

Thus, only a single photon is exchanged. In general, the number of signal modes is different from the number of vacuum modes to which they are coupled. Thus a coupling matrix of the unitary transformation is non-diagonal and rectangular in general.

The matrix can be diagonalized in a basis of new modes applying singular value decomposition to it, that we have assumed in the above expression.

We consider that a single photon is to be detected in the split arm by a detector with limited mode resolution. Its detection operator can be represented as a mixture of single-photon measurements on a set of detection modes whose annihilation operators are denoted:

\[ \hat{D}_m \]

Here the coefficients are efficiencies associated with the detection modes. The index \( \delta \) means that the state is written in the basis of modes. One notes that the detection modes are a priori different from the light splitting modes.

The density matrix of the signal light conditioned on single photon detection can then be computed by performing a partial trace over the single photon output subspace and reads:

\[ \hat{\rho}' = \sum_{n, n'} S_{mm'} \hat{A}_m \hat{A}_n^\dagger / P \]

with \( S_{mm'} = \sum_{m} \gamma_m (\hat{D}_m \hat{B}_n) \theta_{n'}^{\dagger} \theta_n (\hat{D}_m \hat{B}_n) \)

and \( P = \sum_{n, n'} S_{mm'} \Tr \hat{A}_m \hat{A}_n^\dagger \)

The measurement being imperfect, the detection operator should eventually be written in terms of Kraus operators as:

\[ \hat{\Pi} = \sum_i \hat{n}_i \hat{\beta}_i^\dagger \text{ with } \hat{n}_i = \sqrt{\gamma_i} |1_i\rangle \langle 1_i| \]

And the conditioned density matrix is computed as:

\[ \hat{\rho}' = \Tr_{\hat{\Pi}} \left[ \sum_i \hat{n}_i \hat{\beta}_i (\hat{\rho} \otimes |0_i\rangle \langle 0_i|) \hat{\beta}_i \dagger \right] / P \]
The normalization constant $P$ defines the total probability to subtract a single photon for a given setup. The weak splitting condition, assumed during the derivation, ensures that $P \ll 1$. Obtained equations (6–8) are general and applicable to describe the single-photon subtraction from a light in arbitrary quantum state $\hat{\rho}$. In the next sections we analyze the photon subtraction from specific quantum states.

Although distinct physical mechanisms can be applied to split light, equation (6) shows that the subtraction of a single-photon is always characterized by a matrix $S$ of coefficients $S_{m\ell}$ that we will refer to as the subtraction matrix. This matrix describes the whole physical process and is defined as:

$$S = \sum_{n,\ell} \hat{\alpha}_n \hat{\alpha}_\ell^\dagger = \sum_j \sigma_j \hat{v}_j \hat{v}_j^\dagger$$

(9)

where $\hat{\alpha}_n$ and $\hat{v}_j$ represent the dual of vectors $\hat{\alpha}_n$ and $\hat{v}_j$ in the modal Hilbert space. The last expression represents the diagonal decomposition of the matrix with orthonormal eigenvectors $\hat{v}_j$ and nonnegative eigenvalues $\sigma_j$, that exists since the matrix is positive-definite and hermitian according to equation (7). Substituting the decomposition into (6) one gets:

$$\hat{\rho}^{-} = \sum_j \hat{\sigma}_j \hat{\rho} \hat{\sigma}_j / P$$

with: $\hat{\sigma}_j = \sum_n \langle \hat{v}_j | \hat{\alpha}_n \rangle \hat{\Lambda}_n$

(10)

Therefore, the overall single-photon subtraction procedure involving multimode light splitting and detection can be described in terms of orthogonal subtraction modes $\{\hat{v}_j\}$ with their annihilation operators $\{\hat{\sigma}_j\}$ and efficiencies $\{\sigma_j\}$ (see figure 1). Let us stress that the expression (10) is not an eigen-decomposition of the density matrix itself.

The efficiencies $\sigma_j$ are all smaller than unity according to equation (7) and can be interpreted in terms of subtraction probability per photon per subtraction mode, as the total subtraction probability $P$ reads:

$$P = \sum_j \sigma_j \text{Tr}(\hat{\sigma}_j^\dagger \hat{\sigma}_j \hat{\rho})$$

(11)

In general the single-photon subtraction is multimode and the conditioned state (10) is mixed. The efficient number of subtraction modes can be characterized with the following quantity, which definition is similar to the one of a Schmidt number [25]:

$$K = \left( \sum \sigma_j^2 \right)^2 / \sum \sigma_j^2$$

(12)

Additional insight into the properties of the conditioned state (10) can be obtained by calculating its quantum marginals, i.e. reduced density matrices of its subsystems. Further, we consider the properties of the conditioned state under different scenarios for the single-photon subtraction.

3. Properties of the photon subtracted state

Expressions (6) and (10) are general and applicable to describe single-photon subtraction from any arbitrary quantum state of light. From now on, we consider a pure state of light such that there is a basis, defined by mode functions and associated their bosonic operators $\{\hat{u}_k, \hat{a}_k\}$, in which the quantum state of the light is factorized [23]:

$$\hat{\rho} = \bigotimes_{k \geq 0} \hat{\rho}_k$$

(13)

Moreover, we consider that each single-mode state $\hat{\rho}_k$ is pure and has a null mean amplitude. These assumptions are valid for a number of quantum states, such as: squeezed vacuum states, Fock states, cat states [26] and their superpositions. We introduce an estimator of the multimode nature of the light field through the definition of an efficient number $N$ of non-vacuum modes, such that:

$$N = \left( \sum \frac{n_k}{n_k^2} \right)$$

(14)

where $n_k$ is the mean photon number per mode $k$. This definition is inspired by the definition of the Schmidt number. For further analysis, it is useful to introduce an expansion of the subtraction modes over the light modes:
\[ \hat{s}_j = \sum_k c_k \hat{a}_k \quad \text{with} \quad c_k = \langle \tilde{v}_j, \tilde{u}_k \rangle \]  

\[ 3.1. \text{State purity} \]

In general the photon subtraction is multimode and the output state \( \hat{\rho}^- \) is mixed. The state purity \( \pi = \text{Tr}( (\hat{\rho}^-)^2 ) \) can be calculated using the expression (6) and the following conditions: (1) the input state of the light is pure, i.e. \( \hat{\rho}_1 = |\psi\rangle \langle \psi| \); (2) \( \hat{A}_n = \sum_k \langle \tilde{a}_m, \tilde{u}_k \rangle \hat{a}_k \); (3) input modes of the light have null mean amplitudes, i.e. \( \langle \psi| \hat{a}_k |\psi\rangle = 0 \); (4) \( \langle \tilde{u}_k, \hat{S}_k \rangle = \sum_n \langle \tilde{a}_k, \tilde{a}_n \rangle S_m n \). The resulting expression for the purity can be written in two ways:

\[ \pi = \sum_{k,k'} n_k n_{k'} |\langle \tilde{u}_k, \hat{S}_k \rangle|^2 / P^2 \]

\[ = \sum_{j,k} |\langle \gamma_j, \gamma_k \rangle|^2 n_k / P^2 \]

The last expression provides a way to calculate the purity if the subtraction modes and associated efficiencies are known. In general, state purity depends both on the initial quantum state of the light and on the applied subtraction procedure, described by the matrix \( S \). From (8) one also gets expressions to calculate the photon subtraction probability:

\[ P = \sum_k n_k |\langle \tilde{u}_k, \hat{S}_k \rangle|^2 \]

\[ = \sum_{j,k} |\langle \gamma_j, \gamma_k \rangle|^2 n_k \]

One can distinguish two extreme cases for the output state depending on the relation between the numbers of modes \( N \) of the signal light and the number of subtraction modes \( K \). In the case of non-selective photon subtraction, when \( K \gg N \) (i.e. \( S \propto \textbf{1} \)) the state purity satisfies:

\[ \pi_{K \gg N} = \frac{1}{N} \]

In the opposite case when \( N \gg K \) (i.e. \( n_k = \text{const} \)), one can show that the purity reads:

\[ \pi_{N \gg K} = \frac{1}{K} \]

The general statement is that the output state is pure when the detected photon belongs with certainty to a particular mode. In a trivial situation, the light field is single mode (\( N = 1 \)) [23], in an interesting case, the subtraction procedure is single-mode itself (\( K = 1 \)) and there is only one term in the sum (10). According to our model of the photon subtraction (7), the subtraction procedure is single-mode in the following cases:

- the light splitting is single mode and the photon is subtracted from a mode \( \tilde{\nu} = \tilde{\omega} \);  
- the detector is mode resolving. Photon detection in the mode \( \tilde{\omega} \) heralds the photon subtraction from the mode \( \tilde{\nu} \propto \sum_n \langle \tilde{\omega}_n, \tilde{\omega} \rangle \theta_n \).

\[ 3.2. \text{Matched and non-matched photon subtraction} \]

Consider a case where the subtraction procedure is single-mode itself—there is only one term in the sum (10). Denoting the corresponding annihilation operator of the subtraction with \( \hat{s} \), the resulting pure state simply reads:

\[ \hat{\rho}^- = \hat{s} \hat{\rho} \hat{s}^+ / \text{Tr}( \hat{s}^+ \hat{s} \hat{\rho} ) \]

Nevertheless, while the state is pure, the photon subtraction does not necessarily happen in a specific mode of signal light (13). Then the following scenarios schematically shown in figure 2 can be realized:

- the subtraction mode is matched to one of the signal light modes (13) that we denote with index `\( k \)` so that:
  \[ \hat{s} = \hat{a}_k \]
  \[ \hat{\rho}^- = \hat{\rho}^0 \hat{\rho}^0 \]
  where \( \hat{\rho}^- = \hat{a}_k \hat{a}_k^+ / n_k \)

  \[ \text{and } n_k \text{ is a mean photon number in the mode.} \]

- the subtraction mode is not matched to a particular mode of the signal light—a photon is subtracted from a linear combination of the light modes (13):
A non-matched single-photon subtraction leads to the entanglement of the light modes in general. It can be used to perform entanglement distillation [12, 26, 27] and study the decoherence of the multimode state or to achieve a non-Gaussian gate on a controllable optical mode such as the node of an optically implemented cluster state [4, 28].

### 3.3. A two-mode example

Here we consider the subtraction of a photon from a two-mode signal light whose initial quantum state can be written as follows:

$$|\psi_\text{in}\rangle = |\psi_{11}\rangle_1 |\psi_{22}\rangle_2$$

We illustrate the difference between non-pure and pure scenarios of the photon subtraction.

In the first case, the photon is subtracted with equal probabilities from each mode. In the above formalism, it means that there are two subtraction modes: $s_{12}$ and $s_{21}$. Then

$$c_{12} c_{21} = c_{11} c_{22}$$

and the purity of the output state, according to (17), becomes:

$$\pi = \frac{n_1^2 + n_2^2}{(m_1 + n_2)^2}$$

A near unit purity is then only achieved if one of the two modes totally overcomes the other with a much higher photon number i.e., if the input state is almost single mode.

In the second case, the single-photon subtraction is pure and the photon is subtracted coherently from a superposition of modes, for example: $s = \hat{a}_1 + \hat{a}_2$. Then $c_{11} = c_{22}$ and the purity of the output state is then necessarily equal to unity. The output state is given by the following superposition of states:

$$|\psi_\text{out}\rangle \propto (\hat{a}_1 |\psi_{11}\rangle_1 |\psi_{22}\rangle_2 + |\psi_{11}\rangle_1 \hat{a}_2 |\psi_{22}\rangle_2)$$

In general, the state is no longer factorizable in the original basis and the modes of the signal light are entangled. What is at stake is the interplay between the light modes and modes of the subtraction procedure.

### 3.4. State of a single mode of light

Here we consider a quantum state of a mode ‘k’ of the multimode signal light (13) after the subtraction of a photon from the light. The conditional probability that the subtracted photon belongs to the mode ‘k’ is given by the expression:

$$P_k = n_k \langle \hat{u}_k, S \hat{u}_k \rangle / P$$

$$= n_k \sum_j |c_j|^2 / P$$

The conditional density matrix of the mode, denoted as $\hat{\rho}^{-1}|_k$, is then a statistical mixture of two density matrices representing two possibilities—either the photon was subtracted from the mode or not:

$$\hat{\rho}^{-1}|_k = P_k \hat{\rho}_k^- + (1 - P_k) \hat{\rho}_k$$

This result can be also formally obtained by tracing out all the modes except the given one in the multimode state (10). One can then show that the subtraction probability $p_k$ defines the fidelity [29] of the resulting state $\hat{\rho}^{-1}|_k$ with the single-photon subtracted state $\hat{\rho}_k^-: $
\[ F(\hat{\rho}|k, \hat{\rho}_{\alpha}) = \text{Tr}(\hat{\rho}|k\hat{\rho}_{\alpha}) = p_k \]  

(31)

The purity of the state (30) is calculated as:

\[ n_k = \text{Tr}(\hat{\rho}|k\hat{\rho}_{\alpha}) = p_k^2 + (1 - p_k)^2 \leq 1 \]  

(32)

There are then two reasons for the single-mode state to be mixed. Firstly, when the single-photon subtraction itself is not pure and results in a non-pure multimode state \( \hat{\rho} \). Secondly, if the subtraction mode is not matched to the given light mode, (i.e. \( |\epsilon^2| < 1 \)) it results in entanglement with the other light modes.

### 4. Photon subtraction from multimode squeezed vacuum

From now on, we consider the photon subtraction from light in a particular quantum state, i.e. pure multimode squeezed vacuum, described by the expression:

\[ \hat{\rho} = \prod_k \exp\left[ \frac{\xi_k}{2} \hat{a}_k^+ \hat{a}_k - \frac{\xi_k^*}{2} \hat{a}_k \hat{a}_k^+ \right] |0\rangle \otimes \text{h.c.} \]  

(33)

where \( \xi_k = r_k e^{i\phi} \) is a complex squeezing parameter of \( k \)-th mode.

#### 4.1. Negativity of the Wigner function

Photon subtraction from squeezed light provides a way to produce a quantum state of light with non-Gaussian, e.g. negative, Wigner function [20]. When the subtraction is performed on the multimode squeezed light (33), the expression (30) allows to calculate the Wigner function of the state of a given light mode \( \alpha \) after the procedure:

\[ W^{-}|k(\alpha, \alpha^g) = p_k W_k^{-}(\alpha, \alpha^g) + (1 - p_k) W_k(\alpha, \alpha^g) \]  

(34)

where \( W_k \) and \( W_k^{-} \) are respectively the Wigner functions of the squeezed vacuum state and the photon subtracted squeezed vacuum state [30], also known as a ‘squeezed single-photon state’:

\[ W_k(\alpha, \alpha^g) = \frac{2}{\pi} e^{-2|\alpha|^2}, \]  

(35)

\[ W_k^{-}(\alpha, \alpha^g) = \frac{2}{\pi} e^{-2|\alpha|^2} (4|\alpha|^2 - 1) \]  

(36)

where \( \alpha = \alpha \cosh r_k - \alpha^g e^{i\phi} \sinh r_k \) is a squeeze coordinate transformation. Both the Wigner functions have extrema at the origin of the phase-space: \( W_k'(0, 0) = -W_k^{-}(0, 0) = 2/\pi \). In turn, the total Wigner function (34) of the state embedded in mode \( \alpha \) possesses a negative value at the origin of the phase space only when the probability to subtract a photon from the mode is higher than one-half: \( p_k > 1/2 \). This value constitutes a benchmark for the single-mode subtraction probability.

#### 4.2. Photon subtraction from weakly squeezed multimode light

In this section, we consider the single-photon subtraction from a multimode light in a weakly squeezed vacuum state, that we approximate as a single mode state, that we approximate as a superposition of mostly vacuum state and two photon states:

\[ \hat{\rho} \approx |0\rangle + \frac{1}{\sqrt{2}} \sum_k \xi_k |2k\rangle_u \]  

(37)

where we defined Fock states of the modes \( \hat{u}_k \): \( |n_k\rangle_u = |0, \ldots, 0, n, 0, \ldots\rangle_u = (\hat{a}_k^+)^n |0\rangle / \sqrt{n!} \). Under this approximation, one can get explicit results for the single-photon subtracted state that can be treated as an approximation of the general case for arbitrary squeezing. For instance, one can consider a single mode state which probability to measure a two-photon state is much greater than the probability to measure a four-photon (or higher even number) state. A ratio of 10 corresponds to about 3.17 dB of squeezing.

The conditional subtraction of a photon heralds a single-photon state. The corresponding state is derived from equations (10), (15), (37) and can be written as:

\[ \hat{\rho}^- = \sum_{k,k'} L_{kk'} \langle 1_{kk'} \rangle |1_{kk'}\rangle / P \]  

(38)

with the coefficients:

\[ L_{kk'} = \xi_k \xi_{k'}^* \langle \hat{u}_k, \hat{u}_{k'} \rangle \]  

\[ = \xi_k \xi_{k'}^* \sum_j \sigma_j \langle \hat{u}_k, \hat{v}_j \rangle \langle \hat{v}_j, \hat{u}_{k'} \rangle \]  

(39)
We first consider the case where the single-photon subtraction is single-mode and happens in mode $\tilde{v}$. One can show that expression (38) is reduced to a pure single-photon state:

$$|1\rangle \propto \sum_k \xi_k (\tilde{v}, \tilde{u}_k) |1_k\rangle_u$$

(40)

The state means that the single photon is heralded in a mode defined by the squeezing parameter of each input mode as well as its overlap with the subtraction mode:

$$\tilde{w} \propto \sum_k \xi_k (\tilde{v}, \tilde{u}_k) \tilde{u}_k,$$

(41)

As an illustration, we consider the photon subtraction from a continuous-wave squeezed light produced in a degenerate optical parametric oscillator operating below its oscillation threshold [31]. In particular, we consider the photon subtraction from a single longitudinal mode of the oscillator. Such a regime is achieved experimentally by placing an additional spectral filter in subtraction channel, so that the filter removes the nondegenerate cavity modes. The light after the filter possesses sideband squeezing in the mode bandwidth $\gamma$. We model this physical situation as frequency dependent squeezing $\xi_k \rightarrow |\xi_k| \propto (1 + 4\omega^2/\gamma^2)^{-1}$ of a set of continuous modes of squeezing $\tilde{u}_k \rightarrow u_k(t) \propto e^{i\omega t}$. Furthermore we assume that a single-photon detector placed in the subtraction channel has a temporal resolution better than $\gamma^{-1}$. Under this condition the detector click heralds instant subtraction of a single photon from the light. We model the subtraction mode as a Dirac function in the time domain $\tilde{v} \rightarrow \delta(t)$. Then the mode of the heralded photon reads $\tilde{w} \rightarrow \int \text{d} \omega \, e^{i\omega t}/(1 + 4\omega^2/\gamma^2) \propto e^{-i\omega t}$, according to (41). It is a double-sided exponential reproduced in several works on the photon subtraction [32, 7, 31, 33]. Here we have illustrated its derivation using the multimode approach presented above.

Conversely, when the subtraction is not mode-selective (i.e. $S \propto \hat{1}$ in equation (39)), the heralded single-photon state of equation (38) reads: $\hat{\rho}^- = \sum_k \xi_k^2 |1_k\rangle \langle 1_k|_w / P$. The single-photon is then heralded with the highest probability in the most squeezed mode.

To treat the intermediate situation, we introduce a hermitian matrix $L$ of the single-photon state similarly to the subtraction matrix $S$:

$$L = \sum_{k,k'} L_{kk'} \tilde{u}_k \tilde{u}_k^{\dagger} = \sum_j \lambda_j \tilde{w}_j \tilde{w}_j^{\dagger}$$

(42)

The second expression is an eigen-decomposition of $L$ in a basis of orthonormal eigenvectors $\{\tilde{w}_j\}$. By substituting this decomposition into equation (38), we get the single-photon density matrix in a diagonal form, i.e. as a sum of single-photon states in orthogonal modes:

$$\hat{\rho}^- = \sum_j \lambda_j |1_j\rangle \langle 1_j|_w / P,$$

where: $|1_j\rangle_w \propto \sum_k \langle \tilde{w}_k, \tilde{u}_k | 1_k\rangle_u$

(43)

The highest eigenvalue $\lambda_j$ in the decomposition, denoted with the index ‘s’, defines the mode where the photon is heralded with the highest probability. In general, this mode is different from the modes of squeezing of the signal light. The state embedded in this mode is a statistical mixture of a single-photon and vacuum state:

$$\hat{\rho}_s = \text{Tr}_{\tilde{w}^\perp}(\hat{\rho}^-) = \rho_s |1\rangle \langle 1|_v + (1 - \rho_s) |0\rangle \langle 0|_v,$$

where: $\rho_s = \lambda_s / P$

(44)

The coefficient $\rho_s$ gives the conditional probability to herald a single photon in the mode ‘s’. It is also the fidelity of the state in mode ‘s’ with the single-photon–Fock state. The Wigner function of the state in mode ‘s’ can be written using the expressions (34, 36), where one have to set $\sigma = \alpha_s$. It leads to a statistical mixture of vacuum and a single-photon state. The Wigner function is negative at the origin of the phase space only when $\rho_s > 1/2$. Furthermore, comparing (28) and (39) alongside with (44), one sees that $\rho_s$ is a diagonal element of $L$, while $\rho_s$ is the maximal eigenvalue of the matrix. The Courant–Fischer theorem guarantees the relation: $p_s \geq \rho_s$. Thus the mode ‘s’ is the best candidate to find negativity of Wigner function of the corresponding quantum state, when the weak squeezing approximation holds.

5. Conclusion

We have introduced a general framework that allows the full description of the subtraction of a single photon from a multimode quantum light through the computation of the process subtraction modes and efficiencies. The strength of this formalism is its adaptability to any subtraction schemes. Furthermore, by choosing different subtraction methods applied to the same multimode quantum light, one can tailor the conditional quantum
state of light. It is a special possibility provided by multimode quantum light. Such an approach is promising to engineer the quantum state of light and perform various quantum information protocols on the same quantum resource in a scalable way.

More specifically, we have shown that the developed framework predicts the properties of the single-photon subtracted multimode state given the modal structure of the quantum state of light, of the splitting mechanism and of the single-photon detection. In particular, we have calculated the purity of the multimode state and have derived the conditions to get a pure resulting state. We have also calculated the reduced density matrices of different light modes. For the particular case of weakly squeezed light, we have found a general recipe to calculate a mode where a single-photon state is heralded with the highest conditional probability.

We anticipate that the formalism developed here is suitable to analyze the existing experiments on conditional quantum state preparation performed via the photon subtraction. It is also useful as a theoretical background for future works aimed at manipulating the quantum state of a multimode light.

Finally, the developed formalism can perfectly describe a single-photon addition as well as multiphoton conditional operations on a multimode light resource.

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