Rasmussen invariant and the divisibility of Lee’s class

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Abstract

We define a link invariant \( s'_c(L) \) by a combination of some classical properties of its diagram \( D \), and the \( c \)-divisibility of a specific class (generalized Lee’s class) in the Khovanov-type link homology \( H_c(D; R) \) determined by \( D \) and an element \( c \) in the base ring \( R \). This invariant has many properties common to the Rasmussen’s \( s \)-invariant: \( s'_c \) is a link-concordance invariant, provides a lower bound for the slice genus, and the equality is sharp for positive knots. Similarly an alternative proof for the Milnor conjecture follows. In particular if the base ring is the polynomial ring \( Q[h] \) and \( c = h \), then \( s'_h \) coincides with the \( s \)-invariant for knots.

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1 Introduction

Almost two decades have passed since Khovanov introduced in [14] a link homology theory that categorifies the Jones polynomial, now known as Khovanov homology. In [28] Rasmussen introduced a knot invariant $s(K)$ based on Lee homology over $\mathbb{Q}$, which is a modified version of Khovanov homology introduced by Lee in [20]. The invariant defines a homomorphism from the knot concordance group in $S^3$ to $2\mathbb{Z}$, provides a lower bound for the slice genus, and the equality is sharp for positive knots. As a corollary, an alternative proof for the Milnor conjecture follows. The result was notable since it provided for the first time, a purely combinatorial proof of an important fact of 4-dimensional topology. The well-definedness of $s(K)$ is based on the invariance of the “canonical generators” of Lee homology, a specific set of classes introduced by Lee that is defined combinatorially from a link diagram, and which Rasmussen proved that is invariant (up to unit) under the Reidemeister moves.

Our research started from considering the Lee homology over $\mathbb{Z}$. Do Lee’s classes generate the homology over $\mathbb{Z}$? Computational experiments showed that the answer is “No”. In fact, for all knot diagrams that we have computed, the components of the classes with respect to a computed basis, were 2-powers. Also we can find in the proof Rasmussen’s, that the units being multiplied by the Reidemeister moves are actually 2-powers. Where does ‘2’ come from?

Khovanov introduced a general framework in [16] that unifies the original theory, Lee’s theory and other variant theories, by considering a general quadratic polynomial $X^2 - hX - t$ with $h, t$ elements in the base ring $R$, and a Frobenius algebra defined by $A = R[X]/(X^2 - hX - t)$. From arguments given by Mackaay, Turner and Vaz in [31], when the polynomial splits as product of linear polynomials, then Lee’s classes can be generalized. For Lee homology the polynomial is $X^2 - 1$, and 2 comes from the difference of the two roots. In general we denote the difference of the roots by $c$. If $c$ is invertible, then
similar arguments follow as in Lee theory. If it is not, then we can consider the maximal $c$-divisibility of the class (modulo torsions), which we denote by $k_c(D)$. By inspecting the variance of this value under the Reidemeister moves, we see that the following combination of values determine a link invariant:

$$s'_c(L) = 2k_c(D) - r(D) + w(D) + 1,$$

where $r(D)$ is the number of Seifert circles of $D$, and $w(D)$ is the writhe of $D$. Also by inspecting the behaviour of $s'_c(L)$ under cobordisms between links, we see that $s'_c(L)$ possesses properties common to those of the $s$-invariant: it is a link concordance invariant, it provides a lower bound for the slice genus, and the equality is sharp for positive knots. Again, the Milnor conjecture follows as a corollary. If the base ring is the polynomial ring $\mathbb{Q}[h]$ (which corresponds to the bigraded Bar-Natan homology over $\mathbb{Q}$), then we see that $s'_h$ coincides with the $s$-invariant for knots (Theorem 5.32):

$$s'_h(K) = s(K).$$

The proof is accomplished by normalizing Lee’s classes so that they correspond exactly under the Reidemeister moves. Thus the normalized generators are link invariants, and we denote them by $[\zeta(K)], X[\zeta(K)]$. $s$-invariant can be characterized as

$$s(K) = \text{qdeg}[\zeta(K)] - 1.$$

Not only that $[\zeta(K)]$ is a knot invariant, it is a knot concordance invariant, which reflects the knot concordance invariance of $s(K)$. The above construction works for any field $F$ of char $\neq 2$, and for Lee theory over $\mathbb{Z}$. In these cases we see that the corresponding homology theories (modulo torsions) are functorial with respect to cobordisms between knots (without sign ambiguity), and in particular they are invariant under knot concordance. We do not know at the time of writing, whether there exists any $c$ such that $s'_c$ is distinct from $s$.

Outline

In Section 2, we review the definitions of Khovanov homology, Lee homology and the $s$-invariant. In Section 3, we first review the generalization of Khovanov homology theory based on its relation with TQFT. Then we generalize Lee’s classes, which we call $\alpha$-classes. Proposition 3.10 states the variance of $\alpha$-classes under the Reidemeister moves, and is essential in the following sections. In Section 4, we define the $c$-divisibility $k_c(D)$ of the $\alpha$-class (modulo torsion), and define the link invariant $s'_c(L)$. By inspecting the behaviour of $s'_c$ under cobordisms, we obtain many properties common to the $s$-invariant. In Section 5, we focus on knots, and specialize to the Lee theory over $\mathbb{Z}$, and to the (bigraded) Bar-Natan theory over a field $F$ of char $\neq 2$. We normalize the $\alpha$-classes so that they correspond exactly under the Reidemeister moves. Using this generator we obtain some functorial properties of the theory, and also conclude that $s'_h(K; F[h])$ coincides with $s(K; F)$. Section 6 gives further remarks and questions. Postponed proofs are given in the final section.
Conventions

In this paper, we refer to a link as an isotopy class of a disjoint union of smoothly embedded circles in $\mathbb{R}^3$, always equipped with an orientation. A knot is a link with one component. We refer to a diagram of a link as the image under a regular projection onto $\mathbb{R}^2$ of the link. For any link $L$, we denote by $|L|$ the number of components of $L$, by $-L$ the link obtained from $L$ by reversing the orientation on each of its components, and by $\overline{L}$ the mirror image of $L$. For any pair of links $L, L'$, the disjoint union $L \sqcup L'$ is the union of $L$ and $L'$ after a translation so that they are disjoint, in a sense that we can take two disjoint 3-balls containing the links separately. The connected sum $L \# L'$ is not uniquely determined unless they are both knots, but we assume it represents one of the links obtained by the usual connected sum operation between $L$ and $L'$. When $L \# L'$ appears in a statement, we assume that it holds for any choice of such operation. Corresponding notations for diagrams should be understood obviously.

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2 Review: Khovanov homology theory

In this section, we give a brief review of the construction of Khovanov homology, Lee homology, and the definition of Rasmussen’s $s$-invariant.

2.1 Khovanov homology

We follow the original construction given in [14]. Let $D$ be a link diagram with $n$ crossings. Each crossing admits a 0-resolution and a 1-resolution, as depicted in Figure 1.

A simultaneous choice of resolutions for all crossings is called a state. The weight of a state $s$ is the number of 1-resolved crossings, and is denoted by $|s|$. Any state $s$ yields a diagram consisting of disjoint circles by resolving all crossings accordingly. We call these circles $s$-circles, and denote the number of the $s$-circles by $r(D, s)$.

Two states are adjacent if one is obtained from the other by changing the resolution of a single crossing. We write $s \prec s'$ when $s$ and $s'$ are adjacent and
Figure 1: 0-, 1-resolution of a crossing.

$|s| + 1 = |s'|$. Passing between adjacent states can be seen as performing a band sum to the circle(s) along the crossing, resulting in either two $s$-circles merging into one $s'$-circle, or one $s$-circle splitting into two $s'$-circles.

Prior to constructing the chain complex, we construct a commutative cube $C$ associated to $D$. Let $R$ be a commutative ring with unity, and $A$ be a free $R$-module generated by 1 and $X$. To each state $s$ we assign a vertex module $V(D, s)$ defined by the $r(D, s)$-fold tensor product of $A$. An element in $V(D, s)$ of the form:

$$x = e_1 \otimes \cdots \otimes e_r, \quad e_i \in \{1, X\}$$

is called an enhanced state. Each enhanced state is identified with a labeling of 1 or $X$ on the $s$-circles, and all together they states form a basis of $V(D, s)$. To each pair of adjacent states $s \prec s'$, we assign an edge map from $V(D, s)$ to $V(D, s')$ defined as follows. Depending on whether the $s$-circle(s) merge or split, apply the multiplication $m: A \otimes A \to A$ or the comultiplication $\Delta: A \to A \otimes A$ to the tensor factor(s) corresponding to the $s$-circle(s) being transformed, while leaving other factors unchanged. $m, \Delta$ are given by:

- $m(X \otimes X) = 0$, $\Delta(X) = X \otimes X$,
- $m(X \otimes 1) = X$, $\Delta(1) = X \otimes 1 + 1 \otimes X$,
- $m(1 \otimes X) = X$,
- $m(1 \otimes 1) = 1$.

These modules and maps can be placed on a 1-skeleton of an $n$-dimensional cube. It can be shown by direct calculation, or by relating the construction to a specific TQFT (detail discussion is given in the next section), that every square (a face with 4 vertices) of the cube commutes.

Next we turn this cube skew-commutative. Let $X$ be the set of crossings of $D$, and consider the exterior algebra formally generated by $X$ over $R$. Any subset $Y \subseteq X$ determines a rank 1 free submodule that is spanned by the wedge product of crossings in $Y$. Any $Y \subseteq X$ corresponds one-to-one to a state $s$, so we obtain a skew commutative cube $E$ by placing to each vertex the rank 1 submodule, and to each edge a map defined by wedging the missing crossing from the right (see Figure 2). Tensoring this to the previously constructed $C$ (vertex-wise and edge-wise) yields a skew-commutative cube $C' = C \otimes E$.

For actual calculation there is an easier way by fixing the ordering of the crossings of $D$. An ordering determines the preferred generator for each vertex.
module of \( \mathcal{E} \), and thus gives an isomorphism from the vertex module to the base ring \( R \). Each map \( \wedge x_i \) corresponds to a multiplication by \( \pm 1 \). With this correspondence, tensoring \( \mathcal{E} \) reduces to adding signs to the edges of \( C \) accordingly. With the ordering fixed, it is convenient to identify each state with a word of length \( n \) consisting of two letters \( \{0, 1\} \), and denote the edge between states \((\cdots 0 \cdots) \prec (\cdots 1 \cdots)\) by \((\cdots \star \cdots)\). The sign is determined by the parity of the number of 1's appearing after the star \( \star \).

Now we fold the cube into a chain complex. The \( i \)-th chain group \( \overline{C}_i^{\mathrm{Kh}}(D; R) \) is defined as the direct sum of the vertex modules associated to states of weight \( i \):

\[
\overline{C}_i^{\mathrm{Kh}}(D; R) = \bigoplus_{|s| = i} V(D, s).
\]

The \( i \)-th differential is defined by summing up the corresponding edge maps:

\[
d^i : \overline{C}_i^{\mathrm{Kh}}(D; R) \to \overline{C}_{i+1}^{\mathrm{Kh}}(D; R).
\]

\( d \circ d = 0 \) follows from the skew commutativity of the cube. Thus we obtain a chain complex \( (\overline{C}_\cdot^{\mathrm{Kh}}(D; R), \ d^\cdot) \).

We also endow a secondary grading on \( \overline{C}_\cdot^{\mathrm{Kh}}(D; R) \). Let \( \deg(1) = 0, \deg(X) = -2 \). We define the degree of an enhanced state \( x \) by the sum of the degrees of the factors, and its unnormalized quantum degree by \( 2 \)

\[
\overline{\mathrm{qdeg}}(x) = \deg(x) + |s| + r(D, s).
\]

For any element \( x \) in \( V(D, s) \), its degree \( \deg(x) \) is defined by the minimum degree among its terms. We see that \( d \) preserves \( \overline{\mathrm{qdeg}} \). Thus we obtain a bigraded chain complex \( (\overline{C}_\cdot^{\mathrm{Kh}}(D; R), \ d^\cdot) \) with a differential of bidegree \((1, 0)\). We call this the unnormalized Khovanov chain complex, and its homology \( \overline{\Pi}_\cdot^{\mathrm{Kh}}(D; R) \) the unnormalized Khovanov homology of \( D \). In order to obtain a bigraded link invariant, we need to shift the bidegree of \( \overline{C}_\cdot^{\mathrm{Kh}}(D; R) \). Let \( n^+, n^- \) be the number of positive, negative crossings of \( D \) respectively. Define the (normalized)

\[\text{In [5] the sign is taken by the parity of the number of 1's appearing before the star \( \star \).}
\]
\[\text{In [14] the q-degree is defined by \( \deg(1) = 1, \deg(X) = -1 \), and \( \overline{\mathrm{qdeg}}(x) = \deg(x) + |s| \), which gives the same result.}
\]

Figure 2: The skew commutative cube \( \mathcal{E} \).
Figure 3: $\mathcal{C}_{Kh}(D; R)$ for the left hand trefoil.

Khovanov chain complex of $D$ by:

$$C_{Kh}(D; R) = C_{Kh}(D; R)[-n^-, n^+ - 2n^-].$$

where $[p, q]$ denotes the $(p, q)$ bidegree shift on a bigraded module\(^3\). The (normalized) Khovanov homology of $D$ is its homology:

$$H_{Kh}(D; R) = H(C_{Kh}(D; R)).$$

We call the first degree the homological degree (or $h$-degree), and the second degree the quantum degree (or $q$-degree). The $q$-degree of an element $x$ in $V(D, s)$ is described as:

$$q\text{deg}(x) = \text{deg}(x) + |s| + r(D, s) + n^+ - 2n^-.$$

The following are the two main results of [14].

**Theorem 2.1** ([14, Theorem 1]). Let $L$ be a link. For any diagram $D$ of $L$, the isomorphism class of $H_{Kh}(D; R)$ (as a bigraded $R$-module) is an invariant of $L$.

**Proposition 2.2** ([14, Proposition 9]). The graded Euler characteristic of $H_{Kh}(L; \mathbb{Q})$ gives the (unnormalized) Jones polynomial of $L$:

$$\sum_{i,j} (-1)^i q^j \dim \mathbb{Q}(H_{Kh}^{ij}(L; \mathbb{Q})) = (q + q^{-1})V(L)|_{\sqrt{q} = -q}.\footnote{This is opposite to the notation used in [14]}$$
A proof of Theorem 2.1 is given in Section 7.1 with a more generalized setting. We cite the following propositions without proofs.

**Proposition 2.3** ([14, (167), Proposition 32]).

1. \(C_{Kh}(-D) = C_{Kh}(D)\)
2. \(C_{Kh}(D \sqcup D') \cong C_{Kh}(D) \otimes C_{Kh}(D')\)
3. \(C_{Kh}(D) \cong C_{Kh}^*(D)\), where \((C_{Kh}^*)^{ij} = \text{Hom}_R(C_{Kh}^{-i-j}, R)\).

**Proposition 2.4.** Let \(D\) be a link diagram, and \(x\) be any crossing of \(D\). Let \(D_0, D_1\) be diagrams obtained from \(D\) by 0-, 1-resolving the crossing \(x\) respectively. By fixing a crossing order of \(D\) such that \(x\) is placed at the last, \(C_{Kh}(D)\) decomposes (as an \(R\)-module) as the direct sum of two chain complexes:
\[
\overline{C}_{Kh}(D) \cong \overline{C}_{Kh}(D_0) \oplus \overline{C}_{Kh}(D_1)[1, 1]
\]
and the differential \(d\) is given by:
\[
d = d_0 + d_1^0 - d_1,
\]
where \(d_0, d_1\) are the differentials of \(\overline{C}_{Kh}(D_0), \overline{C}_{Kh}(D_1)\) respectively, and \(d_1^0\) is the chain map defined by the sum of the edge maps in the cube of \(D\) mapping from \(D_0\) to \(D_1\).

**Proposition 2.5.** With the assumption of Proposition 2.4, there is a short exact sequence of chain complexes:
\[
0 \longrightarrow \overline{C}_{Kh}(D_1)[1, 1] \overset{i}{\longrightarrow} \overline{C}_{Kh}(D) \overset{j}{\longrightarrow} \overline{C}_{Kh}(D_0) \longrightarrow 0
\]
where \(i\) is the inclusion, \(j\) is the projection. The connecting homomorphism in the long homology exact sequence is the induced homomorphism of \(d_1^0\).

### 2.2 Lee homology and \(\alpha\)-classes

In [20], Lee defined a variant Khovanov-type homology theory by modifying the (co)multiplication of \(A\). It was originally introduced to prove the conjectures proposed in [5] about the patterns of Khovanov homology of alternating knots. However the theory also led to Rasmussen’s \(s\)-invariant and to other variants of Khovanov-type homology.

As in the original theory, let \(A\) be the free \(R\)-module generated by 1 and \(X\). In Lee’s theory, the multiplication and the comultiplication are given by:
\[
\begin{align*}
m(X \otimes X) &= 1, & \Delta(X) &= X \otimes X + 1 \otimes 1, \\
m(X \otimes 1) &= X, & \Delta(1) &= X \otimes 1 + 1 \otimes X \\
m(1 \otimes X) &= X \\
m(1 \otimes 1) &= 1
\end{align*}
\]
The underlined terms indicate the difference from the ones given in the original theory. Following the construction of the original theory, we obtain the Lee chain complex \((C_{\text{Lee}}(D; R), d)\), and its homology \(H_{\text{Lee}}(D; R)\) the Lee homology.

Note that the underlying module of Lee’s chain complex is the same as Khovanov’s. The q-degree is similarly defined on the chain level, but since the underlined terms of \(m, \Delta\) increase the q-degree by 4, \(d\) does not preserve the q-degree and hence \(H_{\text{Lee}}(D; R)\) cannot be bigraded. However, \(d\) is q-degree non-decreasing, so we may define a filtration on \(C_{\text{Lee}}(D; R)\) as

\[
F^j C^i_{\text{Lee}}(D; R) := \{ x \in C^i_{\text{Lee}}(D; R) \mid \text{qdeg}(x) \geq j \}.
\]

This filtration induces a filtration on \(H_{\text{Lee}}(D; R)\). The q-degree of a homology class \(\gamma\) is given by the maximum q-degree among its representatives

\[
\text{qdeg}(\gamma) := \max \{ \text{qdeg}(x) \mid [x] = \gamma \}.
\]

The invariance of Lee homology is proved in [20, Subsection 4.3.2] over \(R = \mathbb{Q}\), but it holds in general. Our proof in Section 7.1 also includes the following.

**Theorem 2.6.** Let \(L\) be a link. For any diagram \(D\) of \(L\), the isomorphism class of \(H_{\text{Lee}}(D; R)\) (as a graded \(R\)-module) is an invariant of \(L\).

The remarkable fact about Lee homology over \(\mathbb{Q}\) is that the isomorphism class of \(H_{\text{Lee}}(D; \mathbb{Q})\) is determined solely by the number of components of \(D\) and the linking number between two components. We prove this fact in a more general setting.

First, define a pair of elements in \(A\) by:

\[
a = X + 1, \quad b = X - 1.
\]

By direct calculation, the (co)multiplication diagonalizes on \(a, b\) as

\[
\begin{align*}
m(a \otimes a) &= 2a, & \Delta(a) &= a \otimes a, \\
m(a \otimes b) &= 0, & \Delta(b) &= b \otimes b \\
m(b \otimes a) &= 0 \\
m(b \otimes b) &= -2b
\end{align*}
\]

We call \(a\) and \(b\) colors, and refer to an assignment of \(a\) or \(b\) to a state circle as coloring. A coloring on all \(s\)-circles defines an element in \(V(D, s)\), which we call a colored state. Now, recall that an oriented link diagram possesses a unique orientation preserving state \(s_0\), where every state circle admits an orientation coherent with the given orientation of \(D\). Such a state can be obtained by 0-resolving the positive crossings, and 1-resolving the negatives. The corresponding state circles are called Seifert circles. We color each Seifert circle by either \(a\) or \(b\) according to the following algorithm:
Algorithm 2.7. Color the regions of $\mathbb{R}^2$ divided by the Seifert circles in the checkerboard fashion, with the unbounded region colored white. Color a circle $a$ if it sees a black region to the left with respect to the given orientation, otherwise color $b$.

Lemma 2.8. Every crossing of $D$ connects differently colored circles. In particular, no crossing connects a circle to itself.

Proof. There are only two coloring patterns that can be seen at a neighbourhood of a crossing, and from the coloring rule the two oriented strands must be colored differently in either case. 

Denote by $\alpha(D) \in V(D,s_0)$ the element obtained by Algorithm 2.7. If we forget the given orientation of $D$, there are $2^{|D|}$ possible orientations on the underlying unoriented diagram of $D$, which we call alternative orientations of $D$. For each alternative orientation $o$, there is an orientation preserving state with respect to $o$, and an element $\alpha(D,o)$ defined by the same procedure.

Lemma 2.9. Every $\alpha(D,o)$ is a cycle in $C_{\text{Lee}}(D;R)$.

Proof. From Lemma 2.8, all edge maps going out from the orientation preserving state of $o$ are merge maps, which annihilate differently colored circles.

Definition 2.10 ($\alpha$-cycles, $\alpha$-classes). We call the cycles $\{\alpha(D,o)\}$ the $\alpha$-cycles of $D$, and the homology classes $\{[\alpha(D,o)]\}$ the $\alpha$-classes of $D$. In particular if $o$ is the given orientation of $D$, we simply call them the $\alpha$-cycle (class) of $D$ and denote by $\alpha(D)$ and $[\alpha(D)]$.

Proposition 2.11. If $2$ is invertible in $R$, then $H_{\text{Lee}}(D;R)$ is freely generated by $\{[\alpha(D,o)]\}$ over $R$. 

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Lee proved this proposition in [20] for $R = \mathbb{Q}$ using Hodge theory. Shumakovich proved in [30] that this also holds for $R = \mathbb{F}_p$ with any prime $p \geq 3$. Here we take a different approach which is also applicable to non-field $R$’s, the admissible coloring decomposition of $C_{\text{Lee}}$, proposed by Wehrli in [32].

A coloring of a diagram $D$ is an assignment of either $a$ or $b$ on each arc of $D$. A coloring is admissible if each crossing admits a resolution that the arc segments can be colored accordingly. Local coloring patterns at a crossing of an admissibly colored diagram is shown in Figure 6.

![Figure 6: Local colorings of an admissibly colored diagram.](image)

**Proof of Proposition 2.11.** If 2 is invertible in $R$, then $\{a, b\}$ forms a basis of $A$. Thus colored states form a basis of $C_{\text{Lee}}(D; R)$. Denote by $D_c$ the diagram $D$ colored by $c$. Obviously $d$ is closed under a coloring, so we denote by $C_{\text{Lee}}(D_c; R)$ be the subcomplex spanned by the colored states matching $c$. Colorings of $D$ gives a partition of colored states, and since there is no generator matching a non-admissible coloring, we have a decomposition:

$$C_{\text{Lee}}(D; R) = \bigoplus_{c: \text{admissible}} C_{\text{Lee}}(D_c; R).$$

In each $C_{\text{Lee}}(D_c; R)$, for each state $s$ either there exists a single generator matching $c$ or there is none. Admissible colorings are grouped into two types: (I) ones that contain a crossing that is locally unicolored, and (II) ones which all crossings are locally bicolored. We claim: (i) a subcomplex associated to a type I coloring is acyclic, (ii) type II colorings are in one-to-one correspondence with the alternative orientations, and (iii) a subcomplex associated to a type II coloring is singly generated by the corresponding $\alpha$-cycle.

(i) For a type I coloring, take a crossing that is locally unicolored. Consider the exact sequence of Proposition 2.5, which holds for Lee homology too. The connecting homomorphism is induced from the direct sum of merge & split maps, and these are isomorphisms since 2 is invertible.

(ii) Given an orientation, Algorithm 2.7 yields a type II admissible coloring of $D$. Conversely, given a type II admissible coloring, every crossing has a unique admissible resolution, and thus there is a unique state where every state circle can be colored accordingly. Again by the checkerboard coloring on $\mathbb{R}^2$ divided by these circles, at a neighbourhood of each crossing we may locally orient the arcs so that the arc colored $a$ sees black to the left, and $b$ sees black to the right. This extends to a global orientation on $D$. 

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(iii) Under this correspondence, the single colored state associated to a type II admissible coloring is exactly the $\alpha$-cycle of the corresponding orientation. Since the chain complex is isolated, its homology is the chain complex itself. \hfill \qed

**Proposition 2.12.** Let $D_1, \cdots, D_\ell$ be the components of $D$. Let $o$ be an alternative orientation of $D$. Let $I \subset \{1, \ldots, \ell\}$ be the set of $i$’s where $o$ is opposite on $D_i$ with respect to the original orientation. The $h$-degree of $\alpha(D,o)$ is given by:

$$2 \sum_{i \in I, j \notin I} \text{lk}(D_i, D_j).$$

In particular, $\alpha(D)$ has $h$-degree zero.

**Proof.** The $h$-degree of $\alpha(D,o)$ is given by the difference between the negative crossing number with respect to $o$ and that of the original orientation. For each pair of components $(D_i, D_j)$, the contribution to the degree occurs only when one of the component is reversed by $o$, and the amount is exactly $2\text{lk}(D_i, D_j)$.

\hfill \Box

### 2.3 Rasmussen’s $s$-invariant

In [29], Rasmussen introduced the $s$-invariant, a knot invariant derived from Lee homology over $\mathbb{Q}$. Later in [7], Beliakova and Wehrli extended $s$ to links. Here we review the definition of the $s$-invariant of a link.

**Proposition 2.13 ([29, Proposition 2.3]).** Suppose $2$ is invertible in $R$. Let $D, D'$ be two diagrams related by a single Reidemeister move. There is an isomorphism $\rho : H_{\text{Lee}}(D; R) \to H_{\text{Lee}}(D'; R)$, such that for any alternative orientation $o$ on $D$ (and the the induced orientation $o'$ on $D'$), $\rho$ maps $[\alpha(D,o)]$ to $[\alpha(D',o')]$ up to multiplication by unit.

From Proposition 2.11 and 2.13, Rasmussen calls $\{ [\alpha(D,o)] \}$ the canonical generators of $H_{\text{Lee}}(L; \mathbb{Q})$. A generalized proposition is given in the next section, so we skip the proof for now.

**Lemma 2.14 ([14, Proposition 24]).** $C_{K_h}^{ij}(D; R)$ is supported only where $j \equiv |D|$ mod 2.

**Proof.** Since $\deg(1) = 0$, $\deg(X) = -2$, we have $\deg(x) \equiv 0$ mod 2 for any enhanced state $x$. Passing between adjacent states changes $|s|$ by $\pm 1$ and also $r(D,s)$ by $\pm 1$, thus $q\deg$ mod 2 are equal among all enhanced states of $D$. Let $s_0$ be the orientation preserving state of $D$, and denote $r(D) = r(D,s_0)$. Since an orientation preserving resolution corresponds to performing a band-sum on the link, every time a crossing is resolved the number of components changes by $\pm 1$. Thus $|D| \equiv r(D) + n$ mod 2, and

$$q\deg \equiv r(D) + |s_0| + n^+ - 2n^- \equiv r(D) + n \equiv |D| \text{ mod } 2$$

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Lemma 2.15. \( C_{\text{Lee}}(D; R) \) can be decomposed into subcomplexes:

\[
C_{\text{Lee}}(D; R)' = \text{Span}_R \{ x \mid \text{qdeg}(x) \equiv |D| \mod 4 \}
\]
\[
C_{\text{Lee}}(D; R)'' = \text{Span}_R \{ x \mid \text{qdeg}(x) \equiv |D| + 2 \mod 4 \}
\]

where \( x \) runs over the enhanced state of \( D \).

Proof. Obvious from Lemma 2.14, and the fact that \( d \) preserves q-degree modulo 4. \(\square\)

Definition 2.16 (\( \beta \)-cycle). Let \( D \) be a link diagram. For any alternative orientation \( o \) of \( D \), define \( \beta(D, o) = \alpha(D, -o) \) where \(-o\) is the reversed orientation of \( o \). In particular if \( o \) is the given orientation of \( D \), then we simply denote it by \( \beta(D) \).

By definition \( \beta(D, o) \) is obtained by interchanging \( a \) and \( b \) on the tensor factors of \( \alpha(D, o) \). In the following we only consider the given orientation, since same arguments apply to any alternative one.

Lemma 2.17. Let \((\alpha, \beta) = (\alpha(D), \beta(D))\). There are two elements in \( C_{\text{Lee}}(D; R) \):

\[
\xi = (\alpha + \beta)/2, \\
\eta = (\alpha - \beta)/2.
\]

One of \( \xi, \eta \) is contained in \( C_{\text{Lee}}(D; R)' \), and the other is in \( C_{\text{Lee}}(D; R)'' \).

Proof. By extending all terms of \( \alpha \) and \( \beta \), we see that \( \xi \) (resp. \( \eta \)) is the part of \( \alpha \) with even (resp. odd) numbers of 1’s in its tensor factors. Every term of \( \alpha \) consists of \( r(D) \) factors, and replacing one \( X \) with 1 increases the q-degree by 2, so the q-degree modulo 4 of each term is determined by the parity of the number of 1’s. \(\square\)

A fusion move on a diagram is a local modification of arcs that preserves the global orientation, as depicted in Figure 7:

![Figure 7: A fusion move.](image)

Lemma 2.18. If \( D' \) is obtained by applying a fusion move on \( D \), there is a chain map from \( C_{\text{Lee}}(D; R) \) to \( C_{\text{Lee}}(D'; R) \) of filtered degree \(-1\), which maps \( \alpha(D) \) to \( \alpha(D') \) multiplied by either 1 or \( \pm 2 \).
Proof. Consider a diagram $\tilde{D}$ by replacing the fusion site with a crossing, so that $D, D'$ can be considered as the 0-, 1-resolved diagrams of $\tilde{D}$ at the crossing respectively. The desired chain map is the one given in Proposition 2.4 (which also holds for $C_{Lee}$). Since the two arcs in the fusion site have different directions, the corresponding factors of the $\alpha(D)$ must have the same color. Since Seifert circles of $D$ and $D'$ are identical outside the fusion site, we see that $\alpha(D)$ is applied either $m$ or $\Delta$, and mapped to $\alpha(D')$ multiplied by $\pm 2$ or 1. \qed

**Proposition 2.19** ([29, Proposition 3.5]). If 2 is invertible in $R$, the (filtered) $q$-degree of the two classes $[\xi]$ and $[\eta]$ differ exactly by 2.

Proof. The decomposition of the chain complex of Lemma 2.15 descends to homology, and from Lemma 2.17 we see that $q$-degree of the two classes $[\xi], [\eta]$ differ by 2 modulo 4. We show that the difference is exactly 2. Take any fusion site in $D$. From the proof of Lemma 2.18, applying the fusion map twice maps $a \to 2a$ and $b \to -2b$. Thus $\xi$ and $\eta$ are interchanged up to multiplication by $\pm 2$. The fusion map has filtered degree $-1$, and multiplication by an invertible value is an isomorphism hence preserves the filtered degree. Thus we have $qdeg[\xi] - 2 \leq qdeg[\eta]$ and $qdeg[\eta] - 2 \leq qdeg[\xi]$, so the difference cannot be larger than 2. \qed

**Proposition 2.20** ([29, Corollary 3.6]).

\[ qdeg[\alpha] = qdeg[\beta] = \min\{ qdeg[\xi], qdeg[\eta] \}. \]

Proof. Suppose $[\xi]$ has the lower $q$-degree (the proof is similar for the other case). Since $\alpha = \xi + \eta$, we have $qdeg[\alpha] \geq qdeg[\xi]$. If $qdeg[\alpha] > qdeg[\xi]$ then $qdeg[\alpha] \geq qdeg[\eta]$ so we have $qdeg[\xi] = qdeg((\alpha - [\eta]) \geq qdeg[\eta]$ which is a contradiction. The proof is similar for $[\beta]$. \qed

Thus the following definition is justified:

**Definition 2.21** ([7, Definition 7.1]). Let $R$ be a commutative ring such that $2 \in R$ is invertible. The Rasmussen invariant $s(L; R)$ of an oriented link $L$ over $R$ is defined by:

\[ s(L; R) = \frac{qdeg[\xi] + qdeg[\eta]}{2} \]

where $\xi, \eta$ are defined as in Lemma 2.17 for any diagram $D$ of $L$.

We end this subsection by citing some properties of the $s$-invariant without proofs.

**Proposition 2.22** ([29, Theorem 2]). $s$ defines a homomorphism

\[ s: Conc(S^3) \to 2\mathbb{Z}. \]

where $Conc(S^3)$ denotes the concordance group of knots in $S^3$. 

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Proposition 2.23.

\[ s(L) \equiv |L| - 1 \mod 2. \]

Proposition 2.24 ([7, Subsection 6.2]). Let \( S \) be a oriented cobordism between two links \( L, L' \) such that every connected component of \( S \) has boundary in both \( L \) and \( L' \). Then

\[ |s(L) - s(L')| \leq -\chi(S). \]

Proposition 2.25 ([7, Lemma 6.1]). Let \( L, L' \) be links.

1. \( 2 - 2|L| \leq s(L) + s(L') \leq 2 \)
2. \( s(L \sqcup L') = s(L) + s(L') - 1 \)
3. \( s(L \# L') = s(L) + s(L') \) or \( s(L) + s(L') - 2 \)

3 Generalizations of Khovanov homology and \( \alpha \)-classes

In this section, we first review the generalization of Khovanov homology proposed by Khovanov in [16] based on the relation with Frobenius algebras. \( \alpha \)-classes also generalizes as elements in the generalized homology group. Proposition 3.10 states the behaviour of the \( \alpha \)-classes under the Reidemeister moves, and the results will be essential in the coming sections. We also state the module structure of the generalized Khovanov homology.

3.1 Frobenius algebra and Khovanov homology

In [16], Khovanov introduced a homology theory that unifies: the original theory, Lee theory, and Bar-Natan theory (which was introduced by Bar-Natan in [6]).

Let \( R \) be a commutative ring with unity. A Frobenius algebra over \( R \) is a quintuple \((A, m, \iota, \Delta, \varepsilon)\) satisfying the following:

1. \((A, m, \iota)\) is an associative \( R \)-algebra with multiplication \( m : A \otimes A \to A \) and unit \( \iota : R \to A \),
2. \((A, \Delta, \varepsilon)\) is a cocommutative \( R \)-coalgebra with comultiplication \( \Delta : A \to A \otimes A \) and counit \( \varepsilon : A \to R \), and
3. the Frobenius relation holds:

\[ \Delta \circ m = (id \otimes m) \circ (\Delta \otimes id) = (m \otimes id) \circ (id \otimes \Delta). \]

Given a (co)commutative Frobenius algebra \( A \), there is an associated 1+1 TQFT \( F_A \), a tensor functor from \( \text{Cob}_2 \) – the category of oriented 2-dimensional cobordisms, to \( \text{Mod}_R \) – the category of \( R \)-modules, that maps a disjoint union
of $r$ circles to the $r$-fold tensor product of $A$, and an elementary cobordism between circles to an appropriate operation of $A$ as depicted in Figure 8.

Given a link diagram $D$, we construct a commutative cube in $\text{Cob}_2$ consisting of cobordisms between sets of state circles of $D$. By applying the TQFT $\mathcal{F}_A$ we obtain a cube in $\text{Mod}_R$, and the functoriality assures that the resulting cube is also commutative. Following the remaining procedure as in the original theory, we obtain a chain complex $C_A(D)$ and its homology group $H_A(D)$. We call such functor $H_A$ a Khovanov-type homology theory determined by $A$.

It is stated in [16], that for $H_A$ to be an invariant (as graded $R$-modules) of links, it is necessary that $A$ (as an $R$-module) must be free with rank 2. Let $h,t$ be two elements of $R$, and define $A_{h,t} = R[X]/(X^2 - hX - t)$. Let $(m,\iota)$ be the multiplication and the unit given by the ring structure of $A_{h,t}$. Define the counit $\varepsilon : A_{h,t} \to R$ by:

$$\varepsilon(X) = 1, \quad \varepsilon(1) = 0.$$ 

Then the comultiplication $\Delta$ is uniquely determined so that $(A_{h,t}, m, \iota, \Delta, \varepsilon)$ becomes a Frobenius algebra. Explicitly, the multiplication and the comultiplication of $A_{h,t}$ are:

$$m(X \otimes X) = hX + t, \quad \Delta(X) = X \otimes X + t1 \otimes 1,$$

$$m(X \otimes 1) = X, \quad \Delta(1) = X \otimes 1 + 1 \otimes X - h1 \otimes 1,$$

$$m(1 \otimes X) = X$$

$$m(1 \otimes 1) = 1$$

We denote the corresponding chain complex by $C_{h,t}(D; R)$ and its homology group by $H_{h,t}(D; R)$. If $R$ is a graded ring with $h,t$ having degree $-2,-4$ (or being zero) respectively, then $d$ preserves the $q$-degree and $H_{h,t}(D; R)$ is bigraded. Otherwise if $\text{deg}(h) \geq -2$, $\text{deg}(t) \geq -4$, then $d$ is $q$-degree non-decreasing and $H_{h,t}(D; R)$ is filtered. Khovanov’s original theory is given by $(h,t) = (0,0)$, Lee theory by $(h,t) = (0,1)$, and Bar-Natan theory over a field $F$ by $R = F[h]$ and $(h,t) = (h,0)$.

It is stated in [16] that any rank 2 Frobenius algebra can be obtained from the following Frobenius algebra:

$$R = \mathbb{Z}[h,t], \quad A_{h,t} = R[X]/(X^2 - hX - t),$$

$$\text{deg}(h) = -2, \quad \text{deg}(t) = -4$$
with a composition of the following two operations. A **base change** of $A$ is another Frobenius algebra $A'$ obtained by a ring homomorphism $\psi: R \to R'$. A **twist** of a Frobenius algebra $A$ by an invertible element $\theta$ in $A$, is another Frobenius algebra $(A, m, \iota, \Delta', \varepsilon')$ with the same algebra structure as $A$ but with a different coalgebra structure given by:

$$\Delta'(x) = \Delta(\theta^{-1}x), \quad \varepsilon'(x) = \varepsilon(\theta x).$$

A base change obviously induces a chain map between the corresponding complexes. Similarly for the twisting we have:

**Lemma 3.1** ([16, Proposition 2]). Let $A$ be a (co)commutative Frobenius algebra, and $A'$ be $A$ twisted by an invertible element $\theta \in A$. For any link diagram $D$, there is an isomorphism between the corresponding chain complexes $C_A(D)$ and $C_{A'}(D)$.

**Proof.** From the Frobenius relation, we have

$$\Delta'(x) = \Delta(\theta^{-1}x) = \theta^{-1}(\Delta x) = (\Delta x)\theta^{-1}$$

These correspond to merging a circle labeled $\theta^{-1}$ to a circle appearing before or after the splitting. There is a commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow{id} & & \downarrow{\theta^{-1}} \\
A & \xrightarrow{\Delta'} & A \otimes A
\end{array}
\]

We construct an isomorphism between the corresponding cubes, i.e. a set of isomorphisms between corresponding vertex modules such that each square spanned any two corresponding edges commutes. Starting with the identity map on the source state (the state with all 0-resolutions), we extend the map arrow by arrow by multiplying $\theta^{-1}$ to the appropriate factor at every tail of a split map. That these maps are well-defined (i.e. it is independent from the path from the source to a state) can be seen by considering the corresponding cobordism in $\text{Cob}_2$, where we can freely slide a tube with one end attached to a surface along its connected component. \(\square\)

Thus for a fixed ring $R$, considering $H_{h,t}(-; R)$ for an arbitrary pair $(h, t) \in R^2$ gives the most general setting.

**Theorem 3.2.** Let $L$ be a link. For any diagram $D$ of $L$, the isomorphism class of $H_{h,t}(D; R)$ (as a graded $R$-module) is an invariant of $L$. If also $H_{h,t}(D; R)$ is bigraded, then its isomorphism class (as a bigraded $R$-module) is an invariant of $L$.

Khovanov proves this by relying on the invariance of the “formal complex of a tangle” ([6, Theorem 1]). In this paper we give a direct proof in Section 7.1.
by following the original proof of [14]. The constructed isomorphisms will be used lately.

The following properties hold as in the original theory:

**Proposition 3.3.**

1. \( C_{h,t}(-D) = C_{h,t}(D) \)
2. \( C_{h,t}(D \sqcup D') \cong C_{h,t}(D) \otimes C_{h,t}(D') \)
3. \( C_{h,t}(\overline{D}) \cong C_{-h,t}^*(D) \)

Note the \(-h\) in the right hand side of (3). This is due to the fact that \( A_{h,t} \) is “almost self dual”, namely \( A_{h,t} \cong A_{-h,t}^* \) under the correspondence:

\[
1 \longleftrightarrow X^* \\
X \longleftrightarrow 1^*.
\]

If \( 2h = 0 \) in \( R \), then \( A_{h,t} \) is self dual, which is the case for Khovanov theory, Lee theory, and \( F_2 \) Bar-Natan theory.

**3.2 Generalized \( \alpha \)-classes**

Following the discussion given by Mackaay, Turner, Vaz in [24], we may extend the definition of the \( \alpha \)-classes as elements in \( H_{h,t}(D; R) \). Throughout this section, we assume the following condition holds:

**Condition 3.4.** There exists \( c \in R \) such that \( h^2 + 4t = c^2 \) and \( (h \pm c)/2 \in R \).

Under this condition, take \( c = \sqrt{h^2 + 4t} \) (fix one such square root), and let

\[
u = (h - c)/2, \quad v = (h + c)/2 \in R.
\]

Then \( X^2 - hX - t \) factors as \( (X-u)(X-v) \) in \( R[X] \). Also let

\[
a = X - u, \quad b = X - v \in A.
\]

Then \( ab = ba = 0 \), and with \( a - b = v - u = c \), we have:

\[
m(a \otimes a) = ca, \quad \Delta(a) = a \otimes a, \\
m(a \otimes b) = 0, \quad \Delta(b) = b \otimes b \\
m(b \otimes a) = 0 \\
m(b \otimes b) = -cb
\]

Lee theory corresponds to \( (h,t) = (0,1) \), \( c = 2 \). Bar-Natan theory corresponds to \( (h,t) = (h,0), \ c = h \).

For any alternative orientation \( o \) of \( D \), by Algorithm 2.7 we similarly obtain a colored state \( \alpha(D,o) \) in \( C_{h,t}(D; R) \). As in Lemma 2.8 we see that all \( \alpha(D,o) \) are actually cycles.
Definition 3.5 ($\alpha$-cycles, $\alpha$-classes). We call the cycles \(\{\alpha(D, o)\}\) the $\alpha$-cycles of $D$, and the homology classes \(\{[\alpha(D, o)]\}\) the $\alpha$-classes of $D$. In particular if $o$ is the given orientation of $D$, we simply call them the $\alpha$-cycle (class) of $D$ and denote by $\alpha(D)$ and $[\alpha(D)]$.

Definition 3.6. For any alternative orientation $o$ of $D$, define $\beta(D, o) = \alpha(D, -o)$. In particular if $o$ is the given orientation of $D$, we denote it by $\beta(D)$.

If also $c$ is invertible, then $a, b$ forms a basis of $A_{h,t}$ and $C_{h,t}(D; R)$ admits an admissible coloring decomposition. Thus the proof of Proposition 2.11 works verbatim, and we obtain the following:

Proposition 3.7. If $c = \sqrt{h^2 + 4t}$ is invertible, then $H_{h,t}(D; R)$ is freely generated by $\{[\alpha(D, o)]\}$ over $R$.

The above description for the filtered Bar-Natan theory (Bar-Natan theory collapsed with $h = 1$) over $\mathbb{F}_2$ is given in [31]. In fact, since $c = 1$, this description holds over any $R$.

Our main concern is when $c$ is not invertible.

Corollary 3.8. If $c = \sqrt{h^2 + 4t} \neq 0$, then $H_{h,t}(D; R)$ contains only $c$-torsions, i.e. all torsions are annihilated by multiplying some power of $c$.

Proof. Let $R_c = R[c^{-1}]$ be the ring of $R$ localized by powers of $c$. Since $R_c$ is flat over $R$ and $c$ is invertible in $R_c$, we have $H_{h,t}(D; R) \otimes R_c \cong H_{h,t}(D; R_c)$. The result being is free (thus torsion-free) implies that $H_{h,t}(D; R)$ has only $c$-torsions. \qed

Remark 3.9. The situation is apparently different when $c = 0$. In [26], it is shown that for any $2 \leq n \leq 8$ there are infinite families of links whose $\mathbb{Z}$-Khovanov homology have $\mathbb{Z}_n$-summands.

3.3 Behaviour under Reidemeister moves

The following proposition is a generalization of Proposition 2.13, which states the behaviour of the $\alpha$-classes under the Reidemeister moves.

Proposition 3.10. Let $(h, t)$ be a pair satisfying Condition 3.4, and let $c = \sqrt{h^2 + 4t}$. Suppose $D, D'$ are two diagrams related by a single Reidemeister move. Then there is an isomorphism $\rho : H_{h,t}(D; R) \rightarrow H_{h,t}(D'; R)$ such that for any alternative orientation $o$ of $D$ (and the corresponding orientation $o'$ of $D'$), there exists some $j \in \{0, \pm 1\}$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ satisfying $\varepsilon \varepsilon' = (-1)^j$ such that for the pairs $\alpha(D, o), \beta(D, o)$ and $\alpha(D', o'), \beta(D', o')$, those homology classes are related as:

$$[\alpha(D', o')] = \varepsilon \varepsilon' \rho[\alpha(D, o)],$$

$$[\beta(D', o')] = \varepsilon' \varepsilon \rho[\beta(D, o)].$$
(Here \( c \) is not necessarily invertible, so the equation \( z = c^j w \) is to be understood as \( c^{-j} z = w \) when \( j < 0 \).) Moreover \( j \) is determined as in Table 1 by the type of the move and the difference of the number of Seifert circles (with respect to \( o, o' \)).

| Type   | \( \Delta r \) | \( j \) |
|--------|----------------|--------|
| RM1_\(L\) | 1             | 0      |
| RM1_\(R\) | 1             | 1      |
| RM2    | 0             | 0      |
|        | 2             | 1      |
| RM3    | 0             | 0      |
|        | 2             | 1      |
|        | -2            | -1     |

Table 1: Exponent of \( c \) corresponding to the Reidemeister moves

The proof occupies Section 7.2. Here we only note that \( \rho \) is the isomorphism constructed for the proof of Theorem 3.2 (in Section 7.1). This proposition implies the \( \alpha \)-classes are not invariant up to unit when \( c \) is not invertible.

We see from Table 1 that \( \Delta r = 2j \) for RM2 and RM3. Denote by \( w(D, o) \) the writhe of \( D \) (with respect to \( o \)), and let \( \Delta w = w(D', o') - w(D, o) \). Since \( \Delta w = \pm 1 \) for RM1 and \( \Delta w = 0 \) for RM2 and RM3, we see that the following relation holds.

**Corollary 3.11.**

\[
j = \frac{\Delta r - \Delta w}{2}
\]

**Remark 3.12.** From this we also see that the classes \( [\alpha(D, o)] \) are transverse link invariants. Such discussion will be given in Section 6.

### 3.4 Relations among different pairs

Next we inspect the relations among \( H_{h,t}(D; R) \) for different \((h, t)\) satisfying Condition 3.4.

**Lemma 3.13.** Let \((h, t), (h', t')\) be pairs satisfying Condition 3.4. Let \( c = \sqrt{h^2 + 4t}, \ c' = \sqrt{h'^2 + 4t'} \). If \( c = \theta c' \) for some invertible \( \theta \in R \), then there is a Frobenius algebra isomorphism from \( A_{h,t} \) to another Frobenius algebra \( B \), that gives \( A_{h',t'} \) by the \( \theta \)-twisting.

\[
A_{h,t} \xrightarrow{\cong} B \xrightarrow{\theta \text{-twist}} A_{h',t'}
\]

These maps satisfy the cocycle condition: For any three pairs such that the following three arrows exist, the diagram commutes.
Proof. Denote $A = A_{h,t}$, $A' = A_{h',t'}$ and let $B$ be $A'$ twisted by $\theta^{-1}$. Define a ring homomorphism $f : R[X] \to R[X]$ by

$$X \mapsto \theta(X - u') + u.$$ 

$f$ maps $X - u$, $X - v$ to $\theta(X - u')$, $\theta(X - v')$ respectively, thus descends to an $R$-algebra homomorphism $f : A \to A''$. Obviously $f$ is invertible, so $f$ is an $R$-algebra isomorphism. We may take \{1, $f(X)$\} as a basis of $A''$. With:

$$\varepsilon''f(1) = 0, \quad \varepsilon''(f(X)) = \varepsilon'((\theta^{-1}f(X)) = 1,$$

it follows that $f$ commutes with the comultiplication $\Delta''$ of $A''$. Thus $f$ is a Frobenius algebra isomorphism. The cocycle condition is obvious from the definition.

Proposition 3.14. Suppose the assumption of Lemma 3.13 holds. Then for any link diagram $D$, there is an isomorphism from $C_{h,t}(D; R)$ to $C_{h',t'}(D; R)$ under which each $\alpha$-cycle in $C_{h,t}(D; R)$ is mapped to the $\alpha$-cycle in $C_{h',t'}(D; R)$ corresponding to the same alternative orientation, multiplied by a power of $\theta$. In particular if $\theta = 1$, then the $\alpha$-cycles correspond one-to-one.

Proof. That the chain complexes are isomorphic is immediate from Lemma 3.13 and 3.1. The $R$-algebra isomorphism $f$ maps the $\alpha$-cycle a $\alpha$-cycle multiplied by a power of $\theta$. Then the $\theta$ twisting multiplies each $\alpha$-cycle by a power of $\theta^{-1}$.

Corollary 3.15. Suppose $(h, t)$ satisfies Condition 3.4. Let $c = \sqrt{h^2 + 4t}$. For any link diagram $D$,

1. $C_{h,t}(D; R) \cong C_{c,0}(D; R)$.
2. $C_{h,t}(D; R) \cong C_{0,(c/2)^2}(D; R)$, if $c/2 \in R$ (or equivalently $h/2 \in R$).

In both cases, the $\alpha$-cycles correspond one-to-one.

Given any $c \in R$ we may define a Frobenius algebra:

$$A_c = \bigoplus_{h,t} A_{h,t} \bigg/ \sim$$

where $(h, t)$ runs over pairs satisfying $c = \sqrt{h^2 + 4t}$, and the equivalence relation $\sim$ is given by the Frobenius algebra isomorphisms of Lemma 3.13. Each inclusion $A_{h,t} \hookrightarrow \bigoplus A_{h',t'}$ induces an isomorphism $A_{h,t} \cong A_c$. We define $C_c(-; R)$ by the
Khovanov-type chain complex determined by $A_c$. This is obviously equivalent to defining:

$$C_c(D; R) = \bigoplus_{h,t} C_{h,t}(D; R) / \sim$$

where the equivalence relation $\sim$ is given by the isomorphism of Proposition 3.14 (since the identification holds on the cube level). Note that $C_0, C_1, C_2$ is represented by the three theories: $C_{Kh} = C_{0,0}$, $C_{fBN} = C_{1,0}$, $C_{Lee} = C_{0,1}$ respectively.

**Proposition 3.16.** Let $(h, t), (h', t')$ be pairs satisfying Condition 3.4 with $\sqrt{h^2 + 4t} = \sqrt{h'^2 + 4t'} = c$. For any two diagrams $D, D'$ related by a single Reidemeister move, the following diagram commutes:

$$\begin{array}{ccc}
H_{h, t}(D) & \xrightarrow{\rho} & H_{h', t'}(D') \\
\downarrow f & & \downarrow f \\
H_{h', t'}(D) & \xrightarrow{\rho'} & H_{h', t'}(D')
\end{array}$$

where $\rho, \rho'$ are the corresponding isomorphisms of Proposition 3.10, and $f$ is the isomorphism of Proposition 3.14. Thus there is a well-defined isomorphism:

$$\rho : H_c(D; R) \rightarrow H_c(D'; R),$$

and the $\alpha$-classes are related as stated in Proposition 2.19.

**Proof.** $\rho$ is given explicitly in the proof of Theorem 3.2 (Section 7.1). For RM1,

$$\rho(X) = X \otimes X - hX \otimes 1 - t1 \otimes 1,$$

$$\rho(1) = 1 \otimes X - X \otimes 1$$

We compute

$$\rho(1) = f \rho(1),$$
and
\[ \rho(X - u) = (X - u) \otimes (X - v) \]
so \( \rho \) commutes with \( f \). For RM2 and RM3, the commutativity is obvious since \( f \) and \( \gamma \) (in the definition of \( \rho \)) commutes.

**Remark 3.17.** We might also expect that \( \rho \) also commutes when \( c = \theta c' \) for some invertible \( \theta \), but they do not in general, due to the effect of twisting. Consider the two unknot diagrams depicted in Figure 10.

![Figure 10](image_url)

Take \( R = \mathbb{Q} \), \( c = 2 \), \( c' = 1 \), and consider the following diagram:

\[
\begin{array}{ccc}
H_{2,0}(D) & \xrightarrow{\rho} & H_{2,0}(D') \\
\downarrow f & & \downarrow \quad \downarrow \\
H_{1,0}(D) & \xrightarrow{\rho'} & H_{1,0}(D')
\end{array}
\]

\[
[\alpha_2] \xrightarrow{f} [\alpha_2'] \quad [\alpha_1] \xrightarrow{4} [\alpha_1']
\]

Denote the \( \alpha \)-classes from the upper left: \( [\alpha_2], [\alpha_2'], [\alpha_1], [\alpha_1'] \). In the horizontal direction, since \( \Delta r = 0 \) we know from Proposition 3.10 that \( [\alpha_2] \) maps to \( [\alpha_2'] \) and \( [\alpha_1] \) maps to \( [\alpha_1'] \). In the vertical direction, both sides are multiplied by 4 by the map of Lemma 3.13, but the right side has also the effect of twisting and is multiplied by \( 1/2 \).

### 3.5 Module structure

Khovanov introduced a module structure for the Khovanov homology of a knot in [15]. Hedden and Ni extended the construction to links but over \( \mathbb{F}_2 \) in [11]. Alishahi and Dowlin extended the construction to Bar-Natan theory over \( \mathbb{F}_2 \) in [3], and to Lee theory over \( \mathbb{Q} \) in [4]. We construct a module structure for a general Khovanov-type homology \( H_{h,t}(L; R) \) by following the construction of [11].

Let \( D \) be a link diagram. Let \( p \) be a point on an arc of \( D \), and take a small circle \( \odot \) near \( p \). Merging \( \odot \) into a neighbourhood of \( p \) corresponds to the multiplication:

\[ m_p : A_{h,t} \otimes C_{h,t}(D) \rightarrow C_{h,t}(D) \]

Define an endomorphism \( X_p \) by

\[ X_p = m_p(X \otimes -) : C_{h,t}(D) \rightarrow C_{h,t}(D) \]

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Now mark a point \( p_i \) for each component \( D_i \) of \( D \). Each \( m_{p_i} \) defines an \( A_{h,t} \)-module structure on \( C_{h,t}(D) \). Two such multiplications are obviously commutative. Let

\[
A_{h,t} = R[X_1, \ldots, X_\ell]/(p_{h,t}(X_1), \ldots, p_{h,t}(X_\ell))
\]

where \( \ell = |D| \) and \( p_{h,t}(X) = X^2 - hX - t \). We obtain an \( A_{h,t} \)-module structure on \( C_{h,t}(D) \):

\[
A_{h,t} \otimes C_{h,t}(D) \longrightarrow C_{h,t}(D)
\]

and this descends to:

\[
A_{h,t} \otimes H_{h,t}(D) \longrightarrow H_{h,t}(D).
\]

The following proposition is a generalization of [11, Proposition 2.2].

**Proposition 3.18.** Let \( D, D' \) be diagrams of the same link. Let \( p_i \in D_i, p'_i \in D'_i \) be marked points, one chosen on each component of the diagram. Then there is a commutative diagram:

\[
\begin{array}{ccc}
A_{h,t} \otimes H_{h,t}(D) & \xrightarrow{\varphi} & H_{h,t}(D) \\
\downarrow{\psi \otimes \rho} & & \downarrow{\rho} \\
A_{h,t} \otimes H_{h,t}(D') & \xrightarrow{\varphi'} & H_{h,t}(D')
\end{array}
\]

where \( \psi \) is a ring automorphism, \( \rho \) is a composition of isomorphisms given in Proposition 3.10, and \( \varphi, \varphi' \) are the module structures determined by the marked points.

First we prove the following lemma, which is a generalization of [11, Lemma 2.3], [3, Lemma 2.1] and [4, Lemma 3.3].

**Lemma 3.19.** If \( p, q \) are two marked points on a strand of \( D \) separated by a crossing \( x \), then \( X_p + X_q \) is chain homotopic to multiplication by \( h \).

**Proof.** Denote \( C(D) = C_{h,t}(D) \). Consider the decomposition of Proposition 2.4 \( C(D) = C(D_0) \oplus C(D_1) \) where \( D_0, D_1 \) are the 0-, 1-resolved diagrams of \( D \) at \( x \) respectively. If we consider another diagram \( D' \) with the crossing changed at \( x \), we have \( C(D') = C(D_1) \oplus C(D_0) \). There are chain maps \( d_+, d_- \) in both directions:

\[
\begin{array}{ccc}
d_0 & \xrightarrow{d_+} & d_1 \\
\downarrow{d_0} & & \downarrow{d_1} \\
C(D_0) & \xrightarrow{d_-} & C(D_1)
\end{array}
\]

and we have

\[
d = d_0 + d_+ - d_1, \\
d' = d_1 + d_- - d_0.
\]
From $d^2 = 0$, $d'^2 = 0$, we have

$$d_+d_0 = d_1d_+,$$

$$d_1d_+ = d_0d_-. $$

Define a homomorphism $H : C(D) \rightarrow C(D)$ by

$$H = \begin{cases} 
0 & \text{on } C(D_0), \\
 d_- & \text{on } C(D_1). 
\end{cases}$$

We claim

$$X_p + X_q - h = dH + Hd.$$ 

For any $(z, w) \in C(D_0) \oplus C(D_1)$,

$$(dH + Hd)(z, w) = d_-d_+(z) + (d_0 + d_+)d_-(w) - d_-d_1(w)$$

$$= d_-d_+(z) + d_+d_-(w).$$

It suffices to prove that the equation holds on each vertex module $V(D,s) \subset C(D)$. Suppose a state $s$ has a 0-resolution at $x$. Then $V(D,s) \subset C(D_0)$ so $d_+d_-|_{V(D,s)} = 0$. If $p, q$ belongs to the same $s$-circle, then

$$d_-d_+|_{V(D,s)} = m\Delta.$$ 

We have

$$m\Delta(1) = 2X - h,$$

$$m\Delta(X) = hX + 2t.$$ 

On the other hand,

$$(X_p + X_q - h)(1) = 2X - h,$$

$$(X_p + X_q - h)(X) = 2(hX + t) - hX,$$

so the desired equation holds. If $p, q$ belongs to different $s$-circles, then

$$d_-d_+|_{V(D,s)} = \Delta m,$$

and again we see that the equation holds by direct calculation. If $s$ has a 1-resolution at $x$, then we consider $d_+d_-$ instead of $d_-d_+$ and the proof is identical.

**Proof of Proposition 3.18.** $(D, \{p_i\})$ can be transformed into $(D', \{p'_i\})$ by a sequence of Reidemeister moves, followed by sliding each marked point $p_i$ along its component until it meets $p'_i$. We decompose the transformation into sequence of moves so that each move corresponds to either: (i) a Reidemeister move performed in the complement of all neighborhoods of the marked points, or (ii) passing a marked point through a crossing. Then we decompose the square of
Proposition 3.18 into vertically stacked squares accordingly. It suffices to show that each square commutes.

For a type (i) Reidemeister move, it is obvious from the construction of $\rho$ that the square commutes, with $\psi = id$. For a type (ii) move, let $p_i$ be the marked point passing through a crossing, and define a ring isomorphism by:

$$\psi : A_{h,t} \to A_{h,t}, \quad X_i \mapsto -X_i + h, \quad X_j \mapsto X_j \quad (j \neq i).$$

Lemma 3.19 implies that the square commutes with $\psi = id$.

**Remark 3.20.** If $h = 2 = 0$ in $R$, which is the case for Khovanov theory over $\mathbb{F}_2$, we have $\psi = id$ and the module structure on $H_{h,t}(L; R)$ is independent of the choice of the marked points.

**Proposition 3.21.** Let $c \in R$, and $(h, t), (h', t')$ be pairs satisfying Condition 3.4 and $c = \sqrt{h^2 + 4t} = \sqrt{h'^2 + 4t'}$. The isomorphism $f$ of Lemma 3.13 induces isomorphisms which is natural with respect to the commutative diagram of Proposition 3.18, i.e. every face of the following cube commutes:

$$\begin{array}{cccc}
A \otimes H(D) & \xrightarrow{\varphi} & H(D) \\
\downarrow \psi \otimes \rho & & \downarrow \rho \\
A \otimes H(D') & \xrightarrow{\varphi'} & H(D') \\
\downarrow \rho & & \downarrow \rho \\
A' \otimes H'(D) & \xrightarrow{\varphi'} & H'(D) \\
\downarrow \psi' \otimes \rho & & \downarrow \rho \\
A' \otimes H'(D') & \xrightarrow{\varphi'} & H'(D')
\end{array}$$

where $A_{h,t}$, $A_{h',t'}$ are denoted by $A$, $A'$, and $H_{h,t}, H_{h',t'}$ are denoted by $H, H'$ respectively.

**Proof.** Proposition 3.21 gives the commutativity of the top and bottom faces, and Proposition 3.16 gives that of the right face. $f$ induces a ring isomorphism:

$$\tilde{f} : A_{h,t} \to A_{h,t}, \quad X_i \mapsto X_i - u' + u \quad (1 \leq i \leq \ell).$$

When a marked point $p_i$ passes through a crossing, $\psi$ maps $X_i - u$ to $-(X_i - v)$. Thus we have

$$\begin{array}{cccc}
X_i - u & \xrightarrow{\psi} & -(X_i - v) \\
\downarrow \tilde{f} & & \downarrow \tilde{f} \\
X_i - u' & \xrightarrow{\psi'} & -(X_i - v')
\end{array}$$

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With Proposition 3.16 we see that the left face commutes. That the front and back faces commute follows from the fact that $f$ commutes with multiplication.

Define

$$A_c = \bigoplus_{h,t} \mathcal{A}_{h,t} / \sim$$

where $(h, t)$ runs over pairs satisfying $c = \sqrt{h^2 + 4t}$, and the equivalence relation $\sim$ is given by the above isomorphism $\tilde{f}$. The above proposition implies that $H_c(D; R)$ admits an $A_c$-module structure:

$$A_c \otimes H_c(D; R) \rightarrow H_c(D; R),$$

and in particular when $D$ is a knot diagram, an $A_c$-module structure:

$$A_c \otimes H_c(D; R) \rightarrow H_c(D; R).$$

## 4 Link invariant from the divisibility of the $\alpha$-class

In this section, we define the $c$-divisibility $k_c(D)$ of the $\alpha$-class of a link diagram $D$, and the link invariant $s'_c(L)$ given by a combination of $k_c(D)$ and some classical knot diagram properties of $D$. By inspecting the behaviour of $s'_c$ under cobordisms, we see that $s'_c$ possesses many properties common to the $s$-invariant.

Throughout this section, we assume that $R$ is an integral domain.

### 4.1 Definition and basic properties

**Definition 4.1.** Let $M$ be an $R$-module, and $c$ be an element in $R$. Define the $c$-divisibility of an element $z$ in $M$ by:

$$k_c(z) = \sup_{k \geq 0} \{ z \in c^k M \} \in [0, \infty].$$

Note that if $c$ is invertible or $z = 0$, then $k_c(z) = \infty$.

**Lemma 4.2.** If $n \geq 0$ then

$$k_c(c^n z) \geq k_c(z) + n.$$

Moreover if $M$ is torsion-free, then the equality holds.

**Proof.** $z \in c^k$ implies $c^n z \in c^{k+n} M$, so we have the inequality. Suppose $M$ is torsion free. If $k_c(c^n z)$ is infinite then so is $k_c(z)$. Suppose $k' = k_c(c^n z)$ is finite. From the maximality of $k'$ we have $n \leq k'$, and $c^n z = c^{k'} w$ for some $w \in M$. $c^n z - c^{k'} w = c^n (z - c^{k'-n} w) = 0$ implies $z = c^{k'-n} w \in c^{k'-n} M$ so $k_c(z) \geq k_c(c^n z) - n$. \qed
Remark 4.3. The equality does not hold if $M$ is not torsion free. Consider the case $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}_2$, $c = 2$ and $z = (2, 1)$. In this case $k_2(z) = 0$, but $2z = (4, 0)$ so $k_2(2z) = 2$.

Lemma 4.4. Let $\psi : R \to R'$ be a ring homomorphism between integral domains, and $\phi : M \to \psi^*M'$ be an $R$-module homomorphism. Then for any $z \in M$,

$$k_c(z) \leq k_{\psi(c)}(\phi(z)).$$

Moreover if $\psi, \phi$ are isomorphisms, then the equality holds.

Proof. $z \in c^kM$ implies $\phi(z) \in \phi(c^kM) = \psi(c)^k\phi(M) \subset \psi(c)^kM'$.

Now we return to link homology. Denote by $H_c(D; R)$ the free part of $H_c(D; R)$, i.e. the quotient of $H_c(D; R)$ by its torsion submodule. By abuse of notation, we denote the image of an element $[z] \in H_c(D; R)$ by the same symbol $[z] \in H_c(D; R_f)$.

Definition 4.5. For any link diagram $D$, define:

$$k_c(D; R) = k_c([\alpha(D)])$$

where $[\alpha(D)]$ is the (image of the) $\alpha$-class of $D$ in $H_c(D; R)_f$. $k_c(D, o; R)$ is defined similarly for any alternative orientation $o$ of $D$.

Divisibility is uninteresting when $c = 0$ or when $c$ is invertible, so in the following we assume that $c$ is non-zero non-invertible. Note that $[\alpha(D, o)] \neq 0$, and in particular when $R$ is a PID we see that $k_c(D, o)$ is finite. In the following we only consider the $\alpha$-class corresponding to the given orientation, since same arguments apply for any alternative orientation.

Example 4.6. $k_c(\bigcirc) = 0$ since $H(\bigcirc) = C(\bigcirc) = \mathbb{Z}/1, X$.

Example 4.7. $D =$ (unknot with a negative twist). With $(R, h, t) = (\mathbb{Z}, 0, 1)$ and $c = 2$, we have

$$C(D) = \{ A \xrightarrow{\Delta} A^{\otimes 2} \}$$

and the $\alpha$-cycle is $\alpha(D) = (X - 1) \otimes (X + 1)$. From $\Delta(X) = X \otimes X + 1 \otimes 1$, $\Delta(1) = 1 \otimes X + X \otimes 1$, we see that $\alpha(D)$ is homologous to $2(1 \otimes X - 1 \otimes 1)$. Since $\{[[1 \otimes X], [1 \otimes 1]]\}$ form a basis of $H(D)_f$, we have $k_2(D) = 1$.

These examples show that $k_c(D)$ is not a link invariant. In fact, the following proposition shows that the difference of $k_c$ of two diagrams of the same link can be computed easily. Recall that $r(D)$ denotes the number of Seifert circles of $D$, and $w(D)$ denotes the writhe of $D$.

Proposition 4.8. Let $D, D'$ be two diagrams of the same link. Then

$$\Delta k_c = \frac{\Delta r - \Delta w}{2},$$

where the prefixed $\Delta$ denotes the difference of the corresponding values of $D, D'$. 

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Proof. Take any sequence of Reidemeister moves that transforms $D$ to $D'$. Let $\rho$ be the composition of the isomorphisms corresponding to the Reidemeister moves given in Proposition 3.10. Let $J$ be the sum of the $c$-exponents occurring at each move. From Corollary 3.11 we have $2J = \Delta r - \Delta w$. Since $[\alpha(D')] = \pm c'\rho([\alpha(D)])$, with Lemma 4.2 and 4.4 we obtain the desired result.

Thus we obtain a link invariant:

**Definition 4.9.** For any link $L$, define:

$$s'_c(L;R) = 2k_c(D;R) - r(D) + w(D) + 1,$$

where $D$ is any diagram of $L$.

First we state some basic properties of $k_c$. Obviously $k_c$ is bounded below, while it is unbounded among diagrams of the same link, since $k_c$ increases by 1 as we add one negative twist.

**Proposition 4.10.** If $D$ is a positive diagram (i.e. a diagram with only positive crossings), then $k_c(D) = 0$.

**Proof.** The orientation preserving state of $D$ is $s_o = (0 \cdots 0)$. By 0-resolving the crossings one by one, we obtain a sequence of chain maps:

$$C(D) \to C(D_0) \to \cdots \to C(D_{0\cdots 0})$$

The rightmost diagram has no crossing, so $H(D_{0\cdots 0}) = C(D_{0\cdots 0})$. Its $\alpha$-cycle is non $c$-divisible, since it has a term $X \otimes \cdots \otimes X$ with coefficient 1. Under the composition of the chain maps, $\alpha(D)$ is mapped to $\alpha(D_{0\cdots 0})$, so from Lemma 4.4 we have $k_c(D) \leq 0$.

$k_c$ may be regarded as a measuring the “non-positivity” of a diagram.

**Proposition 4.11.**

$$k_c(D) = k_c(-D).$$

**Proof.** Consider a Frobenius algebra isomorphism induced from:

$$R[X] \to R[X], \quad X \mapsto -(X - u') + u.$$<ref></ref>

This maps $a, b$ to $-b, -a$ respectively. The induced chain automorphism on $C_c(D)$ maps $\alpha(D)$ to $(-1)^{r(D)}\beta(D) = (-1)^{r(D)}\alpha(-D)$. 

**Proposition 4.12.**

$$k_c(D \sqcup D') \geq k_c(D) + k_c(D')$$

Moreover, if $R$ is a PID and $c$ is prime in $R$, then the equality holds.
**Proof.** From Proposition 3.3 we have \( C(D \cup D') \cong C(D) \otimes C(D') \). From a general statement of Lemma 7.2, there is a homomorphism:

\[
h : H(D)_f \otimes H(D')_f \to H(D \cup D')_f,
\]

that maps \([z] \otimes [w]\) to \([z \otimes w]\), and if \( R \) is a PID this is an isomorphism. Let \((\alpha, \alpha') = (\alpha(D), \alpha(D'))\). The \( \alpha \)-cycle of \( D \cup D' \) is given by \( \alpha'' = \alpha \otimes \alpha' \). Let \([\alpha] = c^k[\alpha_0], [\alpha'] = c^{k'}[\alpha'_0] \) with maximal \( k, k' \). Then

\[
[\alpha''] = [\alpha \otimes \alpha'] = h([\alpha] \otimes [\alpha']) = c^{k+k'} h([\alpha_0] \otimes [\alpha'_0]),
\]

so the inequality follows.

Now suppose \( R \) is a PID and \( c \) is prime in \( R \). Let \([\alpha''] = c^{k''}[\beta]\). We have

\[
h([\alpha_0] \otimes [\alpha'_0]) = c^{l} \beta \text{ where } l = k'' - (k + k').
\]

Let \([\{z_i\}], \{[z'_j]\}\) be the bases of \( H(D)_f, H(D')_f \) respectively. Since \( h \) is an isomorphism, \( \{[z_i] \otimes [z'_j]\} \) forms a basis of \( h((D \cup D')_f) \). Let \([\alpha_0] = \sum_i a_i[z_i], [\alpha'_0] = \sum_j a'_j[z'_j] \). Then

\[
h([\alpha_0] \otimes [\alpha'_0]) = \sum_{i,j} a_i a'_j[z_i \otimes z'_j] \in c^l H(D \cup D')_f.
\]

If \( l > 0 \) then one of \([\alpha_0], [\alpha'_0]\) must be \( c \)-divisible. Indeed, we have \( c \mid a_i a'_j \) for all \( i, j \), and if there is one \( a_i \) that is not \( c \)-divisible then all \( a'_j \) must be \( c \)-divisible. However this contradicts the maximality of \( k' \), so \( l = 0 \) and we obtain the equality. □

**Proposition 4.13.**

\[
k_c(D \# D') \leq k_c(D \cup D') \leq k_c(D \# D') + 1
\]

**Proof.** \( D \# D' \) and \( D \cup D' \) are related by fusion moves, and the \( \alpha \)-classes correspond as:

\[
\alpha(D \# D') \xrightarrow{\Delta} \alpha(D \cup D') \xrightarrow{m} \pm c \alpha(D \# D').
\]

□

Next we state some basic properties of \( s'_c \).

**Lemma 4.14.** Let \( L \) be a link, \( D \) be a diagram of \( L \). Let \( S \) be the Seifert surface of \( L \) obtained by applying the Seifert’s algorithm to \( D \). Then

\[
\chi(S) = 2 - 2g(S) - |L| = r(D) - n(D).
\]

**Proof.** \( S \) deformation retracts to the Seifert graph of \( D \). □

**Proposition 4.15.**

\[
s'_c(L; R) \equiv |L| - 1 \mod 2.
\]

**Proof.** Immediate from Lemma 4.14. □
Proposition 4.16. Let $L$ be a positive link, and $D$ be a positive diagram of $L$. Let $S$ be the Seifert surface of $L$ obtained by applying the Seifert’s algorithm to $D$. Then
\[ s'_c(L) = 2g(S) + |L| - 1. \]
In particular for a positive knot $K$,
\[ s'_c(K) = 2g(S). \]

Proof. Immediate from $k_c(D) = 0$, $w(D) = n(D)$ and Lemma 4.14. \qed

The following properties can be easily obtained from those of $k_c$.

Proposition 4.17.

1. $s'_c(\emptyset) = 0$.
2. $s'_c(L) = s'_c(-L)$.
3. $s'_c(L \sqcup L') \geq s'_c(L) + s'_c(L') - 1$.
4. $s'_c(L \# L') = s'_c(L \sqcup L') \pm 1$.

If $R$ is a PID and $c$ is prime in $R$, then we have instead:

3'. $s'_c(L \sqcup L') = s'_c(L) + s'_c(L') - 1$.
4'. $s'_c(L \# L') = s'_c(L) + s'_c(L')$ or $s'_c(L) + s'_c(L') - 2$.

\qed

4.2 Behaviour under cobordisms

By following the arguments given by Rasmussen in [29] and [28], we construct a homomorphism corresponding to a cobordism between links, and state the behaviour of the $\alpha$-classes.

Let $L, L'$ be two links in $\mathbb{R}^3$, and $S \subset \mathbb{R}^3 \times [0, 1]$ be an (oriented smooth) cobordism between $L$ and $L'$ with $\partial S = (-L) \times \{0\} \cup L' \times \{1\}$. Let $D, D'$ be diagrams of $L, L'$. We construct a homomorphism
\[ \phi : H_c(D; R) \to H_c(D'; R). \]
as follows. By modifying $S$ by a small isotopy, we may assume that $S$ can be decomposed into a union of elementary cobordisms, and that the image of the cross section $S \cap (\mathbb{R}^3 \times \{t\})$ under the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ are regular except for finite many $t$’s. Thus there is a decomposition of $S$ into a union of $T_i = S \cap (\mathbb{R}^3 \times [t_{i-1}, t_i])$ ($i = 1, \ldots, N$) such that the boundary of each $T_i$ is a Reidemeister move or a Morse move. Let $L_i = S \cap (\mathbb{R}^3 \times \{t_i\})$ with $L_0 = L$, $L_N = L'$, and $D_i = p(L_i)$ with $D = D_0$, $D' = D_N$.

We define homomorphisms $\phi_i : H_c(D_{i-1}) \to H_c(D_i)$ correspondingly, namely, if $T_i$ corresponds to a Reidemeister move $\phi_i$ is the isomorphism $\rho$ given in the
proof of Theorem 3.2 (see Section 7.1). If $S_i$ corresponds to a 0-, 1-, 2- handle move, then $\phi_i$ is $\iota, f, \varepsilon$ respectively, where $\iota, \varepsilon$ are the (co)units, and $f$ is the homomorphism corresponding to the fusion move. Define $\phi$ by the composition of all $\phi_i$’s.

**Proposition 4.18.** In addition to the above setting, suppose every component of $S$ has a boundary in $L$. Then the induced homomorphism

$$\phi : H_c(D; R) \to H_c(D'; R)$$

maps

$$\phi[\alpha(D)] = \pm c^l[\alpha(D')], \quad l = \frac{1}{2}(-\Delta r + \Delta w - \chi(S)),$$

where the prefixed $\Delta$ denotes the difference of the corresponding values of $D, D'$, and $\chi(S)$ is the Euler number of $S$.

**Proof.** It suffices to prove the equation assuming that $c$ is invertible, by localizing by $c$. Let $S_i = S \cap (\mathbb{R}^3 \times [0, t_i])$. Each $S_i$ may be given alternative orientations, i.e. the possible orientations on the underlying unoriented surface of $S_i$. Within these alternative orientations, there are ones that agrees with the given orientation of $L$ on the bottom boundary. We call such orientations to be permissible. An orientation on $S_i$ induces an orientation on the upper boundary $L_i$. We call the orientation on $L_i$ induced from a permissible orientation on $S_i$ to be permissible. There are no two permissible orientations on $S_i$ that give the same orientation on $L_i$. Indeed, suppose $o, o'$ are permissible orientations that differ on a component $T$ of $S_i$ but induces the same orientation on $L_i$. $T$ has no boundary in $L$, otherwise $o, o'$ must be equal on $T$. Similarly, $T$ does not have a boundary in $L_i$. This implies that $T$ is a closed component, which contradicts the hypothesis. Thus we may identify permissible orientations on $S_i$ and the permissible orientations on $L_i$.

---

Figure 11: Cobordism map
Claim 1. For each permissible orientation $o$ on $L_i$,

$$\phi_i[\alpha(D_i, o)] = \pm c^j \sum_{o'} \alpha(D_{i+1}, o')$$

where $j \in \{0, \pm 1\}$, and the sum runs over a (possibly empty) set of permissible orientations of $S_{i+1}$ that coincide with $o$ on $S_i$.

For Reidemeister moves, $o$ extends uniquely to a permissible orientation $o'$ and we know the claim is true from Proposition 3.10. For a 0-handle move, there are two permissible orientations that extends $o$, and the corresponding classes are $[\alpha(D_i, o) \otimes a], [\alpha(D_i, o) \otimes b]$. Since $\phi_i = (- \otimes 1) = c^{-1}(- \otimes (a - b))$, the claim holds with $j = -1$.

For a 1-handle move there are several cases to consider. If the move splits a component of $L_i$, then $o$ uniquely extends to a permissible orientation $o'$ of $S_{i+1}$. If the move splits one of the Seifert circles (with respect to $o$) then $f$ maps $[\alpha(D_i, o)]$ to $[\alpha(D_{i+1}, o')]$, or or if it merges two circles then to $\pm c[\alpha(D_{i+1}, o')]$. Note that the Seifert circle(s) may split or merge, regardless of the splitting or the merging of the link. Next, suppose the move merges two components of $L_i$ into one. If the orientations on the components are coherent with respect to the handle, then the situation is similar. Otherwise $[\alpha(D_i, o)]$ is mapped to 0, since the two strands where the handle is attached point to the same direction and must be colored differently.

For a 2-handle move, $o$ uniquely extends to a permissible orientation $o'$. Since $\varepsilon(a) = \varepsilon(b) = 1$, we have $\phi_i[\alpha(D_i, o)] = [\alpha(D_{i+1}, o')]$. Thus the claim holds for all cases.

Claim 2. Suppose $x$ is an element in $H(D_i)$ written as

$$x = \sum_o (\pm c^{k_o})[\alpha(D_i, o)], \quad k_o \in \mathbb{Z}$$

where $o$ runs over a set of permissible orientations of $S_i$. Then the image of $x$ under $\phi_i$ has the same form (possibly zero).

This is obvious from the previous claim, and from the observation that no two permissible orientations extends to the same one.

Claim 3.

$$\phi[\alpha(D)] = \pm c^l[\alpha(D')]$$

for some integer $l$.

We see that the successive images of $[\alpha(D)]$ are of the form of Claim 2. $[\alpha(D_i)]$ (with the orientation induced from $S$) maps to $\pm c^l[\alpha(D_{i+1})]$ (modulo other terms), since mapping to 0 can happen only when a 1-handle merges inconsistently oriented components. At the end, there is only one permissible orientation on $S_N = S$, that is the given orientation of $S$. Thus $[\alpha(D)]$ maps to some $c$-power multiple of $[\alpha(D')]$. The right side of Figure 11 depicts the successive images under $\phi_i$. 

\[33\]
Proof continued. Now it remains to describe $l$. From the previous observation, we see that $l$ is given by the sum of the $c$-exponents each appearing at the coefficient of $[\alpha(D_{i+1})]$ in $\phi_i[\alpha(D_i)]$. Let $n_0, n_1, n_2$ be the numbers of 0-, 1-, 2-handle moves respectively. Also let $n_1 = n_{1,m} + n_{1,s}$, where $n_{1,m}$ ($n_{1,s}$, resp.) is the number of times the Seifert circles of $D_i$ are merged (split, resp.) by the 1-handle move. Let $J$ be the sum of $c$-exponent occurring at each $\phi_i([\alpha(D_i)])$ that corresponds to a Reidemeister move. For the Morse moves, $j = -1$ occurs only by the 0-handle move, and $j = +1$ occurs only when the Seifert circles merge. Thus we have

$$l = J - n_0 + n_{1,m}.$$ 

Let $\Delta r = \Delta r_R + \Delta r_M$, where $\Delta r_R$ (resp. $\Delta r_M$) is the sum of the differences of $r$ at each step corresponding to the Reidemeister move (resp. Morse move). Since $w$ is constant under the Morse moves, from Corollary 3.11 we have

$$-2J = \Delta r_R - \Delta w,$$

and obviously we have

$$\Delta r_M = n_0 - n_{1,m} + n_{1,s} - n_2.$$ 

Thus

$$l = -\frac{1}{2}(\Delta r_R - \Delta w) - n_0 + n_{1,m}$$

$$= \frac{1}{2}(-\Delta r + \Delta r_M + \Delta w - 2n_0 + 2n_{1,m})$$

$$= \frac{1}{2}(-\Delta r + \Delta w - n_0 + (n_{1,m} + n_{1,s}) - n_2)$$

$$= \frac{1}{2}(-\Delta r + \Delta w - \chi(S)).$$

\[\square\]

Proposition 4.19. If $S$ is a oriented cobordism between links $L, L'$ such that every component of $S$ has a boundary in $L$, then

$$s'_c(L') - s'_c(L) \geq \chi(S)$$

If also every component of $S$ has a boundary in both $L$ and $L'$. Then

$$|s'_c(L') - s'_c(L)| \leq -\chi(S).$$

Remark 4.20. Note that in the latter case each component $T$ of $S$ has at least two boundary components, so $\chi(T) \leq 0$ and $-\chi(S) = \sum_T -\chi(T) \geq 0$.

Proof. From Proposition 4.18, we have

$$k_c([\alpha(D)]) \leq k_c(\phi[\alpha(D)]) = k_c([\alpha(D')]) + l$$

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hence
$$\Delta k_c \geq -l = \frac{1}{2}(\Delta r - \Delta w + \chi(S)).$$

and
$$\Delta s' = 2\Delta k_c - \Delta r + \Delta w \geq \chi(S).$$

With the latter assumption, $S$ may also be regarded as a cobordism from $L'$ to $L$ satisfying the required condition of Proposition 4.18. Thus
$$s_c'(L) - s_c'(L') \geq \chi(S).$$

\[\square\]

### 4.3 Consequences

From Proposition 4.19 we obtain many properties of $s_c'$ common to the $s$-invariant; it is a link concordance invariant, gives a lower bound of the slice genus, and provides an alternative proof of the Milnor conjecture.

Recall that two links $L, L'$ are concordant if and only if $L, L'$ are cobordant by a union of annuli.

**Theorem 4.21.** $s_c'$ is invariant under link concordance in $S^3$.\[\square\]

**Lemma 4.22.** If a link $L$ in $S^3$ bounds a surface $S$ in $B^4$, then
$$|s_c'(L)| \leq -\chi(S).$$

**Proof.** Removing a small disk in $S$ gives an oriented connected cobordism between $L$ and $\Circle$.\[\square\]

**Proposition 4.23.** Let $L$ be a weakly slice link in $S^3$, i.e. there exists an oriented smooth connected surface $S$ in $B^4$ of genus zero that is bounded by $L$. Then
$$|s_c'(L)| \leq |L| - 1.$$

In particular for a slice knot $K$,
$$s_c'(K) = 0.$$

\[\square\]

**Proposition 4.24.** For a knot $K$,
$$|s_c'(K)| \leq 2g_*(K),$$

where $g_*(K)$ is the slice genus of $K$.\[\square\]

**Proposition 4.25.** If $K$ is a positive knot, then
$$s_c'(K) = 2g_*(K) = 2g(K),$$

where $g(K)$ is the knot genus of $K$.\[\square\]
Proof. From Proposition 4.16 we have $s'_c(K) = 2g(S)$, where $g(S)$ is the genus of the Seifert surface of $K$ obtained by applying the Seifert’s algorithm to a positive diagram of $K$. From Proposition 4.24,

$$2g_*(K) \leq 2g(K) \leq s'_c(K) \leq 2g_*(K),$$

and all values above are equal. \hfill \Box

**Corollary 4.26 (The Milnor Conjecture).** The slice genus and the unknotting number of the $(p,q)$ torus knot are both equal to $(p-1)(q-1)/2$.

We prove some more properties of $s'_c$ and $k_c$ that easily follows from Proposition 4.19.

**Proposition 4.27.** The following relations hold:

- For any link $L$,

  $$1 - 2|L| \leq s'_c(L \sqcup \overline{L}) \leq 1.$$

- For any link diagram $D$,

  $$r(D) - |D| \leq k_c(D \sqcup \overline{D}) \leq r(D).$$

Proof. For a link $L$, there is a cobordism consisting of $|L|$ saddles connecting $L \sqcup \overline{L}$ to the $|L|$-component unlink, so we have

$$|s'_c(L \sqcup \overline{L}) - (|L| - 1)| \leq |L|$$

$$\Leftrightarrow \quad 1 - 2|L| \leq s'_c(L \sqcup \overline{L}) \leq 1.$$

The inequality for $k_c$ follows from definition. \hfill \Box

**Corollary 4.28.** Suppose $R$ is a PID and $c$ is prime in $R$. Then the following relations hold:

- For any link $L$,

  $$2 - 2|L| \leq s'_c(L) + s'_c(\overline{L}) \leq 2.$$

- For any link diagram $D$,

  $$r(D) - |D| \leq k_c(D) + k_c(\overline{D}) \leq r(D).$$

\hfill \Box

**Proposition 4.29.** Let $D$ be a link diagram. Let $D'$ be the diagram obtained from $D$ by removing one crossing in the orientation preserving way. If the removed crossing is positive, then

$$k_c(D) \leq k_c(D') \leq k_c(D) + 1.$$

If it is negative, then

$$k_c(D) - 1 \leq k_c(D') \leq k_c(D).$$
Proof. Removing a crossing can be realized by attaching a 1-handle near the crossing. So we have
\[ |\Delta s'_c| = |2\Delta k_c + \Delta w| \leq 1 \]
where \( \Delta w = -1 \) if the removed crossing is positive, and \( \Delta w = 1 \) if negative.

Corollary 4.30. For any link diagram \( D \),
\[ 0 \leq k_c(D) \leq n^-(D) \]
where \( n^-(D) \) is the number of negative crossings of \( D \).

Proof. Denote by \( D' \) the diagram obtained from \( D \) by removing all negative crossings. Since \( D' \) is positive we have \( k_c(D') = 0 \), and from the previous corollary we have \( k_c(D) - n^-(D) \leq 0 \).

5 \quad s\text{-invariant and the canonical generator}

Rasmussen called the \( \alpha \)-classes in \( H_{Lee}(L; \mathbb{Q}) \) the canonical generators of \( H_{Lee}(L; \mathbb{Q}) \), from the fact that they form a basis of \( H_{Lee}(L; \mathbb{Q}) \) (Proposition 2.11) and that they are invariant (up to unit) under the Reidemeister moves (Proposition 2.13).

We have seen that for a general \((R, c)\), these classes do not generate \( H_c(D; R) \) and that they are not invariant under the moves. In this section, we focus on knots, and restrict \((R, c)\) to the integral Lee theory and the (bigraded) Bar-Natan theory over a field \( F \) of char \( F \neq 2 \). For these two theories, we can normalize \([\alpha(D)]\) and \([\beta(D)]\) so that they form a basis of \( H_c(D; R) \approx \mathbb{R}^2 \), and that they are invariant under the Reidemeister moves. With the normalized generator \([\zeta(K)]\) and \( X[\zeta(K)]\), we see that \( s'_c \) defines a homomorphism
\[ s'_c: \text{Conc}(S^3) \to 2\mathbb{Z} \]
and in particular for \((R, c) = (F[h], h)\), we see that \( s'_c \) coincides with the \( s \)-invariant, and is characterised by
\[ s(K; F) = q\deg [\zeta(K)] - 1. \]

Throughout this section, we assume \((R, c)\) is either \((\mathbb{Z}, 2)\) or \((F[h], h)\) with char \( F \neq 2 \) and \( \deg h = -2 \).

5.1 Construction of the generator

To commonalize the arguments, denote by \( R_0 \) the subring of \( R \) spanned by homogeneous elements of degree \( \equiv 0 \mod 4 \) \((R_0 = \mathbb{Z} \text{ for } R = \mathbb{Z}, \text{ and } R_0 = F[h^2] \text{ for } R = F[h])\). In both cases \( R \) and \( R_0 \) are PIDs, \( c/2 \in R \) and \( \deg (c/2)^2 \equiv 0 \mod 4 \), so \((c/2)^2 \in R_0 \). For a link diagram \( D \), denote
\[ C(D; R) = C_{0, (c/2)^2}(D; R), \]
\[ C(D; R_0) = C_{0, (c/2)^2}(D; R_0). \]
Since $H(D; R_0)_f$ is torsion-free, the natural map

$$H(D; R_0)_f \rightarrow H(D; R_0)_f \otimes_{R_0} R$$

is injective, and since $R$ is flat over $R_0$ (for $R$ is torsion-free and $R_0$ is a PID), we have

$$H(D; R_0)_f \otimes R \cong (H(D; R_0) \otimes R)_f \cong H(D; R)_f.$$ Under this correspondence we regard $H(D; R_0)_f \subset H(D; R)_f$.

Let \((\alpha, \beta) = (\alpha(D), \beta(D))\). Note that \(\alpha, \beta \notin C(D; R_0)\) when \(\deg c = -2\). The following two lemmas are generalizations of Lemma 2.15, 2.17.

**Lemma 5.1.** $C(D; R_0)$ decomposes into a direct sum of two subcomplexes:

$$C(D; R_0)' = \text{Span}_{R_0} \{ x \mid \text{qdeg}(x) \equiv |D| \text{ mod } 4 \}$$

$$C(D; R_0)'' = \text{Span}_{R_0} \{ x \mid \text{qdeg}(x) \equiv |D| + 2 \text{ mod } 4 \}$$

where $x$ runs over the enhanced states of $D$.

**Proof.** Obvious from Lemma 2.14, and that $d$ preserves qdeg mod 4. \(\square\)

**Lemma 5.2.** There are two elements in $C(D; R_0) \subset C(D; R)$:

$$\xi = (\alpha + \beta)/2,$$

$$\eta = (\alpha - \beta)/c.$$

Either one is contained in $C(D; R_0)'$ and the other is in $C(D; R_0)''$.

**Proof.** \(\xi = (\alpha + \beta)/2\) is the part of $\alpha$ with even numbers of \((c/2)\)'s in its tensor factors. \((\alpha - \beta)/2\) is the odd part, and dividing it by \(c/2\) gives \(\eta\). Both $\xi, \eta$ belong to $C(D; R_0)$, since each of them can be written as a linear combination of enhanced states with each coefficient a power of \((c/2)^2 \in R_0\). If \(\deg c = 0\) regarding q-degree modulo 4 we may ignore the coefficients and the discussion is same as in the proof of Lemma 2.17. If \(\deg c = -2\), then $\alpha, \beta$ are homogeneous and dividing by \(c\) increases the degree by 2. \(\square\)

**Lemma 5.3.** Take any point $p$ on an arc of $D$. Let $X_p$ be the endomorphism of Lemma 3.19. Define $\varepsilon_p = 1$ if $\alpha(D)$ is colored $a$ on the circle containing $p$, otherwise define $\varepsilon_p = -1$. Then the induced endomorphism $\varepsilon_p X_p$ on $H(D; R)$ is independent of the choice of the point $p$.

**Proof.** It suffices to prove $\varepsilon_p X_p = \varepsilon_q X_q$ on $H(D; R)$ when $p, q$ are two marked points on a strand of $D$ separated by a crossing. From Lemma 3.19, $X_p$ and $-X_q$ are chain homotopic, and from Lemma 2.8 we have $\varepsilon_p = -\varepsilon_q$. \(\square\)

Thus $H(D; R)$ admits an $A_{0,(c/2)^2}$-module structure by

$$X \cdot x = \varepsilon_p X_p(x).$$

**Lemma 5.4.** $X$ maps $C(D; R_0)'$ to $C(D; R_0)''$ and vice versa.
Proof. Obvious since $X_p$ is a map of q-degree 2 mod 4. \hfill \Box

Lemma 5.5. $X$ maps:

\[
\begin{align*}
\alpha & \mapsto (c/2)\alpha, \\
\beta & \mapsto -(c/2)\beta, \\
\xi & \mapsto (c/2)^2\eta, \\
\eta & \mapsto \xi.
\end{align*}
\]

Proof. If $\varepsilon_p = 1$, then $\alpha = a \otimes \alpha'$ and $\beta = b \otimes \beta'$ for some $\alpha', \beta'$, where the factor corresponding to the circle containing $p$ is placed at the first. We have

\[
\begin{align*}
\alpha &= (X + c/2) \otimes \sigma \xrightarrow{X_p} ((c/2)^2 + (c/2)X) \otimes \sigma = (c/2)\alpha \\
\beta &= (X - c/2) \otimes \bar{\beta} \xrightarrow{X_p} ((c/2)^2 - (c/2)X) \otimes \bar{\beta} = -(c/2)\beta
\end{align*}
\]

The computation is similar when $\varepsilon_p = -1$. The images of $\xi, \eta$ are obvious from definition. \hfill \Box

Corollary 5.6. $C_{Lee}(D; R) = C_{0,1}(D; R)$ splits into two mutually isomorphic subcomplexes.

Proof. $X$ is an involution mapping $C(D; R_0)'$ isomorphically onto $C(D; R_0)''$. \hfill \Box

In the remaining we assume that $D$ is a knot diagram.

Proposition 5.7. For any knot diagram $D$, there is a unique class $[\zeta] \in H(D; R_0)_f$ such that:

- $\{ [\zeta], X[\zeta] \}$ is a basis of $H(D; R_0)_f$ and of $H(D; R)_f$, and,
- $[\alpha], [\beta] \in H(D; R)_f$ can be written as

\[
\begin{align*}
[\alpha] &= c^k (X[\zeta] + (c/2)[\zeta]) \\
[\beta] &= (-c)^k (X[\zeta] - (c/2)[\zeta])
\end{align*}
\]

where $k = k_c(D)$.

Proof. The decomposition $C(D; R_0) = C(D; R_0)' \oplus C(D; R_0)''$ descends to homology, and since two independent classes $[\xi], [\eta]$ live separately in the summands, there is a basis $\{[\xi_0], [\eta_0]\}$ of $H(D; R_0)_f \cong (R_0)^2$ such that

$[\xi] = x'[\xi_0], \quad [\eta] = y'[\eta_0]$

for some $x', y' \in R_0$. With $H(D; R_0)_f \otimes R \cong H(D; R)_f$, the two elements also form a basis of $H(D; R)_f$. Now

\[
\begin{align*}
[\alpha] &= [\xi] + (c/2)[\eta] = x'[\xi_0] + (c/2)y'[\eta_0], \\
[\beta] &= [\xi] - (c/2)[\eta] = x'[\xi_0] - (c/2)y'[\eta_0]
\end{align*}
\]
and since

\[ [\alpha] \in c^k H(D; R) = R \langle c^k[\xi_0], c^k[\eta_0] \rangle, \]

there are \( x, y \in R \) such that

\( x' = c^k x, \quad (c/2)y' = c^k y. \)

so we may write

\[ [\alpha] = c^k( x[\xi_0] + y[\eta_0] ), \]

\[ [\beta] = c^k( x[\xi_0] - y[\eta_0] ). \]

First, \( x, y \) is not commonly divisible by \( c \) from the maximality of \( k \). With the endomorphism \( X \), we have:

\[ [\xi] = c^k x[\xi_0] \Rightarrow (c/2)[\eta] = (c/2)c^k y[\eta_0] = c^k xX[\xi_0] \]

\[ (c/2)[\eta] = c^k y[\eta_0] \Rightarrow (c/2)[\xi] = (c/2)c^k x[\xi_0] = c^k yX[\eta_0] \]

so

\[ (c/2)y[\eta_0] = xX[\xi_0] \quad (a) \]

\[ (c/2)x[\xi_0] = yX[\eta_0] \quad (b) \]

and this implies

\[ x \mid (c/2)y, \quad y \mid (c/2)x. \quad (c) \]

We define \([\zeta]\) as:

\[ [\zeta] = \begin{cases} 
(c/2)^{-1} y[\eta_0] & \text{if } k \text{ is even} \\
(c/2)^{-1} x[\xi_0] & \text{if } k \text{ is odd} 
\end{cases} \]

and check that this definition is valid.
**Case 1** \(((R, c) = (\mathbb{Z}, 2))\). (c) implies \(x = \pm y\). First \(x, y\) is not commonly divisible by 2. If there is a prime \(p\) that divides both \(x, y\), then by tensoring \(\mathbb{F}_p\) we see that \([\alpha] = [\beta] = 0 \in H(D; \mathbb{F}_p)\), which contradicts Proposition 2.11. Thus \(x, y \in \{\pm 1\}\), and the definition is valid. 

**Case 2** \(((R, c) = (F[h], h))\). From \(x' = h^k x, \quad y' = 2h^{k-1} y \in F[h^2]\), we must have
\[
\begin{aligned}
x &\sim 1, \quad y \sim h \quad \text{if } k \text{ is even} \\
x &\sim h, \quad y \sim 1 \quad \text{if } k \text{ is odd}
\end{aligned}
\]
so the definition is valid. 

*Proof continued.* Next we check that \([\zeta]\) satisfies the required conditions. If \(k\) is even, \([\zeta]\) is associated to \([\eta_0]\) and from (b) we have
\[x[\xi_0] = (c/2)^{-1} y X[\eta_0] = X[\zeta].\]
Thus \(X[\zeta]\) is associated to \([\xi_0]\) and we have
\[
[\alpha] = c^k ( X[\zeta] + (c/2)[\zeta] ),
[\beta] = c^k ( X[\zeta] - (c/2)[\zeta] ).
\]
If \(k\) is odd, then \([\zeta]\) is associated to \([\xi_0]\) and from (b) we have \(y[\eta_0] = X[\zeta]\). Thus \(X[\zeta]\) is associated to \([\eta_0]\) and we have
\[
[\alpha] = c^k ( (c/2)[\zeta] + X[\zeta] ),
[\beta] = c^k ( (c/2)[\zeta] - X[\zeta] ).
\]
Hence in both cases \{\([\zeta], X[\zeta]\)\} form a basis of \(H(D; R_0)_f\) and of \(H(D; R)_f\), and satisfies the desired description of \([\alpha], [\beta]\). Uniqueness follows by comparing the descriptions of \([\alpha]\) and \([\beta]\). 

**Corollary 5.8.**
\([\eta] = (2X)^k [\zeta], \quad [\xi] = (2X)^k X[\zeta]\)

*Proof.* Follows from Proposition 5.7 and \(c^2 = (2X)^2\). 

Thus \(k_c(D)\) may also be regarded as the \(2X\)-divisibility of \([\eta(D)]\). 

**Question 5.9.** Can Proposition 5.7 be extended to link diagrams? 

If we try to apply the proof directly, there is an ambiguity of choosing the generator of \(H(D; R_0)_f\), which was unique (up to unit) when \(|L| = 1\). An induction on the number of components might work, but the exactness of the long exact sequence of Proposition 2.4 collapses since we have modded out the torsions.
Next we prove the invariance of the class $[\zeta(D)]$. First, we adjust the signs of the isomorphisms corresponding to the Reidemeister moves given in Proposition 3.10, so that each $\rho : H(D) \to H(D')$ maps

$$[\alpha(D')] = c^j \rho[\alpha(D)],$$
$$[\beta(D')] = (-c)^j \rho[\beta(D)].$$

This is done by redefining $\rho$ by $\varepsilon \rho$. Under this modification, the following holds:

**Proposition 5.10.** Let $D, D'$ be knot diagrams related by a single Reidemeister move, and $\rho : H(D) \to H(D')$ be the corresponding isomorphism. Then $\rho$ descends to an $A_0, (c/2)^2$-module isomorphism

$$\rho : H(D)_f \to H(D')_f,$$
that maps $[\zeta(D)]$ to $[\zeta(D')]$.

*Proof.* We have $k' = k + j$, where $k = k_c(D)$ and $k' = k_c(D')$. Since $\rho$ preserves the splitting, we see that $[\zeta], X[\zeta]$ is mapped to $[\zeta'], X[\zeta']$ by comparing the images of $[\alpha]$ and $[\beta]$. \qed

**Remark 5.11.** This implies that $H_c(K)_f$ has trivial monodromy group, i.e. the automorphism induced from any sequence of Reidemeister moves from $D$ returning back to $D$ is always the identity. This is not the case for Khovanov homology, where in [12, Theorem 1] a non-trivial monodromy for the diagram of 8_{18} is given.

With the commutative diagram of Proposition 3.21, for any knot $K$ we may define an $A_c$-module:

$$H_c(K; R)_f = \bigoplus_D H_c(D; R)_f / \rho$$

where $D$ runs over all diagrams of $K$. We denote by $[\zeta(K)]$ the equivalence class represented by any $[\zeta(D)]$. As a summary, we obtain the following:

**Theorem 5.12.** For any knot $K$, $H_c(K; R)_f$ is generated by $[\zeta(K)]$ over $A_c$.

**Question 5.13.** Is there a geometric interpretation for the class $[\zeta(K)] \in H(K)_f$? Can we explicitly (combinatorially) construct a representative cycle $\zeta(D) \in C(D)$ from $D$?

The answer when $D$ is either positive or negative is given in Section 5.4.

### 5.2 Behaviour under cobordisms

Next we inspect the behaviour of $[\zeta(D)]$ under cobordisms. In Section 4.2, we defined a homomorphism corresponding to a cobordism by decomposing it into
elementary cobordisms and associating them to homomorphisms between the homology groups. As we have adjusted the signs of $\rho$, we adjust the signs of the homomorphisms corresponding to the Morse moves, so that at each level the $\alpha$-class (corresponding to the orientation induced from that of the cobordism) maps to the $\alpha$-class of the next level (modulo other terms) without a sign. This can be done by adjusting the signs of the homomorphisms corresponding to the 0-handle move, and the 1-handle moves that merge two Seifert circles. Thus for a connected cobordism $S$ between two knots $K, K'$, we obtain a homomorphism

$$\phi : H(D) \to H(D')$$

such that:

$$\phi[\alpha] = e^l[\alpha'], \quad l = \frac{1}{2}(-\Delta r + \Delta w - \chi(S)).$$

As for $[\beta]$, $\phi$ yields negative signs at every Reidemeister move with $j = \pm 1$, every 0-handle move, and every 1-handle move that merges two Seifert circles. With the symbols used in the proof of Proposition 4.18, the overall sign is given by

$$(-1)^{J+n_0+n_1,m} = (-1)^l.$$

Thus we have

$$\phi[\beta] = (-e)^l[\beta'].$$

Under this modification, the following holds:

**Proposition 5.14.** Let $S$ be a connected oriented cobordism between two knots $K, K'$. Let $D, D'$ be knot diagrams of $K, K'$, and $\phi : H_c(D) \to H_c(D')$ be a homomorphism constructed from $S$. Then $\phi$ is an $A_e$-module homomorphism, and

$$\phi[\zeta] = (2X)^{\Delta s'_e - \chi(S)}[\zeta']$$

where $\Delta s'_e = s'_e(K') - s'_e(K)$.

**Remark 5.15.** Both $\Delta s'_e, \chi(S)$ are even, and from Proposition 4.19 we have $\Delta s'_e - \chi(S) \geq 0$.

**Proof.** With the description of Proposition 5.7, we have

$$\phi[\zeta] = \phi( e^{-k-1}(\alpha) + (-1)^k[\beta] )$$

$$= e^{l-k-1}(\alpha') + (-1)^{l+k}[\beta']$$

$$= e^{l+k'-k-1}((c/2)[\zeta'] + X[\zeta'] + (-1)^{l+k'-k}((c/2)[\zeta] - X[\zeta])),$$

and

$$\phi(X[\zeta]) = \phi( (c^{-k}/2)(\alpha) - (-1)^k\phi[\beta] )$$

$$= (e^{l+k'-k}/2)((c/2)[\zeta'] + X[\zeta'] - (-1)^{l+k'-k}((c/2)[\zeta] - X[\zeta])),$$

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With
\[ l + k' - k = \frac{1}{2} (2\Delta k - \Delta r + \Delta w - \chi(S)) \]
and \( c^2 = (2X)^2 \), we obtain
\[ \phi \left( \left[ \zeta \right] \right) = (2X) \frac{\Delta s' - \chi(S)}{2} \left[ \zeta' \right] . \]

Thus \( \phi \) commutes with \( X \) and we obtain the desired result.

This also shows that \( \phi \) is functorial, since \( \Delta s' \) and \( \chi(S) \) are both additive under the composition of cobordisms. Thus we obtain

**Theorem 5.16.** \( H_c(-; R)_f \) is a functor from the category of knots (with morphisms cobordisms between knots) to the category of \( A_c \)-modules.

**Remark 5.17.** This functor has no sign ambiguity, and the image of a cobordism is determined by its Euler number (thus invariant up to homeomorphism). This is again different from Khovanov homology, where the functor is defined only up to sign, and the image of a cobordism is invariant only up to isotopy relative to the boundary. See [12, Theorem 2].

Moreover if we consider cobordisms such that \( \chi(S) = 0 \), then from Proposition 4.19 we have \( \Delta s' = 0 \), and the classes \( [\zeta(K)] \) correspond one-to-one. In particular, \( [\zeta(K)] \) is invariant under knot concordance. For any knot concordance class \( K \in Conc(S^3) \) we define

\[ H_c(K; R)_f = \bigoplus_K H_c(K; R)_f \bigg/ \phi \]

where \( K \) runs over the representative knots of \( K \). We denote by \( [\zeta(K)] \) the equivalence class represented by \( [\zeta(K)] \) for any \( K \in K \).

**Theorem 5.18.** For any knot concordance class \( K \in Conc(S^3) \), \( H_c(K; R)_f \) is generated by \( [\zeta(K)] \) over \( A_c \).

**Remark 5.19 (An interpretation of \( 2X \)).** \( 2X \) appears in Corollary 5.8 and in Proposition 5.14. We have
\[ m\Delta = 2X \]
in \( A_{0,t} \), so multiplication by \( 2X \) is equivalent to the endomorphism \( \phi : H(K) \to H(K) \) given by the connected sum of the identity cobordism and a torus.
5.3 Homomorphism property of $s'_c$

There is unimodular pairing

$$\langle -, - \rangle : C(D) \otimes C(D) \to R$$

defined by the composition of the isomorphism $T : C(D) \to C(D)^*$ of Proposition 3.3 and the standard pairing between $C(D)$ and its dual $C(D)^*$. From a general statement of Lemma 7.1, this descends to

$$\langle -, - \rangle : H(D)_f \otimes H(D)_f \to R,$$

and since $R$ is a PID, it is unimodular.

**Notation 5.20.** Any bilinear form of $R$-modules

$$\langle -, - \rangle : M \otimes N \to R$$

induces a bilinear map

$$\langle -, - \rangle : M^m \otimes N^n \to \text{Mat}(m, n; R).$$

In particular, we write

$$\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c & d \end{pmatrix} \rangle = \begin{pmatrix} \langle a, c \rangle & \langle a, d \rangle \\ \langle b, c \rangle & \langle b, d \rangle \end{pmatrix}.$$

**Lemma 5.21.** Let $(\alpha, \beta), (\overline{\alpha}, \overline{\beta})$ be the pairs of $\alpha$-cycles of $D, \overline{D}$ respectively.

$$\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \overline{\alpha} & \overline{\beta} \end{pmatrix} \rangle = c^r(D) \begin{pmatrix} (-1)^b & 0 \\ 0 & (-1)^a \end{pmatrix}$$

where $a, b$ are the numbers of $a$’s and $b$’s in the tensor factors of $\alpha$.

**Proof.** From

$$a = X + (c/2)1, \quad b = X - (c/2)1$$

$$T(\overline{\alpha}) = 1^* + (c/2)X^*, \quad T(\overline{\beta}) = 1^* - (c/2)X^*$$

we have:

$$\langle \begin{pmatrix} a \\ b \end{pmatrix}, (\overline{\alpha} & \overline{\beta}) \rangle = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$

The result follows since the Seifert circles of $D$ and $\overline{D}$ are identical. $\square$

**Proposition 5.22** (Mirror formula). Let $D$ be a knot diagram.

$$k_c(D) + k_c(\overline{D}) = r(D) - 1.$$
Proof. With the description of Proposition 5.7,

\[
\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \pi \\ \tilde{\beta} \end{pmatrix} \right\rangle = c_k + k' \left( c/2 \mp c/2 \pm 1 \right) \left\langle \begin{pmatrix} \zeta \\ X\zeta \end{pmatrix}, \begin{pmatrix} \zeta, X\zeta \end{pmatrix} \right\rangle \left( c/2 \mp c/2 \pm 1 \right)
\]

Since the pairing is unimodular, the middle matrix on the right hand side must have unital determinant. Combined with Lemma 5.21, by comparing the determinants on both side we have

\[2r(D) = 2(k + k') + 2.\]

\[\square\]

Corollary 5.23. For a negative knot diagram \(D\),

\[k_c(D) = r(D) - 1.\]

\[\square\]

Corollary 5.24.

\[\left\langle \begin{pmatrix} \zeta \\ X\zeta \end{pmatrix}, \begin{pmatrix} \zeta, X\zeta \end{pmatrix} \right\rangle = (-1)^b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\]

where \(|\zeta| \in H(D)_f, |\tilde{\zeta}| \in H(\overline{D})_f\) are the unique classes of Proposition 5.7 for \(D, \overline{D}\) respectively, and \(b\) is the numbers of \(b\)'s in the tensor factors of \(\alpha\).

\[\square\]

Proposition 5.25. Let \(D, D'\) be knot diagrams.

\[k_c(D \# D') = k_c(D) + k_c(D').\]

Proof. Since \(R\) is a PID, from Proposition 4.12, 4.13, for any \(D, D'\) we have

\[k_c(D \# D') \leq k_c(D) + k_c(D').\]

With Proposition 5.22 we have

\[k_c(D \# D') = -k_c(\overline{D} \# \overline{D'}) + r(D \# D') - 1 \geq -(k_c(\overline{D}) + k_c(\overline{D'})) + (r(D) + r(D') - 1) - 1 = k_c(D) + k_c(D').\]

\[\square\]

Theorem 5.26. \(s'_c\) defines a homomorphism from the concordance group of knots in \(S^3\) to \(2\mathbb{Z}\),

\[s'_c: \text{Conc}(S^3) \to 2\mathbb{Z}.\]
Proof. Well-definedness of the map follows from Theorem 4.21. That this is a homomorphism follows from Proposition 5.22, 5.25.

The following is due to Livingston [23], which holds for any knot invariant satisfying the properties of Theorem 5.26 and Proposition 4.25.

**Corollary 5.27.** If \( K^+ \) and \( K^- \) differ by a single crossing change, from positive to negative, then
\[
s'_c(K^+) - s'_c(K^-) = 0 \text{ or } 2.
\]

5.4 Representative of the generator

**Proposition 5.28.** If \( D \) is a positive knot diagram, then the class \([\zeta(D)]\) is represented by the cycle \( \eta(D) \) of Lemma 5.2.

**Proof.** Obvious from \( k_c(D) = 0 \) and the definition of \([\zeta(D)]\). □

Next we consider when \( D \) is negative. In [2], a procedure to construct a representative cycle of \([\alpha(D)]\) in \( H_{Lee}(D; \mathbb{Q}) \) that has strictly higher \( q \)-degree than that of \( \alpha(D) \) is given. Such cycle gives a lower bound for the \( s \)-invariant. This procedure can also be used to construct a cycle \( \alpha' \) such that \( \alpha \sim c^k \alpha' \) for some \( k \geq 0 \).

We say a Seifert circle is negative if it is touched by and only by negative crossings.

**Lemma 5.29.** Let \( D \) be a non-split link diagram. Denote by \( r^-(D) \) the numbers of negative Seifert circles of \( D \). Define
\[
k = \min\{r^-(D), r(D) - 1\}.
\]

Let \( \alpha = \alpha(D) \) and \( s_0 \) be the orientation preserving state of \( D \). There is a cycle \( \alpha' \in V(D, s_0) \) such that
\[
\alpha \sim (-1)^b c^k \alpha' \ (\text{homologous}),
\]
and \( \alpha' \) has the form:
\[
\alpha' = \underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes \alpha''
\]
where the factors corresponding to the negative circles are placed at first, \( b \) is the number of \( b \)'s in the leading \( k \) factors of \( \alpha \), and \( \alpha'' \) is the remaining \( r(D) - k \) factors of \( \alpha \).

**Proof.** From Lemma 2.8, the edge maps into \( s_0 \) are split maps and those out of \( s_0 \) are merge maps. Let \( \gamma^- \) be a negative circle, and \( \gamma \) be any circle adjacent to \( \gamma^- \) by a negative crossing \( x \). Let \( s \) be the state obtained from \( s_0 \) by changing the resolution at \( x \) to 0. Let \( \tilde{\gamma} \) be the circle that splits into \( \gamma^- \) and \( \gamma \) by the resolution change at \( x \).
Suppose \( \gamma^- \) is colored \( a \) in \( \alpha \). Then from Lemma 2.8, \( \gamma \) is colored \( b \). We write \( \alpha = a \otimes b \otimes a'' \in V(D, s_0) \). Let \( z = b \otimes a'' \in V(D, s) \), where the first factor corresponds to \( \gamma \). Then \( dz = b \otimes b \otimes a'' \), since the edge map from \( s \) to \( s_0 \) is a split map, and others are merge maps that annihilates \( z \), again from Lemma 2.8. We have

\[
dz = b \otimes b \otimes a'' \\
= X \otimes b \otimes a'' - v \otimes b \otimes a''
\]

so:

\[
\alpha = a \otimes b \otimes a'' \\
= X \otimes b \otimes a'' - u \otimes b \otimes a'' \\
= (v - u) \otimes b \otimes a'' \\
= c \otimes b \otimes a''
\]

Define \( \alpha' = 1 \otimes b \otimes a'' \), then \( \alpha' \) is a cycle satisfying \( \alpha \sim \alpha' \). If \( \gamma^- \) is colored \( b \), then \( \alpha = b \otimes a \otimes a'' \), and with \( \alpha' = 1 \otimes a \otimes a'' \) and we have \( \alpha \sim -\alpha' \). This procedure can be repeated for each negative cycle, except when all circles are negative and all but one negative circles have been consumed, since there will be no circle left as its counterpart \( \gamma \).

\[\square\]

**Corollary 5.30.** For a non-split link diagram \( D \),

\[
k_c(D) \geq \min\{r^{-}(D), \, r(D) - 1\}.
\]

**Proposition 5.31.** If \( D \) is a negative knot diagram, then \( [\zeta(D)] \) is represented by \( (-1)^b 1 \otimes \cdots \otimes 1 \in V(D, s_0) \) where \( b \) is the number of \( b \)'s in the tensor factors of \( \alpha \).

*Proof.* Let \( k = r(D) - 1 \). Since \( D \) is negative we have \( k_c(D) = k \). Let \( z = 1 \otimes \cdots \otimes 1 \in V(D, s_0) \). This is a cycle, since \( s_0 \) is at the last end of the cube. From Lemma 5.29, there is a cycle \( \alpha_0 \) satisfying \( \alpha \sim \pm c^k \alpha_0 \) having the form \( 1 \otimes \cdots \otimes 1 \otimes x \) with \( x \in \{a, b\} \). Take a point \( p \) in an arc of the circle corresponding to the last factor. Suppose \( x = a \). The sign is given by \( (-1)^b \). We have \( Xz = 1 \otimes \cdots \otimes 1 \otimes X \), so \( \alpha_0 = 1 \otimes \cdots \otimes 1 \otimes (X - u) = Xz + (c/2)z \).

Suppose \( x = b \). The sign is given by \( (-1)^{b-1} \), and \( Xz = 1 \otimes \cdots \otimes 1 \otimes X \) so \( \alpha_0 = 1 \otimes \cdots \otimes 1 \otimes (X - v) = -Xz - (c/2)z \). Thus in both cases we have \( \alpha \sim (-1)^b c^k (Xz + (c/2)z) \). Since the description of \([\alpha]\) given in Proposition 5.7 is unique, \( (-1)^b z \) must be a representative of \([\zeta]\).

\[\square\]

In general the situation seems to be more complicated. In [2], a “non-state cycle” representing the \( \alpha \)-class of the standard pretzel diagram of \( P(3, -5, -7) \) is given. Another one for the diagram of \( 8_{19} \) is given in [1]. A “non-state cycle” is a cycle that does not belong to a single state, in particular one that cannot be obtained from the above procedure. It is stated that such explicit presentation is not known in general, even for an alternating diagram. Finding an explicit representative for \([\zeta(D)]\) in general would be as equally difficult.

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5.5 Coincidence of $s$ and $s'_h$

Finally we prove that $s'_c$ coincides with the $s$-invariant when $(R, c) = (F[h], h)$.

**Theorem 5.32.** For any knot $K$,

$$s(K; F) = s'_h(K; F[h]).$$

**Proof.** It suffices to prove the inequality:

$$s(K; F) \geq s'_h(K; F[h]).$$

Denote $C(D; F) = C_{0, 1}(D; F)$ and $C_h(D; F[h]) = C_{0, (h/2)^2}(D; F[h])$. Let $\alpha, \alpha_h$ be the $\alpha$-cycles of $D$ in $C(D; F)$, $C_h(D; F[h])$ respectively. From Proposition 2.19 we have $s(K; F) = q\deg([\alpha]) + 1$, so the proposition is equivalent to:

$$q\deg([\alpha]) \geq 2k_h(D; F[h]) + w(D) - r(D).$$

$\alpha_h$ is homogeneous with $q\deg(\alpha_h) = w(D) - r(D)$. Let $[\alpha_h] = h^k[\alpha'_h]$ with maximal $k$. Since $\deg(h) = -2$, we have $q\deg([\alpha'_h]) = 2k + w(D) - r(D)$. A ring homomorphism $\pi : F[h] \to F$, $h \mapsto 2$ induces a $q$-degree non-decreasing chain map $\pi : C_h(D; F[h]) \to C(D; F)$. We have $\pi(\alpha_h) = \alpha$ from the definition of the $\alpha$-cycle. Since multiplication by 2 preserves the $q$-degree in $H(D; F)$, we have

$$q\deg([\alpha]) = q\deg(\pi_*[\alpha_h])$$
$$= q\deg(\pi_*[\alpha'_h])$$
$$\geq q\deg([\alpha'_h])$$
$$= 2k + w(D) - r(D).$$

We have also proved the following:

**Corollary 5.33.** Let $D$ be a knot diagram and $\alpha$ be the $\alpha$-cycle of $D$ in $C_2(D; F)$. The increase of $q$-degree in homology is given by:

$$q\deg[\alpha] - q\deg \alpha = 2k_h(D; F[h]).$$

Several characterizations of the $s$-invariant follow.

**Proposition 5.34.** For any knot $K$,

$$s(K; F) = q\deg[\zeta(K)] - 1,$$

where $[\zeta(K)] \in H_h(K; F[h])$. 49
Proof. With the symbols used in the previous proof, the (bigraded) q-degree of \([\alpha_h'] = h^{-k}[\alpha_h] \in H_{h,0}(D;F[h])\) is equal to the (filtered) q-degree of \([\alpha_2] \in H_{0,1}(K;F)\). Together with Proposition 5.7, we have

\[
\text{qdeg} [\zeta] = \text{qdeg} [\alpha_h] + 2(k + 1) = \text{qdeg} [\alpha_h'] + 2 = \text{qdeg} [\alpha_2] + 2 = s(K;F) + 1.
\]

The following is stated in [16, Proposition 8] (without a proof) for \(F = \mathbb{Q}\).

**Corollary 5.35.** Let \(t\) be a formal variable of degree \(-4\). For any knot \(K\), \(H_{0,t}(K;F[t])\) admits a free \(F[X]\)-module structure, with a bigrading preserving isomorphism

\[
H_{0,t}(K;F[t]) \cong (F[X])[0,s(K;F) + 1].
\]

**Proof.** If we consider \((R,c) = (\mathbb{Q}[\sqrt{t}], 2\sqrt{t})\), then the subring \(R_0\) is \(F[t]\) and \(H_{0,(c/2)^2}(D;R_0) = H_{0,t}(D;F[t])\). From Proposition 5.7, \(H_{0,t}(D;F[t])\) is freely generated by \(\{\zeta, X[\zeta]\}\) over \(F[t]\). The endomorphism \(X\) gives \(H_{0,t}(D;F[t])\) an \(F[X]\)-module structure, and with \(X^2 = (c/2)^2 = t\) we see that it is freely generated by \(\zeta\) over \(F[X]\).

The following is stated in [18] with \((R, c) = (\mathbb{Q}[\lambda], \lambda^2), \deg \lambda = -2\).

**Corollary 5.36.** Let \(S\) be a connected cobordism from the unknot to a knot \(K\). Let \(D\) be a diagram of \(K\), and \(\phi : H(\bigcirc) \to H(D)\) be the homomorphism corresponding to \(S\). Let \(m^+\), \(m^-\) be the \(h\)-divisibility of \(\phi(1), \phi(X)\) respectively, and let \(m = (m^+ + m^-)/2\). We have

\[
s(K;F) = 2m + \chi(S).
\]

**Proof.** Since \(\zeta(\bigcirc) = 1\), from Proposition 5.14 we either have

\[
m_+ = m_- = (s_\phi'(K) - \chi(S))/2,
\]

or

\[
m_+ + 1 = m_- - 1 = (s_\phi'(K) - \chi(S))/2.
\]

We end this section with some questions.

**Question 5.37.** Does \(s(-;F) = s_\phi'(-;F[h])\) also hold for links?

**Question 5.38.** Is \(s_2'(-;\mathbb{Z})\) distinct from any of \(s(-;F) = s_\phi'(-;F[h])\)?
Computational results\(^4\) show that \(s'(K; \mathbb{Z})\) coincides with \(s(K; \mathbb{Q})\) for knot diagrams of crossing number up to 11.

**Conjecture 5.39.** \(s'(K; \mathbb{Z})\) coincides with \(s(K; \mathbb{Q}) = s_h(K; \mathbb{Q}[h])\).

Theoretically, the relation between the divisibility of the \(\alpha\)-class in \(H_2(D; \mathbb{Z})\) and the increase of the (filtered) \(q\)-degree in homology is ambiguous. We give two statements that imply \(s'_2(K; \mathbb{Z})\) is equal to any \(s(K; F)\) for a field \(F\) of \(\text{char } F \neq 2\).

**Statement 5.40.** Let \(\alpha\) be the \(\alpha\)-cycle of a knot diagram \(D\) in \(C_2(D; \mathbb{Z})\), and \(k = k_2(D; \mathbb{Z})\). There is a cycle \(\alpha'\) such that \([\alpha] = 2^k[\alpha']\) and \(q\deg \alpha' \geq q\deg \alpha + 2k\).

**Proposition 5.41.** If Statement 5.40 is true, then for any field \(F\) of \(\text{char } F \neq 2\), we have
\[
k_2(D; \mathbb{Z}) = k_h(D; F[h]),
\]
where \(D\) is any knot diagram and \(h\) is a formal variable of \(\text{deg } h = -2\).

**Proof.** Let \(\alpha'\) be a cycle of Statement 5.40. The natural map
\[
C_2(D; \mathbb{Z}) \xrightarrow{\otimes F} C_2(D; F)
\]
is \(q\)-degree non-decreasing, and since \(\alpha'\) is homologous to \(\alpha\) in \(C_2(D; F)\), we have
\[
2k_h(D; F[h]) = q\deg [\alpha] - q\deg \alpha \\
\geq q\deg \alpha' - q\deg \alpha \\
\geq 2k_2(D; \mathbb{Z}).
\]
The reverse inequality follows from the mirror formula.

Another approach is to consider \((R, c) = (\mathbb{Z}[h], h)\). Since \(\mathbb{Z}[h]\) is not a PID, we do not know if the following holds:

**Statement 5.42.** For \((R, c) = (\mathbb{Z}[h], h)\), \(k_c\) satisfies the mirror formula.

**Proposition 5.43.** If Statement 5.42 is true, then for any \((R, c)\) such that \(k_c\) also satisfies the mirror formula, we have
\[
k_h(D; \mathbb{Z}[h]) = k_c(D; R)
\]
for any knot diagram \(D\).

---

\(^4\)Computation are done by a program created by the author, with the planar diagram codes of Knot Atlas as the input data.

- **Input Data:** [http://katlas.org/wiki/The_Take_Home_Database](http://katlas.org/wiki/The_Take_Home_Database)
- **Program:** [https://github.com/taketo1024/SwiftyMath](https://github.com/taketo1024/SwiftyMath)
- **Results:** [https://git.io/fphro](https://git.io/fphro)
Proof. The ring homomorphism
\[
\mathbb{Z}[h] \to R, \quad h \mapsto c
\]
gives the inequality \( k_h(D; \mathbb{Z}[h]) \leq k_c(D; R) \), and the mirror formula gives the reverse inequality. \( \square \)

Corollary 5.44. If either Statement 5.40 or 5.42 is true, then for any field \( F \) of \( \text{char} \ F \neq 2 \), for any knot diagram \( D \)
\[
k_2(D; \mathbb{Z}) = k_h(D; F[h]),
\]
and for any knot \( K \)
\[
s_2(K; \mathbb{Z}) = s(K; F).
\]
In particular, all \( s(K; F) \) for fields of \( \text{char} \ F \neq 2 \) are equal.

Remark 5.45. In [22], the \( s \)-invariant for knots over a field \( F \) is defined by
\[
s(K; F) = \frac{q_{\min} + q_{\max}}{2}
\]
where
\[
q_{\min} = \min \{ \text{qdeg} \ x \mid x \in H_{fBN}(D; F) \setminus 0 \},
q_{\max} = \max \{ \text{qdeg} \ x \mid x \in H_{fBN}(D; F) \setminus 0 \}.
\]
This definition coincides with Definition 2.21 when \( \text{char} \ F \neq 2 \). Direct computations done by Seed showed that \( K = K14n19265 \) has \( s(K; \mathbb{Q}) = 0 \) but \( s(K; \mathbb{F}_2) = -2 \) (see [22, Remark 6.1]).

6 Further remarks and questions

• Implication from the Jones conjecture

\( k_c(D) \) can be related with a classical conjecture in knot theory, proposed by Jones in [13] and proved by Dynnikov, Prasolov in [10], and by LaFountain, Menasco in [19]. It is reformulated by Malešić, Traczyk in [25] as follows:

Theorem 6.1 (Jones conjecture). If \( D_0 \) is a diagram of an oriented link \( L \) which has the minimum number of Seifert circles \( r_0 \) among all diagrams of \( L \), then:

1. \( w(D_0) \) is uniquely determined.
2. For any diagram \( D \) of \( L \) with \( r_0 + m \) Seifert circles, \( w(D) \) is bounded as:
\[
w(D_0) - m \leq w(D) \leq w(D_0) + m.
\]

Combining this result with the invariance of \( \hat{s}_c \), we obtain
Proposition 6.2. With the assumption of Theorem 6.1:

1. $k_c(D_0)$ is uniquely determined.

2. For any diagram $D$ of $L$ with $r_0 + m$ Seifert circles, $k_c(D)$ is bounded as:
   \[ k_c(D_0) \leq k_c(D) \leq k_c(D_0) + m. \]

Thus we see that $k_c$ takes the minimum value whenever $r$ is minimum. The converse does not hold, since $k_c$ stays constant by adding a positive twist, while $r$ is incremented. We define

\[ k_c(L; R) = \min_D k_c(D; R) = k_c(D_0; R). \]

If $L$ is a positive link, then any positive diagram $D$ of $L$ satisfies $k_c(D) = 0$. Thus $k_c(L; R)$ gives the obstruction to $L$ possessing a positive diagram.

- **Transverse link invariants**

A transverse link is a link in $\mathbb{R}^3$ that is everywhere transverse to the standard contact structure $\ker(dz - ydx)$. From a work of Bennequin [8], given a braid representation $B$ of a transverse link $T$, the self-linking number of $T$ is given by

\[ \text{sl}(T) = -b(B) + e(B) \]

where $b(B)$ is the number of strings of $B$ and $e(B)$ is the exponent sum. Denoting by $D$ the closure of $B$, we have $b(B) = r(D)$ and $e(B) = w(D)$. Combined with Proposition 3.10 (after the adjustment of signs), we obtain

**Proposition 6.3.** $[\alpha(D)] \in H_c(D; R)$ is an invariant of $T$.

The case for Khovanov homology $(R, c) = (\mathbb{Z}, 0)$ gives Plamenevskaya’s invariant ([27]): $\psi(D) \in H_{Kh}(D; \mathbb{Z})$. We also see that a transverse stabilization annihilates $\psi(D)$, since it corresponds to RM1$_R$ and $c = 0$ is multiplied. The corresponding invariant for filtered Bar-Natan homology over $\mathbb{Z}$ is given in [21], and one for a general Frobenius algebra is given in [9].

Numerical transverse link invariants can be derived from the divisibility of $[\alpha(D)]$. We define

\[ \tilde{k}_c(D; R) = k_c([\alpha(D)]), \]

where $[\alpha(D)] \in H_c(D; R)$. Note that $\tilde{k}_c$ measures the divisibility in $H_c(D; R)$, whereas $k_c$ measures in the free part $H_c(D; R)_f$. We obviously have $\tilde{k}_c \leq k_c$. From Proposition 6.3, we have

**Proposition 6.4.** Both $k_c(D), \tilde{k}_c(D)$ are non-negative invariants of $T$.
Thus we denote them by \( k_c(T) \) and \( \tilde{k}_c(T) \) for bigraded Bar-Natan homology over a field \( F \): \((R,c) = (F[h],h)\) is given in [9], where it is called the \( c\)-invariant of a transverse link.

\( \hat{s}_c \) is in particular an invariant of a transverse link, so we obtain another description of the self-linking number:

\[
sl(T) = \hat{s}_c(T) - 2k_c(T) - 1.
\]

With Corollary 4.30 we obtain a bound

**Proposition 6.5.**

\[
\hat{s}_c(T) + 2e^-(T) - 1 \leq sl(T) \leq \hat{s}_c(T) - 1.
\]

where

\[
e^-(T) = \max_B \{ e^-(B) \} \leq 0
\]

is the maximum negative exponent sum among all braids representing \( T \).

With Theorem 5.32, the above \( \hat{s}_c \) can be replaced by \( s(-;F) \) for \( \text{char } F \neq 2 \).

- **Quasi-positive links / knots**

Proposition 4.10 can be extended to quasi-positive links. A link \( L \) is quasi-positive if it is the closure of a braid \( B \) of the form

\[
B = \prod_k \omega_k \sigma_i \omega_k^{-1}
\]

where each \( \omega_k \) is a word in the braid group.

**Lemma 6.6.** Let \( D \) be the closure of a braid \( B \) of the above form. Then \( k_c(D) = 0 \).

**Proof.** Let \( D_0 \) be the diagram obtained from \( D \) by removing all crossings corresponding to \( \sigma_i \). Then from Proposition 4.29, we have

\[
0 \leq k_c(D) \leq k_c(D').
\]

The corresponding braid is given by

\[
B' = \prod_k \omega_k \sigma_i \omega_k^{-1} = 1,
\]

so \( D' \) is a disjoint union of circles, and \( k_c(D') = 0 \).

Thus Proposition 4.16, 4.25 extends to quasi-positive links / knots as:

**Proposition 6.7.** If \( K \) is a quasi-positive knot, then

\[
\hat{s}_c(K) = 2g_+(K) = 2g(K).
\]
• Crossing change and the nugatory crossing conjecture

Let $K, K'$ be knots, where $K'$ is obtained from $K$ by a changing a positive crossing $x$ to a negative one. A crossing change can be realized by performing a 1-handle move near the crossing twice, thus corresponds to a cobordism $S$ of $\chi(S) = -2$. Let $D, D'$ be knot diagrams of $K, K'$ respectively. From $\Delta r = 0, \Delta w = -2$, we have $l = 0$ and $\phi[\alpha] = [\alpha'], \phi[\beta] = [\beta']$. On the other hand if $(R, c) = (\mathbb{Z}, 2)$ or $(F[h], h)$ with char $F \neq 2$, then from Corollary 5.27 we have $\Delta \hat{s} = -2$ or 0 (corresponding to $\Delta k = 0$ or 1 respectively), so

$$\phi[\zeta] = \begin{cases} [\zeta'] & \text{if } \Delta k = 0, \\ 2X[\zeta'] & \text{if } \Delta k = 1. \end{cases}$$

If $\Delta k = 0$ then $\phi$ is an isomorphism (with a q-degree shift of $-2$). If $\Delta k = 1$ then with $\phi X[\zeta] = 2(c/2)^2[\zeta']$ we see how the $c$-divisibility increases.

![Diagram](https://via.placeholder.com/150)

There is an open question in knot theory, the nugatory crossing conjecture [17, Problem 1.58], whether a crossing change yields an isotopic knot if and only if the crossing is nugatory. We have a diagram of implications:

$$K \approx K' \iff x \text{ is nugatory} \iff \Delta \hat{s} = 0 \iff \Delta k = 1$$

Thus to prove the conjecture, it suffices to prove that $k_c$ increases only when the crossing is nugatory.

• Torsions of $H_c(D; R)$

We have discarded the torsions for the definition of $k_c(D)$, or else it would be uneasy to handle (see Remark 4.3). If we have a better understanding of the torsions, we might obtain a more accurate invariant. Inductive arguments on the number of crossings might be available (see Question 5.9). Computational results showed that for any knot diagram $D$ of crossing number up to 9, the torsion components of $[\alpha(D)], [\beta(D)]$ (with respect to a computed basis of $H_2(D; \mathbb{Z}) \cong \mathbb{Z}^2 \oplus (2$-tors)) were all zero. For instance, $D = 9_{40}$ has
$H(D) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2^{12}$ and the components of $[\alpha], [\beta]$ are computed as:

\[
(4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \\
(-4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]

With the fact that the splitting is not canonical (it is easy to change the basis so that the torsion components are non-zero), we can expect there is a “natural” splitting of $H_c(D; R)$ into the free part and the torsion part (at least there seems to be something natural for the computer), and that the $\alpha$-classes live exclusively in the free part.

- **Varying $c$ in $H_c(D; R)$**

Many of the results were obtained under the assumption that $c = \sqrt{h^2 + 4t} \in R$ and $c$ is prime. Still it is interesting to explore how $H_{c,0}(D; Z)$ varies with $(h, t)$. In Table 2 we list some results of $H_{c,0}(D; Z)$ with $c$ taken in $0 \leq c \leq 6$.

Recall that $c = 0, 1, 2$ corresponds to Khovanov homology, filtered Bar-Natan homology, and Lee homology.

| $c$ | -3 | -2 | -1 | 0   | -2 | -1 | 0   | 1   | 2   |
|-----|----|----|----|-----|----|----|-----|-----|-----|
| 0   | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | 0 | $\mathbb{Z}^2$ | 0   | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 1   | 0   | 0   | 0   | $\mathbb{Z}^2$ | 1   | 0   | 0   | $\mathbb{Z}^2$ | 0   | 0   |
| 2   | 0   | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}^2$ | 2   | 0   | $\mathbb{Z}_2$ | $\mathbb{Z}^2$ | 0   | $\mathbb{Z}_2^2$ |
| 3   | 0   | $\mathbb{Z}_9$ | 0   | $\mathbb{Z}^2$ | 3   | 0   | $\mathbb{Z}_9$ | $\mathbb{Z}^2$ | 0   | $\mathbb{Z}_9$ |
| 4   | 0   | $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ | 0   | $\mathbb{Z}^2$ | 4   | 0   | $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ | $\mathbb{Z}^2$ | 0   | $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ |
| 5   | 0   | $\mathbb{Z}_{25}$ | 0   | $\mathbb{Z}^2$ | 5   | 0   | $\mathbb{Z}_{25}$ | $\mathbb{Z}^2$ | 0   | $\mathbb{Z}_{25}$ |
| 6   | 0   | $\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$ | 0   | $\mathbb{Z}^2$ | 6   | 0   | $\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$ | $\mathbb{Z}^2$ | 0   | $\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$ |

(a) $D = 3_1$  
(b) $D = 4_1$

Table 2: $H_c(D; Z)$ for $0 \leq c \leq 6$

- **Twisting**

In Remark 3.17 we have stated that $\rho$ does not commute with the $c$-exchanging map $f$ when there is a twisting. Within the cube of a link diagram $D$, take any path from the source state $s_+$ to the orientation preserving state $s_0$, and let $q(D)$ be the number of splitting edges within the path (this is independent of the choice of the path). $q(D)$ satisfies

\[
r(D, s_0) - r(D, s_+) = q(D) - (n^-(D) - q(D))
\]

so we have

\[
q(D) = \frac{r(D, s_0) - r(D, s_+) + n^-(D)}{2}.
\]
From the definition of $f$ in Lemma 3.13 and the $\theta$-twisting chain map in Lemma 3.1, $[\alpha]$ maps to $[\alpha']$ multiplied by a power of $\theta$ with exponent:

$$r(D) - q(D) = \frac{r(D, s_0) + r(D, s_+)}{2} - n_-(D).$$

Can we relate $q(D)$ and $k_c(D)$? Computational results showed that for knot diagrams of crossing number up to 10, most $D$ satisfies $q(D) = k_2(D)$ except for the following ten: 9_{42}, 10_{132}, 10_{136}, 10_{145}, 10_{152}, 10_{155}, 10_{156}, 10_{161}, 10_{165}. For $n = 11$ there are 84 exceptions. It is noticeable that all of these are non-alternating diagrams. In all of these cases $q(D) < k_2(D)$.

- $F = \mathbb{F}_2$

In Section 5 we have excluded the case $F = \mathbb{F}_2$ since the decomposition of Lemma 5.1 cannot be applied. In [33] a splitting of $H_h(\cdot; \mathbb{F}_2[h])$ is given. We might find some relations between the two types of splitting.

7 Proofs

7.1 Proof of Theorem 3.2

We show the invariance of $H_{h,t}(-; R)$ by following the original proof given in [14]. We assume that $h, t$ are elements of degree $-2, -4$ respectively, since the non-graded case follows by collapsing the degrees. To each Reidemeister move between two diagrams we associate a bidegree preserving quasi-isomorphism between the corresponding complexes. In the following we suppress the subscript $(h, t)$ and the base ring $R$. Recall that the bar $\overline{C}, \overline{H}$ indicates that the bidegree is unnormalized.

- $\text{RM1}_L$: Left twist

Let $D'$ be a diagram obtained by performing a left-twist on an arc of $D$. Let $a$ be the added crossing of $D'$. Fix any crossing-order of $D$, and for $D'$ append $a$ as the last one. Denote by $D'_0, D'_1$ the 0-, 1- resolved diagram of $D'$ at $a$ respectively.

![Figure 13: A left twist](image)
Recall that $\overline{C}(D')$ decomposes as $\overline{C}(D'_0) \oplus \overline{C}(D'_1)[1,1]$ and the differential $d$ can be written as

$$d = d_0 + d_0^1 - d_1$$

where $d_0, d_1$ are the differentials of $\overline{C}(D'_0), \overline{C}(D'_1)$ respectively and $d_0^1$ is the chain map given by the resolution change at $a$. $\overline{C}(D'_0), \overline{C}(D'_1)$ are obviously isomorphic to $\overline{C}(D) \otimes A$, $\overline{C}(D)$ respectively. We decompose $\overline{C}(D')$ into subcomplexes $X_1 \oplus X_2$, where $X_1$ is isomorphic to $\overline{C}(D)$ and $X_2$ is acyclic.

Define a bidegree $(-1,0)$ chain map by

$$\iota : \overline{C}(D'_1)[1,1] \longrightarrow \overline{C}(D'_0) \cong \overline{C}(D'_1) \otimes A$$

$$x \mapsto x \otimes 1.$$

$\iota$ satisfies:

$$d_0^1 \circ \iota = \text{id}_{\overline{C}(D'_1)}$$

since all edge maps consisting $d_0^1$ are merge maps. Also define a bidegree preserving chain map:

$$\gamma = \iota \circ d_0^1 : \overline{C}(D'_0) \rightarrow \overline{C}(D'_0)$$

and two subsets of $\overline{C}(D')$:

$$X_1 = \{x - \gamma(x) \mid x \in \overline{C}(D'_0)\}$$

$$X_2 = \{\iota(y) + z \mid y, z \in \overline{C}(D'_1)[1,1]\}.$$

Claim 1. $X_1, X_2$ are subcomplexes of $\overline{C}(D')$.

Proof. $d(x - \gamma(x)) = (d_0 + d_0^1)(x - \gamma(x))$

$$= d_0x + d_0^1x - d_0d_0^1x - d_0^1d_0^1x$$

$$= d_0x - \gamma(d_0x)$$

$$d(\iota y + z) = (d_0 + d_0^1)\iota y - d_1z$$

$$= \iota d_1 y + (y - d_1 z)$$

$\square$
Claim 2. $\overline{C}(D') = X_1 \oplus X_2$.

Proof. It suffices to prove that this decomposition holds on the level of underlying modules. Since $X_2 = \text{Im} \iota \oplus \overline{C}(D_1'[1,1])$, it suffices to prove $\overline{C}(D_0') = X_1 \oplus \text{Im} \iota$. We may assume $D = \bigcirc$.

By definition $X_1 = (id - \gamma)\overline{C}(D_0')$. Recall that $\gamma = \iota m$, so $X_1$ is generated by:

\[
\begin{align*}
X \otimes X & \mapsto X \otimes X - hX \otimes 1 - t1 \otimes 1 \\
X \otimes 1 & \mapsto 0 \\
1 \otimes X & \mapsto 1 \otimes X - X \otimes 1 \\
1 \otimes 1 & \mapsto 0
\end{align*}
\]

$\text{Im} \iota$ is generated by:

\[
\begin{align*}
X & \mapsto X \otimes 1 \\
1 & \mapsto 1 \otimes 1
\end{align*}
\]

Thus:

\[
\begin{align*}
X_1 \oplus \text{Im} \iota = (X \otimes X - t1 \otimes 1 - hX \otimes 1, 1 \otimes X - X \otimes 1, X \otimes 1, 1 \otimes 1) \\
&= (X \otimes X, X \otimes 1, 1 \otimes X, 1 \otimes 1) \\
&= \overline{C}(D_0')
\end{align*}
\]

Claim 3. $X_2$ is acyclic.

Proof. There is an isomorphism (as modules):

\[
X_2 = \text{Im} \iota \oplus \overline{C}(D_1') \cong \overline{C}(D_1') \oplus \overline{C}(D_1'),
\]

since $\iota$ is injective. $d$ maps $\iota y + z$ to $\iota(d_1 y) + (y - d_1 z)$, which corresponds to $(d_1 / d - d_1)$ on the right. Thus $X_2$ is isomorphic to the cone of the identity map, hence acyclic.

Claim 4. $\overline{C}(D) \cong X_1[0,1]$.

Proof. Define a map $\rho : \overline{C}(D) \to X_1$ of bidegree $(0,-1)$ by mapping the factor corresponding to the circle appearing in $\overline{C}$ as:

\[
\begin{align*}
X & \mapsto X \otimes X - hX \otimes 1 - t1 \otimes 1, \\
1 & \mapsto 1 \otimes X - X \otimes 1
\end{align*}
\]

$\rho$ is obviously a module isomorphism, and by direct calculation we can see that it commutes with $d$. 

\[\square\]
Claim 5. \( \rho_* : H(D) \xrightarrow{\sim} H(D') \), bidegree preserving.

Proof. With \( n^+(D') = n^+(D) + 1 \) and \( n^-(D) = n^-(D') \),

\[
C(D) = C(D)[-n^-(D), n^+(D) - 2n^-(D)] \\
\cong X_1[-n^-(D), n^+(D) - 2n^-(D) + 1] \\
\cong C(D')[-n^-(D'), n^+(D') - 2n^-(D')] \\
= C(D')
\]

\[\square\]

- **RM1\(_R\)**: Right twist

A right twist is accomplished by a tangency move (RM2) followed by a left untwist (RM1\(_L^{-1}\)), thus the invariance follows from those of the two moves.

- **RM2**: Tangency move

Let \( D, D' \) be two diagrams as depicted as below.

\[
\begin{array}{c}
\text{D} & \xrightarrow{\text{RM1}_R} & \text{D}' \\
\text{D}' & \xleftarrow{\text{RM1}_L^{-1}} & \text{D} \\
\end{array}
\]

Figure 14: A right twist

Figure 15: A tangency move

Fix any crossing-order of \( D \), and for \( D' \) append \( a, b \) in this order. \( C(D') \) can be decomposed as in the following figure. \( D'_{00}, D'_{01}, D'_{10}, D'_{11} \) are four diagrams obtained from \( D' \) by resolving \( a, b \) accordingly. \( d_{ij} \) are the differentials
on $\overline{C}(D'_ij)$, and $d^{kl}_{ij} : \overline{C}(D'_ij) \to \overline{C}(D'_{kl})$ are the chain maps that corresponds to the resolution changes. Note that $D'_{01}$ is isotopic to $D$.

\[
\begin{array}{c}
\includegraphics[width=0.8\textwidth]{diagram}
\end{array}
\]

Define chain maps as in the previous proof:

\[
\begin{align*}
\iota : \overline{C}(D'_{11})[2,2] &\to \overline{C}(D'_{10})[1,1] \\
\gamma = \iota \circ d_{01}^{11} : \overline{C}(D'_{01})[1,1] &\to \overline{C}(D'_{10})[1,1]
\end{align*}
\]

Also define three subsets of $\overline{C}(D')$:

\[
\begin{align*}
X_1 &= \{x + \gamma(x) \mid x \in \overline{C}(D'_{01})[1,1]\} \\
X_2 &= \{x + dy \mid x, y \in \overline{C}(D'_{00})\} \\
X_3 &= \{x + \iota y \mid x, y \in \overline{C}(D'_{11})[2,2]\}.
\end{align*}
\]

**Claim 1.** $\overline{C}(D') = X_1 \oplus X_2 \oplus X_3$.

**Claim 2.** $X_2, X_3$ are acyclic.

**Proof.** We only check that $X_1$ is a subcomplex of $\overline{C}(D')$.

\[
\begin{align*}
d(x + \gamma(x)) &= (-d_{01} - d_{01}^{11})x + (-d_{10} + d_{10}^{11})\gamma(x) \\
&= -d_{01}x - d_{01}^{11}x - \gamma d_{01}x + d_{01}^{11}x \\
&= -d_{01}x + \gamma(-d_{01}x)
\end{align*}
\] (1)

**Claim 3.** $\overline{C}(D)[1,1] \cong X_1$.

**Proof.** With the identification $\overline{C}(D) \cong \overline{C}(D'_{01})$, we define a bidegree preserving isomorphism:

\[
\begin{align*}
\rho : \overline{C}(D)[1,1] &\to X_1 \\
x &\mapsto (-1)^{\text{bideg}(x)}(x + \gamma(x))
\end{align*}
\]
where $\text{hdeg}(x)$ denotes the homological degree of $x$. This is a module isomorphism since $X_1 = (\text{id} + \gamma)\overline{C}(D) \cong \overline{C}(D)$, and that it commutes with $d$ follows from (1).

**Claim 4.** $\rho_* : H(D) \xrightarrow{\cong} H(D')$, bidegree preserving.

**Proof.** With $n^+(D') = n^+(D) + 1$ and $n^-(D') = n^-(D) + 1$,

\[
C(D) = \overline{C}(D)[-n^-(D), n^+(D) - 2n^-(D)] \\
\cong X_1[-n^-(D) - 1, n^+(D) - 2n^-(D) - 1] \\
\simeq \overline{C}(D')[-n^-(D'), n^+(D') - 2n^-(D')] \\
= C(D')
\]

\[\square\]

- **RM3 : Triple point move**

Let $D, D'$ be two diagrams as depicted below:

![Figure 16: A triple point move](image)

We decompose both $\overline{C}(D), \overline{C}(D')$ as in the following figure. The notations should be obvious from the previous case.
Define maps

$\{ \iota : \overline{C}(D_{110})[2, 2] \rightarrow \overline{C}(D_{010})[1, 1] \}
\{ \gamma : \overline{C}(D_{100})[1, 1] \rightarrow \overline{C}(D_{010})[1, 1] \}$

$\{ \iota' : \overline{C}(D'_{110})[2, 2] \rightarrow \overline{C}(D'_{010})[1, 1] \}
\{ \gamma' : \overline{C}(D'_{010})[1, 1] \rightarrow \overline{C}(D'_{100})[1, 1] \}$

and subsets of $\overline{C}(D), \overline{C}(D')$ by

$\{ X_1 = \{ x + \gamma(x) + y \mid x \in \overline{C}(D_{100})[1, 1], y \in \overline{C}(D_{110}) \} \}
\{ X_2 = \{ x + dy \mid x, y \in \overline{C}(D_{000}) \} \}
\{ X_3 = \{ x + dy \mid x, y \in \overline{C}(D_{110})[2, 2] \} \}$

$\{ X'_1 = \{ x + \gamma'(x) + y \mid x \in \overline{C}(D'_{010})[1, 1], y \in \overline{C}(D'_{110}) \} \}
\{ X'_2 = \{ x + d'y \mid x, y \in \overline{C}(D'_{000}) \} \}
\{ X'_3 = \{ x + d'y \mid x, y \in \overline{C}(D'_{110})[2, 2] \} \}$
Claim 1. $\overline{C}(D) = X_1 \oplus X_2 \oplus X_3$, and $\overline{C}(D') = X'_1 \oplus X'_2 \oplus X'_3$.

Claim 2. $X_2, X_3, X'_2, X'_3$ are acyclic.

Proof. We only check that $X_1, X'_1$ are subcomplexes of $\overline{C}(D), \overline{C}(D')$ respectively.

$$d(x + \gamma(x) + y) = (-d_{100} + d_{101}^{10} + d_{010}^{10})x + (-d_{010} - d_{100}^{10} + d_{010}^{01})\gamma(x) + dy$$

$$= -d_{100}x + d_{101}^{10}x + d_{100}^{01}x - \gamma d_{100}x - d_{010}^{10}x + d_{010}^{01}x + dy$$

$$= -d_{100}x + \gamma (-d_{100}x) + (d_{101}^{01}x + d_{010}^{01}x + dy)$$

$$d'(x + \gamma'(x) + y) = (-d_{010} - d_{101}^{01} + d_{010}^{11})x + (-d_{100} + d_{101}^{10} + d_{101}^{01})\gamma'(x) + d'y$$

$$= -d_{010}x - d_{101}^{01}x + d_{101}^{01}x - \gamma' d_{010}x + d_{100}^{10}x + d_{101}^{01}x + dy$$

$$= -d_{010}x + \gamma' (-d_{010}x) + (d_{101}^{10}x + d_{010}^{11}x + d'y)$$

\[\Box\]

Claim 3. $X_1 \cong X'_1$.

Proof. There are obvious isomorphisms $\overline{C}(D_{100}) \cong \overline{C}(D'_{010})$ and $\overline{C}(D_{*1}) \cong \overline{C}(D'_{*1})$. With these identifications, the desired chain isomorphism is given by:

$$\rho : X_1 \longrightarrow X'_1$$

$$x + \gamma(x) + y \longmapsto x + \gamma'(x) + y$$

\[\Box\]

Claim 4. $\rho_* : H(D) \cong H(D')$, bidegree preserving.

Proof. With $n^+(D') = n^+(D), n^-(D') = n^-(D)$,

$$\begin{align*}
C(D) &\cong X_1[-n^-(D), n^+(D) - 2n^-(D)] \\
&\cong X'_1[-n^-(D'), n^+(D') - 2n^-(D')] \\
&\cong C(D')
\end{align*}$$

\[\Box\]

Thus the proof of Theorem 3.2 is complete.

7.2 Proof of Proposition 3.10

Let $D, D'$ be diagrams related by a Reidemeister move. Denote by $\alpha, \alpha'$ the $\alpha$-cycles of $D, D'$, and by $s, s'$ the orientation preserving states of $D, D'$ respectively. Let $\Delta r = r(D') - r(D)$. For each Reidemeister move we prove:

$$\alpha' \sim \varepsilon \varepsilon^j \rho(\alpha),$$

$$\beta' \sim \varepsilon' \varepsilon^j \rho(\beta)$$

with $j \in \{\pm 1\}$ as in Table 1, and $\varepsilon \varepsilon' = (-1)^j$. 

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We restate the definition of the quasi-isomorphism $\rho$.

$$\rho: C(D) \to X_1$$

$$X \mapsto X \otimes X - hX \otimes 1 - t1 \otimes 1$$

$$1 \mapsto 1 \otimes X - X \otimes 1$$

We have $s' = (s0)$. The Seifert circle of $D$ containing the arc in $\bigcirc$ is either colored $a$ or $b$. In the first case $\alpha = \cdots \otimes a$, and $\alpha' = \cdots \otimes a \otimes b$.

Recall that $a = X - u$, $b = X - v$, and with $h = u + v$, $t = -uv$, we see that $\rho$ maps:

$$\alpha \mapsto a \otimes b$$

$$\beta \mapsto b \otimes a$$

Thus $\alpha' = \rho(\alpha)$, $\beta' = \rho(\beta)$. The above argument also proves the other case, where the circle is colored $b$. In the following, we only state one of the two possible colorings of the Seifert circle of $D$.

$\rho$ is given by:

$$\rho: C(D) \to X_1$$

$$x \mapsto x + \gamma(x)$$

We divide cases by the direction of the two strands in $\bigcirc$ of $D$.

**Case 1** (\(\uparrow\uparrow\) or \(\downarrow\downarrow\)). In this case $s' = (s01)$ and $\Delta r = 0$. Since $C(D) = C(D_{01}')$ and $\gamma(\alpha) = \gamma(\beta) = 0$, we have $\alpha' = \rho(\alpha), \beta' = \rho(\beta)$.

**Case 2** (\(\uparrow\downarrow\) or \(\downarrow\uparrow\)). In this case $s' = (s10)$. We must divide into subcases to determine $\Delta r$, whether the two arcs belong to the same $s$-circle or to different $s$-circles.

**Case 2.1.** In this case $\Delta r = 2$. Suppose $\alpha, \alpha'$ are colored as follows:

$$\alpha = \begin{array}{c}
\cdots
\end{array}$$

$$\alpha' = \begin{array}{c}
\cdots
\end{array}$$

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Take an element $x$ in $C(D_{00}')$ as depicted below:

From $a = b + c$, we have
\[ dx = (\alpha' + c\gamma(\alpha)) + c\alpha \Rightarrow \alpha' \sim -c\rho(\alpha). \]

Let $x$ be the chain obtained from $x$ by flipping $a$'s and $b$'s. Similarly we have:
\[ d\bar{x} = (\beta' - c\gamma(\beta)) - c\beta \Rightarrow \beta' \sim c\rho(\beta) \]

**Case 2.2.** In this case $\Delta r = 0$. From the coloring rule two Seifert circles of $D$ appearing in $\circ$ are colored the same. Similarly as in the previous case, we compute:

\[ dx = (\alpha' + \gamma(\alpha)) + \alpha \Rightarrow \alpha' \sim -\rho(\alpha) \]
\[ d\bar{x} = (\beta' + \gamma(\beta)) + \beta \Rightarrow \alpha' \sim -\rho(\beta) \]
• RM3

\[ \rho : X_1 \rightarrow X'_1 \]
\[ x + h(x) + y \rightarrow x + h'(x) + y \]

This move is point-symmetric in \( \odot \), so regarding local orientations on the strands, we may assume that the top-most strand of \( D \) points upward. There are four possible cases for the direction of the three strands.

Case 1 (↑↑↑). \( s = s' = (*111) \).

Case 2 (↑↑↓). \( s = s' = (*001) \).

Case 3 (↑↓↓). \( s = (*100), s' = (*010) \).

For the above three cases we see that \( \Delta r = 0 \), and \( \alpha' = \rho(\alpha), \beta' = \rho(\beta) \) follows from the definition of \( \rho \).

Case 4 (↑↓↑). \( s = (*010), s' = (*100) \). The possible connections of the arcs are the following five:

Case 4.1. In this case \( \Delta r = -2 \). Suppose \( \alpha, \alpha' \) are colored as follows:

Define elements \( x, y, z \in C(D) \) as follows:
Then we have:
\[ dx = cy + (\alpha + c\gamma(y)) + cz \Rightarrow \alpha \sim -cy - c\gamma(y) - cz \]
\[ dx = -c\bar{y} + (\beta - c\gamma(\bar{y})) - cz \Rightarrow \beta \sim c\bar{y} + c\gamma(y) + cz \]

Similarly in \( C(D') \), define chains \( x', y', z' \) as:

Then we have:
\[ dx' = (\alpha' + \gamma(y')) + y' + z' \Rightarrow \alpha' \sim -y' - \gamma(y') - z' \]
\[ dx' = (\beta' + \gamma(\bar{y}')) + \bar{y}' + z' \Rightarrow \beta' \sim -\bar{y}' - \gamma(\bar{y}') - z' \]

Thus from the definition of \( \rho \), we have:
\[ c\alpha' \sim \rho(\alpha) \]
\[ -c\beta' \sim \rho(\beta) \]

The diagram of \( \alpha, \alpha' \) shows \( \Delta r = -2 \). The remaining cases proceeds verbatim, so we just list the diagrams and the equations.
Case 4.2.

\[ dx = y + (α + γ(y)) + cz \Rightarrow α \sim -y - γ(y) - cz \]
\[ dx = y + (β + γ(y)) - cz \Rightarrow β \sim -y - γ(y) + cz \]

\[ dx' = (α' + γ(y')) + y' + cz' \Rightarrow α' \sim -y' - γ(y') - cz' \]
\[ dx' = (β' + γ(y')) + y' - cz' \Rightarrow β' \sim -y' - γ(y') + cz' \]

\[ ∴ ∆r = 0, \quad α' \sim ρ(α), \quad β' \sim ρ(β). \]
Case 4.3.

\[
\begin{align*}
    dx &= y + (\alpha + \gamma(y)) + z \Rightarrow \alpha \sim -y - \gamma(y) - z \\
    dx' &= (\alpha' + c\gamma(y')) + cy' + cz' \Rightarrow \alpha' \sim -cy' - c\gamma(y') - cz'
\end{align*}
\]

\[
\begin{align*}
    dx &= y + (\alpha + \gamma(y)) + z \Rightarrow \alpha \sim -y - \gamma(y) - z \\
    dx' &= (\beta' - c\gamma(y')) - cy' - cz' \Rightarrow \beta' \sim cy' + c\gamma(y') + cz'
\end{align*}
\]

\[\therefore \Delta r = 2, \quad \alpha' \sim c\rho(\alpha), \quad \beta' \sim -c\rho(\beta).\]
Case 4.4.

\[ dx = cy + (\alpha + c\gamma(y)) + cz \Rightarrow \alpha \sim -cy - c\gamma(y) - cz \]

\[ dx' = (\alpha' + c\gamma(y')) + cy' + cz' \Rightarrow \alpha' \sim -cy' - c\gamma(y') - cz' \]

\[ dx = -c\bar{y} + (\beta - c\gamma(y)) - cz \Rightarrow \beta \sim c\bar{y} + c\gamma(y) + cz \]

\[ dx' = (\beta' - c\gamma(y')) - c\bar{y}' - cz' \Rightarrow \beta' \sim c\bar{y}' + c\gamma(y') + cz' \]

\[ \therefore \Delta r = 0, \quad \alpha' \sim \rho(\alpha), \quad \beta' \sim \rho(\beta). \]
Case 4.5.

\[ dx = cy + (\alpha + c\gamma(y)) + z \Rightarrow \alpha \sim -cy - c\gamma(y) - z \]
\[ d\mathfrak{x} = -cg + (\beta - c\gamma(y)) + z \Rightarrow \beta \sim cg + c\gamma(y) - z \]

\[ dx' = (\alpha' + c\gamma(y')) + cy' + z' \Rightarrow \alpha' \sim -cy' - c\gamma(y') - z' \]
\[ d\mathfrak{x}' = (\beta' - c\gamma(y')) - cg' + z' \Rightarrow \beta' \sim cg' + c\gamma(y') - z' \]

\[ \therefore \Delta r = 0, \quad \alpha' \sim \rho(\alpha), \quad \beta' \sim \rho(\beta). \]

\[ \square \]
7.3 Lemmas on homological algebra

**Lemma 7.1.** Let $C$ be a chain complex over an integral domain $R$. Denote by $H(C)_f$ the quotient of $H(C)$ by the torsion submodule. The canonical pairing between $H(C)$ and $H(C^*)$ induces:

$$\langle -, - \rangle : H(C)_f \otimes H(C^*)_f \to R.$$ 

Moreover if $R$ a PID, $C$ is free over $R$ and $H(C)$ is finitely generated, then the pairing is unimodular.

**Proof.** The first statement is obvious, since torsions are annihilated by the pairing. Assume the latter condition. We claim $H(C^*)_f \cong (H(C)_f)^*$. From the universal coefficient theorem, there is a canonical surjection $h : H(C^*) \to \text{Hom}_R(H(C), R) \cong \text{Hom}_R(H(C)_f, R) = (H(C)_f)^*$ with $\ker h \cong \text{Ext}_R(H(C), R)$. Torsions are annihilated by $h$ so $H(C^*_f) \subset \ker h$. From the structure theorem for finitely generated modules over a PID, we may write $H(C) \cong R^r \oplus_1 R/(a_i)$, and $\ker h \cong \text{Ext}_R(H(C), R) \cong \bigoplus_i R/(a_i)$ so all elements of $\ker h$ are torsional. Thus $\ker h = H(C^*_f) \subset \ker h$ and $h$ induces an isomorphism $h : H(C^*_f) \to (H(C)_f)^*$. Take a basis $\{[z_i]\}$ of $H(C)_f \cong R^r$, and its dual basis $\{f_j\}$ of $(H(C)_f)^*$. The pullback $\{h^{-1}(f_j)\}$ forms a basis of $H(C^*_f)$, and we have

$$\langle [z_i], h^{-1}(f_j) \rangle = h(h^{-1}(f_j))[z_i] = f_j[z_i] = \delta_{ij},$$

hence the pairing is unimodular. \[\square\]

**Lemma 7.2.** Let $C, C'$ be chain complexes over an integral domain $R$. The canonical map:

$$H(C) \otimes H(C') \to H(C \otimes C')$$

induces:

$$H(C)_f \otimes H(C')_f \to H(C \otimes C')_f.$$ 

Moreover if $R$ is a PID, and both $C, C'$ are free and finitely generated over $R$, then the induced homomorphism is an isomorphism.

**Proof.** The first statement is obvious. Assume the latter condition. Let $\{[z_i]\}_{i=1}^r$, $\{[w_j]\}_{j=1}^s$ be bases of $H(C)_f, H(C')_f$ respectively. We claim that $\{[z_i \otimes w_j]\}$ is a basis of $H(C \otimes C')_f$. Let $F$ be the fraction field of $R$, then from $H(C ; F) \otimes H(C' ; F) \cong H(C \otimes C' ; F)$ we have $\text{rank}_R H(C \otimes C')_f = rr'$. Let $\{[f_i]\}, \{[g_j]\}$ be the dual bases of $\{[z_i]\}, \{[w_j]\}$ with respect to the pairing given in the previous lemma. With $(C \otimes C')^* \cong C^* \otimes C'^*$, there is a unimodular pairing

$$\langle -, - \rangle : H(C \otimes C')_f \otimes H(C^* \otimes C'^*)_f \to R.$$ 

and we have

$$\langle [z_i \otimes w_j], [f_k \otimes g_l]\rangle = \langle z_i \otimes w_j, f_k \otimes g_l\rangle = \langle z_i, f_k \rangle \langle w_j, g_l\rangle = \delta_{ik} \delta_{jl},$$

so

$$\det(\langle [z_i \otimes w_j], [f_k \otimes g_l]\rangle) = \det(I \otimes I) = 1.$$ 

Thus in particular $\{[z_i \otimes w_j]\}$ must be a basis of $H(C \otimes C')_f$ otherwise the above determinant would have a non-unital factor. \[\square\]
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