Fuzzy Regularly Closed Sets in Michálek’s Fuzzy Topological Space

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Abstract. In this paper the Fuzzy Regularly closed subsets have been discussed with the help of the Fuzzy Topological Space defined by Michálek. Also we have discussed some characteristics of Fuzzy Regularly Closed sets.

1. Introduction

The commonly used concept of set can be expressed in a more general form by the Fuzzy sets introduced by Zadeh [5] in 1965. To introduce fuzzy sets Zadeh used the function form of an ordinary set that is, any subset A of X is characterized by its characteristic function \( \chi_A : X \to \{0, 1\} \). Using the fuzzy sets introduced by Zadeh, C L Chang [2] defined Fuzzy Topological space in 1968 for the first time. According to Chang, any family of fuzzy sets in X which include empty set and the complete set and which are closed under arbitrary union and finite intersection forms a fuzzy topology. But this definition is just a natural translation of the ordinary definition of topological space to fuzzy sets. In 1975 J Michálek [4] in his paper defined fuzzy topological space in a quite different manner. To define Fuzzy Topological Space Michálek first defined ordinary topological space in terms of functions and then defined Fuzzy Topological Space as an extension of these functions.

From the introduction of Chang’s Fuzzy Topology in 1968 this area caught the attention of researchers and many papers appeared there after in this area. But not much work has been carried out in the area of Fuzzy Topology by Michálek. In 2001, Francisco Galligo Lupranetz in his paper [3] studied few characteristics of the Fuzzy Topological Space as given by Michálek. In 1981 K K Azad [1] came up with an idea of Fuzzy Regularly Closed subsets in Changs Fuzzy Topological Space. This work presents the concept of Fuzzy Regularly Closed Sets in Michálek sense and study some of its properties.

In the next section of this paper we give the necessary preliminary results required for the development of the main concepts.

2. Preliminaries

Definition 1. (Michálek [4]) Consider the set X and \( P(X) \) its power set, function \( u: P(X) \rightarrow P(X) \) satisfies the conditions below

1. If \( A \subseteq X \), contains atmost one element then \( uA = A \).
2. If \( A_1 \subseteq X, A_2 \subseteq X \), then \( u(A_1 \cup A_2) = uA_1 \cup uA_2 \).

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Then the pair \((X, u)\) is a topological space.

**Definition 2.** (Michálek [4]) Consider the set \(X\) and its power set \(P(X)\). Let \(F\) be the system of every fuzzy set on \(X\). Then the pair \((X, u)\) where \(u: P(X) \to F\) is a fuzzy topological space if and only if it satisfies the conditions below
1. If \(A \subseteq X\), then \(uA(x) = 1\) for all \(x \in A\).
2. If \(A \subseteq X\) contains at most one element, then \(uA(x) = \chi_A(x)\) where \(\chi_A\) is the characteristic function of the set \(A\).
3. If \(A_1 \subseteq X, A_2 \subseteq X\) then \(u(A_1 \cup A_2) = \max\{uA_1(x), uA_2(x)\}\).

**Definition 3.** (Michálek [4]) Let \((X, u)\) be the fuzzy topological space. Then for any subset \(A\) of \(X\) the fuzzy set \(\mu_A\) defined by \(\mu_A(x) = 1 - (uA^c)(x)\) is called the Fuzzy Interior of the set \(A\).

**Definition 4.** (Michálek [4]) Let \((X, u)\) be the fuzzy topological space. Then for any subset \(A\) of \(X\) the fuzzy set \(\mu_{\delta A}\) defined by \(\mu_{\delta A}(x) = \min\{uA(x), uA^c(x)\}\) is the Fuzzy Boundary of the subset \(A\) of \(X\).

**Definition 5.** (Michálek [4]) Let \((X, u)\) be the fuzzy topological space. Then \(A \subseteq X\), is said to be fuzzy closed if \(\chi_A(x) \geq \min\{uA(x), uA^c(x)\}\) for all \(x \in X\).

**Definition 6.** (Michálek [4]) Let \((X, u)\) be the fuzzy topological space. Then a subset \(A\) of \(X\) is called fuzzy open when for every \(x \in X\), \(\min\{\chi_A(x), \mu_{\delta A}(x)\} = 0\) where \(\mu_{\delta A}\) is the fuzzy boundary of the set \(A\). Equivalently, a subset \(A\) of \(X\) is fuzzy closed if \(uA = \chi_A\) and fuzzy open if \(\mu_A \ni \chi_A\). Also a subset \(A\) of \(X\) is fuzzy open if \(A^c\) is fuzzy closed.

### 3. Fuzzy Regularly Closed Sets and Its Properties

While in Chang's fuzzy topology, the interior of a fuzzy set is defined, Michálek discussed the fuzzy interior of a subset of \(X\). Here we introduce the concepts of fuzzy closure of the subset of \(X\) and fuzzy regularly closed subset of \(X\) in Michálek sense.

**Definition 7.** Let \((X, u)\) be the fuzzy topological space as defined by. Then \(\mu \in I^X\) is fuzzy closed if there exist a fuzzy closed subset \(A\) of \(X\) such that \(\mu = u_A\). \(\mu \in I^X\) is said to be fuzzy open if there is a fuzzy open subset \(B\) of \(X\) such that \(\mu = u_B\).

**Definition 8.** Let \(\mu \in I^X\) and \((X, u)\) be the fuzzy topological space as defined by Michálek. Then the closure of fuzzy set \(\mu, \bar{\mu} = \inf\{\gamma \in I^X : \gamma \geq \mu, \gamma\text{ is fuzzy closed}\}\). This definition is equivalent to \(\bar{\mu}(x) = \min\{u_A(x) : u_A(x) \geq \mu(x)\}\) for all \(x\) in \(X\) and \(A\) is fuzzy closed in \(X\). The interior of a fuzzy set \(\mu^c(x) = \sup\{\delta \in I^X : \delta \leq \mu, \delta\text{ is fuzzy open}\}\). This definition is equivalent to \(\mu^c(x) = \max\{u_A(x) : u_A(x) \leq \mu(x)\}\) for all \(x\) in \(X\) and \(A\) is fuzzy open in \(X\).

**Definition 9.** Let \((X, u)\) be the fuzzy topological space as defined by Michálek and \(A\) be a fuzzy closed subset of \(X\). Then the fuzzy set \(\mu_A\) defined by \(\mu_A(x) = \min\{u_A(x) : u_A(x) \geq \chi_A(x)\}\) is termed as the fuzzy closure of the set \(A\).

**Note 1.** To note that \(A\) is fuzzy closed in Michálek sense.
Definition 10. Let \((X, u)\) be the fuzzy topological space as defined by Michálek and let \(A\) be a fuzzy closed subset of \(X\). Then \(A\) is said to be fuzzy regularly closed if the closure of the fuzzy interior of \(A = \chi_A\), i.e., \(\text{cl}(f\text{ int} A) = \chi_A\).

Definition 11. Let \((X, u)\) be the fuzzy topological space as defined by Michálek and let \(A\) be a fuzzy open subset of \(X\). Then \(A\) is said to be fuzzy regularly open if the interior of the fuzzy closure of \(A = \chi_A\), i.e., \(\text{int}(f\text{ cl} A) = \chi_A\).

Note 2. We use the notation \(\text{FRC}_M(X)\) and \(\text{FRO}_M(X)\) to represent Fuzzy Regularly Closed and Fuzzy Regularly Open subset of \(X\).

Definition 12. For any two subsets \(A, B\) of \(\text{FRC}_M(X)\), define \(A \subseteq B\) if and only if \(u_A(x) \leq u_B(x)\) for every \(x\) in \(X\) then \(\subseteq\) is a partial order on \(\text{FRC}_M(X)\) and hence \(\text{FRC}_M(X)\) is a poset.

If \(A\) and \(B\) are in \(\text{FRC}_M(X)\) then there union \(A \cup B\) also in \(\text{FRC}_M(X)\), but \(A \cap B\) need not belongs to \(\text{FRC}_M(X)\). Therefore we define the union and intersection in \(\text{FRC}_M(X)\), as \(A \cup B\) and \(\text{cl}(f - \text{int}(A \cap B))\) respectively. Then \((\text{FRC}_M(X), \subseteq)\) is a lattice.

Note 3. We use the notations \(\lor\) and \(\land\) to represent lattice join and meet and \(\cup\) and \(\cap\) to represent the union and intersection of elements of \(\text{FRC}_M(X)\).

Theorem 1. For a Michálek Fuzzy Topological Space \(X\), \((\text{FRC}_M(X), \subseteq)\) is a complemented lattice with complement defined as for \(A \in \text{FRC}_M(X), A^c = f\text{ cl}(X - A) = X - (f\text{ int}(A))\).

Proof. For \(A \in \text{FRC}_M(X),\) take \(A^c = f\text{ cl}(X - A) = X - (f\text{ int}(A))\).

1. \((A^c)^c = f\text{ cl}(X - A^c) = f\text{ cl}\left(X - f\text{ cl}(X - A)\right) = f\text{ cl}\left(X - (X - f\text{ int}A)\right) = f\text{ cl}(f\text{ int}A) = A.

2. \((A \land B)^c = f\text{ cl}(X - (A \land B)) = f\text{ cl}\left(X - f\text{ cl}(f\text{ int}(A \land B))\right) = f\text{ cl}\left(f\text{ int}(X - f\text{ int}A \cup (X - f\text{ int}A))\right) = f\text{ cl}(f\text{ int}(A^c \cup B^c)) = (A^c \cup B^c).

3. \((A \lor B)^c = f\text{ cl}(X - (A \lor B)) = f\text{ cl}\left(X - f\text{ cl}(f\text{ int}(A \lor B))\right) = f\text{ cl}(X - f\text{ cl}(f\text{ int}(A \lor B))) = f\text{ cl}(X - f\text{ cl}(f\text{ int}A) \cup f\text{ cl}(f\text{ int}B)) = f\text{ cl}(f\text{ int}(X - f\text{ int}A \cap (X - f\text{ int}B))) = f\text{ cl}(X - f\text{ int}A \land (1 - f\text{ int}B)) = f\text{ cl}(f\text{ int}(A^c \cap B^c)) = A^c \land B^c.

Theorem 2. \(\text{FRC}_M(X)\) is a complete lattice.

Proof. Let \(A_\alpha\) be members of \(\text{FRC}_M(X)\). Then we show that,

\[(1) \land_\alpha A_\alpha = f - \text{cl}(f - \text{int}(\land_\alpha A_\alpha)) \quad (2) \lor_\alpha A_\alpha = f - \text{cl}(f - \text{int}(\lor_\alpha A_\alpha))\]
1. Put $\beta = f - \text{cl}(f - \text{int}(\cap \alpha A_\alpha))$. Then $\beta \in \text{FRC}_M(X)$

$\cap \alpha A_\alpha \leq A_\alpha$ for every $\alpha$.

Therefore, $f - \text{cl}(f - \text{int}(\cap \alpha A_\alpha)) \leq A_\alpha$. i.e., $\beta \leq A_\alpha$.

Therefore $\beta$ is a lower bound of $A_\alpha$.

Let $\gamma \in \text{FRC}_M(X)$ such that $\gamma \leq A_\alpha$ for every $\alpha$

Therefore $\gamma \leq \cap \alpha A_\alpha$.

Therefore $f - \text{cl}(f - \text{int} \gamma) \leq f - \text{cl}(f - \text{int} \cap \alpha A_\alpha)$.

i.e., $\gamma \leq f - \text{cl}(f - \text{int} \cap \alpha A_\alpha)$

i.e., $\gamma \leq \beta$. Therefore, $\beta$ is the greatest lower bound.

Hence, $\cap \alpha A_\alpha = f - \text{cl}(f - \text{int}(\cap \alpha A_\alpha))$.

2. Now put $\delta = f - \text{cl}(f - \text{int}(\cup \alpha A_\alpha))$. Then $\delta \in \text{FRC}_M(X)$

$\cup \alpha A_\alpha \geq A_\alpha$ for every $\alpha$

i.e., $f - \text{cl}(f - \text{int}(A_\alpha)) \leq f - \text{cl}(f - \text{int}(\cup \alpha A_\alpha))$. i.e., $A_\alpha \leq \delta$.

Therefore, $\delta$ is an upper bound of $A_\alpha$

Let $\eta \in \text{FRC}_M(X)$ such that $A_\alpha \leq \eta$ for every $\alpha$. Then $(\cup A_\alpha) \leq \eta$.

Therefore, $f - \text{cl}(f - \text{int}(\cup \alpha A_\alpha)) \leq f - \text{cl}(f - \text{int} \eta) = \eta$.

i.e., $\delta \leq \eta$.

Therefore $\delta$ is the least upper bound.

Therefore, $\vee \alpha A_\alpha = f - \text{cl}(f - \text{int}(\cup \alpha A_\alpha))$.

Therefore, $\text{FRC}_M(X)$ is a complete lattice.

Note 3. Similarly we can prove that $\text{FRO}_M(X)$ is a complete lattice.

4. Conclusion

As $\text{FRC}_M(X)$ and $\text{FRO}_M(X)$ are complete lattices, they can be used as the basic structure for the construction of topological spaces such as fuzzy extensions and fuzzy absolutes.

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